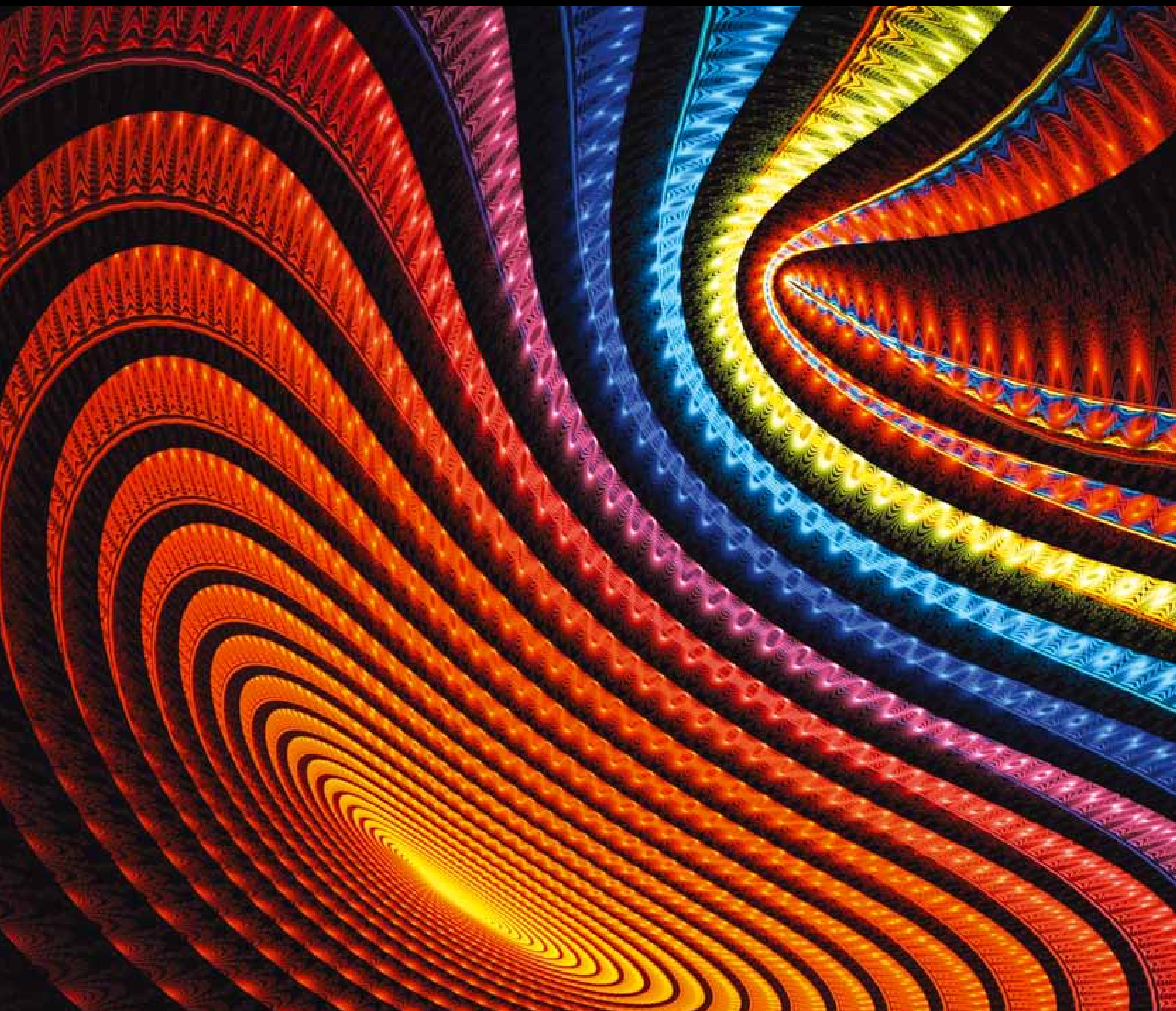
Abstract and Applied Analysis

Analysis of Nonlinear Dynamics of Neural Networks

Guest Editors: Sabri Arik, Juhyun Park, Tingwen Huang, and José J. Oliveira



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Editorial

Analysis of Nonlinear Dynamics of Neural Networks

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Dynamical neural networks proved to be an important tool to solve some practical engineering problems in the areas such as optimization, image and signal processing, control systems, associative memories, and pattern recognition. When employing neural networks to solve practical engineering problems, it is crucial to be able to completely characterize the dynamical properties of neural networks. There are many various classes of neural network models that can be described in the form of nonlinear systems. Therefore, neural networks may exhibit extremely different complex dynamical behaviors depending on the model and network parameters. Hence, the analysis of nonlinear dynamics of neural networks still possesses new challenges to researchers.

The aim of this special issue is to solicit theoretical and application research in the fields of neural networks exploiting their nonlinear dynamics. We believe it provides a good opportunity for reflection on current developments in the nonlinear analysis of dynamical behaviors of neural networks. The papers submitted to this special issue represent a mixture of cross-cutting investigations and provide deep insight into the current developments in the field. The accepted papers in this special issue addressed the following topics:

(i) stability analysis of dynamical neural networks,

(ii) almost periodic solution of of neutral-type neural networks,

(iii) impulsive control of stochastic synchronization of reaction-diffusion neural networks,

(iv) neural network model for predicting peak ground acceleration,

(v) dynamical analysis of high-order neural networks,

(vi) synchronization of nonlinear coupled complex networks.

As we mentioned above, the special issue aimed to reveal new ideas in the area of nonlinear dynamics of neural networks, which would be helpful for the scientists and researchers who share the common interest in neural networks. We hope that the readers will agree with us that the published papers reflect convincingly the issue's objectives with its variety of presented topics, investigated at both theoretical and application levels.

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Research Article

Exponential Stability and Periodicity of Fuzzy Delayed Reaction-Diffusion Cellular Neural Networks with Impulsive Effect

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This paper considers dynamical behaviors of a class of fuzzy impulsive reaction-diffusion delayed cellular neural networks (FIRDDCNNs) with time-varying periodic self-inhibitions, interconnection weights, and inputs. By using delay differential inequality, M-matrix theory, and analytic methods, some new sufficient conditions ensuring global exponential stability of the periodic FIRDDCNN model with Neumann boundary conditions are established, and the exponential convergence rate index is estimated. The differentiability of the time-varying delays is not needed. An example is presented to demonstrate the efficiency and effectiveness of the obtained results.

1. Introduction

The fuzzy cellular neural networks (FCNNs) model, which combines fuzzy logic with the structure of traditional neural networks (CNNs) [1–3], has been proposed by Yang et al. [4, 5]. Unlike previous CNNs structures, the FCNNs model has fuzzy logic between its template and input and/or output besides the "sum of product" operation. Studies have shown that the FCNNs model is a very useful paradigm for image processing and pattern recognition [6–8]. These applications heavily depend on not only the dynamical analysis of equilibrium points but also on that of the periodic oscillatory solutions. In fact, the human brain is naturally in periodic oscillatory [9], and the dynamical analysis of periodic oscillatory solutions is very important in learning theory [10, 11], because learning usually requires repetition. Moreover, an equilibrium point can be viewed as a special periodic solution of neural networks with arbitrary period. Stability analysis problems for FCNNs with and without delays have recently been probed; see [12–22] and the references therein. Yuan et al. [13] have investigated stability of FCNNs by linear matrix inequality approach, and several criteria have been provided for checking the periodic solutions for FCNNs with time-varying delays. Huang [14] has probed exponential stability of fuzzy cellular neural networks with distributed delay, without considering reaction-diffusion effects.

Strictly speaking, reaction-diffusion effects cannot be neglected in both biological and man-made neural networks [19–32], especially when electrons are moving in noneven electromagnetic field. In [19], stability is considered for FCNNs with diffusion terms and time-varying delay. Wang and Lu [20] have probed global exponential stability of FCNNs with delays and reaction-diffusion terms. Song and Wang [21] have studied dynamical behaviors of fuzzy reaction-diffusion periodic cellular neural networks with variable coefficients and delays without considering pulsing effects. Wang et al. [22] have discussed exponential stability of impulsive stochastic fuzzy reaction-diffusion Cohen-Grossberg neural networks with mixed delays. Zhao and Mao [30] have investigated boundedness and stability of nonautonomous cellular neural networks with reaction-diffusion terms. Zhao and Wang [31] have considered existence of periodic oscillatory solution of reaction-diffusion neural networks with delays without fuzzy logic and impulsive effect.

As we all know, many practical systems in physics, biology, engineering, and information science undergo abrupt changes at certain moments of time because of impulsive inputs [33]. Impulsive differential equations and impulsive neural networks have been received much interest in recent years; see, for example, [34–42] and the references therein. Yang and Xu [36] have investigated existence and exponential stability of periodic solution for impulsive delay differential equations and applications. Li and Lu [38] have discussed global exponential stability and existence of periodic solution of Hopfield-type neural networks with impulses without reaction-diffusion. To the best of our knowledge, few authors have probed the existence and exponential stability of the periodic solutions for the FIRDDCNN model with variable coefficients, and time-varying delays. As a result of the simultaneous presence of fuzziness, pulsing effects, reaction-diffusion phenomena, periodicity, variable coefficients and delays, the dynamical behaviors of this kind of model become much more complex and have not been properly addressed, which still remain important and challenging.

Motivated by the above discussion, we will establish some sufficient conditions for the existence and exponential stability of periodic solutions of this kind of FIRDDCNN model, applying delay differential inequality, M-matrix theory, and analytic methods. An example is employed to demonstrate the usefulness of the obtained results.

Notations. Throughout this paper, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n-dimensional Euclidean space and the set of all $n\times m$ real matrices. The superscript "T" denotes matrix transposition and the notation $X \geq Y$ (resp., $X > Y$), where X and Y are symmetric matrices, means that $X - Y$ is positive semidefinite (resp., positive definite). $\Omega = \{x = (x_1,\dots,x_m)^{\mathrm{T}}, |x_i| < \mu\}$ is a bounded compact set in space $\mathbb{R}^m$ with smooth boundary $\partial\Omega$ and measure $\operatorname{mes}\Omega > 0$; Neumann boundary condition $\partial u_i/\partial n = 0$ is the outer normal to $\partial\Omega$; $L^2(\Omega)$ is the space of real functions Ω which are L^2 for the Lebesgue measure. It is a Banach space with the norm $\|u(t,x)\|_2 = (\sum_{i=1}^{n}\|u_i(t,x)\|_2^2)^{1/2}$, where $u(t,x) = (u_1(t,x),\dots,u_n(t,x))^{\mathrm{T}}$, $\|u_i(t,x)\|_2 = (\int_\Omega |u_i(t,x)|^2dx)^{1/2}$, $|u(t,x)| = (|u_1(t,x)|,\dots,|u_n(t,x)|)^{\mathrm{T}}$. For function $g(x)$ with positive period ω, we denote $\overline{g} = \max_{t\in[0,\omega]}g(t)$, $\underline{g} = \min_{t\in[0,\omega]}g(t)$. Sometimes, the arguments of a function or a matrix will be omitted in the analysis when no confusion can arise.

2. Preliminaries

Consider the impulsive fuzzy reaction-diffusion delayed cellular neural networks (FIRDDCNN) model:

$$\begin{aligned}
\frac{\partial u_i(t,x)}{\partial t} &= \sum_{l=1}^{m}\frac{\partial}{\partial x_l}\left(D_i\frac{\partial u_i(t,x)}{\partial x_l}\right) \\
&\quad - c_i(t)u_i(t,x) + \sum_{j=1}^{n}a_{ij}(t)f_j\left(u_j(t,x)\right) \\
&\quad + \sum_{j=1}^{n}b_{ij}(t)v_j(t) + J_i(t) \\
&\quad + \bigwedge_{j=1}^{n}\alpha_{ij}(t)g_j\left(u_j\left(t-\tau_j(t),x\right)\right) \\
&\quad + \bigvee_{j=1}^{n}\beta_{ij}(t)g_j\left(u_j\left(t-\tau_j(t),x\right)\right) \\
&\quad + \bigwedge_{j=1}^{n}T_{ij}(t)v_j(t) + \bigvee_{j=1}^{n}H_{ij}(t)v_j(t), \\
&\qquad t\neq t_k,\ x\in\Omega, \\
u_i(t_k^+,x) - u_i(t_k^-,x) &= I_{ik}\left(u_i(t_k^-,x)\right), \\
&\qquad t = t_k,\ k\in\mathbb{Z}_+,\ x\in\Omega, \\
\frac{\partial u_i(t,x)}{\partial n} &= 0, \quad t\geq t_0,\ x\in\partial\Omega, \\
u_i(t_0+s,x) &= \psi_i(s,x), \quad -\tau_j\leq s\leq 0,\ x\in\Omega,
\end{aligned} \tag{1}$$

where $n \geq 2$ is the number of neurons in the network and $u_i(t,x)$ corresponds to the state of the ith neuron at time t and in space x; $D = \operatorname{diag}(D_1, D_2,\dots,D_n)$ is the diffusion-matrix and $D_i \geq 0$; $\Delta = \sum_{k=1}^{m}(\partial^2/\partial x_k^2)$ is the Laplace operator; $f_j(u_j(t,x))$ denotes the activation function of the jth unit and $v_j(t)$ the activation function of the jth unit; $J_i(t)$ is an input at time t; $c_i(t) > 0$ represents the rate with which the ith unit will reset its potential to the resting state in isolation when disconnected from the networks and external inputs at time t; $a_{ij}(t)$ and $b_{ij}(t)$ are elements of feedback template and feed forward template at time t, respectively. Moreover, in model (1), $\alpha_{ij}(t)$, $\beta_{ij}(t)$, $T_{ij}(t)$, and $H_{ij}(t)$ are elements of fuzzy feedback MIN template, fuzzy feedback MAX template, fuzzy feed forward MIN template, and fuzzy feed forward MAX template at time t, respectively; the symbols "$\bigwedge$" and "$\bigvee$" denote the fuzzy AND and fuzzy OR operation, respectively; time-varying delay $\tau_j(t)$ is the transmission delay along the axon of the jth unit and satisfies $0 \leq \tau_j(t) \leq \tau_j$ (τ_j is a constant); the initial condition $\phi_i(s,x)$ is bounded and continuous on $[-\tau,0]\times\Omega$, where $\tau = \max_{1\leq j\leq n}\tau_j$. The fixed moments t_k satisfy $0 = t_0 < t_1 < t_2\dots$, $\lim_{k\to+\infty}t_k = +\infty$, $k\in\mathbb{N}$. $u_i(t_k^+,x)$ and $u_i(t_k^-,x)$ denote the right-hand and left-hand limits at t_k, respectively. We always assume $u_i(t_k^+,x) = u_i(t_k,x)$, for all $k\in N$. The initial value functions $\psi(s,x)$ belong to $\mathrm{PC}_\Omega([-\tau,0]\times\Omega;R^n)$. $\mathrm{PC}_\Omega(J\times\Omega, L^2(\Omega)) = \{\psi : J\times\Omega \to L^2(\Omega) \mid$ for every $t\in J$, $\psi(t,x)\in L^2(\Omega)$; for any fixed $x\in\Omega$, $\psi(t,x)$ is continuous for all but at most countable points $s\in J$ and at these points, $\psi(s^+,x)$ and $\psi(s^-,x)$ exist, $\psi(s^+,x) = \psi(s^-,x)\}$, where $\psi(s^+,x)$ and $\psi(s^-,x)$ denote the right-hand and left-hand limit of the function $\psi(s,x)$, respectively. Especially, let $\mathrm{PC}_\Omega = \mathrm{PC}([-\tau,0]\times\Omega, L^2(\Omega))$. For any $\psi(t,x) = (\psi_1(t,x),\dots,\psi_n(t,x))\in\mathrm{PC}_\Omega$, suppose that $|\psi_i(t,x)|_\tau = \sup_{-\tau<s\leq 0}|\psi_i(t+s,x)|$ exists as a finite number

and introduce the norm $\|\psi(t)\|_2 = (\sum_{i=1}^{n} \|\psi_i(t)\|_2^2)^{1/2}$, where $\|\psi_i(t)\|_2 = (\int_\Omega |\psi_i(t,x)|^2 dx)^{1/2}$.

Throughout the paper, we make the following assumptions.

(**H1**) There exists a positive diagonal matrix $F = \operatorname{diag}(F_1, F_2, \ldots, F_n)$, and $G = \operatorname{diag}(G_1, G_2, \ldots, G_n)$ such that

$$F_j = \sup_{x \neq y} \left| \frac{f_j(x) - f_j(y)}{x - y} \right|, \qquad G_j = \sup_{x \neq y} \left| \frac{g_j(x) - g_j(y)}{x - y} \right| \tag{2}$$

for all $x \neq y$, $j = 1, 2, \ldots, n$.

(**H2**) $c_i(t) > 0$, $a_{ij}(t), b_{ij}(t), \alpha_{ij}(t), \beta_{ij}(t), T_{ij}(t), H_{ij}(t), v_i(t)$, $I_i(t)$, and $\tau_j(t) \geq 0$ are periodic function with a common positive period ω for all $t \geq t_0$, $i, j = 1, 2, \ldots, n$.

(**H3**) For $\omega > 0$, $i = 1, 2, \ldots, n$, there exists $q \in Z_+$ such that $t_k + \omega = t_{k+q}$, $I_{ik}(u_i) = I_{i(k+q)}(u_i)$ and $I_{ik}(u_i(t_k, x))$ are Lipschitz continuous in $\mathbb{R}^n$.

Definition 1. The model in (1) is said to be globally exponentially periodic if (i) there exists one ω-periodic solution and (ii) all other solutions of the model converge exponentially to it as $t \to +\infty$.

Definition 2 (see [26]). Let $\mathbf{C} = ([t-\tau, t], \mathbb{R}^n)$, where $\tau \geq 0$ and $F(t, x, y) \in \mathbf{C}(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbf{C}, \mathbb{R}^n)$. Then the function $F(t, x, y) = (f_1(t, x, y), f_2(t, x, y), \ldots, f_n(t, x, y))^{\mathrm{T}}$ is called an M-function, if (i) for every $t \in \mathbb{R}^+$, $x \in \mathbb{R}^n$, $y^{(1)} \in \mathbf{C}$, there holds $F(t, x, y^{(1)}) \leq F(t, x, y^{(2)})$, for $y^{(1)} \leq y^{(2)}$, where $y^{(1)} = (y_1^{(1)}, \ldots, y_n^{(1)})^{\mathrm{T}}$ and $y^{(2)} = (y_1^{(2)}, \ldots, y_n^{(2)})^{\mathrm{T}}$; (ii) every ith element of F satisfies $f_i(t, x^{(1)}, y) \leq f_i(t, x^{(2)}, y)$ for any $y \in \mathbf{C}$, $t \geq t_0$, where arbitrary $x^{(1)}$ and $x^{(2)}$ ($x^{(1)} \leq x^{(2)}$) belong to $\mathbb{R}^n$ and have the same ith component $x_i^{(1)} = x_i^{(2)}$. Here, $x^{(1)} = (x_1^{(1)}, \ldots, x_n^{(1)})^{\mathrm{T}}$, $x^{(2)} = (x_1^{(2)}, \ldots, x_n^{(2)})^{\mathrm{T}}$.

Definition 3 (see [26]). A real matrix $A = (a_{ij})_{n \times n}$ is said to be a nonsingular M-matrix if $a_{ij} \leq 0$ $(i \neq j; i, j = 1, \ldots, n)$ and all successive principal minors of A are positive.

Lemma 4 (see [13]). *Let u and u^* be two states of the model in* (1)*, then we have*

$$\left| \bigwedge_{j=1}^{n} \alpha_{ij}(t) f_j(u_j) - \bigwedge_{j=1}^{n} \alpha_{ij}(t) f_j(u_j^*) \right| \leq \sum_{j=1}^{n} |\alpha_{ij}(t)| \cdot |f_j(u_j) - f_j(u_j^*)|,$$

$$\left| \bigvee_{j=1}^{n} \beta_{ij}(t) f_j(u_j) - \bigvee_{j=1}^{n} \beta_{ij}(t) f_j(u_j^*) \right| \leq \sum_{j=1}^{n} |\beta_{ij}(t)| \cdot |f_j(u_j) - f_j(u_j^*)|. \tag{3}$$

Lemma 5 (see [26]). *Assume that $F(t, x, y)$ is an M-function, and (i) $x(t) < y(t)$, $t \in [t-\tau, t_0]$, (ii) $D^+ y(t) > F(t, y(t), y^s(t))$, $D^+ x(t) \leq F(t, x(t), x^s(t))$, $t \geq t_0$, where $x^s(t) = \sup_{-\tau \leq s \leq 0} x(t+s)$, $y^s(t) = \sup_{-\tau \leq s \leq 0} y(t+s)$. Then $x(t) < y(t)$, $t \geq t_0$.*

3. Main Results and Proofs

We should first point out that, under assumptions (**H1**), (**H2**), and (**H3**), the FIRDDCNN model (1) has at least one ω-periodic solution of [26]. The proof of the existence of the ω-periodic solution of (1) can be carried out similar to [26, 28] by the nonlinear functional analysis methods such as topological degree and here is omitted. We will mainly discuss the uniqueness of the periodic solution and its exponential stability.

Theorem 6. *Assume that* (**H1**)–(**H3**) *holds. Furthermore, assume that the following conditions hold*

(**H4**) *$\underline{C} - \overline{A}F - (\overline{\alpha} + \overline{\beta})G$ is a nonsingular M-matrix.*

(**H5**) *The impulsive operators $h_k(u) = u + I_k(u)$ is Lipschitz continuous in $\mathbb{R}^n$; that is, there exists a nonnegative diagnose matrix $\Gamma_k = \operatorname{diag}(\gamma_{1k}, \ldots, \gamma_{nk})$ such that $|h_k(u) - h_k(u^*)| \leq \Gamma_k |u - u^*|$ for all $u, u^* \in \mathbb{R}^n$, $k \in N^+$, where $|h_k(u)| = (|h_{1k}(u_1)|, \ldots, |h_{nk}(u_n)|)^{\mathrm{T}}$, $I_k(u) = (I_{1k}(u_1), \ldots, I_{nk}(u_n))^{\mathrm{T}}$.*

(**H6**) *$\eta = \sup_{k \in N^+} \{\ln \eta_k / (t_k - t_{k-1})\} < \lambda$, where $\eta_k = \max_{1 \leq i \leq n} \{1, \gamma_{ik}\}$, $k \in N^+$.*

Then the model (1) is global exponential periodic and the exponential convergence rate index $\lambda - \eta$ and λ can be estimated by

$$\xi_i (\lambda - \underline{c}_i) + \sum_{j=1}^{n} \xi_j \left(|\overline{a}_{ij}| F_j + e^{\tau\lambda} \left(|\overline{\alpha}_{ij}| + |\overline{\beta}_{ij}| \right) G_j \right) < 0 \tag{4}$$

$$i = 1, \ldots, n,$$

where $\underline{C} = \operatorname{diag}(\underline{c}_1, \ldots, \underline{c}_n)$ and $\xi_i > 0$, $\overline{A} = (|\overline{a}_{ij}|)_{n \times n}$, $\overline{\alpha} = (|\overline{\alpha}_{ij}|)_{n \times n}$, $\overline{\beta} = (|\overline{\beta}_{ij}|)_{n \times n}$, satisfies $-\xi_i \underline{c}_i + \sum_{j=1}^{n} \xi_i (|\overline{a}_{ij}| F_i + (|\overline{\alpha}_{ij}| + |\overline{\beta}_{ij}|) G_i) < 0$.

Proof. For any $\phi, \psi \in \mathrm{PC}_\Omega$, let $u(t, x, \phi) = (u_1(t, x, \phi), \ldots, u_n(t, x, \phi))^{\mathrm{T}}$ be a periodic solution of the system (1) starting from ϕ and $u(t,x,\psi)=(u_1(t,x,\psi),\dots,u_n(t,x,\psi))^{\mathrm{T}}$, a solution of the system (1) starting from ψ. Define

$$u_t(\phi,x)=u(t+s,x,\phi),\qquad u_t(\psi,x)=u(t+s,x,\psi),\quad s\in[-\tau,0],\tag{5}$$

and we can see that $u_t(\phi,x),u_t(\psi,x)\in \mathrm{PC}_\Omega$ for all $t>0$. Let $U_i=u_i(t,x,\phi)-u_i(t,x,\psi)$, then from (1) we get

$$\begin{aligned}\frac{\partial U_i}{\partial t}&=\sum_{l=1}^{m}\frac{\partial}{\partial x_l}\left(D_i\frac{\partial U_i}{\partial x_l}\right)-c_i(t)U_i+\sum_{j=1}^{n}a_{ij}(t)\\&\quad\times\left[f_j\left(u_j(t,x,\phi)\right)-f_j\left(u_j(t,x,\psi)\right)\right]\\&\quad+\left[\bigwedge_{j=1}^{n}\alpha_{ij}(t)f_j\left(u_j\left(t-\tau_j(t),x,\phi\right)\right)-\bigwedge_{j=1}^{n}\alpha_{ij}(t)f_j\left(u_j\left(t-\tau_j(t),x,\psi\right)\right)\right]\\&\quad+\left[\bigvee_{j=1}^{n}\beta_{ij}(t)f_j\left(u_j\left(t-\tau_j(t),x,\phi\right)\right)-\bigvee_{j=1}^{n}\beta_{ij}(t)f_j\left(u_j\left(t-\tau_j(t),x,\psi\right)\right)\right]\end{aligned}\tag{6}$$

for all $t\neq t_k$, $x\in\Omega$, $i=1,\dots,n$.

Multiplying both sides of (6) by U_i and integrating it in Ω, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega U_i^2\mathrm{d}x&=\int_\Omega U_i\sum_{l=1}^{m}\frac{\partial}{\partial x_l}\left(D_i\frac{\partial U_i}{\partial x_l}\right)\mathrm{d}x\\&\quad-c_i(t)\int_\Omega U_i^2dx+\sum_{j=1}^{n}a_{ij}(t)\int_\Omega U_i\\&\quad\times\left[f_j\left(u_j(t,x,\phi)-f_j\left(u_j(t,x,\psi)\right)\right)\right]\mathrm{d}x\\&\quad+\int_\Omega U_i\left[\bigwedge_{j=1}^{n}\alpha_{ij}(t)f_j\left(u_j\left(t-\tau_j(t),x,\phi\right)\right)-\bigwedge_{j=1}^{n}\alpha_{ij}(t)f_j\left(u_j\left(t-\tau_j(t),x,\psi\right)\right)\right]\mathrm{d}x\\&\quad+\int_\Omega U_i\left[\bigvee_{j=1}^{n}\beta_{ij}(t)f_j\left(u_j\left(t-\tau_j(t),x,\phi\right)\right)-\bigvee_{j=1}^{n}\beta_{ij}(t)f_j\left(u_j\left(t-\tau_j(t),x,\psi\right)\right)\right]\mathrm{d}x\end{aligned}\tag{7}$$

for $t\neq t_k$, $x\in\Omega$, $i=1,\dots,n$. By boundary condition and Green Formula, we can get

$$\int_\Omega U_i\sum_{l=1}^{m}\frac{\partial}{\partial x_l}\left(D_i\frac{\partial U_i}{\partial x_l}\right)dx\le -D_i\int_\Omega(\nabla U_i)^2\mathrm{d}x.\tag{8}$$

Then, from (8), (9), **(H1)-(H2)**, Lemma 4, and the Holder inequality,

$$\begin{aligned}\frac{d}{dt}\|U_i\|_2^2&\le -2\underline{c}_i\|U_i\|_2^2+2\sum_{j=1}^{n}\left|\overline{a}_{ij}\right|F_j\|U_i\|_2\left\|U_j\right\|_2\\&\quad+2\sum_{j=1}^{n}\left(\left|\overline{\alpha}_{ij}\right|+\left|\overline{\beta}_{ij}\right|\right)G_j\|U_i\|_2\\&\quad\times\left\|u_j\left(t-\tau_j(t),x,\phi\right)-u_j\left(t-\tau_j(t),x,\psi\right)\right\|_2,\quad t\neq t_k.\end{aligned}\tag{9}$$

Thus,

$$\begin{aligned}D^+\|U_i\|_2&\le -\underline{c}_i\|U_i\|_2+\sum_{j=1}^{n}\left|\overline{\alpha}_{ij}\right|F_j\left\|U_j\right\|_2\\&\quad+\sum_{j=1}^{n}\left(\left|\overline{\alpha}_{ij}\right|+\left|\overline{\beta}_{ij}\right|\right)G_j\\&\quad\times\left\|u_j\left(t-\tau_j(t),x,\phi\right)-u_j\left(t-\tau_j(t),x,\psi\right)\right\|_2,\quad t\neq t_k\end{aligned}\tag{10}$$

for $i=1,\dots,n$. Since $\underline{C}-(\overline{A}F+(\overline{\alpha}+\overline{\beta})G)$ is a nonsingular M-matrix, there exists a vector $\xi=(\xi_1,\dots,\xi_n)^{\mathrm{T}}>0$ such that

$$-\xi_i\underline{c}_i+\sum_{j=1}^{n}\xi_j\left(\left|\overline{a}_{ij}\right|F_j+\left(\left|\overline{\alpha}_{ij}\right|+\left|\overline{\beta}_{ij}\right|\right)G_j\right)<0.\tag{11}$$

Considering functions

$$\begin{aligned}\Psi_i(y)&=\xi_i(y-\underline{c}_i)\\&\quad+\sum_{j=1}^{n}\xi_j\left(\left|\overline{a}_{ij}\right|F_j+e^{\tau y}\left(\left|\overline{\alpha}_{ij}\right|+\left|\overline{\beta}_{ij}\right|\right)G_j\right),\quad i=1,\dots,n,\end{aligned}\tag{12}$$

we know from (11) that $\Psi_i(0)<0$ and $\Psi_i(y)$ is continuous. Since $\mathrm{d}\Psi_i(y)/\mathrm{d}y>0$, $\Psi_i(y)$ is strictly monotonically increasing, there exists a scalar $\lambda_i>0$ such that

$$\begin{aligned}\Psi_i(\lambda_i)&=\xi_i(\lambda_i-\underline{c}_i)\\&\quad+\sum_{j=1}^{n}\xi_j\left(\left|\overline{a}_{ij}\right|F_j+e^{\tau\lambda_i}\left(\left|\overline{\alpha}_{ij}\right|+\left|\overline{\beta}_{ij}\right|\right)G_j\right)=0,\quad i=1,\dots,n.\end{aligned}\tag{13}$$

Choosing $0 < \lambda < \min\{\lambda_1, \ldots, \lambda_n\}$, we have

$$\xi_i(\lambda_i - \underline{c}_i) + \sum_{j=1}^{n} \xi_j \left(|\overline{a}_{ij}| F_j + e^{\tau\lambda_i} \left(|\overline{\alpha}_{ij}| + |\overline{\beta}_{ij}| \right) G_j \right) < 0, \quad i = 1, \ldots, n. \tag{14}$$

That is,

$$\lambda\xi - \left(\underline{C} - \overline{A}F \right) \xi + \left(\overline{\alpha} + \overline{\beta} \right) G\xi e^{-\lambda t} < 0. \tag{15}$$

Furthermore, choose a positive scalar p large enough such that

$$pe^{-\lambda t}\xi > (1, 1, \ldots, 1)^{\mathrm{T}}, \quad t \in [-\tau, 0]. \tag{16}$$

For any $\varepsilon > 0$, let

$$r(t) = pe^{-\lambda t} \left(\|\phi - \psi\|_2 + \varepsilon \right) \xi, \quad t_0 \le t < t_1. \tag{17}$$

From (15)–(17), we obtain

$$\begin{aligned} D^+ r(t) &> -\left(\underline{C} - \overline{A}F \right) r(t) + \left(\overline{\alpha} + \overline{\beta} \right) G r^s(t) \\ &=: V\left(t, r(t), r^s(t)\right), \quad t_0 \le t < t_1, \end{aligned} \tag{18}$$

where $r^s(t) = (r_1^s(t), \ldots, r_n^s(t))^{\mathrm{T}}$ and $r_i^s(t) = \sup_{-\tau \le s \le 0} pe^{-\lambda(t+s)} (\|\phi - \varphi\|_2 + \varepsilon)\xi_i$. It is easy to verify that $V(t, r(t), r^s(t))$ is an M-function. It follows also from (16) and (17) that

$$\begin{aligned} \|U_i\|_2 &\le \|\phi - \varphi\|_2 < pe^{-\lambda t}\xi_i \|\phi - \varphi\|_2 \\ &< r_i(t), \quad t \in [-\tau, 0], \ i = 1, 2, \ldots, n. \end{aligned} \tag{19}$$

Denote

$$\begin{aligned} U^{\diamond} &:= \left(\|u_1(t, x, \phi) - u_1(t, x, \psi)\|_2, \ldots, \|u_n(t, x, \phi) - u_n(t, x, \psi)\|_2 \right)^{\mathrm{T}}, \\ U^{\diamond(s)} &:= \left(\|u_1(t, x, \phi) - u_1(t, x, \psi)\|_2^{(s)}, \ldots, \|u_n(t, x, \phi) - u_n(t, x, \psi)\|_2^{(s)} \right)^{\mathrm{T}}, \end{aligned} \tag{20}$$

where $\|U_i\|_2^{(s)} = \sup_{-\tau \le s \le 0} \|u_i(t+s, x, \phi) - u_i(t+s, x, \psi)\|_2$, then

$$U^{\diamond} < r(t), \quad t \in [-\tau, 0]. \tag{21}$$

From (10), we can obtain

$$\begin{aligned} D^+ U^{\diamond} &\le -\left(\underline{C} - \overline{A}F \right) U^{\diamond} + \left(\overline{\alpha} + \overline{\beta} \right) G U^{\diamond(s)} \\ &= V\left(t, U^{\diamond}, U^{\diamond(s)}\right), \quad t \ne t_k. \end{aligned} \tag{22}$$

Now, it follows from (18)–(22) and Lemma 5 that

$$U^{\diamond} < r(t) = pe^{-\lambda t} \left(\|\phi - \psi\|_2 + \varepsilon \right) \xi, \quad t_0 \le t < t_1. \tag{23}$$

Letting $\varepsilon \to 0$, we have

$$U^{\diamond} \le p\xi \|\phi - \psi\|_2 e^{-\lambda t}, \quad t_0 \le t < t_1. \tag{24}$$

And moreover, from (24), we get

$$\left(\sum_{i=1}^{n} \|U_i\|_2^2 \right)^{1/2} \le p \left(\sum_{i=1}^{n} \xi_i^2 \right)^{1/2} \|\phi - \psi\|_2 e^{-\lambda t}, \quad t_0 \le t < t_1. \tag{25}$$

Let $\widetilde{M} = p(\sum_{i=1}^{n} \xi_i^2)^{1/2}$, then $\widetilde{M} \ge 1$. Define $W(t) = \|u_t(x, \phi) - u_t(x, \psi)\|_2$; it follows from (25) and the definitions of $u_t(\phi, x)$ and $u_t(\psi, x)$ that

$$\begin{aligned} W(t) &= \|u_t(x, \phi) - u_t(x, \psi)\|_2 \\ &\le \widetilde{M} \|\phi - \psi\|_2 e^{-\lambda t}, \quad t_0 \le t < t_1. \end{aligned} \tag{26}$$

It is easily observed that

$$W(t) \le \widetilde{M} \|\phi - \psi\|_2 e^{-\lambda t}, \quad -\tau \le t \le t_0 = 0. \tag{27}$$

Because (26) holds, we can suppose that for $l \le k$ inequality

$$W(t) \le \eta_0 \cdots \eta_{l-1} \widetilde{M} \|\phi - \psi\|_2 e^{-\lambda t}, \quad t_{l-1} \le t < t_l \tag{28}$$

holds, where $\eta_0 = 1$. When $l = k + 1$, we note **(H5)** that

$$\begin{aligned} W(t_k) &= \left\| u_{t_k}(x, \phi) - u_{t_k}(x, \psi) \right\|_2 \\ &= \left\| h_k\left(u_{t_k}^-(x, \phi) \right) - h_k\left(u_{t_k}^-(x, \psi) \right) \right\|_2 \\ &\le \rho\left(\Gamma_k^2 \right) \left\| u_{t_k}^-(x, \phi) - u_{t_k}^-(x, \psi) \right\|_2 \\ &= \rho\left(\Gamma_k^2 \right) W\left(t_k^- \right) \\ &\le \eta_0 \cdots \eta_{l-1} \rho\left(\Gamma_k^2 \right) \widetilde{M} \|\phi - \psi\|_2 e^{-\lambda t_k} \\ &\le \eta_0 \cdots \eta_{l-1} \eta_k \rho\left(\Gamma_k^2 \right) \widetilde{M} \|\phi - \psi\|_2 e^{-\lambda t_k}, \end{aligned} \tag{29}$$

where $\rho(\Gamma_k^2)$ is the spectral radius of Γ_k^2. Let $M = \max\{\widetilde{M}, \rho(\Gamma_k^2)\widetilde{M}\}$, by (28), (29), and $\eta \ge 1$, we obtain

$$W(t) \le \eta_0 \cdots \eta_{l-1} \eta_k M \|\phi - \psi\|_2 e^{-\lambda t}, \quad t_k - \tau \le t \le t_k. \tag{30}$$

Combining (10), (17), (30), and Lemma 5, we get

$$W(t) \le \eta_0 \cdots \eta_{l-1} \eta_k M \|\phi - \psi\|_2 e^{-\lambda t}, \quad t_k \le t < t_{k+1}, \ k \in N^+. \tag{31}$$

Applying mathematical induction, we conclude that

$$W(t) \le \eta_0 \cdots \eta_{l-1} M \|\phi - \psi\|_2 e^{-\lambda t}, \quad t_{k-1} \le t < t_k, \ k \in N^+. \tag{32}$$

From **(H6)** and (32), we have

$$\begin{aligned} W(t) &\le e^{\eta t_1} e^{\eta(t_2-t_1)} \cdots e^{\eta(t_{k-1}-t_{k-2})} \\ &\quad \times M\|\phi-\psi\|_2 e^{-\lambda t} \le M\|\phi-\psi\|_2 e^{\eta t} e^{-\lambda t} \\ &= M\|\phi-\psi\|_2 e^{-(\lambda-\eta)t}, \\ &\qquad t_{k-1} \le t < t_k,\ k \in N^+. \end{aligned} \tag{33}$$

This means that

$$\begin{aligned} &\|u_t(x,\phi)-u_t(x,\psi)\|_2 \\ &\quad \le M\|\phi-\psi\|_2 e^{-(\lambda-\eta)t} \\ &\quad \le M\|\phi-\psi\|_2 e^{-(\lambda-\eta)(t-\tau)}, \quad t \ge t_0, \end{aligned} \tag{34}$$

choosing a positive integer N such that

$$Me^{-(\lambda-\eta)(N\omega-\tau)} \le \frac{1}{6}. \tag{35}$$

Define a Poincare mapping $\mathfrak{D} : \Gamma \to \Gamma$ by

$$\mathfrak{D}(\phi) = u_\omega(x,\phi), \tag{36}$$

Then

$$\mathfrak{D}^N(\phi) = u_{N\omega}(x,\phi). \tag{37}$$

Setting $t = N\omega$ in (34), from (35) and (37), we have

$$\left\|\mathfrak{D}^N(\phi)-\mathfrak{D}^N(\psi)\right\|_2 \le \frac{1}{6}\|\phi-\psi\|_2, \tag{38}$$

which implies that $\mathfrak{D}^N$ is a contraction mapping. Thus, there exists a unique fixed point $\phi^* \in \Gamma$ such that

$$\mathfrak{D}^N(\mathfrak{D}(\phi^*)) = \mathfrak{D}\left(\mathfrak{D}^N(\phi^*)\right) = \mathfrak{D}(\phi^*). \tag{39}$$

From (37), we know that $\mathfrak{D}(\phi^*)$ is also a fixed point of $\mathfrak{D}^N$, and then it follows from the uniqueness of the fixed point that

$$\mathfrak{D}(\phi^*) = \phi^*, \quad \text{that is, } u_\omega(x,\phi^*) = \phi^*. \tag{40}$$

Let $u(t,x,\phi^*)$ be a solution of the model (1), then $u(t+\omega,x,\phi^*)$ is also a solution of the model (1). Obviously,

$$u_{t+\omega}(x,\phi^*) = u_t(u_\omega(x,\phi^*)) = u_t(x,\phi^*), \tag{41}$$

for all $t \ge t_0$. Hence, $u(t+\omega,x,\phi^*) = u(t,x,\phi^*)$, which shows that $u(t,x,\phi^*)$ is exactly one ω-periodic solution of model (1). It is easy to see that all other solutions of model (1) converge to this periodic solution exponentially as $t \to +\infty$, and the exponential convergence rate index is $\lambda-\eta$. The proof is completed. □

Remark 7. When $c_i(t) = c_i$, $a_{ij}(t) = a_{ij}$, $b_{ij}(t) = b_{ij}$, $\alpha_{ij}(t) = \alpha_{ij}$, $\beta_{ij}(t) = \beta_{ij}$, $T_{ij}(t) = T_{ij}$, $H_{ij}(t) = H_{ij}$, $v_i(t) = v_i$, $I_i(t) = I_i$, and $\tau_t = \tau_i(c_i, a_{ij}, b_{ij}, \alpha_{ij}, \beta_{ij}, T_{ij}, H_{ij}, v_i, I_i$, and τ_i are constants), then the model (1) is changed into

$$\begin{aligned} \frac{\partial u_i(t,x)}{\partial t} &= \sum_{l=1}^{m} \frac{\partial}{\partial x_l}\left(D_i \frac{\partial u_i(t,x)}{\partial x_l}\right) \\ &\quad - c_i u_i(t,x) + \sum_{j=1}^{n} a_{ij} f_j(u_j(t,x)) \\ &\quad + \sum_{j=1}^{n} b_{ij} v_j + J_i \\ &\quad + \bigwedge_{j=1}^{n} \alpha_{ij} g_j\left(u_j\left(t-\tau_j(t),x\right)\right) \\ &\quad + \bigvee_{j=1}^{n} \beta_{ij} g_j\left(u_j\left(t-\tau_j(t),x\right)\right) \\ &\quad + \bigwedge_{j=1}^{n} T_{ij} v_j + \bigvee_{j=1}^{n} H_{ij} v_j, \quad t \ne t_k,\ x \in \Omega, \\ u_i(t_k^+,x) - u_i(t_k^-,x) &= I_{ik}(u_i(t_k^-,x)), \\ &\qquad t = t_k,\ k \in \mathbb{Z}_+,\ x \in \Omega, \\ \frac{\partial u_i(t,x)}{\partial n} &= 0, \quad t \ge t_0,\ x \in \partial\Omega, \\ u_i(t_0+s,x) &= \psi_i(s,x), \quad -\tau_j \le s \le 0,\ x \in \Omega. \end{aligned} \tag{42}$$

For any positive constant $\omega \ge 0$, we have $c_i(t+\omega) = c_i(t)$, $a_{ij}(t+\omega) = a_{ij}(t)$, $b_{ij}(t+\omega) = b_{ij}(t)$, $\alpha_{ij}(t+\omega) = \alpha_{ij}(t)$, $\beta_{ij}(t+\omega) = \beta_{ij}(t)$, $T_{ij}(t+\omega) = T_{ij}(t)$, $H_{ij}(t+\omega) = H_{ij}(t)$, $v_i(t+\omega) = v_i(t)$, $I_i(t+\omega) = I_i(t)$, and $\tau_i(t+\omega) = \tau_i(t)$ for $t \ge t_0$. Thus, the sufficient conditions in Theorem 6 are satisfied.

Remark 8. If $I_k(\cdot) = 0$, the model (1) is changed into

$$\begin{aligned} \frac{\partial u_i(t,x)}{\partial t} &= \sum_{l=1}^{m} \frac{\partial}{\partial x_l}\left(D_i \frac{\partial u_i(t,x)}{\partial x_l}\right) \\ &\quad - c_i(t) u_i(t,x) + \sum_{j=1}^{n} a_{ij}(t) f_j(u_j(t,x)) \\ &\quad + \sum_{j=1}^{n} b_{ij}(t) v_j(t) + J_i(t) \\ &\quad + \bigwedge_{j=1}^{n} \alpha_{ij}(t) g_j\left(u_j\left(t-\tau_j(t),x\right)\right) \\ &\quad + \bigvee_{j=1}^{n} \beta_{ij}(t) g_j\left(u_j\left(t-\tau_j(t),x\right)\right) \end{aligned}$$

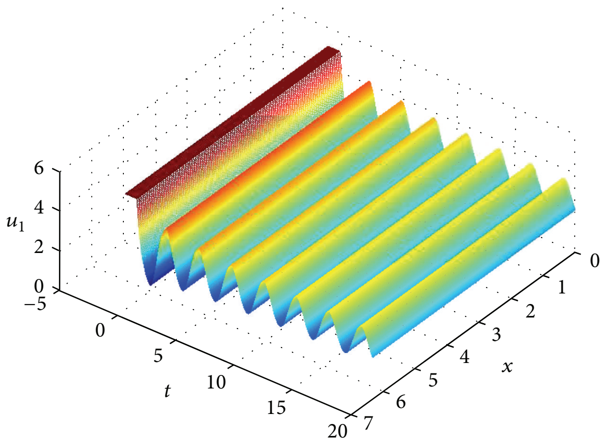

FIGURE 1: State response $u1(t,x)$ of model (44) without impulsive effects.

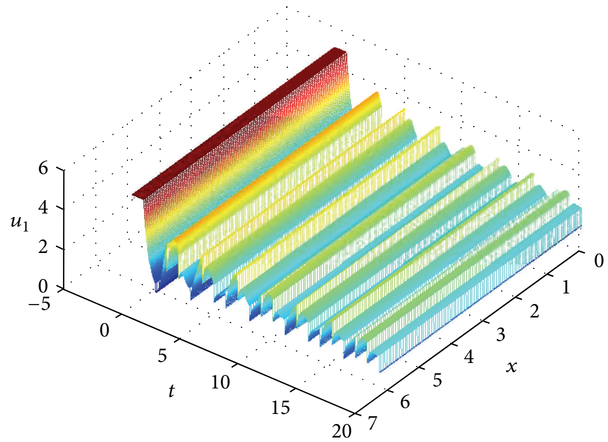

FIGURE 2: State response $u1(t,x)$ of model (44) with impulsive effects.

$$
\begin{aligned}
&+\bigwedge_{j=1}^{n} T_{ij}(t)\,v_j(t)+\bigvee_{j=1}^{n} H_{ij}(t)\,v_j(t),\\
&\qquad t\neq t_k,\ x\in\Omega,\\
&\frac{\partial u_i(t,x)}{\partial n}=0,\quad t\geq t_0,\ x\in\partial\Omega,\\
&u_i\left(t_0+s,x\right)=\psi_i(s,x),\quad -\tau_j\leq s\leq 0,\ x\in\Omega,
\end{aligned}
\tag{43}
$$

which has been discussed in [22]. As Song and Wang have pointed out, the model (43) is more general than some well-studied fuzzy neural networks. For example, when $c_i(t)>0, a_{ij}(t), b_{ij}(t), \alpha_{ij}(t), \beta_{ij}(t), T_{ij}(t), H_{ij}(t), v_i(t)$, and $I_i(t)$ are all constants, the model in (43) reduces the model which has been studied by Huang [19]. Moreover, if $D_i=0$, $\tau_i(t)=0$, $f_i(\theta)=g_i(\theta)=(1/2)(|\theta+1|-|\theta-1|)$, $(i=1,\ldots,n)$, then model (42) covers the model studied by Yang et al. [4, 5] as a special case. If $D_i=0$ and $\tau_j(t)$ is assumed to be differentiable for $i,j=1,2,\ldots,n$, then model (43) can be specialized to the model investigated in Liu and Tang [12] and Yuan et al. [13]. Obviously, our results are less conservative than that of the above-mentioned literature, because they do not consider impulsive effects.

4. Numerical Examples

Example 9. Consider a two-neuron FIRDDCNN model:

$$
\begin{aligned}
\frac{\partial u_i(t,x)}{\partial t}&=\sum_{l=1}^{m}\frac{\partial}{\partial x_l}\left(D_i\frac{\partial u_i(t,x)}{\partial x_l}\right)\\
&\quad-c_i(t)\,u_i(t,x)\\
&\quad+\sum_{j=1}^{2}a_{ij}(t)\,f_j\left(u_j(t,x)\right)\\
&\quad+\sum_{j=1}^{2}b_{ij}(t)\,v_j(t)+J_i(t)\\
&\quad+\bigwedge_{j=1}^{2}\alpha_{ij}(t)\,g_j\left(u_j\left(t-\tau_j(t),x\right)\right)\\
&\quad+\bigvee_{j=1}^{2}\beta_{ij}(t)\,g_j\left(u_j\left(t-\tau_j(t),x\right)\right)\\
&\quad+\bigwedge_{j=1}^{2}T_{ij}(t)\,v_j(t)+\bigvee_{j=1}^{2}H_{ij}(t)\,v_j(t),\\
&\qquad t\neq t_k,\ x\in\Omega,\\
u_i\left(t_k^{+},x\right)&=\left(1-\gamma_{ik}\right)u_i\left(t_k^{-},x\right),\\
&\qquad t=t_k,\ k\in\mathbb{Z}_+,\ x\in\Omega,\\
\frac{\partial u_i(t,x)}{\partial n}&=0,\quad t\geq t_0,\ x\in\partial\Omega,\\
u_i\left(t_0+s,x\right)&=\psi_i(s,x),\\
&\quad-\tau_j\leq s\leq 0,\ x\in\Omega,
\end{aligned}
\tag{44}
$$

where $i=1,2$. $c_1(t)=26$, $c_2(t)=20.8$, $a_{11}(t)=-1-\cos(t)$, $a_{12}(t)=1+\cos(t)$, $a_{21}(t)=1+\sin(t)$, $a_{22}(t)=-1-\sin(t)$, $D_1=8$, $D_2=4$, $\partial u_i(t,x)/\partial n=0$ $(t\geq t_0, x=0,2\pi)$, $\gamma_{1k}=0.4$, $\gamma_{2k}=0.2$, $\psi_1(\cdot)=\psi_1(\cdot)=5$, $b_{11}(t)=b_{21}(t)=\cos(t)$, $b_{12}(t)=b_{22}(t)=-\cos(t)$, $J_1(t)=J_2(t)=1$, $H_{11}(t)=H_{21}(t)=\sin(t)$, $H_{12}(t)=H_{22}(t)=-1+\sin(t)$, $T_{11}(t)=T_{21}(t)=-\sin(t)$, $T_{12}(t)=T_{22}(t)=2+\sin(t)$, $\tau_1(t)=\tau_2(t)=1$, $f_j(u_j)=u_j(t,x)$ $(j=1,2)$, $g_j(u_j(t-1,x))=u_j(t-1,x)e^{-u_j(t-1,x)}$ $(j=1,2)$, $\alpha_{11}(t)=-12.8$, $\alpha_{21}(t)=\alpha_{12}(t)=-1+\cos(t)$, $\alpha_{22}(t)=-10$, $\beta_{11}(t)=12.8$, $\beta_{12}(t)=-1+\sin(t)=\beta_{21}(t)$, $\beta_{22}(t)=$

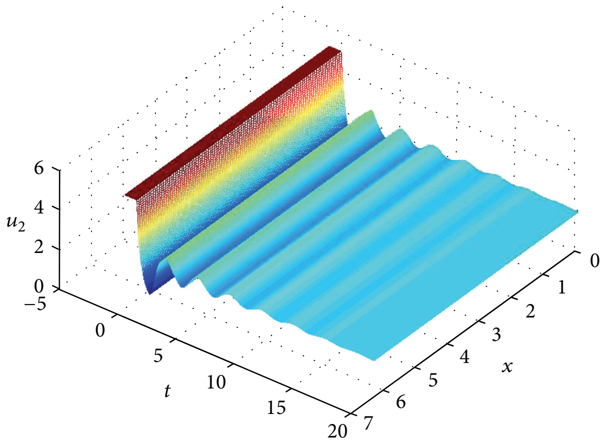

FIGURE 3: State response $u2(t, x)$ of model (44) without impulsive effects.

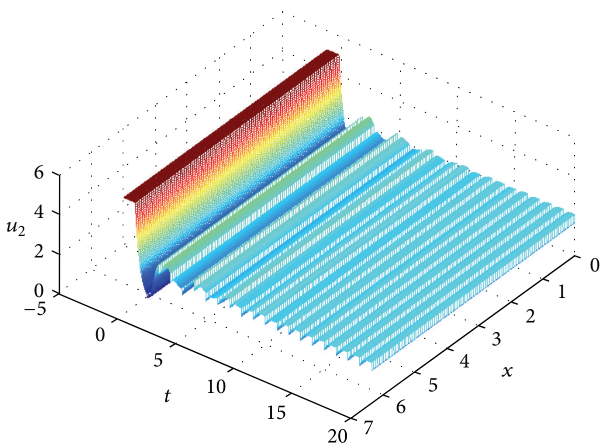

FIGURE 4: State response $u2(t, x)$ of model (44) with impulsive effects.

10, $v_j(t) = \sin(t)$. We assume that there exists $q = 6$ such that $t_k + 2\pi = t_{k+q}$. Obviously, f_1, f_2, g_1, and g_2 satisfy the assumption **(H1)** with $F_1 = F_2 = G_1 = G_2 = 1$ and **(H2)** and **(H3)** are satisfied with a common positive period 2π

$$\underline{C} - \left(\overline{A} + \overline{\alpha} + \overline{\beta}\right) F = \begin{bmatrix} \frac{2}{5} & 0 \\ 0 & \frac{4}{5} \end{bmatrix} \tag{45}$$

is a nonsingular M-matrix. The conditions of Theorem 6 are satisfied, hence there exists exactly one 2ω-periodic solution of the model and all other solutions of the model converge exponentially to it as $t \rightarrow +\infty$. Furthermore, the exponential converging index can be calculated as $\lambda = 0.021$, because here $\eta_k = 1$ and $\eta = 0$. The simulation results are shown in Figures 1, 2, 3, and 4, respectively.

5. Conclusions

In this paper, periodicity and global exponential stability of a class of FIRDDCNN model with variable both coefficients and delays have been investigated. By using Halanay's delay differential inequality, M-matrix theory, and analytic methods, some new sufficient conditions have been established to guarantee the existence, uniqueness, and global exponential stability of the periodic solution. Moreover, the exponential convergence rate index can be estimated. An example and its simulation have been given to show the effectiveness of the obtained results. In particular, the differentiability of the time-varying delays has been removed. The dynamic behaviors of fuzzy neural networks with the property of exponential periodicity are of great importance in many areas such as learning systems.

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Research Article

Delay-Dependent Exponential Optimal H^{∞} Synchronization for Nonidentical Chaotic Systems via Neural-Network-Based Approach

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A novel approach is presented to realize the optimal H^{∞} exponential synchronization of nonidentical multiple time-delay chaotic (MTDC) systems via fuzzy control scheme. A neural-network (NN) model is first constructed for the MTDC system. Then, a linear differential inclusion (LDI) state-space representation is established for the dynamics of the NN model. Based on this LDI state-space representation, a delay-dependent exponential stability criterion of the error system derived in terms of Lyapunov's direct method is proposed to guarantee that the trajectories of the slave system can approach those of the master system. Subsequently, the stability condition of this criterion is reformulated into a linear matrix inequality (LMI). According to the LMI, a fuzzy controller is synthesized not only to realize the exponential synchronization but also to achieve the optimal H^{∞} performance by minimizing the disturbance attenuation level at the same time. Finally, a numerical example with simulations is given to demonstrate the effectiveness of our approach.

1. Introduction

The stability analysis and stabilization of time-delay systems are problems of considerable theoretical and practical significance and have attracted the interest of many investigators for several years. Furthermore, time delays often appear in various engineering systems [1], such as the structure control of tall buildings, hydraulics, or electronic networks. Notably, the introduction of a time-delay factor tends to complicate the analysis. Consequently, convenient methods to check stability have long been sought later. The stability criteria of time-delay systems so far have been approached from two main directions based on the dependence on the size of delay. One method is to contrive stability conditions which do not include information on the delay, while the other method takes time delay into account. The former case is often referred to as delay-independent criterion and generally gives good algebraic conditions. Nevertheless, the abandonment of information on the size of the time delay necessarily causes conservativeness of the criteria, especially when the delay is comparatively small. Hence, delay-dependent criteria are derived to deal with the stability problem in this study.

Moreover, time delays have gained increasing interest in chaotic systems, ever since chaotic phenomenon in time-delay systems was first found by Mackey and Glass [2]. Chaotic phenomena have been observed in numerous physical systems, which can lead to irregular performance and possibly catastrophic failures [3]. Chaos is a well-known nonlinear phenomenon, and it is the seemingly random behavior of a deterministic system that is characterized by sensitive dependence on initial conditions [4]. Besides, chaos is occasionally preferable but usually intrinsically unpredictable as it can restrict the operating range of many physical devices and reduce performance. Therefore, the ability to control chaos is of much practical importance. According to these properties, chaos has received a great deal of interest among scientists from various research fields [5, 6]. One of the research fields for communication, chaotic synchronization, has been investigated extensively.

The chaotic synchronization of identical systems with different initial conditions was first introduced by Pecora and Carroll in 1990 [7]. They are intended to control one chaotic system to follow another. Since the introduction of this concept, various synchronization approaches have been widely developed in the past two decades. Chaotic synchronization can be applied in the vast areas of physics and engineering science, especially in secure communication [8]. Consequently, chaotic synchronization has become a popular study [9, 10]. However, all of them are focused on synchronizing two identical chaotic systems with different initial conditions [11]. In fact, experimental and even more real systems are often not fully identical; in particular, there are mismatches in parameters of the systems [11]. Also, in many real world applications, there are no exactly two identical chaotic systems. As a result, the problem of chaos synchronization between two different uncertain chaotic systems is an important research issue [12]. For instance, He et al. [13] investigate synchronization of two nonidentical chaotic systems with time-varying delay and parameter mismatches via impulsive control. To synchronize nonidentical chaotic systems with unknown parameters, Li et al. [14] proposed an approach based on the invariance principle of differential equations, and employing a combination of feedback control and adaptive control. Li and Ge [15] presented a new fuzzy model to simulate and synchronize two totally different and complicated chaotic systems.

In general, some noise or disturbances always exist that may cause instability. The influence of the external disturbance will worsen the performance of chaotic systems. Therefore, how to reduce the effect of external disturbances in the synchronization process for chaotic systems is an important issue [16, 17]. The H^{∞} control has been conferred for synchronization in chaotic systems over the last few years [16–20], and the H^{∞} synchronization problem has been investigated extensively for time-delay chaotic systems (e.g., see [21–23]). Accordingly, the purpose of this study is to realize the exponential synchronization of nonidentical multiple time-delay chaotic (MTDC) systems and attenuate the effect of external disturbances on the control performance to a minimum level at the same time.

Neural-network-(NN-) based modeling has become an active research field in the past few years due to its unique merits in solving complex nonlinear system identification and control problems [24–29]. Neural networks consist of simple elements operating in parallel; these elements are inspired by biological nervous systems. As a result, we can train an NN to represent a particular function by adjusting the weights between elements. As in nature, the connections between elements largely determine the network function. Individuals can train a neural network to perform a particular function by adjusting the values of the connections (weights) between elements. Hence, the nonlinear systems can be approximated as close as desired by the NN models via repetitive training. Recently, numerous reports on the success of NN applications in control systems have appeared in the literature (see [30–35]). For instance, Limanond et al. [30] applied neural networks to the optimal etch time control design for a reactive ion etching process. Enns and Si [32] advanced an NN-based approximate dynamic programming control mechanism to helicopter flight control. Despite several promising empirical results and its nonlinear mapping approximation properties, the rigorous closed-loop stability results for systems using NN-based controllers are still difficult to establish. Therefore, an LDI state-space representation was introduced to deal with the stability analysis of NN models (see [36]).

In the past few years, significant research efforts have been devoted to fuzzy control, which has attracted a great deal of attention from both the academic and industrial communities, and there have been many successful applications. For example, Wang et al. [37] presented a new measurement system that comprises a model-based fuzzy logic controller, an arterial tonometer, and a micro syringe device for the noninvasive monitoring of the continuous blood pressure wave form in the radial artery. A good tracking performance control scheme, a hybrid fuzzy neural-network control for nonlinear motor-toggle servomechanisms, was given by Wai [38]; Hwang et al. [39] developed the trajectory tracking of a car-like mobile robot using network-based fuzzy decentralized sliding-mode control; a hybrid fuzzy-PI speed controller for permanent magnet synchronous motors was proposed in Sant [40]; Spatti et al. [41] introduced a fuzzy control strategy for voltage regulation in electric power distribution systems—this real-time controller would act on power transformers equipped with under-load tap changers.

In spite of the successes of fuzzy control, many basic problems remain to be solved. Stability analysis and systematic design are certainly among the most important issues for fuzzy control systems. Recently, significant research efforts have been devoted to these issues (see [42–45] and the references therein). However, all of them have neglected the modeling errors between the fuzzy models and the nonlinear systems. In fact, the existence of modeling errors may be a potential source of instability for control designs based on the assumption that the fuzzy model exactly matches the nonlinear plant [46]. In recent years, novel approaches to overcome the influence of modeling errors in the field of model-based fuzzy control for nonlinear systems have been proposed by Kiriakidis [46], Chen et al. [47, 48], and Cao et al. [49, 50].

Almost all the existing research works of synchronization method made use of fuzzy models to approximate the chaotic systems (see [3, 4, 28, 42] and the references therein). Although using fuzzy models to approximate the chaotic systems is more simple than the neural-networks (NNs), the NN models will approach the chaotic systems by iterative training and adjusting the weights. In other words, the modeling errors of NN models will be much less than those of fuzzy models. With a view to the abovementioned, a novel approach is proposed via the neural-network-(NN-) based technique to realize the optimal H^{∞} exponential synchronization of nonidentical multiple time-delay chaotic (MTDC) systems such that the trajectories of the slave systems can approach those of the master systems and the effect of external disturbances on the control performance can be attenuated to a minimum level. First, the NN model is constructed for the chaotic systems with multiple time delays. Then, a linear differential inclusion (LDI) state-space representation is established for the dynamics of the NN model. Next, in terms of Lyapunov's direct method, a delay-dependent criterion is derived to guarantee the exponential stability of the error system between the master system and slave system. Subsequently, the stability condition of this criterion is reformulated into a linear matrix inequality (LMI). According to the LMI, a fuzzy controller is synthesized not only to realize the exponential synchronization but also to achieve the optimal H^{∞} performance by minimizing the disturbance attenuation level at the same time.

The remainder of this paper is organized as follows. The system description is arranged in Section 2. In Section 3, a robustness design of fuzzy control and a delay-dependent stability criterion are proposed to realize the optimal H^{∞} exponential synchronization. The design algorithm is given in Section 4. In Section 5, the effectiveness of the proposed approach is illustrated by a numerical simulation. Finally, the conclusions are drawn in Section 6.

2. Problem Formulation

Consider two different multiple time-delay chaotic (MTDC) systems in master-slave configuration. The dynamics of the master system (N_m) and slave system (N_s) are described as follows:

$$N_m: \quad \dot{X}(t) = f(X(t)) + \sum_{k=1}^{g} H_k(X(t-\tau_k)), \tag{1}$$

$$N_s: \quad \dot{\widehat{X}}(t) = \widehat{f}(\widehat{X}(t)) + \sum_{k=1}^{g} \widehat{H}_k(\widehat{X}(t-\tau_k)) + BU(t) + D(t), \tag{2}$$

where $f(\cdot)$, $\widehat{f}(\cdot)$, $H_k(\cdot)$, and $\widehat{H}_k(\cdot)$ are the nonlinear vector-valued functions, $\tau_k (k = 1, 2, \ldots, g)$ are the time delays, $U(t)$ is the control input, and $D(t)$ denotes the external disturbance. Besides, $X(t)$ and $\widehat{X}(t)$ are the state vectors of N_m and N_s, respectively.

In this section, a neural-network (NN) model is first constructed for the MTDC system. The dynamics of the NN model are then converted into a linear differential inclusion (LDI) state-space representation. Finally, based on the LDI state-space representation, a fuzzy controller is synthesized to realize the synchronization of nonidentical MTDC systems.

2.1. Neural-Network (NN) Model. The MTDC system can be approximated by an NN model, as shown in Figure 1, that has S layers with $J^{\sigma} (\sigma = 1, 2, \ldots, S)$ neurons for each layer, in which $x_1(t) \sim x_\delta(t)$ are the state variables and $x_1(t-\tau_1) \sim x_1(t-\tau_g)$, $x_2(t-\tau_1) \sim x_\delta(t-\tau_g)$ are the state variables with delays.

To distinguish among these layers, the superscripts are used for identification. Specifically, the number of the layer is appended as a superscript to the names for each of these variables. Thus, the weight matrix for the σth layer is written as W^{σ}. Furthermore, it is assumed that $v_{\varsigma}^{\sigma}(t)(\varsigma = 1, 2, \ldots, J^{\sigma}; \sigma = 1, 2, \ldots, S)$ is the net input and $T(v_{\varsigma}^{\sigma}(t))$ is the transfer

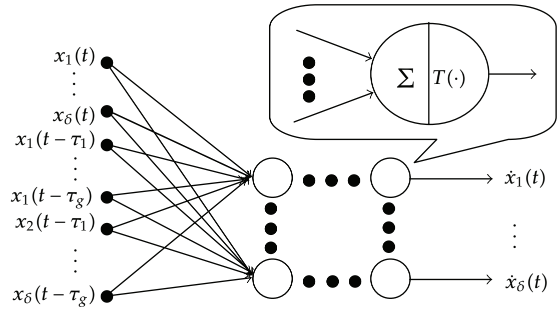

FIGURE 1: An NN model for N_d.

function of the neuron. Subsequently, the transfer function vector of the σth layer is defined as

$$\Psi^{\sigma}(v_{\varsigma}^{\sigma}(t)) \equiv \left[T(v_1^{\sigma}(t))\, T(v_2^{\sigma}(t)) \cdots T(v_{J^{\sigma}}^{\sigma}(t))\right]^T, \quad \sigma = 1, 2, \ldots, S, \tag{3}$$

where $T(v_{\varsigma}^{\sigma}(t))$ $(\varsigma = 1, 2, \ldots, J^{\sigma})$ is the transfer function of the ςth neuron. The final output of NN model can then be inferred as follows:

$$\dot{X}(t) = \Psi^S\left(W^S\Psi^{S-1}\left(W^{S-1}\Psi^{S-2} \times \left(\cdots\Psi^2\left(W^2\Psi^1\left(W^1\Lambda(t)\right)\right)\cdots\right)\right)\right), \tag{4}$$

where $\Lambda^T(t) = [X^T(t)X^T(t-\tau_k)]$ with $X(t) = [x_1(t)x_2(t)\cdots x_\delta(t)]^T$,

$$X(t-\tau_k) = \left[x_1(t-\tau_1)\cdots x_1(t-\tau_g)\; x_2(t-\tau_1)\cdots x_\delta(t-\tau_g)\right]^T, \quad \text{for } k = 1, 2\ldots, g. \tag{5}$$

2.2. Linear Differential Inclusion (LDI). To handle the synchronization problem of MTDC systems, this study establishes the following LDI state-space representation for the dynamics of the NN model, described as [36, 51]

$$\dot{O}(t) = A(a(t))\,O(t), \quad A(a(t)) = \sum_{i=1}^{\phi} h_i(a(t))\,\widetilde{A}_i, \tag{6}$$

where ϕ is a positive integer, $a(t)$ is a vector signifying the dependence of $h_i(\cdot)$ on its elements, $\widetilde{A}_i$ $(i = 1, 2, \ldots, \phi)$ are constant matrices, and $O(t) = [o_1(t)o_2(t)\cdots o_{\aleph}(t)]^T$. Moreover, it is assumed that $h_i(a(t)) \geq 0$ and $\sum_{i=1}^{\phi} h_i(a(t)) = 1$. According to the properties of LDI, without loss of generality, $h_i(t)$ can be replaced by $h_i(a(t))$. The following procedure represents the dynamics of the NN model (4) using the LDI state-space representation [36].

To begin with, notice that the output $T(v_\varsigma^\sigma(t))$ satisfies

$$\begin{aligned} g_{\varsigma 0}^\sigma v_\varsigma^\sigma(t) \le T\left(v_\varsigma^\sigma(t)\right) \le g_{\varsigma 1}^\sigma v_\varsigma^\sigma(t), \quad v_\varsigma^\sigma(t) \ge 0, \\ g_{\varsigma 1}^\sigma v_\varsigma^\sigma(t) \le T\left(v_\varsigma^\sigma(t)\right) \le g_{\varsigma 0}^\sigma v_\varsigma^\sigma(t), \quad v_\varsigma^\sigma(t) < 0, \end{aligned} \tag{7}$$

where $g_{\varsigma 0}^\sigma$ and $g_{\varsigma 1}^\sigma$ denote the minimum and maximum of the derivative of $T(v_\varsigma^\sigma(t))$, respectively, and are given in the following:

$$g_{\varsigma\varphi}^\sigma = \begin{cases} \min\limits_{v} \dfrac{dT\left(v_\varsigma^\sigma(t)\right)}{dv_\varsigma^\sigma(t)}, & \text{when } \varphi = 0, \\ \max\limits_{v} \dfrac{dT\left(v_\varsigma^\sigma(t)\right)}{dv_\varsigma^\sigma(t)}, & \text{when } \varphi = 1. \end{cases} \tag{8}$$

Subsequently, the min-max matrix G^σ of the σth layer is defined as follows:

$$G^\sigma \equiv \operatorname{diag}\left[g_{\varsigma\varphi_\varsigma}^\sigma\right] = \begin{bmatrix} g_{1\varphi_1}^\sigma & 0 & 0 & \cdots & 0 \\ 0 & g_{2\varphi_2}^\sigma & 0 & \ddots & 0 \\ 0 & 0 & g_{3\varphi_3}^\sigma & 0 & \vdots \\ \vdots & \ddots & 0 & \ddots & 0 \\ 0 & 0 & \cdots & 0 & g_{J^\sigma\varphi_J}^\sigma \end{bmatrix}. \tag{9}$$

Besides, on the basis of the interpolation method, the transfer function $T(v_\varsigma^\sigma(t))$ can be represented as follows [36]:

$$\begin{aligned} T\left(v_\varsigma^\sigma(t)\right) &= \left(h_{\varsigma 0}^\sigma(t)\, g_{\varsigma 0}^\sigma + h_{\varsigma 1}^\sigma(t)\, g_{\varsigma 1}^\sigma\right) v_\varsigma^\sigma(t) \\ &= \left(\sum_{\varphi=0}^{1} h_{\varsigma\varphi}^\sigma(t)\, g_{\varsigma\varphi}^\sigma\right) v_\varsigma^\sigma(t), \end{aligned} \tag{10}$$

where the interpolation coefficients $h_{\varsigma\varphi}^\sigma(t) \in [0,1]$ and $\sum_{\varphi=0}^{1} h_{\varsigma\varphi}^\sigma(t) = 1$. Equations (3) and (10) show that

$$\begin{aligned} &\Psi^\sigma\left(v_\varsigma^\sigma(t)\right) \\ &\quad \equiv \left[T\left(v_1^\sigma(t)\right)\ T\left(v_2^\sigma(t)\right)\ \cdots\ T\left(v_{J^\sigma}^\sigma(t)\right)\right]^T \\ &\quad = \left[\left(\sum_{\varphi_1=0}^{1} h_{1\varphi_1}^\sigma(t)\, g_{1\varphi_1}^\sigma\right) v_1^\sigma(t)\ \left(\sum_{\varphi_2=0}^{1} h_{2\varphi_2}^\sigma(t)\, g_{2\varphi_2}^\sigma\right) v_2^\sigma(t)\ \cdots\ \left(\sum_{\varphi_J=0}^{1} h_{J^\sigma\varphi_J}^\sigma(t)\, g_{J^\sigma\varphi_J}^\sigma\right) v_{J^\sigma}^\sigma(t)\right]^T. \end{aligned} \tag{11}$$

Hence, the final output of the NN model (4) can be reformulated as follows:

$$\begin{aligned} &\dot{X}(t) \\ &\quad = \sum_{p=0}^{1} h_{\varsigma p}^S(t)\, G^S \\ &\qquad \times \left(W^S\left[\cdots\left[\sum_{n=0}^{1} h_{\varsigma n}^2(t)\, G^2\right.\right.\right. \\ &\qquad\qquad \times \left(W^2\left[\sum_{b=0}^{1} h_{\varsigma b}^1(t)\, G^1\right.\right. \\ &\qquad\qquad\qquad \left.\left.\left.\left.\left. \times \left(W^1 \Lambda(t)\right)\right]\right)\right]\cdots\right]\right) \\ &\quad = \sum_{p=0}^{1}\cdots\sum_{n=0}^{1}\sum_{b=0}^{1} h_{\varsigma p}^S(t)\cdots h_{\varsigma n}^2(t)\, h_{\varsigma b}^1(t)\, G^S W^S \\ &\qquad \cdots G^2 W^2 G^1 W^1 \Lambda(t) \\ &\quad = \sum_{\Omega} h_{\varsigma\Omega}^\sigma(t)\, C_\Omega^\sigma \Lambda(t), \end{aligned} \tag{12}$$

where

$$\begin{aligned} \sum_{b=0}^{1} h_{\varsigma b}^1(t) &\equiv \sum_{b_1=0}^{1} h_{1b_1}^1(t) \sum_{b_2=0}^{1} h_{2b_2}^1(t) \cdots \sum_{b_J=0}^{1} h_{J^1 b_J}^1(t), \\ \sum_{n=0}^{1} h_{\varsigma n}^2(t) &\equiv \sum_{n_1=0}^{1} h_{1n_1}^2(t) \sum_{n_2=0}^{1} h_{2n_2}^2(t) \cdots \sum_{n_J=0}^{1} h_{J^2 n_J}^2(t), \\ &\ \ \vdots \\ \sum_{p=0}^{1} h_{\varsigma p}^S(t) &\equiv \sum_{p_1=0}^{1} h_{1p_1}^S(t) \sum_{p_2=0}^{1} h_{2p_2}^S(t) \cdots \sum_{p_J=0}^{1} h_{J^S p_J}^S(t), \\ \sum_{\Omega} h_{\varsigma\Omega}^\sigma(t) &\equiv \sum_{p=0}^{1}\cdots\sum_{n=0}^{1}\sum_{b=0}^{1} h_{\varsigma p}^S(t)\cdots h_{\varsigma n}^2(t)\, h_{\varsigma b}^1(t), \\ &\qquad \varsigma = 1, 2, \ldots, J^\sigma, \\ C_\Omega^\sigma &\equiv G^S W^S \cdots G^2 W^2 G^1 W^1, \end{aligned} \tag{13}$$

and b_ς, n_ς, p_ς ($\varsigma = 1, 2, \ldots, J$) represent the variables φ of the ςth neuron of the first, second, and Sth layer, respectively.

Finally, based on (6), the dynamics of the NN model (12) can be rewritten as the following LDI state-space representation:

$$\dot{X}(t) = \sum_{i=1}^{\phi} h_i(t) C_i \Lambda(t), \tag{14}$$

where $h_i(t) \geq 0$, $\sum_{i=1}^{\phi} h_i(t) = 1$, ϕ is a positive integer and C_i is a constant matrix with appropriate dimension associated with C_{Ω}^{σ}. Furthermore, the LDI state-space representation (14) can be rearranged as follows:

$$\dot{X}(t) = \sum_{i=1}^{\phi} h_i(t) \left\{ A_i X(t) + \sum_{k=1}^{g} \overline{A}_{ik} X(t-\tau_k) \right\}, \tag{15}$$

where A_i and $\overline{A}_{ik}$ are the partitions of C_i corresponding to the partitions of $\Lambda^T(t)$.

From the abovementioned, the NN models of the master and slave chaotic systems are described by the following LDI state-space representations (16) and (17), respectively:

$$\text{master}: \quad \dot{X}(t) = \sum_{i=1}^{\phi} h_i(t) \left\{ A_i X(t) + \sum_{k=1}^{g} \overline{A}_{ik} X(t-\tau_k) \right\}, \tag{16}$$

$$\text{slave}: \quad \dot{\widehat{X}}(t) = \sum_{j=1}^{\phi} \widehat{h}_j(t) \left[\widehat{A}_j \widehat{X}(t) + \sum_{k=1}^{g} \widehat{\overline{A}}_{jk} \widehat{X}(t-\tau_k) \right] + BU(t). \tag{17}$$

2.3. Fuzzy Controller. On the basis of the state-feedback control scheme, a fuzzy controller is utilized to make the slave system synchronize with the master system. The fuzzy controller is in the following form:

Control Rule l : IF $e_1(t)$ is M_{l1} and $\cdots$ and $e_\delta(t)$ is $M_{l\delta}$,

$$\text{THEN} \quad U(t) = -K_l E(t), \tag{18}$$

where $l = 1, 2, \ldots, \rho$, and ρ is the number of IF-THEN rules of the fuzzy controller and $M_{l\eta}$ $(\eta = 1, 2, \ldots, \delta)$ are the fuzzy sets. Therefore, the final output of this fuzzy controller can be inferred as follows:

$$U(t) = \frac{-\sum_{l=1}^{\rho} w_l(t) K_l E(t)}{\sum_{l=1}^{\rho} w_l(t)} = -\sum_{l=1}^{\rho} \overline{h}_l(t) K_l E(t), \tag{19}$$

with $w_l(t) \equiv \prod_{\eta=1}^{\delta} M_{l\eta}(e_\eta(t))$, $M_{l\eta}(e_\eta(t))$ is the grade of membership of $e_\eta(t)$ in $M_{l\eta}$.

3. Stability Analysis and Chaotic Synchronization via Fuzzy Control

In this section, the synchronization of nonidentical multiple time-delay chaotic (MTDC) systems is examined under the influence of modeling error. The exponential synchronization scheme of the multiple time-delay chaotic systems is described as follows.

3.1. Error Systems. From (1) and (2), the synchronization error is defined as $E(t) \equiv \widehat{X}(t) - X(t) = [e_1(t), e_2(t), \ldots, e_\delta(t)]^T$, and then the dynamics of the error system under the fuzzy control (19) can be described as follows:

$$\begin{aligned}
\dot{E}(t) &= \widehat{\Psi} + D(t) - \Psi \\
&\quad + \sum_{i=1}^{\phi} \sum_{j=1}^{\phi} \sum_{l=1}^{\rho} h_i(t) \widehat{h}_j(t) \overline{h}_l(t) \\
&\quad \times \Bigg\{ G_{il} E(t) + \left(\widehat{A}_j - A_i\right) \widehat{X}(t) \\
&\qquad + \sum_{k=1}^{g} \left(\widehat{\overline{A}}_{jk} - \overline{A}_{ik}\right) \widehat{X}(t-\tau_k) \\
&\qquad + \sum_{k=1}^{g} \overline{A}_{ik} E(t-\tau_k) \Bigg\} \\
&\quad - \sum_{i=1}^{\phi} \sum_{j=1}^{\phi} \sum_{l=1}^{\rho} h_i(t) \widehat{h}_j(t) \overline{h}_l(t) \\
&\quad \times \Bigg\{ G_{il} E(t) + \left(\widehat{A}_j - A_i\right) \widehat{X}(t) \\
&\qquad + \sum_{k=1}^{g} \left(\widehat{\overline{A}}_{jk} - \overline{A}_{ik}\right) \widehat{X}(t-\tau_k) \\
&\qquad + \sum_{k=1}^{g} \overline{A}_{ik} E(t-\tau_k) \Bigg\} \\
&= \sum_{i=1}^{\phi} \sum_{l=1}^{\rho} h_i(t) \overline{h}_l(t) \left\{ G_{il} E(t) + \sum_{k=1}^{g} \overline{A}_{ik} E(t-\tau_k) \right\} \\
&\quad + D(t) + \Phi(t),
\end{aligned} \tag{20}$$

where

$$\begin{gathered}
G_{il} \equiv A_i - BK_l, \\
\widehat{\Psi} \equiv \widehat{f}\left(\widehat{X}(t)\right) + \sum_{k=1}^{g} \widehat{H}_k\left(\widehat{X}(t-\tau_k)\right) + U(t), \\
\Psi \equiv f(X(t)) + \sum_{k=1}^{g} H_k(X(t-\tau_k)),
\end{gathered} \tag{21}$$

with

$$\begin{aligned}
U(t) &= -\sum_{l=1}^{\rho} \overline{h}_l(t) K_l E(t), \\
\Phi(t) &\equiv \widehat{\Psi} - \Psi \\
&\quad - \left\{ \sum_{i=1}^{\phi} \sum_{l=1}^{\rho} h_i(t) \overline{h}_l(t) \left[G_{il} E(t) + \sum_{k=1}^{g} \overline{A}_{ik} E(t-\tau_k) \right] \right\}.
\end{aligned} \tag{22}$$

Suppose that there exists a bounding matrix ΘR_{il} such that

$$\|\Phi(t)\| \leq \left\| \sum_{i=1}^{\phi}\sum_{l=1}^{\rho} h_i(t)\,\bar{h}_l(t)\,\Theta R_{i\ell}E(t) \right\| \tag{23}$$

for the trajectory $E(t)$, and the bounding matrix ΘR_{il} can be described as follows:

$$\Theta R_{il} = \varepsilon_{il} R, \tag{24}$$

where R is the specified structured bounding matrix and $\|\varepsilon_{il}\| \leq 1$, for $i = 1, 2, \ldots, \phi$; $l = 1, 2, \ldots, \rho$. Equations (23) and (24) show that

$$\begin{aligned}\Phi^T(t)\,\Phi(t) &\leq \sum_{i=1}^{\phi}\sum_{l=1}^{\rho} h_i(t)\,\bar{h}_l(t)\,\|RE(t)\|\,\|\varepsilon_{il}\| \\ &\quad\times \sum_{i=1}^{\phi}\sum_{l=1}^{\rho} h_i(t)\,\bar{h}_l(t)\,\|\varepsilon_{il}\|\,\|RE(t)\| \\ &\leq [RE(t)]^T\,[RE(t)]\,.\end{aligned} \tag{25}$$

Namely, $\Phi(t)$ is bounded by the specified structured bounding matrix R.

Remark 1 (see [47]). The following simple example describes the procedures for determining ε_{il} and R. First, assume that the possible bounds for all elements in ΘR_{il} are

$$\Theta R_{il} = \begin{bmatrix} \Theta r_{il}^{11} & \Theta r_{il}^{12} & \Theta r_{il}^{13} \\ \Theta r_{il}^{21} & \Theta r_{il}^{22} & \Theta r_{il}^{23} \\ \Theta r_{il}^{31} & \Theta r_{il}^{32} & \Theta r_{il}^{33} \end{bmatrix}, \tag{26}$$

where $-r^{qs} \leq \Delta r_{il}^{qs} \leq r^{qs}$ for some r_{il}^{qs} with $q, s = 1, 2, 3$; $i = 1, 2, \ldots, \phi$, and $l = 1, 2, \ldots, \rho$.

A possible depiction for the bounding matrix ΘR_{il} is

$$\Theta R_{il} = \begin{bmatrix} \varepsilon_{il}^{11} & 0 & 0 \\ 0 & \varepsilon_{il}^{22} & 0 \\ 0 & 0 & \varepsilon_{il}^{33} \end{bmatrix} \begin{bmatrix} r^{11} & r^{12} & r^{13} \\ r^{21} & r^{22} & r^{23} \\ r^{31} & r^{32} & r^{33} \end{bmatrix} = \varepsilon_{il} R, \tag{27}$$

where $-1 \leq \varepsilon_{il}^{qq} \leq 1$ for $q = 1, 2, 3$. Notice that ε_{il} can be chosen by other forms as long as $\|\varepsilon_{il}\| \leq 1$. The validity of (23) is then checked in the simulation. If it is not satisfied, we can expand the bounds for all elements in ΘR_{il} and repeat the design procedure until (23) holds.

3.2. Delay-Dependent Stability Criterion for Exponential H^∞ Synchronization. In this subsection, a delay-dependent criterion is proposed to guarantee the exponential stability of the error system described in (20). Moreover, in general, some noises or disturbances always exist that may cause instability. The influence of the external disturbance $D(t)$ will worsen the performance of chaotic systems. To reduce the effect of the external disturbance, an optimal H^∞ scheme is used to design the fuzzy control so that the effect of external disturbance on control performance can be attenuated to a minimum level. In other words, the fuzzy controller (19) realizes exponential synchronization and at the same time achieves the optimal H^∞ control performance in this study.

Before examination of the stability of the error system, some definitions and a lemma are given follows.

Lemma 2 (see [52]). *For the real matrices A and B with appropriate dimension,*

$$A^T B + B^T A \leq \lambda A^T A + \lambda^{-1} B^T B, \tag{28}$$

where λ is a positive constant.

Definition 3 (see [51]). The slave system (2) can exponentially synchronize with the master system (1) (i.e., the error system (20) is exponentially stable) if there exist two positive numbers α and β such that the synchronization error satisfies

$$\|E(t)\| \leq \alpha \exp\left(-\beta\left(t - t_0\right)\right), \quad \forall t \geq 0, \tag{29}$$

where the positive number β is called the exponential convergence rate.

Definition 4 (see [19–23]). The master system (1) and slave system (2) are said to be exponential H^∞ synchronization if the following conditions are satisfied:

(i) with zero disturbance (i.e., $D(t) = 0$), the error system (20) with the fuzzy controller (19) is exponentially stable;

(ii) under the zero initial conditions (i.e., $E(t) = 0$ for $t \in [-\tau_{\max}, 0]$, in which $\tau_{\max}$ is the maximal value of τ_k's) and a given constant $\kappa > 0$, the following condition holds:

$$\begin{aligned}\Theta(E(t), \partial(t)) &= \int_0^\infty E^T(t)\,E(t)\,dt - \kappa^2 \int_0^\infty D^T(t)\,D(t)\,dt \\ &\leq 0,\end{aligned} \tag{30}$$

where the parameter κ is called the H^∞ norm bound or the disturbance attenuation level. If the minimum κ is found (i.e., the error system can reject the external disturbance as strong as possible) to satisfy the previous conditions, the fuzzy controller (19) is an optimal H^∞ synchronizer [18].

Theorem 5. *For given positive constants a and n, if there exist two symmetric positive definite matrices P, ψ_k and positive constants ξ, κ so that the following inequalities hold, then the exponential H^∞ synchronization with the disturbance attenuation κ is guaranteed via the fuzzy controller* (19) *consider.*

$$\begin{aligned}\Delta_{il} \equiv{}& \sum_{k=1}^{g} \tau_k P G_{il} + \sum_{k=1}^{g} \tau_k G_{il}^T P + \sum_{k=1}^{g} \psi_k + ngR^T R + I \\ &+ \sum_{k=1}^{g} \tau_k^2 P^2 \left(\xi^{-1} + n^{-1} + ga^{-1}\right) < 0,\end{aligned} \tag{31a}$$

$$\nabla_{ik} \equiv ga\overline{A}_{ik}^{T}\overline{A}_{ik} - \psi_k < 0, \tag{31b}$$

$$\kappa > \sqrt{\xi g}, \tag{31c}$$

where $G_{il} \equiv A_i - BK_l$, *for* $i = 1, 2, \ldots, \phi$; $k = 1, 2, \ldots, g$, *and* $l = 1, 2, \ldots, \rho$.

Proof. Let the Lyapunov function for the error system (20) be defined as

$$V(t) = \sum_{k=1}^{g} E^T(t)\tau_k P E(t) + \sum_{k=1}^{g}\int_0^{\tau_k} E^T(t-\pi)\psi_k E(t-\pi)\,d\pi, \tag{32}$$

where the weighting matrices $P = P^T > 0$ and $\psi_k = \psi_k^T > 0$. We then evaluate the time derivative of $V(t)$ on the trajectories of (20) to obtain

$$\begin{aligned}
\dot{V}(t) &= \sum_{k=1}^{g}\tau_k\left[\dot{E}^T(t)PE(t) + E^T(t)P\dot{E}(t)\right] \\
&\quad + \sum_{k=1}^{g}\left[E^T(t)\psi_k E(t) - E^T(t-\tau_k)\psi_k E(t-\tau_k)\right] \\
&= \sum_{k=1}^{g}\tau_k\left\{\sum_{i=1}^{\phi}\sum_{l=1}^{\rho}h_i(t)\overline{h}_l(t)\left[G_{il}E(t) + \sum_{d=1}^{g}\overline{A}_{id}E(t-\tau_d)\right]\right. \\
&\qquad \left. + D(t) + \Phi(t)\right\}^T PE(t) + \sum_{k=1}^{g}\tau_k E^T(t)P \\
&\quad \times\left\{\sum_{i=1}^{\phi}\sum_{l=1}^{\rho}h_i(t)\overline{h}_l(t)\left[G_{il}E(t) + \sum_{d=1}^{g}\overline{A}_{id}E(t-\tau_d)\right.\right. \\
&\qquad \left.\left. + D(t) + \Phi(t)\right]\right\} \\
&\quad + \sum_{k=1}^{g}\left[E^T(t)\psi_k E(t) - E^T(t-\tau_k)\psi_k E(t-\tau_k)\right] \\
&= \sum_{k=1}^{g}\sum_{i=1}^{\phi}\sum_{l=1}^{\rho}h_i(t)\overline{h}_l(t)E^T(t)\left[\tau_k G_{il}^T P + \tau_k PG_{il} + \psi_k\right]E(t) \\
&\quad + \sum_{k=1}^{g}\sum_{i=1}^{\phi}\sum_{d=1}^{g}h_i(t)\left[E^T(t-\tau_d)\tau_k\overline{A}_{id}^T PE(t)\right. \\
&\qquad \left. + E^T(t)\tau_k P\overline{A}_{id}E(t-\tau_d)\right] \\
&\quad + \sum_{k=1}^{g}\tau_k\left[D^T(t)\tau_k PE(t) + E^T(t)\tau_k PD(t)\right. \\
&\qquad \left. + \Phi^T(t)\tau_k PE(t) + E^T(t)\tau_k P\Phi(t)\right] \\
&\quad - \sum_{k=1}^{g}\left[E^T(t-\tau_k)\psi_k E(t-\tau_k)\right].
\end{aligned} \tag{33}$$

According to Lemma 2 and (33), we have

$$\begin{aligned}
\dot{V}(t) &\le \sum_{k=1}^{g}\sum_{i=1}^{\phi}\sum_{l=1}^{\rho}h_i(t)\overline{h}_l(t)E^T(t)\left[\tau_k G_{il}^T P + \tau_k PG_{il} + \psi_k\right]E(t) \\
&\quad + \sum_{k=1}^{g}\sum_{i=1}^{\phi}\sum_{d=1}^{g}h_i(t)\left[aE^T(t-\tau_d)\overline{A}_{id}^T\overline{A}_{id}E(t-\tau_d)\right. \\
&\qquad \left. + a^{-1}E^T(t)\tau_k^2P^2E(t)\right] \\
&\quad + \sum_{k=1}^{g}\left[\xi D^T(t)D(t) + \xi^{-1}E^T(t)\tau_k^2P^2E(t)\right. \\
&\qquad \left. + n\Phi^T(t)\Phi(t) + n^{-1}E^T(t)\tau_k^2P^2E(t)\right] \\
&\quad - \sum_{k=1}^{g}\left[E^T(t-\tau_k)\psi_k E(t-\tau_k)\right]
\end{aligned} \tag{34}$$

$$\begin{aligned}
&\le \sum_{k=1}^{g}\sum_{i=1}^{\phi}\sum_{l=1}^{\rho}h_i(t)\overline{h}_l(t)E^T(t) \\
&\quad \times\left[\tau_k G_{il}^T P + \tau_k PG_{il} + \psi_k\right]E(t) \\
&\quad + \sum_{k=1}^{g}\sum_{i=1}^{\phi}\sum_{d=1}^{g}h_i(t)\left[aE^T(t-\tau_d)\overline{A}_{id}^T\overline{A}_{id}E(t-\tau_d)\right. \\
&\qquad \left. + a^{-1}E^T(t)\tau_k^2P^2E(t)\right] \\
&\quad + \sum_{k=1}^{g}\left[\xi D^T(t)D(t) + \xi^{-1}E^T(t)\tau_k^2P^2E(t)\right. \\
&\qquad \left. + nE^T(t)R^TRE(t) + n^{-1}E^T(t)\tau_k^2P^2E(t)\right] \\
&\quad - \sum_{k=1}^{g}\left[E^T(t-\tau_k)\psi_k E(t-\tau_k)\right] \quad (\text{by } (25))
\end{aligned} \tag{35}$$

$$\begin{aligned}
&= \sum_{i=1}^{\phi}\sum_{l=1}^{\rho}h_i(t)\overline{h}_l(t)E^T(t) \\
&\quad \times\left[\sum_{k=1}^{g}\tau_k PG_{il} + \sum_{k=1}^{g}\tau_k G_{il}^T P\right. \\
&\qquad + \sum_{k=1}^{g}\psi_k + ngR^TR \\
&\qquad \left. + \sum_{k=1}^{g}\tau_k P^2\left(\xi^{-1} + n^{-1} + ga^{-1}\right)\right]E(t) \\
&\quad + \sum_{k=1}^{g}\sum_{i=1}^{\phi}h_i(t)E^T(t-\tau_k)\left[ga\overline{A}_{ik}^T\overline{A}_{ik} - \psi_k\right] \\
&\quad \times E(t-\tau_k) + \xi g D^T(t)D(t).
\end{aligned} \tag{36}$$

From (36), we have

$$\begin{aligned}
&\dot{V}(t)+E^T(t)E(t)-\kappa^2D^T(t)D(t)\\
&\quad\le\sum_{i=1}^{\phi}\sum_{l=1}^{\rho}h_i(t)\overline{h}_l(t)E^T(t)\Delta_{il}E(t)\\
&\quad\quad+\sum_{i=1}^{\phi}\sum_{k=1}^{g}h_i(t)E^T(t-\tau_k)\nabla_{ik}E(t-\tau_k)\\
&\quad\quad+\left(\xi g-\kappa^2\right)D^T(t)D(t)\\
&\quad\le\sum_{i=1}^{\phi}\sum_{l=1}^{\rho}h_i(t)\overline{h}_l(t)\lambda_{\max}(\Delta_{il})E^T(t)E(t)\\
&\quad\quad+\sum_{i=1}^{\phi}\sum_{k=1}^{g}h_i(t)\lambda_{\max}(\nabla_{ik})E^T(t-\tau_k)E(t-\tau_k)\\
&\quad\quad+\left(\xi g-\kappa^2\right)D^T(t)D(t)<0,
\end{aligned}\tag{37}$$

where

$$\begin{aligned}
\Delta_{il}&\equiv\sum_{k=1}^{g}\tau_kPG_{il}+\sum_{k=1}^{g}\tau_kG_{il}^TP+\sum_{k=1}^{g}\psi_k+ngR^TR+I\\
&\quad+\sum_{k=1}^{g}\tau_k^2P^2\left(\xi^{-1}+n^{-1}+ga^{-1}\right)\quad(\text{see (31a)}),\\
\nabla_{ik}&\equiv ga\overline{A}_{ik}^T\overline{A}_{ik}-\psi_k\quad(\text{see (31b)}).
\end{aligned}\tag{38}$$

Integrating (37) from $t=0$ to $t=\infty$, the following inequality is obtained as

$$V(\infty)-V(0)+\int_0^{\infty}E^T(t)E(t)\,dt-\kappa^2\int_0^{\infty}D^T(t)D(t)\,dt\le0.\tag{39}$$

With zero initial conditions (i.e., $E(t)\equiv0$ for $t\in[-\tau_{\max},0]$), we have

$$\int_0^{\infty}E^T(t)E(t)\,dt\le\kappa^2\int_0^{\infty}D^T(t)D(t)\,dt.\tag{40}$$

That is, (30) and the H^{∞} control performance are achieved with a prescribed attenuation κ.

Since

$$\begin{aligned}
&\sum_{k=1}^{g}\tau_k\lambda_{\min}(P)E^T(t)E(t)\\
&\quad\le\sum_{k=1}^{g}\tau_kE^T(t)PE(t)\\
&\quad=V(t)-\sum_{k=1}^{g}\int_0^{\tau_k}E^T(t-\pi)\psi_kE(t-\pi)\,d\pi\\
&\quad<V(t)
\end{aligned}\tag{41}$$

(from (32)), we can get the following inequality from (37):

$$\begin{aligned}
&\dot{V}(t)+E^T(t)E(t)-\kappa^2D^T(t)D(t)\\
&\quad<\sum_{i=1}^{\phi}\sum_{l=1}^{\rho}h_i(t)\overline{h}_l(t)\frac{\lambda_{\max}(\Delta_{il})}{\sum_{k=1}^{g}\tau_k\lambda_{\min}(P)}V(t)<0.
\end{aligned}\tag{42}$$

Then, we can easily obtain

$$V(t)|_{\partial(t)=0}<V(t_0)\exp\overline{\beta}(t-t_0),\tag{43}$$

where $\overline{\beta}=\sum_{i=1}^{\phi}\sum_{l=1}^{\rho}h_i(t)\overline{h}_l(t)[\lambda_{\max}(\Delta_{il})/\sum_{k=1}^{g}\tau_k\lambda_{\min}(P)]<0$.

Equations (32) and (43) show that

$$\begin{aligned}
&\sum_{k=1}^{g}\tau_k\lambda_{\min}(P)E^T(t)E(t)\\
&\quad\le\sum_{k=1}^{g}E^T(t)\tau_kPE(t)\\
&\quad<V(t_0)\exp\overline{\beta}(t-t_0)\\
&\quad\quad-\sum_{k=1}^{g}\int_0^{\tau_k}E^T(t-\pi)\psi_kE(t-\pi)\,d\pi\\
&\quad<V(t_0)\exp\overline{\beta}(t-t_0).
\end{aligned}\tag{44}$$

That is, $\|E(t)\|^2\le(V(t_0)/\sum_{k=1}^{g}\tau_k\lambda_{\min}(P))\exp\overline{\beta}(t-t_0)$. Therefore, we conclude that

$$\begin{aligned}
&\|E(t)\|\le\alpha\exp\left(-\beta(t-t_0)\right),\\
&\text{with }\alpha\equiv\sqrt{\frac{V(t_0)}{\sum_{k=1}^{g}\tau_k\lambda_{\min}(P)}}>0,\ \beta\equiv-\frac{1}{2}\overline{\beta}>0.
\end{aligned}\tag{45}$$

Hence, on basis of the **Definition 3**, the error system (20) with the fuzzy controller (19) is exponentially stable for $D(t)=0$. □

Corollary 6. *Equations* (31a) *and* (31b) *can be reformulated into LMIs via the following procedure.*

By introducing the new variables $Q=P^{-1}$, $F_l=K_lQ$, *and* $\overline{\psi}_k=Q\psi_kQ$, (31a) *and* (31b) *can be rewritten as follows:*

$$\begin{aligned}
&\sum_{k=1}^{g}\tau_k\left\{A_iQ-BF_l+QA_i^T-F_l^TB^T\right\}\\
&\quad+\sum_{k=1}^{g}\overline{\psi}_k+ngQR^TRQ+QIQ\\
&\quad+\sum_{k=1}^{g}\tau_k^2\left(\xi^{-1}+n^{-1}+ga^{-1}\right)I<0,
\end{aligned}\tag{46a}$$

$$gaQ\overline{A}_{ik}^T\overline{A}_{ik}Q-\overline{\psi}_k<0,\tag{46b}$$

for $i=1,2,\ldots,\phi$; $k=1,2,\cdots,g$ *and* $l=1,2,\cdots,\rho$. *According to Schur's complement [36], it is easy to show that the linear* *matrix inequalities in* (46a) *and* (46b) *are equivalent to the following LMIs in* (47a) *and* (47b):

$$\begin{bmatrix} \Xi & QR^T & Q \\ RQ^T & -(ng)^{-1}I & 0 \\ Q & 0 & -I \end{bmatrix} < 0, \tag{47a}$$

$$\begin{bmatrix} -\overline{\psi}_k & Q\overline{A}_{ik}^T \\ \overline{A}_{ik}Q & -(ga)^{-1}I \end{bmatrix} < 0, \tag{47b}$$

where

$$\begin{aligned} \Xi \equiv & \sum_{k=1}^{g} \tau_k A_i Q - \sum_{k=1}^{g} \tau_k B F_l \\ & + \sum_{k=1}^{g} \tau_k Q A_i^T - \sum_{k=1}^{g} \tau_k F_l^T B^T \\ & + \sum_{k=1}^{g} \overline{\psi}_k + \sum_{k=1}^{g} \tau_k^2 \left(\xi^{-1} + n^{-1} + ga^{-1}\right) I. \end{aligned} \tag{48}$$

Hence, Theorem 5 can be transformed into an LMI problem, and efficient interior-point algorithms are now available in Matlab LMI Solver to solve this problem.

