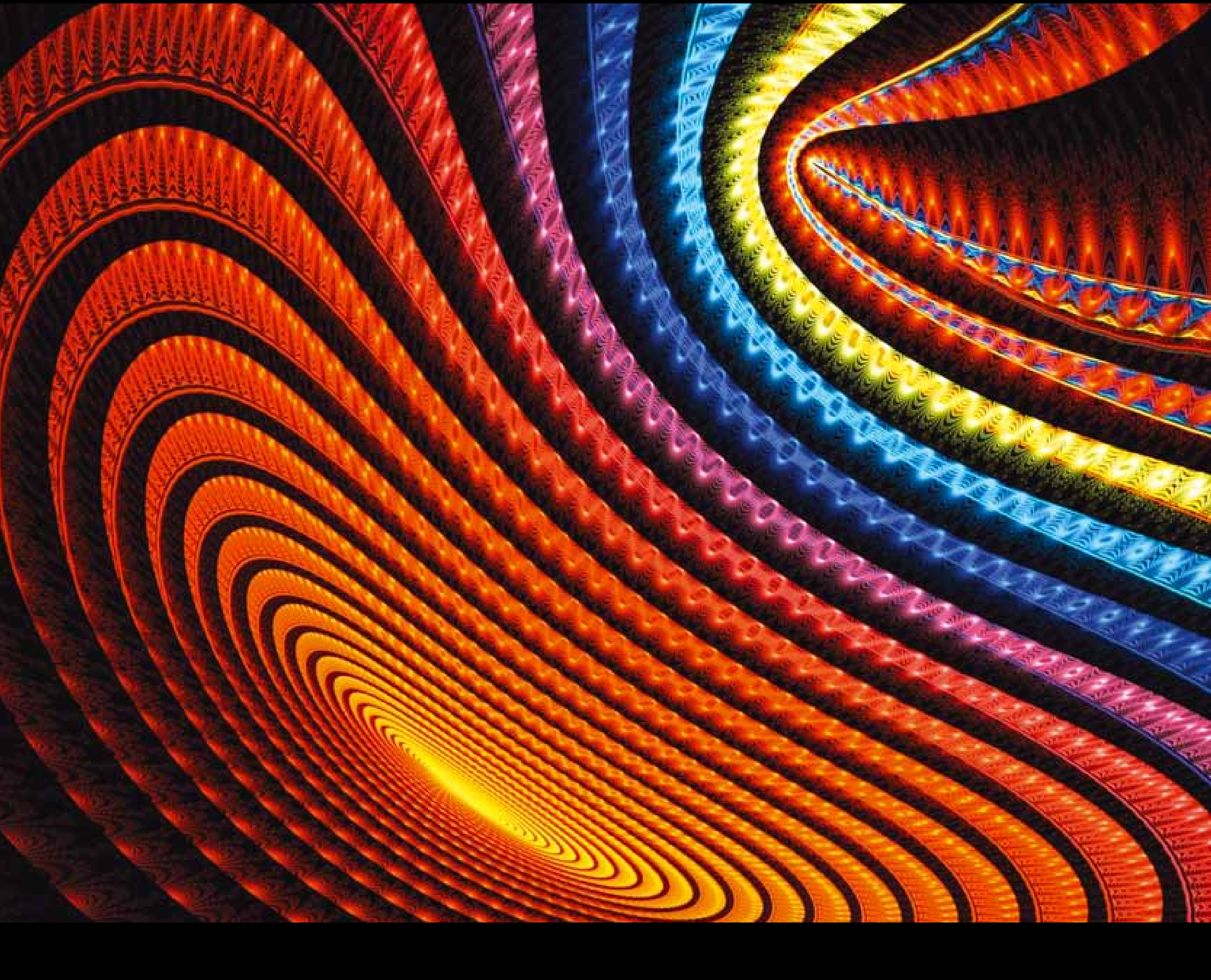
Abstract and Applied Analysis

Trends in Classical Analysis, Geometric Function Theory, and Geometry of Conformal Invariants

Guest Editors: Árpád Baricz, Saminathan Ponnusamy, Matti Vuorinen, and Karl Joachim Wirths



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Editorial Board

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Editorial

Trends in Classical Analysis, Geometric Function Theory, and Geometry of Conformal Invariants

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The present special issue of the Journal of *Abstract and Applied Analysis* is devoted to *trends in classical analysis, geometric function theory, and geometry of conformal mappings*. In the sense of the title of this journal, we wanted to present a spectrum of research themes reaching from applied analysis to pure analysis and from applications of analysis in geometry to applications in differential equations and integral equations.

Further, our aim was to find articles on classical function theory as well as on its generalizations in several directions.

One additional aspect, that was paid attention to, is the tendency of mathematics to use computers to solve problems by new and effective algorithms. In detail, we addressed to the following themes.

That we did not forget classical pure analysis is proved by an article that considers harmonic functions on Riemann manifolds and by an article on geometry and topology of Banach spaces.

The classical geometric function theory is represented by an article on univalence criterions associated with the nth derivative.

The relationships between conformal mappings and integral equations are in the scope of two articles, where algorithms are proved to compute the mappings of unbounded multiply connected as well as bounded multiply connected regions onto slit regions.

Other old themes of classical function theory are entire functions for which we publish a paper on uniqueness theorems for monomials of entire functions.

Concerning the generalizations of classical analysis, we incorporated a paper on the stability of solutions of fractional differential equations and two papers on harmonic mappings, specially one on the general theory of log-harmonic mappings and one on certain classes of harmonic mappings defined by convolutions.

The papers on the generalizations of holomorphic functions are completed by a longer article on the distribution of zeros and poles of the rational approximants of a non-holomorphic function on an interval.

The aspect of new algorithms is addressed in an article on Hermite interpolation using Möbius transformations of planar Pythagorean-hodograph cubics and in a paper where algorithms to calculate inverse Z-transforms on the unit disc by number-theoretical methods are proved.

We hope that in this broad variety of analytic themes many researchers can find something interesting and new.

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Research Article

On Certain Classes of Biharmonic Mappings Defined by Convolution

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We introduce a class of complex-valued biharmonic mappings, denoted by $BH^0(\phi_k;\sigma,a,b)$, together with its subclass $TBH^0(\phi_k;\sigma,a,b)$, and then generalize the discussions in Ali et al. (2010) to the setting of $BH^0(\phi_k;\sigma,a,b)$ and $TBH^0(\phi_k;\sigma,a,b)$ in a unified way.

1. Introduction

A four times continuously differentiable complex-valued function $F = u + iv$ in a domain $D \subset \mathbb{C}$ is *biharmonic* if ΔF, the Laplacian of F, is harmonic in D. Note that ΔF is *harmonic* in D if F satisfies the biharmonic equation $\Delta(\Delta F) = 0$ in D, where Δ represents the Laplacian operator

$$\Delta = 4\frac{\partial^2}{\partial z \partial \overline{z}} := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{1.1}$$

It is known that, when D is simply connected, a mapping F is biharmonic if and only if F has the following representation:

$$F(z) = \sum_{k=1}^{2} |z|^{2(k-1)} G_k(z), \tag{1.2}$$

where G_k are complex-valued harmonic mappings in D for $k \in \{1,2\}$ (cf. [1–6]). Also it is known that G_k can be expressed as the form

$$G_k = h_k + \overline{g_k} \tag{1.3}$$

for $k \in \{1,2\}$, where all h_k and g_k are analytic in D (cf. [7, 8]).

Biharmonic mappings arise in a lot of physical situations, particularly, in fluid dynamics and elasticity problems, and have many important applications in engineering and biology (cf. [9–11]). However, the investigation of biharmonic mappings in the context of geometric function theory is a recent one (cf. [1–6]).

In this paper, we consider the biharmonic mappings in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Let $BH^0(\mathbb{D})$ denote the set of all biharmonic mappings F in $\mathbb{D}$ with the following form:

$$\begin{aligned} F(z) &= \sum_{k=1}^{2} |z|^{2(k-1)} \left(h_k(z) + \overline{g_k(z)} \right) \\ &= \sum_{k=1}^{2} |z|^{2(k-1)} \left(\sum_{j=1}^{\infty} a_{k,j} z^j + \sum_{j=1}^{\infty} \overline{b_{k,j} z^j} \right), \end{aligned} \tag{1.4}$$

with $a_{1,1} = 1$, $a_{2,1} = 0$, $b_{1,1} = 0$, and $b_{2,1} = 0$.

In [12], Qiao and Wang proved that for each $F \in BH^0(\mathbb{D})$, if the coefficients of F satisfy the following inequality:

$$\sum_{k=1}^{2} \sum_{j=1}^{\infty} (2(k-1) + j) \left(|a_{k,j}| + |b_{k,j}| \right) \le 2, \tag{1.5}$$

then F is sense preserving, univalent, and starlike in $\mathbb{D}$ (see [12, Theorems 3.1 and 3.2]).

Let S_H denote the set of all univalent harmonic mappings f in $\mathbb{D}$, where

$$f(z) = h(z) + \overline{g(z)} = z + \sum_{j=2}^{\infty} a_j z^j + \sum_{j=1}^{\infty} \overline{b_j z^j}, \tag{1.6}$$

with $|b_1| < 1$. In particular, we use S_H^0 to denote the set of all mappings in S_H with $b_1 = 0$. Obviously, $S_H^0 \subset BH^0(\mathbb{D})$.

In 1984, Clunie and Sheil-Small [7] discussed the class S_H and its geometric subclasses. Since then, there have been many related papers on S_H and its subclasses (see [13, 14] and the references therein). In 1999, Jahangiri [15] studied the class $S_H^*(\alpha)$ consisting of all mappings $f = h + \overline{g}$ such that h and g are of the form

$$h(z) = z - \sum_{j=2}^{\infty} |a_j| z^j, \qquad g(z) = \sum_{j=1}^{\infty} |b_j| z^j \tag{1.7}$$

and satisfy the condition

$$\frac{\partial}{\partial\theta}\left(\arg f\left(re^{i\theta}\right)\right) = \operatorname{Re}\left\{\frac{zh' - \overline{zg'}}{h+\overline{g}}\right\} > \alpha \tag{1.8}$$

in $\mathbb{D}$, where $0 \le \alpha < 1$.

For two analytic functions f_1 and f_2, if

$$f_1(z) = \sum_{j=1}^{\infty} a_j z^j, \qquad f_2(z) = \sum_{j=1}^{\infty} A_j z^j, \tag{1.9}$$

then the *convolution* of f_1 and f_2 is defined by

$$(f_1 * f_2)(z) = f_1(z) * f_2(z) = \sum_{j=1}^{\infty} a_j A_j z^j. \tag{1.10}$$

By using the convolution, in [16], Ali et al. introduced the class $S_H^0(\phi,\sigma,\alpha)$ of harmonic mappings in the form of (1.6) such that

$$\operatorname{Re}\left\{\frac{z(h*\phi)'(z) - \sigma\overline{z(g*\phi)'(z)}}{(h*\phi)(z) + \sigma\overline{(g*\phi)(z)}}\right\} > \alpha \tag{1.11}$$

and the class $SP_H^0(\phi,\sigma,\alpha)$ such that

$$\operatorname{Re}\left\{\left(1+e^{i\gamma}\right)\frac{z(h*\phi)'(z) - \sigma\overline{z(g*\phi)'(z)}}{(h*\phi)(z) + \sigma\overline{(g*\phi)(z)}} - e^{i\gamma}\right\} > \alpha, \tag{1.12}$$

where $\sigma \in \mathbb{R}$ and $\alpha \in [0,1)$ are constants, $\gamma \in \mathbb{R}$ and $\phi(z) = z + \sum_{n=2}^{\infty}\phi_n z^n$ is analytic in $\mathbb{D}$.

Now we consider a class of biharmonic mappings, denoted by $BH^0(\phi_k;\sigma,a,b)$, as follows: $F \in BH^0(\mathbb{D})$ with the form (1.4) is said to be in $BH^0(\phi_k;\sigma,a,b)$ if and only if

$$\operatorname{Re}\left\{a\frac{\Phi(z)}{\Psi(z)} - b\right\} > 0, \tag{1.13}$$

where

$$\begin{aligned}\Phi(z) &= z\left[\left(\sum_{k=1}^{2}|z|^{2(k-1)}(h_k*\phi_k)(z)\right)' + \sigma\left(\sum_{k=1}^{2}|z|^{2(k-1)}\overline{(g_k*\phi_k)(z)}\right)'\right],\\ \Psi(z) &= z'\sum_{k=1}^{2}|z|^{2(k-1)}\left((h_k*\phi_k)(z) + \sigma\overline{(g_k*\phi_k)(z)}\right),\end{aligned} \tag{1.14}$$

$\phi_k(z) = z + \sum_{j=2}^{\infty} \phi_{k,j} z^j$ are analytic in $\mathbb{D}$ for $k \in \{1,2\}$, $\sigma \in \mathbb{R}$ is a constant, $a = p + \rho e^{i\gamma}$, $b = q + \rho e^{i\gamma}$, $p,\ q,\ \rho \in [0,+\infty)$ are constants with $a - b > 0$, $\gamma \in \mathbb{R}$, and $z = re^{i\theta}$. Here and in what follows, "$'$" always stands for "$\partial/\partial\theta$".

Obviously, if $\phi_2 = 0$, $a = 1$ and $b = \alpha$, then $BH^0(\phi_k;\sigma,a,b)$ reduces to $S_H^0(\phi,\sigma,\alpha)$, and if $\phi_2 = 0$, $a = 1 + e^{i\gamma}$ and $b = \alpha + e^{i\gamma}$, then $BH^0(\phi_k;\sigma,a,b)$ reduces to $SP_H^0(\phi,\sigma,\alpha)$.

Further, we use $TBH^0(\phi_k;\sigma,a,b)$ to denote the class consisting of all mappings F in $BH^0(\phi_k;\sigma,a,b)$ with the form

$$F(z) = \sum_{k=1}^{2} |z|^{2(k-1)} \left(h_k(z) + \overline{g_k(z)} \right), \tag{1.15}$$

where

$$\begin{aligned} h_k(z) &= a_{k,1} z - \sum_{j=2}^{\infty} a_{k,j} z^j, \quad a_{k,j} \geq 0,\ a_{1,1} = 1,\ a_{2,1} = 0, \\ g_k(z) &= \sigma \sum_{j=1}^{\infty} b_{k,j} z^j, \quad b_{k,j} \geq 0,\ b_{1,1} = b_{2,1} = 0. \end{aligned} \tag{1.16}$$

The object of this paper is to generalize the discussions in [16] to the setting of $BH^0(\phi_k;\sigma,a,b)$ and $TBH^0(\phi_k;\sigma,a,b)$ in a unified way. The organization of this paper is as follows. In Section 2, we get a convolution characterization for $BH^0(\phi_k;\sigma,a,b)$. As a corollary, we derive a sufficient coefficient condition for mappings in $BH^0(\mathbb{D})$ to belong to $BH^0(\phi_k;\sigma,a,b)$. The main results are Theorems 2.1 and 2.3. In Section 3, first, we get a coefficient characterization for $TBH^0(\phi_k;\sigma,a,b)$, and then find the extreme points of $TBH^0(\phi_k;\sigma,a,b)$. The corresponding results are Theorems 3.1 and 3.6.

2. A Convolution Characterization

We begin with a convolution characterization for $BH^0(\phi_k;\sigma,a,b)$.

Theorem 2.1. *Let $F \in BH^0(\mathbb{D})$. Then $F \in BH^0(\phi_k;\sigma,a,b)$ if and only if*

$$\begin{aligned} &\sum_{k=1}^{2} |z|^{2(k-1)} (h_k * \phi_k)(z) * \left(\frac{z + ((ax - a + 2b)/(2a-2b))z^2}{(1-z)^2} \right) \\ &\quad - \sigma \sum_{k=1}^{2} |z|^{2(k-1)} \overline{(g_k * \phi_k)(z)} * \left(\frac{((ax+b)/(a-b))\overline{z} - ((ax - a + 2b)/(2a-2b))\overline{z}^2}{(1-\overline{z})^2} \right) \neq 0, \end{aligned} \tag{2.1}$$

for all $z \in \mathbb{D} \setminus \{0\}$ and all $x \in \mathbb{C}$ with $|x| = 1$.

Proof. By definition, a necessary and sufficient condition for a mapping F in $BH^0(\mathbb{D})$ to be in $BH^0(\phi_k;\sigma,a,b)$ is given by (1.13). Let

$$G(z) = \frac{1}{a-b}\left(a\frac{\Phi(z)}{\Psi(z)} - b\right). \tag{2.2}$$

Then $G(0) = 1$, and so the condition (1.13) is equivalent to

$$G(z) \neq \frac{x-1}{x+1}, \tag{2.3}$$

for all $z \in \mathbb{D} \setminus \{0\}$ and all $x \in \mathbb{C}$ with $|x| = 1$ and $x \neq -1$. Obviously, (2.3) holds if and only if

$$a(x+1)\Phi(z) - b(x+1)\Psi(z) - (a-b)(x-1)\Psi(z) \neq 0. \tag{2.4}$$

Straightforward computations show that

$$\begin{aligned}
&a(x+1)\Phi(z) - b(x+1)\Psi(z) - (a-b)(x-1)\Psi(z)\\
&= a(x+1)z'\sum_{k=1}^{2}|z|^{2(k-1)}\left(z + \sum_{j=2}^{\infty} j a_{k,j}\phi_{k,j}z^j - \sigma\sum_{j=2}^{\infty} j\overline{b_{k,j}\phi_{k,j}z^j}\right)\\
&\quad - (ax-a+2b)z'\sum_{k=1}^{2}|z|^{2(k-1)}\left(z + \sum_{j=2}^{\infty} a_{k,j}\phi_{k,j}z^j + \sigma\sum_{j=2}^{\infty}\overline{b_{k,j}\phi_{k,j}z^j}\right)\\
&= z'\sum_{k=1}^{2}|z|^{2(k-1)}(h_k * \phi_k)(z) * \left(\frac{2(a-b)z + (ax-a+2b)z^2}{(1-z)^2}\right)\\
&\quad - \sigma z'\sum_{k=1}^{2}|z|^{2(k-1)}\overline{(g_k * \phi_k)(z)} * \left(\frac{2(ax+b)\overline{z} - (ax-a+2b)\overline{z}^2}{(1-\overline{z})^2}\right),
\end{aligned} \tag{2.5}$$

from which we see that (2.3) is true if and only if so is (2.1). The proof is complete. □

Remark 2.2. If $h_2 = g_2 = 0$, $a = 1$ and $b = \alpha$, then Theorem 2.1 coincides with Theorem 2.1 in [16], and if $h_2 = g_2 = 0$, $a = 1 + e^{i\gamma}$, and $b = \alpha + e^{i\gamma}$, then Theorem 2.1 coincides with Theorem 2.3 in [16].

As an application of Theorem 2.1, we derive a sufficient condition for mappings in $BH^0(\mathbb{D})$ to be in $BH^0(\phi_k;\sigma,a,b)$ in terms of their coefficients.

Theorem 2.3. *Let $F \in BH^0(\mathbb{D})$. Then $F \in BH^0(\phi_k;\sigma,a,b)$ if*

$$\sum_{k=1}^{2}\sum_{j=2}^{\infty}\frac{j\|a\|_{\max} - \|b\|_{\max}}{a-b}\left|\phi_{k,j}a_{k,j}\right| + |\sigma|\sum_{k=1}^{2}\sum_{j=2}^{\infty}\frac{j\|a\|_{\max} + \|b\|_{\max}}{a-b}\left|\phi_{k,j}b_{k,j}\right| \le 1, \tag{2.6}$$

here and in the following, $\|z\|_{\max} = \max_{\gamma\in R}\{|x+ye^{i\gamma}|\} = x+y$*, where* $z = x+ye^{i\gamma}$*,* x *and* $y \in [0,+\infty)$ *are constants.*

Proof. For F given by (1.4), we see that

$$\begin{aligned}
L(z) &\triangleq \left|\sum_{k=1}^{2}|z|^{2(k-1)}(h_k * \phi_k)(z) * \left(\frac{z+((ax-a+2b)/(2a-2b))z^2}{(1-z)^2}\right)\right.\\
&\qquad \left. -\sigma\sum_{k=1}^{2}|z|^{2(k-1)}\overline{(g_k * \phi_k)(z)} * \left(\frac{((ax+b)/(a-b))\overline{z}-((ax-a+2b)/(2a-2b))\overline{z}^2}{(1-\overline{z})^2}\right)\right|\\
&= \left|z+\sum_{k=1}^{2}|z|^{2(k-1)}\sum_{j=2}^{\infty}\left(j+(j-1)\frac{ax-a+2b}{2a-2b}\right)\phi_{k,j}a_{k,j}z^j\right.\\
&\qquad \left. -\sigma\sum_{k=1}^{2}|z|^{2(k-1)}\sum_{j=2}^{\infty}\left(j\frac{ax+b}{a-b}-(j-1)\frac{ax-a+2b}{2a-2b}\right)\overline{\phi_{k,j}b_{k,j}z^j}\right|.
\end{aligned} \tag{2.7}$$

If F is the identity, obviously, $L(z) = |z|$.

If F is not the identity, then

$$L(z) > |z|\left(1-\sum_{k=1}^{2}\sum_{j=2}^{\infty}\frac{j\|a\|_{\max}-\|b\|_{\max}}{a-b}\left|\phi_{k,j}a_{k,j}\right| - |\sigma|\sum_{k=1}^{2}\sum_{j=2}^{\infty}\frac{j\|a\|_{\max}+\|b\|_{\max}}{a-b}\left|\phi_{k,j}b_{k,j}\right|\right). \tag{2.8}$$

Hence the assumption implies that $L(z) > 0$ for all $z \in \mathbb{D}\setminus\{0\}$ and all $x \in \mathbb{C}$ with $|x| = 1$. It follows from Theorem 2.1 that $F \in BH^0(\phi_k;\sigma,a,b)$. □

Remark 2.4. If $h_2 = g_2 = 0$, $a = 1$ and $b = \alpha$, then Theorem 2.3 coincides with Theorem 2.2 in [16], and if $h_2 = g_2 = 0$, $a = 1+e^{i\gamma}$ and $b = \alpha+e^{i\gamma}$, then Theorem 2.3 coincides with Theorem 2.4 in [16].

3. A Coefficient Characterization and Extreme Points

We start with a coefficient characterization for $TBH^0(\phi_k;\sigma,a,b)$.

Theorem 3.1. *Let* $\phi_k(z) = z+\sum_{j=2}^{\infty}\phi_{k,j}z^j$ *with* $\phi_{k,j} \geq 0$*, and let* F *be of the form* (1.15)*. Then* $F \in TBH^0(\phi_k;\sigma,a,b)$ *if and only if*

$$\sum_{k=1}^{2}\sum_{j=2}^{\infty}\frac{j\|a\|_{\max}-\|b\|_{\max}}{a-b}\phi_{k,j}a_{k,j}+\sigma^2\sum_{k=1}^{2}\sum_{j=2}^{\infty}\frac{j\|a\|_{\max}+\|b\|_{\max}}{a-b}\phi_{k,j}b_{k,j} \leq 1. \tag{3.1}$$

Proof. By similar arguments as in the proof of Theorem 2.3, we see that it suffices to prove the "only if" part. For $F \in TBH^0(\phi_k;\sigma,a,b)$, obviously, (1.13) is equivalent to

$$\operatorname{Re}\left\{\frac{P(z)-Q(z)}{z-\sum_{k=1}^{2}|z|^{2(k-1)}\left(\sum_{j=2}^{\infty}a_{k,j}\phi_{k,j}z^j-\sigma^2\sum_{j=2}^{\infty}b_{k,j}\phi_{k,j}\overline{z}^j\right)}\right\}>0 \tag{3.2}$$

in $\mathbb{D}$, where

$$\begin{aligned}P(z)&=(a-b)z-\sum_{k=1}^{2}|z|^{2(k-1)}\sum_{j=2}^{\infty}(aj-b)a_{k,j}\phi_{k,j}z^j,\\ Q(z)&=\sigma^2\sum_{k=1}^{2}|z|^{2(k-1)}\sum_{j=2}^{\infty}(aj+b)b_{k,j}\phi_{k,j}\overline{z}^j.\end{aligned} \tag{3.3}$$

Letting $z \to 1^-$ through real values leads to the desired inequality. So the proof is complete. □

Remark 3.2. If $h_2 = g_2 = 0$, $a = 1$, and $b = \alpha$, then Theorem 3.1 coincides with Theorem 3.1 in [16].

It follows from Theorem 3.1 that we have the following.

Corollary 3.3. *Let $\phi_k(z) = z + \sum_{j=2}^{\infty}\phi_{k,j}z^j$ with $\phi_{k,j} \ge \phi_{1,2} > 0$ $(k \in \{1,2\},\ j \ge 2)$ and $|\sigma| \ge (2\|a\|_{\max} - \|b\|_{\max})/(2\|a\|_{\max} + \|b\|_{\max})$. If $F \in TBH^0(\phi_k;\sigma,a,b)$, then for $|z| = r < 1$, one has*

$$r-\frac{a-b}{(2\|a\|_{\max}-\|b\|_{\max})\phi_{1,2}}r^2\le|F(z)|\le r+\frac{a-b}{(2\|a\|_{\max}-\|b\|_{\max})\phi_{1,2}}r^2. \tag{3.4}$$

The result is sharp with equality for mappings

$$F(z)=z-\frac{a-b}{(2\|a\|_{\max}-\|b\|_{\max})\phi_{1,2}}z^2. \tag{3.5}$$

Theorem 3.1 and Corollary 3.3 imply the following

Corollary 3.4. *Under the hypotheses of Corollary 3.3, one has that $TBH^0(\phi_k;\sigma,a,b)$ is closed under the convex combination.*

Definition 3.5. Let X be a topological vector space over the field of complex numbers, and let E be a subset of X. A point $x \in E$ is called an extreme point of E if it has no representation of the form $x = ty + (1-t)z$ $(0 < t < 1)$ as a proper convex combination of two distinct points y and z in E (cf. [17]).

We now determine the extreme points of $TBH^0(\phi_k;\sigma,a,b)$.

Theorem 3.6. *Let*

(1) $h_{11}(z) = z$,

(2) $h_{21}(z) = g_{11}(z) = g_{21}(z) = 0$,

(3) $h_{kj}(z) = z - |z|^{2(k-1)}((a-b)/(j\|a\|_{\max} - \|b\|_{\max})\phi_{k,j})z^j$ *for* $k \in \{1,2\}$ *and all* $j \geq 2$,

(4) $g_{kj}(z) = z + |z|^{2(k-1)}((a-b)/\sigma(j\|a\|_{\max} + \|b\|_{\max})\phi_{k,j})\overline{z}^j$ *for* $k \in \{1,2\}$ *and all* $j \geq 2$.

Under the hypotheses of Corollary 3.3, one has that $F \in TBH^0(\phi_k;\sigma,a,b)$ *if and only if it can be expressed as*

$$F(z) = \sum_{k=1}^{2}\sum_{j=1}^{\infty}\left(x_{kj}h_{kj}(z) + y_{kj}g_{kj}(z)\right), \tag{3.6}$$

where $x_{21} = y_{11} = y_{21} = 0$, *all other* x_{kj} *and* y_{kj} *are nonnegative, and* $\sum_{k=1}^{2}\sum_{j=1}^{\infty}(x_{kj} + y_{kj}) = 1$.

In particular, the extreme points of $TBH^0(\phi_k;\sigma,a,b)$ *are all mappings* h_{kj} *and* g_{kj} *listed in* (1), (3), *and* (4) *above.*

Proof. It follows from the assumptions that

$$\begin{aligned} F(z) &= \sum_{k=1}^{2}\sum_{j=1}^{\infty}\left(x_{kj}h_{kj}(z) + y_{kj}g_{kj}(z)\right) \\ &= z - \sum_{k=1}^{2}\sum_{j=2}^{\infty}|z|^{2(k-1)}\frac{a-b}{(j\|a\|_{\max} - \|b\|_{\max})\phi_{k,j}}x_{kj}z^j \\ &\quad + \sigma\sum_{k=1}^{2}\sum_{j=2}^{\infty}|z|^{2(k-1)}\frac{a-b}{\sigma^2(j\|a\|_{\max} + \|b\|_{\max})\phi_{k,j}}y_{kj}\overline{z}^j, \end{aligned} \tag{3.7}$$

whence

$$\begin{aligned} &\sum_{k=1}^{2}\sum_{j=2}^{\infty}\frac{(j\|a\|_{\max} - \|b\|_{\max})}{a-b}\phi_{k,j}\cdot\frac{a-b}{(j\|a\|_{\max} - \|b\|_{\max})\phi_{k,j}}x_{kj} \\ &\quad + \sigma^2\sum_{k=1}^{2}\sum_{j=2}^{\infty}\frac{(j\|a\|_{\max} + \|b\|_{\max})}{a-b}\phi_{k,j}\cdot\frac{a-b}{\sigma^2(j\|a\|_{\max} + \|b\|_{\max})\phi_{k,j}}y_{kj} \\ &= \sum_{k=1}^{2}\sum_{j=2}^{\infty}x_{kj} + \sum_{k=1}^{2}\sum_{j=2}^{\infty}y_{kj} \\ &\leq 1, \end{aligned} \tag{3.8}$$

and so Theorem 3.1 implies that $F \in TBH^0(\phi_k;\sigma,a,b)$.

Conversely, assume $F \in TBH^0(\phi_k;\sigma,a,b)$, and let

$$x_{21} = y_{11} = y_{21} = 0, \qquad x_{11} = 1 - \sum_{k=1}^{2}\sum_{j=2}^{\infty} x_{kj} - \sum_{k=1}^{2}\sum_{j=2}^{\infty} y_{kj},$$
$$x_{kj} = \frac{\left(j\|a\|_{\max} - \|b\|_{\max}\right)\phi_{k,j}a_{k,j}}{a-b}, \tag{3.9}$$
$$y_{kj} = \frac{\sigma^2\left(j\|a\|_{\max} + \|b\|_{\max}\right)\phi_{k,j}b_{k,j}}{a-b},$$

for $k \in \{1,2\}$ and all $j \geq 2$. Then

$$F(z) = z - \sum_{k=1}^{2}|z|^{2(k-1)}\sum_{j=2}^{\infty}a_{k,j}z^j + \sigma\sum_{k=1}^{2}|z|^{2(k-1)}\sum_{j=2}^{\infty}b_{k,j}\overline{z}^j. \tag{3.10}$$

The proof of the theorem is complete. □

Remark 3.7. If $h_2 = g_2 = 0$, $a = 1$ and $b = \alpha$, then Theorem 3.6 coincides with Theorem 3.2 in [16].

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Research Article

Conformal Mapping of Unbounded Multiply Connected Regions onto Canonical Slit Regions

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We present a boundary integral equation method for conformal mapping of unbounded multiply connected regions onto five types of canonical slit regions. For each canonical region, three linear boundary integral equations are constructed from a boundary relationship satisfied by an analytic function on an unbounded multiply connected region. The integral equations are uniquely solvable. The kernels involved in these integral equations are the modified Neumann kernels and the adjoint generalized Neumann kernels.

1. Introduction

In this paper, we present a new method for numerical conformal mapping of unbounded multiply connected regions onto five types of canonical slit regions. A canonical region in conformal mapping is known as a set of finitely connected regions S such that each finitely connected nondegenerate region is conformally equivalent to a region in S. With regard to conformal mapping of multiply connected regions, there are several types of canonical regions as listed in [1–4]. The five types of canonical slit regions are disk with concentric circular slits U_d, annulus with concentric circular slits U_a, circular slit regions U_c, radial slit regions U_r, and parallel slit regions U_p. One major setback in conformal mapping is that only for certain regions exact conformal maps are known. One way to deal with this limitation is by numerical computation. Trefethen [5] has discussed several methods for computing conformal mapping numerically. Amano [6] and DeLillo et al. [7] have successfully map unbounded regions onto circular and radial slit regions. Boundary integral equation related to a boundary relationship satisfied by a function which is analytic in a simply or doubly connected region bounded by closed smooth Jordan curves has been given by Murid [8] and Murid and Razali [9]. Special realizations of this integral equation are the integral equations related to the Szegö kernel, Bergmann kernel, Riemann map, and Ahlfors map. The kernels arise in these integral equations are the Neumann kernel and the Kerzman-Stein kernel. Murid and Hu [10] managed to construct a boundary integral equation for numerical conformal mapping of a bounded multiply connected region onto a unit disk with slits. However, the integral equation involves unknown radii which lead to a system of nonlinear algebraic equations upon discretization of the integral equation. Nasser [11–13] produced another technique for numerical conformal mapping of bounded and unbounded multiply connected regions by expressing the mapping function in terms of the solution of a uniquely solvable Riemann-Hilbert problem. This uniquely solvable Riemann Hilbert problem can be solved by means of boundary integral equation with the generalized Neumann kernel. Recently, Sangawi et al. [14–17] have constructed new linear boundary integral equations for conformal mapping of bounded multiply region onto canonical slit regions, which improves the work of Murid and Hu [10] where in [10], the system of algebraic equations are nonlinear. In this paper, we extend the work of [14–17] for numerical conformal mapping of unbounded multiply connected regions onto all five types of canonical slit regions. These boundary integral equations are constructed from a boundary relationship satisfied by an analytic function on an unbounded multiply connected region.

The plan of this paper is as follows: Section 2 presents some auxiliary material. Section 3 presents a boundary integral equation related to a boundary relationship. In Sections 4–8, we present the derivations for numerical conformal mapping for all five types of canonical regions. In Section 9, we give some examples to illustrate the effectiveness of our method. Finally, Section 10 presents a short conclusion.

2. Auxiliary Material

Let Ω^- be an unbounded multiply connected region of connectivity m. The boundary Γ consists of m smooth Jordan curves Γ_j, $j = 1, 2, \ldots, m$ and will be denoted by $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_m$. The boundaries Γ_j are assumed in clockwise orientation (see Figure 1). The curve Γ_j is parameterized by 2π-periodic twice continuously differentiable complex function $\eta_j(t)$ with nonvanishing first derivative, that is,

$$\eta_j'(t) = \frac{d\eta_j(t)}{dt} \neq 0, \quad t \in J_j = [0, 2\pi], \quad k = 1, \ldots, m. \tag{2.1}$$

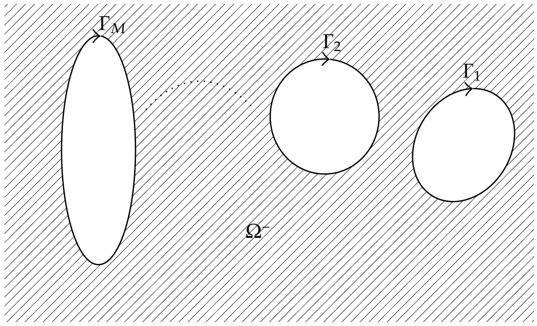

Figure 1: An unbounded multiply connected region Ω^- with connectivity m.

The total parameter domain J is the disjoint union of m intervals $J_1, \ldots, J_m$. We define a parameterization η of the whole boundary Γ on J by

$$\eta(t) = \begin{cases} \eta_1(t), & t \in J_1 = [0, 2\pi], \\ \vdots & \\ \eta_m(t), & t \in J_m = [0, 2\pi]. \end{cases} \tag{2.2}$$

Let $\Phi(z)$ be the conformal mapping function that maps Ω^- onto U^-, where U^- represents any canonical region mentioned above, z_1 is a prescribed point located inside Γ_1, z_m is a prescribed point inside Γ_m and β is a prescribed point located in Ω^-. In this paper, we determine the mapping function $\Phi(z)$ by computing the derivatives of the mapping function $\Phi'(\eta(t))$ and two real functions on J, that is, the unknown function $\varphi(t)$ and a piecewise constant real function $R(t)$. Let H be the space of all real Hölder continuous 2π-periodic functions and S be the subspace of H which contains the piecewise real constant functions $R(t)$. The piecewise real constant function $R(t)$ can be written as

$$R(t) = \begin{cases} R_1, & t \in J_1 = [0, 2\pi], \\ \vdots & \\ R_m, & t \in J_m = [0, 2\pi], \end{cases} \tag{2.3}$$

briefly written as $R(t) = (R_1, \ldots, R_m)$. Let $A(t)$ be a complex continuously differentiable 2π-periodic function for all $t \in J$. We define two real kernels formed with A as [18]

$$\begin{aligned} N(s,t) &= \frac{1}{\pi} \operatorname{Im}\left(\frac{A(s)}{A(t)} \frac{\eta'(t)}{\eta(t) - \eta(s)} \right), \\ M(s,t) &= \frac{1}{\pi} \operatorname{Re}\left(\frac{A(s)}{A(t)} \frac{\eta'(t)}{\eta(t) - \eta(s)} \right). \end{aligned} \tag{2.4}$$

The kernel $N(s,t)$ is known as the generalized Neumann kernel formed with complex-valued functions A and η. The kernel $N(s,t)$ is continuous with

$$N(t,t) = \frac{1}{\pi}\operatorname{Im}\left(\frac{\eta''(t)}{\eta'(t)} - \frac{A'(t)}{A(t)}\right). \tag{2.5}$$

The kernel $M(s,t)$ has a cotangent singularity

$$M(s,t) = -\frac{1}{2\pi}\cot\frac{s-t}{2} + M_1(s,t), \tag{2.6}$$

where the kernel $M_1(s,t)$ is continuous with

$$M_1(t,t) = \frac{1}{\pi}\operatorname{Re}\left(\frac{1}{2\pi}\frac{\eta''(t)}{\eta'(t)} - \frac{A'(t)}{A(t)}\right). \tag{2.7}$$

The adjoint function $\tilde{A}$ of A is defined by

$$\tilde{A} = \frac{\eta'(t)}{A(t)}. \tag{2.8}$$

The generalized Neumann kernel $\widetilde{N}(s,t)$ and the real kernel $\widetilde{M}$ formed with $\tilde{A}$ are defined by

$$\begin{aligned} \widetilde{N}(s,t) &= \frac{1}{\pi}\operatorname{Im}\left(\frac{\tilde{A}(s)}{\tilde{A}(t)}\frac{\eta'(t)}{\eta(t)-\eta(s)}\right), \\ \widetilde{M}(s,t) &= \frac{1}{\pi}\operatorname{Re}\left(\frac{\tilde{A}(s)}{\tilde{A}(t)}\frac{\eta'(t)}{\eta(t)-\eta(s)}\right). \end{aligned} \tag{2.9}$$

Then,

$$\widetilde{N}(s,t) = -N^*(s,t), \qquad \widetilde{M}(s,t) = -M^*(s,t), \tag{2.10}$$

where $N^*(s,t) = N(t,s)$ is the adjoint kernel of the generalized Neumann kernel $N(s,t)$. We define the Fredholm integral operators $\mathbf{N}^*$ by

$$\mathbf{N}^*v(t) = \int_J N^*(t,s)v(s)ds, \quad t \in J. \tag{2.11}$$

Integral operators M^*, $\widetilde{N}$, and $\widetilde{M}$ are defined in a similar way. Throughout this paper, we will assume the functions A and $\tilde{A}$ are given by

$$A(t) = 1, \qquad \tilde{A}(t) = \eta'(t). \tag{2.12}$$

It is known that $\lambda = 1$ is not an eigenvalue of the kernel N and $\lambda = -1$ is an eigenvalue of the kernel N with multiplicity m [18]. The eigenfunctions of N corresponding to the eigenvalue $\lambda = -1$ are $\{\chi^{[1]}, \chi^{[2]}, \ldots, \chi^{[m]}\}$, where

$$\chi^{[j]}(\xi) = \begin{cases} 1, & \xi \in \Gamma_j, \\ 0, & \text{otherwise}. \end{cases} \quad j = 1, 2, \ldots, m. \tag{2.13}$$

We also define an integral operator $\mathbf{J}$ by (see [14])

$$\mathbf{J}\mu(s) := \int_J \frac{1}{2\pi} \sum_{j=0}^{m} \chi^{[j]}(s)\chi^{[j]}(t)\mu(t)dt. \tag{2.14}$$

The following theorem gives us a method for calculating the piecewise constant real function $h(t)$ in connection with conformal mapping later. This theorem can be proved by using the approach as in [19, Theorem 5].

Theorem 2.1. *Let* $\mathrm{i} = \sqrt{-1}$, γ, $\mu \in H$ *and* $h \in S$ *such that*

$$Af = \gamma + h + \mathrm{i}\mu \tag{2.15}$$

are the boundary values of a function $f(z)$ *analytic in* Ω^-. *Then the function* $h = (h_1, h_2, \ldots, h_m)$ *has each element given by*

$$h_j = \left(\gamma, \rho^{[j]}\right) = \frac{1}{2\pi} \int_\Gamma \gamma(t)\rho^{[t]}dt, \tag{2.16}$$

where $\rho^{[t]}$ *is the unique solution of the integral equation*

$$(\mathbf{I} + \mathbf{N}^* + \mathbf{J})\rho^{[j]} = -\chi^{[j]}, \quad j = 1, 2, \ldots, m. \tag{2.17}$$

3. The Homogeneous Boundary Relationship

Suppose we are given a function $D(z)$ which is analytic in Ω^-, continuous on $\Omega^- \cup \Gamma$, Hölder continuous on Γ and $D(\infty)$ is finite. The boundary Γ_j is assumed to be a smooth Jordan curve. The unit tangent to Γ at the point $\eta(t) \in \Gamma$ will be denoted by $T(\eta(t)) = \eta'(t)/|\eta'(t)|$. Suppose further that $D(\eta(t))$ satisfies the exterior homogeneous boundary relationship

$$D(\eta(t)) = c(t)\frac{\overline{T(\eta(t))}^2 \overline{D(\eta(t))}}{\overline{P(\eta(t))}}, \tag{3.1}$$

where $c(t)$ and P are complex-valued functions with the following properties:

(1) $P(z)$ is analytic in Ω^- and does not have zeroes on $\Omega^- \cup \Gamma$,

(2) $P(\infty) \neq 0$, $D(\infty)$ is finite,

(3) $c(t) \neq 0$, $P(\eta(t)) \neq 0$.

Note that the boundary relationship (3.1) also has the following equivalent form:

$$P(\eta(t)) = \overline{c(t)}T(\eta(t))^2 \frac{D(\eta(t))^2}{|D(\eta(t))|^2}. \tag{3.2}$$

Under these assumptions, an integral equation for $D(\eta(t))$ can be constructed by means of the following theorem.

Theorem 3.1. *If the function $D(\eta(t))$ satisfies the exterior homogeneous boundary relationship* (3.1), *then*

$$\phi(t) + \int_J K(s,t)\phi(s)ds = \nu(t), \tag{3.3}$$

where

$$\begin{aligned} \phi(t) &= D(\eta(t))\eta'(t), \\ K(s,t) &= \frac{1}{2\pi \mathrm{i}}\left[\frac{\eta'(t)}{\eta(t)-\eta(s)} - \frac{c(t)}{c(s)}\frac{\overline{\eta'(t)}}{\left(\overline{\eta(t)}-\overline{\eta(s)}\right)}\right], \\ \nu(t) &= D(\infty)\eta'(t) + c(t)\overline{\eta'(t)}\frac{\overline{D(\infty)}}{\overline{P(\infty)}}. \end{aligned} \tag{3.4}$$

Proof. Consider the integral $I_1(\eta(t))$,

$$I_1(\eta(t)) = \frac{1}{2\pi \mathrm{i}}\int_J \frac{D(\eta(s))}{\eta(s)-\eta(t)}ds. \tag{3.5}$$

Since the boundary is in clockwise orientation and D is analytic in Ω^-, then by [20, p. 2] we have

$$I_1(\eta(t)) = \frac{D(\eta(t))}{2} - D(\infty). \tag{3.6}$$

Now, let the integral $I_2(\eta(t))$ be defined as

$$I_2(\eta(t)) = \frac{1}{2\pi \mathrm{i}}\int_J \frac{c(t)\overline{T(\eta(t))}^2 D(\eta(s))}{c(s)\left(\overline{\eta(s)-\eta(t)}\right)\overline{T(\eta(s))}}|ds|. \tag{3.7}$$

Using the boundary relationship (3.1) and the fact that $\overline{T(\eta(t))}|dt| = dt$ and $P(\eta(t))$ does not contain zeroes, then by [20, p. 2] we obtain

$$I_2(\eta(t)) = -c(t)\overline{T(\eta(t))}^2 \overline{\frac{D(\eta(t))}{P(\eta(t))}} + c(t)\overline{T(\eta(t))}^2 \overline{\frac{D(\infty)}{P(\infty)}}. \tag{3.8}$$

Next, by taking $I_2(\eta(t)) - I_1(\eta(t))$ with further arrangement yields

$$\begin{aligned} &D(\eta(t)) + \frac{1}{2\pi \mathrm{i}}\int_J \left[\frac{1}{\eta(t)-\eta(s)} - \frac{c(t)}{c(s)}\frac{\overline{T(\eta(t))}^2}{\overline{\eta(t)-\eta(s)}}\right] D(\eta(s))|ds| \\ &= D(\infty) + c(t)\overline{T(\eta)}^2 \overline{\frac{D(\infty)}{P(\infty)}}. \end{aligned} \tag{3.9}$$

Then multiplying (3.9) with $T(\eta(t))$ and $|\eta'(t)|$, subsequently yields (3.3). □

Theorem 3.2. *The kernel $K(s,t)$ is continuous with*

$$K(t,t) = \frac{1}{2\pi}\operatorname{Im}\frac{\eta''(t)}{\eta'(t)} - \frac{1}{2\pi \mathrm{i}}\frac{c'(t)}{c(t)}. \tag{3.10}$$

Proof. Let the kernel $K(s,t)$ be written as

$$K(s,t) = K_1(s,t) + K_2(s,t), \tag{3.11}$$

where

$$\begin{aligned} K_1(s,t) &= \frac{1}{2\pi \mathrm{i}}\left[\frac{\eta'(t)}{\eta(t)-\eta(s)} - \frac{\overline{\eta'(t)}}{\overline{\eta(t)-\eta(s)}}\right], \\ K_2(s,t) &= \frac{1}{2\pi \mathrm{i}}\left[-\frac{c(t)}{c(s)}\frac{\overline{\eta'(t)}}{\overline{\eta(t)-\eta(s)}} + \frac{\overline{\eta'(t)}}{\overline{\eta(t)-\eta(s)}}\right] = \frac{1}{2\pi \mathrm{i}}\frac{\overline{\eta'(t)}}{c(s)}\left[\frac{c(s)-c(t)}{\overline{\eta(t)-\eta(s)}}\right]. \end{aligned} \tag{3.12}$$

Notice that $K_1(s,t)$ is the classical Neumann kernel with $K_1(t,t) = 1/(2\pi)\operatorname{Im}((\eta''(t))/(\eta'(t)))$. Now for $K_2(s,t)$, as we take the limit $s \to t$ we have,

$$\begin{aligned} K_2(t,t) &= \frac{1}{2\pi \mathrm{i}}\lim_{s\to t}\frac{\overline{\eta'(t)}}{c(s)}\lim_{s\to t}\left[\frac{c(s)-c(t)}{\overline{\eta(t)-\eta(s)}}\right] \\ &= -\frac{1}{2\pi \mathrm{i}}\frac{c'(t)}{c(t)}. \end{aligned} \tag{3.13}$$

Hence, by combining $K_1(t,t)$ and $K_2(t,t)$, we obtain

$$K(t,t) = \frac{1}{2\pi}\left(\operatorname{Im}\frac{\eta''(t)}{\eta'(t)} - \frac{1}{\mathrm{i}}\frac{c'(t)}{c(t)}\right). \tag{3.14}$$

□

Note that when $c(t) = 1$, the kernel $K(s,t)$ reduces to the classical Neumann kernel.

We define the Fredholm integral operator $\mathbf{K}$ by

$$\mathbf{K}\upsilon(t) = \frac{1}{2\pi\mathrm{i}}\int_J K(s,t)\upsilon(s)ds, \quad t \in J. \tag{3.15}$$

Hence, (3.3) becomes

$$(\mathbf{I} + \mathbf{K})\phi(t) = \nu(t). \tag{3.16}$$

The solvability of the integral equation (3.16) depends on the possibility of $\lambda = -1$ being an eigenvalue of the kernel $K(s,t)$. For the numerical examples considered in this paper, $\lambda = -1$ is always an eigenvalue of the kernel $K(s,t)$. Although there is no theoretical proof yet, numerical evidence suggests that $\lambda = -1$ is an eigenvalue of $K(s,t)$. If the multiplicity of the eigenvalue $\lambda = -1$ is $\hat{m}$, then one need to add $\hat{m}$ conditions to the integral equation to ensure the integral equation is uniquely solvable.

4. Exterior Unit Disk with Circular Slits

The canonical region U_d is the exterior unit disk along with $m-1$ arcs of circles. We assume that $w = \Phi(z)$ maps the curve Γ_1 onto the unit circle $|w| = 1$, the curve Γ_j, where $j = 2,3,\ldots,m$, onto circular slit on the circle $|w| = R_j$, where $R_2, R_3, \ldots, R_m$ are unknown real constants. The circular slits are traversed twice. The boundary values of the mapping function Φ are given by

$$\Phi(\eta(t)) = R(t)e^{\mathrm{i}\theta(t)}, \tag{4.1}$$

where $\theta(t)$ represents the boundary correspondence function and $R(t) = (1, R_2, \ldots, R_m)$. By differentiating (4.1) with respect to t and dividing the result obtained by its modulus, we have

$$\frac{\Phi'(\eta(t))\eta'(t)}{|\Phi'(\eta(t))\eta'(t)|} = \mathrm{i}\ \operatorname{sign}(\theta'(t))e^{\mathrm{i}\theta(t)}. \tag{4.2}$$

Using the fact that unit tangent $T(\eta(t)) = \eta'(t)/|\eta'(t)|$ and $e^{\mathrm{i}\theta(t)} = \Phi(\eta(t))/R(t)$, it can be shown that

$$\Phi(\eta(t)) = \operatorname{sign}(\theta'(t))\frac{R(t)}{\mathrm{i}}T(\eta(t))\frac{\Phi'(\eta(t))}{|\Phi'(\eta(t))|}. \tag{4.3}$$

Boundary relationship (4.3) is useful for computing the boundary values of $\Phi(z)$ provided $\theta'(t)$, $R(t)$, and $\Phi'(\eta(t))$ are all known. By taking logarithmic derivative on (4.1), we obtain

$$\eta'(t)\frac{\Phi'(\eta(t))}{\Phi(\eta(t))} = \mathrm{i}\theta'(t). \tag{4.4}$$

The mapping function $\Phi(z)$ can be uniquely determined by assuming

$$\Phi(\infty) = \infty, \qquad c = \Phi'(\infty) = \lim_{z\to\infty}\frac{\Phi(z)}{z} > 0. \tag{4.5}$$

Thus, the mapping function can be expressed as [12]

$$\Phi(z) = c(z - z_1)e^{F(z)}, \tag{4.6}$$

where $F(z)$ is an analytic function and $F(\infty) = 0$. By taking logarithm on both sides of (4.6), we obtain

$$F(\eta(t)) = \ln\frac{R(t)}{c} + \mathrm{i}\theta - \log(\eta(t) - z_1). \tag{4.7}$$

Hence (4.7) satisfies boundary values (2.15) in Theorem 2.1 with $A(t) = 1$,

$$h(t) = \left(\ln\frac{1}{c}, \ln\frac{R_2}{c}, \ldots, \ln\frac{R_m}{c}\right), \qquad \gamma(t) = -\ln\left|\eta(t) - z_1\right|. \tag{4.8}$$

Hence, the values of R_j can be calculated by

$$R_j = e^{h_j - h_1} \quad \text{for } j = 1, 2, \ldots, m. \tag{4.9}$$

To find $\theta'(t)$, we began by differentiating (4.7) and comparing with (4.4) which yields

$$\mathrm{i}\theta'(t) = F'(\eta(t))\eta'(t) - \frac{\eta'(t)}{\eta(t) - z_1}. \tag{4.10}$$

In view of $\widetilde{A} = \eta'(t)$ and letting $f(z) = F'(z) - 1/(z - z_1)$, where $f(z)$ is analytic function in Ω^-, (4.10) becomes

$$\widetilde{A}f(\eta(t)) = \mathrm{i}\theta'(t). \tag{4.11}$$

By [18, Theorem 2(c)], we obtain $(\mathbf{I} - \widetilde{\mathbf{N}})\theta' = 0$ which implies

$$(\mathbf{I} + \mathbf{N}^*)\theta' = 0. \tag{4.12}$$

However, this integral equation is not uniquely solvable according to [18, Theorem 12]. To overcome this, since the image of the curve Γ_1 is clockwise oriented and the images of the curves $\Gamma_j, j = 2, 3, \ldots, m$ are slits so we have $\theta_1(2\pi) - \theta_1(0) = -2\pi$ and $\theta_j(2\pi) - \theta_j(0) = 0$, which implies

$$\mathbf{J}\theta'(t) = \tilde{h}(t) = (-1, 0, \ldots, 0). \tag{4.13}$$

By adding this condition to (4.12), the unknown function $\theta'(t)$ is the unique solution of the integral equation

$$(\mathbf{I} + \mathbf{N}^* + \mathbf{J})\theta'(t) = \tilde{h}(t). \tag{4.14}$$

Next, the presence of $\operatorname{sign}(\theta'(t))$ in (4.3) can be eliminated by squaring both sides of (4.3), that is,

$$\Phi(\eta(t))^2 = -R(t)^2 T(\eta(t))^2 \frac{\Phi'(\eta(t))^2}{|\Phi'(\eta(t))|^2}. \tag{4.15}$$

Upon comparing (4.15) with (3.2), this leads to a choice of $P(\eta(t)) = \Phi(\eta(t))^2$, $P(\infty) = \infty$, $D(\eta(t)) = \Phi'(\eta(t))$, $D(\infty) = c$, $c(t) = -R(t)^2$. Hence,

$$\phi(t) = \Phi'(\eta(t))\eta'(t) \tag{4.16}$$

satisfies the integral equation (3.16) with

$$\nu(t) = \Phi'(\infty)\eta'(t). \tag{4.17}$$

Numerical evidence shows that $\lambda = -1$ is an eigenvalue of $K(s, t)$ of multiplicity m, which means one needs to add m conditions. Since $\Phi(z)$ is assumed to be single-valued, it is also required that the unknown mapping function $\Phi'(z)$ satisfies [4]

$$\int_{J_j} \phi(t)dt = \int_{\Gamma_j} \Phi'(\eta)d\eta = 0, \quad j = 1, 2, \ldots, m, \tag{4.18}$$

that is,

$$\mathbf{J}\phi = 0. \tag{4.19}$$

Then, $\phi(t)$ is the unique solution of the following integral equation

$$(\mathbf{I} + \mathbf{K} + \mathbf{J})\phi(t) = \nu(t). \tag{4.20}$$

By obtaining $\phi(t)$, the derivatives of the mapping function, $\Phi'(t)$ can be found using

$$\Phi'(t) = \frac{\phi(t)}{\eta'(t)}. \tag{4.21}$$

By obtaining the values of $R(t)$, $\theta'(t)$ and $\Phi'(\eta(t))$, the boundary value of $\Phi(\eta(t))$ can be calculated by using (4.3).

