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Integral and Differential Systems in Function Spaces and Related Problems

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Editorial Board

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Editorial

Integral and Differential Systems in Function Spaces and Related Problems

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As has been seen, integral and differential systems in function spaces are ideal mathematical models in many applied problems stemmed from the real world. They have in recent years been major objects of investigations with fast increasing interest. This special issue is dedicated to the dissemination of current significant progresses and new trends in this field.

This issue is composed of papers that emphasize different aspects of the theory of the integral and differential systems in function spaces and related issues. The topics addressed by these published papers in the special issue include the stability of delay differential systems; the Wiener product on a bosonic Connes space associated to a bilaplacian and the formal Wiener chaos on the path space; multiplicative and additive perturbation of convoluted C-regularized operator families, convoluted C-cosine operator families, and convoluted C-semigroups related to the differential equations in Banach spaces; the local Gevrey regularity of the solutions of the linearized spatially homogeneous Boltzmann equations; the boundedness of some rough bilinear fractional integral on Morrey spaces and modified Morrey spaces; the global bifurcation of positive solutions for semilinear elliptic equations with asymptotically linear function on a unit ball; hybrid gradient-projection algorithm for solving constrained convex minimization problems with generalized mixed equilibrium problems; the pointwise estimates for the sharp function of the maximal multilinear commutators and maximal iterated commutator generalized by m-linear Calderón-Zygmund singular integral operator; and the existence of nontrivial solutions of a quasilinear elliptic equation.

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We would like to thank all the authors who submitted manuscripts for consideration in this special issue and regret that many could not be accepted. Special thanks go to all the reviewers for their valuable comments, criticism, and suggestions.

Ti-Jun Xiao
James H. Liu
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Research Article

On Perturbation of Convoluted C-Regularized Operator Families

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Of concern are two classes of convoluted C-regularized operator families: convoluted C-cosine operator families and convoluted C-semigroups. We obtain new and general multiplicative and additive perturbation theorems for these convoluted C-regularized operator families. Two examples are given to illustrate our abstract results.

1. Introduction

It is well known that the cosine operator families (resp., the C_0 semigroups) and the fractionally integrated C-cosine operator families (resp., integrated C-semigroups) are important tools in studying incomplete second-order (resp., first-order) abstract Cauchy problems (cf., e.g., [1–17]). As an extension of the cosine operator families (resp., the C_0 semigroups) as well as the fractionally integrated C-cosine operator families (resp., integrated C-semigroups), the convoluted C-cosine operator families (resp., convoluted C-semigroups) (cf., e.g., [15, 18, 19]) are also good operator families in dealing with ill-posed incomplete second order (resp. first order) abstract Cauchy problems.

In last two decades, there are many works on the perturbations on the C-regularized operator families (cf., e.g., [16, 20–24]). In the present paper, we will study the multiplicative and additive perturbation for two classes of convoluted C-regularized operator families: convoluted C-cosine operator families and convoluted C-semigroups, and our purpose is to obtain some new and general perturbation theorems for these convoluted C-regularized operator families and to make the results new even for convoluted n-times integrated C-cosine operator families (resp., convoluted n-times integrated C-semigroups) ($n \in \mathbb{N}_0$, where $\mathbb{N}_0$ denotes the nonnegative integers).

Throughout this paper, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$ denote the set of positive integers, the real numbers, and the complex plane, respectively. X denotes a nontrivial complex Banach space, and $L(X)$ denotes the space of bounded linear operators from X into X. In the sequel, we assume that $C \in L(X)$ is an injective operator. $\mathbf{C}([a,b],X)$ denotes the space of all continuous functions from $[a,b]$ to X. For a closed linear operator A on X, its domain, range, resolvent set, and the C-resolvent set are denoted by $D(A)$, $R(A)$, $\rho(A)$, and $\rho_c(A)$, respectively, where $\rho_c(A)$ is defined by

$$\rho_c(A) := \{\lambda \in \mathbb{C} : R(C) \subset R(\lambda - A), \\ \lambda - A \text{ is injective}\}. \tag{1}$$

$K \in \mathbf{C}([0,\infty),\mathbb{C})$ is an exponentially bounded function, and for $\beta \in \mathbb{R}$,

$$\mathscr{L}[K(t)](\lambda) \neq 0 \quad (\operatorname{Re}\lambda > \beta), \tag{2}$$

where $\mathscr{L}[K(t)](\lambda)$ is the Laplace transform of $K(t)$ as in the monograph [15]. We define

$$\Theta(t) := \int_0^t K(s)\,ds, \quad t \geq 0. \tag{3}$$

Next, we recall some notations and basic results from [15, 19] about the convoluted C-cosine operator families and convoluted C-semigroups.

The following definition is the convoluted version of [15, Chapter 1, Definition 4.1].

Definition 1. Let $\omega \geq 0$ and $(\omega^2, \infty) \subset \rho_c(A)$. Let $\{C_K(t)\}_{t\geq 0}(C_K(t) \in L(X), t \geq 0)$ be a strongly continuous operator family such that

$$\|C_K(t)\| \leq Me^{\omega t}, \quad t \geq 0, \tag{4}$$

for some $M > 0$, and

$$\lambda\left(\lambda^2 - A\right)^{-1}Cx = \frac{1}{\mathscr{L}[K(t)](\lambda)}\int_0^\infty e^{-\lambda t}C_K(t)\,xdt, \quad \operatorname{Re}\lambda > \max(\omega, \beta), \ x \in X. \tag{5}$$

Then, A is called a subgenerator of the exponentially bounded K-convoluted C-cosine operator family $\{C_K(t)\}_{t\geq 0}$. Moreover, the operator $\overline{A} := C^{-1}AC$ is called the generator of the $\{C_K(t)\}_{t\geq 0}$.

Proposition 2. *Let A be a closed operator and $\{C_K(t)\}_{t\geq 0}$ a strongly continuous, exponentially bounded operator family. Then A is the subgenerator of a K-convoluted C-cosine operator family $\{C_K(t)\}_{t\geq 0}$ if and only if*

(1) $C_K(t)C = CC_K(t)$, $t \geq 0$;

(2) $C_K(t)A \subset AC_K(t)$, $t \geq 0$, *and*

$$A\int_0^t\int_0^s C_K(\sigma)\,xd\sigma ds = C_K(t)\,x - \Theta(t)\,Cx, \quad t \geq 0, \ x \in X. \tag{6}$$

Remark 3. If A is the subgenerator of a K-convoluted C-cosine operator family, then $CA \subseteq AC$.

Definition 4. Let $0 \leq \omega < \infty$ and $(\omega, \infty) \subset \rho_c(A)$. Let $\{T_K(t)\}_{t\geq 0}$ be a strongly continuous operator family such that

$$\|T_K(t)\| \leq Me^{\omega t}, \quad t \geq 0, \tag{7}$$

for some $M > 0$, and

$$(\lambda - A)^{-1}Cx = \frac{1}{\mathscr{L}[K(t)](\lambda)}\int_0^\infty e^{-\lambda t}T_K(t)\,xdt, \quad \operatorname{Re}\lambda > \max\{\omega, \beta\}, \ x \in X. \tag{8}$$

Then, A is called a subgenerator of an exponentially bounded K-convoluted C-semigroup $\{T_K(t)\}_{t\geq 0}$. Moreover, the operator $\overline{A} := C^{-1}AC$ is called the generator of the $\{T_K(t)\}_{t\geq 0}$.

Proposition 5. *Let A be a closed operator, and $\{T_K(t)\}_{t\geq 0}$ a strongly continuous, exponentially bounded operator family. Then, A is the subgenerator of a K-convoluted C-semigroup $\{T_K(t)\}_{t\geq 0}$ if and only if*

(1) $T_K(t)C = CT_K(t)$, $t \geq 0$;

(2) $T_K(t)A \subset AT_K(t)$, $t \geq 0$, *and*

$$A\int_0^t T_K(s)\,xds = T_K(t)\,x - \Theta(t)\,Cx, \quad t \geq 0, \ x \in X. \tag{9}$$

Remark 6. From [15], we know that the C-cosine operator families (resp., C-semigroups) are exactly the 0-times integrated C-cosine operator families (resp., the 0-times integrated C-semigroups). Let $\Gamma(\cdot)$ be the well-known Gamma function, and

$$K(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}. \tag{10}$$

Then, by Propositions 2 and 5, we get results for the α-times integrated C-cosine operator families (resp., α-times integrated C-semigroups) as well as C-cosine operator families (resp., C-semigroups). For more information on various C operator families, we refer the reader to, for example, [3, 6–8, 14, 15, 17, 22] and references therein.

2. Multiplicative Perturbation Theorems

Lemma 7. *Suppose that A is a subgenerator of an exponentially bounded K-convoluted C-cosine operator family on X. If $\rho(A) \neq \emptyset$, then $C^{-1}AC = A$.*

Proof. For any $\lambda_0 \in \rho(A)$ and $x \in D(C^{-1}AC)$, let

$$y = \lambda_0 x - C^{-1}ACx. \tag{11}$$

Then,

$$\left(\lambda_0 - A\right)^{-1}C = C\left(\lambda_0 - A\right)^{-1}, \qquad Cx = \left(\lambda_0 - A\right)^{-1}Cy = C\left(\lambda_0 - A\right)^{-1}y. \tag{12}$$

Therefore,

$$x = \left(\lambda_0 - A\right)^{-1}y \in D(A). \tag{13}$$

This means that $C^{-1}AC \subseteq A$. Thus, by Remark 3, we see that $C^{-1}AC = A$. □

Theorem 8. *Let A be a closed linear operator on X and $\mathscr{R} \in L(X)$. Assume that there exists an injective operator C on X satisfying $CA \subseteq AC$, $\mathscr{R}C = C\mathscr{R}$. Then, the following statements hold.*

(1) *If $\mathscr{R}A$ subgenerates an exponentially bounded K-convoluted C-cosine operator family on X, then $A\mathscr{R}$ subgenerates an exponentially bounded K-convoluted C-cosine operator family on X.*

(2) *If $A\mathscr{R}$ subgenerates an exponentially bounded K-convoluted C-cosine operator family on X and $\rho(\mathscr{R}A) \neq \emptyset$, then $\mathscr{R}A$ generates an exponentially bounded K-convoluted C-cosine operator family on X.*

Proof. (1) Assume that $\mathscr{R}A$ subgenerates an exponentially bounded K-convoluted C-cosine operator family $\{C_K(t)\}_{t\geq 0}$ on X.

In this case, it is easy to see that for any $t \geq 0$, the operator

$$x \longmapsto A \int_0^t S_K(s) \mathscr{R} x ds \tag{14}$$

is bounded, since

$$\int_0^t S_K(s) \mathscr{R} x ds \in D(\mathscr{R} A), \tag{15}$$

where $S_K(t) = \int_0^t C_K(s) ds$. Now, for each $t \geq 0$, we define a bounded linear operator as follows:

$$\widehat{C}_K(t) x = \Theta(t) C x + A \int_0^t S_K(s) \mathscr{R} x ds. \tag{16}$$

Clearly, the graph norms of $\mathscr{R}A$ and A are equivalent. Therefore, noting that $\mathscr{R}A$ subgenerates an exponentially bounded K-convoluted C-cosine operator family $\{C_K(t)\}_{t\geq 0}$ on X, we obtain, for every $t_1, t_2 \geq 0$, and $x \in X$, that there exists a constant M_1 such that

$$\begin{aligned}
&\left\|\widehat{C}_K(t_1) x - \widehat{C}_K(t_2) x\right\| \\
&\quad \leq \|\Theta(t_1) C x - \Theta(t_2) C x\| \\
&\qquad + \left\| A \left(\int_0^{t_1} S_K(s) \mathscr{R} x ds - \int_0^{t_2} S_K(s) \mathscr{R} x ds \right) \right\| \\
&\quad \leq \|\Theta(t_1) C x - \Theta(t_2) C x\| \\
&\qquad + M_1 \left(\left\| \mathscr{R} A \left(\int_0^{t_1} S_K(s) \mathscr{R} x ds - \int_0^{t_2} S_K(s) \mathscr{R} x ds \right) \right\| \right. \\
&\qquad\qquad \left. + \left\| \int_0^{t_1} S_K(s) \mathscr{R} x ds - \int_0^{t_2} S_K(s) \mathscr{R} x ds \right\| \right) \\
&\quad = (M_1 + 1) \|\Theta(t_1) C x - \Theta(t_2) C x\| \\
&\qquad + M_1 \|C_K(t_1) x - C_K(t_2) x\| \\
&\qquad + \left\| \int_0^{t_1} S_K(s) \mathscr{R} x ds \right. \\
&\qquad\quad \left. - \int_0^{t_2} S_K(s) \mathscr{R} x ds \right\| \longrightarrow 0, \quad \text{as } t_1 \longrightarrow t_2.
\end{aligned} \tag{17}$$

Hence, $\widehat{C}_K(\cdot)$ is strongly continuous.

Similarly, we can prove that $\widehat{C}_K(\cdot)$ is exponentially bounded; that is, there exists a constant $\widehat{M} > 0$ such that

$$\left\|\widehat{C}_K(t)\right\| \leq \widehat{M} e^{\omega t}, \quad t \geq 0. \tag{18}$$

As in the monograph [15], we write

$$\begin{gathered}
\mathscr{L}\left[\widehat{C}_K(t)\right](\lambda) x = \int_0^\infty e^{-\lambda t} \widehat{C}_K(t) x dt, \\
\text{for } \operatorname{Re} \lambda > \max(\omega, \beta), \ x \in X.
\end{gathered} \tag{19}$$

Then, by (16), we have

$$\begin{aligned}
\mathscr{L}\left[\widehat{C}_K(t)\right](\lambda) x = {} & \frac{\mathscr{L}[K(t)](\lambda)}{\lambda} C x \\
& + A \frac{\mathscr{L}[K(t)](\lambda)}{\lambda} \left(\lambda^2 - \mathscr{R} A\right)^{-1} C \mathscr{R} x.
\end{aligned} \tag{20}$$

Hence,

$$\begin{aligned}
&\mathscr{R} \mathscr{L}\left[\widehat{C}_K(t)\right](\lambda) x \\
&\quad = \frac{\mathscr{L}[K(t)](\lambda)}{\lambda} \\
&\qquad \times C\left[\mathscr{R} x + \mathscr{R} A\left(\lambda^2 - \mathscr{R} A\right)^{-1} C \mathscr{R} x\right] \\
&\quad = \lambda \mathscr{L}[K(t)](\lambda)\left(\lambda^2 - \mathscr{R} A\right)^{-1} C \mathscr{R} x \\
&\quad \in D(A).
\end{aligned} \tag{21}$$

Furthermore,

$$\begin{aligned}
&\left(\lambda^2 - A \mathscr{R}\right) \mathscr{L}\left[\widehat{C}_K(t)\right](\lambda) x \\
&\quad = \lambda^2 \mathscr{L}\left[\widehat{C}_K(t)\right](\lambda) x \\
&\qquad - \lambda \mathscr{L}[K(t)](\lambda) A\left(\lambda^2 - \mathscr{R} A\right)^{-1} C \mathscr{R} x \\
&\quad = \lambda \mathscr{L}[K(t)](\lambda) C x.
\end{aligned} \tag{22}$$

On the other hand, for each $x \in D(A\mathscr{R})$, $\operatorname{Re} \lambda > \max(\omega, \beta)$, we obtain

$$\begin{aligned}
&\frac{\mathscr{L}[K(t)](\lambda)}{\lambda}\left[C + A\left(\lambda^2 - \mathscr{R} A\right)^{-1} C \mathscr{R}\right]\left(\lambda^2 - A \mathscr{R}\right) x \\
&\quad = \lambda \mathscr{L}[K(t)](\lambda) C x.
\end{aligned} \tag{23}$$

Therefore,

$$\lambda\left(\lambda^2 - A \mathscr{R}\right)^{-1} C = \frac{1}{\lambda}\left[I + A\left(\lambda^2 - \mathscr{R} A\right)^{-1} \mathscr{R}\right] C. \tag{24}$$

It follows from (20) that

$$\mathscr{L}\left[\widehat{C}_K(t)\right](\lambda) x = \lambda \mathscr{L}[K(t)](\lambda)\left(\lambda^2 - A \mathscr{R}\right)^{-1} C x. \tag{25}$$

Thus, by Definition 1, we know that $A\mathscr{R}$ subgenerates an exponentially bounded K-convoluted C-cosine operator family on X.

(2) Assume that $A\mathscr{R}$ subgenerates an exponentially bounded K-convoluted C-cosine operator family on X and $\rho(\mathscr{R}A) \neq \emptyset$, and let

$$\begin{gathered}
\lambda_0 \in \rho(\mathscr{R} A), \\
E = \left(\lambda_0 - \mathscr{R} A\right) \mathscr{R}, \qquad F = A\left(\lambda_0 - \mathscr{R} A\right)^{-1}.
\end{gathered} \tag{26}$$

It is not hard to see that E is closed operator on X and

$$F \in L(X), \qquad CE \subseteq EC, \qquad FC = CF. \tag{27}$$

Since $FE = A\mathscr{R}$ subgenerates an exponentially bounded K-convoluted C-cosine operator family on X, we know from (1) that the operator $EF = \mathscr{R}A$ subgenerates an exponentially bounded K-convoluted C-cosine operator family on X.

Noting that $\rho(\mathscr{R}A) \neq \emptyset$ and in view of Lemma 7, we see that $\mathscr{R}A$ generates an exponentially bounded K-convoluted C-cosine operator family on X. □

Theorem 9. *Let A be a subgenerator of an exponentially bounded K-convoluted C-cosine operator family $\{C_K(t)\}_{t\geq 0}$ on X,*

$$S_K(t) = \int_0^t C_K(s)\,ds, \quad t \geq 0, \tag{28}$$

$B \in L(X)$, and $R(B) \subset R(C)$. Suppose that

(H1) *there exists an operator $\mathscr{F} : X \to X$ such that*

$$\mathscr{F} S_K(t)\,x := G_K(t)\,x \in \mathbf{C}([0,\infty), X) \tag{29}$$

is Laplace transformable, and

$$\mathscr{L}(G_K)(\lambda) = \left(\lambda^2 - A\right)^{-1} Cx, \quad x \in X; \tag{30}$$

(H2) *for any $\Phi \in \mathbf{C}([0,\infty), X)$, $\int_0^t G_K(t-s)C^{-1}B\Phi(s)ds \in D(A)$, and*

$$\left\| A \int_0^t G_K(t-s)\,C^{-1}B\Phi(s)\,ds \right\| \leq \widetilde{M} \int_0^t e^{\omega(t-s)} \|\Phi(s)\|\,ds, \quad t \geq 0, \tag{31}$$

where $\widetilde{M}$ is a constant;

(H3) *there exists an injective operator $C_1 \in L(X)$ such that $R(C_1) \subset R(C)$ and $C_1A(I+B) \subset A(I+B)C_1$.*

Then,

(1) *$A(I+B)$ subgenerates an exponentially bounded K-convoluted C_1-cosine operator family,*

(2) *if $\rho(A) \neq \emptyset$, then $A(I+B)$ generates an exponentially bounded K-convoluted C_1-cosine operator family;*

(3) *if $\rho((I+B)A) \neq \emptyset$ and $BC_1 = C_1B$, $C_1A \subseteq AC_1$, then $(I+B)A$ generates an exponentially bounded K-convoluted C_1-cosine operator family on X.*

Proof. (1) For each $x \in X, t \geq 0$, define

$$\overline{C}_0(t)\,x = C_K(t)\,x,$$
$$\overline{C}_n(t)\,x = A\int_0^t G_K(t-s)\,C^{-1}B\overline{C}_{n-1}(s)\,x\,ds, \quad n = 1, 2, \ldots. \tag{32}$$

Then, the operator family $\{\overline{C}_n(t)\}_{t\geq 0}$ has the following properties:

(i) for any $x \in X$, $\overline{C}_n(t)x \in \mathbf{C}([0,\infty), X)$;

(ii) $\|\overline{C}_n(t)\| \leq (M\widetilde{M}^n t^n/n!)e^{\omega t}$, $t \geq 0, \forall n \in \mathbb{N}_0$.

Therefore, the following series

$$\sum_{n=0}^{\infty} \overline{C}_n(t)\,C^{-1}C_1, \quad t \geq 0, \tag{33}$$

is uniformly convergent on every compact interval in t, and we set

$$h(t) = \sum_{n=0}^{\infty} \overline{C}_n(t)\,C^{-1}C_1, \quad t \geq 0. \tag{34}$$

Clearly,

$$\|h(t)\| \leq M_1 e^{(\omega+\widetilde{M})t}, \quad t \geq 0, \tag{35}$$

where $M_1 = M\left\|C^{-1}C_1\right\|$, and

$$t \longrightarrow h(t)\,x \text{ is continuous on } [0,\infty) \text{ for any } x \in X. \tag{36}$$

Moreover,

$$h(t)\,x = C_K(t)\,C^{-1}C_1x + A\int_0^t G_K(t-s)\,C^{-1}Bh(s)\,x\,ds, \quad x \in X,\ t \geq 0. \tag{37}$$

As in the monograph [15], we write, for sufficiently large λ,

$$\mathscr{L}[h(t)](\lambda)\,x = \int_0^{\infty} e^{-\lambda t} h(t)\,x\,dt, \quad x \in X. \tag{38}$$

Thus, by (5), we have

$$\mathscr{L}[h(t)](\lambda)\,x = \lambda\mathscr{L}[K(t)](\lambda)\left(\lambda^2 - A\right)^{-1}C_1x + A\left(\lambda^2 - A\right)^{-1}B\mathscr{L}[h(t)](\lambda)\,x, \quad x \in X. \tag{39}$$

This implies that

$$\begin{aligned} &R((I+B)\,\mathscr{L}[h(t)](\lambda)) \subseteq D(A), \\ &\left(\lambda^2 - A(I+B)\right)\mathscr{L}[h(t)](\lambda)\,x \\ &\quad = \lambda^2\mathscr{L}[h(t)](\lambda)\,x - \lambda\mathscr{L}[K(t)](\lambda)\,A\left(\lambda^2 - A\right)^{-1}C_1x \\ &\qquad - \lambda^2 A\left(\lambda^2 - A\right)^{-1}B\mathscr{L}[h(t)](\lambda)\,x \\ &\quad = \lambda\mathscr{L}[K(t)](\lambda)\,C_1x, \quad x \in X. \end{aligned} \tag{40}$$

Let

$$U(t)x = A\int_0^t G_K(t-s)C^{-1}Bxds, \quad x \in X,\ t \geq 0. \tag{41}$$

Then, for large λ, we have

$$\|\lambda\mathscr{L}[U(t)](\lambda)\| = \left\|\lambda\int_0^\infty e^{-\lambda t}U(t)\,dt\right\| \leq \frac{\widetilde{M}}{\lambda-\omega}. \tag{42}$$

So, for sufficiently large λ,

$$\|\lambda\mathscr{L}[K(t)](\lambda)\| = \left\|A\left(\lambda^2-A\right)^{-1}B\right\| < 1. \tag{43}$$

This means that the operator $I - A(\lambda^2-A)^{-1}B$ is invertible.

On the other hand, since $\lambda^2 - A$ and $I - A(\lambda^2-A)^{-1}B$ are injective, and

$$\left(\lambda^2-A\right)\left(I-A\left(\lambda^2-A\right)^{-1}B\right)x = \left(\lambda^2-A(I+B)\right)x, \quad x \in D(A(I+B)), \tag{44}$$

we infer that $\lambda^2 - A(I+B)$ is injective. This together with (40) implies that

$$\lambda\left(\lambda^2-A(I+B)\right)^{-1}C_1x = \frac{1}{\mathscr{L}[K(t)](\lambda)}\int_0^\infty e^{-\lambda t}h(t)\,xdt. \tag{45}$$

By Definition 1, we know that $A(I+B)$ subgenerates an exponentially bounded K-convoluted C_1-cosine operator family on X.

(2) By the proof of (1), we see that the operator $I - A(\lambda^2 - A)^{-1}B$ is invertible, and $\rho(A) \neq \emptyset$ implies that

$$\rho(A(I+B)) \neq \emptyset. \tag{46}$$

In view of Lemma 7, we get

$$C_1^{-1}A(I+B)C_1 = A(I+B). \tag{47}$$

(3) By virtue of Theorem 8 (2), we have the conclusion. □

Remark 10. (1) It is easy to see that if we take

$$\mathscr{F}S_K(t)x := \left(\mathscr{L}^{-1}\left(\frac{1}{\mathscr{L}[K(t)](\lambda)}\right) * S_K\right)(t)x, \tag{48}$$

then (H1) is satisfied.

(2) In Theorem 9, if we take

$$K(t) = \frac{t^{n-1}}{\Gamma(n)}, \qquad \mathscr{F} := \frac{d^n}{dt^n}, \quad n \in \mathbb{N}, \tag{49}$$

then we obtain the perturbations for n-times integrated C-cosine operator families.

(3) In Theorem 9, if we take

$$K(t) \equiv \frac{1}{t} \quad (t \neq 0) \tag{50}$$

and $\mathscr{F} := I$, then we have the multiplicative perturbations on the exponentially bounded C-cosine operator families.

By Theorem 9, we can immediately deduce the following theorem on K-convoluted C-semigroups.

Theorem 11. *Let A be a subgenerator of an exponentially bounded K-convoluted C-semigroup $\{T_K(t)\}_{t\geq 0}$ on X, $B \in L(X)$ and $R(B) \subset R(C)$. Suppose that*

(H1) *there exists an operator $\mathscr{F} : X \to X$ such that*

$$\mathscr{F}T_K(t)x := H_K(t)x \in \mathbf{C}([0,\infty), X) \tag{51}$$

is Laplace transformable, and

$$\mathscr{L}(H_K)(\lambda) = (\lambda - A)^{-1}Cx, \quad x \in X; \tag{52}$$

(H2) *for any $\Phi \in \mathbf{C}([0,\infty), X)$, $\int_0^t H_K(t-s)C^{-1}B\Phi(s)ds \in D(A)$, and*

$$\left\|A\int_0^t H_K(t-s)C^{-1}B\Phi(s)\,ds\right\| \leq \overline{M}\int_0^t e^{\omega(t-s)}\|\Phi(s)\|\,ds, \quad t \geq 0, \tag{53}$$

where $\overline{M}$ is a constant;

(H3) *there exists an injective operator $C_1 \in L(X)$ such that $R(C_1) \subset R(C)$ and $C_1A(I+B) \subset A(I+B)C_1$.*

Then,

(1) *$A(I+B)$ subgenerates an exponentially bounded K-convoluted C_1-semigroup on X;*

(2) *if $\rho(A) \neq \emptyset$, then $A(I+B)$ generates an exponentially bounded K-convoluted C_1-semigroup on X.*

(3) *if $\rho((I+B)A) \neq \emptyset$, then $(I+B)A$ generates an exponentially bounded K-convoluted C_1-semigroup on X.*

Remark 12. (1) In Theorem 11, if we take

$$K(t) := \frac{t^{n-1}}{\Gamma(n)}, \qquad \mathscr{F} := \frac{d^n}{dt^n}, \quad n \in \mathbb{N}, \tag{54}$$

then we obtain the perturbations for n-times integrated C-semigroups.

(2) In Theorem 11, if we take

$$K(t) := \frac{1}{t} \quad (t \neq 0) \tag{55}$$

and $\mathscr{F} := I$, then we have the multiplicative perturbations on the exponentially bounded C-semigroups.

3. Additive Perturbation Theorem

Theorem 13. *Let $B \in L(X)$, $R(B) \subset R(C)$, and there exists an injective operator $C_1 \in L(X)$ such that $R(C_1) \subset R(C)$ and $C_1(A+B) \subset (A+B)C_1$.*

(i) *Suppose that A is a subgenerator of an exponentially bounded K-convoluted C-cosine operator family $\{C_K(t)\}_{t\geq 0}$ on X. If there exists an operator $\mathscr{F} : X \to X$ such that*

$$\mathscr{F}C_K(t)x := G_K(t)x \in \mathbf{C}([0,\infty), X) \tag{56}$$

is Laplace transformable, and

$$\mathscr{L}(G_K)(\lambda) = \left(\lambda^2 - A\right)^{-1} Cx, \quad x \in X, \tag{57}$$

then $A+B$ subgenerates an exponentially bounded K-convoluted C_1-cosine operator family $\{\widehat{C}_K(t)\}_{t\geq 0}$ on X, where

$$\widehat{C}_K(t)x = C_K(t)C^{-1}C_1 x + \int_0^t S_K(t-s)C^{-1}B\widehat{C}_K(t)x\,ds, \quad t \geq 0,\ x \in X,$$
$$S_K(t)x = \int_0^t C_K(s)x\,ds, \quad t \geq 0,\ x \in X. \tag{58}$$

(ii) *Suppose that A is a subgenerator of an exponentially bounded K-convoluted C-semigroup $\{T_K(t)\}_{t\geq 0}$ on X. If there exists an operator $\mathscr{F}: X \rightarrow X$ such that*

$$\mathscr{F}T_K(t)x := H_K(t)x \in \mathbf{C}([0,\infty), X) \tag{59}$$

is Laplace transformable, and

$$\mathscr{L}(H_K)(\lambda) = (\lambda - A)^{-1} Cx, \ x \in X, \tag{60}$$

then $A+B$ subgenerates an exponentially bounded K-convoluted C_1-semigroup $\{\widehat{T}_K(t)\}_{t\geq 0}$ on X, where

$$\widehat{T}_K(t)x = T_K(t)C^{-1}C_1 x + \int_0^t T_K(t-s)C^{-1}B\widehat{T}_K(s)x\,ds, \quad t \geq 0,\ x \in X. \tag{61}$$

Proof. Replacing (37) with the following equality:

$$h(t)x = C_K(t)C^{-1}C_1 x + \int_0^t G_K(t-s)C^{-1}Bh(s)x\,ds, \quad x \in X,\ t \geq 0, \tag{62}$$

and by the arguments similar to those in the proof of Theorem 9, we can prove (i).

Point (ii) can also be deduced by a similar way. □

Remark 14. In Theorem 13, if we take

$$K(t) = \frac{t^{n-1}}{\Gamma(n)}, \qquad \mathscr{F} := \frac{d^n}{dt^n}, \quad n \in \mathbb{N}_0, \tag{63}$$

then we obtain an additive perturbation theorem for the exponentially bounded n-times integrated C_1-cosine operator families (resp., n-times integrated C_1-semigroups) as well as C_1-cosine operator families (resp., 0-times integrated C_1-semigroup).

4. Examples

Example 1. Let

$$\begin{aligned} X &:= C_0(\mathbb{R}) \oplus C_0(\mathbb{R}) \oplus C_0(\mathbb{R}), \\ A(f,g,h)(\cdot) &:= \left(f', g', \left(\chi_{[0,\infty)} - \chi_{(-\infty,0]}\right)h\right), \end{aligned} \tag{64}$$

where

$$\begin{aligned} &(f,g,h) \in D(A) \\ &= \left\{(f,g,h) \in X : f' \in C_0(\mathbb{R}), g' \in C_0(\mathbb{R}), h(0) = 0\right\}, \\ &C(f,g,h) := (f, g, \sin(\cdot)h(\cdot)), \quad f,g,h \in C_0(\mathbb{R}). \end{aligned} \tag{65}$$

As in [22, Examples 8.1 and 8.2], we can prove that A is the generator of an exponentially bounded once integrated C-semigroup ([15]).

Define

$$\begin{aligned} &B(f,g,h)(t) \\ &= \left(e^{-t}\cos t\int_0^t f(s)\,ds,\ e^{-2t}\cos 2t \times \int_0^t g(s)\,ds,\ te^{-6t}\sin t\cdot h(t)\right), \end{aligned} \tag{66}$$

for every $t \in \mathbb{R}$, and $f,g,h \in C_0(\mathbb{R})$. Then, we can simply verify $B \in L(X)$, $R(B) \subset C(D(A))$, and

$$BC(f,g,h) = CB(f,g,h), \quad (f,g,h) \in X. \tag{67}$$

Therefore, taking

$$K(t) \equiv 1, \qquad \mathscr{F} := \frac{d}{dt} \tag{68}$$

and using Remark 12 (1), we know that $A(I+B)$ subgenerates an exponentially bounded once integrated C-semigroup on X.

Example 2. Let $X_1 = L^\infty(\mathbb{R})$, $X_2 = L^2(\mathbb{R})$,

$$\begin{aligned} A_1 &= \frac{d^2}{d\xi^2}, \qquad D(A_1) = W^{2,\infty}(\mathbb{R}), \\ A_2 &= \frac{d^2}{d\xi^2}, \qquad D(A_2) = H^2(\mathbb{R}). \end{aligned} \tag{69}$$

It follows from [15] that A_1 generates an exponentially bounded C_1-cosine operator family $C_1(\cdot)$ on X_1, where $C_1 = (1 - d^2/d\xi^2)^{-1}$. Moreover, it is well known that A_2 generates a strongly continuous cosine operator family $C_2(\cdot)$ on X_2.

Let

$$b_1(\cdot) \in W^{4,\infty}(\mathbb{R}), \qquad b_2(\cdot) \in H^2(\mathbb{R}), \tag{70}$$

and define $B_1: X_2 \rightarrow X_1$, $B_2: X_1 \rightarrow X_2$ as follows:

$$\begin{aligned} (B_1\phi)(\xi) &= b_1(\xi)\int_0^1 \phi(\sigma)\,d\sigma, \\ (B_2\phi)(\xi) &= b_2(\xi)\int_0^1 \phi(\sigma)\,d\sigma. \end{aligned} \tag{71}$$

Set $X = X_1 \times X_2$,

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \qquad D(A) := D(A_1) \times D(A_2),$$
$$B = \begin{pmatrix} 0 & B_1 \\ B_2 & 0 \end{pmatrix}, \qquad D(B) := X. \tag{72}$$

Clearly, $\rho(A) \neq \emptyset$ and $D(A_1) = R(C_1)$. Take

$$\lambda_0 \in \rho(A), \qquad C = (\lambda_0 - A)^{-1}. \tag{73}$$

Then, A generates an exponentially bounded C-cosine operator family $C(\cdot)$ on X, where

$$C(t) = \begin{pmatrix} C_1(t) C_1^{-1}(\lambda_0 - A_1)^{-1} & 0 \\ 0 & C_2(t)(\lambda_0 - A_2)^{-1} \end{pmatrix}. \tag{74}$$

Hence,

$$S(t) = \begin{pmatrix} S_1(t) C_1^{-1}(\lambda_0 - A_1)^{-1} & 0 \\ 0 & S_2(t)(\lambda_0 - A_2)^{-1} \end{pmatrix}, \tag{75}$$

where

$$S(t) := \int_0^t C(s)\,ds, \qquad S_1(t) := \int_0^t C_1(s)\,ds, \tag{76}$$

$$S_2(t) := \int_0^t C_2(s)\,ds. \tag{77}$$

Therefore, we have, for each $f = \binom{f_1}{f_2} \in \mathbf{C}([0,\infty), X)$,

$$\int_0^t S(t-s) C^{-1} B f(s)\,ds = \begin{pmatrix} A_1 \int_0^t S_1(t-s) C_1^{-1} B_1 f_2(s)\,ds \\ A_2 \int_0^t S_2(t-s) B_2 f_1(s)\,ds \end{pmatrix}. \tag{78}$$

Since

$$R(B_1) \subset D(A_1 C_1^{-1}), \qquad R(B_2) \subset D(A_2), \tag{79}$$

we see that there exist $M, \omega > 0$ such that

$$\left\| \int_0^t S(t-s) C^{-1} B f(s)\,ds \right\| \leq M \int_0^t e^{\omega(t-s)} \|f(s)\|\,ds, \quad t \geq 0. \tag{80}$$

Consequently, if there exists an injective operator $\widetilde{C} \in L(X)$ such that $R(\widetilde{C}) \subset R(C)$ and $\widetilde{C}(A+B) \subset (A+B)\widetilde{C}$, then taking

$$K(t) \equiv \frac{1}{t}, \quad \mathscr{F} := I \tag{81}$$

and using Remark 14, we know that $A + B$ subgenerates a $\widetilde{C}$-cosine operator family on X.

Moreover, it is not hard to see that there exist $\widehat{M}, \omega > 0$ such that

$$\left\| A \int_0^t S(t-s) C^{-1} B f(s)\,ds \right\| \leq \widehat{M} \int_0^t e^{\omega(t-s)} \|f(s)\|\,ds, \quad t \geq 0. \tag{82}$$

Hence, if there exists an injective operator $\widehat{C} \in L(X)$ such that $R(\widehat{C}) \subset R(C)$ and $\widehat{C}A(I+B) \subset A(I+B)\widehat{C}$, then by Remark 10 (3) ($\rho(A) \neq \emptyset$), we know that $A(I + B)$ generates a $\widehat{C}$-cosine operator family on X.

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Research Article

Multiplicity and Bifurcation of Solutions for a Class of Asymptotically Linear Elliptic Problems on the Unit Ball

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This paper mainly dealt with the exact number and global bifurcation of positive solutions for a class of semilinear elliptic equations with asymptotically linear function on a unit ball. As byproducts, some existence and multiplicity results are also obtained on a general bounded domain.

1. Introduction

In this paper, we are concerned with positive solutions of the following elliptic equation subject to homogeneous Dirichlet boundary condition

$$\begin{aligned} -\Delta u &= \lambda f(u), \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned} \qquad (P_\lambda)$$

where Ω is a smooth bounded domain in R^N, λ is a positive parameter, $u \in C^2(\Omega) \cap C(\overline{\Omega})$, and the function f satisfies the following.

(F1) $f : [0, +\infty) \to (0, +\infty)$ is a positive C^1 function, and f is strictly convex; that is, $f'(t)$ is strictly increasing in $t \in (0, \infty)$.

(F2) f is asymptotically linear, that is,

$$\lim_{t \to \infty} \frac{f(t)}{t} = a \in (0, +\infty). \qquad (1)$$

For the past years, this problem attracted attentions of many authors. It was studied in [1–4] with f being strictly increasing and was studied in [5–7] with a specific function $f(u) = \sqrt{(u-b)^2 + \epsilon}$ which is not increasing.

The main goal of this paper is to study the exact number and bifurcation structure of the solutions of (P_λ) on a unit ball Ω, with a general asymptotically linear function f. Some results in this paper (see Section 3) can be viewed as an extension and improvement of that in [7], but the argument approach here is very different to that in [7]. As byproducts, we also get some new results which also hold for general domain Ω (see Section 2). The paper is organized as follows. In Section 2, we study the existence and multiplicity of solutions for problem (P_λ) on a general bounded domain, with some new results complementing those existing in the literature. In Section 3, we study the exact number and global bifurcation structure of positive solutions of (P_λ) on a unit ball.

2. Multiplicity of Positive Solutions on a General Domain

Throughout this section, we assume that Ω is a smooth bounded domain in R^N, and f satisfies (F1) and (F2). We also note that, by maximum principle, all solutions of (P_λ) are positive on Ω.

Before the statement of our main result, we derive some preliminary lemmas. Though some of them may be known, we provide their proofs for reader's convenience.

Lemma 1. *For any $\lambda \in (0, \lambda_1/a)$, (P_λ) is solvable.*

Proof. Consider the functional

$$J_\lambda(u) = \int_\Omega \left(\frac{|\nabla u|^2}{2} - \lambda F(u) \right) dx, \tag{2}$$

where $F(u) = \int_0^u f(t)dt$.

From (F1) and (F2), it is easy to see that

$$f'(t) < a, \tag{3}$$

so

$$F(u) \le \frac{au^2}{2} + f(0)u. \tag{4}$$

Poincàre's inequality $\int_\Omega u^2 \le (1/\lambda_1)\int_\Omega |\nabla u|^2$, and the imbedding theorem of $L^2(\Omega)$ to $L^1(\Omega)$ yield

$$\begin{aligned} J_\lambda(u) &\ge \int_\Omega \frac{|\nabla u|^2}{2} dx - \frac{a\lambda}{2}\int_\Omega u^2 dx - \lambda f(0)\int_\Omega u\, dx \\ &\ge \int_\Omega \frac{|\nabla u|^2}{2} dx - \frac{a\lambda}{2\lambda_1}\int_\Omega |\nabla u|^2 dx - \lambda f(0)\int_\Omega u\, dx \\ &\ge \frac{1}{2}\left(1 - \frac{a\lambda}{\lambda_1}\right)\int_\Omega |\nabla u|^2 dx - \lambda f(0)\, C\left(\int_\Omega |\nabla u|^2\right)^{1/2} dx, \end{aligned} \tag{5}$$

so $J_\lambda(u) \to \infty$ as $\|u\|_{H_0^1(\Omega)} \to \infty$, where $\|u\|_{H_0^1(\Omega)} = (\int_\Omega |\nabla u|^2)^{1/2} dx$, and then $J_\lambda(u)$ is coercive and bounded from below. It is also easy to see that $J_\lambda(u)$ is weakly lower semi-continuous [8, page 446, Theorem 1]. By applying direct variational methods [9, page 4, Theorem 1.2], we can get the desired result; that is, $\min_{u \in H_0^1(\Omega)} J_\lambda(u)$ is reached at some point $u(\lambda)$, and $u(\lambda)$ is a solution of (P_λ) when $\lambda \in (0, \lambda_1/a)$. □

Lemma 2. *For any $\lambda > \lambda_1/m$, (P_λ) has no solution, where $m = \inf_{t>0}(f(t)/t)$.*

Proof. If not, assume that u is a solution of (P_λ) for some $\lambda > \lambda_1/m$. Multiplying (P_λ) by $\varphi_1 > 0$, the normalized positive eigenfunction with respect to the first eigenvalue λ_1 of $-\Delta$ subject to homogenous Dirichlet boundary condition, and then integrating by parts, we get

$$\begin{aligned} \lambda_1 \int_\Omega u\varphi_1 dx &= \int_\Omega -\Delta u \varphi_1 dx \\ &= \lambda \int_\Omega f(u)\varphi_1 dx > \lambda_1 \int_\Omega u\varphi_1 dx, \end{aligned} \tag{6}$$

which is a contradiction. □

We begin by show the following.

Lemma 3. *There exists a number $\lambda_1/a \le \Lambda \le \lambda_1/m$, such that (P_λ) has at least a solution for $\lambda < \Lambda$ and has no solution for $\lambda > \Lambda$.*

Proof. Let

$$\Lambda = \{\lambda : (P_\lambda) \text{ has a solution}\}. \tag{7}$$

By Lemmas 1 and 2, $\lambda_1/a \le \Lambda \le \lambda_1/m$. We need just to prove that if (P_μ) has a solution, then (P_λ) also has a solution for all $0 < \lambda < \mu$. This can be done by a simple argument of sub-sup solution method, since it is easy to see that any solution of (P_μ) is a super solution of (P_λ) and $u \equiv 0$ a subsolution.

It is easy to see that $u_* \equiv 0$ is a subsolution of (P_λ), then a standard sub-super solution method's argument and comparison theorems give the following lemma. □

Lemma 4. *If (P_λ) is solvable, then one has a minimal solution u_λ, that is, for any solution v of (P_λ), $u_\lambda \le v$. Moreover, u_λ is increasing with respect to λ.*

Lemma 5. *If $\lambda \in (0, \lambda_1/a)$, then the solution of (P_λ) is unique.*

Proof. Suppose that v_1 and v_2 are solutions of (P_λ). Let $v = v_1 - v_2$, then

$$\begin{gathered} -\Delta v = \lambda\left[f(v_1) - f(v_2)\right], \quad \text{in } \Omega, \\ v = 0, \quad \text{on } \partial\Omega. \end{gathered} \tag{8}$$

By mean value theorem, v satisfies

$$-\Delta v = f'(\overline{v})\, v, \tag{9}$$

where $\overline{v}$ lies between v_1 and v_2. Multiplying v and integrating, we get

$$\begin{aligned} \int_\Omega |\nabla v|^2 dx &= \lambda \int_\Omega f'(\overline{v})\, v^2 dx \\ &\le a\lambda \int_\Omega v^2 dx \le \frac{a\lambda}{\lambda_1}\int_\Omega |\nabla v|^2 dx, \end{aligned} \tag{10}$$

which implies that $v \equiv 0$. The proof is complete. □

Lemma 6. *The minimal solution u_λ is stable, that is, $\lambda_1(-\Delta - \lambda f'(u_\lambda)) \ge 0$, where $\lambda_1(-\Delta - \lambda f'(u_\lambda))$ denotes the first eigenvalue of the following problem:*

$$\begin{gathered} -\Delta w - \lambda f'(u_\lambda)\, w = \mu w, \quad \text{in } \Omega, \\ w = 0, \quad \text{on } \partial\Omega. \end{gathered} \tag{11}$$

Proof. Suppose on the contrary that $\lambda_1(-\Delta - \lambda f'(u_\lambda)) = \mu < 0$, and $w > 0$ is the corresponding eigenvector. Let $v_\varepsilon = u_\lambda - \varepsilon\varphi$, then by (P_λ) and (11), we have

$$\begin{aligned} -\Delta v_\varepsilon - \lambda f(v_\lambda) &= \lambda f(u_\lambda) - \lambda\varepsilon f'(u_\lambda)\varphi \\ &\quad - \lambda f(u_\lambda - \varepsilon\varphi) - \mu\varepsilon\varphi \\ &= -\mu\varepsilon\varphi + o(\varepsilon\varphi) > 0, \end{aligned} \tag{12}$$

when ε is small enough, and hence $v_\varepsilon = u_\lambda - \varepsilon\varphi$ is a super solution of problem (P_λ). On the other hand, 0 is a subsolution of (P_λ), and Hopf's boundary lemma implies that

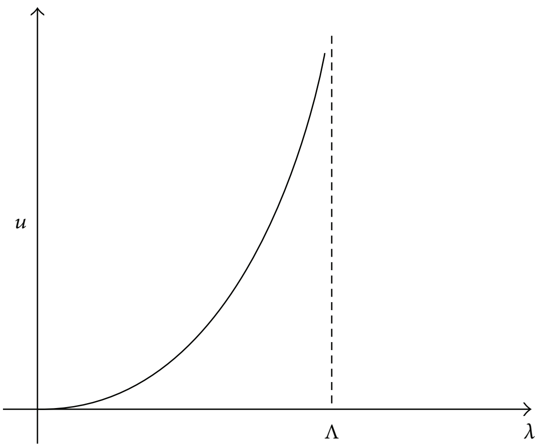

Figure 1: Diagram for $\Lambda = \lambda_1/a$.

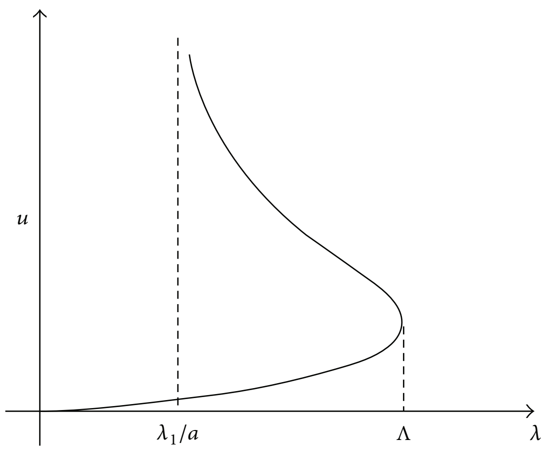

Figure 2: Minimal diagram for $\Lambda > \lambda_1/a$.

$0 < v_\varepsilon$ for $\varepsilon > 0$ small. An application of sub-sup solution method guarantees that there is a solution $\overline{u}$ of (P_λ) satisfying $0 < \overline{u} \le u_\lambda - \varepsilon\varphi$ in Ω, which is a contradiction with the minimality of u_λ. The proof is complete. □

Now we state our main result.

Theorem 7. *Suppose that f satisfies (F1) and (F2), then there exists $\Lambda \in [\lambda_1/a, \lambda_1/m]$ (where $m = \inf_{t>0}(f(t)/t)$) such that problem (P_λ)*

(i) *has at least one solution for $\lambda \in (0, \Lambda)$ and a unique solution for $\lambda \in (0, \lambda_1/a)$;*

(ii) *has no solution for $\lambda \in (\Lambda, +\infty)$;*

(iii) (a) *if $\Lambda = \lambda_1/a$, then problem (P_λ) has no solution at $\lambda = \Lambda$, and $\lim_{\lambda\to\Lambda-0} u_\lambda(x) = +\infty$ for all $x \in \Omega$, where u_λ denotes the unique solution of (P_λ) for $\lambda \in (0, \Lambda)$ (see Figure 1),*

(b) *if $\Lambda > \lambda_1/a$, then problem (P_λ) has a unique solution for $\lambda \in (0, \lambda_1/a]$ and $\lambda = \Lambda$, has at least two solutions for $\lambda \in (\lambda_1/a, \Lambda)$ (see Figure 2 for a minimal diagram).*

Proof. Statement (i) follows from Lemmas 3 and 5. Statement (ii) follows from Lemma 3. Now we give the proof of statement (iii).

(a) Suppose $\Lambda = \lambda_1/a$. The solution (P_λ) bifurcates at infinity near $\Lambda = \lambda_1/a$ (see [2, 10] for details). On the other hand, (P_λ) has a unique solution u_λ for $\lambda \in (0, \lambda_1/a)$, and no solution for $\lambda > \lambda_1/a$. Therefore the bifurcation curve from infinity is on the left of $\lambda = \lambda_1/a$, and hence $\lim_{\lambda\to\Lambda-0} u_\lambda(x) = +\infty$ for all $x \in \Omega$ by the expression of the bifurcation solution in Theorem 13 in Section 3.

If (P_Λ) has a solution, let u_Λ denote the minimal solution of (P_λ). By Lemma 4, $u_\lambda \le u_\Lambda$ for $\lambda \in (0, \Lambda)$, contradicting $\lim_{\lambda\to\Lambda-0} \| u_\lambda \|_\infty = \infty$.

(b) For clarity, the proof will be divided into 3 steps.

Step 1. The existence and uniqueness of solutions of (P_λ) for $\lambda = \lambda_1/a$.

The existence follows directly from Lemma 4. Note that $f' < a$, and the uniqueness can be proved in a similar way as in the proof of Lemma 5.

Step 2. The existence and uniqueness of solutions of (P_λ) for $\lambda = \Lambda$.

By Lemmas 3 and 4, (P_λ) has a minimal solution u_λ for any $\lambda \in (0, \Lambda)$, and u_λ is increasing in λ. Let $(\lambda_n) \subset (\lambda_1/a, \Lambda)$ be any sequence such that $\lim_{n\to\infty} \lambda_n = \Lambda$. Firstly we insure that case (u_{λ_n}) is $L^2(\Omega)$ bounded. Suppose the contrary that $\lim_{n\to\infty} \| u_{\lambda_n} \|_{L^2(\Omega)} = \infty$. Let $c_n = \| u_{\lambda_n} \|_{L^2(\Omega)}$ and $v_{\lambda_n} = u_{\lambda_n}/c_n$, then

$$\begin{gathered} -\Delta v_{\lambda_n} = \frac{\lambda_n}{c_n} f\left(c_n v_{\lambda_n}\right), \quad \text{in } \Omega, \\ v_{\lambda_n} = 0, \quad \text{on } \partial\Omega. \end{gathered} \tag{13}$$

Since $f(c_n v_{\lambda_n})/c_n$ is bounded in $L^2(\Omega)$, it follows from (13) that v_{λ_n} is bounded in $H_0^1(\Omega)$. Then subject to a subsequence, we may suppose that there exits v^*, such that

$$\begin{gathered} v_{\lambda_n} \rightharpoonup v^* \quad \text{weakly in } H_0^1(\Omega), \\ v_{\lambda_n} \longrightarrow v^* \quad \text{strongly in } L^2(\Omega), \\ v_{\lambda_n} \longrightarrow v^* \quad \text{a.e. in } \Omega. \end{gathered} \tag{14}$$

Then by letting $n \to \infty$, we get from (13) in the weak sense that

$$\begin{gathered} -\Delta v^* = a\Lambda v^*, \quad \text{in } \Omega, \\ v^* = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{15}$$

with $\| v^* \|_{L^2(\Omega)} = 1$, and $v^* > 0$ by strong maximum principle. Hence $a\Lambda = \lambda_1$, that is, $\Lambda = \lambda_1/a$, a desired contradiction.

Now in a similar way, the boundedness of (u_{λ_n}) in $L^2(\Omega)$ implies that (u_{λ_n}) is bounded in $H_0^1(\Omega)$. Then subject to a subsequence, we may suppose that there exits u^*, such that

$$\begin{gathered} u_{\lambda_n} \rightharpoonup u^* \quad \text{weakly in } H_0^1(\Omega), \\ u_{\lambda_n} \longrightarrow u^* \quad \text{strongly in } L^2(\Omega), \\ u_{\lambda_n} \longrightarrow u^* \quad \text{a.e. in } \Omega. \end{gathered} \tag{16}$$

Then by letting $n \to \infty$, we get

$$-\Delta u^* = \Lambda f(u^*), \quad \text{in } \Omega, \qquad u^* = 0, \quad \text{on } \partial\Omega, \tag{17}$$

and the existence is proved.

Now we prove the uniqueness. Let u_Λ be the minimal solution of (P_Λ) and $\overline{u}$ a different solution. Then $w := \overline{u} - u_\Lambda > 0$ satisfies

$$-\Delta v = \Lambda f'(u_\Lambda + \theta w) w, \quad \text{in } \Omega, \qquad v = 0, \quad \text{on } \partial\Omega, \tag{18}$$

where $\theta : \Omega \to \mathbb{R}$ satisfying $0 < \theta < 1$. It follows that $\lambda_1(-\Delta - \Lambda f'(u_\Lambda + \theta w)) = 0$, where $\lambda_1(-\Delta - \Lambda f'(u_\Lambda + \theta w))$ denotes the first eigenvalue of the operator $-\Delta - \Lambda f'(u_\Lambda + \theta w)$ subject to the Dirichlet boundary condition, as defined in Lemma 1. Since $f'(u_\Lambda) < f'(u_\Lambda + \theta w)$ in Ω, we have that $\lambda_1(-\Delta - \Lambda f'(u_\Lambda)) > \lambda_1(-\Delta - \Lambda f'(u_\lambda + \theta w)) = 0$, which implies that the operator $-\Delta - \Lambda f'(u_\Lambda)$ is nondegenerate. Then by the Implicit Function Theorem, the solution of (P_λ) forms a cure in a neighborhood of (Λ, u_Λ), which is clearly contradicted to the definition of Λ in (7).

Step 3. Prove that (P_λ) has at least two solutions for $\lambda \in (\lambda_1/a, \Lambda)$.

Following the argument in [5], we prove it by variational method of Nehari type (see [11]). As we have known (Lemma 5), there exists a minimal solution u_λ of (P_λ) when $\lambda \in (\lambda_1/a, \Lambda)$. Now we must look for another solution $u (> u_\lambda)$. Assuming that $u = v + u_\lambda$, with $v > 0$, then v satisfies

$$-\Delta v = \lambda [f(v + u_\lambda) - f(u_\lambda)], \quad \text{in } \Omega, \qquad v = 0, \quad \text{on } \partial\Omega. \tag{19}$$

For convenience, let $g(v) = f(v + u_\lambda) - f(u_\lambda)$ and $G(v) = \int_0^v g(t)dt$, then we have

$$-\Delta v = \lambda g(v), \quad \text{in } \Omega, \qquad v = 0, \quad \text{on } \partial\Omega. \tag{20}$$

Define

$$J_\lambda(v) = \int_\Omega \left(\frac{|\nabla v|^2}{2} - \lambda G(v) \right) dx, \qquad I_\lambda(v) = \int_\Omega \left(|\nabla v|^2 - \lambda v g(v) \right) dx, \tag{21}$$

and the solution manifold

$$M_\lambda = \left\{ v \in H_0^1(\Omega) : v > 0 \text{ in } \Omega, I_\lambda(v) = 0 \right\}. \tag{22}$$

Firstly we show that $M_\lambda \neq \phi$ for any $\lambda \in (\lambda_1/a, \Lambda)$. Let φ_1 be the first eigenfunction of $-\Delta$ in Ω subject to Dirichlet boundary condition and $\int_\Omega \varphi_1^2 dx = 1$, then

$$\begin{aligned} I_\lambda(t\varphi_1) &= \lambda_1 t^2 - \lambda \int_\Omega t\varphi_1 g(t\varphi_1) dx \\ &= t^2 \left(\lambda_1 - \lambda \int_\Omega \frac{\varphi_1 g(t\varphi_1)}{t} dx \right), \end{aligned} \tag{23}$$

$$\lim_{t\to\infty} \int_\Omega \frac{\varphi_1 g(t\varphi_1)}{t} dx = \lim_{t\to\infty} \int_\Omega \varphi_1^2 \cdot \frac{g(t\varphi_1)}{t\varphi_1} dx = a.$$

It follows from (23) that

$$I_\lambda(t\varphi_1) < 0, \tag{24}$$

for sufficiently large t if $\lambda \in (\lambda_1/a, \Lambda)$.

On the other hand, let ω_1 be the eigenfunction with $\int_\Omega \omega_1^2 dx = 1$ of the first eigenvalue μ_1 of the following equation:

$$-\Delta \omega_1 - \lambda f'(u_\lambda)\omega_1 = \mu_1 \omega_1, \quad \text{in } \Omega, \qquad \omega_1 = 0, \quad \text{on } \partial\Omega. \tag{25}$$

Since u_λ is the minimal solution, it follows from Lemmas 4 and 6 that $\mu_1 > 0$. Then

$$\begin{aligned} I_\lambda(s\omega_1) &= s^2 \int_\Omega |\nabla \omega_1|^2 dx - \lambda s \int_\Omega \omega_1 g(s\omega_1) dx \\ &= s^2 \int_\Omega |\nabla \omega_1|^2 dx - \lambda s \int_\Omega \left[f'(u_\lambda) s\omega_1^2 + o(s^2) \right] dx \\ &= s^2 \left[\int_\Omega \left(|\nabla \omega_1|^2 - \lambda f'(u_\lambda)\omega_1^2 \right) dx + o(1) \right] \\ &= s^2 (\mu_1 + o(1)). \end{aligned} \tag{26}$$

Hence $I_\lambda(s\omega_1) > 0$ when s is small enough. Now it is easy to see that M_λ is not empty. In fact, take $w_* = t\varphi_1$ for some large t, and $w^* = s\omega$ for some small $s > 0$, such that

$$I_\lambda(w_*) < 0, \qquad I_\lambda(w^*) > 0, \tag{27}$$

respectively. Define a continuous function G on $[0, 1]$, namely,

$$G(\xi) = I_\lambda(\xi w_* + (1 - \xi) w^*). \tag{28}$$

Then $G(0) > 0$, $G(1) < 0$, and hence there exist $\xi_0 \in (0, 1)$ such that $G(\xi_0) = 0$, that is, $I_\lambda(\xi_0 w_* + (1 - \xi_0)w^*) = 0$, and $M_\lambda \neq \phi$, a desired conclusion.

