# Recent Advances in Symmetry Groups and Conservation Laws for Partial Differential Equations and Applications 

Guest Editors: Maria Candarias, Mariano Torrisi, Maria Bruzón,
Rita Tracinà, and Chaudry Masood Khalipue

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## Abstract and Applied Analysis

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## Editorial

# Recent Advances in Symmetry Groups and Conservation Laws for Partial Differential Equations and Applications 

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Differential equations govern many natural phenomena and play an important role in the progress of engineering and technology. Essentially a lot of the fundamental equations are nonlinear and in general such nonlinear equations are often very difficult to solve explicitly. Symmetry group techniques provide methods to obtain solutions of these equations. These methods have several applications in the studies of partial differential equation. They are also useful in the search for conservation laws which arises in many fields of the applied sciences. Recent studies have shown that infinitely many nonlocal symmetries of various integrable models are related to their Lax pairs. Moreover symmetry method is one of the most powerful tools that give new integrable models from known ones. Integrable models have played an important role in applied sciences and are one of the central topics in soliton theory. In order to know if a system is integrable, it is very important to study Lax pairs of the system.

A symmetry can be considered as an equivalence transformation which leaves invariant not only the differential structure of equation but also the form of the arbitrary elements. This fact made Ovsiannikov search for equivalence transformations in a systematic way by using an algorithm based on the extension of the Lie infinitesimal criterion.

When an equation contains an arbitrary function, it reflects the individual characteristic of the phenomena belonging to a large class. In this sense, the knowledge
of equivalence transformations can provide us with certain relations between the solutions of different phenomena of the same class.

Nowadays several branches of the theoretical and applied sciences such as mathematics, physics, biology, economy, and finance rely on processes which are usually modeled by nonlinear differential equations. Often it is difficult to obtain reductions and exact solutions for these models. Our aim is to highlight applications of symmetry methods to nonlinear models in physics, engineering, and the applied science as well as to show recent theoretical developments in symmetry groups and geometric methods.

The authors of this special issue had been invited to submit original research articles as well as review articles in the following topics: advanced researches and theoretical analyses in group transformations and differential equations; Noether symmetries, applications, and conservation laws; numerical algorithms concerning the symmetry groups for partial differential equations; new and direct methods to obtain exact explicit solutions for differential equations; applications: novel applications in sciences, including engineering, physics, biology, and finance; and reviews: lucid surveys and review articles dealing with modern and classical topics.

However, we received 20 papers in these research fields. After a rigorous reviewing process, twelve articles were finally
accepted for publication. These articles contain some new and innovative techniques and ideas that may stimulate further researches in several branches of theory and applications of the transformation groups.

## Acknowledgments

We would like to thank all the authors who sent their works and all the referees for the time spent in reviewing the papers. Their contributions and their efforts have been very important for the publication of this special issue.

Maria Gandarias
Mariano Torrisi
Maria Bruzón Rita Tracinà
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## Research Article

# Invariant Inhomogeneous Bianchi Type-I Cosmological Models with Electromagnetic Fields Using Lie Group Analysis in Lyra Geometry 

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#### Abstract

We find a new class of invariant inhomogeneous Bianchi type-I cosmological models in electromagnetic field with variable magnetic permeability. For this, Lie group analysis method is used to identify the generators that leave the given system of nonlinear partial differential equations (NLPDEs) (Einstein field equations) invariant. With the help of canonical variables associated with these generators, the assigned system of PDEs is reduced to ordinary differential equations (ODEs) whose simple solutions provide nontrivial solutions of the original system. A new class of exact (invariant-similarity) solutions have been obtained by considering the potentials of metric and displacement field as functions of coordinates $x$ and $t$. We have assumed that $F_{12}$ is only nonvanishing component of electromagnetic field tensor $F_{i j}$. The Maxwell equations show that $F_{12}$ is the function of $x$ alone whereas the magnetic permeability $\bar{\mu}$ is the function of $x$ and $t$ both. The physical behavior of the obtained model is discussed.


## 1. Introduction

The inhomogeneous cosmological models play a significant role in understanding some essential features of the universe, such as the formation of galaxies during the early stages of evolution and process of homogenization. Therefore, it will be interesting to study inhomogeneous cosmological models. The best-known inhomogeneous cosmological model is the Lemaitre-Tolman model (or LT model) which deals with the study of structure in the universe by means of exact solutions of Einstein's field equations. Some other known exact solutions of inhomogeneous cosmological models are the Szekeres metric, Szafron metric, Stephani metric, KantowskiSachs metric, Barnes metric, Kustaanheimo-Qvist metric, and Senovilla metric [1].

Einstein's general theory relativity is based on Riemannian geometry. If one modifies the Riemannian geometry, then Einstein's field equations will be changed automatically from its original form. Modifications of Riemannian geometry have developed to solve the problems such as unification of gravitation with electromagnetism, problems
arising when the gravitational field is coupled to matter fields, and singularities of standard cosmology. In recent years, there has been considerable interest in alternative theory of gravitation to explain the above-unsolved problems. Long ago, since 1951, Lyra [2] proposed a modification of Riemannian geometry by introducing a gauge function into the structureless manifold that bears a close resemblance to Weyl's geometry.

Using the above modification of Riemannian geometry Sen [3, 4] and Sen and Dunn [5] proposed a new scalar tensor theory of gravitation and constructed it very similar to Einstein field equations. Based on Lyra's geometry, the field equations can be written as $[3,4]$

$$
\begin{equation*}
R_{i j}-\frac{1}{2} g_{i j} R+\frac{3}{2} \phi_{i} \phi_{j}-\frac{3}{4} g_{i j} \phi_{k} \phi^{k}=-\chi T_{i j}, \tag{1}
\end{equation*}
$$

where $\phi_{i}$ is the displacement vector and other symbols have their usual meaning as in Riemannian geometry.

Halford [6] has argued that the nature of constant displacement field $\phi_{i}$ in Lyra's geometry is very similar to cosmological constant $\Lambda$ in the normal general relativistic
theory. Halford also predicted that the present theory will provide the same effects within observational limits, as far as the classical solar system tests are concerned, as well as tests based on the linearized form of field equations. For a review on Lyra geometry, one can see [7].

Recently, Pradhan et al. [8-14], Casana et al. [15], Rahaman et al. [16], Bali and Chandnani [17, 18], Kumar and Singh [19], Yadav et al. [20], Rao et al. [21], Zia and Singh [22], and Ali and Rahaman [23] have studied cosmological models based on Lyra's geometry in various contexts.

To study the nonlinear physical phenomena [24-27], it is important to search the exact solutions of nonlinear PDEs. Ovsiannikov [28] is the pioneer who had observed that the usual Lie infinitesimal invariance approach could as well be employed in order to construct symmetry groups [2931]. The symmetry groups of a differential equation could be defined as the groups of continuous transformations that lead a given family of equations invariant [32-35] and are proved to be important to solve the nonlinear equations of the models to describe complex physical phenomena in various fields of science, especially in fluid mechanics, solid state physics, plasma physics, plasma wave, and general relativity.

In this paper, we have obtained exact solutions of Einstein's modified field equations in inhomogeneous space-time Bianchi type-I cosmological model within the frame work of Lyra's geometry in the presence of magnetic field with variable magnetic permeability and time varying displacement vector $\beta(x, t)$ using the so-called symmetry analysis method. Since the field equations are highly nonlinear differential equations, therefore symmetry analysis method can be successfully applied to nonlinear differential equations. The similarity (invariant) solutions help to reduce the independent variables of the problem, and therefore we employ this method in the investigation of exact solution of the field equations. In general, invariant solutions will transform the system of nonlinear PDEs into a system of ODEs. We attempted to find a new class of exact (invariant) solutions for the field equations based on Lyra geometry.

The scheme of the paper is as follows. Magnetized inhomogeneous Bianchi type-I cosmological model with variable magnetic permeability based on Lyra geometry is introduced in Section 2. In Section 3, we have performed symmetry analysis and have obtained isovector fields for Einstein field equations under consideration. In Section 4, we found new class of exact (invariant) solutions for Einstein field equations. Section 5 is devoted to study of some physical and geometrical properties of the model.

## 2. The Metric and Field Equations

We consider Bianchi type-I metric, with the convention ( $x^{0}=$ $t, x^{1}=x, x^{2}=y, x^{3}=z$ ), in the form

$$
\begin{equation*}
d s^{2}=d t^{2}-A^{2} d x^{2}-B^{2} d y^{2}-C^{2} d z^{2} \tag{2}
\end{equation*}
$$

where $A$ is a function of $t$ only while $B$ and $C$ are functions of $x$ and $t$. Without loss of generality, we can put the following transformation:

$$
\begin{equation*}
B=A f, \quad C=A g \tag{3}
\end{equation*}
$$

where $f$ and $g$ are functions of $x$ and $t$. The volume element of model (2) is given by

$$
\begin{equation*}
V=\sqrt{-g}=A^{3} f g . \tag{4}
\end{equation*}
$$

The four-acceleration vector, the rotation, the expansion scalar, and the shear scalar characterizing the four-velocity vector field, $u^{i}$, satisfying the relation in comoving coordinate system

$$
\begin{equation*}
g_{i j} u^{i} u^{j}=1, \quad u^{i}=u_{i}=(1,0,0,0), \tag{5}
\end{equation*}
$$

respectively, have the usual definitions as given by Raychaudhuri [36]

$$
\begin{align*}
& \dot{u}_{i}=u_{i ; j} u^{j} \\
& \omega_{i j}=u_{[i ; j]}+\dot{u}_{[i} u_{j]}, \\
& \Theta=u_{i ;}^{i}  \tag{6}\\
& \sigma^{2}=\frac{1}{2} \sigma_{i j} \sigma^{i j}
\end{align*}
$$

where

$$
\begin{equation*}
\sigma_{i j}=u_{(i, j)}+\dot{u}_{(i} u_{j)}-\frac{1}{3} \Theta\left(g_{i j}+u_{i} u_{j}\right) . \tag{7}
\end{equation*}
$$

In view of metric (2), the four-acceleration vector, the rotation, the expansion scalar, and the shear scalar given by (6) can be written in a comoving coordinates system as

$$
\begin{align*}
\dot{u}_{i}= & 0 \\
\omega_{i j}= & 0 \\
\Theta= & \frac{3 \dot{A}}{A}+\frac{f_{t}}{f}+\frac{g_{t}}{g}  \tag{8}\\
\sigma^{2}= & \frac{\dot{A}}{A}\left(\frac{2 \dot{A}}{A}+\frac{4 f_{t}}{3 f}+\frac{4 g_{t}}{3 g}\right)+\frac{5 f_{t}^{2}}{9 f^{2}} \\
& +\frac{f_{t} g_{t}}{9 f g}+\frac{5 g_{t}^{2}}{9 g^{2}}
\end{align*}
$$

where the nonvanishing components of the shear tensor $\sigma_{i}^{j}$ are

$$
\begin{array}{ll}
\sigma_{1}^{1}=-\frac{f_{t}}{3 f}-\frac{g_{t}}{3 f}, & \sigma_{2}^{2}=\frac{2 f_{t}}{3 f}-\frac{g_{t}}{3 g}, \\
\sigma_{3}^{3}=\frac{2 g_{t}}{3 g}-\frac{f_{t}}{3 f}, & \sigma_{4}^{4}=-\frac{2 \dot{A}}{A}-\frac{2 f_{t}}{3 f}-\frac{2 g_{t}}{3 g} . \tag{9}
\end{array}
$$

To study the cosmological model, we use the field equations in Lyra geometry given in (1) in which the displacement field vector $\phi_{i}$ is given by

$$
\begin{equation*}
\phi_{i}=(\beta(x, t), 0,0,0) \tag{10}
\end{equation*}
$$

$T_{i j}$ is the energy momentum tensor given by

$$
\begin{equation*}
T_{i j}=(\rho+p) u_{i} u_{j}-p g_{i j}+E_{i j}, \tag{11}
\end{equation*}
$$

where $E_{i j}$ is the electromagnetic field given by Lichnerowicz [37]:

$$
\begin{equation*}
E_{i j}=\bar{\mu}\left[h_{l} h^{l}\left(u_{i} u_{j}-\frac{1}{2} g_{i j}\right)+h_{i} h_{j}\right] . \tag{12}
\end{equation*}
$$

Here $\rho$ and $p$ are the energy density and isotropic pressure, respectively, while $\bar{\mu}$ is the magnetic permeability and $h_{i}$ is the magnetic flux vector defined by

$$
\begin{equation*}
h_{i}=\frac{\sqrt{-g}}{2 \bar{\mu}} \epsilon_{i j k l} F^{k l} u^{j} \tag{13}
\end{equation*}
$$

$F_{i j}$ is the electromagnetic field tensor and $\epsilon_{i j k l}$ is a Levi-Civita tensor density. If we consider the current flow along $z$-axis, then $F_{12}$ is only nonvanishing component of $F_{i j}$. Then the Maxwell equations

$$
\begin{gather*}
F_{i j ; k}+F_{j k ; i}+F_{k i ; j}=0, \\
{\left[\frac{1}{\bar{\mu}} F^{i j}\right]_{; j}=J^{i}} \tag{14}
\end{gather*}
$$

require $F_{12}$ to be function of $x$ alone [38]. We assume the magnetic permeability as a function of both $x$ and $t$. Here the semicolon represents a covariant differentiation.

For the line element (2), field equation (1) can be reduced to the following system of NLPDEs:

$$
\begin{gather*}
E_{1}=\frac{f_{x t}}{f}+\frac{g_{x t}}{g}=0, \\
E_{2}=\frac{f_{t t}}{f}+\frac{f_{t} g_{t}}{f g}+\frac{1}{A^{2}}\left(\frac{g_{x x}}{g}-\frac{f_{x} g_{x}}{f g}\right)+\frac{3 \dot{A} f_{t}}{A f}=0,  \tag{15}\\
\chi \rho+\frac{3}{4} \beta^{2}= \\
\frac{g_{t t}}{2 g}+\frac{3 f_{t} g_{t}}{2 f g}-\frac{1}{A^{2}}\left(\frac{f_{x x}}{2 f}+\frac{g_{x x}}{g}+\frac{3 f_{x} g_{x}}{2 f g}\right) \\
\\
+\frac{\dot{A}}{A}\left(\frac{2 f_{t}}{f}+\frac{g_{t}}{2 g}+\frac{3 \dot{A}}{A}\right), \\
\chi p+\frac{3}{4} \beta^{2}=  \tag{16}\\
\frac{1}{2 A^{2}}\left(\frac{f_{x x}}{f}+\frac{f_{x} g_{x}}{f g}\right)-\frac{f_{t t}}{f}-\frac{g_{t t}}{2 g}-\frac{f_{t} g_{t}}{2 f g} \\
\\
\quad-\frac{\dot{A}}{A}\left(\frac{3 f_{t}}{f}+\frac{\dot{A}}{A}\right)-\frac{2 \ddot{A}}{A}, \\
\frac{\chi F_{12}^{2}}{\bar{\mu} A^{4} f^{2}}=\frac{1}{A^{2}}\left(\frac{f_{x x}}{f}-\frac{f_{x} g_{x}}{f g}\right)+\frac{g_{t t}}{g}+\frac{f_{t} g_{t}}{f g}+\frac{3 \dot{A} g_{t}}{A g} .
\end{gather*}
$$

## 3. Symmetry Analysis Method

Equations (15)-(16) are highly nonlinear PDEs and hence it is so difficult to handle since there exist no standard methods for obtaining analytical solution. The system (15) is nonlinear

PDEs of second order for the two unknowns $f$ and $g$. If we solve this system, then we can get the solution of the field equations. In order to obtain an exact solution of the system of nonlinear PDEs (15), we will use the symmetry analysis method. For this we write
as the infinitesimal Lie point transformations. We have assumed that the system (15) is invariant under the transformations given in (17). The corresponding infinitesimal generator of Lie groups (symmetries) is given by

$$
\begin{equation*}
X=\sum_{i=1}^{2} \xi_{i} \frac{\partial}{\partial x_{i}}+\sum_{\alpha=1}^{2} \eta_{\alpha} \frac{\partial}{\partial u_{\alpha}} \tag{18}
\end{equation*}
$$

where $x_{1}=x, x_{2}=t, u_{1}=f$, and $u_{2}=g$. The coefficients $\xi_{1}, \xi_{2}, \eta_{1}$, and $\eta_{2}$ are the functions of $x, t, f$, and $g$. These coefficients are the components of infinitesimals symmetries corresponding to $x, t, f$, and $g$, respectively, to be determined from the invariance conditions:

$$
\begin{equation*}
\left.\operatorname{Pr}^{(2)} X\left(E_{m}\right)\right|_{E_{m}=0}=0 \tag{19}
\end{equation*}
$$

where $E_{m}=0, m=1,2$ are the system (15) under study and $\operatorname{Pr}^{(2)}$ is the second prolongation of the symmetries $X$. Since our equations (15) are at most of order two, therefore, we need second order prolongation of the infinitesimal generator in (19). It is worth noting that the 2 nd order prolongation is given by

$$
\begin{equation*}
\operatorname{Pr}^{(2)} X=X+\sum_{i=1}^{2} \sum_{\alpha=1}^{2} \eta_{\alpha, i} \frac{\partial}{\partial u_{\alpha, i}}+\sum_{j=1}^{2} \sum_{i=1}^{2} \sum_{\alpha=1}^{2} \eta_{\alpha, i j} \frac{\partial}{\partial u_{\alpha, i j}} \tag{20}
\end{equation*}
$$

where

$$
\begin{align*}
\eta_{\alpha, i} & =D_{i}\left[\eta_{\alpha}-\sum_{j=1}^{2} \xi_{j} u_{\alpha, j}\right]+\sum_{j=1}^{2} \xi_{j} u_{\alpha, i j}  \tag{21}\\
\eta_{\alpha, i j} & =D_{i j}\left[\eta_{\alpha}-\sum_{k=1}^{2} \xi_{k} u_{\alpha, k}\right]+\sum_{k=1}^{2} \xi_{k} u_{\alpha, i j k}
\end{align*}
$$

The operator $D_{i}\left(D_{i j}\right)$ is called the total derivative (Hach operator) and takes the following form:

$$
\begin{equation*}
D_{i}=\frac{\partial}{\partial x_{i}}+\sum_{\alpha=1}^{2} u_{\alpha, i} \frac{\partial}{\partial u_{\alpha}}+\sum_{j=1}^{2} \sum_{\alpha=1}^{2} u_{\alpha, i j} \frac{\partial}{\partial u_{\alpha, j}} \tag{22}
\end{equation*}
$$

where $D_{i j}=D_{i}\left(D_{j}\right)=D_{j}\left(D_{i}\right)=D_{j i}$ and $u_{\alpha, i}=\partial u_{\alpha} / \partial x_{i}$.
Expanding the system (19) with the aid of Mathematica program, along with the original system (15) to eliminate $f_{x t}$ and $g_{x x}$ while we set the coefficients involving $f_{x}, f_{t}, f_{x x}, f_{t t}$, $g_{x}, g_{t}, g_{x t}$, and $g_{t t}$ and various products equal zero, these gives rise to the essential set of overdetermined equations. Solving the set of these determining equations, the components of symmetries take the following form:

$$
\begin{gather*}
\xi_{1}=c_{1} x+c_{2}, \quad \xi_{2}=c_{3} t+c_{4},  \tag{23}\\
\eta_{1}=0, \quad \eta_{2}=c_{5} g,
\end{gather*}
$$

where $c_{i}, i=1,2, \ldots, 5$, are arbitrary constants and the function $A(t)$ must equal

$$
\begin{equation*}
A(t)=c_{6} \xi_{2}(t) \exp \left[-c_{1} \int \frac{d t}{\xi_{2}(t)}\right] \tag{24}
\end{equation*}
$$

Therefore, $A(t)$ becomes

$$
\begin{align*}
& A(t)=c_{6}\left(c_{3} t+c_{4}\right)^{1-\left(c_{1} / c_{3}\right)}, \quad \text { if } c_{3} \neq 0 \\
& A(t)=c_{7} \exp \left[-\frac{c_{1}}{c_{4}} t\right], \quad \text { if } c_{3}=0 \tag{25}
\end{align*}
$$

where $c_{6}$ and $c_{7}=c_{4} c_{6}$ are arbitrary constants.

## 4. Invariant Solutions

The characteristic equations corresponding to the symmetries (23) are given by

$$
\begin{equation*}
\frac{d x}{c_{1} x+c_{2}}=\frac{d t}{c_{3} t+c_{4}}=\frac{d f}{0}=\frac{d g}{c_{5} g} \tag{26}
\end{equation*}
$$

By solving the above system, we have the following four cases.
Case 1. When $c_{1} \neq 0$ and $c_{3} \neq 0$, the similarity variable and similarity functions can be written as follows:

$$
\begin{gather*}
\xi=\frac{x+a}{(t+b)^{c}}, \quad f(x, t)=\Psi(\xi)  \tag{27}\\
g(x, t)=(x+a)^{d} \Phi(\xi)
\end{gather*}
$$

where $a=c_{2} / c_{1}, b=c_{4} / c_{3}, c=c_{1} / c_{3}$, and $d=c_{5} / c_{1}$ are arbitrary constants. In this case, $A(t)=q(t+b)^{1-c}$, where $q=c_{6} c_{3}^{1-c}$. Substituting the transformations (27) in the field (15) leads to the following system of ODEs:

$$
\begin{gather*}
\frac{\xi \Psi^{\prime \prime}+\Psi^{\prime}}{\Psi}+\frac{\xi \Phi^{\prime \prime}+(1+d) \Phi^{\prime}}{\Phi}=0  \tag{28}\\
\frac{\left(c^{2} q^{2} \xi^{2}-1\right) \xi \Phi^{\prime} \Psi^{\prime}}{\Phi \Psi}+\frac{2 d \Phi^{\prime}+\xi \Phi^{\prime \prime}}{\Phi} \\
+\frac{c^{2} q^{2} \xi^{3} \Psi^{\prime \prime}-\left[d+2 c(1-2 c) q^{2} \xi^{2}\right] \Psi^{\prime}}{\Psi}=\frac{d(1-d)}{\xi} . \tag{29}
\end{gather*}
$$

If one solves the system of second order NLPDEs (28)-(29), one can obtain the exact solutions of the original Einstein field equations (15) corresponding to reduction (30).

Case 2. When $c_{1} \neq 0$ and $c_{3}=0$, the similarity variable and similarity functions can be written as follows:

$$
\begin{gather*}
\xi=(x+a) \exp [b t], \quad f(x, t)=\Psi(\xi),  \tag{30}\\
g(x, t)=(x+a)^{c} \Phi(\xi),
\end{gather*}
$$

where $a=c_{2} / c_{1}, b=-c_{1} / c_{4}$, and $c=c_{5} / c_{1}$ are arbitrary constants. In this case, $A(t)=d \exp [b t]$, where $d=c_{7}$.

Substituting transformations (30) in the field equations (15) leads to the following system of ODEs:

$$
\begin{equation*}
\frac{\xi \Psi^{\prime \prime}+\Psi^{\prime}}{\Psi}+\frac{\xi \Phi^{\prime \prime}+(1+c) \Phi^{\prime}}{\Phi}=0 \tag{31}
\end{equation*}
$$

$$
\begin{align*}
& \frac{\left(b^{2} d^{2} \xi^{2}-1\right) \xi \Phi^{\prime} \Psi^{\prime}}{\Phi \Psi}+\frac{2 c \Phi^{\prime}+\xi \Phi^{\prime \prime}}{\Phi} \\
& +\frac{b^{2} d^{2} \xi^{3} \Psi^{\prime \prime}+\left(4 b^{2} d^{2} \xi^{2}-c\right) \Psi^{\prime}}{\Psi}=\frac{c(1-c)}{\xi} \tag{32}
\end{align*}
$$

If one solves the system of second order NLPDEs (31)-(32), one can obtain the exact solutions of the original Einstein field equations (15) corresponding to reduction (30).

Case 3. When $c_{1}=0$ and $c_{3} \neq 0$, the similarity variable and similarity functions can be written as follows:

$$
\begin{gather*}
\xi=\frac{\exp [a x]}{t+b}, \quad f(x, t)=\Psi(\xi),  \tag{33}\\
g(x, t)=\Phi(\xi) \exp [a x]
\end{gather*}
$$

where $a=c_{3} / c_{2}$ and $b=c_{4} / a_{3}$ are arbitrary constants. In this case we have $A(t)=c(t+b)$, where $c=c_{6}$. Substituting the transformations (33) in the field equations (15) leads to the following system of ODEs:

$$
\begin{gather*}
\frac{\xi \Psi^{\prime \prime}+\Psi^{\prime}}{\Psi}+\frac{\xi \Phi^{\prime \prime}+2 \Phi^{\prime}}{\Phi}=0 \\
\xi\left[\frac{\left(c^{2}-a^{2}\right) \xi \Phi^{\prime} \Psi^{\prime}}{\Phi \Psi}+\frac{a^{2}\left(3 \Phi^{\prime}+\xi \Phi^{\prime \prime}\right)}{\Phi}\right.  \tag{34}\\
\left.+\frac{c^{2} \xi \Psi^{\prime \prime}-\left(c^{2}+a^{2}\right) \Psi^{\prime}}{\Psi}\right]+a^{2}=0
\end{gather*}
$$

Without loss of generality, we can take the following useful transformation:

$$
\begin{gather*}
\xi=\exp [\theta], \quad \frac{d \Psi}{d \xi}=\exp [-\theta] \frac{d \Psi}{d \theta} \\
\frac{d^{2} \Psi}{d \xi^{2}}=\exp [-2 \theta]\left(\frac{d^{2} \Psi}{d \theta^{2}}-\frac{d \Psi}{d \theta}\right) \tag{35}
\end{gather*}
$$

Then the system of ODEs (34) transforms to

$$
\begin{gather*}
\frac{\ddot{\Psi}}{\Psi}+\frac{\ddot{\Phi}+\dot{\Phi}}{\Phi}=0  \tag{36}\\
\frac{\left(c^{2}-a^{2}\right) \dot{\Phi} \dot{\Psi}}{\Phi \Psi}+\frac{a^{2}(2 \dot{\Phi}+\ddot{\Phi})}{\Phi} \\
+\frac{c^{2} \ddot{\Psi}-\left(2 c^{2}+a^{2}\right) \dot{\Psi}}{\Psi}+a^{2}=0 \tag{37}
\end{gather*}
$$

Equation (36) can be written in the following form:

$$
\begin{equation*}
\ddot{\Psi}=-\frac{\Psi}{\Phi}(\ddot{\Phi}+\dot{\Phi}) . \tag{38}
\end{equation*}
$$

From the above equation, if we substitute $\ddot{\Psi}$ in (37), we can obtain the following form:

$$
\begin{gather*}
\left(c^{2}-a^{2}\right)\left[\frac{\dot{\Phi} \dot{\Psi}}{\Phi \Psi}-\frac{\ddot{\Phi}}{\Phi}\right]+\left(2 a^{2}-c^{2}\right)\left(\frac{\dot{\Phi}}{\Phi}\right) \\
-\left(2 c^{2}+a^{2}\right)\left(\frac{\dot{\Psi}}{\Psi}\right)+a^{2}=0 \tag{39}
\end{gather*}
$$

Equation (39) is a nonlinear ODE which is very difficult to solve. However, it is worth noting that this equation is easy to solve when $c=a$. In this case, we can integrate (39) and obtain the following:

$$
\begin{equation*}
\Phi(\theta)=a_{1} \Psi^{3}(\theta) \exp [-\theta] \tag{40}
\end{equation*}
$$

where $a_{1}$ is an arbitrary constant of integration. Substituting (40) in (36), we have the following ODE of the function $\Psi$ only:

$$
\begin{equation*}
\Psi(4 \ddot{\Psi}-3 \dot{\Psi})+6 \dot{\Psi}^{2}=0 \tag{41}
\end{equation*}
$$

The general solution of the above equation is

$$
\begin{equation*}
\Psi(\theta)=a_{3}\left(a_{2}+\exp \left[\frac{3 \theta}{4}\right]\right)^{2 / 5} \tag{42}
\end{equation*}
$$

where $a_{2}$ and $a_{3}$ are arbitrary constants of integration. Now, by using (40) and the inverse of transformations (35) and (33), we can find the solution as follows:

$$
\begin{align*}
A(t) & =a(t+b) \\
f(x, t) & =a_{3}\left(a_{2}+\left[\frac{\exp [a x]}{t+b}\right]^{3 / 4}\right)^{2 / 5}  \tag{43}\\
g(x, t) & =a_{1} a_{3}(t+b)\left(a_{2}+\left[\frac{\exp [a x]}{t+b}\right]^{3 / 4}\right)^{6 / 5} .
\end{align*}
$$

It is observed from (43) and (3) that the line element (2) can be written in the following form:

$$
\begin{align*}
d s^{2}= & d t^{2}-a^{2}(t+b)^{2} d x^{2}-d^{2}(t+b)^{2} \\
& \times\left(a_{2}+\left[\frac{\exp [a x]}{t+b}\right]^{3 / 4}\right)^{4 / 5} d y^{2}  \tag{44}\\
& -q^{2}(t+b)^{4}\left(a_{2}+\left[\frac{\exp [a x]}{t+b}\right]^{3 / 4}\right)^{12 / 5} d z^{2}
\end{align*}
$$

where $a, b, d=a a_{3}, q=a a_{1} a_{3}$, and $a_{2}$ are arbitrary constants.
Case 4. When $c_{1}=c_{3}=0$, the similarity variable and similarity functions can be written as follows:

$$
\begin{align*}
& \xi=a x+b t, \quad f(x, t)=\Psi(\xi)  \tag{45}\\
& g(x, t)=\Phi(\xi) \exp [c x]
\end{align*}
$$

where $a=c_{4}, b=-c_{2}$, and $c=c_{5} / c_{2}$ are arbitrary constants. In this case we have $A(t)=c_{7}$. Substituting transformations (45) in field equations (15) leads to the following system of ODEs:

$$
\begin{gather*}
\frac{a \Psi^{\prime \prime}}{\Psi}+\frac{a \Phi^{\prime \prime}+c \Phi^{\prime}}{\Phi}=0  \tag{46}\\
\frac{\left(b^{2} r^{2}-a^{2}\right) \Phi^{\prime} \Psi^{\prime}}{\Phi \Psi}+\frac{a\left(2 c \Phi^{\prime}+a \Phi^{\prime \prime}\right)}{\Phi}  \tag{47}\\
+\frac{b^{2} r^{2} \Psi^{\prime \prime}-a c \Psi^{\prime}}{\Psi}+c^{2}=0
\end{gather*}
$$

Equation (46) can be written in the following form:

$$
\begin{equation*}
\Psi^{\prime \prime}=-\frac{\Psi}{a \Phi}\left(a \Phi^{\prime \prime}+c \Phi^{\prime}\right) \tag{48}
\end{equation*}
$$

From the above equation, if we substitute $\Psi \ddot{\Psi}$ in (47), we can obtain the following form:

$$
\begin{align*}
& \left(b^{2} r^{2}-a^{2}\right)\left[\frac{\Phi^{\prime} \Psi^{\prime}}{\Phi \Psi}-\frac{\Phi^{\prime \prime}}{\Phi}\right]+\left(2 a^{2}-b^{2} r^{2}\right)\left(\frac{c \Phi^{\prime}}{a \Phi}\right)  \tag{49}\\
& \quad-a c\left(\frac{\Psi^{\prime}}{\Psi}\right)+c^{2}=0
\end{align*}
$$

Equation (49) is a nonlinear ODE which is very difficult to solve. However, it is worth noting that this equation is easy to solve when $a=b r$. In this case, we can integrate (49) and obtain the following:

$$
\begin{equation*}
\Phi(\xi)=a_{1} \Psi(\xi) \exp \left[-\frac{c \xi}{b r}\right] \tag{50}
\end{equation*}
$$

where $a_{1}$ is an arbitrary constant of integration. Substituting (50) in (46), we have the following ODE of function $\Psi$ only:

$$
\begin{equation*}
2 b r \Psi^{\prime \prime}=c \Psi \tag{51}
\end{equation*}
$$

The general solution of the above equation is

$$
\begin{equation*}
\Psi(\xi)=a_{3}+a_{2} \exp \left[\frac{c \xi}{2 b r}\right] \tag{52}
\end{equation*}
$$

where $a_{2}$ and $a_{3}$ are arbitrary constants of integration. Now, by using (50) and (33), we can find the solution as follows:

$$
\begin{align*}
A(t) & =r \\
f(x, t) & =a_{3}+a_{2} \exp \left[\frac{c(t+r x)}{2 b r}\right]  \tag{53}\\
g(x, t) & =a_{1} \exp \left[-\frac{c t}{r}\right]\left(a_{3}+a_{2} \exp \left[\frac{c(t+r x)}{2 b r}\right]\right)
\end{align*}
$$

It is observed from (53) and (3) that the line element (2) can be written in the following form:

$$
\begin{align*}
d s^{2}= & d t^{2}-r^{2} d x^{2}-r^{2}\left(a_{3}+a_{2} \exp \left[\frac{c(t+r x)}{2 b r}\right]\right)^{2}  \tag{54}\\
& \times\left(d y^{2}+a_{1}^{2} \exp \left[-\frac{2 c t}{r}\right] d z^{2}\right)
\end{align*}
$$

where $r, c, b, a_{1}, a_{2}$, and $a_{3}$ are arbitrary constants.

## 5. Physical Properties of the Model

The field equations (15)-(16) constitute a system of five highly nonlinear differential equations with seven unknowns variables, $A, f, g, p, \rho, \bar{\mu}$, and $\beta$. The symmetries give one condition (24) for function $A$. Therefore, one physically reasonable condition amongst these parameters is required to obtain explicit solutions of the field equations. Let us assume that the density $\rho$ and the pressure $p$ are related by barotropic equation of state:

$$
\begin{equation*}
p=\lambda \rho, \quad 0 \leq \lambda \leq 1 . \tag{55}
\end{equation*}
$$

5.1. For Model (44). Using (43) in the Einstein field equations (16), with taking into account condition (55), the expressions for density $\rho$, pressure $p$, magnetic permeability $\bar{\mu}$, and displacement field $\beta$ are given by

$$
\begin{align*}
& \rho(x, t)=\frac{9}{5 \chi(1-\lambda)} \\
& \times\left[\frac{5 a_{2}+[\exp [a x] /(t+b)]^{3 / 4}}{(t+b)^{2}\left(a_{2}+[\exp [a x] /(t+b)]^{3 / 4}\right)}\right], \\
& p(x, t)=\frac{9 \lambda}{5 \chi(1-\lambda)} \\
& \times\left[\frac{5 a_{2}+[\exp [a x] /(t+b)]^{3 / 4}}{(t+b)^{2}\left(a_{2}+[\exp [a x] /(t+b)]^{3 / 4}\right)}\right], \\
& \bar{\mu}(x, t)=\frac{\chi F_{12}^{2}(x)}{3 a^{2} a_{2} d^{2}(t+b)^{2}}\left(a_{2}+\left[\frac{\exp [a x]}{t+b}\right]^{3 / 4}\right)^{1 / 5}, \\
& \beta^{2}(x, t)=\frac{2}{15}\left[\left(5 a_{2}(5+13 \lambda)\right.\right. \\
& \left.+2(2+7 \lambda)\left[\frac{\exp [a x]}{t+b}\right]^{3 / 4}\right) \\
& \times\left((\lambda-1)(t+b)^{2}\right. \\
& \left.\left.\times\left(a_{2}+\left[\frac{\exp [a x]}{t+b}\right]^{3 / 4}\right)\right)^{-1}\right] . \tag{56}
\end{align*}
$$

For the line element (44), using (4), (8), and (9), we have the following physical properties. The volume element is

$$
\begin{equation*}
V=a d q(t+b)^{4}\left(a_{2}+\left[\frac{\exp [a x]}{t+b}\right]^{3 / 4}\right)^{8 / 5} \tag{57}
\end{equation*}
$$

The expansion scalar, which determines the volume behavior of the fluid, is given by

$$
\begin{equation*}
\Theta=\frac{2}{5}\left[\frac{10 a_{2}+7[\exp [a x] /(t+b)]^{3 / 4}}{(t+b)\left(a_{2}+[\exp [a x] /(t+b)]^{3 / 4}\right)}\right] \tag{58}
\end{equation*}
$$

The nonvanishing components of the shear tensor, $\sigma_{i}^{j}$, are

$$
\begin{align*}
& \sigma_{1}^{1}=\frac{[\exp [a x] /(t+b)]^{3 / 4}-5 a_{2}}{15(t+b)\left(a_{2}+[\exp [a x] /(t+b)]^{3 / 4}\right)} \\
& \sigma_{2}^{2}=-\frac{1}{30}\left[\frac{10 a_{2}+[\exp [a x] /(t+b)]^{3 / 4}}{(t+b)\left(a_{2}+[\exp [a x] /(t+b)]^{3 / 4}\right)}\right]  \tag{59}\\
& \sigma_{3}^{3}=\frac{4 a_{2}+7[\exp [a x] /(t+b)]^{3 / 4}}{6(t+b)\left(a_{2}+[\exp [a x] /(t+b)]^{3 / 4}\right)} \\
& \sigma_{4}^{4}=-\frac{4}{15}\left[\frac{10 a_{2}+7[\exp [a x] /(t+b)]^{3 / 4}}{(t+b)\left(a_{2}+[\exp [a x] /(t+b)]^{3 / 4}\right)}\right]
\end{align*}
$$

Hence the shear scalar $\sigma$ is given by

$$
\begin{align*}
\sigma^{2}= & \left(3500 a_{2}^{2}+4630 a_{2}[\exp [a x] /(t+b)]^{3 / 4}\right. \\
& \left.+1607[\exp [a x] /(t+b)]^{3 / 2}\right)  \tag{60}\\
& \times\left(900(t+b)^{2}\left(a_{2}+[\exp [a x] /(t+b)]^{3 / 4}\right)^{2}\right)^{-1} .
\end{align*}
$$

The model does not admit acceleration and rotation, since $\dot{u}_{i}=0$ and $\omega_{i j}=0$. We can see that

$$
\begin{equation*}
\frac{\sigma_{4}^{4}}{\Theta}=-\frac{2}{3} \tag{61}
\end{equation*}
$$

which is a constant of proportional. We found also that

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \frac{\sigma^{2}}{\Theta}=\frac{5 \sqrt{35}}{2} \neq 0 \tag{62}
\end{equation*}
$$

and this means that there is no possibility that the universe may got isotropized in some later time; that is, it remains anisotropic for all times.
5.2. For Model (54). Using (53) in the Einstein field equations (16), with taking into account condition (55), the expressions
for density $\rho$, pressure $p$, magnetic permeability $\bar{\mu}$, and displacement field $\beta$ are given by

$$
\begin{align*}
\rho(x, t)= & \frac{c^{2}}{\chi r^{2}(\lambda-1)}\left(\frac{c_{2} \exp [c(t+r x) / 2 r]-c_{3}}{c_{2} \exp [c(t+r x) / 2 r]+c_{3}}\right) \\
p(x, t)= & \frac{\lambda c^{2}}{\chi r^{2}(\lambda-1)}\left(\frac{c_{2} \exp [c(t+r x) / 2 r]-c_{3}}{c_{2} \exp [c(t+r x) / 2 r]+c_{3}}\right), \\
\bar{\mu}(x, t)= & \frac{\chi F_{12}^{2}(x)}{c_{3} c^{2} r^{2}}\left(c_{2} \exp \left[\frac{c(t+r x)}{2 r}\right]+c_{3}\right)^{-1} \\
\beta^{2}(x, t)= & \frac{2 c^{2}}{3 r^{2}(\lambda-1)} \\
& \times\left(\frac{c_{3}(1+\lambda)-2 \lambda c_{2} \exp [c(t+r x) / 2 r]}{c_{3}+c_{2} \exp [c(t+r x) / 2 r]}\right) \tag{63}
\end{align*}
$$

For the line element (54), using (4), (8), and (9), we have the following physical properties. The volume element is

$$
\begin{equation*}
V=c_{1} r^{3} \exp \left[-\frac{c t}{r}\right]\left(c_{3}+c_{2} \exp \left[\frac{c(t+r x)}{2 r}\right]\right)^{2} \tag{64}
\end{equation*}
$$

The expansion scalar, which determines the volume behavior of the fluid, is given by

$$
\begin{equation*}
\Theta=-\frac{c c_{3}}{r}\left(c_{3}+c_{2} \exp \left[\frac{c(t+r x)}{2 r}\right]\right)^{-1} \tag{65}
\end{equation*}
$$

The nonvanishing components of the shear tensor, $\sigma_{i}^{j}$, satisfy

$$
\begin{gather*}
\frac{\sigma_{1}^{1}}{\Theta}=-\frac{1}{3} \\
\frac{\sigma_{2}^{2}}{\Theta}=-\left(\frac{1}{3}+\frac{c_{2}}{2 c_{3}} \exp \left[\frac{c(t+r x)}{2 r}\right]\right) \\
\frac{\sigma_{3}^{3}}{\Theta}=\frac{2}{3}+\frac{c_{2}}{2 c_{3}} \exp \left[\frac{c(t+r x)}{2 r}\right]  \tag{66}\\
\frac{\sigma_{4}^{4}}{\Theta}=-\frac{2}{3}
\end{gather*}
$$

Hence the shear scalar $\sigma$ is given by

$$
\begin{equation*}
\sigma^{2}=\frac{c^{2}}{4 r^{2}}+\frac{11 c_{3}^{2} c^{2}}{36 r^{2}}\left(c_{3}+c_{2} \exp \left[\frac{c(t+r x)}{2 r}\right]\right)^{-2} \tag{67}
\end{equation*}
$$

The model does not admit acceleration and rotation, since $\dot{u}_{i}=0$ and $\omega_{i j}=0$.

## Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Derivation of Conservation Laws for the Magma Equation Using the Multiplier Method: Power Law and Exponential Law for Permeability and Viscosity 

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The derivation of conservation laws for the magma equation using the multiplier method for both the power law and exponential law relating the permeability and matrix viscosity to the voidage is considered. It is found that all known conserved vectors for the magma equation and the new conserved vectors for the exponential laws can be derived using multipliers which depend on the voidage and spatial derivatives of the voidage. It is also found that the conserved vectors are associated with the Lie point symmetry of the magma equation which generates travelling wave solutions which may explain by the double reduction theorem for associated Lie point symmetries why many of the known analytical solutions are travelling waves.

## 1. Introduction

The one-dimensional migration of melt upwards through the mantle of the Earth is governed by the third order nonlinear partial differential equation

$$
\begin{equation*}
\frac{\partial \phi}{\partial t}+\frac{\partial}{\partial z}\left[K(\phi)\left(1-\frac{\partial}{\partial z}\left(G(\phi) \frac{\partial \phi}{\partial t}\right)\right)\right]=0 \tag{1}
\end{equation*}
$$

where $\phi(t, z)$ is the voidage or volume fraction of melt, $t$ is time, $z$ is the vertical spatial coordinate, $K$ is the permeability of the medium, and $G$ is the viscosity of the matrix phase. The variables $\phi, t$, and $z$ and the physical quantities $K(\phi)$ and $G(\phi)$ in (1) are dimensionless. The voidage $\phi(t, z)$ is scaled by the background voidage $\phi_{0}$. The background state is therefore defined by $\phi=1$. The characteristic length in the $z$-direction, which is vertically upwards, is the compaction length $\delta_{c}$ defined by

$$
\begin{equation*}
\delta_{c}=\left[\frac{K\left(\phi_{0}\right) G\left(\phi_{0}\right)}{\mu}\right]^{1 / 2} \tag{2}
\end{equation*}
$$

where $\mu$ is the coefficient of shear viscosity of the melt. The characteristic time is $t_{0}$ defined by

$$
\begin{equation*}
t_{0}=\frac{\phi_{0}}{g \Delta \rho}\left[\frac{\mu G\left(\phi_{0}\right)}{K\left(\phi_{0}\right)}\right]^{1 / 2} \tag{3}
\end{equation*}
$$

where $g$ is the acceleration due to gravity and $\Delta \rho$ is the difference between the density of the solid matrix and the density of the melt. The permeability is scaled by $K\left(\phi_{0}\right)$ and therefore

$$
\begin{equation*}
K(1)=1 . \tag{4}
\end{equation*}
$$

When the voidage is zero the permeability must also be zero and therefore

$$
\begin{equation*}
K(0)=0 . \tag{5}
\end{equation*}
$$

The viscosity $G(\phi)$ is scaled by $G\left(\phi_{0}\right)$ so that

$$
\begin{equation*}
G(1)=1 \tag{6}
\end{equation*}
$$

and $G(0)$ will be infinite because the matrix viscosity is infinite when the voidage vanishes. In the derivation of (1) it is assumed that the background voidage satisfies $\phi_{0} \ll 1$.

The partially molten medium consists of a solid matrix and a fluid melt which are modelled as two immiscible fully connected fluids of constant but different densities. The density of the melt is less than the density of the solid matrix and the melt migrates through the compacting medium by the buoyancy force due to the difference in density between the melt and the solid matrix. Changes of phase are not included in the model. It is assumed that the melting has occurred and only migration of the melt under gravity is described by (1) [1].

In the model proposed by Scott and Stevenson [2], consider

$$
\begin{equation*}
K(\phi)=\phi^{n}, \quad G(\phi)=\phi^{-m} \tag{7}
\end{equation*}
$$

where $n \geq 0$ and $m \geq 0$. Harris and Clarkson [3] have investigated this model using Painleve analysis. Mindu and Mason [4] showed that the magma equation also admits Lie point symmetries other than translations in time and space if the permeability is in the form of an exponential law:

$$
\begin{equation*}
K(\phi)=\exp [n(\phi-1)] \tag{8}
\end{equation*}
$$

Conservation laws for (1) when the permeability and matrix viscosity satisfy the power laws (7) have been obtained using the direct method by Barcilon and Richter [5] and Harris [6] and using Lie point symmetry generators by Maluleke and Mason [7].

In this paper we will derive the conservation laws for the partial differential equation (1) using the multiplier method. We will consider power laws given by (7) and also the exponential laws

$$
\begin{equation*}
K(\phi)=\exp [n(\phi-1)], \quad G(\phi)=\exp [-m(\phi-1)] \tag{9}
\end{equation*}
$$

where $n \geq 0$ and $m \geq 0$, relating the permeability and matrix viscosity to the voidage. The permeability increases as the voidage increases while the viscosity of the matrix decreases as the voidage increases. The exponential laws are not suitable models when the voidage $\phi$ is small because $K(0)=\exp (-n) \neq 0$ and $G(0)=\exp (m) \neq \infty$. They are suitable for describing rarefaction for which $\phi>1$.

An outline of the paper is as follows. In Section 2 we present the formulae and theory that we will use in the paper. In Section 3 conservation laws for the magma equation, with power laws relating the permeability and viscosity to the voidage, are derived using the multiplier method. Further in Section 4 conservation laws for the magma equation, with exponential laws relating the permeability and viscosity to the voidage, are derived using the multiplier method. Finally the conclusions are summarized in Section 5.

## 2. Formulae and Theory

Consider an $s$ th order partial differential equation

$$
\begin{equation*}
F\left(x, \phi, \phi_{(1)}, \ldots, \phi_{(s)}\right)=0 \tag{10}
\end{equation*}
$$

in the variables $x=\left(x^{1}, x^{2}, \ldots, x^{p}\right)$, where $\phi_{(p)}$ denotes the collection of $p$ th-order partial derivatives of $\phi$. The equation

$$
\begin{equation*}
D_{i} T^{i}=0 \tag{11}
\end{equation*}
$$

evaluated on the surface given by (10), where $i$ runs from 1 to $r$ and $D_{i}$ is the total derivative defined by

$$
\begin{equation*}
D_{i}=\frac{\partial}{\partial x_{i}}+\phi_{i} \frac{\partial}{\partial \phi}+\phi_{k i} \frac{\partial}{\partial \phi_{k}}+\cdots \tag{12}
\end{equation*}
$$

is called a conservation law for the differential equation (10). The vector $T=\left(T^{1}, \ldots, T^{r}\right)$ is a conserved vector for the partial differential equation and $T^{1}, \ldots, T^{r}$ are its components. Thus, a conserved vector gives rise to a conservation law. A Lie point symmetry generator

$$
\begin{equation*}
X=\xi^{i}(x, u) \frac{\partial}{\partial x^{i}}+\eta(x, \phi) \frac{\partial}{\partial \phi}, \tag{13}
\end{equation*}
$$

where $i$ runs from 1 to $r$, is said to be associated with the conserved vector $T=\left(T^{1}, \ldots, T^{r}\right)$ for the partial differential equation (10) if $[8,9]$

$$
\begin{equation*}
X\left(T^{i}\right)+T^{i} D_{k}\left(\xi^{k}\right)-T^{k} D_{k}\left(\xi^{i}\right)=0, \quad i=1,2, \ldots r . \tag{14}
\end{equation*}
$$

The association of a Lie point symmetry with a conserved vector can be used to integrate the partial differential equation twice by the double reduction theorem of Sjöberg [10].

Conserved vectors for a partial differential equation can be generated from known conserved vectors and Lie point symmetries of the partial differential equation. For

$$
\begin{equation*}
T_{*}^{i}=X\left(T^{i}\right)+T^{i} D_{k}\left(\xi^{k}\right)-T^{k} D_{k}\left(\xi^{i}\right), \quad i=1,2, \ldots r \tag{15}
\end{equation*}
$$

where $k$ runs from 1 to $r$, is a conserved vector for the partial differential equation although it may be a linear combination of known conserved vectors $[8,9]$.

We now present the multiplier method for the derivation of conservation laws for partial differential equations. We will outline its application to the partial differential equation (1) in two independent variables.
(1) Multiply the partial differential equation (1) by the multiplier, $\Lambda$, to obtain the conservation law

$$
\begin{equation*}
\Lambda F=D_{1} T^{1}+D_{2} T^{2} \tag{16}
\end{equation*}
$$

where $F=0$ is the partial differential equation (1), $x^{1}=t$ and $x^{2}=z$, and

$$
\begin{align*}
D_{1} & =D_{t}=\frac{\partial}{\partial t}+\phi_{t} \frac{\partial}{\partial \phi}+\phi_{t t} \frac{\partial}{\partial_{t}}+\phi_{z t} \frac{\partial}{\partial_{z}}+\cdots \\
D_{2} & =D_{z}=\frac{\partial}{\partial z}+\phi_{z} \frac{\partial}{\partial \phi}+\phi_{t z} \frac{\partial}{\partial_{t}}+\phi_{z z} \frac{\partial}{\partial_{z}}+\cdots \tag{17}
\end{align*}
$$

The multiplier depends on $t, z, \phi$, and the partial derivatives of $\phi$. The more derivatives included in the multiplier the wider the range of conserved vectors that can be derived.
(2) The determining equation for the multiplier is obtained by operating on (16) by the Euler operator $E_{\phi}$ defined by [11]

$$
\begin{align*}
E_{\phi}= & \frac{\partial}{\partial \phi}-D_{t} \frac{\partial}{\partial \phi_{t}}-D_{z} \frac{\partial}{\partial \phi_{z}}+D_{t}^{2} \frac{\partial}{\partial \phi_{t t}} \\
& +D_{t} D_{z} \frac{\partial}{\partial \phi_{t z}}+D_{z}^{2} \frac{\partial}{\partial \phi_{z z}}-\cdots \tag{18}
\end{align*}
$$

Since the Euler operator annihilates divergence expressions this gives [11]

$$
\begin{equation*}
E_{\phi}[\Lambda F]=0 \tag{19}
\end{equation*}
$$

(3) The determining equation (19) is separated by equating the coefficients of like powers and products of the derivatives of $\phi$ because $\phi$ is an arbitrary function.
(4) When $\phi$ is a solution of the partial differential equation, $F=0$, (16) becomes a conservation law. The condition $F=0$ is imposed on (16). The product of the multiplier and the partial differential equation is then written in conserved form by elementary manipulations. This yields the conserved vectors by setting all the constants equal to zero except one in turn.

## 3. Conservation Laws for the Magma Equation with Power Law Permeability and Viscosity by the Multiplier Method

When the permeability and viscosity are related to the voidage by the power laws (7) the magma equation becomes

$$
\begin{equation*}
\frac{\partial \phi}{\partial t}+\frac{\partial}{\partial z}\left[\phi^{n}\left(1-\frac{\partial}{\partial z}\left(\phi^{-m} \frac{\partial \phi}{\partial t}\right)\right)\right]=0 \tag{20}
\end{equation*}
$$

3.1. Lower Order Conservation Laws. In order to derive conservation laws for (20) consider first a multiplier of the form

$$
\begin{equation*}
\Lambda=\Lambda(\phi) \tag{21}
\end{equation*}
$$

A multiplier for the partial differential equation has the property

$$
\begin{equation*}
\Lambda(\phi) F\left(\phi, \phi_{t}, \phi_{z}, \phi_{t z}, \phi_{z z}, \phi_{t z z}\right)=D_{1} T^{1}+D_{2} T^{2} \tag{22}
\end{equation*}
$$

where

$$
\begin{align*}
F(\phi, & \phi_{t}, \\
, & \left., \phi_{t z}, \phi_{z z}, \phi_{t z z}\right)  \tag{23}\\
= & \phi_{t}+n \phi^{n-1} \phi_{z}+m(n-m-1) \phi^{n-m-2} \phi_{z}^{2} \phi_{t} \\
& +m \phi^{n-m-1} \phi_{z z} \phi_{t}+(2 m-n) \phi^{n-m-1} \phi_{z} \phi_{t z} \\
& \quad-\phi^{n-m} \phi_{t z z} .
\end{align*}
$$

The determining equation for the multiplier is

$$
\begin{equation*}
E_{\phi}\left[\Lambda(\phi) F\left(\phi, \phi_{t}, \phi_{z}, \phi_{t z}, \phi_{z z}, \phi_{t z z}\right)\right]=0 \tag{24}
\end{equation*}
$$

where $E_{\phi}$ is defined by (18). Separating (24) with respect to products and powers of the partial derivatives of $\phi$ we obtain the following system of equations:

$$
\begin{align*}
& \phi_{z} \phi_{t z}: \phi \frac{d^{2} \Lambda}{d \phi^{2}}+(m+n) \frac{d \Lambda}{d \phi}=0  \tag{25}\\
& \phi_{t} \phi_{z z}: \phi \frac{d^{2} \Lambda}{d \phi^{2}}+(m+n) \frac{d \Lambda}{d \phi}=0  \tag{26}\\
& \phi_{t} \phi_{z}^{2}: \phi^{2} \frac{d^{3} \Lambda}{d \phi^{3}}+2 n \phi \frac{d^{2} \Lambda}{d \phi^{2}}-(m+n)(m-n+1) \frac{d \Lambda}{d \phi}=0 \tag{27}
\end{align*}
$$



Figure 1: The $(m, n)$-plane. The conservation laws are constrained to the region $m \geq 0, n \geq 0$. The special cases are the straight lines $n+m-1=0, n+m-2=0, m=1$, and $n=m-1$.

Equation (26) is the same as (25). It is readily verified that every solution of (25) is a solution of (27). We therefore need to consider only (25). The general solution of (25) is

$$
\begin{align*}
& \Lambda(\phi)=c_{2} \phi^{1-m-n}+c_{1}, \quad \text { if } n+m-1 \neq 0  \tag{28}\\
& \Lambda(\phi)=c_{2} \ln \phi+c_{1}, \quad \text { if } n+m-1=0 \tag{29}
\end{align*}
$$

There are several cases to consider depending on the values of $m$ and $n$. The special cases are illustrated as lines and points in the $(m, n)$ plane in Figure 1.
(i) $n+m-1 \neq 0, n+m-2 \neq 0, m \neq 1$. From (22) and (28),

$$
\begin{align*}
\left(c_{1}+c_{2} \phi^{1-m-n}\right)( & \phi_{t}+n \phi^{n-1} \phi_{z}+m(n-m-1) \\
& \times \phi^{n-m-2} \phi_{z}^{2} \phi_{t}+m \phi^{n-m-1} \phi_{z z} \phi_{t} \\
& \left.+(2 m-n) \phi^{n-m-1} \phi_{z} \phi_{t z}-\phi^{n-m} \phi_{t z z}\right) \\
=D_{1}\left[c_{1} \phi+\right. & c_{2}\left(\frac{1}{2-m-n}\left(\phi^{2-m-n}-1\right)\right. \\
& \left.\left.+\frac{(1-m-n)}{2} \phi^{-2 m} \phi_{z}^{2}\right)\right] \\
+D_{2}[ & c_{1}\left(\phi^{n}\left(1+m \phi^{-m-1} \phi_{t} \phi_{z}-\phi^{-m} \phi_{t z}\right)\right) \\
& \left.+c_{2}\left(\frac{n}{m-1} \phi^{1-m}-\phi^{1-2 m} \phi_{t z}+m \phi^{-2 m} \phi_{t} \phi_{z}\right)\right] \tag{30}
\end{align*}
$$

Equation (30) is satisfied for arbitrary functions $\phi(t, z)$. When $\phi(t, z)$ is a solution of the partial differential equation (20), then

$$
\begin{align*}
D_{1}\left[c_{1} \phi+c_{2}\right. & \left.\left(\frac{1}{2-m-n}\left(\phi^{2-m-n}-1\right)+\frac{(1-m-n)}{2} \phi^{-2 m} \phi_{z}^{2}\right)\right] \\
+D_{2} & {\left[c_{1}\left(\phi^{n}\left(1+m \phi^{-m-1} \phi_{t} \phi_{z}-\phi^{-m} \phi_{t z}\right)\right)\right.} \\
& \left.\quad+c_{2}\left(\frac{n}{m-1} \phi^{1-m}-\phi^{1-2 m} \phi_{t z}+m \phi^{-2 m} \phi_{t} \phi_{z}\right)\right]=0 . \tag{31}
\end{align*}
$$

Hence, any conserved vector of the partial differential equation (20) with $m$ and $n$ satisfying the conditions of this case and with multiplier of the form $\Lambda=\Lambda(\phi)$ is a linear combination of the two conserved vectors

$$
\begin{gather*}
T^{1}=\phi, \quad T^{2}=\phi^{n}\left(1+m \phi^{-m-1} \phi_{t} \phi_{z}-\phi^{-m} \phi_{t z}\right)  \tag{32}\\
T^{1}=\frac{1}{2-m-n}\left(\phi^{2-m-n}-1\right)+\frac{(1-m-n)}{2} \phi^{-2 m} \phi_{z}^{2} \\
T^{2}=\frac{n}{1-m} \phi^{1-m}-\phi^{1-2 m} \phi_{t z}+m \phi^{-2 m} \phi_{t} \phi_{z} \tag{33}
\end{gather*}
$$

The conserved vector (32) is the elementary conserved vector.
(ii) $n+m=1, m \neq 1$. Proceeding as before we obtain

$$
\begin{gather*}
T^{1}=\phi, \quad T^{2}=\phi^{1-m}\left(1+m \phi^{-m-1} \phi_{t} \phi_{z}-\phi^{-m} \phi_{t z}\right),  \tag{34}\\
T^{1}=-\frac{1}{2} \phi^{-2 m} \phi_{z}^{2}+\ln \phi \\
T_{2}=\phi^{1-2 m} \ln \phi-\frac{1}{1-m} \phi^{1-m}  \tag{35}\\
-\left(\phi^{1-2 m} \ln \phi\right) \phi_{t z}+\left(m \phi^{-2 m} \ln \phi\right) \phi_{t} \phi_{z} .
\end{gather*}
$$

The conserved vector (34) is the elementary conserved vector with $n=1-m$. The multiplier for (35) is, from (29),

$$
\begin{equation*}
\Lambda(\phi)=\ln \phi \tag{36}
\end{equation*}
$$

(iii) $n+m=2, m \neq 1$. We find that

$$
\begin{gather*}
T^{1}=\phi, \quad T^{2}=\phi^{2-m}\left(1+m \phi^{-m-1} \phi_{t} \phi_{z}-\phi^{-m} \phi_{t z}\right),  \tag{37}\\
T^{1}=-\frac{1}{2} \phi^{-2 m} \phi_{z}^{2}+\ln \phi,  \tag{38}\\
T^{2}=\frac{2-m}{1-m} \phi^{1-m}-\phi^{1-2 m} \phi_{t z}+m \phi^{-2 m} \phi_{t} \phi_{z} .
\end{gather*}
$$

The conserved vector (37) is the elementary conserved vector with $n=2-m$. The multiplier for (38) is, from (28),

$$
\begin{equation*}
\Lambda(\phi)=\frac{1}{\phi} \tag{39}
\end{equation*}
$$

(iv) $m=1, n=0$. We obtain

$$
\begin{gather*}
T^{1}=\phi, \quad T^{2}=1+\phi^{-2} \phi_{t} \phi_{z}-\phi^{-1} \phi_{t z}  \tag{40}\\
T^{1}=-\frac{1}{2} \phi^{-2} \phi_{z}^{2}+\phi \ln \phi-\phi  \tag{41}\\
T^{2}=-\left(\phi^{-1} \ln \phi\right) \phi_{t z}+\left(\phi^{-2} \ln \phi\right) \phi_{t} \phi_{z}
\end{gather*}
$$

The conserved vector (40) is the elementary conserved vector with $m=1, n=0$. The multiplier for (41) is given by (29).
(v) $m=n=1$. We obtain

$$
\begin{gather*}
T^{1}=\phi, \quad T^{2}=\phi\left(1+\phi^{-2} \phi_{t} \phi_{z}-\phi^{-1} \phi_{t z}\right),  \tag{42}\\
T^{1}=-\frac{1}{2} \phi^{-2} \phi_{z}^{2}+\ln \phi  \tag{43}\\
T^{2}=\ln \phi-\phi^{-1} \phi_{t z}+\phi^{-2} \phi_{t} \phi_{z} .
\end{gather*}
$$

The conserved vector (42) is the elementary conserved vector with $m=n=1$. The multiplier for (43) is (39).
(vi) $m=1, n \neq 0, n \neq 1$. We obtain

$$
\begin{gather*}
T^{1}=\phi, \quad T^{2}=\phi^{n}\left(1+\phi^{-2} \phi_{t} \phi_{z}-\phi^{-1} \phi_{t z}\right)  \tag{44}\\
T^{1}=\frac{1}{1-n} \phi^{1-n}-\frac{n}{2} \phi^{-2} \phi_{z}^{2}  \tag{45}\\
T^{2}=\ln \phi-\phi^{-1} \phi_{t z}+\phi^{-2} \phi_{t} \phi_{z}
\end{gather*}
$$

The conserved vector (44) is the elementary conserved vector with $m=1$. The multiplier for (45) is

$$
\begin{equation*}
\Lambda(\phi)=\phi^{-n} \tag{46}
\end{equation*}
$$

3.2. The Search for Higher Order Conservation Laws. We now consider a multiplier of the form

$$
\begin{equation*}
\Lambda=\Lambda\left(\phi, \phi_{z}\right) \tag{47}
\end{equation*}
$$

As before the determining equation for the multiplier is

$$
\begin{equation*}
E_{\phi}\left[\Lambda\left(\phi, \phi_{z}\right) F\left(\phi, \phi_{t}, \phi_{z}, \phi_{t z}, \phi_{z z}, \phi_{t z z}\right)\right]=0 \tag{48}
\end{equation*}
$$

where $F$ is given by (23). By equating the coefficient of the highest order derivative term, $\phi_{t z z z}$, to zero in (48) we have

$$
\begin{equation*}
\frac{\partial \Lambda}{\partial \phi_{z}}=0 \tag{49}
\end{equation*}
$$

and therefore

$$
\begin{equation*}
\Lambda\left(\phi, \phi_{z}\right)=\Lambda(\phi) \tag{50}
\end{equation*}
$$

Hence, (47) does not give a new multiplier or a new conserved vector.

Consider next the multiplier

$$
\begin{equation*}
\Lambda=\Lambda\left(\phi, \phi_{z}, \phi_{z z}\right) \tag{51}
\end{equation*}
$$

The determining equation for the multiplier is

$$
\begin{equation*}
E_{\phi}\left[\Lambda\left(\phi, \phi_{z}, \phi_{z z}\right) F\left(\phi, \phi_{t}, \phi_{z}, \phi_{t z}, \phi_{z z}, \phi_{t z}\right)\right]=0 \tag{52}
\end{equation*}
$$

where $F$ is given by (23). By Equating the coefficients of $\phi_{t z} \phi_{z z z z}, \phi_{t} \phi_{z z z z}$, and $\phi_{z z z z}^{2}$ to zero in (52), we obtain the following system of equations:

$$
\begin{align*}
\phi_{t z} \phi_{z z z z}: & (2 m-n) \phi_{z} \frac{\partial^{2} \Lambda}{\partial \phi_{z z}^{2}}+\phi \frac{\partial^{2} \Lambda}{\partial \phi_{z} \phi_{z z}}=0  \tag{53}\\
\phi_{t} \phi_{z z z z}: & \left(\phi^{m+2}+m(n-m-1) \phi^{n}+m \phi^{n+1}\right) \frac{\partial^{2} \Lambda}{\partial \phi_{z z}^{2}}  \tag{54}\\
& +(m+n) \phi^{n+1} \frac{\partial \Lambda}{\partial \phi_{z z}}+\phi^{n+2} \frac{\partial^{2} \Lambda}{\partial \phi \partial \phi_{z z}}=0 \\
\phi_{z z z z}^{2}: & \frac{\partial^{2} \Lambda}{\partial \phi_{z z}^{2}}=0 \tag{55}
\end{align*}
$$

From (55) it follows that

$$
\begin{equation*}
\Lambda\left(\phi, \phi_{z}, \phi_{z z}\right)=A\left(\phi, \phi_{z}\right) \phi_{z z}+B\left(\phi, \phi_{z}\right) \tag{56}
\end{equation*}
$$

Substituting (56) into (53) we find that

$$
\begin{equation*}
\frac{\partial A}{\partial \phi_{z}}=0 \tag{57}
\end{equation*}
$$

and therefore

$$
\begin{equation*}
A\left(\phi, \phi_{z}\right)=A(\phi) . \tag{58}
\end{equation*}
$$

Thus, (51) becomes

$$
\begin{equation*}
\Lambda\left(\phi, \phi_{z}, \phi_{z z}\right)=A(\phi) \phi_{z z}+B\left(\phi, \phi_{z}\right) \tag{59}
\end{equation*}
$$

Now substitute (59) into (54) which gives

$$
\begin{equation*}
\phi \frac{d A}{d \phi}+(m+n) A=0 \tag{60}
\end{equation*}
$$

The solution to (60) is

$$
\begin{equation*}
A=c_{1} \phi^{-(m+n)} \tag{61}
\end{equation*}
$$

where $c_{1}$ is a constant. Equation (59) becomes

$$
\begin{equation*}
\Lambda\left(\phi, \phi_{z}, \phi_{z z}\right)=c_{1} \phi^{-(m+n)} \phi_{z z}+B\left(\phi, \phi_{z}\right) \tag{62}
\end{equation*}
$$

Now substitute (62) into (48) and then equate the coefficient of $\phi_{t z z z}$ in (48) to zero. This gives

$$
\begin{equation*}
\frac{\partial B}{\partial \phi_{z}}=c_{1}(n-2 m) \phi^{-(m+n+1)} \phi_{z} \tag{63}
\end{equation*}
$$

and integrating (63) we have

$$
\begin{equation*}
B\left(\phi, \phi_{z}\right)=\frac{1}{2} c_{1}(n-2 m) \phi^{-(m+n+1)} \phi_{z}^{2}+P(\phi) . \tag{64}
\end{equation*}
$$

Thus, (62) becomes

$$
\begin{align*}
\Lambda\left(\phi, \phi_{z}, \phi_{z z}\right)= & c_{1} \phi^{-(m+n)} \phi_{z z} \\
& +\frac{1}{2} c_{1}(n-2 m) \phi^{-(m+n+1)} \phi_{z}^{2}+P(\phi) . \tag{65}
\end{align*}
$$

Lastly, substitute (65) into (48) and equate the coefficient of $\phi_{t} \phi_{z}^{2} \phi_{z z}$ to zero, which gives

$$
\begin{equation*}
(m-n-1) c_{1}=0 \tag{66}
\end{equation*}
$$

It follows from (66) that there are two cases.
Case $1\left(c_{1}=0\right)$. If $m-n-1 \neq 0$, then from (66) we have $c_{1}=0$. Therefore,

$$
\begin{equation*}
\Lambda\left(\phi, \phi_{z}, \phi_{z z}\right)=P(\phi) \tag{67}
\end{equation*}
$$

Thus, $\Lambda\left(\phi, \phi_{z}, \phi_{z z}\right)$ does not give a new multiplier and therefore new conservation laws will not be derived.

Case $2(n=m-1)$. If $c_{1} \neq 0$, then from (66), we have $n=m-1$. Thus, (65) becomes

$$
\begin{equation*}
\Lambda\left(\phi, \phi_{z}, \phi_{z z}\right)=c_{1} \phi^{1-2 m} \phi_{z z}-\frac{1}{2}(m+1) c_{1} \phi^{-2 m} \phi_{z}^{2}+P(\phi) \tag{68}
\end{equation*}
$$

Now substitute (68) into (48) and equate the coefficient of $\phi_{z} \phi_{t z}$ in (48) to zero, which gives

$$
\begin{equation*}
\frac{d^{2} P}{d \phi^{2}}+\frac{(2 m-1)}{\phi} \frac{d P}{d \phi}=(m-2) c_{1} \phi^{1-2 m} \tag{69}
\end{equation*}
$$

Solving (69) we have

$$
\begin{equation*}
P(\phi)=\frac{m-2}{3-2 m} \phi^{3-2 m} c_{1}+\frac{\phi^{2(1-m)}}{2(1-m)} c_{2}+c_{3} \tag{70}
\end{equation*}
$$

provided that $m \neq 3 / 2$ and $m \neq 1$. Since $n=m-1$, these two special cases correspond to the points $(3 / 2,1 / 2)$ and $(1,0)$ on the $(m, n)$ plane in Figure 1.

Consider first the general case $n=m-1$ excluding the points ( $3 / 2,1 / 2$ ) and ( 1,0 ). Equation (68) becomes

$$
\begin{align*}
\Lambda\left(\phi, \phi_{z}, \phi_{z z}\right)= & c_{1} \phi^{1-2 m} \phi_{z z}-\frac{1}{2}(m+1) c_{1} \phi^{-2 m} \phi_{z}^{2} \\
& +\frac{m-2}{3-2 m} \phi^{3-2 m} c_{1}+\frac{\phi^{2(1-m)}}{2(1-m)} c_{2}+c_{3} . \tag{71}
\end{align*}
$$

Substituting (71) into (48) we find that $c_{2}=0$. Hence,

$$
\begin{align*}
\Lambda\left(\phi, \phi_{z}, \phi_{z z}\right)= & c_{1} \phi^{1-2 m} \phi_{z z}-\frac{1}{2}(m+1) c_{1} \phi^{-2 m} \phi_{z}^{2}  \tag{72}\\
& +\frac{m-2}{3-2 m} \phi^{3-2 m} c_{1}+c_{3}
\end{align*}
$$

Since the multiplier (72) contains two constants, $c_{1}$ and $c_{3}$, it leads to two conserved vectors. The conserved vector
corresponding to $c_{3}=1, c_{1}=0$ is the elementary conserved vector (32). The constants $c_{1}=1, c_{3}=0$ lead to the conserved vector

$$
\begin{align*}
T_{1}= & \frac{1}{2} \phi^{-2 m} \phi_{z z}^{2}-\frac{1}{6} m(m+2) \phi^{-2 m-2} \phi_{z}^{4}  \tag{73}\\
& +\frac{1}{2}(3-m) \phi^{1-2 m} \phi_{z}^{2}+\frac{1}{2(3-2 m)} \phi^{4-2 m}, \\
T_{2}= & {\left[\frac{1}{6} m(m+5) \phi^{-2 m-2} \phi_{z}^{3}-\frac{(m-3)(m-1)}{(3-2 m)} \phi^{1-2 m} \phi_{z}\right] \phi_{t} } \\
& -\left[\frac{1}{2}(m+1) \phi^{-2 m-1} \phi_{z}^{2}+\frac{(2-m)}{(3-2 m)} \phi^{2-2 m}\right] \phi_{t z} \\
& -\frac{1}{2}(m-1) \phi^{-m-1} \phi_{z}^{2}+\frac{(m-1)}{(3-2 m)} \phi^{2-m} . \tag{74}
\end{align*}
$$

The case $n=m-1$ with $m=1$ and $n=0$ has already been considered. The multiplier is given by (29) and the conserved vectors by (40) and (41).

Consider $n=m-1$ with $m=3 / 2$ and $n=1 / 2$. The differential equation (69) becomes

$$
\begin{equation*}
\frac{d^{2} P}{d \phi^{2}}+\frac{2}{\phi} \frac{d P}{d \phi}=-\frac{1}{2} \frac{c_{1}}{\phi^{2}} \tag{75}
\end{equation*}
$$

The general solution of (75) is

$$
\begin{equation*}
P(\phi)=-\frac{1}{2} c_{1} \ln \phi+\frac{c_{3}}{\phi}+c_{4} . \tag{76}
\end{equation*}
$$

When $m=3 / 2$ the multiplier (68) becomes

$$
\begin{equation*}
\Lambda\left(\phi, \phi_{z}, \phi_{z z}\right)=c_{1} \phi^{-2} \phi_{z z}-\frac{5}{4} c_{1} \phi^{-3} \phi_{z}^{2}-\frac{1}{2} c_{1} \ln \phi+\phi^{-1} c_{3}+c_{4} . \tag{77}
\end{equation*}
$$

On substituting (77) into the determining equation (48) we find that $c_{3}=0$ and the multiplier reduces to

$$
\begin{equation*}
\Lambda=c_{1} \phi^{-2} \phi_{z z}^{2}-\frac{5}{4} c_{1} \phi^{-3} \phi_{z}^{2}-\frac{1}{2} c_{1} \ln \phi+c_{4} . \tag{78}
\end{equation*}
$$

The multiplier (78) again contains two arbitrary constants, $c_{1}$ and $c_{4}$. Setting the constants $c_{4}=1, c_{1}=0$ gives the elementary conserved vector (32). Setting $c_{1}=1, c_{4}=0$ leads to the conserved vector

$$
\begin{align*}
T_{1}= & \frac{1}{2} \phi^{-3} \phi_{z z}^{2}-\frac{7}{8} \phi^{-5} \phi_{z}^{4}+\frac{3}{4} \phi^{-2} \phi_{z}^{2} \frac{1}{2} \phi \ln \phi-\frac{1}{2} \phi,  \tag{79}\\
T^{2}= & {\left[\frac{13}{8} \phi^{-5} \phi_{z}^{3}-\phi^{-2} \phi_{z}+\frac{3}{4}\left(\phi^{-2} \ln \phi\right) \phi_{z}\right] \phi_{t} } \\
& -\left[\frac{5}{4} \phi^{-4} \phi_{z}^{2}+\frac{1}{2} \phi^{-1} \ln \phi\right] \phi_{t z}-\frac{1}{4} \phi^{-5 / 2} \phi_{z}^{2}+\frac{1}{2} \phi^{1 / 2} \ln \phi . \tag{80}
\end{align*}
$$

Consider next multipliers of the form

$$
\begin{equation*}
\Lambda\left(\phi, \phi_{z}, \phi_{z z}, \phi_{z z z}\right) \tag{81}
\end{equation*}
$$

The determining equation for the multiplier is

$$
\begin{equation*}
E_{\phi}\left(\Lambda\left(\phi, \phi_{z}, \phi_{z z}, \phi_{z z z}\right) F\left(\phi, \phi_{t}, \phi_{z}, \phi_{t z}, \phi_{z z}, \phi_{t z z}\right)\right]=0, \tag{82}
\end{equation*}
$$

where $F$ is given by (23). Equating the coefficient of $\phi_{t z z z z z}$ in (82) to zero, we find that

$$
\begin{equation*}
\frac{\partial \Lambda}{\partial \phi_{z z z}}=0 \tag{83}
\end{equation*}
$$

and therefore

$$
\begin{equation*}
\Lambda=\Lambda\left(\phi, \phi_{z}, \phi_{z z}\right) \tag{84}
\end{equation*}
$$

which has already been considered.
Harris [6] proved that, except possibly for the two special cases $n=m-1$ with $m \neq 1$ and $m=1$ with $n \neq 0$, there are no more independent conserved vectors. She proved this result using the direct method for conservation laws.

The multipliers and the corresponding conserved vectors for the partial differential equation (20) are listed in Table 1. This table was presented by Maluleke and Mason [7] without the multipliers. These conserved vectors agree with the results obtained by Barcilon and Richter [5] and Harris [6].
3.3. Association of Lie Point Symmetries with Conserved Vectors. The Lie point symmetries for the partial differential equation (20) are listed in Table 2. These Lie point symmetries were derived by Maluleke and Mason [7, 12]. Using (14) we will investigate which of the Lie point symmetries are associated with the conserved vectors for the Magma equation (20).
(i) $0 \leq n<\infty, 0 \leq m<\infty$. Consider first the Lie symmetry generator

$$
\begin{align*}
X= & \left(c_{1}+(2-m-n) c_{3} t\right) \frac{\partial}{\partial t} \\
& +\left(c_{2}+(n-m) c_{3} z\right) \frac{\partial}{\partial z}+2 c_{3} \phi \frac{\partial}{\partial \phi} \tag{85}
\end{align*}
$$

and the elementary conserved vector (32). Applying (14) we find that (85) is associated with the conserved vector (32) provided that $c_{3}=0$, that is, provided that

$$
\begin{equation*}
X=c_{1} \frac{\partial}{\partial t}+c_{2} \frac{\partial}{\partial z} . \tag{86}
\end{equation*}
$$

(ii) $n+m \neq 2, m \neq 1, n+m \neq 1$. Consider next the Lie point symmetry generator (85), with the conserved vector (33). Applying (14) we find that (85) is associated with (33) provided that $c_{3}=0$, that is, provided $X$ is given by (86).
(iii) $m=1, n \neq 0, n \neq 1$. Now consider

$$
\begin{equation*}
X=\left(-c_{3} t+c_{1}\right) \frac{\partial}{\partial t}+\left(c_{3} z+c_{2}\right) \frac{\partial}{\partial z}+\frac{2 c_{3} \phi}{n-1} \frac{\partial}{\partial \phi} \tag{87}
\end{equation*}
$$

Table 1: Multipliers and conserved vectors for the partial differential equation (20).