5. Annulus with Circular Slits

The canonical region U_a consists of an annulus centered at the origin together with $m-2$ circular arcs. We assume that $\Phi(z)$ maps the curve Γ_1 onto the unit circle $|\Phi| = 1$, the curve Γ_m onto the circle $|\Phi| = R_m$ and Γ_j onto circular slit $|\Phi| = R_j$, where $j = 2, 3, \ldots, m-1$. The slit are traversed twice. The boundary values of the mapping function Φ are given by

$$\Phi(\eta(t)) = R(t)e^{i\theta(t)}, \tag{5.1}$$

where $\theta(t)$ represents the boundary correspondence function and $R(t) = (1, R_2, \ldots, R_m)$ is a piecewise real constant function. By using the same reasoning as in Section 4, we get

$$\Phi(\eta_j(t)) = \operatorname{sign}(\theta'(t)) \frac{R_j(t)}{i} T(\eta_j(t)) \frac{\Phi'(\eta_j(t))}{|\Phi'(\eta_j(t))|}, \tag{5.2}$$

$$\eta'(t) \frac{\Phi'(\eta(t))}{\Phi(\eta(t))} = i\theta'(t). \tag{5.3}$$

The mapping function $\Phi(z)$ can be uniquely determined by assuming $c = \Phi(\infty) > 0$. Thus, the mapping function can be expressed as [12]

$$\Phi(z) = c\frac{z - z_m}{z - z_1} e^{F(z)}. \tag{5.4}$$

By taking logarithm on both sides of (5.4), we obtain

$$F(\eta(t)) = \ln \frac{R(t)}{c} + i\theta - \log \frac{\eta(t) - z_m}{\eta(t) - z_1}. \tag{5.5}$$

Hence, (5.5) satisfies boundary values (2.15) in Theorem 2.1 with $A(t)$=1,

$$h(t) = \left(\ln \frac{1}{c}, \ln \frac{R_2}{c}, \ldots, \ln \frac{R_m}{c}\right), \qquad \gamma(t) = -\ln \left| \frac{\eta(t) - z_m}{\eta(t) - z_1} \right|. \tag{5.6}$$

By obtaining h_j, the values of R_j can be computed by

$$R_j = e^{h_j - h_1}, \quad j = 2, 3, \ldots, m. \tag{5.7}$$

By differentiating (5.5) and comparing with (5.3) yields

$$\mathrm{i}\theta'(t) = F'(\eta(t))\eta'(t) + \frac{\eta'(t)}{\eta(t) - z_m} - \frac{\eta'(t)}{\eta(t) - z_1}. \tag{5.8}$$

In view of $\widetilde{A} = \eta'(t)$ and $f(\eta(t)) = F'(\eta(t)) + 1/(\eta(t) - z_m) - 1/(\eta(t) - z_1)$, where $f(z)$ is analytic in Ω^-, (5.8) is equivalent to

$$\widetilde{A} f(\eta(t)) = \mathrm{i}\theta'(t). \tag{5.9}$$

By [18, Theorem 2(c)], we obtain

$$(\mathbf{I} + \mathbf{N}^*)\theta'(t) = 0. \tag{5.10}$$

Note that the image of the curve Γ_1 is counterclockwise oriented, Γ_m is clockwise oriented and the images of the curves Γ_j, $j = 2, 3, \ldots, m-1$ are slits so we have $\theta_1(2\pi) - \theta_1(0) = 2\pi$, $\theta_m(2\pi) - \theta_m(0) = -2\pi$ and $\theta_j(2\pi) - \theta_j(0) = 0$, which implies

$$\mathbf{J}\theta'(t) = \widetilde{h}(t) = (1, 0, \ldots, -1). \tag{5.11}$$

Hence, the unknown function $\theta'(t)$ is the unique solution of the integral equation

$$(\mathbf{I} + \mathbf{N}^* + \mathbf{J})\theta'(t) = \widetilde{h}(t). \tag{5.12}$$

Next, the presence of $\operatorname{sign}(\theta'(t))$ in (5.2) can be eliminated by squaring both sides of the equation, that is,

$$\Phi(\eta(t))^2 = -R(t)^2 T(\eta(t))^2 \frac{\Phi'(\eta(t))^2}{|\Phi'(\eta(t))|^2}. \tag{5.13}$$

Comparing (5.13) with (3.2) leads to a choice of $P(\eta(t)) = \Phi(\eta(t))^2$, $P(\infty) = c^2$, $D(\eta(t)) = \Phi'(\eta(t))$, $D(\infty) = 0$, $c(t) = -R(t)^2$. Hence,

$$\phi(t) = \Phi'(\eta(t))\eta'(t) \tag{5.14}$$

satisfies the integral equation (3.16) with

$$\nu(t) = (0, \ldots, 0). \tag{5.15}$$

Numerical evidence shows that $\lambda = -1$ is an eigenvalue of $K(s,t)$ of multiplicity $m+1$, which implies one needs to add $m+1$ conditions. Since $\Phi(z)$ is assumed to be single-valued, hence it is also required that the unknown mapping function $\Phi'(z)$ satisfies [4]

$$\int_{J_j} \phi(s)ds = \int_{\Gamma_j} \Phi'(\eta)d\eta = 0, \quad j = 1,2,\ldots,m, \tag{5.16}$$

that is,

$$\mathbf{J}\phi = 0. \tag{5.17}$$

Since we assume the mapping function $\Phi(z)$ can be uniquely determined by $c = \Phi(\infty) > 0$, hence by [20]

$$\Phi(\infty) = c = \frac{1}{2\pi}\int_J \frac{\eta'(t)}{\theta'(t)(\eta - z_1)}\phi(t)dt. \tag{5.18}$$

If we define the Fredholm operator $\mathbf{G}$ as

$$\mathbf{G}\mu(s) = \frac{1}{2\pi}\int_J \frac{\eta'(t)}{\theta'(t)(\eta - z_1)}\mu(t)dt, \tag{5.19}$$

then $\phi(t)$ is the unique solution of the following integral equation:

$$(\mathbf{I} + \mathbf{K} + \mathbf{J} + \mathbf{G})\phi(t) = c. \tag{5.20}$$

By obtaining $\phi(\eta(t))$, the derivatives of the mapping function, $\Phi'(t)$ can be obtained by

$$\Phi'(t) = \frac{\phi(\eta(t))}{\eta'(t)}. \tag{5.21}$$

6. Circular Slits

The canonical region U_c consists of m slits along the circle $|\Phi| = R_j$ where $j = 1,2,\ldots,m$ and $R_1, R_2, \ldots, R_m$ are unknown real constants. The boundary values of the mapping function Φ are given by

$$\Phi(\eta(t)) = R(t)e^{i\theta(t)}, \tag{6.1}$$

where $\theta(t)$ represents the boundary correspondence function and $R(t) = (1, R_2, \ldots, R_m)$. By using the same reasoning as in Section 4, we get

$$\Phi(\eta_j(t)) = \operatorname{sign}(\theta'(t)) \frac{R_j(t)}{\mathrm{i}} T(\eta_j(t)) \frac{\Phi'(\eta_j(t))}{|\Phi'(\eta_j(t))|}, \tag{6.2}$$

$$\eta'(t) \frac{\Phi'(\eta(t))}{\Phi(\eta(t))} = \mathrm{i}\theta'(t). \tag{6.3}$$

The mapping function $\Phi(z)$ can be uniquely determined by assuming $\Phi(\beta) = 0$, $\Phi(\infty) = \infty$ and $\Phi'(\infty) = 1$. Thus, the mapping function Φ can be expressed as [12]

$$\Phi(z) = (z - \beta) e^{F(z)}. \tag{6.4}$$

By taking logarithm on both sides of (6.4), we obtain

$$F(\eta(t)) = \ln R(t) + \mathrm{i}\theta - \log(\eta(t) - \beta). \tag{6.5}$$

Hence, (6.5) satisfies boundary values in Theorem 2.1 with $A(t) = 1$,

$$h(t) = (\ln R_1, \ln R_2, \ldots, \ln R_m), \quad \gamma(t) = -\ln|\eta(t) - \beta|. \tag{6.6}$$

By obtaining h_j, the values of R_j can be obtained by

$$R_j = e^{h_j}. \tag{6.7}$$

By differentiating (6.5) and comparing with (6.3) yields

$$\mathrm{i}\theta'(t) = F'(\eta(t))\eta'(t) + \frac{\eta'(t)}{\eta(t) - \beta}. \tag{6.8}$$

In view of $\widetilde{A} = \eta'(t)$, $f(t) = F'(\eta(t))$ and $g(t) = 1/(\eta(t) - \beta)$ where $f(z)$ is analytic in Ω^- and $g(z)$ is analytic in Ω^+, we rewrite (6.8) as

$$\widetilde{A} f(\eta(t)) = \mathrm{i}\theta'(t) - \widetilde{A} g(\eta(t)). \tag{6.9}$$

Let $\widetilde{A} g(t) = \psi + \mathrm{i}\varphi$. Then by [18, Theorems 2(c) and 2(d)], we obtain

$$(\mathbf{I} + \mathbf{N}^*)(\theta'(t) - \varphi(t)) = \widetilde{\mathbf{M}}\psi(t), \tag{6.10}$$

$$(\mathbf{I} - \mathbf{N}^*)\varphi(t) = \widetilde{\mathbf{M}}\psi(t). \tag{6.11}$$

Subtracting (6.11) from (6.10), we get

$$(\mathbf{I} + \mathbf{N}^{*})\theta'(t) = 2\varphi(t). \tag{6.12}$$

Since the images of the curve $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$ are slits, we have $\theta_j(2\pi) - \theta_j(0) = 0$. Thus

$$\mathbf{J}\theta'(t) = (0, 0, \ldots, 0). \tag{6.13}$$

Hence the unknown function $\theta'(t)$ is the unique solution of the integral equation

$$(\mathbf{I} + \mathbf{N}^{*} + \mathbf{J})\theta'(t) = 2\varphi(t), \tag{6.14}$$

where

$$\varphi(t) = \operatorname{Im}\left[\widetilde{A}(t) g(\eta(t))\right] = \operatorname{Im}\left[\frac{\eta'(t)}{\eta(t) - \beta}\right]. \tag{6.15}$$

By squaring both sides of (6.2) and dividing the result by $(\eta(t) - \beta)^2$, we obtain

$$\frac{\Phi(\eta(t))^2}{(\eta(t) - \beta)^2} = -\frac{R(t)^2}{(\eta(t) - \beta)^2} T(\eta(t))^2 \frac{\Phi'(\eta(t))^2}{|\Phi'(\eta(t))|^2}. \tag{6.16}$$

Upon comparing (6.16) with (3.2) leads to a choice of $P(\eta(t)) = \Phi(\eta(t))^2/(\eta(t) - \beta)^2$, $P(\infty) = 1$, $D(\eta(t)) = \Phi'(\eta(t))$, $D(\infty) = 1$, $c(t) = -R(t)^2/(\overline{\eta(t) - \beta}^2)$. Hence,

$$\phi(t) = \Phi'(\eta(t))\eta'(t) \tag{6.17}$$

satisfies the integral equation (3.16) with

$$\nu(t) = -\frac{R(t)^2}{\overline{(\eta(t) - \beta)^2}}\overline{\eta'(t)} + \eta'(t). \tag{6.18}$$

Numerical evidence shows that $\lambda = -1$ is an eigenvalue of $K(s,t)$ of multiplicity m, thus one needs to add m conditions. Since $\Phi(z)$ is assumed to be single-valued, it is also required that the unknown mapping function $\Phi'(z)$ satisfies [4]

$$\int_{J_j} \phi(t)dt = \int_{\Gamma_j} \Phi'(\eta)d\eta = 0, \quad j = 1, 2, \ldots, m, \tag{6.19}$$

that is,

$$\mathbf{J}\phi = 0. \tag{6.20}$$

Then, $\phi(t)$ is the unique solution of the integral equation

$$(\mathbf{I}+\mathbf{K}+\mathbf{J})\phi(t) = \nu(t). \tag{6.21}$$

By obtaining $\phi(\eta(t))$, the derivatives of the mapping function, $\Phi'(t)$ can be obtained by

$$\Phi'(t) = \frac{\phi(\eta(t))}{\eta'(t)}. \tag{6.22}$$

7. Radial Slits

The canonical region U_r consists of m slits along m segments of the rays with $\arg(\Phi) = \theta_j$, $j = 1, 2, \ldots, m$. Then, the boundary values of the mapping function Φ are given by

$$\Phi(\eta(t)) = r(t)e^{\mathrm{i}\theta(t)} = e^{R(t)}e^{\mathrm{i}\theta(t)}, \tag{7.1}$$

where the boundary correspondence function $\theta(t) = (\theta_1, \theta_2, \ldots, \theta_m)$ now becomes real constant function and $R(t)$ is an unknown function. By taking logarithmic derivative on (7.1), we obtain

$$\eta'(t)\frac{\Phi'(\eta(t))}{\Phi(\eta(t))} = R'(t). \tag{7.2}$$

It can be shown that the mapping function $\Phi(z)$ can be determined using

$$\Phi(\eta(t)) = \frac{\eta'(t)\Phi'(\eta(t))}{R'(t)}. \tag{7.3}$$

The mapping function $\Phi(z)$ can be uniquely determined by assuming $\Phi(\beta) = 0$, $\Phi(\infty) = \infty$ and $\Phi'(\infty) = 1$. Thus, the mapping function $\Phi(z)$ can be expressed as [12]

$$\Phi(z) = (z-\beta)e^{\mathrm{i}F(z)}. \tag{7.4}$$

By taking logarithm on both sides of (7.4) and multiplying the result by $-\mathrm{i}$, we obtain

$$F(\eta(t)) = \theta - \mathrm{i}R(t) + \mathrm{i}\log(\eta(t)-\beta). \tag{7.5}$$

Hence, (7.5) satisfies boundary values in Theorem 2.1 with $A(t) = 1$,

$$h(t) = (\theta_1, \theta_2, \ldots, \theta_m), \qquad \gamma(t) = -\operatorname{Arg}(\eta(t)-\beta). \tag{7.6}$$

By obtaining $h(t)$, one can obtain the values of $\theta(t)$ by

$$\theta_j = h_j \quad \text{for } j = 1, 2, \ldots, m. \tag{7.7}$$

Then, by differentiating (7.5) and comparing with (7.2) yields

$$\mathrm{i}R'(t) = -F'(\eta(t))\eta'(t) + \mathrm{i}\frac{\eta'(t)}{\eta(t)-\beta}. \tag{7.8}$$

In view of $\widetilde{A} = \eta'(t)$, $f(t) = F'(\eta(t))$ and $g(t) = \mathrm{i}/(\eta(t)-\beta)$ where $f(z)$ is analytic in Ω^- and $g(z)$ is analytic in Ω^+, we rewrite (7.8) as

$$\widetilde{A}f(\eta(t)) = \mathrm{i}R'(t) - \widetilde{A}g(\eta(t)). \tag{7.9}$$

Let $\widetilde{A}g(t) = \psi + \mathrm{i}\varphi$. Then by [18, Theorems 2(c) and 2(d)], we obtain

$$\begin{gathered}(\mathbf{I}+\mathbf{N}^*)(\varphi(t) - R'(t)) = -\widetilde{\mathbf{M}}\psi(t),\\ (\mathbf{I}-\mathbf{N}^*)\varphi(t) = \widetilde{\mathbf{M}}\psi(t).\end{gathered} \tag{7.10}$$

Adding these equations, we get

$$(\mathbf{I}+\mathbf{N}^*)R'(t) = 2\varphi(t). \tag{7.11}$$

Since the images of the curve $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$ are slits, we have $R_j(2\pi) - R_j(0) = 0$. Therefore

$$\mathbf{J}R'(t) = (0, 0, \ldots, 0). \tag{7.12}$$

Hence, the unknown function $R'(t)$ is the unique solution of the integral equation

$$(\mathbf{I}+\mathbf{N}^* + \mathbf{J})R'(t) = 2\varphi(t), \tag{7.13}$$

where

$$\varphi(t) = \operatorname{Im}\left[\widetilde{A}(t)g(\eta(t))\right] = \operatorname{Im}\left[\frac{\mathrm{i}\eta'(t)}{\eta(t)-\beta}\right]. \tag{7.14}$$

The boundary relationship (7.3) can be rewritten as

$$\Phi'(\eta(t)) = \pm e^{\mathrm{i}\theta_j}\overline{T(\eta(t))}\left|\Phi'(\eta(t))\right|. \tag{7.15}$$

Squaring both sides of (7.15) yields

$$\Phi'(\eta(t)) = e^{2\mathrm{i}\theta_j}\overline{T(\eta(t))^2\Phi'(\eta(t))}. \tag{7.16}$$

Upon comparing (7.16) with (3.1) leads to a choice of $P(\eta(t)) = 1$, $P(\infty) = 1$, $D(\eta(t)) = \Phi'(\eta(t))$, $D(\infty) = 1$, $c(t) = e^{i2\theta_j}$. Hence,

$$\phi(t) = \Phi'(\eta(t))\eta'(t) \tag{7.17}$$

satisfies the integral equation (3.16) with

$$\nu(t) = e^{2i\theta_j(t)}\overline{\eta'(t)} + \eta'(t). \tag{7.18}$$

Numerical evidence shows that $\lambda = -1$ is an eigenvalue of $K(s,t)$ of multiplicity m, which suggests one needs to add m conditions. Since $\Phi(z)$ is assumed to be single-valued, hence it is also required that the unknown mapping function $\Phi'(z)$ satisfies [4]

$$\int_{J_j} \phi(t)dt = \int_{\Gamma_j} \Phi'(\eta)d\eta = 0, \quad j = 1,2,\ldots,m, \tag{7.19}$$

that is,

$$\mathbf{J}\phi = 0. \tag{7.20}$$

Then, $\phi(t)$ is the solution of the following integral equation

$$(\mathbf{I} + \mathbf{K} + \mathbf{J})\phi(t) = \nu(t). \tag{7.21}$$

By obtaining $\phi(\eta(t))$, the derivatives of the mapping function $\Phi'(t)$ can be found using

$$\Phi'(t) = \frac{\phi(\eta(t))}{\eta'(t)}. \tag{7.22}$$

8. Parallel Slits

The canonical region U_p consists of a m parallel straight slits on the w-plane. Let $B = e^{i(\pi/2-\theta)}$, then the boundary values of the mapping function Φ are given by

$$B\Phi(\eta(t)) = R(t) + i\delta(t), \tag{8.1}$$

where θ is the angle of intersection between the lines $\mathrm{Re}[B\Phi] = R_j$ and the real axis. $R(t) = (R_1(t), R_2(t), \ldots, R_M(t))$ is a piecewise real constant function and $\delta(t)$ is an unknown function. It can be shown that (8.1) can be written as

$$B\Phi'(\eta(t))\eta'(t) = i\delta'(t). \tag{8.2}$$

The mapping function $\Phi(z)$ can be uniquely determined by assuming $\Phi(\infty) = \infty$ and $\lim_{z\to\infty}\Phi(z) - z = 0$. Thus, the mapping function Φ can be expressed as [12]

$$\Phi(z) = z + \overline{B}F(z), \tag{8.3}$$

where $F(z)$ is an analytic function with $F(\infty) = 0$. By multiplying both sides of (8.3) with B, we obtain

$$F(\eta(t)) = B\Phi(\eta(t)) - B\eta(t). \tag{8.4}$$

Hence, (8.4) satisfies the boundary values in Theorem 2.1 with $A(t) = 1$,

$$h(t) = (R_1, R_2, \ldots, R_m) \qquad \gamma(t) = -B\eta(t). \tag{8.5}$$

Differentiating (8.4) and comparing the result with (8.2) yield

$$\mathrm{i}\delta'(t) = B\eta'(t) + \eta'(t)F'(\eta(t)). \tag{8.6}$$

In view of $\widetilde{A} = \eta'(t)$, $f(t) = F'(\eta(t))$ and $g(t) = B$, where $f(z)$ is analytic in Ω^- and $g(z)$ is analytic in Ω^+, we rewrite (8.6) as

$$\widetilde{A}f(\eta(t)) = \mathrm{i}\delta'(t) - \widetilde{A}g(\eta(t)). \tag{8.7}$$

Assuming $\widetilde{A}g(t) = \psi + \mathrm{i}\varphi$, then by [18, Theorems 2(c) and 2(d)], we obtain

$$\begin{gathered}(\mathbf{I} + \mathbf{N}^*)(\delta'(t) - \varphi(t)) = \widetilde{\mathbf{M}}\psi(t),\\ (\mathbf{I} - \mathbf{N}^*)\varphi(t) = \widetilde{\mathbf{M}}\psi(t).\end{gathered} \tag{8.8}$$

These two equations yields

$$(\mathbf{I} + \mathbf{N}^*)\delta'(t) = 2\varphi(t). \tag{8.9}$$

Note that, the images of the curves $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$ are slits, so we have $\delta_j(2\pi) - \delta_j(0) = 0$, which implies

$$\mathbf{J}\delta'(t) = (0, 0, \ldots, 0). \tag{8.10}$$

Hence, the unknown function $\delta'(t)$ is the unique solution of the integral equation

$$(\mathbf{I} + \mathbf{N}^* + \mathbf{J})\delta'(t) = 2\varphi(t), \tag{8.11}$$

where

$$\varphi(t) = \operatorname{Im}\left[\tilde{A}(t)g(\eta(t))\right] = \operatorname{Im}\left[\eta'(t)B\right]. \tag{8.12}$$

From (8.1), we introduce an analytic function $\vartheta(\eta(t))$ such that

$$\vartheta(\eta(t)) = e^{F(z)} = e^{-B\eta(t)}e^{R(t)+\mathrm{i}\delta(t)}, \tag{8.13}$$

where $\vartheta(\infty) = 1$. By differentiating (8.13) with respect to t, we obtain

$$\vartheta'(\eta(t))\eta'(t) = \left(\mathrm{i}\delta'(t) - B\eta'(t)\right)\vartheta(\eta(t)). \tag{8.14}$$

Let $\sigma(t)$ be an analytic function such that it has the following representation

$$\sigma(\eta(t)) = \vartheta'(\eta(t)) + B\vartheta(\eta(t)), \tag{8.15}$$

where $\sigma(\infty) = B$. From (8.13)–(8.15) it can be shown that, the function $\vartheta(\eta(t))$ can be rewritten as

$$\vartheta(\eta(t)) = e^{R_j}e^{(-\operatorname{Re}[B\eta(t)])}\frac{\operatorname{sign}(\delta'(t))}{\mathrm{i}}T(\eta(t))\frac{\sigma(\eta(t))}{|\sigma(\eta(t))|}. \tag{8.16}$$

By squaring both sides of (8.16), the sign $\delta'(t)$ is eliminated, that is,

$$\vartheta(\eta(t))^2 = -e^{2R_j}e^{(-2\operatorname{Re}[B\eta(t)])}T(\eta(t))^2\frac{\sigma(\eta(t))^2}{|\sigma(\eta(t))|^2}. \tag{8.17}$$

Comparing (8.17) with (3.2) leads to a choice of $P(z) = \vartheta(z)^2$, $P(\infty) = 1$, $D(z) = \sigma(z)$, $D(\infty) = B$, $c(t) = -e^{2R_j}e^{(-2\operatorname{Re}[B\eta(t)])}$. Hence,

$$\phi(t) = \sigma(\eta(t))\eta'(t) \tag{8.18}$$

satisfies the integral equation (3.16) with

$$\nu(t) = -e^{2R_j}e^{(-2\operatorname{Re}[B\eta(t)])}\overline{\eta'(t)B} + B\eta'(t). \tag{8.19}$$

Numerical evidence shows that $\lambda = -1$ is an eigenvalue of $K(s,t)$ of multiplicity m, which suggests one needs to add m conditions. Since $e^{B\eta_j(t)}\vartheta(\eta_j(t))|_0^{2\pi} = 0$, hence we can have m additional conditions for the integral equation above as in the following:

$$\begin{gathered}
\int_{J_j} \frac{d}{dt}\left(e^{B\eta_j(t)}\vartheta_j(\eta_j(t))\right)dt = 0, \\
\int_0^{2\pi} e^{B\eta_j(t)}\left(\vartheta_j'(\eta_j(t)) + B\vartheta(\eta_j(t))\right)\eta'(t)dt = 0, \\
\int_0^{2\pi} e^{B\eta_j(t)}(\sigma(\eta_j(t)))\eta'(t)dt = 0, \\
\int_0^{2\pi} e^{B\eta_j(t)}\phi(\eta_j(t))dt = 0, \quad q = 1, 2, \ldots, m.
\end{gathered} \tag{8.20}$$

We define Fredholm operator $\mathbf{L}$ as

$$\mathbf{L}\mu(s) = \int_{J_j} e^{B\eta_j(t)}\mu(t)dt. \tag{8.21}$$

Then, $\phi(t)$ is the solution of the following integral equation:

$$(\mathbf{I} + \mathbf{K} + \mathbf{L})\phi(t) = \nu(t). \tag{8.22}$$

Hence, by obtaining $\phi(\eta(t))$, the function $\sigma(t)$ can be found by

$$\sigma(t) = \frac{\phi(\eta(t))}{\eta'(t)}. \tag{8.23}$$

This allows the value for $\vartheta(\eta(t))$ to be calculated from (8.16), which in turn allows the boundary values for the mapping function $\Phi(\eta(t))$ to be calculated by

$$\Phi(\eta(t)) = \eta(t) + \overline{B}\log\vartheta(\eta(t)). \tag{8.24}$$

9. Numerical Examples

Since the boundaries Γ_j are parameterized by $\eta(t)$ which are 2π-periodic functions, the reliable method to solve the integral equations are by means of Nyström method with trapezoidal rule [21]. Each boundary will be discretized by n number of equidistant points. The resulting linear systems are then solved by using Gaussian elimination. For numerical examples, we choose regions with connectivities one, two, three and four. For the region

Table 1: Error norm $\|w_j - \widehat{w}_j\|_\infty$ for Example 9.1.

n	U_c	U_r	$U_{p,\pi/3}$
32	6.2×10^{-12}	4.6×10^{-14}	8.8×10^{-16}
64	9.1×10^{-14}	9.6×10^{-15}	—

with connectivity one, we compare our result with the analytic solution given in [12]. All the computations were done by using MATLAB R2008a software.

Example 9.1. Consider an unbounded region Ω^- bounded by a unit circle

$$\Gamma_1(t) = e^{-\mathrm{i}t}, \quad (0 \le t \le 2\pi). \tag{9.1}$$

We choose the special point $\beta = 2.5 + 1.5\mathrm{i}$. The exact mapping function for U_c, U_r, and U_p are given respectively by [12]

$$\begin{aligned} w_c &= z - \beta + \frac{\beta - z}{1 - \overline{\beta} z}, \\ w_r &= z - \beta - \frac{z - \beta}{\overline{\beta} z}, \\ w_p &= z + \frac{e^{2\mathrm{i}\theta}}{z}. \end{aligned} \tag{9.2}$$

For this example, we compare the error for each boundary value between our method and the exact mapping function. See Table 1 for Error Norm of $\|w_j - \widehat{w}_j\|_\infty$.

Example 9.2. Consider an unbounded region Ω^- bounded by a circle and an ellipse

$$\begin{aligned} \Gamma_1(t) &= 2 + \mathrm{i} + e^{-\mathrm{i}t}, \quad (0 \le t \le 2\pi), \\ \Gamma_2(t) &= -2 + \cos t - 2\mathrm{i} \sin t, \quad (0 \le t \le 2\pi). \end{aligned} \tag{9.3}$$

Figure 2 shows the region and its five canonical images by using our proposed method.

Example 9.3. Consider an unbounded region Ω^- bounded by 3 circles

$$\begin{aligned} \Gamma_1(t) &= 2 + e^{-\mathrm{i}t}, \quad (0 \le t \le 2\pi), \\ \Gamma_2(t) &= -1 + \mathrm{i}\sqrt{3} + 0.5e^{-\mathrm{i}t}, \quad (0 \le t \le 2\pi), \\ \Gamma_3(t) &= -1 - \mathrm{i}\sqrt{3} + 1.5e^{-\mathrm{i}t}, \quad (0 \le t \le 2\pi). \end{aligned} \tag{9.4}$$

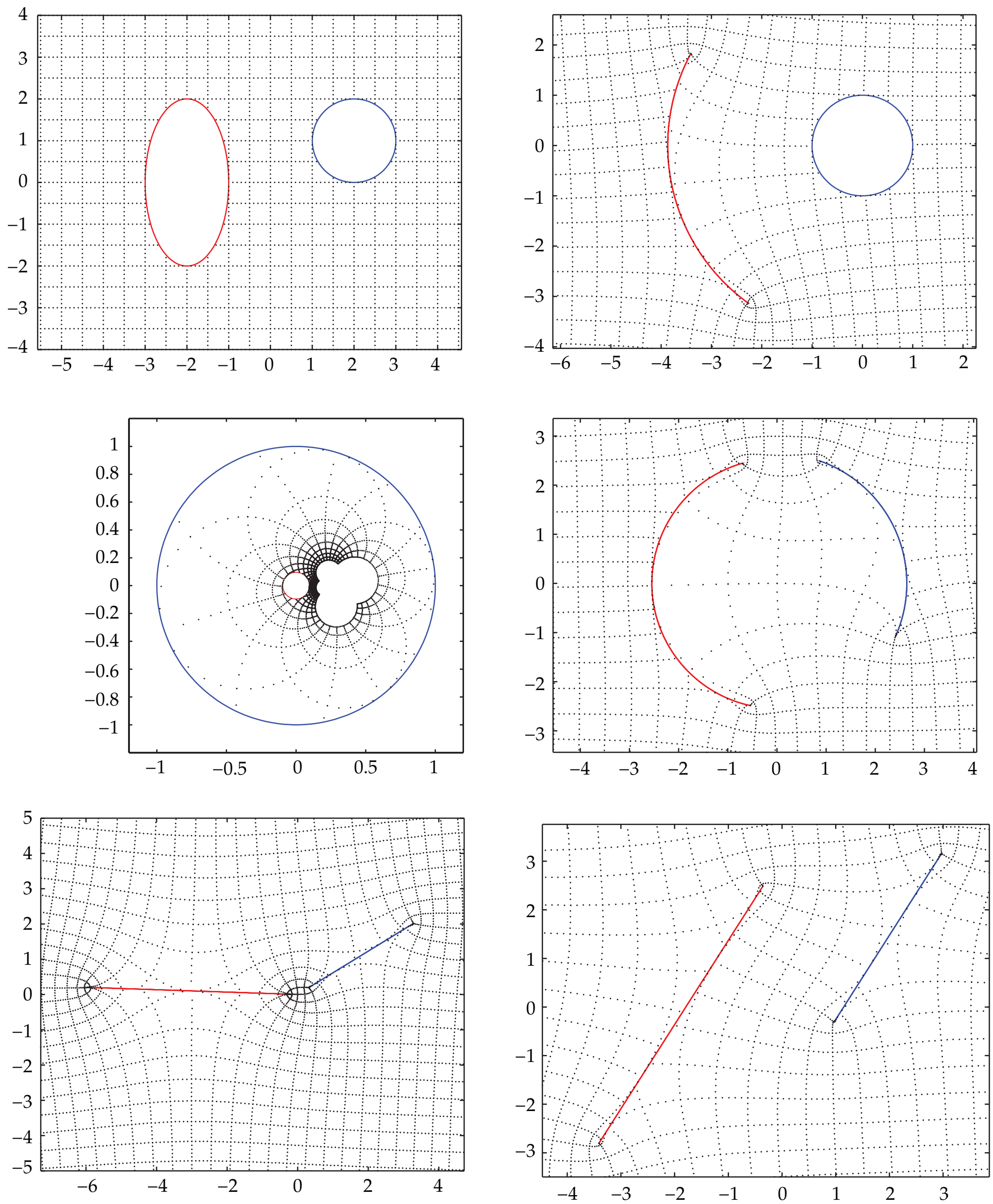

Figure 2: The original region Ω^{-} and its canonical images with $\theta_p = \pi/3$ for parallel slits.

This example has also been considered in [6, 12]. Figure 3 shows the regions and its five canonical images by using our proposed method. See Table 3 for numerical comparison between our parameter values (see Table 2) those in [12]. Note that our method has considered exterior unit disk with slits as a canonical region while [12] has considered interior

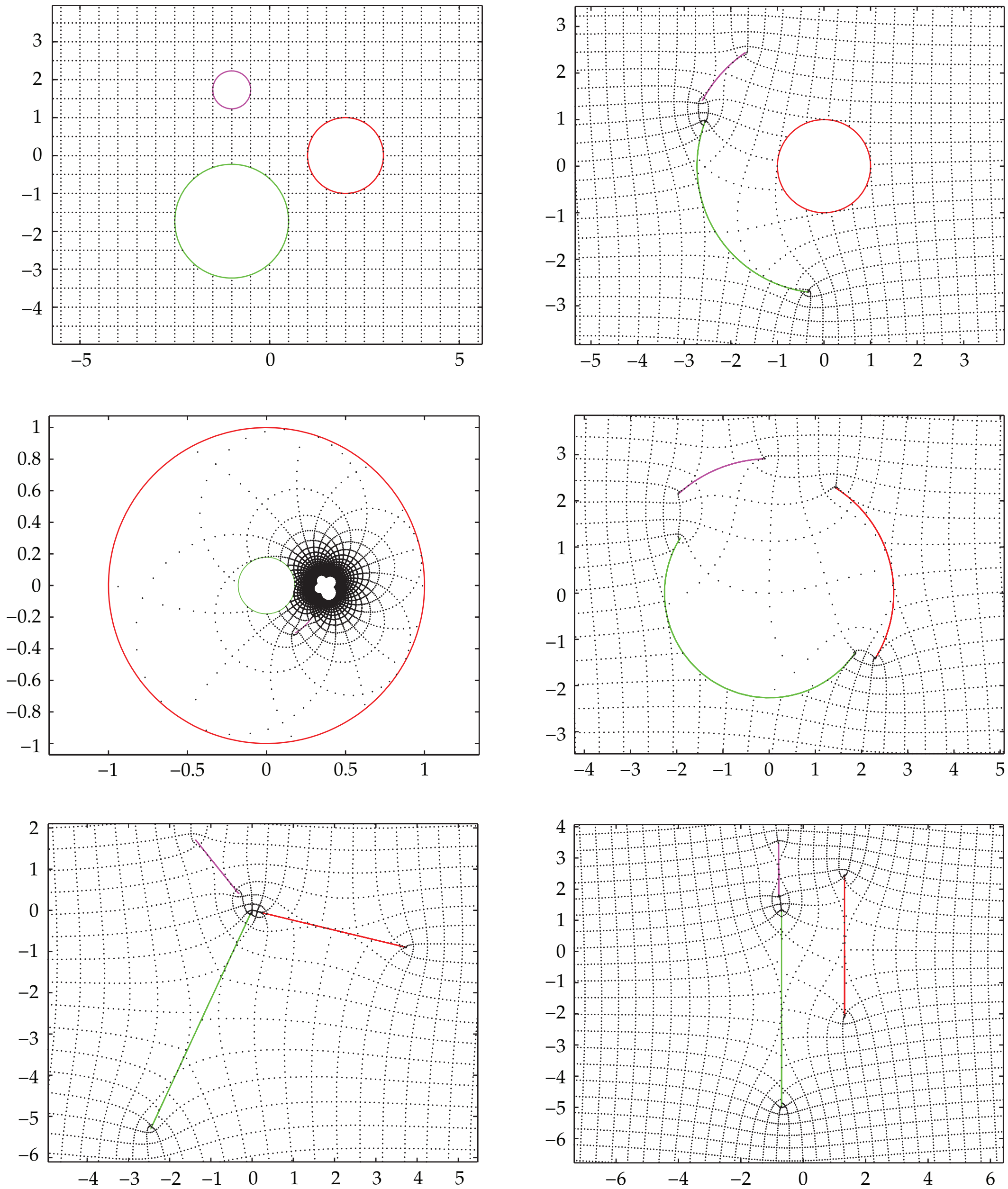

Figure 3: The original region Ω^- and its canonical images with $\theta_p = \pi/2$ for parallel slits.

unit disk with slits. Thus, in computing the error for U_d, we need to change the values for U_d to $1/|\Phi(z)|$. See Table 4 for Error Norm of $\max_{1\le j\le 3}\|w_j - \widehat{w}_j\|_\infty$. We also compared the condition number of our linear system for each n with [6, 12], see Figures 4 and 5. The results show that for our integral equations for finding $\Phi'(z)$ that is, (4.20), (5.20), (6.21), (7.21) and (8.22), the condition numbers are almost constant except for (5.20). This is because the kernel

Table 2: The values for approximated parameters in Example 9.3 with $n = 256$.

j	$R_j(U_d)$	$R_j(U_a)$	$R_j(U_c)$	$\theta_j(U_r)$	$R_j(U_{p,\pi/2})$
1	1.0000000000	1.0000000000	2.6958524041	−0.23582974094	1.32203173492
2	2.9672620504	0.3515929850	2.9121788457	2.24673051228	−0.78705294688
3	2.7249636495	0.1792099292	2.2653736950	−2.00502589294	−0.69725704737

Table 3: Error norm $\max_{1\le j\le 3} \|R_j - \widehat{R}_j\|_\infty$ of our method with [12] for Example 9.3.

n	U_d	U_a	U_c	U_r	$U_{p,\pi/2}$
32	2.7×10^{-11}	3.4×10^{-11}	2.9×10^{-01}	1.6×10^{-01}	2.0×10^{-10}
64	5.6×10^{-17}	1.1×10^{-16}	4.4×10^{-05}	2.0×10^{-05}	2.2×10^{-16}
128	7.8×10^{-16}	4.7×10^{-16}	2.2×10^{-09}	4.1×10^{-10}	8.9×10^{-16}
256	1.6×10^{-15}	9.99×10^{-16}	3.8×10^{-12}	5.2×10^{-13}	1.3×10^{-15}

Table 4: Error norm $\max_{1\le j\le 3} \|w_j - \widehat{w}_j\|_\infty$ of our method with [12] for Example 9.3.

n	U_d	U_a	U_c	U_r	$U_{p,\pi/2}$
32	1.1×10^{-07}	1.1×10^{-07}	0.15	0.16	1.5×10^{-06}
64	6.1×10^{-14}	2.9×10^{-14}	6.2×10^{-05}	2.0×10^{-05}	1.1×10^{-13}
128	7.5×10^{-14}	2.1×10^{-14}	2.2×10^{-09}	4.2×10^{-10}	4.4×10^{-12}
256	2.2×10^{-13}	4.7×10^{-13}	3.3×10^{-12}	4.1×10^{-12}	9.5×10^{-12}

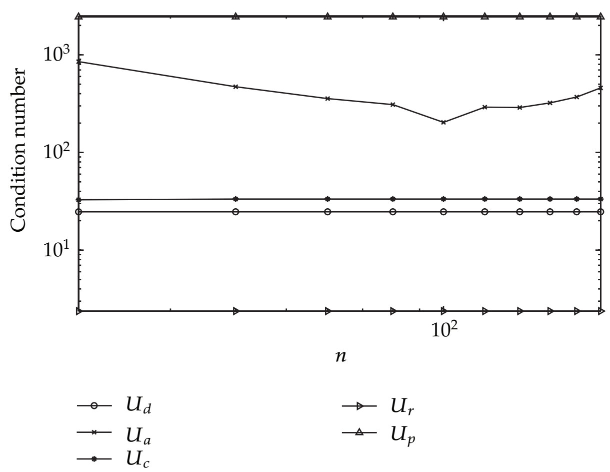

Figure 4: Condition numbers of the matrices for our method for U_d, U_a, U_c, U_r and U_p.

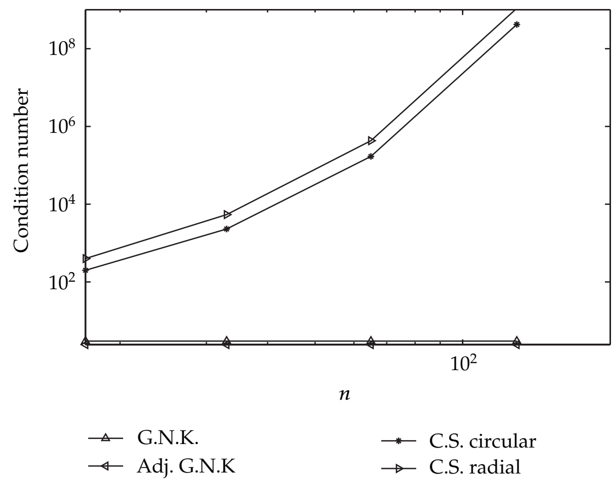

Figure 5: Condition number of the matrices for generalized Neumann kernel (G.N.K), adjoint generalized Neumann kernel (Adj.G. N. K), charge simulation for circular slits (C.S. circular) and charge simulation for radial slits (C.S. radial).

involves $\theta'(t)$, which varies with the number of collocation points, n. However, this does not have any effect on the accuracy of the method.

Example 9.4. Consider an unbounded region Ω^- of 4-connectivity with boundaries

$$\Gamma_1(t) = 3 + 2\mathrm{i} + e^{-\mathrm{i}t}, \quad (0 \le t \le 2\pi), \tag{9.5}$$

$$\Gamma_2(t) = -3 + 2\mathrm{i} + e^{-\mathrm{i}t}, \quad (0 \le t \le 2\pi), \tag{9.6}$$

$$\Gamma_3(t) = -3 - 2\mathrm{i} + 0.7\cos t - 1.4\mathrm{i}\,\sin t, \quad (0 \le t \le 2\pi), \tag{9.7}$$

$$\Gamma_4(t) = 3 - 2\mathrm{i} + 0.7\cos t - 1.4\mathrm{i}\,\sin t, \quad (0 \le t \le 2\pi). \tag{9.8}$$

Figure 6 shows the region and its five canonical images by using our proposed method.

10. Conclusion

In this paper, we have constructed a unified method for numerical conformal mapping of unbounded multiply connected regions onto canonical slit regions. The advantage of this method is that the integral equations are all linear which overcomes the nonlinearity problem encountered in [10]. From the numerical experiments, we can conclude that our method works on any finite connectivity with high accuracy. By computing the boundary values of the mapping function, the exterior points will be calculated by means of Cauchy's integral formula.

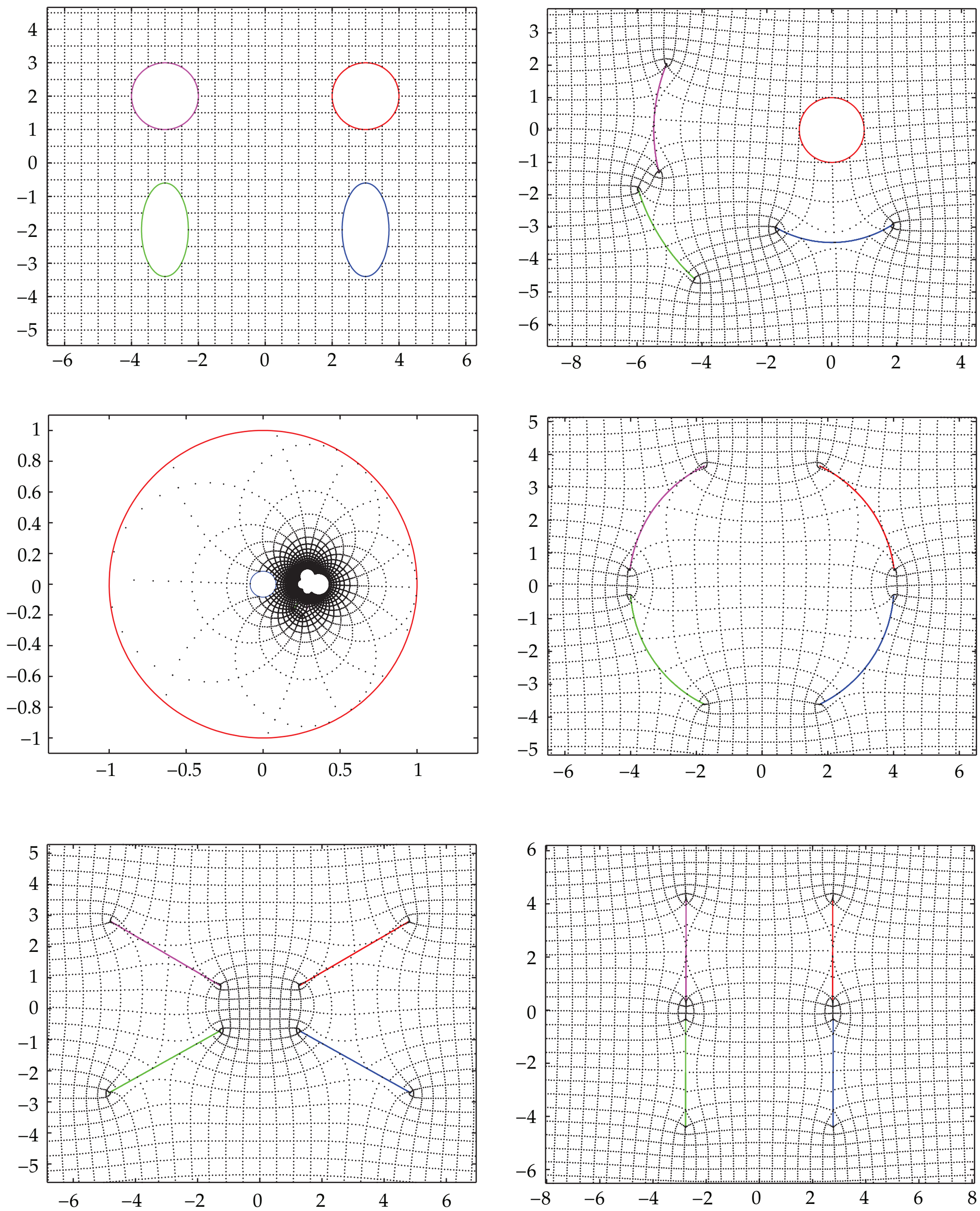

Figure 6: The original region Ω^- and its canonical images with $\theta_p = \pi/2$ for parallel slits.

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Research Article

Uniqueness Theorems on Difference Monomials of Entire Functions

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The aim of this paper is to discuss the uniqueness of the difference monomials $f^n f(z+c)$. It assumed that f and g are transcendental entire functions with finite order and $E_{k)}(1, f^n f(z+c)) = E_{k)}(1, g^n g(z+c))$, where c is a nonzero complex constant and n, k are integers. It is proved that if one of the following holds (i) $n \geq 6$ and $k = 3$, (ii) $n \geq 7$ and $k = 2$, and (iii) $n \geq 10$ and $k = 1$, then $fg = t_1$ or $f = t_2 g$ for some constants t_2 and t_3 which satisfy $t_2^{n+1} = 1$ and $t_3^{n+1} = 1$. It is an improvement of the result of Qi, Yang and Liu.

1. Introduction and Main Results

In this paper, a meromorphic (respectively entire) function always means meromorphic (respectively, analytic) in the complex plane $\mathbb{C}$. It is also assumed that the reader is familiar with the basic concepts of the Nevanlinna theory. We adopt the standard notations in the Nevanlinna value distribution theory of meromorphic functions as explained in [1, 2].

Let f and g be two nonconstant meromorphic functions, and let a be a value in the extended plane. We say that f and g share the value a CM, provided that f and g have the same a-pints with the same multiplicities. We say that f and g share the value a IM, provided that f and g have the same a-points ignoring multiplicities. The order of f is defined by

$$\sigma(f) = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r}. \tag{1.1}$$

Let f be a nonconstant meromorphic function on $\mathbb{C}$, let $a \in \mathbb{C}$ be a finite value, and let k be a positive integer or infinity. We denote by $E(a, f)$ the set of zeros of $f - a$ and count multiplicities, while by $\overline{E}(a, f)$ the set of zeros of $f - a$ but ignore multiplicities. Also, we denote by $E_{k)}(a, f)$ the set of zeros of $f - a$ with multiplicities less than or equal to k and count multiplicities. For $a \in \mathbb{C}\bigcup\{\infty\}$, we denote by $N_{k)}(r, 1/(f - a))$ the counting function corresponding to the set $E_{k)}(a, f)$ while by $N_{(k+1}(r, 1/(f - a))$ the counting function corresponding to the set $E_{(k+1}(a, f) := E(a, f) \setminus E_{k)}(a, f)$. If $E_k(a, f) = E_k(a, g)$, we say that f, g share the value a with weight k.

The definition implies that if f and g share a value a with weight k, then z_0 is a zero of $f - a$ with multiplicity $m(\leq k)$ if and only if it is a zero of $g - a$ with multiplicity $m(\leq k)$ and z_0 is a zero of $f - a$ with multiplicity $m(> k)$ if and only if it is a zero of $g - a$ with multiplicity $n(> k)$ where m is not necessarily equal to n.

Also, we denote by $\overline{N}_{k)}(r, 1/(f - a))$ and $\overline{N}_{(k+1}(r, 1/(f - a))$ the reduced forms of $N_{k)}(r, 1/(f - a))$ and $N_{(k+1}(r, 1/(f - a))$, respectively. At last, we set

$$N_k(r, f) = \overline{N}(r, f) + \overline{N}_{(2}(r, f) + \cdots + \overline{N}_{(k}(r, f). \tag{1.2}$$

Hayman proposed the well-known conjecture in [3].

Hayman Conjecture

If an entire function f satisfies $f^n f' \neq 1$ for all $n \in \mathbb{N}$, then f is a constant.

In fact, Hayman has proved that the conjecture holds in the cases $n \geq 2$ in [4] while Clunie proved the cases $n = 1$ in [5], respectively. In 1997, Yang and Hua [6] studied the uniqueness theorem of the differential monomials and obtained the following result.

Theorem A. *Let f and g be nonconstant entire function, and let $n \geq 6$ be an integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$ where c_1, c_2 and c are constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$, or $f = tg$ for a constant t such that $t^{n+1} = 1$.*

In 2010, Qi et al. [7] studied the uniqueness of the difference monomials and obtained the following result.

Theorem B. *Let f and g be transcendental entire functions with finite order, c a non-zero complex constant, and $n \geq 6$ an integer. If $E(1, f^n f(z + c)) = E(1, g^n g(z + c))$, then $fg = t_1$ or $f = t_2 g$ for some constants t_1 and t_2 which satisfy $t_1^{n+1} = 1$ and $t_2^{n+1} = 1$.*

In this paper, we will obtain the following results.

Theorem 1.1. *Let f and g be transcendental entire functions with finite order, c a non-zero complex constant, and $n \geq 6$ an integer. If $E_{3)}(1, f^n f(z + c)) = E_{3)}(1, g^n g(z + c))$, then the assertion of Theorem B holds.*

Theorem 1.2. *Let f and g be transcendental entire functions with finite order, c a non-zero complex constant, and $n \geq 7$ an integer. If $E_{2)}(1, f^n f(z + c)) = E_{2)}(1, g^n g(z + c))$, then the assertion of Theorem B holds.*

Theorem 1.3. *Let f and g be transcendental entire functions with finite order, c a non-zero complex constant, and $n \geq 10$ an integer. If $E_{1)}(1, f^n f(z+c)) = E_{1)}(1, g^n g(z+c))$, then the assertion of Theorem B holds.*

2. Auxiliary Results

Lemma 2.1 (see [8, Corollary 2.5]). *Let $f(z)$ be a meromorphic function in the complex plane with finite order $\sigma = \sigma(f)$, and let η be a fixed non-zero complex number. Then for each $\varepsilon > 0$, one has*

$$m\left(r, \frac{f(z+\eta)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+\eta)}\right) = O\left(r^{\sigma-1+\varepsilon}\right). \tag{2.1}$$

Lemma 2.2 (see [8, Theorem 2.1]). *Let $f(z)$ be a meromorphic function in the complex plane with finite order $\sigma = \sigma(f)$, and let η be a fixed non-zero complex number. Then for each $\varepsilon > 0$, one has*

$$T(r, f(z+\eta)) = T(r, f(z)) + O\left(r^{\sigma-1+\varepsilon}\right) + O(\log r). \tag{2.2}$$

Lemma 2.3. *Let $f(z)$ be an entire function with finite order $\sigma = \sigma(f)$, c a fixed non-zero complex number, and*

$$P(z) = a_n f(z)^n + a_{n-1} f(z)^{n-1} + \cdots + a_1 f(z) + a_0, \tag{2.3}$$

where $a_j (j = 0, 1, \ldots, n)$ are constants. If $F(z) = P(z) f(z+c)$, then

$$T(r, F) = (n+1) T(r, f) + O\left(r^{\sigma(f)-1+\varepsilon}\right) + O(\log r). \tag{2.4}$$

Proof. Since $f(z)$ is an entire transcendental function with finite order, we can deduce from Lemma 2.1 and the standard Valiron-Mohon'ko theorem that

$$\begin{aligned}
(n+1) T(r, f(z)) &= T(r, f(z) P(z)) + O(1) \\
&= m(r, f(z) P(z)) + O(1) \\
&\leq m\left(r, \frac{f(z) P(z)}{F(z)}\right) + m(r, F(z)) + O(1) \\
&= m\left(r, \frac{f(z)}{f(z+c)}\right) + m(r, F(z)) + O(1) \\
&\leq T(r, F(z)) + O\left(r^{\sigma-1+\varepsilon}\right) + O(1).
\end{aligned} \tag{2.5}$$

Therefore

$$T(r, F) \geq (n+1) T(r, f) + O\left(r^{\sigma-1+\varepsilon}\right) + O(1). \tag{2.6}$$

On the other hand, Lemma 2.2 implies that

$$\begin{aligned}T(r,F(z)) &\le T(r,P(z)) + T(r,f(z+c))\\ &= nT(r,f(z)) + T(r,f(z)) + O\left(r^{\sigma-1+\varepsilon}\right) + O(\log r)\\ &= (n+1)T(r,f) + O\left(r^{\sigma-1+\varepsilon}\right) + O(\log r).\end{aligned} \tag{2.7}$$

We will obtain the conclusion of Lemma 2.3. □

Remark 2.4. The condition "entire" cannot be replaced by "meromorphic" in Lemma 2.3, as is shown by the following example.

Example 2.5. Let $f(z) = (e^z - 1)/(e^z + 1)$, $c = \pi i$, and $F(z) = f(z)f(z+c)$, we can see

$$T(r,F) \not\equiv 2T(r,f) + O\left(r^{\sigma(f)-1+\varepsilon}\right) + O(\log r) \tag{2.8}$$

for every set of $\{r_n\}$ with infinite measure.

Lemma 2.6 (see [9, Lemma 2.1]). *Let f and g be two nonconstant meromorphic functions satisfying $E_{k)}(1,f) = E_{k)}(1,g)$ for some positive integer $k \in \mathbb{N}$. Define H as follows:*

$$H = \left(\frac{f''}{f'} - \frac{2f'}{f-1}\right) - \left(\frac{g''}{g'} - \frac{2g'}{g-1}\right). \tag{2.9}$$

If $H \not\equiv 0$, then

$$\begin{aligned}N(r,H) \le\ & \overline{N}_{(2}(r,f) + \overline{N}_{(2}\left(r,\frac{1}{f}\right) + \overline{N}_{(2}(r,g) + \overline{N}_{(2}\left(r,\frac{1}{g}\right) + \overline{N}_0\left(r,\frac{1}{f'}\right)\\ &+ \overline{N}_0\left(r,\frac{1}{g'}\right) + \overline{N}_{(k+1}\left(r,\frac{1}{f-1}\right) + \overline{N}_{(k+1}\left(r,\frac{1}{g-1}\right)\\ &+ S(r,f) + S(r,g),\end{aligned} \tag{2.10}$$

where $\overline{N}_0(r,1/f')$ denotes the counting function of zeros of f' but not zeros of $f(f-1)$ and $\overline{N}_0(r,1/g')$ is similarly defined.

Lemma 2.7 (see [10]). *Under the condition of Lemma 2.6, one has*

$$N_{1)}\left(r,\frac{1}{f-1}\right) = N_{1)}\left(r,\frac{1}{g-1}\right) \le N(r,H) + S(r,f) + S(r,g). \tag{2.11}$$

Lemma 2.8 (see [10]). *Let H be defined as Lemma 2.6. If $H \equiv 0$, then either $f \equiv g$ or $fg \equiv 1$ provided that*

$$\limsup_{r\to\infty, r\in I} \frac{\overline{N}(r,f)+\overline{N}(r,g)+\overline{N}(r,1/f)+\overline{N}(r,1/f)}{T(r)} < 1, \tag{2.12}$$

where $T(r) := \max\{T(r,f), T(r,g)\}$ and I is a set with infinite linear measure.

Lemma 2.9 (see [11, Lemma 2.2]). *Let $T : (0,+\infty) \to (0,+\infty)$ be a nondecreasing continuous function, $s > 0$, $0 < \alpha < 1$, and let $F \subset R_+$ be the set of all r such that*

$$T(r) \le \alpha T(r+s). \tag{2.13}$$

If the logarithmic measure of F is infinite, then

$$\limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} = \infty. \tag{2.14}$$

3. Proof of Theorem 1.1

We define

$$\begin{aligned} F &:= f^n f(z+c), \\ G &:= g^n g(z+c). \end{aligned} \tag{3.1}$$

First of all, suppose that $H \not\equiv 0$. We replace f and g by F and G, respectively, in Lemma 2.7 and Lemma 2.8. Thus,

$$\begin{aligned} N_{1)}\left(r,\frac{1}{F-1}\right) = N_{1)}\left(r,\frac{1}{G-1}\right) &\le N(r,H) + S(r,f) + S(r,g) \\ &\le \overline{N}_{(2}\left(r,\frac{1}{F}\right) + \overline{N}_{(2}\left(r,\frac{1}{G}\right) + \overline{N}_{0}\left(r,\frac{1}{F'}\right) + \overline{N}_{0}\left(r,\frac{1}{G'}\right) \\ &\quad + \overline{N}_{(4}\left(r,\frac{1}{F-1}\right) + \overline{N}_{(4}\left(r,\frac{1}{G-1}\right) + S(r,f) + S(r,g). \end{aligned} \tag{3.2}$$

Applying the second main theorem to F and G jointly implies that

$$\begin{aligned} T(r,F) + T(r,G) \le{}& \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{F-1}\right) + \overline{N}\left(r,\frac{1}{G}\right) + \overline{N}\left(r,\frac{1}{G-1}\right) \\ & - \overline{N}_{0}\left(r,\frac{1}{F'}\right) - \overline{N}_{0}\left(r,\frac{1}{G'}\right) + S(r,f) + S(r,g). \end{aligned} \tag{3.3}$$

Noting that

$$\begin{aligned}
&\overline{N}\left(r,\frac{1}{F-1}\right)-\frac{1}{2}N_{1)}\left(r,\frac{1}{F-1}\right)+\overline{N}_{(4}\left(r,\frac{1}{F-1}\right)\le\frac{1}{2}N\left(r,\frac{1}{F-1}\right)\le\frac{1}{2}T(r,F),\\
&\overline{N}\left(r,\frac{1}{G-1}\right)-\frac{1}{2}N_{1)}\left(r,\frac{1}{G-1}\right)+\overline{N}_{(4}\left(r,\frac{1}{G-1}\right)\le\frac{1}{2}N\left(r,\frac{1}{G-1}\right)\le\frac{1}{2}T(r,G).
\end{aligned}\tag{3.4}$$

According to Lemma 2.9 and (3.2)–(3.4), we can obtain that

$$\begin{aligned}
T(r,F)+T(r,G)&\le 2N_2\left(r,\frac{1}{F}\right)+2N_2\left(r,\frac{1}{G}\right)+S(r,f)+S(r,g)\\
&\le 4\overline{N}\left(r,\frac{1}{f}\right)+4\overline{N}\left(r,\frac{1}{g}\right)+2N\left(r,\frac{1}{f(z+c)}\right)\\
&\quad+2N\left(r,\frac{1}{g(z+c)}\right)+S(r,f)+S(r,g)\\
&\le 6N\left(r,\frac{1}{f}\right)+6N\left(r,\frac{1}{g}\right)+S(r,f)+S(r,g)\\
&\le 6T\left(r,\frac{1}{f}\right)+6T\left(r,\frac{1}{g}\right)+S(r,f)+S(r,g).
\end{aligned}\tag{3.5}$$

Lemma 2.3 shows that

$$\begin{aligned}
T(r,F)&=(n+1)T(r,f)+O\left(r^{\sigma(f)-1+\varepsilon}\right)+O(\log r),\\
T(r,G)&=(n+1)T(r,g)+O\left(r^{\sigma(g)-1+\varepsilon}\right)+O(\log r).
\end{aligned}\tag{3.6}$$

We can deduce that

$$(n-5)(T(r,f)+T(r,g))\le O\left(r^{\sigma(f)-1+\varepsilon}\right)+O\left(r^{\sigma(g)-1+\varepsilon}\right)+S(r,f)+S(r,g),\tag{3.7}$$

which is impossible since $n\ge 6$. Therefore, we have $H\equiv 0$. Noting that

$$N\left(r,\frac{1}{F}\right)+N\left(r,\frac{1}{G}\right)\le 3T(r,f)+3T(r,g)+S(r,f)+S(r,g)\le T(r),\tag{3.8}$$

where $T(r)=\max\{T(r,F),T(r,G)\}$. Together with Lemma 2.8, it shows that either $F\equiv G$ or $FG\equiv 1$. We will consider the following two cases.

Case 1. Suppose that $F(z)=G(z)$. Therefore

$$f(z)^n f(z+c)=g(z)^n g(z+c).\tag{3.9}$$

Let $h_1(z) = f(z)/g(z)$; we have

$$h_1(z)^n h_1(z+c) \equiv 1. \tag{3.10}$$

If $h_1(z)$ is not a constant, then Lemma 2.2 and (3.10) imply that

$$nT(r,h_1) = T(r,h_1(z+c)) + O(1) = T(r,h_1) + O\left(r^{\sigma(h_1)-1+\varepsilon}\right) + O(\log r), \tag{3.11}$$

which is a contraction with $n \geq 6$. Thus, $h_1(z) \equiv t_1$, where t_1 is a constant. From (3.10), we have $f(z) = t_1 g(z)$ and $t_1^{n+1} = 1$.

Case 2. Suppose that $F(z)G(z) \equiv 1$. Therefore

$$f(z)^n f(z+c) g(z)^n g(z+c) \equiv 1. \tag{3.12}$$

Let $h_2(z) = f(z)g(z)$; we have

$$h_2(z)^n h_2(z+c) \equiv 1. \tag{3.13}$$

By the same way as Case 1, we can obtain that h_2 is a constant. Therefore, $f(z)g(z) = t_2$ and $t_2^{n+1} = 1$.

4. Proof of Theorem 1.2

Noting that

$$\begin{aligned} &\overline{N}\left(r,\frac{1}{F-1}\right) - \frac{2}{5}N_{1)}\left(r,\frac{1}{F-1}\right) + \frac{4}{5}\overline{N}_{(3}\left(r,\frac{1}{F-1}\right) \leq \frac{3}{5}N\left(r,\frac{1}{F-1}\right) \leq \frac{3}{5}T(r,F), \\ &\overline{N}\left(r,\frac{1}{G-1}\right) - \frac{2}{5}N_{1)}\left(r,\frac{1}{G-1}\right) + \frac{4}{5}\overline{N}_{(3}\left(r,\frac{1}{G-1}\right) \leq \frac{3}{5}N\left(r,\frac{1}{G-1}\right) \leq \frac{3}{5}T(r,G). \end{aligned} \tag{4.1}$$

According to (3.1) and (4.1), we can obtain the conclusion of Theorem 1.2 by the same way as Section 3.

5. Proof of Theorem 1.3

Noting that

$$\begin{aligned} &\overline{N}\left(r,\frac{1}{F-1}\right) - \frac{1}{4}N_{1)}\left(r,\frac{1}{F-1}\right) + \frac{1}{2}\overline{N}_{(2}\left(r,\frac{1}{F-1}\right) \leq \frac{3}{4}N\left(r,\frac{1}{F-1}\right) \leq \frac{3}{4}T(r,F), \\ &\overline{N}\left(r,\frac{1}{G-1}\right) - \frac{1}{4}N_{1)}\left(r,\frac{1}{G-1}\right) + \frac{1}{2}\overline{N}_{(2}\left(r,\frac{1}{G-1}\right) \leq \frac{3}{4}N\left(r,\frac{1}{G-1}\right) \leq \frac{3}{4}T(r,G). \end{aligned} \tag{5.1}$$

According to (3.1) and (5.1), we can obtain the conclusion of Theorem 1.2 by the same way as Section 3.

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Research Article

On the Distribution of Zeros and Poles of Rational Approximants on Intervals

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The distribution of zeros and poles of best rational approximants is well understood for the functions $f(x) = |x|^{\alpha}$, $\alpha > 0$. If $f \in C[-1,1]$ is not holomorphic on $[-1,1]$, the distribution of the zeros of best rational approximants is governed by the equilibrium measure of $[-1,1]$ under the additional assumption that the rational approximants are restricted to a bounded degree of the denominator. This phenomenon was discovered first for polynomial approximation. In this paper, we investigate the asymptotic distribution of zeros, respectively, a-values, and poles of best real rational approximants of degree at most n to a function $f \in C[-1,1]$ that is real-valued, but not holomorphic on $[-1,1]$. Generalizations to the lower half of the Walsh table are indicated.

1. Introduction

Let B be a subset of $\mathbb{C}$; we denote by

$$m_1(B) := \inf \sum_{\nu} |U_{\nu}| \tag{1.1}$$

the m_1-*measure of* B, where the infimum is taken over all coverings $\{U_{\nu}\}$ of B by disks U_{ν} and $|U_{\nu}|$ is the radius of the disk U_{ν}.

Let D be a region in $\mathbb{C}$ and φ a function defined in D with values in $\overline{\mathbb{C}}$. A sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ of meromorphic functions in D is said to *converge to a function* φ *with* *respect to the m_1-measure inside D* if for every $\varepsilon > 0$ and any compact set $K \subset D$ we have

$$m_1(\{z \in K : |(\varphi - \varphi_n)(z)| \geq \varepsilon\}) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty \tag{1.2}$$

(cf. Gončar [1]).

The sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ is said to converge to φ *m_1-almost geometrically inside D* if for any $\varepsilon > 0$ there exists a set $\Omega(\varepsilon)$ in $\mathbb{C}$ with $m_1(\Omega(\varepsilon)) < \varepsilon$ such that

$$\limsup_{n\to\infty} \|\varphi - \varphi_n\|_{K\setminus\Omega(\varepsilon)}^{1/n} < 1 \tag{1.3}$$

for any compact set $K \subset D$. We note that $\|\cdot\|_B$ is the supremum norm on a subset B of $\mathbb{C}$.

For $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, we denote by $\mathcal{P}_n$ the collection of all polynomials of degree at most n, and let

$$\mathcal{R}_{n,m} := \left\{ r = \frac{p}{q} : p \in \mathcal{P}_n, q \in \mathcal{P}_m, q \not\equiv 0 \right\}. \tag{1.4}$$

In [2], sequences $\{r_n\}_{n\in\mathbb{N}}$, $r_n \in \mathcal{R}_{n,n}$, on a region D were investigated if the number of poles of r_n in D is bounded. It turns out that the geometric convergence of $\{r_n\}_{n\in\mathbb{N}}$ on a continuum $S \subset D$ implies that the sequence converges m_1-almost geometrically inside D to a meromorphic function f in D with at most a finite number of poles in D.

To be precise, let $B \subset \mathbb{C}$ and let $\mathcal{M}_m(B)$ denote the subset of meromorphic functions in B with at most m poles in B, each pole counted with its multiplicity. The main result of [2] can be stated as follows.