Since f is convex, $g(v)$ is convex with respect to $v > 0$ such that

$$g(v) = g(v) - g(0) \leq g'(v) v. \tag{29}$$

Integrating (29) with respect to v from 0 to v, we get

$$2G(v) \leq g(v) v. \tag{30}$$

Therefore, on M_λ

$$J_\lambda(v) = \frac{\lambda}{2}\int_\Omega [g(v)v - 2G(v)]\,dx \ge 0, \tag{31}$$

that is, $J_\lambda(v)$ is bounded from below.

And then we obtain a nonminimal positive solution of (P_λ) by using the *Nehari* variational method. The proof is complete. □

Remark 8. The solutions that we get from the above discussion are weak ones, but a standard elliptic regularity argument shows that they are indeed classical solutions.

In view of Theorem 7, we want to know what conditions ensure that $\Lambda = \lambda_1/a$ or $\Lambda > \lambda_1/a$. Following [4], we consider the function $L(t) = at - f(t)$. It is easy to see that $L(t)$ is strictly increasing, and hence $\lim_{t\to\infty} L(t) = L_\infty$ exists (may be $+\infty$). Also note that $L(0) = -f(0) < 0$.

Theorem 9. *If $L_\infty \le 0$, then $\Lambda = \lambda_1/a$; if $L_\infty > 0$, then $\Lambda > \lambda_1/a$.*

Proof. (i) If $L_\infty \le 0$, then $f(t) \ge at$ for all $t \ge 0$. We prove that (P_λ) has no solution and hence $\Lambda = \lambda_1/a$. Suppose the contrary that u is a solution (P_λ) for $\lambda = \lambda_1/a$, then

$$-\Delta u = \frac{\lambda_1}{a} f(u) \ge \lambda_1 u. \tag{32}$$

Let φ be a positive eigenfunction of the first eigenvalue λ of $-\Delta$ on Ω with Dirichlet boundary condition, that is

$$\begin{gathered} \Delta\varphi + \lambda_1\varphi = 0, \quad \text{in } \Omega, \\ \varphi = 0, \quad \text{on } \partial\Omega. \end{gathered} \tag{33}$$

Multiplying (32) by $\varphi > 0$, and integrating by parts, we get

$$\int_\Omega (f(u) - au)\varphi\,dx = 0, \tag{34}$$

which yields that $f(u) = au$, contradicting the fact that $f(0) > 0$.

(ii) If $L_\infty > 0$, we prove that $\Lambda > \lambda_1/a$.

Let $(\lambda(s), u(s))$ be the bifurcation curve as described in Theorem 13 in Section 3, then

$$\begin{gathered} \Delta u(s) + \lambda(s) f(u(s)) = 0, \quad \text{in } \Omega, \\ u(s) = 0, \quad \text{on } \partial\Omega. \end{gathered} \tag{35}$$

It follows from (33) and (35) that

$$\begin{aligned} \lambda(s)\int_\Omega f(u(s))\varphi\,dx &= \lambda_1\int_\Omega u(s)\varphi\,dx \\ &= \frac{\lambda_1}{a}\int_\Omega au(s)\varphi\,dx. \end{aligned} \tag{36}$$

By the fact that $u(s)(x) = s\varphi(x) + z(s)(x) \to \infty$ $(s \to \infty)$ a.e. in Ω, we have

$$\begin{aligned} &\int_\Omega au(s)\varphi\,dx - \int_\Omega f(u(s))\varphi\,dx \\ &\quad = \int_\Omega (au(s) - f(u(s)))\varphi\,dx > 0, \end{aligned} \tag{37}$$

for s sufficiently large. It follows from (36) that $\lambda(s) > \lambda_1/a$ when s is sufficiently large, which means that the bifurcation curve $(\lambda(s), u(s))$ from infinity is on the right of $\lambda = \lambda_1/a$, and hence $\Lambda > \lambda_1/a$ by the definition of Λ in (7). The proof is complete. □

Now we define another function which is also crucial in studying exact multiplicity in the next section. Let

$$K(t) = tf'(t) - f(t), \tag{38}$$

then $K'(t) = tf''(t) > 0$ a.e. in $(0, +\infty)$, and $K(t)$ is strictly increasing, and $K(0) = -f(0) < 0$. Denote

$$\lim_{t\to\infty} K(t) = K_\infty \in (-\infty, +\infty]. \tag{39}$$

Theorem 10. *If $K_\infty \le 0$, then $\Lambda = \lambda_1/a$; if $K_\infty > 0$, then $\Lambda > \lambda_1/a$.*

Proof. If $K_\infty \le 0$, then $(f(t)/t)' = K(t)/t^2 < 0$ for all $t > 0$. It follows that $f(t)/t$ is strictly decreasing and hence $f(t)/t > a$, which implies that $L_\infty \le 0$.

On the other hand, if $K_\infty > 0$, by

$$L(t) - K(t) = t\left(a - f'(t)\right) > 0, \quad \forall t > 0, \tag{40}$$

we get that $L_\infty > 0$. Then the conclusion follows for Theorem 9. □

3. Exact Number and Global Bifurcation of Solutions on a Unit Ball

From Theorem 7, the exact number of solutions (P_λ) is now clear in the case of $\Lambda = \lambda_1/a$; that is, the solution is unique if it exists. On the other hand, it is far from known in general exactly how may solutions of (P_λ) for $\lambda \in (\lambda_1/a, \Lambda)$ if $\Lambda > \lambda_1/a$. Using the bifurcation approach developed in [12–14], and also the idea and techniques developed in [7], we solve this problem on the unit ball under some conditions.

Throughout this section, we suppose that Ω is the unit ball in R^N centered with the origin.

The next remarkable results regarding (P_λ) are due to Gidas et al. [15] and Lin and Ni [16].

Lemma 11. *(1) If f is locally Lipschitz continuous in $[0,\infty)$, then all positive solutions of (P_λ) are radially symmetric, that is, $u(x) = u(r)$, $r = |x|$, and satisfies*

$$\begin{gathered} u'' + \frac{n-1}{r}u' + \lambda f(u) = 0, \quad r \in (0,1), \\ u'(0) = u(1) = 0. \end{gathered} \tag{41}$$

Moreover, $u'(r) < 0$ for all $r \in (0,1]$, and hence $u(0) = \max_{0\le r\le 1} u(r)$.

(2) Suppose $f \in C^1(R)$. If u is a positive solution to (P_λ), and w is a solution of the linearized problem (43) *(if it exists), then w is also radially symmetric and satisfies*

$$\begin{gathered} w'' + \frac{n-1}{r} + \lambda f'(u)w = 0, \quad r \in (0,1), \\ w'(0) = w(1) = 0. \end{gathered} \tag{42}$$

The next lemma also plays a key role in this section.

Lemma 12. *(1) For any $d > 0$, there is at most one $\lambda_d > 0$ such that (P_λ) have a positive solution $u(\cdot)$ with $\lambda = \lambda_d$ and $u(0) = d$.*

(2) Let $T = \{d > 0 : (P_\lambda)$ have a positive solution with $u(0) = d\}$, then T is open; $\lambda(d) = \lambda_d$ is a well-defined continuous function from T to R^+.

Lemma 12 is well known; see, for example, [13, 17, 18]. A simple proof of the first part of the lemma can be found in [14]. Because of Lemma 12, we call $R^+ \times R^+ = \{(\lambda, d) : \lambda > 0, d > 0\}$ the phase space, $\{(\lambda(d), d) : d \in T\}$ the bifurcation curve, and the phase space with bifurcation curve the bifurcation diagram.

We will also need the following theorem of bifurcation from infinity.

Theorem 13 (see [10, 19]). *Suppose $f \in C^1(R)$. Let $\lim_{u\to\infty} f(u)/u = a \in (0,\infty)$ and $\lambda_\infty = \lambda_1/a$. Then all positive solutions of (P_λ) near (λ_∞, ∞) have the form of $(\lambda(s), s\varphi + z(s))$ for $s \in (\delta, \infty)$ and some $\delta > 0$, where φ is a positive eigenfunction of the first eigenvalue λ_1 of $-\Delta$ on Ω subjected to Dirichlet boundary condition, $\lim_{s\to\infty}\lambda(s) = \lambda_\infty$, and $\| z(s)\|_{C^{2,\alpha}(\overline{B}^n)} = o(s)$ as $s \to \infty$.*

To make bifurcation argument work, a crucial thing is the following result.

Let u be a solution of problem (P_λ), then u is called a degenerate solution if the corresponding linearized equation

$$\begin{gathered} -\Delta w = \lambda f'(u)\, w, \quad \text{in } \Omega, \\ w = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{43}$$

has a nontrivial solution.

Now suppose that f satisfies (F1), (F2). As in the end of Section 2, let

$$\begin{gathered} K(t) = tf'(t) - f(t) \\ K_\infty = \lim_{t\to\infty} K(t). \end{gathered} \tag{44}$$

If $K_\infty > 0$, then there exists a unique real number $\beta > 0$, such that

$$\begin{gathered} K(t) < 0 \quad \text{for } t \in [0, \beta); \\ K(t) > 0 \quad \text{for } t \in (\beta, \infty);\ K(\beta) = 0. \end{gathered} \tag{45}$$

Lemma 14. *Suppose that $K_\infty > 0$. If u is a degenerate solution of (P_λ), then $u(0) > \beta$.*

Proof. Suppose the contrary that $u(0) \le \beta$, then

$$K(u) = uf'(u) - f(u) < 0, \quad \text{in } \Omega \setminus \{0\}. \tag{46}$$

Let w be a nontrivial solution of the corresponding linearized equation (43). From (P_λ) and (43), we get

$$0 = \int_\Omega (-\Delta w u + \Delta u w)\, dx = \lambda \int_\Omega \left(uf'(u) - f(u)\right) w\, dx. \tag{47}$$

It appears from (46) and (47) that w must change sign in Ω.

In view of Lemma 11(2), we suppose that $|x| = r_1$ is a maximal zero in $(0, 1)$. We may also suppose that $w(x) > 0$, for all $r_1 < |x| < 1$. Then

$$\begin{aligned} &\int_{\Omega\setminus B(r_1)} (-\Delta w u + \Delta u w)\, dx \\ &\quad = \lambda \int_\Omega \left(uf'(u) - f(u)\right) w\, dx < 0, \end{aligned} \tag{48}$$

where $B(r_1)$ denotes the ball of radius r_1 centered with the origin.

On the other hand, using integration by parts, we have

$$\int_{\Omega\setminus B(r_1)} (-\Delta w u + \Delta u w)\, dx = -\int_{\partial(\Omega\setminus B(r_1))} \frac{\partial w}{\partial \nu} u\, ds > 0. \tag{49}$$

a contradiction. □

Theorem 15. *Suppose that f satisfies (F1)-(F2) with $0 < K_\infty < a\beta$. If u is a degenerate solution of (P_λ), then any nontrivial solution of the corresponding linearized equation (43) does not change sign in Ω.*

Proof. By Lemma 14, $\max_{x\in\overline{\Omega}} u(x) = u(0) > \beta$. In view of Lemma 11, there exists $r^* \in (0, 1)$, such that $u(r^*) = \beta$. Let w be a non-trivial solution of the corresponding linearized equation (43), then $w(0) \ne 0$.

We assert that $w(r)$ has no zeroes on $[r^*, 1)$. Suppose the contrary and let r_1 be the largest zero of w on $[r^*, 1)$. We may suppose that $w > 0$ in $(r_1, 1)$. Note that $u(r) < \beta$ for $r \in (r_1, 1)$, a similar argument as in the proof of Lemma 14 yields a contradiction.

Now we prove that $w(r)$ has no zeroes on $(0, r^*)$. Suppose the contrary and let r_0 be the smallest zero of $w(r)$ on $(0, r^*)$. We may suppose that $w > 0$ in $B(r_0)$. Multiplying (P_λ) by $u - \beta$, (43) by w, subtracting, and integrating on $B(r_0)$, we get

$$\begin{aligned} &\int_{B(r_0)} [-\Delta w (u - \beta) + \Delta u w]\, dx \\ &\quad = \lambda \int_{B(r_0)} \left[(u - \beta) f'(u) - f(u)\right] w\, dx. \end{aligned} \tag{50}$$

Let $J(t) = (t - \beta) f'(t) - f(t)$, then $J(0) = -f(0) < 0$, $J(\infty) = \lim_{t\to\infty} J(u) = K_\infty - a\beta < 0$, and $J'(t) = (t - \beta) f''(t) > 0$ for $t > \beta$. Hence $J(u) = (u-\beta) f'(u) - f(u) < 0$ for $x \in B(r_0)$. Then

$$\int_{B(r_0)} \left[(u - \beta) f'(u) - f(u)\right] w\, dx < 0. \tag{51}$$

On the other hand, by Green formula,

$$\begin{aligned} &\int_{B(r_0)} [-\Delta w (u - \beta) + \Delta u w]\, dx \\ &\quad = -\int_{\partial(B(r_0))} \frac{\partial w}{\partial \nu} (u - \beta)\, dx > 0. \end{aligned} \tag{52}$$

A contradiction occurs from (50), (51), and (52). Hence $w(r)$ has no zeroes in $(0, 1)$, that is to say, w does not change sign in Ω. The proof is complete. □

Now define $F : C_0^{2,\alpha}(\overline{\Omega}) \rightarrow C^{\alpha}(\overline{\Omega})$, by

$$Fu = \Delta u + \lambda f(u), \tag{53}$$

then the linearized operator (Frechèt derivative) is

$$F_u(\lambda, u) w = \Delta w + \lambda f'(u) w. \tag{54}$$

From the maximum principle, all solutions of (P_λ) are positive on Ω. Moreover, if (λ^*, u^*) is degenerate solution of (P_λ), then by Theorem 15, the nontrivial solution w of (43) does not change sign in Ω, and hence w can be chosen to be positive. Then by Krein-Rutman's Theorem, $N(F_u(\lambda^*, u^*)) = \operatorname{span}\{w\}$, and it follows from Fredholm alternative theorem that $\operatorname{codim} R(F_u(\lambda^*, u^*)) = 1$. Now we prove that $F_\lambda(\lambda^*, u^*) \notin R(F_u(\lambda^*, u^*))$. If it is not the case, then there exists $v \in C_0^{2,\alpha}(\overline{\Omega})$, such that

$$\Delta v + \lambda^* f'(u^*) v = f(u^*). \tag{55}$$

We also have

$$\Delta w + \lambda^* f'(u^*) w = 0. \tag{56}$$

Multiplying (55) by w, (56) by v, subtracting, and integrating, we obtain

$$\int_\Omega f(u^*) w \, dx = 0, \tag{57}$$

a contradiction. As all the conditions of Crandall-Rabinowitz's bifurcation theorem [20] are satisfied, the solutions of (P_λ) near the degenerate solution (λ^*, u^*) form a smooth curve which is expressed in the form

$$(\lambda(s), u(s)) = (\lambda^* + \tau(s), u_0 + sw + z(s)), \tag{58}$$

where $s \rightarrow (\tau(s), z(s)) \in R \times Z$ is a smooth function near $s = 0$ with $\tau(0) = \tau'(0) = 0$, $z(0) = z'(0) = 0$, where Z is a complement of $\operatorname{span}\{w\}$ in X, and w is the positive solution of (43), which is unique if normalized.

Substituting u and λ by expression (58), then differentiating the equation (P_λ) twice, and evaluating at $s = 0$, we have

$$\begin{gathered} \Delta u_{ss} + \lambda f(u) u_{ss} + 2\lambda' f'(u) u_s + \lambda f''(u) u_s^2 + \lambda'' f(u) = 0, \\ \Delta u_{ss} + \lambda^* f'(u) u_{ss} + \lambda^* f''(u) w^2 + \lambda''(0) f(u) = 0. \end{gathered} \tag{59}$$

Multiplying (59) by w, (43) by u_{ss}, subtracting, and integrating, we obtain

$$\tau''(0) = -\lambda^* \frac{\int_\Omega f''(u^*) w^3 dx}{\int_\Omega f(u^*) w \, dx} < 0. \tag{60}$$

By (60) and the Taylor expansion formula of $\tau(s)$ at $s = 0$, we conclude that at any degenerate solution (λ^*, u^*) of (P_λ), the solution curve turns left, that is to say, there is no any solution (λ, u) on the right near (λ^*, u^*). This observation is very important to our proof of the following theorem.

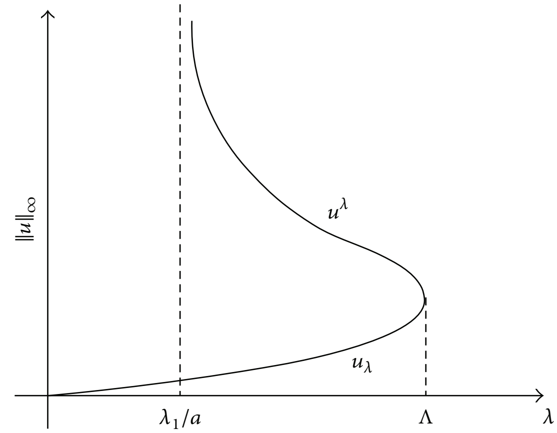

FIGURE 3: Precise bifurcation diagram on a unit ball.

Theorem 16. *Suppose that Ω is the unit ball in R^n, f satisfies (F1)-(F2), and $0 < K_\infty < a\beta$. Then for problem (P_λ),*

(1) *there exist no solutions for $\lambda > \Lambda$,*

(2) *there exists exactly one solution for $\lambda \in (0, \lambda_1/a] \cup \{\Lambda\}$,*

(3) *there exist exactly two solutions for $\lambda \in (\lambda_1/a, \Lambda)$.*

Moreover, the solution set $\{(\lambda, u)\}$ of (P_λ) forms a smooth curve in the space $R \times C(\overline{\Omega})$, which can be roughly described as in Figure 3.

Proof. By Theorem 10, $\Lambda > \lambda_1/a$, and Theorem 7 tells us that (P_λ) has a unique solution (Λ, u_Λ) for $\lambda = \Lambda$, and Implicit Function Theorem implies that (Λ, u_Λ) is a degenerate solution. By Theorem 15, non-trivial solution w of the corresponding linearized equation (43) does not change sign in Ω, and we may suppose that w is positive in Ω. Then Crandall-Rabinowitz's bifurcation theorem [20] and the discussion prior to this theorem imply that the solutions near (Λ, u_Λ) form a smooth curve which turns to the left in the phase space. We may call the part of the smooth solution curve $\{(\lambda, u)\}$ with $u(0) > u_\Lambda(0)$ the upper branch, and the rest the lower branch. We denote the upper branch by u^λ and the lower branch by u_λ.

For the upper branch, as long as (λ, u^λ) nondegenerate, the Implicit Function Theorem ensures that we can continue to extend this solution curve in the direction of decreasing λ. We still denote the extension by (λ, u^λ). This process of continuation towards smaller values of λ will not encounter any other degenerate solutions. This is because, if, say, (λ, u^λ) becomes degenerate at $\lambda = \lambda_0$, the discussion prior to this theorem implies that all the solutions near $(\lambda_0, u^{\lambda_0})$ must lie to the left side of it, which is a contradiction. Lemma 12 tells us that $\lambda \rightarrow u^\lambda(0)$ is decreasing. So in the progress of extension of (λ, u^λ) towards smaller values of λ, there are only the following two possibilities.

(i) The upper branch (λ, u^λ) stops at some $(0, u_0)$, and $u_0(0) > u_\Lambda(0)$.

(ii) $\| u_\lambda \|_\infty$ goes to infinity as $\lambda \rightarrow \widetilde{\lambda} + 0$, $0 \le \widetilde{\lambda} < \Lambda$.

But case (i) cannot happen, since $(0, u_0)$ is obviously not a solution of (P_λ). Hence case (ii) happens. We assert that $\tilde{\lambda} = \lambda_1/a$. In fact, let $\{\lambda_n\}$ be an arbitrary sequence such that $\lambda_n \to \tilde{\lambda}$. Denote $M_n = \| u_n\|_\infty$, $v_n = u_n/M_n$, then $M_n \to \infty$ and

$$\Delta v_n + \lambda_n \frac{f(M_n v_n)}{M_n} = 0, \quad \text{in } \Omega, \qquad v = 0, \quad \text{on } \partial\Omega. \tag{61}$$

Since $f(M_n v_n)/M_n$ is bounded, by Sobolev Imbedding Theorems and standard regularity of elliptic equation, it is easy to see that $\{v_n\}$ has a subsequence, still denoted by $\{v_n\}$, such that $v_n \to v$ in $C^{2,\alpha}(\Omega)$ $(n \to \infty)$, for some $v \in C^{2,\alpha}(\Omega)$, $v > 0$ in Ω. Letting $n \to \infty$ in (61), we get

$$\Delta v + \tilde{\lambda} a v = 0, \quad \text{in } \Omega, \qquad v = 0, \quad \text{on } \partial\Omega, \tag{62}$$

which implies that $\tilde{\lambda} = \lambda_1/a$.

Now we study the structure of the lower branch. As in the case of upper branch, as long as (λ, u_λ) nondegenerate, the Implicit Function Theorem ensures that we can continue to extend this solution curve in the direction of decreasing λ. We still denote the extension by (λ, u_λ). This process of continuation towards smaller values of λ will not encounter any other degenerate solutions. Lemma 12 implies that $\lambda \to u_\lambda(0)$ is increasing. So in the progress of extension of (λ, u_λ) towards smaller values of λ, there are only the following two possibilities.

(i) The lower branch (λ, u_λ) stops at some $(0, u_0)$ with $u_0(0) > 0$.

(ii) The lower branch (λ, u_λ) stops at some $(\lambda_0, 0)$ with $0 \le \lambda_0 < \Lambda$.

As before, case (i) will not happen. Then case (ii) happens. By $f(0) > 0$, it is easy to see that $\lambda_0 = 0$. That is to say, the lower branch of solutions extends till the origin $(0, 0)$ in the phase plane.

By the above argument, we obtain a smooth positive solution curve which consists of an upper branch $\{(\lambda, u^\lambda)\}$ and a lower branch $\{(\lambda, u_\lambda)\}$. The lower branch starts from (Λ, u_Λ) and stops at $(0, 0)$, and $\lambda \to u_\lambda(0)$ is a strictly increasing function. The upper branch $\{(\lambda, u^\lambda)\}$ starts from (Λ, u_Λ) and stops at $(\lambda_1/a, \infty)$, and $\lambda \to u^\lambda(0)$ is a strictly decreasing function with $u^\lambda(0)$ blowing up as $\lambda \to \lambda_1/a + 0$. By Lemma 12, all solutions of (P_λ) are contained in this smooth solution curve, and the complete bifurcation diagram can be described as in Figure 3. The proof is complete. □

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Research Article

Stability for a Class of Differential Equations with Nonconstant Delay

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Stability is investigated for the following differential equations with nonconstant delay $x'(t) = q(t)F(x(t)) - p(t)f(x(t-\tau(t)))$, where $p : [0,+\infty) \to [0,+\infty)$, $q : [0,+\infty) \to R$, $\tau : [0,+\infty) \to [0,r]$, and F and $f : R \to R$ with $xf(x) > 0$ for $x \neq 0$ and $|x| \leq a$ (a is a positive constant) are continuous functions. A criterion is given for the zero solution of this delay equation being uniformly stable and asymptotically stable.

1. Introduction

Delays are inherent in many physical and technological systems. In particular, pure delays are often used to ideally represent the effects of transmission, transportation, and inertia phenomena. Delay differential equations constitute basic mathematical models of real phenomena, for instance in biology, mechanics, and economics (cf., e.g., [1–17] and references therein). Stability analysis of delay differential equations is particularly relevant in control theory, where one cause of delay is the finite speed of communication. There have been a lot of results on the study of stability of delay differential equations. For example, we can see many earlier results on this issue from Burton's book [2]. Recently, in 2004, Butcher et al. [4] studied the stability properties of delay differential equations with time-periodic parameters. By employing a shifted Chebyshev polynomial approximation in each time interval with length equal to the delay and parametric excitation period, the system is reduced to a set of linear difference equations for the Chebyshev expansion coefficients of the state vector in the previous and current intervals. In 2005, Wahi and Chatterjee [16] used Galerkin-projection to reduce the infinite dimensional dynamics of a delay differential equation to one occurring on a finite number of modes. In 2009, Kalmár-Nagy [7] demonstrated that the method of steps for linear delay differential equation together with the inverse Laplace transform can be used to find a converging sequence of polynomial approximants to the transcendental function determining stability of the delay equation. Most recently, Berezansky and Braverman [3] gave some explicit conditions of asymptotic and exponential stability for the scalar nonautonomous linear delay differential equation with several delays and an arbitrary number of positive and negative coefficients.

This paper is concerned with the following differential equations with nonconstant delay:

$$x'(t) = q(t)F(x(t)) - p(t)f(x(t-\tau(t))), \tag{1}$$

where $a : [0,+\infty) \to [0,+\infty)$, $q : [0,+\infty) \to R$, $\tau : [0,+\infty) \to [0,r]$, and F and $f : R \to R$ with

$$xf(x) > 0 \quad \text{for } x \neq 0,\ |x| \leq a \tag{2}$$

(a is a positive constant) are continuous functions. We aim at giving general criterion for the zero solution of this delay equation being uniformly stable and asymptotically stable.

2. Main Result

Denote by $C[t_0 - r, t_0]$ the Banach space of continuous functions from $[t_0 - r, t_0]$ to R with the sup-norm

$$\|\varphi\|_{C[t_0-r,t_0]} = \max_{s\in[t_0-r,t_0]} \|\varphi(s)\|, \quad \text{for every } \varphi \in C[t_0-r,t_0]. \tag{3}$$

We consider (1) for $t \geq t_0$ with the initial conditions (for any $t_0 \geq 0$)

$$x(t) = \varphi(t), \quad t_0 - r \leq t \leq t_0, \tag{4}$$

where $\varphi \in C[t_0 - r, t_0]$.

For an initial function $\varphi \in C[t_0 - r, t_0]$, we denote by $x(t; t_0, \varphi)$ the solution of (1) such that (4) holds.

Definition 1. The zero solution of (1) is said to be stable if for any $\varepsilon > 0$ and $t_0 \geq 0$, there exists $\delta(t_0, \varepsilon) > 0$ such that if

$$\|\varphi\|_{C[t_0-r,t_0]} < \delta, \tag{5}$$

then

$$|x(t; t_0, \varphi)| < \varepsilon \quad \forall t \geq t_0. \tag{6}$$

The zero solution of (1) is uniformly stable if the above δ is independent of t_0.

Definition 2. The zero solution of (1) is said to be asymptotically stable if it is stable and if for any $t_0 \geq 0$, there exists $\delta(t_0) > 0$ such that if

$$\|\varphi\|_{C[-r,0]} < \delta, \tag{7}$$

then

$$|x(t; t_0, \varphi)| \longrightarrow 0, \quad \text{as } t \longrightarrow +\infty. \tag{8}$$

Theorem 3. *Assume that*

(1) *the zero solution to* (1) *is unique;*

(2) *if q is nontrivial function and $F(\cdot)$ is nontrivial in any interval $[-b, b]$ $(b > 0)$, then*

$$\lim_{t\to+\infty} q(t) = 0, \qquad \lim_{t\to+\infty} \int_{t-\tau(t)}^{t} |q(s)|\, ds = 0, \quad p(t) \geq \mu > 0, \quad t \geq 0, \tag{9}$$

for a constant μ;

(3) $\lim_{t\to+\infty} \int_{t-\tau(t)}^{t} p(s)ds = A$;

(4) *if $A \neq 0$, then*

$$|f(x)| \leq \frac{\lambda |x|}{2A}, \quad \text{for } x \in R, \tag{10}$$

where $0 < \lambda < 1$.

Then the zero solution of (1) *is uniformly stable.*

Proof. For each $\varepsilon > 0$, we set

$$S(f, \varepsilon) := \sup\{|f(x)|; |x| \leq \varepsilon\}, \tag{11}$$

and when q is a nontrivial function and $F(\cdot)$ is nontrivial in any interval $[-b, b]$ $(b > 0)$, we set

$$S(F, \varepsilon) := \sup\{|F(x)|; |x| \leq \varepsilon\}, \tag{12}$$

$$I(\varepsilon) := \inf\left\{xf(y); xy > 0, \frac{1-\lambda}{2}\varepsilon \leq |x| \leq \varepsilon, \frac{1-\lambda}{2}\varepsilon \leq |y| \leq \varepsilon\right\}. \tag{13}$$

From (3) and (2), it follows that for every $\varepsilon > 0$, there exists $t(\varepsilon) > 0$ such that

$$\int_{t-\tau(t)}^{t} p(s)\, ds < \begin{cases} \dfrac{1-\lambda}{4(S(f,\varepsilon)+1)}\varepsilon, & \text{if } A = 0, \\ \dfrac{1-\lambda}{4(S(f,\varepsilon)+1)} \min\{A, 1\}\varepsilon + A, & \text{if } A \neq 0, \end{cases} \quad \forall t > t(\varepsilon), \tag{14}$$

and when q is a nontrivial function and $F(\cdot)$ is nontrivial in any interval $[-b, b]$ $(b > 0)$, such that

$$\int_{t-\tau(t)}^{t} |q(s)|\, ds < \frac{1-\lambda}{4(S(F,\varepsilon)+1)}\varepsilon, \quad \forall t > t(\varepsilon), \tag{15}$$

$$|q(t)| \leq \mu \frac{I(\varepsilon)}{2(S(F,\varepsilon)+1)(\varepsilon+1)}, \quad \forall t > t(\varepsilon). \tag{16}$$

We claim that for any $\varepsilon > 0$ and $t_0 \geq t(\varepsilon)$, if

$$\|\varphi\|_{C[t_0-r,t_0]} < \frac{1-\lambda}{2}\varepsilon, \tag{17}$$

then

$$|x(t; t_0, \varphi)| < \varepsilon \quad \forall t \geq t_0, \tag{18}$$

which means that the zero solution of (1) is eventually uniformly stable. Actually, if this is not true, then there exist

$$\varepsilon_0 \leq \min\{a, 1\} \tag{19}$$

and a solution

$$x(t) := x(t; t_0, \varphi) \tag{20}$$

to (1) with $\|\varphi\|_{C[t_0-r,t_0]} < ((1-\lambda)/2)\varepsilon$ and

$$t_0 > t(\varepsilon_0) \tag{21}$$

such that there is a $\bar{t} > t_0$,

$$|x(\bar{t})| \geq \varepsilon_0. \tag{22}$$

Define

$$t_2 := \inf \{t \geq t_0; |x(t)| = \varepsilon_0\}, \tag{23}$$

$$t_1 := \sup \left\{t_0 \leq t < t_2; |x(t)| = \frac{1-\lambda}{2}\varepsilon_0\right\}, \qquad V(x) = x^2, \quad x \in R. \tag{24}$$

Then, together with (21) and (22), we obtain

$$t(\varepsilon_0) < t_1 < t_2, \qquad V(x(t_1)) = \frac{(1-\lambda)^2}{4}\varepsilon_0^2, \qquad V(x(t_2)) > \varepsilon_0^2, \tag{25}$$

and, for $t \in (t_1, t_2)$,

$$\frac{(1-\lambda)^2}{4}\varepsilon_0^2 < V(x(t)) < \varepsilon_0^2, \tag{26}$$

and for arbitrary $\eta > 0$, there exists $\xi \in [t_2 - \eta, t_2]$ such that

$$V'(x(\xi)) > 0. \tag{27}$$

Therefore,

$$V'(x(t_2)) \geq 0. \tag{28}$$

This implies that

$$t_1 \geq t_2 - \tau(t_2). \tag{29}$$

In fact, if

$$t_1 < t_2 - \tau(t_2), \tag{30}$$

then by (23)–(25), we have

$$\frac{1-\lambda}{2}\varepsilon_0 \leq |x(t_2 - \tau(t_2))| \leq \varepsilon_0, \qquad t_2 - \tau(t_2) > t(\varepsilon_0). \tag{31}$$

It is not hard to see that we can choose t_1 and t_2 above to make $x(t)$ have constant sign in $[t_1, t_2]$.

Case I. When $q(t) \equiv 0$ or

$$F(x) \equiv 0 \quad \text{for } |x| \leq b, \tag{32}$$

where b is a positive real number.

In this case, if $q(t) \equiv 0$, then

$$V'(x(t_2)) = -2p(t_2)x(t_2)f(x(t_2 - \tau(t_2))) < 0, \tag{33}$$

which contradicts with (28). Moreover, if

$$F(x) \equiv 0 \quad \text{for } |x| \leq b, \tag{34}$$

for a positive real number b, then it is clear that we can require $\varepsilon_0 < b$. Hence,

$$V'(x(t_2)) = -2p(t_2)x(t_2)f(x(t_2 - \tau(t_2))) < 0, \tag{35}$$

which contradicts with (28) too.

Consequently, in this case we have the following observation.

Case I-1. If $A = 0$, then we deduce by (23), (24), (1), and (11) that

$$\begin{aligned}
\frac{\varepsilon_0}{2} + \frac{\lambda}{2}\varepsilon_0 &= |x(t_2)| - |x(t_1)| \\
&\leq |x(t_2) - x(t_1)| \\
&\leq \int_{t_1}^{t_2} p(s)\,|f(x(s - \tau(s)))|\,ds \\
&\quad + \int_{t_1}^{t_2} |q(s)|\,|F(x(s))|\,ds \\
&\leq S(f, \varepsilon_0)\int_{t_1}^{t_2} p(s)\,ds \\
&\leq S(f, \varepsilon_0)\int_{t_2-\tau(t_2)}^{t_2} p(s)\,ds \\
&< \frac{\varepsilon_0}{2}.
\end{aligned} \tag{36}$$

This is clearly impossible.

Case I-2. If $A \neq 0$, then we deduce by (23), (24), (1), (11), and (14) that

$$\begin{aligned}
\frac{\varepsilon_0}{2} + \frac{\lambda}{2}\varepsilon_0 &= |x(t_2)| - |x(t_1)| \\
&\leq |x(t_2) - x(t_1)| \\
&\leq \int_{t_1}^{t_2} p(s)\,|f(x(s - \tau(s)))|\,ds \\
&\quad + \int_{t_1}^{t_2} |q(s)|\,|F(x(s))|\,ds \\
&\leq \frac{\lambda\varepsilon_0}{2A}\int_{t_1}^{t_2} p(s)\,ds \\
&\leq \frac{\lambda\varepsilon_0}{2A}\int_{t_2-\tau(t_2)}^{t_2} p(s)\,ds \\
&\leq \frac{\lambda\varepsilon_0}{2A}\left(\frac{1-\lambda}{4(S(f, \varepsilon_0)+1)}\min\{A, 1\}\varepsilon_0 + A\right) \\
&< \frac{1-\lambda}{4}\varepsilon + \frac{\lambda}{2}\varepsilon \\
&< \frac{\varepsilon}{2}.
\end{aligned} \tag{37}$$

This is clearly impossible too.

Therefore, in this case, the zero solution of (1) is eventually uniformly stable. This, together with assumption (1), implies that the zero solution of (1) is uniformly stable.

Case II. q is a nontrivial function and $F(\cdot)$ is nontrivial in any interval $[-b, b]$ ($b > 0$).

In this case, by virtue of (1), and assumption (2), (12), (13), and (16), we get

$$\begin{aligned} V'(x(t_2)) &= -2p(t_2)x(t_2)f(x(t_2-\tau(t_2))) \\ &\quad + 2x(t_2)q(t_2)F(x(t_2)) \\ &\le -2\mu I(\varepsilon_0) + 2\varepsilon_0\mu \frac{I(\varepsilon_0)}{2(S(F,\varepsilon_0)+1)(\varepsilon_0+1)} S(F,\varepsilon) \\ &\le -\mu I(\varepsilon) \\ &< 0, \end{aligned} \tag{38}$$

which contradicts with (28).

Consequently, in this case we have the following observation,

Case II-1. If $A = 0$, then we deduce by (23), (24), (1), (11), (12), (14), and (15) that

$$\begin{aligned} \frac{\varepsilon_0}{2} + \frac{\lambda}{2}\varepsilon_0 &= |x(t_2)| - |x(t_1)| \\ &\le |x(t_2) - x(t_1)| \\ &\le \int_{t_1}^{t_2} p(s)\left|f(x(s-\tau(s)))\right| ds \\ &\quad + \int_{t_1}^{t_2} |q(s)|\,|F(x(s))|\, ds \\ &\le S(f,\varepsilon)\int_{t_1}^{t_2} p(s)\,ds \\ &\quad + S(F,\varepsilon)\int_{t_1}^{t_2} |q(s)|\,ds \\ &\le S(f,\varepsilon)\int_{t_2-\tau(t_2)}^{t_2} p(s)\,ds \\ &\quad + S(F,\varepsilon)\int_{t_2-\tau(t_2)}^{t_2} |q(s)|\,ds \\ &< \frac{1-\lambda}{4}\varepsilon + \frac{1-\lambda}{4}\varepsilon \\ &< \frac{\varepsilon}{2}. \end{aligned} \tag{39}$$

This is a contradiction.

Case II-2. If $A \ne 0$, then we deduce by (23), (24), (1), (11), (12), (14), and (15) that

$$\begin{aligned} \frac{\varepsilon}{2} + \frac{\lambda}{2}\varepsilon &= |x(t_2)| - |x(t_1)| \\ &\le |x(t_2) - x(t_1)| \\ &\le \int_{t_1}^{t_2} p(s)\left|f(x(s-\tau(s)))\right| ds \\ &\quad + \int_{t_1}^{t_2} |q(s)|\,|F(x(s))|\, ds \\ &\le \frac{\lambda\varepsilon_0}{2A}\int_{t_1}^{t_2} p(s)\,ds + S(F,\varepsilon)\int_{t_1}^{t_2} |q(s)|\,ds \\ &\le \frac{\lambda\varepsilon_0}{2A}\int_{t_2-\tau(t_2)}^{t_2} p(s)\,ds + S(F,\varepsilon)\int_{t_2-\tau(t_2)}^{t_2} |q(s)|\,ds \\ &\le \frac{\lambda\varepsilon_0}{2A}\left(\frac{1-\lambda}{4(S(f,\varepsilon_0)+1)}\min\{A,1\}\varepsilon_0 + A\right) \\ &\quad + \frac{1-\lambda}{4}\varepsilon \\ &< \frac{1-\lambda}{4}\varepsilon + \frac{\lambda}{2}\varepsilon + \frac{1-\lambda}{4}\varepsilon \\ &= \frac{\varepsilon}{2}. \end{aligned} \tag{40}$$

This is a contradiction too.

Therefore, in this case, the zero solution of (1) is eventually uniformly stable. This, together with assumption (1), implies that the zero solution of (1) is uniformly stable. □

Theorem 4. *Assume that*

(1) *the zero solution to* (1) *is unique;*

(2) *if* $q(t) \equiv 0$ *or*

$$F(x) \equiv 0 \quad \text{for } |x| \le b, \tag{41}$$

for a positive real number b, *then*

$$\int_0^{+\infty} p(s)\,ds = +\infty; \tag{42}$$

(3) *if* q *is nontrivial function and* $F(\cdot)$ *is nontrivial in any interval* $[-b,b]$ $(b>0)$, *then*

$$\begin{gathered} \lim_{t\to+\infty} q(t) = 0, \qquad \lim_{t\to+\infty}\int_{t-\tau(t)}^{t} |q(s)|\,ds = 0, \\ \int_0^{+\infty} q(s)\,ds < +\infty, \\ p(t) \ge \mu > 0, \quad t \ge 0, \end{gathered} \tag{43}$$

for a constant μ;

(4) $\lim_{t\to+\infty}\int_{t-\tau(t)}^{t} p(s)ds = A$;

(5) *if* $A \ne 0$, *then*

$$|f(x)| \le \frac{\lambda|x|}{2A}, \quad \text{for } x \in R, \tag{44}$$

where $0 < \lambda < 1$. *Then the zero solution of* (1) *is asymptotically stable.*

Proof. It follows from Theorem 3 that the zero solution of (1) is uniformly stable; that is, for arbitrarily given $\varepsilon > 0$ and $t_0 \geq 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if

$$\|\varphi\|_{C[t_0-r,t_0]} < \delta, \tag{45}$$

then

$$\left|x\left(t; t_0, \varphi\right)\right| < \varepsilon \quad \forall t \geq t_0. \tag{46}$$

Next, we will prove that

$$\left|x\left(t; t_0, \varphi\right)\right| \longrightarrow 0, \quad \text{as } t \longrightarrow +\infty. \tag{47}$$

First, we show that

$$\liminf_{t \to +\infty} \left|x\left(t; t_0, \varphi\right)\right| = 0. \tag{48}$$

Suppose that this is not true. Then

$$\liminf_{t \to +\infty} \left|x\left(t; t_0, \varphi\right)\right| > 0. \tag{49}$$

Hence, for the arbitrarily given

$$0 < \varepsilon < \min\{a, b\}, \tag{50}$$

there exist $0 < \varepsilon_0 < \varepsilon$ and $T > t_0$ such that

$$x\left(t; t_0, \varphi\right) > \varepsilon_0 \quad \forall t \geq T, \tag{51}$$

or

$$x\left(t; t_0, \varphi\right) < -\varepsilon_0 \quad \forall t \geq T. \tag{52}$$

Let us now consider

$$x\left(t; t_0, \varphi\right) > \varepsilon_0 \quad \forall t \geq T. \tag{53}$$

Case I. When $q(t) \equiv 0$ or

$$F(x) \equiv 0 \quad \text{for } |x| \leq b, \tag{54}$$

for a positive real number b, we obtain by assumption (2), (46), (50), and (53)

$$\begin{aligned} x(t) &= x(T+r) - \int_{T+r}^{t} p(s) f(x(s-\tau(s))) \, ds \\ &\quad + \int_{T+r}^{t} q(s) F(x(s)) \, ds \\ &\leq x(T+r) - \inf\left\{f(x); x \in \left[\varepsilon_0, \varepsilon\right]\right\} \int_{T+r}^{t} p(s) \, ds. \end{aligned} \tag{55}$$

This implies that

$$x(t) \longrightarrow -\infty \quad \text{as } t \longrightarrow +\infty, \tag{56}$$

which contradicts with (53).

Case II. When q is a nontrivial function and $F(\cdot)$ is nontrivial in any interval $[-b, b]$ $(b > 0)$, we obtain by assumptions (3), (46), (50), and (53)

$$\begin{aligned} x(t) &= x(T+r) - \int_{T+r}^{t} p(s) f(x(s-\tau(s))) \, ds \\ &\quad + \int_{T+r}^{t} q(s) F(x(s)) \, ds \\ &\leq x(T+r) - \mu \inf\left\{f(x); x \in \left[\varepsilon_0, \varepsilon\right]\right\}(t-T-r) \\ &\quad + \sup\left\{|F(x)|; x \in \left(\varepsilon_0, \varepsilon\right)\right\} \int_{T+r}^{t} |q(s)| \, ds. \end{aligned} \tag{57}$$

This, together with assumption (2), implies that

$$x(t) \longrightarrow -\infty \quad \text{as } t \longrightarrow +\infty, \tag{58}$$

which contradicts with (53).

Moreover, in a similar way, we can prove that

$$x\left(t; t_0, \varphi\right) < -\varepsilon \quad \forall t \geq T \tag{59}$$

is impossible.

Therefore, (48) is true.

Based on (48), we will show that

$$\limsup_{t \to +\infty} \left|x\left(t; t_0, \varphi\right)\right| = 0. \tag{60}$$

Actually, if this is not true, that is,

$$\limsup_{t \to +\infty} \left|x\left(t; t_0, \varphi\right)\right| > 0, \tag{61}$$

then by (48) we see that there are ε_0 with

$$0 < \varepsilon_0 < \min\{a, b, 1\}, \tag{62}$$

and two sequences $\{\theta_n\}$ and $\{t_n\}$ such that

$$\begin{gathered} \theta_n < t_n, \quad n = 1, 2, \ldots, \\ \theta_n \longrightarrow +\infty \qquad t_n \longrightarrow +\infty \quad \text{as } n \longrightarrow +\infty, \\ V\left(x\left(\theta_n\right)\right) = \frac{(1-\lambda)^2}{4} \varepsilon_0^2, \qquad V\left(x\left(t_n\right)\right) > \varepsilon_0^2, \\ V'\left(x\left(t_n\right)\right) > 0, \end{gathered} \tag{63}$$

and for $t \in (\theta_n, t_n)$,

$$\frac{(1-\lambda)^2}{4} \varepsilon_0^2 < V(x(t)) < \varepsilon_0^2. \tag{64}$$

By the same reason as that in the proof of Theorem 3, we know that

$$t_n - \tau\left(t_n\right) \leq \theta_n \leq t_n. \tag{65}$$

Define $S(f, \varepsilon)$, $S(F, \varepsilon)$, $I(\varepsilon)$, and $t(\varepsilon)$ as those in the proof of Theorem 3. Then when n is large enough, we have

$$t_n > t(\varepsilon). \tag{66}$$

Case I. When $q(t) \equiv 0$ or

$$F(x) \equiv 0 \quad \text{for } |x| \le b, \tag{67}$$

where b is a positive real number.

Case I-1. If $A = 0$, then we deduce that

$$\begin{aligned}
\frac{\varepsilon_0}{2} + \frac{\lambda}{2}\varepsilon_0 &= |x(t_n)| - |x(\theta_n)| \\
&\le |x(t_n) - x(\theta_n)| \\
&\le \int_{\theta_n}^{t_n} p(s)\,|f(x(s-\tau(s)))|\,ds \\
&\quad + \int_{\theta_n}^{t_n} |q(s)|\,|F(x(s))|\,ds \\
&\le S(f,\varepsilon_0)\int_{\theta_n}^{t_n} p(s)\,ds \\
&\le S(f,\varepsilon_0)\int_{t_n-\tau(t_n)}^{t_n} p(s)\,ds \\
&< \frac{\varepsilon_0}{2}.
\end{aligned} \tag{68}$$

This is impossible.

Case I-2. If $A \ne 0$, then we obtain

$$\begin{aligned}
\frac{\varepsilon_0}{2} + \frac{\lambda}{2}\varepsilon_0 &= |x(t_n)| - |x(\theta_n)| \\
&\le |x(t_n) - x(\theta_n)| \\
&\le \int_{\theta_n}^{t_n} p(s)\,|f(x(s-\tau(s)))|\,ds \\
&\quad + \int_{\theta_n}^{t_n} |q(s)|\,|F(x(s))|\,ds \\
&\le \frac{\lambda\varepsilon_0}{2A}\int_{\theta_n}^{t_n} p(s)\,ds \\
&\le \frac{\lambda\varepsilon_0}{2A}\int_{t_n-\tau(t_n)}^{t_n} p(s)\,ds \\
&\le \frac{\lambda\varepsilon_0}{2A}\left(\frac{1-\lambda}{4(S(f,\varepsilon_0)+1)}\min\{A,1\}\varepsilon_0 + A\right) \\
&< \frac{1-\lambda}{4}\varepsilon + \frac{\lambda}{2}\varepsilon \\
&< \frac{\varepsilon}{2}.
\end{aligned} \tag{69}$$

This is clearly impossible too.

Consequently, (60) is true in this case.

Case II. When q is nontrivial function and $F(\cdot)$ is nontrivial in any interval $[-b,b]$ $(b > 0)$.

Case II-1. If $A = 0$, then we deduce that

$$\begin{aligned}
\frac{\varepsilon_0}{2} + \frac{\lambda}{2}\varepsilon_0 &= |x(t_n)| - |x(\theta_n)| \\
&\le |x(t_n) - x(\theta_n)| \\
&\le \int_{\theta_n}^{t_n} p(s)\,|f(x(s-\tau(s)))|\,ds \\
&\quad + \int_{\theta_n}^{t_n} |q(s)|\,|F(x(s))|\,ds \\
&\le S(f,\varepsilon)\int_{\theta_n}^{t_n} p(s)\,ds \\
&\quad + S(F,\varepsilon)\int_{\theta_n}^{t_n} |q(s)|\,ds \\
&\le S(f,\varepsilon)\int_{t_n-\tau(t_n)}^{t_n} p(s)\,ds \\
&\quad + S(F,\varepsilon)\int_{t_n-\tau(t_n)}^{t_n} |q(s)|\,ds \\
&< \frac{1-\lambda}{4}\varepsilon + \frac{1-\lambda}{4}\varepsilon \\
&< \frac{\varepsilon}{2}.
\end{aligned} \tag{70}$$

This is a contradiction.

Case II-2. If $A \ne 0$, then we obtain

$$\begin{aligned}
\frac{\varepsilon_0}{2} + \frac{\lambda}{2}\varepsilon_0 &= |x(t_n)| - |x(\theta_n)| \\
&\le |x(t_n) - x(\theta_n)| \\
&\le \int_{\theta_n}^{t_n} p(s)\,|f(x(s-\tau(s)))|\,ds \\
&\quad + \int_{\theta_n}^{t_n} |q(s)|\,|F(x(s))|\,ds \\
&\le \frac{\lambda\varepsilon_0}{2A}\int_{\theta_n}^{t_n} p(s)\,ds + S(F,\varepsilon)\int_{\theta_n}^{t_n} |q(s)|\,ds \\
&\le \frac{\lambda\varepsilon_0}{2A}\int_{t_n-\tau(t_n)}^{t_n} p(s)\,ds \\
&\quad + S(F,\varepsilon)\int_{t_n-\tau(t_n)}^{t_n} |q(s)|\,ds \\
&\le \frac{\lambda\varepsilon_0}{2A}\left(\frac{1-\lambda}{4(S(f,\varepsilon_0)+1)}\min\{A,1\}\varepsilon_0 + A\right)
\end{aligned}$$

$$\begin{aligned} &+ \frac{1-\lambda}{4}\varepsilon \\ &< \frac{1-\lambda}{4}\varepsilon + \frac{\lambda}{2}\varepsilon + \frac{1-\lambda}{4}\varepsilon \\ &= \frac{\varepsilon}{2}. \end{aligned} \tag{71}$$

This is a contradiction too.

Therefore, (60) is true in this case. So, (60) holds truly. This means that the zero solution of (4) is asymptotically stable. □

Remark 5. Our results are new comparing with the results in [2, 3] since $\tau(t)$ could go to 0 or a big number as $t \to +\infty$ and in this case $p(t)$ also could be very large in our theorems. Moreover, for the case of $A = 0$, the condition on f in our results is very weak.

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Research Article

Multiplicative Perturbations of Convoluted *C*-Cosine Functions and Convoluted *C*-Semigroups

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We obtain the multiplicative perturbation theorems for convoluted C-cosine functions (resp., convoluted C-semigroups) and n-times integrated C-cosine functions (resp., n-times integrated C-semigroups) for $n \in \mathbb{N}$. Moreover, we obtain some new results for perturbations on C-cosine functions (resp., C-semigroups). Some examples are presented.

1. Introduction and Preliminaries

The α-times integrated C-semigroups, α-times integrated C-cosine functions ($\alpha > 0$) [1–6], 0-times integrated semigroups (i.e., C-semigroups), and 0-times integrated C-cosine functions (i.e., C-cosine functions) [5, 7–11] are powerful tools in studying ill-posed abstract Cauchy problems. The convoluted C-cosine functions (resp., convoluted C-semigroups) are the extension of α-times integrated C-cosine functions (resp., α-times integrated C-semigroups), they can be used to deal with more complicated ill-posed abstract Cauchy problems of evolution equations [5, 12–16].

Many researchers studied the perturbations on C-cosine functions and C-semigroups [17–22]. In [16], Kostić studied the additive perturbations of convoluted C-cosine functions and convoluted C-semigroups. However, to the authors' knowledge, few papers can be found in the literature for the multiplicative perturbations on the convoluted C-cosine functions (resp., convoluted C-semigroups).

In this paper, based on the previously mentioned works we study the multiplicative perturbations on the convoluted C-cosine functions and convoluted C-semigroups. Moreover, we obtain the corresponding new results for n-times integrated C-semigroups (resp., n-times integrated C-cosine functions) ($n \in \mathbb{N}_0$, $\mathbb{N}_0$ denotes the nonnegative integers).

Throughout this paper, $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$, and $\mathbb{C}$ denote the positive integers, the nonnegative integers, the real numbers, the complex plane, respectively. X denotes a nontrivial complex Banach space, $L(X)$ denotes the space of bounded linear operators from X into X. In the sequel, we assume that $C \in L(X)$ is an injective operator. $\mathbf{C}([a,b],X)$ denotes the space of all continuous functions from $[a,b]$ to X. For a closed linear operator A on X, its domain, range, resolvent set, and the C-resolvent set are denoted by $D(A)$, $R(A)$, $\rho(A)$, and $\rho_c(A)$, respectively, where $\rho_c(A)$ is defined by

$$\rho_c(A) := \{\lambda \in \mathbb{C} : R(C) \subset R(\lambda - A) \text{ and } \lambda - A \text{ is injective}\}. \tag{1}$$

$K \in \mathbf{C}([0,\infty),\mathbb{C})$ is an exponentially bounded function and for $\beta \in \mathbb{R}$, $\widehat{K}(\lambda) \neq 0$ ($\operatorname{Re}\lambda > \beta$), where $\widehat{K}(\lambda)$ is the Laplace transform of $K(t)$. We define $\vartheta(t) := \int_0^t K(s)ds$.

The next definition is the convoluted version of Definition 4.1 in Chapter 1 of [5].

Definition 1 (see [5, 13, 15]). Let $\omega \geq 0$. If $\{\lambda^2 : \operatorname{Re}\lambda > \max(\omega,\beta)\} \subset \rho_c(A)$ and there exists a strongly continuous operator family $\{C_K(t)\}_{t\geq 0}(C_K(t) \in L(X), t \geq 0)$ such that $\|C_K(t)\| \leq Me^{\omega t}, t \geq 0$ for some $M > 0$, and

$$\lambda\left(\lambda^2 - A\right)^{-1}Cx = \frac{1}{\widehat{K}(\lambda)}\int_0^\infty e^{-\lambda t}C_K(t)\,x\,dt, \qquad \operatorname{Re}\lambda > \max(\omega,\beta),\ x \in X, \tag{2}$$

then it is said that A is a subgenerator of an exponentially bounded K-convoluted C-cosine function $\{C_K(t)\}_{t\geq 0}$. The operator $\overline{A} := C^{-1}AC$ is called the generator of $\{C_K(t)\}_{t\geq 0}$.

Theorem 2 (see [13–15]). *Let $\{C_K(t)\}_{t\geq 0}$ be a strongly continuous, exponentially bounded operator family, and let A be a closed operator. Then the statements (i) and (ii) are equivalent, where*

(i) *A is the subgenerator of a K-convoluted C-cosine function $\{C_K(t)\}_{t\geq 0}$,*

(ii) (1) $C_K(t)C = CC_K(t)$, $t \geq 0$,

(2) $C_K(t)A \subset AC_K(t)$, $t \geq 0$ *and*

$$A \int_0^t \int_0^s C_K(\sigma)\, x\, d\sigma\, ds = C_K(t)\, x - \vartheta(t)\, Cx, \quad t \geq 0,\ x \in X. \tag{3}$$

Definition 3. Let $0 \leq \omega < \infty$. If $\{\lambda : \operatorname{Re}\lambda > \max(\omega, \beta)\} \subset \rho_c(A)$ and there exists a strongly continuous operator family $\{T_K(t)\}_{t\geq 0}$ such that $\|T_K(t)\| \leq Me^{\omega t}$, $t \geq 0$ for some $M > 0$, and

$$(\lambda - A)^{-1}Cx = \frac{1}{\widehat{K}(\lambda)} \int_0^\infty e^{-\lambda t} T_K(t)\, x\, dt, \quad \operatorname{Re}\lambda > \max(\omega, \beta),\ x \in X, \tag{4}$$

then it is said that A is a subgenerator of an exponentially bounded K-convoluted C-semigroup $\{T_K(t)\}_{t\geq 0}$. The operator $\overline{A} := C^{-1}AC$ is called the generator of $\{T_K(t)\}_{t\geq 0}$.

Theorem 4. *Let $\{T_K(t)\}_{t\geq 0}$ be a strongly continuous, exponentially bounded operator family, and let A be a closed operator. Then the assertions (i) and (ii) are equivalent, where*

(i) *A is the subgenerator of a K-convoluted C-semigroup $\{T_K(t)\}_{t\geq 0}$,*

(ii) (1) $T_K(t)C = CT_K(t)$, $t \geq 0$,

(2) $T_K(t)A \subset AT_K(t)$, $t \geq 0$ *and*

$$A \int_0^t T_K(s)\, x\, ds = T_K(t)\, x - \vartheta(t)\, Cx, \quad t \geq 0,\ x \in X. \tag{5}$$

Remark 5 (see [16]). In Theorems 2 and 4, putting $K(t) = t^{r-1}/\Gamma(r)$, where $\Gamma(\cdot)$ denotes the Gamma function, one obtains the classes of r-times integrated C-cosine functions and r-times integrated C-semigroups; a 0-times integrated C-cosine function (resp., 0-times integrated C-semigroup) is defined to be a C-cosine function (resp., C-semigroup). More knowledge for them, we refer the reader to, for example, [1–3, 5, 7–11, 18] and references there in.

Next, we recall the definitions of r-times integrated C-semigroup and r-times integrated C-cosine functions ($r \geq 0$).

