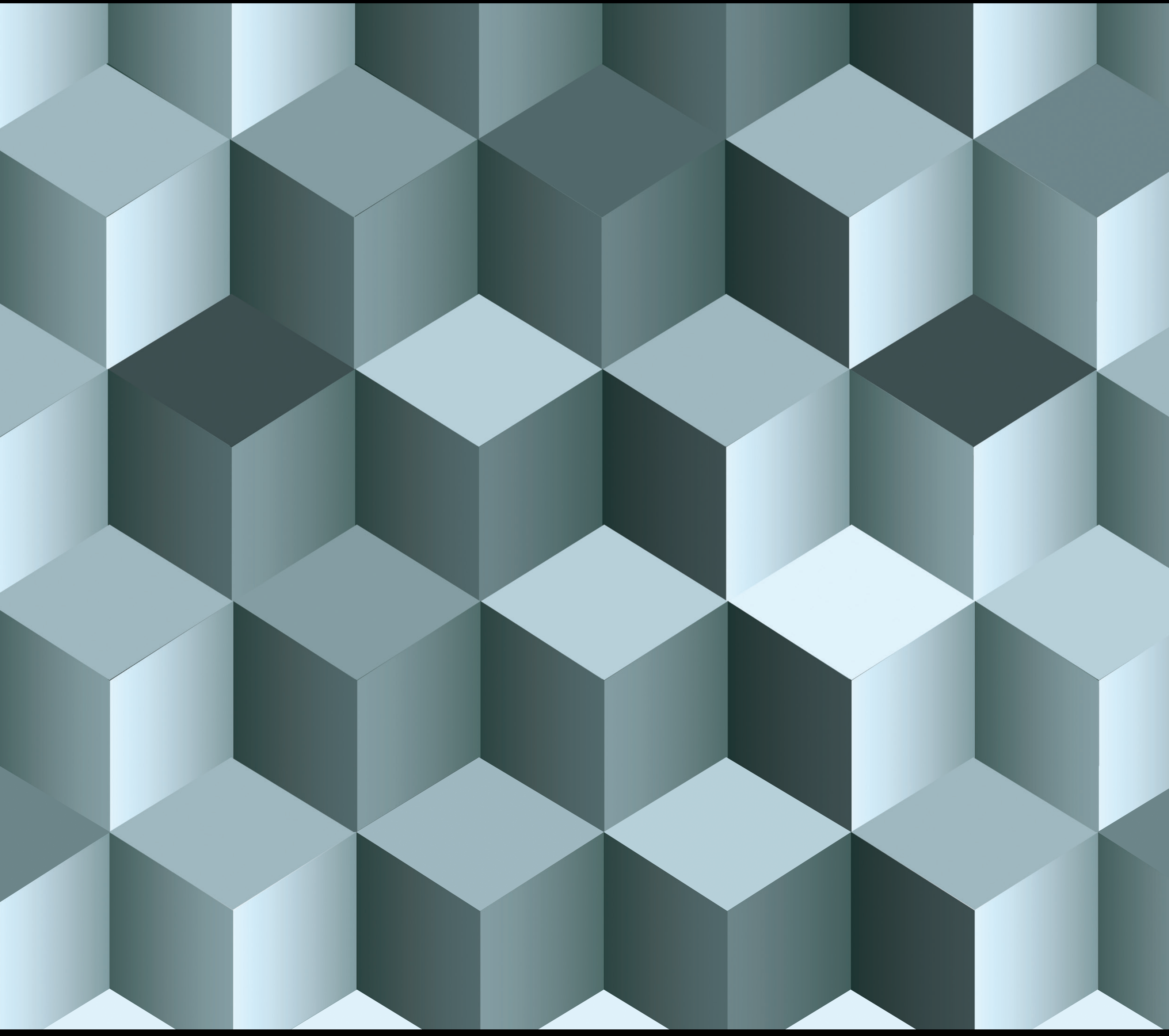
Journal of Function Spaces

Function Spaces, Approximation Theory, and Their Applications

Guest Editors: Carlo Bardaro, Ioan Rasa, Rudolf L. Stens, and Gianluca Vinti



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Editorial Board

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Editorial

Function Spaces, Approximation Theory, and Their Applications

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The purpose of this special issue was to present new developments in the theory of function spaces, along with the deep interconnections with approximation theory and the applications in various fields of pure and applied mathematics. The reaction of the mathematical community was very satisfactory. We collected thirty-five submissions, covering a wide range of mathematical topics, ten of which were found to be suitable for publications in this issue. The major part of the accepted papers treats function spaces and their applications. In this respect, in the article by X Yang et al. a new class of function spaces, named "multi-β-normed spaces", is introduced, in connection with stability properties of certain type of functional equations, while, in the paper by A. A. Bakery, sequential spaces of Orlicz type are studied and connected with the theory of summability. In the review paper by L. Angeloni and G. Vinti, the approximation theory in the space of functions with bounded variation is developed, in view of applications to signal processing. Different notions of variation are considered and several approximation theorems for families of integral or discrete type operators are given. In the more theoretical article by S. Wulede et al., a new class of Banach spaces which generalizes the class of uniformly extremely convex Banach spaces is introduced, and some characterizations of these spaces are given. Another paper by N. Khan treats the convergence of new type of double sequences, here introduced, in n-normed spaces. An interesting abstract approach to the theory of filter convergence is given in the article by A. Boccuto and X. Dimitriou, in which the links with function spaces and approximation theory are also dealt with. Other aspects of the theory of function spaces and their interconnections with calculus of variations, numerical analysis, complex variables, and stochastic processes are discussed, respectively, in the articles by T. Ma and Y. Feng, H. Wang et al., S. Wang and T. Zhan, and finally P. Duan. These four papers point out how certain methods of general approximation theory in function spaces can be employed in order to solve problems coming from a large variety of mathematical fields. We think that these contributions may represent starting points for new researches in the field of function spaces and approximation theory.

Acknowledgments

The guest editors wish to express their deep gratitude to all the contributors for the interest showed for our special issue and their interesting articles.

Carlo Bardaro
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Gianluca Vinti

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Review Article

A Review on Approximation Results for Integral Operators in the Space of Functions of Bounded Variation

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We present a review on recent approximation results in the space of functions of bounded variation for some classes of integral operators in the multidimensional setting. In particular, we present estimates and convergence in variation results for both convolution and Mellin integral operators with respect to the Tonelli variation. Results with respect to a multidimensional concept of φ-variation in the sense of Tonelli are also presented.

1. Introduction

The aim of the present paper is to give a review on recent results about convergence of integral operators of convolution type with respect to some concepts of multidimensional variation. We will consider the case of classical convolution integral operators of the form

$$(T_w f)(\mathtt{s}) = \int_{\mathbb{R}^N} K_w(\mathtt{t}) f(\mathtt{s}-\mathtt{t})\, d\mathtt{t}, \qquad \mathtt{s} \in \mathbb{R}^N,\ w > 0, \tag{I}$$

where $f \in L^1(\mathbb{R}^N)$ and $\{K_w\}_{w>0}$ is a family of approximate identities, as well as the case of Mellin integral operators of the form

$$(M_w f)(\mathtt{s}) = \int_{\mathbb{R}^N_+} K_w(\mathtt{t}) f(\mathtt{st}) \langle \mathtt{t} \rangle^{-1} d\mathtt{t}, \qquad \mathtt{s} \in \mathbb{R}^N_+,\ w > 0, \tag{II}$$

where $\mathtt{st} := (s_1t_1, \ldots, s_Nt_N)$, $\mathtt{s}, \mathtt{t} \in \mathbb{R}^N_+$, and $\langle \mathtt{t} \rangle := \prod_{i=1}^N t_i$. The above operators (II) are of convolution type with respect to the homothetic operator and the measure $\mu(A) = \int_A (d\mathtt{x}/\langle \mathtt{x} \rangle)$, where A is a Borel subset of $\mathbb{R}^N_+$ (which is an invariant measure with respect to the multiplicative operation).

An important tool in order to frame the results of the paper is the setting of the functional spaces we deal with. The BV-spaces, apart from the well-known importance from the mathematical point of view, also play an important role in problems of Image Reconstruction where some of the various approaches make use of integral operators of convolution type (see, e.g., sampling-type operators).

The working space will be the space of functions of bounded multidimensional variation in the sense of Tonelli (defined in Section 2) and, as further extension, in order to deal with a larger class of functions, we will consider the space BV^{φ}, where φ is a φ-function (see Section 2). We point out that, due to the necessary assumptions on the φ-function φ (Assumption ii), the case of BV cannot be obtained as particular case of BV^{φ}. This is the reason why the two settings have to be treated independently.

For the above classes of operators we will provide estimates, convergence results, and also a characterization of the absolutely (φ-absolutely) continuous functions in terms of the respective convergence in variation. Moreover, the rate of approximation has been considered and examples of kernel functions to which the results can be applied are also furnished. Finally, also the nonlinear case for both the convolution and the Mellin-type operators has been considered.

Apart from the well-known importance of the classical convolution integral operators, the Mellin operators are very interesting and widely studied in approximation theory (for the basic theory see [1, 2] while, for results about similar homothetic-type operators, see, e.g., [3–16]), also because of their important applications in several fields. Among them, for example, we recall that Mellin analysis is deeply connected with some problems of Signal Processing, in particular with the so-called Exponential Sampling, which have applications in various problems of engineering and optical physics (see, e.g., [17–20]).

Concerning now the multidimensional concept of φ-variation introduced in [21], we point out that, due to the lack of an integral representation of the φ-variation for φ-absolutely continuous functions as happens for the classical variation, the major results about convergence require a different approach and suitable techniques. In particular, this holds for the convergence of the φ-modulus of continuity. Again, a similar problem occurs in case of Mellin integral operators on $BV(\mathbb{R}_+^N)$, where the Tonelli integrals are defined via the log-measure. Moreover, in the latter case, in order to prove that the operators are absolutely continuous, in case of regular kernels (this is a crucial point to obtain the characterization of absolute continuity), one has to pass through an equivalent notion of absolute continuity (for the log-absolute continuity, see Section 4) compatible with the setting of $\mathbb{R}_+^N$ equipped with the log-measure.

We finally remark that, of course, all the results of the paper contain, in particular, the one-dimensional case (see [22–27]).

2. Preliminaries and Some Concepts of Variation

We will now recall the multidimensional concept of variation in the sense of Tonelli. Such definition was introduced by Tonelli [28] for functions of two variables and then extended to dimension $N > 2$ by Radó [29] and Vinti [30].

Let us introduce some notations. If we are interested in the jth coordinate of a vector $\mathtt{x} = (x_1, \dots, x_N) \in \mathbb{R}^N$, we will write

$$\mathtt{x}'_j = \left(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_N\right) \in \mathbb{R}^{N-1}, \qquad \mathtt{x} = \left(\mathtt{x}'_j, x_j\right), \tag{1}$$

so that, for a function $f : \mathbb{R}^N \to \mathbb{R}$, there holds

$$f(\mathtt{x}) = f\left(\mathtt{x}'_j, x_j\right). \tag{2}$$

Given an N-dimensional interval $I = \prod_{i=1}^{N} [a_i, b_i]$, by $I'_j = [\mathtt{a}'_j, \mathtt{b}'_j]$ we will denote the $(N-1)$-dimensional interval obtained by deleting the jth coordinate from I; namely,

$$I = \left[\mathtt{a}'_j, \mathtt{b}'_j\right] \times \left[a_j, b_j\right], \quad j = 1, \dots, N. \tag{3}$$

Definition 1. A function $f : \mathbb{R}^N \to \mathbb{R}$ is said to be of bounded variation if the sections of f are a.e. of bounded variation on $\mathbb{R}$ and their variation is summable; that is, $V_{\mathbb{R}}[f(\mathtt{x}'_j, \cdot)]$ (the usual Jordan one-dimensional variation of the jth section of f) is finite a.e. $\mathtt{x}'_j \in \mathbb{R}^{N-1}$ and $\int_{\mathbb{R}^{N-1}} V_{\mathbb{R}}[f(\mathtt{x}'_j, \cdot)] d\mathtt{x}'_j < +\infty$, for every $j = 1, \dots, N$.

In order to compute the variation of f on an interval I, the first step is to define the $(N-1)$-dimensional integrals (the so-called *Tonelli integrals*)

$$\Phi_j(f, I) := \int_{\mathtt{a}'_j}^{\mathtt{b}'_j} V_{[a_j, b_j]}\left[f\left(\mathtt{x}'_j, \cdot\right)\right] d\mathtt{x}'_j, \quad j = 1, \dots, N. \tag{4}$$

Let now $\Phi(f, I)$ be the Euclidean norm of the vector $(\Phi_1(f, I), \dots, \Phi_N(f, I))$; that is,

$$\Phi(f, I) := \left\{\sum_{j=1}^{N} \Phi_j^2(f, I)\right\}^{1/2}, \tag{5}$$

where we put $\Phi(f, I) = \infty$ if $\Phi_j(f, I) = \infty$ for some $j = 1, \dots, N$.

The variation of f on an interval $I \subset \mathbb{R}^N$ is defined as

$$V_I[f] := \sup \sum_{k=1}^{m} \Phi(f, J_k), \tag{6}$$

where the supremum is taken over all the families of N-dimensional intervals $\{J_1, \dots, J_m\}$ which form partitions of I.

Finally, the variation of f over the whole $\mathbb{R}^N$ is defined as

$$V_{\mathbb{R}^N}[f] := \sup_{I \subset \mathbb{R}^N} V_I[f], \tag{7}$$

where the supremum is taken over all the intervals $I \subset \mathbb{R}^N$.

By

$$BV\left(\mathbb{R}^N\right) := \left\{f \in L^1\left(\mathbb{R}^N\right) : V_{\mathbb{R}^N}[f] < +\infty\right\} \tag{8}$$

we will denote the *space of functions of bounded variation on* $\mathbb{R}^N$.

We recall that it can be proved that if $f \in BV(\mathbb{R}^N)$ then ∇f exists a.e. in $\mathbb{R}^N$ and $\nabla f \in L^1(\mathbb{R}^N)$ (see, e.g., [29, 30]).

We point out that, in the multidimensional setting, it is natural to consider functions of bounded variation within the Lebesgue space $L^1(\mathbb{R}^N)$: indeed this is analogous to the distributional concept of variation given by Cesari [31] and, in equivalent forms, by Krickeberg [32], De Giorgi [33], Giusti [34], and Serrin [35]. We notice that the definition of variation in the sense of Tonelli is equivalent to the distributional one in the class of functions which satisfy some approximate continuity properties (see, e.g., [30]).

We will now recall the concept of absolute continuity in sense of Tonelli.

Definition 2. A function $f : \mathbb{R}^N \to \mathbb{R}$ is locally absolutely continuous ($f \in AC_{loc}(\mathbb{R}^N)$) if, for every interval $I = \prod_{i=1}^{N} [a_i, b_i]$ and for every $j = 1, 2, \dots, N$, the jth section $f(\mathtt{x}'_j, \cdot) : [a_j, b_j] \to \mathbb{R}$ is (uniformly) absolutely continuous for almost every $\mathtt{x}'_j \in [\mathtt{a}'_j, \mathtt{b}'_j]$.

Similarly to the one-dimensional case, it is possible to prove that if $f \in BV(\mathbb{R}^N) \cap AC_{loc}(\mathbb{R}^N)$, then

$$V_{\mathbb{R}^N}[f] = \int_{\mathbb{R}^N} |\nabla f(\mathtt{x})| \, d\mathtt{x} \tag{9}$$

(see, e.g., [29, 30]).

We will denote by $AC(\mathbb{R}^N)$ the space of all the functions $f \in BV(\mathbb{R}^N) \cap AC_{loc}(\mathbb{R}^N)$.

In the following we will present also results about convergence for a family of Mellin integral operators. In order to study such kind of operators the most natural way is to consider $\mathbb{R}_+^N :=]0, +\infty)^N$ (the domain of the functions where Mellin operators act), as a group with the multiplicative operation (instead of the additive operation on $\mathbb{R}^N$) and equipped with the logarithmic Haar-measure $\mu(A) = \int_A (d\mathtt{x}/\langle \mathtt{x} \rangle)$ (A is a Borel subset of $\mathbb{R}_+^N$ and $\langle \mathtt{x} \rangle := \prod_{i=1}^N x_i$, $\mathtt{x} = (x_1, \dots, x_N) \in \mathbb{R}_+^N$), instead of the usual Lebesgue measure. For this reason, in order to obtain results in BV-spaces for such kind of operators, it seems natural to adapt the definition of the Tonelli variation to this frame: we therefore introduced in [36] a new concept of multidimensional variation in which, in the Tonelli integrals, the Lebesgue measure is replaced by the logarithmic measure μ.

Definition 3. One will say that $f \in \widetilde{L}^1(\mathbb{R}_+^N)$ is of bounded variation on $\mathbb{R}_+^N$ if the sections $f(\mathtt{x}_j', \cdot)$ are of bounded variation on $\mathbb{R}_+$ a.e. $\mathtt{x}_j' \in \mathbb{R}_+^{N-1}$ and $V_{\mathbb{R}_+}[f(\mathtt{x}_j', \cdot)] \in \widetilde{L}^1(\mathbb{R}_+^{N-1})$, where $\widetilde{L}^1(\mathbb{R}_+^N)$ denotes the space of the functions $f : \mathbb{R}_+^N \to \mathbb{R}$ such that $\int_{\mathbb{R}_+^N} |f(\mathtt{t})| \langle \mathtt{t} \rangle^{-1} d\mathtt{t} < +\infty$.

In order to define the multidimensional variation on $\mathbb{R}_+^N$, for a fixed interval $I := \prod_{i=1}^N [a_i, b_i] \subset \mathbb{R}_+^N$ we consider the $(N-1)$-dimensional integrals

$$\Phi_j(f, I) := \int_{\mathtt{a}_j'}^{\mathtt{b}_j'} V_{[a_j, b_j]}\left[f\left(\mathtt{x}_j', \cdot\right)\right] \frac{d\mathtt{x}_j'}{\left\langle \mathtt{x}_j' \right\rangle}, \tag{10}$$

where $\langle \mathtt{x}_j' \rangle$ denotes the product $\prod_{i=1, i \neq j}^N x_i$. The remaining steps for the definition of the variation follow as before.

For the sake of simplicity, we will use the same notations for the variation in both the cases of $\mathbb{R}^N$ and $\mathbb{R}_+^N$: as it is natural, when one works on $\mathbb{R}_+^N$, it is intended that the measure used is the logarithmic one.

The classical definition of Jordan variation [37] was extended in the literature in several directions: one of the first generalizations was the quadratic variation introduced by Wiener [38], extended to the p-variation, $p \geq 1$ [39, 40], and later to the concept of φ-variation. The φ-variation was first introduced by Young [41] and then extensively studied by Musielak and Orlicz and their school (see, e.g., [22, 42–49]).

From now on we will assume that $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is as follows.

Assumption i. φ is a convex φ-function, where a φ-function is a continuous, nondecreasing function on $\mathbb{R}_0^+$, such that $\varphi(0) = 0$, $\varphi(u) > 0$ for $u > 0$, and $\lim_{u \to +\infty} \varphi(u) = +\infty$.

Assumption ii. $u^{-1}\varphi(u) \to 0$ as $u \to 0^+$.

We recall that (see [22]) the φ-variation of $f : \mathbb{R} \to \mathbb{R}$ on $[a, b] \subset \mathbb{R}$ is defined as

$$V_{[a,b]}^{\varphi}[f] := \sup_D \sum_{i=1}^n \varphi\left(|f(s_i) - f(s_{i-1})|\right), \tag{11}$$

where the supremum is taken over all the partitions $D = \{s_0 = a, s_1, \dots, s_n = b\}$ of the interval $[a, b]$, and

$$V_{\mathbb{R}}^{\varphi}[f] := \sup_{[a,b] \subset \mathbb{R}} V_{[a,b]}^{\varphi}[f]. \tag{12}$$

Definition 4. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be of bounded φ-variation ($f \in BV^{\varphi}(\mathbb{R})$) if $V_{\mathbb{R}}^{\varphi}[\lambda f] < +\infty$, for some $\lambda > 0$.

The Musielak-Orlicz φ-variation was generalized to the multidimensional frame in [50] following the approach of Vitali. However, in order to study approximation problems, the approach of the Tonelli variation seems to be the most natural in this context. For such reason in [21] we introduced a concept of multidimensional φ-variation inspired by the Tonelli and C. Vinti approach.

Again, the crucial point is to define, for $j = 1, \dots, N$, the Tonelli integrals: in this case we put

$$\Phi_j^{\varphi}(f, I) := \int_{\mathtt{a}_j'}^{\mathtt{b}_j'} V_{[a_j, b_j]}^{\varphi}\left[f\left(\mathtt{x}_j', \cdot\right)\right] d\mathtt{x}_j', \tag{13}$$

where $V_{[a_j, b_j]}^{\varphi}[f(\mathtt{x}_j', \cdot)]$ is the (one-dimensional) Musielak-Orlicz φ-variation of the jth section of f.

Putting now

$$\Phi^{\varphi}(f, I) := \left\{\sum_{j=1}^N \left[\Phi_j^{\varphi}(f, I)\right]^2\right\}^{1/2}, \tag{14}$$

the multidimensional φ-variation of $f : \mathbb{R}^N \to \mathbb{R}$ on an interval $I \subset \mathbb{R}^N$ is defined as

$$V_I^{\varphi}[f] := \sup \sum_{k=1}^m \Phi^{\varphi}(f, J_k), \tag{15}$$

(the supremum is taken over all the partitions $\{J_1, \dots, J_m\}$ of I) and, finally,

$$V_{\mathbb{R}^N}^{\varphi}[f] := \sup_{I \subset \mathbb{R}^N} V_I^{\varphi}[f]. \tag{16}$$

By

$$\begin{aligned} &BV^{\varphi}\left(\mathbb{R}^N\right) \\ &\quad = \left\{f \in L^1\left(\mathbb{R}^N\right) : \exists \lambda > 0 \text{ s.t. } V^{\varphi}[\lambda f] < +\infty\right\} \end{aligned} \tag{17}$$

we will denote the *space of functions of bounded φ-variation* over $\mathbb{R}^N$.

Similarly to the classical variation, it is natural to introduce a concept of multidimensional φ-absolute continuity.

Definition 5. One says that $f : \mathbb{R}^N \to \mathbb{R}$ is locally φ-absolutely continuous ($AC^{\varphi}_{loc}(\mathbb{R}^N)$) if it is φ-absolutely continuous in the sense of Tonelli; that is, for every $I = \prod_{i=1}^{N}[a_i, b_i] \subset \mathbb{R}^N$ and $j = 1, 2, \ldots, N$, the section $f(\mathtt{x}'_j, \cdot) : [a_j, b_j] \to \mathbb{R}$ is (uniformly) φ-absolutely continuous for almost every $\mathtt{x}'_j \in [\mathtt{a}'_j, \mathtt{b}'_j]$.

Here (see [22]) a function $g : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is φ-absolutely continuous if there exists $\lambda > 0$ such that the following property holds:

for every $\varepsilon > 0$, *there exists* $\delta > 0$ *for which*

$$\sum_{\nu=1}^{n} \varphi\left(\lambda \left|g\left(\beta^{\nu}\right) - g\left(\alpha^{\nu}\right)\right|\right) < \varepsilon, \tag{18}$$

for all the finite collections of nonoverlapping intervals $[\alpha^{\nu}, \beta^{\nu}] \subset [a, b]$, $\nu = 1, \ldots, n$, *such that*

$$\sum_{\nu=1}^{n} \varphi\left(\beta^{\nu} - \alpha^{\nu}\right) < \delta. \tag{19}$$

By $AC^{\varphi}(\mathbb{R}^N)$ we will denote the space of all the functions $f \in BV^{\varphi}(\mathbb{R}^N) \cap AC^{\varphi}_{loc}(\mathbb{R}^N)$.

As before, in order to obtain results for Mellin integral operators in BV^{φ}-spaces in the multidimensional frame, we adapted the previous definition of φ-variation in the sense of Tonelli to the case of functions defined on $\mathbb{R}^N_+$ equipped with the logarithmic measure μ. In such concept of multidimensional φ-variation, introduced in [51], the Tonelli integrals (13) are replaced by

$$\Phi^{\varphi}_{j}(f, I) := \int_{\mathtt{a}'_j}^{\mathtt{b}'_j} V^{\varphi}_{[a_j, b_j]}\left[f\left(\mathtt{x}'_j, \cdot\right)\right] \frac{d\mathtt{x}'_j}{\left\langle \mathtt{x}'_j \right\rangle}. \tag{20}$$

Again, we will use the same notations, as in the case of the variation, for functions defined on both $\mathbb{R}^N$ and $\mathbb{R}^N_+$.

3. Approximation Results for Convolution Integral Operators

In this section we will present results about approximation in variation by means of the convolution integral operators, namely, (I), for $f \in L^1(\mathbb{R}^N)$. Here $\{K_w\}_{w>0}$ is a family of approximate identities (see, e.g., [52]); that is,

(K_w.1) $K_w : \mathbb{R}^N \to \mathbb{R}$ is a measurable essentially bounded function such that $K_w \in L^1(\mathbb{R}^N)$, $\|K_w\|_1 \leq A$ for an absolute constant $A > 0$ and $\int_{\mathbb{R}^N} K_w(\mathtt{t}) d\mathtt{t} = 1$, for every $w > 0$;

(K_w.2) for any fixed $\delta > 0$, $\int_{|\mathtt{t}|>\delta} |K_w(\mathtt{t})| d\mathtt{t} \to 0$, as $w \to +\infty$.

In the following we will say that $\{K_w\}_{w>0} \subset \mathcal{K}_w$ if (K_w.1) and (K_w.2) are satisfied.

Of course the operators (I) are well-defined for every $f \in L^1(\mathbb{R}^N)$ and therefore in particular for every function $f \in BV(\mathbb{R}^N)$, since

$$\left|\left(T_w f\right)(\mathtt{x})\right| \leq \|f\|_1 \left\|K_w\right\|_{\infty}, \quad \forall \mathtt{x} \in \mathbb{R}^N. \tag{21}$$

We first recall that the family of operators (I) map $BV(\mathbb{R}^N)$ into itself. Indeed, the following estimate holds.

Proposition 6 (see [25]). *Let $f \in BV(\mathbb{R}^N)$. If $\{K_w\}_{w>0}$ satisfies (K_w.1), then*

$$V_{\mathbb{R}^N}\left[T_w f\right] \leq A V_{\mathbb{R}^N}[f], \tag{22}$$

$w > 0$, *where A is the constant of Assumption* (K_w.1).

Remark 7. In the case of nonnegative kernels $\{K_w\}_{w>0}$, Proposition 6 gives the "variation diminishing property" for the operators $T_w f$: indeed in this case $A = \|K_w\|_1 = 1$, $w > 0$.

In order to obtain the main result about convergence in variation, the following estimate of the error of approximation is essential.

Proposition 8 (see [25]). *If $f \in BV(\mathbb{R}^N)$ then, for every $w > 0$,*

$$\begin{aligned} &V_{\mathbb{R}^N}\left[T_w f - f\right] \\ &\quad \leq \int_{\mathbb{R}^N} V_{\mathbb{R}^N}\left[f(\cdot - \mathtt{t}) - f(\cdot)\right] \left|K_w(\mathtt{t})\right| d\mathtt{t}. \end{aligned} \tag{23}$$

Another important tool is a characterization of the convergence for the *modulus of smoothness* of f, defined as

$$\omega(f, \delta) := \sup_{|\mathtt{t}|\leq\delta} V_{\mathbb{R}^N}\left[\tau_{\mathtt{t}} f - f\right], \tag{24}$$

where $(\tau_{\mathtt{t}} f)(\mathtt{s}) := f(\mathtt{s} - \mathtt{t})$, for every $\mathtt{s}, \mathtt{t} \in \mathbb{R}^N$, is the translation operator (see, e.g., [6, 53]).

Theorem 9 (see [25]). *Let $f \in BV(\mathbb{R}^N)$. Then $f \in AC(\mathbb{R}^N)$ if and only if*

$$\lim_{\delta\to 0} \omega(f, \delta) = 0. \tag{25}$$

The proof of the sufficient part of this result is a consequence of integral representation (9) of the variation for absolutely continuous functions and of the continuity in L^1 of the translation operator. For the necessary part, in [25] it is proved that if the kernel functions K_w are absolutely continuous (as it happens in the most common cases), then also the integral operators $T_w f$ belong to $AC(\mathbb{R}^N)$. Then, since $AC(\mathbb{R}^N)$ is a closed subspace of $BV(\mathbb{R}^N)$ with respect to the convergence in variation [25] and by estimate (23), in case of regular kernels, the convergence of the modulus of smoothness implies that $f \in AC(\mathbb{R}^N)$.

By means of Proposition 8 and Theorem 9 it is possible to obtain the main result about convergence for absolutely continuous functions.

Theorem 10 (see [25]). *If $f \in AC(\mathbb{R}^N)$ and $\{K_w\}_{w>0} \subset \mathcal{K}_w$, then*

$$\lim_{w\to+\infty} V_{\mathbb{R}^N}\left[T_w f - f\right] = 0. \tag{26}$$

Remark 11. We point out that the assumption of absolute continuity of the function is crucial to obtain the main convergence theorem and such result does not hold, in general, if, for example, $f \in \mathrm{BV}(\mathbb{R}^N)\setminus \mathrm{AC}(\mathbb{R}^N)$. For example, in the case $N = 1$, let us consider $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1, & |x| \le 1, \\ 0, & |x| > 1. \end{cases} \tag{27}$$

First of all we point out that $V[\tau_t f - f] \nrightarrow 0$ as $t \to 0$. Let us now consider the Poisson-Cauchy kernel defined as

$$K_w(t) = \sqrt{\frac{2}{\pi}}\frac{w}{1+w^2t^2}, \quad w > 0,\ t \in \mathbb{R}. \tag{28}$$

Then

$$(T_w f)(s) = \sqrt{\frac{2}{\pi}}\left[\arctan(w(s+1)) - \arctan(w(s-1))\right], \quad s \in \mathbb{R}, \tag{29}$$

and therefore

$$\begin{aligned} V_{\mathbb{R}}\left[T_w f - f\right] &\ge V_{(-\infty,-1]}\left[T_w f - f\right] \\ &= \left|(T_w f)(-1) - \lim_{s\to-\infty}(T_w f)(s)\right| \\ &= -\sqrt{\frac{2}{\pi}}\arctan(-2w). \end{aligned} \tag{30}$$

This implies that

$$\liminf_{w\to+\infty} V_{\mathbb{R}}\left[T_w f - f\right] \ge \sqrt{\frac{\pi}{2}}, \tag{31}$$

and hence $V_{\mathbb{R}}[T_w f - f] \nrightarrow 0$ as $w \to +\infty$.

In case of regular kernels, by the closure of $\mathrm{AC}(\mathbb{R}^N)$ in $\mathrm{BV}(\mathbb{R}^N)$, the converse of Theorem 10 is also true. Hence we obtain the following characterization of the space of the absolutely continuous functions, similarly to what happens in the one-dimensional case for the Jordan variation.

Theorem 12 (see [25]). *Assume that $f \in BV(\mathbb{R}^N)$ and $\{K_w\}_{w>0} \subset \mathcal{K}_w \cap AC(\mathbb{R}^N)$. Then $f \in AC(\mathbb{R}^N)$ if and only if*

$$\lim_{w\to+\infty} V_{\mathbb{R}^N}\left[T_w f - f\right] = 0. \tag{32}$$

The previous results were generalized to the case of the multidimensional φ-variation in [21]. In particular, besides a kind of variation diminishing property, in [21] the following estimate for the error of approximation is obtained.

Proposition 13 (see [21]). *Let $f \in BV^{\varphi}(\mathbb{R}^N)$ and let $\{K_w\}_{w>0}$ be such that $(K_w.1)$ is satisfied. Then, for every $\lambda, \delta > 0$ and for every $w > 0$,*

$$\begin{aligned} &V^{\varphi}_{\mathbb{R}^N}\left[\lambda\left(T_w f - f\right)\right] \\ &\quad\le A^{-1}\left\{\omega^{\varphi}(\lambda A f, \delta)\int_{|\mathtt{t}|\le\delta}\left|K_w(\mathtt{t})\right| d\mathtt{t} \right. \\ &\quad\left. + V^{\varphi}_{\mathbb{R}^N}\left[2\lambda A f\right]\int_{|\mathtt{t}|>\delta}\left|K_w(\mathtt{t})\right| d\mathtt{t}\right\}, \end{aligned} \tag{33}$$

where $\omega^{\varphi}(f,\delta) := \sup_{|\mathtt{t}|\le\delta} V^{\varphi}_{\mathbb{R}^N}[\tau_{\mathtt{t}} f - f]$ is the φ-modulus of smoothness of f.

Using the previous estimate, the main convergence result follows by the singularity assumption on the kernel functions $(K_w.2)$ and by the convergence for the φ-modulus of smoothness.

Theorem 14 (see [54]). *Let $f \in BV^{\varphi}(\mathbb{R}^N)$. Then there exists $\lambda > 0$ such that*

$$\lim_{\delta\to0^+}\omega^{\varphi}(\lambda f, \delta) = 0, \tag{34}$$

if and only if $f \in AC^{\varphi}_{\mathrm{loc}}(\mathbb{R}^N)$.

The convergence for the modulus of smoothness, in case of the Tonelli variation, is a direct consequence of the integral representation of the variation for absolutely continuous functions (9). On the contrary, for the φ-variation there are no results of this kind and, in order to get the convergence in φ-variation of the φ-modulus of smoothness, it is necessary to use a different technique. In particular (see [54]), the crucial point is to construct a kind of "step" functions that approximate the function f and for which a convergence result can be proved.

Using Theorem 14, it is possible to obtain the main result of convergence in φ-variation for the convolution integral operators (I).

Theorem 15 (see [21]). *If $f \in AC^{\varphi}(\mathbb{R}^N)$ and $\{K_w\}_{w>0} \subset \mathcal{K}_w$, then there exists $\lambda > 0$ such that*

$$\lim_{w\to+\infty} V^{\varphi}_{\mathbb{R}^N}\left[\lambda\left(T_w f - f\right)\right] = 0. \tag{35}$$

To give a sketch of the proof, the starting point is the estimate of Proposition 13 for the error of approximation $V^{\varphi}_{\mathbb{R}^N}[\lambda(T_w f - f)]$. By Theorem 14, we have that $\omega^{\varphi}(\lambda f, \delta)$ tends to 0, for sufficiently small $\delta > 0$, while, by Assumption $(K_w.2)$ on the kernel functions, in correspondence with such small δ, $\int_{|\mathtt{t}|>\delta}|K_w(\mathtt{t})|d\mathtt{t}$ converges to 0 for w large enough; hence the result follows, taking into account $(K_w.1)$ and the fact that $f \in \mathrm{BV}^{\varphi}(\mathbb{R}^N)$.

As before, in case of regular kernels the converse of Theorem 15 is also true.

Theorem 16 (see [21]). *Let $f \in BV^{\varphi}(\mathbb{R}^N)$ and let $\{K_w\}_{w>0} \subset \mathcal{K}_w \cap AC^{\varphi}(\mathbb{R}^N)$. Then $f \in AC^{\varphi}(\mathbb{R}^N)$ if and only if there exists $\lambda > 0$ such that*

$$\lim_{w\to+\infty} V^{\varphi}_{\mathbb{R}^N}\left[\lambda\left(T_w f - f\right)\right] = 0. \tag{36}$$

Remark 17. We point out that it is easy to find examples of kernel functions to which the previous results can be applied. Among them, for example, the Gauss-Weierstrass kernel is defined as

$$G_w(\mathtt{t}) = \frac{w^N}{\pi^{N/2}} e^{-w^2|\mathtt{t}|^2}, \quad \mathtt{t} \in \mathbb{R}^N, \ w > 0, \tag{37}$$

or the Picard kernel

$$P_w(\mathtt{t}) = \frac{w^N \Gamma(N/2)}{2\pi^{N/2}\Gamma(N)} e^{-w|\mathtt{t}|}, \quad \mathtt{t} \in \mathbb{R}^N, \ w > 0, \tag{38}$$

where Γ is the Gamma Euler function (see Figure 1).

4. Approximation Results for Mellin Integral Operators

We now turn our attention to Mellin integral operators, defined as (II), where $\mathtt{st} := (s_1t_1, \dots, s_Nt_N)$, $\mathtt{s}, \mathtt{t} \in \mathbb{R}^N_+$. On the kernel functions $\{K_w\}_{w>0}$ we assume that

($\widetilde{K}_w$.1) $K_w : \mathbb{R}^N_+ \to \mathbb{R}$ is a measurable essentially bounded function such that $K_w \in \widetilde{L}^1(\mathbb{R}^N_+)$, $\|K_w\|_{\widetilde{L}^1} \leq A$ for an absolute constant $A > 0$ and $\int_{\mathbb{R}^N_+} K_w(\mathtt{t})\langle \mathtt{t}\rangle^{-1} d\mathtt{t} = 1$, for every $w > 0$;

($\widetilde{K}_w$.2) for every fixed $0 < \delta < 1$, $\int_{|\mathbf{1}-\mathtt{t}|>\delta} |K_w(\mathtt{t})|\langle \mathtt{t}\rangle^{-1} d\mathtt{t} \to 0$, as $w \to +\infty$,

that is, the assumptions of approximate identities, adapted to the present setting of $\mathbb{R}^N_+$. If $\{K_w\}_{w>0}$ satisfy ($\widetilde{K}_w$.1) and ($\widetilde{K}_w$.2), we will write $\{K_w\}_{w>0} \subset \widetilde{\mathcal{K}}_w$.

For Mellin integral operators it is possible to develop an "approximation theory" similar to the case of the convolution integral operators; however, one of the main differences is the homothetic structure of $\mathbb{R}^N_+$ which leads to the choice of the logarithmic measure μ and also to the necessity to adapt some definitions. For example, the modulus of smoothness of $f \in BV(\mathbb{R}^N_+)$ has to be now defined as

$$\omega(f, \delta) := \sup_{|\mathbf{1}-\mathtt{t}|\leq\delta} V_{\mathbb{R}^N_+}[\sigma_{\mathtt{t}} f - f], \tag{39}$$

$0 < \delta < 1$, where $(\sigma_{\mathtt{t}} f)(\mathtt{s}) := f(\mathtt{st})$, for every $\mathtt{s}, \mathtt{t} \in \mathbb{R}^N_+$, is the homothetic operator and $\mathbf{1} = (1, \dots, 1)$ is the unit vector of $\mathbb{R}^N_+$. Such notion of modulus of smoothness is the natural generalization, in the present frame of $BV(\mathbb{R}^N_+)$, of the classical modulus of continuity (see, e.g., [6, 21, 25]).

Of course, due to the presence of the logarithmic measure, we cannot use the integral representation of the Tonelli variation; nevertheless, it is possible to directly prove a result of convergence in variation for the modulus of smoothness in case of AC-functions, using a kind of "separated" variations ($V^j[f]$, $j = 1, \dots, N$) which take into account just a single direction instead of all the N directions.

Theorem 18 (see [36]). *If $f \in AC(\mathbb{R}^N_+)$, then*

$$\lim_{\delta\to 0^+} \omega(f, \delta) = 0. \tag{40}$$

By means of the previous result and an estimate for the error of approximation $(M_w f - f)$ analogous to Proposition 13, it is possible to prove the following convergence result.

Theorem 19. *Let $f \in AC(\mathbb{R}^N_+)$ and $\{K_w\}_{w>0} \subset \widetilde{\mathcal{K}}_w$. Then*

$$\lim_{w\to+\infty} V_{\mathbb{R}^N_+}[M_w f - f] = 0. \tag{41}$$

A natural question now is whether, at least in case of AC-kernels, the converse of the previous result holds, as in the case of convolution operators. Actually, due to the form of the operators $M_w f$, such question is now much more delicate and direct approach cannot be used. In order to solve the problem, it is necessary to use another concept of absolute continuity (the *log-absolute continuity*), equivalent to the classical one, which takes into account the logarithmic measure μ. We first present the definition in the one-dimensional case.

Definition 20 (see [55]). One says that $f : \mathbb{R}_+ \to \mathbb{R}$ is *log-absolutely continuous* on $[a, b] \subset \mathbb{R}_+$ ($f \in AC_{\log}([a, b])$) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for every collection of nonoverlapping intervals $[\alpha^\nu, \beta^\nu]_{\nu=1}^n$ in $[a, b]$ such that

$$\sum_{\nu=1}^{n} \left|\log(\beta^\nu) - \log(\alpha^\nu)\right| < \delta, \tag{42}$$

then

$$\sum_{\nu=1}^{n} \left|f(\beta^\nu) - f(\alpha^\nu)\right| < \varepsilon. \tag{43}$$

By $AC_{\log}(\mathbb{R}_+)$ we will denote the space of functions which are of bounded variation on $\mathbb{R}_+$ and log-absolutely continuous on $[a, b]$, for every $[a, b] \subset \mathbb{R}_+$.

Now, in the general multidimensional frame, $f : \mathbb{R}^N_+ \to \mathbb{R}$ is *log-absolutely continuous* on $I = \prod_{i=1}^N [a_i, b_i] \subset \mathbb{R}^N_+$ if, for every $j = 1, 2, \dots, N$, the jth sections of f, $f(\mathtt{x}'_j, \cdot) : [a_j, b_j] \to \mathbb{R}$, are (uniformly) log-absolutely continuous for almost every $\mathtt{x}'_j \in \mathbb{R}^{N-1}_+$.

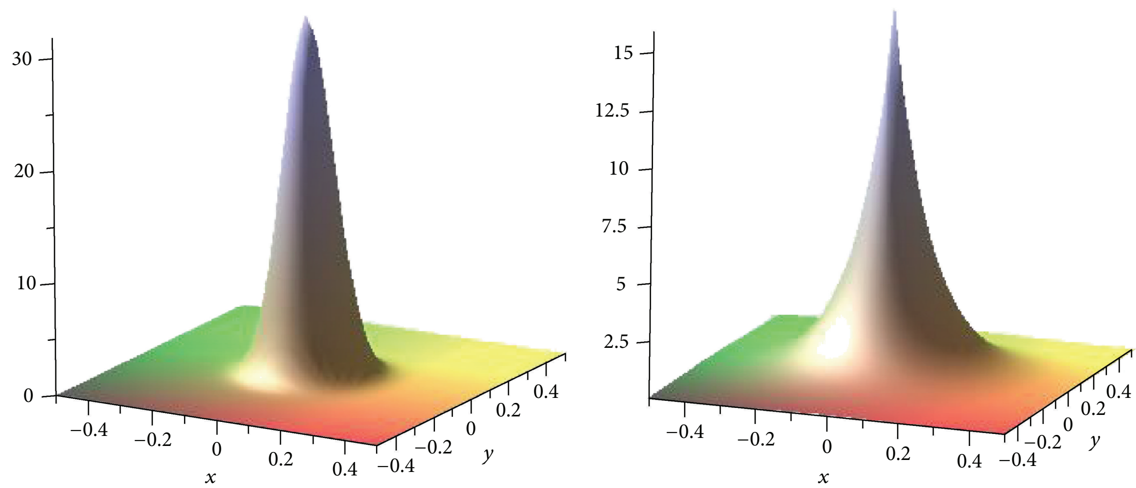
By means of the definition of the log-absolute continuity, in [55] it is proved that Mellin integral operators, as the classical convolution operators, preserve absolute continuity: this, together with the fact that the set of the absolutely continuous functions is a closed subspace of the set of the BV-functions, allows us to obtain the following characterization.

Theorem 21 (see [55]). *Let $f \in BV(\mathbb{R}^N_+)$ and $\{K_w\}_{w>0} \subset \widetilde{\mathcal{K}}_w \cap AC(\mathbb{R}^N_+)$. Then $f \in AC(\mathbb{R}^N_+)$ if and only if* $\lim_{w\to+\infty} V_{\mathbb{R}^N_+}[M_w f - f] = 0$.

In [56] approximation properties for Mellin integral operators were studied in the frame of BV^φ-spaces, using the multidimensional version of the φ-variation on $\mathbb{R}^N_+$ introduced in [51]. In particular the following theorem is obtained.

Theorem 22 (see [56]). *Let $f \in AC^\varphi(\mathbb{R}^N_+)$ and $\{K_w\}_{w>0} \subset \widetilde{\mathcal{K}}_w$. Then there exists a constant $\mu > 0$ such that*

$$\lim_{w\to+\infty} V^\varphi_{\mathbb{R}^N_+}[\mu(M_w f - f)] = 0. \tag{44}$$



(a) (b)

FIGURE 1: Examples of Gauss-Weierstrass kernel $G_w(x, y)$ and Picard kernel $P_w(x, y)$ with $w = 10$.

The proof of Theorem 22 is based on the estimate of the error of approximation [56],

$$\begin{aligned} V^{\varphi}\left[\lambda\left(M_w f - f\right)\right] &\\ \leq \omega^{\varphi}\left(\lambda A f, \delta\right) &\\ + A^{-1} V^{\varphi}\left[2\lambda A f\right] \int_{|\mathbf{1}-\mathtt{t}|>\delta} \left|K_w(\mathtt{t})\right| \langle \mathtt{t}\rangle^{-1} d\mathtt{t}, & \end{aligned} \tag{45}$$

and on a convergence result for the φ-modulus of smoothness

$$\omega^{\varphi}(f, \delta) := \sup_{|\mathbf{1}-\mathtt{t}|\leq\delta} V^{\varphi}_{\mathbb{R}^N_+}\left[\sigma_{\mathtt{t}} f - f\right], \tag{46}$$

$\delta > 0$. This last result was obtained in [51] by means of a direct approach: the function f is approximated in φ-variation by two auxiliary functions, constructed on a grid on which their sections are piecewise constant.

In order to prove the converse of Theorem 22, it is again necessary to use a concept of logarithmic φ-absolute continuity, which takes into account the homothetic structure of $\mathbb{R}^N_+$. We report below the definition in the one-dimensional case, while for the multidimensional case it is sufficient to proceed as for the log-absolute continuity.

Definition 23 (see [57]). One says that $f : [a, b] \to \mathbb{R}$ is log-φ-absolutely continuous on $[a, b] \subset \mathbb{R}_+$ if there exists $\lambda > 0$ such that, for every $\varepsilon > 0$, there exists $\delta > 0$ for which

$$\sum_{\nu=1}^{n} \varphi\left(\lambda \left|f\left(\beta^{\nu}\right) - f\left(\alpha^{\nu}\right)\right|\right) < \varepsilon, \tag{47}$$

for all finite collections of nonoverlapping intervals $[\alpha^{\nu}, \beta^{\nu}] \subset [a, b]$, $\nu = 1, \ldots, n$, such that

$$\sum_{\nu=1}^{n} \varphi\left(\log\left(\beta^{\nu}\right) - \log\left(\alpha^{\nu}\right)\right) < \delta. \tag{48}$$

The log-φ-absolute continuity is equivalent to the φ-absolute continuity and allows obtaining the characterization of $AC^{\varphi}(\mathbb{R}^N_+)$ in terms of convergence in φ-variation of Mellin integral operators.

Theorem 24 (see [57]). *Let $f \in BV^{\varphi}(\mathbb{R}^N_+)$ and $\{K_w\}_{w>0} \subset \widetilde{\mathcal{K}}_w \cap AC^{\varphi}(\mathbb{R}^N_+)$. Then $f \in AC^{\varphi}(\mathbb{R}^N_+)$ if and only if there exists $\lambda > 0$ such that $\lim_{w\to+\infty} V^{\varphi}_{\mathbb{R}^N_+}[\lambda(M_w f - f)] = 0$.*

Remark 25. We point out that taking $N = 1$ as particular case of Theorem 24 we obtain the characterization of φ-absolute continuity in the one-dimensional case, namely, for the classical Musielak-Orlicz φ-variation.

Remark 26. It is not difficult to find examples of kernel functions which fulfill Assumptions ($\widetilde{K}_w$.1) and ($\widetilde{K}_w$.2). Among them, for example, the moment-type kernels (or average kernels) are defined as

$$A_w(\mathtt{t}) := w^N \langle \mathtt{t}\rangle^{w} \chi_{]0,1[^N}(\mathtt{t}), \quad \mathtt{t} \in \mathbb{R}^N_+,\ w > 0. \tag{49}$$

It is easy to see that they fulfill Assumption ($\widetilde{K}_w$.1). Moreover, for every $\delta \in]0, 1[$, $|\mathbf{1} - \mathtt{t}| > \delta$ implies that there exists an index $j = 1, \ldots, N$ such that $|1 - t_j| > \delta/\sqrt{N}$; hence $\{\mathtt{t} \in]0, 1[^N : |\mathbf{1} - \mathtt{t}| > \delta\} \subset \bigcup_{j=1}^{N} \{\mathtt{t} \in \mathbb{R}^N_+ : 0 < t_j < 1 - \delta/\sqrt{N}, 0 < t_i < 1, \forall i \neq j\}$. Therefore

$$\begin{aligned} &\int_{|\mathbf{1}-\mathtt{t}|>\delta} \left|A_w(\mathtt{t})\right| \langle \mathtt{t}\rangle^{-1} d\mathtt{t} \\ &\quad \leq \sum_{j=1}^{N} \left\{ \left(\prod_{i\neq j} \int_0^1 w t_i^{w-1} dt_i \right) \int_0^{1-\delta/\sqrt{N}} w t_j^{w-1} dt_j \right\} \\ &\quad = N\left(1 - \frac{\delta}{\sqrt{N}}\right)^{w} \longrightarrow 0, \end{aligned} \tag{50}$$

as $w \to +\infty$; that is, also ($\widetilde{K}_w$.2) is satisfied.

Other families of kernel functions to which the previous results can be applied are the Mellin-Gauss-Weierstrass kernels, defined as

$$\widetilde{G}_w(\mathtt{t}) := \frac{w^N}{\pi^{N/2}} e^{-w^2|\log \mathtt{t}|^2}, \quad \mathtt{t} \in \mathbb{R}_+^N,\ w > 0, \tag{51}$$

or the Mellin-Picard kernels, which are defined as

$$\widetilde{P}_w(\mathtt{t}) := \frac{w^N}{2\pi^{N/2}} \frac{\Gamma(N/2)}{\Gamma(N)} e^{-w|\log \mathtt{t}|}, \quad \mathtt{t} \in \mathbb{R}_+^N,\ w > 0. \tag{52}$$

We point out that these definitions are the natural reformulations, in the present multiplicative setting of $\mathbb{R}_+^N$, of the classical Gauss-Weierstrass kernels and Picard kernels, respectively (see Remark 17).

5. Further Results

We will now give some hints about further approximation results that were obtained in BV-spaces.

First of all, an interesting problem is to study the nonlinear versions of operators (I) and (II). We point out that the nonlinear case is much more delicate than the linear one and requires some ad hoc assumptions; on the other side, it not only is interesting from a mathematical point of view, being of course more general than the linear one, but also is important from the point of view of the applications. Indeed, there are several applicative problems that cannot be faced by means of linear processes; an example is furnished by some problems of Signal Processing.

The nonlinear version of the convolution integral operators (II) is

$$\left(\overline{T}_w f\right)(\mathtt{s}) = \int_{\mathbb{R}^N} \overline{K}_w(\mathtt{t}, f(\mathtt{s} - \mathtt{t}))\, d\mathtt{t}, \quad w > 0,\ \mathtt{s} \in \mathbb{R}^N, \tag{III}$$

where $\{\overline{K}_w\}_{w>0}$ is a family of measurable functions $\overline{K}_w : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ of the form

$$\overline{K}_w(\mathtt{t}, u) = L_w(\mathtt{t})\, H_w(u) \tag{53}$$

for every $\mathtt{t} \in \mathbb{R}^N$, $u \in \mathbb{R}$. Here $L_w : \mathbb{R}^N \to \mathbb{R}$ and $H_w : \mathbb{R} \to \mathbb{R}$ with $H_w(0) = 0$ is a Lipschitz kernel for every $w > 0$; that is, there exists $K > 0$ such that

$$\left|H_w(u) - H_w(v)\right| \leq K|u - v|, \quad \forall u, v \in \mathbb{R}. \tag{54}$$

Moreover we assume that

($\overline{K}_w$.1) $L_w : \mathbb{R}^N \to \mathbb{R}$ is a measurable function such that $L_w \in L^1(\mathbb{R}^N)$, $\|L_w\|_1 \leq A$, for some $A > 0$ and for every $w > 0$, and $\int_{\mathbb{R}^N} L_w(\mathtt{t})d\mathtt{t} = 1$, for every $w > 0$;

($\overline{K}_w$.2) for any fixed $\delta > 0$, $\int_{|\mathtt{t}|>\delta} |L_w(\mathtt{t})| d\mathtt{t} \to 0$, as $w \to +\infty$;

($\overline{K}_w$.3) denoted by $G_w(u) := H_w(u) - u$, $u \in \mathbb{R}$, $w > 0$,

$$\frac{V_J[G_w]}{m(J)} \longrightarrow 0, \quad \text{as } w \longrightarrow +\infty, \tag{55}$$

uniformly with respect to every (not trivial) bounded interval $J \subset \mathbb{R}$; that is, for every $\varepsilon > 0$ there exists $\overline{w} > 0$ such that $V_J[G_w]/m(J) < \varepsilon$, for every $w \geq \overline{w}$ and for every bounded interval $J \subset \mathbb{R}$.