Theorem A. *Let S be a continuum in $\mathbb{C}$ and D a region with $S \subset D$. Let $\{r_n\}_{n\in\mathbb{N}}$, $r_n \in \mathcal{R}_{n,n}$, be a sequence of rational functions converging geometrically to a function f on S, that is,*

$$\limsup_{n\to\infty} \|f - r_n\|_S^{1/n} < 1, \tag{1.5}$$

and assume that $f \not\equiv 0$ on S. If there exists a fixed integer $m \in \mathbb{N}$ such that $r_n \in \mathcal{M}_m(D)$ for all n and

$$N_0(r_n, K) = o(n) \quad \text{as } n \longrightarrow \infty \tag{1.6}$$

for each compact set $K \subset D$, then the sequence $\{r_n\}_{n\in\mathbb{N}}$ converges m_1-almost geometrically inside D to a meromorphic function $f \in \mathcal{M}_m(D)$.

Here, the number $N_0(r_n, K)$ denotes the number of zeros of r_n in K, each zero counted with its multiplicity.

The above result was applied in [2] to Chebyshev approximation on $[-1,1]$. Let $G(z,\infty)$ be the Green function of $\Omega = \overline{\mathbb{C}} \setminus [-1,1]$ with pole at ∞, and let

$$\mathcal{E}_\rho := \{z \in \mathbb{C} : G(z,\infty) < \log \rho\}, \quad \rho > 1, \tag{1.7}$$

be the Green domain to the parameter ρ, that is, $\mathcal{E}_\rho$ is the open Joukowski-ellipse with foci at $+1$ and -1 and major axis $\rho + 1/\rho$.

Let $f \in C[-1,1]$ be real-valued on $[-1,1]$. For abbreviation, we will write $\|\cdot\|$ for $\|\cdot\|_{[-1,1]}$. Given $n, m \in \mathbb{N}_0$, let $r^*_{n,m} = r^*_{n,m}(f) \in \mathcal{R}_{n,m}$ denote the real rational function of best uniform approximation to $f \in C[-1,1]$ with respect to $\mathcal{R}_{n,m}$, that is,

$$E_{n,m}(f) := \|f - r^*_{n,m}\| = \inf\{\|f - r\| : r \in \mathcal{R}_{n,m},\ r \text{ real-valued on } \mathbb{R}\}. \tag{1.8}$$

Moreover, let $\{m_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{N}$ with

$$\lim_{n\to\infty} m_n = \infty, \quad m_n = o\left(\frac{n}{\log n}\right) \quad \text{as } n \longrightarrow \infty, \tag{1.9}$$

and let us consider a function $f \in C[-1,1]$ that can be continued meromorphically into $\mathcal{E}_\rho$ for some $\rho > 1$. Then the sequence $\{r^*_{n,m_n}\}_{n\in\mathbb{N}}$ converges m_1-almost geometrically inside $\mathcal{E}_\rho$ to f [3]. Using Theorem A, we obtain results about the distribution of the a-values in the neighborhood of a point $z_0 \in \partial E_\rho$. For $a \in \overline{\mathbb{C}}$ and $B \subset \mathbb{C}$, we denote by

$$N_a(r, B) := \#\{z \in B : r(\mathrm{z}) = a\} \tag{1.10}$$

the number of a-values of the rational function r in B and each a-value is counted with its multiplicity. If f cannot be continued meromorphically to z_0, then for any neighborhood U of z_0 and any $a \in \overline{\mathbb{C}}$, with at most one exception,

$$\limsup_{n\to\infty} N_a(r^*_{n,m_n}, U) = \infty. \tag{1.11}$$

Particulary, such a point z_0 is either an accumulation point of zeros or of poles of r^*_{n,m_n}.

On the other hand, if f is not holomorphic on $[-1,1]$, so far results about the distribution of the zeros of $r^*_{n,m_n}(f)$ are only known in the case that $m_n = 0$ for all $n \in \mathbb{N}$ (polynomial approximation) or in the case that $m_n = m \in \mathbb{N}$ is fixed (rational approximation with a bounded number of free poles). In the polynomial case, the normalized zero counting measures of $r^*_{n,0}(f)$ converge in the weak*-sense to the equilibrium measure of $[-1,1]$, at least for a subsequence $n \in \Lambda \subset \mathbb{N}$ [4]. This result was generalized to rational approximation with a bounded number of poles (cf. [5, Theorem 4.1]). Moreover, Stahl [6] and Saff and Stahl [7] have investigated for the function $f(x) = |x|^\alpha$, $\alpha > 0$, the distribution of zeros and poles of rational approximants, as well as the alternation points of the optimal error function.

In contrast to the distribution of zeros of r^*_{n,m_n}, the behavior of the alternation points of $f - r^*_{n,m_n}$ for $f \in C[-1,1]$ is well understood, not only in the polynomial case (cf. [8, 9]), but also for rational approximations (cf. [10–14]). The aim of the present paper is to investigate the distribution of the zeros of the rational approximants via the distribution of the alternation points.

2. Main Results

Let f be continuous on $[-1,1]$, possibly complex-valued. It is well known that the rate of approximation by rational functions does not guarantee the holomorphy of the function f. Gončar ([15], p. 101) pointed out the example

$$f(z) = \sum_{n=1}^{\infty} \frac{A_n}{z - \alpha_n}, \tag{2.1}$$

where the points α_n are situated in $\mathbb{C} \setminus [-1,1]$ such that any point of $[-1,1]$ is a limit point of the sequence $\{\alpha_n\}$ and the coefficients A_n converge to zero sufficiently fast. Hence, it is possible that there exists a sequence $\{r_n\}_{n\in\mathbb{N}}$, $r_n \in \mathcal{R}_{n,n}$, such that

$$\limsup_{n\to\infty} \|f - r_n\|^{1/n} < 1, \tag{2.2}$$

and f is continuous on $[-1,1]$, but nowhere holomorphic on $[-1,1]$.

But it turns out that in this case Theorem A immediately yields the following.

Theorem 2.1. *Let $f \in C[-1,1]$ be not holomorphic on $[-1,1]$, and let $\{r_n\}_{n\in\mathbb{N}}$, $r_n \in \mathcal{R}_{n,n}$, be a sequence such that*

$$\limsup_{n\to\infty} \|f - r_n\|^{1/n} < 1. \tag{2.3}$$

Then for any non holomorphic point $z_0 \in [-1,1]$ of f any neighborhood U of z_0 either

$$\limsup_{n\to\infty} N_\infty(r_n, U) = \infty \tag{2.4}$$

or

$$\limsup_{n\to\infty} \frac{N_a(r_n, U)}{n} > 0 \tag{2.5}$$

for all $a \in \mathbb{C}$.

In the following we consider functions $f \in C[-1,1]$ that are always real-valued on $[-1,1]$. Then the case that

$$\limsup_{n\to\infty} E_{n,n}^{1/n}(f) = 1 \tag{2.6}$$

is not covered by Theorem 2.1. By Bernstein's theorem, condition (2.6) implies that $f \in C[-1,1]$ is not holomorphic on $[-1,1]$. Examples for (2.6) are functions which are piecewise analytic on $[-1,1]$ (Newman [16], Gončar [15]).

In the following, we assume that $\{m_n\}_{n\in\mathbb{N}}$ is a sequence with

$$m_n \le n, \qquad m_n \le m_{n+1} \le m_n + 1. \tag{2.7}$$

For abbreviation, let

$$E_n := E_{n,m_n}(f), \qquad r_n^* := r_{n,m_n}^*(f) = \frac{p_n^*}{q_n^*}, \tag{2.8}$$

where $p_n^* \in \mathcal{P}_n$ and $q_n^* \in \mathcal{P}_{m_n}$ have no common factor. We define

$$\delta_n := \min(n - \deg p_n^*, m_n - \deg q_n^*) \tag{2.9}$$

as the defect of r_n^* and $d_n := n+m_n+1-\delta_n$. According to the alternation theorem of Chebyshev (cf. Meinardus [17], Theorem 98) there exist $d_n + 1$ points $x_k^{(n)}$,

$$-1 \le x_0^{(n)} < x_1^{(n)} < \cdots < x_{d_n}^{(n)} \le 1, \tag{2.10}$$

which satisfy

$$\lambda_n(-1)^k(f - r_n^*)\left(x_k^{(n)}\right) = \|f - r_n^*\|_{[-1,1]}, \quad 0 \le k \le d_n, \tag{2.11}$$

where $\lambda_n = +1$ or $\lambda_n = -1$ is fixed. For each pair (n, m_n) let

$$A_n = A_n(f) := \left\{x_k^{(n)}\right\}_{k=0}^{d_n} \tag{2.12}$$

denote an arbitrary, but fixed alternation set for the best approximation $r_n^* \in \mathcal{R}_{n,m_n}$, and let ν_n denote the normalized counting measure of A_n, that is,

$$\nu_n([\alpha,\beta]) := \frac{\#\left\{x_k^{(n)} : \alpha \le x_k^{(n)} \le \beta\right\}}{d_n + 1} \tag{2.13}$$

for any interval $[\alpha,\beta] \subset [-1,1]$. Since ν_n is a probability measure on $[-1,1]$, there exists a subsequence $\Lambda \subset \mathbb{N}$ such that

$$\nu_n \xrightarrow{*} \nu \quad \text{as } n \longrightarrow \infty, n \in \Lambda, \tag{2.14}$$

in the weak*-topology and ν is again a probability measure on $[-1,1]$.

Theorem 2.2. *Let $f \in C[-1,1]$ be real-valued, and let* (2.6) *hold. Moreover, let f be approximated with respect to $\mathcal{R}_{n,m_n}$, where the sequence $\{m_n\}_{n\in\mathbb{N}}$ satisfies* (2.7)*. Then there exists a subsequence $\Lambda \subset \mathbb{N}$ with the following properties:*

(i)

$$\nu_n \overset{*}{\longrightarrow} \nu \quad \text{as } n \longrightarrow \infty, n \in \Lambda, \tag{2.15}$$

(ii) *let $z_0 \in \operatorname{supp}(\nu)$, $a \in \mathbb{C}$, and let U be a neighborhood of z_0 with $f(z) \not\equiv a$ on $U \cap [-1,1]$; then*

$$\begin{aligned} &\text{either} \quad \limsup_{n\in\Lambda, n\to\infty} N_\infty(r_n^*, U) = \infty \\ &\text{or} \quad \limsup_{n\in\Lambda, n\to\infty} \frac{N_a(r_n^*, U)}{n} > 0. \end{aligned} \tag{2.16}$$

Applying to the approximation in the upper half of the Walsh table, we obtain the following.

Corollary 2.3. *Let $f \in C[-1,1]$ with* (2.6) *and let the subsequence $\{m_n\}_{n\in\mathbb{N}}$ satisfy*

$$m_n \le cn \quad \text{with } 0 \le c < 1, \qquad m_n \le m_{n+1} \le m_n + 1. \tag{2.17}$$

Then there exists a subsequence $\Lambda \subset \mathbb{N}$ with the following property: Let $a \in \mathbb{C}$, $z_0 \in [-1,1]$, and let U be a neighborhood of z_0 with $f(z) \not\equiv a$ on $U \cap [-1,1]$; then either (i) *or* (ii) *holds.*

3. Auxiliary Tools

One of the essential tools for proving Theorem 2.2 is the interaction between alternation points and poles of best rational approximants.

Let τ_n denote the normalized counting measure of the poles of r_n^*, counted with their multiplicities, and let us denote by $\widehat{\tau}_n$ the balayage measure of τ_n onto $[-1,1]$. Then the following distribution results hold for the interaction between the alternation points of A_n and the poles of r_n^* and r_{n+1}^*.

Theorem B (See [11]). *Let f be not a rational function, and let $\{m_n\}_{n\in\mathbb{N}}$ satisfy* (2.7)*. Then there exists a subsequence $\Lambda \subset \mathbb{N}$ such that*

$$\nu_n - \alpha_n(\widehat{\tau}_n + \widehat{\tau}_{n+1}) - (1-\alpha_n)\mu \overset{*}{\longrightarrow} 0 \quad \text{as } n \longrightarrow \infty, n \in \Lambda, \tag{3.1}$$

where

$$\alpha_n = \frac{\deg q_n^* + \deg q_{n+1}^*}{d_n + 1} \tag{3.2}$$

and μ is the equilibrium distribution of $[-1,1]$.

We remark that in the proof of Theorem B in [11], the subsequence $\Lambda \subset \mathbb{N}$ is defined by

$$\Lambda := \left\{ n \in \mathbb{N} : \frac{E_n + E_{n+1}}{E_n - E_{n+1}} \leq n^2 \right\}. \tag{3.3}$$

Inspecting the proof of (3.1) in [11], it turns out that we can modify the definition of Λ by

$$\Lambda := \left\{ n \in \mathbb{N} : E_{n+1} \leq \left(1 - \frac{1}{n^2}\right) E_n \right\}. \tag{3.4}$$

The existence of such sequences Λ is based on the divergence of the infinite product

$$\prod_{n=0}^{\infty} \frac{E_{n+1}}{E_n} = \prod_{n=0}^{\infty} \left(1 - \frac{E_n - E_{n+1}}{E_n}\right) \tag{3.5}$$

to 0 if f is not a rational function. This argument has already been used by Kadec [9] in his proof for the distribution of the alternation points in polynomial approximation.

Concerning the distribution of the zeros of best polynomial approximations p_n^* to f,

$$p_n^*(z) = a_n z^n + \cdots, \tag{3.6}$$

the asymptotic behavior of the highest coefficient a_n plays an essential role, namely,

$$\limsup_{n \to \infty} |a_n|^{1/n} = \frac{1}{\operatorname{cap}([-1,1]) \limsup_{n \to \infty} e_n^{1/n}}, \tag{3.7}$$

where

$$e_n = \|f - p_n^*\| = \inf_{p_n \in \mathcal{P}_n} \|f - p_n\| \tag{3.8}$$

and $\operatorname{cap}([-1,1]) = 1/2$ is the logarithmic capacity of $[-1,1]$.

If $f \in C[-1,1]$ is not holomorphic on $[-1,1]$, then $\limsup_{n \to \infty} e_n^{1/n} = 1$ and we can choose a subsequence $\Lambda \subset \mathbb{N}$ such that

$$\lim_{n \in \Lambda, n \to \infty} e_n^{1/n} = 1 \tag{3.9}$$

and moreover,

$$\lim_{n \to \Lambda, n \to \infty} |a_n|^{1/n} = 2. \tag{3.10}$$

If $e_n \neq e_{n+1}$, then the polynomial

$$p_n(z) := \frac{p_n^*(z)}{a_n} \tag{3.11}$$

is monic and satisfies

$$\|p_n\| \le \left(\frac{1}{2-\varepsilon}\right)^n \tag{3.12}$$

for all $n \in \Lambda$ which are sufficiently large, where $\varepsilon > 0$ can be chosen arbitrarily. Then the Erdős-Turán Theorem [18] (cf. [19]) implies a weak*-version of Kadec's result, namely, the weak*-convergence of the normalized counting measures of alternation sets of $f - p_n^*$ to the equilibrium measure μ of $[-1,1]$, at least for a subsequence Λ, $n \in \Lambda$.

The objective of this section is to show that there exists a subsequence $\Lambda \subset \mathbb{N}$ such that (3.4) and the analogue of (3.9) for rational approximation hold simultaneously with consequences for the behavior of the difference of two consecutive best approximants.

Lemma 3.1. *Let $f \in C[-1,1]$ with* (2.6)*. Then there exists a subsequence $\Lambda \subset \mathbb{N}$ such that*

$$\begin{gathered} E_{n+1} \le \left(1 - \frac{1}{n^2}\right) E_n \quad \text{for } n \in \Lambda, \\ \lim_{n\in\Lambda, n\to\infty} E_n^{1/n} = 1. \end{gathered} \tag{3.13}$$

Moreover, let $\{\xi_n\}_{n\in\Lambda}$ be a sequence in $[-1,1]$ with $|(f - r_n^*)(\xi_n)| = \|f - r_n^*\|$; then

$$\lim_{n\in\Lambda, n\to\infty} \left|\left(r_n^* - r_{n+1}^*\right)(\xi_n)\right|^{1/n} = 1. \tag{3.14}$$

Proof. Using the above arguments of the beginning of this section, there exists a subsequence $\Lambda_1 \subset \mathbb{N}$ such that

$$E_{n+1} \le \left(1 - \frac{1}{n^2}\right) E_n \quad \text{for } n \in \Lambda_1. \tag{3.15}$$

First, we show that there exists $\Lambda \subset \mathbb{N}$ such that (3.13) holds.

For proving this, we define

$$\widetilde{\Lambda} := \left\{ n \in \mathbb{N} : E_{n+1} \le \left(1 - \frac{1}{n^2}\right) E_n \right\}. \tag{3.16}$$

Since $\Lambda_1 \subset \widetilde{\Lambda}$, $\widetilde{\Lambda} \ne \emptyset$, and $\widetilde{\Lambda}$ is not finite, hence the complement

$$\widetilde{\Lambda}^c := \mathbb{N} \setminus \widetilde{\Lambda} \tag{3.17}$$

of $\widetilde{\Lambda}$ in $\mathbb{N}$ has the property that

$$E_{n+1} > \left(1 - \frac{1}{n^2}\right) E_n \quad \text{for } n \in \widetilde{\Lambda}^c. \tag{3.18}$$

If $\widetilde{\Lambda}^c$ is a finite set, then there exists $m \in \mathbb{N}$ such that

$$\Lambda := \{n \in \mathbb{N} : n \geq m\} \tag{3.19}$$

satisfies property (3.13).

If $\widetilde{\Lambda}^c$ is an infinite set, then observing that $\widetilde{\Lambda}$ is not a finite set, we can define subsequences $\{m_j\}_{j\in\mathbb{N}}$ and $\{n_j\}_{j\in\mathbb{N}}$ of $\mathbb{N}$ such that

$$\begin{gathered} n_{j-1} < m_j \leq n_j < m_{j+1}, \\ \widetilde{\Lambda} = \left\{n \in \mathbb{N} : m_j \leq n \leq n_j, j \geq 1\right\}. \end{gathered} \tag{3.20}$$

Next, we consider a fixed integer $m \geq m_1$. If

$$n_{j-1} < m < m_j, \quad j \geq 2, \tag{3.21}$$

then $m \notin \widetilde{\Lambda}$ and we deduce

$$\begin{aligned} E_{m_j} &> \left(1 - \frac{1}{(m_j - 1)^2}\right) E_{m_j-1} > \left(1 - \frac{1}{(m_j - 1)^2}\right)\left(1 - \frac{1}{(m_j - 2)^2}\right) E_{m_j-2} \\ &> \cdots > \prod_{k=0}^{m_j-m-1} \left(1 - \frac{1}{(m+k)^2}\right) E_m. \end{aligned} \tag{3.22}$$

Since the infinite product

$$S = \prod_{n=2}^{\infty}\left(1 - \frac{1}{n^2}\right) \tag{3.23}$$

converges, there exists a constant β, $0 < \beta < 1$, such that all partial products

$$S_{\nu,\mu} := \prod_{n=\nu}^{\mu}\left(1 - \frac{1}{n^2}\right), \quad 2 \leq \nu < \mu, \tag{3.24}$$

of S are bounded by β from below, that is, $S_{\nu,\mu} \geq \beta$.

By (3.22), $E_{m_j} > \beta E_m$ and

$$E_{m_j}^{1/m_j} \geq E_{m_j}^{1/m} > \beta^{1/m} E_m^{1/m} \text{ for } E_{m_j} \leq 1. \tag{3.25}$$

Let us define for $m \geq m_1$

$$\nu(m) := \begin{cases} m, & \text{if } m \in \widetilde{\Lambda} \\ m_j, & \text{if } n_{j-1} < m < m_j. \end{cases} \tag{3.26}$$

Next, we choose a subsequence $\Lambda_2 = \{k_j\}_{j\in\mathbb{N}}$ of $\mathbb{N}$ such that $k_1 \geq m_1$ and

$$\lim_{j\to\infty} E_{k_j}^{1/k_j} = 1. \tag{3.27}$$

If $\Lambda_2 \subset \widetilde{\Lambda}$, then we are done. As for the general case, let us define

$$\Lambda := \bigcup_{j=1}^{\infty} \{\nu(k_j)\}; \tag{3.28}$$

then $\Lambda \subset \widetilde{\Lambda}$ and (3.25)–(3.27) imply

$$\lim_{j\to\infty} E_{\nu(k_j)}^{1/\nu(k_j)} = 1. \tag{3.29}$$

Hence, (3.13) is proved.

Moreover, for $n \in \Lambda$,

$$\begin{aligned}
\left|(r_n^* - r_{n+1}^*)(\xi_n)\right| &\geq \left|(f - r_n^*)(\xi_n)\right| - \left|(f - r_{n+1}^*)(\xi_n)\right| \\
&\geq E_n - E_{n+1} \geq E_n - \left(1 - \frac{1}{n^2}\right)E_n = \frac{1}{n^2}E_n, \\
1 \geq \limsup_{n\to\infty}\left|(r_n^* - r_{n+1}^*)(\xi_n)\right|^{1/n} &\geq \limsup_{n\in\Lambda, n\to\infty}\left|(r_n^* - r_{n+1}^*)(\xi_n)\right|^{1/n} \\
&\geq \limsup_{n\in\Lambda, n\to\infty}\left(\left(\frac{1}{n^2}\right)^{1/n} E_n^{1/n}\right) = \lim_{n\in\Lambda, n\to\infty} E_n^{1/n} = 1.
\end{aligned} \tag{3.30}$$

Hence,

$$\lim_{n\in\Lambda, n\to\infty}\left|(r_n^* - r_{n+1}^*)(\xi_n)\right|^{1/n} = 1, \tag{3.31}$$

and (3.14) is proved. □

4. Proofs

Proof of Theorem 2.2. First we will prove the theorem for $a = 0$.

According to the lemma in Section 3, there exists a subsequence $\Lambda \subset \mathbb{N}$ such that (3.13)–(3.14) hold. Then Theorem B applies and (3.1) holds for $n \in \Lambda$. Because ν_n are probability measures on $[-1, 1]$, we may assume that

$$\nu_n \xrightarrow{*} \nu \quad \text{as } n \longrightarrow \infty, n \in \Lambda. \tag{4.1}$$

Let $z_0 \in \operatorname{supp}(\nu)$ and U a neighborhood of z_0 such that $f(z) \not\equiv 0$ on $U \cap [-1, 1]$.

Let us assume that (ii) of Theorem 2.2 does not hold. Hence, there exists $m \in \mathbb{N}$

$$N_\infty(r_n^*, U) \le m \quad \forall n \in \mathbb{N}, \tag{4.2}$$

$$N_0(r_n^*, U) = o(n) \quad \text{as } n \longrightarrow \infty. \tag{4.3}$$

Of course, we may assume that U is a bounded symmetric region with respect to the real axis. Let l_n be the number of poles $\xi_{n,i}$ of r_n^* in U counted with their multiplicities. Then we define

$$q_n(z) := \begin{cases} \prod_{i=1}^{l_n}(z - \xi_{n,i}), & l_n \ge 1, \\ 1, & l_n = 0. \end{cases} \tag{4.4}$$

Because $q_n, q_{n+1} \in \mathcal{P}_m$, there exists a subsequence $\Lambda_1 \subset \Lambda$ and $\tilde{q}_0, \tilde{q}_1 \in \mathcal{P}_m$ such that

$$\lim_{n \in \Lambda_1, n \to \infty} q_{n+i} = \tilde{q}_i \quad \text{for } i = 0, 1. \tag{4.5}$$

Together with $f(z) \not\equiv 0$ for $z \in U \cap [-1, 1]$, this implies that there exists an interval $[\alpha, \beta] \subset U \cap [-1, 1]$, $\alpha \ne \beta$, and a constant $\kappa > 0$ such that

$$\left|\tilde{q}_i(x)\right| \ge \kappa \quad \text{for } x \in [\alpha, \beta], i = 0, 1, \tag{4.6}$$

$$\left|f(x)\right| \ge \kappa \quad \text{for } x \in [\alpha, \beta]. \tag{4.7}$$

Let k_n be the number of zeros (with multiplicities) of r_n^* in U. If $k_n \ge 1$, let $\eta_{n,i}$, $1 \le i \le k_n$, be the zeros of r_n^* in U and let

$$\pi_n(z) := \begin{cases} \prod_{i=1}^{k_n}(z - \eta_{n,i}), & k_n > 0 \\ 1, & k_n = 0. \end{cases} \tag{4.8}$$

Because of (4.3), $k_n = o(n)$ as $n \to \infty$ and we obtain

$$\limsup_{n \to \infty} \|\pi_n\|_K^{1/n} \le 1 \tag{4.9}$$

for any compact set K in $\mathbb{C}$. Now, let us define

$$h_n(z) := \frac{1}{n} \log|\Phi_n(z)| \tag{4.10}$$

with

$$\Phi_n(z) := \frac{\pi_n(z)}{r_n^*(z) q_n(z)}. \tag{4.11}$$

Then Φ_n is holomorphic in U and h_n harmonic in U.

Consider $z \in [\alpha, \beta]$ and Λ_1 as before. Then by (4.5)–(4.7) there exists $\tilde{n} \in \mathbb{N}$ such that

$$\left| r_{n+i}^*(z) \right| \geq \frac{\kappa}{2}, \qquad \left| q_{n+i}(z) \right| \geq \frac{\kappa}{2} \tag{4.12}$$

for $z \in [\alpha, \beta]$, $i = 0, 1$, and $n \in \Lambda_1$, $n \geq \tilde{n}$. Then for $i = 0, 1$

$$\left\| \frac{\pi_{n+i}}{r_{n+i}^* q_{n+i}} \right\|_{[\alpha,\beta]} \leq \frac{4(d+1)^{k_{n+i}}}{\kappa^2}, \tag{4.13}$$

where

$$d = \sup_{z \in U} |z|. \tag{4.14}$$

According to a Lemma of Gončar [20, Lemma 1, page 153], for any compact set $K \subset U$ there exists a constant $\lambda = \lambda([\alpha, \beta], U, K) > 1$ such that

$$\left\| \frac{\pi_{n+i}}{r_{n+i}^* q_{n+i}} \right\|_{K} \leq \lambda^{n+i} \left\| \frac{\pi_{n+i}}{r_{n+i}^* q_{n+i}} \right\|_{[\alpha,\beta]} \tag{4.15}$$

for $i = 0, 1$. For example, $\lambda([\alpha, \beta], U, K)$ can be chosen as

$$\lambda\big([\alpha, \beta], U, K\big) := \max_{z \in K} \sup_{t \in \overline{\mathbb{C}} \setminus U} \exp\big(G_{[\alpha,\beta]}(z, t)\big), \tag{4.16}$$

where $G_{[\alpha,\beta]}(z, t)$ is the Green function of $\overline{\mathbb{C}} \setminus [\alpha, \beta]$ with pole at t.

Next, we choose a region $W \subset U$, W symmetric to the real axis, with $z_0 \in W$, $\overline{W} \subset U$ and $[\alpha, \beta] \subset W$, then

$$h_{n+i}(z) \leq \lambda\Big([\alpha, \beta], U, \overline{W}\Big) + \frac{1}{n+i} \log \frac{4}{\kappa^2} + \frac{k_n + i}{n+i} \log(1+d) \tag{4.17}$$

for $i = 0, 1$. Hence for $i = 0, 1$, the sequences $\{h_{n+i}\}_{n \in \Lambda_1}$ are uniformly bounded in W from above as $n \to \infty$, $n \in \Lambda_1$, $i = 0, 1$. By Harnack's theorem, either

$$h_n(z) \longrightarrow -\infty \quad \text{locally uniformly in } W \text{ as } n \longrightarrow \infty, n \in \Lambda_1, \tag{4.18}$$

or there exists a subsequence $\Lambda_2 \subset \Lambda_1$ such that $\{h_n\}_{n \in \Lambda_2}$ converges locally uniformly to h_0 as $n \to \infty$, $n \in \Lambda_2$, in the region W and the function h_0 is harmonic in W.

Next, let us show that the first situation cannot occur: if $C > 0$ is such that

$$\max_{z\in[\alpha,\beta]} h_n(z) \le -C \tag{4.19}$$

for $n \in \Lambda_1$ and n sufficiently large, then

$$n \cdot \max_{z\in[\alpha,\beta]} |h_n(z)| \le \max_{z\in[\alpha,\beta]} \log\left|\frac{\pi_n(z)}{r_n^*(z)q_n(z)}\right| \le -nC. \tag{4.20}$$

Hence, by (4.5)–(4.7) there exists a constant $c_1 > 0$ such that

$$\max_{z\in[\alpha,\beta]} |\pi_n(z)| \le c_1 e^{-nC}. \tag{4.21}$$

Since $\pi_n \in \wp_{k_n}$ is a monic polynomial and $k_n = o(n)$ as $n \to \infty$, this is a contradiction to

$$\|\pi_n\|_{[\alpha,\beta]} \ge 2\left(\frac{(\beta-\alpha)}{4}\right)^{k_n}. \tag{4.22}$$

Next, we consider (4.17) for $i = 1$. Again by Harnack's theorem, either

$$h_{n+1}(z) \longrightarrow -\infty \quad \text{locally uniformly in } W \text{ as } n \longrightarrow \infty, n \in \Lambda_2, \tag{4.23}$$

or there exists a subsequence $\Lambda_3 \subset \Lambda_2$ such that $\{h_{n+1}\}_{n\in\Lambda_3}$ converges locally uniformly to a function h_1 in W and h_1 is harmonic in W.

As above for $\{h_n\}_{n\in\Lambda_1}$, the first situation cannot occur. Consequently,

$$\max_{z\in[\alpha,\beta]} h_i(z) \ge 0 \quad \text{for } i = 0, 1. \tag{4.24}$$

On the other hand, using (4.13) we deduce for $i = 0, 1$ that

$$\limsup_{n\in\Lambda_1, n\to\infty} \max_{z\in[\alpha,\beta]} h_{n+i}(z) \le 0. \tag{4.25}$$

Summarized, we have for $i = 0, 1$ that

$$h_i(z) \equiv 0 \quad \text{for } z \in [\alpha,\beta]. \tag{4.26}$$

By definition, the regions U, W are symmetric to $\mathbb{R}$ as well as the functions

$$\left|r_{n+i}^*(z)\right|, \qquad |\pi_{n+i}(z)|, \qquad \left|q_{n+i}(z)\right| \tag{4.27}$$

for $i = 0, 1$. This symmetry, together with (4.26), implies that

$$h_i(z) \equiv 0 \quad \forall z \in W \tag{4.28}$$

for $i = 0, 1$. Hence,

$$\lim_{n \in \Lambda_3, n \to \infty} \left\| r^*_{n+i} q_{n+i} \right\|_K^{1/n} \le 1 \tag{4.29}$$

for all compact sets K in W, $i = 0, 1$.

Combining (4.29) for $i = 0, 1$, we obtain

$$\lim_{n \in \Lambda_3, n \to \infty} \left\| \left(r^*_n - r^*_{n+1} \right) q_n q_{n+1} \right\|_K^{1/n} \le 1 \tag{4.30}$$

for all compact sets $K \subset W$. Hence, the function $V(z) \equiv 0$ is a harmonic majorant for the sequence $\{F_n\}_{n \in \Lambda_3}$ of subharmonic functions in W, where

$$F_n(z) := \frac{1}{n} \log\left| \left(r^*_n - r^*_{n+1} \right)(z) q_n(z) q_{n+1}(z) \right|, \quad n \in \mathbb{N}. \tag{4.31}$$

Next, we want to show that $V(z) \equiv 0$ is an *exact* harmonic majorant for $\{F_n\}_{n \in \Lambda_3}$ and also for any $\{F_n\}_{n \in \Lambda_4}$ for any subsequence $\Lambda_4 \subset \Lambda_3$.

Let us assume that this assertion would be false: then there exists a subsequence $\Lambda_4 \subset \Lambda_3 \subset \Lambda$ (Λ as in the Corollary of Section 3) and a continuum $K \subset W$ such that

$$\limsup_{n \in \Lambda_4, n \to \infty} \max_{z \in K} F_n(z) < 0. \tag{4.32}$$

Since $V(z) \equiv 0$ is a harmonic majorant for $\{F_n\}_{n \in \Lambda_4}$ in W, then (4.32) implies that the inequality (4.32) holds for any continuum $K \subset W$.

First, let us note that under the condition (4.2) a point $\xi \in U \cap [-1, 1]$ cannot be an isolated point of $\operatorname{supp}(\nu)$.

To prove this, let us denote by δ_z the Dirac measure of the point $z \in \overline{\mathbb{C}}$, and let $\widehat{\delta}_z$ be the associated balayage measure of δ_z to the interval $[-1, 1]$. For $z \notin [-1, 1]$ the density of the balayage measure $\widehat{\delta}_z$ at the point $x \in (-1, 1)$ is given by

$$\frac{d}{dx} \widehat{\delta}_z(x) = \frac{\partial}{\partial_{n_+}} G(x, z) + \frac{\partial}{\partial_{n_-}} G(x, z), \tag{4.33}$$

where n_+ (resp., n_-) denotes the normal at the point x to the upper half (resp., lower half) plane and $G(\xi, z)$ is the Green function for $\xi \in \overline{\mathbb{C}} \backslash [-1, 1]$ with pole at z, continuously extended by $G(x, z) = 0$ to $\xi = x \in [-1, 1]$.

Then for any interval $[\alpha, \beta] \subset [-1, 1]$

$$0 \leq \widehat{\delta}_z\big([\alpha, \beta]\big) \leq 1,$$
$$\lim_{\substack{z \to \eta \\ \eta \in [-1,1] \setminus [\alpha, \beta]}} \widehat{\delta}_z\big([\alpha, \beta]\big) = 0. \tag{4.34}$$

Let $z \in \overline{\mathbb{C}} \setminus [-1, 1]$, $\xi \in U \cap [-1, 1]$, and $\varepsilon > 0$; then

$$\lim_{\varepsilon \to 0} \widehat{\delta}_z([\xi - \varepsilon, \xi + \varepsilon]) = 0. \tag{4.35}$$

Consider the exterior of the ε-neighborhood of $[-1, 1]$; that is, let

$$W_\varepsilon := \left\{ z \in \overline{\mathbb{C}} : \operatorname{dist}(z, [-1, 1]) \geq \varepsilon \right\}. \tag{4.36}$$

Then we can obtain a sharpening of (4.35), namely,

$$\lim_{\varepsilon \to 0} \max_{z \in W_\varepsilon \setminus U} \widehat{\delta}_z([\xi - \varepsilon, \xi + \varepsilon]) = 0. \tag{4.37}$$

Since $\xi \in U \cap [-1, 1]$ and (4.2) holds, (4.34)–(4.37) imply

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \widetilde{\tau}_n([\xi - \varepsilon, \xi + \varepsilon]) = 0. \tag{4.38}$$

Because (3.1) and (4.1) hold for $n \in \Lambda$, ξ cannot be an isolated point of $\operatorname{supp}(\nu)$.

Consequently, since $z_0 \in \operatorname{supp}(\nu)$ there exists a sequence $\{\xi_k\}_{k \in \mathbb{N}}$ in U, $\xi_k \in \operatorname{supp}(\nu)$, such that

$$z_0 = \lim_{k \to \infty} \xi_k \tag{4.39}$$

and each ξ_k is not an isolated point of $\operatorname{supp}(\nu)$. Hence, for any $k \in \mathbb{N}$ and any open interval (α, β) with $\xi_k \in (\alpha, \beta)$ we have $\nu((\alpha, \beta)) > 0$. Taking into account (4.39) and the fact that the zero set

$$Z := \left\{ z \in \mathbb{C} : \widetilde{p}_0(z) = 0 \text{ or } \widetilde{p}_i(z) = 0 \right\} \tag{4.40}$$

of the polynomials $\widetilde{p}_0$, $\widetilde{p}_1$ in (4.5) is finite, there exists an interval $[\widetilde{\alpha}, \widetilde{\beta}] \subset U \cap [-1, 1]$, $\widetilde{\alpha} < \widetilde{\beta}$, with

$$\nu\Big(\big[\widetilde{\alpha}, \widetilde{\beta}\big]\Big) > 0, \qquad \big[\widetilde{\alpha}, \widetilde{\beta}\big] \cap Z = \emptyset. \tag{4.41}$$

Using (4.5) we conclude that there exists $n_1 \in \mathbb{N}$ and a constant $\tilde{\kappa} > 0$ such that

$$\left|q_{n+i}(z)\right| \geq \tilde{\kappa} \quad \text{for } z \in \left[\tilde{\alpha}, \tilde{\beta}\right], \tag{4.42}$$

where $n \in \Lambda_1$, $n \geq n_1$, and $i = 0, 1$.

Let us choose for K in (4.32) the interval $[\tilde{\alpha}, \tilde{\beta}]$. Then there exists, by definition of $F_n(z)$ in (4.31), a constant δ, $0 < \delta < 1$, and $n_2 \in \mathbb{N}$, $n_2 \geq n_1$, such that

$$\max_{z \in [\tilde{\alpha}, \tilde{\beta}]} \left|\left(r_n^* - r_{n+1}^*\right)(z) q_n(z) q_{n+1}(z)\right| \leq \delta^n. \tag{4.43}$$

for all $n \in \Lambda_4$, $n \geq n_2$. By (4.42) we obtain

$$\max_{z \in [\tilde{\alpha}, \tilde{\beta}]} \left|\left(r_n^* - r_{n+1}^*\right)(z)\right| \leq \frac{\delta^n}{\tilde{\kappa}^2}, \tag{4.44}$$

$$\limsup_{n \in \Lambda_4, n \to \infty} \left\|r_n^* - r_{n+1}^*\right\|_{[\tilde{\alpha}, \tilde{\beta}]}^{1/n} \leq \delta < 1 \tag{4.45}$$

contradicting the property (3.14) and $\nu([\tilde{\alpha}, \tilde{\beta}]) > 0$.

Hence, $V(z) \equiv 0$ is an exact harmonic majorant for $\{F_n\}_{n \in \Lambda_3}$ and for any subsequence $\{F_n\}_{n \in \Lambda_4}$, $\Lambda_4 \subset \Lambda_3$, in the region W.

This is now the situation that a distribution result of Walsh about the zeros of the sequence

$$\left\{\left(r_n^* - r_{n+1}^*\right) q_n q_{n+1}\right\}_{n \in \Lambda_3} \tag{4.46}$$

of holomorphic functions in W can be applied (Walsh [21], Theorem 16, page 221): for every compact set K in W we have

$$N_0\left(\left(r_n^* - r_{n+1}^*\right) q_n q_{n+1}, K\right) = o(n) \quad \text{as } n \in \Lambda_3, n \longrightarrow \infty. \tag{4.47}$$

Choosing for K the interval $[\tilde{\alpha}, \tilde{\beta}]$, then the number of alternations of $f - r_n^*$ in $[\tilde{\alpha}, \tilde{\beta}]$ is a lower bound for the number

$$N_0\left(\left(r_n^* - r_{n+1}^*\right) q_n q_{n+1}, \left[\tilde{\alpha}, \tilde{\beta}\right]\right) \tag{4.48}$$

of zeros of $(r_n^* - r_{n+1}^*) q_n q_{n+1}$ in $[\tilde{\alpha}, \tilde{\beta}]$. Because of (4.1) and $\nu([\tilde{\alpha}, \tilde{\beta}]) > 0$,

$$\lim_{n \in \Lambda_3, n \to \infty} \nu_n\left(\left[\tilde{\alpha}, \tilde{\beta}\right]\right) = \nu\left(\left[\tilde{\alpha}, \tilde{\beta}\right]\right) > 0 \tag{4.49}$$

which contradicts (4.47).

Hence, the theorem is proved for $a = 0$. The case $a \neq 0$ can be reduced to $a = 0$ by defining

$$\widetilde{f}(z) := f(z) + a, \quad z \in [-1,1], \tag{4.50}$$

$$\widetilde{r}(z) := r(z) + a \quad \text{for } r \in \mathcal{R}_{n,n}, z \in \mathbb{C}. \tag{4.51}$$

If $a \in \mathbb{C}$, we note that the inequality (4.30) is equivalent to

$$\lim_{n\in\Lambda_3, n\to\infty} \left\| (\widetilde{r}_n^* - \widetilde{r}_{n+1}^*) q_n q_{n+1} \right\|_K^{1/n} \leq 1 \tag{4.52}$$

and (3.14) is equivalent to

$$\lim_{n\in\Lambda, n\to\infty} \left| (\widetilde{r}_n^* - \widetilde{r}_{n+1}^*)(\xi_n) \right|^{1/n} = 1, \tag{4.53}$$

where $\{\xi_n\}_{n\in\Lambda}$, $\xi_n \in [-1,1]$, and $|(\widetilde{f} - \widetilde{r}_n^*)(\xi_n)| = \|\widetilde{f} - \widetilde{r}_n^*\|$. Therefore, all arguments for the sequence $\{F_n\}$ are invariant by replacing in definition (4.10) the functions r_n^*, r_{n+1}^* by $\widetilde{r}_n^*, \widetilde{r}_{n+1}^*$. Hence, Theorem 2.2 is true for all $a \in \mathbb{C}$. □

Proof of the Corollary. In the proof of Theorem 2.2, the subsequence Λ was chosen such that

$$\nu_n - \alpha_n(\widehat{\tau}_n + \widehat{\tau}_{n+1}) - (1-\alpha_n)\mu \xrightarrow{*} 0 \quad \text{as } n \longrightarrow \infty, n \in \Lambda, \tag{4.54}$$

where

$$\alpha_n = \frac{\deg q_n^* + \deg q_{n+1}^*}{d_n + 1}. \tag{4.55}$$

Since $\{m_n\}$ fulfills (2.17), we obtain

$$\begin{aligned} \alpha_n &= \frac{\deg q_n^* + \deg q_{n+1}^*}{n + m_n + 1 - \delta_n} \\ &\leq \frac{\deg q_n^* + \deg q_{n+1}^*}{n + m_n + 1 - (m_n - \deg q{*}_n)} \\ &= 1 - \frac{n + 1 - \deg q_{n+1}^*}{n + 1 + \deg q_n^*} \\ &< 1 - \frac{n+1-c(n+1)}{n+1+c(n+1)} = 1 - \frac{1-c}{1+c}. \end{aligned} \tag{4.56}$$

Hence, by (3.1)

$$\nu_n([\alpha,\beta]) \geq \frac{1-c}{1+c}\mu([\alpha,\beta]) \tag{4.57}$$

for any interval $[\alpha,\beta] \subset C[-1,1]$. Therefore, property (i) of Theorem 2.2 implies that

$$\nu_n \xrightarrow{*} \nu, \quad \operatorname{supp}(\nu) = [-1,1] \tag{4.58}$$

and Theorem 2.2 holds for all $z_0 \in [-1,1]$. □

5. Generalization to the Lower-Half of the Walsh Table

Theorem 2.2 restricts the approximation to the upper half of the Walsh table. In the following, we also want to allow approximations in the lower half of the Walsh table. We assume that the pairs

$$(n(s), m(s)) \in \mathbb{N}_0 \times \mathbb{N}_0 \tag{5.1}$$

depend on parameters $s \in \mathbb{N}$. For abbreviation, let

$$E_s := E_{n(s),m(s)}(f), \qquad r_s^* = r_{n(s),m(s)}^*(f) = \frac{p_s^*}{q_s^*}, \tag{5.2}$$

where p_s^* and q_s^* have no common factor. As above, let

$$\delta_s := \min\left(n(s) - \deg p_s^*, m(s) - \deg p_s^*\right) \tag{5.3}$$

be the defect of r_s^*, and let $A_s = A_s(f) = \{x_k^{(s)}\}_{k=0}^{d(s)}$ be an alternation point set to $f - r_s^*$, where

$$d_s = n(s) + m(s) + 1 - \delta_s. \tag{5.4}$$

We denote by ν_s the normalized counting measure of A_s. Then Theorem 2.2 can be generalized in the following way.

Theorem 5.1. *Let $(n(s), m(s))$, $s \in \mathbb{N}$, be a strictly increasing subsequence of $\mathbb{N}_0 \times \mathbb{N}_0$ with*

$$n(s) \le n(s+1) \le n(s) + 1, \qquad m(s) \le m(s+1) \le m(s) + 1, \tag{5.5}$$

and let us approximate $f \in C[-1,1]$, with respect to $\mathcal{R}_{n(s),m(s)}$, where

$$\begin{gathered} m(s) \le n(s) + \kappa(s), \quad s \in \mathbb{N}, \\ \kappa(s) = o\left(\frac{s}{\log s}\right) \quad \text{as } s \longrightarrow \infty. \end{gathered} \tag{5.6}$$

If $f \in C[-1,1]$ satisfies (2.6), then there exists a subset $\Lambda \subset \mathbb{N}$ with the following properties:

(i) $\nu_s \xrightarrow{*} \nu$ *as* $s \to \infty$, $s \in \Lambda$.

(ii) *let* $a \in \mathbb{C}$*; then for any* $z_0 \in \operatorname{supp}(\nu)$ *and any neighborhood* U *of* z_0 *with* $f(z) \not\equiv a$ *on* $U \cap [-1,1]$ *either*

$$\limsup_{s\to\infty} N_\infty(r_s^*, U) = \infty \tag{5.7}$$

or

$$\limsup_{s\to\infty} \frac{N_a(r_s^*, U)}{s} > 0. \tag{5.8}$$

For the proof, we use a generalization of Theorem B to the previous situation (see [10]): if (5.5) and (5.6) hold, then there exists a subsequence $\Lambda \subset \mathbb{N}$ such that

$$\nu_s - \alpha_s(\widehat{\tau}_s + \widehat{\tau}_{s+1}) - (1-\alpha_s)\mu \overset{*}{\to} 0 \quad \text{as } s \longrightarrow \infty, s \in \Lambda. \tag{5.9}$$

Again, we use in (5.9) the balayage measures of the normalized pole counting measures τ_s and τ_{s+1} of r_s^*, respectively, r_{s+1}^*, onto $[-1,1]$ and

$$\alpha_s := \frac{\deg q_s^* + \deg q_{s+1}^*}{d_s + 1}. \tag{5.10}$$

Then the proof of (5.7) and (5.8) follows the same lines as the proof of Theorem 2.2 if

$$\limsup_{s\to\infty} E_s^{1/s} = 1. \tag{5.11}$$

Because of (5.5), the index $n(s)$ runs from $n(1)$ to ∞. Moreover, let $M(s) := \max(n(s), m(s))$, $s \in \mathbb{N}$; then $M(s)$ runs from $M(1)$ to ∞ and

$$\begin{aligned}\limsup_{s\to\infty} E_s^{1/s} &= \limsup_{s\to\infty} E_{n(s),m(s)}^{1/s} \\ &\geq \limsup_{s\to\infty}\left(E_{M(s),M(s)}^{1/M(s)}\right)^{M(s)/s} = 1,\end{aligned} \tag{5.12}$$

since

$$s \geq M(s) - M(1). \tag{5.13}$$

6. Remarks

For the function $f(x) = |x|^\alpha$, $\alpha > 0$, the distribution of alternation points of the optimal error curves, as well as the zeros and poles of $r^*_{n,m}$ is very well investigated [7].

Let $\alpha \in \mathbb{R}_+ \setminus 2\mathbb{N}$, and let $(n, m_n) \in \mathbb{N} \times \mathbb{N}$ with

$$\lim_{n\to\infty} \frac{m_n}{n} = c \leq 1, \quad n \geq m_n + 2\left[\frac{\alpha}{2}\right]. \tag{6.1}$$

Since all best approximants of $f(x) = |x|^\alpha$ are even functions, we can assume that $n, m_n \in \mathbb{N}$ are even. Moreover, the error function $f - r^*_{n,m_n}$ has always exactly $n + m_n + 3$ points [7]. By $\nu_{A_n} = \nu_n$ we denote the normalized alternation counting measure and ν_{P_n} denotes the normalized pole counting measure of r^*_{n,m_n} and ν_{Z_n} the normalized zero counting measure of r^*_{n,m_n}. Then

$$\nu_{A_n} \underset{n\to\infty}{\overset{*}{\longrightarrow}} \frac{2c}{1+c}\delta_0 + \frac{1-c}{1+c}\mu, \tag{6.2}$$

$$\nu_{P_n} \underset{n\to\infty}{\overset{*}{\longrightarrow}} \delta_0, \tag{6.3}$$

$$\nu_{Z_n} \underset{n\to\infty}{\overset{*}{\longrightarrow}} c\delta_0 + (1-c)\mu \tag{6.4}$$

(cf. Theorems 1.6 and 1.7 in [7]).

For $c < 1$, we would obtain by (3.1) and by the corollary of Theorem 2.2 that any point of $[-1, 1]$ is either a limit point of poles or of a-values of r^*_{n,m_n}, $a \in \mathbb{C}$, as $n \to \infty$. Since by (6.3) the normalized pole counting measures converge to the Dirac measure at 0, any point of $[-1, 1]$, with 0 as only possible exception, is a limit point of a-values.

For $c = 1$, $\nu_{A_n} \overset{*}{\to} \delta_0$. Hence Theorem 2.2 can only tell us that the point 0 is either a limit point of poles or of a-values, $a \in \mathbb{C}$. But (6.3) and (6.4) show that 0 is as well a limit point of zeros as of poles of r^*_{n,m_n}. Hence, the investigations in [7] for the special functions $f(x) = |x|^\alpha$ lead to deeper results for the zeros and poles of the best approximants.

But the example of $f(x) = |x|^\alpha$ shows an interesting area for further investigations, namely, a weak*-type analogue of relation (3.1) for the distribution of zeros, respectively, a-values, and poles of rational approximation would be desirable. Moreover, the approximation problem should be moved from the interval $[-1, 1]$ to more general compact sets E in $\mathbb{C}$.

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Research Article

General Univalence Criterion Associated with the *n*th Derivative

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For normalized analytic functions $f(z)$ with $f(z) \neq 0$ for $0 < |z| < 1$, we introduce a univalence criterion defined by sharp inequality associated with the nth derivative of $z/f(z)$, where $n \in \{3, 4, 5, \ldots\}$.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the following form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are normalized analytic in the open unit disk $\mathbb{U} := \{z : |z| < 1\}$.

In [1], Aksentev proved that the condition

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| \leq 1 \tag{1.2}$$

or equivalently $\mathrm{Re}(f^2(z)/z^2 f'(z)) \geq 1/2$, for $z \in \mathbb{U}$, is sufficient for $f(z) \in \mathcal{A}$ to be univalent in $\mathbb{U}$. By virtue of the aforementioned result of Aksentev, the class of functions defined by (1.2) was extensively studied by Obradović and Ponnusamy [2, 3], Ozaki and Nunokawa [4], Obradović et al. [5], and others. Afterwards, Nunokawa et al. [6] proved for $f(z) \in \mathcal{A}$ with $f(z) \neq 0$ when $0 < |z| < 1$ that

$$\left| \left(\frac{z}{f(z)} \right)'' \right| \leq 1 \tag{1.3}$$

implies $|z^2 f'(z)/f^2(z) - 1| \leq 1$ for $z \in \mathbb{U}$, and hence $f(z)$ is univalent in $\mathbb{U}$. Later, Yang and Liu [7] extended this result for $f(z) \in \mathcal{A}$:

$$\left| \left(\frac{z}{f(z)} \right)'' \right| \leq 2 \tag{1.4}$$

with $f(z) \neq 0$ when $0 < |z| < 1$ implies that $f(z)$ is univalent in $\mathbb{U}$ and the bound 2 is best possible for univalence. This result was also given first in the preprint of reports of the Department of Mathematics, University of Helsinki: M. Obradović, S. Ponnusamy, New criteria, and distortion theorems for univalent functions, Preprint 190, June 1998. Later, under the same name, the paper was published in Complex Variables Theory Application (see [3]). Corresponding to the functions defined by (1.4), Yang and Liu in [7] studied a class of analytic univalent functions $f(z)$ satisfying $|(z/f(z))''| \leq \beta (0 < \beta \leq 2)$ and denoted by $S(\beta)$. The class $S(\beta)$ is extensively studied in the recent years (see [2, 3, 8–10]).

In this work, we introduce a univalence criteria defined by the conditions $f(z) \neq 0$ for $0 < |z| < 1$ and

$$\sum_{k=2}^{n-1} \frac{k-1}{k!} |\beta_k| + \frac{n-1}{n!} \left| \frac{\mathrm{d}^n}{\mathrm{d}z^n} \left(\frac{z}{f(z)} \right) \right| \leq 1 \quad \text{for } |z| < 1, \tag{1.5}$$

where $f(z)$ is normalized analytic in $\mathbb{U}$ and $\beta_k = (\mathrm{d}^k/\mathrm{d}z^k)(z/f(z))|_{z=0}$, $n \in \{3, 4, \ldots\}$. The sharpness occurs for the Koebe function. Indeed, all functions satisfying the condition (1.5) are univalent in $\mathbb{U}$ and the bound 1 in the inequality is best possible for univalence. Letting $n = 2$ in (1.5) gives the univalence criterion defined by (1.4). Some special cases and examples for functions satisfying (1.5) are given.

2. Sufficient Conditions for Univalence

Let us prove the following theorem.

Theorem 2.1. *Let $f(z) \in \mathcal{A}$ with $f(z) \neq 0$ for $0 < |z| < 1$ and let $g(z) \in \mathcal{A}$ be bounded in $\mathbb{U}$ and satisfy*

$$m = \inf \left\{ \left| \frac{g(z_1) - g(z_2)}{z_1 - z_2} \right| : z_1, z_2 \in \mathbb{U} \right\} > 0. \tag{2.1}$$

For any $n \in \{3, 4, \ldots\}$, if

$$\left| \frac{\mathrm{d}^n}{\mathrm{d}z^n} \left(\frac{z}{f(z)} - \frac{z}{g(z)} \right) \right| \leq K \quad (z \in \mathbb{U}), \tag{2.2}$$

where

$$K = \frac{n!}{n-1}\left(\frac{m}{M^2} - \sum_{k=2}^{n-1}\frac{k-1}{k!}|\alpha_k|\right), \qquad \alpha_k = \frac{\mathrm{d}^k}{\mathrm{d}z^k}\left(\frac{z}{g(z)} - \frac{z}{f(z)}\right)\bigg|_{z=0}, \tag{2.3}$$

and $M = \sup\{|g(z)| : z \in \mathbb{U}\}$, *then* $f(z)$ *is univalent in* $\mathbb{U}$.

Proof. If we put

$$h(z) = \frac{\mathrm{d}^n}{\mathrm{d}z^n}\left(\frac{z}{f(z)} - \frac{z}{g(z)}\right), \tag{2.4}$$

then the function h is analytic in $\mathbb{U}$ and, by integration from 0 to z, we get

$$\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}\left(\frac{z}{f(z)} - \frac{z}{g(z)}\right) = \alpha_{n-1} + \int_0^z h(u_1)du_1. \tag{2.5}$$

Integrating both sides of the previous equation $(n-1)$-times from 0 to z gives

$$\frac{z}{f(z)} - \frac{z}{g(z)} = \sum_{k=1}^{n-1}\frac{\alpha_k}{k!}z^k + \int_0^z du_n \int_0^{u_n} du_{n-1} \cdots \int_0^{u_3} du_2 \int_0^{u_2} h(u_1)du_1. \tag{2.6}$$

Thus, we have

$$f(z) = \frac{g(z)}{1 + g(z)\sum_{k=1}^{n-1}(\alpha_k/k!)z^{k-1} + g(z)(\varphi(z)/z)}, \tag{2.7}$$

where

$$\varphi(z) = \int_0^z du_n \int_0^{u_n} du_{n-1} \cdots \int_0^{u_3} du_2 \int_0^{u_2} h(u_1)du_1. \tag{2.8}$$

Next, for $n = 3$, we have

$$z^2\left(\frac{\varphi(z)}{z}\right)' = \int_0^z u\varphi''(u)\mathrm{d}u = \int_0^z u\,\mathrm{d}u \int_0^u h(u_1)\mathrm{d}u_1, \tag{2.9}$$

and for $n = 4$,

$$z^2\left(\frac{\varphi(z)}{z}\right)' = \int_0^z u\varphi''(u)du = \int_0^z u\ \mathrm{d}u \int_0^u \mathrm{d}u_2 \int_0^{u_2} h(u_1)\mathrm{d}u_1. \tag{2.10}$$

In general, for $n \in \{3, 4, \ldots\}$,

$$\begin{aligned}
z^2\left(\frac{\varphi(z)}{z}\right)' &= \int_0^z u\varphi''(u)du \\
&= \int_0^z u\ du \int_0^u du_{n-2} \int_0^{u_{n-2}} du_{n-3} \cdots \int_0^{u_2} h(u_1)du_1 \\
&= \int_0^1 z^2 t\ dt \int_0^{zt} du_{n-2} \int_0^{u_{n-2}} du_{n-3} \cdots \int_0^{u_2} h(u_1)du_1 \quad (\text{by setting } u = zt) \\
&= \int_0^1 z^3 t^2\ dt \int_0^1 ds_1 \int_0^{u_{n-2}} du_{n-3} \cdots \int_0^{u_2} h(u_1)du_1 \quad (\text{by setting } u_{n-2} = zts_1) \\
&= \int_0^1 z^4 t^3\ dt \int_0^1 s_1 ds_1 \int_0^1 ds_2 \cdots \int_0^{u_2} h(u_1)du_1 \quad (\text{by setting } u_{n-3} = zts_1 s_2) \\
&= \int_0^1 z^n t^{n-1}\ dt \int_0^1 s_1^{n-3} ds_1 \int_0^1 s_2^{n-4} ds_2 \cdots \\
&\quad \int_0^1 s_{n-3} ds_{n-3} \int_0^1 h(zts_1 \cdot s_{n-2}) ds_{n-2} \quad (\text{by setting } u_1 = zts_1 s_2 \cdots s_{n-2}),
\end{aligned} \tag{2.11}$$

therefore

$$\left|\left(\frac{\varphi(z)}{z}\right)'\right| \le \frac{|z|^{n-2}}{n} \cdot \frac{1}{n-2} \cdot \frac{1}{n-3} \cdots \frac{1}{2} \int_0^1 |h(zts_1 s_2 \cdots s_{n-2})|\ ds_{n-2} \le \frac{n-1}{n!} K, \tag{2.12}$$

and so

$$\left|\frac{\varphi(z_2)}{z_2} - \frac{\varphi(z_1)}{z_1}\right| = \left|\int_{z_1}^{z_2} \left(\frac{\varphi(z)}{z}\right)' dz\right| \le \frac{n-1}{n!} K |z_2 - z_1| \tag{2.13}$$

for $z_1, z_2 \in \mathbb{U}$ and $z_1 \neq z_2$. If $z_1 \neq z_2$, then $g(z_1) \neq g(z_2)$, and it follows, from (2.7) and (2.13), that

$$\begin{aligned}
&\left|f(z_1)-f(z_2)\right| \\
&= \frac{\left|g(z_1)-g(z_2)+g(z_1)g(z_2)\sum_{k=2}^{n-1}(\alpha_k/k!)\left(z_2^{k-1}-z_1^{k-1}\right)+g(z_1)g(z_2)(\psi(z_2)/z_2-\psi(z_1)/z_1)\right|}{\left|1+g(z_1)\sum_{k=1}^{n-1}(\alpha_k/k!)z_1^{k-1}+g(z_1)(\psi(z_1)/z_1)\right|\left|1+g(z_2)\sum_{k=1}^{n-1}(\alpha_k/k!)z_2^{k-1}+g(z_2)(\psi(z_2)/z_2)\right|} \\
&> \frac{\left|g(z_1)-g(z_2)\right|-M^2|z_1-z_2|\sum_{k=2}^{n-1}(|\alpha_k|/k!)\left|\sum_{t=0}^{k-2} z_1^t z_2^{k-1-t}\right|-((n-1)/n!)KM^2|z_1-z_2|}{\left|1+g(z_1)\sum_{k=1}^{n-1}(\alpha_k/k!)z_1^{k-1}+g(z_1)(\psi(z_1)/z_1)\right|\left|1+g(z_2)\sum_{k=1}^{n-1}(\alpha_k/k!)z_2^{k-1}+g(z_2)(\psi(z_2)/z_2)\right|} \\
&> \frac{\left|g(z_1)-g(z_2)\right|-M^2|z_1-z_2|\sum_{k=2}^{n-1}(|\alpha_k|(k-1)/k!)-((n-1)/n!)KM^2|z_1-z_2|}{\left|1+g(z_1)\sum_{k=1}^{n-1}(\alpha_k/k!)z_1^{k-1}+g(z_1)(\psi(z_1)/z_1)\right|\left|1+g(z_2)\sum_{k=1}^{n-1}(\alpha_k/k!)z_2^{k-1}+g(z_2)(\psi(z_2)/z_2)\right|} \\
&\geq 0.
\end{aligned} \tag{2.14}$$

Hence, $f(z)$ is univalent in $\mathbb{U}$. □

Corollary 2.2. *Let $f(z) \in \mathcal{A}$ with $f(z) \neq 0$ when $0 < |z| < 1$. For any $n \in \{3, 4, \dots\}$, if*

$$\sum_{k=2}^{n-1} \frac{k-1}{k!}\left|\beta_k\right| + \frac{n-1}{n!}\left|\frac{d^n}{dz^n}\left(\frac{z}{f(z)}\right)\right| \leq 1 \quad (z \in \mathbb{U}), \tag{2.15}$$

where $\beta_k = (d^k/dz^k)(z/f(z))|_{z=0}$, then $f(z)$ is univalent in $\mathbb{U}$. The result is sharp, where equality occurs for the Koebe function $k(z) = z/(1-z)^2$ and also for functions of the following form:

$$f(z) = \frac{z}{1+az+z^2}, \quad (|a| \leq 2), \qquad f_n(z) = \frac{z}{(1 \pm (1/(n-2))z)^{n-1}}. \tag{2.16}$$

Proof. Setting $g(z) = z$ in Theorem 2.1 immediately yields (2.15). To show that the result is sharp for $n \geq 3$, we consider

$$f(z) = \frac{z}{(1+(1/(n-2))z)^{n+\epsilon-1}} \quad (\epsilon > 0). \tag{2.17}$$

A computation shows, for $1 \leq k \leq n-1$, that

$$\frac{d^k}{dz^k}\left(\frac{z}{f(z)}\right) = (n-2)^{-k}(\epsilon+n-1)(\epsilon+n-2)\cdots(\epsilon+n-k)\left(1+\frac{1}{n-2}z\right)^{\epsilon+n-k-1}. \tag{2.18}$$

Letting $\epsilon = 0$ in (2.17) and (2.18) implies, respectively, that $(d^n/dz^n)(z/f(z)) = 0$ and

$$|\beta_k| = \frac{(n-1)!}{(n-k-1)!(n-2)^k}. \tag{2.19}$$

This satisfies the equality in (2.15), because for $x \in \mathbb{R}$ and $n \geq 3$, an application of the binomial theorem gives

$$(1+x)^{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} x^k, \tag{2.20}$$

and so

$$\begin{aligned} \sum_{k=2}^{n-1} (k-1)\binom{n-1}{k} x^k &= 1 + (n-1)(1+x)^{n-2}x - (1+x)^{n-1} \\ &= 1 + (1+x)^{n-2}[x(n-2) - 1]. \end{aligned} \tag{2.21}$$

Choosing $x = 1/(n-2)$ in assertion (2.21) gives the equality. However, for every $\epsilon > 0$, we have

$$f'\left(\frac{n-2}{n-2+\epsilon}\right) = 0. \tag{2.22}$$

Hence f is not univalent in $\mathbb{U}$ and the result is sharp. Moreover it can be easily checked that the equality in (2.15) holds for the given functions and the proof is complete. □

3. Special Cases and Examples

Letting $n = 2$ in inequality (2.15) gives the univalence criterion defined by (1.4), which is due to Yang and Liu [7]. Next, we reduce the result for some values of n by computing the corresponding values of β_k in terms of the coefficients. More precisely, for $n = 3$ and $n = 4$, Corollary 2.2 reduces at once to the following two remarks.