Definition 6 (see [5]). Let $0 \leq \omega < \infty$ and let $r \in [0, \infty)$. If $(\omega^2, \infty) \subset \rho_c(A)$ (resp., $(\omega, \infty) \subset \rho_c(A)$) and there exists a strongly continuous operator family $\{C_r(t)\}_{t\geq 0}$ (resp., $\{T_r(t)\}_{t\geq 0}$) such that $\|C_r(t)\| \leq Me^{\omega t}$, $t \geq 0$ (resp., $\|T_r(t)\| \leq Me^{\omega t}$, $t \geq 0$) for some $M > 0$, and

$$\lambda\left(\lambda^2 - A\right)^{-1}Cx = \lambda^r \int_0^\infty e^{-\lambda t} C_r(t)\, x\, dt, \quad \lambda > \omega,\ x \in X,$$
$$\left(\text{resp. } (\lambda - A)^{-1}Cx = \lambda^r \int_0^\infty e^{-\lambda t} T_r(t)\, x\, dt,\ \lambda > \omega,\ x \in X, \right), \tag{6}$$

then it is said that A is a subgenerator of an exponentially bounded r-times integrated C-cosine function $\{C_r(t)\}_{t\geq 0}$ (resp., r-times integrated C-semigroup $\{T_r(t)\}_{t\geq 0}$) on X. If $r = 0$, then $\{C_r(t)\}_{t\geq 0}$ (resp., $\{T_r(t)\}_{t\geq 0}$) is called an exponentially bounded 0-times integrated C-cosine function (resp., 0-times integrated C-semigroup).

We present the definition of C-cosine functions which will be used in the proof of Theorem 12.

Definition 7 (see [1, 5]). A strongly continuous family $\{C(t)\}_{t\geq 0}$ of bounded linear operators on X is called a C-cosine function on X, if $CC(\cdot) = C(\cdot)C$, $C(0) = C$ and $C(t+s)C + C(|t-s|)C = 2C(t)C(s)$, for all $t, s \geq 0$.

2. Main Results

Suppose that A is a subgenerator of an exponentially bounded K-convoluted C-cosine function $\{C_K(t)\}_{t\geq 0}$ on X, $S_K(t) = \int_0^t C_K(s)ds$, for any $\Psi \in \mathbf{C}([0,\infty), L(X))$ with $\|\Psi(t)\| = O(e^{\omega t})$, we set

$$L(\lambda) := \sup\left\{ \int_0^a e^{-\lambda t} \left\| \int_0^t \delta(\lambda)\, \Psi(s)\, C^{-1}PAC_K(t-s)\, x\, ds \right\| dt, \quad x \in D(A),\ \|x\| \leq 1 \right\} < \infty, \tag{7}$$

for some $a \in (0, +\infty]$ and $\lambda > \max(\omega, \beta)$, where $\delta(\lambda)$ is some function and $P = B/\widehat{K}(\lambda)$, $B \in L(X)$ with $R(B) \subset R(C)$.

We have the following multiplicative perturbation theorem.

Theorem 8. *Suppose that A is a subgenerator of an exponentially bounded K-convoluted C-cosine function $\{C_K(t)\}_{t\geq 0}$ on X. Let $BC = CB$, and $D(A)$ is dense in X,*

$$\left\{\lambda^2 : \lambda > \max(\omega, \beta)\right\} \subset \rho\left((I + \delta(\lambda)\, B)\, A\right). \tag{8}$$

If $\lim_{\lambda\to\infty} L(\lambda)e^{\lambda t} = 0$ for all $t \geq 0$, then $(I + \delta(\lambda)B)A$ subgenerates an exponentially bounded K-convoluted C-cosine function on X.

Proof. For all $x \in D(A)$, $\|x\| \leq 1$, λ is large enough and ε is small enough, we have

$$\begin{aligned}\left\|\int_0^t \delta(\lambda)\Psi(s)C^{-1}PAS_K(t-s)\,x\,ds\right\| &= \left\|\int_0^t\int_0^s \delta(\lambda)\Psi(\sigma)C^{-1}PAC_K(s-\sigma)\,x\,d\sigma\, ds\right\| \\ &\leq e^{\lambda t}\int_0^t e^{-\lambda s}\left\|\int_0^s \delta(\lambda)\Psi(\sigma)C^{-1}PAC_K(s-\sigma)\,x\,d\sigma\right\| ds \\ &\leq e^{\lambda t}L(\lambda) < \varepsilon < 1, \quad t \geq 0.\end{aligned} \tag{9}$$

Let $\mathscr{V} : [0,\infty) \to L(X)$ be any strongly continuous function with $\|\mathscr{V}(t)\| = O(e^{\omega t})$; we define

$$(\mathscr{M}\mathscr{V})(t)\,x = \int_0^t \delta(\lambda)\mathscr{V}(s)C^{-1}PAS_K(t-s)\,x\,ds, \quad x \in D(A),\ t \geq 0. \tag{10}$$

Obviously, $(\mathscr{M}\mathscr{V})(t)x$ is continuous on $t \geq 0$, from (9) and the denseness of $D(A)$, $\mathscr{M}$ maps $\mathbf{C}([0,\infty), L(X))$ into $\mathbf{C}([0,\infty), L(X))$.

It follows from (9) that $(I-\mathscr{M})^{-1}$ is bounded. For each $t \geq 0$, set

$$\widehat{C}_K(t)\,x := (I-\mathscr{M})^{-1}\left[C_K(\cdot)\,x\right](t), \quad x \in X. \tag{11}$$

Then, $\widehat{C}_K(t)C = C\widehat{C}_K(t)$, and there exists a constant $\widehat{M}$ such that $\|\widehat{C}_K(t)\| \leq \widehat{M}e^{\omega t}$,

$$\widehat{C}_K(t)\,x = C_K(t)\,x + \delta(\lambda)\int_0^t \widehat{C}_K(s)C^{-1}PAS_K(t-s)\,x\,ds. \tag{12}$$

For sufficiently large λ, we set

$$\mathscr{L}(\lambda)\,x = \int_0^\infty e^{-\lambda t}\widehat{C}_K(t)\,x\,dt, \quad x \in X. \tag{13}$$

Taking Laplace transform of (12), we have

$$\begin{aligned}\mathscr{L}(\lambda)\,x &= \lambda\widehat{K}(\lambda)\left(\lambda^2 - A\right)^{-1}Cx \\ &\quad + \delta(\lambda)\mathscr{L}(\lambda)C^{-1}BA\left(\lambda^2 - A\right)^{-1}Cx, \quad x \in X.\end{aligned} \tag{14}$$

Therefore for $x \in D(A)$,

$$\mathscr{L}(\lambda)\left(\lambda^2 - (I+\delta(\lambda)B)A\right)x = \lambda\widehat{K}(\lambda)Cx. \tag{15}$$

Noting (8), for $x \in X$, we have

$$\begin{aligned}&\mathscr{L}(\lambda)\left(\lambda^2 - (I+\delta(\lambda)B)A\right)\left(\lambda^2 - (I+\delta(\lambda)B)A\right)^{-1}x \\ &\quad = \lambda\widehat{K}(\lambda)\left(\lambda^2 - (I+\delta(\lambda)B)A\right)^{-1}Cx,\end{aligned} \tag{16}$$

that is

$$\begin{aligned}\frac{1}{\widehat{K}(\lambda)}\int_0^\infty e^{-\lambda t}\widehat{C}_K(t)\,x\,dt &= \frac{1}{\widehat{K}(\lambda)}\mathscr{L}(\lambda)\,x \\ &= \lambda\left(\lambda^2 - (I+\delta(\lambda)B)A\right)^{-1}Cx.\end{aligned} \tag{17}$$

Then from Definition 1, $(I+\delta(\lambda)B)A$ subgenerates an exponentially bounded K-convoluted C-cosine function $\{\widehat{C}_K(t)\}_{t\geq 0}$. □

Theorem 9. *Suppose A is a subgenerator of an exponentially bounded K-convoluted C-cosine function $\{C_K(t)\}_{t\geq 0}$ on X, $S_K(t) = \int_0^t C_K(s)ds$. Let $B \in L(X)$ with $BC = CB$ and let $R(B) \subset R(C)$, and $D(A)$ is dense in X. If for any $\Phi \in \mathbf{C}([0,\infty), L(X))$,*

$$\begin{aligned}\left\|\int_0^t \Phi(s)C^{-1}BAS_K(t-s)\,x\,ds\right\| &\leq \widetilde{M}\int_0^t e^{\omega(t-s)}\|\Phi(s)\|\,ds\cdot\|x\|, \\ &\qquad x \in D(A),\ t \geq 0,\end{aligned} \tag{18}$$

where $\widetilde{M}$ is a constant, then for some (and all) λ, $\operatorname{Re}\lambda > \max(\omega,\beta)$, $(I+\widehat{K}(\lambda)B)A$ subgenerates an exponentially bounded K-convoluted C-cosine function on X.

Proof. Define the operator functions $\{\overline{C}_n(t)\}_{t\geq 0}$ as follows:

$$\begin{aligned}\overline{C}_0(t)\,x &= C_K(t)\,x, \\ \overline{C}_n(t)\,x &= \int_0^t \overline{C}_{n-1}(s)C^{-1}BAS_K(t-s)\,x\,ds, \\ &\qquad x \in D(A),\quad t \geq 0,\ n = 1,2,\ldots.\end{aligned} \tag{19}$$

By induction, we obtain

(i) for any $x \in X$, $\overline{C}_n(t)x \in \mathbf{C}([0,\infty), X)$,

(ii) $\|\overline{C}_n(t)x\| \leq (M\widetilde{M}^n t^n/n!)e^{\omega t}\|x\|$, $t \geq 0$, $x \in X$, for all $n \geq 0$.

Define the operator function

$$h(t) = \sum_{n=0}^{\infty}\overline{C}_n(t), \quad t \geq 0. \tag{20}$$

Noting that the series $\sum_{n=0}^{\infty}(M\widetilde{M}^n t^n/n!)e^{\omega t}$ is uniformly converge on every compact interval in t, we can see that the series (20) is uniformly converge on every compact interval in t, so does the operator $h(t)$. It is obvious that $\|h(t)\| \leq Me^{(\omega+\widetilde{M})t}$ and $t \to h(t)x$ is continuous on $[0,\infty)$ for any $x \in X$. Moreover,

$$h(t)\,x = C_K(t)\,x + \int_0^t h(s)C^{-1}BAS_K(t-s)\,x\,ds, \quad x \in X,\ t \geq 0. \tag{21}$$

For $\operatorname{Re}\lambda$ sufficiently large, we set

$$\mathscr{L}(\lambda)x=\int_0^\infty e^{-\lambda t}h(t)x\,dt,\quad x\in X. \tag{22}$$

Next, we show that the following equalities hold:

$$\mathscr{L}(\lambda)\left[\lambda^2-\left(I+\widehat{K}(\lambda)B\right)A\right]x=\lambda\widehat{K}(\lambda)Cx,\quad x\in D(A), \tag{23}$$

$$\left[\lambda^2-\left(I+\widehat{K}(\lambda)B\right)A\right]\mathscr{L}(\lambda)x=\lambda\widehat{K}(\lambda)Cx,\quad x\in X. \tag{24}$$

By induction, it is not difficult to see that

$$\begin{aligned}\int_0^\infty & e^{-\lambda t}\overline{C}_n(t)x\,dt\\ &=\lambda\widehat{K}(\lambda)\left(\lambda^2-A\right)^{-1}\left[\widehat{K}(\lambda)BA\left(\lambda^2-A\right)^{-1}\right]^nCx,\\ &\qquad x\in X,\ n\ge 0.\end{aligned} \tag{25}$$

Let

$$Q(t)x=\int_0^t C^{-1}BAS_K(t-s)x\,ds,\quad x\in D(A). \tag{26}$$

By hypothesis, $Q(t)$ can be extended to X and satisfies

$$\|Q(t)\|\le\frac{\widetilde{M}}{\omega}\left(e^{\omega t}-1\right),\quad t\ge 0. \tag{27}$$

Set

$$\widehat{Q}(\lambda)x=\int_0^\infty e^{-\lambda t}Q(t)x\,dt,\quad x\in X. \tag{28}$$

Then from (25) and (27), $\|\lambda\widehat{Q}(\lambda)\|=\|C^{-1}\widehat{K}(\lambda)BA(\lambda^2-A)^{-1}C\|<1$ for $|\lambda|$ sufficiently large. Therefore, the series

$$\begin{aligned}\sum_{n=0}^\infty&\left[\widehat{K}(\lambda)BA\left(\lambda^2-A\right)^{-1}\right]^nC\\ &=\sum_{n=0}^\infty C\left[C^{-1}\widehat{K}(\lambda)BA\left(\lambda^2-A\right)^{-1}C\right]^n\end{aligned} \tag{29}$$

converges.

For $x\in D(A)$ and $\operatorname{Re}\lambda>\max(\omega,\beta)$, from (25), we have

$$\begin{aligned}\mathscr{L}&(\lambda)\left[\lambda^2-\left(I+\widehat{K}(\lambda)B\right)A\right]x\\
&=\int_0^\infty e^{-\lambda t}\sum_{n=0}^\infty\overline{C}_n(t)\left[\lambda^2-\left(I+\widehat{K}(\lambda)B\right)A\right]x\,dt\\
&=\widehat{K}(\lambda)\sum_{n=0}^\infty\lambda\left(\lambda^2-A\right)^{-1}\left[\widehat{K}(\lambda)BA\left(\lambda^2-A\right)^{-1}\right]^n\\
&\quad\times C\left[\lambda^2-\left(I+\widehat{K}(\lambda)B\right)A\right]x\\
&=\lambda\widehat{K}(\lambda)Cx-\lambda\left(\widehat{K}(\lambda)\right)^2\left(\lambda^2-A\right)^{-1}CBAx\\
&\quad+\sum_{n=1}^\infty\lambda\widehat{K}(\lambda)\left(\lambda^2-A\right)^{-1}\left[\widehat{K}(\lambda)BA\left(\lambda^2-A\right)^{-1}\right]^n\\
&\quad\times C\left(\lambda^2-A\right)x\\
&\quad-\sum_{n=1}^\infty\lambda\widehat{K}(\lambda)\left(\lambda^2-A\right)^{-1}\left[\widehat{K}(\lambda)BA\left(\lambda^2-A\right)^{-1}\right]^n\\
&\quad\times C\widehat{K}(\lambda)BAx\\
&=\lambda\widehat{K}(\lambda)Cx+\sum_{n=2}^\infty\lambda\widehat{K}(\lambda)\left(\lambda^2-A\right)^{-1}\\
&\quad\times\left[\widehat{K}(\lambda)BA\left(\lambda^2-A\right)^{-1}\right]^nC\left(\lambda^2-A\right)x\\
&\quad-\sum_{n=1}^\infty\lambda\widehat{K}(\lambda)\left(\lambda^2-A\right)^{-1}\left[\widehat{K}(\lambda)BA\left(\lambda^2-A\right)^{-1}\right]^n\\
&\quad\times C\widehat{K}(\lambda)BAx\\
&=\lambda\widehat{K}(\lambda)Cx.\end{aligned} \tag{30}$$

Similarly, we can prove (24). Now, from Definition 1, we conclude that $(I+\widehat{K}(\lambda)B)A$ subgenerates an exponentially bounded K-convoluted C-cosine function on X. □

By the proof of Theorems 8 and 9, we immediately obtain the following results for K-convoluted C-semigroups.

Theorem 10. *Suppose that A is a subgenerator of an exponentially bounded K-convoluted C-semigroup $\{T_K(t)\}_{t\ge0}$ on X. $D(A)$ is dense in X. Let $B\in L(X)$ with $BC=CB$ and let $R(B)\subset R(C)$.*

(i) *One sets*

$$\begin{aligned}L(\lambda):=\sup&\left\{\int_0^a e^{-\lambda s}\left\|\delta(\lambda)C^{-1}PAT_K(s)x\right\|ds,\right.\\ &\left.\quad x\in D(A),\|x\|\le1\right\}<\infty,\end{aligned} \tag{31}$$

for some $a\in(0,+\infty]$ and $\lambda>\max(\omega,\beta)$, where $\delta(\lambda)$ is a function and $P=B/\widehat{K}(\lambda)$. If $\{\lambda:\lambda>\max(\omega,\beta)\}\subset\rho((I+\delta(\lambda)B)A)$, then $(I+\delta(\lambda)B)A$ subgenerates an *exponentially bounded K-convoluted C-semigroup on X provided that $\lim_{\lambda\to\infty}L(\lambda)e^{\lambda t}=0$ for all $t\geq 0$.*

(ii) *If for any $\Phi\in\mathbf{C}([0,\infty),L(X))$,*

$$\left\|\int_0^t \Phi(s)C^{-1}BAT_K(t-s)x\,ds\right\| \leq \widetilde{M}\int_0^t e^{\omega(t-s)}\|\Phi(s)\|\,ds\cdot\|x\|, \quad x\in D(A),\ t\geq 0, \tag{32}$$

where $\widetilde{M}$ is a constant, then for some (and all) λ, $\operatorname{Re}\lambda>\max(\omega,\beta)$, *$(I+\widehat{K}(\lambda)B)A$ subgenerates an exponentially bounded K-convoluted C-semigroup on X.*

Proof. (i) For any $\Psi\in\mathbf{C}([0,\infty),L(X))$ with $\|\Psi(t)\|=O(e^{\omega t})$, sufficiently large λ and sufficiently small ε, we have

$$\begin{aligned}\left\|\int_0^t \delta(\lambda)\Psi(s)C^{-1}PAT_K(t-s)x\,ds\right\| &\leq M^*e^{(\lambda+\omega)t}\int_0^t e^{-\lambda s}\left\|\delta(\lambda)C^{-1}PAT_K(s)x\right\|ds \\ &\leq M^*e^{(\lambda+\omega)t}L(\lambda)<\varepsilon<1, \quad t\geq 0,\ x\in D(A),\ \|x\|\leq 1,\end{aligned}\tag{33}$$

where M^* is a constant. The rest part of the proof is exactly the same as the corresponding part of the proof of Theorem 8.

The proof of (ii) is similar to the one of Theorem 9. □

In Theorems 8–10, take $K(t)=t^{n-1}/\Gamma(n)$, we have the following result for n-times integrated C-cosine function (resp., n-times integrated C-semigroup).

Theorem 11. *Suppose A is a subgenerator of an exponentially bounded n-times integrated C-cosine function $\{C_n(t)\}_{t\geq 0}$ (resp., n-times integrated C-semigroup $\{T_n(t)\}_{t\geq 0}$) on X. Let $B\in L(X)$ with $BC=CB$ and let $R(B)\subset R(C)$, and $D(A^{n+1})$ is dense in X.*

(i) *One sets*

$$L(\lambda):=\sup\left\{\int_0^a e^{-\lambda t}\left\|\int_0^t \delta(\lambda)\Psi(s)C^{-1}BA\times\left(\frac{d^n}{dt^n}C_n(t-s)x\right)ds\right\|dt,\ x\in D\left(A^{n+1}\right),\|x\|\leq 1\right\}<\infty, \tag{34}$$

for any $\Psi\in\mathbf{C}([0,\infty),L(X))$ with $\|\Psi(t)\|=O(e^{\omega t})$,

$$\left(\text{resp. } L(\lambda):=\sup\left\{\int_0^a e^{-\lambda s}\left\|\delta(\lambda)C^{-1}BA\left(\frac{d^n}{ds^n}T_n(s)x\right)\right\|ds,\ x\in D\left(A^{n+1}\right),\|x\|\leq 1\right\}<\infty\right) \tag{35}$$

for some $a\in(0,+\infty]$ and $\lambda>\omega$, where $\delta(\lambda)$ is a function. If $(\omega^2,\infty)\subset\rho((I+\delta(\lambda)B)A)$ (resp., $(\omega,\infty)\subset\rho((I+\delta(\lambda)B)A)$), then $(I+\delta(\lambda)B)A$ subgenerates an exponentially bounded n-times integrated C-cosine function (resp., n-times integrated C-semigroup) on X provided that $\lim_{\lambda\to\infty}L(\lambda)e^{\lambda t}=0$ for all $t\geq 0$.

(ii) *If for any $\Phi\in\mathbf{C}([0,\infty),L(X))$,*

$$\left\|\int_0^t \Phi(s)C^{-1}BA\left(\frac{d^n}{dt^n}S_n(t-s)x\right)ds\right\| \leq \widetilde{M}\int_0^t e^{\omega(t-s)}\|\Phi(s)\|\,ds\cdot\|x\|, \quad x\in D\left(A^{n+1}\right),\ t\geq 0, \tag{36}$$

where $S_n(t)=\int_0^t C_n(s)ds$ and $\widetilde{M}$ is a constant,

$$\left(\text{resp. } \left\|\int_0^t \Phi(s)C^{-1}BA\left(\frac{d^n}{dt^n}T_n(t-s)x\right)ds\right\| \leq \widetilde{M}\int_0^t e^{\omega(t-s)}\|\Phi(s)\|\,ds\cdot\|x\|, \quad x\in D\left(A^{n+1}\right),t\geq 0\right) \tag{37}$$

then for some (and all) λ, $\lambda>\omega$, $(I+\widehat{K}(\lambda)B)A$ subgenerates an exponentially bounded n-times integrated C-cosine function (resp., n-times integrated C-semigroup) on X.

When $n=0$, from Theorem 11(ii), we immediately obtain the result of 0-times integrated C-cosine function (resp., 0-times integrated C-semigroup).

Theorem 12. *Let $B\in L(X)$ with $BC=CB$ and let $R(B)\subset R(C)$, and $D(A)$ is dense in X. Suppose that A is an exponentially bounded generator of a C-cosine function $\{C(t)\}_{t\geq 0}$ (resp., C-semigroup $\{T(t)\}_{t\geq 0}$) on X. If for any $\Phi\in$ $\mathbf{C}([0,\infty),L(X))$,*

$$\left\|\int_0^t \Phi(s)C^{-1}BAS(t-s)x\,ds\right\| \leq M\int_0^t e^{\omega(t-s)}\|\Phi(s)\|\,ds\cdot\|x\|, \quad x\in D(A),\ t\geq 0, \tag{38}$$

where $S(t)=\int_0^t C(s)ds$.

$$\left(\text{resp. } \left\|\int_0^t \Phi(s)C^{-1}BAT(t-s)x\,ds\right\| \leq M\int_0^t e^{\omega(t-s)}\|\Phi(s)\|\,ds\cdot\|x\|, \quad x\in D(A),\ t\geq 0\right) \tag{39}$$

for some $a \in (0, +\infty]$ and $\lambda > \omega$, then $(I+B)A$ subgenerates an exponentially bounded C-cosine function (resp., C-semigroup) on X.

Noting the Definition 7 and the special properties of C-cosine functions (resp., C-semigroups), we obtain a different result from Theorem 11(i) (when $n = 0$).

Theorem 13. *Let $B \in L(X)$ with $BC = CB$ and let $R(B) \subset R(C)$, and $D(A)$ is dense in X, $(\omega^2, \infty) \subset \rho((I+B)A)$ (resp., $(\omega, \infty) \subset \rho((I+B)A)$). Suppose that A is an exponentially bounded generator of a C-cosine function $\{\mathrm{C}(t)\}_{t\geq 0}$ (resp., C-semigroup $\{\mathrm{T}(t)\}_{t\geq 0}$) on X. If*

$$L(\lambda) := \sup \left\{ \int_0^a e^{-\lambda t} \left\| \int_0^t C^{-1} BA\mathrm{C}(t-s)\, x ds \right\| dt, \quad x \in D(A), \|x\| \leq 1 \right\} < \infty, \tag{40}$$

$$\left(\text{resp. } L(\lambda) := \sup \left\{ \int_0^a e^{-\lambda s} \left\| C^{-1} BA\mathrm{T}(s)\, x \right\| ds, \quad x \in D(A), \|x\| \leq 1 \right\} < \infty \right) \tag{41}$$

for some $a \in (0, +\infty]$ and $\lambda > \omega$, letting $L(\infty) = \lim_{\lambda\to\infty} L(\lambda)$, then for any $\varepsilon < (L(\infty))^{-1}$, $(I+\varepsilon B)A$ subgenerates an exponentially bounded C-cosine function (resp., C-semigroup) on X.

Proof. We prove only for C-cosine functions. Choose $0 < \mu < \mu_1 < \mu_2 < 1$ such that $|\varepsilon| = \mu(L(\infty))^{-1}$. For any $\Psi \in \mathbf{C}([0,t], L(X))$, pick a λ large enough such that $L(\lambda)/L(\infty) < \mu_1/\mu$, and then pick a $\tau \in (0, a)$ small enough such that $e^{\lambda\tau} \sup_{s\in[0,\tau]} \|\Psi(s)\| \leq \mu_2/\mu_1$, then for all $x \in D(A)$, $\|x\| \leq 1$, we have

$$\begin{aligned}
&\left\| \int_0^t \varepsilon \Psi(s)\, C^{-1} BA\mathrm{S}(t-s)\, x ds \right\| \\
&\quad = \left\| \int_0^t \int_0^s \varepsilon \Psi(\sigma)\, C^{-1} BA\mathrm{C}(s-\sigma)\, x d\sigma ds \right\| \\
&\quad \leq e^{\lambda t} \int_0^t e^{-\lambda s} \left\| \int_0^s \varepsilon C^{-1} BA\mathrm{C}(s-\sigma)\, x d\sigma \right\| ds \cdot \sup_{s\in[0,\tau]} \|\Psi(s)\| \\
&\quad \leq e^{\lambda\tau} |\varepsilon|\, L(\lambda) \cdot \sup_{s\in[0,\tau]} \|\Psi(s)\| < \mu_2 < 1, \quad t \in [0, \tau],
\end{aligned} \tag{42}$$

where $\mathrm{S}(t) = \int_0^t \mathrm{C}(s) ds$.

Let $\mathscr{V} : [0,\tau] \to L(X)$ be any strongly continuous function; we define

$$\begin{aligned}
&(\mathscr{M}\mathscr{V})(t)\, x \\
&\quad = \int_0^t \varepsilon \mathscr{V}(s)\, C^{-1} BA\mathrm{S}(t-s)\, x ds, \quad x \in D(A),\ t \in [0,\tau].
\end{aligned} \tag{43}$$

Obviously, $(\mathscr{M}\mathscr{V})(t)x$ is continuous on $t \geq 0$, from (42) and the denseness of $D(A)$, $\mathscr{M}$ maps $\mathbf{C}([0,\tau], L(X))$ into $\mathbf{C}([0,\tau], L(X))$.

It follows from (42) that $(I - \mathscr{M})^{-1}$ is bounded. For each $t \in [0,\tau]$, set

$$V(t)\, x := (I - \mathscr{M})^{-1} [\mathrm{C}(\cdot)\, x](t), \quad x \in X. \tag{44}$$

Then, $V(t)C = CV(t)$, and there exists a constant $\overline{M}$ such that $\|V(t)\| \leq \overline{M} e^{\omega t}$:

$$V(t)\, x = \mathrm{C}(t)\, x + \int_0^t \varepsilon V(s)\, C^{-1} BA\mathrm{S}(t-s)\, x ds, \quad t \in [0,\tau]. \tag{45}$$

For $t \in ((n-1)\tau, n\tau]$, $n = 2, 3, \ldots$, we define inductively

$$\begin{aligned}
V(t) := &-V(2(n-1)\tau - t) \\
&+ 2C^{-1} V(t-(n-1)\tau)\, V((n-1)\tau).
\end{aligned} \tag{46}$$

Next, we will prove by induction that for any $n \in \mathbb{N}$, $R(V(\sigma)V((n-1)\tau)) \subset R(C)$, for $\sigma \in [0,\tau]$, and that for every $n \in \mathbb{N}$, $V(\cdot)$ is strongly continuous in $[0, n\tau]$ and

$$V(t)\, x = \mathrm{C}(t)\, x + \int_0^t \varepsilon V(s)\, C^{-1} BA\mathrm{S}(t-s)\, x ds, \quad x \in X,\ t \in [0, n\tau]. \tag{47}$$

Indeed for $n = 1$, this is true. Assume that (47) holds for n. Then for $x \in X$, $\sigma \in [0,\tau]$ we get

$$\begin{aligned}
&2V(\sigma)\, V(n\tau)\, x \\
&\quad = 2\mathrm{C}(\sigma)\, \mathrm{C}(n\tau)\, x \\
&\qquad + 2\int_0^{n\tau} \varepsilon V(\sigma)\, V(s)\, C^{-1} BA\mathrm{S}(n\tau - s)\, x ds \\
&\qquad + 2\int_0^{\sigma} \varepsilon V(s)\, C^{-1} BA\mathrm{S}(\sigma - s)\, \mathrm{C}(n\tau)\, x ds \\
&\quad = C[\mathrm{C}(\sigma + n\tau)\, x + \mathrm{C}(n\tau - \sigma)\, x] \\
&\qquad + 2\int_0^{n\tau} \varepsilon V(\sigma)\, V(s)\, C^{-1} BA\mathrm{S}(n\tau - s)\, x ds \\
&\qquad + 2\int_0^{\sigma} \varepsilon V(s)\, C^{-1} BA\mathrm{S}(\sigma - s)\ \mathrm{C}(n\tau)\, x ds \\
&\quad = 2\mathscr{M}[V(\sigma)\, V(\cdot)](n\tau)\, x + C[\mathrm{C}(\sigma + n\tau)x + \mathrm{C}(n\tau - \sigma)\, x] \\
&\qquad + C\int_0^{\sigma} \varepsilon V(s)\, C^{-1} BA \\
&\qquad\quad \times [\mathrm{S}(n\tau + \sigma - s)\, x - \mathrm{S}(n\tau - \sigma + s)\, x]\, ds.
\end{aligned} \tag{48}$$

Then for $x \in X, \sigma \in [0, \tau]$,

$$2V(\sigma)V(n\tau)x = C(I-\mathscr{M})^{-1}\{C(\sigma+n\tau)x + C(n\tau-\sigma)x + \int_0^\sigma \varepsilon V(s)C^{-1}BA[S(n\tau+\sigma-s)x - S(n\tau-\sigma+s)x]\,ds\}. \quad (49)$$

Hence, $V(\sigma)V(n\tau) \subset R(C)$, $\sigma \in [0,\tau]$, and $\sigma \to C^{-1}V(\sigma)V(n\tau)x$ is continuous in $[0,\tau]$ for each $x \in X$. From (48), we have

$$\begin{aligned}
2V(\sigma)V(n\tau)x &= C[C(\sigma+n\tau)x + C(n\tau-\sigma)x] \\
&\quad + 2\int_0^{n\tau} \varepsilon V(\sigma)V(s)C^{-1}BAS(n\tau-s)x\,ds \\
&\quad + C\int_0^\sigma \varepsilon V(s)C^{-1}BA[S(n\tau+\sigma-s)x \\
&\quad - S(n\tau-\sigma+s)x]\,ds \\
&= C[C(\sigma+n\tau)x + C(n\tau-\sigma)x] + C\int_0^{n\tau} \varepsilon[V(\sigma+s) \\
&\quad + V(|s-\sigma|)]C^{-1}BAS(n\tau-s)x\,ds \\
&\quad + C\int_0^\sigma \varepsilon V(s)C^{-1}BA[S(n\tau+\sigma-s)x \\
&\quad - S(n\tau-\sigma+s)x]\,ds \\
&= C[C(\sigma+n\tau)x + C(n\tau-\sigma)x] \\
&\quad + C\int_\sigma^{\sigma+n\tau} \varepsilon V(s)C^{-1}BAS(n\tau+\sigma-s)x\,ds \\
&\quad + C\int_0^\sigma \varepsilon V(s)C^{-1}BAS(n\tau-\sigma+s)x\,ds \\
&\quad + C\int_0^{n\tau-\sigma} \varepsilon V(s)C^{-1}BAS(n\tau-\sigma-s)x\,ds \\
&\quad + C\int_0^\sigma \varepsilon V(s)C^{-1}BA[S(n\tau+\sigma-s)x \\
&\quad - S(n\tau-\sigma+s)x]\,ds \\
&= C[C(\sigma+n\tau)x + C(n\tau-\sigma)x] \\
&\quad + C\int_0^{\sigma+n\tau} \varepsilon V(s)C^{-1}BAS(n\tau+\sigma-s)x\,ds \\
&\quad + C\int_0^{n\tau-\sigma} \varepsilon V(s)C^{-1}BAS(n\tau-\sigma-s)x\,ds \\
&= CV(\sigma+n\tau)x + CV(n\tau-\sigma)x.
\end{aligned} \quad (50)$$

Therefore, $V(\cdot)$ is strongly continuous in $[0,\infty)$ and (47) holds for all $t \geq 0$. Taking Laplace transform of (47), then the conclusion can be proved in a similar way in the proof of Theorem 8.

We can prove the case of C-semigroups in a similar way. □

3. Examples

Example 14. Let

$$X := \left\{ f \in C^\infty[0,1] : \|f\| := \sup_{p\geq 0} \frac{\|f^{(p)}\|_\infty}{p!^2} < \infty \right\},$$
$$A := -\frac{d}{dx}, \qquad D(A) := \{ f \in X : f' \in X, f(0) = 0 \}. \quad (51)$$

It is well known that there exist positive real numbers m and M such that

$$\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq 0\} \subset \rho(A), \qquad \|R(\lambda, A)\| \leq Me^{m\sqrt{|\lambda|}}, \quad \operatorname{Re}\lambda \geq 0. \quad (52)$$

Moreover, A generates an exponentially bounded K_a-convoluted semigroup $\{T_K(t)\}_{t\geq 0}$ for some $a > \sqrt{2}m$, where $K(t) = (a/(2\sqrt{\pi t^3}))e^{-a^2/(4t)}$, $t \geq 0$, then $\widehat{K}_a(\lambda) = e^{-a\sqrt{\lambda}}$, $\operatorname{Re}\lambda > 0$ [14, 23]. We set

$$Bf(x) := \sum_{n=1}^{j} \int_0^x \frac{(x-s)^{n-1}}{(n-1)!} f(s)\,ds, \quad j \in \mathbb{N}, \ x \in [0,1], \quad f \in X. \quad (53)$$

Obviously, $B \in L(X)$ and $BA \subset AB$. Then from Theorem 10(ii) ($C = I$), $(I + e^{-a\sqrt{\lambda}}B)A$ subgenerates an exponentially bounded K_a-convoluted semigroup $\{\widetilde{T}_K(t)\}_{t\geq 0}$ on X.

Example 15. Let $X := C_0(\mathbb{R}) \oplus C_0(\mathbb{R}) \oplus C_0(\mathbb{R})$,

$$A(f,g,h)(\cdot) := (f', g', (\chi_{[0,\infty)} - \chi_{(-\infty,0]})h),$$
$$(f,g,h) \in D(A) = \{(f,g,h) \in X : f' \in C_0(\mathbb{R}), g' \in C_0(\mathbb{R}), h(0) = 0\} \quad (54)$$

and $C(f,g,h) := (f, g, \sin(\cdot)h(\cdot))$, $f,g,h \in C_0(\mathbb{R})$. Arguing as in [3, Examples 8.1 and 8.2], one gets that A is a generator of an exponentially bounded once integrated C-semigroup [16].

For $f,g,h \in C_0(\mathbb{R})$ and $t \in \mathbb{R}$, we set

$$B(f,g,h)(t) = \left(e^{-t}\int_0^t f(s)\,ds,\ e^{-2t}\int_0^t g(s)\,ds,\ te^{-3t}\sin t \cdot h(t) \right). \quad (55)$$

Then one can simply verify that $B \in L(X), R(B) \subset C(D(A))$, and $BC(f,g,h) = CB(f,g,h)$, $(f,g,h) \in X$. Then from Theorem 11(i), $(I + e^{-\lambda^2}B)A$ subgenerates an exponentially bounded once integrated C-semigroup on X.

Example 16. Let $X_1 = L^2(\mathbb{R}^3)$, $X_2 = L^p(\mathbb{R}^3)$ $(1 \le p \le \infty)$,

$$A_1 = \Delta, \qquad D(A_1) = H^2(\mathbb{R}^3),$$
$$A_2 = a\Delta + \sum_{i=1}^{3} c_i \frac{\partial}{\partial x_1} + c_4 \ \ (a > 0,\ c_i \in \mathbb{R},\ i = 1,2,3,4),$$
$$D(A_2) = W^{2,p}(\mathbb{R}^3). \tag{56}$$

Then A_1 generates a strongly continuous cosine function $\mathrm{C}_1(\cdot)$ on X_1. It follows from [5] that A_2 generates an exponentially bounded C_2-cosine function $\mathrm{C}_2(\cdot)$ on X_2, where $C_2 = (1-\Delta)^{-1}$.

Set $r_1(\cdot) \in H^2(\mathbb{R}^3)$, $r_2(\cdot) \in W^{2,p}(\mathbb{R}^3)$, $q_1(\cdot) \in C_c^2(\mathbb{R}^3)$, $q_2(\cdot) \in C_c(\mathbb{R}^3)$. Define bounded linear operators $B_1 : X_2 \to X_1$, $B_2 : X_1 \to X_2$ as follows:

$$(B_1\phi)(\xi) = r_1(\xi)\int_{\mathbb{R}^3} q_1(\sigma)\phi(\sigma)\,d\sigma,$$
$$(B_2\phi)(\xi) = r_2(\xi)\int_{\mathbb{R}^3} q_2(\sigma)\phi(\sigma)\,d\sigma. \tag{57}$$

Let $X = X_1 \times X_2$,

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \qquad D(A) := D(A_1)\times D(A_2),$$
$$B = \begin{pmatrix} 0 & B_1 \\ B_2 & 0 \end{pmatrix}, \qquad D(B) := X. \tag{58}$$

Taking $\lambda_0 \in \rho(A)$ and putting $C = (\lambda_0 - A)^{-1}$, then A generates an exponentially bounded C-cosine function $\mathrm{C}(\cdot)$ on X, where

$$\mathrm{C}(t) = \begin{pmatrix} \mathrm{C}_1(t)(\lambda_0 - A_1)^{-1} & 0 \\ 0 & \mathrm{C}_2(t)C_2^{-1}(\lambda_0 - A_2)^{-1} \end{pmatrix}. \tag{59}$$

We denote $\mathrm{S}_1(t) := \int_0^t \mathrm{C}_1(s)ds$, $\mathrm{S}_2(t) := \int_0^t \mathrm{C}_2(s)ds$, $\mathrm{S}(t) := \int_0^t \mathrm{C}(s)ds$, then

$$\mathrm{S}(t) = \begin{pmatrix} \mathrm{S}_1(t)(\lambda_0 - A_1)^{-1} & 0 \\ 0 & \mathrm{S}_2(t)C_2^{-1}(\lambda_0 - A_2)^{-1} \end{pmatrix}, \tag{60}$$

and for any $x = \binom{x_1}{x_2} \in D(A)$, $0 \le s \le t < \infty$,

$$C^{-1}BA\mathrm{S}(t-s)x = \begin{pmatrix} (\lambda_0 - A_1)B_1A_2\mathrm{S}_2(t-s)C_2^{-1}(\lambda_0 - A_2)^{-1}x_2 \\ (\lambda_0 - A_2)B_2A_1\mathrm{S}_1(t-s)(\lambda_0 - A_1)^{-1}x_1 \end{pmatrix}. \tag{61}$$

It follows from $R(B_1) \subset D(A_1)$ and $R(B_2) \subset D(A_2)$ that there exist $M, \omega > 0$ such that

$$e^{-\lambda t}\left\|\int_0^t C^{-1}BA\mathrm{C}(t-s)x\,ds\right\| \le \frac{M}{\omega}e^{-\lambda t}\left(e^{\omega t} - 1\right)\|x\|,\ x \in D(A), \tag{62}$$

then

$$L(\lambda) := \sup\left\{\int_0^a e^{-\lambda t}\left\|\int_0^t C^{-1}BA\mathrm{C}(t-s)x\,ds\right\|dt,\ x \in D(A), \|x\| \le 1\right\} < \infty, \tag{63}$$

and then (40) is satisfied.

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Research Article

Nontrivial Solutions for a Modified Capillary Surface Equation

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A negative solution and a positive solution are obtained for a modified capillary surface equation by variational methods.

1. Introduction

In this paper, we study the existence of nontrivial solutions to the following quasilinear elliptic equation:

$$-\operatorname{div}\left(\frac{|\nabla u|^{2p-2}\nabla u}{\sqrt{1+|\nabla u|^{2p}}}\right)=f(x,u)\quad \text{in } \Omega,\qquad u=0\quad \text{on } \partial\Omega, \tag{1}$$

where $p>1$, and Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary. The function $f\in C(\overline{\Omega}\times\mathbb{R},\mathbb{R})$ with the subcritical growth

(f)

$$\left|f(x,t)\right|\le c\left(1+|t|^{q-1}\right),\quad t\in\mathbb{R},\ x\in\Omega, \tag{2}$$

where $q\in[1,Np/(N-p))$ if $1<p<N$ or $q\in[1,+\infty)$ if $1<N\le p$, and c is a positive constant.

In the case that $p=1$, (1) is the mean curvature equation or the capillary surface equation; when $f(x,u)\equiv u$, it describes the equilibrium shape of a liquid surface with constant surface tension in a uniform gravity field, and this is the shape of a pendent drop [1]. When $p>1$, one calls (1) a modified capillary surface equation which is also worth considering even though it is not exactly the capillary surface equation [2]. For the capillary surface equation, radially symmetric solutions in the case that Ω is a ball or entire space have been investigated precisely; See, for example, [3–5] and the references therein. In [2], by minimization sequence method and the Ambrosetti-Rabinowitz mountain pass lemma without Palais-Smale condition, positive solutions were obtained to nonlinear eigenvalue problem for the modified capillary surface equation which is of the form

$$-\operatorname{div}\left(\frac{|\nabla u|^{2p-2}\nabla u}{\sqrt{1+|\nabla u|^{2p}}}\right)=\lambda f(x,u)\quad \text{in } \Omega,\qquad u\ge 0\quad \text{in } \Omega,\qquad u=0\quad \text{on } \partial\Omega, \tag{3}$$

where λ is a positive parameter. In the proof of the main results of [2], λ is crucial not only to the existence of global or local minimizer but also to the construction of mountain pass geometry. In our paper, one object is to find existence conditions of solutions to (1) without the constraint of λ. Since

$$\sqrt{1+|\nabla u|^{2p}}-1\sim|\nabla u|^{p}\quad \text{as } |\nabla u|\longrightarrow\infty, \tag{4}$$

the other object is to investigate the probability to present the property of f by the eigenvalue of the problem

$$-\Delta_p u=\lambda|u|^{p-2}u\quad \text{in } \Omega,\qquad u=0\quad \text{on } \partial\Omega, \tag{5}$$

where $\Delta_p u=\operatorname{div}(|\nabla u|^{p-2}\nabla u)$.

In the following, we recall some known facts about problem (5). Let $\lambda_1 > 0$ be the first eigenvalue of the problem (5). It is known that λ_1 is characterized by

$$\lambda_1 := \inf \left\{ \int_\Omega |\nabla u|^p dx : \int_\Omega |u|^p dx = 1, u \in W_0^{1,p}(\Omega) \setminus \{0\} \right\}, \tag{6}$$

where $W_0^{1,p}(\Omega)$ is the reflexive Banach space defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\| := (\int_\Omega |\nabla u|^p dx)^{1/p}$. Also, λ_1 is single and has an associated eigenfunction $\varphi_1 > 0$ in Ω and $\|\varphi_1\| = 1$. The reader is referred to [6, 7] for details.

By a solution u of (1), we mean that u satisfies (1) in the weak sense; that is, for all $\varphi \in W_0^{1,p}(\Omega)$,

$$\int_\Omega \frac{|\nabla u|^{2p-2} \nabla u \nabla \varphi}{\sqrt{1 + |\nabla u|^{2p}}} dx = \int_\Omega f(x, u) \varphi dx. \tag{7}$$

A solution such that $u(x) \geq 0$ in Ω and $u \neq 0$, respectively, $u(x) \leq 0$ in Ω and $u \neq 0$, is a positive, respectively, negative, solution.

Define

$$\begin{aligned} J(u) &= \frac{1}{p} \int_\Omega \left(\sqrt{1 + |\nabla u|^{2p}} - 1 \right) dx, \quad u \in W_0^{1,p}(\Omega), \\ K(u) &= \int_\Omega F(x, u) dx, \quad u \in W_0^{1,p}(\Omega), \\ I(u) &= J(u) - K(u), \quad u \in W_0^{1,p}(\Omega), \end{aligned} \tag{8}$$

where $F(x,t) = \int_0^t f(x,s) ds$. From a variational stand point, finding solutions of (1) in $W_0^{1,p}(\Omega)$ is equivalent to finding critical points of the C^1 functional I. As to the differentiability of the functional I, one can consult [2] for details. Since f satisfies the subcritical growth condition (f_0), stand proofs show that K is weakly continuous. Since the function $\varphi(t) = \sqrt{1 + t^{2p}}$ is convex, the functional J is also convex. In addition, J belongs to C^1. Hence, J is weakly lower semi-continuous. Thus, we have shown that I is weakly lower semi-continuous.

Now, let us state the main results of this paper.

Theorem 1. *Let* (f) *hold. Furthermore, assume that* f *satisfies the following conditions.*

(f_0) *There is some* $r > 0$ *small such that*

$$pF(x,t) \geq \lambda_1 |t|^p, \quad |t| \leq r, \ x \in \Omega, \tag{9}$$

(f_1) $\limsup_{|t| \to \infty} (pF(x,t)/|t|^p) < \lambda_1$ *uniformly for* $x \in \Omega$.

Then, (1) *has at least a negative solution and a positive solution which correspond to negative critical values of the associated functional given by* (8).

Theorem 2. *Let* (f) *and* (f_0) *hold. Furthermore, assume that* f *satisfies the following conditions.*

(f_2) $\lim_{|t| \to \infty} (pF(x,t)/|t|^p) = \lambda_1$ *uniformly for* $x \in \Omega$,

(f_3) $\lim_{|t| \to \infty} (f(x,t) - pF(x,t)) = +\infty$ *uniformly for* $x \in \Omega$.

Then, (1) *has at least a negative solution and a positive solution which correspond to negative critical values of the associated functional given by* (8).

Remark 3. With the conditions (f_0)–(f_3), Liu and Su in [8] have studied the existence of solutions to p-Laplacian quasilinear elliptic equation

$$\begin{aligned} -\Delta_p u &= f(x, u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{10}$$

Under the conditions (f_0) and (f_1), (10) may be resonant at the eigenvalue λ_1 near the origin. With the conditions (f_2) and (f_3), it may be resonant at λ_1 both near the origin and near infinity. In fact, the condition (f_0) allows (10) to be resonant near the origin from the right side of λ_1, while the conditions (f_2) and (f_3) allow it to be resonant at infinity from the left side of λ_1.

Remark 4. Theorems 1 and 2 have shown a new fact that the interaction between the first eigenvalue of $-\Delta_p$ with zero Dirichlet boundary data and nonlinearity f can influence the existence of nontrivial solutions to (1).

Before concluding this section, we explain some notations used in the paper. $|\Omega|$ is the Lebesgue measure of Ω. c_i $(i \in \mathbb{N})$ is always a positive constant independent of functions. $\langle \cdot, \cdot \rangle$ is the duality between $(W_0^{1,p}(\Omega))^*$ and $W_0^{1,p}(\Omega)$. In addition, we use $|\cdot|$ to denote the usual norm of $\mathbb{R}^N$.

2. The Proof of the Main Results

In this section, we prove Theorems 1 and 2.

Proof of Theorem 1. The proof consists of two steps.

(i) To obtain a positive solution, cut-off techniques are used. Define

$$\begin{aligned} \widehat{f}(x,t) &= \begin{cases} f(x,t), & t \geq 0, \\ 0, & t < 0, \end{cases} \\ \widehat{F}(x,t) &= \int_0^t \widehat{f}(x,s) ds, \\ \widehat{I}(u) &= J(u) - \int_\Omega \widehat{F}(x,u) dx, \quad u \in W_0^{1,p}(\Omega). \end{aligned} \tag{11}$$

Since $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ and (f_1) holds, for any given $\varepsilon > 0$, there exists $c_1 > 0$ such that

$$\widehat{F}(x,t) \leq \frac{1}{p} (\lambda_1 - \varepsilon) |t|^p + c_1, \quad t \in \mathbb{R}, \ x \in \Omega. \tag{12}$$

By the Poincaré inequality, for $u \in W_0^{1,p}(\Omega)$,

$$\begin{aligned}\widehat{I}(u) &= \frac{1}{p}\int_\Omega \sqrt{1+|\nabla u|^{2p}}dx - \int_\Omega \widehat{F}(x,u)\,dx - \frac{|\Omega|}{p} \\ &\geq \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{1}{p}(\lambda_1-\varepsilon)\int_\Omega |u|^p dx \\ &\quad - \left(c_1+\frac{1}{p}\right)|\Omega| \\ &\geq \frac{1}{p}\left(1-\frac{\lambda_1-\varepsilon}{\lambda_1}\right)\|u\|^p - \left(c_1+\frac{1}{p}\right)|\Omega| \\ &= \frac{\varepsilon}{p\lambda_1}\|u\|^p - \left(c_1+\frac{1}{p}\right)|\Omega|.\end{aligned} \tag{13}$$

Hence, $\widehat{I}$ is coercive; that is, $\widehat{I}(u) \to \infty$ as $n \to \infty$. In addition, since $\widehat{f}$ also satisfies the condition (f), $\widehat{I}$ is weakly lower semi-continuous. So, it has a global minimizer.

Take a number $t_0 > 0$ such that $0 < t_0\varphi_1 \leq r$ in Ω. By the condition (f_0), we have that

$$\begin{aligned}\widehat{I}(t_0\varphi_1) &= \frac{1}{p}\int_\Omega\left(\sqrt{1+t_0^{2p}|\nabla\varphi_1|^{2p}}-1\right)dx \\ &\quad - \int_\Omega F(x,t_0\varphi_1)\,dx \\ &< \frac{1}{p}t_0^p\int_\Omega|\nabla\varphi_1|^p dx - \frac{1}{p}\lambda_1 t_0^p\int_\Omega \varphi_1^p dx \\ &= 0.\end{aligned} \tag{14}$$

Thus, the global minimizer of $\widehat{I}$ is a nontrivial critical point, denoted by u_1 which satisfies $\widehat{I}(u_1) < 0$. Putting $u_1^-(x) = \min\{u_1(x),0\}$, we have that

$$\left\langle \widehat{I}'(u_1), u_1^- \right\rangle = \int_\Omega \frac{|\nabla u_1^-|^{2p}}{\sqrt{1+|\nabla u_1^-|^{2p}}}dx = 0. \tag{15}$$

Hence, $u_1^- = 0$. So, u_1 is a positive solution of (1), and $I(u_1) < 0$.

(ii) To obtain a negative solution, we only need to replace $\widehat{f}$ with

$$\widetilde{f}(x,t) = \begin{cases} 0, & t > 0, \\ f(x,t), & t \leq 0. \end{cases} \tag{16}$$

Similar to step (i), it is shown that (1) has a negative solution u_2 with $I(u_2) < 0$.

The proof is completed. □

Proof of Theorem 2. We adopt the notations in the proof of Theorem 1.

First of all, we show that the functional $\widehat{I}$ is also coercive under the conditions (f_2) and (f_3). Write

$$\begin{aligned}\widehat{F}(x,t) &= \frac{1}{p}\lambda_1(t^+)^p + \widehat{G}(x,t), \\ \widehat{f}(x,t) &= \lambda_1|t|^{p-2}t^+ + \widehat{g}(x,t),\end{aligned} \tag{17}$$

where $t^+ = \max\{t,0\}$. Given $x \in \Omega$, we have that

$$\begin{aligned}&\lim_{t\to+\infty}\frac{p\widehat{G}(x,t)}{t^p} = 0, \\ &\lim_{t\to+\infty}\left(\widehat{g}(x,t)t - p\widehat{G}(x,t)\right) = +\infty.\end{aligned} \tag{18}$$

Thus, for every $M > 0$, there exists $R_M > 0$ such that

$$\widehat{g}(x,t)t - p\widehat{G}(x,t) \geq M, \quad t \geq R_M,\ x \in \Omega. \tag{19}$$

Integrating the equality

$$\frac{d}{dt}\left(\frac{\widehat{G}(x,t)}{t^p}\right) = \frac{\widehat{g}(x,t)t - p\widehat{G}(x,t)}{t^{p+1}} \tag{20}$$

over the interval $[t,T] \subset [R_M,+\infty)$,

$$\frac{\widehat{G}(x,T)}{T^p} - \frac{\widehat{G}(x,t)}{t^p} \geq \frac{M}{p}\left(\frac{1}{t^p}-\frac{1}{T^p}\right). \tag{21}$$

Letting $T \to +\infty$, we show that $\widehat{G}(x,t) \leq -M/p$, $t \geq R_M$.

Suppose that $\{u_n\} \subset W_0^{1,p}(\Omega)$ satisfies $\|u_n\| \to \infty$ and $\widehat{I}(u_n) \leq C$ for some constant $C \in \mathbb{R}$. Let $v_n = u_n/\|u_n\|$. Up to subsequence if necessary, we may assume that there exists $v_0 \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned}v_n &\rightharpoonup v_0 \quad \text{in } E, \\ v_n &\longrightarrow v_0 \quad \text{in } L^p(\Omega), \\ v_n(x) &\longrightarrow v_0(x) \quad \text{a.e. } x \in \Omega.\end{aligned} \tag{22}$$

Given $M = 1$ in (19), we have that

$$\widehat{G}(x,t) \leq -\frac{1}{p}, \quad t \geq R_1. \tag{23}$$

Let $c_2 = \max_{(x,t)\in\overline{\Omega}\times[-R_1,R_1]}|\widehat{G}(x,t)|$. Thus,

$$\begin{aligned}\frac{C}{\|u_n\|^p} &\geq \frac{1}{p\|u_n\|^p}\left(\int_\Omega\sqrt{1+|\nabla u|^{2p}}dx - \lambda_1\int_\Omega|u_n|^p dx\right) \\ &\quad - \frac{1}{\|u_n\|^p}\int_\Omega\widehat{G}(x,u_n)\,dx - \frac{|\Omega|}{p\|u_n\|^p} \\ &\geq \frac{1}{p}\int_\Omega\left(|\nabla v_n|^p - \lambda_1|v_n|^p\right)dx - \frac{|\Omega|}{p\|u_n\|^p} \\ &\quad - \frac{1}{\|u_n\|^p}\int_{|u_n|\geq R_1}\widehat{G}(x,u_n)\,dx \\ &\quad - \frac{1}{\|u_n\|^p}\int_{|u_n|\leq R_1}\widehat{G}(x,u_n)\,dx \\ &\geq \frac{1}{p}\int_\Omega\left(|\nabla v_n|^p - \lambda_1|v_n|^p\right)dx - \frac{|\Omega|}{p\|u_n\|^p} - \frac{c_2|\Omega|}{\|u_n\|^p} \\ &= \frac{1}{p}\int_\Omega\left(|\nabla v_n|^p - \lambda_1|v_n|^p\right)dx + \frac{c_3}{\|u_n\|^p},\end{aligned} \tag{24}$$

where $c_3 = (1/p+c_2)|\Omega|$. It follows from (22) and the previous inequality that

$$\limsup_{n\to\infty}\int_\Omega |\nabla v_n|^p dx \le \lambda_1 \int_\Omega |v_0|^p dx. \tag{25}$$

Because the norm is weakly lower semi-continuous, using Poincaré inequality, we get that

$$\begin{aligned}\limsup_{n\to\infty}\int_\Omega |\nabla v_n|^p dx &\le \lambda_1 \int_\Omega |v_0|^p dx \le \int_\Omega |\nabla v_0|^p dx \\ &\le \liminf_{n\to\infty}\int_\Omega |\nabla v_n|^p dx \\ &\le \limsup_{n\to\infty}\int_\Omega |\nabla v_n|^p dx.\end{aligned} \tag{26}$$

Hence, $\int_\Omega |\nabla v_0|^p dx = \lambda_1 \int_\Omega |v_0|^p dx$ and $v_n \to v_0$ in $W_0^{1,p}(\Omega)$ with $\|v_0\| = 1$. So, v_0 is the corresponding eigenfunction to λ_1. Without loss of generality, we may assume that $v_0 = \varphi_1$. Thus, $u_n \to +\infty$ a.e. $x \in \Omega$. Consequently, $\widehat{G}(x, u_n(x)) \to -\infty$ a.e. $x \in \Omega$. Therefore,

$$C \ge -\int_\Omega \widehat{G}(x, u_n)\, dx \longrightarrow +\infty, \tag{27}$$

which contradicts the fact that $C \in \mathbb{R}$. From the fact that $\widehat{I}$ is weakly low semi-continuous, we know that it has a global minimizer u_1. As in the proof of Theorem 1, u_1 is a positive solution of (1) with $I(u_1) < 0$. In a similar way, we can obtain a negative solution with negative critical value.

The proof is completed. □

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Research Article

A Path-Integral Approach to the Cameron-Martin-Maruyama-Girsanov Formula Associated to a Bilaplacian

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We define the Wiener product on a bosonic Connes space associated to a Bilaplacian and we introduce formal Wiener chaos on the path space. We consider the vacuum distribution on the bosonic Connes space and show that it is related to the heat semigroup associated to the Bilaplacian. We deduce a Cameron-Martin quasi-invariance formula for the heat semigroup associated to the Bilaplacian by using some convenient coherent vector. This paper enters under the Hida-Streit approach of path integral.