Corollary 7 (see [53]). *In order to verify the feasibility of solving the inequalities in* (47a) *and* (47b) *using LMI Solver (Matlab), the interior-point optimization techniques are utilized to compute feasible solutions. Such techniques require that the system of LMI is constrained to be strictly feasible; that is, the feasible set has a nonempty interior. For feasibility problems, the LMI Solver by feasp (feasp is the syntax used to test feasibility of a system of LMIs in MATLAB) is shown as follows:*

$$\text{find } x \text{ such that the LMI } L(x) < 0, \tag{49a}$$

(in this study, (49a) *can be represented as* (47a) *and* (47b)*) and*

$$\text{minimize } t \text{ subject to } \; L(x) < t \times I. \tag{49b}$$

From the abovementioned, the LMI constraint is always strictly feasible in x, t and the original LMI (49a) is feasible if and only if the global minimum $t_{\min}$ (the global minimum $t_{\min}$ is the scalar value returned as the output argument by feasp) of (49b) satisfies $t_{\min} < 0$. In other words, if $t_{\min} < 0$ will satisfy (47a) and (47b) then the stability conditions (31a) and (31b) in Theorem 5 can be met. Then, the obtained fuzzy controller (19) can exponentially stabilize the error system, and the H^∞ control performance is achieved at the same time.

Corollary 8. *In order to achieve optimal H^∞ exponential synchronization, the fuzzy control design is formulated as the following constrained optimization problem:*

$$\begin{gathered} \text{minimize} \quad \kappa > \sqrt{\xi g} \\ \text{subject to} \quad Q = Q^T > 0, \\ \overline{\psi}_k = \overline{\psi}_k^T > 0, \quad \text{(47a) and (47b).} \end{gathered} \tag{50}$$

More details to search the minimum κ are given as follows.

The positive constant ξ is minimized by the mincx function of Matlab LMI toolbox. Therefore, the minimum disturbance attenuation level $\kappa_{\min} > \sqrt{\xi_{\min} g}$ can be obtained.

Remark 9. In order to reduce the computational burden, this study sets the positive constants a and n as unity.

Remark 10. It is an important issue to reduce the effect of external disturbances in the synchronization process. The H^∞ norm bound κ is generally chosen as a positive small value less than unity for attenuation of disturbance. A smaller κ is desirable as this yields better performance. However, a smaller κ will result in a smaller ξ, making the stability conditions (31a) more difficult to satisfy.

Remark 11. According to (25), the modeling error $\Phi(t)$ is assumed to be bounded by the specified structured bounding matrix R, and then a larger $\Phi(t)$ results in a larger R. Since the matrices Δ_{il} must be negative definite to meet the stability condition (31a), a larger R will make Theorem 5 more difficult to satisfy.

4. Algorithm

The complete design procedure can be summarized as follows.

Problem 1. Given two different multiple time-delay chaotic systems with different initial conditions, the problem is centered on how to synthesize a fuzzy controller to realize the optimal H^∞ exponential synchronization.

We can solve this problem based on the following steps.

Step 1. Construct the neural-network (NN) models of the master system (1) and the slave system (2), respectively. According to the interpolation method, the NN models are then converted into LDI state-space representations.

Step 2. On the basis of the state-feedback control scheme, a fuzzy controller (19) is synthesized to exponentially stabilize the error system.

Step 3. Define the synchronization error $E(t) = \widehat{X}(t) - X(t)$, and then the dynamics of the error system (20) can be obtained.

Step 4. Based on Corollary 8, the positive constant ξ is minimized by the mincx function of Matlab LMI toolbox, and then we have the minimum disturbance attenuation level.

Step 5. The matrices Q, F_l, and $\overline{\psi}_k$ can be obtained with the minimum disturbance attenuation $\kappa_{\min}$.

5. Numerical Example

The following example illustrates the effectiveness of the previous algorithm.

Problem 2. The purpose of this example is to synthesize a fuzzy controller to achieve optimal H^{∞} exponential synchronization. Consider the modified multiple time-delay Genesio and Lorenz chaotic systems in master-slave configuration, described as follows:

$$\begin{aligned}\dot{x}_1(t) &= x_2(t),\\ \dot{x}_2(t) &= x_3(t),\\ \dot{x}_3(t) &= -6x_1(t) - 2.92x_2(t-0.015)\\ &\quad - 1.2x_3(t) + x_1^2(t-0.13)\end{aligned} \tag{51}$$

$$\begin{aligned}\dot{\hat{x}}_1(t) &= 10\left(\hat{x}_2(t) - \hat{x}_1(t)\right) + D(t) + u_1(t),\\ \dot{\hat{x}}_2(t) &= 28\hat{x}_1(t) - \hat{x}_2(t-0.13)\\ &\quad - \hat{x}_1(t)\hat{x}_3(t) + D(t) + u_2(t),\\ \dot{\hat{x}}_3(t) &= \hat{x}_1(t)\hat{x}_2(t) - \left(\frac{8}{3}\right)\hat{x}_3(t-0.015) + D(t) + u_3(t),\end{aligned} \tag{52}$$

where $[x_1(t)\ x_2(t)\ x_3(t)]^T$ and $[\hat{x}_1(t)\ \hat{x}_2(t)\ \hat{x}_3(t)]^T$ are the state vectors of master and slave systems, respectively. Let the different initial conditions of master and slave systems be $[x_1(0) = -0.5x_2(0) = 2x_3(0) = 6]$ and $[\hat{x}_1(0) = 0.2\hat{x}_2(0) = -1.5\hat{x}_3(0) = 5]$, and the external disturbance $D(t) = 0.5\sin(2.3t)$.

Figures 2(a) and 2(b) show the chaotic behaviors of the master (51) and slave (52) systems, respectively.

Solution 1. We can solve the previous problem based on the following steps.

Step 1. Establish the NN models for master and slave systems via back propagation algorithm, respectively. First, the NN model to approximate the master chaotic system is constructed by 7–3, and the transfer functions of the hidden layer are chosen as follows:

$$T\left(v_{\varsigma}^{\sigma}(t)\right) = \left\{\frac{2}{\left[1+\exp\left(-v_{\varsigma}^{\sigma}(t)/0.5\right)\right]} - 1\right\}, \quad \text{for } \sigma = 1. \tag{53}$$

On the other hand, the transfer functions of the output layer are chosen as follows:

$$T\left(v_{\varsigma}^{\sigma}(t)\right) = v_{\varsigma}^{\sigma}(t), \quad \text{for } \sigma = 2. \tag{54}$$

After training, we can obtain the following connection weights (the indices in $W_{\varsigma\vartheta}^{\sigma}$ state that the weight of the σth layer in the NN model represents the connection to the ςth neuron from the ϑth source):

$$\begin{aligned}W^1 = \left[W_{\varsigma\vartheta}^1\right] &= 10^{-3} \times \begin{bmatrix} -1.03122 & 5.94314 & -20.9809 & 0.13627 & 507.458 & 868.021 & 588.569 & 0.2062 & 651.633\\ 8.37089 & 26.8407 & 21.6151 & 0.00088 & -239.108 & -740.187 & -377.569 & -0.01391 & 76.6848\\ 501.958 & -3.80717 & 132.938 & 0.80211 & 135.643 & 137.647 & 57.0662 & 0.06242 & 992.269\\ 1963.99 & -273.63 & 359.637 & 8.01727 & -848.291 & -61.2187 & -668.702 & 5.75107 & -843.648\\ -2.69396 & -2.90578 & -10.7761 & 0.02579 & -892.099 & -976.195 & 203.963 & 0.06003 & -114.643\\ -770.561 & 146.747 & -194.79 & 1.70179 & 61.5951 & -325.754 & -474.057 & 0.70796 & -786.694\\ -495.801 & 7.53321 & -127.132 & -0.59639 & 558.334 & -675.635 & 308.158 & -0.00742 & 923.796 \end{bmatrix},\\ W^2 = \left[W_{\varsigma\vartheta}^2\right] &= 10^{2} \times \begin{bmatrix} 0.22075 & 0.37482 & -0.15363 & -0.00174 & 0.2211 & -0.00355 & -0.14954\\ -0.12996 & -0.05835 & -0.00991 & 0.00027 & -0.84887 & -0.00075 & -0.40655\\ 11.2915 & -7.58331 & 4.85989 & -0.05542 & -35.5864 & -0.22732 & 5.1942 \end{bmatrix}.\end{aligned} \tag{55}$$

Then, the net inputs of the σth ($\sigma = 1, 2$) layer are as follows (the symbol v_{ς}^{σ} denotes the net input of the ςth neuron of the σth layer in the NN model, and the indices σ and ς shown in $h_{\varsigma\varphi}^{\sigma}$ ($\varphi = 1, 2$) indicate the same thing):

$$\begin{aligned}v_{\varsigma}^1(t) &= W_{\varsigma1}^1 x_1(t) + W_{\varsigma2}^1 x_2(t) + W_{\varsigma3}^1 x_3(t) + W_{\varsigma4}^1 x_1(t-0.13)\\ &\quad + W_{\varsigma5}^1 \cdot 0 + W_{\varsigma6}^1 \cdot 0 + W_{\varsigma7}^1 \cdot 0 + W_{\varsigma8}^1 x_2(t-0.015)\\ &\quad + W_{\varsigma9}^1 \cdot 0, \quad \varsigma = 1, 2, 3, 4, 5, 6, 7,\end{aligned} \tag{56a}$$

$$\begin{aligned}v_{\varsigma}^2(t) &= W_{\varsigma1}^2 T\left(v_1^1(t)\right) + W_{\varsigma2}^2 T\left(v_2^1(t)\right)\\ &\quad + W_{\varsigma3}^2 T\left(v_3^1(t)\right) + W_{\varsigma4}^2 T\left(v_4^1(t)\right) + W_{\varsigma5}^2 T\left(v_5^1(t)\right)\\ &\quad + W_{\varsigma6}^2 T\left(v_6^1(t)\right) + W_{\varsigma7}^2 T\left(v_7^1(t)\right), \quad \varsigma = 1, 2, 3,\end{aligned} \tag{56b}$$

$$\dot{X}(t) = \begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\end{bmatrix} = \begin{bmatrix}T\left(v_1^2(t)\right)\\ T\left(v_2^2(t)\right)\\ T\left(v_3^2(t)\right)\end{bmatrix}. \tag{57}$$

Based on (8), the minimum and maximum of the derivative of each transfer function shown in (53) and (54) can be obtained as follows:

$$\begin{aligned} g_{\varsigma0}^1 = 0, &\qquad g_{\varsigma0}^2 = 1,\\ g_{\varsigma1}^1 = g_{\varsigma1}^2 = 1, &\quad \text{for } \varsigma = 1, 2, \ldots, J^{\sigma}.\end{aligned} \tag{58}$$

In order to simplify the notation, we let $g_{\varsigma0}^1 = g_0^1$, $g_{\varsigma1}^1 = g_1^1$, $g_{\varsigma0}^2 = g_0^2$ and $g_{\varsigma1}^2 = g_1^2$. Then, according to the interpolation method, we have

$$\begin{aligned}
\dot{x}_1(t) &= \sum_{d=0}^{1} h_{1d}^2(t)\, g_d^2 \sum_{\varsigma=1}^{7} W_{1\varsigma}^2 T\left(v_\varsigma^1(t)\right) \\
&= \sum_{d=0}^{1} h_{1d}^2(t)\, g_d^2 \sum_{\varsigma=1}^{7} W_{1\varsigma}^2 \left(h_{\varsigma 0}^1(t)\, g_0^1 + h_{\varsigma 1}^1(t)\, g_1^1\right) v_\varsigma^1(t) \\
&= \sum_{d=0}^{1} h_{1d}^2(t)\, g_d^2 \\
&\quad \times \sum_{s=0}^{1}\sum_{p=0}^{1}\sum_{r=0}^{1}\sum_{o=0}^{1}\sum_{c=0}^{1}\sum_{l=0}^{1}\sum_{k=0}^{1} h_{1s}^1(t)\, h_{2p}^1(t)\, h_{3r}^1(t) \\
&\qquad \times h_{4o}^1(t)\, h_{5c}^1(t)\, h_{6l}^1(t)\, h_{7k}^1(t) \\
&\quad \cdot \left(g_s^1 W_{\varsigma 1}^2 v_1^1(t) + g_p^1 W_{\varsigma 2}^2 v_2^1(t) + g_r^1 W_{\varsigma 3}^2 v_3^1(t)\right. \\
&\qquad + g_o^1 W_{\varsigma 4}^2 v_4^1(t) + g_c^1 W_{\varsigma 5}^2 v_5^1(t) + g_l^1 W_{\varsigma 6}^2 v_6^1(t) \\
&\qquad \left. + g_k^1 W_{\varsigma 7}^2 v_7^1(t)\right), \\
\dot{x}_2(t) &= \sum_{e=0}^{1} h_{2e}^2(t)\, g_e^2 \sum_{\varsigma=1}^{7} W_{2\varsigma}^2 T\left(v_\varsigma^1(t)\right) \\
&= \sum_{e=0}^{1} h_{2e}^2(t)\, g_e^2 \sum_{\varsigma=1}^{7} W_{1\varsigma}^2 \left(h_{\varsigma 0}^1(t)\, g_0^1 + h_{\varsigma 1}^1(t)\, g_1^1\right) v_\varsigma^1(t) \\
&= \sum_{e=0}^{1} h_{2e}^2(t)\, g_e^2 \\
&\quad \times \sum_{s=0}^{1}\sum_{p=0}^{1}\sum_{r=0}^{1}\sum_{o=0}^{1}\sum_{c=0}^{1}\sum_{l=0}^{1}\sum_{k=0}^{1} h_{1s}^1(t)\, h_{2p}^1(t)\, h_{3r}^1(t) \\
&\qquad \times h_{4o}^1(t)\, h_{5c}^1(t)\, h_{6l}^1(t)\, h_{7k}^1(t) \\
&\quad \cdot \left(g_s^1 W_{\varsigma 1}^2 v_1^1(t) + g_p^1 W_{\varsigma 2}^2 v_2^1(t) + g_r^1 W_{\varsigma 3}^2 v_3^1(t)\right. \\
&\qquad + g_o^1 W_{\varsigma 4}^2 v_4^1(t) + g_c^1 W_{\varsigma 5}^2 v_5^1(t) \\
&\qquad \left. + g_l^1 W_{\varsigma 6}^2 v_6^1(t) + g_k^1 W_{\varsigma 7}^2 v_7^1(t)\right), \\
\dot{x}_3(t) &= \sum_{f=0}^{1} h_{3f}^2(t)\, g_f^2 \sum_{\varsigma=1}^{7} W_{2\varsigma}^2 T\left(v_\varsigma^1(t)\right) \\
&= \sum_{f=0}^{1} h_{3f}^2(t)\, g_f^2 \sum_{\varsigma=1}^{7} W_{1\varsigma}^2 \left(h_{\varsigma 0}^1(t)\, g_0^1 + h_{\varsigma 1}^1(t)\, g_1^1\right) v_\varsigma^1(t) \\
&= \sum_{f=0}^{1} h_{3f}^2(t)\, g_f^2 \\
&\quad \times \sum_{s=0}^{1}\sum_{p=0}^{1}\sum_{r=0}^{1}\sum_{o=0}^{1}\sum_{c=0}^{1}\sum_{l=0}^{1}\sum_{k=0}^{1} h_{1s}^1(t)\, h_{2p}^1(t) \\
&\qquad \times h_{3r}^1(t)\, h_{4o}^1(t)\, h_{5c}^1(t)\, h_{6l}^1(t)\, h_{7k}^1(t) \\
&\quad \cdot \left(g_s^1 W_{\varsigma 1}^2 v_1^1(t) + g_p^1 W_{\varsigma 2}^2 v_2^1(t)\right. \\
&\qquad + g_r^1 W_{\varsigma 3}^2 v_3^1(t) + g_o^1 W_{\varsigma 4}^2 v_4^1(t) + g_c^1 W_{\varsigma 5}^2 v_5^1(t) \\
&\qquad \left. + g_l^1 W_{\varsigma 6}^2 v_6^1(t) + g_k^1 W_{\varsigma 7}^2 v_7^1(t)\right).
\end{aligned} \tag{59}$$

On the basis of (9), let

$$G^1 = \begin{bmatrix} g_s^1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & g_p^1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & g_r^1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & g_o^1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & g_c^1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & g_l^1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & g_k^1 \end{bmatrix}, \quad G^2 = \begin{bmatrix} g_d^2 & 0 & 0 \\ 0 & g_e^2 & 0 \\ 0 & 0 & g_f^2 \end{bmatrix}, \tag{60}$$

then, $E_{defsproclk} \equiv G^2 W^2 G^1 W^1 = [\Upsilon_{\mathfrak{R}\aleph}]_{3\times 9}$, $\mathfrak{R} = 1, 2, 3$; $\aleph = 1, 2 \ldots, 9$.

Plugging (56a) and (56b) into (59) leads to

$$\begin{aligned}
\dot{X}(t) &= \sum_{d=0}^{1}\sum_{e=0}^{1}\sum_{f=0}^{1}\sum_{s=0}^{1}\sum_{p=0}^{1}\sum_{r=0}^{1}\sum_{o=0}^{1}\sum_{c=0}^{1}\sum_{l=0}^{1}\sum_{k=0}^{1} h_{1d}^2(t) \\
&\quad \times h_{2e}^2(t)\, h_{3f}^2(t)\, h_{1s}^1(t)\, h_{2p}^1(t)\, h_{3r}^1(t)\, h_{4o}^1(t) \\
&\quad \times h_{5c}^1(t)\, h_{6l}^2(t)\, h_{7k}^2(t) \left\{A_{defsproclk} X(t)\right. \\
&\qquad + \overline{A}_{defsproclk1} X(t - 0.13) \\
&\qquad \left. + \overline{A}_{defsproclk2} X(t - 0.015)\right\},
\end{aligned} \tag{61}$$

where $X(t) = [x_1(t)\, x_2(t)\, x_3(t)]^T$, $X(t - 0.13) = [x_1(t - 0.13)\, 0\, 0]^T$, $X(t - 0.015) = [0\, x_2(t - 0.015)\, 0]^T$,

$$\begin{aligned}
A_{defsproclk} &= \begin{bmatrix} \Upsilon_{11} & \Upsilon_{12} & \Upsilon_{13} \\ \Upsilon_{21} & \Upsilon_{22} & \Upsilon_{23} \\ \Upsilon_{31} & \Upsilon_{32} & \Upsilon_{33} \end{bmatrix}, \\
\overline{A}_{defsproclk1} &= \begin{bmatrix} \Upsilon_{14} & \Upsilon_{15} & \Upsilon_{16} \\ \Upsilon_{24} & \Upsilon_{25} & \Upsilon_{26} \\ \Upsilon_{34} & \Upsilon_{35} & \Upsilon_{36} \end{bmatrix}, \\
\overline{A}_{defsproclk2} &= \begin{bmatrix} \Upsilon_{17} & \Upsilon_{18} & \Upsilon_{19} \\ \Upsilon_{27} & \Upsilon_{28} & \Upsilon_{29} \\ \Upsilon_{37} & \Upsilon_{38} & \Upsilon_{39} \end{bmatrix}.
\end{aligned} \tag{62}$$

Next, by renumbering the matrices shown in (61), the NN model of master system can be rewritten as the following LDI state-space representation:

$$\dot{X}(t) = \sum_{i=1}^{1024} h_i(t) \left\{A_i X(t) + \sum_{k=1}^{2} \overline{A}_{ik} X(t - \tau_k)\right\}, \tag{63}$$

where $\tau_1 = 0.13, \tau_2 = 0.015$,

$$\begin{gathered} A_1 = A_{0000000000}, \ldots, \qquad A_{1023} = A_{1111111110}, \\ A_{1024} = A_{1111111111}, \\ \overline{A}_{11} = A_{0000000000\,1}, \ldots, \qquad \overline{A}_{1023\,1} = A_{1111111110\,1}, \\ \overline{A}_{1024\,1} = A_{1111111111\,1}, \\ \overline{A}_{12} = A_{0000000000\,2}, \ldots, \qquad \overline{A}_{1023\,2} = A_{1111111110\,2}, \\ \overline{A}_{1024\,2} = A_{1111111111\,2}. \end{gathered} \tag{64}$$

Similarly, the connection weights of the NN model for the slave system are obtained as follows:

$$\begin{gathered} \widehat{W}^1 = \left[\widehat{W}^1_{\varsigma\vartheta}\right] = 10^{-3} \times \begin{bmatrix} 152.414 & -108.845 & 5.89316 & -2.46010 & 141.365 & -68.6751 & 379.275 & 0.92354 & -435.589 \\ 25.9179 & -1.25408 & 0.23133 & -0.03779 & 143.659 & -441.921 & -736.338 & 0.04626 & 951.915 \\ 16.4571 & -32.4571 & -0.07041 & -0.03095 & -427.963 & 350.75 & -752.998 & 0.05790 & -927.148 \\ 185.863 & -168.015 & 63.4102 & 0.77781 & 398.267 & 807.329 & -618.194 & 2.26166 & -347.51 \\ 30.1031 & -6.65419 & 20.3791 & -0.18987 & 592.515 & 817.051 & -708.535 & 0.22644 & 946.027 \\ -29.6884 & 2.93641 & -19.0384 & 0.14757 & -116.821 & 494.393 & 170.087 & -0.19531 & -269.934 \\ -30.3835 & -2.8972 & 14.6662 & -0.05355 & -107.568 & -478.976 & -853.276 & 0.04245 & -381.7 \end{bmatrix}, \\ \widehat{W}^2 = \left[\widehat{W}^2_{\varsigma\vartheta}\right] = 10^{-2} \times \begin{bmatrix} -0.02461 & -2.39353 & -2.95221 & 0.00325 & -0.54634 & -0.78523 & -0.23624 \\ 0.36735 & -35.2973 & 3.22363 & -0.06269 & -32.8677 & -48.5154 & -16.6456 \\ -2.20513 & 8.93374 & -5.80935 & -1.25235 & 164.85 & 164.933 & -10.6894 \end{bmatrix}. \end{gathered} \tag{65}$$

Step 2. The procedures of constructing the NN model for the slave system are similar to those for that of the master system, and then we have the NN model of the slave system as follows:

$$\dot{\widehat{X}}(t) = \sum_{j=1}^{1024} \widehat{h}_j(t) \left\{ \widehat{A}_j \widehat{X}(t) + \sum_{k=1}^{2} \widehat{\overline{A}}_{jk} \widehat{X}(t - \tau_k) \right\} + BU(t), \tag{66}$$

where $\widehat{X}(t) = [\hat{x}_1(t) \quad \hat{x}_2(t) \quad \hat{x}_3(t)]^T$, $\widehat{X}(t-0.13) = [0 \quad \hat{x}_2(t-0.13) \quad 0]^T$, $\widehat{X}(t-0.015) = [0 \quad 0 \quad \hat{x}_3(t-0.015)]^T$ and B is identity matrix. The responses of $\dot{X}(t)$ and $\dot{\widehat{X}}(t)$ for original systems and *NN* models are shown in Figures 3(a) and 3(b).

Step 3. In order to synchronize the master and slave systems, a fuzzy controller is synthesized as follows:

$$\begin{gathered} \text{Control Rule 1: IF } e_1(t) \text{ is } M_1, \text{ THEN } U(t) = -K_1 E(t), \\ \text{Control Rule 2: IF } e_1(t) \text{ is } M_2, \text{ THEN } U(t) = -K_2 E(t), \end{gathered} \tag{67}$$

where M_1 and M_2 are the membership functions for each e_1 (see Figure 4) as follows:

$$M_1(e_1(t)) = \frac{1}{2}\left(1 + \frac{e_1(t)}{q}\right), \tag{68a}$$

$$M_2(e_1(t)) = \frac{1}{2}\left(1 - \frac{e_1(t)}{q}\right). \tag{68b}$$

Based on (19), we have the overall fuzzy controller

$$U(t) = -\frac{\sum_{l=1}^{2} w_l(t) K_l E(t)}{\sum_{l=1}^{2} w_l(t)} = -\sum_{l=1}^{2} \overline{h}_l(t) K_l E(t), \tag{69}$$

with $w_l(t) \equiv M_l(e_1(t))$, $\overline{h}_l(t) \equiv w_l(t) / \sum_{l=1}^{2} w_l(t)$.

Based on (20), the dynamics of the error system are obtained as follows:

$$\begin{aligned} \dot{E}(t) = & \sum_{i=1}^{1024} \sum_{k=1}^{2} \sum_{l=1}^{2} h_i(t) \overline{h}_l(t) \\ & \times \left\{ G_{il} E(t) + \overline{A}_{ik} E(t - \tau_k) \right\} + D(t) + \Phi(t), \end{aligned} \tag{70}$$

where $G_{il} \equiv A_i - BK_l$, $\widehat{\Psi} \equiv f(\widehat{X}(t)) + \sum_{k=1}^{2} H_k(\widehat{X}(t-\tau_k)) + U(t)$, with $U(t) = -\sum_{l=1}^{2} h_l(t) K_l E(t)$, $\Psi \equiv \widehat{f}(X(t)) + \sum_{k=1}^{2} \widehat{H}_k(X(t-\tau_k))$, $\Phi(t) \equiv \widehat{\Psi} - \Psi - \{\sum_{i=1}^{1024} \sum_{k=1}^{2} \sum_{l=1}^{2} h_i(t) \overline{h}_l(t) [G_{il} E(t) + \overline{A}_{ik} E(t - \tau_k)\}$.

Step 4. According to (55) and (61)–(70), the LMIs in (47a) and (47b) can be solved via the Matlab LMI toolbox. In accordance with Remark 1, the specified structured bounding matrix R and ε_{il} are set as

$$R = \begin{bmatrix} 18000 & 0 & 0 \\ 0 & 18000 & 0 \\ 0 & 0 & 18000 \end{bmatrix}, \qquad \varepsilon_{il} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{71}$$

Based on Corollary 8, the positive constant ξ is minimized by the mincx function of Matlab LMI toolbox $\xi_{\min} = 0.0000125$, and then we have the minimum disturbance attenuation level $\rho_{\min} = 0.006$.

Step 5. The common solutions Q, F_1, F_2, $\overline{\psi}_1$, and $\overline{\psi}_2$ of the stability conditions (31a) and (31b) can be obtained with the best value *tmin* of LMI Solver (Matlab) as -2.202477×10^{-7} as follows:

$$Q = 10^{-6} \times \begin{bmatrix} 0.7456 & 0 & 0 \\ 0 & 0.7454 & -0.0001 \\ 0 & -0.0001 & 0.7452 \end{bmatrix}, \tag{72}$$

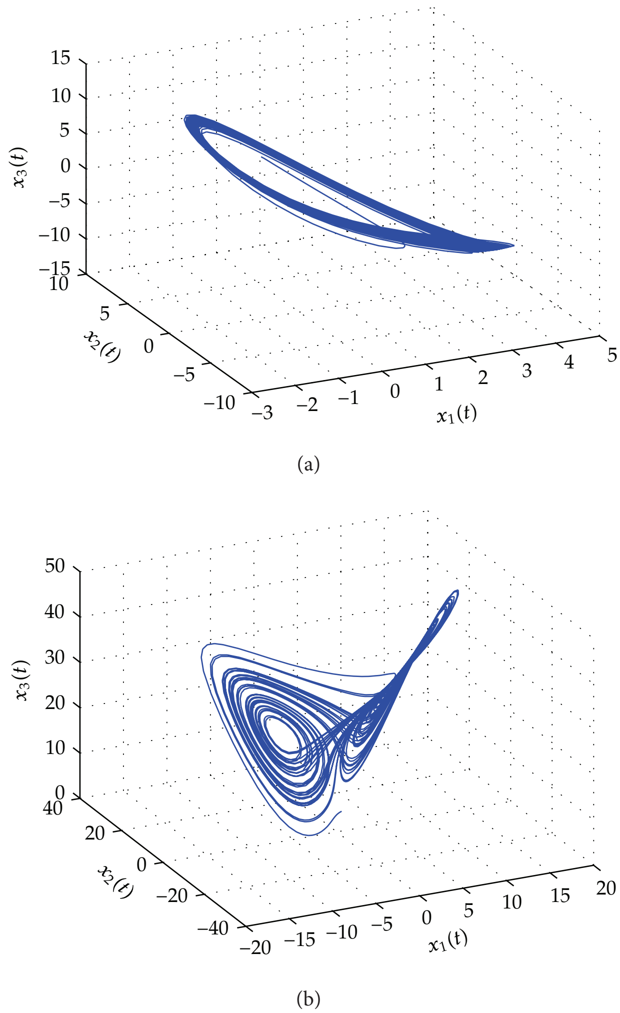

FIGURE 2: (a) Chaotic behavior of the master system (51). (b) Chaotic behavior of the slave system (52) without control.

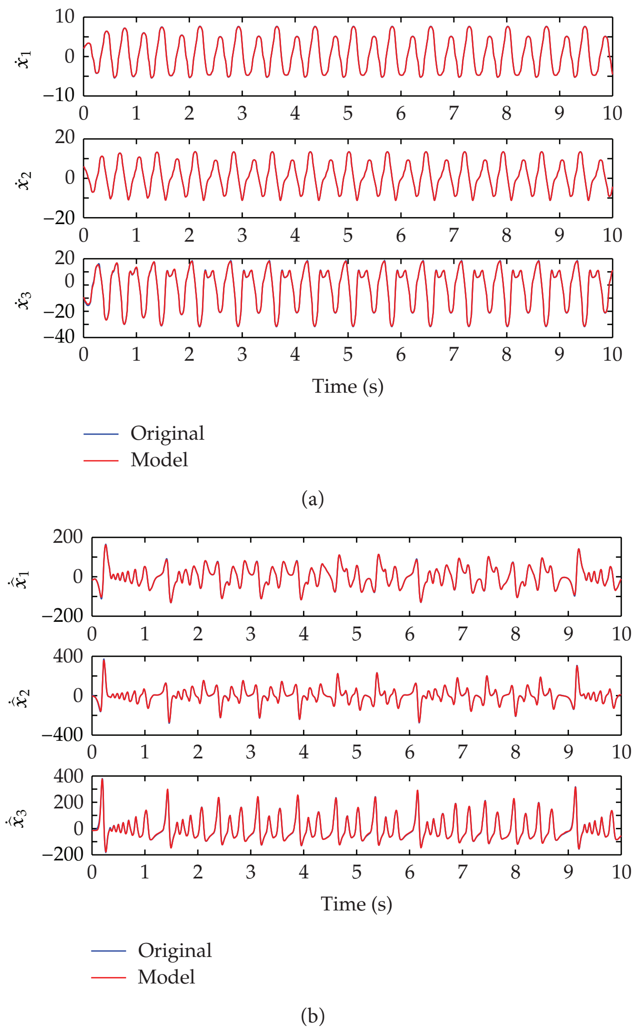

FIGURE 3: (a) The responses of $\dot{X}(t)$ for original system and NN model. (b) The responses of $\dot{\widehat{X}}(t)$ for original system and NN model.

$$F_1 = \begin{bmatrix} 0.0071 & 0 & 0 \\ 0 & 0.0071 & 0 \\ 0 & 0 & 0.0071 \end{bmatrix}, \tag{73a}$$

$$F_2 = \begin{bmatrix} 0.0071 & 0 & 0 \\ 0 & 0.0071 & 0 \\ 0 & 0 & 0.0071 \end{bmatrix}. \tag{73b}$$

In addition, the resulting controller gains are

$$\begin{gathered} K_1 = 10^3 \times \begin{bmatrix} 9.4701 & -0.0004 & -0.0055 \\ 0.0004 & 9.4701 & 0.0014 \\ 0.0055 & -0.0014 & 9.4701 \end{bmatrix}, \\ K_2 = 10^3 \times \begin{bmatrix} 9.4701 & 0.0001 & 0.0013 \\ -0.0001 & 9.4701 & -0.0053 \\ -0.0013 & 0.0053 & 9.4701 \end{bmatrix}, \\ \overline{\psi}_1 = \overline{\psi}_2 = \begin{bmatrix} 1.0344 & 0 & -0.0032 \\ 0 & 1.0345 & 0.0002 \\ -0.0032 & 0.0002 & 1.0345 \end{bmatrix}. \end{gathered} \tag{74}$$

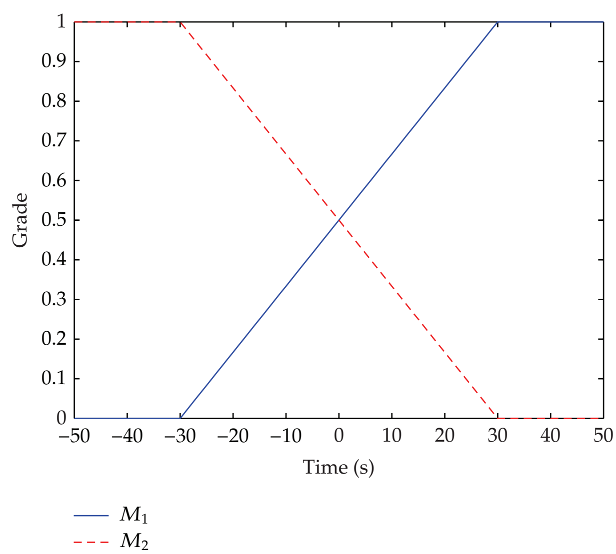

FIGURE 4: Membership functions of the fuzzy controller.

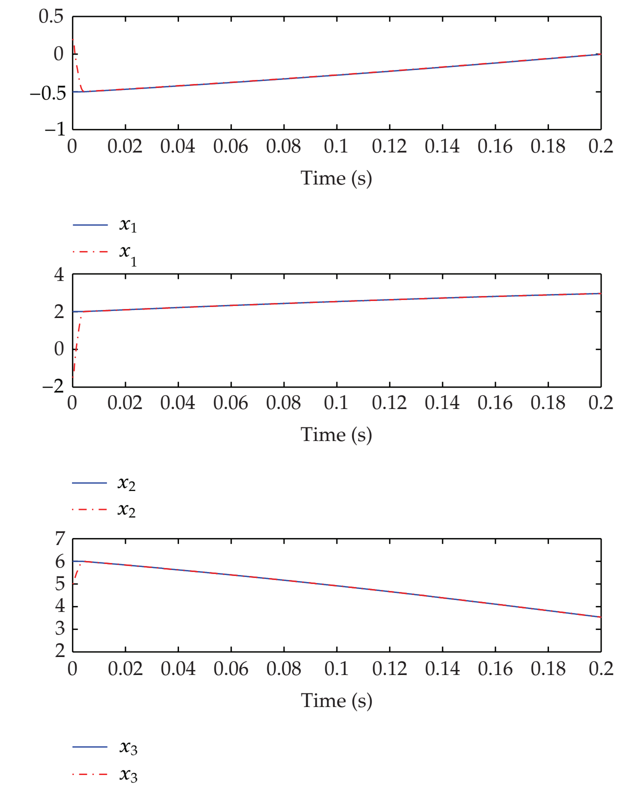

FIGURE 5: State responses of both master and slave systems.

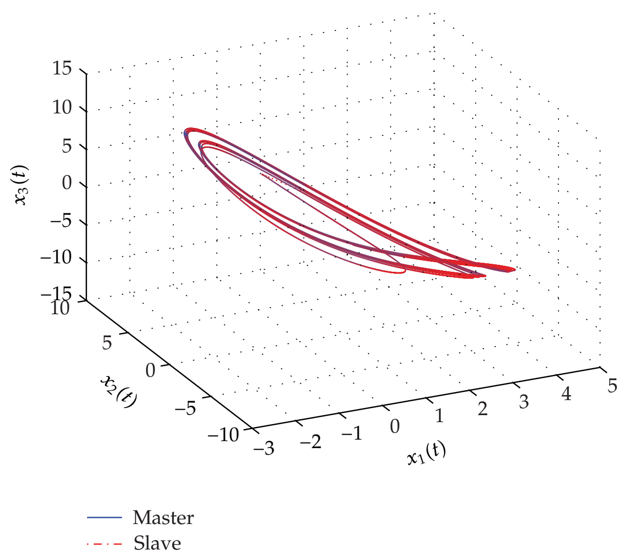

FIGURE 6: The chaotic behaviors of the master and slave systems.

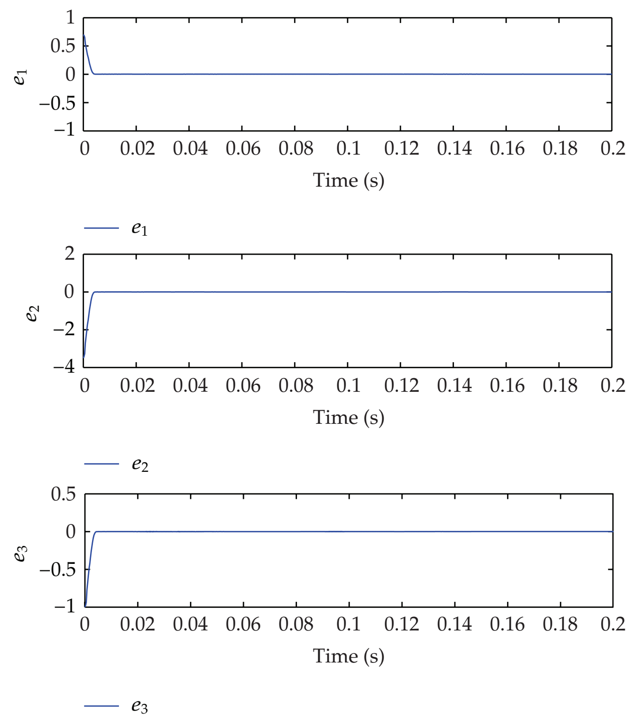

FIGURE 7: State responses of the error system.

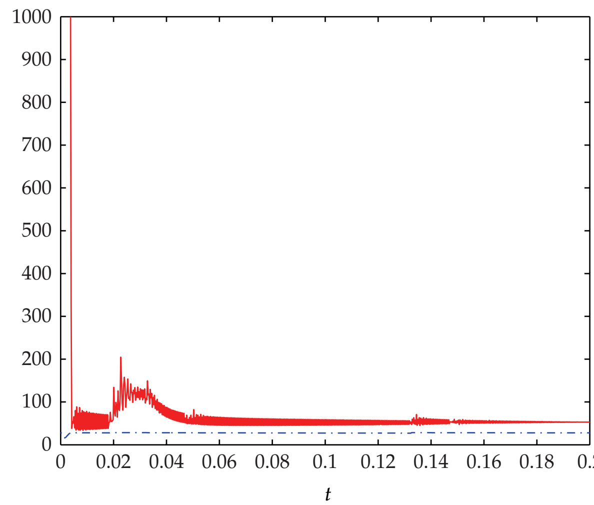

FIGURE 8: Plots of $\|\Phi(t)\|$ (blue line) and $\| \sum_{i=1}^{1024} \sum_{l=1}^{2} h_i(t) h_l(t) \Theta R_{il} E(t) \|$ (red line).

Figure 5 displays the state responses of both master and slave systems. The chaotic behaviors of the master and slave systems are shown in Figure 6. Besides, Figure 7 illustrates the synchronization errors (e_1, e_2, and e_3) which converge to zero. Moreover, the assumption of $\|\Phi(t)\| \leq \| \sum_{i=1}^{1024} \sum_{l=1}^{2} h_i(t) h_l(t) \Theta R_{il} E(t) \|$ is satisfied from the illustration shown in Figure 8.

6. Conclusion

This study proposes a novel approach not only to realize the exponential synchronization of nonidentical multiple time-delay chaotic (MTDC) systems but also to achieve the optimal H^{∞} performance at the same time. First, a neural-network (NN) model is employed to approximate the MTDC system. Then, a linear differential inclusion (LDI) state-space representation is established for the dynamics of the NN model. Next, in terms of Lyapunov's direct method, a delay-dependent stability criterion is derived to ensure that the slave system can exponentially synchronize with the master system. Subsequently, the stability condition of this criterion is reformulated into a linear matrix inequality (LMI). On the basis of the Lyapunov stability theory and LMI approach, a fuzzy controller is synthesized to realize the exponential H^{∞} synchronization of the chaotic master-slave systems and reduce the H^{∞} norm from disturbance to synchronization error at the lowest level. Finally, the simulation results demonstrate that the exponential H^{∞} synchronization of two different MTDC systems can be achieved by the designed fuzzy controller. algorithm, respectively. First, the NN model to approximate the master chaotic

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Research Article

Robust Synchronization Criterion for Coupled Stochastic Discrete-Time Neural Networks with Interval Time-Varying Delays, Leakage Delay, and Parameter Uncertainties

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The purpose of this paper is to investigate a delay-dependent robust synchronization analysis for coupled stochastic discrete-time neural networks with interval time-varying delays in networks coupling, a time delay in leakage term, and parameter uncertainties. Based on the Lyapunov method, a new delay-dependent criterion for the synchronization of the networks is derived in terms of linear matrix inequalities (LMIs) by constructing a suitable Lyapunov-Krasovskii's functional and utilizing Finsler's lemma without free-weighting matrices. Two numerical examples are given to illustrate the effectiveness of the proposed methods.

1. Introduction

In recent years, the problem of synchronization of coupled neural networks which is one of hot research fields of complex networks has been a challenging issue due to its potential applications such as physics, information sciences, biological systems, and so on. Here, complex networks, which are a set of interconnected nodes with specific dynamics, have been studied from various fields of science and engineering such as the World Wide Web, social networks, electrical power grids, global economic markets, and so on. Many mathematical models were proposed to describe various complex networks [1, 2]. Also, in the real applications of systems, there exists naturally time delay due to the finite information processing speed and the finite switching speed of amplifiers. It is well known that time delay often causes undesirable dynamic behaviors such as performance degradation and instability of the systems. So, some sufficient conditions for synchronization of coupled neural networks with time delay have been proposed in [3–5]. Moreover, the synchronization of delayed systems was applied in practical systems such as secure communication [6]. Furthermore, these days, most systems use digital computers (usually microprocessor or microcontrollers) with the necessary input/output hardware to implement the systems. The fundamental character of the digital computer is that it takes compute answers at discrete steps. Therefore, discrete-time modeling with time delay plays an important role in many fields of science and engineering applications. In this regard, various approaches to synchronization stability criterion for discrete-time complex networks with time delay have been investigated in the literature [7–9].

On the other hand, in implementation of many practical systems such as aircraft, chemical and biological systems, and electric circuits, there exist occasionally stochastic perturbations. It is not less important than the time delay as a considerable factor affecting dynamics in the fields of science and engineering applications. Therefore, the study on the problems for various forms of stochastic systems with time-delay has been addressed. For more details, see the literature

[10–13] and references therein. Furthermore, on the problem of synchronization of coupled stochastic neural networks with time delay, various researches have been conducted [14–17]. Li and Yue [14] studied the synchronization stability problem for a class of complex networks with Markovian jumping parameters and mixed time delays. The model considered in [14] has stochastic coupling terms and stochastic disturbances to reflect more realistic dynamical behaviors of the complex networks that are affected by noisy environment. In [15], by utilizing novel Lyapunov-Krasovskii's functional with both lower and upper delay bounds, the synchronization criteria for coupled stochastic discrete-time neural networks with mixed delays were presented. Tang and Fang [16] derived several sufficient conditions for the synchronization of delayed stochastically coupled fuzzy cellular neural networks with mixed delays and uncertain hybrid coupling based on adaptive control technique and some stochastic analysis methods. In [17], by using Kronecker product as an effective tool, robust synchronization problem of coupled stochastic discrete-time neural networks with time-varying delay was investigated. Moreover, Song [18–20] addressed synchronization problem for the array of asymmetric, chaotic, and coupled connected neural networks with time-varying delay or nonlinear coupling. Also, in [21], robust exponential stability analysis of uncertain delayed neural networks with stochastic perturbation and impulse effects was investigated.

Very recently, a time delay in leakage term of the systems is being put to use in the problem of stability for neural networks as a considerable factor affecting dynamics for the worse in the systems [22, 23]. Li et al. [22] studied the existence and uniqueness of the equilibrium point of recurrent neural networks with time delays in the leakage term. By use of the topological degree theory, delay-dependent stability conditions of neural networks of neutral type with time delays in the leakage term were proposed in [23]. Unfortunately, to the best of authors' knowledge, delay-dependent synchronization analysis of coupled stochastic discrete-time neural networks with time-varying delay in network coupling and leakage delay has not been investigated yet. Thus, by attempting the synchronization analysis for the model of coupled stochastic discrete-time neural networks with time delay in the leakage term, the model for coupled neural networks and its applications are closed to the practical networks. Here, delay-dependent analysis has been paid more attention than delay-independent one because the sufficient conditions for delay-dependent analysis make use of the information on the size of time delay [24]. That is, the former is generally less conservative than the latter.

Motivated by the above discussions, the problem of a new delay-dependent robust synchronization criterion for coupled stochastic discrete-time neural networks with interval time-varying delays in network coupling, the time delay in leakage term, and parameter uncertainties is considered for the first time. The coupled stochastic discrete-time neural networks are represented as a simple mathematical model by the use of Kronecker product technique. Then, by construction of a suitable Lyapunov-Krasovskii's functional and utilization of Finsler's lemma without free-weighting matrices, a new synchronization criterion is derived in terms of LMIs. The LMIs can be formulated as convex optimization algorithms which are amenable to computer solution [25]. In order to utilize Finsler's lemma as a tool of getting less conservative synchronization criteria on the number of decision variables, it should be noted that a new zero equality from the constructed mathematical model is devised. The concept of scaling transformation matrix will be utilized in deriving zero equality of the method. In [26], the effectiveness of Finsler's lemma was illustrated by the improved passivity criteria of uncertain neural networks with time-varying delays. Finally, two numerical examples are included to show the effectiveness of the proposed method.

Notation. $\mathbb{R}^n$ is the n-dimensional Euclidean space, and $\mathbb{R}^{m\times n}$ denotes the set of all $m \times n$ real matrices. For symmetric matrices X and Y, $X > Y$ (resp., $X \geq Y$) means that the matrix $X - Y$ is positive definite (resp., nonnegative). $X^{\perp}$ denotes a basis for the null-space of X. I_n and 0_n and $0_{m\times n}$ denote $n \times n$ identity matrix and $n\times n$ and $m\times n$ zero matrices, respectively. $\|\cdot\|$ refers to the Euclidean vector norm or the induced matrix norm. $\lambda_{\max}(\cdot)$ means the maximum eigenvalue of a given square matrix. $\mathrm{diag}\{\cdots\}$ denotes the block diagonal matrix. $\star$ represents the elements below the main diagonal of a symmetric matrix. Let $(\Omega, \mathcal{F}, \{F_t\}_{t\geq 0}, \mathcal{P})$ be complete probability space with a filtration $\{F_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all $\mathcal{P}$-pull sets). $\mathbb{E}\{\cdot\}$ stands for the mathematical expectation operator with respect to the given probability measure $\mathcal{P}$.

2. Problem Statements

Consider the following discrete-time delayed neural networks:

$$\begin{aligned} y(k+1) &= (A + \Delta A)\, y(k-\tau) + (W_1 + \Delta W_1)\, g(y(k)) \\ &\quad + (W_2 + \Delta W_2)\, g(y(k - h(k))) + b, \end{aligned} \tag{1}$$

where n denotes the number of neurons in a neural network, $y(\cdot) = [y_1(\cdot), \ldots, y_n(\cdot)]^T \in \mathbb{R}^n$ is the neuron state vector, $g(\cdot) = [g_1(\cdot), \ldots, g_n(\cdot)]^T \in \mathbb{R}^n$ denotes the neuron activation function vector, $b = [b_1, \ldots, b_n]^T \in \mathbb{R}^n$ means a constant external input vector, $A = \mathrm{diag}\{a_1, \ldots, a_n\} \in \mathbb{R}^{n\times n}$ $(0 < a_q < 1, q = 1, \ldots, n)$ is the state feedback matrix, $W_q \in \mathbb{R}^{n\times n}$ $(q = 1, 2)$ are the connection weight matrices, and ΔA and ΔW_q $(q = 1, 2)$ are the parameter uncertainties of the form

$$[\Delta A, \Delta W_1, \Delta W_2] = DF(k)\,[E_a, E_1, E_2], \tag{2}$$

where $F(k)$ is a real uncertain matrix function with Lebesgue measurable elements satisfying

$$F^T(k)\,F(k) \leq I. \tag{3}$$

The delays $h(k)$ and τ are interval time-varying delays and leakage delay, respectively, satisfying

$$0 < h_m \leq h(k) \leq h_M, \quad 0 < \tau, \tag{4}$$

where h_m and h_M are positive integers.

The neuron activation functions, $g_p(y_p(\cdot))$ $(p = 1, \ldots, n)$, are assumed to be nondecreasing, bounded, and globally Lipschitz; that is,

$$l_p^- \leq \frac{g_p(\xi_p) - g_p(\xi_q)}{\xi_p - \xi_q} \leq l_p^+, \quad \forall \xi_p, \xi_q \in \mathbb{R},\ \xi_p \neq \xi_q, \tag{5}$$

where l_p^- and l_p^+ are constant values.

For simplicity, in stability analysis of the network (1), the equilibrium point $y^* = [y_1^*, \ldots, y_n^*]^T$ is shifted to the origin by the utilization of the transformation $\tilde{y}(\cdot) = \tilde{y}(\cdot) - y^*$, which leads the network (1) to the following form:

$$\begin{aligned} \tilde{y}(k+1) &= (A + \Delta A)\,\tilde{y}(k-\tau) + (W_1 + \Delta W_1)\,\tilde{g}(\tilde{y}(k)) \\ &\quad + (W_2 + \Delta W_2)\,\tilde{g}(\tilde{y}(k - h(k))), \end{aligned} \tag{6}$$

where $\tilde{y}(\cdot) = [\tilde{y}_1(\cdot), \ldots, \tilde{y}_n(\cdot)]^T \in \mathbb{R}^n$ is the state vector of the transformed network, and $\tilde{g}(\tilde{y}(\cdot)) = [\tilde{g}_1(\tilde{y}_1(\cdot)), \ldots, \tilde{g}_n(\tilde{y}_n(\cdot))]^T$ is the transformed neuron activation function vector with $\tilde{g}_q(\tilde{y}_q(\cdot)) = g_q(\tilde{y}_q(\cdot) + y_q^*) - g_q(y_q^*)$ $(q = 1, \ldots, n)$ satisfies, from (5), $l_p^- \leq \tilde{g}_p(\xi_p)/\xi_p \leq l_p^+, \forall \xi_p \neq 0$, which is equivalent to

$$\left[\tilde{g}_p(\tilde{y}_p(k)) - l_p^- \tilde{y}_p(k)\right]\left[\tilde{g}_p(\tilde{y}_p(k)) - l_p^+ \tilde{y}_p(k)\right] \leq 0. \tag{7}$$

In this paper, a model of coupled stochastic discrete-time neural networks with interval time-varying delays in network coupling, leakage delay, and parameter uncertainties is considered as

$$\begin{aligned} \tilde{y}_i(k+1) &= (A + \Delta A)\,\tilde{y}_i(k-\tau) + (W_1 + \Delta W_1)\,\tilde{g}(\tilde{y}_i(k)) \\ &\quad + (W_2 + \Delta W_2)\,\tilde{g}(\tilde{y}_i(k - h(k))) \\ &\quad + \sum_{j=1}^{N} g_{ij}\Gamma \tilde{y}_j(k - h(k))(1 + \omega_1(k)) \\ &\quad + \sigma_i(k, \tilde{y}_i(k), \tilde{y}_i(k - h(k)))\,\omega_2(k), \\ &\qquad i = 1, 2, \ldots, N, \end{aligned} \tag{8}$$

where N is the number of couple nodes, $\tilde{y}_i(k) = [\tilde{y}_{i1}(k), \ldots, \tilde{y}_{in}(k)]^T \in \mathbb{R}^n$ is the state vector of the ith node, $\Gamma \in \mathbb{R}^{n \times n}$ is the constant inner-coupling matrix of nodes, which describe the individual coupling between the subnetworks, $G = [g_{ij}]_{N \times N}$ is the outer-coupling matrix representing the coupling strength and the topological structure of the network satisfies the diffusive coupling connections

$$\begin{aligned} g_{ij} &= g_{ji} \geq 0 \quad (i \neq j), \\ g_{ii} &= -\sum_{j=1, i \neq j}^{N} g_{ij} \quad (i, j = 1, 2, \ldots, N), \end{aligned} \tag{9}$$

and $\omega_q(k)$ $(q = 1, 2)$ are m-dimensional Wiener processes (Brownian Motion) on $(\Omega, \mathcal{F}, \{F_t\}_{t \geq 0}, \mathcal{P})$ which satisfy

$$\begin{aligned} &\mathbb{E}\{\omega_q(k)\} = 0, \\ &\mathbb{E}\{\omega_q^2(k)\} = 1, \\ &\mathbb{E}\{\omega_q(i)\,\omega_q(j)\} = 0 \quad (i \neq j). \end{aligned} \tag{10}$$

Here, $\omega_1(k)$ and $\omega_2(k)$, which are mutually independent, are the coupling strength disturbance and the system noise, respectively. And the nonlinear uncertainties $\sigma_i(\cdot, \cdot, \cdot) \in \mathbb{R}^{n \times m}$ $(i = 1, \ldots, N)$ are the noise intensity functions satisfying the Lipschitz condition and the following assumption:

$$\begin{aligned} &\sigma_i^T(k, \tilde{y}_i(k), \tilde{y}_i(k - h(k)))\,\sigma_i(k, \tilde{y}_i(k), \tilde{y}_i(k - h(k))) \\ &\quad \leq \|H_1 \tilde{y}_i(k)\|^2 + \|H_2 \tilde{y}_i(k - h(k))\|^2, \end{aligned} \tag{11}$$

where H_q $(q = 1, 2)$ are constant matrices with appropriate dimensions.

Remark 1. According to the graph theory [27], the outer-coupling matrix G is called the negative Laplacian matrix of undirected graph. A physical meaning of the matrix G is the bilateral connection between node i and j. If the matrix G cannot satisfy symmetric, the unidirectional connection between nodes i and j is expressed. At this time, the matrix G is called the negative Laplacian matrix of directed graph. Therefore, new numerical model and strong sufficient condition guaranteed to the stability for networks are needed. Moreover, in order to analyze the consensus problem for multiagent systems, the Laplacian matrix of directed graph was used [28].

For the convenience of stability analysis for the network (8), the following Kronecker product and its properties are used.

Lemma 2 (see [29]). *Let $\otimes$ denote the notation of Kronecker product. Then, the following properties of Kronecker product are easily established:*

(i) $(\alpha A) \otimes B = A \otimes (\alpha B)$,

(ii) $(A + B) \otimes C = A \otimes C + B \otimes C$,

(iii) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$,

(iv) $(A \otimes B)^T = A^T \otimes B^T$.

Let us define

$$\begin{aligned} x(k) &= [\tilde{y}_1(k), \ldots, \tilde{y}_N(k)]^T, \\ f(x(k)) &= [\tilde{g}(\tilde{y}_1(k)), \ldots, \tilde{g}(\tilde{y}_N(k))]^T, \\ \sigma(t) &= [\sigma_1(\cdot, \cdot, \cdot), \ldots, \sigma_N(\cdot, \cdot, \cdot)]^T. \end{aligned} \tag{12}$$

Then, with Kronecker product in Lemma 2, the network (8) can be represented as

$$\begin{aligned} x(k+1) &= (I_N \otimes A(k))\,x(k-\tau) + (I_N \otimes W_1(k))\,f(x(k)) \\ &\quad + (I_N \otimes W_2(k))\,f(x(k - h(k))) \\ &\quad + (G \otimes \Gamma)\,x(k - h(k))(1 + \omega_1(k)) + \sigma(t)\,\omega_2(t), \end{aligned} \tag{13}$$

where $A(k) = A + DF(k)E_a$, $W_1(k) = W_1 + DF(k)E_1$, and $W_2(k) = W_2 + DF(k)E_2$.
In addition, for stability analysis, (13) can be rewritten as follows:

$$x(k+1) = \eta(k) + \varrho(k)\omega(k), \tag{14}$$

where

$$\begin{aligned}
\eta(k) &= (I_N \otimes A)x(k-\tau) + (I_N \otimes W_1) f(x(k)) \\
&\quad + (I_N \otimes W_2) f(x(k-h(k))) + (G \otimes \Gamma) x(k-h(k)) \\
&\quad + (I_N \otimes D) p(k), \\
p(k) &= (I_N \otimes F(k)) q(k), \\
q(k) &= (I_N \otimes E_a) x(k-\tau) + (I_N \otimes E_1) f(x(k)) \\
&\quad + (I_N \otimes E_2) f(x(t-h(k))), \\
\varrho(k) &= [(G \otimes \Gamma) x(k-h(k)), \sigma(k)], \\
\omega^T(k) &= [\omega_1^T(k), \omega_2^T(k)].
\end{aligned} \tag{15}$$

The aim of this paper is to investigate the delay-dependent synchronization stability analysis of the network (14) with interval time-varying delays in network coupling, leakage delay, and parameter uncertainties. In order to do this, the following definition and lemmas are needed.

Definition 3 (see [7]). The network (8) is said to be asymptotically synchronized if the following condition holds:

$$\lim_{t\to\infty} \left\| x_i(k) - x_j(k) \right\| = 0, \quad i, j = 1, 2, \ldots, N. \tag{16}$$

Lemma 4 (see [3]). *Let $U = [u_{ij}]_{N\times N}$, $P \in \mathbb{R}^{n\times n}$, $x^T = [x_1, x_2, \ldots, x_n]^T$, and $y^T = [y_1, y_2, \ldots, y_n]^T$. If $U = U^T$ and each row sum of U is zero, then*

$$x^T(U \otimes P) y = -\sum_{1\le i<j\le N} u_{ij} (x_i - x_j)^T P (y_i - y_j). \tag{17}$$

Lemma 5 (see [30]). *For any constant matrix $0 < M = M^T \in \mathbb{R}^{n\times n}$, integers h_m and h_M satisfying $1 \le h_m \le h_M$, and vector function $x(k) \in \mathbb{R}^n$, the following inequality holds:*

$$\begin{aligned}
&-(h_M - h_m + 1)\sum_{k=h_m}^{h_M} x^T(k) M x(k) \\
&\quad \le -\left(\sum_{k=h_m}^{h_M} x(k)\right)^T M \left(\sum_{k=h_m}^{h_M} x(k)\right).
\end{aligned} \tag{18}$$

Lemma 6 (see [31] (Finsler's lemma)). *Let $\zeta \in \mathbb{R}^n$, $\Phi = \Phi^T \in \mathbb{R}^{n\times n}$, and $\Upsilon \in \mathbb{R}^{m\times n}$ such that rank$(\Upsilon) < n$. The following statements are equivalent:*

(i) $\zeta^T \Phi \zeta < 0$, $\forall \Upsilon\zeta = 0$, $\zeta \ne 0$,

(ii) ${\Upsilon^{\perp}}^T \Phi \Upsilon^{\perp} < 0$.

3. Main Results

In this section, a new synchronization criterion for the network (14) will be proposed. For the sake of simplicity on matrix representation, e_i $(i = 1, \ldots, 9) \in \mathbb{R}^{9n\times n}$ are defined as block entry matrices (e.g., $e_2 = [0_n, I_n, 0_n, 0_n, 0_n, 0_n, 0_n, 0_n, 0_n]^T$). The notations of several matrices are defined as follows:

$$\begin{aligned}
\zeta^T(k) &= \left[x^T(k), x^T(k-\tau), x^T(k-h_m), x^T(k-h(k)),\right. \\
&\qquad x^T(k-h_M), (\eta(k) - x(k))^T, f^T(x(k)), \\
&\qquad \left. f^T(x(k-h(k))), p^T(k)\right], \\
z_{ij}(k) &= x_i(k) - x_j(k), f(z_{ij}(k)) = f(x_i(k)) - f(x_j(k)), \\
\eta_{ij}(k) &= \eta_i(k) - \eta_j(k), p_{ij}(k) = p_i(k) - p_j(k), \\
\zeta_{ij}^T(k) &= \left[z_{ij}^T(k), z_{ij}^T(k-\tau), z_{ij}^T(k-h_m), z_{ij}^T(k-h(k)),\right. \\
&\qquad z_{ij}^T(k-h_M), (\eta_{ij}(k) - z_{ij}(k))^T, f^T(z_{ij}(k)), \\
&\qquad \left. f^T(z_{ij}(k-h(k))), p_{ij}^T(k)\right], \\
\Upsilon_{ij} &= \left[-I_n, A, 0_n, -(Ng_{ij}\Gamma), 0_n, -I_n, W_1, W_2, D\right], \\
\Sigma &= P + h_m^2 R_1 + (h_M - h_m)^2 R_2 + \tau^2 S_2, \\
\Xi_1 &= e_1 P e_6^T + e_6 P e_1^T + e_6 P e_6^T, \\
\Xi_2 &= e_1 Q_1 e_1^T - e_3 (Q_1 - Q_2) e_3^T - e_5 Q_2 e_5^T, \\
\Xi_3 &= e_6 \left(h_m^2 R_1 + (h_M - h_m)^2 R_2\right) e_6^T - (e_1 - e_3) R_1 (e_1 - e_3)^T \\
&\quad - (e_3 - e_4) R_2 (e_3 - e_4)^T - (e_4 - e_5) R_2 (e_4 - e_5)^T \\
&\quad - (e_3 - e_4) T^T (e_4 - e_5)^T - (e_4 - e_5) T (e_3 - e_4)^T, \\
\Xi_4 &= e_1 S_1 e_1^T - e_2 S_1 e_2^T + e_6 (\tau^2 S_2) e_6^T - (e_1 - e_2) S_2 (e_1 - e_2)^T, \\
\Xi_5 &= e_4 \left(N \sum_{l=1}^{N} g_{il} g_{lj} \Gamma^T \Sigma \Gamma\right) e_4^T + e_1 \left(\rho H_1^T H_1\right) e_1^T \\
&\quad + e_4 \left(\rho H_2^T H_2\right) e_4^T, \\
\Xi_6 &= -e_1 \left(2 L_m D_1 L_p\right) e_1^T + e_1 \left(L_m + L_p\right) D_1 e_7^T \\
&\quad + \left(e_1 \left(L_m + L_p\right) D_1 e_7^T\right)^T - e_7 (2D_1) e_7^T \\
&\quad - e_4 \left(2 L_m D_2 L_p\right) e_4^T + e_4 \left(L_m + L_p\right) D_2 e_8^T \\
&\quad + \left(e_4 \left(L_m + L_p\right) D_2 e_8^T\right)^T - e_8 (2D_2) e_8^T, \\
\Xi_7 &= -e_9 (\epsilon I_n) e_9^T, \\
\Psi &= \left[0_n, E_a, 0_n, 0_n, 0_n, 0_n, E_1, E_2, 0_n\right].
\end{aligned} \tag{19}$$

Then, the main result of this paper is presented as follows.

Theorem 7. *For given positive integers h_m, h_M and τ, diagonal matrices $L_m = \mathrm{diag}\{l_1^-, \ldots, l_n^-\}$ and $L_p = \mathrm{diag}\{l_1^+, \ldots, l_n^+\}$, the network* (14) *is asymptotically synchronized for $h_m \le h(k) \le h_M$, if there exist positive scalars ρ, ϵ, positive definite matrices $P, Q_1, Q_2, R_1, R_2, S_1, S_2$, positive diagonal matrices D_1, D_2, and any matrix T satisfying the following LMIs for $1 \le i < j \le N$:*

$$\Sigma - \rho I_n \le 0, \tag{20}$$

$$\begin{bmatrix} R_2 & T \\ \star & R_2 \end{bmatrix} \ge 0, \tag{21}$$

$$\begin{aligned} &\left[\begin{array}{c|c} \left[(j-i)\Upsilon_{ij}\right]^{\perp} & 0_{9n\times n} \\ \hline 0_{n\times 8n} & I_n \end{array}\right]^T \left[\begin{array}{c|c} \sum_{l=1}^{7}\Xi_l & \epsilon\Psi^T \\ \hline \star & -\epsilon I_n \end{array}\right] \\ &\times \left[\begin{array}{c|c} \left[(j-i)\Upsilon_{ij}\right]^{\perp} & 0_{9n\times n} \\ \hline 0_{n\times 8n} & I_n \end{array}\right] < 0, \end{aligned} \tag{22}$$

where Σ, Υ_{ij}, Ξ_l $(l = 1, \ldots, 7)$, and Ψ are defined in (19).

Proof. Define a matrix U as

$$U = \left[u_{ij}\right]_{N\times N} = \begin{bmatrix} N-1 & -1 & \cdots & -1 \\ -1 & N-1 & -1 & \vdots \\ \vdots & -1 & \ddots & -1 \\ -1 & \cdots & -1 & N-1 \end{bmatrix} \tag{23}$$

and the forward difference of $x(k)$ and $V(k)$ as

$$\begin{aligned} \Delta x(k) &= x(k+1) - x(k) = \eta(k) - x(k) + \varrho(k)\omega(t), \\ \Delta V(k) &= V(k+1) - V(k). \end{aligned} \tag{24}$$

Let us consider the following Lyapunov-Krasovskii's functional candidate as

$$V(k) = V_1(k) + V_2(k) + V_3(k) + V_4(k), \tag{25}$$

where

$$\begin{aligned} V_1(k) &= x^T(k)(U\otimes P)x(k), \\ V_2(k) &= \sum_{s=k-h_m}^{k-1} x^T(s)(U\otimes Q_1)x(s) \\ &\quad + \sum_{s=k-h_M}^{k-h_m-1} x^T(s)(U\otimes Q_2)x(s), \\ V_3(k) &= h_m \sum_{s=-h_m}^{-1}\sum_{u=k+s}^{k-1} \Delta x^T(u)(U\otimes R_1)\Delta x(u) \\ &\quad + (h_M - h_m)\sum_{s=-h_M}^{-h_m-1}\sum_{u=k+s}^{k-1} \Delta x^T(u)(U\otimes R_2)\Delta x(u), \\ V_4(k) &= \sum_{s=k-\tau}^{k-1} x^T(s)(U\otimes S_1)x(s) \\ &\quad + \tau\sum_{s=-\tau}^{-1}\sum_{u=k+s}^{k-1} \Delta x^T(u)(U\otimes S_2)\Delta x(u). \end{aligned} \tag{26}$$

The mathematical expectation of $\Delta V(k)$ is calculated as follows:

$$\begin{aligned} &\mathbb{E}\{\Delta V_1(k)\} \\ &\quad = \mathbb{E}\Big\{x^T(k+1)(U\otimes P)x(k+1) \\ &\qquad - x^T(k)(U\otimes P)x(k)\Big\} \\ &\quad = \mathbb{E}\Big\{(\Delta x(k) + x(k))^T(U\otimes P)(\Delta x(k) + x(k)) \\ &\qquad - x^T(k)(U\otimes P)x(k)\Big\} \\ &\quad = \mathbb{E}\Big\{\Delta x^T(k)(U\otimes P)\Delta x(k) \\ &\qquad + 2\Delta x^T(k)(U\otimes P)x(k)\Big\} \\ &\quad = \mathbb{E}\Big\{(\eta(k) - x(k))^T(U\otimes P)(\eta(k) - x(k)) \\ &\qquad + (\varrho(k)\omega(k))^T(U\otimes P)(\varrho(k)\omega(k)) \\ &\qquad + 2(\eta(k) - x(k))^T(U\otimes P)x(k)\Big\} \\ &\quad = \mathbb{E}\Big\{(\eta(k) - x(k))^T(U\otimes P)(\eta(k) - x(k)) \\ &\qquad + \underbrace{x^T(t-h(k))(G\otimes\Gamma)^T(U\otimes P)(G\otimes\Gamma)x(t-h(k))}_{\Theta_1} \\ &\qquad + \underbrace{\sigma^T(k)(U\otimes P)\sigma(k)}_{\Omega_1} \\ &\qquad + 2(\eta(k) - x(k))^T(U\otimes P)x(k)\Big\}, \\ &\mathbb{E}\{\Delta V_2(k)\} \\ &\quad = \mathbb{E}\Big\{x^T(k)(U\otimes Q_1)x(k) \\ &\qquad - x^T(k-h_m)(U\otimes(Q_1 - Q_2))x(k-h_m) \\ &\qquad - x^T(k-h_M)(U\otimes Q_2)x(k-h_M)\Big\}, \\ &\mathbb{E}\{\Delta V_3(k)\} \\ &\quad = \mathbb{E}\Bigg\{\Delta x^T(k)\left(U\otimes\left(h_m^2 R_1 + (h_M - h_m)^2 R_2\right)\right)\Delta x(k) \\ &\qquad - h_m\sum_{s=k-h_m}^{k-1}\Delta x^T(s)(U\otimes R_1)\Delta x(s) \\ &\qquad - (h_M - h_m)\sum_{s=k-h_M}^{k-h_m-1}\Delta x^T(s)(U\otimes R_2)\Delta x(s)\Bigg\} \end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\Bigg\{\left(\eta(k)-x(k)\right)^T\left(U\otimes\left(h_m^2R_1+\left(h_M-h_m\right)^2R_2\right)\right)\\
&\quad\times\left(\eta(k)-x(k)\right)\\
&\quad+\underbrace{\begin{pmatrix} x^T(t-h(k))(G\otimes\Gamma)^T\\ \times\left(U\otimes\left(h_m^2R_1+\left(h_M-h_m\right)^2R_2\right)\right)\\ \times(G\otimes\Gamma)\,x(t-h(k))\end{pmatrix}}_{\Theta_2}\\
&\quad+\underbrace{\sigma^T(k)\left(U\otimes\left(h_m^2R_1+\left(h_M-h_m\right)^2R_2\right)\right)\sigma(k)}_{\Omega_2}\\
&\quad-h_m\sum_{s=k-h_m}^{k-1}\Delta x^T(s)\left(U\otimes R_1\right)\Delta x(s)\\
&\quad-\left(h_M-h_m\right)\sum_{s=k-h_M}^{k-h_m-1}\Delta x^T(s)\left(U\otimes R_2\right)\Delta x(s)\Bigg\},\\
&\mathbb{E}\left\{\Delta V_4(k)\right\}\\
&= \mathbb{E}\Bigg\{x^T(k)\left(U\otimes S_1\right)x(k)-x^T(k-\tau)\left(U\otimes S_1\right)x(k-\tau)\\
&\quad+\Delta x^T(k)\left(U\otimes\tau^2S_2\right)\Delta x(k)\\
&\quad-\tau\sum_{s=k-\tau}^{k-1}\Delta x^T(s)\left(U\otimes S_2\right)\Delta x(s)\Bigg\}\\
&= \mathbb{E}\Bigg\{x^T(k)\left(U\otimes S_1\right)x(k)-x^T(k-\tau)\left(U\otimes S_1\right)x(k-\tau)\\
&\quad+\left(\eta(k)-x(k)\right)^T\left(U\otimes\tau^2S_2\right)\left(\eta(k)-x(k)\right)\\
&\quad+\underbrace{x^T(t-h(k))(G\otimes\Gamma)^T\left(U\otimes\tau^2S_2\right)(G\otimes\Gamma)\,x(t-h(k))}_{\Theta_3}\\
&\quad+\underbrace{\sigma^T(k)\left(U\otimes\tau^2S_2\right)\sigma(k)}_{\Omega_3}\\
&\quad-\tau\sum_{s=k-\tau}^{k-1}\Delta x^T(s)\left(U\otimes S_2\right)\Delta x(s)\Bigg\}.
\end{aligned}
\tag{27}
$$

By Lemmas 4 and 5, the sum terms of $\mathbb{E}\{\Delta V_3(k)\}$ are bounded as follows:

$$
\begin{aligned}
&-h_m\sum_{s=k-h_m}^{k-1}\Delta x^T(s)\left(U\otimes R_1\right)\Delta x(s)\\
&\leq-\left(\sum_{s=k-h_m}^{k-1}\Delta x(s)\right)^T\left(U\otimes R_1\right)\left(\sum_{s=k-h_m}^{k-1}\Delta x(s)\right)\\
&=-\sum_{1\leq i<j\leq N}\zeta_{ij}^T(k)\left(e_1^T-e_3^T\right)^TR_1\left(e_1^T-e_3^T\right)\zeta_{ij}(k),
\end{aligned}
\tag{28}
$$

$$
\begin{aligned}
&-\left(h_M-h_m\right)\sum_{s=k-h_M}^{k-h_m-1}\Delta x^T(s)\left(U\otimes R_2\right)\Delta x(s)\\
&\leq-\begin{bmatrix}\sum_{s=k-h_M}^{k-h(k)-1}\Delta x(s)\\ \sum_{s=k-h(k)}^{k-h_m-1}\Delta x(s)\end{bmatrix}^T\begin{bmatrix}\frac{1}{\alpha_k}\left(U\otimes R_2\right) & 0_{Nn}\\ 0_{Nn} & \frac{1}{1-\alpha_k}\left(U\otimes R_2\right)\end{bmatrix}\\
&\quad\times\begin{bmatrix}\sum_{s=k-h_M}^{k-h(k)-1}\Delta x(s)\\ \sum_{s=k-h(k)}^{k-h_m-1}\Delta x(s)\end{bmatrix}\\
&=-\sum_{1\leq i<j\leq N}\zeta_{ij}^T(k)\begin{bmatrix}e_4^T-e_5^T\\ e_3^T-e_4^T\end{bmatrix}^T\\
&\quad\times\begin{bmatrix}\frac{1}{\alpha_k}R_2 & 0_n\\ 0_n & \frac{1}{1-\alpha_k}R_2\end{bmatrix}\begin{bmatrix}e_4^T-e_5^T\\ e_3^T-e_4^T\end{bmatrix}\zeta_{ij}(k),
\end{aligned}
\tag{29}
$$

where $\alpha_k=(h_M-h(k))(h_M-h_m)^{-1}$, which satisfies $0<\alpha_k<1$.

Also, by Theorem 7 in [32], the following inequality for any matrix T holds

$$
\begin{aligned}
&\begin{bmatrix}\sqrt{\frac{1-\alpha_k}{\alpha_k}}I_n & 0_n\\ 0_n & -\sqrt{\frac{\alpha_k}{1-\alpha_k}}I_n\end{bmatrix}\begin{bmatrix}R_2 & T\\ \star & R_2\end{bmatrix}\\
&\quad\times\begin{bmatrix}\sqrt{\frac{1-\alpha_k}{\alpha_k}}I_n & 0_n\\ 0_n & -\sqrt{\frac{\alpha_k}{1-\alpha_k}}I_n\end{bmatrix}\geq 0,
\end{aligned}
\tag{30}
$$

which implies

$$
\begin{bmatrix}\frac{1}{\alpha_k}R_2 & 0_n\\ 0_n & \frac{1}{1-\alpha_k}R_2\end{bmatrix}\geq\begin{bmatrix}R_2 & T\\ \star & R_2\end{bmatrix},
\tag{31}
$$

then, an upper bound of the sum term (29) of $\mathbb{E}\{\Delta V_3(k)\}$ can be rebounded as

$$
\begin{aligned}
&-\left(h_M-h_m\right)\sum_{s=k-h_M}^{k-h_m-1}\Delta x^T(s)\left(U\otimes R_2\right)\Delta x(s)\\
&\leq-\sum_{1\leq i<j\leq N}\zeta_{ij}^T(k)\begin{bmatrix}e_4^T-e_5^T\\ e_3^T-e_4^T\end{bmatrix}^T\begin{bmatrix}R_2 & T\\ \star & R_2\end{bmatrix}\\
&\quad\times\begin{bmatrix}e_4^T-e_5^T\\ e_3^T-e_4^T\end{bmatrix}\zeta_{ij}(k).
\end{aligned}
\tag{32}
$$

Similarly, the sum term of $\mathbb{E}\{\Delta V_4(k)\}$ is bounded as

$$-\tau\sum_{s=k-\tau}^{k-1}\Delta x^T(s)(U\otimes S_2)\Delta x(s) \le -\sum_{1\le i<j\le N}\zeta_{ij}^T(k)(e_1-e_2)S_2(e_1-e_2)^T\zeta_{ij}(k). \tag{33}$$

Also, by properties of Kronecker product in Lemma 2 and $UG = GU = NG$, the terms Θ_q $(q = 1,2,3)$ in (27) are calculated as follows:

$$\begin{aligned}\sum_{l=1}^{3}\Theta_l &= x^T(t-h(k))(G\otimes\Gamma)^T(U\otimes\Sigma)(G\otimes\Gamma)x(t-h(k))\\ &= x^T(t-h(k))\left(NG^TG\otimes\Gamma^T\Sigma\Gamma\right)x(t-h(k)),\end{aligned} \tag{34}$$

where Σ is defined in (19), and, if $\Sigma\le\rho I_n$, then, from (11), the upper bound of terms Ω_q $(q = 1,2,3)$ in (27) is calculated as follows:

$$\begin{aligned}\sum_{l=1}^{3}\Omega_l &= \sigma^T(k)(U\otimes\Sigma)\sigma(k)\\ &\le \rho\left\{x^T(k)\left(U\otimes H_1^TH_1\right)x(k)\right.\\ &\quad\left.+x^T(t-h(k))\left(U\otimes H_2^TH_2\right)x(t-h(k))\right\}.\end{aligned} \tag{35}$$

Then, by utilizing Lemma 4, an upper bound of $\mathbb{E}\{\Delta V(k) = \sum_{l=1}^{4}\Delta V_l(k)\}$ can be written as follows:

$$\mathbb{E}\{\Delta V(k)\}\le\mathbb{E}\left\{\sum_{1\le i<j\le N}\zeta_{ij}^T(k)\left(\sum_{l=1}^{5}\Xi_l\right)\zeta_{ij}(k)\right\}. \tag{36}$$

From (7), for any positive diagonal matrices D_q $(q = 1,2)$, the following inequalities hold.

$$0\le\sum_{1\le i<j\le N}\zeta_{ij}^T(k)\,\Xi_6\zeta_{ij}(k). \tag{37}$$

Since the relational expression between $p(k)$ and $q(k)$, $p^T(k)p(k)\le q^T(k)q(k)$, holds from the second equality of the system (14), there exists a positive scalar ϵ satisfying the following inequality:

$$0\le\sum_{1\le i<j\le N}\zeta_{ij}^T(k)\left(\epsilon\Psi^T\Psi+\Xi_7\right)\zeta_{ij}(k). \tag{38}$$

From (36)–(38), by S-procedure [25], the $\mathbb{E}\{\Delta V(k)\}$ has a new upper bound as follows:

$$\mathbb{E}\{\Delta V(k)\}\le\mathbb{E}\left\{\sum_{1\le i<j\le N}\zeta_{ij}^T(k)\left(\sum_{l=1}^{7}\Xi_l+\epsilon\Psi^T\Psi\right)\zeta_{ij}(k)\right\}. \tag{39}$$

Also, the network (14) with the augmented matrix $\zeta_{ij}(k)$ can be rewritten as follows:

$$\mathbb{E}\left\{\sum_{1\le i<j\le N}(j-i)\Upsilon_{ij}\zeta_{ij}(k)\right\}=0_{n\times1}. \tag{40}$$

Here, in order to illustrate the process of obtaining (40), let us define the following:

$$\Lambda=[\Lambda_1,\Lambda_2,\ldots,\Lambda_N]=[N,N-1,\ldots,1]\otimes I_n\in\mathbb{R}^{n\times Nn}. \tag{41}$$

By (14), (23), and properties of Kronecker product in Lemma 2, we have the following zero equality:

$$\begin{aligned}0_{n\times1} = \mathbb{E}\{&\Lambda(U\otimes A)x(k-\tau)+\Lambda(NG\otimes\Gamma)x(k-h(k))\\ &-\Lambda(U\otimes I_n)(\eta(k)-x(k))+\Lambda(U\otimes W_1)f(x(k))\\ &+\Lambda(U\otimes W_2)f(x(k-h(k)))+\Lambda(U\otimes D)p(k)\}.\end{aligned} \tag{42}$$

By Lemma 4, the first term of (42) can be obtained as follows:

$$\begin{aligned}&\Lambda(U\otimes A)x(k-\tau)\\ &=\underbrace{[NI_n,\ldots,I_n]}_{n\times Nn}\underbrace{(U\otimes A)}_{Nn\times Nn}\underbrace{[x_1(k-\tau),\ldots,x_N(k-\tau)]^T}_{Nn\times1}\\ &=-\sum_{1\le i<j\le N}u_{ij}\left(\Lambda_i-\Lambda_j\right)A\left(x_i(k-\tau)-x_j(k-\tau)\right)\\ &=\sum_{1\le i<j\le N}\left(\Lambda_i-\Lambda_j\right)Az_{ij}(k-\tau)\\ &=\sum_{1\le i<j\le N}\left((N+1-i)I_n-(N+1-j)I_n\right)Az_{ij}(k-\tau)\\ &=\sum_{1\le i<j\le N}(j-i)Az_{ij}(k-\tau).\end{aligned} \tag{43}$$

Similarly, the other terms of (42) are calculated as follows:

$$\begin{aligned}&\Lambda(NG\otimes\Gamma)x(k-h(k))\\ &=-\sum_{1\le i<j\le N}Ng_{ij}\left(\Lambda_i-\Lambda_j\right)\Gamma\\ &\qquad\times\left(x_i(t-h(k))-x_j(t-h(k))\right)\\ &=-\sum_{1\le i<j\le N}(j-i)\left(Ng_{ij}\Gamma\right)z_{ij}(k-h(k)),\\ &-\Lambda(U\otimes I_n)(\eta(k)-x(k))\\ &=\sum_{1\le i<j\le N}u_{ij}\left(\Lambda_i-\Lambda_j\right)I_n\\ &\qquad\times\left((\eta_i(k)-x_i(k))-\left(\eta_j(k)-x_j(k)\right)\right)\\ &=-\sum_{1\le i<j\le N}(j-i)\left(\eta_{ij}(k)-z_{ij}(k)\right),\end{aligned}$$

$$\begin{aligned}
&\Lambda(U\otimes W_1) f(x(k)) \\
&\quad = -\sum_{1\le i<j\le N} u_{ij}(\Lambda_i-\Lambda_j) W_1 \left(f(x_i(k)) - f\left(x_j(k)\right)\right) \\
&\quad = \sum_{1\le i<j\le N} (j-i) W_1 f\left(z_{ij}(k)\right), \\
&\Lambda(U\otimes W_2) f(x(k-h(k))) \\
&\quad = -\sum_{1\le i<j\le N} u_{ij}(\Lambda_i-\Lambda_j) W_2 \\
&\qquad \times \left(f(x_i(t-h(k))) - f\left(x_j(t-h(k))\right)\right) \\
&\quad = \sum_{1\le i<j\le N} (j-i) W_2 f\left(z_{ij}(k-h(k))\right), \\
&\Lambda(U\otimes D) p(k) \\
&\quad = -\sum_{1\le i<j\le N} u_{ij}(\Lambda_i-\Lambda_j) D\left(p_i(k) - p_j(k)\right) \\
&\quad = \sum_{1\le i<j\le N} (j-i) D p_{ij}(k).
\end{aligned} \tag{44}$$

Then, (42) can be rewritten as follows:

$$\begin{aligned}
0_{n\times 1} = \mathbb{E}\Bigg\{ &\sum_{1\le i<j\le N} (j-i) \\
&\times \underbrace{\left[-I_n, A, 0_n, -\left(N g_{ij}\Gamma\right), 0_n, -I_n, W_1, W_2, D\right]}_{\Upsilon_{ij}} \\
&\times \zeta_{ij}(k) \Bigg\}.
\end{aligned} \tag{45}$$

Therefore, if the zero equality (40) holds, then a synchronization condition for the network (14) is

$$\mathbb{E}\left\{\sum_{1\le i<j\le N} \zeta_{ij}^T(k)\left(\sum_{l=1}^{7}\Xi_l + \epsilon\Psi^T\Psi\right)\zeta_{ij}(k)\right\} < 0 \tag{46}$$

subject to

$$\mathbb{E}\left\{\sum_{1\le i<j\le N} (j-i)\Upsilon_{ij}\zeta_{ij}(k)\right\} = 0_{n\times 1}. \tag{47}$$

Here, if inequality (47) holds, then there exists a positive scalar ε such that $\sum_{l=1}^{7}\Xi_l + \epsilon\Psi^T\Psi < -\varepsilon I_{9n}$. From (39) and (47), we have $\mathbb{E}\{\Delta V(k)\} \le \mathbb{E}\{-\varepsilon\sum_{1\le i<j\le N}\|x_i(k) - x_j(k)\|^2\}$. Thus, by Lyapunov theorem and Definition 3, it can be guaranteed that the subnetworks in the coupled discrete-time neural networks (14) are asymptotically synchronized. Also, condition (47) is equivalent to the following inequality:

$$\sum_{1\le i<j\le N} \zeta_{ij}^T(k)\left(\sum_{l=1}^{7}\Xi_l + \epsilon\Psi^T\Psi\right)\zeta_{ij}(k) < 0 \tag{48}$$

subject to

$$\sum_{1\le i<j\le N} (j-i)\Upsilon_{ij}\zeta_{ij}(k) = 0_{n\times 1}. \tag{49}$$

Finally, by the use of Lemma 6, condition (49) is equivalent to the following inequality:

$$\sum_{1\le i<j\le N} \left[(j-i)\Upsilon_{ij}\right]^{\perp T}\left(\sum_{l=1}^{7}\Xi_l + \epsilon\Psi^T\Psi\right)\left[(j-i)\Upsilon_{ij}\right]^{\perp} < 0, \tag{50}$$

and applying Schur complement [25] leads to

$$\sum_{1\le i<j\le N} \left[\begin{array}{c|c} \left[(j-i)\Upsilon_{ij}\right]^{\perp T}\left(\sum_{l=1}^{7}\Xi_l\right)\left[(j-i)\Upsilon_{ij}\right]^{\perp} & \star \\ \hline \epsilon\Psi\left[(j-i)\Upsilon_{ij}\right]^{\perp} & -\epsilon I_n \end{array}\right] < 0, \tag{51}$$

which can be rewritten by

$$\begin{aligned}
\sum_{1\le i<j\le N} &\left[\begin{array}{c|c} \left[(j-i)\Upsilon_{ij}\right]^{\perp} & 0_{9n\times n} \\ \hline 0_{n\times 8n} & I_n \end{array}\right]^T \left[\begin{array}{c|c} \sum_{l=1}^{7}\Xi_l & \epsilon\Psi^T \\ \hline \star & -\epsilon I_n \end{array}\right] \\
&\times \left[\begin{array}{c|c} \left[(j-i)\Upsilon_{ij}\right]^{\perp} & 0_{9n\times n} \\ \hline 0_{n\times 8n} & I_n \end{array}\right] < 0.
\end{aligned} \tag{52}$$

From inequality (52), if the LMIs (22) are satisfied, then stability condition (47) holds. This completes our proof. □

Remark 8. In order to induce a new zero equality (40), the matrix Λ in (41) was defined. It is inspired by the concept of scaling transformation matrix. To reduce the decision variable, Finsler's lemma (ii) $\Upsilon^{\perp T}\Phi\Upsilon^{\perp} < 0$ without free-weighting matrices was used. At this time, a zero equality is required. If the matrix Λ is not considered, then the following description (see only (43) as an example)

$$\begin{aligned}
&\{\,\}(U\otimes A)\,x(k-\tau) \\
&\quad = \{\,\}(U\otimes A)\left[x_1(k-\tau),\ldots,x_N(k-\tau)\right]^T \\
&\quad = \sum_{1\le i<j\le N}\{\cdot\}\,A\left(x_i(k-\tau) - x_j(k-\tau)\right)
\end{aligned} \tag{53}$$

as shown in (53) does not hold. Thus, the derivation of zero equality in (40) is impossible. Here, to use Lemma 4, a suitable vector or matrix in the empty parentheses $\{\,\}$ is needed. Therefore, by defining the matrix Λ, the induction of the zero equality (40) is possible.

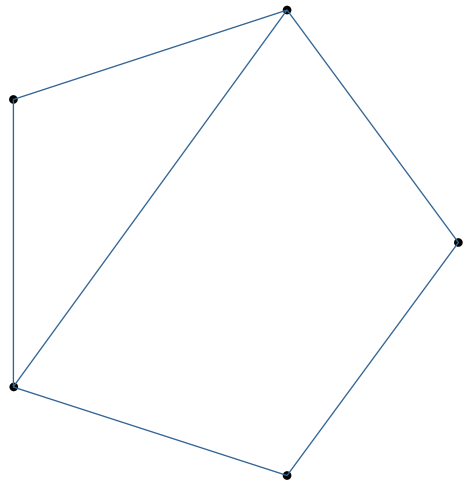


Figure 1: The structure of complex networks with $N = 5$ (Example 10).

Table 1: Maximum allowable delay bounds, h_M, with different h_m and fixed $\tau = 3$ (Example 10).

h_m	1	5	10	50	100
h_M	3	7	12	52	102

Table 2: The conditions of simulation in Example 10.

Number	τ	h_m	h_M	$h(k)$
Cl-1	3 15 30	5	7	$\sin(k\pi/2) + 6$
Cl-2	3	50	52	$\sin(k\pi/2) + 51$

Remark 9. In this paper, the problem of new delay-dependent synchronization for coupled stochastic discrete-time neural networks with leakage delay and parameter uncertainties is considered. By using Finsler's lemma without free-weighting matrices, the proposed robust synchronization criterion for the network is established in terms of LMIs. Here, as mentioned in the Introduction, the leakage delay is the time delay in leakage or forgetting term of the systems and a considerable factor affecting dynamics for the worse in the network. The effect of the leakage delay which cannot be negligible is shown in Figure 2. Also, the stochastic discrete-time systems with parameter uncertainties do not formulate like as the network (14) in any other literature. To do this, the vector $(\eta(k) - x(k))$ is added in the augmented vector $\zeta(k)$. It is just like as $\dot{x}(t)$ in continuous-time systems. This form for the systems may give more less conservative results for stability analysis. As a case of stochastic continuous-time systems with parameter uncertainties, Kwon [13] derived the delay-dependent stability criteria for uncertain stochastic dynamic systems with time-varying delays via the Lyapunov-Krasovskii's functional approach with two delay fraction numbers.

4. Numerical Examples

In this section, we provide two numerical examples to illustrate the effectiveness of the proposed synchronization criterion in this paper.

Example 10. Consider the following coupled neural networks by complex model in Figure 1:

$$\begin{aligned}\tilde{y}_i(k+1) &= (A + \Delta A)\,\tilde{y}_i(k-\tau) + (W_1 + \Delta W_1)\,\tilde{g}(\tilde{y}_i(k)) \\ &\quad + (W_2 + \Delta W_2)\,\tilde{g}(\tilde{y}_i(k - h(k))) \\ &\quad + \sum_{j=1}^{5} g_{ij}\Gamma\tilde{y}_j(k - h(k))\,(1 + \omega_1(k)) \\ &\quad + \sigma_i(k, \tilde{y}_i(k), \tilde{y}_i(k - h(k)))\,\omega_2(k),\end{aligned} \tag{54}$$

with $\tilde{g}(x) = 0.5\tanh(x)$, where

$$\begin{gathered} A = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix}, \\ W_1 = \begin{bmatrix} 0.001 & 0 \\ 0 & 0.005 \end{bmatrix}, \qquad W_2 = \begin{bmatrix} -0.1 & 0.01 \\ -0.2 & -0.1 \end{bmatrix}, \\ \Gamma = 0.01 I_2, \\ G = \begin{bmatrix} -2 & 1 & 0 & 0 & 1 \\ 1 & -3 & 1 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 1 & 1 & -3 & 1 \\ 1 & 0 & 0 & 1 & -2 \end{bmatrix}, \\ L_m = 0_2, \qquad L_p = 0.5 I_2, \qquad D = 0.1 I_2, \\ E_a = \begin{bmatrix} 0.3 & 0 \\ 0 & -0.1 \end{bmatrix}, \qquad E_1 = \begin{bmatrix} -0.4 & 0 \\ 0.3 & -0.7 \end{bmatrix}, \\ E_2 = E_1, \qquad H_1 = 0.2 I_2, \qquad H_2 = H_1. \end{gathered} \tag{55}$$

For the network above, the maximum allowable delay bounds with different h_m and fixed $\tau = 3$ by Theorem 7 are listed in Table 1. In order to confirm the obtained results with the conditions of the time delays as listed in Table 2, the simulation results for the trajectories of state responses, $x_i(k)$ $(i = 2, 3, 4, 5)$, and synchronization errors, $z_{i1}(k) = x_i(k) - x_1(k)$, of the network (54) are shown in Figures 2, 3, 4, and 5. These figures show that the network with the errors converge to zero for given initial values of the state by $x_1^T(0) = [1, -3]$, $x_2^T(0) = [-1, 2]$, $x_3^T(0) = [4, -5]$, $x_4^T(0) = [3, -1]$, and $x_5^T(0) = [4, 2]$. Specially, the simulation results in Figure 2 show state response trajectories for the values of leakage delay, τ, by 3, 15, and 30 with fixed values $h_m = 5$ and $h_M = 7$. It is easy to illustrate that the larger value of leakage delay gives the worse dynamic behaviors of the network (54).

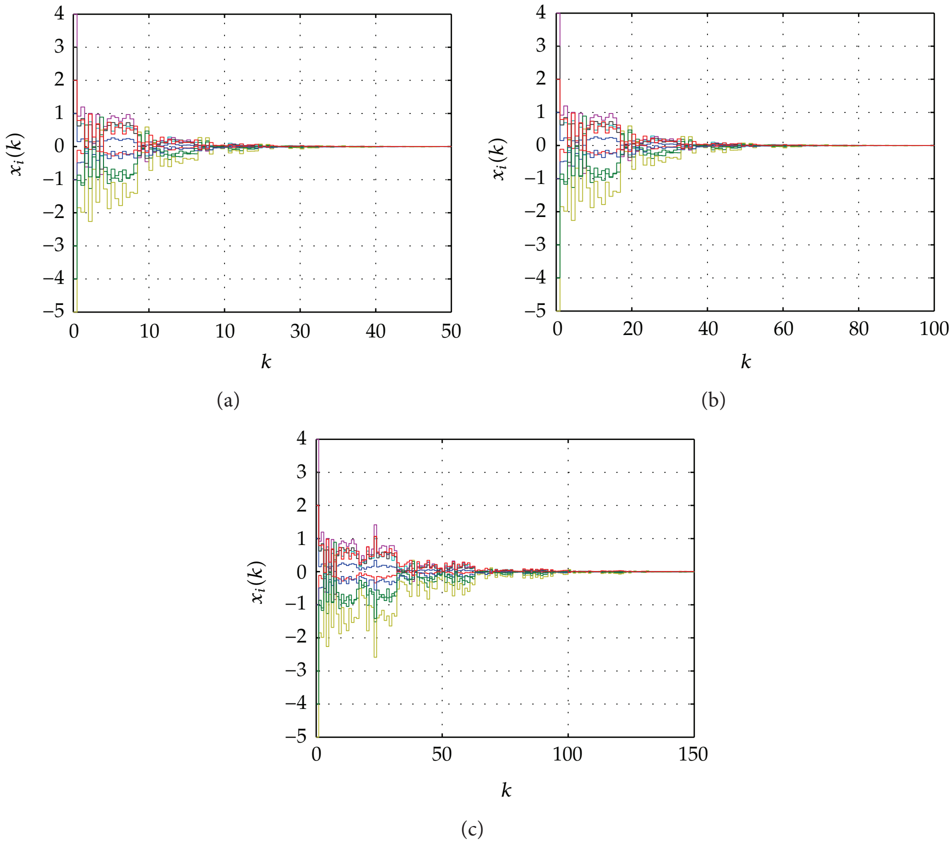


Figure 2: State responses with C1-1 (Example 10): (a) $\tau = 3$, (b) $\tau = 15$, and (c) $\tau = 30$.

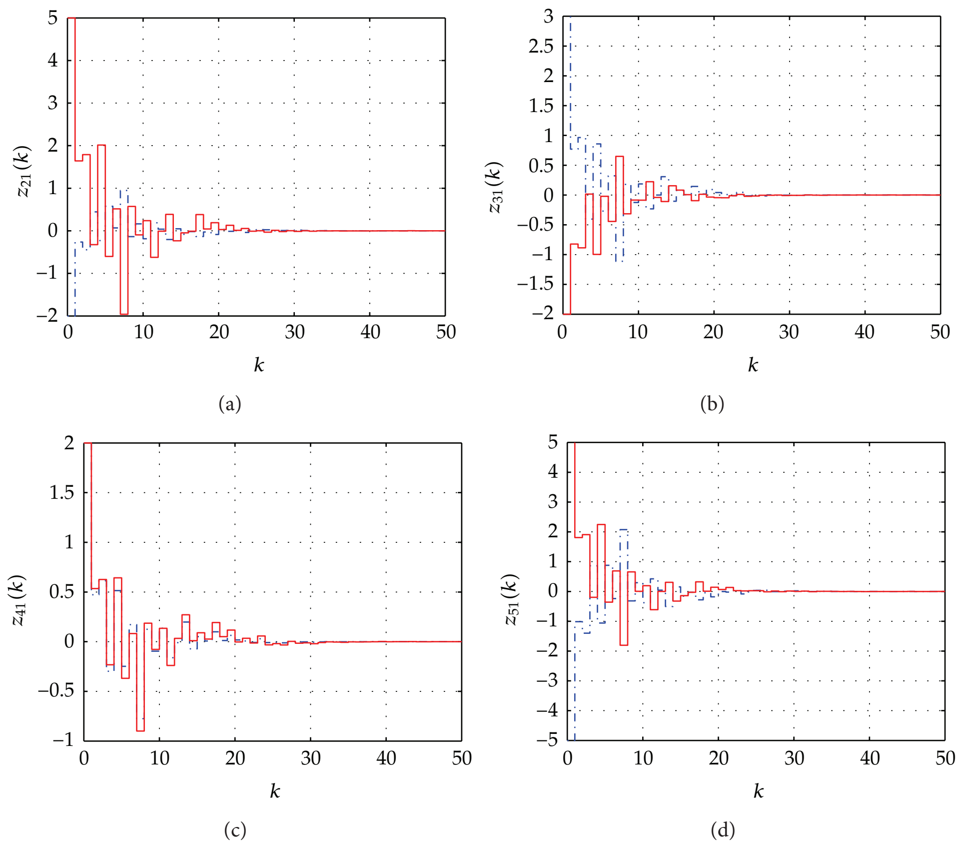


Figure 3: Synchronization errors trajectories with C1-1 ($\tau = 3$) (Example 10).

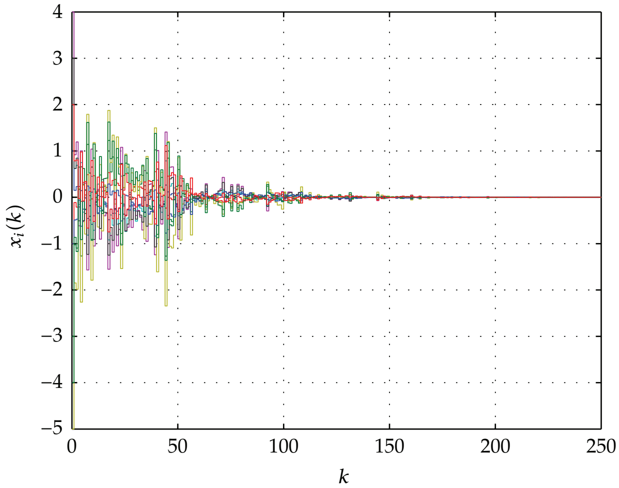


FIGURE 4: State responses with C1-2 (Example 10).

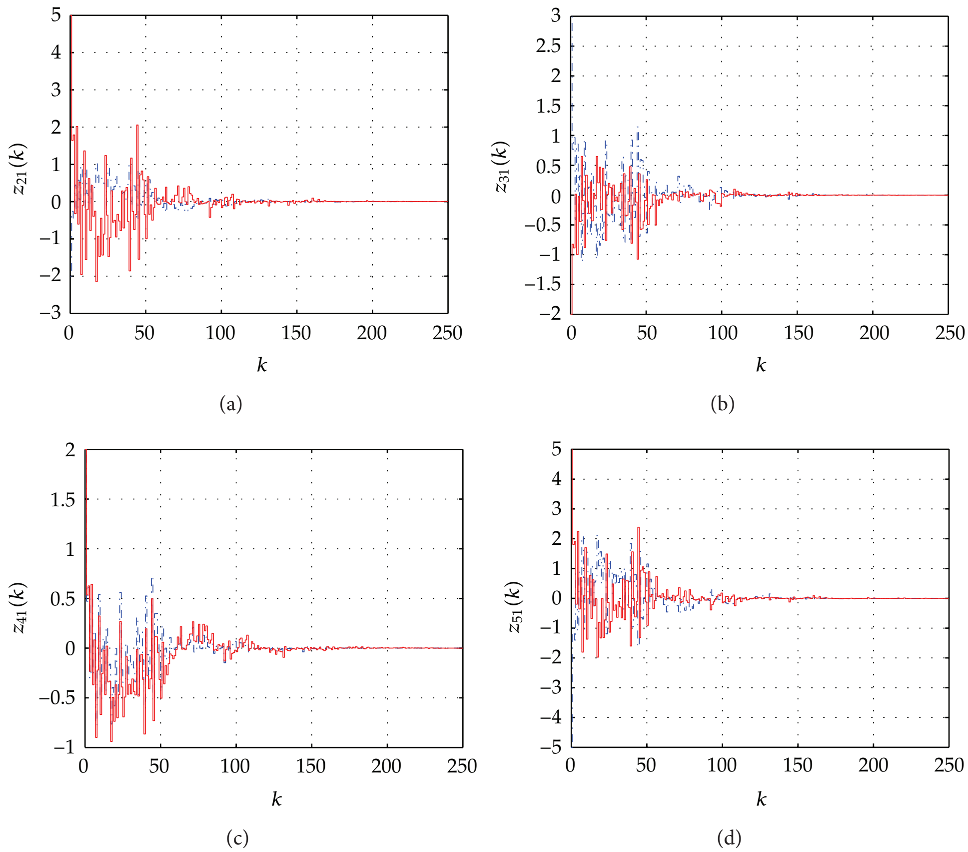


FIGURE 5: Synchronization errors trajectories with C1-2 (Example 10).