Remark 3.1. Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ with $f(z) \neq 0$ when $0 < |z| < 1$ satisfy

$$\left|\left(\frac{z}{f(z)}\right)'''\right| \leq 3 - 3\left|a_2^2 - a_3\right| \quad (z \in \mathbb{U}). \tag{3.1}$$

Then $f(z)$ is univalent in $\mathbb{U}$. The bound in (3.1) is best possible, where equality occurs for the Koebe function and for functions of the following form:

$$f(z) = \frac{z}{1 + az + z^2} \qquad (|a| \leq 2). \tag{3.2}$$

Proof. The result follows from taking $n = 3$ in Corollary 2.2 and that $|\beta_2| = 2|a_2^2 - a_3|$. □

Remark 3.2. Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ with $f(z) \neq 0$ for $0 < |z| < 1$ satisfy

$$\left|\frac{d^4}{dz^4}\left(\frac{z}{f(z)}\right)\right| \leq 8 - 8\left|a_2^2 - a_3\right| - 16\left|a_4 - 2a_2a_3 + a_2^3\right| \quad (z \in \mathbb{U}). \tag{3.3}$$

Then $f(z)$ is univalent in $\mathbb{U}$. The bound in (3.3) is best possible, where equality occurs for the Koebe function and also for functions of the following form:

$$f(z) = \frac{z}{1 + az + z^2} \quad (|a| \leq 2), \qquad f(z) = \frac{z}{(1 \pm (1/2)z)^3}. \tag{3.4}$$

Proof. The result follows from taking $n = 4$ in Corollary 2.2 and that $|\beta_3| = 6|a_4 - 2a_2a_3 + a_2^3|$, and $|\beta_2| = 2|a_2^2 - a_3|$. □

To understand the behavior of the extremal functions for our criterion (2.15), let us consider, for example, $f(z) = z/(1 - (1/2)z)^3$, which is an extremal function for the case $n = 4$. Figures 1(a) and 1(b) show the images of the unit circle under the functions $f(z)$ and $g(z) = z/(1 - (1/2)z)^{3.05}$, respectively. If we restrict the images around the cusps as shown in Figures 1(c) and 1(d), we see that the image of g is a curve that intersects itself in some purely real point u. This means that there are two different points z_1 and z_2 that lie on the unit circle such that $g(z_1) = g(z_2) = u$. In fact, each purely real point lies inside the closed curve of Figures 1(c) and 1(d) which is an image for two different points in $\mathbb{U}$ having the same modulus but different arguments. However, we cannot find such points for the function f, and this interprets why f is an extremal function for univalence, since the closed curve of Figure 1(d) vanishes whenever the power in the function g approaches to 3 as shown in Figure 1(c).

From Corollary 2.2, we have the following.

Corollary 3.3. *Let*

$$f(z) = \frac{z}{1 + \sum_{k=1}^{\infty} b_k z^k} \in \mathcal{A} \tag{3.5}$$

with $f(z) \neq 0$ *for* $0 < |z| < 1$ *and*

$$\sum_{k=2}^{n}(k-1)|b_k| + (n-1)\sum_{k=n+1}^{\infty}\binom{k}{n}|b_k| \leq 1, \tag{3.6}$$

for some $n \in \{2, 3, \ldots\}$. *Then* $f(z)$ *is univalent in* $\mathbb{U}$.

Proof. In view of (3.5) and by simple computation we have

$$\frac{d^n}{dz^n}\left(\frac{z}{f(z)}\right) = n!b_n + \sum_{k=n+1}^{\infty}\frac{k!}{(k-n)!}b_k z^{k-n}, \tag{3.7}$$

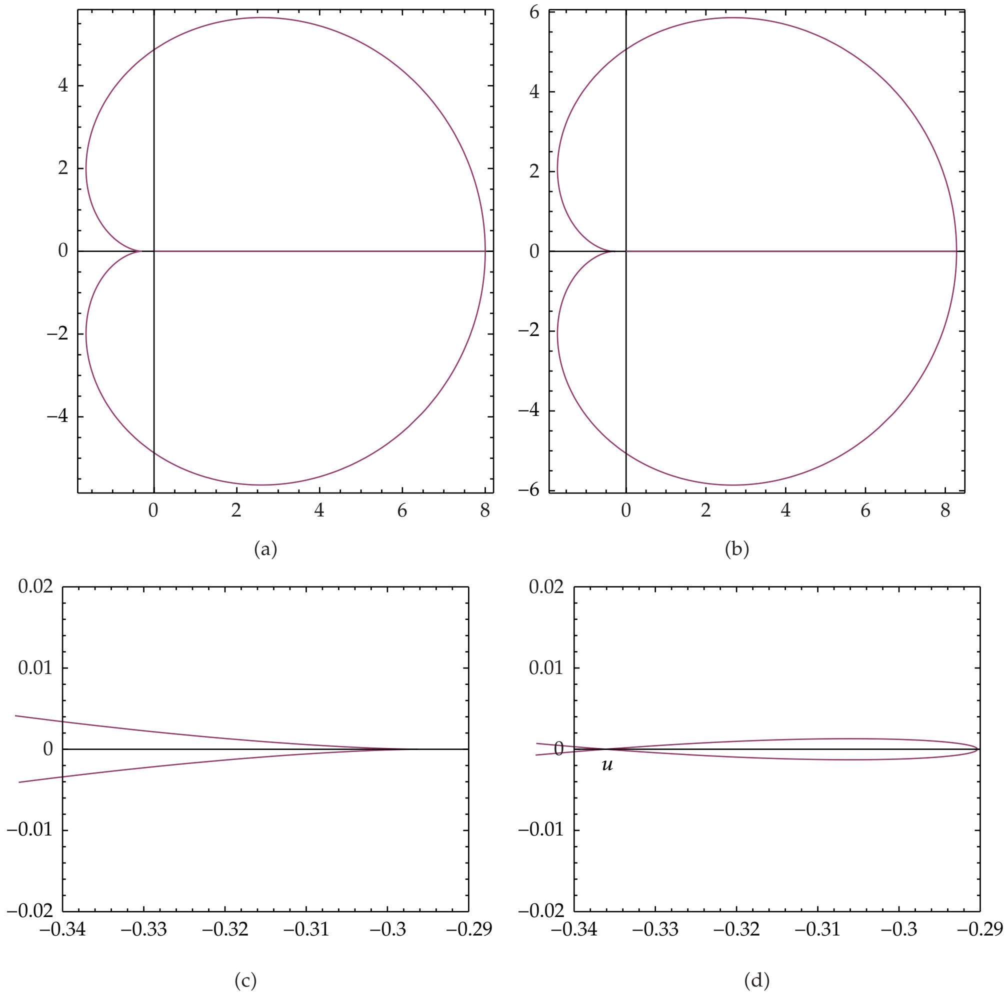

Figure 1: Geometric description for the sharpness of the case $n = 4$.

and so $\beta_m = m! b_m$, for $1 \le m \le n-1$. It follows that

$$\left| \frac{d^n}{dz^n}\left(\frac{z}{f(z)}\right) \right| \le \sum_{k=n}^{\infty} \frac{k!|b_k|}{(k-n)!}. \tag{3.8}$$

Hence, by applying Corollary 2.2, we get the desired result. □

Remark 3.4. Taking $n = 2$ in Corollary 3.3 gives a result of Yang and Liu [7].

Example 3.5. From Corollary 3.3, it can be easily seen that the functions

$$f(z) = \frac{z}{1 + \sum_{k=1}^{n} b_k z^k}, \tag{3.9}$$

with $f(z) \neq 0$ for $0 < |z| < 1$ and $\sum_{k=2}^{n} (k-1)|b_k| \le 1$, are univalent in $\mathbb{U}$.

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Research Article

Harmonic Morphisms Projecting Harmonic Functions to Harmonic Functions

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For Riemannian manifolds M and N, admitting a submersion ϕ with compact fibres, we introduce the projection of a function via its decomposition into horizontal and vertical components. By comparing the Laplacians on M and N, we determine conditions under which a harmonic function on $U = \phi^{-1}(V) \subset M$ projects down, via its horizontal component, to a harmonic function on $V \subset N$.

1. Introduction and Preliminaries

Harmonic morphisms are the maps between Riemannian manifolds which preserve germs of harmonic functions, that is, these (locally) pull back harmonic functions to harmonic functions. The aim of this paper is to analyse the converse situation and to investigate the class of harmonic morphisms that (locally) projects or pushes forward harmonic functions to harmonic functions, in the sense of Definition 2.4. If such a class exists, another interesting question arises "*to what extent does the pull back of the projected function preserve the original function.*"

The formal theory of harmonic morphisms between Riemannian manifolds began with the work of Fuglede [1] and Ishihara [2].

Definition 1.1. A smooth map $\phi : M^m \to N^n$ between Riemannian manifolds is called a *harmonic morphism* if, for every real-valued function f which is harmonic on an open subset V of N with $\phi^{-1}(V)$ nonempty, $f \circ \phi$ is a harmonic function on $\phi^{-1}(V)$.

These maps are related to horizontally (weakly) conformal maps which are a natural generalization of Riemannian submersions.

For a smooth map $\phi : M^m \to N^n$, let $C_\phi = \{x \in M \mid \text{rank } d\phi_x < n\}$ be its *critical set*. The points of the set $M \setminus C_\phi$ are called *regular points*. For each $x \in M \setminus C_\phi$, the *vertical space* at x is defined by $T_x^V M = \text{Ker } d\phi_x$. The *horizontal space* $T_x^H M$ at x is given by the orthogonal complement of $T_x^V M$ in $T_x M$.

Definition 1.2 (see [3, Section 2.4]). A smooth map $\phi : (M^m, \mathbf{g}) \to (N^n, \mathbf{h})$ is called *horizontally (weakly) conformal* if $d\phi = 0$ on C_ϕ and the restriction of ϕ to $M \setminus C_\phi$ is a conformal submersion, that is, for each $x \in M \setminus C_\phi$, the differential $d\phi_x : T_x^H M \to T_{\phi(x)} N$ is conformal and surjective. This means that there exists a function $\lambda : M \setminus C_\phi \to \mathbb{R}^+$ such that

$$\mathbf{h}\big(d\phi(X), d\phi(Y)\big) = \lambda^2 \mathbf{g}(X, Y), \quad \forall X, Y \in T_x^H M. \tag{1.1}$$

By setting $\lambda = 0$ on C_ϕ, we can extend $\lambda : M \to \mathbb{R}_0^+$ to a continuous function on M such that λ^2 is smooth. The extended function $\lambda : M \to \mathbb{R}_0^+$ is called the *dilation* of the map.

For $x \in M \setminus C_\phi$, the assignments $x \mapsto T_x^H M$ and $x \mapsto T_x^V M$ define smooth distributions $T^H M$ and $T^V M$ on $M \setminus C_\phi$ or subbundles of $TM|_{M \setminus C_\phi}$, the tangent bundle of $M \setminus C_\phi$. The distributions $T^H M$ and $T^V M$ are, respectively, called horizontal distribution (or horizontal subbundle) and vertical distribution (or vertical subbundle) defined by ϕ.

Recall that a map $\phi : M^m \to N^n$ is said to be *harmonic* if it extremizes the associated energy integral $E(\phi) = (1/2) \int_\Omega \|\phi_*\|^2 dv^M$ for every compact domain $\Omega \subset M$. It is well known that a map (ϕ) is harmonic if and only if its tension field vanishes.

Harmonic morphisms can be viewed as a subclass of harmonic maps in the light of the following characterization, obtained in [1, 2].

A smooth map is a harmonic morphism if and only if it is harmonic and horizontally (weakly) conformal.

What is special about this characterization of harmonic morphism is that it equips them with geometric as well as analytic features. For instance, the following result of Baird and Eells [4, Riemannian case] and Gudmundsson [5, semi-Riemannian case] reflects such properties of harmonic morphisms.

Theorem 1.3. *Let $\phi : M^m \to N^n$ be a horizontally conformal submersion with dilation λ. If*

(1) $n = 2$, *then ϕ is a harmonic map if and only if it has minimal fibres;*

(2) $n \geq 3$, *then two of the following imply the other:*

- (a) *ϕ is a harmonic map,*
- (b) *ϕ has minimal fibres,*
- (c) $\mathbf{grad}^H \lambda^2 = 0$ *where $\mathbf{grad}^H \lambda^2$ denotes the projection of $\mathbf{grad}\, \lambda^2$ on the horizontal subbundle of TM, obtained through the unique orthogonal decomposition into vertical and horizontal parts.*

For the fundamental results and properties of harmonic morphisms, the reader is referred to [1, 3, 6, 7] and for an updated *online* bibliography to [8].

2. The Projection of a Function via a Submersion

Given a smooth map $\phi : M^m \to N^n$ with compact fibres $\phi^{-1}(\phi(x))$ for $x \in M \setminus C_\phi$, we can use fibre integration to define the horizontal and vertical components of every integrable function f on $U \subset M$ at regular points.

Definition 2.1. Let $\phi : M^m \to N^n$ be a smooth map between Riemannian manifolds with compact fibres. Define the horizontal component of an integrable function f, on M, at a regular point x as the average of f taken over the fibre $\phi^{-1}(\phi(x))$. Precisely, for any $V \subset N$ and integrable function $f : U = \phi^{-1}(V) \subset M \to \mathbb{R}$, the *horizontal component* of f at a regular point x is defined as

$$(\mathcal{H} f)(x) = \frac{1}{\operatorname{vol}(\phi^{-1}(y))}\left(\int_{\phi^{-1}(y)} f\, dv^{\phi^{-1}(y)}\right)(\phi(x)), \tag{2.1}$$

where $x \in U$, $\phi(x) = y$, $dv^{\phi^{-1}(y)}$ is the volume element of the fibre $\phi^{-1}(y)$, $\operatorname{vol}(\phi^{-1}(y))$ is the volume of the fibre $\phi^{-1}(y)$, and $(\int_{\phi^{-1}(y)} f\, dv^{\phi^{-1}(y)})(\phi(x))$ denotes the integral of $f|_{\phi^{-1}(\phi(x))}$.

The *vertical component* of f is given by

$$(\mathcal{V} f)(x) = (f - \mathcal{H} f)(x). \tag{2.2}$$

Note that the horizontal component of a function depends only on the fibre $\phi^{-1}(y)$ and not the choice of $x \in \phi^{-1}(y)$.

Definition 2.2. Let $\phi : M^m \to N^n$ be a submersion. A function $f : U \subset M \to \mathbb{R}$ is called *horizontally homothetic* if the vector field $\mathbf{grad}(f)$ is vertical, that is, at each point $\mathbf{grad}(f)$ is tangent to the fibre.

The components $\mathcal{H} f$ and $\mathcal{V} f$ have the following basic properties for submersions.

Lemma 2.3. *Let $\phi : M^m \to N^n$ be a submersion with compact fibres. Suppose that the fibres $\phi^{-1}(y)$, $y \in N$ are minimal submanifolds of M. Consider $x \in U$ and a function $f : U \subset M \to \mathbb{R}$.*

(1) *If f is horizontally homothetic at x, then $\mathcal{H} f$ is also horizontally homothetic at x.*

(2) *If $\mathcal{H} f$ is horizontally homothetic at x and either $X_i(f) \geq 0$ or $X_i(f) \leq 0$ (for all i) on the fibre through x, then f is horizontally homothetic, where $\{X_i\}_{i=1}^n$ is a local orthonormal frame for the horizontal distribution.*

(3) *If f is constant along the fibre through x then $\mathcal{V} f = 0$.*

Proof. The proof can be completed by following the calculations in Proposition 3.1. □

Definition 2.4. Let $\phi : M^m \to N^n$ be a submersion with compact fibres, and let $f : U = \phi^{-1}(V) \subset M \to \mathbb{R}$ be an integrable function. The horizontal component of f defines a function $\tilde{f} : V \subset N \to \mathbb{R}$ as

$$\tilde{f}(y) = (\mathcal{H} f)(x), \tag{2.3}$$

where $x \in U$ and $y = \phi(x)$. The function $\tilde{f}$ is called the *projection* of f on N, *via the map* ϕ.

We next focus on projection of harmonic functions to harmonic functions via harmonic morphisms.

3. Harmonic Morphisms Projecting Harmonic Functions

The conditions under which harmonic morphisms project harmonic functions to harmonic functions can be obtained by employing an identity relating the Laplacian on the fibre with the Laplacians on the domain and target manifolds.

Recalling that for a submersion $\phi : M^m \to N^n$, the vector fields X on M and Y on N are said to be *ϕ-related* if $d\phi(X_x) = Y_{\phi(x)}$ for every $x \in M$. A horizontal vector field X on M is called *basic* if it is ϕ-related to some vector field X' on N, and X is called *horizontal lift* of X'. It is well known that for a given vector field X' on N, there exists a unique horizontal lift X of X' such that X and X' are ϕ-related.

Proposition 3.1. *Let $\phi : (M^m, \mathbf{g}) \to (N^n, \mathbf{h})$ $(n > 2)$ be a nonconstant submersive harmonic morphism with dilation λ, having compact, connected, and minimal fibres. Then for any $V \subset N$ and $f : U = \phi^{-1}(V) \subset M \to \mathbb{R}$,*

$$\begin{aligned}\Delta^N \tilde{f} &= \frac{1}{\operatorname{vol}(\phi^{-1}(y))}\int_{\phi^{-1}(y)} \frac{1}{\lambda^2}\left(\Delta^M f - \Delta^{\phi^{-1}(y)} f\right) dv^{\phi^{-1}(y)} \\ &\quad + \frac{1}{\operatorname{vol}(\phi^{-1}(y))}\sum_{i=1}^{n}\int_{\phi^{-1}(y)} \left(\nabla^M_{X_i} X_i\right)^V f\, dv^{\phi^{-1}(y)},\end{aligned} \tag{3.1}$$

where $x \in U$, $\phi(x) = y$, $\tilde{f}$ is as defined in Definition 2.4 and Δ^M, Δ^N, $\Delta^{\phi^{-1}(y)}$ are the Laplacians on M, N, $\phi^{-1}(y)$, respectively, ∇^M is the Levi-Civita connection on M, $(\nabla^M_{X_i} X_i)^V$ denotes the vertical component of $\nabla^M_{X_i} X_i$, and $\{X_i\}_{i=1}^n$ denote the horizontal lift of a local orthonormal frame $\{X'_i\}_{i=1}^n$ for TN.

Proof. First notice from Theorem 1.3 that λ is horizontally homothetic, a fact which will be used repeatedly in the proof.

Choose a local orthonormal frame $\{X'_i\}_{i=1}^n$ for TN. If X_i denotes the horizontal lift of X'_i for $i = 1, \ldots, n$, then $\{\lambda X_i\}_{i=1}^n$ is a local orthonormal frame for the horizontal distribution. Let $\{X_\alpha\}_{\alpha=n+1}^m$ be a local orthonormal frame for the vertical distribution. Then we can write the Laplacian Δ^M on M as

$$\begin{aligned}\Delta^M &= \sum_{i=1}^{n}\left\{\lambda X_i \circ \lambda X_i - \nabla^M_{\lambda X_i} \lambda X_i\right\} + \sum_{\alpha=n+1}^{m}\left\{X_\alpha \circ X_\alpha - \nabla^M_{X_\alpha} X_\alpha\right\} \\ &= \lambda^2 \sum_{i=1}^{n}\left\{X_i \circ X_i - \nabla^M_{X_i} X_i\right\} + \sum_{\alpha=n+1}^{m}\left\{X_\alpha \circ X_\alpha - \nabla^M_{X_\alpha} X_\alpha\right\}.\end{aligned} \tag{3.2}$$

Now the Laplacian of the fibre $\phi^{-1}(y)$ is

$$\Delta^{\phi^{-1}(y)} = \sum_{\alpha=n+1}^{m}\left\{X_\alpha \circ X_\alpha - \nabla^{\phi^{-1}(y)}_{X_\alpha} X_\alpha\right\}. \tag{3.3}$$

If B is the second fundamental form of the fibre $\phi^{-1}(y)$ as a submanifold in M, we can write $\nabla^M_{X_\alpha} X_\alpha$ as

$$\nabla^M_{X_\alpha} X_\alpha = \nabla^{\phi^{-1}(y)}_{X_\alpha} X_\alpha + B(X_\alpha, X_\alpha). \tag{3.4}$$

Let μ denote the mean curvature vector of $\phi^{-1}(y)$ given by

$$\mu = \frac{1}{m-n} \sum_{\alpha=n+1}^{m} B(X_\alpha, X_\alpha). \tag{3.5}$$

Setting $H = (m-n)\mu$, we obtain from (3.2)

$$\begin{aligned} \Delta^M &= \lambda^2 \sum_{i=1}^{n} \left\{ X_i \circ X_i - \left(\nabla^M_{X_i} X_i\right)^H \right\} + \Delta^{\phi^{-1}(y)} - H - \lambda^2 \sum_{i=1}^{n} \left(\nabla^M_{X_i} X_i\right)^V \\ &= \lambda^2 \sum_{i=1}^{n} \left\{ X_i \circ X_i - \left(\nabla^M_{X_i} X_i\right)^H \right\} + \Delta^{\phi^{-1}(y)} - \lambda^2 \sum_{i=1}^{n} \left(\nabla^M_{X_i} X_i\right)^V, \end{aligned} \tag{3.6}$$

where X^H, X^V denote the orthogonal projections of a vector field X on the horizontal and vertical subbundles of TM, respectively.

Since X_i is the horizontal lift of X_i' $(i = 1, \dots, n)$, we have

$$\begin{aligned} X_i'\left(\tilde{f}\right) &= \frac{1}{\operatorname{vol}(\phi^{-1}(y))} \left\{ \int_{\phi^{-1}(y)} X_i(f)\, dv^{\phi^{-1}(y)} + \int_{\phi^{-1}(y)} f \mathcal{L}_{X_i}\left(dv^{\phi^{-1}(y)}\right) \right\} \\ &= \frac{1}{\operatorname{vol}(\phi^{-1}(y))} \left\{ \int_{\phi^{-1}(y)} X_i(f)\, dv^{\phi^{-1}(y)} + \sum_{\alpha=n+1}^{m} \int_{\phi^{-1}(y)} f \mathbf{g}\left(\nabla^M_{X_\alpha} X_i, X_\alpha\right) dv^{\phi^{-1}(y)} \right\} \\ &= \frac{1}{\operatorname{vol}(\phi^{-1}(y))} \left\{ \int_{\phi^{-1}(y)} X_i(f)\, dv^{\phi^{-1}(y)} - \int_{\phi^{-1}(y)} f \mathbf{g}(H, X_i)\, dv^{\phi^{-1}(y)} \right\}, \end{aligned} \tag{3.7}$$

where $\mathcal{L}_{X_i}$ denotes the Lie derivative along X_i. The volume of the fibres does not vary in the horizontal direction because of the relation $X_i'(\operatorname{vol}(\phi^{-1}(y))) = -\int_{\phi^{-1}(y)} \mathbf{g}(H, X_i)\, dv^{\phi^{-1}(y)}$ and the fact that the fibres are minimal.

Similarly, we obtain

$$\begin{aligned} X_i' \circ X_i'\left(\tilde{f}\right) = &\frac{1}{\operatorname{vol}(\phi^{-1}(y))} \left\{ \int_{\phi^{-1}(y)} X_i \circ X_i(f)\, dv^{\phi^{-1}(y)} - \int_{\phi^{-1}(y)} X_i(f) \cdot \mathbf{g}(H, X_i)\, dv^{\phi^{-1}(y)} \right\} \\ &- \frac{1}{\operatorname{vol}(\phi^{-1}(y))} \left\{ \int_{\phi^{-1}(y)} X_i(f \mathbf{g}(H, X_i))\, dv^{\phi^{-1}(y)} - \int_{\phi^{-1}(y)} f(\mathbf{g}(H, X_i))^2\, dv^{\phi^{-1}(y)} \right\}. \end{aligned} \tag{3.8}$$

The horizontal homothety of the dilation implies that $(\nabla^M_{X_i} X_i)^H$ is the horizontal lift of $\nabla^N_{X'_i} X'_i$, cf. [9, Lemma 3.1]; therefore, we have

$$\nabla^N_{X'_i} X'_i\left(\tilde{f}\right) = \frac{1}{\operatorname{vol}(\phi^{-1}(y))} \times \left\{ \int_{\phi^{-1}(y)} \left(\nabla^M_{X_i} X_i\right)^H (f) dv^{\phi^{-1}(y)} - \int_{\phi^{-1}(y)} f \cdot \mathbf{g}\left(H, \left(\nabla^M_{X_i} X_i\right)^H\right) dv^{\phi^{-1}(y)} \right\}. \tag{3.9}$$

Now using (3.7), (3.8), (3.9), along with the condition that the fibres are minimal, in (3.6) completes the proof. □

From the above proposition, we see that it suffices to take λ constant to have both f and $\tilde{f}$ harmonic on M and N, respectively. In this case, by a homothety of M we may suppose that $\lambda \equiv 1$ and ϕ is a harmonic Riemannian submersion. We then have the following consequence.

Theorem 3.2. *Let $\phi : (M^m, \mathbf{g}) \to (N^n, \mathbf{h})$ $(n \geq 2)$ be a harmonic Riemannian submersion with compact, connected fibres. Then the projection $\tilde{f} : V \subset N \to \mathbb{R}$ (via ϕ) of any harmonic function $f : U = \phi^{-1}(V) \subset M \to \mathbb{R}$ is a harmonic function. Moreover, $\mathscr{H} f = \tilde{f} \circ \phi$. If $[f_{\mathscr{H}}]$ denotes the class of harmonic functions on $U = \phi^{-1}(V)$ having the same horizontal component then each class $[f_{\mathscr{H}}]$ has a unique representative in the space of harmonic functions on V.*

Proof. Since $\Delta^M f = 0$ and the dilation $\lambda \equiv 1$, Proposition 3.1 leads to

$$\Delta^N \tilde{f} = \frac{1}{\operatorname{vol}(\phi^{-1}(y))} \sum_{i=1}^{n} \int_{\phi^{-1}(y)} \left(\nabla^M_{X_i} X_i\right)^V f dv^{\phi^{-1}(y)}, \tag{3.10}$$

where we have also used the fact that

$$\int_{\phi^{-1}(y)} \Delta^{\phi^{-1}(y)} f dv^{\phi^{-1}(y)} = 0 \tag{3.11}$$

for compact fibres.

Let $\{X'_i\}_{i=1}^n$ be a local orthonormal frame for TN and X_i be the horizontal lift of X'_i for $i = 1, \ldots, n$. Then $\{X_i\}_{i=1}^n$ is a local orthonormal frame for the horizontal distribution. Let $\{X_\alpha\}_{\alpha=n+1}^m$ be a local orthonormal frame for the vertical distribution. Then using the standard expression for Levi-Civita connection, we have

$$\begin{aligned}
\left(\nabla^M_{X_i} X_i\right)^V &= \sum_{\alpha=n+1}^{m} \mathbf{g}\left(\nabla^M_{X_i} X_i, X_\alpha\right) X_\alpha \\
&= \frac{1}{2} \sum_{\alpha=n+1}^{m} \{X_i(\mathbf{g}(X_i, X_\alpha)) + X_i(\mathbf{g}(X_\alpha, X_i)) - X_\alpha(\mathbf{g}(X_i, X_i)) \\
&\qquad -\mathbf{g}(X_i, [X_i, X_\alpha]) + \mathbf{g}(X_i, [X_\alpha, X_i]) + \mathbf{g}(X_\alpha, [X_i, X_i])\} X_\alpha.
\end{aligned} \tag{3.12}$$

Because X_i are basic, X_α are vertical we have $[X_i, X_\alpha]$ vertical and therefore

$$\left(\nabla^M_{X_i} X_i\right)^V = 0. \tag{3.13}$$

Hence, from (3.10), $\tilde{f}$ is harmonic. The rest of the proof follows from the construction of $\tilde{f}$. □

As an application, we give a description of harmonic functions on manifolds admitting harmonic Riemannian submersions with compact fibres.

Corollary 3.3. *Let M^m be a Riemannian manifold admitting a harmonic Riemannian submersion $\phi : M^m \to N^n$ with compact fibres. Then*

(1) *every horizontally homothetic harmonic function on $U \subset M$ is horizontal, that is, $\mathcal{V}f = 0$, and so in particular is constant;*

(2) *every nonhorizontally homothetic harmonic function f on $U \subset M$ satisfies one of the following:*

(a) $\mathcal{V}f \neq 0$;

(b) $\mathcal{V}f = 0$ *and* $X_i(\mathcal{H}f) \neq 0$ *for at least one* $i \in \{1, \ldots, n\}$;

(c) $\mathcal{V}f = 0$, $X_i(\mathcal{H}f) = 0$ *(for all i) and* $X_i(f)$ *changes sign on the fibre, for at least one* $i \in \{1, \ldots, n\}$.

Proof. Equation (3.6) implies that a horizontally homothetic harmonic function on M is harmonic on the fibre and hence is constant on the fibre. Now using Lemma 2.3 we get the proof. □

Remark 3.4. (1) Since an $\mathbb{R}^N$-valued map $f = (f^1, \ldots, f^N)$ is harmonic if and only if each of its component is harmonic, we see that Riemannian submersions with compact fibres project $\mathbb{R}^N$-valued harmonic maps from $\phi^{-1}(V)$ to $\mathbb{R}^N$-valued harmonic maps from V.

(2) Given a Lie group G and a compact subgroup H of G, the standard projection $\phi : G \to G/H$ with G-invariant metric provides many examples satisfying the hypothesis of Theorem 3.2. Further examples can be obtained from Bergery's construction $\phi : G/K \to G/H$ with $K \subset H \subset G$ and K, H compact; see [10] for the details of the metrics for which ϕ is a harmonic morphism. Another reference for such examples is [11, Chapter 6].

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Research Article

Hermite Interpolation Using Möbius Transformations of Planar Pythagorean-Hodograph Cubics

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We present an algorithm for C^1 Hermite interpolation using Möbius transformations of planar polynomial Pythagoreanhodograph (PH) cubics. In general, with PH cubics, we cannot solve C^1 Hermite interpolation problems, since their lack of parameters makes the problems overdetermined. In this paper, we show that, for each Möbius transformation, we can introduce an *extra parameter* determined by the transformation, with which we can reduce them to the problems determining PH cubics in the complex plane $\mathbb{C}$. Möbius transformations preserve the PH property of PH curves and are biholomorphic. Thus the interpolants obtained by this algorithm are also PH and preserve the topology of PH cubics. We present a condition to be met by a Hermite dataset, in order for the corresponding interpolant to be simple or to be a loop. We demonstrate the improved stability of these new interpolants compared with PH quintics.

1. Introduction

Farouki and Sakkalis [1] introduced Pythagorean-hodograph (PH) curves, which are a special class of polynomial curves with a polynomial speed function. These curves have many computationally attractive features: in particular, their arc lengths and offset curves can be determined exactly. Farouki [2] reviews the abundant results on these curves obtained by many researchers. Hermite interpolation with PH curves is one of the main topics in this research (Farouki and Neff [3], Albrecht and Farouki [4], Jüttler [5], Jüttler and Mäurer [6], Farouki et al. [7], Pelosi et al. [8], and Šír et al. [9]).

In this paper, we solve the C^1 Hermite interpolation problem using the Möbius transformations of polynomial PH cubics in the plane. The use of Möbius transformation has been demonstrated in recent publications [10, 11]. In [11], Bartoň et al. used a *general* Möbius transformation in $\mathbb{R}^3$, which is defined as a composition of an arbitrary number of inversions with respect to spheres or planes. They showed that $(\mu \circ \mathbf{x})(t)$ is a rational PH curve for any general Möbius transformation $\mu(x_1, x_2, x_3)$, if $\mathbf{x}(t)$ is a polynomial PH space curve in $\mathbf{R}^3$. (The preservation of PH properties under transformation is first studied by Ueda [12].) They also presented an algorithm for G^1 Hermite interpolation. In this work, we use the *classical* Möbius transformation, a bijective linear fractional transformation in the extended complex plane $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$, that is,

$$\Phi(z) = \frac{az+b}{cz+d}, \tag{1.1}$$

for some complex numbers a, b, c, and d for which $ad - bc \neq 0$ [13]. Using this transformation, we can solve the C^1 Hermite interpolation problems with PH cubics. In general, with PH cubics, we cannot solve C^1 Hermite interpolation problems, since their lack of parameters makes the problems overdetermined. But we can show that, for a C^1 Hermite interpolation problem, we are always able to obtain four interplants which are constructed by PH cubics. The Möbius transformation makes this possible, since it permits the introduction of a new extra parameter into the problem, which is to be reduced to a simple problem to determine PH cubics as follows: here we adapt the complex representation method [14] to solve the C^1 Hermite interpolation problem. The original problem is, for a Hermite dataset $(\mathbf{p}_i, \mathbf{p}_f, \mathbf{v}_i, \mathbf{v}_f)$, to find a polynomial PH curve $\mathbf{r}(t)$ and a Möbius transformation $\Phi(z)$, which satisfy

$$(\Phi \circ \mathbf{r})(0) = \mathbf{p}_i, \qquad (\Phi \circ \mathbf{r})(1) = \mathbf{p}_f, \qquad (\Phi \circ \mathbf{r})'(0) = \mathbf{v}_i, \qquad (\Phi \circ \mathbf{r})'(1) = \mathbf{v}_f. \tag{1.2}$$

Next, by an appropriate translation, rotation, and scaling of the dataset, we can arrange that $\mathbf{p}_i = \mathbf{0}$ and $\mathbf{p}_f = 1$ and take a Möbius transformation

$$\Phi(z) = \frac{\alpha z}{(\alpha - 1)z + 1}, \tag{1.3}$$

which fixes 0 and 1, for some nonzero complex number α. Then the inverse image of the C^1 Hermite dataset under a Möbius transformation Φ makes (1.2) into

$$\mathbf{r}(0) = 0, \qquad \int_0^1 \mathbf{r}'(t)\,dt = 1, \qquad \mathbf{r}'(0) = \frac{1}{\alpha}\mathbf{v}_i, \qquad \mathbf{r}'(1) = \alpha \mathbf{v}_f, \tag{1.4}$$

which are suitable forms for adapting the complex representation method (for details, see Section 4). Farouki and Neff [3] already solved the C^1 Hermite interpolation problem with PH quintics. According to (1.2), this is exactly the case in which $\mathbf{r}(t)$ is a quintic and $\Phi(z)$ is the identity, that is, $\alpha = 1$. On the other hand, our work in this paper is the case just when $\Phi(z)$ is not the identity, that is, $\alpha \neq 1$. At the end of this paper, we will compare our interpolants with PH quintic ones for the same C^1 Hermite dataset.

The interpolants obtained by our method are specific rational curves represented by complex rational functions. For planar rational curves, there already exists a general theory, which were introduced by Pottmann [15] and Fiorot and Gensane [16]: they studied rational plane curves with rational offets. These curves are represented in the dual form, in which curves are specified using line coordinates instead of point coordinates. Pottmann showed how to design rational PH curves segments by G^1 and G^2 Hermite interpolations [17, 18]. However, in our work, what we need is only a suitable PH cubic and a PH-preserving transformation which is algebraically simple as possible and which can generate an extra-parameter, and the latter is completely settled by the classical Möbius transformation. Moreover, the transformation is *biholomorphic*. Thus it preserves the topology of the preimage curve (PH cubic). Therefore, the interpolants obtained by our method should have no cusp, although cusps are a generic feature of rational PH curves. They are simple curves or else loops. Hence, to obtain these, even avoiding the easy shortcut, there is no need to follow up the lengthy path with a far starting point. We just use the classical Möbius transformation of PH cubics, that is all.

The rest of this paper is organized as follows. In Sections 2 and 3, we review some basic properties of Möbius transformations and planar PH cubics. In Section 4, we solve the C^1 Hermite interpolation problem using the Möbius transformations of planar PH cubics. In Section 5, we present the condition on a Hermite dataset, which determine whether the corresponding Hermite interpolant has a loop, we also compare these new interpolants with PH quintics and show that the former have improved stability. We conclude this paper in Section 6.

2. Möbius Transformations

A *Möbius transformation* $\Phi(z)$ is a bijective linear fractional transformation in the extended complex plane $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$, that is,

$$\Phi(z) = \frac{az+b}{cz+d}, \tag{2.1}$$

for some complex numbers a, b, c, and d for which $ad - bc \neq 0$ [13]. Then $\Phi(z)$ is a one-to-one correspondence on the extended complex plane $\mathbb{C}_\infty$ with its inverse

$$\Phi^{-1}(z) = \frac{dz-b}{-cz+a}. \tag{2.2}$$

The derivative of $\Phi(z)$ is

$$\Phi'(z) = \frac{ad-bc}{(cz+d)^2}. \tag{2.3}$$

For any Möbius transformations $\Phi(z)$ and $\Psi(z)$, $(\Psi \circ \Phi)(z)$ is also a Möbius transformation. Thus the set $\mathcal{M}$ of all Möbius transformations forms a group under composition.

A rational plane curve $\mathbf{r}(t) = x(t) + \sqrt{-1}y(t)$ is called a *Pythagorean-hodograph* (PH) curve [1] if there exists a rational function $\sigma(t)$ such that

$$x'(t)^2 + y'(t)^2 = \sigma(t)^2. \tag{2.4}$$

Lemma 2.1. *Let $\Phi(z)$ be a Möbius transformation and $\mathbf{r}(t)$ be a polynomial PH curve. Then $\mathbf{s}(t) = (\Phi \circ \mathbf{r})(t)$ is a rational PH curve.*

Proof. Since $\mathbf{s}'(t) = \Phi'(\mathbf{r}(t))\mathbf{r}'(t)$, we have

$$\left|\mathbf{s}'(t)\right| = \frac{|ad - bc|}{|c\mathbf{r}(t) + d|^2}\left|\mathbf{r}'(t)\right| = \frac{|ad - bc|}{\operatorname{Re}(c\mathbf{r}(t) + d)^2 + \operatorname{Im}(c\mathbf{r}(t) + d)^2}\left|\mathbf{r}'(t)\right|. \tag{2.5}$$

This completes the proof. □

Lemma 2.1 means that Möbius transformations preserve the PH property, which is a special case of the result of Bartoň et al. [11].

For a polynomial PH curve $\mathbf{r}(t)$ of degree n, a Möbius transformation of $\mathbf{r}(t)$

$$(\Phi \circ \mathbf{r})(t) = \frac{a\mathbf{r}(t) + b}{c\mathbf{r}(t) + d} \tag{2.6}$$

is a rational curve, also of degree n, with coefficients in the complex plane $\mathbb{C}$. However, if we associate the complex plane $\mathbb{C}$ with $\mathbb{R}^2$, and express $(\Phi \circ \mathbf{r})(t)$ as a rational curve with real coefficients in $\mathbb{R}^2$ then the result is generally a rational PH curve of degree $2n$ or n. If we perform a further Möbius transformation $\Psi(z)$, the rational curve $(\Psi \circ (\Phi \circ \mathbf{r}))(t) = ((\Psi \circ \Phi) \circ \mathbf{r})(t)$ retains a degree of $2n$ or n, since $(\Psi \circ \Phi)(z)$ is a Möbius transformation.

Lemma 2.2. *Let $(\Phi \circ \mathbf{r})(t)$ be a Möbius transformation $\Phi(z)$ of a polynomial curve $\mathbf{r}(t)$, such that $(\Phi \circ \mathbf{r})(0) = 0$ and $(\Phi \circ \mathbf{r})(1) = 1$. Then there exist a polynomial curve $\mathbf{s}(t)$ and a Möbius transformation $\Psi(z)$, such that $\mathbf{s}(0) = 0$, $\mathbf{s}(1) = 1$, and $(\Phi \circ \mathbf{r})(t) = (\Psi \circ \mathbf{s})(t)$.*

Proof. We can find a Möbius transformation $\Phi_1(z) = az + b$ for some complex constants a and b such that $a \neq 0$ and $\mathbf{s}(t) = (\Phi_1 \circ \mathbf{r})(t)$ is a polynomial curve with $\mathbf{s}(0) = 0$ and $\mathbf{s}(1) = 1$. Consequently, we can obtain the Möbius transformation $\Psi(z) = (\Phi \circ \Phi_1^{-1})(z)$ such that $(\Phi \circ \mathbf{r})(t) = (\Psi \circ \mathbf{s})(t)$. □

A Möbius transformation $\Psi(z)$ of this sort also fixes 0 and 1.

Lemma 2.3. *Let $\Phi(z)$ be a Möbius transformation with $\Phi(0) = 0$ and $\Phi(1) = 1$. Then there exists a nonzero complex constant α such that*

$$\Phi(z) = \frac{\alpha z}{(\alpha - 1)z + 1}. \tag{2.7}$$

If $\tau = |\alpha|$ and $\eta = \arg(\alpha)$, then $\Phi(z) = (\Phi_\tau \circ \Phi_\eta)(z) = (\Phi_\eta \circ \Phi_\tau)(z)$, where

$$\Phi_\tau(z) = \frac{\tau z}{(\tau - 1)z + 1}, \qquad \Phi_\eta(z) = \frac{e^{i\eta} z}{(e^{i\eta} - 1)z + 1}. \tag{2.8}$$

Proof. Let $\Psi(z) = (az + b)/(cz + d)$ be a Möbius transformation with $\Psi(0) = 0$ and $\Psi(\infty) = \infty$. Then from $\Psi(0) = 0$ we get $b = 0$, and from $\Psi(\infty) = \infty$ we get $c = 0$. Thus $\Psi(z) = \alpha z$, where $\alpha = a/d$. Let $\tau = |\alpha|$ and $\eta = \arg(\alpha)$. Then we obtain $\Psi(z) = (\Psi_\tau \circ \Psi_\eta)(z) = (\Psi_\eta \circ \Psi_\tau)(z)$, where $\Psi_\tau(z) = \tau z$ and $\Psi_\eta(z) = e^{i\eta} z$.

Now let $\Phi(z)$ be a Möbius transformation with $\Phi(0) = 0$ and $\Phi(1) = 1$. Let $S(z) = z/(z-1)$, so that $S(0) = 0$ and $S(\infty) = 1$. Then, since $(S^{-1} \circ \Phi \circ S)(0) = 0$ and $(S^{-1} \circ \Phi \circ S)(\infty) = \infty$, we get $(S^{-1} \circ \Phi \circ S)(z) = \Psi(z)$, where $\Psi(z) = \alpha z$ for some nonzero α. Thus we obtain

$$\Phi(z) = \left(S \circ \Psi \circ S^{-1}\right)(z) = \frac{\alpha z}{(\alpha - 1)z + 1}. \tag{2.9}$$

Moreover, in the same way, we can obtain $\Phi_\tau(z) = (S \circ \Psi_\tau \circ S^{-1})(z)$ and $\Phi_\eta(z) = (S \circ \Psi_\eta \circ S^{-1})(z)$, so that $\Phi(z) = (\Phi_\tau \circ \Phi_\eta)(z) = (\Phi_\eta \circ \Phi_\tau)(z)$. □

Since $\Phi'(z) = \alpha/((\alpha-1)z+1)^2$, we have $\Phi'(0) = \alpha$ and $\Phi'(1) = 1/\alpha$.

3. Planar Pythagorean-Hodograph Cubics

A planar polynomial curve $\mathbf{r}(t) = x(t) + \sqrt{-1}y(t)$ is a PH curve [19] if and only if there exist polynomials $h(t)$, $u(t)$, and $v(t)$, which satisfy

$$x'(t) = h(t)\left[u(t)^2 - v(t)^2\right], \qquad y'(t) = h(t)[2u(t)v(t)]. \tag{3.1}$$

Note that, if $\gcd(u(t), v(t)) = 1$, then $\gcd(u(t)^2 - v(t)^2, 2u(t)v(t)) = 1$. In this paper, we will assume that $h(t)$ is monic, meaning that its leading coefficient is 1.

A polynomial curve $\mathbf{r}(t)$ is a PH curve [14] if and only if there exists a polynomial $h(t)$ and a polynomial curve $\mathbf{w}(t)$ such that

$$\mathbf{r}'(t) = h(t)\mathbf{w}(t)^2. \tag{3.2}$$

Suppose that the PH cubic $\mathbf{r}(t)$ is a line. Then the hodograph $\mathbf{r}'(t)$ can be expressed as $h(t)(x_0 + \sqrt{-1}y_0)$, where $x_0 + \sqrt{-1}y_0$ is a nonzero point and $h(t)$ is the quadratic monic polynomial

$$h(t) = h_0(1-t)^2 + h_1 2(1-t)t + h_2 t^2, \tag{3.3}$$

and h_0, h_1, and h_2 are real constants such that $h_0 + h_2 \neq 2h_1$.

Let $\mathbf{r}(t)$ be a PH cubic for which $\mathbf{r}'(t) = \mathbf{w}(t)^2$. Since $\mathbf{w}(t)$ is linear, we can write $\mathbf{w}(t)$ in Bernstein form:

$$\mathbf{w}(t) = \mathbf{w}_0(1-t) + \mathbf{w}_1 t, \tag{3.4}$$

where $\mathbf{w}_0$ and $\mathbf{w}_1$ are distinct complex constants. The hodograph $\mathbf{r}'(t)$ can then be expressed as

$$\mathbf{r}'(t) = \mathbf{w}_0^2(1-t)^2 + \mathbf{w}_0\mathbf{w}_1 2(1-t)t + \mathbf{w}_1^2 t^2. \tag{3.5}$$

If we represent the PH cubic $\mathbf{r}(t)$ in the Bernstein form

$$\mathbf{r}(t) = \mathbf{p}_0(1-t)^3 + \mathbf{p}_1 3(1-t)^2 t + \mathbf{p}_2 3(1-t)t^2 + \mathbf{p}_3 t^3, \tag{3.6}$$

then we obtain

$$\mathbf{p}_1 = \mathbf{p}_0 + \frac{1}{3}\mathbf{w}_0^2, \qquad \mathbf{p}_2 = \mathbf{p}_1 + \frac{1}{3}\mathbf{w}_0\mathbf{w}_1, \qquad \mathbf{p}_3 = \mathbf{p}_2 + \frac{1}{3}\mathbf{w}_1^2, \tag{3.7}$$

where $\mathbf{p}_0$ can be chosen arbitrarily.

4. First-Order Hermite Interpolation

We will now solve the C^1 Hermite interpolation problem using Möbius transformations of PH cubics.

Let $\mathbf{p}_i$ and $\mathbf{p}_f$ be the initial and final points to be interpolated, where $\mathbf{p}_i \neq \mathbf{p}_f$. Let $\mathbf{v}_i = r_i e^{\sqrt{-1}\theta_i}$ and $\mathbf{v}_f = r_f e^{\sqrt{-1}\theta_f}$, respectively, be the initial vector at $\mathbf{p}_i$ and the final vector at $\mathbf{p}_f$, where $r_i > 0$ and $r_f > 0$. For this Hermite dateset $(\mathbf{p}_i, \mathbf{p}_f, \mathbf{v}_i, \mathbf{v}_f)$, we want to find planar PH cubics $\mathbf{r}(t)$ and Möbius transformations $\Phi(z)$ which satisfy (1.2), which are equivalent to

$$\begin{gathered}(\Phi \circ \mathbf{r})(0) = \mathbf{p}_i, \qquad \int_0^1 (\Phi \circ \mathbf{r})'(t)\,dt = \mathbf{p}_f - \mathbf{p}_i, \\ (\Phi \circ \mathbf{r})'(0) = \mathbf{v}_i, \qquad (\Phi \circ \mathbf{r})'(1) = \mathbf{v}_f.\end{gathered} \tag{4.1}$$

By an appropriate translation, rotation, and scaling of the data-set, we can arrange that $\mathbf{p}_i = \mathbf{0}$ and $\mathbf{p}_f = 1$. Then, from Lemmas 2.2 and 2.3, we seek some nonzero constants α and PH cubics $\mathbf{r}(t)$, which satisfy (1.4).

4.1. Case of $r'(t) = h(t) = h_0(1-t)^2 + h_1 2(1-t)t + h_2 t^2$

In this case, (4.1) become

$$\mathbf{r}(0) = 0, \qquad h_0 + h_1 + h_2 = 3, \qquad \alpha h_0 = \mathbf{v}_i, \qquad \frac{h_2}{\alpha} = \mathbf{v}_f. \tag{4.2}$$

From the second and third of these equations, we can see that Hermite interpolants $\mathbf{r}(t)$ exist if and only if $\theta_i + \theta_f = m\pi$ for some integers m. In this case, for $\alpha = \tau e^{\sqrt{-1}\theta_i}$ or $\alpha = \tau e^{\sqrt{-1}(\theta_i+\pi)}$, where τ is any positive number, we have

$$h_0 = \frac{\mathbf{v}_i}{\alpha}, \qquad h_2 = \alpha \mathbf{v}_f, \qquad h_1 = 3 - h_0 - h_2. \tag{4.3}$$

Consequently, we can obtain the PH cubics

$$\mathbf{r}(t) = \frac{h_0}{3}3(1-t)^2 t + \frac{h_0 + h_1}{3}3(1-t)t^2 + \frac{h_0 + h_1 + h_2}{3}t^3 \tag{4.4}$$

and their Möbius transformations

$$(\Phi \circ \mathbf{r})(t) = \frac{\alpha \mathbf{r}(t)}{(\alpha - 1)\mathbf{r}(t) + 1}. \tag{4.5}$$

4.2. Case of $r'(t) = w(t)^2$ Where $w(t) = w_0(1-t) + w_1 t$

From (3.5) and (3.6), (4.1) become

$$\mathbf{r}(0) = 0, \qquad \mathbf{w}_0^2 + \mathbf{w}_0\mathbf{w}_1 + \mathbf{w}_1^2 = 3, \qquad \alpha\mathbf{w}_0^2 = \mathbf{v}_i, \qquad \frac{\mathbf{w}_1^2}{\alpha} = \mathbf{v}_f. \tag{4.6}$$

If we let

$$\mathbf{v} = \frac{\sqrt{r_i r_f}}{3} e^{\sqrt{-1}(\theta_i + \theta_f)/2}, \tag{4.7}$$

and also let $\mathbf{a} = \mathbf{w}_0^2/3$, $\mathbf{b} = \mathbf{w}_1^2/3$ and $\mathbf{k} = \mathbf{w}_0\mathbf{w}_1/3$, then second and third equations in (4.6) imply that $\mathbf{k} = \mathbf{v}$ or $\mathbf{k} = -\mathbf{v}$, and so we have

$$\mathbf{ab} = \mathbf{k}^2, \qquad \mathbf{a} + \mathbf{b} = 1 - \mathbf{k}. \tag{4.8}$$

Now let

$$\begin{aligned} \mathbf{a}_m &= \frac{1 - \mathbf{k} + \sqrt{(1+\mathbf{k})(1-3\mathbf{k})}}{2}, \\ \mathbf{b}_m &= \frac{1 - \mathbf{k} - \sqrt{(1+\mathbf{k})(1-3\mathbf{k})}}{2}, \end{aligned} \tag{4.9}$$

where $m = 1$ if $\mathbf{k} = \mathbf{v}$, and $m = -1$ if $\mathbf{k} = -\mathbf{v}$. Then we have $\mathbf{a} = \mathbf{a}_m$ and $\mathbf{b} = \mathbf{b}_m$, or $\mathbf{a} = \mathbf{b}_m$ and $\mathbf{b} = \mathbf{a}_m$. Consequently, we can obtain the four PH cubics

$$\begin{aligned} \mathbf{r}_{m,1}(t) &= \mathbf{a}_m 3(1-t)^2 t + (\mathbf{a}_m + \mathbf{k})3(1-t)t^2 + (\mathbf{a}_m + \mathbf{k} + \mathbf{b}_m)t^3, \\ \mathbf{r}_{m,2}(t) &= \mathbf{b}_m 3(1-t)^2 t + (\mathbf{b}_m + \mathbf{k})3(1-t)t^2 + (\mathbf{b}_m + \mathbf{k} + \mathbf{a}_m)t^3, \end{aligned} \tag{4.10}$$

where $\mathbf{k} = \mathbf{v}$ or $\mathbf{k} = -\mathbf{v}$. Note that $\mathbf{r}_{m,1} = \mathbf{r}_{m,2}$ if and only if $\mathbf{k}$ is -1 or $1/3$. From the PH cubics $\mathbf{r}_{m,j}(t)$ we can obtain the Möbius transformations of the PH cubics $\Phi_{m,j}(\mathbf{r}_{m,j}(t))(m = 1, -1$, and $j = 1, 2)$, where

$$\Phi_{m,j}(z) = \frac{\alpha_{m,j} z}{(\alpha_{m,j} - 1)z + 1}, \quad \alpha_{m,1} = \frac{1}{3}\frac{\mathbf{v}_i}{\mathbf{a}_m}, \ \alpha_{m,2} = \frac{1}{3}\frac{\mathbf{v}_i}{\mathbf{b}_m}. \tag{4.11}$$

If $\mathbf{k}$ is nonreal, then both $\mathbf{r}_{m,1}(t)$ and $\mathbf{r}_{m,2}(t)$ are nonlinear. But if $\mathbf{k}$ is a real number, then $-1 \le \mathbf{k} \le 1/3$ if and only if both $\mathbf{r}_{m,1}(t)$ and $\mathbf{r}_{m,2}(t)$ are linear.

We can summarize these results.

Theorem 4.1. *Let* $(0, 1, \mathbf{v}_i = r_i e^{\sqrt{-1}\theta_i}, \mathbf{v}_f = r_f e^{\sqrt{-1}\theta_f})$ *be a* C^1 *Hermite data-set such that* $r_i > 0$ *and* $r_f > 0$.

(a) *Let* $\mathbf{v}$ *be the vector given by* (4.7), *and let* $\mathbf{k}$ *be* $\mathbf{v}$ *or* $-\mathbf{v}$. *Then all* C^1 *Hermite interpolants using Möbius transformations of planar PH cubics* $\mathbf{r}(t)$, *such that* $\mathbf{r}'(t) = \mathbf{w}(t)^2$ *for some linear curve* $\mathbf{w}(t)$, *are* $\Phi_{m,j}(\mathbf{r}_{m,j}(t))(m = 1, -1,$ *and* $j = 1, 2)$, *from* (4.10) *and* (4.11), *where* $\mathbf{a}_m$ *and* $\mathbf{b}_m$ *are given by* (4.9).

(b) C^1 *Hermite interpolants using Möbius transformations of planar PH cubics* $\mathbf{r}(t)$, *such that* $\mathbf{r}'(t) = h_0(1-t)^2 + h_1 2(1-t)t + h_2 t^2$ *for some real number* h_0, h_1, *and* h_2 *such that* $h_0 + h_2 \neq h_1$, *exist if and only if* $\theta_i + \theta_f = m\pi$ *for some integers* m. *In this case, the interpolants* $(\Phi \circ \mathbf{r})(t)$ *are given by* (4.5), *where* $\mathbf{r}(t)$ *is given by* (4.3) *and* (4.4), *where* $\alpha = \tau e^{\sqrt{-1}\theta_i}$ *or* $\alpha = \tau e^{\sqrt{-1}(\theta_i + \pi)}$ *for any positive number* τ.

5. Best Interpolant

In this section we consider how to choose the best interpolant for a given Hermite data-set $(\mathbf{p}_i = \mathbf{0},\ \mathbf{p}_f = 1, \mathbf{v}_i, \mathbf{v}_f)$.

We will begin by presenting a condition under which the Möbius transformation of a PH cubic $(\Phi \circ \mathbf{r})(t)$ has a loop, where $\mathbf{r}'(t) = (\mathbf{w}_0(1-t) + \mathbf{w}_1 t)^2$ for some distinct complex constants $\mathbf{w}_0$ and $\mathbf{w}_1$. Since $\Phi(z)$ represents a one-to-one correspondence on the extended complex plane, the condition that $\mathbf{r}(t)$ has a loop is both necessary and sufficient to establish that $(\Phi \circ \mathbf{r})(t)$ has a loop. Under the conditions $\mathbf{r}(0) = 0$ and $\mathbf{r}(1) = 1$, the PH cubic $\mathbf{r}(t)$ is given by $\mathbf{r}(t) = A(t-B)^3 + C$, where

$$A = \frac{(\mathbf{w}_0 - \mathbf{w}_1)^2}{3}, \qquad B = \frac{\mathbf{w}_0}{\mathbf{w}_0 - \mathbf{w}_1}, \qquad C = \frac{\mathbf{w}_0^3}{3(\mathbf{w}_0 - \mathbf{w}_1)}. \tag{5.1}$$

The condition that there exist constants t_1 and t_2, such that $0 \leq t_1 < t_2 \leq 1$ and $\mathbf{r}(t_1) = \mathbf{r}(t_2)$, is necessary and sufficient to establish that $\mathbf{r}(t)$ has a loop. From $\mathbf{r}(t_1) = \mathbf{r}(t_2)$, we can obtain $3B^2 - 3(t_1 + t_2)B + (t_1^2 + t_1 t_2 + t_2^2) = 0$, which implies

$$B = \frac{3(t_1 + t_2) \pm \sqrt{-1}(t_2 - t_1)\sqrt{3}}{6}. \tag{5.2}$$

This equation is equivalent to

$$t_1 + t_2 = 2\operatorname{Re} B, \qquad t_2 - t_1 = 2\sqrt{3}|\operatorname{Im} B|, \tag{5.3}$$

and hence

$$t_1 = \operatorname{Re} B - \sqrt{3}|\operatorname{Im} B|, \qquad t_2 = \operatorname{Re} B + \sqrt{3}|\operatorname{Im} B|. \tag{5.4}$$

Consequently, $\mathbf{r}(t)$ has a loop if and only if $B \in \Omega_1 \cup \Omega_2$, (see Figure 1) where

$$\begin{aligned} \Omega_1 &= \left\{ z \in \mathbb{C} 0 \leq \operatorname{Re} z - \sqrt{3}\operatorname{Im} z < \operatorname{Re} z + \sqrt{3}\operatorname{Im} z \leq 1 \right\}, \\ \Omega_2 &= \left\{ z \in \mathbb{C} \mid 0 \leq \operatorname{Re} z + \sqrt{3}\operatorname{Im} z < \operatorname{Re} z - \sqrt{3}\operatorname{Im} z \leq 1 \right\}. \end{aligned} \tag{5.5}$$

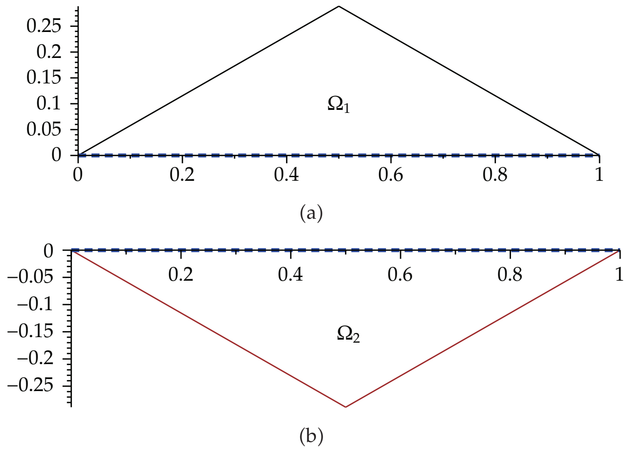


Figure 1: Areas of Ω_1 and Ω_2: B belongs to $\Omega_1 \cup \Omega_2$ if and only if $\mathbf{r}(t)$ has a loop.