Case A. $0 \leq n<\infty, 0 \leq m<\infty$
Multiplier: $\Lambda=1$
$T^{1}=\phi$
$T^{2}=\phi^{n}\left(1+m \phi^{-m-1} \phi_{t} \phi_{z}-\phi^{-m} \phi_{t z}\right)$

Case B.1. $m \neq 1, m+n \neq 1, m+n \neq 2$
Multiplier: $\Lambda=\phi^{1-m-n}$
$T^{1}=\frac{1}{2-m-n}\left(\phi^{2-m-n}-1\right)+\frac{1}{2}(1-m-n) \phi^{-2 m} \phi_{z}^{2}$
$T^{2}=\frac{n}{1-m} \phi^{1-m}-\phi^{1-2 m} \phi_{t z}+m \phi^{-2 m} \phi_{t} \phi_{z}$
Case B.2. $m=1, n \neq 0, n \neq 1$
Multiplier: $\Lambda=\phi^{-n}$
$T^{1}=\frac{1}{1-n}-\frac{n}{2} \phi^{-2} \phi_{z}^{2}$
$T^{2}=n \ln \phi-\phi^{-1} \phi_{t z}+\phi^{-2} \phi_{t} \phi_{z}$
Case B.3. $m=n=1$
Multiplier: $\Lambda=\frac{1}{\phi}$
$T^{1}=-\frac{1}{2} \phi^{-2} \phi_{z}^{2}+\ln \phi$
$T^{2}=\ln \phi-\phi^{-1} \phi_{t z}+\phi^{-2} \phi_{t} \phi_{z}$

Case B.4. $m=1, n=0$
Multiplier: $\Lambda=\phi^{2}$
$T^{1}=\frac{1}{2} \phi^{-2} \phi_{z}^{2}+\phi \ln \phi-\phi$
$T^{2}=-\left(\phi^{-1} \ln \phi\right) \phi_{t z}+\left(\phi^{-2} \ln \phi\right) \phi_{t} \phi_{z}$

> Case B.5. $n+m=1, m \neq 1$ Multiplier: $\Lambda=\phi^{-2(n-2)}$ $T^{1}=-\frac{1}{2} \phi^{-2 m} \phi_{z}^{2}+\ln \phi$ $T^{2}=\phi^{1-m} \ln \phi-\frac{1}{1-m} \phi^{1-m}-\left(\phi^{1-2 m} \ln \phi\right) \phi_{t z}$ $\quad+\left(m \phi^{-2 m} \ln \phi\right) \phi_{t} \phi_{z}$

Case B.6. $n+m=2, m \neq 1$
Multiplier: $\Lambda=\frac{1}{\phi}$
$T^{1}=-\frac{1}{2} \phi^{-2 m} \phi_{z}^{2}+\ln \phi$
$T^{2}=\frac{2-m}{1-m} \phi^{1-m}-\phi^{1-2 m} \phi_{t z}+m \phi^{-2 m} \phi_{t} \phi_{z}$
Case C.1. $n=m-1, m \neq \frac{3}{2}, m \neq 1$
Multiplier: $\Lambda=\phi^{1-2 m} \phi_{z z}-\frac{1}{2}(m+1) \phi^{-2 m} \phi_{z}^{2}$

$$
\begin{aligned}
&+\frac{m-2}{3-2 m} \phi^{3-2 m} \\
& T^{1}= \frac{1}{1-n} \phi^{1-n}-\frac{n}{2} \phi^{-2} \phi_{z}^{2} \\
&+\frac{1}{2}(3-m) \phi^{1-2 m} \phi_{z}^{2}+\frac{1}{2(3-2 m)} \phi^{4-2 m} \\
& T^{2}= {\left[\frac{1}{6} m(m+5) \phi^{-2 m-2} \phi_{z}^{3}-\frac{(m-3)(m-1)}{(3-2 m)} \phi^{1-2 m} \phi_{z}\right] \phi_{t} } \\
&-\left[\frac{1}{2}(m+1) \phi^{-2 m-1} \phi_{z}^{2}+\frac{(2-m)}{(3-2 m)} \phi^{2-2 m}\right] \phi_{t z} \\
&-\frac{1}{2}(m-1) \phi^{-m-1} \phi_{z}^{2}+\frac{(m-1)}{(3-2 m)} \phi^{2-m} \\
& \text { Case C.2. } n=m-1, m=\frac{3}{2}, n=\frac{1}{2} \\
& \text { Multiplier: } \Lambda=\phi^{-2} \phi_{z z}^{2}-\frac{5}{4} \phi^{-3} \phi_{z}^{2}-\frac{1}{2} \ln \phi \\
& T^{1}= \frac{1}{2} \phi^{-3} \phi_{z z}^{2}-\frac{7}{8} \phi^{-5} \phi_{z}^{4}+\frac{3}{4} \phi^{-2} \phi_{z}^{2} \\
&+\frac{1}{2} \phi \ln \phi-\frac{1}{2} \phi \\
& T^{2}=\left.\frac{13}{8} \phi^{-5} \phi_{z}^{3}-\phi^{-2} \phi_{z}+\frac{3}{4}\left(\phi^{-2} \ln \phi\right) \phi_{z}\right] \phi_{t}-\left[\frac{5}{4} \phi^{-4} \phi_{z}^{2}+\frac{1}{2} \phi^{-1} \ln \phi\right] \phi_{t z} \\
&-\frac{1}{4} \phi^{-(5 / 2)} \phi_{z}^{2}+\frac{1}{2} \phi^{1 / 2} \ln \phi
\end{aligned}
$$

and the conserved vector (45). Applying (14) we find that (87) is associated with (45) provided that $c_{3}=0$, that is, provided that $X$ is given by (86).
(iv) $m=n=1$. Consider next

$$
\begin{equation*}
X=c_{1} \frac{\partial}{\partial t}+c_{2} \frac{\partial}{\partial z}+2 c_{3} \phi \frac{\partial}{\partial \phi} \tag{88}
\end{equation*}
$$

and the conserved vector (42). Applying (14) we find that (88) is associated with (42) provided that $c_{3}=0$, that is, provided that $X$ is (86).
(v) $m=1, n=0$. Consider

$$
\begin{equation*}
X=\xi^{1}(t, z) \frac{\partial}{\partial t}+\left(c_{2}+c_{3} z\right) \frac{\partial}{\partial z}-2 c_{3} \phi \frac{\partial}{\partial \phi} \tag{89}
\end{equation*}
$$

and the conserved vector (41). Applying (14) we find that (89) is associated with (41) provided that $c_{3}=0$, that is, provided that

$$
\begin{equation*}
X=\xi^{1}(t, z) \frac{\partial}{\partial t}+c_{2} \frac{\partial}{\partial z} \tag{90}
\end{equation*}
$$

(vi) $n+m=1, m \neq 1$. Consider next

$$
\begin{equation*}
X=\left(c_{1}+c_{3} t\right) \frac{\partial}{\partial t}+\left(c_{2}+(1-2 m) c_{3} z\right) \frac{\partial}{\partial z}+2 c_{3} \phi \frac{\partial}{\partial \phi} \tag{91}
\end{equation*}
$$

and the conserved vector (35). Applying (14) we find that (91) is associated with (35) provided that $c_{3}=0$, that is, provided that $X$ is given by (86).
(vii) $n+m=2, m \neq 1$. Consider

$$
\begin{equation*}
X=c_{1} \frac{\partial}{\partial t}+\left(c_{2}+2(1-m) c_{3} z\right) \frac{\partial}{\partial z}+2 c_{3} \phi \frac{\partial}{\partial \phi} \tag{92}
\end{equation*}
$$

with the conserved vector (38). Applying (14) we find that (92) is associated with (38) provided that $c_{3}=0$, that is, provided that $X$ is given by (86).
(viii) $n=m-1, m \neq 3 / 2, m \neq 1$. Consider

$$
\begin{equation*}
X=\left(c_{1}+(3-2 m) c_{3}\right) \frac{\partial}{\partial t}+\left(c_{2}-c_{3} z\right) \frac{\partial}{\partial z}+2 c_{3} \phi \frac{\partial}{\partial \phi} \tag{93}
\end{equation*}
$$

with the conserved vector given by (73) and (74). Applying (14) we find that (93) is associated with the conserved vector with components (73) and (74) provided that $c_{3}=0$, that is, provided that $X$ is given by (86).
(ix) $n=m-1, m=3 / 2, n=1 / 2$. Finally consider the Lie point symmetry (93) and the conserved vector with components (79) and (80). Applying (14) we find that (93) is associated with this conserved vector provided that $c_{3}=0$, that is, provided that $X$ is given by (86).

Except for the conserved vector (41) ( $n=0, m=1$ ) the conserved vectors are all associated with the Lie point symmetry which generates travelling wave solutions. The Lie point symmetry with which the conserved vector (41) is associated contains (86) as a special case. In all cases new conserved vectors are not generated by (15).

Next we derive the conservation laws for the magma equation with an exponential law for the permeability and viscosity using the multiplier method.

## 4. Conservation Laws for the Magma Equation with an Exponential Law for the Permeability and Viscosity by the Multiplier Method

When the permeability and viscosity are related to the voidage by exponential laws the magma equation becomes

$$
\begin{align*}
\frac{\partial \phi}{\partial t}+ & \frac{\partial}{\partial z}\left[\exp [n(\phi-1)]\left(1-\frac{\partial}{\partial z}\left(\exp [-m(\phi-1)] \frac{\partial \phi}{\partial t}\right)\right)\right] \\
& =0 \tag{94}
\end{align*}
$$

4.1. Lower Order Conservation Laws. In order to derive conservation laws for (94) consider a multiplier of the form
(21). A multiplier for the partial differential equation has the property (22), where now

$$
\begin{align*}
& F\left(\phi, \phi_{t}, \phi_{z}, \phi_{t z}, \phi_{z z}, \phi_{t z z}\right) \\
&= \phi_{t}+n \phi_{z} \exp [n(\phi-1)] \\
&+m n \phi_{t} \phi_{z}^{2} \exp [(n-m)(\phi-1)] \\
&-n \phi_{z} \phi_{t z} \exp [(n-m)(\phi-1)]  \tag{95}\\
&-m^{2} \phi_{t} \phi_{z}^{2} \exp [(n-m)(\phi-1)] \\
&+m \phi_{t} \phi_{z z} \exp [(n-m)(\phi-1)] \\
&+2 m \phi_{z} \phi_{t z} \exp [(n-m)(\phi-1)] \\
& \quad-\exp [(n-m)(\phi-1)] \phi_{t z z} .
\end{align*}
$$

The determining equation for the multiplier is given by (24), where $E_{\phi}$ is given by (18). Separating (24) with respect to products and powers of the partial derivatives of $\phi$ we obtain the following system of equations:

$$
\begin{align*}
& \phi_{z} \phi_{t z}: \frac{d^{2} \Lambda}{d \phi^{2}}+(m+n) \frac{d \Lambda}{d \phi}=0,  \tag{96}\\
& \phi_{t} \phi_{z z}: \frac{d^{2} \Lambda}{d \phi^{2}}+(m+n) \frac{d \Lambda}{d \phi}=0,  \tag{97}\\
& \phi_{t} \phi_{z}^{2}: \frac{d^{3} \Lambda}{d \phi^{3}}+2 n \frac{d^{2} \Lambda}{d \phi^{2}}+\left(n^{2}-m^{2}\right) \frac{d \Lambda}{d \phi}=0 . \tag{98}
\end{align*}
$$

Equation (96) is the same as (97). It is readily verified that every solution of (96) is a solution of (98). We therefore need to consider only (96). The general solution of (96) is

$$
\begin{gather*}
\Lambda(\phi)=c_{2} \exp [-(m+n) \phi]+c_{1}, \quad \text { if } n+m \neq 0  \tag{99}\\
\Lambda(\phi)=c_{2} \phi+c_{1}, \quad \text { if } n+m=0 \tag{100}
\end{gather*}
$$

We are considering $n \geq 0$ and $m \geq 0$ and therefore $n+m=$ 0 only if $n=0$ and $m=0$. Proceeding as before we have for various combinations of $m$ and $n$ different conserved vectors.
(i) $n+m \neq 0, m \neq 0$. This gives the conserved vectors

$$
\begin{gather*}
T^{1}=\phi, \\
T^{2}=\exp [n(\phi-1)]+m \exp [(n-m)(\phi-1)] \phi_{t} \phi_{z} \\
-\exp [(n-m)(\phi-1)] \phi_{t z}  \tag{101}\\
T^{1}=-\frac{1}{m+n} \exp [-(m+n) \phi] \\
\quad-\frac{(n+m)}{2} \phi_{z}^{2} \exp [-2 m \phi+m-n]  \tag{102}\\
T^{2}= \\
-\frac{n}{m} \exp [-(m \phi+n)] \\
\quad+\exp [-2 m \phi+m-n]\left(m \phi_{t} \phi_{z}-\phi_{t z}\right)
\end{gather*}
$$

Table 2: Lie point symmetries of the partial differential equation (20).
Case 1. $m \neq 1, m \neq n, n \neq 0$
$X_{1}=(2-n-m) t \frac{\partial}{\partial t}+(n-m) z \frac{\partial}{\partial z}+2 \phi \frac{\partial}{\partial \phi}$
$X_{2}=\frac{\partial}{\partial t}$
$X_{3}=\frac{\partial}{\partial z}$
Case 2. $n \neq 0, m=1, n \neq 1$
$X_{1}=-t \frac{\partial}{\partial t}+z \frac{\partial}{\partial z}+\frac{2 \phi}{n-1} \frac{\partial}{\partial \phi}$
$X_{2}=\frac{\partial}{\partial t}$
$X_{3}=\frac{\partial}{\partial z}$
Case 3. $m=\frac{4}{3}, n=0$
$X_{1}=-\frac{1}{3} z^{2} \frac{\partial}{\partial z}+z \phi \frac{\partial}{\partial \phi}$
$X_{2}=\xi(t) \frac{\partial}{\partial t}$
$X_{3}=z \frac{\partial}{\partial z}-\frac{3}{2} \phi \frac{\partial}{\partial \phi}$
$X_{4}=\frac{\partial}{\partial z}$

The conserved vector (101) is the elementary conserved vector. The multiplier for (102) is, from (99),

$$
\begin{equation*}
\Lambda(\phi)=\exp [-(m+n) \phi] \tag{103}
\end{equation*}
$$

(ii) $n=m=0$. We obtain two conserved vectors

$$
\begin{array}{cl}
T^{1}=\phi, & T^{2}=1-\phi_{t z} \\
T^{1}=\frac{\phi^{2}}{2}, & T^{2}=\phi_{t}-\phi_{t z} \tag{105}
\end{array}
$$

The conserved vector (104) is the elementary conserved vector with multiplier $c_{1}$ and $m=n=0$. The multiplier for (105) is, from (100),

$$
\begin{equation*}
\Lambda(\phi)=\phi . \tag{106}
\end{equation*}
$$

(iii) $m=0$ and $n>0$. We again obtain two conserved vectors

$$
\begin{gather*}
T^{1}=\phi, \quad T^{2}=\exp [n(\phi-1)]\left(1-\phi_{t z}\right),  \tag{107}\\
T^{1}=-\frac{1}{n} \exp [-n \phi]-\frac{1}{2} n \exp [-n \phi] \phi_{z}^{2},  \tag{108}\\
T^{2}=n \exp [-n \phi]-\phi_{t z} .
\end{gather*}
$$

The conserved vector (107) is the elementary conserved vector with multiplier $c_{1}$ and $m=0$. The multiplier of the conserved vector (108) is by (99)

$$
\begin{equation*}
\Lambda(\phi)=\exp [-n \phi] \tag{109}
\end{equation*}
$$

(iv) $n=0, m>0$. Finally we obtain the conserved vectors

$$
\begin{gather*}
T^{1}=\phi, \\
T^{2}=1+m \exp [-m(\phi-1)] \phi_{t} \phi_{z}-\exp [-m(\phi-1)] \phi_{t z} \tag{110}
\end{gather*}
$$

$$
\begin{gather*}
T_{1}=-\frac{1}{m} \exp (-m \phi)-\frac{1}{2} m \exp [m(1-2 \phi)] \phi_{z}^{2}  \tag{111}\\
T_{2}=\exp [m(1-2 \phi)]\left(m \phi_{t} \phi_{z}-\phi_{t z}\right)
\end{gather*}
$$

The conserved vector (110) is the elementary conserved vector with multiplier $c_{1}$ and $n=0$, while the multiplier for (111) is by (99):

$$
\begin{equation*}
\Lambda=\exp [-m \phi] \tag{112}
\end{equation*}
$$

4.2. The Search for Higher Order Conservation Laws. We now consider a multiplier of the form

$$
\begin{equation*}
\Lambda=\Lambda\left(\phi, \phi_{z}\right) \tag{113}
\end{equation*}
$$

The determining equation for the multiplier is (48), where $F$ is given by (95). By equating the coefficient of the highest order derivative term, $\phi_{t z z z}$, to zero in (48) we obtain again (49), so that $\Lambda\left(\phi, \phi_{z}\right)=\Lambda(\phi)$. The multiplier therefore reduces to that of the previous case and new conserved vectors are not derived.

Consider next the multiplier

$$
\begin{equation*}
\Lambda=\Lambda\left(\phi, \phi_{z}, \phi_{z z}\right) \tag{114}
\end{equation*}
$$

As before, the determining equation for the multiplier is (48), where $F$ is given by (95). By equating the coefficients of $\phi_{t z} \phi_{z z z z}, \phi_{t} \phi_{z z z z}$, and $\phi_{z z z z}^{2}$ to zero in (48), the following system of equations is obtained:

$$
\begin{align*}
\phi_{t z} \phi_{z z z z}: & (2 m-n) \phi_{z} \frac{\partial^{2} \Lambda}{\partial \phi_{z z}^{2}}+\frac{\partial^{2} \Lambda}{\partial \phi_{z} \phi_{z z}}=0,  \tag{115}\\
\phi_{t} \phi_{z z z z}: & m\left((n-m) \phi_{z}^{2}+\phi_{z z}\right) \frac{\partial^{2} \Lambda}{\partial \phi_{z z}^{2}}  \tag{116}\\
& +(m+n) \frac{\partial \Lambda}{\partial \phi_{z z}}+\frac{\partial^{2} \Lambda}{\partial \phi \partial \phi_{z z}}=0, \\
\phi_{z z z z}^{2}: & \frac{\partial^{2} \Lambda}{\partial \phi_{z z}^{2}}=0 . \tag{117}
\end{align*}
$$

By using (115) and (117), it is readily shown that (56) again holds. Substituting (56) into (116) gives

$$
\begin{equation*}
\frac{d A}{d \phi}+(m+n) A=0 \tag{118}
\end{equation*}
$$

and therefore

$$
\begin{equation*}
A(\phi)=c_{1} \exp [-(m+n) \phi] . \tag{119}
\end{equation*}
$$

Equation (56) now becomes

$$
\begin{equation*}
\Lambda\left(\phi, \phi_{z}, \phi_{z z}\right)=c_{1} \exp [-(m+n) \phi] \phi_{z z}+B\left(\phi, \phi_{z}\right) . \tag{120}
\end{equation*}
$$

Substituting (120) into the determining equation (48) and then equating the coefficients of $\phi_{t z z z}$ in (48) to zero gives

$$
\begin{equation*}
\frac{\partial B}{\partial \phi_{z}}=\frac{1}{2}(n-2 m) c_{1} \exp [-(m+n) \phi] \phi_{z} \tag{121}
\end{equation*}
$$

and hence

$$
\begin{equation*}
B\left(\phi, \phi_{z}\right)=\frac{1}{4}(n-2 m) c_{1} \exp [-(m+n) \phi] \phi_{z}^{2}+P(\phi) . \tag{122}
\end{equation*}
$$

The multiplier becomes

$$
\begin{align*}
\Lambda\left(\phi, \phi_{z}, \phi_{z z}\right)= & c_{1} \phi^{-(m+n)} \phi_{z z}+\frac{1}{4}(n-2 m)  \tag{123}\\
& \times c_{1} \exp [-(m+n) \phi] \phi_{z}^{2}+P(\phi) .
\end{align*}
$$

Finally we substitute (123) back into (48) and equate the coefficient of $\phi_{t} \phi_{z}^{2} \phi_{z z}$ to zero. This yields

$$
\begin{equation*}
m(m-n) c_{1}=0 . \tag{124}
\end{equation*}
$$

There are three cases to consider, $m=0, m=n$, and $c_{1}=0$. The conserved vectors for $m=0, n>0$ are given by (107) and (108). We now consider the two remaining cases.

Case $1(m \neq n, m>0)$. Then, $c_{1}=0$ and (123) reduces to

$$
\begin{equation*}
\Lambda\left(\phi, \phi_{z}, \phi_{z z}\right)=P(\phi) . \tag{125}
\end{equation*}
$$

The multiplier is therefore a function of $\phi$ only which does not yield new conserved vectors.

Case $2\left(c_{1} \neq 0, m>0\right)$. Then, $m=n$ and the multiplier (123) becomes

$$
\begin{align*}
\Lambda\left(\phi, \phi_{z}, \phi_{z z}\right)= & c_{1} \exp [-2 n \phi] \phi_{z z} \\
& -\frac{1}{4} n c_{1} \exp [-2 n \phi] \phi_{z}^{2}+P(\phi) . \tag{126}
\end{align*}
$$

Equation (126) is substituted back into the determining equation (48) and by equating the coefficient of $\phi_{z} \phi_{t z}$ to zero we obtain

$$
\begin{equation*}
\frac{d^{2} P}{d \phi^{2}}+2 n \frac{d P}{d \phi}=n \exp [-n(2 \phi+1)+1] c_{1} . \tag{127}
\end{equation*}
$$

The general solution to (127) is

$$
\begin{align*}
P(\phi)= & -\frac{c_{1}}{4 n}(1+2 n \phi) \exp [1-n-2 n \phi]  \tag{128}\\
& +c_{2} \exp [-2 n \phi]+c_{3},
\end{align*}
$$

where $n \neq 0$ since $n=m$ and $m \neq 0$. Thus, (123) becomes

$$
\begin{align*}
\Lambda\left(\phi, \phi_{z}, \phi_{z z}\right)= & c_{1} \exp [-2 n \phi] \phi_{z z}-\frac{1}{4} n c_{1} \exp [-2 n \phi] \phi_{z}^{2} \\
& -\frac{c_{1}}{4 n}(1+2 n \phi) \exp [1-n-2 n \phi] \\
& +c_{2} \exp [-2 n \phi]+c_{3} . \tag{129}
\end{align*}
$$

Finally substituting (129) into the determining equation (48) gives $c_{2}=0$ and therefore

$$
\begin{align*}
\Lambda\left(\phi, \phi_{z}, \phi_{z z}\right)= & c_{1} \exp [-2 n \phi] \phi_{z z}-\frac{1}{4} n c_{1} \exp [-2 n \phi] \phi_{z}^{2} \\
& -\frac{c_{1}}{4 n}(1+2 n \phi) \exp [1-n-2 n \phi]+c_{3} \tag{130}
\end{align*}
$$

Two conserved vectors are obtained since the multiplier (130) contains two arbitrary constants. The constant $c_{3}$ gives the elementary conserved vector (100) while the constant $c_{1}$ gives the new conserved vector

$$
\begin{aligned}
T^{1}= & \frac{1}{2} \exp [-2 n \phi] \phi_{z z}^{2}-\frac{1}{2} n^{2} \exp [-2 n \phi] \phi_{z}^{4} \\
& +\frac{1}{4} n(1+2 n \phi) \phi_{z}^{2}+\frac{1}{8 n^{2}}(1+2 n \phi) \exp [1-n-2 n \phi]
\end{aligned}
$$

$$
\begin{align*}
T_{2}= & {\left[n^{2} \exp [-2 n \phi] \phi_{z}^{3}-\frac{n}{4}(2 n \phi+1) \exp [1-n-2 n \phi] \phi_{z}\right] }  \tag{131}\\
& \times \phi_{t}-n \exp [-n(\phi+1)] \phi_{t z} \\
& -\frac{1}{4}(2 n \phi+1) \exp [1-n-2 n \phi], \tag{132}
\end{align*}
$$

which exists if $m=n$ and $m>0, n>0$.

Table 3: Multipliers and conserved vectors for the partial differential equation (94).

```
Case A. \(0 \leq n<\infty, 0 \leq m<\infty\)
Multiplier: \(\Lambda(\phi)=1\)
\(T^{1}=\phi\)
\(T^{2}=\exp [n(\phi-1)]+m \exp [(n-m)(\phi-1)] \phi_{t} \phi_{z}-\exp [(n-m)(\phi-1)] \phi_{t z}\)
```

Case B.1. $m+n \neq 0, m>0, n>0$
Multiplier: $\Lambda(\phi)=\exp [-(m+n) \phi]$
$T^{1}=-\frac{1}{m+n} \exp [-(m+n) \phi]-\frac{1}{2}(n+m) \exp [m-n-2 m \phi] \phi_{z}^{2}$
$T^{2}=-\frac{n}{m} \exp [-(m \phi+n)]+\exp [m-n-2 m \phi]\left(m \phi_{t} \phi_{z}-\phi_{t z}\right)$

Case B.2. $m=0, n=0$
Multiplier: $\Lambda(\phi)=\phi$
$T^{1}=\frac{1}{2} \phi^{2}$
$T^{2}=\phi_{t}-\phi_{t z}$
Case B.3. $m=0, n>0$
Multiplier: $\Lambda(\phi)=\exp (-n \phi)$
$T^{1}=-\frac{1}{n} \exp (-n \phi)-\frac{1}{2} n \exp [-n \phi] \phi_{z}^{2}$
$T^{2}=n \exp [-n \phi]-\phi_{t z}$
Case B.4. $n=0, m>0$
Multiplier: $\Lambda(\phi)=\exp [-m \phi]$
$T^{1}=-\frac{1}{m} \exp [-m \phi]-\frac{1}{2} m \exp [m(1-2 \phi)] \phi_{z}^{2}$
$T^{2}=\exp [m(1-2 \phi)]\left(m \phi_{t} \phi_{z}-\phi_{t z}\right)$
Case C. $m=n, n \neq 0$
Multiplier: $\Lambda\left(\phi, \phi_{z}, \phi_{z z}\right)=\exp [-2 n \phi] \phi_{z z}-\frac{1}{2} n \exp [-2 n \phi] \phi_{z}^{2}-\frac{1}{4 n}(2 n \phi+1) \exp [1-n-2 n \phi]$
$T_{1}=\frac{1}{2} \exp [-2 n \phi] \phi_{z z}^{2}-\frac{1}{2} n^{2} \exp [-2 n \phi] \phi_{z}^{4}+\frac{1}{4} n(2 n \phi+1) \phi_{z}^{2}+\frac{1}{8 n^{2}}(2 n \phi+1) \exp [1-n-2 n \phi]$
$\left.T_{2}=\left[n^{2} \exp (-2 n \phi) \phi_{z}^{3}-\frac{1}{4} n(2 n \phi+1) \exp [1-n-2 n \phi)\right] \phi_{z}\right] \phi_{t}-n \exp (-n(\phi+1)) \phi_{t z}$
$-\frac{1}{4}(2 n \phi+1) \exp [1-n-2 n \phi]$

Consider next multipliers of the form (81). The determining equation for the multiplier is (82), where $F$ is given by (95). By equating to zero the coefficient of $\phi_{t z z z z z}$ in (82) we again derive (83) and the multiplier therefore reduces to the form (84) which has already been considered.

The multipliers and the corresponding conserved vectors for the partial differential equation (94) are listed in Table 3. The $(m, n)$ plane is illustrated in Figure 2.

### 4.3. Association of Lie Point Symmetries with Conserved

 Vectors. The Lie point symmetries of the partial differential equation (94) are given in Table 4. We use (14) to investigate which Lie point symmetries of (94) are associated with the conserved vectors for (94).(i) $m \neq 0, n>0, m>0$. Consider first the Lie point symmetry generator

$$
\begin{equation*}
X=\left(c_{3} \frac{m+n}{m-n} t+c_{1}\right) \frac{\partial}{\partial t}+\left(c_{3} z+c_{2}\right) \frac{\partial}{\partial z}-\frac{2 c_{3}}{m-n} \frac{\partial}{\partial \phi} \tag{133}
\end{equation*}
$$

and the elementary conserved vector (101). We find that (133) is associated with the elementary conserved vector provided that $c_{3}=0$, that is, provided that $X$ is given by (86).
(ii) $m+n \neq 0, m>0, n>0$. Consider next the Lie point symmetry generator (133) and the conserved vector (102). It can be verified that (102) is associated with (133) provided that $c_{3}=0$, that is, provided that $X$ is given by (86).
(iii) $m=0, n>0$. Now consider the Lie point symmetry

$$
\begin{equation*}
X=\left(c_{1}-c_{3} t\right) \frac{\partial}{\partial t}+\left(c_{3} z+c_{2}\right) \frac{\partial}{\partial z}+\frac{2}{n} c_{3} \frac{\partial}{\partial \phi} \tag{134}
\end{equation*}
$$

and the conserved vector (108). Using again (14) we find that (134) is associated with (108) provided that $c_{3}=0$, that is, provided that $X$ is given by (86).
(iv) $n=0, m>0$. Consider the Lie point symmetry

$$
\begin{equation*}
X=\xi^{1}(t, z) \frac{\partial}{\partial t}+\left(c_{2}+c_{3} z\right) \frac{\partial}{\partial z}-\frac{2 c_{3}}{m} \frac{\partial}{\partial \phi} \tag{135}
\end{equation*}
$$



Figure 2: The ( $m, n$ )-plane. The special cases lie on the line $n=m$.
and the conserved vector (111). We find that (135) is associated with (111) provided that $c_{3}=0$, that is, provided that

$$
\begin{equation*}
X=\xi^{1}(t, z)+c_{2} \frac{\partial}{\partial z} . \tag{136}
\end{equation*}
$$

We see that, except for $(135)(n=0, m>0)$, the conserved vector is associated with the Lie point symmetry (86) which generates a travelling wave solution. The conserved vector (111) is associated with (136) which includes (86) as a special case. In all cases, (15) does not yield a new conserved vector.

## 5. Conclusion

In this paper the multiplier method was used to derive the conservation laws for the magma equation for the case in which the permeability and viscosity satisfy a power law. The results agree with those of Harris [6], who derived the conserved vectors using the direct method. Unlike the direct method the functional form of the conserved vector does not need to be assumed with the multiplier method. Instead the variables on which the multiplier depend have to be chosen but this can be done by starting with a simple form and including higher order partial derivatives later to derive higher order conservation laws. The determining equation for the multiplier is readily obtained with the aid of the Euler operator.

Conserved vectors for the magma equation when the permeability and matrix viscosity depend on the voidage by exponential laws were derived using the multiplier method. Their properties are similar to the properties of the conserved vectors for the power law relations.

We investigated the association of Lie point symmetries of the magma equation with the conserved vectors. For all

Table 4: Lie point symmetries of the partial differential equation (94).

| Case 1. $m, n \neq 0, m \neq n$ | Case 2. $n=0, m \neq 0$ |
| :--- | :--- |
| $X_{1}=\frac{m+n}{m-n} t \frac{\partial}{\partial t}+z \frac{\partial}{\partial z}-\frac{2}{m-n} \frac{\partial}{\partial \phi}$ | $X_{1}=\xi(t) \frac{\partial}{\partial t}$ |
| $X_{2}=\frac{\partial}{\partial t}$ | $X_{2}=\frac{\partial}{\partial z}$ |
| $X_{3}=\frac{\partial}{\partial z}$ | $X_{3}=z \frac{\partial}{\partial z}-\frac{2}{m} \frac{\partial}{\partial \phi}$ |
| Case $3 . n \neq 0, m \neq 0, m=n$ | Case $4 . m=0, n \neq 0$ |
| $X_{1}=t \frac{\partial}{\partial t}-\frac{1}{n} \frac{\partial}{\partial \phi}$ | $X_{1}=-t \frac{\partial}{\partial t}+z \frac{\partial}{\partial z}+\frac{2}{n} \frac{\partial}{\partial \phi}$ |
| $X_{2}=\frac{\partial}{\partial t}$ | $X_{2}=\frac{\partial}{\partial z}$ |
| $X_{3}=\frac{\partial}{\partial z}$ | $X_{3}=\frac{\partial}{\partial t}$ |

conserved vectors considered except two the associated Lie point symmetry was the Lie point symmetry which generates travelling wave solutions $[4,5]$.

We were not able to derive new conservation laws for the partial differential equation (20) or determine if the number of conservation laws for (20) is finite or infinite. Harris [6] has proved that except possibly for the two special cases, $n=m-1$ with $m \neq 1$ and $m=1$ with $n \neq 0$, there are no more independent conserved vectors. Our results derived using multipliers are consistent with the results of Harris. All known conserved vectors of (20) and also the new conserved vectors for (94) can be derived from multipliers which depend only on $\phi$ and the partial derivatives of $\phi$ with respect to $z$. We find that the multipliers $\Lambda\left(\phi, \phi_{z}\right)$ and $\Lambda\left(\phi, \phi_{z}, \phi_{z z}, \phi_{z z z}\right)$ whose variables ended in odd order partial derivatives of $\phi$ with respect to $z$ did not generate new conserved vectors but instead reduced to the multipliers $\Lambda(\phi)$ and $\Lambda\left(\phi, \phi_{z}, \phi_{z z}\right)$, respectively. This also applies to the multipliers for the conserved vectors for (94).

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Nonlinear Self-Adjoint Classification of a Burgers-KdV Family of Equations 

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The concepts of strictly, quasi, weak, and nonlinearly self-adjoint differential equations are revisited. A nonlinear self-adjoint classification of a class of equations with second and third order is carried out.

## 1. Introduction

Since Ibragimov [1] proposed an extension to the Noether theorem, overcoming the major deficiency of that result, the existence of a Lagrangian, a considerable number of researchers have been applying his ideas for constructing conservation laws for equations without classical Lagrangians.

However, the price for applying what Ibragimov proposed in [1] is the obtainment, a priori, of nonlocal conservation laws instead of local ones.

In $[1,2]$, it was introduced the concept of self-adjoint differential equation, latter receiving a new designation in [3, 4], where it was called strictly self-adjoint equation. This last one will be adopted in this work. Although such concept was not necessarily new; see [5], the works [1, 2] were the start point of an intense research in this kind of ideas, giving rise to new developments $[3,4,6]$ and providing local conservation laws for equations once its symmetries are known.

One of the first papers dealing with some kind of classification was [7]. There, the authors considered a class of fourth-order evolution equations and found the self-adjoint subclasses. Then, in [8], the same class was enlarged by considering nonlinear dispersion as well as source terms.

Weak self-adjointness of some classes of equations was discussed by Gandarias and coauthors in [9-11].

In regard to third-order equations, in [12], a KdV type family was considered. However, at the time of this last reference, the general concept of nonlinear self-adjointness was not already introduced. In [13], a class of third-order dispersive equations was considered from the quasi selfadjoint point of view. More recently, in [14] a general family of dispersive evolution equations was classified with respect to quasi self-adjointness.

Recently [15], we considered a class of time dependent equations up to fifth-order and we obtained necessary and sufficient conditions for determining the nonlinearly selfadjoint subclasses. Nonlinear self-adjointness of equations up to fifth-order can also be found in [16-18].

In [19], a general class of first order $(1+1)$ PDE was classified with respect to strictly and quasi self-adjointness. Later, in [20], the subclass of the Riemman, or inviscid Burgers equation, was reconsidered from the point of view of nonlinear self-adjointness. Recently, in the paper [21], the last class was studied incorporating damping and conservation laws were established.

The number of works dealing with systems and selfadjointness is reduced compared with those ones considering scalar equations. To cite some of them; for instance, up to our knowledge, the first paper dealing with some self-adjointness and systems of PDEs was [22]. Nonlinear self-adjointness of
a system of coupled modified KdV equations was studied in [23]. Further examples can be found in [4].

In [24] quasi self-adjointness of a class of wave equation was considered. Sometime ago, in [25], a class of wave equation with dissipation was considered from the nonlinear self-adjoint point of view.

The vast majority of the papers deal with $(1+1)$ equations. However, some results considering PDEs with more independent variables have been communicated in the literature. In [26-28], diffusion equations with more than one spatial dimensions were considered. A generalization of Kuramoto-Sivashinsky equation was discussed in [29]. In [30] an extension of KdV equation, the so-called ZakharovKuznetsov equation, was studied. All of these papers dealt with nonlinear self-adjointness.

The concepts of self-adjoint differential equations will be better discussed in Section 2. In fact, this is a threefold purpose paper: the first is to provide a review on some works dealing with conservation laws and using the concepts introduced in $[1-4,6]$. The second one is to explore the concepts of strict, quasi, weak, and nonlinear self-adjoint differential equations, as the reader can check in Section 2. Although these concepts are commonly, and in fact, powerfully employed for constructing local conservation laws, such concepts have interest by themselves. Finally, it is common to classify equations under certain properties; see, for instance, [31, 32]. Then, in this work we consider, in Section 4, a nonlinear self-adjoint classification of the equation

$$
\begin{align*}
u_{t}= & r(x, t, u) u_{x x x}+s(x, t, u) u_{x x}  \tag{1}\\
& +f(x, t, u) u_{x}+h(x, t, u)
\end{align*}
$$

Such equation includes a great number of important equations in mathematical physics. To cite a few number of them, we mention KdV, Burgers, Burgers-KdV, and Riemman equations. More equations belonging to this class will be considered in the next sections.

Moreover, the self-adjoint classification carried out here will be used in [33] for constructing local conservation laws for equations without Lagrangians.

## 2. Preliminaries

Before presenting the procedure, it is convenient to leave clear that in the present paper we only consider scalar differential equations. In the current section, unless it is explicitly announced, $x=\left(x^{1}, \ldots, x^{n}\right)$ is an independent variable, while $u=u(x)$ is a dependent one. The set of first order derivatives of $u$ is denoted by $u_{(1)}$ and equal convention is employed for referring to higher order derivatives; for example, $u_{(k)}$ means the set of $k$ th derivatives of $u$.

We assume the summation over the repeated indices. By differential functions we mean locally analytic functions of a finite number of variables $x, u$ and $u$ derivatives. The highest order of derivatives appearing in a differential function is called its order. The vector space of all differential functions of finite order is denoted by $\mathscr{A}$.

Let us now show the algorithm for constructing conservation laws. Given a PDE

$$
\begin{equation*}
F=F\left(x, u, u_{(1)}, \ldots, u_{(m)}\right)=0, \quad F \in \mathscr{A} \tag{2}
\end{equation*}
$$

Step 1. We construct the formal Lagrangian $\mathscr{L}=v F$.
Step 2. From the Euler-Lagrange equations, the following system is obtained:

$$
\begin{gather*}
F\left(x, u, u_{(1)}, \ldots, u_{(m)}\right)=0  \tag{3}\\
F^{*}\left(x, u, v, u_{(1)}, v_{(1)}, \ldots, u_{(m)}, v_{(m)}\right)=0 \tag{4}
\end{gather*}
$$

where the second equation of the system (3)-(4) is called adjoint equation to $F=0$.

Step 3. A conserved vector for such system is $C=\left(C^{i}\right)$, where

$$
\begin{align*}
C^{i}= & \xi^{i} \mathscr{L}+W\left[\frac{\partial \mathscr{L}}{\partial u_{i}}-D_{j}\left(\frac{\partial \mathscr{L}}{\partial u_{i j}}\right)+D_{j} D_{k}\left(\frac{\partial \mathscr{L}}{\partial u_{i j k}}\right)-\cdots\right] \\
& +D_{j}(W)\left[\frac{\partial \mathscr{L}}{\partial u_{i j}}-D_{k}\left(\frac{\partial \mathscr{L}}{\partial u_{i j k}}\right)+\cdots\right]+\cdots \tag{5}
\end{align*}
$$

and $W=\eta-\xi^{i} u_{i}$.
Of course, it is clear that components (5) depend explicitly on the new variable $v$, which is not a "natural" variable from the original equation. For this reason, the conservation laws provided by the developments [1] are, a priori, nonlocal conservation laws and, consequently, the conserved vectors are nonlocal ones.

As it was previously pointed out at the beginning, the points related with conservation laws will be retaken soon, in [33]. However, for those more anxious, we invite them to consult the books [34-37] for the discussion between symmetries and conservation laws. We also guide the interested readers to [38-42] for discussions on conservation laws.

The question is: would it be possible to construct, from the nonlocal conservation laws (5), local ones? This point is essentially related with: would it be possible to replace the nonlocal function $v$ by an expression depending of $x, u$ and eventually derivatives of $u$ ? This lead us to the main subject of the paper: "self-adjointness", which will just be revisited. We firstly begin with the following.

Definition 1. Equation (3) is said to be strictly self-adjoint if the equation obtained from the adjoint equation (4) by the substitution $v=u$ is identical with the original equation (3); that is,

$$
\begin{equation*}
\left.F^{*}\right|_{v=u}=\lambda(x, u, \ldots) F \tag{6}
\end{equation*}
$$

for some $\lambda \in \mathscr{A}$.
This concept was first introduced in [1, 2] as self-adjoint differential equations. More recently, in [3, 4], Ibragimov himself changed the designation and he referred to this
concept as strictly self-adjoint differential equation. Then we use the last definition proposed by Ibragimov.

Some examples are now welcomed. We start with the following.

Example 2. Consider the Riemman or inviscid Burgers equation:

$$
\begin{equation*}
u_{t}+a(u) u_{x}=0 \tag{7}
\end{equation*}
$$

where we assume $a^{\prime}(u) \neq 0$. In this case, the adjoint equation to $(7)$ is

$$
\begin{equation*}
v_{t}+a(u) v_{x}=0 \tag{8}
\end{equation*}
$$

Clearly, setting $v=u$ into (8), (7) is obtained. Therefore, Riemman equation is strictly self-adjoint. For further details, see [19, 20].

Example 3. Consider now KdV equation:

$$
\begin{equation*}
u_{t}=u_{x x x}+u u_{x} . \tag{9}
\end{equation*}
$$

Its adjoint equation is

$$
\begin{equation*}
v_{t}=v_{x x x}+u v_{x} . \tag{10}
\end{equation*}
$$

Then, setting $v=u$ into (10), one obtains (9). Therefore, KdV equation is strictly self-adjoint. For further details, see [1, 16].

Example 4. Consider Harry-Dym equation:

$$
\begin{equation*}
u_{t}=u^{3} u_{x x x} \tag{11}
\end{equation*}
$$

Its adjoint equation is given by $[14,15,43]$

$$
\begin{align*}
v_{t}= & u^{3} v_{x x x}+9\left(u^{2} v_{x}+2 u v u_{x}\right) u_{x x} \\
& +9 u^{2} u_{x} v_{x x}+18 u v_{x} u_{x}^{2}+6 v u_{x}^{3} . \tag{12}
\end{align*}
$$

Equation (12) is not strictly self-adjoint, as it can easily be checked directly from (12) setting $v=u$ or consulting [43].

In [43] the concept of quasi self-adjoint differential equation was proposed, which is recalled at the following.

Definition 5. Equation (3) is said to be quasi self-adjoint if the equation obtained from adjoint equation (4) by the substitution $v=\phi(u)$, for a certain $\phi$ such that $\phi(u) \neq 0$, is identical with the original equation (3); that is,

$$
\begin{equation*}
\left.F^{*}\right|_{v=\phi(u)}=\lambda(x, u, \ldots) F, \tag{13}
\end{equation*}
$$

for some $\lambda \in \mathscr{A}$.
Originally, the notion of quasi self-adjointness was slightly different. In fact, in its first formulation [43], it was required that function $\phi$ satisfies condition $\phi^{\prime}(u) \neq 0$. However, such condition was relaxed in [4] and we adopt here the last Ibragimov's formulation.

We now analyse our previous examples taking Definition 5 into account.

Example 6. Since (7) is strictly self-adjoint, consequently, it is also quasi self-adjoint. However, let $\phi=\phi(u)$ be a smooth function such that $\phi^{\prime \prime}(u) \neq 0$. Substituting $v=\phi(u)$ into the left side of (8) the following is obtained:

$$
\begin{equation*}
v_{t}+\left.a(u) v_{x}\right|_{v=\phi(u)}=\phi^{\prime}(u)\left[u_{t}+a(u) u_{x}\right] . \tag{14}
\end{equation*}
$$

This shows that (7) is quasi self-adjoint admitting an arbitrary nonlinear substitution $\phi=\phi(u)$. For further details and discussion, see [19, 20].

Example 7. Consider KdV equation (9) again. Substituting $v=\phi(u)$ into (10), we obtain $v=c_{1} u+c_{2}$, where $c_{1}$ and $c_{2}$ are arbitrary constants. This lead us to two different substitutions: $v_{1}=u$ and $v_{2}=1$. Therefore, $K d V$ equation is quasi selfadjoint. For further details, see [4].

Example 8. Consider Harry-Dym equation (11). As it was already pointed out, it is not strictly self-adjoint. However, in [43] Ibragimov showed that the adjoint equation to (11) is equivalent to the original one if the substitution is taken as follows:

$$
\begin{equation*}
v_{1}=\frac{1}{u^{3}} . \tag{15}
\end{equation*}
$$

In [14], Torrisi and Tracinà discovered the new substitution:

$$
\begin{equation*}
v_{2}=\frac{1}{u^{2}} \tag{16}
\end{equation*}
$$

Therefore, (11) is quasi self-adjoint.
Our next definition was formulated by Gandarias in [6].
Definition 9. Equation (3) is said to be weak self-adjoint if the equation obtained from adjoint equation (4) by the substitution $v=\phi(x, u)$ for a certain function $\phi$ such that $\phi_{u} \neq 0$ and $\phi_{x^{i}} \neq 0$, for some $x^{i}$, is identical with the original equation (3); that is,

$$
\begin{equation*}
\left.F^{*}\right|_{v=\phi(x, u)}=\lambda(x, u, \ldots) F, \tag{17}
\end{equation*}
$$

for some $\lambda \in \mathscr{A}$.
While strictly self-adjointness implies quasi selfadjointness, weak self-adjointness does not imply neither strictly nor quasi self-adjointness. In fact, Definition 9 is stronger than both Definitions 1 and 5. We illustrate now this fact.

Example 10. Consider again (7). Although it is clear that such equation is strictly and quasi self-adjoint, neither $v=u$ nor $v=\phi(u)$ are substitutions satisfying Definition 9. However, let $\varphi=\varphi(z)$ be a smooth real valued function and define $v=$ $\varphi(x-\operatorname{ta}(u))$. Then, substituting this $v$ into (10), we arrive at

$$
\begin{array}{rl}
v_{t}+a & \left.a(u) v_{x}\right|_{v=\varphi(x-t a(u))}  \tag{18}\\
& =-a \varphi^{\prime}(x-t a(u))+a \varphi(x-t a(u)) \equiv 0 .
\end{array}
$$

Thus, if $\varphi^{\prime} \neq 0$, it means that $v=\varphi(x-t a(u))$ is a substitution satisfying Definition 9.

Example 11. KdV equation (9) is weak self-adjoint. In fact, one can take the substitution $v=x+t u$ and easily check that (10) is equivalent to (9) with this substitution. For further details, see $[4,16]$.

Example 12. Considering Harry-Dym equation, it is now clear that it is not strictly self-adjoint, but it is quasi selfadjoint. In [15], we proved that adjoint equation (12) to (11) is also equivalent to itself by considering the substitutions:

$$
\begin{equation*}
v_{3}=\frac{x}{u^{3}}, \quad v_{4}=\frac{x^{2}}{u^{3}} . \tag{19}
\end{equation*}
$$

While substitutions (15) and (16) show that (11) is quasi self-adjoint, substitutions (19) are enough to prove weak selfadjointness. On the other hand, neither (15) nor (16) are substitutions that satisfy what is required in Definition 9.

Finally, we arrived at the state of the art in this field: nonlinear self-adjointness.

Definition 13. Equation (3) is said to be nonlinearly selfadjoint if the equation obtained from the adjoint equation (4) by the substitution $v=\phi(x, u)$ with a certain function $\phi(x, u) \neq 0$ is identical with the original equation (3); that is,

$$
\begin{equation*}
\left.F^{*}\right|_{\nu=\phi(x, u)}=\lambda(x, u, \ldots) F . \tag{20}
\end{equation*}
$$

for some $\lambda \in \mathscr{A}$.
Definition 13 generalizes all of the previous ones. The substitution required on Definition 13 can be generalized, allowing dependence on the derivatives of function $u$, that is, a substitution of the type $v=\phi\left(x, u, u_{(1)}, \ldots\right)$. In the last case, condition (20) is replaced to

$$
\begin{align*}
& \left.F^{*}\right|_{v=\phi\left(x, u, u_{(1)}, \ldots\right)} \\
& \quad=\lambda(x, u, \ldots) F+\lambda^{i_{1} \ldots i_{j}}\left(x, u, u_{(1)}, \ldots\right) D_{i_{1}} \ldots D_{i_{j}} F, \tag{21}
\end{align*}
$$

where

$$
\begin{equation*}
D_{i}=\frac{\partial}{\partial x^{i}}+u_{i} \frac{\partial}{\partial u}+u_{i j} \frac{\partial}{\partial u_{j}}+\cdots \tag{22}
\end{equation*}
$$

are the total derivative operators.
Example 14. It is clear that all of the previous discussed equations provide examples of nonlinear self-adjointness. Let us now give a different example due to Ibragimov [4], with explicit dependence on the differential variables.

Consider the equation

$$
\begin{equation*}
u_{x y}-\sin u=0 \tag{23}
\end{equation*}
$$

Its adjoint equations are given by

$$
\begin{equation*}
v_{x y}-v \cos u=0 \tag{24}
\end{equation*}
$$

Consider the differential function $\phi=u_{y}$. Then, substituting it into the left side of (24), the following is obtained:

$$
\begin{equation*}
v_{x y}-\left.v \cos u\right|_{v=u_{y}}=D_{y}\left(u_{x y}-\sin u\right) . \tag{25}
\end{equation*}
$$

Finally, we would like to guide the interested reader to [4], which is the real and complete reference on this subject. Therefore, it is an obligatory reading for everyone interested in this field.

## 3. Nonlinear Self-Adjoint Classification of (1)

Here, we follow Steps 1 and 2 of the algorithm presented at the beginning of Section 2. We start obtaining the formal Lagrangian that in this case is

$$
\begin{gather*}
\mathscr{L}=v\left(u_{t}-r(x, t, u) u_{x x x}-s(x, t, u) u_{x x}\right.  \tag{26}\\
\left.-f(x, t, u) u_{x}-h(x, t, u)\right) .
\end{gather*}
$$

Since the easiest step was overcome, we move on to Step 2. Consider now Euler-Lagrange operators, given by the formal sums,

$$
\begin{gather*}
\frac{\delta}{\delta u}=\frac{\partial}{\partial u}+\sum_{j=1}^{\infty}(-1)^{j} D_{i_{1}} \cdots D_{i_{j}} \frac{\delta}{\delta u_{i_{1} \cdots i_{j}}}  \tag{27}\\
\frac{\delta}{\delta v}=\frac{\partial}{\partial v}+\sum_{j=1}^{\infty}(-1)^{j} D_{i_{1}} \cdots D_{i_{j}} \frac{\partial}{\partial v_{i_{1} \cdots i_{j}}}
\end{gather*}
$$

Considering the particular Lagrangian (26), our EulerLagrange operators can be simplified to

$$
\begin{gather*}
\frac{\delta}{\delta u}=\frac{\partial}{\partial u}-D_{t} \frac{\partial}{\partial u_{t}}-D_{x} \frac{\partial}{\partial u_{x}}+D_{x}^{2} \frac{\partial}{\partial u_{x x}}-D_{x}^{3} \frac{\partial}{\partial u_{x x x}}  \tag{28}\\
\frac{\delta}{\delta v}=\frac{\partial}{\partial v}
\end{gather*}
$$

Then, we have

$$
\begin{align*}
\frac{\delta \mathscr{L}}{\delta v}= & u_{t}-r(x, t, u) u_{x x x}-s(x, t, u) u_{x x} \\
& -f(x, t, u) u_{x}-h(x, t, u) \equiv F  \tag{29}\\
\frac{\delta \mathscr{L}}{\delta u}= & -v\left(f_{u} u_{x}+s_{u} u_{x x}+r_{u} u_{x x x}+h_{u}\right) \\
& -D_{t}(v)+D_{x}(v f)-D_{x}^{2}(v s)+D_{x}^{3}(v r) \equiv F^{*}
\end{align*}
$$

In order to avoid a tiring notation, we omit from now the dependence on $(x, t, u)$ of the functions involved in the calculations. Thus, the expression $F^{*}$ is given by

$$
\begin{align*}
F^{*}= & -v\left(f_{u} u_{x}+s_{u} u_{x x}+r_{u} u_{x x x}+h_{u}\right) \\
& -v_{t}+v_{x} f+v f_{x}+v f_{u} u_{x}-v_{x x} s-2 v_{x} s_{x} \\
& -2 v_{x} s_{u} u_{x}-v s_{x x}-2 v s_{x u} u_{x}-v s_{u u} u_{x}^{2}-v s_{u} u_{x x} \\
& +v_{x x x} r+3 v_{x x} r_{x}+3 v_{x x} r_{u} u_{x}+3 v_{x} r_{x x}  \tag{30}\\
& +6 v_{x} r_{x u} u_{x}+3 v_{x} r_{u u} u_{x}^{2}+3 v r_{x x u} u_{x}+3 v r_{x u u} u_{x}^{2} \\
& +3 v_{x} r_{u} u_{x x}+3 v r_{u u} u_{x} u_{x x}+3 v r_{x u} u_{x x} \\
& +v r_{x x x}+v r_{u u u} u_{x}^{3}+v r_{u} u_{x x x} .
\end{align*}
$$

Replacing $v=\phi(x, t, u)$ into (30), we obtain

$$
\begin{align*}
\left.F^{*}\right|_{v=\phi(x, t, u)}= & -\phi_{u} F \\
& +\left(-\phi_{t}-(\phi h)_{u}+(\phi f)_{x}-(\phi s)_{x x}+(\phi r)_{x x x}\right) \\
& +\left(3(\phi r)_{x x u}-2(\phi s)_{x u}\right) u_{x} \\
& +\left(3(\phi r)_{x u u}-(\phi s)_{u u}\right) u_{x}^{2}+(\phi r)_{u u u} u_{x}^{3} \\
& +\left(3(\phi r)_{x u}-2(\phi s)_{u}\right) u_{x x}+3(\phi r)_{u u} u_{x} u_{x x} . \tag{31}
\end{align*}
$$

From Definition 13, in order to (1) be nonlinearly selfadjoint, we must have $\lambda=-\phi_{u}$ and

$$
\begin{align*}
\left(-\phi_{t}\right. & \left.-(\phi h)_{u}+(\phi f)_{x}-(\phi s)_{x x}+(\phi r)_{x x x}\right) \\
& +\left(3(\phi r)_{x x u}-2(\phi s)_{x u}\right) u_{x} \\
& +\left(3(\phi r)_{x u u}-(\phi s)_{u u}\right) u_{x}^{2}+(\phi r)_{u u u} u_{x}^{3}  \tag{32}\\
& +\left(3(\phi r)_{x u}-2(\phi s)_{u}\right) u_{x x} \\
& +3(\phi r)_{u u} u_{x} u_{x x}=0 .
\end{align*}
$$

Since the set $\left\{1, u_{x}, u_{x}^{2}, u_{x}^{3}, u_{x x}, u_{x} u_{x x}\right\}$ is linearly independent, we obtain the following system of equations:

$$
\begin{gather*}
-\phi_{t}-(\phi h)_{u}+(\phi f)_{x}-(\phi s)_{x x}+(\phi r)_{x x x}=0,  \tag{33}\\
3(\phi r)_{x x u}-2(\phi s)_{x u}=0,  \tag{34}\\
3(\phi r)_{x u u}-(\phi s)_{u u}=0,  \tag{35}\\
(\phi r)_{u u u}=0,  \tag{36}\\
3(\phi r)_{x u}-2(\phi s)_{u}=0,  \tag{37}\\
3(\phi r)_{u u}=0 . \tag{38}
\end{gather*}
$$

From (38) and (35), we conclude that $(\phi s)_{u u}=0$.
Equations (38) and (37) imply, respectively, (36) and (34). Thus, we arrived at the following system:

$$
\begin{gather*}
(\phi r)_{u u}=0, \quad 3(\phi r)_{x u}-2(\phi s)_{u}=0, \quad(\phi s)_{u u}=0, \\
(\phi f)_{x}-\phi_{t}-(\phi h)_{u}-(\phi s)_{x x}+(\phi r)_{x x x}=0 \tag{39}
\end{gather*}
$$

## 4. Examples of Nonlinearly Self-Adjoint Equations of the Type (1)

In this section, we present two examples of nonlinearly selfadjoint equations of type (1). We consider some particular cases of the equations studied in [31]. First, let us consider the equation

$$
\begin{equation*}
u_{t}+p(t) e^{k x} u u_{x}+q(t) u_{x x x}=0 \tag{40}
\end{equation*}
$$

where $p(t)$ and $q(t)$ are nonzero functions and $k=$ constant. From (39), we obtain

$$
\begin{gather*}
q(t) \phi_{u u}=0, \quad q(t) \phi_{x u}=0,  \tag{41}\\
\left(p(t) e^{k x} u \phi\right)_{x}+\phi_{t}+q(t) \phi_{x x x}=0 . \tag{42}
\end{gather*}
$$

From (41), since $q(t)$ is nonzero, we conclude that the function $\phi=\phi(x, t, u)$ is $\phi=A(t) u+B(x, t)$, for certain functions $A=A(t)$ and $B=B(x, t)$. Substituting the expression of $\phi$ into (42), we have

$$
\begin{align*}
k p(t) & e^{k x} A(t) u^{2} \\
& +\left[A^{\prime}(t)+p(t) e^{k x} B_{x}(x, t)+k p(t) e^{k x} B(x, t)\right] u \\
& +B_{t}(x, t)+q(t) B_{x x x}(x, t)=0 . \tag{43}
\end{align*}
$$

Since the set $\left\{1, u, u^{2}\right\}$ is linearly independent, we have the following system of equations:

$$
\begin{gather*}
k p(t) e^{k x} A(t)=0  \tag{44}\\
A^{\prime}(t)+p(t) e^{k x} B_{x}(x, t)+k p(t) e^{k x} B(x, t)=0  \tag{45}\\
B_{t}(x, t)+q(t) B_{x x x}(x, t)=0 \tag{46}
\end{gather*}
$$

Considering the case $k=0$ and $p(t) A(t) \neq 0$ in (44). Thus, (45) is simplified to

$$
\begin{equation*}
A^{\prime}(t)+p(t) B_{x}(x, t)=0 \tag{47}
\end{equation*}
$$

and, therefore,

$$
\begin{equation*}
B(x, t)=-\frac{A^{\prime}(t)}{p(t)} x+C(t) \tag{48}
\end{equation*}
$$

Equation (48) implies that $B_{x x x}(x, t)=0$. Then, from (46), we conclude that

$$
\begin{equation*}
A(t)=c_{1} \int p(t) d t+c_{2}, \quad B(x, t)=-c_{1} x+c_{3}, \tag{49}
\end{equation*}
$$

where $c_{1}, c_{2}$ and $c_{3}$ are arbitrary constants. Therefore, we obtain

$$
\begin{equation*}
\phi(x, t, u)=c_{1}\left[\left(\int p(t) d t\right) u-x\right]+c_{2} u+c_{3} . \tag{50}
\end{equation*}
$$

Whenever $p=q=-1$, and under the change $c_{1} \mapsto-c_{1}$, it is concluded that $\phi=c_{1}(x+t u)+c_{2} u+c_{3}$, a well-known result on KdV equation; see $[4,16]$.

Now consider the case $k \neq 0$. Then we must consider three subcases: $A=0$ and $p \neq 0, A \neq 0$ and $p=0$, and $A=p=0$.

Case $A=0$ and $p \neq 0$. From (44)-(46), we conclude that $B(x, t)=b_{0} e^{-k x+k^{3} Q(t)}$, with $Q^{\prime}(t)=q(t)$, and then, choose $b_{0}=1$;

$$
\begin{equation*}
\phi(x, t, u)=e^{-k x+k^{3} Q(t)} \tag{51}
\end{equation*}
$$

Case $A \neq 0$ and $p=0$. From (45), we easily arrive at $A(t)=c_{1}$. Then, the substitution is

$$
\begin{equation*}
\phi(x, t, u)=c_{1} u+B(x, t), \tag{52}
\end{equation*}
$$

where $B$ is a solution of (46). The third subcase $A=p=0$ is a particular case of this last one taking $c_{1}=0$ into (52).

As a second example, consider the equation

$$
\begin{equation*}
u_{t}+\frac{1}{x} u u_{x}+g(t) u_{x x x}=0 \tag{53}
\end{equation*}
$$

with $g(t) \neq 0$. In this case, system (39) reads

$$
\begin{align*}
g(t) \phi_{u u} & =0, \quad g(t) \phi_{x u} \tag{54}
\end{align*}=0, ~=~\left(\frac{u}{x} \phi\right)_{x}+\phi_{t}+g(t) \phi_{x x x}=0 . ~ \$
$$

From (54), we again obtain

$$
\begin{equation*}
\phi=A(t) u+B(x, t) . \tag{56}
\end{equation*}
$$

Replacing the function expression in (55) and grouping in terms of $u$ and $u^{2}$ we have

$$
\begin{gather*}
-\frac{A(t)}{x^{2}} u^{2}+\left(A^{\prime}(t)+\frac{B_{x}(x, t)}{x}-\frac{B(x, t)}{x^{2}}\right) u  \tag{57}\\
+B_{t}(x, t)+g(t) B_{x x x}(x, t)=0 .
\end{gather*}
$$

The system

$$
\begin{gather*}
-\frac{A(t)}{x^{2}}=0  \tag{58}\\
A^{\prime}(t)+\frac{B_{x}(x, t)}{x}-\frac{B(x, t)}{x^{2}}=0  \tag{59}\\
B_{t}(x, t)+g(t) B_{x x x}(x, t)=0 \tag{60}
\end{gather*}
$$

follows from (57). Equation (58) implies $A(t)=0$. From (59) and (60), we conclude that

$$
\begin{equation*}
\phi(x, t, u)=c x \tag{61}
\end{equation*}
$$

with $c=$ constant .

## 5. Conclusion

In this paper we discussed about some ideas introduced in $[1,3,4,6,43]$. These ideas are very recent and until now the applications are mainly restricted to the obtainment of local conservation laws using the approach suggested in [1]. However, some recent facts show that there is more to selfadjointness than meets the eye.

In fact, in [4], Ibragimov began to consider the concept of approximated nonlinear self-adjointness. Recently [44] explored deeper these concepts. In [45] approximate conservation laws for a nonlinear filtration equation were established.

Nowadays, the fractional calculus seems to be a new branch in Mathematics. In [46] fractional conservation laws
using the approach proposed in [1] were presented, which means that the concepts of self-adjoint differential equations must be considered in the sense of fractional differential equations.

Finally, we would like to mention that recently some possible connections between integrable equations, strictly self-adjointness and scale invariance have been reported in [47], although this relation is not clear yet.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Combinatorial Properties and Characterization of Glued Semigroups 

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This work focuses on the combinatorial properties of glued semigroups and provides its combinatorial characterization. Some classical results for affine glued semigroups are generalized and some methods to obtain glued semigroups are developed.

## 1. Introduction

Let $S=\left\langle n_{1}, \ldots, n_{l}\right\rangle$ be a finitely generated commutative semigroup with zero element which is reduced (i.e., $S \cap(-S)=\{0\}$ ) and cancellative (if $m, n, n^{\prime} \in S$ and $m+n=m+n^{\prime}$ then $n=n^{\prime}$ ). Under these settings if $S$ is torsion-free, then it is isomorphic to a subsemigroup of $\mathbb{N}^{p}$ which means it is an affine semigroup (see [1]). From now on assume that all the semigroups appearing in this work are finitely generated, commutative, reduced, and cancellative, but not necessarily torsionfree.

Let $\mathbb{K}$ be a field and $\mathbb{K}\left[X_{1}, \ldots, X_{l}\right]$ the polynomial ring in $l$ indeterminates. This polynomial ring is obviously an $S$ graded ring (by assigning the $S$-degree $n_{i}$ to the indeterminate $X_{i}$, the $S$-degree of $X^{\alpha}=X_{1}^{\alpha_{1}} \cdots X_{l}^{\alpha_{l}}$ is $\sum_{i=1}^{l} \alpha_{i} n_{i} \in S$ ). It is well known that the ideal $I_{S}$ generated by

$$
\begin{equation*}
\left\{X^{\alpha}-X^{\beta} \mid \sum_{i=1}^{l} \alpha_{i} n_{i}=\sum_{i=1}^{l} \beta_{i} n_{i}\right\} \subset \mathbb{K}\left[X_{1}, \ldots, X_{l}\right] \tag{1}
\end{equation*}
$$

is an $S$-homogeneous binomial ideal called semigroup ideal (see [2] for details). If $S$ is torsion-free, the ideal obtained defines a toric variety (see [3] and the references therein). By Nakayama's lemma, all the minimal generating sets of $I_{S}$ have the same cardinality and the $S$-degrees of its elements can be determinated.

The main goal of this work is to study the semigroups which result from the gluing of other two. This concept was
introduced by Rosales in [4] and it is closely related to complete intersection ideals (see [5] and the references therein). A semigroup $S$ minimally generated by $A_{1} \sqcup A_{2}$ (with $A_{1}=$ $\left\{n_{1}, \ldots, n_{r}\right\}$ and $\left.A_{2}=\left\{n_{r+1}, \ldots, n_{l}\right\}\right)$ is the gluing of $S_{1}=\left\langle A_{1}\right\rangle$ and $S_{2}=\left\langle A_{2}\right\rangle$ if there exists a set of generators $\rho$ of $I_{S}$ of the form $\rho=\rho_{1} \cup \rho_{2} \cup\left\{X^{\gamma}-X^{\gamma^{\prime}}\right\}$, where $\rho_{1}, \rho_{2}$ are generating sets of $I_{S_{1}}$ and $I_{S_{2}}$, respectively, $X^{\gamma}-X^{\gamma^{\prime}} \in I_{S}$, and the supports of $\gamma$ and $\gamma^{\prime}$ verify $\operatorname{supp}(\gamma) \subset\{1, \ldots, r\}$ and $\operatorname{supp}\left(\gamma^{\prime}\right) \subset\{r+1, \ldots, l\}$. Equivalently, $S$ is the gluing of $S_{1}$ and $S_{2}$ if $I_{S}=I_{S_{1}}+I_{S_{2}}+\left\langle X^{\gamma}-\right.$ $\left.X^{\gamma^{\prime}}\right\rangle$. A semigroup is a glued semigroup when it is the gluing of other two.

As seen, glued semigroups can be determinated by the minimal generating sets of $I_{S}$ which can be studied by using combinatorial methods from certain simplicial complexes (see [6-8]). In this work the simplicial complexes used are defined as follows: for any $m \in S$, set

$$
\begin{equation*}
C_{m}=\left\{X^{\alpha}=X_{1}^{\alpha_{1}} \cdots X_{l}^{\alpha_{l}} \mid \sum_{i=1}^{l} \alpha_{i} n_{i}=m\right\} \tag{2}
\end{equation*}
$$

and the simplicial complex

$$
\begin{equation*}
\nabla_{m}=\left\{F \subseteq C_{m} \mid \operatorname{gcd}(F) \neq 1\right\} \tag{3}
\end{equation*}
$$

with $\operatorname{gcd}(F)$ as the greatest common divisor of the monomials in $F$.

Furthermore, some methods which require linear algebra and integer programming are given to obtain examples of glued semigroups.

The content of this work is organized as follows. Section 2 presents the tools to generalize to nontorsion-free semigroups a classical characterization of affine gluing semigroups (Proposition 2). In Section 3, the nonconnected simplicial complexes $\nabla_{m}$ associated with glued semigroups are studied. By using the vertices of the connected components of these complexes we give a combinatorial characterization of glued semigroups as well as their glued degrees (Theorem 6). Besides, in Corollary 7 we deduce the conditions for the ideal of a glued semigroup to have a unique minimal system of generators. Finally, Section 4 is devoted to the construction of glued semigroups (Corollary 10) and affine glued semigroups (Section 4.1).

## 2. Preliminaries and Generalizations about Glued Semigroups

A binomial of $I_{S}$ is called indispensable if it is an element of all systems of generators of $I_{S}$ (up to a scalar multiple). This kind of binomials was introduced in [9] and they have an important role in Algebraic Statistics. In [10] the authors characterize indispensable binomials by using simplicial complexes $\nabla_{m}$. Note that if $I_{S}$ is generated by its indispensable binomials then it is minimally generated, up to scalar multiples, in an unique way.

With the above notation, the semigroup $S$ is associated with the lattice $\operatorname{ker} S$ formed by the elements $\alpha=$ $\left(\alpha_{1}, \ldots, \alpha_{l}\right) \in \mathbb{Z}^{l}$ such that $\sum_{i=1}^{l} \alpha_{i} n_{i}=0$. Given $G$ a system of generators of $I_{S}$, the lattice ker $S$ is generated by the elements $\alpha-\beta$ with $X^{\alpha}-X^{\beta} \in G$ and ker $S$ also verifies that $\operatorname{ker} S \cap \mathbb{N}^{l}=$ $\{0\}$ if and only if $S$ is reduced. If $\mathscr{M}\left(I_{S}\right)$ is a minimal generating set of $I_{S}$, denote by $\mathscr{M}\left(I_{S}\right)_{m} \subset \mathscr{M}\left(I_{S}\right)$ the set of elements whose $S$-degree is equal to $m \in S$ and by $\operatorname{Betti}(S)$ the set of the $S$-degrees of the elements of $\mathscr{M}\left(I_{S}\right)$. When $I_{S}$ is minimally generated by $\operatorname{rank}(\operatorname{ker} S)$ elements, the semigroup $S$ is called a complete intersection semigroup.

Let $\mathscr{C}\left(\nabla_{m}\right)$ be the number of connected components of $\nabla_{m}$. The cardinality of $\mathscr{M}\left(I_{S}\right)_{m}$ is equal to $\mathscr{C}\left(\nabla_{m}\right)-1$ (see Remark 2.6 in [6] and Theorem 3 and Corollary 4 in [8]) and the complexes associated with the elements in $\operatorname{Betti}(S)$ are nonconnected.

Construction 1 (see [7, Proposition 1]). For each $m \in \operatorname{Betti}(S)$ the set $\mathscr{M}\left(I_{S}\right)_{m}$ is obtained by taking $\mathscr{C}\left(\nabla_{m}\right)-1$ binomials with monomials in different connected components of $\nabla_{m}$ satisfying that two different binomials do not have their corresponding monomials in the same components and fulfilling that there is at least a monomial of every connected component of $\nabla_{m}$. This let us construct a minimal generating set of $I_{S}$ in a combinatorial way.

Let $S$ be minimally (we consider a minimal generator set of $S$ and in the other case $S$ is trivially the gluing of the semigroup generated by one of its nonminimal generators and the semigroup generated by the others) generated by
$A_{1} \sqcup A_{2}$ with $A_{1}=\left\{a_{1}, \ldots, a_{r}\right\}$ and $A_{2}=\left\{b_{1}, \ldots, b_{t}\right\}$. From now on, identify the sets $A_{1}$ and $A_{2}$ with the matrices $\left(a_{1}|\cdots| a_{r}\right)$ and $\left(b_{1}|\cdots| b_{t}\right)$. Denote by $\mathbb{K}\left[A_{1}\right]$ and $\mathbb{K}\left[A_{2}\right]$ the polynomial rings $\mathbb{K}\left[X_{1}, \ldots, X_{r}\right]$ and $\mathbb{K}\left[Y_{1}, \ldots, Y_{t}\right]$, respectively. A monomial is a pure monomial if it has indeterminates only in $X_{1}, \ldots, X_{r}$ or only in $Y_{1}, \ldots, Y_{t}$; otherwise it is a mixed monomial. If $S$ is the gluing of $S_{1}=\left\langle A_{1}\right\rangle$ and $S_{2}=\left\langle A_{2}\right\rangle$, then the binomial $X^{\gamma_{X}}-Y^{\gamma_{Y}} \in I_{S}$ is a glued binomial if $\mathscr{M}\left(I_{S_{1}}\right) \cup \mathscr{M}\left(I_{S_{2}}\right) \cup\left\{X^{\gamma_{X}}-Y^{\gamma_{Y}}\right\}$ is a generating set of $I_{S}$ and in this case the element $d=S$-degree $\left(X^{\gamma_{X}}\right) \in S$ is called a glued degree.

It is clear that if $S$ is a glued semigroup, the lattice $\operatorname{ker} S$ has a basis of the form

$$
\begin{equation*}
\left\{L_{1}, L_{2},\left(\gamma_{X},-\gamma_{Y}\right)\right\} \subset \mathbb{Z}^{r+t} \tag{4}
\end{equation*}
$$

where the supports of the elements in $L_{1}$ are in $\{1, \ldots, r\}$, the supports of the elements in $L_{2}$ are in $\{r+1, \ldots, r+t\}, \operatorname{ker} S_{i}=$ $\left\langle L_{i}\right\rangle(i=1,2)$ by considering only the coordinates in $\{1, \ldots$, $r\}$ or $\{r+1, \ldots, r+t\}$ of $L_{i}$, and $\left(\gamma_{X}, \gamma_{Y}\right) \in \mathbb{N}^{r+t}$. Moreover, since $S$ is reduced, one has that $\left\langle L_{1}\right\rangle \cap \mathbb{N}^{r+t}=\left\langle L_{2}\right\rangle \cap \mathbb{N}^{r+t}=\{0\}$. Denote by $\left\{\rho_{1 i}\right\}_{i}$ the elements in $L_{1}$ and by $\left\{\rho_{2 i}\right\}_{i}$ the elements in $L_{2}$.

The following proposition generalizes [4, Theorem 1.4] to nontorsion-free semigroups.

Proposition 2. The semigroup $S$ is the gluing of $S_{1}$ and $S_{2}$ if and only if there exists $d \in\left(S_{1} \cap S_{2}\right) \backslash\{0\}$ such that $G\left(S_{1}\right) \cap$ $G\left(S_{2}\right)=d \mathbb{Z}$, where $G\left(S_{1}\right), G\left(S_{2}\right)$, and $d \mathbb{Z}$ are the associated commutative groups of $S_{1}, S_{2}$, and $\{d\}$, respectively.

Proof. Assume that $S$ is the gluing of $S_{1}$ and $S_{2}$. In this case, $\operatorname{ker} S$ is generated by the set (4). Since $\left(\gamma_{X},-\gamma_{Y}\right) \in \operatorname{ker} S$, the element $d$ is equal to $A_{1} \gamma_{X}=A_{2} \gamma_{Y} \in S$ and $d \in S_{1} \cap S_{2} \subset$ $G\left(S_{1}\right) \cap G\left(S_{2}\right)$. Let $d^{\prime}$ be in $G\left(S_{1}\right) \cap G\left(S_{2}\right)$; then there exists $\left(\delta_{1}, \delta_{2}\right) \in \mathbb{Z}^{r} \times \mathbb{Z}^{t}$ such that $d^{\prime}=A_{1} \delta_{1}=A_{2} \delta_{2}$. Therefore $\left(\delta_{1},-\delta_{2}\right) \in \operatorname{ker} S$ because $\left(A_{1} \mid A_{2}\right)\left(\delta_{1},-\delta_{2}\right)=0$ and so there exist $\lambda, \lambda_{i}^{\rho_{1}}, \lambda_{i}^{\rho_{2}} \in \mathbb{Z}$ satisfying

$$
\begin{align*}
& \left(\delta_{1}, 0\right)=\sum_{i} \lambda_{i}^{\rho_{1}} \rho_{1 i}+\lambda\left(\gamma_{X}, 0\right) \\
& \left(0, \delta_{2}\right)=-\sum_{i} \lambda_{i}^{\rho_{2}} \rho_{2 i}+\lambda\left(0, \gamma_{Y}\right) \tag{5}
\end{align*}
$$

and $d^{\prime}=A_{1} \delta_{1}=\sum_{i} \lambda_{i}^{\rho_{1}}\left(A_{1} \mid 0\right) \rho_{1 i}+\lambda A_{1} \gamma_{X}=\lambda d$. We conclude that $G\left(S_{1}\right) \cap G\left(S_{2}\right)=d \mathbb{Z}$ with $d \in S_{1} \cap S_{2}$.

Conversely, suppose that there exists $d \in\left(S_{1} \cap S_{2}\right) \backslash\{0\}$ such that $G\left(S_{1}\right) \cap G\left(S_{2}\right)=d \mathbb{Z}$. We see that $I_{S}=I_{S_{1}}+I_{S_{2}}+\left\langle X^{\gamma_{X}}-\right.$ $\left.Y^{\gamma_{Y}}\right\rangle$. Trivially, $I_{S_{1}}+I_{S_{2}}+\left\langle X^{\gamma_{X}}-Y^{\gamma_{Y}}\right\rangle \subset I_{S}$. Let $X^{\alpha} Y^{\beta}-X^{\gamma} Y^{\delta}$ be a binomial in $I_{S}$. Its $S$-degree is $A_{1} \alpha+A_{2} \beta=A_{1} \gamma+A_{2} \delta$. Using $A_{1}(\alpha-\gamma)=A_{2}(\beta-\delta) \in G\left(S_{1}\right) \cap G\left(S_{2}\right)=d \mathbb{Z}$, there exists $\lambda \in \mathbb{Z}$ such that $A_{1} \alpha=A_{1} \gamma+\lambda d$ and $A_{2} \delta=A_{2} \beta+\lambda d$. We have the following cases.
(i) If $\lambda=0$,

$$
\begin{align*}
X^{\alpha} Y^{\beta}-X^{\gamma} Y^{\delta} & =X^{\alpha} Y^{\beta}-X^{\gamma} Y^{\beta}+X^{\gamma} Y^{\beta}-X^{\gamma} Y^{\delta} \\
& =Y^{\beta}\left(X^{\alpha}-X^{\gamma}\right)+X^{\gamma}\left(Y^{\beta}-Y^{\delta}\right) \in I_{S_{1}}+I_{S_{2}} \tag{6}
\end{align*}
$$

(ii) If $\lambda>0$,

$$
\begin{align*}
X^{\alpha} Y^{\beta}-X^{\gamma} Y^{\delta}= & X^{\alpha} Y^{\beta}-X^{\gamma} X^{\lambda \gamma_{X}} Y^{\beta} \\
& +X^{\gamma} X^{\lambda \gamma_{X}} Y^{\beta}-X^{\gamma} X^{\lambda \gamma_{Y}} Y^{\beta}+X^{\gamma} X^{\lambda \gamma_{Y}} Y^{\beta} \\
& -X^{\gamma} Y^{\delta}=Y^{\beta}\left(X^{\alpha}-X^{\gamma} X^{\lambda \gamma_{X}}\right) \\
& +X^{\gamma} Y^{\beta}\left(X^{\lambda \gamma_{X}}-Y^{\lambda \gamma_{Y}}\right) \\
& +X^{\gamma}\left(Y^{\lambda \gamma_{Y}} Y^{\beta}-Y^{\delta}\right) . \tag{7}
\end{align*}
$$

Using that

$$
\begin{equation*}
X^{\lambda \gamma_{X}}-Y^{\lambda \gamma_{Y}}=\left(X^{\gamma_{X}}-Y^{\gamma_{Y}}\right)\left(\sum_{i=0}^{\lambda-1} X^{(\lambda-1-i) \gamma_{X}} Y^{i \gamma_{Y}}\right) \tag{8}
\end{equation*}
$$

the binomial $X^{\alpha} Y^{\beta}-X^{\gamma} Y^{\delta}$ belongs to $I_{S_{1}}+I_{S_{2}}+\left\langle X^{\gamma_{X}}-\right.$ $\left.Y^{\gamma_{Y}}\right\rangle$.
(iii) The case $\lambda<0$ is solved similarly.