1. Introduction

Let us recall some basic tools of Wiener analysis. Let B_t be a one-dimensional Brownian motion starting from 0. It is classically related to the heat equation on $\mathbb{R}$:

$$\frac{\partial}{\partial t}E\left[f(B_t)\right] = \frac{1}{2}E\left[\Delta f(B_t)\right], \tag{1.1}$$

where $\Delta = \partial^2/\partial x^2$ is the standard Laplacian and f is a smooth function with bounded derivatives at each order. Associated to the heat equation there is a convenient probability measure on a convenient path space. Almost surely, the trajectory of B is continuous. We construct by this way the Wiener measure dP on the continuous path space endowed with its Borelian σ-algebra. Let $\mathbb{H}$ be the Hilbert space $L^2([0,1];\mathbb{R})$. We consider the symmetric tensor product $\mathbb{H}^{\hat{\otimes} n}$ of this Hilbert space. It is constituted of maps $h_n(s_1, \ldots, s_n)$ symmetric in s_i such that

$$\|h_n\|^2 = \int_{[0,1]^n} h_n^2(s_1, \ldots, s_n) ds_1 \cdots ds_n < \infty. \tag{1.2}$$

We consider the symmetric Fock space $F(\mathbb{H})$ of set $\sigma = \sum_{n=0}^{\infty} h_n$ such that

$$\|\sigma\|^2 = \sum n! \|h_n\|^2 < \infty. \tag{1.3}$$

We consider the vacuum expectation.

$$\mu[\sigma] = h_0. \tag{1.4}$$

With an element h_n of $\mathbb{H}^{\hat{\otimes} n}$ is associated the Wiener chaos

$$\Psi(h_n) = \int_{[0,1]^n} h_n(s_1, \ldots, s_n) dB_{s_1} \cdots dB_{s_n}. \tag{1.5}$$

The mat Ψ realizes a isomorphism between $F(\mathbb{H})$ and $L^2(dP)$. On the level of the Fock space some important elements are constituted by coherent vectors:

$$\sigma = \sum \frac{h^{\otimes n}}{n!}. \tag{1.6}$$

The functional associated to such a coherent vector is a so-called exponential martingale

$$\Psi(\sigma) = \exp\left[\int_0^1 h_s dB_s - \frac{\|h\|^2}{2}\right]. \tag{1.7}$$

We refer to the books of Hida et al. [1], to the book of Obata [2], and to the book of Meyer [3] for an extensive study on that subject. Especially on the Fock space, we can define the Wiener product:

$$\Psi(\sigma_1 \cdot \sigma_2) = \Psi(\sigma_1)\Psi(\sigma_2), \tag{1.8}$$

where we consider the ordinary product of the two $\Psi(\sigma_i)$. For that, we use **the Itô table for the Laplacian**

$$\begin{aligned} dB_s \cdot dB_s &= \frac{1}{2} ds, \\ dB_s \cdot ds &= ds \cdot ds = 0 \end{aligned} \tag{1.9}$$

which reflect algebraically the Itô formula for the Brownian motion. From this Itô table, we deduce classically that if σ is an exponential vector, $\Psi(\sigma) = \exp[\int_0^1 h_s dB_s - \|h\|^2/2]$ and not $\exp[\int_0^1 h_s dB_s]$.

The law of $B_t + \int_0^t h_s ds$ is absolutely continuous with respect of the law of B_t, and the Radon-Nikodym derivative between these two laws is $\Psi(\sigma) = \exp[-\int_0^1 h_s dB_s - \|h\|^2/2]$. It is the subject of the Cameron-Martin formula.

The construction of a full path probability measure associated to a semi-group is related to Hunt theory: the generator L of the semi-group has to satisfy maximum principle. We are motivated where we take others type of generator. To simplify the computations we take the simplest of such operators $L = -\partial^4/\partial x^4$. We have implemented recently some stochastic tools for semi-groups whose generators do not simplify maximum principle ([4–10]). We construct in [8, 9] the Wiener distribution associated to a Bilaplacian using the Hida-Streit approach of path integrals as distribution. We refer to the works of Funaki [11], Hochberg [12], Krylov [13], and the review paper of Mazzucchi [14] for other approaches. We refer to the review paper of Albeverio [15] for various approach of path integrals.

In the Hida-Streit approach of path integral, there are basically 3 objects:

(i) an algebraic space, generally a kind of Fock space;

(ii) a map Ψ from this algebraic space into a set of functionals on a mapping space;

(iii) the path integral is continuous on the level of the algebraic set. We say that it is an Hida-type distribution.

Generally, people were considering map Ψ as the map Wiener chaos. A breakdown was performed by Getzler [16] motivated by the works of Atiyah-Bismut-Witten relating the structure of the free loop space and the Index theory. Developments were done by Léandre in [17, 18]. Especially, in [8, 9] we were using map Ψ as related to cylindrical functional to define a path integral associated to the Bilaplacian and to state some properties related to this path integral.

In this paper, we come back to the original map Ψ of Wiener, by using Wiener chaos. But we use **formal** Wiener chaos. We consider a continuous path w_s. We consider a map $h_n^{i_1,\dots,i_n}(s_1,\dots,s_n)$ $s_1 < s_2 < \cdots < s_n < 1$ with values in $\mathbb{R}$. We consider the formal Wiener chaos:

$$\Psi(h_n) = \int_{0<s_1<\cdots<s_n<1} h_n^{i_1,\dots,i_n}(s_1,\dots,s_n)\, dw_{s_1}^{i_1} \cdots dw_{s_n}^{i_n}. \tag{1.10}$$

We put

$$dw_s^4 = 24ds. \tag{1.11}$$

If $i > 4$, $dw_s^i = 0$. We use in order to define the Wiener product on formal chaos associated to the Bilaplacian L the **Itô table for the Bilaplacian:**

$$dw_s^i dw_s^j = dw_s^{i+j}. \tag{1.12}$$

In order to simplify the exposition, we use in the sequel Connes space and not a Hida Fock space. We consider L^∞ the set of map h from $[0,1]$ into $\mathbb{R}^3$ such that

$$\sup_s |h(s)| = \|h\|_\infty. \tag{1.13}$$

We introduce the bosonic Connes space $CO_{\infty-}(L^\infty)$ (a refinement of the traditional bosonic Fock space). To $\sigma \in CO_{\infty-}(L^\infty)$, we associate a formal Wiener chaos $\Psi(\sigma)$. We use the Itô table for the Bilaplacian in order to define a Wiener product on the bosonic Connes space:

$$\Psi(\sigma_1 \cdot \sigma_2) = \Psi(\sigma_1)\Psi(\sigma_2). \tag{1.14}$$

The bosonic Connes space becomes a commutative topological algebra for the Wiener product (For similar consideration for the case of the standard Laplacian, we refer to the book of Meyer [3]).

We consider as classical the vacuum expectation on the bosonic Connes space, and we state a kind of Itô-Segal-Bargmann-Wiener isomorphism, but in this case there is no Hilbert space involved. We show that for the vacuum expectation w_s has in some sense independent increments. We consider a type of generalization of the exponential martingale of the Brownian motion:

$$\Psi(\sigma_t) = \sum \int_{0<s_1<\cdots<s_n<t} h_{s_1} dw^1_{s_1} \cdots h_{s_n} dw^1_{s_n}. \tag{1.15}$$

We suppose that h is continuous. Let f be a polynomial on $\mathbb{R}$. We put

$$Q^h_t[f] = \mu\left[f\left(w^1_t\right)\Psi(\sigma_t)\right]. \tag{1.16}$$

We show the following Cameron-Martin-Maruyama-Girsanov type formula:

$$\frac{\partial}{\partial t} Q^h_t[f] = Q^h_t[L_{h,t} f], \tag{1.17}$$

where

$$L_{h,t} = L + \text{lowerterm}. \tag{1.18}$$

2. Formal Wiener Chaos Associated to a Bilaplacian

We consider the set L^∞. $(L^\infty)^{\otimes n}$ is constituted of maps:

$$\sum_{i_1,\ldots,i_n} h^{i_1,\ldots,i_n}(s_1,\ldots,s_n) e_{i_1} \otimes \cdots \otimes e_{i_n} = h_n(s_1,\ldots,s_n), \tag{2.1}$$

where e_i is the standard basis of $\mathbb{R}^3$. On $(L^\infty)^{\otimes n}$, we consider the natural supremum norm $\|h_n\|_\infty$. Moreover, there is a natural action of the symmetric group on $(L^\infty)^{\otimes n}$. Elements which are invariant under this action of the symmetric group are called elements of the **symmetric** tensor product $(L^\infty)^{\widehat{\otimes} n}$. $CO_{C,r}(L^\infty)$ $(r > 0, C > 0)$ is constituted of formal series $\sigma = \sum h_n$ where h_n belongs to $(L^\infty)^{\widehat{\otimes} n}$ such that

$$\|\sigma\|_C = \sum C^n n! \|h_n\|_\infty < \infty. \tag{2.2}$$

Definition 2.1. The intersection of all $CO_C(L^\infty)$ is called the bosonic Connes space $CO_{\infty-}(L^\infty)$.

Remark 2.2. In the sequel we could choose an Hida Fock space.

Definition 2.3. The vacuum expectation μ on $CO_{\infty-}(L^\infty)$ is defined by

$$\mu(\sigma) = h_0. \tag{2.3}$$

If h_n belongs to $(L^\infty)^{\widehat{\otimes} n}$, we consider the formal Wiener chaos:

$$\Psi(h_n) = \sum_{i_1,\dots,i_n} \int_{0<s_1<\cdots<s_n<1} h_n^{i_1,\dots,i_n}(s_1,\dots s_n) dw_{s_1}^{i_1} \cdots dw_{s_n}^{i_n}. \tag{2.4}$$

We could do the same expression if h_n belongs to $(L^\infty)^{\otimes n}$.

Definition 2.4. The map Ψ defined on $CO_{\infty-}(L^\infty)$ is called the map formal Wiener chaos.

Let $\{1,\dots,n\}$, $\{n+1,\dots,n+m\}$. Let $\{l\}$ be a concatenation (or pairing). It is an increasing injective map from a set with l element in $\{1,\dots,n\}$ into $\{n+1,\dots,n+m\}$. There is at most C^{n+m} pairing of length l. We consider $h_n^1 \otimes_{\{l\},sh_{\{l\}}} h_m^2$ where we concatain the time in h_n and in h_m according the pairing, and we shuffle according to the shuffle sh_l and the time in h_n^1 and h_m^1 between two continuous times in the pairing. When we concatenate two times, we use the Itô table for the Bilaplacian, and we symmetrized the expression in the time.

The classical product of $\Psi(h_n^1)\Psi(h_m^2)$ is equal to $\sum_{\{l\},sh_{\{l\}}} \Psi(h_n^1 \otimes_{\{l\},sh_{\{l\}}} h_m^2)$ and generalized with this new Itô table the standard formula which gives the product of two Wiener chaos in the Brownian case. There are at most $C^{n+m} C_n^l C_m^l$ pairing $\{l\}$ and shuffle according to the pairing $\{l\}$.

Definition 2.5. The Wiener product of h_n^1 and h_m^2 is defined by

$$\Psi\left(h_n^1 \cdot h_m^2\right) = \Psi\left(h_n^1\right)\Psi\left(h_n^2\right). \tag{2.5}$$

Theorem 2.6. *The Wiener product endows the symmetric Connes space with a structure of topological commutative algebra.*

Proof. Let us show first of all that the Wiener product is continuous. We have

$$\left\| h_n^1 \otimes_{\{l\},sh\{l\}} h_m^2 \right\|_\infty \le C^{n+m} \left\| h_n^1 \right\|_\infty \left\| h_m^2 \right\|_\infty. \tag{2.6}$$

Therefore,

$$\left\| h_n^1 \cdot h_m^2 \right\|_C \le C_1^{n+m} C^{n+m} \left\| h_n^1 \right\|_\infty \left\| h_m^2 \right\|_\infty \sum_{\{l\}, sh_{\{l\}}} C^{-l} ((n+m-2l)!). \tag{2.7}$$

But

$$\sum_{\{l\}, sh_{\{l\}}} C^{-l} \le \sum_l C_n^l C_m^l C_3^{n+m} C^{-l} \le C_2^{n+m} \left(1 + C^{-1}\right)^{n+m} \le C_4^{n+m}. \tag{2.8}$$

On the other hand, by the Stirling formula,

$$(n!)^{-1} (m!)^{-1} (n+m-2l)! \le C_3^{n+m}. \tag{2.9}$$

We deduce that

$$\|\sigma_1 \cdot \sigma_2\|_C \le K \|\sigma_1\|_{C'} \|\sigma_2\|_{C'} \tag{2.10}$$

and therefore the Wiener product is continuous on the bosonic Connes space.

Let h_{n_1}, h_{n_2}, and h_{n_3} be 3 elements of the bosonic Connes space.

Let $sh_{1,2,3}$ be a shuffle between the 3 sets $\{1, n_1\}$, $\{n_1+1, n_1+n_2\}$, and $\{n_1+n_2+1, n_1+n_2+n_3\}$.

We perform two concatenations between the times when the shuffle is done:

(i) either we concatain 2 contiguous times in $\{1, n_1\}$ and in $\{n_1+1, n_1+n_2\}$ and two contiguous time in $\{1, n_1\}$ and in $\{n_1+n_2+1, n_1+n_2+n_3\}$;

(ii) either we concatain 2 contiguous times in $\{n_1+1, n_1+n_2\}$ and in $\{1, n_1\}$ and two contiguous times in $\{n_1+1, n_1+n_2\}$ and in $\{n_1+n_2+1, n_1+n_2+n_3\}$;

(iii) either we concatain 2 contiguous times in $\{n_1+n_2+1, n_1+n_2+n_3\}$ and in $\{1, n_1\}$ and two contiguous times in $\{n_1+n_2+1, n_1+n_2+n_3\}$ and in $\{n_1+1, n_1+n_2\}$;

(iv) or we concatain 3 contiguous times in $\{1, n_1\}$, in $\{n_1+1, n_1+n_2\}$ and in $\{n_1+n_2+1, n_1+n_2+n_3\}$.

When we concatain time, we use the iterated Itô rule:

$$\left(dw_s^{i_1} \cdot dw_s^{i_2}\right) \cdot dw_s^{i_3} = dw_s^{i_1+i_2+i_3}. \tag{2.11}$$

Such a concatenation is called $l_{1,2,3}$ and the final result is called $h_{n_1} \otimes_{sh_{1,2,3}, l_{1,2,3}} h_{n_2} \otimes_{sh_{1,2,3} l_{1,2,3}} h_{n_3}$. We deduce the formula

$$(h_{n_1} \cdot h_{n_2}) \cdot h_{n_3} = \sum_{l_{1,2,3}, sh_{1,2,3}} h_{n_1} \otimes_{sh_{1,2,3}, l_{1,2,3}} h_{n_2} \otimes_{sh_{1,2,3} l_{1,2,3}} h_{n_3}. \tag{2.12}$$

From this formula we deduce the associativity of the Wiener product. □

From the product formula, we deduce easily the next theorem.

Theorem 2.7 (Itô-Bargmann-Wiener-Segal). *Let $h_{n_1}^{i_1,\dots,i_{n_1}}$ and $h_{n_2}^{j_1,\dots,j_{n_2}}$ be elements of the bosonic Connes space. They are seen as a function on the involved simplices. Then*

$$\mu[\Psi(h_{n_1})\Psi(h_{n_2})] = \delta_{n_1,n_2}\prod\delta_{i_l+j_l=4}24^{n_1} \times \int_{0<s_1<\cdots<s_n<1} h_{n_1}^{i_1,\dots,i_n}(s_1,\dots,s_n)h_{n_1}^{j_1,\dots,j_n}(s_1,\dots,s_n)ds_1\cdots ds_n. \tag{2.13}$$

Remark 2.8. In the case of the classical Laplacian, this formula justifies the choice of $\mathbb{H}$ instead of L^∞. But in the previous formula, only a prehilbert space appears. So it is not obviously justified to choose $\mathbb{H}$ instead of L^∞ to perform our computations. We have chosen L^∞ because the estimates are simpler with this space.

We say that h_n belongs to $CO_{\infty-,t]}(L^\infty)$ if h_n vanishes as soon as one of the $s_i \geq t$. We say that h_n belongs to $CO_{\infty-,[t}(L^\infty)$ if h_n vanishes as soon as one of the $s_i \leq t$. We get the next theorem whose proof is obvious.

Theorem 2.9. *$CO_{\infty-,t]}(L^\infty)$ and $CO_{\infty-,[t}(L^\infty)$ are subalgebras of $CO_{\infty-}(L^\infty)$ for the Wiener product. Moreover, if $\sigma_1 \in CO_{\infty-,t]}(L^\infty)$ and if $\sigma_2 \in CO_{\infty-,[t}(L^\infty)$,*

$$\mu[\Psi(\sigma_1)\Psi(\sigma_2)] = \mu[\Psi(\sigma_1)]\mu[\Psi(\sigma_2)]. \tag{2.14}$$

Remark 2.10. Let us justify heuristically this part. Let Q_t^0 be the semi-group generated by L. Let us suppose that there is a formal measure $d\mu$ on a path space $t \to w_t$ such that

$$Q_t^0[f] = \int f(w_t)d\mu. \tag{2.15}$$

(In the case of the standard Laplacian it is the measure of the Brownian motion). We refer to [19] for a physicist way to construct this measure. We have

$$Q_t^0\left[x^4\right] = 24t \tag{2.16}$$

So the infinitesimal increment $(dw_t)^i$ of w_t should satisfy the Itô table (1.12) and the formal Wiener chaos should be an extension of the classical Wiener chaos in the Brownian case.

3. A Cameron-Martin-Maruyama-Girsanov Formula Associated to a Bilaplacian

We put if f is a polynomial,

$$Q_t^h[f] = \mu\left[f\left(w_t^1\right)\Psi(\sigma_t)\right], \tag{3.1}$$

where

$$\Psi(\sigma_t) = \sum \int_{0<s_1<\cdots<s_n<t} h_{s_1} dw^1_{s_1} \cdots h_{s_n} dw^1_{s_n}. \tag{3.2}$$

We suppose that h is continuous. In this formula, only finite sums appear due to (2.13). We get the following.

Theorem 3.1 (Cameron-Martin-Maruyama-Girsanov). *If f is a polynomial,*

$$\frac{\partial}{\partial t} Q^h_t[f] = Q^h_t[L_{h,t} f], \tag{3.3}$$

where

$$L_h = -\frac{\partial^4}{\partial x^4} + \alpha h_t \frac{\partial^3}{\partial x^3}. \tag{3.4}$$

Proof. Let us consider the case where $f(x) = x^n$. We use $w^1_t = \int_0^t dw^1_s$ and the fact that the Wiener product is associative. We get

$$\left(w^1 + w^1_{t+\Delta t} - w^1_t\right)^n = \sum C^k_n \left(w^1_t\right)^{n-k} \left(w^1_{t+\Delta t} - w^1_t\right)^k. \tag{3.5}$$

We put

$$\sigma_{\Delta t} = \sum \frac{\mathbb{I}^{\otimes n}_{[t,t+\Delta t]}}{n!} \tag{3.6}$$

such that by the Itô rules on $[t, t+\Delta t]$ for $\Delta t > 0$:

$$\sigma_{t+\Delta t} = \sigma_t \cdot \sigma_{\Delta_t}. \tag{3.7}$$

We use Theorem 2.9 and the Itô table on $[t, t+\Delta t]$. We deduce that

$$\begin{aligned} \mu\left[\left(w^1_{t+\Delta t}\right)^n \Psi(\sigma_{t+\Delta t})\right] &= \mu\left[\left(w^1_t\right)^n \Psi(\sigma_t)\right] \\ &\quad + n(n-1)(n-2)(n-3)\mu\left[\left(w^1_t\right)^{n-4} \Psi(\sigma_t)\right] \Delta t \\ &\quad + \alpha h_t n(n-1)(n-2)\mu\left[\left(w^1_t\right)^{n-3} \Psi(\sigma_t)\right] \Delta_t + o(\Delta t). \end{aligned} \tag{3.8}$$

Therefore, the result is obtained. $\square$

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Research Article

Local Gevrey Regularity for Linearized Homogeneous Boltzmann Equation

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The local Gevrey regularity of the solutions of the linearized spatially homogeneous Boltzmann equation has been shown in the non-Maxwellian case with mild singularity.

1. Introduction

This paper focuses on the Gevrey class smoothing property of solutions of the following linear Cauchy problems of the spatially homogeneous Boltzmann equation:

$$\frac{\partial f}{\partial t} = Lf = Q(\mu, f) + Q(f, \mu), \quad v \in \mathbb{R}^3,\ t > 0, \quad \mu(v) = (2\pi)^{-3/2} e^{-|v|^2/2}, \qquad f|_{t=0} = f_0, \tag{1.1}$$

where the initial datum $f_0 \not\equiv 0$ satisfies the natural boundedness on mass, energy, and entropy:

$$f_0 \geq 0, \quad \int_{\mathbb{R}^3} f_0(v) \left\{ 1 + |v|^2 + \log\left(1 + f_0(v)\right) \right\} dv < +\infty. \tag{1.2}$$

$Q(g, f)$ is the Boltzmann quadratic operator which has the following form:

$$Q(g, f) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - v_*, \sigma) \{ g(v'_*) f(v') - g(v_*) f(\mathrm{v}) \} d\sigma\, dv_*, \tag{1.3}$$

where $\sigma \in \mathbb{S}^2$ (unit sphere of $\mathbb{R}^3$); the post- and precollisional velocities are given as follows:

$$v' = \frac{v+v_*}{2} + \frac{|v+v_*|}{2}\sigma, \qquad v'_* = \frac{v+v_*}{2} - \frac{|v+v_*|}{2}\sigma. \tag{1.4}$$

The Boltzmann collision cross-section $B(|z|,\sigma)$ is a nonnegative function which depends only on $|z|$ and the scalar product $\langle z/|z|,\sigma\rangle$. To capture its main properties, we usually assume

$$B(|v-v_*|,\sigma) = \Phi(|v-v_*|)b(\cos\theta), \quad \cos\theta = \left\langle \frac{v-v_*}{|v-v_*|},\sigma \right\rangle,\ \theta \in \left[0,\frac{\pi}{2}\right]. \tag{1.5}$$

μ is called the normalized Maxwellian distribution in (1.1). Notice that $Q(\mu,\mu) \equiv 0$.

Recall that the inverse power law potential $1/\rho^s$, where $s > 1$, and ρ denotes the distance between two particles, has the form (1.5) with the corresponding kinetic factors:

$$\begin{gathered} \Phi(|v-v_*|) \approx |v-v_*|^{1-4/s}, \\ b(\cos\theta) \approx \frac{K}{\theta^{2+\nu}}, \quad \theta \longrightarrow 0, \end{gathered} \tag{1.6}$$

for a constant $K > 0$ and $0 < \nu = 2/s < 2$. The cases $1 < s < 4$, $s = 4$, and $s > 4$ correspond to so-called soft, Maxwellian, and hard potentials, respectively.

We will concentrate on the modified hard potentials as follows:

$$\begin{gathered} \Phi(|v-v_*|) = \left(1+|v-v_*|^2\right)^{\gamma/2}, \quad 0 < \gamma < 1, \\ b(\cos\theta) \approx \frac{K}{\theta^{2+\nu}}, \quad \theta \longrightarrow 0,\ 0 < \nu < 2, \end{gathered} \tag{1.7}$$

where the singularity is called the mild singularity when $0 < \nu < 1$ and the strong singularity when $1 \le \nu < 2$. In this paper, we consider only the case of the mild singularity. Before making the discussion, we start by introducing the norms of the weighted function spaces:

$$\|f\|_{L^p_r} = \left\|\langle |v|\rangle^r f(v)\right\|_{L^p}, \qquad \|f\|_{H^s_r} = \left\|\langle |D|\rangle^s \langle |v|\rangle^r f(v)\right\|_{L^2}, \tag{1.8}$$

where $\langle |v|\rangle = (1+|v|^2)^{1/2}$ and $\langle |D|\rangle$ is the corresponding pseudodifferential operator. And then, we list the definition of the weak solution in the Cauchy problem (1.1); compare [1].

Definition 1.1. For an initial datum $f_0(v) \in L^1_2(\mathbb{R}^3)$, $f(t,v)$ is called a weak solution of the Cauchy problem (1.1) if it satisfies

$$f(t,v) \in C\left(\mathbb{R}^+;\mathcal{D}'\left(\mathbb{R}^3\right)\right) \cap L^2\left([0,T];L^1_2\left(\mathbb{R}^3\right)\right) \cap L^\infty\left([0,T];L^1\left(\mathbb{R}^3\right)\right), \qquad f(0,v) = f_0,$$
$$\int_{\mathbb{R}^3} f(t,v)\varphi(t,v)dv - \int_{\mathbb{R}^3} f(0,v)\varphi(0,v)dv - \int_0^t d\tau \int_{\mathbb{R}^3} f(\tau,v)\partial_\tau\varphi(\tau,v)dv \tag{1.9}$$
$$= \int_0^t d\tau \int_{\mathbb{R}^3} L(f)(\tau,v)\varphi(\tau,v)dv,$$

for any test function $\varphi \in L^\infty([0,T];W^{2,\infty}(\mathbb{R}^3))$.

For the definition of the Gevrey class functions, compare [1–5].

Definition 1.2. Suppose that W is a bounded open set on $\mathbb{R}^3$, for $s \geq 1$, $u \in G^s(W)$ which is the Gevrey class function space with index s, if $u \in C^\infty(W)$ and for any compact subset $U \subset W$, there exists a constant $C = C(U) > 0$ such that for any $k \in \mathbb{N}$,

$$\left\|D^k u\right\|_{L^2(U)} \leq C^{k+1}(k!)^s, \tag{1.10}$$

or equivalently,

$$\left\|\langle|D|\rangle^k u\right\|_{L^2(U)} \leq C^{k+1}(k!)^s, \tag{1.11}$$

where

$$\left\|D^k u\right\|^2_{L^2(U)} = \sum_{|\beta|=k}\left\|D^\beta u\right\|^2_{L^2(U)}, \qquad \langle|D|\rangle = \left(1+|D_v|^2\right)^{1/2}. \tag{1.12}$$

Particularly, $u \in G^s(\mathbb{R}^3)$, that is, $||D^k u||_{L^2(\mathbb{R}^3)} \leq C^{k+1}(k!)^s$, is equivalent to the fact that there exists $\epsilon_0 > 0$ such that $e^{\epsilon_0\langle|D|\rangle^{1/s}}u \in L^2(\mathbb{R}^3)$.

Notice that $G^1(\mathbb{R}^3)$ is the usual analytic function space. When $0 < s < 1$, we call $G^s(\mathbb{R}^3)$ the ultra-analytic function space, cpmpare [4, 5].

There have been some results about the Gevrey regularity of the solutions for the Boltzmann equation; compare [1, 4, 6–8]. Among them, unique local solutions having the same Gevrey regularity as the initial data are first constructed in [8]. This implies the propagation of the Gevrey regularity. In 2009, Desvillettes et al. improved this result for the nonlinear spatially homogeneous Boltzmann equation, they showed in [6] that, for the Maxwellian molecules model, the Gevrey regularity can propagate globally in time. Other results for the nonlinear case can be found in [4], where the Gevrey regularity of the radially symmetric weak solutions has been proved. Meanwhile, this issue is also considered in [7] for the Maxwellian decay solutions. For the linear case, the best result so far is obtained by the work of Morimoto et al. in [1]; they proved the propagation of Gevrey regularity of the solutions, without any extra assumption for the initial data. We mention that the crucial tools in [1, 6] are the following pseudodifferential operator:

$$G_\delta(t, D_v) = \frac{1}{\delta + e^{-t\langle |D_v|\rangle^{\nu/2}}}, \quad 0 < \nu < 2. \tag{1.13}$$

In the Maxwellian case, this pseudodifferential operator can be used successfully, but it seems unsuitable for the non-Maxwellian model. The difficulty comes from the commutator of the kinetic factor Φ and the pseudodifferential operator (1.13) which lacks of the effective estimations. In this paper, we apply a new method which is based on the mathematical induction to overcome it. Compared with [7], we consider only the local space; however, we discuss this issue by using the much weaker preconditions (actually, we do not need any smooth assumption for the initial data). Concerning the same issue for the other related equations, such as the Landau equation and the Kac equation, compare [2–5].

Now we can state our main result.

Theorem 1.3. *Suppose Φ, b have the forms in (1.7), $0 < \nu < 1$. Let W be a bounded open set of $\mathbb{R}^3$, and $f(t, v)$ be the weak solution of the Cauchy problem (1.1) satisfying*

$$\sup_{t\in(0,T]} \|f(t,\cdot)\|_{L^2(W)} < +\infty. \tag{1.14}$$

Then for any $t \in (0, T]$, there exists a number $s = s(t) > 3$ satisfying $f(t,\cdot) \in G^s(W)$. More precisely, for any fixed $0 < t_0 \le T$ and compact subset $U \subset W$, there exists a constant $C = C(U) > 0$ and a number $s > 3$ such that for any $k \in \mathbb{N}$,

$$\sup_{t\in[t_0,T]} \left\|D^k f(t,\cdot)\right\|_{L^2(U)} \le C^{k+1}(k!)^s. \tag{1.15}$$

From Theorem 1.3, we have the following remark.

Remark 1.4. Suppose that Φ, b have the forms in (1.7), $0 < \nu < 1$. If the weak solution $f(t, v)$ satisfies that

$$\sup_{t\in(0,T]} \|f(t,\cdot)\|_{L^2(\mathbb{R}^3)} < +\infty, \tag{1.16}$$

then for any $t \in (0, T]$, any bounded open set $U \subset \mathbb{R}^3$, there exists a constant $s = s(t)$ satisfying $f(t,\cdot) \in G^s(U)$.

2. Useful Lammas for the Main Result

In order to gain the main result, we need to prove the following lemmas in this section.

Lemma 2.1. *Suppose* $\Phi(v) = \langle |v| \rangle^{\gamma} = (1 + |v|^2)^{\gamma/2}$ *where* $\gamma \in (0,1), v \in \mathbb{R}^n$, *and* $n \in \mathbb{N}$. *Then the* k*th order derivative of* Φ *satisfies*

$$\left|\Phi^{(k)}(v)\right| \leq 4^k k! \Phi(v) \langle |v| \rangle^{-k}. \tag{2.1}$$

Proof. Without loss of generality, we only consider the case of $n = 1$; the other cases are similar. By direct calculation, we have

$$\begin{aligned}
\Phi^{(2m)}(v) &= \sum_{i=0}^{m} C_{i,2m} \gamma(\gamma - 2) \cdots (\gamma - 2i - 2m + 2) \left(1 + v^2\right)^{\gamma/2 - i - m} v^{2i}, \\
\Phi^{(2m+1)}(v) &= \sum_{i=0}^{m} A_{i,2m+1} \gamma(\gamma - 2) \cdots (\gamma - 2i - 2m) \left(1 + v^2\right)^{\gamma/2 - i - m - 1} v^{2i+1}.
\end{aligned} \tag{2.2}$$

In addition,

$$\begin{aligned}
C_{i,2m} + 2(i+1) C_{i+1,2m} &= A_{i,2m+1}, \\
(2i+1) A_{i,2m+1} + A_{i-1,2m+1} &= C_{i,2m+2}.
\end{aligned} \tag{2.3}$$

Thus we obtain

$$C_{i,2m+2} = C_{i-1,2m} + (4i+1) C_{i,2m} + (2i+1)(2i+2) C_{i+1,2m} \tag{2.4}$$

and then we will prove the following inequality:

$$|C_{i,2m}| \leq 2^{2m} \left| (\gamma - 2i - 2m) \cdots (\gamma - 4m + 2) \right|. \tag{2.5}$$

The inequality is obviously true for $m = 1$. Suppose it is valid for $1 \leq m \leq M$, then

$$\begin{aligned}
|C_{i,2M+2}| &= |C_{i-1,2M} + (4i+1) C_{i,2M} + (2i+1)(2i+2) C_{i+1,2M}| \\
&\leq 2^{2M} \left[(4i+1) \left| \gamma - 2i - 2M \right| + (\gamma - 2i + 2 - 2M) \left| \gamma - 2i - 2M \right| \right. \\
&\qquad \left. + (2i+1)(2i+2) \right] \left| (\gamma - 2i - 2 - 2M) \cdots (\gamma - 4M + 2) \right| \\
&\leq 2^{2(M+1)} \left| (\gamma - 2i - 2 - 2M) \cdots (\gamma - 4M)(\gamma - 4M - 2) \right|
\end{aligned} \tag{2.6}$$

which proves (2.5) by induction. Therefore, we have

$$\begin{aligned}\left|\Phi^{(2m)}(v)\right| &\le 2^{2m}\left|\gamma(\gamma-2)\cdots(\gamma-4m+2)\right|\sum_{i=0}^{m}\left(1+v^2\right)^{\gamma/2-i-m}v^{2i}\\ &\le (m+1)2^{4m-1}(2m-1)!\Phi(v)\langle|v|\rangle^{-2m}\\ &\le 4^{2m}(2m)!\Phi(v)\langle|v|\rangle^{-2m}.\end{aligned}\tag{2.7}$$

The case of $(2m+1)$th order derivative is similar. This completes the proof of Lemma 2.1. □

Setting $M_N(\xi)=(1+|\xi|^2)^{Nt/2}$ for any $\xi\in\mathbb{R}^3$ and $N\in\mathbb{N}$, by using the similar technique of Lemma 2.1, we conclude the following.

Remark 2.2. For $t\in(0,1]$,

$$\begin{aligned}\left|\partial_\xi^k M_N(\xi)\right| &\le 4^k\langle|\xi|\rangle^{Nt-k}|N(N-1)\cdots(N-k+1)|\\ &\le 4^k\langle|\xi|\rangle^{(N-k)t}|N(N-1)\cdots(N-k+1)|,\end{aligned}\tag{2.8}$$

where $k\in\mathbb{N}, 1\le k\le N$.

Lemma 2.3. *There exists a constant C such that for any $k\in\mathbb{N}$,*

$$\left|\partial_v^k\mu(v)\right| \le C^k\cdot k!\cdot\max\left(1,|v|^k\right)\cdot\mu(v),\tag{2.9}$$

where μ is the absolute Maxwellian distribution in (1.1).

Proof. Without loss of generality, we also only consider the case in the real space $\mathbb{R}^1$. Putting

$$\begin{aligned}\partial_v^k\mu(v) &= \partial_v^k\left(e^{-v^2/2}\right)=\sum_{j=0}^{k}a'_{j,k}v^je^{-v^2/2},\\ \partial_v^k\mu^{-1}(v) &= \partial_v^k\left(e^{v^2/2}\right)=\sum_{j=0}^{k}a_{j,k}v^je^{v^2/2}.\end{aligned}\tag{2.10}$$

Evidently,

$$\begin{gathered}0\le\left|a'_{j,k}\right|\le a_{j,k},\\ \left|a'_{k,k}\right|=a_{k,k}\equiv 1,\\ \sum_{j=0}^{1}a_{j,1}=1\le 8^1\cdot 1!,\\ a_{j,k+1}=a_{j-1,k}+(j+1)a_{j+1,k}.\end{gathered}\tag{2.11}$$

Therefore, fixed a number $m \geq 0$, together with the following assumption (F_m):

$$\sum_j a_{j,m} \leq 8^m \cdot m!, \tag{2.12}$$

we can obtain (F_{m+1})

$$\begin{aligned}\sum_j a_{j,m+1} &= \sum_j \left(a_{j-1,m} + (j+1) a_{j+1,m}\right) \leq (m+2) \cdot \left(\sum_j a_{j-1,m} + \sum_j a_{j+1,m}\right) \\ &\leq 2(m+2) \cdot 8^m \cdot m! \leq 8^{m+1} \cdot (m+1)!.\end{aligned} \tag{2.13}$$

This completes the proof of Lemma 2.3 by induction. □

Setting

$$H^*(v) = \left(1 + |v_*|^2\right)^4 \mu^*(v) = \left(1 + |v_*|^2\right)^4 \cdot \mu(v + v_*), \tag{2.14}$$

where v is belong to a bounded set U. Then we state Lemma 2.4 as below.

Lemma 2.4. *There exists a constant $C = C(U) > 0$, which satisfies that for any $k \in \mathbb{N}$,*

$$\sup_{v_*} \left|\partial_v^k H^*(v)\right| \leq C^k \cdot (k!)^2. \tag{2.15}$$

Proof. Since $e^{-(v+v_*)^2/4} \leq e^{v^2/4} \cdot e^{-v_*^2/8}$, and the fact that when $|v| \geq 1$,

$$\left|v^k e^{-v^2/4}\right| = \frac{|v|^k}{\left(\sum_{n=0}^{+\infty} |v|^{2n} / 2^{2n} \cdot n!\right)} \leq \frac{|v|^k}{\left(|v|^k / 2^k \cdot (k/2)!\right)} \leq 2^k \cdot k!, \tag{2.16}$$

by using Lemma 2.3, we have

$$\begin{aligned}\left|\partial_v^k H^*(v)\right| &= \left|\left(1 + |v_*|^2\right)^4 \partial_v^k \mu^*(v)\right| \\ &\leq \left(1 + |v_*|^2\right)^4 \cdot C^k \cdot k! \cdot \max\left(1, |v + v_*|^k\right) \cdot \mu(v + v_*) \\ &\leq \left(1 + |v_*|^2\right)^4 \cdot C^k \cdot k! \cdot \left[\max\left(1, |v + v_*|^k\right) \cdot e^{-(v+v_*)^2/4}\right] \cdot e^{-(v+v_*)^2/4} \\ &\leq \left[\left(1 + |v_*|^2\right)^4 e^{-v_*^2/8}\right] \cdot C^k \cdot k! \cdot \left[\max\left(1, |v + v_*|^k\right) \cdot e^{-(v+v_*)^2/4}\right] \cdot e^{v^2/4} \\ &\leq \left(C^k \cdot k!\right)^2 \leq [C(U)]^k \cdot (k!)^2.\end{aligned} \tag{2.17}$$

This completes the proof of Lemma 2.4. □

By applying the Cauchy integral theorem, we will prove the helpful estimates as follows.

Lemma 2.5. *Suppose the Fourier transform for v_*,*

$$\mathcal{F}\left(\Phi(|v-v_*|)\mu(v_*)\right)(\xi) = h(v,\xi)\widehat{\mu}(\xi), \tag{2.18}$$

where μ is the absolute Maxwellian distribution in (1.1)*. Then we have*

$$h(v,\xi) = (2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{-|v_*|^2/2}\left[1+|v-v_*|^2-|\xi|^2+2i(v-v_*)\cdot\xi\right]^{\gamma/2}dv_*. \tag{2.19}$$

Proof. First we consider the case of $n=1$,

$$\begin{aligned}
\mathcal{F}\left(\Phi(|v-v_*|)\mu(v_*)\right)(\xi) &= \int_{\mathbb{R}^1}\left(1+|v-v_*|^2\right)^{\gamma/2}(2\pi)^{-3/2}e^{-|v_*|^2/2-iv_*\cdot\xi}dv_* \\
&= (2\pi)^{-3/2}e^{-\xi^2/2}\int_{\mathbb{R}^1}\left[1+(v-v_*)^2\right]^{\gamma/2}e^{-(v_*+i\xi)^2/2}dv_* \\
&= (2\pi)^{-3/2}e^{-\xi^2/2}\int_C e^{-z^2/2}\left[1+(v-z+i\xi)^2\right]^{\gamma/2}dz,
\end{aligned}\tag{2.20}$$

where $z=v_*+i\xi$, and C denotes the curve: $v_*+i\xi$, $-\infty<v_*<\infty$. By Cauchy integral theorem [9], it follows that

$$\int_C e^{-z^2/2}\left[1+(v-z+i\xi)^2\right]^{\gamma/2}dz = \int_{\mathbb{R}^1} e^{-|v_*|^2/2}\left[1+(v-v_*+i\xi)^2\right]^{\gamma/2}dv_*. \tag{2.21}$$

Now we turn to consider the case of $n=3$. Letting $v=(v_1,v_2,v_3)$, and $v_*=(v_{*1},v_{*2},v_{*3})$ and using the previous result, we have

$$\begin{aligned}
&\mathcal{F}\left(\Phi(|v-v_*|)\mu(v_*)\right)(\xi) \\
&\quad= \int_{\mathbb{R}^3}\left(1+|v-v_*|^2\right)^{\gamma/2}(2\pi)^{-3/2}e^{-|v_*|^2/2-iv_*\cdot\xi}dv_* \\
&\quad= (2\pi)^{-3/2}\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^1}\left[1+(v_1-v_{*1})^2+(v_2-v_{*2})^2+(v_3-v_{*3})^2\right]^{\gamma/2}e^{-v_{*1}^2/2-iv_{*1}\xi_1}dv_{*1}\right) \\
&\qquad\times e^{-(v_{*2}^2+v_{*3}^2)/2-i(v_{*2}\xi_2+v_{*3}\xi_3)}dv_{*2}dv_{*3}
\end{aligned}$$

$$
\begin{aligned}
&= (2\pi)^{-3/2} \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^1} \left[1 + (v_1 - v_{*1} + i\xi_1)^2 + (v_2 - v_{*2})^2 + (v_3 - v_{*3})^2 \right]^{\gamma/2} e^{-v_{*1}^2/2 - \xi_1^2/2} dv_{*1} \right) \\
&\quad \times e^{-(v_{*2}^2 + v_{*3}^2)/2 - i(v_{*2}\xi_2 + v_{*3}\xi_3)} dv_{*2} dv_{*3} \\
&= (2\pi)^{-3/2} e^{-|\xi|^2/2} \int_{\mathbb{R}^3} e^{-|v_*|^2/2} \left[1 + \sum_{j=1}^{3} (v_j - v_{*j} + i\xi_j)^2 \right]^{\gamma/2} dv_* \\
&= (2\pi)^{-3/2} \hat{\mu}(\xi) \int_{\mathbb{R}^3} e^{-|v_*|^2/2} \left[1 + |v - v_*|^2 - |\xi|^2 + 2i(v - v_*) \cdot \xi \right]^{\gamma/2} dv_*.
\end{aligned}
\tag{2.22}
$$

Thus we conclude the result of Lemma 2.5. □

Lemma 2.6. *For the expression of $h(v, \xi)$ in Lemma 2.5, we have*

$$
\begin{gathered}
|h(v, \xi)| \le C \cdot \langle |v| \rangle^{\gamma} \langle |\xi| \rangle^{\gamma}, \\
\left| \nabla_{\xi}^2 h(v, \xi) \right| \le C \cdot \langle |v| \rangle^{\gamma} \langle |\xi| \rangle^{\gamma}, \\
|h(v, \xi^+) - h(v, \xi)| \le C \cdot \langle |v| \rangle^{\gamma} \langle |\xi| \rangle^{1+\gamma} \sin \frac{\theta}{2}, \quad \theta = \arccos \left\langle \frac{\xi}{|\xi|}, \sigma \right\rangle,
\end{gathered}
\tag{2.23}
$$

where $\xi^+ = (\xi + |\xi|\sigma)/2$, and C is a constant independent of v and ξ.

Proof. The first inequality is obvious. To prove the third one, set $\xi = (\xi_1, \xi_2, \xi_3)$. Since

$$
\begin{aligned}
h(v, \xi)\hat{\mu}(\xi) &= \mathcal{F}\left(\Phi(|v - v_*|) \mu(v_*) \right)(\xi) \\
&= \int_{\mathbb{R}^3} \left(1 + |v - v_*|^2 \right)^{\gamma/2} (2\pi)^{-3/2} e^{-|v_*|^2/2 - iv_* \cdot \xi} dv_*,
\end{aligned}
\tag{2.24}
$$

proceeding as in the proof of Lemma 2.5, we can get

$$
\begin{aligned}
\partial_{\xi_i} \left(h(v, \xi)\hat{\mu}(\xi) \right) &= \hat{\mu}(\xi) \left[\partial_{\xi_i} h(v, \xi) - \xi_i h(v, \xi) \right] \\
&= \int_{\mathbb{R}^3} \left(1 + |v - v_*|^2 \right)^{\gamma/2} (2\pi)^{-3/2} e^{-|v_*|^2/2 - iv_* \cdot \xi} (-iv_{*i}) dv_* \\
&= (2\pi)^{-3/2} \hat{\mu}(\xi) \int_{\mathbb{R}^3} \left[1 + |v - v_*|^2 - |\xi|^2 + 2i(v - v_*) \cdot \xi \right]^{\gamma/2} \\
&\qquad \cdot e^{-|v_*|^2/2} \cdot (-\xi_i - iv_{*i}) dv_*.
\end{aligned}
\tag{2.25}
$$

Therefore,

$$\begin{aligned}\partial_{\xi_i} h(v,\xi) &= (2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{-|v_*|^2/2}\left[1+|v-v_*|^2-|\xi|^2+2i(v-v_*)\cdot\xi\right]^{\gamma/2} \\ &\quad\cdot(-\xi_i - iv_{*i})dv_* + \xi_i h(v,\xi) \\ &= (2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{-|v_*|^2/2}\left[1+|v-v_*|^2-|\xi|^2+2i(v-v_*)\cdot\xi\right]^{\gamma/2} \\ &\quad\cdot(-iv_{*i})dv_*\end{aligned} \tag{2.26}$$

which implies that

$$\left|\nabla_\xi h(v,\xi)\right| \le C\cdot\langle|v|\rangle^{\gamma}\langle|\xi|\rangle^{\gamma}. \tag{2.27}$$

By the mean value theorem of differentials, we have

$$\begin{aligned}|h(v,\xi^+)-h(v,\xi)| &\le C\cdot\left|\nabla_\xi h(v,\eta)\right|\cdot|\xi^+-\xi| \\ &\le C'\cdot\langle|v|\rangle^{\gamma}\langle\left|\eta\right|\rangle^{\gamma}|\xi^+-\xi| \le C''\cdot\langle|v|\rangle^{\gamma}\langle|\xi|\rangle^{1+\gamma}\sin\frac{\theta}{2},\end{aligned} \tag{2.28}$$

where $\theta = \arccos\langle\xi/|\xi|,\sigma\rangle$. Thus the third inequality has been obtained.

Finally, the above way can also be used in estimating the second one. Similarly,

$$\begin{aligned}\partial^2_{\xi_i\xi_j}\left(h(v,\xi)\widehat{\mu}(\xi)\right) &= \partial_{\xi_j}\left(\partial_{\xi_i}\left(h(v,\xi)\widehat{\mu}(\xi)\right)\right) \\ &= -\int_{\mathbb{R}^3}\left(1+|v-v_*|^2\right)^{\gamma/2}(2\pi)^{-3/2}e^{-|v_*|^2/2-iv_*\cdot\xi}v_{*i}v_{*j}dv_* \\ &= -(2\pi)^{-3/2}\widehat{\mu}(\xi)\int_{\mathbb{R}^3}\left[1+|v-v_*|^2-|\xi|^2+2i(v-v_*)\cdot\xi\right]^{\gamma/2} \\ &\quad\cdot e^{-|v_*|^2/2}(v_{*i}-i\xi_i)(v_{*j}-i\xi_j)dv_*.\end{aligned} \tag{2.29}$$

On the other hand,

$$\begin{aligned}\partial^2_{\xi_i\xi_j}\left(h(v,\xi)\widehat{\mu}(\xi)\right) &= \partial_{\xi_j}\left\{\widehat{\mu}(\xi)\left[\partial_{\xi_i}h(v,\xi)-\xi_i h(v,\xi)\right]\right\} \\ &= \widehat{\mu}(\xi)\left[\xi_j\xi_i h(v,\xi)+\partial^2_{\xi_j\xi_i}h(v,\xi)-\xi_j\partial_{\xi_i}h(v,\xi)-\xi_i\partial_{\xi_j}h(v,\xi)\right].\end{aligned} \tag{2.30}$$

Combining with the above expressions of $h(v,\xi)$ and $\partial_{\xi_i} h(v,\xi)$, we get

$$\begin{aligned}\partial^2_{\xi_j\xi_i} h(v,\xi) &= \xi_i \partial_{\xi_j} h(v,\xi) + \xi_j \partial_{\xi_i} h(v,\xi) - \xi_j\xi_i h(v,\xi) \\ &\quad - (2\pi)^{-3/2} \int_{\mathbb{R}^3} \left[1 + |v - v_*|^2 - |\xi|^2 + 2i(v - v_*)\cdot\xi\right]^{\gamma/2} \\ &\quad \cdot e^{-|v_*|^2/2} (v_{*i} - i\xi_i)(v_{*j} - i\xi_j)\, dv_* \\ &= - (2\pi)^{-3/2} \int_{\mathbb{R}^3} \left[1 + |v - v_*|^2 - |\xi|^2 + 2i(v - v_*)\cdot\xi\right]^{\gamma/2} \\ &\quad \cdot e^{-|v_*|^2/2} v_{*i} v_{*j}\, dv_*.\end{aligned} \tag{2.31}$$

Therefore,

$$\left|\nabla^2_\xi h(v,\xi)\right| \le C \cdot \langle |v|\rangle^\gamma \langle |\xi|\rangle^\gamma. \tag{2.32}$$

This completes the proof of the second inequality. □

Lemma 2.7. *Suppose that $0 < \nu < 1$ in (1.7). Then for any $r > 0$, $f \in L^1_{2+\gamma}(\mathbb{R}^3) \cap H^{+\infty}(\mathbb{R}^3)$, there exists a constant C independent of r satisfying*

$$I_0(\tau) = \left(Q(f,\mu), \langle |D|\rangle^r f\right)_{L^2} \le C \|f\|_{L^1_{2+\gamma}} \|f\|_{L^1} (r+3)!. \tag{2.33}$$

Proof. Let $\xi^\pm = (\xi \pm |\xi|\sigma)/2$, from Bobylev's formula (see [10]), we have

$$\begin{aligned}I_0(\tau) &= \int_{\mathbb{R}^3} \langle |\xi|\rangle^r \overline{\widehat{f}(\xi)} \left[\iint_{\mathbb{R}^6} \int_{\mathbb{S}^2} \Phi(|v - v_*|) b\left(\frac{\xi}{|\xi|}\cdot\sigma\right) \mu(v_*) f(v)\right. \\ &\qquad \left. \times \left(e^{-i(v_*\cdot\xi^+ + v\cdot\xi^-)} - e^{-iv_*\cdot\xi}\right) d\sigma\, dv\, dv_*\right] d\xi \\ &= \int_{\mathbb{R}^3} \langle |\xi|\rangle^r \overline{\widehat{f}(\xi)} \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} \left[\mathcal{F}\left(\Phi(|v - v_*|)\mu(v_*)\right)(\xi^+) b f(v) e^{-iv\cdot\xi^-}\right. \\ &\qquad \left. - \mathcal{F}\left(\Phi(|v - v_*|)\mu(v_*)\right)(\xi) b f(v)\right] d\sigma\, dv\, d\xi \\ &= \int_{\mathbb{R}^3} \langle |\xi|\rangle^r \overline{\widehat{f}(\xi)} \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} [h(v,\xi^+) - h(v,\xi)] \widehat{\mu}(\xi^+) b f(v) e^{-iv\cdot\xi^-} d\sigma\, dv\, d\xi \\ &\quad + \int_{\mathbb{R}^3} \langle |\xi|\rangle^r \overline{\widehat{f}(\xi)} \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} h(v,\xi) [\widehat{\mu}(\xi^+) - \widehat{\mu}(\xi)] b f(v) e^{-iv\cdot\xi^-} d\sigma\, dv\, d\xi \\ &\quad + \int_{\mathbb{R}^3} \langle |\xi|\rangle^r \overline{\widehat{f}(\xi)} \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} h(v,\xi) \widehat{\mu}(\xi) b f(v) \left[e^{-iv\cdot\xi^-} - e^0\right] d\sigma\, dv\, d\xi \\ &= I_{01} + I_{02} + I_{03}.\end{aligned} \tag{2.34}$$

In [1], it is shown that

$$\begin{gathered}\left|\widehat{\mu}(\xi^{+})-\widehat{\mu}(\xi)\right| \le \widehat{\mu}(\xi^{+})|\xi|^{2}\sin^{2}\frac{\theta}{2}, \qquad e^{-|\xi|^{2}/2}=\widehat{\mu}(\xi)\le\widehat{\mu}(\xi^{+})\le e^{-|\xi|^{2}/4},\\ \langle|\xi|\rangle^{r+\gamma+2}\le(r+3)!e^{\langle|\xi|\rangle^{(r+\gamma+2)/(r+3)}}\le(r+3)!e^{\langle|\xi|\rangle}.\end{gathered} \tag{2.35}$$

Together with Lemma 2.6, we have

$$\begin{aligned}|I_{02}| &\le C\cdot\|f\|_{L^1_{1+\gamma}}\int_{\mathbb{R}^3}\langle|\xi|\rangle^{r+\gamma+2}e^{-|\xi|^2/4}\overline{\widehat{f}(\xi)}d\xi\\ &\le C\cdot\|f\|_{L^1_{1+\gamma}}\|f\|_{L^1}\int_{\mathbb{R}^3}\langle|\xi|\rangle^{r+\gamma+2}e^{-|\xi|^2/4}d\xi\\ &\le C\cdot\|f\|_{L^1_{1+\gamma}}\|f\|_{L^1}(r+3)!\int_{\mathbb{R}^3}e^{\langle|\xi|\rangle-|\xi|^2/4}d\xi\\ &\le C'\|f\|_{L^1_{1+\gamma}}\|f\|_{L^1}(r+3)!.\end{aligned} \tag{2.36}$$

Now we turn to estimate the terms in I_{01} and I_{03}. For the case $0<\nu<1$ in (1.7), it is easy to see that

$$\left|e^{-iv\cdot\xi^-}-e^0\right|=\left|-2\sin\frac{v\cdot\xi^-}{2}\left(\sin\frac{v\cdot\xi^-}{2}+i\cos\frac{v\cdot\xi^-}{2}\right)\right|\le C|v|\left|\xi^-\right|\le C|v||\xi|\sin\frac{\theta}{2}. \tag{2.37}$$

Therefore, applying the above estimates and Lemma 2.6, we also conclude that

$$|I_{0i}|\le C'\|f\|_{L^1_{1+\gamma}}\|f\|_{L^1}(r+3)! \tag{2.38}$$

for any $i\in\{1,3\}$. This completes the proof of Lemma 2.7. □

3. Related Analysis

Let f be the weak solution of the Cauchy problem (1.1). For any $k\in\mathbb{N}$, the compact support

$$\operatorname{supp}(M_k f)\subseteq\operatorname{supp}(f), \tag{3.1}$$

which implies that for any compact subset $U\subset W$,

$$f^*=\begin{cases}f, & \text{if } v\in U,\\ 0, & \text{if } v\notin U,\end{cases} \tag{3.2}$$

is also a weak solution of the following equation

$$\left(\frac{\partial f}{\partial t}, M_k^2 f\right)_{L^2(\mathbb{R}^3)} = \left(Q(\mu, f) + Q(f, \mu), M_k^2 f\right)_{L^2(\mathbb{R}^3)}. \tag{3.3}$$

Since Theorem 1.3 is mainly concerned with the Gevrey smoothness property of the solution f on W, we need only to study the solution of the above equation on any fixed compact subset of W. That is, we can suppose that f has compact support in U for any $t \in [0, T]$,

$$\operatorname{supp}(f) \subseteq U, \qquad f(U^c) \equiv 0. \tag{3.4}$$

Thus, for any $p \geq 0$,

$$\begin{gathered} \|f\|_{L^1_p(\mathbb{R}^3)} \leq O(1)\|f\|_{L^1(U)} < +\infty, \\ \|f\|_{H^p(\mathbb{R}^3)} = \|f\|_{H^p(U)}. \end{gathered} \tag{3.5}$$

Together with Lemma 2.6, we can get the fact that $f \in H^{+\infty}(\mathbb{R}^3)$. This proof is similar as the proof of [11, Theorem 1.1] and hence omitted. Clearly, $\|f\|_{H^r(\mathbb{R}^3)} = \|f\|_{H^r(U)}$. Moreover, without loss of generality, we restrict $T \leq 1$, then for any $k \in \mathbb{N}$, it is assumed that

$$(E_k)\text{: for any } i \in [0, k-1], \quad \sup_{t \in (0,T]} \|f(t, \cdot)\|_{H^{it}} \leq C_0^{i+1}(i!)^s, \tag{3.6}$$

where C_0 is a sufficiently large constant satisfying

$$C_0 \geq 16^6 \max\left(\sup_{t \in (0,T]} \|f\|_{L^i},\ i = 1, 2\right). \tag{3.7}$$

In the following discussion, we will use C and C_i, $i \in \mathbb{N}$ to denote the positive constants independent of k and t. Let $M_k(D_v) = \langle |D_v| \rangle^{kt}$ and $\Phi^*(v) = \langle |v - v_*| \rangle^{\gamma}$. In order to prove Theorem 1.3, we need the propositions as below.

Proposition 3.1. *One has*

$$\sup_{t\in(0,T]} \left\| [M_k(D_v), \Phi^*] f(t,v) \right\|_{L^2} \leq C \cdot C_0^{k+1} (k!)^s. \tag{3.8}$$

Proposition 3.2. *One has*

$$\sup_{t\in(0,T]} \left\| \nabla_v [M_k(D_v), \Phi^*] f(t,v) \right\|_{L^2} \leq C \Big\{ (k+1) \left\| M_k f(t,v) \right\|_{L^2} + C_0^{k+1} (k!)^s \Big\}, \tag{3.9}$$

$$\sup_{t\in(0,T]} \left\| [M_k(D_v), H^*] f(t,v) \right\|_{L^2} \leq C \cdot C_0^{k+1} (k!)^s, \tag{3.10}$$

where H^* is the function which has the form (2.14).

The proof of the above propositions will be given in Section 5.