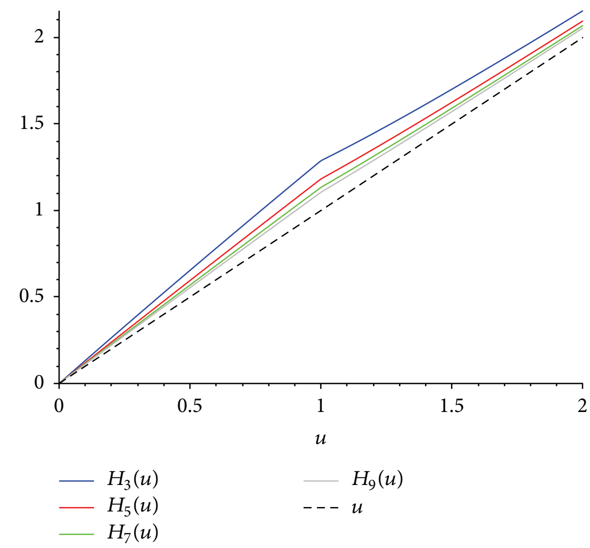


Figure 2: Example of kernels $H_w(u)$, $w = 3, 5, 7, 9$.

Remark 27. We point out that Assumption ($\overline{K}_w$.3) is due to the nonlinear frame and of course it is obviously satisfied in the linear case ($H_w(u) = u$, $u \in \mathbb{R}$). Moreover it is not difficult to provide examples of kernels which fulfill all the previous assumptions. For example, we can consider the kernel functions $\overline{K}_w(t, u) = L_w(t)H_w(u)$, $t \in \mathbb{R}_0^+$, $u \in \mathbb{R}$, $w > 0$, where $\{L_w\}_{w>0}$ are approximate identities,

$$H_w(u) = \begin{cases} u + \log\left(1 + \dfrac{u}{w}\right), & 0 \leq u < 1, \\ u + \log\left(1 + \dfrac{1}{wu}\right), & u \geq 1, \end{cases} \tag{56}$$

and the definition of $H_w(u)$ is extended in odd way for $u < 0$ (see Figure 2).

The problem of the convergence in variation for the nonlinear integral operators (III) was faced in [58, 59]; in particular, the main convergence result reads as follows.

Theorem 28 (see [58]). *If $f \in AC(\mathbb{R}^N)$ and $\{\overline{K}_w\}_{w>0}$ satisfy ($\overline{K}_w$.i), $i = 1, 2, 3$, then*

$$\lim_{w\to+\infty} V_{\mathbb{R}^N}\left[\overline{T}_w f - f\right] = 0. \tag{57}$$

Similar approximation results were obtained in [60] for the nonlinear convolution integral operators (III) in the frame of $BV^{\varphi}(\mathbb{R}^N)$ and in [61] for the nonlinear version of the Mellin integral operators (II) in $BV(\mathbb{R}_+^N)$.

We finally point out that, besides the problem of convergence, the rate of approximation has been also studied in all the previously mentioned settings. In order to do it, as it is natural, one has to introduce suitable Lipschitz classes which take into account the variation functional. We point out that, in order to approach the mentioned problem, the assumptions on kernels have to be slightly modified.

For example, let us consider the case of the convolution integral operators (I) in the setting of $BV(\mathbb{R}^N)$. In this frame the Lipschitz class is defined as

$$V\mathrm{Lip}^N(\alpha) := \left\{ f \in BV\left(\mathbb{R}^N\right) : V_{\mathbb{R}^N}\left[\tau_{\mathtt{t}} f - f\right] = O\left(|\mathtt{t}|^\alpha\right), \text{ as } |\mathtt{t}| \longrightarrow 0 \right\}, \tag{58}$$

$\alpha > 0$, and Assumption (K_w.2) has to be replaced by the following:

(K'_w.2) for any fixed $\delta > 0$, $\int_{|\mathtt{t}|>\delta} |L_w(\mathtt{t})| d\mathtt{t} = O(w^{-\alpha})$, as $w \to +\infty$.

Moreover we will say that $\{K_w\}_{w>0}$ is an *α-singular kernel*, for $0 < \alpha \leq 1$, if

$$\int_{|\mathtt{t}|>\delta} \left|K_w(\mathtt{t})\right| d\mathtt{t} = O\left(w^{-\alpha}\right), \quad \text{as } w \longrightarrow +\infty, \tag{59}$$

for every $\delta > 0$. Then it is possible to obtain the following result about the order of approximation for the convolution integral operators (I).

Theorem 29 (see [25]). *Let $f \in V\mathrm{Lip}^N(\alpha)$ and let $\{K_w\}_{w>0} \subset \mathcal{K}_w$ be an α-singular kernel satisfying (K_w.1) and (K'_w.2). Moreover assume that there exists $0 < \tilde{\delta} < 1$ such that*

$$\int_{|\mathtt{t}|\leq\tilde{\delta}} \left|K_w(\mathtt{t})\right| |\mathtt{t}|^\alpha d\mathtt{t} = O\left(w^{-\alpha}\right), \quad \text{as } w \longrightarrow +\infty. \tag{60}$$

Then

$$V_{\mathbb{R}^N}\left[T_w f - f\right] = O\left(w^{-\alpha}\right), \tag{61}$$

as $w \to +\infty$.

Similar results in the nonlinear case were obtained in [58], while, for results about the rate of approximation for convolution integral operators with respect to the multidimensional φ-variation, see [21] and [60] (nonlinear case).

The case of Mellin integral operators was studied in [36] and in [61] (nonlinear case) with respect to the Tonelli variation, while the case of the multidimensional φ-variation was studied in [56]. We finally refer to [62, 63] for approximation results in the slightly different setting of $BV^\varphi((\mathbb{R}_0^+)^N)$.

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this article.

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Research Article

Mappings of Type Special Space of Sequences

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We give sufficient conditions on a special space of sequences defined by Mohamed and Bakery (2013) such that the finite rank operators are dense in the complete space of operators whose approximation numbers belong to this sequence space. Hence, under a few conditions, every compact operator would be approximated by finite rank operators. We apply it on the sequence space defined by Tripathy and Mahanta (2003). Our results match those known for p-absolutely summable sequences of reals.

1. Introduction and Basic Definitions

By ω and $L(V, W)$, we will denote the spaces of all real sequences and all bounded linear operators between two Banach spaces V into W, respectively. In [1], Pietsch, by using the approximation numbers and p-absolutely summable sequences of real numbers, formed the operator ideals. In [2], Mohamed and Bakery have considered the space ℓ_M, when $M(t) = t^p$ $(0 < p < \infty)$, which matches especially ℓ^p. A subclass U of $L = \{L(V, W)\}$ is an operator ideal if its components verify the following conditions:

(i) The space $F(V, W)$ of all finite rank operators is a subset of $U(V, W)$.

(ii) The space $U(V, W)$ is linear.

(iii) For two Banach spaces V_0 and W_0, if $T \in L(V_0, V)$, $S \in U(V, W)$, and $R \in L(W, W_0)$, then $RST \in U(V_0, W_0)$. See [3, 4].

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is convex, positive, nondecreasing, and continuous, where $M(0) = 0$ and $\lim_{x\to\infty} M(x) = \infty$. An Orlicz function M is said to satisfy Δ_2-condition for all values of $x \geq 0$ if there exists a constant $k > 0$, such that $M(2x) \leq kM(x)$. Lindenstrauss and Tzafriri [5] used the idea of an Orlicz function to define Orlicz sequence spaces as follows:

$$\ell_M = \left\{ x \in \omega : \exists \lambda > 0 \text{ with } \rho(\lambda x) = \sum_{k=1}^{\infty} M(|\lambda x(k)|) < \infty \right\}. \tag{1}$$

$(\ell_M, \|x\|)$ is a Banach space, where $\|x\| = \inf\{\lambda > 0 : \rho(x/\lambda) \leq 1\}$. The space ℓ^p is an Orlicz sequence space with $M(x) = x^p$ for $1 \leq p < \infty$.

Remark 1. For any Orlicz function M, we have $M(\lambda x) \leq \lambda M(x)$, for all λ with $0 < \lambda < 1$.

Let P_s be the class of all subsets of $\mathbb{N} = \{0, 1, 2, \ldots\}$ that do not contain more than s number of elements and let $\{\phi_n\}$ be a nondecreasing sequence of positive reals such that $n\phi_{n+1} \leq (n+1)\phi_n$, for all $n \in \mathbb{N}$. Tripathy and Mahanta [6] defined and studied the following sequence space:

$$m(\phi, M) = \left\{ x = (x_k) \in \omega : \exists \zeta > 0 \text{ with } \sup_{s\geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M\left(\frac{|x_k|}{\zeta}\right) < \infty \right\}, \tag{2}$$

with the norm

$$\rho(x) = \inf\left\{\zeta > 0 : \sup_{s\geq 1, \sigma\in P_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M\left(\frac{|x_k|}{\zeta}\right) \leq 1\right\}. \quad (3)$$

Lemma 2. *(i) $\ell_M \subseteq m(\phi, M)$.*
(ii) $\ell_M = m(\phi, M)$ if and only if $\sup_s \phi_s < \infty$.

As of late, different classes of sequences have been presented using Orlicz functions by Braha [7], Raj and Sharma Sunil [8], Raj et al. [9], and many others ([10–13]).

Definition 3 (see [14, 15]). A special space of sequences (sss) is a linear space E with the following:

(1) $e_n \in E$ for all $n \in \mathbb{N}$, where $e_n = \{0, 0, \ldots, 1, 0, 0, \ldots\}$ with 1 appearing at nth place for all $n \in \mathbb{N}$.

(2) E is solid$''$.

(3) $(x_0, x_0, x_1, x_1, \ldots) \in E$, if $(x_n)_{n\in\mathbb{N}} \in E$.

A premodular (sss) E_ρ is a (sss) and there is a function $\rho : E \to [0, \infty[$ with the following:

(i) $\rho(x) \geq 0$, for each $x \in E$ and $\rho(x) = 0 \Leftrightarrow x = \theta$, where θ is the zero element of E.

(ii) ρ satisfies Δ_2-condition.

(iii) For each $x, y \in E$, $\rho(x + y) \leq k(\rho(x) + \rho(y))$ holds for some $k \geq 1$.

(iv) The space E is ρ-solid; that is, $\rho((x_n)) \leq \rho((y_n))$, whenever $|x_n| \leq |y_n|$, for all $n \in \mathbb{N}$.

(v) For some numbers $k_0 \geq 1$, the inequality $\rho((x_n)_{n\in\mathbb{N}}) \leq \rho((x_0, x_0, x_1, x_1, \ldots)) \leq k_0\rho((x_n)_{n\in\mathbb{N}})$ holds.

(vi) $\overline{F} = E_\rho$; that is, the set of all finite sequences F is ρ-dense in E.

(vii) For each $\lambda > 0$, there is a constant $\xi > 0$ such that $\rho(\lambda, 0, 0, 0, \ldots) \geq \xi\lambda\rho(1, 0, 0, 0, \ldots)$.

Condition (ii) says that ρ is continuous at θ. The function ρ defines a metrizable topology in E and the linear space E enriched with this topology is denoted by E_ρ.

Definition 4 (see [16]). Consider the following:

$$U_E^{\text{app}} := \{U_E^{\text{app}}(V, W)\}, \quad (4)$$

where

$$U_E^{\text{app}}(V, W) := \left\{T \in L(V, W) : (\alpha_n(T))_{n\in\mathbb{N}} \in E\right\}. \quad (5)$$

Theorem 5 (see [2]). *If E is a (sss), then U_E^{app} is an operator ideal.*

We explain some results related to the operator spaces.

2. Main Results

In this part, we give sufficient conditions on E such that the finite rank operators are dense in the complete space of operators $U_E^{\text{app}}(V, W)$.

Lemma 6. *If E_ρ is a premodular (sss) and $(x_n) \in E_\rho$ is a decreasing sequence of positive reals, then*

$$\begin{aligned}&\rho\left(\overbrace{0, 0, 0, \ldots, 0}^{2n}, x_n, x_{n+1}, x_{n+2}, \ldots\right)\\&\quad\leq k_0\rho\left(\overbrace{0, 0, 0, \ldots, 0}^{n}, x_n, x_{n+1}, x_{n+2}, \ldots\right).\end{aligned} \quad (6)$$

Proof. By using Definition 3, conditions (iv) and (v), and since the elements of E are decreasing, we get

$$\begin{aligned}&\rho\left(\overbrace{0, 0, 0, \ldots, 0}^{2n}, x_n, x_{n+1}, x_{n+2}, \ldots\right)\\&\quad\leq \rho\left(\overbrace{0, 0, 0, \ldots, 0}^{2n}, x_n, x_n, x_{n+1}, x_{n+1}, \ldots\right)\\&\quad\leq k_0\rho\left(\overbrace{0, 0, 0, \ldots, 0}^{n}, x_n, x_{n+1}, x_{n+2}, \ldots\right).\end{aligned} \quad (7)$$

□

Theorem 7. *Let E_ρ be a premodular (sss); then $\overline{F(V, W)}^g = U_{E_\rho}^{app}(V, W)$, where $g(T) = \rho(\alpha_n(T)_{n\in\mathbb{N}})$.*

Proof. To prove that $F(V, W) \subseteq U_E^{\text{app}}(V, W)$, since $e_m \in E$ for each $m \in \mathbb{N}$, from the linearity of E and $T \in F(V, W)$, then finitely many elements of $(\alpha_n(T))_{n\in\mathbb{N}}$ are different from zero. Hence, $T \in U_E^{\text{app}}(V, W)$. For the other inclusion $U_E^{\text{app}}(V, W) \subseteq \overline{F(V, W)}$, let $T \in U_E^{\text{app}}(V, W)$ and, from the definition of approximation numbers, there is $N \in \mathbb{N}$, $N > 0$, A_N with $\text{rank}(A_N) \leq N$ and also

$$\|T - A_N\| \leq 2\alpha_N(T). \quad (8)$$

Since $\alpha_N(T) \to 0$ as $N \to \infty$, then $\|T - A_N\| \to 0$ as $N \to \infty$; we have to prove that $\rho((\alpha_n(T - A_N))_{n\in\mathbb{N}}) \to 0$ as

$N \to \infty$, by taking $N = 8\eta$, where η is a natural number. From Definition 3, condition (iii), we have

$$\begin{aligned} d(T, A_N) &= \rho\left((\alpha_n(T - A_N))_{n\in\mathbb{N}}\right) \\ &= \rho\Bigg[\Big(\alpha_0(T-A_N), \alpha_1(T-A_N), \dots, \\ &\alpha_{8\eta-1}(T-A_N), 0,0,0,\dots\Big) + \Big(\overbrace{0,0,0,\dots,0}^{8\eta}, \\ &\alpha_{8\eta}(T-A_N), \alpha_{8\eta+1}(T-A_N), \dots, \alpha_{12\eta-1}(T-A_N), \\ &0,0,0,\dots\Big) + \Big(\overbrace{0,0,0,\dots,0}^{12\eta}, \alpha_{12\eta}(T-A_N), \\ &\alpha_{12\eta+1}(T-A_N), \dots\Big)\Bigg] \le k^2\Bigg[\rho\Big(\alpha_0(T-A_N), \\ &\alpha_1(T-A_N), \dots, \alpha_{8\eta-1}(T-A_N), 0,0,0,\dots\Big) \\ &+ \rho\Big(\overbrace{0,0,0,\dots,0}^{8\eta}, \alpha_{8\eta}(T-A_N), \alpha_{8\eta+1}(T-A_N), \\ &\dots, \alpha_{12\eta-1}(T-A_N), 0,0,0,\dots\Big) + \rho\Big(\overbrace{0,0,0,\dots,0}^{12\eta}, \\ &\alpha_{12\eta}(T-A_N), \alpha_{12\eta+1}(T-A_N), \dots\Big)\Bigg] = k^2\left[I_1(N)\right. \\ &\left.+ I_2(\eta) + I_3(\eta)\right]. \end{aligned} \tag{9}$$

Since $\alpha_n(A_N) = 0$ for $n \ge N$, then

$$\alpha_n(T - A_N) \le \alpha_{n-N}(T). \tag{10}$$

By using Lemma 6, inequality (10), and Definition 3, condition (v), we get

$$\begin{aligned} I_3(\eta) &= \rho\Big(\overbrace{0,0,0,\dots,0}^{12\eta}, \alpha_{12\eta}(T-A_N), \\ &\alpha_{12\eta+1}(T-A_N), \dots\Big) \le \rho\Big(\overbrace{0,0,0,\dots,0}^{12\eta}, \alpha_{4\eta}(T), \\ &\alpha_{4\eta+1}(T), \dots\Big) \le k_0\rho\Big(\overbrace{0,0,0,\dots,0}^{6\eta}, \alpha_{4\eta}(T), \\ &\alpha_{4\eta+1}(T), \dots\Big) \le k_0^2\rho\Big(\overbrace{0,0,0,\dots,0}^{3\eta}, \alpha_{4\eta}(T), \\ &\alpha_{4\eta+1}(T), \dots\Big) \le k_0^2\rho\Big(\overbrace{0,0,0,\dots,0}^{3\eta}, \alpha_{3\eta}(T), \\ &\alpha_{3\eta+1}(T), \dots\Big) = k_0^2\rho\left((\alpha_n(T))_{n=3\eta}^{\infty}\right) \longrightarrow 0 \\ &\text{as } \eta \longrightarrow \infty. \end{aligned} \tag{11}$$

Now, using Lemma 6 and Definition 3, condition (v), we have

$$\begin{aligned} I_2(\eta) &= \rho\Big(\overbrace{0,0,0,\dots,0}^{8\eta}, \alpha_{8\eta}(T-A_N), \\ &\alpha_{8\eta+1}(T-A_N), \dots, \alpha_{12\eta-1}(T-A_N), 0,0,0,\dots\Big) \\ &\le k_0\rho\Big(\overbrace{0,0,0,\dots,0}^{4\eta}, \alpha_{8\eta}(T-A_N), \\ &\alpha_{8\eta+1}(T-A_N), \dots, \alpha_{12\eta-1}(T-A_N), 0,0,0,\dots\Big) \\ &\le k_0\rho\Big(\alpha_0(T-A_N), \alpha_1(T-A_N), \dots, \\ &\alpha_{8\eta-1}(T-A_N), 0,0,0,\dots\Big) = k_0 I_1(N). \end{aligned} \tag{12}$$

Finally, we have to show that $I_1(N) \to 0$ as $N \to \infty$. Since $T \in U_E^{\mathrm{app}}(V, W)$ and ρ is continuous at θ, we have $\rho(\alpha_k(T))_{k=n}^{\infty} \to 0$ as $n \to \infty$. Then, for each $\varepsilon > 0$, there exists $N_0(\varepsilon)$ such that for all $n \ge N_0(\varepsilon)$ we have

$$\rho\left((\alpha_k(T))_{k=n}^{\infty}\right) < \varepsilon. \tag{13}$$

By taking $\varepsilon_1 = \varepsilon/3lk$ for each $n \ge N_0(\varepsilon_1)$ and using inequality (13) and Definition 3, conditions (ii) and (iii), then we have

$$\begin{aligned} I_1(N) &= \rho\big(\alpha_0(T-A_N), \alpha_1(T-A_N), \dots, \alpha_{N-1}(T \\ &- A_N), 0,0,0,\dots\big) \le k\Bigg[\rho\Big(\alpha_0(T-A_N), \\ &\alpha_1(T-A_N), \dots, \alpha_{N_0-1}(T-A_N), 0,0,0,\dots\Big) \\ &+ \rho\Big(\overbrace{0,0,0,\dots,0}^{N_0}, \alpha_{N_0}(T-A_N), \alpha_{N_0+1}(T-A_N), \\ &\dots, \alpha_{N-1}(T-A_N), 0,0,0,\dots\Big)\Bigg] \le k\Bigg[\rho\big(\|T-A_N\|, \\ &\|T-A_N\|, \dots, \|T-A_N\|, 0,0,0,\dots\big) \\ &+ \rho\Big(\overbrace{0,0,0,\dots,0}^{N_0}, 2\alpha_N(T), 2\alpha_N(T), \dots, 2\alpha_N(T), \\ &0,0,0,\dots\Big)\Bigg] \le k\Bigg[\|T-A_N\| l\rho\Big(\overbrace{1,1,1,\dots,1}^{N_0}, 0,0,0, \\ &\dots\Big) + 2l\rho\Big(\overbrace{0,0,0,\dots,0}^{N_0}, \alpha_{N_0}(T), \alpha_{N_0+1}(T), \dots, \\ &\alpha_N(T), 0,0,0,\dots\Big)\Bigg] \le k\left[\|T-A_N\| lk_1(\varepsilon) + 2l\varepsilon_1\right], \end{aligned} \tag{14}$$

where $k_1(\varepsilon) = \rho(\overbrace{1,1,1,\dots,1}^{N_0}, 0,0,0,\dots)$, and since $\|T - A_N\| \to 0$ as $N \to \infty$, then for each $\varepsilon > 0$ there exists N

such that $\|T - A_N\| k_1(\varepsilon) \leq \varepsilon_1$, for that we have $I_1(N) \leq k[l\varepsilon_1 + 2l\varepsilon_1] = 3kl\varepsilon_1 = \varepsilon$. This completes the proof. □

We give here the sufficient conditions on the sequence spaces $m(\phi, M)$ such that the class of all bounded linear operators between any arbitrary Banach spaces with $(\alpha_n(T))_{n\in\mathbb{N}}$ in these sequence spaces form an ideal operator; the ideal of the finite rank operators in the class of Banach spaces is dense in $U^{\text{app}}_{m(\phi,M)}(V, W)$.

Theorem 8. *Let M be an Orlicz function satisfying Δ_2-condition. Then,*

(a) *$U^{app}_{m(\phi,M)}$ is an operator ideal,*

(b) *$\overline{F(V, W)} = U^{app}_{m(\phi,M)}(V, W)$.*

Proof. We first prove that the space $m(\phi, M)$ is a (sss).

(1) Let $\lambda_1, \lambda_2 \in \mathbb{R}$ and $x, y \in m(\phi, M)$; then there exist $\zeta_1 > 0$ and $\zeta_2 > 0$ such that

$$\sup_{s\geq 1,\sigma\in P_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M\left(\frac{|x_k|}{\zeta_1}\right) < \infty, \qquad \sup_{s\geq 1,\sigma\in P_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M\left(\frac{|x_k|}{\zeta_2}\right) < \infty. \tag{15}$$

Let $\zeta_3 = \max(2|\lambda_1|\zeta_1, 2|\lambda_2|\zeta_2)$. Since M is nondecreasing convex function with Δ_2-condition, we have

$$\sum_{k\in\sigma} M\left(\frac{|\lambda_1 x_k + \lambda_2 y_k|}{\zeta_3}\right) \leq \frac{1}{2}\left[\sum_{k\in\sigma} M\left(\frac{|x_k|}{\zeta_1}\right) + \sum_{k\in\sigma} M\left(\frac{|y_k|}{\zeta_2}\right)\right]. \tag{16}$$

So, we get

$$\sup_{s\geq 1,\sigma\in P_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M\left(\frac{|\lambda_1 x_k + \lambda_2 y_k|}{\zeta_3}\right) \leq \frac{1}{2}\left[\sup_{s\geq 1,\sigma\in P_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M\left(\frac{|x_k|}{\zeta_1}\right) + \sup_{s\geq 1,\sigma\in P_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M\left(\frac{|y_k|}{\zeta_2}\right)\right]. \tag{17}$$

Thus, $\lambda_1 x + \lambda_2 y \in m(\phi, M)$. Hence, $m(\phi, M)$ is a linear space over the field of real numbers. Also, since $e_n \in \ell_M$ and $\ell_M \subseteq m(\phi, M)$, we have $e_n \in m(\phi, M)$ for all $n \in \mathbb{N}$.

(2) Let $x \in \omega$ and $y = (y_k)^{\infty}_{k=0} \in m(\phi, M)$ with $|x_k| \leq |y_k|$ for each $k \in \mathbb{N}$; since M is nondecreasing, then we get

$$\sup_{s\geq 1,\sigma\in P_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M\left(\frac{|x_k|}{\zeta}\right) \leq \sup_{s\geq 1,\sigma\in P_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M\left(\frac{|y_k|}{\zeta}\right) < \infty; \tag{18}$$

then $x = (x_k)^{\infty}_{k=0} \in m(\phi, M)$.

(3) Let $x = (x_k)^{\infty}_{k=0} \in m(\phi, M)$; then we have

$$\sup_{s\geq 1,\sigma\in P_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M\left(\frac{|x_{[k/2]}|}{\zeta}\right) \leq 2 \sup_{s\geq 1,\sigma\in P_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M\left(\frac{|x_k|}{\zeta}\right) < \infty; \tag{19}$$

then $x = (x_{[k/2]})^{\infty}_{k=0} \in m(\phi, M)$.

Finally, we have proved that the space $m(\phi, M)$ with $\rho(x)$ is a premodular (sss).

(i) Clearly, $\rho(x) \geq 0$ for all $x \in m(\phi, M)$ and $\rho(x) = 0 \Leftrightarrow x = \theta$.

(ii) Let $\lambda \in \mathbb{R}$ and $x \in m(\phi, M)$; then for $\lambda \neq 0$ we have

$$\begin{aligned}\rho(\lambda x) &= \inf\left\{\zeta > 0 : \sup_{s\geq 1,\sigma\in P_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M\left(\frac{|\lambda x_k|}{\zeta}\right) \leq 1\right\} \\ &= \inf\left\{|\lambda| \mu > 0 : \sup_{s\geq 1,\sigma\in P_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M\left(\frac{|x_k|}{\mu}\right) \leq 1\right\},\end{aligned} \tag{20}$$

where $\mu = \zeta/|\lambda|$. Thus, $\rho(\lambda x) = |\lambda| \inf\{\mu > 0 : \sup_{s\geq 1,\sigma\in P_s}(1/\phi_s)\sum_{k\in\sigma} M(|x_k|/\mu) \leq 1\} = |\lambda|\rho(x)$.

Also, for $\lambda = 0$, we have $\rho(\lambda x) = \lambda\rho(x) = 0$.

(iii) Let $x, y \in m(\phi, M)$; then there exist $\zeta_1 > 0$ and $\zeta_2 > 0$ such that

$$\sup_{s\geq 1,\sigma\in P_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M\left(\frac{|x_k|}{\zeta_1}\right) \leq 1, \qquad \sup_{s\geq 1,\sigma\in P_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M\left(\frac{|x_k|}{\zeta_2}\right) \leq 1. \tag{21}$$

Let $\zeta = \zeta_1 + \zeta_2$, and since M is nondecreasing and convex, then we have

$$\begin{aligned}&\sup_{s\geq 1,\sigma\in P_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M\left(\frac{|x_k + y_k|}{\zeta}\right) \leq \sup_{s\geq 1,\sigma\in P_s} \frac{1}{\phi_s} \cdot \sum_{k\in\sigma} M\left(\frac{|x_k| + |y_k|}{\zeta_1 + \zeta_2}\right) \leq \sup_{s\geq 1,\sigma\in P_s} \frac{1}{\phi_s} \\ &\cdot \sum_{k\in\sigma}\left[\left(\frac{\zeta_1}{\zeta_1 + \zeta_2}\right) M\left(\frac{|x_k|}{\zeta_1}\right) + \left(\frac{\zeta_2}{\zeta_1 + \zeta_2}\right) M\left(\frac{|y_k|}{\zeta_2}\right)\right] \leq \left(\frac{\zeta_1}{\zeta_1 + \zeta_2}\right) \sup_{s\geq 1,\sigma\in P_s} \frac{1}{\phi_s} \\ &\cdot \sum_{k\in\sigma} M\left(\frac{|x_k|}{\zeta_1}\right) + \left(\frac{\zeta_2}{\zeta_1 + \zeta_2}\right) \sum_{k\in\sigma} M\left(\frac{|y_k|}{\zeta_2}\right) \leq 1.\end{aligned} \tag{22}$$

Since ζ's are nonnegative, we have

$$\begin{aligned}
&\rho(x+y)\\
&= \inf\left\{\zeta>0 : \sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{k\in\sigma}M\left(\frac{|x_k+y_k|}{\zeta}\right)\leq 1\right\}\\
&\leq \inf\left\{\zeta_1>0 : \sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{k\in\sigma}M\left(\frac{|x_k|}{\zeta_1}\right)\leq 1\right\}\\
&\quad+\inf\left\{\zeta_2>0 : \sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{k\in\sigma}M\left(\frac{|y_k|}{\zeta_2}\right)\leq 1\right\}\\
&= \rho(x)+\rho(y).
\end{aligned} \tag{23}$$

(iv) Let $|x_k| \leq |y_k|$ for each $k \in \mathbb{N}$, and since M is nondecreasing, then we get

$$\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{k\in\sigma}M\left(\frac{|x_k|}{\zeta}\right)\leq \sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{k\in\sigma}M\left(\frac{|y_k|}{\zeta}\right); \tag{24}$$

thus,

$$\begin{aligned}
&\inf\left\{\zeta>0 : \sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{k\in\sigma}M\left(\frac{|x_k|}{\zeta}\right)\right\}\\
&\quad\leq \inf\left\{\zeta>0 : \sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{k\in\sigma}M\left(\frac{|y_k|}{\zeta}\right)\right\}.
\end{aligned} \tag{25}$$

So, $\rho(x)\leq\rho(y)$.

(v) Since

$$\begin{aligned}
&\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{k\in\sigma}M\left(\frac{|x_{[k/2]}|}{\zeta}\right)\\
&\quad\leq 2\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{k\in\sigma}M\left(\frac{|x_k|}{\zeta}\right),
\end{aligned} \tag{26}$$

we have

$$\begin{aligned}
&\inf\left\{\zeta>0 : \sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{k\in\sigma}M\left(\frac{|x_{[k/2]}|}{\zeta}\right)\right\}\\
&\quad\leq 2\inf\left\{\zeta>0 : \sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{k\in\sigma}M\left(\frac{|x_k|}{\zeta}\right)\right\}.
\end{aligned} \tag{27}$$

So, $\rho((x_k))\leq\rho((x_{[k/2]}))\leq 2\rho((x_k))$.

(vi) For each $x=(x_k)_{k=0}^{\infty}\in m(\phi,M)$, then

$$\begin{aligned}
&\rho\left((x_k)_{k=0}^{\infty}\right)\\
&= \inf\left\{\zeta>0 : \sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{k\in\sigma}M\left(\frac{|x_{[k/2]}|}{\zeta}\right)<\infty\right\};
\end{aligned} \tag{28}$$

we can find $t\in\mathbb{N}$ such that $\rho((x_k)_{k=t}^{\infty})<\infty$. This means the set of all finite sequences is ρ-dense in $m(\phi,M)$.

(vii) For any $\lambda>0$, there exists a constant $\zeta\in]0,1]$ such that

$$\rho(\lambda,0,0,0,\ldots)\geq\zeta\lambda\rho(1,0,0,0,\ldots). \tag{29}$$

By using Theorems 5 and 7, the proof follows. □

As special cases of the above theorem, we obtain the following corollaries.

Corollary 9. *If* $\sup_s\phi_s<\infty$, *one gets that*

(a) $U_{\ell_M}^{app}$ *is an operator ideal,*

(b) $\overline{F(V,W)}=U_{\ell_M}^{app}(V,W)$.

Corollary 10. *If* $\sup_s\phi_s<\infty$ *and* $M(t)=t^p$ *with* $0<p<\infty$, *one gets that*

(a) $U_{\ell^p}^{app}$ *is an operator ideal,*

(b) $\overline{F(V,W)}=U_{\ell^p}^{app}(V,W)$. *See [1].*

Theorem 11. *If* E_ρ *is a premodular (sss), then* $U_{E_\rho}^{app}(V,W)$ *is complete.*

Proof. Let (T_m) be a Cauchy sequence in $U_{E_\rho}^{\mathrm{app}}(V,W)$; then, by using Definition 3, condition (vii), and since $U_{E_\rho}^{\mathrm{app}}(V,W)\subseteq L(V,W)$, we get

$$\begin{aligned}
\rho\left(\left(\alpha_n\left(T_i-T_j\right)\right)_{n\in\mathbb{N}}\right)&\geq\rho\left(\alpha_0\left(T_i-T_j\right),0,0,0,\ldots\right)\\
&=\rho\left(\left\|T_i-T_j\right\|,0,0,0,\ldots\right)\\
&\geq\xi\left\|T_i-T_j\right\|\rho(1,0,0,0,\ldots);
\end{aligned} \tag{30}$$

then (T_m) is also a Cauchy sequence in $L(V,W)$. Since the space $L(V,W)$ is a Banach space, then there exists $T\in L(V,W)$ such that $\|T_m-T\|\to 0$, as $m\to\infty$, and since $(\alpha_n(T_m))_{n\in\mathbb{N}}\in E$, for each $m\in\mathbb{N}$, then from Definition 3, conditions (iii) and (v), and since ρ is continuous at θ, we have

$$\begin{aligned}
\rho\left(\left(\alpha_n(T)\right)_{n\in\mathbb{N}}\right)&=\rho\left(\alpha_n\left(T-T_m+T_m\right)\right)_{n\in\mathbb{N}}\\
&\leq k\rho\left(\alpha_{[n/2]}\left(T-T_m\right)\right)_{n\in\mathbb{N}}\\
&\quad+k\rho\left(\alpha_{[n/2]}\left(T_m\right)\right)_{n\in\mathbb{N}}\\
&\leq k\rho\left(\left\|T_m-T\right\|\right)_{n\in\mathbb{N}}\\
&\quad+k\rho\left(\alpha_n\left(T_m\right)\right)_{n\in\mathbb{N}}<\varepsilon;
\end{aligned} \tag{31}$$

we get $(\alpha_n(T))_{n\in\mathbb{N}}\in E$, and then $T\in U_{E_\rho}^{\mathrm{app}}(V,W)$. This finishes the proof. □

By applying Theorem 11 on $m(\phi,M)$, we can easily conclude the next corollaries.

Corollary 12. *Pick up an Orlicz function* M *which satisfies* Δ_2*-condition. Then,* M *is continuous from the right at* 0 *and* $U_{m(\phi,M)}^{app}(V,W)$ *is complete.*

Corollary 13. *Pick up an Orlicz function M which satisfies Δ_2-condition with $\sup_s \phi_s < \infty$. Then, M is continuous from the right at 0 and $U_{l_M}^{app}(X, Y)$ is complete.*

Corollary 14. *$U_{\ell^p}^{app}(V, W)$ is complete if $M(t) = t^p$ and $p \in (0, \infty)$.*

Competing Interests

The author declares that he has no competing interests.

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Research Article

On the Stability of Alternative Additive Equations in Multi-β-Normed Spaces

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We introduce the notion of multi-β-normed space ($0 < \beta \le 1$) and study the stability of the alternative additive functional equation of two forms in this type of space.

1. Introduction

In 1940, Ulam [1] proposed the following stability problem: given a metric group $G(\cdot, \rho)$, a number $\varepsilon > 0$, and mapping $f : G \to G$ which satisfies the inequality $\rho(f(x \cdot y), f(x) \cdot f(y)) < \varepsilon$ for all x, y in G, does there exist an automorphism a of G and a constant $k > 0$, depending only on G, such that $\rho(a(x), f(x)) \le k\varepsilon$ for all x in G? If the answer is affirmative, we call the equation $a(x \cdot y) = a(x) \cdot a(y)$ of automorphism stable. One year later, Hyers [2] provided a positive partial answer to Ulam's problem. In 1978, a generalized version of Hyers' result was proved by Rassias in [3]. Since then, the stability problems of several functional equations have been extensively investigated by a number of authors [4–12]. In particular, we also refer the readers to the survey paper [13] for recent developments in Ulam's type stability, [14] for recent developments of the conditional stability of the homomorphism equation, and books [15–18] for the general understanding of the stability theory.

The notion of multinormed space was introduced by Dales and Polyakov [19]. This concept is somewhat similar to operator sequence space and has some connections with operator spaces. Because of its applications in and outside of mathematics, the study on the stability of various functional equations has become one of the most important research subjects in the field of functional equations and attracts much attention from many researchers worldwide. Many examples of multinormed spaces can be found in [19], and further development of the stability in multinormed spaces can be found in papers [20–24].

In order to study the stability problem in more general setting, in this paper we introduce the notion of multi-β-normed spaces which are the combination of multinormed spaces and β-normed spaces, and the definition is given as follows.

In this paper we will use the following notations. Let $(E, \|\cdot\|)$ be a complex β-normed space with $0 < \beta \le 1$, and let $k \in N$. We denote by E^k the linear space $E \oplus \cdots \oplus E$ consisting of k-tuples $(x_1, \ldots, x_k)$, where $x_1, \ldots, x_k \in E$. The linear operations on E^k are defined coordinatewise. The zero element of either E or E^k is denoted by 0. We denote the set $N_k = \{1, 2, \ldots, k\}$ and denote by S_k the group of permutation on N_k.

Definition 1. A multi-β-norm on $\{E^k : k \in N\}$ is a sequence $(\|\cdot\|_k) = (\|\cdot\|_k : k \in N)$ such that $\|\cdot\|_k$ is β-norm on E^k for each $k \in N$, $\|x\|_1 = \|x\|$ for each $x \in E$, and the following axioms are satisfied for each $k \in N$ with $k \ge 2$:

($A1$) $\|(x_{\sigma(1)}, \ldots, x_{\sigma(k)})\|_k = \|(x_1, \ldots, x_k)\|_k$, ($\sigma \in S_k$, $x_1, \ldots, x_k \in E$);

($A2$) $\|(\alpha_1 x_1, \ldots, \alpha_k x_k)\|_k \le (\max_{i \in N_k} |\alpha_i|^\beta) \|(x_1, \ldots, x_k)\|_k$, ($\alpha_1, \ldots, \alpha_k \in C$, $x_1, \ldots, x_k \in E$);

($A3$) $\|(x_1, \ldots, x_{k-1}, 0)\|_k = \|(x_1, \ldots, x_{k-1})\|_{k-1}$, ($x_1, \ldots, x_k \in E$);

(A4) $\|(x_1,\ldots,x_{k-1},x_{k-1})\|_k = \|(x_1,\ldots,x_{k-1})\|_{k-1}$, $(x_1,\ldots,x_k \in E)$.

In this case, we say that $((E^k,\|\cdot\|_k): k \in N)$ is a multi-β-normed space.

The following two properties of multi-β-normed spaces are easily obtained:

$$\|(x,\ldots,x)\|_k = \|x\|, \quad (x \in E); \tag{1}$$

$$\begin{aligned}\max_{i\in N_k}\|x_i\| &\le \|(x_1,\ldots,x_k)\|_k \le \sum_{i=1}^{k}\|x_i\| \\ &\le k\max_{i\in N_k}\|x_i\|, \quad (x_1,\ldots,x_k \in E).\end{aligned} \tag{2}$$

It follows from (2) that if $(E,\|\cdot\|)$ is a complete β-normed space, then $(E^k,\|\cdot\|_k)$ is complete β-normed space for each $k \in N$; in this case $((E^k,\|\cdot\|_k): k \in N)$ is a complete multi-β-normed space. In particular, if $\beta = 1$ $(E,\|\cdot\|)$ is *Banach* space, then the space $((E^k,\|\cdot\|_k): k \in N)$ is multi-Banach space. Now we give one example of multi-β-normed space.

Example 2. Let E be an arbitrary β-normed space. The sequence $(\|\cdot\|_k,\ k \in N)$ on $E^k: k \in N$ defined by

$$\|(x_1,\ldots,x_k)\|_k = \max_{i\in N_k}\|x_i\|, \quad (x_1,\ldots,x_k \in E), \tag{3}$$

is a multi-β-norm.

Lemma 3. *Let $k \in N$ and $(x_1,\ldots,x_k) \in E^k$. For each $j \in \{1,\ldots,k\}$, let $(x_n^j)_{n=1,2,\ldots}$ be a sequence in E such that $\lim_{n\to\infty}x_n^j = x_j$. Then for each $(y_1,\ldots,y_k) \in E^k$ one has*

$$\lim_{n\to\infty}\left(x_n^1 - y_1,\ldots,x_n^k - y_k\right) = (x_1 - y_1,\ldots,x_k - y_k). \tag{4}$$

Definition 4. Let $((E^k,\|\cdot\|_k): k \in N)$ be a multi-β-normed space. A sequence (x_n) in E is a multinull sequence if, for each $\varepsilon > 0$, there exists $n_0 \in N$ such that

$$\sup_{k\in N}\|(x_n,\ldots,x_{n+k-1})\|_k < \varepsilon \tag{5}$$

for all $n \ge n_0$. Let $x \in E$; we say that the sequence (x_n) is multiconvergent to x in E if $(x_n - x)$ is a multinull sequence. In this case, x is called the limit of the sequence (x_n) and we denote it by $\lim_{n\to\infty}x_n = x$.

In this paper we will study the stability in the multi-β-normed space of alternative additive equation of the two forms, which were further studied in the normed spaces in paper [25], and their definitions are presented as follows.

Definition 5 (see [25]). Let X, Y be linear spaces and let A be mapping from X to Y. The equation is called alternative additive of the first form if A satisfies the functional equation

$$A(x_1 + x_2) + A(x_1 - x_2) = -2A(-x_1). \tag{6}$$

Obviously (6) is equivalent to the alternative Jensen equation

$$A\left(-\frac{x+y}{2}\right) = -\frac{1}{2}\left[A(x) + A(y)\right]. \tag{7}$$

Definition 6 (see [25]). Let X, Y be linear spaces and let A be mapping from X to Y. The equation is called alternative additive of the second form if A satisfies the functional equation

$$A(x_1 + x_2) + A(x_1 - x_2) = -2A(-x_2). \tag{8}$$

Obviously (6) is equivalent to the alternative Jensen equation

$$A\left(-\frac{x-y}{2}\right) = -\frac{1}{2}\left[A(x) - A(y)\right]. \tag{9}$$

2. Stability of Alternative Additive Equation of the First Form

In this section we will study the stability of the alternative additive equation of the first form in multi-β-normed space and on the restricted domain. First, we investigate the general case where the domain of the mapping is the whole space. The following theorem is obtained.

Theorem 7. *Let X be a real normed space, and let $((Y^n,\|\cdot\|): n \in N)$ be a complete real multi-β-normed space. Suppose that $\delta \ge 0$; mapping $f: X \to Y$ satisfies*

$$\begin{aligned}\sup_{k\in N}\|(&f(x_1+y_1) + f(x_1-y_1) \\ &+ 2f(-x_1),\ldots,f(x_k+y_k) + f(x_k-y_k) \\ &+ 2f(-x_k))\|_k \le \delta;\end{aligned} \tag{10}$$

$$\sup_{k\in N}\|(f(-z_1) + f(z_1),\ldots,f(-z_k) + f(z_k))\|_k \le \frac{\delta}{2^\beta} \tag{11}$$

for all $x_1,\ldots,x_k,y_1,\ldots,y_k,z_1,\ldots,z_k \in X$. Then there exists unique alternative additive mapping of the first form $A: X \to Y$ satisfying

$$\begin{aligned}\sup_{k\in N}&\|(f(z_1) - A(z_1),\ldots,f(z_k) - A(z_k))\|_k \\ &\le \left(2\delta + \frac{\delta}{4^\beta}\right)\cdot\frac{1}{2^\beta - 1}\end{aligned} \tag{12}$$

for all $z_1,\ldots,z_k \in X$.

Proof. Letting $x_i = y_i = 0$ $(i = 1,\ldots,k)$ in (10) yields

$$\sup_{k\in N}\|(f(0),\ldots,f(0))\|_k \le \frac{\delta}{4^\beta}. \tag{13}$$

Setting $x_i = y_i = z_i$ $(i = 1,\ldots,k)$ yields

$$\begin{aligned}\sup_{k\in N}\|(&f(2z_1) + f(0) + 2f(-z_1),\ldots,f(2z_k) + f(0) \\ &+ 2f(-z_k))\|_k \le \delta.\end{aligned} \tag{14}$$

It follows from (11), (13), and (14) that

$$\begin{aligned}
&\sup_{k\in N}\|(f(2z_1)-2f(z_1),\dots,f(2z_k)-2f(z_k))\|_k \\
&\quad\le \sup_{k\in N}\|(f(2z_1)+f(0)+2f(-z_1),\dots,f(2z_k) \\
&\quad + f(0)+2f(-z_k))\|_k \\
&\quad + \sup_{k\in N}\|(-2(f(-z_1)+f(z_1)),\dots, \\
&\quad -2(f(-z_k)+f(z_k)))\|_k + \sup_{k\in N}\|(-f(0),\dots, \\
&\quad -f(0))\|_k \le 2\delta+\frac{\delta}{4^\beta}.
\end{aligned} \tag{15}$$

Therefore, we have

$$\begin{aligned}
&\sup_{k\in N}\left\|\left(\frac{1}{2^n}f(2^nz_1)-f(z_1),\dots,\frac{1}{2^n}f(2^nz_k)\right.\right. \\
&\quad \left.\left.-f(z_k)\right)\right\|_k \le \left(2\delta+\frac{\delta}{4^\beta}\right)\sum_{k=1}^{n}\frac{1}{2^{k\beta}},
\end{aligned} \tag{16}$$

$$\begin{aligned}
&\sup_{k\in N}\left\|\left(\frac{1}{2^{n+m}}f(2^{n+m}z_1)\right.\right. \\
&\quad -\frac{1}{2^n}f(2^nz_1),\dots,\frac{1}{2^{n+m}}f(2^{n+m}z_k) \\
&\quad \left.\left.-\frac{1}{2^n}f(2^nz_k)\right)\right\|_k \le \left(2\delta+\frac{\delta}{4^\beta}\right)\sum_{k=n+1}^{n+m}\frac{1}{2^{k\beta}}
\end{aligned} \tag{17}$$

for all $m,n\in N$, $m\ge 1$.

It follows from $(A2)$ and (17) that

$$\begin{aligned}
&\sup_{k\in N}\left\|\left(\frac{1}{2^{n+m}}f(2^{n+m}x)-\frac{1}{2^n}f(2^nx),\dots,\frac{1}{2^{n+m+k-1}}\right.\right. \\
&\quad \left.\left.\cdot f\left(2^{n+m+k-1}x\right)-\frac{1}{2^{n+k-1}}f\left(2^{n+k-1}x\right)\right)\right\|_k \\
&\quad = \sup_{k\in N}\left\|\left(\frac{1}{2^{n+m}}f(2^{n+m}x)-\frac{1}{2^n}f(2^nx),\dots,\right.\right. \\
&\quad \frac{1}{2^{k-1}}\left(\frac{1}{2^{n+m}}f\left(2^{n+m}\left(2^{k-1}x\right)\right)\right. \\
&\quad \left.\left.\left.-\frac{1}{2^n}f\left(2^n\left(2^{k-1}x\right)\right)\right)\right)\right\|_k \le \sup_{k\in N}\left\|\left(\frac{1}{2^{n+m}}f(2^{n+m}x)\right.\right. \\
&\quad -\frac{1}{2^n}f(2^nx),\dots,\frac{1}{2^{n+m}}f\left(2^{n+m}\left(2^{k-1}x\right)\right)-\frac{1}{2^n} \\
&\quad \left.\left.\cdot f\left(2^n\left(2^{k-1}x\right)\right)\right)\right\|_k \le \left(2\delta+\frac{\delta}{4^\beta}\right)\sum_{k=n+1}^{n+m}\frac{1}{2^{k\beta}}.
\end{aligned} \tag{18}$$

Hence $\{(1/2^n)f(2^nx)\}$ is Cauchy sequence, which must be convergent in complete real multi-β-normed space; that is, there exists mapping $A:X\to Y$ such that $A(x):=\lim_{n\to\infty}(1/2^n)f(2^nx)$. Hence, for arbitrary $\varepsilon>0$, there exists $n_0\in N$; if $n\ge n_0$, then we have

$$\begin{aligned}
&\sup_{k\in N}\left\|\left(\frac{1}{2^n}f(2^nx)-A(x),\dots,\frac{1}{2^{n+k-1}}f\left(2^{n+k-1}x\right)\right.\right. \\
&\quad \left.\left.-A(x)\right)\right\|_k<\varepsilon.
\end{aligned} \tag{19}$$

Considering (2), we obtain

$$\lim_{n\to\infty}\left\|\frac{1}{2^n}f(2^nx)-A(x)\right\|=0,\quad x\in X. \tag{20}$$

If we let $n=0$ in (17), then we have

$$\begin{aligned}
&\sup_{k\in N}\left\|\left(\frac{1}{2^m}f(2^mz_1)-f(z_1),\dots,\frac{1}{2^m}f(2^mz_k)\right.\right. \\
&\quad \left.\left.-f(z_k)\right)\right\|_k \le \left(2\delta+\frac{\delta}{4^\beta}\right)\sum_{k=1}^{m}\frac{1}{2^{k\beta}}.
\end{aligned} \tag{21}$$

Letting $m\to\infty$ and making use of Lemma 3 and (20), we know that mapping A satisfies (12).

Let $x,y\in X$. Setting $x_1=\cdots=x_k=2^nx$, $y_1=\cdots=y_k=2^ny$ in (10) and dividing both sides by $2^{n\beta}$ yield

$$\begin{aligned}
&\sup_{k\in N}\left\|\left(\frac{1}{2^n}f(2^n(x+y))+\frac{1}{2^n}f(2^n(x-y))+2\right.\right. \\
&\quad \cdot\frac{1}{2^n}f(-2^nx),\dots,\frac{1}{2^n}f(2^n(x+y)) \\
&\quad \left.\left.+\frac{1}{2^n}f(2^n(x-y))+2\cdot\frac{1}{2^n}f(-2^nx)\right)\right\|_k\le\frac{\delta}{2^{n\beta}},
\end{aligned} \tag{22}$$

which together with (1) implies

$$\begin{aligned}
&\left\|\frac{1}{2^n}f(2^n(x+y))+\frac{1}{2^n}f(2^n(x-y))+2\right. \\
&\quad \left.\cdot\frac{1}{2^n}f(-2^nx)\right\|\le\frac{\delta}{2^{n\beta}}.
\end{aligned} \tag{23}$$

Taking limit as $n\to\infty$, we have

$$A(x+y)+A(x-y)+2A(-x)=0,\quad (x,y\in X). \tag{24}$$

So A is the alternative additive mapping of the first form. It remains to show that A is uniquely determined. Let $A':X\to Y$ be another alternative additive mapping of the first form that satisfies (12). It follows from (24) that some properties of mapping A are obtained:

(1) If we let $y=0$, we get $A(x)=-A(x)$, so A is odd mapping.

(2) If we let $x=y=0$, we have $A(0)=0$.

(3) Putting $y=x$ yields $A(2x)=2A(x)$; that is, $A(x)=(1/2)A(2x)$.

(4) Replacing x,y with $2x$, respectively, yields $A(2^2x)=2^2A(x)$; hence $A(x)=(1/2^2)A(2^2x)$.

(5) Replacing x, y with 2^2x, respectively, yields $A(2^3x) = 2^3A(x)$; that is, $A(x) = (1/2^3)A(2^3x)$.

Proceeding in an obvious fashion yields $A(x) = (1/2^n)A(2^nx)$. Similarly, we have $A'(x) = (1/2^n)A'(2^nx)$. Letting $z_1 = \cdots = z_k = x$ in (12) and in view of (1) we obtain

$$\|f(x) - A(x)\| \le \left(2\delta + \frac{\delta}{4^\beta}\right) \cdot \frac{1}{2^\beta - 1}. \tag{25}$$

Similarly we have

$$\|f(x) - A'(x)\| \le \left(2\delta + \frac{\delta}{4^\beta}\right) \cdot \frac{1}{2^\beta - 1}. \tag{26}$$

Therefore,

$$\begin{aligned} \|A(x) - A'(x)\| &= \left\|\frac{1}{2^n}A(2^nx) - \frac{1}{2^n}A'(2^nx)\right\| \\ &\le \frac{1}{2^{n\beta}}\|A(2^nx) - A'(2^nx)\| \\ &\le \frac{1}{2^{n\beta}}\|A(2^nx) - f(2^nx)\| \\ &\quad + \frac{1}{2^{n\beta}}\|f(2^nx) - A'(2^nx)\| \\ &\le \frac{1}{2^{n\beta}}\left(2\delta + \frac{\delta}{4^\beta}\right) \cdot \frac{2}{2^\beta - 1}. \end{aligned} \tag{27}$$

Taking limit as $n \to \infty$, we have $A' = A$. □

It is a time to study the stability of this type mapping on the local domain. We only prove the stability result when the target spaces are real multi-Banach spaces, that is, the special case of real multi-β-normed space when $\beta = 1$. For $0 < \beta < 1$, it is an interesting open problem. The following are our results.

