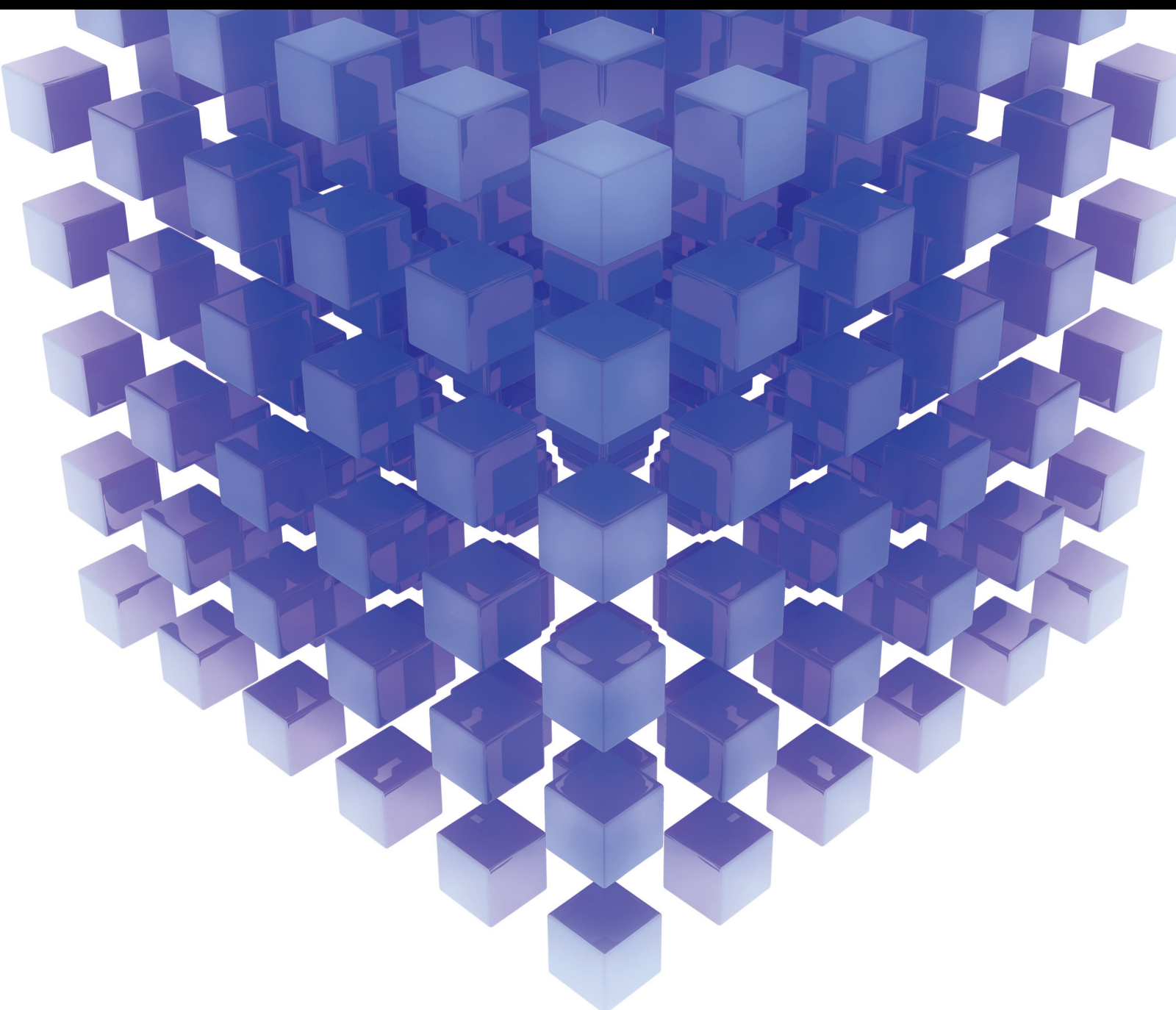


Mathematical Problems in Engineering

Stochastic Optimal Control and its Applications

Lead Guest Editor: Zhuo Jin

Guest Editors: Jiaqin Wei, Linyi Qian, and Xiaofeng Zong





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Corrigendum

Corrigendum to “The Principle-Agent Conflict Problem in a Continuous-Time Delegated Asset Management Model”

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In the article titled “The Principle-Agent Conflict Problem in a Continuous-Time Delegated Asset Management Model” [1], the author Dr. Siyuan Hao has been added to the authors’ list, who contributed to data analysis. With the agreement of all authors, the corrected authors’ list is shown above.

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Research Article

The Principle-Agent Conflict Problem in a Continuous-Time Delegated Asset Management Model

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This paper considers the principle-agent conflict problem in a continuous-time delegated asset management model when the investor and the fund manager are all risk-averse with risk sensitivity coefficients γ_f and c_m , respectively. Suppose that the investor entrusts his money to the fund manager. The return of the investment is determined by the manager's effort level and incentive strategy, but the benefit belongs to the investor. In order to encourage the manager to work hard, the investor will determine the manager's salary according to the terminal income. This is a stochastic differential game problem, and the distribution of income between the manager and the investor is a key point to be solved in the custody model. The uncertain form of the incentive strategy implies that the problem is different from the classical stochastic optimal control problem. In this paper, we first express the investor's incentive strategy in term of two auxiliary processes and turn this problem into a classical one. Then, we employ the dynamic programming principle to solve the problem.

1. Introduction

Since professional asset management institutions can make efficient investment decisions, save investors' time and effort, and simplify the investment process, more and more investors now entrust their money to fund managers, securities firms, and other asset management organizations. Nowadays, scholars pay more and more attention to asset management problems. We can refer to [1–5] to name just a few.

The whole asset management process involves two parties: the investor and the manager. The return of the investment is closely related to the manager's effort level and investment strategy, but the interests belong to the investor. So, the investor and manager's relation poses a principal-agent conflict. An important part of discussing the asset management problem is finding the investor's optimal incentive mode under the principle agent conflict.

There are many papers committed to solving principal-agent conflict problems. Most of the early literature studies investigate the discrete-time case (we can refer to [6–8] or a summary book [9]). The problem in continuous-time

models is discussed for the first time in [10]. It points out that the investor's optimal incentive mode is linear. See references [11–14] for further work. In recent years, the maximum principle or the martingale representation theorem is often used to solve this problem in continuous-time models. For the literature using the maximum principle, we can refer to [15, 16], and for the literature of using the martingale representation theorem, we can refer to [17, 18]. However, since this problem often needs to solve a backward stochastic differential equation (BSDE) that rarely has explicit solutions, there are few articles which give analytical solutions to this problem. In order to get explicit solutions of principal-agent conflict problems, the authors of [19] express the investor's incentive strategy in terms of two auxiliary processes and turn the principle agent problem into a classical stochastic differential game problem.

Although there are many papers committed to solving principal-agent conflict problems in continuous-time models, the delegated asset management problems are usually investigated in discrete-time models for the sake of simplicity. Thus, there are some contributions in this paper:

- (i) This paper considers the delegated asset management problem in a continuous-time model
- (ii) Learning from [19], this paper gives explicit value functions and the optimal strategies of both sides by expressing the investor's incentive strategy in terms of two auxiliary processes and turning the problem into a classical stochastic differential game problem
- (iii) In order to make the model more realistic, this paper brings in risk sensitivity coefficients to represent the subjects' risk aversion attitudes

This paper is organized as follows. In Section 2, we establish a continuous-time model of the fund management problem. In Section 3, we discuss the manager's optimization problem under fixed investor's incentive strategy. By substituting the manager's optimal strategy into the investor's optimal problem, both the investor and the manager's optimal strategies are obtained in Section 4.

2. The Principal-Agent Conflict Model

Similar to the model in [20], let us assume that the investor employs a professional fund controller (manager) to invest and the investor will get a profit and pay the manager at the terminal moment T . Since the manager's effort level cannot be observed, the investor will determine the manager's salary according to the terminal profit of the investment. The investor's return is determined by the terminal investment profit and the manager's salary. The terminal investment profit is related to the manager's investment strategy and effort level, and the incentive mechanism largely determines the manager's strategy. Therefore, the investor needs to find the optimal incentive mechanism (the manager's salary) to maximize his terminal net income. Meanwhile, according to the investor's incentive mechanism, the manager shall decide his investment strategy and the best effort level to maximize his net salary (terminal salary minus effort cost). This is a non-cooperative game problem. Next, let us build a mathematical model of this problem in probability space (Ω, \mathcal{F}, P) .

Similar to the model in [18], we suppose that the manager's effort will affect the fund income R_t^n which satisfies

$$dR_t^n = R_t^n[(r + \mu + n_t)dt + \sigma dW(t)], \quad (1)$$

where $\mu \geq 0$, $\sigma \geq 0$, and $r > 0$ is the risk-free interest rate, $W(t)$ is a Brownian motion on (Ω, \mathcal{F}, P) , and $\{n_t\}_{t \geq 0}$ is the manager's effort level. Here, for the convenience of calculation, we assume that the drift coefficient of R_t^n is a linear function of the manager's effort level. In fact, as long as the drift coefficient of R_t^n has the form of $R_t^n(r + f(n_t))$ for some function $f(n)$, the same method in this paper can be used after replacing n with $f(n)$. For more general forms of the drift coefficient of R_t^n , the existence of the time value makes it hard to obtain explicit solutions.

Considering the manager's strategy $\pi = (b_t^\pi, n_t^\pi)$, where b_t^π represents the wealth that the manager decides to operate at moment t (The manager may not want to operate all the

wealth since the cost of the effort will increase with the wealth operated increases. The money left will get a risk-free return.) and n_t^π represents the manager's effort level at t . By some simple calculations, we can get that the investment income under this strategy satisfies

$$dX_t^\pi = (rX_t^\pi + b_t^\pi(\mu + n_t^\pi))dt + b_t^\pi\sigma dW(t). \quad (2)$$

Define the natural filtration produced by $W(t)$ as $\{\mathcal{F}_t^W\}_{t \geq 0}$. Now, let us give the definition of both the manager and the investor's admissible strategies. Considering the manager's strategy $\pi = (b_t^\pi, n_t^\pi)$. If b_t^π and n_t^π are bounded positive predictable stochastic processes, under the strategy π , (2) has a unique solution.

We call that strategy $\pi = (b_t^\pi, n_t^\pi)$ is admissible. Denote the set of all the manager's admissible strategies by Π .

Remark 1. Here, we do not consider the case when $b = 0$ or $n = 0$ since in that case, the model is meaningless.

Suppose that the investor's incentive strategy is a function of the investment income at T and denote it by $w(\cdot)$. If $\sup_{\pi \in \Pi} E[w(X_T^\pi)] < \infty$, the manager's value function under $w(\cdot)$ is a decreasing convex function with respect to the initial wealth, we say that $w(\cdot)$ is the investor's admissible strategy. Denote the set of all the investor's admissible strategies by $\tilde{\Pi}$.

Now, let us analyze the whole game process. Referring to [15], we know that investors play a leading role in the game. Managers need to decide their effort level and investment strategy according to the investors' incentive strategy. Therefore, first, we need to fix $w(\cdot)$ and investigate the manager's optimal problem. We can get the manager's optimal effort and investment strategy in terms of $w(\cdot)$ as a byproduct. Then, by substituting the manager's optimal strategy into the wealth process, we can solve the investor's optimal problem by using the dynamic programming principle.

Therefore, firstly, we fix the investor's incentive strategy $w(\cdot)$ and consider the manager's optimal problem. Suppose that the manager is risk-averse and denote his risk sensitivity coefficient by $\gamma_m < 0$. Referring to [18], we suppose that the manager needs to pay $(\theta n^2 b/2)$ to manage b units of capital in unit time under the effort level n . Here, $\theta > 0$ is a constant which represents the effort cost parameter. The objective of the manager is to find the optimal effort level and investment strategy to maximize his net income (salary minus effort cost), which is equivalent to minimize

$$J_m^\pi(t, x; w) = E \left[e^{\gamma_m \left(w(X_T^\pi) - \int_t^T e^{r(T-t)} (\theta (n_t^\pi)^2 / 2) b_t^\pi dt \right)} \middle| X_t^\pi = x \right]. \quad (3)$$

Denote the manager's optimal strategy by π^w , then the value function is

$$V_m(t, x; w) = \inf_{\pi \in \Pi} J_m^\pi(t, x; w) = J_m^{\pi^w}(t, x; w). \quad (4)$$

Suppose that the investor is risk-averse too, his risk-sensitive coefficient is $\gamma_f < 0$. Next, we consider the investor's optimal problem.

If the manager's salary is too high, the investor's income will be reduced. If the manager's salary is too low, the manager's enthusiasm wanes, which also deduces the investor's terminal income. Therefore, the investor needs to find a reasonable incentive strategy to maximize his net income, that is, minimize

$$J_f^w(t, x) = E \left[e^{\gamma_f (X_T^w - w(X_T^w))} | X_t^w = x \right], \quad (5)$$

where X_t^w is the investment income process under strategy π^w . Thus, the investor's value function is

$$V_f(t, x) = \inf_{w \in \Pi} J_f^w(t, x). \quad (6)$$

Remark 2. The problem discussed above is not a standard stochastic optimal control problem since the form of $w(\cdot)$ is uncertain, and we cannot solve it directly by using standard stochastic optimal methods. In Section 3, we give another form of the incentive strategy and transform the game problem into a classical one. Then, we can use the dynamic programming principle to solve the problem.

3. The Manager's Optimization Problem

Define $D_t = e^{r(T-t)}$, $\beta(t, \pi) = \gamma_m D_t (\theta n_t^{\pi^2}/2) b_t^\pi$, and $\Gamma(t, T, \pi) = e^{-\int_t^T \beta(u, \pi) du}$. Then, $J_m^\pi(t, x; w)$ can be denoted by

$$J_m^\pi(t, x; w) = E \left[\Gamma(t, T, \pi) e^{\gamma_m w(X_T^\pi)} | X_t^\pi = x \right]. \quad (7)$$

Using the results of Section 3.4 in [21], we know that, under the incentive strategy $w(\cdot)$, the manager's value function $V_m(t, x; w)$ satisfies the HJB equation:

$$-V_{mt}(t, x; w) = \inf_{\pi \in \Pi} \{ -\beta(t, \pi) V_m(t, x; w) + [rx + b_t^\pi (\mu + n_t^\pi)]$$

$$V_{mx}(t, x; w) + \frac{b_t^{\pi^2} \sigma^2}{2} V_{mxx}(t, x; w) \} \quad (8)$$

and the boundary condition

$$V_m(T, x; w) = e^{\gamma_m w(x)}. \quad (9)$$

Since $V_m(t, x; w)$ is a decreasing convex function of x , for $\forall (t, x, y, z, \gamma) \in [0, T] \times \mathbb{R} \times [0, \infty) \times (-\infty, 0) \times (0, \infty)$, we can define the Hamiltonian function:

$$H(t, x, y, z, \gamma) = \inf_{n>0, b>0} h(t, x, y, z, \gamma, n, b), \quad (10)$$

where

$$h(t, x, y, z, \gamma, n, b) = -D_t \frac{\gamma_m \theta n^2 b}{2} y + (rx + b(\mu + n))z + \frac{b^2 \sigma^2}{2} \gamma. \quad (11)$$

Theorem 1.

$$n_t^{*y, z, \gamma} = \frac{z}{\theta \gamma_m y D_t}, \quad (12)$$

$$b_t^{*y, z, \gamma} = \frac{-(\mu + (n_t^{*y, z, \gamma}/2))}{\sigma^2} \frac{z}{\gamma}, \quad (13)$$

is the minimum point of h in (10).

Proof. According to the definition, we know that h is a convex function of (n, b) . So, the minimum point of h in (10) is the stable point under constraint conditions $n > 0, b > 0$. By some simple calculations, we have

$$\begin{aligned} h_n(n, b; t, x, y, z, \gamma) &= -\theta D_t b n \gamma_m y + bz, \\ h_b(n, b; t, x, y, z, \gamma) &= \sigma^2 \gamma b + (\mu + n)z - \frac{D_t \theta n \gamma_m y}{2}. \end{aligned} \quad (14)$$

Combining the above two equations, we can obtain the stable point of h :

$$\begin{aligned} n_t^{*y, z, \gamma} &= \frac{z}{\theta \gamma_m y D_t} > 0, \\ b_t^{*y, z, \gamma} &= \frac{-(\mu + (n_t^{*y, z, \gamma}/2))}{\sigma^2} \frac{z}{\gamma} > 0. \end{aligned} \quad (15)$$

The proof is done. \square

Remark 3. In this case, the optimal investment strategy is similar to that without principal-agent relationships. The only difference is that the numerator of the optimal investment strategy is changed from $(\mu + n_t^{*y, z, \gamma})$ into $(\mu + (n_t^{*y, z, \gamma}/2))$. Clearly, this is due to the existence of the agency relationship.

Apparently, the investor's incentive strategy and the manager's value function are one-to-one. In the following, we will use auxiliary stochastic processes (Z_t, Γ_t) to determine the manager's value function and transform the investor's incentive strategy into (Z_t, Γ_t) . Then, the problem in Section 2 can be translated into a classical stochastic optimal control problem.

First, let us give the space of auxiliary stochastic processes (Z, Γ) . Fix $t \in [0, T]$, let $Z: [t, T] \times \Omega \rightarrow (-\infty, 0), \Gamma: [t, T] \times \Omega \rightarrow (0, \infty)$ be \mathcal{F}^W -predictable processes which satisfy

$$E \left[\int_t^T (Z_s^2 \sigma_s^2 + \Gamma_s \sigma_s^2) ds \right] < +\infty. \quad (16)$$

Denote the set of all the processes satisfying the above conditions by $\mathcal{V}(t)$.

For some $(Z, \Gamma) \in \mathcal{V}(t)$ and $Y_t \geq 0$, define the \mathcal{F}^W -progressively measurable process $Y^{Z, \Gamma}$ on the filtration space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t^W\}_{t \geq 0})$ by

$$Y_s^{Z, \Gamma} = Y_t - \int_t^s H(r, X_r, Y_r^{Z, \Gamma}, Z_r, \Gamma_r) dr + \int_t^s Z_r dX_r + \frac{1}{2} \int_t^s \Gamma_r d\langle X \rangle_r, \quad s \in [t, T], \quad (17)$$

where X_r is the investment income process. Clearly, for fixed Y_t, Z, Γ , $Y_T^{Z, \Gamma}$ is only related to the investment income process and is \mathcal{F}_T measurable, suppose that it is an incentive strategy (we prove it in Corollary 1). In the following, we give the relationship between $Y_s^{Z, \Gamma}$ and the manager's value function. First, we give the following lemma.

Lemma 1. Define

$$\begin{aligned} \pi^{*Z, \Gamma} &= (b^{*Z, \Gamma}, n^{*Z, \Gamma}) \\ &= \left(\left\{ b_t^{*Y_t^{Z, \Gamma}, Z_t, \Gamma_t} \right\}_{t \geq 0}, \left\{ n_t^{*Y_t^{Z, \Gamma}, Z_t, \Gamma_t} \right\}_{t \geq 0} \right), \end{aligned} \quad (18)$$

and then we have $\pi^{*Z, \Gamma} \in \Pi$.

Proof. On the one hand, since $Z, \Gamma, Y^{Z, \Gamma}$ are all predictable stochastic processes, referring to (12) and (13), we can get that $b^{*Z, \Gamma}$ and $n^{*Z, \Gamma}$ are bounded positive predictable stochastic processes. On the other hand, $b_t^{*Y_t^{Z, \Gamma}, Z_t, \Gamma_t}$ and $n_t^{*Y_t^{Z, \Gamma}, Z_t, \Gamma_t}$ are independent of x . Taking $b^{*Z, \Gamma}$ and $n^{*Z, \Gamma}$ into (2), we can get the Lipschitz continuity and linear growth of the coefficients in (2) with respect to X_t ; then, (2) has a unique solution. The proof is done.

Denote the investment income process under $\pi^{*Z, \Gamma}$ by $X^{*Z, \Gamma}$. We also have the following theorem. \square

Theorem 2. Denote the manager's value function with a terminal return $(\ln Y_T^{Z, \Gamma} / \gamma_m)$ by $V_m(t, x; Y_T^{Z, \Gamma})$. We can obtain that

$$Y_t = V_m(t, x; Y_T^{Z, \Gamma}). \quad (19)$$

Furthermore, the manager's optimal strategy is $\pi^{*Z, \Gamma}$.

Proof. $\forall \pi \in \Pi, s \in [t, T]$, we have

$$\begin{aligned} Y_s^{Z, \Gamma} &= Y_t - \int_t^s H(r, X_r^\pi, Y_r^{Z, \Gamma}, Z_r, \Gamma_r) dr \\ &\quad + \int_t^s Z_r dX_r^\pi + \frac{1}{2} \int_t^s \Gamma_r d\langle X^\pi \rangle_r. \end{aligned} \quad (20)$$

Using Ito's formula, we have

$$\begin{aligned} de^{-\int_t^r \beta(u, \pi) du} Y_r^{Z, \Gamma} &= e^{-\int_t^r \beta(u, \pi) du} \left[-H(r, X_r^\pi, Y_r^{Z, \Gamma}, Z_r, \Gamma_r) \right. \\ &\quad \left. + (rX_r^\pi + b_r^\pi(\mu + n_r^\pi))Z_r \right. \\ &\quad \left. + \frac{b_r^{\pi^2} \sigma^2}{2} \Gamma_r - \beta(r, \pi) \right] dr \\ &\quad + e^{-\int_t^r \beta(u, \pi) du} \sigma Z_r dW(r). \end{aligned} \quad (21)$$

It follows from (16) that $e^{-\int_t^r \beta(u, \pi) du} \sigma Z_r dW(r)$ is a martingale. Integrating and taking expectations on both sides of (21), we can get

$$Y_t \geq E \left[e^{-\int_t^T \beta(u, \pi) du} Y_T^{Z, \Gamma} | X_t^\pi = x \right] = J_m^\pi(t, x; Y_T^{Z, \Gamma}). \quad (22)$$

Furthermore, by simple calculations, under $\pi^{*Z, \Gamma} \in \Pi$, we have

$$dY_t^{Z, \Gamma} = \beta(t, \pi^{*Z, \Gamma}) Y_t^{Z, \Gamma} dt + b_t^{*Y_t^{Z, \Gamma}, Z_t, \Gamma_t} Z_t \sigma dW_t. \quad (23)$$

Using (23) and Ito's formula, we can obtain

$$de^{-\int_t^r \beta(u, \pi^{*Z, \Gamma}) du} Y_r^{Z, \Gamma} = e^{-\int_t^r \beta(u, \pi^{*Z, \Gamma}) du} b_t^{*Y_t^{Z, \Gamma}, Z_t, \Gamma_t} Z_t \sigma dW_t. \quad (24)$$

With similar methods, integrating and taking expectations on both sides of (24), we have

$$\begin{aligned} Y_t &= E \left[e^{-\int_t^T \beta(u, \pi^{*Z, \Gamma}) du} Y_T^{Z, \Gamma} | X_t^{*Z, \Gamma} = x \right] \\ &= J_m^{\pi^{*Z, \Gamma}}(t, x; Y_T^{Z, \Gamma}) \geq J_m^\pi(t, x; Y_T^{Z, \Gamma}). \end{aligned} \quad (25)$$

This implies that $\pi^{*Z, \Gamma}$ is the manager's optimal strategy and

$$Y_t = V_m(t, x; Y_T^{Z, \Gamma}). \quad (26)$$

Up till now, fixing $(Z, \Gamma) \in \mathcal{V}(t)$, we can get the manager's optimal strategy and represent the manager's value function. In Section 4, we begin to consider the investor's optimization problem. That is, finding the optimal $(Z, \Gamma) \in \mathcal{V}(t)$ to maximize the investor's net profit. \square

4. The Investor's Optimization Problem

Suppose that the investor's wealth is x at t . Apparently, the investor's value function is uniquely determined by the wealth process and the manager's value function. So, the

objective of the investor is to find the optimal $(Z, \Gamma) \in \mathcal{V}(t)$ to minimize his value function. Define

$$v(t, x, y) = \inf_{(Z, \Gamma) \in \mathcal{V}(t)} E \left[e^{\gamma_f (X_t^{*Z, \Gamma} - (\ln Y_t^{Z, \Gamma} / \gamma_m))} | X_t^\pi = x, Y_t^{Z, \Gamma} = y \right]. \quad (27)$$

Referring to Theorem 4.1 in [19], we know that if Assumption 3.2, Assumption 4.3, and Assumption 4.4 in [19] hold, the investor's value function satisfies

$$V_f(t, x) = \inf_{y \in [0, e^{\gamma_m R}]} v(t, x, y). \quad (28)$$

Here, R is the minimum pay in order to make sure that the manager takes the job.

Section 4.1 gives the verification of the three assumptions.

4.1. The Verification of Assumptions

Assumption 1 (Assumption 3.2 in [19]). H has at least one extreme point $(b_t^{*y, z, \gamma}, n_t^{*y, z, \gamma})$. For any $t \in [0, T]$, $(Z, \Gamma) \in \mathcal{V}(t)$, we have $\pi^{*Z, \Gamma} \in \Pi$.

Proof. This is the result of Theorem 1 and Lemma 1.

The Hamiltonian function can be expressed as

$$H(t, x, y, z, \gamma) = \inf_{b > 0} \left\{ F(t, x, y, z, b) + \frac{b^2 \sigma^2}{2} \gamma \right\}. \quad (29)$$

Here,

$$F(t, x, y, z, b) = \inf_{n > 0} \left\{ -D_t \frac{\gamma_m \theta n^2 b}{2} y + (rx + b(\mu + n))z \right\}. \quad (30)$$

Define

$$Y_s^Z = Y_t - \int_t^s F(r, X_r, Y_r^Z, Z_r) dr + \int_t^s Z_r dX_r, \quad s \in [t, T], \quad (31)$$

and we have the following assumption. \square

Assumption 2 (Assumption 4.3 in [19]). F has at least one extreme point $n_t^{*y, z, b}$; furthermore, $(b, n^{*Y^Z, Z, b}) \in \Pi$.

Proof. On the one hand, the right hand of F is a parabola with an opening up with respect to n ; so, the minimum point is attained at the axis of the parabola $(z/D_t \gamma_m \theta y)$, that is, $n_t^{*y, z, b} = (z/D_t \gamma_m \theta y)$. On the other hand, since $Z < 0$ is predictable, we can get that $n_t^{*Y_t^Z, Z_t, b} = (bZ_t/D_t \gamma_m \theta b Y_t^Z)$ is a positive predictable process. Furthermore, b and $n_t^{*y, z, b}$ are independent of x . This implies the Lipschitz continuity and linear growth of the coefficients in (2) with respect to the investment income process; then, (2) has a unique solution. \square

Assumption 3 (Assumption 4.4 in [19]). $\forall b > 0$, $(1/b^2 \sigma^2)$ is bounded.

Proof. We can get the result directly from $\sigma > 0, b > 0$. \square

4.2. The Investor's Value Function. Clearly, as soon as we get $v(t, x, y)$, we can obtain $V_f(t, x)$. The following theorem gives the partial differential equation satisfied by $v(t, x, y)$.