We conclude that $I_{S}=I_{S_{1}}+I_{S_{2}}+\left\langle X^{\gamma_{X}}-Y^{\gamma_{Y}}\right\rangle$.
From the above proof it is deduced that given the partition of the system of generators of $S$ the glued degree is unique.

## 3. Glued Semigroups and Combinatorics

Glued semigroups by means of nonconnected simplicial complexes are characterized. For any $m \in S$, redefine $C_{m}$ from (2) as
$C_{m}=\left\{X^{\alpha} Y^{\beta}=X_{1}^{\alpha_{1}} \cdots X_{r}^{\alpha_{r}} Y_{1}^{\beta_{1}} \cdots Y_{t}^{\beta_{t}} \mid \sum_{i=1}^{r} \alpha_{i} a_{i}+\sum_{i=1}^{t} \beta_{i} b_{i}=m\right\}$,
and consider the sets of vertices and the simplicial complexes

$$
\begin{gather*}
C_{m}^{A_{1}}=\left\{X_{1}^{\alpha_{1}} \cdots X_{r}^{\alpha_{r}} \mid \sum_{i=1}^{r} \alpha_{i} a_{i}=m\right\}, \\
\nabla_{m}^{A_{1}}=\left\{F \subseteq C_{m}^{A_{1}} \mid \operatorname{gcd}(F) \neq 1\right\} \\
C_{m}^{A_{2}}=\left\{Y_{1}^{\beta_{1}} \cdots Y_{t}^{\beta_{t}} \mid \sum_{i=1}^{t} \beta_{i} b_{i}=m\right\}  \tag{10}\\
\nabla_{m}^{A_{2}}=\left\{F \subseteq C_{m}^{A_{2}} \mid \operatorname{gcd}(F) \neq 1\right\}
\end{gather*}
$$

where $A_{1}=\left\{a_{1}, \ldots, a_{r}\right\}$ and $A_{2}=\left\{b_{1}, \ldots, b_{t}\right\}$ as in Section 2. Trivially, the relations between $\nabla_{m}^{A_{1}}, \nabla_{m}^{A_{2}}$, and $\nabla_{m}$ are

$$
\begin{equation*}
\nabla_{m}^{A_{1}}=\left\{F \in \nabla_{m} \mid F \subset C_{m}^{A_{1}}\right\}, \quad \nabla_{m}^{A_{2}}=\left\{F \in \nabla_{m} \mid F \subset C_{m}^{A_{2}}\right\} . \tag{11}
\end{equation*}
$$

The following result shows an important property of the simplicial complexes associated with glued semigroups.

Lemma 3. Let $S$ be the gluing of $S_{1}$ and $S_{2}$ and $m \in \operatorname{Betti}(S)$. Then all the connected components of $\nabla_{m}$ have at least a pure monomial. In addition, all mixed monomials of $\nabla_{m}$ are in the same connected component.

Proof. Suppose that there exists $C$, a connected component of $\nabla_{m}$ only with mixed monomials. By Construction 1 in all generating sets of $I_{S}$ there is at least a binomial with a mixed monomial, but this does not occur in $\mathscr{M}\left(I_{S_{1}}\right) \cup \mathscr{M}\left(I_{S_{2}}\right) \cup\left\{X^{\gamma_{X}}-\right.$ $\left.Y^{\gamma_{Y}}\right\}$ with $X^{\gamma_{X}}-Y^{\gamma_{Y}}$ as a glued binomial.

Since $S$ is a glued semigroup, $\operatorname{ker} S$ has a system of generators as (4). Let $X^{\alpha} Y^{\beta}, X^{\gamma} Y^{\delta} \in C_{m}$ be two monomials such that $\operatorname{gcd}\left(X^{\alpha} Y^{\beta}, X^{\gamma} Y^{\delta}\right)=1$. In this case, $(\alpha, \beta)-(\gamma, \delta) \in$ $\operatorname{ker} S$ and there exist $\lambda, \lambda_{i}^{\rho_{1}}, \lambda_{i}^{\rho_{2}} \in \mathbb{Z}$ satisfying

$$
\begin{align*}
& (\alpha-\gamma, 0)=\sum_{i} \lambda_{i}^{\rho_{1}} \rho_{1 i}+\lambda\left(\gamma_{X}, 0\right) \\
& (0, \beta-\delta)=\sum_{i} \lambda_{i}^{\rho_{2}} \rho_{2 i}-\lambda\left(0, \gamma_{Y}\right) \tag{12}
\end{align*}
$$

(i) If $\lambda=0, \alpha-\gamma \in \operatorname{ker} S_{1}$, and $\beta-\delta \in \operatorname{ker} S_{2}$, then $A_{1} \alpha=A_{1} \gamma, A_{2} \beta=A_{2} \delta$, and $X^{\alpha} Y^{\delta} \in C_{m}$.
(ii) If $\lambda>0,(\alpha, 0)=\sum_{i} \lambda_{i}^{\rho_{1}} \rho_{1 i}+\lambda\left(\gamma_{X}, 0\right)+(\gamma, 0)$, and

$$
\begin{equation*}
A_{1} \alpha=\sum_{i} \lambda_{i}^{\rho_{1}}\left(A_{1} \mid 0\right) \rho_{1 i}+\lambda A_{1} \gamma_{X}+A_{1} \gamma=\lambda d+A_{1} \gamma \tag{13}
\end{equation*}
$$

then $X^{\lambda \gamma_{X}} X^{\gamma} Y^{\beta} \in C_{m}$.
(iii) The case $\lambda<0$ is solved likewise.

In any case, $X^{\alpha} Y^{\beta}$ and $X^{\gamma} Y^{\delta}$ are in the same connected component of $\nabla_{m}$.

We now describe the simplicial complexes that correspond to the $S$-degrees which are multiples of the glued degree.

Lemma 4. Let $S$ be the gluing of $S_{1}$ and $S_{2}, d \in S$ the glued degree, and $d^{\prime} \in S \backslash\{d\}$. Then $C_{d^{\prime}}^{A_{1}} \neq \emptyset \neq C_{d^{\prime}}^{A_{2}}$ if and only if $d^{\prime} \in(d \mathbb{N}) \backslash\{0\}$. Furthermore, the simplicial complex $\nabla_{d^{\prime}}$ has at least one connected component with elements in $C_{d^{\prime}}^{A_{1}}$ and $C_{d^{\prime}}^{A_{2}}$.

Proof. If there exist $X^{\alpha}, Y^{\beta} \in C_{d^{\prime}}$, then $d^{\prime}=\sum_{i=1}^{r} \alpha_{i} a_{i}=$ $\sum_{i=1}^{t} \beta_{i} b_{i} \in S_{1} \cap S_{2} \subset G\left(S_{1}\right) \cap G\left(S_{2}\right)=d \mathbb{Z}$. Hence, $d^{\prime} \in d \mathbb{N}$.

Conversely, let $d^{\prime}=j d$ with $j \in \mathbb{N}$ and $j>1$ and let $X^{\gamma_{X}}-Y^{\gamma_{Y}} \in I_{S}$ be a glued binomial. It is easy to see that $X^{j \gamma_{X}}, Y^{j \gamma_{Y}} \in C_{d^{\prime}}$ and thus $\left\{X^{j \gamma_{X}}, X^{(j-1) \gamma_{X}} Y^{\gamma_{X}}\right\}$ and $\left\{X^{(j-1) \gamma_{X}} Y^{\gamma_{Y}}, Y^{j \gamma_{Y}}\right\}$ belong to $\nabla_{d^{\prime}}$.

The following lemma is a combinatorial version of [11, Lemma 9] and it is a necessary condition of Theorem 6.

Lemma 5. Let $S$ be the gluing of $S_{1}$ and $S_{2}$ and $d \in S$ the glued degree. Then the elements of $C_{d}$ are pure monomials and $d \in$ Betti(S).

Proof. The order $\preceq_{S}$ defined by $m^{\prime} \leq_{S} m$ if $m-m^{\prime} \in S$ is a partial order on $S$.

Assume that there exists a mixed monomial $T \in C_{d}$. By Lemma 3, there exists a pure monomial $Y^{b}$ in $C_{d}$ such that $\left\{T, Y^{b}\right\} \in \nabla_{d}$ (the proof is analogous if we consider $X^{a}$ with $\left\{T, X^{a}\right\} \in \nabla_{d}$. Now take $T_{1}=\operatorname{gcd}\left(T, Y^{b}\right)^{-1} T$ and $Y^{b_{1}}=$ $\operatorname{gcd}\left(T, Y^{b}\right)^{-1} Y^{b}$. Both monomials are in $C_{d^{\prime}}$, where $d^{\prime}$ is equal to $d$ minus the $S$-degree of $\operatorname{gcd}\left(T, Y^{b}\right)$. By Lemma 4, if $C_{d^{\prime}}^{A_{1}} \neq \emptyset$, then $d^{\prime} \in d \mathbb{N}$, but since $d^{\prime}<_{S} d$ this is not possible. So, if $T_{1}$ is a mixed monomial and $C_{d^{\prime}}^{A_{1}}=\emptyset$, then $C_{d^{\prime}}^{A_{2}} \neq \emptyset$. If there exists a pure monomial in $C_{d^{\prime}}^{A_{2}}$ connected to a mixed monomial in $C_{d^{\prime}}$, we perform the same process obtaining $T_{2}, Y^{b_{2}} \in$ $C_{d^{\prime \prime}}$ with $T_{2}$ as a mixed monomial and $d^{\prime \prime} \prec_{s} d^{\prime}$. This process can be repeated if there existed a pure monomial and a mixed monomial in the same connected component. By degree reasons this cannot be performed indefinitely and an element $d^{(i)} \in \operatorname{Betti}(S)$ verifying that $\nabla_{d^{(i)}}$ is not connected having a connected component with only mixed monomials is found. This contradicts Lemma 3.

After examining the structure of the simplicial complexes associated with glued semigroups, we enunciate a combinatorial characterization by means of the nonconnected simplicial complexes $\nabla_{m}$.

Theorem 6. The semigroup $S$ is the gluing of $S_{1}$ and $S_{2}$ if and only if the following conditions are fulfilled.
(1) For all $d^{\prime} \in \operatorname{Betti}(S)$, any connected component of $\nabla_{d^{\prime}}$ has at least a pure monomial.
(2) There exists a unique $d \in \operatorname{Betti}(S)$ such that $C_{d}^{A_{1}} \neq$ $\emptyset \neq C_{d}^{A_{2}}$ and the elements in $C_{d}$ are pure monomials.
(3) For all $d^{\prime} \in \operatorname{Betti}(S) \backslash\{d\}$ with $C_{d^{\prime}}^{A_{1}} \neq \emptyset \neq C_{d^{\prime}}^{A_{2}}, d^{\prime} \in d \mathbb{N}$.

Besides, the above $d \in \operatorname{Betti}(S)$ is the glued degree.
Proof. If $S$ is the gluing of $S_{1}$ and $S_{2}$, the result is obtained from Lemmas 3, 4, and 5.

Conversely, by hypotheses 1 and 3, given that $d^{\prime} \in$ $\operatorname{Betti}(S) \backslash\{d\}$, the set $\mathscr{M}\left(I_{S_{1}}\right)_{d^{\prime}}$ is constructed from $C_{d^{\prime}}^{A_{1}}$ and $\mathscr{M}\left(I_{S_{2}}\right)_{d^{\prime}}$ from $C_{d^{\prime}}^{A_{2}}$ as in Construction 1. Analogously, if $d \in \operatorname{Betti}(S)$, the set $\mathscr{M}\left(I_{S}\right)_{d}$ is obtained from the union of $\mathscr{M}\left(I_{S_{1}}\right)_{d}, \mathscr{M}\left(I_{S_{2}}\right)_{d}$ and the binomial $X^{\gamma_{X}}-Y^{\gamma_{Y}}$ with $X^{\gamma_{X}} \in C_{d}^{A_{1}}$ and $Y^{\gamma_{Y}} \in C_{d}^{A_{2}}$. Finally

$$
\begin{equation*}
\coprod_{m \in \operatorname{Betti}(S)}\left(\mathscr{M}\left(I_{S_{1}}\right)_{m} \sqcup \mathscr{M}\left(I_{S_{2}}\right)_{m}\right) \sqcup\left\{X^{\gamma_{X}}-Y^{\gamma_{Y}}\right\} \tag{14}
\end{equation*}
$$

is a generating set of $I_{S}$ and $S$ is the gluing of $S_{1}$ and $S_{2}$.
From Theorem 6 we obtain an equivalent property to Theorem 12 in [11] by using the language of monomials and binomials.

Corollary 7. Let $S$ be the gluing of $S_{1}$ and $S_{2}$ and $X^{\gamma_{X}}-Y^{\gamma_{Y}} \in I_{S}$ a glued binomial with $S$-degree $d$. The ideal $I_{S}$ is minimally
generated by its indispensable binomials if and only if the following conditions are fulfilled.
(i) The ideals $I_{S_{1}}$ and $I_{S_{2}}$ are minimally generated by their indispensable binomials.
(ii) The element $X^{\gamma_{X}}-Y^{\gamma_{Y}}$ is an indispensable binomial of $I_{S}$.
(iii) For all $d^{\prime} \in \operatorname{Betti}(S)$, the elements of $C_{d^{\prime}}$ are pure monomials.

Proof. Suppose that $I_{S}$ is generated by its indispensable binomials. By [10, Corollary 6], for all $m \in \operatorname{Betti}(S)$ the simplicial complex $\nabla_{m}$ has only two vertices. By Construction $1 \nabla_{d}=\left\{\left\{X^{\gamma_{x}}\right\},\left\{Y^{\gamma_{Y}}\right\}\right\}$ and by Theorem 6 for all $d^{\prime} \in \operatorname{Betti}(S) \backslash$ $\{d\}$ the simplicial $\nabla_{d^{\prime}}$ is equal to $\nabla_{d^{\prime}}^{A_{1}}$ or $\nabla_{d^{\prime}}^{A_{2}}$. In any case, $X^{\gamma_{X}}-Y^{\gamma_{Y}} \in I_{S}$ is an indispensable binomial, and $I_{S_{1}}, I_{S_{2}}$ are generated by their indispensable binomials.

Conversely, suppose that $I_{S}$ is not generated by its indispensable binomials. Then, there exists $d^{\prime} \in \operatorname{Betti}(S) \backslash\{d\}$ such that $\nabla_{d^{\prime}}$ has more than two vertices in at least two different connected components. By hypothesis, there are not mixed monomials in $\nabla_{d^{\prime}}$ and thus
(i) if $\nabla_{d^{\prime}}$ is equal to $\nabla_{d^{\prime}}^{A_{1}}$ (or $\nabla_{d^{\prime}}^{A_{2}}$ ), then $I_{S_{1}}$ (or $I_{S_{2}}$ ) is not generated by its indispensable binomials;
(ii) otherwise, $C_{d^{\prime}}^{A_{1}} \neq \emptyset \neq C_{d^{\prime}}^{A_{2}}$ and by Lemma 4, $d^{\prime}=j d$ with $j \in \mathbb{N}$, therefore $X^{(j-1) \gamma_{X}} Y^{\gamma_{Y}} \in C_{d^{\prime}}$ which contradicts the hypothesis.

We conclude that $I_{S}$ is generated by its indispensable binomials.

The following example taken from [5] illustrates the above results.

Example 8. Let $S \subset \mathbb{N}^{2}$ be the semigroup generated by the set

$$
\begin{equation*}
\{(13,0),(5,8),(2,11),(0,13),(4,4),(6,6),(7,7),(9,9)\} . \tag{15}
\end{equation*}
$$

In this case, $\operatorname{Betti}(S)$ is

$$
\begin{align*}
& \{(15,15),(14,14),(12,12),(18,18)  \tag{16}\\
& \quad(10,55),(15,24),(13,52),(13,13)\} .
\end{align*}
$$

Using the appropriated notation for the indeterminates in the polynomial ring $\mathbb{K}\left[x_{1}, \ldots, x_{4}, y_{1}, \ldots, y_{4}\right]\left(x_{1}, x_{2}, x_{3}\right.$, and $x_{4}$ for the first four generators of $S$ and $y_{1}, y_{2}, y_{3}, y_{4}$ for the others), the simplicial complexes associated with the elements in $\operatorname{Betti}(S)$ are those that appear in Figure 1. From Figure 1 and by using Theorem 6, the semigroup $S$ is the gluing of $\langle(13,0),(5,8),(2,11),(0,13)\rangle$ and $\langle(4,4),(6,6),(7,7),(9,9)\rangle$ and the glued degree is $(13,13)$. From Corollary 7, the ideal $I_{S}$ is not generated by its indispensable binomials ( $I_{S}$ has only four indispensable binomials).


Figure 1: Nonconnected simplicial complexes associated with Betti(S).

## 4. Generating Glued Semigroups

In this section, an algorithm to obtain examples of glued semigroups is given. Consider $A_{1}=\left\{a_{1}, \ldots, a_{r}\right\}$ and $A_{2}=$ $\left\{b_{1}, \ldots, b_{t}\right\}$ as two minimal generator sets of the semigroups $T_{1}$ and $T_{2}$ and let $L_{j}=\left\{\rho_{j i}\right\}_{i}$ be a basis of $\operatorname{ker} T_{j}$ with $j=1,2$. Assume that $I_{T_{1}}$ and $I_{T_{2}}$ are nontrivial proper ideals of their corresponding polynomial rings. Consider $\gamma_{X}$ and $\gamma_{Y}$ be two nonzero elements in $\mathbb{N}^{r}$ and $\mathbb{N}^{t}$, respectively, (note that $\gamma_{X} \notin$ $\operatorname{ker} T_{1}$ and $\gamma_{Y} \notin \operatorname{ker} T_{2}$ because these semigroups are reduced) and the integer matrix

$$
A=\left(\begin{array}{c|c}
L_{1} & 0  \tag{17}\\
\hline 0 & L_{2} \\
\hline \gamma_{X} & -\gamma_{Y}
\end{array}\right)
$$

Let $S$ be a semigroup such that $\operatorname{ker} S$ is the lattice generated by the rows of matrix $A$. This semigroup can be computed by using the Smith Normal Form (see [1, Chapter 3]). Denote by $B_{1}, B_{2}$ two sets of cardinality $r$ and $t$, respectively, satisfying $S=\left\langle B_{1}, B_{2}\right\rangle$ and $\operatorname{ker}\left(\left\langle B_{1}, B_{2}\right\rangle\right)$ is generated by the rows of $A$.

The following proposition shows that the semigroup $S$ satisfies one of the necessary conditions to be a glued semigroup.

Proposition 9. The semigroup $S$ verifies $G\left(\left\langle B_{1}\right\rangle\right) \cap G\left(\left\langle B_{2}\right\rangle\right)=$ $\left(B_{1} \gamma_{X}\right) \mathbb{Z}=\left(B_{2} \gamma_{Y}\right) \mathbb{Z}$ with $d=B_{1} \gamma_{X} \in\left\langle B_{1}\right\rangle \cap\left\langle B_{2}\right\rangle$.

Proof. Use that ker $S$ has a basis as (4) and proceed as in the proof of the necessary condition of Proposition 2.

Because $B_{1} \cup B_{2}$ may not be a minimal generating set, this condition does not assure that $S$ is a glued semigroup. For instance, taking the numerical semigroups $T_{1}=\langle 3,5\rangle$, $T_{2}=\langle 2,7\rangle$, and $\left(\gamma_{X}, \gamma_{Y}\right)=(1,0,2,0)$, the matrix obtained from formula (17) is

$$
\left(\begin{array}{cc|cc}
5 & -3 & 0 & 0  \tag{18}\\
\hline 0 & 0 & 7 & -2 \\
\hline 1 & 0 & -2 & 0
\end{array}\right)
$$

and $B_{1} \cup B_{2}=\{12,20,6,21\}$ is not a minimal generating set. The following result solves this issue.

Corollary 10. The semigroup $S$ is a glued semigroup if

$$
\begin{equation*}
\sum_{i=1}^{r} \gamma_{X i}>1, \quad \sum_{i=1}^{t} \gamma_{Y i}>1 \tag{19}
\end{equation*}
$$

Proof. Suppose that the set of generators $B_{1} \cup B_{2}$ of $S$ is nonminimal and thus one of its elements is a natural combination of the others. Assume that this element is the first of $B_{1} \cup B_{2}$ and then there exist $\lambda_{2}, \ldots, \lambda_{r+t} \in \mathbb{N}$ such that $B_{1}\left(1,-\lambda_{2}, \ldots,-\lambda_{r}\right)=B_{2}\left(\lambda_{r+1}, \ldots, \lambda_{r+t}\right) \in G\left(\left\langle B_{1}\right\rangle\right) \cap$ $G\left(\left\langle B_{2}\right\rangle\right)$. By Proposition 9, there exists $\lambda \in \mathbb{Z}$ satisfying $B_{1}\left(1,-\lambda_{2}, \ldots,-\lambda_{r}\right)=B_{2}\left(\lambda_{r+1}, \ldots, \lambda_{r+t}\right)=B_{1}\left(\lambda \gamma_{X}\right)$. Since $B_{2}\left(\lambda_{r+1}, \ldots, \lambda_{r+t}\right) \in S, \lambda \geq 0$ and thus

$$
\begin{align*}
\nu & =(1-\lambda \gamma_{X 1}, \underbrace{, \lambda_{2}-\lambda \gamma_{X 2}, \ldots,-\lambda_{r}-\lambda \gamma_{X r}}_{\leq 0})  \tag{20}\\
& \in \operatorname{ker}\left(\left\langle B_{1}\right\rangle\right)=\operatorname{ker} T_{1},
\end{align*}
$$

with the following cases.
(i) If $\lambda \gamma_{X 1}=0$, then $T_{1}$ is not minimally generated which it is not possible by hypothesis.
(ii) If $\lambda \gamma_{X 1}>1$, then $0>v \in \operatorname{ker} T_{1}$, but this is not possible because $T_{1}$ is a reduced semigroup.
(iii) If $\lambda \gamma_{X 1}=1$, then $\lambda=\gamma_{X 1}=1$ and

$$
\begin{equation*}
\nu=(0, \underbrace{-\lambda_{2}-\gamma_{X 2}, \ldots,-\lambda_{r}-\gamma_{X r}}_{\leq 0}) \in \operatorname{ker} T_{1} . \tag{21}
\end{equation*}
$$

If $\lambda_{i}+\gamma_{X i} \neq 0$ for some $i=2, \ldots, r$, then $T_{1}$ is not a reduced semigroup. This implies that $\lambda_{i}=\gamma_{X i}=0$ for all $i=2, \ldots, r$.

We have just proved that $\gamma_{X}=(1,0, \ldots, 0)$. In the general case, if $S$ is not minimally generated it is because either $\gamma_{X}$ or $\gamma_{Y}$ are elements in the canonical bases of $\mathbb{N}^{r}$ or $\mathbb{N}^{t}$, respectively. To avoid this situation, it is sufficient to take $\gamma_{X}$ and $\gamma_{Y}$ satisfying $\sum_{i=1}^{r} \gamma_{X i}>1$ and $\sum_{i=1}^{t} \gamma_{Y i}>1$.

From the above result we obtain a characterization of glued semigroups: $S$ is a glued semigroup if and only if $\operatorname{ker} S$ has a basis as (4) satisfies Condition (19).

Example 11. Let $T_{1}=\langle(-7,2),(11,1),(5,0),(0,1)\rangle \subset \mathbb{Z}^{2}$ and $T_{2}=\langle 3,5,7\rangle \subset \mathbb{N}$ be two reduced affine semigroups. We compute their associated lattices

$$
\begin{align*}
& \operatorname{ker} T_{1}=\langle(1,2,-3,-4),(2,-1,5,-3)\rangle,  \tag{22}\\
& \operatorname{ker} T_{2}=\langle(-4,1,1),(-7,0,3)\rangle .
\end{align*}
$$

If we take $\gamma_{X}=(2,0,2,0)$ and $\gamma_{Y}=(1,2,1)$, the matrix $A$ is

$$
\left(\begin{array}{ccccccc}
1 & 2 & -3 & -4 & 0 & 0 & 0  \tag{23}\\
2 & -1 & 5 & -3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -4 & 1 & 1 \\
0 & 0 & 0 & 0 & -7 & 0 & 3 \\
2 & 0 & 2 & 0 & -1 & -2 & -1
\end{array}\right)
$$

and the semigroup $S \subset \mathbb{Z}_{4} \times \mathbb{Z}^{2}$ is generated by

$$
\begin{align*}
& \{\underbrace{(1,-5,35),(3,12,-55),(1,5,-25),(0,1,0)}_{B_{1}},  \tag{24}\\
& \underbrace{(2,0,3),(2,0,5),(2,0,7)}_{B_{2}}\} .
\end{align*}
$$

The semigroup $S$ is the gluing of the semigroups $\left\langle B_{1}\right\rangle$ and $\left\langle B_{2}\right\rangle$ and $\operatorname{ker} S$ is generated by the rows of the above matrix. The ideal $I_{S} \subset \mathbb{C}\left[x_{1}, \ldots, x_{4}, y_{1}, \ldots, y_{3}\right]$ is generated (see [12] to compute $I_{S}$ when $S$ has torsion) by

$$
\left\{\begin{array}{c}
x_{1} x_{3}^{8} x_{4}-x_{2}^{3}, x_{1} x_{2}^{2}-x_{3}^{3} x_{4}^{4}, x_{1}^{2} x_{3}^{5}-x_{2} x_{4}^{3}, x_{1}^{3} x_{2} x_{3}^{2}-x_{7}^{7} \\
y_{1} y_{3}-y_{2}^{2}, y_{1}^{3} y_{2}-y_{3}^{2}, y_{1}^{4}-y_{2} y_{3}, \underbrace{x_{1}^{2} x_{3}^{2}-y_{1}^{5} y_{2}}_{\text {glued binomial }}\} \tag{25}
\end{array}\right.
$$

then $S$ is really a glued semigroup.
4.1. Generating Affine Glued Semigroups. From Example 11 it be can deduced that the semigroup $S$ is not necessarily torsion-free. In general, a semigroup $T$ is affine (or equivalently it is torsion-free) if and only if the invariant factors (the invariant factors of a matrix are the diagonal elements of its Smith Normal Form (see [13, Chapter 2] and [1, Chapter 2])) of the matrix whose rows are a basis of $\operatorname{ker} T$ are equal to one. Assume that zero-columns of the Smith Normal Form of a matrix are located on its right side. We now show conditions for $S$ being torsion-free.

Take $L_{1}$ and $L_{2}$ as the matrices whose rows form a basis of $\operatorname{ker} T_{1}$ and $\operatorname{ker} T_{2}$, respectively, and let $P_{1}, P_{2}, Q_{1}$, and $Q_{2}$ be some matrices with determinant $\pm 1$ (i.e., unimodular matrices) such that $D_{1}=P_{1} L_{1} Q_{1}$ and $D_{2}=P_{2} L_{2} Q_{2}$ are the Smith Normal Form of $L_{1}$ and $L_{2}$, respectively. If $T_{1}$ and $T_{2}$ are two affine semigroups, the invariant factors of $L_{1}$ and $L_{2}$ are equal to 1 . Then

$$
\left(\begin{array}{c|c}
D_{1} & 0  \tag{26}\\
\hline 0 & D_{2} \\
\hline \gamma_{X}^{\prime} & \gamma_{Y}^{\prime}
\end{array}\right)=\left(\begin{array}{c|c|c}
P_{1} & 0 & 0 \\
\hline 0 & P_{2} & 0 \\
\hline 0 & 0 & 1
\end{array}\right) \underbrace{\left(\begin{array}{c|c}
L_{1} & 0 \\
\hline 0 & L_{2} \\
\hline \gamma_{X} & -\gamma_{Y}
\end{array}\right)}_{=: A}\left(\begin{array}{c|c}
Q_{1} & 0 \\
\hline 0 & Q_{2}
\end{array}\right),
$$

where $\gamma_{X}^{\prime}=\gamma_{X} Q_{1}$ and $\gamma_{Y}^{\prime}=-\gamma_{Y} Q_{2}$. Let $s_{1}$ and $s_{2}$ be the numbers of zero-columns of $D_{1}$ and $D_{2}\left(s_{1}, s_{2}>0\right.$ because $T_{1}$ and $T_{2}$ are reduced, see [1, Theorem 3.14]).

Lemma 12. The semigroup $S$ is an affine semigroup if and only if

$$
\begin{equation*}
\operatorname{gcd}\left(\left\{\gamma_{X i}^{\prime}\right\}_{i=r-s_{1}}^{r} \cup\left\{\gamma_{Y i}^{\prime}\right\}_{i=t-s_{2}}^{t}\right)=1 \tag{27}
\end{equation*}
$$

Proof. With the conditions fulfilled by $T_{1}, T_{2}$, and $\left(\gamma_{X}, \gamma_{Y}\right)$, the necessary and sufficient condition for the invariant factors of $A$ to be all equal to one is $\operatorname{gcd}\left(\left\{\gamma_{X i}^{\prime}\right\}_{i=r-s_{1}}^{r} \cup\left\{\gamma_{Y i}^{\prime}\right\}_{i=t-s_{2}}^{t}\right)=1$.

The following corollary gives the explicit conditions that $\gamma_{X}$ and $\gamma_{Y}$ must satisfy to construct an affine semigroup.

Corollary 13. The semigroup $S$ is an affine glued semigroup if and only if
(1) $T_{1}$ and $T_{2}$ are two affine semigroups;
(2) $\left(\gamma_{X}, \gamma_{Y}\right) \in \mathbb{N}^{r+t}$;
(3) $\sum_{i=1}^{r} \gamma_{X i}, \sum_{i=1}^{t} \gamma_{Y i}>1$;
(4) there exist $f_{r-s_{1}}, \ldots, f_{r}, g_{t-s_{2}}, \ldots, g_{t} \in \mathbb{Z}$ such that

$$
\begin{align*}
& \left(f_{r-s_{1}}, \ldots, f_{r}\right) \cdot\left(\gamma_{X\left(r-s_{1}\right)}^{\prime}, \ldots, \gamma_{X r}^{\prime}\right)  \tag{28}\\
& \quad+\left(g_{t-s_{2}}, \ldots, g_{t}\right) \cdot\left(\gamma_{Y\left(t-s_{2}\right)}^{\prime}, \ldots, \gamma_{Y t}^{\prime}\right)=1
\end{align*}
$$

Proof. It is trivial by the given construction, Corollary 10 and Lemma 12.

Therefore, to obtain an affine glued semigroup it is enough to take two affine semigroups and any solution ( $\gamma_{X}$, $\gamma_{Y}$ ) of the equations of the above corollary.

Example 14. Let $T_{1}$ and $T_{2}$ be the semigroups of Example 11. We compute two elements $\gamma_{X}=\left(a_{1}, a_{2}, a_{3}, a_{4}\right)$ and $\gamma_{Y}=$ $\left(b_{1}, b_{2}, b_{3}\right)$ in order to obtain an affine semigroup. First of all, we perform a decomposition of the matrix as (26) by computing the integer Smith Normal Form of $L_{1}$ and $L_{2}$ :

$$
\left.\begin{array}{l}
\left(\begin{array}{cccc|cccc}
1 & 0 & 0 & 0 & 0 & 0 & & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & & 0 \\
\hline 0 & 0 & 0 & & 0 & 1 & 0 & \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & \\
\hline a_{1} & a_{1}-2 a_{2}-a_{3} & -7 a_{1}+11 a_{2}+5 a_{3} & 2 a_{1}+a_{2}+a_{4} & -b_{1} & b_{1}+2 b_{2}+3 b_{3} & -3 b_{1}-5 b_{2}-7 b_{3}
\end{array}\right) \\
=\left(\begin{array}{cc|cc|c}
1 & 0 & 0 & 0 & 0 \\
2 & -1 & 0 & 0 & 0 \\
\hline 0 & 0 & -2 & 1 & 0 \\
0 & 0 & 7 & -4 & 0 \\
\hline 0 & 0 & 0 & 0 & 1
\end{array}\right)\left(\begin{array}{cccc|ccc}
1 & 2 & -3 & -4 & 0 & 0 & 0 \\
2 & -1 & 5 & -3 & 0 & 0 & 0 \\
\hline 0 & 0 & 0 & 0 & -4 & 1 & 1 \\
0 & 0 & 0 & 0 & -7 & 0 & 3 \\
\hline a_{1} & a_{2} & a_{3} & a_{4} & -b_{1} & -b_{2} & -b_{3}
\end{array}\right)\left(\begin{array}{cccccc}
1 & 1 & -7 & 2 & 0 & 0 \\
0 & -2 & 11 & 1 & 0 & 0 \\
0 \\
0 & -1 & 5 & 0 & 0 & 0 \\
0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 \\
\hline 0 & 0 & 0 & 0 & 0 & -2 \\
0 & 5 \\
0 & 0 & 0 & 0 & 0 & -3
\end{array}\right) \tag{29}
\end{array}\right) .
$$

Second, by Corollary 13, we must find a solution to the system:

$$
\begin{gather*}
a_{1}+a_{2}+a_{3}+a_{4}>1 \\
b_{1}+b_{2}+b_{3}>1 \\
f_{1}, f_{2}, g_{1} \in \mathbb{Z}  \tag{30}\\
f_{1}\left(-7 a_{1}+11 a_{2}+5 a_{3}\right)+f_{2}\left(2 a_{1}+a_{2}+a_{4}\right) \\
+g_{1}\left(-3 b_{1}-5 b_{2}-7 b_{3}\right)=1
\end{gather*}
$$

with $a_{1}, a_{2}, a_{3}, a_{4}, b_{1}, b_{2}, b_{3} \in \mathbb{N}$. Such solution is computed (in less than a second) using FindInstance of Wolfram Mathematica (see [14]):

$$
\begin{align*}
& \text { FindInstance } \begin{aligned}
& \left(-7 a_{1}+11 a_{2}+5 a_{3}\right) * f_{1} \\
& +\left(2 a_{1}+a_{2}+a_{4}\right) * f_{2} \\
& +\left(-3 b_{1}-5 b_{2}-7 b_{3}\right) * g_{1}==1 \\
& \& \& a 1+a 2+a 3+a 4>1 \\
& \& \& b_{1}+b_{2}+b_{3}>1 \& \& a_{1} \geq 0 \& \& a_{2} \geq 0 \\
& \& \& a_{3} \geq 0 \& \& a_{4} \geq 0 \& \& b_{1} \geq 0 \\
& \& \& b_{2} \geq 0 \& \& b_{3} \geq 0, \\
& \left\{a_{1}, a_{2}, a_{3}, a_{4}, b_{1}, b_{2}, b_{3}, f_{1}, f_{2}, g_{1}\right\}, \\
& \text { Integers }] \\
& \downarrow \downarrow \\
\left\{\left\{a_{1} \longrightarrow 0, a_{2}\right.\right. & \longrightarrow 0, a_{3} \longrightarrow 3, a_{4} \longrightarrow 0, b_{1} \longrightarrow 1, \\
b_{2} \longrightarrow 1, b_{3} & \left.\left.\longrightarrow 0, f_{1} \longrightarrow 1, f_{2} \longrightarrow 0, g_{1} \longrightarrow 0\right\}\right\}
\end{aligned}
\end{align*}
$$

We now take $\gamma_{X}=(0,0,3,0)$ and $\gamma_{Y}=(1,1,0)$, and construct the matrix

$$
A=\left(\begin{array}{ccccccc}
1 & 2 & -3 & -4 & 0 & 0 & 0  \tag{32}\\
2 & -1 & 5 & -3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -4 & 1 & 1 \\
0 & 0 & 0 & 0 & -7 & 0 & 3 \\
0 & 0 & 3 & 0 & -1 & -1 & 0
\end{array}\right)
$$

We have the affine semigroup $S \subset \mathbb{Z}^{2}$ which is minimally generated by

$$
\begin{equation*}
\{\underbrace{(2,-56),(1,88),(0,40),(1,0)}_{B_{1}}, \underbrace{(0,45),(0,75),(0,105)}_{B_{2}}\} \tag{33}
\end{equation*}
$$

satisfying that $\operatorname{ker} S$ is generated by the rows of $A$ and it is the result of gluing the semigroups $\left\langle B_{1}\right\rangle$ and $\left\langle B_{2}\right\rangle$. The ideal $I_{S}$ is generated by

$$
\left\{\begin{array}{l}
\left\{x_{1} x_{3}^{8} x_{4}-x_{2}^{3}, x_{1} x_{2}^{2}-x_{3}^{3} x_{4}^{4}, x_{1}^{2} x_{3}^{5}-x_{2} x_{4}^{3}, x_{1}^{3} x_{2} x_{3}^{2}-x_{4}^{7}\right. \\
y_{1} y_{3}-y_{2}^{2}, y_{1}^{3} y_{2}-y_{3}^{2}, y_{1}^{4}-y_{2} y_{3}, \underbrace{x_{3}^{3}-y_{1} y_{2}}_{\text {glued binomial }}\} \tag{34}
\end{array}\right.
$$

therefore, $S$ is a glued semigroup.
All glued semigroups have been computed by using our program Ecuaciones which is available in [15] (this program requires Wolfram Mathematica 7 or above to run).

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Conservation Laws, Symmetry Reductions, and New Exact Solutions of the ( $2+1$ )-Dimensional Kadomtsev-Petviashvili Equation with Time-Dependent Coefficients 

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#### Abstract

The $(2+1)$-dimensional Kadomtsev-Petviashvili equation with time-dependent coefficients is investigated. By means of the Lie group method, we first obtain several geometric symmetries for the equation in terms of coefficient functions and arbitrary functions of $t$. Based on the obtained symmetries, many nontrivial and time-dependent conservation laws for the equation are obtained with the help of Ibragimov's new conservation theorem. Applying the characteristic equations of the obtained symmetries, the $(2+1)$-dimensional KP equation is reduced to $(1+1)$-dimensional nonlinear partial differential equations, including a special case of $(2+1)$-dimensional Boussinesq equation and different types of the KdV equation. At the same time, many new exact solutions are derived such as soliton and soliton-like solutions and algebraically explicit analytical solutions.


## 1. Introduction

The Lie group method is a powerful tool to perform Lie symmetry analysis, study conservation laws, and look for exact solutions of nonlinear partial differential equations (NLPDEs) [1-4]. The notion of conservation laws, which plays an important role in the study of nonlinear science, is used for the development of appropriate numerical methods and for mathematical analysis, in particular, existence, uniqueness, and stability analysis [5, 6]. In addition, the existence of a large number of conservation laws of a partial differential equation (system) is a strong indication of its integrability. On the other hand, seeking exact solutions of NLPDEs has become one central theme of perpetual interest in mathematical physics as explicit solutions will be helpful to better understand the phenomena described by the equations. To get exact solutions of NLPDEs, many effective methods have been presented such as inverse scattering method [7], Hirota's bilinear method [8], and Painlevé expansion method [9]. Among them the Lie group method offers a systematic algorithmic procedure to find the symmetry reductions and exact solutions of a partial differential
equation. In this paper, we use the Lie group method to consider a time-dependent Kadomtsev-Petviashvili equation:

$$
\begin{equation*}
E_{1} \equiv u_{x t}+6 u_{x}^{2}+6 u u_{x x}+u_{x x x x}+e(t) u_{x}+n(t) u_{y y}=0 \tag{1}
\end{equation*}
$$

with time-dependent coefficient functions $e(t), n(t)$, and $n(t) \neq 0$.

The above equation was also called "a 2D KdV equation with time-dependent coefficients" by Hereman and Zhuang [10]; they performed Painlevé analysis for (1) and found that (1) was Painlevé integrable when $e_{t}+2 e^{2}=0, n_{t}+4 n e=$ 0 . Equation (1) can be reduced to the KdV equation $(e(t)=$ $0, n(t)=0)$ or the KP equation $(e(t)=0, n(t)= \pm 1)$. Equation (1) can also be reduced to the cylindrical KdV equation

$$
\begin{equation*}
u_{t}+6 u u_{x}+u_{x x x}+\frac{1}{2 t} u=0 \tag{2a}
\end{equation*}
$$

when $e(t)=1 / 2 t, n(t)=0$ or the cylindrical KP equation

$$
\begin{equation*}
u_{x t}+6 u_{x}^{2}+6 u u_{x x}+u_{x x x x}+\frac{1}{2 t} u_{x} \pm 3 \frac{1}{t^{2}} u_{y y}=0 \tag{2b}
\end{equation*}
$$

when $e(t)=1 / 2 t, n(t)= \pm 3 / t^{2}$. The KdV and KP equations and their cylindrical generalizations (2a) and (2b) are all known to be completely integrable [10]. Zhang et al. [11] performed Painlevé analysis for (1) and constructed bilinear auto-Bäcklund, analytic solutions in the Wronskian form. Soliton-like solutions, Jacobi elliptic function-like solutions, and other exact solutions have been obtained by the method of auxiliary equations [12-15]. Elwakil et al. [16] used the homogeneous balance method to study the exact solutions of (1). Based on the homogeneous balance method and Clarkson-Kruskal method, direct reduction and exact solutions have been obtained in [17] by Moussa and El-Shiekh. The bilinear formalism, bilinear Bäcklund transformation, and Lax pair of (1) have been obtained by the binary Bell polynomial approach in [18]. As far as we know, conservation laws and symmetry reductions for (1) have not been studied.

The rest of the paper is organized as follows. In Section 2, the Lie group method is applied to the time-dependent Kadomtsev-Petviashvili equation (1) and thus Lie symmetries of (1) are obtained. In Section 3, using the obtained symmetries and the general theorem on conservation laws by Ibragimov, nontrivial and time-dependent conservation laws are derived. In Section 4, we use the symmetry to get symmetry reductions and new exact solutions of (1). The last section is a short summary and discussion.

## 2. Lie Symmetry Analysis of (1)

Generally speaking, Lie symmetry denotes a transformation that leaves the solution manifold of a system invariant; that is, it maps any solution of the system into a solution of the same system, so it is also called geometric symmetry. In this section, we will perform Lie symmetry analysis for (1) by the classical Lie group method. Suppose that Lie symmetry of (1) is expressed as follows:

$$
\begin{equation*}
V=\xi \frac{\partial}{\partial x}+\eta \frac{\partial}{\partial y}+\tau \frac{\partial}{\partial t}+\phi \frac{\partial}{\partial u} \tag{3}
\end{equation*}
$$

where $\xi, \eta, \tau$, and $\phi$ are undetermined functions with respect to $x, y, t$, and $u$. According to the procedures of Lie group method, the vector field (3) can be determined by applying the fourth prolongation of $V$ to (1) and thus the undetermined functions $\xi, \eta, \tau$, and $\phi$ must satisfy the following invariant condition:

$$
\begin{align*}
& \phi^{x t}+12 u_{x} \phi^{x}+6 u_{x x} \phi+6 u \phi^{x x}+\phi^{x x x x} \\
& \quad+e^{\prime}(t) \tau u_{x}+e(t) \phi^{x}+n^{\prime}(t) \tau u_{y y}+n(t) \phi^{y y}=0 \tag{4}
\end{align*}
$$

where

$$
\begin{gathered}
\phi^{x}=D_{x}\left(\phi-\xi u_{x}-\eta u_{y}-\tau u_{t}\right)+\xi u_{x x}+\eta u_{x y}+\tau u_{x t} \\
\phi^{x t}=D_{x t}\left(\phi-\xi u_{x}-\eta u_{y}-\tau u_{t}\right)+\xi u_{x x t}+\eta u_{x t y}+\tau u_{x t t}
\end{gathered}
$$

$$
\begin{align*}
\phi^{x x}= & D_{x x}\left(\phi-\xi u_{x}-\eta u_{y}-\tau u_{t}\right) \\
& +\xi u_{x x x}+\eta u_{x x y}+\tau u_{x x t}, \\
\phi^{y y}= & D_{y y}\left(\phi-\xi u_{x}-\eta u_{y}-\tau u_{t}\right) \\
& +\xi u_{x y y}+\eta u_{y y y}+\tau u_{y y t} \\
\phi^{x x x x}= & D_{x x x x}\left(\phi-\xi u_{x}-\eta u_{y}-\tau u_{t}\right) \\
& +\xi u_{x x x x x}+\eta u_{x x x x y}+\tau u_{x x x x t} . \tag{5}
\end{align*}
$$

Substituting (5) into (4) with $u$ being a solution of (1), that is,

$$
\begin{equation*}
u_{x x x x}=-u_{x t}-6 u_{x}^{2}-6 u u_{x x}-e(t) u_{x}-n(t) u_{y y} \tag{6}
\end{equation*}
$$

we obtain the determining equations of symmetry (3). Solving the determining equations with the aid of Maple, we can get the following cases.

Case 1. When $e(t)$ and $n(t)$ are arbitrary functions,

$$
\begin{gather*}
\xi=-\frac{g_{t} y}{2 n(t)}+f(t), \quad \eta=g(t), \quad \tau=0 \\
\phi=\frac{f_{t}}{6}-\frac{g_{t t}}{12 n(t)} y+\frac{g_{t} n_{t}}{12 n^{2}(t)} y \tag{7}
\end{gather*}
$$

where $f(t)$ and $g(t)$ are arbitrary functions. It shows that (1) admits an infinite-dimensional Lie algebra of symmetries

$$
\begin{equation*}
V=V_{f}+V_{g} \tag{8}
\end{equation*}
$$

where

$$
\begin{gather*}
V_{f}=f(t) \frac{\partial}{\partial x}+\frac{f_{t}}{6} \frac{\partial}{\partial u}, \\
V_{g}=-\frac{g_{t} y}{2 n(t)} \frac{\partial}{\partial x}+g(t) \frac{\partial}{\partial y}+\left(\frac{g_{t} n_{t}}{12 n^{2}(t)} y-\frac{g_{t t}}{12 n(t)} y\right) \frac{\partial}{\partial u} . \tag{9}
\end{gather*}
$$

Case 2. When $e(t)=0, n(t)=(t-m)^{p} C_{1}, p \neq 0, C_{1} \neq 0$, and $C_{2} \neq 0$,

$$
\begin{gather*}
\xi=\frac{C_{2} x}{3 p}-\frac{g_{t} y}{2 C_{1}(t-m)^{p}}+f(t), \\
\eta=\left(\frac{2 C_{2}}{3 p}+\frac{C_{2}}{2}\right) y+g(t), \quad \tau=\frac{C_{2}(t-m)}{p}, \\
\phi=-\frac{2 C_{2}}{3 p} u+\frac{g_{t}}{12 C_{1}(t-m)^{p+1}} y p-\frac{g_{t t}}{12 C_{1}(t-m)^{p}} y+\frac{f_{t}}{6}, \tag{10}
\end{gather*}
$$

where $m, p, C_{1}$, and $C_{2}$ are constants and $f(t)$ and $g(t)$ are arbitrary functions. This shows that the symmetries of equation

$$
\begin{equation*}
u_{x t}+6 u_{x}^{2}+6 u u_{x x}+u_{x x x x}+C_{1}(t-m)^{p} u_{y y}=0 \tag{11}
\end{equation*}
$$

have the form of

$$
\begin{equation*}
V=V_{1}+V_{f}+V_{g} \tag{12}
\end{equation*}
$$

where

$$
\begin{equation*}
V_{1}=\frac{x}{3 p} \frac{\partial}{\partial x}+\left(\frac{2}{3 p}+\frac{1}{2}\right) y \frac{\partial}{\partial y}+\frac{(t-m)}{p} \frac{\partial}{\partial t}-\frac{2}{3 p} u \frac{\partial}{\partial u} \tag{13}
\end{equation*}
$$

is a one-dimensional Lie algebra of symmetries and $V_{f}$ and $V_{g}$ are two infinite-dimensional Lie algebra of symmetries as expressed by (9) with $n(t)=(t-m)^{p} C_{1}$.

Case 3. When $e(t)=0, n(t)=$ Const., and $\tau(t) \neq 0$,

$$
\begin{gather*}
\xi=\frac{\tau_{t}}{3} x-\frac{\tau_{t t}}{6 n} y^{2}-\frac{g_{t}}{2 n} y+f(t), \\
\eta=\frac{2}{3} \tau_{t} y+g(t), \quad \tau=\tau(t),  \tag{14}\\
\phi=-\frac{2 \tau_{t}}{3} u+\frac{\tau_{t t}}{18} x-\frac{\tau_{t t t}}{36 n} y^{2}-\frac{g_{t t}}{12 n} y+\frac{f_{t}}{6},
\end{gather*}
$$

where $f(t)$ and $g(t)$ are arbitrary functions. It shows that the KP equation

$$
\begin{equation*}
u_{x t}+6 u_{x}^{2}+6 u u_{x x}+u_{x x x x}+C u_{y y}=0 \tag{15}
\end{equation*}
$$

admits an infinite-dimensional Lie algebra of symmetries

$$
\begin{equation*}
V=V_{f}+V_{g}+V_{\tau} \tag{16}
\end{equation*}
$$

where $C$ is a constant and $C \neq 0 ; V_{f}$ and $V_{g}$ are expressed by (9) with $n(t)=$ Const.,

$$
\begin{align*}
V_{\tau}= & \left(\frac{\tau_{t}}{3} x-\frac{\tau_{t t}}{6 n} y^{2}\right) \frac{\partial}{\partial x}+\frac{2}{3} \tau_{t} y \frac{\partial}{\partial y}+\tau \frac{\partial}{\partial t}  \tag{17}\\
& +\left(-\frac{2 \tau_{t}}{3} u+\frac{\tau_{t t}}{18} x-\frac{\tau_{t t t}}{36 n} y^{2}\right) \frac{\partial}{\partial u}
\end{align*}
$$

Case 4. When $e(t)=-n_{t} / 4 n+C_{3} / \tau(t), \tau(t) \neq 0$, and $n_{t} \neq 0$,

$$
\begin{aligned}
\xi= & \frac{\tau_{t}}{3} x-\frac{\tau_{t t}}{6 n(t)} y^{2}-\frac{g_{t}}{2 n(t)} y-\frac{\tau_{t} n_{t}}{8 n^{2}(t)} y^{2} \\
& -\frac{\tau(t) n_{t t}}{8 n^{2}(t)} y^{2}+\frac{\tau(t) n_{t}^{2}}{8 n^{3}(t)} y^{2}+f(t)
\end{aligned}
$$

$$
\begin{align*}
& \eta=\left(\frac{\tau(t) n_{t}}{2 n(t)}+\frac{2}{3} \tau_{t}\right) y+g(t), \\
& \phi=-\frac{2 \tau_{t}}{3} u+\frac{\tau_{t t}}{18} x+\frac{\tau(t) n_{t t} n_{t}}{12 n^{3}(t)} y^{2} \\
&-\frac{\tau(t) n_{t t t}}{48 n^{2}(t)} y^{2}-\frac{\tau(t) n_{t}^{3}}{16 n^{4}(t)} y^{2}+\frac{\tau_{t t} n_{t}}{144 n^{2}(t)} y^{2} \\
&-\frac{\tau_{t t t}}{36 n(t)} y^{2}+\frac{\tau_{t} n_{t}^{2}}{16 n^{3}(t)} y^{2}-\frac{\tau_{t} n_{t t}}{24 n^{2}(t)} y^{2} \\
&+\frac{f_{t}}{6}-\frac{g_{t t}}{12 n(t)} y+\frac{g_{t} n_{t}}{12 n^{2}(t)} y, \tag{18}
\end{align*}
$$

where $f(t)$ and $g(t)$ are arbitrary functions, $C_{3}$ is an integral constant, and $n(t)$ and $\tau(t)$ satisfy the following ordinary differential equation:

$$
\begin{align*}
n_{t t t} & +\frac{2 n_{t t} \tau_{t}}{\tau(t)}-\frac{3 \tau_{t} n_{t}^{2}}{n(t) \tau(t)}+\frac{3 n_{t}^{3}}{n^{2}(t)}  \tag{19}\\
& -\frac{4 n_{t t} n_{t}}{n(t)}-\frac{4 C_{3} n(t) \tau_{t t}}{3 \tau^{2}(t)}=0
\end{align*}
$$

This shows that, under the condition (19), the equation

$$
\begin{align*}
u_{x t} & +6 u_{x}^{2}+6 u u_{x x}+u_{x x x x} \\
& +\left(-\frac{n_{t}}{4 n}+\frac{C_{3}}{\tau(t)}\right) u_{x}+n(t) u_{y y}=0 \tag{20}
\end{align*}
$$

admits an infinite-dimensional Lie algebra of symmetries

$$
\begin{equation*}
V=V_{f}+V_{g}+V_{0 \tau} \tag{21}
\end{equation*}
$$

where $V_{f}$ and $V_{g}$ are expressed by (9):

$$
\begin{align*}
& V_{0 \tau}=\left(\frac{\tau_{t}}{3} x\right.-\frac{\tau_{t t}}{6 n} y^{2}-\frac{\tau_{t} n_{t}}{8 n^{2}(t)} y^{2}-\frac{\tau(t) n_{t t}}{8 n^{2}(t)} y^{2} \\
&\left.+\frac{\tau(t) n_{t}^{2}}{8 n^{3}(t)} y^{2}\right) \frac{\partial}{\partial x}+\left(\frac{\tau(t) n_{t}}{2 n(t)}+\frac{2}{3} \tau_{t}\right) y \frac{\partial}{\partial y}+\tau \frac{\partial}{\partial t} \\
&+\left(-\frac{2 \tau_{t}}{3} u+\frac{\tau_{t t}}{18} x+\frac{\tau(t) n_{t t} n_{t}}{12 n^{3}(t)} y^{2}-\frac{\tau(t) n_{t t t}}{48 n^{2}(t)} y^{2}\right. \\
& \quad-\frac{\tau(t) n_{t}^{3}}{16 n^{4}(t)} y^{2}+\frac{\tau_{t t} n_{t}}{144 n^{2}(t)} y^{2}-\frac{\tau_{t t t}}{36 n(t)} y^{2} \\
&\left.+\frac{\tau_{t} n_{t}^{2}}{16 n^{3}(t)} y^{2}-\frac{\tau_{t} n_{t t}}{24 n^{2}(t)} y^{2}\right) \frac{\partial}{\partial u} . \tag{22}
\end{align*}
$$

## 3. Conservation Laws for (1)

3.1. A General Theorem on Conservation Laws. As expressed through the famous Noether theorem, for a given differential equation, there is a close connection between Lie symmetries and conservation laws. To derive conservation laws of (1), we use the following conclusion proved by Ibragimov in [19].

Theorem 1. Every Lie point, Lie-Bäcklund, and nonlocal symmetry

$$
\begin{equation*}
V=\xi^{i}\left(x, u, u_{(1)}, \ldots\right) \frac{\partial}{\partial x^{i}}+\eta^{s}\left(x, u, u_{(1)}, \ldots\right) \frac{\partial}{\partial u^{s}} \tag{23}
\end{equation*}
$$

of a system of m equations

$$
\begin{equation*}
F_{s}\left(x, u, u_{(1)}, \ldots, u_{(N)}\right)=0, \quad s=1, \ldots, m \tag{24}
\end{equation*}
$$

with $n$ independent variables $x=\left(x^{1}, \ldots, x^{n}\right)$ and $m$ dependent variables; $u=\left(u^{1}, \ldots, u^{m}\right)$ provides a conservation law for system (24) and the corresponding adjoint system

$$
\begin{gather*}
F_{s}^{*}\left(x, u, v, u_{(1)}, v_{(1)}, \ldots, u_{(N)}, v_{(N)}\right) \\
\quad \equiv \frac{\delta\left(v^{i} F_{i}\right)}{\delta u^{s}}=0, \quad s=1, \ldots, m \tag{25}
\end{gather*}
$$

Then the elements of the conservation vector $T=\left(T^{1}, \ldots, T^{n}\right)$ are defined by the following expression:

$$
\begin{align*}
T^{i}= & \xi^{i} L+W^{s} \\
& \times\left[\frac{\partial L}{\partial u_{i}^{s}}-D_{x^{j}}\left(\frac{\partial L}{\partial u_{i j}^{s}}\right)+D_{x^{j}} D_{x^{k}}\left(\frac{\partial L}{\partial u_{i j k}^{s}}\right)-\cdots\right] \\
& +D_{x^{j}}\left(W^{s}\right) \\
& \times\left[\frac{\partial L}{\partial u_{i j}^{s}}-D_{x^{k}}\left(\frac{\partial L}{\partial u_{i j k}^{s}}\right)+D_{x^{k}} D_{x^{r}}\left(\frac{\partial L}{\partial u_{i j k r}^{s}}\right)-\cdots\right] \\
& +D_{x^{j}} D_{x^{k}}\left(W^{s}\right)\left[\frac{\partial L}{\partial u_{i j k}^{s}}-D_{x^{r}}\left(\frac{\partial L}{\partial u_{i j k r}^{s}}\right)+\cdots\right]+\cdots, \tag{26}
\end{align*}
$$

with

$$
\begin{equation*}
W^{s}=\eta^{s}-\xi^{i} u_{i}^{s}, \quad s=1, \ldots, m \tag{27}
\end{equation*}
$$

3.2. Conservation Laws for (1). To search for conservation laws of (1) by Theorem 1, adjoint equation and formal Lagrangian of (1) must be known. We first construct its adjoint equation. Following the idea in [19], the adjoint equation of (1) is

$$
\begin{equation*}
E_{1}^{*} \equiv v_{x t}+6 u v_{x x}+v_{x x x x}-e(t) v_{x}+n(t) v_{y y}=0 \tag{28}
\end{equation*}
$$

where $v$ is a new dependent variable with respect to $x, y$, and $t$.

According to the method of constructing Lagrangian in [19], the formal Lagrangian for the system consisting of (1) and (28) is

$$
\begin{equation*}
L=v\left(u_{x t}+6 u_{x}^{2}+6 u u_{x x}+u_{x x x x}+e(t) u_{x}+n(t) u_{y y}\right) . \tag{29}
\end{equation*}
$$

By means of the symmetries of (1), conservation laws of the system consisting of (1) and (28) can be derived by

Theorem 1. However, we are only interested in the conservation laws of (1). Therefore one has to eliminate the nonlocal variable $v$ which is introduced in the adjoint equation. To solve this problem, the concepts of self-adjointness, quasi-self-adjointness, and nonlinear self-adjointness are developed [20-24]. In the following, we will discuss the adjointness and nonlinear adjointness using these definitions.

Equation (1) is said to be self-adjoint if the equation obtained from the adjoint equation (28) by the substitution $v=u$ is identical with the original equation (1). It is easy to see that (28) is not identical with (1) when $v=u$, so (1) is not a self-adjoint equation. According to the definition of nonlinear self-adjointness [24], (1) is said to be nonlinearly self-adjoint if its adjoint equation (28) is satisfied for all solutions $u$ of (1) upon a substitution

$$
\begin{equation*}
v=H(x, y, t, u), \quad H(x, y, t, u) \neq 0 . \tag{30}
\end{equation*}
$$

In other words, (1) is nonlinearly self-adjoint if and only if

$$
\begin{equation*}
\left.E_{1}^{*}\right|_{v=H(x, y, t, u)}=\lambda\left(x, y, t, u, u_{x}, u_{y}, u_{t}, u_{x x}, \ldots\right) E_{1} \tag{31}
\end{equation*}
$$

where $\lambda$ is an undetermined and smooth function.
From (31), we can get the following equation:

$$
\begin{align*}
\left(H_{u}\right. & -\lambda) u_{x x x x}+n(t)\left(H_{u}-\lambda\right) u_{y y}+\left(H_{u}-\lambda\right) u_{x t} \\
& +4 H_{u u} u_{x} u_{x x x}+4 H_{x u} u_{x x x}+2 n(t) H_{y u} u_{y} \\
& +u_{x}^{2}\left(6 u H_{u u}+6 H_{u u u} u_{x x}-6 \lambda+6 H_{x x u u}\right) \\
& +u\left(12 u_{x} H_{x u}+6 H_{u} u_{x x}-6 \lambda u_{x x}+6 H_{x x}\right)+H_{t u} u_{x} \\
& -\lambda e(t) u_{x}-e(t) u_{x} H_{u}+n(t) u_{y}^{2} H_{u u} \\
& +12 H_{x u u} u_{x} u_{x x}+u_{x} u_{t} H_{u u}+6 H_{x x u} u_{x x}+4 H_{x x x u} u_{x} \\
& +H_{x u} u_{t}+4 H_{x u u u} u_{x}^{3}+3 H_{u u} u_{x x}^{2}+H_{u u u u} u_{x}^{4} \\
& +\left(-e(t) H_{x}+n(t) H_{y y}+H_{x t}+H_{x x x x}\right)=0 . \tag{32}
\end{align*}
$$

Solving the above system with the aid of Maple, the final results read as

$$
\begin{gather*}
\lambda=0  \tag{33}\\
H=(a(t) y+b(t)) x-\frac{a_{t} y^{3}}{6 n(t)}-\frac{b_{t} y^{2}}{2 n(t)}  \tag{34}\\
+\frac{e(t) a(t) y^{3}}{6 n(t)}+\frac{e(t) b(t) y^{2}}{2 n(t)}+k(t) y+l(t)
\end{gather*}
$$

where $a(t), b(t), k(t)$, and $l(t)$ are arbitrary functions. In summary, we have the following statements.

Theorem 2. The time-dependent KP equation (1) is nonlinearly self-adjoint.

In the following, we first construct the conservation laws for the system consisting of the initial equation (1) and its adjoint (28).

For the symmetry in Case 1, the corresponding components of the conservation laws are

$$
\begin{align*}
& X_{1}=f(t) u_{x} v_{t}+f_{t} u_{x} v-g(t) u_{x x x y} v-g(t) u_{x y} v_{x x} \\
& -f_{t} v_{x} u+f(t) u_{x t} v+g(t) u_{y} v_{x x x}+\frac{f_{t} e(t) v}{6} \\
& +g(t) u_{x x y} v_{x}+g(t) u_{y} v_{t} \\
& +f(t) u_{x} v_{x x x}+f(t) v_{x} u_{x x x} \\
& -f(t) u_{x x} v_{x x}+\frac{g_{t} y u_{x x} v_{x x}}{2 n(t)}-\frac{1}{6} f_{t} v_{x x x}-\frac{g_{t} y u_{y y} v}{2} \\
& +f(t) n(t) u_{y y} v+\frac{g_{t t} y v_{t}}{12 n(t)}+\frac{g_{t t} y v_{x x x}}{12 n(t)}+6 f(t) u_{x} v_{x} u \\
& -g(t) e(t) u_{y} v-6 g(t) u_{y} u_{x} v+6 g(t) u_{y} v_{x} u \\
& -6 g(t) u_{x y} u v-\frac{1}{6} f_{t} v_{t}-\frac{g_{t t} y e(t) v}{12 n(t)}-\frac{g_{t} y u_{x} v_{t}}{2 n(t)} \\
& -\frac{g_{t} y u_{x t} v}{2 n(t)}-\frac{g_{t t} y u_{x} v}{2 n(t)}+\frac{g_{t t} y u v_{x}}{2 n(t)}+\frac{g_{t} y n_{t} e(t) v}{12 n^{2}(t)} \\
& +\frac{g_{t} y n_{t} u_{x} v}{2 n^{2}(t)}-\frac{g_{t} y n_{t} v_{x} u}{2 n^{2}(t)}-\frac{g_{t} y n_{t} v_{t}}{12 n^{2}(t)}-\frac{g_{t} y n_{t} v_{x x x}}{12 n^{2}(t)} \\
& -\frac{3 g_{t} y u_{x} v_{x} u}{n(t)}-\frac{g_{t} y u_{x} v_{x x x}}{2 n(t)}-\frac{g_{t} y u_{x x x} v_{x}}{2 n(t)}, \\
& Y_{1}=-\frac{n(t) f_{t} v_{y}}{6}+\frac{y g_{t t} v_{y}}{12}-\frac{g_{t} y n_{t} v_{y}}{12 n(t)}-\frac{1}{2} g_{t} y u_{x} v_{y} \\
& +f(t) n(t) u_{x} v_{y}+g(t) n(t) u_{y} v_{y}-\frac{g_{t t} v}{12}+\frac{g_{t} n_{t} v}{12 n(t)} \\
& +\frac{1}{2} g_{t} u_{x} v+\frac{1}{2} g_{t} y u_{x y} v-f(t) n(t) u_{x y} v \\
& -g(t) v n(t) u_{y y}, \\
& T_{1}=\frac{g_{t} y u_{x x} v}{2 n(t)}-f(t) u_{x x} v-g(t) u_{x y} v . \tag{35}
\end{align*}
$$

For the symmetry in Case 2, the corresponding components of the conservation laws are

$$
\begin{aligned}
X_{2}= & -6 \frac{C_{2} m u_{t} v_{x} u}{p}-\frac{1}{6} f_{t} v_{x x x}-\frac{1}{6} f_{t} v_{t}+\frac{g_{t} y}{2 n(t)} u_{x x} v_{x x} \\
& +\frac{C_{2} m u_{t x x x} v}{p}-\frac{g_{t t} y}{2 n(t)} u_{x} v+\frac{g_{t t} y}{12 n(t)} v_{t}+\frac{g_{t t} y}{12 n(t)} v_{x x x} \\
& +\frac{C_{2} x}{3 p} u_{x} v_{t}+\frac{C_{2} x}{3 p} u_{x} v_{x x x}-\frac{3 g_{t} y}{n(t)} u_{x} v_{x} u-\frac{g_{t} y}{2 n(t)} u_{x} v_{t} \\
& +4 \frac{C_{2} y}{p} u_{y} v_{x} u+\frac{2 C_{2} y}{3 p} u_{y} v_{t}+\frac{2 C_{2} y}{3 p} u_{y} v_{x x x}
\end{aligned}
$$

$$
\begin{aligned}
& -3 C_{2} y u_{y} u_{x} v+3 C_{2} y u_{y} v_{x} u-6 \frac{C_{2} t}{p} u_{t} u_{x} v \\
& +6 \frac{C_{2} t}{p} u_{t} u_{x} u+\frac{C_{2} t}{p} u_{t} v_{t}+\frac{C_{2} t}{p} u_{t} v_{x x x}+6 \frac{C_{2} m u_{t} u_{x} v}{p} \\
& -\frac{C_{2} m u_{t} v_{x x x}}{p}-\frac{C_{2} x}{3 p} u_{x x} v_{x x}-\frac{2 C_{2} y}{3 p} u_{x y} v_{x x} \\
& -3 C_{2} y u v u_{x y}-6 \frac{C_{2} t}{p} u_{x t} u v-\frac{C_{2} t}{p} u_{x t} v_{x x} \\
& -\frac{g_{t} y p}{12 n(t)(t-m)} v_{t}-\frac{10 C_{2} u u_{x} v}{p}-4 \frac{C_{2} y u_{y} u_{x} v}{p} \\
& +\frac{C_{2} x u_{x t} v}{3 p}+\frac{C_{2} x n(t) u_{y y} v}{3 p}-\frac{g_{t} y}{2 n(t)} u_{x t} v \\
& +f(t) v n(t) u_{y y}+\frac{g_{t} y p}{2 n(t)(t-m)} u_{x} v \\
& -\frac{g_{t} y p}{2 n(t)(t-m)} v_{x} u+\frac{g_{t t} y}{2 n(t)} v_{x} u \\
& -\frac{g_{t} y p}{12 n(t)(t-m)} v_{x x x}+6 \frac{C_{2} m}{p} u_{x t} u v \\
& +\frac{C_{2} m}{p} u_{x t} v_{x x}+\frac{C_{2} x}{3 p} v_{x} u_{x x x}-\frac{g_{t} y}{2 n(t)} u_{x x x} v_{x} \\
& +\frac{2 C_{2} y}{3 p} u_{x x y} v_{x}+\frac{C_{2} t}{p} u_{x x t} v_{x}-\frac{2 C_{2} y u_{x x x y} v}{3 p} \\
& -\frac{C_{2} t u_{x x x t} v}{p}-\frac{C_{2} m u_{t} v_{t}}{p}+g(t) u_{y} v_{x x x} \\
& -f_{t} v_{x} u-g(t) u_{x x x y} v-f(t) u_{x x} v_{x x}+f(t) u_{x x x} v_{x} \\
& +g(t) u_{x x y} v_{x}+f_{t} u_{x} v+f(t) u_{x} v_{x x x}-g(t) u_{x y} v_{x x} \\
& +v f(t) u_{x t}+g(t) u_{y} v_{t}+f(t) u_{x} v_{t}+\frac{2 C_{2}}{3 p} u v_{t} \\
& +\frac{4 C_{2}}{p} u^{2} v_{x}-\frac{g_{t} y}{2} u_{y y} v-\frac{C_{2} u_{x} v_{x x}}{p}+\frac{2 C_{2} x}{p} u_{x} v_{x} u \\
& +6 g(t) u_{y} v_{x} u-6 g(t) u_{y} u_{x} v+\frac{C_{2} y}{2} u_{y} v_{x x x} \\
& +\frac{C_{2} y}{2} u_{y} v_{t}+6 f(t) u_{x} v_{x} u+\frac{2 C_{2}}{3 p} u v_{x x x} \\
& -\frac{C_{2} y}{2} v u_{x x x y}-\frac{5 C_{2} u_{x x x} v}{3 p} \\
& +\frac{C_{2} y}{2} v_{x} u_{x x y}+\frac{4 C_{2}}{3 p} u_{x x} v_{x}-6 g(t) u_{x y} v u \\
& -\frac{C_{2} y}{2} v_{x x} u_{x y}-\frac{4 C_{2} y u_{x y} u v}{p}-\frac{C_{2} m u_{x x t} v_{x}}{p},
\end{aligned}
$$

$$
\begin{align*}
Y_{2}= & -\frac{g_{t t} v}{12}-\frac{C_{2} t n(t) u_{y t} v}{p}+\frac{C_{2} m n(t) u_{y t} v}{p} \\
& -f(t) n(t) v u_{x y}+\frac{g_{t} v p}{12(t-m)}+\frac{2 C_{2} n(t) u v_{y}}{3 p} \\
& -\frac{g_{t} y p v_{y}}{12(t-m)}+\frac{C_{2} x n(t) u_{x} v_{y}}{3 p}+f(t) n(t) u_{x} v_{y} \\
& +\frac{2 C_{2} y n(t) u_{y} v_{y}}{3 p}+\frac{C_{2} y n(t)}{2} u_{y} v_{y}+g(t) n(t) u_{y} v_{y} \\
& +\frac{C_{2} t n(t) u_{t} v_{y}}{p}-\frac{C_{2} m n(t) u_{t} v_{y}}{p}+\frac{g_{t}}{2} u_{x} v \\
& -\frac{4 C_{2} n(t) u_{y} v}{3 p}-\frac{C_{2} x n(t) u_{x y} v}{3 p}+\frac{g_{t t}}{12} y v_{y}+\frac{g_{t} y u_{x y} v}{2} \\
& -\frac{g_{t} y u_{x} v_{y}}{2}-\frac{1}{6} f_{t} n(t) v_{y}-\frac{C_{2} n(t) u_{y} v}{2} \\
& -g(t) v n(t) u_{y y}-\frac{2 C_{2} y v n(t)}{3 p} u_{y y}-\frac{C_{2} y v n(t)}{2} u_{y y} \\
T_{2}= & -\frac{C_{2} u_{x} v}{p}-\frac{C_{2} x u_{x x} v}{3 p}+\frac{g_{t} y u_{x x} v}{2 n(t)}-f(t) u_{x x} v \\
& -\frac{2 C_{2} y u_{x y} v}{3 p}-\frac{C_{2} y u_{x y} v}{2}-g(t) u_{x y} v-\frac{C_{2} v t u_{x t}}{p} \\
& +\frac{C_{2} v m u_{x t}}{p} . \tag{36}
\end{align*}
$$

Here we should note that the coefficient function $n(t)$ in the expression of $X_{2}, Y_{2}$, and $T_{2}$ satisfies $n(t)=(t-m)^{p} C_{1}, m, p$, and $C_{1}$ are constants, and $p \neq 0, C_{1} \neq 0$.