On the other hand, the PH cubic $\mathbf{r}(t)$ can be represented by

$$\mathbf{r}(t) = \mathbf{a}3(1-t)^2 t + (\mathbf{a}+\mathbf{k})3(1-t)t^2 + (\mathbf{a}+\mathbf{k}+\mathbf{b})t^3, \tag{5.6}$$

where $\mathbf{a} = \mathbf{w}_0^2/3$, $\mathbf{k} = \mathbf{w}_0\mathbf{w}_1/3$, and $\mathbf{b} = \mathbf{w}_1^2/3$. From $\mathbf{k} = \mathbf{w}_0\mathbf{w}_1/3$ and $B = \mathbf{w}_0/(\mathbf{w}_0 - \mathbf{w}_1)$, we can obtain

$$B^2 - B = \frac{\mathbf{k}}{1-3\mathbf{k}} = \frac{1}{1/\mathbf{k} - 3}, \tag{5.7}$$

and hence

$$\mathbf{k} = \frac{1}{3} - \frac{1}{3(3B^2 - 3B + 1)}. \tag{5.8}$$

Note that

$$\begin{aligned}
&\left\{ \frac{1}{3} - \frac{1}{3(3\mathbf{z}^2 - 3\mathbf{z} + 1)} \in \mathbb{C} \mid 0 \le \operatorname{Re}\mathbf{z} - \sqrt{3}\operatorname{Im}\mathbf{z} < \operatorname{Re}\mathbf{z} + \sqrt{3}\operatorname{Im}\mathbf{z} \le 1 \right\} \\
&\quad = \left\{ \frac{1}{3} - \frac{1}{3(3\mathbf{z}^2 - 3\mathbf{z} + 1)} \in \mathbb{C} \mid 0 \le \operatorname{Re}\mathbf{z} + \sqrt{3}\operatorname{Im}\mathbf{z} < \operatorname{Re}\mathbf{z} - \sqrt{3}\operatorname{Im}\mathbf{z} \le 1 \right\}.
\end{aligned} \tag{5.9}$$

Therefore we conclude as following.

Theorem 5.1. *Suppose that $(\Phi \circ \mathbf{r})(t)$ is a Möbius transformation of a planar PH cubic, such that $\mathbf{r}(0) = \mathbf{0}$, $\mathbf{r}(1) = 1$, and $\mathbf{r}'(t) = (\mathbf{w}_0(1-t) + \mathbf{w}_1 t)^2$ for some distinct complex constants $\mathbf{w}_0$ and $\mathbf{w}_1$ (see Figure 2). Then $(\Phi \circ \mathbf{r})(t)(0 \le t \le 1)$ is a simple curve if and only if $\mathbf{w}_0\mathbf{w}_1/3 \notin D$, where*

$$D = \left\{ \frac{1}{3} - \frac{1}{3(3\mathbf{z}^2 - 3\mathbf{z} + 1)} \in \mathbb{C} \mid 0 \le \operatorname{Re}\mathbf{z} - \sqrt{3}\operatorname{Im}\mathbf{z} < \operatorname{Re}\mathbf{z} + \sqrt{3}\operatorname{Im}\mathbf{z} \le 1 \right\}. \tag{5.10}$$

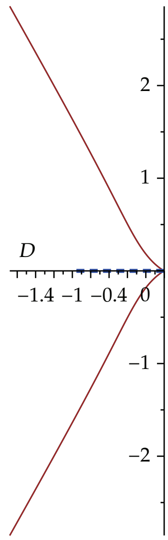


Figure 2: Area of D: $\mathbf{w}_0\mathbf{w}_1/3 \notin D$ if and only if $(\Phi \circ \mathbf{r})(t)$ in Theorem 5.1 is a simple curve.

For a given Hermite data-set $(\mathbf{0}, 1, \mathbf{v}_i, \mathbf{v}_f)$, the term $\mathbf{v}$ in (4.7) belongs to D if and only if $(\Phi_{1,1} \circ \mathbf{r}_{1,1})(t)$ and $(\Phi_{1,2} \circ \mathbf{r}_{1,2})(t)$ have a loop; and $-\mathbf{v}$ belongs to D if and only if $(\Phi_{-1,1} \circ \mathbf{r}_{-1,1})(t)$ and $(\Phi_{-1,2} \circ \mathbf{r}_{-1,2})(t)$ have a loop. Note that D is a subset of the left half-plane, that is, $D \subset \{z \in \mathbb{Z} : \operatorname{Re} z < 0\}$. Thus we can deduce that both $(\Phi_{1,1} \circ \mathbf{r}_{1,1})(t)$ and $(\Phi_{1,2} \circ \mathbf{r}_{1,2})(t)$, or both $(\Phi_{-1,1} \circ \mathbf{r}_{-1,1})(t)$ and $(\Phi_{-1,1} \circ \mathbf{r}_{-1,2})(t)$ are simple curves. From these simple curves we can choose a best interpolant, which is that with the least bending energy

$$E((\Phi \circ \mathbf{r})(t)) = \int_{\Phi\circ\mathbf{r}} \kappa^2 ds = \int_0^1 \kappa(t)^2 \left|(\Phi \circ \mathbf{r})'(t)\right| dt, \tag{5.11}$$

where κ is the curvature of $(\Phi \circ \mathbf{r})(t)$.

Example 5.2. Consider a Hermite data-set $(0, 1, 2e^{-\sqrt{-1}\pi/4}, 2e^{-\sqrt{-1}\pi/8})$. *Then the vector* $\mathbf{v}$ *becomes*

$$\mathbf{v} = \frac{\sqrt{r_i r_f}}{3} e^{\sqrt{-1}(\theta_i+\theta_f)/2} = \frac{2}{3} e^{-\sqrt{-1}3\pi/16}. \tag{5.12}$$

Thus $\mathbf{v} \notin D$ and $-\mathbf{v} \in D$, which implies that $(\Phi_{1,1} \circ \mathbf{r}_{1,1})(t)$ and $(\Phi_{1,2} \circ \mathbf{r}_{1,2})(t)$ are simple but $(\Phi_{-1,1} \circ \mathbf{r}_{-1,1})(t)$ and $(\Phi_{-1,2} \circ \mathbf{r}_{-1,2})(t)$ each have a loop. See Figure 3.

Example 5.3. In the case of a Hermite data-set $(0, 1, e^{-\sqrt{-1}3\pi/5}, e^{-\sqrt{-1}\pi/5})$, *the vector* $\mathbf{v}$ *becomes*

$$\mathbf{v} = \frac{\sqrt{r_i r_f}}{3} e^{\sqrt{-1}(\theta_i+\theta_f)/2} = \frac{1}{3} e^{-\sqrt{-1}2\pi/5}. \tag{5.13}$$

Thus $\mathbf{v} \notin D$ and $-\mathbf{v} \notin D$, which implies that $(\Phi_{1,1} \circ \mathbf{r}_{1,1})(t)$, $(\Phi_{1,2} \circ \mathbf{r}_{1,2})(t)$, $(\Phi_{-1,1} \circ \mathbf{r}_{-1,1})(t)$, and $(\Phi_{-1,2} \circ \mathbf{r}_{-1,2})(t)$ are all simple. See Figure 4.

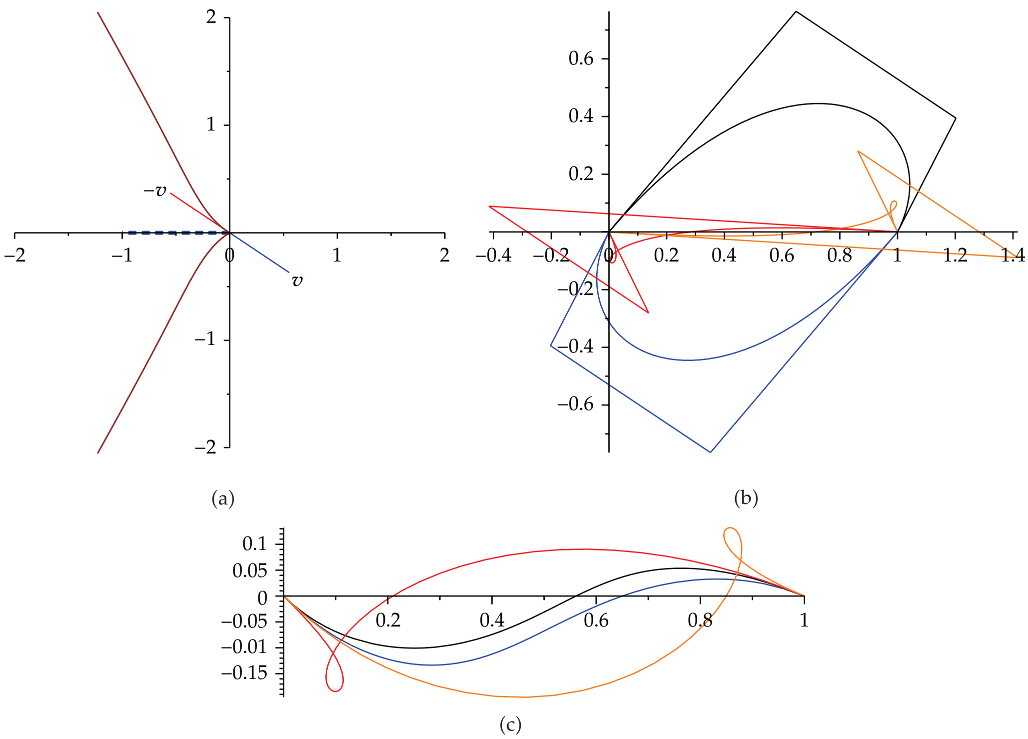
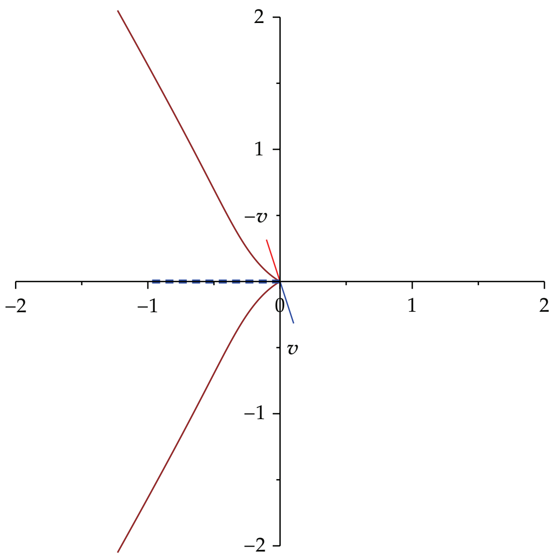


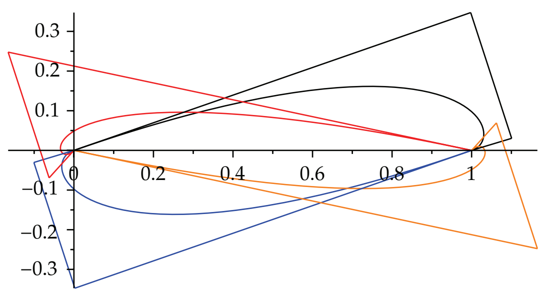
Figure 3: For the Hermite dataset $(0, 1, 2e^{-\sqrt{-1}\pi/4}, 2e^{-\sqrt{-1}\pi/8})$, the graph on the left shows that $-\mathbf{v} \in D$ and $\mathbf{v} \notin D$; the central graph shows the PH cubics $\mathbf{r}(t)$ with their control polygons; the graph on the right shows the four interpolants.

Example 5.4. Consider a family of C^1 Hermite data-sets $(0, 2, k(1+\sqrt{-1}), 1+2\sqrt{-1})$, where $k = 1, 5, 10, 20$. We construct C^1 Hermite interpolants that satisfy these data-sets using Möbius transformations of PH cubics, and also PH quintics, all shown in Figure 5. The Möbius transformations of the PH cubics always provide two S-shaped simple curves and two other curves; the latter are C-shaped simple curves when $k = 1$ or 5 and have a single loop in the other cases. As the parametric speed of the initial Hermite condition increases, the C-shaped interpolants change from simple curves to single loops, while the simple S-shaped interpolants retain their original shape characteristics. We also observe that, unlike the S-shaped interpolants produced by Möbius transformations of PH cubics, the S-shaped PH quintic interpolants may be simple (like the curve labeled 4 in Figure 5), or have one or two loops (some PH quintics labeled 2 in Figure 5 are S-shaped double loops).

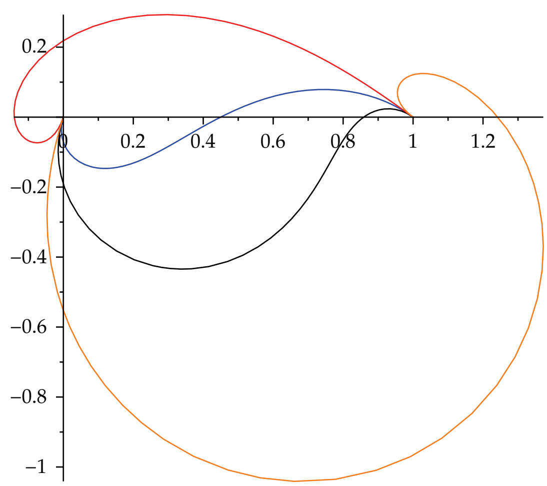
We observed the behavior of these interpolants as the parametric speed at the endpoints changes. As this speed increases, the arc-lengths of PH quintics increase rapidly, but the arc-length, of Möbius transformations of PH cubics are generally less affected. In particular, the simple S-shaped interpolants, produced by Möbius transformations of PH cubics show little change in arc-length. Table 1 shows that these latter interpolants have both lower bending energies and shorter arc-lengths, than all the other interpolants we are considering. If we look at Table 1 and identify the most shapely interpolants with the lowest bending energies, we find that the best Möbius transformation of a PH cubic is always S-shaped and



(a)



(b)



(c)

Figure 4: For the Hermite data-set $(0, 1, e^{-\sqrt{-1}3\pi/5}, e^{-\sqrt{-1}\pi/5})$, the graph on the left shows that $-\mathbf{v} \notin D$ and $\mathbf{v} \notin D$; the central graph shows the PH cubics $\mathbf{r}(t)$ with their control polygons; the graph on the right shows the four interpolants.

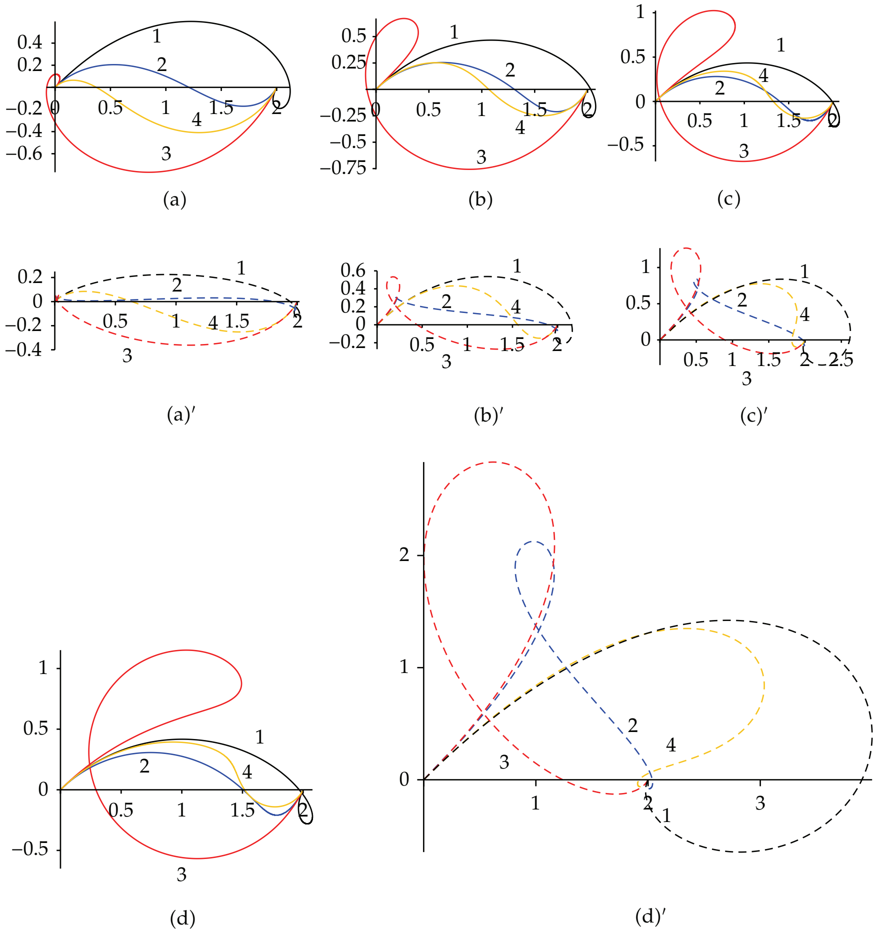


Figure 5: Comparison of pairs of PH interpolants, satisfying the same C^1 Hermite data-set $(0, 2, k(1+\sqrt{-1}), 1+2\sqrt{-1})$, when $k = 1, 5, 10, 20$: (a), (b), (c), and (d), respectively, show the Möbius transformations of the PH cubics MC_i when $k = 1, 5, 10, 20$; and (a)′, (b)′, (c)′, and (d)′ show the correspoding PH quintics Q_i.

simple. However, the merit of the PH quintic interpolants depends on the parametric speeds at their end points. For example, in Figures 5(a)′ and 5(b)′, the curves labeled 4 are best, while the curves labeled 1 are best in Figures 5(c)′ and 5(d)′. Looking closely at the PH quintic interpolants, we see that the simple S-shaped curve with the best shape when $k = 1$ becomes less and less acceptable as the parametric speeds at the end-points increase. But the interpolants labeled 1 in Figures 5(a)′, 5(b)′, 5(c)′, and 5(d)′ exhibit the opposite behavior: initially these curves are C-shaped loops with high bending energies when $k = 1$; but as the parametric speed increases, they become C-shaped simple curves with lower bending energies. When k reaches 20, it has the best shape but the greatest arc-length. This suggests that the best-shaped interpolants, produced by Möbius transformations of PH cubics are more stable than the corresponding PH quintics, in the sense that the former largely achieve a lower arc-length and bending energy than the latter, except when the end-point speeds are significantly asymmetric, as we see when $k = 20$ in this example.

Table 1: Comparison of arc-length and bending energy for the interpolants of Figure 5.

$k=1$	MC_1	MC_2	MC_3	MC_4	Q_1	Q_2	Q_3	Q_4
arc-length	3.03	2.19	3.10	2.29	2.34	2.16	2.34	2.16
BE	45.0	5.5	72.8	6.8	149	3106	273	5.3
$k=5$	MC_1	MC_2	MC_3	MC_4	Q_1	Q_2	Q_3	Q_4
arc-length	2.93	2.28	4.50	2.31	3.05	2.40	3.05	2.40
BE	50.2	6.5	20.9	5.7	36.1	762	47.3	10.0
$k=10$	MC_1	MC_2	MC_3	MC_4	Q_1	Q_2	Q_3	Q_4
arc-length	2.89	2.31	5.47	2.36	4.42	3.02	4.42	3.02
BE	54.03	8.2	16.6	7.5	14.4	345.9	19.3	36.9
$k=20$	MC_1	MC_2	MC_3	MC_4	Q_1	Q_2	Q_3	Q_4
arc-length	2.85	2.34	6.13	2.40	7.91	5.39	7.91	5.39
BE	60.1	11.9	17.7	11.3	8.0	136	10.7	97.9

6. Conclusions

Möbius transformations preserve Pythagorean-hodograph properties. For any C^1 Hermite data-set, we can generally obtain four C^1 Hermite interpolants as Möbius transformations of PH cubics. We have proved that these interpolants are always simple curves or single loops, and that at least two of them must be simple. We have also presented the condition that an interpolant must meet if it is to be a simple curve.

We compared the shape characteristics of C^1 Hermite interpolants, produced by Möbius transformations of PH cubics, together with their response to changes of parametric speed at their end points, with the same data for PH quintic interpolants satisfying an identical C^1 Hermite dataset: we found that interpolants produced by Möbius transformations of PH cubics generally have lower bending energies and shorter arc-lengths than PH quintics.

One avenue for further research is to look for ways of predicting how the geometry of Möbius transformation of PH cubics will be determined by a particular C^1 Hermite data-set. Another avenue to explore would be the application of Möbius transformations to other interpolation problems involving PH (or MPH) curves, in both two and three dimensions. In particular, we might look to complete the geometric characterization of Möbius transformation of PH cubics in C^1 Hermite interpolation.

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Research Article

Circular Slits Map of Bounded Multiply Connected Regions

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We present a boundary integral equation method for the numerical conformal mapping of bounded multiply connected region onto a circular slit region. The method is based on some uniquely solvable boundary integral equations with adjoint classical, adjoint generalized, and modified Neumann kernels. These boundary integral equations are constructed from a boundary relationship satisfied by a function analytic on a multiply connected region. Some numerical examples are presented to illustrate the efficiency of the presented method.

1. Introduction

In general, the exact conformal mapping functions are unknown except for some special regions. It is well known that every multiply connected regions can be mapped conformally onto the circle with concentric circular slits, the circular ring with concentric circular slits, the circular slit region, the radial slit region, and the parallel slit region as described in Nehari [1, page 334]. Several methods for numerical approximation for the conformal mapping of multiply connected regions have been proposed in [2–16]. Recently, reformulations of conformal mappings from bounded and unbounded multiply connected regions onto the five canonical slit regions as Riemann-Hilbert problems are discussed in Nasser [12, 13, 17]. An integral equation with the generalized Neumann kernel is then used to solve the RH problem as developed in [18]. The integral equation however involves singular integral which is calculated by Wittich's method. Murid and Hu [11] formulated an integral equation method based on another form of generalized Neumann kernel for conformal mapping of bounded doubly connected regions onto a disk with circular slit but the kernel of the integral equation involved the unknown circular radii. Discretization of the integral equation yields a system of nonlinear equations which they solved using an optimization method. To overcome this nonlinear problem, Sangawi et al. [19] have developed linear integral equations for conformal mapping of bounded multiply connected regions onto a disk with circular slits. In this paper, we describe an integral equation method for computing the conformal mapping function f of bounded multiply connected regions onto a circular slit region. This boundary integral equation is constructed from a boundary relationship that relates the mapping function f on a multiply connected region with f', $\theta'(t)$, and $|f|$, where θ is the boundary correspondence function.

The plan of the paper is as follows. Section 2 presents some auxiliary materials. Derivations of two integral equations related to f' and $\theta'(t)$ are given in Sections 3 and 4, respectively. Section 5 presents a method to calculate the modulus of f. In Section 6, we give some examples to illustrate our boundary integral equation method. Finally, Section 7 presents a short conclusion.

2. Notations and Auxiliary Material

Let Ω be a bounded multiply connected region of connectivity $M+1$. The boundary Γ consists of $M+1$ smooth Jordan curves Γ_j, $j = 0, 1, \ldots, M$, such that $\Gamma_{\hat{j}}$, $\hat{j} = 1, \ldots, M$, lies in the interior of Γ_0, where the outer curve Γ_0 has counterclockwise orientation and the inner curves $\Gamma_{\hat{j}}$, $\hat{j} = 1, \ldots, M$, have clockwise orientation. The positive direction of the contour $\Gamma = \bigcup_{j=0}^{M} \Gamma_j$ is usually that for which Ω is on the left as one traces the boundary (see Figure 1). The curve Γ_k is parametrized by 2π-periodic twice continuously differentiable complex function $z_k(t)$ with nonvanishing first derivative

$$z_k'(t) = \frac{dz_k(t)}{dt} \neq 0, \quad t \in J_k = [0, 2\pi], \ k = 0, 1, \ldots, M. \tag{2.1}$$

The total parameter domain J is the disjoint union of $M+1$ intervals $J_0, \ldots, J_M$. We define a parametrization z of the whole boundary Γ on J by

$$z(t) = \begin{cases} z_0(t), & t \in J_0 = [0, 2\pi], \\ \vdots & \\ z_M(t), & t \in J_M = [0, 2\pi]. \end{cases} \tag{2.2}$$

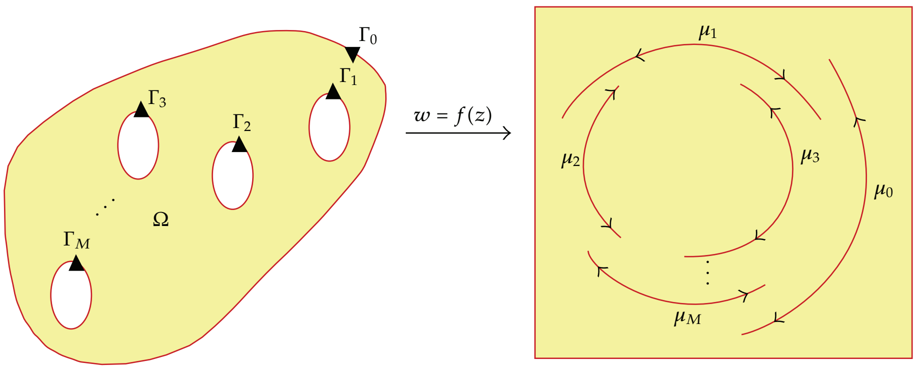


Figure 1: Mapping of the bounded multiply connected region Ω of connectivity $M+1$ onto a circular slit region.

Let H^* be the space of all real Hölder continuous 2π-periodic functions $\omega(t)$ of the parameter t on J_k for $k = 0, 1, \ldots, M$, that is,

$$\omega(t) = \begin{cases} \omega_0(t), & t \in J_0, \\ \omega_1(t), & t \in J_1, \\ \vdots & \\ \omega_M(t), & t \in J_M. \end{cases} \tag{2.3}$$

Let $\theta(t)$ (the boundary corresponding function) be given for $t \in J$ by

$$\theta(t) = \begin{cases} \theta_0(t), & t \in J_0, \\ \vdots & \\ \theta_M(t), & t \in J_M. \end{cases} \tag{2.4}$$

Let μ (a piecewise constant real function) be given for $t \in J$ by

$$\mu(t) = (\mu_0, \mu_1, \ldots, \mu_M) = \begin{cases} \mu_0, & t \in J_0, \\ \vdots & \\ \mu_M, & t \in J_M. \end{cases} \tag{2.5}$$

Let $\widehat{A}(t)$ be a complex continuously differentiable 2π-periodic function for all $t \in J$. The generalized Neumann kernel formed with $\widehat{A}$ is defined by

$$\widehat{N}(t,s) = \frac{1}{\pi} \operatorname{Im}\left(\frac{\widehat{A}(t)}{\widehat{A}(s)} \frac{z'(s)}{z(s) - z(t)} \right). \tag{2.6}$$

The kernel $\widehat{N}$ is continuous with

$$\widehat{\mathrm{N}}(t,t) = \frac{1}{\pi}\left(\frac{1}{2}\operatorname{Im}\frac{z''(t)}{z'(t)} - \operatorname{Im}\frac{\widehat{A}'(t)}{\widehat{A}(t)}\right). \tag{2.7}$$

Define also the kernel $\widehat{M}$ by

$$\widehat{M}(t,s) = \frac{1}{\pi}\operatorname{Re}\left(\frac{\widehat{A}(t)}{\widehat{A}(s)}\frac{z'(s)}{z(s)-z(t)}\right), \tag{2.8}$$

which has a cotangent singularity type (see [18] for more detail). The classical Neumann kernel is the generalized Neumann kernel formed with $\widehat{A}(t) = 1$, that is,

$$N(t,s) = \frac{1}{\pi}\operatorname{Im}\left(\frac{z'(s)}{z(s)-z(t)}\right). \tag{2.9}$$

The adjoint kernel $N^*(s,t)$ of the classical Neumann kernel is given by

$$N^*(t,s) = N(s,t) = \frac{1}{\pi}\operatorname{Im}\left(\frac{z'(t)}{z(t)-z(s)}\right). \tag{2.10}$$

The adjoint function to the function $\widehat{A}$ is given by

$$\widetilde{A}(t) = \frac{z'(t)}{\widehat{A}(t)} = z'(t). \tag{2.11}$$

The generalized Neumann kernel $\widetilde{N}(s,t)$ formed with $\tilde{A}$ is given by

$$\widetilde{N}(t,s) = \frac{1}{\pi}\operatorname{Im}\left(\frac{\widetilde{A}(t)}{\widetilde{A}(s)}\frac{z'(s)}{z(s)-z(t)}\right). \tag{2.12}$$

If $\widehat{A} = 1$, then

$$\widetilde{N}(t,s) = -N^*(t,s). \tag{2.13}$$

We define the Fredholm integral operators $\mathbf{N}, \widetilde{\mathbf{N}}, \mathbf{N}^*$ by

$$\mathbf{N}v(t) = \int_J N(t,s)v(s)ds, \quad t \in J, \tag{2.14}$$

$$\widetilde{\mathbf{N}}v(t) = \int_J \widetilde{N}(t,s)v(s)ds, \quad t \in J, \tag{2.15}$$

$$\mathbf{N}^*v(t) = \int_J N(s,t)v(s)ds, \quad t \in J. \tag{2.16}$$

Note that $\widetilde{\mathbf{N}} = -\mathbf{N}^*$, if $\widehat{A} = 1$.

It is known that $\lambda = 1$ is an eigenvalue of the kernel N with multiplicity 1 and $\lambda = -1$ is an eigenvalue of the kernel N with multiplicity M [18]. We define the piecewise constant functions

$$\chi^{[j]}(\xi) = \begin{cases} 1, & \xi \in \Gamma_j, \ j = 0,1,2,\ldots,M. \\ 0, & \text{otherwise.} \end{cases} \tag{2.17}$$

Then, we have from [18]

$$\mathrm{Null}(\mathbf{I} - \mathbf{N}) = \mathrm{span}\{1\}, \qquad \mathrm{Null}(\mathbf{I} - \mathbf{N}) = \mathrm{span}\left\{\chi^{[1]}, \chi^{[2]}, \ldots, \chi^{[M]}\right\}. \tag{2.18}$$

Lastly, we define integral operators $\mathbf{J}$ and $\widehat{\mathbf{J}}$ by

$$\begin{aligned} \mathbf{J}v &= \int_J \frac{1}{2\pi} \sum_{j=1}^{M} \chi^{[j]}(s)\chi^{[j]}(t)v(s)ds, \\ \widehat{\mathbf{J}}v &= \int_J \frac{1}{2\pi} \sum_{j=0}^{M} \chi^{[j]}(s)\chi^{[j]}(t)v(s)ds, \end{aligned} \tag{2.19}$$

which are required for uniqueness of solution in a later section.

3. Homogenous and Nonhomogenous Boundary Relationship

3.1. Nonhomogeneous Boundary Relationship for Conformal Mapping

Suppose that $c(z)$, $Q(z)$, and $H(z)$ are complex-valued functions defined on Γ such that $c(z) \neq 0$, $H(z) \neq 0$, $Q(z) \neq 0$, and $\overline{H(z)}/(T(z)Q(z))$ satisfies the Hölder condition on Γ. Then, the interior relationship is defined as follows.

A complex-valued function $P(z)$ is said to satisfy the interior relationship if $P(z)$ is analytic in Ω and satisfies the nonhomogeneous boundary relationship

$$P(z) = c(z)\frac{\overline{T(z)Q(z)}}{\overline{G(z)}}\overline{P(z)} + \overline{H(z)}, \quad z \in \Gamma, \tag{3.1}$$

where $G(z)$ analytic in Ω, Hölder continuous on Γ, and $G(z) \neq 0$ on Γ. The boundary relationship (3.1) also has the following equivalent form:

$$G(z) = \overline{c(z)}T(z)Q(z)\frac{P(z)^2}{|P(z)|^2} + \frac{G(z)H(z)}{\overline{P(z)}}, \quad z \in \Gamma. \tag{3.2}$$

Let the function $L_R(\tilde{z})$ be defined in the region $C \cup \{\infty\} \setminus \Gamma$ by

$$L_R(\tilde{z}) = \frac{1}{2\pi i}\int_\Gamma \frac{\overline{c(\tilde{z})}H(w)}{\overline{c(w)}(w-\tilde{z})Q(w)T(w)}dw, \quad \tilde{z} \in \Omega^-, \tag{3.3}$$

where Ω^- is the complement of Ω. The following theorem gives an integral equation for an analytic function satisfying the interior nonhomogeneous boundary relationship (3.1) or (3.2). This theorem generalizes the results of Murid and Razali [9] and can be proved by using the approach used in proving Theorem 3.1 in [20, page 45].

Theorem 3.1. *Let U and V be any complex-valued functions that are defined on Γ. If the function $P(z)$ satisfies the interior nonhomogeneous boundary relationship* (3.1) *or* (3.2), *then*

$$\begin{aligned}\frac{1}{2}\left[V(z) + \frac{U(z)}{\overline{T(z)Q(z)}}\right]P(z) + \mathrm{PV}\int_\Gamma K(z,w)P(w)|dw| + c(z)U(z)\\ \times\left[\sum_{a_j\in\Omega}\operatorname*{Res}_{w=a_j}\frac{P(w)}{(w-z)G(w)}\right]^{\mathrm{conj}} = -U(z)\overline{L_R^-(z)}, \quad z \in \Gamma,\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}K(z,w) &= \frac{1}{2\pi i}\left[\frac{c(z)U(z)}{\overline{c(w)(\overline{w}-\overline{z})Q(w)}} - \frac{V(z)T(w)}{w-z}\right],\\ L_R^-(z) &= \frac{-1}{2}\frac{H(z)}{Q(z)T(z)} + \mathrm{PV}\frac{1}{2\pi i}\int_\Gamma \frac{\overline{c(z)}H(w)}{\overline{c(w)}(w-z)Q(w)T(w)}dw.\end{aligned} \tag{3.5}$$

The symbol "conj" in the superscript denotes complex conjugate, while the minus sign in the superscript denotes limit from the exterior. The sum in (3.4) *is over all those zeros $a_1, a_2, \ldots, a_M$ of G that lie inside Ω. If G has no zeros in Ω, then the term containing the residue in* (3.4) *will not appear.*

Proof. Suppose that $P(z)$ and $G(z)$ are analytic functions in Ω and G has a finite number of zeros at $a_1, a_2, \ldots, a_M$ in Ω. Then, by the calculus of residues, we have

$$\frac{1}{2\pi i}\int_\Gamma \frac{P(w)}{(w-\tilde{z})G(w)}dw = \sum_{a_j\in\Omega}\operatorname*{Res}_{w=a_j}\frac{P(w)}{(w-\tilde{z})G(w)}, \quad \tilde{z} \in \Omega^-. \tag{3.6}$$

Since P and G satisfy the Hölder condition on Γ and $G(z) \neq 0$ on Γ, then P/G also satisfies the Hölder condition on Γ. Taking the limit $\Omega^{-} \ni \tilde{z} \to z \in \Gamma$ and applying Sokhotski formula [5], we get

$$-\frac{1}{2}\frac{P(z)}{G(z)} + \mathrm{PV}\frac{1}{2\pi i}\int_{\Gamma}\frac{P(z)}{(w-z)G(w)}dw = \sum_{a_j \in \Omega}\operatorname*{Res}_{w=a_j}\frac{P(w)}{(w-z)G(w)}, \quad z \in \Gamma. \tag{3.7}$$

By taking conjugate to both sides and using (3.1), we get

$$\begin{aligned}
&-\frac{1}{2}\frac{P(z)}{c(z)\overline{Q(z)}\mathrm{T}(z)} + \frac{1}{2}\frac{\overline{H(z)}}{c(z)\overline{Q(z)}T(z)} - \mathrm{PV}\frac{1}{2\pi i}\int_{\Gamma}\frac{P(z)}{c(w)(\overline{w}-\overline{z})\overline{Q(w)}}\frac{\overline{dw}}{\overline{T(w)}} \\
&\quad + \mathrm{PV}\frac{1}{2\pi i}\int_{\Gamma}\frac{\overline{H(z)dw}}{c(w)(\overline{w}-\overline{z})\overline{Q(z)T(z)}} = \left[\sum_{a_j \in \Omega}\operatorname*{Res}_{w=a_j}\frac{P(w)}{(w-z)G(w)}\right]^{\mathrm{conj}}, \quad z \in \Gamma.
\end{aligned} \tag{3.8}$$

Multiplying both sides by $-c(z)$ and the fact that $dw = T(w)|dw|$, after some arrangement, yield

$$\begin{aligned}
&\frac{1}{2}\frac{P(z)}{\overline{Q(z)T(z)}} + \mathrm{PV}\frac{1}{2\pi i}\int_{\Gamma}\frac{c(z)P(z)}{c(w)(\overline{w}-\overline{z})\overline{Q(w)}}|dw| + c(z)\left[\sum_{a_j \in \Omega}\operatorname*{Res}_{w=a_j}\frac{P(w)}{(w-z)G(w)}\right]^{\mathrm{conj}} \\
&\quad = -\left[-\frac{1}{2}\frac{H(z)}{Q(z)T(z)} + \mathrm{PV}\frac{1}{2\pi i}\int_{\Gamma}\frac{\overline{c(z)}H(z)}{\overline{c(w)}(w-z)Q(z)T(z)}dw\right]^{\mathrm{conj}}, \quad z \in \Gamma.
\end{aligned} \tag{3.9}$$

Applying Sokhotski formulas again to the expression inside the bracket of the right-hand side yields

$$\begin{aligned}
&\frac{1}{2}\frac{P(z)}{\overline{Q(z)T(z)}} + \mathrm{PV}\frac{1}{2\pi i}\int_{\Gamma}\frac{c(z)P(z)}{c(w)(\overline{w}-\overline{z})\overline{Q(w)}}|dw| + c(z)\left[\sum_{a_j \in \Omega}\operatorname*{Res}_{w=a_j}\frac{P(w)}{(w-z)G(w)}\right]^{\mathrm{conj}} \\
&\quad = -\overline{L_R^{-}(z)}, \quad z \in \Gamma.
\end{aligned} \tag{3.10}$$

Since $P(z)$ is analytic in Ω, then by Cauchy integral formula, we have

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{P(z)}{w-\tilde{z}}dw = 0, \quad z \in \Omega^{-}. \tag{3.11}$$

Taking the limit $\omega^{-} \ni \tilde{z} \to z \in \Gamma$ and applying Sokhotiski formulas, we get

$$-\frac{1}{2}P(z) + \mathrm{PV}\frac{1}{2\pi i}\int_{\Gamma}\frac{T(w)P(z)}{w-z}|dw| = 0, \quad z \in \Gamma. \tag{3.12}$$

Multiplying (3.12) by $v(z)$ and subtracting it from (3.10) multiplied by $u(z)$ yield (3.4). □

3.2. Homogeneous Boundary Relationship for Conformal Mapping

Let $w = f(z)$ be the analytic function which maps Ω in the z-plane onto a canonical region of the circular slit region in the w-plane. Let 0 and a be a fixed point in Ω such that $a \neq 0$. Then, the mapping function is made uniquely determined by assuming that $f(a) = 0$ and $f(0) = \infty$ such that the residue of the function f at 0 is equal to 1 [1]. Hence, the function f can be written in the form

$$f(z) = \left(\frac{1}{z} - \frac{1}{a}\right)e^{zg(z)}, \tag{3.13}$$

where g is analytic in Ω [12, 13]. Note that the boundary value of f can be represented in the form

$$f\left(z_p(t)\right) = \mu_p e^{\mathrm{i}\theta_p(t)}, \qquad \Gamma_p : z = z_p(t), \quad 0 \leq t \leq \beta_p,\ p = 0, 1, \ldots, M, \tag{3.14}$$

where θ_p is a boundary correspondence function of Γ_p and μ_p is the radius of the circular slit. The unit tangent to Γ at $z(t)$ is denoted by $T(z(t)) = z'(t)/|z'(t)|$. Thus, it can be shown that

$$f(z) = \frac{\left|f(z)\right|}{i}T(z)\frac{\left|\theta_p'(t)\right|}{\theta_p'(t)}\frac{f'(z)}{\left|f'(z)\right|}, \qquad z \in \Gamma. \tag{3.15}$$

4. Integral Equation Method for Computing $F'(Z)$

Note that the value of $\theta_p'(t)$ may be positive or negative since each circular slit $f(\Gamma_p)$ is traversed twice. Thus, $|\theta_p'|/\theta_p' = \pm 1$. Hence, the boundary relationship (3.15) can be written as

$$f(z) = \pm T(z)\frac{\left|f(z)\right|}{i}\frac{f'(z)}{\left|f'(z)\right|}, \quad z \in \Gamma. \tag{4.1}$$

To eliminate the $\pm$ sign, we square both sides of the boundary relationship (4.1) to get

$$f(z)^2 = -T(z)^2\left|f(z)\right|^2\frac{f'(z)^2}{\left|f'(z)\right|^2}, \quad z \in \Gamma. \tag{4.2}$$

Then, the function $E(z)$ defined by

$$D(z) = z^2 f'(z) = z^2 f(z)\left[zg'(z) + g(z)\right] - e^{zg(z)} \tag{4.3}$$

is analytic in Ω.

Combining (4.3), (4.2), and (3.13), we obtain the following boundary relationship:

$$\frac{ze^{2zh(z)}}{a^2} = -\frac{\overline{z}|z|^2}{(a-z)^2}\left|f(z)\right|^2 T(z)^2\frac{D(z)^2}{|D(z)|^2}, \quad z \in \Gamma. \tag{4.4}$$

Comparison of (4.4) and (3.2) leads to a choice of $P(z) = D(z)$, $c(z) = -z|z|^2|f(z)|^2/(\overline{a}-\overline{z})^2$, $Q(z) = T(z)$, $G(z) = ze^{2zh(z)}/a^2$, $H(z) = 0$. Setting $U(z) = \overline{T(z)Q(z)}$ and $V(z) = 1$, Theorem 3.1 yields

$$\begin{aligned} &T(z)D(z) + \mathrm{PV}\frac{1}{2\pi \mathrm{i}}\int_\Gamma \left[\frac{z|z|^2\left|f(z)\right|^2\overline{(a-w)^2}\,\overline{T(z)}}{w|w|^2\left|f(w)\right|^2\overline{(a-z)^2}(\overline{w}-\overline{z})} - \frac{T(z)}{w-z}\right]T(w)D(w)|dw| \\ &\quad = \frac{z|z|^2\left|f(z)\right|^2}{(\overline{a}-\overline{z})^2}\overline{T(z)}\left[\sum_{a_j\in\Omega}\operatorname*{Res}_{w=a_j}\frac{a^2D(w)}{(w-z)we^{2wh(w)}}\right]^{\mathrm{conj}}, \quad z\in\Gamma. \end{aligned} \tag{4.5}$$

Note that $a^2D(w)/(w-z)w^2$ has a simple pole at $w = 0$. To evaluate the residue in (4.5), we use the fact that if $L(z) = d(z)/q(z)$ where $d(z)$ and $q(z)$ are analytic at z_0 and $d(z_0) \neq 0$, $q(z_0) = 0$ and $q'(z_0) \neq 0$, which means z_0 is a simple pole of $L(z)$, then

$$\operatorname*{Res}_{w=z_0} L(w) = \frac{d(z_0)}{q'(z_0)}. \tag{4.6}$$

Applying (4.6) to the residue in (4.5) and after several algebraic manipulations, we obtain

$$\sum_{a_j\in\Omega}\operatorname*{Res}_{w=a_j}\frac{a^2D(w)}{(w-z)we^{2wg(w)}} = \frac{a^2}{z}. \tag{4.7}$$

Thus, integral equation (4.5) becomes

$$F(Z) + \int_\Gamma N^+(z,w)F(w)|dw| = \frac{\overline{a^2}z^2\left|f(z)\right|^2}{(\overline{a}-\overline{z})^2}\overline{T(z)}, \qquad z\in\Gamma, \tag{4.8}$$

where

$$\begin{aligned} F(z) &= T(z)D(z), \\ D(z) &= z^2 f'(z), \\ N^+(z,w) &= \frac{1}{2\pi i}\left[\frac{T(z)}{z-w} - \frac{z|z|^2\left|f(z)\right|^2\overline{(a-w)^2}\,\overline{T(z)}}{w|w|^2\left|f(w)\right|^2\overline{(a-z)^2}(\overline{z}-\overline{w})}\right], \\ N^+(t,t) &= \frac{1}{2\pi|z'(t)|}\operatorname{Im}\frac{z''(t)}{z'(t)} + \frac{1}{\pi i|z'(t)|}\left[\frac{\overline{z'(t)}}{\overline{z(t)}-\overline{a}} - \operatorname{Re}\left(\frac{z'(t)}{z(t)}\right)\right] \\ &\quad - \frac{1}{2\pi i|z'(t)|}\frac{z'(t)}{z(t)}. \end{aligned} \tag{4.9}$$

By using single valuedness of the mapping function f leads to the following condition:

$$\frac{1}{2\pi}\int_{-\Gamma_q}\frac{F(w)}{w^2}|dw| = 0, \qquad q = 0,1,\ldots,M. \tag{4.10}$$

By means of Cauchy's integral formula, we can get the following condition:

$$\frac{1}{2\pi}\int_{\Gamma}\frac{F(w)}{w}|dw| = -i. \tag{4.11}$$

Thus, the integral equation (4.8) with the conditions (4.10) and (4.11) should give a unique solution provided the parameters μ_p, $p = 0,1,\ldots,M$ that appear in $N^+(z,w)$ are known.

Integral equation methods for computing μ_p and θ'_p are discussed in the next two sections.

5. Integral Equation for Computing $|f(z)|$

Note that, from (3.13) and (3.14), we get the following equation:

$$z(t)g(z(t)) = \log\left|f(z(t))\right| - \log\left|\frac{1}{z(t)} - \frac{1}{a}\right| - i\arg\left(\frac{1}{z(t)} - \frac{1}{a}\right) + \theta_p(t). \tag{5.1}$$

Since $g(z)$ is analytic in Ω, thus

$$\widehat{A}(t)g(z(t)) = \gamma(t) + h(t) + iv, \tag{5.2}$$

from (5.1) and (5.2), yields

$$\widehat{A}(t) = z(t), \tag{5.3}$$

$$\gamma(t) = -\log\left(\frac{1}{z(t)} - \frac{1}{a}\right), \tag{5.4}$$

$$h(t) = \log\mu(t) = \left(\log\mu_0, \log\mu_1, \ldots, \log\mu_M\right). \tag{5.5}$$

The following theorem from [22] gives a method for calculating $h(t)$, and hence $\mu_p = |f(z_p)|$.

Theorem 5.1 (see [22, Theorem 5]). *The function h is given by $h = (h_0, h_1, \ldots, h_M)$, where*

$$h_j = \left(\gamma, \phi^{[j]}\right) = \frac{1}{2\pi}\int_J \gamma(t)\phi^{[j]}(t)dt, \tag{5.6}$$

and where $\phi^{[j]}$ is the unique solution of the following integral equation

$$\left(\mathbf{I} + \widehat{\mathbf{N}}^* + \widehat{\mathbf{J}}\right)\phi^{[j]} = -\chi^{[j]}, \qquad j = 0,1,\ldots,M, \tag{5.7}$$

where the kernel $\widehat{N}^*(s,t)$ is the adjoint kernel of the kernel $\widehat{N}(s,t)$ which is formed with $\widehat{A}(t) = z(t)$.

By obtaining $h_0, h_1, \ldots, h_M$ from (5.6), in view of (5.5), we obtain

$$\mu_j = e^{h_j}, \qquad j = 0, 1, \ldots, M. \tag{5.8}$$

6. Integral Equation Method for Computing $\theta'_p(t)$

This section gives another application of Theorem 3.1 for computing f'/f. Let f be the mapping function as described in Section 3.2. Note that (4.2) can be written in the following form:

$$\left|\frac{f'(z)}{\mathrm{f}(z)}\right|^2 = -T(z)^2\left(\frac{f'(z)}{f(z)}\right)^2, \qquad z \in \Gamma. \tag{6.1}$$

Taking the derivative of both sides of (3.13) together with some elementary calculations yields

$$\frac{f'(z)}{f(z)} + \frac{a}{z(a-z)} = zg'(z) + g(z). \tag{6.2}$$

Let $E(z) = (f'(z)/f(z)) + (a/z(a-z)) \; = zg'(z) + g(z)$ be analytic in Ω. Then,

$$\frac{f'(z)}{f(z)} = E(z) + \frac{a}{z(z-a)}, \qquad z \in \Gamma. \tag{6.3}$$

Equations (6.1) and (6.3) together with some elementary calculations yield

$$E(z) = -\overline{T(z)}^2\overline{E(z)} - \frac{\overline{a}\overline{T(z)}^2}{\overline{z}(\overline{z}-\overline{a})} - \frac{a}{z(z-a)}, \qquad z \in \Gamma. \tag{6.4}$$

Comparison of (6.4) and (3.1) leads to a choice of $P(z) = E(z)$, $c(z) = -1$, $Q(z) = T(z)$, $G(z) = 1$, $H(z) = -(aT(z)^2/z(z-a)) - (\bar{a}/\bar{z}(\overline{z}-\overline{a}))$. Setting $U(z) = \overline{T(z)Q(z)}$ and $V(z) = 1$, Theorem 3.1 yields

$$E(z)T(z) + \mathrm{PV}\frac{1}{2\pi i}\int_\Gamma\left[\frac{\overline{T(z)}}{\overline{w}-\overline{z}} - \frac{T(z)}{w-z}\right]E(w)T(w)|dw| = -\overline{T(z)}\overline{L_R^-(z)}, \quad z \in \Gamma, \tag{6.5}$$

where

$$\begin{aligned} T(z)L_R^-(z) = &-\frac{1}{2}\left[\frac{-aT(z)}{z(z-a)} - \frac{\overline{a}\overline{T(z)}}{\overline{z}(\overline{z}-\overline{a})}\right] + T(z)\mathrm{PV}\frac{1}{2\pi i}\int_\Gamma \frac{a}{w(w-z)(w-a)}dw \\ &- T(z)\mathrm{PV}\frac{1}{2\pi i}\int_\Gamma \frac{A\overline{a}\overline{T(w)}^2}{\overline{w}(\overline{w}-\overline{a})(w-z)}dw, \quad z \in \Gamma. \end{aligned} \tag{6.6}$$

Then, it follows from [5, page 91] that

$$\mathrm{PV}\frac{1}{2\pi i}\int_{\Gamma}\frac{a}{w(w-z)(w-a)}dw = -\frac{1}{2}\frac{a}{z(z-a)}. \tag{6.7}$$

From (6.5),(6.6), (6.7), and (6.3), we obtain the integral equation

$$\frac{f'(z)}{f(z)}T(z) + \mathrm{PV}\frac{1}{2\pi i}\int_{\Gamma}\left[\frac{T(z)}{z-w} - \frac{\overline{T(z)}}{\overline{z}-\overline{w}}\right]\frac{f'(w)}{f(w)}T(w)|dw| = 2i\,\mathrm{Im}\left[\frac{aT(z)}{z(z-a)}\right], \quad z \in \Gamma. \tag{6.8}$$

In the above integral equation, let $z = z(t)$ and $w = z(s)$. Then, by multiplying both sides of (6.8) by $|z'(t)|$ and using the fact that

$$\frac{f'(z)}{f(z)}z'(t) = i\theta_p'(t), \qquad z \in \Gamma, \tag{6.9}$$

the above integral equation can also be written as

$$\theta_p'(t) + \int_J N(s,t)\theta_p'(s)ds = 2\,\mathrm{Im}\left[\frac{az'(t)}{z(t)(z(t)-a)}\right]. \tag{6.10}$$

Since $N(s,t) = N^*(t,s)$, the integral equation can be written as an integral equation in operator form

$$(\mathbf{I} + \mathbf{N}^*)\theta_p' = \widetilde{\varphi}, \tag{6.11}$$

where

$$\widetilde{\varphi} = 2\,\mathrm{Im}\left[\frac{az'(t)}{z(t)(z(t)-a)}\right]. \tag{6.12}$$

However, $\lambda = -1$ is an eigenvalue of N^* with multiplicity M, by [18, Theorem 12]. Therefore, the integral equation (6.11) is not uniquely solvable. To overcome this problem, note that

$$\int_{J_j}\theta_p'(t)dt = 0, \quad j = 1,2,\ldots,M, \tag{6.13}$$

which implies

$$\mathbf{J}\theta_p' = 0. \tag{6.14}$$

By adding (6.14) to (6.11), we obtain the integral equation

$$(\mathbf{I} + \mathbf{N}^* + \mathbf{J})\theta_p' = \widetilde{\varphi}. \tag{6.15}$$

The integral equation (6.15) is uniquely solvable in view of the following theorem which can be proved by using the approach used in proving [22, Theorem 4].

Theorem 6.1.

$$\mathrm{Null}(\mathbf{I}+\mathbf{N}^{*}+\mathbf{J}) = \{0\}. \tag{6.16}$$

Proof. Let $\upsilon \in \mathrm{Null}(\mathbf{I}+\mathbf{N}^{*}+\mathbf{J})$, that is, υ is a solution of the integral equation

$$(\mathbf{I}+\mathbf{N}^{*}+\mathbf{J})\upsilon = 0. \tag{6.17}$$

Then, it follows from the definition of the operator $\mathbf{J}$, (2.18), and the Fredholm alternative theorem that

$$\begin{gathered}
\mathbf{J} = \mathbf{J}^{*} = \mathbf{J}^{2}, \\
\mathrm{Range}(\mathbf{J}) = \mathrm{span}\left\{\chi^{[1]},\dots,\chi^{[M]}\right\} = \mathrm{Null}(\mathbf{I}+\mathbf{N}), \\
\mathrm{Null}(\mathbf{J}) = \left(\mathrm{span}\left\{\chi^{[1]},\dots,\chi^{[M]}\right\}\right)^{\perp} = \mathrm{Null}(\mathbf{I}+\mathbf{N})^{\perp} = \mathrm{Range}(\mathbf{I}+\mathbf{N}^{*}).
\end{gathered} \tag{6.18}$$

Hence, we have $\mathbf{NJ} = -\mathbf{J}$ and $\mathbf{JN}^{*} = \mathbf{J}^{*}\mathbf{N}^{*} = (\mathbf{NJ})^{*} = -\mathbf{J}$. By multiplying (6.17) by $\mathbf{J}$, we obtain

$$\mathbf{J}\upsilon = 0, \quad (\mathbf{I}+\mathbf{N}^{*})\upsilon = 0. \tag{6.19}$$

Thus,

$$\upsilon \in \mathrm{Null}(\mathbf{J}) \cap \mathrm{Null}(\mathbf{I}+\mathbf{N}^{*}) = \mathrm{Range}(\mathbf{I}+\mathbf{N}^{*}) \cap \mathrm{Null}(\mathbf{I}+\mathbf{N}^{*}). \tag{6.20}$$

Since $\widehat{A} = 1$, thus the index of the function $\widehat{A}$ is given by (see [18] for the definition of the index)

$$\kappa_j = 0, \quad j = 0,1,\dots,m, \ \kappa = 0. \tag{6.21}$$

The space S^{+} defined in [18, Equation (30)] is then given by $S^{+} = \mathrm{span}\{1\}$. Then, it follows from [18, Equation (92)] that the dimension of the space $\widetilde{S}^{+}$ defined in [18, Equation (32)] is given by $\dim(\widetilde{S}^{+}) = M$. Similarly, it follows from [18, Equation (105)] that

$$\dim(\mathrm{Null}(\mathbf{I}+\mathbf{N}^{*})) = \dim\left(\mathrm{Null}\left(\mathbf{I}-\widetilde{\mathbf{N}}\right)\right) = M. \tag{6.22}$$

Thus, it follows from [18, Lemma 20(b)] that $\mathrm{Null}(\mathbf{I}+\mathbf{N}^{*}) = \widetilde{S}^{+}$ and the space $\widetilde{R}^{+} \cap \widetilde{S}^{-}$ in [18, Lemma 20(a)] contains only the zero function, that is, $\widetilde{R}^{+} \cap \widetilde{S}^{-} = \{0\}$. Thus, it follows from

[18, Equation (103)] (applied to the adjoint function $\widetilde{A}(t) = \widehat{A}(t)/z'(t)$ instead of $\widehat{A}(t)$) and from [18, Equation (100)] that

$$\operatorname{Range}(\mathbf{I} + \mathbf{N}^*) \cap \operatorname{Null}(\mathbf{I} + \mathbf{N}^*) = \{0\}. \tag{6.23}$$

Hence, it follows from (6.20) that $\upsilon = 0$. □

By solving the integral equation (6.15), we get $\theta_p(t)$. And solving the integral equation (5.7), we get $\phi^{[j]}$, $j = 0, 1, \ldots, M$, which gives h_j through (5.6) which in turn gives μ_j through (5.8). By solving integral equation (4.8), (4.10), and (4.11) with the known values of μ_j, we get $F(z)$. From the definition of $F(z)$, we get

$$f'(z(t)) = \frac{F(z(t))}{z^2(t)z'(t)}. \tag{6.24}$$

Finally, from (3.14) and (6.24), the approximate boundary value of $f(z)$ is given by

$$f(z) = \frac{|f(z)|}{i} T(z) \frac{|\theta_p'(t)|}{|\theta_p'(t)|} \frac{f'(z)}{|f'(z)|}, \quad z \in \Gamma. \tag{6.25}$$

The approximate interior value of the function $f(z)$ is calculated by the Cauchy integral formula

$$f(z) = \frac{a - z}{az} \frac{1}{2\pi i} \int_\Gamma \frac{awf(w)}{a - w} \frac{1}{w - z} dw, \quad z \in \Gamma. \tag{6.26}$$

For points z which are not close to the boundary, the integral in (6.26) is approximated by the trapezoidal rule. However, for the points z closed to the boundary Γ, the numerical integration in (6.26) is nearly singular. This difficulty is overcome by using the fact that $(1/2\pi i) \int_\Gamma (1/(w - z))\, dw = 1$, and rewrite $f(z)$ as

$$f(z) = \frac{((a - z)/az)(1/2\pi i) \int_\Gamma \big(awf(w)/(a - w)\big)(1/(w - z))dw}{\int_\Gamma (1/(w - z))dw}, \quad z \in \Omega. \tag{6.27}$$

This idea has the advantage that the denominator in this formula compensates for the error in the numerator (see [23]). The integrals in (6.27) are approximated by the trapezoidal rule.

7. Numerical Examples

Since the function $z_p(t)$ is 2π-periodic, a reliable procedure for solving the integral equations (6.15), (5.7), and (4.8) with the conditions (4.10) and (4.11) numerically is by using the Nyström's method with the trapezoidal rule [24]. The trapezoidal rule is the most accurate method for integrating periodic functions numerically [25, page 134–142]. Thus, solving the integral equations numerically reduces to solving linear systems of the form

$$AX = B. \tag{7.1}$$

Table 1: Error norm (unit circle).

n	$\|\mu - \mu_n\|_\infty$	$\|f - f_n(t)\|_\infty$
8	1.8×10^{-05}	2.2×10^{-02}
16	3.7×10^{-10}	5.0×10^{-06}
32	8.8×10^{-16}	3.4×10^{-14}

Table 2: The numerical values of μ_0 for Example 7.2.

n	μ_0
16	3.5383174719052
32	3.5355590602433
64	3.5355585660566
128	—

The above linear system (7.1) is uniquely solvable for sufficiently large number of collocation points on each boundary component, since the integral equations (6.15), (5.7), and (4.8) with the conditions (4.10) and (4.11) are uniquely solvable [26]. The computational details are similar to [6, 11–13].

7.1. Regions of Connectivity One

For numerical experiments, we have used some test regions of connectivity two, three, four, and five based on the examples given in [2, 4, 7, 12, 13, 15, 27–29]. All the computations were done using MATLAB 7.8.0.347(R2009a)(double precision floating point number). The number of points used in the discretization of each boundary component Γ_j is n.

In this section, we have used three test regions of connectivity one. Only the first test region has known exact mapping function. The results for sup norm error between the exact values of f, μ_1 and approximate values f_n, μ_{1n} are shown in Table 1.

Example 7.1. Consider a region Ω bounded by the unit circle

$$\Gamma : \left\{ z(t) = e^{it} \right\}, \qquad a = -0.2 + 0.2i, \tag{7.2}$$

Then, the exact mapping function is given by [1, page 340]

$$g(z) = \frac{(a - z)}{az(1 - \overline{a}z)}, \qquad r = \frac{1}{|a|}. \tag{7.3}$$

Figure 2 shows the region and its image based on our method. See Table 1 for results.

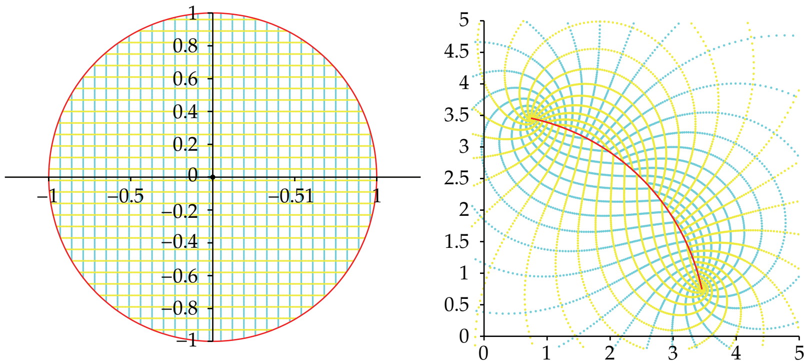
Example 7.2. Consider the elliptical region bounded by the ellipse

$$\Gamma : \{z(t) = 4\cos t + 2i\sin t\}, \qquad a = -0.2 - 0.2i. \tag{7.4}$$

Figure 3 shows the region and its image based on our method. See Table 2 for our computed value of μ_0.