Theorem 3. $v(t, x, y)$ is the viscosity solution of

$$-v_t(t, x, y) = \inf_{(Z, \Gamma) \in \mathcal{V}(t)} G(t, x, y, Z, \Gamma), \quad (32)$$

$$v(T, x, y) = e^{\gamma_f x} y^{(-\gamma_f / \gamma_m)}, \quad (33)$$

where

$$\begin{aligned} G(t, x, y, Z, \Gamma) = & [rx + b_t^{*Z, \Gamma}(\mu + n_t^{*Z, \Gamma})]v_x \\ & + \frac{\sigma^2 (b_t^{*Z, \Gamma})^2}{2} v_{xx} + \frac{D_t \gamma_m \theta (n_t^{*Z, \Gamma})^2}{2} b_t^{*Z, \Gamma} y v_y \\ & + \frac{\sigma^2 (b_t^{*Z, \Gamma})^2}{2} Z^2 v_{yy} + \sigma^2 (b_t^{*Z, \Gamma})^2 Z v_{xy}. \end{aligned} \quad (34)$$

Proof. By the definition of $v(t, x, y)$, we can obtain that it satisfies (33). Furthermore, according to the dynamic programming principle, we have

$$v(t, x, y) = \inf_{(Z, \Gamma) \in \mathcal{V}(t)} v(t+h, X_{t+h}^{*Z, \Gamma}, Y_{t+h}^{Z, \Gamma}). \quad (35)$$

By using Ito's formula with respect to $v(s, X_s^{*Z, \Gamma}, Y_s^{Z, \Gamma})$ from t to $t+h$, we have

$$\begin{aligned} v(t+h, X_{t+h}^{*Z, \Gamma}, Y_{t+h}^{Z, \Gamma}) = & v(t, x, y) + \int_t^{t+h} v_t(s, X_s^{*Z, \Gamma}, Y_s^{Z, \Gamma}) ds \\ & + G(s, X_s^{*Z, \Gamma}, Y_s^{Z, \Gamma}, Z_s, \Gamma_s) ds. \end{aligned} \quad (36)$$

Combining with the above two equations, we can get

$$v_t(t, x, y) + \inf_{(Z, \Gamma) \in \mathcal{V}(t)} G(t, x, y, Z, \Gamma) = 0. \quad (37)$$

That is, $v(t, x, y)$ satisfies (32). The proof is done.

Next, we are going to solve (32) and (33). Considering the boundary condition, we guess

$$v(t, x, y) = e^{\gamma_f D_t x} y^{(-\gamma_f / \gamma_m)} E(t), \quad (38)$$

where $E(t)$ is a function of t which satisfies $E(T) = 1$.

If the variables in the solution can be separated from each other, (32) can be easily solved. However, (32) contains $e^{\gamma_f D_t x}$, which is a cross term of t and x . To cancel the cross term, we introduce $z_t = D_t X_t^{*Z, \Gamma}$. Using Ito's formula, we can get

$$\begin{aligned} dz_t = & -r D_t X_t^{*Z, \Gamma} dt + D_t dX_t^{*Z, \Gamma} \\ = & D_t b_t^{*Z, \Gamma} [(\mu + n_t^{*Z, \Gamma}) dt + \sigma dW(t)]. \end{aligned} \quad (39)$$

We can also obtain $z_T = X_T^{*Z,\Gamma}$. Define

$$\begin{aligned} V(t, z, y) &= \inf_{(Z,\Gamma) \in \mathcal{V}(t)} E \left[e^{\gamma_f (z_T - (\ln Y_T^{Z,\Gamma} / \gamma_m))} | z_t = z \right] \\ &= \inf_{(Z,\Gamma) \in \mathcal{V}(t)} E \left[e^{\gamma_f (X_T^{*Z,\Gamma} - (\ln Y_T^{Z,\Gamma} / \gamma_m))} | X_t^{*Z,\Gamma} = \frac{z}{D_t} \right] \\ &= v \left(t, \frac{z}{D_t}, y \right). \end{aligned} \quad (40)$$

Obviously, solving $v(t, x, y)$ is equivalent to solving $V(t, z, y)$. Using a similar method as the one in Theorem 3, we can get that

$$\begin{aligned} -V_t &= \inf_{(Z,\Gamma) \in \mathcal{V}(t)} \left\{ -\frac{(\mu + (n_t^{*Z,\Gamma}/2))(\mu + n_t^{*Z,\Gamma})}{\sigma^2} D_t \frac{Z}{\Gamma} V_z \right. \\ &\quad - \frac{\gamma_m \theta (n_t^{*Z,\Gamma})^2 (\mu + (n_t^{*Z,\Gamma}/2))}{2\sigma^2} \gamma D_t \frac{Z}{\Gamma} V_y \\ &\quad + \frac{(\mu + (n_t^{*Z,\Gamma}/2))^2}{2\sigma^2} D_t^2 \frac{Z^2}{\Gamma^2} V_{zz} \\ &\quad + \frac{(n_t^{*Z,\Gamma})^2 (\mu + (n_t^{*Z,\Gamma}/2))^2}{2\sigma^2} (\gamma_m \theta \gamma D_t)^2 \frac{Z^2}{\Gamma^2} V_{yy} \\ &\quad \left. + \frac{n_t^{*Z,\Gamma} (\mu + (n_t^{*Z,\Gamma}/2))^2}{\sigma^2} \gamma_m \theta \gamma D_t^2 \frac{Z^2}{\Gamma^2} V_{zy} \right\}, \end{aligned} \quad (41)$$

$$V(T, z, y) = e^{\gamma_f z} y^{-(\gamma_f / \gamma_m)}. \quad (42)$$

The first step in solving (41) is to find its minimum point. Define $M^{Z,\Gamma} = (Z/\Gamma)$, it is shown in Section 3 that (Z, Γ) and $(M^{Z,\Gamma}, n^{*Z,\Gamma})$ are one-to-one. Then, (41) is transformed into

$$\begin{aligned} -V_t &= \inf_{(n,M) \in \mathbb{R}^+ \times \mathbb{R}^-} \left\{ -\frac{(\mu + (n/2))(\mu + n)}{\sigma^2} D_t M V_z \right. \\ &\quad + \frac{(\mu + (n/2))^2}{2\sigma^2} D_t^2 M^2 V_{zz} - \frac{\gamma_m \theta n^2 (\mu + (n/2))}{2\sigma^2} \gamma D_t M V_y \\ &\quad + \frac{n^2 (\mu + (n/2))^2}{2\sigma^2} (\gamma_m \theta \gamma D_t)^2 M^2 V_{yy} \\ &\quad \left. + \frac{n(\mu + (n/2))^2}{\sigma^2} \gamma_m \theta \gamma D_t^2 M^2 V_{zy} \right\}. \end{aligned} \quad (43)$$

Now, the problem of finding the minimum point in (41) is changed into a problem of finding the minimum point in (43).

According to (38), we suppose that $V(t, z, y) = E(t) e^{\gamma_f z} y^{-(\gamma_f / \gamma_m)}$. By some simple calculations, we can get that

$$\begin{aligned} V_z(t, z, y) &= \gamma_f V(t, z, y), \\ V_{zz}(t, z, y) &= \gamma_f^2 V(t, z, y), \\ \gamma V_y(t, z, y) &= -\frac{\gamma_f}{\gamma_m} V(t, z, y), \\ \gamma^2 V_{yy}(t, z, y) &= \frac{\gamma_f (\gamma_f + \gamma_m)}{\gamma_m^2} V(t, z, y), \\ \gamma V_{zy}(t, z, y) &= -\frac{\gamma_f^2}{\gamma_m} V(t, z, y). \end{aligned} \quad (44)$$

Taking them into (43), we have

$$\begin{aligned} -E'(t) V(t, z, y) &= \inf_{(n,M) \in \mathbb{R}^+ \times \mathbb{R}^-} \left\{ -\frac{(\mu + (n/2))(\mu + n)}{\sigma^2} D_t M \gamma_f \right. \\ &\quad + \frac{(\mu + (n/2))^2}{2\sigma^2} D_t^2 M^2 \gamma_f^2 + \frac{\gamma_f \theta n^2 (\mu + (n/2))}{2\sigma^2} D_t M \\ &\quad + \frac{n^2 (\mu + (n/2))^2}{2\sigma^2} \theta^2 D_t^2 M^2 \gamma_f (\gamma_f + \gamma_m) \\ &\quad \left. - \frac{n(\mu + (n/2))^2}{\sigma^2} \gamma_f^2 \theta D_t^2 M^2 \right\} E(t) V(t, z, y). \end{aligned} \quad (45)$$

Since the right hand of (45) is continuous, the minimum point can only be attained at the stable points or the boundary points, which depends on the parameter values. Denote the minimum point of (45) by (n_t^*, M_t^*) , and denote the corresponding minimum point of (41) by (Z_t^*, Γ_t^*) . It is shown from the Appendix that n_t^* and $D_t M_t^*$ are constants concerning μ, θ, γ_f , and γ_m . Let $n_t^* = n^*$. \square

Remark 4. On the one hand, the exponential form of the objective function implies that b_t^* is independent of X_t^* . On the other hand, the benefit and the cost brought by the manager's effort are only related to b_t^* , so n_t^* is independent of X_t^* . Furthermore, in this paper, we consider the discounted benefit and cost brought by the manager's effort; so, n_t^* is independent of t .

Remark 5. It is shown from figures in the Appendix that n^* decreases with an increase in μ (the drift coefficient of the fund wealth process), θ (the effort cost coefficient), and $|\gamma_m|$ (the manager's risk aversion level). It increases with an increase in $|\gamma_f|$ (the investor's risk aversion level).

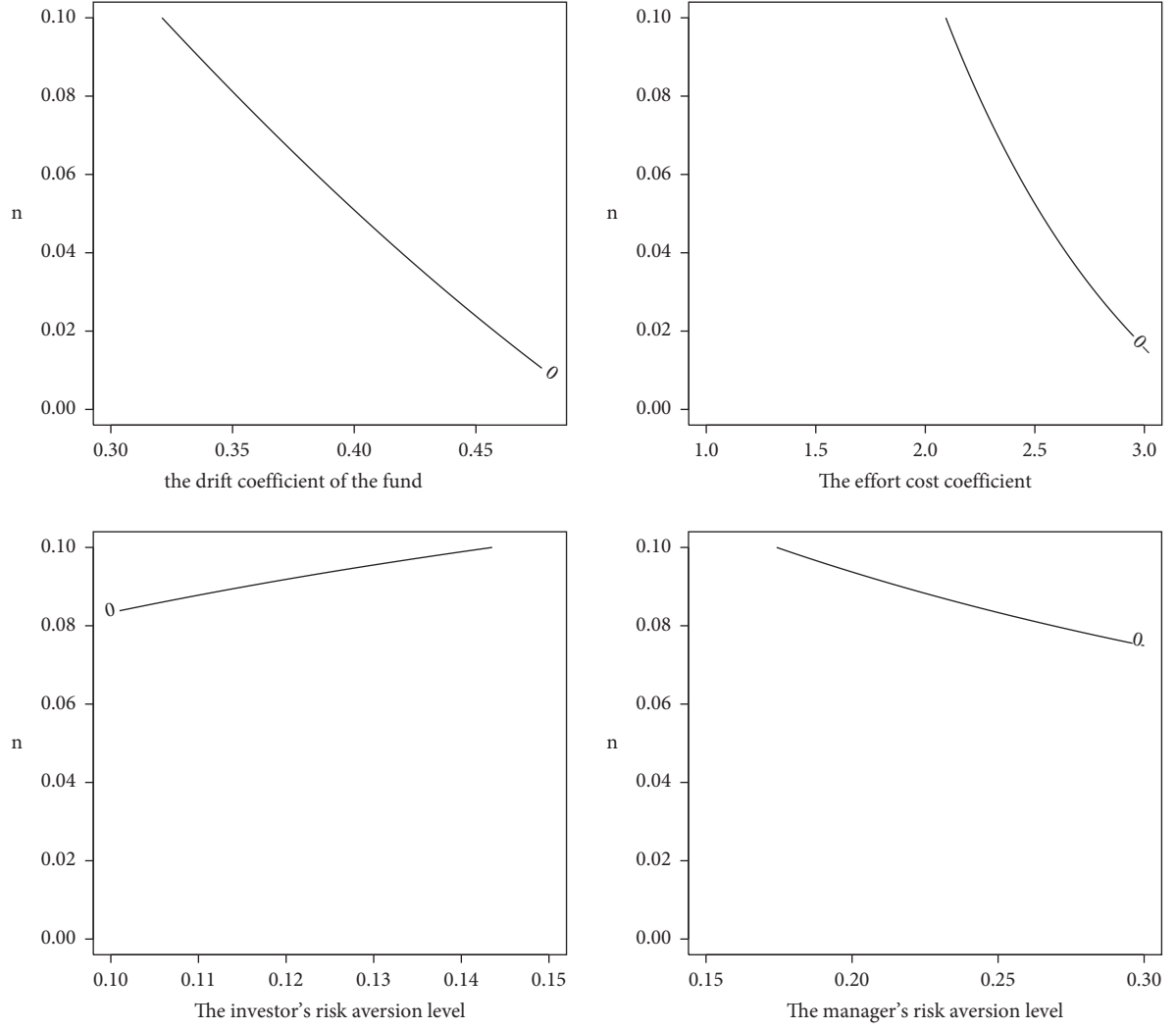


FIGURE 1: The effects of parameters on the optimal effort level.

Remark 6. Define $Y_t^* = V_m(t, x; Y_T^{Z^*, \Gamma^*})$, considering (12) and (13), we can get that $(Z_t^*/Y_t^*) = \theta \gamma_m D_t n^*$ and $D_t b_t^* = (- (\mu + (n^*/2))/\sigma^2) D_t M_t^*$ are constants.

$$V(t, z, y) = e^{B(T-t)} e^{\gamma_f z} y^{- (\gamma_f / \gamma_m)}. \quad (46)$$

Here,

Taking the minimum point into (45) and solving it, we can get

$$\begin{aligned} B = & -\frac{(\mu + (n^*/2))(\mu + n^*)}{\sigma^2} D_t M_t^* \gamma_f \\ & + \frac{(\mu + (n^*/2))^2}{2\sigma^2} D_t^2 M_t^{*2} \gamma_f^2 + \frac{\gamma_f \theta n^{*2} (\mu + (n^*/2))}{2\sigma^2} D_t M_t^* \\ & + \frac{n^{*2} (\mu + (n^*/2))^2}{2\sigma^2} \theta^2 D_t^2 M_t^{*2} \gamma_f (\gamma_f + \gamma_m) \\ & - \frac{n^* (\mu + (n^*/2))^2}{\sigma^2} \gamma_f^2 \theta D_t^2 M_t^{*2} \end{aligned} \quad (47)$$

is a constant. As a consequence, we can also get the following results.

$$\begin{aligned} v(t, x, y) &= e^{B(T-t)} e^{\gamma_f D_t x} y^{(-\gamma_f/\gamma_m)}, \\ V_f(t, x) &= v(t, x, e^{\gamma_m R}) = e^{B(T-t)} e^{\gamma_f (D_t x - R)}. \end{aligned} \quad (48)$$

4.3. The Investor's Excitation Mechanism. In this section, let us analyze the investor's excitation mechanism. Denote $Y_t^* = Y_t^{Z^*, I^*}$. From the above analysis, we know that

$$\begin{aligned} dY_t^* &= \frac{D_t \gamma_m \theta n^{*2} b_t^* Y_t^*}{2} dt + b_t^* Z_t^* \sigma dW_t, \\ Y_t^* &= e^{\gamma_m R}. \end{aligned} \quad (49)$$

Using Ito's formula, we have

$$d \ln Y_t^* = \frac{D_t \gamma_m \theta n^{*2} b_t^*}{2} dt + b_t^* \frac{Z_t^*}{Y_t^*} \sigma dW_t - b_t^{*2} \sigma^2 \frac{Z_t^{*2}}{2Y_t^{*2}} dt. \quad (50)$$

Furthermore, we can get that the investment income under n^* and b_t^* satisfies

$$dX_t^* = (rX_t^* + b_t^* (n^* + \mu)) dt + b_t^* \sigma dW_t, \quad (51)$$

which implies

$$\begin{aligned} d \ln Y_t^* &= \left(\frac{D_t \gamma_m \theta n^{*2} b_t^*}{2} - b_t^{*2} \sigma^2 \frac{Z_t^{*2}}{2Y_t^{*2}} \right) dt + \frac{Z_t^*}{Y_t^*} (dX_t^* - (rX_t^* + b_t^* (n^* + \mu)) dt) \\ &= \gamma_m \left(\frac{D_t b_t^* \theta n^{*2}}{2} - \frac{1}{2} D_t^2 b_t^{*2} \theta^2 \gamma_m n^{*2} \sigma^2 + D_t b_t^* \theta n^* (n^* + \mu) \right) dt \\ &\quad + \gamma_m \theta n^* D_t (dX_t^* - rX_t^* dt). \end{aligned} \quad (52)$$

Define constant

$$A = \frac{D_t b_t^* \theta n^{*2}}{2} - \frac{1}{2} D_t^2 b_t^{*2} \theta^2 \gamma_m n^{*2} \sigma^2 + D_t b_t^* \theta n^* (n^* + \mu) > 0, \quad (53)$$

and then we can obtain

$$d \ln Y_t^* = \gamma_m A dt + \gamma_m \theta n^* D_t (dX_t^* - rX_t^* dt) = \gamma_m (A dt + \theta n^* dD_t X_t^*). \quad (54)$$

So,

$$\ln Y_T^* - \gamma_m R = \gamma_m [A(T-t) + n^* \theta (X_T^* - D_t x)] \quad (55)$$

can be deduced immediately. Since $\ln Y_T = \gamma_m w(X_T)$, we can get the strategy

$$w(X_T) = A(T-t) + n^* \theta (X_T^* - D_t x) + R. \quad (56)$$

It is a linear function of $X_T^* - D_t x$, which is the discounted profit of the investment.

Remark 7. It follows from the above results that the manager's wages increase with the increase of the cost coefficient, the effort level and the discounted profit of the investment. Furthermore, the longer the work, the higher the salary. It is consistent with reality.

We can also get the following corollary.

Corollary 1

$$Y_s^* = V_m(s, X_s^*; Y_T^*) = e^{\gamma_m [R + A(T-t) + n^* \theta (X_s^* - D_t x)]}. \quad (57)$$

This implies that $V_m(s, X_s^*; Y_T^*)$ is a decreasing convex function of X_s^* . Thus, the assumption in Section 2 that Y_T^* is an incentive strategy is proved.

Appendix

Define

$$\begin{aligned} I(n, M; t) &= -\frac{(\mu + (n/2))(\mu + n)}{\sigma^2} D_t M \gamma_f \\ &\quad + \frac{(\mu + (n/2))^2}{2\sigma^2} D_t^2 M^2 \gamma_f^2 + \frac{\gamma_f \theta n^2 (\mu + (n/2))}{2\sigma^2} D_t M \\ &\quad + \frac{n^2 (\mu + (n/2))^2}{2\sigma^2} \theta^2 D_t^2 M^2 \gamma_f (\gamma_f + \gamma_m) \\ &\quad - \frac{n(\mu + (n/2))^2}{\sigma^2} \gamma_f^2 \theta D_t^2 M^2. \end{aligned} \quad (A.1)$$

We know that there are three kinds of points which may be the minimum point of (45):

(i) The points which satisfy $I_n(n, M; t) = 0, I_M$

$$(n, M; t) = 0$$

(ii) The points which satisfy $n = 0, I_M(0, M, t) = 0$

(iii) The points which satisfy $M = 0, I_n(n, 0, t) = 0$

With parameters fixed, we can easily decide which is the minimum point of (45). In the following, we will investigate the form of those points.

The first kind of points (n_{1t}, M_{1t}) is the solution of the following equations:

$$\begin{aligned} I_n(n, M; t) = & -\frac{D_t \gamma_f}{\sigma^2} \left(\frac{3}{2} \mu + n \right) M \\ & + \frac{D_t \gamma_f \theta}{2\sigma^2} \left(\frac{3}{2} n^2 + 2\mu n \right) M + \frac{D_t^2 \gamma_f^2}{2\sigma^2} \left(\mu + \frac{n}{2} \right) M^2 \\ & + \frac{\theta^2 D_t^2 \gamma_f (\gamma_f + \gamma_m)}{2\sigma^2} (n^3 + 3\mu n^2 + 2\mu^2 n) M^2 \\ & - \frac{\gamma_f^2 \theta D_t^2}{\sigma^2} \left(\frac{3n^2}{4} + 2\mu n + \mu^2 \right) M^2 = 0, \end{aligned}$$

$$\begin{aligned} I_M(n, M; t) = & -\frac{D_t \gamma_f}{\sigma^2} (\mu + n) \left(\mu + \frac{n}{2} \right) \\ & + \frac{D_t \gamma_f \theta}{2\sigma^2} n^2 \left(\mu + \frac{n}{2} \right) + \frac{D_t^2 \gamma_f^2}{\sigma^2} \left(\mu + \frac{n}{2} \right)^2 M \\ & + \frac{\theta^2 D_t^2 \gamma_f (\gamma_f + \gamma_m)}{\sigma^2} n^2 \left(\mu + \frac{n}{2} \right)^2 M \\ & - \frac{2\gamma_f^2 \theta D_t^2}{\sigma^2} n \left(\mu + \frac{n}{2} \right)^2 M = 0. \end{aligned} \quad (A.3)$$

We can deduce from (A.2) that

$$D_t M_{1t} = \frac{n_{1t} + \mu - (n_{1t}^2 \theta / 2) + (\mu / 2) - (\theta n_{1t}^2 / 4) - \theta \mu n_{1t}}{((n_{1t} / 2) + \mu) [2n_{1t} \theta \gamma_f + (n_{1t})^2 \theta^2 (\gamma_m + \gamma_f) + \gamma_f - (\gamma_f / 2) + \gamma_f \theta \mu - (\gamma_f \theta n_{1t} / 2) + \theta^2 \mu (\gamma_f + \gamma_m) n_{1t}]}. \quad (A.4)$$

It also follows from (A.3) that

$$D_t M_{1t} = \frac{n_{1t} + \mu - (n_{1t})^2 \theta / 2}{((n_{1t} / 2) + \mu) [2n_{1t} \theta \gamma_f + (n_{1t})^2 \theta^2 (\gamma_m + \gamma_f) + \gamma_f]}. \quad (A.5)$$

Combining the above two equations, we can get that

$$\begin{aligned} \left(n_{1t} + \mu - \frac{(n_{1t})^2 \theta}{2} \right) \left[-\frac{\gamma_f}{2} + \gamma_f \theta \mu - \frac{\gamma_f \theta n_{1t}}{2} + \theta^2 \mu (\gamma_f + \gamma_m) n_{1t} \right] \\ + \left(\frac{\theta n_{1t}^2}{4} + \theta \mu n_{1t} - \frac{\mu}{2} \right) [2n_{1t} \theta \gamma_f + (n_{1t})^2 \theta^2 (\gamma_m + \gamma_f) + \gamma_f] = 0. \end{aligned} \quad (A.6)$$

Clearly, by solving (A.5) and (A.6), we can get that n_1 and $D_t M_{1t}$ are constants.

Denote the second kind of point by $(0, M_{2t})$. Thus, M_{2t} satisfies (A.5) with n replaced with 0 and we can get that $D_t M_{2t}$ is a constant.

Denote the third kind of points by $(n_{3t}, 0)$. They satisfy (A.4). By solving it, we can get that n_{3t} is a constant.

Denote the minimum point of (45) by (n_t^*, M_t^*) . It follows from the above analysis that n_t^* and $D_t M_t^*$ are all constants. For different μ, θ, γ_f , and γ_m , by calculating (A.6), (A.2), or (A.3), we can get different n^* .

By using R, we plot the following figures which indicate the effect of μ, θ, γ_f , and γ_m on n^* (Figure 1).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Near Optimality of Linear Delayed Doubly Stochastic Control Problem

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In this paper, we study a kind of near optimal control problem which is described by linear quadratic doubly stochastic differential equations with time delay. We consider the near optimality for the linear delayed doubly stochastic system with convex control domain. We discuss the case that all the time delay variables are different. We give the maximum principle of near optimal control for this kind of time delay system. The necessary condition for the control to be near optimal control is deduced by Ekeland's variational principle and some estimates on the state and the adjoint processes corresponding to the system.

1. Introduction

As known to all, stochastic differential equations and stochastic analysis develop rapidly. The theory of stochastic differential equations is widely used in economy, biology, physics, financial mathematics, and other fields. In order to give the probabilistic expression of stochastic partial differential equations, Pardoux and Peng [1] gave a class of double stochastic differential equations. Due to the wide applications of this kind of equation in many fields, more and more people pay attention to it. Han et al. [2] deduced the maximum principle for the backward doubly stochastic control system. Zhu and Shi [3] discussed the optimal control problem of the backward doubly stochastic system with partial information. And then they studied a type of forward-backward doubly stochastic differential equations with random jumps and applied their results to related games [4]. Many scholars have discussed the maximum principle of optimal control for different control systems [5].

With the further exploration of stochastic problems, we find that many problems in the objective world are not only affected by the current state but also influenced by the past history. This kind of problem is called time delay problem. Time delay exists in many fields such as the latent period of infectious diseases, genetic problems, advertising effects,

network transmission, and so on. The equation describing this kind of problem is called delay equation. Because of the importance of time delay, people try to study this kind of problem. Chen and Wu [6] considered the delayed backward stochastic system and obtained the maximum principle for this problem. Wu and Wang [7] studied the optimal control problem of the backward stochastic differential delay equation under partial information. Lv et al. [8] considered the maximum principle for optimal control of the anticipated forward-backward stochastic delayed system with regime switching. Wang and Wu [9] concerned with the optimal control problems of the forward-backward delay system involving impulse controls and established the stochastic maximum principle for this kind of system. Zhou [10] investigated the maximum principle for stochastic optimal control problems of the delay system with random coefficients involving both continuous and impulse controls. In previous work, we mainly studied the theory of doubly stochastic differential equations with time delay. We deduced the maximum principle for the double stochastic control system when all variables contain time delay variables [11]. And we concerned the expression of optimal control and value function by the solution of the Riccati equation for a special delayed doubly stochastic linear quadratic control system [12].

Functions f and g can be defined in different forms according to different problems. In this paper, we mainly

investigate the delayed doubly stochastic linear quadratic control system, that is,

$$\begin{cases} dx(t) = [A_1(t)x(t) + B_1(t)x(t - \delta_1) + C_1(t)y(t) + D_1(t)y(t - \delta_2) \\ + E_1(t)u(t) + F_1(t)u(t - \delta_3)]dt + [A_2(t)x(t) + B_2(t)x(t - \delta_1) \\ + C_2(t)y(t) + D_2(t)y(t - \delta_2) + E_2(t)u(t) + F_2(t)u(t - \delta_3)]d\overrightarrow{W}(t) - y(t)d\overleftarrow{B}(t), & t \in [0, T], \\ x(t) = \varphi(t), & t \in [-\delta_1, 0], \\ y(t) = \psi(t), & t \in [-\delta_2, 0], \\ u(t) = 0, & t \in [-\delta_3, 0], \end{cases} \quad (2)$$

where the delayed variables δ_1 , δ_2 , and δ_3 are not equal.

Remark 1. In this delayed doubly stochastic control system, the state variables and the control variables contain time delay at the same time, and the three delay variables are different. Time delay exists all the time in the system. However, we do nothing before the initial time. So, we give the assumption that $u(t) = 0$ when the time t belongs to the interval before the intervention of the control variable.

The cost functional can be written as

$$J(u(\cdot)) = E \left\{ \int_0^T l(t, x(t), y(t), u(t))dt + \Phi(x(T)) \right\}. \quad (3)$$

For better analysis and research, we give some definitions similar to these in reference [16].