For the symmetry in Case 3 , the corresponding components of the conservation laws are

$$
\begin{aligned}
X_{3}= & -\frac{x}{18} \tau_{t t} v_{x x x}+f(t) v_{x} u_{x x x}+g(t) v_{x} u_{x x y} \\
& +f(t) u_{x} v_{x x x}+\tau(t) u_{t} v_{t}-\tau_{t} u_{x} v_{x x}+g(t) u_{y} v_{x x x} \\
& +g(t) u_{y} v_{t}+\frac{2}{3} \tau_{t} u v_{t}-\tau(t) u_{x t} v_{x x}+\frac{2}{3} \tau_{t} u v_{x x x} \\
& +\tau(t) v_{x} u_{t x x}-\frac{5}{3} \tau_{t} v u_{x x x}-\tau(t) v u_{t x x x}+4 \tau_{t} u^{2} v_{x} \\
& -f_{t} v_{x} u-g(t) u_{x y} v_{x x}+\frac{1}{3} \tau_{t t} u v+f_{t} u_{x} v \\
& -f(t) u_{x x} v_{x x}-\frac{1}{6} f_{t} v_{x x x}-g(t) v u_{x x x y}+f(t) v u_{t x} \\
& +f(t) v_{t} u_{x}+\tau(t) u_{t} v_{x x x}-\frac{x}{18} \tau_{t t} v_{t}+\frac{y^{2}}{6 n} \tau_{t t t} u v_{x} \\
& +\frac{y}{2 n} g_{t t} u v_{x}-\frac{y^{2}}{6 n} \tau_{t t} v u_{x t}-\frac{1}{6} f_{t} v_{t}+2 x \tau_{t} u_{x} v_{x} u
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{1}{18} \tau_{t t} v_{x x}-\frac{y^{2}}{6 n} \tau_{t t t} u_{x} v+\frac{x}{3} n \tau_{t} v u_{y y}-\frac{y}{2 n} g_{t t} u_{x} v \\
& +\frac{x}{3} \tau_{t} v u_{x t}-\frac{y^{2}}{6} \tau_{t t} v u_{y y}-\frac{y}{2} g_{t} v u_{y y}+f(t) v n u_{y y} \\
& -10 \tau_{t} u u_{x} v+\frac{x}{3} \tau_{t t} u_{x} v-\frac{x}{3} \tau_{t t} v_{x} u+\frac{y^{2}}{36 n} \tau_{t t t} v_{t} \\
& +\frac{y^{2}}{36 n} \tau_{t t t} v_{x x x}+\frac{y}{12 n} g_{t t} v_{t}+\frac{y}{12 n} g_{t t} v_{x x x}+\frac{x}{3} \tau_{t} u_{x} v_{t} \\
& +\frac{x}{3} \tau_{t} u_{x} v_{x x x}+6 f(t) u_{x} v_{x} u+\frac{2 y}{3} \tau_{t} u_{y} v_{t} \\
& +\frac{2 y}{3} \tau_{t} u_{y} v_{x x x}-6 g(t) u_{x} u_{y} v+6 g(t) v_{x} u_{y} u \\
& -6 \tau(t) u_{t} u_{x} v+6 \tau(t) u_{t} v_{x} u-\frac{x}{3} \tau_{t} u_{x x} v_{x x} \\
& -\frac{2 y}{3} \tau_{t} u_{x y} v_{x x}-6 g(t) u_{x y} u v-6 \tau(t) u_{x t} u v \\
& +\frac{x}{3} \tau_{t} u_{x x x} v_{x}+\frac{2 y}{3} \tau_{t} u_{x x y} v_{x}-\frac{2 y}{3} \tau_{t} u_{x x x y} v \\
& -\frac{y}{2 n} g_{t} u_{t x} v-\frac{y^{2}}{6 n} \tau_{t t} v_{x} u_{x x x}-\frac{y}{2 n} g_{t} u_{x x x} v_{x} \\
& -\frac{y^{2}}{n} \tau_{t t} u_{x} v_{x} u-\frac{y^{2}}{6 n} \tau_{t t} u_{x} v_{t}-\frac{y^{2}}{6 n} \tau_{t t} u_{x} v_{x x x} \\
& -\frac{3 y}{n} g_{t} u_{x} v_{x} u-\frac{y}{2 n} g_{t} u_{x} v_{t}-\frac{y}{2 n} g_{t} u_{x} v_{x x x} \\
& -4 y \tau_{t} u_{y} u_{x} v+4 y \tau_{t} u_{y} v_{x} u+\frac{y^{2}}{6 n} \tau_{t t} u_{x x} v_{x x} \\
& +\frac{y}{2 n} g_{t} u_{x x} v_{x x}-4 y \tau_{t} u_{x y} u v+\frac{4}{3} \tau_{t} u_{x x} v_{x}, \\
& Y_{3}=\frac{1}{2} g_{t} v u_{x}+\frac{y}{12} g_{t t} v_{y}-\frac{y}{18} \tau_{t t t} v+\frac{y^{2}}{36} \tau_{t t t} v_{y} \\
& -\frac{1}{12} g_{t t} v+\frac{x}{3} n \tau_{t} u_{x} v_{y}+\frac{y}{3} \tau_{t t} u_{x} v+\frac{y^{2}}{6} \tau_{t t} u_{x y} v \\
& +\frac{2}{3} n \tau_{t} u v_{y}-\frac{x}{18} n \tau_{t t} v_{y}+n f(t) u_{x} v_{y}+g(t) n v_{y} u_{y} \\
& +n \tau(t) u_{t} v_{y}-\frac{4}{3} n \tau_{t} v u_{y}-n f(t) v u_{x y}-\tau(t) n v u_{t y} \\
& -\frac{y^{2}}{6} \tau_{t t} u_{x} v_{y}-\frac{y}{2} g_{t} u_{x} v_{y}+\frac{y}{2} g_{t} v u_{x y}-\frac{1}{6} n f_{t} v_{y} \\
& +\frac{2 y}{3} n \tau_{t} u_{y} v_{y}-\frac{x}{3} n \tau_{t} v u_{x y}-\frac{2 y}{3} \tau_{t} v n u_{y y} \\
& -g(t) v n u_{y y},
\end{aligned}
$$

$$
\begin{align*}
T_{3}= & -\tau(t) v u_{x t}-\tau_{t} v u_{x}+\frac{1}{18} \tau_{t t} v-\frac{x}{3} \tau_{t} v u_{x x} \\
& +\frac{y^{2}}{6 n} \tau_{t t} u_{x x} v+\frac{y}{2 n} g_{t} u_{x x} v-f(t) v u_{x x} \\
& -\frac{2 y}{3} \tau_{t} u_{x y} v-\tau(t) v u_{x y} \tag{37}
\end{align*}
$$

For the fourth symmetry, the two functions $\tau(t)$ and $n(t)$ are determined by the differential equation (19) and they have many explicit solutions. For simplicity, we take $\tau(t)=1$; then $n(t)=1+\tan ^{2} t$ and $e(t)=(-\tan t / 2)+C_{3}$. When $f(t)=$ $g(t)=0$, the corresponding Lie symmetry is

$$
\begin{equation*}
V=-\frac{y^{2}}{4} \frac{\partial}{\partial x}+y \tan t \frac{\partial}{\partial y}+\frac{\partial}{\partial t}+0 \frac{\partial}{\partial u}, \tag{38}
\end{equation*}
$$

and the components of the conservation laws are

$$
\begin{align*}
& X_{4}=-v u_{t x x x}-\frac{y^{2}}{4} u_{x x x} v_{x}-6 u v u_{x t}+\frac{y^{2}}{4} u_{x x} v_{x x} \\
&+6 u u_{t} v_{x}-6 v u_{t} u_{x}-C_{3} u_{t} v+v_{x} u_{t x x}+u_{t} v_{t} \\
&+\frac{\tan t}{2} u_{t} v-\frac{y^{2}}{4} u_{x} v_{x x x}-\frac{y^{2}}{4} u_{x} v_{t}-\frac{y^{2}}{4} u_{y y} v \\
&-\frac{y^{2}}{4} v u_{x t}+u_{t} v_{x x x}-\frac{3 y^{2}}{2} u_{x} v_{x} u+\frac{\tan ^{2} t}{2} y v u_{y} \\
&+y u_{y} v_{x x x} \tan t+y u_{y} v_{t} \tan t+y v_{x} u_{x x y} \tan t \\
&-6 y v u_{y} u_{x} \tan t-C_{3} y v u_{y} \tan t-\frac{\tan ^{2} t}{4} y^{2} v u_{y y} \\
&-u_{x t} v_{x x}-y v_{x x} u_{x y} \tan t-y v u_{x x x y} \tan t \\
&-6 y u v u_{x y} \tan t+6 y u u_{y} v_{x} \tan t, \\
& Y_{4}=-\frac{y^{2}}{4} u_{x} v_{y}-\frac{y^{2}}{4} v_{y} u_{x} \tan ^{2} t+y v_{y} u_{y} \tan t \\
&+y v_{y} u_{y} \tan ^{3} t+v_{y} u_{t}+v_{y} u_{t} \tan { }^{2} t+\frac{1}{2} v y u_{x} \\
&+\frac{y^{2}}{4} v u_{x y}+\frac{y^{2}}{4} v u_{x y} \tan ^{2} t-v u_{y} \tan t-v u_{y} \tan ^{3} t \\
&-v u_{t y}-v u_{t y} \tan ^{2} t+\frac{1}{2} v y u_{x} \tan { }^{2} t \\
&-y v \tan t u_{y y}-y v \tan ^{3} t u_{y y} \\
& T_{4}=\frac{y^{2}}{4} v u_{x x}-y v u_{x y} \tan t-v u_{x t} \tag{39}
\end{align*}
$$

We should mention that in the above components of the conservation laws for (1) and (28), $u$ is a solution of (1) and $v$ is a solution of the adjoint equation (28). Making use of the
explicit solutions of (28), local conservation laws for (1) can be obtained. For example, when $a(t)=0$ and $b(t)=0$ in (34),

$$
\begin{equation*}
v=k(t) y+l(t) \tag{40}
\end{equation*}
$$

where $k(t)$ and $l(t)$ are arbitrary functions, is an exact solution of (28). Substituting (40) into the above four conservation laws, we can obtain time-dependent and local conservation laws for (1). Here we take $\left(X_{4}, Y_{4}, T_{4}\right)$ as an illustrative example; when $v=k(t) y+l(t)$, the components of the conservation laws $\left(X_{4}, Y_{4}, T_{4}\right)$ become

$$
\begin{align*}
& \bar{X}_{4}=-C_{3} y^{2} u_{y} k(t) \tan t-C_{3} l(t) y u_{y} \tan t \\
& -6 k(t) y^{2} u_{y} u_{x} \tan t-6 l(t) y u_{y} u_{x} \tan t+l^{\prime}(t) u_{t} \\
& -l(t) u_{x x x t}-6 k(t) y^{2} u_{x y} u \tan t-\frac{y^{3}}{4} k(t) u_{y y} \\
& -\frac{y^{2}}{4} l^{\prime}(t) u_{x}-C_{3} l(t) u_{t}+\frac{1}{2} l(t) u_{t} \tan t-6 l(t) u_{t} u_{x} \\
& -6 l(t) u_{x t} u-\frac{y^{2}}{4} l(t) u_{y y}-k(t) y u_{x x x t}-\frac{y^{3}}{4} k^{\prime}(t) u_{x} \\
& +k^{\prime}(t) y u_{t}-\frac{y^{2}}{4} l(t) u_{x t}-\frac{y^{3}}{4} k(t) u_{x t} \\
& -y l(t) u_{x x x y} \tan t-k(t) y^{2} u_{x x x y} \tan t \\
& -\frac{y^{3}}{4} k(t) u_{y y} \tan ^{2} t+\frac{y}{2} k(t) u_{t} \tan t+l^{\prime}(t) y u_{y} \tan t \\
& -6 k(t) y u u_{x t}+\frac{y}{2} l(t) u_{y} \tan ^{2} t+\frac{y^{2}}{2} k(t) u_{y} \tan ^{2} t \\
& -6 k(t) y u_{t} u_{x}-C_{3} k(t) y u_{t}-\frac{y^{2} \tan ^{2} t}{4} l(t) u_{y y} \\
& +k^{\prime}(t) y^{2} u_{y} \tan t-6 l(t) y u_{x y} u \tan t, \\
& \bar{Y}_{4}=-l(t) y u_{y y} \tan ^{3} t-l(t) y u_{y y} \tan t-l(t) u_{y t} t+k(t) u_{t} \\
& -k(t) y^{2} \tan ^{3} t u_{y y}-k(t) y \tan ^{2} t u_{y t}+\frac{y}{2} l(t) u_{x} \tan ^{2} t \\
& +\frac{y^{2}}{4} l(t) u_{x y} \tan ^{2} t+\frac{y^{3}}{4} k(t) u_{x y}+k(t) u_{t} \tan ^{2} t \\
& +\frac{y^{2}}{4} k(t) u_{x}-l(t) u_{y t} \tan ^{2} t-y k(t) u_{y t}-l(t) u_{y} \tan ^{3} t \\
& -l(t) u_{y} \tan t+\frac{y^{2}}{4} l(t) u_{x y}+\frac{y}{2} l(t) u_{x} \\
& +\frac{y^{2}}{4} k(t) u_{x} \tan ^{2} t+\frac{y^{3}}{4} k(t) u_{x y} \tan ^{2} t \\
& -k(t) y^{2} u_{y y} \tan t, \\
& \bar{T}_{4}=\frac{1}{4}(k(t) y+l(t))\left(y^{2} u_{x x}-4 y u_{x y} \tan t-4 u_{x t}\right) . \tag{41}
\end{align*}
$$

These are local and explicit conservation laws of (1). Next we show that the above conservation laws $\left(\bar{X}_{4}, \bar{Y}_{4}, \bar{T}_{4}\right)$ are nontrivial:

$$
\begin{aligned}
D_{x}\left(\bar{X}_{4}\right) & +D_{y}\left(\bar{Y}_{4}\right)+D_{t}\left(\bar{T}_{4}\right) \\
= & -C_{3} y^{2} k(t) u_{x y} \tan t-l(t) u_{x x x x t}-l(t) u_{x t t} \\
& -k(t) y u_{x t t}-12 l(t) u_{x} u_{x t}-2 l(t) u_{y y} \tan ^{3} t \\
& +\frac{1}{2} l(t) u_{x} \tan ^{2} t-2 l(t) u_{y y} \tan t+\frac{1}{2} y k(t) u_{x} \\
& -k(t) y u_{x x x x t}-6 l(t) u u_{x x t}-6 l(t) u_{t} u_{x x} \\
& +\frac{1}{2} l(t) u_{x t} \tan t-C_{3} l(t) u_{x t}-l(t) u_{y y t} \tan ^{2} t \\
& -k(t) y u_{y y t}-6 y^{2} k(t) u u_{x x y} \tan t \\
& -12 y^{2} k(t) u_{x} u_{x y} \tan t-12 y l(t) u_{x} u_{x y} \tan t \\
& -6 l(t) y u u_{x x y} \tan t-6 k(t) y^{2} u_{y} u_{x x} \tan t
\end{aligned}
$$

$$
-6 l(t) y u_{y} u_{x x} \tan t-l(t) u_{y y t}-C_{3} l(t) y u_{x y} \tan t
$$

$$
+\frac{1}{2} l(t) u_{x}-k(t) y u_{y y t} \tan ^{2} t-l(t) y u_{y y y} \tan ^{3} t
$$

$$
+\frac{1}{2} y l(t) u_{x y} \tan ^{2} t-C_{3} k(t) y u_{x t}-6 y k(t) u_{t} u_{x x}
$$

$$
+\frac{1}{2} y^{2} k(t) u_{x y} \tan ^{2} t-l(t) y u_{x x x x y} \tan t
$$

$$
-12 k(t) y u_{x} u_{x t}-6 k(t) y u u_{x x t}+\frac{1}{2} y k(t) u_{x t} \tan t
$$

$$
-k(t) y^{2} u_{x x x x y} \tan t-k(t) y^{2} u_{x y t} \tan t
$$

$$
-l(t) y u_{x y t} \tan t-2 k(t) y u_{y y} \tan ^{3} t
$$

$$
+\frac{1}{2} y k(t) u_{x} \tan ^{2} t-2 k(t) y u_{y y} \tan t
$$

$$
-k(t) y^{2} u_{y y y} \tan t-l(t) y u_{y y y} \tan t
$$

$$
\begin{equation*}
-k(t) y^{2} u_{y y y} \tan ^{3} t \tag{42}
\end{equation*}
$$

Obviously, if $k(t), l(t)$ are not zero at the same time, $D_{x}\left(\bar{X}_{4}\right)+$ $D_{y}\left(\bar{Y}_{4}\right)+D_{t}\left(\bar{T}_{4}\right) \neq 0$. And we can easily check that

$$
\begin{align*}
& \left(D_{x}\left(X_{4}\right)+D_{y}\left(Y_{4}\right)\right. \\
& \left.\quad+D_{t}\left(T_{4}\right)\right)\left.\right|_{u_{x x x x}=-u_{x t}-6 u_{x}^{2}-6 u u_{x x}-e(t) u_{x}-n(t) u_{y y}} \equiv 0 \tag{43}
\end{align*}
$$

## 4. Symmetry Reductions and New Exact Solutions of (1)

In Section 2, we obtain the Lie symmetries of (1). In this section, we will investigate the symmetry reductions and exact solutions for the equation. Using the obtained symmetries (3), similarity variables and symmetry reductions can be found by solving the corresponding characteristic equation:

$$
\begin{equation*}
\frac{d x}{\xi}=\frac{d y}{\eta}=\frac{d t}{\tau}=\frac{d u}{\phi} \tag{44}
\end{equation*}
$$

For the four different cases, we determine the following symmetry reductions and exact solutions of (1).
4.1. For the Symmetry in Case 1, Where e $(t)$ and $n(t)(n(t) \neq 0)$ Are Arbitrary Functions.
(i) When $g(t)=0, f(t) \neq 0$, we can obtain

$$
\begin{equation*}
u=\frac{f_{t} x}{6 f}+\Omega(y, t) \tag{45}
\end{equation*}
$$

and $\Omega(y, t)$ is a solution of the following reduction equation:

$$
\begin{equation*}
\frac{f_{t t}}{6 f}+\frac{e f_{t}}{6 f}+n \Omega_{y y}=0 \tag{46}
\end{equation*}
$$

From the above equation, we can obtain an algebraically explicit analytical solution for (1):

$$
\begin{equation*}
u=\frac{f_{t} x}{6 f}-\frac{f_{t t}+e f_{t}}{12 n f} y^{2}+F_{1}(t) y+F_{2}(t) \tag{47}
\end{equation*}
$$

where $F_{1}(t)$ and $F_{2}(t)$ are arbitrary functions of $t$.
(ii) When $f(t)=0, g(t)=t$, the corresponding symmetry is

$$
\begin{equation*}
V=-\frac{y}{2 n} \frac{\partial}{\partial x}+t \frac{\partial}{\partial y}+0 \frac{\partial}{\partial t}+\frac{n_{t}}{12 n^{2}} y \frac{\partial}{\partial u} . \tag{48}
\end{equation*}
$$

By the characteristic equations of the symmetry, we have $u=\Omega(\theta, t), \theta=y^{2} / 2+2 n x t$. Substituting it into (1), we get a symmetry reduction of (1):

$$
\begin{align*}
\Omega_{\theta t} & +\frac{\theta}{t} \Omega_{\theta \theta}+12 n t\left(\Omega_{\theta} \Omega\right)_{\theta}+8 n^{3} t^{3} \Omega_{\theta \theta \theta \theta} \\
& +\left(\frac{3}{2 t}-\frac{n_{t}}{n}+e(t)\right) \Omega_{\theta}+\frac{n_{t}^{2}}{6 n^{3} t}-\frac{n_{t t}}{12 n^{2} t}  \tag{49}\\
& -\frac{e(t) n_{t}}{12 n^{2} t}=0
\end{align*}
$$

If the coefficient functions $e(t)=0, n(t)=$ Const., the obtained symmetry reduction can be simplified to

$$
\begin{equation*}
\Omega_{\theta t}+\frac{\theta}{t} \Omega_{\theta \theta}+\frac{3}{2 t} \Omega_{\theta}+12 n t\left(\Omega_{\theta} \Omega\right)_{\theta}+8 n^{3} t^{3} \Omega_{\theta \theta \theta \theta}=0 \tag{50}
\end{equation*}
$$

Integrating (50) with respect to $\theta$ and taking the constant of integration to zero, we get the following equation:

$$
\begin{equation*}
\Omega_{t}+12 n t \Omega_{\theta} \Omega+8 n^{3} t^{3} \Omega_{\theta \theta \theta}+\frac{\theta}{t} \Omega_{\theta}+\frac{1}{2 t} \Omega=0 \tag{51}
\end{equation*}
$$

Equation (51) is the $(1+1)$-dimensional generalized KdV equation with variable coefficients. To the best of our knowledge, exact solutions of (51) have not been studied up to now. Solving (51) by the method in [25], we can get the following solutions for (1):

$$
\begin{gather*}
u=\Omega(\theta, t)=\frac{\theta}{24 n t^{2}}+\frac{M_{3}}{24 n t M_{1}} \\
-\frac{8 n^{2} M_{1}^{2} c_{2}}{3 t}-\frac{8 n^{2} M_{1}^{2} c_{4}}{t} P^{2}(\varphi)  \tag{52}\\
\varphi=M_{1} \theta t^{-3 / 2}+M_{3} t^{-1 / 2}+M_{2}
\end{gather*}
$$

where $M_{1}, M_{2}$, and $M_{3}$ are arbitrary constants and the function $P(\varphi)$ satisfies

$$
\begin{equation*}
P^{\prime 2}=c_{0}+c_{2} P^{2}+c_{4} P^{4} \tag{53}
\end{equation*}
$$

where $c_{0}, c_{2}$, and $c_{4}$ are constants; solutions of (53) have been given in [26]. By means of the solutions of (53), plenty of solutions for (1) can be obtained; for example,

$$
\begin{align*}
u_{1}= & \frac{y^{2} / 2+2 n x t}{24 n t^{2}}+\frac{M_{3}}{24 n t M_{1}} \\
& -\frac{8 n^{2} M_{1}^{2}\left(-k^{2}-1\right)}{3 t}-\frac{8 n^{2} M_{1}^{2} k^{2} \mathrm{sn}^{2}(\varphi)}{t}, \\
& \left(c_{0}=1, c_{2}=-1-k^{2}, c_{4}=k^{2}\right), \\
u_{2}= & \frac{y^{2} / 2+2 n x t}{24 n t^{2}}+\frac{M_{3}}{24 n t M_{1}} \\
& -\frac{8 n^{2} M_{1}^{2}\left(-k^{2}-1\right)}{3 t}-\frac{8 n^{2} M_{1}^{2} n^{2}(\varphi)}{t}, \\
u_{3}= & \frac{y^{2} / 2+2 n x t}{24 n t^{2}}+\frac{M_{3}}{24 n t M_{1}}-\frac{8 n^{2} M_{1}^{2} c_{2}}{3 t} \\
& +\frac{8 n^{2} M_{1}^{2} c_{2} \operatorname{sech}^{2}(\varphi)}{t}, \quad\left(c_{0}^{2}=0, c_{2}>0, c_{4}<0\right), \\
u_{4}= & \frac{y^{2} / 2+2 n x t}{24 n t^{2}}+\frac{M_{3}}{24 n t M_{1}}-\frac{8 n^{2} M_{1}^{2} c_{2}}{3 t} \\
& +\frac{4 n^{2} M_{1}^{2} c_{2} \tanh ^{2}(\varphi)}{t}, \quad\left(c_{0}=\frac{c_{2}^{2}}{4 c_{4}}, c_{2}<0, c_{4}>0\right),
\end{align*}
$$

where $k(0<k<1)$ denotes the modulus of the Jacobi elliptic function.
(iii) When $e(t)=0, n(t)=(t-m)^{p} C_{1}, p \neq 0, C_{1} \neq 0, f(t)=$ $M_{0}$, and $g(t)=1$, we can get

$$
\begin{equation*}
u=\Omega(\theta, t), \quad \theta=x-M_{0} y . \tag{55}
\end{equation*}
$$

And $\Omega(\theta, t)$ satisfies the following reduction equation:

$$
\begin{align*}
& \Omega_{\theta t}+6\left(\Omega_{\theta}^{2}+\Omega \Omega_{\theta \theta}\right)+\Omega_{\theta \theta \theta \theta}  \tag{56}\\
& \quad+M_{0}^{2} C_{1}(t-m)^{p} \Omega_{\theta \theta}=0
\end{align*}
$$

The above equation can be integrated by $\theta$ and, when we take the constant of integration to zero, we get a reduced reduction equation:

$$
\begin{equation*}
\Omega_{t}+6 \Omega \Omega_{\theta}+\Omega_{\theta \theta \theta}+M_{0}^{2} C_{1}(t-m)^{p} \Omega_{\theta}=0 \tag{57}
\end{equation*}
$$

Equation (57) is variable coefficient KdV equation and soliton-like solutions have been obtained in [27]. By means of the known solutions, many explicit solutions of (1) can be obtained. For example,

$$
\begin{gather*}
u_{1}=k_{1}+2 c k_{4}^{2} \operatorname{sech}^{2}(\sqrt{c} \varphi), \\
\varphi=k_{4}\left(x-M_{0} y\right)-6 k_{1} k_{4} t-4 c k_{4}^{3} t \\
-\frac{M_{0}^{2} C_{1} k_{4}}{p+1}(t-m)^{p+1}, \\
u_{2}=k_{1}-2 c k_{4}^{2} \tanh ^{2}(\varphi),  \tag{58}\\
\varphi=k_{4}\left(x-M_{0} y\right)-6 k_{1} k_{4} t+8 k_{4}^{3} t \\
-\frac{M_{0}^{2} C_{1} k_{4}}{p+1}(t-m)^{p+1},
\end{gather*}
$$

where $k_{1}, k_{4}$, and $c$ are constants.
(iv) When $e(t) \neq 0$ and $n(t)=N_{0} \exp \left(\left(\int\left(e_{t}-2 e^{2}\right) / e\right) d t\right)$, $f(t)=N_{1}, g(t)=1$. By the corresponding characteristic equation of the symmetry, we have

$$
\begin{equation*}
u=\Omega(\theta, t), \quad \theta=x-N_{1} y . \tag{59}
\end{equation*}
$$

Substituting it into (1), we get the following symmetry reduction of (1):

$$
\begin{align*}
& \Omega_{\theta t}+6\left(\Omega_{\theta}^{2}+\Omega \Omega_{\theta \theta}\right)+\Omega_{\theta \theta \theta \theta}+e(t) \Omega_{\theta} \\
& \quad+N_{1}^{2} N_{0} \exp \left(\int \frac{e_{t}-2 e^{2}}{e} d t\right) \Omega_{\theta \theta} \tag{60}
\end{align*}
$$

Integrating the above equation with respect to $\theta$ and taking the constant of integration to zero, the obtained reduction equation becomes

$$
\begin{align*}
& \Omega_{t}+6 \Omega \Omega_{\theta}+\Omega_{\theta \theta \theta}+e(t) \Omega \\
& \quad+N_{1}^{2} N_{0} \exp \left(\int \frac{e_{t}-2 e^{2}}{e} d t\right) \Omega_{\theta} \tag{61}
\end{align*}
$$

Equation (61) is a variable coefficient $K d V$ equation $[28,29]$.
4.2. For the Symmetry in Case $2, e(t)=0, n(t)=(t-m)^{p} C_{1}, p \neq 0$, $C_{1} \neq 0$. When $f(t)=g(t)=0, m=0, p=C_{2}=2 / 3$, then
$n(t)=C_{1} t^{2 / 3}$, and $C_{1} \neq 0$; the corresponding symmetry of (1) is

$$
\begin{equation*}
V=\frac{x}{3} \frac{\partial}{\partial x}+y \frac{\partial}{\partial y}+t \frac{\partial}{\partial t}-\frac{2}{3} u \frac{\partial}{\partial u} . \tag{62}
\end{equation*}
$$

By the characteristic equations of the symmetry, we can get the explicit solutions for (1)

$$
\begin{equation*}
u=\Omega(\theta, \delta) t^{-2 / 3}, \quad \theta=\frac{x^{3}}{t}, \delta=\frac{y}{t} \tag{63}
\end{equation*}
$$

where the function $\Omega(\theta, \delta)$ satisfies the following reduction equation:

$$
\begin{align*}
- & 3 \theta^{5 / 3} \Omega_{\theta \theta}-3 \theta^{2 / 3} \delta \Omega_{\theta \delta}-5 \theta^{2 / 3} \Omega_{\theta} \\
& +54 \theta^{4 / 3}\left(\Omega_{\theta}^{2}+\Omega_{\theta \theta} \Omega\right)+36 \theta^{1 / 3} \Omega_{\theta} \Omega  \tag{64}\\
& +81 \theta^{8 / 3} \Omega_{\theta \theta \theta \theta}+324 \theta^{5 / 3} \Omega_{\theta \theta \theta}+180 \theta^{2 / 3} \Omega_{\theta \theta} \\
& +C_{1} \Omega_{\delta \delta}=0
\end{align*}
$$

Equation (64) is difficult to solve and we will study its exact solutions in a future paper.
4.3. For the Symmetry in Case 3, $e(t)=0, n(t)=$ Const., and $\tau(t) \neq 0$. When $f(t)=0, g(t)=0$, the corresponding symmetry is

$$
\begin{align*}
V= & \left(\frac{\tau_{t}}{3} x-\frac{\tau_{t t}}{6 n} y^{2}\right) \frac{\partial}{\partial x}+\frac{2}{3} \tau_{t} y \frac{\partial}{\partial y}+\tau(t) \frac{\partial}{\partial t}  \tag{65}\\
& +\left(-\frac{2 \tau_{t}}{3} u+\frac{\tau_{t t}}{18} x-\frac{\tau_{t t t}}{36 n} y^{2}\right) \frac{\partial}{\partial u} .
\end{align*}
$$

By the characteristic equation of the symmetry, we have

$$
\begin{array}{r}
u=\frac{1}{18 \tau} x \tau_{t}-\frac{1}{36 n \tau} y^{2} \tau_{t t}+\frac{1}{54 n \tau^{2}} y^{2} \tau_{t}^{2}+\Omega(\theta, \delta) \tau^{-2 / 3}, \\
\theta=x \tau^{-1 / 3}+\frac{1}{6 n} y^{2} \tau_{t} \tau^{-4 / 3}, \delta=y \tau^{-2 / 3} \tag{66}
\end{array}
$$

Substituting it into (1), we get a symmetry reduction of (1):

$$
\begin{equation*}
6 \Omega_{\theta}^{2}+6 \Omega_{\theta \theta} \Omega+\Omega_{\theta \theta \theta \theta}+n \Omega_{\delta \delta}=0 \tag{67}
\end{equation*}
$$

Equation (67) is the special case of $(2+1)$-dimensional Boussinesq equation and exact solutions of (67) have been studied by Chen and Zhang in [30] (with $a=0, b=0, r=$ $-3 / n$, and $s=-1 / n$ ). With the help of the known solutions in [30], many explicit solutions of (1) can be obtained. We
list the following soliton solutions $\left(u_{1}-u_{4}\right)$ and Jacobi elliptic function solutions $\left(u_{5}-u_{17}\right)$ :

$$
\begin{aligned}
& u_{1}=\left(\frac{-n \omega^{2}}{6 \alpha^{2}}+\frac{4}{3} \alpha^{2}\right) \tau^{-2 / 3}-2 \alpha^{2} \tau^{-2 / 3} \tanh ^{2}(\varphi) \\
& +\frac{\tau_{t} x}{18 \tau}-\frac{\tau_{t t} y^{2}}{36 n \tau}+\frac{\tau_{t}^{2} y^{2}}{54 n \tau^{2}}, \\
& u_{2}=\left(\frac{-n \omega^{2}}{6 \alpha^{2}}-\frac{2}{3} \alpha^{2}\right) \tau^{-2 / 3}+2 \alpha^{2} \tau^{-2 / 3} \operatorname{sech}^{2}(\varphi) \\
& +\frac{\tau_{t} x}{18 \tau}-\frac{\tau_{t t} y^{2}}{36 n \tau}+\frac{\tau_{t}^{2} y^{2}}{54 n \tau^{2}}, \\
& u_{3}=\left(\frac{-n \omega^{2}}{6 \alpha^{2}}+\frac{1}{3} \alpha^{2}\right) \tau^{-2 / 3} \\
& -\frac{\alpha^{2}}{2} \tau^{-2 / 3} \frac{\varepsilon \tanh ^{4}(\varphi)+\beta(1+\operatorname{sech}(\varphi))^{4}}{\tanh ^{2}(\varphi)(1+\operatorname{sech}(\varphi))^{2}} \\
& +\frac{\tau_{t} x}{18 \tau}-\frac{\tau_{t t} y^{2}}{36 n \tau}+\frac{\tau_{t}^{2} y^{2}}{54 n \tau^{2}}, \\
& u_{4}=\left(\frac{-n \omega^{2}}{6 \alpha^{2}}-\frac{2}{3} \alpha^{2}\right) \tau^{-2 / 3}-2 \alpha^{2} \tau^{-2 / 3} \frac{\operatorname{sech}^{2}(\varphi)}{\tanh ^{2}(\varphi)} \\
& +\frac{\tau_{t} x}{18 \tau}-\frac{\tau_{t t} y^{2}}{36 n \tau}+\frac{\tau_{t}^{2} y^{2}}{54 n \tau^{2}}, \\
& u_{5}=\left(\frac{-n \omega^{2}}{6 \alpha^{2}}+\frac{2}{3} \alpha^{2}+\frac{2}{3} \alpha^{2} m^{2}\right) \tau^{-2 / 3}-2 \alpha^{2} \tau^{-2 / 3} \\
& \times \frac{\varepsilon+\beta m^{2} \mathrm{sn}^{4}(\varphi)}{\mathrm{sn}^{2}(\varphi)}+\frac{\tau_{t} x}{18 \tau}-\frac{\tau_{t t} y^{2}}{36 n \tau}+\frac{\tau_{t}^{2} y^{2}}{54 n \tau^{2}}, \\
& u_{6}=\left(\frac{-n \omega^{2}}{6 \alpha^{2}}+\frac{2}{3} \alpha^{2}+\frac{2}{3} \alpha^{2} m^{2}\right) \tau^{-2 / 3}-2 \alpha^{2} \tau^{-2 / 3} \\
& \times \frac{\varepsilon \operatorname{dn}^{4}(\varphi)+\beta m^{2} \mathrm{cn}^{4}(\varphi)}{\mathrm{cn}^{2}(\varphi) \mathrm{dn}^{2}(\varphi)}+\frac{\tau_{t} x}{18 \tau}-\frac{\tau_{t t} y^{2}}{36 n \tau}+\frac{\tau_{t}^{2} y^{2}}{54 n \tau^{2}}, \\
& u_{7}=\left(\frac{-n \omega^{2}}{6 \alpha^{2}}-\frac{4}{3} \alpha^{2} m^{2}+\frac{2}{3} \alpha^{2}\right) \tau^{-2 / 3}-2 \alpha^{2} \tau^{-2 / 3} \\
& \times \frac{\varepsilon\left(1-m^{2}\right)-\beta m^{2} \mathrm{cn}^{4}(\varphi)}{\mathrm{cn}^{2}(\varphi)}+\frac{\tau_{t} x}{18 \tau}-\frac{\tau_{t t} y^{2}}{36 n \tau}+\frac{\tau_{t}^{2} y^{2}}{54 n \tau^{2}}, \\
& u_{8}=\left(\frac{-n \omega^{2}}{6 \alpha^{2}}-\frac{4}{3} \alpha^{2}+\frac{2}{3} \alpha^{2} m^{2}\right) \tau^{-2 / 3}+2 \alpha^{2} \tau^{-2 / 3} \\
& \times \frac{\varepsilon\left(1-m^{2}\right)+\beta \mathrm{dn}^{4}(\varphi)}{\operatorname{dn}^{2}(\varphi)}+\frac{\tau_{t} x}{18 \tau}-\frac{\tau_{t t} y^{2}}{36 n \tau}+\frac{\tau_{t}^{2} y^{2}}{54 n \tau^{2}},
\end{aligned}
$$

$$
\begin{aligned}
& u_{9}=\left(\frac{-n \omega^{2}}{6 \alpha^{2}}-\frac{4}{3} \alpha^{2}+\frac{2}{3} \alpha^{2} m^{2}\right) \tau^{-2 / 3} \\
& -2 \alpha^{2} \tau^{-2 / 3} \frac{\varepsilon\left(1-m^{2}\right) \mathrm{sn}^{4}(\varphi)+\beta \mathrm{cn}^{4}(\varphi)}{\operatorname{sn}^{2}(\varphi) \mathrm{cn}^{2}(\varphi)} \\
& +\frac{\tau_{t} x}{18 \tau}-\frac{\tau_{t t} y^{2}}{36 n \tau}+\frac{\tau_{t}^{2} y^{2}}{54 n \tau^{2}}, \\
& u_{10}=\left(\frac{-n \omega^{2}}{6 \alpha^{2}}-\frac{4}{3} \alpha^{2} m^{2}+\frac{2}{3} \alpha^{2}\right) \tau^{-2 / 3} \\
& -2 \alpha^{2} \tau^{-2 / 3} \frac{\varepsilon \operatorname{dn}^{4}(\varphi)-\beta m^{2}\left(1-m^{2}\right) \mathrm{sn}^{4}(\varphi)}{\operatorname{sn}^{2}(\varphi) \mathrm{dn}^{2}(\varphi)} \\
& +\frac{\tau_{t} x}{18 \tau}-\frac{\tau_{t t} y^{2}}{36 n \tau}+\frac{\tau_{t}^{2} y^{2}}{54 n \tau^{2}}, \\
& u_{11}=\left(\frac{-n \omega^{2}}{6 \alpha^{2}}-\frac{1}{3} \alpha^{2}+\frac{2}{3} \alpha^{2} m^{2}\right) \tau^{-2 / 3} \\
& -\frac{\alpha^{2}}{2} \tau^{-2 / 3} \frac{\varepsilon \mathrm{sn}^{4}(\varphi)+\beta(1 \pm \mathrm{cn}(\varphi))^{4}}{\operatorname{sn}^{2}(\varphi)(1 \pm \mathrm{cn}(\varphi))^{2}} \\
& +\frac{\tau_{t} x}{18 \tau}-\frac{\tau_{t t} y^{2}}{36 n \tau}+\frac{\tau_{t}^{2} y^{2}}{54 n \tau^{2}}, \\
& u_{12}=\left(\frac{-n \omega^{2}}{6 \alpha^{2}}-\frac{1}{3} \alpha^{2}+\frac{2}{3} \alpha^{2} m^{2}\right) \tau^{-2 / 3} \\
& -\frac{\alpha^{2}}{2} \tau^{-2 / 3} \frac{\varepsilon \mathrm{cn}^{4}(\varphi)+\beta\left(\sqrt{1-m^{2}} \operatorname{sn}(\varphi) \pm \operatorname{dn}(\varphi)\right)^{4}}{\mathrm{cn}^{2}(\varphi)\left(\sqrt{1-m^{2}} \operatorname{sn}(\varphi) \pm \operatorname{dn}(\varphi)\right)^{2}} \\
& +\frac{\tau_{t} x}{18 \tau}-\frac{\tau_{t t} y^{2}}{36 n \tau}+\frac{\tau_{t}^{2} y^{2}}{54 n \tau^{2}}, \\
& u_{13}=\left(\frac{-n \omega^{2}}{6 \alpha^{2}}-\frac{1}{3} \alpha^{2}-\frac{1}{3} \alpha^{2} m^{2}\right) \tau^{-2 / 3} \\
& +\frac{\alpha^{2}}{2}\left(1-m^{2}\right) \tau^{-2 / 3} \frac{\varepsilon \operatorname{dn}^{4}(\varphi)+\beta(1 \pm m \operatorname{sn}(\varphi))^{4}}{\operatorname{dn}^{2}(\varphi)(1 \pm m \operatorname{sn}(\varphi))^{2}} \\
& +\frac{\tau_{t} x}{18 \tau}-\frac{\tau_{t t} y^{2}}{36 n \tau}+\frac{\tau_{t}^{2} y^{2}}{54 n \tau^{2}}, \\
& u_{14}=\left(\frac{-n \omega^{2}}{6 \alpha^{2}}-\frac{1}{3} \alpha^{2}-\frac{1}{3} \alpha^{2} m^{2}\right) \tau^{-2 / 3} \\
& -\frac{\alpha^{2}}{2}\left(1-m^{2}\right) \tau^{-2 / 3} \frac{\varepsilon \mathrm{cn}^{4}(\varphi)+\beta(1 \pm \operatorname{sn}(\varphi))^{4}}{\mathrm{cn}^{2}(\varphi)(1 \pm \operatorname{sn}(\varphi))^{2}} \\
& +\frac{\tau_{t} x}{18 \tau}-\frac{\tau_{t t} y^{2}}{36 n \tau}+\frac{\tau_{t}^{2} y^{2}}{54 n \tau^{2}}, \\
& u_{15}=\left(\frac{-n \omega^{2}}{6 \alpha^{2}}-\frac{1}{3} \alpha^{2}-\frac{1}{3} \alpha^{2} m^{2}\right) \tau^{-2 / 3}
\end{aligned}
$$

$$
+\frac{\alpha^{2}}{2} \tau^{-2 / 3} \frac{\varepsilon\left(1-m^{2}\right)^{2}+\beta(m \mathrm{cn}(\varphi) \pm \operatorname{dn}(\varphi))^{4}}{(m \operatorname{cn}(\varphi) \pm \operatorname{dn}(\varphi))^{2}}
$$

$$
+\frac{\tau_{t} x}{18 \tau}-\frac{\tau_{t t} y^{2}}{36 n \tau}+\frac{\tau_{t}^{2} y^{2}}{54 n \tau^{2}}
$$

$$
u_{16}=\left(\frac{-n \omega^{2}}{6 \alpha^{2}}-\frac{1}{3} \alpha^{2}-\frac{2}{3} \alpha^{2} m^{2}\right) \tau^{-2 / 3}
$$

$$
-\frac{\alpha^{2}}{2} \tau^{-2 / 3} \frac{\varepsilon\left(1-m^{2}\right)^{2} \operatorname{sn}^{4}(\varphi)+\beta(\operatorname{dn}(\varphi) \pm \mathrm{cn}(\varphi))^{4}}{\operatorname{sn}^{2}(\varphi)(\operatorname{dn}(\varphi) \pm \mathrm{cn}(\varphi))^{2}}
$$

$$
+\frac{\tau_{t} x}{18 \tau}-\frac{\tau_{t t} y^{2}}{36 n \tau}+\frac{\tau_{t}^{2} y^{2}}{54 n \tau^{2}}
$$

$$
u_{17}=\left(\frac{-n \omega^{2}}{6 \alpha^{2}}-\frac{1}{3} \alpha^{2} m^{2}+\frac{2}{3} \alpha^{2}\right) \tau^{-2 / 3}
$$

$$
-\frac{\alpha^{2}}{2} \tau^{-2 / 3} \frac{\varepsilon m^{4} \mathrm{cn}^{4}(\varphi)+\beta\left(\sqrt{1-m^{2}} \pm \operatorname{dn}(\varphi)\right)^{4}}{\mathrm{cn}^{2}(\varphi)\left(\sqrt{1-m^{2}} \pm \operatorname{dn}(\varphi)\right)^{2}}
$$

$$
\begin{equation*}
+\frac{\tau_{t} x}{18 \tau}-\frac{\tau_{t t} y^{2}}{36 n \tau}+\frac{\tau_{t}^{2} y^{2}}{54 n \tau^{2}} \tag{68}
\end{equation*}
$$

where $\varphi=\alpha\left(x \tau^{-1 / 3}+(1 / 6 n) y^{2} \tau_{t} \tau^{-4 / 3}\right)+\omega\left(y \tau^{-2 / 3}\right), \alpha$ and $\omega$ are constants, $k(0<k<1)$ denotes the modulus of the Jacobi elliptic function, and $\varepsilon$ and $\beta$ are arbitrary elements of $\{0,1\}$. We should mention that the soliton solution $u_{1}$ is the limit of $u_{5}$ when $m \rightarrow 1, \varepsilon=0, \beta=1$. The solutions $u_{2}, u_{3}$, and $u_{4}$ are the limit of $u_{7}, u_{11}$, and $u_{9}$, respectively, when $m \rightarrow 1$, $\beta=1$.
4.4. For the Symmetry in Case $4, e(t)=-n_{t} / 4 n+C_{3} / \tau(t), n(t)$, and $\tau(t)$ Satisfy (19). For simplicity, we take $f(t)=g(t)=0$, $\tau(t)=1$; then $n(t)=1+\tan ^{2} t$ and $e(t)=-\tan t / 2+C_{3}$. Solving the corresponding characteristic equation, we get

$$
\begin{equation*}
u=\Omega(\theta, \delta), \quad \theta=x+\frac{y^{2}}{4} \sin t \cos t, \delta=y \cos t \tag{69}
\end{equation*}
$$

Substituting it into (1), we get a symmetry reduction of (1):

$$
\begin{equation*}
\frac{\delta^{2}}{4} \Omega_{\theta \theta}+6 \Omega_{\theta \theta} \Omega+6 \Omega_{\theta}^{2}+\Omega_{\theta \theta \theta \theta}+C_{3} \Omega_{\theta}+\Omega_{\delta \delta}=0 \tag{70}
\end{equation*}
$$

Obviously, $\Omega=-\left(C_{3} / 6\right) \theta+N_{1} \delta+N_{2}$ is a solution of (70). From that, we can get an algebraically explicit analytical solution for (1) as follows:

$$
\begin{equation*}
u=-\frac{C_{3}}{6}\left(x+\frac{y^{2}}{4} \sin t \cos t\right)+N_{1} y \cos t+N_{2} \tag{71}
\end{equation*}
$$

where $N_{1}$ and $N_{2}$ are integral constants. And, if $C_{3}=0$, (70) becomes the following $(2+1)$-dimensional variable coefficient Boussinesq equation:

$$
\begin{equation*}
\frac{\delta^{2}}{4} \Omega_{\theta \theta}+6 \Omega_{\theta \theta} \Omega+6 \Omega_{\theta}^{2}+\Omega_{\theta \theta \theta \theta}+\Omega_{\delta \delta}=0 \tag{72}
\end{equation*}
$$

Remark 3. To the best of our knowledge, the symmetry reductions obtained in this paper have not been reported in the existent literature, so they are completely new. The exact solutions of (1) obtained here are all different from the known solutions and they are also new. All the solutions and conservation laws obtained in this paper for (1) have been checked by Maple software.

## 5. Conclusions

In summary, by performing Lie symmetry analysis to (1), four cases of geometric symmetries are obtained when the coefficient functions satisfy four different constraint conditions. According to the relationship between symmetry and conservation laws given by Ibragimov, many explicit and nontrivial conservation laws, which includes arbitrary functions of $t$, are derived. These conservation laws may be useful for the explanation of some practical physical problems. Using the associated vector fields of the obtained symmetry, ( 1 ) is reduced to $(1+1)$-dimensional nonlinear partial differential equations including different types of variable coefficient KdV equation (see (51), (57), and (61)), special case of $(2+1)$-dimensional Boussinesq equation (see (67) and (72)), and other reduction equations (see (64) and (70)). Many new explicit solutions of (1) have been derived by solving the reduction equations. These solutions, including soliton solutions, Jacobi doubly periodic solutions, and algebraically explicit analytical solutions, can make one discuss the behavior of solutions and also provide mathematical foundation for the explanation of some interesting physical phenomena.

## Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Nonlinearly Self-Adjoint, Conservation Laws and Solutions for a Forced BBM Equation 

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#### Abstract

We study a forced Benjamin-Bona-Mahony (BBM) equation. We prove that the equation is not weak self-adjoint; however, it is nonlinearly self-adjoint. By using a general theorem on conservation laws due to Nail Ibragimov and the symmetry generators, we find conservation laws for these partial differential equations without classical Lagrangians. We also present some exact solutions for a special case of the equation.


## 1. Introduction

In a recent paper [1], Eloe and Usman have considered the damped externally excited Benjamin-Bona-Mahony (BBM) type equation given by

$$
\begin{equation*}
u_{t}+u_{x}+2 b u u_{x}-c u_{x x}-d u-a u_{x x t}=\eta \cos k(x+\lambda t) \tag{1}
\end{equation*}
$$

where $c$ and $d$ are nonnegative constants that are proportional to the strength of the damping effect. Equation (1) was introduced to model long waves in nonlinear dispersive systems. Some special cases of (1) are studied in [2, 3]. If $b=1 / 2$ and $a=1, c=0, d=0$, and $\eta=0$, then (1) reduces to the celebrated Benjamin-Bona-Mahony (BBM) equation

$$
\begin{equation*}
u_{t}+u_{x}+u u_{x}-u_{x x t}=0 \tag{2}
\end{equation*}
$$

The well-known BBM equation (2) was derived in [4] for moderately long wave equations in nonlinear dispersive systems. The authors derived three conservation laws for (2) and also considered the forcing equation. In [5], it was proved that these conservation laws are the only conservation laws admitted by the BBM equation. In [6], a family of BBM equations with strong nonlinear dispersive term was
considered from the point of view of symmetry analysis. The symmetry reductions were derived from the optimal system of subalgebras and lead to systems of ordinary differential equations. For special values of the parameters of this equation, many exact solutions are expressed by various single and combined nondegenerative Jacobi elliptic function solutions and their degenerative solutions (soliton, kink, and compactons). In [7], nonlocal symmetries of a family of Benjamin-Bona-Mahony-Burgers equations were studied. In [8] for a family of Benjamin-Bona-Mahony equations with strong nonlinear dispersion, the subclass of equations which are self-adjoint was determined and some nontrivial conservation laws were derived. In [9], da Silva and Freire showed that the BBM equation is strictly self-adjoint and a conservation law obtained from the scaling invariance was established.

In [1], the authors have obtained an analytic steady state solution of (1) and they have studied properties of some travelling wave solutions using a perturbation method.

In [10], the first author of this paper introduced the definition of weak self-adjointness and showed that the substitution $v=h(u)$ can be replaced with a more general substitution, where $h$ involves not only the variable $u$ but also the independent variables $h=h(x, t, u)$. In [11], Ibragimov
pointed out that, in constructing conservation laws, it is only important that $v$ does not vanish identically and introduced the definition of nonlinearly self-adjoint equation; that is, the substitution $v=h(u)$ can be replaced with a more general substitution, where $h$ involves not only the variable $u$ but also its derivatives as well as the independent variables; that is, $v=h\left(x, t, u, u_{t}, u_{x}, \ldots\right)$.

In this paper, we consider a generalization of the damped externally excited Benjamin-Bona-Mahony type equation (1), that is, the forced BMM type equation

$$
\begin{equation*}
u_{t}+u_{x}+2 b u u_{x}-c u_{x x}-d u-a u_{x x t}=f(x, t) \tag{3}
\end{equation*}
$$

where $c$ and $d$ are nonnegative constants that are proportional to the strength of the damping effect and $f(x, t)$ is an arbitrary function of the variables $x$ and $t$.

The aim of this paper is to prove that (3) is nonlinearly self-adjoint. We determine, by using the Lie generators of (3) and the notation and techniques of [12], some nontrivial conservation laws for (3). Finally, we present some exact solutions for a special case of (3).

## 2. Self-Adjoint and Nonlinearly Self-Adjoint Equations

Consider an $s$ th-order partial differential equation

$$
\begin{equation*}
F\left(x, u, u_{(1)}, \ldots, u_{(s)}\right)=0 \tag{4}
\end{equation*}
$$

with independent variables $x=\left(x^{1}, \ldots, x^{n}\right)$ and a dependent variable $u$, where $u_{(1)}=\left\{u_{i}\right\}, u_{(2)}=\left\{u_{i j}\right\}, \ldots$, denote the sets of the partial derivatives of the first, second, and so forth orders, $u_{i}=\partial u / \partial x^{i}, u_{i j}=\partial^{2} u / \partial x^{i} \partial x^{j}$. The adjoint equation to $(4)$ is

$$
\begin{equation*}
F^{*}\left(x, u, v, u_{(1)}, v_{(1)}, \ldots, u_{(s)}, v_{(s)}\right)=0 \tag{5}
\end{equation*}
$$

with

$$
\begin{equation*}
F^{*}\left(x, u, v, u_{(1)}, v_{(1)}, \ldots, u_{(s)}, v_{(s)}\right)=\frac{\delta(v F)}{\delta u} \tag{6}
\end{equation*}
$$

where

$$
\begin{equation*}
\frac{\delta}{\delta u}=\frac{\partial}{\partial u}+\sum_{s=1}^{\infty}(-1)^{s} D_{i_{1}} \cdots D_{i_{s}} \frac{\partial}{\partial u_{i_{1} \cdots i_{s}}} \tag{7}
\end{equation*}
$$

denotes the variational derivatives (the Euler-Lagrange operator) and $v$ is a new dependent variable. Here,

$$
\begin{equation*}
D_{i}=\frac{\partial}{\partial x^{i}}+u_{i} \frac{\partial}{\partial u}+u_{i j} \frac{\partial}{\partial u_{j}}+\cdots \tag{8}
\end{equation*}
$$

are the total differentiations.
Definition 1. Equation (4) is said to be self-adjoint if the equation obtained from the adjoint equation (5) by the substitution $v=u$,

$$
\begin{equation*}
F^{*}\left(x, u, u, u_{(1)}, u_{(1)}, \ldots, u_{(s)}, u_{(s)}\right)=0 \tag{9}
\end{equation*}
$$

is identical to the original equation (4).

Definition 2. Equation (4) is said to be weak self-adjoint if the equation obtained from the adjoint equation (5) by the substitution $v=h(x, t, u)$, with a certain function $h(x, t, u)$ such that $h_{x}(x, t, u) \neq 0,\left(\right.$ or $\left.h_{t}(x, t, u) \neq 0\right)$ and $h_{u}(x, t, u) \neq 0$, is identical to the original equation.

Definition 3. Equation (4) is said to be nonlinearly selfadjoint if the equation obtained from the adjoint equation (5) by the substitution $v=h\left(x, t, u, u_{(1)}, \ldots\right)$, with a certain function $h\left(x, t, u, u_{(1)}, \ldots\right)$ such that $h\left(x, t, u, u_{(1)}, \ldots\right) \neq 0$, is identical to the original equation (4).
2.1. The Subclass of Nonlinearly Self-Adjoint Equations. Let us single out some nonlinearly self-adjoint equations from the equations of the form (3). Equation (6) yields

$$
\begin{align*}
F^{*}= & \frac{\delta}{\delta u}\left[v \left(u_{t}+u_{x}+2 b u u_{x}-c u_{x x}\right.\right. \\
& \left.\left.\quad-d u-a u_{x x t}-f(x, t)\right)\right]  \tag{10}\\
= & -c v_{x x}-2 b u v_{x}-v_{x}+a v_{t x x}-v_{t}-d v
\end{align*}
$$

Setting $v=h(x, t, u)$ in (10), we get

$$
\begin{align*}
& a h_{u u} u_{t} u_{x x}-c h_{u} u_{x x}+a h_{t u} u_{x x}+a h_{u u u} u_{t}\left(u_{x}\right)^{2} \\
& \quad-c h_{u u}\left(u_{x}\right)^{2}+a h_{t u u}\left(u_{x}\right)^{2}+2 a h_{u u} u_{t x} u_{x} \\
& \quad+2 a h_{u u x} u_{t} u_{x}-2 b h_{u} u u_{x}-2 c h_{u x} u_{x}-h_{u} u_{x}  \tag{11}\\
& \quad+2 a h_{t u x} u_{x}+a h_{u} u_{t x x}+2 a h_{u x} u_{t x}+a h_{u x x} u_{t} \\
& \quad-h_{u} u_{t}-2 b h_{x} u-c h_{x x}-h_{x}+a h_{t x x}-h_{t}-d h=0 .
\end{align*}
$$

Now, we assume that

$$
\begin{equation*}
F^{*}-\lambda\left(u_{t}+u_{x}+2 b u u_{x}-c u_{x x}-d u-a u_{x x t}-f(x, t)\right)=0 \tag{12}
\end{equation*}
$$

where $\lambda$ is an undetermined coefficient. Condition (12) reads

$$
\begin{align*}
& c u_{x x} \lambda-2 b u u_{x} \lambda-u_{x} \lambda+a u_{t x x} \lambda-u_{t} \lambda \\
& \quad+d u \lambda+f \lambda+a h_{u u} u_{t} u_{x x}-c h_{u} u_{x x} \\
& \quad+a h_{t u} u_{x x}+a h_{u u u} u_{t}\left(u_{x}\right)^{2}-c h_{u u}\left(u_{x}\right)^{2} \\
& \quad+a h_{t u u}\left(u_{x}\right)^{2}+2 a h_{u u} u_{t x} u_{x}+2 a h_{u u x} u_{t} u_{x}  \tag{13}\\
& \quad-2 b h_{u} u u_{x}-2 c h_{u x} u_{x}-h_{u} u_{x}+2 a h_{t u x} u_{x} \\
& \quad+a h_{u} u_{t x x}+2 a h_{u x} u_{t x}+a h_{u x x} u_{t} \\
& \quad-h_{u} u_{t}-2 b h_{x} u-c h_{x x}-h_{x}+a h_{t x x}-h_{t}-d h=0
\end{align*}
$$

Comparing the coefficients for the different derivatives of $u$, we obtain

$$
\begin{gather*}
\lambda=-h_{u} \\
h=c_{2} e^{2 c t / a} u+\beta \tag{14}
\end{gather*}
$$

where $f=f(x, t)$ and $\beta(x, t)$ satisfy the following conditions:

$$
\begin{gather*}
-2 c_{2} d e^{2 c t / a}-\frac{2 c c_{2} e^{2 c t / a}}{a}-2 b \beta_{x}=0  \tag{15}\\
-c_{2} f e^{2 c t / a}-\beta d-\beta_{x x} c-\beta_{x}+a \beta_{t x x}-\beta_{t}=0
\end{gather*}
$$

From above, we get that

$$
\begin{equation*}
\beta=f_{3}-\frac{\left(a c_{2} d+c c_{2}\right) e^{2 c t / a} x}{a b} \tag{16}
\end{equation*}
$$

with $f_{3}=f_{3}(t)$ and the following condition must be satisfied:

$$
\begin{align*}
& \frac{c_{2} d^{2} e^{2 c t / a} x}{b}+\frac{3 c c_{2} d e^{2 c t / a} x}{a b}+\frac{2 c^{2} c_{2} e^{2 c t / a} x}{a^{2} b}  \tag{17}\\
& -c_{2} f e^{2 c t / a}+\frac{c_{2} d e^{2 c t / a}}{b}+\frac{c c_{2} e^{2 c t / a}}{a b}-f_{3 t}-d f_{3}=0 .
\end{align*}
$$

We can now state the following theorem.
Theorem 4. Equation (3) is nonlinearly self-adjoint with

$$
\begin{equation*}
h=-\frac{\left(a c_{2} d+c c_{2}\right) e^{2 c t / a} x}{a b}+c_{2} e^{2 c t / a} u+f_{3} \tag{18}
\end{equation*}
$$

for any functions $f=f(x, t)$ and $f_{3}(t)$ satisfying condition (17).

In particular, we can state the following theorem.
Theorem 5. Equation (3) is nonlinearly self-adjoint for any arbitrary function $f=f(x, t)$ with

$$
\begin{equation*}
h=c_{3} e^{-d t} \tag{19}
\end{equation*}
$$

## 3. Conservation Laws: General Theorem

We use the following theorem on conservation laws proved in [12].

Theorem 6. Any Lie point, Lie-Bäcklund, or non-local symmetry

$$
\begin{equation*}
X=\xi^{i}\left(x, u, u_{(1)}, \ldots\right) \frac{\partial}{\partial x^{i}}+\eta\left(x, u, u_{(1)}, \ldots\right) \frac{\partial}{\partial u} \tag{20}
\end{equation*}
$$

of (4) provides a conservation law $D_{i}\left(C^{i}\right)=0$ for system (4), (5). The conserved vector is given by

$$
\begin{align*}
C^{i}= & \xi^{i} \mathscr{L}+W\left[\frac{\partial \mathscr{L}}{\partial u_{i}}-D_{j}\left(\frac{\partial \mathscr{L}}{\partial u_{i j}}\right)+D_{j} D_{k}\left(\frac{\partial \mathscr{L}}{\partial u_{i j k}}\right)-\cdots\right] \\
& +D_{j}(W)\left[\frac{\partial \mathscr{L}}{\partial u_{i j}}-D_{k}\left(\frac{\partial \mathscr{L}}{\partial u_{i j k}}\right)+\cdots\right] \\
& +D_{j} D_{k}(W)\left[\frac{\partial \mathscr{L}}{\partial u_{i j k}}-\cdots\right]+\cdots \tag{21}
\end{align*}
$$

where $W$ and $\mathscr{L}$ are defined as follows:

$$
\begin{equation*}
W=\eta-\xi^{j} u_{j}, \quad \mathscr{L}=v F\left(x, u, u_{(1)}, \ldots, u_{(s)}\right) . \tag{22}
\end{equation*}
$$

Let us apply Theorem 6 to the nonlinearly self-adjoint equation:

$$
\begin{equation*}
u_{t}+u_{x}+2 b u u_{x}-c u_{x x}-d u-a u_{x x t}=f(x, t), \tag{23}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathscr{L}=\left(u_{t}+u_{x}+2 b u u_{x}-c u_{x x}-d u-a u_{x x t}-f(x, t)\right) v, \tag{24}
\end{equation*}
$$

provided by the generator

$$
\begin{equation*}
\mathbf{v}=k_{1} \frac{\partial}{\partial t}+k_{2} \frac{\partial}{\partial x} \tag{25}
\end{equation*}
$$

Here, $f=f(x, t)$ must satisfy $k_{1} f_{t}+k_{2} f_{x}=0$. We get the conservation law

$$
\begin{equation*}
D_{t}\left(C^{1}\right)+D_{x}\left(C^{2}\right)=0 \tag{26}
\end{equation*}
$$

with

$$
\begin{gather*}
C^{1}=-k k_{1} e^{-d t}(d u+f)+D_{x}(B), \\
C^{2}=k e^{-d t}\left(c d k_{1} u_{x}+a d k_{1} u_{t x}-b d k_{1} u^{2}\right.  \tag{27}\\
\left.-d k_{1} u-f k_{2}\right)-D_{t}(B),
\end{gather*}
$$

where

$$
\begin{gather*}
B=\left(k e ^ { - d t } \left(a k_{2} u_{x x}-3 c k_{1} u_{x}-2 a k_{1} u_{t x}+3 b k_{1} u^{2}\right.\right.  \tag{28}\\
\left.\left.-3 k_{2} u+3 k_{1} u\right)\right)(3)^{-1}
\end{gather*}
$$

We simplify the conserved vector by transferring the terms of the form $D_{x}(\cdots)$ from $C^{1}$ to $C^{2}$ and obtain

$$
\begin{gather*}
C^{1}=-k k_{1} e^{-d t}(d u+f) \\
C^{2}=k e^{-d t}\left(c d k_{1} u_{x}+a d k_{1} u_{t x}\right.  \tag{29}\\
\left.-b d k_{1} u^{2}-d k_{1} u-f k_{2}\right)
\end{gather*}
$$

## 4. Exact Solutions

In this section, we obtain exact solutions of (3) when $d=0$ and $f=0$; that is, we consider the following equation:

$$
\begin{equation*}
u_{t}+u_{x}+2 b u u_{x}-c u_{x x}-a u_{x x t}=0 . \tag{30}
\end{equation*}
$$

This equation has two translation symmetries; namely, $X_{1}=$ $\partial / \partial x$ and $X_{2}=\partial / \partial t$. We first use these two symmetries and transform (30) into an ordinary differential equation. Then, employing the simplest equation method, we obtain exact solutions.
4.1. Symmetry Reduction of (30). The symmetry $\nu X_{1}+X_{2}$ gives rise to the group-invariant solution

$$
\begin{equation*}
u=F(z) \tag{31}
\end{equation*}
$$

where $z=x-v t$ is an invariant of $v X_{1}+X_{2}$. Substitution of (31) into (30) results in the nonlinear third-order ordinary differential equation

$$
\begin{equation*}
a v F^{\prime \prime \prime}(z)-c F^{\prime \prime}(z)+2 b F(z) F^{\prime}(z)+(1-v) F^{\prime}(z)=0 \tag{32}
\end{equation*}
$$

4.2. Exact Solutions Using Simplest Equation Method. Let us briefly recall the simplest equation method $[13,14]$ here. Consider the solutions of (32) in the form

$$
\begin{equation*}
F(z)=\sum_{i=0}^{M} A_{i}(H(z))^{i} \tag{33}
\end{equation*}
$$

where $H(z)$ satisfies a Bernoulli or Riccati equation, $M$ is a positive integer that can be determined by a balancing procedure [14], and the coefficients $A_{0}, \ldots, A_{M}$ are parameters to be determined.

The Bernoulli equation we consider here is given by

$$
\begin{equation*}
H^{\prime}(z)=\alpha H(z)+\beta H^{2}(z) \tag{34}
\end{equation*}
$$

which has a solution in the form

$$
\begin{equation*}
H(z)=\alpha\left\{\frac{\cosh [\alpha(z+C)]+\sinh [\alpha(z+C)]}{1-\beta \cosh [\alpha(z+C)]-\beta \sinh [\alpha(z+C)]}\right\} \tag{35}
\end{equation*}
$$

For the Riccati equation

$$
\begin{equation*}
H^{\prime}(z)=\alpha H^{2}(z)+\beta H(z)+\gamma \tag{36}
\end{equation*}
$$

we will use the two solutions

$$
\begin{align*}
H(z) & =-\frac{\beta}{2 \alpha}-\frac{\theta}{2 \alpha} \tanh \left[\frac{1}{2} \theta(z+C)\right] \\
H(z) & =-\frac{\beta}{2 \alpha}-\frac{\theta}{2 \alpha} \tanh \left(\frac{1}{2} \theta z\right)  \tag{37}\\
& +\frac{\operatorname{sech}(\theta z / 2)}{C \cosh (\theta z / 2)-(2 \alpha / \theta) \sinh (\theta z / 2)}
\end{align*}
$$

where $\theta^{2}=\beta^{2}-4 \alpha \gamma$ and $C$ is a constant of integration.
4.2.1. Solutions of (30) Using Bernoulli Equation as the Simplest Equation. In this case, the balancing procedure [14] gives $M=2$ and therefore the solutions of (32) are of the form

$$
\begin{equation*}
F(z)=A_{0}+A_{1} H+A_{2} H^{2} \tag{38}
\end{equation*}
$$

Now, substituting (38) into (32) and making use of (34) and then equating all coefficients of the functions $H^{i}$ to zero, we obtain an algebraic system of equations in terms of $A_{0}, A_{1}$, and $A_{2}$.

Solving this system of algebraic equations, with the aid of Mathematica, we obtain

$$
\begin{gather*}
b=-\frac{6 a v \beta^{2}}{A_{2}}, \quad c=-5 a v \alpha \\
A_{0}=\frac{A_{2}\left(-v+1+6 a v \alpha^{2}\right)}{12 a v \beta^{2}}, \quad A_{1}=\frac{2 A_{2} \alpha}{\beta} . \tag{39}
\end{gather*}
$$

Thus, a solution of (30) is

$$
\begin{align*}
& u(t, x) \\
& =A_{0}+A_{1} \alpha\left\{\frac{\cosh [\alpha(z+C)]+\sinh [\alpha(z+C)]}{1-\beta \cosh [\alpha(z+C)]-\beta \sinh [\alpha(z+C)]}\right\} \\
& \quad+A_{2} \alpha^{2}\left\{\frac{\cosh [\alpha(z+C)]+\sinh [\alpha(z+C)]}{1-\beta \cosh [\alpha(z+C)]-\beta \sinh [\alpha(z+C)]}\right\}^{2} \tag{40}
\end{align*}
$$

where $z=x-v t$ and $C$ is a constant of integration.
4.2.2. Solutions of (30) Using Riccati Equation as the Simplest Equation. The balancing procedure yields $M=2$ so the solutions of (32) are of the form

$$
\begin{equation*}
F(z)=A_{0}+A_{1} H+A_{2} H^{2} \tag{41}
\end{equation*}
$$

Again substituting (41) into (32) and making use of the Riccati equation (36), we obtain, as before, an algebraic system of equations in terms of $A_{0}, A_{1}, A_{2}$. Solving the algebraic system of equations, one obtains

$$
\begin{gathered}
b=-\frac{6 a v \alpha^{2}}{A_{2}}, \quad c=-\frac{5 a v\left(A_{1} \alpha-A_{2} \beta\right)}{A_{2}}, \\
\gamma=-\frac{A_{1}\left(A_{1} \alpha-2 A_{2} \beta\right)}{4 A_{2}^{2}}
\end{gathered}
$$

$$
\begin{align*}
& A_{0} \\
& =-\frac{3 a v \alpha^{2} A_{1}^{2}-12 a v A_{1} \alpha \beta A_{2}+6 a v A_{2}^{2} \beta^{2}+v A_{2}^{2}-A_{2}^{2}}{12 a v A_{2} \alpha^{2}}, \tag{42}
\end{align*}
$$

and hence solutions of (30) are

$$
\begin{align*}
u(t, x)= & A_{0}+A_{1}\left\{-\frac{\beta}{2 \alpha}-\frac{\theta}{2 \alpha} \tanh \left[\frac{1}{2} \theta(z+C)\right]\right\} \\
& +A_{2}\left\{-\frac{\beta}{2 \alpha}-\frac{\theta}{2 \alpha} \tanh \left[\frac{1}{2} \theta(z+C)\right]\right\}^{2},  \tag{43}\\
u(t, x)= & A_{0}+A_{1}\left\{-\frac{\beta}{2 \alpha}-\frac{\theta}{2 \alpha} \tanh \left(\frac{1}{2} \theta z\right)\right. \\
& \left.+\frac{\operatorname{sech}(\theta z / 2)}{C \cosh (\theta z / 2)-(2 \alpha / \theta) \sinh (\theta z / 2)}\right\} \\
+ & A_{2}\left\{-\frac{\beta}{2 \alpha}-\frac{\theta}{2 \alpha} \tanh \left(\frac{1}{2} \theta z\right)\right. \\
& \left.+\frac{\operatorname{sech}(\theta z / 2)}{C \cosh (\theta z / 2)-(2 \alpha / \theta) \sinh (\theta z / 2)}\right\}^{2}, \tag{44}
\end{align*}
$$

where $z=x-v t$ and $C$ is a constant of integration.

## 5. Conclusions

We have proved that the generalized forced BBM equation (3) is nonlinearly self-adjoint. We have determined, by using
the Lie generators of (3) and the notation and techniques of [12], some nontrivial conservation laws for (3). Finally, we presented some exact solutions for a special case of (3).

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Weak Equivalence Transformations for a Class of Models in Biomathematics 

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A class of reaction-diffusion systems unifying several Aedes aegypti population dynamics models is considered. Equivalence transformations are found. Extensions of the principal Lie algebra are derived for some particular cases.

## 1. Introduction

In this paper we focus our attention on the following class of nonlinear advection-reaction-diffusion systems:

$$
\begin{align*}
u_{t}=\left(f(u) u_{x}\right)_{x} & +g\left(u, v, u_{x}\right), \quad g_{u_{x}} \neq 0 \\
v_{t} & =h(u, v) . \tag{1}
\end{align*}
$$

These systems can describe the evolution of the densities $u$ and $v$ of two interacting populations where the balance equation for $u$ takes into account not only the reaction-diffusion effects but also some advection effects while the balance equation for density $v$ takes into account only the so-called reaction terms. The advection effects are due to the presence in the function $g$ of the gradient $u_{x}$ and appear when the individuals of population $u$ feel external stimuli as, for instance, wind effects or water currents. Class (1) can be considered a generalization of the equations with the typical properties of the already known Aedes aegypti mathematical models [1-3]. We recall them shortly in the following.

The Aedes aegypti mosquitos are the main vector of dengue, a viral disease that causes the so-called dengue hemorrhagic fever characterized by coagulation problems often leading the infected individual to death. Since the subtropical zone climate and environmental conditions are favorable to the development of Aedes aegypti, dengue is a serious public
health problem in many countries around the world. However, due to the global warming, the interest in such considered mosquitoes is not restricted to those places affected by the disease, but it is also of interest for those countries whose weather, in the next decades, can become similar to the current environmental found in the subtropical zone. Therefore, the interest in modeling such a vector is not only a theoretical deal but also a way for finding methods and alternatives to overcome and control the problems arising from the dispersal dynamics of the mosquitos and, consequently, the propagation of the disease.

The following system

$$
\begin{gather*}
u_{t}=\left(u^{p} u_{x}\right)_{x}-2 v u^{q} u_{x}+\frac{\gamma}{k} v(1-u)-m_{1} u,  \tag{2}\\
v_{t}=k(1-v) u-\left(m_{2}+\gamma\right) v,
\end{gather*}
$$

where $p, q \in \mathbb{R}$, belongs to class (1). It was introduced in [2] as a generalization of a model studied in [1].

We recall that in (2), as well as in [1], $u$ and $v$ are, respectively, nondimensional densities of winged and aquatic populations of mosquitoes and $k, \gamma, m_{1}$, and $m_{2}$ are nondimensional, in general, positive parameters, $\nu \in \mathbb{R}$. Specifically $k$ is the ratio between two constants $\bar{k}_{1}$ and $\bar{k}_{2}$, which are, respectively, the carrying capacity related to the amount of findable nutrients and the carrying capacity effect dependent on the occupation of the available breeder and $\gamma$ denotes the
specific rate of maturation of the aquatic form into winged female mosquitoes, while $m_{1}$ and $m_{2}$ are, respectively, the mortality of winged and the mortality of aquatic populations. Finally $v$ denotes a constant of velocity for flux due to wind currents that, in general, generate an advection motion of large masses of the winged population and consequently can facilitate a quick advance of the infestation. For further details, see [4] and references therein.

Another system belonging to (1) is

$$
\begin{gather*}
u_{t}=\left(u^{p} u_{x}\right)_{x}-2 v u^{q} u_{x}+\frac{\gamma}{k} v+\left(\frac{\gamma}{k}-m_{1}\right) u  \tag{3}\\
v_{t}=k u+\left(k-m_{2}-\gamma\right) v
\end{gather*}
$$

which was also introduced in [2] starting from (2) and, due to the weak interaction between the aquatic and winged populations, by modifying the source terms.

The first equation of (3) gives the time rate of change of the mosquitoes density as a sum of the growth terms $(\gamma / k) v$, of the per capita death rate $\left((\gamma / k)-\mu_{1}\right) u$ and the diffusiveadvective flux due to the movement of mosquito population.

The second equation gives the corresponding time rate of change of the density of aquatic population as a sum of the growth term $k u$ of aquatic population, due to the new egg depositions of female mosquitoes, with per capita death rate $\left(k-\mu_{2}\right) v$ of aquatic population. The term $-\gamma v$ represents the loss due to the change of the aquatic into winged form; see [4].

The family of system (1) contains arbitrary functions or numerical parameters, which specifies the individual characteristics of phenomena belonging to large subclasses. In this sense, the knowledge of equivalence transformations can provide us with certain relations between the solutions of different phenomena of the same class and allows us to get symmetries in a quite direct way.

Following [5], an equivalence transformation is a nondegenerated change of independent and dependent variables $t$, $x, u$, and $v$ into $\widehat{t}, \widehat{x}, \widehat{u}$, and $\widehat{v}$ :

$$
\begin{align*}
x & =x(\widehat{x}, \widehat{t}, \widehat{u}, \widehat{v}), \\
t & =t(\widehat{x}, \widehat{t}, \widehat{u}, \widehat{v}), \\
u & =u(\widehat{x}, \widehat{t}, \widehat{u}, \widehat{v}),  \tag{4}\\
v & =v(\widehat{x}, \widehat{t}, \widehat{u}, \widehat{v}),
\end{align*}
$$

which maps a system of class (1) in another one of the same class, that is, in an equation preserving differential structure but, in general, with

$$
\begin{gather*}
\widehat{f}(\widehat{u}) \neq f(u), \quad \widehat{g}\left(\widehat{u}, \widehat{v}, \widehat{u}_{\widehat{x}}\right) \neq g\left(u, v, u_{x}\right),  \tag{5}\\
\widehat{h}(\widehat{u}, \widehat{v}) \neq h(u, v) .
\end{gather*}
$$

Of course in the case

$$
\begin{gather*}
\widehat{f}(\widehat{u})=f(u), \quad \widehat{g}\left(\widehat{u}, \widehat{v}, \widehat{u}_{\widehat{x}}\right)=g\left(u, v, u_{x}\right), \\
\widehat{h}(\widehat{u}, \widehat{v})=h(u, v) \tag{6}
\end{gather*}
$$

an equivalence transformation becomes a symmetry.

In this paper we look for certain equivalence transformations for the class of systems (1) in order to find symmetries for special systems belonging to (1) and to get information about constitutive parameters $f, g$, and $h$ appearing there. Moreover we wish to stress that, as it is known, an equivalence transformation maps solutions of an equation in solutions of the transformed equation [6]. Then in order to find solutions for a certain equation one can look for the equivalence transformations that bring the equation in simpler other ones whose solutions are well studied; see, for example, $[7,8]$ and references inside.

The plan of the paper is as follows. In the next section we provide some elements about equivalence transformations. In Section 3 we apply these concepts in order to obtain a set of weak equivalence generators. In Section 4, after having introduced a projection theorem, we show how to apply it to find symmetries of (1). In Section 5, after having introduced a special structure of the advection-reaction function $g$ that generalizes that one used in (3), we find extensions with respect to the principal Lie algebra. Conclusions and final remarks are given in Section 6.

## 2. Elements on Equivalence Transformations

In the past differential equation literature it is possible to find several examples of equivalence transformations. The direct search for the most general equivalence transformations through the finite form of the transformation is connected with considerable computational difficulties and quite often leads to partial solutions of the problem (e.g., $[9,10]$ ).

A systematic treatment to look for continuous equivalence transformations by using the Lie infinitesimal criterion was suggested by Ovsiannikov [11].

In general, the equivalence transformations for class (1) can be considered as transformations acting on point of the basic augmented space

$$
\begin{equation*}
A \equiv\left\{t, x, u, v, u_{t}, u_{x}, v_{t}, v_{x}, f, g, h\right\} \tag{7}
\end{equation*}
$$

The previous elements allow us to consider, in the following, the one-parameter equivalence transformations as a group of transformations, acting on the basic augmented space $A$, of the type

$$
\begin{gather*}
x=x(\widehat{x}, \widehat{t}, \widehat{u}, \widehat{v}, \varepsilon), \\
t=t(\widehat{x}, \widehat{t}, \widehat{u}, \widehat{v}, \varepsilon), \\
u=u(\widehat{x}, \widehat{t}, \widehat{u}, \widehat{v}, \varepsilon), \\
v=v(\widehat{x}, \widehat{t}, \widehat{u}, \widehat{v}, \varepsilon),  \tag{8}\\
f=f\left(\widehat{x}, \widehat{t}, \widehat{u}, \widehat{v}, \widehat{u}_{t}, \widehat{u}_{\widehat{x}}, \widehat{v}_{t}, \widehat{v}_{\widehat{x}}, \widehat{f}, \widehat{g}, \widehat{h}, \varepsilon\right), \\
g=g\left(\widehat{x}, \widehat{t}, \widehat{u}, \widehat{v}, \hat{u}_{\hat{t}}, \widehat{u}_{\widehat{x}}, \hat{v}_{t}, \hat{v}_{\widehat{x}}, \widehat{f}, \widehat{g}, \widehat{h}, \varepsilon\right), \\
h=h\left(\widehat{x}, \widehat{t}, \widehat{u}, \widehat{v}, \hat{u}_{\hat{t}}, \widehat{u}_{\widehat{x}}, \widehat{v}_{t}, \widehat{v}_{\widehat{x}}, \widehat{f}, \widehat{g}, \widehat{h}, \varepsilon\right),
\end{gather*}
$$

which is locally a $C^{\infty}$-diffeomorphism, depending analytically on the parameter $\varepsilon$ in a neighborhood of $\varepsilon=0$, and reduces to the identity transformation for $\varepsilon=0$.

Following [6, 11-14] (see also, e.g., [5, 15-17] ) we consider the infinitesimal generator of the equivalence transformations (8) of the systems (1) that reads as follows:

$$
\begin{equation*}
Y=\xi^{1} \partial_{x}+\xi^{2} \partial_{t}+\eta^{1} \partial_{u}+\eta^{2} \partial_{v}+\mu^{1} \partial_{f}+\mu^{2} \partial_{g}+\mu^{3} \partial_{h} \tag{9}
\end{equation*}
$$

where the infinitesimal components $\xi^{1}, \xi^{2}, \eta^{1}$, and $\eta^{2}$ are sought depending on $x, t, u$, and $v$, while the infinitesimal components $\mu^{i}(i=1,2,3)$ are sought, at least in principle, depending on $x, t, u, v, u_{t}, u_{x}, u_{x}, v_{t}, v_{x}, f, g$, and $h$. In order to obtain the determining system which allows us to get the infinitesimal coordinates $\xi^{i}, \eta^{i}$, and $\mu^{j}(i=1,2$ and $j=$ $1,2,3$ ), we apply the Lie-Ovsiannikov infinitesimal criterion by requiring the invariance, with respect to suitable prolongations $Y^{(1)}$ and $Y^{(2)}$ of (9), of the following equations:

$$
\begin{gather*}
u_{t}-\left(f u_{x}\right)_{x}-g=0  \tag{10}\\
v_{t}-h=0
\end{gather*}
$$

together with the invariance of the auxiliary conditions $[13,14$, $18,19]$ :

$$
\begin{align*}
f_{t} & =f_{x}=f_{v}=f_{u_{x}}=f_{u_{t}}=f_{v_{x}}=f_{v_{t}} \\
& =g_{t}=g_{x}=g_{u_{t}}=g_{v_{t}}=g_{v_{x}}=0,  \tag{11}\\
h_{t} & =h_{x}=h_{u_{x}}=h_{u_{t}}=h_{v_{x}}=h_{v_{t}}=0,
\end{align*}
$$

where $u$ and $v$ are $(t, x)$ functions while $f, g$, and $h$ are considered as functions depending, a priori, on ( $t, x, u$, $\left.v, u_{t}, u_{x}, v_{t}, v_{x}\right)$. All of these functions are assumed to be analytical. The constraints, given by (11), characterize the functional dependence of $f, g$, and $h$.

In this paper, instead, in view of further applications and following [20], we modify the previous classical procedure by looking for equivalence transformations whose generators are got by solving the determining system obtained from the following invariance conditions:

$$
\begin{align*}
& \left.Y^{(2)}\left[u_{t}-\left(f(u) u_{x}\right)_{x}-g\left(u, v, u_{x}\right)\right]\right|_{\substack{u_{t}-\left(f(u) u_{x}\right)_{x}-g\left(u, v, u_{x}\right)=0, v_{t}-h(u, v)=0}} \\
& \quad=0, \tag{12}
\end{align*}
$$

$$
\begin{equation*}
\left.Y^{(1)}\left[v_{t}-h(u, v)\right]\right|_{\substack{u_{t}-\left(f(u) u_{x}\right)_{x}-g\left(u, v, u_{x}\right)=0, v_{t}-h(u, v)=0}}=0 . \tag{13}
\end{equation*}
$$

As the functional dependences of the parameters $f, g$, and $h$ are known a priori we do not require the invariance of the auxiliary conditions (11). In this way we work in a basic augmented space $A \equiv\left\{t, x, u, v, u_{x}, f, g, h\right\}$. Therefore the $\mu^{i}$ components must be sought, at least in principle, depending on $x, t, u, v, u_{x}, f, g$, and $h$.

The infinitesimal operators, obtained by following this shortening procedure, can generate transformations that map equations of our class into new equations of the same class where the transformed arbitrary functions may have new additional functional dependencies. Such transformations are called weak equivalence transformations [13, 14].

With respect to the application in biomathematical models, equivalence and weak equivalence transformations were applied not only to study of tumor models [21, 22] but also to the population dynamics in $[20,23,24]$.

## 3. Calculation of Weak <br> Equivalence Transformations

In order to avoid long formulas and write $Y^{(1)}$ and $Y^{(2)}$ in a compact way, we put

$$
\begin{gather*}
x=x^{1}, \quad t=x^{2}, \\
u=u^{1}, \quad v=u^{2},  \tag{14}\\
f=h^{1}, \quad g=h^{2}, \quad h=h^{3} .
\end{gather*}
$$

For this reason system (1) is rewritten as

$$
\begin{gather*}
u_{x^{2}}^{1}-h^{2}-h_{u^{1}}^{1}\left(u_{x^{1}}^{1}\right)^{2}-h^{1} u_{x^{1} x^{1}}^{1}=0  \tag{15}\\
u_{x^{2}}^{2}-h^{3}=0
\end{gather*}
$$

while the equivalence generator assumes the following form:

$$
\begin{equation*}
Y=\xi^{i} \partial_{x^{i}}+\eta^{\alpha} \partial_{u^{\alpha}}+\mu^{A}, \partial_{h^{A}}, \tag{16}
\end{equation*}
$$

where $i=1,2, \alpha=1,2$, and $A=1,2,3$. Here the summation over the repeated indices is presupposed.