Theorem 8. *Let X be a real normed space, let $((Y^n, \|\cdot\|) : n \in N)$ be a real multi-Banach space, and let $d > 0$, $\delta \ge 0$. Suppose that mapping $f : X \to Y$ satisfies*

$$\begin{aligned} &\|(f(x_1 + y_1) + f(x_1 - y_1) \\ &\quad + 2f(-x_1), \dots, f(x_k + y_k) + f(x_k - y_k) \\ &\quad + 2f(-x_k))\|_k \le \delta, \end{aligned} \tag{28}$$
$$\|(f(z_1) + f(-z_1), \dots, f(z_k) + f(-z_k))\|_k \le \frac{\delta}{2}$$

for all $x_1, \dots, x_k, y_1, \dots, y_k, z_1, \dots, z_k \in X$ that satisfy $\|(x_1, \dots, x_k)\|_k + \|(y_1, \dots, y_k)\|_k \ge d$ and $\|(z_1, \dots, z_k)\|_k \ge d$. Then there exists unique alternative additive mapping of the first form $A : X \to Y$ such that

$$\sup_{k\in N}\|(f(z_1) - A(z_1), \dots, f(z_k) - A(z_k))\|_k \le \frac{95}{4}\delta \tag{29}$$

for all $z_1, \dots, z_k \in X$.

Proof. Fix $k \in N$. Let $\mathbf{X} = (x_1, \dots, x_k)$ and $\mathbf{Y} = (y_1, \dots, y_k)$ satisfy $\|(x_1, \dots, x_k)\|_k + \|(y_1, \dots, y_k)\|_k < d$. If $\mathbf{X} = \mathbf{Y} = \mathbf{0}$, then let $\mathbf{T} = (t_1, \dots, t_k) \in X^k$ and $\|\mathbf{T}\|_k = d$. If $\mathbf{X} \ne \mathbf{0}$ or $\mathbf{Y} \ne \mathbf{0}$, let

$$\mathbf{T} = \begin{cases} \left(1 + \dfrac{d}{\|\mathbf{X}\|_k}\right)\mathbf{X}, & \|\mathbf{X}\|_k \ge \|\mathbf{Y}\|_k; \\ \left(1 + \dfrac{d}{\|\mathbf{Y}\|_k}\right)\mathbf{Y}, & \|\mathbf{X}\|_k < \|\mathbf{Y}\|_k. \end{cases} \tag{30}$$

If $\|\mathbf{X}\|_k \ge \|\mathbf{Y}\|_k$, we get $\|\mathbf{T}\|_k = \|\mathbf{X}\|_k + d > d$. If $\|\mathbf{X}\|_k < \|\mathbf{Y}\|_k$, we have $\|\mathbf{T}\|_k = \|\mathbf{Y}\|_k + d > d$. Therefore,

$$\begin{aligned} &\|\mathbf{X} - \mathbf{T}\|_k + \|\mathbf{Y} + \mathbf{T}\|_k \ge 2\|\mathbf{T}\|_k - (\|\mathbf{X}\|_k + \|\mathbf{Y}\|_k) \ge d; \\ &\|\mathbf{X} - \mathbf{T}\|_k + \|\mathbf{Y} - \mathbf{T}\|_k \ge 2\|\mathbf{T}\|_k - (\|\mathbf{X}\|_k + \|\mathbf{Y}\|_k) \ge d; \\ &\|\mathbf{X} - 2\mathbf{T}\|_k + \|\mathbf{Y}\|_k \ge 2\|\mathbf{T}\|_k - (\|\mathbf{X}\|_k + \|\mathbf{Y}\|_k) \ge d; \\ &\|\mathbf{X} \pm \mathbf{T}\|_k \ge \|\mathbf{T}\|_k - \|\mathbf{X}\|_k \\ &\qquad = (\|\mathbf{X}\|_k + d) - \|\mathbf{X}\|_k = d, \\ &\qquad \text{for } \|\mathbf{X}\|_k \ge \|\mathbf{Y}\|_k; \\ &\|\mathbf{X} \pm \mathbf{T}\|_k \ge \|\mathbf{T}\|_k - \|\mathbf{X}\|_k \\ &\qquad = (\|\mathbf{Y}\|_k + d) - \|\mathbf{X}\|_k = d, \\ &\qquad \text{for } \|\mathbf{Y}\|_k \ge \|\mathbf{X}\|_k; \\ &\|\mathbf{T} - \mathbf{X}\|_k + \|\mathbf{T}\|_k \ge d. \end{aligned} \tag{31}$$

It follows from (28) that

$$\begin{aligned} &\|(f(x_1 + y_1) + f(x_1 - y_1) \\ &\quad + 2f(-x_1), \dots, f(x_k + y_k) + f(x_k - y_k) \\ &\quad + 2f(-x_k))\|_k \le \|(f(x_1 + y_1) \\ &\quad + f(x_1 - y_1 - 2t_1) \\ &\quad + 2f(-(x_1 - t_1)), \dots, f(x_k + y_k) \\ &\quad + f(x_k - y_k - 2t_k) + 2f(-(x_k - t_k)))\|_k \\ &\quad + \|(f(x_1 + y_1 - 2t_1) + f(x_1 - y_1) \\ &\quad + 2f(-(x_1 - t_1)), \dots, f(x_k + y_k - 2t_k) \\ &\quad - f(x_k - y_k) + 2f(-(x_k - t_k)))\|_k \\ &\quad + \|(f(x_1 + y_1 - 2t_1) + f(x_1 - y_1 - 2t_1) \\ &\quad + 2f(-(x_1 - 2t_1)), \dots, f(x_k + y_k - 2t_k) \\ &\quad + f(x_k - y_k - 2t_k) + 2f(-(x_k - 2t_k)))\|_k \\ &\quad + \|(2f((t_1 - x_1) + t_1) + 2f(-x_1) \\ &\quad + 4f(x_1 - t_1), \dots, 2f((t_k - x_k) + t_k) + 2f(-x_k) \\ &\quad + 4f(x_k - t_k))\|_k + \|(4f(x_1 - t_1) \end{aligned}$$

$$+4f(-(x_1-t_1)),\dots,4f(x_k-t_k)$$
$$+4f(-(x_k-t_k)))\|_k \le 7\delta,$$
$$\|(2f(z_1)+2f(-z_1),\dots,2f(z_k)+2f(-z_k))\|_k$$
$$\le \|(2f(z_1)+f(-z_1+t_1)$$
$$+f(-z_1-t_1),\dots,2f(z_k)+f(-z_k+t_k)$$
$$+f(-z_k-t_k))\|_k + \|(-f(-z_1-t_1)$$
$$-f(z_1+t_1),\dots,-f(-z_k-t_k)-f(z_k+t_k))\|_k$$
$$+\|(-f(z_1-t_1)-f(-z_1+t_1),\dots,-f(z_k-t_k)$$
$$-f(-z_k+t_k))\|_k + \|(2f(-z_1)+f(z_1-t_1)$$
$$+f(z_1+t_1),\dots,2f(-z_k)+f(z_k-t_k)$$
$$+f(z_k+t_k))\|_k \le 15\delta. \tag{32}$$

It follows from Theorem 7 that there exists unique alternative additive mapping of the first form $A: X \to Y$ satisfying (29) for all $z_1,\dots,z_k \in X$. □

Corollary 9. *Let $((X^n,\|\cdot\|): n \in N)$ be a real multinormed space, and let $((Y^n,\|\cdot\|): n \in N)$ be a multi-Banach space. Mapping $f: X \to Y$ satisfies alternative additive equation of the first form if and only if, for each $k \in N$, if $\|(x_1,\dots,x_k)\|_k + \|(y_1,\dots,y_k)\|_k \to \infty$ and $\|(z_1,\dots,z_k)\|_k \to \infty$, one has*

$$\|(f(x_1+y_1)+f(x_1-y_1)$$
$$+2f(-x_1),\dots,f(x_k+y_k)+f(x_k-y_k) \tag{33}$$
$$+2f(-x_k))\|_k \longrightarrow 0;$$
$$\|(f(z_1)+f(-z_1),\dots,f(z_k)+f(-z_k))\|_k \longrightarrow 0.$$

3. Stability of Alternative Additive Equation of the Second Form

In this section we will study the stability of the alternative additive equation of the second form in multi-β-normed space and on the restricted domain. First, we investigate the general case where the domain of the mapping is the whole space. The following theorem is obtained.

Theorem 10. *Let X be a real normed space, and let $((Y^n,\|\cdot\|): n \in N)$ be a complete real multi-β-normed space. Suppose that $\delta \ge 0$; mapping $f: X \to Y$ satisfies*

$$\sup_{k\in N}\|(f(x_1+y_1)-f(x_1-y_1)$$
$$+2f(-y_1),\dots,f(x_k+y_k)-f(x_k-y_k) \tag{34}$$
$$+2f(-y_k))\|_k \le \delta$$

for all $x_1,\dots,x_k,y_1,\dots,y_k \in X$. Then there exists unique alternative additive mapping of the second form $A: X \to Y$ such that

$$\sup_{k\in N}\|(f(x_1)-A(x_1),\dots,f(x_k)-A(x_k))\|_k$$
$$\le \frac{2^{2\beta}+2^{\beta}+1}{2^{\beta}(2^{\beta}-1)}\delta \tag{35}$$

for all $x_1,\dots,x_k \in X$.

Proof. Let $x_i = y_i = 0$ $(i = 1,\dots,k)$ in (34); we get

$$\sup_{k\in N}\|(f(0),\dots,f(0))\|_k \le \frac{\delta}{2^{\beta}}. \tag{36}$$

Replacing y_i $(i=1,\dots,k)$ with x_i, we obtain

$$\sup_{k\in N}\|(f(2x_1)-f(0)+2f(-x_1),\dots,f(2x_k)$$
$$-f(0)+2f(-x_k))\|_k \le \delta. \tag{37}$$

Let $x_1 = \cdots = x_k = 0$ and replace y_i $(i=1,\dots,k)$ with x_i; we obtain

$$\sup_{k\in N}\|(f(x_1)+f(-x_1),\dots,f(x_k)+f(-x_k))\|_k \le \delta. \tag{38}$$

Hence for all $x_1,\dots,x_k \in X$ we have

$$\sup_{k\in N}\|(f(2x_1)-2f(x_1),\dots,f(2x_k)-2f(x_k))\|_k$$
$$\le \sup_{k\in N}\|(f(2x_1)-f(0)+2f(-x_1),\dots,f(2x_k)$$
$$-f(0)+2f(-x_k))\|_k$$
$$+\sup_{k\in N}\|(2(f(x_1)+f(-x_1)),\dots, \tag{39}$$
$$2(f(x_k)+f(-x_k)))\|_k + \sup_{k\in N}\|(f(0),\dots,f(0))\|_k$$
$$\le \frac{2^{2\beta}+2^{\beta}+1}{2^{\beta}}\delta.$$

Therefore, for all $m, n \in N$, $m \ge 1$, we have

$$\sup_{k\in N}\left\|\left(\frac{1}{2^n}f(2^n x_1)-f(x_1),\dots,\frac{1}{2^n}f(2^n x_k)\right.\right.$$
$$\left.\left.-f(x_k)\right)\right\|_k \le \left(\frac{2^{2\beta}+2^{\beta}+1}{2^{\beta}}\delta\right)\sum_{k=1}^{n}\frac{1}{2^{k\beta}},$$
$$\sup_{k\in N}\left\|\left(\frac{1}{2^{n+m}}f(2^{n+m}x_1)\right.\right. \tag{40}$$
$$-\frac{1}{2^n}f(2^n x_1),\dots,\frac{1}{2^{n+m}}f(2^{n+m}x_k)$$
$$\left.\left.-\frac{1}{2^n}f(2^n x_k)\right)\right\|_k \le \left(\frac{2^{2\beta}+2^{\beta}+1}{2^{\beta}}\delta\right)\sum_{k=n+1}^{n+m}\frac{1}{2^{k\beta}}.$$

We omit the following arguments because they are similar to that of Theorem 7. □

Theorem 11. *Let X be a real normed space, let $((Y^n, \|\cdot\|) : n \in N)$ be a complete real multi-Banach space, and let $d > 0$, $\delta \geq 0$. Suppose that $f : X \to Y$ satisfies*

$$\begin{aligned} &\|(f(x_1 + y_1) - f(x_1 - y_1) \\ &\quad + 2f(-y_1), \ldots, f(x_k + y_k) - f(x_k - y_k) \\ &\quad + 2f(-y_k))\|_k \leq \delta, \\ &\|(f(z_1) + f(-z_1), \ldots, f(z_k) + f(-z_k))\|_k \leq \frac{\delta}{2} \end{aligned} \tag{41}$$

for $x_1, \ldots, x_k, y_1, \ldots, y_k, z_1, \ldots, z_k \in X$ that satisfy $\|(x_1, \ldots, x_k)\|_k + \|(y_1, \ldots, y_k)\|_k \geq d$ and $\|(z_1, \ldots, z_k)\|_k \geq d$. Then there exists unique alternative additive mapping of the second form $A : X \to Y$ such that

$$\sup_{k \in N} \|(f(x_1) - A(x_1), \ldots, f(x_k) - A(x_k))\|_k \leq \frac{49}{2}\delta \tag{42}$$

for all $x_1, \ldots, x_k \in X$.

Proof. Fix $k \in N$; choose $\mathbf{X} = (x_1, \ldots, x_k)$ and $\mathbf{Y} = (y_1, \ldots, y_k)$ with $\|(x_1, \ldots, x_k)\|_k + \|(y_1, \ldots, y_k)\|_k < d$. If $\mathbf{X} = \mathbf{Y} = \mathbf{0}$, then let $\mathbf{T} = (t_1, \ldots, t_k) \in X^k$ and $\|\mathbf{T}\|_k = d$. If $\mathbf{X} \neq \mathbf{0}$ or $\mathbf{Y} \neq \mathbf{0}$, then let

$$\mathbf{T} = \begin{cases} \left(1 + \dfrac{d}{\|\mathbf{X}\|_k}\right)\mathbf{X}, & \|\mathbf{X}\|_k \geq \|\mathbf{Y}\|_k; \\ \left(1 + \dfrac{d}{\|\mathbf{Y}\|_k}\right)\mathbf{Y}, & \|\mathbf{X}\|_k < \|\mathbf{Y}\|_k. \end{cases} \tag{43}$$

If $\|\mathbf{X}\|_k \geq \|\mathbf{Y}\|_k$, then $\|\mathbf{T}\|_k = \|\mathbf{X}\|_k + d > d$. If $\|\mathbf{X}\|_k < \|\mathbf{Y}\|_k$, then $\|\mathbf{T}\|_k = \|\mathbf{Y}\|_k + d > d$. Therefore,

$$\begin{aligned} &\|\mathbf{X} - \mathbf{T}\|_k + \|\mathbf{Y} + \mathbf{T}\|_k \geq 2\|\mathbf{T}\|_k - (\|\mathbf{X}\|_k + \|\mathbf{Y}\|_k) \geq d; \\ &\|\mathbf{X} - \mathbf{T}\|_k + \|\mathbf{Y} - \mathbf{T}\|_k \geq 2\|\mathbf{T}\|_k - (\|\mathbf{X}\|_k + \|\mathbf{Y}\|_k) \geq d; \\ &\|\mathbf{X} - 2\mathbf{T}\|_k + \|\mathbf{Y}\|_k \geq 2\|\mathbf{T}\|_k - (\|\mathbf{X}\|_k + \|\mathbf{Y}\|_k) \geq d; \\ &\|\mathbf{T}\|_k + \|\mathbf{Y}\|_k \geq d; \\ &\|\mathbf{T} - \mathbf{Y}\|_k \geq d; \\ &\|\mathbf{T} + \mathbf{Y}\|_k \geq d. \end{aligned} \tag{44}$$

It follows from (41) that

$$\begin{aligned} &\|(f(x_1 + y_1) - f(x_1 - y_1) \\ &\quad + 2f(-y_1), \ldots, f(x_k + y_k) - f(x_k - y_k) \\ &\quad + 2f(-y_k))\|_k \leq \|(f(x_1 + y_1) \\ &\quad - f(x_1 - y_1 - 2t_1) \\ &\quad + 2f(-(y_1 + t_1)), \ldots, f(x_k + y_k) \\ &\quad - f(x_k - y_k - 2t_k) + 2f(-(y_k + t_k)))\|_k \\ &\quad + \|(f(x_1 + y_1 - 2t_1) - f(x_1 - y_1) \\ &\quad + 2f(-(y_1 - t_1)), \ldots, f(x_k + y_k - 2t_k) \\ &\quad - f(x_k - y_k) + 2f(-(y_k - t_k)))\|_k \\ &\quad + \|(f(x_1 + y_1 - 2t_1) - f(x_1 - y_1 - 2t_1) \\ &\quad + 2f(-y_1), \ldots, f(x_k + y_k - 2t_k) \\ &\quad - f(x_k - y_k - 2t_k) + 2f(-y_k))\|_k \\ &\quad + \|(2f(t_1 + y_1) - 2f(t_1 - y_1) \\ &\quad + 4f(-y_1), \ldots, 2f(t_k + y_k) - 2f(t_k - y_k) \\ &\quad + 4f(-y_k))\|_k + \|(2f(t_1 + y_1) \\ &\quad - 2f(-(t_1 + y_1)), \ldots, 2f(t_k + y_k) \\ &\quad + 2f(-(t_k - y_k)))\|_k \leq 7\delta, \\ &\|(2f(z_1) + 2f(-z_1), \ldots, 2f(z_k) + 2f(-z_k))\|_k \\ &\quad \leq \|(2f(z_1) + f(-z_1 + t_1) \\ &\quad - f(z_1 + t_1), \ldots, 2f(z_k) + f(-z_k + t_k) \\ &\quad - f(z_k + t_k))\|_k + \|(2f(-z_1) + f(z_1 + t_1) \\ &\quad - f(-z_1 + t_1), \ldots, 2f(-z_k) + f(z_k + t_k) \\ &\quad - f(-z_k + t_k))\|_k \leq 14\delta. \end{aligned} \tag{45}$$

According to Theorem 10, there exists unique alternative additive mapping of the second form $A : X \to Y$ such that (42) holds true. □

Corollary 12. *Let $((X^n, \|\cdot\|) : n \in N)$ be a real multinormed space and let $((Y^n, \|\cdot\|) : n \in N)$ be a multi-Banach space. Mapping $f : X \to Y$ satisfies alternative additive equation of the second form if and only if, for each $k \in N$ if $\|(x_1, \ldots, x_k)\|_k + \|(y_1, \ldots, y_k)\|_k \to \infty$ and $\|(z_1, \ldots, z_k)\|_k \to \infty$, one has*

$$\begin{aligned} &\|(f(x_1 + y_1) - f(x_1 - y_1) \\ &\quad + 2f(-y_1), \ldots, f(x_k + y_k) - f(x_k - y_k) \\ &\quad + 2f(-y_k))\|_k \longrightarrow 0; \\ &\|(f(z_1) + f(-z_1), \ldots, f(z_k) + f(-z_k))\|_k \longrightarrow 0. \end{aligned} \tag{46}$$

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All authors conceived of the study, participated its design and coordination, drafted the paper, participated in the sequence alignment, and read and approved the final paper.

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Research Article

SPDIEs and BSDEs Driven by Lévy Processes and Countable Brownian Motions

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The paper is devoted to solving a new class of backward stochastic differential equations driven by Lévy process and countable Brownian motions. We prove the existence and uniqueness of the solutions to the backward stochastic differential equations by constructing Cauchy sequence and fixed point theorem. Moreover, we give a probabilistic solution of stochastic partial differential integral equations by means of the solution of backward stochastic differential equations. Finally, we give an example to illustrate.

1. Introduction

The backward stochastic differential equations (BSDEs for short), in the nonlinear cases, were firstly introduced by Pardoux and Peng [1] in order to give a probabilistic interpretation for the solution of semilinear parabolic partial differential equations. In the past decades, the equations have been extensively considered because of the applications in mathematic finance [2, 3], stochastic games [4–6], and partial differential equations (PDEs for short) [7–10].

As the applications developed, different settings of BSDEs have been introduced. Pardoux and Peng [11] proposed a new class of BSDEs driven by two Brownian motions, which are called backward doubly stochastic differential equations (BDSDEs for short), in order to give a probabilistic interpretation for the solution of quasi-linear stochastic partial differential equations (SPDEs for short). Since then, many authors discussed various settings of BDSDEs, for example, Bally and Matoussi [12], Matoussi and Scheutzow [13], Zhang and Zhao [14, 15], and the references therein.

In 2000, Nualart and Schoutens [16] gave a martingale representation of Lévy process. Furthermore, they [17] discussed the BSDEs driven by Lévy process and the application in finance. Following it, many authors were devoted to the BSDEs driven by Lévy process. Bahlali et al. [18] generalized the results [17] to the BSDEs driven by Teugels martingales associated with Lévy process and a Brownian motion. Also, they gave the application in partial differential integral equations (PDIEs for short). Ren et al. [19] introduced a class of BDSDEs driven by Teugels martingales associated with Lévy process and two Brownian motions. They obtained the existence and uniqueness of solution and gave the probabilistic interpretation for solutions of stochastic partial differential integral equations (SPDIEs for short). Later, Hu and Ren [20] discussed BDSDEs driven by Teugels martingales associated with Lévy process and an adapted continuous increasing process. Recently, Duan et al. [21] made further discussion of reflected backward stochastic differential equations driven by countable Brownian motions under Lipschitz conditions. Owo [22] studied the equations with continuous coefficients.

To the best of our knowledge, there are no works on the BSDEs driven by Teugels martingales associated with Lévy processes and countably many Brownian motions. Thus, we will make the first attempt to study such problem in this paper.

The structure of this paper is organized as follows. In Section 2, we present some basic notions and assumptions. Section 3 is devoted to the existence and uniqueness of solutions for BSDEs driven by Teugels martingales associated with Lévy processes and countably many Brownian motions by means of martingale representation theorem, fixed point theorem, and constructing Cauchy sequence. In Section 4, we discuss the connection between the BSDEs and SPDIEs.

2. Notations

Let $T > 0$ be a fixed terminal time. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, P)$ be a complete probability space, let $\{\beta_j(t), t \in [0,T], j = 1,2,\ldots\}$ and $\{L_t, t \in [0,T]\}$ be mutually independent processes, where $\{\beta_j(t)\}$ is a sequence of $\mathbb{R}$-valued standard Brownian motion and mutually independent, and $\{L_t\}$ is $\mathbb{R}$-valued Lévy process corresponding to a standard Lévy measure ν such that $\int_{\mathbb{R}}(1 \wedge y)\nu(dy) < \infty$.

Let $\mathcal{N}$ denote the totality of P-null sets of $\mathcal{F}$. For each $t \in [0,T]$, we define

$$\mathcal{F}_t = \left(\bigvee_{j=1}^{\infty} \mathcal{F}_{t,T}^{\beta_j}\right) \vee \mathcal{F}_t^L, \tag{1}$$

where for any process η_t, $\mathcal{F}_{s,t}^{\eta} = \sigma\{\eta_r - \eta_s, s \le r \le t\} \vee \mathcal{N}$, $\mathcal{F}_t^{\eta} = \mathcal{F}_{0,t}^{\eta}$.

Let us introduce some spaces which will be carried out in the following parts.

(i) $L^2(\Omega, \mathcal{F}_T, P)$ denotes the set of all $\mathcal{F}_T$-measurable random variables ξ such that $E|\xi|^2 < \infty$.

(ii) $\mathcal{H}^2$ denotes the space of $\mathbb{R}$-valued, square integrable, and $\mathcal{F}_t$- progressively measurable processes $\{\varphi_t : t \in [0,T]\}$ such that

$$|\varphi|^2 = E\int_0^T |\varphi_t|^2\, dt < \infty. \tag{2}$$

And we denote by $\mathcal{P}^2$ the subspace of $\mathcal{H}^2$ formed by the predictable processes.

(iii) $\mathcal{S}^2$ denotes the set of $\mathbb{R}$-valued, $\mathcal{F}_t$-measurable processes $\{\varphi_t : t \in [0,T]\}$ such that

$$E\left(\sup_{0\le t\le T} |\varphi_t|^2\right) < \infty. \tag{3}$$

Let l^2 be the space of $\mathbb{R}$-valued sequences $\{x_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} x_i^2 < \infty$. $\mathcal{H}^2(l^2)$ and $\mathcal{P}^2(l^2)$ denote the corresponding space of l^2-valued processes endowed with the norm

$$\|\varphi\|^2 = \sum_{i=1}^{\infty} E\int_0^T \left|\varphi_t^{(i)}\right|^2 dt. \tag{4}$$

Now, we give the definition of the Teugels martingales denoted by $\{H_t^{(i)}\}$, associated with the Lévy processes $\{L_t, t \in [0,T]\}$, which is given by

$$H_t^{(i)} = c_{i,i}Y_t^{(i)} + c_{i,i-1}Y_t^{(i-1)} + \cdots + c_{i,1}Y_t^{(1)}, \tag{5}$$

where $Y_t^{(i)} = L_t^{(i)} - E[L_t^{(i)}] = L_t^{(i)} - tE[L_1^{(i)}]$ for all $i \ge 1$ and $L_t^{(i)}$ are power-jump processes. That is, $L_t^{(1)} = L_t$ and $L_t^{(i)} = \sum_{0<s\le t}(\Delta L_s)^i$ for $i \ge 2$, where $\Delta L_t = L_t - L_{t^-}$ and $L_{t^-} = \lim_{s\uparrow t} L_s$. The coefficients $c_{i,k}$ correspond to the orthonormalization of the polynomials $1, x, x^2, \ldots$ with respect to the measure $\mu(dx) = x^2\nu(dx) + \sigma^2\delta_0(dx)$:

$$q_{i-1} = c_{i,i}x^{i-1} + c_{i,i-1}x^{i-2} + \cdots + c_{i,1}. \tag{6}$$

We set

$$p_i(x) = xq_{i-1}(x) = c_{i,i}x^i + c_{i,i-1}x^{i-1} + \cdots + c_{i,1}x. \tag{7}$$

For more details on Teugels martingales associated with the Lévy process $\{L_t, t \in [0,T]\}$, we can refer to [16, 17].

In this paper, we will discuss the following backward stochastic differential equations driven by Lévy process and countably many Brownian motions:

$$\begin{aligned} Y_t = \xi &+ \int_t^T f(s, Y_{s-}, Z_s)\, ds \\ &+ \sum_{j=1}^{\infty}\int_t^T g_j(s, Y_{s-}, Z_s)\, d\beta_j(s) - \sum_{i=1}^{\infty}\int_t^T Z_s^{(i)} dH_s^{(i)}, \\ &\qquad t \in [0,T], \end{aligned} \tag{8}$$

where the integral with respect to $\beta_j(s)$ is the classical backward Itô integral and the integral with respect to $H_t^{(i)}$ is standard forward Itô integral.

With the above preparation, we introduce the definition of solution of (8).

Definition 1. A pair of processes $(Y_t, Z_t)_{t\in[0,T]} \in \mathcal{S}^2 \times \mathcal{P}^2(l^2)$ is a solution to (8), if it satisfies (8).

In order to get the solution of (8), we propose the following assumptions:

(H1) $\xi \in L^2(\Omega, \mathcal{F}_T, P)$.

(H2) The functions $f : [0,T] \times \Omega \times \mathbb{R} \times l^2 \to \mathbb{R}$ and $g : [0,T] \times \Omega \times \mathbb{R} \times l^2 \to \mathbb{R}$ are progressively measurable such that

$$\begin{gathered} E\int_0^T |f(t,0,0)|^2\, dt < \infty, \\ \sum_{j=1}^{\infty} E\int_0^T |g_j(t,0,0)|^2\, dt < \infty. \end{gathered} \tag{9}$$

(H3) There exist some nonnegative constants C, C_j, α_j with $\sum_{j=1}^{\infty} C_j < \infty$ and $\alpha = \sum_{j=1}^{\infty}\alpha_j < 1$ such that, for any $t \in [0,T]$, $y_1, y_2 \in \mathbb{R}$, $z_1, z_2 \in l^2$,

$$\begin{aligned} &|f(t,y_1,z_1) - f(t,y_2,z_2)|^2 \\ &\quad \le C\left(|y_1 - y_2|^2 + |z_1 - z_2|^2\right), \\ &|g_j(t,y_1,z_1) - g_j(t,y_2,z_2)|^2 \\ &\quad \le C_j|y_1 - y_2|^2 + \alpha_j|z_1 - z_2|^2. \end{aligned} \tag{10}$$

Our conclusions depend on the extensive Itô formula in [19].

Lemma 2. *Let $\alpha \in \mathcal{S}^2, \beta, \gamma, \eta$ and $\zeta^{(i)} \in \mathcal{H}^2$ such that*

$$\alpha_t = \alpha_0 + \int_0^t \beta_s ds + \int_0^t \gamma_s dB_s + \int_0^t \eta_s dW_s + \sum_{i=1}^{\infty} \int_0^t \zeta_s^{(i)} dH_s^{(i)}, \quad 0 \leq t \leq T. \tag{11}$$

Then

$$\begin{aligned} |\alpha_t|^2 &= |\alpha_0|^2 + 2\int_0^t \alpha_s \beta_s ds + 2\int_0^t \alpha_s \gamma_s dB_s \\ &\quad + 2\int_0^t \alpha_s \eta_s dW_s + 2\sum_{i=1}^{\infty}\int_0^t \alpha_s \zeta_s^{(i)} dH_s^{(i)} \\ &\quad - \int_0^t |\gamma_s|^2 ds + \int_0^t |\eta_s|^2 ds \\ &\quad + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_0^t \zeta_s^{(i)}\zeta_s^{(j)} d\left[H^{(i)}, H^{(j)}\right]_s. \end{aligned} \tag{12}$$

Noting that $\langle H^{(i)}, H^{(j)}\rangle_t = \delta_{ij} t$, we have

$$\begin{aligned} E|\alpha_t|^2 &= |\alpha_0|^2 + 2E\int_0^t \alpha_s \beta_s ds - E\int_0^t |\gamma_s|^2 ds \\ &\quad + E\int_0^t |\eta_s|^2 ds + \sum_{i=1}^{\infty} E\int_0^t \left(\zeta_s^{(i)}\right)^2 ds. \end{aligned} \tag{13}$$

3. Existence and Uniqueness

In this section, we begin with establishing the existence and uniqueness of (8) in the case that f and g do not depend on Y and Z with finite noise; that is,

$$\begin{aligned} Y_t &= \xi + \int_t^T f(s) ds + \sum_{j=1}^{n} \int_t^T g_j(s) d\beta_j(s) \\ &\quad - \sum_{i=1}^{\infty}\int_t^T Z_s^{(i)} dH_s^{(i)}, \quad t \in [0,T]. \end{aligned} \tag{14}$$

Theorem 3. *Assume that (H1)–(H3) hold. Then, there exists a unique solution $(Y_t, Z_t) \in \mathcal{S}^2 \times \mathcal{P}^2(l^2)$ satisfying (14).*

Proof. For $E\int_0^T |f(s)|^2 ds < \infty$, $\sum_{j=1}^{\infty}\int_0^T |g_j(s)|^2 ds < \infty$, we set the filtration $\{\mathcal{C}_t : t \in [0,T]\}$ as follows:

$$\mathcal{C}_t = \mathcal{F}_t^L \vee \left(\bigvee_{j=1}^{n} \mathcal{F}_t^{\beta_j}\right) \tag{15}$$

and the $\mathcal{C}_t$-square integrable martingale is as follows:

$$M_t = E\left[\xi + \int_0^T f(s) ds + \sum_{j=1}^{n}\int_0^T g_j(s) d\beta_j(s) \mid \mathcal{C}_t\right]. \tag{16}$$

By the predictable representation property, there exists $Z \in \mathcal{P}^2(l^2)$ such that

$$M_t = M_0 + \sum_{i=1}^{\infty}\int_0^t Z_s^{(i)} dH_s^{(i)}. \tag{17}$$

So, we have

$$M_T = M_t + \sum_{i=1}^{\infty}\int_t^T Z_s^{(i)} dH_s^{(i)}. \tag{18}$$

Let

$$\begin{aligned} Y_t &= M_t - \int_0^t f(s) ds - \sum_{j=1}^{n}\int_0^t g_j(s) d\beta_j(s) \\ &= M_T - \int_0^t f(s) ds - \sum_{j=1}^{n}\int_0^t g_j(s) d\beta_j(s) \\ &\quad - \sum_{i=1}^{\infty}\int_t^T Z_s^{(i)} dH_s^{(i)}. \end{aligned} \tag{19}$$

Hence, we have

$$\begin{aligned} Y_t &= \xi + \int_t^T f(s) ds + \sum_{j=1}^{n}\int_t^T g_j(s) d\beta_j(s) \\ &\quad - \sum_{i=1}^{\infty}\int_t^T Z_s^{(i)} dH_s^{(i)}. \end{aligned} \tag{20}$$

From the above equality, we can deduce the existence of solution of (14). The proof of uniqueness is a procedure similar to that in [11]; we omit it. □

With the preparation of above, we consider the following BSDEs with finite noise:

$$\begin{aligned} Y_t &= \xi + \int_t^T f(s, Y_{s-}, Z_s) ds \\ &\quad + \sum_{j=1}^{n}\int_t^T g_j(s, Y_{s-}, Z_s) d\beta_j(s) \\ &\quad - \sum_{i=1}^{\infty}\int_t^T Z_s^{(i)} dH_s^{(i)}, \quad t \in [0,T]. \end{aligned} \tag{21}$$

Theorem 4. *Assume that (H1)–(H3) hold. Then, there exists a unique solution $(Y_t, Z_t) \in \mathcal{S}^2 \times \mathcal{P}^2(l^2)$ satisfying (21).*

Proof. From Theorem 3, for each $(\overline{Y}_t, \overline{Z}_t)$, there exists (Y_t, Z_t) satisfying

$$\begin{aligned} Y_t &= \xi + \int_t^T f\left(s, \overline{Y}_{s-}, \overline{Z}_s\right) ds \\ &\quad + \sum_{j=1}^{n}\int_t^T g_j\left(s, \overline{Y}_{s-}, \overline{Z}_s\right) d\beta_j(s) \\ &\quad - \sum_{i=1}^{\infty}\int_t^T Z_s^{(i)} dH_s^{(i)}, \quad t \in [0,T]. \end{aligned} \tag{22}$$

Following it, we define a map Φ from $\mathcal{S}^2\times\mathcal{P}^2(l^2)$ to itself; that is, $\Phi(\overline{Y}_t,\overline{Z}_t)=(Y_t,Z_t)$. In the following parts, we will show that Φ is a strict contraction with the norm

$$\|(Y,Z)\|_\beta^2=E\int_0^T e^{\beta s}\left[\left|Y_{s-}\right|^2+\left\|Z_s\right\|^2\right]ds, \tag{23}$$

for suitable constant $\beta>0$. In addition, $\mathcal{S}^2\times\mathcal{P}^2(l^2)$ is a Banach space.

Set $(Y,Z)=\Phi(\widetilde{Y},\widetilde{Z})$, $(Y',Z')=\Phi(\widetilde{Y}',\widetilde{Z}')$, where (Y,Z) and (Y',Z') are the solutions of (22) associated with $(\xi,f(t,\widetilde{Y},\widetilde{Z}),g_j(t,\widetilde{Y},\widetilde{Z}))$ and $(\xi,f(t,\widetilde{Y}',\widetilde{Z}'),g_j(t,\widetilde{Y}',\widetilde{Z}'))$, respectively. Let $(\overline{Y}_t,\overline{Z}_t)=(Y_t-Y_t',Z_t-Z_t')$. Applying Itô formula to $e^{\beta t}|Y_t-Y_t'|^2$, we have

$$\begin{aligned}e^{\beta t}\left(Y_t-Y_t'\right)^2&=2\int_t^T e^{\beta s}\left(Y_{s-}-Y_{s-}'\right)\left[f\left(s,\widetilde{Y}_{s-},\widetilde{Z}_s\right)\right.\\&\quad\left.-f\left(s,\widetilde{Y}_{s-}',\widetilde{Z}_s'\right)\right]ds+2\sum_{j=1}^n\int_t^T e^{\beta s}\left(Y_{s-}-Y_{s-}'\right)\\&\quad\cdot\left[g_j\left(s,\widetilde{Y}_{s-},\widetilde{Z}_s\right)-g_j\left(s,\widetilde{Y}_{s-}',\widetilde{Z}_s'\right)\right]d\beta_j(s)\\&\quad-2\sum_{i=1}^\infty\int_t^T e^{\beta s}\left(Y_{s-}-Y_{s-}'\right)\left(Z_s^{(i)}-Z_s'^{(i)}\right)dH_s^{(i)}\\&\quad-\beta\int_t^T e^{\beta s}\left(Y_{s-}-Y_{s-}'\right)^2ds\\&\quad+\sum_{j=1}^n\int_t^T e^{\beta s}\left|g_j\left(s,\widetilde{Y}_{s-},\widetilde{Z}_s\right)-g_j\left(s,\widetilde{Y}_{s-}',\widetilde{Z}_s'\right)\right|^2ds\\&\quad-\sum_{i=1}^\infty\sum_{k=1}^\infty\int_t^T e^{\beta s}\left(Z_s^{(i)}-Z_s'^{(i)}\right)\\&\quad\cdot\left(Z_s^{(k)}-Z_s'^{(k)}\right)d\left[H^{(i)},H^{(i)}\right]_s.\end{aligned} \tag{24}$$

Taking mathematical expectation on both sides, we obtain

$$\begin{aligned}&Ee^{\beta t}\left(Y_t-Y_t'\right)^2+\sum_{i=1}^\infty E\int_t^T e^{\beta s}\left(Z_s^{(i)}-Z_s'^{(i)}\right)^2ds\\&\quad+\beta E\int_t^T e^{\beta s}\left(Y_{s-}-Y_{s-}'\right)^2ds=2E\int_t^T e^{\beta s}\left(Y_{s-}\right.\\&\quad\left.-Y_{s-}'\right)\left[f\left(s,\widetilde{Y}_{s-},\widetilde{Z}_s\right)-f\left(s,\widetilde{Y}_{s-}',\widetilde{Z}_s'\right)\right]ds\\&\quad+\sum_{j=1}^n E\int_t^T e^{\beta s}\left|g_j\left(s,\widetilde{Y}_{s-},\widetilde{Z}_s\right)-g_j\left(s,\widetilde{Y}_{s-}',\widetilde{Z}_s'\right)\right|^2ds.\end{aligned} \tag{25}$$

With the conditions of (H1)–(H3), it follows that

$$\begin{aligned}&Ee^{\beta t}\left(Y_t-Y_t'\right)^2+\sum_{i=1}^\infty E\int_t^T e^{\beta s}\left|Z_s^{(i)}-Z_s'^{(i)}\right|^2ds\\&\quad+\beta E\int_t^T e^{\beta s}\left(Y_{s-}-Y_{s-}'\right)^2ds\le\frac{2C}{1-\sum_{j=1}^\infty\alpha_j}\\&\quad\cdot E\int_t^T e^{\beta s}\left|Y_{s-}-Y_{s-}'\right|^2ds+\frac{1-\sum_{j=1}^\infty\alpha_j}{2}\\&\quad\cdot E\int_t^T e^{\beta s}\left(\left|\widetilde{Y}_{s-}-\widetilde{Y}_{s-}'\right|^2+\left\|\widetilde{Z}_s-\widetilde{Z}_s'\right\|^2\right)ds\\&\quad+\left(\sum_{j=1}^n C_j\right)E\int_t^T e^{\beta s}\left|\widetilde{Y}_{s-}-\widetilde{Y}_{s-}'\right|^2ds+\left(\sum_{j=1}^n\alpha_j\right)\\&\quad\cdot E\int_t^T e^{\beta s}\left\|\widetilde{Z}_s-\widetilde{Z}_s'\right\|^2ds\le\frac{2C}{1-\alpha}\\&\quad\cdot E\int_t^T e^{\beta s}\left|Y_{s-}-Y_{s-}'\right|^2ds+\left(\sum_{j=1}^\infty C_j+\frac{1-\alpha}{2}\right)\\&\quad\cdot E\int_t^T e^{\beta s}\left|\widetilde{Y}_{s-}-\widetilde{Y}_{s-}'\right|^2ds+\frac{1+\alpha}{2}\\&\quad\cdot E\int_t^T e^{\beta s}\left\|\widetilde{Z}_s-\widetilde{Z}_s'\right\|^2ds.\end{aligned} \tag{26}$$

Furthermore, we have

$$\begin{aligned}&Ee^{\beta t}\left(Y_t-Y_t'\right)^2+E\int_t^T e^{\beta s}\left\|Z_s-Z_s'\right\|^2ds\\&\quad+\left(\beta-\frac{2C}{1-\alpha}\right)E\int_t^T e^{\beta s}\left(Y_{s-}-Y_{s-}'\right)^2ds\\&\quad\le\left(\sum_{j=1}^\infty C_j+\frac{1-\alpha}{2}\right)E\int_t^T e^{\beta s}\left|\widetilde{Y}_{s-}-\widetilde{Y}_{s-}'\right|^2ds\\&\quad+\frac{1+\alpha}{2}E\int_t^T e^{\beta s}\left\|\widetilde{Z}_s-\widetilde{Z}_s'\right\|^2ds.\end{aligned} \tag{27}$$

Let $\gamma=2C/(1-\alpha)$, $\overline{C}=2(\sum_{j=1}^\infty C_j+(1-\alpha)/2)/(1+\alpha)$, and $\beta=\gamma+\overline{C}$, and we have

$$\begin{aligned}&Ee^{\beta t}\left|Y_t-Y_t'\right|^2+E\int_t^T e^{\beta s}\left\|Z_s-Z_s'\right\|^2ds\\&\quad+\overline{C}E\int_t^T e^{\beta s}\left(Y_{s-}-Y_{s-}'\right)^2ds\\&\quad\le\frac{1+\alpha}{2}E\int_t^T e^{\beta s}\left(\overline{C}\left|\widetilde{Y}_{s-}-\widetilde{Y}_{s-}'\right|^2+\left\|\widetilde{Z}_s-\widetilde{Z}_s'\right\|^2\right)ds.\end{aligned} \tag{28}$$

Moreover,

$$E\int_t^T e^{\beta s}\overline{C}\left|Y_{s-}-Y'_{s-}\right|^2 ds + E\int_t^T e^{\beta s}\left\|Z_s-Z'_s\right\|^2 ds \leq \frac{1+\alpha}{2}E\int_t^T e^{\beta s}\left(\overline{C}\left|\widetilde{Y}_{s-}-\widetilde{Y}'_{s-}\right|^2+\left\|\widetilde{Z}_s-\widetilde{Z}'_s\right\|^2\right)ds. \tag{29}$$

That is,

$$\left\|(Y_t,Z_t)\right\|_\beta^2 \leq \frac{1+\alpha}{2}\left\|(\overline{Y}_t,\overline{Z}_t)\right\|_\beta^2. \tag{30}$$

It follows that Φ is a strict contraction with the norm $\|\cdot\|_\beta$. Then, from B-D-G inequality, Φ has a unique fixed point $(Y_t,Z_t) \in \mathcal{S}^2\times\mathcal{P}^2(l^2)$, which is the unique solution of (21). □

Theorem 5. *Under the conditions (H1)–(H3), there exists a unique solution $(Y_t,Z_t)\in\mathcal{S}^2\times\mathcal{P}^2(l^2)$ satisfying (8).*

Proof (existence). From Theorem 4, for each n, there exists a unique solution of (21) under the conditions (H1)–(H3) denoted by (Y_t^n,Z_t^n):

$$\begin{aligned}Y_t^n &= \xi + \int_t^T f\left(s,Y_{s-}^n,Z_s^n\right)ds \\ &\quad + \sum_{j=1}^n\int_t^T g_j\left(s,Y_{s-}^n,Z_s^n\right)d\beta_j(s) \\ &\quad - \sum_{i=1}^\infty\int_t^T Z_s^{n(i)}dH_s^{(i)}, \quad t\in[0,T].\end{aligned} \tag{31}$$

In the following part, we claim that (Y_t^n,Z_t^n) is Cauchy sequence in $\mathcal{S}^2\times\mathcal{P}^2(l^2)$. Applying Itô formula to $|Y_t^n-Y_t^m|^2$, without loss of generality, we let $n<m$; then

$$\begin{aligned}&\left|Y_t^n-Y_t^m\right|^2+\int_t^T\left\|Z_s^n-Z_s^m\right\|^2ds = 2\int_t^T\left(Y_{s-}^n-Y_{s-}^m\right) \\ &\cdot\left(f\left(s,Y_{s-}^n,Z_s^n\right)-f\left(s,Y_{s-}^m,Z_s^m\right)\right)ds \\ &+2\sum_{j=n+1}^m\int_t^T\left|g_j\left(s,Y_{s-}^n,Z_s^n\right)-g_j\left(s,Y_{s-}^m,Z_s^m\right)\right|^2ds \\ &+2\sum_{j=n+1}^m\int_t^T\left(Y_{s-}^n-Y_{s-}^m\right)\left(g_j\left(s,Y_{s-}^n,Z_s^n\right)\right. \\ &\left.-g_j\left(s,Y_{s-}^m,Z_s^m\right)\right)d\beta_j(s) \\ &-2\sum_{i=1}^\infty\int_t^T\left(Y_{s-}^n-Y_{s-}^m\right)\left(Z_s^{n(i)}-Z_s^{m(i)}\right)dH_s^{(i)} \\ &-\int_t^T\sum_{i=1}^\infty\sum_{j=1}^\infty\left(Z_s^{n(i)}-Z_s^{m(i)}\right) \\ &\cdot\left(Z_s^{n(j)}-Z_s^{m(j)}\right)d\left[H^{(i)},H^{(j)}\right]_s.\end{aligned} \tag{32}$$

Taking mathematical expectation on both sides, we have

$$\begin{aligned}&E\left|Y_t^n-Y_t^m\right|^2+E\int_t^T\left\|Z_s^n-Z_s^m\right\|^2ds \\ &= 2E\int_t^T\left(Y_{s-}^n-Y_{s-}^m\right) \\ &\cdot\left(f\left(s,Y_{s-}^n,Z_s^n\right)-f\left(s,Y_{s-}^m,Z_s^m\right)\right)ds \\ &+2\sum_{j=n+1}^m E\int_t^T\left|g_j\left(s,Y_{s-}^n,Z_s^n\right)-g_j\left(s,Y_{s-}^m,Z_s^m\right)\right|^2ds \\ &\leq \frac{2C}{1-\alpha}E\int_t^T\left|Y_{s-}^n-Y_{s-}^m\right|^2ds+\frac{1-\alpha}{2} \\ &\cdot E\int_t^T\left|Y_{s-}^n-Y_{s-}^m\right|^2ds+\frac{1+\alpha}{2} \\ &\cdot E\int_t^T\left\|Z_s^n-Z_s^m\right\|^2ds+\left(\sum_{j=n+1}^m\alpha_j\right) \\ &\cdot E\int_t^T\left\|Z_s^n-Z_s^m\right\|^2ds+\left(\sum_{j=n+1}^m C_j\right) \\ &\cdot E\int_t^T\left|Y_{s-}^n-Y_{s-}^m\right|^2ds \leq \left(\frac{2C}{1-\alpha}+\frac{1-\alpha}{2}\right. \\ &\left.+\sum_{j=1}^m C_j\right)E\int_t^T\left|Y_{s-}^n-Y_{s-}^m\right|^2ds+\frac{1+\alpha}{2} \\ &\cdot E\int_t^T\left\|Z_s^n-Z_s^m\right\|^2ds.\end{aligned} \tag{33}$$

Therefore, we obtain

$$\begin{aligned}&E\left|Y_t^n-Y_t^m\right|^2+\frac{1-\alpha}{2}E\int_t^T\left\|Z_s^n-Z_s^m\right\|^2ds \\ &\leq\left(\frac{2C}{1-\alpha}+\frac{1-\alpha}{2}+\sum_{j=1}^m C_j\right)E\int_t^T\left|Y_{s-}^n-Y_{s-}^m\right|^2ds.\end{aligned} \tag{34}$$

By the Gronwall inequality and B-D-G inequality, we have

$$E\left[\sup_{0\leq t\leq T}\int_t^T\left|Y_{s-}^n-Y_{s-}^m\right|^2ds\right]\longrightarrow 0. \tag{35}$$

Denote its limit by (Y_t,Z_t); from the continuity of f and g and Lebesgue dominated convergence theorem, we can imply that it is the solution of (8).

Uniqueness. We set

$$\begin{aligned}\Psi_M(x) &= x^2 1_{\{-M\leq x\leq M\}}+M(2x-M)1_{\{x>M\}} \\ &\quad - M(2x+M)1_{\{x<-M\}}.\end{aligned} \tag{36}$$

If we define $\Psi'_M(x)/x=2$, when $x=0$, then, $0\leq\Psi'_M(x)/x\leq 2$. Let (Y_t^i,Z_t^i) $(i=1,2)$ be two solutions of (8); we apply Itô formula to $e^{\beta t}\Psi_M(\overline{Y}_t)$, where $(\overline{Y}_t, \overline{Z}_t) = (Y_t^1 - Y_t^2, Z_t^1 - Z_t^2)$, and β is constant:

$$\begin{aligned}
&e^{\beta t}\Psi_M\left(\overline{Y}_t\right) + \beta\int_t^T e^{\beta s}\Psi_M\left(\overline{Y}_{s-}\right)ds \\
&\quad + \int_t^T e^{\beta s}1_{\{-M\le\overline{Y}_s\le M\}}\left\|\overline{Z}_s\right\|^2 ds = \int_t^T e^{\beta s}\Psi_M'\left(\overline{Y}_{s-}\right) \\
&\quad \cdot\left(f\left(s, Y_{s-}^1, Z_s^1\right) - f\left(s, Y_{s-}^2, Z_s^2\right)\right)ds \\
&\quad + \sum_{j=1}^{\infty}\int_t^T e^{\beta s}1_{\{-M\le\overline{Y}_s\le M\}}\left|g_j\left(s, Y_{s-}^1, Z_s^1\right)\right. \\
&\quad \left.- g_j\left(s, Y_{s-}^2, Z_s^2\right)\right|^2 ds - \sum_{j=1}^{\infty}\int_t^T e^{\beta s}\Psi_M'\left(\overline{Y}_{s-}\right) \\
&\quad \cdot\left(g_j\left(s, Y_{s-}^1, Z_s^1\right) - g_j\left(s, Y_{s-}^2, Z_s^2\right)\right)d\beta_j(s) \\
&\quad - \sum_{i=1}^{\infty}\int_t^T e^{\beta s}\left(Z_s^{1(i)} - Z_s^{2(i)}\right)dH_s^{(i)} \\
&\quad - \int_t^T\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}e^{\beta s}\left(Z_s^{1(i)} - Z_s^{1(j)}\right) \\
&\quad \cdot\left(Z_s^{2(i)} - Z_s^{2(j)}\right)d\left[H^{(i)}, H^{(j)}\right]_s.
\end{aligned} \tag{37}$$

Taking expectation on both sides,

$$\begin{aligned}
&Ee^{\beta t}\Psi_M\left(\overline{Y}_t\right) + \beta E\int_t^T e^{\beta s}\Psi_M\left(\overline{Y}_{s-}\right)ds \\
&\quad + E\int_t^T e^{\beta s}1_{\{-M\le\overline{Y}_s\le M\}}\left\|\overline{Z}_s\right\|^2 ds \\
&\quad = E\int_t^T e^{\beta s}\Psi_M'\left(\overline{Y}_{s-}\right)\left(f\left(s, Y_{s-}^1, Z_s^1\right)\right. \\
&\quad \left.- f\left(s, Y_{s-}^2, Z_s^2\right)\right)ds \\
&\quad + \sum_{j=1}^{\infty}E\int_t^T e^{\beta s}1_{\{-M\le\overline{Y}_s\le M\}}\left|g_j\left(s, Y_{s-}^1, Z_s^1\right)\right. \\
&\quad \left.- g_j\left(s, Y_{s-}^2, Z_s^2\right)\right|^2 ds \le \left(\frac{2C}{1-\alpha} + \sum_{j=1}^{\infty}C_j\right. \\
&\quad \left.+ \frac{1-\alpha}{2}\right)E\int_t^T e^{\beta s}\left|\overline{Y}_{s-}\right|^2 ds + \frac{1+\alpha}{2} \\
&\quad \cdot E\int_t^T e^{\beta s}\left\|\overline{Z}_s\right\|^2 ds.
\end{aligned} \tag{38}$$

Let $M \to \infty$, β is large enough, and we have

$$\begin{aligned}
&Ee^{\beta t}\left|\overline{Y}_t\right|^2 \\
&\quad + \left(\beta - \frac{2C}{1-\alpha} - \sum_{j=1}^{\infty}C_j - \frac{1-\alpha}{2}\right)E\int_t^T e^{\beta s}\left|\overline{Y}_{s-}\right|^2 ds \\
&\quad + \frac{1-\alpha}{2}E\int_t^T e^{\beta s}\left\|\overline{Z}_s\right\|^2 ds \le 0.
\end{aligned} \tag{39}$$

So, we complete the proof of uniqueness. □

4. Application to SPDIEs

In this section, we consider the application of BSDEs driven by Lévy processes and countably many Brownian motions to the solution of a class of SPDIEs. Suppose that our Lévy processes have the form of $L_t = bt + \int_{|z|<1}(z(N_t(\cdot, dz)) - t\nu(dz))$, where $N_t(\omega, dz)$ denotes the random measure such that $\int_\Lambda N_t(\cdot, dz)$ is a Poisson process with parameter $\nu(\Lambda)$ for all the set Λ where $0 \notin \overline{\Lambda}$.

Consider the following SDE:

$$X_t = x + \int_0^t \sigma\left(X_{s-}\right)dL_s, \quad t \in [0, T]. \tag{40}$$

Under adequate conditions, there exists a unique solution of (40).