Definition 1. The optimal control problem of the delayed doubly stochastic system can be described as minimizing the cost functional over $U[0, T]$ to obtain the optimal control $u^*(\cdot)$ satisfying

$$J(u^*(\cdot)) = V = \inf_{u(\cdot) \in U[0, T]} J(u(\cdot)), \quad (4)$$

and the corresponding $(x^*(\cdot), y^*(\cdot), u^*(\cdot))$ is called an optimal triple.

Definition 2. For a given $\varepsilon > 0$, an admissible triple $(x^\varepsilon(\cdot), y^\varepsilon(\cdot), u^\varepsilon(\cdot))$ or simply $u^\varepsilon(\cdot)$ is called ε -optimal if $|J(u^\varepsilon) - V| \leq \varepsilon$.

Definition 3. A family of admissible triples $(x^\varepsilon(\cdot), y^\varepsilon(\cdot), u^\varepsilon(\cdot))$ or simply $u^\varepsilon(\cdot)$ parameterized by $\varepsilon > 0$ is called near optimal if $|J(u^\varepsilon) - V| \leq r(\varepsilon)$ holds for sufficiently small ε , where $r(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. The estimate $r(\varepsilon)$ is called an error bound. If $r(\varepsilon) = c\varepsilon^\gamma$ for some $\gamma > 0$ independent of the constant c , then u^ε is called near optimal with order ε^γ .

We assume that the following conditions hold:

- (A1) Assume that the coefficient matrices A_i , B_i , C_i , D_i , E_i , and F_i ($i = 1, 2$) are bounded matrix processes with proper dimensions, ($i = 1, 2$)
- (A2) The function Φ is continuously differentiable in x , and the partial derivative of Φ is bounded
- (A3) The function l is continuously differentiable in (x, y, u) , and every partial derivative is bounded

Corresponding to the delayed doubly stochastic control system, the adjoint equation can be written as

$$\begin{cases} -dp(t) = \{A_1^\top(t)p(t) + E^{\mathcal{F}_t} [B_1^\top(t + \delta_1)p(t + \delta_1)] + A_2^\top(t)q(t) - l_x(t) \\ + E^{\mathcal{F}_t} [B_2^\top(t + \delta_1)q(t + \delta_1)]\}dt + \{l_y(t) - C_1^\top(t)p(t) - C_2^\top(t)q(t) \\ - E^{\mathcal{F}_t} [D_1^\top(t + \delta_2)p(t + \delta_2)] - E^{\mathcal{F}_t} [D_2^\top(t + \delta_2)q(t + \delta_2)]\}d\overleftarrow{B}(t) - q(t)d\overrightarrow{W}(t), & t \in [0, T], \\ p(T) = -\Phi_x(x(T)), \\ p(t) = 0, & t \in (T, T + \delta], \\ q(t) = 0, & t \in (T, T + \delta], \end{cases} \quad (5)$$

where the variable $\delta = \max\{\delta_1, \delta_2, \delta_3\}$.

Remark 2. According to Theorem 3.1 in [11], the delayed doubly stochastic differential equation (2) admits a unique solution.

Lemma 1. Under the assumption (A1), the adjoint equation (5) admits a unique solution $(p(t), q(t))$ for any $u \in U[0, T]$. And there exists a positive constant $C > 0$ such that

$$E \left[\sup_{0 \leq t \leq T} |p^u(t)|^2 + \int_0^T |q^u(t)|^2 dt \right] \leq C, \quad \forall u \in U[0, T]. \quad (6)$$

Proof. Adjoint equation (5) is a new kind of equation which is similar to the anticipated backward stochastic differential equation in [22]. We call it anticipated backward doubly stochastic differential equation. Theorem 2.2 in reference [23] introduces the conditions for the existence and

uniqueness of solution of general anticipated backward doubly stochastic differential equation. In this paper, we only discuss the linear system, which is a special case in reference [23]. Characteristics of the linear system and boundedness of coefficient from assumption (A1) satisfy the condition of Theorem 2.3. We can directly deduce the existence and uniqueness of the solution from this theorem.

Under the premise of the existence of solutions, Theorem 2.5 in reference [24] gives the boundedness of solutions in general cases. When we discuss the linear system, the term $\int_0^T (|f(t, 0, 0, 0, 0)|^2 + |g(t, 0, 0, 0, 0)|^2) dt = 0$. And the delayed terms $p(t) = 0, q(t) = 0$ when $t \in [T, T + \delta]$. Then, we can deduce inequality (6) by Theorem 2.5 in reference [24] directly. \square

Definition 4. Let us define a metric d on U by $d(u, v) = [E \int_0^T |u(t) - v(t)|^2 dt]^{1/2}$.

Obviously, (U, d) is a complete metric space. Next, we will discuss the relation by using the metric d .

Lemma 2. Assume (A1), then there exists a constant $C > 0$ satisfying

$$E \left[\sup_{0 \leq t \leq T} |x^u(t) - x^v(t)|^2 + \int_0^T |y^u(t) - y^v(t)|^2 dt \right] \leq C d(u, v)^2. \quad (7)$$

Proof. Applying Itô's formula and Jensen inequality for the general delayed doubly stochastic system (1), we have

$$\begin{aligned} & E \left[|x^u(t) - x^v(t)|^2 + \int_0^T |y^u(t) - y^v(t)|^2 dt \right] \\ & \leq E \int_0^T |f(t, x^u(t), x^u(t - \delta_1), y^u(t), y^u(t - \delta_2), u(t), u(t - \delta_3)) \\ & \quad - f(t, x^v(t), x^v(t - \delta_1), y^v(t), y^v(t - \delta_2), v(t), v(t - \delta_3))|^2 dt \\ & \quad + E \int_0^T |g(t, x^u(t), x^u(t - \delta_1), y^u(t), y^u(t - \delta_2), u(t), u(t - \delta_3)) \\ & \quad - g(t, x^v(t), x^v(t - \delta_1), y^v(t), y^v(t - \delta_2), v(t), v(t - \delta_3))|^2 dt. \end{aligned} \quad (8)$$

For the liner system (2), we can deal with the first term in (8) as the following:

$$\begin{aligned} & E \int_0^T |f(t, x^u(t), x^u(t - \delta_1), y^u(t), y^u(t - \delta_2), u(t), u(t - \delta_3)) \\ & \quad - f(t, x^v(t), x^v(t - \delta_1), y^v(t), y^v(t - \delta_2), v(t), v(t - \delta_3))|^2 dt \\ & \leq E \int_0^T [A_1(t) |x^u(t) - x^v(t)|^2 + B_1(t) |x^u(t - \delta_1) - x^v(t - \delta_1)|^2 + C_1(t) |y^u(t) - y^v(t)|^2 \\ & \quad + D_1(t) |y^u(t - \delta_2) - y^v(t - \delta_2)|^2 + E_1(t) |u(t) - v(t)|^2 + F_1(t) |u(t - \delta_3) - v(t - \delta_3)|^2] dt. \end{aligned} \quad (9)$$

Using variable substitution and paying attention to the initial conditions, we can get the following conclusions:

$$\begin{aligned} & E \int_0^T |x^u(t - \delta_1) - x^v(t - \delta_1)| dt \\ & = E \int_{-\delta_1}^{T-\delta_1} |x^u(t) - x^v(t)| dt \\ & \leq E \int_0^T |x^u(t) - x^v(t)| dt. \end{aligned} \quad (10)$$

Similarly,

$$\begin{aligned} & E \int_0^T |y^u(t - \delta_2) - y^v(t - \delta_2)| dt \\ & = E \int_{-\delta_2}^{T-\delta_2} |y^u(t) - y^v(t)| dt \\ & \leq E \int_0^T |y^u(t) - y^v(t)| dt, \end{aligned} \quad (11)$$

$$\begin{aligned} & E \int_0^T |u(t - \delta_3) - v(t - \delta_3)| dt \\ & = E \int_{-\delta_3}^{T-\delta_3} |u(t) - v(t)| dt \\ & \leq E \int_0^T |u(t) - v(t)| dt. \end{aligned} \quad (12)$$

Then, substitute inequalities (10)–(12) into (9). Under the assumption (A1), there is a constant $C > 0$ such that

$$\begin{aligned}
 & E \int_0^T |f(t, x^u(t), x^u(t - \delta_1), y^u(t), y^u(t - \delta_2), u(t), u(t - \delta_3)) \\
 & \quad - f(t, x^v(t), x^v(t - \delta_1), y^v(t), y^v(t - \delta_2), v(t), v(t - \delta_3))|^2 dt \\
 & \leq C \left\{ E \int_0^T |x^u(t) - x^v(t)|^2 dt + E \int_0^T |y^u(t) - y^v(t)|^2 dt + |u(t) - v(t)|^2 \right\} \\
 & = C \left\{ E \int_0^T |x^u(t) - x^v(t)|^2 dt + E \int_0^T |y^u(t) - y^v(t)|^2 dt + d(u, v)^2 \right\}.
 \end{aligned} \tag{13}$$

Similarly, for the second term in (8), we have

$$\begin{aligned}
 & E \int_0^T |g(t, x^u(t), x^u(t - \delta_1), y^u(t), y^u(t - \delta_2), u(t), u(t - \delta_3)) \\
 & \quad - g(t, x^v(t), x^v(t - \delta_1), y^v(t), y^v(t - \delta_2), v(t), v(t - \delta_3))|^2 dt \\
 & \leq C \left\{ E \int_0^T |x^u(t) - x^v(t)|^2 dt + E \int_0^T |y^u(t) - y^v(t)|^2 dt + d(u, v)^2 \right\}.
 \end{aligned} \tag{14}$$

Using Gronwall's inequality and Lemma 3.1 in [9], we can deduce conclusion (7) directly. \square

Similarly, using the same method and Proposition 2.5 in reference [24], we can deduce the following conclusion directly.

Lemma 3. Assume (A1), then there exists a constant $C > 0$ satisfying

$$E \left[\sup_{0 \leq t \leq T} |p^u(t) - q^v(t)|^2 + \int_0^T |p^u(t) - q^v(t)|^2 dt \right] \leq C d(u, v)^2. \tag{15}$$

Lemma 4. Assume (A1–A3), then there exists a constant $C > 0$ satisfying

$$|J(u) - J(v)| \leq C d(u, v), \tag{16}$$

for all $u, v \in U$.

Proof. From (3) and the elementary inequality, we have

$$\begin{aligned}
 |J(u) - J(v)| & \leq \left| E \int_0^T \{l^u(x^u(t), y^u(t), u(t)) - l^v(x^v(t), y^v(t), v(t))\} dt \right| \\
 & \quad + |\Phi(x^u(T)) - \Phi(x^v(T))|.
 \end{aligned} \tag{17}$$

From condition (A2), Lemma 2, and Cauchy–Schwartz inequality, we find that

$$\begin{aligned}
 |\Phi(x^u(T)) - \Phi(x^v(T))| & = \left| \int_0^1 \langle \Phi_x(x^v(T) + \lambda(x^u(T) - x^v(T))), x^u(T) - x^v(T) \rangle d\lambda \right| \\
 & \leq C d(u, v).
 \end{aligned} \tag{18}$$

For the convenience of proof, we denote symbol

$$\varsigma(t) = (t, x^v(t) + \lambda(x^u(t) - x^v(t)), y^v(t) + \lambda(y^u(t) - y^v(t)), v(t) + \lambda(u(t) - v(t))). \quad (19)$$

Then, we have

$$\begin{aligned} & |l^u(x^u(t), y^u(t), u(t)) - l^v(x^v(t), y^v(t), v(t))| \\ &= \left| \int_0^1 \langle l_x(\varsigma(t)), x^u(t) - x^v(t) \rangle + \langle l_y(\varsigma(t)), y^u(t) - y^v(t) \rangle + \langle l_u(\varsigma(t)), u(t) - v(t) \rangle d\lambda \right|. \end{aligned} \quad (20)$$

By using the same method, from the assumption (A3), Lemmas 2 and 3, and the Definition 4, we can deduce that

$$\left| E \int_0^T \{l^u(x^u(t), y^u(t), u(t)) - l^v(x^v(t), y^v(t), v(t))\} dt \right| \leq Cd(u, v). \quad (21)$$

Combining inequalities (18) and (21), we can prove the conclusion directly. \square

Ekeland's variational principle is an important tool for our study which can be seen in [25].

Lemma 5. (Ekeland's variational principle). Let (S, d) be a complete metric space and $\rho(\cdot): S \rightarrow \mathbb{R}^1$ be a lower-semi-continuous and bounded from below. For $\varepsilon \geq 0$, suppose $u^\varepsilon \in S$ satisfies

$$\rho(u^\varepsilon) \leq \inf_{u \in S} \rho(u) + \varepsilon. \quad (22)$$

Then, for any $\lambda > 0$, there exists $u^\lambda \in S$ such that

$$\begin{aligned} & \rho(u^\lambda) \leq \rho(u^\varepsilon), \\ & d(u^\lambda, u^\varepsilon) \leq \lambda, \\ & \rho(u^\lambda) \leq \rho(u) + \frac{\varepsilon}{\lambda} d(u, u^\lambda), \quad \text{for all } u \in S. \end{aligned} \quad (23)$$

Assume that $u^\varepsilon \in U$ is a ε -optimal control; from Definition 2, we have $|J(u^\varepsilon) - V| \leq \varepsilon$, that is, $J(u^\varepsilon) \leq V + \varepsilon$. Then, from Definition 1, we have $J(u^\varepsilon) \leq \inf_{v \in U} J(v) + \varepsilon$. From assumption (A2), we know that $J(\cdot)$ is a continuous bounded function and (U, d) is a complete metric space. From

Lemma 5, we know that there is a $u^\lambda \in S$, such that $J(u^\lambda) \leq J(u^\varepsilon)$, $\forall \lambda > 0$. Take $\lambda = \sqrt{\varepsilon}$ and then $u^\lambda = \tilde{u}^\varepsilon$. We have $J(\tilde{u}^\varepsilon) \leq J(u^\varepsilon)$ and $d(\tilde{u}^\varepsilon, u^\varepsilon) \leq \lambda = \sqrt{\varepsilon}$.

Then, we have

$$J(\tilde{u}^\varepsilon) \leq J(u) + \sqrt{\varepsilon} d(u, \tilde{u}^\varepsilon), \quad \forall u \in U. \quad (24)$$

We discuss \tilde{u}^ε first and pay attention to u^ε . Let $u \in M^2(-\delta', T)$ ($\delta' = \min\{\delta_1, \delta_2, \delta_3\}$) satisfy $\tilde{u}^\varepsilon + u \in U$. In the previous assumptions, we know that $u(t) = 0$ for $-\delta_3 \leq t \leq 0$. Define $u^\theta = \tilde{u}^\varepsilon + \theta u$, $\theta \in [0, 1]$. Then, we have $u^\theta = \tilde{u}^\varepsilon + \theta(u - \tilde{u}^\varepsilon) = (1 - \theta)\tilde{u}^\varepsilon + \theta u$. From the convexity of U , we can deduce that $u^\theta \in U$ for any $\theta \in [0, 1]$. Then, $d(u^\theta, \tilde{u}^\varepsilon) = [E \int_0^T (u^\theta - \tilde{u}^\varepsilon)^2 dt]^{(1/2)} = [E \int_0^T (\theta u)^2 dt]^{(1/2)} = \theta [E \int_0^T (u)^2 dt]^{(1/2)}$. From the bounded of U , we know that there exist a constant β independent of ε and θ , such that $d(u^\theta, \tilde{u}^\varepsilon) \leq \beta\theta$.

From inequality (24), we have

$$\begin{aligned} J(\tilde{u}^\varepsilon) & \leq J(u^\theta) + \sqrt{\varepsilon} d(u^\theta, \tilde{u}^\varepsilon) \\ & \leq J(u^\theta) + \beta\sqrt{\varepsilon}\theta. \end{aligned} \quad (25)$$

That is,

$$J(u^\theta) - J(\tilde{u}^\varepsilon) \geq -\beta\sqrt{\varepsilon}\theta. \quad (26)$$

Let us introduce variational equations.

$$\begin{cases} dx_1(t) = [A_1(t)x_1(t) + B_1(t)x_1(t - \delta_1) + C_1(t)y_1(t) + D_1(t)y_1(t - \delta_2) \\ + E_1(t)u(t) + F_1(t)u(t - \delta_3)]dt + [A_2(t)x_1(t) + B_2(t)x_1(t - \delta_1) \\ + C_2(t)y_1(t) + D_2(t)y_1(t - \delta_2) + E_2(t)u(t) + F_2(t)u(t - \delta_3)]d\overrightarrow{W}(t) - y_1(t)d\overleftarrow{B}(t), & t \in [0, T], \\ x_1(t) = 0, & t \in [-\delta_1, 0], \\ y_1(t) = 0, & t \in [-\delta_2, 0]. \end{cases} \quad (27)$$

In order to simplify the symbols in proof, we denote $\xi^\theta(t) := (t, x^\theta(t), y^\theta(t), u^\theta(t))$ and $\tilde{\xi}^\varepsilon(t) := (t, \tilde{x}^\varepsilon(t), \tilde{y}^\varepsilon(t), \tilde{u}^\varepsilon(t))$.

3. Main Results

Theorem 1. Let (A1)–(A3) hold. Then, there exists a constant $\beta > 0$ independent of ε , such that

$$\begin{aligned} E \int_0^T \langle E^{\mathcal{F}_t} [F_1^\top(t + \delta_3)\tilde{p}^\varepsilon(t + \delta_3) + F_2^\top(t + \delta_3)\tilde{q}^\varepsilon(t + \delta_3)] \\ + E_1^\top(t)\tilde{p}^\varepsilon(t) + E_2^\top(t)\tilde{q}^\varepsilon(t) - l_u(\tilde{\xi}^\varepsilon(t)), v - \tilde{u}^\varepsilon(t) \rangle dt \leq \beta\sqrt{\varepsilon}, \quad \forall v \in U. \end{aligned} \quad (28)$$

Proof. From the definition of function $J(\cdot)$ and inequality (26), we have

$$\begin{aligned} & \lim_{\theta \rightarrow 0} \frac{J(u^\theta) - J(\tilde{u}^\varepsilon)}{\theta} \\ &= \lim_{\theta \rightarrow 0} \frac{E \int_0^T [l(\xi^\theta(t)) - t \ln(\tilde{\xi}^\varepsilon(t))] dt}{\theta} + \lim_{\theta \rightarrow 0} \frac{E\{\Phi(x^\theta(T)) - \Phi(\tilde{x}^\varepsilon(T))\}}{\theta} \\ &= E \left\{ \int_0^T [\langle l_x(\tilde{\xi}^\varepsilon(t)), x_1(t) \rangle + \langle l_y(\tilde{\xi}^\varepsilon(t)), y_1(t) \rangle + \langle l_u(\tilde{\xi}^\varepsilon(t)), u(t) \rangle] dt + \langle \Phi_x(\tilde{x}^\varepsilon(T)), x_1(T) \rangle \right\} \\ &\geq -\beta\sqrt{\varepsilon}. \end{aligned} \quad (29)$$

Next, we will deal with the term $\langle \Phi_x(\tilde{x}^\varepsilon(T)), x_1(T) \rangle$. We connect it with the solution of the adjoint equation. Using the Itô–Doebelin formula, we have

$$\begin{aligned} & E \langle x_1(T), -\Phi_x(\tilde{x}^\varepsilon(T)) \rangle \\ &= E \left\{ \int_0^T \langle -x_1(t), E^{\mathcal{F}_t} [B_1^\top(t + \delta_1)\tilde{p}^\varepsilon(t + \delta_1)] + E^{\mathcal{F}_t} [B_2^\top(t + \delta_1)\tilde{q}^\varepsilon(t + \delta_1)] - l_x^*(\tilde{\xi}^\varepsilon(t)) \rangle \right. \\ &\quad + \langle B_1(t)x_1(t - \delta_1) + D_1(t)y_1(t - \delta_2) + E_1(t)u(t) + F_1(t)u(t - \delta_3), \tilde{p}^\varepsilon(t) \rangle \\ &\quad + \langle B_2(t)x_1(t - \delta_1) + D_2(t)y_1(t - \delta_2) + E_2(t)u(t) + F_2(t)u(t - \delta_3), \tilde{q}^\varepsilon(t) \rangle \\ &\quad \left. + \langle y_1(t), l_y^*(\tilde{\xi}^\varepsilon(t)) - E^{\mathcal{F}_t} [D_1^\top(t + \delta_2)\tilde{p}^\varepsilon(t + \delta_2)] - E^{\mathcal{F}_t} [D_2^\top(t + \delta_2)\tilde{q}^\varepsilon(t + \delta_2)] \rangle \right\} dt. \end{aligned} \quad (30)$$

Let us deal with the first term.

$$\begin{aligned}
& E \int_0^T \langle -x_1(t), E^{\mathcal{F}_t} [B_1^\top(t + \delta_1) \tilde{p}^\varepsilon(t + \delta_1)] \rangle dt \\
&= E \int_{\delta_1}^{T+\delta_1} \langle -x_1(t - \delta_1), E^{\mathcal{F}_{t-\delta_1}} [B_1^\top(t) \tilde{p}^\varepsilon(t)] \rangle dt = E \int_{\delta_1}^{T+\delta_1} \langle -x_1(t - \delta_1), B_1^\top(t) \tilde{p}^\varepsilon(t) \rangle dt \\
&= E \left\{ \int_0^T \langle -x_1(t - \delta_1), B_1^\top(t) \tilde{p}^\varepsilon(t) \rangle dt - \int_0^{\delta_1} \langle -x_1(t - \delta_1), B_1^\top(t) \tilde{p}^\varepsilon(t) \rangle dt \right. \\
&\quad \left. + \int_T^{T+\delta_1} \langle -x_1(t - \delta_1), B_1^\top(t) \tilde{p}^\varepsilon(t) \rangle dt \right\}.
\end{aligned} \tag{31}$$

From the definition of adjoint equation (5) and variation equation (10), we have

$$E \int_0^T \langle -x_1(t - \delta_1), B_1^\top(t) \tilde{p}^\varepsilon(t) \rangle dt = 0, \tag{32}$$

$$E \int_T^{T+\delta_1} \langle -x_1(t - \delta_1), B_1^\top(t) \tilde{p}^\varepsilon(t) \rangle dt = 0. \tag{33}$$

Combining equalities (31)–(33), we deduce the following equality:

$$E \int_0^T \left[\langle -x_1(t), E^{\mathcal{F}_t} [B_1^\top(t + \delta_1) \tilde{p}^\varepsilon(t + \delta_1)] \rangle + \langle B_1(t) x_1(t - \delta_1), \tilde{p}^\varepsilon(t) \rangle \right] dt = 0. \tag{34}$$

Similarly, we have

$$E \int_0^T \left[\langle -x_1(t), E^{\mathcal{F}_t} [B_2^\top(t + \delta_1) \tilde{q}^\varepsilon(t + \delta_1)] \rangle + \langle B_2(t) x_1(t - \delta_1), \tilde{q}^\varepsilon(t) \rangle \right] dt = 0. \tag{35}$$

In the same way, we have

$$E \int_0^T \left\{ \langle D_1(t) y_1(t - \delta_2), \tilde{p}^\varepsilon(t) \rangle + \langle y_1(t), -E^{\mathcal{F}_t} [D_1^\top(t + \delta_2) \tilde{p}^\varepsilon(t + \delta_2)] \rangle \right\} dt = 0, \tag{36}$$

$$E \int_0^T \left\{ \langle D_2(t) y_1(t - \delta_2), \tilde{q}^\varepsilon(t) \rangle + \langle y_1(t), -E^{\mathcal{F}_t} [D_2^\top(t + \delta_2) \tilde{q}^\varepsilon(t + \delta_2)] \rangle \right\} dt = 0. \tag{37}$$

Substituting (34)–(37) into equality (30), we have

$$\begin{aligned}
& E \langle x_1(T), -\Phi_x(x(T)) \rangle \\
&= E \left\{ \int_0^T \langle x_1(t), l_x(\tilde{\xi}^\varepsilon(t)) \rangle + \langle E_1(t) u(t) + F_1(t) u(t - \delta_3), \tilde{p}^\varepsilon(t) \rangle \right. \\
&\quad \left. + \langle E_2(t) u(t) + F_2(t) u(t - \delta_3), \tilde{q}^\varepsilon(t) \rangle + \langle y_1(t), l_y(\tilde{\xi}^\varepsilon(t)) \rangle \right\} dt.
\end{aligned} \tag{38}$$

Let us deal with delayed control variables.

$$\begin{aligned}
& \int_0^T \langle F_1(t)u(t - \delta_3), \tilde{p}^\varepsilon(t) \rangle dt \\
&= \int_{-\delta_3}^{T-\delta_3} \langle F_1(t + \delta_3)u(t), \tilde{p}^\varepsilon(t + \delta_3) \rangle dt \\
&= \int_{-\delta_3}^0 \langle F_1^\top(t + \delta_3)\tilde{p}^\varepsilon(t + \delta_3), u(t) \rangle dt + \int_0^{T-\delta_3} \langle F_1^\top(t + \delta_3)\tilde{p}^\varepsilon(t + \delta_3), u(t) \rangle dt.
\end{aligned} \tag{39}$$

From the remark, we know that $u(t) = 0$ when $-\delta_3 \leq t \leq 0$. From the adjoint equation (5), we have the

terminal condition that $p(t) = 0$ for $T \leq t \leq T + \delta$, $\delta = \max\{\delta_1, \delta_2, \delta_3\}$. Then, we have

$$\begin{aligned}
& E \int_0^T \langle F_1(t)u(t - \delta_3), \tilde{p}^\varepsilon(t) \rangle dt \\
&= E \left\{ \int_0^T \langle F_1^\top(t + \delta_3)\tilde{p}^\varepsilon(t + \delta_3), u(t) \rangle dt - \int_{T-\delta_3}^T \langle F_1^\top(t + \delta_3)\tilde{p}^\varepsilon(t + \delta_3), u(t) \rangle dt \right\} \\
&= E \int_0^T \langle E^{\mathcal{F}_t} [F_1^\top(t + \delta_3)] \tilde{p}^\varepsilon(t + \delta_3), u(t) \rangle dt.
\end{aligned} \tag{40}$$

Equation (38) can be written as

$$\begin{aligned}
& E \langle x_1(T), -\Phi_x(x(T)) \rangle \\
&= E \left\{ \int_0^T \langle x_1(t), l_x(\tilde{\xi}^\varepsilon(t)) \rangle + \langle E_1(t)u(t), \tilde{p}^\varepsilon(t) \rangle + \langle E_2(t)u(t), \tilde{q}^\varepsilon(t) \rangle + \langle y_1(t), l_y(\tilde{\xi}^\varepsilon(t)) \rangle \right. \\
&\quad \left. + \langle E^{\mathcal{F}_t} [F_1^\top(t + \delta_3)] \tilde{p}^\varepsilon(t + \delta_3), u(t) \rangle + \langle E^{\mathcal{F}_t} [F_2^\top(t + \delta_3)] \tilde{q}^\varepsilon(t + \delta_3), u(t) \rangle \right\} dt. \\
&= E \left\{ \int_0^T \langle x_1(t), l_x(\tilde{\xi}^\varepsilon(t)) \rangle + \langle E_1^\top(t) \tilde{p}^\varepsilon(t) + E_2^\top(t) \tilde{q}^\varepsilon(t), u(t) \rangle + \langle y_1(t), l_y(\tilde{\xi}^\varepsilon(t)) \rangle \right. \\
&\quad \left. + \langle E^{\mathcal{F}_t} [F_1^\top(t + \delta_3) \tilde{p}^\varepsilon(t + \delta_3) + F_2^\top(t + \delta_3) \tilde{q}^\varepsilon(t + \delta_3)], u(t) \rangle \right\} dt.
\end{aligned} \tag{41}$$

According to inequality (29) and equation (41), we can deduce that

$$\begin{aligned}
& E \int_0^T \langle l_u(\tilde{\xi}^\varepsilon(t)) - E_1^\top(t) \tilde{p}^\varepsilon(t) - E_2^\top(t) \tilde{q}^\varepsilon(t) - E^{\mathcal{F}_t} [F_1^\top(t + \delta_3) \tilde{p}^\varepsilon(t + \delta_3) \\
&\quad - F_2^\top(t + \delta_3) \tilde{q}^\varepsilon(t + \delta_3)], u(t) \rangle dt \geq \beta \sqrt{\varepsilon}.
\end{aligned} \tag{42}$$

We know that u is a variable such that $u^\varepsilon + u \in U$. Assume that $u^\varepsilon + u = v \in U$, then the desired conclusion (28) is deduced directly, that is,

$$\begin{aligned}
& E \int_0^T \langle E^{\mathcal{F}_t} [F_1^\top(t + \delta_3) \tilde{p}^\varepsilon(t + \delta_3) + F_2^\top(t + \delta_3) \tilde{q}^\varepsilon(t + \delta_3)] \\
&\quad + E_1^\top(t) \tilde{p}^\varepsilon(t) + E_2^\top(t) \tilde{q}^\varepsilon(t) - l_u(\tilde{\xi}^\varepsilon(t)), v - \tilde{u}^\varepsilon(t) \rangle dt \leq \beta \sqrt{\varepsilon}, \quad \forall v \in U.
\end{aligned} \tag{43}$$

Theorem 1 is proved. \square

First, we give the definition of the Hamiltonian function of general delayed doubly stochastic system (1).