After putting

$$
\begin{gather*}
\left(z^{1}, z^{2}, z^{3}, z^{4}, z^{5}\right)=\left(x^{1}, x^{2}, u^{1}, u^{2}, u_{1}^{1}\right), \\
\left(v^{1}, v^{2}, v^{3}, v^{4}, v^{5}\right)=\left(\xi^{1}, \xi^{2}, \eta^{1}, \eta^{2}, \zeta_{1}^{1}\right),  \tag{17}\\
u_{i}^{\alpha}=u_{x^{i}}^{\alpha}, \quad h_{a}^{A}=h_{z^{a}}^{A},  \tag{18}\\
\widetilde{D}_{a}^{e}=\partial_{z^{a}}+h_{a}^{A} \partial_{h^{A}},  \tag{19}\\
D_{j}^{e}=\partial_{x^{j}}+u_{j}^{\alpha} \partial_{u^{\alpha}}+u_{i j}^{\alpha} \partial_{u_{i}^{\alpha}}+\cdots, \tag{20}
\end{gather*}
$$

the prolongations $Y^{(1)}$ and $Y^{(2)}$ assume the following form:

$$
\begin{gather*}
Y^{(1)}=Y+\zeta_{j}^{\alpha} \partial_{u_{j}^{\alpha}}+\omega_{a}^{A} \partial_{h_{a}^{A}} \\
Y^{(2)}=Y^{(1)}+\zeta_{11}^{1} \partial_{u_{11}^{1}}=Y+\zeta_{j}^{\alpha} \partial_{u_{j}^{\alpha}}+\omega_{a}^{A} \partial_{h_{a}^{A}}+\zeta_{11}^{1} \partial_{u_{11}^{1}} \tag{21}
\end{gather*}
$$

where

$$
\begin{align*}
\zeta_{j}^{\alpha} & =D_{j}^{e} \eta^{\alpha}-u_{k}^{\alpha} D_{j}^{e} \xi^{k} \\
\zeta_{i j}^{\alpha} & =D_{j}^{e} \zeta_{j}^{\alpha}-u_{i k}^{\alpha} D_{j}^{e} \xi^{k}  \tag{22}\\
\omega_{a}^{A} & =\widetilde{D}_{a}^{e} \mu^{A}-h_{b}^{A} \widetilde{D}_{a}^{e} \nu^{b}
\end{align*}
$$

The invariant conditions read

$$
\begin{align*}
Y^{(2)} F_{1}= & \zeta_{2}^{1}-\mu^{2}-2 \zeta_{1}^{1} h_{u^{1}}^{1} u_{1}^{1}-\left(\omega_{3}^{1}\right)\left(u_{1}^{1}\right)^{2}  \tag{23}\\
& -\left(\mu^{1}\right) u_{11}^{1}-h^{1} \zeta_{11}^{1}=0, \\
& Y^{(1)} F_{2}=\zeta_{2}^{2}-\mu^{3}=0, \tag{24}
\end{align*}
$$

both under the constraints (15), which are after (18)

$$
\begin{gather*}
u_{2}^{1}=h^{2}+h_{u^{1}}^{1}\left(u_{1}^{1}\right)^{2}+h^{1} u_{11}^{1},  \tag{25}\\
u_{2}^{2}=h^{3} . \tag{26}
\end{gather*}
$$

The coefficients $\zeta_{1}^{1}, \zeta_{2}^{1}, \zeta_{2}^{2}$, and $\omega_{3}^{1}$ are given, respectively, by

$$
\begin{align*}
\zeta_{1}^{1}= & \eta_{1}^{1}+\left(\eta_{u^{1}}^{1}-\xi_{1}^{1}\right) u_{1}^{1}-\xi_{1}^{2} u_{2}^{1}+\eta_{u^{2}}^{1} u_{1}^{2} \\
& -\xi_{u^{1}}^{1}\left(u_{1}^{1}\right)^{2}-\xi_{u^{2}}^{1} u_{1}^{1} u_{1}^{2}-\xi_{u^{1}}^{2} u_{1}^{1} u_{2}^{1}-\xi_{u^{2}}^{2} u_{2}^{1} u_{1}^{2}, \\
\zeta_{2}^{1}= & \eta_{2}^{1}-\xi_{2}^{1} u_{1}^{1}+\left(\eta_{u^{1}}^{1}-\xi_{2}^{2}\right) u_{2}^{1}+\eta_{u^{2}}^{1} u_{2}^{2} \\
& -\xi_{u^{1}}^{1} u_{1}^{1} u_{2}^{1}-\xi_{u^{2}}^{1} u_{1}^{1} u_{2}^{2}-\xi_{u^{1}}^{2}\left(u_{2}^{1}\right)^{2}-\xi_{u^{2}}^{2} u_{2}^{1} u_{2}^{2}  \tag{27}\\
\zeta_{2}^{2}= & \eta_{2}^{2}+\eta_{u^{1}}^{2} u_{2}^{1}+\left(\eta_{u^{2}}^{2}-\xi_{2}^{2}\right) u_{2}^{2}-\xi_{2}^{1} u_{1}^{2} \\
& -\xi_{u^{1}}^{1} u_{1}^{2} u_{2}^{1}-\xi_{u^{2}}^{1} u_{1}^{2} u_{2}^{2}-\xi_{u^{1}}^{2} u_{2}^{1} u_{2}^{2}-\xi_{u^{2}}^{2}\left(u_{2}^{2}\right)^{2} \\
\omega_{3}^{1}= & \mu_{u^{1}}^{1}+h_{u^{1}}^{1} \mu_{h^{1}}^{1}+h_{u^{1}}^{2} \mu_{h^{2}}^{1}+h_{u^{1}}^{3} u_{h^{3}}^{1}-h_{u^{1}}^{1} \eta_{u^{1}}^{1}
\end{align*}
$$

Taking into account (24) and (27), we can write

$$
\begin{align*}
Y^{(1)} F_{2}= & \eta_{2}^{2}+\eta_{u^{1}}^{2} u_{2}^{1}+\left(\eta_{u^{2}}^{2}-\xi_{2}^{2}\right) u_{2}^{2}-\xi_{2}^{1} u_{1}^{2}-\xi_{u^{1}}^{1} u_{1}^{2} u_{2}^{1}  \tag{28}\\
& -\xi_{u^{2}}^{1} u_{1}^{2} u_{2}^{2}-\xi_{u^{1}}^{2} u_{2}^{1} u_{2}^{2}-\xi_{u^{2}}^{2}\left(u_{2}^{2}\right)^{2}-\mu^{3}=0,
\end{align*}
$$

with the constraints (25) and (26).
Then, substituting (25) and (26) into (28), we get

$$
\begin{align*}
\eta_{2}^{2}+ & \eta_{u^{1}}^{2}\left(h^{2}+h_{u^{1}}^{1}\left(u_{1}^{1}\right)^{2}+h^{1} u_{11}^{1}\right)+\left(\eta_{u^{2}}^{2}-\xi_{2}^{2}\right) h^{3} \\
& -\xi_{2}^{1} u_{1}^{2}-\xi_{u^{1}}^{1} u_{1}^{2}\left(h^{2}+h_{u^{1}}^{1}\left(u_{1}^{1}\right)^{2}+h^{1} u_{11}^{1}\right)  \tag{29}\\
& -\xi_{u^{2}}^{1} u_{1}^{2} h^{3}-\xi_{u^{1}}^{2}\left(h^{2}+h_{u^{1}}^{1}\left(u_{1}^{1}\right)^{2}+h^{1} u_{11}^{1}\right) h^{3} \\
& -\xi_{u^{2}}^{2}\left(h^{3}\right)^{2}-\mu^{3}=0 .
\end{align*}
$$

From this condition we obtain the following determining equations:

$$
\begin{gather*}
\eta_{2}^{2}+\left(\eta_{u^{2}}^{2}-\xi_{2}^{2}\right) h^{3}-\xi_{u^{2}}^{2}\left(h^{3}\right)^{2}-\mu^{3}=0 \\
\xi_{2}^{1}=0, \quad \xi_{u^{1}}^{1}=0, \quad \xi_{u^{2}}^{1}=0  \tag{30}\\
\eta_{u^{1}}^{2}=0, \quad \xi_{u^{1}}^{2}=0
\end{gather*}
$$

Then it follows that

$$
\begin{align*}
& \xi^{1}=\xi^{1}\left(x^{1}\right), \quad \xi^{2}=\xi^{2}\left(x^{1}, x^{2}, u^{2}\right), \quad \eta^{2}=\eta^{2}\left(x^{1}, x^{2}, u^{2}\right), \\
& \eta^{1}=\eta^{1}\left(x^{1}, x^{2}, u^{1}, u^{2}\right), \quad \mu^{3}=\eta_{2}^{2}+\left(\eta_{u^{2}}^{2}-\xi_{2}^{2}\right) h^{3}-\xi_{u^{2}}^{2}\left(h^{3}\right)^{2} \tag{31}
\end{align*}
$$

and, consequently, $\mu_{u_{1}^{1}}^{3}=0$.

Following the same procedure we can write the invariance condition (23) as

$$
\begin{align*}
& Y^{(2)} F_{1}= \eta_{2}^{1}+\left(\eta_{u^{1}}^{1}-\xi_{2}^{2}\right)\left(h^{2}+h_{u^{1}}^{1}\left(u_{1}^{1}\right)^{2}+h^{1} u_{11}^{1}\right) \\
&+ \eta_{u^{2}}^{1} h^{3}-\xi_{u^{2}}^{2}\left(h^{2}+h_{u^{1}}^{1}\left(u_{1}^{1}\right)^{2}+h^{1} u_{11}^{1}\right) h^{3} \\
&- \mu^{2}-2\left\{\eta_{1}^{1}+\left(\eta_{u^{1}}^{1}-\xi_{1}^{1}\right) u_{1}^{1}\right. \\
& \quad-\xi_{1}^{2}\left(h^{2}+h_{u^{1}}^{1}\left(u_{1}^{1}\right)^{2}+h^{1} u_{11}^{1}\right)+\eta_{u^{2}}^{1} u_{1}^{2}+ \\
&\left.\quad-\xi_{u^{2}}^{2}\left(h^{2}+h_{u^{1}}^{1}\left(u_{1}^{1}\right)^{2}+h^{1} u_{11}^{1}\right) u_{1}^{2}\right\} h_{u^{1}}^{1} u_{1}^{1} \\
&-\left(\mu_{u^{1}}^{1}+h_{u^{1}}^{1} \mu_{h^{1}}^{1}+h_{u^{1}}^{2} \mu_{h^{2}}^{1}+h_{u^{1}}^{3} \mu_{h^{3}}^{1}-h_{u^{1}}^{1} \eta_{u^{1}}^{1}\right) \\
& \times\left(u_{1}^{1}\right)^{2}-\left(\mu^{1}\right) u_{11}^{1} \\
&- h^{1}\left(D_{1}^{e} \eta_{1}^{1}+D_{1}^{e}\left(\eta_{u^{1}}^{1}-\xi_{1}^{1}\right) u_{1}^{1}+\left(\eta_{u^{1}}^{1}-\xi_{1}^{1}\right) u_{11}^{1}\right. \\
&\left.\quad+\left(D_{1}^{e} \eta_{u^{2}}^{1}\right) u_{1}^{2}+\eta_{u^{2}}^{1} u_{11}^{2}\right) \\
&- h^{1}\left(-D_{1}^{e}\left(\xi_{1}^{2}\right)\left(h^{2}+h_{u^{1}}^{1}\left(u_{1}^{1}\right)^{2}+h^{1} u_{11}^{1}\right)\right. \\
& \quad \xi_{1}^{2} u_{21}^{1}-D_{1}^{e} \xi_{u^{2}}^{2}\left(h^{2}+h_{u^{1}}^{1}\left(u_{1}^{1}\right)^{2}+h^{1} u_{11}^{1}\right) u_{1}^{2} \\
& \quad-\xi_{u^{2}}^{2} u_{21}^{1} u_{1}^{2}-\xi_{u^{2}}^{2}\left(h^{2}+h_{u^{1}}^{1}\left(u_{1}^{1}\right)^{2}+h^{1} u_{11}^{1}\right) u_{11}^{2} \\
&\left.\quad u_{11}^{1} \xi_{1}^{1}\right)=0 . \tag{32}
\end{align*}
$$

Collecting the terms with $\left(h_{u^{1}}^{1}\right)^{2}$ and with $h_{u^{1}}^{1}$ and equating their respective coefficients to zero it is obtained that,

$$
\begin{gather*}
\xi_{1}^{2}=0, \quad \xi_{u^{2}}^{2}=0, \quad \eta_{1}^{1}=0  \tag{33}\\
\mu_{h^{1}}^{1}=2 \xi_{1}^{1}-\xi_{2}^{2}, \quad \eta_{u^{2}}^{1}=0 \quad \eta_{1}^{1}=0
\end{gather*}
$$

Thus, from the coefficients of $h_{u^{1}}^{2}$ and $h_{u^{1}}^{3}$

$$
\begin{equation*}
\mu_{h^{2}}^{1}=\mu_{h^{3}}^{1}=0 \tag{34}
\end{equation*}
$$

Considering the coefficients of $u_{11}^{1}$ we get

$$
\begin{equation*}
\mu^{1}=\left(2 \xi_{1}^{1}-\xi_{2}^{2}\right) h^{1}=0 \Longrightarrow \mu_{u^{1}}^{1}=0 \tag{35}
\end{equation*}
$$

The remaining terms of (32) give the following form for the infinitesimal component of $h^{2}$ :

$$
\begin{equation*}
\mu^{2}=\eta_{2}^{1}+\left(\eta_{u^{1}}^{1}-\xi_{2}^{2}\right) h^{2}+\left(\xi_{11}^{1}-\eta_{u^{1} u^{1}}^{1} u_{1}^{1}\right) h^{1} . \tag{36}
\end{equation*}
$$

Therefore, once having taken into account all restrictions obtained we are finally able to write the following infinitesimal components for the weak equivalence generators:

$$
\begin{gather*}
\xi^{1}=\alpha\left(x^{1}\right), \quad \xi^{2}=\beta\left(x^{2}\right) \\
\eta^{1}=\delta\left(x^{2}, u^{1}\right), \quad \eta^{2}=\lambda\left(x^{1}, x^{2}, u^{2}\right)  \tag{37}\\
\mu^{1}=\left(2 \alpha^{\prime}-\beta^{\prime}\right) h^{1}, \quad \mu^{3}=\left(\lambda_{u^{2}}-\beta^{\prime}\right) h^{3}+\lambda_{x^{2}}
\end{gather*}
$$

while, from (36), we get

$$
\begin{equation*}
\mu^{2}=\delta_{x^{2}}+\left(\delta_{u^{1}}-\beta_{x^{2}}\right) h^{2}+\left(\alpha_{x^{1} x^{1}}-\delta_{u^{1} u^{1}} u_{x^{1}}^{1}\right) h^{1}, \tag{38}
\end{equation*}
$$

where $\alpha\left(x^{1}\right), \beta\left(x^{2}\right), \delta\left(x^{2}, u^{1}\right)$, and $\lambda\left(x^{1}, x^{2}, u^{2}\right)$ are arbitrary real functions of their arguments. Then, going back to the original variables, the most general operator of these continuous weak equivalence transformations reads

$$
\begin{align*}
Y= & \alpha(x) \partial_{x}+\beta(t) \partial_{t}+\delta(t, u) \partial_{u} \\
& +\lambda(x, t, v) \partial_{v}+\left(2 \alpha^{\prime}-\beta^{\prime}\right) f \partial_{f} \\
& +\left(\delta_{t}+\left(\delta_{u}-\beta_{t}\right) g+\left(\alpha^{\prime \prime}-\delta_{u u} u_{x}\right) f\right) \partial_{g}  \tag{39}\\
& +\left(\left(\lambda_{v}-\beta^{\prime}\right) h+\lambda_{t}\right) \partial_{h} .
\end{align*}
$$

## 4. Symmetries for the System (1)

In the next sections in order to carry out symmetries for the system (1) we do not use the classical Lie approach. Instead of the mentioned method we apply the projection theorem, introduced in [25] and eventually reconsidered in [13, 14, 18, 19]. In agreement with these references, we can affirm the following.

## Theorem 1. Let

$$
\begin{align*}
Y= & \alpha(x) \partial_{x}+\beta(t) \partial_{t}+\delta(t, u) \partial_{u} \\
& +\lambda(x, t, v) \partial_{v}+\left(2 \alpha^{\prime}-\beta^{\prime}\right) f \partial_{f} \\
& +\left(\delta_{t}+\left(\delta_{u}-\beta_{t}\right) g+\left(\alpha^{\prime \prime}-\delta_{u u} u_{x}\right) f\right) \partial_{g}  \tag{40}\\
& +\left(\left(\lambda_{v}-\beta^{\prime}\right) h+\lambda_{t}\right) \partial_{h}
\end{align*}
$$

be an infinitesimal equivalence generator for the system (1); then the operator

$$
\begin{equation*}
X=\alpha(x) \partial_{x}+\beta(t) \partial_{t}+\delta(t, u) \partial_{u}+\lambda(x, t, v) \partial_{v} \tag{41}
\end{equation*}
$$

which corresponds to the projection of $Y$ on the space $(x, t, u, v)$, is an infinitesimal symmetry generator of the system (1) if and only if the constitutive equations, specifying the forms of $f, h$, and $g$, are invariant with respect to $Y$.

For the system under consideration, in general, the constitutive equations whose invariance must be requested are

$$
\begin{gather*}
f=D(u), \\
g=G\left(u, v, u_{x}\right),  \tag{42}\\
h=F(u, v)
\end{gather*}
$$

The request of invariance

$$
\begin{align*}
& \left.Y(f-D(u))\right|_{(42)}=0,\left.\quad Y^{(1)}\left(g-G\left(u, v, u_{x}\right)\right)\right|_{(42)}=0, \\
& \left.Y(h-F(u, v))\right|_{(42)}=0 \tag{43}
\end{align*}
$$

brings us to the following equations:

$$
\begin{gather*}
\mu^{1}-\eta^{1} D_{u}=0, \\
\mu^{2}-\zeta_{1}^{1} G_{u_{x}}-\eta^{1} G_{u}-\eta^{2} G_{v}=0,  \tag{44}\\
\mu^{3}-\eta^{1} F_{u}-\eta^{2} F_{v}=0,
\end{gather*}
$$

under the restrictions (42).
Then substituting

$$
\begin{equation*}
\zeta_{1}^{1}=\left(\eta_{u^{1}}^{1}-\xi_{1}^{1}\right) u_{1}^{1}=\left(\delta_{u}-\alpha^{\prime}\right) u_{x} \tag{45}
\end{equation*}
$$

and taking into account the constraints we can write (44) as

$$
\begin{gather*}
\left(2 \alpha^{\prime}-\beta^{\prime}\right) D(u)-\delta(t, u) D_{u}=0  \tag{46}\\
\delta_{t}+\left(\delta_{u}-\beta_{t}\right) G-\left(\delta_{u u} u_{x}^{2}-\alpha_{x x} u_{x}\right) D-\left(\delta_{u}-\alpha^{\prime}\right) u_{x} G_{u_{x}} \\
-\delta G_{u}-\lambda(t, x, v) G_{v}=0 \tag{47}
\end{gather*}
$$

$$
\begin{equation*}
\left(\left(\lambda_{v}-\beta^{\prime}\right) F+\lambda_{t}\right)-\delta F_{u}-\lambda(t, x, v) F_{v}=0 \tag{48}
\end{equation*}
$$

We recall here that the principal Lie algebra $L_{\mathscr{P}}[5,12]$ is the Lie algebra of the principal Lie group, that is, the group of the all Lie point symmetries

$$
\begin{align*}
X= & \xi(x, t, u, v) \frac{\partial}{\partial x}+\tau(x, t, u, v) \frac{\partial}{\partial t} \\
& +\eta^{1}(x, t, u, v) \frac{\partial}{\partial u}+\eta^{2}(x, t, u, v) \frac{\partial}{\partial v} \tag{49}
\end{align*}
$$

that leave the system (1) invariant for any form of the functions $D(u), G\left(u, v, u_{x}\right)$, and $F(u, v)$. In other words we can remark that the principal Lie algebra is the subalgebra of the equivalence algebra such that any operator $Y$ of this subalgebra leaves the equations $f=D(u), g=G\left(u, v, u_{x}\right)$, and $h=$ $F(u, v)$ invariant for any form of the functions $D(u)$, $G\left(u, v, u_{x}\right)$, and $F(u, v)$. Then we can say [5] the following.

Corollary 2. An equivalence operator for the system (1) belongs to the principal Lie algebra $L_{\mathscr{P}}$ if and only if $\eta^{i}=0, \mu^{j}=0$, $i=1,2$, and $j=1,2,3$.

Taking Corollary 2 into account from the previous equations (46)-(48) it is a simple matter to ascertain that the $L_{\mathscr{P}}$ [5, 12] is spanned by the following translation generators:

$$
\begin{equation*}
X_{0}=\partial_{t}, \quad X_{1}=\partial_{x} \tag{50}
\end{equation*}
$$

## 5. Some Extensions of $L_{\mathscr{P}}$

In order to show some extensions of the principal algebra, which could be of interest in biomathematics, we assume that the advection-reaction function is of the form

$$
\begin{equation*}
G=\rho u^{r} u_{x}^{s}+\Gamma_{1} u^{a}+\Gamma_{2} v^{b} \tag{51}
\end{equation*}
$$

where the parameters $\rho, \Gamma_{1}, \Gamma_{2}, r, s, a$, and $b$ are constitutive parameters of the considered phenomena.

This form of $G$ is a generalization of

$$
\begin{equation*}
G=-2 v u^{q} u_{x}+\frac{\gamma}{k} v+\left(\frac{\gamma}{k}-m_{1}\right) u \tag{52}
\end{equation*}
$$

appearing in (3), where

$$
\begin{equation*}
\rho=-2 \nu, \quad \Gamma_{1}=\left(\frac{\gamma}{k}-m_{1}\right), \quad \Gamma_{2}=\frac{\gamma}{k} \tag{53}
\end{equation*}
$$

Consequently in (51) we must consider $\Gamma_{2}>0$ and as limit cases $\Gamma_{1}=0$ and $a=0$. Moreover, in this section, we assume that the value $s=0$ will not be considered because in this case the advective effects disappear. We also assume that $b \neq 0$. This last restriction implies that the balance equation of the density $u$ depends on the density $v$. Finally for the sake of simplicity we omit the limit case $\Gamma_{1}=0$ and assume that the diffusion is only nonlinear; that is, $D_{u} \neq 0$.

In the following we continue the discussion of invariance conditions written in the previous section.

From (46), by deriving with respect to $x$, we get

$$
\begin{equation*}
\alpha^{\prime \prime}=0 \tag{54}
\end{equation*}
$$

Then

$$
\begin{equation*}
\alpha=\alpha_{1} x+\alpha_{0} \tag{55}
\end{equation*}
$$

with $\alpha_{1}$ and $\alpha_{0}$ arbitrary constants, so (46) becomes

$$
\begin{equation*}
\left(2 \alpha_{1}-\beta^{\prime}\right) D(u)-\delta(t, u) D_{u}=0 \tag{56}
\end{equation*}
$$

while, after having taken (51) into account, (47) reads

$$
\begin{align*}
\delta_{t} & +\left(\delta_{u}-\beta_{t}\right)\left(\rho u^{r} u_{x}^{s}+\Gamma_{1} u^{a}+\Gamma_{2} v^{b}\right) \\
& -D \delta_{u u} u_{x}^{2}-\left(\delta_{u}-\alpha_{1}\right) \rho s u^{r} u_{x}^{s}+  \tag{57}\\
& -\delta\left(\rho r u^{r-1} u_{x}^{s}+\Gamma_{1} a u^{a-1}\right)-\lambda(t, x, v) \Gamma_{2} b v^{b-1}=0 .
\end{align*}
$$

From (57) we get immediately

$$
\begin{equation*}
\lambda=\lambda(t, v) . \tag{58}
\end{equation*}
$$

In the following we analyze separately the case $s \neq 2$ and the case $s=2$.
5.1. $s \neq 2$. From $u_{x}^{2}$ coefficient we get $\delta_{u u}=0$; that is,

$$
\begin{equation*}
\delta=\delta_{1}(t) u+\delta_{0}(t) \tag{59}
\end{equation*}
$$

Therefore, from the remaining terms we have

$$
\begin{align*}
& \rho u^{r-1} u_{x}^{s}\left[u\left((1-s-r) \delta_{1}+s \alpha_{1}-\beta^{\prime}\right)-r \delta_{0}\right] \\
& \quad+\delta_{0}^{\prime}+\delta_{1}^{\prime} u+\Gamma_{1}\left((1-a) \delta_{1}-\beta^{\prime}\right)+  \tag{60}\\
& \quad-a \Gamma_{1} \delta_{0} u^{a-1}+\Gamma_{2}\left(\delta_{1}-\beta^{\prime}\right) v^{b}-\Gamma_{2} \lambda b v^{b-1}=0
\end{align*}
$$

As we assumed $s \neq 0$, from the coefficient of $u_{x}^{s}$ in (60) we conclude that $\delta_{0}(t)=0$ and

$$
\begin{equation*}
\beta^{\prime}=(1-s-r) \delta_{1}+s \alpha_{1} \tag{61}
\end{equation*}
$$

Then, still from (60) we have the following constraints to consider:

$$
\begin{gather*}
\delta_{1}^{\prime} u+\Gamma_{1} u^{a}\left((1-a) \delta_{1}-\beta^{\prime}\right)=0  \tag{62}\\
\left(\delta_{1}-\beta^{\prime}\right) v^{b}-\lambda b v^{b-1} \tag{63}
\end{gather*}
$$

From (62) two cases are obtained.
(i) Case $a \neq 1$. Then, from (62) we conclude that $\delta_{1}=$ const and it follows that

$$
\begin{equation*}
\beta^{\prime}=(1-a) \delta_{1} \tag{64}
\end{equation*}
$$

From (64) and (61) we obtain

$$
\begin{gather*}
\alpha_{1}=\frac{s+r-a}{s} \delta_{1},  \tag{65}\\
\beta=(1-a) \delta_{1} t+\beta_{0}
\end{gather*}
$$

with $\beta_{0}$ and $\delta_{1}$ arbitrary constants.
The analysis of (63) leads to the following two subcases.
(1) Consider $\lambda(v)=\lambda_{0} v$, with

$$
\begin{equation*}
\lambda_{0}=\frac{a}{b} \delta_{1} \tag{66}
\end{equation*}
$$

Taking into account the previous results and going back to (56) and (48) we get that the system (3) with $G$ of the form (51) admits the 3 -dimensional Lie algebra spanned by the translations in space and time and by the following additional generator:

$$
\begin{equation*}
X_{3}=(1-a) \partial_{t}+\frac{1}{s}(s+r-a) x \partial_{x}+u \partial_{u}+\frac{a}{b} v \partial_{v} \tag{67}
\end{equation*}
$$

provided that $D$ and $F$ are solutions of the following differential equations:

$$
\begin{gather*}
u D_{u}=\left((1+a)+2 \frac{r-a}{s}\right) D  \tag{68}\\
b u F_{u}+a v F_{v}=(a-b(1-a)) F
\end{gather*}
$$

(2) Consider $\lambda(v)=\delta_{1}=0$. In this case the only symmetries admitted are translations of the independent variables and the form of $D$ and $F$ is arbitrary, so there is not an extension of the principal Lie algebra.
(ii) Case $a=1$. In this case from (62) it follows that

$$
\begin{equation*}
\delta_{1}^{\prime}-\Gamma_{1} \beta^{\prime}=0 \tag{69}
\end{equation*}
$$

After having substituted (69) into (61) we obtain

$$
\begin{gather*}
\beta(t)=-\beta_{0}+c_{1} e^{(1-s-r) \Gamma_{1} t}, \\
\delta_{1}(t)=\frac{s}{s+r-1} \alpha_{1}+c_{1} \Gamma_{1} e^{(1-s-r) \Gamma_{1} t}, \tag{70}
\end{gather*}
$$

with $\alpha_{1}, \beta_{0}$, and $c_{1}$ arbitrary constants and once assumed $s+r-$ $1 \neq 0$. Finally we analyze the contribution of (63) from where the following two subcases arise.
(1) Consider $\lambda(t, v)=\lambda_{0}(t) v$, with

$$
\begin{equation*}
\lambda_{0}(t)=\frac{1}{b} \delta_{1}(t) \tag{71}
\end{equation*}
$$

Then taking into account the previous results it reads

$$
\begin{equation*}
\lambda_{0}(t)=\frac{1}{b}\left[\frac{s}{s+r-1} \alpha_{1}+c_{1} \Gamma_{1} e^{(1-s-r) \Gamma_{1} t}\right] . \tag{72}
\end{equation*}
$$

Going to put the previous result in the condition (46) and by separating the variable we get

$$
\begin{gather*}
\beta(t)=-\beta_{0}, \\
\delta_{1}(t)=\frac{s}{s+r-1} \alpha_{1},  \tag{73}\\
\lambda_{0}(t)=\frac{1}{b}\left[\frac{s}{s+r-1} \alpha_{1}\right],
\end{gather*}
$$

with $\alpha_{1}$ and $\beta_{0}$ arbitrary constants and provided that the diffusion coefficient $D$ is solution of

$$
\begin{equation*}
u D_{u}=2\left(1+\frac{r-1}{s}\right) D \tag{74}
\end{equation*}
$$

From condition (48) we do not get further restrictions on the infinitesimal components of the symmetry generator but only the following constraint on the reaction function $F$ :

$$
\begin{equation*}
b u F_{u}+v F_{v}=F . \tag{75}
\end{equation*}
$$

Taking into account the arbitrariness of $\alpha_{0}$ in this case we have got a 3-dimensional Lie algebra. The additional generator is

$$
\begin{equation*}
X_{3}=x \partial_{x}+\frac{s}{s+r-1} u \partial_{u}+\frac{1}{b} \frac{s}{s+r-1} v \partial_{v} . \tag{76}
\end{equation*}
$$

In particular, from (75), setting $F=k u+\left(k-m_{2}-\gamma\right) v$, we obtain $b=1$, while from (75) we obtain the power function $D=D_{0} u^{2((r+s-1) / s)}$.

Remark 3. By putting $r=1$, the corresponding system of class (1) admits still a 3-dimensional symmetry Lie algebra generated by translations in time and in space and by $X_{3}$. These results are in agreement with the ones obtained in [2].

By considering $s+r-1=0$ we do not get extension of $L_{\mathscr{P}}$ for $D$ and $F$ arbitrary. We get, instead, the following extension:

$$
\begin{equation*}
X_{3}=s t \partial_{t}+x \partial_{x}+\Gamma_{1} s t u \partial_{u}+\frac{\Gamma_{1} s t}{b} v \partial_{v} \tag{77}
\end{equation*}
$$

provided that $D=D_{0}=$ const. and $F=\left(\Gamma_{1} / b\right) v$.
(2) Consider $\lambda(t, v)=\delta_{1}(t)=0$. In this case the only symmetries admitted are the translations in space and time and the form of $D$ and $F$ is arbitrary.
5.2. $s=2$. We analyze this case by beginning with the discussion of (56) from which two cases arise.
(1) $D(u)$ Is Arbitrary. It follows, of course, that $\delta=0$ and $\beta^{\prime}=$ $2 \alpha_{1}$ so

$$
\begin{equation*}
\beta=2 \alpha_{1} t+\beta_{0} \tag{78}
\end{equation*}
$$

with $\beta_{0}$ arbitrary constant.
Moreover (46) and (48) become

$$
\begin{gather*}
-2 \alpha_{1} G+\alpha_{1} u_{x} G_{u_{x}}-\lambda(v) G_{v}=0,  \tag{79}\\
\left(\lambda_{v}-2 \alpha_{1}\right) F-\lambda(v) F_{v}=0 . \tag{80}
\end{gather*}
$$

But taking into account that

$$
\begin{equation*}
G=\rho u^{r} u_{x}^{2}+\Gamma_{1} u^{a}+\Gamma_{2} v^{b} \tag{81}
\end{equation*}
$$

(79) becomes

$$
\begin{equation*}
-2 \alpha_{1}\left(\rho u^{r} u_{x}^{2}+\Gamma_{1} u^{a}+\Gamma_{2} v^{b}\right)+2 \alpha_{1} \rho u^{r} u_{x}^{2}-\lambda(v) b \Gamma_{2} v^{b-1}=0 \tag{82}
\end{equation*}
$$

and gives us the following conditions:

$$
\begin{equation*}
2 \alpha_{1} \Gamma_{1}=0, \quad 2 \alpha_{1} v+\lambda(v) b=0 \tag{83}
\end{equation*}
$$

from where, taking into account the work hypotheses at the beginning of this section, we get $\alpha_{1}=\lambda=0$ with $F$ arbitrary function of $u$ and $v$.

In this case the only admitted symmetries are translations in time and space.
(2) Consider $D_{u} / D=\left(2 \alpha_{1}-\beta^{\prime}\right) / \delta(t, u)$. Then by requiring

$$
\begin{equation*}
\frac{\partial}{\partial t}\left(\frac{2 \alpha_{1}-\beta^{\prime}}{\delta(t, u)}\right)=0 \tag{84}
\end{equation*}
$$

we get

$$
\begin{equation*}
-\delta(t, u) \beta^{\prime \prime}-\left(2 \alpha_{1}-\beta^{\prime}\right) \delta_{t}=0 \tag{85}
\end{equation*}
$$

from where we derive
(a) $\delta=\delta(t, u)$ arbitrary function and $2 \alpha_{1}-\beta^{\prime}=0$ and then

$$
\begin{equation*}
\beta=2 \alpha_{1} t+\beta_{0} \tag{86}
\end{equation*}
$$

and $D_{u}=0$, so we omit this case,
(b)

$$
\begin{equation*}
\delta(t, u)=\left(2 \alpha_{1}-\beta^{\prime}\right) A(u) \tag{87}
\end{equation*}
$$

In this case

$$
\begin{equation*}
\frac{D_{u}}{D}=\frac{1}{A(u)} \tag{88}
\end{equation*}
$$

which implies

$$
\begin{equation*}
D(u)=D_{0} e^{\int d u / A(u)} \equiv D_{0} e^{a(u)} \tag{89}
\end{equation*}
$$

where

$$
\begin{equation*}
a^{\prime}(u)=\frac{1}{A(u)} . \tag{90}
\end{equation*}
$$

Equation (47) after (54), (58), and (81) becomes

$$
\begin{align*}
& -\beta^{\prime \prime} A(u)+\left(\left(2 \alpha_{1}-\beta^{\prime}\right) A^{\prime}(u)-\beta_{t}\right)\left(\rho u^{r} u_{x}^{2}+\Gamma_{1} u^{a}+\Gamma_{2} v^{b}\right) \\
& -\left(\delta_{u u} u_{x}^{2}\right) D_{0} e^{\int d u / A(u)}+ \\
& -\left(\left(2 \alpha_{1}-\beta^{\prime}\right) A^{\prime}(u)-\alpha_{1}\right) u_{x} 2 \rho u^{r} u_{x} \\
& -\left(2 \alpha_{1}-\beta^{\prime}\right) A(u)\left(r \rho u^{r-1} u_{x}^{2}+a \Gamma_{1} u^{a-1}\right) \\
& -\lambda(t, v) b \Gamma_{2} v^{b-1}=0 . \tag{91}
\end{align*}
$$

From terms in $v$ we get the following condition:

$$
\begin{equation*}
\left(\left(2 \alpha_{1}-\beta^{\prime}\right) A^{\prime}(u)-\beta^{\prime}\right) v-\lambda(t, v) b=0 \tag{92}
\end{equation*}
$$

which gives us $A^{\prime \prime}(u)=0$ and then $A=A_{1} u+A_{0}$. Therefore, from (87)

$$
\begin{equation*}
\delta(t, u)=\left(2 \alpha_{1}-\beta^{\prime}\right)\left(A_{1} u+A_{0}\right), \tag{93}
\end{equation*}
$$

which implies that

$$
\begin{equation*}
\delta_{u u}=0 . \tag{94}
\end{equation*}
$$

From (92) arise two cases.
(i) Consider $\lambda(t, v)=0$ and $\left(2 \alpha_{1}-\beta^{\prime}\right) A_{1}-\beta^{\prime}=0$. In this case after having derived from (91) the following additional conditions

$$
\begin{gather*}
A_{0}=0, \quad \beta^{\prime \prime} A_{1}=0 \\
2 A_{1}\left(2 \alpha_{1}-\beta^{\prime}\right)(2-r)-2 \alpha_{1}=0, \quad A_{1}\left(2 \alpha_{1}-\beta^{\prime}\right) a \Gamma_{1}=0, \tag{95}
\end{gather*}
$$

it is a simple matter to ascertain that there does not exist extension of $L_{\mathscr{P}}$.
(ii) Consider $\lambda(t, v)=\lambda_{0}(t) v$ with

$$
\begin{equation*}
\lambda_{0}(t)=\frac{1}{b}\left(\left(2 \alpha_{1}-\beta^{\prime}\right) A_{1}-\beta^{\prime}\right) \tag{96}
\end{equation*}
$$

Then (91) assumes the following form:

$$
\begin{align*}
& -\beta^{\prime \prime}\left(A_{1} u+A_{0}\right)+\left(\left(2 \alpha_{1}-\beta^{\prime}\right) A_{1}-\beta_{t}\right)\left(\rho u^{r} u_{x}^{2}+\Gamma_{1} u^{a}\right)+ \\
& -\left(\left(2 \alpha_{1}-\beta^{\prime}\right) A_{1}-\alpha_{1}\right) u_{x} 2 \rho u^{r} u_{x} \\
& -\left(2 \alpha_{1}-\beta^{\prime}\right)\left(A_{1} u+A_{0}\right)\left(r \rho u^{r-1} u_{x}^{2}+a \Gamma_{1} u^{a-1}\right)=0 . \tag{97}
\end{align*}
$$

For $r \neq 1$, from (97) we obtain the following:

$$
\begin{gather*}
A_{0}=0, \quad \beta=\beta_{1} t+\beta_{0}, \\
\left(2 \alpha_{1}-\beta_{1}\right)\left(1-A_{1}(1+r)\right)=0,  \tag{98}\\
\Gamma_{1}\left[\left(2 \alpha_{1}-\beta_{1}\right) A_{1}(1-a)-\beta_{1}\right]=0 .
\end{gather*}
$$

From the previous conditions we consider the following subclasses.
(A) Consider $2 \alpha_{1}-\beta_{1}=0$ and $\left(2 \alpha_{1}-\beta_{1}\right) A_{1}(1-a)-\beta_{1}=0$. As a consequence we get

$$
\begin{equation*}
\beta_{1}=\alpha_{1}=\delta=\lambda=0, \tag{99}
\end{equation*}
$$

and then there is no extension of $L_{\mathscr{P}}$.
(B) For $1-A_{1}(1+r)=0$ and $\left(2 \alpha_{1}-\beta_{1}\right) A_{1}(1-a)-\beta_{1}=0$ we get, for $a \neq 1$,

$$
\begin{gather*}
A_{1}=\frac{1}{1+r}, \quad \alpha=\frac{\beta_{1}}{2}(2-a-r) x+\alpha_{0}, \quad \beta=\beta_{1} t+\beta_{0} \\
\delta=\frac{\beta_{1}}{1-a} u, \quad \lambda=\beta_{1} \frac{1}{b} \frac{a}{1-a} v \tag{100}
\end{gather*}
$$

and then in this subcase we got an extension by one of $L_{\mathscr{P}}$ given by

$$
\begin{equation*}
X_{3}=t \partial_{t}+\frac{2-a-r}{2} x \partial_{x}+\frac{u}{1-a} \partial_{t} u+\frac{a v}{(1-a) b} \partial_{v} \tag{101}
\end{equation*}
$$

provided that $D(v)$ and $F(u, v)$ are solutions of the following equations:

$$
\begin{gather*}
\frac{u}{1-a} D_{u}=(1-a-r) D  \tag{102}\\
(a-b(1-a)) F-b u F_{u}-a v F_{v}=0 .
\end{gather*}
$$

For $a=1$, instead, we conclude that

$$
\begin{equation*}
\beta_{1}=0, \quad \delta=\frac{2 \alpha_{1}}{1+r}, \quad \lambda=\frac{1}{b} \frac{2 \alpha_{1}}{1+r} v . \tag{103}
\end{equation*}
$$

Therefore, the extension is given by

$$
\begin{equation*}
X_{3}=x \partial_{x}+\frac{2}{1+r} u \partial_{u}+\frac{1}{b} \frac{2}{1+r} v \tag{104}
\end{equation*}
$$

provided that $D(u)$ and $F(u, v)$ are solutions of the following differential equations:

$$
\begin{gather*}
\frac{u}{1+r} D_{u}=D,  \tag{105}\\
F-b u F_{u}-v F_{v}=0 .
\end{gather*}
$$

## 6. Conclusions

In this paper we have considered a class of reaction-diffusion systems with an additional advection term. The studied class includes, as particular cases, all partial differential equation models concerned with the Aedes aegypti mosquito that have been proposed until now. We have investigated such a system from the point of view of equivalence transformations in the spirit of the Lie-Ovsiannikov algorithm based on the Lie infinitesimal criterion. In agreement with some modifications of the Lie-Ovsiannikov algorithm, introduced in [20], we obtained a group of weak equivalence transformations. We
have applied these transformations in order to obtain symmetry generators by using a projection theorem. In particular after having obtained the principal Lie algebra, we investigated a specific, but quite general, form for the advectionreaction term $G$ and we derived some extensions of the principal Lie algebra. The specializations of the results for the systems studied in [2] are in agreement with those ones obtained there.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Global Existence and Large Time Behavior of Solutions to the Bipolar Nonisentropic Euler-Poisson Equations 

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#### Abstract

We study the one-dimensional bipolar nonisentropic Euler-Poisson equations which can model various physical phenomena, such as the propagation of electron and hole in submicron semiconductor devices, the propagation of positive ion and negative ion in plasmas, and the biological transport of ions for channel proteins. We show the existence and large time behavior of global smooth solutions for the initial value problem, when the difference of two particles' initial mass is nonzero, and the far field of two particles' initial temperatures is not the ambient device temperature. This result improves that of Y.-P. Li, for the case that the difference of two particles' initial mass is zero, and the far field of the initial temperature is the ambient device temperature.


## 1. Introduction

In this paper we study the following 1D bipolar nonisentropic Euler-Poisson equations:

$$
\begin{align*}
& n_{1 t}+j_{1 x}=0, \\
& j_{1 t}+\left(\frac{j_{1}^{2}}{n_{1}}\right)_{x}+\left(n_{1} T_{1}\right)_{x}=n_{1} E-j_{1}, \\
& T_{1 t}+\frac{j_{1}}{n_{1}} T_{1 x}+\frac{2}{3} T_{1}\left(\frac{j_{1}}{n_{1}}\right)_{x}-\frac{2}{3 n_{1}} T_{1 x x}=\frac{1}{3}\left(\frac{j_{1}}{n_{1}}\right)^{2}-\left(T_{1}-T^{*}\right), \\
& n_{2 t}+j_{2 x}=0, \\
& j_{2 t}+\left(\frac{j_{2}^{2}}{n_{2}}\right)_{x}+\left(n_{2} T_{2}\right)_{x}=-n_{2} E-j_{2}, \\
& T_{2 t}+\frac{j_{2}}{n_{2}} T_{2 x}+\frac{2}{3} T_{2}\left(\frac{j_{2}}{n_{2}}\right)_{x}-\frac{2}{3 n_{2}} T_{2 x x}=\frac{1}{3}\left(\frac{j_{2}}{n_{2}}\right)^{2}-\left(T_{2}-T^{*}\right), \\
& E_{x}=n_{1}-n_{2}, \tag{1}
\end{align*}
$$

where $n_{i}>0, j_{i}, T_{i},(i=1,2)$, and $E$ denote the particle densities, current densities, temperatures, and the electric field, respectively, and $T^{*}>0$ stands for the ambient device temperature. The system (1) models various physical phenomena, such as the propagation of electron and hole in submicron semiconductor derives, the propagation of positive ion and negative ion in plasmas, and the biological transport of ions for channel proteins. When the temperature $T_{i}(i=$ $1,2)$ is the function of the density $n_{i}(i=1,2)$, the system (1) reduces to the isentropic bipolar Euler-Poisson equations. For more details on the bipolar isentropic and nonisentropic Euler-Poisson equations (hydrodynamic models), we can see [1-3] and so forth.

Due to their physical importance, mathematical complexity, and wide rang, of applications, many results concerning the existence and uniqueness of (weak, strong, or smooth) solutions for the bipolar Euler-Poisson equations can be found in $[4-14]$ and the references cited therein. However, the study of the corresponding nonisentropic bipolar EulerPoisson equation is very limited in the literature. In [15] Li investigated the global existence and nonlinear diffusive waves of smooth solutions for the initial value problem of the one-dimensional nonisentropic bipolar hydrodynamic model when the difference of two particles' initial mass is
zero, and the far field of two particles' initial temperatures is the ambient device temperature. We also mention that there are some results about the relaxation limit and quasineutral limit of the bipolar Euler-Poisson system see [16-19]. In this paper, we will show the existence and large time behavior of global smooth solutions for the initial value problem of (1), when the difference of two particles' initial mass is nonzero and the far field of the initial temperatures is not the ambient device temperature. We now prescribe the following initial data:

$$
\begin{gather*}
\left(n_{i}, j_{i}, T_{i}\right)(x, t=0)=\left(n_{i 0}, j_{i 0}, T_{i 0}\right)(x), \\
n_{i 0}>0, \quad i=1,2, \\
\lim _{x \rightarrow \pm \infty}\left(n_{i 0}, j_{i 0}, T_{i 0}\right)(x)=\left(n_{ \pm}, j_{i \pm}, T_{i \pm}\right),  \tag{2}\\
n_{ \pm}>0, T_{i \pm}>0, \quad i=1,2,
\end{gather*}
$$

and ( $n_{ \pm}, j_{i \pm}, T_{i \pm}$ ) are the state constants. We also give the electric field as $x=-\infty$; that is,

$$
\begin{equation*}
E(-\infty, t)=0 . \tag{3}
\end{equation*}
$$

The nonlinear diffusive phenomena both in smooth and weak senses were also observed for the bipolar isentropic and nonisentropic by Gasser et al. [4], Huang and Li [5], and Li [15], respectively. Namely, according to the Darcy's law, it is expected that the solutions $\left(n_{1}, j_{1}, T_{1}, n_{2}, j_{2}, T_{2}, E\right)(x, t)$ converge in $L^{\infty}$-sense to $\left(\bar{n}, \bar{j}, T^{*}, \bar{n}, \bar{j}, T^{*}, 0\right)(x, t)$; here $(\bar{n}, \bar{j})=$ $(\bar{n}, \bar{j})\left(\left(x+x_{0}\right) / \sqrt{1+t}\right)\left(x_{0}\right.$ is a shift constants) is the nonlinear diffusion waves, which is self-similar solutions to the following equations:

$$
\begin{gather*}
\bar{n}_{t}-\left(\bar{n} T^{*}\right)_{x x}=0, \\
\bar{j}:=-\left(\bar{n} T^{*}\right)_{x}  \tag{4}\\
(\bar{n}, \bar{j}) \longrightarrow\left(n_{ \pm}, 0\right), \quad \text { as } x \longrightarrow \pm \infty
\end{gather*}
$$

Note that in [15], the author assumed that

$$
\begin{equation*}
j_{i+}=j_{i-}, \quad T_{i \pm}=T^{*}, \quad i=1,2 \tag{5}
\end{equation*}
$$

which lead to the difference of two particles' initial mass to be zero; that is,

$$
\begin{equation*}
\int_{\mathbb{R}}\left[n_{10}(x)-n_{20}(x)\right] d x=0, \quad i=1,2 \tag{6}
\end{equation*}
$$

This implies, from the last equation of (1), that

$$
\begin{equation*}
E(+\infty, t)-E(-\infty, t)=0 . \tag{7}
\end{equation*}
$$

In this paper, we try to drop off these too stiff conditions. That is, $j_{i+} \neq j_{i-}, T_{i \pm} \neq T^{*}(i=1,2)$. Moreover, for stating our results, set for $i=1,2$,

$$
\begin{align*}
& \left(\varphi_{i 0}, \psi_{i 0}, \theta_{i 0}\right)(x) \\
& =\left(\int_{-\infty}^{x}\left[n_{i 0}(\xi)-\widehat{n}_{i}(\xi, t)-\bar{n}\left(\xi+x_{0}, t=0\right)\right] d \xi\right. \\
& \left.\quad j_{i 0}(x)-\widehat{j}_{i 0}(x)-\bar{j}\left(x+x_{0}, t=0\right), T_{i 0}(x)-\widehat{T}_{i 0}(x)-T^{*}\right), \tag{8}
\end{align*}
$$

where ( $\widehat{n}_{1}, \widehat{j}_{1}, \widehat{T}_{1}, \widehat{n}_{2}, \widehat{j}_{2}, \widehat{T}_{2}, \widehat{E}$ ) are the gap functions (or say correction functions) which will be given in Section 2, and $(\bar{n}, \bar{j})=(\bar{n}, \bar{j})\left(x+x_{0}, t\right)$ are the shifted diffusion waves with $x_{0}:=\left(1 /\left(n_{+}-n_{-}\right)\right) \int_{\mathbb{R}}\left[n_{i 0}(x)-\widehat{n}_{i 0}(x)-\bar{n}(x)\right] d x$ for $i=1,2$.

Throughout this paper, the diffusion waves are always denoted by $(\bar{n}, \bar{j})(x / \sqrt{1+t})$. $C$ denotes the generic positive constant. $L^{p}(\mathbb{R})(1 \leq p<\infty)$ denotes the space of measurable functions whose $p$-powers are integrable on $\mathbb{R}$, with the norm $\|\cdot\|_{L^{p}}=\left(\int_{\mathbb{R}}|\cdot|^{p} d x\right)^{1 / p}$, and $L^{\infty}$ is the space of bounded measurable functions on $\mathbb{R}$, with the norm $\|\cdot\|_{L^{\infty}}=\operatorname{esssup}_{x}|\cdot|$. Without confusion, we also denote the norm of $L^{2}(\mathbb{R})$ by $\|\cdot\|$ for brevity. $H^{k}(\mathbb{R})\left(H^{k}\right.$ without any ambiguity) denotes the usual Sobolev space with the norm $\|\cdot\|_{k}$, especially $\|\cdot\|_{0}=\|\cdot\|$.

Now we state our main results as follows.
Theorem 1. Let $\left(\phi_{i 0}, \psi_{i 0}, \theta_{i 0}\right)(i=1,2) \in H^{3}(\mathbb{R}) \times H^{2}(\mathbb{R}) \times$ $H^{3}(\mathbb{R})$, and set $\delta:=\left|j_{1+}\right|+\left|j_{1-}\right|+\left|j_{2+}\right|+\left|j_{2-}\right|+\left|T_{1+}-T^{*}\right|+\mid T_{1-}-$ $T^{*}\left|+\left|T_{2+}-T^{*}\right|+\left|T_{2-}-T^{*}\right|+\left|n_{+}-n_{-}\right|\right.$and $\Phi_{0}:=\left\|\left(\phi_{10}, \phi_{20}\right)\right\|_{3}+$ $\left\|\left(\psi_{10}, \psi_{20}\right)\right\|_{2}+\left\|\left(\theta_{10}, \theta_{20}\right)\right\|_{3}$. Then, there is a $\delta_{0}>0$ such that if $\Phi_{0}+\delta \leq \delta_{0}$ the solutions ( $\left.n_{1}, n_{2}, j_{1}, j_{2}, \theta_{1}, \theta_{2}, E\right)$ of IVP (1)-(3) uniquely and globally exist and satisfy

$$
\begin{align*}
& n_{1}-\widehat{n}_{1}-\bar{n}, n_{2}-\widehat{n}_{2}-\bar{n} \\
& \in C\left([0,+\infty), H^{2}(\mathbb{R})\right) \cap C^{1}\left([0,+\infty), H^{1}(\mathbb{R})\right), \\
& j_{1}-\widehat{j_{1}}-\bar{j}, j_{2}-\widehat{j_{2}}-\bar{j} \\
& \in C\left([0,+\infty), H^{2}(\mathbb{R})\right) \cap C^{1}\left([0,+\infty), H^{1}(\mathbb{R})\right), \\
& T_{1}-\widehat{T}_{1}-T^{*}, T_{2}-\widehat{T}_{2}-T^{*} \\
& \quad C\left([0,+\infty), H^{3}(\mathbb{R})\right) \cap C^{1}\left([0,+\infty), H^{1}(\mathbb{R})\right) \\
& E-\widehat{E} \in C\left([0,+\infty), H^{3}(\mathbb{R})\right) \\
& \cap C^{1}\left([0,+\infty), H^{2}(\mathbb{R})\right) \cap C^{2}\left([0,+\infty), H^{1}(\mathbb{R})\right) . \tag{9}
\end{align*}
$$

Moreover, it holds that

$$
\begin{aligned}
& \sum_{k=0}^{2}(1+t)^{k+1} \| \partial_{x}^{k}\left(n_{1}-\widehat{n}_{1}-\bar{n}, T_{1}-\widehat{T}_{1}-T^{*}\right. \\
& \left.n_{2}-\widehat{n}_{2}-\bar{n}, T_{2}-\widehat{T}_{2}-T^{*}\right)(t) \|^{2} \\
& +\sum_{k=0}^{2}(1+t)^{k+2}\left\|\partial_{x}^{k}\left(j_{1}-\hat{j}_{1}-\bar{j}, j_{2}-\widehat{j}_{2}-\bar{j}\right)(t)\right\|^{2} \\
& \quad+(1+t)^{3}\left\|\partial_{x}^{3}\left(T_{1}-\widehat{T}_{1}-T^{*}, T_{2}-\widehat{T}_{2}-T^{*}\right)(t)\right\|^{2} \\
& \leq C\left(\delta+\Phi_{0}\right),
\end{aligned}
$$

$$
\begin{align*}
\|\left(n_{1}\right. & \left.-\widehat{n}_{1}-n_{2}+\widehat{n}_{2}\right)(t) \|_{1}^{2} \\
& +\left\|\left(j_{1}-\hat{j}_{1}-j_{2}+\widehat{j}_{2}\right)(t)\right\|_{1}^{2} \\
& +\left\|\left(T_{1}-\widehat{T}_{1}-T_{2}+\widehat{T}_{2}\right)(t)\right\|_{2}^{2} \\
& +\|(E-\widehat{E})(t)\|_{2}^{2} \\
\leq & C\left(\Phi_{0}+\delta\right) e^{-\alpha t} \tag{10}
\end{align*}
$$

for some constant $\alpha>0$.
Remark 2. It is more important and interesting that we should discuss the existence and large time behavior of global smooth solution for the bipolar nonisentropic Euler-Poisson system with the general ambient device temperature functions, instead of the constant ambient device temperature, as in [20]. Moreover, we also should consider the similar problem for the corresponding multi-dimensional bipolar non-isentropic Euler-Poisson systems. These are left for the forthcoming future.

The rest of this paper is arranged as follows. In Section 2, we make some necessary preliminaries. That is, we first give some well-known results on the diffusion waves and one key inequality will be used later; then we trickly construct the correction functions to delete the gaps between the solutions and the diffusion waves at the far field. We reformulate the original problem in terms of a perturbed variable and state local-in-time existence of classical solutions in Section 3. Section 4 is used to establish the uniformly a priori estimate and to show the global existence of smooth solutions, while we prove the algebraic convergence rate of smooth solutions in Section 5.

## 2. Some Preliminaries

In this section, we state the nonlinear diffusive wave and then construct the correction functions. First of all, we list some known results concerning the self-similar solution of the nonlinear parabolic equation (4). Let us recall that the nonlinear parabolic equation

$$
\begin{gather*}
\bar{n}_{t}-\left(\bar{n} T^{*}\right)_{x x}=0, \\
\bar{n} \longrightarrow n_{ \pm}, \quad \text { as } x \longrightarrow \pm \infty, \tag{11}
\end{gather*}
$$

possesses a unique self-similar solution $\bar{n}(x, t) \triangleq \bar{n}(\xi), \xi=$ $x / \sqrt{1+t}$, which is increasing if $n_{-}<n_{+}$and decreasing if $n_{-}>n_{+}$. The corresponding Darcy law is $\bar{j}:=-\left(\bar{n} T^{*}\right)_{x}$ satisfying $\bar{j} \rightarrow 0$ as $x \rightarrow \pm \infty$.

Lemma 3 (see $[4,15,21]$ ). For the self-similar solution of (11), it holds

$$
\begin{gathered}
\left|\bar{n}(\xi)-n_{+}\right|_{\xi>0}+\left|\bar{n}(\xi)-n_{-}\right|_{\xi<0} \\
\leq C\left|n_{+}-n_{-}\right| e^{-v_{0} \xi^{2}}
\end{gathered}
$$

$$
\begin{align*}
& \left|\partial_{x}^{k} \partial_{t}^{l} \bar{n}(x, t)\right| \\
& \quad \leq C\left|n_{+}-n_{-}\right|(1+t)^{-(k+2 l) / 2} e^{-v_{0} \xi^{2}} \quad k+l \geq 1, \quad k, l \geq 0 \\
& \int_{-\infty}^{0}\left|\bar{n}(x, t)-n_{-}\right|^{2} d x+\int_{0}^{+\infty}\left|\bar{n}(x, t)-n_{+}\right|^{2} d x \\
& \quad \leq C\left|n_{+}-n_{-}\right|^{2}(1+t)^{1 / 2} \\
& \int_{\mathbb{R}}\left|\partial_{x}^{k} \partial_{t}^{l} \bar{n}\right|^{2} d x \\
& \quad \leq C\left|n_{+}-n_{-}\right|^{2}(1+t)^{(1 / 2)-2 l-k}, \quad k+l \geq 1 \tag{12}
\end{align*}
$$

where $\nu_{0}>0$ is a constant.
Next, we construct the gap function, which will be used in Sections 3 and 4. First of all, motivated by [6, 22], let us look into the behaviors of the solutions to (1)(3) at the far fields $x= \pm \infty$. Then we may understand how big the gaps are between the solution and the diffusion waves at the far fields. Let $\left(n_{i}^{ \pm}(t), j_{i}^{ \pm}(t), T_{i}^{ \pm}(t)\right) \quad:=$ $\left(n_{i}( \pm \infty, t), j_{i}( \pm \infty, t), T_{i}( \pm \infty, t)\right), i=1,2$ and $E^{ \pm}(t) \quad:=$ $E( \pm \infty, t)$. From $(1)_{1}$ and $(1)_{4}$, since $\left.\partial_{x} j_{i}\right|_{x= \pm \infty}=0$ for $i=1,2$, it can be easily seen that

$$
\begin{equation*}
n_{i}^{ \pm}(t)=n_{i}( \pm \infty, t) \equiv n_{ \pm} . \tag{13}
\end{equation*}
$$

Differentiating $(1)_{7}$ with respect to $t$ and applying $(1)_{1}$ and $(1)_{4}$, we have $E_{t x}=\left(n_{1}-n_{2}\right)_{t}=-\left(j_{1}-j_{2}\right)_{x}$, which implies

$$
\begin{align*}
\frac{d}{d t} E^{+} & (t)-\frac{d}{d t} E^{-}(t)  \tag{14}\\
& =-\left[j_{1}^{+}(t)-j_{2}^{+}(t)\right]+\left[j_{1}^{-}(t)-j_{2}^{-}(t)\right]
\end{align*}
$$

Taking $x= \pm \infty$ to $(1)_{2,3}$ and $(1)_{5,6}$, we also formally have

$$
\begin{gather*}
\frac{d}{d t} j_{1}^{ \pm}(t)=n_{ \pm} E^{ \pm}(t)-j_{1}^{ \pm}(t),  \tag{15}\\
\frac{d}{d t} j_{2}^{ \pm}(t)=-n_{ \pm} E^{ \pm}(t)-j_{2}^{ \pm}(t),  \tag{16}\\
\frac{d}{d t} T_{i}^{ \pm}(t)=\frac{1}{3}\left(\frac{j_{i}^{ \pm}}{n_{i}^{ \pm}}\right)^{2}-\left(T_{i}^{ \pm}-T^{*}\right), \quad i=1,2 . \tag{17}
\end{gather*}
$$

It can be easily seen that (14)-(17) can uniquely determine the unknown state functions $j_{i}^{ \pm}(t), T_{i}^{ \pm}(t)(i=1,2)$, and $E^{+}(t)$ since we have known $E^{-}(t)=E(-\infty, t)=0$. Solving these
O.D.E and noticing (13), there exists some constant $0<\beta_{0}<$ $1 / 2$ such that

$$
\begin{gather*}
n_{i}( \pm \infty, t)=n_{ \pm}, \quad i=1,2, \\
\left|j_{i}(+\infty, t)\right|=O(1) e^{-\beta_{0} t}, \quad i=1,2, \\
j_{i}(-\infty, t)=O(1) e^{-t}, \quad i=1,2, \\
T_{i}( \pm \infty, t)=T^{*}+\left(T_{i \pm}-T^{*}\right) e^{-t}+O(1) e^{-\beta_{0} t}, \quad i=1,2, \\
|E(+\infty, t)|=O(1) e^{-\beta_{0} t}, \\
E(-\infty, t)=0 . \tag{18}
\end{gather*}
$$

Obviously, there are some gaps between $j_{i}( \pm \infty, t)$ and $\bar{j}( \pm \infty, t), T_{i}( \pm \infty, t)$ and $T^{*}$, and $E(+\infty, t)$ and $\bar{E} \equiv 0$, which lead to $j_{i}(x, t)-\bar{j}(x, t), T_{i}(x, t)-T^{*}, E(x, t) \notin L^{2}(\mathbb{R})$. To delete these gaps, we need to introduce the correction functions $\left(\widehat{n}_{1}, \widehat{n}_{2}, \widehat{j}_{1}, \widehat{j}_{2}, \widehat{T}_{1}, \widehat{T}_{2}, \widehat{E}\right)(x, t)$. As those done in $[6,22]$, we can construct these gap functions. That is, we can choose $\left(\widehat{n}_{1}, \widehat{n}_{2}, \widehat{j}_{1}, \widehat{j}_{2}, \widehat{E}\right)(x, t)$, which solve the system

$$
\begin{align*}
& \widehat{n}_{1 t}+\hat{j}_{1 x}=0 \\
& \widehat{j}_{1 t}=\breve{n} \widehat{E}-\widehat{j}_{1} \\
& \widehat{n}_{2 t}+\widehat{j}_{2 x}=0  \tag{19}\\
& \hat{j}_{2 t}=-\breve{n} \widehat{E}-\hat{j}_{2} \\
& \widehat{E}_{x}=\widehat{n}_{1}-\widehat{n}_{2}
\end{align*}
$$

with $\hat{j}_{i}(x, t) \rightarrow j_{i}^{ \pm}(t)$ as $x \rightarrow \pm \infty, \widehat{E}(x, t) \rightarrow 0$ as $x \rightarrow-\infty$, and $\widehat{E}(x, t) \rightarrow E^{+}(t)$ as $x \rightarrow+\infty$. Here, $\breve{n}(x)=n_{-}+\left(n_{+}-n_{-}\right) \int_{-\infty}^{x+2 L_{0}} m_{0}(y) d y$ with $m_{0}(x) \geq 0, m_{0} \in$ $C_{0}^{\infty}(\mathbb{R})$, supp $m_{0} \subseteq\left[-L_{0}, L_{0}\right]$, and $\int_{-\infty}^{+\infty} m_{0}(y) d y=1$. Moreover, we take $\widehat{T}_{i}(x, t)=\widehat{T}_{i}^{-}(t)(1-g(x))+\widehat{T}_{i}^{+}(t) g(x)(i=$ $1,2)$ with $g(x)=\int_{-\infty}^{x} m_{0}(y) d y$, which together with (17) implies

$$
\begin{aligned}
\frac{\partial}{\partial t} \widehat{T}_{i}(x, t)= & -\widehat{T}_{i}(x, t)+\frac{1}{3}\left(\frac{j_{i}^{-}(t)}{n_{-}}\right)^{2}(1-g(x)) \\
& +\frac{1}{3}\left(\frac{j_{i}^{+}(t)}{n_{+}}\right)^{2} g(x) \\
= & -\widehat{T}_{i}(x, t)+S_{i}(x, t), \quad i=1,2
\end{aligned}
$$

In conclusion, we have constructed the required correction functions ( $\widehat{n}_{1}, \widehat{n}_{2}, \widehat{j}_{1}, \widehat{j}_{2}, \widehat{T}_{1}, \widehat{T}_{2}, \widehat{E}$ ) which satisfy

$$
\begin{gather*}
\widehat{n}_{1 t}+\widehat{j}_{1 x}=0, \\
\hat{j}_{1 t}=\breve{n} \widehat{E}-\widehat{j}_{1}, \\
\widehat{n}_{2 t}+\widehat{j}_{2 x}=0, \\
\widehat{j}_{2 t}=-\breve{n} \widehat{E}-\widehat{j}_{2}, \\
\widehat{T}_{i t}=-\widehat{T}_{i}+S_{i}(x, t),  \tag{21}\\
\widehat{E}_{x}=\widehat{n}_{1}-\widehat{n}_{2},
\end{gather*}
$$

$$
\text { with } \begin{cases}\widehat{j}_{i}(x, t) \rightarrow j_{i}^{ \pm}(t), & \text { as } x \rightarrow \pm \infty \\ \widehat{T}_{i}(x, t) \rightarrow T_{i}^{ \pm}(t)-T^{*}, & \text { as } x \rightarrow \pm \infty \\ \widehat{E}(x, t) \rightarrow 0, & \text { as } x \rightarrow-\infty \\ \widehat{E}(x, t) \rightarrow E^{+}(t), & \text { as } x \rightarrow+\infty\end{cases}
$$

Since these details can be found in [6, 22], we only give the following decay time-exponentially of $\left(\widehat{n}_{1}, \widehat{n}_{2}, \widehat{j}_{1}, \widehat{j}_{2}, \widehat{T}_{1}, \widehat{T}_{2}, \widehat{E}\right)(x, t)$.

Lemma 4. There exist positive constants $C$ and $v<1 / 2$ independent of $t$, such that

$$
\begin{equation*}
\left\|\left(\widehat{n}_{i}, \widehat{j}_{i}, \widehat{T}_{i}, \widehat{E}\right)(t)\right\|_{L^{\infty}(\mathbb{R})} \leq C \delta e^{-\nu t}, \quad i=1,2 \tag{22}
\end{equation*}
$$

and $\operatorname{supp} \widehat{n}_{i}=\operatorname{supp} m_{0} \subseteq\left[-L_{0}, L_{0}\right], i=1,2$.

## 3. Reformulation of Original Problem

In this section, we first reformulate the original problem in terms of the perturbed variables. Setting for $i=1,2$,

$$
\begin{align*}
& \left(\varphi_{i}, \psi_{i}, \theta_{i}, \mathscr{H}\right)(x, t) \\
& \quad:=\left(\int_{-\infty}^{x}\left[n_{i}(\xi, t)-\widehat{n}_{i}(\xi, t)-\bar{n}\left(\xi+x_{0}, t\right)\right] d \xi\right. \\
& \quad j_{i}(x, t)-\widehat{j}_{i}(x, t)-\bar{j}\left(x+x_{0}, t\right), T_{i}(x, t)  \tag{23}\\
& \left.\quad-\widehat{T}_{i}(x, t)-T^{*}, E(x, t)-\widehat{E}(x, t)\right),
\end{align*}
$$

$$
\varphi_{i t}+\psi_{i}=0
$$

$$
\begin{align*}
& \psi_{i t}+\left(\frac{\left(-\varphi_{i t}+\widehat{j}_{i}+\bar{j}\right)^{2}}{\varphi_{i x}+\widehat{n}_{i}+\bar{n}}+\left(\varphi_{i x}+\widehat{n}_{i}+\bar{n}\right)\right. \\
&\left.\times\left(\theta_{i}+\widehat{T}_{i}+T^{*}\right)-\bar{n} T^{*}\right)_{x} \\
&=(-1)^{i-1}\left(\varphi_{i x}+\widehat{n}_{i}+\bar{n}\right) \mathscr{H} \\
&+(-1)^{i-1}\left(\varphi_{i x}+\widehat{n}_{i}+\bar{n}-\breve{n}\right) \widehat{E} \\
&-\psi_{i}+\left(\bar{n} T^{*}\right)_{t x} \tag{24}
\end{align*}
$$

$$
\begin{aligned}
& \theta_{i t}+ \frac{-\varphi_{i t}+\widehat{j}_{i}+\bar{j}}{\varphi_{i x}+\widehat{n}_{i}+\bar{n}}\left(\theta_{i}+\widehat{T}_{i}+T^{*}\right)_{x} \\
&+\frac{2}{3}\left(\frac{-\varphi_{i t}+\widehat{j}_{i}+\bar{j}}{\varphi_{i x}+\widehat{n}_{i}+\bar{n}}\right)_{x}\left(\theta_{i}+\widehat{T}_{i}+T^{*}\right) \\
&-\frac{2}{3\left(\varphi_{i x}+\widehat{n}_{i}+\bar{n}\right)}\left(\theta_{i}+\widehat{T}_{i}+T^{*}\right)_{x x} \\
&= {\left[\frac{1}{3}\left(\frac{-\varphi_{i t}+\widehat{j}_{i}+\bar{j}}{\varphi_{i x}+\widehat{n}_{i}+\bar{n}}\right)^{2}-S_{i}(x, t)\right]-\theta_{i} } \\
& \mathscr{H}=\varphi_{1}-\varphi_{2},
\end{aligned}
$$

with the initial data $\left(\varphi_{i}, \psi_{i}, \theta_{i}\right)(x, 0)=\left(\varphi_{10}, \psi_{i 0}, \theta_{i 0}\right)(x), i=$ 1,2 . Further, we have

$$
\begin{gather*}
\varphi_{1 t t}+\varphi_{1 t}-\left(\left(\theta_{1}+\widehat{T}_{1}+T^{*}\right) \varphi_{1 x}+\bar{n} \theta_{1}\right)_{x}+\left(\varphi_{1 x}+\widehat{n}_{1}+\bar{n}\right) \mathscr{H} \\
=-f_{1}+g_{1 x}-\left(\bar{n} T^{*}\right)_{t x} \\
\varphi_{2 t t}+\varphi_{2 t}-\left(\left(\theta_{2}+\widehat{T}_{2}+T^{*}\right) \varphi_{2 x}+\bar{n} \theta_{1}\right)_{x}-\left(\varphi_{2 x}+\widehat{n}_{2}+\bar{n}\right) \mathscr{H} \\
=f_{2}+g_{2 x}-\left(\bar{n} T^{*}\right)_{t x}  \tag{25}\\
\theta_{1 t}+\theta_{1}-\frac{2}{3\left(\varphi_{1 x}+\widehat{n}_{1}+\bar{n}\right)} \theta_{1 x x}-\frac{2}{3}\left(\theta_{1}+\widehat{T}_{1}+T^{*}\right)\left(\frac{\varphi_{1 t}}{\varphi_{1 x}+\widehat{n}_{1}+\bar{n}}\right)_{x}=G_{1} \\
\theta_{2 t}+\theta_{2}-\frac{2}{3\left(\varphi_{2 x}+\widehat{n}_{2}+\bar{n}\right)} \theta_{2 x x}-\frac{2}{3}\left(\theta_{2}+\widehat{T}_{2}+T^{*}\right)\left(\frac{\varphi_{2 t}}{\varphi_{2 x}+\widehat{n}_{2}+\bar{n}}\right)_{x}=G_{2}
\end{gather*}
$$

with the initial data

$$
\begin{gather*}
\varphi_{i}(x, 0)=\varphi_{i 0}(x) \\
\varphi_{i t}(x, 0)=-\psi_{i 0}(x),  \tag{26}\\
\theta_{i}(x, 0)=\theta_{i 0}(x) \\
i=1,2
\end{gather*}
$$

Here

$$
\begin{gathered}
f_{i}=\left(\varphi_{i x}+\widehat{n}_{i}+\bar{n}-\breve{n}\right) \widehat{E}-\left(\left(\theta_{i}+\widehat{T}_{i}+T^{*}\right) \widehat{n}_{i}+\bar{n} \widehat{T}_{i}\right)_{x} \\
g_{i}=\frac{\left(-\varphi_{i t}+\widehat{j}_{i}+\bar{j}\right)^{2}}{\varphi_{i x}+\widehat{n}_{i}+\bar{n}} \\
G_{i}=-\frac{-\varphi_{i t}+\widehat{j}_{i}+\bar{j}}{\varphi_{i x}+\widehat{n}_{i}+\bar{n}}\left(\theta_{i}+\widehat{T}_{i}+T^{*}\right)_{x}
\end{gathered}
$$

$$
\begin{align*}
& -\frac{2}{3}\left(\frac{\widehat{j}_{i}+\bar{j}}{\varphi_{i x}+\widehat{n}_{i}+\bar{n}}\right)_{x}\left(\theta_{i}+\widehat{T}_{i}+T^{*}\right) \\
& -\frac{2}{3}\left(\frac{\widehat{j_{i}}+\bar{j}}{\varphi_{i x}+\widehat{n}_{i}+\bar{n}}\right)_{x}\left(\theta_{i}+\widehat{T}_{i}+T^{*}\right)-S_{i}(x, t) \tag{27}
\end{align*}
$$

By the standard iteration methods (see [23]), we can prove the local existence of classical solutions of the IVP (25) and (26). Here for the sake of clarity, we only state result and omit the proof.

Lemma 5. Suppose that $\left(\varphi_{i 0},-\psi_{i 0}, \theta_{i 0}\right) \in H^{3}(\mathbb{R}) \times H^{2}(\mathbb{R}) \times$ $H^{3}(\mathbb{R})$ for $i=1,2$. Then there is a $C_{1}>0$ such that if

$$
\begin{equation*}
\left\|\left(\varphi_{10}, \theta_{10}, \varphi_{20}, \theta_{20}\right)\right\|_{3}^{2}+\left\|\left(\psi_{10}, \psi_{20}\right)\right\|_{2}^{2} \leq C_{1} \tag{28}
\end{equation*}
$$

then there is a positive number $T_{0}$ such that the initial value problems (25) and (26) have a unique solution $\left(\varphi_{1}, \theta_{1}, \varphi_{2}, \theta_{2}\right)$ satisfying $\varphi_{i} \in C\left(\left[0, T_{0}\right] ; H^{3}(\mathbb{R})\right) \cap$
$C^{1}\left(\left[0, T_{0}\right] ; H^{2}(\mathbb{R})\right) \cap C^{2}\left(\left[0, T_{0}\right] ; H^{1}(\mathbb{R})\right), \theta_{i} \in C\left(\left[0, T_{0}\right] ;\right.$ $\left.H^{3}(\mathbb{R})\right) \cap C^{1}\left(\left[0, T_{0}\right] ; H^{1}(\mathbb{R})\right)(i=1,2)$, and

$$
\begin{aligned}
& \left\|\left(\varphi_{1}, \theta_{1}, \varphi_{2}, \theta_{2}\right)(\cdot, t)\right\|_{3}^{2}+\left\|\left(\varphi_{1 t}, \varphi_{2 t}\right)(\cdot, t)\right\|_{2}^{2} \\
& \quad+\left\|\left(\theta_{1 t}, \theta_{2 t}\right)(\cdot, t)\right\|_{1}^{2} \\
& \quad \leq
\end{aligned}
$$

for some positive constant $C$.
To end this section, we also derive

$$
\begin{aligned}
\mathscr{H}_{t t} & +\mathscr{H}_{t}+2 \bar{n} \mathscr{H}-\left(\bar{n}\left(\theta_{1}-\theta_{2}\right)+\left(\theta_{1}+\widehat{T}_{1}+T^{*}\right) \mathscr{H}_{x}\right)_{x} \\
& =h_{1 x}-h_{2}-h_{3}+h_{4 x}, \\
\left(\theta_{1}-\right. & \left.\theta_{2}\right)_{t}+\left(\theta_{1}-\theta_{2}\right)-\frac{2}{3\left(\varphi_{1 x}+\widehat{n}_{1}+\bar{n}\right)}\left(\theta_{1}-\theta_{2}\right)_{x x} \\
& -\frac{2\left(\theta_{1}+\widehat{T}_{1}+T^{*}\right)}{3\left(\varphi_{1 x}+\widehat{n}_{1}+\bar{n}\right)} \mathscr{H}_{t x} \\
= & G_{3}
\end{aligned}
$$

where

$$
\begin{gathered}
h_{1}:=\left(\theta_{1}+\widehat{T}_{1}+T^{*}\right)\left(\widehat{n}_{1}-\widehat{n}_{2}\right)+\left(\varphi_{2 x}+\widehat{n}_{2}\right)\left(\theta_{1}-\theta_{2}\right) \\
+\left(\varphi_{2 x}+\widehat{n}_{2}+\bar{n}\right)\left(\widehat{T}_{1}-\widehat{T}_{2}\right), \\
h_{2}:=\left(\varphi_{1 x}+\varphi_{2 x}+\widehat{n}_{1}+\widehat{n}_{2}\right) \mathscr{H}, \\
h_{3}:=\left[\varphi_{1 x}+\varphi_{2 x}+\widehat{n}_{1}+\widehat{n}_{2}+2(\bar{n}-\breve{n})\right] \widehat{E}, \\
G_{3} \quad h_{4}:=\frac{\left(-\varphi_{1 t}+\hat{j}_{1}+\bar{j}\right)^{2}}{\varphi_{1 x}+\widehat{n}_{1}+\bar{n}}-\frac{\left(-\varphi_{2 t}+\widehat{j}_{2}+\bar{j}\right)^{2}}{\varphi_{2 x}+\widehat{n}_{2}+\bar{n}}, \\
:=G_{1}-G_{2}+\left[\frac{2}{3\left(\varphi_{1 x}+\widehat{n}_{1}+\bar{n}\right)}-\frac{2}{3\left(\varphi_{2 x}+\widehat{n}_{2}+\bar{n}\right)}\right] \theta_{2 x x} \\
+ \\
+\left[\frac{2\left(\theta_{1}+\widehat{T}_{1}+T^{*}\right)}{3\left(\varphi_{1 x}+\widehat{n}_{1}+\bar{n}\right)}-\frac{2\left(\theta_{2}+\widehat{T}_{2}+T^{*}\right)}{3\left(\varphi_{2 x}+\widehat{n}_{2}+\bar{n}\right)}\right] \varphi_{2 t x} \\
\\
-\frac{2\left(\theta_{1}+\widehat{T}_{1}+T^{*}\right)}{3\left(\varphi_{1 x}+\widehat{n}_{1}+\bar{n}\right)^{2}} \varphi_{1 t} \mathscr{H}_{x x}-\frac{2\left(\theta_{1}+\widehat{T}_{1}+T^{*}\right)}{3\left(\varphi_{1 x}+\widehat{n}_{1}+\bar{n}\right)^{2}} \widehat{n}_{1 x} \mathscr{H}_{t}
\end{gathered}
$$

$$
\begin{align*}
& +\left[\frac{2\left(\theta_{2}+\widehat{T}_{2}+T^{*}\right)}{3\left(\varphi_{2 x}+\widehat{n}_{2}+\bar{n}\right)^{2}} \varphi_{2 t}-\frac{2\left(\theta_{1}+\widehat{T}_{1}+T^{*}\right)}{3\left(\varphi_{1 x}+\widehat{n}_{1}+\bar{n}\right)^{2}} \varphi_{1 t}\right] \\
& \times\left(\varphi_{2 x}+\bar{n}\right)_{x} \\
& -\left[\frac{2\left(\theta_{2}+\widehat{T}_{2}+T^{*}\right)}{3\left(\varphi_{2 x}+\widehat{n}_{2}+\bar{n}\right)^{2}} \widehat{n}_{1 x}-\frac{2\left(\theta_{1}+\widehat{T}_{1}+T^{*}\right)}{3\left(\varphi_{1 x}+\widehat{n}_{1}+\bar{n}\right)^{2}} \widehat{n}_{2 x}\right] \varphi_{2 t} . \tag{32}
\end{align*}
$$

## 4. Global Existence of Smooth Solutions

In this section we mainly prove global existence of smooth solutions for the initial value problems (25) and (26). To begin with, we focus on the a priori estimates of $\left(\varphi_{1}, \theta_{1}, \varphi_{2}, \theta_{2}\right)$. For this purpose, letting $T \in(0,+\infty)$, we define

$$
X(T)
$$

$$
\begin{align*}
=\{ & \left(\varphi_{i}, \varphi_{i t}, \theta_{i}, \theta_{i t}\right): \partial_{t}^{j} \varphi_{i} \in C\left([0, T] ; H^{3-j}(\mathbb{R})\right), \\
& \theta_{i} \in C\left([0, T] ; H^{3}(\mathbb{R})\right), \theta_{i t} \in C\left([0, T] ; H^{1}(\mathbb{R})\right), \\
& i=1,2, j=0,1\}, \tag{33}
\end{align*}
$$

with the norm

$$
\begin{gather*}
N(T)^{2}=\max _{0 \leq t \leq T}\left\{\left\|\left(\varphi_{1}, \varphi_{2}, \theta_{1}, \theta_{2}\right)(t)\right\|_{3}^{2}+\left\|\left(\varphi_{1 t}, \varphi_{2 t}\right)(t)\right\|_{2}^{2}\right.  \tag{34}\\
\left.+\left\|\left(\theta_{1 t}, \theta_{2 t}\right)(t)\right\|_{1}^{2}\right\} .
\end{gather*}
$$

Let $N(T)^{2} \leq \varepsilon^{2}$, where $\varepsilon$ is sufficiently small and will be determined later. Then, by Sobolev inequality, we have for $i=1,2$,

$$
\begin{equation*}
\left\|\left(\varphi_{i}, \varphi_{i x}, \varphi_{i x x}, \theta_{i}, \theta_{i x}, \theta_{i t}, \theta_{i x x}, \varphi_{i t}, \varphi_{i t x}\right)(t)\right\|_{L^{\infty}} \leq C \varepsilon . \tag{35}
\end{equation*}
$$

Clearly, there exists a positive constant $c_{1}, c_{2}$ such that

$$
\begin{align*}
& 0<\frac{1}{c_{1}} \leq n_{i}=\varphi_{i x}+\widehat{n}_{i}+\bar{n} \leq c_{1} \\
& 0<\frac{1}{c_{2}} \leq T_{i}=\theta_{i}+\widehat{T}_{i}+T^{*} \leq c_{2} \tag{36}
\end{align*}
$$

$$
i=1,2 .
$$

Further, from $(24)_{7}$, we also have $\partial_{t}^{j} \mathscr{H} \in C\left(0, T ; H^{2-j}(\mathbb{R})\right)$ and

$$
\begin{equation*}
\left\|\left(\mathscr{H}, \mathscr{H}_{x}, \mathscr{H}_{t}\right)(t)\right\|_{L^{\infty}} \leq C \varepsilon . \tag{37}
\end{equation*}
$$

Now we first establish the following basic energy estimate.