Example 11. Consider the following coupled neural networks by BA scale-free model [33] in Figure 6:

$$\begin{aligned}\tilde{y}_i(k+1) &= (A+\Delta A)\,\tilde{y}_i(k-\tau) + (W_1+\Delta W_1)\,\tilde{g}(\tilde{y}_i(k)) \\ &\quad + (W_2+\Delta W_2)\,\tilde{g}(\tilde{y}_i(k-h(k))) \\ &\quad + \sum_{j=1}^{50} g_{ij}\Gamma\tilde{y}_j(k-h(k))\,(1+\omega_1(k)) \\ &\quad + \sigma_i(k,\tilde{y}_i(k),\tilde{y}_i(k-h(k)))\,\omega_2(k),\end{aligned} \tag{56}$$

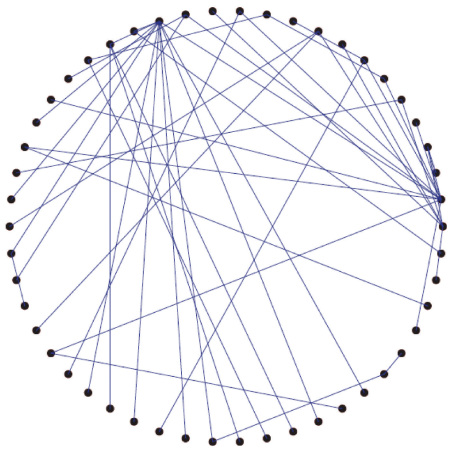
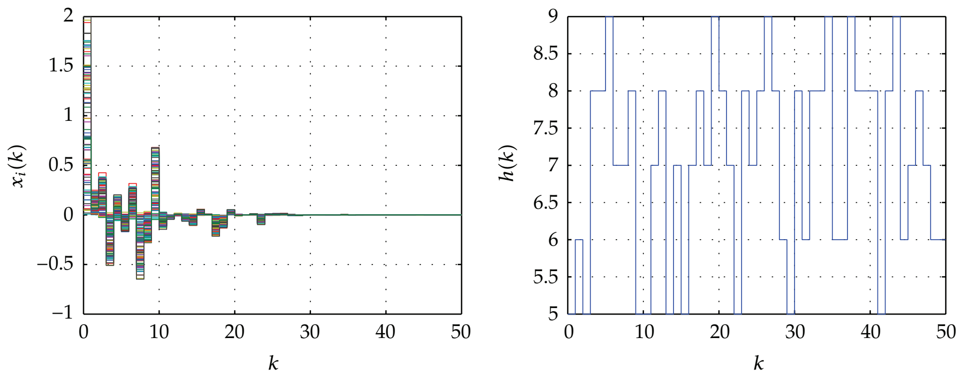


Figure 6: The structure of BA scale-free networks with $N = 50$ (Example 11).



(a) (b)

Figure 7: State responses and time-delay $h(k)$ with C2-1 (Example 11).

with $\tilde{g}(x) = 0.1\tanh(x)$, where

$$A = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.02 \end{bmatrix},$$

$$W_1 = \begin{bmatrix} 0.2 & -0.1 \\ 0.3 & -0.2 \end{bmatrix}, \qquad W_2 = \begin{bmatrix} 0.3 & 0.1 \\ -0.3 & 0.2 \end{bmatrix},$$

$$\Gamma = 0.001I_2, \qquad L_m = 0_2, \qquad L_p = 0.1I_2,$$

$$D = 0.1I_2,$$

$$E_a = \begin{bmatrix} 0.7 & -0.2 \\ 0 & 0.4 \end{bmatrix}, \qquad E_1 = \begin{bmatrix} 0.2 & -0.5 \\ 0 & 0.3 \end{bmatrix},$$

$$E_2 = E_1, H_1 = 0.2I_2, H_2 = H_1. \tag{57}$$

The results of maximum allowable delay bounds with different h_m and fixed $\tau = 3$ by Theorem 7 are listed in Table 3. For lack of space, the outer-coupling matrix G is omitted. It is easy that the matrix G was expressed from Figure 6. Figures 7 and 8 show the state response trajectories, $x_i(t)$ $(i = 1, \ldots, 50)$, of the network (56) with the condition of the time delays as listed in Table 4 for random initial values of the state. These figures show that the network (56) with the state responses converge to zero. This means the synchronization stability of the network (56).

Table 3: Maximum allowable delay bounds, h_M, with different h_m and fixed $\tau = 5$ (Example 11).

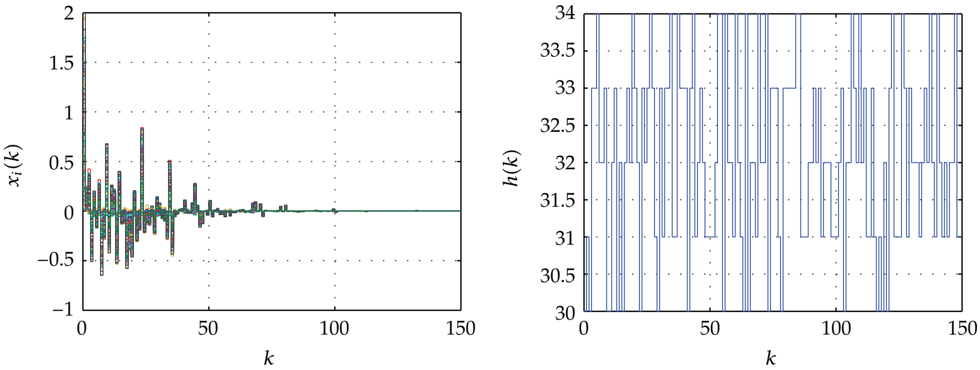
h_m	1	5	10	25	30
h_M	5	9	14	29	34

Table 4: The conditions of simulation in Example 11.

Number	h_m	h_M	$h(k)$
C2-1	5	9	Random integer variable with $5 \le h(k) \le 9$
C2-2	30	34	Random integer variable with $30 \le h(k) \le 34$

5. Conclusions

In this paper, the delay-dependent robust synchronization criterion for the coupled stochastic discrete-time neural



(a) (b)

Figure 8: State responses and time-delay $h(k)$ with C2-2 (Example 11).

networks with interval time-varying delays in network coupling, leakage delay, and parameter uncertainties has been proposed. To do this, the suitable Lyapunov-Krasovskii's functional was used to investigate the feasible region of stability criterion. By utilization of Finsler's lemma with a new zero equality, a sufficient condition for guaranteeing asymptotic synchronization for the concerned networks has been derived in terms of LMIs. Two numerical examples have been given to show the effectiveness and usefulness of the presented criterion.

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Research Article

Input-to-State Stability for Dynamical Neural Networks with Time-Varying Delays

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A class of dynamical neural network models with time-varying delays is considered. By employing the Lyapunov-Krasovskii functional method and linear matrix inequalities (LMIs) technique, some new sufficient conditions ensuring the input-to-state stability (ISS) property of the nonlinear network systems are obtained. Finally, numerical examples are provided to illustrate the efficiency of the derived results.

1. Introduction

Recently, the dynamical neural networks (DNNs), which are firstly introduced by Hopfield in [1], have been extensively studied due to its wide applications in various areas such as associative memory, parallel computation, signal processing, optimization, and moving object speed detection. Since time delay is inevitably encountered in implementation of DNNs and is frequently a source of oscillation and instability, neural networks with time delays have become a topic of great theoretical and practical importance, and many interesting results have been derived (see, e.g., [2–5] and [6–9]). Furthermore, in practical evolutionary processes of the networks, absolute constant delay may be scarce and is only the poetic approximation of the time-varying delays. Delays are generally varied with time because information transmission from one neuron to another neuron may make the response of networks with time-varying delays. Accordingly, dynamical behaviors of neural networks with time-varying delays have been discussed in the last decades (see, e.g., [3, 8–11], etc.).

It is well known that neural networks are often influenced by external disturbances and input errors. Thus many dissipative properties such as robustness [12], passivity [13], and input-to-state stability [4, 10, 11, 14–19] are apparently significant to analyze its dynamical behaviors of the networks. For instance, Ahn incorporated robust training law in switched Hopfield neural networks with external disturbances to study boundedness and exponentially stability [12], and studied passivity in [13]. Especially, the ISS implies not only that the unperturbed system is asymptotically stable in the Lyapunov sense but also that its behavior remains bounded when its inputs are bounded. It is one of the useful classes of dissipative properties for nonlinear systems, which is firstly introduced in nonlinear control systems by Sontag in [20], and then extended by Praly and Jiang [21] and Angeli et al. [19 and Ahn (see [17, 19], and references therein). Due to these research background, the ISS properties of neural networks are investigated in recent years (see, e.g., [16–19] and references therein). For example, by using the Lyapunov function method, some nonlinear feedback matrix norm conditions for ISS have been developed for recurrent neural networks ([16]). Moreover, Ahn utilized Lyapunov function method to discuss robust stability problem for a class of recurrent neural networks, and also some LMI sufficient conditions have been proposed to guarantee the ISS (see [17]). In [18], by employing a suitable Lyapunov function, some results on boundedness, ISS, and convergence are established. Also, in [19] a new sufficient condition is derived to guarantee ISS of Takagi-Sugeno fuzzy Hopfield neural networks with time delay. However, there is few results to deal with the ISS of dynamical neural networks (DNNs) with time-varying delays ([11]).

Motivated by the above discussions, we discuss the ISS properties of DNNs with time-varying delays in this paper. By using Lyapunov-Krasovskii functional technique, ISS conditions for the considered dynamical neural networks are given in terms of LMIs, which can be easily calculated by certain standard numerical packages. We also provide two illustrative examples to demonstrate the effectiveness of the proposed stability results.

The organization of this paper is as follows. In Section 2, our mathematical model of dynamical neural networks is presented and some preliminaries are given. In Section 3, the main results for both ISS and asymptotically stability of dynamical neural networks with time-varying delays are proposed. In Section 4, two numerical examples are illustrated to demonstrate the effectiveness of the theoretical results. Concluding remarks are collected in Section 5. Proof of Lemma 2.4 is given in the appendix.

Notions

Let R^n denote the n-dimensional Euclidean space and $|\cdot|$ denote the usual Euclidean norm. Denote $C = C([-\tau, 0], R^n)$ and designate the norm of an element in C by $\|\phi\|_\tau = \sup_{-\tau\le\vartheta\le 0}\|\phi(\vartheta)\|$. $R^{n\times n}$ is the set of all $n\times n$ real matrices. Let B^T, B^{-1}, $\lambda_{\max}(B)$, $\lambda_{\min}(B)$, and $\|B\| = \sqrt{\lambda_{\max}(B^TB)}$ denote the transpose, the inverse, the largest eigenvalue, the smallest eigenvalue, and the Euclidean norm of a square matrix B, respectively. The notation $P > 0$ (≥ 0) means that P is real symmetric and positive definite (positive semidefinite). The notion $X > Y$ ($X \ge Y$), where X and Y are symmetric matrices, means that $X - Y$ is positive definite (positive semidefinite). I denotes the element matrix. The set of all measurable locally essentially bounded functions $u : R^+ \to R^n$, endowed with (essential) supremum norm $\|u\|_\infty = \sup\{\|u(t)\|, t \ge 0\}$, is denoted by L_∞^m. In addition, denote u_t the truncation of u at t; that is, $u_t(s) = u(s)$ if $s \le t$, and $u(s) = 0$ if $s > t$. We recall that a function $\gamma : R^+ \to R^+$ is a K-function if it is continuous, strictly increasing, and $\gamma(0) = 0$; it will be a K_∞-function if it is a K-function and also $\gamma(s) \to \infty$ as $s \to \infty$. A function $\beta : R^+ \times R^+ \to R^+$ is a KL-function if for each fixed $t \ge 0$ the function $\beta(\cdot, t)$ is a K-function, and, for each fixed $s \ge 0$, it is decreasing to zero as $t \to \infty$.

2. Mathematical Model and Preliminaries

Consider the following nonlinear time-delay system

$$\dot{x} = f(t, x_t, u(t)), \tag{2.1}$$

where $x(t) \in R^n$ is the state vector, $u(t) \in R^n$ is the input function; $x_t \in C$ is the standard function given by $x_t(\tau) = x(t+\tau)$. Without loss of generality, we suppose that $f(0,0,0) = 0$, which ensuring that $x(t) = 0$ is the trivial solution for the unforced system $\dot{x}(t) = f(t, x_t, 0)$. Define $x(t) \triangleq x(t, t_0, \phi)$ is a solution of system with initial value ϕ at time t_0.

Given a continuous functional $V : R^+ \times C \to R^+$, the upper right-hand derivative $\dot{V}$ of the function V is given by

$$\dot{V}(t, \phi) = \limsup_{h \to 0^+} \frac{1}{h}\left[V(t+h, x_{t+h}(t, x_t)) - V(t, x_t)\right]. \tag{2.2}$$

For delayed dynamical system, we first give the input-to-state stable (ISS) definition as usual case.

Definition 2.1. System (2.1) is ISS if there exist a KL-function β and a K-function γ such that, for each input $u \in L_\infty^m$ and each $\xi_{t_0} \in C$, it satisfies

$$|x(t; \xi, u)| \le \beta\left(\|\xi_{t_0}\|_\tau, t - t_0\right) + \gamma(\|u_t\|_\infty), \quad \forall t \ge t_0. \tag{2.3}$$

Note that, by causality, the same definition would result if one could replace $\|u_t\|_\infty$ by $\|u\|_\infty$.

Definition 2.2. A continuous differentiable functional $V(t, \phi) : R^+ \times C \to R^+$ is called the ISS Lyapunov-Krasovskii functional if there exist functions α_1, α_2 of class K_∞, a function χ of class K and a continuous positive definite function W such that

$$\alpha_1\left(|\phi(0)|\right) \le V(t, \phi) \le \alpha_2\left(\|\phi\|_\tau\right), \tag{2.4}$$

$$\dot{V}(t, \phi) \le -W\left(|\phi(0)|\right) \quad \text{if } |\phi(0)| \ge \chi(\|u\|_\infty),\ \forall \phi \in C,\ u \in L_\infty^m. \tag{2.5}$$

Remark 2.3. A continuous differential functional $V(t, \phi) : R^+ \times C \to R^+$ is an ISS Lyapunov-Krasovskii functional if and only if there exist $\alpha_3, \alpha_4 \in K_\infty$ such that (2.4) holds and

$$\dot{V}(t, \phi) \le -\alpha_3\left(|\phi(0)|\right) + \alpha_4(\|u\|_\infty). \tag{2.6}$$

The proof is similarly to one of Remark 2.4 in [23]. We omit it here.

Similarly to the case of ordinary differential equation (ODE), we will establish a link between the ISS property and the ISS Lyapunov-Krasovskii functional for time-delay systems in the following Lemma.

Lemma 2.4. *The system* (2.1) *is ISS if it admits an ISS Lyapunov-Krasovskii functional.*

For completeness, the proof is given in appendix.

To obtain our results, we need the following two useful lemmas.

Lemma 2.5 (Schur Complement [24]). *For given symmetric matrix* $S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$, *where* $S_{11} \in R^{r\times r}$, $S_{21} = S_{12}^T$, *the following three conditions are equivalent:*

(i) $S < 0$;

(ii) $S_{11} < 0, S_{22} - S_{12}^T S_{11}^{-1} S_{12}$;

(iii) $S_{22} < 0, S_{11} - S_{12} S_{22}^{-1} S_{12}^T$.

Lemma 2.6 (see [25]). *Given any matrix* X, Y, *and* Λ *with appropriate dimensions such that* $\Lambda = \Lambda^T$ *and any scalar* $\varepsilon > 0$, *then*

$$X^T Y + Y^T X \le \varepsilon X^T \Lambda X + \frac{1}{\varepsilon} Y^T \Lambda^{-1} Y. \tag{2.7}$$

In this paper, we consider the following dynamical neural networks with time-varying delays

$$\frac{dx_i(t)}{dt} = -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} g_j\left(x_j(t)\right) + \sum_{j=1}^{n} c_{ij} g_j\left(x_j(t-\tau(t))\right) + \sum_{i=1}^{n} u_i(t), \quad i = 1, 2, \ldots, n, \tag{2.8}$$

or equivalently

$$\frac{dx(t)}{dt} = -Ax(t) + Wg(x(t)) + W_1 g(x(t-\tau(t))) + u(t), \tag{2.9}$$

where $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ is the neuron state, $u(t) = (u_1(t), u_2(t), \ldots, u_n(t))^T$ is the input, and $g(x(t)) = (g_1(x_1(t)), g_2(x_2(t)), \ldots, g_n(x_n(t)))^T$ denotes the nonlinear neuron activation function. $A = \operatorname{diag}\{a_1, a_2, \ldots, a_n\}$ is the positive diagonal matrix. $W = (b_{ij})_{n\times n}$ and $W_1 = (c_{ij})_{n\times n}$ are the interconnection matrices representing the weighting coefficients of neurons. $\tau(t)$ is the time-varying delays.

Throughout this paper, we always suppose that

$$\left|g_i(x_i(t))\right| \le l_i |x_i|, \quad \forall i, \forall x_i \in R,\ i = 1, 2, \ldots, n, \tag{$A1$}$$

$$0 \le \tau(t) \le \tau, \qquad 0 \le \dot{\tau}(t) \le 1. \tag{$A2$}$$

From $(A1)$, we easily see that $x(t) = 0$ is the solution of (2.9) with $u(t) = 0$.

3. ISS Analysis

In this section, we give two theorems on ISS in form of LMIs.

Theorem 3.1. *Let* $(A1)$ *and* $(A2)$ *hold. If there exist a positive definite matrix* P *and a positive diagonal matrix* D *such that*

$$\begin{pmatrix} -A^TP-PA & M & I & PW_1 \\ M & D^{-2}+W^TPW & 0 & 0 \\ I & 0 & P & 0 \\ W_1^TP & 0 & 0 & -D^{-2} \end{pmatrix} < 0, \tag{3.1}$$

where $M = \operatorname{diag}\{l_1, l_2, \ldots, l_n\}$, *and then the system* (2.9) *is ISS.*

Proof. We consider the following functional:

$$V(x(t)) = x^T(t)Px(t) + \int_{t-\tau(t)}^{t} g^T(x(\zeta))D^{-2}g(x(\zeta))d\zeta. \tag{3.2}$$

Its derivative along the solution $x(t)$ of (2.9) is given as

$$\begin{aligned}
\dot{V}(x(t)) &= 2x^T(t)P\dot{x}(t) + g^T(x(t))D^{-2}g(x(t)) \\
&\quad - (1-\dot{\tau}(t))g^T(x(t-\tau(t)))D^{-2}g(x(t-\tau(t))) \\
&= 2x^T(t)P\left[-Ax(t) + Wg(x(t)) + W_1g(x(t-\tau(t))) + u(t)\right] \\
&\quad + g^T(x(t))D^{-2}g(x(t)) - (1-\dot{\tau}(t))g^T(x(t-\tau(t)))D^{-2}g(x(t-\tau(t))) \\
&= 2x^T(t)\left(-A^TP - PA\right)x(t) + 2x^T(t)PWg(x(t)) \\
&\quad + 2x^T(t)PW_1g(x(t-\tau(t))) + g^T(x(t))D^{-2}g(x(t)) \\
&\quad - (1-\dot{\tau}(t))g^T(x(t-\tau(t)))D^{-2}g(x(t-\tau(t))) + 2x^T(t)Pu(t).
\end{aligned} \tag{3.3}$$

We have

$$\begin{aligned}
&- (1-\dot{\tau}(t))g^T(x(t-\tau(t)))D^{-2}g(x(t-\tau(t))) + 2x^T(t)PW_1g(x(t-\tau(t))) \\
&\quad = -\left[\sqrt{(1-\dot{\tau}(t))}D^{-1}g(x(t-\tau(t))) - DW_1^TPx(t)\right]^T \\
&\quad\quad \cdot \left[\sqrt{(1-\dot{\tau}(t))}D^{-1}g(x(t-\tau(t))) - DW_1^TPx(t)\right] \\
&\quad\quad + x^T(t)PW_1D^2W_1^TPx(t).
\end{aligned} \tag{3.4}$$

Since the first term of the right-hand side of (3.4) is negative semidefinite, we obtain

$$\begin{aligned}
&- (1-\dot{\tau}(t))g^T(x(t-\tau(t)))D^{-2}g(x(t-\tau(t))) \\
&\quad + 2x^T(t)PW_1g(x(t-\tau(t))) \le x^T(t)PW_1D^2W_1^TPx(t).
\end{aligned} \tag{3.5}$$

From $(A1)$, we obtain

$$g^T(x(t))D^{-2}g(x(t)) \leq x^T(t)MD^{-2}Mx(t), \tag{3.6}$$

where $M = \text{diag}\{l_1, l_2, \ldots, l_n\}$.

Then by Lemma 2.6, we have

$$\begin{aligned}
2x^T(t)PWg(x(t)) &\leq x^T(t)Px(t) + \left[PWg(x(t))\right]^T P^{-1}PWg(x(t)) \\
&= x^T(t)Px(t) + g^T(x(t))W^TPWg(x(t)) \\
&\leq x^T(t)Px(t) + x^T(t)MW^TPWMx(t), \\
2x^T(t)Pu(t) &\leq x^T(t)Px(t) + [Pu(t)]^T P^{-1}Pu(t) \\
&= x^T(t)Px(t) + u^T(t)Pu(t).
\end{aligned} \tag{3.7}$$

Substituting (3.5), (3.6), and (3.7) into (3.3), we finally obtain

$$\begin{aligned}
\dot{V}(x(t)) &\leq x^T(t)Gx(t) + u^T(t)Pu(t) \\
&\leq \lambda_{\min}(G)|x(t)|^2 + \lambda_{\max}(P)|u(t)|^2,
\end{aligned} \tag{3.8}$$

where $G = -A^TP - PA + M(D^{-2} + W^TPW)M + PW_1D^2W_1^TP + P$.

Define $\alpha_3(r) = -\lambda_{\min}(G) \cdot r^2$, $\alpha_4(r) = \lambda_{\max}(P) \cdot r^2$, then we can obtain that

$$\dot{V}(x(t)) \leq -\alpha_3(|x(t)|) + \alpha_4(\|u\|_\infty). \tag{3.9}$$

Note that $G < 0$ is equivalent to (3.1) by Lemma 2.5. Then the defined V is an ISS Lyapunov-Krasovskii functional. It follows from Lemma 2.4 and Remark 2.3 that the delayed neural networks (2.9) are ISS. The proof is complete. □

Remark 3.2. Theorem 3.1 reduces to asymptotically stability condition for dynamical neural networks with time-varying delays when $u(t) = 0$.

Remark 3.3. Recently, some results on ISS or IOSS were obtained in [10, 17–19, 26]. However, these results were restricted to nondelay or constant delay. In contrast to the results [10, 17–19, 26], we consider dynamical neural networks with time-varying delays and propose a set of delay-independent criteria for asymptotically convergent state estimation of these neural networks in this paper.

In the following, we give a delay-dependent sufficient criterion.

Theorem 3.4. *Let* $(A1)$ *and* $(A2)$ *hold. The system* (2.9) *is ISS if there exist a symmetric positive definite matrix* P *and a positive definite matrix* Q *such that*

$$-A^T P - PA + \tau Q = -\mu I, \quad \mu > 0, \forall u \in L_\infty^m, \tag{3.10}$$

$$\left(\frac{2\|M\| \|W_1\|}{\sqrt{1-\dot{\tau}(t)}} + \|M\|^2 \|W\|^2 + 2 \right) < \mu, \tag{3.11}$$

where $M = \operatorname{diag}\{l_1, l_2, \ldots, l_n\}$.

Proof. We consider the following functional:

$$V(x(t)) = x^T(t)Px(t) + \int_{-\tau(t)}^0 \int_{t+\zeta}^t x^T(\eta)Qx(\eta)d\eta\, d\zeta + \int_{t-\tau(t)}^t g^T(x(\zeta))Rg(x(\zeta))d\zeta, \tag{3.12}$$

where R is a positive definite matrix.

The derivative of (3.12) along the trajectories of the system is obtained as follows:

$$\begin{aligned}
\dot{V}(x(t)) &= 2x^T(t)P\big[-Ax(t) + Wg(x(t)) + W_1 g(x(t-\tau(t))) + u(t)\big] \\
&\quad + \tau(t)x^T(t)Qx(t) - (1-\dot{\tau}(t)) \int_{t-\tau(t)}^t x^T(\zeta)Qx(\zeta)d\zeta \\
&\quad + g^T(x(t))Rg(x(t)) - (1-\dot{\tau}(t))g^T(x(t-\tau(t)))Rg(x(t-\tau(t))) \\
&\le x^T(t)\Big(-A^T P - AP + \tau Q\Big)x(t) + 2x^T(t)PWg(x(t)) + 2x^T(t)Pu(t) \\
&\quad + 2x^T(t)PW_1 g(x(t-\tau(t))) - (1-\dot{\tau}(t)) \int_{t-\tau(t)}^t x^T(\zeta)Qx(\zeta)d\zeta \\
&\quad + g^T(x(t))Rg(x(t)) - (1-\dot{\tau}(t))g^T(x(t-\tau(t)))Rg(x(t-\tau(t))).
\end{aligned} \tag{3.13}$$

From (3.10), which reduces to

$$\begin{aligned}
\dot{V}(x(t)) \le &-\mu x^T(t)x(t) + 2x^T(t)PWg(x(t)) + 2x^T(t)Pu(t) \\
&+ 2x^T(t)PW_1 g(x(t-\tau(t))) - (1-\dot{\tau}(t)) \int_{t-\tau(t)}^t x^T(\zeta)Qx(\zeta)d\zeta \\
&+ g^T(x(t))Rg(x(t)) - (1-\dot{\tau}(t))g^T(x(t-\tau(t)))Rg(x(t-\tau(t))).
\end{aligned} \tag{3.14}$$

From $(A1)$, we obtain that

$$g^T(x(t))Rg(x(t)) \le x^T(t)MRMx(t) \le \|R\| \|M\|^2 |x(t)|^2, \tag{3.15}$$

where $M = \operatorname{diag}\{l_1, l_2, \ldots, l_n\}$.

From Lemma 2.6, we have

$$\begin{aligned}
2x^T(t)PWg(x(t)) &\le x^T(t)Px(t) + [PWg(x(t))]^T P^{-1} PWg(x(t)) \\
&= x^T(t)Px(t) + g^T(x(t))W^T PWg(x(t)) \\
&\le x^T(t)Px(t) + x^T(t)MW^T PWMx(t) \\
&\le \left(1 + \|M\|^2\|W\|^2\right)\|P\||x(t)|^2, \\
2x^T(t)PW_1 g(x(t-\tau(t))) &\le (1-\dot{\tau}(t))g^T(x(t-\tau(t)))Rg(x(t-\tau(t))) \\
&\quad + \frac{1}{1-\dot{\tau}(t)} x^T(t)PW_1R^{-1}W_1^T Px(t).
\end{aligned} \tag{3.16}$$

Then

$$\begin{aligned}
&2x^T(t)PW_1 g(x(t-\tau(t))) - (1-\dot{\tau}(t))g^T(x(t-\tau(t)))Rg(x(t-\tau(t))) \\
&\quad \le \frac{1}{1-\dot{\tau}(t)} x^T(t)PW_1R^{-1}W_1^T Px(t) \le \frac{1}{1-\dot{\tau}(t)}\|R\|^{-1}\|W_1\|^2\|P\|^2|x(t)|^2.
\end{aligned} \tag{3.17}$$

For the third term of (3.14), we have

$$\begin{aligned}
2x^T(t)Pu(t) &\le x^T(t)Px(t) + [Pu(t)]^T P^{-1} Pu(t) \\
&= x^T(t)Px(t) + u^T(t)Pu(t) \\
&\le \|P\||x(t)|^2 + \|P\||u(t)|^2.
\end{aligned} \tag{3.18}$$

Substituting (3.15), (3.16), (3.17), and (3.18) into (3.14), we can obtain the following inequality:

$$\dot{V}(x(t)) \le \lambda|x(t)|^2 + \|P\||u(t)|^2, \tag{3.19}$$

where we denote that

$$\begin{aligned}
\lambda &= \left(-\mu + \|R\|\|M\|^2 + \|P\|\|M\|^2\|W\|^2 + 2\|P\| + \frac{1}{1-\dot{\tau}(t)}\|R\|^{-1}\|W_1\|^2\|P\|^2\right), \\
\|R\| &= \frac{\|W_1\|\|P\|}{\sqrt{1-\dot{\tau}(t)}\|M\|}.
\end{aligned} \tag{3.20}$$

From (3.11), we easily obtain that $\lambda < 0$.

Define K_∞-functions $\alpha_3(r) = -\lambda r^2$, $\alpha_4(r) = \|P\| \cdot r^2$. Then we can obtain that

$$\dot{V}(x(t)) \le \alpha_3(|x(t)|) + \alpha_4(\|u\|_\infty). \tag{3.21}$$

From Lemma 2.4 and Remark 2.3, the system (2.9) is ISS. The proof is complete. □

4. Illustrative Examples

In this section, we will give two examples to show the efficiency of the results derived in Section 3.

Example 4.1. Consider a 3-dimension dynamical neural network (2.9) with parameters defined as

$$A = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \qquad W = W_1 = \begin{pmatrix} 1 & -0.1 & -0.2 \\ -0.1 & 1 & -0.3 \\ -0.2 & -0.3 & 1 \end{pmatrix}. \tag{4.1}$$

Letting $g_i(x_i) = 1/(1+e^{-x_i})$ and the time-varying delay is chosen as $\tau(t) = 0.6|\sin t|$. They satisfy assumptions $(A1)$ and $(A2)$, respectively. Obviously, there exist $l_1 = l_2 = l_3 = 1$ and $\tau = 0.5, 0 < \dot{\tau}(t) < 0.6$ that satisfy the conditions. Then $M = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$.

By using MATLAB to solve the LMIs (3.1), we have

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \dfrac{1}{2} & 0 \\ 0 & 0 & \dfrac{1}{3} \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.2}$$

From Theorem 3.1, we can see that delayed neural network (2.9) achieves ISS.

Example 4.2. Consider a 3-dimension dynamical neural network (2.9) with parameters followed as

$$A = \begin{pmatrix} 7.6 & 0 & 0 \\ 0 & 9.5 & 0 \\ 0 & 0 & 8.8 \end{pmatrix}, \qquad W = W_1 = \begin{pmatrix} 0.40 & 0.12 & 0.32 \\ 0.45 & 0.02 & 0.10 \\ 0.12 & 0.04 & 0.42 \end{pmatrix}. \tag{4.3}$$

Letting $g_i(\theta) = (|\theta+1| - |\theta-1|)/2, \theta \in R$ and the time-varying delay is chosen as $\tau(t) = 1/(t+2)$. We can check the assumptions $(A1)$ and $(A2)$ with $l_1 = l_2 = l_3 = 1$ and $0 \le \tau(t) \le 1, 0 \le \dot{\tau}(t) \le 0.5$ for any $t > 0$. Also we have $M = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$.

By solving (3.10) and (3.11), we get

$$P = \begin{pmatrix} 0.7 & 0.1 & 0.3 \\ 0.1 & 0.7 & 0.1 \\ 0.3 & 0.1 & 0.7 \end{pmatrix}, \qquad Q = \begin{pmatrix} 2.82 & 0.855 & 2.46 \\ 0.855 & 4.15 & 0.915 \\ 2.46 & 0.915 & 3.16 \end{pmatrix}, \qquad R = \begin{pmatrix} 0.927 & -0.03 & -0.02 \\ -0.03 & 9.28 & -0.08 \\ -0.01 & -0.02 & 9.25 \end{pmatrix}. \tag{4.4}$$

From Theorem 3.4, we can see that delayed neural network (2.9) obtains ISS.

However, the above results cannot be obtained by using criteria on ISS in existing publications (e.g., [10, 11, 17–19, 26]).

5. Conclusions

In this paper, dynamical neural networks with time-varying delays were considered. By using Lyapunov-Krasovskii functional method and linear matrix inequalities (LMIs) techniques, several theorems with regarding to judging the ISS property of DNNs with time-varying delays have been obtained. It is shown that the ISS can be determined by solving a set of LMIs, which can be checked by using some standard numerical packages in MATLAB. At last, two numerical examples were given to illustrate the theoretical results.

Appendix

Proof of Lemma 2.4. We divided into four parts to prove this lemma.
Claim 1. (i) The solution $x = 0$ of the system (2.9) is uniformly asymptotically stable if and only if there exists a function β of class KL and a positive number c independent of t_0 such that for for all $t \geq t_0$, for all $\|x_{t_0}\|_\tau \leq c$ it satisfies that

$$|x(t)| \leq \beta(\|x_{t_0}\|_\tau, t - t_0). \tag{A.1}$$

Particularly, the system (2.9) is uniformly global asymptotically stable if and only if (A.1) admits for any $x_{t_0} \in C$.

The Claim is so trivial that we omit the proof here. □

Claim 2. For each $(t, \phi) \in R^+ \times C$, if there exist a continuous functional $V(t, \phi) : R^+ \times C \to R^+$, functions α_1, α_2 of class K_∞, and a continuous positive definite function W such that

$$\alpha_1(|\phi(0)|) \leq V(t, \phi) \leq \alpha_2(\|\phi\|_\tau), \tag{A.2}$$

$$\dot{V}(t, \phi) \leq -W(|\phi(0)|), \tag{A.3}$$

then the solution $x(t) = 0$ is globally uniformly asymptotically stable, and there exist a $\beta \in KL$ such that

$$|x(t)| \leq \beta(\|x_{t_0}\|_\tau, t - t_0), \quad \forall (t_0, x_{t_0}) \in R^+ \times C,\ t \geq t_0. \tag{A.4}$$

Proof. From [27], the solution $x(t) = 0$ is globally uniformly asymptotically stable. Then by Claim 1, we obtain (A.4). The proof is complete. □

Claim 3. Let (A.3) in Claim 2 replaced by

$$\dot{V}(t, \phi) \leq -W(|\phi(0)|), \quad \forall |\phi(0)| \geq \mu > 0. \tag{A.5}$$

Then for any $x_{t_0} \in C$, there exist $\beta \in KL, T \triangleq T(x_{t_0}, \mu)$, such that

$$\begin{aligned} |x(t)| &\leq \beta(\|x_{t_0}\|_\tau, t - t_0), \quad \forall t \in [t_0, t_0 + T], \\ |x(t)| &\leq \alpha_1^{-1}(\alpha_2(\mu)), \quad \forall t \geq t_0 + T. \end{aligned} \tag{A.6}$$

Proof. Let $B_\mu \triangleq \{x \in R^n \mid \|x_{t_0}\|_\tau \le \mu\}$, $B_\mu^c \triangleq R^n - B_\mu$, $D_\mu \triangleq \{x \in R^n \mid \|x_{t_0}\|_\tau \le \alpha^{-1}(\alpha_2(\mu))\}$ (no loss generality, we assume that $\alpha_1(\mu) \le \alpha_2(\mu)$, then $\mu \le \alpha_1^{-1}(\alpha_2(\mu))$). Then $B_\mu \subseteq D_\mu$. In the following, we divided $x_{t_0} \in C$ into two parts.

Case 1. $x_{t_0} \in B_\mu$.

We make the claim that $x(t)$ will be always remain in B_μ. Define $t_1 = \inf\{t \ge t_0 : |x(t)| = \mu\}$, if $|x(t)| > \mu$, $t > t_1$, then $\|x(t)\| \ge \mu$, $t \ge t_1$ and $\dot{V}(t,\phi) \le -\alpha_3(|\phi(0)|) < 0$, and we have $\alpha_1(|x(t)|) \le V(t,x_t) \le V(t,x_{t_1}) \le \alpha_2(\|x_{t_1}\|_\tau)$, $t \ge t_1$. Then $|x(t)| \le \alpha_1^{-1}(\alpha_2(\|x_{t_1}\|_\tau)) = \alpha_1^{-1}(\alpha_2(\mu))$. If $|x(t)| < \mu$, $t > t_1$, let $t_2 = \inf\{t \ge t_1 : |x(t)| = \mu\}$, we will analyze them as the above. Then we obtain $|x(t)| \le \alpha_1^{-1}(\alpha_2(\mu))$, $t \ge t_0$.

Case 2. $x_{t_0} \in B_\mu^c$, that is, $\|x_{t_0}\|_\tau > \mu$.

Let $t_0 + T(x_{t_0},\mu) = \inf\{t \ge t_0 : |x(t)| = \mu\}$ and $T = T(x_{t_0},\mu)$. We prove that $t_0 + T$ is limit. From $|x(t)| \ge \mu$, $t \in [t_0, t_0+T]$, (A.2), (A.5), and Case 2, we have $|x(t)| \le \beta(\|x_{t_0}\|_\tau, t-t_0)$, where $\beta \in KL$. Since $\beta(\|x_{t_0}\|_\tau, t-t_0)$ is strictly decreasing, and $\beta(\|x_{t_0}\|_\tau, \cdot) \to 0$ as $t \to \infty$, $t_0 + T$ is limit. Then from Case 1, $x(t)$ will be always remain in D_μ if arrive the boundary of B_μ. Then we obtain (A.6). The proof is complete. □

Claim 4. Let (A.3) in Claim 2 replaced by

$$\dot{V}(t,\phi) \le -W(\|\phi(0)\|), \quad \text{if } \|\phi(0)\| \ge \rho(\|u\|_\infty), \tag{A.7}$$

where $\rho \in K$. Then the system is ISS.

Proof. From Claim 3, we have

$$|x(t)| \le \beta(\|x_{t_0}\|_\tau, t-t_0) + \alpha_1^{-1}(\alpha_2(\rho(\|u\|_\infty))), \quad t \ge t_0. \tag{A.8}$$

Since $x(t)$ only depends on the $u(s)$ defined on $[t_0, t]$, we obtain

$$|x(t)| \le \beta(\|x_{t_0}\|_\tau, t-t_0) + \alpha_1^{-1}(\alpha_2(\rho(\|u_t\|_\infty))), \quad t \ge t_0. \tag{A.9}$$

Then

$$|x(t)| \le \beta(\|x_{t_0}\|_\tau, t-t_0) + \gamma(\|u_t\|_\infty), \quad t \ge t_0, \tag{A.10}$$

where $\gamma \triangleq \alpha_1^{-1}(\alpha_2(\rho))$. This proves that the system is ISS. □

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Research Article

Global Stability of Almost Periodic Solution of a Class of Neutral-Type BAM Neural Networks

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A class of BAM neural networks with variable coefficients and neutral delays are investigated. By employing fixed-point theorem, the exponential dichotomy, and differential inequality techniques, we obtain some sufficient conditions to insure the existence and globally exponential stability of almost periodic solution. This is the first time to investigate the almost periodic solution of the BAM neutral neural network and the results of this paper are new, and they extend previously known results.

1. Introduction

Neural networks have been extensively investigated by experts of many areas such as pattern recognition, associative memory, and combinatorial optimization, recently, see [1–10]. Up to now, many results about stability of bidirectional associative memory (BAM) neural networks have been derived. For these BAM systems, periodic oscillatory behavior, almost periodic oscillatory properties, chaos, and bifurcation are their research contents; generally speaking, almost periodic oscillatory property is a common phenomenon in the real world, and in some aspects, it is more actual than other properties, see [11–21].

Time delays cannot be avoided in the hardware implementation of neural networks because of the finite switching speed of amplifiers and the finite signal propagation time in biological networks. The existence of time delay may lead to a system's instability or oscillation, so delay cannot be neglected in modeling. It is known to all that many practical delay systems can be modelled as differential systems of neutral type, whose differential expression concludes not only the derivative term of the current state, but also concludes the derivative of the past state. It means that state's changing at the past time may affect the current state. Practically, such phenomenon always appears in the study of automatic control, population dynamics, and so forth, and it is natural and important that systems will contain some information about the derivative of the past state to further describe and model the dynamics for such complex neural reactions [22]. Authors in [18–29] added neutral delay into the neural networks. In these papers, only [18–20] studied the almost periodic solution of the neutral neural networks. For example, in [19] the following network was studied:

$$\dot{x}_i(t) = -c_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t) f_j\left(x_j\left(t - \tau_{ij}(t)\right)\right) + \sum_{j=1}^{n} b_{ij}(t) g_j\left(\dot{x}_j\left(t - \sigma_{ij}(t)\right)\right) + I_i(t). \tag{1.1}$$

Some sufficient conditions are obtained for the existence and globally exponential stability of almost periodic solution by employing fixed-point theorem and differential inequality techniques. References [21–26] studied the global asymptotic stability of equilibrium point, where [22] investigated the equilibrium point of the following BAM neutral neural network with constant coefficients:

$$\begin{aligned} \dot{u}_i(t) &= -a_i u_i(t) + \sum_{j=1}^{m} w_{1ji} g_j\left(v_j(t-d)\right) + \sum_{j=1}^{n} w_{2ij} \dot{u}_j(t-h) + I_i, \\ \dot{v}_j(t) &= -b_j v_j(t) + \sum_{i=1}^{n} r_{1ij} g_i(u_i(t-h)) + \sum_{i=1}^{m} r_{2ji} \dot{v}_i(t-d) + J_i. \end{aligned} \tag{1.2}$$

By using the Lyapunov method and linear matrix inequality techniques, a new stability criterion was derived. References [27–29] studied the exponential stability of equilibrium point.

It is obviously that men always studied the stability of the equilibrium point of the neutral neural networks, and there is little result for the almost periodic solution of neutral neural networks, especially, for the BAM neutral type neural networks. Besides, in papers [11, 23, 27, 28], time delay must be differentiable, and its derivative is bounded, which we think is a strict condition.

Motivated by the above discussions, in this paper, we consider the almost periodic solution of a class of BAM neural networks with variable coefficients and neutral delays. By fixed-point theorem and differential inequality techniques, we obtain some sufficient conditions to insure the existence and globally exponential stability of almost periodic solution. To the best of the authors' knowledge, this is the first time to investigate the almost periodic solution of the BAM neutral neural network, and we can remove delay's derivable condition, so the results of this paper are new, and they extend previously known results.

2. Preliminaries

In this paper, we consider the following system:

$$\begin{aligned} \dot{x}_i(t) &= -c_i(t)x_i(t) + \sum_{j=1}^{m} a_{ij}(t) f_{1j}\left(y_j\left(t - \tau_{ij}(t)\right)\right) + \sum_{j=1}^{n} b_{ji}(t) f_{2j}\left(\dot{x}_j\left(t - \overline{\delta}_{ji}(t)\right)\right) + I_i(t), \\ \dot{y}_j(t) &= -d_j(t)y_j(t) + \sum_{i=1}^{n} p_{ji}(t) g_{1i}\left(x_i\left(t - \delta_{ji}(t)\right)\right) + \sum_{i=1}^{m} q_{ij}(t) g_{2i}\left(\dot{y}_j\left(t - \overline{\tau}_{ij}(t)\right)\right) + J_j(t), \end{aligned} \tag{2.1}$$

where $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, m$. $x_i(t)$, $y_j(t)$ are the states of the ith neuron of X layer and the jth neuron of Y layer, respectively; $a_{ij}(t)$, $p_{ji}(t)$ and $b_{ji}(t)$, $q_{ij}(t)$ are the delayed strengths of connectivity and the neutral delayed strengths of connectivity, respectively; f_{1j}, f_{2j}, g_{1i}, g_{2i} are activation functions; $I_i(t)$, $J_j(t)$ stands for the external inputs; $\tau_{ij}(t)$, $\overline{\tau}_{ij}(t)$, $\delta_{ji}(t)$, and $\overline{\delta}_{ji}(t)$ correspond to the delays, they are nonnegative; $c_i(t)$, $d_j(t) > 0$ represent the rate with which the ith neuron of X layer and the jth neuron of Y layer will reset its potential to the resting state in isolation when disconnected from the networks.

Throughout this paper, we assume the following.

(H_1) $c_i(t)$, $d_j(t)$, $a_{ij}(t)$, $p_{ji}(t)$, $b_{ji}(t)$, $q_{ij}(t)$, $\tau_{ij}(t)$, $\overline{\tau}_{ij}(t)$, $\delta_{ji}(t)$, $\overline{\delta}_{ji}(t)$, $I_i(t)$, and $J_j(t)$ are continuous almost periodic functions. Moreover, we let

$$
\begin{gathered}
c_i^+ = \sup_{t\in R}\{c_i(t)\}, \qquad c_i^- = \inf_{t\in R}\{c_i(t)\} > 0, \qquad d_j^+ = \sup_{t\in R}\{d_j(t)\}, \qquad d_j^- = \inf_{t\in R}\{d_j(t)\} > 0, \\
a_{ij} = \sup_{t\in R}\{|a_{ij}(t)|\} < \infty, \qquad b_{ji} = \sup_{t\in R}\{|b_{ji}(t)|\} < \infty, \qquad p_{ji} = \sup_{t\in R}\{|p_{ji}(t)|\} < \infty, \\
q_{ij} = \sup_{t\in R}\{|q_{ij}(t)|\} < \infty, \qquad I_i = \sup_{t\in R}\{|I_i(t)|\} < \infty, \qquad J_j = \sup_{t\in R}\{|J_j(t)|\} < \infty.
\end{gathered}
\tag{2.2}
$$

(H_2) f_{1j}, f_{2j}, g_{1i}, and g_{2i} are Lipschitz continuous with the Lipschitz constants F_{1j}, F_{2j}, G_{1i}, G_{2i}, and $f_{1j}(0) = f_{2j}(0) = g_{1i}(0) = g_{2i}(0) = 0$.

(H_3) Consider

$$
\begin{aligned}
\alpha = \max\Bigg\{ &\max_{1\le i\le n} \max\left\{\frac{1}{c_i^-}, 1 + \frac{c_i^+}{c_i^-}\right\}\left(\sum_{j=1}^{m} a_{ij}F_{1j} + \sum_{j=1}^{n} b_{ji}F_{2j}\right), \\
&\max_{1\le j\le m} \max\left\{\frac{1}{d_j^-}, 1 + \frac{d_j^+}{d_j^-}\right\}\left(\sum_{i=1}^{n} p_{ji}G_{1i} + \sum_{i=1}^{m} q_{ij}G_{2i}\right)\Bigg\} < 1.
\end{aligned}
\tag{2.3}
$$

The initial conditions of system (2.1) are of the following form:

$$
\begin{aligned}
x_i(t) &= \varphi_i(t), \quad t \in [-\delta, 0], \ \delta = \sup_{t\in R} \max_{i,j} \max\left\{\delta_{ji}(t), \overline{\delta}_{ji}(t)\right\}, \\
y_j(t) &= \phi_j(t), \quad t \in [-\tau, 0], \ \tau = \sup_{t\in R} \max_{i,j} \max\{\tau_{ij}(t), \overline{\tau}_{ij}(t)\},
\end{aligned}
\tag{2.4}
$$

where $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, m$; $\varphi_i(t)$, $\phi_j(t)$ are continuous almost periodic functions.

Let $X = \{\psi \mid \psi = (\varphi_1, \varphi_2, \ldots, \varphi_n, \phi_1, \phi_2, \ldots, \phi_m)^T$, where φ_i, $\phi_j : R \to R$ are continuously differentiable almost periodic functions. For any $\psi \in X$, $\psi(t) = (\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t), \phi_1(t), \phi_2(t), \ldots, \phi_m(t))^T$. We define $\|\psi(t)\|_1 = \max\{\|\psi(t)\|_0, \|\dot{\psi}(t)\|_0\}$, where $\|\psi(t)\|_0 = \max\{\max_{1\le i\le n}\{|\varphi_i(t)|\}, \max_{1\le j\le m}\{|\phi_i(t)|\}\}$, and $\dot{\psi}(t)$ is the derivative of ψ at t. Let $\|\psi\| = \sup_{t\in R}\|\psi(t)\|_1$, then X is a Banach space.

The following definitions and lemmas will be used in this paper.

Definition 2.1 (see [11]). Let $x(t) : R \rightarrow R^n$ be continuous in t. $x(t)$ is said to be almost periodic on R, if for any $\varepsilon > 0$, the set $T(x,\varepsilon) = \{w | x(t+w) - x(t) < \varepsilon, \text{for all } t \in R\}$ is relatively dense, that is, for all $\varepsilon > 0$, it is possible to find a real number $l = l(\varepsilon) > 0$, for any interval length $l(\varepsilon)$, there exists a number $\tau = \tau(\varepsilon)$ in this interval such that $|x(t+\tau) - x(t)| < \varepsilon$, for all $t \in R$.

Definition 2.2 (see [11]). Let $x \in C(R, R^n)$ and $Q(t)$ be $n \times n$ continuous matrix defined on R. The following linear system:

$$\dot{x}(t) = Q(t)x(t) \tag{2.5}$$

is said to admit an exponential dichotomy on R if there exist constants K, α, projection P, and the fundamental solution $X(t)$ of (2.5) satisfying

$$\begin{gathered}\left|X(t)PX^{-1}(s)\right| \leq Ke^{-\alpha(t-s)}, \quad t \geq s, \\ \left|X(t)(I-P)X^{-1}(s)\right| \leq Ke^{-\alpha(s-t)}, \quad t \leq s.\end{gathered} \tag{2.6}$$

Definition 2.3. Let $z^*(t) = (x^*(t), y^*(t))^T = (x_1^*(t), \ldots, x_n^*(t), y_1^*(t), \ldots, y_m^*(t))^T$ be a continuously differentiable almost periodic solution of (2.1) with initial value $\psi^* = (\varphi^*, \phi^*)^T = (\varphi_1^*, \ldots, \varphi_n^*, \phi_1^*, \ldots, \phi_m^*)^T$. If there exist constants $\lambda > 0$, $M > 1$ such that for every solution $z(t) = (x(t), y(t))^T = (x_1(t), \ldots, x_n(t), y_1(t), \ldots, y_m(t))^T$ of (2.1) with any initial value $\psi = (\varphi, \phi)^T = (\varphi_1, \ldots, \varphi_n, \phi_1, \ldots, \phi_m)^T$, if

$$\|z(t) - z^*(t)\|_1 \leq Me^{\lambda t}\|\psi - \psi^*\|, \quad \text{for } t > 0, \tag{2.7}$$

where $\varphi_i^*(\mathrm{t}), \phi_j^*(t), \varphi_i(t)$, and $\phi_j(t)$ are almost periodic functions. Then $z^*(t)$ is said to be globally exponentially stable.

Lemma 2.4 (see [11]). *If the linear system* (2.5) *admits an exponential dichotomy, then the almost periodic system*

$$\dot{x}(t) = Q(t)x(t) + f(t) \tag{2.8}$$

has a unique almost periodic solution

$$\psi(t) = \int_{-\infty}^{t} X(t)PX^{-1}(s)f(s)ds - \int_{t}^{+\infty} X(t)(I-P)X^{-1}f(s)ds. \tag{2.9}$$

Lemma 2.5 (see [11]). *Let $q_i(t)$ be an almost periodic function on R and*

$$M[q_i] = \lim_{T \to +\infty} \frac{1}{T} \int_{t}^{t+T} q_i(t)ds > 0, \quad i = 1, 2, \ldots, n, \tag{2.10}$$

then the linear system $\dot{z}(t) = \operatorname{diag}\{-q_1(t), \ldots, -q_n(t)\}z(t)$ admits exponential dichotomy on R.

3. Existence and Uniqueness of Almost Periodic Solutions

In this section, we consider the existence and uniqueness of almost periodic solutions by fixed-point theorem.

Theorem 3.1. *Under the assumptions* (H_1) − (H_3)*, the system* (2.1) *has a unique almost periodic solution in the region* $\|\varphi - \varphi_0\| \leq \alpha\beta/(1-\alpha)$.

(H_4) *If*

$$M[c_i] = \lim_{T\to+\infty}\frac{1}{T}\int_t^{t+T} c_i(s)ds > 0, \quad i = 1,2,\ldots,n,$$
$$M[d_j] = \lim_{T\to+\infty}\frac{1}{T}\int_t^{t+T} d_j(s)ds > 0, \quad j = 1,2,\ldots,m \tag{3.1}$$

holds, where

$$\beta = \max\left\{\max_{1\leq i\leq n}\max\left\{\frac{I_i}{c_i^-}, I_i + \frac{I_i c_i^+}{c_i^-}\right\}, \max_{1\leq j\leq m}\max\left\{\frac{J_j}{d_j^-}, J_j + \frac{J_j d_j^+}{d_j^-}\right\}\right\},$$
$$\varphi_0(t) = \left(\int_{-\infty}^t e^{-\int_s^t c_1(u)du} I_1(s)ds, \ldots, \int_{-\infty}^t e^{-\int_s^t c_n(u)du} I_n(s)ds,\right.$$
$$\left.\int_{-\infty}^t e^{-\int_s^t d_1(u)du} J_1(s)ds, \ldots, \int_{-\infty}^t e^{-\int_s^t d_m(u)du} J_m(s)ds\right)^T. \tag{3.2}$$

Proof. For any $(\varphi,\phi)^T = (\varphi_1,\ldots,\varphi_n,\phi_1,\ldots,\phi_m)^T \in X$, we consider the the following system:

$$\dot{x}_i(t) = -c_i(t)x_i(t) + \sum_{j=1}^m a_{ij}(t)f_{1j}(\phi_j(t-\tau_{ij}(t))) + \sum_{j=1}^n b_{ji}(t)f_{2j}\left(\dot{\varphi}_j\left(t-\overline{\delta}_{ji}(t)\right)\right) + I_i(t),$$
$$\dot{y}_j(t) = -d_j(t)y_j(t) + \sum_{i=1}^n p_{ji}(t)g_{1i}(\varphi_i(t-\delta_{ji}(t))) + \sum_{i=1}^m q_{ij}(t)g_{2i}(\dot{\phi}_i(t-\overline{\tau}_{ij}(t))) + J_j(t). \tag{3.3}$$

From (H_4) and Lemma 2.5, we know the following linear system:

$$\dot{x}_i(t) = -c_i(t)x_i(t),$$
$$\dot{y}_j(t) = -d_j(t)y_j(t) \tag{3.4}$$

admits an exponential dichotomy on R. By Lemma 2.4, System (3.3) has an almost periodic solution $z_{(\varphi,\phi)^T}(t)$ which can be expressed as follows:

$$\begin{aligned} z_{(\varphi,\phi)^T}(t) = \Bigg(&\int_{-\infty}^{t} e^{-\int_s^t c_1(u)du}(A_1(s)+I_1(s))ds,\dots,\int_{-\infty}^{t} e^{-\int_s^t c_n(u)du}(A_n(s)+I_n(s))ds, \\ &\int_{-\infty}^{t} e^{-\int_s^t d_1(u)du}\left(\overline{A}_1(s)+J_1(s)\right)ds,\dots,\int_{-\infty}^{t} e^{-\int_s^t d_m(u)du}\left(\overline{A}_m(s)+J_m(s)\right)ds\Bigg)^T, \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} A_i(s) &= \sum_{j=1}^{m} a_{ij}(s) f_{1j}(\phi_j(s-\tau_{ij}(s))) + \sum_{j=1}^{n} b_{ji}(s) f_{2j}\left(\dot{\varphi}_j\left(s-\overline{\delta}_{ji}(s)\right)\right), \quad i=1,2,\dots,n, \\ \overline{A}_j(s) &= \sum_{i=1}^{n} p_{ji}(s) g_{1i}(\varphi_i(s-\delta_{ji}(s))) + \sum_{i=1}^{m} q_{ij}(s) g_{2i}(\dot{\phi}_i(s-\overline{\tau}_{ij}(s))), \quad j=1,2,\dots,m. \end{aligned} \tag{3.6}$$

So, we can define a mapping $T: X \to X$, by letting

$$T(\varphi,\phi)^T(t) = z_{(\varphi,\phi)^T}(t), \quad \forall (\varphi,\phi)^T \in X. \tag{3.7}$$

Set $X_0 = \{\psi \mid \psi \in X, \|\psi - \varphi_0\| \le \alpha\beta/(1-\alpha)\}$; clearly, X_0 is a closed convex subset of X, so we have

$$\begin{aligned} \|\varphi_0\| = \max\Bigg\{ &\sup_{t\in R}\max_{1\le i\le n}\left|\int_{-\infty}^{t} e^{-\int_s^t c_i(u)du} I_i(s)ds\right|, \sup_{t\in R}\max_{1\le i\le n}\left|\left(\int_{-\infty}^{t} e^{-\int_s^t c_i(u)du} I_i(s)ds\right)'\right|, \\ &\sup_{t\in R}\max_{1\le j\le m}\left|\int_{-\infty}^{t} e^{-\int_s^t d_j(u)du} J_j(s)ds\right|, \sup_{t\in R}\max_{1\le j\le m}\left|\left(\int_{-\infty}^{t} e^{-\int_s^t d_j(u)du} J_j(s)ds\right)'\right|\Bigg\} \\ \le \max\Bigg\{ &\max_{1\le i\le n}\max\left\{\frac{I_i}{c_i^-}, I_i + \frac{I_i c_i^+}{c_i^-}\right\}, \max_{1\le j\le m}\max\left\{\frac{J_j}{d_j^-}, J_j + \frac{J_j d_j^+}{d_j^-}\right\}\Bigg\} = \beta. \end{aligned} \tag{3.8}$$

Therefore,

$$\|\psi\| \le \|\psi - \varphi_0\| + \|\varphi_0\| \le \frac{\alpha\beta}{1-\alpha} + \beta = \frac{\beta}{1-\alpha}, \quad \forall \psi \in X_0. \tag{3.9}$$

First, we prove that the mapping T is a self-mapping from X_0 to X_0. In fact, for any $\psi = (\overline{\varphi}_1, \dots, \overline{\varphi}_n, \overline{\phi}_1, \dots, \overline{\phi}_m)^T \in X_0$, let

$$\begin{aligned}
B_i(s) &= \sum_{j=1}^{m} a_{ij}(s) f_{1j}\left(\overline{\phi}_j(s - \tau_{ij}(s))\right) + \sum_{j=1}^{n} b_{ji}(s) f_{2j}\left(\dot{\overline{\varphi}}_j\left(s - \overline{\delta}_{ji}(s)\right)\right), \quad i = 1, 2, \dots, n, \\
\overline{B}_j(s) &= \sum_{i=1}^{n} p_{ji}(s) g_{1i}(\overline{\varphi}_i(s - \delta_{ji}(s))) + \sum_{i=1}^{m} q_{ij}(s) g_{2i}\left(\dot{\overline{\phi}}_i(s - \overline{\tau}_{ij}(s))\right), \quad j = 1, 2, \dots, m.
\end{aligned} \tag{3.10}$$

From (H_2) and (H_3), we have

$$\begin{aligned}
\|T\psi - \psi_0\| = \max\Bigg\{ & \sup_{t\in R} \max_{1\le i\le n} \left\{ \left| \int_{-\infty}^{t} e^{-\int_s^t c_i(u)du} B_i(s) ds \right| \right\}, \\
& \sup_{t\in R} \max_{1\le i\le n} \left\{ \left| -c_i(t) \int_{-\infty}^{t} e^{-\int_s^t c_i(u)du} B_i(s) ds + B_i(t) \right| \right\}, \\
& \sup_{t\in R} \max_{1\le j\le m} \left\{ \left| \int_{-\infty}^{t} e^{-\int_s^t d_j(u)du} \overline{B}_j(s) ds \right| \right\}, \\
& \sup_{t\in R} \max_{1\le j\le m} \left\{ \left| -d_j(t) \int_{-\infty}^{t} e^{-\int_s^t d_j(u)du} \overline{B}_j(s) ds + \overline{B}_j(t) \right| \right\} \Bigg\} \\
\le \max\Bigg\{ & \sup_{t\in R} \max_{1\le i\le n} \left\{ \int_{-\infty}^{t} e^{c_i^-(s-t)} |B_i(s)| ds \right\}, \\
& \sup_{t\in R} \max_{1\le i\le n} \left\{ c_i^+ \int_{-\infty}^{t} e^{c_i^-(s-t)} |B_i(s)| ds + |B_i(t)| \right\}, \\
& \sup_{t\in R} \max_{1\le j\le m} \left\{ \int_{-\infty}^{t} e^{d_j^-(s-t)} \left|\overline{B}_j(s)\right| ds \right\}, \\
& \sup_{t\in R} \max_{1\le j\le m} \left\{ d_j^+ \int_{-\infty}^{t} e^{d_j^-(s-t)} \left|\overline{B}_j(s)\right| ds + \left|\overline{B}_j(t)\right| \right\} \Bigg\} \\
\le \max\Bigg\{ & \max_{1\le i\le n} \left\{ \frac{1}{c_i^-} \left(\sum_{j=1}^{m} a_{ij} F_{1j} + \sum_{j=1}^{n} b_{ji} F_{2j} \right) \right\}, \\
& \max_{1\le i\le n} \left\{ \left(1 + \frac{c_i^+}{c_i^-}\right) \left(\sum_{j=1}^{m} a_{ij} F_{1j} + \sum_{j=1}^{n} b_{ji} F_{2j} \right) \right\}, \\
& \max_{1\le j\le m} \left\{ \frac{1}{d_j^-} \left(\sum_{i=1}^{n} p_{ji} G_{1i} + \sum_{i=1}^{m} q_{ij} G_{2i} \right) \right\}, \\
& \max_{1\le j\le m} \left\{ \left(1 + \frac{d_j^+}{d_j^-}\right) \left(\sum_{i=1}^{n} p_{ji} G_{1i} + \sum_{i=1}^{m} q_{ij} G_{2i} \right) \right\} \Bigg\} \|\psi\|
\end{aligned}$$

$$
\begin{aligned}
&= \max\left\{ \max_{1\le i\le n} \max\left\{ \frac{1}{c_i^-}, 1+\frac{c_i^+}{c_i^-} \right\} \left(\sum_{j=1}^{m} a_{ij}F_{1j} + \sum_{j=1}^{n} b_{ji}F_{2j} \right),\right. \\
&\qquad \left. \max_{1\le j\le m} \max\left\{ \frac{1}{d_j^-}, 1+\frac{d_j^+}{d_j^-} \right\} \left(\sum_{i=1}^{n} p_{ji}G_{1i} + \sum_{i=1}^{m} q_{ij}G_{2i} \right) \right\} \|\varphi\| \\
&= \alpha\|\varphi\| \le \frac{\alpha\beta}{1-\alpha}.
\end{aligned}
\tag{3.11}
$$

This implies that $T(\varphi) \in X_0$, so T is a self-mapping from X_0 to X_0.

Finally, we prove that T is a contraction mapping. In fact, for any $\varphi_1 = (\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m)^T$, $\varphi_2 = (\overline{\alpha}_1, \dots, \overline{\alpha}_n, \overline{\beta}_1, \dots, \overline{\beta}_m)^T \in X_0$. Let

$$
\begin{aligned}
H_i(s) &= \sum_{j=1}^{m} a_{ij}(s)\left[f_{1j}(\beta_j(s-\tau_{ij}(s))) - f_{1j}\left(\overline{\beta}_j(s-\tau_{ij}(s))\right)\right] \\
&\quad + \sum_{j=1}^{n} b_{ji}(s)\left[f_{2j}\left(\dot{\alpha}_j\left(s-\overline{\delta}_{ji}(s)\right)\right) - f_{2j}\left(\dot{\overline{\alpha}}_j\left(s-\overline{\delta}_{ji}(s)\right)\right)\right], \quad i = 1,2,\dots,n, \\
\overline{H}_j(s) &= \sum_{i=1}^{n} p_{ji}(s)\left[g_{1i}(\alpha_i(s-\delta_{ji}(s))) - g_{1i}(\overline{\alpha}_i(s-\delta_{ji}(s)))\right] \\
&\quad + \sum_{i=1}^{m} q_{ij}(s)\left[g_{2i}(\dot{\beta}_i(s-\overline{\tau}_{ij}(s))) - g_{2i}\left(\dot{\overline{\beta}}_i(s-\overline{\tau}_{ij}(s))\right)\right], \quad j = 1,2,\dots,m.
\end{aligned}
\tag{3.12}
$$

We have

$$
\begin{aligned}
\|T\varphi_1 - T\varphi_2\| = &\max\left\{ \sup_{t\in R} \max_{1\le i\le n} \left\{ \left| \int_{-\infty}^{t} e^{-\int_s^t c_i(u)du} H_i(s)ds \right| \right\},\right. \\
&\quad \sup_{t\in R} \max_{1\le i\le n} \left\{ \left| -c_i(t) \int_{-\infty}^{t} e^{-\int_s^t c_i(u)du} H_i(s)ds + H_i(t) \right| \right\}, \\
&\quad \sup_{t\in R} \max_{1\le j\le m} \left\{ \left| \int_{-\infty}^{t} e^{-\int_s^t d_j(u)du} \overline{H}_j(s)ds \right| \right\}, \\
&\quad \left. \sup_{t\in R} \max_{1\le j\le m} \left\{ \left| -d_j(t) \int_{-\infty}^{t} e^{-\int_s^t d_j(u)du} \overline{H}_j(s)ds + \overline{H}_j(t) \right| \right\} \right\} \\
&\le \max\left\{ \max_{1\le i\le n} \max\left\{ \frac{1}{c_i^-}, 1+\frac{c_i^+}{c_i^-} \right\} \left(\sum_{j=1}^{m} \mathrm{a}_{ij}F_{1j} + \sum_{j=1}^{n} b_{ji}F_{2j} \right),\right. \\
&\qquad \left. \max_{1\le j\le m} \max\left\{ \frac{1}{d_j^-}, 1+\frac{d_j^+}{d_j^-} \right\} \left(\sum_{i=1}^{n} p_{ji}G_{1i} + \sum_{i=1}^{m} q_{ij}G_{2i} \right) \right\} \|\varphi_1 - \varphi_2\| \\
&= \alpha\|\varphi_1 - \varphi_2\|.
\end{aligned}
\tag{3.13}
$$

Notice that $\alpha < 1$, it means that the mapping T is a contraction mapping. By Banach fixed-point theorem, there exists a unique fixed-point $\varphi^* \in X_0$ such that $T\varphi^* = \varphi^*$, which implies system (2.1) has a unique almost periodic solution. □

4. Global Exponential Stability of the Almost Periodic Solution

In this section, we consider the exponential stability of almost periodic solution, and we give two corollaries.

Theorem 4.1. *Under the assumptions* $(H_1) - (H_4)$, *then system* (2.1) *has a unique almost periodic solution which is global exponentially stable.*

Proof. It follows from Theorem 3.1 that system (2.1) has a unique almost periodic solution $z^*(t) = (x^*(t), y^*(t))^T = (x_1^*(t), \dots, x_n^*(t), y_1^*(t), \dots, y_m^*(t))^T$ with the initial value $\varphi^* = (\varphi^*, \phi^*)^T = (\varphi_1^*, \dots, \varphi_n^*, \phi_1^*, \dots, \phi_m^*)^T$. Set $z(t) = (x(t), y(t))^T = (x_1(t), \dots, x_n(t), y_1(t), \dots, y_m(t))^T$ is an arbitrary solution of system (2.1) with initial value $\varphi = (\varphi, \phi)^T = (\varphi_1, \dots, \varphi_n, \phi_1, \dots, \phi_m)^T$. Let $u_i(t) = x_i(t) - x_i^*(t)$, $v_j(t) = y_j(t) - y_j^*(t)$, $\Psi_i = \varphi_i - \varphi_i^*$, $\Phi_j = \phi_j - \phi_j^*$. Then $z(t) - z^*(t) = (u_1(t), \dots, u_n(t), v_1(t), \dots, v_m(t))^T$, where $i = 1, 2, \dots, n$; $j = 1, 2, \dots, m$. Then system (2.1) is equivalent to the following system:

$$\begin{aligned} &\dot{u}_i(s) + c_i(s) u_i(s) = F_i(s), \quad s > 0, \\ &\dot{v}_j(s) + d_j(s) v_j(s) = \overline{F}_j(s), \quad s > 0, \end{aligned} \tag{4.1}$$

with the initial value

$$\begin{aligned} &\Psi_i(s) = \varphi_i(s) - \varphi_i^*(s), \quad s \in [-\delta, 0], \\ &\Phi_j(s) = \phi_j(s) - \phi_j^*(s), \quad s \in [-\tau, 0], \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} F_i(s) = {} & \sum_{j=1}^{m} a_{ij}(s) \Big[f_{1j}\Big(y_j^*(s - \tau_{ij}(s)) + v_j(s - \tau_{ij}(s)) \Big) - f_{1j}\Big(y_j^*(s - \tau_{ij}(s)) \Big) \Big] \\ & + \sum_{j=1}^{n} b_{ji}(s) \Big[f_{2j}\Big(\dot{x}_j^*\Big(s - \overline{\delta}_{ji}(s)\Big) + \dot{u}_j\Big(s - \overline{\delta}_{ji}(s)\Big) \Big) - f_{2j}\Big(\dot{x}_j^*\Big(s - \overline{\delta}_{ji}(s)\Big) \Big) \Big], \\ \overline{F}_j(s) = {} & \sum_{i=1}^{n} p_{ji}(s) \left[g_{1i}(x_i^*(s - \delta_{ji}(s)) + u_i(s - \delta_{ji}(s))) - g_{1i}(x_i^*(s - \delta_{ji}(s))) \right] \\ & + \sum_{i=1}^{m} q_{ij}(s) \left[g_{2i}(\dot{y}_i^*(s - \overline{\tau}_{ij}(s)) + \dot{v}_i(s - \overline{\tau}_{ij}(s))) - g_{2i}(\dot{y}_i^*(s - \overline{\tau}_{ij}(s))) \right]. \end{aligned} \tag{4.3}$$

Let

$$\Gamma_i(\xi_i) = c_i^- - \xi_i - \sum_{j=1}^{m} a_{ij} F_{1j} e^{\tau\xi_i} - \sum_{j=1}^{n} b_{ji} F_{2j} e^{\delta\xi_i},$$
$$\overline{\Gamma}_i\left(\overline{\xi}_i\right) = c_i^- - \overline{\xi}_i - \left(c_i^+ + c_i^-\right)\left(\sum_{j=1}^{m} a_{ij} F_{1j} e^{\tau\overline{\xi}_i} + \sum_{j=1}^{n} b_{ji} F_{2j} e^{\delta\overline{\xi}_i}\right), \tag{4.4}$$

where $\xi_i, \overline{\xi}_i \geq 0$, $i = 1, 2, \ldots, n$. From (H_3), we know $\Gamma_i(0) > 0$, $\overline{\Gamma}_i(0) > 0$. Since $\Gamma_i(\cdot)$ and $\overline{\Gamma}_i(\cdot)$ are continuous on $[0, \infty]$ and $\Gamma_i(\xi_i), \overline{\Gamma}_i(\overline{\xi}_i) \to -\infty$ as $\xi_i, \overline{\xi}_i \to +\infty$, so there exist $\xi_i^*, \overline{\xi}_i^* > 0$ such that $\Gamma_i(\xi_i^*) = \overline{\Gamma}_i(\overline{\xi}_i^*) = 0$ and $\Gamma_i(\xi_i) > 0$ for $\xi_i \in (0, \xi_i^*)$, $\overline{\Gamma}_i(\overline{\xi}_i) > 0$ for $\overline{\xi}_i \in (0, \overline{\xi}_i^*)$. By choosing $\xi = \min\{\xi_1^*, \ldots, \xi_n^*, \overline{\xi}_1^*, \ldots, \overline{\xi}_n^*\}$, we obtain $\Gamma_i(\xi), \overline{\Gamma}_i(\xi) \geq 0$. So we can choose a positive constant λ_1, $0 < \lambda_1 < \min\{\xi, c_i^-, \ldots, c_n^-\}$ such that $\Gamma_i(\lambda_1), \overline{\Gamma}_i(\lambda_1) > 0$. For the same reason, we define

$$G_j(\eta_j) = d_j^- - \eta_j - \sum_{i=1}^{n} p_{ji} G_{1i} e^{\delta\eta_j} - \sum_{i=1}^{m} q_{ij} G_{2i} e^{\tau\eta_j},$$
$$\overline{G}_j\left(\overline{\eta}_j\right) = d_j^- - \overline{\eta}_j - \left(d_j^- + d_j^+\right)\left(\sum_{i=1}^{n} p_{ji} G_{1i} e^{\delta\overline{\eta}_j} + \sum_{i=1}^{m} q_{ij} G_{2i} e^{\tau\overline{\eta}_j}\right). \tag{4.5}$$

There exists λ_2, $0 < \lambda_2 < d_j^-$, $j = 1, 2, \ldots, m$, such that $G_j(\lambda_2), \overline{G}_j(\lambda_2) > 0$. Taking $\lambda = \min\{\lambda_1, \lambda_2\}$, since $\Gamma_i(\cdot)$, $\overline{\Gamma}_i(\cdot)$, $G_j(\cdot)$, and $\overline{G}_j(\cdot)$ are strictly monotonous decrease functions, therefore, $\Gamma_i(\lambda), \overline{\Gamma}_i(\lambda), G_j(\lambda), \overline{G}_j(\lambda) > 0$, which implies

$$r_i := \frac{1}{c_i^- - \lambda}\left(\sum_{j=1}^{m} a_{ij} F_{1j} e^{\tau\lambda} + \sum_{j=1}^{n} b_{ji} F_{2j} e^{\delta\lambda}\right) < 1,$$
$$\overline{r}_i := \left(1 + \frac{c_i^+}{c_i^- - \lambda}\right)\left(\sum_{j=1}^{m} a_{ij} F_{1j} e^{\tau\lambda} + \sum_{j=1}^{n} b_{ji} F_{2j} e^{\delta\lambda}\right) < 1, \quad i = 1, 2, \ldots, n;$$
$$\frac{1}{d_j^- - \lambda}\left(\sum_{i=1}^{n} p_{ji} G_{1i} e^{\delta\lambda} + \sum_{i=1}^{m} q_{ij} G_{2i} e^{\tau\lambda}\right) < 1,$$
$$\left(1 + \frac{d_j^+}{d_j^- - \lambda}\right)\left(\sum_{i=1}^{n} p_{ji} G_{1i} e^{\delta\lambda} + \sum_{i=1}^{m} q_{ij} G_{2i} e^{\tau\lambda}\right) < 1, \quad j = 1, 2, \ldots, m. \tag{4.6}$$

Multiplying the two equations of system (4.1) by $e^{\int_0^s c_i(u)du}$ and $e^{\int_0^s d_j(u)du}$, respectively, and integrating on $[0,t]$, we get

$$u_i(t) = u_i(0)e^{-\int_0^t c_i(u)du} + \int_0^t e^{-\int_s^t c_i(u)du} F_i(s)ds,$$
$$v_j(t) = v_j(0)e^{-\int_0^t d_j(u)du} + \int_0^t e^{-\int_s^t d_j(u)du} \overline{F}_j(s)ds. \tag{4.7}$$

Taking

$$M = \max\left\{ \max_{1\le i\le n} \frac{c_i^-}{\sum_{j=1}^m a_{ij}F_{1j} + \sum_{j=1}^n b_{ji}F_{2j}}, \max_{1\le j\le m} \frac{d_j^-}{\sum_{i=1}^n p_{ji}G_{1i} + \sum_{i=1}^m q_{ij}G_{2i}} \right\}, \tag{4.8}$$

then $M > 1$, thus

$$\|z(t) - z^*(t)\|_1 = \left\|\varphi(t) - \varphi^*(t)\right\|_1 \le \left\|\varphi - \varphi^*\right\| \le M\left\|\varphi - \varphi^*\right\|e^{\lambda t}, \quad t \le 0, \tag{4.9}$$

where $\lambda > 0$ as in (4.6). We claim that

$$\|z(t) - z^*(t)\|_1 \le M\left\|\varphi - \varphi^*\right\|e^{\lambda t}, \quad t > 0. \tag{4.10}$$

To prove (4.10), we first show for any $p > 1$, the following inequality holds:

$$\|z(t) - z^*(t)\|_1 < pM\left\|\varphi - \varphi^*\right\|e^{\lambda t}, \quad t > 0. \tag{4.11}$$

If (4.11) is false, then there must be some $t_1 > 0$ and some $i, l \in \{1,2,\ldots,n\}$, $j, k \in \{1,2,\ldots,m\}$, such that

$$\begin{aligned}\|z(t_1) - z^*(t_1)\|_1 &= \max\left\{|u_i(t_1)|, |\dot{u}_l(t_1)|, \left|v_j(t_1)\right|, |\dot{v}_k(t_1)|\right\} \\ &= pM\left\|\varphi - \varphi^*\right\|e^{\lambda t_1},\end{aligned} \tag{4.12}$$

$$\|z(t) - z^*(t)\|_1 < pM\left\|\varphi - \varphi^*\right\|e^{\lambda t}, \quad 0 < t < t_1. \tag{4.13}$$

By (4.3)–(4.8), (4.12), and (4.13), we have

$$
\begin{aligned}
|u_i(t_1)| &= \left| u_i(0)e^{-\int_0^{t_1} c_i(u)du} + \int_0^{t_1} e^{-\int_s^{t_1} c_i(u)du} F_i(s)ds \right| \\
&\le e^{-c_i^- t_1}\|\varphi-\varphi^*\| + \int_0^{t_1} e^{-c_i^-(t_1-s)}|F_i(s)|ds \\
&\le e^{-c_i^- t_1}\|\varphi-\varphi^*\| + \int_0^{t_1} e^{-c_i^-(t_1-s)}\left(\sum_{j=1}^{m} a_{ij}F_{1j}pM\|\varphi-\varphi^*\|e^{-\lambda(s-\tau_{ij}(s))}\right. \\
&\qquad\qquad \left. +\sum_{j=1}^{n} b_{ji}F_{2j}pM\|\varphi-\varphi^*\|e^{-\lambda(s-\overline{\delta}_{ji}(s))}\right)ds \\
&< pM\|\varphi-\varphi^*\|e^{-\lambda t_1}\left[\frac{e^{t_1(\lambda-c_i^-)}}{M} + \frac{1-e^{t_1(\lambda-c_i^-)}}{c_i^- - \lambda}\left(\sum_{j=1}^{m} a_{ij}F_{1j}e^{\lambda\tau} + \sum_{j=1}^{n} b_{ji}F_{2j}e^{\lambda\delta}\right)\right] \\
&= pM\|\varphi-\varphi^*\|e^{-\lambda t_1}\left[\left(\frac{1}{M}-r_i\right)e^{t_1(\lambda-c_i^-)} + r_i\right] \\
&< pM\|\varphi-\varphi^*\|e^{-\lambda t_1}; \\
|\dot{u}_l(t_1)| &= \left| -c_l(t_1)u_l(0)e^{-\int_0^{t_1} c_l(u)du} - c_l(t_1)\int_0^{t_1} e^{-\int_s^{t_1} c_l(u)du}F_l(s)ds + F_l(t_1) \right| \\
&\le c_l^+ e^{-c_l^- t_1}\|\varphi-\varphi^*\| + c_l^+\int_0^{t_1} e^{-c_l^-(t_1-s)}|F_l(s)|ds + |F_l(t_1)| \\
&\le c_l^+ e^{-c_l^- t_1}\|\varphi-\varphi^*\| + c_l^+\int_0^{t_1} e^{-c_l^-(t_1-s)}\left(\sum_{j=1}^{m} a_{lj}F_{1j}pM\|\varphi-\varphi^*\|e^{-\lambda(s-\tau_{lj}(s))}\right. \\
&\qquad\qquad \left. +\sum_{j=1}^{n} b_{jl}F_{2j}pM\|\varphi-\varphi^*\|e^{-\lambda(s-\overline{\delta}_{jl}(s))}\right)ds \\
&\quad + \sum_{j=1}^{m} a_{lj}F_{1j}pM\|\varphi-\varphi^*\|e^{-\lambda(t_1-\tau_{lj}(t_1))} + \sum_{j=1}^{n} b_{jl}F_{2j}pM\|\varphi-\varphi^*\|e^{-\lambda(t_1-\overline{\delta}_{jl}(t_1))} \\
&< pM\|\varphi-\varphi^*\|e^{-\lambda t_1}\left[\left(\frac{1}{M}-r_l\right)e^{t_1(\lambda-c_l^-)} + \overline{r}_l\right] \\
&< pM\|\varphi-\varphi^*\|e^{-\lambda t_1}.
\end{aligned}
\tag{4.14}
$$

We also can get

$$
\begin{aligned}
|v_j(t_1)| &< pM\|\varphi-\varphi^*\|e^{-\lambda t_1}, \\
|\dot{v}_k(t_1)| &< pM\|\varphi-\varphi^*\|e^{-\lambda t_1}.
\end{aligned}
\tag{4.15}
$$

From (4.14)–(4.15), we have

$$\|z(t_1) - z^*(t_1)\|_1 = \max\{|u_i(t_1)|, |\dot{u}_l(t_1)|, |v_j(t_1)|, |\dot{v}_k(t_1)|\} < pM\|\varphi - \varphi^*\|e^{-\lambda t_1}, \tag{4.16}$$

which contradicts the equality (4.12), so (4.11) holds. Letting $p \to 1$, then (4.10) holds. The almost periodic solution of system (2.1) is globally exponentially stable. □

Corollary 4.2. *Let $b_{ji}(t) = q_{ij}(t) = 0$. Under assumptions (H_1), (H_2), and (H_4), if, (H_5)*

$$\alpha_1 = \max\left\{\max_{1\le i\le n}\left\{\frac{1}{c_i^-}\sum_{j=1}^{m} a_{ij}F_{1j}\right\}, \max_{1\le j\le m}\left\{\frac{1}{d_j^-}\sum_{i=1}^{n} p_{ji}G_{1i}\right\}\right\} < 1 \tag{4.17}$$

holds, then system

$$\begin{aligned}\dot{x}_i(t) &= -c_i(t)x_i(t) + \sum_{j=1}^{m} a_{ij}(t)f_{1j}\left(y_j\left(t - \tau_{ij}(t)\right)\right) + I_i(t),\\ \dot{y}_j(t) &= -d_j(t)y_j(t) + \sum_{i=1}^{n} p_{ji}(t)g_{1i}\left(x_i\left(t - \delta_{ji}(t)\right)\right) + J_j(t)\end{aligned} \tag{4.18}$$

has a unique almost periodic solution in the region $\|\varphi - \varphi_0\| \le \alpha_1\beta/(1-\alpha_1)$, which is global exponentially stable.

In fact, Zhang and Si [11, 16] and Chen et al. [17] studied system (4.18). This Corollary 4.2 is the Theorem 3.1 in [11], Theorem 1.1 in [16], and Theorem 1 in [17]. Especially, in [17], authors let

(H_5')

$$\overline{\alpha}_1 = \max_{1\le i\le n}\left\{\frac{1}{c_i^-}\sum_{j=1}^{m} a_{ij}F_{1j}\right\} + \max_{1\le j\le m}\left\{\frac{1}{d_j^-}\sum_{i=1}^{n} p_{ji}G_{1i}\right\} < 1. \tag{4.19}$$

Therefore, we extend and improve previously known results.

Remark 4.3. Let $c_i(t) = d_j(t)$, $a_{ij}(t) = p_{ji}(t)$, $b_{ji} = q_{ij}(t)$, $I_i(t) = J_j(t)$, $\tau_{ij}(t) = \delta_{ji}(t)$, $\overline{\tau}_{ij}(t) = \overline{\delta}_{ji}(t)$, $n = m$. Then system (2.1) is reduced to be system (1.1), hence we have the following.

Corollary 4.4. *Under assumptions* (H_1), (H_2), *and* (H_4), *if* (H_6)

$$\alpha_2 = \max_{1\leq i\leq n} \max\left\{1+\frac{c_i^+}{c_i^-}\right\}\sum_{j=1}^{m}\left(a_{ij}F_{1j}+b_{ji}F_{2j}\right) < 1, \tag{4.20}$$

holds, then system (1.1) *has a unique almost periodic solution in the region* $\|\varphi-\varphi_0\| \leq \alpha_2\beta/(1-\alpha_2)$, *which is global exponentially stable.*

This Corollary 4.4 is the result of [19].

5. An Example

In this section, we give an example to illustrate the effectiveness of our results.

Let $n = m = 2$, $f_1(y_1) = y_1/10$, $f_2(y_2) = \sin y_2/10$, $g_1(x_1) = x_1/12$, $g_2(x_2) = |x_2|/8$, $\tau_{ij}(t) = \overline{\tau}_{ij}(t) = \cos^2 t$, $\delta_{ji}(t) = \overline{\delta}_{ji}(t) = 0.5$, $I_1(t) = 1+\sin^2(t)$, $I_2(t) = 1+\cos^2 t$, $J_1(t) = 1+|\sin t|$, and $J_2(t) = \sin 2t + 0.5$, then we consider the following almost periodic system:

$$\begin{aligned}
\dot{x}_1(t) &= -c_1(t)x_1(t) + \sum_{j=1}^{2} a_{1j}(t)f_j\left(y_j\left(t-\cos^2 t\right)\right) + \sum_{j=1}^{2} b_{j1}(t)\dot{x}_j(t-0.5) + I_1(t),\\
\dot{x}_2(t) &= -c_2(t)x_2(t) + \sum_{j=1}^{2} a_{2j}(t)f_j\left(y_j\left(t-\cos^2 t\right)\right) + \sum_{j=1}^{2} b_{j2}(t)\dot{x}_j(t-0.5) + I_2(t),\\
\dot{y}_1(t) &= -d_1(t)y_1(t) + \sum_{i=1}^{2} p_{1i}(t)g_i(x_i(t-0.5)) + \sum_{i=1}^{2} q_{i1}(t)\dot{y}_i\left(t-\cos^2 t\right) + J_1(t),\\
\dot{y}_2(t) &= -d_2(t)y_2(t) + \sum_{i=1}^{2} p_{2i}(t)g_i(x_i(t-0.5)) + \sum_{i=1}^{2} q_{i2}(t)\dot{y}_i\left(t-\cos^2 t\right) + J_2(t),
\end{aligned} \tag{5.1}$$

where $c_1(t) = 1+\cos^2 t$, $c_2(t) = 1+\sin^2 t$, $d_1(t) = 1+|\cos t|$, $d_2(t) = 1+|\sin t|$, $a_{11}(t) = |\sin t|/4$, $a_{12}(t) = \cos^2 t/8$, $a_{21}(t) = \cos^2 t/6$, $a_{22}(t) = |\sin t|/4$, $b_{11}(t) = \cos 2t/8$, $b_{12}(t) = 0$, $b_{21}(t) = 0$, $b_{22}(t) = \sin 2t/10$, $p_{11}(t) = \cos 2t/4$, $p_{12}(t) = \sin 2t/9$, $p_{21}(t) = \sin^2 t/8$, $p_{22}(t) = |\cos t|/6$, $q_{11}(t) = \cos t/8$, $q_{12}(t) = 0$, $q_{21}(t) = 0$, and $q_{22}(t) = \cos^2 t/10$. By simple calculation, we obtain $\alpha = \max\{39/80, 51/120, 69/144, 63/160\} < 1$, hence this system has a unique almost periodic solution, which is global exponentially stable by Theorem 4.1. Figure 1 depicts the time responses of state variables of $x_1(t)$, $x_2(t)$, $y_1(t)$, and $y_2(t)$ with step $h = 0.005$ and initial states $[-0.2, 0.2, -0.3, 0.4]^T$ for $t \in [-1, 0]$, and Figures 2, 3, and 4 depict the phase orbits of $x_1(t)$ and $y_1(t)$, $x_1(t)$, and $x_2(t)$, $y_1(t)$ and $y_2(t)$. It confirms that our results are effective for (5.1).

6. Conclusions

In this paper, a class of BAM neural networks with variable coefficients and neutral time-varying delays are investigated. By employing Banach fixed-point theorem, the exponential

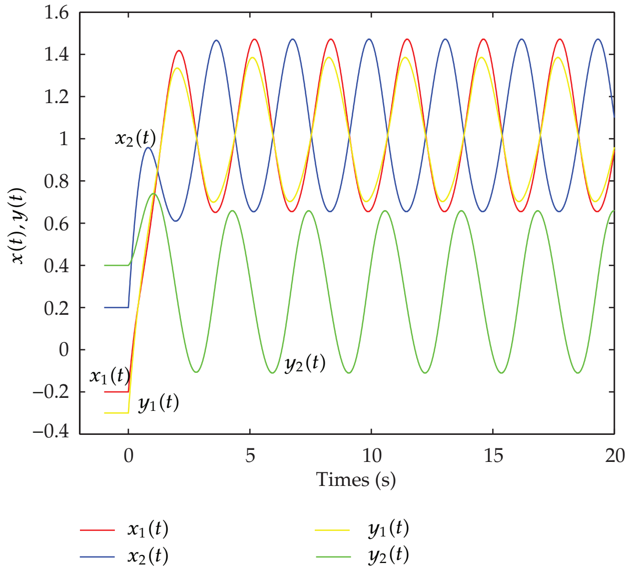

Figure 1: Transient response of state variable $x_1(t)$, $x_2(t)$, $y_1(t)$ and $y_2(t)$.

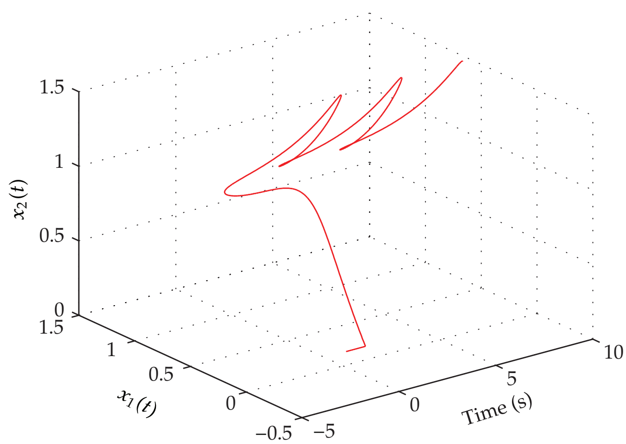

Figure 2: Phase response of state variable $x_1(t)$ and $x_2(t)$.

dichotomy and differential inequality techniques, some sufficient conditions are obtained to ensure the existence, uniqueness, and stability of the almost periodic solution. As is known to all, neural networks with neutral delays are studied rarely, and most authors solve these problems by linear matrix inequality techniques. In addition, BAM neural networks are much more complicated than the one-layer neural network. In a word, this paper is original, and novel. It also extends and improves other previously known results (see [11, 16, 17, 19]).

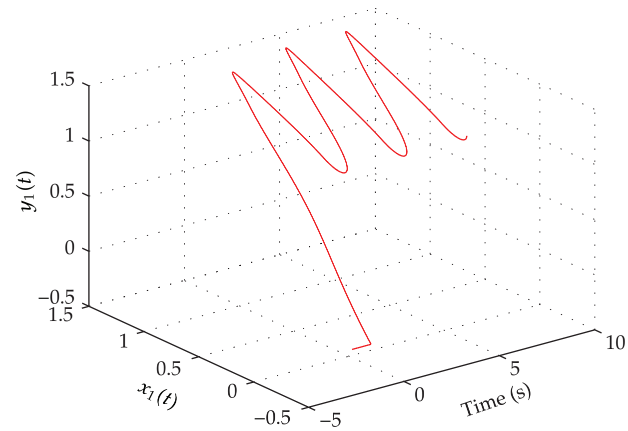

Figure 3: Phase response of state variable $x_1(t)$ and $y_1(t)$.

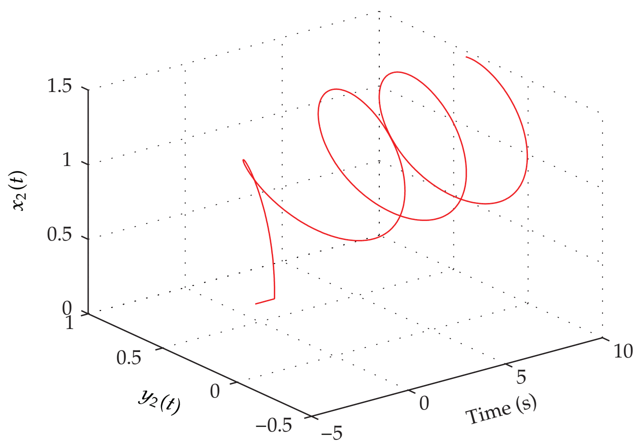

Figure 4: Phase response of state variable $x_2(t)$ and $y_2(t)$.

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Research Article

Stochastic Synchronization of Reaction-Diffusion Neural Networks under General Impulsive Controller with Mixed Delays

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This paper investigates drive-response synchronization of a class of reaction-diffusion neural networks with time-varying discrete and distributed delays via general impulsive control method. Stochastic perturbations in the response system are also considered. The impulsive controller is assumed to be nonlinear and has multiple time-varying discrete and distributed delays. Compared with existing nondelayed impulsive controller, this general impulsive controller is more practical and essentially important since time delays are unavoidable in practical operation. Based on a novel impulsive differential inequality, the properties of random variables and Lyapunov functional method, sufficient conditions guaranteeing the global exponential synchronization in mean square are derived through strict mathematical proof. In our synchronization criteria, the distributed delays in both continuous equation and impulsive controller play important role. Finally, numerical simulations are given to show the effectiveness of the theoretical results.

1. Introduction

Since the pioneering work of Pecora and Carroll [1], the issue of synchronization and chaos control has been extensively studied [2] due to its potential engineering applications such as secure communication, biological systems, and information processing (see [3–10]). It is shown that neural networks exhibit chaotic behavior and provided that parameters and delays are appropriately chosen (see [11, 12]). Therefore, in recent years, synchronization and control of neural networks has been one of the hot research topics (see [13–15], etc.).

It is known that many pattern formation and wave propagation phenomena that appear in nature can be described by systems of coupled nonlinear differential equations, generally known as reaction-diffusion equations. These wave propagation phenomena are exhibited by systems belonging to very different scientific disciplines. The reaction-diffusion effects, therefore, cannot be neglected in both biological and man-made neural networks, especially when electrons are moving in noneven electromagnetic field [16]. So we must consider that the activations vary in space as well as in time, and in this case the model should be expressed by partial differential equations. There are some published papers concerning stability or synchronization of neural networks with reaction-diffusion terms and delays (see [17–25]). In [22], the authors investigated synchronization of reaction-diffusion neural networks with discrete and unbounded distributed delays. In [24], the authors investigated the boundedness and exponential stability for nonautonomous fuzzy cellular neural networks with unbounded distributed delays and reaction-diffusion terms. The authors of [25] studied exponential stability of reaction-diffusion Cohen-Grossberg neural networks with time-varying discrete delays and stochastic perturbations.

Time delays usually exist in neural networks due to finite speeds of switching of amplifiers and transmission of signals in hardware implementation. Ignoring them when studying dynamics of neural networks may lead to impractical results. Moreover, delays are commonly time varying and unknown [26]. Therefore, papers concerning synchronization or stability of neural networks with or without reaction-diffusion terms have considered various time delays. The authors in [11] studied exponential synchronization problem for coupled neural networks with constant time delay. In [27], both constant and time-varying discrete delays were considered for the synchronization of a class of delayed neural networks. In [28–31] several types of synchronization for neural networks with discrete and bounded distributed delays were studied. However, the delay kernel of the bounded distributed delays in [28–31] has to be 1 because the well-known Jensen's inequality [32] is not applicable anymore if the delay kernel is not 1. In the case of unbounded distributed delay, it is necessary to consider the delay kernel, which satisfies the condition that its integral from zero to infinite is bounded [22, 33, 34]. But the authors in [22, 33, 34] had to use algebraic approach instead of matrix method to derive their main results which has more complex form and is more conservative than those obtained by matrix method. In [21], Wang and Zhang studied global asymptotic stability of reaction-diffusion Cohen-Grossberg neural networks with unbounded distributed delays by using a matrix decomposition method, and the obtained results were in terms of linear matrix inequality (LMI). But the Lyapunov functional and proof process used in [21] are relatively complex. Recently, authors in [35] studied global asymptotic synchronization in an array of coupled neural networks with probabilistic interval time-varying coupling delays and unbounded distributed delays; a novel integral inequality including the Jensen's inequality as a special case was developed. By using the developed integral inequality, one can use LMI method to solve the problem of distributed delays with not-equal-to-1 delay kernel instead of the matrix decomposition method used in [21].

It should be noted that control method is of great significance to realize synchronization. Specially, in [29], the output feedback controller which has time-varying discrete and distributed delays was considered. On the other hand, impulsive control, as one of the most effective and economic control methods, has recently attracted great interests of many researchers in different fields, since it needs small control gains and acts only at discrete times, thus control cost and the amount of transmitted information can be reduced drastically (see [3, 9, 26, 36–40] and references cited therein). As for neural networks with reaction-diffusion terms, there are several results on synchronization via control. For instance, state feedback control technique is utilized in [20] to realize exponential synchronization of stochastic fuzzy cellular neural networks with time delay in the leakage term and reaction diffusion. In [22], global exponential stability and synchronization of delayed reaction-diffusion neural networks under hybrid state feedback control and impulsive control. However, to the authors knowledge, impulsive control has not been considered in the literature to realize synchronization of reaction-diffusion neural networks. Moreover, the impulsive controllers in [3, 9, 36–40] were nondelayed. Recently, in [41], global exponential stability of fuzzy reaction-diffusion cellular neural networks with time-varying discrete delays and unbounded distributed delays and impulsive perturbations were studied. Nevertheless, to the best of our knowledge, there are no results on stability or synchronization of reaction-diffusion neural networks with time-varying discrete delays and distributed delays under impulsive controller which has multiple time-varying delays, let alone impulsive controller with distributed delays. If these delays are considered in impulsive controller, the analysis methods used in [3, 9, 26, 36–40] are not applicable anymore. Considering the fact that both discrete delays and distributed delays are unavoidable in practice, it is of great importance to consider delayed impulsive control to synchronize-delayed neural neural networks.

Being motivated by the above discussions, this paper aims to study the global exponential derive-response synchronization of reaction-diffusion neural networks with multiple time-varying discrete delays and unbounded distributed delays via general impulsive control. The general impulsive controller is assumed to be nonlinear and has multiple time-varying discrete and distributed delays. Since time delays are always vary and unavoidable in practical operation, the general impulsive controller is essentially important and more practical than existing nondelayed impulsive controller. Stochastic perturbations in the response system are also considered. By using a novel integral inequality in [35], the problem of distributed delays with not-equal-to-1 delay kernel can be solved by matrix method. By utilizing the novel integral inequality, the properties of random variables and Lyapunov functional method, sufficient conditions guaranteeing the considered drive-response systems to realize synchronization in mean square are derived through strict mathematical proof. The proof process and the results are very simple. Finally, numerical simulations are given to show the effectiveness of the theoretical results.

The rest of this paper is organized as follows. In Section 2, the considered model of coupled reaction-diffusion neural networks with delays is presented. Some necessary assumptions, definitions, and lemmas are also given in this section. In Section 3, synchronization for the proposed model is studied. Then, in Section 4, simulation examples are presented to show the effectiveness of the theoretical results. Finally, Section 5 provides some conclusions.

Notations. In the sequel, if not explicitly stated, matrices are assumed to have compatible dimensions. $\mathbb{N}_+$ denotes the set of positive integers. I_n denotes the $n \times n$ identity matrix. $\mathbb{R}^n$ denotes the Euclidean space, and $\mathbb{R}^{n\times m}$ is the set of all $n \times m$ real matrix. $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ mean the largest and smallest eigenvalues of matrix A, respectively, $\|A\| = \sqrt{\lambda_{\max}(A^T A)}$, where T denotes transposition. $C = \operatorname{diag}(c_1, c_2, \dots, c_n)$ means C is a diagonal matrix. Moreover, let $(S, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., the filtration contains all P-null sets and is right continuous). Denote by $L^p_{\mathcal{F}_0}((-\infty, 0]; \mathbb{R}^n)$ the family of all $\mathcal{F}_0$-measurable $C((-\infty, 0]; \mathbb{R}^n)$-valued random variables $\xi = \{\xi(s) : s \leq 0\}$ such that $\sup_{s\leq 0}\mathbb{E}(\|\xi(s)\|^p) < \infty$, where $\mathbb{E}\{\cdot\}$ stands for mathematical expectation operator with respect to the given probability measure P. Sometimes, the arguments of a function or a matrix will be omitted in the analysis when no confusion can arise.