In order to get the main result, we give a technical lemma that appears in [17].

Lemma 6. *Let $c : \Omega\times[0, T]\times\mathbb{R} \to \mathbb{R}$ be a measurable function such that*

$$c(s, y) \le a_s\left(y^2 \wedge |y|\right), \quad a.s., \tag{41}$$

where $\{a_s : s \in [0, T]\}$ is a nonnegative predictable process such that $E\int_0^T a_s^2 ds < \infty$. Then, for each $0 \le t \le T$, we have

$$\begin{aligned}
\sum_{t\le s\le T}c\left(s, \Delta L_s\right) &= \sum_{i=1}^{\infty}\int_t^T\left\langle c(s, \cdot), p_i\right\rangle_{L^2(\nu)}dH_s^{(i)} \\
&\quad + \int_t^T\int_{\mathbb{R}}c(s, y)\,\nu(dy)\,ds.
\end{aligned} \tag{42}$$

Consider the following BSDEs driven by Lévy processes and countably many Brownian motions:

$$\begin{aligned}
Y_t &= h\left(L_T\right) + \int_t^T f\left(s, X_{s-}, Y_{s-}, Z_s\right)ds \\
&\quad + \sum_{j=1}^{\infty}\int_t^T g_j\left(s, X_{s-}, Y_{s-}, Z_s\right)d\beta_j(s) \\
&\quad - \sum_{i=1}^{\infty}\int_t^T Z_s^{(i)}dH_s^{(i)}, \quad t \in [0, T],
\end{aligned} \tag{43}$$

where $E|h(L_T)|^2 < \infty$.

Define

$$u^1(t, x, y) = u(t, x+y) - u(t, x) - \frac{\partial u}{\partial x}(t, x)\,y, \tag{44}$$

where u is the solution of the following SPDIEs:

$$\begin{aligned}
&\frac{\partial u}{\partial t}(t, x) + \frac{1}{2}\sigma^2(x)\frac{\partial^2 u}{\partial x^2}(t, x) + a'\frac{\partial u}{\partial x}(t, x) \\
&\quad + \int_{\mathbb{R}}u^1(t, x, y)\,\nu(dy) \\
&\quad + f\left[t, u(t, x), \frac{\partial u}{\partial x}(t, x), \left\{u^{(i)}(t, x)\right\}_{i=1}^{\infty}\right] \\
&\quad + \sum_{j=1}^{\infty}g_j\left[t, u(t, x), \frac{\partial u}{\partial x}(t, x), \left\{u^{(i)}(t, x)\right\}_{i=1}^{\infty}\right]\dot{\beta}_j(t)\,dt \\
&\quad = 0,
\end{aligned} \tag{45}$$

with $u(T,x)=h(x)$, $a'=a+\int_{|y|\le 1} y\nu(dy)$, and

$$u^1(t,x)=\int_{\mathbb{R}} u^1(t,x,y)\,p_1(y)\,\nu(dy)+\frac{\partial u}{\partial x}(t,x)\left(\int_{\mathbb{R}} y^2\nu(dy)\right)^{1/2}, \tag{46}$$

and for $i\ge 2$

$$u^{(i)}(t,x)=\int_{\mathbb{R}} u^1(t,x,y)\,p_i(y)\,\nu(dy). \tag{47}$$

In order to give the meaning of $\dot{\beta}_j(t)dt$, we write the above SPDIEs in the following integral form:

$$\begin{aligned}u(t,x)&=h(x)+\int_t^T\left[\frac{1}{2}\sigma^2(x)\frac{\partial^2 u}{\partial x^2}(s,x)\right.\\&\quad\left.+\int_{\mathbb{R}} u^1(s,x,y)\,\nu(dy)+a'\frac{\partial u}{\partial x}(s,x)\right]ds\\&\quad+\int_t^T f\left[s,u(s,x),\frac{\partial u}{\partial x}(s,x),\left\{u^{(i)}(s,x)\right\}_{i=1}^{\infty}\right]ds\\&\quad+\sum_{j=1}^{\infty}\int_t^T g_j\left[s,u(s,x),\frac{\partial u}{\partial x}(s,x),\right.\\&\quad\left.\left\{u^{(i)}(s,x)\right\}_{i=1}^{\infty}\right]d\beta_j(s).\end{aligned} \tag{48}$$

Suppose that u is $\mathscr{C}^{1,2}$ function that $\partial u/\partial t$ and $\partial^2u/\partial x^2$ are bounded by a polynomial function of x, uniformly in t. Next, we give the main result of this section.

Theorem 7. *The unique adapted solution of (43) is given by*

$$\begin{aligned}Y_t&=u(t,L_t),\\Z_t^{(i)}&=\int_{\mathbb{R}} u^1(t,L_{t-},y)\,p_i(y)\,\nu(dy),\quad i\ge 2,\\Z_t^{(1)}&=\int_{\mathbb{R}} u^1(t,L_{t-},y)\,p_1(y)\,\nu(dy)\\&\quad+\frac{\partial u}{\partial x}(t,L_{t-})\left(\int_{\mathbb{R}} y^2\nu(dy)\right)^{1/2}.\end{aligned} \tag{49}$$

Proof. Applying Itô formula to $u(s,L_s)$, we obtain

$$\begin{aligned}&u(T,L_T)-u(t,L_t)\\&=\int_t^T\frac{\partial u}{\partial s}(s,L_s)\,ds+\frac{1}{2}\int_t^T\sigma^2(L_s)\frac{\partial^2 u}{\partial x^2}(s,L_s)\,ds\\&\quad+\int_t^T\frac{\partial u}{\partial x}(s,L_{s-})\,dL_s\\&\quad+\sum_{t\le s\le T}\left[u(s,L_s)-u(s,L_{s-})-\frac{\partial u}{\partial x}(s,L_{s-})\,\Delta L_s\right].\end{aligned} \tag{50}$$

We apply Lemma 6 to $u(s,L_{s-}+y)-u(s,L_{s-})-(\partial u/\partial x)(s,L_{s-})y$, and then

$$\begin{aligned}&\sum_{t\le s\le T}\left[u(s,L_s)-u(s,L_{s-})-\frac{\partial u}{\partial x}(s,L_{s-})\,\Delta L_s\right]\\&\quad=\sum_{i=1}^{\infty}\int_t^T\left(\int_{\mathbb{R}} u^1(s,L_{s-},y)\,p_i(y)\,\nu(dy)\right)dH_s^{(i)}\\&\quad\quad+\int_t^T\int_{\mathbb{R}} u^1(s,L_{s-},y)\,\nu(dy)\,ds.\end{aligned} \tag{51}$$

Note that

$$L_t=Y_t^{(1)}+tEL_1=\left(\int_{\mathbb{R}} y^2\nu(dy)\right)^{1/2}H_t^{(1)}+tEL_1, \tag{52}$$

where $EL_1=a+\int_{\{|y|\ge 1\}} y\nu(dy)$.

Substituting (51) and (52) into (50), we obtain

$$\begin{aligned}h(L_T)-u(t,L_t)&=\int_t^T\frac{\partial u}{\partial s}(s,L_{s-})\,ds+\frac{1}{2}\\&\quad\cdot\int_t^T\sigma^2(L_s)\frac{\partial^2 u}{\partial x^2}(s,L_s)\,ds\\&\quad+\sum_{i=1}^{\infty}\int_t^T\left(\int_{\mathbb{R}} u^1(s,L_{s-},y)\,p_i(y)\,\nu(dy)\right)dH_s^{(i)}\\&\quad+\int_t^T\int_{\mathbb{R}} u^1(s,L_{s-},y)\,\nu(dy)\,ds+\int_t^T\frac{\partial u}{\partial s}(s,L_{s-})\\&\quad\cdot\left[\left(\int_{\mathbb{R}} y^2\nu(dy)\right)^{1/2}dH_s^{(1)}+a'ds\right]\end{aligned} \tag{53}$$

$$\begin{aligned}h(L_T)-u(t,L_t)&=\int_t^T\left[\frac{\partial u}{\partial s}(s,L_{s-})+\frac{1}{2}\right.\\&\quad\left.\cdot\sigma^2(L_s)\frac{\partial^2 u}{\partial x^2}(s,L_s)+a'\frac{\partial u}{\partial s}(s,L_{s-})\right]ds\\&\quad+\int_t^T\int_{\mathbb{R}} u^1(s,L_{s-},y)\,\nu(dy)\,ds+\int_t^T\left[\frac{\partial u}{\partial s}(s,L_{s-})\right.\\&\quad\cdot\int_{\mathbb{R}} y^2\nu(dy)^{1/2}\\&\quad\left.+\int_{\mathbb{R}} u^1(s,L_{s-},y)\,p_1(y)\,\nu(dy)\right]dH_s^{(1)}\\&\quad+\sum_{i=2}^{\infty}\int_{\mathbb{R}} u^1(s,L_{s-},y)\,p_i(y)\,\nu(dy)\,dH_s^{(i)}.\end{aligned} \tag{54}$$

From (54), we can derive the result. □

In the following, we give an example of SPDIEs.

Example 8. Suppose that the Lévy process L has the form of $L_t=at+\sum_{i=1}^{\infty}(N_t^{(i)}-\alpha_i t)$, where $\{N^{(i)}\}_{i=1}^{\infty}$ is a sequence of independent Poisson processes with parameters $\{\alpha_i\}_{i=1}^{\infty}$ ($\alpha_i>0$).

Its Lévy measure is $\nu(dx) = \sum_{i=1}^{\infty} \alpha_i \delta_{\beta_i}(dx)$, where $\delta_{\beta_i}(dx)$ denotes the positive mass measure at $\beta_i \in \mathbb{R}$ of size 1. Moreover, we assume that $\sum_{i=1}^{\infty} \alpha_i |\beta_i|^2 < \infty$. Note that $H_t^{(1)} = \sum_{i=1}^{\infty} (\beta_1/\sqrt{\alpha_i})(N_t^{(i)} - \alpha_1 t)$ and $H_t^{(i)} = 0$, $i \geq 2$ (see [17]). Let (Y, Z) be the solution of the following equation:

$$\begin{aligned} Y_t = h(L_T) &+ \int_t^T f(s, X_{s-}, Y_{s-}, Z_s)\, ds \\ &+ \sum_{j=1}^{\infty} \int_t^T g_j(s, X_{s-}, Y_{s-}, Z_s)\, d\beta_j(s) \\ &- \sum_{i=1}^{\infty} \int_t^T Z_s^{(i)} d\left(N_s^{(i)} - \alpha_i s\right). \end{aligned} \tag{55}$$

Then

$$\begin{aligned} Y_t &= u(t, L_t), \\ Z_t^{(1)} &= \alpha_1 u^1(t, L_{t-}, \beta_1)\, p_1(\beta_1) \\ &\quad + \left(\sum_{i=1}^{\infty} \alpha_i |\beta_i|^2\right)^{1/2} \frac{\partial u}{\partial x}(t, L_{t-}), \\ Z_t^{(1)} &= \alpha_i u^1(t, L_{t-}, \beta_1)\, p_i(\beta_i), \quad i \geq 2, \end{aligned} \tag{56}$$

where u is the solution of the following SPDIEs:

$$\begin{aligned} &\frac{\partial u}{\partial t}(t, x) + \frac{1}{2}\sigma^2(x) \frac{\partial^2 u}{\partial x^2}(t, x) + a' \frac{\partial u}{\partial x}(t, x) \\ &\quad + \int_{\mathbb{R}} u^1(t, x, y)\, \nu(dy) \\ &\quad + f\left[t, u(t, x), \frac{\partial u}{\partial x}(t, x), \left\{u^{(i)}(t, x)\right\}_{i=1}^{\infty}\right] \\ &\quad + \sum_{j=1}^{\infty} g_j \left[t, u(t, x), \frac{\partial u}{\partial x}(t, x), \left\{u^{(i)}(t, x)\right\}_{i=1}^{\infty}\right] \\ &\quad \cdot \dot{\beta}_j(t)\, dt = 0, \\ &u(T, x) = h(x). \end{aligned} \tag{57}$$

Competing Interests

The author declares that he has no competing interests.

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Research Article

Optimal Bounds for Gaussian Arithmetic-Geometric Mean with Applications to Complete Elliptic Integral

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We present the best possible parameters $\alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{R}$ and $\alpha_3, \beta_3 \in (1/2, 1)$ such that the double inequalities $Q^{\alpha_1}(a,b)A^{1-\alpha_1}(a,b) < AG[A(a,b), Q(a,b)] < Q^{\beta_1}(a,b)A^{1-\beta_1}(a,b)$, $\alpha_2 Q(a,b) + (1-\alpha_2)A(a,b) < AG[A(a,b), Q(a,b)] < \beta_2 Q(a,b) + (1-\beta_2)A(a,b)$, $Q[\alpha_3 a + (1-\alpha_3)b, \alpha_3 b + (1-\alpha_3)a] < AG[A(a,b), Q(a,b)] < Q[\beta_3 a + (1-\beta_3)b, \beta_3 b + (1-\beta_3)a]$ hold for all $a, b > 0$ with $a \neq b$, where $A(a,b)$, $Q(a,b)$, and $AG(a,b)$ are the arithmetic, quadratic, and Gauss arithmetic-geometric means of a and b, respectively. As applications, we find several new bounds for the complete elliptic integrals of the first and second kind.

1. Introduction

Let $r \in (0,1)$ and $a, b > 0$. Then the elliptic elliptic integral of the first kind $\mathscr{K}(r)$ and second kind $\mathscr{E}(r)$, Gaussian arithmetic-geometric mean $AG(a,b)$, arithmetic mean $A(a,b)$, and quadratic mean $Q(a,b)$ are, respectively, given by

$$\mathscr{K}(r) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - r^2 \sin^2 t}} dt, \qquad \mathscr{E}(r) = \int_0^{\pi/2} \sqrt{1 - r^2 \sin^2 t}\, dt, \tag{1}$$

$$AG(a,b) = \frac{\pi}{2\int_0^{\pi/2} \left(dt / \sqrt{a^2 \cos^2 t + b^2 \sin^2 t} \right)}, \qquad A(a,b) = \frac{a+b}{2}, \qquad Q(a,b) = \sqrt{\frac{a^2+b^2}{2}}. \tag{2}$$

The Gauss identity [1–3] shows that

$$AG(1, r') = \frac{\pi}{2\mathscr{K}(r)} \tag{3}$$

for all $r \in (0,1)$, where and in what follows $r' = \sqrt{1-r^2}$.

It is well known that the elliptic elliptic integrals $\mathscr{K}(r)$ and $\mathscr{E}(r)$ and the Gaussian arithmetic-geometric mean $AG(a,b)$ have many applications in mathematics, physics, mechanics, and engineering [4–9]. Recently, the bounds for the Gaussian arithmetic-geometric mean $AG(a,b)$ have attracted the attention of many researchers.

The inequalities

$$\frac{1+\sqrt{r}}{2} AG(1, \sqrt{r}) < AG(1, r) < \frac{\pi}{2\log(4/r)}, \tag{4}$$

$$L(a,b) < AG(a,b) < L_{3/2}(a,b) \tag{5}$$

for all $r \in (0,1)$ and $a, b > 0$ with $a \neq b$ can be found in the literature [10–12], where $L(a,b) = (a-b)/(\log a - \log b)$ and $L_p(a,b) = L^{1/p}(a^p, b^p)$ are, respectively, the logarithmic and pth generalized logarithmic means of a and b. The first inequality of (5) is due to Carlson and Vuorinen [13].

By using a variant of L'Hospital's rule and representation theorems with elliptic integrals, Vamanamurthy and Vuorinen [14] proved, among other results, the inequalities

$$AG(a,b) < \sqrt{A(a,b)L(a,b)}, \tag{6}$$

$$L(a,b) < AG(a,b) < \frac{\pi}{2}L(a,b), \tag{7}$$

$$AG(a,b) < I(a,b) < A(a,b), \tag{8}$$

$$AG(a,b) < \frac{A(a,b)+G(a,b)}{2}, \tag{9}$$

$$A(a,b) < \frac{AG(a^2,b^2)}{AG(a,b)} < Q(a,b), \tag{10}$$

$$AG(a,b) > L^{1/\lambda}(a^\lambda,b^\lambda) \tag{11}$$

for all $a,b>0$ with $a\neq b$ and $\lambda\in(0,1]$, where $I(a,b)=(b^b/a^a)^{1/(b-a)}/e$ is the identric mean of a and b.

By use of the homogeneity of the above means and a series representation of $AG(a,b)$ due to Gauss, Sándor [15] obtained, among other results, new proofs for inequalities (7), (8) and a counterpart of inequality (9):

$$AG(a,b) > \sqrt{A(a,b)G(a,b)}, \tag{12}$$

for all $a,b>0$ with $a\neq b$, where $G(a,b)=\sqrt{ab}$ is the geometric mean of a and b. Inequalities (9) and (12) show that AG lies between the arithmetic and geometric means of A and G. In [16], Sándor provided new proofs for inequalities (6) and (8), (9), (10), and (12) by using only elementary methods for recurrent sequences and found much stronger forms of these results.

Neuman and Sándor [17] gave the comparison of the Gaussian arithmetic-geometric mean and the Schwab-Borchardt mean.

The upper bounds $\pi/[2\log(4/r)]$ for $AG(1,r)$ in (4) were replaced by $\pi(1-r^2/9)/[2\log(4/r)]$ due to Kühnau [18].

Qiu and Vamanamurthy [19] presented that $4\pi/[(9-r^2)(2\log 2-\log r)]$ and $(9-r^2)\pi/[18.192\times(2\log 2-\log r)]$ are, respectively, the lower and upper bounds for $AG(1,r)$ with $r\in(0,1)$. Alzer and Qiu [20] proved that $\lambda=3/4$ and $\mu=2/\pi$ are the best possible parameters such that the double inequality

$$\begin{aligned}\frac{1}{\lambda/L(a,b)+(1-\lambda)/A(a,b)} &< AG(a,b)\\ &< \frac{1}{\mu/L(a,b)+(1-\mu)/A(a,b)}\end{aligned} \tag{13}$$

holds for all $a,b>0$ with $a\neq b$.

Chu and Wang [21] proved that the double inequality

$$S_p(a,b) < AG(a,b) < S_q(a,b) \tag{14}$$

holds for all $a,b>0$ with $a\neq b$ if and only if $p\le 1/2$ and $q\ge 1$, where $S_p(a,b)=[(a^{p-1}+b^{p-1})/(a+b)]^{1/(p-2)}$ $(p\neq 2)$ and $S_2(a,b)=(a^ab^b)^{1/(a+b)}$ is the pth Gini mean of a and b. In [22], Yang et al. proved that the inequalities

$$\begin{aligned}S_{7/4,-1/4}(a,b) &< AG(a,b) < A^{1/4}(a,b)L^{3/4}(a,b),\\ AG(a,b) &< \sqrt{S_{p,1}(a,b)S_{1-p,1}(a,b)}\end{aligned} \tag{15}$$

hold for all $p\in(1/2,1)$ and $a,b>0$ with $a\neq b$, where $S_{p,q}(a,b)=[q(a^p-b^p)/(p(a^q-b^q))]^{1/(p-q)}$ is the Stolarsky mean [23] of a and b.

Let $a,b>0$ with $a\neq b$ and $x\in[1/2,1]$. Then it is not difficult to verify that the function $f(x)=Q[xa+(1-x)b, xb+(1-x)a]$ is continuous and strictly increasing on the interval $[1/2,1]$. Note that

$$\begin{aligned}f\left(\frac{1}{2}\right) &= A(a,b)=\min\{A(a,b),Q(a,b)\}\\ &< AG[A(a,b),Q(a,b)],\end{aligned} \tag{16}$$

$$\begin{aligned}AG[A(a,b),Q(a,b)] &< \max\{A(a,b),Q(a,b)\}\\ &= Q(a,b)=f(1).\end{aligned} \tag{17}$$

Inequalities (16) give us the motivation to deal with the best possible parameters $\alpha_1,\beta_1,\alpha_2,\beta_2\in\mathbb{R}$ and $\alpha_3,\beta_3\in(1/2,1)$ such that the double inequalities

$$\begin{aligned}&Q^{\alpha_1}(a,b)A^{1-\alpha_1}(a,b) < AG[A(a,b),Q(a,b)]\\ &\quad< Q^{\beta_1}(a,b)A^{1-\beta_1}(a,b),\\ &\alpha_2Q(a,b)+(1-\alpha_2)A(a,b) < AG[A(a,b),Q(a,b)]\\ &\quad< \beta_2Q(a,b)+(1-\beta_2)A(a,b),\\ &Q[\alpha_3a+(1-\alpha_3)b,\alpha_3b+(1-\alpha_3)a]\\ &\quad< AG[A(a,b),Q(a,b)]\\ &\quad< Q[\beta_3a+(1-\beta_3)b,\beta_3b+(1-\beta_3)a]\end{aligned} \tag{18}$$

hold for all $a,b>0$ with $a\neq b$.

2. Lemmas

In order to prove our main results we need several derivative formulas and particular values for $\mathscr{K}(r)$ and $\mathscr{E}(r)$, which we present in this section.

$\mathscr{K}(r)$ and $\mathscr{E}(r)$ satisfy the formulas (see [24])

$$\begin{aligned}&\frac{d\mathscr{K}(r)}{dr}=\frac{\mathscr{E}(r)-r'^2\mathscr{K}(r)}{rr'^2},\\ &\frac{d\mathscr{E}(r)}{dr}=\frac{\mathscr{E}(r)-\mathscr{K}(r)}{r},\\ &\mathscr{K}(0^+)=\mathscr{E}(0^+)=\frac{\pi}{2},\\ &\mathscr{K}(1^-)=\infty,\\ &\mathscr{E}(1^-)=1,\end{aligned}$$

$$\mathscr{K}\left(\frac{\sqrt{2}}{2}\right)=\frac{\Gamma^2(1/4)}{4\sqrt{\pi}}=1.85407467\ldots,$$
$$\mathscr{E}\left(\frac{\sqrt{2}}{2}\right)=\frac{4\Gamma^2(3/4)+\Gamma^2(1/4)}{8\sqrt{\pi}}=1.35064388\ldots, \tag{19}$$

where $\Gamma(x)=\int_0^{\infty}t^{x-1}e^{-t}dt$ $(x>0)$ is the classical Euler gamma function.

Lemma 1 (see [24, Theorem 1.25]). *Let $-\infty<a<b<\infty$, $f,g:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) and $g'(x)\neq 0$ on (a,b). Then both functions*

$$\frac{f(x)-f(a)}{g(x)-g(a)},\qquad \frac{f(x)-f(b)}{g(x)-g(b)} \tag{20}$$

are increasing (decreasing) on (a,b) if $f'(x)/g'(x)$ is increasing (decreasing) on (a,b). If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

Lemma 2 (see [24, Theorem 3.21(1), Theorem 3.21(7), and Exercises 3.43(32)]). *The following statements are true:*

(1) *The function $r\to[\mathscr{E}(r)-r'^2\mathscr{K}(r)]/r^2$ is strictly increasing from $(0,1)$ onto $(\pi/4,1)$.*

(2) *The function $r\to r'^{\lambda}\mathscr{K}(r)$ is strictly decreasing from $(0,1)$ onto $(0,\pi/2)$ if $\lambda\geq 1/2$.*

(3) *The function $r\to[\mathscr{K}(r)-\mathscr{E}(r)]/[r^2\mathscr{K}(r)]$ is strictly increasing from $(0,1)$ onto $(1/2,1)$.*

Lemma 3. *Let $p=1/2+\sqrt{2}\sqrt{\pi^2-2\mathscr{K}^2(\sqrt{2}/2)}/[4\mathscr{K}(\sqrt{2}/2)]=0.8299\ldots$ and $h(r)$ be defined by*

$$h(r)=\frac{\mathscr{E}(r)-r'^2\mathscr{K}(r)}{r^2r'^2\mathscr{K}^3(r)}-\frac{16p(1-p)}{\pi^2}. \tag{21}$$

Then there exists $r_0\in(0,\sqrt{2}/2)$ such that $h(r)<0$ for $r\in(0,r_0)$ and $h(r)>0$ for $r\in(r_0,\sqrt{2}/2)$.

Proof. From (21) we clearly see that $h(r)$ can be rewritten as

$$h(r)=\frac{\mathscr{E}(r)-r'^2\mathscr{K}(r)}{r^2}\left[r'^{2/3}\mathscr{K}(r)\right]^{-3}-\frac{16p(1-p)}{\pi^2}. \tag{22}$$

It follows from Lemma 2(1) and (2) together with (22) that $h(r)$ is strictly increasing on $(0,\sqrt{2}/2)$.

Numerical computations show that

$$\begin{aligned}h(0^+)&=\frac{2\left[\pi^2-3\mathscr{K}^2\left(\sqrt{2}/2\right)\right]}{\pi^2\mathscr{K}^2\left(\sqrt{2}/2\right)}=-0.02612\ldots\\&<0,\\h\left(\frac{\sqrt{2}}{2}^-\right)&=\frac{4\left[\pi^2\mathscr{E}\left(\sqrt{2}/2\right)-2\mathscr{K}^3\left(\sqrt{2}/2\right)\right]}{\pi^2\mathscr{K}^3\left(\sqrt{2}/2\right)}\\&=0.03708\ldots>0.\end{aligned} \tag{23}$$

Therefore, Lemma 3 follows easily from (23) and the monotonicity of $h(r)$ on the interval $(0,\sqrt{2}/2)$. □

3. Main Results

Theorem 4. *The double inequality*

$$Q^{\alpha_1}(a,b)A^{1-\alpha_1}(a,b)<AG[A(a,b),Q(a,b)]<Q^{\beta_1}(a,b)A^{1-\beta_1}(a,b) \tag{24}$$

holds for all $a,b>0$ with $a\neq b$ if and only if $\alpha_1\leq 1/2$ and $\beta_1\geq 2[\log\pi-\log\mathscr{K}(\sqrt{2}/2)]/\log 2-1=0.5215\ldots$.

Proof. Since $A(a,b)$, $Q(a,b)$, and $AG(a,b)$ are symmetric and homogenous of degree 1, without loss of generality, we assume that $a>b>0$. Let $r=(a-b)/\sqrt{2(a^2+b^2)}\in(0,\sqrt{2}/2)$. Then (2) and (3) lead to

$$AG[A(a,b),Q(a,b)]=\frac{\pi A(a,b)}{2r'\mathscr{K}(r)},\qquad Q(a,b)=\frac{A(a,b)}{r'}, \tag{25}$$

$$\frac{\log AG[A(a,b),Q(a,b)]-\log A(a,b)}{\log Q(a,b)-\log A(a,b)}=\frac{\log\mathscr{K}(r)+\log r'+\log 2-\log\pi}{\log r'}. \tag{26}$$

Let

$$f_1(r)=\log\mathscr{K}(r)+\log r'+\log 2-\log\pi,\qquad f_2(r)=\log r',\qquad f(r)=\frac{f_1(r)}{f_2(r)}. \tag{27}$$

Then simple computations give

$$f_1(0^+)=f_2(0^+)=0,\qquad \frac{f_1'(r)}{f_2'(r)}=\frac{\mathscr{K}(r)-\mathscr{E}(r)}{r^2\mathscr{K}(r)}. \tag{28}$$

It follows from Lemmas 1, 2(3) and (27) and (28) that

$$f(0^+)=\frac{1}{2} \tag{29}$$

and $f(r)$ is strictly increasing on the interval $(0,\sqrt{2}/2)$.

Note that

$$f\left(\frac{\sqrt{2}^-}{2}\right) = \frac{2\left[\log\pi - \log\mathscr{K}\left(\sqrt{2}/2\right)\right]}{\log 2} - 1. \quad (30)$$

Therefore, Theorem 4 follows easily from (26), (27), (29), and (30) and the monotonicity of $f(r)$ on the interval $(0, \sqrt{2}/2)$. □

Remark 5. The left side inequality of Theorem 4 for $\alpha_1 \le 1/2$ can be derived directly from the fact that $AG(a,b) > G(a,b) = \sqrt{ab}$ and $Q(a,b) > A(a,b)$ for all $a, b > 0$ with $a \ne b$.

Theorem 6. *The double inequality*

$$\begin{aligned}\alpha_2 Q(a,b) + (1-\alpha_2) A(a,b) &< AG[A(a,b), Q(a,b)] \\ &< \beta_2 Q(a,b) + (1-\beta_2) A(a,b)\end{aligned} \quad (31)$$

holds for all $a, b > 0$ with $a \ne b$ if and only if $\alpha_2 \le [\pi - \sqrt{2}\mathscr{K}(\sqrt{2}/2)]/[(2-\sqrt{2})\mathscr{K}(\sqrt{2}/2)] = 0.4783\ldots$ and $\beta_2 \ge 1/2$.

Proof. Without loss of generality, we assume that $a > b > 0$. Let $r = (a-b)/\sqrt{2(a^2+b^2)} \in (0, \sqrt{2}/2)$. Then it follows from (2) and (3) that

$$\frac{AG[A(a,b), Q(a,b)] - A(a,b)}{Q(a,b) - A(a,b)} = \frac{\pi/2\mathscr{K}(r) - r'}{1-r'}. \quad (32)$$

Let

$$\begin{aligned} g_1(r) &= \frac{\pi}{2\mathscr{K}(r)} - r', \\ g_2(r) &= 1 - r', \\ g(r) &= \frac{g_1(r)}{g_2(r)}. \end{aligned} \quad (33)$$

Then simple computations lead to

$$\begin{aligned} &g_1(0^+) = g_2(0^+) = 0, \\ &\frac{g_1'(r)}{g_2'(r)} = 1 - \frac{\pi}{2}\frac{\mathscr{E}(r) - r'^2\mathscr{K}(r)}{r^2}\left[r'^{1/2}\mathscr{K}(r)\right]^{-2}. \end{aligned} \quad (34)$$

It follows from Lemmas 1, 2(1) and (2) together with (33) and (34) that

$$g(0^+) = \frac{1}{2} \quad (35)$$

and $g(r)$ is strictly decreasing on the interval $(0, \sqrt{2}/2)$.

Note that

$$g\left(\frac{\sqrt{2}^-}{2}\right) = \frac{\pi - \sqrt{2}\mathscr{K}\left(\sqrt{2}/2\right)}{\left(2-\sqrt{2}\right)\mathscr{K}\left(\sqrt{2}/2\right)}. \quad (36)$$

Therefore, Theorem 6 follows easily from (32), (33), (35), and (36) and the monotonicity of $g(r)$ on the interval $(0, \sqrt{2}/2)$. □

Remark 7. The right side inequality of Theorem 6 for $\beta_2 \ge 1/2$ can be derived directly from the fact that $AG(a,b) < A(a,b) = (a+b)/2$ and $Q(a,b) > A(a,b)$ for all $a, b > 0$ with $a \ne b$.

Theorem 8. *Let $\alpha_3, \beta_3 \in (1/2, 1)$. Then the double inequality*

$$\begin{aligned} &Q[\alpha_3 a + (1-\alpha_3) b, \alpha_3 b + (1-\alpha_3) a] \\ &\quad < AG[A(a,b), Q(a,b)] \\ &\quad < Q[\beta_3 a + (1-\beta_3) b, \beta_3 b + (1-\beta_3) a] \end{aligned} \quad (37)$$

holds for all $a, b > 0$ with $a \ne b$ if and only if $\alpha_3 \le 1/2 + \sqrt{2}\sqrt{\pi^2 - 2\mathscr{K}^2(\sqrt{2}/2)}/[4\mathscr{K}(\sqrt{2}/2)] = 0.8299\ldots$ and $\beta_3 \ge 1/2 + \sqrt{2}/4 = 0.8535\ldots$.

Proof. Without loss of generality, we assume that $a > b > 0$. Let $r = (a-b)/\sqrt{2(a^2+b^2)} \in (0, \sqrt{2}/2)$ and $p \in (1/2, 1)$. Then (2) and (3) lead to

$$\begin{aligned} &Q[pa + (1-p) b, pb + (1-p) a] \\ &\quad = \frac{\sqrt{1 - 4p(1-p) r^2}}{r'} A(a,b), \end{aligned} \quad (38)$$

$$\begin{aligned} &Q[pa + (1-p) b, pb + (1-p) a] \\ &\quad - AG[A(a,b), Q(a,b)] \\ &\quad = \frac{A(a,b)}{\left[\sqrt{1 - 4p(1-p) r^2} + \pi/[2\mathscr{K}(r)]\right] r'} H(r), \end{aligned} \quad (39)$$

where

$$H(r) = 1 - 4p(1-p) r^2 - \frac{\pi^2}{4\mathscr{K}^2(r)}, \quad (40)$$

$$H(0^+) = 0, \quad (41)$$

$$H\left(\frac{\sqrt{2}^-}{2}\right) = 1 - 2p(1-p) - \frac{\pi^2}{4\mathscr{K}^2\left(\sqrt{2}/2\right)}, \quad (42)$$

$$H'(r) = \frac{\pi^2 r}{2} h(r), \quad (43)$$

where $h(r)$ is defined by (21).

We divide the proof into four cases.

Case 1 ($p = p_0 = 1/2 + \sqrt{2}\sqrt{\pi^2 - 2\mathscr{K}^2(\sqrt{2}/2)}/[4\mathscr{K}(\sqrt{2}/2)]$). Then (42) becomes

$$H\left(\frac{\sqrt{2}^-}{2}\right) = 0. \quad (44)$$

It follows from Lemma 3 and (43) that there exists $r_0 \in (0, \sqrt{2}/2)$ such that $H(r)$ is strictly decreasing on $(0, r_0]$ and strictly increasing on $[r_0, \sqrt{2}/2)$. Therefore,

$$\begin{aligned} &Q[p_0 a + (1-p_0) b, p_0 b + (1-p_0) a] \\ &\quad < AG[A(a,b), Q(a,b)] \end{aligned} \quad (45)$$

follows from (39), (41), and (44) together with the piecewise monotonicity of $H(r)$ on the interval $(0, \sqrt{2}/2)$.

Case 2 ($p = p_0^* = 1/2 + \sqrt{2}/4$). Then we clearly see that

$$\begin{aligned} &Q\left[p_0^* a + \left(1 - p_0^*\right) b, p_0^* b + \left(1 - p_0^*\right) a\right] \\ &\quad = \sqrt{\frac{A^2(a,b) + Q^2(a,b)}{2}} = Q\left[A(a,b), Q(a,b)\right] \\ &\quad > A\left[A(a,b), Q(a,b)\right] > AG\left[A(a,b), Q(a,b)\right]. \end{aligned} \tag{46}$$

Case 3 ($1/2 + \sqrt{2}\sqrt{\pi^2 - 2\mathcal{K}^2(\sqrt{2}/2)}/[4\mathcal{K}(\sqrt{2}/2)] < p < 1$). Then (42) leads to

$$H\left(\frac{\sqrt{2}}{2}^{-}\right) > 0. \tag{47}$$

Equation (39) and inequality (47) imply that there exists small enough $0 < \delta_1 < \sqrt{2}/2$ such that

$$\begin{aligned} &Q\left[pa + (1-p)b, pb + (1-p)a\right] \\ &\quad > AG\left[A(a,b), Q(a,b)\right] \end{aligned} \tag{48}$$

for all $a, b > 0$ with $|a-b|/\sqrt{2(a^2+b^2)} \in (\sqrt{2}/2 - \delta_1, \sqrt{2}/2)$.

Case 4 ($1/2 < p < 1/2 + \sqrt{2}/4$). Then (40) leads to

$$H(r) = \left[4\left(p - \frac{1}{2}\right)^2 - \frac{1}{2}\right] r^2 + o\left(r^2\right). \tag{49}$$

Note that

$$\left[4\left(p - \frac{1}{2}\right)^2 - \frac{1}{2}\right] r^2 < 0. \tag{50}$$

Equations (39) and (49) together with inequality (50) imply that there exists small enough $0 < \delta_2 < \sqrt{2}/2$ such that

$$\begin{aligned} &Q\left[pa + (1-p)b, pb + (1-p)a\right] \\ &\quad < AG\left[A(a,b), Q(a,b)\right] \end{aligned} \tag{51}$$

for all $a, b > 0$ with $|a-b|/\sqrt{2(a^2+b^2)} \in (0, \delta_2)$. □

4. Applications

In this section, we use Theorems 4, 6, and 8 to present several bounds for the complete elliptic integrals $\mathcal{K}(r)$ and $\mathcal{E}(r)$.

From Theorems 4, 6, and 8 we get Theorem 9 immediately.

Theorem 9. *Let $\lambda_1 = 2[\log\pi - \log\mathcal{K}(\sqrt{2}/2)]/\log 2 - 1 = 0.5215\ldots$, $\lambda_2 = [\pi - \sqrt{2}\mathcal{K}(\sqrt{2}/2)]/[(2-\sqrt{2})\mathcal{K}(\sqrt{2}/2)] = 0.4783\ldots$, and $\lambda_3 = 2 - \pi^2/[2\mathcal{K}^2(\sqrt{2}/2)] = 0.5644\ldots$. Then the double inequalities*

$$\begin{aligned} &\frac{\pi}{2\left(1-r^2\right)^{(1-\lambda_1)/2}} < \mathcal{K}(r) \\ &\qquad < \frac{\pi}{2\left[\lambda_2 + (1-\lambda_2)\sqrt{1-r^2}\right]}, \\ &\frac{\pi}{2\sqrt{1-(1/2)r^2}} < \mathcal{K}(r) < \frac{\pi}{2\sqrt{1-\lambda_3 r^2}} \end{aligned} \tag{52}$$

hold for all $r \in (0, \sqrt{2}/2)$.

It follows from the inequality

$$\frac{\pi^2}{4} < \mathcal{E}(r)\mathcal{K}(r) < \frac{\pi^2}{4\sqrt{r'}} \tag{53}$$

given in [24] that

$$\frac{\pi^2}{4\mathcal{K}(r)} < \mathcal{E}(r) < \frac{\pi^2}{4\sqrt{r'}\mathcal{K}(r)}. \tag{54}$$

Theorem 9 and (54) lead to the following.

Theorem 10. *Let $\lambda_1 = 2[\log\pi - \log\mathcal{K}(\sqrt{2}/2)]/\log 2 - 1 = 0.5215\ldots$, $\lambda_2 = [\pi - \sqrt{2}\mathcal{K}(\sqrt{2}/2)]/[(2-\sqrt{2})\mathcal{K}(\sqrt{2}/2)] = 0.4783\ldots$, and $\lambda_3 = 2 - \pi^2/[2\mathcal{K}^2(\sqrt{2}/2)] = 0.5644\ldots$. Then the double inequalities*

$$\begin{aligned} &\frac{\pi}{2}\left[\lambda_2 + (1-\lambda_2) r'\right] < \mathcal{E}(r) < \frac{\pi}{2} r'^{(1/2)-\lambda_1}, \\ &\frac{\pi}{2}\sqrt{1-\lambda_3 r^2} < \mathcal{E}(r) < \frac{\pi}{2}\sqrt{\frac{2-r^2}{2r'}} \end{aligned} \tag{55}$$

hold for all $r \in (0, \sqrt{2}/2)$.

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Abstract Theorems on Exchange of Limits and Preservation of (Semi)continuity of Functions and Measures in the Filter Convergence Setting

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We give necessary and sufficient conditions for exchange of limits of double-indexed families, taking values in sets endowed with an abstract structure of convergence, and for preservation of continuity or semicontinuity of the limit family, with respect to filter convergence. As a consequence, we give some filter limit theorems and some characterization of continuity and semicontinuity of the limit of a pointwise convergent family of set functions. Furthermore, we pose some open problems.

1. Introduction

A widely investigated problem in convergence theory and topology is to find necessary and/or sufficient conditions for continuity and/or semicontinuity of the limit of a pointwise convergent net of functions or measures. There have been many recent related studies in abstract structures, like topological spaces, lattice groups, metric semigroups, and cone metric spaces, with respect to usual, statistical, or filter/ideal convergence and associated with the notions of equicontinuity, filter exhaustiveness, and filter continuous convergence (see also [1–9]). The study of semicontinuous functions is associated with quasimetric spaces, that is, spaces endowed with an asymmetric distance function (for a related literature, see, e.g., [3–5, 10–13]).

A concept associated with these topics is that of *strong uniform continuity*, which is used to study the problem of finding a topology with respect to which the set of the continuous functions is closed, and pointwise convergence of continuous functions implies convergence in this topology (see also [1, 14, 15]).

Another related field is the study of convergence theorems for measures taking values in abstract structures. When dealing with the classical convergence, it is possible to prove σ-additivity, (s)-boundedness, and absolute continuity of the limit measure directly from pointwise convergence (with respect to a single order sequence of regulator) of the involved measures, without requiring additional hypotheses. This is not always true in the setting of filter convergence. A historical comprehensive overview, together with a survey on the most recent results and developments, can be found in [16] (see also its bibliography).

In this paper we present a unified axiomatic approach and extend results of this kind to double-indexed families, taking values in abstract structures, whose particular cases are lattice groups, topological groups, (quasi)metric semigroups, and cone (quasi)metric spaces. To include both continuity and semicontinuity, we assume the existence of a "generalized distance" function, which is assumed to satisfy only the triangular property and takes values in a group endowed with a suitable system of "intervals" or "half lines" containing its neutral element 0. Thus, both topological groups and lattice groups endowed with (r)-, (D)-, or order convergence are particular cases of these abstract structures. We prove some results on exchange of limits in the setting of filter convergence. Observe that the involved "distance" can be symmetric or asymmetric (for a literature, see also [3, 5, 10] and their bibliographies). Furthermore, in our setting, both sequences and nets of functions/measures are included, and note that it is possible to consider them as families endowed with filters (see also [17–19]).

As applications, we give some necessary and sufficient conditions for continuity from above/below and absolute continuity and semicontinuity of the limit measure in the context of filter convergence, which include the cases of σ-additivity and (s)-boundedness, showing, by means of related examples, that they are not always satisfied, differently from the classical case. For a literature on measures satisfying upper/lower semicontinuity conditions or similar properties and related applications, see, for instance, [20] and the bibliography therein. Finally, we pose some open problems.

2. Assumptions and Examples

We begin with giving our axiomatic approach, which deals with abstract convergence with respect to filters, without using necessarily nets. For a literature about these topics, see, for instance, [16, 17, 19, 21–24] and their bibliographies.

Definition 1. (a) Let Λ be any nonempty set, and let $\mathscr{P}(\Lambda)$ be the class of all subsets of Λ. A family of sets $\mathscr{F} \subset \mathscr{P}(\Lambda)$ is called a *filter* of Λ iff $\mathscr{F} \neq \emptyset$, $\emptyset \notin \mathscr{F}$, and $A \cap B \in \mathscr{F}$ for each $A, B \in \mathscr{F}$, and $B \in \mathscr{F}$ whenever $B \supset A$ and $A \in \mathscr{F}$.

Some examples are the filter $\mathscr{F}_{\text{cofin}}$ of all subsets of $\mathbb{N}$ whose complement is finite and the filter $\mathscr{F}_{\text{st}}$ of all subsets of $\mathbb{N}$ having asymptotic density one. Some other classes of filters can be found in [16].

(b) Let R be a nonempty set, and let $Y = (Y, +)$ be an abelian group with neutral element 0. Given $k \in \mathbb{N}$ and $U_1, U_2, \ldots, U_k \subset Y$, put $U_1 + U_2 + \cdots + U_k := \{u_1 + u_2 + \cdots + u_k : u_j \in U_j,\ j = 1, 2, \ldots, k\}$, and $kU := U + \cdots + U$ (k times).

(c) Let Π be a nonempty set. A Π-*system* $\mathscr{U}$ is a class of families $\mathbf{U} = (U_\pi)_{\pi\in\Pi}$ of subsets of Y, with $0 \in \bigcap_{\pi\in\Pi} U_\pi$ for each $\mathbf{U} = (U_\pi)_{\pi\in\Pi}$, such that for every $\mathbf{U} = (U_\pi)_{\pi\in\Pi}$ and $\mathbf{V} = (V_\pi)_{\pi\in\Pi} \in \mathscr{U}$ there is $\mathbf{W} = (W_\pi)_{\pi\in\Pi} \in \mathscr{U}$ such that $U_\pi + V_\pi \subset W_\pi$ for every $\pi \in \Pi$. Let $\rho : R \times R \to Y$ be a function, and suppose that

($\mathscr{H}1$) for every $\mathbf{U} = (U_\pi)_\pi$, $\mathbf{V} = (V_\pi)_\pi \in \mathscr{U}$ and for each $\pi \in \Pi$ and $a, b, c \in R$, if $\rho(a, b) \in U_\pi$ and $\rho(b, c) \in V_\pi$, then $\rho(a, c) \in U_\pi + V_\pi$.

(d) Fix a Π-system $\mathscr{U}$ on Y and a filter $\mathscr{F}$ of Λ. A family b_λ, $\lambda \in \Lambda$, of elements of R is said to $(\mathscr{U}\mathscr{F})$-*backward* (resp., $(\mathscr{U}\mathscr{F})$-*forward*) *converge* to $b \in R$ iff there is a family $(U_\pi)_{\pi\in\Pi} \in \mathscr{U}$, such that for every $\pi \in \Pi$ there is a set $F \in \mathscr{F}$ with $\rho(b_\lambda, b) \in U_\pi$ (resp., $\rho(b, b_\lambda) \in U_\pi$) for any $\lambda \in F$. We say that $(b_\lambda)_\lambda$ $(\mathscr{U}\mathscr{F})$-*converges* to $b \in R$ iff it $(\mathscr{U}\mathscr{F})$-converges both backward and forward to b, and in this case we write $(\mathscr{U}\mathscr{F})\lim_{\lambda\in\Lambda} b_\lambda = b$.

(e) Let Ξ be a nonempty set. Given two families $(a_{\lambda,\xi})_{\lambda\in\Lambda,\xi\in\Xi}$ and $(a_\xi)_{\xi\in\Xi}$ of elements of R, we say that $(a_{\lambda,\xi})_{\lambda,\xi}$ $(\Xi\mathscr{U}\mathscr{F})$-*backward* (resp., $(\Xi\mathscr{U}\mathscr{F})$-*forward*) converges to $(a_\xi)_\xi$ iff there is a family $(U_\pi)_{\pi\in\Pi} \in \mathscr{U}$, such that for each $\pi \in \Pi$ and $\xi \in \Xi$ there is $F \in \mathscr{F}$ with $\rho(a_{\lambda,\xi}, a_\xi) \in U_\pi$ (resp., $\rho(a_\xi, a_{\lambda,\xi}) \in U_\pi$) for any $\lambda \in F$. Analogously as above it is possible to formulate the notions of $(\Xi\mathscr{U}\mathscr{F})$-convergence and $(\Xi\mathscr{U}\mathscr{F})$-limit.

Remark 2. Observe that, in our context, we will consider filters without dealing explicitly with nets, and this is not a restriction. A *net* on R is a function $\mathscr{N} : \Lambda \to R$, where $\Lambda = (\Lambda, \geq)$ is a *directed* set, namely, a partially ordered set such that for any $\lambda_1, \lambda_2 \in \Lambda$ there exists $\lambda_0 \in \Lambda$ with $\lambda_0 \geq \lambda_j$, $j = 1, 2$. Given a directed set $(\Lambda, \geq)$, it is possible to associate the filter $\mathscr{F}_\Lambda$ generated by the family $\mathscr{C}' := \{\{\lambda' \in \Lambda : \lambda' \geq \lambda\} : \lambda \in \Lambda\}$. Note that $\mathscr{C}'$ is a *filter base* of Λ; that is, for every $A, B \in \mathscr{C}'$ there is an element $C \in \mathscr{C}'$ with $C \subset A \cap B$. The filter generated by a filter base $\mathscr{C}$ is the family $\{A \subset \Lambda :$ there is $B \in \mathscr{C}$ with $B \subset A\}$. Conversely, given a filter base $\mathscr{C} := \{C_\lambda : \lambda \in \Lambda\}$, it is possible to associate a directed partial order $\geq$ on Λ, by setting $\lambda_1 \geq \lambda_2$ if and only if $C_{\lambda_1} \subset C_{\lambda_2}$, $\lambda_1, \lambda_2 \in \Lambda$ (see also [18, 19]).

Example 3. We now present some kinds of abstract space in which our approach can be applied, including both symmetric and asymmetric distance functions (for a literature, see also [3, 5, 10–13]).

(a) Let R be a Dedekind complete lattice group, $Y = R$, and let $\rho(a, b) := |a - b|$, $a, b \in R$, be the *absolute value* of $a - b$. It is possible to define different kinds of convergences, as follows (see also [16]).

Let $\Pi_1 := \mathbb{R}^+$ be endowed with the usual order, $\mathscr{U}_1 := \{([-\varepsilon u, \varepsilon u])_{\varepsilon\in\mathbb{R}^+} : u \in R,\ u > 0\}$ ((r)-*convergence*); let $\Pi_2 := \mathbb{N}$ be with the usual order, $\mathscr{U}_2 := \{([-\sigma_p, \sigma_p])_{p\in\mathbb{N}} : (\sigma_p)_p$ is an (O)-sequence$\}$, where an (O)-*sequence* is a decreasing sequence in R whose infimum is equal to 0 (*order convergence* of (O)-*convergence*); let $\Pi_3 := \mathbb{N}^\mathbb{N}$ be directed with the pointwise order, $\mathscr{U}_3 := \{([-\bigvee_{t=1}^\infty a_{t,\varphi(t)}, \bigvee_{t=1}^\infty a_{t,\varphi(t)}])_{\varphi\in\mathbb{N}^\mathbb{N}} : (a_{t,l})_{t,l}$ is a (D)-sequence$\}$, where a (D)-*sequence* or *regulator* is a bounded double sequence in R such that $(a_{t,l})_l$ is an (O)-sequence for each $t \in \mathbb{N}$ ((D)-*convergence*). The (D)-convergence was presented in [25] to give direct proofs of extension theorems for vector lattice-valued functionals and replaces the ε-technique in dealing with suprema and infima of lattice group- or vector lattice-valued families. For technical reasons, sometimes the (D)-convergence is easier to handle than (O)-convergence, and in particular it is very useful when one replaces a sequence of regulators with a single (D)-sequence (for a literature about these topics, see also [16, 23, 26, 27]).

It is not difficult to check that $\mathscr{U}_j$, $j = 1, 2, 3$, are Π_j-systems, satisfying ($\mathscr{H}1$).

(b) We can extend the examples given in (a) to the case in which R is a *cone metric space* (with respect to Y); that is, R is a nonempty set and $(Y, +)$ is a Dedekind complete lattice group endowed with a *distance function* $\rho : R \times R \to Y$, satisfying the following axioms:

(i) $\rho(a, b) \geq 0$ and $\rho(a, b) = 0$ if and only if $a = b$.

(ii) $\rho(a, b) = \rho(b, a)$ (*symmetric property*).

(iii) $\rho(a, c) \leq \rho(a, b) + \rho(b, c)$ (*triangular property*) for every $a, b, c \in R$.

(See also [6, 28].) When a cone metric space R is a semigroup, we say that R is a *cone metric semigroup*, a cone metric semigroup in which $Y = \mathbb{R}$ is said to be a *metric semigroup*. Note that the set of fuzzy numbers is a metric semigroup, but not a group (see also [20]). If ρ satisfies the first and the third of the above axioms, but not the symmetric property, then we say that ρ is an *asymmetric distance function* and that R is a *cone asymmetric metric space* or *cone quasimetric space* (see also [3, 5, 10]). For example, let $\mathcal{T}$ be a nonempty set, $R = \{f : \mathcal{T} \to \mathbb{R}, f \text{ is bounded}\}$, and let $a_0 \neq 1$ be a fixed positive real number and let u be a fixed element of R with $u > 0$. For each $f_1, f_2 \in R$ and $t \in \mathcal{T}$, set

$$d^{(u)}_{a_0,t}(f_1(t), f_2(t)) = \begin{cases} (f_2(t) - f_1(t))\, u, & \text{if } f_1(t) \leq f_2(t), \\ a_0 (f_1(t) - f_2(t))\, u, & \text{if } f_1(t) > f_2(t), \end{cases} \tag{1}$$

and let $\rho^{(u)}_{a_0}(f_1, f_2) = \bigvee_{t\in\mathcal{T}} d^{(u)}_{a_0,t}(f_1(t), f_2(t))$. It is not difficult to see that ρ_{a_0} is an asymmetric distance function (see also [3, 10]).

(c) When R is a lattice group and $Y = R$, it is advisable to deal not only with continuity, but also with upper or lower semicontinuity (see also [4]). In this setting we take $\rho(a,b) := b - a$, $a, b \in R$, Π_j, $j = 1,2,3$, as in (a), and $\mathcal{U}^{(0)}_1 := \{(\{r \in R : r \leq \varepsilon u\})_{\varepsilon\in\mathbb{R}^+} : u \in R, u > 0\}$; $\mathcal{U}^{(0)}_2 := \{(\{r \in R : r \leq \sigma_p\})_{p\in\mathbb{N}} : (\sigma_p)_p \text{ is an } (O)\text{-sequence}\}$; $\mathcal{U}^{(0)}_3 := \{(\{r \in R : r \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}\})_{\varphi\in\mathbb{N}^{\mathbb{N}}} : (a_{t,l})_{t,l} \text{ is a } (D)\text{-sequence}\}$.