Table 3: Error norm for Example 7.3.

n	$\|\mu_0 - \mu_{0n}\|_\infty$
8	1.0×10^{-02}
16	7.2×10^{-05}
32	1.1×10^{-08}
64	4.6×10^{-15}



(a) (b)

Figure 2: Mapping a region Ω bounded by unit circle onto a circular slit region.

Example 7.3. Consider a region Ω bounded by

$$\Gamma : \left\{ z(t) = (10 + 3\cos 3t)e^{it} \right\}, \qquad a = 0.1 - 0.6i. \tag{7.5}$$

Figure 4 shows the region and its image based on our method. See Table 3 for comparison between our computed values of μ_0 with those computed values μ_{0n} of Nasser [12, 13].

7.2. Regions of Connectivity Two

In this section, we have used two test regions of connectivity two whose exact mapping functions are unknown. The first and second test regions are circular frame, and the third test region is bounded by an ellipse and circle. Figures 5–7 show the region and its image based on our method, and approximate values of μ_0 and μ_1 are shown in Tables 4–6.

Example 7.4 (circular frame). Consider a pair of circles [28]

$$\begin{gathered}\Gamma_0 : \left\{ z(t) = e^{it} \right\}, \\ \Gamma_1 : \left\{ z(t) = -0.6 + 0.2e^{-it} \right\}, \qquad t : 0 \le t \le 2\pi,\ a = 0.25 + 0.25i,\end{gathered} \tag{7.6}$$

such that the region bounded by Γ_0 and Γ_1 is the region between a unit circle and a circle centered at -0.6 with radius 0.2. Then, Figure 5 shows the region and its image based on our

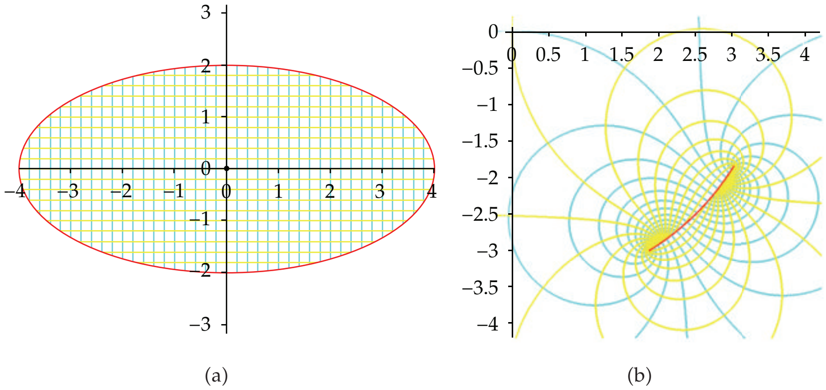


Figure 3: Mapping for Example 7.2.

Table 4: Error norm for Example 7.4.

$2n$	$\lVert\mu_0 - \mu_{0n}\rVert_\infty$	$\lVert\mu_1 - \mu_{1n}\rVert_\infty$
32	3.2×10^{-03}	5.8×10^{-03}
64	2.4×10^{-06}	5.1×10^{-06}
128	1.7×10^{-12}	3.5×10^{-12}
256	8.8×10^{-16}	2.2×10^{-15}

method. See Table 4 for comparison between our computed values of μ_0 and μ_1 with those computed values μ_{0n} and μ_{1n} of Nasser [12, 13].

Example 7.5 (ellipse with one circle). Consider a region Ω bounded by an ellipse and a circle

$$\Gamma_0 : \{z(t) = 4\cos t + i\sin t\},$$
$$\Gamma_1 : \left\{z(t) = -1 + 0.25e^{-it}\right\}, \quad t : 0 \le t \le 2\pi, \; a = -1.4, \tag{7.7}$$

such that the region bounded by Γ_0 and Γ_1 is the region between an ellipse and a circle centered at -1 with radius 0.25. Then, Figure 6 shows the region and its image based on our method. See Table 5 for comparison between our computed values of μ_0 and μ_1 with those computed values μ_{0n} and μ_{1n} of Nasser [12, 13].

Example 7.6 (two ellipses). Consider a region Ω bounded by pair of ellipses

$$\Gamma_0 : \{z(t) = 4\cos t + i\sin t\},$$
$$\Gamma_1 : \{z(t) = 1 + 0.7\cos t - 0.3i\sin t\}, \quad t : 0 \le t \le 2\pi, \; a = 2.3. \tag{7.8}$$

Figure 7 shows the region and its image based on our method. See Table 6 for comparison between our computed values of μ_0 and μ_1 with those computed values μ_{0n} and μ_{1n} of Nasser [12, 13].

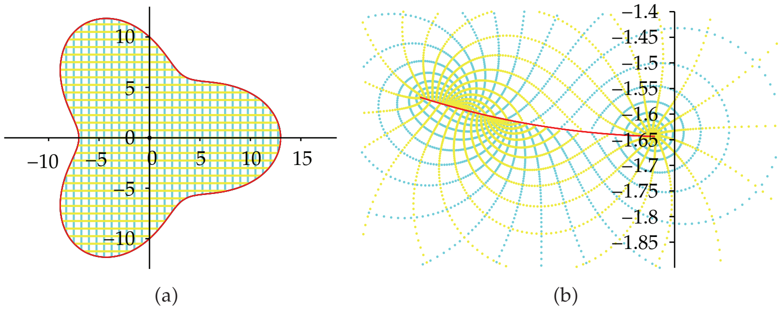


Figure 4: Mapping an original region and its image.

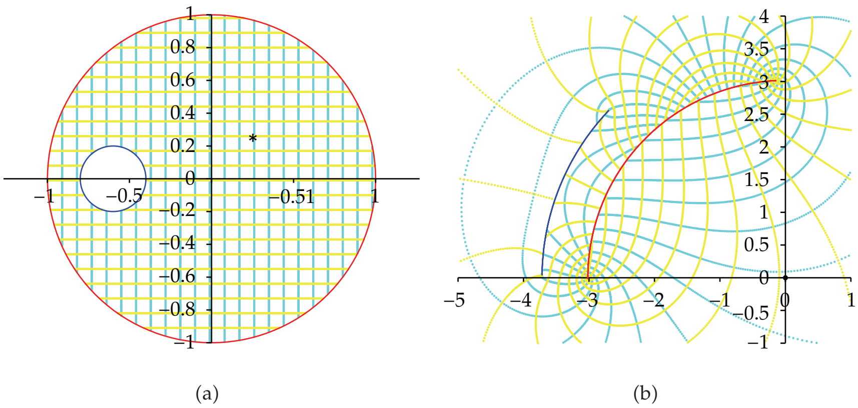


Figure 5: Mapping a region Ω bounded by two circles onto a circular slit region.

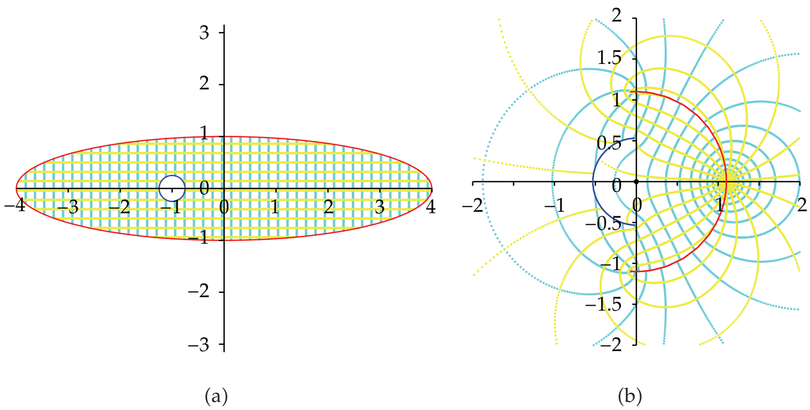


Figure 6: Mapping a region Ω bounded by an ellipse and a circle onto a circular slit region.

Table 5: Error norm for Example 7.5.

$2n$	$\|\mu_0 - \mu_{0n}\|_\infty$	$\|\mu_1 - \mu_{1n}\|_\infty$
64	1.5×10^{-03}	6.2×10^{-04}
128	4.9×10^{-07}	8.5×10^{-10}
256	7.1×10^{-14}	3.5×10^{-14}

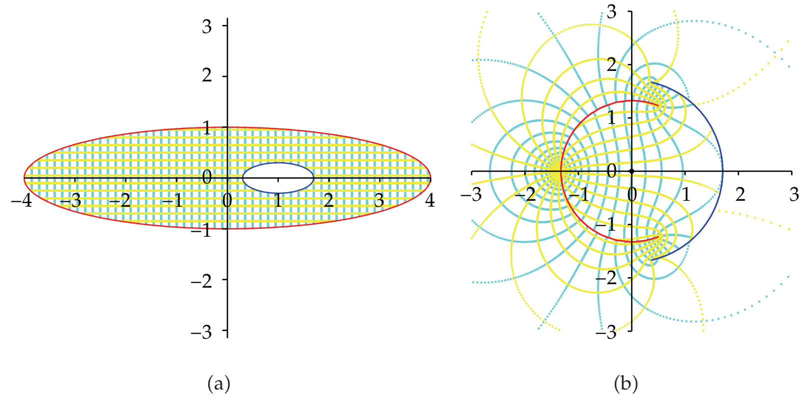


Figure 7: Mapping a region Ω bounded by two ellipses onto a circular slit region.

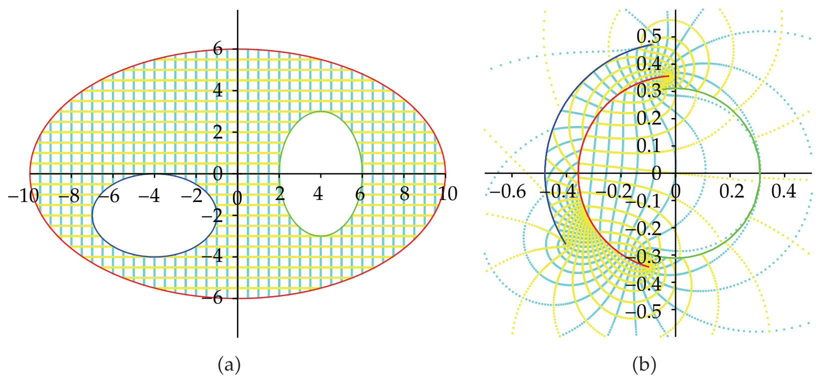


Figure 8: Mapping a region Ω bounded by three ellipses onto a circular slit region.

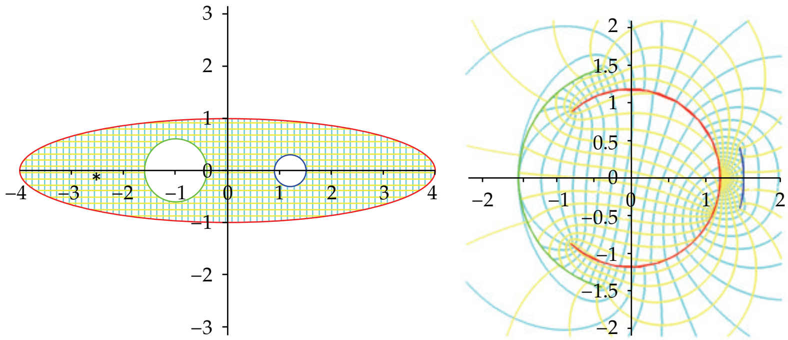
7.3. Regions of Connectivity Three

In this section, we have used three test regions of connectivity three. The first test region is bounded by three ellipses, the second test region is bounded by an ellipse and two circles, and the third test region is a circular region. The results for sup norm error between the our numerical values of μ_0, μ_1, μ_2 and the computed values of μ_{0n}, μ_{1n}, μ_{2n} obtained from [12, 13] are shown in Tables 7–9.

Example 7.7 (three ellipses). Let Ω be the region bounded by

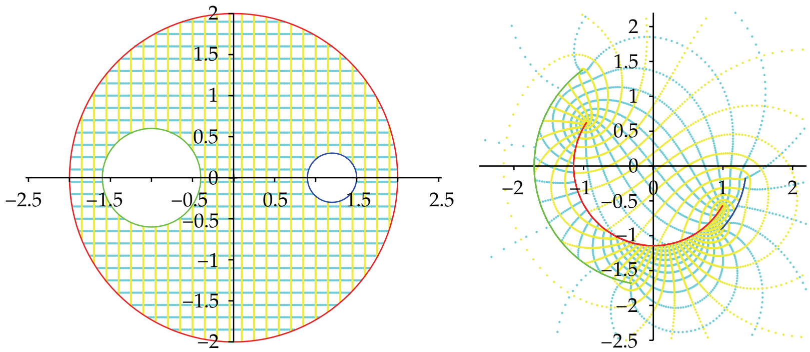
$$\begin{aligned} \Gamma_0 &: \{z(t) = 10\cos t + 6i\sin t\}, \\ \Gamma_1 &: \{z(t) = -4 - 2i + 3\cos t - 2i\sin t\}, \\ \Gamma_2 &: \{z(t) = 4 + 2\cos t - 3i\sin t\}, \quad 0 \le t \le 2\pi, \ a = 7. \end{aligned} \tag{7.9}$$

Figure 8 shows the region and its image based on our method. See Table 7 for comparison between our computed values of μ_0, μ_1, and μ_2 with those computed values of Nasser [12].



(a) (b)

Figure 9: Mapping a region Ω bounded by an ellipse and two circles onto a circular slit region.



(a) (b)

Figure 10: Mapping a region Ω bounded by three circles onto a circular slit region.

Example 7.8 (ellipse with two circles). Let Ω be the region bounded by [7, 13, 15]

$$\begin{aligned} \Gamma_0 &: \{z(t) = 4\cos t + i\sin t\}, \\ \Gamma_1 &: \{z(t) = 1.2 + 0.3(\cos t - i\sin t)\}, \\ \Gamma_2 &: \{z(t) = -1 + 0.6(\cos t - i\sin t)\}, \quad 0 \le t \le 2\pi, \ a = -2.5 - 0.1i. \end{aligned} \tag{7.10}$$

Figure 9 shows the region and its image based on our method. See Table 8 for comparison between our computed values of μ_0, μ_1, and μ_2 with those computed values of Nasser [13].

Example 7.9 (three circles). Let Ω be the region bounded by

$$\begin{aligned} \Gamma_0 &: \left\{z(t) = 2e^{it}\right\}, \\ \Gamma_1 &: \left\{z(t) = 1.2 + 0.3e^{-it}\right\}, \\ \Gamma_2 &: \left\{z(t) = -1 + 0.6e^{-it}\right\}, \quad 0 \le t \le 2\pi, \ a = 0.5 - 1.25i. \end{aligned} \tag{7.11}$$

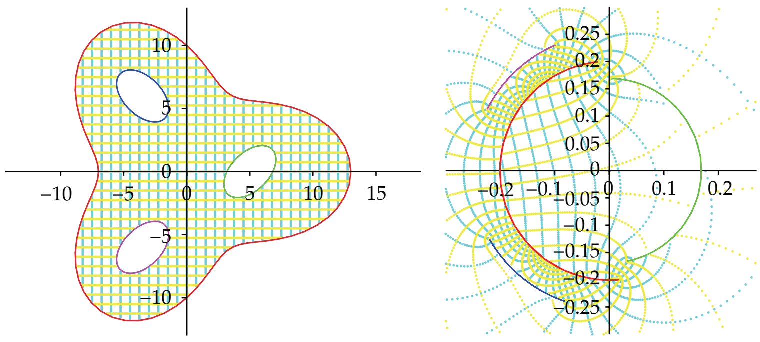


Figure 11: Mapping for Example 7.10.

Table 6: Error norm for Example 7.6.

$2n$	$\Vert\mu_0 - \mu_{0n}\Vert_\infty$	$\Vert\mu_1 - \mu_{1n}\Vert_\infty$
64	2.3×10^{-03}	2.4×10^{-03}
128	7.4×10^{-07}	9.5×10^{-07}
256	7.3×10^{-14}	9.9×10^{-14}

Table 7: Error norm for Example 7.7.

$3n$	$\Vert\mu_0 - \mu_{0n}\Vert_\infty$	$\Vert\mu_1 - \mu_{1n}\Vert_\infty$	$\Vert\mu_2 - \mu_{2n}\Vert_\infty$
48	5.1×10^{-04}	1.3×10^{-03}	4.7×10^{-04}
96	2.8×10^{-06}	7.5×10^{-06}	3.9×10^{-06}
192	2.4×10^{-10}	6.3×10^{-10}	3.1×10^{-10}
384	5.5×10^{-17}	2.7×10^{-16}	4.9×10^{-16}

Table 8: Error norm for Example 7.8.

$3n$	$\Vert\mu_0 - \mu_{0n}\Vert_\infty$	$\Vert\mu_1 - \mu_{1n}\Vert_\infty$	$\Vert\mu_2 - \mu_{2n}\Vert_\infty$
96	1.6×10^{-05}	1.0×10^{-03}	4.9×10^{-03}
192	2.7×10^{-06}	2.8×10^{-06}	8.6×10^{-07}
384	1.2×10^{-11}	1.4×10^{-11}	1.2×10^{-11}

Table 9: The numerical values of μ_0, μ_1, and μ_2 for Example 7.9.

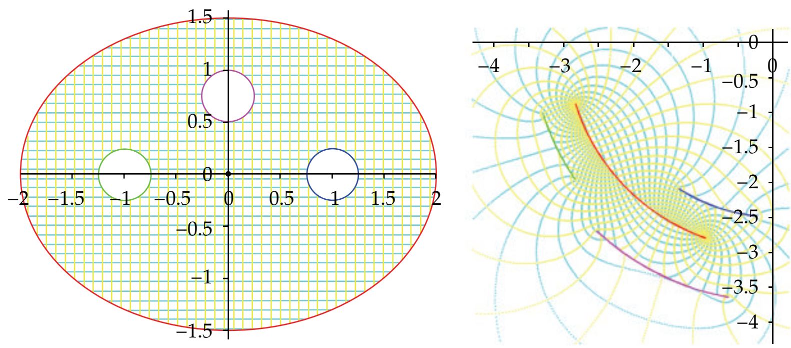
$3n$	μ_0	μ_1	μ_2
96	1.144844712112	1.333447560114	1.711779222648
192	1.144844080644	1.333446944282	1.711778670173
384	—	1.333446944281	—

Table 10: Error norm for Example 7.10.

$4n$	$\Vert\mu_0 - \mu_{0n}\Vert_\infty$	$\Vert\mu_1 - \mu_{1n}\Vert_\infty$	$\Vert\mu_2 - \mu_{2n}\Vert_\infty$	$\Vert\mu_3 - \mu_{3n}\Vert_\infty$
64	6.7×10^{-05}	7.2×10^{-05}	9.9×10^{-05}	2.2×10^{-05}
128	6.4×10^{-09}	5.0×10^{-08}	1.8×10^{-09}	4.5×10^{-08}
256	6.8×10^{-13}	1.0×10^{-12}	9.8×10^{-13}	9.7×10^{-13}
512	1.3×10^{-16}	1.2×10^{-15}	3.0×10^{-16}	4.4×10^{-16}

Table 11: The numerical values of μ_0, μ_1, μ_2, and μ_3 for Example 7.11.

$4n$	μ_0	μ_1	μ_2	μ_3
64	2.97316998311	2.50170500411	3.45373711618	3.69125205510
128	2.96757277502	2.49923061605	3.45041067650	3.69904161729
256	2.96756361086	2.49922735100	3.45040617845	3.69905124306
512	2.96756361085	2.49922735099	3.45040617844	3.69905124308



(a) (b)

Figure 12: Mapping a region Ω bounded by an ellipse and three circles onto a circular slit region.

Figure 10 shows the region and its image based on our method. See Table 9 for our computed values of μ_0, μ_1, and μ_2.

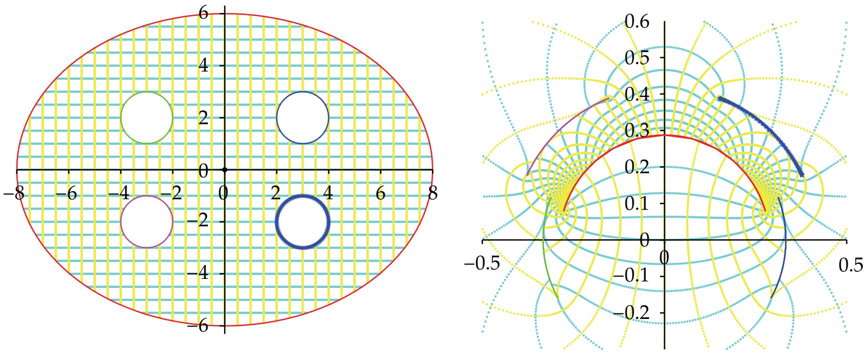
7.4. Regions of Connectivity Four and Five

In this section, we have used four test regions for multiply connected regions whose exact mapping functions are unknown. The results for sup norm error for first and third regions between the our numerical values of μ_0, μ_1, μ_2, μ_3, μ_4 and the computed values of μ_{0n}, μ_{1n}, μ_{2n}, μ_{3n}, μ_{4n} obtained from [12] are shown in Tables 10 and 12.

Example 7.10. Let Ω be the region bounded by [12]

$$\begin{gathered}\Gamma_0 : \left\{z(t) = (10 + 3\ \cos 3t\ )e^{it}\right\},\\ \Gamma_1 : \left\{z(t) = -3.5 + 6i + 0.5e^{-i\pi/4}\left(e^{it} + 4e^{-it}\right)\right\},\\ \Gamma_2 : \left\{z(t) = 5 + 0.5e^{i\pi/4}\left(e^{it} + 4e^{-it}\right)\right\},\\ \Gamma_3 : \left\{z(t) = -3.5 - 6i + 0.5e^{i\pi/4}\left(e^{it} + 4e^{-it}\right)\right\}, \quad 0 \le t \le 2\pi,\ \ a = 8.5 + 0.1i.\end{gathered} \tag{7.12}$$

Figure 11 shows the region and its image based on our method. See Table 10 for comparison between our computed values of μ_0, μ_1, μ_2, and μ_3 with those computed values of Nasser [12].



(a) (b)

Figure 13: Mapping a region Ω bounded by an ellipse and four circles onto a circular slit region.

Table 12: Error norm for Example 7.12.

$5n$	$\Vert\mu_0 - \mu_{0n}\Vert_\infty$	$\Vert\mu_1 - \mu_{1n}\Vert_\infty$	$\Vert\mu_2 - \mu_{2n}\Vert_\infty$	$\Vert\mu_3 - \mu_{3n}\Vert_\infty$	$\Vert\mu_4 - \mu_{4n}\Vert_\infty$
80	4.2×10^{-05}	4.5×10^{-05}	4.5×10^{-05}	4.4×10^{-05}	4.3×10^{-05}
160	1.1×10^{-07}	3.2×10^{-08}	3.2×10^{-08}	6.6×10^{-08}	6.6×10^{-08}
320	1.6×10^{-13}	5.7×10^{-14}	5.7×10^{-14}	1.2×10^{-13}	1.2×10^{-13}
400	9.9×10^{-16}	0	9.9×10^{-16}	0	0

Example 7.11 (ellipse with three circles). Let Ω be the region bounded by

$$\begin{aligned}
\Gamma_0 &: \{z(t) = 2\cos t + 1.5i\sin t\},\\
\Gamma_1 &: \{z(t) = 1 + 0.25(\cos t - i\sin t)\},\\
\Gamma_2 &: \{z(t) = -1 + 0.25(\cos t - i\sin t)\},\\
\Gamma_3 &: \{z(t) = 0.75i + 0.25(\cos t - i\sin t)\}, \qquad 0 \le t \le 2\pi,\ a = 0.25 - 0.25i.
\end{aligned} \tag{7.13}$$

Figure 12 shows the region and its image based on our method. See Table 11 for our computed values of μ_0, μ_1, μ_2, and μ_3.

Example 7.12 (ellipse with four circles). Let Ω be the region bounded by

$$\begin{aligned}
\Gamma_0 &: \{z(t) = 0.2 + 8\cos t + 6i\sin t\},\\
\Gamma_1 &: \{z(t) = 3 + 2i + \cos t - i\sin t\},\\
\Gamma_2 &: \{z(t) = -3 + 2i + \cos t - i\sin t\},\\
\Gamma_3 &: \{z(t) = -3 - 2i + \cos t - i\sin t\},\\
\Gamma_4 &: \{z(t) = 3 - 2i + \cos t - i\sin t\}, \qquad 0 \le t \le 2\pi,\ a = 4i.
\end{aligned} \tag{7.14}$$

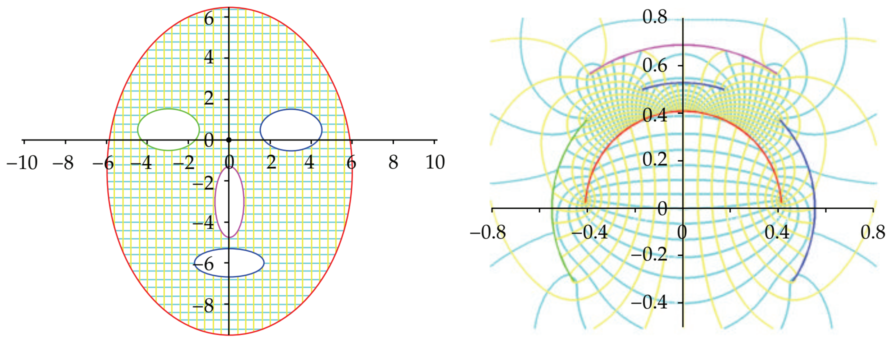


Figure 14: Mapping a region Ω bounded by five ellipses onto a circular slit region.

Table 13: The numerical values of μ_0, μ_1, μ_2, μ_3, and μ_4 for Example 7.13.

$5n$	μ_0	μ_1	μ_2	μ_3	μ_4
160	0.4081769461	0.5470254751	0.5470254751	0.6850879289	0.5258641902
320	0.4081097591	0.5470505181	0.5470505181	0.6850466360	0.5258066821
400	0.4081097885	0.5470505071	0.5470505071	0.6850466537	0.5258067072

Figure 13 shows the region and its image based on our method. See Table 12 for comparison between our computed values of μ_0, μ_1, μ_2, μ_3, and μ_4 with those computed values of Nasser [12].

Example 7.13 (five ellipses). Let Ω be the region bounded by

$$\begin{gathered}
\Gamma_0 : \{z(t) = -1.5i + 6\cos t + 8i\sin t\}, \\
\Gamma_1 : \{z(t) = 3 + 0.5i + 1.5\cos t - i\sin t\}, \\
\Gamma_2 : \{z(t) = -3 + 0.5i + 1.5\cos t - i\sin t\}, \\
\Gamma_3 : \{z(t) = -3i + 0.7\cos t - 1.7i\sin t\}, \\
\Gamma_4 : \{z(t) = -6i + 1.7\cos t - 0.7i\sin t\}, \quad 0 \le t \le 2\pi, \ a = 0.4i.
\end{gathered} \tag{7.15}$$

Figure 14 shows the region and its image based on our method. See Table 13 for our computed values of μ_0, μ_1, μ_2, μ_3, and μ_4.

8. Conclusion

In this paper, we have constructed new boundary integral equations for conformal mapping of multiply regions onto a circular slit region. We have also constructed a new method to find the values of modulus of $f(z)$. The advantage of our method is that our boundary integral equations are all linear. Several mappings of the test regions of connectivity one, two, three, four, and five were computed numerically using the proposed method. After the boundary values of the mapping function are computed, the interior mapping function is calculated by the means of Cauchy integral formula. The numerical examples presented have illustrated that our boundary integral equation method has high accuracy.

Acknowledgments

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Research Article

A New Class of Banach Spaces and Its Relation with Some Geometric Properties of Banach Spaces

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By introducing the concept of L-limited sets and then L-limited Banach spaces, we obtain some characterizations of it with respect to some well-known geometric properties of Banach spaces, such as Grothendieck property, Gelfand-Phillips property, and reciprocal Dunford-Pettis property. Some complementability of operators on such Banach spaces are also investigated.

1. Introduction and Preliminaries

A subset A of a Banach space X is called limited (resp., Dunford-Pettis (DP)), if every weak* null (resp., weak null) sequence (x_n^*) in X^* converges uniformly on A, that is,

$$\lim_{n\to\infty} \sup_{a\in A} |\langle a, x_n^* \rangle| = 0. \tag{1.1}$$

Also if $A \subseteq X^*$ and every weak null sequence (x_n) in X converges uniformly on A, we say that A is an L-set.

We know that every relatively compact subset of X is limited and clearly every limited set is DP and every DP subset of a dual Banach space is an L-set, but the converse of these assertions, in general, are false. If every limited subset of a Banach space X is relatively compact, then X has the Gelfand-Phillips property (GP). For example, the classical Banach spaces c_0 and ℓ_1 have the GP property and every reflexive space, every Schur space (i.e., weak and norm convergence of sequences in X coincide), and dual of spaces containing no copy of ℓ_1 have the same property.

Recall that a Banach space X is said to have the DP property if every weakly compact operator $T : X \to Y$ is completely continuous (i.e., T maps weakly null sequences into norm null sequences) and X is said to have the reciprocal Dunford-Pettis property (RDP) if every completely continuous operator on X is weakly compact.

So the Banach space X has the DP property if and only if every relatively weakly compact subset of X is DP and it has the RDP property if and only if every L-set in X^* is relatively weakly compact.

A stronger version of DP property was introduced by Borwein et al. in [1]. In fact, a Banach space X has the DP* property if every relatively weakly compact subset of X is limited. But if X is a Grothendieck space (i.e., weak and weak* convergence of sequences in X^* coincide), then these properties are the same on X. The reader can find some useful and additional properties of limited and DP sets and Banach spaces with the GP, DP, or RDP property in [2–6].

We recall from [7] that a bounded linear operator $T : X \to Y$ is limited completely continuous (lcc) if it carries limited and weakly null sequences in X to norm null ones in Y. We denote the class of all limited completely continuous operators from X to Y by $\mathrm{Lcc}(X, Y)$. It is clear that every completely continuous operator is lcc and we showed in [7] that every weakly compact operator is limited completely continuous.

Here, by introducing a new class of subsets of Banach spaces that are called L-limited sets, we obtain some characterizations of Banach spaces that every L-limited set is relatively weakly compact and then we investigate the relation between these spaces with the GP, DP, RDP and Grothendieck properties.

The notations and terminologies are standard. We use the symbols X, Y, and Z for arbitrary Banach spaces. We denoted the closed unit ball of X by B_X, absolutely closed convex hull of a subset A of X by $a\overline{\mathrm{co}}(A)$, the dual of X by X^*, and T^* refers to the adjoint of the operator T. Also we use $\langle x, x^* \rangle$ for the duality between $x \in X$ and $x^* \in X^*$. We denote the class of all bounded linear, weakly compact, and completely continuous operators from X to Y by $L(X, Y)$, $W(X, Y)$, and $Cc(X, Y)$ respectively. We refer the reader for undefined terminologies to the classical references [8, 9].

2. *L*-Limited Sets

Definition 2.1. A subset A of dual space X^* is called an L-limited set, if every weak null and limited sequence (x_n) in X converges uniformly on A.

It is clear that every L-set in X^* is L-limited and every subset of an L-limited set is the same. Also, it is evident that every L-limited set is weak* bounded and so is bounded. The following theorem gives additional properties of these sets.

Theorem 2.2. (a) *Absolutely closed convex hull of an L-limited set is L-limited.*

(b) *Relatively weakly compact subsets of dual Banach spaces are L-limited.*

(c) *Every weak* null sequence in dual Banach space is an L-limited set.*

Proof. Let $A \subseteq X^*$ be an L-limited set, and the sequence (x_n) in X is weak null and limited. Since

$$\sup\{|\langle x_n, x^* \rangle| : x^* \in a\overline{\mathrm{co}}(A)\} = \sup\{|\langle x_n, x^* \rangle| : x^* \in A\}, \tag{2.1}$$

the proof of (a) is clear. For the proof of (b) suppose $A \subset X^*$ is relatively weakly compact but it is not an L-limited set. Then there exists a weakly null and limited sequence (x_n) in X, a sequence (a_n) in A and an $\epsilon > 0$ such that $|\langle x_n, a_n\rangle| > \epsilon$ for all integer n. Since A is relatively weakly compact, there exists a subsequence (a_{n_k}) of (a_n) that converges weakly to an element $a \in X^*$. Since

$$|\langle x_{n_k}, a_{n_k}\rangle| \leq |\langle x_{n_k}, a_{n_k} - a\rangle| + |\langle x_{n_k}, a\rangle| \longrightarrow 0, \tag{2.2}$$

we have a contradiction.

Finally, for (c), suppose (x_n^*) is a weak* null sequence in X^*. Define the operator $T : X \to c_0$ by $T(x) = (\langle x, x_n^*\rangle)$. Since c_0 has the GP property by [7], T is lcc. So for each weakly null and limited sequence (x_m) in X, we have

$$\sup_n |\langle x_m, x_n^*\rangle| = \|T(x_m)\| \longrightarrow 0, \tag{2.3}$$

as $m \to \infty$. Hence (x_n^*) is an L-limited set. □

Note that the converse of assertion (b) in general is false. In fact, the following theorem show that the closed unit ball of ℓ_1 is an L-limited set, while the standard unit vectors (e_n) in c_0, as a weakly null sequence, shows that the closed unit ball of ℓ_1 is neither an L-set nor a relatively weakly compact. The following Theorem 2.4, give a necessary and sufficient condition for Banach spaces that L-sets and L-limited sets in its dual coincide.

Theorem 2.3. *A Banach space X has the GP property if and only if every bounded subset of X^* is an L-limited set.*

Proof. Since the Banach space X has the GP property if and only if every limited and weakly null sequence (x_n) in X is norm null [10], the proof is clear. □

Theorem 2.4. *A Banach space X has the DP* property if and only if each L -limited set in X^* is an L-set.*

Proof. Suppose X has the DP* property. Since every weakly null sequence in X is limited so every L-limited set in X^* is L-set.

Conversely, it is enough to show that, for each Banach space Y, $Cc(X, Y) = \mathrm{Lcc}(X, Y)$ [7, Theorem 2.8]. If $T : X \to Y$ is lcc, it is clear that $T^*(B_{Y^*})$ is an L-limited set. So by hypothesis, it is an L-set and we know that the operator $T : X \to Y$ is completely continuous if and only if $T^*(B_{Y^*})$ is an L-set. □

The following two corollaries extend Theorem 3.3 and Corollary 3.4 of [1].

Corollary 2.5. *For a Banach space X, the following are equivalent.*

(a) *X has the DP* property,*

(b) *If Y has the Gelfand-Phillips property, then each operator $T : X \to Y$ is completely continuous.*

Proof. (a) $\Rightarrow$ (b). Suppose that Y has the Gelfand-Phillips property. By [7, Theorem 2.2], every operator $T : X \to Y$ is lcc, thus $T^*(B_{Y^*})$ is an L-limited set and by Theorem 2.3, it is an L-set. Hence T is completely continuous.

(b) $\Rightarrow$ (a). If X does not have the DP* property, there exists a weakly null sequence (x_n) in X that is not limited. So there is a weak * null sequence (x_n^*) in X^* such that $|\langle x_n, x_n^*\rangle| > \epsilon$, for all integer n and some positive ϵ [10]. Now the bounded operator $T : X \to c_0$ defined by $T(x) = (\langle x, x_n^*\rangle)$ is not completely continuous, since (x_n) is weakly null and $\|Tx_n\| > \epsilon$ for all n. This is a contradiction. □

Corollary 2.6. *A Gelfand-Phillips space has the DP* property if and only if it has the Schur property.*

Proof. It is clear that the Banach space X has the Schur property if and only if every bounded subset of X^* is an L-set. Now, if X is a GP space with the DP* property, then by Theorem 2.3, unit ball X^* is L-limited and so it is an L-set. The converse of the assertion is also clear. □

Definition 2.7. A Banach space X has the L-limited property, if every L-limited set in X^* is relatively weakly compact.

Theorem 2.8. *For a Banach space X, the following are equivalent:*

(a) *X has the L-limited property,*

(b) *for each Banach space Y, $Lcc(X, Y) = W(X, Y)$,*

(c) $Lcc(X, \ell_\infty) = W(X, \ell_\infty)$.

Proof. (a) $\Rightarrow$ (b). Suppose that X has the L-limited property and $T : X \to Y$ is lcc. Thus $T^*(B_{Y^*})$ is an L-limited set in X^*. So by hypothesis, it is relatively weakly compact and T is a weakly compact operator.

(b) $\Rightarrow$ (c). It is clear.

(c) $\Rightarrow$ (a). If X does not have the L-limited property, there exists an L-limited subset A of X^* that is not relatively weakly compact. So there is a sequence $(x_n^*) \subseteq A$ with no weakly convergent subsequence. Now we show that the operator $T : X \to \ell_\infty$ defined by $T(x) = (\langle x, x_n^*\rangle)$ for all $x \in X$ is limited completely continuous but it is not weakly compact. As $(x_n^*) \subseteq A$ is L-limited set, for every weakly null and limited sequence (x_m) in X we have

$$\|T(x_m)\| = \sup_n |\langle x_m, x_n^*\rangle| \longrightarrow 0 \quad \text{as } m \longrightarrow \infty, \tag{2.4}$$

thus T is a limited completely continuous operator. It is easy to see that $T^*(e_n^*) = x_n^*$, for all $n \in \mathbb{N}$. Thus T^* is not a weakly compact operator and neither is T. This finishes the proof. □

The following corollary shows that the Banach spaces c_0 and ℓ_1 do not have the L-limited property.

Corollary 2.9. *A Gelfand-Phillips space has the L-limited property if and only if it is reflexive.*

Proof. If a Banach space X has the GP property, then by [7], the identity operator on X is lcc and so is weakly compact, thanks to the L-limited property of X. Hence X is reflexive. □

Theorem 2.10. *If a Banach space X has the L-limited property, then it has the RDP and Grothendieck properties.*

Proof. At the first, we show that X has the RDP property. For arbitrary Banach space Y, let $T : X \to Y$ be a completely continuous operator. Thus it is limited completely continuous and so by Theorem 2.8, T is weakly compact. Hence X has the RDP property.

By [11], we know that a Banach space X is Grothendieck if and only if $W(X, c_0) = L(X, c_0)$. Since c_0 has the GP property, by [7], $\mathrm{Lcc}(X, c_0) = L(X, c_0)$ and by hypothesis on X, $W(X, c_0) = \mathrm{Lcc}(X, c_0)$. So X is Grothendieck. □

We do not know the converse of Theorem 2.10, in general, is true or false. In the following, we show that in Banach lattices that are Grothendieck and have the DP property, the converse of this theorem is correct.

Theorem 2.11. *If a Banach lattice X has both properties of Grothendieck and DP, then it has the L-limited property.*

Proof. Suppose that $T : X \to Y$ is limited completely continuous. We know, that in Grothendieck Banach spaces, DP and DP* properties are equivalent. Thus by [7], T is completely continuous. On the other hand, ℓ_1 is not a Grothendieck space and Grothendieck property is carried by complemented subspaces. Hence the Grothendieck space X does not have any complemented copy of ℓ_1. Since X is a Banach lattice, by [12], it has the RDP property and so the completely continuous operator $T : X \to Y$ is weakly compact. Thus X has the L-limited property, thanks to Theorem 2.8. □

As a corollary, since ℓ_∞ is a Banach lattice that has Grothendieck and DP properties, it has the L-limited property. This shows that the L-limited property on Banach spaces is not hereditary, since c_0 does not have this property. In the following, we show that the L-limited property is carried by every complemented subspace.

Theorem 2.12. *If a Banach space X has the L-limited property, then every complemented subspace of X has the L-limited property.*

Proof. Consider a complemented subspace X_0 of X and a projection map $P : X \to X_0$. Suppose $T : X_0 \to \ell_\infty$ is a limited completely continuous operator, so $TP : X \to \ell_\infty$ is also lcc. Since X has the L-limited property, by Theorem 2.8, TP is weakly compact. Hence T is weakly compact. □

As another corollary, for infinite compact Hausdorff space K, we have the following corollary for the Banach space $C(K)$ of all continuous functions on K with supremum norm.

Corollary 2.13. *$C(K)$ has the L-limited property if and only if it contains no complemented copy of c_0.*

Proof. We know that $C(K)$ is a Banach lattice with the DP property. On the other hand, $C(K)$ is a Grothendieck space if and only if it contains no complemented copy of c_0 [13]. So the direct implication is an application of Theorem 2.12 and the opposite implication is also an easy conclusion of Theorem 2.11. □

3. Complementation in Lcc Operators

In [11], Bahreini investigated the complementability of $W(X, \ell_\infty)$ and $Cc(X, \ell_\infty)$ in $L(X, \ell_\infty)$. She showed that if X is not a reflexive Banach space, then $W(X, \ell_\infty)$ is not complemented in $L(X, \ell_\infty)$ and if X is not a Schur space, $Cc(X, \ell_\infty)$ is not complemented in $L(X, \ell_\infty)$. In the following, we investigate the complementability of $W(X, \ell_\infty)$ and $Cc(X, \ell_\infty)$ in $\mathrm{Lcc}(X, \ell_\infty)$. We need the following lemma of [14].

Lemma 3.1. *Let X be a separable Banach space, and $\phi : \ell_\infty \to L(X, \ell_\infty)$ is a bounded linear operator with $\phi(e_n) = 0$ for all n. Then there is an infinite subset M of $\mathbb{N}$ such that for each $\alpha \in \ell_\infty(M)$, $\phi(\alpha) = 0$, where $\ell_\infty(M)$ is the set of all $\alpha = (\alpha_n) \in \ell_\infty$ with $\alpha_n = 0$ for each $n \notin M$.*

Theorem 3.2. *If X does not have the L-limited property, then $W(X, \ell_\infty)$ is not complemented in $Lcc(X, \ell_\infty)$.*

Proof. Consider a subset A of X^* that is L-limited but it is not relatively weakly compact. So there is a sequence (x_n^*) in A that has no weakly convergent subsequence. Hence $S : X \to \ell_\infty$ defined by $S(x) = (\langle x, x_n^* \rangle)$ is an lcc operator but it is not weakly compact. Choose a bounded sequence (x_n) in B_X such that $S(x_n)$ has no weakly convergent subsequence. Let $X_0 = [x_n]$, the closed linear span of the sequence (x_n) in X. It follows that X_0 is a separable subspace of X such that $S_{|X_0}$ is not a weakly compact operator. If $y_n^* = x_{n|X_0}^*$, we have $(y_n^*) \subseteq X_0^*$ is bounded and has no weakly convergent subsequence.

Now define $T : \ell_\infty \to \mathrm{Lcc}(X, \ell_\infty)$ by $T(\alpha)(x) = (\alpha_n \langle x, x_n^* \rangle)$, where $x \in X$ and $\alpha = (\alpha_n) \in \ell_\infty$. Then

$$\|T(\alpha)(x)\| = \sup_n |\alpha_n \langle x, x_n^* \rangle| \leq \|\alpha\| \cdot \|x_n^*\| \cdot \|x\| < \infty. \tag{3.1}$$

We claim that for each $\alpha = (\alpha_n) \in \ell_\infty$, $T(\alpha) \in \mathrm{Lcc}(X, \ell_\infty)$.

Fix $\alpha = (\alpha_n) \in \ell_\infty$ and a weakly null and limited sequence (x_m) in X. Since (x_n^*) is an L-limited set, $\sup_n |\langle x_m, x_n^* \rangle| \to 0$. So we have

$$\|T(\alpha)(x_m)\| = \sup_n |\alpha_n \langle x_m, x_n^* \rangle| \leq \|\alpha\| \sup_n |\langle x_m, x_n^* \rangle| \longrightarrow 0, \tag{3.2}$$

as $m \to \infty$. This finishes the proof of the claim and so T is a well-defined operator into $\mathrm{Lcc}(X, \ell_\infty)$.

Let $R : \mathrm{Lcc}(X, \ell_\infty) \longrightarrow \mathrm{Lcc}(X_0, \ell_\infty)$ be the restriction map and define

$$\phi : \ell_\infty \longrightarrow \mathrm{Lcc}(X_0, \ell_\infty) \text{ by } \phi = RT. \tag{3.3}$$

Now suppose that $W(X, \ell_\infty)$ is complemented in $\mathrm{Lcc}(X, \ell_\infty)$ and

$$P : \mathrm{Lcc}(X, \ell_\infty) \longrightarrow W(X, \ell_\infty) \tag{3.4}$$

is a projection. Define $\psi : \ell_\infty \to W(X_0, \ell_\infty)$ by $\psi = RPT$. Note that as $T(e_n)$ is a rank one operator, we have $T(e_n) \in W(X, \ell_\infty)$. Hence

$$\psi(e_n) = RPT(e_n) = RT(e_n) = \phi(e_n) \tag{3.5}$$

for all $n \in \mathbb{N}$. By Lemma 3.1, there is an infinite set $M \subseteq \mathbb{N}$ so that $\psi(\alpha) = \phi(\alpha)$ for all $\alpha \in \ell_\infty(M)$. Thus $\phi(\chi_M)$ is a weakly compact operator. On the other hand, if (e_n^*) is the standard unit vectors of ℓ_1, for each $x \in X_0$ and each $n \in \mathbb{N}$, we have

$$\left\langle (\phi(\chi_M))^*(e_n^*), x \right\rangle = \langle x_n^*, x \rangle. \tag{3.6}$$

Therefore $(\phi(\chi_M))^*(e_n^*) = x^*_{n|X_0} = y_n^*$ for all $n \in M$. Thus $(\phi(\chi_M))^*$ is not a weakly compact operator and neither is $\phi(\chi_M)$. This contradiction ends the proof. □

Corollary 3.3. *Let X be a Banach space. Then the following are equivalent:*

(a) *X has the L-limited property,*

(b) $W(X, \ell_\infty) = \mathrm{Lcc}(X, \ell_\infty)$,

(c) $W(X, \ell_\infty)$ *is complemented in* $\mathrm{Lcc}(X, \ell_\infty)$.

We conclude this paper with another complementation theorem. Recall from [11] that a closed operator ideal $\mathcal{O}$ has the property $(*)$ whenever X is a Banach space and S is not in $\mathcal{O}(X, \ell_\infty)$, then there is an infinite subset $M_0 \subseteq \mathbb{N}$ such that S_M is not in $\mathcal{O}(X, \ell_\infty)$ for all infinite subsets $M \subseteq M_0$, where $S_M : X \to \ell_\infty$ is the operator defined by $S_M(x) = \Sigma_{m\in M} e_m^*(Sx)e_m$, for all $x \in X$.

Theorem 3.4. *If a Banach space X does not have the DP* property, then* $Cc(X, \ell_\infty)$ *is not complemented in* $\mathrm{Lcc}(X, \ell_\infty)$.

Proof. By hypothesis, there is a weakly null sequence (x_m) in X that is not limited. So there exists a weak* null sequence (x_n^*) in X^* such that

$$\lim_{m\to\infty} \sup_n |\langle x_m, x_n^* \rangle| \neq 0. \tag{3.7}$$

Now define the operator $S : X \to \ell_\infty$ by $S(x) = (\langle x, x_n^* \rangle)$. By Theorem 2.2, (x_n^*) is an L-limited set, but S is not completely continuous. So for $X_0 = [x_n]$, $S|_{X_0}$ is not completely continuous. Since $Cc(X_0, \ell_\infty)$ has the property (*) [11, Theorem 4.12], one can choose $M_0 \subseteq \mathbb{N}$ so that for each infinite subset M of M_0, $S_M \notin Cc(X_0, \ell_\infty)$. Define $T : \ell_\infty \to \mathrm{Lcc}(X, \ell_\infty)$ by $T(\alpha)(x) = (\alpha_n \langle x, x_n^* \rangle)$, where $x \in X$ and $\alpha = (\alpha_n) \in \ell_\infty$. As shown in the proof of the preceding theorem, T is well defined.

Let $R : \mathrm{Lcc}(X, \ell_\infty) \to \mathrm{Lcc}(X_0, \ell_\infty)$ be the restriction map and define

$$\phi : \ell_\infty \longrightarrow \mathrm{Lcc}(X_0, \ell_\infty) \text{ by } \phi = RT. \tag{3.8}$$

Now suppose $Cc(X, \ell_\infty)$ is complemented in $\mathrm{Lcc}(X, \ell_\infty)$ and

$$P : \mathrm{Lcc}(X, \ell_\infty) \longrightarrow Cc(X, \ell_\infty) \tag{3.9}$$

is a projection. Define $\psi : \ell_\infty \to Cc(X_0, \ell_\infty)$ by $\psi = RPT$. Since

$$\psi(e_n) = RPT(e_n) = RT(e_n) = \phi(e_n), \tag{3.10}$$

for all $n \in \mathbb{N}$, one can use Lemma 3.1 to select an infinite subset M of M_0 such that $\psi(\alpha) = \phi(\alpha)$ for all $\alpha \in \ell_\infty(M)$. Thus $\phi(\alpha) = RT(\alpha)$ belongs to $Cc(X_0, \ell_\infty)$ for each $\alpha \in \ell_\infty(M)$. But $T(\chi_M)|_{X_0} = S_M \notin Cc(X_0, \ell_\infty)$, so we have a contradiction. □

Corollary 3.5. *Let X be a Banach space. Then the following are equivalent:*

(a) *X has the DP* property,*

(b) $Cc(X, \ell_\infty) = \mathrm{Lcc}(X, \ell_\infty)$,

(c) $Cc(X, \ell_\infty)$ *is complemented in* $\mathrm{Lcc}(X, \ell_\infty)$.

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Research Article

On Generalized Hyers-Ulam Stability of Admissible Functions

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We consider the Hyers-Ulam stability for the following fractional differential equations in sense of Srivastava-Owa fractional operators (derivative and integral) defined in the unit disk: $D_z^{\beta} f(z) = G(f(z), D_z^{\alpha} f(z), zf'(z); z), 0 < \alpha < 1 < \beta \leq 2$, in a complex Banach space. Furthermore, a generalization of the admissible functions in complex Banach spaces is imposed, and applications are illustrated.

1. Introduction

A classical problem in the theory of functional equations is the following: if a function f approximately satisfies functional equation $\mathcal{E}$, when does there exist an exact solution of $\mathcal{E}$ which f approximates? In 1940, Ulam [1, 2] imposed the question of the stability of Cauchy equation, and in 1941, Hyers solved it [3]. In 1978, Rassias [4] provided a generalization of Hyers theorem by proving the existence of unique linear mappings near approximate additive mappings. The problem has been considered for many different types of spaces (see [5–7]). Li and Hua [8] discussed and proved the Hyers-Ulam stability of spacial type of finite polynomial equation, and Bidkham et al. [9] introduced the Hyers-Ulam stability of generalized finite polynomial equation. Rassias [10] imposed a Cauchy type additive functional equation and investigated the generalized Hyers-Ulam "product-sum" stability of this equation.

Recently, Jung presented a book [11], which complements the books of Hyers, Isac, and Rassias (Stability of Functional Equations in Several Variables, Birkhäuser, 1998) and of Czerwik (Functional Equations and Inequalities in Several Variables, World Scientific, 2002) by covering and offering almost all classical results on the Hyers-Ulam-Rassias stability such as the Hyers-Ulam-Rassias stability of the additive Cauchy equation, generalized additive functional equations, Hosszú's functional equation, Hosszú's equation of Pexider type, homogeneous functional equation, Jensen's functional equation, the quadratic functional equations, the exponential functional equations, Wigner equation, Fibonacci functional equation, the gamma functional equation, and the multiplicative functional equations. Furthermore, the concept of superstability for some problems is defined and studied.

The Ulam stability and data dependence for fractional differential equations in sense of Caputo derivative has been posed by Wang et al. [12] while in sense of Riemann-Liouville derivative has been discussed by Ibrahim [13]. Finally, the author generalized the Ulam-Hyers stability for fractional differential equation including infinite power series [14, 15].

The class of fractional differential equations of various types plays important roles and tools not only in mathematics but also in physics, control systems, dynamical systems and engineering to create the mathematical modeling of many physical phenomena. Naturally, such equations required to be solved. Many studies on fractional calculus and fractional differential equations, involving different operators such as Riemann-Liouville operators [16], Erdèlyi-Kober operators [17], Weyl-Riesz operators [18], Grünwald-Letnikov operators [19] and Caputo fractional derivative [20–24], have appeared during the past three decades. The existence of positive solution and multipositive solutions for nonlinear fractional differential equation are established and studied [25]. Moreover, by using the concepts of the subordination and superordination of analytic functions, the existence of analytic solutions for fractional differential equations in complex domain is suggested and posed in [26–28].

2. Preliminaries

Let $U := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$ and $\mathcal{H}$ denote the space of all analytic functions on U. Here we suppose that $\mathcal{H}$ as a topological vector space endowed with the topology of uniform convergence over compact subsets of U. Also for $a \in \mathbb{C}$ and $m \in \mathbb{N}$, let $\mathcal{H}[a,m]$ be the subspace of $\mathcal{H}$ consisting of functions of the form

$$f(z) = a + a_m z^m + a_{m+1} z^{m+1} + \cdots, \quad z \in U. \tag{2.1}$$

Let $\mathcal{A}$ be the class of functions f, analytic in U and normalized by the conditions $f(0) = f'(0) - 1 = 0$. A function $f \in \mathcal{A}$ is called univalent ($\mathcal{S}$) if it is one-one in U. A function $f \in \mathcal{A}$ is called convex if it satisfies the following inequality:

$$\Re\left\{ \frac{zf''(z)}{f'(z)} + 1 \right\} > 0, \quad (z \in U). \tag{2.2}$$

We denoted this class $\mathcal{C}$.

In [29], Srivastava and Owa, posed definitions for fractional operators (derivative and integral) in the complex z-plane $\mathbb{C}$ as follows.

Definition 2.1. The fractional derivative of order α is defined, for a function $f(z)$ by

$$D_z^\alpha f(z) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^\alpha} \, d\zeta, \tag{2.3}$$

where the function $f(z)$ is analytic in simply connected region of the complex z-plane $\mathbb{C}$ containing the origin, and the multiplicity of $(z-\zeta)^{-\alpha}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta)>0$.

Definition 2.2. The fractional integral of order $\alpha>0$ is defined, for a function $f(z)$, by

$$I_z^{\alpha} f(z) := \frac{1}{\Gamma(\alpha)} \int_0^z f(\zeta)(z-\zeta)^{\alpha-1}\, d\zeta; \quad \alpha>0, \tag{2.4}$$

where the function $f(z)$ is analytic in simply connected region of the complex z-plane ($\mathbb{C}$) containing the origin, and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta)>0$.

Remark 2.3. We have the following:

$$\begin{aligned} D_z^{\alpha} z^{\mu} &= \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} z^{\mu-\alpha}, \quad \mu>-1, \\ I_z^{\alpha} z^{\mu} &= \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} z^{\mu+\alpha}, \quad \mu>-1. \end{aligned} \tag{2.5}$$

In [27], it was shown the relation

$$I_z^{\alpha} D_z^{\alpha} f(z) = D_z^{\alpha} I_z^{\alpha} f(z) = f(z), \qquad f(0)=0. \tag{2.6}$$

More details on fractional derivatives and their properties and applications can be found in [30, 31].

We next introduce the generalized Hyers-Ulam stability depending on the properties of the fractional operators.

Definition 2.4. Let $p \in (0,1)$. We say that

$$\sum_{n=0}^{\infty} a_n z^{n+\alpha} = f(z) \tag{2.7}$$

has the generalized Hyers-Ulam stability if there exists a constant $K>0$ with the following property: for every $\epsilon>0$, $w \in \overline{U} = U \bigcup \partial U$, if

$$\left| \sum_{n=0}^{\infty} a_n w^{n+\alpha} \right| \le \epsilon \left(\sum_{n=0}^{\infty} \frac{|a_n|^p}{p(n+1)^2} \right), \tag{2.8}$$

then there exists some $z \in \overline{U}$ that satisfies (2.7) such that

$$\left| z^i - w^i \right| \le \epsilon K, \quad \left(z, w \in \overline{U},\ i \in \mathbb{N} \right). \tag{2.9}$$

In the present paper, we study the generalized Hyers-Ulam stability for holomorphic solutions of the fractional differential equation in complex Banach spaces X and Y

$$D_z^{\beta} f(z) = G\big(f(z), D_z^{\alpha} f(z), zf'(z); z\big), \quad 0 < \alpha < 1 < \beta \leq 2, \tag{2.10}$$

where $G : X^3 \times U \to Y$ and $f : U \to X$ are holomorphic functions such that $f(0) = \Theta$ (Θ is the zero vector in X).

3. Generalized Hyers-Ulam Stability

In this section we present extensions of the generalized Hyers-Ulam stability to holomorphic vector-valued functions. Let X, Y represent complex Banach space. The class of admissible functions $\mathcal{G}(X, Y)$ consists of those functions $g : X^3 \times U \to Y$ that satisfy the admissibility conditions:

$$\big\|g(r, ks, lt; z)\big\| \geq 1, \quad \text{when } \|r\| = \|s\| = \|t\| = 1, \ (z \in U, \ k, l \geq 1). \tag{3.1}$$

We need the following results.