2. Preliminaries

Consider a delayed neural network with reaction-diffusion terms which is described as follows:

$$\begin{aligned}\frac{\partial y_i(t,x)}{\partial t} &= \sum_{l=1}^{m}\frac{\partial}{\partial x_l}\left(r_{il}\frac{\partial y_i(t,x)}{\partial x_l}\right) - c_i y_i(t,x) + \sum_{j=1}^{n} a_{ij} f_j\left(y_j(t,x)\right) \\ &\quad + \sum_{j=1}^{n} b_{ij} f_j\left(y_j(t-\tau_1(t),x)\right) + \sum_{j=1}^{n} d_{ij}\int_{-\infty}^{t} K(t-s) f_j\left(y_j(s,x)\right)\mathrm{d}s + I_i(t),\end{aligned} \tag{2.1}$$

or in a compact form

$$\begin{aligned}\frac{\partial y(t,x)}{\partial t} &= \sum_{l=1}^{m}\frac{\partial}{\partial x_l}\left(R_l\frac{\partial y(t,x)}{\partial x_l}\right) - Cy(t,x) + Af\left(y(t,x)\right) + Bf\left(y(t-\tau_1(t),x)\right) \\ &\quad + D\int_{-\infty}^{t} K(t-s) f\left(y(s,x)\right)\mathrm{d}s + I(t),\end{aligned} \tag{2.2}$$

where $i = 1,2,\ldots,n$, $R_l = \operatorname{diag}(r_{1l}, r_{2l}, \ldots, r_{nl})$, $l = 1,2,\ldots,m$, $r_{il} \geq 0$ means the transmission diffusion coefficient along the ith neuron; $x = (x_1, x_2, \ldots, x_m)^T \in \Omega \subset \mathbb{R}^m$, $\Omega = \{x \mid |x_k| \leq z_l,\ l = 1,2,\ldots,m\}$, and z_l is a constant. $y(t,x) = (y_1(t,x), y_2(t,x), \ldots, y_n(t,x))^T \in \mathbb{R}^n$ represents the state vector of the network at time t and in space x; n corresponds to the number of neurons; $f(y(t,x)) = (f_1(y_1(t,x)), \ldots, f_n(y_n(t,x)))^T$ is the neuron activation function at time t and in space x; $C = \operatorname{diag}(c_1, c_2, \ldots, c_n)$ with $c_i > 0$; $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$ and $D = (d_{ij})_{n\times n}$ are the connection weight matrix; $I(t) = (I_1(t), I_2(t), \ldots, I_n(t))^T \in \mathbb{R}^n$ is an external input vector. The bounded function $\tau_1(t)$ represents unknown time-varying discrete delay of the system with $0 < \tau_1(t) \leq \overline{\tau}_1$, in which $\overline{\tau}_1$ is a constant, $K(t)$ is a nonnegative bounded scalar function defined on $[0, +\infty)$ describing the delay kernel of the unbounded distributed delay.

We suppose that system (2.2) has an unique continuous solution for any initial condition of the following form: $y(s,x) = \phi(s,x) \in C([-\infty,0]\times\Omega, \mathbb{R}^n)$, where $C([-\infty,0]\times\Omega, \mathbb{R}^n)$ denotes the Banach space of all continuous functions from $[-\infty,0]\times\Omega$ to $\mathbb{R}^n$ with the norm

$$\|\phi(s,x)\| = \left[\int_{\Omega}\phi^T(s,x)\phi(s,x)\mathrm{d}x\right]^{1/2}. \tag{2.3}$$

It is assumed that (2.2) satisfies the following Dirichlet boundary condition:

$$y(t,x) = 0, \quad (t,x) \in [-\infty, +\infty] \times \partial\Omega. \tag{2.4}$$

Based on the concept of drive-response synchronization, we take (2.2) as the driver system and design the following controlled response system:

$$\begin{aligned}
\mathrm{d}u(t,x) = &\left[\sum_{l=1}^{m}\frac{\partial}{\partial x_l}\left(R_l\frac{\partial u(t,x)}{\partial x_l}\right) - Cu(t,x) + Af(u(t,x)) + Bf(u(t-\tau_1(t),x))\right. \\
&\left. + D\int_{-\infty}^{t}K(t-s)f(u(s,x))\mathrm{d}s + I(t) + \sum_{k=1}^{+\infty}\delta(t-t_k)U_k(t,x)\right]\mathrm{d}t + \sigma(t,x)\mathrm{d}\omega(t),
\end{aligned} \tag{2.5}$$

where $e(t,x) = u(t,x) - y(t,x)$, $\delta(t)$ is the Dirac delta function, the time sequence $\{t_k\}$ satisfies $0 = t_0 < t_1 < t_2 < \cdots < t_{k-1} < t_k < \cdots$, and $\lim_{k\to+\infty}t_k = +\infty$. $U_k(t,x)$ is the control input. $\omega(t) = (\omega_1(t),\ldots,\omega_n(t))^T \in \mathbb{R}^n$ is a n-dimensional Brown motion defined on $(S,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},P)$. Here, the white noise $\mathrm{d}\omega_i(t)$ is independent of $\mathrm{d}\omega_j(t)$ for $i\neq j$, and $\sigma(t,x) \triangleq \sigma(t,e(t,x),e(t-\tau_2(t),x),\int_{t-\tau_3(t)}^{t}e(s,x)\mathrm{d}s)$ is the noise intensity function matrix, in which the bounded functions $\tau_2(t)$ and $\tau_3(t)$ represent unknown discrete and distributed delays of the system in the stochastic perturbation with $0 < \tau_i(t) \leq \overline{\tau}_i$, $i = 2,3$. This type of stochastic perturbation can be regarded as a result from the occurrence of random uncertainties during the process of transmission. We assume that the output signals of (2.2) can be received by (2.5).

In the present paper, the control input $U_k(t,x)$ is assumed to be the following form:

$$U_k(t,x) = h_k\left(e(t,x), e(t-\eta_1(t),x),\ldots,e(t-\eta_q(t),x),\int_{t-\eta_{q+1}(t)}^{t}e(s,x)\mathrm{d}s\right) - e(t,x), \tag{2.6}$$

where $\eta_i(t)$ $i = 1,2,\ldots,q+1$ are unknown time-varying delays with $0 < \eta_i(t) \leq \overline{\eta}_i$.

Integrating from $t_k - \varepsilon$ to $t_k + \varepsilon$ ($\varepsilon > 0$ is a sufficient small constant) on both sides of system (2.5) and letting $\varepsilon \to 0^+$, one gets from the property of the Dirac delta function that

$$\begin{aligned}
u(t_k^+,x) - u(t_k^-,x) = &h_k\left(e(t_k,x), e(t_k-\eta_1(t_k),x),\ldots,e(t_k-\eta_q(t_k),x),\int_{t_k-\eta_{q+1}(t_k)}^{t_k}e(s,x)\mathrm{d}s\right) \\
&- e(t_k,x),
\end{aligned} \tag{2.7}$$

where $u(t_k^+,x) = \lim_{t\to t_k^+}u(t,x)$, $u(t_k^-,x) = \lim_{t\to t_k^-}u(t,x)$. In the following, we use $h_k(t_k,x)$ to denote $h_k(e(t_k,x), e(t_k-\eta_1(t_k),x),\ldots,e(t_k-\eta_q(t_k),x),\int_{t_k-\eta_{q+1}(t_k)}^{t_k}e(s,x)\mathrm{d}s)$ for short.

Remark 2.1. Equation (2.7) is actually the impulsive controller of response system (2.5). To the best of our knowledge, result on synchronization of reaction-diffusion neural networks under impulsive control is seldom. In [22], global exponential synchronization of delayed reaction-diffusion neural networks was studied. However, the control scheme in [22] is hybrid nondelayed state feedback control and nondelayed impulsive control, and the continuous state feedback controller is indispensable. Moreover, the impulsive controller (2.7) is very general, since it includes information of multiple time-varying discrete delays and time-varying distributed delays. Nevertheless, most of published paper concerning impulsive control including [3, 9, 26, 36–40] did not consider time delay in the impulsive function, let alone multiple time-varying discrete delays and time-varying distributed delays. It is known that both discrete delays and distributed delays are unavoidable and often time-varying in neural networks, hence considering impulsive control with time-varying discrete delays and time-varying distributed delays is essentially important. However, when time-varying discrete delays and time-varying distributed delays are considered in impulsive control, the results in [3, 9, 26, 36–40] is not applicable anymore.

From (2.7), the controlled system (2.5) can be rewritten as

$$\begin{aligned} \mathrm{d}u(t,x) &= \left[\sum_{l=1}^{m}\frac{\partial}{\partial x_l}\left(R_l\frac{\partial u(t,x)}{\partial x_l}\right) - Cu(t,x) + Af(u(t,x)) + Bf(u(t-\tau_1(t),x))\right. \\ &\quad \left. +D\int_{-\infty}^{t} K(t-s)f(u(s,x))\mathrm{d}s + I(t)\right]\mathrm{d}t + \sigma(t,x)\mathrm{d}\omega(t), \quad t \neq t_k, \\ u\left(t_k^+,x\right) &= u\left(t_k^-,x\right) + h_k(t_k,x) - e(t_k,x), \quad t = t_k,\ k \in \mathbb{N}_+. \end{aligned} \tag{2.8}$$

To maintain consistency with above definitions, the initial value and the boundary condition of (2.8) are given in the following form:

$$u(s,x) = \varphi(s,x) \in C([-\infty,0] \times \Omega, \mathbb{R}^n), \tag{2.9}$$

$$u(t,x) = 0, \quad (t,x) \in [-\infty,+\infty] \times \partial\Omega. \tag{2.10}$$

Throughout this paper, we always assume that $u(t,x)$ is left continuous at t_k, that is, $u(t_k^-,x) = u(t_k,x)$. Then subtracting (2.2) from (2.8) gets the following error dynamical system:

$$\begin{aligned} \mathrm{d}e(t,x) &= \left[\sum_{l=1}^{m}\frac{\partial}{\partial x_l}\left(R_l\frac{\partial e(t,x)}{\partial x_l}\right) - Ce(t,x) + Ag(e(t,x)) + Bg(e(t-\tau_1(t),x))\right. \\ &\quad \left. +D\int_{-\infty}^{t} K(t-s)g(e(s,x))\mathrm{d}s\right]\mathrm{d}t + \sigma(t,x)\mathrm{d}\omega(t), \quad t \neq t_k, \\ e\left(t_k^+,x\right) &= h_k(t_k,x), \quad t = t_k,\ k \in \mathbb{N}_+, \end{aligned} \tag{2.11}$$

where $g(e(t,x)) = f(u(t,x)) - f(y(t,x))$.

It is obvious that system (2.11) satisfies the Dirichlet boundary condition, and its initial condition is

$$e(s,x) = \varphi(s,x) - \phi(s,x) = \overline{\varphi}(s,x) \in C([-\infty,0] \times \Omega, \mathbb{R}^n), \quad i = 1,2,\ldots,N. \tag{2.12}$$

It is easy to see that the error system (2.11) admits a zero solution. Clearly, if the zero solution is globally exponentially stable, then the controlled system (2.8) is globally exponentially synchronized with system (2.2).

Throughout this paper, we assume that

(H_1) for any $u, v \in \mathbb{R}$, there exist constants μ_i $(i = 1, 2, \ldots, n)$ such that $|f_i(u) - f_i(v)| \leq \mu_i |u - v|$;

(H_2) there is a positive constant $\overline{k}$ such that $\int_0^{+\infty} K(u)\mathrm{d}u = \overline{k}$;

(H_3) there exist positive constants ρ_1, ρ_2 and ρ_3 such that

$$\operatorname{trace}\left[\sigma^T(t,x)\sigma(t,x)\right] \leq \rho_1 e^T(t,x)e(t,x) + \rho_2 e^T(t-\tau_2(t),x)e(t-\tau_2(t),x) + \rho_3 \int_{t-\tau_3(t)}^{t} e^T(s,x)e(s,x)\mathrm{d}s; \tag{2.13}$$

(H_4) there exist nonnegative constants α_k, β_k^j, $j = 1, 2, \ldots, q+1$ such that

$$h_k^T(t_k,x)h_k(t_k,x) \leq \alpha_k e^T(t_k,x)e(t_k,x) + \beta_k^1 e^T(t_k - \eta_1(t_k),x)e(t_k - \eta_1(t_k),x) + \cdots + \beta_k^q e^T(t_k - \eta_q(t_k))e(t_k - \eta_q(t_k)) + \beta_k^{q+1}\int_{t_k-\eta_{q+1}(t_k)}^{t_k} e^T(s,x)e(s,x)\mathrm{d}s. \tag{2.14}$$

The following basic definitions and lemmas are needed in this paper to get main results.

Definition 2.2 (see [9]). The dynamical network (2.9) is said to be globally exponentially synchronized with system (2.2) in mean square if there exist constants $M > 1$ and $\theta > 0$ such that for any initial values (2.12)

$$\mathbb{E}\left\{\|e(t,x)\|^2\right\} \leq \max_{s\leq 0}\mathbb{E}\left\{\|\overline{\varphi}(s,x)\|^2\right\}Me^{-\theta t} \tag{2.15}$$

hold for $t \geq 0$.

Lemma 2.3 (see [17]). *Let Ω be a cube $|x_k| < l_k$ $(k = 1, 2, \ldots, m)$, and let $v(x)$ be a real-valued function belonging to $C^1(\Omega)$ which vanish on the boundary $\partial\Omega$ of Ω, that is, $v(x)|_{\partial\Omega} = 0$. Then*

$$\int_\Omega v^2(x)\mathrm{d}x \leq l_k^2 \int_\Omega \left|\frac{\partial v(x)}{\partial x_k}\right|^2 \mathrm{d}x. \tag{2.16}$$

Lemma 2.4 (see [42]). *If X, Y are real matrices with appropriate dimensions, then there exist number $\varepsilon > 0$ such that*

$$X^T Y + Y^T X \leq \varepsilon X^T Y + \frac{1}{\varepsilon} Y^T Y. \tag{2.17}$$

Lemma 2.5 (see [35]). *Suppose that $K(t)$ is a nonnegative bounded scalar function defined on $[0,+\infty)$, and there exists a positive constant k such that $\int_0^{+\infty} K(u)\,\mathrm{d}u = k$. For any constant matrix $D \in \mathbb{R}^{n\times n}$, $D > 0$, and vector function $x : (-\infty, t] \to \mathbb{R}^n$ for $t \geq 0$, one has*

$$k\int_{-\infty}^{t} K(t-s)x^T(s)Dx(s)\,\mathrm{d}s \geq \left(\int_{-\infty}^{t} K(t-s)x(s)\,\mathrm{d}s\right)^T D\int_{-\infty}^{t} K(t-s)x(s)\,\mathrm{d}s \tag{2.18}$$

provided the integrals are all well defined.

Remark 2.6. When there is a positive bounded function $k(t)$ such that $\int_0^{\theta(t)} K(u)\mathrm{d}u = k(t)$, where $0 < \theta(t) \leq \theta$, then the inequality (2.18) becomes the following from:

$$k(t)\int_{t-\theta(t)}^{t} K(t-s)x^T(s)Dx(s)\mathrm{d}s \geq \left(\int_{t-\theta(t)}^{t} K(t-s)x(s)\mathrm{d}s\right)^T D\int_{t-\theta(t)}^{t} K(t-s)x(s)\mathrm{d}s. \tag{2.19}$$

Specially, when $K(t) = 1$ for $t \geq 0$, then $k(t) = \theta(t)$ in (2.19). In this case, the inequality (2.19) turns out to the well-known Jensen's inequality [32]. In the literature, there were many results concerning stability or synchronization of neural networks with bounded distributed delays, for instance, see [28–31]. However, the delay kernels in [28–31] were all assumed to be 1. Obviously, the unbounded distributed delays in this paper include those [28–31] as a special case. It is easy to see from inequalities (2.18) and (2.19) that results of this paper are also applicable to neural networks with bounded distributed delays, no matter whether $K(t)$ is equal to 1 or not. In this sense, models in this paper are more general than those those in [28–31].

Lemma 2.7. *Consider the following impulsive differential inequalities:*

$$\begin{gathered} D^+v(t) \leq av(t) + b_1[v(t)]_{\tau_1} + b_2[v(t)]_{\tau_2} + \cdots + b_m[v(t)]_{\tau_m}, \quad t \neq t_k,\ t \geq t_0, \\ v(t_k^+) \leq p_k v(t_k^-) + q_k^1\left[v(t_k^-)\right]_{\tau_1} + q_k^2\left[v(t_k^-)\right]_{\tau_2} + \cdots + q_k^m\left[v(t_k^-)\right]_{\tau_m}, \quad k \in \mathbb{N}_+, \\ v(t) = \phi(t), \quad t \in [t_0 - \tau, t_0], \end{gathered} \tag{2.20}$$

where a, b_i, p_k, q_k^i, and τ_i are constants, $b_i \geq 0$, $p_k \geq 0$, $q_k^i \geq 0$, $\tau_i \geq 0$, $i = 1,2,\ldots,m$, $v(t) \geq 0$, $[v(t)]_{\tau_i} = \sup_{t-\tau_i\leq s\leq t} v(s)$, $[v(t_k^-)]_{\tau_i} = \sup_{t_k-\tau_i(t_k)\leq s<t_k} v(s)$, $\phi(t)$ is continuous on $[t_0-\tau, t_0]$, and $v(t)$ is continuous except t_k, $k \in \mathbb{N}_+$, where it has jump discontinuities. The consequence $\{t_k\}$ satisfies $0 = t_0 < t_1 < t_2 < \cdots < t_k < t_{k+1} < \cdots$, and $\lim_{k\to+\infty} t_k = +\infty$. Suppose that

$$p_k + \sum_{i=1}^{m} q_k^i < 1, \tag{2.21}$$

$$a + \frac{\sum_{i=1}^{m} b_i}{p_k + \sum_{j=1}^{m} q_k^j} + \frac{\ln\left(p_k + \sum_{j=1}^{m} q_k^j\right)}{t_{k+1} - t_k} < 0. \tag{2.22}$$

Then there exist constants $\beta > 1$ and $\lambda > 0$ such that

$$v(t) \leq \|\phi\|_{\tau} \beta e^{-\lambda(t-t_0)}, \quad t \geq t_0, \tag{2.23}$$

where $\|\phi\|_{\tau} = \sup_{t_0-\tau \leq s \leq t_0} \|\phi(s)\|$, $\tau = \max\{\tau_i, i = 1, 2, \ldots, m\}$.

The proof of Lemma 2.7 is given in the appendix, which is partly similarly to that of Lemma 1 in [43].

Remark 2.8. Lemma 2.7 actually provides stability criterion for impulsive differential equations with multiple time-varying delays, and impulsive function is related to the same multiple time-varying delays. Actually, Lemma 2.7 can be written in a more general form. Let $b_i = 0$, $i = h+1, \ldots, m$, $q_k^j = 0$, $j = 1, \ldots, h$, $1 < h < m-1$, $\tau_{h+1} = \sigma_1, \ldots, \tau_{h+m-h} = \sigma_{m-h} = \sigma_r$, $q_k^{h+1} = \tilde{q}_k^1, \ldots, q_k^{h+m-h} = \tilde{q}_k^{m-h} = \tilde{q}_k^r$, the other parameters are the same as those in Lemma 2.7. Then one can get the following Lemma 2.9.

Lemma 2.9. *Consider the following impulsive differential inequality:*

$$\begin{aligned}
D^+ v(t) &\leq a v(t) + b_1 [v(t)]_{\tau_1} + b_2 [v(t)]_{\tau_2} + \cdots + b_h [v(t)]_{\tau_h}, \quad t \neq t_k,\ t \geq t_0, \\
v(t_k^+) &\leq p_k v(t_k^-) + \tilde{q}_k^1 [v(t_k^-)]_{\sigma_1} + \tilde{q}_k^2 [v(t_k^-)]_{\sigma_2} + \cdots + \tilde{q}_k^r [v(t_k^-)]_{\sigma_r}, \quad k \in \mathbb{N}_+, \\
v(t) &= \phi(t), \quad t \in [t_0 - \tau, t_0].
\end{aligned} \tag{2.24}$$

Suppose that

$$p_k + \sum_{i=1}^{r} \tilde{q}_k^i < 1, \qquad a + \frac{\sum_{i=1}^{h} b_i}{p_k + \sum_{j=1}^{r} \tilde{q}_k^j} + \frac{\ln\left(p_k + \sum_{j=1}^{r} \tilde{q}_k^j\right)}{t_{k+1} - t_k} < 0. \tag{2.25}$$

Then there exist constants $\beta > 1$ and $\lambda > 0$ such that

$$v(t) \leq \|\phi\|_{\tau} \beta e^{-\lambda(t-t_0)}, \quad t \geq t_0, \tag{2.26}$$

where $\|\phi\|_{\tau} = \sup_{t_0-\tau \leq s \leq t_0} \|\phi(s)\|$, $\tau = \max\{\tau_i, \sigma_j,\ i = 1, 2, \ldots, h,\ j = 1, 2, \ldots, r\}$.

Remark 2.10. Lemmas 2.7 and 2.9 are general. Specially, if $\tilde{q}_k^i = 0$, $i = 1, 2, \ldots, r$, then the inequalities in (2.25) becomes

$$p_k < 1, \qquad a + \frac{\sum_{i=1}^{h} b_i}{p_k} + \frac{\ln p_k}{t_{k+1} - t_k} < 0. \tag{2.27}$$

Take $p = \max\{p_k, k \in \mathbb{N}_+\}$, $\rho = \sup_{k \in \mathbb{N}^+}\{t_k - t_{k-1}\}$. Then $p < 1$ and

$$a + \frac{\sum_{i=1}^{h} b_i}{p_k} + \frac{\ln p_k}{t_{k+1} - t_k} \leq a + \frac{\sum_{i=1}^{h} b_i}{p_k} + \frac{\ln p}{t_{k+1} - t_k} \leq a + \frac{\sum_{i=1}^{h} b_i}{p_k} + \frac{\ln p}{\rho}. \tag{2.28}$$

Therefore,

$$p < 1, \qquad a + \frac{\sum_{i=1}^{h} b_i}{p} + \frac{\ln p}{\rho} < 0 \tag{2.29}$$

implies (2.27); that is, the inequality (2.27) is less conservative than (2.29). In fact, the inequality (2.29) is exactly the inequalities (5) and (6) in Theorem 3.1 of [26]. (In the proof in Theorem 3.1 in [26], one can get from $b_i = (L_i/\alpha)\sqrt{\lambda_{\max}(P)/\lambda_{\min}(P)}$ that $(L_i^2\lambda_{\max}(P))/(b_i\lambda_{\min}(P)) = \alpha L_i\sqrt{\lambda_{\max}(P)/\lambda_{\min}(P)}$. By comparing the coefficients in the first two inequalities in the proof of Theorem 3.1 in [26] with those in the inequalities (5) and (6) in [26], the conclusion can be easily achieved). Hence, Lemmas 2.7 and 2.9 improve and extend the Theorem 3.1 in [26]. In the literature, many results including those in [3, 9, 38, 40] were derived by using similar method used in [26]. Since Lemmas 2.7 and 2.9 include corresponding results in [26] as a special case and are less conservative than them, Lemmas 2.7 and 2.9 are very useful for stabilization and synchronization of impulsive control system.

3. Main Results

In this section, the global exponential synchronization criteria for system (2.8) and (2.2) are derived through strict mathematical reasoning.

Theorem 3.1. *Suppose that conditions* (H_1)–(H_4) *hold. If there exists constants* $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ *and* $\varepsilon_3 > 0$ *such that*

$$0 < \alpha_k + \sum_{i=1}^{q} \beta_k^i + \beta_k^{q+1}\overline{\eta}_{q+1} < 1, \quad k \in \mathbb{N}_+, \tag{3.1}$$

$$a + \frac{\varepsilon_2\mu + \rho_2 + \varepsilon_3\overline{k}^2\mu + \rho_3\overline{\tau}_3}{\alpha_k + \sum_{i=1}^{q}\beta_k^i + \beta_k^{q+1}\overline{\eta}_{q+1}} + \frac{\ln\left(\alpha_k + \sum_{i=1}^{q}\beta_k^i\right) + \beta_k^{q+1}\overline{\eta}_{q+1}}{t_{k+1} - t_k} < 0, \tag{3.2}$$

where $a = -2\lambda_{\min}(\widetilde{R} + C) + \varepsilon_1^{-1}\|A\|^2 + \varepsilon_1\mu + \varepsilon_2^{-1}\|B\|^2 + \varepsilon_3^{-1}\|D\|^2 + \rho_1$, $\widetilde{R} = \operatorname{diag}(\sum_{l=1}^{m}(r_{1l}/z_l^2), \sum_{l=1}^{m}(r_{2l}/z_l^2), \ldots, \sum_{l=1}^{m}(r_{nl}/z_l^2))$, $\mu = \max\{\mu_i^2,\ i = 1, 2, \ldots, n\}$. *Then, under the impulsive controller* (2.7)*, the controlled system* (2.8) *is globally exponentially synchronized with system* (2.2) *in mean square.*

Proof. Consider the following Lyapunov function:

$$V(t) = \int_{\Omega} \frac{1}{2} e^T(t,x)e(t,x)\mathrm{d}x. \tag{3.3}$$

We use $\mathcal{L}V(t)$ to denote the infinitesimal operator of $V(t)$ [44], which is defined as

$$\mathcal{L}V(t) = \lim_{\Delta \to 0^+} \Delta^{-1}[\mathbb{E}\{V(t+\Delta) \mid t\} - V(t)]. \tag{3.4}$$

Based on the property of Wiener process [11], differentiating $V(t)$ along the solution of the error system (2.11) for $t \in (t_{k-1}, t_k]$, $k \in \mathbb{N}_+$ obtains that

$$\mathrm{d}V(t) = \mathcal{L}V(t)\mathrm{d}t + e(t,x)\sigma(t,x)\mathrm{d}\omega(t), \tag{3.5}$$

where

$$\begin{aligned}\mathcal{L}V(t) = \int_\Omega \Bigg[& e^T(t,x)\sum_{l=1}^{m}\frac{\partial}{\partial x_l}\left(R_l\frac{\partial e(t,x)}{\partial x_l}\right) - e^T(t,x)Ce(t,x) + e^T(t,x)Ag(e(t,x)) \\ & + e^T(t,x)Bg(e(t-\tau_1(t),x)) + e^T(t,x)D\int_{-\infty}^{t}K(t-s)g(e(s,x))\mathrm{d}s \\ & +\frac{1}{2}\operatorname{trace}\left[\sigma^T(t,x)\sigma(t,x)\right]\Bigg]\mathrm{d}x.\end{aligned} \tag{3.6}$$

From the Green's formula and the Dirichlet boundary condition, we have (see [17–19])

$$\begin{aligned}\int_\Omega e^T(t,x)\sum_{l=1}^{m}\frac{\partial}{\partial x_l}\left(R_l\frac{\partial e(t,x)}{\partial x_l}\right)\mathrm{d}x &= \int_\Omega \sum_{i=1}^{n}e_i(t,x)\sum_{l=1}^{m}\frac{\partial}{\partial x_l}\left(r_{il}\frac{\partial e_i(t,x)}{\partial x_l}\right)\mathrm{d}x \\ &= \sum_{i=1}^{n}\int_\Omega e_i(t,x)\nabla\left(r_{il}\frac{\partial e_i(t,x)}{\partial x_l}\right)_{l=1}^{m}\mathrm{d}x \\ &= \sum_{i=1}^{n}\int_\Omega \nabla\left(e_i(t,x)r_{il}\frac{\partial e_i(t,x)}{\partial x_l}\right)_{l=1}^{m}\mathrm{d}x \\ &\quad -\sum_{j=1}^{n}\int_\Omega \left(r_{il}\frac{\partial e_i(t,x)}{\partial x_l}\right)_{l=1}^{m}\nabla e_i(t,x)\mathrm{d}x \\ &= \sum_{i=1}^{n}\int_{\partial\Omega}\left(e_i(t,x)r_{il}\frac{\partial e_i(t,x)}{\partial x_l}\right)_{l=1}^{m}\mathrm{d}x \\ &\quad -\sum_{i=1}^{n}\int_\Omega \sum_{l=1}^{m}r_{il}\left(\frac{\partial e_i(t,x)}{\partial x_l}\right)^2\mathrm{d}x \\ &= -\sum_{i=1}^{n}\int_\Omega \sum_{l=1}^{m}r_{il}\left(\frac{\partial e_i(t,x)}{\partial x_l}\right)^2\mathrm{d}x,\end{aligned} \tag{3.7}$$

in which $\nabla = (\partial/\partial x_1, \partial/\partial x_2, \ldots, \partial/\partial x_m)$ is the gradient operator, and

$$\left(r_{il}\frac{\partial e_i(t,x)}{\partial x_l}\right)_{l=1}^{m} = \left(r_{i1}\frac{\partial e_i(t,x)}{\partial x_1}, r_{i2}\frac{\partial e_i(t,x)}{\partial x_2}, \ldots, r_{im}\frac{\partial e_i(t,x)}{\partial x_m}\right)^T. \tag{3.8}$$

In view of Lemma 2.3, it is derived that

$$-\sum_{i=1}^{n}\int_{\Omega}\sum_{l=1}^{m}r_{il}\left(\frac{\partial e_i(t,x)}{\partial x_l}\right)^2 \mathrm{d}x \le -\sum_{i=1}^{n}\int_{\Omega}\sum_{l=1}^{m}\frac{r_{il}}{z_l^2}e_i^2(t,x)\mathrm{d}x = -\int_{\Omega}e^T(t,x)\widetilde{R}e(t,x)\mathrm{d}x. \quad (3.9)$$

For any positive constants ε_1, ε_2, and ε_3, it follows from (H_1) and Lemma 2.4 that

$$\begin{aligned} e^T(t,x)Ag(e(t,x)) &\le \frac{1}{2}\varepsilon_1^{-1}e^T(t,x)AA^Te(t,x)+\frac{1}{2}\varepsilon_1 g^T(e(t,x))g(e(t,x)) \\ &\le \frac{1}{2}\left(\varepsilon_1^{-1}\|A\|^2+\varepsilon_1\mu\right)e^T(t,x)e(t,x), \end{aligned} \quad (3.10)$$

$$e^T(t,x)Bg(e(t-\tau_1(t),x)) \le \frac{1}{2}\varepsilon_2^{-1}\|B\|^2e^T(t,x)e(t,x)+\frac{1}{2}\varepsilon_2\mu e^T(t-\tau_1(t),x)e(t-\tau_1(t),x), \quad (3.11)$$

$$\begin{aligned} e^T(t,x)D\int_{-\infty}^{t}K(t-s)g(e(s,x))\mathrm{d}s &\le \frac{1}{2}\varepsilon_3\left(\int_{-\infty}^{t}K(t-s)g(e(s,x))\mathrm{d}s\right)^T \\ &\quad\times\int_{-\infty}^{t}K(t-s)g(e(s,x))\mathrm{d}s+\frac{1}{2}\varepsilon_3^{-1}\|D\|^2e^T(t,x)e(t,x). \end{aligned} \quad (3.12)$$

By using condition (H_2) and Lemma 2.5, one obtains from (3.12) that

$$\begin{aligned} e^T(t,x)D\int_{-\infty}^{t}K(t-s)g(e(s,x))\mathrm{d}s &\le \frac{1}{2}\varepsilon_3\overline{k}\int_{-\infty}^{t}K(t-s)g^T(e(s,x))g(e(s,x))\mathrm{d}s \\ &\quad+\frac{1}{2}\varepsilon_3^{-1}\|D\|^2e^T(t,x)e(t,x) \\ &\le \frac{1}{2}\varepsilon_3\overline{k}\mu\int_{-\infty}^{t}K(t-s)e^T(s,x)e(s,x)\mathrm{d}s \\ &\quad+\frac{1}{2}\varepsilon_3^{-1}\|D\|^2e^T(t,x)e(t,x). \end{aligned} \quad (3.13)$$

Considering condition (H_3) and substituting (3.9)–(3.11) and (3.13) into (3.6) derive that

$$\begin{aligned} \mathcal{L}V(t) \le \int_{\Omega}\Bigg[&\frac{a}{2}e^T(t,x)e(t,x)+\frac{1}{2}\varepsilon_2\mu e^T(t-\tau_1(t),x)e(t-\tau_1(t),x) \\ &+\frac{\rho_2}{2}e^T(t-\tau_2(t),x)e(t-\tau_2(t),x)+\frac{1}{2}\varepsilon_3\overline{k}\mu\int_{-\infty}^{t}K(t-s)e^T(s,x)e(s,x)\mathrm{d}s \\ &+\frac{\rho_3}{2}\int_{t-\tau_3(t)}^{t}e^T(s,x)e(s,x)ds\Bigg]\mathrm{d}x \end{aligned}$$

$$= aV(t) + \varepsilon_2\mu V(t-\tau_1(t)) + \rho_2 V(t-\tau_2(t)) + \varepsilon_3\overline{k}\mu\int_{-\infty}^{t} K(t-s)V(s)\mathrm{d}s + \rho_3\int_{t-\tau_3(t)}^{t} V(s)\mathrm{d}s. \tag{3.14}$$

Taking mathematical expectations on both sides of (3.5), it can be derived from inequations (3.14) and (H_2) that

$$\begin{aligned}\frac{\mathrm{d}\mathbb{E}\{V(t)\}}{\mathrm{d}t} \leq{}& a\mathbb{E}\{V(t)\} + \varepsilon_2\mu[\mathbb{E}\{V(t)\}]_{\overline{\tau}_1} + \rho_2[\mathbb{E}\{V(s)\}]_{\overline{\tau}_2} + \varepsilon_3\overline{k}^2\mu[\mathbb{E}\{V(s)\}]_{-\infty} \\ &+ \rho_3\overline{\tau}_3[\mathbb{E}\{V(s)\}]_{\overline{\tau}_3}, \quad t \in (t_{k-1}, t_k],\ k \in \mathbb{N}_+,\end{aligned} \tag{3.15}$$

where $[\mathbb{E}\{V(s)\}]_{-\infty} = \max_{s\leq t}\mathbb{E}\{V(s)\}$.

On the other hand, it is obtained from (H_4) and the second equation of (2.11) that

$$\begin{aligned}V(t_k^+) &= \int_\Omega \frac{1}{2}e^T(t_k^+, x)e(t_k^+, x)\mathrm{d}x = \int_\Omega \frac{1}{2}h_k^T(t_k, x)h_k(t_k, x)\mathrm{d}x \\ &\leq \alpha_k V(t_k) + \beta_k^1 V(t_k - \eta_1(t_k)) + \cdots + \beta_k^q V(t_k - \eta_q(t_k)) + \beta_k^{q+1}\int_{t-\eta_{q+1}(t)}^{t} V(s)\mathrm{d}s,\end{aligned} \tag{3.16}$$

which means that

$$\mathbb{E}\{V(t_k^+)\} \leq \alpha_k\mathbb{E}\{V(t_k)\} + \beta_k^1[\mathbb{E}\{V(t_k)\}]_{\overline{\eta}_1} + \cdots + \beta_k^q[\mathbb{E}\{V(t_k)\}]_{\overline{\eta}_q} + \beta_k^{q+1}\overline{\eta}_{q+1}[V(s)]_{\overline{\eta}_{q+1}}. \tag{3.17}$$

By virtue of Lemma 2.7, if the inequalities (3.1) and (3.2) hold, then it follows from (3.15) and (3.17) that there exist constants $M > 1$ and $\theta > 0$ such that

$$\mathbb{E}\{V(t)\} \leq \max_{s\leq 0}\mathbb{E}\left\{\|\overline{\varphi}_1(s,x)\|^2\right\}Me^{-\theta t}, \quad t \geq 0. \tag{3.18}$$

By Definition 2.2, the controlled system (2.8) is globally exponentially synchronized with system (2.2) in mean square. This completes the proof. □

Note that there are three uncertain positive constants ε_1, ε_2, and ε_3. Not making a good choice of the three constants may lead to the conservativeness of Theorem 3.1 in practical application. In order to hit off this fault, our next aim is to determine the constants ε_1, ε_2, and ε_3 such that the conservativeness of Theorem 3.1 can be reduced as much as possible. We present the following Theorem 3.2.

Theorem 3.2. *Suppose that conditions (H_1)–(H_4). Then, under the impulsive controller* (2.7), *the controlled system* (2.8) *is globally exponentially synchronized with system* (2.2) *in mean square if the following inequalities hold*

$$0 < b_k < 1, \quad k \in \mathbb{N}_+, \tag{3.19}$$

$$\begin{aligned} \xi_k = & -\lambda_{\min}\left(\widetilde{R} + C\right) + \|A\|\sqrt{\mu} + \|B\|\sqrt{\frac{\mu}{b_k}} + \overline{k}\|D\|\sqrt{\frac{\mu}{b_k}} \\ & + \frac{1}{2}\left(\rho_1 + \frac{\rho_2}{b_k} + \frac{\rho_3\overline{\tau}_3}{b_k} + \frac{\ln b_k}{t_{k+1} - t_k}\right) < 0, \quad k \in \mathbb{N}_+, \end{aligned} \tag{3.20}$$

where $b_k = \alpha_k + \sum_{i=1}^{q} \beta_k^i + \beta_k^{q+1}\overline{\eta}_{q+1}$, *the other parameters are defined as those in Theorem 3.1.*

Proof. Define the function $H(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ with positive variables ε_1, ε_2, and ε_3 as follows:

$$H(\varepsilon_1, \varepsilon_2, \varepsilon_3) = a + \frac{\varepsilon_2\mu + \rho_2 + \varepsilon_3\overline{k}^2\mu + \rho_3\overline{\tau}_3}{\alpha_k + \sum_{i=1}^{q}\beta_k^i + \beta_k^{q+1}\overline{\eta}_{q+1}} + \frac{\ln\left(\alpha_k + \sum_{i=1}^{q}\beta_k^i\right) + \beta_k^{q+1}\overline{\eta}_{q+1}}{t_{k+1} - t_k}. \tag{3.21}$$

In order that the result of Theorem 3.1 is less conservative, we only need to find out three constants ε_1^0, ε_2^0, and ε_3^0 such that the inequality (3.2) is less conservative. To achieve this goal, we will find a point $(\varepsilon_1^0, \varepsilon_2^0, \varepsilon_3^0)$ such that $H(\varepsilon_1^0, \varepsilon_2^0, \varepsilon_3^0)$ takes the minimum value and $H(\varepsilon_1^0, \varepsilon_2^0, \varepsilon_3^0) < 0$. By simple computation, one derives that $\partial H/\partial\varepsilon_1 = \mu - (\|A\|^2/\varepsilon_1^2)$, $\partial H/\partial\varepsilon_2 = (\mu/b_k) - (\|B\|^2/\varepsilon_2^2)$, $\partial H/\partial\varepsilon_3 = (\overline{k}^2\mu/b_k) - (\|D\|^2/\varepsilon_3^2)$. Let $\partial H/\partial\varepsilon_1 = \partial H/\partial\varepsilon_2 = \partial H/\partial\varepsilon_3 = 0$, one gets $(\varepsilon_1^0, \varepsilon_2^0, \varepsilon_3^0) = (\|A\|/\sqrt{\mu}, \|B\|\sqrt{b_k/\mu}, \|D\|/\overline{k}\sqrt{b_k/\mu})$. It is obvious that the Hesse matrix of $H(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ at $(\varepsilon_1^0, \varepsilon_2^0, \varepsilon_3^0)$ is positive definite. Hence, $H(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ takes the minimum value at $(\varepsilon_1^0, \varepsilon_2^0, \varepsilon_3^0)$ according to the extreme value theory of multivariate function. Taking $H(\varepsilon_1^0, \varepsilon_2^0, \varepsilon_3^0) < 0$ arrives at the condition (3.20). This completes the proof. □

Remark 3.3. Theorems 3.1 and 3.2 are not dependent on discrete delays of both continuous equation and impulsive controller, which is consistent with results of [3, 9, 26, 38], though they did not consider delays in impulses. It should be noted that the inequalities in Theorems 3.1 and 3.2 are related to $\overline{k}^2$, $\overline{\tau}_3$, and $\overline{\eta}_{q+1}$, which mean that distributed delays in both continuous equation and impulsive controller have important effects on synchronization criteria in our results. This new discovery is completely different from existing results including those in [3, 9, 26, 37–40]. As was pointed out in Remark 2.8, results in [3, 9, 38, 40] were derived by using similar method used in [26], hence results of this paper improve those in [3, 9, 26, 38, 40] even when $D = 0$, $\sigma(t, x) = \sigma(t, e(t, x), e(t-\tau_2(t), x))$ and $h_k(t_k, x) = h(e(t_k, x))$ in (2.11). To sum up, results of this paper are new and improve and extend most of known corresponding ones.

Remark 3.4. Lemma 2.5 is utilized in (3.13), which makes the proof process more simple than those in [21, 22, 34]. In [21], matrix decomposition method was used to deal with not-equal-to-1 delay kernel, hence the Lyapunov functional and proof process are relatively complex. Authors in [22, 34] had to utilize algebraic approach instead of matrix method to derive their main results. It is well known that results derived from algebraic approach have more complex form and is more conservative than those obtained by matrix method. Therefore, results of this paper improve those in [21, 22, 34] to some extent.

Remark 3.5. Stochastic perturbations are unavoidable in real applications of neural networks. In this paper, we synchronize a class of reaction-diffusion neural networks with stochastic perturbations via impulsive control. Although there were several results on stability of reaction-diffusion neural with stochastic perturbations [45, 46], seldom published papers considered synchronization of this kind of neural networks under impulsive control. Moreover, the stochastic perturbations of this paper are more general than those in [45, 46], since they include information of distributed delays.

4. Examples and Simulations

As applications of the the theoretical results derived above, in this section, we give numerical simulations to demonstrate that our synchronization criteria are effective.

Consider the following reaction-diffusion neural network with both discrete and unbounded distributed delays

$$\begin{aligned}\frac{\partial y(t,x)}{\partial t} &= \frac{\partial}{\partial x}\left(R\frac{\partial y(t,x)}{\partial x}\right) - Cy(t,x) + Af\left(y(t,x)\right) + Bf\left(y(t-\tau_1(t),x)\right)\\ &\quad + D\int_{-\infty}^{t} K(t-s)f\left(y(s,x)\right)\mathrm{d}s + I(t),\end{aligned} \tag{4.1}$$

where $y(t,x) = (y_1(t,x), y_2(t,x))^T$, $x \in \Omega = [-2,2]$, $f(y(t,x)) = (\tanh(x_1(t,x)), \tanh(x_2(t,x)))^T$, $\tau_1(t) = 1$, $K(t) = e^{-0.5t}$, $R = \mathrm{diag}(0.1, 0.1)$, $I(t) = (1, 1.2)^T$,

$$C = \begin{pmatrix} 1.2 & 0 \\ 0 & 1 \end{pmatrix}, \qquad A = \begin{pmatrix} 3 & -0.3 \\ 4 & 5 \end{pmatrix}, \qquad B = \begin{pmatrix} -1.4 & 0.1 \\ 0.3 & -8 \end{pmatrix}, \qquad D = \begin{pmatrix} -1.2 & 0.1 \\ -2.8 & -1 \end{pmatrix}. \tag{4.2}$$

Take the boundary condition of (4.1) as $y(t,x) = 0$, $(t,x) \in (-\infty, +\infty) \times \partial\Omega$. In the case that initial condition is chosen as $y(s,x) = (0.4, 0.6)^T$, $(s,x) \in [-3,0] \times \Omega$ and $y(s,x) = 0$, $(s,x) \in (-\infty,-3) \times \Omega$, the chaotic-like trajectory of (4.1) is shown in Figures 1, 2, and 3. Taking $R = 0$, then we get the chaotic-like trajectory of (4.1) without reaction-diffusion terms shown in Figure 4.

Let system (4.1) be the driver network, we design a response system as

$$\begin{aligned}\mathrm{d}u(t,x) &= \left[\frac{\partial}{\partial x}\left(R\frac{\partial u(t,x)}{\partial x}\right) - Cu(t,x) + Af(u(t,x)) + Bf(u(t-\tau_1(t),x))\right.\\ &\quad \left. +D\int_{-\infty}^{t} K(t-s)f(u(s,x))\mathrm{d}s + I(t)\right]\mathrm{d}t + \sigma(t,x)\mathrm{d}\omega(t), \quad t \neq t_k,\\ u(t_k^+,x) &= u(t_k^-,x) + h_k(t_k,x) - e(t_k,x), \quad t = t_k,\ k \in \mathbb{N}_+,\end{aligned} \tag{4.3}$$

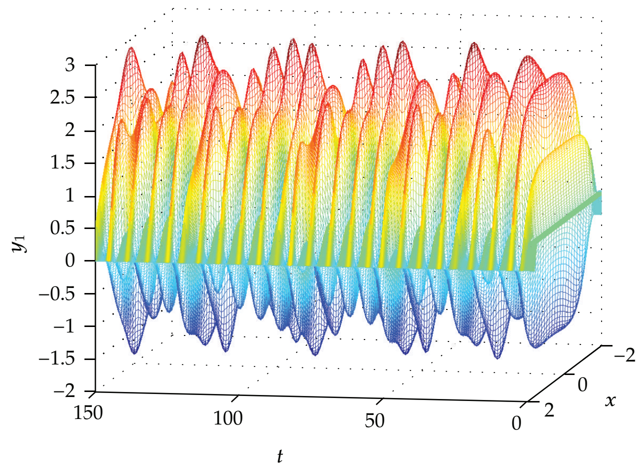


Figure 1: Chaotic behavior of the state $y_1(t, x)$ in system (4.1).

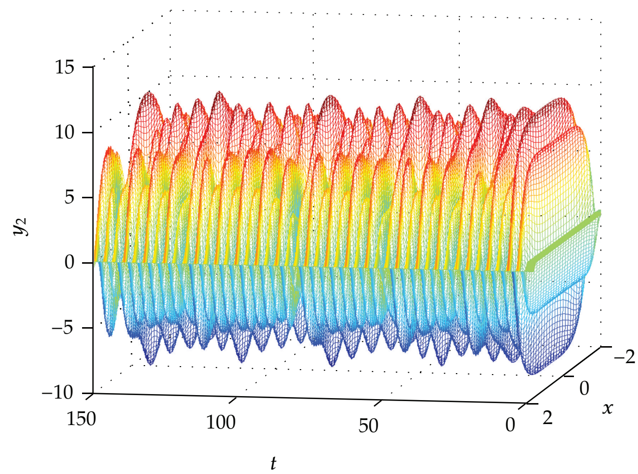


Figure 2: Chaotic behavior of the state $y_2(t, x)$ in system (4.1).

where $e(t, x) = u(t, x) - y(t, x)$, $h_k(t_k, x) = ae(t_k, x) + be(t_k - 0.5|\sin t_k|, x) + c\int_{t-0.5}^{t} e(s, x)\mathrm{d}s$ with positive constants a, b, and c, the noise intensity function matrix is

$$\sigma(t, x) = 0.1\begin{pmatrix} e_1(t, x) & e_2(t-1, x) \\ \int_{t-0.3}^{t} e_1(s, x)\mathrm{d}s & e_2(t, x) \end{pmatrix}. \tag{4.4}$$

By Jensen's inequality (which is a special case of inequality (2.19)), one has

$$\left(\int_{t-0.3}^{t} e_1(s, x)\mathrm{d}s\right)^2 \leq 0.3\int_{t-0.3}^{t} (e_1(s, x))^2\mathrm{d}s \leq 0.3\int_{t-0.3}^{t} e^T(s, x)e(s, x)\mathrm{d}s. \tag{4.5}$$

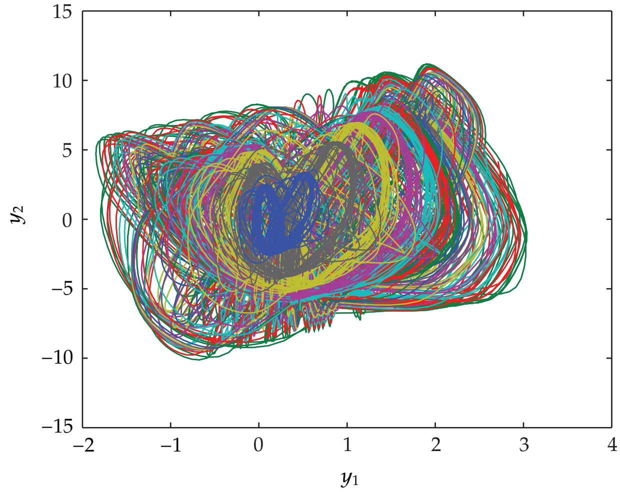


Figure 3: Chaotic behavior of system (4.1).

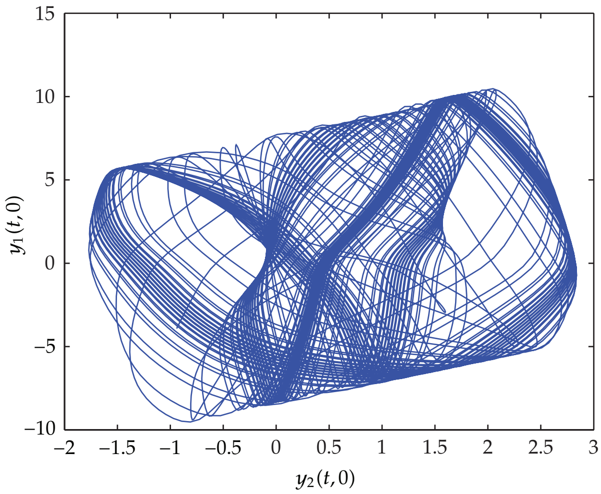
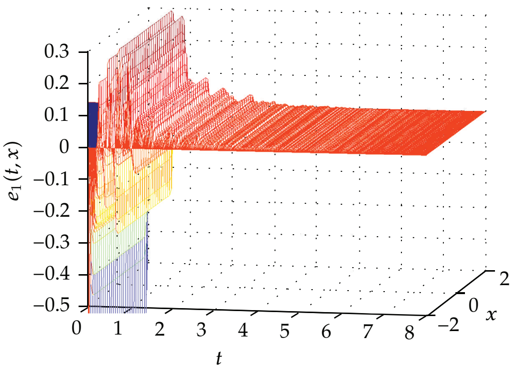


Figure 4: Chaotic behavior of system (4.1) with $R = 0$.

From (4.5) one gets

$$\begin{aligned}\operatorname{trace}\left(\sigma^T(t,x)\sigma(t,x)\right) \le {} & 0.01e^T(t,x)e(t,x) + 0.01e^T(t-1,x)e(t-1,x)\\ & + 0.003\int_{t-0.3}^{t} e^T(s,x)e(s,x)\mathrm{d}s.\end{aligned} \tag{4.6}$$



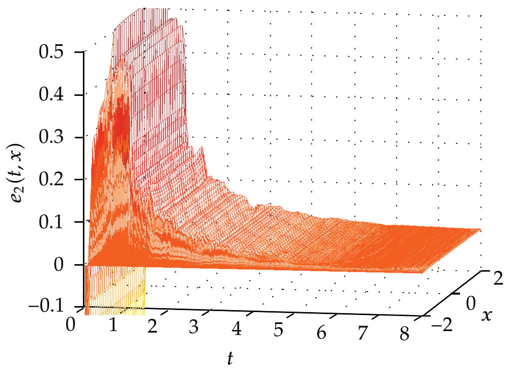


Figure 5: Dynamical behavior of synchronization errors $e_1(t,x)$ (a) and $e_2(t,x)$ (b).

Similarly, by using Jensen's inequality one derives that

$$h_k^T(t_k,x)h_k(t_k,x) \leq (a+b+c)\Bigg[ae^T(t_k,x)e(t_k,x)+be^T(t_k-0.5|\sin t_k|,x)e(t_k-0.5|\sin t_k|,x)$$
$$+c\int_{t-0.5}^{t} e^T(s,x)e(s,x)\mathrm{d}s\Bigg]. \tag{4.7}$$

Obviously, $\mu_1=\mu_2=1, \overline{k}=2, \rho_1=\rho_2=0.01, \rho_3=0.003, \alpha_k=a(a+b+c), \beta_k^1=b(a+b+c)$, $\beta_k^2=c(a+b+c)$, $q=1$, $\overline{\tau}_1=\overline{\tau}_2=1$, $\overline{\tau}_3=0.3$, and $\overline{\eta}_1=\overline{\eta}_2=0.5$. Therefore, (H_1)–(H_4) are satisfied. Choose $a=0.2, b=0.15, c=0.2$, and $t_k-t_{k-1}=0.02$. Then the inequalities (3.19) and (3.20) are satisfied with $b_k=0.2475$, $\xi_k=-0.4086<0$, respectively. According to Theorem 3.2, the controlled system (4.3) is globally exponentially synchronized with system (4.1) in mean square. Figure 5 presents the dynamical behavior of synchronization errors $e_1(t,x)$ and $e_2(t,x)$, which close to zero quickly as time increases.

5. Conclusion

Delays are unavoidable in practical systems, and they are always unknown and time-varying. This paper studies stochastic synchronization of reaction-diffusion neural networks with both time-varying discrete and distributed delays via delayed impulsive control. The impulsive controller has multiple time-varying discrete and distributed delays which is very general. Based on a novel integral inequality, the problem of distributed delays with not-equal-to-1 delay kernel is well handled with matrix method. Sufficient synchronization criteria are given to guarantee the global exponential synchronization in mean square of the considered system. The function extreme value theorem is utilized to get a less conservative result. It is discovered that, in our synchronization criteria, the distributed delays in both continuous equation and impulsive controller have important effects. At last, numerical simulations show the validity of the obtained criteria.

Appendix

Proof of Lemma 2.7. Without loss of generality, we assume that $\tau = \tau_1 \geq \tau_2 \geq \cdots \geq \tau_m$. Consider the following scalar function:

$$g_k(\lambda) = 2\lambda + a + \frac{\sum_{i=1}^m b_i e^{\lambda\tau_i}}{p_k + \sum_{j=1}^m q_k^j e^{\lambda\tau_j}} + \frac{\ln\left(p_k + \sum_{j=1}^m q_k^j e^{\lambda\tau_j}\right)}{t_{k+1} - t_k}. \tag{A.1}$$

It follows from inequality (2.22) that $g_k(0) = a + (\sum_{i=1}^m b_i)/(p_k + \sum_{j=1}^m q_k^j) + (\ln(p_k + \sum_{j=1}^m q_k^j))/(t_{k+1} - t_k) < 0$. Since $g_k'(\lambda) = 2 + \sum_{i=1}^m (p_k b_i \lambda e^{\lambda\tau_i}/(p_k + \sum_{j=1}^m q_k^j e^{\lambda\tau_j})^2) + (\lambda \sum_{j=1}^m q_k^j e^{\lambda\tau_j})/((t_{k+1} - t_k)(p_k + \sum_{j=1}^m q_k^j e^{\lambda\tau_j})) > 0$ for $\lambda > 0$ and $g_k(\lambda)$ is continuous on $(0, +\infty)$, there exists a positive constant λ such that $g_k(\lambda) < 0$ and $p_k + \sum_{j=1}^m q_k^j e^{\lambda\tau_j} \leq 1$ for all $k \in \mathbb{N}_+$.

Let $\gamma = \sup_{k\in\mathbb{N}_+}\{1/(p_k + \sum_{j=1}^m q_k^j e^{\lambda\tau_j})\} \geq 1$. Then we can select a constant $\sigma > 0$ such that for all $k \in \mathbb{N}_+$,

$$a + \sum_{i=1}^m \gamma b_i e^{\lambda\tau_i} \leq \sigma - \lambda, \tag{A.2}$$

$$(\sigma + \lambda)(t_{k+1} - t_k) < -\ln\left(p_k + \sum_{j=1}^m q_k^j e^{\lambda\tau_j}\right) \leq \ln\gamma. \tag{A.3}$$

From (A.3), we can choose $\beta = \beta_1 \geq \beta_2 \geq \cdots \geq \beta_m > 1$ such that

$$1 < e^{(\sigma+\lambda)(t_1 - t_0)} \leq \beta_i \leq \gamma e^{\lambda\tau_i}. \tag{A.4}$$

It follows from the above inequality that

$$\|\phi\|_\tau < \|\phi\|_\tau e^{\sigma(t_1 - t_0)} \leq \|\phi\|_\tau \beta_1 e^{-\lambda(t_1 - t_0)}. \tag{A.5}$$

Next we will prove that

$$v(t) \leq \|\phi\|_\tau \beta_1 e^{-\lambda(t - t_0)}, \quad t \in [t_{k-1}, t_k),\ k \in \mathbb{N}_+. \tag{A.6}$$

We use mathematical induction to prove that (A.6) holds. Firstly, we prove that (A.6) holds for $k = 1$. To do this, we only need to prove that

$$v(t) \leq \|\phi\|_\tau \beta_1 e^{-\lambda(t_1 - t_0)}, \quad t \in [t_0, t_1). \tag{A.7}$$

If the inequality (A.7) is not true, then there exists some $\bar{t} \in (t_0, t_1)$ such that

$$\begin{aligned} v\left(\bar{t}\right) &> \|\phi\|_\tau \beta_1 e^{-\lambda(t_1 - t_0)} \geq \|\phi\|_\tau \beta_2 e^{-\lambda(t_1 - t_0)} \geq \cdots \geq \|\phi\|_\tau \beta_m e^{-\lambda(t_1 - t_0)} \\ &\geq \|\phi\|_\tau e^{\sigma(t_1 - t_0)} > \|\phi\|_\tau \geq v(t_0 + s), \quad s \in [-\tau, 0], \end{aligned} \tag{A.8}$$

which implies that there exists $\widetilde{t}_i \in (t_0, \bar{t})$ such that $\widetilde{t}_m \le \widetilde{t}_{m-1} \le \cdots \le \widetilde{t}_1$ and

$$v\left(\widetilde{t}_i\right) = \|\phi\|_\tau \beta_i e^{-\lambda(t_1-t_0)}, \quad v(t) \le v\left(\widetilde{t}_i\right), \quad t \in \left[t_0-\tau, \widetilde{t}_i\right], \tag{A.9}$$

and there exists $\widehat{t} \in [t_0, \widetilde{t}_m)$ such that

$$v\left(\widehat{t}\right) = \|\phi\|_\tau, \quad v\left(\widehat{t}\right) \le v(t) \le v\left(\widetilde{t}_i\right), \quad t \in \left[\widehat{t}, \widetilde{t}_i\right]. \tag{A.10}$$

Therefore, one gets from (A.4), (A.9), and (A.10) that, for any $s \in [-\tau_i, 0]$,

$$v(t+s) \le \|\phi\|_\tau \beta_i e^{-\lambda(t_1-t_0)} \le \|\phi\|_\tau \gamma e^{\lambda\tau_i} e^{-\lambda(t_1-t_0)} \le \gamma e^{\lambda\tau_i} v\left(\widehat{t}\right) \le \gamma e^{\lambda\tau_i} v(t), \quad t \in \left[\widehat{t}, \widetilde{t}_i\right]. \tag{A.11}$$

Thus, one has from (A.2) and (A.11) that

$$\begin{aligned} D^+v(t) &\le av(t) + b_1[v(t)]_{\tau_1} + b_2[v(t)]_{\tau_2} + \cdots + b_m[v(t)]_{\tau_m} \\ &\le \left(a + \sum_{i=1}^{m} \gamma b_i e^{\lambda\tau_i}\right) v(t) \le (\sigma-\lambda)v(t), \quad t \in \left[\widehat{t}, \widetilde{t}_1\right]. \end{aligned} \tag{A.12}$$

It follows from (A.5), (A.9), (A.10), and (A.12) that

$$\begin{aligned} v\left(\widetilde{t}_1\right) &\le v\left(\widehat{t}\right) e^{(\sigma-\lambda)(\widetilde{t}_1-\widehat{t})} = \|\phi\|_\tau e^{(\sigma-\lambda)(\widetilde{t}_1-\widehat{t})} < \|\phi\|_\tau e^{\sigma(t_1-t_0)} \\ &\le \|\phi\|_\tau \beta_1 e^{-\lambda(t_1-t_0)} = v\left(\widetilde{t}_1\right), \end{aligned} \tag{A.13}$$

which is a contradiction. Hence (A.6) holds for $k = 1$.

Now we assume that (A.6) holds for $k = 1, 2, \ldots, n$, $n \in \mathbb{N}_+$, $n \ge 1$, that is,

$$v(t) \le \|\phi\|_\tau \beta_1 e^{-\lambda(t-t_0)}, \quad t \in [t_{k-1}, t_k),\ k = 1, 2, \ldots, n. \tag{A.14}$$

Next, we will show that (A.6) holds for $k = n+1$, that is,

$$v(t) \le \|\phi\|_\tau \beta_1 e^{-\lambda(t-t_0)}, \quad t \in [t_n, t_{n+1}). \tag{A.15}$$

For the sake of contradiction, suppose that (A.15) does not hold. Define $\check{t} = \inf\{t \in [t_n, t_{n+1}] \mid v(t) > \|\phi\|_\tau \beta_1 e^{-\lambda(t-t_0)}\}$. Then one obtains from (A.3) and (A.14) that

$$\begin{aligned}
v(t_n^+) &\le p_n v(t_n^-) + q_n^1 \left[v(t_n^-)\right]_{\tau_1} + q_n^2 \left[v(t_n^-)\right]_{\tau_2} + \cdots + q_n^m \left[v(t_n^-)\right]_{\tau_m} \\
&\le p_n \|\phi\|_\tau \beta_1 e^{-\lambda(t_n-t_0)} + q_n^1 \|\phi\|_\tau \beta_1 e^{-\lambda(t_n-\tau_1-t_0)} + q_n^2 \|\phi\|_\tau \beta_1 e^{-\lambda(t_n-\tau_2-t_0)} \\
&\quad + \cdots + q_n^m \|\phi\|_\tau \beta_1 e^{-\lambda(t_n-\tau_m-t_0)} \\
&= \left(p_n + \sum_{j=1}^{m} q_n^j e^{\lambda\tau_j}\right) \|\phi\|_\tau \beta_1 e^{\lambda(\check{t}-t_n)} e^{-\lambda(\check{t}-t_0)} \\
&< \left(p_n + \sum_{j=1}^{m} q_n^j e^{\lambda\tau_j}\right) e^{\lambda(t_{n+1}-t_n)} \|\phi\|_\tau \beta_1 e^{-\lambda(\check{t}-t_0)} \\
&< e^{-(\sigma+\lambda)(t_{n+1}-t_n)} e^{\lambda(t_{n+1}-t_n)} \|\phi\|_\tau \beta_1 e^{-\lambda(\check{t}-t_0)} \\
&= e^{-\sigma(t_{n+1}-t_n)} \|\phi\|_\tau \beta_1 e^{-\lambda(\check{t}-t_0)} < \|\phi\|_\tau \beta_1 e^{-\lambda(\check{t}-t_0)},
\end{aligned} \tag{A.16}$$

which implies that $\check{t} \ne t_n$. From the continuity of $v(t)$ in the interval $[t_n, t_{n+1})$, one has

$$v(\check{t}) = \|\phi\|_\tau \beta_1 e^{-\lambda(\check{t}-t_0)}, \quad v(t) \le v(\check{t}),\ t \in [t_n, \check{t}]. \tag{A.17}$$

On the other hand, one can deduce from (A.16) that there exists $t^* \in (t_n, \check{t})$ such that

$$v(t^*) = \left(p_n + \sum_{j=1}^{m} q_n^j e^{\lambda\tau_j}\right) e^{\lambda(t_{n+1}-t_n)} \|\phi\|_\tau \beta_1 e^{-\lambda(\check{t}-t_0)}, \quad v(t^*) \le v(t) \le v(\check{t}),\ t \in [t^*, \check{t}]. \tag{A.18}$$

For any $t \in [t^*, \check{t}]$, $s \in [-\tau_i, 0]$, either $t + s \in [t_0 - \tau_i, t_n)$ or $t + s \in [t_n, \check{t}]$. Two cases will be discussed as follows.

Case 1. If $t + s \in [t_0 - \tau_i, t_n)$, then one obtains from (A.14) that

$$\begin{aligned}
v(t+s) &\le \|\phi\|_\tau \beta_1 e^{-\lambda(t-t_0)} e^{-\lambda s} \le \|\phi\|_\tau \beta_1 e^{-\lambda(\check{t}-t_0)} e^{\lambda(\check{t}-t)} e^{\lambda\tau_i} \\
&\le \|\phi\|_\tau \beta_1 e^{-\lambda(\check{t}-t_0)} e^{\lambda(t_{n+1}-t_n)} e^{\lambda\tau_i}.
\end{aligned} \tag{A.19}$$

Case 2. If $t + s \in [t_n, \check{t}]$, then it follows from (A.17) that

$$v(t+s) \le \|\phi\|_\tau \beta_1 e^{-\lambda(\check{t}-t_0)} \le \|\phi\|_\tau \beta_1 e^{-\lambda(\check{t}-t_0)} e^{\lambda(t_{n+1}-t_n)} e^{\lambda\tau_i}. \tag{A.20}$$

In any case, one has from (A.18), (A.19), and (A.20) that, for any $s \in [-\tau_i, 0]$,

$$
\begin{aligned}
v(t+s) &\le \|\phi\|_\tau \beta_1 e^{-\lambda(\check{t}-t_0)} e^{\lambda(t_{n+1}-t_n)} e^{\lambda\tau_i} = \frac{e^{\lambda\tau_i}}{p_n + \sum_{j=1}^m q_n^j e^{\lambda\tau_j}} v(t^*) \\
&\le \frac{e^{\lambda\tau_i}}{p_n + \sum_{j=1}^m q_n^j e^{\lambda\tau_j}} v(t) \le \gamma e^{\lambda\tau_i} v(t), \quad t \in [t^*, \check{t}].
\end{aligned}
\tag{A.21}
$$

Hence, one obtains from (A.2) and (A.21) that

$$
\begin{aligned}
D^+ v(t) &\le a v(t) + b_1 [v(t)]_{\tau_1} + b_2 [v(t)]_{\tau_2} + \cdots + b_m [v(t)]_{\tau_m} \\
&\le \left(a + \sum_{i=1}^m \gamma b_i e^{\lambda\tau_i}\right) v(t) \le (\sigma - \lambda) v(t), \quad t \in [t^*, \check{t}].
\end{aligned}
\tag{A.22}
$$

It follows from inequalities (A.3), (A.17), (A.18), and (A.19) that

$$
\begin{aligned}
v(\check{t}) &\le v(t^*) e^{(\sigma-\lambda)(\check{t}-t^*)} \\
&= \left(p_n + \sum_{j=1}^m q_n^j e^{\lambda\tau_j}\right) e^{\lambda(t_{n+1}-t_n)} \|\phi\|_\tau \beta_1 e^{-\lambda(\check{t}-t_0)} e^{(\sigma-\lambda)(\check{t}-t^*)} \\
&< e^{-(\sigma+\lambda)(t_{n+1}-t_n)} e^{\lambda(t_{n+1}-t_n)} \|\phi\|_\tau \beta_1 e^{-\lambda(\check{t}-t_0)} e^{(\sigma-\lambda)(\check{t}-t^*)} \\
&= e^{-\sigma(t_{n+1}-t_n)} \|\phi\|_\tau \beta_1 e^{-\lambda(\check{t}-t_0)} e^{(\sigma-\lambda)(\check{t}-t^*)} \\
&\le \|\phi\|_\tau \beta_1 e^{-\lambda(\check{t}-t_0)} = v(\check{t}),
\end{aligned}
\tag{A.23}
$$

which is a contradiction. Therefore the assumption that the inequality (A.15) does not hold is not true, and hence the inequality (A.6) holds for $k = n + 1$. According to the theory of mathematical induction method, the inequality (A.6) holds for all $k \in \mathbb{N}_+$. This completes the proof. □

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Research Article

Global Synchronization of Neutral-Type Stochastic Delayed Complex Networks

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This paper is concerned with the delay-dependent synchronization criterion for neutral-type stochastic delayed complex networks. Firstly, expectations of stochastic crossterms containing the Itô integral are investigated. In fact, for stochastic delay systems, if we want to obtain the delay-dependent condition with less conservatism, how to deal with expectations of stochastic cross terms properly is of vital importance, and many existing results did not deal with expectations of these stochastic cross terms correctly. Then, based on this, this paper establishes a novel delay-dependent synchronization criterion for neutral-type stochastic delayed complex networks. In the derivation process, the mathematical development avoids bounding stochastic cross terms. Thus, this method shows less conservatism. Finally, a numerical example is provided to demonstrate the effectiveness of the proposed approach.

1. Introduction

In the real world, many systems can be described as complex networks such as Internet networks, biological networks, epidemic spreading networks, collaborative networks, social networks, neural networks, and so forth [1–4]. Thus, during the past years, the study of complex networks has become a very active area, see, for example, [5, 6] and the references therein. In particular, for complex networks, the major collective behavior is the synchronization phenomena, because many problems in practice have close relationships with synchronization [7]. Recently, growing research results, that focused on synchronization problems for complex networks, have been reported in [8–12] and the references therein.

Up to now, it has been well realized that in spreading information through complex networks, there always exist time delays caused by the finite speed of information transmission and the limit of bandwidth, which often decrease the quality of the system and even lead to oscillation, divergence, and instability. Accordingly, synchronization problems for many delayed complex networks have been studied in [13–17]. It is worth mentioning that in the above results for delayed complex networks, each dynamical node is modeled as a retarded functional differential equation coupling with other nodes. However, in some cases, in order to reflect dynamical behaviors for some realistic networks models, the information about derivatives of the past state variables of the networks should be utilized. Therefore, the dynamic of the complex networks should be described by a group of neutral-type functional differential equations. This kind of delayed complex network is termed as the neutral-type delayed complex network. As a matter of fact, neutral-type delays exist in many fields such as the population ecology, distributed networks containing lossless transmission lines, and a typical neutral-type delayed complex network example which is the stock transaction system [18]. Consequently, synchronization problems of neutral-type delayed complex networks were studied in [18–20]. For instance, a delay-dependent synchronization criterion for complex networks with neutral-type coupling delay was presented in [18], and the robust synchronization criterion for a class of uncertain neutral-type delayed complex networks was given in [19]. And [20] discussed the synchronization problem for the neutral-type complex networks with coupling time-varying delays.

On the other hand, in the real world, complex networks are often subject to stochastic disturbances. For example, the signal transfer in a real complex network could be perturbed randomly from the release of probabilistic causes such as neurotransmitters and packet dropouts [21]. Hence, such a stochastic disturbance phenomenon that typically occurs in complex networks has attracted considerable attention during the past years, and synchronization problems for delayed complex networks with stochastic disturbances have been investigated in [21–24]. For instance, the synchronization problems of discrete-time delayed complex networks with stochastic disturbances were investigated in [21, 22]. Reference [24] designed an adaptive feedback controller to solve the synchronization problem for an array of linearly stochastically coupled networks with time delays. Although the above results have discussed delayed complex networks under the influence of stochastic noises, it should be pointed out that as to the neutral-type delayed complex networks, there is still *no* paper to investigate the influence of stochastic disturbances on this kind of complex networks.

Moreover, for delay systems including delayed complex networks, a very active research topic is to obtain the delay-dependent conditions. The reason is that the delay-dependent condition makes use of the information on the size of time delays, and the delay-dependent condition is generally less conservative than the delay-independent one [25–27]. However, when we used the existing effective methods, such as the model transformation method [25, 26] and the free-weighting matrix method [27], to give the delay-dependent condition for stochastic delay systems including stochastic delayed complex (or neural) networks, the following stochastic cross terms containing the Itô integral will appear:

$$x(t)^T \mathbb{J} \int_{t-h}^{t} \mu(s, x_s) dw(s), \qquad x(t-h)^T \mathbb{K} \int_{t-h}^{t} \mu(s, x_s) dw(s),$$
$$\left(\int_{t-h}^{t} \kappa(s, x_s) ds\right)^T \mathbb{L} \int_{t-h}^{t} \mu(s, x_s) dw(s). \tag{1.1}$$

It is still very difficult to calculate expectations of these stochastic cross terms up to now. The results in [28–31] resorted to bounding techniques, which obviously can bring the conservatism. Some papers such as [32–34] considered that expectations of these stochastic cross terms are all equal to zero. However, these results are not given by strict mathematical proofs, and we can find examples to illustrate that expectations of some stochastic cross terms are not equal to zero in Remark 3.3. Therefore, in order to obtain the delay-dependent synchronization criterion with less conservatism for neutral-type stochastic delayed complex networks, there is a strong need to investigate the expectations of stochastic cross terms containing the Itô integral firstly.

Motivated by the discussion mentioned above, this paper investigates the delay-dependent synchronization problem for neutral-type stochastic delayed complex networks. The main contributions of this paper are summarized as follows. (1) Expectations of stochastic cross terms containing the Itô integral are investigated by stochastic analysis techniques in Lemma 3.1 and Corollary 3.2. We prove that the expectation of $x(t-h)^T\mathbb{K}\int_{t-h}^{t}\mu(s,x_s)dw(s)$ is equal to zero and expectations of other stochastic cross terms are not. (2) Based on this conclusion, this paper establishes a delay-dependent synchronization criterion that guarantees the globally asymptotic synchronization of neural-type stochastic delayed complex networks. In the derivation process, the mathematical development avoids bounding stochastic cross terms. Thus, this method leads to a criterion with less conservatism. Finally, a numerical example is provided to demonstrate the effectiveness of the proposed approach.

Notation. Throughout the paper, unless otherwise specified, we will employ the following notation. Let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$ be a complete probability space with a natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$, and let $\mathcal{E}(\cdot)$ be the expectation operator with respect to the probability measure. If A is a vector or matrix, its transpose is denoted by A^T. If P is a square matrix, then $P>0$ $(P<0)$ means that it is a symmetric positive (negative) definite matrix of appropriate dimensions while $P\geq 0$ $(P\leq 0)$ is a symmetric positive (negative) semidefinite matrix. I stands for the identity matrix of appropriate dimensions. Denote by $\lambda_{\min}(\cdot)$ the minimum eigenvalue of a given matrix. Let $|\cdot|$ denote the Euclidean norm of a vector and its induced norm of a matrix. Unless explicitly specified, matrices are assumed to have real entries and compatible dimensions. $L^2(\Omega)$ denotes the space of all random variables X with $\mathcal{E}|X|^2<\infty$, it is a Banach space with norm $\|X\|_2=(\mathcal{E}|X|^2)^{1/2}$. Let $h>0$ and $C([-h,0];\mathcal{R}^n)$ denote the family of all continuous $\mathcal{R}^n$-valued functions φ on $[-h,0]$ with the norm $\|\varphi\|=\sup\{|\varphi(\theta)|:-h\leq\theta\leq 0\}$. Let $L^2_{\mathcal{F}_0}([-h,0];\mathcal{R}^n)$ be the family of all $\mathcal{F}_0$-measurable $C([-h,0];\mathcal{R}^n)$-valued random variables ϕ such that $\mathcal{E}(\|\phi\|^2)<\infty$, and let $\mathcal{L}^2([a,b];\mathcal{R}^n)$ be the family of all $\mathcal{R}^n$-valued $\mathcal{F}_t$-adapted processes $\{f(t)\}_{a\leq t\leq b}$ such that $\int_a^b|f(t)|^2dt<\infty$ a.s. Let $\mathcal{M}^2([a,b];\mathcal{R}^n)$ be the family of processes $\{f(t)\}_{a\leq t\leq b}$ in $\mathcal{L}^2([a,b];\mathcal{R}^n)$ such that $\mathcal{E}(\int_a^b|f(t)|^2dt)<\infty$, and $\mathcal{M}^2([a,b])$ is the 1-dimensional case of $\mathcal{M}^2([a,b];\mathcal{R}^n)$.

2. Problem Formulation and Preliminaries

In this paper, we consider the following neutral-type stochastic delayed complex networks consisting of N identical nodes:

$$
\begin{aligned}
&d[x_i(t) - Dx_i(t-h)] \\
&\quad = \left[Ax_i(t) + Bf(x_i(t)) + Cf(x_i(t-h)) + \sum_{j=1}^{N} g_{ij}\Gamma x_j(t) + \sum_{j=1}^{N} h_{ij}\Upsilon x_j(t-h)\right]dt \\
&\qquad + \sigma_i(t, x_i(t), x_i(t-h))dw(t), \quad i = 1, 2, \ldots, N,
\end{aligned}
\tag{2.1}
$$

where $x_i(t) = [x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t)]^T \in \mathcal{R}^n$ represents the state vector of the ith node; the scalar $h > 0$ is the time delay; A is a known connection matrix; B and C denote, respectively, the connection weight matrix and the delayed connection weight matrix; $\Gamma, \Upsilon \in \mathcal{R}^{n\times n}$ are matrices describing the inner coupling between the subsystems at time t and $t-h$, respectively; $G = (g_{ij})_{N\times N}$ and $H = (h_{ij})_{N\times N}$ are called the outer-coupling configuration matrices representing the coupling strength and the topological structure of the complex networks; D is a known real matrix, and the spectrum radius of the matrix D, $\rho(D)$, satisfies $\rho(D) < 1$. $\sigma_i(\cdot,\cdot,\cdot) : \mathcal{R} \times \mathcal{R}^n \times \mathcal{R}^n \to \mathcal{R}^n$ which is the noise intensity function vector; $w(t)$ is a scalar standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with a natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$. $f(x_i(t)) = (f_1(x_{i1}(t)), \ldots, f_n(x_{in}(t)))^T$, is an unknown but sector-bounded nonlinear function.

The initial conditions associated with system (2.1) are given by

$$
x_i(s) = \varphi_i(s), \quad -h \leq s \leq 0, \; i = 1, 2, \ldots, N,
\tag{2.2}
$$

where $\varphi_i(\cdot) \in L^2_{\mathcal{F}_0}([-h, 0]; \mathcal{R}^n)$.

Let

$$
\begin{aligned}
x(t) &= \left(x_1(t)^T, \ldots, x_N(t)^T\right)^T, \\
F(x(t)) &= \left(f(x_1(t))^T, \ldots, f(x_N(t))^T\right)^T, \\
F(x(t-h)) &= \left(f(x_1(t-h))^T, \ldots, f(x_N(t-h))^T\right)^T, \\
\sigma(t) &= \left(\sigma_1(t, x_1(t), x_1(t-h))^T, \ldots, \sigma_N(t, x_N(t), x_N(t-h))^T\right)^T, \\
\overline{D} &= \operatorname{diag}\left(\overbrace{D, D, \ldots, D}^{N}\right).
\end{aligned}
\tag{2.3}
$$

With the Kronecker product “$\otimes$” for matrices, system (2.1) can be rearranged as

$$
\begin{aligned}
d\left[x(t) - \overline{D}x(t-h)\right] = {}& [(I_N \otimes A + G \otimes \Gamma)x(t) + (H \otimes \Upsilon)x(t-h) \\
&+ (I_N \otimes B)F(x(t)) + (I_N \otimes C)F(x(t-h))]dt + \sigma(t)dw(t).
\end{aligned}
\tag{2.4}
$$

Before stating our main results, we need the following definitions, assumptions, and propositions.

Definition 2.1. The neutral-type stochastic delayed complex network (2.1) is globally asymptotically synchronized in the mean square if, for all $\varphi_i(\cdot), \varphi_j(\cdot) \in L^2_{\mathcal{F}_0}([-h,0];\mathcal{R}^n)$, the following holds:

$$\lim_{t\to\infty} \mathcal{E}\left\{ \left| x_i(t,\varphi_i) - x_j(t,\varphi_j) \right|^2 \right\} = 0, \quad 1 \le i < j \le N. \tag{2.5}$$

Definition 2.2 (see [35]). If a stochastic process $\{\nu(t)\}_{a\le t\le b}$ belongs to $\mathcal{M}^2([a,b])$, then its Itô integral (from a to b) is defined by

$$\int_a^b \nu(t)dw(t) = \lim_{n\to\infty} \int_a^b \nu_n(t)dw(t) \quad \left(\lim \text{in } L^2(\Omega)\right), \tag{2.6}$$

where $\{\nu_n(t)\}_{a\le t\le b}$ $(n = 1,2,\ldots)$ are the step stochastic processes and belong to $\mathcal{M}^2([a,b])$ such that

$$\lim_{n\to\infty} \mathcal{E}\left(\int_a^b |\nu(t) - \nu_n(t)|^2 dt \right) = 0. \tag{2.7}$$

Definition 2.3 (see [36]). Let $\{\mathcal{F}_t\}_{t\in T}$ be an increasing family of σ-algebras of subset of Ω. A stochastic process $\{X_t\}_{t\in T}$ is said to be adapted to $\{\mathcal{F}_t\}_{t\in T}$ if for each t, the random variable X_t is $\mathcal{F}_t$-measurable.

Assumption 2.4. The outer-coupling configuration matrices of the complex networks (2.1) satisfy

$$\begin{gathered} g_{ij} = g_{ji} \ge 0, \quad h_{ij} = h_{ji} \ge 0, \quad (i \ne j), \\ g_{ii} = -\sum_{j=1, j\ne i}^{N} g_{ij}, \quad h_{ii} = -\sum_{j=1, j\ne i}^{N} h_{ij}, \quad i,j = 1,2,\ldots,N. \end{gathered} \tag{2.8}$$

Assumption 2.5. The noise intensity function vector $\sigma_i : \mathcal{R} \times \mathcal{R}^n \times \mathcal{R}^n \to \mathcal{R}^n$ satisfies the Lipschitz condition, that is, there exist constant matrices W_1 and W_2 of appropriate dimensions such that

$$\left| \sigma_i(t,x_1,y_1) - \sigma_j(t,x_2,y_2) \right|^2 \le |W_1(x_1 - x_2)|^2 + \left| W_2(y_1 - y_2) \right|^2, \tag{2.9}$$

for all $i,j = 1,2,\ldots,N$ and $x_1, y_1, x_2, y_2 \in \mathcal{R}^n$.

Assumption 2.6. For all $x, y \in \mathcal{R}^n$, the nonlinear function $f(\cdot)$ is assumed to satisfy the following condition:

$$(f(x) - f(y) - U(x-y))^T (f(x) - f(y) - V(x-y)) \le 0, \tag{2.10}$$

where U and V are real constant matrices with U-V being symmetric and positive definite.

Proposition 2.7 (see [14]). *The Kronecker product has the following properties:*

$$
\begin{gathered}
(\alpha A) \otimes B = A \otimes (\alpha B), \\
(A + B) \otimes C = A \otimes C + B \otimes C, \\
(A \otimes B)(C \otimes D) = (AC) \otimes (BD), \\
(A \otimes B)^T = A^T \otimes B^T.
\end{gathered}
\tag{2.11}
$$

Proposition 2.8 (see [19]). *Let* $\mathcal{U} = (\alpha_{ij})_{n\times n}$, $P \in \mathcal{R}^{m\times m}$, $x = (x_1^T, x_2^T, \ldots, x_n^T)^T$, $y = (y_1^T, y_2^T, \ldots, y_n^T)^T$, *where* $x_i = (x_{i1}, x_{i2}, \ldots, x_{im})^T \in \mathcal{R}^m$, $y_i = (y_{i1}, y_{i2}, \ldots, y_{im})^T \in \mathcal{R}^m$ $(i = 1, 2, \ldots, n)$. *If* $\mathcal{U} = \mathcal{U}^T$ *and each row sum of* $\mathcal{U}$ *is equal to zero, then*

$$
x^T(\mathcal{U} \otimes P)y = -\sum_{1\le i<j\le n} \alpha_{ij}(x_i - x_j)^T P(y_i - y_j).
\tag{2.12}
$$

Proposition 2.9 (see [35]). *Let* $\{\vartheta(t)\}_{a\le t\le b}$ *be a stochastic process and belong to* $\mathcal{M}^2([a,b])$, *then*

$$
\mathcal{E}\left(\int_a^b \vartheta(t)dw(t)\right) = 0.
\tag{2.13}
$$

3. Main Results

Then, we give the following lemma and corollary which will play a key role in the proof of our main results.

Lemma 3.1. *If a stochastic process* $\{\nu(t)\}_{a\le t\le b} \in \mathcal{M}^2([a,b])$ *and* ϖ *is a bounded and* $\mathcal{F}_a$*-measurable random variable, then*

$$
\mathcal{E}\left(\varpi \int_a^b \nu(t)dw(t)\right) = 0.
\tag{3.1}
$$

Proof. Firstly, in order to prove the above results, we will prove that if $\{\nu(t)\}_{a\le t\le b} \in \mathcal{M}^2([a,b])$ and ϖ is a bounded and $\mathcal{F}_a$-measurable random variable, then

$$
\varpi \int_a^b \nu(t)dw(t) = \int_a^b \varpi\nu(t)dw(t).
\tag{3.2}
$$

Most important of all, since ϖ is a bounded and $\mathcal{F}_a$-measurable random variable, it is easy to verify $\{\varpi\nu(t)\}_{a\le t\le b} \in \mathcal{M}^2([a,b])$. Then, we will prove (3.2) by the following two steps.

Step 1. If $\{\nu(t)\}_{a\le t\le b}$ is a step stochastic process, then we let, without loss of generality,

$$
\nu(t) = \sum_{i=1}^{n} \varsigma_{i-1} 1_{[t_{i-1},t_i)}(t),
\tag{3.3}
$$

where $t_0 = a, t_n = b, \varsigma_{i-1}$ is $\mathcal{F}_{t_{i-1}}$-measurable and $\mathcal{E}(\varsigma_{i-1}^2) < \infty$. In this case,

$$\int_a^b \varpi\nu(t)dw(t) = \sum_{i=1}^{n} \varpi\varsigma_{i-1}(w(t_i) - w(t_{i-1})) = \varpi\sum_{i=1}^{n}\varsigma_{i-1}(w(t_i) - w(t_{i-1})) = \varpi\int_a^b \nu(t)dw(t). \tag{3.4}$$

Step 2. If $\{\nu(t)\}_{a\le t\le b} \in \mathcal{M}^2([a,b])$ is not a step stochastic process, then by Definition 2.2, we can find a sequence of step stochastic processes in $\mathcal{M}^2([a,b])$: $\{\nu_1(t)\}_{a\le t\le b}, \{\nu_2(t)\}_{a\le t\le b}, \ldots, \{\nu_n(t)\}_{a\le t\le b}, \ldots$ such that

$$\int_a^b \nu(t)dw(t) = \lim_{n\to\infty}\int_a^b \nu_n(t)dw(t) \quad \left(\lim \text{in } L^2(\Omega)\right), \tag{3.5}$$

where $\{\nu(t)\}_{a\le t\le b}$ and $\{\nu_n(t)\}_{a\le t\le b}$ satisfy

$$\lim_{n\to\infty}\mathcal{E}\left(\int_a^b |\nu(t) - \nu_n(t)|^2 dt\right) = 0. \tag{3.6}$$

Because ϖ is bounded, by Definition 2.2 and (3.5)-(3.6), it is easy to prove that

$$\begin{aligned}\int_a^b \varpi\nu(t)dw(t) &= \lim_{n\to\infty}\int_a^b \varpi\nu_n(t)dw(t) \quad \left(\lim \text{in } L^2(\Omega)\right),\\ \varpi\int_a^b \nu(t)dw(t) &= \lim_{n\to\infty}\varpi\int_a^b \nu_n(t)dw(t) \quad \left(\lim \text{in } L^2(\Omega)\right).\end{aligned} \tag{3.7}$$

From Step 1, it follows that for each step stochastic process $\{\nu_n(t)\}_{a\le t\le b}$, we have

$$\int_a^b \varpi\nu_n(t)dw(t) = \varpi\int_a^b \nu_n(t)dw(t). \tag{3.8}$$

Therefore, it is easy to obtain

$$\lim_{n\to\infty}\int_a^b \varpi\nu_n(t)dw(t) = \lim_{n\to\infty}\varpi\int_a^b \nu_n(t)dw(t) \quad \left(\lim \text{in } L^2(\Omega)\right). \tag{3.9}$$

Then, we can get by (3.7) and (3.9) that

$$\int_a^b \varpi\nu(t)dw(t) = \varpi\int_a^b \nu(t)dw(t). \tag{3.10}$$

Due to $\{\varpi v(t)\}_{a\le t\le b} \in \mathcal{M}^2([a,b])$, then by Proposition 2.9, we can know that

$$\mathcal{E}\left(\varpi \int_a^b v(t)dw(t)\right) = \mathcal{E}\left(\int_a^b \varpi v(t)dw(t)\right) = 0. \tag{3.11}$$

This completes the proof. □

Corollary 3.2. *Let one consider the following neutral stochastic functional differential equation:*

$$d[x(t) - \mathfrak{D}x(t-h)] = \kappa(t,x_t)dt + \mu(t,x_t)dw(t), \tag{3.12}$$

on $t \ge 0$ with the initial data $x_0 = \xi \in L^2_{\mathcal{F}_0}([-h,0];\mathcal{R}^n)$. $\kappa(\cdot,\cdot)$ and $\mu(\cdot,\cdot)$ satisfy the local Lipschitz condition and the linear growth condition. If $x(t)$ is the solution of (3.12) *and $\mathbb{K}$ is any compatible dimensional matrix, then*

$$\mathcal{E}\left(x(t-h)^T\mathbb{K}\left[\int_{t-h}^t \mu(s,x_s)dw(s)\right]\right) = 0, \quad t \ge h. \tag{3.13}$$

Especially when $\mathfrak{D} = 0$ in (3.12)*, that is,*

$$dx(t) = \kappa(t,x_t)dt + \mu(t,x_t)dw(t). \tag{3.14}$$

Equation (3.14) *is a common stochastic functional equation. For this case,* (3.13) *is also tenable.*

Proof. Since $\kappa(\cdot,\cdot)$ and $\mu(\cdot,\cdot)$ satisfy the local Lipschitz condition and the linear growth condition, we can know that, for all $T > 0$, (3.12) has a unique continuous solution on $[-h,T]$ denoted by $\{x(t)\}_{-h\le t\le T}$ that is adapted to $\{\mathcal{F}_t\}_{-h\le t\le T}$ and $\{x(t)\}_{-h\le t\le T} \in \mathcal{M}^2([-h,T])$ [37]. Therefore, it can be derived that for $t \ge h$, $x(t-h)$ is a bounded random variable and $x(t-h)$ is $\mathcal{F}_{t-h}$-measurable. Then, by Lemma 3.1, it is easy to obtain (3.13). If $\mathfrak{D} = 0$ in (3.12) that is a common stochastic functional equation, then we can easily prove that (3.13) is also tenable for this case. □

Remark 3.3. Lemma 3.1 has proved

$$\mathcal{E}\left(x(t-h)^T\mathbb{K}\left[\int_{t-h}^t \mu(s,x_s)dw(s)\right]\right) = 0, \quad t \ge h. \tag{3.15}$$

However, for any compatible dimensional matrix $\mathbb{J}$ or $\mathbb{L}$, the following results are *not* correct:

$$\begin{aligned}
&\mathcal{E}\left(x(t)^T\mathbb{J}\left[\int_{t-h}^{t}\mu(s,x_s)dw(s)\right]\right)=0,\\
&\mathcal{E}\left(\left(\int_{t-h}^{t}\kappa(s,x_s)ds\right)^T\mathbb{L}\left[\int_{t-h}^{t}\mu(s,x_s)dw(s)\right]\right)=0,
\end{aligned}\qquad t\geq h. \tag{3.16}$$

We will give two examples to illustrate it.

Example 3.4. Consider the following one-dimensional Langevin equation in [36] that can be regarded as a special class of neutral stochastic delay systems as follows:

$$d[x(t)-0x(t-h)]=\kappa(t,x_t)dt+\mu(t,x_t)dw(t),\qquad x(0)=\xi, \tag{3.17}$$

where $\kappa(t,x_t)=-\beta x(t),\mu(t,x_t)=\alpha$ and $\alpha>0,\beta>0$. This equation has a solution

$$x(t)=e^{-\beta(t-u)}x(u)+\alpha\int_{u}^{t}e^{-\beta(t-s)}dw(s),\quad u\leq t. \tag{3.18}$$

Then by (3.18), we can know that

$$\begin{aligned}
\mathcal{E}\left(x(t)\mathbb{J}\int_{t-h}^{t}\mu(s,x_s)dw(s)\right)&=\mathcal{E}\left(\left(e^{-\beta h}x(t-h)+\alpha\int_{t-h}^{t}e^{-\beta(t-s)}dw(s)\right)\right.\\
&\qquad\left.\times\mathbb{J}\left[\int_{t-h}^{t}\alpha\,dw(s)\right]\right)\\
&=e^{-\beta h}\mathcal{E}\left(x(t-h)\mathbb{J}\left[\int_{t-h}^{t}\alpha\,dw(s)\right]\right)\\
&\quad+\mathcal{E}\left(\alpha\int_{t-h}^{t}e^{-\beta(t-s)}dw(s)\mathbb{J}\int_{t-h}^{t}\alpha\,dw(s)\right)\\
&=0+\alpha^2\mathbb{J}e^{-\beta t}\int_{t-h}^{t}e^{\beta s}ds\\
&=\frac{\alpha^2\mathbb{J}}{\beta}\left(1-e^{-\beta h}\right)\neq 0,\quad\forall\mathbb{J}\neq 0,\\
\mathcal{E}\left(\int_{t-h}^{t}\kappa(s,x_s)ds\mathbb{L}\left[\int_{t-h}^{t}\mu(s,x_s)dw(s)\right]\right)&=\mathcal{E}\left(\left(x(t)-x(t-h)-\int_{t-h}^{t}\mu(s,x_s)dw(s)\right)\right.\\
&\qquad\left.\times\mathbb{L}\left[\int_{t-h}^{t}\mu(s,x_s)dw(s)\right]\right)\\
&=\mathcal{E}\left(x(t)\mathbb{L}\int_{t-h}^{t}\mu(s,x_s)dw(s)\right)
\end{aligned}$$

$$
\begin{aligned}
&\quad - \mathcal{E}\left(x(t-h)\mathbb{L}\int_{t-h}^{t}\mu(s,x_s)dw(s)\right) \\
&\quad - \mathcal{E}\left(\int_{t-h}^{t}\mu(s,x_s)dw(s)\mathbb{L}\int_{t-h}^{t}\mu(s,x_s)dw(s)\right) \\
&= \frac{\alpha^2\mathbb{L}}{\beta}\left(1-e^{-\beta h}\right) - 0 - \mathbb{L}\int_{t-h}^{t}\alpha^2 ds \\
&= \frac{\alpha^2\mathbb{L}}{\beta}\left(1-e^{-\beta h}-\beta h\right) \neq 0, \quad \forall \mathbb{L} \neq 0.
\end{aligned}
\tag{3.19}
$$

Example 3.5. Consider the following one-dimensional stochastic equation:

$$
d[x(t) - 0x(t-h)] = dw(t), \tag{3.20}
$$

which has a one solution $x(t) = w(t)$. However, we can easily verify that

$$
\mathcal{E}\left(x(t)^T\mathbb{J}\int_{t-h}^{t}\mu(s,x_s)dw(s)\right) = \mathcal{E}\left(w(t)\mathbb{J}\int_{t-h}^{t}dw(s)\right) = \mathbb{J}h \neq 0, \quad \forall \mathbb{J} \neq 0. \tag{3.21}
$$

We should point out that in recent years, some papers such as [32–34] considered that the expectations of these stochastic terms are all equal to zero. However, this is not the case. From the above examples and Corollary 3.2, we can see that $x(t-h)^T\mathbb{K}\int_{t-h}^{t}\mu(s,x_s)dw(s)$ is the only one whose expectation is equal to zero.

Then, we are in the position to present our main result for the synchronization criterion of the neutral-type delayed complex networks with stochastic disturbances.