(d) Let R be a Hausdorff topological group with neutral element 0 satisfying the first axiom of countability, $Y = R$, $\Pi^* = \mathbb{N}$, $\mathcal{U}^* := \{(U_p)_{p\in\mathbb{N}} : (U_p)_{p\in\mathbb{N}}$ is a base of closed symmetric neighborhoods of 0$\}$, and $\rho(a,b) = b - a$. It is not difficult to see that $\mathcal{U}^*$ is a Π^*-system (see also [16, 29]).

(e) Let $\mathcal{F}$ be a filter of Λ. When we consider (r)-convergence and R is a cone quasimetric space, a family $(b_\lambda)_\lambda$ of elements of R is said to *$(r\mathcal{F})$-backward converge to b* iff there is $u \in Y$, $u > 0$, with $\{\lambda \in \Lambda : \rho(b_\lambda, b) \leq \varepsilon u\} \in \mathcal{F}$ for all $\varepsilon > 0$. When we deal with (O)-sequences, we say that $(b_\lambda)_\lambda$ *$(O\mathcal{F})$-backward converges to b* iff there exists an (O)-sequence $(\sigma_p)_p$ in Y with $\{\lambda \in \Lambda : \rho(b_\lambda, b) \leq \sigma_p\} \in \mathcal{F}$ for every $p \in \mathbb{N}$. When we consider (D)-sequences, we say that the net $(b_\lambda)_\lambda$ *$(D\mathcal{F})$-backward converges to b* iff there exists a regulator $(a_{t,l})_{t,l}$ in Y with

$$\left\{\lambda \in \Lambda : \rho(b_\lambda, b) \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}\right\} \in \mathcal{F} \quad \text{for each } \varphi \in \mathbb{N}^{\mathbb{N}}. \tag{2}$$

When $\Lambda = \mathbb{N}$ and $\mathcal{F} = \mathcal{F}_{\text{cofin}}$, we have the classical (r)-, (O)-, and (D)-(backward, forward) convergence (see also [3, 16]). If $(R, +)$ is a Hausdorff topological group and $Y = R$, then we say that a net b_λ, $\lambda \in \Lambda$, in R, *$\mathcal{F}$-backward converges to $b \in R$* iff $\{\lambda \in \Lambda : b_\lambda - b \in U\} \in \mathcal{F}$ for each neighborhood U of 0. Similarly as above it is possible to formulate the corresponding notions of $(r\mathcal{F})$-, $(O\mathcal{F})$-, and $(D\mathcal{F})$-(forward) convergences and limits.

(f) When R is a Dedekind complete lattice group, $(a_{\lambda,\xi})_{\lambda\in\Lambda,\xi\in\Xi}$ and $(a_\xi)_{\xi\in\Xi}$ are two families in R and $\mathcal{U}$ is the Π-system associated with (r)-convergence (resp., (O)-convergence, (D)-convergence); we say that $(\Xi r\mathcal{F})\lim_{\lambda\in\Lambda} a_{\lambda,\xi} = a_\xi$ (resp., $(\Xi O\mathcal{F})\lim_{\lambda\in\Lambda} a_{\lambda,\xi} = a_\xi$, $(\Xi D\mathcal{F})\lim_{\lambda\in\Lambda} a_{\lambda,\xi} = a_\xi$) iff $(\Xi\mathcal{U}\mathcal{F})\lim_{\lambda\in\Lambda} a_{\lambda,\xi} = a_\xi$. Analogously it is possible to formulate the corresponding concepts of backward and forward convergences (see also [3, 5, 10]). In particular, when $R = \mathbb{R}$ endowed with the usual convergence, since it coincides with (r)- (O)-, and (D)-convergence, we will denote by $(\mathcal{F})$- and $(\Xi\mathcal{F})$-(backward, forward) convergence the usual filter (backward, forward) convergence and the ordinary pointwise filter (backward, forward) convergence. When R is a Hausdorff topological group, $\mathcal{U}^*$, Π^* are as in (d), and we get that the $(\Xi\mathcal{U}^*\mathcal{F})$-convergence is equivalent to the pointwise $(\mathcal{F})$-convergence, and hence we write $(\mathcal{F})\lim_{\lambda\in\Lambda} a_{\lambda,\xi} = a_\xi$ for every $\xi \in \Xi$, or $(\Xi\mathcal{F})\lim_{\lambda\in\Lambda} a_{\lambda,\xi} = a_\xi$.

(g) Observe that, in general, a family $(b_\lambda)_\lambda$ can be backward (resp., forward) convergent to more than one element. For example, if R is a Dedekind complete lattice group, Λ is a nonempty set, $\mathcal{F}$ is any filter of Λ, $\rho(a,b) = b - a$ for every $a, b \in R$, $b_\lambda = 0$ for every $\lambda \in \Lambda$, and b is *any* element of R with $b \leq 0$ (resp., $b \geq 0$), then it is not difficult to see that $(b_\lambda)_\lambda$ $(r\mathcal{F})$-backward (resp., $(r\mathcal{F})$-forward) converges to b.

(h) In general, backward and forward convergence are not equivalent. For example, similarly as in (1), let $\mathcal{T}$ be a nonempty set, let $\Lambda := [1, +\infty[$ be endowed with the usual order, let $\mathcal{F}$ be a filter of Λ containing all half lines $[c, +\infty[$ with $c \geq 1$, pick $R = \{f : \mathcal{T} \to \mathbb{R}, f \text{ is bounded}\}$, and let $\mathbf{0}$, $\mathbf{1}$ be those functions which associate with every element of $\mathcal{T}$ the real constants 0, 1, respectively. For any $f_1, f_2 \in R$ and $t \in \mathcal{T}$, set

$$d'_t(f_1(t), f_2(t)) = \begin{cases} (f_2(t) - f_1(t)) \cdot \mathbf{1}, & \text{if } f_1(t) \leq f_2(t), \\ \mathbf{1}, & \text{if } f_1(t) > f_2(t), \end{cases} \tag{3}$$

and put $\rho'(f_1, f_2) := \bigvee_{t\in\mathcal{T}} d'_t(f_1(t), f_2(t))$. It is not difficult to check that ρ' is an asymmetric distance function (see also [3, 10]). For each $\lambda \in \Lambda$, set $f_\lambda := 1/\lambda \cdot \mathbf{1}$ and $h_\lambda := -f_\lambda = -1/\lambda \cdot \mathbf{1}$. Note that $d(\mathbf{0}, f_\lambda) = f_\lambda$, $d(f_\lambda, \mathbf{0}) = \mathbf{1}$, $d(h_\lambda, \mathbf{0}) = f_\lambda$, and $d(\mathbf{0}, h_\lambda) = \mathbf{1}$. From this it is not difficult to deduce that the family $(f_\lambda)_\lambda$ $(r\mathcal{F})$-forward converges to $\mathbf{0}$ and $(h_\lambda)_\lambda$ $(r\mathcal{F})$-backward converges to $\mathbf{0}$, while $(f_\lambda)_\lambda$ does not $(r\mathcal{F})$-backward converge to $\mathbf{0}$ and $(h_\lambda)_\lambda$ does not $(r\mathcal{F})$-forward converge to $\mathbf{0}$.

However, if Λ is any nonempty set, $\mathcal{F}$ is any filter of Λ, $\rho^{(u)}_{a_0}$ is as in (1), and $C_{a_0} = \max\{a_0, 1/a_0\}$, then it is not difficult to see that $\rho^{(u)}_{a_0}(f_1, f_2) \leq C_{a_0}\, \rho^{(u)}_{a_0}(f_2, f_1)$ whenever $f_1, f_2 \in R$. From this it follows that a family $(f_\lambda)_{\lambda\in\Lambda}$ in R is $(r\mathcal{F})$-backward convergent if and only if it is $(r\mathcal{F})$-forward convergent. We claim that, in this case, the involved limit coincides. Indeed, if $(f_\lambda)_\lambda$ $(r\mathcal{F})$-backward converges to f_0 and $(r\mathcal{F})$-forward converges to h_0 with respect to $\rho^{(u)}_{a_0}$, then there exist $v, w \in R$, such that for every $\varepsilon > 0$ there are $F_1, F_2 \in \mathcal{F}$ with $\rho^{(u)}_{a_0}(h_0, f_\lambda) \leq \varepsilon v$ for every $\lambda \in F_1$, $\rho^{(u)}_{a_0}(f_\lambda, f_0) \leq \varepsilon w$ whenever $\lambda \in F_2$. Note that $F_1 \cap F_2 \in \mathcal{F}$. If λ_0 is any fixed element of

$F_1 \cap F_2$, then from the triangular property of $\rho_{a_0}^{(u)}$ we deduce that

$$\rho_{a_0}^{(u)}\left(h_0, f_0\right) \le \rho_{a_0}^{(u)}\left(h_0, f_{\lambda_0}\right) + \rho_{a_0}^{(u)}\left(f_{\lambda_0}, f_0\right) \le \varepsilon\left(v + w\right). \tag{4}$$

Thus, by arbitrariness of ε, we get $\rho_{a_0}^{(u)}(h_0, f_0) = 0$, and hence $h_0 = f_0$, getting the claim.

3. The Main Results

In this section we give the fundamental results of the paper in our unified setting, which includes lattice groups, cone metric spaces, metric groups and topological groups, symmetric and asymmetric distances, continuity and semicontinuity of the limit, and families of functions and of measures. We first present the notion of weak filter backward and forward exhaustiveness in our abstract context, which extends the corresponding ones given in the literature and the classical concept of equicontinuity (see also [4, 8, 16, 30]).

Definition 4. (a) Let Ξ be a nonempty set; fix $\xi \in \Xi$ and let $\mathcal{S}_\xi$ be a filter of Ξ. One says that the family $(a_{\lambda,\xi})_{\lambda,\xi}$ is *weakly* $(\mathcal{U}\mathcal{F})$*-backward* (resp., *forward*) *exhaustive at* ξ iff there exists a family $(U_\pi)_{\pi\in\Pi} \in \mathcal{U}$ such that for each $\pi \in \Pi$ there is a set $S \in \mathcal{S}_\xi$ such that for every $\zeta \in S$ there is a set $F_\zeta \in \mathcal{F}$ with $\rho(a_{\lambda,\zeta}, a_{\lambda,\xi}) \in U_\pi$ (resp., $\rho(a_{\lambda,\xi}, a_{\lambda,\zeta}) \in U_\pi$) for any $\lambda \in F_\zeta$. The family $(a_{\lambda,\xi})_{\lambda,\xi}$ is said to be *weakly* $(\mathcal{U}\mathcal{F})$*-exhaustive at* ξ iff it is both weakly $(\mathcal{U}\mathcal{F})$-backward and weakly $(\mathcal{U}\mathcal{F})$-forward exhaustive at ξ.

(b) Let $\mathcal{S}_\xi$, $\xi \in \Xi$, be a family of filters of Ξ. One says that $(a_{\lambda,\xi})_{\lambda,\xi}$ is *weakly* $(\mathcal{U}\mathcal{F})$*- (backward, forward) exhaustive on* Ξ iff it is weakly $(\mathcal{U}\mathcal{F})$- (backward, forward) exhaustive at every $\xi \in \Xi$ with respect to a single family $\mathbf{U} \in \mathcal{U}$, independent of ξ.

Example 5. We now show that, in general, weak $(\mathcal{U}\mathcal{F})$-backward and forward exhaustiveness do not coincide. Let $\Lambda = R = \Xi = \mathbb{R}$, $Y = \mathbb{R}$, be equipped with the usual convergence; that is, let $\Pi := \mathbb{R}^+$ be endowed with the usual order, and $\mathcal{U} := \{([-\varepsilon u, \varepsilon u])_{\varepsilon\in\mathbb{R}^+} : u \in \mathbb{R}^+\}$. Let us define $\rho : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ by

$$\rho(\xi,\zeta) := \begin{cases} \zeta - \xi, & \text{if } \xi \le \zeta, \\ 1, & \text{if } \xi > \zeta, \end{cases} \qquad \xi, \zeta \in \mathbb{R}. \tag{5}$$

It is not difficult to see that ρ is an asymmetric distance function (see also [10, Example 5.3]). Let $\mathcal{F}$ be any filter of Λ, and, for every $\xi \in \Xi$, let $\mathcal{S}_\xi$ be the filter of all neighborhoods of ξ with respect to the topology generated by ρ. Set $a_{\lambda,\xi} := \xi + \lambda$, $\xi, \lambda \in \mathbb{R}$. We claim that the family $(a_{\lambda,\xi})_{\lambda,\xi}$ is weakly $(\mathcal{U}\mathcal{F})$-forward exhaustive at ξ. Indeed, in correspondence with $\varepsilon > 0$, take $\eta := \min\{\varepsilon, 1/2\}$, and set $F_\zeta := \Lambda$ for any $\zeta \in [\xi, \xi + \eta] = S_\rho(\xi,\eta)$, where $S_\rho(\xi,\eta)$ denotes the ball of center ξ and radius η with respect to ρ. For every $\lambda \in \Lambda$ and $\zeta \in [\xi, \xi+\eta]$ we get $\rho(a_{\lambda,\xi}, a_{\lambda,\zeta}) = \zeta+\lambda-(\xi+\lambda) = \zeta-\xi \in [-\eta, \eta]$, getting the claim.

Now, in correspondence with every $\xi \in \mathbb{R}$ and $\theta > 0$, let $\eta = \min\{\theta, 1\}$ and take $\zeta = \xi + \eta$. Note that $\zeta \in S_\rho(\xi, \theta)$. Choose arbitrarily $F \in \mathcal{F}$. It is not hard to see that $\rho(a_{\lambda,\zeta}, a_{\lambda,\xi}) = \rho(a_{\lambda,\xi+\eta}, a_{\lambda,\xi}) = 1$ for every $\lambda \in F$. Hence, the family $(a_{\lambda,\xi})_{\lambda,\xi}$ is not weakly $(\mathcal{U}\mathcal{F})$-backward exhaustive at ξ. Furthermore note that, analogously as in (3), it is not difficult to check that $(\mathcal{U}\mathcal{F})$-forward (resp., backward) convergence does not imply $(\mathcal{U}\mathcal{F})$-backward (resp., forward) convergence with respect to ρ.

The following result deals with characterizations and properties of the limit family and extends [3, Theorem 3.1], [4, Theorems 2.5, 2.6], and [6, Theorem 3.1] to the abstract context.

Theorem 6. *Assume that $(a_{\lambda,\xi})_{\lambda,\xi}$ $(\Xi\mathcal{U}\mathcal{F})$-converges to $(a_\xi)_\xi$, fix $\xi \in \Xi$, and let $\mathcal{S}_\xi$ be a filter of Ξ. Then the following are equivalent:*

(i) *$(a_{\lambda,\xi})_{\lambda,\xi}$ is weakly $(\mathcal{U}\mathcal{F})$-backward (resp., forward) exhaustive at ξ.*

(ii) *$(a_\zeta)_\zeta$ $(\mathcal{U}\mathcal{S}_\xi)$-backward (resp., forward) converges to a_ξ as $\zeta \to \xi$.*

Proof. We give the proof only in the "backward" case, since the other case is analogous.

(i) $\Rightarrow$ (ii) Let $(U_\pi)_{\pi\in\Pi} \in \mathcal{U}$ be a family related to $(\mathcal{U}\mathcal{F})$-backward exhaustiveness of $(a_{\lambda,\xi})_{\lambda,\xi}$ at ξ. By hypothesis, for each $\pi \in \Pi$, there exists a set $S \in \mathcal{S}_\xi$, associated with weak $(\mathcal{U}\mathcal{F})$-backward exhaustiveness. Pick arbitrarily $\zeta \in S$. There is a set $F_1 \in \mathcal{F}$ with $\rho(a_{\lambda,\zeta}, a_{\lambda,\xi}) \in U_\pi$ for any $\lambda \in F_1$. Moreover, thanks to $(\Xi\mathcal{U}\mathcal{F})$-convergence, there is a family $(U^*_\pi)_\pi \in \mathcal{U}$ such that for every $\pi \in \Pi$ there exists $F_2 \in \mathcal{F}$ with $\rho(a_\zeta, a_{\lambda,\zeta}) \in U^*_\pi$ and $\rho(a_{\lambda,\xi}, a_\xi) \in U^*_\pi$ whenever $\lambda \in F_2$. From this and $(\mathcal{H}1)$ it follows that $\rho(a_\zeta, a_\xi) \in 2U^*_\pi + U_\pi$, getting (ii).

(ii) $\Rightarrow$ (i) By hypothesis, there exists a family $(U_\pi)_\pi \in \mathcal{U}$ such that for each $\pi \in \Pi$ there is a set $S \in \mathcal{S}_\xi$ with

$$\rho\left(a_\zeta, a_\xi\right) \in U_\pi \quad \text{whenever } \zeta \in S. \tag{6}$$

Choose $\zeta \in S$. By $(\Xi\mathcal{U}\mathcal{F})$-convergence of $(a_{\lambda,\xi})_{\lambda,\xi}$ to $(a_\xi)_\xi$, there is a family $(U^*_\pi)_\pi \in \mathcal{U}$ such that for every $\pi \in \Pi$ there is a set $F^* \in \mathcal{F}$ with

$$\rho\left(a_\xi, a_{\lambda,\xi}\right) \in U^*_\pi, \qquad \rho\left(a_{\lambda,\zeta}, a_\zeta\right) \in U^*_\pi \tag{7}$$

for each $\lambda \in F^*$. From (6), (7), and $(\mathcal{H}1)$ we get that for every $\pi \in \Pi$ there is $S \in \mathcal{S}_\xi$ such that for each $\zeta \in S$ there exists $F^* \in \mathcal{F}$ with $\rho(a_{\lambda,\zeta}, a_{\lambda,\xi}) \in 2U^*_\pi + U_\pi$ whenever $\lambda \in F^*$. Thus the family $(a_{\lambda,\xi})_{\lambda,\xi}$ is weakly $(\mathcal{U}\mathcal{F})$-backward exhaustive at ξ. This ends the proof. □

Remark 7. Observe that Theorem 6 holds also if $(\Xi\mathcal{U}\mathcal{F})$-convergence is replaced by $(\Xi\mathcal{U}\mathcal{F})$-forward convergence, under the hypothesis that forward convergence implies backward convergence (see also [10]). In general this last condition is essential. Indeed, let $\Lambda := [1, +\infty[$ be endowed with the usual order, let $\mathscr{F}$ be a filter of Λ containing all half lines $[c,+\infty[$ with $c \geq 1$, let $\Xi := [0,1]$ be equipped with the usual distance, let $\mathscr{S}_\xi$, $\xi \in \Xi$, be the filter of all neighborhoods of ξ, let $Y = \mathbb{R}$ be endowed with the usual convergence, $R = [0,1] \times [0,1]$, and let $\rho^* : R \times R \to \mathbb{R}$ be defined by

$$\rho^*((\xi_1,\xi_2),(\zeta_1,\zeta_2)) = \begin{cases} 0, & \text{if } (\xi_1,\xi_2) = (\zeta_1,\zeta_2), \\ \max\{|\xi_1-\zeta_1|,|\xi_2-\zeta_2|\}, & \text{if } \xi_1 \leq \zeta_1,\ \zeta_1 > 0, \\ 1, & \text{otherwise.} \end{cases} \tag{8}$$

It is not difficult to check that ρ^* is an asymmetric distance function. For every $\lambda \in \Lambda$ and $\xi \in \Xi$, set $a^*_{\lambda,\xi} := (1/\lambda, \xi)$. Observe that $\rho^*((0,\xi),(1/\lambda,\xi)) = 1/\lambda$ and $\rho^*((1/\lambda,\xi),(0,\xi)) = 1$ for every $\lambda \in \Lambda$ and $\xi \in \Xi$. It is not difficult to see that the family $(a^*_{\lambda,\xi})_{\lambda,\xi}$ $(\Xi\mathscr{U}\mathscr{F})$-forward converges to $(a^*_\xi)_{\xi\in\Xi}$, where $a^*_\xi = (0,\xi)$, $\xi \in \Xi$, but does not $(\Xi\mathscr{U}\mathscr{F})$-backward converge. Moreover, since $\rho^*((0,\zeta),(0,0)) = \rho^*((0,0),(0,\zeta)) = 1$, for every $\zeta \in \Xi, \zeta \neq 0$, the family $(a^*_\zeta)_{\zeta\in\Xi}$ is neither $(\mathscr{U}\mathscr{S}_\xi)$-backward nor $(\mathscr{U}\mathscr{S}_\xi)$-forward convergent to a^*_0 as $\zeta \to 0$. Furthermore, we get

$$\begin{aligned}\rho^*\left(a^*_{\lambda,\zeta}, a^*_{\lambda,0}\right) &= \rho^*\left(\left(\frac{1}{\lambda},\zeta\right),\left(\frac{1}{\lambda},0\right)\right) = \zeta \\ &= \rho^*\left(\left(\frac{1}{\lambda},0\right),\left(\frac{1}{\lambda},\zeta\right)\right) = \rho^*\left(a^*_{\lambda,0}, a^*_{\lambda,\zeta}\right)\end{aligned} \tag{9}$$

for every $\lambda \in \Lambda$ and $\zeta \in \Xi$. From (9) it is not difficult to deduce that the family $(a^*_{\lambda,\xi})_{\lambda,\xi}$ is both weakly $(\mathscr{U}\mathscr{F})$-forward and weakly $(\mathscr{U}\mathscr{F})$-backward exhaustive at 0 (see also [3, Example 3.7], [10, Example 5.10]).

We now give some kinds of convergences for families, which are some necessary and sufficient conditions for exchange of limits, which extend to our context some results proved in [1, 2, 4, 6, 9] about necessary and sufficient conditions for continuity of the pointwise limit of continuous functions. We extend to our setting the concepts of Arzelà, Alexandroff, and strong uniform convergence given in [1, 15, 31, 32].

Definition 8. (a) Fix $\xi \in \Xi$, and let $\mathscr{S}_\xi$ be a filter of Ξ. One says that $(a_{\lambda,\xi})_{\lambda,\xi}$ *$(\mathscr{U}\mathscr{F})$-forward strongly uniformly converges to* $(a_\xi)_\xi$ *at* ξ (shortly, $a_{\lambda,\xi} \xrightarrow{\mathscr{U}\mathscr{F}\ fw-\mathscr{T}^s} a_\xi$) iff there exists a family $(U_\pi)_\pi \in \mathscr{U}$ such that for each $\pi \in \Pi$ there is $F \in \mathscr{F}$ such that for every $\lambda \in F$ there is a set $S_\lambda \in \mathscr{S}_\xi$ with $\rho(a_\zeta, a_{\lambda,\zeta}) \in U_\pi$ whenever $\zeta \in S_\lambda$.

(b) One says that $(a_{\lambda,\xi})_{\lambda,\xi}$ is *$(\mathscr{U}\mathscr{F})$-forward Arzelà convergent to* $(a_\xi)_\xi$ *at* ξ (in brief, $a_{\lambda,\xi} \xrightarrow{\mathscr{U}\mathscr{F}\ fw-Arz.} a_\xi$) iff there exists a family $(U_\pi)_{\pi\in\Pi} \in \mathscr{U}$ such that for every $\pi \in \Pi$ and $F \in \mathscr{F}$ there are a finite set $\{\lambda_1, \lambda_2, \ldots, \lambda_q\} \subset F$ and a set $S \in \mathscr{S}_\xi$, such that for each $\zeta \in S$ there is $j \in [1,q]$ with $\rho(a_\zeta, a_{\lambda_j,\zeta}) \in U_\pi$.

(c) If $\mathscr{S}_\xi$, $\xi \in \Xi$, is a family of filters of Ξ, then one says that a *finitely uniform cover* of Ξ is a family $\mathscr{V}$ of subsets of Ξ such that $\Xi = \bigcup_{V\in\mathscr{V}} V$, and for every $\xi \in \Xi$ there are a set $S_\xi \in \mathscr{S}_\xi$ and a finite subset $\mathscr{Y} := \{V_{l_1}, \ldots, V_{l_q}\}$ of $\mathscr{V}$, such that for each $\zeta \in S_\xi$ there exists $j \in [1,q]$ with $\zeta \in V_{l_j}$.

(d) The family $(a_{\lambda,\xi})_{\lambda,\xi}$ is said to *$(\mathscr{U}\mathscr{F})$-forward strongly uniformly* (resp., *$(\mathscr{U}\mathscr{F})$-forward Arzelà*) *converge to* $(a_\xi)_\xi$ *on* Ξ iff it $(\mathscr{U}\mathscr{F})$-strongly uniformly (resp., $(\mathscr{U}\mathscr{F})$-Arzelà) converges to $(a_\xi)_\xi$ at ξ for every $\xi \in \Xi$ with respect to a single family $\mathbf{U} \in \mathscr{U}$, independent of ξ.

(e) One says that $(a_{\lambda,\xi})_{\lambda,\xi}$ is *$(\mathscr{U}\mathscr{F})$-forward Alexandroff convergent to* $(a_\xi)_\xi$ *on* Ξ (shortly, $a_{\lambda,\xi} \xrightarrow{\mathscr{U}\mathscr{F}\ fw-Al.} a_\xi$ on Ξ) iff there exists a family $(U_\pi)_\pi \in \mathscr{U}$ such that for each $\pi \in \Pi$ and $F \in \mathscr{F}$ there are a nonempty set $\Lambda_0 \subset F$ and a finitely uniform cover $\{V_\lambda : \lambda \in \Lambda_0\}$ of Ξ with $\rho(a_\zeta, a_{\lambda,\zeta}) \in U_\pi$ for any $\lambda \in \Lambda_0$ and $\zeta \in V_\lambda$.

Note that, analogously as above, it is possible to formulate the corresponding concepts of (backward) filter strong uniform, Arzelà, and Alexandroff convergence.

The next result extends [2, Theorem 3.9], [4, Theorems 2.9, 2.11 and Corollary 2.10], and [9, Proposition 3.5].

Theorem 9. *Let $\xi \in \Xi$ be fixed, let $\mathscr{S}_\xi$ be a filter of Ξ, and suppose that*

(3.6.1) $(\Lambda\mathscr{U}\mathscr{S}_\xi)\lim_{\zeta\to\xi} a_{\lambda,\zeta} = a_{\lambda,\xi}$;

(3.6.2) *the family $(a_{\lambda,\zeta})_{\lambda,\zeta}$ $(\Xi\mathscr{U}\mathscr{F})$-converges to $(a_\zeta)_\zeta$.*

Then the following are equivalent:

(i) *$(a_\zeta)_\zeta$ $(\mathscr{U}\mathscr{S}_\xi)$-backward converges to a_ξ as $\zeta \to \xi$.*

(ii) $a_{\lambda,\xi} \xrightarrow{\mathscr{U}\mathscr{F}\ fw-\mathscr{T}^s} a_\xi$ *at* ξ.

(iii) $a_{\lambda,\xi} \xrightarrow{\mathscr{U}\mathscr{F}\ fw-Arz.} a_\xi$ *at* ξ.

Proof. (i) $\Rightarrow$ (ii) Let $(U_\pi)_\pi$, $(Y_\pi)_\pi$, and $(U^*_\pi)_\pi \in \mathscr{U}$ be three families associated with (i), (3.6.1), and (3.6.2), respectively, and take arbitrarily $\pi \in \Pi$. By (3.6.2), there is $F \in \mathscr{F}$ with $\rho(a_\xi, a_{\lambda,\xi}) \in U^*_\pi$ for all $\lambda \in F$. By (3.6.1) and (i), for each $\lambda \in F$ there is $S_\lambda \in \mathscr{S}_\xi$ with $\rho(a_{\lambda,\xi}, a_{\lambda,\zeta}) \in Y_\pi$ and $\rho(a_\zeta, a_\xi) \in U_\pi$ for any $\zeta \in S_\lambda$. For such ζ's, taking into account $(\mathscr{H}1)$, we have $\rho(a_\zeta, a_{\lambda,\zeta}) \in Y_\pi + U^*_\pi + U_\pi$, getting (ii).

(ii) $\Rightarrow$ (iii) Let $(U_\pi)_\pi \in \mathscr{U}$ be a family, according to $(\mathscr{U}\mathscr{F})$-strong uniform convergence. Choose arbitrarily $\pi \in \Pi$ and $F \in \mathscr{F}$, and let $F_0 \in \mathscr{F}$ be associated with $(\mathscr{U}\mathscr{F})$-strong uniform convergence. Pick any finite set $W := \{\lambda_1, \ldots, \lambda_q\} \subset F \cap F_0 \in \mathscr{F}$: since $\mathscr{F}$ is a filter, W does exist. For every $j \in [1,q]$, let $S_{\lambda_j} \in \mathscr{S}_\xi$ be related to $(\mathscr{U}\mathscr{F})$-strong uniform convergence, and set $S = \bigcap_{j=1}^q S_{\lambda_j}$. Note that $S \in \mathscr{S}_\xi$. By construction, for each $\zeta \in S$ and $j \in [1,q]$, we get $\rho(a_\zeta, a_{\lambda_j,\zeta}) \in U_\pi$. Thus, we obtain (iii).

(iii) $\Rightarrow$ (i) Let $(U_\pi)_\pi$, $(Y_\pi)_\pi$, and $(Z_\pi)_\pi \in \mathscr{U}$ be families related to (iii), (3.6.1), and (3.6.2), respectively. By (3.6.2), there is a set $F \in \mathscr{F}$ with

$$\rho\left(a_{\lambda,\xi}, a_\xi\right) \in Z_\pi \quad \forall \lambda \in F. \tag{10}$$

Choose arbitrarily $\pi \in \Pi$. By (iii), in correspondence with π and F, there exist a finite set $\{\lambda_1, \lambda_2, \ldots, \lambda_q\} \subset F$ and a set $S \in \mathcal{S}_\xi$ such that for every $\zeta \in S_\xi$ there is $j \in [1, q]$ with

$$\rho\left(a_\zeta, a_{\lambda_j,\zeta}\right) \in U_\pi. \tag{11}$$

Thanks to (3.6.1), we find a set $W \in \mathcal{S}_\xi$, without loss of generality $W \subset S$, with

$$\rho\left(a_{\lambda_j,\zeta}, a_{\lambda_j,\xi}\right) \in Y_\pi \tag{12}$$

for each $\zeta \in W$. From (10), (11), (12), and ($\mathscr{H}1$) it follows that $\rho(a_\zeta, a_\xi) \in U_\pi + Y_\pi + Z_\pi$ for every $\zeta \in W_\xi$, getting (i). □

Remark 10. (a) In general, Theorem 9 does not hold, when the involved "forward" convergences are replaced by the corresponding "backward" ones. Indeed, for example, let $\Lambda := \mathbb{N}$, $\mathscr{F}$ be any filter of $\mathbb{N}$, let $\Xi = [0, 1]$ be endowed with the usual metric, $\xi = 1$, and let $\mathcal{S}_\xi$ be the filter of all neighborhoods of 1 contained in $[0, 1]$, $R = Y = \mathbb{R}$, $\mathscr{U} := \{(\{\zeta \in \mathbb{R} : \zeta \leq \varepsilon u\})_{\varepsilon\in\mathbb{R}^+} : u \in \mathbb{R}^+\}$, $\rho(a, b) = b - a$, $a, b \in \mathbb{R}$. Put $a_{n,\zeta} := \zeta^n$, $n \in \mathbb{N}$, $\zeta \in [0, 1]$. We get $\lim_{\zeta\to\xi} a_{n,\zeta} = 1$ for every $n \in \mathbb{N}$, and

$$a_\zeta := \lim_n a_{n,\zeta} = \begin{cases} 0, & \text{if } 0 \leq \zeta < 1, \\ 1, & \text{if } \zeta = 1. \end{cases} \tag{13}$$

Note that for each $\varepsilon > 0$ and $n \in \mathbb{N}$ we get

$$\rho\left(a_{n,\zeta}, a_\zeta\right) = a_\zeta - a_{n,\zeta} = \begin{cases} -\zeta^n < \varepsilon, & \text{if } 0 \leq \zeta < 1, \\ 0 < \varepsilon, & \text{if } \zeta = 1. \end{cases} \tag{14}$$

Hence, $a_{n,\xi} \xrightarrow{\mathscr{U}\mathscr{F}\ bw-\mathscr{T}^s} a_\xi$ at ξ. On the other hand, for every $n \in \mathbb{N}$ and for each neighborhood S of 1 contained in $[0, 1]$ there is a real number $\zeta \in S \cap]0, 1[$, close enough to 1, with $\zeta > 1/2^{1/n}$, and hence

$$\rho\left(a_\zeta, a_{n,\zeta}\right) = a_{n,\zeta} - a_\zeta = \zeta^n > \frac{1}{2}. \tag{15}$$

Thus, $a_{n,\xi} \overset{\mathscr{U}\mathscr{F}\ fw-\mathscr{T}^s}{\not\longrightarrow} a_\xi$ at ξ. The family $(a_\zeta)_\zeta$ $(\mathscr{U}\mathcal{S}_\xi)$-forward, but not backward, converges to a_ξ as $\zeta \to \xi$: indeed for every $\zeta \in [0, 1[$ we have $\rho(a_\xi, a_\zeta) = a_\zeta - a_\xi = -1 < \varepsilon$ for each $\varepsilon > 0$, but $\rho(a_\zeta, a_\xi) = a_\xi - a_\zeta = 1$. Note that the function $\zeta \mapsto a_\zeta$, $\zeta \in [0, 1]$, is upper semicontinuous, but not lower semicontinuous, at 1.

(b) Observe that Theorem 9 does not hold, where in (3.6.1) the involved convergence is replaced by the corresponding backward or forward convergence (see also [2, Example 3.3]).

Let $\Lambda, \mathscr{F}, R, Y, \mathscr{U}$, and ρ be as in (a), let $\Xi := \mathbb{R}$ be endowed with the usual metric, $\xi = 0$, and let $\mathcal{S}_\xi$ be the filter of all neighborhoods of 0. Set

$$a_{n,\zeta} := \begin{cases} 0, & \text{if } \zeta \in \left]-\infty, -\dfrac{1}{n}\right] \cup \{0\} \cup \left[\dfrac{1}{n}, +\infty\right[, \\ 1, & \text{otherwise.} \end{cases} \tag{16}$$

Observe that $a_\zeta := \lim_n a_{n,\zeta} = 0$ for every $\zeta \in \mathbb{R}$, so that (3.6.2) holds, and condition (i) of Theorem 9 is fulfilled. Moreover it is not difficult to see that, for each $n \in \mathbb{N}$, $a_{n,\zeta}$ converges backward, but not forward, to $a_{n,0} = 0$ as ζ tends to 0, and hence (3.6.1) is not verified. However, note that for every $n \in \mathbb{N}$ and for every neighborhood U of 0 there is $\zeta \in U$ with $a_{n,\zeta} = 1$, and hence $\rho(a_\zeta, a_{n,\zeta}) = a_{n,\zeta} - a_\zeta = 1$. Thus, condition (ii) of Theorem 9 is not satisfied.

Furthermore, if we define $b_{n,\zeta}$, $n \in \mathbb{N}$, $\zeta \in \mathbb{R}$, by

$$b_{n,\zeta} := \begin{cases} 1, & \text{if } \zeta \in \left]-\infty, -\dfrac{1}{n}\right] \cup \left[\dfrac{1}{n}, +\infty\right[, \\ 2, & \text{if } \zeta = 0, \\ 0, & \text{otherwise,} \end{cases} \tag{17}$$

then

$$b_\zeta := \lim_n b_{n,\zeta} = \begin{cases} 1, & \text{if } \zeta \neq 0, \\ 2, & \text{if } \zeta = 0. \end{cases} \tag{18}$$

Hence, (3.6.2) is satisfied, but condition (i) of Theorem 9 does not hold. Observe that, for any $n \in \mathbb{N}$, $b_{n,\zeta}$ converges forward, but not backward, to $b_{n,0} = 2$ as ζ tends to 0, and hence (3.6.1) is not satisfied. On the other hand, since $\rho(b_\zeta, b_{n,\zeta}) = b_{n,\zeta} - b_\zeta \leq 0$ for any $n \in \mathbb{N}$ and $\zeta \in \mathbb{R}$, we get that condition (ii) of Theorem 9 is fulfilled.

We now turn to the main theorem in our abstract setting, which extends [1, Theorems 4.7, 4.11], [2, Theorem 3.10], [4, Theorem 2.12], and [6, Corollary 3.5] to our abstract unified setting.

Theorem 11. *Let $\mathcal{S}_\xi$, $\xi \in \Xi$, be a family of filters of Ξ, with the property that $\xi \in S$ for every $\xi \in \Xi$ and $S \in \mathcal{S}_\xi$. Suppose that (3.6.2) holds and that*

(3.8.1) $(\Lambda\mathscr{U}\mathcal{S}_\xi)\lim_{\zeta\to\xi} a_{\lambda,\zeta} = a_{\lambda,\xi}$ *for each $\xi \in \Xi$ with respect to a single family $\mathbf{Y} \in \mathscr{U}$, independent of both λ and ξ.*

Then the following are equivalent:

(i) $(a_\zeta)_\zeta(\mathscr{U}\mathcal{S}_\xi)$*-backward converges to a_ξ as $\zeta \to \xi$ for every $\xi \in \Xi$, with respect to a single family $\mathbf{U} \in \mathscr{U}$, independent of ξ.*

(ii) $a_{\lambda,\xi} \xrightarrow{\mathscr{U}\mathscr{F}\ fw-\mathscr{T}^s} a_\xi$ *on Ξ.*

(iii) $a_{\lambda,\xi} \xrightarrow{\mathscr{U}\mathscr{F}\ fw-Al.} a_\xi$ *on Ξ.*

(iv) $a_{\lambda,\xi} \xrightarrow{\mathscr{U}\mathscr{F}\ fw-Arz.} a_\xi$ *on Ξ.*

(v) $(a_{\lambda,\xi})_{\lambda,\xi}$ *is weakly $(\mathscr{U}\mathscr{F})$-backward exhaustive on Ξ.*

Proof. (i) ⇔ (v) It is similar to Theorem 6.

(i) ⇔ (ii) ⇔ (iv) It is similar to Theorem 9.

(ii) ⇒ (iii) Let $(W_\pi)_\pi \in \mathscr{U}$ be a family associated with $\mathscr{U}\mathscr{F}$-$\mathscr{T}^s$-convergence of $(a_{\lambda,\xi})_{\lambda,\xi}$ to $(a_\xi)_\xi$. Choose arbitrarily $\pi \in \Pi$ and $F_0 \in \mathscr{F}$. By (ii), for every $\xi \in \Xi$, there exists a set $F_\xi \in \mathscr{F}$, such that for every $\lambda \in F_\xi$ there is $S_{\lambda,\xi} \in \mathcal{S}_\xi$ with

$\rho(a_\zeta, a_{\lambda,\zeta}) \in W_\pi$ for any $\zeta \in S_{\lambda,\xi}$. Set $F = \bigcup_{\xi\in\Xi} F_\xi$: note that $\Lambda_0 \in \mathcal{F}$, where $\Lambda_0 := F \cap F_0 \neq \emptyset$. For every $\lambda \in \Lambda_0$, let

$$E_\lambda := \left\{\xi \in \Xi : \rho\left(a_\zeta, a_{\lambda,\zeta}\right) \in W_\pi \text{ for every } \zeta \in S_{\lambda,\xi}\right\}. \quad (19)$$

Pick arbitrarily $\xi \in \Xi$ and choose $\lambda \in \Lambda_0$. We have $\rho(a_\zeta, a_{\lambda,\zeta}) \in W_\pi$ whenever $\zeta \in S_{\lambda,\xi}$. Thus, $\Xi = \bigcup_{\lambda\in\Lambda_0} E_\lambda$. For each $\lambda \in \Lambda_0$, set $S_\lambda = \bigcup_{\xi\in E_\lambda} S_{\lambda,\xi}$. Note that $\mathcal{V} := \{S_\lambda : \lambda \in \Lambda_0\}$ is a cover of Ξ. For every $\lambda \in \Lambda_0$ and $\zeta \in S_\lambda$ there is $\xi \in E_\lambda$ with $\zeta \in S_{\lambda,\xi}$, and hence $\rho(a_\zeta, a_{\lambda,\zeta}) \in W_\pi$. Now, in correspondence with $\xi \in \Xi$, choose an element $\lambda \in F_\xi \cap F_0$ and pick $S_{\lambda,\xi}$. Note that $S_{\lambda,\xi} \in \mathcal{S}_\xi$ and $S_{\lambda,\xi} \subset S_\lambda$. Thus, $\mathcal{V}$ is a finitely uniform cover of Ξ, with $S_\xi = S_{\lambda,\xi}$ and $\mathcal{Y} = \{S_\lambda\}$. Therefore, $(a_{\lambda,\xi})_{\lambda,\xi}$ $(\mathcal{U}\mathcal{F})$-Alexandroff converges to $(a_\xi)_\xi$.

(iii) $\Rightarrow$ (iv) Let $(U_\pi)_{\pi\in\Pi} \in \mathcal{U}$ be a family associated with $(\mathcal{U}\mathcal{F})$-Alexandroff convergence of $(a_{\lambda,\xi})$ to $(a_\xi)_\xi$. Pick arbitrarily $\xi \in \Xi, \pi \in \Pi$, and $F \in \mathcal{F}$. By (iii), there are a nonempty set $\Lambda_0 \subset F$ and a finitely uniform cover $\mathcal{V} := \{V_\lambda : \lambda \in \Lambda_0\}$ of Ξ, with $\rho(a_\zeta, a_{\lambda,\zeta}) \in U_\pi$ for each $\lambda \in \Lambda_0$ and $\xi \in V_\lambda$. Since $\mathcal{V}$ is a finitely uniform cover, in correspondence with ξ, there exist a set $S_\xi \in \mathcal{S}_\xi$ and a finite subset $\{\lambda_1, \lambda_2, \dots, \lambda_q\} \subset \Lambda_0$, such that for every $\zeta \in S_\xi$ there is $j \in [1, q]$ with $\zeta \in V_{\lambda_j}$. Thus $\rho(a_\zeta, a_{\lambda_j,\zeta}) \in U_\pi$, and so we obtain (iv). This ends the proof. □

Remark 12. Observe that when the function ρ is symmetric, Theorems 6, 9, and 11 can be viewed as necessary and sufficient conditions in order to have exchange of limits (for a related literature, see also [16, 26, 33]).

4. Applications to Set Functions

In this section, as consequences of Theorems 6, 9, and 11, we will give some necessary and sufficient conditions for some kind of continuity and semicontinuity of the limit of set functions. We begin with proving a result on continuity from below the limit measure. Note that, thanks to the limit theorems existing in the literature, these conditions are often fulfilled (for a comprehensive historical survey, see [16] and its bibliography). However, we give an example in which these properties do not hold in the setting of filter convergence.

Let Λ be any nonempty set, let $\mathcal{F}$ be any filter of Λ, let G be any infinite set, let Σ be σ-algebra of subsets of G, and let (R, Y) be a (symmetric) cone metric semigroup, $\Xi := \mathbb{N} \cup \{+\infty\}$, $\xi := +\infty$, $\mathcal{S}_\xi := \{F \cup \{+\infty\} : F \in \mathcal{F}_{\text{cofin}}\}$. It is not difficult to check that $\mathcal{S}_\xi$ is a filter of Ξ. Moreover, let $\mathcal{U}$ be a fixed Π-system associated with (R, Y).

A set function $m : \Sigma \to R$ is said to be *$\mathcal{U}$-continuous from below* (resp., *from above*) on Σ iff $(\mathcal{U}\mathcal{S}_\xi)\lim_k \rho(m(C_k), m(C)) = 0$ for every increasing (resp., decreasing) sequence $(C_k)_k$ in Σ whose union (resp., intersection) is equal to C. A consequence of Theorems 6 and 9 is the following.

Theorem 13. *Let $m_\lambda : \Sigma \to R$, $\lambda \in \Lambda$, be a family of set functions, $\mathcal{U}$-continuous from below on Σ, with respect to a family $\mathbf{U} \in \mathcal{U}$ independent of λ. Suppose that*

(4.1.1) *$m(E) := (\mathcal{U}\mathcal{F})\lim_\lambda m_\lambda(E)$, $E \in \Sigma$, exists in R with respect to a family $\mathbf{V} \in \mathcal{U}$ independent of $E \in \Sigma$.*

Then the following are equivalent:

(i) *m is $\mathcal{U}$-continuous from below on Σ.*

(ii) *For every increasing sequence $(C_k)_k$ in Σ there is a family $(W_\pi)_\pi \in \mathcal{U}$ such that for any $\pi \in \Pi$ there is $\overline{k} \in \mathbb{N}$ such that, for every $k \geq \overline{k}$, there is a set $F \in \mathcal{F}$ with $\rho(m_\lambda(C_k), m_\lambda(C)) \in W_\pi$ for each $\lambda \in F$.*

(iii) *For any increasing sequence $(C_k)_k$ in Σ there is a family $(U_\pi)_\pi \in \mathcal{U}$ such that for every $\pi \in \Pi$ there is $F \in \mathcal{F}$ such that for each $\lambda \in F$ there exists a positive integer k_λ with $\rho(m(C_k), (m_\lambda(C_k))$ for any $k \geq k_\lambda$.*

(iv) *For every increasing sequence $(C_k)_k$ in Σ there is a family $(Y_\pi)_\pi \in \mathcal{U}$ such that for each $\pi \in \Pi$ and $F \in \mathcal{F}$ there are $\lambda_1, \dots, \lambda_q \in F$ and $\overline{k} \in \mathbb{N}$ such that for each $k \geq \overline{k}$ there exists $j \in [1, q]$ with $\rho(m(C_k), m_{\lambda_j}(C_k)) \in Y_\pi$.*

Indeed, it is enough to take

$$\begin{aligned} a_{\lambda,\zeta} &= m_\lambda(C_k), \\ a_\zeta &= m(C_k), \\ a_{\lambda,\xi} &= m_\lambda(C), \\ a_\xi &= m(C), \end{aligned} \quad (20)$$

where $(C_k)_k$ is a fixed increasing sequence in Σ, whose union is C. Conditions (i) of Theorem 6 and (ii) and (iii) of Theorem 9 become conditions (ii), (iii), and (iv) of Theorem 13, respectively.

Remark 14. (a) Observe that results analogous to Theorem 13 hold when the involved set functions m_λ, $\lambda \in \Lambda$, are $\mathcal{U}$-continuous from above or *$\mathcal{U}$-(s)-bounded* on Σ, that is, if $(\mathcal{U})\lim_k \rho(m_\lambda(A_k), 0) = 0$ for every disjoint sequence $(A_k)_k$ in Σ.

(b) Note that conditions (ii)–(iv) of Theorem 13 are just satisfied, for example, when $R = Y$ is a Dedekind complete lattice group, $\rho(a, b) = |a - b|$, $a, b \in \mathbb{R}$, $\Lambda = \mathbb{N}$, $\mathcal{F} = \mathcal{F}_{\text{cofin}}$, and $(m_n)_n$ is a sequence of σ-additive positive R-valued measures, thanks to the classical limit theorems (see also [16, 34, 35]).

The next step is to give necessary and sufficient conditions for absolute continuity of the limit measure.

Let $\nu : \Sigma \to \mathbb{R}_0^+$ be a finitely additive measure. We endow Σ with the Fréchet-Nikodým topology generated by the pseudometric $\rho_\nu(D, E) := |\nu(D) - \nu(E)|$, $D, E \in \Sigma$. Pick now $\Xi = \Sigma$, and for each $E \in \Sigma$ let $\mathcal{S}_E$ be the filter generated by the base $\mathcal{W} := \{\{D \in \Sigma : \rho_\nu(D, E) < \eta\} : \eta > 0\}$.

We say that $(m_\lambda)_\lambda$ is *weakly $(\mathcal{U}\mathcal{F})$-ν-exhaustive at $E \in \Sigma$* iff there is a family $(U_\pi)_\pi \in \mathcal{U}$ (depending on E) such that for each $\pi \in \Pi$ there is $\eta > 0$ such that for every $D \in \Sigma$ with $\rho_\nu(D, E) < \eta$ there is a set $F_D \in \mathcal{F}$ with $\rho(m_\lambda(D), m_\lambda(E)) \in U_\pi$ whenever $\lambda \in F$. We say that $(m_\lambda)_\lambda$ is *weakly $(\mathcal{U}\mathcal{F})$-ν-exhaustive on* Σ iff it is weakly $(\mathcal{U}\mathcal{F})$-ν-exhaustive at every $E \in \Sigma$ with respect to a family $\mathbf{X} \in \mathcal{U}$ independent of $E \in \Sigma$.

A measure $m : \Sigma \to R$ is said to be *$\mathscr{U}$-ν-continuous at* $E \in \Sigma$ iff there is a family $(U_\pi)_\pi \in \mathscr{U}$ (depending on E) such that for every $\pi \in \Pi$ there is $\eta > 0$ with $\rho(m(D), m(E)) \in U_\pi$ whenever $\rho_\nu(D, E) < \eta$. We say that m is *globally $\mathscr{U}$-ν-continuous on* Σ with respect to ν iff it is $\mathscr{U}$-ν-continuous at E with respect to ν for each $E \in \Sigma$, relative to a family $\mathbf{T} \in \mathscr{U}$, independent of $E \in \Sigma$.

The next result is a consequence of Theorem 11.

Theorem 15. *Let $m_\lambda : \Sigma \to R$, $\lambda \in \Lambda$, be a family of measures $\mathscr{U}$-ν-continuous at a fixed set $E \in \Sigma$ (resp., globally $\mathscr{U}$-ν-continuous on Σ) with respect to a family $\mathbf{Z} \in \mathscr{U}$ independent of λ and $(\Xi\mathscr{U}\mathscr{F})$-convergent to a measure $m_0 : \Sigma \to R$. Then the following are equivalent:*

(i) *The limit measure m_0 is $\mathscr{U}$-ν-continuous at E (resp., globally $\mathscr{U}$-ν-continuous on Σ).*

(ii) *The net m_λ, $\lambda \in \Lambda$, is weakly $(\mathscr{U}\mathscr{F})$-exhaustive at E (resp., on Σ).*

(iii) *There is a family $(U_\pi)_\pi \in \mathscr{U}$, depending on $E \in \Sigma$ (resp., independent of $E \in \Sigma$), such that for each $\pi \in \Pi$ there is $F \in \mathscr{F}$ such that for every $\lambda \in F$ there is $\eta > 0$ with $\rho(m_0(D), m_\lambda(D)) \in U_\pi$ for each $D \in \Sigma$ with $\rho_\nu(D, E) < \eta$.*

(iv) *There is a family $(Y_\pi)_\pi \in \mathscr{U}$, depending on $E \in \Sigma$ (resp., independent of $E \in \Sigma$), such that for every $\pi \in \mathbb{N}$ and $F \in \mathscr{F}$ there are $\lambda_1, \lambda_2, \ldots, \lambda_q \in F$ and a positive real number η such that for any $D \in \Sigma$ with $\rho_\nu(D, E) < \eta$ there exists $j \in [1, q]$ with $\rho(m_0(D), m_{\lambda_j}(D)) \in Y_\pi$.*

Moreover, if m_λ's are globally $\mathscr{U}$-ν-continuous, statements (i)–(iv) are equivalent to the following:

(v) *There is a family $(W_\pi)_\pi \in \mathscr{U}$ such that for any $\pi \in \Pi$ and $F \in \mathscr{F}$ there exist a nonempty set $\Lambda_0 \subset F$ and a finitely uniform cover $\{V_\lambda : \lambda \in \Lambda_0\}$ of Σ with $\rho(m_0(D), m_\lambda(D)) \in W_\pi$ whenever $\lambda \in \Lambda_0$ and $D \in V_\lambda$.*

Remark 16. (a) Observe that when $\Lambda = \mathbb{N}$, $\mathscr{F} = \mathscr{F}_{\text{cofin}}$, m_n, $n \in \mathbb{N}$, are positive σ-additive measures, R is a Dedekind complete lattice group, and $Y = R$, $\rho(a, b) = |b - a|$, $a, b \in R$, we get that conditions (ii)–(v) of Theorem 15 are fulfilled, thanks to the limit theorems existing in the literature (see also [16, 34, 35]).