Next, we will show the necessary condition for the near optimal control of the delayed doubly stochastic control system.

$$\begin{aligned} H(t, x(t), x(t - \delta_1), y(t), y(t - \delta_2), u(t), u(t - \delta_3)) \\ = f^\top(t, x(t), x(t - \delta_1), y(t), y(t - \delta_2), u(t), u(t - \delta_3))p(t) \\ + g^\top(t, x(t), x(t - \delta_1), y(t), y(t - \delta_2), u(t), u(t - \delta_3))q(t). \end{aligned} \quad (44)$$

For linear system (2), we have

$$\begin{aligned} H_u &= E_1^\top(t)p(t) + E_2^\top(t)q(t), \\ H_{u_\delta} &= F_1^\top(t)p(t) + F_2^\top(t)q(t), \\ H_{u_\delta}(t + \delta_3) &= F_1^\top(t + \delta_3)p(t + \delta_3) + F_2^\top(t + \delta_3)q(t + \delta_3). \end{aligned} \quad (45)$$

Assume that

$$\tilde{H} = H_u + E^{\mathcal{F}_t} [H_{u_\delta}(t + \delta_3)]. \quad (46)$$

Then, we have the following conclusion.

control triple $(x^\varepsilon, y^\varepsilon, u^\varepsilon)$ of the delayed doubly stochastic control problems (2)–(4), we have

$$E \int_0^T \langle \widetilde{H}_u^\varepsilon, v - u_t^\varepsilon \rangle dt \leq \beta \varepsilon^\gamma, \quad \forall v \in U. \quad (47)$$

Theorem 2. Assume (A1)–(A3). There exists a constant $\beta > 0$ such that for any $\varepsilon > 0$, $\gamma \in [0, (1/2)]$, and the ε -optimal

Proof. From the definition of function \tilde{H} , we have

$$\tilde{H}_u^\varepsilon = E_1^\top(t)p^\varepsilon(t) + E_2^\top(t)q^\varepsilon(t) - l_u(\xi^\varepsilon(t)) + E^{\mathcal{F}_t} [F_1^\top(t + \delta_3)p^\varepsilon(t + \delta_3) + F_2^\top(t + \delta_3)q^\varepsilon(t + \delta_3)]. \quad (48)$$

Then, inequality (47) can be written as

$$\begin{aligned} E \int_0^T \langle E^{\mathcal{F}_t} [F_1^\top(t + \delta_3)p^\varepsilon(t + \delta_3) + F_2^\top(t + \delta_3)q^\varepsilon(t + \delta_3)] \\ + E_1^\top(t)p^\varepsilon(t) + E_2^\top(t)q^\varepsilon(t) - l_u(\xi^\varepsilon(t)), v - u^\varepsilon(t) \rangle dt \leq \beta \sqrt{\varepsilon}, \quad \forall v \in U. \end{aligned} \quad (49)$$

We find that inequalities (28) and (49) are very similar. We need to focus on the differences between them.

We denote

$$\begin{aligned} \Delta_1 &= E \int_0^T [\langle E_1^\top(t)p^\varepsilon(t), v - u^\varepsilon(t) \rangle - \langle E_1^\top(t)\tilde{p}^\varepsilon(t), v - \tilde{u}^\varepsilon(t) \rangle] dt \\ &= E \int_0^T [\langle E_1^\top(t)(p^\varepsilon(t) - \tilde{p}^\varepsilon(t)), v - \tilde{u}^\varepsilon(t) \rangle + \langle E_1^\top(t)p^\varepsilon(t), \tilde{u}^\varepsilon(t) - u^\varepsilon(t) \rangle] dt \\ &= \Delta_{11} + \Delta_{12}, \end{aligned} \quad (50)$$

where

$$\begin{aligned}\Delta_{11} &= E \int_0^T \langle E_1^\top(t)(p^\varepsilon(t) - \tilde{p}^\varepsilon(t)), v - \tilde{u}^\varepsilon(t) \rangle dt, \\ \Delta_{12} &= E \int_0^T \langle E_1^\top(t)p^\varepsilon(t), \tilde{u}^\varepsilon(t) - u^\varepsilon(t) \rangle dt.\end{aligned}\quad (51)$$

Next, we will deal with these two terms. From the assumption (A1), Lemma 3, and the bounded of the control domain, there exist a series of constants $C', C'', C_1, C_{11}, C_{12}, C_2, \dots$, which are all independent of ε . We have

$$\begin{aligned}\Delta_{11} &\leq C' E \int_0^T E_1^\top(t)(p^\varepsilon(t) - \tilde{p}^\varepsilon(t)) dt \\ &\leq C_{11} E \int_0^T |p^\varepsilon(t) - \tilde{p}^\varepsilon(t)| dt \\ &\leq C_{11} d(u^\varepsilon, \tilde{u}^\varepsilon) \\ &\leq C_{11} \sqrt{\varepsilon}.\end{aligned}\quad (52)$$

And then from the assumption (A1), Lemmas 1 and 3, and the Cauchy-Schwartz inequality, we can deduce that

$$\begin{aligned}\Delta_{12} &\leq C'' E \int_0^T |p^\varepsilon(t)| |u^\varepsilon(t) - \tilde{u}^\varepsilon(t)| dt \\ &\leq C_{12} \sqrt{\varepsilon}.\end{aligned}\quad (53)$$

Combining (52) and (53), we have

$$\Delta_1 = \Delta_{11} + \Delta_{12} \leq C_1 \sqrt{\varepsilon}, \quad \text{where } C_1 = \max\{C_{11}, C_{12}\}.\quad (54)$$

We denote Δ_2 and prove it like Δ_1 . Then, we have

$$\begin{aligned}\Delta_2 &= E \int_0^T [\langle E_2^\top(t)q^\varepsilon(t), v - u^\varepsilon(t) \rangle - \langle E_2^\top(t)\tilde{q}^\varepsilon(t), v - \tilde{u}^\varepsilon(t) \rangle] dt \\ &\leq C_2 \sqrt{\varepsilon}.\end{aligned}\quad (55)$$

Set

$$\begin{aligned}\Delta_3 &= E \int_0^T [\langle E^{\mathcal{F}_t}(F_1^\top(t + \delta_3))p^\varepsilon(t + \delta_3), v - u^\varepsilon(t) \rangle \\ &\quad - \langle E^{\mathcal{F}_t}(F_1^\top(t + \delta_3))\tilde{p}^\varepsilon(t + \delta_3), v - \tilde{u}^\varepsilon(t) \rangle] dt.\end{aligned}\quad (56)$$

Using variable substitution, we can deduce that

$$\begin{aligned}\Delta_3 &= E \int_{\delta_3}^{T+\delta_3} [\langle F_1^\top(t)p^\varepsilon(t), v(t - \delta_3) - u^\varepsilon(t - \delta_3) \rangle - \langle F_1^\top(t)\tilde{p}^\varepsilon(t), v(t - \delta_3) - \tilde{u}^\varepsilon(t - \delta_3) \rangle] dt \\ &= E \int_{\delta_3}^T [\langle F_1^\top(t)p^\varepsilon(t), v(t - \delta_3) - u^\varepsilon(t - \delta_3) \rangle - \langle F_1^\top(t)\tilde{p}^\varepsilon(t), v(t - \delta_3) - \tilde{u}^\varepsilon(t - \delta_3) \rangle] dt \\ &= E \int_{\delta_3}^T [\langle F_1^\top(t)(p^\varepsilon(t) - \tilde{p}^\varepsilon(t)), v(t - \delta_3) - \tilde{u}^\varepsilon(t - \delta_3) \rangle + \langle F_1^\top(t)p^\varepsilon(t), \tilde{u}^\varepsilon(t - \delta_3) - u^\varepsilon(t - \delta_3) \rangle] dt.\end{aligned}\quad (57)$$

Similar to the proof of Δ_1 , the results can be obtained by using the boundedness of control domain and coefficients. We can deduce the result directly, that is,

$$\Delta_3 \leq C_3 \sqrt{\varepsilon}.\quad (58)$$

Similarly, we have

$$\begin{aligned}\Delta_4 &= E \int_0^T [\langle E^{\mathcal{F}_t}(F_2^\top(t + \delta_3))p^\varepsilon(t + \delta_3), v - u^\varepsilon(t) \rangle \\ &\quad - \langle E^{\mathcal{F}_t}(F_2^\top(t + \delta_3))\tilde{p}^\varepsilon(t + \delta_3), v - \tilde{u}^\varepsilon(t) \rangle] dt \\ &\leq C_4 \sqrt{\varepsilon}.\end{aligned}\quad (59)$$

Then, we have

$$\begin{aligned}\Delta_5 &= E \int_0^T [\langle -l_u(\xi^\varepsilon(t)), v - u^\varepsilon(t) \rangle - \langle -l_u(\tilde{\xi}^\varepsilon(t)), v - \tilde{u}^\varepsilon(t) \rangle] dt \\ &= E \int_0^T [\langle l_u(\tilde{\xi}^\varepsilon(t)) - l_u(\xi^\varepsilon(t)), v - \tilde{u}^\varepsilon(t) \rangle - \langle l_u(\xi^\varepsilon(t)), \tilde{u}^\varepsilon(t) - u^\varepsilon(t) \rangle] dt.\end{aligned}\quad (60)$$

Set

$$\begin{aligned}\Delta_{51} &= E \int_0^T \langle l_u(\tilde{\xi}^\varepsilon(t)) - l_u(\xi^\varepsilon(t)), v - \tilde{u}^\varepsilon(t) \rangle dt, \\ \Delta_{52} &= E \int_0^T \langle l_u(\xi^\varepsilon(t)), u^\varepsilon(t) - \tilde{u}^\varepsilon(t) \rangle dt.\end{aligned}\quad (61)$$

Then, $\Delta_5 = \Delta_{51} + \Delta_{52}$.

We deal with the term Δ_{51} firstly. Similar to the previous proof, by the boundedness of U and inequality of (21), we have

$$\Delta_{51} \leq C_{51} d(u^\varepsilon, \tilde{u}^\varepsilon) \leq C_{51} \sqrt{\varepsilon}. \quad (62)$$

From assumption (A3), we know that the partial derivative of function l is bounded, so we have

$$\Delta_{52} \leq C_{52} \sqrt{\varepsilon}, \quad (63)$$

$$\Delta_5 \leq C_5 \sqrt{\varepsilon}. \quad (64)$$

According to inequalities (54), (55), (58), (59), and (65), we can deduce that

$$\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 + \Delta_5 \leq C \sqrt{\varepsilon}, \quad \text{where } C = \max\{C_1, C_2, C_3, C_4, C_5\}. \quad (65)$$

Applying Theorem 2, we can deduce the conclusion directly. \square

Generally speaking, optimal control is limited by many conditions. Near optimal control is relatively easy to obtain and can be selected, analyzed, and applied to a wider range of fields. When the time delay variables $\delta_1 = \delta_2 = \delta_3$, this is a special near optimal control problem with the same time delay variables. When the time delay variables $\delta_1 = \delta_2 = \delta_3 = 0$, we can deduce the conclusions directly for the common system which is described by doubly stochastic differential equations. In either case, we find that the results depending on the adjoint equation of the system. The adjoint equation is a new kind of equation which can be called anticipated double stochastic equations. Using the properties of this kind of equation, we deal with the delay terms reasonably. In the future, we should pay attention to the study of this kind of equation which can help us solve such problems relatively easy.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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Research Article

The Optimal Time to Merge Two First-Line Insurers with Proportional Reinsurance Policies

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We examine the optimal time to merge two first-line insurers with proportional reinsurance policies. The problem is considered in a diffusion approximation model. The objective is to maximize the survival probability of the two insurers. First, the verification theorem is verified. Then, we divide the problem into two cases. In case 1, never merging is optimal and the two insurers follow the optimal reinsurance policies that maximize their survival probability. In case 2, the two insurers follow the same reinsurance policies as those in case 1 until the sum of their surplus processes reaches a boundary. Then, they merge and apply the merged company's optimal reinsurance strategy.

1. Introduction

Mergers of companies bring a range of benefits, such as diversification, management and operational risk decentralization, elimination of competition, tax reduction, and optimization of resource allocation. The topic has attracted more and more attention from scholars in recent years. The authors in [1] listed a number of advantages from mergers. The authors in [2] deemed that, in contrast to acquisition, little cash is paid during a merger and the merger is realized through the exchange of shares. The authors in [3] examined the effect of mergers on the wealth of firms' shareholders. To learn more about companies' mergers, see [4–6] and so on.

However, the above analysis is all qualitative and only little quantitative work has been done. Only the authors in [7] considered the problem of a merger of two companies with dividend policies. Their objective was to maximize the sum of the two companies' expected discounted value. They constructed a situation in which the merger of the two companies results in a gain and gave a useful guideline on corporate governance. An open problem of finding the optimal time to merge in a more realistic situation was raised at the end of this paper. The authors in [8] solved this problem with some additional conditions. In this paper, we

also determine the optimal time to merge, but it is different from what was found in [7, 8]:

- (i) In this paper, we seek to find the optimal time to merge to maximize the survival probability of two first-line insurers. The problem is a mixed regular control/two-dimensional optimal stopping problem (for optimal stopping problems, see [9–11]).
- (ii) The problem is considered with proportional reinsurance (for optimal reinsurance problems, see [12–15]).

In Theorem 2, we give the verification theorem of this problem. To find the optimal strategy and the value function, we focus on two critical inequalities and consider the problem separately in two cases. In case 1, never merging is optimal and the two insurers apply the optimal reinsurance strategies that maximize their survival probability. The calculations in case 2 are more complex. First, we construct a function $M(x)$. In Lemma 2, we analyze the property of this function. Then, the constructed function is shown to satisfy the conditions in Theorem 2. Finally, we prove that the constructed function is exactly the value function. The optimal policy can be obtained as a by product. The two

insurers follow the optimal reinsurance policies that maximize their survival probability until the sum of their surpluses reaches a boundary c , and then they merge and apply the merged company's optimal reinsurance strategy.

This paper is organized as follows. Section 2 presents the formulation. In Section 3, we analyze the reinsurance problem of the two first-line insurers without a merger and the reinsurance problem of the merged company, respectively. In Section 4, the conditions for a function to be greater than the value function are given. The value function and the optimal policy are derived in Section 5. Section 6 reveals the effects of all parameters on the optimal strategy and shows that the results are consistent with economic phenomena. Conclusions are presented in Section 7.

2. Problem Formulation

In this section, we set up the mathematical model of the problem. The problem is considered on a probability space (Ω, \mathcal{F}, P) . Suppose there are two insurers labeled 1 and 2. Their safety loadings are η_1 and η_2 , and their risk processes are governed by compound Poisson processes. Similar to the procedure in [16], we suppose that the reserve processes of the two insurers are

$$\begin{aligned} X_1(t) &= x_1 + (\lambda + \lambda_1)(1 + \eta_1)\mu t - \sum_{i=1}^{N_1(t)+N(t)} u_i, \\ X_2(t) &= x_2 + (\lambda + \lambda_2)(1 + \eta_2)\mu t - \sum_{j=1}^{N_2(t)+N(t)} u_j, \end{aligned} \quad (1)$$

where I is the cost of the merger, x_1 is the reserve of insurer 1 at the time to merge, x_2 is the reserve of insurer 2 at the time to merge, and η_m is the safety loading of the merged company. Here, we assume that $\eta_m \leq \theta$.

where $N_1(t)$, $N_2(t)$, and $N(t)$ are three independent Poisson processes defined on (Ω, \mathcal{F}, P) . Their intensities are $\lambda_1 > 0$, $\lambda_2 > 0$, and $\lambda > 0$, respectively. The claim sizes $\{u_i, i = 1, 2, \dots\}$ and $\{u_j, j = 1, 2, \dots\}$ are i.i.d. positive random variables with expectation μ and variance σ^2 . Let $\{\mathcal{F}_t\}_{t \geq 0}$ be the underlying filtration.

Let θ be the reinsurance safety loading, where $\theta > \eta_i, i = 1, 2$. With self-retention rate b_1 , insurer 1's reserve process becomes

$$X_1^{b_1}(t) = x_1 + (\lambda_1 + \lambda)\mu[(1 + \theta)b_1 - (\theta - \eta_1)]t - b_1 \sum_{i=1}^{N_1(t)+N(t)} u_i. \quad (2)$$

With self-retention rate b_2 , insurer 2's reserve process becomes

$$X_2^{b_2}(t) = x_2 + (\lambda_2 + \lambda)\mu[(1 + \theta)b_2 - (\theta - \eta_2)]t - b_2 \sum_{j=1}^{N_2(t)+N(t)} u_j. \quad (3)$$

If the two insurers merge, the merged company's reserve process satisfies

$$X_m(t) = x_1 + x_2 - I + (\lambda_1 + \lambda_2 + 2\lambda)(1 + \eta_m)\mu t - \sum_{i=1}^{N_1(t)+N(t)} u_i - \sum_{j=1}^{N_2(t)+N(t)} u_j, \quad (4)$$

With self-retention rate b_m , the merged company's reserve process becomes

$$X_m^{b_m}(t) = x_1 + x_2 - I + (\lambda_1 + \lambda_2 + 2\lambda)\mu[(1 + \theta)b_m - (\theta - \eta_m)]t - b_m \left(\sum_{i=1}^{N_1(t)+N(t)} u_i + \sum_{j=1}^{N_2(t)+N(t)} u_j \right). \quad (5)$$

The martingale central limit theorem tells us that the diffusion approximation is a good approximation of a compound Poisson process provided the number of insurance contracts is large enough. Therefore, from now

on, we consider the problem under the diffusion approximation model. According to [16], the approximated diffusion process of $X_1^{b_1}(t) + X_2^{b_2}(t)$ satisfies the following:

$$X_1^{b_1}(t) + X_2^{b_2}(t) = x_1 + x_2 + (\mu_1 + \mu_2)t + \sqrt{\gamma_1^2 + \gamma_2^2 + 2\rho\gamma_1\gamma_2}B(t), \quad (6)$$

where $B(t)$ is a standard Brownian motion on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$, and

$$\begin{aligned} \mu_1 &= (\lambda_1 + \lambda)\mu[\theta b_1 - (\theta - \eta_1)], \\ \mu_2 &= (\lambda_2 + \lambda)\mu[\theta b_2 - (\theta - \eta_2)], \\ \gamma_1 &= \sqrt{(\lambda_1 + \lambda)(\mu^2 + \sigma^2)}b_1, \\ \gamma_2 &= \sqrt{(\lambda_2 + \lambda)(\mu^2 + \sigma^2)}b_2, \\ \rho &= \frac{\lambda}{\gamma_1\gamma_2}b_1b_2\mu^2. \end{aligned} \quad (7)$$

The approximated diffusion process of $X_m^{b_m}(t)$ satisfies the following:

$$X_m^{b_m}(t) = x_1 + x_2 - I + (\lambda_1 + \lambda_2 + 2\lambda)\mu[\theta b_m - (\theta - \eta_m)]t + \sqrt{(\lambda_1 + \lambda_2 + 2\lambda)(\mu^2 + \sigma^2) + 2\lambda\mu^2b_m}B(t). \quad (8)$$

Considering a policy $\pi = (T^\pi, b_1^\pi, b_2^\pi, b_m^\pi)$, where the control component T^π represents the time of the merger, $b_{i,t}^\pi$ ($i = 1, 2$) represent the proportions of risks undertaken by insurer i before the merger, and b_m^π represents the

proportion of risk undertaken by the merged company after the merger. Denote the total surplus of the two companies at time t with policy $\pi = (T^\pi, b_1^\pi, b_2^\pi, b_m^\pi; t \geq 0)$ by $X^\pi(t)$. Then, we can get

$$X^\pi(t) = x_1 + x_2 - I1_{\{t \geq T^\pi\}} + \int_0^{t \wedge T^\pi} d(X_1^\pi(s) + X_2^\pi(s)) + \int_{t \wedge T^\pi}^t dX_m^\pi(s), \quad (9)$$

where x_1 and x_2 are the initial values of the two insurers. Let $\tau_0^\pi = \inf\{t \geq 0: X^\pi(t) \leq 0\}$. A control policy $\pi = (T^\pi, b_1^\pi, b_2^\pi, b_m^\pi; t \geq 0)$ is said to be admissible if

- (i) $b_{i,t}^\pi \in [0, 1]$ ($i = 1, 2, m$) are progressively measurable
- (ii) T^π is an \mathcal{F}_t -stopping time and $T^\pi \leq \tau_0^\pi$
- (iii) There exists a unique nonnegative solution of equation (9) under the policy π

We denote the set of all admissible controls by Π .

The two insurers want to determine an admissible control policy to maximize their survival probability (i.e., if the merger occurs, they want to maximize the survival probability of the merged company); that is, they want to maximize

$$\delta^\pi(x) = P(\tau_0^\pi = \infty | X_1^\pi(0) + X_2^\pi(0) = x). \quad (10)$$

Denote the value function by

$$\delta(x) = \sup_{\pi \in \Pi} \delta^\pi(x). \quad (11)$$

Remark 1. The bankruptcy occurs if and only if the sum of the two insurers' values reaches zero. So, in reality, the two insurers can be regarded as two subsidiaries of a company.

3. Preliminaries

First, let us analyze the optimal proportional reinsurance problem of the merged insurer m . Denote the survival probability of insurer m with reinsurance policy b by $\delta_m^b(x)$. Then, the value function is

$$\delta_m(x) = \sup_{b \in [0,1]} \delta_m^b(x). \quad (12)$$

According to [12], we know that $\delta_m(x)$ satisfies

$$\sup_{b \in [0,1]} \mathcal{L}_m^b \delta_m(x) = 0, \quad (13)$$

where

$$\mathcal{L}_m^b \delta_m(x) = (\lambda_1 + \lambda_2 + 2\lambda)[\mu(\theta b_m - (\theta - \eta_m))] \delta_m'(x) + \frac{1}{2}[(\lambda_1 + \lambda_2 + 2\lambda)(\mu^2 + \sigma^2) + 2\lambda\mu^2]b_m^2 \delta_m''(x). \quad (14)$$

By some simple calculations, we can obtain that the optimal proportional reinsurance policy is

$$b_m^* = 2 \left(1 - \frac{\eta_m}{\theta} \right) \wedge 1, \quad (15)$$

and the optimal survival probability is

$$\delta_m(x) = 1 - e^{-k_m x}, \quad (16)$$

where

$$k_m = \begin{cases} \frac{A\theta^2}{2(\theta - \eta_m)}, & \eta_m \leq \theta \leq 2\eta_m, \\ 2A\eta_m, & \theta \geq 2\eta_m. \end{cases} \quad (17)$$

$$\begin{aligned} \mathcal{L}_{1,2}^{(b_1, b_2)} g(x) &= \mu[(\lambda_1 + \lambda)(\theta b_1 - (\theta - \eta_1)) + (\lambda_2 + \lambda)(\theta b_2 - (\theta - \eta_2))]g'(x) \\ &+ \frac{1}{2}[(\lambda_1 + \lambda)(\mu^2 + \sigma^2)b_1^2 + (\lambda_2 + \lambda)(\mu^2 + \sigma^2)b_2^2 + 2\lambda b_1 b_2 \mu^2]g''(x). \end{aligned} \quad (19)$$

Define

$$\tau^{(b_1, b_2)} = \inf\{t \geq 0 \mid X_1^{b_1}(t) + X_2^{b_2}(t) = 0\}. \quad (20)$$

Here, b_1 is the proportional reinsurance policy of insurer 1 and b_2 is the proportional reinsurance policy of insurer 2. Let $\delta_{1,2}^{(b_1, b_2)}(x)$ be the survival probability of the two insurers with policy (b_1, b_2) if the merger does not occur. That is,

$$\delta_{1,2}^{(b_1, b_2)}(x) = P\left(\tau^{(b_1, b_2)} = \infty \mid X_1^{b_1}(0) + X_2^{b_2}(0) = x\right). \quad (21)$$

Define

$$\delta_{1,2}(x) = \sup_{0 \leq b_i \leq 1 \ (i=1,2)} \delta_{1,2}^{(b_1, b_2)}(x). \quad (22)$$

The same methods used in [12] show that

$$\sup_{b_i \in [0,1], i=1,2} \mathcal{L}_{1,2}^{(b_1, b_2)} \delta_{1,2}(x) = 0. \quad (23)$$

Let

$$\begin{aligned} b_1^* &= \frac{(\lambda + \lambda_1)(\mu^2 + \sigma^2) - \lambda\mu^2}{\lambda + \lambda_1} B, \\ b_2^* &= \frac{(\lambda + \lambda_2)(\mu^2 + \sigma^2) - \lambda\mu^2}{\lambda + \lambda_2} B, \end{aligned} \quad (24)$$

where

$$B = \frac{2[(\theta - \eta_1)(\lambda_1 + \lambda) + (\theta - \eta_2)(\lambda_2 + \lambda)]}{\theta[(\sigma^2 + \mu^2)(\lambda_1 + \lambda_2) + 2\lambda\sigma^2]}. \quad (25)$$

Because considering proportional reinsurance policy 1 makes no sense, we can make b_1^* and b_2^* less than 1 by taking

Here,

$$A = \frac{(\lambda_1 + \lambda_2 + 2\lambda)\mu}{(\lambda_1 + \lambda_2 + 2\lambda)(\mu^2 + \sigma^2) + 2\lambda\mu^2}. \quad (18)$$

Next, let us analyze the optimal proportional reinsurance policies of the two insurers if they do not merge. Define

$$\delta_{1,2}(x) = 1 - e^{-k_{1,2}x}, \quad (26)$$

where

$$k_{1,2} = \frac{\theta\mu(\lambda_2 + \lambda)}{2b_2^*(\mu^2 + \sigma^2)(\lambda_2 + \lambda) + 2b_1^*\lambda\mu^2}. \quad (27)$$

In Section 4, we will consider two cases:

- (i) $k_{1,2} \geq k_m$
- (ii) $k_{1,2} < k_m$

We will show in case 1 that the two insurers do not merge; in case 2, the two insurers follow the reinsurance policy (b_1^*, b_2^*) until the sum of their reserve processes reaches a boundary c , and then they merge and follow reinsurance policy b_m^* .