4. Proof of Theorem 1.3

Now we will prove the main result in this section. For any $t \in (0,T]$, we state the following identity from [11]:

$$\Big(Q(\mu, f), M_k^2 f \Big)_{L^2} - \left(Q(\mu, M_k f), M_k f \right)_{L^2} = I_1 + I_2 + I_3, \tag{4.1}$$

where

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^6} \int_{\mathbb{S}^2} b(\cos\theta) \mu(v_*) (M_k(\xi) - M_k(\xi^+)) \widehat{\Phi^* f}(\xi^+) e^{-iv_*\xi^-} \overline{M_k(\xi) \widehat{f}(\xi)} d\sigma\, dv_* d\xi, \\ I_2 &= \int_{\mathbb{R}^6} \int_{\mathbb{S}^2} b(\cos\theta) \mu(v_*) \{ [M_k, \Phi^*] f(v') \cdot M_k f(v') - [M_k, \Phi^*] f(v) \cdot M_k f(v) \} d\sigma\, dv_* dv, \\ I_3 &= \int_{\mathbb{R}^6} \int_{\mathbb{S}^2} b(\cos\theta) \mu(v_*) ([M_k, \Phi^*] f(v) - [M_k, \Phi^*] f(v')) M_k f(v') d\sigma\, dv_* dv. \end{aligned} \tag{4.2}$$

Our purpose is to obtain the estimations of I_1, I_2 and I_3. Setting $\eta = |\xi|^2$ and $\eta^+ = |\xi^+|^2$, since $|\xi^+| = |\xi| \cos(\theta/2)$ and $|\xi^+|^2 - |\xi|^2 = |\xi|^2 \sin^2(\theta/2)$, applying the mean value theorem and the fact that $0 < t \leq T \leq 1$, we have

$$\begin{aligned} |M_k(\xi) - M_k(\xi^+)| &= \left| (1+\eta)^{kt/2} - (1+\eta^+)^{kt/2} \right| \\ &\leq C \cdot kt \cdot (\eta - \eta^+) (1+\eta_0)^{kt/2-1} \\ &\leq C \cdot kt \cdot \sin^2 \frac{\theta}{2} (1+\eta)^{kt/2} \\ &\leq C' \cdot k \cdot \sin^2\theta M_k(\xi), \end{aligned} \tag{4.3}$$

where η_0 is a number between η an η^+. Therefore,

$$
\begin{aligned}
|I_1| &\le C\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b\sin^2\theta M_k(\xi)\left|\int_{\mathbb{R}^6}\Phi(|v-v_*|)\mu(v_*)f(v)e^{-iv_*\cdot\xi^- - iv\cdot\xi^+}dv_*dv\right| \\
&\quad\times\left|kM_k(\xi)\widehat{f}(\xi)\right|d\sigma\,d\xi \\
&= C\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b\sin^2\theta M_k(\xi)\left|\int_{\mathbb{R}^6}\Phi(|v_*|)\mu(v_*+v)f(v)e^{-iv_*\cdot\xi^- - iv\cdot\xi}dv_*dv\right| \\
&\quad\times\left|kM_k(\xi)\widehat{f}(\xi)\right|d\sigma\,d\xi \\
&\le C\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b\sin^2\theta\langle|v_*|\rangle^{\gamma-8}\cdot\left|M_k(\xi)\widehat{H^* f}(\xi)\right|\cdot\left|kM_k(\xi)\widehat{f}(\xi)\right|d\sigma\,dv_*d\xi \\
&\le C\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b\sin^2\theta\langle|v_*|\rangle^{\gamma-8}\cdot\left|M_k(\xi)\widehat{H^* f}(\xi)\right|^2 d\sigma\,dv_*d\xi \\
&\quad+C\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b\sin^2\theta\langle|v_*|\rangle^{\gamma-8}\cdot\left|kM_k(\xi)\widehat{f}(\xi)\right|^2 d\sigma\,dv_*d\xi = I_{11}+I_{12}.
\end{aligned}
\tag{4.4}
$$

Here H^* is the function which has the form (2.14). It is clear that

$$
\begin{aligned}
I_{12} &\le C'k^2\|M_k f\|_{L^2}^2, \\
I_{11} &\le C\int_{\mathbb{R}^3}\langle|v_*|\rangle^{\gamma-8}\cdot\|M_k(D_v)H^* f\|_{L^2}^2 dv_* \\
&\le C\int_{\mathbb{R}^3}\langle|v_*|\rangle^{\gamma-8}\|[M_k,H^*]f\|_{L^2}^2 dv_* \\
&\quad+C\int_{\mathbb{R}^3}\langle|v_*|\rangle^{\gamma-8}\|H^* M_k f\|_{L^2}^2 dv_* \\
&= I_{111}+I_{112}.
\end{aligned}
\tag{4.5}
$$

By the hypothesis (3.4), f has compact support in $\overline{U}$, we obtain

$$
\begin{aligned}
I_{112} &\le C\int_{\mathbb{R}^3}\langle|v_*|\rangle^{\gamma+8}\cdot\|\mu(v+v_*)M_k f\|_{L^2}^2 dv_* \\
&\le C\int_{\mathbb{R}^3}\langle|v_*|\rangle^{\gamma+8}e^{-|v_*|^2/4}\cdot\left\|e^{|v|^2/2}M_k f\right\|_{L^2}^2 dv_* \\
&\le C'\|M_k f\|_{L^2}^2.
\end{aligned}
\tag{4.6}
$$

Here we use the fact that $e^{-|v+v_*|^2/2} \le e^{|v|^2/2} \cdot e^{-|v_*|^2/4}$. By (3.10) of Proposition 3.2, we get

$$I_{111} \le C' \cdot \left[C_0^{k+1}(k!)^s\right]^2. \tag{4.7}$$

This, together with (4.5)-(4.6), implies

$$|I_1| \le C_1 \cdot \left(k^2\|M_k f\|_{L^2}^2 + \left[C_0^{k+1}(k!)^s\right]^2\right). \tag{4.8}$$

The cancellation lemma gives (cf. [10, 11])

$$I_2 = S\int_{\mathbb{R}^6} \mu(v_*)[M_k, \Phi^*]f(v) \cdot M_k f(v) dv\, dv_*, \tag{4.9}$$

where S is a constant function. Therefore,

$$\begin{aligned} |I_2| &\le C\|\mu\|_{L^1} \cdot \|[M_k, \Phi^*]f\|_{L^2} \cdot \|M_k f\|_{L^2} \\ &\le C \cdot C_0^{k+1}(k!)^s\|\mu\|_{L^1} \cdot \|M_k f\|_{L^2} \\ &\le C_2\left\{\left[C_0^{k+1}(k!)^s\right]^2 + \|M_k f\|_{L^2}^2\right\}. \end{aligned} \tag{4.10}$$

Since $|v' - v| \le C\langle|v'|\rangle\langle|v_*|\rangle \sin(\theta/2)$, by using (3.4), Proposition 3.2, and the change of variables

$$v \longrightarrow z = v' + \tau(v - v') \tag{4.11}$$

whose Jacobian is bounded uniformly for v_*, σ, τ (see [11]), we have

$$\begin{aligned} |I_3| \le{}& C\int_0^1\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b(\cos\theta)|v' - v| \cdot |M_k f(v')| \cdot |\nabla_v[M_k, \Phi^*]f(v' + \tau(v - v'))| \\ &\times |\mu(v_*)| d\sigma\, dv\, dv_* d\tau \\ \le{}& C'\int_0^1\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b(\cos\theta)\sin\frac{\theta}{2}|M_k f(v')| \cdot |\nabla_v[M_k, \Phi^*]f(v' + \tau(v - v'))| \\ &\times \langle|v_*|\rangle|\mu(v_*)| d\sigma\, dv\, dv_* d\tau \end{aligned}$$

$$
\begin{aligned}
&\leq C' \int_0^1 \int_{\mathbb{R}^6} \int_{\mathbb{S}^2} b(\cos\theta) \sin\frac{\theta}{2} \langle |v_*| \rangle \mu(v_*) \\
&\qquad \times \left\{ \left| M_k f(v') \right|^2 + \left| \nabla_v [M_k, \Phi^*] f(v' + \tau(v - v')) \right|^2 \right\} d\sigma dv dv_* d\tau \\
&\leq C'' \left(\| M_k f \|_{L^2}^2 + \| \nabla_v [M_k, \Phi^*] f \|_{L^2}^2 \right) \\
&\leq C_3 \left\{ \left[C_0^{k+1} (k!)^s \right]^2 + k^2 \| M_k f \|_{L^2}^2 \right\}.
\end{aligned}
\tag{4.12}
$$

Combining (4.8), (4.10), and (4.12), we obtain

$$
\left(Q(\mu, f), M_k^2 f \right)_{L^2} - \left(Q(\mu, M_k f), M_k f \right)_{L^2} \leq C_4 \left\{ \left[C_0^{k+1} (k!)^s \right]^2 + k^2 \| M_k f \|_{L^2}^2 \right\}. \tag{4.13}
$$

Moreover, by [11, Lemma 2.2] and [11, page 467], we have

$$
\begin{aligned}
\left\| \langle \cdot \rangle^{\gamma/2} M_k f \right\|_{H^{\nu/2}}^2 &= O(1) \left\| \langle |D| \rangle^{\nu/2} M_k f \right\|_{L^2_{\gamma/2}}^2, \\
\left(Q(\mu, M_k f), M_k f \right)_{L^2} &\leq -C_{\mu,1} \left\| \langle \cdot \rangle^{\gamma/2} M_k f \right\|_{H^{\nu/2}}^2 + C_{\mu,2} \| M_k f \|_{L^2_{\gamma/2}}^2,
\end{aligned}
\tag{4.14}
$$

where $C_{\mu,1}$ and $C_{\mu,2}$ are the constants depending only on μ. Therefore, by (3.4) and (4.14), we get

$$
\left(Q(\mu, M_k f), M_k f \right)_{L^2} \leq -C_5 \| M_k f \|_{H^{\nu/2}}^2 + C_6 \| M_k f \|_{L^2}^2. \tag{4.15}
$$

Together with (4.13), we thus have

$$
\left(Q(\mu, f), M_k^2 f \right)_{L^2} \leq C_7 \cdot \left[C_0^{k+1} (k!)^s \right]^2 + C_8 \cdot k^2 \| M_k f \|_{L^2}^2 - C_5 \| M_k f \|_{H^{\nu/2}}^2. \tag{4.16}
$$

Let $M_k^2 f$ be the test function in the Cauchy problem (1.1), for any $t \in (0, T]$, we have

$$
\begin{aligned}
\| M_k f(t, v) \|_{L^2}^2 &= \| f_0(v) \|_{L^2}^2 + 2 \int_0^t \int_{\mathbb{R}^3} L(f)(\tau, v) M_k^2 f(\tau, v) dv\, d\tau \\
&\quad + \int_0^t \int_{\mathbb{R}^3} f(\tau, v) \left(\partial_\tau M_k^2(\tau) \right) f(\tau, v) dv\, d\tau \\
&= 2 \int_0^t \left\{ \left(Q(\mu, f), M_k^2 f \right)_{L^2} + \left(Q(f, \mu), M_k^2 f \right)_{L^2} \right\} d\tau \\
&\quad + \int_0^t \int_{\mathbb{R}^3} f(\tau, v) \left(\partial_\tau M_k^2(\tau) \right) f(\tau, v) dv\, d\tau + \| f_0(v) \|_{L^2}^2.
\end{aligned}
\tag{4.17}
$$

Since

$$
\begin{gathered}
O(1)(k!)^{2s} \geq 2^{2(k+2)}[(k+2)!]^2 \geq (2k+3)!, \\
\partial_t M_k^2(t,\xi) = 2k M_k^2(t,\xi)\log\langle\xi\rangle,
\end{gathered} \tag{4.18}
$$

by Lemma 2.7 and (4.16), it holds that

$$
\begin{aligned}
&\|M_k f(t,v)\|_{L^2}^2 + C_5 \int_0^t \|M_k f\|_{H^{\nu/2}}^2 d\tau \\
&\quad \leq 2\mathrm{k} \int_0^t \left\| (\log\langle D_v\rangle)^{1/2} (M_k f)(\tau) \right\|_{L^2}^2 d\tau \\
&\qquad + C_8 \int_0^t k^2 \|M_k f\|_{L^2}^2 d\tau + C_9 \left[C_0^{k+1}(k!)^s\right]^2 + \|f_0(v)\|_{L^2}^2.
\end{aligned} \tag{4.19}
$$

The Young's inequality gives

$$
\begin{aligned}
C_8 k^2 &\leq \left[\frac{4C_8}{C_5\cdot(\nu+2)}\right]^{2/\nu} \cdot \frac{C_8\nu}{\nu+2} \cdot k^{2+4/\nu}\langle|\xi|\rangle^{-2} + \frac{C_5}{2}\langle|\xi|\rangle^{\nu}, \\
2k\log\langle|\xi|\rangle &= \frac{4k}{\nu}\log\langle|\xi|\rangle^{\nu/2} \\
&\leq \frac{4k}{\nu}\cdot\langle|\xi|\rangle^{\nu/2} \\
&\leq \left[\frac{4\cdot(4+\nu)}{C_8\cdot\nu(\nu+2)}\right]^{(4+\nu)/\nu} \frac{2}{\nu+2}\cdot k^{2+4/\nu}\langle|\xi|\rangle^{-2} + \frac{C_5}{2}\langle|\xi|\rangle^{\nu},
\end{aligned} \tag{4.20}
$$

which implies

$$
\begin{aligned}
&2k\int_0^t \left\| (\log\langle D_v\rangle)^{1/2}(M_k f)(\tau) \right\|_{L^2}^2 d\tau + C_8 \int_0^t \|k M_k f\|_{L^2}^2 d\tau \\
&\quad \leq C_5 \int_0^t \|M_k f\|_{H^{\nu/2}}^2 d\tau + C_{10}\cdot k^{2+4/\nu}\cdot \int_0^t \|M_{k-1} f\|_{L^2}^2 d\tau.
\end{aligned} \tag{4.21}
$$

Taking (4.21) into (4.19), and applying the assumption (E_k), we have

$$
\begin{aligned}
\|M_k f(t,v)\|_{L^2}^2 &\leq \|f_0(v)\|_{L^2}^2 + C_9\left[C_0^{k+1}(k!)^s\right]^2 + C_{10}\cdot k^{2+4/\nu}\int_0^t \|M_{k-1} f\|_{L^2}^2 d\tau \\
&\leq \|f_0(v)\|_{L^2}^2 + C_9\left[C_0^{k+1}(k!)^s\right]^2 + C_{10}\cdot k^{2+4/\nu}\left\{C_0^k\cdot[(k-1)!]^s\right\}^2 \\
&\leq C_{11}\left[C_0^{k+1}(k!)^s\right]^2,
\end{aligned} \tag{4.22}
$$

which implies that

$$\sup_{t\in(0,T]} \|M_k f(t,v)\|_{L^2} = \sup_{t\in(0,T]} \|f(t,\cdot)\|_{H^{kt}} \le C_{11}\cdot C_0^{k+1}(k!)^s. \tag{4.23}$$

In other words, it follows from (E_k) that

$$(E_{k+1})\text{: for any } i\in[0,k],\quad \sup_{t\in(0,T]} \|f(t,\cdot)\|_{H^{it}} \le C_{11}\cdot C_0^{i+1}(i!)^s. \tag{4.24}$$

Taking the same procedures as above, we can also gain (E_{k+2}) from (E_{k+1}), which is described as below:

$$(E_{k+2})\text{: for any } i\in[0,k+1],\quad \sup_{t\in(0,T]} \|f(t,\cdot)\|_{H^{it}} \le C_{11}^2\cdot C_0^{i+1}(i!)^s, \tag{4.25}$$

that is,

$$\begin{aligned}
\sup_{t\in(0,T]} \|f(t,\cdot)\|_{H^{0t}} \le C_0^1(0!)^s &\Longrightarrow \sup_{t\in(0,T]} \|f(t,\cdot)\|_{H^{1t}} \le C_{11}^1\cdot C_0^2(1!)^s \\
&\Longrightarrow \sup_{t\in(0,T]} \|f(t,\cdot)\|_{H^{2t}} \le C_{11}^2\cdot C_0^3(2!)^s \\
&\quad\vdots \\
&\Longrightarrow \sup_{t\in(0,T]} \|f(t,\cdot)\|_{H^{kt}} \le C_{11}^k\cdot C_0^{k+1}(k!)^s.
\end{aligned} \tag{4.26}$$

Let $C_{12} = C_0\cdot C_{11}$, we thus conclude that for any $k\in\mathbb{N}$,

$$\sup_{t\in(0,T]} \|f(t,\cdot)\|_{H^{kt}(U)} \le C_{12}^{k+1}(k!)^s. \tag{4.27}$$

For any fixed number $0 < t \le T \le 1$, suppose that

$$\begin{aligned}
s_0 &= \left(\left[\frac{1}{t}\right]+1\right)s, \\
s_1 &= \sum_{i=0}^{[1/t]} (1-it),
\end{aligned} \tag{4.28}$$

where $\lceil 1/t \rceil$ denotes the smallest integer bigger than $1/t$. With a convention that $k! = 1$ if $0 \geq k \in \mathbb{Z}$, we have

$$\begin{aligned} 2^{ks_1} &= 2^k \cdot 2^{k(1-t)} \cdot 2^{k(1-2t)} \cdots 2^{k(1-\{1/t\}t)} \\ &\geq \frac{k!}{(kt)![(1-t)k]!} \cdot \frac{[(1-t)k]!}{(kt)![(1-2t)k]!} \cdots \frac{[(1-\lceil 1/t\rceil t)k]!}{(kt)![(1-(\lceil 1/t\rceil+1)t)k]!} \\ &\geq \frac{k!}{[(kt)!]^{(\lceil 1/t\rceil+1)}} \\ &\geq \frac{k!}{[(kt)!]^{s_0/s}}. \end{aligned} \tag{4.29}$$

This, together with (4.27), implies that

$$\sup_{t\in(0,T]} \|f(t,\cdot)\|_{H^{kt}(U)} \leq C_{12}^{k+1}(k!)^s \leq 2^{kss_1}C_{12}^{k+1}[(kt!)]^{s_0} \leq C_{13}^{k+1}[(kt!)]^{s_0}, \tag{4.30}$$

where $k \in \mathbb{N}$, and C_{16} is a constant only depending on t. Furthermore, for any fixed number $t_0 > 0$, put

$$\begin{aligned} s_0' &= \left(\left\lceil \frac{1}{t_0} \right\rceil + 1\right)s, \\ s_1' &= \sum_{i=0}^{\lceil 1/t_0\rceil} (1 - it_0). \end{aligned} \tag{4.31}$$

Then for any $k \in \mathbb{N}$, we can choose $C_{13}' = 2^{ss_1'}C_{12}$ and have the fact that

$$\sup_{t\in[t_0,T]} \|f(t,\cdot)\|_{H^{kt}(U)} \leq \left(C_{13}'\right)^{k+1}[(k!)]^{s_0'}. \tag{4.32}$$

This completes the proof of Theorem 1.3.

5. Proof of Propositions 3.1 and 3.2

Proof of Proposition 3.1. We first notice that

$$\begin{aligned} [M_k(D_v), \Phi^*]f(v) &= M_k(D_v)(\Phi^* f)(v) - \Phi^*(v)M_k(D_v)f(v) \\ &= \left(\mathcal{F}^{-1}M_k(\xi) * \Phi^* f\right)(v) - \Phi^*(v)\left(\mathcal{F}^{-1}M_k(\xi) * f\right)(v) \\ &= \int_{\mathbb{R}^6} e^{i(v-y)\xi}M_k(\xi)d\xi f(y)\left(\Phi^*(y) - \Phi^*(v)\right)dy. \end{aligned} \tag{5.1}$$

Using the Taylor formula of order $k+6$, we get

$$\Phi^*(y) - \Phi^*(v) = \sum_{j=1}^{k+5} \frac{(y-v)^j}{j!} \partial_v^j \Phi^*(v) + \frac{(y-v)^{k+6}}{(k+6)!} \partial_v^{k+6} \Phi^*(c) \tag{5.2}$$

for some $c \in (y, v)$. Hence,

$$[M_k(D_v), \Phi^*] f(v) = \sum_{j=1}^{k+5} \Gamma_j f(v) + \Gamma_{k+6} f, \tag{5.3}$$

where

$$\begin{aligned}
\Gamma_j f(v) &= \int_{\mathbb{R}^6} e^{i(v-y)\xi} M_k(\xi) d\xi f(y) \frac{(y-v)^j}{j!} \partial_v^j \Phi^*(v) dy \\
&= \frac{(-i)^j}{j!} \int_{\mathbb{R}^6} e^{i(v-y)\xi} \partial_\xi^j M_k(\xi) d\xi f(y) \partial_v^j \Phi^*(v) dy \\
&= \frac{(-i)^j}{j!} \left(\mathcal{F}^{-1} \partial_\xi^j M_k * f \right)(v) \cdot \partial_v^j \Phi^*(v), \\
\Gamma_{k+6} f &= \int_{\mathbb{R}^6} e^{i(v-y)\xi} M_k(\xi) d\xi f(y) \frac{(y-v)^{k+6}}{(k+6)!} \partial_v^{k+6} \Phi^*(c) dy \\
&= \frac{(-i)^{k+6}}{(k+6)!} \int_{\mathbb{R}^6} e^{i(v-y)\xi} \partial_\xi^{k+6} M_k(\xi) d\xi f(y) \partial_v^{k+6} \Phi^*(c) dy.
\end{aligned} \tag{5.4}$$

From Lemma 2.1, Remark 2.2, (3.6), and (3.7), it follows that

$$\begin{aligned}
\sum_{j=1}^{k} \|\Gamma_j f(v)\|_{L^2} &= \sum_{j=1}^{k} \left\| \frac{(-i)^j}{j!} \left(\mathcal{F}^{-1} \partial_\xi^j M_k * f \right)(v) \cdot \partial_v^j \Phi^*(v) \right\|_{L^2} \\
&\le \sum_{j=1}^{k} \frac{1}{j!} \left\| \partial_\xi^j M_k(\xi) \mathcal{F} f(\xi) \right\|_{L^2} \cdot \left\| \partial_v^j \Phi^*(v) \right\|_{L^\infty} \\
&\le C \cdot \sum_{j=1}^{k} 16^j k \cdots (k-j+1) \{(k-j)!\}^s \cdot C_0^{k-j+1} \\
&\le C \cdot C_0^{k+1} \{(k-1)!\}^s k^2 \\
&\le C_1 \cdot C_0^{k+1} (k!)^s,
\end{aligned}$$

$$\begin{aligned}
\sum_{j=k+1}^{k+5} \left\| \Gamma_j f(v) \right\|_{L^2} &\leq \sum_{j=k+1}^{k+5} \frac{1}{j!} \left\| \partial_\xi^j M_k(\xi) \mathcal{F} f(\xi) \right\|_{L^2} \cdot \left\| \partial_v^j \Phi^*(v) \right\|_{L^\infty} \\
&\leq C \cdot (k+5)! \sup_{t\in(0,T]} \left\| f(t,v) \right\|_{L^2} \\
&\leq C_2 \cdot C_0^{k+1} (k!)^s, \\
\left\| \Gamma_{k+6} f \right\|_{L^2} &= \left\| \frac{(-i)^{k+6}}{(k+6)!} \int_{\mathbb{R}^6} e^{i(v-y)\xi} \partial_\xi^{k+6} M_k(\xi) d\xi f(y) \partial_v^{k+6} \Phi^*(c) dy \right\|_{L^2} \\
&\leq \frac{1}{(k+6)!} \left\| \int_{\mathbb{R}^6} \left| \partial_\xi^{k+6} M_k(\xi) \right| d\xi \cdot \left| f(y) \right| \cdot \left| \partial_v^{k+6} \Phi^*(c) \right| dy \right\|_{L^2} \\
&\leq C \cdot C_0^{k+1} (k!)^s \int_{\mathbb{R}^3} \left(1 + |\xi|^2\right)^{-3} d\xi \\
&\leq C_3 \cdot C_0^{k+1} (k!)^s.
\end{aligned} \tag{5.5}$$

Combining (5.5), we complete the proof of Proposition 3.1. □

Proof of Proposition 3.2. One has

$$\begin{aligned}
\left\| \nabla_v [M_k(D_v), \Phi^*] f(t,v) \right\|_{L^2} &\leq C \cdot \left\| \langle |\xi| \rangle \mathcal{F}\left([M_k, \Phi^*] f \right)(t,\xi) \right\|_{L^2} \\
&\leq C \cdot \left\| [\langle D_v \rangle M_k, \Phi^*] f(t,v) - [\langle D_v \rangle, \Phi^*] M_k f(t,v) \right\|_{L^2}.
\end{aligned} \tag{5.6}$$

Similar to the proof of Proposition 3.1, we obtain

$$\begin{aligned}
\left\| [\langle D_v \rangle M_k, \Phi^*] f(t,v) \right\|_{L^2} &\leq C \left\{ (k+1) \left\| M_k f(t,v) \right\|_{L^2} + C_0^{k+1} (k!)^s \right\}, \\
\left\| [\langle D_v \rangle, \Phi^*] M_k f(t,v) \right\|_{L^2} &\leq C \left\{ \left\| M_k f(t,v) \right\|_{L^2} + C_0^{k+1} (k!)^s \right\}.
\end{aligned} \tag{5.7}$$

Then (3.9) is obtained. The proof of (3.10) is similar so is omitted. This completes the proof of Proposition 3.2. □

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Research Article

Some Estimates of Rough Bilinear Fractional Integral

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We study the boundedness of rough bilinear fractional integral $B_{\Omega,\alpha}$ on Morrey spaces $L^{p,\lambda}(\mathbb{R}^n)$ and modified Morrey spaces $\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$ and obtain some sufficient and necessary conditions on the parameters. Furthermore, we consider the boundedness of $B_{\Omega,\alpha}$ on generalized central Morrey space $\dot{B}^{p,\varphi}(\mathbb{R}^n)$. These extend some known results.

1. Introduction

In recent years, multilinear analysis becomes a very active research topic in studying harmonic analysis. As one of the most important operators, the multilinear fractional integral has also attracted much attention. In this note, we will consider the multilinear fractional integral with rough kernel. For fixed distinct and nonzero real numbers $\theta_1, \dots, \theta_m$, and $0 < \alpha < n$, the m-linear fractional with rough kernel is defined by

$$I_{\Omega,\alpha}\left(\vec{f}\right) = \int_{\mathbb{R}^n} \prod_{i=1}^{m} f_i(x - \theta_i y) \frac{\Omega(y)}{|y|^{n-\alpha}} dy, \tag{1.1}$$

where $\Omega \in L^s(S^{n-1})$ $(s \geq 1)$ is homogeneous of degree zero on $\mathbb{R}^n$, and S^{n-1} denotes the unit sphere of $\mathbb{R}^n$.

When $\Omega \equiv 1$, The L^p boundedness of operator $I_{1,\alpha}$ has been well studied in [1, 2]. Recently, Hendar and Idha discussed the boundedness property of $I_{1,\alpha}$ on generalized Morrey space in [3].

Here, without loss of generality, we will study the case $m = 2$. More specifically, we will study the rough bilinear fractional integral:

$$B_{\Omega,\alpha}(f,g)(x) = \int_{\mathbb{R}^n} f(x-y)g(x+y)\frac{\Omega(y)}{|y|^{n-\alpha}}dy, \quad 0<\alpha<n. \tag{1.2}$$

The study of the operators $B_{\Omega,\alpha}$ and its related operators with rough kernel Ω recently attracted many attentions. In 2002, Ding and Chin first discussed its $L^p(\mathbb{R}^n)$ boundedness. The following theorem is their main result:

Theorem A (see [4]). *Let $0<\alpha<n$, $1 \le s' < n/\alpha$ and $1 \le p_1, p_2 \le \infty$. If*

$$\frac{1}{p_1}+\frac{1}{p_2} \ge \frac{\alpha}{n}, \qquad \frac{1}{q}=\frac{1}{p_1}+\frac{1}{p_2}-\frac{\alpha}{n}, \tag{1.3}$$

there exists a positive constant C such that for any $f \in L^{p_1}(\mathbb{R}^n)$, $g \in L^{p_2}(\mathbb{R}^n)$,

(1) when $s' < \min\{p_1,p_2\}$,

$$\|B_{\Omega,\alpha}(f,g)\|_{L^q(\mathbb{R}^n)} \le C\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}; \tag{1.4}$$

(2) when $s' = \min\{p_1,p_2\}$,

$$\|B_{\Omega,\alpha}(f,g)\|_{L^{q,\infty}(\mathbb{R}^n)} \le C\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}. \tag{1.5}$$

Later, when $q > n/(n-\alpha)$, Chen and Fan in [5] relaxed the conditions of Ω in Theorem A using Hölder inequality. Their main result is as follows.

Theorem B. *Let $q > n/(n-\alpha)$, $0<\alpha<n$, $p_1,p_2>1$ and*

$$\frac{1}{q}=\frac{1}{p_1}+\frac{1}{p_2}-\frac{\alpha}{n}. \tag{1.6}$$

If $\Omega \in L^{n/(n-\alpha)}(S^{n-1})$, then there exists a positive constant C such that

$$\|B_{\Omega,\alpha}(f,g)\|_{L^q(\mathbb{R}^n)} \le C\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}. \tag{1.7}$$

We note that when $q \le n/(n-\alpha)$, Hölder inequality is not sufficient in Theorem B. So how to relax the index of q is left. In fact, in [6, 7] the authors have obtained the necessary and sufficient conditions on the parameters for the m-linear fractional integral operator $I_{\Omega,\alpha}$ with rough kernel from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \times \cdots L^{p_m}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ by using the pointwise rearrangement estimate of the m-linear convolution.

Theorem C. *Let $0<\alpha<n$, Ω and be homogeneous of degree zero on $\mathbb{R}^n$, $\Omega \in L^{n/(n-\alpha)}(S^{n-1})$, let p be the harmonic mean of $p_1,p_2,\ldots,p_m>1$, and $n/(n-\alpha) \le p < n/\alpha$. Then the condition $1/q = 1/p-\alpha/n$ is necessary and sufficient for the boundedness of $I_{\Omega,\alpha}$ from $L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)\times\cdots L^{p_m}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.*

This paper is organized as follows: in the second part of this work we prove some boundedness properties of $B_{\Omega,\alpha}$ on Morrey space and extend Theorem C to Morrey spaces; in the third part, we obtain the sufficient and necessary conditions on the parameters for the boundedness of $B_{\Omega,\alpha}$ on modified Morrey space; in the last part, we find the sufficient condition on the pair (φ, ν) which ensures the boundedness of the operators $B_{\Omega,\alpha}$ on the generalized center Morrey space. Since Morrey space, modified Morrey space and central Morrey space all can be seen as generalized L^p space.

2. The Boundedness of $B_{\Omega,\alpha}$ on Morrey Space

The classical Morrey spaces $L^{p,\lambda}(\mathbb{R}^n)$ were originally introduced by Morrey in [8] to study the local behavior of solutions to second-order elliptic partial differential equations. The reader can find more details in [9].

For $x \in \mathbb{R}^n$ and $t > 0$, let $B(x,t)$ denotes the open ball centered at x of radius t, and $|B(x,t)|$ is the Lebesgue measure of the ball $B(x,t)$. When $1 \le p < \infty$ and $\lambda \ge 0$, Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ is defined by

$$L^{p,\lambda}(\mathbb{R}^n) = \left\{ f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{L^{p,\lambda}(\mathbb{R}^n)} < \infty \right\}, \tag{2.1}$$

where

$$\|f\|_{L^{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, t>0} \left(\frac{1}{t^\lambda} \int_{B(x,t)} |f(x)|^p dx \right)^{1/p}. \tag{2.2}$$

If $1 \le p < \infty$, then $L^{p,0}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $L^{p,n}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$. When $\lambda > n$, $L^{p,\lambda}(\mathbb{R}^n) = \{0\}$. So we only consider the case $0 < \lambda < n$.

Since Morrey space can be seen as the generalized L^p space, we will be interested in the boundedness of $B_{\Omega,\alpha}$ on Morry space $L^{p,\lambda}(\mathbb{R}^n)$. In order to prove our results, we need the following bilinear maximal function:

$$M(f,g)(x) = \sup_{r>0} \frac{1}{r^n} \int_{|y|<r} |f(x-y)||g(x+y)| dy. \tag{2.3}$$

Lemma 2.1. *Let $p > 1$, $0 < \lambda < n$ and $1/p = 1/p_1 + 1/p_2$. If*

$$\frac{\lambda}{p} = \frac{\lambda_1}{p_1} + \frac{\lambda_2}{p_2}, \quad 0 < \lambda_1,\ \lambda_2 < n, \tag{2.4}$$

then there exists a positive constant C such that

$$\|M(f,g)\|_{L^{p,\lambda}(\mathbb{R}^n)} \le C\|f\|_{L^{p_1,\lambda}(\mathbb{R}^n)} \|g\|_{L^{p_2,\lambda}(\mathbb{R}^n)}. \tag{2.5}$$

Proof. In [10], Fefferman and Stein have proved that for every p, $1 < p < \infty$, there is a constant $C_p > 0$ such that for any measurable functions f on $\mathbb{R}^n$ and $\varphi \geq 0$, the following inequality holds,

$$\int_{\mathbb{R}^n} (Mf(x))^p \varphi(x)dx \leq C_p \int_{\mathbb{R}^n} |f(x)|^p M\varphi(x)dx, \tag{2.6}$$

where M is the Hardy-LittleWood maximal function. Set $\varphi(x)$ be the characteristic function $\chi(x)$, when $1 \leq \delta < p$, by the above inequality, we can get

$$\int_{\mathbb{R}^n} (M_\delta f(x))^p \chi(x)dx \leq C_p \int_{\mathbb{R}^n} |f(x)|^p M\chi(x)dx, \tag{2.7}$$

where $M_\delta f(x) = (Mf^\delta)^{1/\delta}(x)$.

Taking $f \in L^{p,\lambda}(\mathbb{R}^n)$, $0 < \lambda < n$, $\chi(x)$ is the characteristic function of a ball $B(x_0, r)$ in $\mathbb{R}^n$, by simple calculating,

$$\int_{B(x_0,r)} (M_\delta f(x))^p dx \leq C\|f\|^p_{L^{p,\lambda}(\mathbb{R}^n)} r^\lambda, \tag{2.8}$$

that is, $\|M_\delta f\|_{L^{p,\lambda}(\mathbb{R}^n)} \leq C\|f\|_{L^{p,\lambda}(\mathbb{R}^n)}$. For More details, see [11] about the boundedness of Hardy-Littlewood maximal function on Morrey space.

So when $p > 1$, $1/p = 1/p_1 + 1/p_2$, $\lambda/p = \lambda_1/p_1 + \lambda_2/p_2$, we have

$$\begin{aligned} \|M(f,g)\|_{L^{p,\lambda}(\mathbb{R}^n)} &\leq \|M_{p_1/p}(f) M_{p_2/p}(g)\|_{L^{p,\lambda}(\mathbb{R}^n)} \\ &\leq \|M_{p_1/p}(f)\|_{L^{p_1,\lambda_1}(\mathbb{R}^n)} \|M_{p_2/p}(g)\|_{L^{p_2,\lambda_2}(\mathbb{R}^n)} \\ &\leq C\|f\|_{L^{p_1,\lambda_1}(\mathbb{R}^n)} \|g\|_{L^{p_2,\lambda_2}(\mathbb{R}^n)}. \end{aligned} \tag{2.9}$$

□

Theorem 2.2. *Suppose $0 < \alpha < n$, and let $\Omega \in L^s(S^{n-1})$ be homogeneous of degree zero on $\mathbb{R}^n$, let p be the harmonic mean of p_1 and p_2, $1 < p < n/\alpha$, $0 < \lambda < n - \alpha p$ and $s' < p$. If*

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n-\lambda}, \quad \frac{\lambda}{p} = \frac{\lambda_1}{p_1} + \frac{\lambda_2}{p_2}, \quad 0 < \lambda_1, \lambda_2 < n, \tag{2.10}$$

then there exists a positive constant C such that

$$\|B_{\Omega,\alpha}(f,g)\|_{L^{q,\lambda}(\mathbb{R}^n)} \leq C\|f\|_{L^{p_1,\lambda_1}(\mathbb{R}^n)} \|g\|_{L^{p_2,\lambda_2}(\mathbb{R}^n)}. \tag{2.11}$$

Proof. Let $f \in L^{p_1,\lambda_1}(\mathbb{R}^n)$, $g \in L^{p_2,\lambda_2}(\mathbb{R}^n)$, $\sigma = (n - \alpha s' + \lambda)/2$, for $s' < p$ and $0 < \lambda < n - \alpha p$, we can get $\lambda < \sigma < n - \alpha s'$, $(n-\lambda)/p > \alpha > (n-\sigma)/s'$. First, $|B_{\Omega,\alpha}(f,g)(x)|$ is decomposed by

$$\begin{aligned} \left|B_{\Omega,\alpha}(f,g)(x)\right| &= \left(\int_{|y|\le\varepsilon} + \int_{|y|\ge\varepsilon}\right) f(x-y)g(x+y)\frac{\Omega(y)}{|y|^{n-\alpha}} \\ &=: I_1(x) + I_2(x). \end{aligned} \tag{2.12}$$

Estimate of $I_1(x)$ is

$$\begin{aligned} I_1(x) &= \sum_{m=1}^{\infty}\int_{|y|\sim\varepsilon 2^{-m}} |f(x-y)g(x+y)|\frac{|\Omega(y)|}{|y|^{n-\alpha}}dy \\ &\le \sum_{m=1}^{\infty}(\varepsilon 2^{-m})^{\alpha-n}\int_{|y|\sim\varepsilon 2^{-m}} |f(x-y)g(x+y)||\Omega(y)|dy \\ &\le \sum_{m=1}^{\infty}(\varepsilon 2^{-m})^{\alpha} M\left(f^{s'},g^{s'}\right)^{1/s'}(x) \\ &\le C\varepsilon^{\alpha} M\left(f^{s'},g^{s'}\right)^{1/s'}(x) \\ &=: C\varepsilon^{\alpha} M_{s'}(f,g)(x), \end{aligned} \tag{2.13}$$

and estimate of $I_2(x)$ is

$$\begin{aligned} I_2(x) &\le \left(\int_{|y|\ge\varepsilon}\frac{f^{s'}(x-y)g^{s'}(x+y)}{|y|^{\sigma}}dy\right)^{1/s'}\left(\int_{|y|\ge\varepsilon}|y|^{(\sigma/s'+\alpha-n)s}|\Omega(y)|^{s}dy\right)^{1/s} \\ &\le C\varepsilon^{(\sigma/s'+\alpha-n)+n/s}\left(\int_{|y|\ge\varepsilon}\frac{f^{s'}(x-y)g^{s'}(x+y)}{|y|^{\sigma}}dy\right)^{1/s'} \\ &=: C\varepsilon^{(\sigma/s'+\alpha-n)+n/s}F_{\sigma}(f,g)(x). \end{aligned} \tag{2.14}$$

For $F_{\sigma}(f,g)(x)$, we have the following estimates:

$$\begin{aligned} F_{\sigma}(f,g)(x) &\le \left(\sum_{k=0}^{\infty}\int_{|y|\sim\varepsilon 2^{k}}\frac{\left|f^{s'}(x-y)g^{s'}(x+y)\right|}{|y|^{\sigma}}dy\right)^{1/s'} \\ &\le \sum_{k=0}^{\infty}\left(\int_{|y|\sim\varepsilon 2^{k}}\frac{\left|f^{s'}(x-y)g^{s'}(x+y)\right|}{|y|^{\sigma}}dy\right)^{1/s'} \\ &\le \sum_{k=0}^{\infty}\left(\varepsilon 2^{k}\right)^{-\sigma/s'}\left(\int_{|y|\sim\varepsilon 2^{k}}\left|f^{s'}(x-y)g^{s'}(x+y)\right|dy\right)^{1/s'} \end{aligned}$$

$$\le \sum_{k=0}^{\infty}\left(\varepsilon 2^k\right)^{(n-\sigma)/s'-n/p}\left(\int_{|y|\sim\varepsilon 2^k}\left|f^{p_1}(x-y)\right|dy\right)^{1/p_1}\left(\int_{|y|\sim\varepsilon 2^k}\left|g^{p_2}(x-y)\right|dy\right)^{1/p_2}$$
$$\le \sum_{k=0}^{\infty}\left(\varepsilon 2^k\right)^{(n-\sigma)/s'-n/p+\lambda_1/p_1+\lambda_2/p_2}\|f\|_{L^{p_1,\lambda_1}(\mathbb{R}^n)}\|g\|_{L^{p_2,\lambda_2}(\mathbb{R}^n)}$$
$$\le C(\varepsilon)^{(n-\sigma)/s'-(n-\lambda)/p}\|f\|_{L^{p_1,\lambda_1}(\mathbb{R}^n)}\|g\|_{L^{p_2,\lambda_2}(\mathbb{R}^n)}. \tag{2.15}$$

Combining the above estimates, we have

$$\left|B_{\Omega,\alpha}(f,g)(x)\right| \le C\varepsilon^{\alpha}M_{s'}(f,g)(x) + C\varepsilon^{(\lambda-n)/p+\alpha}\|f\|_{L^{p_1,\lambda_1}(\mathbb{R}^n)}\|g\|_{L^{p_2,\lambda_2}(\mathbb{R}^n)}. \tag{2.16}$$

Let $\varepsilon^{\alpha}M_{s'}(f,g)(x) = \varepsilon^{((\lambda-n)/p)+\alpha}\|f\|_{p_1,\lambda_1}\|g\|_{p_2,\lambda_2}$, then

$$\left|B_{\Omega,\alpha}(f,g)(x)\right| \le C\left(M_{s'}(f,g)(x)\right)^{p/q}\|f\|^{1-p/q}_{L^{p_1,\lambda_1}(\mathbb{R}^n)}\|g\|^{1-p/q}_{L^{p_2,\lambda_2}(\mathbb{R}^n)}. \tag{$\star$}$$

By computation, we get

$$\left(\frac{1}{r^{\lambda}}\int_{B(x,r)}\left(M_{s'}(f,g)(x)\right)^{(p/q)\times q}dx\right)^{1/q}$$
$$= \left(\frac{1}{r^{\lambda}}\int_{B(x,r)}\left(M\left(f^{s'},g^{s'}\right)(x)\right)^{p/s'}dx\right)^{1/p\times p/q}$$
$$\le \left(\frac{1}{r^{\lambda_1}}\int_{B(x,r)}f(x)^{p_1}dx\right)^{1/p_1\times p/q}\left(\frac{1}{r^{\lambda_2}}\int_{B(x,r)}g(x)^{p_2}dx\right)^{1/p_2\times p/q}$$
$$\le \|f\|^{p/q}_{L^{p_1,\lambda_1}(\mathbb{R}^n)}\|g\|^{p/q}_{L^{p_2,\lambda_2}(\mathbb{R}^n)}. \tag{2.17}$$

Taking the supremum of r, we have

$$\left\|\left(M_{s'}(f,g)\right)^{p/q}\right\|_{L^{q,\lambda}(\mathbb{R}^n)} \le \|f\|^{p/q}_{L^{p_1,\lambda_1}(\mathbb{R}^n)}\|g\|^{p/q}_{L^{p_2,\lambda_2}(\mathbb{R}^n)}. \tag{2.18}$$

Hence

$$\left\|B_{\Omega,\alpha}(f,g)\right\|_{L^{q,\lambda}(\mathbb{R}^n)} \le C\|f\|_{L^{p_1,\lambda_1}(\mathbb{R}^n)}\|g\|_{L^{p_2,\lambda_2}(\mathbb{R}^n)}. \tag{2.19}$$

□

Theorem 2.3. *Suppose $0 < \alpha < n$, and let $\Omega \in L^s(S^{n-1})$ be homogeneous of degree zero on $\mathbb{R}^n$, let p be the harmonic mean of p_1 and p_2, $1 < p < n/\alpha$, $0 < \lambda < n-\alpha p$, $s' < p$ and $\lambda/p = \lambda_1/p_1+\lambda_2/p_2$, $0 < \lambda_1, \lambda_2 < n$, then the condition $1/q = 1/p - \alpha/(n-\lambda)$ is necessary and sufficient for the boundedness of $B_{\Omega,\alpha}$ from $L^{p_1,\lambda_1}(\mathbb{R}^n) \times L^{p_2,\lambda_2}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$.*

Proof. Sufficiency part of Theorem 2.3 is proved in Theorem 2.2.

Necessity. Let $1 < p < n/\alpha$ and $f \in L^{p_1,\lambda_1}(\mathbb{R}^n)$, $g \in L^{p_2,\lambda_2}(\mathbb{R}^n)$. Denote $f_t(x) =: f(tx)$ and $g_t(x) =: g(tx)$. Then we have

$$\|f_t\|_{L^{p_1,\lambda_1}(\mathbb{R}^n)} = t^{-n/p_1+\lambda_1/p_1}\|f\|_{L^{p_1,\lambda_1}(\mathbb{R}^n)}, \qquad \|g_t\|_{L^{p_2,\lambda_2}(\mathbb{R}^n)} = t^{-n/p_2+\lambda_2/p_2}\|g\|_{L^{p_2,\lambda_2}(\mathbb{R}^n)},$$
$$B_{\Omega,\alpha}(f_t,g_t)(x) = t^{-\alpha}B_{\Omega,\alpha}(f,g)(tx), \qquad \|B_{\Omega,\alpha}(f_t,g_t)\|_{L^{q,\lambda}(\mathbb{R}^n)} = t^{-\alpha-n/q+\lambda/q}\|B_{\Omega,\alpha}(f,g)\|_{L^{q,\lambda}(\mathbb{R}^n)}. \tag{2.20}$$

Since $B_{\Omega,\alpha}$ is bounded from $L^{p_1,\lambda_1}(\mathbb{R}^n) \times L^{p_2,\lambda_2}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$, it is true that

$$\begin{aligned}\|B_{\Omega,\alpha}(f,g)\|_{L^{q,\lambda}(\mathbb{R}^n)} &= t^{\alpha+n/q-\lambda/q}\|B_{\Omega,\alpha}(f_t,g_t)\|_{L^{q,\lambda}(\mathbb{R}^n)}\\ &\le Ct^{\alpha+n/q-\lambda/q}\|f_t\|_{L^{p_1,\lambda_1}(\mathbb{R}^n)}\|g_t\|_{L^{p_2,\lambda_2}(\mathbb{R}^n)}\\ &\le Ct^{\alpha+n/q-\lambda/q-n/p+\lambda/p}\|f\|_{L^{p_1,\lambda_1}(\mathbb{R}^n)}\|g\|_{L^{p_2,\lambda_2}(\mathbb{R}^n)},\end{aligned} \tag{2.21}$$

where C depends only on p, q, λ, and n.

If $1/q < 1/p - \alpha/(n-\lambda)$, then in the case $t \to 0$, for all $f \in L^{p_1,\lambda_1}(\mathbb{R}^n)$, $g \in L^{p_2,\lambda_2}(\mathbb{R}^n)$, we have $\|B_{\Omega,\alpha}(f,g)\|_{L^{q,\lambda}(\mathbb{R}^n)} = 0$.

If $1/q > 1/p - \alpha/(n-\lambda)$, then in the case $t \to \infty$, for all $f \in L^{p_1,\lambda_1}(\mathbb{R}^n)$, $g \in L^{p_2,\lambda_2}(\mathbb{R}^n)$, we have $\|B_{\Omega,\alpha}(f,g)\|_{L^{q,\lambda}(\mathbb{R}^n)} = 0$.

Therefore, we get $1/q = 1/p - \alpha/(n-\lambda)$. □

Corollary 2.4. *Let $0 < \alpha < n$, $\Omega \in L^s(S^{n-1})$ be homogeneous of degree zero on $\mathbb{R}^n$, p be the harmonic mean of p_1 and p_2, $1 < p < n/\alpha$, $0 < \lambda < n - \alpha p$, and $s' < p$. If*

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}; \quad \frac{\mu}{q} = \frac{\lambda}{p} = \frac{\lambda_1}{p_1} + \frac{\lambda_2}{p_2}, \quad 0 < \lambda_1, \lambda_2 < n, \tag{2.22}$$

then there exists a positive constant C such that

$$\|B_{\Omega,\alpha}(f,g)\|_{L^{q,\mu}(\mathbb{R}^n)} \le C\|f\|_{L^{p_1,\lambda_1}(\mathbb{R}^n)}\|g\|_{L^{p_2,\lambda_2}(\mathbb{R}^n)}. \tag{2.23}$$

Proof. By Hölder inequality, it is easy to know when $t = (n-\lambda)q/(n-\mu)$, we have $L^{t,\lambda}(\mathbb{R}^n) \subseteq L^{q,\mu}(\mathbb{R}^n)$, through the given condition, $1/t = 1/p - \alpha/(n-\lambda)$. Applying Theorem 2.2, we get

$$\|B_{\Omega,\alpha}(f,g)\|_{L^{q,\mu}(\mathbb{R}^n)} \le \|B_{\Omega,\alpha}(f,g)\|_{L^{t,\lambda}(\mathbb{R}^n)} \le C\|f\|_{L^{p_1,\lambda_1}(\mathbb{R}^n)}\|g\|_{L^{p_2,\lambda_2}(\mathbb{R}^n)}. \tag{2.24}$$

From the inequality ($\star$) and Theorem 2.2, we obtain an Olsen inequality involving a multiplication operator. □

Corollary 2.5. *Suppose $0 < \alpha < n$, and let $\Omega \in L^s(S^{n-1})$ be homogeneous of degree zero on $\mathbb{R}^n$, let p be the harmonic mean of p_1 and p_2, $1 < p < n/\alpha$, $0 < \lambda < n - \alpha p$, $s' < p$, and $W \in L^{(n-\lambda)/\alpha,\lambda}(\mathbb{R}^n)$. If*

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n-\lambda}; \quad \frac{\lambda}{p} = \frac{\lambda_1}{p_1} + \frac{\lambda_2}{p_2}, \quad 0 < \lambda_1, \lambda_2 < n. \tag{2.25}$$

One has

$$\|W \cdot B_{\Omega,\alpha}(f,g)\|_{L^{p,\lambda}(\mathbb{R}^n)} \leq C\|W\|_{L^{(n-\lambda)/\alpha,\lambda}(\mathbb{R}^n)} \|f\|_{L^{p_1,\lambda_1}(\mathbb{R}^n)} \|g\|_{L^{p_2,\lambda_2}(\mathbb{R}^n)}. \tag{2.26}$$

3. The Boundedness of $B_{\Omega,\alpha}$ on Modified Morrey Space

After studying Morrey spaces in detail, people are led to considering the local and global counterpart. There are many famous work by V. I. Burenkov, H. V. Guliyev and V. S. Guliyev, and so forth and (see [12–20]). Recently, Guliyev et al. have considered the following modified Morrey spaces $\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$ in [21].

Definition 3.1. Let $1 \leq p < \infty$, $0 \leq \lambda \leq n$ and $[t]_1 = \min\{1, t\}$. $\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$ is defined as the set of all functions $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$, with the finite norms

$$\|f\|_{\widetilde{L}^{p,\lambda}} = \sup_{x\in\mathbb{R}^n, t>0} \left(\frac{1}{[t]_1^\lambda} \int_{B(x,t)} |f(y)|^p dy \right)^{1/p}. \tag{3.1}$$

Note that

$$\begin{gathered} \widetilde{L}^{p,0}(\mathbb{R}^n) = L^{p,0}(\mathbb{R}^n) = L^p(\mathbb{R}^n), \\ \widetilde{L}^{p,\lambda}(\mathbb{R}^n) \subset_{>} L^{p,\lambda}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n), \qquad \max\{\|f\|_{L^{p,\lambda}}, \|f\|_{L^p}\} \leq \|f\|_{\widetilde{L}^{p,\lambda}}, \end{gathered} \tag{3.2}$$

and if $\lambda < 0$ or $\lambda > n$, then $\widetilde{L}^{p,\lambda}(\mathbb{R}^n) = L^{p,\lambda}(\mathbb{R}^n) = \{0\}$.

In [21], the authors discussed the boundedness of maximal function in modified Morrey spaces $\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$ and obtained the following generalized Hardy-Littlewood-Sobolev inequalities in modified Morrey spaces.

Theorem D. *Let $0 < \alpha < n$ and $0 \leq \lambda < n - \alpha$. If $1 < p < (n-\lambda)/\alpha$, then condition $\alpha/n \leq 1/p - 1/q \leq \alpha/(n-\lambda)$ is necessary and sufficient for the boundedness of the operator I_α from $\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$ to $\widetilde{L}^{q,\lambda}(\mathbb{R}^n)$.*

We also can extend Theorem D to the multilinear case.

Lemma 3.2. *Let $p > 1$, $0 < \lambda < n$ and $1/p = 1/p_1 + 1/p_2$. If*

$$\frac{\lambda}{p} = \frac{\lambda_1}{p_1} + \frac{\lambda_2}{p_2}, \quad 0 < \lambda_1, \lambda_2 < n, \tag{3.3}$$

then there exists a positive constant C such that

$$\|M(f,g)\|_{\widetilde{L}^{p,\lambda}(\mathbb{R}^n)} \leq C\|f\|_{\widetilde{L}^{p_1,\lambda}(\mathbb{R}^n)}\|g\|_{\widetilde{L}^{p_2,\lambda}(\mathbb{R}^n)}. \tag{3.4}$$

Proof. When $1 \leq \delta < p$, the following inequality:

$$\int_{\mathbb{R}^n} (M_\delta f(x))^p \chi(x)dx \leq C_p \int_{\mathbb{R}^n} |f(x)|^p M\chi(x)dx \tag{3.5}$$

holds, where M is the Hardy-littlewood maximal function and $M_\delta f(x) = (Mf^\delta)^{1/\delta}(x)$.

Taking $f \in \widetilde{L}^{p,\lambda}(\mathbb{R}^n)$, $0 < \lambda < n$. Using the method in [21], we get $\|M_\delta f\|_{\widetilde{L}^{p,\lambda}(\mathbb{R}^n)} \leq C\|f\|_{\widetilde{L}^{p,\lambda}(\mathbb{R}^n)}$.

Hence, with the same arguments in Lemma 2.1, we complete the proof of Lemma 3.2. □

Theorem 3.3. *Suppose $0 < \alpha < n$, $\Omega \in L^s(S^{n-1})$ and let be homogeneous of degree zero on $\mathbb{R}^n$, let p be the harmonic mean of p_1 and p_2, $1 < p < n/\alpha$, $0 < \lambda < n - \alpha p$, $s' < p$ and $\lambda/p = \lambda_1/p_1 + \lambda_2/p_2$, $0 < \lambda_1, \lambda_2 < n$. Then the condition $\alpha/n \leq 1/p - 1/q \leq \alpha/(n-\lambda)$ is necessary and sufficient for the boundedness of $B_{\Omega,\alpha}$ from $\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n) \times \widetilde{L}^{p_2,\lambda_2}(\mathbb{R}^n)$ to $\widetilde{L}^{q,\lambda}(\mathbb{R}^n)$.*

Proof. (1) *Sufficiency*. Let $f \in \widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)$, $g \in \widetilde{L}^{p_2,\lambda_2}(\mathbb{R}^n)$, $\sigma = (n - \alpha s' + \lambda)/2$, since $s' < p$ and $0 < \lambda < n - \alpha p$, we can get $\lambda < \sigma < n - \alpha s'$, $(n-\lambda)/p > \alpha > (n-\sigma)/s'$ and $\lambda < n - ((n-\sigma)/s')p < n - \alpha p$.

Do the same decomposition of $B_{\Omega,\alpha}(f,g)(x)$ in the proof of Theorem 2.2, then we only need to estimate $F_\sigma(f,g)(x)$. We can easily obtain

$$\begin{aligned} F_\sigma(f,g)(x) &\leq \left(\sum_{k=0}^{\infty}\int_{|y|\sim\varepsilon 2^k} \frac{\left|f^{s'}(x-y)g^{s'}(x+y)\right|}{|y|^\sigma}dy\right)^{1/s'} \\ &\leq \sum_{k=0}^{\infty}\left(\varepsilon 2^k\right)^{(n-\sigma)/s'-n/p}\left(\int_{|y|\sim\varepsilon 2^k}\left|f^{p_1}(x-y)\right|dy\right)^{1/p_1} \\ &\quad\times\left(\int_{|y|\sim\varepsilon 2^k}\left|g^{p_2}(x-y)\right|dy\right)^{1/p_2} \\ &\leq (\varepsilon)^{(n-\sigma)/s'-n/p}\sum_{k=0}^{\infty}\left(2^k\right)^{(n-\sigma)/s'-n/p}\left[\varepsilon 2^k\right]_1^{\lambda/p}\|f\|_{\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)}\|g\|_{\widetilde{L}^{p_2,\lambda_2}(\mathbb{R}^n)}. \end{aligned} \tag{3.6}$$

For $0 < \varepsilon < 1/2$, we get

$$\begin{aligned}\sum_{k=0}^{\infty}\left(2^k\right)^{(n-\sigma)/s'-n/p}\left[\varepsilon 2^k\right]_1^{\lambda/p} &= \sum_{k=0}^{[\log_2(1/2\varepsilon)]}\varepsilon^{\lambda/p}\left(2^k\right)^{(n-\sigma)/s'-n/p+\lambda/p} \\ &\quad + \sum_{k=[\log_2(1/2\varepsilon)]+1}^{\infty}\left(2^k\right)^{(n-\sigma)/s'-n/p} \\ &\le C\left(\varepsilon^{\lambda/p}+\varepsilon^{(n-\sigma)/s'-n/p}\right) \le C\varepsilon^{\lambda/p}.\end{aligned} \tag{3.7}$$

While $\varepsilon \ge 1/2$, we obtain

$$\sum_{k=0}^{\infty}\left(2^k\right)^{(n-\sigma)/s'-n/p}\left[\varepsilon 2^k\right]_1^{\lambda/p} = \sum_{k=0}^{\infty}\left(2^k\right)^{(n-\sigma)/s'-n/p} \le C. \tag{3.8}$$

Thus, we obtain

$$\begin{aligned}F_\sigma(f,g)(x) &\le C(\varepsilon)^{((n-\sigma)/s')-(n/p)}[2\varepsilon]_1^{\lambda/p}\|f\|_{\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)}\|g\|_{\widetilde{L}^{p_2,\lambda_2}(\mathbb{R}^n)}, \\ \left|B_{\Omega,\alpha}(f,g)(x)\right| &\le C\left(\varepsilon^{\alpha}M_{s'}(f,g)(x)+\varepsilon^{\alpha-(n/p)}[\varepsilon]_1^{\lambda/p}\|f\|_{\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)}\|g\|_{\widetilde{L}^{p_2,\lambda_2}(\mathbb{R}^n)}\right) \\ &\le C\min\Big\{\varepsilon^{\alpha}M_{s'}(f,g)(x)+\varepsilon^{\alpha-n/p}\|f\|_{\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)}\|g\|_{\widetilde{L}^{p_2,\lambda_2}(\mathbb{R}^n)}, \\ &\qquad \varepsilon^{\alpha}M_{s'}(f,g)(x)+\varepsilon^{\alpha-(n-\lambda)/p}\|f\|_{\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)}\|g\|_{\widetilde{L}^{p_2,\lambda_2}(\mathbb{R}^n)}\Big\}.\end{aligned} \tag{3.9}$$

Set

$$\begin{aligned}\varepsilon &= \left(M_{s'}(f,g)(x)^{-1}\|f\|_{\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)}\|g\|_{\widetilde{L}^{p_2,\lambda_2}(\mathbb{R}^n)}\right)^{p/(n-\lambda)}, \\ \varepsilon &= \left(M_{s'}(f,g)(x)^{-1}\|f\|_{\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)}\|g\|_{\widetilde{L}^{p_2,\lambda_2}(\mathbb{R}^n)}\right)^{p/n},\end{aligned} \tag{3.10}$$

we have

$$\begin{aligned}&\left|B_{\Omega,\alpha}(f,g)(x)\right| \\ &\quad\le C\min\left\{\left(\frac{M_{s'}(f,g)(x)}{\|f\|_{\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)}\|g\|_{\widetilde{L}^{p_2,\lambda_2}(\mathbb{R}^n)}}\right)^{1-p\alpha/(n-\lambda)},\left(\frac{M_{s'}(f,g)(x)}{\|f\|_{\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)}\|g\|_{\widetilde{L}^{p_2,\lambda_2}(\mathbb{R}^n)}}\right)^{1-p\alpha/n}\right\} \\ &\qquad\times\|f\|_{\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)}\|g\|_{\widetilde{L}^{p_2,\lambda_2}(\mathbb{R}^n)} \\ &\quad\le C\left(M_{s'}(f,g)(x)\right)^{p/q}\|f\|_{\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)}^{1-p/q}\|g\|_{\widetilde{L}^{p_2,\lambda_2}(\mathbb{R}^n)}^{1-p/q}.\end{aligned} \tag{3.11}$$

Hence, by the boundedness of $M(f,g)(x)$ in Lemma 3.2, we prove that $B_{\Omega,\alpha}$ is bounded from $\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)\times\widetilde{L}^{p_2,\lambda_2}(\mathbb{R}^n)$ to $\widetilde{L}^{q,\lambda}(\mathbb{R}^n)$.