Lemma 3.1 (see [32]). *If $f : D \to X$ is holomorphic, then $\|f\|$ is a subharmonic of $z \in D \subset \mathbb{C}$. It follows that $\|f\|$ can have no maximum in D unless $\|f\|$ is of constant value throughout D.*

Lemma 3.2 (see [33]). *Let $f : U \to X$ be the holomorphic vector-valued function defined in the unit disk U with $f(0) = \Theta$ (the zero element of X). If there exists a $z_0 \in U$ such that*

$$\big\|f(z_0)\big\| = \max_{|z|=|z_0|} \big\|f\big\|, \tag{3.2}$$

then

$$\big\|z_0 f'(z_0)\big\| = \kappa \big\|f(z_0)\big\|, \quad \kappa \geq 1. \tag{3.3}$$

Lemma 3.3 (see [34, page 88]). *If the function $f(z)$ is in the class $\mathcal{S}$, then*

$$\big|D_z^{\alpha+n} f(z)\big| \leq \frac{(n + \alpha + |z|)\Gamma(n + \alpha + 1)}{(1 - |z|)^{n+\alpha+2}}, \quad (z \in U; n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; \ 0 \leq \alpha < 1). \tag{3.4}$$

Lemma 3.4 (see [29, page 225]). *If the function $f(z)$ is in the class $\mathcal{C}$, then*

$$\big|D_z^{\alpha+n} f(z)\big| \leq \frac{\Gamma(n + \alpha + 1)}{(1 - |z|)^{n+\alpha+1}}, \quad (z \in U; n \in \mathbb{N}_0; \ 0 \leq \alpha < 1). \tag{3.5}$$

Theorem 3.5. *Let $G \in \mathcal{G}(X, Y)$ and $f : U \to X$ be a holomorphic vector-valued function defined in the unit disk U, with $f(0) = \Theta$. If $f \in \mathcal{S}$, then*

$$\big\|G\big(f(z), D_z^{\alpha} f(z); z\big)\big\| < 1 \Longrightarrow \big\|f(z)\big\| < 1. \tag{3.6}$$

Proof. Since $f \in \mathcal{S}$, then from Lemma 3.3, we observe that

$$\left|D_z^\alpha f(z)\right| \leq \frac{(\alpha + |z|)\Gamma(\alpha+1)}{(1-|z|)^{\alpha+2}}. \tag{3.7}$$

Assume that $\|f(z)\| \nless 1$ for $z \in U$. Thus, there exists a point $z_0 \in U$ for which $\|f(z_0)\| = 1$. According to Lemma 3.1, we have

$$\begin{gathered}\|f(z)\| < 1, \quad z \in U_{r_0} = \{z : |z| < |z_0| = r_0\},\\ \max_{|z|\leq|z_0|}\|f(z)\| = \|f(z_0)\| = 1.\end{gathered} \tag{3.8}$$

In view of Lemma 3.2, at the point z_0, there is a constant $\kappa \geq 1$ such that

$$\|z_0 f'(z_0)\| = \kappa\|f(z_0)\| = \kappa. \tag{3.9}$$

Consequently, we obtain that

$$\|f(z_0)\| = \frac{(1-|z|)^{\alpha+2}}{(\alpha+|z|)\Gamma(\alpha+1)} \qquad \|D_{z_0}^\alpha f(z_0)\| = \frac{1}{\kappa}\|z_0 f'(z_0)\| = 1. \tag{3.10}$$

We put $k := ((\alpha + |z|)\Gamma(\alpha+1)/(1-|z|)^{\alpha+2}) \geq 1$, for some $0 < \alpha < 1$ and $z \in U$ and $l := \kappa \geq 1$; hence from (3.1), we deduce

$$\|G(f(z_0), D_{z_0}^\alpha f(z_0), z_0 f'(z_0); z_0)\| = \left\|G\left(f(z_0), k\left[\frac{D_{z_0}^\alpha f(z_0)}{k}\right], l\left[\frac{z_0 f'(z_0)}{l}\right]; z_0\right)\right\| \geq 1, \tag{3.11}$$

which contradicts the hypothesis in (3.6) that we must have $\|f(z)\| < 1$. □

Corollary 3.6. *Assume the problem* (2.10). *If $G \in \mathcal{G}(X,Y)$ is a holomorphic univalent vector-valued function defined in the unit disk U, then*

$$\|G(f(z), D_z^\alpha f(z), zf'(z); z)\| < 1 \Longrightarrow \left\|I_z^\beta G(f(z), D_z^\alpha f(z), zf'(z); z)\right\| < 1. \tag{3.12}$$

Proof. By univalency of G, the fractional differential equation (2.10) has at least one holomorphic univalent solution f. Thus, according to Remark 2.3, the solution $f(z)$ of the problem (2.10) takes the form

$$f(z) = I_z^\beta G(f(z), D_z^\alpha f(z), zf'(z); z). \tag{3.13}$$

Therefore, in virtue of Theorem 3.5, we obtain the assertion (3.12). □

Theorem 3.7. *Let $G \in \mathcal{G}(X,Y)$ be holomorphic univalent vector-valued functions defined in the unit disk U then* (2.10) *has the generalized Hyers-Ulam stability for $z \to \partial U$.*

Proof. Assume that

$$G(z) := \sum_{n=0}^{\infty} \varphi_n z^n, \quad z \in U \tag{3.14}$$

therefore, by Remark 2.3, we have

$$I_z^{\alpha} G(z) = \sum_{n=0}^{\infty} a_n z^{n+\alpha} = f(z). \tag{3.15}$$

Also, $z \to \partial U$ and thus $|z| \to 1$. According to Theorem 3.5, we have

$$\|f(z)\| < 1 = |z|. \tag{3.16}$$

Let $\epsilon > 0$ and $w \in \overline{U}$ be such that

$$\left|\sum_{n=1}^{\infty} a_n w^{n+\alpha}\right| \leq \epsilon\left(\sum_{n=1}^{\infty} \frac{|a_n|^p}{p(n+1)^2}\right). \tag{3.17}$$

We will show that there exists a constant K independent of ϵ such that

$$\left|w^i - u^i\right| \leq \epsilon K, \quad w \in \overline{U},\ u \in U \tag{3.18}$$

and satisfies (2.7). We put the function

$$f(w) = \frac{-1}{\lambda a_i} \sum_{n=1, n \neq i}^{\infty} a_n w^{n+\alpha}, \quad a_i \neq 0,\ 0 < \lambda < 1, \tag{3.19}$$

thus, for $w \in \partial U$, we obtain

$$\begin{aligned}
\left|w^i - u^i\right| &= \left|w^i - \lambda f(w) + \lambda f(w) - u^i\right| \\
&\leq \left|w^i - \lambda f(w)\right| + \lambda\left|f(w) - u^i\right| \\
&< \left|w^i - \lambda f(w)\right| + \lambda\left|w^i - u^i\right| \\
&= \left|w^i + \frac{1}{a_i} \sum_{n=1, n \neq i}^{\infty} a_n w^{n+\alpha}\right| + \lambda\left|w^i - u^i\right| \\
&= \frac{1}{|a_i|}\left|\sum_{n=1}^{\infty} a_n w^{n+\alpha}\right| + \lambda\left|w^i - u^i\right|.
\end{aligned} \tag{3.20}$$

Without loss of generality, we consider $|a_i| = \max_{n\geq 1}(|a_n|)$ yielding

$$\begin{aligned}
\left|w^i - u^i\right| &\leq \frac{1}{|a_i|(1-\lambda)}\left|\sum_{n=1}^{\infty} a_n w^{n+\alpha}\right| \\
&\leq \frac{\epsilon}{|a_i|(1-\lambda)}\left(\sum_{n=0}^{\infty}\frac{|a_n|^p}{p(n+1)^2}\right) \\
&\leq \frac{\epsilon|a_i|^{p-1}}{p(1-\lambda)}\left(\sum_{n=0}^{\infty}\frac{1}{(n+1)^2}\right) \\
&= \frac{\pi^2\epsilon|a_i|^{p-1}}{6p(1-\lambda)} \\
&:= K\epsilon.
\end{aligned} \tag{3.21}$$

This completes the proof. □

In the same manner of Theorem 3.5, and by using Lemma 3.4, we have the following result.

Theorem 3.8. *Let $G \in \mathcal{G}(X, Y)$ and $f : U \to X$ be a holomorphic vector-valued function defined in the unit disk U, with $f(0) = \Theta$. If $f \in \mathcal{C}$, then*

$$\left\|G\left(f(z), D_z^{\alpha} f(z), zf'(z); z\right)\right\| < 1 \Longrightarrow \left\|f(z)\right\| < 1. \tag{3.22}$$

4. Applications

In this section, we introduce some applications of functions to achieve the generalized Hyers-Ulam stability.

Example 4.1. Consider the function $G : X^3 \times U \to \mathbb{R}$ by

$$G(r, s, t; z) = a(\|r\| + \|s\| + \|t\|)^n + b|z|^2, \quad n \in \mathbb{R}_+ \tag{4.1}$$

with $a \geq 0.5$, $b \geq 0$ and $G(\Theta, \Theta, \Theta;\ 0) = 0$. Our aim is to apply Theorem 3.5, this follows since

$$\|G(r, ks, lt; z)\| = a(\|r\| + k\|s\| + l\|t\|)^n + b|z|^2 = a(1 + k + l)^n + b|z|^2 \geq 1, \tag{4.2}$$

when $\|r\| = \|s\| = \|t\| = 1$, $z \in U$. Hence by Theorem 3.5, we have the following. If $a \geq 0.5$, $b \geq 0$ and $f : U \to X$ is a holomorphic univalent vector-valued function defined in U, with $f(0) = \Theta$, then

$$a\left(\left\|f(z)\right\| + \left\|D_z^{\alpha} f(z)\right\| + \left\|zf'(z)\right\|\right)^n + b|z|^2 < 1 \Longrightarrow \left\|f(z)\right\| < 1. \tag{4.3}$$

Consequently, $\|I_z^{\alpha} G(f(z), D_z^{\alpha} f(z), zf'(z); z)\| < 1$, thus in view of Theorem 3.7, f has the generalized Hyers-Ulam stability.

Example 4.2. Assume that the function $G : X^3 \longrightarrow X$ by

$$G(r,s,t;z) = G(r,s,t) = re^{\|s\|\|t\|-1}, \tag{4.4}$$

with $G(\Theta,\Theta,\Theta) = \Theta$. By applying Corollary 3.6, we need to show that $G \in \mathcal{G}(X,X)$. Since

$$\|G(r,ks,lt)\| = \left\|re^{\|ks\|\|lt\|-1}\right\| = e^{kl-1} \geq 1, \tag{4.5}$$

when $\|r\| = \|s\| = \|t\| = 1$, $k \geq 1$ and $l \geq 1$. Hence by Corollary 3.6, we have the following. For $f : U \rightarrow X$ is a holomorphic vector-valued function defined in U, with $f(0) = \Theta$, then

$$\left\|f(z)e^{\|D_z^\alpha f(z)\|\|zf'(z)\|-1}\right\| < 1 \Longrightarrow \|f(z)\| < 1. \tag{4.6}$$

Consequently, $\|I_z^\alpha G(f(z), D_z^\alpha f(z), zf'(z); z)\| < 1$, thus in view of Theorem 3.7, f has the generalized Hyers-Ulam stability.

Example 4.3. Let $a,b,c : U \rightarrow \mathbb{C}$ satisfy the following:

$$|a(z) + \mu b(z) + \nu c(z)| \geq 1, \tag{4.7}$$

for every $\mu \geq 1$, $\nu > 1$ and $z \in U$. Consider the function $G : X^3 \longrightarrow Y$ by

$$G(r,s,t;z) = a(z)r + \mu b(z)s + \nu c(z)t, \tag{4.8}$$

with $G(\Theta,\Theta,\Theta) = \Theta$. Now for $\|r\| = \|s\| = \|t\| = 1$, we have

$$\|G(r,\mu s,\nu t;z)\| = |a(z) + \mu b(z) + \nu c(z)| \geq 1 \tag{4.9}$$

and thus $G \in \mathcal{G}(X,Y)$. If $f : U \rightarrow X$ is a holomorphic vector-valued function defined in U, with $f(0) = \Theta$, then

$$\left\|a(z)f(z) + b(z)D_z^\alpha f(z) + zc(z)f'(z)\right\| < 1 \Longrightarrow \|f(z)\| < 1. \tag{4.10}$$

Hence according to Theorem 3.7, f has the generalized Hyers-Ulam stability.

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Research Article

Dirichlet Characters, Gauss Sums, and Inverse Z Transform

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A generalized Möbius transform is presented. It is based on Dirichlet characters. A general algorithm is developed to compute the inverse Z transform on the unit circle, and an error estimate is given for the truncated series representation.

1. Introduction

We consider a causal, linear, time-invariant system with an infinite impulse response $\{c_j\}_{j=1}^{\infty}$. The system is assumed to be stable and the Z transform $X(z) = \sum_{j=1}^{\infty} c_j z^{-j}$ is convergent for $|z| > r$, where $r < 1$. The frequency response of the system is obtained by evaluating the Z transform on the unit circle.

The arithmetic Fourier transform (AFT) offers a convenient method, based on the construction of weighted averages, to calculate the Fourier coefficients of a periodic function. It was discovered by Bruns [1] at the beginning of the last century. Similar algorithms were studied by Wintner [2] and Sadasiv [3] for the calculation of the Fourier coefficients of even periodic functions. This method was extended in [4] to calculate the Fourier coefficients of both the even and odd components of a periodic function. The Bruns approach was incorporated in [5] resulting in a more computationally balanced algorithm. In [6, 7], Knockaert presented the theory of the generalized Möbius transform and gave a general formulation.

In [8], Schiff et al. applied Wintner's algorithm for the computation of the inverse Z-transform of an infinite causal sequence. Hsu et al. [9] applied two special Möbius inversion formulae to the inverse Z-transform.

The transform pairs play a central part in the arithmetic Fourier transform and inverse Z-transform. In this paper, based on Dirichlet characters, we presented a generalized Möbius transform of which all the transform pairs used in the mentioned papers are the special cases. A general algorithm was developed in Section 2 to compute the inverse Z transform on the unit circle. The algorithm computes each term c_j of the infinite impulse response from sampled values of the Z transform taken at a countable set of points on the unit circle. An error estimate is given in Section 3 for the truncated series representation. A numerical example is given in Section 4. Number theory and Dirichlet characters [10] play an important role in the paper.

2. The Algorithm

According to the Möbius inversion formula for finite series [4], if n is a positive integer and $f(n)$, $g(n)$ are two number-theoretic functions, then

$$g(n) = \sum_{k=1}^{[N/n]} f(kn) \quad \text{iff } f(n) = \sum_{m=1}^{[N/n]} \mu(m) g(mn), \tag{2.1}$$

where $[y]$ denotes the integer part of real number y and $\mu(n)$ is the Möbius function:

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^r & \text{if } n \text{ includes } r \text{ distinct prime factors,} \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

Knockaert [6] extended the Möbius inversion formula and proved the following proposition.

Proposition 2.1. *Let $f_1, f_2, \ldots$ be a sequence of real numbers and $\alpha(n)$, $\beta(n)$ two arithmetical functions. For the transform pair*

$$s_n = \sum_{k=1}^{\infty} \alpha(k) f_{kn}, \qquad f_n = \sum_{k=1}^{\infty} \beta(k) s_{kn} \tag{2.3}$$

to be valid for all sequences f_n, it is necessary and sufficient that

$$\sum_{kl=m} \alpha(k)\beta(l) = \sum_{k|m} \alpha(k)\beta\left(\frac{m}{k}\right) = \delta_{1m} = \begin{cases} 1, & m = 1, \\ 0, & m \neq 1. \end{cases} \tag{2.4}$$

Let G be the group of reduced residue classes modulo q. Corresponding to each character f of G, we define an arithmetical function $\chi = \chi_f$ as follows:

$$\chi(n) = f(\hat{n}) \quad \text{if } (n, q) = 1, \qquad \chi(n) = 0 \quad \text{if } (n, q) > 1, \tag{2.5}$$

where $\hat{n} = \{x : x \equiv n (\text{mod } q)\}$ and (a, b) denotes the greatest common divisor of a and b.

The function χ is called a Dirichlet character modulo q. The principal character χ_1 is that which has the properties

$$\chi_1(n) = \begin{cases} 1 & \text{if } (n,q) = 1, \\ 0 & \text{if } (n,q) > 1. \end{cases} \tag{2.6}$$

If $q \geq 1$, the Euler's totient $\phi(q)$ is defined to be the number of positive integers not exceeding q that are relatively prime to q. There are $\phi(q)$ distinct Dirichlet characters modulo q, each of which is completely multiplicative and periodic with period q. That is, we have

$$\chi(mn) = \chi(m)\chi(n) \quad \forall m, n, \tag{2.7}$$

$$\chi(n+q) = \chi(n) \quad \forall n. \tag{2.8}$$

Conversely, if χ is completely multiplicative and periodic with period q, and if $\chi(n) = 0$ if $(n,q) > 1$, then χ is one of the Dirichlet characters modulo q.

Let $f(n)$ be an arithmetical function. Series of the form $\sum_{n=1}^{\infty} f(n)/n^s$ are called Dirichlet series with coefficients $f(n)$. If $f(n) = \chi(n)$, then the series are called Dirichlet L-functions. For any Dirichlet character χ mod q, the sum

$$G(n,\chi) = \sum_{m=1}^{q} \chi(m)\, e^{2\pi i m n/q} \tag{2.9}$$

is called the Gauss sums associated with χ. If $\chi = \chi_1$, then the Gauss sums reduce to Ramanujan's sum

$$G(n,\chi_1) = \sum_{\substack{m=1 \\ (m,q)=1}}^{q} e^{2\pi i m n/q} = c_q(n). \tag{2.10}$$

See [10].

Let χ be a Dirichlet character modulo q. We have

$$\sum_{k|m} \chi(k)\mu\left(\frac{m}{k}\right)\chi\left(\frac{m}{k}\right) = \chi(m)\sum_{k|m}\mu\left(\frac{m}{k}\right) = \delta_{1m}. \tag{2.11}$$

In this way, we have defined a generalized Möbius transform pair.

Lemma 2.2. *Let χ be a Dirichlet character modulo q; then transform pair*

$$s_n = \sum_{k=1}^{\infty} \chi(k) f_{kn}, \qquad f_n = \sum_{k=1}^{\infty} \mu(k)\chi(k) s_{kn} \tag{2.12}$$

is valid for all q.

Remarks 1. The transform pairs play a central part in the arithmetic Fourier transform and inverse Z-transform. It is not hard to show that all the transform pairs used in the mentioned papers are the special cases of our generalized Möbius transform. In fact,

(a) let $q = 1$ in Lemma 2.2; we have

$$s_n = \sum_{k=1}^{[N/n]} f_{kn}, \qquad f_n = \sum_{k=1}^{[N/n]} \mu(k) s_{kn}, \tag{2.13}$$

which is Theorem 3 in [4] and Lemma 1 in [8];

(b) let $q = 2^\alpha$ and $\chi = \chi_1$ in Lemma 2.2, where $\alpha \geq 1$ is a positive integer; we have

$$s_n = \sum_{k=1,3,5,\ldots}^{[N/n]} f_{kn}, \qquad f_n = \sum_{k=1,3,5,\ldots}^{[N/n]} \mu(k) s_{kn}, \tag{2.14}$$

which is Case 1 of Lemma 1 in [9];

(c) let $q = 2^\alpha$, $\alpha \geq 2$, and

$$\chi_2(k) = \begin{cases} (-1)^{(k-1)/2} & \text{if } (k, q) = 1, \\ 0 & \text{if } (k, q) > 1 \end{cases} \tag{2.15}$$

in Lemma 2.2, then χ_2 is one of the Dirichlet characters modulo q since $\chi_2(k)$ is completely multiplicative and periodic with period q. We have

$$s_n = \sum_{k=1,3,5,\ldots}^{[N/n]} f_{kn} (-1)^{(k-1)/2}, \qquad f_n = \sum_{k=1,3,5,\ldots}^{[N/n]} \mu(k) s_{kn} (-1)^{(k-1)/2}, \tag{2.16}$$

which is Case 2 of Lemma 1 in [9];

(d) let $\chi = \chi_1$ in Lemma 2.2; we have

$$s_n = \sum_{(k,q)=1} f_{kn}, \qquad f_n = \sum_{(k,q)=1} \mu(k) s_{kn}, \tag{2.17}$$

which is transform pair I of Theorem 4 in [7];

(e) let $q = 4$, p^α or $2p^\alpha$, and $\chi_3(k) = (k/q)$ in Lemma 2.2, where p is an odd prime, $\alpha \geq 1$, and (k/q) is the Legendre's symbol defined as follows:

$$\left(\frac{k}{q}\right) = \begin{cases} 1 & \text{if } (k, q) = 1 \text{ and } n \text{ is a quadratic residue } \bmod q, \\ -1 & \text{if } (k, q) = 1 \text{ and } n \text{ is not a quadratic residue } \bmod q, \\ 0 & \text{if } (k, q) > 1. \end{cases} \tag{2.18}$$

From [10], we know that q admits a primitive root and $(k/q) = (-1)^{\mathrm{ind}(k)}$. We have

$$s_n = \sum_{(k,q)=1} (-1)^{\mathrm{ind}(k)} f_{kn}, \qquad f_n = \sum_{(k,q)=1} \mu(k)(-1)^{\mathrm{ind}(k)} s_{kn}, \tag{2.19}$$

which is transform pair II of Theorem 4 in [7].

From these facts, we claim that Lemma 2.2 is actually an important extension on the Möbius inversion formula. In practice, we can choose the best possible transform pair.

We do not discuss the convergence of the transform pair since in practice it is used only on a truncated series. Next we establish our main theorem.

Theorem 2.3. *Let $X(z) = \sum_{j=1}^{\infty} c_j z^{-j}$ be convergent for $|z| > r$, where $r < 1$. For any fixed $q \geq 1$ and Dirichlet character χ modulo q, the coefficients are given by*

$$c_n = \frac{1}{qn} \sum_{k=1}^{\infty} \frac{\mu(k)\chi(k)}{k} \sum_{r=1}^{q} G(r,\chi) \sum_{l=1}^{kn} X\left(e^{(2\pi i/kn)(l+r/q)}\right). \tag{2.20}$$

Proof. On $|z| = 1$, let us write $X(\theta) = X(e^{i\theta}) = \sum_{j=1}^{\infty} c_j e^{-ij\theta}$.

Define

$$s_n = \frac{1}{q} \sum_{r=1}^{q} G(r,\chi) \left[\frac{1}{n} \sum_{l=1}^{n} X\left(e^{(2\pi i/n)(l+r/q)}\right) \right]. \tag{2.21}$$

Note that for a positive integer k

$$\frac{1}{n} \sum_{m=1}^{n} e^{2\pi i k m/n} = \begin{cases} 1 & \text{if } n \text{ divides } k, \\ 0 & \text{if } n \text{ does not divide } k, \end{cases} \tag{2.22}$$

we have

$$\begin{aligned} s_n &= \frac{1}{q} \sum_{r=1}^{q} G(r,\chi) \left[\frac{1}{n} \sum_{l=1}^{n} X\left(e^{(2\pi i/n)(l+r/q)}\right) \right] = \frac{1}{q} \sum_{r=1}^{q} G(r,\chi) \left[\frac{1}{n} \sum_{l=1}^{n} \sum_{j=1}^{\infty} c_j e^{-2\pi i j (l+r/q)/n} \right] \\ &= \frac{1}{q} \sum_{r=1}^{q} G(r,\chi) \left[\frac{1}{n} \sum_{j=1}^{\infty} c_j \sum_{l=1}^{n} e^{-2\pi i j l/n} e^{-2\pi i j r/nq} \right] = \frac{1}{q} \sum_{r=1}^{q} G(r,\chi) \sum_{l=1}^{\infty} c_{ln} e^{-2\pi i l r/q}. \end{aligned} \tag{2.23}$$

Let $l = qk + s$; then

$$s_n = \frac{1}{q} \sum_{r=1}^{q} G(r,\chi) \sum_{k=0}^{\infty} \sum_{s=1}^{q} c_{n(qk+s)} e^{-2\pi i s r/q} = \frac{1}{q} \sum_{k=0}^{\infty} \sum_{s=1}^{q} c_{n(qk+s)} \sum_{m=1}^{q} \chi(m) \sum_{r=1}^{q} e^{2\pi i r(m-s)/q}. \tag{2.24}$$

Note that $1 \leq m,\ s \leq q$, so $q \mid (m-s)$ if and only if $m = s$; therefore,

$$s_n = \sum_{k=0}^{\infty} \sum_{s=1}^{q} \chi(s) c_{n(qk+s)} = \sum_{t=1}^{\infty} \chi(t) c_{nt}. \tag{2.25}$$

By Lemma 2.2, we have

$$c_n = \sum_{k=1}^{\infty} \mu(k)\chi(k) s_{nk} = \frac{1}{qn} \sum_{k=1}^{\infty} \frac{\mu(k)\chi(k)}{k} \sum_{r=1}^{q} G(r,\chi) \sum_{l=1}^{kn} X\left(e^{(2\pi i/kn)(l+r/q)}\right). \tag{2.26}$$

This completes the proof of Theorem 2.3. □

Remarks 2. Let $q = 1$ in Theorem 2.3; we have

$$c_n = \sum_{k=1}^{\infty} \frac{\mu(k)}{kn} \sum_{l=1}^{kn} X\left(e^{(2\pi i/kn)(l+1)}\right), \tag{2.27}$$

which is the theorem in [8].

Let $q = 2$ in Theorem 2.3 or $q = 4$ and $\chi = \chi_1$ in Theorem 2.3; we easily have

$$c_n = \sum_{k=1,3,5,\ldots}^{\infty} \frac{\mu(k)}{2kn} \left[\sum_{l=1}^{kn} X\left(e^{(2\pi i/kn)(l+1)}\right) - \sum_{l=1}^{kn} X\left(e^{(2\pi i/kn)(l+1/2)}\right) \right]. \tag{2.28}$$

Let $q = 4$ and $\chi = \chi_2$ in Theorem 2.3; we have

$$c_n = \sum_{k=1,3,5,\ldots}^{\infty} \frac{\mu(k)(-1)^{(k-1)/2} i}{2kn} \left[\sum_{l=1}^{kn} X\left(e^{(2\pi i/kn)(l+1/4)}\right) - \sum_{l=1}^{kn} X\left(e^{(2\pi i/kn)(l+3/4)}\right) \right]. \tag{2.29}$$

In practice, a large number of coefficients c_n may be calculated. We suppose that a truncation is employed. Next we estimate the error due to the truncation of the series.

3. Error Estimate

In order to estimate the error due to truncation of the series representation of the coefficients c_n, we require the following lemma.

Lemma 3.1. *If f is a function of period 2π, with $f' \in \mathrm{Lip}_1([0, 2\pi])$, then*

$$\left| \int_0^{2\pi} f(\theta) d\theta - \frac{1}{n} \sum_{m=1}^{n} f\left(\theta + \frac{2\pi m}{n}\right) \right| \leq \frac{C}{n^2}, \tag{3.1}$$

uniformly in θ, where C is the Lipschitz constant.

Proof. This is Lemma 3 of [8]. □

Taking $X(z)$ as in Theorem 2.3, we maintain the following theorem.

Theorem 3.2. *The truncation error satisfies*

$$\left|c_n - \frac{1}{qn}\sum_{k=1}^{N}\frac{\mu(k)\chi(k)}{k}\sum_{r=1}^{q}G(r,\chi)\sum_{l=1}^{kn}X\left(e^{(2\pi i/kn)(l+r/q)}\right)\right| \leq \frac{C\phi(q)}{n^2 N}, \tag{3.2}$$

where C is the Lipschitz constant.

Proof. Note that we have

$$0 = c_0 = \frac{1}{2\pi}\int_0^{2\pi} X\left(e^{i\varphi}\right)d\varphi. \tag{3.3}$$

Moreover, $X' \in \mathrm{Lip}_1([0,2\pi])$ by the analyticity of X. By Theorem 2.3 and Lemma 3.1, we have

$$\begin{aligned}
&\left|c_n - \frac{1}{qn}\sum_{k=1}^{N}\frac{\mu(k)\chi(k)}{k}\sum_{r=1}^{q}G(r,\chi)\sum_{l=1}^{kn}X\left(e^{(2\pi i/kn)(l+r/q)}\right)\right| \\
&\quad = \left|\frac{1}{qn}\sum_{k=N+1}^{\infty}\frac{\mu(k)\chi(k)}{k}\sum_{r=1}^{q}G(r,\chi)\sum_{l=1}^{kn}X\left(e^{(2\pi i/kn)(l+r/q)}\right)\right| \\
&\quad \leq \left|\frac{1}{q}\sum_{k=N+1}^{\infty}\mu(k)\chi(k)\sum_{r=1}^{q}G(r,\chi)\frac{C}{n^2k^2}\right| \leq \frac{C\phi(q)}{n^2}\sum_{k=N+1}^{\infty}\frac{1}{k^2} \leq \frac{C\phi(q)}{n^2N}.
\end{aligned} \tag{3.4}$$

This completes the proof of Theorem 3.2. □

Table 1: Calculation of the Z-transform coefficients of the function: $X(z) = e^{1/z} + 1/(z - 1/2) - 1,\ q = 1$.

k	c_1	c_2	c_3
1	$3.718281828 + 0.000000720i$	$1.209747301 + 0.000000338i$	$0.453772595 + 0.000000199i$
2	$2.508534526 + 0.000000381i$	$1.034722511 + 0.000000208i$	$0.420637706 + 0.000000128i$
3	$2.054761931 + 0.000000182i$	$1.001587622 + 0.000000138i$	$0.416721106 + 0.000000087i$
4	$2.054761931 + 0.000000182i$	$1.001587622 + 0.000000138i$	$0.416721106 + 0.000000087i$
5	$1.981912218 + 0.000000089i$	$0.999632365 + 0.000000100i$	$0.416660127 + 0.000000062i$
6	$2.015047107 + 0.000000160i$	$1.000120712 + 0.000000131i$	$0.416667696 + 0.000000082i$
7	$1.999100704 + 0.000000103i$	$0.999998692 + 0.000000105i$	$0.416666804 + 0.000000065i$
8	$1.999100704 + 0.000000103i$	$0.999998692 + 0.000000105i$	$0.416666804 + 0.000000065i$
9	$1.999100704 + 0.000000103i$	$0.999998692 + 0.000000105i$	$0.416666804 + 0.000000065i$
10	$2.001055961 + 0.000000140i$	$1.000000538 + 0.000000123i$	$0.416666743 + 0.000000077i$

Table 2: Calculation of the Z-transform coefficients of the function: $X(z) = e^{1/z} + 1/(z-1/2) - 1,\ q = 2,\ q = 4$, or $\chi = \chi_1$.

k	c_1	c_2	c_3
1	$2.508534526 + 0.000000785i$	$1.034722484 + 0.000000226i$	$0.420637664 + 0.000000130i$
3	$2.087896862 + 0.000000654i$	$1.002075958 + 0.000000213i$	$0.416728633 + 0.000000129i$
5	$2.017002434 + 0.000000624i$	$1.000122546 + 0.000000212i$	$0.416667589 + 0.000000126i$
7	$2.001178059 + 0.000000618i$	$1.000000467 + 0.000000209i$	$0.416666630 + 0.000000121i$
9	$2.001178059 + 0.000000618i$	$1.000000467 + 0.000000209i$	$0.416666630 + 0.000000121i$
11	$2.000201462 + 0.000000617i$	$0.999999985 + 0.000000205i$	$0.416666627 + 0.000000117i$
13	$1.999957311 + 0.000000615i$	$0.999999950 + 0.000000200i$	$0.416666625 + 0.000000113i$
15	$2.000018355 + 0.000000618i$	$0.999999956 + 0.000000204i$	$0.416666626 + 0.000000117i$
17	$2.000003089 + 0.000000614i$	$0.999999953 + 0.000000200i$	$0.416666625 + 0.000000114i$
19	$1.999999268 + 0.000000610i$	$0.999999951 + 0.000000196i$	$0.416666624 + 0.000000111i$

4. An Example

Consider the function

$$X(z) = e^{1/z} + \frac{1}{z - 1/2} - 1, \quad |z| > \frac{1}{2}. \tag{4.1}$$

The few first coefficients are $c_1 = 2$, $c_2 = 1$, and $c_3 = 5/12$. Employing formulae (2.27), (2.28), and (2.29), we obtain the results given in Tables 1, 2, and 3. The results show that formulae (2.28) and (2.29) is quite more accurate than formula (2.27). Choosing carefully the modulo q and the Dirichlet character, we will greatly improve the algorithm.

5. Conclusion

A general algorithm offers a general way to compute the inverse Z transform. It is based on generalized Möbius transform, Dirichlet characters, and Gauss sums. The algorithm computes each term c_j of the infinite impulse response from sampled values of the Z transform taken at a countable set of points on the unit circle. An error estimate and a numerical example are given for the truncated series representation. Choosing carefully the

Table 3: Calculation of the Z-transform coefficients of the function: $X(z) = e^{1/z} + 1/(z - 1/2) - 1$, $q = 4$, and $\chi = \chi_2$.

k	c_1	c_2	c_3
1	$1.641470945 + 0.000000164i$	$0.969199603 + 0.000000195i$	$0.412817735 + 0.000000128i$
3	$2.054288680 + 0.000000292i$	$1.001830856 + 0.000000204i$	$0.416726726 + 0.000000119i$
5	$1.983516336 + 0.000000263i$	$0.999877456 + 0.000000214i$	$0.416665686 + 0.000000128i$
7	$1.999338790 + 0.000000262i$	$0.999999531 + 0.000000205i$	$0.416666645 + 0.000000122i$
9	1.999338790 + 0.000000262i	$0.999999531 + 0.000000205i$	$0.416666645 + 0.000000122i$
11	$2.000315379 + 0.000000252i$	$1.000000013 + 0.000000200i$	$0.416666650 + 0.000000120i$
13	$2.000071234 + 0.000000261i$	$0.999999978 + 0.000000204i$	$0.416666646 + 0.000000122i$
15	$2.000010194 + 0.000000270i$	$0.999999971 + 0.000000207i$	$0.416666642 + 0.000000123i$
17	$1.999994930 + 0.000000277i$	$0.999999967 + 0.000000209i$	$0.416666639 + 0.000000124i$
19	$1.999998750 + 0.000000271i$	$0.999999971 + 0.000000208i$	$0.416666642 + 0.000000123i$

modulo q and the Dirichlet character we will greatly improve the algorithm. But this is not exhaustive. Dirichlet characters and Gauss sums play an important role in number theory, and there are so many methods and results associated with them. Any development on the Dirichlet character and Gauss sums may be applied to the inverse Z transform.

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Research Article

Univalent Logharmonic Mappings in the Plane

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This paper surveys recent advances on univalent logharmonic mappings defined on a simply or multiply connected domain. Topics discussed include mapping theorems, logharmonic automorphisms, univalent logharmonic extensions onto the unit disc or the annulus, univalent logharmonic exterior mappings, and univalent logharmonic ring mappings. Logharmonic polynomials are also discussed, along with several important subclasses of logharmonic mappings.

1. Introduction

Let D be a domain in the complex plane $\mathbb{C}$. Denote by $H(D)$ (resp., by $M(D)$) the linear space of all analytic (resp., meromorphic) functions in D, and let $B(D)$ be the set of all functions $a \in H(D)$ satisfying $|a(z)| < 1$, $z \in D$. A nonconstant function f is logharmonic in D if f is the solution of the nonlinear elliptic differential equation

$$\overline{f_{\overline{z}}} = a\frac{\overline{f}}{f} f_z, \tag{1.1}$$

$a \in B(D)$. The function a is called the second dilatation of f. In contrast to the linear space $H(D)$ consisting of analytic functions, translations in the image do not preserve logharmonicity, and the inverse of a logharmonic function is not necessarily logharmonic. If f_1 and f_2 are two logharmonic functions with respect to $a \in B(D)$, then $f_1 \cdot f_2$ is logharmonic with respect to the same a. If, in addition, $0 \notin f_2(D)$, then f_1/f_2 is also logharmonic. The composition $f \circ \phi$ of a logharmonic mapping f with a conformal premapping ϕ is also logharmonic with respect to $a \circ \phi$. However, the composition $\phi \circ f$ of a conformal postmapping ϕ with a logharmonic mapping f is in general not logharmonic. If f is a logharmonic mapping in D, then f is a nonconstant locally quasiregular mapping, and, therefore, it is continuous, open, and light. It follows that f can be represented as a composition of two functions $f = A \circ \chi$, where χ is a locally quasiconformal homeomorphism in D and $A \in H(\chi(D))$. As an immediate consequence, the maximum principle, the identity principle, and the argument principle all still hold for logharmonic mappings.

The study of logharmonic mappings was initiated in the main by Abdulhadi, Bshouty, and Hengartner in the last century, and the basic theory of logharmonic mappings was developed in [1–8].

A local representation for logharmonic mappings was given by Abdulhadi and Bshouty in [1]. In particular, they obtained the following result.

Theorem 1.1. *Let f be a logharmonic mapping in D with respect to $a \in B(D)$. Suppose that $f(z_0) = 0$ and $B(z_0,\rho)\backslash\{z_0\} \subset D\backslash Z(f)$, where $B(z_0,\rho) = \{z : |z - z_0| < \rho\}$ and $Z(f) = \{z \in D : f(z) = 0\}$. Then f admits the representation*

$$f(z) = (z - z_0)|z - z_0|^{2\beta n} h(z)\overline{g(z)}, \qquad z \in B(z_0,\rho), \tag{1.2}$$

where $n \in \mathbb{N}$, $\beta = n\overline{a(z_0)}(1 + a(z_0))/(1 - |a(z_0)|^2)$ and, therefore, $\mathrm{Re}(\beta) > -n/2$. The functions h and g are in $H(B(z_0,\rho))$, with $h(z_0) \neq 0$ and $g(z_0) = 1$.

As a direct consequence of Theorem 1.1, we have the following global representation for logharmonic mappings.

Corollary 1.2. *Let D be a simply connected domain in $\mathbb{C}$ and f a logharmonic mapping in D. If f has exactly p zeros $\{z_k\}_{k=1}^{p}$ in D (counting multiplicities), then f admits a global representation given by*

$$f(z) = \left[\prod_{k=1}^{p}(z - z_k)|z - z_k|^{2\beta_k}\right] h(z)\overline{g(z)}, \tag{1.3}$$

where $\beta_k = \overline{a(z_k)}(1 + a(z_k))/(1 - |a(z_k)|^2)$ and, therefore, $\mathrm{Re}(\beta_k) > -1/2$. The functions h and g are in $H(D)$, and $0 \notin h \cdot g(D)$.

For the converse, Abdulhadi and Hengartner [2] proved the following theorem.

Theorem 1.3. *Suppose that $f(z) = h(z)\overline{g(z)}$ is defined in a domain D, where h and g are in $H(D)$, such that $f(D)$ does not lie on a logarithmic spiral. Then either $f = \overline{g}$ or f is a solution of*

$$\overline{f_{\overline{z}}(z)} = a\frac{\overline{f(z)}}{f(z)} f_z(z), \qquad a \in M(D),\ |a| \neq 1. \tag{1.4}$$

Remark 1.4. The converse of Theorem 1.3 does not hold. Indeed, consider the partial differential equation $\overline{f_{\overline{z}}} = (1/3)(\overline{f}/f)f_z$. Then $f_1(z) = z^6\overline{z}^2$ and $f_2(z) = z|z|$ are solutions of this equation. The function f_1 can be written in the form $h\overline{g}$ while f_2 could not.

Remark 1.5. The function $g_w(z) = f(z) - w,\ w \in \mathbb{C}$, cannot be written in the form $h\overline{g}$ unless $w = 0$ or f is a constant. However, it is a solution of the second Beltrami equation

$$\overline{\left(\frac{\partial g_w(z)}{\partial \overline{z}}\right)} = \mu_w(z, g_w)\frac{\partial g_w(z)}{\partial z}, \tag{1.5}$$

where

$$\mu_w(z, g_w) = a(z)\frac{\overline{g_w(z) + w}}{g_w(z) + w}. \tag{1.6}$$

Hence, $|\mu_w| \equiv |a|$ in D and is independent of w.

Corollary 1.6. *The image $f(D)$ of a nonconstant function $f(z) = h(z)\overline{g(z)}$ lies on a logarithmic spiral if and only if f is a solution of* (1.1) *with* $|a| \equiv 1$.

In the theory of quasiconformal mappings, it is proved that, for any measurable function μ with $|\mu| < 1$, the solution of Beltrami equation $f_{\overline{z}} = \mu f_z$ can be factorized in the form $f = \psi \circ F$, where F is a univalent quasiconformal mapping and ψ is an analytic function (see [9]). For sense-preserving harmonic mappings, the answer is negative. In [10], Duren and Hengartner gave a necessary and sufficient condition on sense-preserving harmonic mappings f for the existence of such a factorization. Moreover, for logharmonic mappings, such a factorization need not exist. For example, the function $f(z) = z^2/|1 - z|^4$ is a sense-preserving logharmonic mapping with respect to $a(z) = z$, and f has no decomposition of the desired form (see [11]). The following factorization theorem was proved in [11].

Theorem 1.7. *Let f be a nonconstant logharmonic mapping defined in a domain $D \subset \mathbb{C}$, and let a be its second dilatation function. Then f can be factorized in the form $f = F \circ \varphi$, for some analytic function φ and some univalent logharmonic mapping F if and only if*

(a) $|a(z)| \neq 1$ *in* D,

(b) $f(z_1) = f(z_2)$ *implies* $a(z_1) = a(z_2)$.

Under these conditions, the representation is unique up to a conformal mapping; any other representation $f = F_1 \circ \varphi_1$ has the form $F_1 = F \circ \psi^{-1}$ and $\varphi_1 = \psi \circ \varphi$ for some conformal mapping defined in $\varphi(D)$.

Consider now the logharmonic mapping $f(z) = ze^{1/z}\overline{e^{-1/z}}$. The point $z = 0$ is an isolated singularity of f, and f is continuous at this point. However, f does not admit a logharmonic-continuation to $\mathbb{C}$. A further restriction is needed.

Theorem 1.8 (see [2] (logharmonic-continuation across an isolated singularity)). *Let D_r be the point disc $D_r = \{z : 0 < |z| < r\}$, and let $f = h\overline{g}$ defined in D_r be a logharmonic mapping with respect to $a \in B(D)$ satisfying* $\lim_{z\to 0} f(z) = 0$. *Then f admits a logharmonic-continuation across the origin and has the representation*

$$f(z) = z^{n_0}\overline{z}^{m_0}h_0(z)\overline{g_0(z)}, \tag{1.7}$$

where n_0 *and* m_0 *are nonnegative integers,* $0 \leq m_0 < n_0$*, and* h_0 *and* g_0 *are analytic functions on* $|z| < r$ *satisfying* $h_0(0)g_0(0) \neq 0$.

Liouville's theorem does not hold for entire logharmonic functions. The function $f(z) = \exp(z)\exp(-\overline{z})$ is a nonconstant bounded logharmonic in $\mathbb{C}$. Its dilatation is $a(z) \equiv -1$. However, the following modified version of Liouville's theorem was given in [2].

Theorem 1.9 (modified Liouville's theorem). *Let* $f = h\overline{g}$ *be a bounded logharmonic function in* $\mathbb{C}$*. Then either the image* $f(\mathbb{C})$ *is a circle centered at the origin with dilatation function* $a(z) \equiv -1$ or f *is a constant.*

Let $f(z) = h(z)\overline{g(z)}$ be a logharmonic mapping defined in a domain D with respect to $a \in B(D)$ satisfying $|a(z)| \not\equiv 1$. Let

(1) $S_G(D) = \{z \in D : |a(z)| > 1\}$,

(2) $S_L(D) = \{z \in D : |a(z)| < 1\}$,

(3) $S_E(D) = \{z \in D : |a(z)| = 1\}$,

(4) $NZ(f - w, D)$ be the cardinality of $Z(f - w, D)$, that is, the number of zeros of $f - w$ in D, multiplicity is not counted,

(5) $VZ(f - w, G)$ be the number of zeros of $f - w$ in $S_G(D)$, multiplicity counted.

The following argument principle for logharmonic mappings in D is shown in [2].

Theorem 1.10 (generalized argument principle for logharmonic mappings). *Let D be a Jordan domain, and let* $f = h\overline{g}$ *be a logharmonic mapping defined in the closure* $\overline{D}$ *with respect to* $a \in B(D)$ *satisfying* $|a(z)| \not\equiv 1$*. Fix* $w \in \mathbb{C}$ *such that* $Z(f - w, \overline{D}) \cap (\partial D \cup S_E(D))$ *is empty. Then*

$$VZ(f - w, S_L(D)) - VZ(f - w, S_G(D)) = \frac{1}{2\pi}\oint_{\partial D} d\arg(f - w). \tag{1.8}$$

As a consequence of the argument principle, the following result is obtained.

Theorem 1.11. *Let* f_n *be a sequence of logharmonic mappings defined in* U *with respect to a given* $a_n \in B(U)$*, where* U *is the unit disc. Suppose that* a_n *converges locally uniformly to* $a \in B(U)$ *and that* f_n *converges locally uniformly to a logharmonic mapping* f *with respect to* a*. If* $w_0 \notin f_n(U)$ *for all* $n \in \mathbb{N}$*, then* $w_0 \notin f(U)$.

In Section 2, a survey is given on univalent logharmonic mappings defined in a simply connected domain D of $\mathbb{C}$. Section 3 deals with univalent logharmonic mappings defined on multiply connected domains, while Section 4 considers logharmonic polynomials. The final section of the survey discusses several important subclasses of logharmonic mappings.

2. Univalent Logharmonic Mappings in a Simply Connected Domain

2.1. Motivation

Let Ω be a domain in the complex plane $\mathbb{C}$, and let S be a nonparametric minimal surface lying over Ω. Then S can be represented by a function $s = G(u, v)$, $w = u + iv \in \Omega$, and there is a univalent orientation-preserving harmonic mapping $w = F(z)$ from an appropriate domain D of $\mathbb{C}$ onto Ω which determines S in the following sense. The mapping F is a solution of the system of linear elliptic partial differential equation

$$\overline{F_{\overline{z}}} = AF_z, \tag{2.1}$$

where $A \in H(D)$. Since F is orientation preserving, it follows that $|A(z)| < 1$ in D. The function A is the second dilatation of F. The value $(1 + |A(z)|)/(1 - |A(z)|)$ is the quotient of the maximum value and the minimum value of the differential $|dF(z)|$ when dz varies on the unit circle (see, e.g., [12, 13]). The representation of the minimal surface S is given by three real-valued harmonic functions (see, e.g., [13, 14]),

$$u(z) = \operatorname{Re}(F(z)), \qquad v(z) = \operatorname{Im}(F(z)), \qquad s(z) = \operatorname{Im}\int^{z} \sqrt{A}F_z\, dz. \tag{2.2}$$

Since $(s_z)^2 = -\overline{F_{\overline{z}}}F_z = -A(F_z)^2$ in D, it follows that $\sqrt{A}$ belongs to $H(D)$. In particular, each zero of A is of even order. Since the Riemannian metric of S is $ds^2 = |F_z|^2(1+|A|)^2|dz|^2$, it follows that $x = \operatorname{Re}(z)$ and $y = \operatorname{Im}(z)$ are isothermal parameters for S. Moreover, the exterior unit normal vector $\vec{n}(z) = (n_1(z), n_2(z), n_3(z))$, $n_3(z) \geq 0$, to the minimal surface S (known as the Gauss mapping) depends only on the second dilatation function A of F. More precisely,

$$\vec{n} = \left(2\operatorname{Im}\left(\sqrt{A}\right), 2\operatorname{Re}\left(\sqrt{A}\right), \frac{1-A}{1+A}\right). \tag{2.3}$$

The inverse of the stereographic projection of the Gauss mapping $\vec{n}$, $i/\sqrt{A(z)}$, is called the Weierstrass parameter.

The following question arises: What are the domains D? If φ is univalent and analytic and if F is univalent and harmonic, then the composition $F \circ \varphi$ (whenever well defined) is a univalent harmonic mapping but $\varphi \circ F$ need not be harmonic. Hence, if F represents a minimal surface over Ω (in the sense of relation (2.2)), then $F(\varphi)$ represents the same minimal surface but in other isothermal parameters.

Suppose that Ω is a proper simply connected domain in $\mathbb{C}$. Then, we may choose for D any proper simply connected domain in $\mathbb{C}$. In particular, $D = U$ or $D = \Omega$ are appropriate choices.

Consider now the left half-plane $D = \{z : \operatorname{Re}(z) < 0\}$, and let F be a univalent harmonic and orientation-preserving map defined in D satisfying the relation

$$F(z + \alpha i) = F(z) + \beta \quad \forall z \in D, \tag{2.4}$$

where α and β are real constants. Applying the transformation $(2\pi/\beta)F(2\alpha/2\pi)$, it may be assumed without loss of generality that $\alpha = \beta = 2\pi$, that is,

$$F(z + 2\pi i) = F(z) + 2\pi i \quad \forall z \in D. \tag{2.5}$$

Whenever $\lim_{x\to-\infty} \operatorname{Re}(F(z)) = c$ for some $c \in [-\infty, \infty)$, we will write $\operatorname{Re}(F(-\infty)) = c$. Similarly, $A(-\infty) = c$ means that $\lim_{x\to-\infty} A(z) = c$.

Let UHP denote the class of all univalent harmonic orientation-preserving mappings defined on the left half-plane $D = \{z : \mathrm{Re}(z) < 0\}$ satisfying

$$\begin{gathered} F(z+2\pi i) = F(z) + 2\pi i \quad \forall z \in D, \\ \mathrm{Re}(F(-\infty)) = -\infty. \end{gathered} \tag{2.6}$$

It follows that the second dilatation function A is periodic, that is, $A(z+2\pi i) = A(z)+2\pi i$ in D, and therefore the Gauss map is also periodic. Observe that $A(-\infty)$ exists. Furthermore, it was shown in [6] that mappings in the class UHP admit the representation

$$F(z) = z + \beta x + H(z) + \overline{G(z)}, \tag{2.7}$$

where

(a) H and G are in $H(D)$ such that

(i) $G(-\infty) = 0$ and $H(-\infty)$ exists and finite in $\mathbb{C}$,

(ii) $H(z+2\pi i) = H(z)$ and $G(z+2\pi i) = G(z)$ for all $z \in D$;

(b)

$$\begin{gathered} \left| \frac{G'(z)+\overline{\beta}}{1+\beta+H'(z)} \right| < 1 \quad \text{on } D, \\ \beta = \frac{\overline{A(-\infty)}(1-A(-\infty))}{1-|A(-\infty)|^2}, \text{ and hence } \mathrm{Re}(\beta) > -1. \end{gathered} \tag{2.8}$$

Define

$$f(z) = e^{F(\log(z))}, \qquad z \in U. \tag{2.9}$$

Then f is a univalent logharmonic mapping in U with respect to $a(z) = A(\log(z))$ and hence $a \in B(U)$. Observe that the family of all univalent logharmonic and orientation-preserving mappings f defined in U satisfying $f(0) = 0$ is isomorphic to the class UHP. It was shown in [4, 7] that it is easier to work with logharmonic mappings even if the differential equation becomes nonlinear.

2.2. Univalent Logharmonic Mappings

Let D be a simply connected domain in $\mathbb{C}$, $D \neq \mathbb{C}$, and suppose that f is a univalent logharmonic mapping defined in D. If $0 \notin f(D)$, then $\log(f(z))$ is a univalent and harmonic mapping in D. This mapping has been extensively studied in [15–18]. If $f(0) = 0$ and f is a univalent logharmonic mapping defined in D, then the representation (1.2) of f becomes

$$f(z) = z|z|^{2\beta} h(z)\overline{g(z)}, \tag{2.10}$$

for every $z \in U$, where

(a) $\beta = \overline{a(0)}(1 + a(0))/(1 - |a(0)|^2)$, and so $\operatorname{Re}(\beta) > -1/2$,

(b) h and g are in $H(U)$ satisfying $g(0) = 1$ and $0 \notin h \cdot g(U)$.

It follows that f is locally quasiconformal. The analogue of Caratheodory's Kernel Theorem might fail for univalent logharmonic mappings. Indeed, each function

$$f_r(z) = \frac{z}{(1-z)^2} \exp\left(-2r\left(\operatorname{Re}\int_0^z \frac{(1+z)}{(1+rz)(1-z)} dz\right)\right), \quad 0 < r < 1, \tag{2.11}$$

which is univalent and logharmonic with respect to $a_r(z) = -rz$, satisfies the normalization $f_r(0) = 0$, $(f_r)_z(0) = 1$, and maps the unit disc U onto the slit domain $\mathbb{C} \setminus (-\infty, -p_r)$. The tip p_r of the omitted slit varies monotonically from $-1/4$ to -1 as r varies from 0 to 1. The limit function $\lim_{r\to 1} f_r(z) = f_1(z) = (z(1-\overline{z}))/(1-z)$ is univalent and logharmonic and maps U onto U. It has the boundary value $f(e^{it}) = -1$ for $0 < |t| \le \pi$, and the cluster set of f_1 at the point 1 is the unit circle.

Let D be a simply connected domain in $\mathbb{C}$ and $z_0 \in D$. The following characterization theorem was proved in [1].

Theorem 2.1. *Let f be a univalent mapping defined in D such that $f(z_0) = 0$. Then f is of the form $h\overline{g}$ if and only if f is a logharmonic mapping with respect to $a \in B(D)$ satisfying $a(z_0) = m/(1+m)$, $m \in \mathbb{N} \cup \{0\}$.*

Univalent logharmonic mappings have the following properties.

Theorem 2.2 (see [1]). *Let D be a simply connected domain in $\mathbb{C}$ and f a univalent logharmonic mapping defined in D with respect to $a \in B(D)$.*

(a) *Then $f_z(z) \neq 0$ for all $z \in D$ whenever $f(z) \neq 0$.*

(b) *If $f(z_0) = 0$, then $\lim_{z\to z_0}(z - z_0)f_z(z)/f(z)$ exists and is in $\mathbb{C} \setminus \{0\}$. Therefore, $(z - z_0)f_z(z)/f(z)$ is a nonvanishing function in $H(D)$.*

(c) *Let α be a complex number such that $\operatorname{Re}(\alpha) > -1/2$. Then $F = f|f|^{2\alpha}$ is a univalent logharmonic mapping with respect to*

$$a^* = \frac{1+\overline{\alpha}}{1+\alpha}\frac{a + (\overline{\alpha}/(1+\overline{\alpha}))}{1 + a(\alpha/(1+\alpha))} \in B(D). \tag{2.12}$$

There are few logharmonic mappings that are univalent on the whole complex plane $\mathbb{C}$. Indeed, Abdulhadi and Bshouty [1] showed the following.

Theorem 2.3. *A function f is a univalent logharmonic mapping defined in $\mathbb{C}$ with respect to $a \in U$ if and only if*

$$f(z) = \text{const} \cdot (z - z_0)|z - z_0|^{2\beta}, \quad \beta = \frac{\overline{a}(1+a)}{\left(1 - |a|^2\right)}, \; z_0 \in \mathbb{C}. \tag{2.13}$$

Now let D be a simply connected proper domain in $\mathbb{C}$ and f a univalent logharmonic function in D with respect to $a \in B(D)$. Denote by φ a conformal mapping from the unit disc U onto D. Then $f \circ \varphi$ is univalent logharmonic in U with respect to $a^* = a \circ \varphi \in B(U)$. Therefore, we may assume that $D = U$ and $f(0) = 0$.

Analogous to the analytic case, we denote

$$S_{Lh} = \Big\{ f(z) = z|z|^{2\beta} h\overline{g} : f \text{ is a univalent logharmonic mapping defined in } U \\ \text{with } h(0) = g(0) = 1 \Big\}. \tag{2.14}$$

Now $1^{2\beta} = 1$, and S_{Lh} is not compact with respect to the topology of normal convergence. Indeed, the sequence $f_n(z) = z|z|^{(1-n)/n}$ is in S_{Lh}, and it converges uniformly to $f(z) = z|z|^{-1}$ not in S_{Lh}. Our next result deals with the subclass S_{Lh}^0 of S_{Lh} defined by $S_{Lh}^0 = \{f \in S_{Lh} : a(0) = 0 \text{ (resp., } \beta = 0)\}$. The following result was proved in [1].

Theorem 2.4. *S_{Lh}^0 is compact in the topology of normal (locally uniform) convergence.*

Remark 2.5. In contrast to univalent harmonic mappings, S_{Lh} is not a normal family. Indeed,

$$f_n(z) = \frac{z}{(1-z)^2} \left| \frac{z}{(1-z)^2} \right|^{2n} \tag{2.15}$$

is not locally uniformly bounded for n sufficiently large.

The following interesting distortion theorem is due to Abdulhadi and Bshouty [1], and it was used in the proof of the mapping theorem.

Theorem 2.6. *If $f \in S_{Lh}^0$, then $|f(z)| \geq |z|/4(1+|z|)^2$. In particular, the disc $\{w : |w| < 1/16\}$ is in $f(U)$.*

2.3. Mapping Theorem

We look for an analogue of the Riemann Mapping Theorem. Let $\Omega \neq \mathbb{C}$ be a simply connected domain in $\mathbb{C}$, and let $a \in B(U)$ be given. Fix $z_0 \in U$ and $w_0 \in \Omega$. We are interested in the existence of a univalent logharmonic function f from U into Ω with respect to the given function a and normalized by $f(z_0) = w_0$ and $f_z(z_0) > 0$. If $|a| \leq k < 1$ for all $z \in U$, then the univalent logharmonic mappings are quasiconformal, and therefore the problem is solvable.

Suppose that we want to find a univalent logharmonic mapping f with $a(z) = -z$, normalized by $f(0) = 0$ and $f_z(0) > 0$ such that f maps U onto $\Omega = \mathbb{C} \setminus (-\infty, -1]$. Assume that such a function exists. Then, using Theorem 5.1 ($\alpha = 0$), it follows that f must be of the form

$$f = \text{const} \cdot \frac{z(1-\overline{z})}{(1-z)}. \tag{2.16}$$

Observe that f is univalent in U, but maps U onto a disc, and not onto a slit domain. In other words, there is no univalent logharmonic mapping defined in U with respect to $a(z) = -z$

satisfying $f(0) = 0$, $f_z(0) > 0$, and $f(U) = \Omega$. However, the following mapping theorem was proved in [1].

Theorem 2.7. *Let Ω be a bounded simply connected domain in $\mathbb{C}$ containing the origin, and whose boundary is locally connected. Let $a \in B(U)$ be given. Then there is a univalent logharmonic function defined in U with the following properties.*

(i) *f is a solution of* (1.1).

(ii) *$f(U) \subset \Omega$, normalized at the origin by $f(z) = cz|z|^{2\beta}(1 + o(1))$, where $\beta = \overline{a(0)}(1 + a(0))/(1 - |a(0)|^2)$ and $c > 0$.*

(iii) *$\lim_{z\to e^{it}} f(z) = \widehat{f}(e^{it})$ exists and is in $\partial\Omega$ for each $t \in \partial U \setminus E$, where E is a countable set.*

(iv) *For each $e^{it_0} \in \partial U$, $f_*(e^{it_0}) = \operatorname{ess\,lim}_{t\uparrow t_0} \widehat{f}(e^{it})$ and $f^*(e^{it_0}) = \operatorname{ess\,lim}_{t\downarrow t_0} \widehat{f}(e^{it})$ exist and are in $\partial\Omega$.*

(v) *For $e^{it_0} \in E$, the cluster set of f at e^{it_0} lies on a helix joining the point $f^*(e^{it_0})$ to the point $f_*(e^{it_0})$.*

Remark 2.8. In the case where $\|a\| = \sup_{z\in U}|a(z)| < 1$, properties (ii) and (iii) imply that $f(U) = \Omega$.

Remark 2.9. If $e^{it_0} \in E$ and $f_*(e^{it_0}) = f^*(e^{it_0})$, then the cluster set at e^{it_0} is a circle. Suppose that $A = f_*(e^{it_0}) \neq f^*(e^{it_0}) = B$, then there are infinitely many helices joining A and B. But the cluster set of f at e^{it_0} lies on one of them. For example, the cluster set of

$$f(z) = z\frac{(1-\overline{z})}{(1-z)} \exp\left(-2\arg\frac{1-iz}{1-z}\right) \tag{2.17}$$

at $z = 1$ lies on the helix, $\gamma(\tau) = \exp[-\tau + i(\pi/2 + \tau)]$ joining the points $f^*(1) = -e^{-\pi/2}$ and $f_*(1) = -e^{3\pi/2}$, where the cluster set of f at $z = -i$ is the straight line segment from $f^*(-i) = -e^{-\pi/2}$ and $f_*(-i) = -e^{3\pi/2}$.

The uniqueness of the mapping theorem was proved in [6] for the special case Ω is a strictly starlike and bounded domain; that is, every ray starting at the origin intersects $\partial\Omega$ at exactly one point.

Theorem 2.10 (uniqueness in the mapping theorem). *Let $a \in B(U)$ be given such that $\|a\| = \sup_{z\in U}|a(z)| < 1$. Let Ω be a strictly starlike and bounded domain. Then there exists a unique univalent logharmonic function $f(z) = z|z|^{2\beta}h(z)\overline{g(z)}$ with respect to a such that $f(U) = \Omega$ and $h(0) > 0$.*

2.4. Logharmonic Automorphisms

We consider univalent logharmonic mappings from U onto U. With no loss of generality, it is assumed that $f(0) = 0$ and $h(0) > 0$. Otherwise, we consider an appropriate Möbius transformation of the preimage. Let $\mathrm{AUT}_{Lh}(U)$ denote the class of such mappings. The following two theorems established in [8] characterize completely mappings in $\mathrm{AUT}_{Lh}(U)$.

Theorem 2.11. *Let h and g be two nonvanishing analytic functions in U. Then $f(z) = z|z|^{2\beta}h(z)\overline{g(z)}$ is in $AUT_{Lh}(U)$ satisfying $h(0) > 0$ and $g(0) = 1$ if and only if $g = 1/h$, $\text{Re}(\beta) > -1/2$, and $\text{Re}(zh'(z))/h(z) > -1/2$ in U.*

We now associate to each $f(z) = z|z|^{2\beta}h(z)/\overline{h(z)}$ in $\text{AUT}_{Lh}(U)$ with the mapping $\varphi(z) = z(h(z))^2 \in S^*$.

Theorem 2.12. (a) *For each $\varphi \in S^*$ and for each β, $\text{Re}(\beta) > -1/2$, there is one and only one $f \in AUT_{Lh}(U)$ such that $f(z)/(\varphi(z)|z|^{2\beta}) > 0$ for every $z \in U$ and $h(0) = 1$.*

(b) *For each $a \in B(U)$, there is a unique solution of* (1.1) *which is in $AUT_{Lh}(U)$.*

Remark 2.13. Part (a) of Theorem 2.12 is quite surprising. Indeed, consider $\varphi(z) = z/(1-z)^2$ and $\beta = 0$. Then $\arg(f(e^{it})) = \arg(\varphi(e^{it})) = \pm\pi$, almost everywhere; however, $f(U) = U$. To be more precise, the corresponding mapping is $f(z) = z(1-\overline{z})/(1-z)$ satisfying $f(e^{it}) = -1$ for all $0 < |t| \leq \pi$, where the cluster set of f at the point 1 is the unit circle.

2.5. Univalent Logharmonic Mappings Extensions onto the Unit Disc

In 1926 Kneser [19] obtained the following result.

Theorem 2.14. *Let Ω be a bounded simply connected Jordan domain, and let f^* be an orientation-preserving homeomorphism from the unit disc circle ∂U onto $\partial\Omega$. Then, if $f(U) = \Omega$, the solution of the Dirichlet problem* (*the Poisson integral*) *is univalent on the unit disc U.*

Since $f(U)$ always contains Ω and lies in the convex hull of Ω, Kneser used Theorem 2.14 to obtain the following solution to a problem posed by Rado in [20].

Theorem 2.15. *Let f^* be a homeomorphism from ∂U onto $\partial\Omega$, where Ω is a bounded convex domain. Then the Dirichlet solution f is univalent on U.*

In 1945, Choquet [21] independently gave another proof of Theorem 2.15, and he pointed out that it holds whenever Ω is not a convex domain.

We will use the following definition.

Definition 2.16. Let D be the unit disc U or the annulus $A(r,1)$, $r \in (0,1)$, and suppose that f^* is a continuous function defined on ∂D. One says that f is a logharmonic solution of the Dirichlet problem if

(a) f is a solution of the form (1.1),

(b) f is continuous in D,

(c) $f|_{\partial D} \equiv f^*$.

The next two theorems proved in [6] deal with the solutions of the Dirichlet problem for logharmonic mappings of the form (2.10).

Theorem 2.17. *Let f^* be a nonvanishing continuous complex-valued function defined on ∂U. Then there exist h and g analytic in U which are independent of β, such that*

$$f(z) = z|z|^{2\beta}h(z)\overline{g(z)}, \quad \text{Re}(\beta) > -\frac{1}{2} \tag{2.18}$$

is a logharmonic solution of the Dirichlet problem (i.e., $f(e^{it}) \equiv f^*(e^{it})$). Furthermore, if $g(0) = 1$, then h and g are uniquely determined.