In the following, we give two basic equations that are critical to find the value function. If the two insurers apply policy $\pi^m = (0, 0, 0, b_m^*)$, then

$$\delta(x) \geq \delta^{\pi^m}(x) = \delta_m(x - I). \quad (28)$$

If the two insurers apply policy $\pi^0 = (\infty, b_1^*, b_2^*, 0)$, then

$$\delta(x) \geq \delta^{\pi^0}(x) = \delta_{1,2}(x). \quad (29)$$

4. The HJB Equation and the Verification Theorem

In this section, we give a verification result about $\delta(x)$. This result will help us find the optimal strategy and the value

function of our problem. The following theorem gives a crucial equation to prove the verification result.

Theorem 1. *The value function $\delta(x)$ satisfies*

$$\delta(x) = \sup_{\pi \in \Pi} E_x [\delta_m(X_1^\pi(T^\pi) + X_2^\pi(T^\pi) - I)]. \quad (30)$$

Proof. First, since for any $\pi \in \Pi$, we have

$$\begin{aligned} \delta^\pi(x) &= E_x \left[1_{\{\tau_0^\pi = \infty\}} \right] = E_x \left[E \left[1_{\{\tau_0^\pi = \infty\}} \mid X_1^\pi(T^\pi) + X_2^\pi(T^\pi) \right] \right] \\ &\leq E_x [\delta_m(X_1^\pi(T^\pi) + X_2^\pi(T^\pi) - I)]. \end{aligned} \quad (31)$$

$$\sup_{\pi \in \Pi} E_x [\delta_m(X_1^\pi(T^\pi) + X_2^\pi(T^\pi) - I)] = \sup_{\pi \in \bar{\Pi}} E_x [\delta_m(X_1^\pi(T^\pi) + X_2^\pi(T^\pi) - I)]. \quad (34)$$

Since

$$\delta(x) \geq \delta^{\bar{\pi}}(x), \quad (35)$$

taking supremums on both sides and combining with equations (33) and (34), we can obtain

$$\delta(x) \geq \sup_{\pi \in \Pi} E_x [\delta_m(X_1^\pi(T^\pi) + X_2^\pi(T^\pi) - I)]. \quad (36)$$

Then, the proof is finished.

Next, we give a verification result about $\delta(x)$. \square

Taking supremums on both sides of equation (31) with respect to π , we can get

$$\delta(x) \leq \sup_{\pi \in \Pi} E_x [\delta_m(X_1^\pi(T^\pi) + X_2^\pi(T^\pi) - I)]. \quad (32)$$

On the other hand, $\forall \pi \in \Pi$, construct a new policy $\bar{\pi} = (T^\pi, b_1^\pi, b_2^\pi, b_m^*; t \geq 0)$, and we can easily get

$$\delta^{\bar{\pi}}(x) = E_x [\delta_m(X_1^\pi(T^\pi) + X_2^\pi(T^\pi) - I)]. \quad (33)$$

Let $\bar{\Pi} = \{\bar{\pi} = (T^\pi, b_1^\pi, b_2^\pi, b_m^*; t \geq 0): \pi \in \Pi\}$, then

Theorem 2. *Suppose that we can find a nonnegative function $w(x)$, piecewise twice continuously differentiable on $[0, \infty)$ with bounded derivative and satisfying the following:*

- (1) $\sup_{b_i \in [0,1], i=1,2} \mathcal{L}_{1,2}^{(b_1, b_2)} w(x) \leq 0$
- (2) $w(x) \geq \delta_m(x - I)$

With the initial condition, $w(0) = 0$. Then, $w(x) \geq \delta(x)$ for all $x \geq 0$.

Proof. For any control policy $\pi \in \Pi$, suppose $X_1^\pi(0) + X_2^\pi(0) = x$ and consider $w(X_1^\pi(t \wedge \tau_0^\pi) + X_2^\pi(t \wedge \tau_0^\pi))$. Using a generalized Itô's formula from 0 to T^π , we can get

$$\begin{aligned} w(X_1^\pi(T^\pi) + X_2^\pi(T^\pi)) &= w(x) + \int_0^{T^\pi} \mathcal{L}_{1,2}^\pi w(X_1^\pi(t) + X_2^\pi(t)) dt \\ &\quad + \int_0^{T^\pi} \sqrt{\gamma_1^2 + \gamma_2^2 + 2\rho\gamma_1\gamma_2} w'(X_1^\pi(t) + X_2^\pi(t)) dB(t). \end{aligned} \quad (37)$$

Since $w'(x)$ is bounded, taking expectations on both sides and using the two conditions in this theorem, we can get

$$w(x) \geq E_x [w(X_1^\pi(T^\pi) + X_2^\pi(T^\pi))] \geq E_x [\delta_m(X_1^\pi(T^\pi) + X_2^\pi(T^\pi) - I)]. \quad (38)$$

Taking supremums with respect to π on both sides and referring to Theorem 1, we can obtain the result. \square

5. The Value Function and the Optimal Strategy

The following theorem tells us that if $k_{1,2} \geq k_m$, the two insurers never merge and follow reinsurance policy (b_1^*, b_2^*) .

Theorem 3. *If $k_{1,2} \geq k_m$, then*

$$\delta(x) = \delta_{1,2}(x). \quad (39)$$

Proof. Using equations (16) and (26), we can see that if $k_{1,2} \geq k_m$, then

$$\delta_{1,2}(x) \geq \delta_m(x - I). \quad (40)$$

On the other hand,

$$\sup_{b_i \in [0,1], i=1,2} \mathcal{L}_{1,2}^{(b_1, b_2)} \delta_{1,2}(x) = \mathcal{L}_{1,2}^{(b_1^*, b_2^*)} \delta_{1,2}(x) = 0. \quad (41)$$

Therefore, $\delta_{1,2}(x)$ satisfies the conditions in Theorem 2; thus,

$$\delta_{1,2}(x) \geq \delta(x). \quad (42)$$

Since

$$\delta_{1,2}(x) \leq \delta(x), \quad (43)$$

the proof is completed.

The following lemma defines a function $M(x)$. For $k_{1,2} < k_m$, we will prove that $M(x)$ is the value function in Theorem 4. \square

Lemma 1. Let

$$M(x) = \sup_{\tau \in \mathcal{T}} E \left[\delta_m \left(X_1^{\pi^0}(\tau) + X_2^{\pi^0}(\tau) - I \right) \right]. \quad (44)$$

Then,

$$M(x) = \begin{cases} k\delta_{1,2}(x), & x < c, \\ \delta_m(x - I), & x \geq c, \end{cases} \quad (45)$$

where c satisfies

$$\begin{aligned} k_m e^{(k_{1,2} - k_m)c} + (k_{12} - k_m) e^{-k_m c} &= k_{1,2} e^{-k_m I}, \\ k &= \frac{k_m \exp\{c(k_{1,2} - k_m) + k_m I\}}{k_{1,2}}. \end{aligned} \quad (46)$$

Proof. Using the optimal stopping theorem, we can obtain that

$$\max \left\{ \mathcal{L}_{1,2}^{(b_1^*, b_2^*)} M(x), \delta_m(x - I) - M(x) \right\} = 0. \quad (47)$$

Furthermore, there exists a $c \geq 0$, for $x < c$,

$$\mathcal{L}_{1,2}^{(b_1^*, b_2^*)} M(x) = 0, \quad (48)$$

and for $x \geq c$,

$$M(x) = \delta_m(x - I). \quad (49)$$

Solving equation (48), we can obtain

$$M(x) = k\delta_{1,2}(x), \quad x < c, \quad (50)$$

where k is the undetermined coefficient. Using the smooth fit principle, we know that k and c are determined by

$$k\delta_{1,2}(c) = \delta_m(c - I), \quad (51)$$

$$k\delta'_{1,2}(c) = \delta'_m(c - I). \quad (52)$$

By simple calculations, we can get

$$k_m e^{(k_{1,2} - k_m)c} + (k_{12} - k_m) e^{-k_m c} = k_{1,2} e^{-k_m I},$$

$$k = \frac{k_m \exp\{c(k_{1,2} - k_m) + k_m I\}}{k_{1,2}}. \quad (53)$$

Lemma 2 is used to prove that $M(x)$ satisfies condition 2 in Theorem 2. \square

Lemma 2. If $k_{1,2} < k_m$, for $x > c$, we have

$$\sup_{b_i \in [0,1], i=1,2} \mathcal{L}_{1,2}^{(b_1, b_2)} M(x) = \sup_{b_i \in [0,1], i=1,2} \mathcal{L}_{1,2}^{(b_1, b_2)} \delta_m(x - I) \leq 0. \quad (54)$$

Proof. Since

$$\mathcal{L}_{1,2}^{(b_1^*, b_2^*)} \delta_m(c - I) \leq k \mathcal{L}_{1,2}^{(b_1^*, b_2^*)} \delta_{1,2}(c) = 0, \quad (55)$$

combining with equation (52), we can obtain

$$\delta_m''(c - I) \leq k\delta_{1,2}''(c). \quad (56)$$

According to equations (16) and (26), define

$$G(x) = \frac{\delta_m''(x - I)}{k\delta_{1,2}''(x)} = \frac{e^{k_m I} k_m^2}{k k_{1,2}^2} e^{-(k_m - k_{1,2})x}. \quad (57)$$

Clearly, if $k_m > k_{1,2}$, $G(x)$ is strictly decreasing. For $x > c$, we have

$$\frac{\delta_m''(x - I)}{k\delta_{1,2}''(x)} < \frac{\delta_m''(c - I)}{k\delta_{1,2}''(c)} \leq 1. \quad (58)$$

This implies that

$$\delta_m''(x - I) < k\delta_{1,2}''(x), \quad x > c. \quad (59)$$

Furthermore,

$$\delta_m'(c - I) = k\delta_{1,2}'(c), \quad (60)$$

and then we have

$$\delta_m'(x - I) < k\delta_{1,2}'(x), \quad x > c. \quad (61)$$

Thus, $\forall (b_1, b_2)$,

$$\mathcal{L}_{1,2}^{(b_1, b_2)} \delta_m(x - I) < k \mathcal{L}_{1,2}^{(b_1, b_2)} \delta_{1,2}(x) \leq 0, \quad x > c. \quad (62)$$

Taking supermums on both sides, we complete the proof. \square

Theorem 4. If $k_{1,2} < k_m$, then $\delta(x) = M(x)$. The optimal strategy is that the two insurers follow the reinsurance policies that maximize their survival probability until the sum of their surplus processes reaches c , and then they merge and apply the merged company's optimal reinsurance strategy.

Proof. First, by the definition of $M(x)$, we know that

$$M(x) \geq \delta_m(x - I). \quad (63)$$

For $x \leq c$,

$$M(x) = k\delta_{1,2}(x), \quad (64)$$

which implies that

$$\sup_{b_i \in [0,1], i=1,2} \mathcal{L}_{1,2}^{(b_1, b_2)} M(x) = k \sup_{b_i \in [0,1], i=1,2} \mathcal{L}_{1,2}^{(b_1, b_2)} \delta_{1,2}(x) = k \mathcal{L}_{1,2}^{(b_1^*, b_2^*)} \delta_{1,2}(x) = 0. \quad (65)$$

Combining with Lemma 2, we have for $x \geq 0$,

$$\sup_{b_i \in [0,1], i=1,2} \mathcal{L}_{1,2}^{(b_1, b_2)} M(x) \leq 0. \quad (66)$$

Thus, the two conditions in Theorem 2 are satisfied, and we can obtain

$$\delta(x) \leq M(x). \quad (67)$$

On the other hand,

$$\delta(x) \geq M(x). \quad (68)$$

Then, we have

$$\delta(x) = M(x). \quad (69)$$

Clearly, by the definition of $M(x)$, the optimal strategy is that the two insurers follow the reinsurance policies that maximize their survival probability until the sum of their surplus processes reaches c , and then they merge and apply the merged company's optimal reinsurance strategy.

In this case, the optimal merge time is as follows:

$$T^* = \inf \left\{ t | X_1^{b_1^*}(t) + X_2^{b_2^*}(t) = c \right\}. \quad (70)$$

□

6. Illustration of the Results

In this section, we discuss the effects of all the parameters on the optimal policy. $k_m - k_{1,2}$ determines whether or not to merge, so in Section 6.1, let us show the effects of the parameters on the symbol of $k_m - k_{1,2}$.

6.1. Effects of all the Parameters on $k_m - k_{1,2}$. Figures 1–7 give the effects of all the parameters on $k_m - k_{1,2}$.

Figure 1 shows that η_m has a positive effect on k_m but has no effect on $k_{1,2}$. So, $k_m - k_{1,2}$ increases as η_m increases. At the beginning, $k_m < k_{1,2}$; they are equal near $\eta_m = 0.24$; for $\eta_m > 0.24$, $k_m > k_{1,2}$.

Figures 2 and 3 show that η_1 and η_2 have positive effects on $k_{1,2}$ but have no effect on k_m . So, $k_m - k_{1,2}$ decreases as η_1 or η_2 increases. At the beginning, $k_{1,2} < k_m$; they are equal near $\eta_1 = 0.26$ and $\eta_2 = 0.2$, respectively. The two figures also indicate that the merged company has greater survival probability with a smaller safety loading.

Figure 4 shows that θ has a negative effect on both k_m and $k_{1,2}$. Furthermore, $k_{1,2}$ decreases more quickly than k_m as θ

increases. At the beginning, $k_m < k_{1,2}$; they are equal near $\theta = 0.49$; for $\theta > 0.49$, $k_m > k_{1,2}$. This indicates the following:

- (i) The merger has more and more advantages as θ increases
- (ii) The merged company is better at resisting reinsurance rate risk

Figure 5 shows that λ has negative effects on both k_m and $k_{1,2}$. Furthermore, k_m decreases more quickly than $k_{1,2}$ as λ increases. This indicates the following:

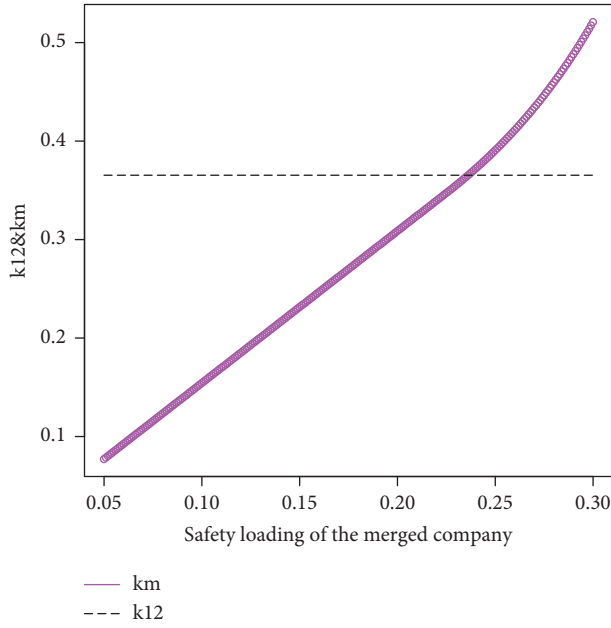
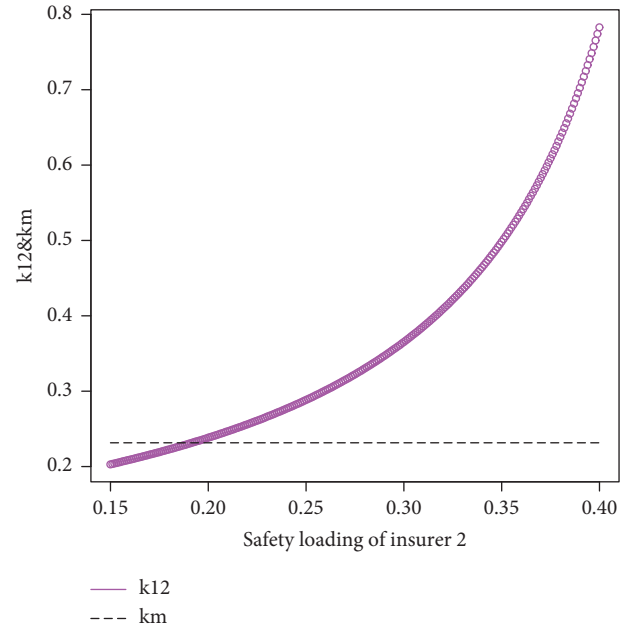
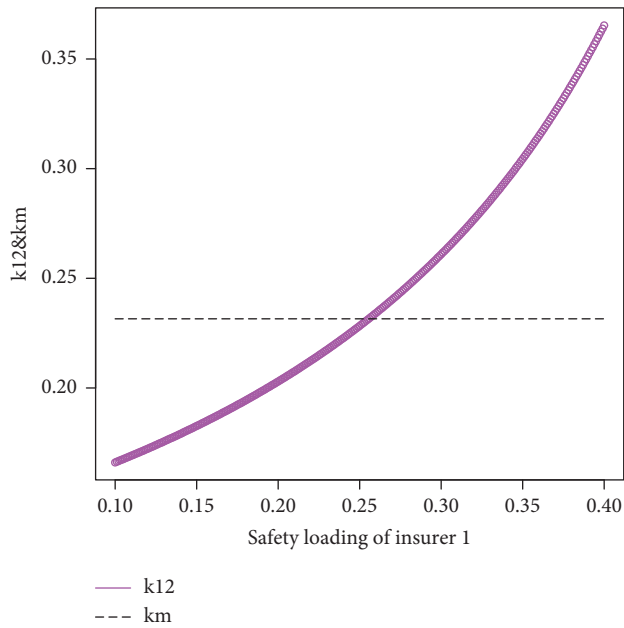
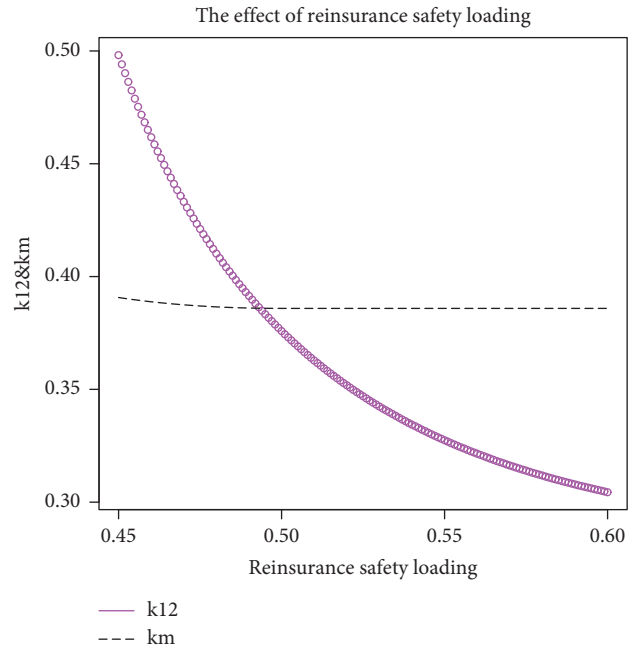
- (i) The stronger the risk correlation (λ), the smaller the survival probability (refer to catastrophic insurance).
- (ii) The merged company's survival probability is more sensitive to the risk correlation (λ). So, the merger has more and more disadvantages as λ increases.
- (iii) $k_m - k_{1,2}$ is sensitive to η_m . A small increase in η_m results in a change in the symbol of $k_m - k_{1,2}$. So, for different λ , we can set different η_m to get a better merged result.

Let $\eta_1 = 0.4$ and $\eta_2 = 0.35$; that is, the safety loading of insurer 1 is greater than the safety loading of insurer 2. It implies that insurer 1 is an insurer with a better reputation and service. We plot Figures 6 and 7 to illustrate the effect of different insurers' idiosyncratic claim intensities on their optimal policy.

Figure 6 shows that λ_1 has positive effects on both k_m and $k_{1,2}$. Furthermore, $k_{1,2}$ increases more quickly than k_m as λ_1 increases. So, $k_m - k_{1,2}$ decreases as λ_1 increases. At the beginning, $k_{1,2} < k_m$; they are equal near $\lambda_1 = 3$; for $\lambda_1 > 3$, $k_m < k_{1,2}$. This indicates the following:

- (i) The business expansion of insurer 1 results in greater survival probabilities regardless of whether merger occurs (this is clear because the business expansion of a better insurer will bring more profits than risks)
- (ii) The merger has more and more disadvantages with the business expansion of insurer 1

Figure 7 shows that λ_2 has a positive effect on k_m and has a negative effect on $k_{1,2}$. At the beginning, $k_m < k_{1,2}$; they are equal near $\lambda_2 = 3$; for $\lambda_2 > 3$, $k_m > k_{1,2}$. This indicates that the business expansion of the bad insurer (insurer 2) decreases the survival probability, but if it is merged with some good insurer (insurer 1), the business expansion increases survival probability.

FIGURE 1: The effect of η_m on k_{12} and k_m .FIGURE 3: The effect of η_2 on k_{12} and k_m .FIGURE 2: The effect of η_1 on k_{12} and k_m .FIGURE 4: The effect of θ on k_{12} and k_m .

If we know $k_m - k_{1,2}$, we can decide whether to merge. Thus, in this section, we have determined whether or not to merge for different situations. In the next section, we consider, for $k_m \geq k_{1,2}$, the effects of k_m , $k_{1,2}$, and I on the time to merge. This is equivalent to analyzing the effects of k_m , $k_{1,2}$, and I on c .

6.2. Effects of $k_{1,2}$, k_m , and I on c . Figures 8–10 present the results of the problem when $k_m \geq k_{1,2}$.

Figure 8 shows that k_m has a negative effect on c . This indicates the following:

- (i) As k_m increases, the gap between $k_{1,2}$ and k_m becomes larger and larger, synergy becomes more and

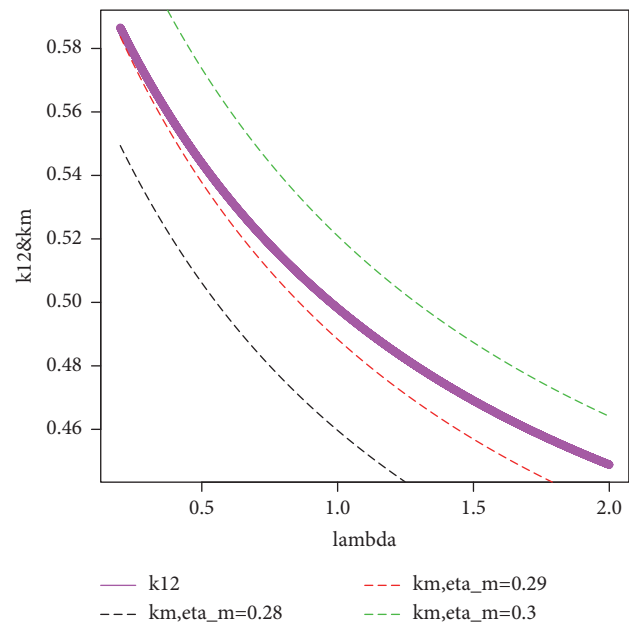


FIGURE 5: The effect of λ on k_{12} and k_m .

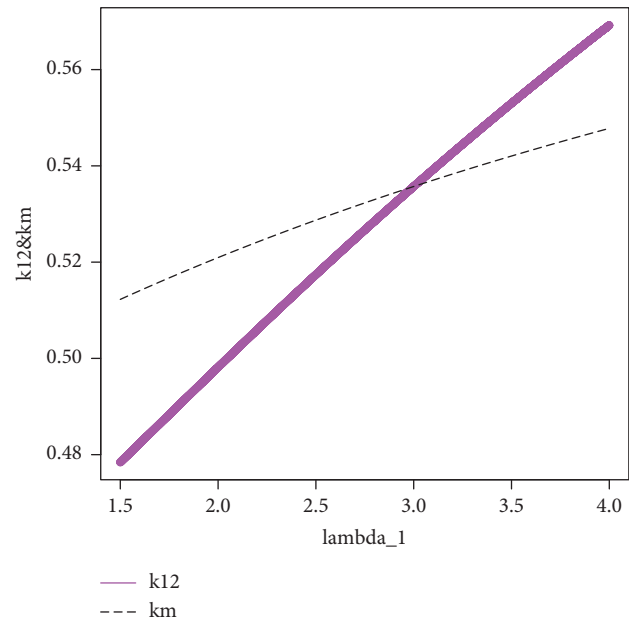
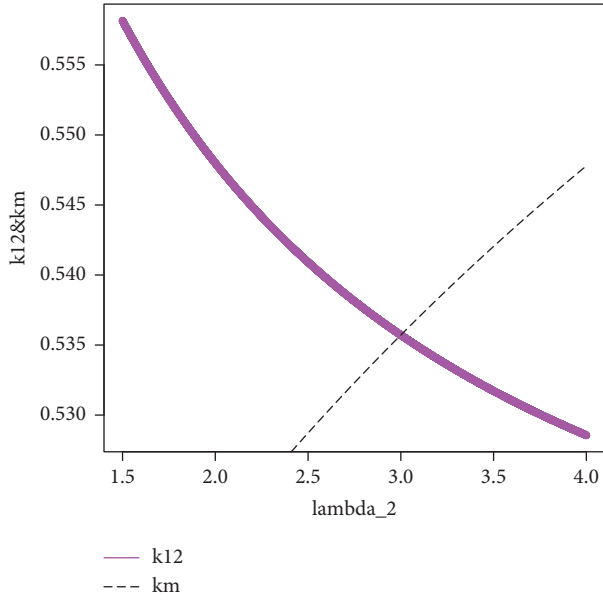
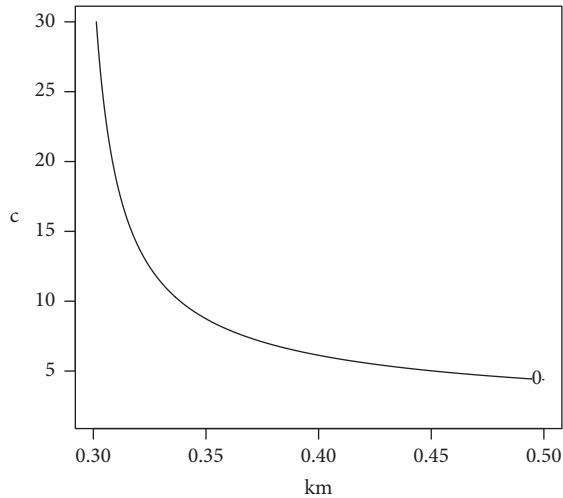


FIGURE 6: The effect of λ_1 on k_{12} and k_m .