(2) *Necessity*. Let $1<p<n/\alpha$ and $f\in\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)$, $g\in\widetilde{L}^{p_2,\lambda_2}(\mathbb{R}^n)$. Denote $f_t(x)=:f(tx)$, $g_t(x)=:g(tx)$, and $[t]_{1,+}=\max\{1,t\}$. Then from [21], we have

$$\|f_t\|_{\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)}=t^{-n/p_1}[t]_{1,+}^{\lambda_1/p_1}\|f\|_{\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)},\qquad \|g_t\|_{\widetilde{L}^{p_2,\lambda_2}(\mathbb{R}^n)}=t^{-n/p_2}[t]_{1,+}^{\lambda_2/p_2}\|f\|_{\widetilde{L}^{p_2,\lambda_2}(\mathbb{R}^n)},$$

$$B_{\Omega,\alpha}(f_t,g_t)(x)=t^{-\alpha}B_{\Omega,\alpha}(f,g)(tx),\qquad \|B_{\Omega,\alpha}(f_t,g_t)\|_{\widetilde{L}^{q,\lambda}(\mathbb{R}^n)}=t^{-\alpha-n/q}[t]_{1,+}^{\lambda/q}\|B_{\Omega,\alpha}(f,g)\|_{\widetilde{L}^{q,\lambda}(\mathbb{R}^n)}. \tag{3.12}$$

By the boundedness of $B_{\Omega,\alpha}$, we have

$$\begin{aligned}\|B_{\Omega,\alpha}(f,g)\|_{\widetilde{L}^{q,\lambda}(\mathbb{R}^n)}&=t^{\alpha+n/q}[t]_{1,+}^{-\lambda/q}\|B_{\Omega,\alpha}(f_t,g_t)\|_{\widetilde{L}^{q,\lambda}(\mathbb{R}^n)}\\&\le Ct^{\alpha+n/q}[t]_{1,+}^{-\lambda/q}\|f_t\|_{\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)}\|g_t\|_{\widetilde{L}^{p_2,\lambda_2}(\mathbb{R}^n)}\\&\le Ct^{\alpha+n/q-(n/p)}[t]_{1,+}^{\lambda/p-\lambda/q}\|f_t\|_{\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)}\|g_t\|_{\widetilde{L}^{p_2,\lambda_2}(\mathbb{R}^n)}.\end{aligned} \tag{3.13}$$

If $1/q>1/p-\alpha/(n-\lambda)$, then in the case $t\to 0$, for all $f\in\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)$, $g\in\widetilde{L}^{p_2,\lambda_2}(\mathbb{R}^n)$, we have $\|B_{\Omega,\alpha}(f,g)\|_{\widetilde{L}^{q,\lambda}(\mathbb{R}^n)}=0$.

If $1/q<1/p-\alpha/(n-\lambda)$, then in the case $t\to\infty$, for all $f\in\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)$, $g\in\widetilde{L}^{p_2,\lambda_2}(\mathbb{R}^n)$, we have $\|B_{\Omega,\alpha}(f,g)\|_{L^{q,\lambda}(\mathbb{R}^n)}=0$.

Therefor $\alpha/n\le 1/p-1/q\le\alpha/(n-\lambda)$. □

4. The Boundedness of $B_{\Omega,\alpha}$ on Generalized Center Morrey Space

Definition 4.1. Let $\varphi(r)$ be a positive measurable function on $\mathbb{R}_+$ and $1\le p<\infty$. We denote by $\dot{B}^{p,\varphi}(\mathbb{R}^n)$ the generalized central Morrey space, the space of all functions $f\in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{\dot{B}^{p,\varphi}(\mathbb{R}^n)}=\sup_{r>0}\varphi(r)^{-1}|B(0,r)|^{-1/p}\|f\|_{L^p(B(0,r))}, \tag{4.1}$$

where $B(0,r)$ denotes a ball centered at 0 with side length r and $|B(0,r)|$ is the Lebesgue measure of the ball $B(0,r)$.

According to this definition, we recover the spaces $\dot{B}^{p,\lambda}(\mathbb{R}^n)$ under the choice $\varphi(r)=r^{n\lambda}$. About the $\dot{B}^{p,\lambda}(\mathbb{R}^n)$ space, the readers can refer to [22], In fact, we can easily check that $\dot{B}^{p,\lambda}(\mathbb{R}^n)$ is a Banach space, $\dot{B}^{p,\lambda}(\mathbb{R}^n)$ reduce to $\{0\}$ when $\lambda<-1/p$, $\dot{B}^{p,(-1/p)}(\mathbb{R}^n)=L^p(\mathbb{R}^n)$ and $\dot{B}^{p,0}(\mathbb{R}^n)=\dot{B}^p(\mathbb{R}^n)$.

There are many papers that discussed the conditions on φ to obtain the boundedness of fractional integral on the generalized Morrey spaces, see [23, 24]. In [25] the following condition was imposed on the pair (φ_1, φ_2):

$$\int_r^\infty \frac{\operatorname{ess\,inf}_{t<s<\infty} \varphi_1(s) s^{n/p}}{t^{n/q+1}} \leq C\varphi_2(r) \tag{4.2}$$

for the fractional integral I_α, where $1/q = 1/p - \alpha/n$ and C (> 0) does not depend on r.

Theorem E (see [26]). *The inequality*

$$\operatorname*{ess\,sup}_{t>0} \omega(t) Hg(t) \leq c \operatorname*{ess\,sup}_{t>0} \nu(t) g(t) \tag{4.3}$$

holds for all nonnegative and nonincreasing g on $(0, \infty)$ if and only if

$$A := \sup_{t>0} \frac{\omega(t)}{t} \int_0^t \frac{dr}{\operatorname{ess\,sup}_{t>0} \nu(s)} < \infty, \tag{4.4}$$

and $c \approx A$, where the H is the Hardy operator

$$Hg(t) := \frac{1}{t} \int_0^t g(r) dr, \quad 0 < t < \infty. \tag{4.5}$$

In this section we are going to discuss the boundedness of $B_{\Omega,\alpha}$ on generalized central Morrey space.

Lemma 4.2. *Suppose $0 < \alpha < n$, $1/p = 1/p_1 + 1/p_2$, $1/q = 1/p - \alpha/n$, and $s \geq p'$, then for $1 < p < n/\alpha$, the inequality*

$$\begin{aligned} &\|B_{\Omega,\alpha}(f, g)\|_{L^q(B(0,r))} \\ &\quad \leq C r^{n/q} \left(\int_{2r}^\infty \|f\|_{L^{p_1}(B(0,t))}^{p_1/p} \frac{dt}{t^{n/q+1}} \right)^{p/p_1} \left(\int_{2r}^\infty \|g\|_{L^{p_2}(B(0,t))}^{p_2/p} \frac{dt}{t^{n/q+1}} \right)^{p/p_2} \end{aligned} \tag{4.6}$$

holds for any ball $B(0,r)$ and for all $f \in L_{p_1}^{\mathrm{loc}}(\mathbb{R}^n)$ and $g \in L_{p_2}^{\mathrm{loc}}(\mathbb{R}^n)$.

Proof. Let $1 < p < n/\alpha$, $1/p = 1/p_1 + 1/p_2$, $1/q = 1/p - \alpha/n$ and $s \geq p'$. For any $r > 0$, set $B = B(0,r)$, we write

$$\begin{aligned} f(x) &= f(x)\chi_{3B}(x) + f(x)\chi_{(3B)^c}(x) := f_1(x) + f_2(x), \\ g(x) &= g(x)\chi_{3B}(x) + g(x)\chi_{(3B)^c}(x) := g_1(x) + g_2(x). \end{aligned} \tag{4.7}$$

Hence

$$\begin{aligned} \|B_{\Omega,\alpha}(f, g)\|_{L^q(B)} \leq{}& \|B_{\Omega,\alpha}(f_1, g_1)\|_{L^q(B)} + \|B_{\Omega,\alpha}(f_1, g_2)\|_{L^q(B)} \\ &+ \|B_{\Omega,\alpha}(f_2, g_1)\|_{L^q(B)} + \|B_{\Omega,\alpha}(f_2, g_1)\|_{L^q(B)}. \end{aligned} \tag{4.8}$$

Since $B_{\Omega,\alpha}$ is bounded from $L^{p_1} \times L^{p_2}$ to L^q, we have

$$\begin{aligned}\left\|B_{\Omega,\alpha}(f_1,g_1)\right\|_{L^q(B)} &\leq \left\|B_{\Omega,\alpha}(f_1,g_1)\right\|_{L^q(\mathbb{R}^n)} \leq C\left\|f_1\right\|_{L^{p_1}(\mathbb{R}^n)}\left\|g_1\right\|_{L^{p_2}(\mathbb{R}^n)} \\ &\leq C\|f\|_{L^{p_1}(3B)}\|g\|_{L^{p_2}(3B)},\end{aligned} \tag{4.9}$$

where the constant $C > 0$ is independent of f and g.

To estimate $B_{\Omega,\alpha}(f_1, g_2)$, it follows that

$$\begin{aligned}\left|B_{\Omega,\alpha}(f_1,g_2)\right| &= \left|\int_{\mathbb{R}^n}\frac{f_1(x-y)g_2(x+y)\Omega(y)}{|y|^{n-\alpha}}dy\right| \\ &\leq \left(\int_{\mathbb{R}^n}\left|f_1^{p_1/p}(x-y)\Omega(y)\right|dy\right)^{p/p_1}\left(\int_{\mathbb{R}^n}\frac{\left|g_2^{p_2/p}(x-y)\Omega(y)\right|}{|y|^{(n-\alpha)p_2/p}}dy\right)^{p/p_2} \\ &\leq \left(\int_{4B}\left|f^{p_1}(y)\right|dy\right)^{1/p_1}\left(\int_{4B}\left|\Omega^{p'}(x-y)\right|dy\right)^{p/p_1p'} \\ &\quad\times\left(\int_{(2B)^c}\frac{\left|g^{p_2/p}(y)\Omega(x-y)\right|}{|y|^{(n-\alpha)p_2/p}}dy\right)^{p/p_2} \\ &\leq Cr^{pn/p_1p'}\left(r^{n/q}\|f\|_{L^{p_1}(4B)}^{p_1/p}\int_{4r}^{\infty}\frac{dt}{t^{n/q+1}}\right)^{p/p_1} \\ &\quad\times\left(\int_{(2B)^c}\left|g^{p_2/p}(y)\Omega(x-y)\right|\int_{|y|}^{\infty}\frac{dt}{|t|^{(n-\alpha)p_2/p+1}}dy\right)^{p/p_2} \\ &\leq Cr^{pn/p_1p'+np/qp_1}\left(\int_{4r}^{\infty}\|f\|_{L^{p_1}(B(0,t))}^{p_1/p}\frac{dt}{t^{n/q+1}}\right)^{p/p_1} \\ &\quad\times\left(\int_{2r}^{\infty}\int_{2r\leq|y|<t}\left|g^{p_2/p}(y)\Omega(x-y)\right|dy\frac{dt}{|t|^{(n-\alpha)p_2/p+1}}\right)^{p/p_2} \\ &\leq Cr^{(p/p_1)(n-\alpha)}\left(\int_{2r}^{\infty}\|f\|_{L^{p_1}(B(0,t))}^{p_1/p}\frac{dt}{t^{n/q+1}}\right)^{p/p_1} \\ &\quad\times\left(\int_{2r}^{\infty}\|g\|_{L^{p_2}(B(0,t))}^{p_2/p}\frac{dt}{|t|^{(n-\alpha)p_2/p+1-(n/p')}}\right)^{p/p_2} \\ &\leq C\left(\int_{2r}^{\infty}\|f\|_{L^{p_1}(B(0,t))}^{p_1/p}\frac{dt}{t^{n/q+1}}\right)^{p/p_1} \\ &\quad\times\left(\int_{2r}^{\infty}|t|^{(p/p_1)(n-\alpha)}\|g\|_{L^{p_2}(B(0,t))}^{p_2/p}\frac{dt}{|t|^{(n-\alpha)p_2/p+1-(n/p')}}\right)^{p/p_2}\end{aligned}$$

$$\leq C\left(\int_{2r}^{\infty}\|f\|_{L^{p_1}(B(0,t))}^{p_1/p}\frac{dt}{t^{n/q+1}}\right)^{p/p_1}$$
$$\times\left(\int_{2r}^{\infty}\|g\|_{L^{p_2}(B(0,t))}^{p_2/p}\frac{dt}{|t|^{n/q+1}}\right)^{p/p_2}. \tag{4.10}$$

So

$$\|B_{\Omega,\alpha}(f_1,g_2)\|_{L^q(B(0,r))}\leq Cr^{n/q}\left(\int_{2r}^{\infty}\|f\|_{L^{p_1}(B(0,t))}^{p_1/p}\frac{dt}{t^{(n/q)+1}}\right)^{p/p_1}\left(\int_{2r}^{\infty}\|g\|_{L^{p_2}(B(0,t))}^{p_2/p}\frac{dt}{t^{(n/q)+1}}\right)^{p/p_2}. \tag{4.11}$$

By the same estimating, we also can obtain

$$\|B_{\Omega,\alpha}(f_2,g_1)\|_{L^q(B(0,r))}\leq Cr^{n/q}\left(\int_{2r}^{\infty}\|f\|_{L^{p_1}(B(0,t))}^{p_1/p}\frac{dt}{t^{(n/q)+1}}\right)^{p/p_1}\left(\int_{2r}^{\infty}\|g\|_{L^{p_2}(B(0,t))}^{p_2/p}\frac{dt}{t^{(n/q)+1}}\right)^{p/p_2}. \tag{4.12}$$

To estimate $B_{\Omega,\alpha}(f_2,g_2)$, we get

$$\begin{aligned}
|B_{\Omega,\alpha}(f_2,g_2)| &= \left|\int_{\mathbb{R}^n}\frac{f_1(x-y)g_2(x+y)\Omega(y)}{|y|^{n-\alpha}}dy\right| \\
&\leq\left(\int_{\mathbb{R}^n}\frac{\left|f_2^{p_1/p}(x-y)\Omega(y)\right|}{|y|^{n-\alpha}}dy\right)^{p/p_2} \\
&\quad\times\left(\int_{\mathbb{R}^n}\frac{\left|g_2^{p_2/p}(x-y)\Omega(y)\right|}{|y|^{n-\alpha}}dy\right)^{p/p_2} \\
&\leq\left(\int_{(2B)^c}\frac{\left|f^{p_1/p}(y)\Omega(x-y)\right|}{|y|^{n-\alpha}}dy\right)^{p/p_1} \\
&\quad\times\left(\int_{(2B)^c}\frac{\left|g^{p_2/p}(y)\Omega(x-y)\right|}{|y|^{n-\alpha}}dy\right)^{p/p_2} \\
&\leq C\left(\int_{(2B)^c}\left|f^{p_1/p}(y)\Omega(x-y)\right|\int_{|y|}^{\infty}\frac{dt}{|t|^{n-\alpha+1}}dy\right)^{p/p_1} \\
&\quad\times\left(\int_{(2B)^c}\left|g^{p_2/p}(y)\Omega(x-y)\right|\int_{|y|}^{\infty}\frac{dt}{|t|^{n-\alpha+1}}dy\right)^{p/p_2}
\end{aligned}$$

$$
\begin{aligned}
&\leq C\left(\int_{2r}^{\infty}\int_{2r\leq|y|<t}\left|f^{p_1/p}(y)\Omega(x-y)\right|dy\frac{dt}{|t|^{n-\alpha+1}}\right)^{p/p_1}\\
&\quad\times\left(\int_{2r}^{\infty}\int_{2r\leq|y|<t}\left|g^{p_2/p}(y)\Omega(x-y)\right|dy\frac{dt}{|t|^{n-\alpha+1}}\right)^{p/p_2}\\
&\leq C\left(\int_{2r}^{\infty}\|f\|_{L^{p_1}(B(0,t))}^{p_1/p}\frac{dt}{t^{n/q+1}}\right)^{p/p_1}\left(\int_{2r}^{\infty}\|g_2\|_{L^{p_2}(B(0,t))}^{p_2/p}\frac{dt}{|t|^{n/q+1}}\right)^{p/p_2}.
\end{aligned}
\tag{4.13}
$$

Combining the above estimates, we end the proof of Lemma 4.2. □

Theorem 4.3. *Suppose* $0<\alpha<n$, $1/p=1/p_1+1/p_2$, $1<p<n/\alpha$, $1/q=1/p-\alpha/n$, *and* $s\geq p'$. *If* (φ_1,ν_1) *satisfies the condition*

$$
\int_r^{\infty}\frac{\operatorname{ess\,inf}_{t<s<\infty}\varphi_1^{p_1/p}(s)s^{n/p}}{t^{n/q+1}}\leq C\nu_1^{p_1/p}(r),
\tag{4.14}
$$

and (φ_2,ν_2) *satisfies the condition*

$$
\int_r^{\infty}\frac{\operatorname{ess\,inf}_{t<s<\infty}\varphi_2^{p_1/p}(s)s^{n/p}}{t^{n/q+1}}\leq C\nu_2^{p_1/p}(r),
\tag{4.15}
$$

where the constant $C>0$ *does not depend on r. Let* $\varphi=\nu_1\nu_2$, *then* $B_{\Omega,\alpha}$ *is bounded from* $\dot{B}^{p_1,\varphi_1}\times\dot{B}^{p_2,\varphi_2}$ *to* $\dot{B}^{q,\varphi}$.

Proof. By Theorem E and Lemma 4.2, we have

$$
\begin{aligned}
\|B_{\Omega,\alpha}(f,g)\|_{\dot{B}^{q,\varphi}(\mathbb{R}^n)}&\leq C\sup_{r>0}\varphi(r)^{-1}\left(\int_r^{\infty}\|f\|_{L^{p_1}(B(0,t))}^{p_1/p}\frac{dt}{t^{n/q+1}}\right)^{p/p_1}\\
&\quad\times\left(\int_r^{\infty}\|g_2\|_{L^{p_2}(B(0,t))}^{p_2/p}\frac{dt}{|t|^{n/q+1}}\right)^{p/p_2}\\
&=C\sup_{r>0}\left(\nu_1(r)^{-p_1/p}\int_0^{r^{-n/q}}\|f\|_{L^{p_1}(B(0,t^{-q/n}))}^{p_1/p}dt\right)^{p/p_1}\\
&\quad\times\left(\nu_2(r)^{-p_2/p}\int_0^{r^{-n/q}}\|g_2\|_{L^{p_2}(B(0,t^{-q/n}))}^{p_2/p}dt\right)^{p/p_2}\\
&=C\sup_{r>0}\left(\nu_1\left(r^{-q/n}\right)^{-p_1/p}\int_0^{r}\|f\|_{L^{p_1}(B(0,t^{-q/n}))}^{p_1/p}dt\right)^{p/p_1}\\
&\quad\times\left(\nu_2(r^{-q/n})^{-p_2/p}\int_0^{r}\|g_2\|_{L^{p_2}(B(0,t^{-q/n}))}^{p_2/p}dt\right)^{p/p_2}
\end{aligned}
$$

$$
\begin{aligned}
&\leq C \sup_{r>0}\left(\varphi_1\left(r^{-q/n}\right)^{-p_1/p} r^{q/p}\|f\|_{L^{p_1}(B(0,r^{-q/n}))}^{p_1/p}\right)^{p/p_1} \\
&\quad\times \sup_{r>0}\left(\varphi_2\left(r^{-q/n}\right)^{-p_2/p} r^{q/p}\|g\|_{L^{p_2}(B(0,r^{-q/n}))}^{p_2/p}\right)^{p/p_2} \\
&\leq C\|f\|_{\dot{B}^{p_1,\varphi_1}(\mathbb{R}^n)}\|g\|_{\dot{B}^{p_2,\varphi_2}(\mathbb{R}^n)}.
\end{aligned}
\tag{4.16}
$$

□

Corollary 4.4. *Suppose* $0 < \alpha < n$, $1/p = 1/p_1 + 1/p_2$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $s \geq p'$, $\lambda_1 < -\alpha p/np_1$, $\lambda_2 < -\alpha p/np_2$, *and* $\lambda < \lambda_1 + \lambda_2 + \alpha/n$, *then* $B_{\Omega,\alpha}$ *is bounded from* $\dot{B}^{p_1,\lambda_1} \times \dot{B}^{p_2,\lambda_2}$ *to* $\dot{B}^{q,\lambda}$.

Remark 4.5. Although we worked on the bilinear case. Applying same ideas in the argument, we may obtain similar extension of $I_{\Omega,\alpha}(\vec{f})$.

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Research Article

Hybrid Gradient-Projection Algorithm for Solving Constrained Convex Minimization Problems with Generalized Mixed Equilibrium Problems

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It is well known that the gradient-projection algorithm (GPA) for solving constrained convex minimization problems has been proven to have only weak convergence unless the underlying Hilbert space is finite dimensional. In this paper, we introduce a new hybrid gradient-projection algorithm for solving constrained convex minimization problems with generalized mixed equilibrium problems in a real Hilbert space. It is proven that three sequences generated by this algorithm converge strongly to the unique solution of some variational inequality, which is also a common element of the set of solutions of a constrained convex minimization problem, the set of solutions of a generalized mixed equilibrium problem, and the set of fixed points of a strict pseudocontraction in a real Hilbert space.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H and let P_C be the metric projection of H onto C. Recall that a ρ-Lipschitz continuous mapping $T : C \to H$ is a mapping on C such that

$$\|Tx - Ty\| \leq \rho \|x - y\|, \quad \forall x, y \in C, \tag{1.1}$$

where $\rho \geq 0$ is a constant. In particular, if $\rho \in [0, 1)$ then T is called a contraction on C; if $\rho = 1$ then T is called a nonexpansive mapping on C. A mapping $A : C \to H$ is called monotone if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C. \tag{1.2}$$

A mapping $A : C \to H$ is called α-inverse strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C; \tag{1.3}$$

see, for example, [1]. A self-mapping $S : C \to C$ is called k-strictly pseudocontractive if there exists a constant $k \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + k\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C; \tag{1.4}$$

see, for example, [2]. In particular, if $k = 0$, then S reduces to a nonexpansive self-mapping on C.

Consider the following constrained convex minimization problem:

$$\text{minimize}\{f(x) : x \in C\}, \tag{1.5}$$

where $f : C \to \mathbf{R}$ is a real-valued convex function. If f is (Frechet) differentiable, then the gradient-projection method (for short, GPM) generates a sequence $\{x_n\}$ via the recursive formula

$$x_{n+1} = P_C(x_n - \lambda \nabla f(x_n)), \quad \forall n \geq 0, \tag{1.6}$$

or more generally,

$$x_{n+1} = P_C(x_n - \lambda_n \nabla f(x_n)), \quad \forall n \geq 0, \tag{1.7}$$

where in both (1.6) and (1.7), the initial guess x_0 is taken from C arbitrarily, the parameters, λ or λ_n, are positive real numbers, and P_C is the metric projection from H onto C. The convergence of the algorithms (1.6) and (1.7) depends on the behavior of the gradient ∇f. As a matter of fact, it is known that if ∇f is strongly monotone and Lipschitzian; namely, there are constants $\eta, L > 0$ satisfying the properties

$$\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \eta \|x - y\|^2, \tag{1.8}$$

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\| \tag{1.9}$$

for all $x, y \in C$, then, for $0 < \lambda < 2\eta/L^2$, the operator

$$T := P_C(I - \lambda \nabla f) \tag{1.10}$$

is a contraction; hence, the sequence $\{x_n\}$ defined by algorithm (1.6) converges in norm to the unique solution of the minimization (1.5). More generally, if the sequence $\{\lambda_n\}$ is chosen to satisfy the property

$$0 < \liminf_{n \to \infty} \lambda_n \leq \limsup_{n \to \infty} \lambda_n < \frac{2\eta}{L^2} \tag{1.11}$$

then the sequence $\{x_n\}$ defined by algorithm (1.7) converges in norm to the unique minimizer of (1.5). However, if the gradient ∇f fails to be strongly monotone, the operator T defined in (1.10) would fail to be contractive; consequently, the sequence $\{x_n\}$ generated by algorithm (1.6) may fail to converge strongly (see Section 4 in Xu [3]). The following theorem states that if the Lipschitz condition (1.9) holds, then the algorithms (1.6) and (1.7) can still converge in the weak topology.

Theorem 1.1 (see [3, Theorem 3.2]). *Assume the minimization* (1.5) *is consistent and let* Ω *denote its solution set. Assume the gradient* ∇f *satisfies the Lipschitz condition* (1.9). *Let the sequence of parameters,* $\{\lambda_n\}$, *satisfy the condition*

$$0 < \liminf_{n\to\infty}\lambda_n \le \limsup_{n\to\infty}\lambda_n < \frac{2}{L}. \tag{1.12}$$

Then the sequence $\{x_n\}$ *generated by the gradient-projection algorithm* (1.7) *converges weakly to a minimizer of* (1.5).

From the above theorem, it is known that the gradient-projection algorithm has weak convergence, in general, unless the underlying Hilbert space is finite dimensional. This gives naturally rise to a question how to appropriately modify the gradient-projection algorithm so as to have strong convergence. Xu [3] gave two such modifications, one of which is simply a convex combination of a contraction with the point generated by the projected gradient algorithm.

Theorem 1.2 (see [3, Theorem 5.2]). *Assume the minimization* (1.5) *is consistent and let* Ω *denote its solution set. Assume the gradient* ∇f *satisfies the Lipschitz condition* (1.9). *Let* $Q : C \to C$ *be a* ρ*-contraction with* $\rho \in [0,1)$. *Let a sequence* $\{x_n\}$ *be generated by the following hybrid gradient-projection algorithm:*

$$x_{n+1} = \alpha_n Q x_n + (1-\alpha_n) P_C(x_n - \lambda_n \nabla f(x_n)), \quad \forall n \ge 0. \tag{1.13}$$

Assume the sequence $\{\lambda_n\}$ *satisfies the condition* (1.12) *and, in addition, the following conditions are satisfied for* $\{\lambda_n\}$ *and* $\{\alpha_n\} \subset [0,1]$:

(i) $\alpha_n \to 0$;

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;

(iv) $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.

Then the sequence $\{x_n\}$ *converges in norm to a minimizer of* (1.5) *which is also the unique solution of the variational inequality of finding* $x^* \in \Omega$ *such that*

$$\langle (I-Q)x^*, x - x^* \rangle \ge 0, \quad \forall x \in \Omega. \tag{1.14}$$

In other words, x^* *is the unique fixed point of the contraction* $P_\Omega Q$, $x^* = P_\Omega Q x^*$.

On the other hand, Peng and Yao [4] recently introduced the following generalized mixed equilibrium problem of finding $\overline{x} \in C$ such that

$$\Theta(\overline{x}, y) + \varphi(y) - \varphi(\overline{x}) + \langle F\overline{x}, y - \overline{x} \rangle \ge 0, \quad \forall y \in C, \tag{1.15}$$

where $F : C \to H$ is a nonlinear mapping and $\varphi : C \to \mathbf{R}$ is a function and $\Theta : C \times C \to \mathbf{R}$ is a bifunction. The set of solutions of problem (1.15) is denoted by GMEP. Subsequently, Yao et al. [5] and Ceng and Yao [6] also considered problem (1.15).

The problem (1.15) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problems in noncooperative games, and others; see, for example, [7–15]. Here some special cases of problem (1.15) are stated as follows.

If $F = 0$, then problem (1.15) reduces to the following mixed equilibrium problem of finding $\overline{x} \in C$ such that

$$\Theta(\overline{x}, y) + \varphi(y) - \varphi(\overline{x}) \geq 0, \quad \forall y \in C, \tag{1.16}$$

which was considered by Ceng and Yao [7] and Bigi et al. [16]. Very recently, Peng [10] further discussed this problem. The set of solutions of this problem is denoted by MEP.

If $\varphi = 0$, then problem (1.15) reduces to the following generalized equilibrium problem of finding $\overline{x} \in C$ such that

$$\Theta(\overline{x}, y) + \langle F\overline{x}, y - \overline{x} \rangle \geq 0, \quad \forall y \in C, \tag{1.17}$$

which was studied by S. Takahashi and W. Takahashi [8].

If $\varphi = 0$ and $F = 0$, then problem (1.15) reduces to the following equilibrium problem of finding $\overline{x} \in C$ such that

$$\Theta(\overline{x}, y) \geq 0, \quad \forall y \in C. \tag{1.18}$$

If $\Theta = 0$, $\varphi = 0$ and $F = A$, then problem (1.15) reduces to the following classical variational inequality of finding $\overline{x} \in C$ such that

$$\langle A\overline{x}, y - \overline{x} \rangle \geq 0, \quad \forall y \in C, \tag{1.19}$$

whose solution set is denoted by $\mathrm{VI}(C, A)$.

The variational inequalities have been extensively studied in the literature; see [14, 17–27] and the references therein. In 2006, Nadezhkina and Takahashi [22, 25] and Zeng and Yao [18] proposed some variants of Korpelevič's extragradient method [17] for finding an element of $\mathrm{Fix}(S) \cap \mathrm{VI}(C, A)$, where S is a nonexpansive self-mapping on C.

Very recently, Peng [10] also introduced a variant of Korpelevič's extragradient method [17] for finding a common element of the set of solutions of a mixed equilibrium problem, the set of fixed points of a strict pseudocontraction, and the set of solutions of a variational inequality for a monotone, Lipschitz continuous mapping.

Theorem 1.3 (see [10, Theorem 3.1]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\Theta : C \times C \to \mathbf{R}$ be a bifunction satisfying conditions (H1)–(H4) and $\varphi : C \to \mathbf{R}$ a lower semicontinuous and convex function with assumptions (A1) or (A2), where*

(H1) $\Theta(x, x) = 0$, *for all* $x \in C$;

(H2) Θ *is monotone, that is,* $\Theta(x, y) + \Theta(y, x) \leq 0$, *for all* $x, y \in C$;

(H3) *for each* $y \in C$, $x \mapsto \Theta(x,y)$ *is weakly upper semicontinuous;*

(H4) *for each* $x \in C$, $y \mapsto \Theta(x,y)$ *is convex and lower semicontinuous;*

(A1) *for each* $x \in H$ *and* $r > 0$, *there exists a bounded subset* $D_x \subset C$ *and* $y_x \in C$ *such that for any* $z \in C \setminus D_x$,

$$\Theta(z,y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0; \tag{1.20}$$

(A2) C *is a bounded set.*

Let $A : C \to H$ *be a monotone and* L*-Lipschitz-continuous mapping and* $S : C \to C$ *be a* k*-strict pseudocontraction for some* $0 \le k < 1$ *such that* $\mathrm{Fix}(S) \cap \mathrm{VI}(C,A) \cap \mathrm{MEP} \neq \emptyset$. *For given* $x_0 \in H$ *arbitrarily, let* $\{x_n\}$, $\{t_n\}$, $\{y_n\}$, $\{u_n\}$, $\{z_n\}$ *be sequences generated by*

$$\begin{gathered}
\Theta(t_n,y) + \varphi(y) - \varphi(t_n) + \frac{1}{r_n}\langle y - t_n, t_n - x_n\rangle \ge 0, \quad \forall y \in C,\\
y_n = P_C(t_n - \lambda_n A t_n),\\
u_n = P_C(t_n - \lambda_n A y_n),\\
z_n = \alpha_n u_n + (1-\alpha_n) S u_n,\\
C_n = \left\{ z \in C : \|z_n - z\|^2 \le \|x_n - z\|^2 - (1-\alpha_n)(\alpha_n - \varepsilon)\|t_n - S t_n\|^2 \right\},\\
Q_n = \{ z \in H : \langle x_n - z, x - x_n\rangle \ge 0\},\\
x_{n+1} = P_{C_n \cap Q_n} x, \quad \forall n \ge 0.
\end{gathered} \tag{1.21}$$

Assume that $\{\lambda_n\} \subset [a,b]$ *for some* $a,b \in (0,1/L)$, $\{\alpha_n\} \subset [c,d]$ *for some* $c,d \in (k,1)$ *and let* $\{r_n\} \subset (0,\infty)$ *satisfy* $\liminf_{n\to\infty} r_n > 0$. *Then,* $\{x_n\}$, $\{t_n\}$, $\{y_n\}$, $\{u_n\}$, $\{z_n\}$ *converge strongly to* $w = P_{\mathrm{Fix}(S)\cap \mathrm{VI}(C,A)\cap \mathrm{MEP}}x$.

Furthermore, related iterative methods for solving fixed point problems, variational inequalities, equilibrium problems, and optimization problems can be found in [1, 2, 6, 11, 13–16, 19, 20, 24, 26–35].

In this paper, let C be a nonempty closed convex subset of a real Hilbert space H. Let $\Theta : C \times C \to \mathbf{R}$ be a bifunction satisfying conditions (H1)–(H4) and $\varphi : C \to \mathbf{R}$ a lower semicontinuous and convex function with assumptions (A1) or (A2). Suppose the minimization (1.5) is consistent and let Ω denote its solution set. Let the gradient ∇f be L-Lipschitzian with constant $L > 0$ and $F : C \to H$ be an α-inverse strongly monotone mapping. Let $S : C \to C$ be a k-strictly pseudocontractive mapping such that $\mathrm{Fix}(S) \cap \Omega \cap \mathrm{GMEP} \neq \emptyset$. Let $Q : C \to C$ be a ρ-contraction with $\rho \in [0, 1/2)$. For given $x_0 \in C$ arbitrarily, let the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be generated iteratively by

$$\begin{gathered}
\Theta(z_n,y) + \varphi(y) - \varphi(z_n) + \langle F x_n, y - z_n\rangle + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \ge 0, \quad \forall y \in C,\\
y_n = \alpha_n Q x_n + (1-\alpha_n) P_C(z_n - \lambda_n \nabla f(z_n)),\\
x_{n+1} = \beta_n x_n + \gamma_n P_C(z_n - \lambda_n \nabla f(z_n)) + \delta_n S y_n, \quad \forall n \ge 0,
\end{gathered} \tag{1.22}$$

where $\{\lambda_n\} \subset (0, 2/L], \{r_n\} \subset (0, 2\alpha]$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ are four sequences in $[0,1]$ such that $\beta_n + \gamma_n + \delta_n = 1$ for all $n \geq 0$. It is proven that under very mild conditions, the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to the unique solution of the variational inequality of finding $x^* \in \mathrm{Fix}(S) \cap \Omega \cap \mathrm{GMEP}$ such that

$$\langle (I - Q)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \mathrm{Fix}(S) \cap \Omega \cap \mathrm{GMEP}. \tag{1.23}$$

In other words, x^* is the unique fixed point of the contraction $P_{\mathrm{Fix}(S)\cap\Omega\cap\mathrm{GMEP}}Q$, $x^* = P_{\mathrm{Fix}(S)\cap\Omega\cap\mathrm{GMEP}}Qx^*$. The result presented in this paper generalizes and improves some well-known results in the literature. Indeed, compared with some well-known results in the literature, our result improves and extends them in the following aspects.

(i) Compared with Xu [3, Theorem 3.2], a weak convergence result, our result is a strong convergence result.

(ii) Our problem of finding an element of $\mathrm{Fix}(S) \cap \Omega \cap \mathrm{GMEP}$ is more general than the problem of finding an element of $\mathrm{Fix}(S) \cap \mathrm{VI}(C, A)$ in [14, 18, 22, 23, 25].

(iii) In our algorithm (1.22), Xu's modified gradient-projection algorithm in [3, Theorem 5.2] is rewritten as the second iteration step

$$y_n = \alpha_n Q x_n + (1 - \alpha_n) P_C\big(z_n - \lambda_n \nabla f(z_n)\big). \tag{1.24}$$

Here the main purpose of the reason why we use such an iteration step is to play a convenience and efficiency role in the computation of an element of Ω. Therefore, Xu's algorithm (1.13) is extended to develop our algorithm (1.22).

(iv) Our problem of finding an element of $\mathrm{Fix}(S) \cap \Omega \cap \mathrm{GMEP}$ is more general than the problem of finding an element of Ω in Xu [3]. In addition, it is worth pointing out that Xu's conditions $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ and $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$ in the above Theorem 1.2 are replaced by the weaker conditions $\lim_{n\to\infty}(\alpha_n - \alpha_{n+1}) = 0$ and $\lim_{n\to\infty}(\lambda_n - \lambda_{n+1}) = 0$ in our result (see Theorem 3.2 in Section 3).

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$ and C a nonempty closed convex subset of H. We write $\to$ to indicate that the sequence $\{x_n\}$ converges strongly to x and $\rightharpoonup$ to indicate that the sequence $\{x_n\}$ converges weakly to x. Moreover, we use $\omega_w(x_n)$ to denote the weak ω-limit set of the sequence $\{x_n\}$, that is,

$$\omega_w(x_n) := \big\{x : x_{n_i} \rightharpoonup x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\big\}. \tag{2.1}$$

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall x \in C. \tag{2.2}$$

P_C is called the metric projection of H onto C. We know that P_C is a firmly nonexpansive mapping of H onto C; that is, there holds the following relation:

$$\langle P_C x - P_C y, x - y \rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \tag{2.3}$$

Consequently, P_C is nonexpansive and monotone. It is also known that P_C is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, P_C x - y \rangle \geq 0, \tag{2.4}$$

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \tag{2.5}$$

for all $x \in H$, $y \in C$; see [36] for more details. Let $A : C \to H$ be a monotone mapping. In the context of the variational inequality, this implies that

$$x \in \mathrm{VI}(C, A) \Longleftrightarrow x = P_C(x - \lambda A x) \quad \forall \lambda > 0. \tag{2.6}$$

A set-valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle f - g, x - y \rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is called maximal if its graph $G(T)$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle f - g, x - y \rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$.

Let $A : C \to H$ be a monotone, k-Lipschitz-continuous mapping and let $N_C v$ be the normal cone to C at $v \in C$, that is, $N_C v = \{w \in H : \langle v - u, w \rangle \geq 0, \text{ for all } u \in C\}$. Define

$$Tv = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases} \tag{2.7}$$

Then, T is maximal monotone and $0 \in Tv$ if and only if $v \in \mathrm{VI}(C, A)$; see [37].

Recall that a mapping $S : C \to C$ is called a strict pseudocontraction if there exists a constant $0 \leq k < 1$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + k\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C. \tag{2.8}$$

In this case, we also say that S is a k-strict pseudocontraction. A mapping $A : C \to H$ is called α-inverse strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{2.9}$$

It is obvious that any α-inverse strongly monotone mapping is Lipschitz continuous. Meantime, observe that (2.8) is equivalent to

$$\langle Sx - Sy, x - y \rangle \leq \|x - y\|^2 - \frac{1 - k}{2}\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C. \tag{2.10}$$

It is easy to see that if S is a k-strictly pseudocontractive mapping, then $I - S$ is $((1 - k)/2)$-inverse strongly monotone and hence $(2/(1 - k))$-Lipschitz continuous. Thus, S is Lipschitz continuous with constant $(1 + k)/(1 - k)$. We denote by $\mathrm{Fix}(S)$ the set of fixed points of S. It is clear that the class of strict pseudocontractions strictly includes the one of nonexpansive mappings which are mappings $S : C \to C$ such that $\|Sx - Sy\| \leq \|x - y\|$ for all $x, y \in C$.

In order to prove our main result in the next section, we need the following lemmas and propositions.

Lemma 2.1 (see [7]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\Theta : C \times C \to \mathbf{R}$ be a bifunction satisfying conditions (H1)–(H4) and let $\varphi : C \to \mathbf{R}$ be a lower semicontinuous and convex function. For $r > 0$ and $x \in H$, define a mapping $T_r^{(\Theta,\varphi)} : H \to C$ as follows:*

$$T_r^{(\Theta,\varphi)}(x) = \left\{ z \in C : \Theta(z,y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C \right\} \tag{2.11}$$

for all $x \in H$. Assume that either (A1) or (A2) holds. Then the following conclusions hold:

(i) $T_r^{(\Theta,\varphi)}(x) \neq \emptyset$ *for each* $x \in H$ *and* $T_r^{(\Theta,\varphi)}$ *is single-valued;*

(ii) $T_r^{(\Theta,\varphi)}$ *is firmly nonexpansive, that is, for any* $x, y \in H$,

$$\left\| T_r^{(\Theta,\varphi)}x - T_r^{(\Theta,\varphi)}y \right\|^2 \leq \left\langle T_r^{(\Theta,\varphi)}x - T_r^{(\Theta,\varphi)}y, x - y \right\rangle; \tag{2.12}$$

(iii) $\mathrm{Fix}(T_r^{(\Theta,\varphi)}) = \mathrm{MEP}(\Theta,\varphi)$;

(iv) $\mathrm{MEP}(\Theta,\varphi)$ *is closed and convex.*

Remark 2.2. If $\varphi = 0$, then $T_r^{(\Theta,\varphi)}$ is rewritten as T_r^{Θ}.

The following lemma is an immediate consequence of an inner product.

Lemma 2.3. *In a real Hilbert space H, there holds the inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in H. \tag{2.13}$$

Proposition 2.4 (see [6, Proposition 2.1]). *Let C, H, Θ, φ, and $T_r^{(\Theta,\varphi)}$ be as in Lemma 2.1. Then the following relation holds:*

$$\left\| T_s^{(\Theta,\varphi)}x - T_t^{(\Theta,\varphi)}x \right\|^2 \leq \frac{s-t}{s}\left\langle T_s^{(\Theta,\varphi)}x - T_t^{(\Theta,\varphi)}x, T_s^{(\Theta,\varphi)}x - x \right\rangle \tag{2.14}$$

for all $s, t > 0$ and $x \in H$.

Recall that $S : C \to C$ is called a quasi-strict pseudocontraction if the fixed point set of S, $\mathrm{Fix}(S)$, is nonempty and if there exists a constant $0 \leq k < 1$ such that

$$\|Sx - p\|^2 \leq \|x - p\|^2 + k\|x - Sx\|^2 \quad \forall x \in C,\ p \in \mathrm{Fix}(S). \tag{2.15}$$

We also say that S is a k-quasi-strict pseudocontraction if condition (2.15) holds.

Proposition 2.5 (see [2, Proposition 2.1]). *Assume C is a nonempty closed convex subset of a real Hilbert space H and let $S : C \to C$ be a self-mapping on C.*

(i) *If S is a k-strict pseudocontraction, then S satisfies the Lipschitz condition*

$$\|Sx - Sy\| \leq \frac{1+k}{1-k}\|x - y\|, \quad \forall x, y \in C. \tag{2.16}$$

(ii) *If S is a k-strict pseudocontraction, then the mapping $I - S$ is demiclosed (at 0). That is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup \tilde{x}$ and $(I - S)x_n \to 0$, then $(I - S)\tilde{x} = 0$, that is, $\tilde{x} \in \mathrm{Fix}(S)$.*

(iii) *If S is a k-quasi-strict pseudocontraction, then the fixed point set $\mathrm{Fix}(S)$ of S is closed and convex so that the projection $P_{\mathrm{Fix}(S)}$ is well defined.*

The following lemma was proved by Suzuki [30].

Lemma 2.6 (see [30]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty}\|y_n - x_n\| = 0$.*

Lemma 2.7 (see [34]). *Let $\{a_n\}$ be a sequence of nonnegative numbers satisfying the condition*

$$a_{n+1} \leq (1 - \delta_n)a_n + \delta_n\sigma_n, \quad \forall n \geq 0, \tag{2.17}$$

where $\{\delta_n\}$, $\{\sigma_n\}$ are sequences of real numbers such that

(i) *$\{\delta_n\} \subset [0, 1]$ and $\sum_{n=0}^{\infty}\delta_n = \infty$, or equivalently,*

$$\prod_{n=0}^{\infty}(1 - \delta_n) := \lim_{n\to\infty}\prod_{j=0}^{n}(1 - \delta_j) = 0; \tag{2.18}$$

(ii) *$\limsup_{n\to\infty}\sigma_n \leq 0$, or,*

(iii) *$\sum_{n=0}^{\infty}\delta_n\sigma_n$ is convergent.*

Then $\lim_{n\to\infty}a_n = 0$.

3. Strong Convergence Theorem

In order to prove our main result, we shall need the following lemma given in [21].

Lemma 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $S : C \to C$ be a k-strictly pseudocontractive mapping. Let γ and δ be two nonnegative real numbers. Assume $(\gamma + \delta)k \leq \gamma$. Then*

$$\|\gamma(x - y) + \delta(Sx - Sy)\| \leq (\gamma + \delta)\|x - y\|, \quad \forall x \cdot y \in C. \tag{3.1}$$

We are now in a position to state and prove our main result.

Theorem 3.2. *Let C be a nonempty bounded closed convex subset of a real Hilbert space H. Let $\Theta : C \times C \to \mathbf{R}$ be a bifunction satisfying conditions (H1)–(H4) and $\varphi : C \to \mathbf{R}$ a lower semicontinuous* *and convex function with assumptions (A1) or (A2). Suppose the minimization* (1.5) *is consistent and let* Ω *denote its solution set. Assume the gradient* ∇f *is* L*-Lipschitzian with constant* $L > 0$ *and* $F : C \to H$ *is an* α*-inverse strongly monotone mapping. Let* $S : C \to C$ *be a* k*-strictly pseudocontractive mapping such that* $\mathrm{Fix}(S) \cap \Omega \cap \mathrm{GMEP} \neq \emptyset$. *Let* $Q : C \to C$ *be a* ρ*-contraction with* $\rho \in [0, 1/2)$. *For given* $x_0 \in C$ *arbitrarily, let the sequences* $\{x_n\}$, $\{y_n\}$, *and* $\{z_n\}$ *be generated iteratively by*

$$\Theta(z_n, y) + \varphi(y) - \varphi(z_n) + \langle F x_n, y - z_n \rangle + \frac{1}{r_n} \langle y - z_n, z_n - x_n \rangle \geq 0, \quad \forall y \in C,$$
$$y_n = \alpha_n Q x_n + (1 - \alpha_n) P_C(z_n - \lambda_n \nabla f(z_n)),$$
$$x_{n+1} = \beta_n x_n + \gamma_n P_C(z_n - \lambda_n \nabla f(z_n)) + \delta_n S y_n, \quad \forall n \geq 0, \tag{3.2}$$

where $\{\lambda_n\} \subset (0, 2/L]$, $\{r_n\} \subset (0, 2\alpha]$, *and* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ *are four sequences in* $[0,1]$ *such that*

(i) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 2/L$ *and* $\lim_{n\to\infty}(\lambda_n - \lambda_{n+1}) = 0$;

(ii) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < 2\alpha$ *and* $\lim_{n\to\infty}(r_n - r_{n+1}) = 0$;

(iii) $\beta_n + \gamma_n + \delta_n = 1$ *and* $(\gamma_n + \delta_n)k \leq \gamma_n < (1 - 2\rho)\delta_n$ *for all* $n \geq 0$;

(iv) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(v) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *and* $\liminf_{n\to\infty} \delta_n > 0$;

(vi) $\lim_{n\to\infty}(\gamma_{n+1}/(1 - \beta_{n+1}) - \gamma_n/(1 - \beta_n)) = 0$.

Then the sequences $\{x_n\}$, $\{y_n\}$, *and* $\{z_n\}$ *converge strongly to the unique solution of the variational inequality of finding* $x^* \in \mathrm{Fix}(S) \cap \Omega \cap \mathrm{GMEP}$ *such that*

$$\langle (I - Q)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \mathrm{Fix}(S) \cap \Omega \cap \mathrm{GMEP}. \tag{3.3}$$

In other words, x^* *is the unique fixed point of the contraction* $P_{\mathrm{Fix}(S)\cap\Omega\cap\mathrm{GMEP}} Q$, $x^* = P_{\mathrm{Fix}(S)\cap\Omega\cap\mathrm{GMEP}} Q x^*$.

Proof. First it is obvious that there hold the following assertions:

(a) $x^* \in C$ solves the minimization (1.5);

(b) x^* solves the fixed point equation

$$x^* = P_C(I - \lambda \nabla f)x^*, \tag{3.4}$$

where $\lambda > 0$ is any fixed positive number;

(c) x^* solves the variational inequality of finding $x^* \in C$ such that

$$\langle \nabla f(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C, \tag{3.5}$$

where its solution set is denoted by $\mathrm{VI}(C, \nabla f)$.

We divide the proof into several steps.

Step 1. We claim that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

Indeed, first, we can write (3.2) as $x_{n+1} = \beta_n x_n + (1-\beta_n)u_n$, for all $n \geq 0$, where $u_n = (x_{n+1} - \beta_n x_n)/(1-\beta_n)$. It follows that

$$\begin{aligned}
u_{n+1} - u_n &= \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1-\beta_n} \\
&= \frac{\gamma_{n+1}P_C(z_{n+1} - \lambda_{n+1}\nabla f(z_{n+1})) + \delta_{n+1}Sy_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n P_C(z_n - \lambda_n \nabla f(z_n)) + \delta_n S y_n}{1-\beta_n} \\
&= \frac{\gamma_{n+1}\left[P_C(z_{n+1} - \lambda_{n+1}\nabla f(z_{n+1})) - P_C(z_n - \lambda_n\nabla f(z_n))\right] + \delta_{n+1}(Sy_{n+1} - Sy_n)}{1-\beta_{n+1}} \\
&\quad + \left(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}\right)P_C(z_n - \lambda_n\nabla f(z_n)) + \left(\frac{\delta_{n+1}}{1-\beta_{n+1}} - \frac{\delta_n}{1-\beta_n}\right)Sy_n.
\end{aligned} \tag{3.6}$$

From Lemma 3.1 and (3.2), we get

$$\begin{aligned}
&\left\|\gamma_{n+1}\left[P_C(z_{n+1} - \lambda_{n+1}\nabla f(z_{n+1})) - P_C(z_n - \lambda_n\nabla f(z_n))\right] + \delta_{n+1}(Sy_{n+1} - Sy_n)\right\| \\
&\quad \leq \left\|\gamma_{n+1}(y_{n+1} - y_n) + \delta_{n+1}(Sy_{n+1} - Sy_n)\right\| \\
&\qquad + \gamma_{n+1}\left\|\left[P_C(z_{n+1} - \lambda_{n+1}\nabla f(z_{n+1})) - y_{n+1}\right] + \left[y_n - P_C(z_n - \lambda_n\nabla f(z_n))\right]\right\| \\
&\quad \leq (\gamma_{n+1} + \delta_{n+1})\left\|y_{n+1} - y_n\right\| + \gamma_{n+1}\alpha_{n+1}\left\|Qx_{n+1} - P_C(z_{n+1} - \lambda_{n+1}\nabla f(z_{n+1}))\right\| \\
&\qquad + \gamma_{n+1}\alpha_n\left\|Qx_n - P_C(z_n - \lambda_n\nabla f(z_n))\right\|.
\end{aligned} \tag{3.7}$$

Let $\{T_{r_n}^{(\Theta,\varphi)}\}$ be a sequence of mappings defined as in Lemma 2.1. Note that the L-Lipschitz continuity of ∇f implies that the gradient ∇f is $(1/L)$-ism [31]. Since ∇f and F are $(1/L)$-inverse strongly monotone mapping and α-inverse strongly monotone mapping, respectively, then we have

$$\begin{aligned}
&\left\|(I - \lambda\nabla f)x - (I - \lambda\nabla f)y\right\|^2 \\
&\quad = \|x - y\|^2 - 2\lambda\langle\nabla f(x) - \nabla f(y), x - y\rangle + \lambda^2\left\|\nabla f(x) - \nabla f(y)\right\|^2 \\
&\quad \leq \|x - y\|^2 + \lambda\left(\lambda - \frac{2}{L}\right)\left\|\nabla f(x) - \nabla f(y)\right\|^2, \\
&\left\|(I - \mu F)x - (I - \mu F)y\right\|^2 \leq \|x - y\|^2 + \mu(\mu - 2\alpha)\|Fx - Fy\|^2.
\end{aligned} \tag{3.8}$$

It is clear that if $0 \leq \lambda \leq 2/L$ and $0 \leq \mu \leq 2\alpha$, then $(I - \lambda\nabla f)$ and $(I - \mu F)$ are nonexpansive. It follows from that

$$\begin{aligned}
&\left\|P_C(z_{n+1} - \lambda_{n+1}\nabla f(z_{n+1})) - P_C(z_n - \lambda_n\nabla f(z_n))\right\| \\
&\quad \leq \left\|z_{n+1} - \lambda_{n+1}\nabla f(z_{n+1}) - (z_n - \lambda_n\nabla f(z_n))\right\| \\
&\quad \leq \left\|z_{n+1} - \lambda_{n+1}\nabla f(z_{n+1}) - (z_n - \lambda_{n+1}\nabla f(z_n))\right\| + |\lambda_{n+1} - \lambda_n|\left\|\nabla f(z_n)\right\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \|z_{n+1} - z_n\| + |\lambda_{n+1} - \lambda_n| \|\nabla f(z_n)\| \\
&= \left\| T_{r_{n+1}}^{(\Theta,\varphi)}(x_{n+1} - r_{n+1} F x_{n+1}) - T_{r_n}^{(\Theta,\varphi)}(x_n - r_n F x_n) \right\| + |\lambda_{n+1} - \lambda_n| \|\nabla f(z_n)\| \\
&\leq \left\| T_{r_{n+1}}^{(\Theta,\varphi)}(x_{n+1} - r_{n+1} F x_{n+1}) - T_{r_{n+1}}^{(\Theta,\varphi)}(x_n - r_n F x_n) \right\| \\
&\quad + \left\| T_{r_{n+1}}^{(\Theta,\varphi)}(x_n - r_n F x_n) - T_{r_n}^{(\Theta,\varphi)}(x_n - r_n F x_n) \right\| + |\lambda_{n+1} - \lambda_n| \|\nabla f(z_n)\| \\
&\leq \|(x_{n+1} - r_{n+1} F x_{n+1}) - (x_n - r_n F x_n)\| \\
&\quad + \left\| T_{r_{n+1}}^{(\Theta,\varphi)}(x_n - r_n F x_n) - T_{r_n}^{(\Theta,\varphi)}(x_n - r_n F x_n) \right\| + |\lambda_{n+1} - \lambda_n| \|\nabla f(z_n)\| \\
&\leq \|(x_{n+1} - r_{n+1} F x_{n+1}) - (x_n - r_{n+1} F x_n)\| + |r_{n+1} - r_n| \|F x_n\| \\
&\quad + \left\| T_{r_{n+1}}^{(\Theta,\varphi)}(x_n - r_n F x_n) - T_{r_n}^{(\Theta,\varphi)}(x_n - r_n F x_n) \right\| + |\lambda_{n+1} - \lambda_n| \|\nabla f(z_n)\| \\
&\leq \|x_{n+1} - x_n\| + \left\| T_{r_{n+1}}^{(\Theta,\varphi)}(x_n - r_n F x_n) - T_{r_n}^{(\Theta,\varphi)}(x_n - r_n F x_n) \right\| \\
&\quad + |r_{n+1} - r_n| \|F x_n\| + |\lambda_{n+1} - \lambda_n| \|\nabla f(z_n)\|.
\end{aligned}
\tag{3.9}
$$

Then,

$$
\begin{aligned}
&\|y_{n+1} - y_n\| \\
&\quad \leq \|P_C(z_{n+1} - \lambda_{n+1} \nabla f(z_{n+1})) - P_C(z_n - \lambda_n \nabla f(z_n))\| \\
&\qquad + \alpha_{n+1} \|Q x_{n+1} - P_C(z_{n+1} - \lambda_{n+1} \nabla f(z_{n+1}))\| + \alpha_n \|Q x_n - P_C(z_n - \lambda_n \nabla f(z_n))\| \\
&\quad \leq \|x_{n+1} - x_n\| + \left\| T_{r_{n+1}}^{(\Theta,\varphi)}(x_n - r_n F x_n) - T_{r_n}^{(\Theta,\varphi)}(x_n - r_n F x_n) \right\| \\
&\qquad + |r_{n+1} - r_n| \|F x_n\| + |\lambda_{n+1} - \lambda_n| \|\nabla f(z_n)\| \\
&\qquad + \alpha_n \|Q x_n - P_C(z_n - \lambda_n \nabla f(z_n))\| + \alpha_{n+1} \|Q x_{n+1} - P_C(z_{n+1} - \lambda_{n+1} \nabla f(z_{n+1}))\|.
\end{aligned}
\tag{3.10}
$$

So, from (3.6), (3.7), and (3.10), we have

$$
\begin{aligned}
\|u_{n+1} - u_n\| \leq \|x_{n+1} - x_n\| &+ \left(1 + \frac{\gamma_{n+1}}{1 - \beta_{n+1}}\right) \alpha_n \|Q x_n - P_C(z_n - \lambda_n \nabla f(z_n))\| \\
&+ \left(1 + \frac{\gamma_{n+1}}{1 - \beta_{n+1}}\right) \alpha_{n+1} \|Q x_{n+1} - P_C(z_{n+1} - \lambda_{n+1} \nabla f(z_{n+1}))\| \\
&+ \left| \frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n} \right| (\|P_C(z_n - \lambda_n \nabla f(z_n))\| + \|S y_n\|) \\
&+ \left\| T_{r_{n+1}}^{(\Theta,\varphi)}(x_n - r_n F x_n) - T_{r_n}^{(\Theta,\varphi)}(x_n - r_n F x_n) \right\| \\
&+ |r_{n+1} - r_n| \|F x_n\| + |\lambda_{n+1} - \lambda_n| \|\nabla f(z_n)\|.
\end{aligned}
\tag{3.11}
$$

Utilizing Proposition 2.4 and condition (ii), we have

$$\lim_{n\to\infty}\left\|T_{r_{n+1}}^{(\Theta,\varphi)}(x_n - r_n F x_n) - T_{r_n}^{(\Theta,\varphi)}(x_n - r_n F x_n)\right\| = 0. \tag{3.12}$$

This implies that

$$\limsup_{n\to\infty}(\|u_{n+1} - u_n\| - \|x_{n+1} - x_n\|) \le 0. \tag{3.13}$$

Hence by Lemma 2.6, we get $\lim_{n\to\infty}\|u_n - x_n\| = 0$. Consequently,

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1-\beta_n)\|u_n - x_n\| = 0. \tag{3.14}$$

Step 2. We claim that $\lim_{n\to\infty}\|\nabla f(z_n) - \nabla f(x^*)\| = 0$ and $\lim_{n\to\infty}\|Fx_n - Fx^*\| = 0$.