Theorem 2.18. *Let f^* be an orientation-preserving homeomorphism from ∂U onto ∂U, that is, $f(e^{it}) = e^{i\lambda(t)}$, where λ is continuous and strictly monotonically increasing on $[0, 2\pi)$. Furthermore, suppose that $\lambda(2\pi) = \lambda(0) + 2\pi$. Then, for a given β with $\mathrm{Re}(\beta) > -1/2$, the logharmonic solution of the Dirichlet problem which is of the form $f(z) = z|z|^{2\beta}h(z)/\overline{h(z)}$ is univalent in U.*

2.6. Boundary Behavior

Let f be a univalent logharmonic mapping in the unit disc U with respect to $a \in B(U)$. If $|a(z)| \leq k < 1$ for all $z \in U$, then f is a quasiconformal map, and its boundary behavior is the same as for conformal mappings. However, if $|a|$ approaches one as z tends to the boundary, then the boundary behavior of f is quite different. It may happen that the boundary values are constant on an interval of ∂U, or that there are jumps as the following example shows.

Example 2.19. The function $f(z) = z(1-\overline{z})/(1-z)$ is a univalent logharmonic mapping in the unit disc U with respect to $a(z) = -z$, such that $f(U) = U$. It follows that $f(e^{it}) = -1$ for all $0 < |t| \leq \pi$ and that the cluster set of f at the point 1 is the unit circle.

The following theorem was stated in [1].

Theorem 2.20. *Let Ω be a simply connected domain of $\mathbb{C}$ whose boundary $\partial\Omega$ is locally connected, and $a \in B(U)$. Let f be a univalent logharmonic mapping from U onto Ω satisfying $f(0) = 0$. Then the nonrestricted limit $f^*(e^{it})$ of f at e^{it} exists on $\partial U \setminus E$, where E is a countable set. If $e^{it} \in E$, then f^* jumps at e^{it}, and the cluster set at e^{it} is a subinterval of a logarithmic spiral.*

The next theorem [22] shows that the boundary values of f depend strongly on the values of $a(e^{it})$.

Theorem 2.21. *Let Ω be a simply connected domain of $\mathbb{C}$ whose boundary $\partial\Omega$ is locally connected and $a \in B(U)$. Suppose that the function a has an analytic extension across an open subinterval $I = \{e^{it} : \sigma < t < \sigma + 2\pi\}$ of the unit circle ∂U, such that $|a(z)| \equiv 1$ in I. Let f be a univalent logharmonic mapping with respect to a which maps U onto Ω and satisfies $f(0) = 0$. Then the following relations hold in I.*

(a) *Let $\sigma < t < \sigma + 2\pi$ and $\arg(f(z))$ be a continuous function on the set $\Upsilon := |z : 1/2 < |z| < 1,\ \sigma < \arg(z) < \tau\}$. If $\sigma < t < t + h < \tau$, then*

$$\begin{aligned}\log\left(f^*\left(e^{i(t+h)}\right)\right) - \overline{a(e^{i(t+h)})\log(f^*(e^{i(t+h)}))} - \log\left(f^*\left(e^{it}\right)\right)\\ + \overline{a(e^{it})\log(f^*(e^{it}))} + \overline{\int_t^{t+h} \log(f^*(e^{i\phi}))da(e^{i\phi})} \equiv 0.\end{aligned} \tag{2.19}$$

(b) *If f^* is continuous at e^{it}, then*

$$\lim_{t\downarrow 0} \mathrm{Im}\sqrt{a(e^{it})}\frac{f^*(e^{i(t+h)})/f^*(e^{i(t-h)}) - 1}{h} = 0. \tag{2.20}$$

(c) *If f^* jumps at e^{it}, which must and can happen only when $f^*(I)$ lies on a segment of a logarithmic spiral, for $q \in f^*(I)$, then*

$$\arg\left(\log \frac{f^*\left(e^{i(t+0)}\right)}{q}\right) = -\frac{1}{2}\arg\left(a\left(e^{it}\right)\right) \bmod \pi. \tag{2.21}$$

(d) *If f^* is not constant on a subinterval of I, then the right limit*

$$\lim_{t\downarrow 0}\arg\left(\frac{f^*\left(e^{i(t+h)}\right)}{f^*\left(e^{i(t-h)}\right)} - 1\right) = -\frac{1}{2}\arg\left(a\left(e^{it}\right)\right) \bmod \pi \tag{2.22}$$

exists everywhere on I.

2.7. A Constructive Method

In this section, a method is introduced for constructing univalent logharmonic mappings from the unit disc onto a strictly starlike domain Ω, which has been successfully applied to conformal mappings (see, e.g., [23–25]), as well as for univalent harmonic mappings (see, e.g., [26, 27]).

Let Ω be a strictly starlike domain of $\mathbb{C}$. Then $\partial\Omega$ can be expressed in the parametric form

$$w(t) = R(t)e^{it}, \quad 0 \le t \le 2\pi, \tag{2.23}$$

where R is a positive continuous function on $[0, 2\pi]$. The following notations will be used:

$$\begin{gathered} \|f\|_\infty = \sup\{|f(z)|; z \in U\}, \\ \|\Omega\|_\infty = \sup\{|w|; w \in \Omega\}, \\ d(\partial\Omega) = \text{distance from the origin to } \partial\Omega. \end{gathered} \tag{2.24}$$

For all $w \in \mathbb{C}$, define

$$\lambda_\Omega(w) = \begin{cases} \dfrac{|w|}{R(t)}, & 0 \ne w = |w|e^{it}, \\ 0, & w = 0. \end{cases} \tag{2.25}$$

Then

$$\begin{gathered} \lambda_\Omega(w) < 1 \Longleftrightarrow w \in \Omega, \\ \lambda_\Omega(w) = 1 \Longleftrightarrow w \in \partial\Omega, \\ \lambda_\Omega(w) > 1 \Longleftrightarrow w \in \mathbb{C} \setminus \overline{\Omega}, \\ \lambda_\Omega(w) = 0 \Longleftrightarrow w = 0. \end{gathered} \tag{2.26}$$

For any complex-valued function f in U, define

$$\mu_\Omega(f) = \sup\{\lambda_\Omega(w) : w \in f(U)\}. \tag{2.27}$$

The following properties are due to Bshouty et al. [26].

Lemma 2.22. (a) $\mu_\Omega(f) \le 1 \Leftrightarrow f(U) \subset \Omega$,
(b) $\mu_\Omega(tf) = t\mu_\Omega(f)$ *for all* $t \ge 0$,
(c) $\mu_\Omega(f) \le \|f\|_\infty / d(\partial\Omega)$,
(d) $\|f\|_\infty \le \mu_\Omega(f)\|\Omega\|_\infty$,
(e) $\mu_\Omega(f_1 + f_2) \le (\mu_\Omega(f_1) + \mu_\Omega(f_2))(\|f\|_\infty / d(\partial\Omega))$.

The next lemma shows that μ_Ω is lower semicontinuous with respect to the point-wise convergence; this was proved in [28].

Lemma 2.23. *Let Ω be a strictly starlike domain of $\mathbb{C}$, and let f_n be a sequence of mappings from U into $\mathbb{C}$ which converges pointwise to f. Then $\lim_{n\to\infty} \inf(\mu_\Omega(f_n)) \ge \mu_\Omega(f)$. Strict inequality can hold even in the case of locally uniform convergence.*

Let Ω be a fixed strictly starlike domain of $\mathbb{C}$, and let $a \in H(U)$, $a(0) = 0$, $|a| \le k < 1$ be a given (second) dilatation function. Denote by N the set of all logharmonic mappings $f(z) = zh(z)\overline{g(z)}$ with respect to the given dilatation function which are normalized by $g(0) = h(0) = 1$. Observe that $\beta = 0$ since it is assumed that $a(0) = 0$. Hengartner and Nadeau [27] solved the following optimization problem.

Theorem 2.24. *Let Ω be a strictly starlike domain of $\mathbb{C}$, and let $a \in H(U)$, $a(0) = 0$, $|a| \le k < 1$ be given. Denote by $F(z) = zH(z)\overline{G(z)}$ the univalent logharmonic mapping satisfying $F(U) = \Omega$, $G(0) = 1$, and $H(0) > 0$. Then there exists a unique $f^* \in N$ such that $\mu_\Omega(f^*) \le \mu_\Omega(f)$ for all $f \in N$ and $f^* = F/H(0)$.*

Theorem 2.24 allows us to solve the following mathematical program:

$$\min M, \qquad \lambda(f(z)) \le M \quad \forall z \in U,\ \forall f \in N. \tag{2.28}$$

For $f \in N$, $f(z) = zh(z)\overline{g(z)}$, where $h(z) = \exp(\sum_{k=1}^{\infty} a_k z^k)$ and

$$g(z) = \exp\left(\int_0^z \frac{a(s)}{s} ds + \sum_{k=1}^{\infty} k a_k \int_0^z a(s) s^{k-1} ds\right). \tag{2.29}$$

Furthermore, each $f \in N$ is an open mapping. Denote by V_n the set of all mappings $f \in N$ of the form

$$f(z) = z \exp\left(\overline{\int_0^z \frac{a(s)}{s} ds} + \sum_{k=1}^{\infty}\left[a_k z^k + k\overline{a_k}\overline{\int_0^z a(s) s^{k-1} ds}\right]\right) \tag{2.30}$$

and by f_n^* any solution of the optimization problem

$$\min\{\mu_\Omega(f); f \in V_n\}. \tag{2.31}$$

As a consequence of Theorem 2.24, Hengartner and Rostand [28] obtained the following result.

Theorem 2.25. *Let a be a polynomial such that $\|a\|_\infty \le k < 1$ in U. Then the sequence f_n^* of solutions of*

$$\min \mu_\Omega(f), \qquad f \in V_n, \tag{2.32}$$

converges locally uniformly to the univalent solution f^* of

$$\min \mu_\Omega(f), \qquad f \in N. \tag{2.33}$$

The question remains how big could n be. It follows from Theorem 5.24 that $|a_n| \le 2 + n^{-1}$ and $|b_n| \le 2 - n^{-1}$. Suppose that Ω is a Jordan domain whose boundary $\partial\Omega$ is rectifiable and piecewise smooth. Hengartner and Nadeau [27] obtained the following additional estimate for the coefficients.

Theorem 2.26. *Let*

$$F(z) = z|z|^{2\beta} \exp\left(\sum_{k=1}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k}\right) \tag{2.34}$$

be a univalent logharmonic mapping from U onto Ω, and let L be the length of $F(|z| = r)$, $0 < r < 1$. Then

$$\begin{gathered} |a_n| \le \frac{\lim_{r\to 1}\inf L(r)}{2\pi d(\partial\Omega) n}, \\ |b_n| \le \frac{\lim_{r\to 1}\inf L(r)}{2\pi d(\partial\Omega) n}. \end{gathered} \tag{2.35}$$

Equality holds for the case $\Omega = U$ and $f(z) = z(1-\overline{z})/(1-z)$.

3. Univalent Logharmonic Mappings on Multiply Connected Domains

3.1. Univalent Logharmonic Exterior Mappings

This section considers univalent logharmonic and orientation-preserving mappings f defined on the exterior of the unit disc U, $\Delta = \{z : |z| > 1\}$, satisfying $f(\infty) = \infty$. These mappings are called univalent logharmonic exterior mappings. If f does not vanish on Δ, then $\Psi(z) = 1/f(1/z)$ is a univalent logharmonic mapping defined in U normalized by $\Psi(0) = 0$. Moreover, $F(\zeta) = \log f(e^\zeta)$ is a univalent harmonic mapping defined on the right half-plane $\{\zeta : \operatorname{Re}(\zeta) > 0\}$ satisfying the relation $F(\zeta + 2\pi i) = F(\zeta) + 2\pi i$ and F is a solution of the linear elliptic partial differential equation

$$\overline{F_{\overline{\zeta}}} = AF_{\zeta}, \tag{3.1}$$

where the second dilatation function $A(\zeta) = a(e^{\zeta})$, $a \in B(\Delta)$, satisfies $A(\zeta + 2\pi i) = A(\zeta)$ on $\{\zeta : \operatorname{Re}(\zeta) > 0\}$. Such mappings were studied in [9, 29–32]. Several authors have also studied harmonic mappings between Riemannian manifolds, and an excellent survey has been given in [33–37].

The next result proved in [4] is a global representation of univalent logharmonic exterior mappings.

Theorem 3.1. *Let f be a univalent logharmonic mapping defined on the exterior Δ of the closed unit disc $\overline{U}$ such that $f(\infty) = \infty$. Suppose that $f(p) = 0$ for some $p \in \Delta$, or if f does not vanish, let $p = 1$. Then there are two complex numbers β and γ, $\operatorname{Re}(\beta) > -1/2$, $\operatorname{Re}(\gamma) > -1/2$, and two nonvanishing analytic functions h and g on $\Delta \cup \{\infty\}$ such that $g(\infty) = 1$, and f is of the form*

$$f(z) = z|z|^{2\beta}\left(\frac{z-p}{1-\overline{p}z}\right)\left|\frac{z-p}{1-\overline{p}z}\right|^{2\gamma} h(z)\overline{g(z)} \tag{3.2}$$

for all $z \in \Delta$.

Remark 3.2. Observe that not each function of the form (3.2) is univalent. Indeed, the function

$$f(z) = \overline{z}|z|^2 \frac{z-4}{1-4\overline{z}} \tag{3.3}$$

is not a univalent logharmonic mapping on Δ, but it can be written in the form (3.2) by putting $\beta = 1$, $\gamma = 0$, $p = 4$, $h(z) = 1/g(z) = (4z-1)/(4z)$.

Let f be a univalent logharmonic exterior mapping defined on the exterior Δ of the closed unit disc $\overline{U}$ such that $f(\infty) = \infty$. Applying an appropriate rotation to the preimage, we may assume that $p \geq 1$.

Definition 3.3. The class $\sum_{Lh}$ consists of all univalent logharmonic mappings defined on Δ which are of the form (3.2), where $p \geq 1$, $\operatorname{Re}(\beta) > -1/2$, $\operatorname{Re}(\gamma) > -1/2$, and h and g are analytic nonvanishing functions on $\Delta \cup \{\infty\}$, normalized by $g(\infty) = 1$ and $|h(\infty)| = 1$.

Let f be a univalent logharmonic mapping in Δ with $f(\infty) = \infty$. Then there is a real number α and a positive constant A such that $Af(e^{-\alpha}z)$ belongs to $\sum_{Lh}$. If f does not vanish on Δ, then the set of omitted values is a continuum. In other words, there is no univalent logharmonic mapping f defined on Δ satisfying $f(\infty) = \infty$ and $f(\Delta) = \mathbb{C} \setminus \{0\}$. Note that 0 is an exceptional point, since, for each $w_0 \in \mathbb{C} \setminus \{0\}$, there are univalent logharmonic mappings f such that $f(\Delta) = \mathbb{C} \setminus \{w_0\}$. Assume that $p > 1$, let $f \in \sum_{Lh}$, and let w_0 be an omitted value of f. Applying a rotation to the image $f(\Delta)$, we may assume that $w_0 = 1$, and we restrict ourselves to the subclass $\sum_{Lh}^{*} = \{f \in \sum_{Lh},\ p > 1,\ w_0 = 1 \notin f(\Delta)\}$.

In the next theorem, Abdulhadi and Hengartner [4] gave a complete characterization of all mappings in the class $\sum_{Lh}^{*}$.

Theorem 3.4. *A mapping f belongs to $\sum_{Lh}^{*}$ and $f(\Delta) = \mathbb{C} \setminus \{1\}$ if and only if f is of the form*

$$f(z) = \overline{z}|z|^{2\beta}\left(\frac{z-p}{1-p\overline{z}}\right)\left|\frac{z-p}{1-pz}\right|^{2\gamma}, \quad p > 1, \tag{3.4}$$

where β and γ satisfy the inequality

$$\left|\frac{\beta(1+\overline{\gamma}) - \gamma\left(1+\overline{\beta}\right)}{1+\gamma+\overline{\gamma}} - \frac{1}{p^2-1}\right| \leq \frac{p}{p^2-1}. \tag{3.5}$$

3.2. Univalent Logharmonic Ring Mappings

In this section we investigate the family A_r of all univalent logharmonic mappings f which map an annulus $A(r,1) = \{z : r < |z| < 1\}$, $0 < r < 1$, onto an annulus $A(R,1)$ for some $R \in [0,1)$ satisfying the condition

$$\frac{1}{2\pi}\int_{|z|=\rho} d\arg f\left(\rho e^{it}\right) = 1 \tag{3.6}$$

for all $\rho \in (r,1)$. The last condition says that the outer boundary corresponds to the outer boundary. We call an element $f \in A_r$ a univalent logharmonic ring mapping.

If $a \equiv 0$, then $R = r$ and $f(z) = e^{i\alpha}z$, $\alpha \in R$, are the only mappings in A_r. In the case of univalent harmonic mappings from $A(r,1)$ onto $A(R,1)$, it may be possible that $R = 0$; for example, $f(z) = (1-r^2)^{-1}(z-(r^2/\overline{z}))$ has this property. However, Nitsche [38] has shown that there is an $R_0(r) < 1$ such that there is no univalent harmonic mapping from $A(r,1)$ onto $A(R,1)$ whenever $R_0 < R < 1$.

There is no univalent logharmonic mappings from $A(r,1)$, $0 < r < 1$, onto $A(0,1)$. This is a direct consequence of Theorem 3.5. But, on the other hand, for R there is neither a positive lower bound nor a uniform upper bound strictly less than one. Indeed, $f(z) = z|z|^{2\beta}$, $\mathrm{Re}(\beta) > -1/2$, is univalent on $A(r,1)$, and its image is $A(r^{1+2\,\mathrm{Re}(\beta)},1)$.

Unlike the case of univalent harmonic mappings, univalent logharmonic mappings need not have a continuous extension onto the closure of $A(r,1)$. Indeed, $f(z) = z(\overline{z}-1)/(z-1)$ is a univalent logharmonic ring mapping from $A(1/2,1)$ onto itself whose cluster sets on the outer boundary are $C(f,e^{it}) = \{-1\}$, if $z = e^{it}$, $0 < t < 2\pi$, and $C(f,1) = \{w : |w| = 1\}$.

Let $S^*(r,1)$ be the set of all univalent analytic functions φ on $A(r,1)$ with the properties

(i) $p(z) = z\varphi'(z)/\varphi(z) \in H(A(r,1))$,

(ii) $\mathrm{Re}(p(z)) > 0$ on $A(r,1)$.

Theorems 3.5 and 3.6 [5] give a complete characterization of univalent logharmonic mappings in A_r.

Theorem 3.5. *A function f belongs to A_r if and only if*

$$f(z) = z|z|^{2\beta}\frac{h(z)}{\overline{h(z)}}, \tag{3.7}$$

where

(a) $h \in H(A(r,1))$ *and* $0 \notin h(A(r,1))$,

(b) $\mathrm{Re}(zh'(z)/h(z)) > -1/2$ *on* $A(r,1)$,

(c) $(1/2\pi)\int_{|z|=\rho} d\arg f(\rho e^{it}) = 0$, $r < \rho < 1$,

(d) $\mathrm{Re}(\beta) > -1/2$.

In particular, functions belonging to A_r map concentric circles onto concentric circles.

Theorem 3.6. *A function f is in A_r if and only if it is of the form*

$$f(z) = \left(\frac{\varphi(z)}{|\varphi(z)|}|z|^{2\gamma}\right), \tag{3.8}$$

where $\mathrm{Re}(\varphi) > 0$ *and* $\varphi \in S^*(r,1)$.

Next we fix the second dilatation function $a \in H(A(r,1))$, $|a(z)| < 1$ for all $z \in A(r,1)$. The following existence and uniqueness theorem was established in [5].

Theorem 3.7. *For a given $a \in H(A(r,1))$, $|a(z)| < 1$ for all $z \in A(r,1)$, and, for a given $z_0 \in A(r,1)$, there exists one and only one univalent solution f of* (1.1) *in A_r such that $f(z_0) > 0$.*

Remark 3.8. Theorem 3.7 is not true for univalent harmonic ring mappings (see [32, Theorem 7.3].)

3.3. *Univalent Logharmonic Mappings Extensions onto the Annulus*

The next two theorems proved in [6] deal with the solution of the Dirichlet problem for ring domains.

Theorem 3.9. *Let f^* be a nonvanishing continuous function defined on the boundary $\partial A(r,1)$ of the annulus $A(r,1)$. Then there exists, for each β, $\mathrm{Re}(\beta) > -1/2$, a unique mapping f of the form* (2.10)*, which is continuous on the closure of $A(r,1)$ and satisfies $f = f^*$ on $\partial A(r,1)$.*

Theorem 3.10. *Let $f^*(e^{it}) = e^{i\lambda(t)}$ and $f^*(re^{it}) = \mathrm{Re}^{i\mu(t)}$, $0 < R < 1$, be a given continuous function on $\partial A(r,1)$, $0 < r < 1$, satisfying*

(a) $d\lambda(t) \geq 0$ *and* $d\mu(t) \geq 0$ *on* $[0, 2\pi]$,

(b) $\int_0^{2\pi} d\lambda(t) = \int_0^{2\pi} d\mu(t) = 2\pi$.

Then the logharmonic solution of the Dirichlet problem with respect to $f^*$and $A(r,1)$ is a univalent mapping from $A(r,1)$ onto $A(R,1)$.

4. Logharmonic Polynomials

Denote by p_n an analytic polynomial of degree n. A logharmonic polynomial is a function of the form $f = p_n\overline{p_m}$. In contrast to the analytic case, there are nonconstant logharmonic polynomials which are not p-valent for every $p > 0$. For example, the function $f(z) = z\overline{z}$

is a logharmonic polynomial in $\mathbb{C}$ with respect to $a = -1$. Moreover, the function $f(z) = (z-1)(\overline{z}+1)$ is a logharmonic polynomial in $\mathbb{C}$ with respect to $a(z) = (z+1)/(z-1)$. This polynomial is two-valent and omits the half-plane $\operatorname{Re}(w) < -1$. On the other hand, they inherit the property $\lim_{z\to\infty} f(z) = \infty$ of analytic polynomials. This follows from the fact that $|f| = |p_n\overline{p_m}| = |p_n p_m|$. However, the converse is not true; there are logharmonic functions $f = h\overline{g}$ defined in $\mathbb{C}$ which are not logharmonic polynomials and have the property $\lim_{z\to\infty} f(z) = \infty$. The function $f(z) = ze^{z}e^{-\overline{z}}$ is such an example. Note that there are harmonic polynomials $p_n(z) + \overline{p_m(z)}$ which do not satisfy $\lim_{z\to\infty} f(z) = \infty$. However, if it is assumed that $a(\infty)$ exists and $|a(\infty)| \neq 1$, then the following result [2] is deduced.

Theorem 4.1. *Let $f = h\overline{g}$ be a logharmonic function in $\mathbb{C}$ such that $\lim_{z\to\infty} f(z) = \infty$. If $\lim_{z\to\infty} a(z) = a(\infty)$ exists and if $|a(\infty)| \neq 1$, then f is a polynomial.*

Denote by $NZ(f-w, D)$ the cardinality of $Z(f-w, D)$, that is, the number of zeros of $f-w$ in D, multiplicity not counted. The polynomial $f(z) = |z|^2$ has the property that $NZ(f-1, \mathbb{C}) = \infty$. On the other hand, using Theorem 2.3, it follows that a univalent logharmonic mapping in $\mathbb{C}$ is necessarily a polynomial which is either of the form $f(z) = \text{const} \cdot (z-a)\overline{(z-a)}^{m}$ or of its conjugate, where const $\neq 0$, $a \in \mathbb{C}$, and $m = 0, 1, 2, \ldots$. There are functions of the form $f = h\overline{g}$ which are not polynomials but have the property that $NZ(f-w, \mathbb{C})$ is finite and uniformly bounded for all $w \in \mathbb{C}$. For example, the function $f(z) = ze^{z}e^{\overline{z}} - w$ has at most two zeros for all fixed $w \in \mathbb{C}$. The following result was shown in [2].

Theorem 4.2. *Let $f = h\overline{g}$ be a logharmonic function in $\mathbb{C}$ such that $NZ(f-w, G)$ is finite for at least two different values of w, $\lim_{z\to\infty} a(z) = a(\infty)$ exists with $|a(\infty)| \neq 1$, then f is a polynomial.*

An upper bound on the number of w-points of a logharmonic polynomial can be readily obtained by using Bezout's theorem [39].

Theorem 4.3 (see [39]). *Let $p(x,y)$ and $q(x,y)$ be polynomials in the real variables x and y with real coefficients. If $\deg(p) = n$ and $\deg(q) = m$, then either p and q have at most nm common zeros or they have infinitely many zeros.*

Wilmshurst [40] has shown that Bezout's theorem gives a sharp upper bound for the number of zeros of a harmonic polynomial and hence for polyanalytic polynomials (see, e.g., [41, 42]). However, this is not true for logharmonic polynomials.

Let $f = p_n\overline{p_m}$ be a logharmonic polynomial of degree $n+m$. Then $f(z) - w = \sum_{k=0}^{n}\sum_{j=0}^{m} a_{kj} z^{k}\overline{z}^{j}$. The functions $P(z) = \operatorname{Re}(f(z))$ and $Q(z) = \operatorname{Im}(f(z))$ are real-valued polynomials in x and y and are of degree $n+m$. Applying Bezout's theorem, we conclude with the following estimate.

Theorem 4.4. *Let $f = p_n\overline{p_m}$ be a logharmonic polynomial defined in $\mathbb{C}$. Then either $f-w$ has infinitely many zeros or $f-w$ has at most $(n+m)^2$ zeros for all $w \in \mathbb{C}$.*

The bound is not the best possible. Indeed, a quadratic polynomial is of the form $f(z) = p_2(z)$, $\overline{f(z)} = p_2(z)$, or $f(z) = a(z+b)\overline{(z+c)}$. In all three cases, $f-w$ has either infinitely many zeros or it has at most two.

Observe that the logharmonic polynomial $f(z) = (z-1)/(\overline{z}+1)$ is 2-valent and omits the half-plane $\operatorname{Re}(w) < -1$ and that $|a| \not\equiv 1$. However, the situation changes if $|a(\infty)| \neq 1$ and we have the following result [2].

Theorem 4.5. *Let $f = p_n\overline{p_m}$ be a logharmonic polynomial defined in $\mathbb{C}$, and suppose that $n > m$. Fix $w \in \mathbb{C}$ such that $Z(f - w, \mathbb{C}) \cap (\partial D \cup S_E(D))$ is empty. Then the number of zeros $VZ(f - w, S_E(\mathbb{C}))$ of $f - w$, and hence also the valency $V(f, \mathbb{C})$ of f in $\mathbb{C}$, is at least $n - m$. The bound is best possible.*

The following result is an immediate consequence of Theorem 4.5.

Corollary 4.6. *Let $f = p_n\overline{p_m}$ be a logharmonic polynomial defined in $\mathbb{C}$, and suppose that $n > m$. Then*

(i) $f(\mathbb{C}) = \mathbb{C}$,

(ii) *for almost all $w \in \mathbb{C}$, the function $f - w$ has at least $n - m$ disjoint zeros.*

The next result characterizes polynomials of finite valency [2].

Theorem 4.7. *Let $f = p_n\overline{p_m}$ be a logharmonic polynomial defined in $\mathbb{C}$, such that $p_n \not\equiv \text{const} \cdot p_m$. Then the cardinality $NZ(f - w, \mathbb{C})$ of the zero set $Z(f - w, \mathbb{C})$ is finite (hence, by Bezout's theorem, uniformly bounded) for all $w \in \mathbb{C}$.*

Remark 4.8. If $p_n \equiv \text{const} \cdot p_m$, then the image lies on a straight line.

5. Subclasses of Logharmonic Mappings

5.1. Spirallike Logharmonic Mappings

Let Ω be a simply connected domain if $\mathbb{C}$ contains the origin. We say that Ω is α-spirallike, $-\pi/2 < \alpha < \pi/2$, if $w \in \Omega$ implies that $w\exp(-te^{i\alpha}) \in \Omega$ for all $t \geq 0$. If $\alpha = 0$, the domain Ω is called starlike (with respect to the origin). We will use the following notations.

(a) S^{α}_{Lh} is the set of all univalent logharmonic mappings f in U satisfying $f(0) = 0$, $h(0) = g(0) = 1$, and $f(U)$ is an α-spirallike domain.

(b) $S^{\alpha} = \{f \in S^{\alpha}_{Lh}$ and $f \in H(U)\}$.

(c) $S^{*}_{Lh} = S^{0}_{Lh}$ and $S^{*} = S^{0}$, for which $f(U)$ is starlike (with respect to the origin).

To each $f(z) = z|z|^{2\beta}h(z)\overline{g(z)} \in S^{\alpha}_{Lh}$, we associate the analytic function $\psi(z) = zh(z)/g(z)^{e^{i\alpha}}$, $\psi(0) = 0$. Abdulhadi and Hengartner [8] gave a representation theorem for mappings in the class S^{α}_{Lh}.

Theorem 5.1. (a) *If $f \in S^{\alpha}_{Lh}$, then $\psi \in S^{\alpha}$.*

(b) *For any given $\psi \in S^{\alpha}$ and $a \in B(U)$, there are h and g in $H(U)$ uniquely determined such that*

(i) $0 \notin h \cdot g(U)$, $h(0) = g(0) = 1$,

(ii) $\psi(z) = zh(z)/g(z)^{e^{i\alpha}}$,

(iii) $f(z) = z|z|^{2\beta}h(z)\overline{g(z)}$ *is a solution of* (1.1) *in S^{α}_{Lh}, where* $\beta = (\overline{a(0)}(1 + a(0)))/(1 - |a(0)|^2)$.

Remark 5.2. Theorem 5.1 has no equivalence for the class of all convex univalent logharmonic mappings. Indeed, $\psi(z) = z$ is a convex mapping, $a(z) = z^4 \in B(U)$, but $f(z) = z/|1 - z^4|^{1/2}$ is not a convex mapping.

Remark 5.3. Theorem 5.1 asserts that the class S^{α}_{Lh}, α fixed in $(-\pi/2, \pi/2)$, is isomorphic to $S^{\alpha} \times B(U)$.

The following result is an immediate consequence of Theorem 5.1.

Corollary 5.4. *If $f \in S^{\alpha}_{Lh}$, then $f(rz)/r \in S^{\alpha}_{Lh}$ for all $r \in (0,1)$. In other words, level sets inherit the property of being α-spirallike.*

The next result is an integral representation for $f \in S^{\alpha}_{Lh}$ [8].

Theorem 5.5. *A function $f \in S^{\alpha}_{Lh}$ if and only if there are two probability measures μ and ν on the Borel sets of ∂U and an $a(0) \in U$ such that*

$$f(z) = z|z|^{2\beta} \cdot \exp\left\{ \int_{\partial U \times \partial U} K(z, \zeta, \xi; a(0)) d\mu(\zeta) d\nu(\xi) \right\}, \tag{5.1}$$

where

$$\beta = \frac{\overline{a(0)}(1 + a(0))}{1 - |a(0)|^2},$$

$$\begin{aligned} K(z, \zeta, \xi; a(0)) &= -2\cos(\alpha) \cdot e^{i\alpha} \cdot \log(1 - \zeta z) + 2e^{i\alpha} \operatorname{Re}\left\{ e^{i\alpha} \log(1 - \zeta z) \right\} + T(z, \zeta, \xi; a(0)), \\ T(z, \zeta, \xi; a(0)) &= 2e^{i\alpha} \operatorname{Re} \frac{e^{i\alpha}(1 + a(0))\left(1 - \overline{a(0)} e^{-2i\alpha}\right)\zeta + e^{-i\alpha}\left(1 + \overline{a(0)}\right)\left(1 - a(0)e^{2i\alpha}\right)\xi}{(\zeta - \xi)|1 - a(0)e^{2i\alpha}|^2} \\ &\quad \times \log \frac{1 - \xi z}{1 - \zeta z}, \end{aligned} \tag{5.2}$$

if $|\zeta| = |\xi| = 1$, $\zeta \neq \xi$, and

$$T(z, \zeta, \zeta; a(0)) = 4\cos(\alpha) \cdot e^{i\alpha} \cdot \operatorname{Re} \frac{\zeta z}{(1 - \zeta z)} \frac{1 - |a(0)|^2}{|1 - a(0)e^{2i\alpha}|^2}. \tag{5.3}$$

Observe that S^{α}_{Lh} is not compact, but Theorem 5.5 can be used to solve extremal problems over the class of mappings in S^{α}_{Lh} with a given $a(0) = 0$.

We have seen in Corollary 5.4 that if f is a univalent logharmonic mapping in U, $f(0) = 0$, and if $f(U)$ is starlike, then $f(|z| < r)$ is starlike (with respect to the origin) for all $r \in (0,1)$. The next result proved in [8] shows that this property may fail whenever $f(0) \neq 0$.

Theorem 5.6. *For each $z_0 \in U \setminus \{0\}$, there are univalent logharmonic mappings f such that $f(z_0) = 0$, $f(U)$ is starlike (with respect to the origin), but no level set $f(|z| < r)$, $|z_0| = \rho < r < 1$, is starlike.*

5.2. Close-to-Starlike Logharmonic Mappings

5.2.1. Logharmonic Mappings with Positive Real Part

Let P_{Lh} be the set of all logharmonic mappings R in U which are of the form $R = H\overline{G}$, where H and G are in $H(U)$, $H(0) = G(0) = 1$, such that $\mathrm{Re}(R(z)) > 0$ for all $z \in U$. In particular, the set P of all analytic functions p in U with $p(0) = 1$ and $\mathrm{Re}(p(z)) > 0$ in U is a subset of P_{Lh}.

The next result [43] describes the connection between P_{Lh} and P.

Theorem 5.7. *A function $R = H\overline{G} \in P_{Lh}$ if and only if $p = H/G \in P$.*

As a consequence of Theorem 5.7, it follows that R admits the representation

$$R(z) = p(z) \exp 2\,\mathrm{Re} \int_0^z \frac{a(s)}{1-a(s)} \frac{p'(s)}{p(s)} ds, \tag{5.4}$$

where $a \in B(U)$ and p is an analytic function with positive real part normalized by $p(0) = 1$.

The following result [43] is a distortion theorem for the class P_{Lh}.

Theorem 5.8. *Let $R(z) = H(z)\overline{G(z)} \in P_{Lh}$, and suppose that $a(0) = 0$. Then for $z \in U$*

(i) $\exp(-2|z|/(1-|z|)) \le |R(z)| \le \exp(2|z|/(1-|z|))$,

(ii) $|R_z(z)| \le (2/(1-|z|)(1-z|^2)) \exp(2|z|/(1-|z|))$,

(iii) $|R_{\overline{z}}(z)| \le (2|z|/(1-|z|)(1-|z|^2)) \exp(2|z|/(1-|z|))$.

Equality occurs for the right inequalities if R is a function of the form $R_0(\zeta z)$, $|\zeta| = 1$, where

$$R_0(z) = \frac{1+z}{1-z} \left| \frac{1+z}{1-z} \right| \exp\left(\mathrm{Re}\, \frac{2z}{1-z} \right), \tag{5.5}$$

and equality occurs for the left inequalities if R is of the form

$$\frac{1}{R_0(\zeta z)}, \qquad |\zeta| = 1. \tag{5.6}$$

5.2.2. Close-to-Starlike Logharmonic Mappings

Let $F(z) = z|z|^{2\beta} h \overline{g}$ be a logharmonic mapping. The function F is close to starlike if F is a product between a starlike logharmonic mapping $f(z) = z|z|^{2\beta} h^* \overline{g^*} \in S^*_{Lh}$ which is a solution of (1.1) with respect to $a \in B(U)$ and a logharmonic mapping with positive real part $R \in P_{Lh}$ with the same second dilatation function a.

The geometric interpretation for a close-to-starlike logharmonic mappings is that the radius vector of the image of $|z| = r < 1$ never turns back by an amount more than π.

Denote by CST_{Lh} the set of all close-to-starlike logharmonic mappings. It contains in particular the set CST of all analytic close-to-starlike functions which was introduced by Reade [44] in 1955. Also, the set S^*_{Lh} of all starlike univalent logharmonic mappings is a subset of CST_{Lh} (take $R(z) \equiv 1$ in the product). Furthermore, if $F(z) = z|z|^{2\beta} h \overline{g}$ is a logharmonic mapping with respect to $a \in B(U)$ satisfying $h(0) = g(0) = 1$ and $\operatorname{Re} F(z)/z|z|^{2\beta} > 0$, then F is a close-to-starlike logharmonic mapping, where

$$f(z) = z|z|^{2\beta} \left| \exp\left(\int_0^z \frac{a(s)/s}{1-a(s)} ds \right) \right|^2. \tag{5.7}$$

On the other hand, a mapping $F \in \mathrm{CST}_{Lh}$ need not necessarily be univalent. For example, take $F(z) = z(1+z)$, where $z \in S^*$ and $1 + z \in P$.

Our next result is a representation theorem for the class CST_{Lh} proved in [43].

Theorem 5.9. (a) *Let $F = z|z|^{2\beta} h\overline{g}$ be in CST_{Lh}. Then $\varphi = zh/g \in CST$.*

(b) *Given any $\varphi \in CST$ and $a \in B(U)$, there are h and g in $H(U)$ uniquely determined such that*

(i) $0 \notin h \cdot g(U)$, $h(0) = g(0) = 1$,

(ii) $\varphi(z) = zh/g$,

(iii) $F = z|z|^{2\beta} h\overline{g}$ *is in CST_{Lh} which is a solution of* (1.1) *with respect to the given a.*

Corollary 5.10. *$F \in CST_{Lh}$ if and only if $F(rz)/r \in CST_{Lh}$ for all $r \in (0,1)$.*

In the next result the radius of univalence and the radius of starlikeness are determined for those mappings in the set CST_{Lh} [43].

Theorem 5.11. *Let $F = z|z|^{2\beta} h\overline{g} \in CST_{Lh}$. Then F maps the disc $|z| < R$, $R \le 2-\sqrt{3}$, onto a starlike domain. The upper bound is best possible for all $a \in B(U)$.*

Combining Theorems 5.8 and 5.11 with $\alpha = 0$, we obtain the following distortion theorem for the class CST_{Lh}.

Theorem 5.12. *Let $F = zh\overline{g} \in CST_{Lh}$. Then, for every $z \in U$,*

(a) $|z| \exp(-2|z|/(1-|z|) - 4|z|/(1+|z|)) \le |F(z)| \le |z| \exp(6|z|/(1-|z|))$,

(b) $|F_z(z)| \le ((|z|^2 + 4|z| + 1)/(1-|z|)^2(1+|z|)) \exp(6|z|/(1-|z|))$,

(c) $|F_{\overline{z}}(z)| \le (|z|(|z|^2 + 4|z| + 1)/(1-|z|)^2(1+|z|)) \exp(6|z|/(1-|z|))$.

Equality holds for the right inequalities if F is a function of the form

$$F_{\eta,\zeta}(z) = \frac{z(1-\overline{\eta z})}{(1-\eta z)} \frac{1+\zeta z}{1-\zeta z} \left| \frac{1-\zeta z}{1+\zeta z} \right| \exp\left(\operatorname{Re}\left[\frac{4\eta z}{1-\eta z} + \frac{2\zeta z}{1-\zeta z} \right] \right), \tag{5.8}$$

where $|\eta| = |\zeta| = 1$, and for the left inequalities if F is a function of the form

$$F_{\eta,\zeta}(z) = \frac{z(1-\overline{\eta z})}{(1-\eta z)} \frac{1+\zeta z}{1-\zeta z} \left| \frac{1-\zeta z}{1+\zeta z} \right| \exp\left(\operatorname{Re}\left[\frac{4\eta z}{1-\eta z} - \frac{2\zeta z}{1-\zeta z} \right] \right), \tag{5.9}$$

where $|\eta| = |\zeta| = 1$.

5.3. Typically Real Logharmonic Mappings

A logharmonic mapping f is said to be typically real if and only if f is real whenever z is real and if f is normalized by $f(0) = 0$ and $h(0)\overline{g(0)} = 1$, or equivalently by $f(0) = 0$ and $h(0) = g(0) = 1$. Denote by T_{Lh} the class of all orientation-preserving typically real logharmonic mappings. Since f is orientation preserving and univalent on the interval $(-1, 1)$, it follows that f is of the form (2.10). Furthermore, if $f \in T_{Lh}$, then β (and hence, also $a(0)$) has to be real and yields the relation

$$\operatorname{Im} z \operatorname{Im} f(z) > 0, \quad \forall\, z \in U \setminus \mathbb{R}. \tag{5.10}$$

The class T_{Lh} is a compact convex set with respect to the topology of locally uniform convergence, and it contains, in particular, the set T of all analytic typically real functions.

5.3.1. Basic Properties of Mappings from T_{Lh}

The following representation theorem for typically real logharmonic mappings was proved in [45].

Theorem 5.13. (a) *If $f(z) = z|z|^{2\beta}h(z)\overline{g(z)}$ is in T_{Lh}, then $\phi = zh/g \in T$.*

(b) *Given $\phi \in T$ and $a \in B(U)$ such that $\beta \in \mathbb{R}$ and $a(0) \in \mathbb{R}$, there are uniquely determined mappings h and g in $H(U)$ such that*

(i) $0 \notin h \cdot g(U)$, $h(0) = g(0) = 1$,

(ii) $\phi(z) = zh/g$,

(iii) $F = z|z|^{2\beta}h\overline{g}$ *is in T_{Lh} which is a solution of* (1.1) *with respect to the given a.*

As a consequence of Theorem 5.13, it follows that

$$f(z) = zh(z)\overline{g(z)} = \phi(z)\left|g(z)\right|^2, \tag{5.11}$$

where

$$\begin{gathered} g(z) = \exp \int_0^z \frac{a(s)\phi'(s)}{(1 - a(s))\phi(s)} ds, \\ zh(z) = \phi(z)g(z). \end{gathered} \tag{5.12}$$

Denote by T_{Lh}^0 the subclass of T_{Lh} ($\beta = 0$) consisting of all mappings F from T_{Lh} for which $\phi = zh/g = z/(1 - z^2)$. Then F is of the form

$$F(z) = \frac{z}{1 - z^2} \exp 2 \operatorname{Re} \int_0^z \frac{a(s)(1 + s^2)}{s(1 - a(s))(1 - s^2)} ds. \tag{5.13}$$

The next theorem links the class T_{Lh} with the class P_{Lh}.

Theorem 5.14 (see [45]). *Let $f(z) = zh(z)\overline{g(z)} \in T_{Lh}$ with respect to $a \in B(U)$, and $a(0) = 0$. Then there exist an $R \in P_{Lh}$ and an $F \in T_{Lh}^0$, such that both functions are logharmonic with respect to the same a and*

$$f(z) = F(z)R(z). \tag{5.14}$$

The next result is a distortion theorem for the class T_{Lh}^0.

Theorem 5.15 (see [45]). *Let $F(z) = zh(z)\overline{g(z)} \in T_{Lh}^0$. Then, for $z \in U$,*

(a) $|F(z)| \le |z| \exp(2|z|/(1-|z|))$,

(b) $|F_z(z)| \le ((1+|z|^2)/(1-|z|^2)(1-|z|)) \exp(2|z|/(1-|z|))$,

(c) $|F_{\overline{z}}(z)| \le (|z|(1+|z|^2)/(1-|z|^2)(1-|z|)) \exp(2|z|/(1-|z|))$.

Equality holds if and only if F is of the form $\overline{\eta}F_0(\eta z)$, $|\eta| = 1$, where

$$F_0(z) = \frac{z}{1-z^2}\left|1-z^2\right| \exp\left(\operatorname{Re}\left(\frac{2z}{1-z}\right)\right). \tag{5.15}$$

Combining Theorems 5.8, 5.14, and 5.15, the following distortion theorem is obtained for the class T_{Lh}.

Theorem 5.16. *Let $f(z) = zh(z)\overline{g(z)} \in T_{Lh}$. Then, for $z \in U$,*

(a) $|f(z)| \le |z| \exp(4|z|/(1-|z|))$,

(b) $|f_z(z)| \le ((1+|z|)/(1-|z|^2)) \exp(4|z|/(1-|z|))$,

(c) $|f_{\overline{z}}(z)| \le (|z|(1+|z|)/(1-|z|^2)) \exp(4|z|/(1-|z|))$.

Equality holds if f is of the form $\overline{\eta}f_0(\eta z)$, $|\eta| = 1$, where

$$f_0(z) = \frac{z(1-\overline{z})}{1-z} \exp\left(\operatorname{Re}\left(\frac{4z}{1-z}\right)\right). \tag{5.16}$$

Remark 5.17. The function f_0 given in (5.16) plays the role of the Koebe mapping in the set of logharmonic mappings (see, e.g., [1, 6]).

The next result gives the radius of univalence and the radius of starlikeness for the mappings in the set T_{Lh} [45].

Theorem 5.18 (see [45]). *Let $f(z) = z|z|^{2\beta}h(z)\overline{g(z)} \in T_{Lh}$. Then f maps the disc $\{z : |z| < R_0\}$, where $R_0 = (1+\sqrt{5}-\sqrt{2+2\sqrt{5}})/2$, onto a starlike domain. The upper bound is the best possible for all $a \in B(U)$.*

5.3.2. Univalent Mappings in T_{Lh}

Now we consider univalent mappings in T_{Lh}. For analytic typically real functions, it is known that if $t(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is univalent in the unit disc U, then t belongs to T if and only if the image $t(U)$ is a domain symmetric with respect to the real axis.

One might consider a similar problem in T_{Lh}. Let $f(z) = z|z|^{2\beta}h(z)\overline{g(z)}$ be a univalent logharmonic mapping in the unit disc U, and $h(0) = g(0) = 1$, $\beta > -1/2$. Observe that β (and hence $a(0)$) is real. Is it true that f belongs to T_{Lh} if and only if the image of $f(U)$ is a symmetric domain with respect to the real axis?

The answer is negative in both directions. Indeed, the function

$$f(z) = z\left(1 + \frac{iz}{8}\right)\left(1 - \frac{i\overline{z}}{8}\right) \tag{5.17}$$

is a normalized univalent logharmonic typically real mapping, but $f(U)$ is not symmetric with respect to the real axis. On the other hand, the function $f(z) = z(1 + i\overline{z})/(1 - iz)$ is a univalent logharmonic mapping from U onto U, and $f(U)$ is symmetric with respect to the real axis, but f is not typically real (for more details, see [45]).

Additional conditions on a and on the image domain $\Omega = f(U)$ are needed in order to get an affirmative answer to the question posed above.

Theorem 5.19 (see [45]). *Let $f(z) = z|z|^{2\beta}h(z)\overline{g(z)}$ be a univalent (orientation-preserving) logharmonic mapping in the unit disc U and normalized by $f(0) = 0$, $h(0) = \overline{g(0)} = 1$. Suppose that the second dilatation function a has real coefficients, that is, $a(z) \equiv \overline{a(\overline{z})}$. (Observe that the condition $a(0)$ real or equivalently β real is automatically satisfied.)*

(a) *If f is typically real, then $f(U)$ is symmetric with respect to the real axis.*

(b) *If $|a| \le k < 1$ in U and $f(U)$ is a strictly starlike Jordan domain symmetric with respect to the real axis, then f is typically real.*

5.4. Starlike Logharmonic Mappings of Order α

Let $f = z|z|^{2\beta}h\overline{g}$ be a univalent logharmonic mapping. We say that f is starlike logharmonic mapping of order α if

$$\frac{\partial \arg f(re^{i\theta})}{\partial\theta} = \operatorname{Re}\ \frac{zf_z - \overline{z}f_{\overline{z}}}{f} > \alpha, \quad 0 \le \alpha < 1, \tag{5.18}$$

for all $z \in U$. Denote by $\mathrm{ST}_{Lh}(\alpha)$ the set of all starlike logharmonic mappings of order α. If $\alpha = 0$, we get the class of starlike logharmonic mappings. Also, let $\mathrm{ST}(\alpha) = \{f \in \mathrm{ST}_{Lh}(\alpha)$ and $f \in H(U)\}$.

In this section, we obtain two representation theorems [46] for functions in $\mathrm{ST}_{Lh}(\alpha)$. In the first, we establish the connection between the classes $\mathrm{ST}_{Lh}(\alpha)$ and $\mathrm{ST}(\alpha)$. The second is an integral representation theorem.

Theorem 5.20. *Let $f(z) = zh(z)\overline{g(z)}$ be a logharmonic mapping in U, $0 \notin hg(U)$. Then $f \in ST_{Lh}(\alpha)$ if and only if $\varphi(z) = zh(z)/g(z) \in ST(\alpha)$.*

Theorem 5.21. *A function $f = zh\overline{g} \in ST_{Lh}(\alpha)$ with $a(0) = 0$ if and only if there are two probability measures μ and ν such that*

$$f(z) = z\exp\left(\int_{\partial U\times\partial U} K(z,\zeta,\xi)\,d\mu(\zeta)\,d\nu(\xi)\right), \tag{5.19}$$

where

$$K(z,\zeta,\xi) = (1-\alpha)\log\left(\frac{1+\overline{\zeta z}}{1-\zeta z}\right) + T(z,\zeta,\xi),$$

$$T(z,\zeta,\xi) = \begin{cases} -2(1-\alpha)\operatorname{Im}\left(\dfrac{\zeta+\xi}{\zeta-\xi}\right)\arg\left(\dfrac{1-\xi z}{1-\zeta z}\right) - 2\alpha\log|1-\xi z|, & \text{if } |\zeta| = |\xi| = 1,\ \zeta\neq\xi, \\ (1-\alpha)\operatorname{Re}\left(\dfrac{4\zeta z}{1-\zeta z}\right) - 2\alpha\log|1-\zeta z|, & \text{if } |\zeta| = |\xi| = 1,\ \zeta = \xi. \end{cases} \tag{5.20}$$

Remark 5.22. Theorem 5.21 can be used to solve extremal problems for the class $\mathrm{ST}_{Lh}(\alpha)$ with $a(0) = 0$. For example, see Theorem 5.23.

The following is a distortion theorem for the class $\mathrm{ST}_{Lh}(\alpha)$ with $a(0) = 0$.

Theorem 5.23 (see [46]). *Let $f = zh(z)\overline{g(z)} \in ST_{Lh}(\alpha)$ with $a(0) = 0$. Then, for $z \in U$,*

$$\frac{|z|}{(1+|z|)^{2\alpha}}\exp\left((1-\alpha)\frac{-4|z|}{1+|z|}\right) \le |f(z)| \le \frac{|z|}{(1-|z|)^{2\alpha}}\exp\left((1-\alpha)\frac{4|z|}{1-|z|}\right). \tag{5.21}$$

Equalities occur if and only if $f(z) = \overline{\zeta} f_0(\zeta z)$, $|\zeta| = 1$, where

$$f_0(z) = z\left(\frac{1-\overline{z}}{1-z}\right)\frac{1}{(1-\overline{z})^{2\alpha}}\exp\left((1-\alpha)\operatorname{Re}\frac{4z}{1-z}\right). \tag{5.22}$$

The next result gives sharp coefficient estimates for functions h and g in the starlike logharmonic mapping $f = zh(z)\overline{g(z)}$.

Theorem 5.24 (see [6]). *Let $f = zh(z)\overline{g(z)} \in ST_{Lh}(0)$ with $a(0) = 0$, and put*

$$h(z) = \exp\left(\sum_{k=1}^{\infty} a_k z^k\right), \qquad g(z) = \exp\left(\sum_{k=1}^{\infty} b_k z^k\right). \tag{5.23}$$

Then

$$|a_n| \le 2 + \frac{1}{n}, \qquad |b_n| \le 2 - \frac{1}{n} \tag{5.24}$$

for all $n \geq 1$. *Equality holds for the mapping*

$$f(z) = z\frac{1 - \overline{ze^{i\alpha}}}{1 - ze^{i\alpha}} \exp\left(\frac{4ze^{i\alpha}}{1 - ze^{i\alpha}}\right), \qquad \alpha \in (0, 2\pi). \tag{5.25}$$

5.5. Functions with Logharmonic Laplacian

We consider the class of all continuous complex-valued functions $F = u + iv$ in a domain $D \subseteq C$ such that the Laplacian of F is logharmonic. Note that $\log(\Delta F)$ is harmonic in D if it satisfies the Laplace equation $\Delta(\log(\Delta F)) = 0$, where

$$\Delta = 4\frac{\partial^2}{\partial z \partial \overline{z}}. \tag{5.26}$$

In any simply connected domain D, we can write

$$F = r^2 L + H, \qquad z = re^{i\theta}, \tag{5.27}$$

where L is logharmonic and H is harmonic in D. It is known that L and H can be expressed as

$$\begin{aligned} L &= h_1 \overline{g_1}, \\ H &= h_2 + \overline{g_2}, \end{aligned} \tag{5.28}$$

where h_1, g_1, h_2, and g_2 are analytic in D. Denote by $L_{Lh}(U)$ the set of all functions of the form (5.27) which are defined in the unit disc U.

Note that the composition $L \circ \phi$ of a logharmonic function L with an analytic function ϕ is logharmonic and, also, the composition $H \circ \phi$ of a harmonic function H with analytic function ϕ is harmonic, while this is not true for the function F. Also, if $F_1(z) = r^2 L_1(z)$ and $F_2(z) = r^2 L_2(z)$ are in $L_{Lh}(U)$, where L_1 and L_2 are logharmonic with respect to the same a, then $F_1^{\alpha} F_2^{\beta}$, $\alpha + \beta = 1$, is also in $L_{Lh}(U)$.

Denote the Jacobian of W by J_W. Then

$$J_W = |W_z|^2 - |W_{\overline{z}}|^2. \tag{5.29}$$

Also let

$$\begin{aligned} \lambda_W &= |W_z| - |W_{\overline{z}}|, \\ \Lambda_W &= |W_z| + |W_{\overline{z}}|. \end{aligned} \tag{5.30}$$

Then $J_W = \lambda_W \Lambda_W$.

5.5.1. The Univalence of Functions with Logharmonic Laplacian

First a lower bound for the area of the range of $F(z) = r^2 L(z)$ is established, where L is a starlike univalent logharmonic mapping.

Theorem 5.25 (see [47]). *Let $F(z) = r^2 L(z)$, where $L = h\overline{g}$ is starlike univalent logharmonic in U, with $g(0) = 1$ and $h'(0) = 1$. Let $A(r, F)$ denote the area of $F(U_r)$, where $U_r = \{z : |z| < r\}$, for $r < 1$. Then*

$$A(r, F) \geq 2\pi \left[-2r + r^2 - \frac{2r^3}{3} + \frac{r^4}{2} - \frac{r^5}{5} + \frac{r^6}{6} - \frac{r^8}{8} + 2\ln(1 + r) \right]. \tag{5.31}$$

Equality holds if and only if $L_0(z) = r^2 z(1 + \overline{z}/2)/(1 + z/2)$ or one of its rotations.

Definition 5.26. Let L be logharmonic function in U. A complex-valued function of the form $F(z) = r^2 L(z)$ is starlike in U if it is orientation preserving, $F(0) = 0$, $F(z) \neq 0$ when $z \neq 0$ and the curve $F(re^{it})$ is starlike with respect to the origin for each $0 < r < 1$. In other words, $\partial \arg F(re^{it})/\partial t = \operatorname{Re}((zF_z - \overline{z}F_{\overline{z}})/F) > 0$.

Remark 5.27. Note that starlike functions are univalent in U.

The following theorem links starlike functions in $L_{Lh}(U)$ with the class of starlike analytic functions.

Theorem 5.28 (see [48]). *Let $F(z) = r^2 L(z)$, where $L(z) = h(z)\overline{g(z)}$, be a logharmonic function in U with respect to a, where $a \in B(U)$ with $a(0) = 0$. Then F is starlike univalent in U if and only if $\varphi(z) = h(z)/g(z)$ is starlike univalent function in U.*

Corollary 5.29. *The function $r^2 L(z)$ is starlike for all conformal starlike functions L.*

A characterization of the logharmonic Laplacian solutions of the Dirichlet problem in the unit disc U is given in [48].

Theorem 5.30. *Let F^* be an orientation-preserving homeomorphism from ∂U onto ∂U, that is, $F^*(e^{it}) = e^{i\lambda(t)}$, where λ is continuous and strictly monotonically increasing on $[0, 2\pi]$. Furthermore, suppose that $\lambda(2\pi) = \lambda(0) + 2\pi$. Then $F(z) = \overline{z}|z|^2 h(z)/\overline{h(z)}$ is a univalent solution of the Dirichlet problem in U.*

For the general case $F(z) = r^2 L(z) + H(z)$, a sufficient condition is obtained that makes F locally univalent.

Theorem 5.31 (see [48]). *Let $F(z) = r^2 h_1(z)\overline{g_1(z)} + h_2(z) + \overline{g_2(z)}$ be in the class $L_{Lh}(U)$. Suppose that $\varphi(z) = h_1(z)/g_1(z)$ is starlike univalent in U, and $|g_2'(z)| < |h_2'(z)|$ for $z \in U$. If*

$$\operatorname{Re}\left[g_2' \overline{(r^2 h_1 \overline{g_1})_{\overline{z}}} \right] < \operatorname{Re}\left[h_2' \overline{(r^2 h_1 \overline{g_1})_{z}} \right], \tag{5.32}$$

then $J_F(z) > 0$ for $z \neq 0$, and F is locally univalent.

5.5.2. Landau's Theorem for Functions with Logharmonic Laplacian

Lewy's famous theorem [49] states that a harmonic function W is locally univalent in D (univalent in some neighborhood of each point in D) if and only if its Jacobian does not vanish in D.

The classical Landau Theorem states that if f is analytic in the unit disc U with $f(0) = 0$, $f'(0) = 1$, and $|f(z)| < M$ for $z \in U$, then f is univalent in the disc $U_{\rho_0} = \{z : |z| < \rho_0\}$ with

$$\rho_0 = \frac{1}{M + \sqrt{M^2 - 1}}, \tag{5.33}$$

and $f(U_{\rho_0})$ contains a disc U_{R_0} with $R_0 = M\rho_0^2$. This result is sharp, with the extremal function $f(z) = Mz(1 - Mz)/(M - z)$ (see [19]).

Chen et al. [50] obtained a version of Landau's Theorem for bounded harmonic mappings of the unit disc. Unfortunately their result is not sharp. Better estimates were given in [51] and later in [52].

Specifically, it was shown in [52] that if f is harmonic in the unit disc U with $f(0) = 0$, $J_f(0) = 1$, and $|f(z)| < M$ for $z \in U$, then f is univalent in the disc $U_{\rho_1} = \{z : |z| < \rho_1\}$ with

$$\rho_1 = 1 - \frac{2\sqrt{2}M}{\sqrt{\pi + 8M^2}}, \tag{5.34}$$

and $f(U_{\rho_1})$ contains a disc U_{R_1} with $R_1 = \pi/4M - 2M(\rho_1^2/(1 - \rho_1))$. This result is the best known, but not sharp.

The following Schwarz lemma for harmonic mappings is due to Grigoryan [52].

Lemma 5.32 (Schwarz lemma). *Let f be a harmonic mapping of the unit disc U with $f(0) = 0$ and $f(U) \subset U$. Then*

$$\begin{gathered} \left|f(z)\right| \leq \frac{4}{\pi}\arctan|z| \leq \frac{4}{\pi}|z|, \\ \Lambda_f(0) \leq \frac{4}{\pi}. \end{gathered} \tag{5.35}$$

Recently Mao et al. [53] established the Schwarz lemma for logharmonic mappings, through which two versions of Landau's theorem for these mappings were obtained.

The next theorem gives Landau's theorem for functions with logharmonic Laplacian of the form $F = r^2L(z)$.

Theorem 5.33 (see [47]). *Let L be logharmonic in U such that $L(0) = 0$, $J_L(0) = 1$, and $|L(z)| < M$ for $z \in U$. Then there is a constant $0 < \rho_2 < 1$ such that $F = r^2L$ is univalent in the disc $|z| < \rho_2$, ρ_2 is the solution of the equation $1 = 2\rho_2M/(1-\rho_2^2) - 2M\rho_2/(1-\rho_2^2)^2$, and $f(U_{\rho_2})$ contains a disc U_{R_2} with $R_2 = \rho_2^2 - 2M\rho_2^4/(1 - \rho_2^2)$. This result is not sharp.*

Finally we give a Landau theorem for functions of logharmonic Laplacian of the form $F = r^2L + K$.

Theorem 5.34 (see [47]). *Let $F = r^2L + K$, $z = re^{i\theta}$ be in $L_{Lh}(U)$, where L is logharmonic and K is harmonic in the unit disc U, such that $L(0) = K(0) = 0$, $J_F(0) = 1$, and $|L|$ and $|K|$ are both bounded by M. Then there is a constant $0 < \rho_3 < 1$ such that F is univalent in $|z| < \rho_3$. Specifically ρ_3 satisfies*

$$\frac{\pi}{4M} - 2\rho_3 M - 2M\left(\frac{\rho_3^3}{\left(1-\rho_3^2\right)^2} + \frac{1}{\left(1-\rho_3\right)^2} - 1\right) = 0, \tag{5.36}$$

and $F(U_{\rho_3})$ contains a disc U_{R_3}, where

$$R_3 = \frac{\pi}{4M}\rho_3 - \rho_3^2 M\frac{1}{1-\rho_3^2} - 2M\frac{\rho_3^2}{1-\rho_3}. \tag{5.37}$$

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