FIGURE 7: The effect of λ_2 on k_{12} and k_m .FIGURE 8: The effect of k_m on c .

more obvious, and the boundary of the merger c becomes lower and lower

- (ii) As k_m increases, the gap between $k_{1,2}$ and k_m becomes larger and larger, and the slope of the line approaches zero (this result is consistent with the diminishing marginal effect)

Figure 9 shows that $k_{1,2}$ has a positive effect on c . This indicates the following:

- (i) As $k_{1,2}$ increases, the gap between $k_{1,2}$ and k_m becomes smaller and smaller and the boundary of the merger c becomes higher and higher
- (ii) As $k_{1,2}$ decreases, the gap between $k_{1,2}$ and k_m becomes larger and larger and the slope of the line approaches zero (this result is consistent with the diminishing marginal effect)

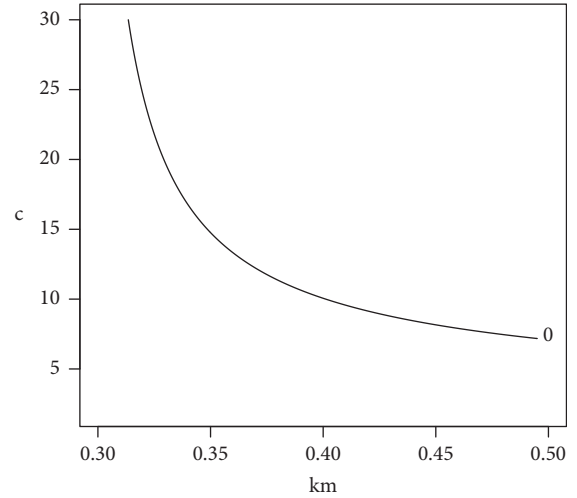
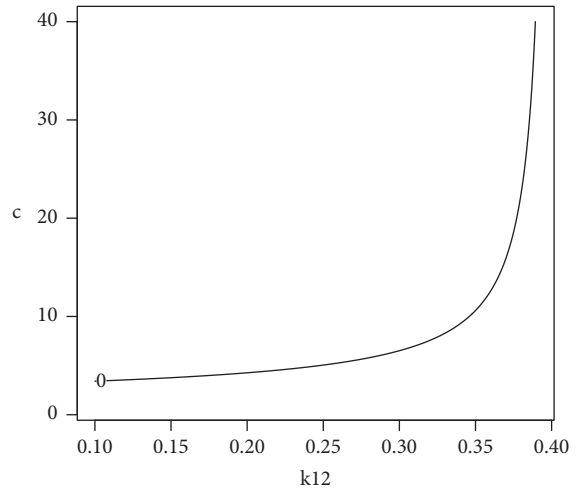
FIGURE 9: The effect of k_{12} on c .FIGURE 10: The effect of I on c .

Figure 10 shows that I has a positive effect on c (the larger the cost of the merger, the higher the boundary of the merger).

7. Conclusion

Economies of scale, competitive advantage theory, and agency theory have led to the rapid development of enterprise merger and acquisition theory, making them one of the most active fields in Western economics. However, the existing research results are mainly address the motivation for mergers and acquisitions. From the perspective of enterprise management and financial analysis, those papers mainly focus on economies of scale, management efficiency, and enterprise pricing. Most of these research results are qualitative analysis and ignore the measurement of enterprise risk.

From the perspective of risk control, this paper gives the optimal merger time and reinsurance strategies of two insurance companies by means of optimal stopping theory and

stochastic optimal control theory. By analyzing the influence of changes in parameters on the merger strategy, we obtain many meaningful conclusions. For example, the merged company is more competitive and more adaptable to changes in reinsurance rate. Expanding the business of the company with a better reputation and service will reduce the bankruptcy probability. These conclusions are in line with the theories of economies of scale and competitive advantage. We also find that the more obvious the advantages of the company's merger, the earlier the merger time; the higher the merger cost, the later the merger time.

This paper gives the optimal strategy on the premise of equal bargaining between two companies. However, the merger of two companies with different bargaining power is a topic worthy of further discussion. As this problem is more complex, it requires more auxiliary tools such as game theory and so on.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

An Annuitization Problem in the Tax-Deferred Annuity Model

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This paper examines the optimal annuitization, investment, and consumption strategies of an individual facing a time-dependent mortality rate in the tax-deferred annuity model and considers both the case when the rate of buying annuities is unrestricted and the case when it is restricted. At the beginning, by using the dynamic programming principle, we obtain the corresponding HJB equation. Since the existence of the tax and the time-dependence of the value function make the corresponding HJB equation hard to solve, firstly, we analyze the problem in a simpler case and use some numerical methods to get the solution and some of its useful properties. Then, by using the obtained properties and Kuhn–Tucker conditions, we discuss the problem in general cases and get the value functions and the optimal annuitization strategies, respectively.

1. Introduction

Referring to [1], we know that, with the aging population, the basic endowment insurance has been unable to meet people's insurance needs. In order to guarantee the sustainable operation of the old age insurance system, many governments begin to adopt preferential tax policies and attract people to buy tax-deferred annuities. In the United States, tax-deferred annuities have benefited more than 60% of families and become the main source of continuous growth of pension assets in the past 30 years. Tax-deferred annuities also reduce the pension burden in many countries such as Germany and France. However, in some countries, the pilot of tax extension type endowment insurance is not successful. Since tax-deferred annuities can not only play a positive role in improving the three pillar endowment insurance system, but can also alleviate social problems caused by the aging population, prosper the commercial endowment insurance market, and provide investment funds for the securities market, the promotion of tax-deferred annuities is extremely essential and urgent in the whole world.

In order to attract more and more people to buy tax-deferred annuities, the government should know how to adopt tax preferential policies, and insurance companies

should know how to design and publicize products. In this paper, by using stochastic optimal methods and the Kuhn–Tucker conditions, we get the public's strategies on how to buy tax-deferred annuities. Then, we can give some suggestions for the government and insurance companies to promote tax-deferred annuities.

Up till now, there have been a lot of papers investigating individuals' optimal annuitization strategies in a taxable annuity model. Reference [2] investigates the optimal annuitization strategy for an infinitely lived individual who faces a choice between voluntary annuitization and discretionary management of assets with systematic withdrawal for consumption purposes. Reference [3] examines the optimal annuitization, investment, and consumption strategies of a utility-maximizing retiree facing a stochastic time of death under an all-or-nothing arrangement and an open-market structure arrangement, respectively. Reference [4] solves a problem of finding the optimal time of annuitization for a retiree having the possibility of choosing her investment and consumption strategy. Reference [5] aims at maximizing the expected utility of consumption with commutable life annuities and finds that the optimal annuitization strategy depends on the size of proportional surrender charge. For a future reference, we can refer to [6–9].

However, there are only a few articles that investigate the tax-deferred annuities. Reference [10] gives some comparisons of different tax regimes applied to private pensions. Reference [11] identifies four potential risks associated with tax-deferred. Reference [12] concerns with the risk of fluctuating tax rates and changing tax brackets; it also examines how the saver's tax credit changes optimal tax-deferral choices of individuals. These papers are all analyzed by using the data processing method, and they are focusing on comparing the tax-deferred annuity and an ordinary one. This paper investigates the optimal annuitization, investment, and consumption strategies for an individual under EET tax payment mode (in which individuals deposit the premium into an independent account, and the insurance company invests it in the financial market and returns the income to the individual in the form of annuities on retirement. Both the taxes of premiums and investment incomes are deferred until receiving annuities) by using stochastic optimal control theory. We consider both the case when the rate of buying annuities is unrestricted and the case when it is restricted. We get the value functions and the optimal annuitization strategies, respectively.

This paper is organized as follows. In Section 2, the mathematical formulation of the problem is presented. In Section 3, we investigate the optimal investment, consumption, and annuitization strategies when the rate of buying annuities is unrestricted. In Section 4, the same problem is considered when the rate of buying annuities is restricted. Finally, the numerical method is provided in Appendix.

2. Problem Formulation

Let us consider the financial market first. Suppose that there are a risk-free asset and a risky asset in the financial market. Their prices per share $B(t)$ and $R(t)$ follow the following equations:

$$\begin{aligned} dB(t) &= rB(t)dt, \\ dR(t) &= R(t)[\mu dt + \sigma dW(t)]. \end{aligned} \quad (1)$$

Here, $r > 0$ represents the risk-free rate of return, $\mu > r$ is the drift coefficient of the risk asset, $\sigma > 0$ is the volatility coefficient of the risk asset, $W(t)$ is a standard Brownian motion on a complete probability space (Ω, \mathcal{F}, P) , and \mathcal{F}_t is the natural filtration generated by $W(t)$.

Let $b_{1,t}$ be the wealth of the individual invested in the risky asset, and p_t the rate of buying tax-deferred annuities. Setting aside taxes, we can get that the individual's wealth process under these policies denoted by $x_t^{b_{1,t}, p}$ satisfies

$$dx_t^{b_{1,t}, p} = [rx_t^{b_{1,t}, p} + b_{1,t}(\mu - r) - p_t]dt + \sigma b_{1,t}dW(t). \quad (2)$$

Like a general tax policy, we suppose that there is a threshold $b > 0$; if the income rate is less than b , nothing will be taxed; otherwise, the part above b will be taxed with a tax rate k . Here, we also assume that the income rate is greater than b since, otherwise, there is no tax preference to buy tax-deferred annuities for the individual. Then, the taxed wealth $dX_t^{b_{1,t}, p}$ satisfies

$$\begin{aligned} dX_t^{b_{1,t}, p} &= [rX_t^{b_{1,t}, p} + b_{1,t}(\mu - r) - p_t]dt + \sigma b_{1,t}dW(t) \\ &\quad - k[(rX_t^{b_{1,t}, p} + b_{1,t}(\mu - r) - p_t - b)dt + \sigma b_{1,t}dW(t)] \\ &= [\hat{r}X_t^{b_{1,t}, p} + b_{1,t}(\hat{\mu} - \hat{r}) - \hat{p}_t]dt + \hat{\sigma}b_{1,t}dW(t) + kbdt. \end{aligned} \quad (3)$$

Here,

$$\begin{aligned} \hat{r} &= (1 - k)r, \\ \hat{\mu} &= (1 - k)\mu, \\ \hat{\sigma} &= (1 - k)\sigma, \\ \hat{p}_t &= (1 - k)p_t. \end{aligned} \quad (4)$$

Taking consumptions into consideration, we have

$$\begin{aligned} dX_t^{b_{1,t}, p, c} &= [\hat{r}X_t^{b_{1,t}, p, c} + b_{1,t}(\hat{\mu} - \hat{r}) - \hat{p}_t]dt \\ &\quad + \hat{\sigma}b_{1,t}dW(t) + (kb - c_t)dt, \end{aligned} \quad (5)$$

where c_t is the consumption rate of the individual at time t .

In the pension market, we assume that the insurance company evaluates the residual lifetime using exponentially distributed random variables. Denote the hazard rate evaluated by the individual at time t by λ_t^s and the hazard rate evaluated by the insurance company at time t by λ_t^o , respectively. The wealth in the pension fund will also be invested in the financial market. The difference is that the income will not be taxed until the retirement time T . So, the income process in the pension fund under investment policy $b_{2,t}$ is

$$dY_t^{b_2} = [rY_t^{b_2} + b_{2,t}(\mu - r) + p_t]dt + \sigma b_{2,t}dW(t). \quad (6)$$

Suppose that the individual has to pay taxes at a tax rate k_2 before he receives annuities. Since the insurance company assumes that the hazard rate of the individual is λ_t^o , we can get that the annuity received by the individual is

$$c_0 = \frac{(1 - k_2)Y_T}{A}, \quad (7)$$

where Y_T is the total pension fund at the retirement time T , $A = \int_T^\infty e^{-(\lambda_u^o + r)} du$. Then, the individual's expected cumulative discounted pension is

$$g(Y_T) = \int_T^\infty e^{-(\lambda_u^s + r)} c_0 du = \frac{B(1 - k_2)Y_T}{A}, \quad (8)$$

where $B = \int_T^\infty e^{-(\lambda_u^s + r)} du$.

In this paper, we assume that the short selling is allowed, and then, $b_{i,t}$, $i = 1, 2$ can be negative. Now, let us give the definition of admissible controls. If $\pi = (b_{1,t}^\pi, b_{2,t}^\pi, p_t^\pi, c_t^\pi)$ satisfies

- (i) $b_{i,t}^\pi$, $i = 1, 2$ are \mathcal{F}_t predictable control processes, and p_t^π , c_t^π are nonnegative \mathcal{F}_t predictable control processes.
- (ii) $\int_0^\infty c_t^\pi dt < \infty$, $\int_0^\infty (b_{i,t}^\pi)^2 dt < \infty$, $i = 1, 2$, $\int_0^\infty p_t^\pi dt < \infty$.

Then, we say that π is admissible. Denote all the admissible strategies by Π .

Next, let us focus on this question: What are the optimal consumption/investment strategies and the best annuitization strategy for the individual to maximize the sum of his expected accumulated discounted consumption before retirement and the expected discounted wealth at retirement?

Suppose that the individual's wealth at time t is x and his money in his pension account at time t is y . Denote the objective function under policy π by $J^\pi(t, x, y)$, and then, we have

$$J^\pi(t, x, y) = E \left[\int_t^T \Gamma(t, s) c_s^\pi ds + \Gamma(t, T) (X_T^\pi + g(Y_T^\pi)) | X_t^\pi = x, Y_t^\pi = y \right], \quad (9)$$

where $\Gamma(t, s) = e^{-\int_t^s (r + \lambda_u^s) du}$. Thus, the value function $V(t, x, y)$ is

$$V(t, x, y) = \sup_{\pi \in \Pi} J^\pi(t, x, y). \quad (10)$$

In the following, we give some initial condition assumptions.

(i) Assumption 1

$$\begin{aligned} V_x(t, 0, y) &> 1, \\ V_x(t, 0, y) &\geq \frac{V_y(t, 0, y)}{1 - k}, \quad \forall y \geq 0. \end{aligned} \quad (11)$$

(ii) Assumption 2

$$\frac{V_y(t, 0, 0)}{1 - k} > 1. \quad (12)$$

3. The Optimal Strategies when p_t Is Unrestricted

In this section, we investigate the optimal strategy of an individual when p_t is unrestricted. According to 3.4.2 in [13], we can get the following theorem.

Theorem 1. $V(t, x, y)$ satisfies

$$\begin{aligned} \min \left\{ (r + \lambda_t^s) V(t, x, y) - \sup_{b_1 \geq 0, b_2 \geq 0} \left[(\tilde{r}x + (\tilde{\mu} - \tilde{r})b_1 + kb)V_x(t, x, y) + \frac{\hat{\sigma}^2 b_1^2}{2} V_{xx}(t, x, y) \right. \right. \\ \left. \left. + \sigma \hat{\sigma} b_1 b_2 V_{xy} + (ry + b_2(\mu - r))V_y(t, x, y) + \frac{\sigma^2 b_2^2}{2} V_{yy}(t, x, y) - V_t(t, x, y) \right] \right\} \\ V_x(t, x, y) - 1, (1 - k)V_x(t, x, y) - V_y(t, x, y) = 0, \quad t < T, \end{aligned} \quad (13)$$

$$V(T, x, y) = x + g(y), \quad (14)$$

$$V(t, 0, 0) = 0, \quad t \geq 0. \quad (15)$$

By taking derivatives with respect to b_1, b_2 in (13), we can get the maximum point:

$$b_1^*(t, x, y) = \frac{(\tilde{\mu} - \tilde{r})(V_{yy}(t, x, y)V_x(t, x, y) - V_y(t, x, y)V_{xy}(t, x, y))}{\hat{\sigma}^2(V_{xy}^2(t, x, y) - V_{xx}(t, x, y)V_{yy}(t, x, y))}, \quad (16)$$

$$b_2^*(t, x, y) = \frac{(\mu - r)(V_{xx}(t, x, y)V_y(t, x, y) - V_x(t, x, y)V_{xy}(t, x, y))}{\sigma^2(V_{xy}^2(t, x, y) - V_{xx}(t, x, y)V_{yy}(t, x, y))}. \quad (17)$$

The following theorem gives us the solution of (13) in the case when $V_x(t, x, y) > 1$, and $V_y(t, x, y) < (1 - k)V_x(t, x, y)$, and it is essential in solving (13).

Theorem 2. For $V_x(t, x, y) > 1$, $V_y(t, x, y) < (1 - k)V_x(t, x, y)$, we have

$$V(t, x, y) = V_1(t, x) + V_2(t, y), \quad 0 \leq t \leq T, \quad (18)$$

where $V_1(t, x)$ satisfies (23) and (20), and $V_2(t, y)$ satisfies (24) and (21).

Proof. Denote the optimal rate of buying tax-deferred annuities by p_t^* and the optimal consumption rate by c_t^* . For $V_x(t, x, y) > 1$, $V_y(t, x, y) < (1 - k)V_x(t, x, y)$, we can deduce that $c_t^* = 0$, $p_t^* = 0$. The individual will never buy annuities in this situation, and there is no interaction between x and y . This implies that

$$V(t, x, y) = V_1(t, x) + V_2(t, y), \quad (19)$$

where $V_1(t, x)$ are the benefits obtained in the financial market, and $V_2(t, y)$ are the benefits obtained from the annuity fund. Then, we can obtain that

$$\begin{aligned} V_1(t, 0) &= 0, \quad t \geq 0, \\ V_1(T, x) &= x, \quad x \geq 0, \end{aligned} \quad (20)$$

$$\begin{aligned} V_2(t, 0) &= 0, \quad t \geq 0, \\ V_2(T, y) &= g(y), \quad y \geq 0. \end{aligned} \quad (21)$$

Taking (16) and (17) into (13), we can get

$$(r + \lambda_t^s)V(t, x, y) - V_t(t, x, y) = \tilde{r}xV_x(t, x, y) + ryV_y(t, x, y)$$

$$(t, x, y) + kbV_x(t, x, y) - \frac{R^2}{2} \left(\frac{V_x^2(t, x, y)}{V_{xx}(t, x, y)} + \frac{V_y^2(t, x, y)}{V_{yy}(t, x, y)} \right), \quad (22)$$

where $R = ((\mu - r)/\sigma)$. Then, the following two differential equations are satisfied immediately:

$$\begin{aligned} &-(r + \lambda_t^s)V_1(t, x) + V_{1t}(t, x) + \tilde{r}xV_{1x}(t, x) + kbV_{1x}(t, x) \\ &-\frac{R^2V_{1x}^2(t, x)}{2V_{1xx}(t, x)} = 0, \end{aligned} \quad (23)$$

$$-(r + \lambda_t^s)V_2(t, y) + V_{2t}(t, y) + ryV_{2y}(t, y) - \frac{R^2V_{2y}^2(t, y)}{2V_{2yy}(t, y)} = 0. \quad (24)$$

According to [14], we know that (23) and (20) or (24) and (21) may be solved by using dual methods. However, the items $(r + \lambda_t^s)V_1(t, x)$, $(r + \lambda_t^s)V_2(t, y)$ in (23) and (24) combined with the time dependence of λ_t^s make those partial differential equations hard to solve. In this paper, we use the finite difference method to get $V_1(t, x)$ and $V_2(t, y)$ in

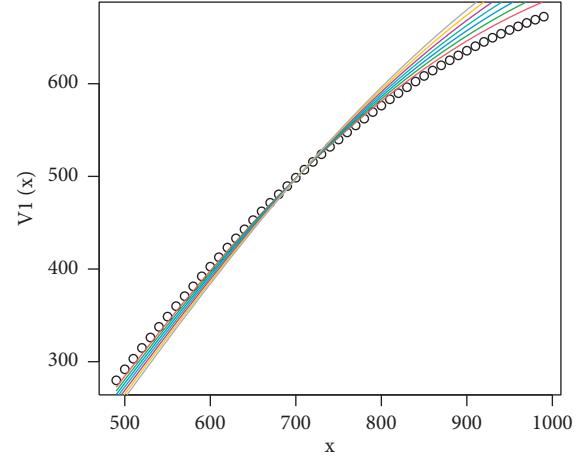


FIGURE 1: $V_1(t, x)$

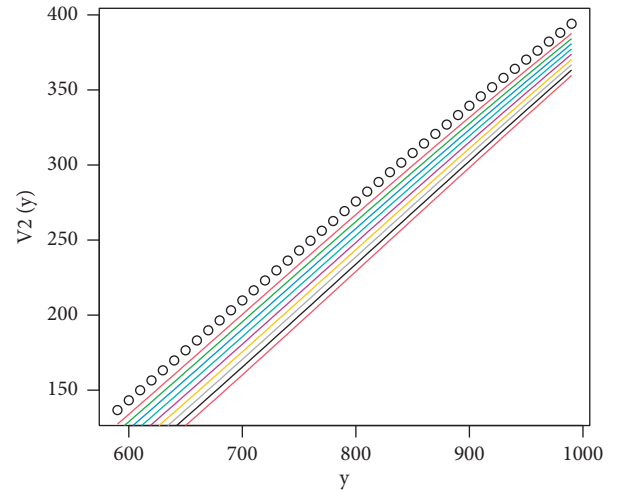


FIGURE 2: $V_2(t, y)$

Appendix A. Considering Figures 1 and 2, we get the following theorem. \square

Theorem 3. $V_1(t, x)$ is a concave increasing positive function with respect to x , $V_2(t, y)$ is a concave increasing positive function with respect to y . The growth rates of $V_1(t, x)$ and $V_2(t, y)$ are increasing with respect to t .

Next, Let us consider the solutions of (13) in other cases. Firstly, the boundlessness of p_t and c_t tells us that the individual's optimal consumption and the optimal amount of buying annuities are lump-sum payments. Denote Δ_1^* to be the individual's optimal consumption at time t , denote Δ_2^* to be the optimal amount of buying annuities at time t , and considering the right hand of (13), we can obtain that

$$V(t, x, y) = V\left(t, x - \Delta_1^* - \Delta_2^*, y + \frac{\Delta_2^*}{1 - k}\right) + \Delta_1^*, \quad (25)$$

$$V_x\left(t, x - \Delta_1^* - \Delta_2^*, y + \frac{\Delta_2^*}{1 - k}\right) = 1, \quad (26)$$

$$V_x \left(t, x - \Delta_1^* - \Delta_2^*, y + \frac{\Delta_2^*}{1-k} \right) = (1-k) \cdot V_y \left(t, x - \Delta_1^* - \Delta_2^*, y + \frac{\Delta_2^*}{1-k} \right). \quad (27)$$

Considering (26) and (27) and using Theorem 2, we know that

$$V \left(t, x - \Delta_1^* - \Delta_2^*, y + \frac{\Delta_2^*}{1-k} \right) = V_1 \left(t, x - \Delta_1^* - \Delta_2^* \right) + V_2 \left(t, y + \frac{\Delta_2^*}{1-k} \right). \quad (28)$$

Combining it with (25), we can get that

$$V(t, x, y) = V_1 \left(t, x - \Delta_1^* - \Delta_2^* \right) + V_2 \left(t, y + \frac{\Delta_2^*}{1-k} \right) + \Delta_1^*. \quad (29)$$

This implies that $V(t, x, y)$ is the solution of the following restricted optimal problem:

$$P: \sup_{\Delta_1 \geq 0, \Delta_2 \geq 0, x - \Delta_1 - \Delta_2 \geq 0} V_1(t, x - \Delta_1 - \Delta_2) + V_2 \left(t, y + \frac{\Delta_2}{1-k} \right) + \Delta_1. \quad (30)$$

Here, Δ_1 is the individual's consumption strategy, and Δ_2 is the wealth of buying annuities. Denote the optimal policy by $(\Delta_1^*(t, x, y), \Delta_2^*(t, x, y))$.

In order to solve Problem P, we need to do some explicit case analysis. Firstly, define $x_0(t), x_1(t, y)$ as

$$\begin{aligned} V_{1x}(t, x_0(t)) &= 1, \\ (1-k)V_{1x}(t, x_1(t, y)) &= V_{2y}(t, y), \quad y \geq 0, \end{aligned} \quad (31)$$

Let

$$\begin{aligned} D_1 &= \{0 \leq t \leq T, y \geq 0 | x_1(t, y) < x_0(t)\}, \\ D_2 &= \{0 \leq t \leq T, y \geq 0 | x_1(t, y) \geq x_0(t)\}. \end{aligned} \quad (32)$$

$$V_1'(t, x - \Delta_1 - \Delta_2) > V_1'(t, x_0(t)) = 1,$$

$$V_2' \left(t, y + \frac{\Delta_2}{1-k} \right) = (1-k)V_1'(t, x_1(t, y)) \leq (1-k)V_1'(t, x - \Delta_1 - \Delta_2). \quad (36)$$

That is,

$$\begin{aligned} \frac{\partial L(\Delta_1, \Delta_2)}{\partial \Delta_1} &< 0, \\ \frac{\partial L(\Delta_1, \Delta_2)}{\partial \Delta_2} &\leq 0. \end{aligned} \quad (37)$$

Clearly, in this case, we have

$$\begin{aligned} \Delta_1^*(t, x, y) &= 0, \\ \Delta_2^*(t, x, y) &= 0. \end{aligned} \quad (38)$$

For $x > x_1(t, y)$, the signs of $(\partial L(\Delta_1, \Delta_2)/\partial \Delta_1)$ and $(\partial L(\Delta_1, \Delta_2)/\partial \Delta_2)$ are uncertain. Since

$$\frac{\partial^2 L(\Delta_1, \Delta_2)}{\partial \Delta_1^2} = \frac{\partial^2 L(\Delta_1, \Delta_2)}{\partial \Delta_2^2} = \frac{\partial^2 L(\Delta_1, \Delta_2)}{\partial \Delta_1 \partial \Delta_2} = V_1''(t, x - \Delta_1 - \Delta_2) \leq 0. \quad (39)$$

We can get that

$$\frac{\partial^2 L(\Delta_1, \Delta_2)}{\partial \Delta_1^2} = \frac{\partial^2 L(\Delta_1, \Delta_2)}{\partial \Delta_2^2} \leq 0, \quad (40)$$

$$\frac{\partial^2 L(\Delta_1, \Delta_2)}{\partial \Delta_1^2} \frac{\partial^2 L(\Delta_1, \Delta_2)}{\partial \Delta_2^2} - \frac{\partial^2 L(\Delta_1, \Delta_2)}{\partial \Delta_1 \partial \Delta_2} = 0.$$

Then, the Hessian matrix of $L(\Delta_1, \Delta_2)$ is seminegative definite, which implies the concavity of $L(\Delta_1, \Delta_2)$. So, the necessary and sufficient conditions for the global maximum point are the corresponding Kuhn–Tucker conditions:

$$\frac{\partial L(\Delta_1^*(t, x, y), \Delta_2^*(t, x, y))}{\partial \Delta_i} \leq 0, \quad \Delta_i \geq 0, \quad (41)$$

$$\frac{\partial L(\Delta_1^*(t, x, y), \Delta_2^*(t, x, y))}{\partial \Delta_i} \Delta_i = 0, \quad i = 1, 2.$$

For $\Delta_1 = 0, \Delta_2 = x - x_1(t, y) > 0$, we can get

$$\begin{aligned} \frac{\partial L(\Delta_1, \Delta_2)}{\partial \Delta_1} &< 0, \\ \frac{\partial L(\Delta_1, \Delta_2)}{\partial \Delta_2} &= 0. \end{aligned} \quad (42)$$

Clearly, the corresponding Kuhn–Tucker conditions are satisfied at $(0, x - x_1(t, y))$. So, it is the global maximum point. \square

Theorem 5. For $(t, y) \in D_2$, we have

$$\begin{aligned} \Delta_1^*(t, x, y) &= \begin{cases} 0, & x \leq x_0(t), \\ x - x_0(t), & x > x_0(t), \end{cases} \\ \Delta_2^*(t, x, y) &= 0. \end{aligned} \quad (43)$$

Proof. The proof is similar to the one in Theorem 4. Now, we omit it. \square

Remark 1. It is shown from the above two theorems that

- (i) In D_1 , the individual will not consume. There is a boundary $x_1(t, y)$, for $x > x_1(t, y)$, putting the wealth above $x_1(t, y)$ to buy annuities is optimal, and for $x \leq x_1(t, y)$, putting all the wealth to invest is optimal.
- (ii) In D_2 , the individual will not buy annuities. There is a boundary $x_0(t)$, for $x > x_0(t)$, putting the wealth above $x_0(t)$ to consume is optimal, and for $x \leq x_0(t)$, putting all the wealth to invest is optimal.
- (iii) People of different ages and wealth have different optimal strategies, so, in order to promote tax-deferred annuities, the government should adopt different tax preferential policies for different people, and insurance companies should take different publicity strategies for different people.