Theorem 3.6. *Under the Assumptions 2.4–2.6, the dynamical system* (2.1) *is globally asymptotically synchronized in the mean square if there exist matrices $P > 0$, $Q_1 > 0$, $Q_2 > 0$, $R > 0$, $Z > 0$, S and scalars $\epsilon > 0$, $\lambda > 0$ such that the following LMIs hold for all $1 \le i < j \le N$:*

$$
P < \lambda I, \tag{3.22}
$$

$$
\Xi = \begin{pmatrix}
\Xi_{11} & \Xi_{12} & 0 & \Xi_{14} & PC & \Xi_{16} & 0 \\
* & \Xi_{22} & \Xi_{23} & -D^TPB & -D^TPC & \Xi_{26} & \Xi_{27} \\
* & * & -Q_2 & 0 & 0 & 0 & 0 \\
* & * & * & R-2\epsilon I & 0 & B^TS^T & 0 \\
* & * & * & * & -R & C^TS^T & 0 \\
* & * & * & * & * & hZ-S^T-S & 0 \\
* & * & * & * & * & * & -hZ
\end{pmatrix} < 0, \tag{3.23}
$$

where

$$\Xi_{11} = PA + A^T P - N g_{ij} P\Gamma - N g_{ij}\Gamma^T P + \lambda W_1^T W_1 + Q_1 + Q_2 - \epsilon U^T V - \epsilon V^T U,$$
$$\Xi_{12} = -A^T PD + N g_{ij}\Gamma^T PD, \qquad \Xi_{14} = PB + \epsilon U^T + \epsilon V^T, \qquad \Xi_{16} = A^T S^T - N g_{ij}\Gamma^T S^T,$$
$$\Xi_{22} = \lambda W_2^T W_2 - Q_1 - N h_{ij} P\Upsilon - N h_{ij}\Upsilon^T P,$$
$$\Xi_{23} = N h_{ij}\Upsilon^T PD, \qquad \Xi_{26} = -N h_{ij}\Upsilon^T S^T, \qquad \Xi_{27} = -hN h_{ij}\Upsilon^T P. \tag{3.24}$$

Proof. Firstly, set

$$y(t) = (I_N \otimes A + G \otimes \Gamma)x(t) + (H \otimes \Upsilon)x(t-h) + (I_N \otimes B)F(x(t)) + (I_N \otimes C)F(x(t-h)), \tag{3.25}$$

then, (2.1) can be rewritten as

$$d\left[x(t) - \overline{D}x(t-h)\right] = y(t)dt + \sigma(t)dw(t). \tag{3.26}$$

From (3.26), we can have

$$\left[x(t) - \overline{D}x(t-h)\right] - \left[x(t-h) - \overline{D}x(t-2h)\right] = \int_{t-h}^{t} y(s)ds + \int_{t-h}^{t} \sigma(s)dw(s). \tag{3.27}$$

Consider the following Lyapunov functional for the system (3.26):

$$\begin{aligned} V(x_t,t) = {} & \left[x(t) - \overline{D}x(t-h)\right]^T (U \otimes P)\left[x(t) - \overline{D}x(t-h)\right] + \int_{t-h}^{t} x(s)^T (U \otimes Q_1)x(s)ds \\ & + \int_{t-2h}^{t} x(s)^T (U \otimes Q_2)x(s)ds + \int_{-h}^{0}\int_{t+\theta}^{t} y(s)^T (U \otimes Z)y(s)ds\, d\theta \\ & + \int_{t-h}^{t} F(x(s))^T (U \otimes R)F(x(s))ds, \quad t \geq h, \end{aligned} \tag{3.28}$$

where

$$U = \begin{pmatrix} N-1 & -1 & \cdots & -1 \\ -1 & N-1 & \cdots & -1 \\ \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & \cdots & N-1 \end{pmatrix}. \tag{3.29}$$

Then, by the Itô's formula, the stochastic differential $dV(x_t,t)$ can be obtained

$$dV(x_t,t) = \mathcal{L}V(x_t,t)dt + 2\left[x(t) - \overline{D}x(t-h)\right]^T (U \otimes P)\sigma(t)dw(t), \tag{3.30}$$

where

$$\begin{aligned}\mathcal{L}V(x_t,t) = {}& 2\left[x(t)-\overline{D}x(t-h)\right]^T(U\otimes P)y(t)+\sigma(t)^T(U\otimes P)\sigma(t)+x(t)^T(U\otimes Q_1)x(t)\\&-x(t-h)^T(U\otimes Q_1)x(t-h)+x(t)^T(U\otimes Q_2)x(t)-x(t-2h)^T(U\otimes Q_2)x(t-2h)\\&+F(x(t))^T(U\otimes R)F(x(t))-F(x(t-h))^T(U\otimes R)F(x(t-h))+hy(t)^T(U\otimes Z)y(t)\\&-\int_{t-h}^{t}\left[y(s)^T(U\otimes Z)y(s)\right]ds.\end{aligned}\tag{3.31}$$

By (3.27), we have

$$\begin{aligned}&2\left[x(t)-\overline{D}x(t-h)\right]^T(U\otimes P)y(t)\\&\quad=2\left[x(t)-\overline{D}x(t-h)\right]^T(U\otimes P)\\&\qquad\times\left[(I_N\otimes A+G\otimes\Gamma)x(t)+(I_N\otimes B)F(x(t))+(I_N\otimes C)F(x(t-h))\right]\\&\qquad+2\left[x(t)-\overline{D}x(t-h)\right]^T(U\otimes P)(H\otimes\Upsilon)x(t-h)\\&\quad=2\left[x(t)-\overline{D}x(t-h)\right]^T(U\otimes P)\\&\qquad\times\left[(I_N\otimes A+G\otimes\Gamma)x(t)+(I_N\otimes B)F(x(t))+(I_N\otimes C)F(x(t-h))\right]\\&\qquad+2\left[x(t-h)-\overline{D}x(t-2h)+\int_{t-h}^{t}y(s)ds+\int_{t-h}^{t}\sigma(s)dw(s)\right]^T(U\otimes P)(H\otimes\Upsilon)x(t-h).\end{aligned}\tag{3.32}$$

From Corollary 3.2, it follows that

$$\begin{aligned}&\mathcal{E}\left(2\left[x(t)-\overline{D}x(t-h)\right]^T(U\otimes P)y(t)\right)\\&\quad=\mathcal{E}\left(2\left[x(t)-\overline{D}x(t-h)\right]^T(U\otimes P)\right.\\&\qquad\times\left[(I_N\otimes A+G\otimes\Gamma)x(t)+(I_N\otimes B)F(x(t))+(I_N\otimes C)F(x(t-h))\right]\\&\qquad\left.+2\left[x(t-h)-\overline{D}x(t-2h)+\int_{t-h}^{t}y(s)ds\right]^T(U\otimes P)(H\otimes\Upsilon)x(t-h)\right).\end{aligned}\tag{3.33}$$

By (3.25), it is easy to know that for any matrix S, we have

$$2y(t)^T(U\otimes S)\left[(I_N\otimes A+G\otimes\Gamma)x(t)+(H\otimes\Upsilon)x(t-h)+(I_N\otimes B)F(x(t))\right.$$
$$\left.+(I_N\otimes C)F(x(t-h))-y(t)\right]=0. \tag{3.34}$$

From (3.31)–(3.34) and by the Propositions 2.7 and 2.8, it is easy to get

$$\begin{aligned}
\mathcal{E}(\mathcal{L}V(x_t,t)) = \mathcal{E}\Bigg(\frac{1}{h}\int_{t-h}^{t}\Big[&2\Big(x(t)-\overline{D}x(t-h)\Big)^T(U\otimes P)\\
&\times[(I_N\otimes A+G\otimes\Gamma)x(t)+(I_N\otimes B)F(x(t))\\
&\quad+(I_N\otimes C)F(x(t-h))]+2\Big(x(t-h)-\overline{D}x(t-2h)+hy(s)\Big)^T\\
&\times(U\otimes P)(H\otimes\Upsilon)x(t-h)\\
&+\sigma(t)^T(U\otimes P)\sigma(t)+x(t)^T(U\otimes Q_1)x(t)-x(t-h)^T\\
&\times(U\otimes Q_1)x(t-h)+x(t)^T(U\otimes Q_2)x(t)\\
&-x(t-2h)^T(U\otimes Q_2)x(t-2h)+F(x(t))^T(U\otimes R)F(x(t))\\
&-F(x(t-h))^T(U\otimes R)F(x(t-h))\\
&+hy(t)^T(U\otimes Z)y(t)-hy(s)^T(U\otimes Z)y(s)+2y(t)^T(U\otimes S)\\
&\times((I_N\otimes A+G\otimes\Gamma)x(t)+(H\otimes\Upsilon)x(t-h)+(I_N\otimes B)F(x(t))\\
&\quad+(I_N\otimes C)F(x(t-h))-y(t))\Big]ds\Bigg)\\
=\mathcal{E}\Bigg(\frac{1}{h}\int_{t-h}^{t}\Bigg[\sum_{1\le i<j\le N}&\Big(2\big(x_i(t)-x_j(t)-D\big(x_i(t-h)-x_j(t-h)\big)\big)^T\\
&\times\big(PA-Ng_{ij}P\Gamma\big)\big(x_i(t)-x_j(t)\big)\\
&+2\big(x_i(t)-x_j(t)-D\big(x_i(t-h)-x_j(t-h)\big)\big)^T\\
&\times PB\big(f(x_i(t))-f(x_j(t))\big)\\
&+2\big(x_i(t)-x_j(t)-D\big(x_i(t-h)-x_j(t-h)\big)\big)^T\\
&\times PC\big(f(x_i(t-h))-f(x_j(t-h))\big)\\
&-2\big(x_i(t-h)-x_j(t-h)-D\big(x_i(t-2h)-x_j(t-2h)\big)\big)^T\\
&\times\big(Nh_{ij}P\Upsilon\big)\big(x_i(t-h)-x_j(t-h)\big)
\end{aligned}$$

$$
\begin{aligned}
&\quad - 2h\big(y_i(s) - y_j(s)\big)^T N h_{ij} P\Upsilon\big(x_i(t-h) - x_j(t-h)\big)\Big) \\
&+ \big(\sigma_i(t, x_i(t), x_i(t-h)) - \sigma_j\big(t, x_j(t), x_j(t-h)\big)\big)^T \\
&\times P\big(\sigma_i(t, x_i(t), x_i(t-h)) - \sigma_j\big(t, x_j(t), x_j(t-h)\big)\big) \\
&+ \big(x_i(t) - x_j(t)\big)^T Q_1\big(x_i(t) - x_j(t)\big) - \big(x_i(t-h) - x_j(t-h)\big)^T \\
&\times Q_1\big(x_i(t-h) - x_j(t-h)\big) \\
&+ \big(x_i(t) - x_j(t)\big)^T Q_2\big(x_i(t) - x_j(t)\big) \\
&- \big(x_i(t-2h) - x_j(t-2h)\big)^T Q_2\big(x_i(t-2h) - x_j(t-2h)\big) \\
&+ \big(f(x_i(t)) - f\big(x_j(t)\big)\big)^T R\big(f(x_i(t)) - f\big(x_j(t)\big)\big) \\
&- \big(f(x_i(t-h)) - f\big(x_j(t-h)\big)\big)^T \\
&\times R\big(f(x_i(t-h)) - f\big(x_j(t-h)\big)\big) \\
&+ h\big(y_i(t) - y_j(t)\big)^T Z\big(y_i(t) - y_j(t)\big) \\
&- \ h\big(y_i(s) - y_j(s)\big)^T Z\big(y_i(s) - y_j(s)\big) \\
&+ 2\big(y_i(t) - y_j(t)\big)^T \big(SA - N g_{ij} S\Gamma\big)\big(x_i(t) - x_j(t)\big) \\
&- 2\big(y_i(t) - y_j(t)\big)^T \big(N h_{ij} S\Upsilon\big)\big(x_i(t-h) - x_j(t-h)\big) \\
&+ 2\big(y_i(t) - y_j(t)\big)^T SB\big(f(x_i(t)) - f\big(x_j(t)\big)\big) \\
&+ 2\big(y_i(t) - y_j(t)\big)^T SC\big(f(x_i(t-h)) - f\big(x_j(t-h)\big)\big) \\
&-2\big(y_i(t) - y_j(t)\big)^T S\big(y_i(t) - y_j(t)\big)\Bigg] ds\Bigg).
\end{aligned} \tag{3.35}
$$

According to Assumptions 2.5 and (3.22), it is clear that

$$
\begin{aligned}
&\big(\sigma_i(t, x_i(t), x_i(t-h)) - \sigma_j\big(t, x_j(t), x_j(t-h)\big)\big)^T P\big(\sigma_i(t, x_i(t), x_i(t-h)) - \sigma_j\big(t, x_j(t), x_j(t-h)\big)\big) \\
&\quad \le \lambda\big(x_i(t) - x_j(t)\big)^T W_1^T W_1\big(x_i(t) - x_j(t)\big) \\
&\qquad + \ \lambda\big(x_i(t-h) - x_j(t-h)\big)^T W_2^T W_2\big(x_i(t-h) - x_j(t-h)\big).
\end{aligned} \tag{3.36}
$$

By Assumption 2.6, we can obtain

$$
\begin{aligned}
0 &\le 2\epsilon\big(x_i(t) - x_j(t)\big)^T U^T\big(f(x_i(t)) - f\big(x_j(t)\big)\big) + 2\epsilon\big(f(x_i(t)) - f\big(x_j(t)\big)\big)^T V\big(x_i(t) - x_j(t)\big) \\
&\quad - 2\epsilon\big(x_i(t) - x_j(t)\big)^T U^T V\big(x_i(t) - x_j(t)\big) - 2\epsilon\big(f(x_i(t)) - f\big(x_j(t)\big)\big)^T\big(f(x_i(t)) - f\big(x_j(t)\big)\big).
\end{aligned} \tag{3.37}
$$

Combining (3.35)–(3.37), we have

$$\mathcal{E}(\mathcal{L}V(x_t,t)) \leq \mathcal{E}\left[\frac{1}{h}\int_{t-h}^{t}\sum_{1\leq i<j\leq N}\xi_{ij}^{T}\Xi\xi_{ij}ds\right], \tag{3.38}$$

where

$$\xi_{ij} = \begin{pmatrix} x_i(t)-x_j(t) \\ x_i(t-h)-x_j(t-h) \\ x_i(t-2h)-x_j(t-2h) \\ f(x_i(t))-f\left(x_j(t)\right) \\ f(x_i(t-h))-f\left(x_j(t-h)\right) \\ y_i(t)-y_j(t) \\ y_i(s)-y_j(s) \end{pmatrix}. \tag{3.39}$$

Since $\Xi < 0$, it is guaranteed that all the subsystems in (2.1) are globally asymptotically synchronized in the mean square. The proof is completed. □

Remark 3.7. We note here that if $D = 0$ in (2.1), then system (2.1) describes a kind of stochastic delayed complex networks considered in [32]. Our result can be applied to this case, and we have pointed out that [32] made a mistake when dealing with expectations of stochastic cross terms in Remark 3.3. If we let A be a diagonal and negative matrix and let $D = 0$ in (2.1), the system (2.1) will be an array of coupled neural networks consisting of N nodes, in which each node is an n-dimensional stochastic delayed Hopfield neural network. As to stochastic Hopfield neural networks with time delays, [30, 38] have investigated the stability problems, respectively. Furthermore, if we don't consider stochastic disturbances and time delays in stochastic delayed Hopfield neural networks, then this kind of neural networks is the famous Hopfield neural network.

Remark 3.8. If we do not consider the stochastic disturbances in (2.1), then the system will be a kind of determinate neutral-type delayed complex networks, that have been considered in the [18–20]. If we let A be a diagonal and negative matrix in this kind of determinate neutral-type delayed complex networks, each node will be an n-dimensional neutral-type delayed neural network. For neutral-type neural networks with time delays, [39, 40] have discussed the stability problems and presented the new and effective stability conditions, respectively.

Remark 3.9. For neutral stochastic delay systems, a very active topic is to obtain the delay-dependent condition. For example, [28, 29] considered delay-dependent stability problems for neutral stochastic delay systems. However, these two papers used bounding techniques including the Jensen inequality to deal with stochastic cross terms contain the Itô integral. Obviously, bounding techniques will increase the conservatism. In the derivation process of Theorem 3.6, we don't use any bounding technique to deal with stochastic cross terms. Therefore, this method can show less conservatism and can also be extended to solve delay-dependent stability problems for neutral stochastic delay systems.

Remark 3.10. In Theorem 3.6, we give a delay-dependent synchronization criterion by the linear matrix inequalities (LMIs), because LMIs can be easily solved by using the Matlab

LMI toolbox and no tuning of parameters is required. Moreover, we can easily get the maximum possible upper bound on the delay by the LMI toolbox. The maximum possible upper bound on the delay is the main criterion for judging the conservatism of a delay-dependent condition.

4. Numerical Example

In this section, we present a simulation example to illustrate the effectiveness of our approach.

Example 4.1. Consider the following complex network consisting of three identical nodes:

$$\begin{aligned} &d[x_i(t) - Dx_i(t-h)] \\ &\quad = \left[Ax_i(t) + Bf(x_i(t)) + Cf(x_i(t-h)) + \sum_{j=1}^{3} g_{ij}\Gamma x_j(t) + \sum_{j=1}^{3} h_{ij}\Upsilon x_j(t-h)\right]dt \\ &\qquad + \sigma_i(t, x_i(t), x_i(t-h))dw(t), \end{aligned} \tag{4.1}$$

for all $i = 1,2,3$, where $x_i(t) = [x_{i1}(t), x_{i2}(t)]^T \in \mathcal{R}^2$ is the state vector of the ith subsystem, and

$$\begin{gathered} A = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} 0.6 & -0.1 \\ -0.3 & 0.5 \end{pmatrix}, \quad C = \begin{pmatrix} -0.5 & -0.1 \\ 0.2 & -1.5 \end{pmatrix}, \quad D = \begin{pmatrix} -0.6 & 0 \\ 0 & -0.6 \end{pmatrix}, \\ G = \begin{pmatrix} -3 & 1 & 2 \\ 1 & -2 & 1 \\ 2 & 1 & -3 \end{pmatrix}, \quad H = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} 0.5 & 0 \\ 0.1 & 0.5 \end{pmatrix}, \quad \Upsilon = \begin{pmatrix} 0.5 & 0.1 \\ 0 & 0.4 \end{pmatrix}, \\ \sigma(t, x(t), x(t-h)) = \begin{pmatrix} \sqrt{0.15} & 0 & \sqrt{0.2} & 0 \\ 0 & \sqrt{0.15} & 0 & \sqrt{0.2} \end{pmatrix} \begin{pmatrix} x(t) \\ x(t-h) \end{pmatrix}, \\ f(x_i(t)) = \left(f_1(x_{i1}(t)), f_2(x_{i2}(t))\right)^T = (\tanh(x_{i1}(t)), \tanh(x_{i2}(t)))^T. \end{gathered} \tag{4.2}$$

Thus, the matrices U, V, W_1, W_2, in the Assumptions 2.5 and 2.6 are

$$U = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad W_1 = \begin{pmatrix} \sqrt{0.3} & 0 \\ 0 & \sqrt{0.3} \end{pmatrix}, \quad W_2 = \begin{pmatrix} \sqrt{0.4} & 0 \\ 0 & \sqrt{0.4} \end{pmatrix}. \tag{4.3}$$

According to Theorem 3.6, the allowable maximum delay h, that can guarantee the globally asymptotic mean-square synchronization of the neutral-type stochastic delayed complex networks, is 0.33. When we randomly choose the the initial states in $[0,1] \times [0,1]$, the synchronization errors are plotted in Figures 1 and 2, which can confirm that the neutral-type stochastic delayed complex system is globally synchronized in the mean square.

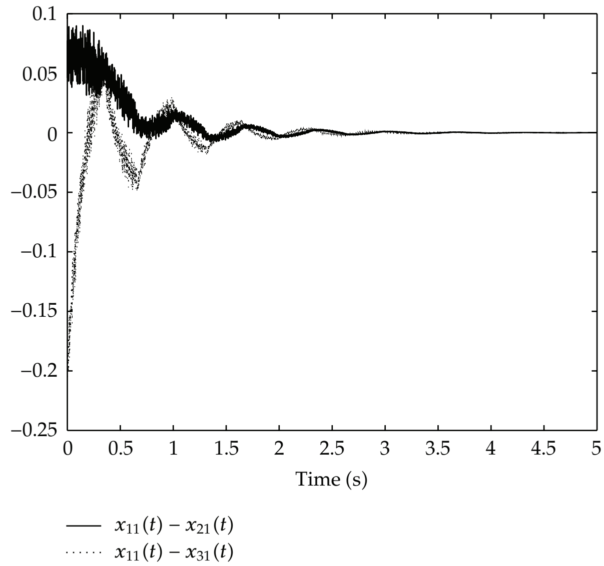

Figure 1: State error of $x_{11}(t) - x_{i1}(t), i = 2, 3$.

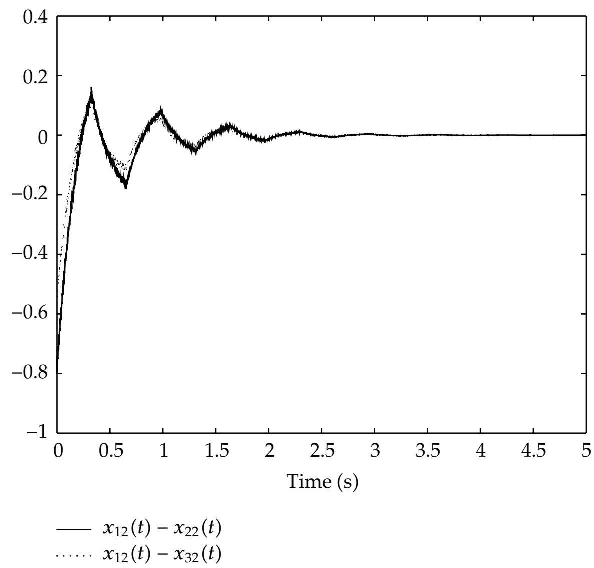

Figure 2: State error of $x_{12}(t) - x_{i2}(t), i = 2, 3$.

5. Conclusions

This paper has investigated the problem of delay-dependent synchronization criterion for neutral-type stochastic delayed complex networks. Most important of all, this paper is concerned with expectations of stochastic cross terms containing the Itô integral. By stochastic analysis techniques, we prove that among these stochastic cross terms, $x(t - h)^T \mathbb{K} \int_{t-h}^{t} \mu(s, x_s) dw(s)$ is the only one whose expectation is equal to zero. Then, this paper has utilized this conclusion to give a delay-dependent synchronization criterion for neutral-type stochastic delayed complex networks. In the derivation process, the mathematical development avoids bounding stochastic cross terms. Thus, the method in our paper can lead to a criterion with less conservatism, and a numerical example is provided to demonstrate the effectiveness of the proposed approach.

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Research Article

Global Convergence for Cohen-Grossberg Neural Networks with Discontinuous Activation Functions

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Cohen-Grossberg neural networks with discontinuous activation functions is considered. Using the property of M-matrix and a generalized Lyapunov-like approach, the uniqueness is proved for state solutions and corresponding output solutions, and equilibrium point and corresponding output equilibrium point of considered neural networks. Meanwhile, global exponential stability of equilibrium point is obtained. Furthermore, by contraction mapping principle, the uniqueness and globally exponential stability of limit cycle are given. Finally, an example is given to illustrate the effectiveness of the obtained results.

1. Introduction

Recently, different types of neural networks with or without time delays have been widely investigated due to their wide applicability [1–32]. Obviously, considerable research interests are focused on the studies of Cohen-Grossberg neural networks (CGNNs) with their various generalizations due to their potential applications in classification, associative memory, and parallel computation and their ability to solve optimization problems. This class of neural networks is proposed by Cohen and Grossberg [1] in 1983, and can be modeled as

$$\frac{du_i(t)}{dt} = -a_i(u_i(t))\left[b_i(u_i(t)) - \sum_{j=1}^{n} w_{ij} f_j(u_j(t)) - I_i\right], \quad i = 1, 2, \dots, n, \tag{1.1}$$

where $n \geq 2$ is the number of neurons in the network, u_i denotes the state variable associated with the ith neuron, a_i represents an amplification function, and b_i is an appropriately behaved function. w_{ij} represents the connection strengths between neurons, and if the output from neuron j excites (resp., inhibits) neuron i, then $w_{ij} \geq 0$ (resp., $w_{ij} \leq 0$). The activation function f_j shows how neurons respond to each other. CGNNs include a lot of famous ecological systems and neural networks as special cases such as the Lotka-Volterra system, the Gilpia-Analg competition system, the Eingen-Schuster system, and the Hopfield neural networks [1–3], where the Hopfield neural networks can be described as follows:

$$\frac{du_i(t)}{dt} = -b_i(u_i(t)) + \sum_{j=1}^{n} w_{ij} f_j(u_j(t)) + I_i, \quad i = 1, 2, \dots, n. \tag{1.2}$$

For CGNNs, dynamics behavior have been studied in literature; we refer to [4–10, 27–29] and the references cited therein. In [4], by using the concept of Lyapunov diagonally stable (LDS) and linear matrix inequality approach, some criteria were given to ensure global stability and global exponential stability. Yuan and Cao in [5] considered global asymptotic stability of delayed Cohen-Grossberg neural networks via nonsmooth analysis. Robust exponentially stability of delayed Cohen-Grossberg neural networks is discussed in [10]. In [27], the authors studied the stochastic stability of a class of Cohen-Grossberg neural networks, in which the interconnections and delays are time varying.

In the above papers, a common feature is that the activation functions are assumed to be continuous and even Lipschitz continuous. However, in [11], Forti and Nistri pointed out that neural networks modeled by differential equations with discontinuous right-hand side are important and do frequently arise in practice. In order to model discrete-time cellular neural networks, a conceptually analogous model based on hard comparators was used [12]. The class of neural networks introduced in [13] to deal with linear and nonlinear programming problems can be considered as another important example. Those networks make use of constraint neurons with a diode-like input-output activations. Once again, in order to ensure satisfaction of the constraints, the diodes are required to have a very high slope in the conducting region; that is, they should approximate the discontinuous characteristic of an ideal diode [14]. When treating with dynamical systems with high-slope nonlinear elements, a system of differential equations with discontinuous right-hand side is often used, rather than the model with high but finite slope [15]. The reason of analyzing the ideal discontinuous case is that such analysis is able to reveal crucial features of the dynamics, such as the possibility that trajectories be confined for some time intervals on discontinuity surfaces. Another interesting phenomenon which is peculiar to discontinuous systems is the possibility that trajectories converge toward an equilibrium point in finite time [16, 17], which is of special interest for designing real-time neural optimization solvers.

In [11], Forti and Nistri discussed the global convergence of neural networks with discontinuous neuron activations by means of the concepts and results of differential equations with discontinuous right-hand side introduced by Filippov [21]. In [18], they extended the results in [11] under the assumption that the interconnection matrix is an M-matrix or H-matrix. In [19], without assuming the boundedness and the continuity of the neuron activations, the authors presented sufficient conditions for the global stability of neural networks with time delay based on linear matrix inequality. Also, in [20], they present some sufficient conditions for the global stability and exponential stability of a class of the CGNNs by using the LDS, and provided an estimate of the convergence rate. In [24–26], the authors discussed the stability or multistability of the neural networks with discontinuous activation functions. However, [11, 24–26] have shown that convergence of the state does not imply convergence of the outputs. In addition, in the practical applications, the result of the neural computation is usually the steady-state neuron output, rather than the asymptotic value of the state. Hence, in this paper, we will study global convergence of CGNNs with discontinuous activation functions, where the interconnection matrix is assumed to be an M-matrix or H-matrix. Firstly, using the property of M-matrix and a generalized Lyapunov-like approach, we prove the uniqueness of state solutions and corresponding output solutions, and equilibrium point and corresponding output equilibrium point for the considered neural networks. Then, global exponential stability of unique equilibrium point is discussed and exponential convergence rate is estimated. Also, by contraction mapping principle, the globally exponential stability of limit cycle is given. Finally, we use a numerical example to illustrate the effectiveness of the theoretical results. The rest of the paper is organized as follows. In Section 2, model description and preliminaries are presented. The main results are stated in Section 3. In Section 4, an example is given to show the validity of the obtained results. Finally, in Section 5, the conclusions are drawn.

Notations. Throughout the paper, the transpose of and inverse of any square matrix A are expressed as A^T and A^{-1}, respectively. For $\alpha = (\alpha_1, \ldots, \alpha_n)^T \in \mathbb{R}^n$, $\alpha > 0$ denotes $\alpha_i > 0$ for $i = 1, 2, \ldots, n$. For $x, y \in \mathbb{R}^n$, $\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$ denotes the scalar product of x, y.

2. Model Description and Preliminaries

In this paper, we consider the CGNNs (1.1) with discontinuous right-hand side. The compact form of model (1.1) is expressed as follows:

$$\frac{du(t)}{dt} = -A(u(t))\left[Bu(t) - Wf(u(t)) - I\right], \tag{2.1}$$

where $u(t) = (u_1(t), u_2(t), \ldots, u_n(t))^T \in \mathbb{R}^n, A(u(t)) = \operatorname{diag}(a_1(u_1(t)), a_2(u_2(t)), \ldots, a_n(u_n(t))), B = \operatorname{diag}(b_1, b_2, \ldots, b_n), W = (w_{ij})_{n\times n}, I = (I_1, I_2, \ldots, I_n)^T \in \mathbb{R}^n$, and $f(u(t)) = (f_1(u_1(t)), \ldots, f_n(u_n(t)))^T$.

Throughout this paper, we make the following assumptions.

(A1) The function $a_i(r)$ is continuous, $0 < \check{a}_i \leq a_i(r) \leq \hat{a}_i$ for all $r \in \mathbb{R}$, where $\check{a}_i$ and $\hat{a}_i$ are positive constants, $i = 1, 2, \ldots, n$.

(A2) The matrix $W = (w_{ij})_{n\times n}$ is nonsingular, that is, $\det W \neq 0$.

Moreover, $f = (f_1, \ldots, f_n)$ is supposed to belong to the following class of discontinuous functions.

Definition 2.1 (see [18] (Function Class $\digamma_D$)). $f(x) \in \digamma_D$ if and only if for $i = 1, 2, \ldots, n$, the following conditions hold:

(i) f_i is bounded on $\mathbb{R}$;

(ii) f_i is piecewise continuous on $\mathbb{R}$; namely, f_i is continuous on $\mathbb{R}$ except a countable set of points of discontinuity p_{ki}, where there exist finite right and left limits $f_i(p_{ki}^+)$ and $f_i(p_{ki}^-)$, respectively; moreover, f_i has finite discontinuous points in any compact interval of $\mathbb{R}$;

(iii) f_i is nondecreasing on $\mathbb{R}$.

Denote the set of discontinuous points of $f_i,\ i = 1, 2, \ldots, n$, by

$$S_i = \{p_{ki} \in \mathbb{R} : f_i(p_{ki}^+) > f_i(p_{ki}^-)\}. \tag{2.2}$$

Sometimes, $f = (f_1, \ldots, f_n)$ is supposed to belong to the next class of discontinuous functions, which is included in $\mathcal{F}_D$.

Definition 2.2 (see [18] (Function Class $\mathcal{F}_{DL}$)). $f(x) \in \mathcal{F}_{DL}$ if and only if $f(x) \in \mathcal{F}_D$ and for $i = 1, 2, \ldots, n$, f_i is locally Lipschitz with Lipschitz constant $l_i(x_i) \geq 0$ for all $x_i \in \mathbb{R} \setminus S_i$. Furthermore, we have $l_i(x_i) \geq L_i < +\infty$ for all $x_i \in \mathbb{R} \setminus S_i$.

For model (1.1) or model (2.1) with discontinuous right-hand side, a solution of Cauchy problem need to be explained. In this paper, solutions in the sense of Filippov [21] are considered whose definition will be given next.

Let $K[f(u)] = (K[f_1(u_1), K[f_2(u_2)], \ldots, K[f_n(u_n)])^T$, where $K[f_i(u_i)] = [f_i(u_i^-), f_i(u_i^+)]$.

Definition 2.3. A function $u(t), t \in [t_1, t_2]$, where $t_1 < t_2 \leq +\infty$ is a solution (in the sense of Filippov) of (2.1) in the interval $[t_1, t_2]$, with initial condition $u(t_1) = u_0 \in \mathbb{R}^n$, if $u(t)$ is absolutely continuous on $[t_1, t_2]$ and $u(t_1) = u_0$, and for almost all (a.a.) $t \in [t_1, t_2]$ we have

$$\frac{du(t)}{dt} \in -A(u(t))\left[Bu(t) - WK\left[f(u(t))\right] - I\right]. \tag{2.3}$$

Let $u(t), t \in [t_1, t_2]$, be a solution of model (2.1). For a.a. $t \in [t_1, t_2]$, one can obtain

$$\frac{du(t)}{dt} = -A(u(t))\left[Bu(t) - W\gamma(t) - I\right], \tag{2.4}$$

where

$$\gamma(t) = W^{-1}\left(A^{-1}(u)\dot{u}(t) + Bu(t) - I\right) \in K\left[f(u(t))\right] \tag{2.5}$$

is the output solution of model (2.1) corresponding to $u(t)$. And $\gamma(t)$ is a bounded measurable function [11], which is uniquely defined by the state solution $u(t)$ for a.a. $t \in [t_1, t_2]$.

Definition 2.4 (equilibrium point). $u^* \in \mathbb{R}^n$ is an equilibrium point of model (2.1) if and only if the following algebraic inclusion is satisfied:

$$0 \in A(u^*)\left(-Bu^* + WK\left[f(u^*)\right] + I\right). \tag{2.6}$$

Definition 2.5 (output equilibrium point). Let u^* be an equilibrium point of model (1.1);

$$\gamma^* = W^{-1}(Bu^* - I) \in K\left[f(u^*)\right] \tag{2.7}$$

is the output equilibrium point of model (2.1) corresponding to u^*.

In this paper, we also need the following definitions and lemma.

Definition 2.6 (see [18]). Let $Q \in \mathbb{R}^{n\times n}$ be a square matrix. Matrix Q is said to be an M-matrix if and only if $Q_{ij} \leq 0$ for each $i \neq j$, and all successive principal minors of Q are positive.

Definition 2.7 (see [18]). Let $Q \in \mathbb{R}^{n\times n}$ be a square matrix. Matrix Q is said to be an H-matrix if and only if the comparison matrix of Q, which is defined by

$$[\mathcal{M}(Q)]_{ij} = \begin{cases} |Q_{ii}|, & i = j, \\ -\left|Q_{ij}\right|, & i \neq j, \end{cases} \tag{2.8}$$

is an M-matrix.

Lemma 2.8 (see [18]). *Suppose that Q is an M-matrix. Then, there exists a vector $\xi > 0$ such that $Q^T\xi > 0$.*

All results of this paper are under one of the following assumptions:

(a) $-W$ is an M-matrix;

(b) $-W$ is an H-matrix such that $W_{ii} < 0$.

(a) and (b) can be applied to cooperative neural networks [22] and cooperative-competitive neural networks, respectively.

From [23], the result that $-W$ is LDS under (a) or (b) can be obtained; hence, all results in [20] hold. So, for any $u_0 \in \mathbb{R}^n$, model (2.1) has a bounded absolutely continuous solution $u(t)$ for $t \geq 0$ which satisfies $u(0) = u_0$. Meanwhile, there exists an equilibrium point $u^* \in \mathbb{R}^n$ of model (2.1).

If $-W$ is an M-matrix, then, there exists $\xi = (\xi_1, \ldots, \xi_n)^T > 0$ such that

$$(-W)^T\xi = \beta > 0. \tag{2.9}$$

If $-W$ is an H-matrix, then, there exists $\xi = (\xi_1, \ldots, \xi_n)^T > 0$ such that

$$[\mathcal{M}(-W)]^T\xi = \beta > 0. \tag{2.10}$$

Using the positive vector ξ, we define the distance in $\mathbb{R}^n$ as follows: for any $x, y \in \mathbb{R}^n$, define

$$\|x - y\|_\xi = \sum_{i=1}^{n} \xi_i \left|x_i - y_i\right|. \tag{2.11}$$

Definition 2.9. The equilibrium point u^* of (2.1) is said to be globally exponentially stable, if there exist constants $\alpha > 0$ and $M > 0$ such that for any solution $u(t)$ of model (2.1), we have

$$\|u(t) - u^*\|_\xi \leq M\|u_0 - u^*\|_\xi \exp\{-\alpha t\}. \tag{2.12}$$

Also, we can consider the CGNNs with periodic input:

$$\frac{du(t)}{dt} = -A(u(t))\left[Bu(t) - Wf(u(t)) - I(t)\right], \tag{2.13}$$

where $I(t) = (I_1(t), I_2(t), \ldots, I_n(t))^T$ is periodic input vectors with period ω.

Definition 2.10. A periodic orbit $u^*(t)$ of Cohen-Grossberg networks is said to be globally exponentially stable, if there exist constants $\alpha > 0$ and $M > 0$ such that such that for any solution $u(t)$ of model (2.13), we have

$$\|u(t) - u^*(t)\| \le M\left\|u_0 - u_0^*\right\|_{\xi} \exp\{-\alpha t\}. \tag{2.14}$$

3. Main Results

In this section, we shall establish some sufficient conditions to ensure the uniqueness of solutions, equilibrium point, output equilibrium point, and limit cycle as well as the global exponential stability of the state solutions.

Because Filippov solution includes set-valued function, in the general case, for a given initial condition, a discontinuous differential equation has multiple solutions starting at it [16]. Next, it will be shown that the uniqueness of solutions of model (2.1) can be obtained under the assumptions (A1) and (A2).

Theorem 3.1. *Under the assumptions (A1) and (A2), if $f \in \mathcal{F}_D$ and $-W$ is an M-matrix or $-W$ is an H-matrix such that $W_{ii} < 0$, then, for any u_0 there is a unique solution $u(t)$ of model* (2.1) *with initial condition $u(0) = u_0$, which is defined and bounded for all $t \ge 0$. Meanwhile, the corresponding output solution $\gamma(t)$ of model* (2.1) *is uniquely defined and bounded for a.a. $t \ge 0$.*

Proof. We only need to prove the uniqueness. Let $u(t)$ and $\tilde{u}(t), t \ge 0$ are two solutions of model (2.1) with the initial condition $u(0) = \tilde{u}(0) = u_0$.

Define

$$V(u - \tilde{u}) = \sum_{i=1}^{n} \xi_i \left| \int_{\tilde{u}_i}^{u_i} \frac{ds}{a_i(s)} \right|. \tag{3.1}$$

Computing the time derivative of V along the solutions of (2.1) gives

$$\frac{dV(u - \tilde{u})}{dt} = \sum_{i=1}^{n} \xi_i \operatorname{sgn}(u_i(t) - \tilde{u}_i(t)) \left(-b_i(u_i(t) - \tilde{u}_i(t)) + \sum_{j=1}^{n} w_{ij}\left(\gamma_j(t) - \tilde{\gamma}_j(t)\right) \right), \tag{3.2}$$

where

$$\operatorname{sgn}(s) = \begin{cases} 1, & s > 0, \\ 0, & s = 0, \\ -1, & s < 0. \end{cases} \tag{3.3}$$

From $f \in \mathcal{F}_D$ and $\gamma_j(t) \in K[f_j(u_j(t))]$, $\tilde{\gamma}_j(t) \in K[f_j(\tilde{u}_j(t))]$, one can have

$$\operatorname{sgn}\left(u_j(t) - \tilde{u}_j(t)\right)\left(\gamma_j(t) - \tilde{\gamma}_j(t)\right) = \left|\gamma_j(t) - \tilde{\gamma}_j(t)\right|. \tag{3.4}$$

Hence

$$\begin{aligned}
\frac{dV(u-\tilde{u})}{dt} &= -\sum_{i=1}^{n}\xi_i b_i|u_i(t)-\tilde{u}_i(t)| + \sum_{i=1}^{n}\xi_i w_{ii}\left|\gamma_i(t)-\tilde{\gamma}_i(t)\right| \\
&\quad + \sum_{i=1}^{n}\xi_i \operatorname{sgn}(u_i(t)-\tilde{u}_i(t)) \sum_{j=1, j\neq i}^{n} w_{ij}\left(\gamma_j(t)-\tilde{\gamma}_j(t)\right) \\
&\le -\sum_{i=1}^{n}\xi_i b_i|u_i(t)-\tilde{u}_i(t)| - \sum_{i=1}^{n}\xi_i |w_{ii}|\left|\gamma_i(t)-\tilde{\gamma}_i(t)\right| \\
&\quad + \sum_{i=1}^{n}\xi_i \sum_{j=1, j\neq i}^{n} \left|w_{ij}\right|\left|\gamma_j(t)-\tilde{\gamma}_j(t)\right| \\
&= -\sum_{i=1}^{n}\xi_i b_i|u_i(t)-\tilde{u}_i(t)| - \sum_{i=1}^{n}\xi_i \Big[|w_{ii}|\left|\gamma_i(t)-\tilde{\gamma}_i(t)\right| \\
&\qquad\qquad + \sum_{j=1, j\neq i}^{n} \left(-\left|w_{ij}\right|\right)\left|\gamma_j(t)-\tilde{\gamma}_j(t)\right|\Big] \\
&= -\sum_{i=1}^{n}\xi_i b_i|u_i(t)-\tilde{u}_i(t)| - \sum_{i=1}^{n}\xi_i \sum_{j=1}^{n}[\mathcal{M}(-W)]_{ij}\left|\gamma_j(t)-\tilde{\gamma}_j(t)\right| \\
&= -\sum_{i=1}^{n}\xi_i b_i|u_i(t)-\tilde{u}_i(t)| - \langle \xi, \mathcal{M}(-W)v(t)\rangle \\
&= -\sum_{i=1}^{n}\xi_i b_i|u_i(t)-\tilde{u}_i(t)| - \left\langle [\mathcal{M}(-W)]^T\xi, v(t)\right\rangle \\
&= -\sum_{i=1}^{n}\xi_i b_i|u_i(t)-\tilde{u}_i(t)| - \langle \beta, v(t)\rangle \le 0,
\end{aligned} \tag{3.5}$$

where $v(t) = (|\gamma_1(t)-\tilde{\gamma}_1(t)|, \ldots, |\gamma_n(t)-\tilde{\gamma}_n(t)|)^T$. Integrating (3.1) between 0 and $t_0 > 0$, we have

$$V(u(t_0)-\tilde{u}(t_0)) \le V(u(0)-\tilde{u}(0)) = V(u_0-u_0) = 0, \tag{3.6}$$

and hence, $u(t_0) = \tilde{u}(t_0)$ for any $t_0 > 0$; that is, the solution of model (2.1) with initial condition $u(0) = u_0$ is unique.

From (2.5), the output solution $\gamma(t)$ corresponding to $u(t)$ is uniquely defined and bounded for a.a. $t \ge 0$. The proof of Theorem 3.1 is completed. □

Remark 3.2. Under the assumptions (A1) and (A2), if $f \in \digamma_D$ and $-W$ is an M-matrix or $-W$ is an H-matrix such that $W_{ii} < 0$, then, for any $I \in \mathbb{R}^n$, model (2.1) has a unique equilibrium point and a unique corresponding output equilibrium point. Because from the assumptions, we have $-W$ is LDS, hence, from Theorem 6 in [20], model (2.1) has a unique equilibrium point. By Definition 2.5, it is easily obtained that corresponding output equilibrium point is unique.

Next, global exponential stability of the equilibrium point of model (2.1) and the uniqueness and global exponential stability of limit cycle of model (2.13) are addressed. The results are given in following theorems.

Theorem 3.3. *Under the assumptions* (*A*1) *and* (*A*2)*, if $f \in \digamma_D$ and $-W$ is an M-matrix or $-W$ is an H-matrix such that $W_{ii} < 0$, then, for any $I \in \mathbb{R}^n$, model* (2.1) *has a unique equilibrium point which is globally exponentially stable.*

Proof. Let $u(t),\ t \geq 0$, be the solution of model (2.1) such that $u(0) = u_0$, and for a.a. $t \geq 0$, let $\gamma(t)$ be the corresponding output solution. For equilibrium point u^*, γ^* is corresponding output equilibrium point.

Since $b_i > 0$, we can choose a small $\varepsilon > 0$ such that

$$b_i - \frac{\varepsilon}{\breve{a}_i} > 0. \tag{3.7}$$

Define

$$\widetilde{V}(u(t) - u^*) = e^{\varepsilon t} \sum_{i=1}^{n} \xi_i \left| \int_{u_i^*}^{u_i(t)} \frac{ds}{a_i(s)} \right|. \tag{3.8}$$

Computing the time derivative of $\widetilde{V}$ along the solutions of (2.1), it follows that

$$\frac{d\widetilde{V}(u(t) - u^*)}{dt} \leq e^{\varepsilon t} \left[-\sum_{i=1}^{n} \xi_i \left(b_i - \frac{\varepsilon}{\breve{a}_i} \right) |u_i(t) - u_i^*| - \langle \beta, \widetilde{v}(t) \rangle \right] \leq 0, \tag{3.9}$$

where $\widetilde{v}(t) = (|\gamma_1(t) - \gamma_1^*|, \dots, |\gamma_n(t) - \gamma_n^*|)^T$.

Hence,

$$\widetilde{V}(u(t) - u^*) \leq \widetilde{V}(u_0 - u^*) \leq \frac{1}{\breve{a}} \|u_0 - u^*\|_{\xi}, \tag{3.10}$$

where $\breve{a} = \min\{\breve{a}_1, \breve{a}_2, \dots, \breve{a}_n\}$.

On the other hand,

$$\widetilde{V}(u(t) - u^*) \geq e^{\varepsilon t} \frac{1}{\widehat{a}} \|u(t) - u^*\|_{\xi}, \tag{3.11}$$

where $\widehat{a} = \max\{\widehat{a}_1, \widehat{a}_2, \dots, \widehat{a}_n\}$.

So, the following inequality holds:

$$\|u(t) - u^*\|_\xi \leq \frac{\widehat{a}}{\check{a}}\|u_0 - u^*\|_\xi e^{-\varepsilon t}, \tag{3.12}$$

that is, u^* is globally exponentially stable. □

Remark 3.4. Since $b_i - \varepsilon/\check{a}_i > 0$, the exponential convergence rate ε can be estimated by means of the maximal allowable value by virtue of inequality $\varepsilon < \check{a}_i b_i,\ i = 1, 2, \ldots, n$. From this, one can see that amplification functions have key effect on the convergence rate of the stability for the considered model.

Next, the uniqueness and the exponentially stability of limit cycle for model (2.13) is given.

Theorem 3.5. *Under the assumptions (A1) and (A2), if $f \in \mathcal{F}_D$ and $-W$ is an M-matrix or $-W$ is an H-matrix such that $W_{ii} < 0$, then model* (2.13) *has a unique globally exponentially stable limit cycle.*

Proof. Let $u(t)$, $\widetilde{u}(t)$ are two solutions of model (2.13), such that $u(0) = u_0,\ \widetilde{u}(0) = \widetilde{u}_0$ respectively.

Define

$$\overline{V}(u(t) - \widetilde{u}(t)) = e^{\varepsilon t}\sum_{i=1}^{n}\xi_i\left|\int_{\widetilde{u}_i(t)}^{u_i(t)}\frac{ds}{a_i(s)}\right|. \tag{3.13}$$

Similar to the proof of Theorem 3.3, the following inequality holds:

$$\|u(t) - \widetilde{u}(t)\|_\xi \leq \frac{\widehat{a}}{\check{a}}\|u_0 - \widetilde{u}_0\|_\xi e^{-\varepsilon t}, \tag{3.14}$$

Define $u^{(t)}(\theta) = u(t+\theta)$. Define a mapping $L : R^n \rightarrow R^n$ by $L(u_0) = u_0^{(\omega)}$, then $L^k(u_0) = u_0^{(k\omega)}$. We can choose a positive integer k, such that for a positive constant $\rho < 1$,

$$\frac{\widehat{a}}{\check{a}}\exp\{-\varepsilon k\omega\} \leq \rho < 1. \tag{3.15}$$

And, from (3.14), we have

$$\left\|L^k(u_0) - L^k(\widetilde{u}_0)\right\|_\xi \leq \frac{\widehat{a}}{\check{a}}\|u_0 - \widetilde{u}_0\|_\xi \exp\{-\varepsilon(k\omega)\} \leq \rho\|u_0 - \widetilde{u}_0\|_\xi. \tag{3.16}$$

By contraction mapping principle, there exists a unique fixed point u_0^* such that $L^k(u_0^*) = u_0^*$. In addition, $L^k(L(u_0^*)) = L(L^k(u_0^*)) = L(u_0^*)$; that is, $L(u_0^*)$ is also a fixed point of L^k. By the uniqueness of the fixed point of the mapping L^k, $L(u_0^*) = u_0^*$; that is, $u_0^{*(\omega)} = u_0^*$. Let $u^*(t)$ be a state of model (1.1) with initial condition u_0^*; we obtain for all $i \in \{1, 2, \dots, n\}$,

$$\frac{du_i^*(t)}{dt} = -a_i(u_i^*(t)) \left[u_i^*(t) - \sum_{j=1}^{n} w_{ij} f_j \left(u_j^*(t) \right) - I_i(t) \right]. \tag{3.17}$$

Then, for all $i \in \{1, 2, \dots, n\}$,

$$\begin{aligned} \frac{du_i^*(t+\omega)}{dt} &= -a_i(u_i^*(t+\omega)) \left[u_i^*(t+\omega) - \sum_{j=1}^{n} w_{ij} f_j \left(u_j^*(t+\omega) \right) - I_i(t+\omega) \right] \\ &= -a_i(u_i^*(t+\omega)) \left[u_i^*(t+\omega) - \sum_{j=1}^{n} w_{ij} f_j \left(u_j^*(t+\omega) \right) - I_i(t) \right], \end{aligned} \tag{3.18}$$

That is, $u^*(t+\omega)^T$ is also a state of the model (2.13) with initial condition $u_0^{*(\omega)}$; here, $u_0^{*(\omega)} = u_0^*$; hence, for all $t \geq 0$, from Theorem 3.1,

$$u^*(t+\omega) = u^*(t). \tag{3.19}$$

Hence, $u^*(t)$ is an isolated periodic orbit of model (2.13) with period ω, that is, a limit cycle of model (2.13). From (3.14), we can obtain that it is globally exponentially stable. The proof of Theorem 3.5 is completed. □

Remark 3.6. Similar to those that are given in [18], global convergence of the output solutions in finite time also can be discussed, which can be embodied in the following example, and the detailed results are omitted.

4. Illustrative Example

In this section, we shall give an example to illustrate the effectiveness of our results.

Example 4.1. Consider the following CGNN model:

$$\begin{aligned} \frac{du_1(t)}{dt} &= (2 + 0.4\cos(u_1(t)))\left[-u_1(t) - 4\,\mathrm{sgn}(u_1(t)) - 2\,\mathrm{sgn}(u_2(t)) + I_1(t)\right], \\ \frac{du_2(t)}{dt} &= (2 + 0.4\cos(u_2(t)))\left[-u_2(t) + 3\,\mathrm{sgn}(u_1(t)) - 2\,\mathrm{sgn}(u_2(t)) + I_2(t)\right], \end{aligned} \tag{4.1}$$

where

$$\mathrm{sgn}(s) = \begin{cases} 1, & s > 0, \\ \text{undefined}, & s = 0, \\ -1, & s < 0. \end{cases} \tag{4.2}$$

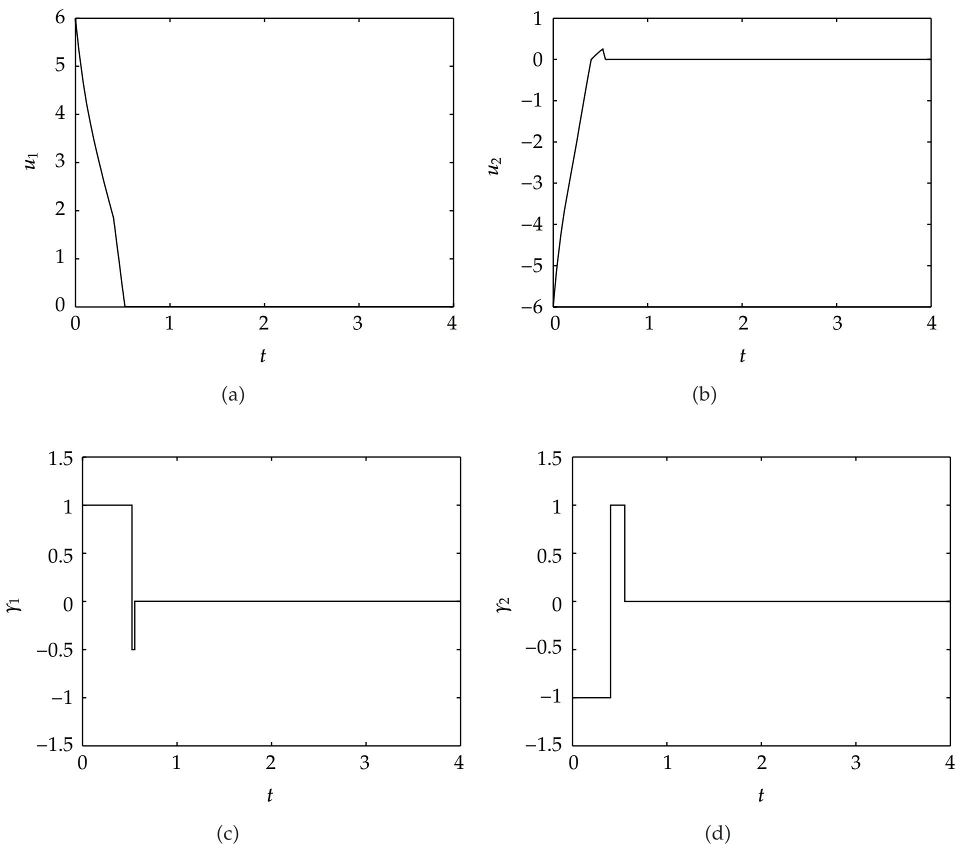

Figure 1: Transient behavior of (u_1, u_2) and (γ_1, γ_2) for $I = (0,0)^T,\ u_0 = (6,-6)^T$.

Obviously, $-W$ is an H-matrix with $w_{ii} < 0$ and

$$\xi = \begin{pmatrix} 1 \\ \dfrac{5}{4} \end{pmatrix}, \qquad \beta = \begin{pmatrix} \dfrac{1}{4} \\ \dfrac{1}{4} \end{pmatrix}. \tag{4.3}$$

Also, the subsets Π^C, Π^D, and Π^{CD} in this example are the same as those in example 1 in [18] which are depicted in detail in Figure 3 in [18].

Firstly, we choose $I = (0,0)^T \in \Pi^D, u_0 = (6,-6)^T$. The equilibrium point u^* of model (4.1) is $(0,0)^T$, and the corresponding output equilibrium point $\gamma^* = (0,0)^T$. Global convergence of $u(t)$ and $\gamma(t)$ in finite time can be obtained. Figure 1 depicts the behavior of state solution $u(t)$ and output solution $\gamma(t)$ with $I = (0,0)^T,\ u_0 = (6,-6)^T$.

Secondly, we choose $I = (4,-6)^T \in \Pi^C,\ u_0 = (-6,6)^T$. Model (4.1) has a unique equilibrium point $u^* = (2,-1)^T$ and a unique output equilibrium point $\gamma^* = (1,-1)^T$. Behavior of state solution and output solution is depicted in Figure 2.

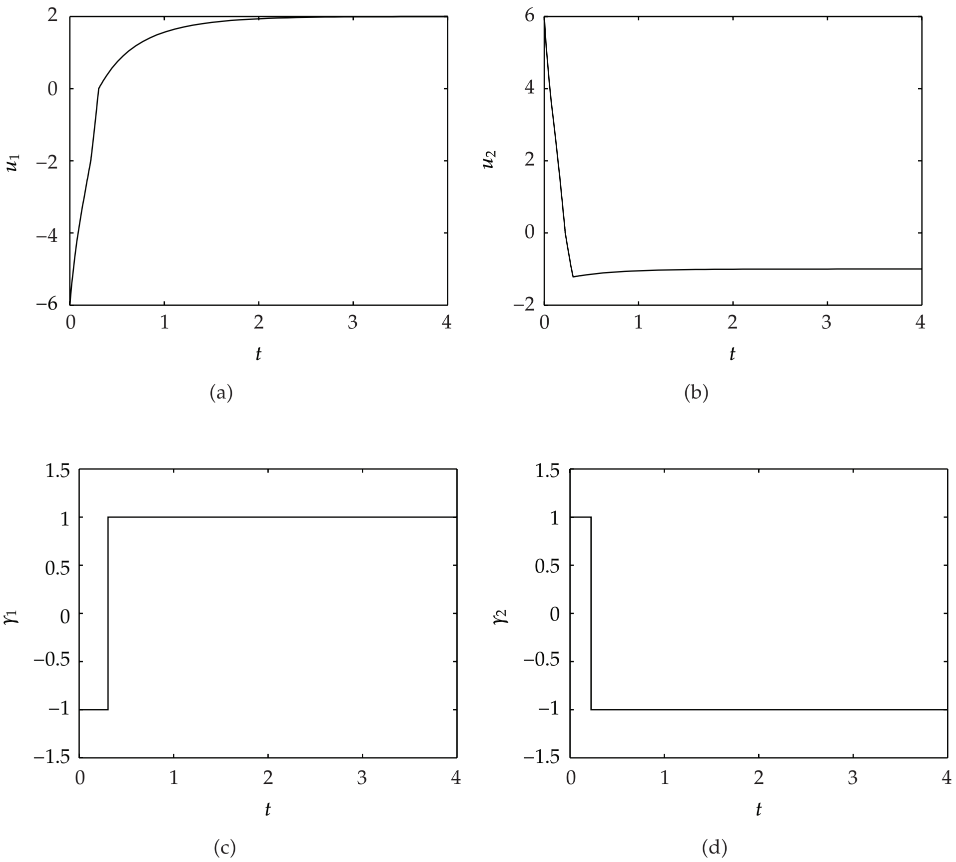

Figure 2: Transient behavior of (u_1, u_2) and (γ_1, γ_2) for $I = (0,0)^T$, $u_0 = (6,-6)^T$.

Then, we choose $I = (0,5)^T \in \Pi^{CD}$, $u_0 = (4,-2)^T$. $u^* = (0,1.5)^T$ and $\gamma^* = (-0.5,1)^T$ are equilibrium point and output equilibrium point of model (4.1), respectively. Simulation results with $I = (0,5)^T$, $u_0 = (4,-2)^T$ about global convergence in finite time of the state solution $u(t)$ and corresponding output solution $\gamma(t)$ are depicted in Figure 3.

5. Conclusions

In this paper, by using the property of M-matrix and a generalized Lyapunov-like approach, global convergence of CGNNs possessing discontinuous activation functions is investigated under the condition that neuron interconnection matrix belongs to the class of M-matrices or H-matrices. The uniqueness is proved for equilibrium point and corresponding output equilibrium point of considered neural networks. It is also proved that for considered model, the solution starting at a given initial condition is unique. Meanwhile, global exponential stability of equilibrium point is obtained for any input. Furthermore, by contraction mapping principle, the uniqueness and the globally exponential stability of limit cycle are given.

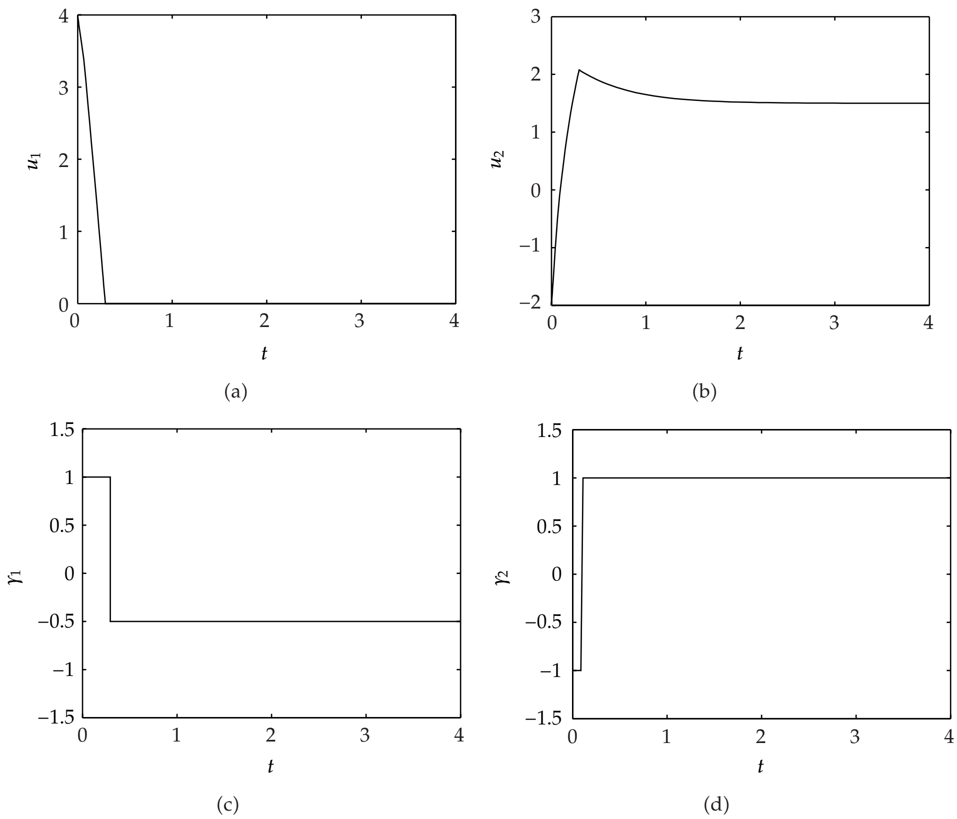

Figure 3: Transient behavior of (u_1, u_2) and (γ_1, γ_2) for $I = (0, 0)^T$, $u_0 = (6, -6)^T$.

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Research Article

New LMI-Based Conditions on Neural Networks of Neutral Type with Discrete Interval Delays and General Activation Functions

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The stability analysis of global asymptotic stability of neural networks of neutral type with both discrete interval delays and general activation functions is discussed. New delay-dependent conditions are obtained by using more general Lyapunov-Krasovskii functionals. Meanwhile, these conditions are expressed in terms of a linear matrix inequality (LMI) and can be verified using the MATLAB LMI toolbox. Numerical examples are used to illustrate the effectiveness of the proposed approach.

1. Introduction

During the past decades, artificial neural networks have received considerable attention due to their applicability in solving signal processing, pattern recognition, associative memories, parallel computation, image processing, and optimization problems [1–6]. Research problems on dynamic behavior such as Chaos control, Hopf bifurcation analysis, and Stability analysis have arisen in such applications and received attention in recent years. In addition, time delays occur frequently in neural networks model [7, 8], which reduce the rate of transmission, as well as cause instability and poor performance of neural networks. Thus, the study of stability of neural networks with time delays is practically required for an engineering system. In recent years, various methods have been proposed to deal with the problem of global stability analysis for neural networks with time delays [9–13]. For example, Singh, 2007 [12], proposed an LMI method for delayed neural networks. Liu et al. 2008 [13] developed a delayed bidirectional associative memory neural network based on

Young's inequality and Hölder's inequality techniques, and several new sufficient criteria are obtained by using a new Lyapunov functional and an-matrix.

In practice, in order to describe the dynamics of some complicated neural networks more precisely, the information about derivatives of the past state has been introduced in the state equations of a considered neural network model [14–16]. This new type of neural networks is often called neural networks of neutral type [17]. In particular, the problem of establishing stability for neural networks of neutral type with discrete time-varying delays has received research attention recently [18–20]. But, unbounded distributed delays were not taken into account in Park et al., 2008 [18]; Park and Kwon, 2009 [19]; Park and Kwon, 2009 [20]. In a real neural system, the presence of distributed delay affects the system stability. More recently, some important results have been obtained on the stability analysis issue for neural networks of neutral type with discrete and unbounded distributed [21, 22]. Nevertheless, in their works, the activation functions of neural networks of neutral type with discrete and unbounded distributed delays have to be Lipschitz continuous to avoid computational complexity. However, in a real system, the activation functions are neither bounded nor monotonous; the functions are also discontinuous and nondifferentiable. Despite important progress made in studies on stability of neutral-type neural networks with discrete delays, due to the lack of the generality of the proposed neural networks model, how to solve the global stability of the proposed model is a challenging and critical issue.

The objective of this paper is to further reduce the conservatism of the stability conditions for neural networks of neutral type with mixed delays (discrete interval delays and unbounded distributed delays) and general activation functions. Based on the Lyapunov-Krasovskii stability theory and the LMI technique, a new sufficient condition is proposed in terms of an LMI. Finally, a numerical example is presented to illustrate the validity of the proposed approach. The rest of this paper is organized as follows. In Section 2, the problem formulation is stated and two assumptions are presented. The proof of the main result of stability analysis is given in Section 3. In Section 4, two numerical examples are provided to demonstrate the effectiveness of the proposed method. The paper is concluded in Section 5.

Throughout this paper, for real symmetric matrices X and Y, the notation $X \geq Y$ (resp., $X > Y$) means that $X - Y$ is positive semidefinite (respectively, positive definite); $\Re^n$ and $\Re^{n\times n}$ denote the n-dimensional Euclidean space and the set of all $n \times n$ real matrices, respectively. The superscripts "T" and "-1" stand for matrix transposition and matrix inverse, respectively. The shorthand $\operatorname{diag}\{X_1, \ldots, X_n\}$ denotes a block diagonal matrix with diagonal blocks being the matrices $X_1, \ldots, X_n$. The symmetric terms in a symmetric matrix are denoted by $(*)$. I is the identity matrix with appropriate dimensions.

2. Problem Description

Consider the following neural networks of neutral-type model:

$$\begin{aligned}
\dot{y}_i(t) = & -c_i y_i(t) + \sum_{j=1}^{n} w_{ij1}\overline{f}_j\left(y_j(t)\right) + \sum_{j=1}^{n} w_{ij2}\overline{g}_j\left(y_j(t-\tau(t))\right) + \sum_{j=1}^{n} a_{ij}\int_{-\infty}^{t} k_j(t-s)\overline{v}_j\left(y_j(s)\right)ds \\
& + \sum_{j=1}^{n} b_{ij}\dot{y}_j(t-h(t)) + I_i, \quad i = 1, \ldots, n,
\end{aligned} \tag{2.1}$$

where $y_i(t)$ is the state of the ith neuron at time$t$, $c_i > 0$ denotes the passive decay rate, w_{ij1}, w_{ij2}, a_{ij}, and b_{ij} are the interconnection matrices representing the weight coefficients of the neurons, $\overline{f}_j(\cdot)$, $\overline{g}_j(\cdot)$, and $\overline{v}_j(\cdot)$ are activation functions, and I_i is an external constant input. The delay k_j is a real valued continuous nonnegative function defined on $[0, +\infty]$, which is assumed to satisfy $\int_0^\infty k_j(s)ds = 1,\ j = 1, \dots, n$.

For system (2.1), the following assumptions are given.

Assumption 2.1. For $i \in \{1, 2, \dots, n\}$, the neuron activation functions in (2.1) satisfy

$$\begin{aligned}
&\widetilde{l}_i^- \le \frac{\overline{f}_i(x_1) - \overline{f}_i(x_2)}{x_1 - x_2} \le \widetilde{l}_i^+, \qquad i = 1, 2, \dots, n,\ x_1, x_2 \in \Re^n,\ x_1 \ne x_2, \\
&\widehat{l}_i^- \le \frac{\overline{g}_i(x_1) - \overline{g}_i(x_2)}{x_1 - x_2} \le \widehat{l}_i^+, \qquad i = 1, 2, \dots, n,\ x_1, x_2 \in \Re^n,\ x_1 \ne x_2, \\
&\bar{l}_i^- \le \frac{\overline{v}_i(x_1) - \overline{v}_i(x_2)}{x_1 - x_2} \le \bar{l}_i^+, \qquad i = 1, 2, \dots, n,\ x_1, x_2 \in \Re^n,\ x_1 \ne x_2,
\end{aligned} \tag{2.2}$$

where $\widetilde{l}_i^-$, $\widetilde{l}_i^+$, $\widehat{l}_i^-$, $\widehat{l}_i^+$, $\bar{l}_i^-$, and $\bar{l}_i^+$ are some constants.

Assumption 2.2. The time-varying delays $\tau(t)$ and $h(t)$ satisfy

$$0 \le \tau_1 \le \tau(t) \le \tau_2, \qquad \dot{\tau}(t) \le \tau_d < 1, \qquad 0 < h(t) \le h, \qquad \dot{h}(t) \le h_d < 1, \tag{2.3}$$

where $\tau_1,\ \tau_2,\ \tau_d,\ h$, and h_d are constants.

Assume $y^* = [y_1^*, y_2^*, \dots, y_n^*]^T$ is an equilibrium point of (2.1). Through $x_i = y_i - y_i^*$, system (2.1) can be transformed into the following system:

$$\dot{x}(t) = -Cx(t) + W_1 f(x(t)) + W_2 g(x(t - \tau(t))) + A \int_{-\infty}^{t} K(t-s) v(x(s)) ds + B\dot{x}(t - h(t)), \tag{2.4}$$

where $x(t) = [x_1(t), \dots, x_n(t)]^T \in \Re^n$ is the neural state vector, $f(x(t)) = [f_1(x_1(t)), \dots, f_n(x_n(t))]^T \in \Re^n$ is the neuron activation function vector with $f(0) = 0$, $g(x(t)) = [g_1(x_1(t)), \dots, g_n(x_n(t))]^T \in \Re^n$ is the neuron activation function vector with $g(0) = 0$, $v(x(t)) = [v_1(x_1(t)), \dots, v_n(x_n(t))]^T \in \Re^n$ is the neuron activation function vector with $v(0) = 0$. $C = \operatorname{diag}\{c_1, \dots, c_n\} > 0$, and $W_1 \in \Re^{n\times n}$, $W_2 \in \Re^{n\times n}$, $A \in \Re^{n\times n}$, and $B \in \Re^{n\times n}$ are the connection weight matrices.

Note that since functions $\overline{f}_i(\cdot)$, $\overline{g}_i(\cdot)$, and $\overline{v}_i(\cdot)$ satisfy Assumption 2.1, $f_i(\cdot)$, $g_i(\cdot)$, and $v_i(\cdot)$ also satisfy

$$\begin{aligned}
&\tilde{l}_i^- \leq \frac{f_i(x_1)-f_i(x_2)}{x_1-x_2} \leq \tilde{l}_i^+, \quad i=1,2,\ldots,n,\ x_1,x_2\in\Re^n,\ x_1\neq x_2,\\
&\hat{l}_i^- \leq \frac{g_i(x_1)-g_i(x_2)}{x_1-x_2} \leq \hat{l}_i^+, \quad i=1,2,\ldots,n,\ x_1,x_2\in\Re^n,\ x_1\neq x_2,\\
&\bar{l}_i^- \leq \frac{v_i(x_1)-v_i(x_2)}{x_1-x_2} \leq \bar{l}_i^+, \quad i=1,2,\ldots,n,\ x_1,x_2\in\Re^n,\ x_1\neq x_2,
\end{aligned} \tag{2.5}$$

where $\tilde{l}_i^-$, $\tilde{l}_i^+$, $\hat{l}_i^-$, $\hat{l}_i^+$, $\bar{l}_i^-$, and $\bar{l}_i^+$ are some constants.

3. Stability Analysis

In order to obtain the main results of stability analysis, the following lemma is introduced.

Lemma 3.1. *For any constant matrix $M>0$, any scalars a and b such that $a<b$, and a vector function $x(t):\ [a,b]\to\Re^n$ such that the integrals concerned are well defined, the following holds:*

$$\left[\int_a^b x(s)ds\right]^T M\left[\int_a^b x(s)ds\right] \leq (b-a)\int_a^b x^T(s)Mx(s)ds. \tag{3.1}$$

To simplify the proofs, the following notations are adopted:

$$\begin{aligned}
&L_1=\operatorname{diag}\left\{\tilde{l}_1^-\tilde{l}_1^+,\tilde{l}_2^-\tilde{l}_2^+,\ldots,\tilde{l}_n^-\tilde{l}_n^+\right\}, \qquad L_2=\operatorname{diag}\left\{\tilde{l}_1^-+\tilde{l}_1^+,\tilde{l}_2^-+\tilde{l}_2^+,\ldots,\tilde{l}_n^-+\tilde{l}_n^+\right\},\\
&L_3=\operatorname{diag}\left\{\hat{l}_1^-\hat{l}_1^+,\hat{l}_2^-\hat{l}_2^+,\ldots,\hat{l}_n^-\hat{l}_n^+\right\}, \qquad L_4=\operatorname{diag}\left\{\hat{l}_1^-+\hat{l}_1^+,\hat{l}_2^-+\hat{l}_2^+,\ldots,\hat{l}_n^-+\hat{l}_n^+\right\},\\
&L_5=\operatorname{diag}\left\{\bar{l}_1^-\bar{l}_1^+,\bar{l}_2^-\bar{l}_2^+,\ldots,\bar{l}_n^-\bar{l}_n^+\right\}, \qquad L_6=\operatorname{diag}\left\{\bar{l}_1^-+\bar{l}_1^+,\bar{l}_2^-+\bar{l}_2^+,\ldots,\bar{l}_n^-+\bar{l}_n^+\right\}.
\end{aligned} \tag{3.2}$$

Then, the following theorem is proposed.

Theorem 3.2. *Under Assumptions 2.1 and 2.2, the origin of system* (2.4) *is globally asymptotically stable, if there exist matrices* $P > 0$, $Q_i = Q_i^T > 0$, $i = 1,2,3,4$, $R_j = R_j^T > 0$, $j = 1,2,3$, $S = S^T > 0$, *diagonal matrices* $Z > 0$, $T_j > 0$, $j = 1,2,\ldots,6$, *and* $E > 0$, *such that the following LMI holds:*

$$\Theta = \begin{bmatrix} \Theta_{1,1} & 0 & \Theta_{1,3} & \Theta_{1,4} & \Theta_{1,5} & 0 & 0 & 0 & \Theta_{1,9} & 0 & \Theta_{1,11} & \Theta_{1,12} \\ * & \Theta_{2,2} & 0 & 0 & 0 & 0 & 0 & \Theta_{2,8} & \Theta_{2,9} & \Theta_{2,10} & 0 & 0 \\ * & * & \Theta_{3,3} & 0 & 0 & 0 & 0 & 0 & \Theta_{3,9} & 0 & \Theta_{3,11} & \Theta_{3,12} \\ * & * & * & \Theta_{4,4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & \Theta_{5,5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \Theta_{6,6} & \Theta_{6,7} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & \Theta_{7,7} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \Theta_{8,8} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & \Theta_{9,9} & 0 & \Theta_{9,11} & \Theta_{9,12} \\ * & * & * & * & * & * & * & * & * & \Theta_{10,10} & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & \Theta_{11,11} & \Theta_{11,12} \\ * & * & * & * & * & * & * & * & * & * & * & \Theta_{12,12} \end{bmatrix} < 0, \tag{3.3}$$

where

$$\begin{gathered} \Theta_{1,1} = -PC - C^T P^T + Q_1 + R_2 + R_3 - L_1T_1 - T_1^T L_1^T - L_3T_3 - T_3^T L_3^T - L_5T_5 - T_5^T L_5^T + C^T \Lambda C, \\ \Theta_{1,3} = PW_1 - C^T Z^T + L_2T_1 - C\Lambda W_1, \qquad \Theta_{1,4} = L_4T_3, \qquad \Theta_{1,5} = L_6T_5, \\ \Theta_{1,9} = PW_2 - C\Lambda W_2, \qquad \Theta_{1,11} = PA - C\Lambda A, \qquad \Theta_{1,12} = PB - C\Lambda B, \\ \Theta_{2,2} = -(1-\tau_d)Q_1 - L_1T_2 - T_2^T L_1^T - L_3T_4 - T_4^T L_3^T - L_5T_6 - T_6^T L_5^T, \\ \Theta_{2,8} = L_2T_2, \qquad \Theta_{2,9} = L_4T_4, \qquad \Theta_{2,10} = L_6T_6, \\ \Theta_{3,3} = ZW_1 + W_1^T Z^T + Q_2 - T_1 - T_1^T + W_1^T \Lambda W_1, \\ \Theta_{3,9} = W_1^T \Lambda W_2 + ZW_2, \qquad \Theta_{3,11} = ZA + W_1^T \Lambda A, \qquad \Theta_{3,12} = ZB + W_1^T \Lambda B, \\ \Theta_{4,4} = Q_3 - T_3 - T_3^T, \qquad \Theta_{5,5} = Q_4 + E - T_5 - T_5^T, \\ \Theta_{6,6} = -R_2 - (\tau_2-\tau_1)^{-1}S, \qquad \Theta_{6,7} = (\tau_2-\tau_1)^{-1}S, \qquad \Theta_{7,7} = -R_3 - (\tau_2-\tau_1)^{-1}S, \\ \Theta_{8,8} = -(1-\tau_d)Q_2 - T_2 - T_2^T, \qquad \Theta_{9,9} = -T_4 - T_4^T - (1-\tau_d)Q_3 + W_2^T \Lambda W_2, \\ \Theta_{9,11} = W_2^T \Lambda A, \qquad \Theta_{9,12} = W_2^T \Lambda B, \qquad \Theta_{10,10} = -T_6 - T_6^T - (1-\tau_d)Q_4, \\ \Theta_{11,11} = -E + A^T \Lambda A, \\ \Theta_{11,12} = A^T \Lambda B, \qquad \Theta_{12,12} = -(1-h_d)R_1 + B^T \Lambda B, \qquad \Lambda = R_1 + (\tau_2-\tau_1)S. \end{gathered} \tag{3.4}$$

Proof. Construct a Lyapunov-Krasovskili functional for system (2.4) as follows:

$$V(x(t),t) = \sum_{i=1}^{5} V_i(x(t),t), \tag{3.5}$$

where

$$V_1(x(t),t) = x^T(t)Px(t) + 2\sum_{i=1}^{n} z_i \int_0^{x_i} f_i(s)ds,$$

$$\begin{aligned} V_2(x(t),t) = {} & \int_{t-\tau(t)}^{t} x^T(s)Q_1x(s)ds + \int_{t-\tau(t)}^{t} \Big[f^T(x(s))Q_2f(x(s)) + g^T(x(s))Q_3g(x(s)) \\ & + v^T(x(s))Q_4v(x(s)) \Big] ds, \end{aligned}$$

$$V_3(x(t),t) = \int_{t-h(t)}^{t} \dot{x}^T(s)R_1\dot{x}(s)ds + \int_{t-\tau_1}^{t} x^T(s)R_2x(s)ds + \int_{t-\tau_2}^{t} x^T(s)R_3x(s)ds,$$

$$V_4(x(t),t) = \sum_{j=1}^{n} e_j \int_0^{\infty} \int_{t-\sigma}^{t} k_j(\sigma)v_j^2(x_j(s))dsd\sigma, \qquad V_5(x(t),t) = \int_{-\tau_2}^{-\tau_1} \int_{t+\theta}^{t} \dot{x}^T(s)S\dot{x}(s)dsd\theta. \tag{3.6}$$

The time derivative of $V(x(t),t)$ along the trajectory of system (2.4) is calculated

$$\dot{V}(x(t),t) = \sum_{i=1}^{5} \dot{V}_i(x(t),t), \tag{3.7}$$

where

$$\begin{aligned} \dot{V}_1(x(t),t) = {} & 2x^T(t)P\Bigg[-Cx(t) + W_1f(x(t)) + W_2g(x(t-\tau(t))) \\ & + A\int_{-\infty}^{t} K(t-s)v(x(s))ds + B\dot{x}(t-h(t)) \Bigg] \\ & + 2f^T(x(t))Z\Bigg[-Cx(t) + W_1f(x(t)) + W_2g(x(t-\tau(t))) \\ & + A\int_{-\infty}^{t} K(t-s)v(x(s))ds + B\dot{x}(t-h(t)) \Bigg], \\ \dot{V}_2(x(t),t) = {} & x^T(t)Q_1x(t) - (1-\dot{\tau}(t))x^T(t-\tau(t))Q_1x(t-\tau(t)) \\ & + f^T(x(t))Q_2f(x(t)) - (1-\dot{\tau}(t))f^T(x(t-\tau(t))) \\ & \times Q_2f(x(t-\tau(t))) + g^T(x(t))Q_3g(x(t)) \\ & - (1-\dot{\tau}(t))g^T(x(t-\tau(t)))Q_3g(x(t-\tau(t))) \\ & + v^T(x(t))Q_4v(x(t)) - (1-\dot{\tau}(t))v^T(x(t-\tau(t)))Q_4v(x(t-\tau(t))) \end{aligned}$$

$$
\begin{aligned}
&\le x^T(t)Q_1x(t) - (1-\tau_d)x^T(t-\tau(t))Q_1x(t-\tau(t)) \\
&\quad + f^T(x(t))Q_2f(x(t)) - (1-\tau_d)f^T(x(t-\tau(t))) \\
&\quad \times Q_2f(x(t-\tau(t))) + g^T(x(t))Q_3g(x(t)) \\
&\quad - (1-\tau_d)g^T(x(t-\tau(t)))Q_3g(x(t-\tau(t))) \\
&\quad + v^T(x(t))Q_4v(x(t)) - (1-\tau_d)v^T(x(t-\tau(t)))Q_4v(x(t-\tau(t))), \\
&\quad \dot{V}_3(x(t),t) = \dot{x}^T(t)R_1\dot{x}(t) - \left(1-\dot{h}(t)\right)\dot{x}^T(t-h(t))R_1\dot{x}(t-h(t)) \\
&\qquad\qquad + x^T(t)R_2x(t) - x^T(t-\tau_1)R_2x(t-\tau_1) \\
&\qquad\qquad + x^T(t)R_3x(t) - x^T(t-\tau_2)R_3x(t-\tau_2) \\
&\qquad\quad \le \dot{x}^T(t)R_1\dot{x}(t) - (1-h_d)\dot{x}^T(t-h(t))R_1\dot{x}(t-h(t)) \\
&\qquad\qquad + x^T(t)R_2x(t) - x^T(t-\tau_1)R_2x(t-\tau_1) \\
&\qquad\qquad + x^T(t)R_3x(t) - x^T(t-\tau_2)R_3x(t-\tau_2), \\
\dot{V}_4(x(t),t) &= \sum_{j=1}^{n} e_j \int_0^{\infty} k_j(\delta)v_j^2\left(x_j(t)\right)d\delta - \sum_{j=1}^{n} e_j \int_0^{\infty} k_j(\delta)v_j^2\left(x_j(t-\delta)\right)d\delta \\
&\quad = v^T(x(t))Ev(x(t)) - \sum_{j=1}^{n} e_j \int_0^{\infty} k_j(\delta)d\delta \int_0^{\infty} k_j(\delta)v_j^2\left(x_j(t-\delta)\right)d\delta \\
&\quad \le v^T(x(t))Ev(x(t)) - \sum_{j=1}^{n} e_j \left(\int_0^{\infty} k_j(\delta)v\left(x_j(t-\delta)\right)d\delta\right)^2, \\
&\quad \dot{V}_5(x(t),t) = (\tau_2-\tau_1)\dot{x}^T(t)S\dot{x}(t) - \int_{t-\tau_2}^{t-\tau_1} \dot{x}^T(s)S\dot{x}(s)ds.
\end{aligned}
\tag{3.8}
$$

By Lemma 3.1, the following inequalities are true:

$$
\begin{aligned}
&-\sum_{j=1}^{n} e_j \left(\int_0^{\infty} k_j(\delta)v\left(x_j(t-\delta)\right)d\delta\right)^2 \\
&\qquad \le -\left(\int_{-\infty}^{t} K(t-s)v(x(s))ds\right)^T E\left(\int_{-\infty}^{t} K(t-s)v(x(s))ds\right), \\
&-\int_{t-\tau_2}^{t-\tau_1} \dot{x}^T(s)S\dot{x}(s)ds = -(\tau_2-\tau_1)^{-1}(\tau_2-\tau_1)\int_{t-\tau_2}^{t-\tau_1} \dot{x}^T(s)S\dot{x}(s)ds
\end{aligned}
$$

$$\begin{aligned}
&\le -(\tau_2-\tau_1)^{-1}\left[\int_{t-\tau_2}^{t-\tau_1}\dot{x}(s)ds\right]^T S\left[\int_{t-\tau_2}^{t-\tau_1}\dot{x}(s)ds\right]\\
&\le -(\tau_2-\tau_1)^{-1}[x(t-\tau_1)-x(t-\tau_2)]^T S[x(t-\tau_1)-x(t-\tau_2)].
\end{aligned}\tag{3.9}$$

From (2.5), the following inequalities can be satisfied

$$\begin{gathered}
\left[f_i(x_i(t))-\widetilde{l}_i^- x_i(t)\right]\left[f_i(x_i(t))-\widetilde{l}_i^+ x_i(t)\right]\le 0,\\
\left[f_i(x_i(t-\tau(t)))-\widetilde{l}_i^- x_i(t-\tau(t))\right]\left[f_i(x_i(t-\tau(t)))-\widetilde{l}_i^+ x_i(t-\tau(t))\right]\le 0,\\
\left[g_i(x_i(t))-\widehat{l}_i^- x_i(t)\right]\left[g_i(x_i(t))-\widehat{l}_i^+ x_i(t)\right]\le 0,\\
\left[g_i(x_i(t-\tau(t)))-\widehat{l}_i^- x_i(t-\tau(t))\right]\left[g_i(x_i(t-\tau(t)))-\widehat{l}_i^+ x_i(t-\tau(t))\right]\le 0,\\
\left[v_i(x_i(t))-\bar{l}_i^- x_i(t)\right]\left[v_i(x_i(t))-\bar{l}_i^+ x_i(t)\right]\le 0,\\
\left[v_i(x_i(t-\tau(t)))-\bar{l}_i^- x_i(t-\tau(t))\right]\left[v_i(x_i(t-\tau(t)))-\bar{l}_i^+ x_i(t-\tau(t))\right]\le 0.
\end{gathered}\tag{3.10}$$

Then, for any $T_j=\operatorname{diag}\{t_{j1},t_{j2},\dots,t_{jn}\}\ge 0,\ j=1,2,\dots,6$, it follows that

$$\begin{aligned}
0\le\ & -2\sum_{i=1}^n t_{1i}\left[f_i(x_i(t))-\widetilde{l}_i^- x_i(t)\right]\left[f_i(x_i(t))-\widetilde{l}_i^+ x_i(t)\right]\\
& -2\sum_{i=1}^n t_{2i}\left[f_i(x_i(t-\tau(t)))-\widetilde{l}_i^- x_i(t-\tau(t))\right]\\
& \times\left[f_i(x_i(t-\tau(t)))-\widetilde{l}_i^+ x_i(t-\tau(t))\right]\\
=\ & -2f^T(x(t))T_1f(x(t))+2x^T(t)L_2T_1f(x(t))\\
& -2x^T(t)L_1T_1x(t)-2f^T(x(t-\tau(t)))T_2f(x(t-\tau(t)))\\
& +2x^T(t-\tau(t))L_2T_2f(x(t-\tau(t)))-2x^T(t-\tau(t))L_1T_2x(t-\tau(t)),\\
0\le\ & -2\sum_{i=1}^n t_{3i}\left[g_i(x_i(t))-\widehat{l}_i^- x_i(t)\right]\left[g_i(x_i(t))-\widehat{l}_i^+ x_i(t)\right]\\
& -2\sum_{i=1}^n t_{4i}\left[g_i(x_i(t-\tau(t)))-\widehat{l}_i^- x_i(t-\tau(t))\right]\\
& \times\left[g_i(x_i(t-\tau(t)))-\widehat{l}_i^+ x_i(t-\tau(t))\right]
\end{aligned}$$

$$
\begin{aligned}
&= -2g^T(x(t))T_3g(x(t)) + 2x^T(t)L_4T_3g(x(t)) \\
&\quad - 2x^T(t)L_3T_3x(t) - 2g^T(x(t-\tau(t)))T_4g(x(t-\tau(t))) \\
&\quad + 2x^T(t-\tau(t))L_4T_4g(x(t-\tau(t))) - 2x^T(t-\tau(t))L_3T_4x(t-\tau(t)),
\end{aligned}
\tag{3.11}
$$

$$
\begin{aligned}
0 \leq &-2\sum_{i=1}^{n} t_{5i}\left[v_i(x_i(t)) - \bar{l}_i^{-}x_i(t)\right]\left[v_i(x_i(t)) - \bar{l}_i^{+}x_i(t)\right] \\
&-2\sum_{i=1}^{n} t_{6i}\left[v_i(x_i(t-\tau(t))) - \bar{l}_i^{-}x_i(t-\tau(t))\right] \\
&\times \left[v_i(x_i(t-\tau(t))) - \bar{l}_i^{+}x_i(t-\tau(t))\right] \\
= &-2v^T(x(t))T_5v(x(t)) + 2x^T(t)L_6T_5v(x(t)) \\
&- 2x^T(t)L_5T_5x(t) - 2v^T(x(t-\tau(t)))T_6v(x(t-\tau(t))) \\
&+ 2x^T(t-\tau(t))L_6T_6v(x(t-\tau(t))) - 2x^T(t-\tau(t))L_5T_6x(t-\tau(t)).
\end{aligned}
\tag{3.12}
$$

Then, combining (3.7)–(3.12), it follows that

$$
\dot{V}(x(t),t) \leq \xi^T(t)\Theta\xi(t), \tag{3.13}
$$

where Θ is given in (3.3) and

$$
\begin{aligned}
&\xi^T(t) \\
&\quad = \begin{bmatrix} x^T(t), x^T(t-\tau(t)), f^T(x(t)), g^T(x(t)), v^T(x(t)), x^T(t-\tau_1), x^T(t-\tau_2), f^T(x(t-\tau(t))), \\ g^T(x(t-\tau(t))), v^T(x(t-\tau(t))), \left(\int_{-\infty}^{t} K(t-s)v(x(s))ds\right)^T, \dot{x}^T(t-h(t)) \end{bmatrix}.
\end{aligned}
\tag{3.14}
$$

It is easy to see that $\dot{V}(x(t),t) < 0$ if $\Theta < 0$ for any $\xi(t) \neq 0$. Thus if the LMI given in (3.3) holds, the system (2.4) is globally asymptotically stable; the proof is completed. □

Remark 3.3. To the best of the authors' knowledge, the problem of global stability for the neural networks of neutral type with both mixed delays (discrete interval and unbounded distributed delays) and general activation functions has not been investigated in the existing literature.

Remark 3.4. In this paper, it is assumed that the resulting activation functions are nonmonotonic and more general than the usual Lipschitz functions. Thus, the advantage of the proposed work lies in the less conservative assumptions of activation functions.

Remark 3.5. It should be noted that when $f(x(t)) = g(x(t)) = v(x(t))$, the system (2.4) is described as

$$\dot{x}(t) = -Cx(t) + W_1 f(x(t)) + W_2 f(x(t-\tau(t))) + A\int_{-\infty}^{t} K(t-s) f(x(s))ds + B\dot{x}(t-h(t)), \tag{3.15}$$

which has been intensively investigated in the literatures [21, 22]. Since the discrete delay are time varying and various in an interval, our work extends and improves the results of [21, 22].

Then the following corollary can be proved directly.

Corollary 3.6. *Under Assumptions 2.1 and 2.2, the origin of system* (3.15) *is globally asymptotically stable, if there exist matrices $P > 0$, $Q_i = Q_i^T > 0$, $i = 1,2$, $R_j = R_j^T > 0$, $j = 1,2,3$, $S = S^T > 0$, diagonal matrices > 0, $T_j > 0$, $j = 1,2$, and $E > 0$, such that the following LMI holds:*

$$\overline{\Theta} = \begin{bmatrix} \overline{\Theta}_{1,1} & 0 & \overline{\Theta}_{1,3} & 0 & 0 & \overline{\Theta}_{1,6} & \Theta_{1,7} & \overline{\Theta}_{1,8} \\ * & \overline{\Theta}_{2,2} & 0 & 0 & 0 & \overline{\Theta}_{2,6} & 0 & 0 \\ * & * & \overline{\Theta}_{3,3} & 0 & 0 & \overline{\Theta}_{3,6} & \Theta_{3,7} & \overline{\Theta}_{3,8} \\ * & * & * & \overline{\Theta}_{4,4} & \overline{\Theta}_{4,5} & 0 & 0 & 0 \\ * & * & * & * & \overline{\Theta}_{5,5} & 0 & 0 & 0 \\ * & * & * & * & * & \overline{\Theta}_{6,6} & \overline{\Theta}_{6,7} & \overline{\Theta}_{6,8} \\ * & * & * & * & * & * & \overline{\Theta}_{7,7} & \overline{\Theta}_{7,8} \\ * & * & * & * & * & * & * & \overline{\Theta}_{8,8} \end{bmatrix} < 0, \tag{3.16}$$

where

$$\overline{\Theta}_{1,1} = -PC - C^T P^T + Q_1 + R_2 + R_3 - 2L_1 T_1 + C^T \Lambda C,$$

$$\overline{\Theta}_{1,3} = PW_1 - C^T Z^T + L_2 T_1 - C^T \Lambda W_1,$$

$$\overline{\Theta}_{1,6} = PW_2 - C^T \Lambda W_2, \qquad \overline{\Theta}_{1,7} = PA - C^T \Lambda A,$$

$$\overline{\Theta}_{1,8} = PB - C^T \Lambda B, \qquad \overline{\Theta}_{2,2} = -(1-\tau_d)Q_1 - 2L_1 T_2,$$

$$\overline{\Theta}_{26} = L_2 T_2, \qquad \overline{\Theta}_{3,3} = E + Q_2 + ZW_1 + W_1^T Z^T - 2T_1 + W_1^T \Lambda W_1,$$

$$\overline{\Theta}_{3,6} = ZW_2 + W_1^T \Lambda W_2,$$

$$\overline{\Theta}_{3,7} = ZA + W_1^T \Lambda B, \qquad \overline{\Theta}_{3,8} = ZB + W_1^T \Lambda A,$$

$$\overline{\Theta}_{4,4} = -R_2 - (\tau_2 - \tau_1)^{-1} S, \qquad \overline{\Theta}_{4,5} = (\tau_2 - \tau_1)^{-1} S,$$

$$\begin{gathered}
\overline{\Theta}_{5,5} = -R_3 - (\tau_2 - \tau_1)^{-1} S, \\
\overline{\Theta}_{6,6} = -(1-\tau_d)Q_2 - 2T_2 + W_2^T \Lambda W_2, \qquad \overline{\Theta}_{6,7} = W_2^T \Lambda B, \\
\overline{\Theta}_{6,8} = W_2^T \Lambda A, \qquad \overline{\Theta}_{7,7} = -E + A^T \Lambda A, \\
\overline{\Theta}_{7,8} = A^T \Lambda B, \qquad \overline{\Theta}_{8,8} = -(1-h_d)R_1 + B^T \Lambda B, \\
\Lambda = R_1 + (\tau_2 - \tau_1) S.
\end{gathered} \tag{3.17}$$

Proof. The proof is similar to that of Theorem 3.2. □

4. Numerical Examples

Example 4.1. Consider the following three-neuron delayed neural networks of neutral type as (2.4), where

$$\begin{gathered}
C = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 10 \end{bmatrix}, \qquad W_1 = \begin{bmatrix} 1.2 & -0.4 & -0.3 \\ -0.12 & -0.81 & -0.1 \\ 0.2 & 0.9 & -0.3 \end{bmatrix}, \\
W_2 = \begin{bmatrix} 1.7 & 0.1 & -0.5 \\ 0.25 & 1.2 & 0.1 \\ -0.1 & 0.65 & 1.2 \end{bmatrix}, \qquad A = \begin{bmatrix} 0.7 & 0.6 & -0.8 \\ -0.1 & 0.1 & 1.1 \\ 0.11 & 0.63 & 0.7 \end{bmatrix}, \\
B = \begin{bmatrix} 0.4 & 0 & 0 \\ 0 & 0.4 & 0 \\ 0 & 0 & 0.4 \end{bmatrix}, \\
\tau(t) = h(t) = 0.3 + 0.3\sin^2(t).
\end{gathered} \tag{4.1}$$

Then, let $\tau_1 = 0.3$, $\tau_2 = 0.6$, $\tau_d = 0.3$, $h_d = 0.3$, $L_1 = 0.09I$, $L_2 = I$, $L_3 = 0.16I$, $L_4 = I$, $L_5 = 0.21I$, and $L_6 = I$. Using MATLAB LMI Control toolbox, by Theorem 3.2, we can find that the system (2.4) is globally asymptotically stable and the solutions of LMI (3.3) are as follows:

$$\begin{gathered}
P = \begin{bmatrix} 41.9798 & 5.3585 & 1.4026 \\ 5.3585 & 74.9441 & -13.7212 \\ 1.4026 & -13.7212 & 60.0629 \end{bmatrix}, \qquad Q_1 = \begin{bmatrix} 85.9041 & 7.5228 & -3.9355 \\ 7.5228 & 123.3913 & -11.8320 \\ -3.9355 & -11.8320 & 93.7664 \end{bmatrix}, \\
Q_2 = \begin{bmatrix} 31.9192 & 3.9335 & 7.9742 \\ 3.9335 & 60.8900 & -3.6718 \\ 7.9742 & -3.6718 & 69.4215 \end{bmatrix}, \qquad Q_3 = \begin{bmatrix} 49.3721 & 2.7949 & -4.1159 \\ 2.7949 & 46.4749 & 1.3294 \\ -4.1159 & 1.3294 & 45.9695 \end{bmatrix},
\end{gathered}$$

$$Q_4 = \begin{bmatrix} 28.1041 & 1.0821 & 0.1583 \\ 1.0821 & 30.9955 & 0.1964 \\ 0.1583 & 0.1964 & 31.1201 \end{bmatrix}, \qquad R_1 = \begin{bmatrix} 4.2784 & 0.8939 & 0.1355 \\ 0.8939 & 9.1183 & -2.1952 \\ 0.1355 & -2.1952 & 6.2238 \end{bmatrix},$$
$$R_2 = \begin{bmatrix} 55.2806 & 3.7360 & 0.1040 \\ 3.7360 & 62.7629 & 1.6388 \\ 0.1040 & 1.6388 & 64.7637 \end{bmatrix}, \qquad R_3 = \begin{bmatrix} 55.2806 & 3.7360 & 0.1040 \\ 3.7360 & 62.7629 & 1.6388 \\ 0.1040 & 1.6388 & 64.7637 \end{bmatrix},$$
$$S = \begin{bmatrix} 3.6864 & 0.0321 & 0.6263 \\ 0.0321 & 6.5233 & -0.9199 \\ 0.6263 & -0.9199 & 5.3944 \end{bmatrix}, \qquad Z = \operatorname{diag}\{8.8438 \;\; 8.8438 \;\; 8.8438\},$$
$$T_1 = \operatorname{diag}\{83.9664 \;\; 83.9664 \;\; 83.9664\}, \qquad T_2 = \{29.3656 \;\; 29.3656 \;\; 29.3656\},$$
$$T_3 = \operatorname{diag}\{54.5299 \;\; 54.5299 \;\; 54.5299\}, \qquad T_4 = \{40.0839 \;\; 40.0839 \;\; 40.0839\},$$
$$T_5 = \operatorname{diag}\{76.6716 \;\; 76.6716 \;\; 76.6716\}, \qquad T_6 = \{31.5403 \;\; 30.5403 \;\; 30.5403\},$$
$$E = \operatorname{diag}\{56.8538 \;\; 56.8538 \;\; 56.8538\}. \tag{4.2}$$

Example 4.2. Consider the following two-neuron delayed neural networks of neutral type as [21], where

$$C = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, \quad W_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 0.6 & -0.12 \\ -0.6 & 0.3 \end{bmatrix}, \qquad A = \begin{bmatrix} 0.2 & -0.1 \\ -0.2 & 0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix},$$
$$\tau(t) \equiv \tau, \qquad h(t) \equiv h. \tag{4.3}$$

Then, let $\tau_1 = 0$, $\tau_2 = 1$, $\tau_d = 0$, $h_d = 0$, $L_1 = 0$, and $L_2 = I$. Using MATLAB LMI Control toolbox, by Corollary 3.6, we can find that the system (3.15) is globally asymptotically stable and the solutions of LMI (3.16) are as follows:

$$\begin{gathered} P = \begin{bmatrix} 201.6082 & 26.2458 \\ 26.2458 & 198.3666 \end{bmatrix}, \qquad Q_1 = \begin{bmatrix} 103.6896 & -2.7859 \\ -2.7859 & 103.2839 \end{bmatrix}, \\ Q_2 = \begin{bmatrix} 93.8975 & -2.3887 \\ -2.3887 & 80.3975 \end{bmatrix}, \qquad R_1 = \begin{bmatrix} 59.2873 & 12.1295 \\ 12.1295 & 57.5817 \end{bmatrix}, \\ R_2 = \begin{bmatrix} 91.0821 & -4.2548 \\ -4.2548 & 91.4235 \end{bmatrix}, \qquad R_3 = \begin{bmatrix} 91.0821 & -4.2548 \\ -4.2548 & 91.4235 \end{bmatrix}, \\ S = \begin{bmatrix} 31.0944 & 7.5926 \\ 7.5926 & 30.4795 \end{bmatrix}, \qquad Z = \operatorname{diag}\{49.1190 \;\; 49.1190\}, \\ T_1 = \operatorname{diag}\{174.5230 \;\; 147.5230\}, \qquad T_2 = \{53.5516 \;\; 53.5516\}, \\ E = \operatorname{diag}\{98.5255 \;\; 98.5255\}. \end{gathered} \tag{4.4}$$

If $\tau_2 = 2$, the conditions in Rakkiyappan and Balasubramaniam, 2008 [21], cannot be satisfied, but by Corollary 3.6 in this paper, one can find that system (3.15) is globally asymptotically stable. Therefore, the proposed result is less conservative than that in Rakkiyappan and Balasubramaniam, 2008 [21].

5. Conclusions

The problem of stability for neural networks of neutral type with discrete interval delays and general activation functions is investigated in this paper. An integrated approach based on a Lyapunov-Krasovskii functional and linear matrix inequality is proposed. In the proposed approach, a corresponding Lyapunov-Krasovskii functional is constructed for neural networks of neutral-type model. Then, by using inequality analysis technique, a reasonably general sufficient condition is obtained in terms of LMI, which can be tested easily using the MATLB LMI toolbox. Moreover, the proposed stability conditions extend and improve the exiting results. Two numerical examples show that the proposed stability result is effective, which can be used to guide engineering design.

In many real world systems, stochastic perturbations often affect the stability of neural networks. Therefore, considering the presence of stochastic perturbations is critical to the stability analysis of networks systems, and some recent progress has been made. In this paper, the proposed neural network of natural type with discrete model was studied by an integrated approach. For future researches, more theoretical analysis should be performed on stochastic neural networks of natural type with mixed delays.

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Research Article

Improved Criteria on Delay-Dependent Stability for Discrete-Time Neural Networks with Interval Time-Varying Delays

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The purpose of this paper is to investigate the delay-dependent stability analysis for discrete-time neural networks with interval time-varying delays. Based on Lyapunov method, improved delay-dependent criteria for the stability of the networks are derived in terms of linear matrix inequalities (LMIs) by constructing a suitable Lyapunov-Krasovskii functional and utilizing reciprocally convex approach. Also, a new activation condition which has not been considered in the literature is proposed and utilized for derivation of stability criteria. Two numerical examples are given to illustrate the effectiveness of the proposed method.

1. Introduction

Neural networks have received increasing attention of researches from various fields of science and engineering such as moving image reconstructing, signal processing, pattern recognition, and fixed-point computation. In the hardware implementation of systems, there exists naturally time delay due to the finite information processing speed and the finite switching speed of amplifiers. It is well known that time delay often causes undesirable dynamic behaviors such as performance degradation, oscillation, or even instability of the systems. Since it is a prerequisite to ensure stability of neural networks before its application to various fields such as information science and biological systems, the problem of stability of neural networks with time delay has been a challenging issue [1–10]. Also, these days, most systems use digital computers (usually microprocessor or microcontrollers) with the necessary input/output hardware to implement the systems. The fundamental character of the digital computer is that it takes computed answers at discrete steps. Therefore, discrete-time modeling with time delay plays an important role in many fields of science and engineering applications. With this regard, various approaches to delay-dependent stability criteria for discrete-time neural networks with time delay have been investigated in the literature [11–16].

In the field of delay-dependent stability analysis, one of the hot issues attracting the concern of the researchers is to increase the feasible region of stability criteria. The most utilized index to check the conservatism of stability criteria is to get maximum delay bounds for guaranteeing the globally exponential stability of the concerned networks. Thus, many researchers put time and efforts into some new approaches to enhance the feasible region of stability conditions. In this regard, Liu et al. [11] proposed a unified linear matrix inequality approach to establish sufficient conditions for the discrete-time neural networks to be globally exponentially stable by employing a Lyapunov-Krasovskii functional. In [12, 13], the existence and stability of the periodic solution for discrete-time recurrent neural network with time-varying delays were studied under more general description on activation functions by utilizing free-weighting matrix method. Based on the idea of delay partitioning, a new stability criterion for discrete-time recurrent neural networks with time-varying delays was derived [14]. Recently, some novel delay-dependent sufficient conditions for guaranteeing stability of discrete-time stochastic recurrent neural networks with time-varying delays were presented in [15] by introducing the midpoint of the time delay's variational interval. Very recently, via a new Lyapunov functional, a novel stability criterion for discrete-time recurrent neural networks with time-varying delays was proposed in [16] and its improvement on the feasible region of stability criterion was shown through numerical examples. However, there are rooms for further improvement in delay-dependent stability criteria of discrete-time neural networks with time-varying delays.

Motivated by the above discussions, the problem of new delay-dependent stability criteria for discrete-time neural networks with time-varying delays is considered in this paper. It should be noted that the delay-dependent analysis has been paid more attention than delay-independent one because the sufficient conditions for delay-dependent analysis make use of the information on the size of time delay [17, 18]. That is, the former is generally less conservative than the latter. By construction of a suitable Lyapunov-Krasovskii functional and utilization of reciprocally convex approach [19], a new stability criterion is derived in Theorem 3.1. Based on the results of Theorem 3.1 and motivated by the work of [20], a further improved stability criterion will be introduced in Theorem 3.4 by applying zero equalities to the results of Theorem 3.1. Finally, two numerical examples are included to show the effectiveness of the proposed method.

Notation. $\mathbb{R}^n$ is the n-dimensional Euclidean space, and $\mathbb{R}^{m\times n}$ denotes the set of all $m \times n$ real matrices. For symmetric matrices X and Y, $X > Y$ (resp., $X \geq Y$) means that the matrix $X - Y$ is positive definite (resp., nonnegative). $X^{\perp}$ denotes a basis for the null space of X. I denotes the identity matrix with appropriate dimensions. $\|\cdot\|$ refers to the Euclidean vector norm or the induced matrix norm. $\mathrm{diag}\{\cdots\}$ denotes the block diagonal matrix. $\star$ represents the elements below the main diagonal of a symmetric matrix.