(b) Let $\Sigma = \mathscr{P}(\mathbb{N})$ be the class of all subsets of $\mathbb{N}$; let $\mathscr{F}$ be a filter containing $\mathscr{F}_{\text{cofin}}$ and $\nu(A) = \sum_{k \in A}(1/2^k)$, $A \in \Sigma$. For each $n \in \mathbb{N}$, let us define the Dirac measure $\delta_n : \Sigma \to \mathbb{R}$ by

$$\delta_n(A) := \begin{cases} 1, & \text{if } n \in A, \\ 0, & \text{if } n \in \mathbb{N} \setminus A. \end{cases} \tag{21}$$

It is not difficult to see that δ_n is σ-additive on Σ. Moreover, δ_n is ν-continuous at $\emptyset$ (i.e., ν-absolutely continuous): indeed, if $\vartheta_n = 1/2^n$ and $\nu(A) < \vartheta_n$, then $n \notin A$, and hence $\delta_n(A) = 0$. We claim that the sequence $(\delta_n)_n$ is not weakly $\mathscr{F}$-exhaustive at $\emptyset$. Indeed, observe that for each $\vartheta > 0$ there is a cofinite set $D_\vartheta \subset \mathbb{N}$ with $\nu(D_\vartheta) < \vartheta$. Note that since $\mathscr{F}$ contains $\mathscr{F}_{\text{cofin}}$, every element of $\mathscr{F}$ is infinite; otherwise $\emptyset \in \mathscr{F}$, which is impossible. Furthermore, observe that for every infinite subset $F \subset \mathbb{N}$, and a fortiori for any $F \in \mathscr{F}$, there is a sufficiently large integer $\overline{n} \in F \cap D_\vartheta$, so that $\delta_{\overline{n}}(D_\vartheta) = 1$. From this we deduce that the sequence $(\delta_n)_n$ is not weakly $\mathscr{F}$-exhaustive at $\emptyset$. If $\mathscr{F}$ is an ultrafilter of $\mathbb{N}$ containing $\mathscr{F}_{\text{cofin}}$ (the existence of such ultrafilters follows from the Axiom of Choice; see also [19, 36]), then for every $A \subset \mathbb{N}$ we have

$$\delta'(A) := (\mathscr{F}) \lim_n \delta_n(A) = \begin{cases} 1, & \text{if } A \in \mathscr{F}, \\ 0, & \text{if } A \notin \mathscr{F}. \end{cases} \tag{22}$$

We claim that δ' is not ν-continuous at $\emptyset$. Indeed, fix arbitrarily $\eta > 0$ and let $\overline{k} \in \mathbb{N}$ be such that $1/2^{\overline{k}-1} \leq \eta$. Let A be any element of $\mathscr{F}$ and set $A^* := A \cap ([\overline{k}, +\infty[)$; then $A^* \in \mathscr{F}$. We get $\nu(A^*) \leq \sum_{k=\overline{k}}^{\infty}(1/2^k) = 1/2^{\overline{k}-1} \leq \eta$ and $\delta'(A^*) = 1$, getting the claim.

Furthermore, in this case, conditions (i)–(iv) in Theorem 13 do not hold. Indeed, choose a filter $\mathscr{F}$ of $\mathbb{N}$ containing $\mathscr{F}_{\text{cofin}}$, and let $C_k := [1, k]$, $k \in \mathbb{N}$. Observe that, as said before, every element of $\mathscr{F}$ is infinite. For every k and for any infinite set $F \subset \mathbb{N}$ there is $\overline{n} \in F \setminus C_k$, and hence we get $\delta_{\overline{n}}(\mathbb{N}) - \delta_{\overline{n}}(C_k) = 1$. Thus, in this case, condition (ii) of Theorem 13 is not fulfilled. If $\mathscr{F}$ is an ultrafilter of $\mathbb{N}$, then the measure δ' defined in (22) is not σ-additive on Σ. Indeed, if A is any element of $\mathscr{F}$, then we get $\sum_{n \in A}\delta'(\{n\}) = 0$ and $\delta'(A) = 1$.

When R is a Dedekind complete lattice group, $Y = R$, $\rho(a, b) = b - a$, and $\mathscr{U}_j^{(0)}$, $j = 1, 2, 3$, are as in Example 3(c); we obtain some results similar to the previous ones also for semicontinuous set functions (for a related literature, see also [20] and the references therein).

In this setting, the concepts of weak backward (resp., forward) filter exhaustiveness and lower (resp., upper) semicontinuity are formulated as follows.

Definition 17. (a) One says that $(m_\lambda)_\lambda$ is *weakly $(\mathscr{U}\mathscr{F})$-ν-backward* (resp., *forward*) *exhaustive at* $E \in \Sigma$ iff there is a family $(U_\pi)_\pi \in \mathscr{U}$ (depending on E) such that for each $\pi \in \Pi$ there is $\eta > 0$ such that for every $D \in \Sigma$ with $\rho_\nu(D, E) < \eta$ there is a set $F_D \in \mathscr{F}$ with $m_\lambda(E) - m_\lambda(D)$ (resp., $m_\lambda(D) - m_\lambda(E)$) $\in U_\pi$ whenever $\lambda \in F$.

(b) One says that $(m_\lambda)_\lambda$ is *weakly $(\mathscr{U}\mathscr{F})$-ν-backward* (resp., *forward*) *exhaustive on* Σ iff it is weakly $(\mathscr{U}\mathscr{F})$-ν-backward (resp., forward) exhaustive at every $E \in \Sigma$ with respect to a family $\mathbf{X} \in \mathscr{U}$ independent of $E \in \Sigma$.

(c) One says that $(m_\lambda)_\lambda$ is *weakly $(\mathscr{U}\mathscr{F})$-ν-exhaustive at* E (resp., *on* Σ) iff it is weakly $(\mathscr{U}\mathscr{F})$-ν-backward and forward exhaustive at E (resp., on Σ).

(d) One says that $m : \Sigma \to R$ is *$\mathscr{U}$-ν-lower* (resp., *upper*) *semicontinuous at* $E \in \Sigma$ iff there is a family $(U_\pi)_\pi \in \mathscr{U}$ (depending on E) such that for every $\pi \in \Pi$ there is $\eta > 0$ with $m(E) - m(D)$ (resp., $m(D) - m(E)$) $\in U_\pi$ whenever $\rho_\nu(D, E) < \eta$. We say that m is *globally $\mathscr{U}$-ν-lower* (resp., *upper*) *semicontinuous on* Σ iff it is $\mathscr{U}$-ν-lower (resp., upper) semicontinuous at E for each $E \in \Sigma$ with respect to a family $\mathbf{T} \in \mathscr{U}$, independent of $E \in \Sigma$.

Similarly as Theorem 15, it is possible to prove the following result about semicontinuity of the limit set function. The next theorem is given in the case of lower semicontinuity; an analogous result holds in the setting of upper semicontinuity.

Theorem 18. *Suppose that $m_\lambda : \Sigma \to R$, $\lambda \in \Lambda$, are globally $\mathscr{U}$-ν-continuous on Σ with respect to a family $\mathbf{S} \in \mathscr{U}$, independent of λ, and $(\Sigma\mathscr{U}\mathscr{F})$-convergent to a set function $m_0 : \Sigma \to R$. Then the following are equivalent:*

(i) *m_0 is $\mathscr{U}$-ν-lower semicontinuous at E (resp., globally $\mathscr{U}$-ν-lower semicontinuous on Σ).*

(ii) *The family $(m_\lambda)_\lambda$ is weakly $(\mathscr{U}\mathscr{F})$-backward exhaustive at E (resp., on Σ).*

(iii) *There is a family $(U_\pi)_\pi \in \mathscr{U}$, depending on E (resp., independent of E), such that for any $\pi \in \Pi$ there is $F \in \mathscr{F}$ such that for every $\lambda \in F$ there is $\eta > 0$ with $m_\lambda(D) - m_0(D) \in U_\pi$ for each $D \in \Sigma$ with $\rho_\nu(D, E) < \eta$.*

(iv) *There exists a family $(V_\pi)_\pi \in \mathscr{U}$, depending on E (resp., independent of E), such that for every $\pi \in \Pi$ and $F \in \mathscr{F}$ there are $\lambda_1, \lambda_2, \ldots, \lambda_q \in F$ and $\eta > 0$ such that for any $D \in \Sigma$ with $\rho_\nu(D, E) < \eta$ there exists $j \in [1, q]$ with $m_{\lambda_j}(D) - m_0(D) \in V_\pi$.*

Moreover, global $\mathscr{U}$-ν-lower semicontinuity of m_0 is equivalent to the following condition:

(v) *There is a family $(W_\pi)_\pi \in \mathscr{U}$, such that for every $\pi \in \Pi$ and $F \in \mathscr{F}$ there exist a nonempty set $\Lambda_0 \subset F$ and a finitely uniform cover $\{V_\lambda : \lambda \in \Lambda_0\}$ of Σ with $m_\lambda(D) - m_0(D) \in W_\pi$ for all $\lambda \in \Lambda_0$ and $D \in V_\lambda$.*

Remark 19. (a) Let $\mathscr{F}$ be an ultrafilter of $\mathbb{N}$ containing $\mathscr{F}_{\text{cofin}}$, let ν be as in Remark 16(b) and let δ', δ_n, $n \in \mathbb{N}$, be as in (22) and (21), respectively. It is not difficult to check that the sequence $(\delta_n)_n$ is weakly $(\mathscr{F})$-ν-backward exhaustive but not weakly $(\mathscr{F})$-ν-forward exhaustive at $\emptyset$ and that δ' is ν-lower semicontinuous but not ν-upper semicontinuous at $\emptyset$.

(b) With the same techniques as above, it is possible to prove similar results even when the involved set Ξ is endowed with a bornology, extending earlier theorems proved in [1, 6]. A *bornology* on Ξ is a family $\mathscr{B}$ of nonempty subsets of Ξ, which covers Ξ, stable under finite unions and with $B' \in \mathscr{B}$ whenever $\emptyset \neq B' \subset B$ and $B \in \mathscr{B}$. Examples of bornologies on Ξ are the classes of all finite nonempty subsets and of all nonempty subsets of Ξ, the collection of all nonempty subsets of Ξ with compact closure when Ξ is a topological space, and, if (Ξ, ρ) is a metric space, the families of all nonempty ρ-bounded subsets of Ξ and of all nonempty ρ-totally bounded subsets of Ξ (see also [1, 14, 15] and the literature therein).

5. Conclusions

We studied the problem of finding conditions for preserving continuity or semicontinuity of the limit family of a double-indexed family of elements of a set endowed with an abstract structure of convergence.

Our axiomatic approach includes symmetric and asymmetric distance functions, topological and lattice groups, cone (quasi)metric spaces, functions and measures, and nets and filters.

We proved some theorems on exchange of limits, giving some necessary and sufficient conditions in terms of weak filter exhaustiveness, Alexandroff, Arzelà, and strong uniform convergence. As a consequence, we proved some necessary and sufficient conditions for continuity from above/below and absolute continuity of the limit set function of a converging family. We showed that, different from the classical cases, these conditions are not always fulfilled.

Open Problems. (a) Prove some similar results in some other abstract contexts and with respect to other types of convergence.

(b) Investigate some other properties of continuous or semicontinuous functions/measures in abstract settings.

Competing Interests

The authors declare that they have no competing interests.

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Research Article

Dual L_p-Mixed Geominimal Surface Area and Related Inequalities

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The integral formula of dual L_p-geominimal surface area is given and the concept of dual L_p-geominimal surface area is extended to dual L_p-mixed geominimal surface area. Properties for the dual L_p-mixed geominimal surface areas are established. Some inequalities, such as analogues of Alexandrov-Fenchel inequalities, Blaschke-Santaló inequalities, and affine isoperimetric inequalities for dual L_p-mixed geominimal surface areas, are also obtained.

1. Introduction

The concept of geominimal surface area was introduced by Petty [1] about 40 years ago, and its L_p-extension was first introduced by Lutwak [2, 3]. They have been proved to be key ingredients in connecting affine differential geometry, relative differential geometry, and Minkowski geometry. The basic theory concerning geominimal surface area is developed, and a close connection is established between this theory and affine differential geometry in [1]. The L_p-geominimal surface area is now thought to be at the core of the rapidly developing L_p-Brunn-Minkowski theory. Hence, it receives a lot of attention and motivates extensions of some known inequalities for geominimal surface areas to L_p-geominimal surface areas. These new inequalities of L_p-type ($p > 1$) are stronger than their classical counterparts.

However, finding an integral expression for the L_p-geominimal surface area seems to be intractable. This also leads to a big obstacle on extending the L_p-geominimal surface area. Until more recently, Zhu et al. [4] provided an integral formula for L_p-geominimal surface area by p-Petty body and introduced L_p-mixed geominimal surface areas which extended the L_p-geominimal surface area. Thereout, they established some new L_p-affine isoperimetric inequalities.

Recently, Wang and Qi [5] introduced a concept of dual L_p-geominimal surface area, which is a dual concept for L_p-geominimal surface area and belongs to the dual L_p-Brunn-Minkowski theory for star bodies also developed by Lutwak (see [6, 7]). The dual L_p-Brunn-Minkowski theory for star bodies and a more extensive dual Orlicz-Brunn-Minkowski theory for star bodies received considerable attention (see, e.g., [8–21]), and they have been proved to be very powerful in solving many geometric problems, for instance, the Busemann-Petty problems (see, e.g., [6, 22–24]).

In this paper, we show that the infimum in the definition of dual L_p-geominimal surface area is a minimum and provide an integral formula for dual L_p-geominimal surface area by dual p-Petty body. Moreover, we define the dual L_p-mixed geominimal surface area and establish some new L_p-affine isoperimetric inequalities for it.

Our paper is organized as follows. In Section 2, we provide the necessary background, such as definitions and known results which will be needed. Section 3 includes the basic theory of dual L_p-geominimal surface area, such as theorem of existence and uniqueness for dual L_p-geominimal surface area, as well as the integral definition of dual L_p-geominimal surface area. In Section 4, we introduce the dual L_p-mixed geominimal surface area and prove some important properties, such as affine invariant properties. We also obtain analogues of Alexandrov-Fenchel inequalities, Blaschke-Santaló inequalities, and affine isoperimetric inequalities for dual L_p-mixed geominimal surface areas.

Finally, we investigate the dual ith L_p-mixed geominimal surface areas and obtain analogues of Blaschke-Santaló and affine isoperimetric inequalities in Section 5.

2. Preliminaries and Notations

Let $\mathscr{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in Euclidean space $\mathbb{R}^n$. For the set of convex bodies containing the origin in their interiors and the set of convex bodies whose centroids lie at the origin in $\mathbb{R}^n$, we write $\mathscr{K}_o^n$ and $\mathscr{K}_c^n$, respectively. Let $V(K)$ denote the n-dimensional volume of a body K, and let B denote the standard Euclidean unit ball in $\mathbb{R}^n$ and write $\omega_n = V(B)$ for its volume, and let S^{n-1} denote the unit sphere for B.

For $K \in \mathscr{K}_o^n$, its support function $h_K = h(K,\cdot) : \mathbb{R}^n \setminus \{o\} \to [0,\infty)$ is defined by $x \in \mathbb{R}^n \setminus \{o\}$, $h(K,x) = \max\{\langle x,y\rangle : y \in K\}$, where $\langle\cdot,\cdot\rangle$ is the standard inner product on $\mathbb{R}^n$. Associated with each $K \in \mathscr{K}_o^n$, one can uniquely define its polar body $K^* \in \mathscr{K}_o^n$ by $K^* = \{x \in \mathbb{R}^n : \langle x,y\rangle \le 1,\ \forall y \in K\}$. It is easily verified that $(K^*)^* = K$ if $K \in \mathscr{K}_o^n$.

For $K, L \in \mathscr{K}^n$ and $\alpha, \beta \ge 0$ (not both zero), the Minkowski linear combination $\alpha\cdot K + \beta\cdot L \in \mathscr{K}^n$ is defined by

$$h(\alpha\cdot K + \beta\cdot L,\cdot) = \alpha h(K,\cdot) + \beta h(L,\cdot). \tag{1}$$

The classical Brunn-Minkowski inequality (see [25]) states that for convex bodies $K, L \in \mathscr{K}^n$ and real $\alpha, \beta \ge 0$ (not both zero), the volume of the bodies and the volume of their Minkowski linear combination $\alpha\cdot K + \beta\cdot L \in \mathscr{K}^n$ are related by

$$V(\alpha\cdot K + \beta\cdot L)^{1/n} \ge \alpha V(K)^{1/n} + \beta V(L)^{1/n}, \tag{2}$$

with equality if and only if K and L are homothetic.

For real $p \ge 1$, $K, L \in \mathscr{K}_o^n$, and $\alpha, \beta \ge 0$ (not both zero), the Firey linear combination, $\alpha\cdot K +_p \beta\cdot L$, is defined by (see [26])

$$h\left(\alpha\cdot K +_p \beta\cdot L,\cdot\right)^p = \alpha h(K,\cdot)^p + \beta h(L,\cdot)^p. \tag{3}$$

For the Firey linear combination $\alpha\cdot K +_p \beta\cdot L$, Firey [26] also established the L_p-Brunn-Minkowski inequality (an inequality that is also known as the Brunn-Minkowski-Firey inequality, see [14]). If $p > 1$, $\alpha, \beta \ge 0$ (not both zero), and $K, L \in \mathscr{K}_o^n$, then

$$V\left(\alpha\cdot K +_p \beta\cdot L\right)^{p/n} \ge \alpha V(K)^{p/n} + \beta V(L)^{p/n}, \tag{4}$$

with equality if and only if K and L are dilates.

A set K in $\mathbb{R}^n$ is star-shaped at o if $o \in K$ and for each $x \in \mathbb{R}^n \setminus \{o\}$, the intersection $K \cap \{cx : c \ge 0\}$ is a (possibly degenerate) compact line segment. If $K \subset \mathbb{R}^n$ is star-shaped at the origin o, we define its radial function ρ_K for $x \in \mathbb{R}^n \setminus \{o\}$ by $\rho(K,x) = \max\{\lambda \ge 0 : \lambda x \in K\}$. If ρ_K is positive and continuous, then K is called a star body about the origin. $\mathscr{S}_o^n$ denotes the set of star bodies (about the origin) in $\mathbb{R}^n$. Two star bodies K and L are dilates of one another if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$. Note that $K \in \mathscr{S}_o^n$ can be uniquely determined by its radial function $\rho_K(\cdot)$ and vice versa. If $\alpha > 0$, we have

$$\begin{aligned}\rho_K(\alpha x) &= \alpha^{-1}\rho_K(x),\\ \rho_{\alpha K}(x) &= \alpha\rho_K(x).\end{aligned} \tag{5}$$

More generally, from the definition of the radial function, it follows immediately that for $\phi \in \mathrm{GL}(n)$ the radial function of the image $\phi K = \{\phi y : y \in K\}$ of $K \in \mathscr{S}_o^n$ is given by (see [27])

$$\rho(\phi K, x) = \rho\left(K, \phi^{-1}x\right), \quad \forall x \in \mathbb{R}^n. \tag{6}$$

Obviously, for $K, L \in \mathscr{S}_o^n$,

$$K \subseteq L \quad \text{iff } \rho_K \le \rho_L. \tag{7}$$

The radial Hausdorff metric between the star bodies K and L is

$$\widetilde{\delta}(K,L) = \max_{u\in S^{n-1}} \left|\rho_K(u) - \rho_L(u)\right|. \tag{8}$$

A sequence $\{K_i\}$ of star bodies is said to be convergent to K if

$$\widetilde{\delta}(K_i,K) \longrightarrow 0, \quad \text{as } i \longrightarrow \infty. \tag{9}$$

Therefore, a sequence of star bodies K_i converges to K if and only if the sequence of radial functions $\rho(K_i,\cdot)$ converges uniformly to $\rho(K,\cdot)$ (see [28, Theorem 7.9]).

According to the definitions of the polar body for convex body, support function, and radial function, it follows that for $K \in \mathscr{K}_o^n$

$$\begin{aligned}h_{K^*}(u)\rho_K(u) &= 1,\\ \rho_{K^*}(u)h_K(u) &= 1,\\ &\forall u \in S^{n-1}.\end{aligned} \tag{10}$$

One of the most important inequalities in convex geometry is the Blaschke-Santaló inequality about polar body (cf. [1, 27, 29]): If $K \in \mathscr{K}_c^n$, then

$$V(K)V\left(K^*\right) \le \omega_n^2, \tag{11}$$

where the equality holds if and only if K is an ellipsoid.

If $K, L \in \mathscr{S}_o^n$ and $\alpha, \beta \ge 0$ (not both zero), then, for $p \ge 1$, the radial harmonic L_p-combination, $\alpha \diamond K \,\widehat{+}_{-p}\, \beta \diamond L \in \mathscr{S}_o^n$, is defined by (see [3])

$$\rho\left(\alpha \diamond K \,\widehat{+}_{-p}\, \beta \diamond L,\cdot\right)^{-p} = \alpha\rho(K,\cdot)^{-p} + \beta\rho(L,\cdot)^{-p}. \tag{12}$$

For $p \ge 1$ and $K, L \in \mathscr{S}_o^n$, the dual harmonic L_p-mixed volume, $\widetilde{V}_{-p}(K,L)$, is defined by

$$-\frac{n}{p}\widetilde{V}_{-p}(K,L) = \lim_{\varepsilon\to 0^+} \frac{V\left(K \,\widehat{+}_{-p}\, \varepsilon \diamond L\right) - V(K)}{\varepsilon}. \tag{13}$$

Let $K, L \in \mathcal{S}_o^n$ and $p \geq 1$. Then, the integral representation of dual harmonic L_p-mixed volume of K and L, $\widetilde{V}_{-p}(K, L)$, is given (see [3]):

$$\widetilde{V}_{-p}(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho_K^{n+p}(u)\, \rho_L^{-p}(u)\, dS(u). \tag{14}$$

With (5) and (14) taken together, we obtain for $\alpha > 0$

$$\widetilde{V}_{-p}(\alpha K, L) = \alpha^{n+p} \widetilde{V}_{-p}(K, L), \\ \widetilde{V}_{-p}(K, \alpha L) = \alpha^{-p} \widetilde{V}_{-p}(K, L). \tag{15}$$

In [3], Lutwak proved the following: For $K, L \in \mathcal{S}_o^n$ and $\alpha, \beta \geq 0$, if $p \geq 1$, then, for $\phi \in \mathrm{GL}(n)$,

$$\phi\left(\alpha \diamond K \widehat{+}_p \beta \diamond L\right) = \alpha \diamond \phi K \widehat{+}_p \beta \diamond \phi L. \tag{16}$$

Since $V(\phi K) = |\det(\phi)| V(K)$, for all $K \in \mathcal{S}_o^n$ and $\phi \in \mathrm{GL}(n)$, the following follows from (6), (14), and (15).

Proposition 1. *If $p \geq 1$ and $K, L \in \mathcal{S}_o^n$, then, for $\phi \in \mathrm{GL}(n)$,*

$$\widetilde{V}_{-p}(\phi K, \phi L) = \left|\det \phi\right| \widetilde{V}_{-p}(K, L). \tag{17}$$

The case $\phi \in \mathrm{SL}(n)$ of Proposition 1 reduces to the following formula:

$$\widetilde{V}_{-p}(\phi K, \phi L) = \widetilde{V}_{-p}(K, L). \tag{18}$$

This integral representation of $\widetilde{V}_{-p}(\cdot, \cdot)$, with Hölder's inequality (see [30, p. 140]) together with the polar coordinate formula, immediately gives the following:

$$\widetilde{V}_{-p}(K, L)^n \geq V(K)^{n+p} V(L)^{-p}, \tag{19}$$

with equality if and only if K and L are dilates.

The following result is an immediate consequence of (19).

Lemma 2. *Suppose that $p \geq 1$ and $\mathcal{U} \subset \mathcal{S}_o^n$ such that $K, L \in \mathcal{U}$. If for all $Q \in \mathcal{U}$*

$$\widetilde{V}_{-p}(K, Q) = \widetilde{V}_{-p}(L, Q) \\ \text{or } \widetilde{V}_{-p}(Q, K) = \widetilde{V}_{-p}(Q, L), \tag{20}$$

then $K = L$.

The continuity of the dual harmonic L_p-mixed volume $\widetilde{V}_{-p} : \mathcal{S}_o^n \times \mathcal{S}_o^n \to (0, \infty)$ is contained.

Lemma 3. *Suppose that sequences $\{K_i\}, \{L_j\} \subset \mathcal{S}_o^n$ and $K_i \to K \in \mathcal{S}_o^n$, $L_j \to L \in \mathcal{S}_o^n$. If $p \geq 1$, then $\lim_{i,j\to\infty} \widetilde{V}_{-p}(K_i, L_j) = \widetilde{V}_{-p}(K, L)$.*

Proof. Since $K_i \to K$ and $L_j \to L$ are equivalent to $\rho_{K_i} \to \rho_K$ and $\rho_{L_j} \to \rho_L$, uniformly on S^{n-1}, and ρ_K, ρ_L are positively continuous on S^{n-1}, then ρ_{K_i} and ρ_{L_j} are uniformly bounded on S^{n-1} (see [28, Theorem 7.9]). Hence,

$$\rho_{K_i}^{n+p} \longrightarrow \rho_K^{n+p}, \quad \text{uniformly on } S^{n-1}, \\ \rho_{L_j}^{-p} \longrightarrow \rho_L^{-p}, \quad \text{uniformly on } S^{n-1}. \tag{21}$$

Hence,

$$\int_{S^{n-1}} \rho_{K_i}^{n+p}(u)\, \rho_{L_j}^{-p}(u)\, dS(u) \\ \longrightarrow \int_{S^{n-1}} \rho_K^{n+p}(u)\, \rho_L^{-p}(u)\, dS(u), \quad \text{if } i, j \longrightarrow \infty. \tag{22}$$

Namely, $\lim_{i,j\to\infty} \widetilde{V}_{-p}(K_i, L_j) = \widetilde{V}_{-p}(K, L)$. □

The volume-normalized dual conical measure $d\widetilde{V}_K^*$ of $K \in \mathcal{S}_o^n$ is defined by $V(K) d\widetilde{V}_K^* = (1/n) \rho_K^n\, dS$, where S is Lebesgue measure on S^{n-1}. We shall make use of the fact that the volume-normalized dual conical measure $\widetilde{V}_K^*$ is a probability measure on S^{n-1}.

The following lemma will be needed.

Lemma 4 (see [3]). *Let $\mathcal{C}^n$ denote the set of compact convex subsets of Euclidean n-space $\mathbb{R}^n$, and suppose $K_i \in \mathcal{K}_o^n$ such that $K_i \to L \in \mathcal{C}^n$. If the sequence $V(K_i^*)$ is bounded, then $L \in \mathcal{K}_o^n$.*

3. The Dual L_p-Geominimal Surface Area

Based on the notion of dual L_p-mixed volumes, Wang and Qi [5] defined the dual L_p-geominimal surface area as follows: For $K \in \mathcal{S}_o^n$, the dual L_p-geominimal surface area, $\widetilde{G}_{-p}(K)$, of K is defined by

$$\omega_n^{-p/n} \widetilde{G}_{-p}(K) \\ = \inf\left\{n \widetilde{V}_{-p}(K, Q)\, V\left(Q^*\right)^{-p/n} : Q \in \mathcal{K}_o^n\right\}. \tag{23}$$

For this notion of L_p-dual geominimal surface area, Wang and Qi in [5] established the following affine isoperimetric inequality and Blaschke-Santaló type inequality: For $K \in \mathcal{S}_o^n$ and $p \geq 1$,

$$\widetilde{G}_{-p}(K) \geq n \omega_n^{-p/n} V(K)^{(n+p)/n}, \tag{24}$$

with equality if and only if K is an ellipsoid centred at the origin.

If $K \in \mathcal{K}_c^n$ and $p \geq 1$, then

$$\widetilde{G}_{-p}(K)\, \widetilde{G}_{-p}(K^*) \leq (n\omega_n)^2, \tag{25}$$

with equality if and only if K is an ellipsoid.

By the homogeneity of volume and dual L_p-mixed volume, the dual L_p-geominimal surface area could also be defined by

$$\widetilde{G}_{-p}(K) \\ = \inf\left\{n \widetilde{V}_{-p}(K, Q) : Q \in \mathcal{K}_o^n,\ V\left(Q^*\right) = \omega_n\right\}. \tag{26}$$

It will be shown that the infimum in the above definition is attained.

Theorem 5. *If $K \in \mathcal{S}_o^n$ and $p \geq 1$, then there exists a unique body $\widetilde{K} \in \mathcal{K}_o^n$ such that*

$$\widetilde{G}_{-p}(K) = n\widetilde{V}_{-p}(K, \widetilde{K}), \quad V(\widetilde{K}^*) = \omega_n. \tag{27}$$

Proof. From the definition of $\widetilde{G}_{-p}(K)$, there exists a sequence $\{M_i\} \subset \mathcal{K}_o^n$ such that $V(M_i^*) = \omega_n$, with $\widetilde{V}_{-p}(K, B) \geq \widetilde{V}_{-p}(K, M_i)$, for all i, and $n\widetilde{V}_{-p}(K, M_i) \to \widetilde{G}_{-p}(K)$. To see that the $M_i \in \mathcal{K}_o^n$, $i = 1, 2, \ldots$, are uniformly bounded, let

$$\begin{aligned} R_i = R(M_i) &= \rho(M_i, u_i) \\ &= \max\{\rho(M_i, u) : u \in S^{n-1}\}, \end{aligned} \tag{28}$$

where u_i is any of the points in S^{n-1} at which this maximum is attained. Let $r_K = \min_{S^{n-1}} \rho_K$. Then, $r_K B \subseteq K$. From definition (14) of dual harmonic L_p-mixed volume and Jensen's inequality, it follows that

$$\begin{aligned} \frac{\widetilde{V}_{-p}(K, B)}{V(K)} &\geq \frac{\widetilde{V}_{-p}(K, M_i)}{V(K)} = \int_{S^{n-1}} \left(\frac{\rho_K(u)}{\rho_{M_i}(u)}\right)^p d\widetilde{V}_K^* \\ &\geq \left(\int_{S^{n-1}} \frac{\rho_K(u)}{\rho_{M_i}(u)} d\widetilde{V}_K^*\right)^p \\ &\geq \left(\int_{S^{n-1}} \frac{\rho_K(u)}{R_i} d\widetilde{V}_K^*\right)^p \\ &= \left(\frac{1}{nV(K) R_i} \int_{S^{n-1}} \rho_K(u)^{n+1} dS(u)\right)^p \\ &\geq \left(\frac{r_K}{nV(K) R_i} \int_{S^{n-1}} \rho_K(u)^n dS(u)\right)^p \\ &= \left(\frac{r_K}{R_i}\right)^p. \end{aligned} \tag{29}$$

Namely,

$$\omega_n r_K^n \left(\frac{r_K}{R_i}\right)^p \leq \widetilde{V}_{-p}(K, M_i) \leq \widetilde{V}_{-p}(K, B) < \infty \tag{30}$$

for a fixed $K \in \mathcal{S}_o^n$; then, the sequence $\{M_i\}$ is uniformly bounded.

Since the sequence $\{M_i\}$ is uniformly bounded, the Blaschke selection theorem guarantees the existence of a subsequence of M_i, which will also be denoted by M_i, and a compact convex $L \in \mathcal{C}^n$, such that $M_i \to L$. Since $V(M_i^*) = \omega_n$, Lemma 4 gives $L \in \mathcal{K}_o^n$. Now, $M_i \to L$ implies that $M_i^* \to L^*$, and since $V(M_i^*) = \omega_n$, it follows that $V(L^*) = \omega_n$. Lemma 3 can now be used to conclude that L will serve as the desired body $\widetilde{K}$.

The uniqueness of the minimizing body is easily demonstrated as follows. Suppose $L_1, L_2 \in \mathcal{K}_o^n$ and $L_1 \neq L_2$, such that $V(L_1^*) = \omega_n = V(L_2^*)$, and

$$\widetilde{V}_{-p}(K, L_1) = \widetilde{V}_{-p}(K, L_2). \tag{31}$$

Define $L \in \mathcal{K}_o^n$ by

$$L = \frac{1}{2} \diamond L_1 \widehat{+}_{-1} \frac{1}{2} \diamond L_2. \tag{32}$$

Since, obviously,

$$L^* = \frac{1}{2} \cdot L_1^* + \frac{1}{2} \cdot L_2^* \tag{33}$$

and $V(L_1^*) = \omega_n = V(L_2^*)$, it follows from Brunn-Minkowski inequality (2) that

$$V(L^*) \geq \omega_n, \tag{34}$$

with equality if and only if $L_1 = L_2$.

By formula (14) of dual L_p-mixed volume, together with the convexity of $\phi(t) = t^p$ $(p \geq 1)$, we have

$$\begin{aligned} \widetilde{V}_{-p}(K, L) &= \frac{1}{n} \int_{S^{n-1}} \left(\frac{\rho_K(u)}{\rho_{1/2\circ L_1 \widetilde{+}_{-1} 1/2\circ L_2}(u)}\right)^p \\ &\cdot \rho_K(u)^n dS(u) = \frac{1}{n} \\ &\cdot \int_{S^{n-1}} \left(\frac{\rho_K(u)}{\left(1/2\rho_{L_1}(u) + 1/2\rho_{L_2}(u)\right)^{-1}}\right)^p \\ &\cdot \rho_K(u)^n dS(u) = \frac{1}{n} \\ &\cdot \int_{S^{n-1}} \left(\frac{\rho_K(u)}{2\rho_{L_1}(u)} + \frac{\rho_K(u)}{2\rho_{L_2}(u)}\right)^p \rho_K(u)^n dS(u) \\ &\leq \frac{1}{2n} \int_{S^{n-1}} \left(\frac{\rho_K(u)}{\rho_{L_1}(u)}\right)^p \rho_K(u)^n dS(u) + \frac{1}{2n} \\ &\cdot \int_{S^{n-1}} \left(\frac{\rho_K(u)}{\rho_{L_2}(u)}\right)^p \rho_K(u)^n dS(u) = \frac{1}{2} \\ &\cdot \widetilde{V}_{-p}(K, L_1) + \frac{1}{2}\widetilde{V}_{-p}(K, L_2) = \widetilde{V}_{-p}(K, L_1) \\ &= \widetilde{V}_{-p}(K, L_2), \end{aligned} \tag{35}$$

with equality if and only if $L_1 = L_2$. Thus,

$$\begin{aligned} \widetilde{V}_{-p}(K, L) V(L^*)^{-p/n} &< \widetilde{V}_{-p}(K, L_1) V(L_1^*)^{-p/n} \\ &= \widetilde{V}_{-p}(K, L_2) V(L_2^*)^{-p/n} \end{aligned} \tag{36}$$

is the contradiction that would arise if it were the case that $L_1 \neq L_2$. This completes the proof. ☐

The unique body whose existence is guaranteed by Theorem 5 will be denoted by $\widetilde{T}_p K$ and will be called the dual p-Petty body of K. The polar body of $\widetilde{T}_p K$ will be denoted by $\widetilde{T}_p^* K$ rather than $(\widetilde{T}_p K)^*$. Thus, for $K \in \mathcal{S}_o^n$, the body $\widetilde{T}_{-p} K$ is defined by

$$\widetilde{G}_{-p}(K) = n\widetilde{V}_{-p}(K, \widetilde{T}_p K), \quad V(\widetilde{T}_p^* K) = \omega_n. \tag{37}$$

For $K \in \mathcal{K}^n$, there exists a unique point $s(K)$ in the interior of K, called the Santaló point of K, such that (see [3])

$$\begin{aligned} & V\left((-s(K)+K)^*\right) \\ & \quad = \min\left\{V\left((-x+K)^*\right) : x \in \operatorname{int} K\right\}, \end{aligned} \tag{38}$$

or, for the unique $s(K) \in K$, this is equivalent to

$$\int_{S^{n-1}} uh(-s(K)+K,u)^{-(n+1)}\, dS(u) = 0. \tag{39}$$

Let $\mathcal{K}_s^n$ denote the set of convex bodies having their Santaló point at the origin in $\mathbb{R}^n$. Thus, we have (see [3])

$$K \in \mathcal{K}_s^n \quad \text{iff } K^* \in \mathcal{K}_c^n. \tag{40}$$

Let

$$\mathcal{T}^n = \left\{\widetilde{T} \in \mathcal{K}^n : s(K) = o,\ V\left(\widetilde{T}^*\right) = \omega_n\right\}. \tag{41}$$

The next result is an immediate consequence of Theorem 5.

Theorem 6. *For each $K \in \mathcal{S}_o^n$, there exists a unique body $\widetilde{T}_pK \in \mathcal{T}^n$ with $\widetilde{G}_{-p}(K) = n\widetilde{V}_{-p}(K, \widetilde{T}_pK)$.*

The unique body $\widetilde{T}_pK$ is called the dual p-Petty body of K.

By Theorem 6 and the integral representation (14) of dual harmonic L_p-mixed volume, we have the following integral formula of $\widetilde{G}_{-p}(K)$.

Proposition 7. *For each $K \in \mathcal{S}_o^n$, there exists a unique convex body $\widetilde{T} = \widetilde{T}_pK \in \mathcal{T}^n$ with*

$$\widetilde{G}_{-p}(K) = \int_{S^{n-1}} \rho_K^{n+p}(p)\,\rho_{\widetilde{T}}^{-p}(u)\, dS(u). \tag{42}$$

4. The Dual L_p-Mixed Geominimal Surface Area

Motivated by the definition of L_p-mixed geominimal surface area of Zhu et al. (see [4]), we now define the dual L_p-mixed geominimal surface area as follow: For each $K_i \in \mathcal{S}_o^n$, $i = 1, \ldots, n$, and $p \geq 1$, there exists a unique convex body (dual p-Petty body of K_i) $\widetilde{T}_i = \widetilde{T}_pK_i \in \mathcal{T}^n$ $(i = 1, \ldots, n)$ with

$$\begin{aligned} \widetilde{G}_{-p}(K_1, \ldots, K_n) = \int_{S^{n-1}} & \left[\rho_{K_1}^{n+p}(p)\,\rho_{\widetilde{T}_1}^{-p}(u) \cdots \rho_{K_n}^{n+p}(u)\right. \\ & \left. \cdot\, \rho_{\widetilde{T}_n}^{-p}(u)\right]^{1/n} dS(u). \end{aligned} \tag{43}$$

$\widetilde{G}_{-p}(K_1, \ldots, K_n)$ will be called the dual L_p-mixed geominimal surface area of $K_1, \ldots, K_n \in \mathcal{S}_o^n$.

Let $g_{-p}(K_i, u) = \rho_{K_i}^{n+p}(p)\rho_{\widetilde{T}_i}^{-p}(u)$. Then, $\widetilde{G}_{-p}(K_1, \ldots, K_n)$ can be written as follows:

$$\begin{aligned} & \widetilde{G}_{-p}(K_1, \ldots, K_n) \\ & \quad = \int_{S^{n-1}} \left[g_{-p}(K_1,u) \cdots g_{-p}(K_n,u)\right]^{1/n} dS(u). \end{aligned} \tag{44}$$

The following propositions will provide that the dual L_p-mixed geominimal surface area is affine invariant.

Proposition 8. *If $K \in \mathcal{S}_o^n$ and every $\phi \in GL(n)$, then*

$$\widetilde{G}_{-p}(\phi K) = |\det \phi|^{(n+p)/n}\, \widetilde{G}_{-p}(K). \tag{45}$$

Proof. From definition (23) of dual L_p-geominimal surface area and (18), for $\phi \in \mathrm{SL}(n)$, we have

$$\begin{aligned} \widetilde{G}_{-p}(\phi K) &= \inf\left\{n\widetilde{V}_{-p}(\phi K, Q) : Q \in \mathcal{K}_o^n,\ V(Q^*)\right. \\ &\left. = \omega_n\right\} = \inf\left\{n\widetilde{V}_{-p}\left(K, \phi^{-1}Q\right) : \phi^{-1}Q\right. \\ &\left. \in \mathcal{K}_o^n,\ V\left(\left(\phi^{-1}Q\right)^*\right) = V\left(\phi^t Q^*\right) = \omega_n\right\} = \widetilde{G}_\phi(K). \end{aligned} \tag{46}$$

On the other hand, for $\lambda > 0$, it follows from (23) that

$$\widetilde{G}_{-p}(\lambda K) = \lambda^{n+p}\widetilde{G}_{-p}(K). \tag{47}$$

Therefore, for every $\phi \in \mathrm{GL}(n)$, we have

$$\widetilde{G}_{-p}(\phi K) = |\det \phi|^{(n+p)/n}\, \widetilde{G}_{-p}(K). \tag{48}$$

□

Proposition 9. *If $K \in \mathcal{S}_o^n$, then, for $\phi \in GL(n)$,*

$$|\det \phi|^{1/n}\, \widetilde{T}_p\phi K = \phi\widetilde{T}_pK. \tag{49}$$

Proof. From the definition of $\widetilde{T}_p$ and Proposition 8, it follows that

$$\begin{aligned} n\widetilde{V}_{-p}\left(K, \widetilde{T}_pK\right) &= \widetilde{G}_{-p}(K) \\ &= |\det \phi|^{-(n+p)/n}\, \widetilde{G}_{-p}(\phi K) \\ &= |\det \phi|^{-(n+p)/n}\, n\widetilde{V}_{-p}\left(\phi K, \widetilde{T}_p\phi K\right). \end{aligned} \tag{50}$$

From the definition of $\widetilde{T}_p$, Proposition 9, and (15),

$$\begin{aligned} \widetilde{V}_{-p}\left(K, \widetilde{T}_pK\right) &= |\det \phi|^{-(n+p)/n}\, \widetilde{V}_{-p}\left(\phi K, \widetilde{T}_p\phi K\right) \\ &= |\det \phi|^{-1}\, \widetilde{V}_{-p}\left(\phi K, |\det \phi|^{1/n}\, \widetilde{T}_p\phi K\right) \\ &= \widetilde{V}_{-p}\left(K, \phi^{-1}\left(|\det \phi|^{1/n}\, \widetilde{T}_p\phi K\right)\right). \end{aligned} \tag{51}$$

Namely, from Lemma 2, for each $\phi \in \mathrm{GL}(n)$,

$$|\det \phi|^{1/n}\, \widetilde{T}_p\phi K = \phi\widetilde{T}_pK. \tag{52}$$

□

Proposition 10. *If $p \geq 1$ and $K_1, \ldots, K_n \in \mathcal{S}_o^n$, then, for $\phi \in GL(n)$,*

$$\begin{aligned} & \widetilde{G}_{-p}(\phi K_1, \ldots, \phi K_n) \\ & \quad = |\det \phi|^{(n+p)/n}\, \widetilde{G}_{-p}(K_1, \ldots, K_n). \end{aligned} \tag{53}$$

In particular, if $\phi \in SL(n)$, then $\widetilde{G}_{-p}(K_1, \ldots, K_n)$ is affine invariant; that is,

$$\widetilde{G}_{-p}(\phi K_1, \ldots, \phi K_n) = \widetilde{G}_{-p}(K_1, \ldots, K_n). \tag{54}$$

Proof. Since $K \in \mathcal{S}_o^n$, for $\phi \in \mathrm{GL}(n)$ and any $u \in S^{n-1}$, we have

$$\begin{aligned} g_{-p}(\phi K, u) &= \rho(\phi K, u)^{n+p} \rho\left(\widetilde{T}_p \phi K, u\right)^{-p} \\ &= \rho(\phi K, u)^{n+p} \rho\left(\left|\det\phi\right|^{-1/n} \phi \widetilde{T}_p K, u\right)^{-p} \\ &= \left|\det\phi\right|^{p/n} \left|\phi^{-1}u\right|^{-n} \rho_K(v)^{n+p} \rho_{\widetilde{T}_p K}(v)^{-p} \\ &= \left|\det\phi\right|^{p/n} \left|\phi^{-1}u\right|^{-n} g_{-p}(K, v), \end{aligned} \tag{55}$$

where $v = (\phi^{-1}u)/|\phi^{-1}u| \in S^{n-1}$. Therefore, for $\phi \in \mathrm{GL}(n)$, we have

$$\begin{aligned} \widetilde{G}_{-p}(\phi K_1, \dots, \phi K_n) &= \left|\det\phi\right|^{p/n} \left|\phi^{-1}u\right|^{-n} \int_{S^{n-1}} \left[g_{-p}(K_1, u) \cdots g_{-p}(K_n, u)\right]^{1/n} \mathrm{d}S\left(\phi\left(\left|\phi^{-1}u\right| \cdot \frac{\phi^{-1}u}{\left|\phi^{-1}u\right|}\right)\right) \\ &= \left|\det\phi\right|^{(n+p)/n} \widetilde{G}_{-p}(K_1, \dots, K_n). \end{aligned} \tag{56}$$

□

The dual mixed volume $\widetilde{V}(K_1, \dots, K_n)$ of sets $K_1, \dots, K_n \in \mathcal{S}_o^n$ is defined by

$$\widetilde{V}(K_1, \dots, K_n) = \frac{1}{n} \int_{S^{n-1}} \rho_{K_1}(u) \cdots \rho_{K_n}(u) \, \mathrm{d}S(u). \tag{57}$$

The classical dual Alexandrov-Fenchel inequalities for dual mixed volumes (cf. [27, 31, 32]) can be written as

$$\widetilde{V}(K_1, \dots, K_n)^m \le \prod_{i=0}^{m-1} \widetilde{V}\left(K_1, \dots, K_{n-m}, \underbrace{K_{n-i}, \dots, K_{n-i}}_{m}\right), \tag{58}$$

with equality if $K_{n-m+1}, \dots, K_n$ are dilates of each other. If $m = 1$, equality holds trivially.

In particular, taking $m = n$ in the above inequality and noticing that $\widetilde{V}(K) = V(K)$, we have

$$\widetilde{V}(K_1, \dots, K_n)^n \le V(K_1) \cdots V(K_n), \tag{59}$$

with equality if and only if $K_1, \dots, K_n$ are dilates.

Take $K_1 = \cdots = K_{n-i} = K$, $K_{n-i+1} = \cdots = K_n = B$ in $\widetilde{V}(K_1, \dots, K_n)$, and

$$\widetilde{V}\left(\underbrace{K, \dots, K}_{n-i}, \underbrace{B, \dots, B}_{i}\right) := \widetilde{W}_i(K), \tag{60}$$

where $\widetilde{W}_i(K)$ is called the ith dual quermassintegral of $K \in \mathcal{S}_o^n$.

The following inequalities are the analogous of dual Alexandrov-Fenchel inequalities for dual L_p-mixed geominimal surface area.

Theorem 11. *If $p \ge 1$ and $K_1, \dots, K_n \in \mathcal{S}_o^n$, then, for $1 \le m \le n$,*

$$\widetilde{G}_{-p}(K_1, \dots, K_n)^m \le \prod_{i=0}^{m-1} \widetilde{G}_{-p}\left(K_1, \dots, K_{n-m}, \underbrace{K_{n-i}, \dots, K_{n-i}}_{m}\right), \tag{61}$$

with equality if $K_{n-m+1}, \dots, K_n$ are dilates of each other. If $m = 1$, equality holds trivially.

In particular, if $m = n$ in the above inequality, then

$$\widetilde{G}_{-p}(K_1, \dots, K_n)^n \le \widetilde{G}_{-p}(K_1) \cdots \widetilde{G}_{-p}(K_n), \tag{62}$$

with equality if K_i $(1 \le i \le n)$ are dilates of each other.

Proof. Let $Y_0(u) = [g_{-p}(K_1, u) \cdots g_{-p}(K_{n-m}, u)]^{1/n}$ and $Y_{i+1}(u) = [g_{-p}(K_{n-i}, u)]^{1/n}$ for $i = 0, \dots, m-1$. By Hölder's inequality (cf. [30]), we have

$$\begin{aligned} \widetilde{G}_{-p}(K_1, \dots, K_n) &= \int_{S^{n-1}} \left[g_{-p}(K_1, u) \cdots g_{-p}(K_n, u)\right]^{1/n} \mathrm{d}S(u) \\ &= \int_{S^{n-1}} Y_0(u) Y_1(u) \cdots Y_m(u) \, \mathrm{d}S(u) \\ &\le \prod_{i=0}^{m-1} \left(Y_0(u) Y_{i+1}(u)^m \, \mathrm{d}S(u)\right)^{1/m} \\ &= \prod_{i=0}^{m-1} \widetilde{G}_{-p}^{1/m}\left(K_1, \dots, K_{n-m}, \underbrace{K_{n-i}, \dots, K_{n-i}}_{m}\right). \end{aligned} \tag{63}$$

The equality in Hölder's inequality holds if and only if $Y_0(u) Y_{i+1}^m = \lambda_{ij}^m Y_0(u) Y_{j+1}^m$ for some $\lambda_{ij} > 0$ and all $0 \le i \ne j \le m-1$. This is equivalent to $\rho_{K_{n-i}}^{n+p}(u) \rho_{\widetilde{T}_{n-i}}^{-p}(u) = \lambda_{ij}^n \rho_{K_{n-j}}^{n+p}(u) \rho_{\widetilde{T}_{n-j}}^{-p}(u)$. From Proposition 9, $\widetilde{T}_p K = \widetilde{T}_p cK$ for constant $c > 0$. Thus, the equality holds if K_{n-i} and K_{n-j} are dilates of each other. □

A lemma of the following type will be needed.

Lemma 12. *If $K \in \mathcal{K}_c^n$ and $p \ge 1$, then*

$$\widetilde{G}_{-p}(K) \le n\omega_n^{(2n-p)/n} V(K^*)^{-(n+p)/n}, \tag{64}$$

with equality if and only if K is a ball centred at the origin.

Proof. From definition (23) and inequality (19), we have

$$\begin{aligned}\omega_n^{-p/n} V(K^*)^{(n+p)/n} \widetilde{G}_{-p}(K) &= \inf\left\{n\widetilde{V}_{-p}(K,Q)\, V(K^*)^{(n+p)/n} V(Q^*)^{-p/n} : Q \in \mathscr{K}_c^n\right\} \\ &\le \inf\left\{n\widetilde{V}_{-p}(K,Q)\, \widetilde{V}_{-p}(K^*,Q^*) : Q \in \mathscr{K}_o^n\right\}.\end{aligned} \tag{65}$$

Since $K \in \mathscr{K}_o^n$, taking $Q = K$, it follows from inequalities (65) and (11) that

$$\begin{aligned}&\omega_n^{-p/n} V(K^*)^{(n+p)/n} \widetilde{G}_{-p}(K) \\ &\quad\le \inf\left\{nV(K)\, V(K^*) : K \in \mathscr{K}_o^n\right\} \le n\omega_n^2.\end{aligned} \tag{66}$$

Namely,

$$\widetilde{G}_{-p}(K) \le n\omega_n^{(2n-p)/n} V(K^*)^{-(n+p)/n}. \tag{67}$$

By the equality condition of (19) and (65), we see that equality holds in (64) if and only if K is a ball centred at the origin. □

Now, we prove the affine isoperimetric inequalities for dual L_p-mixed geominimal surface areas.

Theorem 13. *Let $K_1, \ldots, K_n \in \mathscr{K}_c^n$ and $p \ge 1$; then,*

$$\frac{\widetilde{G}_{-p}(K_1,\ldots,K_n)}{\widetilde{G}_{-p}(B,\ldots,B)} \le \left(\frac{\widetilde{V}(K_1^*,\ldots,K_n^*)}{\widetilde{V}(B,\ldots,B)}\right)^{-(n+p)/n}, \tag{68}$$

with equality if and only if $K_1, \ldots, K_n$ are balls centred at the origin that are dilates of each other.

Proof. From (77) in Section 5, it follows that $\widetilde{T}_p B = B$. Then, $g_{-p}(B,u) = 1$, $\widetilde{G}_{-p}(B) = n\omega_n$, and $\widetilde{G}_{-p}(B,\ldots,B) = n\omega_n$. By inequalities (62), (64), and (59), we have

$$\begin{aligned}\frac{\widetilde{G}_{-p}(K_1,\ldots,K_n)}{\widetilde{G}_{-p}(B,\ldots,B)} &\le \left(\frac{V(K_1^*)}{V(B)} \cdots \frac{V(K_n^*)}{V(B)}\right)^{-(n+p)/n} \\ &\le \left(\frac{\widetilde{V}(K_1^*,\ldots,K_n^*)}{\widetilde{V}(B,\ldots,B)}\right)^{-(n+p)/n}.\end{aligned} \tag{69}$$

By the equality condition of (62), (64), and (59), we see that equality holds in (68) if and only if K is a ball centred at the origin. □

Corollary 14. *Let $K_1, \ldots, K_n \in \mathscr{K}_c^n$ and $p \ge 1$; then,*

$$\begin{aligned}&\widetilde{G}_{-p}(K_1,\ldots,K_n) \\ &\quad\le n\omega_n^{(2n+p)/n} \widetilde{V}(K_1^*,\ldots,K_n^*)^{-(n+p)/n},\end{aligned} \tag{70}$$

with equality if and only if $K_1, \ldots, K_n$ are balls centred at the origin that are dilates of each other.

Take $K_1 = \cdots = K_{n-i} = K$, $K_{n-i+1} = \cdots = K_n = B$ in (70), and we write

$$\widetilde{G}_{-p}\Big(\underbrace{K,\ldots,K}_{n-i},\underbrace{B,\ldots,B}_{i}\Big) := \widetilde{G}_{-p,i}(K). \tag{71}$$

Corollary 15. *Let $K \in \mathscr{K}_c^n$ and $p \ge 1$; then, for $i = 0, 1, \ldots, n-1$,*

$$\widetilde{G}_{-p,i}(K) \le n\omega_n^{(2n+p)/n} \widetilde{W}_i(K^*)^{-(n+p)/n}, \tag{72}$$

with equality if and only if K is a ball centred at the origin.

5. The Dual *i*th L_p-Mixed Geominimal Surface Area

This section is mainly dedicated to investigating the dual *i*th L_p-mixed geominimal surface area.