Clearly, this is practical, and these theorems give us exact barriers to make decisions.

Remark 2. For $k \geq 1 - V_{2y}(t, y)$, we have

$$\frac{V_{2y}(t, y)}{1 - k} \geq 1. \quad (44)$$

The concavity of $V_1(t, x)$ implies that $x_1(t, y) \leq x_0(t)$; thus, the individual will not consume, and he will put all the wealth above $x_1(t, y)$ to buy annuities. This indicates that raising tax rates properly can stimulate people to buy tax-deferred annuities.

Remark 3. According to Theorem 3, we know that as time goes by, $x_1(t, y)$ gets bigger, and the lower bound of purchasing annuities becomes more and more difficult to reach. So, young people are more likely to buy insurance than old ones. Thus, insurance companies should broaden young people's annuity market and design more products for young people.

Now, let us give the value function.

Theorem 6. For $(t, y) \in D_1$, we have

$$V(t, x, y) = \begin{cases} V_1(t, x) + V_2(t, y), & x \leq x_1(t, y), \\ V_1(t, x_1) + V_2\left(t, y + \frac{x - x_1}{1 - k}\right), & x > x_1(t, y), \end{cases} \quad (45)$$

and for $(t, y) \in D_2$, we have

$$V(t, x, y) = \begin{cases} V_1(t, x) + V_2(t, y), & x \leq x_0(t), \\ V_1(t, x_0) + x - x_0(t) + V_2(t, y), & x > x_0(t). \end{cases} \quad (46)$$

Up till now, we have obtained the expression of the value function, the optimal investment, consumption, and annuitization strategies in terms of $V_1(t, x)$ and $V_2(t, y)$. In the next section, we will analyze the problem when the rate of buying tax-deferred annuities is restricted.

4. The Optimal Strategies When $p_t \leq \bar{p}$

The above analysis considers the optimal annuitization strategies under the assumption that buying annuities are not restricted. In fact, in order to ensure the role of Taxation and reduce the financial pressure, many countries stipulate that the number of tax-deferred annuities purchased by a person shall not exceed some point. In this section, let us consider the problem under the assumption that $p_t \leq \bar{p}$ for some constant $\bar{p} > 0$. Using the results in Theorem 1, we can get that the value function satisfies (14) and (15) and

$$\begin{aligned} \min \left\{ (r + \lambda_t^s)V(t, x, y) - V_t(t, x, y) - \sup_{b_1 \geq 0, b_2 \geq 0} [(\hat{r}x + (\hat{\mu} - \hat{r})b_1 + kb)V_x(t, x, y) \right. \\ \left. + \frac{\hat{\sigma}^2 b_1^2}{2}V_{xx}(t, x, y) + \sigma\hat{\sigma}b_1b_2V_{xy} + (ry + b_2(\mu - r))V_y(t, x, y) + \frac{\sigma^2 b_2^2}{2}V_{yy}(t, x, y)] \right. \\ \left. - \sup_{0 \leq p \leq \bar{p}} [pV_y(t, x, y) - \hat{p}V_x(t, x, y)], V_x(t, x, y) - 1 \right\} = 0, \quad t < T. \end{aligned} \quad (47)$$

Thus, in this case, the optimal investment strategies are still determined by (16) and (17).

It follows from (47) that for $(1 - k)V_x(t, x, y) \geq V_y(t, x, y)$, we have

$$\begin{aligned} (r + \lambda_t^s)V(t, x, y) \geq \sup_{b_1 \geq 0, b_2 \geq 0} (\hat{r}x + (\hat{\mu} - \hat{r})b_1 + kb)V_x(t, x, y) + \frac{\hat{\sigma}^2 b_1^2}{2}V_{xx}(t, x, y) + \sigma\hat{\sigma}b_1b_2V_{xy} \\ + (ry + b_2(\mu - r))V_y(t, x, y) + \frac{\sigma^2 b_2^2}{2}V_{yy}(t, x, y) + \sup_{c \geq 0} [c(1 - V_x(t, x, y))] + V_t(t, x, y), \end{aligned} \quad (48)$$

$$V_x(t, x, y) \geq 1,$$

with at least one strict equal sign. For $(1 - k)V_x(t, x, y) < V_y(t, x, y)$, we have

$$\begin{aligned} (r + \lambda_t^s)V(t, x, y) \geq \sup_{b_1 \geq 0, b_2 \geq 0} (\hat{r}x + (\hat{\mu} - \hat{r})b_1 + kb)V_x(t, x, y) + \frac{\hat{\sigma}^2 b_1^2}{2}V_{xx}(t, x, y) + \sigma\hat{\sigma}b_1b_2V_{xy} \\ + (ry + b_2(\mu - r))V_y(t, x, y) + \frac{\sigma^2 b_2^2}{2}V_{yy}(t, x, y) + \sup_{c \geq 0} [c(1 - V_x(t, x, y))] \\ + \bar{p}V_y(t, x, y) - (1 - k)\bar{p}V_x(t, x, y) + V_t(t, x, y), \end{aligned} \quad (49)$$

$$V_x(t, x, y) \geq 1,$$

with at least one strict equal sign.

Let $U_1(t, x)$ satisfy

$$-(r + \lambda_t^s)U_1(t, x) + U_{1t}(t, x) + \hat{r}xU_{1x}(t, x) + (kb - (1 - k)\bar{p})U_{1x}(t, x) - \frac{R^2 U_{1x}^2(t, x)}{2U_{1xx}(t, x)} = 0, \quad (50)$$

$$U_1(T, x) = x, \quad (51)$$

$$U_1(t, 0) = 0. \quad (52)$$

Let $U_2(t, y)$ satisfy

$$-(r + \lambda_t^s)U_2(t, y) + U_{2t}(t, y) + (ry + \bar{p})U_{2y}(t, y) - \frac{R^2 U_{2y}^2(t, y)}{2U_{2yy}(t, y)} = 0, \quad (53)$$

$$U_2(T, y) = g(y), \quad y \geq 0, \quad (54)$$

$$U_2(t, 0) = V_1(t, x_1(t, y)) - U_1(t, x_1(t, y)), \quad t \geq 0. \quad (55)$$

Define $\tilde{x}_0(t)$ by

$$U_{1x}(t, \tilde{x}_0(t)) = 1. \quad (56)$$

Appendix B gives us numerical solutions of $U_1(t, x)$ and $U_2(t, y)$. Figures 3 and 4 give us the following theorem.

Theorem 7. $U_1(t, x)$ is a concave increasing positive function with respect to x , and $U_{1x}(t, x)$ is increasing with respect to t and

$$U_{1x}(t, x) > V_{1x}(t, x), \quad x \geq 0. \quad (57)$$

According to Theorem 7, we can obtain that

$$V_1'(t, \tilde{x}_0) < U_1'(t, \tilde{x}_0) = 1 = V_1'(t, x_0(t)), \quad (58)$$

Then, we can get

$$\tilde{x}_0(t) > x_0(t), \quad \forall t \geq 0. \quad (59)$$

Next, we consider the form of the value function in D_1 and D_2 , respectively.

For $t \in D_1$, define

$$\begin{aligned} w(t, x, y) &= \begin{cases} V_1(t, x) + V_2(t, y), & x \leq x_1(t, y), \\ U_1(t, x) + U_2(t, y), & x_1(t, y) < x < \tilde{x}_0(t), \\ U_1(t, \tilde{x}_0(t)) + x - \tilde{x}_0(t) + U_2(t, y), & x \geq \tilde{x}_0(t), \end{cases} \\ \Delta_1^*(t, x, y) &= \begin{cases} 0, & x \leq \tilde{x}_0(t), \\ x - \tilde{x}_0(t), & x > \tilde{x}_0(t), \end{cases} \\ p^*(t, x, y) &= \begin{cases} 0, & x \leq x_1(t, y), \\ \bar{p}, & x > x_1(t, y). \end{cases} \end{aligned} \quad (60)$$

For $t \in D_2$, define

$$\begin{aligned} w(t, x, y) &= \begin{cases} V_1(t, x) + V_2(t, y), & x \leq x_0(t), \\ V_1(t, x_0(t)) + x - x_0(t) + V_2(t, y), & x > x_0(t), \end{cases} \\ \Delta_1^*(t, x, y) &= \begin{cases} 0, & x \leq x_0(t), \\ x - x_0(t), & x > x_0(t), \end{cases} \\ p^*(t, x, y) &= 0. \end{aligned} \quad (61)$$

Then, we have the following corollary.

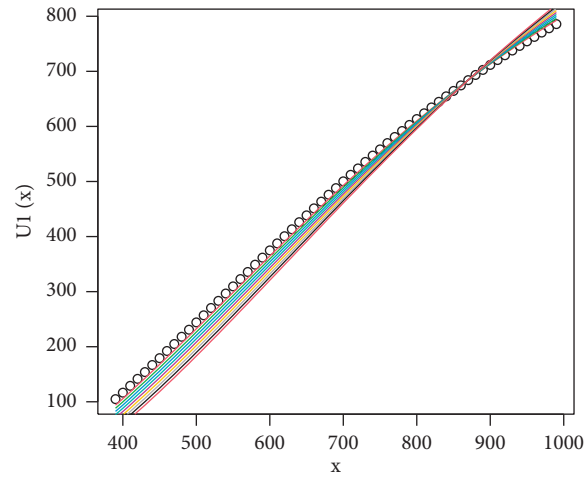
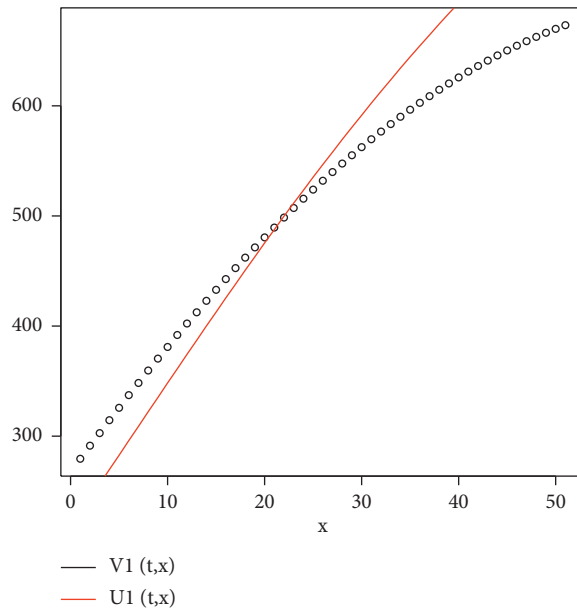
Lemma 1. $w(t, x, y)$ defined above satisfies (14), (15), and (47).

Proof. According to (20), (21), (51), (52), (54), and (55), we can get that $w(t, x, y)$ satisfies (14) and (15).

For $t \in D_1$, $x \leq x_1(t, y)$, we have that

$$\begin{aligned} (1 - k)w_x(t, x, y) &= (1 - k)V_1'(t, x) \geq (1 - k)V_1'(t, x_1(t, y)) = V_2'(t, y) = w_y(t, x, y), \\ w_x(t, x, y) &= V_1'(t, x) > 1. \end{aligned} \quad (62)$$

Thus, we can get that


 FIGURE 3: $U_1(t, x)$

 FIGURE 4: A comparison of $V_1(t, x)$ and $U_1(t, x)$

$$\begin{aligned} c_t^* &= 0, \\ p_t^* &= 0. \end{aligned}$$

(63)

Since $V_1(t, x)$ satisfies (23), and $V_2(t, y)$ satisfies (24), it follows immediately that $w(t, x, y)$ satisfies (47).

For $t \in D_1$, $x_1(t, y) < x < \tilde{x}_0(t)$, we have

$$\begin{aligned} w_x(t, x, y) &= U_1'(t, x) > 1, \\ (1 - k)w_x(t, x, y) &= (1 - k)U_1'(t, x) \leq (1 - k)V_1'(t, x) \leq (1 - k)V_1'(t, x_1(t, y)), \\ (1 - k)V_1'(t, x_1(t, y)) &= V_2'(t, y) = U_2'(t, y) = w_y(t, x, y). \end{aligned} \tag{64}$$

It implies that

$$\begin{aligned} c_t^* &= 0, \\ p_t^* &= \bar{p}. \end{aligned} \quad (65)$$

Since $U_1(t, x)$ satisfies (50), and $U_2(t, y)$ satisfies (53), it follows immediately that $w(t, x, y)$ satisfies (47).

For $t \in D_1$, $x \geq \tilde{x}_0(t)$, we have

$$\begin{aligned} (1-k)w_x(t, x, y) &= (1-k)U_1'(t, \tilde{x}_0(t)) = (1-k)V_1'(t, x_0(t)) < (1-k)V_1'(t, x_1(t, y)), \\ (1-k)V_1'(t, x_1(t, y)) &= V_2'(t, y) = U_2'(t, y) = w_y(t, x, y), \\ w_x(t, x, y) &= 1. \end{aligned} \quad (66)$$

Thus, we can get $p_t^* = \bar{p}$. Clearly, $w(t, x, y)$ is the solution of (47).

The cases for $(t, y) \in D_2$ can be proved with a similar analysis, and we can obtain that $w(t, x, y)$ satisfies (47) finally.

The concavity of $V_1(t, x)$, $U_1(t, x)$, $V_2(t, y)$, and $U_2(t, y)$ indicates that $w(t, x, y)$ satisfies the polynomial growth condition. So, using Theorem 3.5.2 in [13], we can immediately get that $w(t, x, y) = V(t, x, y)$. \square

Remark 4. Obviously, the boundary of buying annuities in this section is the same as the one in Section 3. So, the suggestions proposed in Section 3 can also apply when the rate of buying annuities is restricted.

Remark 5. The form of p^* indicates that properly raising the upper bound \bar{p} can increase the rate of buying annuities.

Appendix

Owing to the existence of tax and the time dependence of the value function, we cannot get explicit solutions of the differential equations satisfied by $V_1(t, x)$, $V_2(t, y)$ and $U_1(t, x)$, $U_2(t, y)$. So, numerical methods should be applied. One of the generally used numerical methods is called the

standard finite difference method. The steps of the method are as follows:

- (i) Gridding the domain of the unknown function.
- (ii) Replacing derivative functions in the differential equation with difference functions expressed by the function values of the grid points.
- (ii) Using the boundary conditions and solving the difference equations to deduce the function values of the grid points.

According to [15], we know that as long as the solution of the differential equation is growing linearly, the method is stable. Thus, we use this method to get the value function of our problem.

In this paper, we suppose that the investor is 22 years old at time 0, and by using the life table data of China Life Insurance and doing regression analysis, we can approximate the hazard rate of the individual at time t with $\lambda_t^s = 0.001 + 0.0006t^2$.

A. The solutions of $V_1(t, x)$ and $V_2(t, y)$

In order to get $V_1(t, x)$, let us discuss the numerical solution of the partial differential equation

$$(r + \lambda_t^s)V_1(t, x) - \hat{r}xV_{1x}(t, x) - kbV_{1x}(t, x) + \frac{R^2V_{1x}^2(t, x)}{2V_{1xx}(t, x)} = V_{1t}(t, x), \quad (A.1)$$

$$V_1(t, 0) = 0.$$

Clearly, the definition domain of $V_1(t, x)$ is $[0, T] \times \mathcal{R}^+ \cup 0$. Fixing $h > 0$, denoting $a_{ij} = V_1((i-1)h, (j-1)h)$, and letting $m-1 = T/h$, $n \in \mathbb{Z}^+$, we construct an $m \times n$ matrix A :

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}. \quad (A.2)$$

Obviously, the matrix is the numerical solution of $V_1(t, x)$ when $h \rightarrow 0$.

Since $V_1(t, 0) = 0$, we can get the first column of A . That is,

$$a_{i1} = 0, \quad i = 1, 2, \dots, m. \quad (A.3)$$

Furthermore, for an initial wealth $x = 0$, we know that the investment strategy is $b_1 = 0$, that is,

$$\frac{V_{1x}^2(t, 0)}{V_{1xx}(t, 0)} = 0. \quad (A.4)$$

Then, for $h \rightarrow 0$, we have

$$\frac{V_{1x}^2(t, h)}{V_{1xx}(t, h)} = 0. \quad (\text{A.5})$$

It means that

$$(r + \lambda_t^s)V_1(t, h) - \hat{r}xV_{1x}(t, h) - kbV_{1x}(t, h) = V_{1t}(t, h), \quad h \longrightarrow 0. \quad (\text{A.6})$$

Using the finite difference method, we can substitute $V_{1x}(t, h)$ with $((V_1(t, h) - V_1(t, 0))/h)$ and get the following differential equation with respect to t

$$(r + \lambda_t^s)V_1(t, h) = V_{1t} + (\hat{r}h + kb)\frac{V_1(t, h) - V_1(t, 0)}{h}. \quad (\text{A.7})$$

By doing some simple calculations, we can obtain that

$$V_1(t, h) = V_1(0, h)e^{\int_0^t L(s)ds}, \quad (\text{A.8})$$

where $L(s) = \lambda_s^s + r - \hat{r} - (kb/h)$.

According to (20), we know that $V_1(T, h) = h$. Combining with (A.8), we can get that

$$V_1(0, h) = he^{-\int_0^T L(s)ds}. \quad (\text{A.9})$$

Then, we obtain the second column of A .

In order to get the first two rows of A , we consider the pair (a_{13}, a_{23}) first. Substituting $V_{1x}(0, 2h)$, V_{1t} and $(V_{1x}^2(0, 2h)/V_{1xx}(0, 2h))$ with $((a_{13} - a_{12})/h)$, $((a_{23} - a_{13})/h)$ and $((a_{13} - a_{12})^2/(a_{13} + a_{11} - 2a_{12}))$, respectively, we can get

$$(r + \lambda_0^s)a_{13} = \frac{a_{23} - a_{13}}{h} + (\hat{r}h + kb)\frac{a_{13} - a_{12}}{h} - \frac{R^2(a_{13} - a_{12})^2}{2(a_{13} + a_{11} - 2a_{12})}. \quad (\text{A.10})$$

Similarly, we can also get

$$(r + \lambda_h^s)a_{23} = \frac{a_{23} - a_{13}}{h} + (\hat{r}h + kb)\frac{a_{23} - a_{22}}{h} - \frac{R^2(a_{23} - a_{22})^2}{2(a_{23} + a_{21} - 2a_{22})}. \quad (\text{A.11})$$

Combining with the two equations, we can get a_{13} and a_{23} .

With the same method, we can also get $a_{14}, a_{24}, a_{15}, a_{25}, \dots$. Then, the first two rows of A are obtained. At last, we can deduce the left elements from left to right, from top to bottom by solving the corresponding difference equations.

Next, let us consider $V_2(t, y)$. We have to find the numerical solution of the partial differential equation

$$(r + \lambda_t^s)V_2(t, y) - ryV_{2y}(t, y) + \frac{R^2V_{1y}^2(t, y)}{2V_{2yy}(t, y)} = V_{2t}(t, y),$$

$$V_2(t, y) = 0,$$

$$V_2(T, y) = g(y). \quad (\text{A.12})$$

Similarly, we construct an $m \times n$ matrix B :

$$\begin{pmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & b_{m3} & \cdots & b_{mn} \end{pmatrix}, \quad (\text{A.13})$$

where $b_{ij} = V_2((i-1)h, (j-1)h)$ and $m-1 = (T/h), n \in \mathbb{Z}^+$. Using the boundary conditions, we can get the first column and the last row of B . That is,

$$b_{i1} = 0, \quad i = 1, 2, \dots, m,$$

$$b_{mj} = g((j-1)h), \quad j = 1, 2, \dots, n. \quad (\text{A.14})$$

What is more, for $h \longrightarrow 0$, we have

$$\frac{V_{2y}^2(t, h)}{V_{2yy}(t, h)} = 0. \quad (\text{A.15})$$

Thus, we can get

$$(r + \lambda_t^s)V_2(t, h) - ryV_{2y}(t, h) = V_{2t}(t, h), \quad h \longrightarrow 0. \quad (\text{A.16})$$

Using the finite difference method, we can substitute $V_{2y}(t, h)$ with $((V_2(t, h) - V_2(t, 0))/h)$ and get the following differential equation with respect to t :

$$(r + \lambda_t^s)V_2(t, h) = V_{2t} + rh\frac{V_2(t, h) - V_2(t, 0)}{h}. \quad (\text{A.17})$$

By doing some simple calculations, we can obtain that

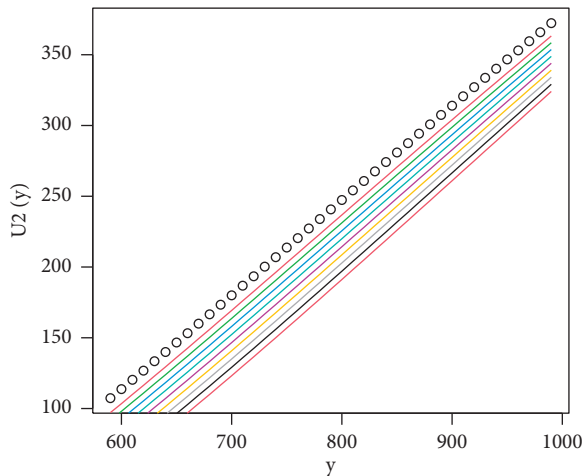
$$V_2(t, h) = V_2(T, h)e^{-\int_t^T \lambda_s^s ds}. \quad (\text{A.18})$$

Then, we get the second column of B . Since the first two columns and the last row of B are obtained, left to right, bottom to top, we can get the elements left.

The R program of solving $V_1(t, x)$ and $V_2(t, y)$ and the R program of plotting Figures 1 and 2 are given in the file A.R in the attachment. Clearly, the image of $V_1(t, x)$ with respect to x for different t illustrates that $V_1(t, x)$ is a concave increasing positive function with respect to x . Similarly, $V_2(t, y)$ is a concave increasing positive function with respect to y .

B. The solutions of $U_1(t, x)$, $U_2(t, y)$ and a comparison of $U_1(t, x)$ and $V_1(t, x)$

Since $U_1(t, x)$, $U_2(t, y)$ satisfy the same forms of differential equations as $V_1(t, x)$, $V_2(t, y)$, respectively, the processes of solving $U_1(t, x)$, $U_2(t, y)$ are the same as the processes of solving $V_1(t, x)$, $V_2(t, y)$. The R program of solving $U_1(t, x)$, $U_2(t, y)$ and the R program of plotting Figures 3–5 are presented in the file B.R in the attachment.

FIGURE 5: $U_2(t, y)$

It is shown from Figure 3 that $U_1(t, x)$ is a concave increasing positive function with respect to x . What is more, Figure 4 tells us that

$$\begin{aligned} U_1(t, x) &\leq V_1(t, x), \\ U_{1x}(t, x) &\leq V_{1x}(t, x). \end{aligned} \quad (\text{B.1})$$

Data Availability

No data were used in this paper.

Conflicts of Interest

The author declares no conflicts of interest.

Acknowledgments

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Research Article

European Option Pricing Formula in Risk-Averse Markets

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In this study, using the method of discounting the terminal expectation value into its initial value, the pricing formulas for European options are obtained under the assumptions that the financial market is risk-averse, the risk measure is standard deviation, and the price process of underlying asset follows a geometric Brownian motion. In particular, assuming the option writer does not need the risk compensation in a risk-neutral market, then the obtained results are degenerated into the famous Black–Scholes model (1973); furthermore, the obtained results need much weaker conditions than those of the Black–Scholes model. As a by-product, the obtained results show that the value of European option depends on the drift coefficient μ of its underlying asset, which does not display in the Black–Scholes model only because $\mu = r$ in a risk-neutral market according to the no-arbitrage opportunity principle. At last, empirical analyses on Shanghai 50 ETF options and S&P 500 options show that the fitting effect of obtained pricing formulas is superior to that of the Black–Scholes model.

1. Introduction

The option pricing theory began in 1900 when the French mathematician Louis Bachelier deduced an option pricing formula under the assumption that underlying asset prices follow a Brownian motion with zero drift. Since then, lots of researchers have contributed to the theory. Black and Scholes [1] present the very famous option pricing formula (i.e., Black–Scholes model) in a risk-neutral market and according to the no-arbitrage opportunity principle. Merton [2] shows the Black–Scholes-type model can be derived from weaker assumptions than in their original formulation and present some pricing methods for non-European options. Bakshi et al. [3] first derive an option pricing model that allows volatility, interest rates, and jumps to be stochastic. Gârleanu et al. [4] model demand-pressure effects on option prices. The model shows that demand pressure in one option contract increases its price by an amount proportional to the variance of the unhedgeable part of the option. Cai and Kou [5] propose a jump diffusion model for asset prices whose jump sizes have a mixed-exponential distribution, which is a weighted average of exponential distributions but with

possibly negative weights, and then they extend the analytical tractability of the Black–Scholes model to alternative models. Bernarda and Czadob [6] investigate the pricing of basket options and more generally of complex exotic contracts depending on multiple indices. Their approach assumes that the underlying assets evolve as dependent GARCH(1, 1) processes. The dependence among the assets is modeled using a copula based on pair copula constructions. Bandi and Bertsimas [7] combine robust optimization and the idea of ε -arbitrage to propose a tractable approach to price a wide variety of options. Bao et al. [8] present a method that there is a possibility to get statistical arbitrage from Black–Scholes's option price.

In the last five years, there are still many researchers contributing to the theory of option pricing. Moretto et al. [9] study option pricing under deformed Gaussian distributions. Leippold and Scharer [10] develop a stochastic liquidity model, and they investigate discrete-time option pricing with stochastic liquidity. Hoka and Chanb [11] develop an option pricing method based on Legendre series expansion of the density function, and approximation formulas for pricing European type options are derived.

Davison and Mamba [12] obtain a solution of the Black–Scholes equation with a nonsmooth boundary condition using symmetry methods. Willems [13] derives a series expansion for the price of a continuously sampled arithmetic Asian option in the Black–Scholes setting. The expansion is based on polynomials that are orthogonal with respect to the log-normal distribution. More literature studies can refer to Liu et al. [14], Friz et al. [15], Dubinsky et al. [16], Huh [17], Liu et al. [18], Siddiqi [19] and their studies.

Although there are a huge number of literature studies on option pricing, they almost assume the financial markets are risk-neutral and complete, especially since Black and Scholes [1]. However, according to the theory and empirical analysis of risk, real financial markets are risk-averse and incomplete. That is, investors need risk compensation for risky assets, and many risky assets cannot be duplicated by any portfolio constructed in real financial markets.

In this study, using the method of discounting the terminal expectation value into its initial value, we obtain European option pricing formula under the assumptions that the financial market is risk-averse, the risk measure is standard deviation, and the price process of underlying asset follows a geometric Brownian motion. In particular, if the option writer does not need the risk compensation in a risk-neutral market, then our obtained results are degenerated into the Black–Scholes model [1]; furthermore, our obtained results need much weaker conditions than those in the Black–Scholes model. At last, we take the Shanghai 50 ETF options, the first floor option in the Chinese financial market, and S&P 500 options as samples to compare the fitting effect. The empirical analyses show that the fitting effect of our pricing formulas is superior to that of the Black–Scholes model.

2. The Black–Scholes Formula

In this study, we will investigate European option pricing and compare our results with those of Black and Scholes [1]. Thus, we first retell the main results of Black & Scholes [1].

Black and Scholes [1] present nine assumptions in the market for the security and for the option and then obtain their famous option pricing formula.

Assumption 1. Security price satisfies a geometric Brownian motion (GBM) model, where its drift coefficient and diffusion coefficient are constant through time. That is, the security price satisfies stochastic differential equation:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t, \quad (1)$$

where μ and σ are constant and $\sigma > 0$.