Lemma 6. Let $\left(\varphi_{1}, \theta_{1}, \varphi_{2}, \theta_{2}\right)(x, t) \in X(T)$ be the solution of the initial value problem (25) and (26). If $\delta+\varepsilon \ll 1$, then it holds that for $0<t<T$,

$$
\begin{gather*}
\sum_{i=0}^{2}\left\|\left(\varphi_{i}, \varphi_{i x}, \varphi_{i t}, \theta_{i}\right)(\cdot, t)\right\|^{2}+\|\mathscr{H}(\cdot, t)\|^{2}+\int_{0}^{t}\|\mathscr{H}(\cdot, \tau)\|^{2} d \tau \\
\quad+\sum_{i=1}^{2} \int_{0}^{t}\left\|\left(\varphi_{i x}, \varphi_{i t}, \theta_{i}, \theta_{i x}\right)(\cdot, \tau)\right\|^{2} d \tau \leq C\left(\Phi_{0}+\delta\right) \tag{38}
\end{gather*}
$$

Proof. Multiplying (25) ${ }_{1}$ and $(25)_{2}$ by $\varphi_{1}$ and $\varphi_{2}$, respectively, and integrating them over $\mathbb{R}$ by parts, we have for $i=1,2$,

$$
\begin{align*}
& \frac{d}{d t} \int_{\mathbb{R}}\left(\varphi_{i} \varphi_{i t}+\frac{1}{2} \varphi_{i}^{2}\right) d x+\int_{\mathbb{R}}\left(\theta_{i}+\widehat{T}_{i}+T^{*}\right) \varphi_{i x}^{2} d x \\
& \quad+(-1)^{i-1} \int_{\mathbb{R}}\left(\varphi_{i x}+\widehat{n}_{i}+\bar{n}\right) \mathscr{H} \varphi_{i} d x-\int_{\mathbb{R}} \varphi_{i t}^{2} d x  \tag{39}\\
& =-\int_{\mathbb{R}} \bar{n} \theta_{i} \varphi_{i x} d x+\int_{\mathbb{R}}\left(\bar{n} T^{*}\right)_{t} \varphi_{i x} d x \\
& \quad+(-1)^{i} \int_{\mathbb{R}} f_{i} \varphi_{i} d x-\int_{\mathbb{R}} g_{i} \varphi_{i x} d x .
\end{align*}
$$

Using Cauchy-Schwartz's inequality, and Lemmas 3 and 4, we have

$$
\begin{align*}
& -\int_{\mathbb{R}} \bar{n} \theta_{i} \varphi_{i x} d x+\int_{\mathbb{R}}\left(\bar{n} T^{*}\right)_{t} \varphi_{i x} d x \\
& \quad \leq \kappa \int_{\mathbb{R}} \varphi_{i x}^{2} d x+C \int_{\mathbb{R}}\left(\theta_{i}^{2}+\bar{n}_{t}^{2}\right) d x \tag{40}
\end{align*}
$$

where and in the subsequent $\kappa>0$ is some proper small constant, and

$$
\begin{equation*}
(-1)^{i} \int_{\mathbb{R}} f_{i} \varphi_{i} d x \leq C \varepsilon \int_{\mathbb{R}} \varphi_{i x}^{2} d x+C \delta^{2}(1+t)^{1 / 4} e^{-\nu t} \tag{41}
\end{equation*}
$$

where we also used the facts

$$
\begin{equation*}
\int_{\mathbb{R}}(\bar{n}-\breve{n})^{2} d x \leq C \delta^{2}(1+t)^{1 / 2} \tag{42}
\end{equation*}
$$

which can be proved from the construction of $\breve{n}(x) \rightarrow n_{ \pm}$, as $x \rightarrow \pm \infty$, and the property of the diffusion wave $\bar{n}((x+$ $\left.\left.x_{0}\right) / \sqrt{(1+t)}\right)$. Similarly, we can show

$$
\begin{align*}
& -\int_{\mathbb{R}} g_{i} \varphi_{i x} d x \\
& \quad=-\int_{\mathbb{R}} \frac{\left(-\varphi_{i t}+\hat{j}_{i}+\bar{j}\right)^{2}}{\varphi_{i x}+\widehat{n}_{i}+\bar{n}} \varphi_{i x} d x \\
& \quad \leq C(\delta+\varepsilon) \int_{\mathbb{R}}\left(\varphi_{i x}^{2}+\varphi_{i t}^{2}\right) d x+C \delta \int_{\mathbb{R}} \bar{n}_{x}^{2} d x+C \delta^{2} e^{-\nu t} \tag{43}
\end{align*}
$$

which together with (39)-(41) implies,

$$
\begin{align*}
& \frac{d}{d t} \int_{\mathbb{R}}\left(\varphi_{i} \varphi_{i t}+\frac{1}{2} \varphi_{i}^{2}\right) d x+\int_{\mathbb{R}}\left[\left(\theta_{i}+\widehat{T}_{i}+T^{*}\right)-\kappa\right] \varphi_{i x}^{2} d x \\
& \quad-\int_{\mathbb{R}} \varphi_{i t}^{2} d x+(-1)^{i-1} \int_{\mathbb{R}}\left(\varphi_{i x}+\widehat{n}_{i}+\bar{n}\right) \mathscr{H} \varphi_{i} d x \\
& \leq C(\delta+\varepsilon) \int_{\mathbb{R}}\left(\varphi_{i x}^{2}+\varphi_{i t}^{2}\right) d x \\
& \quad+C \int_{\mathbb{R}}\left(\theta_{i}^{2}+\bar{n}_{t}^{2}+\bar{n}_{x}^{2}\right) d x+C \delta^{2} e^{-v_{1} t} \tag{44}
\end{align*}
$$

where $0<v_{1}<\nu$. Moreover, for the coupled term with the electric field, we have

$$
\begin{align*}
& \int_{\mathbb{R}}\left(\left(\varphi_{1 x}+\widehat{n}_{1}+\bar{n}\right) \mathscr{H} \varphi_{1}-\left(\varphi_{2 x}+\widehat{n}_{2}+\bar{n}\right) \mathscr{H} \varphi_{2}\right) d x \\
& \quad \geq \int_{\mathbb{R}} \bar{n} \mathscr{H}^{2} d x-C \varepsilon \int_{\mathbb{R}}\left(\mathscr{H}^{2}+\varphi_{1 x}^{2}+\varphi_{2 x}^{2}\right) d x-C \delta^{2} e^{-v t} \tag{45}
\end{align*}
$$

Next, multiplying $(25)_{1}$ and $(25)_{2}$ by $\varphi_{1 t}$ and $\varphi_{2 t}$, respectively, and integrating their sum over $\mathbb{R}$ by parts, we have

$$
\begin{align*}
\frac{d}{d t} \int_{\mathbb{R}} & \left(\frac{1}{2} \varphi_{i t}^{2}+\frac{1}{2}\left(\theta_{i}+\widehat{T}_{i}+T^{*}\right) \varphi_{i x}^{2}\right) d x \\
& +\int_{\mathbb{R}}\left(\varphi_{i t}^{2}+\bar{n} \theta_{i} \varphi_{i t x}+(-1)^{i-1}\left(\varphi_{i x}+\widehat{n}_{i}+\bar{n}\right) \mathscr{H} \varphi_{i t}\right) d x \\
= & (-1)^{i} \int_{\mathbb{R}} f_{i} \varphi_{i t} d x+\int_{\mathbb{R}} g_{i x} \varphi_{i t} d x \\
& -\int_{\mathbb{R}}\left[\left(\bar{n} T^{*}\right)_{t x} \varphi_{i t}-\frac{1}{2}\left(\theta_{i}+\widehat{T}_{i}+T^{*}\right)_{t} \varphi_{i x}^{2}\right] d x . \tag{46}
\end{align*}
$$

Using Schwartz's inequality, (42), and Lemmas 3 and 4, we have

$$
\begin{align*}
& -\int_{\mathbb{R}}\left[\left(\bar{n} T^{*}\right)_{t x} \varphi_{i t}-\frac{1}{2}\left(\theta_{i}+\widehat{T}_{i}+T^{*}\right)_{t} \varphi_{i x}^{2}\right] d x \\
& \quad+(-1)^{i-1} \int_{\mathbb{R}} f_{i} \varphi_{i t} d x \\
& \leq  \tag{47}\\
& \quad \kappa \int_{\mathbb{R}} \varphi_{i t}^{2} d x+C(\delta+\varepsilon) \int_{\mathbb{R}}\left(\varphi_{i t}^{2}+\varphi_{i x}^{2}\right) d x \\
& \quad+C \int_{\mathbb{R}} \bar{n}_{t x}^{2} d x+C \delta^{2}(1+t)^{1 / 4} e^{-\nu t}
\end{align*}
$$

Since

$$
\begin{align*}
g_{i x}= & -\frac{\left(-\varphi_{i t}+\hat{j}_{i}+\bar{j}\right)^{2}}{\left(\varphi_{i x}+\widehat{n}_{i}+\bar{n}\right)^{2}} \varphi_{i x x}-\frac{2\left(-\varphi_{i t}+\hat{j}_{i}+\bar{j}\right)}{\left(\varphi_{i x}+\widehat{n}_{i}+\bar{n}\right)} \varphi_{i x t} \\
+ & O(1)\left[\left(\varphi_{i t}+\hat{j}_{i}+\bar{j}\right)\left(\widehat{n}_{i}+\bar{n}\right)_{t}+\left(\widehat{j}_{i}+\bar{j}\right)^{2}\left(\widehat{n}_{i}+\bar{n}\right)_{x}\right. \\
& \left.\quad+\left(\widehat{n}_{i}+\bar{n}\right)_{x} \varphi_{i t}^{2}\right] \tag{48}
\end{align*}
$$

we obtain, after integration by parts, that

$$
\begin{align*}
& \int_{\mathbb{R}} g_{i x} \varphi_{i t} d x \\
& \leq  \tag{49}\\
& \quad \frac{d}{d t} \int_{\mathbb{R}} \frac{\left(-\varphi_{i t}+\hat{j}_{i}+\bar{j}\right)^{2}}{2\left(\varphi_{i x}+\widehat{n}_{i}+\bar{n}\right)^{2}} \varphi_{i x}^{2} d x \\
& \quad+C(\delta+\varepsilon) \int_{\mathbb{R}}\left(\varphi_{i t}^{2}+\varphi_{i x}^{2}\right) d x \\
& \quad+C \delta \int_{\mathbb{R}}\left(\bar{n}_{t}^{2}+\bar{n}_{x}^{4}\right) d x+C \delta^{2} e^{-\nu t}
\end{align*}
$$

where we have used

$$
\begin{align*}
& \left\|\left(\frac{\left(-\varphi_{i t}+\hat{j}_{i}+\bar{j}\right)^{2}}{\left(\varphi_{i x}+\widehat{n}_{i}+\bar{n}\right)^{2}}\right)\right\|_{x}\left\|_{L^{\infty}},\right\|\left(\frac{\left(-\varphi_{i t}+\hat{j}_{i}+\bar{j}\right)^{2}}{\left(\varphi_{i x}+\widehat{n}_{i}+\bar{n}\right)^{2}}\right)\left\|_{t}\right\|_{L^{\infty}}  \tag{50}\\
& \quad \leq C(\delta+\varepsilon),
\end{align*}
$$

with the aid of $\left|\varphi_{i t t}\right|<C \mid \varphi_{i x x}+\varphi_{i x t}+\varphi_{i x}+\varphi_{i t}+\varphi_{i}+\theta_{i}+\theta_{i x}+$ $\bar{n}_{x t} \mid+C \delta^{2} e^{-v t}$. Putting the above inequality into (46), we have

$$
\begin{align*}
& \frac{d}{d t} \int_{\mathbb{R}}\left[\frac{1}{2} \varphi_{i t}^{2}+\frac{1}{2}\left(\theta_{i}+\widehat{T}_{i}+T^{*}\right) \varphi_{i x}^{2}-\frac{\left(-\varphi_{i t}+\widehat{j}_{i}+\bar{j}\right)^{2}}{2\left(\varphi_{i x}+\widehat{n}_{i}+\bar{n}\right)^{2}} \varphi_{i x}^{2}\right] d x \\
& \quad+(1-\kappa) \int_{\mathbb{R}} \varphi_{i t}^{2} d x+(-1)^{i-1} \int_{\mathbb{R}}\left(\varphi_{i x}+\widehat{n}_{i}+\bar{n}\right) \mathscr{H} \varphi_{i t} d x \\
& \quad+\int_{\mathbb{R}} \bar{n} \theta_{i} \varphi_{i t x} d x \\
& \leq C(\delta+\varepsilon) \int_{\mathbb{R}}\left(\varphi_{i t}^{2}+\varphi_{i x}^{2}\right) d x \\
& \quad+C \int_{\mathbb{R}}\left(\bar{n}_{t x}^{2}+\bar{n}_{t}^{2}+\bar{n}_{x}^{4}\right) d x+C \delta^{2} e^{-v_{1} t} . \tag{51}
\end{align*}
$$

On the other hand, we have

$$
\begin{align*}
& \int_{\mathbb{R}}\left(\left(\varphi_{1 x}+\widehat{n}_{1}+\bar{n}\right) \mathscr{H} \varphi_{1 t}-\left(\varphi_{2 x}+\widehat{n}_{2}+\bar{n}\right) \mathscr{H} \varphi_{2 t}\right) d x \\
& \quad \geq \frac{d}{d t} \int_{\mathbb{R}} \frac{1}{2} \bar{n} \mathscr{H}^{2} d x-\int_{\mathbb{R}} \frac{1}{2} \bar{n}_{t} \mathscr{H}^{2} d x  \tag{52}\\
& \quad-C \varepsilon \int_{\mathbb{R}}\left(\varphi_{1 x}^{2}+\varphi_{1 t}^{2}+\varphi_{2 x}^{2}+\varphi_{2 t}^{2}\right) d x-C \delta e^{-v t}
\end{align*}
$$

Finally, multiplying $(25)_{l}(l=3,4)$ by $\left(3 \bar{n}\left(\varphi_{i x}+\widehat{n}_{i}+\right.\right.$ $\left.\bar{n}) / 2\left(\theta_{i}+\widehat{T}_{i}+T^{*}\right)\right) \theta_{i}(i=1,2)$ and integrating the resultant equation by parts over $\mathbb{R}$, we have

$$
\begin{aligned}
& \frac{d}{d t} \int_{\mathbb{R}} \frac{3 \bar{n}\left(\varphi_{i x}+\widehat{n}_{i}+\bar{n}\right)}{4\left(\theta_{i}+\widehat{T}_{i}+T^{*}\right)} \theta_{i}^{2} d x \\
& \quad \quad+\int_{\mathbb{R}}\left[\frac{3 \bar{n}\left(\varphi_{i x}+\widehat{n}_{i}+\bar{n}\right)}{2\left(\theta_{i}+\widehat{T}_{i}+T^{*}\right)} \theta_{i}^{2}+\frac{\bar{n}}{\theta_{i}+\widehat{T}_{i}+T^{*}} \theta_{i x}^{2}\right] d x,
\end{aligned}
$$

$$
\begin{align*}
-\int_{\mathbb{R}} \bar{n} \theta_{i} \varphi_{i t x} d x= & \int_{\mathbb{R}}\left(\frac{3 \bar{n}\left(\varphi_{i x}+\widehat{n}_{i}+\bar{n}\right)}{4\left(\theta_{i}+\widehat{T}_{i}+T^{*}\right)}\right)_{t} \theta_{i}^{2} d x \\
& -\int_{\mathbb{R}}\left(\frac{\bar{n}}{\theta_{i}+\widehat{T}_{i}+T^{*}}\right)_{x} \theta_{i} \theta_{i x} d x \\
& -\int_{\mathbb{R}} \frac{\bar{n} \theta_{i}}{\varphi_{i x}+\widehat{n}_{i}+\bar{n}} \varphi_{i t}\left(\varphi_{i x}+\widehat{n}_{i}+\bar{n}\right)_{x} d x \\
& +\int_{\mathbb{R}} G_{i} \frac{3 \bar{n}\left(\varphi_{i x}+\widehat{n}_{i}+\bar{n}\right)}{2\left(\theta_{i}+\widehat{T}_{i}+T^{*}\right)} \theta_{i} d x . \tag{53}
\end{align*}
$$

Now we estimate the term of the right hand side of (53), using Cauchy-Schwartz's inequality and Lemmas 3 and 4. First, with the help of the following equality $\widehat{j}_{i}=(1-g(x)) j_{i}^{-}(t)+$ $g(x) j_{i}^{+}(t)$ (see $[6,22]$ ), we have

$$
\begin{align*}
\int_{\mathbb{R}} & \frac{3 \bar{n}\left(\varphi_{i x}+\widehat{n}_{i}+\bar{n}\right)}{2\left(\theta_{i}+\widehat{T}_{i}+T^{*}\right)}\left(\frac{\hat{j}_{i}^{2}}{3\left(\varphi_{i x}+\widehat{n}_{i}+\bar{n}\right)^{2}}-S_{i}(x, t)\right) \theta_{i} d x \\
= & \left\{\int_{-\infty}^{-L_{0}}+\int_{L_{0}}^{+\infty}+\int_{-L_{0}}^{L_{0}}\right\} \frac{3 \bar{n}\left(\varphi_{i x}+\widehat{n}_{i}+\bar{n}\right)}{2\left(\theta_{i}+\widehat{T}_{i}+T^{*}\right)} \\
& \times\left(\frac{\hat{j}_{i}^{2}}{3\left(\varphi_{i x}+\widehat{n}_{i}+\bar{n}\right)^{2}}-S_{i}(x, t)\right) \theta_{i} d x \\
= & \int_{-\infty}^{-L_{0}} \frac{3 \bar{n}\left(\varphi_{i x}+\widehat{n}_{i}+\bar{n}\right)}{2\left(\theta_{i}+\widehat{T}_{i}+T^{*}\right)} \theta_{i} \hat{j}_{i}^{2}\left(\frac{1}{3 n_{i}^{2}}-\frac{1}{3 n_{-}^{2}}\right) d x \\
& +\int_{L_{0}}^{+\infty} \frac{3 \bar{n}\left(\varphi_{i x}+\widehat{n}_{i}+\bar{n}\right)}{2\left(\theta_{i}+\widehat{T}_{i}+T^{*}\right)} \times \theta_{i} \hat{j}_{i}^{2}\left(\frac{1}{3 n_{i}^{2}}-\frac{1}{3 n_{+}^{2}}\right) d x \\
& +\int_{-L_{0}}^{L_{0}} \frac{3 \bar{n}\left(\varphi_{i x}+\widehat{n}_{i}+\bar{n}\right)}{2\left(\theta_{i}+\widehat{T}_{i}+T^{*}\right)} \theta_{i} \\
& \times\left[\frac{\hat{j}_{i}^{2}}{3 n_{i}^{2}}-\frac{(1-g(x))}{3 n_{-}^{2}}\left(j^{-}(t)\right)^{2}-\frac{g(x)}{3 n_{+}^{2}}\left(j^{+}(t)\right)^{2}\right] d x \\
\leq & C \delta \int_{\mathbb{R}}\left(\varphi_{i x}^{2}+\theta_{i}^{2}\right) d x+C \delta^{2} e^{-\nu t} \int_{-\infty}^{-L_{0}}\left(\bar{n}-n_{-}\right)^{2} d x \\
\leq & C \delta \delta_{\mathbb{R}}^{2} e^{-v t} \int_{L_{0}}^{+\infty}\left(\bar{n}-\varphi_{+}\right)^{2} d x+C \delta^{2} e^{-v t} \\
& \left.+\theta_{i}^{2}\right) d x+C \delta^{2}(1+t)^{1 / 2} e^{-v t}, \tag{54}
\end{align*}
$$

which implies

$$
\begin{aligned}
& \int_{\mathbb{R}} \frac{3 \bar{n}\left(\varphi_{i x}+\widehat{n}_{i}+\bar{n}\right)}{2\left(\theta_{i}+\widehat{T}_{i}+T^{*}\right)}\left(\frac{\left(-\varphi_{i t}+\widehat{j}_{i}+\bar{j}\right)^{2}}{3\left(\varphi_{i x}+\widehat{n}_{i}+\bar{n}\right)^{2}}-S_{i}(x, t)\right) \theta_{i} d x \\
& \quad=\int_{\mathbb{R}} \frac{3 \bar{n}\left(\varphi_{i x}+\widehat{n}_{i}+\bar{n}\right)}{2\left(\theta_{i}+\widehat{T}_{i}+T^{*}\right)}\left(\frac{\varphi_{i t}^{2}}{3 n_{i}^{2}}-\frac{2 \varphi_{i t}\left(\hat{j}_{i}+\bar{j}\right)}{3 n_{i}^{2}}\right) \theta_{i} d x
\end{aligned}
$$

$$
\begin{align*}
& \quad+\int_{\mathbb{R}} \frac{3 \bar{n}\left(\varphi_{i x}+\widehat{n}_{i}+\bar{n}\right)}{2\left(\theta_{i}+\widehat{T}_{i}+T^{*}\right)} \frac{\bar{j}^{2}+2 \widehat{j}_{i} \bar{j}_{j}}{3 n_{i}^{2}} \theta_{i} d x \\
& \quad+\int_{\mathbb{R}} \frac{3 \bar{n}\left(\varphi_{i x}+\widehat{n}_{i}+\bar{n}\right)}{2\left(\theta_{i}+\widehat{T}_{i}+T^{*}\right)} \theta_{i}\left(\frac{\widehat{j}_{i}^{2}}{3 n_{i}^{2}}-S_{i}(x, t)\right) d x \\
& \leq C(\delta+\varepsilon) \int_{\mathbb{R}}\left(\varphi_{i x}^{2}+\varphi_{i t}^{2}+\theta_{i}^{2}\right) d x \\
& \quad+C \delta \int_{\mathbb{R}} \bar{n}_{x}^{2} d x+C \delta^{2}(1+t)^{1 / 2} e^{-v t} \tag{55}
\end{align*}
$$

From the definition of $G_{i}(i=1,2)$, and using Schwartz's inequality, we have

$$
\begin{align*}
\int_{\mathbb{R}} G_{i} & \frac{3 \bar{n}\left(\varphi_{i x}+\widehat{n}_{i}+\bar{n}\right)}{2\left(\theta_{i}+\widehat{T}_{i}+T^{*}\right)} \theta_{i} d x \\
\leq & \kappa \int_{\mathbb{R}} \theta_{i}^{2} d x+C(\delta+\varepsilon) \int_{\mathbb{R}}\left(\theta_{i}^{2}+\theta_{i x}^{2}+\varphi_{i x}^{2}+\varphi_{i t}^{2}\right) d x \\
& \quad+C \int_{\mathbb{R}}\left(\bar{n}_{x}^{2}+\bar{n}_{x x}^{2}\right) d x+C \delta^{2} e^{-v_{2} t} \tag{56}
\end{align*}
$$

with $\nu_{1}<\nu_{2}<\nu$. And using Schwartz's inequality and Lemma 3 yields

$$
\begin{align*}
& \int_{\mathbb{R}}\left(\frac{3 \bar{n}\left(\varphi_{i x}+\widehat{n}_{i}+\bar{n}\right)}{4\left(\theta_{i}+\widehat{T}_{i}+T^{*}\right)}\right)_{t} \theta_{i}^{2} d x \\
& \quad-\int_{\mathbb{R}}\left(\frac{\bar{n}}{\theta_{i}+\widehat{T}_{i}+T^{*}}\right)_{x} \theta_{i} \theta_{i x} d x \\
& \leq C(\delta+\varepsilon) \int_{\mathbb{R}}\left(\theta_{i}^{2}+\theta_{i x}^{2}\right) d x  \tag{57}\\
& -\int_{\mathbb{R}} \frac{\bar{n} \theta_{i}}{\varphi_{i x}+\widehat{n}_{i}+\bar{n}} \varphi_{i t}\left(\varphi_{i x}+\widehat{n}_{i}+\bar{n}\right)_{x} d x \\
& \leq \frac{d}{d t} \int_{\mathbb{R}} \frac{\bar{n} \theta_{i}}{2\left(\varphi_{i x}+\widehat{n}_{i}+\bar{n}\right)} \varphi_{i x}^{2} d x \\
& \quad+C(\delta+\varepsilon) \int_{\mathbb{R}}\left(\theta_{i}^{2}+\varphi_{i x}^{2}+\varphi_{i t}^{2}\right) d x
\end{align*}
$$

Putting the above inequalities into (53) yields

$$
\begin{aligned}
& \frac{d}{d t} \int_{\mathbb{R}}\left[\frac{3 \bar{n}\left(\varphi_{i x}+\widehat{n}_{i}+\bar{n}\right)}{4\left(\theta_{1}+\widehat{T}_{i}+T^{*}\right)} \theta_{i}^{2}-\frac{\bar{n} \theta_{i}}{2\left(\varphi_{i x}+\widehat{n}_{i}+\bar{n}\right)} \varphi_{i x}^{2}\right] d x \\
& \quad+\int_{\mathbb{R}}\left[\left(\frac{3 \bar{n}\left(\varphi_{i x}+\widehat{n}_{i}+\bar{n}\right)}{2\left(\theta_{i}+\widehat{T}_{i}+T^{*}\right)}-\kappa\right) \theta_{i}^{2}+\frac{\bar{n}}{\theta_{i}+\widehat{T}_{i}+T^{*}} \theta_{i x}^{2}\right] d x \\
& \quad-\int_{\mathbb{R}} \bar{n} \theta_{i} \varphi_{i t x} d x
\end{aligned}
$$

$$
\begin{align*}
& \leq C(\delta+\varepsilon) \int_{\mathbb{R}}\left(\theta_{i}^{2}+\theta_{i x}^{2}+\varphi_{i t}^{2}+\varphi_{i x}^{2}\right) d x \\
& \quad+C \int_{\mathbb{R}}\left(\bar{n}_{x}^{2}+\bar{n}_{x x}^{2}\right) d x+C \delta^{2} e^{-v_{2} t} \tag{58}
\end{align*}
$$

Combining (44), (45), (51), (52), and (58), we can obtain (38); this completes the proof.

Further, in the completely similar way, we can show the following.

Lemma 7. Let $\left(\varphi_{1}, \theta_{1}, \varphi_{2}, \theta_{2}\right)(x, t) \in X(T)$ be the solution of the initial value problems (25) and (26); then it holds that for $0<t<T$,

$$
\begin{aligned}
& \sum_{i=1}^{2}\left\|\left(\varphi_{i x}, \varphi_{i x x}, \varphi_{i t x}, \varphi_{i t t}, \theta_{i x}, \theta_{i t}, \theta_{i x x}\right)(\cdot, t)\right\|^{2}+\left\|\mathscr{H}_{x}(\cdot, t)\right\|^{2} \\
& \quad+\int_{0}^{t}\left(\sum_{i=1}^{2}\left\|\left(\varphi_{i x x}, \varphi_{i t x}, \varphi_{i t t}, \theta_{i x}, \theta_{i t}, \theta_{i x x}, \theta_{i t x}, \theta_{i x x x}\right)(\cdot, \tau)\right\|^{2}\right. \\
& \left.\quad+\left\|\mathscr{H}_{x}(\cdot, \tau)\right\|^{2}\right) d \tau \\
& \quad \leq C\left(\Phi_{0}+\delta\right)
\end{aligned}
$$

$$
\sum_{i=1}^{2}\left\|\left(\varphi_{i x x}, \varphi_{i x x x}, \varphi_{i t x x}, \varphi_{i t t x}, \theta_{i x x}, \theta_{i t x}, \theta_{i x x x}\right)(\cdot, t)\right\|^{2}
$$

$$
+\left\|\mathscr{H}_{x x}(\cdot, t)\right\|^{2}
$$

$$
+\int_{0}^{t}\left(\sum_{i=1}^{2}\left\|\left(\varphi_{i x x}, \varphi_{i x}, \varphi_{i t x}, \theta_{i x x}, \theta_{i t x}, \theta_{i x x x}\right)(\cdot, \tau)\right\|^{2}\right.
$$

$$
\left.+\left\|\mathscr{H}_{x x}(\cdot, \tau)\right\|^{2}\right) d \tau
$$

$$
\begin{equation*}
\leq C\left(\Phi_{0}+\delta\right) \tag{59}
\end{equation*}
$$

provided that $\varepsilon+\delta \ll 1$.
Based on the local existence given in Lemma 5 and the a priori estimates given in Lemmas 6 and 7, by the standard continuity argument, we can prove the global existence of the unique solutions of the IVP (25) and (26).

Theorem 8. Under the assumption of Theorem 1, the classical solution $\left(\varphi_{1}, \theta_{1}, \varphi_{2}, \theta_{2}, \mathscr{H}\right)(x, t)$ of the solutions of the IVP (25) and (26) exist globally in time if $\Phi_{0}+\delta$ is small enough. Moreover, one has

$$
\begin{aligned}
& \left\|\left(\varphi_{1}, \theta_{1}, \varphi_{2}, \theta_{2}\right)(\cdot, t)\right\|_{3}^{2}+\left\|\left(\varphi_{1 t}, \varphi_{2 t}, \mathscr{H}(\cdot, t)\right)\right\|_{2}^{2} \\
& \quad+\left\|\left(\theta_{1 t}, \theta_{2 t}\right)(\cdot, t)\right\|_{1}^{2} \\
& \quad+\int_{0}^{t}\left(\left\|\left(\varphi_{1 x}, \varphi_{1 t}, \varphi_{2 x}, \varphi_{2 t}, \mathscr{H}\right)(\cdot, \tau)\right\|_{2}^{2}\right.
\end{aligned}
$$

$$
\begin{align*}
& \left.+\left\|\left(\theta_{1}, \theta_{2}\right)(\cdot, \tau)\right\|_{4}^{2}+\left\|\left(\theta_{1 t}, \theta_{2 t}\right)(\cdot, \tau)\right\|_{2}^{2}\right) d \tau \\
& \leq C\left(\left\|\left(\varphi_{10}, \theta_{10}, \varphi_{20}, \theta_{20}\right)\right\|_{3}^{2}+\left\|\left(\psi_{10}, \psi_{20}\right)\right\|_{2}^{2}+\delta\right), \quad t>0 \\
& \left\|\left(\varphi_{1}, \theta_{1}, \varphi_{2}, \theta_{2}\right)(\cdot, t)\right\|_{3}^{2}+\left\|\left(\varphi_{1 t}, \varphi_{2 t}, \mathscr{H}(\cdot, t)\right)\right\|_{2}^{2} \\
& +\left\|\left(\theta_{1 t}, \theta_{2 t}\right)(\cdot, t)\right\|_{1}^{2} \longrightarrow 0, \quad t \longrightarrow \infty \tag{60}
\end{align*}
$$

## 5. The Algebraic Decay Rates

In this section, we prove the time-decay rate of smooth solutions $\left(\varphi_{1}, \theta_{1}, \varphi_{2}, \theta_{2}\right)$ of (25) with the initial data $\left(\varphi_{10},-\psi_{10}, \theta_{10}, \varphi_{20},-\psi_{20}, \theta_{20}\right)$. For this aim, using the idea of $[4,15,24]$, we first prove the exponential decay of $\mathscr{H}$ and $\theta_{1}-\theta_{2}$ to zero then obtain the algebraic convergence of $\left(\varphi_{1}, \theta_{1}, \varphi_{2}, \theta_{2}\right)$. Due to Theorem 8, we know that the global smooth solutions ( $\varphi_{1}, \theta_{1}, \varphi_{2}, \theta_{2}$ ) satisfy

$$
\begin{align*}
& \left\|\left(\varphi_{1}, \theta_{1}, \varphi_{2}, \theta_{2}\right)\right\|_{3}^{2}+\left\|\left(\varphi_{1 t}, \varphi_{2 t}, \mathscr{H}\right)\right\|_{2}^{2}+\left\|\left(\theta_{1 t}, \theta_{2 t}, \mathscr{H}_{t}\right)\right\|_{1}^{2}  \tag{61}\\
& \quad \leq C\left(\Phi_{0}+\delta\right)
\end{align*}
$$

which leads to, in terms of Sobolev embedding theorem, that

$$
\begin{align*}
& \|\left(\varphi_{1}, \varphi_{2}, \varphi_{1 x}, \varphi_{2 x}, \varphi_{1 x x}, \varphi_{2 x x}, \varphi_{1 t}, \varphi_{1 t x}, \varphi_{2 t}, \varphi_{2 t x}\right. \\
& \left.\theta_{1}, \theta_{1 x}, \theta_{1 x x}, \theta_{2}, \theta_{2 x}, \theta_{2 x x}, \mathscr{H}, \mathscr{H}_{x}, \mathscr{H}_{t}\right) \|_{L^{\infty}(\mathbb{R})}  \tag{62}\\
& \quad \leq C\left(\Phi_{0}+\delta\right)
\end{align*}
$$

Further, by (25), we also have

$$
\begin{equation*}
\left\|\left(\varphi_{1 t t}, \varphi_{2 t t}, \theta_{1 t}, \theta_{2 t}\right)\right\|_{L^{\infty}(\mathbb{R})} \leq C\left(\Phi_{0}+\delta\right) \tag{63}
\end{equation*}
$$

Lemma 9. Let $\left(\varphi_{1}, \theta_{1}, \varphi_{2}, \theta_{2}\right)$ be the global classical solutions of IVP (25) and (26) satisfying $\Phi_{0}+\delta \ll 1$. Then it holds for $\mathscr{H}$ and $\theta_{1}-\theta_{2}$ that for $t>0$,

$$
\begin{align*}
& \|\left(\mathscr{H}_{,} \mathscr{H}_{x}, \mathscr{H}_{t}, \mathscr{H}_{x x}, \mathscr{H}_{t x}, \theta_{1}-\theta_{2}\right. \\
& \left.\quad\left(\theta_{1}-\theta_{2}\right)_{x},\left(\theta_{1}-\theta_{2}\right)_{x x}\right)(\cdot, t) \|  \tag{64}\\
& \quad \leq C\left(\Phi_{0}+\delta\right) e^{-\gamma_{0} t} .
\end{align*}
$$

Proof. Multiplying (30) by $\mathscr{H}$ and integrating it by parts over $\mathbb{R}$, we obtain

$$
\begin{align*}
& \frac{d}{d t} \int_{\mathbb{R}}\left(\mathscr{H}_{\mathscr{H}_{t}}+\frac{1}{2} \mathscr{H}^{2}\right) d x-\int_{\mathbb{R}} \mathscr{H}_{t}^{2} d x+\int_{\mathbb{R}} 2 \bar{n} \mathscr{H}^{2} d x \\
& \quad+\int_{\mathbb{R}}\left(\theta_{1}+\widehat{T}_{1}+T^{*}\right) \mathscr{H}_{x}^{2} d x \\
& =-\int_{\mathbb{R}} \bar{n}\left(\theta_{1}-\theta_{2}\right) \mathscr{H}_{x} d x \\
& \quad+\int_{\mathbb{R}}\left(h_{1 x}-h_{2}-h_{3}+h_{4 x}\right) \mathscr{H} d x . \tag{65}
\end{align*}
$$

Using Cauchy-Schwartz's inequality, Lemmas 3 and 4, (62), and (63), we have

$$
\begin{gather*}
-\int_{\mathbb{R}} \bar{n}\left(\theta_{1}-\theta_{2}\right) \mathscr{H}_{x} d x+\int_{\mathbb{R}} h_{1 x} \mathscr{H} d x \\
\leq \kappa \int_{\mathbb{R}} \mathscr{H}_{x}^{2} d x+C \int_{\mathbb{R}}\left(\theta_{1}-\theta_{2}\right)^{2} d x+C \delta e^{-v t}, \\
-\int_{\mathbb{R}}\left(h_{2}+h_{3}\right) \mathscr{H} d x \leq C\left(\Phi_{0}+\delta\right) \int_{\mathscr{R}} \mathscr{H}^{2} d x+C \delta e^{-v_{1} t} \tag{66}
\end{gather*}
$$

Moreover, noticing that

$$
\begin{align*}
h_{4 x}= & -\frac{j_{1}^{2}}{n_{1}^{2}} \mathscr{H}_{x x}-\frac{2 j_{1}}{n_{1}} \mathscr{H}_{t x}+O(1)\left(\widehat{n}_{1 x}+\widehat{n}_{2 x}+\hat{j}_{1 x}+\hat{j}_{2 x}\right) \\
& +O(1)\left(\varphi_{2 x x}+\varphi_{2 t x}+\bar{n}_{x}+\bar{j}_{x}+\widehat{n}_{2 x}+\widehat{j}_{2 x}\right) \\
& \times\left(\mathscr{H}_{x}+\mathscr{H}_{t}+\widehat{n}_{1}+\widehat{n}_{2}+\widehat{j}_{1}+\widehat{j}_{2}\right) \tag{67}
\end{align*}
$$

then

$$
\begin{align*}
& \int_{\mathbb{R}} h_{4 x} \mathscr{H} d x \\
& \quad \leq C\left(\Phi_{0}+\delta\right) \int_{\mathbb{R}}\left(\mathscr{H}^{2}+\mathscr{H}_{x}^{2}+\mathscr{H}_{t}^{2}\right) d x+C \delta e^{-v t} \tag{68}
\end{align*}
$$

Therefore, we have

$$
\begin{align*}
& \frac{d}{d t} \int_{\mathbb{R}}\left(\mathscr{H}_{\mathscr{H}_{t}}+\frac{1}{2} \mathscr{H}^{2}\right) d x-\int_{\mathbb{R}} \mathscr{H}_{t}^{2} d x+2 \int_{\mathbb{R}} \bar{n} \mathscr{H}^{2} d x \\
& \quad+\int_{\mathbb{R}}\left[\left(\theta_{1}+\widehat{T}_{1}+T^{*}\right)-\kappa\right] \mathscr{H}_{x}^{2} d x \\
& \leq C\left(\Phi_{0}+\delta\right) \int_{\mathbb{R}}\left(\mathscr{H}^{2}+\mathscr{H}_{x}^{2}+\mathscr{H}_{t}^{2}\right) d x \\
& \quad+C \int_{\mathbb{R}}\left(\theta_{1}-\theta_{2}\right)^{2} d x+C \delta e^{-v_{1} \mathrm{t}} \tag{69}
\end{align*}
$$

While multiplying (30) by $\mathscr{H}_{t}$ and integrating the resultant equation by parts over $\mathbb{R}$, similarly, we can show

$$
\begin{align*}
& \frac{d}{d t} \int_{\mathbb{R}}\left(\frac{1}{2} \mathscr{H}_{t}^{2}+\bar{n} \mathscr{H}^{2}+\left(\frac{1}{2}\left(\theta_{1}+\widehat{T}_{1}+T^{*}\right)-\frac{j_{1}^{2}}{n_{1}^{2}}\right) \mathscr{H}_{x}^{2}\right) d x \\
& \quad+\int_{\mathbb{R}} \mathscr{H}_{t}^{2} d x+\int_{\mathbb{R}} \bar{n}\left(\theta_{1}-\theta_{2}\right) \mathscr{H}_{t x} d x \\
& \leq C\left(\Phi_{0}+\delta\right) \\
& \quad \times \int_{\mathbb{R}}\left(\left(\theta_{1}-\theta_{2}\right)^{2}+\left(\theta_{1}-\theta_{2}\right)_{x}^{2}+\mathscr{H}^{2}+\mathscr{H}_{x}^{2}+\mathscr{H}_{t}^{2}\right) d x \\
& \quad+C \delta e^{-v_{1} t} \tag{70}
\end{align*}
$$

Next, multiplying (31) by $\left(3 \bar{n}\left(\varphi_{1 x}+\widehat{n}_{1}+\bar{n}\right) / 2\left(\theta_{1}+\widehat{T}_{1}+\right.\right.$ $\left.T^{*}\right)\left(\theta_{1}-\theta_{2}\right)$ and integrating the resultant equation by parts over $\mathbb{R}$, we get

$$
\begin{aligned}
& \frac{d}{d t} \int_{\mathbb{R}} \frac{3 \bar{n}\left(\varphi_{1 x}+\widehat{n}_{1}+\bar{n}\right)}{4\left(\theta_{1}+\widehat{T}_{1}+T^{*}\right)}\left(\theta_{1}-\theta_{2}\right)^{2} d x \\
&+\int_{\mathbb{R}}\left(\frac{3 \bar{n}\left(\varphi_{1 x}+\widehat{n}_{1}+\bar{n}\right)}{2\left(\theta_{1}+\widehat{T}_{1}+T^{*}\right)}\left(\theta_{1}-\theta_{2}\right)^{2}\right. \\
&\left.\quad+\frac{\bar{n}}{\theta_{1}+\widehat{T}_{1}+T^{*}}\left(\theta_{1}-\theta_{2}\right)_{x}^{2}\right) d x \\
&-\int_{\mathbb{R}} \bar{n}\left(\theta_{1}-\theta_{2}\right) \mathscr{H}_{t x} d x \\
&= \int_{\mathbb{R}}\left(\frac{3 \bar{n}\left(\varphi_{1 x}+\widehat{n}_{1}+\bar{n}\right)}{4\left(\theta_{1}+\widehat{T}_{1}+T^{*}\right)}\right)_{t}\left(\theta_{1}-\theta_{2}\right)^{2} d x \\
&-\int_{\mathbb{R}}\left(\frac{\bar{n}}{\theta_{1}+\widehat{T}_{1}+T^{*}}\right)_{x}\left(\theta_{1}-\theta_{2}\right)\left(\theta_{1}-\theta_{2}\right)_{x} d x \\
&+\int_{\mathbb{R}} G_{3} \frac{3 \bar{n}\left(\varphi_{1 x}+\widehat{n}_{1}+\bar{n}\right)}{2\left(\theta_{1}+\widehat{T}_{1}+T^{*}\right)}\left(\theta_{1}-\theta_{2}\right) d x \\
& \leq C\left(\Phi_{0}+\delta\right) \\
& \times \int_{\mathbb{R}}\left(\left(\theta_{1}-\theta_{2}\right)^{2}+\left(\theta_{1}-\theta_{2}\right)_{x}^{2}+\mathscr{H}_{x}^{2}+\mathscr{H}_{t}^{2}\right) d x \\
&+C\left(\Phi_{0}+\delta\right) e^{-v_{2} t},
\end{aligned}
$$

where in the last inequality, we have used

$$
\begin{align*}
& \int_{\mathbb{R}}\left(\frac{3 \bar{n}\left(\varphi_{1 x}+\widehat{n}_{1}+\bar{n}\right)}{4\left(\theta_{1}+\widehat{T}_{1}+T^{*}\right)}\right)_{t}\left(\theta_{1}-\theta_{2}\right)^{2} d x \\
& \quad-\int_{\mathbb{R}}\left(\frac{\bar{n}}{\theta_{1}+\widehat{T}_{1}+T^{*}}\right)_{x}\left(\theta_{1}-\theta_{2}\right)\left(\theta_{1}-\theta_{2}\right)_{x} d x  \tag{72}\\
& \quad \leq C\left(\Phi_{0}+\delta\right) \int_{\mathbb{R}}\left(\left(\theta_{1}-\theta_{2}\right)^{2}+\left(\theta_{1}-\theta_{2}\right)_{x}^{2}\right) d x \\
& \int_{\mathbb{R}} G_{3} \frac{3 \bar{n}\left(\varphi_{1 x}+\widehat{n}_{1}+\bar{n}\right)}{2\left(\theta_{1}+\widehat{T}_{1}+T^{*}\right)}\left(\theta_{1}-\theta_{2}\right) d x \\
& \leq C\left(\Phi_{0}+\delta\right) \int_{\mathbb{R}}\left(\left(\theta_{1}-\theta_{2}\right)^{2}+\left(\theta_{1}-\theta_{2}\right)_{x}^{2}+\mathscr{H}_{x}^{2}+\mathscr{H}_{t}^{2}\right) \\
& \quad+C\left(\Phi_{0}+\delta\right) e^{-v_{2} t}, \tag{73}
\end{align*}
$$

with the aid of

$$
\begin{align*}
\sum_{i=1}^{2}(-1)^{i-1} & \int_{\mathbb{R}}\left(\left(\frac{-\varphi_{i t}+\hat{j}_{i}+\bar{j}}{\varphi_{i x}+\widehat{n}_{i}+\bar{n}}\right)^{2}-S_{i}(x, t)\right) \\
& \times \frac{3 \bar{n}\left(\varphi_{1 x}+\widehat{n}_{1}+\bar{n}\right)}{2\left(\theta_{1}+\widehat{T}_{1}+T^{*}\right)}\left(\theta_{1}-\theta_{2}\right) d x  \tag{74}\\
\leq & C\left(\Phi_{0}+\delta\right) \int_{\mathbb{R}}\left(\left(\theta_{1}-\theta_{2}\right)^{2}+\mathscr{H}_{x}^{2}+\mathscr{H}_{t}^{2}\right) d x \\
& +C\left(\Phi_{0}+\delta\right) e^{-v_{2} t}
\end{align*}
$$

Combine (69), (70), and (71), and choose proper positive constant $\lambda_{1}$ and $\Lambda_{1}$ such that

$$
\begin{align*}
\lambda_{1} \times & (70)+\Lambda_{1} \times((71)+(72)) \\
& \sim \mathscr{H}_{t}^{2}+\mathscr{H}^{2}+\mathscr{H}_{x}^{2}+\left(\theta_{1}-\theta_{2}\right)^{2} \tag{75}
\end{align*}
$$

Then, we have

$$
\begin{align*}
\frac{d}{d t} \| & \left.\| \mathscr{H}_{t}, \mathscr{H}^{\prime}, \mathscr{H}_{x},\left(\theta_{1}-\theta_{2}\right)\right)(\cdot, \tau) \|^{2} \\
& +C\left\|\left(\mathscr{H}_{t}, \mathscr{H}, \mathscr{H}_{x},\left(\theta_{1}-\theta_{2}\right),\left(\theta_{1}-\theta_{2}\right)_{x}\right)(\cdot, t)\right\|^{2}  \tag{76}\\
\leq & C\left(\Phi_{0}+\delta\right) e^{-v_{2} t}
\end{align*}
$$

which, together with Gronwall's inequality, yields

$$
\begin{equation*}
\left\|\left(\mathscr{H}, \mathscr{H}_{x}, \mathscr{H}_{t},\left(\theta_{1}-\theta_{2}\right)\right)(\cdot, t)\right\|^{2} \leq C\left(\Phi_{0}+\delta\right) e^{-\gamma_{1} t} \tag{77}
\end{equation*}
$$

for some positive constants $\gamma_{1}$ and $C$. In the completely same way, treating $\int_{\mathbb{R}} \lambda_{2}(30)_{x} \mathscr{H}_{x}+\Lambda_{2}\left((30)_{x} \mathscr{H}_{t x}+(31)_{x}\left(3 \bar{n}\left(\varphi_{1 x}+\right.\right.\right.$ $\left.\left.\left.\widehat{n}_{1}+\bar{n}\right) / 2\left(\theta_{1}+\widehat{T}_{1}+T^{*}\right)\right)\left(\theta_{1}-\theta_{2}\right)_{x}\right) d x$ for proper positive constants $\lambda_{2}$ and $\Lambda_{2}$, we can show

$$
\begin{equation*}
\|\left(\mathscr{H}_{x}, \mathscr{H}_{x x}, \mathscr{H}_{t x},\left(\theta_{1}-\theta_{2}\right)_{x}(\cdot, t) \|^{2} \leq C\left(\Phi_{0}+\delta\right) e^{-\gamma_{2} t}\right. \tag{78}
\end{equation*}
$$

for some constant $\gamma_{2}$.
Moreover, from (30), (77), and (78), we obtain

$$
\begin{equation*}
\left\|\mathscr{H}_{t t}\right\|^{2} \leq C\left(\Phi_{0}+\delta\right) e^{-\gamma_{3} t} \tag{79}
\end{equation*}
$$

for $\gamma_{3}=\min \left\{\gamma_{1}, \gamma_{2}\right\}$. Finally, by $\int_{\mathbb{R}}(31)_{t}\left(\theta_{1}-\theta_{2}\right)_{t} d x$ and using (77)-(79), there is a positive constant $\gamma_{4}$ such that

$$
\begin{equation*}
\left\|\left(\theta_{1}-\theta_{2}\right)_{t}\right\|^{2} \leq C\left(\Phi_{0}+\delta\right) e^{-\gamma_{4} t} \tag{80}
\end{equation*}
$$

while from (31) and (77)-(80), we have

$$
\begin{equation*}
\left\|\left(\theta_{1}-\theta_{2}\right)_{x x}\right\|^{2} \leq C\left(\Phi_{0}+\delta\right) e^{-\gamma_{5} t} \tag{81}
\end{equation*}
$$

with $\gamma_{5}=\min \left\{\gamma_{3}, \gamma_{4}\right\}$. Combination of (77)-(80) and (81) yields (64). This completes the proof.

In the following, using the idea of $[4,15]$, we turn to derive the time-decay rate of $\left(\varphi_{1}, \theta_{1}, \varphi_{2}, \theta_{2}\right)$ by which we are able to obtain the algebraical decay rate of $\left(\varphi_{1}, \theta_{1}, \varphi_{2}, \theta_{2}\right)$ in large time.

Lemma 10. Let $\left(\varphi_{1}, \theta_{1}, \varphi_{2}, \theta_{2}\right)$ be the global classical solution of the IVP (25) and (26) with initial data satisfying $\Phi_{0}+\delta \ll 1$. If it holds for $\left(\varphi_{1}, \varphi_{2}, \theta_{1}, \theta_{2}\right)(t>0)$ that

$$
\begin{align*}
& \sum_{k=0}^{3}(1+t)^{k}\left\|\partial_{x}^{k}\left(\varphi_{1}, \varphi_{2}\right)\right\|^{2}+\sum_{k=0}^{2}(1+t)^{k+2}\left\|\partial_{x}^{k}\left(\varphi_{1 t}, \varphi_{2 t}\right)\right\|^{2} \\
& \quad+\sum_{k=0}^{1}(1+t)^{k+2}\left\|\partial_{x}^{k}\left(\theta_{1 t}, \theta_{2 t}\right)\right\|^{2}  \tag{82}\\
& \quad+\sum_{k=0}^{2}(1+t)^{k+1}\left\|\partial_{x}^{k}\left(\theta_{1}, \theta_{2}\right)\right\|^{2} \\
& \quad+(1+t)^{3}\left\|\partial_{x}^{3}\left(\theta_{1}, \theta_{2 *}\right)\right\|^{2} \ll 1
\end{align*}
$$

then one has

$$
\begin{align*}
& \sum_{k=0}^{3}(1+t)^{k}\left\|\partial_{x}^{k}\left(\varphi_{1}, \varphi_{2}\right)\right\|^{2}+\sum_{k=1}^{3} \int_{0}^{t}(1+\tau)^{k}\left\|\partial_{x}^{k}\left(\varphi_{1}, \varphi_{2}\right)\right\|^{2} d \tau \\
&+\sum_{k=0}^{2}(1+t)^{k+1}\left\|\partial_{x}^{k}\left(\theta_{1}, \theta_{2}\right)\right\|^{2} \\
&+\sum_{k=0}^{2} \int_{0}^{t}\left((1+\tau)^{k+1}\left\|\partial_{x}^{k}\left(\theta_{1}, \theta_{2}\right)\right\|^{2}\right. \\
&\left.\quad+(1+\tau)^{3}\left\|\partial_{x}^{3}\left(\theta_{1 t}, \theta_{2 t}\right)\right\|^{2}\right) d \tau \\
& \leq C\left(\Phi_{0}+\delta\right), \\
& \sum_{k=0}^{2}(1+t)^{k+2}\left\|\partial_{x}^{k}\left(\varphi_{1 t}, \varphi_{2 t}\right)\right\|^{2} \\
& \quad+\sum_{k=0}^{2} \int_{0}^{t}(1+\tau)^{k+2}\left\|\partial_{x}^{k}\left(\varphi_{1 \tau}, \varphi_{2 \tau}\right)\right\|^{2} d \tau \\
&+\sum_{k=0}^{1}(1+t)^{k+2}\left\|\partial_{x}^{k}\left(\theta_{1 t}, \theta_{2 t}\right)\right\|^{2}+(1+t)^{3}\left\|\partial_{x}^{3}\left(\theta_{1 t}, \theta_{2 t}\right)\right\|^{2} \\
&+\int_{0}^{t}\left(\sum_{k=0}^{1}(1+\tau)^{k+2}\left\|\partial_{x}^{k}\left(\theta_{1}, \theta_{2}\right)\right\|^{2}\right. \\
&\left.\quad+(1+\tau)^{3}\left\|\partial_{x}^{3}\left(\theta_{1 \tau}, \theta_{2 \tau}\right)\right\|^{2}\right) d \tau \\
& \leq C\left(\Phi_{0}+\delta\right) \tag{83}
\end{align*}
$$

Since the proof is similar as that in [15], we can omit the details.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Q-Symmetry and Conditional Q-Symmetries for Boussinesq Equation 

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We study in this paper the $Q$-symmetry and conditional $Q$-symmetries of Boussinesq equation. The solutions which we obtain, in this case, are in the form of convergent power series with easily computable components.

## 1. Introduction

The Boussinesq equation, which belongs to the KdV family of equations and describes motions of long waves in shallow water under gravity propagating in both directions, is given by

$$
\begin{equation*}
u_{t t}+u_{x}^{2}+u u_{x x}+u_{x x x}=0 \tag{1}
\end{equation*}
$$

where $u(x, t)$ is a sufficiently often differentiable function.
A great deal of research work has been invested in recent years for the study of the Boussinesq equation. Many effective methods for obtaining exact solutions of Boussinesq equation have been proposed, such as variational iteration method [1], Travelling wave solutions [2], potential method [3], scattering method [4], the $\left(G^{\prime} / G\right)$ expansion method [5], optimal and symmetry reductions [6], and projective Riccati equations method [7].

The aim of this paper is to calculate and list the Qsymmetry and conditional $Q$-symmetries of Boussinesq equation. We can say today that many mathematicians, mechanicians and physicists, such as Euler, D'Alembert, Poincare, Volterra, Whittaker, Bateman, implicitly used conditional symmetries for the construction of exact solutions of the linear wave equation.

Nontrivial conditional symmetries of a PDE (partial differential equation) allow us to obtain in explicit form such
solutions which cannot be found by using the symmetries of the whole set of solutions of the given PDE [8]. Moreover, conditional symmetries make the class of PDEs reduce to a system of ODEs (ordinary differential equations). As a rule, the reduced equations one obtains from conditional symmetries and from $Q$-symmetry are significantly simpler than those found by reduction using symmetries of the full set of solutions. This allows us to construct exact solutions of the reduced equations.

## 2. Conditional Q-Symmetries

The classical symmetry properties can be extended if one studies (1) together with the invariant surface of the symmetry generator as an overdetermined system of partial differential equations [9]. That is, one studies the Lie symmetry properties of the system

$$
\begin{gather*}
u_{t t}+u_{x}^{2}+u u_{x x}+u_{x x x}=0  \tag{2}\\
\eta(x, t, u)-\xi_{1}(x, t, u) u_{x}-\xi_{2}(x, t, u) u_{t}=0 \tag{3}
\end{gather*}
$$

where (3) is the invariant surfaces corresponding to the Lie symmetry group generator

$$
\begin{equation*}
Z=\xi_{1}(x, t, u) \frac{\partial}{\partial x}+\xi_{2}(x, t, u) \frac{\partial}{\partial t}+\eta(x, t, u) \frac{\partial}{\partial u} \tag{4}
\end{equation*}
$$

The invariance condition leading to conditional $Q$-symmetries for (2) is given by

$$
\begin{equation*}
\left.Z^{(3)} F\right|_{\left\{F^{(j)}=0, Q^{(k)}=0\right\}}=0, \tag{5}
\end{equation*}
$$

where

$$
\begin{gather*}
F_{1}=u_{t t}+u_{x}^{2}+u u_{x x}+u_{x x x} \\
Q=\eta(x, t, u)-\xi_{1}(x, t, u) u_{x}-\xi_{2}(x, t, u) u_{t} . \tag{6}
\end{gather*}
$$

Here $Z^{(3)}$ denotes the second prolongation of $Z$, namely,

$$
\begin{align*}
Z^{(3)}= & Z+\gamma_{1} \frac{\partial}{\partial u_{x}}+\gamma_{2} \frac{\partial}{\partial u_{t}}+\gamma_{11} \frac{\partial}{\partial u_{x x}}  \tag{7}\\
& +\gamma_{22} \frac{\partial}{\partial u_{t t}}+\gamma_{111} \frac{\partial}{\partial u_{x x x}}
\end{align*}
$$

where

$$
\begin{gather*}
\gamma_{1}=D_{x}\left(\eta_{1}\right)-u_{x} D_{x}\left(\xi_{1}\right)-u_{t} D_{x}\left(\xi_{2}\right), \\
\gamma_{2}=D_{t}\left(\eta_{1}\right)-u_{x} D_{t}\left(\xi_{1}\right)-u_{t} D_{t}\left(\xi_{2}\right), \\
\gamma_{11}=D_{x}\left(\gamma_{1}\right)-u_{x x} D_{x}\left(\xi_{1}\right)-u_{x t} D_{x}\left(\xi_{2}\right),  \tag{8}\\
\gamma_{22}=D_{t}\left(\gamma_{2}\right)-u_{x t} D_{t}\left(\xi_{1}\right)-u_{t t} D_{t}\left(\xi_{2}\right), \\
\gamma_{111}=D_{x}\left(\gamma_{11}\right)-u_{x x x} D_{x}\left(\xi_{1}\right)-u_{x x t} D_{x}\left(\xi_{2}\right) .
\end{gather*}
$$

A generator $Z$ which satisfies condition (5) is called a conditional $Q$-symmetry generator, where by the invariant surface (3). The $F^{(j)}$ and $Q^{(k)}$ denote the $j$ th and $k$ th prolongations, respectively. $D_{x}$ and $D_{t}$ denote the total derivative with respect to $x$ and with respect to $t$, respectively.

We now derive the general determining equations for the conditional $Q$-symmetry generators for (2). We set $\xi_{1}=$ $\xi_{1}(x, t, u), \xi_{2}=\xi_{2}(x, t, u)$, and $\eta=\eta(x, t, u)$. The invariance condition (5) leads to the following expression:

$$
\begin{equation*}
\gamma_{22}+u \gamma_{11}+\eta u_{x x}+2 u_{x} \gamma_{1}+\gamma_{111}=0 \tag{9}
\end{equation*}
$$

This leads to

$$
\begin{aligned}
& u_{x x} \eta+u_{t t} \eta_{u}+\eta_{u} u_{x x x}-2 u_{x t} \xi_{t} \\
& \quad-u_{x} u_{t t} \xi_{u}-2 u_{t} u_{x t} \xi_{u}-3 u_{x x}^{2} \xi_{u} \\
& \quad-4 u_{x} u_{x x x} \xi_{u}-3 u_{x x x} \xi_{x} \\
& \quad+2 u_{x} u_{x x}\left(u_{x} \eta_{u}+\eta_{x}-u_{x}^{2} \xi_{u}-u_{x} \xi_{x}\right) \\
& \quad+\eta_{t t}+2 u_{t} \eta_{t u}+u_{t}^{2} \eta_{u u} \\
& \quad+3 u_{x} u_{x x} \eta_{u u}+3 u_{x x} \eta_{x u}-u_{x} \xi_{t t} \\
& \quad-2 u_{t} u_{x} \xi_{t u}-u_{t}^{2} u_{x} \xi_{u u}-6 u_{x}^{2} u_{x x} \xi_{u u} \\
& \quad-9 u_{x} u_{x x} \xi_{x u}-3 u_{x x} \xi_{x x}
\end{aligned}
$$

$$
\begin{align*}
& +u\left(u_{x x} \eta_{u}-3 u_{x} u_{x x} \xi_{u}-2 u_{x x} \xi_{x}+u_{x}^{2} \eta_{u u}\right. \\
& \left.\quad+2 u_{x} \eta_{x u}+\eta_{x x}-u_{x}^{3} \xi_{u u}-2 u_{x}^{2} \xi_{x u}-u_{x} \xi_{x x}\right) \\
& +u_{x}^{3} \eta_{u u u}+3 u_{x}^{2} \eta_{x u u}+3 u_{x} \eta_{x x u}+\eta_{x x x}-u_{x}^{4} \xi_{u u u} \\
& -3 u_{x}^{3} \xi_{x u u}-3 u_{x}^{2} \xi_{x x u}-u_{x} \xi_{x x x}=0 . \tag{10}
\end{align*}
$$

In particular, from $Q=0$ follows

$$
\begin{equation*}
\xi_{2}(x, t, u) u_{t}=\eta(x, t, u)-\xi_{1}(x, t, u) u_{x} . \tag{11}
\end{equation*}
$$

The determining equations for the conditional $Q$-symmetry generator $Z$ are now obtained by equating to zero the coefficients of the independent coordinates. By solving this system of linear partial differential equations for the infinitesimal $\xi_{1}(x, t, u), \xi_{2}(x, t, u)$, and $\eta(x, t, u)$, we obtain

$$
\begin{gather*}
\eta(x, t, u)=-\frac{2 k_{3} u}{3}, \\
\xi_{1}(x, t, u)=k_{1}+\frac{2 k_{3} x}{3}  \tag{12}\\
\xi_{2}(x, t, u)=k_{2}+k_{3} t
\end{gather*}
$$

where $k_{1}, k_{2}$, and $k_{3}$ are arbitrary constants.
The conditional $Q$-symmetry is given by

$$
\begin{equation*}
Z=\left(k_{1}+\frac{2 k_{3} x}{3}\right) \frac{\partial}{\partial x}+\left(k_{2}+k_{3} t\right) \frac{\partial}{\partial t}-\frac{2 k_{3} u}{3} \frac{\partial}{\partial u} . \tag{13}
\end{equation*}
$$

The general solution of the associated invariant surface condition,

$$
\begin{equation*}
\left(k_{1}+\frac{2 k_{3} x}{3}\right) \frac{\partial u}{\partial x}+\left(k_{2}+k_{3} t\right) \frac{\partial u}{\partial t}=-\frac{2 k_{3} u}{3} \tag{14}
\end{equation*}
$$

is

$$
\begin{equation*}
u(x, t)=\frac{\varphi(z)}{\left(3 k_{1}+2 k_{3} x\right)}, \tag{15}
\end{equation*}
$$

where $\varphi(z)$ is arbitrary function of $z$ and

$$
\begin{equation*}
z(x, t)=\frac{3 k_{2}+2 k_{3} x}{\left(k_{2}+k_{3} t\right)^{2 / 3}} . \tag{16}
\end{equation*}
$$

Substituting (15) into (2), we finally obtain the following nonlinear ordinary differential equation for $\varphi(z)$ taking the form

$$
\begin{align*}
& -216 k_{3} \varphi(z)+54 \varphi^{2}(z)+216 k_{3} z \varphi^{\prime}(z)+5 z^{4} \varphi^{\prime}(z) \\
& \quad-72 z \varphi(z) \varphi^{\prime}(z)+18 z^{2} \varphi^{\prime 2}(z)-108 k_{3} z^{2} \varphi^{\prime \prime}(z) \\
& +2 z^{5} \varphi^{\prime \prime}(z)+18 z^{2} \varphi(z) \varphi^{\prime \prime}(z)+36 k_{3} z^{3} \varphi^{\prime \prime \prime}(z)=0, \tag{17}
\end{align*}
$$

where $\varphi^{\prime}(z)=d \varphi_{i} / d z, \varphi^{\prime \prime}(z)=d^{2} \varphi_{i} / d z^{2}$, and $\varphi^{\prime \prime \prime}(z)=$ $d^{3} \varphi_{i} / d z^{3}$.

Solving an ordinary differential equation (17), we have three cases of solutions for $\varphi(z)$.

Case 1. Consider

$$
\begin{equation*}
\varphi(z)=4 k_{3}, \tag{18}
\end{equation*}
$$

where $k_{3}$ is an arbitrary constant.

Case 2. Consider

$$
\begin{equation*}
\varphi(z)=-\frac{1}{4} z^{3}, \quad k_{3}=-3 \tag{19}
\end{equation*}
$$

Case 3. Consider

$$
\begin{equation*}
\varphi(z)=4 k_{3}-\frac{1}{4} z^{3}, \quad k_{3}=-\frac{1}{4} \tag{20}
\end{equation*}
$$

By using (18)-(20) into (15), we have solutions for Boussinesq equation (1) in the following forms.

Family 1. Consider

$$
\begin{equation*}
u(x, t)=\frac{4 k_{3}}{\left(3 k_{1}+2 k_{3} x\right)}, \tag{21}
\end{equation*}
$$

where $k_{1}$ and $k_{3}$ are arbitrary constants.
Family 2. Consider

$$
\begin{equation*}
u(x, t)=\frac{-(1 / 4) z^{3}}{\left(3 k_{1}-6 x\right)} \tag{22}
\end{equation*}
$$

where $z=\left(3 k_{1}-6 x\right) /\left(k_{2}-3 t\right)^{2 / 3}$ and $k_{1}$ is an arbitrary constant.

Family 3. Consider

$$
\begin{equation*}
u(x, t)=-\frac{1+(1 / 4) z^{3}}{\left(3 k_{1}-x / 2\right)} \tag{23}
\end{equation*}
$$

where $z=\left(3 k_{1}-(1 / 2) x\right) /\left(k_{2}-(1 / 4) t\right)^{2 / 3}$ and $k_{1}$ is an arbitrary constant.

## 3. Q-Symmetry Generators

Before we consider conditional symmetries of (1), let us briefly describe the classical Lie approach and introduce our notation [10]. We are concerned with a partial differential equation of order $r$ with $m+1$ independent variables $\left(x_{0}, x_{1}, \ldots, x_{m}\right)$ and one field variable $u$, that is, an equation of the form

$$
\begin{equation*}
F\left(x_{0}, x_{1}, \ldots, x_{m}, u, \frac{\partial u}{\partial x_{0}}, \ldots, \frac{\partial^{r} u}{\partial x_{j_{1}} \cdots \partial x_{j_{r}}}\right)=0 \tag{24}
\end{equation*}
$$

where $0 \leq j_{1} \leq j_{2} \leq \cdots \leq j_{r} \leq m, j=0, \ldots, m$. A Lie transformation group that leaves (24) invariant is generated by a Lie symmetry generator $Z$, defined by

$$
\begin{align*}
Z= & \sum_{j=0}^{m} \xi_{j}\left(x_{0}, x_{1}, \ldots, x_{m}, u, v\right) \frac{\partial}{\partial x_{j}}  \tag{25}\\
& +\eta\left(x_{0}, x_{1}, \ldots, x_{m}, u\right) \frac{\partial}{\partial u}
\end{align*}
$$

$Z_{w}$ is the associated vertical form of (25), defined by

$$
\begin{equation*}
Z_{w}=\left(\eta-\sum_{j=0}^{m} \xi_{j} u_{j}\right) \frac{\partial}{\partial u}, \tag{26}
\end{equation*}
$$

where $\left.Z_{w}\right|_{\theta}=\left.Z\right|_{\theta}$. Here $\theta$ is a differential 1-form, called the contact form, which is defined by

$$
\begin{equation*}
\theta=d u-\sum_{j=0}^{m} u_{j} d x_{j} \tag{27}
\end{equation*}
$$

Equation (24) is called invariant under the prolonged Lie symmetry generators $Z_{w}$ if

$$
\begin{equation*}
L_{\check{Z}_{w}} F=0 \tag{28}
\end{equation*}
$$

$L$ denotes the Lie derivative, and $\breve{Z}_{w}$ is found by prolonging the vertical generator $Z_{w}$; that is,

$$
\begin{equation*}
\breve{Z}_{w}=\sum_{j=0}^{m} D_{j}\left(U_{1}\right) \frac{\partial}{\partial u_{j}}+\cdots+\sum_{j_{1}, \ldots, j_{r}=0}^{m} D_{j_{1}, \ldots, j_{r}}\left(U_{1}\right) \frac{\partial}{\partial u_{j_{1}, \ldots, j_{r}}} \tag{29}
\end{equation*}
$$

where

$$
\begin{equation*}
U=\left(\eta-\sum_{j=0}^{m} \xi_{j} u_{j}\right) \tag{30}
\end{equation*}
$$

and $D_{j}$ is the total derivative operator. We give the definition for conditional invariance of (24) as follows.

Definition 1. Equation (24) is called Q-conditionally invariant if

$$
\begin{equation*}
L_{\check{Z}_{w}} F=0 \tag{31}
\end{equation*}
$$

under the condition

$$
\begin{equation*}
\left.Z_{w}\right|_{\theta}=0 \tag{32}
\end{equation*}
$$

$Z_{w}$ is called the $Q$-symmetry generator and $\breve{Z}_{w}$ is called the prolonged vertical $Q$-symmetry generator. Let us now study (1) by the use of the above definition. From the definition it follows that the Lie derivative (31), for equations

$$
\begin{equation*}
F \equiv u_{t t}+u_{x}^{2}+u u_{x x}+u_{x x x}=0 \tag{33}
\end{equation*}
$$

under the condition

$$
\begin{equation*}
\left.Z_{w}\right|_{\theta}=\eta-\xi_{1} u_{x}-\xi_{2} u_{t}=0 \tag{34}
\end{equation*}
$$

has to be studied. Let us consider the $Q$-symmetry generator in the form

$$
\begin{equation*}
Z=\xi_{1}(x, t, u) \frac{\partial}{\partial x}+\xi_{2}(x, t, u) \frac{\partial}{\partial t}+\eta(x, t, u) \frac{\partial}{\partial u} \tag{35}
\end{equation*}
$$

By applying the Lie derivative (31) and condition (32), we get

$$
\begin{equation*}
D_{t t}(U)+u D_{x x}(U)+\eta u_{x x}+2 u_{x} D_{x}(U)+D_{x x x}(U)=0 \tag{36}
\end{equation*}
$$

where

$$
\begin{align*}
D_{x} & =\frac{\partial}{\partial x}+u_{x} \frac{\partial}{\partial u}+u_{x x} \frac{\partial}{\partial u_{x}}+u_{x t} \frac{\partial}{\partial u_{t}}+\cdots \\
D_{t} & =\frac{\partial}{\partial t}+u_{t} \frac{\partial}{\partial u}+u_{t t} \frac{\partial}{\partial u_{t}}+u_{x t} \frac{\partial}{\partial u_{x}}+\cdots \tag{37}
\end{align*}
$$

The determining equations for the $Q$-symmetry generator $Z$ are now obtained by equating to zero the coefficients of the independent coordinates. By solving this system of linear partial differential equations for the infinitesimal $\xi_{1}, \xi_{2}$, and $\eta$, we obtain

$$
\begin{gather*}
\eta(x, t, u)=-\frac{2 k_{3} u}{3} \\
\xi_{1}(x, t, u)=k_{1}+\frac{2 k_{3} x}{3}  \tag{38}\\
\xi_{2}(x, t, u)=k_{2}+k_{3} t
\end{gather*}
$$

All of the similarity variables associated with the Lie symmetries (38) can be derived by solving the following characteristic equation:

$$
\begin{equation*}
\frac{d x}{\xi_{1}}=\frac{d t}{\xi_{2}}=\frac{d u}{\eta} . \tag{39}
\end{equation*}
$$

Consequently

$$
\begin{equation*}
\frac{d x}{\left(k_{1}+2 k_{3} x / 3\right)}=\frac{d t}{\left(k_{2}+k_{3} t\right)}=\frac{d u}{-2 k_{3} u / 3} . \tag{40}
\end{equation*}
$$

We obtain the following similarity variable:

$$
\begin{equation*}
z(x, t)=\frac{3 k_{1}+2 k_{3} x}{\left(k_{2}+k_{3} t\right)^{2 / 3}}, \tag{41}
\end{equation*}
$$

and the similarity solutions take the form

$$
\begin{equation*}
u(x, t)=\frac{F_{1}(z)}{\left(k_{2}+k_{3} t\right)^{2 / 3}}, \tag{42}
\end{equation*}
$$

where $F_{1}(z)$ is arbitrary functions of $z$. Substituting from (42) into (1), we finally obtain nonlinear ordinary differential equation for $F_{1}(z)$ taking the form

$$
\begin{gather*}
36 k_{3} F_{1}^{\prime \prime \prime}(z)+2 z^{2} F_{1}^{\prime \prime}(z)+18 F_{1}(z) F_{1}^{\prime \prime}(z)  \tag{43}\\
+5 F_{1}(z)+18 F^{\prime 2}(z)+9 z F_{1}^{\prime}(z)=0,
\end{gather*}
$$

where $F_{i}^{\prime}=d \varphi_{i} / d z, F_{i}^{\prime \prime}=d^{2} \varphi_{i} / d z^{2}$ and $F_{i}^{\prime \prime \prime}=d^{3} \varphi_{i} / d z^{3}$; ( $i=1$ ).

Solving a system of an ordinary differential equation (43), we have two cases of solutions for $F_{1}(z)$.

## Case 1. Consider

$$
\begin{equation*}
F_{1}(z)=\frac{4 k_{3}}{z} \tag{44}
\end{equation*}
$$

where $k_{3}$ is an arbitrary constant.
Case 2. Consider

$$
\begin{equation*}
F_{1}(z)=-\frac{1}{4} z^{2} \tag{45}
\end{equation*}
$$

Substitut from (44)-(45) into (42) to obtain the solutions for the Boussinesq equation (1) in the following forms.

Family 1. Consider

$$
\begin{equation*}
u(x, t)=\frac{4 k_{3}}{z\left(k_{2}+k_{3} t\right)^{2 / 3}} \tag{46}
\end{equation*}
$$

where $k_{2}$ and $k_{3}$ are an arbitrary constants.
Family 2. Consider

$$
\begin{equation*}
u(x, t)=-\frac{z^{2}}{4\left(k_{2}+k_{3} t\right)^{2 / 3}}, \tag{47}
\end{equation*}
$$

where $z=\left(3 k_{1}+2 k_{3} x\right) /\left(k_{2}+k_{3} t\right)^{2 / 3}$ and $k_{2}$ and $k_{3}$ are arbitrary constants.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Approximate Symmetry Analysis of a Class of Perturbed Nonlinear Reaction-Diffusion Equations 

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#### Abstract

The problem of approximate symmetries of a class of nonlinear reaction-diffusion equations called Kolmogorov-PetrovskyPiskounov (KPP) equation is comprehensively analyzed. In order to compute the approximate symmetries, we have applied the method which was proposed by Fushchich and Shtelen (1989) and fundamentally based on the expansion of the dependent variables in a perturbation series. Particularly, an optimal system of one-dimensional subalgebras is constructed and some invariant solutions corresponding to the resulted symmetries are obtained.


## 1. Introduction

Nonlinear problems arise widely in various fields of science and engineering mainly due to the fact that most physical systems are inherently nonlinear in nature. But for nonlinear partial differential equations (PDEs), analytical solutions are rare and difficult to obtain. Hence, the investigation of the exact solutions of nonlinear PDEs plays a fundamental role in the analysis of nonlinear physical phenomena. One of the most famous and established procedures for obtaining exact solutions of differential equations is the classical symmetries method, also called group analysis. This method was originated in 1881 from the pioneering work of Sophus Lie [1]. The investigation of symmetries has been manifested as one of the most significant and fundamental methods in almost every branch of science such as in mathematics and physics. Nowadays, the application of Lie group theory for the construction of solutions of nonlinear PDEs can be regarded as one of the most active fields of research in the theory of nonlinear PDEs and many good books have been dedicated to this subject (such as [2-4]). For some nonlinear problems, however, symmetries are not rich to determine useful solutions. Hence, this fact was the motivation for the creation of several generalizations of the classical Lie
group method. Consequently, several alternative reduction methods have been introduced, going beyond Lie's classical procedure and providing further solutions. One of the techniques widely applied in analyzing nonlinear problems is the perturbation analysis. Perturbation theory comprises mathematical methods that are applied to obtain an approximate solution to a problem which cannot be solved exactly. Indeed, this procedure is performed by expanding the dependent variables asymptotically in terms of a small parameter. In order to combine the power of the Lie group theory and perturbation analysis, two different approximate symmetry theories have been developed recently. The first method is due to Baikov et al. [5, 6]. Successively another method for obtaining approximate symmetries was introduced by Fushchich and Shtelen [7].