For $K, L \in \mathscr{S}_o^n$, $p \ge 1$, and $i \in \mathbb{R}$, we define dual *i*th L_p-mixed geominimal surface area, $\widetilde{G}_{-p}(K,L)$, of K, L as

$$\begin{aligned}&\widetilde{G}_{-p,i}(K,L) \\ &\quad= \int_{S^{n-1}} g_{-p}(K,u)^{(n-i)/n} g_{-p}(L,u)^{i/n}\, dS(u)\end{aligned} \tag{73}$$

and write

$$\widetilde{G}_{-p,i}(K,B) = \widetilde{G}_{-p,i}(K). \tag{74}$$

By Theorem 6, we have

$$\widetilde{G}_{-p}(B) = n\widetilde{V}_{-p}\left(B, \widetilde{T}_p B\right), \tag{75}$$

and, obviously,

$$\widetilde{G}_{-p}(B) = n\omega_n = n\widetilde{V}_{-p}(B,B). \tag{76}$$

Thus, the above two equations and the uniqueness part of Theorem 6 show that

$$\widetilde{T}_p B = B. \tag{77}$$

Noticing that $g_{-p}(B,u) = 1$ for $u \in S^{n-1}$, then

$$\widetilde{G}_{-p,i}(K) = \int_{S^{n-1}} g_{-p}(K,u)^{(n-i)/n}\, dS(u). \tag{78}$$

By (44), (73), and (74), we have

$$\begin{aligned}\widetilde{G}_{-p,0}(K) &= \widetilde{G}_{-p}(K), \\ \widetilde{G}_{-p,i}(K,K) &= \widetilde{G}_{-p}(K),\end{aligned} \tag{79}$$

$$\begin{aligned}\widetilde{G}_{-p,0}(K,L) &= \widetilde{G}_{-p}(K), \\ \widetilde{G}_{-p,n}(K,L) &= \widetilde{G}_{-p}(L).\end{aligned} \tag{80}$$

The following cyclic inequality for the dual *i*th L_p-mixed geominimal surface area will be established.

Theorem 16. *For $K, L \in \mathcal{S}_o^n$, $p \geq 1$, $i, j, k \in \mathbb{R}$, and $i < j < k$, we have*

$$\widetilde{G}_{-p,i}(K,L)^{k-j}\widetilde{G}_{-p,k}(K,L)^{j-i} \geq \widetilde{G}_{-p,j}(K,L)^{k-i}, \tag{81}$$

with equality if K and L are dilates of each other.

Proof. From definition (73) and Hölder's inequality, it follows that, for $p \geq 1$,

$$\begin{aligned}
&\widetilde{G}_{-p,i}(K,L)^{(k-j)/(k-i)}\widetilde{G}_{-p,k}(K,L)^{(j-i)/(k-i)} \\
&= \left[\int_{S^{n-1}} g_{-p}(K,u)^{(n-i)/n} g_{-p}(L,u)^{i/n}\,dS(u)\right]^{(k-j)/(k-i)} \\
&\cdot \left[\int_{S^{n-1}} g_{-p}(K,u)^{(n-k)/n} g_{-p}(L,u)^{k/n}\,dS(u)\right]^{(j-i)/(k-i)} \\
&= \left\{\int_{S^{n-1}} \left[g_{-p}(K,u)^{(n-i)(k-j)/n(k-i)}\right.\right. \\
&\left.\left.\cdot\, g_{-p}(L,u)^{i(k-j)/n(k-i)}\right]^{(k-i)/(k-j)} dS(u)\right\}^{(k-j)/(k-i)} \\
&\cdot \left\{\int_{S^{n-1}} \left[g_{-p}(K,u)^{(n-k)(j-i)/n(k-i)}\right.\right. \\
&\left.\left.\cdot\, g_{-p}(L,u)^{k(j-i)/n(k-i)}\right]^{(k-i)/(j-i)} dS(u)\right\}^{(j-i)/(k-i)} \\
&\geq \int_{S^{n-1}} g_{-p}(K,u)^{(n-j)/n(k-i)} g_{-p}(L,u)^{j/n}\,dS(u) \\
&= \widetilde{G}_{-p,j}(K,L).
\end{aligned} \tag{82}$$

That is,

$$\widetilde{G}_{-p,i}(K,L)^{k-j}\widetilde{G}_{-p,k}(K,L)^{j-i} \geq \widetilde{G}_{-p,j}(K,L)^{k-i}. \tag{83}$$

We obtain inequality (81). By the condition of equality in Hölder's inequality, the equality holds in (81) if and only if, for any $u \in S^{n-1}$,

$$\frac{g_{-p}(K,u)^{(n-i)/n} g_{-p}(L,u)^{i/n}}{g_{-p}(K,u)^{(n-k)/n} g_{-p}(L,u)^{k/n}} \tag{84}$$

is a constant; that is, $g_{-p}(K,u)/g_{-p}(L,u)$ is a constant for any $u \in S^{n-1}$. By the same argument in the proof of Theorem 11, we conclude that equality holds if K and L are dilates of each other. □

Taking $L = B$ in Theorem 16 and using (74), we immediately obtain the following.

Corollary 17. *For $K \in \mathcal{S}_o^n$, $p \geq 1$, $i, j, k \in \mathbb{R}$, and $i < j < k$, then*

$$\widetilde{G}_{-p,i}(K)^{k-j}\widetilde{G}_{-p,k}(K)^{j-i} \geq \widetilde{G}_{-p,j}(K)^{k-i}, \tag{85}$$

with equality if K is a ball centered at the origin.

Then, the following Minkowski inequality for the dual ith L_p-mixed geominimal surface area will be obtained.

Theorem 18. *For $K, L \in \mathcal{S}_o^n$, $p \geq 1$, and $i \in \mathbb{R}$ and then for $i < 0$ or $i > n$,*

$$\widetilde{G}_{-p,i}(K,L)^n \geq \widetilde{G}_{-p}(K)^{n-i}\widetilde{G}_{-p}(L)^i, \tag{86}$$

and for $0 < i < n$,

$$\widetilde{G}_{-p,i}(K,L)^n \leq \widetilde{G}_{-p}(K)^{n-i}\widetilde{G}_{-p}(L)^i. \tag{87}$$

Each inequality holds as an equality if K and L are dilates of each other. For $i = 0$ or $i = n$, (86) (or (87)) is identical.

Proof. (i) For $i < 0$, let $(i, j, k) = (i, 0, n)$ in Theorem 16; we obtain

$$\widetilde{G}_{-p,i}(K,L)^n\widetilde{G}_{-p,n}(K,L)^{-i} \geq \widetilde{G}_{-p,0}(K,L)^{n-i}, \tag{88}$$

with equality if K and L are dilates of each other. From (80), we can get

$$\widetilde{G}_{-p,i}(K,L)^n \geq \widetilde{G}_{-p}(K)^{n-i}\widetilde{G}_{-p}(L)^i, \tag{89}$$

with equality if K and L are dilates of each other.

(ii) For $i > n$, let $(i, j, k) = (0, n, i)$ in Theorem 16; we obtain

$$\widetilde{G}_{-p,0}(K,L)^{i-n}\widetilde{G}_{-p,i}(K,L)^n \geq \widetilde{G}_{-p,n}(K,L)^i, \tag{90}$$

with equality if K and L are dilates of each other.

From (80), we can also get inequality (86).

(iii) For $0 < i < n$, let $(i, j, k) = (0, i, n)$ in Theorem 16; we obtain

$$\widetilde{G}_{-p,0}(K,L)^{n-i}\widetilde{G}_{-p,n}(K,L)^i \geq \widetilde{G}_{-p,i}(K,L)^n, \tag{91}$$

with equality if K and L are dilates of each other.

From (80), we can get inequality (87).

(iv) For $i = 0$ (or $i = n$), by (80), one can see (86) (or (87)) is identical. □

Let $L = B$ in Theorem 18, $\widetilde{G}_{-p}(B) = n\omega_n$, and (74) will lead to the following.

Corollary 19. *For $K \in \mathcal{S}_o^n$, $p \geq 1$, and $i \in \mathbb{R}$ and then for $i < 0$ or $i > n$,*

$$\widetilde{G}_{-p,i}(K)^n \geq (n\omega_n)^i\widetilde{G}_{-p}(K)^{n-i}, \tag{92}$$

and for $0 < i < n$,

$$\widetilde{G}_{-p,i}(K)^n \leq (n\omega_n)^i\widetilde{G}_{-p}(K)^{n-i}. \tag{93}$$

Each inequality holds as an equality if K is a ball centered at the origin. For $i = 0$ or $i = n$, (92) (or (93)) is identical.

Now we will give an extended form of inequality (24) as follows.

Theorem 20. *If $K \in \mathcal{S}_o^n$, $p \geq 1$, $i \in \mathbb{R}$, and $i \leq 0$, then*

$$\widetilde{G}_{-p,i}(K) \geq n\omega_n^{((n-p)i-np)/n^2} V(K)^{(n+p)(n-i)/n^2}, \tag{94}$$

with equality if and only if K is an ellipsoid centred at the origin.

Proof. By inequalities (92) and (24), we can immediately obtain inequality (94). □

As the extension of inequality (25), we obtain an analogue of Blaschke-Santaló inequality for the dual ith L_p-mixed geominimal surface area.

Theorem 21. *If $K, L \in \mathcal{S}_o^n$, $p \geq 1$, $i \in \mathbb{R}$, and $0 \leq i \leq n$, then*

$$\widetilde{G}_{-p,i}(K,L)\,\widetilde{G}_{-p,i}(K^*,L^*) \leq (n\omega_n)^2, \tag{95}$$

and equality holds for $0 < i < n$ if K and L are dilated ellipsoids of each other centered at the origin. The inequality holds as an equality for $i = 0$ (or $i = n$) if K (or L) is an ellipsoid centered at the origin.

Proof. Given (87) together with (25), it follows that

$$\begin{aligned}
&\widetilde{G}_{-p,i}(K,L)\,\widetilde{G}_{-p,i}(K^*,L^*) \\
&\quad \leq \left(\widetilde{G}_{-p}(K)\,\widetilde{G}_{-p}(K^*)\right)^{(n-i)/n}\left(\widetilde{G}_{-p}(L)\,\widetilde{G}_{-p}(L^*)\right)^{i/n} \\
&\quad \leq (n\omega_n)^{2(n-i)/n}(n\omega_n)^{2i/n} = (n\omega_n)^2.
\end{aligned} \tag{96}$$

The equality holds for $0 < i < n$ if K and L are dilated ellipsoids of each other. The inequality holds as an equality for $i = 0$ (or $i = n$) if K (or L) is an ellipsoid. □

Recall Ye's isoperimetric inequality (see [33]): If $K \in \mathcal{S}_o^n$, $p \in (0,\infty)\cup(-\infty,-n)$, and the dual L_p-surface area $\widetilde{S}_p(K) = n\widetilde{V}_{-p}(K,B)$, then

$$\frac{\widetilde{S}_p(K)}{\widetilde{S}_p(B)} \geq \left(\frac{V(K)}{V(B)}\right)^{(n+p)/n}. \tag{97}$$

On the other hand, if $p \in (-n, 0)$, then

$$\frac{\widetilde{S}_p(K)}{\widetilde{S}_p(B)} \leq \left(\frac{V(K)}{V(B)}\right)^{(n+p)/n}. \tag{98}$$

The equality in every inequality holds if and only if K is an origin-symmetric Euclidean ball.

We now establish generalized isoperimetric inequalities for $\widetilde{G}_{-p,i}(K)$.

Theorem 22. *If $K \in \mathcal{S}_o^n$, $p \geq 1$, and $i \in \mathbb{R}$, then we have the following.*

(i) If $i \leq 0$,

$$\frac{\widetilde{G}_{-p,i}(K)}{\widetilde{G}_{-p,i}(B)} \geq \left(\frac{V(K)}{V(B)}\right)^{(n+p)(n-i)/n^2}, \tag{99}$$

with equality if K is a ball centered at the origin.

(ii) If $i \geq n$,

$$\frac{\widetilde{G}_{-p,i}(K)}{\widetilde{G}_{-p,i}(B)} \leq \left(\frac{V(K)}{V(B)}\right)^{(n+p)(n-i)/n^2}, \tag{100}$$

with equality if K is a ball centered at the origin.

Proof. (i) For $i = 0$, by (79) and (24), it follows that

$$\frac{\widetilde{G}_{-p}(K)}{\widetilde{G}_{-p}(B)} \geq \left(\frac{V(K)}{V(B)}\right)^{(n+p)/n}. \tag{101}$$

This is Wang's inequality (24).

For $i = n$, by (74), (79), and (80), the equality holds trivially in (100).

For $i < 0$, since $\widetilde{G}_{-p,i}(B) = \widetilde{G}_{-p}(B) = n\omega_n$, by (24), (92), and (94), we have

$$\frac{\widetilde{G}_{-p,i}(K)}{\widetilde{G}_{-p,i}(B)} \geq \left(\frac{\widetilde{G}_{-p}(K)}{\widetilde{G}_{-p}(B)}\right)^{(n-i)/n}. \tag{102}$$

Hence, for $i < 0$ and $p \geq 1$, the L_p-affine isoperimetric inequalities (92) and (24) imply that

$$\begin{aligned}
\frac{\widetilde{G}_{-p,i}(K)}{\widetilde{G}_{-p,i}(B)} &\geq \left(\frac{\widetilde{G}_{-p}(K)}{\widetilde{G}_{-p}(B)}\right)^{(n-i)/n} \\
&\geq \left(\frac{V(K)}{V(B)}\right)^{(n+p)(n-i)/n^2},
\end{aligned} \tag{103}$$

with equality if K is a ball centered at the origin.

(ii) For $i = n$, by (74), (79), and (80), the equality holds trivially in (100). We now prove the case $i > n$. Inequality (93) and the definition of dual L_p-geominimal surface area give the following:

$$\begin{aligned}
\frac{\widetilde{G}_{-p,i}(K)}{\widetilde{G}_{-p,i}(B)} &\leq \left(\frac{\widetilde{G}_{-p}(K)}{\widetilde{G}_{-p}(B)}\right)^{(n-i)/n} \\
&\leq \left(\frac{V(K)}{V(B)}\right)^{(n+p)(n-i)/n^2},
\end{aligned} \tag{104}$$

with equality if K is a ball centered at the origin. □

The following results are interesting.

Theorem 23. *Let $K \in \mathcal{S}_o^n$, $p \geq 1$, $i \in \mathbb{R}$, and $0 \leq i \leq n$; then,*

$$\frac{\widetilde{G}_{-p,i}(K)}{\widetilde{G}_{-p,i}(B)} \leq \left(\frac{\widetilde{S}_p(K)}{\widetilde{S}_p(B)}\right)^{(n-i)/n}, \tag{105}$$

with equality if K is a ball centered at the origin.

Proof. Inequality (93) and the definition of dual L_p-geominimal surface area give the following:

$$\begin{aligned}
\left(\frac{\widetilde{G}_{-p,i}(K)}{\widetilde{G}_{-p,i}(B)}\right)^n &\leq \left(\frac{\widetilde{G}_{-p}(K)}{n\omega_n}\right)^{n-i} \\
&\leq \left(\frac{n\widetilde{V}_{-p}(K,B)}{n\widetilde{V}_{-p}(B,B)}\right)^{n-i} \\
&= \left(\frac{\widetilde{S}_p(K)}{\widetilde{S}_p(B)}\right)^{n-i},
\end{aligned} \tag{106}$$

with equality if K is a ball centered at the origin. □

Competing Interests

The authors declare that they have no competing interests.

Acknowledgments

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Research Article

Classes of I-Convergent Double Sequences over n-Normed Spaces

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I introduce some new classes of $\mathcal{I}$-convergent double sequences defined by a sequence of moduli over n-normed space. Study of their algebraic and topological properties and some inclusion relations has also been done.

1. Introduction

The notion of $\mathcal{I}$-convergence was introduced by Kostyrko et al. in [1]. It is known that $\mathcal{I}$-convergence is generalization of the statistical convergence which was introduced by Fast [2]. It was further studied by Demirci [3], Das et al. [4], Šalát et al. [5], and many others.

For a nonempty set X, the family of sets $\mathcal{I} \subset 2^X$, the power set of X, is said to be an ideal if

(1) $\phi \in \mathcal{I}$;

(2) $\mathcal{I}$ is additive; that is, $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$;

(3) $\mathcal{I}$ is hereditary; that is, $A \in \mathcal{I}, B \subseteq A \Rightarrow B \in \mathcal{I}$.

A nontrivial ideal $\mathcal{I}$ is called admissible if $\{\{x\} : x \in X\} \subseteq \mathcal{I}$. $\mathcal{I}$ is maximal if there cannot exist any nontrivial ideal $J \neq \mathcal{I}$ containing $\mathcal{I}$ as a subset.

Let $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ denote the set of natural, real, and complex numbers, respectively. A double sequence of complex numbers is defined as a function from $\mathbb{N}\times\mathbb{N}$ to $\mathbb{C}$. A number $a \in \mathbb{C}$ is called a limit of a double sequence (x_{ij}) if for every $\epsilon > 0$ there exists some $N = N(\epsilon) \in \mathbb{N}$ such that $|x_{ij} - a| < \epsilon$, $\forall i, j \geq N$. The set of all double sequences is denoted by ${}_2\omega$. Any subset of the ${}_2\omega$ is called double sequence space. A sequence $(x_{ij}) \in {}_2\omega$ is said to be $\mathcal{I}$-convergent to a number L if, for every $\epsilon > 0$, $\{(i, j) \in \mathbb{N}\times\mathbb{N} : |x_{ij}-L| \geq \epsilon\} \in \mathcal{I}$. In this case we write $\mathcal{I} - \lim x_{ij} = L$.

A double sequence space E is said to be solid or normal if $(x_{ij}) \in E$ implies $(\alpha_{ij}x_{ij}) \in E$ for all sequences of scalars (α_{ij}) with $|\alpha_{ij}| < 1$ for all $i, j \in \mathbb{N}$. For more details please see [6–8].

Example 1. Let $\mathcal{I}_2(P)$ be the class of all subsets of $\mathbb{N}\times\mathbb{N}$ such that $D \in \mathcal{I}_2(P)$ implies that there exists $n_0, k_0 \in N$ such that $D \subset \mathbb{N}\times\mathbb{N} \mid \{(n,k) \in \mathbb{N}\times\mathbb{N} : n \geq n_0,\ k \geq k_0\}$.

Then $\mathcal{I}_2(P)$ is an ideal of $\mathbb{N}\times\mathbb{N}$ in the usual Pringsheim sense of convergence of double sequences. If $\mathcal{I}_2(P)$ is replaced by $\mathcal{I}(f)$, the class of finite subsets of $\mathbb{N}$, then we get the usual regular convergence of double sequences.

The theory of 2-normed spaces was first introduced by Gähler [9] in 1964. Later on it was extended to n-normed spaces by Misiak [10]. Since then many mathematicians have worked in this field and obtained many interesting results; for instance see Gunawan [11, 12], Gunawan and Mashadi [13], Mursaleen and Mohiuddine [14], Şahiner et al. [15, 16], and Yamancı and Gürdal [17]. Let $n \in \mathbb{N}$ and let X^n be a linear metric space over the field $\mathbb{K}$ of real or complex numbers of dimension d, where $d \geq n \geq 2$. A real valued function $\|\cdot, \dots, \cdot\|$ on X^n satisfying the following four conditions:

(1) $\|x_1, x_2, \dots, x_n\| = 0$ if and only if $x_1, x_2, \dots, x_n$ are linearly dependent;

(2) $\|x_1, x_2, \dots, x_n\|$ is invariant under permutation;

(3) $\|\alpha x_1, x_2, \dots, x_n\| = |\alpha|\|x_1, x_2, \dots, x_n\|$ for any $\alpha \in \mathbb{K}$;

(4) $\|x + x', x_2, \dots, x_n\| \leq \|x, x_2, \dots, x_n\| + \|x', x_2, \dots, x_n\|$

is called an n-norm on X and the pair $(X, \|\cdot, \ldots, \cdot\|)$ is called an n-normed space over the field $\mathbb{K}$.

Example 2. If we take $X = \mathbb{R}^n$, equipped with Euclidean n-norm $\|x_1, x_2, \ldots, x_n\|_E = \mathrm{Vol}(n - \dim \text{parallelopiped})$ spanned by vectors $(x_1, x_2, \ldots, x_n)$, then the n-norm may be given by the formula $\|x_1, x_2, \ldots, x_n\|_E = |x_{ij}|$, where $(x_{ij}) = (x_{i1}, x_{i2}, \ldots, x_{in})$ for $i = 1, 2, 3, \ldots, n$.

The standard n-norm on X is defined as

$$\|x_1, x_2, \ldots, x_n\|_E = \begin{vmatrix} \langle x_1, x_1 \rangle & \cdots & \langle x_1, x_n \rangle \\ \vdots & \cdots & \vdots \\ \langle x_1, x_1 \rangle & \cdots & \langle x_1, x_1 \rangle \end{vmatrix}^{1/2}, \tag{1}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on X. If $X = \mathbb{R}^n$, then this n-norm is exactly the same as the Euclidean n-norm $\|x_1, x_2, \ldots, x_n\|_E$ mentioned earlier. For $n = 1$ this n-norm is the usual norm $\|x\| = \langle x_1, x_1 \rangle^{1/2}$.

A sequence (x_k) in an n-normed space $(X, \|\cdot, \ldots, \cdot\|)$ is said to converge to some $L \in \mathbb{K}$ if

$$\lim_{k\to\infty} \|x_k - L, z_1, \ldots, z_{n-1}\| = 0 \quad \text{for every } z_1, z_2, \ldots, z_{n-1} \in X. \tag{2}$$

A sequence (x_k) in an n-normed space $(X, \|\cdot, \ldots, \cdot\|)$ is said to be Cauchy if

$$\lim_{k,p\to\infty} \|x_k - x_p, z_1, \ldots, z_{n-1}\| = 0 \quad \text{for every } z_1, z_2, \ldots, z_{n-1} \in X. \tag{3}$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n- normed space is said to be an n-Banach space.

The concept of modulus function was introduced by Nakano [18] in the year 1953. It was further studied by [7, 8, 19–21] and many more. It is defined as a function $f: [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(1) $f(t) = 0$ if and only if $t = 0$,
(2) $f(t + u) \leq f(t) + f(u)$ for all $t, u \geq 0$,
(3) f is increasing,
(4) f is continuous from the right at zero.

Ruckle [22] used the idea of a modulus function f to construct the sequence space

$$X(f) = \left\{ x = (x_k) : \sum_{k=1}^{\infty} f(|x_k|) < \infty \right\}. \tag{4}$$

The space $X(f)$ is closely related to the space l_1 which is an $X(f)$ space with $f(x) = x$ for all real $x \geq 0$. Thus Ruckle [23, 24] proved that, for any modulus f,

$$X(f) \subset l_1, \qquad X(f)^{\alpha} = l_{\infty}. \tag{5}$$

The space $X(f)$ is a Banach space with respect to the norm

$$\|x\| = \sum_{k=1}^{\infty} f(|x_k|) < \infty. \tag{6}$$

After then Kolk [25, 26] gave an extension of $X(f)$ by considering a sequence of modulus functions called the sequence of moduli $F = (f_k)$ and defined the sequence space:

$$X(F) = \{ x = (x_k) : (f_k(|x_k|)) \in X \}. \tag{7}$$

From the above four properties of modulus function it can be clearly seen that f must be continuous everywhere on $[0, \infty)$. For a sequence of moduli, we have further two properties:

(5) $\sup_k f_k(t) < \infty$ for all $t > 0$;
(6) $\lim_{t\to 0} f_k(t) = 0$ uniformly in X and for $k \geq 1$.

Example 3. Let f be a function from $[0, \infty)$ to $[0, \infty)$. If we take $f(x) = x/(x + 1)$, then the function f is a bounded modulus function and if we take $f(x) = x^p$, $0 < p < 1$, then f is an unbounded modulus function.

By a lacunary sequence $\theta = (k_r)$; $r = 0, 1, 2, \ldots$, where $k_0 = 0$, we mean an increasing sequence of nonnegative integers $h_r = (k_r - k_{r-1}) \to \infty$ as $(r \to \infty)$. The intervals determined by θ are denoted by $I_r = (k_r - 1, k_r]$ and the ratio $k_r/(k_r - 1)$ will be denoted by q_r. The space of lacunary strongly convergent sequence N_θ was defined by Freedman et al. [27] as follows:

$$N_\theta = \left\{ x = (x_k) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k \in I_r} |x_k - L| = 0, \text{ for some } L \right\}. \tag{8}$$

The double lacunary sequence was defined by Savaş and Patterson [28]. A double sequence $\theta_{r;s} = \{(k_r, l_s)\}$ is called double lacunary if there exist two increasing sequences of integers such that

$$\begin{aligned} k_0 &= 0, \\ h_r &= k_r - k_{r-1} \longrightarrow \infty \quad \text{as } r \longrightarrow \infty, \\ l_0 &= 0, \\ h_s &= l_s - l_{s-1} \longrightarrow \infty \quad \text{as } s \longrightarrow \infty. \end{aligned} \tag{9}$$

The following interval is determined by θ:

$$I_{r,s} = \{ (k, l) : k_{r-1} < k < k_r,\ l_{s-1} < l < l_s \}. \tag{10}$$

The space of double lacunary strongly convergent sequence is defined as follows:

$$N_{\theta_{r,s}} = \left\{ x = (x_{kl}) : \lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} |x_{kl} - L| = 0, \text{ for some } L \right\}. \tag{11}$$

2. New Classes of Double Sequences

Now, we will define the new classes of double sequences.

Let $\mathcal{I}$ be an admissible ideal, let $F = (f_{ij})$ be a sequence of moduli, let $(X, \|\cdot, \ldots, \cdot\|)$ be an n-normed space, let $p = (p_{ij})$ be a sequence of positive real numbers, let $u = (u_{ij})$ be a sequence of strictly positive real numbers, and let ${}_2w(n-X)$ be the space of all double sequences defined over the n-normed space $(X, \|\cdot, \ldots, \cdot\|)$; then for some $L \in \mathbb{K}$ and every $z_i \in X$, we define

(i) $[{}_2N_\theta, F, u, p, \|\cdot, \ldots, \cdot\|, X_s]^{\mathcal{I}} = \{x = (x_{ij}) \in {}_2w(n-X) : [(r,s) \in \mathbb{N}\times\mathbb{N} : (1/h_{r,s})\sum_{(i,j)\in I_{r,s}} u_{ij}[F(\|x - L, z_1, z_2, \ldots, z_{n-1}\|)]^{p_{ij}} \geq \epsilon] \in \mathcal{I}\}$,

(ii) $[{}_2N_\theta, F, u, p, \|\cdot, \ldots, \cdot\|, X_s]_0^{\mathcal{I}} = \{x = (x_{ij}) \in {}_2w(n-X) : [(r,s) \in \mathbb{N}\times\mathbb{N} : (1/h_{r,s})\sum_{(i,j)\in I_{r,s}} u_{ij}[F(\|x_{ij}, z_1, z_2, \ldots, z_{n-1}\|)]^{p_{ij}} \geq \epsilon] \in \mathcal{I}\}$.

Case 1. If $F(x) = x$, then we get

(i) $[{}_2N_\theta, u, p, \|\cdot, \ldots, \cdot\|, X_s]^{\mathcal{I}} = \{x = (x_{ij}) \in {}_2w(n-X) : [(r,s) \in \mathbb{N}\times\mathbb{N} : (1/h_{r,s})\sum_{(i,j)\in I_{r,s}} u_{ij}(\|x_{ij} - L, z_1, z_2, \ldots, z_{n-1}\|)^{p_{ij}} \geq \epsilon] \in \mathcal{I}\}$,

(ii) $[{}_2N_\theta, u, p, \|\cdot, \ldots, \cdot\|, X_s]_0^{\mathcal{I}} = \{x = (x_{ij}) \in {}_2w(n-X) : [(r,s) \in \mathbb{N}\times\mathbb{N} : (1/h_{r,s})\sum_{(i,j)\in I_{r,s}} u_{ij}(\|x_{ij}, z_1, z_2, \ldots, z_{n-1}\|)^{p_{ij}} \geq \epsilon] \in \mathcal{I}\}$.

Case 2. If $p = (p_{ij}) = 1$, then we get

(i) $[{}_2N_\theta, F, u, \|\cdot, \ldots, \cdot\|, X_s]^{\mathcal{I}} = \{x = (x_{ij}) \in {}_2w(n-X) : [(r,s) \in \mathbb{N}\times\mathbb{N} : (1/h_{r,s})\sum_{(i,j)\in I_{r,s}} u_{ij}[F(\|x_{ij} - L, z_1, z_2, \ldots, z_{n-1}\|)] \geq \epsilon] \in \mathcal{I}\}$,

(ii) $[{}_2N_\theta, F, u, \|\cdot, \ldots, \cdot\|, X_s]_0^{\mathcal{I}} = \{x = (x_{ij}) \in {}_2w(n-X) : [(r,s) \in \mathbb{N}\times\mathbb{N} : (1/h_{r,s})\sum_{(i,j)\in I_{r,s}} u_{ij}[F(\|x_{ij}, z_1, z_2, \ldots, z_{n-1}\|)] \geq \epsilon] \in \mathcal{I}\}$.

Case 3. If $p = (p_{ij}) = 1$ and $u = (u_{ij}) = 1$, then we get

(i) $[{}_2N_\theta, F, \|\cdot, \ldots, \cdot\|, X_s]^{\mathcal{I}} = \{x = (x_{ij}) \in {}_2w(n-X) : [(r,s) \in \mathbb{N}\times\mathbb{N} : (1/h_{r,s})\sum_{(i,j)\in I_{r,s}} [F(\|x_{ij} - L, z_1, z_2, \ldots, z_{n-1}\|)] \geq \epsilon] \in \mathcal{I}\}$,

(ii) $[{}_2N_\theta, F, \|\cdot, \ldots, \cdot\|, X_s]_0^{\mathcal{I}} = \{x = (x_{ij}) \in {}_2w(n-X) : [(r,s) \in \mathbb{N}\times\mathbb{N} : (1/h_{r,s})\sum_{(i,j)\in I_{r,s}} [F(\|x_{ij}, z_1, z_2, \ldots, z_{n-1}\|)] \geq \epsilon] \in \mathcal{I}\}$.

The following inequality will be used throughout the paper. If $0 \leq p_{ij} \leq \sup p_{ij} = H, K = \max(1, 2^{H-1})$, then we have

$$\left|a_{ij} + b_{ij}\right|^{p_{ij}} \leq K\left\{\left|a_{ij}\right|^{p_{ij}} + \left|b_{ij}\right|^{p_{ij}}\right\} \tag{12}$$

for all $a_{ij}, b_{ij} \in \mathbb{C}$ and $(i,j) \in \mathbb{N}\times\mathbb{N}$. Also $|a|^{p_{ij}} \leq \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

3. Main Results

Theorem 4. *The sets* $[{}_2N_\theta, F, u, p, \|\cdot, \ldots, \cdot\|, X_s]^{\mathcal{I}}$ *and* $[{}_2N_\theta, F, u, p, \|\cdot, \ldots, \cdot\|, X_s]_0^{\mathcal{I}}$ *are linear spaces over the field of complex numbers* $\mathbb{C}$*.*

Proof. Let $x = (x_{ij}),\ y = (y_{ij}) \in [{}_2N_\theta, F, u, p, \|\cdot, \ldots, \cdot\|, X_s]^{\mathcal{I}}$, $\alpha, \beta \in \mathbb{C}$, and $C = \max\{1, (|\alpha|/(|\alpha| + |\beta|))^H, (|\beta|/(|\alpha| + |\beta|))^H\}$; then for every $z_i \in X$ we can write

$$\begin{aligned} &\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}} u_{ij}\left[F\left(\left\|x_{ij} - L_1, z_1, z_2, \ldots, z_{n-1}\right\|\right)\right]^{p_{ij}} \\ &\quad \leq \frac{\epsilon}{2KC}, \\ &\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}} u_{ij}\left[F\left(\left\|y_{ij} - L_2, z_1, z_2, \ldots, z_{n-1}\right\|\right)\right]^{p_{ij}} \\ &\quad \leq \frac{\epsilon}{2KC}. \end{aligned} \tag{13}$$

By the use of inequality (12), we have the following inequality:

$$\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}} u_{ij}\left[F\left(\left\|\left(\left(\alpha x_{ij} + \beta y_{ij}\right) - \left(\alpha L_1 + \beta L_2\right)\right), z_1, z_2, \ldots, z_{n-1}\right\|\right)\right]^{p_{ij}} \leq KC\left(\frac{\epsilon}{2KC}\right) < \epsilon. \tag{14}$$

This inequality says to us that the inclusion

$$\begin{aligned} &\Bigg[(r,s) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_{r,s}} \\ &\cdot \sum_{(i,j)\in I_{r,s}} \left[F\left(\left\|\left(\left(\alpha x + \beta y\right) - \left(\alpha L_1 + \beta L_2\right)\right), z_1, z_2, \ldots, z_{n-1}\right\|\right)\right] \\ &\geq \epsilon\Bigg] \subseteq \Bigg[(r,s) \in \mathbb{N}\times\mathbb{N} : KC\frac{1}{h_{r,s}} \\ &\cdot \sum_{(i,j)\in I_{r,s}} u_{ij}\left[F\left(\left\|\left(x - L_1\right), z_1, z_2, \ldots, z_{n-1}\right\|\right)\right]^{p_{ij}} \geq \epsilon\Bigg] \\ &\cup \Bigg[(r,s) \in \mathbb{N}\times\mathbb{N} : KC\frac{1}{h_{r,s}} \\ &\cdot \sum_{(i,j)\in I_{r,s}} u_{ij}\left[F\left(\left\|\left(y - L_2\right), z_1, z_2, \ldots, z_{n-1}\right\|\right)\right]^{p_{ij}} \geq \epsilon\Bigg] \end{aligned} \tag{15}$$

holds. From here, since the right side belongs to $\mathcal{I}$, the left side also belongs to $\mathcal{I}$. This completes the proof. □

Lemma 5. *Let f be a modulus function and let $0 < \delta < 1$. Then for each $x > \delta$, one has*

$$f(x) \le 2f(1)\delta^{-1}x. \quad (16)$$

Theorem 6. *Let $F = (f_{ij})$ be a sequence of moduli and $0 \le \inf_{i,j} p_{ij} = h \le \sup_{i,j} p_{ij} = H < \infty$. Then the following statements hold:*

(i) $[{}_2N_\theta, u, p, \|\cdot,\dots,\cdot\|, X_s]^{\mathcal{I}} \subset [{}_2N_\theta, F, u, p, \|\cdot,\dots,\cdot\|, X_s]^{\mathcal{I}}$,

(ii) $[{}_2N_\theta, u, p, \|\cdot,\dots,\cdot\|, X_s]_0^{\mathcal{I}} \subset [{}_2N_\theta, F, u, p, \|\cdot,\dots,\cdot\|, X_s]_0^{\mathcal{I}}$.

Proof. For some $\delta > 0$, choose $\delta_0 > 0$ such that $\max\{\delta_0^h, \delta_0^H\} < \delta$. By the continuity of $F = (f_{ij})$ for all $(i,j) \in \mathbb{N}\times\mathbb{N}$, we can choose some $\epsilon \in (0,1)$ such that for every t with $0 < t \le \epsilon$ we have

$$F(t) < \delta_0 \quad \forall (i,j) \in \mathbb{N}\times\mathbb{N}. \quad (17)$$

Let $x = (x_{ij}) \in [{}_2N_\theta, u, p, \|\cdot,\dots,\cdot\|, X_s]^{\mathcal{I}}$; then for some $L > 0$, $\delta > 0$ and for every $z_i \in X$, $i = 1,2,3,\dots,(n-1)$, we have

$$A = \left\{ (r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} u_{ij} \left[\left(\|x - L, z_1, z_2, \dots, z_{n-1}\| \right) \right]^{p_{ij}} \ge \epsilon^H \right\} \in \mathcal{I}. \quad (18)$$

Therefore for $(r,s) \notin A$, we have

$$\begin{aligned} &\frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} u_{ij} \left[\left(\|x - L, z_1, z_2, \dots, z_{n-1}\| \right) \right]^{p_{ij}} < \epsilon^H, \\ &\frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} u_{ij} \left[\left(\|x - L, z_1, z_2, \dots, z_{n-1}\| \right) \right]^{p_{ij}} \le \epsilon. \end{aligned} \quad (19)$$

So, by inequality (17), we can write

$$\begin{aligned} &F\left[\frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} \left(\|x - L, z_1, z_2, \dots, z_{n-1}\| \right) \right]^{p_{ij}} \le \delta_0, \\ &\frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} \left[F\left(\|x - L, z_1, z_2, \dots, z_{n-1}\| \right) \right]^{p_{ij}} \\ &\quad < \max\left\{ \delta_0^h, \delta_0^H \right\} < \delta, \end{aligned}$$

$$\left\{ (r,s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} u_{ij} \left[F\left(\|x - L, z_1, z_2, \dots, z_{n-1}\| \right) \right]^{p_{ij}} \ge \delta \right\} \in \mathcal{I}. \quad (20)$$

This implies that $x \in [{}_2N_\theta, F, u, p, \|\cdot,\dots,\cdot\|, X_s]^{\mathcal{I}}$. Hence $[{}_2N_\theta, u, p, \|\cdot,\dots,\cdot\|, X_s]^{\mathcal{I}} \subset [{}_2N_\theta, F, u, p, \|\cdot,\dots,\cdot\|, X_s]^{\mathcal{I}}$. The inclusion is strict as for the reverse inclusion we need the condition given in the next theorem. The other part can be proved similarly. □

Theorem 7. *Let F be a sequence of moduli. If $\lim_t \sup(f_{ij}(t)/t) = A > 0$ for all $(i,j) \in \mathbb{N}\times\mathbb{N}$, then*

(i) $[{}_2N_\theta, u, p, \|\cdot,\dots,\cdot\|, X_s]^{\mathcal{I}} = [{}_2N_\theta, F, u, p, \|\cdot,\dots,\cdot\|, X_s]^{\mathcal{I}}$,

(ii) $[{}_2N_\theta, u, p, \|\cdot,\dots,\cdot\|, X_s]_0^{\mathcal{I}} = [{}_2N_\theta, F, u, p, \|\cdot,\dots,\cdot\|, X_s]_0^{\mathcal{I}}$.

Proof. (i) To prove $[{}_2N_\theta, u, p, \|\cdot,\dots,\cdot\|, X_s]^{\mathcal{I}} = [{}_2N_\theta, F, u, p, \|\cdot,\dots,\cdot\|, X_s]^{\mathcal{I}}$, it is sufficient to show that $[{}_2N_\theta, F, u, p, \|\cdot,\dots,\cdot\|, X_s]^{\mathcal{I}} \subset [{}_2N_\theta, u, p, \|\cdot,\dots,\cdot\|, X_s]^{\mathcal{I}}$. Let $x \in [{}_2N_\theta, F, u, p, \|\cdot,\dots,\cdot\|, X_s]^{\mathcal{I}}$; then, by the definition, we get

$$\frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} u_{ij} \left[F\left(\|x - L, z_1, z_2, \dots, z_{n-1}\| \right) \right]^{p_{ij}} < \delta. \quad (21)$$

By the given condition $\lim_t \sup(f_{ij}(t)/t) = A > 0$ for all $(i,j) \in \mathbb{N}\times\mathbb{N}$, we have $f_{ij}(t) \ge At$ for all (i,j); that is,

$$\begin{aligned} &F\left(\|x - L, z_1, z_2, \dots, z_{n-1}\| \right) \\ &\quad \ge A\left(\|x - L, z_1, z_2, \dots, z_{n-1}\| \right), \end{aligned} \quad (22)$$

$$\begin{aligned} &\frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} u_{ij} \left[F\left(\|x - L, z_1, z_2, \dots, z_{n-1}\| \right) \right]^{p_{ij}} \\ &\quad \ge A^H \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} u_{ij} \left(\|x - L, z_1, z_2, \dots, z_{n-1}\| \right)^{p_{ij}}. \end{aligned} \quad (23)$$

From inequalities (21) and (23), we get

$$\begin{aligned} \delta &> \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} u_{ij} \left[F\left(\|x - L, z_1, z_2, \dots, z_{n-1}\| \right) \right]^{p_{ij}} \\ &\ge A^H \frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} u_{ij} \left(\|x - L, z_1, z_2, \dots, z_{n-1}\| \right)^{p_{ij}} \end{aligned} \quad (24)$$

which consequently implies that

$$\frac{1}{h_{r,s}} \sum_{(i,j)\in I_{r,s}} u_{ij} \left(\|x - L, z_1, z_2, \dots, z_{n-1}\| \right)^{p_{ij}} < \delta. \quad (25)$$

That is,

$$\Big\{(r,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}}u_{ij}\left[\left(\|x-L,z_1,z_2,\dots,z_{n-1}\|\right)\right]^{p_{ij}}\geq\delta\Big\}\in\mathcal{I}. \quad (26)$$

This implies that $x\in[{}_2N_\theta,u,p,\|\cdot,\dots,\cdot\|,X_s]^{\mathcal{I}}$.

$$[{}_2N_\theta,u,p,\|\cdot,\dots,\cdot\|,X_s]^{\mathcal{I}}\subseteq[{}_2N_\theta,F,u,p,\|\cdot,\dots,\cdot\|,X_s]^{\mathcal{I}}. \quad (27)$$

Hence, from the previous theorem and inclusion (27), we get the required result. The other part can be proved similarly. □

Corollary 8. *Let $F'=(f'_{ij})$ and $F''=(f''_{ij})$ be sequences of moduli. If $\lim_t\sup(f'_{ij}(t)/f''_{ij}(t))<\infty$ for all $(i,j)\in\mathbb{N}\times\mathbb{N}$, then*

(i) $[{}_2N_\theta,F',u,p,\|\cdot,\dots,\cdot\|,X_s]^{\mathcal{I}}=[{}_2N_\theta,F'',u,p,\|\cdot,\dots,\cdot\|,X_s]^{\mathcal{I}}$,

(ii) $[{}_2N_\theta,F',u,p,\|\cdot,\dots,\cdot\|,X_s]_0^{\mathcal{I}}=[{}_2N_\theta,F'',u,p,\|\cdot,\dots,\cdot\|,X_s]_0^{\mathcal{I}}$.

Theorem 9. *Let $(X,\|\cdot,\dots,\cdot\|_{X_S})$ and $(X,\|\cdot,\dots,\cdot\|_{X_E})$ be the standard and the Euclidean n-norm spaces, respectively. Then*

$$[{}_2N_\theta,F,u,p,\|\cdot,\dots,\cdot\|,X_S]^{\mathcal{I}}\cap[{}_2N_\theta,F,u,p,\|\cdot,\dots,\cdot\|,X_E]^{\mathcal{I}}\subset[{}_2N_\theta,F,u,p,\|\cdot,\dots,\cdot\|_{X_S}+\|\cdot,\dots,\cdot\|_{X_E}]^{\mathcal{I}}. \quad (28)$$

Proof. The proof of this result is easy, so it is omitted. □

Theorem 10. *Let $F'=(f'_{ij})$ and $F''=(f''_{ij})$ be sequences of moduli; then*

(i) $[{}_2N_\theta,F'',u,p,\|\cdot,\dots,\cdot\|,X_s]^{\mathcal{I}}\subseteq[{}_2N_\theta,F'\circ F'',u,p,\|\cdot,\dots,\cdot\|,X_s]^{\mathcal{I}}$,

(ii) $[{}_2N_\theta,F'',u,p,\|\cdot,\dots,\cdot\|,X_s]_0^{\mathcal{I}}\subseteq[{}_2N_\theta,F'\circ F'',u,p,\|\cdot,\dots,\cdot\|,X_s]_0^{\mathcal{I}}$,

(iii) $[{}_2N_\theta,F',u,p,\|\cdot,\dots,\cdot\|,X_S]^{\mathcal{I}}\cap[{}_2N_\theta,F'',u,p,\|\cdot,\dots,\cdot\|,X_S]^{\mathcal{I}}\subseteq[{}_2N_\theta,F'+F'',u,p,\|\cdot,\dots,\cdot\|,X_S]^{\mathcal{I}}$,

(iv) $[{}_2N_\theta,F',u,p,\|\cdot,\dots,\cdot\|,X_S]_0^{\mathcal{I}}\cap[{}_2N_\theta,F'',u,p,\|\cdot,\dots,\cdot\|,X_S]_0^{\mathcal{I}}\subseteq[{}_2N_\theta,F'+F'',u,p,\|\cdot,\dots,\cdot\|,X_S]_0^{\mathcal{I}}$.

Proof. (i) For some $\epsilon>0$, we choose $\epsilon_0>0$ such that $\max\epsilon_0^H<\epsilon$. Now as F' is a sequence of modulus functions which are always continuous, we can choose $\delta\in(0,1)$ such that, for every $t\in(0,\delta)$, we get $F'(t)<\epsilon_0$:

$$x=(x_{ij})\in[{}_2N_\theta,F'',u,p,\|\cdot,\dots,\cdot\|,X_s]^{\mathcal{I}}. \quad (29)$$

Then, by the definition, we have

$$A=\Big\{(r,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{r,s}}\cdot\sum_{(i,j)\in I_{r,s}}u_{ij}\left[F''\left(\|x-L,z_1,z_2,\dots,z_{n-1}\|\right)\right]^{p_{ij}}\geq\delta^H\Big\}\in\mathcal{I}. \quad (30)$$

Thus, for $(r,s)\notin A$, we get

$$\begin{aligned}&\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}}u_{ij}\left[F''\left(\|x-L,z_1,z_2,\dots,z_{n-1}\|\right)\right]^{p_{ij}}<\delta^H,\\&\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}}u_{ij}\left[F''\left(\|x-L,z_1,z_2,\dots,z_{n-1}\|\right)\right]^{p_{ij}}\leq\delta.\end{aligned} \quad (31)$$

Now, by the continuity of F', we have

$$F'\left[\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}}u_{ij}\left[F''\left(\|x-L,z_1,z_2,\dots,z_{n-1}\|\right)\right]^{p_{ij}}\right]<\epsilon, \quad (32)$$

which further implies that

$$\left[\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}}u_{ij}\left[F'\left(F''\|x-L,z_1,z_2,\dots,z_{n-1}\|\right)\right]^{p_{ij}}\right]<\max\epsilon^H<\epsilon, \quad (33)$$

which implies

$$\Big\{(r,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{r,s}}\cdot\sum_{(i,j)\in I_{r,s}}u_{ij}\left[F'\circ F''\left(\|x-L,z_1,z_2,\dots,z_{n-1}\|\right)\right]^{p_{ij}}\geq\epsilon\Big\}\in\mathcal{I}. \quad (34)$$

Therefore $x\in[{}_2N_\theta,F'\circ F'',u,p,\|\cdot,\dots,\cdot\|,X_s]^{\mathcal{I}}$. This completes the proof. □

(iii) Again consider

$$x\in[{}_2N_\theta,F',u,p,\|\cdot,\dots,\cdot\|,X_s]^{\mathcal{I}}\cap[{}_2N_\theta,F'',u,p,\|\cdot,\dots,\cdot\|,X_s]^{\mathcal{I}}. \quad (35)$$

Then by the definition of both the spaces, we get

$$\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}} u_{ij}\left[F'\left(\|x-L,z_1,z_2,\dots,z_{n-1}\|\right)\right]^{p_{ij}}<\epsilon,$$
$$\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}} u_{ij}\left[F''\left(\|x-L,z_1,z_2,\dots,z_{n-1}\|\right)\right]^{p_{ij}}<\epsilon. \tag{36}$$

Using the fact that $(F'+F'')(x)\le KF'(x)+KF''(x)$, we have

$$\begin{aligned}&\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}} u_{ij}\left[\left(F'+F''\right)\left(\|x-L,z_1,z_2,\dots,z_{n-1}\|\right)\right]^{p_{ij}}\\ &\le K\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}} u_{ij}\left[F'\left(\|x-L,z_1,z_2,\dots,z_{n-1}\|\right)\right]^{p_{ij}}\\ &\quad+K\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}} u_{ij}\left[F''\left(\|x-L,z_1,z_2,\dots,z_{n-1}\|\right)\right]^{p_{ij}}\\ &\le K(\epsilon+\epsilon)=2K\epsilon=\epsilon'\ (\text{say}).\end{aligned} \tag{37}$$

So we get

$$\left\{(r,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{r,s}}\sum_{(i,j)\in I_{r,s}} u_{ij}\left[\left(F'+F''\right)\cdot\left(\|x-L,z_1,z_2,\dots,z_{n-1}\|\right)\right]^{p_{ij}}\ge\epsilon\right\}\in\mathcal{I}. \tag{38}$$

Therefore we have

$$x\in\left[{}_2N_\theta,F'+F'',u,p,\|\cdot,\dots,\cdot\|,X_s\right]^{\mathcal{I}}. \tag{39}$$

Hence we have

$$\begin{aligned}&\left[{}_2N_\theta,F',u,p,\|\cdot,\dots,\cdot\|,X_S\right]^{I}\\ &\quad\cap\left[{}_2N_\theta,F'',u,p,\|\cdot,\dots,\cdot\|,X_S\right]^{\mathcal{I}}\\ &\quad\subseteq\left[{}_2N_\theta,F'+F'',u,p,\|\cdot,\dots,\cdot\|,X_S\right]^{\mathcal{I}}.\end{aligned} \tag{40}$$

4. Conclusion

A detailed study of some new classes of I-convergent double sequences over n-normed spaces has been done. Some algebraic and topological properties and inclusion relations have been proved with supported examples.

Competing Interests

The author declares that he has no competing interests.

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Research Article

A Generalization of Uniformly Extremely Convex Banach Spaces

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We discuss a new class of Banach spaces which are the generalization of uniformly extremely convex spaces introduced by Wulede and Ha. We prove that the new class of Banach spaces lies strictly between either the classes of k-uniformly rotund spaces and k-strongly convex spaces or classes of fully k-convex spaces and k-strongly convex spaces and has no inclusive relation with the class of locally k-uniformly convex spaces. We obtain in addition some characterizations and properties of this new class of Banach spaces. In particular, our results contain the main results of Wulede and Ha.

1. Introduction

Different uniformly convex spaces have been defined between the uniformly convex spaces [1] and the reflexivity of the Banach spaces [2–6]. In the previous paper [7] we introduce a new class of this type, namely, uniformly extremely convex spaces. This new class of Banach spaces lies strictly between either the classes of uniformly convex spaces and strongly convex spaces or the classes of fully k-convex spaces and strongly convex spaces.

Here we consider another new class of this type, namely, k-uniformly extremely convex spaces, as a generalization of uniformly extremely convex spaces and discuss its relation to the drop property, the k-uniformly rotund spaces, the full k-convex spaces, the k-strongly convex spaces, the nearly uniformly convex spaces, and k-nearly uniformly convex spaces. We also give some characterizations of k-uniformly extremely convex spaces and find that this new class of Banach spaces has the following features:

(1) 1-uniformly extremely convex spaces (*indeed* lower case) coincide with uniformly extremely convex spaces;

(2) k-uniformly extremely convex spaces possess the drop linebreak property;

(3) k-uniformly extremely convex spaces are $(k+1)$-uniformly extremely convex spaces, but the converse implication is not true.

Throughout this paper X denotes an infinite-dimensional real Banach space with the norm $\|\cdot\|$. The symbol X^* denotes the dual of the space X. $U(X)$ and $S(X)$ denote the closed unit ball and the unit sphere of X, respectively. The symbol $S(X^*)$ denotes the unit sphere of X^*. The symbol $\sigma(X, X^*)$ denotes the weak topology of X.

Let $x_1, x_2, \ldots, x_{k+1}$ be norm-1 elements in Banach spaces X. The k-dimensional volume enclosed by $x_1, x_2, \ldots, x_{k+1}$ is given by

$$V(x_1, x_2, \ldots, x_{k+1}) = \sup\left\{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ f_1(x_1) & f_1(x_2) & \cdots & f_1(x_{k+1}) \\ \vdots & \vdots & \ddots & \vdots \\ f_k(x_1) & f_k(x_2) & \cdots & f_k(x_{k+1}) \end{vmatrix} : f_1, \ldots, f_k \in S(X^*)\right\}. \quad (1)$$

Here, and throughout the sequel, the symbol $|\cdot|$ denotes the determinant.

Sullivan [6] has introduced the k-uniformly rotund (kUR) spaces and locally k-uniformly rotund (LkUR) spaces. Fan and Glicksberg [2] have introduced the fully k-convex (kR) Banach spaces. It is well known that kUR and kR spaces imply reflexivity. About kUR and kR spaces, we have the following chain of implications [2, 6, 8]:

$$\begin{gathered} \mathrm{UR} = 1\mathrm{UR} \Longrightarrow \cdots \Longrightarrow k\mathrm{UR} \Longrightarrow (k+1)\,\mathrm{UR}; \\ 2\mathrm{R} \Longrightarrow \cdots \Longrightarrow k\mathrm{R} \Longrightarrow (k+1)\,\mathrm{R}; \\ \mathrm{LUR} = \mathrm{L1UR} \Longrightarrow \cdots \Longrightarrow \mathrm{L}k\mathrm{UR} \Longrightarrow L\,(k+1)\,\mathrm{UR}. \end{gathered} \tag{2}$$

A Banach space X is said to be a kUR space ($k \ge 1$) [6] if, for any $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that, for all norm-1 elements $x_1, x_2, \ldots, x_{k+1}$ and $\|x_1 + x_2 + \cdots + x_{k+1}\| > (k+1) - \delta$, we have $V(x_1, x_2, \ldots, x_{k+1}) < \epsilon$.

A Banach space X is said to be a kR space ($k \ge 2$) [2] if, for any sequence $\{x_n\}$ in X such that $\lim_{n_1,\ldots,n_k \to \infty} (1/k)\|x_{n_1} + x_{n_2} + \cdots + x_{n_k}\| = 1$, then $\{x_n\}$ is a Cauchy sequence in X.