2. Problem Statements

Consider the following discrete-time neural networks with interval time-varying delays:

$$y(k+1) = Ay(k) + W_0 g\big(y(k)\big) + W_1 g\big(y(k-h(k))\big) + b, \tag{2.1}$$

where n denotes the number of neurons in a neural network, $y(k) = [y_1(k), \ldots, y_n(k)]^T \in \mathbb{R}^n$ is the neuron state vector, $g(k) = [g_1(k), \ldots, g_n(k)]^T \in \mathbb{R}^n$ denotes the neuron activation function vector, $b = [b_1, \ldots, b_n]^T \in \mathbb{R}^n$ means a constant external input vector, $A = \operatorname{diag}\{a_1, \ldots, a_n\} \in \mathbb{R}^{n\times n}(0 \le a_i < 1)$ is the state feedback matrix, $W_i \in \mathbb{R}^{n\times n}(i = 0, 1)$ are the connection weight matrices, and $h(k)$ is interval time-varying delays satisfying

$$0 < h_m \le h(k) \le h_M, \tag{2.2}$$

where h_m and h_M are known positive integers.

In this paper, it is assumed that the activation functions satisfy the following assumption.

Assumption 2.1. The neurons activation functions, $g_i(\cdot)$, are continuous and bounded, and for any $u, v \in \mathbb{R}$, $u \ne v$,

$$k_i^- \le \frac{g_i(u) - g_i(v)}{u - v} \le k_i^+, \quad i = 1, 2, \ldots, n, \tag{2.3}$$

where k_i^- and k_i^+ are known constant scalars.

As usual, a vector $y^* = [y_1^*, \ldots, y_n^*]^T$ is said to be an equilibrium point of system (2.1) if it satisfies $y^* = Ay^* + W_0 g(y^*) + W_1 g(y^*) + b$. From [10], under Assumption 2.1, it is not difficult to ensure the existence of equilibrium point of the system (2.1) by using Brouwer's fixed-point theorem. In the sequel, we will establish a condition to ensure the equilibrium point y^* of system (2.1) is globally exponentially stable. That is, there exist two constants $\alpha > 0$ and $0 < \beta < 1$ such that $\|y(k) - y^*\| \le \alpha\beta^k \sup_{-h_M \le s \le 0} \|y(s) - y^*\|$. To confirm this, refer to [16]. For simplicity, in stability analysis of the network (2.1), the equilibrium point $y^* = [y_1^*, \ldots, y_n^*]^T$ is shifted to the origin by utilizing the transformation $x(k) = y(k) - y^*$, which leads the network (2.1) to the following form:

$$x(k+1) = Ax(k) + W_0 f(x(k)) + W_1 f(x(k-h(k))), \tag{2.4}$$

where $x(k) = [x_1(k), \ldots, x_n(k)]^T \in \mathbb{R}^n$ is the state vector of the transformed network, and $f(x(k)) = [f_1(x_1(k)), \ldots, f_n(x_n(k))]^T \in \mathbb{R}^n$ is the transformed neuron activation function vector with $f_i(x_i(k)) = g_i(x_i(k) + y_i^*) - g_i(y_i^*)$ and $f_i(0) = 0$. From Assumption 2.1, it should be noted that the activation functions $f_i(\cdot)$ $(i = 1, \ldots, n)$ satisfy the following condition [10]:

$$k_i^- \le \frac{f_i(u) - f_i(v)}{u - v} \le k_i^+, \quad \forall u, v \in \mathbb{R}, u \ne v, \tag{2.5}$$

which is equivalent to

$$\left[f_i(u) - f_i(v) - k_i^-(u-v)\right]\left[f_i(u) - f_i(v) - k_i^+(u-v)\right] \leq 0, \tag{2.6}$$

and if $v = 0$, then the following inequality holds:

$$\left[f_i(u) - k_i^-(u)\right]\left[f_i(u) - k_i^+(u)\right] \leq 0. \tag{2.7}$$

Here, the aim of this paper is to investigate the delay-dependent stability analysis of the network (2.4) with interval time-varying delays. In order to do this, the following definition and lemmas are needed.

Definition 2.2 (see [16]). The discrete-time neural network (2.4) is said to be globally exponentially stable if there exist two constants $\alpha > 0$ and $0 \leq \beta \leq 1$ such that

$$\|x(k)\| \leq \alpha\beta^k \sup_{-h_M \leq s \leq 0} \|x(s)\|. \tag{2.8}$$

Lemma 2.3 ((Jensen inequality) [21]). *For any constant matrix $0 < M = M^T \in \mathbb{R}^{n\times n}$, integers h_m and h_M satisfying $1 \leq h_m \leq h_M$, and vector function $x(k) \in \mathbb{R}^n$, the following inequality holds:*

$$-(h_M - h_m + 1)\sum_{k=h_m}^{h_M} x^T(k)Mx(k) \leq -\left(\sum_{k=h_m}^{h_M} x(k)\right)^T M\left(\sum_{k=h_m}^{h_M} x(k)\right). \tag{2.9}$$

Lemma 2.4 ((Finsler's lemma) [22]). *Let $\zeta \in \mathbb{R}^n$, $\Phi = \Phi^T \in \mathbb{R}^{n\times n}$, and $\Gamma \in \mathbb{R}^{m\times n}$ such that* $\operatorname{rank}(\Gamma) < n$. *The following statements are equivalent:*

(i) $\zeta^T\Phi\zeta < 0$, $\forall\Gamma\zeta = 0$, $\zeta \neq 0$,

(ii) ${\Gamma^{\perp}}^T\Phi\Gamma^{\perp} < 0$,

(iii) $\Phi + \mathcal{X}\Upsilon + \Upsilon^T\mathcal{X}^T < 0$, $\forall\mathcal{X} \in \mathbb{R}^{n\times m}$.

3. Main Results

In this section, new stability criteria for the network (2.4) will be proposed. For the sake of simplicity on matrix representation, $e_i \in \mathbb{R}^{10n\times n} (i = 1, \ldots, 10)$ are defined as block entry matrices (e.g., $e_2 = [0, I, \underbrace{0, \ldots, 0}_{8}]^T$). The notations of several matrices are defined as

$$
\begin{aligned}
h_d &= h_M - h_m, \\
\zeta(k) &= \Big[x^T(k), x^T(k-h_m), x^T(k-h(k)), x^T(k-h_M), \Delta x^T(k), \Delta x^T(k-h_m), \\
&\qquad \Delta x^T(k-h_M), f^T(x(k)), f^T(x(k-h(k))), f^T(x(k+1))\Big]^T, \\
\chi(k) &= \left[x^T(k), x^T(k-h_m), x^T(k-h_M), f^T(x(k))\right]^T, \\
\xi(k) &= \left[x^T(k), \Delta x^T(k)\right]^T, \\
\Gamma &= [(A-I), 0, 0, 0, -I, 0, 0, W_0, W_1, 0], \\
\Pi_1 &= [e_1+e_5, e_2+e_6, e_4+e_7, e_{10}], \qquad \Pi_2 = [e_1, e_2, e_4, e_8], \\
\Pi_3 &= [e_1, e_5], \qquad \Pi_4 = [e_2, e_6], \qquad \Pi_5 = [e_4, e_7], \qquad \Pi_6 = [e_2 - e_3, e_3 - e_4], \\
\Pi_7 &= [e_1, e_8], \qquad \Pi_8 = [e_3, e_9], \qquad \Pi_9 = [e_1 + e_5, e_{10}], \\
\Xi_1 &= \Pi_1 R \Pi_1^T - \Pi_2 R \Pi_2^T, \\
\Xi_2 &= \Pi_3 N \Pi_3^T + \Pi_4 (M - N) \Pi_4^T - \Pi_5 M \Pi_5^T, \\
\Xi_3 &= e_5\left(h_m^2 Q_1\right)e_5^T + e_5\left(h_d^2 Q_2\right)e_5^T + e_1(h_m P_1)e_1^T - e_2(h_m P_1)e_2^T + h_d \sum_{i=2}^{3}\left(e_i P_i e_i^T - e_{i+1} P_i e_{i+1}^T\right), \\
\Xi_4 &= -(e_1 - e_2)(Q_1 + P_1)(e_1 - e_2)^T - \Pi_6 \begin{bmatrix} Q_2 + P_2 & S \\ \star & Q_2 + P_3 \end{bmatrix} \Pi_6^T, \\
\Xi_5 &= \Pi_3\left(h_m^2 Q_3\right)\Pi_3^T + \Pi_3\left(h_d^2 Q_4\right)\Pi_3^T, \\
\Phi &= \sum_{i=1}^{5} \Xi_i, \\
\Theta &= \sum_{i=1}^{3} \Pi_{6+i} \begin{bmatrix} -2K_m H_i K_p & (K_m + K_p) H_i \\ \star & -2H_i \end{bmatrix} \Pi_{6+i}^T.
\end{aligned}
\tag{3.1}
$$

Now, the first main result is given by the following theorem.

Theorem 3.1. *For given positive integers h_m and h_M, diagonal matrices $K_m = \mathrm{diag}\{k_1^-, \ldots, k_n^-\}$ and $K_p = \mathrm{diag}\{k_1^+, \ldots, k_n^+\}$, the network* (2.4) *is globally exponentially stable for $h_m \le h(k) \le h_M$, if there exist positive definite matrices $R \in \mathbb{R}^{4n\times 4n}$, $M \in \mathbb{R}^{2n\times 2n}$, $N \in \mathbb{R}^{2n\times 2n}$, $Q_i \in \mathbb{R}^{n\times n}$,*

$Q_{i+2} \in \mathbb{R}^{2n\times 2n}(i = 1,2)$, *positive diagonal matrices* $H_i \in \mathbb{R}^{n\times n}(i = 1,2,3)$, *any symmetric matrices* $P_i \in \mathbb{R}^{n\times n}(i = 1,2,3)$, *and any matrix* $S \in \mathbb{R}^{n\times n}$ *satisfying the following LMIs:*

$$\left[\Gamma^{\perp}\right]^T (\Phi + \Theta)\left[\Gamma^{\perp}\right] < 0, \tag{3.2}$$

$$\begin{bmatrix} Q_2 + P_2 & S \\ \star & Q_2 + P_3 \end{bmatrix} \geq 0, \tag{3.3}$$

$$Q_3 + \begin{bmatrix} 0 & P_1 \\ \star & 0 \end{bmatrix} > 0, \qquad Q_4 + \begin{bmatrix} 0 & P_2 \\ \star & 0 \end{bmatrix} > 0, \qquad Q_4 + \begin{bmatrix} 0 & P_3 \\ \star & 0 \end{bmatrix} > 0, \tag{3.4}$$

where Φ, Θ, *and* Γ *are defined in* (3.1).

Proof. Define the forward difference of $x(k)$ and $V(k)$ as

$$\begin{aligned} \Delta x(k) &= x(k+1) - x(k), \\ \Delta V(k) &= V(k+1) - V(k). \end{aligned} \tag{3.5}$$

Let us consider the following Lyapunov-Krasovskii functional candidate as

$$V(k) = V_1(k) + V_2(k) + V_3(k) + V_4(k), \tag{3.6}$$

where

$$\begin{aligned} V_1(k) &= \chi^T(k)R\chi(k), \\ V_2(k) &= \sum_{s=k-h_m}^{k-1} \xi^T(s)N\xi(s) + \sum_{s=k-h_M}^{k-h_m-1} \xi^T(s)M\xi(s), \\ V_3(k) &= h_m \sum_{s=-h_m}^{-1} \sum_{u=k+s}^{k-1} \Delta x^T(u)Q_1\Delta x(u) + h_d \sum_{s=-h_M}^{-h_m-1} \sum_{u=k+s}^{k-1} \Delta x^T(u)Q_2\Delta x(u), \\ V_4(k) &= h_m \sum_{s=-h_m}^{-1} \sum_{u=k+s}^{k-1} \xi^T(u)Q_3\xi(u) + h_d \sum_{s=-h_M}^{-h_m-1} \sum_{u=k+s}^{k-1} \xi^T(u)Q_4\xi(u). \end{aligned} \tag{3.7}$$

The forward differences of $V_1(k)$ and $V_2(k)$ are calculated as

$$\begin{aligned}
\Delta V_1(k) &= \chi^T(k+1)R\chi(k+1) - \chi^T(k)R\chi(k) \\
&= \begin{bmatrix} x(k)+\Delta x(k) \\ x(k-h_m)+\Delta x(k-h_m) \\ x(k-h_M)+\Delta x(k-h_M) \\ f(x(k+1)) \end{bmatrix}^T R \begin{bmatrix} x(k)+\Delta x(k) \\ x(k-h_m)+\Delta x(k-h_m) \\ x(k-h_M)+\Delta x(k-h_M) \\ f(x(k+1)) \end{bmatrix} - \chi^T(k)R\chi(k) \\
&= \zeta^T(k)\left(\Pi_1 R\Pi_1^T - \Pi_2 R\Pi_2^T\right)\zeta(k) \\
&= \zeta^T(k)\Xi_1\zeta(k),
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
\Delta V_2(k) &= \xi^T(k)N\xi(k) - \xi^T(k-h_m)N\xi(k-h_m) \\
&\quad + \xi^T(k-h_m)M\xi(k-h_m) - \xi^T(k-h_M)M\xi(k-h_M) \\
&= \zeta^T(k)\left(\Pi_3 N\Pi_3^T + \Pi_4(M-N)\Pi_4^T - \Pi_5 M\Pi_5^T\right)\zeta(k) \\
&= \zeta^T(k)\Xi_2\zeta(k).
\end{aligned} \tag{3.9}$$

By calculating the forward differences of $V_3(k)$ and $V_4(k)$, we get

$$\begin{aligned}
\Delta V_3(k) = h_m^2\Delta x^T(k)Q_1\Delta x(k) - h_m\sum_{s=k-h_m}^{k-1}\Delta x^T(s)Q_1\Delta x(s) \\
+ h_d^2\Delta x^T(k)Q_2\Delta x(k) - h_d\sum_{s=k-h_M}^{k-h_m-1}\Delta x^T(s)Q_2\Delta x(s),
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
\Delta V_4(k) = h_m^2\xi^T(k)Q_3\xi(k) - h_m\sum_{s=k-h_m}^{k-1}\xi^T(s)Q_3\xi(s) \\
+ h_d^2\xi^T(k)Q_4\xi(k) - h_d\sum_{s=k-h_M}^{k-h_m-1}\xi^T(s)Q_4\xi(s).
\end{aligned} \tag{3.11}$$

For any matrix P, integers l_1 and l_2 satisfying $l_1 < l_2$, and a vector function $x(s) : [k-l_2, k-l_1-1] \to \mathbb{R}^n$ where k is the discrete time, the following equality holds:

$$\begin{aligned}
&x^T(k-l_1)Px(k-l_1) - x^T(k-l_2)Px(k-l_2) \\
&\quad = \sum_{s=k-l_2}^{k-1_1-1}\left(x^T(s+1)Px(s+1) - x^T(s)Px(s)\right).
\end{aligned} \tag{3.12}$$

It should be noted that

$$
\begin{aligned}
x^T(s+1)Px(s+1) &- x^T(s)Px(s) \\
&= (\Delta x(s)+x(s))^T P(\Delta x(s)+x(s)) - x^T(s)Px(s) \\
&= \Delta x^T(s)P\Delta x(s) + 2x^T(s)P\Delta x(s).
\end{aligned}
\tag{3.13}
$$

From the equalities (3.12) and (3.13), by choosing (l_1, l_2) as $(0, h_m)$, $(h_m, h(k))$ and $(h(k), h_M)$, the following three zero equations hold with any symmetric matrices P_1, P_2, and P_3:

$$
\begin{aligned}
0 &= x^T(k)(h_m P_1)x(k) - x^T(k-h_m)(h_m P_1)x(k-h_m) \\
&\quad - h_m \sum_{s=k-h_m}^{k-1} \left(\Delta x^T(s)P_1\Delta x(s) + 2x^T(s)P_1\Delta x(s)\right),
\end{aligned}
\tag{3.14}
$$

$$
\begin{aligned}
0 &= x^T(k-h_m)(h_d P_2)x(k-h_m) - x^T(k-h(k))(h_d P_2)x(k-h(k)) \\
&\quad - h_d \sum_{s=k-h(k)}^{k-h_m-1} \left(\Delta x^T(s)P_2\Delta x(s) + 2x^T(s)P_2\Delta x(s)\right),
\end{aligned}
\tag{3.15}
$$

$$
\begin{aligned}
0 &= x^T(k-h(k))(h_d P_3)x(k-h(k)) - x^T(k-h_M)(h_d P_3)x(k-h_M) \\
&\quad - h_d \sum_{s=k-h_M}^{k-h(k)-1} \left(\Delta x^T(s)P_3\Delta x(s) + 2x^T(s)P_3\Delta x(s)\right).
\end{aligned}
\tag{3.16}
$$

By adding three zero equalities into the results of $\Delta V_3(k)$, we have

$$
\begin{aligned}
\Delta V_3(k) &= \zeta^T(k)\Bigg(e_5\left(h_m^2 Q_1\right)e_5^T + e_5\left(h_d^2 Q_2\right)e_5^T + e_1(h_m P_1)e_1^T - e_2(h_m P_1)e_2^T \\
&\qquad\qquad + h_d \sum_{i=2}^{3}\left(e_i P_i e_i^T - e_{i+1}P_i e_{i+1}^T\right)\Bigg)\zeta(k) + \Sigma \\
&= \zeta^T(k)\Xi_3\zeta(k) + \Sigma + \Upsilon,
\end{aligned}
\tag{3.17}
$$

where

$$\begin{aligned}\Sigma = & -h_m \sum_{s=k-h_m}^{k-1} \Delta x^T(s)(Q_1+P_1)\Delta x(s) - h_d \sum_{s=k-h(k)}^{k-h_m-1} \Delta x^T(s)(Q_2+P_2)\Delta x(s) \\ & - h_d \sum_{s=k-h_M}^{k-h(k)-1} \Delta x^T(s)(Q_2+P_3)\Delta x(s),\end{aligned} \tag{3.18}$$

$$\begin{aligned}\Upsilon = & -h_m \sum_{s=k-h_m}^{k-1} 2x^T(s)P_1\Delta x(s) - h_d \sum_{s=k-h(k)}^{k-h_m-1} 2x^T(s)P_2\Delta x(s) \\ & - h_d \sum_{s=k-h_M}^{k-h(k)-1} 2x^T(s)P_3\Delta x(s).\end{aligned} \tag{3.19}$$

By Lemma 2.3, when $h_m < h(k) < h_M$, the sum term Σ in (3.18) is bounded as

$$\begin{aligned}\Sigma \le & -\left(\sum_{s=k-h_m}^{k-1} \Delta x(s)\right)^T (Q_1+P_1)\left(\sum_{s=k-h_m}^{k-1} \Delta x(s)\right) \\ & -\left(\sum_{s=k-h(k)}^{k-h_m-1} \Delta x(s)\right)^T \left(\frac{1}{1-\alpha(k)}\right)(Q_2+P_2)\left(\sum_{s=k-h(k)}^{k-h_m-1} \Delta x(s)\right) \\ & -\left(\sum_{s=k-h_M}^{k-h(k)-1} \Delta x(s)\right)^T \left(\frac{1}{\alpha(k)}\right)(Q_2+P_3)\left(\sum_{s=k-h_M}^{k-h(k)-1} \Delta x(s)\right) \\ = & -\zeta^T(k)(e_1-e_2)(Q_1+P_1)(e_1-e_2)^T\zeta(k) \\ & -\zeta^T(k)\Pi_6 \begin{bmatrix} \frac{1}{1-\alpha(k)}(Q_2+P_2) & 0 \\ \star & \frac{1}{\alpha(k)}(Q_2+P_3) \end{bmatrix} \Pi_6^T\zeta(k),\end{aligned} \tag{3.20}$$

where $\alpha(k) = (h_M - h(k))/h_d$.

By reciprocally convex approach [19], if the inequality (3.3) holds, then the following inequality for any matrix S satisfies

$$\begin{bmatrix} -\sqrt{\frac{\alpha(k)}{1-\alpha(k)}}I & 0 \\ \star & \sqrt{\frac{1-\alpha(k)}{\alpha(k)}}I \end{bmatrix} \begin{bmatrix} Q_2+P_2 & S \\ \star & Q_2+P_3 \end{bmatrix} \begin{bmatrix} -\sqrt{\frac{\alpha(k)}{1-\alpha(k)}}I & 0 \\ \star & \sqrt{\frac{1-\alpha(k)}{\alpha(k)}}I \end{bmatrix} \ge 0, \tag{3.21}$$

which implies

$$\begin{bmatrix} \frac{1}{1-\alpha(k)}(Q_2+P_2) & 0 \\ \star & \frac{1}{\alpha(k)}(Q_2+P_3) \end{bmatrix} \geq \begin{bmatrix} Q_2+P_2 & S \\ \star & Q_2+P_3 \end{bmatrix}. \tag{3.22}$$

It should be pointed out that when $h(k) = h_m$ or $h(k) = h_M$, we have $\sum_{s=k-h(k)}^{k-h_m-1} \Delta x(s) = x(k-h_m) - x(k-h(k)) = 0$ or $\sum_{s=k-h_M}^{k-h(k)-1} \Delta x(s) = x(k-h(k)) - x(k-h_M) = 0$, respectively. Thus, the following inequality still holds:

$$\begin{aligned} \Sigma &\leq \zeta^T(k)\left(-(e_1-e_2)(Q_1+P_1)(e_1-e_2)^T - \Pi_6 \begin{bmatrix} Q_2+P_2 & S \\ \star & Q_2+P_3 \end{bmatrix} \Pi_6^T\right)\zeta(k) \\ &= \zeta^T(k)\Xi_4\zeta(k). \end{aligned} \tag{3.23}$$

Then, $\Delta V_3 + \Delta V_4$ has an upper bound as follows:

$$\begin{aligned} \Delta V_3 + \Delta V_4 \leq \zeta^T(k)&\left(\Xi_3 + \Xi_4 + \underbrace{\Pi_3\left(h_m^2 Q_3\right)\Pi_3^T + \Pi_3\left(h_d^2 Q_4\right)\Pi_3^T}_{\Xi_5}\right)\zeta(k) \\ &- h_m \sum_{s=k-h_m}^{k-1} \xi^T(s)\left\{Q_3 + \begin{bmatrix} 0 & P_1 \\ \star & 0 \end{bmatrix}\right\}\xi(s) \\ &- h_d \sum_{s=k-h(k)}^{k-h_m-1} \xi^T(s)\left\{Q_4 + \begin{bmatrix} 0 & P_2 \\ \star & 0 \end{bmatrix}\right\}\xi(s) \\ &- h_d \sum_{s=k-h_M}^{k-h(k)-1} \xi^T(s)\left\{Q_4 + \begin{bmatrix} 0 & P_3 \\ \star & 0 \end{bmatrix}\right\}\xi(s). \end{aligned} \tag{3.24}$$

Here, if the inequalities (3.4) hold, then $\Delta V_3 + \Delta V_4$ is bounded as

$$\Delta V_3 + \Delta V_4 \leq \zeta^T(k)(\Xi_3 + \Xi_4 + \Xi_5)\zeta(k). \tag{3.25}$$

From (2.7), for any positive diagonal matrices $H_i = \mathrm{diag}\{h_{i1}, \ldots, h_{in}\}$ $(i = 1, 2, 3)$, the following inequality holds:

$$\begin{aligned}
0 \le & -2\sum_{i=1}^{n} h_{1i}\left[f_i(x_i(k)) - k_i^- x_i(k)\right]\left[f_i(x_i(k)) - k_i^+ x_i(k)\right] \\
& -2\sum_{i=1}^{n} h_{2i}\left[f_i(x_i(k-h(k))) - k_i^- x_i(k-h(k))\right]\left[f_i(x_i(k-h(k))) - k_i^+ x_i(k-h(k))\right] \\
& -2\sum_{i=1}^{n} h_{3i}\left[f_i(x_i(k+1)) - k_i^- x_i(k+1)\right]\left[f_i(x_i(k+1)) - k_i^+ x_i(k+1)\right] \\
= & \ \zeta^T(k)\left(\sum_{i=1}^{3}\Pi_{6+i}\begin{bmatrix} -2K_m H_i K_p & (K_m + K_p)H_i \\ \star & -2H_i \end{bmatrix}\Pi_{6+i}^T\right)\zeta(k) \\
= & \ \zeta^T(k)\Theta\zeta(k).
\end{aligned} \tag{3.26}$$

Therefore, from (3.8)–(3.16) and by application of the S-procedure [23], ΔV has a new upper bound as

$$\Delta V \le \zeta^T(k)\left(\underbrace{\sum_{i=1}^{5}\Xi_i}_{\Phi} + \Theta\right)\zeta(k), \tag{3.27}$$

where Φ and Θ are defined in (3.1).

Also, the system (2.4) with the augmented vector $\zeta(k)$ can be rewritten as

$$\Gamma\zeta(k) = 0, \tag{3.28}$$

where Γ is defined in (3.1).

Then, a delay-dependent stability condition for the system (2.4) is

$$\zeta^T(k)(\Phi + \Theta)\zeta(k) < 0 \quad \text{subject to } \Gamma\zeta(k) = 0. \tag{3.29}$$

Finally, by utilizing Lemma 2.4, the condition (3.29) is equivalent to the following inequality

$$\left[\Gamma^{\perp}\right]^T(\Phi + \Theta)\left[\Gamma^{\perp}\right] < 0. \tag{3.30}$$

From the inequality (3.30), if the LMIs (3.2)–(3.4) hold. From (ii) and (iii) of Lemma 2.4, if the stability condition (3.29) holds, then for any free maxrix $\mathcal{X}$ with appropriate dimension, the condition (3.29) is equivalent to

$$\underbrace{\Phi + \Theta + \mathcal{X}\Gamma + \Gamma^T\mathcal{X}^T}_{\Psi} < 0. \tag{3.31}$$

Therefore, from (3.31), there exists a sufficient small scalar $\rho > 0$ such that

$$\Delta V \le \zeta^T(k)\Psi\zeta(k) < -\rho\|x(k)\|^2. \tag{3.32}$$

By using the similar method of [11, 12], the system (2.4) is globally exponentially stable for any time-varying delay $h_m \le h(k) \le h_M$ from Definition 2.2. This completes our proof. □

Remark 3.2. In Theorem 3.1, the stability condition is derived by utilizing a new augmented vector $\zeta(k)$ including $f(x(k+1))$. This state vector $f(x(k+1))$ which may give more information on dynamic behavior of the system (2.4) has not been utilized as an element of augmented vector $\zeta(k)$ in any other literature. Correspondingly, the state vector $f(x(k+1))$ is also included in (3.26).

Remark 3.3. As mentioned in [10], the activation functions of transformed system (2.4) also satisfy the condition (2.6). In Theorem 3.4, by choosing (u,v) in (2.6) as $(x(k), x(k-h(k)))$ and $(x(k-h(k)), f(x(k+1))$, more information on cross-terms among the states $f(x(k))$, $f(x(k-h(k)))$, $f(x(k+1))$, $x(k)$, and $x(k-h(k))$ will be utilized, which may lead to less conservative stability criteria. In stability analysis for discrete-time neural networks with time-varying delays, this consideration has not been proposed in any other literature. Through two numerical examples, it will be shown that the newly proposed activation condition may enhance the feasible region of stability criterion by comparing maximum delay bounds with the results obtained by Theorem 3.1.

As mentioned in Remark 3.3, from (2.6), we add the following new inequality with any positive diagonal matrices $H_i = \operatorname{diag}\{h_{i1}, \ldots, h_{in}\}$ $(i = 4,5,6)$ to be chosen as

$$\begin{aligned}
0 \le\ & -2\sum_{i=1}^{n} h_{4i}\left[f_i(x_i(k)) - f_i(x_i(k-h(k))) - k_i^-(x_i(k) - x_i(k-h(k)))\right] \\
& \times \left[f_i(x_i(k)) - f_i(x_i(k-h(k))) - k_i^+(x_i(k) - x_i(k-h(k)))\right] \\
& -2\sum_{i=1}^{n} h_{5i}\left[f_i(x_i(k-h(k))) - f_i(x_i(k+1)) - k_i^-(x_i(k-h(k)) - x_i(k) - \Delta x_i(k))\right] \\
& \times \left[f_i(x_i(k-h(k))) - f_i(x_i(k+1)) - k_i^+(x_i(k-h(k)) - x_i(k) - \Delta x_i(k))\right] \\
& -2\sum_{i=1}^{n} h_{6i}\left[f_i(x_i(k+1)) - f_i(x_i(k)) - k_i^-\Delta x_i(k)\right] \\
& \times \left[f_i(x_i(k+1)) - f_i(x_i(k)) - k_i^+\Delta x_i(k)\right] \\
& -\zeta^T(k)\left(\sum_{i=1}^{3}\Pi_{9+i}\begin{bmatrix} -2K_m H_{3+i} K_p & (K_m + K_p)H_{3+i} \\ \star & -2H_{3+i}\end{bmatrix}\Pi_{9+i}^T\right)\zeta(k) \\
=\ & \zeta^T(k)\Omega\zeta(k),
\end{aligned} \tag{3.33}$$

where $\Pi_{10} = [e_1 - e_3, e_8 - e_9]$, $\Pi_{11} = [e_3 - e_1 - e_5, e_9 - e_{10}]$, and $\Pi_{12} = [e_5, e_{10} - e_8]$. We will add this inequality (3.33) in Theorem 3.4. Now, we have the following theorem.

Theorem 3.4. *For given positive integers h_m and h_M, diagonal matrices $K_m = \mathrm{diag}\{k_1^-, \dots, k_n^-\}$ and $K_p = \mathrm{diag}\{k_1^+, \dots, k_n^+\}$, the network* (2.4) *is globally exponentially stable for $h_m \le h(k) \le h_M$, if there exist positive definite matrices $R \in \mathbb{R}^{4n\times 4n}$, $M \in \mathbb{R}^{2n\times 2n}$, $N \in \mathbb{R}^{2n\times 2n}$, $Q_i \in \mathbb{R}^{n\times n}$, $Q_{i+2} \in \mathbb{R}^{2n\times 2n}$ $(i = 1, 2)$, positive diagonal matrices $H_i \in \mathbb{R}^{n\times n}$ $(i = 1, \dots, 6)$, any symmetric matrices $P_i \in \mathbb{R}^{n\times n}$ $(i = 1, 2, 3)$, and any matrix $S \in \mathbb{R}^{n\times n}$ satisfying the following LMIs:*

$$\left[\Gamma^{\perp}\right]^T (\Phi + \Theta + \Omega)\left[\Gamma^{\perp}\right] < 0, \tag{3.34}$$

$$\begin{bmatrix} Q_2 + P_2 & S \\ \star & Q_2 + P_3 \end{bmatrix} \ge 0, \tag{3.35}$$

$$Q_3 + \begin{bmatrix} 0 & P_1 \\ \star & 0 \end{bmatrix} > 0, \qquad Q_4 + \begin{bmatrix} 0 & P_2 \\ \star & 0 \end{bmatrix} > 0, \qquad Q_4 + \begin{bmatrix} 0 & P_3 \\ \star & 0 \end{bmatrix} > 0, \tag{3.36}$$

where Φ, Γ, and Ω are defined in (3.1) *and Θ is in* (3.33).

Proof. With the same Lyapunov-Krasovskii functional candidate in (3.6), by using the similar method in (3.8)–(3.16), and considering inequality (3.36), the procedure of deriving the condition (3.34)–(3.36) is straightforward from the proof of Theorem 3.1, so it is omitted. □

4. Numerical Examples

In this section, we provide two numerical examples to illustrate the effectiveness of the proposed criteria in this paper.

Example 4.1. Consider the discrete-time neural networks (2.4) where

$$A = \begin{bmatrix} 0.4 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}, \qquad W_0 = \begin{bmatrix} 0.2 & -0.2 & 0.1 \\ 0 & -0.3 & 0.2 \\ -0.2 & -0.1 & -0.2 \end{bmatrix}, \qquad W_1 = \begin{bmatrix} -0.2 & 0.1 & 0 \\ -0.2 & 0.3 & 0.1 \\ 0.1 & -0.2 & 0.3 \end{bmatrix}. \tag{4.1}$$

The activation functions satisfy Assumption 2.1 with

$$K_m = \mathrm{diag}\{0, -0.4, -0.2\}, \qquad K_p = \mathrm{diag}\{0.6, 0, 0\}. \tag{4.2}$$

For various h_m, the comparison of maximum delay bounds (h_M) obtained by Theorems 3.1 and 3.4 with those of [12, 16] is conducted in Table 1. From Table 1, it can be confirmed that the results of Theorem 3.1 give a larger delay bound than those of [12] and are equal to the results of [16]. However, the results obtained by Theorem 3.4 are better than the results of [16] and Theorem 3.1, which supports the effectiveness of the proposed idea mentioned in Remark 3.3.

Table 1: Maximum bounds h_M with different h_m (Example 4.1).

Methods	2	4	6	10
Song and Wang [12]	6	8	10	14
Wu et al. [16]	12	14	16	20
Theorem 3.1	12	14	16	20
Theorem 3.4	14	16	18	22

Table 2: Maximum bounds h_M with different h_m and $a = 0.9$ (Example 4.2).

Methods	2	4	6	8	10	15
Song and Wang [12]	11	11	12	13	14	17
Zhang et al. [13]	11	12	13	14	16	19
Song et al. [14]	15	16	17	18	19	22
Wu et al. [16]	16	18	18	20	20	22
Theorem 3.1	18	18	19	20	20	23
Theorem 3.4	18	18	19	20	21	23

Example 4.2. Consider the discrete-time neural networks (2.4) having the following parameters:

$$A = \begin{bmatrix} 0.8 & 0 \\ 0 & a \end{bmatrix}, \qquad W_0 = \begin{bmatrix} 0.001 & 0 \\ 0 & 0.005 \end{bmatrix}, \qquad W_1 = \begin{bmatrix} -0.1 & 0.01 \\ -0.2 & -0.1 \end{bmatrix}, \qquad K_m = 0, \qquad K_p = I. \tag{4.3}$$

When $a = 0.9$, for different values of h_m, maximum delay bounds obtained by [12–14, 16] and our Theorems are listed in Table 2. From Table 2, it can be confirmed that all the results of Theorems 3.1 and 3.4 provide larger delay bounds than those of [12–14]. Also, our results are better than or equal to the results of [16]. For the case of $a = 0.7$, another comparison of our results with those of [15, 16] is conducted in Table 3, which shows all the results obtained by Theorems 3.1 and 3.4 give larger delay bounds than those of [15, 16].

5. Conclusions

In this paper, improved delay-dependent stability criteria were proposed for discrete-time neural networks with time-varying delays. In Theorem 3.1, by constructing the suitable Lyapunov-Krasovskii's functional and utilizing some recent results introduced in [19, 20], the sufficient condition for guaranteeing the global exponential stability of discrete-time neural network having interval time-varying delays has been derived. Based on the results of Theorem 3.1, by constructing new inequalities of activation functions, the further improved stability criterion was presented in Theorem 3.4. Via two numerical examples, the improvement of the proposed stability criteria has been successfully verified.

Table 3: Maximum bounds h_M with different h_m and $a = 0.7$ (Example 4.2).

Methods	2	4	6	8	10	15	20	100	1000
Zhang et al. [15]	20	22	24	26	28	33	38	118	1018
Wu et al. [16]	24	26	28	30	32	37	42	122	1022
Theorem 3.1	29	31	32	34	36	41	46	126	1026
Theorem 3.4	29	31	32	34	36	41	46	126	1026

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Research Article

Development of Neural Network Model for Predicting Peak Ground Acceleration Based on Microtremor Measurement and Soil Boring Test Data

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It may not be possible to collect adequate records of strong ground motions in a short period of time; hence microtremor survey is frequently conducted to reveal the stratum structure and earthquake characteristics at a specified construction site. This paper is therefore aimed at developing a neural network model, based on available microtremor measurement and on-site soil boring test data, for predicting peak ground acceleration at a site, in a science park of Taiwan. The four key parameters used as inputs for the model are soil values of the standard penetration test, the medium grain size, the safety factor against liquefaction, and the distance between soil depth and measuring station. The results show that a neural network model with four neurons in the hidden layer can achieve better performance than other models presently available. Also, a weight-based neural network model is developed to provide reliable prediction of peak ground acceleration at an unmeasured site based on data at three nearby measuring stations. The method employed in this paper provides a new way to treat this type of seismic-related problem, and it may be applicable to other areas of interest around the world.

1. Introduction

Earthquake problems are globally considered to be a research topic of importance since many countries are subject to this natural disaster. For instances, the recent big one with magnitude 9.0 on the Richter scale that occurred in Japan on 11 March 2011 and triggered a significant tsunami, caused approximately $35 billion in damage. On 12 January 2010, a devastating earthquake with magnitude 7.0 on the Richter scale struck Haiti in the Caribbean, and claimed more than 200 thousand lives have lost in the capital and surrounding areas.

Other major earthquakes experienced by Chile, China, Indonesia, New Zealand, and Taiwan are listed in the archive of United States Geological Survey [1]. Without exception, all these earthquakes have caused tremendous casualties and property losses, requiring urgent attention to this calamitous problem.

Many seismic-related research issues have been investigated and published previously, with some focused on finding an early warning system, while others are based on records of historical seismic data (e.g., [2–8]). It is quite obvious that strong ground motions data cannot be collected in a short period of time, and also the records for a metropolitan area or a place with high population density are not easy to obtain. Whereas, microtremor surveys can be used to infer the stratum structure and earthquake characteristics at a specified construction site without destroying its ground surface. Thus, this fast and low cost measuring technique is often selected not only to provide useful information for an area which lacks seismic records, but also to effectively analyze potential liquefaction index for the construction site being investigated.

Further, it is worth mentioning that the microtremor measurements with appropriate transformation, such as Fourier transform or Nakamura technique, can estimate the key seismic parameter, that is, peak ground acceleration (PGA), which exhibits a tendency similar to the characteristics of strong ground motion [9, 10]. Note that the above literatures were focused on the development of neural network models based on actual seismic records, and microtremor measurements were used for the sake of comparison. Regarding the development of neural network model in accordance with microtremor measurement and soil profile was not examined in these studies. Also, some important factors such as dominant frequency, shear wave speed, and amplification can be explored by microtremor surveys [11–18], and these can help to determine the distribution of soil layers, liquefaction hazard mapping, and earthquake site response. Although microtremor measurements can be easily carried out at a number of sites, the main limitation is the increase in cost with increased number of measuring stations. Hence, the development of a model for predicting microtremor information for other important but unmeasured sites is useful for economic reasons.

From the references mentioned above and other reports previously published in the field of earthquake engineering and soil dynamics, the microtremor measurements appear to be a function of the soil conditions at a specified site. However, prediction of microtremor information by using soil boring test result has rarely been reported up to now. Therefore, the purpose of this study is to develop a model for mapping soil boring test data to microtremor measurements by using a neural network approach. In particular, three key soil parameters; the standard penetration test value (STP-N), the medium grain size (D50), and the safety factor against liquefaction (FL), and one spatial factor, that is, the distance between soil layer and measuring station (DS), are used to evaluate PGA resulting from microtremor measurements. A weight-based neural network model is also developed to predict PGA at unmeasured sites by using values at three nearby measuring stations. The method developed in this study should provide a new approach for solving problems in the relevant engineering field.

2. Context and Rationale for the Research

Science parks are mainly occupied by many high-tech companies including some world class factories such as ACER, HTC, and TSMC, which play an important economic role in the island of Taiwan. In this study, the chosen Kaohsiung (Luchu) science park is one of the major parks located in the southern part of Taiwan. This park has a total area of 571 hectares, which started construction in the year of 2001 and was completed in 2010. Various high-tech industries, such as integrated circuits, precision machines, optoelectronic components, computer peripherals, communication and biotechnology products, were planned and developed in this park. These types of industries can be affected significantly by strong ground motions and are also sensitive to ambient vibrations. Therefore, it is necessary to consider antiearthquake design and to examine microtremor in the park from time to time to prevent different levels of damages.

Two crucial factors for evaluating the effect of ground motions in the science park are fault distribution and geological condition in Kaohsiung area, which are based on the information from Central Geological Survey shown in Figure 1 [19, 20]. It can be seen that there exists seven faults in this region, which are (1) Chishan fault, (2) Liukuei fault, (3) Tsaujou fault, (4) Hsiaokangshan fault, (5) Yuchang fault, (6) Jenwu fault, and (7) Fengshan fault. These faults may create strong ground motions and endanger the high-tech buildings and instruments. Also from this figure, it can be seen that alluvial soil occupies a large part of Kaohsiung area, particularly at the science park, which can have an influence on microtremor measurements.

The occurrence of strong ground motion is unpredictable, and it can cause serious structural damage within a very short period of time. Thus, a proper antiearthquake design is usually considered for high-tech factory constructions. In contrast, the existence of microtremor is easy to neglect as it is very small, but due to microtremor occurring very often on the earth surface, some precision instruments can be damaged during its operation process due to the constant continuous vibration frequency or peak ground acceleration. Consequently, microtremor can cause, for high-tech manufactures, an unexpected and significant financial loss, and thus environmental ambient vibration survey is a very important consideration for science parks.

Figure 2 shows the Kaohsiung science park, with four microtremor measuring stations MS1, MS2, MS3, and MS4. Also, there are twenty-seven soil boring test sites in the neighborhood of this park. As mentioned previously, this study is focused on the development of a model for predicting peak ground acceleration based on microtremor measurements and soil boring test data. Therefore, the records obtained from these measuring stations and boring test sites can provide useful information for developing the model by using neural network approach. In the next section, the processing of measured and test data is discussed, and then the results obtained from the developed model are presented.

3. Ambient Vibration Measurement and Soil Boring Test Data

Ambient vibration with very low amplitude (about 10^{-6} m) and acceleration (0.8–2.5 gal or cm^2/s), which cannot in general be felt by humans [21, 22], is frequently found on the ground surface of the earth. This vibration, however, can be recorded by using a precise measuring instrument developed recently. In the present study, the results of ambient vibrations are measured and calculated by a set of ultrasensitive seismic accelerometer-Model 731A-made by Wilcoxon with other monitoring instrument and computer software [23–25]. The original microtremor information is stored in the frequency and time domains, but only the data set in time domain is considered for analysis as the characteristics of ground motion is the primary concern in this study.

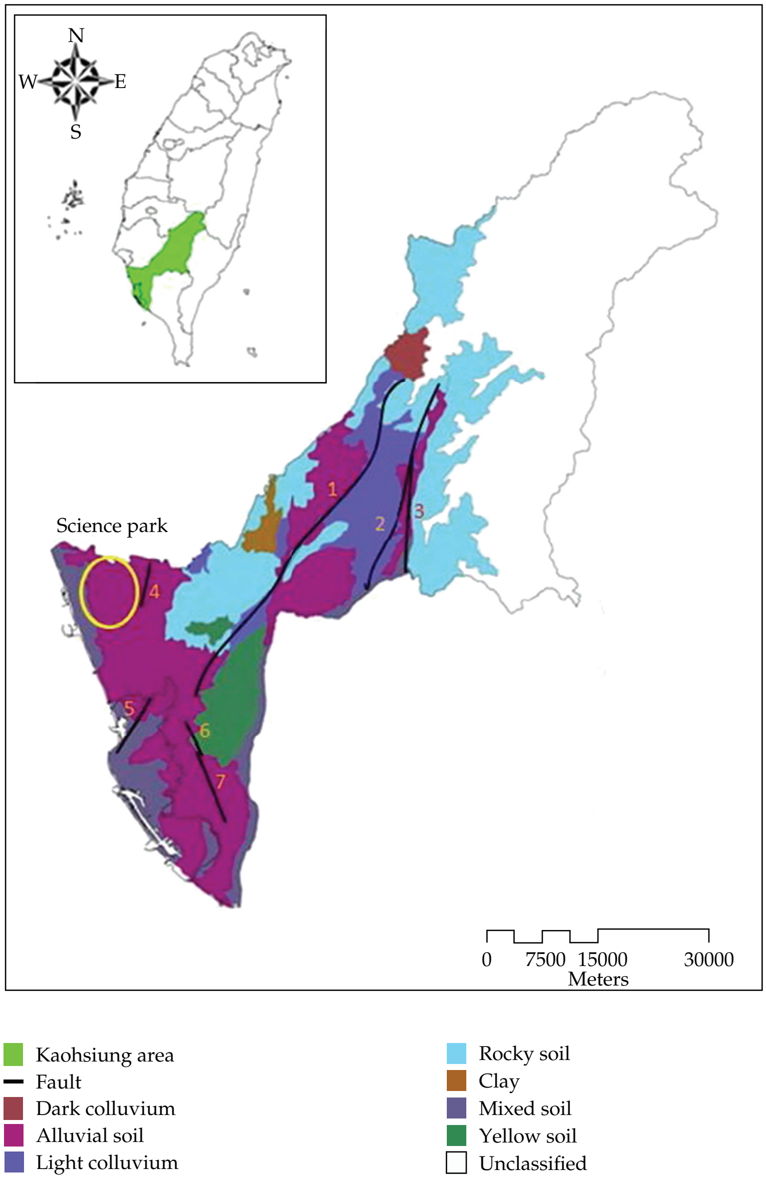

Figure 1: Distribution of faults and geological conditions in the Kaohsiung area.

From microtremor data collected at the four measuring stations, it can be found that the accelerations for measuring station MS1 are basically in the range 0.43 gal to 51.71 gal in both the east-west (EW) and north-south (NS) directions, and the results in vertical (V) direction are all smaller than 0.58 gal. For measuring station MS2, the results are 0.33 gal–24.1 gal, 0.27 gal–47.49 gal, and 0.78 gal–80.19 gal, in EW, NS, and V directions, respectively. For measuring station MS3, the results are 0.33 gal–5.03 gal, 0.27 gal–4.29 gal, and 0.31 gal–6.13 gal, in EW, NS, and V directions, respectively. For measuring station MS4, the results are 0.58 gal–45.85 gal, 0.34 gal–15.46 gal and 1.05 gal–49.52 gal, in EW, NS and V directions, respectively.

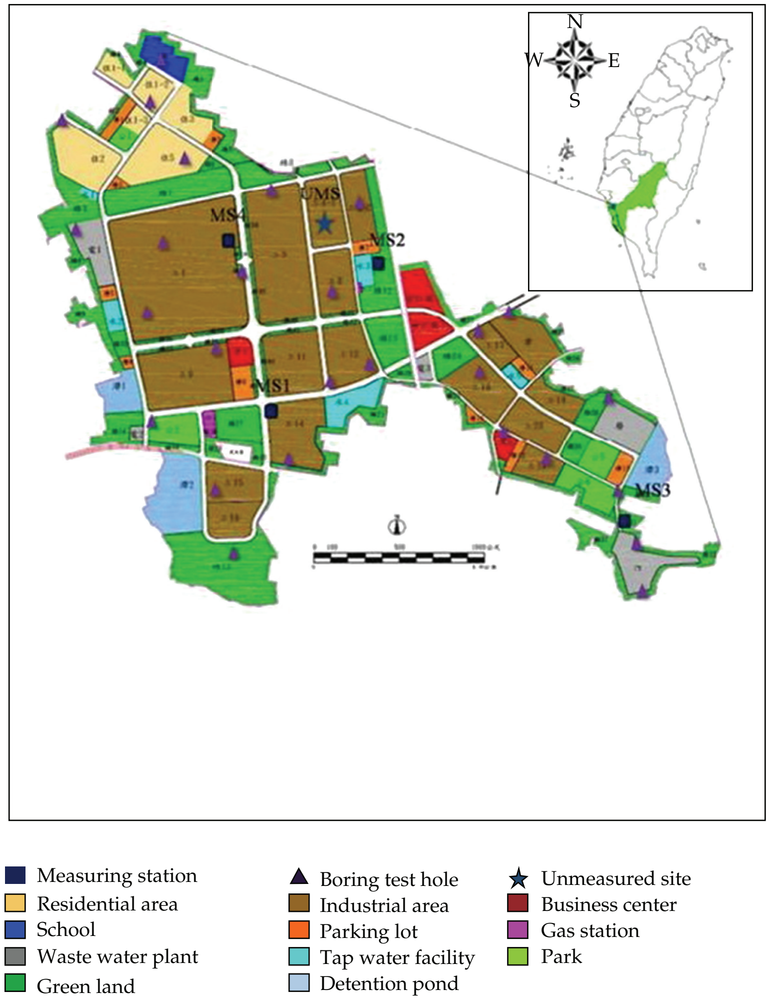

Figure 2: Sketch of research area in Kaohsiung science park.

The above numerical results show that the accelerations in the vertical direction are all relatively higher than those of the other directions, except at the measuring station MS1. As ambient vibrations may result from moving vehicles and construction work, some of the measured data may exhibit much higher values particularly between 8:00 AM and 5.00 PM, and these can affect the true microtremor response. Therefore, in this study, only ambient vibration data collected between 8:00 PM and 7:00 AM are taken for analysis so as to eliminate outside environmental factors as far as possible and increase the accuracy of natural microtremor response measurements.

For soil test data in the boreholes, the samplings are by auger boring method for soils above ground water level and by the method of wash boring for soils below ground water level. The laboratory tests conducted were for general physical, triaxial compression, shear strength, unconfined compressive strength, consolidation, compaction, California bearing ratio, resilient modulus, and groundwater quality. The in-situ tests were for lateral load and

Table 1: Typical values of soil parameters and random PGAs in the four directions.

Measuring station MS1							
STP-N	D50 (mm)	FL	DS (m)	PGA (EW)	PGA (NS)	PGA (V)	PGA (H)
7	0.022	2.20	412.19	1.82	1.69	4.44	2.48
2	0.006	1.75	412.21	2.09	2.02	4.53	2.90
14	0.030	2.00	412.23	1.23	2.38	7.40	2.67
10	0.062	1.27	412.26	1.89	1.74	4.45	2.56
6	0.006	1.57	412.31	8.38	12.6	5.35	15.13
34	0.150	1.17	412.36	1.89	3.61	5.67	4.07
37	0.160	1.17	412.42	1.44	1.96	4.33	2.43
43	0.160	1.22	412.50	0.61	0.84	4.25	1.03
27	0.160	1.01	412.58	0.58	1.29	4.43	1.41
13	0.026	1.79	412.67	2.50	4.84	8.43	5.44
...	...	...	...	...	...	...	...
Total data sets: 50; PGA unit: gal							

permeability. All of these test results can provide soil characteristics in detail for each of the drilling sites.

For a typical soil exploration and testing report in the research area, it can be seen that there are many items such as soil depth, soil profile, USCS (unified soil classification system) classification, standard penetration value, grain size analysis, water content, specific gravity, density, void ratio, liquid limit, plasticity index, and safety factor against liquefaction calculated from shear strength parameters [26]. Note that some of these soil test items may have their own physical meaning and also have a relationship with each other. Previous studies have found that the three important parameters relevant to the problem of actual earthquake response are STP-N, D50, and FL [27–29]. Hence, these three parameters, with ambient vibration surveys and distance between measuring station and test layer within bore hole (DS), are considered for developing a PGA prediction model by the neural network approach. Typical values for STP-N, D50, FL, DS, and PGA in the four directions are shown in Table 1.

4. Neural Network Model and Analysis of Prediction Results

In the field of computational intelligence, neural network approach is widely applied in various engineering applications as it has some attractive features such as easiness to implement, strong pattern recognition capability, and good prediction performance [30–32]. Basics of neural network modeling such as selecting a suitable architecture, learning algorithms, preprocessing of data, training, and testing of models have been comprehensively covered in many publications [33–35]. Thus, further discussions of this method and the use of associated software tool are not included here, but only some of the key points for using this computational technique that are relevant to the present research problem are addressed below.

To develop a neural network model, it is essential to determine the number of neurons in the input layer, the hidden layer, and the output layer. In this study, five soil boring test data in the neighborhood of each microtremor measuring station are used in developing the model. As mentioned in the previous section, the soil input parameters included are STP-N,

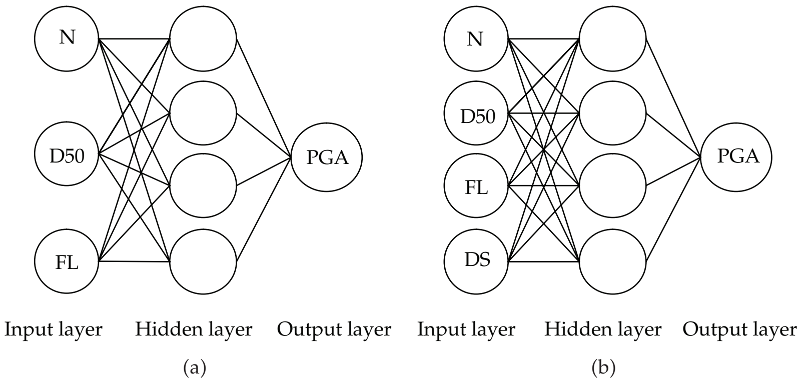

Figure 3: Sketch of neural network models $I_3H_4O_1$ (a) and $I_4H_4O_1$ (b).

D50, and FL. Therefore, a total of 50 data sets are available for the five bore holes, as each bore hole has a 20 m depth, and the soil profile is divided into 10 layers. If the distance parameter DS defined previously is also included, then there are four neurons in the input layer. The PGA in each of the different directions (EW, NS, V, and H) obtained from microtremor measurements is used as the target, resulting in only one neuron in the output layer. The number of neurons in the hidden layer needs to be selected to provide a relatively better performing neural network model.

It is better to examine the input soil data sets in advance to find a suitable neural network prediction model. Initially, we consider a neural network model without the distance parameter as shown in Figure 3(a), and divide the normalized data sets into three groups, where 70% is for training, 20% for verification, and 10% for testing. These three calculation stages are performed in Matlab toolbox with the "train," "adapt," and "simulate" functions [36, 37]. The computational experiments, with the use of correlation coefficient (R) as evaluation index, showed that the training result can achieve high R^2 values (from 0.656 to 0.900) with random data selection in the network calculation, but it has a poor performance in the verification cases (from 0.004 to 0.235) and the testing cases (from 0.001 to 0.361), as seen in Table 2. The poor performance of this model is due to random data selection with no rational basis for the association of the PGA values that are collected over time with the soil properties that are defined spatially within the bore holes. Thus, a rational basis for associating the data values for the input and output variables is required, and available domain knowledge is used as the basis for this association.

Because the STP-N value refers to soil hardness, it can play an important role in influencing the degree of liquefaction during an earthquake. Hence, by taking the STP-N as the primary factor and arranging its data set to increase from small to large values, with corresponding adjustments to the other input parameters, the target PGA data set is then arranged from large to small values. Again, without considering the distance parameter in the input layer, the performances of the neural network models with different number of neurons in the hidden layer were considered, and the model with four neurons in the hidden layer has a relatively better performance than the other models for all three calculation stages.

Note that only 50 data sets were used for developing neural network model in this study, so it is not suitable to choose too many layers or neurons in the hidden layer as it may cause ineffective learning during the training stage. The neural network model with

Table 2: Performance of neural network model ($I_3H_4O_1$) in different calculation stages.

Station	Training (R^2)				Verification (R^2)				Testing (R^2)			
	EW	NS	V	H	EW	NS	V	H	EW	NS	V	H
MS1	0.762	0.682	0.892	0.768	0.145	0.235	0.015	0.018	0.236	0.349	0.028	0.067
MS2	0.656	0.782	0.900	0.834	0.013	0.004	0.020	0.026	0.271	0.361	0.001	0.132
MS3	0.758	0.689	0.821	0.740	0.019	0.120	0.095	0.087	0.235	0.165	0.059	0.017

Table 3: Performance of neural network models ($I_4H_4O_1$) for the three measuring stations.

Station	Training (R^2)				Verification (R^2)				Testing (R^2)			
	EW	NS	V	H	EW	NS	V	H	EW	NS	V	H
MS1	0.996	0.999	0.999	0.999	0.820	0.766	0.897	0.926	0.847	0.760	0.706	0.847
MS2	0.997	0.992	0.995	0.997	0.918	0.656	0.726	0.761	0.815	0.728	0.737	0.601
MS3	0.996	0.999	1.000	0.999	0.877	0.878	0.736	0.893	0.774	0.755	0.697	0.762
Average	0.996	0.997	0.998	0.998	0.872	0.767	0.786	0.860	0.812	0.747	0.714	0.737

four neurons in the hidden layer is found to be more reliable and will also be used with the distance parameter in the input layer.

Table 3 shows the performance of the preferred neural network model $I_4H_4O_1$ (Figure 3(b) for the three microtremor measuring stations. It can be seen that the average R^2 values are quite high and up to 0.998 at the training stage. The average R^2 values range from 0.767 to 0.872 at the verification stage and from 0.714 to 0.812 at the testing stage, all exhibiting reasonably high coefficient of correlation between measurement and estimation. That is, the developed neural network model has a sufficient level of prediction capability and can be used for further investigation.

The above results from the neural network models demonstrate that microtremor measurements may have a relationship to the soil profile. It is crucial to check the capability and apply the developed model for predicting PGA at an unmeasured site. To perform this task, it can be assumed that microtremor measuring station MS4 is an unknown site, then the known PGA values at this station can be used for verifying the ability of neural network model. To estimate PGA in station MS4 from the three known stations MS1, MS2, and MS3, the straightforward method is by distributing the results of these three known stations with weighting factors based on the distances between stations and denoted here by “Model 1.”

Alternatively, a better way to estimate PGA at an unmeasured site is by taking a new set of soil data from five drilling holes nearby, and insert the data set solely to the neural network model developed for each of known measuring stations. Then by summing the results with weighting factors in accordance with the distances between the unknown site to the three known stations, the final estimation is obtained for the unknown site, and this method is denoted here as “Model 2”.

The comparison of prediction results for the two models and microtremor measurements in the different directions are shown in Figure 4. It can be seen that the neural network estimations are not too different for both models, but the results of “Model 2” seem to be slightly closer to the actual measurements. Because the total recorded ambient vibration surveys for MS4 are 70 data sets, the chosen 50 data sets cover a wider range of time interval compared to the other three stations. Therefore, some of the measurements exhibit higher values (PGA > 0.005 g), particularly for the east-west and vertical directions, and these may

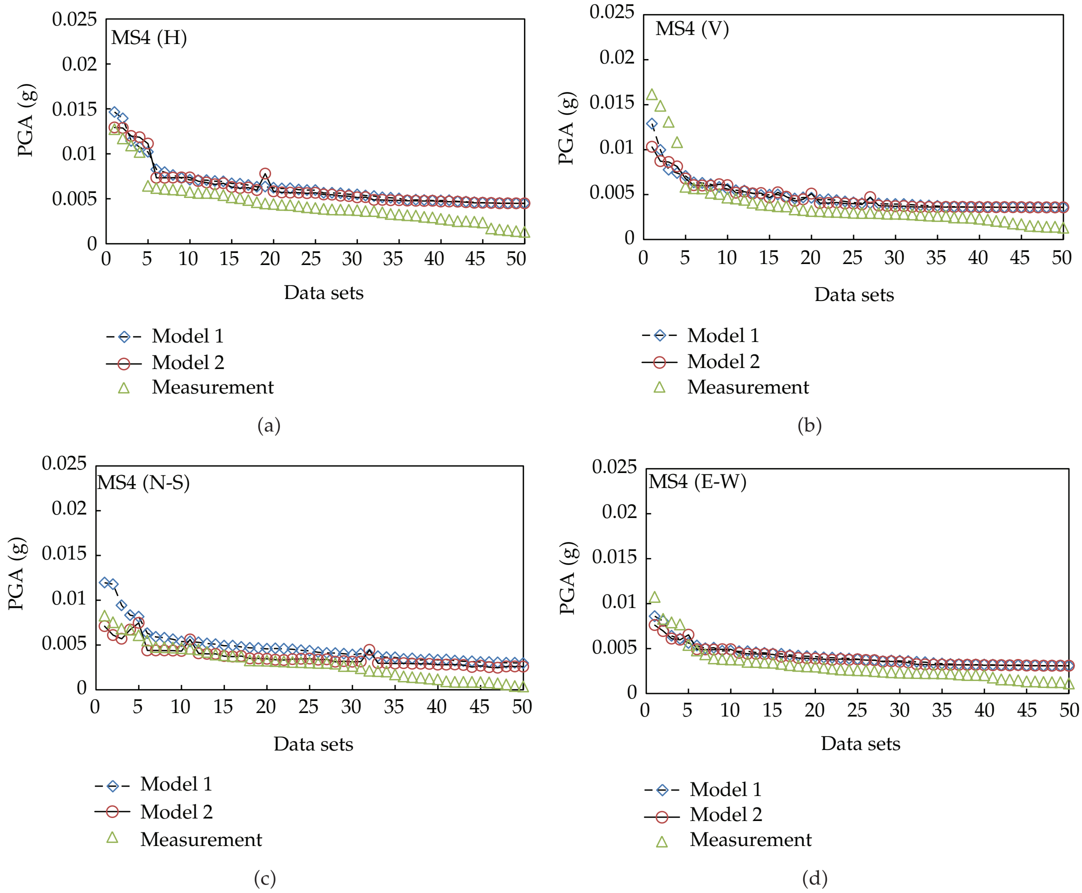

Figure 4: Comparison of neural network estimations and actual measurements.

cause an error in the prediction. In general, the performances of both neural network methods have reasonable accuracy and are acceptable for the problem considered. This comparison of results provides confidence for using this method for prediction of PGA in an unmeasured but important site.

Installation of ambient vibration survey instrument is usually at a place where the density of high-tech buildings is not too high. Thus, this method is suitable for an unmeasured site (UMS) shown in Figure 2, which is closer to several important industrial buildings and is possibly sensitive to natural microtremor. The present approach of developing neural network model is thus very useful for predicting PGA at this site with the use of new soil data sets from nearby five bore holes. Figure 5 shows the prediction result for the unmeasured site. It can be seen that both models exhibit similar predictions, but the curve obtained from "Model 2" is not as smooth as for "Model 1." The reason may be that the soil bore holes used in one of the known measuring stations, MS3, are too far away from the unmeasured site. Actually, this local instability problem is also found in the previous comparison shown in Figure 4, but it is believed that the present neural network model should still be sufficiently reliable. The sequential selection of the data sets for developing the neural network models results in the networks predicting the results for the test data

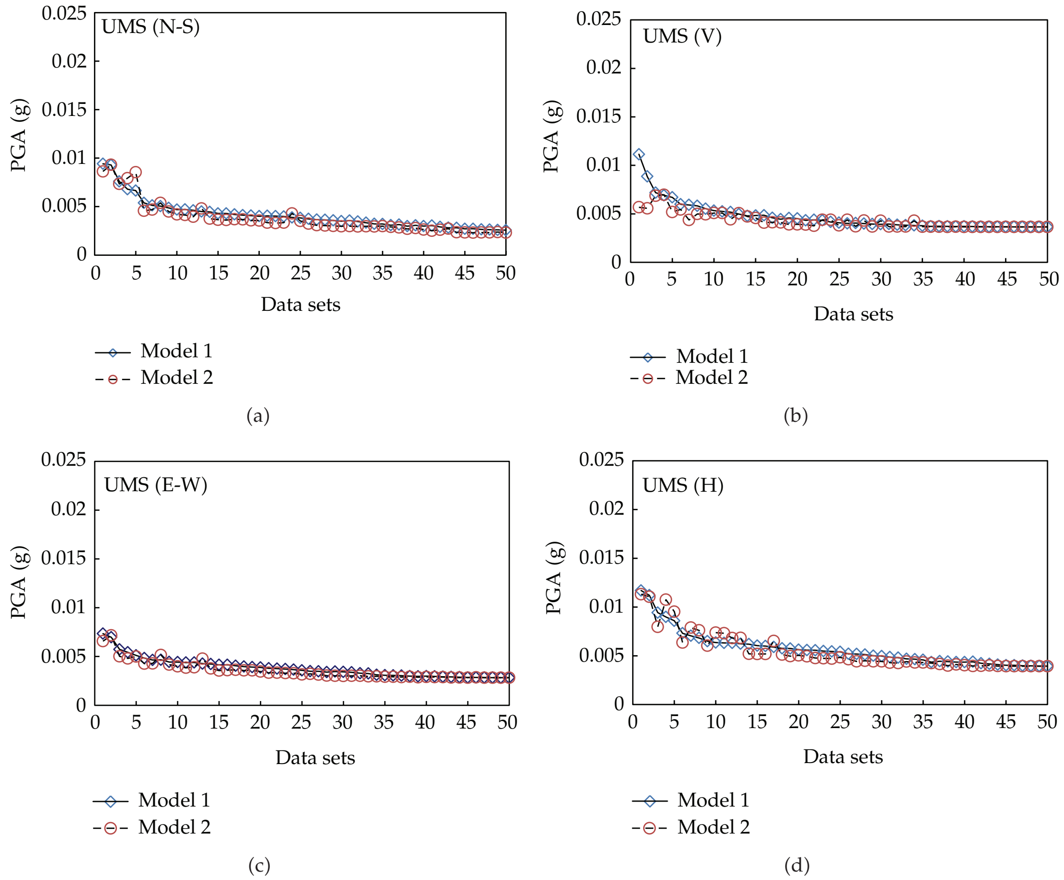

Figure 5: Predicted PGA at an unmeasured site in different directions.

sets by extrapolation. This accounts for the difference between the model predictions and the measurements on the right hand side of the curves in Figure 4. The predictions could be further improved by a random selection of data sets, for training, verification, and testing, after the reordering of the data sets as described previously.

If the measured results in the closest station MS2 are taken as reference, then the prediction result shows that PGA in vertical direction is larger than the other two directions, which is consistent with the measured results, and this is also true for predicted PGA in the horizontal direction. Since this science park is mostly alluvial soil, the prediction result is thus reasonable as there is no significant change of soil conditions in the unmeasured site. Overall, the neural network “Model 2” seems more preferable as the distance parameter and new soil data set are used in the calculation process of this model and, hence, may represent a more true response of the investigating site. In addition, it may be concluded that the model can be applied to predict PGA in any site of interest around the Kaohsiung science park.

5. Conclusion

Without a doubt, the seismic-related problems are very important research topic in the field of disaster prevention technology. This study presented a novel way of using neural network approach to develop a model for learning a relationship for linking two different types of parameters, that is, the ambient vibration measurement and the on-site soil boring test data. In addition, a weight-based neural network model is also developed for predicting peak ground acceleration at an unmeasured site, and is extendable for predicting natural microtremor on any other site of the science park investigated, as long as the soil conditions are suitably distributed.

Due to the limitation of soil boring test data for each microtremor measuring station, the present study picked up only fifty sets of peak ground acceleration from each of microtremor measuring station, to match with soil profiles from five drilling holes nearby for developing neural network model. More data sets might be required to develop a more accurate model for performing the prediction. Nonetheless, the results obtained in this study provide an insight into the seismic-related characteristics in the research area. One of the significant aspects of the present research is that even though the data collected at MS1, MS2, and MS3 are for different time periods, the three neural network models developed for these sites can be combined to predict the PGA distribution at a fourth site (MS4) for a different time period.

Further, it should be mentioned that the predicted results do prove the reliability of the developed model, but the choice of microtremor measurements with commonly found frequency or peak ground acceleration should be further investigated. This may occur repeatedly and hence can affect the accuracy of precision instrument and damage products such as semiconductors or biosensors during the manufacturing process within a high-tech company. Nevertheless, the method used in this study did provide a new way to treat this type of nonlinear problem, and may be applicable in other areas of interest around the world.

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Research Article

Finite-Time Robust Stabilization for Stochastic Neural Networks

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This paper is concerned with the finite-time stabilization for a class of stochastic neural networks (SNNs) with noise perturbations. The purpose of the addressed problem is to design a nonlinear stabilizator which can stabilize the states of neural networks in finite time. Compared with the previous references, a continuous stabilizator is designed to realize such stabilization objective. Based on the recent finite-time stability theorem of stochastic nonlinear systems, sufficient conditions are established for ensuring the finite-time stability of the dynamics of SNNs in probability. Then, the gain parameters of the finite-time controller could be obtained by solving a linear matrix inequality and the robust finite-time stabilization could also be guaranteed for SNNs with uncertain parameters. Finally, two numerical examples are given to illustrate the effectiveness of the proposed design method.

1. Introduction

Since the first paper of Ott et al. [1], a large number of monographs and papers studying the stabilization of the nonlinear systems without or with delays have been published [2–5]. These publications have developed many control techniques including continuous feedback and discontinuous feedback. Take [4] for example, the authors studied the pinning stabilization problem of linearly coupled stochastic neural networks, where a minimum number of controllers are used to force the NNs to the desired equilibrium point by fully utilizing the structure of the network.

On the other hand, the well-known Hopfield neural networks, Cohen-Grossberg neural networks and cellular neural networks [6–18], and so forth have been extensively studied in the past decades and successfully applied in many areas such as signal processing, combinatorial optimization, and pattern recognition. Specially, the stability of Hopfield neural networks has received much research attention since, when applied, the neural network is sometimes assumed to have only one globally stable equilibrium [7–9, 19, 20].

Until now, the stability analysis issues for many kinds of neural networks in the presence of stochastic perturbations and/or parameter uncertainties have attracted a lot of research attention. The reasons include twofold: (a) in real nervous systems, because of random fluctuations from the release of neurotransmitters, and other probabilistic causes, the synaptic transmission is indeed a noisy process; (b) the connection weights of the neurons depend on certain resistance and capacitance values that always exist uncertainties. Therefore, the robust stability has been studied for neural networks with parameter uncertainties [21–24] or external stochastic perturbations [7, 19, 25, 26]. However, to the best of the authors' knowledge, most literature regarding the stability of neural networks is based on the convergence time being large enough, even though we eagerly want the argued network states to become stable as quickly as possible in practical applications. In order to achieve faster stabilization speed and hope to complete stabilization in finite time rather than merely asymptotically [27], an effective method is using finite-time stabilization techniques, which have also demonstrated better robustness and disturbance rejection properties [28].

In this paper, we will focus on the finite-time robust stabilization for neural networks with both stochastic perturbations and parameter uncertainties. The difference of this paper lies in three aspects. First, based on the finite-time stability theorem of stochastic nonlinear systems [29], a new continuous finite-time stabilizator is proposed for a stochastic neural network (SNN). Moreover, in contrast to [30–33], we prove finite-time stabilization by constructing a suitable Lyapunov function and obtain some criteria which are easy to be satisfied. Second, the gain parameters in finite-time stabilizator are designed by solving a linear matrix inequality. Finally, a robust finite-time stabilizator for SNNs with parameter uncertainties is designed as well. Moreover, two illustrative examples are provided to show the effectiveness of the proposed designing.

The notations in this paper are quite standard. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n-dimensional Euclidean space and the set of all $n \times m$ real matrices. The superscript "T" denotes the transpose and the notation $X \geq Y$ (resp., $X > Y$), where X and Y are symmetric matrices, meaning that $X - Y$ is positive semidefinite (resp., positive definite). $\lambda_{\max}(M)$ and $\lambda_{\min}(M)$ denote the maximal and minimal eigenvalues of real matrix M. Let $(\mathbf{\Omega}, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and contains all $\mathbf{P}$-null sets). $\mathbb{E}\{x\}$ stands for the expectation of the stochastic variable x with respect to the given probability measure $\mathbf{P}$. I and 0 represent the identity matrix and a zero matrix, respectively; $\mathrm{diag}(\cdots)$ stands for a block-diagonal matrix. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

2. Model Formulation and Preliminaries

Some preliminary knowledge is presented in this section for the derivation of our main results. The deterministic NN can be described by the following differential equation:

$$\dot{x}(t) = -Ax(t) + Bf(x(t)) + J \tag{2.1}$$

or

$$\dot{x}_i(t) = -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t)) + J_i, \quad i = 1, 2, \ldots, n, \tag{2.2}$$

where $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \in \mathbb{R}^n$ is the vector of neuron states; n represents the number of neurons in the network; $A = \operatorname{diag}(a_1, a_2, \ldots, a_n)$ is an $n \times n$ constant diagonal matrix with $a_i > 0$, $i = 1, 2, \ldots, n$; $B = (b_{ij})_{n\times n}$ is an $n \times n$ interconnection matrix; $f(x) = (f_1(x_1), f_2(x_2), \ldots, f_n(x_n))^T : \mathbb{R}^n \to \mathbb{R}^n$ is a diagonal mapping, where f_i, $i = 1, 2, \ldots, n$ represents the neuron input-output activation and $J = (J_1, J_2, \ldots, J_n)^T$ is a constant external input vector.

To establish our main results, it is necessary to give the following assumption for system (2.1) or (2.2).

Assumption 2.1. The neuron activation function f of the NN (2.1) satisfies the following Lipschitz condition:

$$\|f_i(x) - f_i(y)\| \le M_i \|x - y\|, \quad \forall x, y \in \mathbb{R},\ i = 1, 2, \ldots, n, \tag{2.3}$$

where M_i is a positive constant for $i = 1, 2, \ldots, n$. For convenience, let $M = \operatorname{diag}\{M_1, M_2, \ldots, M_n\}$.

Because of the existence of environmental noise in real neural networks, the stochastic disturbances should be taken into account in the recurrent NN. For this purpose, we modify the system (2.1) as the following SNN:

$$dx(t) = [-Ax(t) + Bf(x(t)) + J]dt + h(t, x(t))d\omega(t), \tag{2.4}$$

where $\omega(t) = (\omega_1(t), \omega_2(t), \ldots, \omega_n(t))^T \in \mathbb{R}^n$ is an n-dimensional Brownian motion defined on the probability space $(\mathbf{\Omega}, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \mathbf{P})$ satisfying the usual conditions (i.e., the filtration contains all $\mathbf{P}$-null sets and is right continuous). The white noise $d\omega_i(t)$ is independent of $d\omega_j(t)$ for $i \ne j$. The intensity function h is the noise intensity function matrix satisfying the following condition:

$$\operatorname{trace}\left[h^T(t, x(t)) \cdot h(t, x(t))\right] \le \|M_h x(t)\|^2, \tag{2.5}$$

where M_h is a known constant matrix with compatible dimensions.

In this paper, we want to control the SNN (2.4) to the desired state x^*, which is an equilibrium point of NN (2.1). Based on the discussions in many other papers, the stochastic perturbation will vanish at this equilibrium point x^*, that is, $h(t, x^*) = 0$. Without loss of generality, one can shift the equilibrium point x^* to the origin by using the translation $y(t) = x(t) - x^*$, which derives the following stochastic dynamical system:

$$dy(t) = [-Ay(t) + Bg(y(t))]dt + h(t, y(t))d\omega(t), \tag{2.6}$$

where $g(y(t)) = f(x(t) + x^*) - f(x(t))$.

Consider the SNN (2.6) with parameter uncertainties: the parameter matrices A and B are unknown but bounded, which are assumed to satisfy

$$A \in A_I, \qquad B \in B_I, \tag{2.7}$$

where $A_I = \{A \mid 0 < \underline{a}_i \le a_i \le \overline{a}_i\}$, $B_I = \{B \mid \underline{b}_{ij} \le b_{ij} \le \overline{b}_{ij}\}$, and $i, j = 1, 2, \ldots, n$.

We denote that $\underline{A} = \operatorname{diag}(\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n)$, $\overline{A} = \operatorname{diag}(\overline{a}_1, \overline{a}_2, \ldots, \overline{a}_n)$, $\underline{B} = (\underline{b}_{ij})_{n\times n}$, $\overline{B} = (\overline{b}_{ij})_{n\times n}$, $A_0 = (1/2)(\overline{A} + \underline{A})$, $B_0 = (1/2)(\overline{B} + \underline{B})$, $A_1 = (1/2)(\overline{A} - \underline{A}) := \operatorname{diag}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n)$, $B_1 = (1/2)(\overline{B} - \underline{B}) := (\tilde{b}_{ij})_{n\times n}$, $E_A = \operatorname{diag}(\sqrt{\tilde{a}_1}, \sqrt{\tilde{a}_2}, \ldots, \sqrt{\tilde{a}_n})$, $E_B = [\sqrt{\tilde{b}_{11}}e_1, \ldots, \sqrt{\tilde{b}_{1n}}e_1, \ldots, \sqrt{\tilde{b}_{n1}}e_n, \ldots, \sqrt{\tilde{b}_{nn}}e_n]_{n\times n^2}$, and $F_B = [\sqrt{\tilde{b}_{11}}e_1, \ldots, \sqrt{\tilde{b}_{1n}}e_n, \ldots, \sqrt{\tilde{b}_{n1}}e_1, \ldots, \sqrt{\tilde{b}_{nn}}e_n]^T_{n^2\times n}$, where $e_i \in \mathbb{R}^n$ is the column vector with the ith element 1 and 0 elsewhere.

For $i, j = 1, 2, \ldots, n$, let

$$\begin{gathered} \boldsymbol{\Delta} = \left\{\Delta \in \mathbb{R}^{n\times n} \mid \Delta = \operatorname{diag}(\delta_1, \delta_2, \ldots, \delta_n),\ |\delta_i| \le 1\right\}, \\ \boldsymbol{\Omega} = \left\{\Omega \in \mathbb{R}^{n^2\times n^2} \mid \Omega = \operatorname{diag}(\omega_{11}, \ldots, \omega_{1n}, \ldots, \omega_{n1}, \ldots, \omega_{nn}),\ \left|\omega_{ij}\right| \le 1\right\}. \end{gathered} \tag{2.8}$$

Then, through simple manipulations, one has

$$A_I = \{A = A_0 + E_A \Delta E_A \mid \Delta \in \boldsymbol{\Delta}\}, \qquad B_I = \{B = B_0 + E_B \Omega F_B \mid \Omega \in \boldsymbol{\Omega}\}. \tag{2.9}$$

In order to stabilize the SNN (2.4) to the equilibrium point x^*, equivalently, one can stabilize the SNN (2.6) to the origin due to the transformation. Hence, in the remainder of this paper, a controller $u(t)$ will be designed for the stabilization of SNN (2.6) in mean square. The controlled SNN can be described by the following stochastic differential equation (SDE):

$$dy(t) = \left[-Ay(t) + Bg\left(y(t)\right) + u(t)\right]dt + h\left(t, y(t)\right)d\omega(t). \tag{2.10}$$

Similar to [30–33], the controller is designed as follows:

$$u(t) = -k_1 y(t) - k_2 \operatorname{sign}\left(y(t)\right)\left|y(t)\right|^{\alpha}, \tag{2.11}$$

where $|y(t)|^{\alpha} = (|y_1(t)|^{\alpha}, |y_2(t)|^{\alpha}, \ldots, |y_n(t)|^{\alpha})^T$, $\operatorname{sign}(y(t)) = \operatorname{diag}(\operatorname{sign}(y_1(t)), \operatorname{sign}(y_2(t)), \ldots, \operatorname{sign}(y_n(t)))$, constants k_1, k_2 are gain coefficients to be determined, and the real number α satisfies $0 < \alpha < 1$. In fact, here the continuous function $u(t)$ in the SNN (2.10) is the key point for ensuring the finite-time stabilization.

Obviously, when $0 < \alpha < 1$, the controller $u(t)$ is a continuous function with respect to y, which leads to the continuity of controlled system (2.10) with respect to the state $y(t)$ [30–33]. If $\alpha = 0$, $u(t)$ turns to be a discontinuous one, which has been considered in [34–36]. If $\alpha = 1$ in the controller (2.11), then it becomes the typical stabilization issues which only can realize an asymptotical stabilization in infinite time [3–5].

Similar to the definition of finite-time stability in probability [29], the finite-time stabilization in probability is given through the following definition.

Definition 2.2. The system (2.6) is said to be finite-time stabilized at the original point by the controller (2.11) in probability, that is, the controlled SNN (2.10) is finite-time stable in probability [37] if, for any initial state $x(0)$, there exists a finite-time function T_0 such that

$$\mathbf{P}\{\|y(t)\| = 0\} = 1, \quad \forall t \geq T_0, \tag{2.12}$$

where $T_0 = T_0(y(0), \omega) = \inf\{T \geq 0 : y(t) = 0, \forall t \geq T\}$ is called the stochastic setting time function satisfying $\mathbb{E}[T_0] < \infty$.

The following lemmas are needed for the derivation of our main results in this paper.

Lemma 2.3 (see [38]. (Itô's formula)). *Let $x(t)$ ba an n-dimensional Itô's process on $t \geq 0$ with the stochastic differential*

$$dx(t) = f(t)dt + g(t)d\omega(t). \tag{2.13}$$

Let $V(x(t), t) \in \mathcal{C}^{2,1}(\mathbb{R}^n \times \mathbb{R}^+; \mathbb{R}^+)$. Then, $V(x(t), t)$ is a real-valued Itô's process with its stochastic differential given by

$$\begin{aligned} dV(x(t), t) &= \mathcal{L}V(x(t), t)dt + V_x(x(t), t)g(t)d\omega(t), \\ \mathcal{L}V(x(t), t) &= V_t(x(t), t) + V_x(x(t), t)f(t) + \frac{1}{2}\,\mathrm{trace}\left(g^T(t)V_{xx}(x(t), t)g(t)\right), \end{aligned} \tag{2.14}$$

where $\mathcal{C}^{2,1}(\mathbb{R}^n \times \mathbb{R}^+)$ denotes the family of all real-valued functions $V(x(t), t)$ such that they are continuously twice differentiable in x and t.

Lemma 2.4 (see [29]). *Consider the stochastic differential equation* (2.13) *with $f(0) = 0$ and $g(0) = 0$ and assume system* (2.13) *has a unique global solution. If there exist real numbers $\eta > 0$ and $0 < \alpha < 1$, such that for the function $V(x)$ in Lemma 2.3,*

$$\mathcal{L}V(x) \leq -\eta(V(x))^{\alpha}, \tag{2.15}$$

then the origin of system (2.13) *is globally stochastically finite-time stable, and $\mathbb{E}[T_0] < (V(x_0))^{1-\alpha}/\eta(1-\alpha)$.*

Lemma 2.5 (see [39]). *If $a_1, a_2, \ldots, a_n$ are positive number and $0 < r < p$, then*

$$\left(\sum_{i=1}^{n} a_i^p\right)^{1/p} \leq \left(\sum_{i=1}^{n} a_i^r\right)^{1/r}. \tag{2.16}$$

Lemma 2.6 (Boyd et al. [40]). *If $\mathcal{U}$, $\mathcal{V}(t)$, and $\mathcal{W}$ are real matrices of appropriate dimension with $\mathcal{N}$ satisfying $\mathcal{N} = \mathcal{N}^T$, then*

$$\mathcal{N} + \mathcal{U}\mathcal{V}(t)\mathcal{W} + \mathcal{W}^T\mathcal{V}^T(t)\mathcal{U}^T < 0 \tag{2.17}$$

for all $\mathcal{V}^T(t)\mathcal{V}(t) \leq I$, if and only if there exists a positive constant λ, such that

$$\mathcal{N} + \lambda^{-1}\mathcal{U}\mathcal{U}^T + \lambda\mathcal{W}^T\mathcal{W} < 0. \tag{2.18}$$

3. Main Results

In this section, we first give some theorems in detail to guarantee that the original point of SNN (2.6) is stabilized in finite time, that is, the controlled system (2.10) with (2.11) is finite-time stable in probability. Then, for SNN (2.6) with parameter uncertainties, we provide a sufficient condition under which the controlled system (2.10) is robust finite-time stable in probability. Finally, the control gains k_1 and k_2 are designed by solving some linear matrix inequalities.

Theorem 3.1. *The controlled system* (2.10) *with* (2.11) *is finite-time stable in probability, if there exist a constant ε and a positive-definite matrix $P \in \mathbb{R}^{n\times n}$ such that*

$$-2PA - 2k_1P + \varepsilon^{-1}PBB^TP + \varepsilon M^TM + \lambda_{\max}(P)M_h^TM_h < 0. \tag{3.1}$$

Moreover, the upper bound of the stochastic settling time for stabilization can be in terms of the initial errors as $(\lambda_{\max}(P)/\lambda_{\min}(P)) \cdot (\|y(0)\|^{1-\alpha}/k_2(1-\alpha))$.

Proof. Consider the controlled system (2.10) with the controller (2.11), we have

$$dy(t) = \left[-(A + k_1I)y(t) + Bg(y(t)) - k_2\operatorname{sign}(y(t))|y(t)|^{\alpha}\right]dt + h(t, y(t))d\omega(t). \tag{3.2}$$

Next, we will prove system (3.2) is finite-time stable in probability based on Definition 2.2. To this end, choose the candidate Lyapunov function $V(y(t)) = y^T(t)Py(t)$ and calculate the time derivative of $V(y(t))$ along the trajectories of the augmented system (3.2). By the Itô's formula, we obtain the stochastic differential as

$$dV(y(t)) = \mathcal{L}V(y(t))dt + 2y^T(t)Ph(t, y(t))d\omega(t), \tag{3.3}$$

where

$$\begin{aligned}\mathcal{L}V(y(t)) &= 2y^T(t)P\left[(-A - k_1I)y(t) + Bg(t) - k_2\operatorname{sign}(y(t))|y(t)|^{\alpha}\right] + \operatorname{trace}\left[h^T(t)Ph(t)\right] \\ &= 2y^T(t)P(-A - k_1I)y(t) + 2y^T(t)PBg(t) + \operatorname{trace}\left[h^T(t)Ph(t)\right] \\ &\quad - 2k_2y^T(t)P\operatorname{sign}(y(t))|y(t)|^{\alpha}.\end{aligned} \tag{3.4}$$

From condition (2.3), using the inequality $x^T y + y^T x \leq \varepsilon x^T x + \varepsilon^{-1} y^T y$, where $\varepsilon > 0$ is an arbitrary constant, we have

$$\begin{aligned} 2y^T(t)PBg(t) &\leq \varepsilon^{-1} y^T(t)PBB^T Py(t) + \varepsilon g^T(t)g(t) \\ &\leq \varepsilon^{-1} y^T(t)PBB^T Py(t) + \varepsilon y^T(t)M^T My(t). \end{aligned} \tag{3.5}$$

Combining (2.5), (3.4)-(3.5) results in

$$\begin{aligned} \mathcal{L}V(y(t)) \leq y^T(t)&\left[-PA - A^T P - 2k_1 P + \varepsilon^{-1} PBB^T P + \varepsilon M^T M + \lambda_{\max}(P) M_h^T M_h\right] \\ &\times y(t) - 2k_2 \lambda_{\min}(P) \sum_{i=1}^{n} \left|y_i(t)\right|^{\alpha+1}. \end{aligned} \tag{3.6}$$

From $0 < \alpha < 1$ and Lemma 2.5, we get

$$\left(\sum_{i=1}^{n} \left|y_i(t)\right|^{\alpha+1}\right)^{1/(\alpha+1)} \geq \left(\sum_{i=1}^{n} \left|y_i(t)\right|^2\right)^{1/2}, \tag{3.7}$$

then,

$$\sum_{i=1}^{n} \left|y_i(t)\right|^{\alpha+1} \geq \left(\sum_{i=1}^{n} \left|y_i(t)\right|^2\right)^{(\alpha+1)/2} = \left[y^T(t)y(t)\right]^{(\alpha+1)/2}. \tag{3.8}$$

Thus, based on condition (3.1), taking the expectations on both sides of (3.3), we have

$$\begin{aligned} \mathbb{E}\{dV(y(t))\} &\leq -2k_2 \lambda_{\min}(P) \mathbb{E}\left\{\left[y^T(t)y(t)\right]^{(\alpha+1)/2}\right\} \\ &\leq -2k_2 \cdot \lambda_{\min}(P) [\lambda_{\max}(P)]^{-(\alpha+1)/2} \mathbb{E}\left\{V(y(t))^{(\alpha+1)/2}\right\}, \\ \text{and} \quad \mathbb{E}\left\{V^{(\alpha+1)/2}(y(0))\right\} &= \left(\mathbb{E}\{V(y(0))\}\right)^{(\alpha+1)/2}. \end{aligned} \tag{3.9}$$

By Lemma 2.4, $V(y(t))$ stochastically converges to zero in a finite time, that is, the controlled system (3.2) is finite-time stable in probability, and the settle time is upper bounded by

$$\begin{aligned} T_P &= \frac{[\lambda_{\max}(P)]^{(\alpha+1)/2} \cdot [V(y(0))]^{(1-\alpha)/2}}{2k_2 \cdot \lambda_{\min}(P) \cdot ((1-\alpha)/2)} \\ &\leq \frac{[\lambda_{\max}(P)]^{(\alpha+1)/2} [\lambda_{\max}(P)]^{(1-\alpha)/2} \|y(0)\|_2^{1-\alpha}}{\lambda_{\min}(P) \cdot k_2(1-\alpha)} \\ &= \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \cdot \frac{\|y(0)\|^{1-\alpha}}{k_2(1-\alpha)}. \end{aligned} \tag{3.10}$$

This completes the proof. □

Remark 3.2. The two gain parameters k_1 and k_2 in the controller $u(t)$ play different roles in ensuring the finite-time stability of the controlled system (3.2). We can see from Theorem 3.1 that, whether or not the controlled system (3.2) could realize the finite-time stability mainly depends on the value of k_1 and satisfies condition (3.1) but nothing on k_2. However, the size of the settle time depends on the value of k_2 but unrelated to k_1, the only requirement for the gain k_1 is satisfying condition (3.1).

Remark 3.3. In [31, 32, 35, 41], the candidate Lyapunov function $V(t)$ was chosen as a simple form of $V(t) = y^T(t)y(t)$ and then the upper bound of settle time turns to be $\|y(0)\|^{1-\alpha}/k_2(1-\alpha)$. In this paper, in order ro reduce some conservation of conditions in Theorem 3.1, a positive definite matrix parameter P is introduced such that condition (3.1) is easier to be satisfied. And the previous conclusions could be included by our results if the matrix $P = pI$ is taken, where p is a arbitrary constant, just as shown in the next corollary.

Corollary 3.4. *The controlled system* (3.2) *is finite-time stable in probability, if there exist two constants ε and p such that*

$$-2pA - 2k_1pI + \varepsilon^{-1}p^2BB^T + \varepsilon M^TM + pM_h^TM_h < 0. \tag{3.11}$$

Moreover, the upper bound of the settle time is

$$\mathbf{T} = \frac{\|y(0)\|^{1-\alpha}}{k_2(1-\alpha)}. \tag{3.12}$$

Our next goal is to deal with the design problem, that is, giving a practical design procedure for the controller gains: k_1 and k_2, such that the inequalities in Theorem 3.1 or Corollary 3.4 are satisfied. Obviously, those inequalities are difficult to solve, since they are nonlinear and coupled. A meaningful approach to tackling such a problem is to convert the nonlinearly coupled matrix inequalities into linear matrix inequalities (LMIs), while the controller gains are designed simultaneously.

Based on the discussion in Remark 3.2, the parameter gain k_2 is one of the primary factors that affect the size of the settle time, which is unrelated to condition (3.11). Hence, in the following discussion, we will fix the gain parameter k_2 and mainly focus on the design of control gain k_1. We claim that the desired controller gain k_1 can be designed if a linear matrix inequality is feasible.

Theorem 3.5. *For a fixed control gain k_2, the finite-time stabilization problem is solvable for the SNN* (2.6)*, if there exist three positive scalars p, K, and ε such that*

$$\begin{pmatrix} -2pA - 2KI + pM_h^TM_h & pB & \varepsilon M^T \\ \star & -\varepsilon I & 0 \\ \star & \star & -\varepsilon I \end{pmatrix} < 0. \tag{3.13}$$

Moreover, the control gain coefficient $k_1 = p^{-1}K$.

Proof. The result can be proved by pre- and post-multiplying the inequality (3.13) by the block-diagonal matrix $\mathrm{diag}\{I, \varepsilon^{-1/2}I, \varepsilon^{-1/2}I\}$ and then following from the famous Schur complement lemma and Corollary 3.4 and we omit it here.

Just as mentioned in Introduction, when modelling a dynamic system, one can hardly obtain an exact model. Specially, in practical implementation of neural networks, the firing rates and the weight coefficients of the neurons depend on certain resistance and capacitance values, which are subject to uncertainties. It is thus necessary to take parameter uncertainties into account in the considered neural network. In the following, we consider the robust finite-time stabilization issue for SNN (2.6) under the parametric uncertainties (2.7). □

Theorem 3.6. *The interval SNN* (3.2) *with uncertain parameters* (2.7) *is robust finite-time stable in probability, if there exist three constants ε, λ_1, λ_2 and a positive-definite matrix $P \in \mathbb{R}^{n\times n}$ such that*

$$\begin{pmatrix} \Phi & PB_0 & \varepsilon M^T & PE_A & PE_B \\ \star & -\varepsilon I + \lambda_2 F_B^T F_B & 0 & 0 & 0 \\ \star & \star & -\varepsilon I & 0 & 0 \\ \star & \star & \star & -\lambda_1 I & 0 \\ \star & \star & \star & \star & -\lambda_2 \mathbf{I} \end{pmatrix} < 0, \tag{3.14}$$

where $\Phi = -2PA_0 - 2k_1 I + PM_h^T M_h + \lambda_1 E_A^T E_A$ *and* $\mathbf{I} = \mathrm{diag}(I, I)$.

Proof. From Theorems 3.1 and 3.5, we know that the SNN (3.2) is finite-time stable in probability, if there exist a constant ε and a positive-definite matrix $P \in \mathbb{R}^{n\times n}$ such that the following LMI holds:

$$\begin{pmatrix} -2PA - 2KI + PM_h^T M_h & PB & \varepsilon M^T \\ \star & -\varepsilon I & 0 \\ \star & \star & -\varepsilon I \end{pmatrix} < 0. \tag{3.15}$$

Thus, for the uncertain parameters satisfying (2.7), we have

$$\begin{aligned} \Psi &= \begin{pmatrix} -2P(A_0 + E_A \Delta E_A) - 2KI + PM_h^T M_h & P(B_0 + E_B \Omega F_B) & \varepsilon M^T \\ \star & -\varepsilon I & 0 \\ \star & \star & -\varepsilon I \end{pmatrix} \\ &= \begin{pmatrix} -2PA_0 - 2KI + PM_h^T M_h & PB_0 & \varepsilon M^T \\ \star & -\varepsilon I & 0 \\ \star & \star & -\varepsilon I \end{pmatrix} + \begin{pmatrix} -2PE_A \Delta E_A & PE_B \Omega F_B & 0 \\ \star & 0 & 0 \\ \star & \star & 0 \end{pmatrix} < 0. \end{aligned} \tag{3.16}$$

For the second term in the above equality, it is easy to have

$$\begin{pmatrix} -2PE_A\Delta E_A & PE_B\Omega F_B & 0 \\ \star & 0 & 0 \\ \star & \star & 0 \end{pmatrix} = \begin{pmatrix} PE_A \\ 0 \\ 0 \end{pmatrix}\Delta\left(E_A \;\; 0 \;\; 0\right) + \begin{pmatrix} E_A \\ 0 \\ 0 \end{pmatrix}\Delta\left(E_A P \;\; 0 \;\; 0\right)$$
$$+ \begin{pmatrix} PE_B \\ 0 \\ 0 \end{pmatrix}\Omega\left(0 \;\; F_B \;\; 0\right) + \begin{pmatrix} 0 \\ F_B^T \\ 0 \end{pmatrix}\Omega\left(E_B^T P \;\; 0 \;\; 0\right). \tag{3.17}$$

Then, based on Lemma 2.6, (3.16) and (3.17), there exist two constants λ_1 and λ_2 such that

$$\Psi = \begin{pmatrix} -2PA_0 - 2KI + PM_h^T M_h & PB_0 & \varepsilon M^T \\ \star & -\varepsilon I & 0 \\ \star & \star & -\varepsilon I \end{pmatrix}$$
$$+ \begin{pmatrix} \lambda_1^{-1} PE_A E_A P + \lambda_1 E_A E_A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \lambda_2^{-1} PE_B E_B^T P & 0 & 0 \\ 0 & \lambda_2 F_B^T F_B & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.18}$$
$$< 0.$$

Then the result can be proved by the famous Schur complement lemma and condition (3.14). □

Corollary 3.7. *For a fixed control gain k_2, the finite-time robust stabilization problem is solvable for the SNN* (2.6) *with* (2.7), *if there exist five positive scalars p, K, ε, λ_1, and λ_2 such that*

$$\begin{pmatrix} \overline{\Phi} & pB_0 & \varepsilon M^T & pE_A & pE_B \\ \star & -\varepsilon I + \lambda_2 F_B^T F_B & 0 & 0 & 0 \\ \star & \star & -\varepsilon I & 0 & 0 \\ \star & \star & \star & -\lambda_1 I & 0 \\ \star & \star & \star & \star & -\lambda_2 \mathbf{I} \end{pmatrix} < 0, \tag{3.19}$$

where $\overline{\Phi} = -2pA_0 - 2KI + pM_h^T M_h + \lambda_1 E_A^T E_A$. Moreover, the control gain coefficient $k_1 = p^{-1}K$.

Proof. Let $P = pI$ and we can prove the result based on Theorem 3.6. □

4. Two Numerical Examples

Example 4.1. Consider the following stochastic neural network:

$$dx(t) = \left[-Ax(t) + Bf(x(t)) + J\right]dt + h(t, x(t))d\omega(t), \tag{4.1}$$

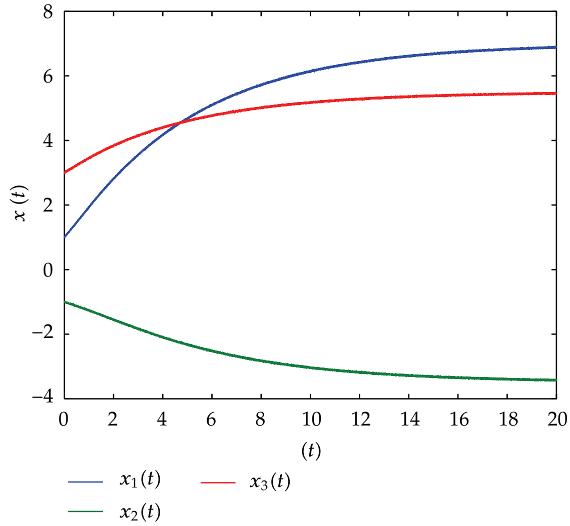

Figure 1: Trajectories of SNN (4.1) without any controller in Example 4.1.

where

$$A = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -0.2 & 0.2 \\ 0.1 & 1 & 0.2 \\ 0.3 & 0.2 & 1 \end{bmatrix}, \qquad J = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{4.2}$$

$h(t, x(t)) = \operatorname{diag}(\tanh(x_1(t)), \tanh(x_2(t)), \tanh(x_3(t)))$, and the activation function is taken as $f(s) = \tanh(s)$. Then, it is obvious that $M = M_h = I_3$, where I_3 is a 3×3 identity matrix. The SNN (4.1) with the above-given parameters is depicted in Figure 1 with initial values $x(0) = [1, -1, 3]^T$.

The stabilization controller is designed as

$$u(t) = -k_1 x(t) - k_2 \operatorname{sign}(x(t))|x(t)|^{\alpha}, \tag{4.3}$$

where the parameter α is chosen as 0.5 and the initial value $x(0) = [1, -1, 3]^T$. Then, $||x(0)|| = 3.3166$.

According to Theorem 3.5 and using Matlab LMI toolbox, we solve the LMI (3.13), and obtain $p = 2.8118$, $K = 10.8900$, and $\varepsilon = 10.1114$. Then by Theorem 3.5, the desired controller parameter can be designed as $k_1 = 3.8730$.

By choosing an arbitrary fixed gain k_2, SNN (4.1) can be stabilized in finite time in probability. Taking $k_2 = 1$, for example, we can obtain the upper bound of the settle time $\mathbf{T} = \|x(0)\|^{1-\alpha} / k_2(1-\alpha) = 3.6423$.

Simulation result is depicted in Figure 2, which shows the states $x_1(t)$, $x_2(t)$, and $x_3(t)$ of the controlled SNN (4.1). The simulation result has confirmed the effectiveness of our main results.

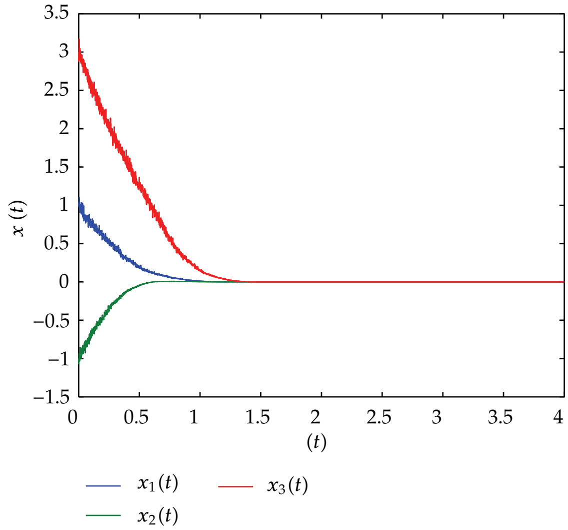

Figure 2: Trajectories of SNN (4.1) under the controller (4.3) with $k_2 = 1$ in Example 4.1.

Example 4.2. Still consider the SNN (4.1) with second-order parameter uncertainties:

$$\underline{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \overline{A} = \begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix}, \quad \underline{B} = \begin{bmatrix} 0.3 & 0.2 \\ 0.2 & 0.3 \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} 0.4 & 0.3 \\ 0.3 & 0.5 \end{bmatrix}. \tag{4.4}$$

The parameter α in the controller (4.3) is chosen as 0.5 and the initial value $x(0) = [1, -1]^T$. Then, $\|x(0)\| = 1.414$. According to Corollary 3.7 and using Matlab LMI toolbox, we solve the LMI (3.19) and obtain $p = 5.3906$, $K = 10.0457$, $\varepsilon = 12.5372$, $\lambda_1 = 21.7115$, and $\lambda_2 = 20.9350$. Then by Corollary 3.7, the desired controller parameter can be designed as $k_1 = 1.8635$.