In the method proposed by Baikov et al. the Lie operator is expanded in a perturbation series other than perturbation for dependent variables as in the usual case. In other words, assume that the perturbed differential equation is in the form $F(z)=F_{0}(z)+\varepsilon F_{1}(z)$, where $z=\left(x, u, u_{(1)}, \ldots, u_{(n)}\right), F_{0}$ is the unperturbed equation, $F_{1}(z)$ is the perturbed term, and $X=X^{0}+\varepsilon X^{1}$ is the corresponding infinitesimal generator. The exact symmetry of the unperturbed equation $F_{0}(z)$ is denoted by $X^{0}$ and can be obtained as $\left.X^{0} F_{0}(z)\right|_{F_{0}(z)=0}=0$.

Then, by applying the auxiliary function $H=(1 / \varepsilon) X^{0}\left(F_{0}(z)+\right.$ $\left.\varepsilon F_{1}(z)\right)\left.\right|_{F_{0}+\varepsilon F_{1}=0}$, vector field $X_{1}$ will be deduced from the following relation:

$$
\begin{equation*}
\left.X^{1} F_{0}(z)\right|_{F_{0}=0}+H=0 \tag{1}
\end{equation*}
$$

Finally, after obtaining the approximate symmetries, the corresponding approximate solutions will be obtained via the classical Lie symmetry method [8].

In the second method due to Fushchich and Shtelen, first of all the dependent variables are expanded in a perturbation series. In the next step, terms are then separated at each order of approximation and as a consequence a system of equations to be solved in a hierarchy is determined. Finally, the approximate symmetries of the original equation are defined to be the exact symmetries of the system of equations resulting from perturbations [7, 9, 10]. Pakdemirli et al. in a recent paper [11] have compared these above two methods. According to their comparison, the expansion of the approximate operator applied in the first method does not reflect well an approximation in the perturbation sense, while the second method is consistent with the perturbation theory and results in correct terms for the approximate solutions. Consequently, the second method is superior to the first one according to the comparison in [11].

Nonlinear reaction-diffusion equations can be regarded as mathematical models which explain the change of the concentration of one or more substances distributed in space. Indeed, this variation occurs under the influence of two main processes including chemical reactions in which the substances are locally transformed into each other and diffusion which makes the substances spread out over a surface in space. From the mathematical point of view, reactiondiffusion systems generally take the form of semilinear parabolic PDEs. It is worth mentioning that the solutions of reaction-diffusion equations represent a wide range of behaviors, such as formation of wave-like phenomena and traveling waves as well as other self-organized patterns.

In this paper, we will apply the method proposed by Fushchich and Shtelen [7] in order to present a comprehensive analysis of the approximate symmetries of a significant class of nonlinear reaction-diffusion equations called Kolmogrov-Petrovsky-Piskounov (KPP) equation [12]. This equation can be regarded as the most simple reactiondiffusion equation concerning the concentration $u$ of a single substance in one spatial dimension and is generally defined as follows:

$$
\begin{equation*}
u_{t}-u_{x x}=R(u) . \tag{2}
\end{equation*}
$$

By inserting different values to the reaction term $R(u)$ of (2), the following significant equations are deduced.
(1) If the reaction term $R(u)$ vanishes, then the resulted equation displays a pure diffusion process and is defined by

$$
\begin{equation*}
u_{t}=u_{x x} \tag{3}
\end{equation*}
$$

Note that the above equation is called Fick's second law [12].
(2) By inserting $R(u)=a u(1-u), a \geq 0$, the Fisher equation (or logistic equation) results as follows:

$$
\begin{equation*}
u_{t}=u_{x x}+a u(1-u) \tag{4}
\end{equation*}
$$

This equation can be regarded as the archetypical deterministic model for the spread of a useful gene in a population of diploid individuals living in a onedimensional habitat [13, 14].
(3) By inserting $R(u)=u^{2}(1-u)$, the Zeldovich equation will be deduced as follows:

$$
\begin{equation*}
u_{t}=u_{x x}+u^{2}(1-u) \tag{5}
\end{equation*}
$$

This equation appears in combustion theory. The unknown $u$ displays temperature, while the last term on the right-hand side is concerned with the generation of heat by combustion [15, 16].
(4) By inserting $R(u)=u\left(1-u^{2}\right)$ the Newell-WhiteheadSegel (NWS) equation (or amplitude equation) results as follows:

$$
\begin{equation*}
u_{t}=u_{x x}+u\left(1-u^{2}\right) \tag{6}
\end{equation*}
$$

This equation arises in the analysis of thermal convection of a fluid heated from below after carrying out a suitable normalization [17].
This paper is organized as follows. Section 2 is devoted to the thorough investigation of the approximate symmetries and approximate solutions of the KPP equation. For this purpose, we will concentrate on the four special and significant forms of the KPP equation described above, that is, Fick's second law, Fisher's equation, Zeldovich equation, and Newell-Whitehead-Segel (NWS) equation. In Section 3, an optimal system of subalgebras is constructed and the corresponding symmetry transformations are obtained. Some concluding remarks are mentioned at the end of the paper.

## 2. Approximate Symmetries of the KPP Equation

In this section, first of all the problem of exact and approximate symmetries of Fick's second law (3) with a small parameter is investigated. Then the approximate symmetries and the exact and approximate invariant solutions corresponding to the perturbed Fisher equation, Zeldovich equation, and Newell-Whitehead-Segel (NWS) equation will be determined.
2.1. Exact Symmetries of the Perturbed Fick Second Law. The perturbed Fick second law is defined as follows:

$$
\begin{equation*}
u_{t}=\varepsilon u_{x x}, \tag{7}
\end{equation*}
$$

where $\varepsilon$ is a small parameter. Let $X$ be the infinitesimal symmetry generator corresponding to (7) which is defined as follows:

$$
\begin{equation*}
X=\xi(x, t, u) \partial_{x}+\tau(x, t, u) \partial_{t}+\varphi(x, t, u) \partial_{u} . \tag{8}
\end{equation*}
$$

Now by acting the second prolongation of the symmetry operator (8) on (7), an overdetermined system of equations for $\xi, \tau$, and $\varphi$ will be obtained. By solving this resulting determining equations, it is inferred that

$$
\begin{gather*}
\xi=\left(c_{1} t x+c_{2} x\right)-2 \varepsilon c_{4} t+c_{6} \\
\tau=c_{1} t^{2}+2 c_{2} t+c_{3}  \tag{9}\\
\varphi=\left(c_{4} x+c_{5}-\frac{c_{1} t}{2}-\frac{c_{1} x^{2}}{4 \varepsilon}\right) u+F(x, t),
\end{gather*}
$$

where $F(x, t)$ is an arbitrary function satisfying the perturbed Fick second law equation (7) and $c_{i}, i=1, \ldots, 6$ are arbitrary constants. Hence, this equation admits a six-dimensional Lie algebra with the following generators:

$$
\begin{gather*}
X_{1}=\partial_{x} \\
X_{4}=-2 \varepsilon t \partial_{x}+x u \partial_{u} \\
X_{2}=\partial_{t} \\
X_{5}=u \partial_{u}  \tag{10}\\
X_{3}=x \partial_{x}+2 t \partial_{t} \\
X_{6}=4 x t \partial_{x}+4 t^{2} \partial_{t}-\left(2 t+\frac{x^{2}}{\varepsilon}\right) u \partial_{u},
\end{gather*}
$$

plus the following infinite dimensional subalgebra which is spanned by $X_{F}=F(x, t) \partial_{u}$, where $F$ satisfies (7).
2.2. Exact Invariant Solutions. In this part, we compute some exact invariant solutions corresponding to the resulting infinitesimal generators.

Case 1. Consider the symmetry operator $X=c X_{1}+X_{2}$, where $c$ is a constant.

Now taking into account [2-4], by applying the Lie symmetry reduction technique the corresponding exact and approximate invariant solutions will be obtained as follows. The characteristic equation associated with the symmetry generator $X$ is given by $d x / c=d t / 1=d u / 0$. By solving the above equation, the following Lie invariants resulting: $x-c t=y, u=v(y)$. By substituting these invariants into (7) we obtain: $\varepsilon v^{\prime \prime}(y)+c v^{\prime}(y)=0$. Consequently, by solving the above resulting ODE, the following solution is deduced for (7): $u(x, t)=c_{1}+c_{2} \exp (-c(x-c t) / \varepsilon)$.

Case 2. For the symmetry generator $X_{3}$, the corresponding characteristic equation is $d x / x=d t / 2 t=d u / 0$. Thus, these Lie invariants are determined: $u=v(y), y=x^{2} / t$. By substituting the above invariants into (7) the following ODE is inferred: $4 \varepsilon y v^{\prime \prime}(y)+v^{\prime}(y)(2 \varepsilon+y)=0$. Hence, another solution is deduced for (7): $u=v(y)=c_{1}+c_{2} \operatorname{erf}(|x| / 2 \sqrt{\varepsilon t})$, where $c_{1}$ and $c_{2}$ are arbitrary constants and erf is the error function given by $\operatorname{erf}(x)=(2 / \sqrt{\pi}) \int_{0}^{x} e^{-t^{2}} d t$.
2.3. Perturbed Fisher's Equation. In this section, a thorough investigation of the symmetries of the perturbed Fisher equation is proposed:

$$
\begin{equation*}
u_{t}=\varepsilon u_{x x}+a u(1-u) . \tag{11}
\end{equation*}
$$

For this purpose, firstly the exact symmetries of the perturbed Fisher's equation (11) will be calculated. Then, the approximate symmetries of this equation will be analyzed.

Now by acting the second prolongation of the symmetry generator (8) on the perturbed Fisher equation and solving the resulting determining equations, it is deduced that $\xi=c_{2}$, $\tau=c_{1}$, and $\varphi=0$, where $c_{1}$ and $c_{2}$ are arbitrary constants. Hence, the following exact trivial symmetries are obtained: $X_{1}=\partial_{x}, X_{2}=\partial_{t}$. For the infinitesimal symmetry generator $X=c \partial_{x}+\partial_{t}$, the corresponding characteristic equation is given by $d x / c=d t / 1=d u / 0$.

Therefore, the Lie invariants resulting as $x-c t=y$ and $u=v(y)$. After substituting these invariants into the perturbed Fisher equation, the following reduced ordinary differential equation is obtained:

$$
\begin{equation*}
\varepsilon v^{\prime \prime}(y)+c v^{\prime}(y)+a v(y)(1-v(y))=0 \tag{12}
\end{equation*}
$$

But it is worth noting that finding an exact solution for the differential equation (12) is difficult. For the particular case $c= \pm 5 / \sqrt{6}$, Ablowitz and Zeppetella [18] used Painleve's singularity structure analysis in order to obtain the first corresponding explicit analytical solution which is given by

$$
\begin{equation*}
v(y)=u(x, t)=\left[1+\frac{\varepsilon}{\sqrt{6}} \exp \left(\sqrt{6} x-\frac{5}{6} t\right)\right]^{-2} \tag{13}
\end{equation*}
$$

2.3.1. Approximate Symmetries of the Perturbed Fisher Equation. In this section, we apply the method proposed in [7] in order to analyze the problem of approximate symmetries of Fisher's equation with an accuracy of order one. First, we expand the dependent variable in perturbation series, and then we separate terms of each order of approximation, so that a system of equations will be formed. The derived system is assumed to be coupled and its exact symmetry will be considered as the approximate symmetry of the original equation.

We expand the dependant variable up to order one as follows:

$$
\begin{equation*}
u=v+\varepsilon w, \quad 0<\varepsilon \leq 1 \tag{14}
\end{equation*}
$$

where $v$ and $w$ are smooth functions of $x$ and $t$. After substitution of (14) into the perturbed Fisher equation (11) and equating to zero the coefficients of $o\left(\varepsilon^{0}\right)$ and $o\left(\varepsilon^{1}\right)$, the following system of partial differential equations results:

$$
\begin{align*}
& O\left(\varepsilon^{0}\right): v_{t}-a v(1-v) w=0  \tag{15}\\
& O(\varepsilon): w_{t}-v_{x x}-a w(1-2 v)=0
\end{align*}
$$

Definition 1. The approximate symmetry of Fisher's equation with a small parameter is called the exact symmetry of the system of differential equations (15).

Now, consider the following symmetry transformation group acting on the PDE system (15):

$$
\begin{align*}
& \widetilde{x}=x+a \xi_{1}(t, x, v, w)+o\left(a^{2}\right) \\
& \tilde{t}=t+a \xi_{2}(t, x, v, w)+o\left(a^{2}\right) \\
& \widetilde{v}=v+a \varphi_{1}(t, x, v, w)+o\left(a^{2}\right)  \tag{16}\\
& \widetilde{w}=w+a \varphi_{2}(t, x, v, w)+o\left(a^{2}\right)
\end{align*}
$$

where $a$ is the group parameter and $\xi_{1}, \xi_{2}$ and $\varphi_{1}, \varphi_{2}$ are the infinitesimals of the transformations for the independent and dependent variables, respectively. The associated vector field is of the form

$$
\begin{align*}
X= & \xi_{1}(t, x, v, w) \partial_{t}+\xi_{2}(t, x, v, w) \partial_{x}  \tag{17}\\
& +\varphi_{1}(t, x, v, w) \partial_{v}+\varphi_{2}(t, x, v, w) \partial_{w} .
\end{align*}
$$

The invariance of the system (15) under the infinitesimal symmetry transformation group (17) leads to the following invariance condition: $p r^{(2)} X[\Delta]=0$ and $\Delta=0$. Hence, the following set of determining equations is inferred:

$$
\begin{align*}
\partial_{w} \xi_{2}=0, & a v^{2} \partial_{w} \xi_{1}+\partial_{w} \varphi_{1}-a v \partial_{w} \xi_{1}=0, \ldots, \\
& 2 \partial_{v x} \xi_{2}-\partial_{v v} \varphi_{1}=0 \tag{18}
\end{align*}
$$

By solving this system of PDEs, it is deduced to $\xi_{2}=C_{1} x+C_{3}$, $\varphi_{1}=0$, and $\xi_{1}=C_{2}, \varphi_{2}=-2 C_{1} w$, where $C_{1}, C_{2}$, and $C_{3}$ are arbitrary constants. Thus, the Lie algebra of the resulting infinitesimal symmetries of the PDE system (15) is spanned by these three vector fields:

$$
\begin{equation*}
X_{1}=\partial_{t}, \quad X_{2}=\partial_{x}, \quad X_{3}=x \partial_{x}-2 w \partial_{w} \tag{19}
\end{equation*}
$$

2.3.2. Approximate Invariant Solutions. In this section, the approximate solutions will be obtained from the approximate symmetries which resulted in the previous section.

Case $1\left(X=x \partial_{x}-2 w \partial_{w}\right)$. By applying the classical Lie symmetry group method, the corresponding characteristic equation is $d x / x=d t / 0=d v / 0=d w /(-2 w)$. So that the resulted invariants are $t=T, v=f(T)$, and $w=g(T) / x^{2}$. After substituting these invariants into the first equation of the PDE system (15), we have

$$
\begin{equation*}
f^{\prime}(T)-a f(T)(1-f(T))=0 \tag{20}
\end{equation*}
$$

Consequently, the following solution is obtained:

$$
\begin{equation*}
f(T)=v=\frac{1}{1+c_{1} e^{-a t}} \tag{21}
\end{equation*}
$$

After substituting $v$ in the second equation of the PDE system (15), this ODE results in $g^{\prime}(T)+\operatorname{ag}(T)\left[2 /\left(1+c_{1} e^{-a t}\right)-1\right]=$ 0 . Therefore, we have $g(T)=c_{2} e^{-a t} /\left(1+c_{1} e^{-a t}\right)^{2}$. Finally, taking into account (14), the following approximate solution is inferred:

$$
\begin{equation*}
u(x, t)=v+\varepsilon w=\frac{1}{1+c_{1} e^{-a t}}+\varepsilon \frac{c_{2} e^{-a t}}{x^{2}\left(1+c_{1} e^{-a t}\right)^{2}} \tag{22}
\end{equation*}
$$

where $c_{1}$ and $c_{2}$ are arbitrary constants.

Case 2. Now consider $X=X_{1}+c X_{2}$, where $c$ is an arbitrary constant. The corresponding characteristic equation is defined by $d x / c=d t / 1=d v / 0=d w / 0$. So, the associated Lie invariants are $x-c t=y, v=f(y)$, and $w=g(y)$. By substituting the resulting invariants into the first equation of the PDE system (15), the reduced equation is determined as $c f^{\prime}(y)+a f(y)(1-f(y))=0$. Therefore, we have $v(x, t)=$ $1 /\left(c_{1} e^{a(x-c t) / c}+1\right)$. Now by substituting $v(x, t)$ into the second equation of the PDE system (15), it is inferred that

$$
\begin{align*}
c g^{\prime}(y) & +\frac{c_{1} a^{2} e^{a y / c}\left(-1+c_{1} e^{a y / c}\right)}{c_{2}\left(1+c_{1} e^{a y / c}\right)^{3}}  \tag{23}\\
& +a g(y)\left(1-\frac{2}{c_{1} e^{a y / c}+1}\right)=0
\end{align*}
$$

By solving the above equation, we have

$$
\begin{align*}
g(y)= & \frac{e^{a y / c}}{\left(c_{1} e^{a y / c}+1\right)^{2}}  \tag{24}\\
& \times\left(c_{1} \frac{a^{2}}{c_{3}} y-\frac{2 a c_{1}}{c_{2}} \ln \left(c_{1} e^{a y / c}+1\right)+c_{2}\right)
\end{align*}
$$

Finally, the following approximate solution results:

$$
\begin{align*}
u(x, t)= & v+\varepsilon w \\
= & \frac{1}{c_{1} e^{a(x-c t) / c}+1} \\
& \times\left\{1+\varepsilon e^{a(x-c t) / c}\right. \\
& \left.\times\left(\frac{c_{1}}{c^{3}}(x-c t)-\frac{2 a c_{1}}{c^{2}} \ln \left(c_{1} e^{a(x-c t) / c}+1\right)+c_{2}\right)\right\} \tag{25}
\end{align*}
$$

Consequently, the approximate solutions corresponding to all the resulted operators were computed.
2.4. Perturbed Zeldovich Equation. In this section, we will investigate the exact and approximate symmetries of the Zeldovich equation with a small parameter:

$$
\begin{equation*}
u_{t}-\varepsilon u_{x x}=u^{2}(1-u) \tag{26}
\end{equation*}
$$

For this purpose, first of all we will compute the exact symmetries and then by applying the classical Lie symmetry method, the perturbed Zeldovich equation would be converted to an ODE.

By acting the symmetry operator (8) on the perturbed Zeldovich equation (26) and solving the resulted determining equations we have $\xi=c_{1}, \tau=c_{2}, \varphi=0$, where $c_{1}$ and $c_{2}$ are arbitrary constants. Hence, the corresponding infinitesimal symmetries will be spanned by these two vector fields $X_{1}=\partial_{t}$ and $X_{2}=\partial_{x}$. The characteristic equation corresponding to the symmetry operator $X=X_{1}+c X_{2}$ is given by $d x / c=d t / 1=d u / 0$. Hence, the Lie invariants are obtained as $x-c t=y$ and $u=f(y)$. After substituting these invariants into (26), the reduced equation is inferred as $\varepsilon f^{\prime \prime}(y)+c f^{\prime}(y)(1-f(y))=0$.
2.4.1. Approximate Symmetries of the Zeldovich Equation. In this section, we use the method proposed in [7] in order to obtain the approximate symmetries of (26) with the accuracy $o(\varepsilon)$. By expanding the dependent variable of this equation in perturbation series we have

$$
\begin{equation*}
u=v+\varepsilon w, \quad 0 \leq \varepsilon \leq 1 \tag{27}
\end{equation*}
$$

Then by substituting the above relation into the perturbed equation (26) and separating terms of each order of approximation, the following equations with respect to $o\left(\varepsilon^{0}\right)$ and $o\left(\varepsilon^{1}\right)$ are deduced:

$$
\begin{gather*}
O\left(\varepsilon^{0}\right): v_{t}-v^{2}(1-v)=0 \\
O\left(\varepsilon^{1}\right): w_{t}-v_{x x}-2 v w(1-v)+v^{2} w=0 \tag{28}
\end{gather*}
$$

It is worth mentioning that the resulting approximate symmetries of the differential equation (26) correspond to the exact symmetries of the PDE system (28).

Now, by acting the second prolongation of the infinitesimal symmetry operator (17) on the PDE system (28) and solving the resulted determining equations, we have $\xi_{1}=c_{2}$, $\xi_{2}=c_{1} x+c_{3}, \varphi_{1}=0$, and $\varphi_{2}=-2 c_{1} w$, where $c_{1}, c_{2}$, and $c_{3}$ are arbitrary constants. Consequently, the Lie algebra of the symmetry generators corresponding to the PDE system (28) is spanned by

$$
\begin{equation*}
X_{1}=\partial_{t}, \quad X_{2}=\partial_{x}, \quad X_{3}=x \partial_{x}-2 w \partial_{w} \tag{29}
\end{equation*}
$$

2.4.2. Approximate Invariant Solutions. Now, we obtain the approximate invariant solutions corresponding to the perturbed equation (26). For the symmetry operator $X_{3}$ the corresponding characteristic equation is given by $d x / x=$ $d t / 0=d v / 0=d w /(-2 w)$. So, the invariants results as $t=T$, $v=f(T)$, and $w=g(T) / x^{2}$. By inserting these invariants into the first equation of the PDE system (28), the reduced equation is $f^{\prime}(T)-f^{2}(T)(1-f(T))=0$. Therefore, we have $v=f(T)=1 / \mathrm{W}\left(-e^{-t-1} / c_{1}\right)$, where the function $\mathrm{W}(z)$ is defined implicitly by this equation $z=\mathrm{W}(z) e^{\mathrm{W}(z)}$. After substituting this resulting solution into the second equation of the PDE system (28), we obtain $g^{\prime}+g(T)\left(3 f^{2}(T)-2 f(T)\right)=$ 0 . The solution of the above equation is

$$
\begin{equation*}
g(T)=\frac{c_{2} \exp \left(-2 \mathrm{~W}\left(-e^{-t-1} / c_{1}\right)\right) \mathrm{W}\left(-e^{-t-1} / c_{1}\right)}{\mathrm{W}\left(-e^{-t-1} / c_{1}\right)+1} \tag{30}
\end{equation*}
$$

Finally, the following approximate invariant solution for the equation (26) is deduced:

$$
\begin{align*}
u(x, t)= & f(T) \\
& +\varepsilon \frac{c_{2} \exp \left(-2 \mathrm{~W}\left(-e^{-t-1} / c_{1}\right)\right) \mathrm{W}\left(-e^{-t-1} / c_{1}\right)}{x^{2}\left(\mathrm{~W}\left(-e^{-t-1} / c_{1}\right)+1\right)} \tag{31}
\end{align*}
$$

2.5. Perturbed NSW Equation. Similar to the previous sections, we will analyze the symmetries of the perturbed NSW equation:

$$
\begin{equation*}
u_{t}-\varepsilon u_{x x}=u\left(1-u^{2}\right) \tag{32}
\end{equation*}
$$

Table 1: The commutator table of the approximate symmetries of the KPP equation.

| $\left[X_{i}, X_{j}\right]$ | $X_{1}$ | $X_{2}$ | $X_{3}$ |
| :--- | :---: | :---: | :---: |
| $X_{1}$ | 0 | 0 | 0 |
| $X_{2}$ | 0 | 0 | $X_{2}$ |
| $X_{3}$ | 0 | $-X_{2}$ | 0 |

By applying the same calculations on this equation, the approximate symmetries are resulted as $X_{1}=\partial_{t}, X_{2}=\partial_{x}$, and $X_{3}=x \partial_{x}-2 w \partial_{w}$. The Lie invariants corresponding to the symmetry operator $X_{3}$ are as $t=T, v=f(T)$, and $w=$ $g(T) / x^{2}$. Consequently, the following approximate invariant solution is deduced:

$$
\begin{equation*}
u(x, t)=\frac{ \pm 1}{\sqrt{1+c_{1} e^{-2 t}}}+\varepsilon \frac{c_{2} e^{-2 t}}{\left(1+c_{1} e^{-2 t}\right)^{3 / 2}} \tag{33}
\end{equation*}
$$

## 3. Optimal System of the KPP Equation

In this section, an optimal system of subalgebras corresponding to the resulting approximate symmetries of the KPP equation is constructed. As it was shown in the previous sections, the Lie algebra of the approximate symmetries corresponding to Fisher's equation, Zeldovich equation, and Newell-Whitehead-Segel (NSW) equation is three-dimensional and spanned by the following generators:

$$
\begin{equation*}
X_{1}=\partial_{t}, \quad X_{2}=\partial_{x}, \quad X_{3}=x \partial_{x}-2 w \partial_{w} \tag{34}
\end{equation*}
$$

The commutation relations corresponding to these vector fields are given in Table 1.

It is worth noting that each $s$-parameter subgroup corresponds to one of the group invariant solutions. Since any linear combination of the infinitesimal generators is also an infinitesimal generator, there are always infinitely many distinct symmetry subgroups for a differential equation. But it is not practical to find the list of all group invariant solutions of a system. Consequently, we need an effective and systematic means of classifying these solutions, leading to an "optimal system" of group invariant solutions from which every other such solutions can results. Let $G$ be a Lie group and let $\mathbf{g}$ denote its Lie algebra. An optimal system of $s$-parameter subgroups is indeed a list of conjugacy inequivalent $s$-parameter subgroups with the property that any other subgroup is conjugate to precisely one subgroup in the list. Similarly, a list of $s$-parameter subalgebras forms an optimal system if every s-parameter subalgebra of $\mathbf{g}$ is equivalent to a unique member of the list under some element of the adjoint representation: $\widetilde{h}=\operatorname{Ad}_{g}(h)$, with $g \in G$.

According to the proposition (3.7) of [3], the problem of finding an optimal system of subgroups is equivalent to that of obtaining an optimal system of subalgebras. For one-dimensional subalgebras, this classification problem is essentially the same as the problem of classifying the orbits of the adjoint representation. Since each onedimensional subalgebra is determined by a nonzero vector in $\mathbf{g}$, this problem is attacked by the naive approach of taking

Table 2: Adjoint representation of the approximate symmetries of the KPP equation.

| Ad | $X_{1}$ | $X_{2}$ | $X_{3}$ |
| :---: | :---: | :---: | :---: |
| $X_{1}$ | $X_{1}$ | $X_{2}$ | $X_{3}$ |
| $X_{2}$ | $X_{1}$ | $X_{2}$ | $X_{3}-\varepsilon X_{2}$ |
| $X_{3}$ | $X_{1}$ | $e^{\varepsilon} X_{2}$ | $X_{3}$ |

a general element $X$ in $\mathbf{g}$ and subjecting it to various adjoint transformations so as to simplify it as much as possible. Thus we will deal with the construction of an optimal system of subalgebras of $\mathbf{g}$. The adjoint action is given by the Lie series: $\operatorname{Ad}\left(\exp \left(\varepsilon X_{i}, X_{j}\right)=X_{j}-\varepsilon\left[X_{i}, X_{j}\right]+\varepsilon^{2} / 2\right)\left[X_{i},\left[X_{i}, X_{j}\right]\right]-\cdots$, where $\left[X_{i}, X_{j}\right]$ denotes the Lie bracket, $\varepsilon$ is a parameter, and $i, j=1,2,3$ [3].

The adjoint representation Ad corresponding to the resulted approximate symmetries is presented in Table 2 with the $(i, j)$ th entry indicating $\operatorname{Ad}\left(\exp \left(\varepsilon x_{i}\right) x_{j}\right)$.

Therefore, we can state the following theorem.
Theorem 2. An optimal system of one-dimensional subalgebras corresponding to the Lie algebra of approximate symmetries of the KPP equation is generated by (i) $X_{1}$, (ii) $\alpha X_{1}+X_{2}$, and (iii) $\beta X_{1}+X_{3}$, where $\alpha, \beta \in \mathbf{R}$ are arbitrary constants.

Proof. Let $F_{i}^{s}: \mathbf{g} \rightarrow \mathbf{g}$ be a linear map defined by $X \rightarrow$ $\operatorname{Ad}\left(\exp \left(s_{i} X_{i}\right) X\right)$ for $i=1, \ldots, 3$. The matrices $M_{i}^{s}$ of $F_{i}^{s}$ with respect to the basis $\left\{X_{1}, X_{2}, X_{3}\right\}$ are given by

$$
\begin{align*}
& M_{1}^{s}=\left(\begin{array}{lll}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right), \\
& M_{2}^{s}=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & -s_{1} & 1
\end{array}\right),  \tag{35}\\
& M_{3}^{s}=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & e^{s_{2}} & 0 \\
0 & 0 & 1
\end{array}\right) .
\end{align*}
$$

Let $X=\sum_{i=1}^{3} a_{i} X_{i}$ then $F_{3}^{s} \circ F_{2}^{s} \circ F_{1}^{s}: X \mapsto a_{1} X_{1}+a_{2} e^{s_{2}} X_{2}+$ $\left(a_{3}-s_{1} a_{2}\right) X_{3}$. In the following, by alternative action of these matrices on a vector field $X$, the coefficients $a_{i}$ of $X$ will be simplified.

If $a_{2} \neq 0$, then we can make the coefficients of $X_{3}$ vanish by $F_{1}^{s}$ by setting $s_{1}=a_{3} / a_{2}$. Scaling $X$ if necessary, we can assume that $a_{2}=1$. So, $X$ is reduced to the case (ii). If $a_{2}=0$ and $a_{3} \neq 0$, by scaling we insert $a_{3}=1$. So $X$ is reduced to the case (iii). Finally, if $a_{2}=a_{3}=0$, then $X$ is reduced to the case (i). There are not any more possible cases for investigating and the proof is complete.

In order to obtain the group transformations which are generated by the resulting infinitesimal symmetry generators (34), we need to solve the following system of first-order
ordinary differential equations $\left(x_{1}=x, x_{2}=t, u_{1}=v, u_{2}=\right.$ $w)$ :

$$
\begin{gather*}
\frac{d \tilde{x}_{j}(s)}{d s}=\xi_{i}^{j}(\widetilde{x}(s), \widetilde{t}(s), \widetilde{v}(s), \widetilde{w}(s)), \\
\tilde{x}_{j}(0)=x_{j}, \quad i=1,2,3, \\
\frac{d \widetilde{u}_{j}(s)}{d s}=\varphi_{i}^{j}(\widetilde{x}(s), \tilde{t}(s), \widetilde{v}(s), \widetilde{w}(s)),  \tag{36}\\
\tilde{u}_{j}(0)=u_{j}, \quad j=1,2 .
\end{gather*}
$$

Hence, by exponentiating the resulting infinitesimal approximate symmetries of the KPP equation, the one-parameter groups $G_{i}(s)$ generated by $X_{i}$ for $i=1,2,3$ are determined as follows:

$$
\begin{align*}
& G_{1}:(t, x, v, w) \longmapsto(t+s, x, v, w), \\
& G_{2}:(t, x, v, w) \longmapsto(t, x+s, v, w),  \tag{37}\\
& G_{3}:(t, x, v, w) \longmapsto\left(t, e^{s} x, v, e^{-2 s} w\right) .
\end{align*}
$$

Consequently, we can state the following theorem.
Theorem 3. If $u=f(t, x)+\varepsilon g(t, x)$ is a solution of the KPP equation, so are the following functions:

$$
\begin{align*}
& G_{1}(s) \cdot u(t, x)=f(t-s, x)+\varepsilon g(t-s, x), \\
& G_{2}(s) \cdot u(t, x)=f(t, x-s)+\varepsilon g(t, x-s),  \tag{38}\\
& G_{3}(s) \cdot u(t, x)=f\left(t, e^{-s} x\right)+\varepsilon e^{-2 s} g\left(t, e^{-s} x\right) .
\end{align*}
$$

## 4. Conclusion

The investigation of the exact solutions of nonlinear PDEs plays an essential role in the analysis of nonlinear phenomena. Lie symmetry method greatly simplifies many nonlinear problems. Exact solutions are nevertheless hard to investigate in general. Furthermore, many PDEs in application depend on a small parameter; hence it is of great significance and interest to obtain approximate solutions. Perturbation analysis method was thus developed and it has a significant role in nonlinear science, particularly in obtaining approximate analytical solutions for perturbed PDEs. This procedure is mainly based on the expansion of the dependent variables asymptotically in terms of a small parameter. The combination of Lie group theory and perturbation theory yields two distinct approximate symmetry methods. The first method due to Baikov et al. generalizes symmetry group generators to perturbation forms $[5,6]$. The second method proposed by Fushchich and Shtelen [7] is based on the perturbation of dependent variables in perturbation series and the approximate symmetry of the original equation is decomposed into an exact symmetry of the system resulting from the perturbation. Taking into account the comparison in [11] the second method is superior to the first one.

As it is well known, the solutions of nonlinear reactiondiffusion equations represent a wide class of behaviors,
including the formation of wave-like phenomena and traveling waves as well as other self-organized patterns. In this paper we have comprehensively analyzed the approximate symmetries of a significant class of nonlinear reactiondiffusion equations called Kolmogorov-Petrovsky-Piskounov (KPP) equation. For this purpose, we have concentrated on four particular and important forms of this equation including Fick's second law, Fisher's equation, Zeldovich equation, and Newell-Whitehead-Segel (NWS) equation. It is worth mentioning that in order to calculate the approximate symmetries corresponding to these equations, we have applied the second approximate symmetry method which was proposed by Fushchich and Shtelen. Meanwhile, we have constructed an optimal system of subalgebras. Also, we have obtained the symmetry transformations and some invariant solutions corresponding to the resulted symmetries.

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## Research Article

# Symmetry Analysis and Exact Solutions to the Space-Dependent Coefficient PDEs in Finance 

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#### Abstract

The variable-coefficients partial differential equations (vc-PDEs) in finance are investigated by Lie symmetry analysis and the generalized power series method. All of the geometric vector fields of the equations are obtained; the symmetry reductions and exact solutions to the equations are presented, including the exponentiated solutions and the similarity solutions. Furthermore, the exact analytic solutions are provided by the transformation technique and generalized power series method, which has shown that the combination of Lie symmetry analysis and the generalized power series method is a feasible approach to dealing with exact solutions to the variable-coefficients PDEs.


## 1. Introduction

Gazizov and Ibragimov [1] studied the Black-Scholes equation of option pricing by Lie equivalence transformations. By the optimal system method, some invariant solutions to heat and Black-Scholes equations are obtained [2]. In [35], the fundamental solutions to the bond pricing equations are considered by Lie symmetry analysis and the integral transform method. In [6], the invariance properties of the bond pricing equation are studied by the group classification method. In [7], the finite element method was adopted to solve the bond pricing type of PDE system, and the numerical implementation was provided, such as system that models the TF convertible bonds with credit risk in bond pricing theory. However, the similarity reductions and exact solutions to such variable-coefficient equations are not considered generally in the aforementioned papers. Recently, we studied some nonlinear PDEs by Lie symmetry analysis and the dynamical system method [8-13]; for example, in [8], we considered Lie group classifications and exact solutions to the space-dependent coefficients hanging chain equation and the simplified bond pricing equation. In [9], we investigated the integrable condition and exact solutions to the timedependent coefficient Gardner equations by the Painlevé test and Lie group analysis method. In [10-13], we developed
the generalized power series method for dealing with exact solutions to some nonlinear PDEs based on the symmetry analysis method.

It is known that the Lie symmetry analysis is a systematic and powerful method for dealing with symmetries and exact solutions to partial differential equations (see, e.g., [1-6, 818] and the references therein). Furthermore, we find that the combination of Lie symmetry analysis and the power series method is a feasible approach to investigating exact solutions to nonlinear PDEs [8-13]. On the other hand, under the perspective of mathematical physics and Lie symmetry analysis, the space-time dependent coefficients system differs greatly from its time-dependent counterpart, and it is more complicated than the latter. However, most of the studies are related to the time-dependent coefficient systems. Moreover, the determination of exact solutions to the variablecoefficients PDEs is a complicated problem that challenges researchers greatly. In the present paper, we consider the symmetry reductions and exact solutions to the general space-dependent coefficients PDEs in finance as follows:

$$
\begin{equation*}
u_{t}+\alpha x^{2} u_{x x}+\beta x u_{x}+\gamma x^{v} u=0, \quad x>0 \tag{1}
\end{equation*}
$$

where $u=u(x, t)$ denotes the unknown function of the space variable $x$ and time $t$ and the parameters $\alpha, \beta, \gamma, \nu \in \mathbb{R}$ are arbitrary constants, $\nu \geq 0$ and $\alpha \neq 0$.

We first note that (1) is the general form of the bond pricing types of equations [1-7]. In particular, if $v=0$, then this equation becomes the following Black-Scholes equation of option pricing:

$$
\begin{equation*}
u_{t}+\alpha x^{2} u_{x x}+\beta x u_{x}+\gamma u=0, \quad x>0 \tag{2}
\end{equation*}
$$

If $v=1$, then (1) is the general bond pricing equation given by

$$
\begin{equation*}
u_{t}+\alpha x^{2} u_{x x}+\beta x u_{x}+\gamma x u=0, \quad x>0 \tag{3}
\end{equation*}
$$

Such equations are called bond pricing types of equations, which are of great importance in financial mathematics and bond pricing theory [3-7]. For dealing with exact solutions to the variable-coefficients PDEs, we will introduce the generalized power series method [10-13] in the present paper. By a generalized power series solution, we mean a generalized power series is of the form

$$
\begin{equation*}
f(\xi)=A(\xi)+\sum_{n=0}^{\infty} c_{n} \xi^{n} \tag{4}
\end{equation*}
$$

which is a solution to a system with respect to the variable $\xi$, where $c_{n}(n=0,1,2, \ldots)$ are constant coefficients to be determined and $A(\xi)$ is the undetermined function with respect to the variable $\xi$. In particular, if $A(\xi) \equiv 0$, then (4) is the regular power series solution. So, the generalized power series solution is the generalization of the regular power series solution and it naturally includes the latter as its special case. If we obtained a generalized power series solution (4) to a system and the convergence of this power series is shown, then the exact generalized power series solution is obtained. This solution sometimes is called the exact analytic solution [10-13, 19].

The main purpose of this paper is to develop the combination of Lie symmetry analysis and the generalized power series method for dealing with symmetries and exact solutions to the variable-coefficients PDEs in finance. The remainder of this paper is organized as follows. In Section 2, we perform Lie symmetry analysis on the bond pricing types of (2) and (3) and give all of the geometric vector fields of the equations in terms of the arbitrary parameters. In Section 3, we consider the symmetry reductions of the equations and provide the exponentiated solutions and the similarity solutions to the equations. In Section 4, we investigate the exact analytic solutions to the variable-coefficient equations by the generalized power series method. In Section 5, we deal with the vector fields and exact solutions to the bond pricing type of (1) for the general case $\nu \neq 0,1$. Finally, the conclusions and some remarks are given in Section 6.

## 2. Lie Symmetry Analysis for (2) and (3)

In this section, we will present a complete list of all possible Lie symmetry algebras for the bond pricing types of equations of the forms (2) and (3).

Recall that the geometric vector fields of such equations are as follows:

$$
\begin{equation*}
V=\xi(x, t, u) \partial_{x}+\tau(x, t, u) \partial_{t}+\phi(x, t, u) \partial_{u} \tag{5}
\end{equation*}
$$

where $\xi(x, t, u), \tau(x, t, u)$, and $\phi(x, t, u)$ are coefficient functions of the vector field to be determined. The symmetry groups of (2) and (3) will be generated by the vector field of the form (5), respectively. Applying the second prolongation $\mathrm{pr}^{(2)} V$ of $V$ to (2) and (3), we find that the coefficient functions $\xi, \tau$, and $\phi$ must satisfy the following Lie symmetry condition:

$$
\begin{equation*}
\left.\operatorname{pr}^{(2)} V(\Delta)\right|_{\Delta=0}=0 \tag{6}
\end{equation*}
$$

where $\Delta=u_{t}+\alpha x^{2} u_{x x}+\beta x u_{x}+\gamma u$ for (2) and $\Delta=$ $u_{t}+\alpha x^{2} u_{x x}+\beta x u_{x}+\gamma x u$ for (3), respectively. Then, the Lie symmetry group calculation method leads to the following conditions on the coefficient functions $\xi$, $\tau$, and $\phi$ :

$$
\begin{align*}
& \xi=\frac{1}{2} x \tau_{t} \log x+x \rho, \quad \phi=r(x, t) u+s(x, t) \\
& r=\frac{1}{8 \alpha} \tau_{t t} \log ^{2} x+\frac{\alpha-\beta}{4 \alpha} \tau_{t} \log x+\frac{1}{2 \alpha} \rho_{t} \log x+\sigma \tag{7}
\end{align*}
$$

for some functions $\tau, \rho$, and $\sigma$. Now the functions $\tau, \rho$, and $\sigma$ depend only on $t$. Moreover, for (2), we have

$$
\begin{align*}
& \frac{1}{8 \alpha} \tau_{t t t} \log ^{2} x+\frac{1}{2 \alpha} \rho_{t t} \log x+\frac{1}{4} \tau_{t t} \\
& -\frac{(\alpha-\beta)^{2}-4 \alpha \gamma}{4 \alpha} \tau_{t}-\frac{\alpha-\beta}{2 \alpha} \rho_{t}+\sigma_{t}=0 \tag{8}
\end{align*}
$$

for (3), we have

$$
\begin{align*}
& \frac{1}{8 \alpha} \tau_{t t t} \log ^{2} x+\frac{1}{2} \gamma \tau_{t} x \log x \\
& \quad+\frac{1}{2 \alpha} \rho_{t t} \log x+\gamma\left(\rho+\tau_{t}\right) x+\frac{1}{4} \tau_{t t}  \tag{9}\\
& \quad-\frac{(\alpha-\beta)^{2}}{4 \alpha} \tau_{t}-\frac{\alpha-\beta}{2 \alpha} \rho_{t}+\sigma_{t}=0
\end{align*}
$$

These equations fix the functions $\xi, \tau, \rho, \sigma$, and $\phi$. Solving the equations, we obtain the vector field of (2) as follows:

$$
\begin{gather*}
V_{1}=\partial_{t}, \quad V_{2}=x \partial_{x}, \quad V_{3}=u \partial_{u} \\
V_{4}=2 \alpha x t \partial_{x}+[\log x+(\alpha-\beta) t] u \partial_{u} \\
V_{5}=2 \alpha x t(\log x) \partial_{x}+4 \alpha t \partial_{t} \\
+\left[(\alpha-\beta) \log x+\left((\alpha-\beta)^{2}-4 \alpha \gamma\right) t\right] u \partial_{u} \\
V_{6}=4 \alpha x t(\log x) \partial_{x}+4 \alpha t^{2} \partial_{t}  \tag{10}\\
+\left[2(\alpha-\beta) t \log x+\log ^{2} x-2 \alpha t\right. \\
\left.+\left((\alpha-\beta)^{2}-4 \alpha \gamma\right) t^{2}\right] u \partial_{u} \\
V_{s}=s \partial_{u}
\end{gather*}
$$

where the parameters $\alpha \neq 0, \beta, \gamma$ are arbitrary constants and the function $s=s(x, t)$ satisfies (2).

For (3), we have the vector field as follows:

$$
\begin{equation*}
V_{1}=\partial_{t}, \quad V_{2}=u \partial_{u}, \quad V_{s}=s \partial_{u}, \tag{11}
\end{equation*}
$$

where the function $s=s(x, t)$ satisfies (3).
Clearly, for (2), a basis of the Lie algebra is $\left\{V_{1}, \ldots, V_{6}, V_{s}\right\}$. For (3), a basis for the Lie algebra is $\left\{V_{1}, V_{2}, V_{s}\right\}$. Thus, the new symmetries cannot be derived from the Lie brackets for the two equations.

Moreover, we can obtain the one-parameter groups generated by $V_{i}$, respectively. In fact, for (2), the one-parameter groups $G_{i}$ generated by $V_{i}(i=1, \ldots, 6, s)$ are given in the following:

$$
\begin{gather*}
G_{1}:(x, t, u) \longrightarrow(x, t+\epsilon, u), \\
G_{2}:(x, t, u) \longrightarrow\left(e^{\epsilon} x, t, u\right), \\
G_{3}:(x, t, u) \longrightarrow\left(x, t, e^{\epsilon} u\right), \\
G_{4}:(x, t, u) \longrightarrow\left(x e^{2 \alpha \epsilon t}, t, x^{\epsilon} u \exp \left[(\alpha-\beta) \epsilon t+\alpha \epsilon^{2} t\right]\right), \\
G_{5}:(x, t, u) \longrightarrow\left(x^{\delta}, \delta^{2} t, u \exp \left[\frac{\alpha-\beta}{2 \alpha}(\delta-1) \log x\right.\right. \\
+\frac{(\alpha-\beta)^{2}-4 \alpha \gamma}{4 \alpha} \\
G_{6}:(x, t, u) \\
\left.\left.\times\left(\delta^{2}-1\right) t\right]\right), \\
x^{1 /(1-4 \alpha \epsilon t)}, \frac{t}{1-4 \alpha \epsilon t}, \\
\\
\begin{array}{l}
u \sqrt{1-4 \alpha \epsilon t} \exp \left\{\left[\frac{\alpha-\beta}{2 \alpha} \log ^{x}\right.\right.
\end{array} \\
\quad+\frac{1}{4 \alpha t} \log ^{2} x \\
G_{s}:(x, t, u) \longrightarrow(x, t, u+\epsilon s), \\
\left.+\frac{(\alpha-\beta)^{2}-4 \alpha \gamma}{4 \alpha} t\right] \tag{12}
\end{gather*}
$$

where $\delta=e^{2 \alpha \epsilon}, \epsilon \ll 1$, and the function $s=s(x, t)$ is an arbitrary solution to (2). For (3), the one-parameter groups are $G_{i}(i=1,3, s)$ as above, while $s=s(x, t)$ is an arbitrary solution to (3).

From the above, we observe that $G_{1}$ is a time translation and $G_{2}$ and $G_{3}$ are trivial scaling transformations, while $G_{i}$ ( $i=4,5,6$ ) are nontrivial local groups of transformations. Their appearances are far from obvious from basic physical
principles, but they are important for us to investigate the exact solutions to PDEs (see, e.g., $[3-5,10]$ ).

## 3. Symmetry Reductions and Exact Solutions to the Bond Pricing Types of Equations

In the preceding section, we obtained the symmetries and symmetry groups of (2) and (3). Now, we deal with the symmetry reductions and exact solutions to the equations.
3.1. The Exponentiated Solutions. Since each $G_{i}(i=$ $1, \ldots, 6, s)$ is a symmetry group, it implies that if $u=f(x, t)$ is a solution to (2), then $u^{(i)}(i=1, \ldots, 6, s)$ are all solutions to the following equation as well:

$$
\begin{gather*}
u^{(1)}=f(x, t-\epsilon),  \tag{13a}\\
u^{(2)}=f\left(e^{-\epsilon} x, t\right),  \tag{13b}\\
u^{(3)}=e^{\epsilon} f(x, t),  \tag{13c}\\
u^{(5)}=x^{\epsilon} \exp \left[(\alpha-\beta) \epsilon t-\alpha \epsilon^{2} t\right] f\left(e^{-2 \alpha \epsilon t} x, t\right),  \tag{13d}\\
\left.+\frac{(\alpha-\beta)^{2}-4 \alpha \gamma}{4 \alpha}\left(1-\delta^{-2}\right) t\right] \\
\times f\left(x^{1 / \delta}, \delta^{-2} t\right),  \tag{13e}\\
u^{(6)}=\frac{\alpha}{\sqrt{1+4 \alpha \epsilon t}} \exp \left\{\left[\frac{\alpha-\beta}{2 \alpha} \log x+\frac{1}{4 \alpha t} \log ^{2} x\right.\right. \\
\times f\left(x^{1 /(1+4 \alpha \epsilon t)}, \frac{t}{1+4 \alpha \epsilon t}\right),
\end{gather*}
$$

$$
\begin{equation*}
u^{(s)}=f(x, t)+\epsilon s, \tag{13~g}
\end{equation*}
$$

where $\delta=e^{2 \alpha \epsilon}, \epsilon$ is an arbitrary real number, and the function $s=s(x, t)$ satisfies (2).

For (3), the exponentiated solutions are $u^{(i)}(i=1,3, s)$ as above while $s=s(x, t)$ satisfies (3).

Such exponentiated solutions are one of group-invariant types of solutions to the PDEs, which are generated from the one-parameter groups and are of importance for studying the exact solutions and investigating the properties of solutions (see Remark 2).

Next, we investigate the symmetry reductions and exact explicit solutions to the two bond pricing equations. Firstly, we consider (2).
3.2. Similarity Solution for $V_{1}$. For the generator $V_{1}$, we have the following reduced ordinary differential equation (ODE):

$$
\begin{equation*}
\alpha \xi^{2} f^{\prime \prime}+\beta \xi f^{\prime}+\gamma f=0 \tag{14}
\end{equation*}
$$

where $f^{\prime}=d f / d \xi$. This is an Euler equation; the corresponding characteristic equation is $\alpha K^{2}-(\alpha-\beta) K+\gamma=0$. Solving this equation, we have $K=((\alpha-\beta) \pm \sqrt{\Delta}) / 2 \alpha$, where $\Delta=(\alpha-\beta)^{2}-4 \alpha \gamma$.

When $\Delta>0$, (14) has the general solution $f=c_{1} \xi^{K_{1}}+$ $c_{2} \xi^{K_{2}}$. Thus, we obtain the exact solution to (2) as follows:

$$
\begin{equation*}
u(x, t)=c_{1} x^{K_{1}}+c_{2} x^{K_{2}} \tag{15}
\end{equation*}
$$

where $c_{1}$ and $c_{2}$ are arbitrary constants and $K_{1,2}=((\alpha-$ $\beta) \pm \sqrt{\Delta}) / 2 \alpha$ are two real roots to the characteristic equation, respectively.

When $\Delta=0$, (14) has the general solution $f=\xi^{K}\left(c_{1}+\right.$ $\left.c_{2} \log \xi\right)$. Thus, we obtain the exact solution to (2) as follows:

$$
\begin{equation*}
u(x, t)=x^{K}\left(c_{1}+c_{2} \log x\right) \tag{16}
\end{equation*}
$$

where $c_{1}$ and $c_{2}$ are arbitrary constants, $K=(\alpha-\beta) / 2 \alpha$ are the real root to the characteristic equation.

When $\Delta<0$, (14) has the general solution $f=$ $\xi^{K}\left(c_{1} \cos (\sqrt{-\Delta} / 2 \alpha) \log \xi+c_{2} \sin (\sqrt{-\Delta} / 2 \alpha) \log \xi\right)$. Thus, we obtain the exact solution to (2) as follows:

$$
\begin{equation*}
u(x, t)=x^{K}\left(c_{1} \cos \frac{\sqrt{-\Delta}}{2 \alpha} \log x+c_{2} \sin \frac{\sqrt{-\Delta}}{2 \alpha} \log x\right) \tag{17}
\end{equation*}
$$

where $c_{1}$ and $c_{2}$ are arbitrary constants, $K=(\alpha-\beta) / 2 \alpha$.
3.3. Similarity Solution for $V_{2}$. For the generator $V_{2}$, we have the following reduced ODE:

$$
\begin{equation*}
f^{\prime}+\gamma f=0 \tag{18}
\end{equation*}
$$

where $f^{\prime}=d f / d \xi$. Solving this equation, we have $f=c e^{-\gamma \xi}$. Thus, we obtain the exact solution to (2) as follows:

$$
\begin{equation*}
u(x, t)=c e^{-\gamma t} \tag{19}
\end{equation*}
$$

where $c$ is an arbitrary constant.
3.4. Similarity Solution for $V_{4}$. For the generator $V_{4}$, we have the following similarity transformation:

$$
\begin{equation*}
\xi=t, \quad \omega=\log u-\frac{1}{4 \alpha t}(\log x+a t)^{2} \tag{20}
\end{equation*}
$$

and the similarity solution is $\omega=f(\xi)$; that is,

$$
\begin{equation*}
u=\exp \left[f(t)+\frac{1}{4 \alpha t}(\log x+a t)^{2}\right] \tag{21}
\end{equation*}
$$

Substituting (21) into (2), we reduce the bond pricing equation to the following ODE:

$$
\begin{equation*}
2 \xi f^{\prime}+2 \gamma \xi+1=0 \tag{22}
\end{equation*}
$$

where $f^{\prime}=d f / d \xi$. It implies that if $\omega=f(\xi)$ is a solution to (22), then (21) is a solution to (2). Solving (22), we get $f(\xi)=$ $-(1 / 2) \log \xi-\gamma \xi+c_{1}$. Thus, we obtain the solution to (2) as follows:

$$
\begin{equation*}
u(x, t)=c \exp \left[\frac{1}{4 \alpha t}(\log x+a t)^{2}-\frac{1}{2} \log t-\gamma t\right] \tag{23}
\end{equation*}
$$

where $c$ is an arbitrary constant.
3.5. Similarity Solution for $V_{5}$. For the generator $V_{5}$, we have the following similarity transformation:

$$
\begin{gather*}
\xi=t^{-1 / 2} \log x \\
\omega=\log u-\frac{\alpha-\beta}{2 \alpha} \log x-\frac{(\alpha-\beta)^{2}-4 \alpha \gamma}{4 \alpha} t \tag{24}
\end{gather*}
$$

and the similarity solution is $\omega=f(\xi)$; that is,

$$
\begin{equation*}
u=\exp \left[f\left(t^{-1 / 2} \log x\right)+\frac{\alpha-\beta}{2 \alpha} \log x+\frac{(\alpha-\beta)^{2}-4 \alpha \gamma}{4 \alpha} t\right] \tag{25}
\end{equation*}
$$

Substituting (25) into (2), we reduce the bond pricing equation to the following ODE:

$$
\begin{equation*}
\alpha f^{\prime \prime}+\alpha f^{\prime 2}-\frac{1}{2} \xi f^{\prime}=0 \tag{26}
\end{equation*}
$$

where $f^{\prime}=d f / d \xi$.
Letting $f^{\prime}=y$, we get the Bernoulli equation

$$
\begin{equation*}
\frac{d y}{d \xi}=\frac{1}{2 \alpha} \xi y-\alpha y^{2} \tag{27}
\end{equation*}
$$

Clearly, $y=0$; that is, $f=c$ is a solution to (26). Thus, we get a solution to (2) as follows:

$$
\begin{equation*}
u(x, t)=\exp \left[\frac{\alpha-\beta}{2 \alpha} \log x+\frac{(\alpha-\beta)^{2}-4 \alpha \gamma}{4 \alpha} t+c\right] \tag{28}
\end{equation*}
$$

for an arbitrary constant number $c$.
When $y \neq 0$, solving the Bernoulli equation, we get $y=$ $e^{(1 / 4 \alpha) \xi^{2}} /\left(\int e^{(1 / 4 \alpha) \xi^{2}} d \xi+c_{1}\right)$. Thus, we obtain the solution to (26) as follows:

$$
\begin{equation*}
f(\xi)=\int \frac{e^{(1 / 4 \alpha) \xi^{2}}}{\int e^{(1 / 4 \alpha) \xi^{2}} d \xi+c_{1}} d \xi+c_{2} \tag{29}
\end{equation*}
$$

where $c_{1}$ and $c_{2}$ are constants of integration. Substituting (29) into (25), we obtain the exact solution to (2) immediately.
3.6. Similarity Solution for $V_{6}$. For the generator $V_{6}$, we have the following similarity transformation:

$$
\begin{align*}
\xi= & t^{-1} \log x \\
\omega= & \log u+\frac{1}{2} \log t-\frac{\alpha-\beta}{2 \alpha} \log x  \tag{30}\\
& \quad-\frac{(\alpha-\beta)^{2}-4 \alpha \gamma}{4 \alpha} t-\frac{1}{4 \alpha} t^{-1} \log ^{2} x
\end{align*}
$$

and the similarity solution is $\omega=f(\xi)$; that is,

$$
\begin{align*}
u=\exp & {\left[f\left(t^{-1} \log x\right)-\frac{1}{2} \log t+\frac{\alpha-\beta}{2 \alpha} \log x\right.}  \tag{31}\\
& \left.+\frac{(\alpha-\beta)^{2}-4 \alpha \gamma}{4 \alpha} t+\frac{1}{4 \alpha} t^{-1} \log ^{2} x\right]
\end{align*}
$$

Substituting (31) into (2), we reduce the bond pricing equation to the following ODE:

$$
\begin{equation*}
f^{\prime \prime}+f^{\prime 2}=0 \tag{32}
\end{equation*}
$$

where $f^{\prime}=d f / d \xi$.
Solving (32), we get $f(\xi)=\log \left|\xi+c_{1}\right|+c_{3}$. Thus, we obtain the solution to (2) as follows:

$$
\begin{align*}
u(x, t)= & c_{2}\left(\frac{1}{t} \log x+c_{1}\right) \\
& \times \exp \left[\frac{\alpha-\beta}{2 \alpha} \log x+\frac{(\alpha-\beta)^{2}-4 \alpha \gamma}{4 \alpha} t\right.  \tag{33}\\
& \left.\quad+\frac{1}{4 \alpha} t^{-1} \log ^{2} x-\frac{1}{2} \log t\right]
\end{align*}
$$

where $c_{1}, c_{2}$ are arbitrary constants.
3.7. Similarity Solution for $V_{1}+v V_{2}$. For the linear combination $V=V_{1}+v V_{2}(v \neq 0$ is an arbitrary constant $)$, we have the following similarity transformation:

$$
\begin{equation*}
\xi=\log x-v t, \quad \omega=u, \tag{34}
\end{equation*}
$$

and the similarity solution is $\omega=f(\xi)$; that is,

$$
\begin{equation*}
u=f(\log x-v t) \tag{35}
\end{equation*}
$$

Substituting (35) into (2), we reduce the bond pricing equation to the following ODE:

$$
\begin{equation*}
\alpha f^{\prime \prime}-(v+\alpha-\beta) f^{\prime}+\gamma f=0 \tag{36}
\end{equation*}
$$

where $f^{\prime}=d f / d \xi$.
This is a second-order linear ODE; the corresponding characteristic equation is $\alpha \lambda^{2}-(v+\alpha-\beta) \lambda+\gamma=0$. Solving the algebraic equation, we have $\lambda_{1}=(v+\alpha-\beta+\sqrt{\Delta}) / 2 \alpha$, $\lambda_{2}=(v+\alpha-\beta-\sqrt{\Delta}) / 2 \alpha$, where $\Delta=(v+\alpha-\beta)^{2}-4 \alpha \gamma$.

When $\Delta>0$, (36) has the solution $f(\xi)=c_{1} e^{\lambda_{1} \xi}+c_{2} e^{\lambda_{2} \xi}$. Thus, we obtain the solution to (2) as follows:

$$
\begin{equation*}
u(x, t)=c_{1} x^{\lambda_{1}} e^{-\lambda_{1} v t}+c_{2} x^{\lambda_{2}} e^{-\lambda_{2} v t} \tag{37}
\end{equation*}
$$

where $c_{1}, c_{2}$ are arbitrary constants.

When $\Delta=0$, (36) has the solution $f(\xi)=\left(c_{1}+c_{2} \xi\right) e^{\lambda \xi}$, where $\lambda=(v+\alpha-\beta) / 2 \alpha$. Thus, we obtain the solution to (2) as follows:

$$
\begin{equation*}
u(x, t)=\left[c_{1}+c_{2}(\log x-v t)\right] x^{\lambda} e^{-\lambda v t} \tag{38}
\end{equation*}
$$

where $c_{1}, c_{2}$ are arbitrary constants.
When $\Delta<0$, (36) has the solution $f(\xi)=$ $\left(c_{1} \cos (\sqrt{-\Delta} / 2 \alpha) \xi+c_{2} \sin (\sqrt{-\Delta} / 2 \alpha) \xi\right) e^{((v+\alpha-\beta) / 2 \alpha) \xi}$. Thus, we obtain the solution to (2) as follows:

$$
\begin{align*}
u(x, t)= & x^{(v+\alpha-\beta) / 2 \alpha} e^{-(v(v+\alpha-\beta) / 2 \alpha) t} \\
& \times\left[c_{1} \cos \frac{\sqrt{-\Delta}}{2 \alpha}(\log x-v t)\right.  \tag{39}\\
& \left.\quad+c_{2} \sin \frac{\sqrt{-\Delta}}{2 \alpha}(\log x-v t)\right]
\end{align*}
$$

where $c_{1}, c_{2}$ are arbitrary constants.
3.8. Similarity Reduction for $V_{1}+v V_{3}$. For the linear combination $V=V_{1}+v V_{3}(v \neq 0$ is an arbitrary constant $)$, we have the following similarity transformation:

$$
\begin{equation*}
\xi=x, \quad \omega=\log u-v t, \tag{40}
\end{equation*}
$$

and the similarity solution is $\omega=f(\xi)$; that is,

$$
\begin{equation*}
u=\exp \{f(x)+v t\} \tag{41}
\end{equation*}
$$

Substituting (41) into (2), we reduce the bond pricing equation to the following ODE:

$$
\begin{equation*}
\alpha \xi^{2} f^{\prime \prime}+\alpha \xi^{2} f^{\prime 2}+\beta \xi f^{\prime}+v+\gamma=0 \tag{42}
\end{equation*}
$$

where $f^{\prime}=d f / d \xi$. This is a nonlinear second-order ODE. In the next section, we will deal with such an equation by the special transformation technique.
3.9. Similarity Solution for $V_{2}+\nu V_{3}$. For the linear combination $V=V_{2}+v V_{3}(v \neq 0$ is an arbitrary constant $)$, we have the following similarity transformation:

$$
\begin{equation*}
\xi=t, \quad \omega=x^{-v} u \tag{43}
\end{equation*}
$$

and the similarity solution is $\omega=f(\xi)$; that is,

$$
\begin{equation*}
u=x^{v} f(t) \tag{44}
\end{equation*}
$$

Substituting (44) into (2), we reduce the bond pricing equation to the following ODE:

$$
\begin{equation*}
f^{\prime}+\alpha v(v-1) f+\beta v f+\gamma f=0 \tag{45}
\end{equation*}
$$

where $f^{\prime}=d f / d \xi$.
Solving (45), we get $f=c \exp \left\{-\left[\alpha v^{2}-(\alpha-\beta) v+\gamma\right] \xi\right\}$. Thus, we obtain the solution to (2) as follows

$$
\begin{equation*}
u(x, t)=c x^{v} \exp \left\{-\left[\alpha v^{2}-(\alpha-\beta) v+\gamma\right] t\right\} \tag{46}
\end{equation*}
$$

where $c$ is an arbitrary constant.
3.10. Similarity Reduction for $V_{1}+v V_{4}$. For the linear combination $V=V_{1}+\nu V_{4}(v \neq 0$ is an arbitrary constant $)$, we have the following similarity transformation:

$$
\begin{gather*}
\xi=\log x-v \alpha t^{2} \\
\omega=\log u-v t \log x+\frac{2}{3} \alpha v^{2} t^{3}-\frac{1}{2}(\alpha-\beta) v t^{2} \tag{47}
\end{gather*}
$$

and the similarity solution is $\omega=f(\xi)$; that is,

$$
\begin{align*}
u=\exp [ & f\left(\log x-v \alpha t^{2}\right)+v t \log x-\frac{2}{3} \alpha v^{2} t^{3} \\
& \left.+\frac{1}{2}(\alpha-\beta) v t^{2}\right] . \tag{48}
\end{align*}
$$

Substituting (48) into (2), we reduce the bond pricing equation to the following ODE:

$$
\begin{equation*}
\alpha f^{\prime \prime}+\alpha f^{\prime 2}-(\alpha-\beta) f^{\prime}+v \xi+\gamma=0 \tag{49}
\end{equation*}
$$

where $f^{\prime}=d f / d \xi$. This is a nonlinear second-order ODE also. In the next section, we will deal with the exact solutions to such equations.

Secondly, we consider (3). In fact, for this equation, we have the nontrivial cases as follows only.
3.11. Similarity Reduction for $V_{1}$ of (3). For the generator $V_{1}$, we have the following reduced ordinary differential equation (ODE):

$$
\begin{equation*}
\alpha \xi f^{\prime \prime}+\beta f^{\prime}+\gamma f=0 \tag{50}
\end{equation*}
$$

where $f^{\prime}=d f / d \xi$. This is a nonlinear second-order ODE as well; there is no general method for tackling it yet. In Section 4, we will deal with such equations by the power series method.
3.12. Similarity Reduction for $V_{1}+v V_{2}$ of (3). For the linear combination $V=V_{1}+v V_{2}(v \neq 0$ is an arbitrary constant $)$, we have the following similarity transformation:

$$
\begin{equation*}
\xi=x, \quad \omega=\log u-v t, \tag{51}
\end{equation*}
$$

and the similarity solution is $\omega=f(\xi)$; that is,

$$
\begin{equation*}
u=\exp [f(x)+v t] . \tag{52}
\end{equation*}
$$

Substituting (52) into (3), we reduce the second bond pricing equation to the following ODE:

$$
\begin{equation*}
\alpha \xi^{2} f^{\prime \prime}+\alpha \xi^{2} f^{\prime 2}+\beta \xi f^{\prime}+\gamma \xi+v=0 \tag{53}
\end{equation*}
$$

where $f^{\prime}=d f / d \xi$. This is a nonlinear second-order ODE also. Similar to the above equations, we will deal with such equations by the generalized power series method in the next section.

## 4. Exact Analytic Solutions in terms of the Generalized Power Series Method

In Section 3, we considered the symmetry reductions and exact solutions to the bond pricing types of (2) and (3). In this section, we will deal with the nonlinear ODEs (42), (49), (50), and (53) by the special transformation technique and generalized power series method. Thus, the exact analytic solutions to (2) and (3) are obtained.
4.1. Exact Solution to (2). Firstly, we consider the ODE (42). Letting $f^{\prime}=y$, we get the Riccati equation

$$
\begin{equation*}
\frac{d y}{d \xi}=-y^{2}-\frac{\beta}{\alpha \xi} y-\frac{v+\gamma}{\alpha \xi^{2}} . \tag{54}
\end{equation*}
$$

Now, we solve the equation by the transformation technique directly. Suppose that (54) has the solution of the form

$$
\begin{equation*}
y=p \xi^{-1} \tag{55}
\end{equation*}
$$

where $p$ is a constant to be determined. Substituting (55) into (54), we have $\alpha p^{2}-(\alpha-\beta) p+v+\gamma=0$. Solving the algebraic equation, we get

$$
\begin{equation*}
p=\frac{(\alpha-\beta) \pm \sqrt{\Delta}}{2 \alpha} \tag{56}
\end{equation*}
$$

where $\Delta=(\alpha-\beta)^{2}-4 \alpha(v+\gamma)$.
Setting $y=z+p \xi^{-1}$ and plugging it into (54), we get

$$
\begin{equation*}
\frac{d z}{d \xi}=-z^{2}-q \frac{z}{\xi}, \quad q=2 p+\frac{\beta}{\alpha} . \tag{57}
\end{equation*}
$$

This is a Bernoulli equation. Solving the equation, we have the following results.

When $q=1$, we get $f(\xi)=p \log \xi+\log \left(\log \xi+c_{1}\right)+c_{2}$. Thus, the exact solution to (2) is

$$
\begin{equation*}
u(x, t)=c_{2} x^{p}\left(\log x+c_{1}\right) e^{v t}, \tag{58}
\end{equation*}
$$

where $c_{1}$ and $c_{2}$ are arbitrary constants; $p$ and $q$ are given by (56) and (57).

When $q \neq 1$, we get $f(\xi)=p \log \xi+(1-q) \int d \xi /(\xi+$ $\left.c_{1} \xi^{q}\right)+c_{2}$. Thus, the exact solution to (2) is

$$
\begin{equation*}
u(x, t)=c_{2} x^{p} \exp \left[(1-q) \int \frac{d x}{x+c_{1} x^{q}}+v t\right] \tag{59}
\end{equation*}
$$

where $c_{1}$ and $c_{2}$ are arbitrary constants and $p$ and $q$ are given by (56) and (57).
4.2. Exact Analytic Solution to (2). Through the transformation technique, we solve the Riccati equation (54), so the exact solutions to (2) are obtained. But for the other equations such as (49), (50), and (53), we cannot get the exact solutions by such special transformation technique. However, we know that the power series can be used to solve nonlinear ODEs, including many complicated differential equations with nonconstant coefficients [10-13, 19, 20].

Now, we consider the power series solution to the reduced equation (49). Letting $f^{\prime}=y$, we get the following Riccati equation:

$$
\begin{equation*}
\alpha y^{\prime}+\alpha y^{2}-(\alpha-\beta) y+\nu \xi+\gamma=0 . \tag{60}
\end{equation*}
$$

We will seek a solution of (60) in a power series of the form

$$
\begin{equation*}
y=\sum_{n=0}^{\infty} c_{n} \xi^{n}=p+\sum_{n=1}^{\infty} c_{n} \xi^{n}, \quad p=c_{0} \tag{61}
\end{equation*}
$$

where the coefficients $c_{n}(n=0,1,2, \ldots)$ are constants to be determined.

Substituting (61) into (60) and comparing coefficients, we obtain

$$
\begin{align*}
& c_{1}=-p^{2}+\frac{\alpha-\beta}{\alpha} p-\frac{\gamma}{\alpha},  \tag{62}\\
& c_{2}=-p c_{1}+\frac{\alpha-\beta}{2 \alpha} c_{1}-\frac{v}{2 \alpha} .
\end{align*}
$$

Generally, for $n \geq 2$, we have

$$
\begin{equation*}
c_{n+1}=\frac{1}{(n+1) \alpha}\left[(\alpha-\beta) c_{n}-\alpha \sum_{k=0}^{n} c_{k} c_{n-k}\right] . \tag{63}
\end{equation*}
$$

Thus, for arbitrarily choosing the parameter $c_{0}$, from (62), we can get $c_{1}$ and $c_{2}$. Furthermore, in view of (63), we have

$$
\begin{align*}
& c_{3}=\frac{\alpha-\beta}{3 \alpha} c_{2}-\frac{1}{3}\left(2 p c_{2}+c_{1}^{2}\right),  \tag{64}\\
& c_{4}=\frac{\alpha-\beta}{4 \alpha} c_{3}-\frac{1}{2}\left(p c_{3}+c_{1} c_{2}\right),
\end{align*}
$$

and so on.
Therefore, the other terms of the sequence $\left\{c_{n}\right\}_{n=0}^{\infty}$ can be determined successively from (63) in a unique manner. This implies that for (60) there exists a power series solution (61) with the coefficients given by (62) and (63). Furthermore, we can show the convergence of the power series solution (61) with the coefficients given by (62) and (63) (see, e.g., [10, 12 , $13,19]$ ); the details are omitted here. So, this solution (61) to (60) is an exact analytic solution.

Hence, the exact power series solution to (49) can be written as follows:

$$
\begin{equation*}
f(\xi)=c+p \xi+\frac{1}{2} c_{1} \xi^{2}+\frac{1}{3} c_{2} \xi^{3}+\sum_{n=2}^{\infty} \frac{1}{n+2} c_{n+1} \xi^{n+2} \tag{65}
\end{equation*}
$$

Substituting (65) into (39), we obtain the exact analytic solution to (2) as follows:

$$
\begin{align*}
u(x, t)=q \exp [ & p\left(\log x-\alpha v t^{2}\right) \\
& +\frac{1}{2} c_{1}\left(\log x-\alpha v t^{2}\right)^{2} \\
& +\frac{1}{3} c_{2}\left(\log x-\alpha v t^{2}\right)^{3} \\
& +\sum_{n=2}^{\infty} \frac{1}{n+2} c_{n+1}\left(\log x-\alpha v t^{2}\right)^{n+2} \\
& \left.+\frac{1}{2}(\alpha-\beta) v t^{2}-\frac{2}{3} \alpha v^{2} t^{3}+v t \log x\right] \tag{66}
\end{align*}
$$

where $p=c_{0}$ and $q$ are arbitrary constants and the other coefficients $c_{n}(n=1,2, \ldots)$ are given by (62) and (63) successively.

Similarly, we can give the exact power series solution to (50) in the power series form (61). So, the exact analytic solution to (3) is obtained. The details are omitted here.
4.3. Exact Analytic Solution to (3). In Section 4.2, we construct the exact analytic solution to (49) by the power series method and obtain the exact analytic solution to (2). Now, we consider (53). Firstly, let $f^{\prime}=y$; then we get the following Riccati type of equation:

$$
\begin{equation*}
\alpha \xi^{2} y^{\prime}+\alpha \xi^{2} y^{2}+\beta \xi y+\gamma \xi+v=0 \tag{67}
\end{equation*}
$$

We will seek a solution of (67) in a generalized power series of the form

$$
\begin{equation*}
y=A \xi^{-1}+\sum_{n=0}^{\infty} c_{n} \xi^{n} \tag{68}
\end{equation*}
$$

where the parameters $A$ and $c_{n}(n=0,1,2, \ldots)$ are constants to be determined.

Substituting (68) into (67) and comparing coefficients, we obtain

$$
\begin{equation*}
A=\frac{(\alpha-\beta) \pm \sqrt{\Delta}}{2 \alpha} \tag{69}
\end{equation*}
$$

where $\Delta=(\alpha-\beta)^{2}-4 \alpha v$, and

$$
\begin{equation*}
c_{0}=\frac{-\gamma}{2 \alpha A+\beta}, \quad 2 \alpha A+\beta \neq 0 \tag{70}
\end{equation*}
$$

Generally, for $n \geq 0$, we have

$$
\begin{equation*}
c_{n+1}=\frac{-\alpha}{(n+1) \alpha+2 \alpha A+\beta} \sum_{k=0}^{n} c_{k} c_{n-k}, \quad n=0,1,2, \ldots . \tag{71}
\end{equation*}
$$

Thus, from (69) and (70), we can get $A$ and $c_{0}$. Furthermore, in view of (71), we have

$$
\begin{gather*}
c_{1}=\frac{-\alpha c_{0}^{2}}{\alpha+2 \alpha A+\beta}, \quad c_{2}=\frac{-2 \alpha c_{0} c_{1}}{2 \alpha+2 \alpha A+\beta}, \\
c_{3}=\frac{-\alpha\left(2 c_{0} c_{2}+c_{1}^{2}\right)}{3 \alpha+2 \alpha A+\beta}, \tag{72}
\end{gather*}
$$

and so on (see Remark 3).
Therefore, the other terms of the sequence $\left\{c_{n}\right\}_{n=0}^{\infty}$ can be determined successively from (71) in a unique manner. This implies that for (67) there exists a generalized power series solution (68) with the coefficients given by (69)-(71). The convergence of the generalized power series solution (68) to (67) is similar to that in Section 4.2; we omit it in this paper. Thus, the power series solution (68) to (67) is also an exact analytic solution.

Hence, the power series solution of (53) can be written as follows:

$$
\begin{equation*}
f(\xi)=\bar{c}+A \log |\xi|+c_{0} \xi+\frac{1}{2} c_{1} \xi^{2}+\sum_{n=1}^{\infty} \frac{1}{n+2} c_{n+1} \xi^{n+2} \tag{73}
\end{equation*}
$$

Substituting (73) into (52), we get the exact analytic solution to (3) as follows:

$$
\begin{align*}
u(x, t)=c x^{A} \exp [ & c_{0} x+\frac{1}{2} c_{1} x^{2} \\
& \left.+\sum_{n=1}^{\infty} \frac{1}{n+2} c_{n+1} x^{n+2}+v t\right] \tag{74}
\end{align*}
$$

where $c$ is an arbitrary constant and $A$ and $c_{n}(n=0,1,2, \ldots)$ are given by (69)-(71) successively.

Remark 1. We note that the generalized power series solution (68) differs from the regular form (61) since $A \neq 0$ in (68). In other words, there is no exact power series solution of the form (61) for (67). In particular, the determination of parameter $A$ depends on the equation greatly (cf. [10, 11] for details).

## 5. Further Discussion about the General Bond Pricing Type of (1)

In the above sections, we considered the symmetries, symmetry reductions, and exact solutions to the general bond pricing type of equation for the cases $\nu=0$ and $\nu=1$, which are the common forms in many practical applications, such as in financial mathematics. In this section, we discuss the generalized bond pricing type of equation of the form

$$
\begin{equation*}
u_{t}+\alpha x^{2} u_{x x}+\beta x u_{x}+\gamma x^{\nu} u=0 \tag{75}
\end{equation*}
$$

where $\nu \neq 0,1$ is an arbitrary positive number. Firstly, by the group classification method, we get the geometric vector field of (75) as follows:

$$
\begin{equation*}
V_{1}=\partial_{t}, \quad V_{2}=u \partial_{u}, \quad V_{s}=s \partial_{u} \tag{76}
\end{equation*}
$$

where the function $s=s(x, t)$ satisfies (75).
Moreover, through the similarity transformation (42), we can reduce this equation to the following equation (ODE):

$$
\begin{equation*}
\alpha \xi^{2} f^{\prime \prime}+\alpha \xi^{2} f^{\prime 2}+\beta \xi f^{\prime}+\gamma \xi^{\eta}+v=0 \tag{77}
\end{equation*}
$$

where $f^{\prime}=d f / d \xi$. Similarly, we can consider the symmetry reductions and exact solutions to the equation. Now, as an example, we study the special case $\nu=2$. In this case, we have

$$
\begin{equation*}
u_{t}+\alpha x^{2} u_{x x}+\beta x u_{x}+\gamma x^{2} u=0 \tag{78}
\end{equation*}
$$

Referring to (77) and setting $f^{\prime}=y$, then we get the following reduced ODE of (78):

$$
\begin{equation*}
\alpha \xi^{2} y^{\prime}+\alpha \xi^{2} y^{2}+\beta \xi y+\gamma \xi^{2}+v=0 \tag{79}
\end{equation*}
$$

Suppose that (79) has the power series solution of the generalized form (68). Then, substituting (68) into (79) and comparing coefficients, we obtain

$$
\begin{equation*}
A=\frac{(\alpha-\beta) \pm \sqrt{\Delta}}{2 \alpha} \tag{80}
\end{equation*}
$$

where $\Delta=(\alpha-\beta)^{2}-4 \alpha v$,

$$
\begin{align*}
& (2 \alpha A+\beta) c_{0}=0  \tag{81}\\
& c_{1}=\frac{-\alpha c_{0}^{2}-\gamma}{\alpha+2 \alpha A+\beta} \tag{82}
\end{align*}
$$

Generally, for $n \geq 1$, we have

$$
\begin{equation*}
c_{n+1}=\frac{-\alpha}{(n+1) \alpha+2 \alpha A+\beta} \sum_{k=0}^{n} c_{k} c_{n-k}, \quad n=1,2, \ldots \tag{83}
\end{equation*}
$$

In view of (81), we have two special cases as follows.
When $2 \alpha A+\beta \neq 0$, from (81), we have $c_{0}=0$. Furthermore, from (82) and (83), we have

$$
\begin{array}{ll}
c_{1}=\frac{-\gamma}{\alpha+2 \alpha A+\beta}, & c_{2}=0 \\
c_{3}=\frac{-\alpha c_{1}^{2}}{3 \alpha+2 \alpha A+\beta}, & c_{4}=0 \tag{84}
\end{array}
$$

and so on. In this case, by induction method, we have

$$
\begin{equation*}
c_{2 n}=0, \quad n=0,1,2, \ldots . \tag{85}
\end{equation*}
$$

When $2 \alpha A+\beta=0$, from (81), we get that $c_{0}$ is an arbitrary constant. Furthermore, from (82) and (83), we have

$$
\begin{array}{ll}
c_{1}=\frac{-\alpha c_{0}^{2}-\gamma}{\alpha+2 \alpha A+\beta}, & c_{2}=\frac{-2 \alpha c_{0} c_{1}}{2 \alpha+2 \alpha A+\beta} \\
c_{3}=\frac{-\alpha\left(2 c_{0} c_{2}+c_{1}^{2}\right)}{3 \alpha+2 \alpha A+\beta}, & c_{4}=\frac{-2 \alpha\left(c_{0} c_{3}+c_{1} c_{2}\right)}{4 \alpha+2 \alpha A+\beta} \tag{86}
\end{array}
$$

and so on (see Remark 3).
Thus, the exact power series solutions to (79) are obtained. In view of (42), the exact analytic solutions to (78) are provided in power series form, respectively. More generally, for $\nu$ is an arbitrary positive integer, the exact power series solutions to (75) can be considered similarly by the generalized power series method; the details are omitted here.