A point $x_0 \in S(X)$ is said to be a denting point of $U(X)$ [8] if $x_0 \notin \overline{\mathrm{co}}(M(x_0, \epsilon))$ for all $\epsilon > 0$, where $M(x_0, \epsilon) = \{y : y \in U(X), \|y - x_0\| \ge \epsilon\}$.

Huff [3] has introduced the nearly uniformly convex (NUC) spaces as a generalization of uniformly convex Banach spaces and showed that the NUC spaces are equivalent to reflexive spaces possessing the uniform Kadec-Klee property. The local version of NUC spaces, namely, locally nearly uniformly convex (LNUC), was studied by Kutzarova and Lin [9]. Kutzarova [4] introduced the k-nearly uniformly convex (kNUC) spaces as a generalization of nearly uniformly convex Banach spaces. In [4, 9], it is pointed out that NUC $\Rightarrow$ LNUC and kNUC $\Rightarrow$ NUC for every $k \ge 2$.

A Banach space X is said to be a NUC [3] space if, for any $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that, for any sequence $\{x_n\} \subset U(X)$, $\mathrm{sep}(x_n) > \epsilon$, we have $\mathrm{co}(\{x_n\}) \cap (1-\delta)U(X) \ne \emptyset$, where $\mathrm{sep}(x_n) = \inf\{\|x_n - x_m\| : n \ne m\}$ and $\mathrm{co}(\{x_n\})$ means the convex hull of $\{x_n\}$.

A Banach space X is said to be a LNUC [9] space if, for any norm-1 element x and $\epsilon > 0$, there exists a $\delta = \delta(\epsilon, x) > 0$ such that, for any sequence $\{x_n\} \subset U(X)$, $\mathrm{sep}(x_n) > \epsilon$, we have $\mathrm{co}(\{x\} \cup \{x_n\}) \cap (1-\delta)U(X) \ne \emptyset$, where $\mathrm{co}(\{x\} \cup \{x_n\})$ means the convex hull of $\{x\}$ and $\{x_n\}$.

A Banach space X is said to be a kNUC [4] space, if for any $\epsilon > 0$ there exists a $0 < \delta(\epsilon) < 1$ such that, for any sequence $\{x_n\} \subset U(X)$, $\mathrm{sep}(x_n) > \epsilon$, there are indices $\{n_i\}$ and scalars $\lambda_i \ge 0$, $i = 1, 2, \ldots, k$, with $\sum_{i=1}^{k} \lambda_i = 1$ so that $\|\sum_{i=1}^{k} \lambda_i x_{n_i}\| \le 1 - \delta$.

Singer [10] has introduced the k-strictly convex spaces. It is well known that k-strictly convex spaces are $(k+1)$-strictly convex spaces; 1-strictly convex spaces (*indeed* lower case) coincide with strictly convex spaces; kR spaces are k-strictly convex spaces and have the drop property. Wu and Li [11] have introduced the strongly convex spaces. Wulede and Wu [12] introduced the k-strongly convex spaces as a generalization of strongly convex Banach spaces and gave an equivalent definition of k-strongly convex spaces (see Theorem 5 in [13]). It is well known that k-strongly convex spaces are k-strictly convex spaces; 1-strongly convex spaces (*indeed* lower case) coincide with strongly convex spaces; k-strongly convex spaces are $(k+1)$-strongly convex spaces, but the converse implication is not true.

A Banach space X is said to be a k-strictly convex space [10] if, for all norm-1 elements $x_1, x_2, \ldots, x_{k+1}$ such that $\|\sum_{i=1}^{k+1} x_i\| = k+1$, then $x_1, x_2, \ldots, x_{k+1}$ are linearly dependent.

A Banach space X is said to be a strongly convex space [11] if, for any $x \in S(X)$, $\{x_n\} \subset S(X)$ and for a certain functional $f \in S_x$ such that $f(x_n) \to 1$ $(n \to \infty)$, then $\|x_n - x\| \to 0$ $(n \to \infty)$, where $S_x = \{f : f \in S(X^*), f(x) = 1\}$.

A Banach space X is said to be a k-strongly convex space [12] if, for any norm-1 element x, $\epsilon > 0$ and for any functional $f \in S_x$, there is a $\delta(x, f, \epsilon) > 0$ such that, for all norm-1 elements $x_1, \ldots, x_k$ and $f(x + x_1 + \cdots + x_k) > (k+1) - \delta$, we have $V(x, x_1, \ldots, x_k) < \epsilon$.

Rolewicz [14] has defined the norm $\|\cdot\|$ to have the drop property, if for every closed set $C \subset X$ disjoint from $U(X)$ there exists $x \in C$ such that $D(x, U(X)) \cap C = \{x\}$, where the set $D(x, U(X))$, the convex hull of x and $U(X)$, is called the drop generated by $x \notin U(X)$.

Lemma 1 (Kadec-Klee property). *If any $x \in S(X)$, $\{x_n\} \subset S(X)$ such that $x_n \xrightarrow{w} x$, $n \to \infty$, and $\|x_n\| \to \|x\|$, $n \to \infty$, then $\|x_n - x\| \to 0$, $n \to \infty$, where $x_n \xrightarrow{w} x$, $n \to \infty$, means that $f(x_n) \to f(x)$, $n \to \infty$, for all $f \in X^*$.*

Lemma 2 (Montesinos [15]). *Let X be a Banach space. Then X has the drop property if and only if X is reflexive and has the Kadec-Klee property.*

Lemma 3 (Nan and Wang [16]). *X is k-strictly convex space if and only if, for any $f \in S(X^*)$, one has $\dim A_f \le k$, where $A_f = \{x : x \in S(X), f(x) = 1\}$.*

Lemma 4 (Wulede and Wu [12], Zhang and Fang [17]). *Let X be a Banach space.*

(i) *If X is k-strongly convex, then X is k-strictly convex and has the Kadec-Klee property.*

(ii) *If X is reflexive, k-strictly convex and has the Kadec-Klee property, then X is k-strongly convex.*

(iii) *If X is k-strongly convex, $\{x_n\} \subset U(X)$, $f \in S(X^*)$, and $f(x_n) \to 1$, $n \to \infty$, then $\mathrm{dist}(x_n, A_f) \to 0$, $n \to \infty$.*

Lemma 5 (Zhang and Fang [17]). *X is k-strongly convex if and only if, for any $x \in S(X)$ and $f \in S_x$, $\epsilon > 0$, there exist $\delta > 0$ and a compact set $C \subset X$ with $\dim C \le k$ such that $F(f, \delta) \subset \{x : x \in X, d(x, C) < \epsilon\}$, where the set $F(f, \delta) = \{x : x \in U(X), f(x) \ge 1 - \delta\}$ is the slice generated by f and δ.*

2. k-Uniformly Extremely Convex Spaces and Drop Property

Definition 6 (see [7]). A Banach space X is said to be a uniformly extremely convex space if, for any sequences

$\{x_n\}, \{y_n\}$ consisting of elements of norm-1 and for a certain functional f of norm-1, $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(y_n) = 1$ holds; then $\lim_{n\to\infty}\|x_n - y_n\| = 0$.

On the base of uniformly extremely convex spaces, now we introduce the notion of k-uniformly extremely convex spaces as a generalization of uniformly extremely convex spaces.

Definition 7. A Banach space X is said to be a k-uniformly extremely convex space if, for any sequences $\{x_n^{(1)}\}, \ldots, \{x_n^{(k+1)}\}$ consisting of elements of norm-1 and for a certain functional f of norm-1, $\lim_{n\to\infty} f(x_n^{(1)}) = \cdots = \lim_{n\to\infty} f(x_n^{(k+1)}) = 1$ holds; then $\lim_{n\to\infty} V(x_n^{(1)}, \ldots, x_n^{(k+1)}) = 0$.

We give first a simple result which shows that the notion of k-uniformly extremely convex space is "coherent."

Theorem 8. *If X is k-uniformly extremely convex space, then X is $(k+1)$-uniformly extremely convex space.*

Proof. If, for any sequences $\{x_n^{(1)}\}, \ldots, \{x_n^{(k+2)}\}$ consisting of elements of norm-1 and for a certain functional f of norm-1, $\lim_{n\to\infty} f(x_n^{(1)}) = \cdots = \lim_{n\to\infty} f(x_n^{(k+2)}) = 1$ holds, then, for all $1 \le j \le k+2$, we have $\lim_{n\to\infty} V(x_n^{(1)}, \ldots, x_n^{(j-1)}, x_n^{(j+1)}, \ldots, x_n^{(k+2)}) = 0$ by the assumption that X is k-uniformly extremely convex space. Furthermore, by the properties of determinant we have

$$\begin{aligned} &\lim_{n\to\infty} V\left(x_n^{(1)}, \ldots, x_n^{(k+2)}\right) \\ &\quad \le \lim_{n\to\infty} \sum_{j=2}^{k+1} V\left(x_n^{(1)}, \ldots, x_n^{(j-1)}, x_n^{(j+1)}, \ldots, x_n^{(k+2)}\right) = 0; \end{aligned} \tag{3}$$

this shows that X is $(k+1)$-uniformly extremely convex space. □

Now we give a simple but useful lemma. By using this lemma we can prove that any k-uniformly extremely convex space has the drop property. And the fact that k-uniformly extremely convex spaces include k-strongly convex spaces can be easily found. We also show that 1-uniformly extremely convex spaces coincide with uniformly extremely convex spaces by using this lemma.

Lemma 9. *X is k-uniformly extremely convex if and only if, for any $\epsilon > 0$, $f \in S(X^*)$, there exists a $\delta(\epsilon) > 0$ such that, for all norm-1 elements $x_1, \ldots, x_{k+1}$ and $f(\sum_{i=1}^{k+1} x_i) > (k+1) - \delta$, one has $V(x_1, \ldots, x_{k+1}) < \epsilon$.*

Proof.

Necessity. Suppose the contrary. Then there exist $\epsilon_0 > 0$, $f_0 \in S(X^*)$ and $\{x_i\}_{i=1}^{k+1} \subset S(X)$ such that, for any $\delta = 1/n$, $n \in N$, we have $f_0(\sum_{i=1}^{k+1} x_i) > (k+1) - 1/n$, but $V(x_1, \ldots, x_{k+1}) \ge \epsilon_0$. Take $x_n^{(i)} = x_i$ $(i = 1, \ldots, k+1)$; then $\{x_n^{(i)}\}_{i=1}^{k+1} \subset S(X)$ and $k + 1 - 1/n < f_0(\sum_{i=1}^{k+1} x_n^{(i)}) \le k+1$. It follows that $\lim_{n\to\infty} f_0(x_n^{(i)}) = 1$. On the other hand, by the definition of the k-uniformly extremely convex space, we have $V(x_1, \ldots, x_{k+1}) \to 0$; this contradicts the statement that $V(x_1, \ldots, x_{k+1}) \ge \epsilon_0$.

Sufficiency. If, for any sequences $\{x_n^{(1)}\}, \ldots, \{x_n^{(k+1)}\}$ consisting of elements of norm-1 and for a certain functional f of norm-1, $\lim_{n\to\infty} f(x_n^{(1)}) = \cdots = \lim_{n\to\infty} f(x_n^{(k+1)}) = 1$ holds, then $\lim_{n\to\infty} f(\sum_{i=1}^{k+1} x_n^{(i)}) = k+1$. Therefore, for any $\delta > 0$, there exists an integer $N_0 \in N$ such that, for all $n \ge N_0$, inequality $f(\sum_{i=1}^{k+1} x_n^{(i)}) > (k+1) - \delta$ holds. For any $\epsilon > 0$, by the conditions given in Lemma 9, we have $V(x_n^{(1)}, \ldots, x_n^{(k+1)}) < \epsilon$; this means that $\lim_{n\to\infty} V(x_n^{(1)}, \ldots, x_n^{(k+1)}) = 0$. □

Remark 10. 1-uniformly extremely convex space (*indeed* lower case) coincides with uniformly extremely convex space.

In fact, by Lemma 9 we know that X is 1-uniformly extremely convex space if and only if, for any $\epsilon > 0$, $f \in S(X^*)$, there exists a $\delta(\epsilon, f) > 0$ such that, for any norm-1 elements x, y and $f(x+y) > 2 - \delta$, we have

$$\begin{aligned} V(x, y) &= \sup \left\{ \left| \begin{matrix} 1 & 1 \\ f(x) & f(y) \end{matrix} \right| : f \in S(X^*) \right\} \\ &= \sup_{f \in S(X^*)} |f(x) - f(y)| = \|x - y\| < \epsilon, \end{aligned} \tag{4}$$

and also if and only if X is uniformly extremely convex space.

Theorem 11. *X is k-uniformly extremely convex space if and only if X is k-strictly convex space and has the drop property.*

Proof.

Necessity. Suppose that X is k-uniformly extremely convex space; by the definition of k-strongly convex space and a condition which characterizes k-uniformly extremely convex space in Lemma 9, it is easy to see that X is k-strongly convex space. From Lemma 4(i), we know that X is k-strictly convex space and has the Kadec-Klee property.

Now we are going to prove that X has the drop property. In fact, from Lemma 2, it is sufficient to prove that X is reflexive. Suppose that X is not reflexive. Using the well-known James' theorem, for each $0 < \epsilon < 1$, we can choose $0 < \theta < 1$ so that $\theta > 1 - \delta(\epsilon)/(k+1)$ and $\theta^k > \epsilon$, and $\{x_1\}, \ldots, \{x_{k+1}\} \subset S(X)$, $\{x_1^*\}, \ldots, \{x_{k+1}^*\} \subset S(X^*)$ so that

$$x_j^*(x_i) = \begin{cases} \theta & \text{if } j \le i \\ 0 & \text{if } j > i. \end{cases} \tag{5}$$

Here $\delta(\epsilon)$ is the function required in Lemma 9.

Now we have that

$$x_1^*(x_1 + x_2 + \cdots + x_{k+1}) = (k+1)\theta > (k+1) - \delta(\epsilon). \tag{6}$$

On the other hand it is easy to check that

$$V(x_1,\dots,x_{k+1}) \geq \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_2^*(x_1) & x_2^*(x_2) & \cdots & x_2^*(x_{k+1}) \\ \vdots & \vdots & \ddots & \vdots \\ x_{k+1}^*(x_1) & x_{k+1}^*(x_2) & \cdots & x_{k+1}^*(x_{k+1}) \end{vmatrix} = \theta^k > \epsilon, \tag{7}$$

which gives the required contradiction.

Sufficiency. From the assumption that X is k-strictly convex space and has the drop property, we can deduce, by Lemma 4(ii), that X is k-strongly convex space and reflexive. Observing the definition of k-strongly convex space and a condition which characterizes k-uniformly extremely convex space in Lemma 9, by the reflexivity of X, it is easy to see that X is k-uniformly extremely convex space. □

Corollary 12 (see [7]). *X is uniformly extremely convex space if and only if X is strictly convex space and has the drop property.*

Noticing the procedure of proving Theorem 11 we can deduce the following.

Corollary 13. *If X is k-uniformly extremely convex space, then X is k-strongly convex space.*

Now we are going to show that the converse to Corollary 13 is not true. In [12], it is proved that LkUR spaces are k-strongly convex spaces. In general, LkUR spaces need not be reflexive since L1UR is just the usual definition of LUR space [18, 19]. It follows that there exists a k-strongly convex space X which is not reflexive. Hence X is not a k-uniformly extremely convex space since X is not reflexive.

Corollary 14. *X is k-uniformly extremely convex space if and only if X is reflexive and, for any $x \in S(X)$ and $f \in S_x$, $\epsilon > 0$, there exist $\delta > 0$ and a compact set $C \subset X$ with $\dim C \leq k$ such that $F(f,\delta) \subset \{x : x \in X, d(x,C) < \epsilon\}$, where the set $F(f,\delta) = \{x : x \in U(X), f(x) \geq 1-\delta\}$ is the slice generated by f and δ.*

Proof. It is immediate from Corollary 13, Theorem 11, and Lemmas 2, 4, and 5. □

Theorem 15. *X is k-uniformly extremely convex space if and only if X is reflexive and, for any $f \in S(X^*)$, one has $\dim A_f \leq k$, $A_f \cap \overline{\mathrm{co}}(U(X) \setminus V_{A_f}) = \emptyset$, where the set V_{A_f}, which includes set A_f, is arbitrary open set with regard to norm topology $(X, \|\cdot\|)$.*

Proof.

Necessity. Suppose that X is k-uniformly extremely convex space; then by Theorem 11 we know that X is k-strictly convex space and reflexive. For any $f \in S(X^*)$, by the reflexivity of X, there exists $x \in S(X)$ such that $f(x) = 1$; hence $x \in A_f$. Combining the fact that X is k-strictly convex space with Lemma 3 we can deduce that $\dim A_f \leq k$.

Now we are going to prove the equality $A_f \cap \overline{\mathrm{co}}(U(X) \setminus V_{A_f}) = \emptyset$.

Firstly, we will prove that, for any $z \notin V_{A_f}$ and every open set V_{A_f} (where $V_{A_f} \supset A_f$) with regard to norm topology $(X, \|\cdot\|)$, there exists a scalar $r > 0$ such that $\mathrm{dist}(z, A_f) \geq r$.

Noticing that A_f is compact set in X, for any $z \notin V_{A_f}$, we can deduce that there exists $x \in A_f$ such that $\|x - z\| = \mathrm{dist}(z, A_f) = r_z$. Now we claim that there exists minimum value of r_z denoted by r, such that $\mathrm{dist}(z, A_f) \geq r$ for any $z \notin V_{A_f}$. In fact, if r_z does not have minimum value, then $1/n$ is impossible to be minimum value for any integer n. Hence, there exist $z_n \notin V_{A_f}$ and $x_n \in A_f$ such that $\|x_n - z_n\| = \mathrm{dist}(z_n, A_f) < 1/n$. Since A_f is compact, the above sequence $\{x_n\}$ has the convergent subsequence; without loss of generality and letting the convergent subsequence be $\{x_n\}$ itself, then $x_n \to x_0$, $x_0 \in A_f$. Noticing that $\|x_n - z_n\| = \mathrm{dist}(z_n, A_f) < 1/n$, we can deduce that $z_n \to x_0$, $x_0 \in A_f \subset V_{A_f}$.

On the other hand, combining the fact that $X \setminus V_{A_f}$ is closed set with $z_n \in X \setminus V_{A_f}$, $z_n \to x_0$, we can deduce that $x_0 \in X \setminus V_{A_f}$. This contradicts $x_0 \subset V_{A_f}$.

Secondly, we will prove that for V_{A_f} there exists a scalar $m > 0$ such that the inequality $f(x) > f(y) + m$ holds for all $x \in A_f$ and $y \in U(X) \setminus V_{A_f}$.

If the above inequality is not true, then there exists $y_n \in U(X) \setminus V_{A_f}$ such that $f(y_n) \to f(x) = 1$, $n \to \infty$. By the condition given in Theorem 15, Corollary 13, and Lemma 4(iii), we have $\mathrm{dist}(y_n, A_f) \to 0$, $n \to \infty$. On the other hand, by $y_n \in U(X) \setminus V_{A_f}$, we can deduce that $\mathrm{dist}(y_n, A_f) \nrightarrow 0$, $n \to \infty$; this contradicts the statement that $\mathrm{dist}(y_n, A_f) \to 0$, $n \to \infty$. Hence we have

$$\begin{aligned} f(x) - m &\geq \sup\left\{f(y) : y \in U(X) \setminus V_{A_f}\right\} \\ &= \sup\left\{f(y) : y \in \overline{\mathrm{co}}\left(U(X) \setminus V_{A_f}\right)\right\}; \end{aligned} \tag{8}$$

this shows that $x \notin \overline{\mathrm{co}}(U(X) \setminus V_{A_f})$. By the arbitrary of $x \in A_f$, we can deduce that $A_f \cap \overline{\mathrm{co}}(U(X) \setminus V_{A_f}) = \emptyset$.

Sufficiency. By Lemmas 2 and 3, Theorem 11, and the condition given in Theorem 15, only we need to prove that X has the Kadec-Klee property. Let $x \in S(X)$, $\{x_n\}_{n=1}^{\infty} \subset S(X)$, and $x_n \xrightarrow{w} x$, $n \to \infty$. By the well-known James' theorem, there exists $f \in S(X^*)$ such that $f(x) = 1$; it follows that $x \in A_f$.

Case 1. If $\{x_n\}_{n=1}^{\infty} \cap A_f = \emptyset$, then $\{x_n\}_{n=1}^{\infty}$ is relatively compact. Otherwise, every point of A_f is not accumulation point of $\{x_n\}_{n=1}^{\infty}$. Hence, for any $x \in A_f$ there exists $\epsilon_0 > 0$ such that $\{y \in X : \|y - x\| < \epsilon_0\}$ does not contain any point of $\{x_n\}_{n=1}^{\infty}$. We construct an open set $V_{A_f} = \bigcup_{x \in A_f}\{y \in X : \|y - x\| < \epsilon_0\}$ with regard to norm topology $(X, \|\cdot\|)$; then $A_f \subset V_{A_f}$ and $V_{A_f} \cap \{x_n\}_{n=1}^{\infty} = \emptyset$. Since $\overline{\mathrm{co}}(U(X) \setminus V_{A_f})$ is bounded closed convex set with regard to norm topology

$(X, \|\cdot\|)$, $\overline{\mathrm{co}}^{w}(U(X) \setminus V_{A_f}) = \overline{\mathrm{co}}(U(X) \setminus V_{A_f})$. Noticing that $\overline{\mathrm{co}}^{w}(U(X) \setminus V_{A_f})$ is bounded set with regard to weak topology $\sigma(X, X^*)$, we know that $\overline{\mathrm{co}}^{w}(U(X) \setminus V_{A_f})$ is compact set with regard to weak topology $\sigma(X, X^*)$. Hence there is a function $g \in X^*$ which separates A_f and $\overline{\mathrm{co}}(U(X) \setminus V_{A_f})$; that is, there is a scalar $l > 0$ such that $g(A_f) - l > \sup g(\overline{\mathrm{co}}(U(X) \setminus V_{A_f}))$. Evidently, $\{x_n\}_{n=1}^{\infty} \subset \overline{\mathrm{co}}(U(X) \setminus V_{A_f})$; it follows that $g(A_f) - g(\{x_n\}_{n=1}^{\infty}) > l$. This contradicts the assumption that $x_n \xrightarrow{w} x$, $n \to \infty$.

Case 2. If $\{x_n\}_{n=1}^{\infty} \cap A_f \neq \emptyset$, then $(\{x_n\}_{n=1}^{\infty} \setminus A_f) \cap A_f = \emptyset$. According to Case 1 we know that $\{x_n\}_{n=1}^{\infty} \setminus A_f$ is relatively compact set. Hence $\{x_n\}_{n=1}^{\infty} \cap A_f$ is compact set since A_f is bounded closed convex set in certain finite dimensional subspace of X. On the other hand, it is obvious that $\{x_n\}_{n=1}^{\infty} = (\{x_n\}_{n=1}^{\infty} \setminus A_f) \cup (\{x_n\}_{n=1}^{\infty} \cap A_f)$; hence $\overline{\{x_n\}_{n=1}^{\infty}} = \overline{(\{x_n\}_{n=1}^{\infty} \setminus A_f)} \cup \overline{(\{x_n\}_{n=1}^{\infty} \cap A_f)}$. This shows that $\{x_n\}_{n=1}^{\infty}$ is relatively compact.

Consequently, in Cases 1 and 2, we always conclude that $\{x_n\}_{n=1}^{\infty}$ is relatively compact. Furthermore, by the assumption that $x_n \xrightarrow{w} x$, $n \to \infty$, we can deduce that $\|x_n - x\| \to 0$, $n \to \infty$. This completes the proof that X has the Kadec-Klee property. □

In particular, considering the special case of Theorem 15 when $k = 1$, we obtained Theorem 2.5 in [7] as a corollary.

Corollary 16. *X is uniformly extremely convex space if and only if X is reflexive and for any $f \in S(X^*)$, one has $\dim A_f = 1$, $A_f \cap \overline{\mathrm{co}}(U(X) \setminus V_{A_f}) = \emptyset$, where the set V_{A_f}, which includes set A_f, is arbitrary open set with regard to norm topology $(X, \|\cdot\|)$. In other words, X is uniformly extremely convex space if and only if X is reflexive and every point of $S(X)$ is a denting point of $U(X)$.*

To show that the converse to Theorem 8 is not true, we consider the following example.

Example 17. There exists a k-uniformly extremely convex space X which is not a $(k-1)$-uniformly extremely convex space.

Let $k \geq 2$ be an integer, and let $i_1 < i_2 < \cdots < i_k$. For each $x = (a_1, a_2, \ldots) \in l_2$, define

$$\|x\|_{i_1,\ldots,i_k}^2 = \left(\sum_{j=1}^{k} \left|a_{i_j}\right|\right)^2 + \sum_{i \neq i_1,\ldots,i_k} a_i^2. \tag{9}$$

From [20] we know that $X_{i_1,\ldots,i_k} = (l_2, \|\cdot\|_{i_1,\ldots,i_k})$ is a kUR space. It is easy to see that kUR space is k-uniformly extremely convex space from the definition of kUR space and a condition which characterizes k-uniformly extremely convex space in Lemma 9. Hence $X_{i_1,\ldots,i_k}$ is a k-uniformly extremely convex space. It follows from Theorem 11 that $X_{i_1,\ldots,i_k}$ is a k-strictly convex space but is not a $(k-1)$-strictly convex space that follows from [16]. Hence $X_{i_1,\ldots,i_k}$ is not a $(k-1)$-uniformly extremely convex space.

3. The Relations between k-Uniformly Extremely Convex Space and Various Other Types of Convex Space

Now we give a list of examples to distinguish k-uniformly extremely convex spaces from kR, kUR, kNUC, and NUC spaces.

(i) We are ready now to distinguish k-uniformly extremely convex and kR spaces.

Since kR spaces are k-strictly convex spaces and have the drop property, it follows from Theorem 11 that kR spaces are k-uniformly extremely convex, but the converse is not true.

Example 18. There exists a k-uniformly extremely convex space X which is not a kR space for every $k \geq 2$.

Let $k \geq 2$ be an integer, and let $i_1 < i_2 < \cdots < i_k$. For each $x = (a_1, a_2, \ldots) \in l_2$, define

$$\|x\|_{i_1,\ldots,i_k}^2 = \left(\sum_{j=1}^{k} \left|a_{i_j}\right|\right)^2 + \sum_{i \neq i_1,\ldots,i_k} a_i^2, \tag{10}$$

and let $X_{i_1,\ldots,i_k} = (l_2, \|\cdot\|_{i_1,\ldots,i_k})$. For $x \in l_2$, let $\|x\|_k = \sup_{i_1 < i_2 < \cdots < i_k} \|x\|_{i_1,\ldots,i_k}$, $X_k = (l_2, \|x\|_k)$. It follows from [20] that X_k is a kUR space but is not a kR space. Hence X_k is a k-uniformly extremely convex space since X is a kUR space.

(ii) We are ready now to distinguish k-uniformly extremely convex and kUR spaces.

Example 19. For all $k \geq 1$, there exists a k-uniformly extremely convex space X which is not a kUR space.

Let $E = (l_2, \|\cdot\|)$; for $x = (a_1, a_2, \ldots) \in E$, define

$$\begin{aligned} \|x\|^2 = {} & \left\{|a_1| + \left(a_2^2 + a_3^2 + \cdots\right)^{1/2}\right\}^2 \\ & + \left\{\left(\frac{a_2}{2}\right)^2 + \cdots + \left(\frac{a_n}{n}\right)^2 + \cdots\right\}^2. \end{aligned} \tag{11}$$

It follows from [2] that $X = (\sum \bigoplus E)_{l_2}$ is a 2R space; furthermore, $X = (\sum \bigoplus E)_{l_2}$ is a k-uniformly extremely convex space but is not a kUR space [20].

(iii) We are ready now to distinguish k-uniformly extremely convex and LkUR spaces.

We consider a nonreflexive LkUR space X. Then X is not a k-uniformly extremely convex space since X is not reflexive. On the other hand, we consider Example 19; then $X = (\sum \bigoplus E)_{l_2}$ is a 2R space and it follows that X is a k-uniformly extremely convex space for all $k \geq 1$. But X is not a LkUR space that follows from [21].

(iv) We are ready now to distinguish k-uniformly extremely convex spaces and NUC or kNUC spaces.

Example 20. For all $k \geq 1$, there exists a k-uniformly extremely convex space X which is neither a NUC nor a kNUC space for all $k \geq 2$.

Let $(X, \|\cdot\|)$ be the l_2-sum of $\{l_n,\ n \geq 2\}$; then $(X, \|\cdot\|)$ is a 2R space with normalized basis $\{e_n\}$. Define, $\forall x = \sum_{n=1}^{\infty} a_n e_n \in X$,

$$|||x||| = \left\{ \left(|a_1| + \left\| \sum_{n=2}^{\infty} a_n e_n \right\| \right)^2 + \sum_{n=2}^{\infty} \left(\frac{a_n}{n} \right)^2 \right\}^{1/2}. \tag{12}$$

By Theorem 4 in [9], we know that $(X, |||\cdot|||)$ is a 2R space but is not a LNUC space. It follows that X is a k-uniformly extremely convex space for all $k \geq 1$ but is neither a NUC nor a kNUC space for all $k \geq 2$.

Remark 21. (i) The class of k-uniformly extremely convex spaces lies strictly between the classes of kUR spaces and the k-strongly convex spaces.

(ii) The class of k-uniformly extremely convex spaces lies strictly between the classes of kR spaces and the class of k-strongly convex spaces.

(iii) The class of k-uniformly extremely convex spaces has no inclusive relation with the class of LkUR spaces.

In particular, considering the special case of Remark 21 when $k = 1$, we obtained the main conclusions of [7], that is, Remarks 3.5 and 3.7 in [7].

Competing Interests

The authors declare that they have no competing interests.

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Research Article

Products of Composition and Differentiation Operators from Bloch into Q_K Spaces

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The boundedness and compactness of the product of differentiation and composition operators from Bloch spaces into Q_K spaces are discussed in this paper.

1. Introduction and Motivation

Let Δ be the open unit disk in the complex plane and let $H(\Delta)$ be the class of all analytic functions on Δ. Let $dA(z)$ be the Euclidean area element on Δ. The Bloch space $\mathscr{B}$ on Δ is the space of all analytic functions f on Δ such that

$$\|f\|_{\mathscr{B}} = |f(0)| + \sup_{z\in\Delta}\left(1-|z|^2\right)\left|f'(z)\right| < \infty. \quad (1)$$

Under the above norm, $\mathscr{B}$ is a Banach space. Let $\mathscr{B}_0$ denote the subspace of $\mathscr{B}$ consisting of those $f \in \mathscr{B}$ for which $(1-|z|^2)f'|z| \to 0$ as $|z| \to 1$. This space is called the little Bloch space.

Throughout this paper, we assume that $K : [0,\infty) \to [0,\infty)$ is a nondecreasing and right-continuous function. A function $f \in H(\Delta)$ is said to belong to Q_K space (see [1]) if

$$\|f\|_K^2 = \sup_{a\in\Delta}\iint_{\Delta}\left|f'(z)\right| K(g(z,a))\, dA(z) < \infty, \quad (2)$$

where $g(z,a)$ is the Green function with logarithmic singularity at a; that is, $g(z,a) = \log(1/|\varphi_a(z)|)$ (φ_a is a conformal automorphism defined by $\varphi_a(z) = (a-z)/(1-a\overline{z})$ for $a \in \Delta$). Q_K is a Banach space under the norm

$$\|f\|_{Q_K}^2 = |f(0)| + \|f\|_K. \quad (3)$$

From [1], we know that $Q_K \subseteq \mathscr{B}$ if

$$\int_0^{1/e} K(-\log r)\, r\, dr < \infty. \quad (4)$$

Let φ denote a nonconstant analytic self-map of Δ. Associated with φ is the composition operator C_φ defined by $C_\varphi(f) = f \circ \varphi$ for $f \in H(\Delta)$. The problem of characterizing the boundedness and compactness of composition operators on many Banach spaces of analytic functions has attracted lots of attention recently, for example, [2] and the reference therein.

Let D be the differentiation operator on $H(\Delta)$; then we have $Df(z) = f'(z)$. For $f \in H(\Delta)$, the products of differentiation and composition operators DC_φ and $C_\varphi D$ are defined by

$$\begin{aligned} DC_\varphi(f) &= (f\circ\varphi)' = f'(\varphi)\varphi', \\ C_\varphi D(f) &= f'(\varphi), \\ &\qquad f \in H(\Delta). \end{aligned} \quad (5)$$

Operators $C_\varphi D$ as well as some other products of linear operators were studied, for example, in [3–9] (see also the references therein).

Recall that a linear operator $T : X \to Y$ is said to be bounded if there exists a constant $M > 0$ such that $\|T(f)\|_Y \leq$

$M\|f\|_X$ for all maps $f \in X$. And $X \to Y$ is compact if it takes bounded sets in X to sets in Y which have compact closure. For Banach spaces X and Y of $H(\Delta)$, T is compact from X to Y if and only if for each sequence $\{x_n\}$ in X; the sequence $\{Tx_n\} \in Y$ contains a subsequence converging to some limit in Y.

Considering the definition of Q_K spaces and $Q_K \subseteq \mathscr{B}$ with some conditions, it is difficult to study the operator $C_\varphi D$ from Bloch spaces to Q_K spaces. In this paper, some sufficient and necessary conditions for the boundedness and compactness of this operator are given.

2. The Boundedness

Lemma 1 (see [10]). *If all $f \in \mathscr{B}$, then*

$$\sup_{z\in\Delta}\left(1-|z|^2\right)^n\left|f^{(n)}\right| + |f(0)| + |f'(0)| + |f''(0)| + \cdots + \left|f^{(n-1)(0)}\right| \approx \|f\|_{\mathscr{B}}, \quad (n = 1, 2, \ldots). \tag{6}$$

Theorem 2. *Let φ be an analytic self-map of Δ. Suppose K is a nondecreasing and right-continuous function on $[0, +\infty)$ such that*

$$\int_0^{1/e} K(-\log r)\, r\, dr < +\infty. \tag{7}$$

Then the following statements are equivalent:

(a) $C_\varphi D : \mathscr{B} \to Q_K$ *is bounded.*
(b) $C_\varphi D : \mathscr{B}_0 \to Q_K$ *is bounded.*
(c)

$$\sup_{a\in\Delta}\iint_\Delta \frac{|\varphi'(z)|^2}{\left(1-|\varphi(z)|^2\right)^4} K(g(z,a))\, dA(z) < \infty. \tag{8}$$

Proof.

(c) ⇒ (a). Suppose that (c) holds. For any $z \in \Delta$ and $f(z) \in \mathscr{B}$, we have

$$\begin{aligned}\left\|C_\varphi Df(z)\right\|_K^2 &= \sup_{a\in\Delta}\iint_\Delta \left|f'(\varphi)'\right|^2 K(g(z,a))\, dA(z) \\ &= \sup_{a\in\Delta}\iint_\Delta \left|(\varphi'(z)) f''(\varphi)\right|^2 K(g(z,a))\, dA(z) \\ &\le \|f\|_{\mathscr{B}}^2 \sup_{a\in\Delta}\iint_\Delta \frac{(\varphi'(z))^2}{\left(1-|\varphi(z)|^2\right)^4} K(g(z,a))\, dA(z) \\ &< +\infty.\end{aligned} \tag{9}$$

Thus (a) holds.

(a) ⇒ (b). It is obvious.

(b) ⇒ (c). Assume (b) holds; that is, there exists a constant C such that $\|C_\varphi Df(z)\|_K \le C\|f\|_{\mathscr{B}}$ for all $f \in \mathscr{B}_0$. Conversely, suppose that $C_\varphi D : \mathscr{B} \to Q_K$ is bounded. Fix $w \in \Delta$ and assume that $\varphi(w) \neq 0$. Consider the function f defined by

$$f_w(z) = \frac{\left(1-|\varphi(w)|^2\right)^2}{\left(1-\overline{\varphi(w)}z\right)^2}, \tag{10}$$

for $z \in \Delta$; then

$$f_w'(z) = 2\overline{\varphi(w)}\frac{\left(1-|\varphi(w)|^2\right)^2}{\left(1-\overline{\varphi(w)}z\right)^3}, \tag{11}$$

for $z \in \Delta$. Since

$$\begin{aligned}\left|f_w'(z)\right| &\le \frac{2\left(1-|\varphi(w)|^2\right)^2}{\left(1-|\varphi(w)||z|\right)^3} \le \frac{2\left(1+|\varphi(w)|^2\right)^2}{1-|z|} \\ &\le \frac{8}{1-|z|}\end{aligned} \tag{12}$$

for all $z \in \Delta$, $f_w \in \mathscr{B}$. Furthermore, it is clear that $f_w \in \mathscr{B}_0$, since

$$\left|f_w(0)\right| = \left|\left(1-|\varphi(w)|^2\right)^2\right| \le 4. \tag{13}$$

Thus

$$M_1 = \sup\{\|f_w\|_{\mathscr{B}} : w \in \Delta\} \le 12. \tag{14}$$

Note that

$$\left|f_w''(\varphi(w))\right| = \frac{2|\varphi(w)|}{\left(1-|\varphi(w)|^2\right)^2}. \tag{15}$$

For this function f_w and this point w we have

$$\begin{aligned}\left|\left(C_\varphi Df_w\right)'(w)\right| &= \left|\varphi'(w) f_w''(\varphi(w))\right| \\ &= \frac{2|\varphi'(w)||\varphi(w)|^2}{\left(1-|\varphi(w)|^2\right)^2}.\end{aligned} \tag{16}$$

So

$$\begin{aligned}&\sup_{a\in\Delta}\iint_\Delta \frac{4|\varphi'(w)|^2|\varphi(w)|^4}{\left(1-|\varphi(w)|^2\right)^4} K(g(z,a))\, dA(z) \\ &\quad\le \left\|C_\varphi Df_w\right\|_K^2 \le M_1\left\|C_\varphi D\right\|_{\mathscr{B}\to Q_K}^2 < +\infty,\end{aligned} \tag{17}$$

for all $w \in \Delta$. Then we can imply

$$\begin{aligned}&\sup_{a\in\Delta}\iint_{|\varphi(w)|>1/2} \frac{|\varphi'(w)|^2}{\left(1-|\varphi(w)|^2\right)^4} K(g(z,a))\, dA(z) \\ &\le 16\sup_{a\in\Delta}\iint_{|\varphi(w)|>1/2} \frac{|\varphi'(w)|^2|\varphi(w)|^4}{\left(1-|\varphi(w)|^2\right)^4} K(g(z,a))\, dA(z) \\ &\le \left\|C_\varphi Df_w\right\|_K^2 \le M_1\left\|C_\varphi D\right\|_{\mathscr{B}\to Q_K}^2 < +\infty.\end{aligned} \tag{18}$$

On the other hand, we note the functions $f(z) \equiv z^2$, which belong to $\mathscr{B}_0$, and we get

$$2 \sup_{a\in\Delta} \iint_{\Delta} \left|\varphi'(z)\right|^2 K(g(z,a))\, dA(z) < \infty. \quad (19)$$

Then

$$\begin{aligned}
&\sup_{a\in\Delta} \iint_{|\varphi(w)|<1/2} \frac{\left|\varphi'(w)\right|^2}{\left(1-\left|\varphi(w)\right|^2\right)^4} K(g(z,a))\, dA(z) \\
&\leq \left(\frac{4}{3}\right)^4 \sup_{a\in\Delta} \iint_{|\varphi(w)|<1/2} \left|\varphi'(z)\right|^2 K(g(z,a))\, dA(z) \\
&\leq \infty.
\end{aligned} \quad (20)$$

So, we get

$$\sup_{a\in\Delta} \iint_{\Delta} \frac{\left|\varphi'(w)\right|^2}{\left(1-\left|\varphi(w)\right|^2\right)^4} K(g(z,a))\, dA < \infty. \quad (21)$$

By the arbitrary of w, we have

$$\sup_{a\in\Delta} \iint_{\Delta} \frac{\varphi'(z)}{\left(1-\left|\varphi(z)\right|^2\right)^4} K(g(z,a))\, dA < \infty, \quad (22)$$

for all $z \in \Delta$.

This completes the proof of this theorem. □

3. The Compactness

The following lemma can be proved similarly to [11].

Lemma 3. *Let φ be an analytic self-map of Δ. Then $C_\varphi D : \mathscr{B} \to Q_K$ (or $\mathscr{B}_0 \to Q_K$) is compact if and only if $C_\varphi D : \mathscr{B} \to Q_K$ (or $\mathscr{B}_0 \to Q_K$) is bounded and for any bounded sequence $(f_n)_{n\in N}$ in $\mathscr{B}$ which converges to zero uniformly on compact subsets of Δ; one has $\|C_\varphi D f_n\|_{Q_K} \to 0$ as $n \to \infty$.*

Lemma 4. *Let φ be an analytic self-map of Δ. Suppose K is a nondecreasing and right-continuous function on $[0, +\infty)$ such that*

$$\int_0^{1/e} K(-\log r)\, r\, dr < +\infty. \quad (23)$$

If $C_\varphi D : \mathscr{B}(\mathscr{B}_0) \to Q_K$ is compact, then for any $\epsilon > 0$ there exists a δ, $0 < \delta < 1$ such that, for all f in E,

$$\sup_{a\in\Delta} \iint_{|\varphi(z)>r|} \left|\left(C_\varphi D f\right)'(z)\right|^2 K(g(z,a))\, dA(z) < \epsilon \quad (24)$$

holds whenever $\delta < r < 1$, where E is the unit ball of $\mathscr{B}(\mathscr{B}_0)$.

Proof. For $f \in \mathscr{B}_0$, let $f_t(z) = f(tz)$ $(0 < t < 1)$. Then $f_t \in \mathscr{B}_0$, and $f_t \to f$ uniformly on compact subsets of Δ as $t \to 1$. Since $C_\varphi D$ is compact, $\|(C_\varphi D f_t - C_\varphi D f)(z)\| \to 0$ as $t \to 1$. That is, for given $\varepsilon > 0$, there exists $t \in (0,1)$ such that

$$\begin{aligned}
&\sup_{a\in\Delta} \iint_{\Delta} \left|\left(\left(C_\varphi D f_t\right)' - \left(C_\varphi D f\right)\right)(z)\right|^2 \\
&\quad \cdot K(g(z,a))\, dA(z) < \epsilon.
\end{aligned} \quad (25)$$

For r $(0 < r < 1)$, the triangle equality gives

$$\begin{aligned}
&\sup_{a\in\Delta} \iint_{|\varphi(z)>r|} \left|\left(C_\varphi D f\right)'(z)\right|^2 K(g(z,a))\, dA(z) \\
&\quad \leq \sup_{a\in\Delta} \iint_{|\varphi(z)>r|} \left|\left(C_\varphi D f_t\right)'(z)\right|^2 \\
&\quad \cdot K(g(z,a))\, dA(z) \\
&\quad + \sup_{a\in\Delta} \iint_{|\varphi(z)>r|} \left|\left(\left(C_\varphi D f_t\right)' - \left(C_\varphi D f\right)\right)(z)\right|^2 \\
&\quad \cdot K(g(z,a))\, dA(z) \\
&\quad \leq \sup_{a\in\Delta} \iint_{|\varphi(z)>r|} \left|f_t''(\varphi(z))\,\varphi'(z)\right|^2 \\
&\quad \cdot K(g(z,a))\, dA(z) + \varepsilon \leq \left\|f_t''\right\|_\infty^2 \\
&\quad \cdot \sup_{a\in\Delta} \iint_{|\varphi(z)>r|} \left|\varphi'(z)\right|^2 K(g(z,a))\, dA(z) \\
&\quad + \varepsilon.
\end{aligned} \quad (26)$$

Then, we prove that for that given $\varepsilon > 0$ and $\|f_t''\|_\infty^2 > 0$ there exists a $\delta \in (0,1)$ such that if $\delta < r < 1$,

$$\left\|f_t''\right\|_\infty^2 \sup_{a\in\Delta} \iint_{|\varphi(z)>r|} \left|\varphi'(z)\right|^2 K(g(z,a))\, dA(z) < \varepsilon. \quad (27)$$

Choose $f_n(z) = z^n \in \mathscr{B}_0$, and we have $n\varphi^{n-1} \in Q_K$. Since $C_\varphi D$ is compact, $\lim_{n\to\infty}\|n\varphi^{n-1}\| = 0$. Thus, for given $\varepsilon > 0$ and $\|f_t''\|_\infty^2 > 0$, there exists an $N \in \mathbf{N}$ such that

$$\begin{aligned}
&\left\|f_t''\right\|_\infty^2 \sup_{a\in\delta} \iint_{\Delta} n^2 (n-1)^2 \left|\varphi^{n-2}(z)\right|^2 \left|\varphi'(z)\right|^2 \\
&\quad \cdot K(g(z,a))\, dA(z) < \varepsilon,
\end{aligned} \quad (28)$$

whenever $n \geq N$. Hence, for $0 < r < 1$,

$$\begin{aligned}
&N^2 (N-1)^2 \sup_{a\in\delta} \iint_{\Delta} \left|\varphi^{N-2}(z)\right|^2 \left|\varphi'(z)\right|^2 \\
&\quad \cdot K(g(z,a))\, dA(z) \geq N^2 (N-1)^2 \\
&\quad \cdot \sup_{a\in\Delta} \iint_{|\varphi(z)>r|} \left|\varphi^{N-2}(z)\right|^2 \left|\varphi'(z)\right|^2 \\
&\quad \cdot K(g(z,a))\, dA(z) \geq N^2 (N-1)^2 \\
&\quad \cdot r^{2(N-2)} \sup_{a\in\Delta} \iint_{|\varphi(z)>r|} \left|\varphi'(z)\right|^2 \\
&\quad \cdot K(g(z,a))\, dA(z).
\end{aligned} \quad (29)$$

Therefore, for $r > (N^2 - N)^{-1/(N-2)}$,

$$\left\|f_t''\right\|_\infty^2 \sup_{a\in\Delta} \iint_{|\varphi(z)>r|} \left|\varphi'(z)\right|^2 K(g(z,a))\, dA(z) < \varepsilon. \quad (30)$$

Thus we have already proved that, for any $\varepsilon > 0$ and $f \in \mathscr{B}_0$, there exists a $\delta = \delta(\varepsilon, f)$ such that

$$\sup_{a\in\Delta} \iint_{|\varphi(z)>r|} \left|f''(\varphi(z))\varphi'(z)\right|^2 K(g(z,a))\,dA(z) < \epsilon \tag{31}$$

holds whenever $\delta < r < 1$.

We finish our proof by showing that the above $\delta = \delta(\varepsilon, f)$, in fact, is independent of $f \in \mathscr{B}_0$. Since $C_\varphi D$ is compact, $C_\varphi D(E)$ is relatively compact in Q_K. It means that there is a finite collection of functions $f_1, f_2, \ldots, f_m$ in E such that, for any $\varepsilon > 0$ and $f \in E$, we can find f_k $(1 \le k \le m)$ satisfying

$$\sup_{a\in\Delta} \iint_{|\varphi(z)>r|} \left|\left(\left(C_\varphi Df_k\right)' - \left(C_\varphi Df\right)\right)(z)\right|^2 \cdot K(g(z,a))\,dA(z) < \varepsilon. \tag{32}$$

On the other hand, if $\max_{1\le k\le m}\delta_k(\varepsilon, f_k) = \delta < r < 1$, we have from the previous observation that, for all $k = 1, 2, \ldots, m$,

$$\sup_{a\in\Delta} \iint_{|\varphi(z)>r|} \left|\left(C_\varphi Df_k\right)'(z)\right|^2 K(g(z,a))\,dA(z) < \varepsilon. \tag{33}$$

By the triangle inequality we obtain that

$$\sup_{a\in\Delta} \iint_{|\varphi(z)>r|} \left|\left(C_\varphi Df\right)'(z)\right|^2 K(g(z,a))\,dA(z) < \varepsilon \tag{34}$$

holds whenever $\delta < r < 1$. The proof is complete. □

Theorem 5. *Let φ be an analytic self-map of Δ. Suppose K is a nondecreasing and right-continuous function on $[0, +\infty)$ such that*

$$\int_0^{1/e} K(-\log r)\, r\, dr < +\infty. \tag{35}$$

Then the following statements are equivalent:

(a) $C_\varphi D : \mathscr{B} \to Q_K$ *is compact.*

(b) $C_\varphi D : \mathscr{B}_0 \to Q_K$ *is compact.*

(c) $C_\varphi D : \mathscr{B} \to Q_K$ *is bounded:*

$$\lim_{t\to 1} \sup_{a\in\Delta} \iint_{|\varphi(z)|>t} \frac{\left|\varphi'(z)\right|^2}{\left(1 - |\varphi(z)|^2\right)^4} K(g(z,a))\,dA(z) = 0. \tag{36}$$

Proof.

(c) ⇒ (a). Assume (c) holds. Without loss of generality, let $\{f_n\}_{n\in N}$ be a sequence in E which converges to 0 uniformly on compact subsets of Δ, as $n \to +\infty$, where E is the unit ball of $\mathscr{B}$. By Cauchy's estimate, we know that $\{f_n''\}_{n\in N}$ also converges to 0 uniformly on compact subsets of Δ. For the sufficiency we will be verifying that $\{C_\varphi Df_n\}$ converges to 0 in Q_K norm. By the assumption, for any $\varepsilon > 0$, there exists a $t_0 \in (0, 1)$ such that

$$\sup_{a\in\Delta} \iint_{|\varphi(z)|>t_0} \frac{\left|\varphi'(z)\right|^2}{\left(1 - |\varphi(z)|^2\right)^4} K(g(z,a))\,dA(z) < \varepsilon. \tag{37}$$

Let $T = \{z \in \Delta : |\varphi(z)| \le t_0\}$; then we have

$$\begin{aligned}
&\left\|C_\varphi Df_n\right\|^2 \\
&= \sup_{a\in\Delta} \iint_{\Delta} \left|\varphi'(z)\, f_n''(\varphi(z))\right|^2 K(g(z,a))\,dA(z) \\
&= \sup_{a\in\Delta} \iint_{T} \left|\varphi'(z)\, f_n''(\varphi(z))\right|^2 K(g(z,a))\,dA(z) \\
&\quad + \sup_{a\in\Delta} \iint_{\Delta/T} \left|\varphi'(z)\, f_n''(\varphi(z))\right|^2 K(g(z,a))\,dA(z) \\
&\le \sup\left\{\left|f_n''(\varphi(z))\right|^2 : z \in T\right\} \|\varphi(z)\|_K + \|f_n\|_{\mathscr{B}}^2 \\
&\quad \cdot \sup_{a\in\Delta} \iint_{\Delta/T} \frac{\left|\varphi'(z)\right|^2}{\left(1 - |\varphi(z)|^2\right)^4} K(g(z,a))\,dA(z) \\
&\le \sup\left\{\left|f_n''(\varphi(z))\right|^2 : z \in T\right\} \|\varphi(z)\|_K + \|f_n\|_{\mathscr{B}}^2 \\
&\quad \cdot \varepsilon.
\end{aligned} \tag{38}$$

By $C_\varphi D$ is bounded, we know that $\varphi \in Q_K$. It follows that $\|C_\varphi Df_n\| \to 0$ since that $\sup\{|f_n''(\varphi(z))|^2 : z \in T\} \to 0$ as $n \to \infty$. By Lemma 1, we can obtain that $C_\varphi D : \mathscr{B} \to Q_K$ is compact.

(a) ⇒ (b). It is obvious.

(b) ⇒ (c). Suppose that $C_\varphi D : \mathscr{B}_0 \to Q_K$ is compact. Then it is clear that $C_\varphi D : \mathscr{B}_0 \to Q_K$ is bounded. We know that $f_\theta(z) = (1/2)\log(1/(1 - e^{-i\theta}z)) \in \mathscr{B}$ for all $\theta \in [0, 2\pi)$. Choose a sequence $\{\lambda_n\}$ in Δ which converges to 1 as $n \to \infty$, and let $f_{\theta,n} = f_\theta(\lambda_n z)$ for $n \in \mathbf{N}$. Thus $f_{\theta,n}$ in E for all $n \in \mathbf{N}$ and $\theta \in [0, 2\pi)$), where E is the unit ball of $\mathscr{B}_0$. By Lemma 4, for any $\varepsilon > 0$

$$\sup_{a\in\Delta} \iint_{|\varphi(z)|>t} \frac{\left|\lambda_n^2\varphi'(z)\right|^2}{\left(1 - e^{-i\theta}\lambda_n\varphi(z)\right)^4} K(g(z,a))\,dA(z) < \varepsilon \tag{39}$$

holds for any $n \in \mathbf{N}$ and $\theta \in [0, 2\pi)$. That is,

$$\lim_{t\to 1} \sup_{a\in\Delta} \iint_{|\varphi(z)|>t} \frac{\left|\lambda_n^2\varphi'(z)\right|^2}{\left(1 - e^{-i\theta}\lambda_n\varphi(z)\right)^4} K(g(z,a))\,dA(z) = 0. \tag{40}$$

Thus, we obtain (c) by integrating, with respect to θ, the Fubini theorem, the Poisson formula, and the Fatou's lemma. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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