By choosing an arbitrary fixed gain k_2, SNN (4.1) can be robustly stabilized in finite time in probability. Taking $k_2 = 1.5$, for example, we can obtain the upper bound of the settle time $\mathbf{T} = \|x(0)\|^{1-\alpha}/k_2(1-\alpha) = 1.5856$.

Simulation result is depicted in Figure 3, which shows the states $x_1(t)$ and $x_2(t)$ of the second-order controlled SNN (4.1). The simulation result has confirmed the effectiveness of our main results.

5. Conclusions

In this paper, we have investigated the issue of finite-time stabilization for SNNs with noise perturbations by constructing a continuous nonlinear stabilizator. Meanwhile, Based on the Lyapunov-Krasovskii functional method combining with the LMI techniques, a sufficient criterion is derived for the states of the augmented system to be global finite-time stable in probability. Subsequently, for SNNs with parameter uncertainties, the robust finite-time stabilizator could be designed well. Finally, two illustrative examples have been used to demonstrate the usefulness of the main results. It is expected that the theory established in

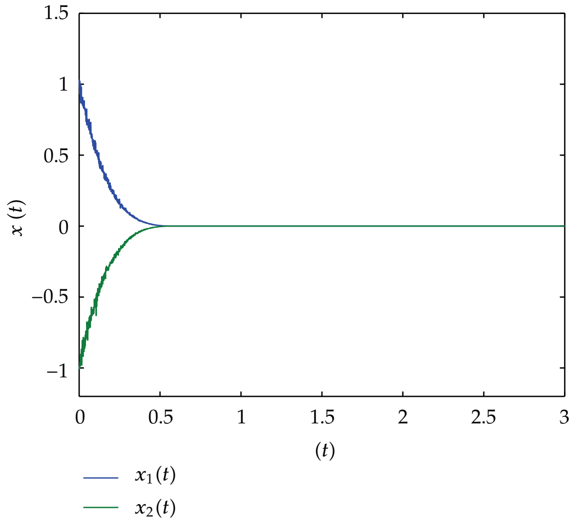

Figure 3: Trajectories of SNN (4.1) under the controller (4.3) with $k_2 = 1.5$ in Example 4.2.

this paper can be widely applied in delayed systems, particularly in those discontinuous cases. It will be an interesting topic in our future research.

Acknowledgments

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Research Article

Dynamical Analysis for High-Order Delayed Hopfield Neural Networks with Impulses

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The global exponential stability and uniform stability of the equilibrium point for high-order delayed Hopfield neural networks with impulses are studied. By utilizing Lyapunov functional method, the quality of negative definite matrix, and the linear matrix inequality approach, some new stability criteria for such system are derived. The results are related to the size of delays and impulses. Two examples are also given to illustrate the effectiveness of our results.

1. Introduction

In the last several years, Hopfield neural networks (HNNs) have received especially considerable attention due to their extensive applications in solving optimization problem, traveling salesman problem, and many other subjects, see [1–17]. However such neural networks are shown to have limitations such as limited capacity when used in pattern recognition problems, see [2, 3]. This led many researchers to use neural networks with high order connections. The high-order neural networks have stronger approximation property, faster convergence rate, greater storage capacity, and higher fault tolerance than lower-order neural networks. Recently, various results on stability of high-order delayed HNN are obtained, see [11–15]. For example, Lou and Cui [13] studied the global asymptotic stability of high-order HNN with time-varying delays by using Lyapunov method, linear matrix inequality (LMI), and analytic technique as follows:

$$\begin{aligned} x_i'(t) = & -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j\left(x_j(t)\right) + \sum_{j=1}^{n} b_{ij} g_j\left(x_j\left(t-\tau_j(t)\right)\right) \\ & + \sum_{j=1}^{n}\sum_{l=1}^{n} T_{ijl}\, g_l\left(x_l\left(t-\tau_l(t)\right)\right) g_j\left(x_j\left(t-\tau_j(t)\right)\right) + I_i, \quad t \ge t_0,\ i = 1,2,\dots,n. \end{aligned} \tag{1.1}$$

But the authors only obtained some global asymptotic stability criteria for the above high-order HNN. Those results cannot ensure the global exponential stability of the equilibrium point. It is well known that global exponential stability plays an important role in many areas such as designs and applications of neural networks and synchronization in secure communication [5, 17–23]. One purpose of this paper is to improve the results in [13]. We obtain several new criteria on global exponential stability and uniform stability for the above high-order HNN.

On the other hand, it is well known that the artificial electronic networks are subject to instantaneous perturbations and experience change of the state abruptly, that is, do exhibit impulsive effects. Such systems are described by impulsive differential systems which have been used successfully in modeling many practical problems arisen in the fields of natural sciences and technology, see [12, 24–30]. Hence, it is very important and, in fact, necessary to investigate the issue of the stability of high-order delayed HNN with impulses. However, to the best of the authors' knowledge, there are few results on the stability of high-order delayed HNN with impulses. In [12], Liu et al. obtained some sufficient conditions for ensuring global exponential stability of impulsive high order HNN with time-varying delays by using the method of Lyapunov functions.

The purpose of this paper is to present some new criteria concerning the global exponential stability and uniform stability for a class of high-order delayed HNN with impulses by utilizing Lyapunov functional method, the quality of negative definite matrix, and the linear matrix inequality approach. The conditions on impulses are different from that presented in [12]. The effects of impulses and delays on the solutions are stressed here. As a special case, several new criteria on global exponential stability and uniform stability for the corresponding high-order HNN without impulses (see [13]) are obtained. To illustrate the validity of those results, two examples are given to illustrate the effectiveness of the results obtained.

2. Preliminaries

Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}_+$ the set of nonnegative real numbers, $\mathbb{Z}_+$ the set of positive integers, and $\mathbb{R}^n$ the n-dimensional real space equipped with the Euclidean norm $\|\cdot\|$.

Consider the following high-order delayed HNN model with impulses

$$\begin{aligned} x_i'(t) = &-c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j\left(x_j(t)\right) + \sum_{j=1}^{n} b_{ij} g_j\left(x_j(t-\tau(t))\right) \\ &+ \sum_{j=1}^{n}\sum_{l=1}^{n} T_{ijl}\, g_l(x_l(t-\tau(t)))\, g_j\left(x_j(t-\tau(t))\right) + I_i, \quad t \neq t_k,\ t \geq t_0, \\ &\Delta x_i|_{t=t_k} = x_i(t_k) - x_i\left(t_k^-\right), \quad i \in \Lambda,\ k \in \mathbb{Z}_+, \end{aligned} \tag{2.1}$$

where $\Lambda = \{1, 2, \dots, n\}$, $n \geq 2$ corresponds to the number of units in a neural network; the impulse times t_k satisfy $0 \leq t_0 < t_1 < \cdots < t_k < \cdots$, $\lim_{k\to+\infty} t_k = +\infty$; x_i corresponds to the membrane potential of the unit i at time t; c_i is positive constant; f_j, g_j denote, respectively, the measures of response or activation to its incoming potentials of the unit j at time t and $t - \tau(t)$; T_{ijl} is the second-order synaptic weights of the neural networks; constant a_{ij} denotes the synaptic connection weight of the unit j on the unit i at time t; constant b_{ij} denotes the synaptic connection weight of the unit j on the unit i at time $t-\tau(t)$; I_i is the input of the unit i; $\tau(t)$ is the transmission delay such that $0<\tau(t)\le\tau$ and $\dot{\tau}(t)\le\rho<1$, $t\ge t_0$; τ, ρ are constants.

The initial conditions associated with system (2.1) are of the form

$$x(s)=\phi(s), \qquad s\in[t_0-\tau,t_0], \tag{2.2}$$

where $x(s)=(x_1(s),x_2(s),\dots,x_n(s))^T$, $\phi(s)=(\phi_1(s),\phi_2(s),\dots,\phi_n(s))^T\in PC([-\tau,0],\mathbb{R}^n)$, $PC([-\tau,0],\mathbb{R}^n)=\{\varphi:[-\tau,0]\to\mathbb{R}^n$ is continuous everywhere except at finite number of points t_k, at which $\varphi(t_k^+)$ and $\varphi(t_k^-)$ exist and $\varphi(t_k^+)=\varphi(t_k)\}$. For $\varphi\in PC([-\tau,0],\mathbb{R}^n)$, the norm of φ is defined by $\|\varphi\|_\tau=\sup_{-\tau\le\theta\le0}|\varphi(\theta)|$. For any $t_0\ge0$, let $PC_\delta(t_0)=\{\varphi\in PC([-\tau,0],\mathbb{R}^n):\|\varphi\|<\delta\}$.

Assume that $x^*=(x_1^*,x_2^*,\dots,x_n^*)^T$ is an equilibrium point of system (2.1). Impulsive operator is viewed as perturbation of the equilibrium point x^* of such system without impulsive effects. We assume that

$$\Delta x_i|_{t=t_k}=x_i(t_k)-x_i(t_k^-)=d_k^{(i)}(x_i(t_k^-)-x_i^*), \quad d_k^{(i)}\in\mathbb{R},\ i\in\Lambda,\ k\in\mathbb{Z}_+. \tag{2.3}$$

Since x^* is an equilibrium point of system (2.1), one can derive from system (2.1)-(2.2) that the transformation $y_i=x_i-x_i^*$, $i\in\Lambda$ transforms such system into the following system (for more details, please see papers [12, 13]):

$$\begin{aligned} y'(t)={}&-Cy(t)+AF(y(t))+BG(y(t-\tau(t)))\\ &+\Gamma^T T^\star G(y(t-\tau(t))),\quad t\ne t_k,\ t\ge t_0,\\ y(t_k)={}&D_k y(t_k^-),\quad k\in\mathbb{Z}_+,\\ y(t_0+\theta)={}&\varphi(\theta),\quad \theta\in[-\tau,0], \end{aligned} \tag{2.4}$$

where

$$\begin{gathered} \varphi(\theta)=x(t_0+\theta)-x^*, \qquad y(t)=(y_1(t),y_2(t),\dots,y_n(t))^T,\\ y(t-\tau(t))=(y_1(t-\tau(t)),y_2(t-\tau(t)),\dots,y_n(t-\tau(t)))^T,\\ F(y(t))=[F_1(y_1(t)),F_2(y_2(t)),\dots,F_n(y_n(t))]^T,\\ G(y(t-\tau(t)))=[G_1(y_1(t-\tau(t))),G_2(y_2(t-\tau(t))),\dots,G_n(y_n(t-\tau(t)))]^T,\\ F_j(y_j(t))=f_j\left(x_j^*+y_j(t)\right)-f_j\left(x_j^*\right), \qquad G_j(y_j(t-\tau(t)))=g_j\left(x_j^*+y_j(t-\tau(t))\right)-g_j\left(x_j^*\right),\\ C=\operatorname{diag}[c_1,c_2,\dots,c_n], \qquad A=(a_{ij})_{n\times n}, \qquad B=(b_{ij})_{n\times n}, \qquad T_i=(T_{ijl})_{n\times n},\\ T^\star=\left(T_1+T_1^T,T_2+T_2^T,\dots,T_n+T_n^T\right)^T, \end{gathered}$$

$$\Gamma = \operatorname{diag}[\varsigma, \varsigma, \dots, \varsigma], \qquad \varsigma = (\varsigma_1, \varsigma_2, \dots, \varsigma_n)^T,$$
$$D_k = \operatorname{diag}\left[1 + d_k^{(1)}, 1 + d_k^{(2)}, \dots, 1 + d_k^{(n)}\right], \tag{2.5}$$

in which ς_l is a real value between $g_l(x_l(t-\tau(t)))$ and $g_l(x_l^*)$, $l \in \Lambda$.

Remark 2.1. Obviously, $(0, 0, \dots, 0)^T$ is an equilibrium point of (2.4). Therefore, there exists at least one equilibrium point of system (2.1). So, the stability analysis of the equilibrium point x^* of (2.1) can now be transformed to the stability analysis of the trivial solution $y = 0$ of (2.4).

In the following, the notations X^T and X^{-1} mean the transpose of and the inverse of a square matrix X. We will use the notation $X > 0$ (or $X < 0$, $X \geq 0$, $X \leq 0$) to denote that the matrix X is a symmetric and positive definite (negative definite, positive semidefinite, negative semidefinite) matrix. Let $\lambda_{\max}(X)$, $\lambda_{\min}(X)$, respectively, denote the largest and smallest eigenvalue of matrix X.

Throughout this paper, we assume that there exist constants $\chi_i > 0, M, N \geq 0$ such that $|g_i(x_i)| \leq \chi_i$, $i \in \Lambda$, $F^T(y)F(y) \leq My^Ty$, $G^T(y)G(y) \leq Ny^Ty$.

We introduce some definitions as follows.

Definition 2.2 (see [5]). Leting $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$, for any $(t, x) \in [t_{k-1}, t_k) \times \mathbb{R}^n$, the upper right-hand Dini derivative of $V(t, x)$ along the solution of (2.4) is defined by

$$D^+V(t,x) = \limsup_{h \to 0^+} \frac{1}{h}\Big\{ V\Big[t + h, x + h\Big(-Cy(t) + AF(y(t)) + BG(y(t-\tau(t))) + \Gamma^T T^\star G(y(t-\tau(t)))\Big)\Big] - V(t,x)\Big\}. \tag{2.6}$$

Definition 2.3 (see [25]). Assume $y(t) = y(t_0, \varphi)(t)$ is the solution of (2.4) through (t_0, φ). Then the zero solution of (2.4) is said to be uniformly stable, if, for any $\varepsilon > 0$ and $t_0 \geq 0$, there exists some $\delta = \delta(\varepsilon) > 0$ such that $\varphi \in PC_\delta(t_0)$ implies $\|y(t)\| < \varepsilon$, $t \geq t_0$.

Definition 2.4 (see [5]). The equilibrium point x^* of the system (2.1) is globally exponentially stable, if there exists constant $\mu > 0, \mathbb{M} \geq 1$ such that, for any initial value ϕ,

$$\left\| x(t_0, \phi)(t) - x^* \right\| < \mathbb{M} \left\| \phi - x^* \right\|_\tau e^{-\mu(t-t_0)}, \quad t \geq t_0. \tag{2.7}$$

Next, in order to obtain our results, we need to establish the following lemma.

Lemma 2.5 (see [13]). *For any vectors $a, b \in \mathbb{R}^n$, the inequality*

$$\pm 2a^Tb \leq a^TXa + b^TX^{-1}b \tag{2.8}$$

holds, in which X is any $n \times n$ matrix with $X > 0$.

Lemma 2.6 (see [31]). *Let $X \in R^{n\times n}$, then*

$$\lambda_{\min}(X)a^T a \leq a^T X a \leq \lambda_{\max}(X)a^T a \tag{2.9}$$

for any $a \in \mathbb{R}^n$ if X is a symmetric matrix.

3. Main Results

In this section, some sufficient delay-dependent conditions of global exponential stability and uniform stability for system (2.1) are obtained.

Theorem 3.1. *Assume that there exist constants $\varepsilon^\star > 0$, $\delta^\star \in [0, \varepsilon^\star)$ and $n\times n$ symmetric and positive definite matrices P, Q_1, Q_2 such that*

(i)

$$\begin{aligned}
&\varepsilon^\star P - PC - CP + PAQ_1^{-1}A^T P + \lambda_{\max}(Q_1)ME + \frac{N\lambda_{\max}\left(Q_2 + T^{\star T}T^\star\right)}{1-\rho}E \\
&\quad + e^{\tau\varepsilon^\star} PBQ_2^{-1}B^T P + e^{\tau\varepsilon^\star}\|\chi\|^2 P^2 \leq 0,
\end{aligned} \tag{3.1}$$

where $\chi = (\chi_1, \chi_2, \ldots, \chi_n)^T$,

(ii) *there exists constant $\mathbb{W} \geq 0$ such that*

$$\sum_{k=1}^{m} \ln \max\{\eta_k, 1\} - \delta^\star(t_m - t_0) \leq \mathbb{W} \quad \forall m \in \mathbb{Z}_+ \text{ holds,} \tag{3.2}$$

where η_k is the largest eigenvalue of $P^{-1}D_k P D_k$, $k \in \mathbb{Z}_+$.

Then the equilibrium point of the system (2.1) is globally exponentially stable and the approximate exponential convergent rate is $(\varepsilon^\star - \delta^\star)/2$.

Proof. We only need to prove that the zero solution of system (2.4) is globally exponentially stable. For any $t_0 \geq 0$, let $y(t) = y(t_0, \varphi)(t)$ be a solution of (2.4) through (t_0, φ).

Consider the Lyapunov functional as follows:

$$V(t) = e^{\varepsilon^\star t} y^T(t) P y(t) + \frac{1}{1-\rho}\int_{t-\tau(t)}^{t} e^{\varepsilon^\star s} G^T(y(s))\left(Q_2 + T^{\star T}T^\star\right)G(y(s))ds, \tag{3.3}$$

then we have

$$\begin{aligned}
\lambda_{\min}(P)e^{\varepsilon^\star t}\|y(t)\|^2 &< V(t)\\
&\le \lambda_{\max}(P)e^{\varepsilon^\star t}\|y(t)\|^2 + \frac{\lambda_{\max}\left(Q_2 + T^{\star T}T^\star\right)Ne^{\varepsilon^\star t}\left(1 - e^{-\varepsilon^\star \tau(t)}\right)}{\varepsilon^\star(1-\rho)}\|y(t)\|_\tau^2\\
&\le \left(\lambda_{\max}(P) + \frac{\lambda_{\max}\left(Q_2 + T^{\star T}T^\star\right)N\left(1 - e^{-\varepsilon^\star \tau}\right)}{\varepsilon^\star(1-\rho)}\right)e^{\varepsilon^\star t}\|y(t)\|_\tau^2.
\end{aligned} \tag{3.4}$$

By Lemma 2.5, we get

$$\begin{aligned}
2y^T(t)PAF(y(t)) &= 2F^T(y(t))A^TPy(t)\\
&\le F^T(y(t))Q_1F(y(t)) + y^T(t)PAQ_1^{-1}A^TPy(t)\\
&\le \lambda_{\max}(Q_1)F^T(y(t))F(y(t)) + y^T(t)PAQ_1^{-1}A^TPy(t)\\
&\le y^T(t)\left[PAQ_1^{-1}A^TP + \lambda_{\max}(Q_1)ME\right]y(t),
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
2y^T(t)PBG(y(t-\tau(t))) &= 2G^T(y(t-\tau(t)))B^TPy(t)\\
&= 2\left[G(y(t-\tau(t)))\sqrt{e^{-\tau\varepsilon^\star}}\right]^T\left(B^TPy(t)\sqrt{e^{\tau\varepsilon^\star}}\right)\\
&\le e^{-\tau\varepsilon^\star}G^T(y(t-\tau(t)))Q_2G(y(t-\tau(t)))\\
&\quad + e^{\tau\varepsilon^\star}y^T(t)PBQ_2^{-1}B^TPy(t).
\end{aligned} \tag{3.6}$$

On the other hand, since $\Gamma^T\Gamma = ||\varsigma||^2E$ and $||\varsigma|| \le ||\chi||$, then we have

$$y^T(t)P\Gamma^T\Gamma Py(t) \le \|\chi\|^2y^T(t)P^2y(t), \tag{3.7}$$

where $\chi = (\chi_1, \chi_2, \dots, \chi_n)^T$.

Thus, we obtain

$$\begin{aligned}
2y^T(t)P\Gamma^TT^\star G(y(t-\tau(t))) &= 2G^T(y(t-\tau(t)))T^{\star T}\Gamma Py(t)\\
&= 2\left[T^\star G(y(t-\tau(t)))\sqrt{e^{-\tau\varepsilon^\star}}\right]^T\left(\Gamma Py(t)\sqrt{e^{\tau\varepsilon^\star}}\right)\\
&\le e^{-\tau\varepsilon^\star}G^T(y(t-\tau(t)))T^{\star T}T^\star G(y(t-\tau(t))) + e^{\tau\varepsilon^\star}y^T(t)P\Gamma^T\Gamma Py(t)\\
&\le e^{-\tau\varepsilon^\star}G^T(y(t-\tau(t)))T^{\star T}T^\star G(y(t-\tau(t))) + e^{\tau\varepsilon^\star}\|\chi\|^2y^T(t)P^2y(t).
\end{aligned} \tag{3.8}$$

Now we consider the derivation of V along the trajectories of system (2.4), for $t \in [t_k, t_{k+1})$, $k \in \mathbb{Z}_+$,

$$\begin{aligned}
D^+V(t)|_{(2.3)} = {} & e^{\varepsilon^\star t}\varepsilon^\star y^T(t)Py(t) + e^{\varepsilon^\star t}\left\{y'^T(t)Py(t) + y^T(t)Py'(t)\right\} \\
& + \frac{1}{1-\rho}e^{\varepsilon^\star t}G^T(y(t))\left(Q_2 + T^{\star T}T^\star\right)G(y(t)) \\
& - \frac{1-\dot{\tau}(t)}{1-\rho}e^{\varepsilon^\star(t-\tau(t))}G^T(y(t-\tau(t)))\left(Q_2 + T^{\star T}T^\star\right)G(y(t-\tau(t))) \\
\leq {} & e^{\varepsilon^\star t}\varepsilon^\star y^T(t)Py(t) + e^{\varepsilon^\star t}\left\{y^T(t)(-CP - PC)y(t) + 2y^T(t)PAF(y(t))\right. \\
& \qquad + 2y^T(t)PBG(y(t-\tau(t))) \\
& \qquad \left. +2y^T(t)P\Gamma^T T^\star G(y(t-\tau(t)))\right\} \\
& + \frac{1}{1-\rho}e^{\varepsilon^\star t}G^T(y(t))\left(Q_2 + T^{\star T}T^\star\right)G(y(t)) \\
& - e^{\varepsilon^\star(t-\tau)}G^T(y(t-\tau(t)))\left(Q_2 + T^{\star T}T^\star\right)G(y(t-\tau(t))) \\
\leq {} & e^{\varepsilon^\star t}y^T(t)\left\{\varepsilon^\star P - PC - CP + PAQ_1^{-1}A^TP + \lambda_{\max}(Q_1)ME\right. \\
& \qquad \left. + \frac{N\lambda_{\max}\left(Q_2 + T^{\star T}T^\star\right)}{1-\rho}E + e^{\tau\varepsilon^\star}PBQ_2^{-1}B^TP + e^{\tau\varepsilon^\star}\|\chi\|^2P^2\right\}y(t) \\
\leq {} & 0.
\end{aligned} \tag{3.9}$$

Moreover, we note

$$\begin{aligned}
V(t_k) &= e^{\varepsilon^\star t_k}y^T(t_k)Py(t_k) + \frac{1}{1-\rho}\int_{t_k-\tau(t_k)}^{t_k} e^{\varepsilon^\star s}G^T(y(s))\left(Q_2 + T^{\star T}T^\star\right)G(y(s))ds \\
&= e^{\varepsilon^\star t_k}y^T(t_k^-)D_kPD_ky(t_k^-) + \frac{1}{1-\rho}\int_{t_k^- -\tau(t_k^-)}^{t_k^-} e^{\varepsilon^\star s}G^T(y(s))\left(Q_2 + T^{\star T}T^\star\right)G(y(s))ds \\
&\leq e^{\varepsilon^\star t_k}\eta_k y^T(t_k^-)Py(t_k^-) + \frac{1}{1-\rho}\int_{t_k^- -\tau(t_k^-)}^{t_k^-} e^{\varepsilon^\star s}G^T(y(s))\left(Q_2 + T^{\star T}T^\star\right)G(y(s))ds \\
&\leq \max\{\eta_k, 1\}V(t_k^-).
\end{aligned} \tag{3.10}$$

By simple induction, considering (3.4)–(3.10), we get, for $k \geq 1$,

$$\lambda_{\min}(P)e^{\varepsilon^{\star}t}\|y(t)\|^2 \leq V(t) \leq V(t_0)\prod_{t_0<t_k\leq t}\max\{\eta_k, 1\}. \tag{3.11}$$

On the other hand, from (3.4), we get

$$V(t_0) \leq \left(\lambda_{\max}(P) + \frac{\lambda_{\max}\left(Q_2 + T^{\star T}T^{\star}\right)N\left(1 - e^{-\varepsilon^{\star}\tau}\right)}{\varepsilon^{\star}(1-\rho)}\right)e^{\varepsilon^{\star}t_0}\|\varphi\|_{\tau}^2. \tag{3.12}$$

Substituting the above inequality into (3.11), we obtain

$$\|y(t)\|^2 \leq \left(\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} + \frac{\lambda_{\max}\left(Q_2 + T^{\star T}T^{\star}\right)N\left(1 - e^{-\varepsilon^{\star}\tau}\right)}{\varepsilon^{\star}(1-\rho)\lambda_{\min}(P)}\right)e^{-\varepsilon^{\star}(t-t_0)}\|\varphi\|_{\tau}^2\prod_{t_0<t_k\leq t}\max\{\eta_k, 1\}. \tag{3.13}$$

In view of condition (ii), we furthermore have

$$\|y(t)\| \leq \mathbb{M}e^{-((\varepsilon^{\star}-\delta^{\star})/2)(t-t_0)}\|\varphi\|_{\tau}, \quad t \geq t_0, \tag{3.14}$$

where

$$\mathbb{M} = \sqrt{\left(\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} + \frac{\lambda_{\max}\left(Q_2 + T^{\star T}T^{\star}\right)N\left(1 - e^{-\varepsilon^{\star}\tau}\right)}{\varepsilon^{\star}(1-\rho)\lambda_{\min}(P)}\right)e^{\mathbb{W}}} \geq 1. \tag{3.15}$$

Hence, the zero solution of system (2.4) is globally exponentially stable; that is, the equilibrium point of system (2.1) is globally exponentially stable and the approximate exponential convergent rate is $(\varepsilon^{\star} - \delta^{\star})/2$. The proof of Theorem 3.1 is therefore complete. □

Remark 3.2. In Theorem 3.1, we find that condition (i) can be replaced by

$$\begin{aligned}&\varepsilon^{\star}P - PC - CP + PAQ_1^{-1}A^TP + \lambda_{\max}(Q_1)ME + \frac{N\lambda_{\max}(Q_2)}{1-\rho}E + \frac{N\lambda_{\max}\left(T^{\star T}T^{\star}\right)}{1-\rho}E \\ &\quad + e^{\tau\varepsilon^{\star}}PBQ_2^{-1}B^TP + e^{\tau\varepsilon^{\star}}\|\chi\|^2P^2 \leq 0.\end{aligned} \tag{3.16}$$

Leting $P = Q_1 = Q_2 = E$ in Theorem 3.1, then we have the following.

Corollary 3.3. *Assume that there exist constants $\varepsilon^\star > 0$, $\delta^\star \in [0, \varepsilon^\star)$ such that*

(i)

$$\lambda_{\max}^\star \le -\varepsilon^\star - M - \frac{N}{1-\rho} - \frac{N\lambda_{\max}\left(T^{\star T}T^\star\right)}{1-\rho} - e^{\tau\varepsilon^\star}\|\chi\|^2, \tag{3.17}$$

where $\lambda_{\max}^\star$ is the largest eigenvalue of $-2C + AA^T + e^{\tau\varepsilon^\star}BB^T$;

(ii) *there exists constant $\mathbb{W} \ge 0$ such that*

$$\sum_{k=1}^{m} \ln \max\left\{\max_{i\in\Lambda}\left(1+d_k^{(i)}\right)^2, 1\right\} - \delta^\star(t_m - t_0) < \mathbb{W} \quad \forall m \in \mathbb{Z}_+ \text{ holds}. \tag{3.18}$$

The equilibrium point of the system (2.1) *is globally exponentially stable and the approximate exponential convergent rate is* $(\varepsilon^\star - \delta^\star)/2$.

Furthermore, if $d_k^{(i)} \in [-2, 0]$ in Corollary 3.3, then we have the following result.

Corollary 3.4. *The equilibrium point of the system* (2.1) *is globally exponentially stable, if $d_k^{(i)} \in [-2, 0]$, and there exists constant $\varepsilon^\star > 0$ such that*

$$-2C + AA^T + e^{\tau\varepsilon^\star}BB^T + \left[\varepsilon^\star + M + \frac{N\left(1+\lambda_{\max}\left(T^{\star T}T^\star\right)\right)}{1-\rho} + e^{\tau\varepsilon^\star}\|\chi\|^2\right]E \le 0. \tag{3.19}$$

Remark 3.5. In fact, Theorem 3.1 implies that if $\sup_{k\in\mathbb{Z}_+}\prod_{s=1}^{k}(1+\beta_s^{(i)})^2 < \infty$, then one may choose $\delta^\star = 0$. On the other hand, Luo an Cui [13] obtained some results on global asymptotic stability. However, those results cannot ensure the global exponential stability. Let $d_k^{(i)} = 0$ (i.e., $D_k = E$) in Corollary 3.4, then we can obtain the desirable result as follows.

Corollary 3.6. *The equilibrium point of the system* (2.1) *without impulses is globally exponentially stable, if there exist $n \times n$ symmetric and positive definite matrices P, Q_1, Q_2 such that*

$$\begin{aligned} &- PC - CP + PAQ_1^{-1}A^TP + \lambda_{\max}(Q_1)ME + \frac{N\lambda_{\max}\left(Q_2 + T^{\star T}T^\star\right)}{1-\rho}E \\ &+ PBQ_2^{-1}B^TP + \|\chi\|^2P^2 < 0. \end{aligned} \tag{3.20}$$

Furthermore, if $P = Q_1 = Q_2 = E$ in Corollary 3.6, then it becomes as follows.

Corollary 3.7. *The equilibrium point of the system* (2.1) *without impulses is globally exponentially stable, if the following condition holds:*

$$-2C + AA^T + BB^T + \frac{N\lambda_{\max}\left(T^{\star T}T^{\star}\right)}{1-\rho} + \left(\frac{N}{1-\rho} + M + \|\chi\|^2\right)E < 0. \tag{3.21}$$

Remark 3.8. Corollaries 3.6 and 3.7 imply that if the above inequality holds, then there exists enough small $\varepsilon^{\star} > 0$ such that all conditions in Corollary 3.4 are satisfied. Hence, Corollaries 3.6 and 3.7 supplied a new criteria for global exponential stability of equilibrium point of the system (2.1) without impulses.

Next we can establish a theorem which provide sufficient conditions for uniform stability of system (2.1) by constructing another Lyapunov functional. Here we shall emphasize the effects of impulses.

Theorem 3.9. *Assume that there exist $n \times n$ symmetric and positive definite matrices P, Q_1, Q_2 such that the following condition*

$$\begin{aligned} &-PC - CP + PAQ_1^{-1}A^TP + \lambda_{\max}(Q_1)ME + \frac{N\lambda_{\max}\left(Q_2 + T^{\star T}T^{\star}\right)}{1-\rho}\left(\prod_{s=1}^{k}\eta_s\right)E \\ &\quad + \left(\prod_{s=1}^{k}\eta_s\right)^{-1} PBQ_2^{-1}B^TP + \left(\prod_{s=1}^{k}\eta_s\right)^{-1}\|\chi\|^2P^2 \le 0 \quad \forall k \in \mathbb{Z}_+ \text{ holds,} \end{aligned} \tag{3.22}$$

where $\sup_{k\in\mathbb{Z}_+}\prod_{s=1}^{k}\eta_s < \infty$, *$\eta_k$ is the largest eigenvalue of $P^{-1}D_kPD_k$.*

Then the equilibrium point of the system (2.1) *is uniformly stable.*

Proof. We only prove the zero solution of system (2.4) is uniformly stable. For any $\varepsilon > 0$, $t_0 \ge 0$, $\varphi \in PC_{\delta}(t_0)$, let $y(t) = y(t_0,\varphi)(t)$ be a solution of (2.4) through (t_0,φ), $t_0 \ge 0$, then we can prove that $\|y(t)\| < \varepsilon$, $t \ge t_0$,

where

$$\delta = \frac{\varepsilon\sqrt{\lambda_{\min}(P)}}{\sqrt{\eta}\sqrt{\lambda_{\max}(P) + \left(\lambda_{\max}\left(Q_2 + T^{\star T}T^{\star}\right)N\eta\tau/(1-\rho)\right)}}, \qquad \eta \doteq \sup_{k\in\mathbb{Z}_+}\prod_{s=1}^{k}\eta_s. \tag{3.23}$$

Consider the following Lyapunov functional

$$V(t) = y^T(t)Py(t) + \frac{1}{1-\rho}\int_{t-\tau(t)}^{t}\left(\prod_{t_s\le t}\eta_s\right)G^T(y(s))\left(Q_2 + T^{\star T}T^{\star}\right)G(y(s))ds, \tag{3.24}$$

then we have

$$\begin{aligned}
\lambda_{\min}(P)\|y(t)\|^2 &< V(t) \\
&\le \lambda_{\max}(P)\|y(t)\|^2 + \frac{\lambda_{\max}\left(Q_2 + T^{\star T}T^{\star}\right)N\eta\tau}{1-\rho}\|y(t)\|_\tau^2 \\
&\le \left(\lambda_{\max}(P) + \frac{\lambda_{\max}\left(Q_2 + T^{\star T}T^{\star}\right)N\eta\tau}{1-\rho}\right)\|y(t)\|_\tau^2.
\end{aligned} \tag{3.25}$$

Applying the same argument as Theorem 3.1, we get

$$\begin{aligned}
2y^T(t)PAF(y(t)) &\le y^T(t)\left[PAQ_1^{-1}A^TP + \lambda_{\max}(Q_1)ME\right]y(t), \\
2y^T(t)PBG(y(t-\tau(t))) &\le \left(\prod_{s=1}^{k}\eta_s\right)G^T(y(t-\tau(t)))Q_2G(y(t-\tau(t))) \\
&\quad + \left(\prod_{s=1}^{k}\eta_s\right)^{-1} y^T(t)PBQ_2^{-1}B^TPy(t), \\
2y^T(t)P\Gamma^TT^{\star}G(y(t-\tau(t))) &\le \left(\prod_{s=1}^{k}\eta_s\right)G^T(y(t-\tau(t)))T^{\star T}T^{\star}G(y(t-\tau(t))) \\
&\quad + \left(\prod_{s=1}^{k}\eta_s\right)^{-1} \|\chi\|^2 y^T(t)P^2y(t).
\end{aligned} \tag{3.26}$$

By simple calculation, we can obtain, for $t \in [t_k, t_{k+1})$, $k \in \mathbb{Z}_+$,

$$\begin{aligned}
D^+V(t)|_{(2.3)} = {} & y^T(t)(-CP - PC)y(t) + 2y^T(t)PAF(y(t)) + 2y^T(t)PBG(y(t-\tau(t))) \\
& + 2y^T(t)P\Gamma^TT^{\star}G(y(t-\tau(t))) + \frac{1}{1-\rho}\left(\prod_{s=1}^{k}\eta_s\right)G^T(y(t))\left(Q_2 + T^{\star T}T^{\star}\right)G(y(t)) \\
& - \frac{1-\dot{\tau}(t)}{1-\rho}\left(\prod_{s=1}^{k}\eta_s\right)G^T(y(t-\tau(t)))\left(Q_2 + T^{\star T}T^{\star}\right)G(y(t-\tau(t)))
\end{aligned}$$

$$
\begin{aligned}
&\leq y^T(t)\Bigg\{-PC - CP + PAQ_1^{-1}A^T P + \lambda_{\max}(Q_1)ME \\
&\qquad + \frac{N\lambda_{\max}\left(Q_2 + T^{\star T}T^{\star}\right)}{1-\rho}\left(\prod_{s=1}^{k}\eta_s\right)E + \left(\prod_{s=1}^{k}\eta_s\right)^{-1} PBQ_2^{-1}B^T P \\
&\qquad + \left(\prod_{s=1}^{k}\eta_s\right)^{-1}\|\chi\|^2 P^2\Bigg\} y(t). \\
&\leq 0.
\end{aligned}
\tag{3.27}
$$

Moreover, we know

$$
\begin{aligned}
V(t_k) &= y^T(t_k)Py(t_k) + \frac{1}{1-\rho}\int_{t_k-\tau(t_k)}^{t_k}\left(\prod_{t_s\leq t_k}\eta_s\right)\Gamma^T(y(s))\left(Q_2 + T^{\star T}T^{\star}\right)\Gamma(y(s))ds \\
&= y^T(t_k^-)D_kPD_ky(t_k^-) + \frac{1}{1-\rho}\int_{t_k^- -\tau(t_k)}^{t_k^-}\left(\prod_{t_s\leq t_k}\eta_s\right)\Gamma^T(y(s))\left(Q_2 + T^{\star T}T^{\star}\right)\Gamma(y(s))ds \\
&\leq \eta_k y^T(t_k^-)Py(t_k^-) + \frac{1}{1-\rho}\eta_k\int_{t_k^- -\tau(t_k)}^{t_k^-}\left(\prod_{t_s\leq t_{k-1}}\eta_s\right)\Gamma^T(y(s))\left(Q_2 + T^{\star T}T^{\star}\right)\Gamma(y(s))ds \\
&= \eta_k V(t_k^-).
\end{aligned}
\tag{3.28}
$$

By simple induction, from (3.27) and (3.28) we may prove that, for $k \geq 1$,

$$
\lambda_{\min}(P)\|y(t)\|^2 \leq V(t) \leq V(t_0)\prod_{t_0<t_k\leq t}\eta_k. \tag{3.29}
$$

Employing the fact (3.25), we obtain

$$
\lambda_{\min}(P)\|y(t)\|^2 \leq \left(\lambda_{\max}(P) + \frac{\lambda_{\max}\left(Q_2 + T^{\star T}T^{\star}\right)N\eta\tau}{1-\rho}\right)\|\varphi\|_\tau^2\eta, \quad t \geq t_0, \tag{3.30}
$$

which implies that

$$
\|y(t)\| < \varepsilon, \quad t \geq t_0. \tag{3.31}
$$

Therefore, the zero solution of system (2.4) is uniformly stable, that is, the equilibrium point of system (2.1) is uniformly stable. The proof of Theorem 3.9 is complete. □

Corollary 3.10. *The equilibrium point of the system* (2.1) *is uniformly stable, if there exist* $n \times n$ *symmetric and positive definite matrices* P, Q_1, Q_2 *such that the following condition holds:*

$$\Xi_k \leq -\lambda_{\max}(Q_1)M - \frac{N\lambda_{\max}\left(Q_2 + T^{\star T}T^{\star}\right)}{1-\rho}\left(\prod_{s=1}^{k}\eta_s\right), \tag{3.32}$$

where Ξ_k *is the largest eigenvalue of* $-PC - CP + PAQ_1^{-1}A^TP + (\prod_{s=1}^{k}\eta_s)^{-1}[PBQ_2^{-1}B^TP + \|\chi\|^2P^2]$.

If $P = Q_1 = Q_2 = E$ in Theorem 3.9, then we have the following.

Corollary 3.11. *The equilibrium point of the system* (2.1) *is uniformly stable, if the following condition*

$$\begin{aligned} &-2C + AA^T + \left[M + \frac{N\lambda_{\max}\left(E + T^{\star T}T^{\star}\right)}{1-\rho}\left(\prod_{s=1}^{k}\max_{i\in\Lambda}\left(1 + d_s^{(i)}\right)^2\right)\right]E \\ &+ \left[\prod_{s=1}^{k}\max_{i\in\Lambda}\left(1 + d_s^{(i)}\right)^2\right]^{-1}\left[BB^T + \|\chi\|^2E\right] \leq 0 \quad \forall k \in \mathbb{Z}_+ \text{ holds,} \end{aligned} \tag{3.33}$$

where $\sup_{k\in\mathbb{Z}_+}\prod_{s=1}^{k}\max_{i\in\Lambda}(1 + d_s^{(i)})^2 < \infty$.

4. Examples

In this section we give two examples to demonstrate our results.

Example 4.1. Consider the following high-order delayed Hopfield-type neural network with impulses

$$\begin{aligned} x_i'(t) = &- c_i x_i(t) + \sum_{j=1}^{3} a_{ij} f_j\left(x_j(t)\right) + \sum_{j=1}^{3} b_{ij} g_j\left(x_j(t-\tau(t))\right) \\ &+ \sum_{j=1}^{3}\sum_{l=1}^{3} T_{ijl} g_l(x_l(t-\tau(t))) g_j\left(x_j(t-\tau(t))\right), \quad t \neq t_k,\ t \geq t_0, \\ \Delta x_i|_{t=t_k} &= x_i(t_k) - x_i\left(t_k^-\right) = \beta_k^{(i)} x_i\left(t_k^-\right), \quad i = 1,2,3, k \in \mathbb{Z}_+, \end{aligned} \tag{4.1}$$

where $\beta_k^{(i)} = \sqrt{1 + (i/k^2)} - 1$, $\tau(t) = \sin t/2$, $f_1(x_1) = \tanh(0.5x_1)$, $f_2(x_2) = \tanh(0.48x_2)$, $f_3(x_3) = \tanh(0.6x_3)$, $g_1(x_1) = \tanh(0.3x_1)$, $g_2(x_2) = \tanh(0.8x_2)$, $g_3(x_3) = \tanh(0.73x_3)$,

$$C = \mathrm{diag}[c_1, c_2, c_3]^T = \begin{bmatrix} 3.2 & 0 & 0 \\ 0 & 2.5 & 0 \\ 0 & 0 & 2.0 \end{bmatrix}, \qquad A = \left(a_{ij}\right)_{3\times 3} = \begin{bmatrix} 0.58 & 0.12 & 0.23 \\ -0.08 & 0.36 & -0.05 \\ -0.04 & 0.04 & -0.37 \end{bmatrix},$$

$$B = \left(b_{ij}\right)_{3\times 3} = \begin{bmatrix} 0.06 & 0 & 0.04 \\ 0.19 & -0.17 & -0.02 \\ -0.03 & 0.13 & 0.44 \end{bmatrix}, \qquad T_1 = \left(T_{1jl}\right)_{3\times 3} = \begin{bmatrix} 0.03 & -0.20 & -0.05 \\ -0.06 & -0.14 & 0.23 \\ 0.27 & 0.03 & -0.20 \end{bmatrix}, \tag{4.2}$$

$$T_2 = \left(T_{2jl}\right)_{3\times 3} = \begin{bmatrix} 0.01 & -0.05 & 0.08 \\ -0.06 & -0.03 & -0.09 \\ 0.15 & -0.04 & 0.11 \end{bmatrix}, \qquad T_3 = \left(T_{3jl}\right)_{3\times 3} = \begin{bmatrix} -0.02 & -0.12 & -0.05 \\ 0.24 & 0.04 & 0.07 \\ -0.02 & 0.08 & 0.01 \end{bmatrix}.$$

In this case, we easily observe that $\tau = \rho = 0.5$, $M = 0.36$, $N = 0.64$, $||\chi||^2 = 0.64$.

For Theorem 3.1, choosing $P = Q_1 = Q_2 = E$, then from

$$\prod_{k=1}^{\infty}\left(1 + \beta_k^{(i)}\right)^2 = \prod_{k=1}^{\infty}\left(1 + \frac{i}{k^2}\right) < \infty, \quad i = 1, 2, 3, \tag{4.3}$$

we may choose $\varepsilon^\star = 0.0976$, $\delta^\star = 0$, $\mathbb{W} = \prod_{k=1}^{\infty}(1 + 3/k^2) < \infty$.

On the other hand, we can compute

$$-2C + AA^T + e^{\tau\varepsilon^\star}BB^T = \begin{bmatrix} -5.9908 & -0.0036 & -0.0869 \\ -0.0036 & -4.7928 & -0.0023 \\ -0.0869 & -0.0023 & -3.6379 \end{bmatrix} = \Theta \tag{4.4}$$

which implies that $\lambda_{\max}(\Theta) = -3.6347$.

Also, we note that

$$T^{\star T}T^\star = \left[T_1 + T_1^T, T_2 + T_2^T, T_3 + T_3^T\right]^T = \begin{bmatrix} 0.2059 & 0.0832 & -0.0535 \\ 0.0832 & 0.2895 & -0.2735 \\ -0.0535 & -0.2735 & 0.4220 \end{bmatrix} \tag{4.5}$$

implies that

$$-\varepsilon^\star - M - \frac{N}{1-\rho} - \frac{N\lambda_{\max}\left(T^{\star T}T^\star\right)}{1-\rho} - e^{\tau\varepsilon^\star}\|\chi\|^2 \approx -3.2501 > -3.6347. \tag{4.6}$$

By Corollary 3.3, the equilibrium point of (4.1) $(0,0,0)^T$ is global exponential stable with the approximate convergence rate 0.0488.

However, the criteria in [12] are invalid here.

Example 4.2. Consider the high-order delayed Hopfield-type neural network with impulses [13]

$$\begin{aligned} x_i'(t) = &- c_i x_i(t) + \sum_{j=1}^{2} a_{ij} f_j\left(x_j(t)\right) + \sum_{j=1}^{2} b_{ij} g_j\left(x_j(t-\tau(t))\right) \\ &+ \sum_{j=1}^{2}\sum_{l=1}^{2} T_{ijl} g_l(x_l(t-\tau(t))) g_j\left(x_j(t-\tau(t))\right) + I_i, \quad t \neq t_k,\ t \geq t_0, \end{aligned} \tag{4.7}$$

and with impulses

$$\Delta x_i|_{t=t_k} = x_i(t_k) - x_i\left(t_k^-\right) = \beta_k^{(i)}\left(x_i\left(t_k^-\right) - x^*\right), \quad i = 1,2,\ k \in \mathbb{Z}_+, \tag{4.8}$$

where $t_k = k$, $t_0 = 0$, $k \in \mathbb{Z}_+$, $f_1(x_1) = g_1(x_1) = \tanh(0.53x_1)$, $f_2(x_2) = g_2(x_2) = \tanh(0.67x_2)$, $\rho = 0.6$, $J_1 = 1.5$, $J_2 = 2$,

$$\begin{gathered} C = \operatorname{diag}[c_1, c_2]^T = \begin{bmatrix} 1.9 & 0 \\ 0 & 1.89 \end{bmatrix}, \qquad A = \left(a_{ij}\right)_{2\times 2} = \begin{bmatrix} 0.05 & 0.14 \\ 0.20 & 0.31 \end{bmatrix}, \\ B = \left(b_{ij}\right)_{2\times 2} = \begin{bmatrix} 0.09 & 0.25 \\ 0.21 & 0.45 \end{bmatrix}, \qquad T_1 = \left(T_{1jl}\right)_{2\times 2} = \begin{bmatrix} 0.05 & 0.14 \\ -0.06 & 0.05 \end{bmatrix}, \\ T_2 = \left(T_{2jl}\right)_{2\times 2} = \begin{bmatrix} 0.29 & 0.10 \\ 0.23 & -0.14 \end{bmatrix}, \\ \beta_k^{(i)} = e^{0.0625} - 1, \quad k \in \mathbb{Z}_+. \end{gathered} \tag{4.9}$$

It is obvious that $M = N = L^2 = 0.4489$, $\|\chi\|^2 = 0.4489$. Here we consider $\tau = 1$. Choose $P = Q_1 = Q_2 = E$, $\varepsilon^\star = 0.25$, $\delta^\star = 0.128$.

Note that

$$\begin{aligned} \sum_{k=1}^{m} \ln \max\left\{\max_{i\in\Lambda}\left(1 + d_k^{(i)}\right)^2, 1\right\} - \delta^\star(t_m - t_0) &= 0.125m - 0.128m \\ &= -0.003m < 0 \quad \forall m \in \mathbb{Z}_+ \text{ holds.} \end{aligned} \tag{4.10}$$

On the other hand, we can compute

$$-2C + AA^T + e^{\tau\varepsilon^\star} BB^T = \begin{bmatrix} -3.7073 & 0.1848 \\ 0.1848 & -3.3973 \end{bmatrix} = \Delta \tag{4.11}$$

which implies that $\lambda_{\max}(\Delta) = -3.3111$.

One can check that

$$-\varepsilon^\star - M - \frac{N}{1-\rho} - \frac{N\lambda_{\max}\left(T^{\star T} T^\star\right)}{1-\rho} - e^{\tau\varepsilon^\star}\|\chi\|^2 \approx -2.9649 > -3.3111. \tag{4.12}$$

By Corollary 3.3, the equilibrium point of (4.1)–(4.7) x^* is global exponential stable with the approximate convergence rate 0.061.

In fact, for above-given impulsive condition, we only need time-delay τ which satisfies the following condition:

$$\lambda_{\max}\left(-2C + AA^T + e^{0.125\tau}BB^T\right) < -0.4489e^{0.125\tau} - 2.2635. \tag{4.13}$$

Remark 4.3. In [13], the author obtained that the equilibrium point of (4.7) without impulses is globally asymptotically stable. From the example, we obtain that the equilibrium point of (4.7) without impulses is global exponential stability. In fact, if $\beta_k^{(i)} = 0$ in (4.7), then we can choose $\delta^\star = 0$, which implies that, for any given $\tau > 0$, there exists corresponding $\varepsilon^\star > 0$ such that all conditions in Corollary 3.6 are satisfied.

5. Conclusions

In this paper, a class of high-order delayed HNN with impulses is considered. We obtain some new criteria ensuring global exponential stability and uniform stability of the equilibrium point for such system by using Lyapunov functional method, the quality of negative definite matrix, and the linear matrix inequality. Our results show delays and impulsive effects on the stability of HNN. Two examples are given to illustrate the feasibility of the results.

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Research Article

Event-Triggered State Estimation for a Class of Delayed Recurrent Neural Networks with Sampled-Data Information

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The paper investigates the state estimation problem for a class of recurrent neural networks with sampled-data information and time-varying delays. The main purpose is to estimate the neuron states through output sampled measurement; a novel event-triggered scheme is proposed, which can lead to a significant reduction of the information communication burden in the network; the feature of this scheme is that whether or not the sampled data should be transmitted is determined by the current sampled data and the error between the current sampled data and the latest transmitted data. By using a delayed-input approach, the error dynamic system is equivalent to a dynamic system with two different time-varying delays. Based on the Lyapunov-krasovskii functional approach, a state estimator of the considered neural networks can be achieved by solving some linear matrix inequalities, which can be easily facilitated by using the standard numerical software. Finally, a numerical example is provided to show the effectiveness of the proposed event-triggered scheme.

1. Introduction

The research of neural networks has been paid much attention during the past few years, due to its potential application in various fields, such as image processing, pattern recognition, and associative memory [1–5]. As a special class of nonlinear dynamical systems, the dynamic behavior of recurrent neural networks has been one of the most important issues. In particular, the analysis problems of stability and synchronization of recurrent neural networks have received great attention and a number of profound results have been proposed [6–12].

In many application, such as signal processing and control engineering, for large-scale neural networks, it is quite common that only partial information can be accessible from the network outputs. Therefore, it is of great significance to estimate the neuron states through available output measurements of the networks and then utilizes the estimated neuron states to achieve certain design objectives; note that state estimation problem for neural networks has been hot reach topics that have drawn considerable attention, and many profound results have been available in the literature [13–25]. The authors in [13] studied the problem of state estimation for a class of delayed neural networks; the traditional monotonicity and smoothness assumption on the activation function had been removed. The design problem of state estimator for a class of neural networks with constant delays was investigated in [14], where a delay-dependent criterion for existence of the estimator was proposed. As an extension, The authors in [14, 15] further discussed state estimation for neural networks with time-varying delays. In practice, sometimes a neural network has finite state modes and modes may switch from one to another at different times. On the other hand, discrete-time neural networks could be more suitable to model digitally transmitted signals in dynamical way; based on the above reason, The authors in [16] investigated state estimation problem for a new class of discrete-time neural networks with Markovian jumping parameters and mode dependent mixed time-delays, where he discrete and distributed delays were mode-dependent. Different from the stuelies in [16, 17] which considered state estimation for Markovian jumping delayed continuous-time recurrent neural networks, where only matrix parameters were mode-dependent. Similar to [16], for continuous-time recurrent neural networks with discrete and distributed delays, state estimation was also investigated in [18]. In [19, 20], synchronization and state estimation had been studied for discrete-time complex networks with distributed delays; it was noticed that in [20], a novel notion of bounded H_∞ synchronization had been first defined to characterize the transient performance of synchronization. Some robust state estimation problems for uncertain neural networks with time-varying delays had been investigated in [21–23], where the parameter uncertainties are assumed to be norm bounded; some sufficient conditions were presented to guarantee the existence of the desired state estimator. Taking into account the stochastic properties of time-varying delays, the authors in [24] discussed state estimation problem for a class of discrete-time stochastic neural networks with random delays; sufficient delay-distribution-dependent conditions were established in terms of linear matrix inequalities (LMIs) that guarantee the existence of the state estimator.

The sampled-data control theory had attracted much attention due to its effectiveness in engineering applications. Especially, a new approach to deal with the sampled-data control problems had been proposed in [26], where the sampling period had been converted into time-varying delay. As its extension, the authors in [27] investigated the sampled-data state estimation problem for a class of recurrent neural networks with time-varying delays, where the sampled measurements had been used to estimate the neuron states. Using a similar approach, where the sampled-data synchronization control problem was investigated in [28] for a class of general delayed complex networks, the sampled-data feedback controllers were designed in terms of the solution to certain linear matrix inequalities. But in the above references, the sampling rate for each signal is the same; but in the actual system, it may be varying from sample to sample owing to unpredictable perturbations; this factor was considered in [29], the problem of robust H_∞ control was investigated for sampled-data systems with probabilistic sampling, where two different sampling periods were considered whose occurrence probabilities were given constants and satisfied Bernoulli distribution. In [30], stochastic sampled-data approach was used for studying the problem of distributed H_∞ filtering in sensor networks, by converting the sampling periods into bounded time-delays, the design problem of H_∞ filters amounted to solving the H_∞ filtering problem for a class of stochastic nonlinear systems with multiple bounded time delays. In [31], the sampled-data synchronization control problem was addressed, where the sampling period was time varying and switched between two different values in a random way. It is worth noting that most of the above results were involved the traditional approach of sampling at pre-specified time instances, which was called time-triggered sampling; this sampling method may lead to an inherently periodic transmission and produce many useless messages if the current sampled signal had not significantly changed in contrast to the previous sampled signal, which led to a conservative usage of the communication resources. Recently, event-triggered scheme provided an effective approach of determining; its main property was that the signal was sampled and only some functions of the system state or output measurement exceeded threshold. Compared with periodic sampling method, the event-triggered scheme could reduce the burden of the communication and also preserve the desired properties of the ideal continuous state feedback system, such as stability and convergence. The utilization on event-triggered scheme could be found in many literatures such as [32–37]. The event-triggered H_∞ control design was investigated in [32] for networked control systems with uncertainties and transmission delays, and a novel event-triggered scheme was proposed. The study in [33] was concerned with the control problem of event-triggered networked systems with both state and control input quantizations. In [34], the problems of exponential stability and L_2-gain analysis of event-triggered networked control systems were studied, where the event-triggered conditions were proposed in the sensor side and controller side. In [35–37], the consensus problems for multiagent systems were investigated by event-triggered control, where different trigger functions were proposed. Unfortunately, as far as we know, up to now, no theoretical results are given for state estimation of recurrent neural networks with time-varying delays based on event-triggered scheme. The purpose of our study is to fill the gap.

Motivated by the above discussion, the paper is concerned with the sampled-data state estimation problem for a class of recurrent neural networks with time-varying delays. The main purpose is to estimate the neuron states through output sampled measurement, and a novel event-triggered scheme is proposed, which can lead to a significant reduction of the information communication burden in the network. By using a delayed-input approach, the error dynamics system is equivalent to a dynamic system with two different time-varying delays. Based on constructing a Lyapunov-Krasovskii functional and employing some analysis techniques, a state estimator of the considered neural networks can be achieved by solving some linear matrix inequalities, which can be easily facilitated by using the standard numerical software. Finally, a numerical example is given to illustrate the effectiveness of the proposed method.

The main contributions of this paper are highlighted as follows.

(1) The novel event-triggered scheme is proposed, compared with a time-triggered periodic communication scheme, since the proposed communication scheme only depends on the state at the sampled instant and the state error between the current sampled instant and the latest transmitted state. Therefore, the number of the transmitted state signals through the network could be reduced apparently.

(2) Sufficient conditions obtained are in the form of linear matrix inequalities which can be readily solved by using the LMI toolbox in Matlab, and the solvability of derived conditions dependents on not only trigger parameters and sampling period but also the size of the delay.

Notation 1. The notation used here is fair standard except where otherwise stated. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space and $\mathbb{R}^{n\times m}$ is the set of real $n\times m$ matrices. The superscript T represents the transpose of the matrix (or vector). I denotes the identity matrix of compatible dimensions. The asterisk represents the symmetric block in one symmetric matrix. diag$\{\cdots\}$ stands for a block-diagonal matrix. The notation $X \geq 0$ ($X > 0$) means that X is positive semi-definite (positive definite). $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$. If they are not explicitly specified, arguments of a function or a matrix will be omitted in the analysis when no confusion can arise.

2. Preliminaries

Consider a class of recurrent neural networks with time-varying delays as follows:

$$\begin{aligned} \dot{x}(t) &= -Ax(t) + W_0 g(x(t)) + W_1 g(x(t-\tau(t))) \\ y(t) &= Cx(t), \end{aligned} \tag{2.1}$$

where $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \in \mathbb{R}^n$ is the state vector associated with n neurons; $A = \text{diag}\{a_1, a_2, \ldots, a_n\}$ is a positive diagonal matrix; $W_0 \in \mathbb{R}^{n\times n}$ and $W_1 \in \mathbb{R}^{n\times n}$ are the connection weight matrix and the delayed connection weight matrix, respectively; $\tau(t) \in [0, \tau]$ is the time-varying bound delay; $C \in \mathbb{R}^{m\times n}$ is a constant matrix; $y(t) = (y_1(t), y_2(t), \ldots, y_m(t))^T \in \mathbb{R}^m$ denotes the output vector; $g(x(t)) = [g_1(x_1(t)), g_2(x_2(t)), \ldots, g_n(x_n(t))]^T \in \mathbb{R}^n$ represents the neuron activation function with $g(0) = 0$.

In this paper, the measurement output is sampled before it enters the estimator; based on the sampling technique and zero-order hold, the actual output can be described as

$$\overline{y}(t) = y(t_k) = Cx(t_k), \quad t \in [t_k, t_{k+1}), \tag{2.2}$$

where $\overline{y}(t) \in \mathbb{R}^m$ is the actual output of the estimator, and t_k denotes the sampling instant satisfying $\lim_{k\to\infty} t_k = \infty$.

Remark 2.1. In practical systems, periodic sampling mechanism may often lead to sending many unnecessary signals through the networks, which will increase the load of network transmission and waste the network bandwidth; therefore, it is significant to introduce a mechanism to determine which sampled signal should be sent out or not. As stated in [32, 33], the event-trigger sampling scheme is effective way because they can reduce the traffic and the power consumption.

The sampled data $y(t_{k+j})$ is transmitted (or released) by the event generator only when the current sampled value $y(t_{k+j})$ and the previously transmitted one $y(t_k)$ satisfy the following judgement algorithm:

$$\left[y(t_{k+j}) - y(t_k)\right]^T W \left[y(t_{k+j}) - y(t_k)\right] < \sigma y^T(t_{k+j}) W y(t_{k+j}), \tag{2.3}$$

where $W \in \mathbb{R}^{m\times m}$ is a positive matrix, and $\sigma \in [0, 1)$ is a positive scalar. The sampled state $y(t_{k+j})$ satisfying the inequality (2.3) will not be transmitted; only the one that exceeds the threshold in (2.3) will be sent to the estimator. Specially, when $\sigma = 0$, the inequality (2.3) is not satisfied for almost all the sampled state $y(t_{k+j})$, and the event-triggered scheme reduces to a periodic release scheme.

Remark 2.2. From event-triggered algorithm (2.3), it is easily seen that all the released signals are subsequences of the sampled data, that is, the set of the release instants $\{t_0, t_1, t_2 \ldots\} \subseteq \{0, 1, 2, \ldots\}$. The amount of $\{t_0, t_1, t_2 \ldots\}$ depends on not only the value of σ but also the variation of the system output.

Suppose that the time-varying delay in network communication is $d_k \in [0, d]$ $(k = 1, 2, \ldots, +\infty)$, the output $\overline{y}(t)$ in (2.2) can be rewritten as

$$\overline{y}(t) = y(t_k) = Cx(t_k), \quad t \in [t_k + d_k, t_{k+1} + d_{k+1}). \tag{2.4}$$

Substituting the output (2.4) into the judgement algorithm (2.3), we can obtain

$$\left[x(t_{k+j}) - x(t_k)\right]^T C^T W C \left[x(t_{k+j}) - x(t_k)\right] < \sigma x^T(t_{k+j}) C^T W C x(t_{k+j}). \tag{2.5}$$

For technical convenience, similar to [32, 33], consider the following two intervals:

$$[t_k + d_k, t_k + h + d), \qquad [t_k + lh + d, t_k + lh + h + d), \tag{2.6}$$

where l is a positive integer and h is a sampling period.

(1) if $t_k + h + d > t_{k+1} + d_{k+1}$, define a function $d(t)$ as follows:

$$d(t) = t - t_k, \quad t \in [t_k + d_k, t_{k+1} + d_{k+1}). \tag{2.7}$$

It can easily be obtained that the following inequality holds:

$$d_k \leq d(t) \leq t_{k+1} - t_k + d_{k+1} \leq h + d. \tag{2.8}$$

(2) if $t_k + h + d < t_{k+1} + d_{k+1}$, there exists a positive integer m, such that

$$t_k + mh + d < t_{k+1} + d_{k+1} < t_k + mh + h + d. \tag{2.9}$$

It can be easily shown that

$$[t_k + d_k, t_{k+1} + d_{k+1}) = I_1 \cup I_2 \cup I_3, \tag{2.10}$$

where

$$\begin{gathered}
I_1 = [t_k + d_k, t_k + h + d) \\
I_2 = \bigcup_{m=1}^{l-1} \{I_2^m\} \\
I_2^m = [t_k + mh + d, t_k + mh + h + d) \\
I_3 = [t_k + lh + d, t_{k+1} + d_{k+1}).
\end{gathered} \tag{2.11}$$

Define a function $d(t)$ as

$$d(t) = \begin{cases} t - t_k & t \in I_1 \\ t - t_k - mh & t \in I_2^m \ (m = 1, 2, \ldots, l-1) \\ t - t_k - lh & t \in I_3. \end{cases} \tag{2.12}$$

From the definition of $d(t)$ defined in (2.12), we can derive

$$\begin{gathered}
0 \le d_k \le d(t) < h + d, \quad t \in I_1 \\
0 \le d_k \le d \le d(t) < h + d, \quad t \in I_2^m \ (m = 1, 2, \ldots, l-1) \\
0 \le d_k \le d \le d(t) < h + d, \quad t \in I_3.
\end{gathered} \tag{2.13}$$

From (2.13), it can be derived that $0 \le d(t) < d_M$, where $d_M = h + d$. For $t \in [t_k + d_k, t_{k+1} + d_{k+1})$, we define

$$e_k(t) = \begin{cases} 0 & t \in I_1 \\ x(t_k + mh) - x(t_k) & t \in I_2^m \ (m = 1, 2, \ldots, l-1) \\ x(t_k + lh) - x(t_k) & t \in I_3. \end{cases} \tag{2.14}$$

Combining the above definitions of $d(t)$ and $e_k(t)$, the algorithm (2.5) can be rewritten as

$$e_k^T(t) C^T W C e_k(t) < \sigma x^T(t - d(t)) C^T W C x(t - d(t)). \tag{2.15}$$

Based on the available sampled measurement $\overline{y}(t)$, the following state estimator is adopted:

$$\dot{\hat{x}}(t) = -A\hat{x}(t) + K(\overline{y}(t) - C\hat{x}(t)), \tag{2.16}$$

where K is feedback gain matrix to be designed, and $\hat{x}(t) = (\hat{x}_1(t), \hat{x}_2(t), \ldots, \hat{x}_n(t))^T \in \mathbb{R}^n$ is estimator state vector.

Setting $e(t) = x(t) - \hat{x}(t)$, the estimation error dynamics can be obtained from (2.1) and (2.16), and it follows that

$$\dot{e}(t) = -(A + KC)e(t) + KCx(t) - KCx(t - d(t)) + KCe_k(t) + W_0 g(x(t)) + W_1 g(x(t - \tau(t))). \tag{2.17}$$

Let $\overline{x}(t) = (x^T(t), e^T(t))^T$, we can get the following augmented system from (2.1) and (2.17)

$$\dot{\overline{x}}(t) = \overline{A}\overline{x}(t) + \overline{B}\overline{x}(t - d(t)) + \overline{W}_0 g(H\overline{x}(t)) + \overline{W}_1 g(H\overline{x}(t - \tau(t))) + \overline{C}e_k(t), \tag{2.18}$$

where

$$\overline{A} = \begin{bmatrix} -A & 0 \\ KC & -A - KC \end{bmatrix} \quad \overline{B} = \begin{bmatrix} 0 & 0 \\ -KC & 0 \end{bmatrix} \quad \overline{W}_0 = \begin{bmatrix} W_0 \\ W_0 \end{bmatrix}$$
$$\overline{W}_1 = \begin{bmatrix} W_1 \\ W_1 \end{bmatrix} \quad H^T = \begin{bmatrix} I \\ 0 \end{bmatrix} \quad \overline{C} = \begin{bmatrix} 0 \\ KC \end{bmatrix}. \tag{2.19}$$

Before giving the main results, the following assumption, definition, and lemma are essential in establishing our main results.

Assumption 2.3 (see, [27]). The activation function $g(\cdot)$ satisfies the following sector-bounded condition:

$$[g(x) - U_1 x]^T [g(x) - U_2 x] \leq 0, \tag{2.20}$$

where U_1 and U_2 are two real constant matrices with $U_2 - U_1 \geq 0$.

Definition 2.4 (see, [27]). The augmented system (2.18) is exponentially stable, if there exist two constants $\alpha > 0$ and $\beta > 0$, such that

$$\|\overline{x}(t)\|^2 \leq \alpha e^{-\beta t} \sup_{-r \leq \theta \leq 0} \|\phi(\theta)\|^2, \tag{2.21}$$

where $\phi(\cdot)$ is in the initial function system (2.18) as $\phi(t) = \overline{x}(t)$, $t \in [-r, 0]$.

Lemma 2.5 (see, [38, 39]). *Suppose $\tau(t) \in [\tau_m, \tau_M]$, Q_i $(i = 1, 2, 3)$ are some constant matrices with appropriate dimensions, then*

$$Q_1 + (\tau_M - \tau(t))Q_2 + (\tau(t) - \tau_m)Q_3 < 0, \tag{2.22}$$

if the following inequalities hold

$$\begin{aligned} &Q_1 + (\tau_M - \tau_m)Q_2 < 0 \\ &Q_1 + (\tau_M - \tau_m)Q_3 < 0. \end{aligned} \tag{2.23}$$

Lemma 2.6 (see, [40]). *For any constant positive matrix $T \in \mathbb{R}^{n\times n}$, scalar $\tau_1 \le \tau(t) < \tau_2$ and vector function $\dot{x}(t) : [-\tau_2, \tau_1] \to \mathbb{R}^n$ such that the following integration is well defined, then it holds that*

$$-(\tau_2 - \tau_1)\int_{t-\tau_2}^{t-\tau_1} \dot{x}^T(v)T\dot{x}(v)dv \le \begin{bmatrix} x(t-\tau_1) \\ x(t-\tau(t)) \\ x(t-\tau_2) \end{bmatrix}^T \begin{bmatrix} -T & T & 0 \\ * & -2T & T \\ * & * & -T \end{bmatrix} \begin{bmatrix} x(t-\tau_1) \\ x(t-\tau(t)) \\ x(t-\tau_2) \end{bmatrix}. \tag{2.24}$$

3. Main Results

In this section, we design a sampled-date estimator with form (2.18) for recurrent neural networks with time-varying delay based event-triggered control.

The system (2.18) can be rewritten as

$$\dot{\overline{x}}(t) = \mathcal{A}\xi(t), \tag{3.1}$$

where

$$\begin{gathered} \xi(t) = \left[\overline{x}^T(t), \overline{x}^T(t-d(t)), \overline{x}^T(t-d_M), \overline{x}^T(t-\tau(t)), \overline{x}^T(t-\tau), g^T(H\overline{x}(t)), g^T(H\overline{x}(t-\tau(t))), e_k^T(t)\right]^T \\ \mathcal{A} = \left[\overline{A}, \overline{B}, 0, 0, 0, \overline{W}_0, \overline{W}_1, \overline{C}\right]. \end{gathered} \tag{3.2}$$

Theorem 3.1. *Suppose that Assumption 2.3 holds, for given estimator gain matrix K, the augmented system* (3.1) *is exponentially stable, if there exist some positive definite matrices $P > 0$, $Q_i > 0$, $R_i > 0$ and S_i, T_i $(i = 1, 2)$ with appropriate dimension, and two positive scalars $\alpha > 0$, $\beta > 0$, such that the following linear matrix inequalities hold:*

$$\Pi_i = \begin{bmatrix} \Pi & \Phi_1 & \Phi_2 & \Phi_3^{(i)} \\ * & -d_M R_1 & 0 & 0 \\ * & * & -\tau R_2 & 0 \\ * & * & * & -\tau R_2 \end{bmatrix} < 0 \quad (i = 1, 2), \tag{3.3}$$

where

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} & 0 & \Pi_{13} & 0 & \Pi_{14} & P\overline{W}_1 & P\overline{C} \\ * & \Pi_{22} & \frac{1}{d_M}R_1 & 0 & 0 & 0 & 0 & 0 \\ * & * & \Pi_{33} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \Pi_{44} & \Pi_{45} & 0 & -\beta\overline{U}_2 & 0 \\ * & * & * & * & \Pi_{55} & 0 & 0 & 0 \\ * & * & * & * & * & -\alpha I & 0 & 0 \\ * & * & * & * & * & * & -\beta I & 0 \\ * & * & * & * & * & * & * & -C^T W C \end{bmatrix}$$

$$\Phi_1 = \left[d_M R_1 \overline{A} \;\; d_M R_1 \overline{B} \;\; 0 \;\; 0 \;\; 0 \;\; d_M R_1 \overline{W}_0 \;\; d_M R_1 \overline{W}_1 \;\; d_M R_1 \overline{C} \right]^T$$

$$\Phi_2 = \left[\tau R_2 \overline{A} \;\; \tau R_2 \overline{B} \;\; 0 \;\; 0 \;\; 0 \;\; \tau R_2 \overline{W}_0 \;\; \tau R_2 \overline{W}_1 \;\; \tau R_2 \overline{C} \right]^T$$

$$\Phi_3^{(1)} = \left[\tau S_1^T \;\; 0 \;\; 0 \;\; \tau S_2^T \;\; 0 \;\; 0 \;\; 0 \;\; 0 \right]^T$$

$$\Phi_3^{(2)} = \left[0 \;\; 0 \;\; 0 \;\; \tau T_1^T \;\; \tau T_2^T \;\; 0 \;\; 0 \;\; 0 \right]^T$$

$$\Pi_{11} = P\overline{A} + \overline{A}^T P + Q_1 + Q_2 - \frac{1}{d_M} R_1 + S_1 + S_1^T - \alpha \overline{U}_1$$

$$\Pi_{12} = P\overline{B} + \frac{1}{d_M} R_1$$

$$\Pi_{13} = S_2^T - S_1$$

$$\Pi_{14} = P\overline{W}_0 - \alpha \overline{U}_2$$

$$\Pi_{22} = -\frac{2}{d_M} R_1 + \sigma\Omega$$

$$\Pi_{33} = -Q_1 - \frac{1}{d_M} R_1$$

$$\Pi_{44} = -S_2 - S_2^T + T_1 + T_1^T - \beta \overline{U}_1$$

$$\Pi_{45} = -T_1 + T_2^T$$

$$\Pi_{55} = -Q_2 - T_2 - T_2^T$$

$$\Omega = \begin{bmatrix} C^T W C & 0 \\ 0 & 0 \end{bmatrix}. \tag{3.4}$$

Proof. Construct the following Lyapunov-Krasovskii functional candidate:

$$V(t, \overline{x}(t)) = V_1(t, \overline{x}(t)) + V_2(t, \overline{x}(t)) + V_3(t, \overline{x}(t)) + V_4(t, \overline{x}(t)), \tag{3.5}$$

where

$$
\begin{aligned}
V_1(t,\overline{x}(t)) &= \overline{x}^T(t)P\overline{x}(t) \\
V_2(t,\overline{x}(t)) &= \int_{t-d_M}^{t} \overline{x}^T(s)Q_1\overline{x}(s)ds + \int_{t-\tau}^{t} \overline{x}^T(s)Q_2\overline{x}(s)ds \\
V_3(t,\overline{x}(t)) &= \int_{t-d_M}^{t}\int_{\theta}^{t} \dot{\overline{x}}^T(s)R_1\dot{\overline{x}}(s)ds\,d\theta \\
V_4(t,\overline{x}(t)) &= \int_{t-\tau}^{t}\int_{\theta}^{t} \dot{\overline{x}}^T(s)R_2\dot{\overline{x}}(s)ds\,d\theta,
\end{aligned}
\tag{3.6}
$$

and $P > 0$, $Q_i > 0$ and $R_i > 0$ $(i = 1,2)$ are matrices to be determined.

The derivative of $V_i(t,\overline{x}(t))$ $(i = 1,2,3,4)$ along the trajectory of system (3.1) can be shown as follows:

$$
\dot{V}_1(t,\overline{x}(t)) = 2\overline{x}^T(t)P\mathcal{A}\xi(t)
\tag{3.7}
$$

$$
\dot{V}_2(t,\overline{x}(t)) = \overline{x}^T(t)(Q_1+Q_2)\overline{x}(t) - \overline{x}^T(t-d_M)Q_1\overline{x}(t-d_M) - \overline{x}^T(t-\tau)Q_2\overline{x}(t-\tau)
\tag{3.8}
$$

$$
\begin{aligned}
\dot{V}_3(t,\overline{x}(t)) &= d_M\dot{\overline{x}}^T(t)R_1\dot{\overline{x}}(t) - \int_{t-d_M}^{t} \dot{\overline{x}}^T(s)R_1\dot{\overline{x}}(s)ds \\
&= d_M\xi^T(t)\mathcal{A}^TR_1\mathcal{A}\xi(t) - \int_{t-d_M}^{t} \dot{\overline{x}}^T(s)R_1\dot{\overline{x}}(s)ds
\end{aligned}
\tag{3.9}
$$

$$
\begin{aligned}
\dot{V}_4(t,\overline{x}(t)) &= \tau\dot{\overline{x}}^T(t)R_2\dot{\overline{x}}(t) - \int_{t-\tau}^{t} \dot{\overline{x}}^T(s)R_2\dot{\overline{x}}(s)ds \\
&= \tau\xi^T(t)\mathcal{A}^TR_2\mathcal{A}\xi(t) - \int_{t-\tau}^{t} \dot{\overline{x}}^T(s)R_2\dot{\overline{x}}(s)ds.
\end{aligned}
\tag{3.10}
$$

Noting that (3.9), it follows from Lemma 2.6 that

$$
-\int_{t-d_M}^{t} \dot{\overline{x}}^T(s)R_1\dot{\overline{x}}(s)ds \le \frac{1}{d_M}\begin{bmatrix} \overline{x}(t) \\ \overline{x}(t-d(t)) \\ \overline{x}(t-d_M) \end{bmatrix}^T \begin{bmatrix} -R_1 & R_1 & 0 \\ * & -2R_1 & R_1 \\ * & * & -R_1 \end{bmatrix} \begin{bmatrix} \overline{x}(t) \\ \overline{x}(t-d(t)) \\ \overline{x}(t-d_M) \end{bmatrix}.
\tag{3.11}
$$

Employing the free matrix method [38, 39], it is easily derived that

$$
\begin{aligned}
2\xi^T(t)S\left[\overline{x}(t) - \overline{x}(t-\tau(t)) - \int_{t-\tau(t)}^{t} \dot{\overline{x}}(s)ds\right] &= 0, \\
2\xi^T(t)T\left[\overline{x}(t-\tau(t)) - \overline{x}(t-\tau) - \int_{t-\tau}^{t-\tau(t)} \dot{\overline{x}}(s)ds\right] &= 0,
\end{aligned}
\tag{3.12}
$$

where

$$S = \left[S_1^T \ \ 0 \ \ 0 \ \ S_2^T \ \ 0 \ \ 0 \ \ 0 \ \ 0\right]^T,$$
$$T = \left[0 \ \ 0 \ \ 0 \ \ T_1^T \ \ T_2^T \ \ 0 \ \ 0 \ \ 0\right]^T. \tag{3.13}$$

It follows that from (3.12) that

$$-2\xi^T(t)S\int_{t-\tau(t)}^{t}\dot{\overline{x}}(s)ds \leq \tau(t)\xi^T(t)SR_2^{-1}S^T\xi(t) + \int_{t-\tau(t)}^{t}\dot{\overline{x}}^T(s)R_2\dot{\overline{x}}(s)ds$$
$$-2\xi^T(t)T\int_{t-\tau}^{t-\tau(t)}\dot{\overline{x}}(s)ds \leq (\tau-\tau(t))\xi^T(t)TR_2^{-1}T^T\xi(t) + \int_{t-\tau}^{t-\tau(t)}\dot{\overline{x}}^T(s)R_2\dot{\overline{x}}(s)ds. \tag{3.14}$$

By Assumption 2.3, the following inequality holds:

$$\begin{bmatrix}\overline{x}(t)\\ g(H\overline{x}(t))\end{bmatrix}^T\begin{bmatrix}\overline{U}_1 & \overline{U}_2\\ \overline{U}_2^T & I\end{bmatrix}\begin{bmatrix}\overline{x}(t)\\ g(H\overline{x}(t))\end{bmatrix} \leq 0, \tag{3.15}$$

where

$$\overline{U}_1 = H^T\widehat{U}_1H, \qquad \overline{U}_2 = H^T\widehat{U}_2$$
$$\widehat{U}_1 = \frac{U_1^TU_2 + U_2^TU_1}{2}, \qquad \widehat{U}_2 = \frac{U_1^T + U_2^T}{2}. \tag{3.16}$$

For all $\alpha, \beta > 0$, it can be derived from (3.15) that

$$-\alpha\begin{bmatrix}\overline{x}(t)\\ g(H\overline{x}(t))\end{bmatrix}^T\begin{bmatrix}\overline{U}_1 & \overline{U}_2\\ \overline{U}_2^T & I\end{bmatrix}\begin{bmatrix}\overline{x}(t)\\ g(H\overline{x}(t))\end{bmatrix} \geq 0$$
$$-\beta\begin{bmatrix}\overline{x}(t-\tau(t))\\ g(H\overline{x}(t-\tau(t)))\end{bmatrix}^T\begin{bmatrix}\overline{U}_1 & \overline{U}_2\\ \overline{U}_2^T & I\end{bmatrix}\begin{bmatrix}\overline{x}(t-\tau(t))\\ g(H\overline{x}(t-\tau(t)))\end{bmatrix} \geq 0. \tag{3.17}$$

Then, (2.15) can be rewritten as

$$\sigma\overline{x}^T(t-d(t))\Omega\overline{x}(t-d(t)) - \begin{bmatrix}e_k^T(t)C^TWCe_k(t) & 0\\ 0 & 0\end{bmatrix} > 0, \tag{3.18}$$

where

$$\Omega = \begin{bmatrix}C^TWC & 0\\ 0 & 0\end{bmatrix}. \tag{3.19}$$

It follows from (3.7)–(3.18) that

$$
\begin{aligned}
\dot{V}(t,\overline{x}(t)) \leq\ & 2\overline{x}^T(t)P\mathcal{A}\xi(t) + \overline{x}^T(t)(Q_1+Q_2)\overline{x}(t) - \overline{x}^T(t-d_M)Q_1\overline{x}(t-d_M) \\
& - \overline{x}^T(t-\tau)Q_2\overline{x}(t-\tau) + d_M\xi^T(t)\mathcal{A}^T R_1\mathcal{A}\xi(t) + \tau\xi^T(t)\mathcal{A}^T R_2\mathcal{A}\xi(t) \\
& + 2\xi^T(t)S(\overline{x}(t) - \overline{x}(t-\tau(t))) + 2\xi^T(t)T(\overline{x}(t-\tau(t)) - \overline{x}(t-\tau)) \\
& + \frac{1}{d_M}\begin{bmatrix} \overline{x}(t) \\ \overline{x}(t-d(t)) \\ \overline{x}(t-d_M) \end{bmatrix}^T \begin{bmatrix} -R_1 & R_1 & 0 \\ * & -2R_1 & R_1 \\ * & * & -R_1 \end{bmatrix} \begin{bmatrix} \overline{x}(t) \\ \overline{x}(t-d(t)) \\ \overline{x}(t-d_M) \end{bmatrix} \\
& - \alpha\begin{bmatrix} \overline{x}(t) \\ g(H\overline{x}(t)) \end{bmatrix}^T \begin{bmatrix} \overline{U}_1 & \overline{U}_2 \\ \overline{U}_2^T & I \end{bmatrix} \begin{bmatrix} \overline{x}(t) \\ g(H\overline{x}(t)) \end{bmatrix} \\
& - \beta\begin{bmatrix} \overline{x}(t-\tau(t)) \\ g(H\overline{x}(t-\tau(t))) \end{bmatrix}^T \begin{bmatrix} \overline{U}_1 & \overline{U}_2 \\ \overline{U}_2^T & I \end{bmatrix} \begin{bmatrix} \overline{x}(t-\tau(t)) \\ g(H\overline{x}(t-\tau(t))) \end{bmatrix} \\
& + \sigma\overline{x}^T(t-d(t))\Omega\overline{x}(t-d(t)) - \begin{bmatrix} e_k^T(t)C^T W C e_k(t) & 0 \\ 0 & 0 \end{bmatrix} \\
& + \tau(t)\xi^T(t)SR_2^{-1}S^T\xi(t) + (\tau-\tau(t))\xi^T(t)TR_2^{-1}T^T\xi(t) \\
=\ & \xi^T(t)\left(\Pi + d_M\mathcal{A}^T R_1\mathcal{A} + \tau\mathcal{A}^T R_2\mathcal{A}\right)\xi(t) + \tau(t)\xi^T(t)SR_2^{-1}S^T\xi(t) \\
& + (\tau-\tau(t))\xi^T(t)TR_2^{-1}T^T\xi(t).
\end{aligned}
\tag{3.20}
$$

By using Schur complement and Lemma 2.5, it can be seen that (3.3) is equivalent to

$$
\Pi + d_M\mathcal{A}^T R_1\mathcal{A} + \tau\mathcal{A}^T R_2\mathcal{A} + \tau(t)SR_2^{-1}S^T + (\tau-\tau(t))TR_2^{-1}T^T < 0 \tag{3.21}
$$

which implies $\dot{V}(t,\overline{x}(t)) < -\varepsilon\|\overline{x}(t)\|^2$; then similar to [41], we can obtain the exponential stability of system (3.1). The proof is completed. □

Remark 3.2. From Theorem 3.1, it can be seen that the trigger parameters σ, W and the upper bound of time delay τ are involved in (3.3); for given σ, the corresponding trigger parameter W and the upper bound of τ can be obtained by using LMI toolbox in Matlab. From the simulation example, it can be derived that the larger the σ, the small the τ; the larger average release period, which means the load of network transmission will be reduced.

Remark 3.3. When the estimator gain matrix K is given, the conditions (3.3) are in the form of linear matrix inequalities, which can be readily solved by using the standard numerical software. The conditions (3.3) are not linear matrix inequalities when the estimator gain matrix K is a matrix variable to be designed, and thus Theorem 3.1 cannot be used to design K directly, a design method will be provided in the following Theorem.

After establishing analysis results in Theorem 3.1, the design problem of state estimator is to be considered and the following results can be readily derived from Theorem 3.1.

Theorem 3.4. *Suppose that Assumption 2.3 holds, the augmented system* (3.1) *is exponentially stable, if there exist* $P = \operatorname{diag}\{\mathbf{P}_1, \mathbf{P}_2\} > 0$, $Q_i = \operatorname{diag}\{\mathbf{Q}_i, \mathbf{Q}_i\} > 0$, $R_i = \operatorname{diag}\{\mathbf{R}_i, \mathbf{R}_i\} > 0$ *and* $S_i = \operatorname{diag}\{\mathbf{S}_i, \mathbf{S}_i\}$, $T_i = \operatorname{diag}\{\mathbf{T}_i, \mathbf{T}_i\}$ $(i = 1, 2)$ *and V with appropriate dimension, and two positive scalars* $\alpha > 0$, $\beta > 0$, *such that the following linear matrix inequalities hold:*

$$\overline{\Pi}_i = \begin{bmatrix} \overline{\Pi} & \overline{\Phi}_1 & \overline{\Phi}_2 & \overline{\Phi}_3^{(i)} \\ * & \overline{\Phi}_4 & 0 & 0 \\ * & * & \overline{\Phi}_5 & 0 \\ * & * & * & -\tau R_2 \end{bmatrix} < 0 \quad (i = 1, 2), \tag{3.22}$$

where

$$\begin{gathered}\overline{\Pi} = \begin{bmatrix} \overline{\Pi}_{11} & \overline{\Pi}_{12} & 0 & \overline{\Pi}_{13} & 0 & \overline{\Pi}_{14} & \overline{\Pi}_{15} & \overline{\Pi}_{16} \\ * & \overline{\Pi}_{22} & \overline{\Pi}_{23} & 0 & 0 & 0 & 0 & 0 \\ * & * & \overline{\Pi}_{33} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \overline{\Pi}_{44} & \overline{\Pi}_{45} & 0 & \overline{\Pi}_{46} & 0 \\ * & * & * & * & \overline{\Pi}_{55} & 0 & 0 & 0 \\ * & * & * & * & * & \overline{\Pi}_{66} & 0 & 0 \\ * & * & * & * & * & * & \overline{\Pi}_{77} & 0 \\ * & * & * & * & * & * & * & \overline{\Pi}_{88} \end{bmatrix}, \\
\overline{\Phi}_1 = \begin{bmatrix} \overline{\Pi}_{17}^T & \overline{\Pi}_{24}^T & 0 & 0 & 0 & \overline{\Pi}_{67}^T & \overline{\Pi}_{78}^T & \overline{\Pi}_{89}^T \end{bmatrix}^T, \\
\overline{\Phi}_2 = \begin{bmatrix} \overline{\Pi}_{18}^T & \overline{\Pi}_{25}^T & 0 & 0 & 0 & \overline{\Pi}_{68}^T & \overline{\Pi}_{79}^T & \overline{\Pi}_{8,10}^T \end{bmatrix}^T, \\
\overline{\Phi}_3^{(1)} = \begin{bmatrix} \overline{\Pi}_{19}^T & 0 & 0 & \overline{\Pi}_{47}^T & 0 & 0 & 0 & 0 \end{bmatrix}^T, \\
\Phi_3^{(2)} = \begin{bmatrix} 0 & 0 & 0 & \widehat{\Pi}_{19}^T & \widehat{\Pi}_{47}^T & 0 & 0 & 0 \end{bmatrix}^T, \\
\overline{\Phi}_4 = \begin{bmatrix} 2d_M\mathbf{P}_1 + d_M\mathbf{R}_1 & 0 \\ 0 & 2d_M\mathbf{P}_2 + d_M\mathbf{R}_1 \end{bmatrix}, \\
\overline{\Phi}_5 = \begin{bmatrix} 2\tau\mathbf{P}_1 + \tau\mathbf{R}_2 & 0 \\ 0 & 2\tau\mathbf{P}_2 + \tau\mathbf{R}_2 \end{bmatrix}, \\
\overline{\Pi}_{11} = \begin{bmatrix} -\mathbf{P}_1 A - A^T\mathbf{P}_1 - \alpha\widehat{U}_1 & C^TV^T \\ VC & -\mathbf{P}_2 A - A^T\mathbf{P}_2 - VC - C^TV^T \end{bmatrix} \\
+ \begin{bmatrix} \mathbf{Q}_1 + \mathbf{Q}_2 - \dfrac{1}{d_M}\mathbf{R}_1 + \mathbf{S}_1 + \mathbf{S}_1^T & 0 \\ 0 & \mathbf{Q}_1 + \mathbf{Q}_2 - \dfrac{1}{d_M}\mathbf{R}_1 + \mathbf{S}_1 + \mathbf{S}_1^T \end{bmatrix}\end{gathered} \tag{3.23}$$

$$\overline{\Pi}_{12} = \begin{bmatrix} \frac{1}{d_M}\mathbf{R}_1 & 0 \\ VC & \frac{1}{d_M}\mathbf{R}_1 \end{bmatrix}, \qquad \overline{\Pi}_{13} = \begin{bmatrix} \mathbf{S}_2^T - \mathbf{S}_1 & 0 \\ 0 & \mathbf{S}_2^T - \mathbf{S}_1 \end{bmatrix}$$

$$\overline{\Pi}_{14} = \begin{bmatrix} \mathbf{P}_1 W_0 - \alpha\widehat{U}_2 \\ \mathbf{P}_2 W_0 \end{bmatrix}, \qquad \overline{\Pi}_{15} = \begin{bmatrix} \mathbf{P}_1 W_1 \\ \mathbf{P}_2 W_1 \end{bmatrix}$$

$$\overline{\Pi}_{16} = \begin{bmatrix} 0 \\ VC \end{bmatrix}, \qquad \overline{\Pi}_{17} = \begin{bmatrix} -d_M A^T\mathbf{P}_1 & d_M C^T V^T \\ 0 & -d_M A^T \mathbf{P}_2 - d_M C^T V^T \end{bmatrix}$$

$$\overline{\Pi}_{18} = \begin{bmatrix} -\tau A^T\mathbf{P}_1 & \tau C^T V^T \\ 0 & -\tau A^T \mathbf{P}_2 - \tau C^T V^T \end{bmatrix}, \qquad \overline{\Pi}_{19} = \begin{bmatrix} \tau\mathbf{S}_1 & 0 \\ 0 & \tau\mathbf{S}_1 \end{bmatrix}$$

$$\widehat{\Pi}_{19} = \begin{bmatrix} \tau\mathbf{T}_1 & 0 \\ 0 & \tau\mathbf{T}_1 \end{bmatrix}, \qquad \overline{\Pi}_{22} = \begin{bmatrix} -\frac{2}{d_M}\mathbf{R}_1 + \sigma C^T W C & 0 \\ 0 & -\frac{2}{d_M}\mathbf{R}_1 \end{bmatrix}$$

$$\overline{\Pi}_{23} = \begin{bmatrix} \frac{1}{d_M}\mathbf{R}_1 & 0 \\ 0 & \frac{1}{d_M}\mathbf{R}_1 \end{bmatrix}, \qquad \overline{\Pi}_{24} = \begin{bmatrix} 0 & -d_M C^T V^T \\ 0 & 0 \end{bmatrix}$$

$$\overline{\Pi}_{25} = \begin{bmatrix} 0 & \tau C^T V^T \\ 0 & 0 \end{bmatrix}, \qquad \overline{\Pi}_{33} = \begin{bmatrix} -\mathbf{Q}_1 - \frac{1}{d_M}\mathbf{R}_1 & 0 \\ 0 & -\mathbf{Q}_1 - \frac{1}{d_M}\mathbf{R}_1 \end{bmatrix}$$

$$\overline{\Pi}_{44} = \begin{bmatrix} -\mathbf{S}_2 - \mathbf{S}_2^T + \mathbf{T}_1 + \mathbf{T}_1^T - \beta\widehat{U}_1 & 0 \\ 0 & -\mathbf{S}_2 - \mathbf{S}_2^T + \mathbf{T}_1 + \mathbf{T}_1^T \end{bmatrix}, \qquad \overline{\Pi}_{45} = \begin{bmatrix} \mathbf{T}_2^T - \mathbf{T}_1 & 0 \\ 0 & \mathbf{T}_2^T - \mathbf{T}_1 \end{bmatrix}$$

$$\overline{\Pi}_{46} = \begin{bmatrix} -\beta\widehat{U}_2 \\ 0 \end{bmatrix}, \qquad \overline{\Pi}_{47} = \begin{bmatrix} \tau\mathbf{S}_2 & 0 \\ 0 & \tau\mathbf{S}_2 \end{bmatrix}, \qquad \widehat{\Pi}_{47} = \begin{bmatrix} \tau\mathbf{T}_2 & 0 \\ 0 & \tau\mathbf{T}_2 \end{bmatrix}$$

$$\overline{\Pi}_{55} = \begin{bmatrix} -\mathbf{Q}_2 - \mathbf{T}_2 - \mathbf{T}_2^T & 0 \\ 0 & -\mathbf{Q}_2 - \mathbf{T}_2 - \mathbf{T}_2^T \end{bmatrix}, \qquad \overline{\Pi}_{66} = -\alpha I$$

$$\overline{\Pi}_{67} = \begin{bmatrix} d_M W_0^T \mathbf{P}_1 & d_M W_0^T \mathbf{P}_2 \end{bmatrix}, \qquad \overline{\Pi}_{68} = \begin{bmatrix} \tau W_0^T \mathbf{P}_1 & \tau W_0^T \mathbf{P}_2 \end{bmatrix}, \qquad \overline{\Pi}_{77} = -\beta I$$

$$\overline{\Pi}_{78} = \begin{bmatrix} d_M W_1^T \mathbf{P}_1 & d_M W_1^T \mathbf{P}_2 \end{bmatrix}, \qquad \overline{\Pi}_{79} = \begin{bmatrix} \tau W_1^T \mathbf{P}_1 & \tau W_1^T \mathbf{P}_2 \end{bmatrix}, \qquad \overline{\Pi}_{88} = -C^T W C$$

$$\overline{\Pi}_{89} = \begin{bmatrix} 0 & -d_M C^T V^T \end{bmatrix}, \qquad \overline{\Pi}_{8,10} = \begin{bmatrix} 0 & -\tau C^T V^T \end{bmatrix}, \tag{3.24}$$

then the desired estimator gain matrix is given as $K = \mathbf{P}_2^{-1} V$.

Proof. By using Schur complement in Theorem 3.1, $\Pi_i < 0$ $(i = 1, 2)$ can be rewritten as

$$\begin{aligned} &\Pi + d_M \mathcal{A}^T R_1 \mathcal{A} + \tau\mathcal{A}^T R_2 \mathcal{A} + \tau S R_2^{-1} S^T < 0 \\ &\Pi + d_M \mathcal{A}^T R_1 \mathcal{A} + \tau\mathcal{A}^T R_2 \mathcal{A} + \tau T R_2^{-1} T^T < 0. \end{aligned} \tag{3.25}$$

By using Lemma 2.5, (3.25) are equivalent to the following matrix inequalities

$$\begin{bmatrix} \overline{\Pi} & \widehat{\Phi}_1 & \widehat{\Phi}_2 & \Phi_3^{(i)} \\ * & -d_M R_1^{-1} & 0 & 0 \\ * & * & -\tau R_2^{-1} & 0 \\ * & * & * & -\tau R_2 \end{bmatrix} < 0 \quad (i = 1, 2), \tag{3.26}$$

where

$$\overline{\Pi} = \begin{bmatrix} \overline{\Pi}_{11} & \overline{\Pi}_{12} & 0 & \overline{\Pi}_{13} & 0 & \overline{\Pi}_{14} & \overline{\Pi}_{15} & \overline{\Pi}_{16} \\ * & \overline{\Pi}_{22} & \overline{\Pi}_{23} & 0 & 0 & 0 & 0 & 0 \\ * & * & \overline{\Pi}_{33} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \overline{\Pi}_{44} & \overline{\Pi}_{45} & 0 & \overline{\Pi}_{46} & 0 \\ * & * & * & * & \overline{\Pi}_{55} & 0 & 0 & 0 \\ * & * & * & * & * & \overline{\Pi}_{66} & 0 & 0 \\ * & * & * & * & * & * & \overline{\Pi}_{77} & 0 \\ * & * & * & * & * & * & * & \overline{\Pi}_{88} \end{bmatrix}$$

$$\widehat{\Phi}_1 = \left[d_M\overline{A} \;\; d_M\overline{B} \;\; 0 \;\; 0 \;\; 0 \;\; d_M\overline{W}_0 \;\; d_M\overline{W}_1 \;\; d_M\overline{C} \right]^T$$

$$\widehat{\Phi}_2 = \left[\tau\overline{A} \;\; \tau\overline{B} \;\; 0 \;\; 0 \;\; 0 \;\; \tau\overline{W}_0 \;\; \tau\overline{W}_1 \;\; \tau\overline{C} \right]^T. \tag{3.27}$$

Then performing a congruence transformation of diag$\{I, P, P, I\}$ to (3.26), it can be derived that

$$\begin{bmatrix} \overline{\Pi} & \widetilde{\Phi}_1 & \widetilde{\Phi}_2 & \Phi_3^{(i)} \\ * & -d_M P R_1^{-1} P & 0 & 0 \\ * & * & -\tau P R_2^{-1} P & 0 \\ * & * & * & -\tau R_2 \end{bmatrix} < 0 \quad (i = 1, 2), \tag{3.28}$$

where

$$\widetilde{\Phi}_1 = \left[d_M P\overline{A} \;\; d_M P\overline{B} \;\; 0 \;\; 0 \;\; 0 \;\; d_M P\overline{W}_0 \;\; d_M P\overline{W}_1 \;\; d_M P\overline{C} \right]^T$$

$$\widetilde{\Phi}_2 = \left[\tau P\overline{A} \;\; \tau P\overline{B} \;\; 0 \;\; 0 \;\; 0 \;\; \tau P\overline{W}_0 \;\; \tau P\overline{W}_1 \;\; \tau P\overline{C} \right]^T. \tag{3.29}$$

Setting $P_2 K = V$ in (3.28) and considering the following inequality:

$$-P R_i^{-1} P \le -2P + R_i \quad (i = 1, 2). \tag{3.30}$$

By using (3.30), we can obtain

$$\begin{bmatrix} \overline{\Pi} & \tilde{\Phi}_1 & \tilde{\Phi}_2 & \Phi_3^{(i)} \\ * & -d_M P R_1^{-1} P & 0 & 0 \\ * & * & -\tau P R_2^{-1} P & 0 \\ * & * & * & -\tau R_2 \end{bmatrix} < \begin{bmatrix} \overline{\Pi} & \tilde{\Phi}_1 & \tilde{\Phi}_2 & \Phi_3^{(i)} \\ * & -2d_M P + d_M R_1 & 0 & 0 \\ * & * & -2\tau P + \tau R_2 & 0 \\ * & * & * & -\tau R_2 \end{bmatrix}. \tag{3.31}$$

Substitute $\overline{A}$, $\overline{B}$, $\overline{W}_0$, H,$\overline{W}_1$, $\overline{C}$, P, Q_i, R_i, S_i, T_i $(i = 1,2)$ into the right of (3.31), combining (3.22), we can obtain

$$\begin{bmatrix} \overline{\Pi} & \tilde{\Phi}_1 & \tilde{\Phi}_2 & \Phi_3^{(i)} \\ * & -2d_M P + d_M R_1 & 0 & 0 \\ * & * & -2\tau P + \tau R_2 & 0 \\ * & * & * & -\tau R_2 \end{bmatrix} < 0 \quad (i = 1,2). \tag{3.32}$$

The rest of the proof follows directly from Theorem 3.1. □

Remark 3.5. When the estimator gain matrix K is a matrix variable to be designed, in order to transform the conditions (3.3) to linear matrix inequalities, and meanwhile reduce the computational complexity (i.e., reduce the number of matrix variables), in Theorem 3.4, matrix variables in Theorem 3.1 are replaced by some diagonal matrices. Then setting $P_2 K = V$, we can obtain (3.22), which is in the form of linear matrix inequalities, which are easy to be verified by LMI toolbox.

Remark 3.6. It is noticed that $d_M = h + d$, if d_M is solved, we can select a sampling period $h < d_M$. For given d, the maximal allowable sampling period $h_{\max}$ can be obtained by the following two-step procedure.

(1) For given τ and d, setting $h_{\max} = h_0$ and step size STEP $=$ STEP$_0$, where h_0 and STEP$_0$ are two specified positive constants.

(2) If LMIs (3.22) are feasible, set $h_{\max} = h_0 + \text{STEP}_0$ and return to step (2): otherwise, h is the maximal allowable sampling period.

4. Numerical Results

In this section, a numerical example is given to verify the effectiveness of the proposed control techniques for estimation of recurrent neural networks with time-varying delays.

Example 4.1. Consider recurrent neural networks (2.1) with the following parameters

$$A = \begin{bmatrix} 1.5 & 0 \\ 0 & 2 \end{bmatrix}, \qquad W_0 = \begin{bmatrix} 0.3 & -0.4 \\ -0.4 & 0.3 \end{bmatrix}, \qquad W_1 = \begin{bmatrix} 0.3 & 0.3 \\ 0.3 & 0.3 \end{bmatrix}, \qquad C = \begin{bmatrix} 0.9 & 0.8 \\ 0.7 & 0.5 \end{bmatrix}. \tag{4.1}$$

The neuron activation function is described as follows:

$$g(x) = \begin{bmatrix} 0.5x_1(t) - \tanh(0.2x_1(t)) + 0.2x_2(t) \\ 0.95x_2(t) - \tanh(0.75x_2(t)) \end{bmatrix}. \tag{4.2}$$

Table 1: $d_M = 0.01$.