Indeed, let $x^* \in \mathrm{Fix}(S)\cap\Omega\cap\mathrm{GMEP}$. Then we have $x^* = Sx^*$, $x^* = P_C(x^* - \lambda_n \nabla f(x^*))$ and

$$x^* = T_{r_n}^{(\Theta,\varphi)}(x^* - r_n F x^*). \tag{3.15}$$

Hence from (3.8), we have

$$\begin{aligned}\left\|P_C(z_n - \lambda_n\nabla f(z_n)) - P_C(x^* - \lambda_n\nabla f(x^*))\right\|^2 &\le \left\|(z_n - \lambda_n\nabla f(z_n)) - (x^* - \lambda_n\nabla f(x^*))\right\|^2\\ &\le \|z_n - x^*\|^2 + \lambda_n\left(\lambda_n - \frac{2}{L}\right)\left\|\nabla f(z_n) - \nabla f(x^*)\right\|^2,\end{aligned} \tag{3.16}$$

$$\begin{aligned}\|z_n - x^*\|^2 &= \left\|T_{r_n}^{(\Theta,\varphi)}(x_n - r_n F x_n) - T_{r_n}^{(\Theta,\varphi)}(x^* - r_n F x^*)\right\|^2\\ &\le \|(x_n - r_n F x_n) - (x^* - r_n F x^*)\|^2\\ &\le \|x_n - x^*\|^2 + r_n(r_n - 2\alpha)\|Fx_n - Fx^*\|^2.\end{aligned} \tag{3.17}$$

It follows from (3.2), (3.16), and (3.17) that

$$\begin{aligned}\left\|y_n - x^*\right\|^2 &\le (1-\alpha_n)\left\|P_C(z_n - \lambda_n\nabla f(z_n)) - P_C(x^* - \lambda_n\nabla f(x^*))\right\|^2 + \alpha_n\|Qx_n - x^*\|^2\\ &\le \alpha_n\|Qx_n - x^*\|^2 + \|z_n - x^*\|^2 + \lambda_n\left(\lambda_n - \frac{2}{L}\right)\left\|\nabla f(z_n) - \nabla f(x^*)\right\|^2\\ &\le \alpha_n\|Qx_n - x^*\|^2 + \|x_n - x^*\|^2 + r_n(r_n - 2\alpha)\|Fx_n - Fx^*\|^2\\ &\quad + \lambda_n\left(\lambda_n - \frac{2}{L}\right)\left\|\nabla f(z_n) - \nabla f(x^*)\right\|^2.\end{aligned} \tag{3.18}$$

Utilizing the convexity of $\|\cdot\|$, we have

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2\\
&= \left\|\beta_n(x_n-x^*)+(1-\beta_n)\frac{1}{1-\beta_n}[\gamma_n(P_C(z_n-\lambda_n\nabla f(z_n))-x^*)+\delta_n(Sy_n-x^*)]\right\|^2\\
&\le \beta_n\|x_n-x^*\|^2+(1-\beta_n)\left\|\frac{\gamma_n}{1-\beta_n}(P_C(z_n-\lambda_n\nabla f(z_n))-x^*)+\frac{\delta_n}{1-\beta_n}(Sy_n-x^*)\right\|^2\\
&= \beta_n\|x_n-x^*\|^2+(1-\beta_n)\left\|\frac{\gamma_n(y_n-x^*)+\delta_n(Sy_n-x^*)}{1-\beta_n}+\frac{\alpha_n\gamma_n}{1-\beta_n}(P_C(z_n-\lambda_n\nabla f(z_n))-Qx_n)\right\|^2\\
&\le \beta_n\|x_n-x^*\|^2+(1-\beta_n)\left\|\frac{\gamma_n(y_n-x^*)+\delta_n(Sy_n-x^*)}{1-\beta_n}\right\|^2+M\alpha_n\\
&\le \beta_n\|x_n-x^*\|^2+(1-\beta_n)\left\|y_n-x^*\right\|^2+M\alpha_n,
\end{aligned} \tag{3.19}$$

where $M>0$ is some appropriate constant. So, from (3.18) and (3.19), it follows that

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 \le{}& \|x_n-x^*\|^2+r_n(r_n-2\alpha)(1-\beta_n)\|Fx_n-Fx^*\|^2\\
&+\lambda_n\left(\lambda_n-\frac{2}{L}\right)(1-\beta_n)\left\|\nabla f(z_n)-\nabla f(x^*)\right\|^2+\left(M+\|Qx_n-x^*\|^2\right)\alpha_n.
\end{aligned} \tag{3.20}$$

Therefore,

$$\begin{aligned}
&\lambda_n\left(\frac{2}{L}-\lambda_n\right)(1-\beta_n)\left\|\nabla f(z_n)-\nabla f(x^*)\right\|^2+r_n(2\alpha-r_n)(1-\beta_n)\|Fx_n-Fx^*\|^2\\
&\quad\le \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\left(M+\|Qx_n-x^*\|^2\right)\alpha_n\\
&\quad\le (\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_n-x_{n+1}\|+\left(M+\|Qx_n-x^*\|^2\right)\alpha_n.
\end{aligned} \tag{3.21}$$

Since $\liminf_{n\to\infty}\lambda_n(2/L-\lambda_n)(1-\beta_n)>0$, $\liminf_{n\to\infty}r_n(2\alpha-r_n)(1-\beta_n)>0$, $\|x_n-x_{n+1}\|\to 0$ and $\alpha_n\to 0$, we have

$$\lim_{n\to\infty}\left\|\nabla f(z_n)-\nabla f(x^*)\right\|=0, \qquad \lim_{n\to\infty}\|Fx_n-Fx^*\|=0. \tag{3.22}$$

Step 3. We claim that $\lim_{n\to\infty}\|Sy_n-y_n\|=0$.

Indeed, set $v_n = P_C(z_n - \lambda_n \nabla f(z_n))$. Noticing the firm nonexpansivity of $T_{r_n}^{(\Theta,\varphi)}$, we have

$$
\begin{aligned}
&\|z_n - x^*\|^2 \\
&\quad = \left\| T_{r_n}^{(\Theta,\varphi)}(x_n - r_n F x_n) - T_{r_n}^{(\Theta,\varphi)}(x^* - r_n F x^*) \right\|^2 \\
&\quad \le \langle (x_n - r_n F x_n) - (x^* - r_n F x^*), z_n - x^* \rangle \\
&\quad = \frac{1}{2}\Big(\|x_n - x^* - r_n(F x_n - F x^*)\|^2 + \|z_n - x^*\|^2 - \|(x_n - x^*) - r_n(F x_n - F x^*) - (z_n - x^*)\|^2 \Big) \\
&\quad \le \frac{1}{2}\Big(\|x_n - x^*\|^2 + \|z_n - x^*\|^2 - \|(x_n - z_n) - r_n(F x_n - F x^*)\|^2 \Big) \\
&\quad = \frac{1}{2}\Big(\|x_n - x^*\|^2 + \|z_n - x^*\|^2 - \|x_n - z_n\|^2 + 2 r_n \langle x_n - z_n, F x_n - F x^* \rangle - r_n^2 \|F x_n - F x^*\|^2 \Big),
\end{aligned} \tag{3.23}
$$

$$
\begin{aligned}
&\|v_n - x^*\|^2 \\
&\quad = \left\| P_C(z_n - \lambda_n \nabla f(z_n)) - P_C(x^* - \lambda_n \nabla f(x^*)) \right\|^2 \\
&\quad \le \left\langle z_n - \lambda_n \nabla f(z_n) - (x^* - \lambda_n \nabla f(x^*)), v_n - x^* \right\rangle \\
&\quad = \frac{1}{2}\Big(\left\| z_n - \lambda_n \nabla f(z_n) - (x^* - \lambda_n \nabla f(x^*)) \right\|^2 + \|v_n - x^*\|^2 \\
&\qquad\quad - \left\| z_n - \lambda_n \nabla f(z_n) - (x^* - \lambda_n \nabla f(x^*)) - (v_n - x^*) \right\|^2 \Big) \\
&\quad \le \frac{1}{2}\Big(\|z_n - x^*\|^2 + \|v_n - x^*\|^2 - \|z_n - v_n\|^2 \\
&\qquad\quad + 2\lambda_n \langle \nabla f(z_n) - \nabla f(x^*), z_n - v_n \rangle - \lambda_n^2 \left\| \nabla f(z_n) - \nabla f(x^*) \right\|^2 \Big) \\
&\quad \le \frac{1}{2}\Big(\|x_n - x^*\|^2 + \|v_n - x^*\|^2 - \|z_n - v_n\|^2 + 2\lambda_n \langle \nabla f(z_n) - \nabla f(x^*), z_n - v_n \rangle \Big).
\end{aligned} \tag{3.24}
$$

Thus, we have

$$
\|z_n - x^*\|^2 \le \|x_n - x^*\|^2 - \|x_n - z_n\|^2 + 2 r_n \langle x_n - z_n, F x_n - F x^* \rangle - r_n^2 \|F x_n - F x^*\|^2, \tag{3.25}
$$

$$
\|v_n - x^*\|^2 \le \|x_n - x^*\|^2 - \|z_n - v_n\|^2 + 2\lambda_n \left\| \nabla f(z_n) - \nabla f(x^*) \right\| \|z_n - v_n\|. \tag{3.26}
$$

It follows that

$$
\begin{aligned}
\left\| y_n - x^* \right\|^2 &\le \alpha_n \|Q x_n - x^*\|^2 + (1 - \alpha_n) \|v_n - x^*\|^2 \\
&\le \alpha_n \|Q x_n - x^*\|^2 + \|v_n - x^*\|^2 \\
&\le \alpha_n \|Q x_n - x^*\|^2 + \|x_n - x^*\|^2 - \|z_n - v_n\|^2 + 2\lambda_n \left\| \nabla f(z_n) - \nabla f(x^*) \right\| \|z_n - v_n\|.
\end{aligned} \tag{3.27}
$$

From (3.18), (3.19), and (3.25), we have

$$
\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le \beta_n\|x_n-x^*\|^2+(1-\beta_n)\alpha_n\|Qx_n-x^*\|^2+(1-\beta_n)\|z_n-x^*\|^2+M\alpha_n\\
&\le \|x_n-x^*\|^2-(1-\beta_n)\|x_n-z_n\|^2+2(1-\beta_n)r_n\|x_n-z_n\|\|Fx_n-Fx^*\|\\
&\quad+\left(M+\|Qx_n-x^*\|^2\right)\alpha_n.
\end{aligned}
\tag{3.28}
$$

It follows that

$$
\begin{aligned}
(1-\beta_n)\|x_n-z_n\|^2 \le{}& (\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_{n+1}-x_n\|+\left(M+\|Qx_n-x^*\|^2\right)\alpha_n\\
&+2(1-\beta_n)r_n\|x_n-z_n\|\|Fx_n-Fx^*\|.
\end{aligned}
\tag{3.29}
$$

Note that $\|x_{n+1}-x_n\| \to 0$, $\alpha_n \to 0$ and $\|Fx_n-Fx^*\| \to 0$. Then we immediately deduce that

$$\lim_{n\to\infty}\|x_n-z_n\|=0. \tag{3.30}$$

From (3.19) and (3.27), we have

$$
\begin{aligned}
\|x_{n+1}-x^*\|^2 \le{}& \|x_n-x^*\|^2-(1-\beta_n)\|z_n-v_n\|^2\\
&+2\lambda_n(1-\beta_n)\left\|\nabla f(z_n)-\nabla f(x^*)\right\|\|z_n-v_n\|+\left(M+\|Qx_n-x^*\|^2\right)\alpha_n.
\end{aligned}
\tag{3.31}
$$

So, we obtain

$$
\begin{aligned}
(1-\beta_n)\|z_n-v_n\|^2 \le{}& (\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_{n+1}-x_n\|\\
&+\left(M+\|Qx_n-x^*\|^2\right)\alpha_n+2\lambda_n(1-\beta_n)\left\|\nabla f(z_n)-\nabla f(x^*)\right\|\|z_n-v_n\|.
\end{aligned}
\tag{3.32}
$$

Note that $\|x_{n+1}-x_n\| \to 0$, $\alpha_n \to 0$ and $\|\nabla f(z_n)-\nabla f(x^*)\| \to 0$. Then we immediately conclude that

$$\lim_{n\to\infty}\|z_n-v_n\|=0. \tag{3.33}$$

This together with $\|y_n-v_n\|\le\alpha_n\|Qx_n-v_n\| \to 0$, implies that

$$\lim_{n\to\infty}\left\|z_n-y_n\right\|=0. \tag{3.34}$$

Thus, from (3.30) and (3.34), we deduce that

$$\lim_{n\to\infty}\left\|x_n-y_n\right\|=0. \tag{3.35}$$

Since

$$\begin{aligned}\left\|\delta_n(Sy_n - x_n)\right\| &\le \|x_{n+1} - x_n\| + \gamma_n \left\|P_C(z_n - \lambda_n \nabla f(z_n)) - x_n\right\| \\ &\le \|x_{n+1} - x_n\| + \gamma_n \left\|y_n - x_n\right\| + \gamma_n \alpha_n \left\|Qx_n - P_C(z_n - \lambda_n \nabla f(z_n))\right\|.\end{aligned} \tag{3.36}$$

Therefore,

$$\lim_{n\to\infty} \left\|Sy_n - x_n\right\| = 0, \qquad \lim_{n\to\infty} \left\|Sy_n - y_n\right\| = 0. \tag{3.37}$$

Step 4. We claim that $\limsup_{n\to\infty} \langle Qx^* - x^*, x_n - x^* \rangle \le 0$ where $x^* = P_{\mathrm{Fix}(S)\cap\Omega\cap\mathrm{GMEP}} Qx^*$.

Indeed, since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup u$ and

$$\limsup_{n\to\infty} \langle Qx^* - x^*, x_n - x^* \rangle = \lim_{i\to\infty} \langle Qx^* - x^*, x_{n_i} - x^* \rangle = \langle Qx^* - x^*, u - x^* \rangle. \tag{3.38}$$

We can obtain that $u \in \mathrm{Fix}(S) \cap \Omega \cap \mathrm{GMEP}$. First, we show $u \in \Omega\ (= \mathrm{VI}(C, \nabla f))$. Since $x_n - z_n \to 0$ and $v_n - z_n \to 0$, we conclude that $z_{n_i} \rightharpoonup u$ and $v_{n_i} \rightharpoonup u$. Let

$$Tv = \begin{cases} \nabla f(v) + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C, \end{cases} \tag{3.39}$$

where $N_C v$ is the normal cone to C at $v \in C$. We have already mentioned that in this case, the mapping T is maximal monotone, and $0 \in Tv$ if and only if $v \in \mathrm{VI}(C, \nabla f)(= \Omega)$; see [37]. Let $G(T)$ be the graph of T and let $(v, w) \in G(T)$. Then, we have $w \in Tv = \nabla f(v) + N_C v$ and hence $w - \nabla f(v) \in N_C v$. So, we have $\langle v - t, w - \nabla f(v) \rangle \ge 0$ for all $t \in C$. On the other hand, from $v_n = P_C(z_n - \lambda_n \nabla f(z_n))$ and $v \in C$, we have

$$\left\langle z_n - \lambda_n \nabla f(z_n) - v_n, v_n - v \right\rangle \ge 0 \tag{3.40}$$

and hence

$$\left\langle v - v_n, \frac{v_n - z_n}{\lambda_n} + \nabla f(z_n) \right\rangle \ge 0. \tag{3.41}$$

From $\langle v - t, w - \nabla f(v) \rangle \ge 0$ for all $t \in C$ and $v_{n_i} \in C$, we have

$$\begin{aligned}&\langle v - v_{n_i}, w \rangle \\ &\quad\ge \left\langle v - v_{n_i}, \nabla f(v) \right\rangle \\ &\quad\ge \left\langle v - v_{n_i}, \nabla f(v) \right\rangle - \left\langle v - v_{n_i}, \frac{v_{n_i} - z_{n_i}}{\lambda_{n_i}} + \nabla f(z_{n_i}) \right\rangle \\ &\quad= \left\langle v - v_{n_i}, \nabla f(v) - \nabla f(v_{n_i}) \right\rangle + \left\langle v - v_{n_i}, \nabla f(v_{n_i}) - \nabla f(z_{n_i}) \right\rangle - \left\langle v - v_{n_i}, \frac{v_{n_i} - z_{n_i}}{\lambda_{n_i}} \right\rangle \\ &\quad\ge \left\langle v - v_{n_i}, \nabla f(v_{n_i}) - \nabla f(z_{n_i}) \right\rangle - \left\langle v - v_{n_i}, \frac{v_{n_i} - z_{n_i}}{\lambda_{n_i}} \right\rangle.\end{aligned} \tag{3.42}$$

Hence, we obtain $\langle v-u,w\rangle \geq 0$ as $i \to \infty$. Since T is maximal monotone, we have $u \in T^{-1}0$ and hence $u \in \mathrm{VI}(C,\nabla f)\ (=\Omega)$.

Secondly, let us show $u \in \mathrm{Fix}(S)$. Since $x_n - y_n \to 0$ and $x_{n_i} \rightharpoonup u$, we have $y_{n_i} \rightharpoonup u$. Also, since $y_n - Sy_n \to 0$, it follows that $y_{n_i} - Sy_{n_i} \to 0$ as $i \to \infty$. So, in terms of Proposition 2.5(ii) we obtain $u \in \mathrm{Fix}(S)$.

Next, let us show $u \in \mathrm{GMEP}$. From $z_n = T_{r_n}^{(\Theta,\varphi)}(x_n - r_nFx_n)$, we know that

$$\Theta(z_n,y)+\varphi(y)-\varphi(z_n)+\langle Fx_n,y-z_n\rangle+\frac{1}{r_n}\langle y-z_n,z_n-x_n\rangle \geq 0, \quad \forall y \in C. \tag{3.43}$$

From (H2), it follows that

$$\varphi(y)-\varphi(z_n)+\langle Fx_n,y-z_n\rangle+\frac{1}{r_n}\langle y-z_n,z_n-x_n\rangle \geq \Theta(y,z_n), \quad \forall y \in C. \tag{3.44}$$

Replacing n by n_i, we have

$$\varphi(y)-\varphi(z_{n_i})+\langle Fx_{n_i},y-z_{n_i}\rangle+\left\langle y-z_{n_i},\frac{z_{n_i}-x_{n_i}}{r_{n_i}}\right\rangle \geq \Theta(y,z_{n_i}), \quad \forall y \in C. \tag{3.45}$$

Put $z_s = sy+(1-s)u$ for all $s \in (0,1]$ and $y \in C$. Then, we have $z_s \in C$. So, from (3.45), we have

$$\begin{aligned}\langle z_s-z_{n_i},Fz_s\rangle &\geq \langle z_s-z_{n_i},Fz_s\rangle-\varphi(z_s)+\varphi(z_{n_i})-\langle z_s-z_{n_i},Fx_{n_i}\rangle\\&\quad-\left\langle z_s-z_{n_i},\frac{z_{n_i}-x_{n_i}}{r_{n_i}}\right\rangle+\Theta(z_s,z_{n_i})\\&=\langle z_s-z_{n_i},Fz_s-Fz_{n_i}\rangle+\langle z_s-z_{n_i},Fz_{n_i}-Fx_{n_i}\rangle-\varphi(z_s)+\varphi(z_{n_i})\\&\quad-\left\langle z_s-z_{n_i},\frac{z_{n_i}-x_{n_i}}{r_{n_i}}\right\rangle+\Theta(z_s,z_{n_i}).\end{aligned} \tag{3.46}$$

Since $\|z_{n_i}-x_{n_i}\| \to 0$, we have $\|Fz_{n_i}-Fx_{n_i}\| \to 0$. Further, from the monotonicity of F, we have $\langle z_s-z_{n_i},Fz_s-Fz_{n_i}\rangle \geq 0$. So, from (H4), the weakly lower semicontinuity of φ, $(z_{n_i}-x_{n_i})/r_{n_i} \to 0$ and $z_{n_i} \rightharpoonup u$, we have

$$\langle z_s-z_{n_i},Fz_s\rangle \geq -\varphi(z_s)+\varphi(u)+\Theta(z_s,u), \tag{3.47}$$

as $i \to \infty$. From (H1), (H4), and (3.47), we also have

$$\begin{aligned}0&=\Theta(z_s,z_s)+\varphi(z_s)+\varphi(z_s)\\&\leq s\Theta(z_s,y)+(1-s)\Theta(z_s,u)+s\varphi(y)+(1-s)\varphi(u)-\varphi(z_s)\\&=s\left[\Theta(z_s,y)+\varphi(y)-\varphi(z_s)\right]+(1-s)\left[\Theta(z_s,u)+\varphi(u)-\varphi(z_s)\right]\\&\leq s\left[\Theta(z_s,y)+\varphi(y)-\varphi(z_s)\right]+(1-s)\langle z_s-u,Fz_s\rangle\\&=s\left[\Theta(z_s,y)+\varphi(y)-\varphi(z_s)\right]+(1-s)s\langle y-u,Fz_s\rangle,\end{aligned} \tag{3.48}$$

and hence

$$0 \le \Theta(z_s, y) + \varphi(y) - \varphi(z_s) + (1-s)\langle y - u, Fz_s\rangle. \tag{3.49}$$

Letting $s \to 0$, we have, for each $y \in C$,

$$0 \le \Theta(u, y) + \varphi(y) - \varphi(u) + \langle y - u, Fu\rangle. \tag{3.50}$$

This shows that $u \in \mathrm{GMEP}$. Therefore, $u \in \mathrm{Fix}(S) \cap \Omega \cap \mathrm{GMEP}$. Hence, it follows from (2.4) that

$$\limsup_{n\to\infty}\langle Qx^* - x^*, x_n - x^*\rangle = \lim_{i\to\infty}\langle Qx^* - x^*, x_{n_i} - x^*\rangle = \langle Qx^* - x^*, u - x^*\rangle \le 0. \tag{3.51}$$

Step 5. We claim that $\lim_{n\to\infty}\|x_n - x^*\| = 0$.

Indeed, from (3.2) and the convexity of $\|\cdot\|$, we have

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\quad = \left\|\beta_n(x_n - x^*) + \gamma_n(y_n - x^*) + \delta_n(Sy_n - x^*) + \gamma_n\alpha_n(P_C(z_n - \lambda_n\nabla f(z_n)) - Qx_n)\right\|^2 \\
&\quad \le \left\|\beta_n(x_n - x^*) + \gamma_n(y_n - x^*) + \delta_n(Sy_n - x^*)\right\|^2 \\
&\qquad + 2\gamma_n\alpha_n\langle P_C(z_n - \lambda_n\nabla f(z_n)) - Qx_n, x_{n+1} - x^*\rangle \\
&\quad \le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\left\|\frac{1}{1-\beta_n}[\gamma_n(y_n - x^*) + \delta_n(Sy_n - x^*)]\right\|^2 \\
&\qquad + 2\gamma_n\alpha_n\langle P_C(z_n - \lambda_n\nabla f(z_n)) - x^*, x_{n+1} - x^*\rangle + 2\gamma_n\alpha_n\langle x^* - Qx_n, x_{n+1} - x^*\rangle.
\end{aligned} \tag{3.52}$$

Utilizing Lemma 3.1, we get from (3.52)

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\quad \le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\left\|y_n - x^*\right\|^2 + 2\gamma_n\alpha_n\left\|P_C(z_n - \lambda_n\nabla f(z_n)) - x^*\right\|\|x_{n+1} - x^*\| \\
&\qquad + 2\gamma_n\alpha_n\langle x^* - Qx_n, x_{n+1} - x^*\rangle \\
&\quad \le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\left[(1-\alpha_n)\|z_n - x^*\|^2 + 2\alpha_n\langle Qx_n - x^*, y_n - x^*\rangle\right] \\
&\qquad + 2\gamma_n\alpha_n\|z_n - x^*\|\|x_{n+1} - x^*\| + 2\gamma_n\alpha_n\langle x^* - Qx_n, x_{n+1} - x^*\rangle.
\end{aligned} \tag{3.53}$$

From (3.17), we note that $\|z_n - x^*\| \le \|x_n - x^*\|$. Hence we have

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\quad \le \beta_n \|x_n - x^*\|^2 + (1-\beta_n)(1-\alpha_n)\|x_n - x^*\|^2 + 2\alpha_n(1-\beta_n)\langle Qx_n - x^*, y_n - x^*\rangle \\
&\qquad + 2\gamma_n\alpha_n\|x_n - x^*\|\|x_{n+1} - x^*\| + 2\gamma_n\alpha_n\langle x^* - Qx_n, x_{n+1} - x^*\rangle \\
&\quad \le [1-(1-\beta_n)\alpha_n]\|x_n - x^*\|^2 + 2\alpha_n\gamma_n\langle Qx_n - x^*, y_n - x_{n+1}\rangle \\
&\qquad + 2\alpha_n\delta_n\langle Qx_n - x^*, y_n - x^*\rangle + 2\alpha_n\gamma_n\|x_n - x^*\|\|x_{n+1} - x^*\| \\
&\quad \le [1-(1-\beta_n)\alpha_n]\|x_n - x^*\|^2 + 2\alpha_n\gamma_n\|Qx_n - x^*\|\|y_n - x_{n+1}\| \\
&\qquad + 2\alpha_n\delta_n\langle Qx_n - x^*, x_n - x^*\rangle + 2\alpha_n\delta_n\langle Qx_n - x^*, y_n - x_n\rangle + 2\alpha_n\gamma_n\|x_n - x^*\|\|x_{n+1} - x^*\| \\
&\quad \le [1-(1-\beta_n)\alpha_n]\|x_n - x^*\|^2 + 2\alpha_n\gamma_n\|Qx_n - x^*\|\|y_n - x_{n+1}\| \\
&\qquad + 2\alpha_n\delta_n\rho\|x_n - x^*\|^2 + 2\alpha_n\delta_n\langle Qx^* - x^*, x_n - x^*\rangle \\
&\qquad + 2\alpha_n\delta_n\|Qx_n - x^*\|\|y_n - x_n\| + 2\alpha_n\gamma_n\|x_n - x^*\|\|x_{n+1} - x^*\| \\
&\quad \le [1-(1-\beta_n)\alpha_n]\|x_n - x^*\|^2 + 2\alpha_n\gamma_n\|Qx_n - x^*\|\|y_n - x_{n+1}\| \\
&\qquad + 2\alpha_n\delta_n\rho\|x_n - x^*\|^2 + 2\alpha_n\delta_n\langle Qx^* - x^*, x_n - x^*\rangle \\
&\qquad + 2\alpha_n\delta_n\|Qx_n - x^*\|\|y_n - x_n\| + \alpha_n\gamma_n\left(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2\right),
\end{aligned} \tag{3.54}$$

that is,

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\quad \le \left[1 - \frac{(1-2\rho)\delta_n - \gamma_n}{1-\alpha_n\gamma_n}\alpha_n\right]\|x_n - x^*\|^2 + \frac{[(1-2\rho)\delta_n - \gamma_n]\alpha_n}{1-\alpha_n\gamma_n} \\
&\qquad \times \left\{\frac{2\gamma_n}{(1-2\rho)\delta_n - \gamma_n}\|Qx_n - x^*\|\|y_n - x_{n+1}\| + \frac{2\delta_n}{(1-2\rho)\delta_n - \gamma_n}\|Qx_n - x^*\|\|y_n - x_n\| \right. \\
&\qquad\quad \left. + \frac{2\delta_n}{(1-2\rho)\delta_n - \gamma_n}\langle Qx^* - x^*, x_n - x^*\rangle\right\}.
\end{aligned} \tag{3.55}$$

Note that $\liminf_{n\to\infty}((1-2\rho)\delta_n - \gamma_n)/(1-\alpha_n\gamma_n) > 0$. It follows that $(\sum_{n=0}^{\infty}((1-2\rho)\delta_n - \gamma_n)/(1-\alpha_n\gamma_n))\alpha_n = \infty$. It is clear that

$$\begin{aligned}
\limsup_{n\to\infty} &\left\{\frac{2\gamma_n}{(1-2\rho)\delta_n - \gamma_n}\|Qx_n - x^*\|\|y_n - x_{n+1}\| + \frac{2\delta_n}{(1-2\rho)\delta_n - \gamma_n}\|Qx_n - x^*\|\|y_n - x_n\| \right. \\
&\quad \left. + \frac{2\delta_n}{(1-2\rho)\delta_n - \gamma_n}\langle Qx^* - x^*, x_n - x^*\rangle\right\} \le 0.
\end{aligned} \tag{3.56}$$

Therefore, all conditions of Lemma 2.7 are satisfied. This immediately implies that $x_n \to x^*$. It is readily seen that both $\{y_n\}$ and $\{z_n\}$ converge strongly to the same point x^*. The proof is complete. □

Utilizing Theorem 3.2, we establish the following corollaries.

Corollary 3.3. *Let C be a nonempty bounded closed convex subset of a real Hilbert space H. Let $\Theta : C \times C \to \mathbf{R}$ be a bifunction satisfying conditions (H1)–(H4) and $\varphi : C \to \mathbf{R}$ be a lower semicontinuous and convex function with assumptions (A1) or (A2). Suppose the minimization* (1.5) *is consistent and let Ω denote its solution set. Assume the gradient ∇f is L-Lipschitzian with constant $L > 0$ and $F : C \to H$ is an α-inverse strongly monotone mapping. Let $S : C \to C$ be a k-strictly pseudocontractive mapping such that* $\mathrm{Fix}(S) \cap \Omega \cap \mathrm{GMEP} \neq \emptyset$. *For fixed $u \in C$ and given $x_0 \in C$ arbitrarily, let the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be generated iteratively by*

$$\begin{gathered}\Theta(z_n, y) + \varphi(y) - \varphi(z_n) + \langle Fx_n, y - z_n\rangle + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \quad \forall y \in C,\\ y_n = \alpha_n u + (1-\alpha_n)P_C(z_n - \lambda_n \nabla f(z_n)),\\ x_{n+1} = \beta_n x_n + \gamma_n P_C(z_n - \lambda_n \nabla f(z_n)) + \delta_n S y_n, \quad \forall n \geq 0,\end{gathered} \tag{3.57}$$

where $\{\lambda_n\} \subset (0, 2/L]$, $\{r_n\} \subset (0, 2\alpha]$, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ are four sequences in $[0,1]$ *such that:*

(i) $0 < \liminf_{n\to\infty}\lambda_n \leq \limsup_{n\to\infty}\lambda_n < 2/L$ *and* $\lim_{n\to\infty}(\lambda_n - \lambda_{n+1}) = 0$;

(ii) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < 2\alpha$ *and* $\lim_{n\to\infty}(r_n - r_{n+1}) = 0$;

(iii) $\beta_n + \gamma_n + \delta_n = 1$ *and* $(\gamma_n + \delta_n)k \leq \gamma_n < \delta_n$ *for all* $n \geq 0$;

(iv) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$;

(v) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$ *and* $\liminf_{n\to\infty}\delta_n > 0$;

(vi) $\lim_{n\to\infty}(\gamma_{n+1}/(1-\beta_{n+1}) - \gamma_n/(1-\beta_n)) = 0$.

Then, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to the same point $x^* = P_{\mathrm{Fix}(S)\cap\Omega\cap\mathrm{GMEP}}u$.

Corollary 3.4. *Let C be a nonempty bounded closed convex subset of a real Hilbert space H. Let $\Theta : C \times C \to \mathbf{R}$ be a bifunction satisfying conditions (H1)–(H4) and $\varphi : C \to \mathbf{R}$ be a lower semicontinuous and convex function with assumptions (A1) or (A2). Suppose the minimization* (1.5) *is consistent and let Ω denote its solution set. Assume the gradient ∇f is L-Lipschitzian with constant $L > 0$ and $F : C \to H$ is an α-inverse strongly monotone mapping. Let $S : C \to C$ be a nonexpansive mapping such that* $\mathrm{Fix}(S) \cap \Omega \cap \mathrm{GMEP} \neq \emptyset$. *Let $Q : C \to C$ be a ρ-contraction with $\rho \in [0, 1/2)$. For given $x_0 \in C$ arbitrarily, let the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be generated iteratively by*

$$\begin{gathered}\Theta(z_n, y) + \varphi(y) - \varphi(z_n) + \langle Fx_n, y - z_n\rangle + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \quad \forall y \in C,\\ y_n = \alpha_n Q x_n + (1-\alpha_n)P_C(z_n - \lambda_n \nabla f(z_n)),\\ x_{n+1} = \beta_n x_n + \gamma_n P_C(z_n - \lambda_n \nabla f(z_n)) + \delta_n S y_n, \quad \forall n \geq 0,\end{gathered} \tag{3.58}$$

where $\{\lambda_n\} \subset (0, 2/L]$, $\{r_n\} \subset (0, 2\alpha]$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ are four sequences in $[0,1]$ *such that*

(i) $0 < \liminf_{n\to\infty}\lambda_n \le \limsup_{n\to\infty}\lambda_n < 2/L$ *and* $\lim_{n\to\infty}(\lambda_n - \lambda_{n+1}) = 0$;

(ii) $0 < \liminf_{n\to\infty} r_n \le \limsup_{n\to\infty} r_n < 2\alpha$ *and* $\lim_{n\to\infty}(r_n - r_{n+1}) = 0$;

(iii) $\beta_n + \gamma_n + \delta_n = 1$ *and* $\gamma_n < (1-2\rho)\delta_n$ *for all* $n \ge 0$;

(iv) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$;

(v) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$ *and* $\liminf_{n\to\infty}\gamma_n > 0$;

(vi) $\lim_{n\to\infty}(\gamma_{n+1}/(1-\beta_{n+1}) - \gamma_n/(1-\beta_n)) = 0$.

Then $\{x_n\}$, $\{y_n\}$, *and* $\{z_n\}$ *converge strongly to the same point* $x^* = P_{\mathrm{Fix}(S)\cap\Omega\cap\mathrm{GMEP}}Qx^*$.

Corollary 3.5. *Let* C *be a nonempty bounded closed convex subset of a real Hilbert space* H. *Let* $\Theta : C\times C \to \mathbf{R}$ *be a bifunction satisfying conditions (H1)–(H4) and* $\varphi : C \to \mathbf{R}$ *a lower semicontinuous and convex function with assumptions (A1) or (A2). Suppose the minimization* (1.5) *is consistent and let* Ω *denote its solution set. Assume the gradient* ∇f *is* L*-Lipschitzian with constant* $L > 0$ *and* $F : C \to H$ *is an* α*-inverse strongly monotone mapping. Let* $S : C \to C$ *be a nonexpansive mapping such that* $\mathrm{Fix}(S)\cap\Omega\cap\mathrm{GMEP} \ne \emptyset$. *For fixed* $u \in C$ *and given* $x_0 \in C$ *arbitrarily, let the sequences* $\{x_n\}$, $\{y_n\}$, *and* $\{z_n\}$ *be generated iteratively by*

$$\begin{gathered}\Theta(z_n, y) + \varphi(y) - \varphi(z_n) + \langle Fx_n, y - z_n\rangle + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \ge 0, \quad \forall y \in C,\\ y_n = \alpha_n u + (1-\alpha_n)P_C(z_n - \lambda_n\nabla f(z_n)),\\ x_{n+1} = \beta_n x_n + \gamma_n P_C(z_n - \lambda_n \nabla f(z_n)) + \delta_n S y_n, \quad \forall n \ge 0,\end{gathered} \tag{3.59}$$

where $\{\lambda_n\} \subset (0, 2/L]$, $\{r_n\} \subset (0, 2\alpha]$, *and* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ *are four sequences in* $[0,1]$ *such that:*

(i) $0 < \liminf_{n\to\infty}\lambda_n \le \limsup_{n\to\infty}\lambda_n < 2/L$ *and* $\lim_{n\to\infty}(\lambda_n - \lambda_{n+1}) = 0$;

(ii) $0 < \liminf_{n\to\infty} r_n \le \limsup_{n\to\infty} r_n < 2\alpha$ *and* $\lim_{n\to\infty}(r_n - r_{n+1}) = 0$;

(iii) $\beta_n + \gamma_n + \delta_n = 1$ *and* $\gamma_n < \delta_n$ *for all* $n \ge 0$;

(iv) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$;

(v) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$ *and* $\liminf_{n\to\infty}\gamma_n > 0$;

(vi) $\lim_{n\to\infty}(\gamma_{n+1}/(1-\beta_{n+1}) - \gamma_n/(1-\beta_n)) = 0$.

Then, $\{x_n\}$, $\{y_n\}$, *and* $\{z_n\}$ *converge strongly to the same point* $x^* = P_{\mathrm{Fix}(S)\cap\Omega\cap\mathrm{GMEP}}u$.

Corollary 3.6. *Let* C *be a nonempty bounded closed convex subset of a real Hilbert space* H. *Suppose the minimization* (1.5) *is consistent and let* Ω *denote its solution set. Assume the gradient* ∇f *is* L*-Lipschitzian with constant* $L > 0$ *and* $A : C \to H$ *is an* α*-inverse strongly monotone mapping. Let* $S : C \to C$ *be a* k*-strictly pseudocontractive mapping such that* $\mathrm{Fix}(S)\cap\Omega\cap\mathrm{VI}(C, A) \ne \emptyset$. *For fixed* $u \in C$ *and given* $x_0 \in C$ *arbitrarily, let the sequences* $\{x_n\}$, $\{y_n\}$, *and* $\{z_n\}$ *be generated iteratively by*

$$\begin{gathered} z_n = P_C(x_n - r_n A x_n),\\ y_n = \alpha_n u + (1-\alpha_n)P_C(z_n - \lambda_n\nabla f(z_n)),\\ x_{n+1} = \beta_n x_n + \gamma_n P_C(z_n - \lambda_n\nabla f(z_n)) + \delta_n S y_n, \quad \forall n \ge 0,\end{gathered} \tag{3.60}$$

where $\{\lambda_n\} \subset (0, 2/L]$, $\{r_n\} \subset (0, 2\alpha]$, *and* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ *are four sequences in* $[0,1]$ *such that:*

(i) $0 < \liminf_{n\to\infty}\lambda_n \le \limsup_{n\to\infty}\lambda_n < 2/L$ *and* $\lim_{n\to\infty}(\lambda_n - \lambda_{n+1}) = 0$;

(ii) $0 < \liminf_{n\to\infty} r_n \le \limsup_{n\to\infty} r_n < 2\alpha$ *and* $\lim_{n\to\infty}(r_n - r_{n+1}) = 0$;

(iii) $\beta_n + \gamma_n + \delta_n = 1$ *and* $(\gamma_n + \delta_n)k \le \gamma_n < \delta_n$ *for all* $n \ge 0$;

(iv) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$;

(v) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$ *and* $\liminf_{n\to\infty}\delta_n > 0$;

(vi) $\lim_{n\to\infty}(\gamma_{n+1}/(1-\beta_{n+1}) - \gamma_n/(1-\beta_n)) = 0$.

Then, $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *converge strongly to the same point* $x^* = P_{\mathrm{Fix}(S)\cap\Omega\cap\mathrm{VI}(C,A)}u$.

Proof. In Theorem 3.2, putting $\Theta = 0$, $\varphi = 0$ and $F = A$, the following relation

$$\Theta(z_n, y) + \varphi(y) - \varphi(z_n) + \langle Fx_n, y - z_n\rangle + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \ge 0, \quad \forall y \in C, \tag{3.61}$$

is reduced to

$$\langle Ax_n, y - z_n\rangle + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \ge 0, \quad \forall y \in C. \tag{3.62}$$

This implies that

$$\langle y - z_n, x_n - r_nAx_n - z_n\rangle \le 0, \quad \forall y \in C. \tag{3.63}$$

So, it follows that $z_n = P_C(x_n - r_nAx_n)$ for all $n \ge 0$. Then, by Theorem 3.2, we obtain the desired result. □

Let $T : C \to C$ be a $\widetilde{k}$-strictly pseudocontractive mapping. For recent convergence result for strictly pseudocontractive mappings, we refer to Zeng et al. [38]. Putting $F = I - T$, we know that

$$\|(I-F)x - (I-F)y\|^2 \le \|x-y\|^2 + \widetilde{k}\|Fx - Fy\|^2, \quad \forall x, y \in C. \tag{3.64}$$

Note that

$$\|(I-F)x - (I-F)y\|^2 = \|x-y\|^2 + \|Fx-Fy\|^2 - 2\langle x-y, Fx-Fy\rangle. \tag{3.65}$$

Hence

$$\langle x-y, Fx-Fy\rangle \ge \frac{1-\widetilde{k}}{2}\|Fx-Fy\|^2, \quad \forall x, y \in C. \tag{3.66}$$

This implies that the mapping $F = I - T$ is $((1-\widetilde{k})/2)$-inverse-strongly monotone.

Corollary 3.7. *Let C be a nonempty bounded closed convex subset of a real Hilbert space H. Let $\Theta : C \times C \to \mathbf{R}$ be a bifunction satisfying conditions (H1)–(H4) and $\varphi : C \to \mathbf{R}$ a lower semicontinuous and convex function with assumptions (A1) or (A2). Suppose the minimization* (1.5) *is consistent and let Ω denote its solution set. Assume the gradient ∇f is L-Lipschitzian with constant $L > 0$ and $T : C \to C$ is a $\widetilde{k}$-strictly pseudocontractive mapping. Let $S : C \to C$ be a k-strictly pseudocontractive mapping such that $\mathrm{Fix}(S) \cap \Omega \cap \mathrm{GMEP} \neq \emptyset$, where $F = I - T$. For fixed $u \in C$ and given $x_0 \in C$ arbitrarily, let the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be generated iteratively by*

$$\begin{aligned} z_n &= T_{r_n}^{(\Theta,\varphi)}((1-r_n)x_n + r_n T x_n), \\ y_n &= \alpha_n u + (1-\alpha_n) P_C(z_n - \lambda_n \nabla f(z_n)), \\ x_{n+1} &= \beta_n x_n + \gamma_n P_C(z_n - \lambda_n \nabla f(z_n)) + \delta_n S y_n, \quad \forall n \geq 0, \end{aligned} \tag{3.67}$$

where $\{\lambda_n\} \subset (0, 2/L]$, $\{r_n\} \subset (0, 1-\widetilde{k}]$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ are four sequences in $[0,1]$ *such that*

(i) $0 < \liminf_{n\to\infty}\lambda_n \leq \limsup_{n\to\infty}\lambda_n < 2/L$ *and* $\lim_{n\to\infty}(\lambda_n - \lambda_{n+1}) = 0$;

(ii) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < 1 - \widetilde{k}$ *and* $\lim_{n\to\infty}(r_n - r_{n+1}) = 0$;

(iii) $\beta_n + \gamma_n + \delta_n = 1$ *and* $(\gamma_n + \delta_n)k \leq \gamma_n < \delta_n$ *for all* $n \geq 0$;

(iv) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$;

(v) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$ *and* $\liminf_{n\to\infty}\delta_n > 0$;

(vi) $\lim_{n\to\infty}(\gamma_{n+1}/(1-\beta_{n+1}) - \gamma_n/(1-\beta_n)) = 0$.

Then, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to the same point $x^* = P_{\mathrm{Fix}(S)\cap\Omega\cap\mathrm{GMEP}} u$.

Proof. Since T is a $\widetilde{k}$-strictly pseudocontractive mapping, the mapping $F = I - T$ is $(1-\widetilde{k})/2$-inverse-strongly monotone. In this case, put $\alpha = (1-\widetilde{k})/2$. Then, we conclude that

$$z_n = T_{r_n}^{(\Theta,\varphi)}(x_n - r_n F x_n) = T_{r_n}^{(\Theta,\varphi)}(x_n - r_n(I-T)x_n) = T_{r_n}^{(\Theta,\varphi)}((1-r_n)x_n + r_n T x_n). \tag{3.68}$$

So, by Theorem 3.2, we obtain the desired result. □

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Research Article

Weighted Estimates for Maximal Commutators of Multilinear Singular Integrals

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This paper is concerned with the pointwise estimates for the sharp function of the maximal multilinear commutators $T^*_{\Sigma b}$ and maximal iterated commutator $T^*_{\Pi b}$, generalized by m-linear operator T and a weighted Lipschitz function b. The $(L^{p_1}(\mu) \times \cdots \times L^{p_m}(\mu), L^r(\mu^{1-r}))$ boundedness and the $(L^{p_1}(\mu) \times \cdots \times L^{p_m}(\mu), L^r(\mu^{1-mr}))$ boundedness are obtained for maximal multilinear commutator $T^*_{\Sigma b}$ and maximal iterated commutator $T^*_{\Pi b}$, respectively.

1. Introduction and Notation

The theory of multilinear Calderón-Zygmund singular integral operators,originated from the work of Coifman and Meyers', has an important role in harmonic analysis. Its study has been attracting a lot of attention in the last few decades. So far, a number of properties for multilinear operators are parallel to those of the classical linear Calderón-Zygmund operators but new interesting phenomena have also been observed. A systematic analysis of many basic properties of such multilinear operators can be found in the articles by Coifman and Meyer [1], Grafakos and Torres [2–4], and Lerner et al. [5]. So we first recall the definition and results of multilinear Calderón-Zygmund operators as well as the corresponding maximal multilinear operators.

Definition 1.1. Let T be a multilinear operator initially defined on the m-fold product of Schwartz space and taking values into the space of tempered distributions:

$$T : \mathcal{S}(\mathbb{R}^n) \times \cdots \times \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n). \tag{1.1}$$

Following [2], we say that T is an m-linear Calderón-Zygmund operator if for some $1 \leq q_j < \infty$, it extends to a bounded multilinear operator from $L^{q_1} \times \cdots \times L^{q_m}$ to L^q, where $1/q = (1/q_1)+\cdots+(1/q_m)$, and if there exists a function K, defined off the diagonal $x = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$, satisfying

$$T(f_1, \dots, f_m)(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \dots, y_m) f_1(y_1) \cdots f_m(y_m) dy_1 \cdots dy_m, \tag{1.2}$$

for all $x \notin \bigcap_{j=1}^m \operatorname{supp} f_j$.

And

$$|K(y_0, y_1, \dots, y_m)| \leq \frac{A}{\left(\sum_{k,l=0}^m |y_k - y_l|\right)^{mn}}, \tag{1.3}$$

$$\left|K(y_0, \dots, y_j, \dots, y_m) - K\left(y_0, \dots, y_j', \dots, y_m\right)\right| \leq \frac{A\left|y_j - y_j'\right|^{\varepsilon}}{\left(\sum_{k,l=0}^m |y_k - y_l|\right)^{mn+\varepsilon}}, \tag{1.4}$$

for some $\varepsilon > 0$ and all $0 \leq j \leq m$, where $|y_j - y_j'| \leq (1/2) \max_{0\leq k\leq m}|y_j - y_k|$.

The maximal multilinear singular integral operator was defined by

$$T^*\left(\vec{f}\right)(x) = \sup_{\delta>0}|T_\delta(f_1, \dots, f_m)(x)|, \tag{1.5}$$

where T_δ is the smooth truncation of T given by

$$T_\delta(f_1, \dots, f_m)(x) = \int_{|x-y_1|^2+\cdots+|x-y_m|^2>\delta^2} K(x, y_1, \dots, y_m) f_1(y_1) \cdots f_m(y_m)\, dy_1 \cdots dy_m. \tag{1.6}$$

As pointed in [4], $T^*(\vec{f})$ is pointwise well defined when $f_j \in L^{q_j}(\mathbb{R}^n)$ with $1 \leq q_j < \infty$.

The study for the multilinear singular integral operator and its maximal operators attracts many authors' attention. For maximal multilinear operator T^*, one can see [4] for details. We list some results for T^* as follows.

Theorem A (see [4]). *Let $1 \leq q_j < \infty$ and q such that $1/q = (1/q_1) + \cdots + (1/q_m)$ and $\omega \in A_{q_1} \cap \cdots \cap A_{q_m}$. Let T be an m-linear Calderón-Zygmund operator. Then there exists a constant $C_{n,q} < \infty$, such that for all $\vec{f} = (f_1, \dots, f_m)$ satisfying*

$$\left\|T^*\left(\vec{f}\right)\right\|_{L^q(\omega)} \leq C_{n,q}(A + W) \prod_{j=1}^m \|f_j\|_{L^{q_j}(\omega)}, \tag{1.7}$$

where W is the norm of T in the mapping T: $L^1 \times \cdots L^1 \rightarrow L^{1/m,\infty}$.

Theorem B (see [4]). *Let T be an m -linear Calderón-Zygmund operator. Then, for all exponents $p, p_1, \ldots, p_m$, satisfying $(1/p_1) + \cdots + (1/p_m) = 1/p$, one has*

$$T^* : L^{p_1} \times \cdots \times L^{p_m} \longrightarrow L^p, \tag{1.8}$$

when $1 < p_1, \ldots, p_m \leq \infty$, one also has

$$T^* : L^{p_1} \times \cdots \times L^{p_m} \longrightarrow L^{p,\infty}, \tag{1.9}$$

when at least one p_j is equal one. In either cases the norm of T^* is controlled by a constant multiple of $A + W$.

Definition 1.2 (see [5] (commutators in the jth entry)). Given a collection of locally integrable function $\vec{b} = (b_1, \ldots, b_m)$, we define the commutators of the m-linear Calderón-Zygmund operator T to be

$$T_{\Sigma b}(f_1, \ldots, f_m) = \sum_{j=1}^{m} T_{b_j}^{j}\left(\vec{f}\right), \tag{1.10}$$

where each term is the commutator of b_j and T in the jth entry of T, that is

$$T_{b_j}^{j}\left(\vec{f}\right) = b_j T(f_1, \ldots, f_j, \ldots, f_m) - T(f_1, \ldots, b_j f_j, \ldots, f_m). \tag{1.11}$$

In [6], the following more general iterated commutators of multilinear Calderón-Zygmund operators and pointwise multiplication with functions in BMO were defined and studied in products of Lebesgue spaces, including strong type and weak end-point estimates with multiple $A_{\vec{p}}$ weights. That is,

$$\begin{aligned} T_{\Pi b}\left(\vec{f}\right)(x) &= \left[b_1, \left[b_2, \ldots, \left[b_{m-1}, [b_m, T]_m\right]_{m-1}\right]_2\right]_1 \\ &= \int_{(\mathbb{R}^n)^m} K(x, y_1, \ldots, y_m) \prod_{j=1}^{m} (b_j(x) - b_j(y_j)) f_1(y_1) \cdots f_m(y_m) dy_1 \cdots dy_m. \end{aligned} \tag{1.12}$$

For the operator $[b, T]$, when T is the Calderón-Zygmund singular integral operator and $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ (the homogeneous Lipschitz spaces), Paluszyński [7] established the (L^p, L^q) boundedness with $1 < p < 1/\beta$ and $1/q = 1/p + \beta/n$. Hu and Gu [8] extended this results to the case: $b \in \mathrm{Lip}_{\beta,\mu}$ with $\mu \in A_1$.

Now we present the definitions of two classes of maximal commutators of multilinear singular integral operators. One is

$$\begin{aligned} &T_{\Sigma b}^{*}\left(\vec{f}\right)(x) \\ &\quad = \sup_{\delta>0} \left| \sum_{j=1}^{m} \int_{\sum_{i=1}^{m} |x-y_i|^2 > \delta^2} K(x, y_1, \ldots, y_m) f_1(y_1) \cdots (b_j(x) - b_j(y_j)) f_j(y_j) \cdots f_m(y_m) d\vec{y} \right|, \end{aligned} \tag{1.13}$$

the other is

$$T^*_{\Pi b}\left(\vec{f}\right)(x) = \sup_{\delta>0}\left|\left[b_1, \left[b_2, \dots, \left[b_{m-1}, \left[b_m, T_\delta\right]_m\right]_{m-1}, \dots\right]_2\right]_1\left(\vec{f}\right)(x)\right|$$
$$= \sup_{\delta>0}\left|\int_{\sum_{i=1}^m |x-y_i|^2>\delta^2} K(x, y_1, \dots, y_m)\prod_{j=1}^m \left(b_j(x) - b_j(y_j)\right) f_1(y_1)\cdots f_m(y_m)\,d\vec{y}\right|, \tag{1.14}$$

where $d\vec{y} = dy_1 \cdots dy_m$. It is obvious to see that

$$T^*_{\Sigma b}\left(\vec{f}\right)(x) \le \sum_{j=1}^m T^{*,j}_{b_j}\left(\vec{f}\right)(x). \tag{1.15}$$

The main purpose of this paper is to extend the results in [8] to the maximal commutators generated by multilinear singular integrals T and $\mathrm{Lip}_{\beta,\mu}$ functions $\vec{b}$.

We can formulate our result as following.

Theorem 1.3. *Assume that the kernel K satisfies* (1.3) *and* (1.4). *Let $1 < q_1, \dots, q_m, q < \infty$ be given numbers satisfying $1/q = (1/q_1) + \cdots + (1/q_m)$. And assume that T maps $L^{q_1}(\mathbb{R}^n) \times \cdots \times L^{q_m}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. For $j \in \{1, \dots, m\}$ and let $1/r = (1/p) - (\beta/n)$, $1 < p < r < \infty$, $0 < \beta < 1$, and $1/p = 1/p_1 + \cdots + 1/p_m$ with $1 < p_i < \infty$, $i = 1, \dots, m$. Given μ such that $\mu \in A_1(\mathbb{R}^n)$ and $b_j \in Lip_{\beta,\mu}(\mathbb{R}^n)$ $(j = 1, \dots, m)$, then one has*

$$\left\|T^{*,j}_{b_j}\left(\vec{f}\right)\right\|_{L^r(\mu^{1-r})} \le C\|b_j\|_{Lip_{\beta,\mu}}\prod_{i=1}^m \|f_i\|_{L^{p_i}(\mu)}, \quad j = 1, \dots, m. \tag{1.16}$$

From (1.15) *and* (1.16), *one can get*

$$\left\|T^*_{\Sigma b}(\vec{f})\right\|_{L^r(\mu^{1-r})} \le C\sum_{j=1}^m \|b_j\|_{Lip_{\beta,\mu}}\prod_{i=1}^m \|f_i\|_{L^{p_i}(\mu)}. \tag{1.17}$$

If $\mu = 1$, one can get the following.

Theorem 1.4. *Assume that the kernel K satisfies* (1.3) *and* (1.4). *Let $1 < q_1, \dots, q_m, q < \infty$ be given numbers satisfying $1/q = (1/q_1) + \cdots + (1/q_m)$. And assume that T maps $L^{q_1}(\mathbb{R}^n) \times \cdots \times L^{q_m}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. For $j \in \{1, \dots, m\}$ and let $1/r = (1/p) - (\beta/n)$, $1 < p < r < \infty$, $0 < \beta < 1$ and $1/p = (1/p_1) + \cdots + (1/p_m)$ with $1 < p_i < \infty$, $i = 1, \dots, m$. Set $b_j \in Lip_\beta(\mathbb{R}^n)(j = 1, \dots, m)$, then one has*

$$\left\|T^{*,j}_{b_j}\left(\vec{f}\right)\right\|_{L^r(\mathbb{R}^n)} \le C\|b_j\|_{Lip_\beta}\prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{R}^n)}, \quad j = 1, \dots, m. \tag{1.18}$$

From (1.15), *one can get*

$$\left\|T^*_{\Sigma b}(\vec{f})\right\|_{L^r(\mathbb{R}^n)} \le C\sum_{j=1}^{m}\|b_j\|_{Lip_\beta}\prod_{i=1}^{m}\|f_i\|_{L^{p_i}(\mathbb{R}^n)}. \tag{1.19}$$

The following theorem states the weighted estimates with two different weights for maximal iterated commutator of multilinear singular integrals.

Theorem 1.5. *Assume that the kernel K satisfies* (1.3) *and* (1.4). *Let $1 < q_1, \dots, q_m, q < \infty$ be given numbers satisfying $1/q = (1/q_1) + \cdots + (1/q_m)$. And assume that T maps $L^{q_1}(\mathbb{R}^n) \times \cdots \times L^{q_m}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. Let $1/r_i = (1/p_i) - (\beta_i/n)$, $1 < p_i < r_i < \infty$, $0 < \beta_i < 1$, $i = 1, \dots, m$ with $1/p = (1/p_1) + \cdots + (1/p_m)$, $1/r = (1/r_1) + \cdots + (1/r_m)$, and $\beta = \beta_1 + \cdots + \beta_m$, $0 < \beta < 1$. Given μ such that $\mu \in A_1(\mathbb{R}^n)$ and $b_i \in Lip_{\beta_i,\mu}(\mathbb{R}^n)(i = 1, \dots, m)$, then one has*

$$\left\|T^*_{\Pi b}\left(\vec{f}\right)\right\|_{L^r(\mu^{1-mr})} \le C\prod_{i=1}^{m}\|b_i\|_{Lip_{\beta_i,\mu}}\|f_i\|_{L^{p_i}(\mu)}. \tag{1.20}$$

Similarly as Theorem 1.4, one also obtains the unweighted estimates of maximal iterated commutators.