## 6. Conclusion and Remarks

In this paper, we investigate the symmetry classifications and exact solutions to the bond pricing types of equations by the combination of Lie symmetry analysis and the generalized power series method; all of the exponentiated solutions and similarity solutions are obtained explicitly for the first time in the literature. Furthermore, for the generalized bond pricing type of equation, the vector field and exact solutions are provided simultaneously. These similarity solutions possess significant features in both financial problems and physical applications. On the other hand, it is known that tackling exact solutions to the vc-PDEs is a difficult problem; from the above discussion, we can see that the combination of Lie symmetry analysis and generalized power series method is a feasible approach and is worthy of further study.

Remark 2. Since there is no space translation $(x, t, u) \rightarrow$ $(x+\epsilon, t, u)$, the bond pricing equations have no traveling wave solutions. However, based on the exponentiated solutions, we can consider the other types of solutions, such as the fundamental solutions and sometimes iterative solutions [3$5,10]$.

Remark 3. In general, we cannot get the exact explicit solutions to the nonlinear equations such as (49), (53), and (79) by the classical analysis method. To tackle these equations, the generalized power series method and special techniques are necessary sometimes. For getting the exact analytic solutions in Sections 4.3 and 5 , the condition $(n+1) \alpha+2 \alpha A+\beta \neq 0$ is necessary for $n=0,1,2, \ldots$.

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## Research Article

# Conservation Laws of Two (2 + 1)-Dimensional Nonlinear Evolution Equations with Higher-Order Mixed Derivatives 

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#### Abstract

In this paper, conservation laws for the $(2+1)$-dimensional ANNV equation and KP-BBM equation with higher-order mixed derivatives are studied. Due to the existence of higher-order mixed derivatives, Ibragimov's "new conservation theorem" cannot be applied to the two equations directly. We propose two modification rules which ensure that the theorem can be applied to nonlinear evolution equations with any mixed derivatives. Formulas of conservation laws for the ANNV equation and KP-BBM equation are given. Using these formulas, many nontrivial and time-dependent conservation laws for these equations are derived.


## 1. Introduction

The construction of explicit forms of conservation laws plays an important role in the study of nonlinear science, as they are used for the development of appropriate numerical methods and for mathematical analysis, in particular, existence, uniqueness, and stability analysis [1-3]. In addition, the existence of a large number of conservation laws of a partial differential equation (system) is a strong indication of its integrability. The famous Noether's theorem [4] has provided a systematic way of determining conservation laws for Euler-Lagrange equations, once their Noether symmetries are known, but this theorem relies on the availability of classical Lagrangians. To find conservation laws of differential equations without classical Lagrangians, researchers have made various generalizations of Noether's theorem [516]. Among those, the new conservation theorem given by Ibragimov [5] is one of the most frequently used methods.

For any linear or nonlinear differential equations, Ibragimov's new conservation theorem offers a procedure for constructing explicit conservation laws associated with the known Lie, Lie-Backlund, or nonlocal symmetries. Furthermore, it does not require the existence of classical Lagrangians. Using the conservation laws formulas given by the theorem, conservation laws for lots of equations have been studied [6-16]. When applying Ibragimov's theorem to
a given nonlinear evolution equation with mixed derivatives, we must be careful with the mixed derivatives. If we apply the conservation laws formulas to equations with mixed derivatives directly, it will result in errors. In [9], we have proposed two modification rules to apply Ibragimov's theorem to study conservation laws of two evolution equations with mixed derivatives, but the mixed derivatives are all second order and not the highest derivative term. In this paper, we will give two new modification rules and then use Ibragimov's theorem to study conservation laws of the following ANNV equation [17-20]:

$$
\begin{equation*}
u_{y t}+u_{x x x y}-3 u_{x x} u_{y}-3 u_{x} u_{x y}=0 \tag{1}
\end{equation*}
$$

and KP-BBM equation [21-24]

$$
\begin{equation*}
u_{x t}+u_{x x}-2 \alpha u_{x}^{2}-2 \alpha u u_{x x}-\beta u_{x x x t}+\gamma u_{y y}=0 \tag{2}
\end{equation*}
$$

where $\alpha, \beta$, and $\gamma$ are constants. Both in (1) and (2), the highest derivative terms $u_{x x x y}$ and $u_{x x x t}$ are mixed. Furthermore, there are other lower-order mixed derivatives in addition to the higher-order mixed derivatives.

The rest of the paper is organized as follows. In Section 2, we recall the main concepts and theorems used in this paper. In Section 3, taking the ANNV equation as an example, we first give two new modification rules which ensure the theorem can be applied to nonlinear evolution equations
with any mixed derivatives. Then formulas of conservation laws and explicit conservation laws for the ANNV equation are obtained. In Section 4, conservation laws for the KPBBM equation with higher-order mixed derivative term are derived by means of Ibragimov's theorem and the two new modification rules. Some conclusions and discussions are given in Section 5.

## 2. Preliminaries

In this section, we briefly present the main notations and theorems [5-7] used in this paper. Consider an sth-order nonlinear evolution equation

$$
\begin{equation*}
F\left(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(s)}\right)=0 \tag{3}
\end{equation*}
$$

with $n$ independent variables $x=\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ and a dependent variable $u$, where $u_{(1)}, u_{(2)}, \ldots, u_{(s)}$ denote the collection of all first-, second-, ..., sth-order partial derivatives. $u_{i}=D_{i}(u), u_{i j}=D_{j} D_{i}(u), \ldots$. Here

$$
\begin{equation*}
D_{i}=\frac{\partial}{\partial x_{i}}+u_{i} \frac{\partial}{\partial u}+u_{i j} \frac{\partial}{\partial u_{j}}+\cdots, \quad i=1,2, \ldots, n \tag{4}
\end{equation*}
$$

is the total differential operator with respect to $x_{i}$.
Definition 1. The adjoint equation of (3) is defined by

$$
\begin{equation*}
F^{*}\left(x, u, v, u_{(1)}, v_{(1)}, u_{(2)}, v_{(2)}, \ldots, u_{(s)}, v_{(s)}\right)=0 \tag{5}
\end{equation*}
$$

with

$$
\begin{equation*}
F^{*}\left(x, u, v, u_{(1)}, v_{(1)}, u_{(2)}, v_{(2)}, \ldots, u_{(s)}, v_{(s)}\right)=\frac{\delta(v F)}{\delta u} \tag{6}
\end{equation*}
$$

where

$$
\begin{equation*}
\frac{\delta}{\delta u}=\frac{\partial}{\partial u}+\sum_{m=1}^{\infty}(-1)^{m} D_{i_{1}} \cdots D_{i_{2}} \frac{\partial}{\partial u_{i_{1} i_{2} \cdots i_{m}}} \tag{7}
\end{equation*}
$$

denotes the Euler-Lagrange operator, $v$ is a new dependent variable, and $v=v(x)$.

Theorem 2. The system consisting of (3) and its adjoint equation (5),

$$
\begin{gather*}
F\left(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(s)}\right)=0 \\
F^{*}\left(x, u, v, u_{(1)}, v_{(1)}, u_{(2)}, v_{(2)}, \ldots, u_{(s)}, v_{(s)}\right)=0 \tag{8}
\end{gather*}
$$

has a formal Lagrangian; namely,

$$
\begin{equation*}
L=v F\left(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(s)}\right) \tag{9}
\end{equation*}
$$

In the following we recall the "new conservation theorem" given by Ibragimov in [5].

Theorem 3. Any Lie point, Lie-Backlund, and nonlocal symmetries,

$$
\begin{equation*}
V=\xi^{i} \frac{\partial}{\partial x_{i}}+\eta \frac{\partial}{\partial u} \tag{10}
\end{equation*}
$$

of (3) provide a conservation law $D_{i}\left(T^{i}\right)=0$ for the system (8). The conserved vector is given by

$$
\begin{align*}
& T^{i}= \xi^{i} L+W\left(\frac{\partial L}{\partial u_{i}}-D_{j}\left(\frac{\partial L}{\partial u_{i j}}\right)+D_{j} D_{k}\left(\frac{\partial L}{\partial u_{i j k}}\right)\right. \\
&\left.-D_{j} D_{k} D_{r}\left(\frac{\partial L}{\partial u_{i j k r}}\right)+\cdots\right) \\
&+D_{j} W\left(\frac{\partial L}{\partial u_{i j}}-D_{k}\left(\frac{\partial L}{\partial u_{i j k}}\right)+D_{k} D_{r}\left(\frac{\partial L}{\partial u_{i j k r}}\right)-\cdots\right) \\
&+D_{j} D_{k} W\left(\frac{\partial L}{\partial u_{i j k}}-D_{r}\left(\frac{\partial L}{\partial u_{i j k r}}\right)+\cdots\right)+\cdots, \tag{11}
\end{align*}
$$

where $L$ is determined by (9), $W$ is the Lie characteristic function, and

$$
\begin{equation*}
W=\eta-\xi^{j} u_{j} \tag{12}
\end{equation*}
$$

## 3. Two Modification Rules and Conservation Laws for the ANNV Equation

The asymmetric Nizhnik-Novikov-Veselov (ANNV) equation (1) is equivalent to the ANNV system [17, 18]

$$
\begin{equation*}
p_{t}+p_{x x x}-3 q_{x} p-3 q p_{x}=0, \quad p_{x}=q_{y} \tag{13}
\end{equation*}
$$

by the transformation $q=u_{x}, p=u_{y}$. A series of new double periodic solutions to the system (13) were derived in [17], and the variable separation solutions of (13) have been given in [18]. The Lie symmetry, reductions, and new exact solutions of the ANNV equation (1) have been studied by us from the point of Lax pair [19]. Optimal system of groupinvariant solutions and conservation laws of (1) have been studied by Wang et al. [20]. In the following, we will study the conservation laws of (1) by Theorem 3.
3.1. Two Modification Rules and Formulas of Conservation Laws for the ANNV Equation. To search for conservation laws of (1) by Theorem 3, Lie symmetry, formal Lagrangian, and adjoint equation of (1) must be known. According to Definition 1, the adjoint equation of (1) is

$$
\begin{equation*}
v_{y t}+v_{y x x x}-6 u_{x y} v_{x}-3 u_{y} v_{x x}-3 u_{x} v_{x y}=0 \tag{14}
\end{equation*}
$$

where $v$ is a new dependent variable with respect to $x, y$, and $t$.

According to Theorem 2, the formal Lagrangian for the system consisting of (1) and (14) is

$$
\begin{equation*}
L=\left(u_{y t}+u_{y x x x}-3 u_{x x} u_{y}-3 u_{x} u_{x y}\right) v \tag{15}
\end{equation*}
$$

where $v$ is a solution of (14).
Suppose that the Lie symmetry for the ANNV equation (1) is as follows:

$$
\begin{equation*}
V=\xi \frac{\partial}{\partial x}+\eta \frac{\partial}{\partial y}+\tau \frac{\partial}{\partial t}+\phi \frac{\partial}{\partial u} \tag{16}
\end{equation*}
$$

From Theorem 3, we get the general formula of conservation laws for the system consisting of (1) and (14):

$$
\begin{align*}
& X=\xi L+W\left(\frac{\partial L}{\partial u_{x}}-D_{x}\left(\frac{\partial L}{\partial u_{x x}}\right)-D_{y}\left(\frac{\partial L}{\partial u_{x y}}\right)\right. \\
& \left.-D_{x x y}\left(\frac{\partial L}{\partial u_{x x x y}}\right)\right) \\
& +D_{x}(W)\left(\frac{\partial L}{\partial u_{x x}}+D_{x y}\left(\frac{\partial L}{\partial u_{x x x y}}\right)\right) \\
& +D_{y}(W)\left(\frac{\partial L}{\partial u_{x y}}+D_{x x}\left(\frac{\partial L}{\partial u_{x x x y}}\right)\right) \\
& +D_{x x}(W)\left(-D_{y}\left(\frac{\partial L}{\partial u_{x x x y}}\right)\right) \\
& +D_{x y}(W)\left(-D_{x}\left(\frac{\partial L}{\partial u_{x x x y}}\right)\right) \\
& +D_{x x y}(W)\left(\frac{\partial L}{\partial u_{x x x y}}\right),  \tag{17}\\
& Y=\eta L+W\left(\frac{\partial L}{\partial u_{y}}-D_{x}\left(\frac{\partial L}{\partial u_{x y}}\right)-D_{t}\left(\frac{\partial L}{\partial u_{y t}}\right)\right. \\
& \left.-D_{x x x}\left(\frac{\partial L}{\partial u_{x x x y}}\right)\right) \\
& +D_{x}(W)\left(\frac{\partial L}{\partial u_{x y}}+D_{x x}\left(\frac{\partial L}{\partial u_{x x x y}}\right)\right) \\
& +D_{t}(W)\left(\frac{\partial L}{\partial u_{y t}}\right) \\
& +D_{x x}(W)\left(-D_{x}\left(\frac{\partial L}{\partial u_{x x x y}}\right)\right) \\
& +D_{x x x}(W)\left(\frac{\partial L}{\partial u_{x x x y}}\right), \\
& T=\tau L+W\left(-D_{y}\left(\frac{\partial L}{\partial u_{y t}}\right)\right)+D_{y}(W)\left(\frac{\partial L}{\partial u_{y t}}\right),
\end{align*}
$$

where $W$ is the Lie characteristic function, $W=\phi-\xi u_{x}-$ $\eta u_{y}-\tau u_{t}$, and $L$ is the formal Lagrangian determined by (15).

In fact, because of the existence of the mixed derivative terms $u_{x y}, u_{y t}$, and $u_{x x x y}$, the general formula of conservation laws must be modified; otherwise the previous $X, Y$, and $T$ do not satisfy

$$
\begin{equation*}
\left.\left(D_{x} X+D_{y} Y+D_{t} T\right)\right|_{u_{x x x}=3 u_{x x} u_{y}+3 u_{x} u_{x y}-u_{y t}}=0 \tag{18}
\end{equation*}
$$

The rules of modifications are as follows.
(1) In one conservation vector $(X, Y$, or $T)$, the time that one derivative with respect to a mixed derivative term appears
is determined by the order of the derivative with respect to its independent variables. For example, whether in $X$ or in $Y, \partial L / \partial u_{x y}$ can only appear once; $\partial L / \partial u_{x x x y}$ can only appear once in $Y$ and can appear three times in $X ; \partial L / \partial u_{x x x t}$ can only appear once in $T$ and can appear three times in $X ; \partial L / \partial u_{x x t}$ can only appear once in $T$ and can appear two times in $X$.
(2) The location that one derivative with respect to a mixed derivative term appears at cannot be the same in different conservation vectors. That is to say, if there is $W\left(-D_{y}\left(\partial L / \partial u_{x y}\right)\right)$ in $X$, then the term appears in $Y$ can only be $D_{x}(W)\left(\partial L / \partial u_{x y}\right)$ and the term $W\left(-D_{x}\left(\partial L / \partial u_{x y}\right)\right)$ cannot appear in $Y$ at the same time. And if there is $W\left(-D_{x x x}\left(\partial L / \partial u_{x x x y}\right)\right)$ in $Y$, then the terms that appear in $X$ contain $D_{x x y}(w)\left(\partial L / \partial u_{x x x y}\right)$ and first and second total derivatives of $\partial L / \partial u_{x x x y}$.

Applying the two rules to the general conservation laws formula in Theorem 3, we can get the following results.

Theorem 4. Suppose that the Lie symmetry of the ANNV equation (1) is expressed as (16). According to the different locations of $\partial L / \partial u_{x y}, \partial L / \partial u_{y t}$, and $\partial L / \partial u_{x x x y}$, the symmetry provides sixteen different conservation laws for the system consisting of (1) and (14). The conserved vectors are given as follows:

$$
\begin{equation*}
\left(X_{i j}, Y_{i j}, T_{i j}\right)=\left(X^{i}, Y^{i}, T^{i}\right)+\left(B_{j}^{X}, B_{j}^{Y}, 0\right), \quad i, j=1,2,3,4 \tag{19}
\end{equation*}
$$

with

$$
\begin{gathered}
X^{1}=\xi L+W\left(\frac{\partial L}{\partial u_{x}}-D_{x}\left(\frac{\partial L}{\partial u_{x x}}\right)\right) \\
+D_{x}(W)\left(\frac{\partial L}{\partial u_{x x}}\right)+W\left(-D_{y}\left(\frac{\partial L}{\partial u_{x y}}\right)\right), \\
Y^{1}=\eta L+W\left(\frac{\partial L}{\partial u_{y}}-D_{t}\left(\frac{\partial L}{\partial u_{y t}}\right)\right)+D_{x}(W)\left(\frac{\partial L}{\partial u_{x y}}\right), \\
T^{1}=\tau L+D_{y}(W)\left(\frac{\partial L}{\partial u_{y t}}\right), \\
+W L+W\left(\frac{\partial L}{\partial u_{x}}-D_{x}\left(\frac{\partial L}{\partial u_{x x}}\right)\right)+D_{x}(W)\left(\frac{\partial L}{\partial u_{x x}}\right) \\
\left.-D_{y}\left(\frac{\partial L}{\partial u_{x y}}\right)\right), \\
Y^{2}= \\
\eta L+W\left(\frac{\partial L}{\partial u_{y}}\right)+D_{x}(W)\left(\frac{\partial L}{\partial u_{x y}}\right) \\
+D_{t}(W)\left(\frac{\partial L}{\partial u_{y t}}\right), \\
T^{2}=\tau L+W\left(-D_{y}\left(\frac{\partial L}{\partial u_{y t}}\right)\right),
\end{gathered}
$$

$$
\begin{aligned}
& X^{3}=\xi L+W\left(\frac{\partial L}{\partial u_{x}}-D_{x}\left(\frac{\partial L}{\partial u_{x x}}\right)\right)+D_{x}(W)\left(\frac{\partial L}{\partial u_{x x}}\right) \\
& +D_{y}(W)\left(\frac{\partial L}{\partial u_{x y}}\right), \\
& Y^{3}=\eta L+W\left(\frac{\partial L}{\partial u_{y}}-D_{x}\left(\frac{\partial L}{\partial u_{x y}}\right)-D_{t}\left(\frac{\partial L}{\partial u_{y t}}\right)\right), \\
& T^{3}=\tau L+D_{y}(W)\left(\frac{\partial L}{\partial u_{y t}}\right), \\
& X^{4}=\xi L+W\left(\frac{\partial L}{\partial u_{x}}-D_{x}\left(\frac{\partial L}{\partial u_{x x}}\right)\right)+D_{x}(W)\left(\frac{\partial L}{\partial u_{x x}}\right) \\
& +D_{y}(W)\left(\frac{\partial L}{\partial u_{x y}}\right), \\
& Y^{4}=\eta L+W\left(\frac{\partial L}{\partial u_{y}}-D_{x}\left(\frac{\partial L}{\partial u_{x y}}\right)\right)+D_{t}(W)\left(\frac{\partial L}{\partial u_{y t}}\right), \\
& T^{4}=\tau L+W\left(-D_{y}\left(\frac{\partial L}{\partial u_{y t}}\right)\right), \\
& B_{1}^{X}=D_{x x y}(W)\left(\frac{\partial L}{\partial u_{x x x y}}\right) \\
& +D_{x y}(W)\left(-D_{x}\left(\frac{\partial L}{\partial u_{x x x y}}\right)\right) \\
& +D_{y}(W)\left(D_{x x}\left(\frac{\partial L}{\partial u_{x x x y}}\right)\right), \\
& B_{1}^{Y}=W\left(-D_{x x x}\left(\frac{\partial L}{\partial u_{x x x y}}\right)\right), \\
& B_{2}^{X}=W\left(-D_{x x y}\left(\frac{\partial L}{\partial u_{x x x y}}\right)\right)+D_{x x y}(W)\left(\frac{\partial L}{\partial u_{x x x y}}\right) \\
& +D_{x y}(W)\left(-D_{x}\left(\frac{\partial L}{\partial u_{x x x y}}\right)\right), \\
& B_{2}^{Y}=D_{x}(W)\left(D_{x x}\left(\frac{\partial L}{\partial u_{x x x y}}\right)\right), \\
& B_{3}^{X}=W\left(-D_{x x y}\left(\frac{\partial L}{\partial u_{x x x y}}\right)\right)+D_{x x y}(W)\left(\frac{\partial L}{\partial u_{x x x y}}\right) \\
& +D_{x}(W)\left(D_{x y}\left(\frac{\partial L}{\partial u_{x x x y}}\right)\right), \\
& B_{3}^{Y}=D_{x x}(W)\left(-D_{x}\left(\frac{\partial L}{\partial u_{x x x y}}\right)\right),
\end{aligned}
$$

$$
\begin{align*}
B_{4}^{X}= & W\left(-D_{x x y}\left(\frac{\partial L}{\partial u_{x x x y}}\right)\right) \\
& +D_{x x}(W)\left(-D_{y}\left(\frac{\partial L}{\partial u_{x x x y}}\right)\right) \\
& +D_{x}(W)\left(D_{x y}\left(\frac{\partial L}{\partial u_{x x x y}}\right)\right) \\
B_{4}^{Y}= & D_{x x x}(W)\left(\frac{\partial L}{\partial u_{x x x y}}\right) \tag{20}
\end{align*}
$$

where $W$ is the Lie characteristic function and $W=\phi-\xi u_{x}-$ $\eta u_{y}-\tau u_{t}, L$ is the formal Lagrangian determined by (15).
3.2. Explicit Conservation Laws of the ANNV Equation. Now, conservation laws of (1) can be derived by Theorem 4 if Lie symmetries of (1) are known. In fact, Lie symmetries of (1) have been obtained in [19] and they are as follows:

$$
\begin{gather*}
V_{1}=g(y) \frac{\partial}{\partial y}, \quad V_{2}=-F(t) \frac{\partial}{\partial u}, \\
V_{3}=f(t) \frac{\partial}{\partial x}-\frac{x f_{t}}{3} \frac{\partial}{\partial u},  \tag{21}\\
V_{4}=\frac{x h_{t}}{3} \frac{\partial}{\partial x}+h(t) \frac{\partial}{\partial t}-\left(\frac{h_{t}}{3} u+\frac{h_{t t}}{18} x^{2}\right) \frac{\partial}{\partial u},
\end{gather*}
$$

where $g(y), F(t), f(t)$, and $h(t)$ are arbitrary functions.
Using the Lie symmetry $V_{1}$ and Theorem 4, we can get sixteen conservation laws for the system consisting of (1) and (14). They are listed as follows:

$$
\begin{aligned}
X_{111}= & -3 g(y) u_{y}^{2} v_{x}-3 g(y) u_{y} u_{x} v_{y}-v g_{y} u_{x x y} \\
& -v g(y) u_{x x y y}-v_{x x} g_{y} u_{y}-v_{x x} g(y) u_{y y} \\
+ & v_{x} g_{y} u_{x y}+v_{x} g(y) u_{x y y} \\
Y_{111}= & g(y) v u_{t y}+g(y) v u_{x x x y}+g(y) u_{y} v_{t} \\
& +g(y) u_{y} v_{x x x} \\
& T_{111}=-v g_{y} u_{y}-v g(y) u_{y y} \\
X_{112}= & -3 g(y) u_{y}^{2} v_{x}-3 g(y) u_{y} u_{x} v_{y}-v g_{y} u_{x x y} \\
- & v g(y) u_{x x y y}+v_{x x y} g(y) u_{y}+v_{x} g_{y} u_{x y} \\
+ & g(y) v_{x} u_{x y y} \\
Y_{112}= & g(y) v u_{t y}+g(y) v u_{x x x y}+g(y) u_{y} v_{t} \\
& -g(y) u_{x y} v_{x x}, \\
& T_{112}=-v g_{y} u_{y}-v g(y) u_{y y}
\end{aligned}
$$

$$
\begin{aligned}
& X_{113}=-3 g(y) u_{y}^{2} v_{x}-3 g(y) u_{y} u_{x} v_{y}-v g_{y} u_{x x y} \\
& -v g(y) u_{x x y y}+v_{x x y} g(y) u_{y}-g(y) u_{x y} v_{x y}, \\
& Y_{113}=g(y) v u_{t y}+g(y) v u_{x x x y}+g(y) u_{y} v_{t} \\
& +g(y) u_{x x y} v_{x}, \\
& T_{113}=-v g_{y} u_{y}-v g(y) u_{y y}, \\
& X_{114}=-3 g(y) u_{y}^{2} v_{x}-3 g(y) u_{y} u_{x} v_{y}+v_{x x y} g(y) u_{y} \\
& +g(y) u_{x x y} v_{y}-g(y) u_{x y} v_{x y}, \\
& Y_{114}=g(y) v u_{t y}+g(y) u_{y} v_{t}, \\
& T_{114}=-v g_{y} u_{y}-v g(y) u_{y y}, \\
& X_{211}=-3 g(y) u_{y}^{2} v_{x}-3 g(y) u_{y} u_{x} v_{y}-v g_{y} u_{x x y} \\
& -v g(y) u_{x x y y}-v_{x x} g_{y} u_{y}-v_{x x} g(y) u_{y y} \\
& +v_{x} g_{y} u_{x y}+v_{x} g(y) u_{x y y}, \\
& Y_{211}=g(y) v u_{x x x y}+g(y) u_{y} v_{x x x}, \\
& T_{211}=g(y) u_{y} v_{y}, \\
& X_{221}=-3 g(y) u_{y}^{2} v_{x}-3 g(y) u_{y} u_{x} v_{y}-v g_{y} u_{x x y} \\
& -v g(y) u_{x x y y}+g(y) u_{y} v_{x x y}+v_{x} g_{y} u_{x y} \\
& +v_{x} g(y) u_{x y y}, \\
& Y_{221}=g(y) v u_{x x x y}-g(y) u_{x y} v_{x x}, \\
& T_{221}=g(y) u_{y} v_{y}, \\
& X_{231}=-3 g(y) u_{y}^{2} v_{x}-3 g(y) u_{y} u_{x} v_{y}-v g_{y} u_{x x y} \\
& -v g(y) u_{x x y y}+g(y) u_{y} v_{x x y}-g(y) u_{x y} v_{x y} \text {, } \\
& Y_{231}=g(y) v u_{x x x y}+g(y) u_{x x y} v_{x}, \\
& T_{231}=g(y) u_{y} v_{y}, \\
& X_{241}=-3 g(y) u_{y}^{2} v_{x}-3 g(y) u_{y} u_{x} v_{y}+g(y) u_{y} v_{x x y} \\
& +g(y) u_{x x y} v_{y}-g(y) u_{x y} v_{x y}, \\
& Y_{241}=0, \\
& T_{241}=g(y) u_{y} v_{y}, \\
& X_{311}=-3 g(y) u_{y}^{2} v_{x}+3 g(y) u_{x y} u_{y} v+3 u_{x} v g_{y} u_{y} \\
& +3 u_{x} v g(y) u_{y y}-v g_{y} u_{x x y}-v g(y) u_{x x y y} \\
& -v_{x x} g_{y} u_{y}-v_{x x} g(y) u_{y y}+v_{x} g_{y} u_{x y}+v_{x} g(y) u_{x y y}, \\
& Y_{311}=g(y) v u_{t y}+g(y) v u_{x x x y}-3 g(y) v u_{x x} u_{y}
\end{aligned}
$$

$$
\left.\begin{array}{rl}
X_{421}= & -3 g(y) u_{y}^{2} v_{x}+3 g(y) u_{x y} u_{y} v+3 u_{x} v g_{y} u_{y} \\
& +3 u_{x} v g(y) u_{y y}-v g_{y} u_{x x y}-v g(y) u_{x x y y} \\
& +g(y) u_{y} v_{x x y}+v_{x} g_{y} u_{x y}+v_{x} g(y) u_{x y y} \\
Y_{421}= & g(y) v u_{x x x y}-3 g(y) v u_{x x} u_{y}-3 g(y) v u_{x} u_{x y} \\
& -3 g(y) u_{y} u_{x} v_{x}-g(y) u_{x y} v_{x x}, \\
T_{421}=g(y) u_{y} v_{y} \\
X_{431}= & -3 g(y) u_{y}^{2} v_{x}+3 g(y) u_{x y} u_{y} v+3 u_{x} v g_{y} u_{y} \\
& +3 u_{x} v g(y) u_{y y}-v g_{y} u_{x x y}-v g(y) u_{x x y y} \\
& +g(y) u_{y} v_{x x y}-g(y) u_{x y} v_{x y}, \\
Y_{431}= & g(y) v u_{x x x y}-3 g(y) v u_{x x} u_{y}-3 g(y) v u_{x} u_{x y} \\
& -3 g(y) u_{y} u_{x} v_{x}+g(y) u_{x x y} v_{x}, \\
T_{431}=g(y) u_{y} v_{y}
\end{array}\right\} \begin{aligned}
& X_{441}=- 3 g(y) u_{y}^{2} v_{x}+3 g(y) u_{x y} u_{y} v+3 u_{x} v g_{y} u_{y} \\
&+3 u_{x} v g(y) u_{y y}+g(y) u_{y} v_{x x y}+g(y) u_{x x y} v_{y} \\
&-g(y) u_{x y} v_{x y}, \\
& Y_{441}=-3 g(y) v u_{x x} u_{y}-3 g(y) v u_{x} u_{x y}-3 g(y) u_{y} u_{x} v_{x}, \\
& T_{441}=g(y) u_{y} v_{y}
\end{aligned}
$$

For the Lie symmetry $V_{2}$, we can also get sixteen conservation laws by Theorem 4. For example, making use of

$$
\begin{equation*}
\left(X_{21}, Y_{21}, T_{21}\right)=\left(X^{2}, Y^{2}, T^{2}\right)+\left(B_{1}^{X}, B_{1}^{Y}, 0\right) \tag{23}
\end{equation*}
$$

we can get

$$
\begin{gather*}
X_{212}=-3 F(t) u_{y} v_{x}-3 F(t) u_{x y} v-3 F(t) u_{x} v_{y} \\
Y_{212}=3 F(t) u_{x x} v-F_{t} v+F(t) v_{x x x}  \tag{24}\\
T_{212}=F(t) v_{y}
\end{gather*}
$$

For the Lie symmetry $V_{3}$, we can also get sixteen conservation laws by Theorem 4. For example, making use of

$$
\begin{equation*}
\left(X_{42}, Y_{42}, T_{42}\right)=\left(X^{4}, Y^{4}, T^{4}\right)+\left(B_{2}^{X}, B_{2}^{Y}, 0\right) \tag{25}
\end{equation*}
$$

we can get

$$
\begin{aligned}
X_{423}= & f(t) v u_{t y}-x f_{t} u_{y} v_{x}-3 f(t) u_{y} v_{x} u_{x}+u_{y} v f_{t} \\
& +\frac{1}{3} x f_{t} v_{x x y}+f(t) u_{x} v_{x x y}+f(t) u_{x x y} v_{x},
\end{aligned}
$$

$$
\begin{gather*}
Y_{423}=-x f_{t} u_{x} v_{x}-3 f(t) u_{x}^{2} v_{x}-\frac{1}{3} x v f_{t t}-v f_{t} u_{x} \\
-v f(t) u_{t x}-\frac{1}{3} f_{t} v_{x x}-f(t) v_{x x} u_{x x}, \\
T_{423}=\frac{1}{3} x f_{t} v_{y}+f(t) v_{y} u_{x} . \tag{26}
\end{gather*}
$$

Using the Lie symmetry $V_{4}$ and Theorem 4, sixteen conservation laws for (1) can be obtained. For example, making use of

$$
\begin{equation*}
\left(X_{13}, Y_{13}, T_{13}\right)=\left(X^{1}, Y^{1}, T^{1}\right)+\left(B_{3}^{X}, B_{3}^{Y}, 0\right) \tag{27}
\end{equation*}
$$

we can get

$$
\begin{align*}
X_{134}= & -h(t) v u_{x x y t}+\frac{1}{3} u h_{t} v_{x x y}-h(t) u_{t x} v_{x y}-h_{t} v u_{x x y} \\
& -x h_{t} u_{x} u_{y} v_{x}-\frac{1}{9} x h_{t t} v_{x y}+h(t) v_{x x y} u_{t}-\frac{2}{3} h_{t} u_{x} v_{x y} \\
& +\frac{1}{18} x^{2} h_{t t} v_{x x y}-2 x h_{t} v u_{x} u_{x y}-h_{t} u u_{y} v_{x}-h_{t} u u_{x y} v \\
& -h_{t} u u_{x} v_{y}-\frac{1}{6} x^{2} h_{t t} u_{y} v_{x}-\frac{1}{6} x^{2} h_{t t} u_{x y} v \\
& -\frac{1}{6} x^{2} h_{t t} u_{x} v_{y}-x h_{t} u_{x}^{2} v_{y}-3 h(t) u_{t} u_{y} v_{x} \\
& -3 h(t) u_{t} u_{x y} v-3 h(t) u_{t} u_{x} v_{y}+2 h_{t} v u_{y} u_{x} \\
& +\frac{1}{3} x v u_{y} h_{t t}+3 h(t) v u_{y} u_{t x}+\frac{1}{3} x h_{t} v_{x x y} u_{x} \\
& -\frac{1}{3} x h_{t} v_{x y} u_{x x}+\frac{1}{3} x h_{t} v u_{t y}, \\
Y_{134}= & h_{t} v u u_{x x}+\frac{1}{6} x^{2} v h_{t t} u_{x x}+2 x h_{t} v u_{x x} u_{x} \\
& +3 h(t) v u_{x x} u_{t}+\frac{1}{3} u v_{t} h_{t}+\frac{1}{18} x^{2} v_{t} h_{t t} \\
& +\frac{1}{3} x h_{t} v_{t} u_{x}+h(t) v_{t} u_{t}+2 h_{t} v u_{x}^{2} \\
& +\frac{1}{3} x h_{t t} v u_{x}+3 h(t) u_{x} v u_{t x}+h_{t} u_{x x} v_{x} \\
& +\frac{1}{9} v_{x} h_{t t}+\frac{1}{3} x h_{t} v_{x} u_{x x x}+h(t) v_{x} u_{t x x} \\
T_{134}= & h(t) v u_{x x x y}-3 h(t) v u_{x x} u_{y}-3 h(t) v u_{x} u_{x y} \\
& -\frac{1}{3} v u_{y} h_{t}-\frac{1}{3} v x u_{x y} h_{t} \tag{28}
\end{align*}
$$

In the previous expressions of conservation laws, $v$ is a solution of (14). If we can find an exact solution $v$ of (14), explicit conservation laws of the ANNV equation (1) can be
obtained by substituting it with the previous expressions. For example,

$$
\begin{equation*}
v=m(y)+n(t) \tag{29}
\end{equation*}
$$

is a solution of (14) with $m(y)$ and $n(t)$ being two arbitrary functions. By that, nontrivial conservation laws of (1) can be obtained.

Remark 5. It is pointed out that the previous conservation laws are all nontrivial. The accuracy of them has been checked by Maple software.

Remark 6. The conservation laws of (1) obtained in this paper are different from each other and are all different from those in [20].

## 4. Formulas of Conservation Laws and Explicit Conservation Laws for the KP-BBM Equation

The solutions of the KP-BBM equation (2) have been studied by Wazwaz in $[21,22]$ who used the sine-cosine method, the tanh method, and the extended tanh method. Abdou [23] used the extended mapping method with symbolic computation to obtain some periodic solutions, solitary wave solution, and triangular wave solution. Exact solutions and conservation laws of (2) have been studied by Adem and Khalique using the Lie group analysis and the simplest equation method [24].
4.1. Formulas of Conservation Laws of the KP-BBM Equation. To search for conservation laws of (2) by Theorem 3, Lie symmetry, formal Lagrangian, and adjoint equation of (2) must be known. According to Definition 1, the adjoint equation of (2) is

$$
\begin{equation*}
v_{x t}+v_{x x}-2 \alpha u v_{x x}-\beta v_{x x x t}+\gamma v_{y y}=0 \tag{30}
\end{equation*}
$$

where $v$ is a new dependent variable with respect to $x, y$, and $t$.

According to Theorem 2, the formal Lagrangian for the system consisting of (2) and (30) is

$$
\begin{equation*}
L=\left(u_{x t}+u_{x x}-2 \alpha u_{x}^{2}-2 \alpha u u_{x x}-\beta u_{x x x t}+\gamma u_{y y}\right) v . \tag{31}
\end{equation*}
$$

Since there are a higher-order mixed derivative $u_{x x x t}$ and a mixed derivative $u_{x t}$ in (2), the two modification rules must be used if we want to get conservation laws of (2) by Theorem 3. Therefore, we can get the following statement.

Theorem 7. Suppose that the Lie symmetry of the KP-BBM equation is as follows:

$$
\begin{equation*}
V=\xi \frac{\partial}{\partial x}+\eta \frac{\partial}{\partial y}+\tau \frac{\partial}{\partial t}+\phi \frac{\partial}{\partial u} . \tag{32}
\end{equation*}
$$

According to the different locations of $\partial L / \partial u_{x t}$ and $\partial L / \partial u_{x x x t}$, the symmetry provides eight different conservation laws for
the system consisting of (2) and (30). The conserved vectors are given as follows:

$$
\begin{array}{r}
\left(X_{i j}, Y_{i j}, T_{i j}\right)=\left(X^{i}, Y^{i}, T^{i}\right)+\left(A_{j}^{X}, 0, A_{j}^{T}\right),  \tag{33}\\
i=1,2, \quad j=1,2,3,4
\end{array}
$$

with

$$
\begin{aligned}
& X^{1}=\xi L+W\left(\frac{\partial L}{\partial u_{x}}-D_{x}\left(\frac{\partial L}{\partial u_{x x}}\right)\right)+D_{x}(W)\left(\frac{\partial L}{\partial u_{x x}}\right) \\
& +W\left(-D_{t}\left(\frac{\partial L}{\partial u_{x t}}\right)\right), \\
& Y^{1}=\eta L+W\left(-D_{y}\left(\frac{\partial L}{\partial u_{y y}}\right)\right)+D_{y}(W)\left(\frac{\partial L}{\partial u_{y y}}\right), \\
& T^{1}=\tau L+D_{x}(W)\left(\frac{\partial L}{\partial u_{x t}}\right), \\
& X^{2}=\xi L+W\left(\frac{\partial L}{\partial u_{x}}-D_{x}\left(\frac{\partial L}{\partial u_{x x}}\right)\right)+D_{x}(W)\left(\frac{\partial L}{\partial u_{x x}}\right) \\
& +D_{t}(W)\left(\frac{\partial L}{\partial u_{x t}}\right), \\
& Y^{2}=Y^{1}, \\
& T^{2}=\tau L+W\left(-D_{x}\left(\frac{\partial L}{\partial u_{x t}}\right)\right), \\
& A_{1}^{X}=D_{x x t}(W)\left(\frac{\partial L}{\partial u_{x x x t}}\right)+D_{x t}(W)\left(-D_{x}\left(\frac{\partial L}{\partial u_{x x x t}}\right)\right) \\
& +D_{t}(W)\left(D_{x x}\left(\frac{\partial L}{\partial u_{x x x t}}\right)\right), \\
& A_{1}^{T}=W\left(-D_{x x x}\left(\frac{\partial L}{\partial u_{x x x t}}\right)\right) \text {, } \\
& A_{2}^{X}=D_{x x t}(W)\left(\frac{\partial L}{\partial u_{x x x t}}\right)+D_{x t}(W)\left(-D_{x}\left(\frac{\partial L}{\partial u_{x x x t}}\right)\right) \\
& +W\left(-D_{x x t}\left(\frac{\partial L}{\partial u_{x x x t}}\right)\right), \\
& A_{2}^{T}=D_{x}(W)\left(D_{x x}\left(\frac{\partial L}{\partial u_{x x x t}}\right)\right) \text {, } \\
& A_{3}^{X}=D_{x x t}(W)\left(\frac{\partial L}{\partial u_{x x x t}}\right)+D_{x}(W)\left(D_{x t}\left(\frac{\partial L}{\partial u_{x x x t}}\right)\right) \\
& +W\left(-D_{x x t}\left(\frac{\partial L}{\partial u_{x x x t}}\right)\right), \\
& A_{3}^{T}=D_{x x}(W)\left(-D_{x}\left(\frac{\partial L}{\partial u_{x x x t}}\right)\right),
\end{aligned}
$$

$$
\begin{align*}
A_{4}^{X}= & D_{x x}(W)\left(-D_{t}\left(\frac{\partial L}{\partial u_{x x x t}}\right)\right) \\
& +D_{x}(W)\left(D_{x t}\left(\frac{\partial L}{\partial u_{x x x t}}\right)\right) \\
& +W\left(-D_{x x t}\left(\frac{\partial L}{\partial u_{x x x t}}\right)\right), \\
A_{4}^{T} & =D_{x x x}(W)\left(\frac{\partial L}{\partial u_{x x x t}}\right), \tag{34}
\end{align*}
$$

where $W$ is the Lie characteristic function, $W=\phi-\xi u_{x}-\eta u_{y}-$ $\tau u_{t}$, and $L$ is the formal Lagrangian determined by (31).
4.2. Explicit Conservation Laws of the KP-BBM Equation. Lie symmetries of (2) have been derived in [24] and are listed as follows:

$$
\begin{align*}
& V_{1}=\frac{\partial}{\partial x}, \quad V_{2}=\frac{\partial}{\partial y}, \quad V_{3}=\frac{\partial}{\partial t}, \\
& V_{4}=-\alpha y \frac{\partial}{\partial y}-2 \alpha t \frac{\partial}{\partial t}+(2 \alpha u-1) \frac{\partial}{\partial u} . \tag{35}
\end{align*}
$$

Applying Theorem 7, we can obtain conservation laws for the system consisting of (2) and (30). For the symmetry $V_{1}$, we can get the following eight conservation laws:

$$
\begin{gathered}
X_{111}=v u_{t x}+\gamma v u_{y y}-2 \alpha u_{x} v_{x} u+u_{x} v_{x}+u_{x} v_{t} \\
+\beta u_{t x} v_{x x}-\beta u_{t x x} v_{x}, \\
T_{111}=-v u_{x x}-\beta u_{x} v_{x x x}, \\
X_{121}=v u_{t x}+\gamma v u_{y y}-2 \alpha u_{x} v_{x} u+u_{x} v_{x}+u_{x} v_{t} \\
-\beta v_{t x x} u_{x}-\beta u_{t x x} v_{x}, \\
T_{121}=-v u_{x x}+\beta u_{x x} v_{x x}, \\
X_{131}=v u_{t x}+\gamma v u_{y y}-2 \alpha u_{x} v_{x} u+u_{x} v_{x}+u_{x} v_{t} \\
\quad-\beta v_{t x x} u_{x}+\beta u_{x x} v_{t x}, \\
T_{131}=-v u_{x x}-\beta u_{x x x} v_{x}, \\
X_{141}=v u_{t x}-\beta u_{t x x x} v+\gamma v u_{y y}-2 \alpha u_{x} v_{x} u+u_{x} v_{x} \\
\quad+u_{x} v_{t}-\beta v_{t x x} u_{x}+\beta u_{x x} v_{t x}-\beta u_{x x x} v_{t}, \\
T_{141}=-v u_{x x}+\beta u_{x x x x} v, \\
X_{211}=\gamma v u_{y y}-2 \alpha u_{x} v_{x} u+u_{x} v_{x}+\beta u_{t x} v_{x x}-\beta u_{t x x} v_{x}, \\
T_{211}=u_{x} v_{x}-\beta u_{x} v_{x x x}, \\
X_{221}=\gamma v u_{y y}-2 \alpha u_{x} v_{x} u+u_{x} v_{x}-\beta v_{t x x} u_{x}-\beta u_{t x x} v_{x}, \\
T_{221}=u_{x} v_{x}+\beta u_{x x} v_{x x}, \\
X_{231}=\gamma v u_{y y}-2 \alpha u_{x} v_{x} u+u_{x} v_{x}-\beta v_{t x x} u_{x}+\beta u_{x x} v_{t x},
\end{gathered}
$$

$$
\begin{gather*}
T_{231}=u_{x} v_{x}-\beta u_{x x x} v_{x}, \\
X_{241}=-\beta u_{t x x x} v+\gamma v u_{y y}-2 \alpha u_{x} v_{x} u+u_{x} v_{x} \\
-\beta v_{t x x} u_{x}+\beta u_{x x} v_{t x}-\beta u_{x x x} v_{t}, \\
T_{241}=u_{x} v_{x}+\beta u_{x x x x} v, \tag{36}
\end{gather*}
$$

where $Y_{i j 1}=\gamma u_{x} v_{y}-\gamma u_{x y} v, i=1,2, j=1,2,3,4$.
For the symmetry $V_{2}$, we can also get eight conservation laws for the system of (2) and (30). For example, making use of

$$
\begin{equation*}
\left(X_{13}, Y_{13}, T_{13}\right)=\left(X^{1}, Y^{1}, T^{1}\right)+\left(A_{3}^{X}, 0, A_{3}^{T}\right) \tag{37}
\end{equation*}
$$

we can get

$$
\begin{gather*}
X_{132}=2 \alpha u_{y} u_{x} v-2 \alpha u_{y} v_{x} u+u_{y} v_{x}-u_{x y} v+2 \alpha u_{x y} v u \\
+u_{y} v_{t}-\beta v_{t x x} u_{y}+\beta u_{t x x y} v+\beta u_{x y} v_{t x}, \\
Y_{132}=v u_{t x}+v u_{x x}-2 \alpha v u_{x}^{2}-2 \alpha v u u_{x x}-\beta v u_{t x x x}+\gamma u_{y} v_{y}, \\
T_{132}=-u_{x y} v-\beta u_{x x y} v_{x} . \tag{38}
\end{gather*}
$$

Similarly, for the symmetry $V_{3}$, we can get eight conservation laws for the system of (2) and (30). For example, making use of

$$
\begin{equation*}
\left(X_{22}, Y_{22}, T_{22}\right)=\left(X^{2}, Y^{2}, T^{2}\right)+\left(A_{2}^{X}, 0, A_{2}^{T}\right) \tag{39}
\end{equation*}
$$

we can get

$$
\begin{gather*}
X_{223}=2 \alpha u_{t} u_{x} v-2 \alpha u_{t} v_{x} u+u_{t} v_{x}-u_{t x} v+2 \alpha u_{t x} v u \\
-u_{t t} v-\beta v_{t x x} u_{t}+\beta u_{t t x x} v-\beta u_{t t x} v_{x}, \\
Y_{223}=\gamma u_{t} v_{y}-\gamma u_{t y} v  \tag{40}\\
T_{223}= \\
u_{t x} v+v u_{x x}-2 \alpha v u_{x}^{2}-2 \alpha v u u_{x x}-\beta v u_{t x x x} \\
+\gamma v u_{y y}+u_{t} v_{x}+\beta u_{t x} v_{x x} .
\end{gather*}
$$

For the symmetry $V_{4}$, we only list the conservation laws derived by

$$
\begin{equation*}
\left(X_{14}, Y_{14}, T_{14}\right)=\left(X^{1}, Y^{1}, T^{1}\right)+\left(A_{4}^{X}, 0, A_{4}^{T}\right), \tag{41}
\end{equation*}
$$

and they are as follows:

$$
\begin{align*}
X_{144}= & 4 \alpha u_{x} v+4 \alpha^{2} u^{2} v_{x}-2 \alpha v_{t} u-4 \alpha v_{x} u-\beta \alpha y v_{t x} u_{x y} \\
& +v_{x}+\beta \alpha y v_{t} u_{x x y}-\beta v_{t x x}+v_{t}-8 \alpha^{2} u u_{x} v \\
& -\alpha y u_{y} v_{x}-2 \alpha t u_{t} v_{x}+\alpha y u_{x y} v+2 \alpha t u_{t x} v-\alpha y v_{t} u_{y} \\
& -2 \alpha t v_{t} u_{t}+2 \beta \alpha v_{t x x} u-2 \beta \alpha v_{t x} u_{x} \\
& -2 \alpha^{2} y u_{y} u_{x} v+2 \beta \alpha v_{t} u_{x x}+2 \alpha^{2} y u_{y} v_{x} u, \\
Y_{144}= & -4 \alpha^{2} t u_{t} u_{x} v+4 \alpha^{2} t u_{t} v_{x} u-2 \alpha^{2} y u_{x y} v u-4 \alpha^{2} t u_{t x} v u \\
& +\beta \alpha y v_{t x x} u_{y}+2 \beta \alpha t v_{t x x} u_{t}-2 \beta \alpha t v_{t x} u_{t x} \\
& +2 \beta \alpha t v_{t} u_{t x x}, \\
T_{144}= & -2 \alpha t v u_{x x}+4 \alpha^{2} t v u_{x}^{2}+4 \alpha^{2} t v u u_{x x}-2 \alpha \gamma t v u_{y y} \\
& +2 \alpha u_{x} v+\alpha y u_{x y} v-2 \beta \alpha v u_{x x x}-\beta \alpha y v u_{x x x y} . \tag{42}
\end{align*}
$$

In the previous expressions of conservation laws, $v$ is a solution of the adjoint equation (30). If we can find an exact solution $v$ of (30), explicit conservation laws for the KPBBM equation (2) can be obtained by substituting it with the previous expressions. For example,

$$
\begin{equation*}
v=(x+M(t)) y+N(t) \tag{43}
\end{equation*}
$$

is a solution of (30) with $M(t)$ and $N(t)$ being two arbitrary functions. By that we can get many infinite conservation laws for (2). Furthermore, the conservation laws are nontrivial and time dependent.

Remark 8. The correctness of the conservation laws of (2) obtained here has been checked by Maple software. The conservation laws obtained here for (2) are much more than those in [24] and different from them.

## 5. Concluding Remarks

Recently, conservation laws of nonlinear evolution equations with mixed derivatives have attracted the interest of mathematical and physical researchers. As shown in [16], when applying Noether's theorem and partial Noether's theorem to obtain conservation laws of nonlinear evolution equations with higher-order mixed derivatives, the obtained conservation laws must be adjusted to satisfy the definition of conservation laws. We face the same problem when applying Ibragimov's new conservation theorem to find conservation laws of nonlinear evolution equations with mixed derivatives. In this paper, we propose two modification rules which ensure that Ibragimov's theorem can be applied to nonlinear evolution equations with higher-order and lower-order mixed derivatives. The two modification rules given in this paper are a generalization of those proposed in [9]. The results are used to study the conservation laws of two partial differential equations with higher-order mixed derivatives:
the ANNV equation and the KP-BBM equation. Many infinite explicit and nontrivial conservation laws are obtained for the two equations. Based on the two modification rules, Ibragimov's new conservation theorem can be used to find conservation laws of other partial differential equations with any mixed derivatives.

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## Research Article

# Symmetry Reductions, Exact Solutions, and Conservation Laws of a Modified Hunter-Saxton Equation 

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#### Abstract

We study a modified Hunter-Saxton equation from the Lie group-theoretic point of view. The Lie point symmetry generators of the underlying equation are derived. We utilize the Lie algebra admitted by the equation to obtain the optimal system of onedimensional subalgebras of the Lie algebra of the equation. These subalgebras are then used to reduce the underlying equation to nonlinear third-order ordinary differential equations. Exact traveling wave group-invariant solutions for the equation are constructed by integrating the reduced ordinary differential equations. Moreover, using the variational method, we construct infinite number of nonlocal conservation laws by the transformation of the dependent variable of the underlying equation.


## 1. Introduction

The nonlinear partial differential equation

$$
\begin{equation*}
\left(u_{t}+u u_{x}\right)_{x x}=\frac{1}{2}\left(u_{x}^{2}\right)_{x} \tag{1}
\end{equation*}
$$

was proposed by Hunter and Saxton [1]. They showed that the weakly nonlinear waves are described by (1), where $u(x, t)$ describes the director field of a nematic liquid crystal, $x$ is a space variable in a reference frame moving with the linearized wave velocity, and $t$ is a slow time variable. Liquid crystals are fluids made up of long rigid molecules. Equation (1) is a high-frequency limit of the CamassaHolm equation [2]. Hunter and Zheng [3] showed that it is bivariational, bi-Hamiltonian, and a member of the Harry Dym hierarchy of integrable flows. It is well known that (1) under reciprocal transformation reduces to the Liouville equation [2]. Also, an interesting geometric interpretation of the Hunter-Saxton equation $[4,5]$ is that, for spatially periodic functions, it describes the geodesic flow on the homogeneous space $\operatorname{Vir}(\mathbb{S}) / \operatorname{Rot}(S)$ of the Virasoro group $\operatorname{Vir}(S)$ modulo the rotations $\operatorname{Rot}(S)$, with respect to the rightinvariant homogeneous $\dot{H}^{1}$ metric: $\langle f, g\rangle=\int_{S} f_{x} g_{x} d x$.

In this paper the modified Hunter-Saxton equation [6]

$$
\begin{equation*}
u_{t}-2 u_{x} u_{x x}-u u_{x x x}-u_{x x t}=0 \tag{2}
\end{equation*}
$$

is studied from the Lie group analysis standpoint. We first obtain the Lie point symmetry generators and then utilize the Lie algebra admitted by the equation to obtain the optimal system of one-dimensional subalgebras of the Lie algebra of the equation. These subalgebras are later used to reduce the modified Hunter-Saxton equation to nonlinear third-order ordinary differential equations. We then construct exact traveling wave group-invariant solutions for the equation by integrating the reduced ordinary differential equations. Finally, using the variational method, we construct infinite number of nonlocal conservation laws by the transformation of the dependent variable of the underlying equation.

The outline of the paper is as follows. In Section 2, we briefly discuss some main operator identities and their relationship. In Section 3, we present the Lie point symmetry generators of (2). In Section 4, the optimal system of onedimensional subalgebras of the Lie symmetry algebra of (2) is constructed. Moreover, using the optimal system of subalgebras, symmetry reductions and exact group-invariant
solutions of (2) are obtained. In Section 5, nonlocal conservation laws of (2) are constructed using Noether's theorem. Finally, in Section 6, concluding remarks are presented.

## 2. Preliminaries

In this section we present the notations that will be used in the sequel. For details, the reader is referred to $[7,8]$.

Consider a $k$ th-order system of partial differential equations (PDEs) of $n$ independent variables $x=\left(x^{1}, x^{2}, \ldots, x^{n}\right)$ and $m$ dependent variables $u=\left(u^{1}, u^{2}, \ldots, u^{m}\right)$; namely,

$$
\begin{equation*}
E^{\alpha}\left(x, u, u_{(1)}, \ldots, u_{(k)}\right)=0, \quad \alpha=1, \ldots, m \tag{3}
\end{equation*}
$$

where $u_{(1)}, u_{(2)}, \ldots, u_{(k)}$ denote the collections of all first, second,...,kth-order partial derivatives; that is, $u_{i}^{\alpha}=D_{i}\left(u^{\alpha}\right)$, $u_{i j}^{\alpha}=D_{j} D_{i}\left(u^{\alpha}\right), \ldots$, respectively, with the total differentiation operator with respect to $x^{i}$ being defined by

$$
\begin{equation*}
D_{i}=\partial_{x^{i}}+u_{i}^{\alpha} \partial_{u^{\alpha}}+u_{i j}^{\alpha} \partial_{u_{j}^{\alpha}}+\cdots, \quad i=1, \ldots, n \tag{4}
\end{equation*}
$$

and the summation convention is used whenever appropriate.
The conserved vector $T=\left(T^{1}, T^{2}, \ldots, T^{n}\right)$ of (3), where each $T^{i} \in \mathscr{A}, \mathscr{A}$ is the space of all differential functions, satisfies the equation

$$
\begin{equation*}
D_{i} T^{i}=0 \tag{5}
\end{equation*}
$$

along the solutions of (3).
The Lie point symmetry generator in the form of infinite formal sum is given by

$$
\begin{equation*}
X=\xi^{i} \partial_{x^{i}}+\eta^{\alpha} \partial_{u^{\alpha}}+\sum_{s \geq 1} \zeta_{i_{1} i_{2} \cdots i_{s}}^{\alpha} \partial_{u_{1 i_{1}-\cdots i i_{s}}^{\alpha}} \tag{6}
\end{equation*}
$$

where $\xi$ and $\eta$ are functions of $x$ and $u$ and are independent of derivatives of $u$, and the additional coefficients are determined uniquely by the prolongation formulae

$$
\begin{gather*}
\zeta_{i}^{\alpha}=D_{i}\left(W^{\alpha}\right)+\xi^{j} u_{i j}^{\alpha} \\
\zeta_{i_{1} \cdots i_{s}}^{\alpha}=D_{i_{1}} \cdots D_{i_{s}}\left(W^{\alpha}\right)+\xi^{j} u_{j i_{1} \cdots i_{s}}^{\alpha}, \quad s>1 \tag{7}
\end{gather*}
$$

in which $W^{\alpha}$ is the Lie characteristic function defined by $W^{\alpha}=\eta^{\alpha}-\xi^{j} u_{j}^{\alpha}$.

Let $L=L\left(x, u, u_{(1)}, \ldots, u_{(l)}\right) \in \mathscr{A}, l \leq k$ be a Lagrangian associated with a Noether symmetry operator $X$. If $B^{1}, \ldots, B^{n}$ are point-dependent gauge terms, then the Noether symmetry operator $X$ is determined by

$$
\begin{equation*}
X L+L D_{i} \xi^{i}=D_{i} B^{i}, \quad i=1, \ldots, n \tag{8}
\end{equation*}
$$

and the conserved vectors, $T^{i}$, corresponding to each $X$

$$
\begin{align*}
T^{i}= & B^{i}-L \xi^{i}-W^{\alpha} \frac{\partial L}{\partial u_{i}^{\alpha}} \\
& -\left[D_{j}\left(W^{\alpha}\right)-W^{\alpha} D_{j}\right] \frac{\partial L}{\partial u_{i j}^{\alpha}}, \quad i=1, \ldots, n \tag{9}
\end{align*}
$$

are obtained via Noether's theorem (see [9]).

## 3. Lie Point Symmetries of (2)

In this section, we discuss in brief the Lie symmetry group method to obtain Lie point symmetry generators admitted by (2). For detailed account of this method see $[10-14]$ and the references therein.

A vector field

$$
\begin{equation*}
X=\tau(t, x, u) \partial_{t}+\xi(t, x, u) \partial_{x}+\eta(t, x, u) \partial u \tag{10}
\end{equation*}
$$

is a generator of point symmetry of (2), if

$$
\begin{equation*}
\left.X^{[3]}\left(u_{t}-2 u_{x} u_{x x}-u u_{x x x}-u_{x x t}\right)\right|_{(2)}=0 \tag{11}
\end{equation*}
$$

where the operator $X^{[3]}$ is the third prolongation of the operator $X$ defined by

$$
\begin{equation*}
X^{[3]}=X+\zeta_{t} \partial_{u_{t}}+\zeta_{x} \partial_{u_{x}}+\zeta_{x x x} \partial_{u_{x x x}}+\zeta_{x x t} \partial_{u_{x x t}} \tag{12}
\end{equation*}
$$

with the coefficients $\zeta_{t}, \zeta_{x}, \zeta_{x x x}$, and $\zeta_{x x t}$ being given by

$$
\begin{gather*}
\zeta_{t}=D_{t}(\eta)-u_{t} D_{t}(\tau)-u_{x} D_{t}(\xi), \\
\zeta_{x}=D_{x}(\eta)-u_{t} D_{x}(\tau)-u_{x} D_{x}(\xi), \\
\zeta_{x x}=D_{x}\left(\zeta_{x}\right)-u_{x t} D_{x}(\tau)-u_{x x} D_{x}(\xi),  \tag{13}\\
\zeta_{x x x}=D_{x}\left(\zeta_{x x}\right)-u_{x x t} D_{x}(\tau)-u_{x x x} D_{x}(\xi), \\
\zeta_{x x t}=D_{t}\left(\zeta_{x x}\right)-u_{x x t} D_{t}(\tau)-u_{x x x} D_{t}(\xi) .
\end{gather*}
$$

Here, $D_{t}$ and $D_{x}$ are the total derivative operators defined by

$$
\begin{align*}
D_{t} & =\partial_{t}+u_{t} \partial_{u}+\cdots \\
D_{x} & =\partial_{x}+u_{x} \partial_{u}+\cdots \tag{14}
\end{align*}
$$

The coefficient functions $\tau, \xi$, and $\eta$ are independent of the derivatives of $u$; thus equating the coefficients of like derivatives of $u$ in the determining equation (11) yields the following over determined system of linear PDEs:

$$
\begin{gather*}
\tau=\tau(t), \quad \xi=\xi(t, x), \quad \eta_{u u}=0 \\
\xi_{x x}-2 \eta_{x u}=0, \quad 2 \xi_{x}-\eta_{x x u}=0 \\
\tau_{t}-3 \xi_{x}+\eta_{u}+\eta_{x x u}=0 \\
\eta_{t}-\eta_{t x x}-u \eta_{x x x}=0  \tag{15}\\
\xi_{t}-u \tau_{t}+3 u \xi_{x}-u \eta_{x x u}-\eta=0 \\
2 \xi_{t x}+3 u \xi_{x x}-2 \eta_{x}-\eta_{t u}-3 u \eta_{x u}=0 \\
\xi_{t x x}+u \xi_{x x x}-\xi_{t}-2 \eta_{x x}-2 \eta_{t x u}-3 u \eta_{x x u}=0
\end{gather*}
$$

Solving the determining equations (15) for $\tau, \xi$, and $\eta$, we obtain the three-dimensional Lie algebra spanned by the following Lie point symmetry generators admitted by (2):

$$
\begin{equation*}
X_{1}=\partial_{t}, \quad X_{2}=\partial_{x}, \quad X_{3}=t \partial_{t}-u \partial_{u} \tag{16}
\end{equation*}
$$

Table 1: Commutator table of the Lie algebra of (2).

|  | $X_{1}$ | $X_{2}$ | $X_{3}$ |
| :---: | :---: | :---: | :---: |
| $X_{1}$ | 0 | 0 | $X_{1}$ |
| $X_{2}$ | 0 | 0 | 0 |
| $X_{3}$ | $-X_{1}$ | 0 | 0 |

Table 2: Adjoint table of the Lie algebra of (2).

| Ad | $X_{1}$ | $X_{2}$ | $X_{3}$ |
| :--- | :---: | :---: | :---: |
| $X_{1}$ | $X_{1}$ | $X_{2}$ | $X_{3}-\epsilon X_{1}$ |
| $X_{2}$ | $X_{1}$ | $X_{2}$ | $X_{3}$ |
| $X_{3}$ | $e^{\epsilon} X_{1}$ | $X_{2}$ | $X_{3}$ |

Table 3: Subalgebras, group invariants, and group-invariant solutions of (2).

| $N$ | $X$ | $\gamma$ | Group-invariant solution |
| :---: | :---: | :---: | :---: |
| 1 | $X_{2}$ | $t$ | $u=h(\gamma)$ |
| 2 | $X_{1}+\epsilon X_{2}$ | $x-\epsilon t$ | $u=h(\gamma)$ |
| 3 | $X_{3}+a X_{2}$ | $x-a \ln t$ | $u=(1 / t) h(\gamma)$ |

Here $\epsilon=0, \pm 1$, and $a$ is an arbitrary constant.

## 4. Symmetry Reductions and Group-Invariant Solutions of (2)

Here we utilize the Lie point symmetry generators (16) of (2) found in Section 3 to obtain symmetry reduction and construct exact group-invariant solutions for (2).

We first present the optimal system of one-dimensional subalgebras of the Lie algebra admitted by (2). The onedimensional subalgebras are then used to reduce (2) to ordinary differential equations (ODEs). Exact group-invariant solutions for the underlying equation (2) are constructed by integrating the reduced ODEs.

The results on the classification of the Lie point symmetries of (2) are summarized in the Tables 1,2 , and 3 . The commutator table of the Lie point symmetries of (2) and the adjoint representations of the symmetry group of (2) on its Lie algebra are given in Tables 1 and 2, respectively. Tables 1 and 2 are then used to construct the optimal system of onedimensional subalgebras for (2) which are given in Table 3 (for more details of the approach see [10]).

Case 1. In this case, the group-invariant solution corresponding to the symmetry generator $X_{2}$ leads to the trivial solution $u(x, t)=C$, where $C$ is a constant.

Case 2. The group-invariant solution arising from the subalgebra $X_{1}+\epsilon X_{2}$ reduces (2) to the following nonlinear thirdorder ODE:

$$
\begin{equation*}
(h-\epsilon) h^{\prime \prime \prime}+2 h^{\prime} h^{\prime \prime}+\epsilon h^{\prime}=0, \quad \epsilon \neq 0 \tag{17}
\end{equation*}
$$

where "prime" denotes differentiation with respect to $\gamma$. Integrating (17) twice with respect to $\gamma$, we obtain

$$
\begin{equation*}
h^{\prime 2}=\frac{\epsilon h^{2}+K_{1} h+K_{2}}{\epsilon-h} \tag{18}
\end{equation*}
$$

where $K_{1}$ and $K_{2}$ are arbitrary constants of integration. Separating the variables in (18), integrating, and reverting back to our original variables, one obtains the solution of (2) in quadrature

$$
\begin{equation*}
\int \sqrt{\frac{\epsilon-u}{\epsilon u^{2}+K_{1} u+K_{2}}} d u=K_{3} \pm(x-\epsilon t) \tag{19}
\end{equation*}
$$

where $K_{3}$ is an arbitrary constant of integration. One can obtain two particular solutions for (17) given by

$$
\begin{gather*}
h(\gamma)=\frac{1}{B}\left(\delta \pm \frac{B \gamma}{2}\right)^{2}-\frac{A}{B}  \tag{20}\\
h(\gamma)=-\frac{\epsilon}{4}(\delta \pm \gamma)^{2}
\end{gather*}
$$

where $A=C_{1}-\epsilon^{2}, B=-\epsilon$, and $C_{1}, \delta$ are arbitrary constants. Hence, we obtain the two special group-invariant solutions for (2) given by

$$
\begin{gather*}
u(x, t)=-\frac{1}{\epsilon}\left[\left(\delta \pm \frac{\epsilon(\epsilon t-x)}{2}\right)^{2}-\left(C_{1}-\epsilon^{2}\right)\right]  \tag{21}\\
u(x, t)=-\frac{\epsilon}{4}[\delta \pm(x-\epsilon t)]^{2}
\end{gather*}
$$

Likewise, one can obtain the following three group-invariant steady state solutions of (2) for the case when $\epsilon=0$ :

$$
\begin{gather*}
u(x, t)=\sqrt{C_{1}}\left(C_{3} \pm x\right), \\
u(x, t)=\left\{\frac{9 C_{2}}{4}\left(C_{3} \pm x\right)^{2}\right\}^{1 / 3}, \\
\sqrt{C_{1} u\left(C_{1} u+C_{2}\right)}-C_{2} \log \left(\sqrt{C_{1}\left(C_{1} u+C_{2}\right)}+C_{1} \sqrt{u}\right) \\
=C_{1}^{3 / 2}\left(C_{3} \pm x\right) \tag{22}
\end{gather*}
$$

Here, $C_{1}, C_{2}$, and $C_{3}$ are arbitrary constants of integration.
Case 3. Substitution of the group-invariant solution corresponding to the subalgebra $X_{3}+a X_{2}$ into (2) results in the reduced nonlinear third-order ODE

$$
\begin{equation*}
(h-a) H^{\prime \prime \prime}-h^{\prime \prime}+2 h^{\prime} h^{\prime \prime}+a h^{\prime}+h=0 \tag{23}
\end{equation*}
$$

where "prime" denotes differentiation with respect to $\gamma$.

## 5. Conservation Laws of (2)

In this section, we obtain nonlocal conservation laws of (2) using the variational approach.

Equation (2) does not have a Lagrangian as it is an evolution equation. By making a substitution $u=v_{x}$, (2) becomes

$$
\begin{equation*}
v_{x t}-2 v_{x x} v_{x x x}-v_{x} v_{x x x x}-v_{x x x t}=0 \tag{24}
\end{equation*}
$$

Equation (24) has a usual Lagrangian

$$
\begin{equation*}
L=-\frac{1}{2}\left(v_{t} v_{x}+v_{x} v_{x x}^{2}+v_{x t} v_{x x}\right) . \tag{25}
\end{equation*}
$$

If $X=\tau \partial_{t}+\xi \partial_{x}+\eta \partial_{v}$ is the Noether symmetry operator, then from the Noether operator determining equation (8) we obtain

$$
\begin{align*}
& -\frac{1}{2} v_{x} \zeta_{t}-\frac{1}{2}\left(v_{t}+v_{x x}^{2}\right) \zeta_{x}-\left(v_{x} v_{x x}+\frac{1}{2} v_{x t}\right) \zeta_{x x} \\
& -\frac{1}{2} v_{x x} \zeta_{x t}+\left(\tau_{t}+\tau_{v} v_{t}\right) L+\left(\xi_{x}+\xi_{v} v_{x}\right) L  \tag{26}\\
& \quad=B_{t}^{1}+B_{v}^{1} v_{t}+B_{x}^{2}+B_{v}^{2} v_{x}
\end{align*}
$$

Expansion of (26) and then equating the coefficients of the various monomials in the first- and second-order partial derivatives of $v$ to zero yield the following determining equations:

$$
\begin{gather*}
\tau_{x}=\tau_{v}=0, \quad \xi_{t}=\xi_{v}=0, \\
\eta_{v}=0, \quad 4 \xi_{x}-\tau_{t}-3 \eta_{v}=0, \\
\eta_{v}-\xi_{x}, \quad \xi_{t}-\eta_{x}=0  \tag{27}\\
B_{v}^{1}=-\frac{1}{2} \eta_{x}, \quad B_{v}^{2}=-\frac{1}{2} \eta_{t} \\
B_{t}^{1}+B_{x}^{2}=0 .
\end{gather*}
$$

The solution of (27) is $\tau=c_{1}, \xi=c_{2}, \eta=a(t), B^{1}=c(t, x)$, and $B^{2}=-1 / 2 a^{\prime}(t) v+d(t, x)$, where $c_{1}, c_{2}$ are arbitrary constants, $a(t)$ is an arbitrary function, and $c_{t}+d_{x}=0$. We set $c=d=0$. Thus, we obtain the following Noether symmetry operators associated with the Lagrangian (25) for (24):

$$
\begin{gather*}
X_{1}=\partial_{t}, \quad B^{1}=0, \\
X_{2}=\partial_{x}, \quad B^{2}=0,  \tag{28}\\
X_{3}=a(t) \partial_{v}, \quad B^{2}=0, \\
B^{2}=0,-\frac{1}{2} a^{\prime}(t) v
\end{gather*}
$$

Hence, by invoking (9), we obtain the following conserved vectors corresponding to the Noether symmetry operators (28):
(i) $X_{1}=\partial_{t}, B^{1}=0, B^{2}=0$,

$$
\begin{gathered}
T^{1}=\frac{1}{2} v_{x} v_{x x}^{2}+\frac{1}{2} v_{t} v_{x x x}, \\
T^{2}=-\frac{1}{2} v_{t}^{2}+\frac{1}{2} v_{t} v_{x x}^{2}+v_{t} v_{x x t}+v_{t} v_{x} v_{x x x} \\
-\frac{1}{2} v_{t t} v_{x x}-v_{x} v_{t x} v_{x x}-\frac{1}{2} v_{t x}^{2} .
\end{gathered}
$$

Hence, from (5), we have

$$
\begin{align*}
D_{t}\left(T^{1}\right)+D_{x}\left(T^{2}\right)= & -v_{t}\left(v_{x t}-2 v_{x x} v_{x x x}-v_{x} v_{x x x x}-v_{x x x t}\right) \\
& +\frac{1}{2} v_{t} v_{x x x t}-\frac{1}{2} v_{x x} v_{x t t} \\
= & \frac{1}{2} v_{t} v_{x x x t}-\frac{1}{2} v_{x x} v_{x t t} \\
= & D_{t}\left(\frac{1}{2} v_{t} v_{x x x}\right)-D_{x}\left(\frac{1}{2} v_{x x} v_{t t}\right) \tag{30}
\end{align*}
$$

Taking these terms across and including them into the conserved flows, we get

$$
\begin{equation*}
D_{t}\left(T^{1}-\frac{1}{2} v_{t} v_{x x x}\right)+D_{x}\left(T^{2}+\frac{1}{2} v_{x x} v_{t t}\right)=0 . \tag{31}
\end{equation*}
$$

If we let

$$
\begin{align*}
& \widetilde{T}^{1}=T^{1}-\frac{1}{2} v_{t} v_{x x x}=\frac{1}{2} v_{x} v_{x x}^{2}, \\
\widetilde{T}^{2}= & T^{2}+\frac{1}{2} v_{x x} v_{t t} \\
= & -\frac{1}{2} v_{t}^{2}+\frac{1}{2} v_{t} v_{x x}^{2}+v_{t} v_{x x t}+v_{t} v_{x} v_{x x x}  \tag{32}\\
& -v_{x} v_{t x} v_{x x}-\frac{1}{2} v_{t x}^{2},
\end{align*}
$$

one can readily verify that the new components of the conserved vector $\widetilde{T}$ satisfy the equation $\left.D_{i} \widetilde{T}^{i}\right|_{(24)}=0$.
(ii) Consider $X_{2}=\partial_{x}, B^{1}=0, B^{2}=0$,

$$
\begin{gather*}
T^{1}=-\frac{1}{2} v_{x}^{2}+\frac{1}{2} v_{x} v_{x x x}-\frac{1}{2} v_{x x}^{2}  \tag{33}\\
T^{2}=v_{x} v_{x x t}+v_{x}^{2} v_{x x x}-\frac{1}{2} v_{x t} v_{x x} .
\end{gather*}
$$

Invoking (5), we obtain

$$
\begin{align*}
D_{t}\left(T^{1}\right)+D_{x}\left(T^{2}\right)= & -v_{x}\left(v_{x t}-2 v_{x x} v_{x x x}-v_{x} v_{x x x x}-v_{x x x t}\right) \\
& +\frac{1}{2} v_{x} v_{x x x t}-\frac{1}{2} v_{x x} v_{x x t} \\
= & \frac{1}{2} v_{x} v_{x x x t}-\frac{1}{2} v_{x x} v_{x x t} \\
= & D_{x}\left(\frac{1}{2} v_{x} v_{x x t}\right)-D_{t}\left(\frac{1}{2} v_{x x}^{2}\right) \tag{34}
\end{align*}
$$

Taking the terms across and adding them into the conserved flows yield

$$
\begin{equation*}
D_{t}\left(T^{1}+\frac{1}{2} v_{x x}^{2}\right)+D_{x}\left(T^{2}-\frac{1}{2} v_{x} v_{x x t}\right)=0 . \tag{35}
\end{equation*}
$$

Considering

$$
\begin{align*}
\widetilde{T}^{1} & =T^{1}+\frac{1}{2} v_{x x}^{2}=\frac{1}{2} v_{x} v_{x x x}-\frac{1}{2} v_{x}^{2} \\
\widetilde{T}^{2} & =T^{2}-\frac{1}{2} v_{x} v_{x x t}  \tag{36}\\
& =\frac{1}{2} v_{x} v_{x x t}+v_{x}^{2} v_{x x x}-\frac{1}{2} v_{x t} v_{x x}
\end{align*}
$$

we obtain the modified new components of the conserved vector $\widetilde{T}$ so that the equation, $\left.D_{i} \widetilde{T}^{i}\right|_{(24)}=0$, is satisfied.
(iii) Consider $X_{3}=a(t) \partial_{v}, B^{1}=0, B^{2}=-(1 / 2) a(t) v$,

$$
\begin{gather*}
T^{1}=\frac{1}{2} a(t) v_{x}-\frac{1}{2} a(t) v_{x x x}, \\
T^{2}=-\frac{1}{2} a^{\prime}(t) v+\frac{1}{2} a(t) v_{t}-\frac{1}{2} a(t) v_{x x}^{2}  \tag{37}\\
-a(t) v_{x x t}-a(t) v_{x} v_{x x x} .
\end{gather*}
$$

In this case, we have infinite number of conservation laws. Using (5), we have

$$
\begin{align*}
D_{t}\left(T^{1}\right)+D_{x}\left(T^{2}\right)= & a(t)\left(v_{x t}-2 v_{x x} v_{x x x}-v_{x} v_{x x x x}-v_{x x x t}\right) \\
& -\frac{1}{2}\left(a^{\prime}(t) v_{x x x}+a(t) v_{x x x t}\right) \\
= & -\frac{1}{2}\left(a^{\prime}(t) v_{x x x}+a(t) v_{x x x t}\right) \\
= & D_{t}\left(-\frac{1}{2} a(t) v_{x x x}\right) . \tag{38}
\end{align*}
$$

Taking the term across and adding it into the conserved flows, we obtain

$$
\begin{equation*}
D_{t}\left(T^{1}+\frac{1}{2} a(t) v_{x x x}\right)+D_{x}\left(T^{2}\right)=0 \tag{39}
\end{equation*}
$$

If we choose

$$
\begin{gather*}
\widetilde{T}^{1}=T^{1}+\frac{1}{2} a(t) v_{x x x}=\frac{1}{2} a(t) v_{x}  \tag{40}\\
\widetilde{T}^{2}=T^{2}
\end{gather*}
$$

we obtain $\left.D_{i} \widetilde{T}^{i}\right|_{(24)}=0$.

## 6. Concluding Remarks

In this paper, we studied the modified Hunter-Saxton equation (2) using the Lie symmetry group of infinitesimal transformations of the equation. We found that the underlying equation admits a three-dimensional Lie algebra spanned by the vector fields of translations in time and space and the scaling of time and the dependent variable. We obtained the optimal system of one-dimensional subalgebras of the Lie algebra of the equation. These subalgebras were then used to reduce the underlying equation to nonlinear third-order ordinary
differential equations. Exact group-invariant solutions called traveling wave solutions were constructed by integrating the reduced ODEs. Furthermore, we constructed infinite number of nonlocal conservation laws for the underlying equation by the transformation of the dependent variable of the equation and making use of Noether's theorem.

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