σ	0	0.01	0.1	0.2	0.3	0.4	0.5
τ	1.2134	1.1966	1.1572	1.1570	1.1569	1.1569	1.1569

Table 2: $\tau = 1$, $d = 0.01$.

σ	0	0.01	0.1	0.15	0.2	0.3	0.99
$h_{\max}$	0.2244	0.2106	0.1998	0.1998	0.1998	0.1998	0.1998

It is easy to verify that the nonlinear function $f(\cdot)$ satisfies Assumption 2.3; by some simple calculations, we can obtain

$$U_1 = \begin{bmatrix} 0.3 & 0.2 \\ 0 & 0.2 \end{bmatrix} \qquad U_2 = \begin{bmatrix} 0.5 & 0.2 \\ 0 & 0.95 \end{bmatrix}. \tag{4.3}$$

Setting $d_M = 0.01$ and $\sigma = 0.1$, by applying Theorem 3.4, it can be obtained the maximum allowable delay $\tau = 1.1572$. More detailed calculation results for different values of σ are given in Table 1. It can be shown that the larger σ, the smaller τ. For given $\tau = 1$ and $d = 0.01$, based on Remark 3.6, we can obtain the maximal allowable sampling period $h_{\max}$, which are shown in Table 2. For given $\tau = 1$, $\sigma = 0.1$ and $d_M = 0.01$, by using LMI Toolbox in LMIs (3.22), the feasible solution can be obtained as follows:

$$\begin{aligned}
&\mathbf{P}_1 = \begin{bmatrix} 5.4676 & -0.1329 \\ -0.1329 & 5.4000 \end{bmatrix}, \quad \mathbf{P}_2 = \begin{bmatrix} 5.0204 & -0.1155 \\ -0.1155 & 4.2212 \end{bmatrix}, \quad \mathbf{Q}_1 = \begin{bmatrix} 2.7863 & -0.0955 \\ -0.0955 & 2.0982 \end{bmatrix} \\
&\mathbf{Q}_2 = \begin{bmatrix} 3.2539 & -0.0133 \\ -0.0133 & 2.8910 \end{bmatrix}, \quad \mathbf{R}_1 = \begin{bmatrix} 0.0222 & 0.0010 \\ 0.0010 & 0.0234 \end{bmatrix}, \quad \mathbf{R}_2 = \begin{bmatrix} 2.2237 & -0.0246 \\ -0.0246 & 1.4752 \end{bmatrix} \\
&\mathbf{S}_1 = \begin{bmatrix} -0.5503 & 0.0195 \\ 0.0478 & -0.4156 \end{bmatrix}, \quad \mathbf{S}_2 = \begin{bmatrix} 1.0185 & 0.0674 \\ 0.1378 & 1.1920 \end{bmatrix}, \quad \mathbf{T}_1 = \begin{bmatrix} -1.0844 & -0.0634 \\ -0.0902 & -1.2378 \end{bmatrix} \\
&\mathbf{T}_2 = \begin{bmatrix} 0.3511 & -0.0221 \\ -0.0195 & 0.3384 \end{bmatrix}, \quad V = \begin{bmatrix} -0.0792 & -0.1282 \\ -0.1755 & -0.0863 \end{bmatrix}, \quad \alpha = 6.4604, \ \beta = 5.7723.
\end{aligned} \tag{4.4}$$

Then the triggered matrix and the desired estimator can be obtained as follows:

$$W = \begin{bmatrix} 4.6153 & -2.7354 \\ -2.7354 & 6.5131 \end{bmatrix}, \qquad K = \begin{bmatrix} -0.0165 & -0.0308 \\ -0.0354 & -0.0214 \end{bmatrix}. \tag{4.5}$$

For giving the sampling period $h = 0.005$, Table 3 gives the relation of the trigger parameter σ, trigger times, the average release period, and the percentage of data transmissions; it can be seen that the larger the σ, the smaller trigger times; the larger average release period, the smaller percentage of data transmission, which are reasonable results. In the following, we provide some simulation results: when $\sigma = 0$, the time varying delay $\tau(t)$ obeys uniform distribution over $[0, 1]$, and the curves of the error dynamics of the neural networks $e_i(t)$ $(i = 1, 2)$ are depicted in Figure 1, from which we can see the errors converge to zero asymptotically. If setting $\sigma = 0.1$, The response of the error dynamics for the delayed

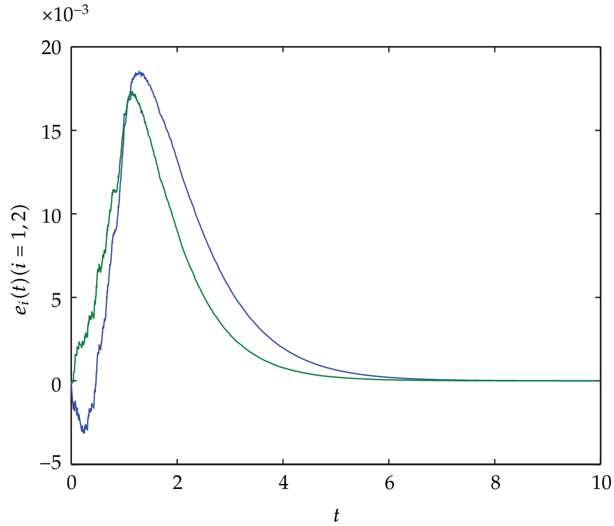

Figure 1: The error curves $e_i(t)$ $(i = 1, 2)$ with trigger parameter $\sigma = 0$ (time-triggered scheme).

Table 3: $h = 0.005$, $d_M = 0.01$, $\tau = 0.1$, $t = 10$.

σ	0	0.01	0.1
Trigger times	2000	188	74
Trigger matrix W	$\begin{bmatrix} 0.7582 & -0.2843 \\ -0.2843 & 0.9490 \end{bmatrix}$	$\begin{bmatrix} 0.7504 & -0.2881 \\ -0.2881 & 0.9444 \end{bmatrix}$	$\begin{bmatrix} 4.6153 & -2.7354 \\ -2.7354 & 6.5131 \end{bmatrix}$
Estimator matrix K	$\begin{bmatrix} -0.0070 & -0.0553 \\ -0.0398 & -0.0257 \end{bmatrix}$	$\begin{bmatrix} -0.0068 & -0.0552 \\ -0.0394 & -0.0255 \end{bmatrix}$	$\begin{bmatrix} -0.0165 & -0.0308 \\ -0.0354 & -0.0214 \end{bmatrix}$
Average release period	0.0050	0.0531	0.1348
Data transmission	100%	9.42%	3.71%

neural networks (2.17) which converge to zero asymptotically in the mean square is given in Figure 2. Figure 3 shows the event-triggered release instants and intervals. It can be seen from Figures 1 and 2 that the simulation results are almost the same, but the percentage of data transmission under even-triggered scheme used much small number than time-triggering scheme. To make this clear, seen the computation results lists in Table 2, from which we can see that data transmission rate with even-triggered scheme ($\sigma = 0.1$) is only 3.71% of sampled measurement output with time-triggered scheme ($\sigma = 0$); from these results, we can draw a conclusion that event-triggered scheme has advantage over the time-triggered scheme in improving the resource utilization.

5. Conclusions

This paper has provided a novel event-triggered scheme to investigate the sampled-data state estimation problem for a class of recurrent neural networks with time-varying delays. This scheme can lead to a significant reduction of the information communication burden in the

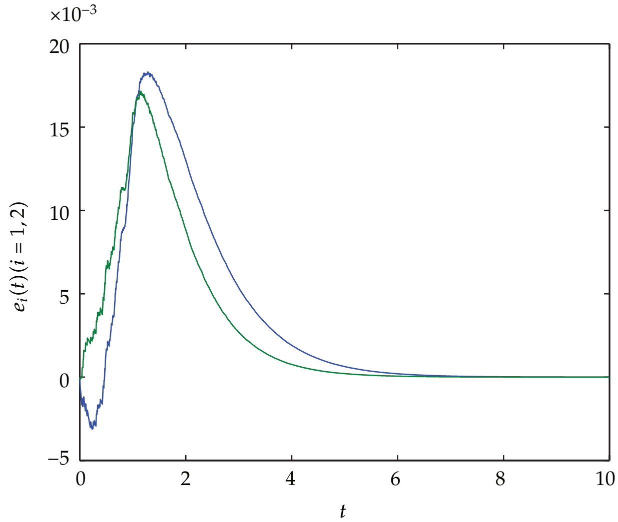

Figure 2: The error curves $e_i(t)$ $(i = 1, 2)$ with trigger parameter $\sigma = 0.1$ (event-triggered scheme).

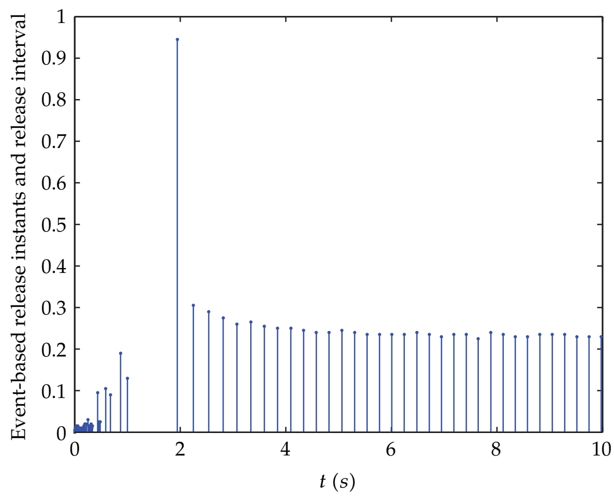

Figure 3: Release instants and release interval by event-triggered scheme.

network. By using a delayed-input approach, the error dynamics system is equivalently to a dynamic system with two different time-varying delays. Based on the Lyapunov-krasovskii functional approach, a state estimator of the considered neural networks can be achieved by solving some linear matrix inequalities, which can be readily solved by using the standard numerical software. Finally, an illustrative example is exploited to show the effectiveness of the event-triggered scheme.

Acknowledgments

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Research Article

Dynamical Behaviors of Impulsive Stochastic Reaction-Diffusion Neural Networks with Mixed Time Delays

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We discuss the dynamical behaviors of impulsive stochastic reaction-diffusion neural networks (ISRDNNs) with mixed time delays. By using a well-known L-operator differential inequality with mixed time delays and combining with the Lyapunov-Krasovkii functional approach, as well as linear matrix inequality (LMI) technique, some novel sufficient conditions are derived to ensure the existence, uniqueness, and global exponential stability of the periodic solutions for ISRDNNs with mixed time delays in the mean square sense. The obtained sufficient conditions depend on the reaction-diffusion terms. The results of this paper are new and improve some of the previously known results. The proposed model is quite general since many factors such as noise perturbations, impulsive phenomena, and mixed time delays are considered. Finally, two numerical examples are provided to verify the usefulness of the obtained results.

1. Introduction

In recent years, neural networks (NNs) with time delays have received considerable attention due to their extensive applications in solving some optimization problems, associative memory, classification of patterns, and other areas. In implementation of NNs, however, time delays are unavoidably encountered. It has been found that the existence of time delays may lead to instability and oscillation in a neural network. Therefore, the analysis of the dynamical behaviors such as stability, periodic oscillation, and chaotic behavior are necessary work for practical design of delayed NNs [1–12]. Zheng and Chen [1] studied the exponential stability for delayed periodic dynamical systems. In [2], the global exponential stability and periodicity of a class of recurrent NNs with time delays are addressed by using Lyapunov functional method and inequality techniques. In the factual operations, however, the diffusion phenomena could not be ignored in NNs when electrons are moving in asymmetric electromagnetic fields. So we must consider that the activations vary in space as well as in time. The NNs with diffusion terms can commonly be expressed by partial differential equations [13–33]. The authors in [13, 19, 20] have dealt with obtaining sufficient conditions for the global exponential stability and periodicity of delayed reaction-diffusion neural networks (RDNNs).

As is well known, besides delays and diffusion effects, impulsive phenomena can be found in a wide variety of evolutionary process, particularly in some biological systems such as biological NNs and bursting rhythm models in pathology, as well as optimal control models in economics, frequency-modulated signal processing systems, and flying object motions, in which many sudden and sharp changes occur instantaneously, in the form of impulse. For example, in implementation of electronic networks, the state of the networks is subject to instantaneous perturbations and experiences abrupt change at certain instants, which may be caused by switching phenomenon, frequency change, or other sudden noise, that is, it exhibits impulsive effects. As artificial electronic system, neural networks are often subject to impulsive perturbations that in turn affect dynamical behaviors of the systems [17, 18, 25–27]. In [17, 26, 27], the global exponential stability for the equilibrium point of impulsive RDNNs with delays was investigated.

However, the models studied in the above mentioned papers have been largely restricted to deterministic RDNNs. In the real world, a real system is usually affected by external perturbations which in many cases are of great uncertainty and hence may be treated as random. As pointed out by Haykin [34] that in real nervous systems synaptic transmission is a noisy process brought on by random fluctuations from the release of neurotransmitters and other probabilistic causes. Hence, it is of significant importance to study stochastic effects for the neural networks. In recent years, the dynamic behavior of stochastic NNs, especially the stability of stochastic NNs, has become a hot study topic. Very recently, several kinds of NNs with delays and stochastic effects have been investigated [22, 28–30]. Lv et al. [22] and Xu et al. [29] have obtained some criteria to guarantee the almost sure exponential stability and mean square exponential stability of an equilibrium solution for RDNNs with continuously distributed delays and stochastic influence, respectively. It is noticed that the authors do not take impulsive phenomena and diffusion effects into account on the dynamic behaviors of RDNNs.

It is well known that not only diffusion effects and delays cannot be avoided but also the existence of impulsive and stochastic effects is extensive in the NNs. Moreover, the interconnection weights b_{ij}, $\widetilde{b}_{ij}$, $\overline{b}_{ij}$, self-inhibition a_i and inputs J_i in the NNs may be variable with time, often periodically. Therefore, it is necessary to consider impulsive and stochastic effects to the stability of RDNNs with mixed time delays and their periodic limits. Unfortunately, to the best of our knowledge, the existence and global exponential stability of periodic solutions have been seldom considered for ISRDNNs with variable coefficients and mixed time delays. Due to the simultaneous presence of impulsive stochastic effects, reaction-diffusion phenomena, periodicity, variable coefficients, and mixed time delays, the dynamical behaviors become much more complex and therefore pose significant difficulties in the analysis.

Based on the above discussions, in this paper, we aim to challenge the analysis problem on dynamical behaviors of ISRDNNs with mixed time delays. By applying a well-known L-operator differential inequality with mixed time delays and combining with the Lyapunov-Krasovkii functional approach, as well as linear matrix inequality (LMI) technique, we have derived some easy-to-test sufficient conditions for the existence and exponential stability of the periodic solutions for ISRDNNs with variable coefficients and mixed time delays. The obtained criteria depend on the reaction-diffusion terms. The results of this paper are new and they complement previously known results. Furthermore, we do not need the differentiability of the time-varying delays. Two examples are employed to demonstrate the effectiveness of the obtained results that are less restrictive than recently known criteria.

Notation. Throughout this paper, the following notations will be used. $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$ denote the n-dimensional Euclidean real space equipped with the norm $|\cdot|$ and the set of all $n \times n$ real matrices, respectively. Trace$(\cdot)$ denotes the trace of the corresponding matrix and I denotes the identity matrix with appropriate dimensions. For square matrices A and B, the notation $A > (\geq, <, \leq)\, B$ denotes $A - B$ is positive-definite (positive-semidefinite, negative, negative-semidefinite) matrix. L denotes the well-known L-operator given by the Ito formula. Let $w(t) = (w_1(t), \ldots, w_n(t))^T$ is an n-dimensional standard Brownian motion defined on a complete probability space $(\Omega, F, \{F_t\}_{t\geq 0}, P)$ with a natural filtration $\{F_t\}_{t\geq 0}$ generated by $\{w(s) : 0 \leq s \leq t\}$. $E(\cdot)$ stands for the mathematical expectation operator. Z^+ is the set of nonnegative integral numbers.

$P\mathbb{C}[(-\infty, 0] \times \Omega, \mathbb{R}^n] = \{\varphi : (-\infty, 0] \times \Omega \to \mathbb{R}^n \mid \varphi(s^+, x) = \varphi(s, x)$ for $s \in (-\infty, 0]$, $\varphi(s^-, x)$ exists for $s \in (-\infty, 0]$, $\varphi(s^-, x) = \varphi(s, x)$ for all but at most countable points $s \in (-\infty, 0]\}$, where $\varphi(t^-, x)$ and $\varphi(t^+, x)$ denote the left-hand limit and the right-hand limit of $\varphi(t, x)$ at time t, respectively. Especially, let $P\mathbb{C} = P\mathbb{C}[(-\infty, 0] \times \Omega, \mathbb{R}^n]$. For $\varphi \in P\mathbb{C}$, we always assume that φ is bounded and introduce the norm $\|\varphi\| = \sup_{-\infty\leq s\leq 0}(\sum_{i=1}^{n} \varphi_i^2(s))^{1/2}$.

Let $P\mathbb{C}^b_{F_0}[(-\infty, 0] \times \Omega, \mathbb{R}^n]$ denote the family of all bounded F_0-measurable, $P\mathbb{C}[(-\infty, 0] \times \Omega, \mathbb{R}^n]$-valued random variables φ, such that $\|\varphi\|_\tau = \sup_{-\infty\leq s\leq 0} E|\varphi(s)|^2 < \infty$. Especially, let $P\mathbb{C}^b_{F_0} = P\mathbb{C}^b_{F_0}[(-\infty, 0] \times \Omega, \mathbb{R}^n]$. Let $u = (u_1, \ldots, u_n)^T \in \mathbb{R}^n$ and $L^2(\Omega)$ is the space of scalar value Lebesgue measurable functions on Ω which is a Banach space for the L_2-norm:

$$\|u\|_2 = \left(\int_\Omega |u(x)|^2 dx\right)^{1/2}, \quad u \in L^2(\Omega), \tag{1.1}$$

where $|\cdot|$ is Euclid norm of a vector $u \in \mathbb{R}^n$.

2. Model Description and Preliminaries

Consider the following ISRDNNs with mixed time delays system:

$$\begin{aligned}
du_i(t,x) ={}& \sum_{l=1}^{m} \frac{\partial}{\partial x_l}\left(D_{il}\frac{\partial u_i(t,x)}{\partial x_l}\right)dt \\
&+ \left[-a_i(t)u_i(t,x) + \sum_{j=1}^{n} b_{ij}(t) f_j\big(u_j(t,x)\big) + \sum_{j=1}^{n} \tilde{b}_{ij}(t)\tilde{f}_j\big(u_j(t-\tau(t),x)\big)\right. \\
&\left. + \sum_{j=1}^{n} \overline{b}_{ij}(t)\int_{-\infty}^{t} k_{ij}(t-s)\overline{f}_j\big(u_j(s,x)\big)ds + J_i(t)\right]dt \\
&+ \sum_{j=1}^{n} \sigma_{ij}\Big(t, x, u(t,x), u(t-\tau(t),x)\Big)dw_j(t), \quad t \geq 0,\ t \neq t_k,\ x \in \Omega,\ k \in Z^+, \\
u_i(t,x) ={}& u_i(t^-,x) - \theta_{ik}u_i(t^-,x), \quad t = t_k,\ x \in \Omega,\ k \in Z^+,
\end{aligned} \tag{2.1}$$

where $i \in N = \{1, 2, \ldots, n\}$, $n \geq 2$, corresponds to the number of units in an NN; the time sequence t_k is called impulsive moment and satisfies $0 < t_0 < t_1 < \cdots < t_k < t_{k+1} < \cdots$,

$\lim_{k\to\infty} t_{k+1} = \infty$; θ_{ik} are some real constants; $x = (x_1, \dots, x_m)^T \in \Omega$, Ω is a compact set with smooth boundary $\partial\Omega$ and $\mathrm{mes}\Omega > 0$ in space $\mathbb{R}^m$, where $\mathrm{mes}\Omega$ is the measure of the set Ω; $u_i(t,x)$ represents the state of the ith neuron at time t and in space x; $b_{ij}(t), \widetilde{b}_{ij}(t)$, and $\overline{b}_{ij}(t)$ denote the strength of the jth neuron on the ith neuron, respectively; f_j, $\widetilde{f}_j$, and $\overline{f}_j$ denote the activation functions of the jth neuron at time t and in space x; J_i denotes the external inputs on the ith neurons; $a_i(t)$ is the rate with which the ith unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs at time t and in space x; $\tau(t)$ represents the transmission delay with $0 \le \tau(t) \le \tau$, τ is a constant; smooth functions $D_{il} > 0$ $(i = 1, 2, \dots, n,\ l = 1, 2, \dots, m)$ stand for the transmission diffusion operators along the ith neuron; the delay kernel $k_{ij}(\cdot)$ is the real value nonnegative continuous function defined on $(0, +\infty)$; $u_i(t^-, x)$ and $u_i(t^+, x)$ denote the left-hand limit and the right-hand limit of $u_i(t,x)$ at time t, respectively. We assume $u_i(t_k, x) = u_i(t_k^+, x)$.

$\sigma_{ij}(t, x, u(t,x), u(t-\tau(t), x))$ $(i, j = 1, 2, \dots, n)$ denotes the weight function of random perturbation.

The boundary conditions and initial conditions are given by

$$\begin{gathered} u_i(t,x) = 0, \quad (t,x) \in [0, +\infty) \times \partial\Omega, \\ u_i(t_0 + s, x) = \varphi_i(s,x), \quad (s,x) \in (-\infty, 0] \times \Omega, \end{gathered} \tag{2.2}$$

where $\varphi = (\varphi_1, \dots, \varphi_n)^T \in P\mathbb{C}^b_{F_0}$.

In fact, some famous NNs models became a special case of system (2.1). For example, when $\sigma_{ij} = 0$, $i, j \in N$, the special case of system (2.1) is the model which has been investigated [25, 27, 32]. When $\theta_{ik} = 0$, $i = 1, 2, \dots, n$, $k \in Z^+$, then system (2.1) becomes stochastic RDNNs with mixed delays, which has been considered in [22, 29]. If $\theta_{ik} = 0$ and $\sigma_{ij} = 0$, $i, j \in N$, $k \in Z^+$, system (2.1) reduces to the deterministic system with mixed time delays:

$$\begin{aligned} \frac{du_i(t)}{dt} = {} & \sum_{l=1}^{m} \frac{\partial}{\partial x_l}\left(D_{il}\frac{\partial u_i(t)}{\partial x_l}\right) - a_i(t)u_i(t) + \sum_{j=1}^{n} b_{ij}(t) f_j(u_j(t,x)) \\ & + \sum_{j=1}^{n} \widetilde{b}_{ij}(t)\widetilde{f}_j(u_j(t-\tau(t),x)) + \sum_{j=1}^{n} \overline{b}_{ij}(t)\int_{-\infty}^{t} k_{ij}(t-s)\overline{f}_j(u_j(s,x))ds + J_i, \end{aligned} \tag{2.3}$$

the dynamical behaviors of the special case for model (2.3) have been discussed by many authors [19, 20]. Therefore, the model (2.1) is new and more general than those discussed in the previous literature.

Throughout this paper, we assume that the following conditions are made.

(A1) Suppose that $a_i(t) > 0$, $b_{ij}(t), \widetilde{b}_{ij}(t), \overline{b}_{ij}(t), \tau(t) \ge 0$ and $J_i(t)$ are all continuously periodic functions defined on $[0, +\infty)$ with common period $\omega > 0$. Moreover,

$$\widehat{a}_i = \min_{t\in[0,\omega]}\{a_i(t)\}, \quad \widehat{b}_{ij} = \max_{t\in[0,\omega]}\{|b_{ij}(t)|\}, \quad \widehat{\widetilde{b}}_{ij} = \max_{t\in[0,\omega]}\left\{\left|\widetilde{b}_{ij}(t)\right|\right\}, \quad \widehat{\overline{b}}_{ij} = \max_{t\in[0,\omega]}\left\{\left|\overline{b}_{ij}(t)\right|\right\}, \quad i, j \in N. \tag{2.4}$$

(A2) There exist positive diagonal matrices $L^f = \mathrm{diag}(L_1^f, \ldots, L_n^f)$, $L^{\tilde{f}} = \mathrm{diag}(L_1^{\tilde{f}}, \ldots, L_n^{\tilde{f}})$, $L^{\overline{f}} = \mathrm{diag}(L_1^{\overline{f}}, \ldots, L_n^{\overline{f}})$, such that for all $\eta_1, \eta_2 \in \mathbb{R}$

$$\begin{gathered}
\left|f_j(\eta_1) - f_j(\eta_2)\right| \leq L_j^f |\eta_1 - \eta_2|, \\
\left|\tilde{f}_j(\eta_1) - \tilde{f}_j(\eta_2)\right| \leq L_j^{\tilde{f}} |\eta_1 - \eta_2|, \\
\left|\overline{f}_j(\eta_1) - \overline{f}_j(\eta_2)\right| \leq L_j^{\overline{f}} |\eta_1 - \eta_2|, \quad j = 1, 2, \ldots, n.
\end{gathered} \tag{2.5}$$

(A3) The delay kernel $k_{ij}(\cdot) : [0, +\infty) \to [0, +\infty)$, $(i, j \in N)$ are real-valued nonnegative continuous functions that satisfy the following conditions:

(i) $\int_0^{+\infty} k_{ij}(s) ds = 1$,

(ii) $k_{ij}(s) \leq \kappa(s)$ for all $s \in [0, +\infty)$, in which $\kappa(s) : [0, +\infty) \to R^+$ is continuous and integral and satisfies $\int_0^{+\infty} \kappa(s) e^{\eta s} ds < +\infty$, where the constant η denotes some positive number.

(A4) For $\omega > 0$, there exists $q \in Z^+$ such that $t_k + \omega = t_{k+q}$ and $\theta_{ik} + \omega = \theta_{i(k+q)}$, $k \in Z^+$, $i \in N$.

(A5) There exist nonnegative constants δ_i and γ_i such that

$$(\sigma_i(t, x, \xi_i', \varsigma_i') - \sigma_i(t, x, \xi_i, \varsigma_i))(\sigma_i(t, x, \xi_i', \varsigma_i') - \sigma_i(t, x, \xi_i, \varsigma_i))^T \leq \delta_i |\xi_i' - \xi_i|^2 + \gamma_i |\varsigma_i' - \varsigma_i|^2, \tag{2.6}$$

for all $\xi_i, \varsigma_i, \xi_i', \varsigma_i' \in \mathbb{R}$, $\sigma_i(t, x, \xi, \varsigma) = (\sigma_{i1}(t, x, \xi, \varsigma), \ldots, \sigma_{in}(t, x, \xi, \varsigma))$ is the ith row vector of $\sigma(t, x, \xi, \varsigma)$, $i \in N$.

For convenience, $u_i(t, x), \varphi_i(s, x)$ are denoted as $u_i(t)$or $u_i, \varphi_i(s)$ or φ_i, respectively, if no confusion should occur.

Definition 2.1. An equilibrium point $u^* = (u_1^*, u_2^*, \ldots, u_n^*)$ of system (2.1)-(2.2) is said to be globally exponentially stable in the mean square sense if there exist positive constants ε and $M \geq 1$ such that

$$E\|u(t, x) - u^*\|_2 \leq M \|\varphi - u^*\|_\tau e^{-\varepsilon(t - t_0)}, \quad t \geq t_0 \geq 0. \tag{2.7}$$

Definition 2.2. The system (2.1)-(2.2) is said to be globally exponentially periodic in the mean square sense if (i) there exist one ω-periodic solutions; (ii) all other solutions of the system converge exponentially to it in the mean square sense as $t \to +\infty$.

Lemma 2.3 (see [24]). *Let Ω be a cube $|x_i| < d_l$ $(l = 1, \ldots, m)$ and let $h(x)$ be a real-valued function belonging to $C^1(\Omega)$ which vanish on the boundary $\partial\Omega$ of Ω, that is, $h(x)|_{\partial\Omega} = 0$. Then*

$$\int_\Omega h^2(x) dx \leq d_l^2 \int_\Omega \left|\frac{\partial h}{\partial x_i}\right| dx. \tag{2.8}$$

Remark 2.4. The boundary conditions of the investigated RDNNs in [22, 24, 26–28, 35] are all the Neumann boundary conditions. The obtained global exponential stability criteria are independent of diffusion term. In other words, these criteria are same whether the diffusion terms exist or not. However, it is also common to consider the diffusion effects in biological systems (such as immigration [36]). In this paper, we investigate dynamical behaviors of ISRDNNs with Dirichlet boundary conditions and mixed delays. The obtained criteria depend on the reaction-diffusion terms. The Lemma 2.3 plays a key role in the reported criteria which are dependent of diffusion terms.

Lemma 2.5 (see [4]). *Let p, q, r, and β_k, $(k \in Z^+)$ be nonnegative constants, and function $V(x) \in P\mathbb{C}^2(\mathbb{R}^n, \mathbb{R}^+)$, LV associated with system* (2.1)*, satisfy the following inequalities:*

$$LV(x(t)) \le -pV(x(t)) + q \sup_{t-\tau \le s \le t} V(x(s)) + r \int_0^{+\infty} \kappa(s) V(x(t-s)) ds, \quad t \ne t_k,\ t \ge 0,$$
$$V(x(t_k)) \le \beta_k V\big(x(t_k^-)\big), \quad k \in Z^+, \tag{2.9}$$

where $\kappa(s)$ is the same as (A3). Assume that

(i) $p > q + r \int_0^{+\infty} \kappa(s) ds$;

(ii) *there exist constants $M > 0$, $\alpha > 0$ such that*

$$\prod_{k=1}^{n} \max\{1, \beta_k\} \le M e^{\alpha t_n}, \quad n \in Z^+. \tag{2.10}$$

Then

$$EV(x(t)) \le MEV_0 e^{-(\lambda-\alpha)t}, \quad t \ge t_0, \tag{2.11}$$

where $EV_0 = \sup_{-\infty < s \le 0} EV(x(s))$, $\lambda \in (0, \eta)$ satisfies $\lambda < p - qe^{\lambda\tau} - r \int_0^{+\infty} \kappa(s) e^{\lambda s} ds$.

Remark 2.6. The above result (2.11) on the impulsive delay differential inequality is an extension of continuous case in [37] and will play an important role in the following qualitative analysis of ISRDNNs with mixed time delays.

Lemma 2.7 (see [38]). *Let $a, b \in \mathbb{R}^n$ and X be an $n \times n$ positive definite matrix, then*

$$2a^T b \le a^T X a + b^T X^{-1} b. \tag{2.12}$$

3. Main Results

This section deals with obtaining sufficient conditions that guarantee the existence and global exponential stability of periodic solution for the system (2.1)-(2.2).

Theorem 3.1. *In addition to (A1)–(A5) and further assume that*

(A6) $p > q + r \int_0^{+\infty} \kappa(s) ds$, (A7) *there exist constants* $M \geq 1, \lambda \in (0, \eta)$ *and* $\alpha \in [0, \lambda)$ *such that*

$$\prod_{k=1}^{n} \max\{1, \beta_k\} \leq M e^{\alpha t_n}, \quad n \in Z^+,$$
$$\lambda < p - q e^{\lambda\tau} - r \int_0^{+\infty} \kappa(s) e^{\lambda s} ds, \tag{3.1}$$

where

$$\begin{aligned}
p = {} & 2\sum_{l=1}^{m} \frac{\min_{i\in N}(D_i)}{d_l^2} + 2\min_{i\in N}(\widehat{a}_i) \\
& - \left[\max_{i\in N}\left(\sum_{j=1}^{n} \left|\widehat{b}_{ij}\right| L_j^f \right) + \sum_{i=1}^{n} \max_{j\in N}\left(\left|\widehat{b}_{ij}\right| L_j^f \right) \right. \\
& \left. \quad + \max_{i\in N}\left(\sum_{j=1}^{n} \left|\widehat{\overline{b}}_{ij}\right| L_j^{\overline{f}} \right) + \max_{i\in N}\left(\sum_{j=1}^{n} \left|\widehat{\overline{b}}_{ij}\right| L_j^{\overline{f}} \right) + \max_{i\in N}\{\delta_i\} \right], \\
q = {} & \left[\sum_{i=1}^{n} \max_{j\in N}\left(\left|\widehat{b}_{ij}\right| L_j^f \right) + \max_{i\in N}\{\gamma_i\} \right], \\
r = {} & \sum_{i=1}^{n} \max_{j\in N}\left(\left|\widehat{\overline{b}}_{ij}\right| L_j^{\overline{f}} \right), \qquad \beta_k = \max_{i\in N}\left\{ (1-\theta_{ik})^2 \right\}, \\
D_i = {} & \min_{1\leq l\leq m}\{D_{il}\},
\end{aligned} \tag{3.2}$$

then system (2.1)-(2.2) *is globally exponentially periodic in the mean square sense.*

Proof. For any $\psi = (\psi_1, \ldots, \psi_n)^T$, $\varphi = (\varphi_1, \ldots, \varphi_n)^T \in P\mathbb{C}_{F_0}^b$, let $\overline{u}(t) = (\overline{u}_1(t), \ldots, \overline{u}_n(t))^T$ and $\underline{u}(t) = (\underline{u}_1(t), \ldots, \underline{u}_n(t))^T$ be the solutions of system (2.1)-(2.2) starting from ψ and φ, respectively.

Let $z_i(t) = \overline{u}_i(t) - \underline{u}_i(t)$, from (2.1), we get

$$\begin{aligned}
dz_i(t) = {} & \sum_{l=1}^{m} \frac{\partial}{\partial x_l}\left(D_{il} \frac{\partial z_i(t)}{\partial x_l} \right) dt + \left[-a_i(t) z_i(t) + \sum_{j=1}^{n} b_{ij}(t)\left(f_j(\overline{u}_j(t)) - f_j\left(\underline{u}_j(t)\right) \right) \right. \\
& + \sum_{j=1}^{n} \widetilde{b}_{ij}(t)\left(\widetilde{f}_j(\overline{u}_j(t-\tau(t))) - \widetilde{f}_j\left(\underline{u}_j(t-\tau(t))\right) \right) \\
& \left. + \sum_{j=1}^{n} \overline{b}_{ij}(t) \int_{-\infty}^{t} k_{ij}(t-s)\left(\overline{f}_j(\overline{u}_j(s)) - \overline{f}_j\left(\underline{u}_j(s)\right) \right) ds \right] dt \\
& + \sum_{j=1}^{n} \left[\sigma_{ij}(t, x, \overline{u}_i(t), \overline{u}_i(t-\tau(t))) - \sigma_{ij}(t, x, \underline{u}_i(t), \underline{u}_i(t-\tau(t))) \right] dw_j(t).
\end{aligned} \tag{3.3}$$

Construct the Lyapunov functional $V(t) = \int_\Omega \sum_{i=1}^{n} z_i^2(t) dx,\ i \in N,$

for $t = t_k$, from (2.1) and (A4), we have

$$\begin{aligned} V(t_k) &= \int_\Omega \sum_{i=1}^n z_i^2(t_k)dx = \int_\Omega \sum_{i=1}^n \left[\overline{u}_i(t_k) - \underline{u}_i(t_k)\right]^2 dx \\ &= \int_\Omega \sum_{i=1}^n \left[\overline{u}_i(t_k+\omega) - \underline{u}_i(t_k)\right]^2 dx = \int_\Omega \sum_{i=1}^n (1-\theta_{ik})^2 \left[\overline{u}_i\left(t_{k+q}^-\right) - \underline{u}_i\left(t_k^-\right)\right]^2 dx \\ &\leq \max_{i\in N}(1-\theta_{ik})^2 \int_\Omega \sum_{i=1}^n \left[\overline{u}_i(t_k^- + \omega) - \underline{u}_i(t_k^-)\right]^2 dx = \max_{i\in N}(1-\theta_{ik})^2 V(t_k^-), \end{aligned} \tag{3.4}$$

when $t \in (t_{k-1}, t_k]$, the infinitesimal operator of $LV(t)$ along with system (3.3) is

$$\begin{aligned} LV(t) = &\int_\Omega 2\sum_{i=1}^n z_i(t) \\ &\times \Bigg\{ \sum_{l=1}^m \frac{\partial}{\partial x_l}\left(D_{il}\frac{\partial z_i(t)}{\partial x_l}\right) - a_i(t)z_i(t) \\ &\quad + \sum_{j=1}^n b_{ij}(t)\left[f_j(\overline{u}_j(t)) - f_j\left(\underline{u}_j(t)\right)\right] + \sum_{j=1}^n \widetilde{b}_{ij}(t)\left(\widetilde{f}_j(\overline{u}_j(t-\tau(t))) - \widetilde{f}_j\left(\underline{u}_j(t-\tau(t))\right)\right) \\ &\quad + \sum_{j=1}^n \overline{b}_{ij}(t)\int_{-\infty}^t k_{ij}(t-s)\left[\overline{f}_j(\overline{u}_j(s)) - \overline{f}_j\left(\underline{u}_j(s)\right)\right]ds \Bigg\} dx \\ &+ \int_\Omega \sum_{i=1}^n \left[\sigma_i(t,x,\overline{u}_i(t),\overline{u}_i(t-\tau(t))) - \sigma_i\left(t,x,\underline{u}_i(t),\underline{u}_i(t-\tau(t))\right)\right] \\ &\times \left[\sigma_i(t,x,\overline{u}_i(t),\overline{u}_i(t-\tau(t))) - \sigma_i\left(t,x,\underline{u}_i(t),\underline{u}_i(t-\tau(t))\right)\right]^T dx. \end{aligned} \tag{3.5}$$

Combining Cauchy inequality with (A2) yields

$$\begin{aligned} &\int_\Omega z_i(t)\int_{-\infty}^t K_{ij}(t-s)\left[\overline{f}_j(\overline{u}_j(t)) - \overline{f}_j\left(\underline{u}_j(t)\right)\right]ds\,dx \\ &\leq \int_\Omega |z_i(t)|\int_0^{+\infty} K_{ij}(s)L_j^{\overline{f}}\left|z_j(t-s)\right|ds\,dx = \int_0^{+\infty} K_{ij}(s)L_j^{\overline{f}}\int_\Omega |z_i(t)|\left|z_j(t-s)\right|dx\,ds \\ &\leq L_j^{\overline{f}}\|z_i(t)\|_2\int_0^{+\infty} K_{ij}(s)L_j^{\overline{f}}\left\|z_j(t-s)\right\|_2 ds \leq \frac{1}{2}L_j^{\overline{f}}\left[\|z_i(t)\|_2^2 + \left(\int_0^{+\infty} K_{ij}(s)\left\|z_j(t-s)\right\|_2 ds\right)^2\right] \\ &= \frac{1}{2}L_j^{\overline{f}}\|z_i(t)\|_2^2 + \frac{1}{2}L_j^{\overline{f}}\left(\int_0^{+\infty} (K_{ij}(s))^{1/2}(K_{ij}(s))^{1/2}\left\|z_j(t-s)\right\|_2 ds\right)^2 \\ &\leq \frac{1}{2}L_j^{\overline{f}}\|z_i(t)\|_2^2 + \frac{1}{2}L_j^{\overline{f}}\left(\int_0^{+\infty} K_{ij}(s)\left\|z_j(t-s)\right\|_2^2 ds\right). \end{aligned} \tag{3.6}$$

According to Green's formula [37] and the Dirichlet boundary condition, we get

$$\int_{\Omega}\sum_{l=1}^{m} z_i(t)\frac{\partial}{\partial x_l}\left(D_{il}\frac{\partial z_i(t)}{\partial x_l}\right)dx = -\sum_{l=1}^{m}\int_{\Omega} D_{il}\left(\frac{\partial z_i(t)}{\partial x_l}\right)^2 dx. \tag{3.7}$$

Moreover, from Lemma 2.3, we have

$$-\sum_{l=1}^{m}\int_{\Omega} D_{il}\left(\frac{\partial z_i(t)}{\partial x_l}\right)^2 dx \leq -\int_{\Omega}\sum_{l=1}^{m}\frac{D_{il}}{d_l^2}(z_i(t))^2 dx \leq -\int_{\Omega}\sum_{l=1}^{m}\frac{\min_{i\in N}(D_i)}{d_l^2}(z_i(t))^2 dx. \tag{3.8}$$

From (A1)–(A3), (A5) and (3.5)–(3.8), we have

$$\begin{aligned}
LV(t) \leq & -2\sum_{l=1}^{m}\sum_{i=1}^{n}\left(\frac{D_{il}}{d_l^2}\|z_i(t)\|_2^2\right) \\
& + 2\sum_{i=1}^{n}\Bigg\{-\widehat{a}_i\|z_i(t)\|_2^2 + \sum_{j=1}^{n}\left(\left|\widehat{b}_{ij}\right|L_j^f\|z_i(t)\|_2\left\|z_j(t)\right\|_2\right) + \frac{1}{2}\sum_{j=1}^{n}\left|\widehat{\overline{b}}_{ij}\right|L_j^{\overline{f}} \\
& \qquad \times\left(\|z_i(t)\|_2^2 + \left(\int_0^{+\infty}K_{ij}(s)\left\|z_j(t-s)\right\|_2^2 ds\right)\right) \\
& \qquad + \sum_{j=1}^{n}\left(\left|\widehat{\widetilde{b}}_{ij}\right|L_j^{\widetilde{f}}\|z_i(t)\|_2\left\|z_j(t-\tau(t))\right\|_2\right)\Bigg\} + \sum_{i=1}^{n}\left(\delta_i\|z_i(t)\|_2^2 + \gamma_i\|z_i(t-\tau(t))\|_2^2\right) \\
\leq & -\sum_{l=1}^{m}\sum_{i=1}^{n}\left(\frac{2D_{il}}{d_l^2}\|z_i(t)\|_2^2\right) \\
& + \sum_{i=1}^{n}\Bigg\{-2\widehat{a}_i\|z_i(t)\|_2^2 + \sum_{j=1}^{n}\left|\widehat{b}_{ij}\right|L_j^f\left(\|z_i(t)\|_2^2 + \left\|z_j(t)\right\|_2^2\right) \\
& \qquad + \sum_{j=1}^{n}\left|\widehat{\overline{b}}_{ij}\right|L_j^{\overline{f}}\left(\|z_i(t)\|_2^2 + \left(\int_0^{+\infty}K_{ij}(s)\left\|z_j(t-s)\right\|_2^2 ds\right)\right) \\
& \qquad + \sum_{j=1}^{n}\left[\left|\widehat{\widetilde{b}}_{ij}\right|L_j^{\widetilde{f}}\left(\|z_i(t)\|_2^2 + \left\|z_j(t-\tau(t))\right\|_2^2\right)\right]\Bigg\} + \sum_{i=1}^{n}\left(\delta_i\|z_i(t)\|_2^2 + \gamma_i\|z_i(t-\tau(t))\|_2^2\right)
\end{aligned}$$

$$\leq \Bigg\{ -2\sum_{l=1}^{m}\frac{\min_{i\in N}(D_i)}{d_l^2} - 2\min_{i\in N}(\widehat{a}_i)$$

$$+ \Bigg[\max_{i\in N}\Bigg(\sum_{j=1}^{n}\left|\widehat{b}_{ij}\right|L_j^f\Bigg) + \sum_{i=1}^{n}\max_{j\in N}\left(\left|\widehat{b}_{ij}\right|L_j^f\right) + \max_{i\in N}\Bigg(\sum_{j=1}^{n}\left|\widehat{\widetilde{b}}_{ij}\right|L_j^{\widetilde{f}}\Bigg)$$

$$+\max_{i\in N}\Bigg(\sum_{j=1}^{n}\left|\widehat{\overline{b}}_{ij}\right|L_j^{\overline{f}}\Bigg)+\max_{i\in N}\{\delta_i\}\Bigg]\Bigg\}\sum_{i=1}^{n}\|z_i(t)\|_2^2 + \Bigg[\sum_{i=1}^{n}\max_{j\in N}\left(\left|\widehat{\widetilde{b}}_{ij}\right|L_j^{\widetilde{f}}\right)+ \max_{i\in N}\{\gamma_i\}\Bigg]$$

$$\times \sum_{i=1}^{n}\|z_i(t-\tau(t))\|_2^2 + \sum_{i=1}^{n}\max_{j\in N}\left(\left|\widehat{\overline{b}}_{ij}\right|L_j^{\overline{f}}\right)\int_0^{+\infty}\kappa(s)\sum_{i=1}^{n}\|z_i(t-s)\|_2^2 ds. \tag{3.9}$$

From (3.4), (3.9), (A6), (A7) and Lemma 2.5, we know

$$EV(t) \leq MEV_0 e^{-(\alpha-\beta)t}, \quad t \geq t_0, \tag{3.10}$$

which means that

$$\int_\Omega \sum_{i=1}^{n} E\left[\overline{u}_i(t) - \underline{u}_i(t)\right]^2 dx \leq M\left\|\varphi - \psi\right\|_\tau^2 e^{-(\alpha-\beta)t}, \quad t \geq t_0. \tag{3.11}$$

By the integral property of measurable functions, we can derive

$$\int_\Omega \sum_{i=1}^{n}\left[\overline{u}_i(t+\omega) - \overline{u}_i(t)\right]^2 dx \leq M\left\|\varphi - \psi\right\|^2 e^{-(\alpha-\beta)t}, \quad t \geq t_0 \text{ a.e.} \tag{3.12}$$

In the light of $(\sum_{i=1}^{n}|z_i|)^2 \leq n\sum_{i=1}^{n}|z_i|^2$, for any $z_i \in \mathbb{R}^+$, we obtain

$$\int_\Omega \sum_{i=1}^{n}|\overline{u}_i(t+\omega) - \overline{u}_i(t)| dx \leq \sqrt{nM}\left\|\varphi - \psi\right\| e^{-0.5(\alpha-\beta)t}, \quad t \geq t_0 \text{ a.e.} \tag{3.13}$$

Noticing that

$$\overline{u}_i(t+k\omega) = \overline{u}_i(t) + \sum_{r=1}^{k}\left[\overline{u}_i(t+r\omega) - \overline{u}_i(t+(r-1)\omega)\right], \quad i \in N. \tag{3.14}$$

For any given $t \geq t_0$, by (3.12), we can see that

$$\begin{aligned}
&\int_{\Omega} \sum_{r=1}^{\infty} [\overline{u}_i(t + r\omega) - \overline{u}_i(t + (r-1)\omega)] dx \\
&\quad = \int_{\Omega} \lim_{k\to\infty} \sum_{r=1}^{k} [(\overline{u}_i(t + r\omega) - \overline{u}_i(t + (r-1)\omega))] dx \leq \sqrt{nM} \|\varphi - \psi\| \lim_{k\to\infty} \sum_{r=1}^{k} e^{-0.5(\alpha-\beta)(t+(r-1)\omega)} \\
&\quad \leq \sqrt{nM} \|\varphi - \psi\| e^{-0.5(\alpha-\beta)t} \lim_{k\to\infty} \sum_{r=1}^{k} e^{-0.5(\alpha-\beta)(r-1)\omega},
\end{aligned} \tag{3.15}$$

therefore, $\lim_{k\to\infty} \overline{u}_i(t + k\omega)$ exists a.e.

Let $\widehat{u}(t) = (\widehat{u}_1(t), \ldots, \widehat{u}_n(t))^T$ be the solution of system (2.1)-(2.2) starting from ϕ, by $\widehat{u}_i(t) = \lim_{k\to\infty} \overline{u}_i(t + k\omega)$, then $\widehat{u}(t)$ is well defined and is a periodic function with period ω. Supposing that $\widehat{v}(t) = (\widehat{v}_1(t), \ldots, \widehat{v}_n(t))^T$ is another ω-periodic solution of system (2.1)-(2.2) starting from ϕ^*, by similar method used before, it is easy to prove

$$\begin{aligned}
\int_{\Omega} \sum_{i=1}^{n} [\widehat{u}_i(t) - \widehat{v}_i(t)]^2 dx &= \int_{\Omega} \sum_{i=1}^{n} [\widehat{u}_i(t + k\omega) - \widehat{v}_i(t + k\omega)]^2 dx \\
&\leq M \|\phi - \phi^*\|^2 e^{-(\alpha-\beta)(t+k\omega)}, \quad t \geq t_0, \text{ a.e.}
\end{aligned} \tag{3.16}$$

Therefore, we can conclude that the system (2.1)-(2.2) is globally exponentially periodic in the mean square sense. This completes the proof of Theorem 3.1. □

Next, omitting condition (A4) and using LMI technique, another sufficient condition ensuring the global exponential stability of periodic solution for the system (2.1)-(2.2) in the mean square sense is derived.

Theorem 3.2. *Suppose that (A1)–(A3) and (A5) hold. If there exists a positive definite diagonal matrix P, positive definite matrices Ξ_1, Ξ_2, nonnegative constants p, q, r, and β_k, $(k \in Z^+)$, such that*

(i) $p > q + r \int_0^{+\infty} \kappa(s) ds$,

(ii) *there exist constants $M \geq 1$, $\lambda \in (0, \eta)$ and $\alpha \in [0, \lambda)$ such that*

$$\prod_{k=1}^{n} \max\{1, \beta_k\} \leq M e^{\alpha t_n}, \quad n \in Z^+, \tag{3.17}$$

and $\lambda < p - q e^{\lambda\tau} - r \int_0^{+\infty} \kappa(s) e^{\lambda s} ds$,

(iii)

$$
\begin{gathered}
-PD^{*}-D^{*T}P-PA-A^{T}P+PBL^{f}+L^{f}B^{T}P+R_{1}+P\widetilde{B}L^{\tilde{f}}\Xi_{1}L^{\tilde{f}}\widetilde{B}^{T}P \\
+\int_{0}^{+\infty}\kappa(s)dsP\overline{B}L^{\overline{f}}\Xi_{2}L^{\overline{f}}\overline{B}^{T}P+pP<0,\quad \Xi_{1}^{-1}+R_{2}-qP<0,\ \Xi_{2}^{-1}-rP<0,\ C_{k}^{T}PC_{k}-\beta_{k}P<0,
\end{gathered}
\tag{3.18}
$$

then the system (2.1)-(2.2) *is global exponential periodic in the mean square sense. Where*

$$
\begin{gathered}
A=\operatorname{diag}(a_{1},\ldots,a_{n}),\qquad B=(b_{ij})_{n\times n},\qquad \widetilde{B}=\left(\tilde{b}_{ij}\right)_{n\times n},\qquad \overline{B}=\left(\overline{b}_{ij}\right)_{n\times n}, \\
\beta_{k}=\max_{i\in N}\left\{(1-\theta_{ik})^{2}\right\},\qquad R_{1}=\operatorname{diag}(\delta_{1},\ldots,\delta_{n}),\qquad R_{2}=\operatorname{diag}(\gamma_{1},\ldots,\gamma_{n}), \\
D^{*}=\operatorname{diag}\left(\sum_{l=1}^{m}\frac{D_{1l}}{d_{l}^{2}},\ldots,\sum_{l=1}^{m}\frac{D_{nl}}{d_{l}^{2}}\right),\qquad C_{k}=\operatorname{diag}(1-\theta_{1k},\ldots,1-\theta_{nk}).
\end{gathered}
\tag{3.19}
$$

Proof. Define the following Lyapunov functional:

$$
V(t)=\int_{\Omega}z^{T}(t)Pz(t)dx,
\tag{3.20}
$$

when $t=t_k$, we have

$$
\begin{aligned}
V(t_{k})-\beta_{k}V(t_{k}^{-})&=\int_{\Omega}z^{T}(t_{k}^{-})C_{k}^{T}PC_{k}z(t_{k}^{-})-z^{T}(t_{k}^{-})\beta_{k}Pz(t_{k}^{-})dx \\
&=\int_{\Omega}z^{T}(t_{k}^{-})\left(C_{k}^{T}PC_{k}-\beta_{k}P\right)z(t_{k}^{-})dx<0.
\end{aligned}
\tag{3.21}
$$

For $t\geq t_0,\ t\neq t_k$, the infinitesimal operator of $LV(t)$ along with (3.16) is

$$
\begin{aligned}
LV(t)&=\int_{\Omega}\left(\frac{\partial}{\partial t}z^{T}Pz+z^{T}P\frac{\partial z}{\partial t}\right)dx+\int_{\Omega}\operatorname{trace}\left(\sigma^{T}P\sigma\right)dx \\
&\leq 2\int_{\Omega}z^{T}P\left(\sum_{l=1}^{m}\frac{\partial}{\partial x_{l}}\left(D_{il}\frac{\partial z}{\partial x_{l}}\right)-Az(t)+Bg(z(t))+\widetilde{B}\tilde{g}(z(t-\tau(t)))\right. \\
&\qquad\left.+\overline{B}\int_{-\infty}^{t}\kappa(t-s)L^{\overline{f}}z(s)ds\right)dx+\int_{\Omega}\operatorname{trace}\left(\tilde{\sigma}^{T}P\tilde{\sigma}\right)dx,
\end{aligned}
\tag{3.22}
$$

where

$$g(z(t)) = \left(g_1(z_1(t)), \dots, g_n(z_n(t))\right)^T, \qquad \widetilde{\sigma} = \left(\sigma_{ij}\left(t, x, \xi_i', \varsigma_i'\right) - \sigma_{ij}(t, x, \xi_i, \varsigma_i)\right)_{n\times n}$$
$$\widetilde{g}(z(t)) = \left(\widetilde{g}_1(z_1(t-\tau(t))), \dots, \widetilde{g}_n(z_n(t-\tau(t)))\right)^T,$$
$$\overline{g}(z(s)) = \left(\overline{g}_1(z_1(s)), \dots, \overline{g}_n(z_n(s))\right)^T, \qquad g_j\left(z_j(t)\right) = f_j\left(\overline{u}_j(t)\right) - f_j\left(\underline{u}_j(t)\right), \tag{3.23}$$
$$\widetilde{g}_j\left(z_j(t-\tau(t))\right) = \widetilde{f}_j\left(\overline{u}_j(t-\tau(t))\right) - \widetilde{f}_j\left(\underline{u}_j(t-\tau(t))\right),$$
$$\overline{g}_j\left(z_j(s)\right) = \overline{f}_j\left(\overline{u}_j(s)\right) - \overline{f}_j\left(\underline{u}_j(s)\right), \quad j = 1, 2, \dots, n.$$

By employing (3.8), (A5) and Lemma 2.7, we have

$$\begin{aligned}
LV \leq{}& 2\int_\Omega \Big(- z^T(t)PD^* z(t) - z^T(t)PAz(t) + z^T(t)PBL^f z(t) \\
&\qquad + z^T(t)P\widetilde{B}L^{\widetilde{f}} z(t-\tau(t)) + z^T(t)P\overline{B}\int_0^{+\infty} \kappa(s)L^{\overline{f}} z(t-s)ds \Big)dx \\
&+ \int_\Omega \Big(z^T(t)R_1 z(t) + z^T(t-\tau(t))R_2 z(t-\tau(t)) \Big)dx \\
\leq{}& \int_\Omega \Big[z^T(t)\Big(-PD^* - D^{*T}P - PA - A^TP + PBL^f + L^fB^TP + R_1\Big)z(t) \\
&\qquad + z^T(t)P\widetilde{B}L^{\widetilde{f}}\Xi_1 L^{\widetilde{f}}\widetilde{B}^T P z(t) + z^T(t-\tau(t))\Xi_1^{-1} z(t-\tau(t)) \\
&\qquad + \int_0^{+\infty} \kappa(s) z^T(t)P\overline{B}L^{\overline{f}}\Xi_2 L^{\overline{f}}\overline{B}^T P z(t)ds \Big]dx \\
&+ \int_\Omega \Big(z^T(t-\tau(t))R_2 z(t-\tau(t)) + \int_0^{+\infty} \kappa(s) z^T(t-s)\Xi_2^{-1} z(t-s)ds \Big)dx \\
\leq{}& \int_\Omega \Big[z^T(t)\Big(- PD^* - D^{*T}P - PA - A^TP + PBL^f \\
&\qquad + L^fB^TP + R_1 + P\widetilde{B}L^{\widetilde{f}}\Xi_1 L^{\widetilde{f}}\widetilde{B}^TP + \int_0^{+\infty}\kappa(s)dsP\overline{B}L^{\overline{f}}\Xi_2 L^{\overline{f}}\overline{B}^TP\Big)z(t) \\
&\qquad + z^T(t-\tau(t))\Big(\Xi_1^{-1} + R_2\Big)z(t-\tau(t)) + \int_0^{+\infty}\kappa(s)z^T(t-s)\Xi_2^{-1}z(t-s)ds\Big]dx \\
\leq{}& \int_\Omega \Big[z^T(t)\Big(- PD^* - D^{*T}P - PA - A^TP + PBL^f + L^fB^TP \\
&\qquad + R_1 + P\widetilde{B}L^{\widetilde{f}}\Xi_1 L^{\widetilde{f}}\widetilde{B}^TP + \int_0^{+\infty}\kappa(s)dsP\overline{B}L^{\overline{f}}\Xi_2 L^{\overline{f}}\overline{B}^TP + pP\Big)z(t)
\end{aligned}$$

$$+z^T(t-\tau(t))\left(\Xi_1^{-1}+R_2-qP\right)z(t-\tau(t))+\int_0^{+\infty}\kappa(s)z^T(t-s)\left(\Xi_2^{-1}-rP\right)z(t-s)ds\Big]dx$$
$$-pV(t)+q\sup_{-\infty\le s\le t}V(s)+r\int_0^{+\infty}\kappa(s)V(t-s)ds. \tag{3.24}$$

It follows from the condition (iii) and (3.24) that we have

$$LV(t)\le -pV(t)+q\sup_{-\infty\le s\le t}V(t)+r\int_0^{+\infty}\kappa(s)V(t-s)ds. \tag{3.25}$$

By Lemma 2.5, we obtain

$$\lambda_{\min}(P)E\|z(t)\|_2^2\le EV(t)\le\lambda_{\max}(P)\|\psi-\varphi\|_\tau^2e^{-(\lambda-\alpha)t}. \tag{3.26}$$

We know that

$$E\|z(t)\|_2\le\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\|\psi-\varphi\|_\tau e^{-((\lambda-\alpha)t)/2}, \tag{3.27}$$

that is,

$$E\left\|\overline{u}_i(t)-\underline{u}_i(t)\right\|_2\le\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\|\psi-\varphi\|_\tau e^{-((\lambda-\alpha)t)/2},\quad\forall t\ge t_0\ge 0, \tag{3.28}$$

where

$$M=\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\ge 1. \tag{3.29}$$

Similar to the proof of Theorem 3.1, we know that the system (2.1)-(2.2) is globally exponentially periodic in the mean square sense. This completes the proof. □

Remark 3.3. In [23], the authors have considered the stability problems of RDNNs, however, they have not considered impulsive stochastic effect and reaction-diffusion terms. To the best of our knowledge, no LMI-based stability results have been reported for ISRDNNs with mixed time delays in the literature.

Since an equilibrium point can be viewed as a special periodic solution of RDNNs with arbitrary period, we can consider ISRDNNs in system (2.1) with parameters $a_i(t)=a_i$, $b_{ij}(t)=b_{ij}$, $\widetilde{b}_{ij}(t)=\widetilde{b}_{ij}$, $\overline{b}_{ij}(t)=\overline{b}_{ij}$, $J_i(t)=J_i$, $\tau(t)=\tau$, $\sigma_{ij}(t,x,u^*,u^*)=0$, where a_i, b_{ij}, $\widetilde{b}_{ij}$, $\overline{b}_{ij}$, J_i are constants. Then, according to the results obtained so far, if the sufficient conditions in Theorems 3.1 or 3.2 are satisfied, a unique periodic solution becomes a periodic solution with arbitrary positive constants as its period. So, the periodic solution reduces to a constant solution, that is, an equilibrium point. Moreover, all other solutions globally exponentially converge to this equilibrium point in the mean square sense as $t \to +\infty$. To this end, by applying Theorems 3.1 or 3.2, we can easily get the following results.

Corollary 3.4. *Suppose that (A1)–(A5) hold for ISRDNNs in* (2.1)-(2.2) *with parameters* $a_i(t) = a_i$, $b_{ij}(t) = b_{ij}$, $\widetilde{b}_{ij}(t) = \widetilde{b}_{ij}$, $\overline{b}_{ij}(t) = \overline{b}_{ij}$, $J_i(t) = J_i$, $\tau(t) = \tau$, $\sigma_{ij}(t,x,u^*,u^*) = 0$, *where* a_i, b_{ij}, $\widetilde{b}_{ij}$, $\overline{b}_{ij}$, J_i *are constants, if* $\theta_{ik} \in [0,2]$, $i \in N$, $k \in Z^+$, *then there exists a unique equilibrium point of system* (2.1)-(2.2), *which is globally exponentially stable in the mean square sense.*

Corollary 3.5. *Suppose that (A2)-(A3), (A5) for system* (2.1)-(2.2) *with* $a_i, b_{ij}, \widetilde{b}_{ij}, \overline{b}_{ij}, J_i$ *being constants and* $\theta_{ik} \in [0,2]$, $i \in N, k \in Z^+$ *hold. If there exist a positive definite diagonal matrix* P, *positive definite matrices* Ξ_1, Ξ_2, *nonnegative constants* p, q, r, *and* $\beta_k, (k \in Z^+)$, *such that*
(i) $p > q + r\int_0^{+\infty} \kappa(s)ds$,
(ii) there exist constants $M \geq 1$, $\lambda \in (0,\eta)$ *and* $\alpha \in [0,\lambda)$ *such that*

$$\prod_{k=1}^{n} \max\{1,\beta_k\} \leq Me^{\alpha t_n}, \quad n \in Z^+, \tag{3.30}$$

and $\lambda < p - qe^{\lambda\tau} - r\int_0^{+\infty} \kappa(s)e^{\lambda s}ds$,
(iii)

$$\begin{gathered} -PD^* - D^{*T}P - PA - A^TP + PBL^f + L^fB^TP + R_1 + P\widetilde{B}L^{\widetilde{f}}\Xi_1L^{\widetilde{f}}\widetilde{B}^TP \\ + \int_0^{+\infty}\kappa(s)dsP\overline{B}L^{\overline{f}}\Xi_2L^{\overline{f}}\overline{B}^TP + pP < 0, \quad \Xi_1^{-1} + R_2 - qP < 0,\ \Xi_2^{-1} - rP < 0, \\ C_k^TPC_k - \beta_kP < 0, \end{gathered} \tag{3.31}$$

then the system (2.1)-(2.2) *has a unique equilibrium point, which is globally exponentially stable in the mean square sense.*

4. Illustrative Examples

Example 4.1. Consider the system (2.1) with two neurons on $\Omega = \{(x_1,x_2)^T \mid 0 < x_l < 1,\ l = 1,2\} \subset \mathbb{R}^2$, the boundary conditions and initial conditions are given by

$$\begin{gathered} u_i(t,x) = 0, \quad (t,x) \in [0,+\infty) \times \partial\Omega, \\ u_i(s,x) = 2\sin \pi x_1 x_2^2, \quad i = 1,2,\ (s,x) \in (-\infty,0] \times \Omega, \end{gathered} \tag{4.1}$$

where $t_k = k$, $k \in Z^+$, $\kappa(s) = k_{ij}(s) = se^{-s}$, $f_j(\eta) = \widetilde{f}_j(\eta) = \overline{f}_j(\eta) = (1/30)(|\eta+1|+|\eta-1|)$, $n = m = 2$, $L_j^f = L_j^{\widetilde{f}} = L_j^{\overline{f}} = 1$, $d_l = \varepsilon_i = 1$, $j,l = 1,2$. $D_{11} = D_{12} = 0.5, D_{21} = 0.3, D_{22} = 0.7$, $\tau(t) =$

$0.02 - 0.01\sin 2\pi t, \tau = \ln 2, a_1(t) = 10.9 - 4\cos 2\pi t, a_2(t) = 11 - \sin 2\pi t, \theta_{ik} = -1 + k,\ k \in Z^+,\ \delta_i = \gamma_i = 1$.

$$\begin{gathered}
\sigma_{ij}(t,x,u_i(t,x),u_i(t-\tau(t),x)) = \frac{\sqrt{2}}{2}(\tanh(u_i(t,x)) + \tanh(u_i(t-\tau(t),x))), \\
b_{11}(t) = 0.3 + 0.1\sin 2\pi t, \quad b_{12}(t) = 0.4 + 0.1\sin 2\pi t, \quad b_{21}(t) = 0.2 + 0.1\cos 2\pi t, \\
b_{22}(t) = 0.3 - 0.1\cos 2\pi t, \quad \widetilde{b}_{11}(t) = 0.2 + 0.1\sin 2\pi t, \quad \widetilde{b}_{12}(t) = 0.3 - 0.2\cos 2\pi t, \\
\widetilde{b}_{21}(t) = 0.5 + 0.1\cos 2\pi t, \quad \widetilde{b}_{22}(t) = 0.4 - 0.1\sin 2\pi t, \quad \overline{b}_{11}(t) = 0.1 - 0.2\sin 2\pi t, \\
\overline{b}_{12}(t) = 0.25 - 0.1\sin 2\pi t, \quad \overline{b}_{21}(t) = 0.2 - 0.1\cos 2\pi t, \quad \overline{b}_{22}(t) = 0.1 - 0.1\cos 2\pi t, \\
J_1(t) = 1 + \sin 2\pi t, \quad J_2(t) = 2 + \cos 2\pi t.
\end{gathered} \tag{4.2}$$

Direct computation shows that $p = 5.65$, $q + r\int_0^{+\infty}\kappa(s)ds = 4.45$. Let $\lambda = 0.2$, $\alpha = 0, M = 1$, and $\tau = \ln 2$ satisfying $\lambda < p - qe^{\lambda\tau} - r\int_0^{+\infty}\kappa(s)e^{\lambda s}ds$. The simulation results are shown in Figures 1–6. When $x_2 = 0.1$, the states surfaces of $u(t,x_1,0.1)$ are shown in Figures 1 and 2, while $x_1 = 0.1$, the states surfaces of $u(t,0.1,x_2)$ are shown in Figures 3 and 4, they are illustrated that the system states in (2.1) and (2.2) converge to periodic solutions. In order to see it clearly, we also draw the curves of the states when $x_1 = 0.1$, $x_2 = 0.1$ in Figures 5 and 6. Hence, it follows from both Theorem 3.1 and the simulation study that system (2.1)-(2.2) is globally exponentially periodic stable in the mean square sense.

Example 4.2. Consider an ISRDNNs in (2.1) with parameters on $\Omega = \{(x_1,x_2)^T \mid 0 < x_l < 1/2,\ l = 1,2\}$,

$$\begin{gathered}
t_k = 0.5k, \quad \kappa(s) = k_{ij}(s) = se^{-s}, \quad D_{il} = \frac{1}{8} \quad (i,j,l = 1,2), \quad J_1(t) = \sin t, \quad J_2(t) = \cos t, \\
A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 & -0.5 \\ 0.5 & 0.5 \end{bmatrix}, \quad \overline{B} = \widetilde{B} = \begin{bmatrix} 0.25 & 0.25 \\ 0.25 & 0.25 \end{bmatrix}, \quad R_1 = R_2 = D^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\
f_j(\eta) = \widetilde{f}_j(\eta) = \overline{f}_j(\eta) = \sin\frac{\eta}{2} + \frac{\eta}{2}, \quad j = 1,2, \quad \tau(t) = 0.1 - 0.1\sin t.
\end{gathered} \tag{4.3}$$

Clearly, $f_j(\eta)$, $\widetilde{f}_j(\eta)$, $\overline{f}_j(\eta)$, $(j = 1,2)$ satisfy the (A2) with $L^f = L^{\widetilde{f}} = L^{\overline{f}} = I_2$, and $\tau(t)$, $J_1(t)$, $J_2(t)$ are continuously periodic functions with a common positive period 2π.

Taking $p = 1$, $q = 0.2$, $r = 0.1$, $\lambda = 0.1$, $\alpha = 0$, $\beta_k = 1$, $P = 2I_2$, $C_k = 0.5I_2$, $\Xi_1 = \Xi_2 = I_2$. By simple calculation, we can easily check (i), (ii), (iii), and (iv) in Theorem 3.2.

To this end, the conditions of Theorem 3.2 are satisfied, therefore, there exists exactly one 2π-periodic solution, and all other solutions converge exponentially to it in the mean square sense as $t \to +\infty$.

Remark 4.3. In Examples 4.1 and 4.2, many factors such as noise perturbations, mixed time delays, and impulsive effects are considered. Therefore, the results reported in [13, 14, 18–20] do not hold in our examples.

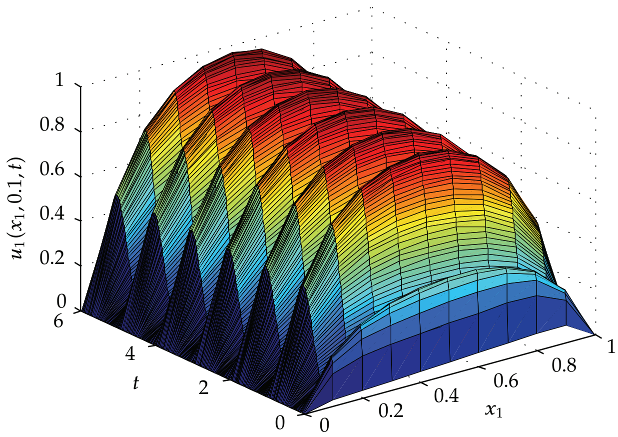

Figure 1: The surface of $u_1(t, x_1, 0.1)$ when $x_2 = 0.1$.

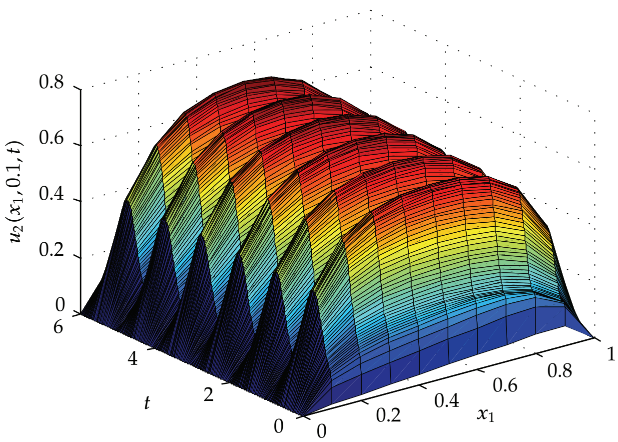

Figure 2: The surface of $u_2(t, x_1, 0.1)$ when $x_2 = 0.1$.

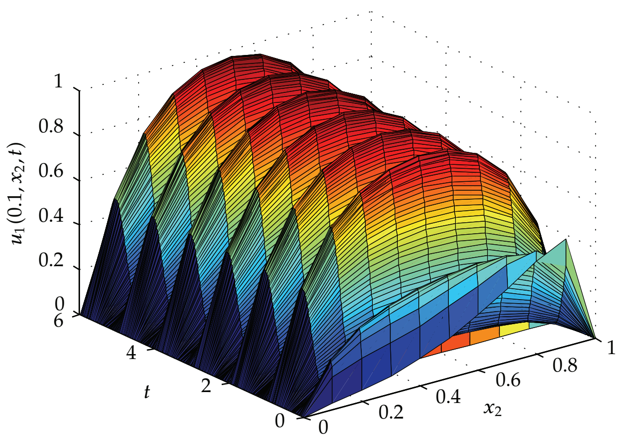

Figure 3: The surface of $u_1(t, 0.1, x_2)$ when $x_1 = 0.1$.

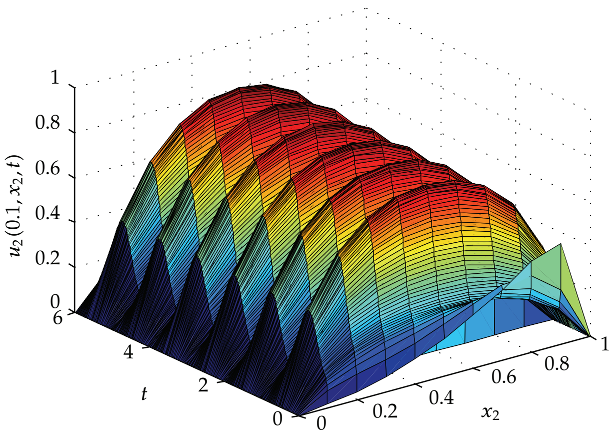

Figure 4: The surface of $u_2(t, 0.1, x_2)$ when $x_1 = 0.1$.

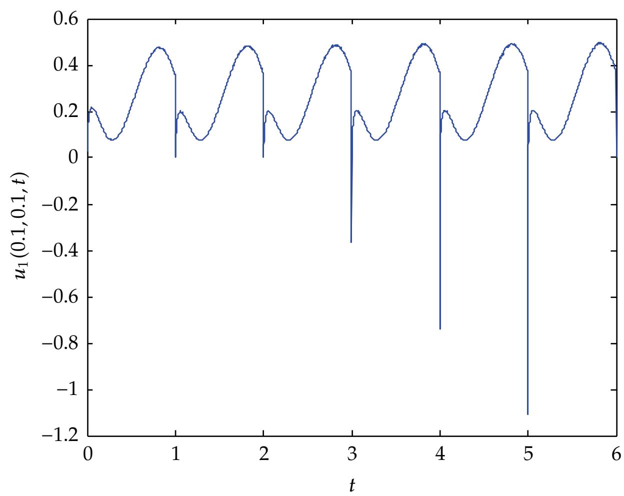

Figure 5: The curve of $u_1(t, 0.1, 0.1)$ when $x_1 = 0.1$, $x_2 = 0.1$.

5. Conclusions

In this paper, the dynamical behaviors for ISRDNNs with mixed time delays have been studied. By using an L-operator differential inequality with impulses and mixed time delays, as well as linear matrix inequality technique, some novel sufficient conditions are derived to guarantee the existence, uniqueness, global exponential stability of the periodic solutions, and the global exponential stability of the equilibrium point in the mean square sense. To the best of our knowledge, the results presented here have been not appeared in the related literature. The obtained sufficient conditions depend on the reaction-diffusion terms. The obtained results generalize and comprise those results with/without reaction-diffusion term, impulsive operators, or noise disturbances in the previous literature. Finally, two numerical examples are also provided in the end of the paper to show the effectiveness of our results.

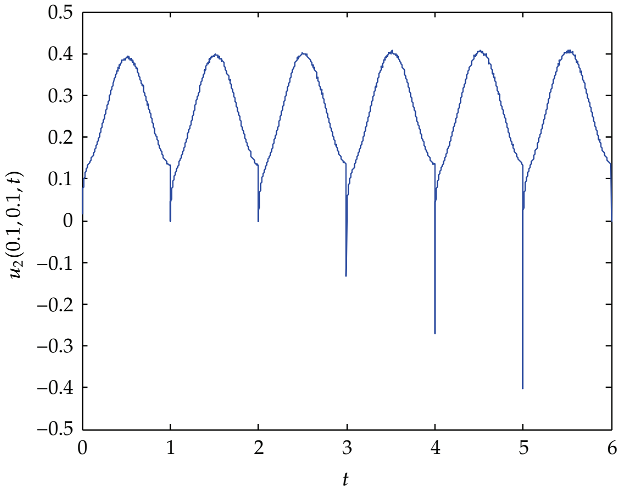

Figure 6: The curve of $u_2(t, 0.1, 0.1)$ when $x_1 = 0.1$, $x_2 = 0.1$.

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Research Article

The Effect of Control Strength on Lag Synchronization of Nonlinear Coupled Complex Networks

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This paper mainly investigates the lag synchronization of nonlinear coupled complex networks using methods that are based on pinning control, where the weight configuration matrix is not necessarily symmetric or irreducible. We change the control strength into a parameter concerning time t, by using the Lyapunov direct method, some sufficient conditions of lag synchronization are obtained. To validate the proposed method, numerical simulation examples are provided to verify the correctness and effectiveness of the proposed scheme.

1. Introduction

In recent years, a great deal of attention has been paid to the investigation of complex networks in various fields. In fact, complex networks are shown to widely exist in our life. Common examples of complex networks include the Internet, the World Wide Web (WWW), food webs, scientific citation webs, as well as many other systems that are made up of a large number of intricately connected parts. Indeed, complex networks are an important part of our daily lives.

Synchronization of complex networks has been one of the focal points in many research and application fields. Synchronization has been studied from various angles and a variety of different synchronization phenomena have been discovered, such as complete synchronization (CS), phase synchronization (PS), lag synchronization (LS), generalized synchronization (GS), anticipatory synchronization, antiphase synchronization, clustering synchronization, projective synchronization, and others [1–15]. It is worth mentioning that, in many practical situations, a propagation delay will appear in the electronic implementation of dynamical systems. Therefore, it is very important to investigate the lag synchronizationa few results have been reported. Guo [16] investigated the lag synchronization of complex networks via pinning control. Without assuming the symmetry and irreducibility of the coupling matrix, sufficient conditions of lag synchronization are obtained by adding controllers to a part of nodes. Particularly, the following two questions are solved: (1) How many controllers are needed to pin a coupled complex network to a homogeneous solution? (2) how should we distribute these controllers? Shahverdiev et al. [17] investigated lag synchronization between unidirectionally coupled Ikeda systems with time delay via feedback control techniques; Yang and Cao [18] studied the exponential lag synchronization of a class of chaotic delayed neural networks with impulsive effects. Some sufficient conditions are established by the stability analysis of impulsive differential equations. Li et al. [19] considered the lag synchronization issue of coupled time-delayed systems with chaos, applied proposed lag synchronization strategies towards the secure communication. Wang and Shi [20] investigated the chaotic bursting lag synchronization of Hindmarsh-Rose system via a single controller. Zhou et al. [21] investigated lag synchronization of coupled chaotic delayed neural networks without noise perturbation by using adaptive feedback control techniques. Wang et al. [22] investigated lag synchronization of chaotic systems with parameter mismatches. Sun and Cao [23] and Yu and Cao [24] researched the adaptive lag synchronization of unknown chaotic delayed neural networks.

It is noticed that almost all the regimes of lag synchronization mentioned above used the method of adding controllers to all the nodes to make complex networks get synchronized. As we know now, the real-world complex networks normally have a large number of nodes. Therefore, for the complexity of the dynamical network, it is difficult to realize the synchronization by adding controllers to all nodes. To reduce the number of the controllers, a natural way is using pinning control method [25–29].

Motivated by the above discussions, in this paper, we work on the lag synchronization of nonlinear coupled complex networks via pinning control method. The main contributions of this paper are three fold. (1) This paper deals with the lay synchronization problem for nonlinear coupled complex networks. We change the control strength into a parameter concerning time t, some sufficient conditions for the synchronization are derived by constructing an effective control scheme. Particularly, the weight configuration matrix is not necessarily symmetric or irreducible. (2) Compared with some similar designs, our pinning controllers are very simple. (3) Generally, previous works require the coupling strength c to be large so that the synchronization of complex networks can be realized. However, there exists a drawback as c becomes larger. This equivalently makes all weights larger simultaneously. This must raise the synchronization cost. In this paper, we show that, as a parameter, $\varepsilon(t) > 0$ can be used to complete the task with a lower cost. Numerical examples are also provided to demonstrate the effectiveness of the theory. This work improves the current results that we have.

The rest of this paper is organized as follows. The network model is introduced and some necessary definitions, lemmas, and hypotheses are given in Section 2. The lag synchronization of the coupled complex networks is discussed in Section 3. Examples and their simulations are obtained in Section 4. Finally, conclusions are drawn in Section 5.

2. Model and Preliminaries

Now we consider the nonlinear coupled complex networks consisting of m identical nodes that are n-dimensional dynamical units. The model is described as

$$\dot{x}_i(t) = f(t, x_i(t)) + c\sum_{j=1}^{m} a_{ij} g(x_j(t)), \quad i = 1, \ldots, m, \tag{2.1}$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t))^T \in R^n$ is the state vector of node i; $f : R^n \to R^n$ standing for the activity of an individual subsystem is a vector value function. $g(\bullet)$ is some nonlinear function reflecting the nonlinear coupling relationship between those nodes. $A = (a_{ij})_{m\times m}$ is the corresponding coupling matrix that satisfies $a_{ij} \geq 0 (i \neq j)$, denoting the coupling coefficients, and $a_{ii} = -\sum_{j=1, j\neq i}^{m} a_{ji}$, for $i, j = 1, 2, \ldots, m$ and c is the coupling strength and will be fixed in this paper.

Based on the system above, we construct a response system whose state variables are denoted by $y_i (i = 1, 2, \ldots, m)$, whereas (2.1) is considered as the drive system with state variables denoted by $x_i (i = 1, 2, \ldots, m)$. In the response network, we add controllers to a part of the nodes which will be much more practical. Without loss of generality, we add the controllers to the first m_1 nodes $(1 \leq m_1 \leq m)$. Therefore, the response system with delay feedback can be described as

$$\begin{aligned} \dot{y}_i(t) &= f(t, y_i(t)) + c\sum_{j=1}^{m} a_{ij} g(y_j(t)) - c\varepsilon(t)\left(g(y_i(t)) - g(x_i(t-\tau))\right), \quad i = 1, \ldots, m_1 \\ \dot{y}_i(t) &= f(t, y_i(t)) + c\sum_{j=1}^{m} a_{ij} g(y_j(t)), \quad i = m_1 + 1, \ldots, m, \end{aligned} \tag{2.2}$$

where $\tau > 0$ is the time delay, $\varepsilon(t) > 0$ and $\dot{\varepsilon}(t) = \sum_{i=1}^{m_1} \delta x_i^T(t) P \delta x_i(t)$. Define $\delta x_i(t) = y_i(t) - x_i(t-\tau)$ and $\delta g(x_i(t)) = g(y_i(t)) - g(x_i(t-\tau))$; then we have the error system as

$$\delta\dot{x}_i(t) = f(t, y_i(t)) - f(t, x_i(t-\tau)) + c\sum_{j=1}^{m} \tilde{a}_{ij} \delta g(x_j(t)), \quad i = 1, \ldots, m, \tag{2.3}$$

where $\tilde{a}_{ii} = a_{ii} - \varepsilon(t)$, $i = 1, \ldots, m_1$ and $\tilde{a}_{ij} = a_{ij}$ otherwise.

Now, we introduce some definitions, assumptions, and lemmas that will be required throughout the paper.

Definition 2.1 (see [30]). The drive system (2.1) is said to lag synchronize with the response system (2.2) at time τ if $y_i(t) - x_i(t-\tau) \to 0$, $t \to \infty$, $i = 1, \ldots, m$, where τ is a given positive time delay.

Lemma 2.2 (see [31]). *Assuming that $A = (a_{ij})_{n\times n}$ satisfies the following conditions.*

(1) $a_{ij} \geq 0, (i \neq j), a_{ii} = -\Sigma_{j=1, i\neq j}^{n} a_{ij}, i, j = 1, 2, \ldots, n.$

(2) *A is irreducible. Then, one has*

(i) real parts of the eigenvalues of A are all negative except an eigenvalue 0 with multiplicity 1,

(ii) A has the right eigenvector $(1,1,\ldots,1)^T$ corresponding to the eigenvalue 0,

(iii) let $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^T$ be the left eigenvector of A corresponding to the eigenvalue 0, $\xi_i > 0$, $i = 1, \ldots n$ for convenience, one writes $\Xi = \operatorname{diag}\{\xi_1, \ldots, \xi_n\}$.

Lemma 2.3 (see [32]). *If $A = (a_{ij})_{n\times n}$ is an irreducible matrix and satisfies $a_{ij} = a_{ji} \geq 0$, for $i \neq j$, and $a_{ii} = -\Sigma_{j=1, i\neq j}^{n} a_{ij}$, $i, j = 1, 2, \ldots, n$ then, all eigenvalues of the matrix $\widetilde{A} = A - \operatorname{diag}(k_1, k_2, \ldots, k_{m_1}, 0, \ldots, 0)$ are negative, where $k_1, k_2, \ldots, k_{m_1}$ are positive constants.*

Assumption 2.4 (see [33]). The function $f(\bullet) \in QUAD(P, \Delta, \eta)$ if there exists a positive definite diagonal matrix $P = \operatorname{diag}(p_1, \ldots, p_n)$, a diagonal matrix $\Delta = \operatorname{diag}(\Delta_1, \ldots, \Delta_n)$, and a scalar $\eta > 0$ such that $(x-y)^T P(f(x) - f(y) - \Delta x + \Delta y) \leq -\eta(x-y)^T(x-y)$ holds for any $x, y \in R_n$, $t > 0$.

Assumption 2.5 (see [34] (Global Lipschitz Condition)). Suppose that there exist nonnegative constants γ, for all $\forall t \in R_+$, such that for any time-varying vectors $x(t), y(t) \in R^n$

$$\|g(x) - g(y)\| \leq \gamma \|x - y\|, \tag{2.4}$$

where $\|\|$ denotes the 2-norm throughout the paper.

For the convenience of later use, we introduce some notations:

$$\begin{gathered}
\delta x(t) = \left[\delta x_1(t)^T, \ldots, \delta x_m(t)^T\right]^T, \qquad \delta\tilde{x}^k(t) = \left[\delta x_1^k(t), \ldots, \delta x_m^k(t)\right]^T, \quad k = 1, \ldots, n, \\
\delta g\left(\tilde{x}^k(t)\right) = \left[\delta g\left(x_1^k(t)\right), \ldots, \delta g\left(x_m^k(t)\right)\right], \quad k = 1, \ldots, n.
\end{gathered} \tag{2.5}$$

3. Main Results

According to proposition in [16], we can get that the matrix ΞA is zero row sum. Moreover, due to A being an irreducible coupling matrix and Ξ a positive diagonal matrix, it is easy to verify that ΞA is also irreducible and the matrix ΞA is negative definite.

Theorem 3.1. *Suppose that Assumptions 2.4 and 2.5 hold and the coupling matrix A is irreducible. If one has*

$$\Delta_k \Xi + c\gamma\left(\Xi\widetilde{A}\right) \leq 0, \quad k = 1, \ldots, n \tag{3.1}$$

then, the drive system (2.1) *lag synchronization with the response system* (2.2) *at time τ.*

Proof. Choose the following Lyapunov functional candidate:

$$V(t) = \frac{1}{2}\sum_{i=1}^{m} \xi_i \delta x_i^T(t) P \delta x_i(t). \tag{3.2}$$

Differentiating $V(t)$ with respect to time along the solution of (2.3) yields

$$\begin{aligned}\dot{V}(t) &= \sum_{i=1}^{m}\xi_i \delta x_i^T(t) P \delta \dot{x}_i^T(t) \\ &= \sum_{i=1}^{m}\xi_i \delta x_i^T(t) P \left[f(t, y_i(t)) - f(t, x_i(t-\tau)) + c\sum_{j=1}^{m}\widetilde{a}_{ij}\delta g(x_j(t)) \right].\end{aligned} \tag{3.3}$$

By the Assumption 2.4 and Lemmas 2.2 and 2.3, we obtain

$$\begin{aligned}\dot{V}(t) \leq & -\eta\sum_{i=1}^{m}\xi_i \delta x_i^T(t)\delta x_i(t) + \sum_{i=1}^{m}\xi_i \delta x_i^T(t) P\Delta\delta x_i(t) + c\sum_{i=1}^{m}\xi_i \delta x_i^T(t) P\sum_{j=1}^{m}\widetilde{a}_{ij}\delta g(x_j(t)) \\ \leq & -\eta\sum_{i=1}^{m}\xi_i \delta x_i^T(t)\delta x_i(t) + \sum_{k=1}^{n}p_k\Delta_k\left(\delta\widetilde{x}^k(t)\right)^T \Xi\delta\widetilde{x}^k(t) + c\sum_{k=1}^{n}p_k\left(\delta\widetilde{x}^k(t)\right)^T\Xi\widetilde{A}\delta g\left(\widetilde{x}^k(t)\right).\end{aligned} \tag{3.4}$$

By the Assumption 2.5, we obtain

$$\dot{V}(t) \leq -\eta\sum_{i=1}^{m}\xi_i \delta x_i^T(t)\delta x_i(t) + \sum_{k=1}^{n}p_k\left(\delta\widetilde{x}^k(t)\right)^T\left[\Delta_k\Xi + c\gamma\left(\Xi\widetilde{A}\right)\right]\delta\widetilde{x}^k(t). \tag{3.5}$$

Therefore, if we have $\Delta_k\Xi + c\gamma(\Xi\widetilde{A}) \leq 0, k = 1, \ldots, n$ then

$$\dot{V}(t) \leq 0. \tag{3.6}$$

Theorem 3.1 is proved completely. □

Theorem 3.2. *Suppose that Assumptions 2.4 and 2.5 hold and the coupling matrix A is reducible. If one has when*

$$\Delta_k\Xi + c\gamma\Xi A - cq\Lambda < 0 \quad k = 1, \ldots, n, \tag{3.7}$$

where $\Lambda = \begin{pmatrix} I_{m_1\times m_1} & 0 \\ 0 & 0\end{pmatrix}_{m\times m}$, then, the drive system (2.1) *lag synchronize with the response system* (2.2) *at time τ.*

Proof. We consider the following system:

$$\begin{aligned}\delta x_i(t) =\ & f(t, y_i(t)) - f(t, x_i(t-\tau)) + c\sum_{j=1}^{m}a_{ij}\delta g(x_j(t)) \\ & -c\varepsilon(t)(g(y_i(t)) - g(x_i(t-\tau))), \quad i = 1, \ldots, m_1 \\ \delta x_i(t) =\ & f(t, y_i(t)) - f(t, x_i(t-\tau)) + c\sum_{j=1}^{m}a_{ij}\delta g(x_j(t)), \quad i = m_1+1, \ldots, m.\end{aligned} \tag{3.8}$$

Choose the following Lyapunov functional candidate:

$$V(t) = \frac{1}{2}\sum_{i=1}^{m}\xi_i\delta x_i^T(t)P\delta x_i(t) + \frac{c}{2}\sum_{i=1}^{m_1}\frac{\left(\gamma\xi_i\varepsilon(t) - q\right)^2}{\gamma\xi_i}, \tag{3.9}$$

where $q > 0$.

Differentiating $V_1(t)$ with respect to time along the solution of (3.8) yields

$$\begin{aligned}
\dot{V}(t) &= \sum_{i=1}^{m}\xi_i\delta x_i^T(t)P\delta\dot{x}_i^T(t) + c\sum_{i=1}^{m_1}\left(\gamma\xi_i\varepsilon(t) - q\right)\dot{\varepsilon}(t) \\
&= \sum_{i=1}^{m}\xi_i\delta x_i^T(t)P\Bigg[f\left(t, y_i(t)\right) - f(t, x_i(t-\tau)) + c\sum_{i=1}^{m}a_{ij}\delta g\left(x_j(t)\right) \\
&\qquad -c\varepsilon(t)\left(g\left(y_i(t)\right) - g(x_i(t-\tau))\right)\Bigg] + c\sum_{i=1}^{m_1}\left(\gamma\xi_i\varepsilon(t) - q\right)\dot{\varepsilon}(t) \\
&= \sum_{i=1}^{m}\xi_i\delta x_i^T(t)P\left[f\left(t, y_i(t)\right) - f(t, x_i(t-\tau)) - \Delta\delta x_i(t)\right] + \sum_{i=1}^{m}\xi_i\delta x_i^T(t)P\Delta\delta x_i(t) \\
&\quad + c\sum_{i=1}^{m}\xi_i\delta x_i^T(t)P\sum_{i=1}^{m}a_{ij}\delta g\left(x_j(t)\right) + c\sum_{i=1}^{m_1}\left(\gamma\xi_i\varepsilon(t) - q\right)\dot{\varepsilon}(t) \\
&\quad - c\sum_{i=1}^{m_1}\xi_i\delta x_i^T(t)P\varepsilon(t)\left(g\left(y_i(t)\right) - g(x_i(t-\tau))\right).
\end{aligned} \tag{3.10}$$

By the Assumption 2.4, we obtain

$$\begin{aligned}
\dot{V}(t) \le& -\eta\sum_{i=1}^{m}\xi_i\delta x_i^T(t)\delta x_i(t) + \sum_{k=1}^{n}p_k\left(\delta\tilde{x}^k(t)\right)^T(\Delta_k\Xi)\delta\tilde{x}^k(t) + c\sum_{i=1}^{m}\xi_i\delta x_i^T(t)P\sum_{i=1}^{m}a_{ij}\delta g\left(x_j(t)\right) \\
&- c\gamma\sum_{i=1}^{m_1}\xi_i\delta x_i^T(t)P\varepsilon(t)\delta x_i(t) + c\gamma\sum_{i=1}^{m_1}\xi_i\varepsilon(t)\dot{\varepsilon}(t) - c\sum_{i=1}^{m_1}q\dot{\varepsilon}(t) \\
\le& -\eta\sum_{i=1}^{m}\xi_i\delta x_i^T(t)\delta x_i(t) + \sum_{k=1}^{n}p_k\left(\delta\tilde{x}^k(t)\right)^T(\Delta_k\Xi)\delta\tilde{x}^k(t) \\
&+ c\sum_{k=1}^{n}p_k\left(\delta\tilde{x}^k(t)\right)^T(\Xi A)\delta g\left(\tilde{x}^k(t)\right) - cq\sum_{k=1}^{n}p_k\left(\delta\tilde{x}^k(t)\right)^T\Lambda\left(\delta\tilde{x}^k(t)\right).
\end{aligned} \tag{3.11}$$

By the Assumption 2.5, we obtain

$$\dot{V}(t) \leq -\eta\sum_{i=1}^{m}\xi_i\delta x_i^T(t)\delta x_i(t) + \sum_{k=1}^{n}p_k\left(\delta\tilde{x}^k(t)\right)^T\left(\Delta_k\Xi + c\gamma\Xi A - cq\Lambda\right)\left(\delta\tilde{x}^k(t)\right). \tag{3.12}$$

Therefore, if we have $\Delta_k\Xi + c\gamma\Xi A - cq\Lambda < 0$, $k = 1, \ldots, n$ then

$$\dot{V}(t) \leq 0, \tag{3.13}$$

Theorem 3.2 is proved completely. □

Remark 3.3. Compared with the control methods in the literature [16], the work requires the coupling strength c and $k_i(u_i(t) = k_i(x_i(t-\tau) - y_i(t))$, where k_i are positive constants) to be large so that the lag synchronization of complex networks can be realized. However, there exists a drawback as c becomes larger. This equivalently makes all weights larger simultaneously. This must raise the synchronization cost. In this paper, we show that, as a parameter, $\varepsilon(t) > 0$ can be used to complete the task with a lower cost.

4. Illustrative Examples

In this section, a numerical example will be given to demonstrate the validity of the lag synchronization criteria obtained in the previous sections. Considering the following network:

$$\begin{aligned}\dot{y}_i(t) &= f\left(t, y_i(t)\right) + c\sum_{j=1}^{m}a_{ij}g\left(y_j(t)\right) - c\varepsilon(t)\left(g\left(y_i(t)\right) - g\left(x_i(t-\tau)\right)\right), \quad i = 1, \ldots, m_1\\ \dot{y}_i(t) &= f\left(t, y_i(t)\right) + c\sum_{j=1}^{m}a_{ij}g\left(y_j(t)\right), \quad i = m_1 + 1, \ldots, m,\end{aligned} \tag{4.1}$$

where $i = 1, 2, \ldots, m$, $f(t, y_i(t)) = Dy_i(t) + h(y_i(t)) + B$, $y_i(t) = (y_{i1}(t), y_{i2}(t), y_{i3}(t))^T$, Here $B = [0, 0, 0.1]^T$, $h(x_i) = (0, 0, y_{i1}y_{i3})^T$, $m_1 = 1$, $c = 0.5$, $\tau = 0.01$, $g(\mathrm{y}) = \cos y + 3y$. And

$$D = \begin{bmatrix} 0 & -1 & -1\\ 1 & 0.1 & 0\\ 1 & 0 & -10 \end{bmatrix}, \qquad \mathrm{A} = \begin{bmatrix} -6 & 1 & 2 & 1 & 1 & 1\\ 1 & -5 & 2 & 1 & 0 & 1\\ 2 & 2 & -7 & 0 & 1 & 2\\ 1 & 1 & 0 & -7 & 2 & 3\\ 1 & 0 & 1 & 2 & -5 & 1\\ 1 & 1 & 2 & 3 & 1 & -8 \end{bmatrix}. \tag{4.2}$$

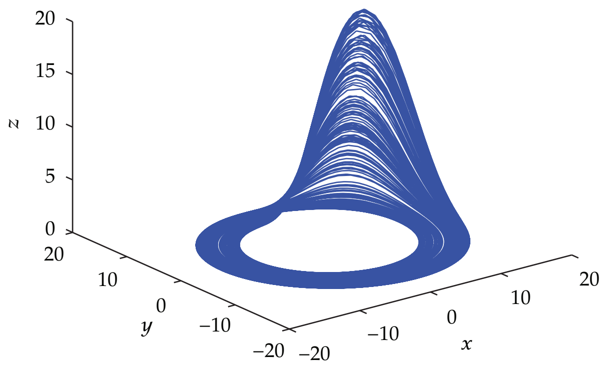

Figure 1: The chaotic behavior of time-delayed Rossler system.

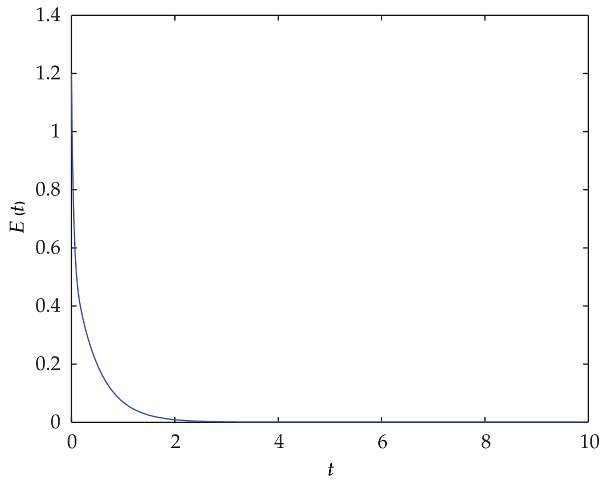

Figure 2: Time evolution of the lag synchronization errors $E(t)$.

The following quantities are utilized to measure the process of lag synchronization

$$E(t) = \sum_{i=1}^{N} \left\| y_i(t) - x_i(t-\tau) \right\|$$
$$e_1(t) = \left\| y_1(t) - x_1(t-\tau) \right\|, \tag{4.3}$$

where $E(t)$ is the error of lag synchronization for this controlled network (2.2); $e_1(t)$ is used to display the synchronization process of the first pinned node. The simulation results are given in Figures 1, 2, 3, and 4. From Figure 4, we see the time evolution of control strength. The numerical results show that the theoretical results are effective.

Remark 4.1. In this paper we designed controllers to ensure that the special networks could get lag synchronization. It indeed provides some new insights for the future practical engineering design.

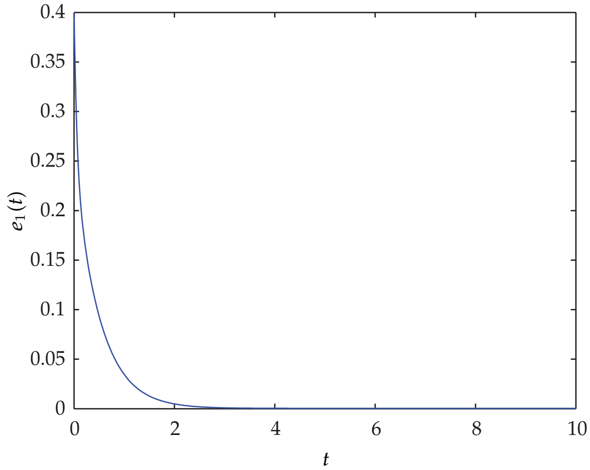

Figure 3: Time evolution of the lag synchronization errors $e_1(t)$.

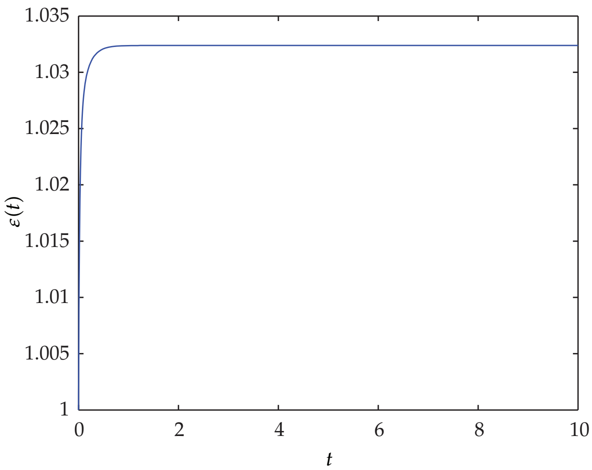

Figure 4: Time evolution of control strength $\varepsilon(t)$.

5. Conclusions

The problems of lag synchronization and pinning control for the nonlinear coupled complex networks are investigated. It is shown that lag synchronization can be realized via pinning controller. The study showed that the use of simple control law helps to derive sufficient criteria which ensure that the lag synchronization of the network model is derived. In addition, numerical simulations were performed to verify the effectiveness of the theoretical results.

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