Theorem 1.6. *Assume that the kernel K satisfies* (1.3) *and* (1.4). *Let $1 < q_1, \dots, q_m, q < \infty$ be given numbers satisfying $1/q = (1/q_1) + \cdots + (1/q_m)$. And assume that T maps $L^{q_1}(\mathbb{R}^n) \times \cdots \times L^{q_m}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. Let $1/r_i = (1/p_i) - (\beta_i/n)$, $1 < p_i < r_i < \infty$, $0 < \beta_i < 1$, $i = 1, \dots, m$ with $1/p = (1/p_1) + \cdots + (1/p_m)$, $1/r = (1/r_1) + \cdots + (1/r_m)$, and $\beta = \beta_1 + \cdots + \beta_m$, $0 < \beta < 1$. Set $b_i \in Lip_{\beta_i}(\mathbb{R}^n)(i = 1, \dots, m)$, then one has*

$$\left\|T^*_{\Pi b}\left(\vec{f}\right)\right\|_{L^r(\mathbb{R}^n)} \le C\prod_{i=1}^{m}\|b_i\|_{Lip_{\beta_i}}\|f_i\|_{L^{p_i}(\mathbb{R}^n)}. \tag{1.21}$$

The rest of this paper is organized as follows. In Section 2, we recall some standard definitions and lemmas. Section 3 is devoted to the proof of our theorems. Throughout this paper, we use the letter C to denote a positive constant that varies line to line, but it is independent of the essential variable. For any $1 \le p \le \infty$, the p' is always used to denote the dual index such that $(1/p) + (1/p') = 1$.

2. Preliminaries

A nonnegative function μ defined on $\mathbb{R}^n$ is called weight if it is locally integrable. A weight μ is said to belong to the Muckenhoupt class $A_p(\mathbb{R}^n)$, $1 < p < \infty$, if there exists a constant C such that

$$\sup_B\left(\frac{1}{|B|}\int_B \mu(x)dx\right)\left(\frac{1}{|B|}\int_B \mu(x)^{-1/(p-1)}\,dx\right)^{p-1} \le C < \infty, \tag{2.1}$$

for every ball $B \subset \mathbb{R}^n$. A weight μ is said to belong to class $A_1(\mathbb{R}^n)$ if

$$\left(\frac{1}{|B|}\int_B \mu(x)dx\right) \le C \inf_{x\in B}\mu(x), \quad \text{almost all } x \in \mathbb{R}^n, \tag{2.2}$$

for every ball $B \ni x$. The class $A_\infty(\mathbb{R}^n)$ can be characterized as $A_\infty = \bigcup_{1\le p<\infty} A_p$.

Many properties of weights can be found in the book [9], we only collect some of them in the following lemma which will be used bellow.

Lemma 2.1. (i) *$A_p \subset A_q$ for $1 \le p < q \le \infty$;*

(ii) *if $\mu \in A_1$, then $\mu^\theta \in A_1$ for $0 \le \theta \le 1$;*

(iii) *for $1 < p < \infty$, $\mu \in A_p$ if and only if $\mu^{1-p'} \in A_{p'}$.*

A locally integrable function f belongs to the weighted Lipschitz space $\mathrm{Lip}^p_{\beta,\mu}(\mathbb{R}^n)$ for $1 \le p \le \infty$, $0 < \beta < 1$ and $\mu \in A_\infty$ if

$$\sup_{B\ni x}\frac{1}{\mu(B)^{\beta/n}}\left(\frac{1}{B}\int_B \left|f(x) - f_B\right|^p \mu(x)^{1-p}\ dx\right)^{1/p} \le C < \infty. \tag{2.3}$$

The smallest bound C satisfying (1.19) is then taken to be the norm of f denoted by $\|f\|_{\mathrm{Lip}^p_{\beta,\mu}}$. Put $\mathrm{Lip}_{\beta,\mu} = \mathrm{Lip}^1_{\beta,\mu}$.

If $\mu \in A_1$, $b \in \mathrm{Lip}_{\beta,\mu}(0 < \beta < 1)$, from the definition of $\|f\|_{\mathrm{Lip}_{\beta,\mu}}$, it is obvious to see

$$|b_B - b_{2^{k+1}B}| \le Ck\mu(x)\mu\left(B\left(x, 2^{k+1}R\right)\right)^{\beta/n}\|b\|_{\mathrm{Lip}_{\beta,\mu}}, \tag{2.4}$$

where $b_B = (1/|B|)\int_B b(y)\ dy$.

The important properties of the weights are the weighted estimates for the maximal function, the sharp maximal function and their variants. One first recalls the maximal function defined by

$$M(f)(x) = \sup_{B\ni x}\frac{1}{|B|}\int_B \left|f(y)\right|dy, \tag{2.5}$$

It is well known that for $1 < p < \infty$, M maps $L^p(\mu)$ into itself if and only if $\mu \in A_p$, see [10].

The sharp maximal function is defined by

$$M^\#(f)(x) = \sup_{B\ni x}\frac{1}{|B|}\int_B \left|f(x) - f_B\right|dy \approx \sup_{B\ni x}\inf_c\frac{1}{|B|}\int_B \left|f(x) - c\right|dy. \tag{2.6}$$

One also recalls the variants $M_\delta(f)(x) = (M(|f|^\delta)(x))^{1/\delta}$, and $M^\#_\delta(f)(x) = (M^\#(|f|^\delta)(x))^{1/\delta}$. We denote the weighted fractional maximal operators by

$$M_{\alpha,\mu,s}(f)(x) = \sup_{B\ni x}\left(\frac{1}{\mu(B)^{1-(s\alpha/n)}}\int_B \left|f(y)\right|^s \mu(y)dy\right)^{1/s}. \tag{2.7}$$

Recall that $M_\alpha := M_{\alpha,1,1}$ is the weighted fractional maximal operators, that is

$$M_\alpha(f)(x) = \sup_{B \ni x} \left(\frac{1}{\mu(B)^{1-(\alpha/n)}} \int_B |f(y)| \mu(y) dy \right). \tag{2.8}$$

The following lemmas are all from [11].

Lemma 2.2 (Kolmogorov's inequality). *Let (X, μ) be a probability measure space and let $0 < p < q < \infty$ then there exists a constant $C = C_{p,q}$ such that for any measurable function f*

$$\|f\|_{L^p(\mu)} \le C \|f\|_{L^{q,\infty}(\mu)}. \tag{2.9}$$

Lemma 2.3. *Let $0 < p, \delta < \infty$ and $\mu \in A_\infty(\mathbb{R}^n)$, there exists $C > 0$ depending on the $A_\infty(\mathbb{R}^n)$ constant of μ such that*

$$\|M_\delta(f)\|_{L^p(\mu)} \le C \left\| M_\delta^\#(f) \right\|_{L^p(\mu)}, \tag{2.10}$$

for any function f for which the left side of the above inequality is finite.

Lemma 2.4. *Suppose that $0 < \alpha < n$, $0 < s < p < \alpha/n$, $1/q = (1/p) - (\alpha/n)$. If $\mu \in A_\infty(\mathbb{R}^n)$, then there exists a constant $C = C_{p,q}$ such that for any measurable function f*

$$\|M_{\alpha,\mu,s}(f)\|_{L^q(\mu)} \le C \|f\|_{L^p(\mu)}. \tag{2.11}$$

Lemma 2.5. *Suppose that $0 < \alpha < n$, $0 < s < p < \alpha/n$, $1/q = (1/p) - (\alpha/n)$. If $\mu \in A_{1+(q/p')}(\mathbb{R}^n)$, then there exists a constant $C = C_{p,q}$ such that for any measurable function f*

$$\|M_\alpha(f)\|_{L^q(\mu)} \le C \|f\|_{L^p(\mu^{p/q})}. \tag{2.12}$$

3. Two Estimates for Maximal Multilinear Commutators

We will prove our theorems in this section. To begin, we prepare another two iterated operators to control the commutators.

Let $\varphi, \psi \in C^\infty([0, +\infty))$ such that $|\varphi'(t)| \le C/t$, $|\psi'(t)| \le C/t$ and satisfying

$$\chi_{[2,\infty)}(t) \le \varphi(t) \le \chi_{[1,\infty)}(t),\ \chi_{[1,2]}(t) \le \varphi(t) \le \chi_{[1/2,3]}(t). \tag{3.1}$$

We define the maximal operators

$$\Phi^*\left(\vec{f}\right)(x) = \sup_{\eta>0}\left|\int_{(\mathbb{R}^n)^m} K(x,y_1,\ldots,y_m)\varphi\left(\frac{\sqrt{|x-y_1|+\cdots+|x-y_m|}}{\eta}\right)\prod_{i=1}^m f_i(y_i)\,d\vec{y}\right|,$$
$$\Psi^*\left(\vec{f}\right)(x) = \sup_{\eta>0}\left|\int_{(\mathbb{R}^n)^m} K(x,y_1,\ldots,y_m)\psi\left(\frac{\sqrt{|x-y_1|+\cdots+|x-y_m|}}{\eta}\right)\prod_{i=1}^m f_i(y_i)\,d\vec{y}\right|. \tag{3.2}$$

For simplicity, we denote $K_{\varphi,\eta}(x,y_1,\ldots,y_m) = K(x,y_1,\ldots,y_m)\varphi(\sqrt{|x-y_1|+\ldots+|x-y_m|}/\eta)$, $K_{\psi,\eta}(x,y_1,\ldots,y_m) = K(x,y_1,\ldots,y_m)\psi(\sqrt{|x-y_1|+\cdots+|x-y_m|}/\eta)$ and

$$\Phi_\eta\left(\vec{f}\right)(x) = \int_{(\mathbb{R}^n)^m} K_{\varphi,\eta}(x,y_1,\ldots,y_m)\prod_{i=1}^m f_i(y_i)\,d\vec{y},$$
$$\Psi_\eta\left(\vec{f}\right)(x) = \int_{(\mathbb{R}^n)^m} K_{\psi,\eta}(x,y_1,\ldots,y_m)\prod_{i=1}^m f_i(y_i)\,d\vec{y}. \tag{3.3}$$

The kernels of Φ_η and Ψ_η satisfy conditions (1.3) and (1.4) uniformly in η, respectively. And by the same argument in [4], both Φ^* and Ψ^* have the same weighted estimates to T^* that appeared in Theorems A and B.

It is easy to see that $T^*(\vec{f}) \le \Phi^*(\vec{f}) + \Psi^*(\vec{f})$. Moreover,

$$T^*_{\Sigma b}\left(\vec{f}\right) \le \Phi^*_{\Sigma b}\left(\vec{f}\right) + \Psi^*_{\Sigma b}\left(\vec{f}\right), \qquad T^*_{\Pi b}\left(\vec{f}\right) \le \Phi^*_{\Pi b}\left(\vec{f}\right) + \Psi^*_{\Pi b}\left(\vec{f}\right), \tag{3.4}$$

where

$$\begin{aligned}
\Phi^*_{\Sigma b}\left(\vec{f}\right)(x) &= \sup_{\eta>0}\left|\sum_{j=1}^m \int_{(\mathbb{R}^n)^m} K_{\varphi,\eta}(x,y_1,\ldots,y_m) f_1(y_1)\right.\\
&\qquad\left.\cdots(b_j(x)-b_j(y_j))f_j(y_j)\cdots f_m(y_m)\,d\vec{y}\right|\\
&\le \sum_{j=1}^m \sup_{\eta>0}\left|\int_{(\mathbb{R}^n)^m} K_{\varphi,\eta}(x,y_1,\ldots,y_m) f_1(y_1)\right.\\
&\qquad\left.\cdots(b_j(x)-b_j(y_j))f_j(y_j)\cdots f_m(y_m)\,d\vec{y}\right|\\
&= \sum_{j=1}^m \Phi^{*,j}_{b_j}\left(\vec{f}\right)(x),
\end{aligned}$$

$$
\begin{aligned}
\Psi^*_{\Sigma b}\left(\vec{f}\right)(x) &= \sup_{\eta>0}\left|\sum_{j=1}^{m}\int_{(\mathbb{R}^n)^m} K_{\varphi,\eta}(x,y_1,\ldots,y_m)f_1(y_1)\right.\\
&\qquad\left.\cdots\left(b_j(x)-b_j(y_j)\right)f_j(y_j)\cdots f_m(y_m)d\vec{y}\right|\\
&\le \sum_{j=1}^{m}\sup_{\eta>0}\left|\int_{(\mathbb{R}^n)^m} K_{\varphi,\eta}(x,y_1,\ldots,y_m)f_1(y_1)\right.\\
&\qquad\left.\cdots\left(b_j(x)-b_j(y_j)\right)f_j(y_j)\cdots f_m(y_m)d\vec{y}\right|\\
&= \sum_{j=1}^{m}\Psi^{*,j}_{b_j}\left(\vec{f}\right)(x),\\
\Phi^*_{\Pi b}\left(\vec{f}\right)(x) &= \sup_{\eta>0}\left|\left[b_1,\left[b_2,\ldots,\left[b_{m-1},\left[b_m,\Phi_\eta\right]_m\right]_{m-1}\cdots\right]_2\right]_1\left(\vec{f}\right)(x)\right|\\
&= \sup_{\eta>0}\left|\int_{(\mathbb{R}^n)^m} K_{\varphi,\eta}(x,y_1,\ldots,y_m)\prod_{j=1}^{m}\left(b_j(x)-b_j(y_j)\right)\prod_{i=1}^{m}f_i(y_i)d\vec{y}\right|,\\
\Psi^*_{\Pi b}\left(\vec{f}\right)(x) &= \sup_{\eta>0}\left|\left[b_1,\left[b_2,\ldots,\left[b_{m-1},\left[b_m,\Psi_\eta\right]_m\right]_{m-1}\cdots\right]_2\right]_1\left(\vec{f}\right)(x)\right|\\
&= \sup_{\eta>0}\left|\int_{(\mathbb{R}^n)^m} K_{\varphi,\eta}(x,y_1,\ldots,y_m)\prod_{j=1}^{m}\left(b_j(x)-b_j(y_j)\right)\prod_{i=1}^{m}f_i(y_i)d\vec{y}\right|.
\end{aligned}
\tag{3.5}
$$

For simplicity, we will only prove for the case $m = 2$. The arguments for the case $m > 2$ are similar. For the similarity to the two commutators $\Phi^*_{\Sigma b}(\vec{f})$ and $\Psi^*_{\Sigma b}(\vec{f})$, we might as well consider the former. We only consider the former. And we establish the following crucial lemma.

Lemma 3.1. *Let $\mu \in A_1(\mathbb{R}^n)$ and $b_j \in Lip_{\beta,\mu}$, with $0 < \beta < 1$, $j = 1, 2$. Let $0 < \delta < 1/2 < 1 < s < n/\beta$. Then one has*

$$
\begin{aligned}
M^{\#}_{\delta}\left[\Phi^{*,j}_{b_j}(f_1,f_2)\right](x) &\le C\mu(x)\|b_j\|_{Lip_{\beta,\mu}} M_{\beta,\mu,s}(\Phi^*(f_1,f_2))(x) + C\mu(x)\|b_j\|_{Lip_{\beta,\mu}}\\
&\quad\times\left(M_{\beta,\mu,s}(f_1)(x)M(f_2)(x) + M(f_1)(x)M_{\beta,\mu,s}(f_2)(x)\right),
\end{aligned}
\tag{3.6}
$$

$$
\begin{aligned}
M^{\#}_{\delta}\left[\Psi^{*,j}_{b_j}(f_1,f_2)\right](x) &\le C\mu(x)\|b_j\|_{Lip_{\beta,\mu}} M_{\beta,\mu,s}(\Psi^*(f_1,f_2))(x) + C\mu(x)\|b_j\|_{Lip_{\beta,\mu}}\\
&\quad\times\left(M_{\beta,\mu,s}(f_1)(x)M(f_2)(x) + M(f_1)(x)M_{\beta,\mu,s}(f_2)(x)\right).
\end{aligned}
\tag{3.7}
$$

Proof. Without loss of generality, we only consider the case $j=1$ and denote b_1 by b for convenience. Fix $x \in \mathbb{R}^n$ and let $B = B(x,R)$, $\lambda = b_{B^*}$ be the average of b on B^*, where $B^* = B(x,2R)$. To proceed, we decompose $f_i = f_i^0 + f_i^\infty$, where $f_i^0 = f_i\chi_{B^*}$, $i = 1,2$. Let c be a constant to be fixed along the proof.

Since $0 < \delta < 1$, we have

$$\begin{aligned}
\left(\frac{1}{|B|}\int_B \left|\left|\Phi_b^{*,1}(f_1,f_2)(y)\right|^\delta - |c|^\delta\right| dy\right)^{1/\delta} &\le \left(\frac{1}{|B|}\int_B \left|\Phi_b^{*,1}(f_1,f_2)(y) - c\right|^\delta dy\right)^{1/\delta} \\
&\le \left(\frac{1}{|B|}\int_B \left|(b(y)-\lambda)\Phi^*(f_1,f_2)(y)\right|^\delta dy\right)^{1/\delta} \\
&\quad + \left(\frac{1}{|B|}\int_B \left|\Phi^*\left((b-\lambda)f_1^0, f_2^0\right)(y)\right|^\delta dy\right)^{1/\delta} \\
&\quad + \left(\frac{1}{|B|}\int_B \left|\Phi^*\left((b-\lambda)f_1^0, f_2^\infty\right)(y)\right|^\delta dy\right)^{1/\delta} \\
&\quad + \left(\frac{1}{|B|}\int_B \left|\Phi^*\left((b-\lambda)f_1^\infty, f_2^0\right)(y)^\delta\right| dy\right)^{1/\delta} \\
&\quad + \left(\frac{1}{|B|}\int_B \sup_{\eta>0}\left|\Phi_\eta((b-\lambda)f_1^\infty, f_2^\infty)(y) - c\right|^\delta dy\right)^{1/\delta} \\
&:= I + II + III + IV + V.
\end{aligned} \tag{3.8}$$

For I, since $0<\delta<1$, $\mu \in A_1$ and $b \in \mathrm{Lip}_{\beta,\mu}$, by Hölder' inequality, we have

$$\begin{aligned}
I &\le \frac{1}{|B|}\int_B \left|(b(y)-\lambda)\Phi^*(f_1,f_2)(y)\right| dy \\
&\le \frac{C}{|B^*|}\int_{B^*} \left|b(y)-b_{B^*}\right|\mu(y)^{-1/s}\Phi^*(f_1,f_2)(y)\mu(y)^{1/s} dy \\
&\le \frac{C}{|B^*|}\int_{B^*} \left(\left|b(y)-b_{B^*}\right|^{s'}\mu(y)^{1-s'} dy\right)^{1/s'} \left(\int_{B^*} \Phi^*(f_1,f_2)(y)^s \mu(y) dy\right)^{1/s} \\
&\le C\frac{1}{\mu(B^*)^{\beta/n}}\left(\frac{1}{\mu(B^*)}\int_{B^*} \left|b(y)-b_{B^*}\right|^{s'}\mu(y)^{1-s'} dy\right)^{1/s'} \frac{\mu(B^*)^{(\beta/n)+(1/s')}}{|B^*|} \\
&\quad \times \left(\frac{1}{\mu(B^*)^{1-(s\beta/n)}}\int_{B^*} \Phi^*(f_1,f_2)(y)^s\mu(y)dy\right)^{1/s} \mu(B^*)^{(1/s)-(\beta/n)} \\
&\le C\|b\|_{\mathrm{Lip}_{\beta,\mu}} M_{\beta,\mu,s}(\Phi^*(f_1,f_2))(x)\frac{\mu(B^*)}{|B^*|} \\
&\le C\mu(x)\|b\|_{\mathrm{Lip}_{\beta,\mu}} M_{\beta,\mu,s}(\Phi^*(f_1,f_2))(x).
\end{aligned} \tag{3.9}$$

To estimate the second term II. Since $0 < \delta < 1/2$, using Kolmogorov's inequality with $p = \delta$, $q = 1/2$, $X = B$, $\omega = dx/|B|$ and the $(L^1(\mathbb{R}^n) \times L^1(\mathbb{R}^n), L^{1/2,\infty}(\mathbb{R}^n))$-boundedness of Φ^*, we derive that

$$\begin{aligned}
II &\le \left\| \Phi^*\left((b-\lambda) f_1^0, f_2^0\right) \right\|_{L^{1/2,\infty}(dy/|B|)} \\
&\le C\left(\frac{1}{|B|}\int_{B^*} \left|(b(y_1) - b_{B^*}) f_1(y_1)\right| dy_1\right)\left(\frac{1}{|B|}\int_{B^*} \left|f_2(y_2)\right| dy_2\right) \\
&\le C\left(\frac{1}{|B^*|}\int_{B^*} \left|(b(y_1) - b_{B^*}) f_1(y_1)\right| dy_1\right)\left(\frac{1}{|B^*|}\int_{B^*} \left|f_2(y_2)\right| dy_2\right) \\
&\le C\mu(x)\|b\|_{\mathrm{Lip}_{\beta,\mu}} M_{\beta,\mu,s}(f_1)(x) M(f_2)(x),
\end{aligned} \tag{3.10}$$

where we have used the analogous technique in I to get the last inequality.

For the term III, using the fact $|y - y_2| \sim |y_2 - x|$ for any $y_2 \in (B^*)^c$, $y \in B$, and note that $K_{\varphi,\eta}$ satisfies (1.3) uniformly in η, we obtain

$$\begin{aligned}
III &\le \frac{1}{|B|}\int_B \left|\Phi^*\left((b-\lambda) f_1^0, f_2^\infty\right)(y)\right| dy \\
&\le \frac{1}{|B|}\int_B \int_{B^* \times (\mathbb{R}^n \setminus B^*)} \frac{A\left|b(y_1) - \lambda\right| \left|f_1(y_1)\right| \left|f_2(y_2)\right|}{\left(\left|y - y_1\right| + \left|y - y_2\right|\right)^{2n}} dy_1\, dy_2\, dy \\
&\le \int_{B^*} \left|b(y_1) - b_{B^*} \mid f_1(y_1)\right| dy_1 \int_{\mathbb{R}^n \setminus B^*} \frac{\left|f_2(y_2)\right|}{\left|y_2 - x\right|^{2n}} dy_2 \\
&\le C\left(\int_{B^*} \left|b(y_1) - b_{B^*}\right| \left|f_1(y_1)\right| dy_1\right)\left(\sum_{k=1}^{\infty} \int_{2^k B^* \setminus 2^{k-1} B^*} \frac{\left|f_2(y_2)\right|}{\left|y_2 - x\right|^{2n}} dy_2\right) \\
&\le C\frac{1}{|B^*|}\left(\int_{B^*} \left|b(y_1) - b_{B^*}\right| \left|f_1(y_1)\right| dy_1\right)\left(\sum_{k=1}^{\infty} 2^{-kn} \frac{1}{\left|2^k B^*\right|} \int_{2^k B^*} \left|f_2(y_2)\right| dy_2\right) \\
&\le C\mu(x)\|b\|_{\mathrm{Lip}_{\beta,\mu}} M_{\beta,\mu,s}(f_1)(x) M(f_2)(x).
\end{aligned} \tag{3.11}$$

For the term IV, using the fact $|y - y_1| \sim |y_1 - x|$ for any $y_1 \in (B^*)^c$, $y \in B$, and note that $K_{\varphi,\eta}$ satisfies (1.3) uniformly in η, and using (2.4), we obtain

$$\begin{aligned}
IV &\le \frac{1}{|B|}\int_B \left|\Phi^*\left((b-\lambda) f_1^\infty, f_2^0\right)(y)\right| dy \\
&\le \frac{1}{|B|}\int_B \int_{(\mathbb{R}^n \setminus B^*) \times B^*} \frac{A\left|b(y_1) - \lambda\right| \left|f_1(y_1)\right| \left|f_2(y_2)\right|}{\left(\left|y - y_1\right| + \left|y - y_2\right|\right)^{2n}} dy_1\, dy_2\, dy \\
&\le C\int_{\mathbb{R}^n \setminus B^*} \frac{\left|b(y_1) - b_{B^*}\right| \left|f_1(y_1)\right|}{\left|x - y_1\right|^{2n}} dy_1 \int_{B^*} \left|f_2(y_2)\right| dy_2
\end{aligned}$$

$$
\begin{aligned}
&\le C\left(\sum_{k=0}^{\infty}\int_{2^{k+1}B^*\setminus 2^kB^*}\frac{|b(y_1)-b_{B^*}||f_1(y_1)|}{|x-y_1|^{2n}}dy_1\right)\left(\int_{B^*}|f_2(y_2)|dy_2\right)\\
&\le C\left(\sum_{k=0}^{\infty}\frac{1}{|2^kB^*|^2}\int_{2^{k+1}B^*}|b(y_1)-b_{B^*}||f_1(y_1)|dy_1\right)\left(\int_{B^*}|f_2(y_2)|dy_2\right)\\
&\le C\left(\sum_{k=0}^{\infty}2^{-kn}\frac{1}{|2^{k+1}B^*|}\int_{2^{k+1}B^*}|b(y_1)-b_{B^*}||f_1(y_1)|dy_1\right)\left(\frac{1}{|B^*|}\int_{B^*}|f_2(y_2)|dy_2\right)\\
&\le CM(f_2)(x)\sum_{k=0}^{\infty}2^{-kn}\frac{1}{|2^{k+1}B^*|}\left(\int_{2^{k+1}B^*}|b(y_1)-b_{2^{k+1}B^*}||f_1(y_1)|dy_1\right.\\
&\qquad\left.+|b_{B^*}-b_{2^{k+1}B^*}|\int_{2^{k+1}B^*}|f_1(y_1)|dy_1\right)\\
&\le CM(f_2)(x)\sum_{k=0}^{\infty}2^{-kn}\left(\mu(x)\|b\|_{Lip_{\beta,\mu}}M_{\beta,\mu,s}(f_1)(x)\right.\\
&\qquad\left.+k\mu(x)\|b\|_{Lip_{\beta,\mu}}\frac{\mu(B(x,2^{k+1}R))^{\beta/n}}{|2^{k+1}B^*|}\int_{2^{k+1}B^*}|f_1(y_1)|dy_1\right)\\
&\le C\mu(x)\|b\|_{\mathrm{Lip}_{\beta,\mu}}M(f_2)(x)M_{\beta,\mu,s}(f_1)(x).
\end{aligned}
\tag{3.12}
$$

For V, fix the value of c by taking $c=\Phi^*((b-\lambda)f_1^{\infty},f_2^{\infty})(x)$, recall that $K_{\varphi,\eta}$ satisfies (1.4) uniformly in η, then we can obtain

$$
\begin{aligned}
V&\le\frac{1}{|B|}\int_B\left|\Phi^*((b-\lambda)f_1^{\infty},f_2^{\infty})(y)-\Phi^*((b-\lambda)f_1^{\infty},f_2^{\infty})(x)\right|dy\\
&\le\frac{1}{|B|}\int_B\sup_{\eta>0}\left|\Phi_\eta((b-\lambda)f_1^{\infty},f_2^{\infty})(y)-\Phi_\eta((b-\lambda)f_1^{\infty},f_2^{\infty})(x)\right|dy\\
&\le\frac{1}{|B|}\int_B\int_{(\mathbb{R}^n\setminus B^*)^2}\sup_{\eta>0}\left|K_{\varphi,\eta}(y,y_1,y_2)-K_{\varphi,\eta}(x,y_1,y_2)\right||b(y_1-\lambda)||f_1(y_1)|\\
&\qquad\times|f_2(y_2)|dy_1\,dy_2\,dy\\
&\le\frac{C}{|B|}\int_B\int_{(\mathbb{R}^n\setminus B^*)^2}\frac{|x-y|^{\varepsilon}}{(|y-y_1|+|y-y_2|)^{2n+\varepsilon}}|(b(y_1)-b_{B^*})f_1(y_1)f_2(y_2)|dy_1\,dy_2\,dy\\
&\le\frac{C}{|B|}\sum_{k=0}^{\infty}\int_{2^{k+1}B^*\setminus 2^kB^*}\frac{|x-y|^{\varepsilon}}{|y_1-x|^{2n+\varepsilon}}|(b(y_1)-b_{B^*})f_1(y_1)||f_2(y_2)|dy_1\,dy_2\,dy\\
&\le C\left(\sum_{k=0}^{\infty}\frac{|B^*|^{\varepsilon/n}}{|2^kB^*|^{2+\varepsilon/n}}\int_{2^{k+1}B^*}|(b(y_1)-b_{B^*})f_1(y_1)|dy_1\right)\left(\int_{2^{k+1}B^*}|f_2(y_2)|dy_2\right)
\end{aligned}
$$

$$\le C\left(\sum_{k=0}^{\infty}2^{-k\varepsilon}\frac{1}{|2^{k+1}B^*|}\int_{2^{k+1}B^*}|(b(y_1)-b_{B^*})f_1(y_1)|dy_1\right)\left(\frac{1}{|2^{k+1}B^*|}\int_{2^{k+1}B^*}|f_2(y_2)|dy_2\right)$$
$$\le C\mu(x)\|b\|_{\mathrm{Lip}_{\beta,\mu}}M_{\beta,\mu,s}(f_1)(x)M(f_2)(x), \tag{3.13}$$

where in the last inequality, we use the same computation in the IV term.

Consequently, combining the estimates of I, II, III, and V, we conclude the proof of Lemma 3.1. □

Now we are ready to return to prove Theorem 1.3.

Proof. First, by Lemma 2.1, we have that $\mu \in A_1 \subset A_{r'}$, and hence $\mu^{1-r} \in A_r \subset A_\infty$. Then by Lemma 2.3, we obtain

$$\begin{aligned}\left\|\Phi_{b_j}^{*,j}(f_1,f_2)(x)\right\|_{L^r(\mu^{1-r})} &\le \left\|M_\delta\left(\Phi_{b_j}^{*,j}(f_1,f_2)\right)(x)\right\|_{L^r(\mu^{1-r})}\\ &\le \left\|M_\delta^{\#}\left(\Phi_{b_j}^{*,j}(f_1,f_2)\right)(x)\right\|_{L^r(\mu^{1-r})}.\end{aligned} \tag{3.14}$$

For $j = 1, 2$, by Lemma 3.1, we reduce to bound the $\|\cdot\|_{L^r(\mu^{1-r})}$ norm of the right-hand side of (3.6). For the first term, since $1/r = (1/p) - (\beta/n)$ and taking s such that $1 < s < p < n/\beta$, by Lemma 2.4, and Theorem B(ii), we have

$$\begin{aligned}\|\mu M_{\beta,\mu,s}(\Phi^*(f_1,f_2))\|_{L^r(\mu^{1-r})} &= \|M_{\beta,\mu,s}(\Phi^*(f_1,f_2))\|_{L^r(\mu)}\\ &\le C\|\Phi^*(f_1,f_2)\|_{L^p(\mu)}\\ &\le C\|f_1\|_{L^{p_1}(\mu)}\|f_2\|_{L^{p_2}(\mu)}.\end{aligned} \tag{3.15}$$

For the second term, we let $1/r = 1/p_2 + 1/l$, and $1/l = 1/p_1 - \beta/n$. Then by Lemma 2.4 again, together with Hölder's inequality, we obtain

$$\begin{aligned}\|\mu M_{\beta,\mu,s}(f_1)M(f_2)\|_{L^r(\mu^{1-r})} &= \|M_{\beta,\mu,s}(f_1)M(f_2)\|_{L^r(\mu)}\\ &\le \|M_{\beta,\mu,s}(f_1)\|_{L^l(\mu)}\|M(f_2)\|_{L^{p_2}(\mu)}\\ &\le C\|f_1\|_{L^{p_1}}(\mu)\|f_2\|_{L^{p_2}}(\mu).\end{aligned} \tag{3.16}$$

We can obtain that

$$\left\|\Phi_b^{*,1}(f_1,f_2)\right\|_{L^r(\mu^{1-r})} \le C\|b\|_{\mathrm{Lip}_{\beta,\mu}}\|f_1\|_{L^{p_1}(\mu)}\|f_2\|_{L^{p_2}(\mu)}\ . \tag{3.17}$$

Similarly, we have

$$\left\|\Psi_b^{*,1}(f_1,f_2)\right\|_{L^r(\mu^{1-r})} \le C\|b\|_{\mathrm{Lip}_{\beta,\mu}}\|f_1\|_{L^{p_1}(\mu)}\|f_2\|_{L^{p_2}(\mu)}. \tag{3.18}$$

Consequently, by the above arguments, we conclude the proof of Theorem 1.3. □

Similarly as the proof of Lemma 3.1 and that of Theorem 1.3, we only consider the case $m = 2$ and establish the following sharp maximal function for $\Phi^*_{\Pi b}$.

Lemma 3.2. *Let $\mu \in A_1(\mathbb{R}^n)$ and $b_i \in Lip_{\beta_1,\mu}$, $i = 1,2$; $\beta = \beta_1 + \beta_2$, and $0 < \beta < 1$. And let $0 < \delta < 1/3 < 1 < s_i < n/\beta_i$, $i = 1,2$. Then one has*

$$M^{\#}_{\delta}(\Phi^*_{\Pi b}(f_1,f_2))(x) \le \mu(x)^2 \prod_{j=1}^{2} \|b_j\|_{Lip_{\beta_j,\mu}} \times (M_{\beta,\mu,s}(\Phi^*(f_1,f_2))(x) + M_{\beta,\mu,s}(f_1)(x) M_{\beta,\mu,s}(f_2)(x)), \tag{3.19}$$

$$M^{\#}_{\delta}(\Psi^*_{\Pi b}(f_1,f_2))(x) \le \mu(x)^2 \prod_{j=1}^{2} \|b_j\|_{Lip_{\beta_j,\mu}} \times (M_{\beta,\mu,s}(\Phi^*(f_1,f_2))(x) + M_{\beta,\mu,s}(f_1)(x) M_{\beta,\mu,s}(f_2)(x)). \tag{3.20}$$

Proof. Fix $x \in \mathbb{R}^n$ and let $B = B(x,R)$ with $n > 0$. Taking $\lambda_i = (b_i)_{B^*}$, the average of b_i on B^*, $i = 1,2$, where $B^* = B(x,2R)$. Let c be a constant to be fixed along the proof. We split $\Phi^*_{\Pi b}(f_1, f_2)(y)$ in the following way:

$$\begin{aligned}\Phi^*_{\Pi b}(f_1,f_2)(y) = \sup_{\eta>0} \big| &(b_1(y)-\lambda_1)(b_2(y)-\lambda_2)\Phi_\eta(f_1,f_2)(y) \\ &- (b_1(y)-\lambda_1)\Phi_\eta(f_1,(b_2-\lambda_2)f_2)(y) - (b_2(y)-\lambda_2)\Phi_\eta \\ &\times ((b_1-\lambda_1)f_1,f_2)(y) + \Phi_\eta((b_1-\lambda_1)f_1,(b_2-\lambda_2)f_2)(y)\big|.\end{aligned} \tag{3.21}$$

Since $0 < \delta < 1/3$, then we have

$$\begin{aligned}&\left(\frac{1}{|B|}\int_B \left| |\Phi^*_{\Pi b}(f_1,f_2)(y)|^\delta - |c|^\delta \right| dy\right)^{1/\delta} \\ &\quad\le \left(\frac{1}{|B|}\int_B |\Phi^*_{\Pi b}(f_1,f_2)(y) - c|^\delta dy\right)^{1/\delta} \\ &\quad\le \left(\frac{1}{|B|}\int_B |(b_1(y)-\lambda_1)(b_2(y)-\lambda_2)\Phi^*(f_1,f_2)(y)|^\delta dy\right)^{1/\delta} \\ &\qquad + \left(\frac{1}{|B|}\int_B \left(\sup_{\eta>0}|(b_1(y)-\lambda_1)\Phi_\eta(f_1,(b_2-\lambda_2)f_2)(y)|\right)^\delta dy\right)^{1/\delta} \\ &\qquad + \left(\frac{1}{|B|}\int_B \left(\sup_{\eta>0}|(b_2(y)-\lambda_2)\Phi_\eta((b_1-\lambda_1)f_1,f_2)(y)|\right)^\delta dy\right)^{1/\delta} \\ &\qquad + \left(\frac{1}{|B|}\int_B \sup_{\eta>0}|\Phi_\eta((b_1-\lambda_1)f_1,(b_2-\lambda_2)f_2) - c|^\delta dy\right)^{1/\delta} \\ &\quad := U_1 + U_2 + U_3 + U_4.\end{aligned} \tag{3.22}$$

For the term U_1, since $0<\delta<1/3$, and $\beta=\beta_1+\beta_2$, then by Hölder's inequality, we have

$$
\begin{aligned}
U_1 &\le \left(\frac{1}{|B|}\int_B \left|b_1(y)-\lambda_1\right|^{3\delta}\,dy\right)^{1/3\delta}\left(\frac{1}{|B|}\int_B \left|b_2(y)-\lambda_2\right|^{3\delta}dy\right)^{1/3\delta}\\
&\quad\times\left(\frac{1}{|B|}\int_B \left|\Phi^*(f_1,f_2)(y)\right|^{3\delta}dy\right)^{1/3\delta}\\
&\le C\left(\frac{1}{|B^*|}\int_{B^*}\left|b_1(y)-(b_1)_{B^*}\right|dy\right)\left(\frac{1}{|B^*|}\int_{B^*}\left|b_2(y)-(b_2)_{B^*}\right|dy\right)\\
&\quad\times\left(\frac{1}{|B^*|}\int_{B^*}\left|\Phi^*(f_1,f_2)(y)\right|dy\right)\\
&\le C\left(\frac{1}{|B^*|}\int_{B^*}\left|b_1(y)-(b_1)_{B^*}\right|^{s'}\mu(y)^{1-s'}dy\right)^{1/s'}\left(\frac{1}{|B^*|}\int_{B^*}\mu(y)dy\right)^{1/s}\\
&\quad\times\left(\frac{1}{|B^*|}\int_{B^*}\left|b_2(y)-(b_2)_{B^*}\right|^{s'}\mu(y)^{1-s'}\,dy\right)^{1/s'}\left(\frac{1}{|B^*|}\int_{B^*}\mu(y)dy\right)^{1/s}\\
&\quad\times\left(\frac{1}{|B^*|}\int_{B^*}\left|\Phi^*(f_1,f_2)(y)\right|^{s}\mu(y)dy\right)^{1/s}\left(\frac{1}{|B^*|}\int_{B^*}\mu(y)^{-s'/s}dy\right)^{1/s'}\\
&\le C\frac{1}{\mu(B^*)^{\beta_1/n}}\left(\frac{1}{\mu(B^*)}\int_{B^*}\left|b_1(y)-(b_1)_{B^*}\right|^{s'}\mu(y)^{1-s'}dy\right)^{1/s'}\frac{\mu(B^*)^{(\beta_1/n)+1}}{|B^*|}\\
&\quad\times\frac{1}{\mu(B^*)^{\beta_2/n}}\left(\frac{1}{\mu(B^*)}\int_{B^*}\left|b_2(y)-(b_2)_{B^*}\right|^{s'}\mu(y)^{1-s'}dy\right)^{1/s'}\frac{\mu(B^*)^{(\beta_2/n)+1}}{|B^*|}\\
&\quad\times\left(\frac{1}{\mu(B^*)^{1-(s\beta/n)}}\int_{B^*}\Phi^*(f_1,f_2)(y)^{s}\mu(y)dy\right)^{1/s}\frac{\mu(B^*)^{(1/s)-(\beta/n)}\mu(B^*)^{(1/s')-1}}{|B^*|}\\
&\le C\mu(x)^2\|b_1\|_{\mathrm{Lip}_{\beta_1,\mu}}\|b_2\|_{\mathrm{Lip}_{\beta_2,\mu}}M_{\beta,\mu,s}(\Phi^*(f_1,f_2))(x).
\end{aligned}
\tag{3.23}
$$

For the term U_2, noting that $0<\delta<1/3$, we use the facts $1=\delta+(1-\delta)$ and $0<\delta/(1-\delta)<1/2$, then by Hölder's inequality and Komolgorov's inequality (Lemma 2.2) and Theorem B, we have

$$
\begin{aligned}
U_2 &\le C\left(\frac{1}{|B^*|}\int_{B^*}\left|b_1(y)-(b_1)_{B^*}\right|dy\right)\left(\frac{1}{|B^*|}\int_{B^*}\sup_{\eta>0}\left|\Phi_\eta(f_1,(b_2-\lambda_2)f_2)(y)\right|^{\delta/(1-\delta)}dy\right)^{(1-\delta)/\delta}\\
&\le \left(\frac{1}{|B^*|}\int_{B^*}\left|b_1(y)-(b_1)_{B^*}\right|dy\right)\left(\frac{1}{|B^*|}\int_{B^*}\left|\Phi^*(f_1,(b_2-\lambda_2)f_2)(y)\right|^{\delta/(1-\delta)}dy\right)^{1-\delta/\delta}\\
&\le \left(\frac{1}{|B^*|}\int_{B^*}\left|b_1(y)-(b_1)_{B^*}\right|dy\right)\left\|\Phi^*(f_1,(b_2-\lambda_2)f_2)\right\|_{L^{1/2,\infty}(dy/|B^*|)}\\
&\le \left(\frac{1}{|B^*|}\int_{B^*}\left|b_1(y)-(b_1)_{B^*}\right|dy\right)\left(\frac{1}{|B^*|}\int_{B^*}\left|f_1(y_1)\right|dy_1\right)
\end{aligned}
$$

$$
\begin{aligned}
&\times\left(\frac{1}{|B^*|}\int_{B^*}\left|(b_2(y_2)-(b_2)_{B^*})f_2(y_2)\right|dy_2\right)\\
&\leq\left(\frac{1}{|B^*|}\int_{B^*}\left|b_1(y)-(b_1)_{B^*}\right|^{s'}\mu(y)^{1-s'}dy\right)^{1/s'}\left(\int_{B^*}\mu(y)dy\right)^{1/s}\\
&\quad\times\left(\frac{1}{|B^*|}\int_{B^*}\left|f_1(y_1)\right|^{s}\mu(y_1)dy_1\right)^{1/s}\left(\frac{1}{|B^*|}\int_{B^*}\mu(y_1)^{-s'/s}dy_1\right)^{1/s'}\\
&\quad\times\frac{1}{|B^*|}\left(\int_{B^*}\left|b_2(y_2)-(b_2)_{B^*}\right|^{s'}\mu(y_2)^{1-s'}dy_2\right)^{1/s'}\left(\int_{B^*}\left|f_2(y_2)\right|^{s}\mu(y_2)dy_2\right)^{1/s}\\
&\leq\frac{1}{\mu(B^*)^{\beta_1/n}}\left(\frac{1}{\mu(B^*)}\int_{B^*}\left|b(y)-b_{B^*}\right|^{s'}\mu(y)^{1-s'}dy\right)^{1/s'}\frac{\mu(B^*)^{(\beta_1/n)+1}}{|B^*|}\\
&\quad\times\left(\frac{1}{\mu(B^*)^{1-(s\beta/n)}}\int_{B^*}\left|f_1(y_1)\right|^{s}\mu(y_1)dy_1\right)^{1/s}\frac{\mu(B^*)^{(1/s)-(\beta/n)}\mu(B^*)^{-1/s}}{|B^*|}\\
&\quad\times\frac{1}{\mu(B^*)^{\beta_2/n}}\left(\frac{1}{\mu(B^*)}\int_{B^*}\left|b_2(y)-(b_2)_{B^*}\right|^{s'}\mu(y)^{1-s'}dy_2\right)^{1/s'}\frac{\mu(B^*)^{(\beta_2/n)+(1/s')}}{|B^*|}\\
&\quad\times\left(\frac{1}{\mu(B^*)^{1-(s\beta/n)}}\int_{B^*}\left|f_2(y_2)\right|^{s}\mu(y_2)dy_2\right)^{1/s}\mu(B^*)^{(1/s)-(\beta/n)}\\
&\leq C\mu(x)^2\|b_1\|_{\mathrm{Lip}_{\beta_1,\mu}}\|b_2\|_{\mathrm{Lip}_{\beta_2,\mu}}M_{\beta,\mu,s}(f_1)(x)M_{\beta,\mu,s}(f_2)(x).
\end{aligned}
\tag{3.24}
$$

Similarly, for the term U_3, we have

$$
U_3\leq C\mu(x)^2\|b_1\|_{\mathrm{Lip}_{\beta,\mu}}\|b_2\|_{\mathrm{Lip}_{\beta,\mu}}M_{\beta,\mu,s}(f_1)(x)M_{\beta,\mu,s}(f_2)(x). \tag{3.25}
$$

Now we turn to estimate the last term U_4. To proceed, we denote that $f_i=f_i^0+f_i^\infty$, where $f_i^0=f_i\chi_{B^*}$, $i=1,2$. Let $c=c_1+c_2+c_3$, where

$$
\begin{aligned}
c_1&=\Phi_\eta\left((b_1-\lambda_1)f_1^0,(b_2-\lambda_2)f_2^\infty\right)(x),\\
c_2&=\Phi_\eta\left((b_1-\lambda_1)f_1^\infty,(b_2-\lambda_2)f_2^0\right)(x),\\
c_3&=\Phi_\eta\left((b_1-\lambda_1)f_1^\infty,(b_2-\lambda_2)f_2^\infty\right)(x).
\end{aligned}
\tag{3.26}
$$

We split IV in the following way:

$$
U_4\leq U_{41}+U_{42}+U_{43}+U_{44}, \tag{3.27}
$$

where

$$U_{41}=\left(\frac{1}{|B|}\int_B\sup_{\eta>0}\left|\Phi_\eta\left((b_1-\lambda_1)f_1^0,(b_2-\lambda_2)f_2^0\right)(y)\right|^\delta dy\right)^{1/\delta},$$
$$U_{42}=\left(\frac{1}{|B|}\int_B\sup_{\eta>0}\left|\Phi_\eta\left((b_1-\lambda_1)f_1^0,(b_2-\lambda_2)f_2^\infty\right)(y)\right.\right.$$
$$\left.\left.-\Phi_\eta\left((b_1-\lambda_1)f_1^0,(b_2-\lambda_2)f_2^\infty\right)(x)\right|^\delta dy\right)^{1/\delta},$$
$$U_{43}=\left(\frac{1}{|B|}\int_B\sup_{\eta>0}\left|\Phi_\eta\left((b_1-\lambda_1)f_1^\infty,(b_2-\lambda_2)f_2^0\right)(y)\right.\right. \tag{3.28}$$
$$\left.\left.-\Phi_\eta\left((b_1-\lambda_1)f_1^\infty,(b_2-\lambda_2)f_2^0\right)(x)\right|^\delta dy\right)^{1/\delta},$$
$$U_{44}=\left(\frac{1}{|B|}\int_B\sup_{\eta>0}\left|\Phi_\eta((b_1-\lambda_1)f_1^\infty,(b_2-\lambda_2)f_2^\infty)(y)\right.\right.$$
$$\left.\left.-\Phi_\eta((b_1-\lambda_1)f_1^\infty,(b_2-\lambda_2)f_2^\infty)(x)\right|^\delta dy\right)^{1/\delta}.$$

For the term U_{41}, we choose $1<p_0<1/2\delta$ and use Kolmogorov's inequality and Theorem B, then we use the same computation as U_2 to deduce that

$$\begin{aligned}
U_{41}&\leq\left(\frac{1}{|B|}\int_B\left|\Phi^*\left((b_1-\lambda_1)f_1^0,(b_2-\lambda_2)f_2^0\right)(y)\right|^{p_0\delta}dy\right)^{1/p_0\delta}\\
&\leq\left\|\Phi^*\left((b_1-\lambda_1)f_1^0,(b_2-\lambda_2)f_2^0\right)\right\|_{L^{1/2,\infty}(dy/|B|)}\\
&\leq\left(\frac{1}{|B|}\int_B\left|(b_1(y_1)-\lambda_1)f_1^0(y_1)\right|dy_1\right)\left(\frac{1}{|B|}\int_B\left|(b_2(y_2)-\lambda_2)f_2^0(y_2)\right|dy_2\right)\\
&\leq C\left(\frac{1}{|B^*|}\int_{B^*}|(b_1(y_1)-(b_1)_{B^*})f_1(y_1)|dy_1\right)\left(\frac{1}{|B^*|}\int_{B^*}|(b_2(y_2)-(b_2)_{B^*})f_2(y_2)|dy_2\right)\\
&\leq C\mu(x)^2\|b_1\|_{\mathrm{Lip}_{\beta_1,\mu}}\|b_2\|_{\mathrm{Lip}_{\beta_2,\mu}}M_{\beta,\mu,s}(f_1)(x)M_{\beta,\mu,s}(f_2)(x).
\end{aligned}\tag{3.29}$$

For U_{42}, by the fact $|y-y_2|\sim|y_2-x|$, for any $y_2\in(B^*)^c$, $y\in B$, and note that $K_{\varphi,\eta}$ satisfies (1.4) uniformly in η, then we get that

$$\begin{aligned}
U_{42}&\leq\left(\frac{1}{|B|}\int_B\sup_{\eta>0}\left|\Phi_\eta\left((b_1-\lambda_1)f_1^0,(b_2-\lambda_2)f_2^\infty\right)(y)\right.\right.\\
&\qquad\left.\left.-\Phi_\eta\left((b_1-\lambda_1)f_1^0,(b_2-\lambda_2)f_2^\infty\right)(x)\right|dy\right)\\
&\leq\frac{C}{|B|}\int_B\left(\int_{B^*}|(b_1(y_1)-\lambda_1)f_1(y_1)|dy_1\int_{(B^*)^c}\frac{|y-x|^\varepsilon|b_2(y_2)-\lambda_2||f_2(y_2)|}{(|y-y_1|+|y-y_2|)^{2n+\varepsilon}}dy_2\right)dy
\end{aligned}$$

$$\begin{aligned}
&\leq \left(\int_{B^*}\left|(b_1(y_1)-(b_1)_{B^*})f_1(y_1)\right|dy_1\right)\\
&\quad\times\left(\sum_{k=0}^{\infty}\int_{2^{k+1}B^*\setminus 2^kB^*}\frac{|y-x|^{\varepsilon}\left|b_2(y_2)-(b_2)_{B^*}\right|\left|f_2(y_2)\right|}{|x-y_2|^{2n+\varepsilon}}dy_2\right)\\
&\leq \left(\int_{B^*}\left|(b_1(y_1)-(b_1)_{B^*})f_1(y_1)\right|dy_1\right)\\
&\quad\times\left(\sum_{k=0}^{\infty}\frac{|B^*|^{\varepsilon/n}}{|2^kB^*|^{2+\varepsilon/n}}\int_{2^{k+1}B^*}\left|b_2(y_2)-(b_2)_{B^*}\right|\left|f_2(y_2)\right|dy_2\right)\\
&\leq \left(\frac{1}{|B^*|}\int_{B^*}\left|(b_1(y_1)-(b_1)_{B^*})f_1(y_1)\right|dy_1\right)\\
&\quad\times\left(\sum_{k=0}^{\infty}2^{-k\varepsilon}\frac{1}{|2^{k+1}B^*|}\int_{2^{k+1}B^*}\left|b_2(y_2)-(b_2)_{B^*}\right|\left|f_2(y_2)\right|dy_2\right)\\
&\leq C\mu(x)\|b_1\|_{\mathrm{Lip}_{\beta_1,\mu}}M_{\beta,\mu,s}(f_1)(x)\sum_{k=1}^{\infty}2^{-k\varepsilon}\frac{1}{|2^{k+1}B^*|}\left(\int_{2^{k+1}B^*}\left|b_2(y_2)-(b_2)_{2^{k+1}B^*}\right|\left|f_2(y_2)\right|dy_2\right.\\
&\qquad\left.+\left|(b_2)_{B^*}-(b_2)_{2^{k+1}B^*}\right|\int_{2^{k+1}B^*}\left|f_2(y_2)\right|dy_2\right)\\
&\leq C\mu(x)^2\|b_1\|_{\mathrm{Lip}_{\beta_1,\mu}}\|b_2\|_{\mathrm{Lip}_{\beta_2,\mu}}M_{\beta,\mu,s}(f_1)(x)M_{\beta,\mu,s}(f_2)(x),
\end{aligned}\tag{3.30}$$

where we have used the same computation of IV to gain the last inequality.

Similarly as U_{42}, we can get the estimates for U_{43},

$$U_{43}\leq C\mu(x)^2\|b_1\|_{\mathrm{Lip}_{\beta,\mu}}\|b_2\|_{\mathrm{Lip}_{\beta,\mu}}M_{\beta,\mu,s}(f_1)(x)M_{\beta,\mu,s}(f_2)(x).\tag{3.31}$$

Now we turn to U_{44}, by the fact $|y-y_1|\sim|y_1-x|$, $|y-y_2|\sim|y_2-x|$ for any $y_1,y_2\in(B^*)^c$, $y\in B$, and recalling that $K_{\varphi,\eta}$ satisfies (1.4) uniformly in η, then we can obtain

$$\begin{aligned}
&\left|\Phi_\eta\left((b_1-\lambda_1)f_1^{\infty},(b_2-\lambda_2)f_2^{\infty}\right)(y)-\Phi_\eta\left((b_1-\lambda_1)f_1^{\infty},(b_2-\lambda_2)f_2^{\infty}\right)(x)\right|\\
&\quad\leq C\sum_{k=0}^{\infty}\int_{(2^{k+1}B^*\setminus 2^kB^*)^2}\frac{|x-y|^{\varepsilon}}{\left(|y-y_1|+|y-y_2|\right)^{2n+\varepsilon}}\\
&\qquad\qquad\left|(b_1(y_1)-\lambda_1)f_1(y_1)(b_2(y-2)-\lambda_2)f_2(y_2)\right|dy_1\,dy_2\\
&\quad\leq C\sum_{k=0}^{\infty}\int_{(2^{k+1}B^*\setminus 2^kB^*)^2}\frac{|x-y|^{\varepsilon}}{|y-y_1|^{2n+\varepsilon}}\left|(b_1(y_1)-\lambda_1)f_1(y_1)(b_2(y-2)-\lambda_2)f_2(y_2)\right|dy_1\,dy_2\\
&\quad\leq C\sum_{k=0}^{\infty}\frac{|B^*|^{\varepsilon/n}}{|2^kB^*|^{2+\varepsilon/n}}\int_{2^{k+1}B^*}\left|(b_1(y_1)-\lambda_1)f_1(y_1)\right|\,dy_1\int_{2^{k+1}B^*}\left|(b_2(y_2)-\lambda_2)f_2(y_2)\right|dy_2
\end{aligned}$$

$$\leq C\sum_{k=0}^{\infty}2^{-k\varepsilon}\left(\frac{1}{|2^{k+1}B^*|}\int_{2^{k+1}B^*}\left|(b_1(y_1)-\lambda_1)f_1(y_1)\right|dy_1\right)$$
$$\times\left(\frac{1}{|2^{k+1}B^*|}\int_{2^{k+1}B^*}\left|(b_2(y_2)-\lambda_2)f_2(y_2)\right|dy_2\right)$$
$$\leq C\mu(x)^2\|b_1\|_{\mathrm{Lip}_{\beta_1,\mu}}\|b_2\|_{\mathrm{Lip}_{\beta_2,\mu}}M_{\beta,\mu,s}(f_1)(x)M_{\beta,\mu,s}(f_2)(x). \tag{3.32}$$

Therefore,

$$U_{44}\leq\frac{1}{|B|}\int_B\sup_{\eta>0}\left|\Phi_\eta\left((b_1-\lambda_1)f_1^\infty,(b_2-\lambda_2)f_2^\infty\right)(y)-\Phi_\eta\left((b_1-\lambda_1)f_1^\infty,(b_2-\lambda_2)f_2^\infty\right)(x)\right|dy$$
$$\leq C\mu(x)^2\|b_1\|_{\mathrm{Lip}_{\beta_1,\mu}}\|b_2\|_{\mathrm{Lip}_{\beta_2,\mu}}M_{\beta,\mu,s}(f_1)(x)M_{\beta,\mu,s}(f_2)(x). \tag{3.33}$$

Consequently, the estimate for U_1, together with those of U_2, U_3, and U_4, can conclude the proof of Lemma 3.2. □

Now we return to prove Theorem 1.5.

Proof. Similarly as the proof of Theorem 1.3,

$$\left\|\Phi^*_{\Pi b}(f_1,f_2)\right\|_{L^r(\mu^{1-2r})}\leq\left\|M_\delta\left(\Phi^*_{\Pi b}(f_1,f_2)\right)\right\|_{L^r(\mu^{1-2r})}\leq\left\|M^\#_\delta\left(\Phi^*_{\Pi b}(f_1,f_2)\right)\right\|_{L^r(\mu^{1-2r})}. \tag{3.34}$$

We reduce to bound the $\|\cdot\|_{L^r(\mu^{1-r})}$ norm of the right-hand side of (3.19). We estimate each term as follows. For the first term, since $1/r=(1/p)-(\beta/n)$ and choosing s such that $1<s<p<n/\beta$, by Lemma 2.4 and Theorem A and observe that $\mu\in A_1$, we obtain

$$\begin{aligned}\left\|\mu^2M_{\beta,\mu,s}\left(\Phi^*(f_1,f_2)\right)\right\|_{L^r(\mu^{1-2r})}&=\left\|M_{\beta,\mu,s}\left(\Phi^*(f_1,f_2)\right)\right\|_{L^r(\mu)}\\&\leq C\left\|\Phi^*(f_1,f_2)\right\|_{L^p(\mu)}\\&\leq C\left\|f_1\right\|_{L^{p_1}(\mu)}\left\|f_2\right\|_{L^{p_2}(\mu)}.\end{aligned} \tag{3.35}$$

For the second term, since $1/r=(1/r_1)+(1/r_2)$, by Hölder's inequality and Lemma 2.4, we get

$$\begin{aligned}\left\|\mu^2M_{\beta,\mu,s}(f_1)M_{\beta,\mu,s}(f_2)\right\|_{L^r(\mu^{1-2r})}&=\left\|M_{\beta,\mu,s}(f_1)M_{\beta,\mu,s}(f_2)\right\|_{L^r(\mu)}\\&\leq\left\|M_{\beta,\mu,s}(f_1)\right\|_{L^{r_1}(\mu)}\left\|M_{\beta,\mu,s}(f_2)\right\|_{L^{r_2}(\mu)}\\&\leq C\left\|f_1\right\|_{L^{p_1}(\mu)}\left\|f_2\right\|_{L^{p_2}(\mu)}.\end{aligned} \tag{3.36}$$

Similarly, we also have

$$\left\|\Psi^{*}_{\Pi b}(f_1, f_2)\right\|_{L^{r}(\mu^{1-2r})} \leq C\left\|f_1\right\|_{L^{p_1}(\mu)}\left\|f_2\right\|_{L^{p_2}(\mu)}. \tag{3.37}$$

This estimate together with that for $\Phi^{*}_{\Pi b}(f_1, f_2)$ finishes the proof of Theorem 1.5. □

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