Assumption 2. The short-term interest rate r is known and is constant through time. That is, the risk-free bond price satisfies ordinary differential equation:

$$\frac{dP_t}{P_t} = r dt, \quad (2)$$

where r is constant.

Assumption 3. The security pays no dividends or other distributions.

Assumption 4. There are no transaction costs in buying or selling the security or the option.

Assumption 5. The security can be continuously transacted.

Assumption 6. The amount of security can be arbitrarily divided.

Assumption 7. It is possible to borrow any fraction of the price of a security to buy it or to hold it, at the short-term interest rate.

Assumption 8. There are no penalties to short selling. A seller who does not own a security will simply accept the price of the security from a buyer and will agree to settle with the buyer on some future date by paying him an amount equal to the price of the security on that date.

Assumption 9. There is no-arbitrage opportunity.

When the above assumptions all hold, Black and Scholes [1] derived the pricing formula for European options, which is the Black–Scholes model.

Theorem 1 (see [1]). *If Assumptions 1 to 9 hold, then the values of European call option and European put option follow as*

$$C(S_0, K, r, \sigma, \tau) = S_0 \Phi(d_2) - Ke^{-r\tau} \Phi(d_1), \quad (3)$$

$$P(S_0, K, r, \sigma, \tau) = Ke^{-r\tau} \Phi(-d_1) - S_0 \Phi(-d_2), \quad (4)$$

where S_0 is the initial price of underlying asset, K is the strike price of option, r is the short-term interest rate, σ is the diffusion coefficient of underlying asset, τ is the left expiration time of option, $\Phi(\cdot)$ is the cumulative density function of standard normal distribution, and

$$d_1 = \frac{1}{\sigma\sqrt{\tau}} \left(\ln \frac{S_0}{K} + \left(r + \frac{1}{2}\sigma^2 \right) \tau \right), \quad (5)$$

$$d_2 = d_1 + \sigma\sqrt{\tau}.$$

3. European Option Pricing in Risk-Averse Markets

Black–Scholes model and its modified versions have some defects. In fact, because real financial markets are incomplete, an option may not be duplicated constantly, so its value deduced by the asset duplication method and no-arbitrage principle may lose the deductive basis. On the other hand, real financial markets are risk-averse. Option seller undertakes the total risk and option buyer has no any risk, so option seller needs a reasonable risk compensation according to the theory of risk. In this section, we will deduce the option pricing formula in risk-averse markets only under three assumptions, i.e., Assumptions 1, 2, and 4 in

Section 2, which is reasonably far more than the Black-Scholes model and its modified versions.

In risk-averse markets, assume the price process of some risky asset X by $\{X_t, t \geq 0\}$. Then the value of European call option at expiration time T with underlying asset X and strike price K follows as

$$(X_T - K)^+, \quad (6)$$

here and in the sequel the operator $(\cdot)^+ = \max\{0, \cdot\}$. The value of European put option at expiration time T with underlying asset X and strike price K follows as

$$(K - X_T)^+. \quad (7)$$

Note that no matter call option or put option, it is always its seller undertakes the total risk and its buyer has no any risk. According to risk theory, the seller reasonably requires some risk compensation $\lambda\rho(X)$, where $\lambda \geq 0$ is the risk-compensation coefficient, $\rho(\cdot)$ is the risk measure, and X is the risk size. After the seller has received the reasonable risk compensation, the seller takes risky asset as equivalent risk-free bond, so it follows from (6) that the value at time t of European call option with underlying asset X , strike price K , and expiration time T follows as

$$C_t(X, K, T, r, \lambda) = e^{-r(T-t)} \{E[(X_T - K)^+] + \lambda\rho((X_T - K)^+)\}, \quad \forall t \leq T, \quad (8)$$

where $r \geq 0$ is the risk-free rate during $[t, T]$. Analogically, it follows from (7) that the value of European put option with underlying asset X , strike price K , and expiration time T follows as

$$P_t(X, K, T, r, \lambda) = e^{-r(T-t)} \{E[(K - X_T)^+] + \lambda\rho((K - X_T)^+)\}, \quad \forall t \leq T. \quad (9)$$

In conclusion, we obtain the following proposition from (8) and (9).

Proposition 1. *In risk-averse market, assume that the risk measure is $\rho(\cdot)$ and the risk-compensation coefficient is $\lambda \geq 0$, and assuming a European option with underlying asset X , strike price K , and expiration time T , then its call-option value at time t follows as*

$$C_t(X, K, T, r, \lambda) = e^{-r(T-t)} \{E[(X_T - K)^+] + \lambda\rho((X_T - K)^+)\}, \quad (10)$$

and its put-option value at time t follows as

$$P_t(X, K, T, r, \lambda) = e^{-r(T-t)} \{E[(K - X_T)^+] + \lambda\rho((K - X_T)^+)\}, \quad (11)$$

where $r \geq 0$ is the risk-free rate during $[t, T]$ and $t \leq T$.

In order to obtain a closed-form solution to Proposition 1, in the following, we always assume the price process of underlying asset follows some geometric Brownian motion model (1), and the risk measure is the standard deviation, i.e., $\rho(Z) = \text{std}(Z)$ for any risk variable Z .

Using Proposition 1, we can deduce the value of European call option in risk-averse markets and under Assumptions 1, 2, and 4.

Theorem 2. *In a risk-averse market, assuming that underlying asset follows a geometric Brownian motion with drift coefficient μ and diffusion coefficient $\sigma > 0$, the current price of underlying asset is S_0 , risk-free interest rate is r through the time, the risk-compensation factor is $\lambda \geq 0$, and the risk measure is standard deviation, then the value of European call option with strike price K and left expiration time τ follows as*

$$C(S_0, K, r, \sigma, \tau, \lambda) = S_0 e^{(\mu-r)\tau} \Phi(d_2) - K e^{-r\tau} \Phi(d_1) + \lambda e^{-r\tau} \cdot \text{sqrt} \left\{ S_0^2 e^{2\mu\tau} \left(e^{\sigma^2\tau} \Phi(d_3) - \Phi^2(d_2) \right) - K S_0 e^{\mu\tau} \Phi(d_2) \Phi(-d_1) + K^2 \Phi(d_1) \Phi(-d_1) \right\}, \quad (12)$$

where

$$d_1 = \frac{1}{\sigma\sqrt{\tau}} \left(\ln \frac{S_0}{K} + \left(\mu - \frac{1}{2}\sigma^2 \right) \tau \right), \quad (13)$$

$$d_m = d_1 + (m-1)\sigma\sqrt{\tau}, \quad m = 2, 3.$$

The Proof of Theorem 2 refers to Appendix A.

If a financial market is risk-neutral, then investors treat expected return and deterministic return equally, so expectation yield μ equals risk-free yield r , that is, $\mu = r$. Otherwise, if $\mu \neq r$, there will exist arbitrage opportunity. Furthermore, in a risk-neutral financial market, the risk-

compensation factor equals zero, that is, $\lambda = 0$. Thus, it yields from Theorem 1 that we have the following corollary.

Corollary 1. *In a risk-neutral market, assuming that underlying asset follows a geometric Brownian motion with drift coefficient μ and diffusion coefficient $\sigma > 0$, the current price of underlying asset is S_0 , and risk-free yield is r through the time, then the value of European call option with strike price K and left expiration time τ follows as*

$$C(S_0, K, r, \sigma, \tau) = S_0 \Phi(d_2) - K e^{-r\tau} \Phi(d_1), \quad (14)$$

where

$$d_1 = \frac{1}{\sigma\sqrt{\tau}} \left(\ln \frac{S_0}{K} + \left(r - \frac{1}{2}\sigma^2 \right) \tau \right), \quad (15)$$

$$d_2 = d_1 + \sigma\sqrt{\tau}.$$

In the following, we will deduce the pricing formula of European put option. Using Proposition 1, we will construct the pricing formula of European put option in risk-averse markets and under Assumptions 1, 2, and 4.

$$P(S_0, K, r, \sigma, \tau, \lambda) = Ke^{-r\tau} \Phi(-d_1) - S_0 e^{(\mu-r)\tau} \Phi(-d_2) + \lambda e^{-r\tau} \cdot \text{sqrt} \left\{ S_0^2 e^{2\mu\tau} \left(e^{\sigma^2\tau} \Phi(-d_3) - \Phi^2(-d_2) \right) - KS_0 e^{\mu\tau} \Phi(-d_2) \Phi(d_1) + K^2 \Phi(-d_1) \Phi(d_1) \right\}, \quad (16)$$

where

$$d_1 = \frac{1}{\sigma\sqrt{\tau}} \left(\ln \frac{S_0}{K} + \left(\mu - \frac{1}{2}\sigma^2 \right) \tau \right), \quad (17)$$

$$d_m = d_1 + (m-1)\sigma\sqrt{\tau}, \quad m = 2, 3.$$

The Proof of Theorem 3 refers to Appendix B.

According to the analysis before Corollary 1, if a financial market is risk-neutral, then $\mu = r$ and $\lambda = 0$. Furthermore, it yields from Theorem 3 that we have the following corollary.

Corollary 2. *In a risk-neutral market, assuming that underlying asset follows a geometric Brownian motion with drift coefficient μ and diffusion coefficient $\sigma > 0$, the current price of underlying asset is S_0 , and risk-free yield is r through the time, then the value of European put option with strike price K and left expiration time τ follows as*

$$P(S_0, K, r, \sigma, \tau) = Ke^{-r\tau} \Phi(-d_1) - S_0 \Phi(-d_2), \quad (18)$$

where

$$d_1 = \frac{1}{\sigma\sqrt{\tau}} \left(\ln \frac{S_0}{K} + \left(r - \frac{1}{2}\sigma^2 \right) \tau \right), \quad (19)$$

$$d_2 = d_1 + \sigma\sqrt{\tau}.$$

Remark 1. Although Corollaries 1 and 2 obtain the same values as those in Theorem 1 for European call option and European put option, Corollaries 1 and 2 need much weaker conditions than those of Theorem 1. That is, Corollaries 1 and 2 improve Theorem 1.

In fact, if Assumptions 1, 2, and 4 in Section 2 hold, Corollaries 1 and 2 hold. However, the conditions that Theorem 1 (i.e., the Black-Scholes model) holds are Assumptions 1 to 9 in Section 2.

Theorem 3. *In a risk-averse market, assuming that underlying asset follows a geometric Brownian motion with drift coefficient μ and diffusion coefficient $\sigma > 0$, the current price of underlying asset is S_0 , risk-free yield is r through the time, the risk-compensation factor is $\lambda \geq 0$, and the risk measure is standard deviation, then the value of European put option with strike price K and left expiration time τ follows as*

4. Empirical Analysis of Shanghai 50 ETF Options

In the section, we will present empirical analysis on Shanghai 50 ETF options, the first floor option in the Chinese financial market, and use the data of September 3 and 4, 2018, to compare the fitting effect of our pricing model and the Black-Scholes pricing model. All used data come from the CSMAR Database, which includes actual option price C or P , trading date t , exercise date T , strike price K , and current price of underlying asset S_0 . In addition, the database also includes the historical volatility of Shanghai 50 ETF $\sigma = 0.1994$ on September 3, 2018, and $\sigma = 0.2005$ on September 4, 2018, and the 1-year deposit benchmark interest rate of Chinese Central Bank $r = 1.5\%$, which is chosen as the reference level of risk-free interest rate in the Chinese financial market. All data analyses in the following are worked out by the software MATLAB R2018b.

4.1. Parameter Estimation. There are 48 call options on September 3, 2018, and the current price of underlying asset $S_0 = 2.512$. We take the annual average return rate of the last month as the drift coefficient of the underlying asset μ . According to simple computation, we obtain $\mu = 2.557\%$ and the left expiration time $\tau = (T - t)/365$ (years), where $t = 2018/09/03$, and then we work out d_1 , d_2 , and d_3 by Theorem 2; see Table 1 for detailed data, where C is the actual closing price of call option, T is the expiration time of call option, K is the strike price of call option, S_0 is the initial price of underlying asset, τ is the left expiration time of call option, and d_1 , d_2 , and d_3 are the parameters in Theorem 2. Furthermore, we obtain the estimated value of the risk-compensation factor $\lambda = 0.0077963$ by the least square method with $R^2 = 0.7058$.

There are 60 put options on September 3, 2018, and the current price of underlying asset $S_0 = 2.512$. Similarly, we obtain $\mu = 2.93\%$ and the left expiration time $\tau = (T - t)/365$ (years), where $t = 2018/09/03$, and then we work out d_1 , d_2 , and d_3 by Theorem 3; see Table 2 for detailed data, where P is

TABLE 1: Parameter estimation of d_1 , d_2 , and d_3 for call options.

C	T	K	S_0	τ	d_1	d_2	d_3
0.0030	2018/09/26	2.75	2.512	0.0630	-1.8013	-1.7512	-1.7012
0.0073	2018/09/26	2.70	2.512	0.0630	-1.4347	-1.3847	-1.3346
0.0146	2018/09/26	2.65	2.512	0.0630	-1.0613	-1.0112	-0.9612
0.0150	2018/10/24	2.75	2.512	0.1397	-1.2038	-1.1293	-1.0547
0.0159	2018/12/26	2.95	2.512	0.3123	-1.4263	-1.3149	-1.2035
0.0220	2018/12/26	2.90	2.512	0.3123	-1.2729	-1.1615	-1.0501
0.0228	2018/10/24	2.70	2.512	0.1397	-0.9576	-0.8831	-0.8086
0.0275	2018/09/26	2.60	2.512	0.0630	-0.6807	-0.6307	-0.5806
0.0287	2018/12/26	2.85	2.512	0.3123	-1.1169	-1.0054	-0.8940
0.0350	2018/10/24	2.65	2.512	0.1397	-0.7068	-0.6323	-0.5578
0.0372	2018/12/26	2.80	2.512	0.3123	-0.9580	-0.8466	-0.7352
0.0451	2018/09/26	2.55	2.512	0.0630	-0.2928	-0.2427	-0.1927
0.0484	2018/12/26	2.75	2.512	0.3123	-0.7964	-0.6849	-0.5735
0.0500	2018/10/24	2.60	2.512	0.1397	-0.4513	-0.3767	-0.3022
0.0601	2018/12/26	2.70	2.512	0.3123	-0.6317	-0.5203	-0.4088
0.0695	2018/09/26	2.50	2.512	0.0630	0.1028	0.1529	0.2029
0.0698	2018/10/24	2.55	2.512	0.1397	-0.1908	-0.1162	-0.0417
0.0742	2019/03/27	2.80	2.512	0.5616	-0.7049	-0.5555	-0.4061
0.0756	2018/12/26	2.65	2.512	0.3123	-0.4640	-0.3525	-0.2411
0.0878	2019/03/27	2.75	2.512	0.5616	-0.5844	-0.4349	-0.2855
0.0938	2018/12/26	2.60	2.512	0.3123	-0.2930	-0.1816	-0.0701
0.0944	2018/10/24	2.50	2.512	0.1397	0.0749	0.1495	0.2240
0.0980	2018/09/26	2.45	2.512	0.0630	0.5064	0.5565	0.6066
0.1026	2019/03/27	2.70	2.512	0.5616	-0.4616	-0.3121	-0.1627
0.1149	2018/12/26	2.55	2.512	0.3123	-0.1188	-0.0073	0.1041
0.1190	2019/03/27	2.65	2.512	0.5616	-0.3365	-0.1870	-0.0376
0.1247	2018/10/24	2.45	2.512	0.1397	0.3460	0.4205	0.4950
0.1361	2018/09/26	2.40	2.512	0.0630	0.9184	0.9684	1.0185
0.1380	2019/03/27	2.60	2.512	0.5616	-0.2090	-0.0596	0.0899
0.1390	2018/12/26	2.50	2.512	0.3123	0.0589	0.1704	0.2818
0.1592	2018/10/24	2.40	2.512	0.1397	0.6226	0.6971	0.7717
0.1623	2019/03/27	2.55	2.512	0.5616	-0.0791	0.0704	0.2198
0.1671	2018/12/26	2.45	2.512	0.3123	0.2402	0.3517	0.4631
0.1770	2018/09/26	2.35	2.512	0.0630	1.3390	1.3891	1.4391
0.1861	2019/03/27	2.50	2.512	0.5616	0.0534	0.2029	0.3523
0.1968	2018/10/24	2.35	2.512	0.1397	0.9051	0.9796	1.0541
0.1981	2018/12/26	2.40	2.512	0.3123	0.4253	0.5367	0.6481
0.2110	2019/03/27	2.45	2.512	0.5616	0.1886	0.3381	0.4875
0.2209	2018/09/26	2.30	2.512	0.0630	1.7687	1.8187	1.8688
0.2297	2018/12/26	2.35	2.512	0.3123	0.6142	0.7256	0.8371
0.2360	2018/10/24	2.30	2.512	0.1397	1.1936	1.2681	1.3427
0.2421	2019/03/27	2.40	2.512	0.5616	0.3266	0.4761	0.6255
0.2660	2018/12/26	2.30	2.512	0.3123	0.8072	0.9186	1.0300
0.2704	2018/09/26	2.25	2.512	0.0630	2.2078	2.2578	2.3079
0.2767	2019/03/27	2.35	2.512	0.5616	0.4675	0.6169	0.7664
0.3074	2018/12/26	2.25	2.512	0.3123	1.0044	1.1158	1.2273
0.3163	2018/09/26	2.20	2.512	0.0630	2.6567	2.7068	2.7568
0.3450	2019/03/27	2.25	2.512	0.5616	0.7585	0.9079	1.0574

the actual closing price of put option, T is the expiration time of put option, K is the strike price of put option, S_0 is the initial price of underlying asset, τ is the left expiration time of put option, and d_1, d_2 , and d_3 are the parameters in Theorem 3. Furthermore, we obtain the estimated value of the risk-compensation factor $\lambda = 0.013306$ by the least square method with $R^2 = 0.6218$.

In addition, we obtain the estimation of risk-free rate $r = 1.768\%$ by minimizing the mean square error of the Black-Scholes model on September 3, 2018. Note that $r =$

1.768% is a little higher than the 1-year deposit benchmark interest rate of Chinese Central Bank 1.5%, so it is very reasonable that we take $r = 1.768\%$ as the risk-free rate. Furthermore, it is far advantageous for the Black-Scholes model to improve its fitting effect.

4.2. Comparison of Pricing Effect. In the section, we will compare the fitting effect of our pricing formulas with the Black-Scholes model.

TABLE 2: Parameter estimation of d_1 , d_2 , and d_3 for put options.

P	T	K	S_0	τ	d_1	d_2	d_3
0.0015	2018/09/26	2.20	2.512	0.0630	2.6567	2.7068	2.7568
0.0027	2018/09/26	2.25	2.512	0.0630	2.2078	2.2578	2.3079
0.0055	2018/09/26	2.30	2.512	0.0630	1.7687	1.8187	1.8688
0.0115	2018/09/26	2.35	2.512	0.0630	1.3390	1.3891	1.4391
0.0165	2018/10/24	2.30	2.512	0.1397	1.1936	1.2681	1.3427
0.0205	2018/09/26	2.40	2.512	0.0630	0.9184	0.9684	1.0185
0.0247	2018/12/26	2.20	2.512	0.3123	1.2061	1.3175	1.4289
0.0255	2018/10/24	2.35	2.512	0.1397	0.9051	0.9796	1.0541
0.0324	2018/12/26	2.25	2.512	0.3123	1.0044	1.1158	1.2273
0.0343	2018/09/26	2.45	2.512	0.0630	0.5064	0.5565	0.6066
0.0371	2018/10/24	2.40	2.512	0.1397	0.6226	0.6971	0.7717
0.0432	2018/12/26	2.30	2.512	0.3123	0.8072	0.9186	1.0300
0.0528	2018/10/24	2.45	2.512	0.1397	0.3460	0.4205	0.4950
0.0531	2018/09/26	2.50	2.512	0.0630	0.1028	0.1529	0.2029
0.0565	2018/12/26	2.35	2.512	0.3123	0.6142	0.7256	0.8371
0.0608	2019/03/27	2.25	2.512	0.5616	0.7585	0.9079	1.0574
0.0730	2019/03/27	2.30	2.512	0.5616	0.6114	0.7609	0.9103
0.0733	2018/10/24	2.50	2.512	0.1397	0.0749	0.1495	0.2240
0.0734	2018/12/26	2.40	2.512	0.3123	0.4253	0.5367	0.6481
0.0791	2018/09/26	2.55	2.512	0.0630	-0.2928	-0.2427	-0.1927
0.0902	2019/03/27	2.35	2.512	0.5616	0.4675	0.6169	0.7664
0.0920	2018/12/26	2.45	2.512	0.3123	0.2402	0.3517	0.4631
0.0989	2018/10/24	2.55	2.512	0.1397	-0.1908	-0.1162	-0.0417
0.1074	2019/03/27	2.40	2.512	0.5616	0.3266	0.4761	0.6255
0.1117	2018/09/26	2.60	2.512	0.0630	-0.6807	-0.6307	-0.5806
0.1140	2018/12/26	2.50	2.512	0.3123	0.0589	0.1704	0.2818
0.1244	2019/03/27	2.45	2.512	0.5616	0.1886	0.3381	0.4875
0.1284	2018/10/24	2.60	2.512	0.1397	-0.4513	-0.3767	-0.3022
0.1401	2018/12/26	2.55	2.512	0.3123	-0.1188	-0.0073	0.1041
0.1483	2019/03/27	2.50	2.512	0.5616	0.0534	0.2029	0.3523
0.1498	2018/09/26	2.65	2.512	0.0630	-1.0613	-1.0112	-0.9612
0.1686	2018/12/26	2.60	2.512	0.3123	-0.2930	-0.1816	-0.0701
0.1750	2019/03/27	2.55	2.512	0.5616	-0.0791	0.0704	0.2198
0.1908	2018/09/26	2.70	2.512	0.0630	-1.4347	-1.3847	-1.3346
0.2003	2018/12/26	2.65	2.512	0.3123	-0.4640	-0.3525	-0.2411
0.2007	2018/10/24	2.70	2.512	0.1397	-0.9576	-0.8831	-0.8086
0.2009	2019/03/27	2.60	2.512	0.5616	-0.2090	-0.0596	0.0899
0.2278	2018/12/26	2.70	2.512	0.3123	-0.6317	-0.5203	-0.4088
0.2312	2019/03/27	2.65	2.512	0.5616	-0.3365	-0.1870	-0.0376
0.2370	2018/09/26	2.75	2.512	0.0630	-1.8013	-1.7512	-1.7012
0.2440	2018/10/24	2.75	2.512	0.1397	-1.2038	-1.1293	-1.0547
0.2635	2019/03/27	2.70	2.512	0.5616	-0.4616	-0.3121	-0.1627
0.2657	2018/12/26	2.75	2.512	0.3123	-0.7964	-0.6849	-0.5735
0.2828	2018/09/26	2.80	2.512	0.0630	-2.1613	-2.1112	-2.0612
0.2976	2019/03/27	2.75	2.512	0.5616	-0.5844	-0.4349	-0.2855
0.3089	2018/12/26	2.80	2.512	0.3123	-0.9580	-0.8466	-0.7352
0.3323	2018/09/26	2.85	2.512	0.0630	-2.5149	-2.4648	-2.4148
0.3330	2019/03/27	2.80	2.512	0.5616	-0.7049	-0.5555	-0.4061
0.3477	2018/12/26	2.85	2.512	0.3123	-1.1169	-1.0054	-0.8940
0.3810	2018/09/26	2.90	2.512	0.0630	-2.8623	-2.8123	-2.7622
0.3939	2018/12/26	2.90	2.512	0.3123	-1.2729	-1.1615	-1.0501
0.4315	2018/09/26	2.95	2.512	0.0630	-3.2039	-3.1538	-3.1037
0.4405	2018/12/26	2.95	2.512	0.3123	-1.4263	-1.3149	-1.2035
0.4810	2018/09/26	3.00	2.512	0.0630	-3.5396	-3.4896	-3.4395
0.5813	2018/09/26	3.10	2.512	0.0630	-4.1947	-4.1447	-4.0946
0.6809	2018/09/26	3.20	2.512	0.0630	-4.8290	-4.7789	-4.7289
0.7798	2018/09/26	3.30	2.512	0.0630	-5.4438	-5.3937	-5.3437
0.8793	2018/09/26	3.40	2.512	0.0630	-6.0402	-5.9901	-5.9401
0.9790	2018/09/26	3.50	2.512	0.0630	-6.6193	-6.5692	-6.5192
1.0807	2018/09/26	3.60	2.512	0.0630	-7.1821	-7.1320	-7.0820

4.2.1. Comparison of Pricing Effect on Call Options. There are 62 call options on September 4, 2018, and the current price of underlying asset $S_0 = 2.55$. Based on the estimated parameters $\mu = 2.557\%$, $\sigma = 0.2005$, $r = 1.768\%$, $\lambda = 0.0077963$ for call options, the call options of Shanghai 50 ETF on September 4, 2018, are priced by our obtained pricing formula in Theorem 3 and the Black–Scholes model, respectively; see Table 3 for detailed data, where T is the expiration time of call option, K is the strike price of call option, S_0 is the initial price of underlying asset, τ is the left expiration time of call option, d_1, d_2 , and d_3 are the parameters in Theorem 3, C_1 is the value of call option computed by the Black–Scholes model, C_2 is the value of call option computed by Theorem 3, and C is the actual closing price of call option.

According to simple computing, the expectation and variance of absolute errors follow as

$$\begin{aligned} E(|C_1 - C|) &= 0.0065574, \text{Var}(|C_1 - C|) = 0.0045044, \\ E(|C_2 - C|) &= 0.0032696, \text{Var}(|C_2 - C|) = 0.001929, \end{aligned} \quad (20)$$

where C is the actual closing price of call option, C_1 is the value of call option computed by the Black–Scholes model, and C_2 is the value of call option computed by Theorem 3. It is obvious that $E(|C_2 - C|) < E(|C_1 - C|)$. In the following, we will support the statement by the hypothesis test (i.e., t -test function of MATLAB R2018b).

In fact, for the hypothesis test,

$$\begin{aligned} H_0: E(|C_2 - C|) - E(|C_1 - C|) &\geq 0, \\ H_1: E(|C_2 - C|) - E(|C_1 - C|) &< 0, \end{aligned} \quad (21)$$

The t -statistics equals -5.9311 with degree of freedom 61, and its p value is 7.6261×10^{-8} . Thus, we accept H_1 , i.e., $E(|C_2 - C|) < E(|C_1 - C|)$. That is, the prices of call options computed by Theorem 3 are far nearer to their actual prices than those computed by the Black–Scholes model.

4.2.2. Comparison of Pricing Effect on Put Options. There are 62 put options on September 4, 2018, and the current price of underlying asset $S_0 = 2.55$. Based on the estimated parameters $\mu = 2.557\%$, $\sigma = 0.2005$, $r = 1.768\%$, and $\lambda = 0.013306$ for put options, the put options of Shanghai 50 ETF on September 4, 2018, are priced by our pricing formula in Theorem 2 and Black–Scholes model, respectively; see Table 4 for detailed data, where T is the expiration time of put option, K is the strike price of put option, S_0 is the initial price of underlying asset, τ is the left expiration time of put option, d_1, d_2 , and d_3 are the parameters in Theorem 3, P_1 is the value of put option computed by the Black–Scholes model, P_2 is the value of put option computed by Theorem 3, and P is the actual closing price of put option.

According to simple computing, the expectation and variance of absolute errors follow as

$$\begin{aligned} E(|P_1 - P|) &= 0.00836, \text{Var}(|P_1 - P|) = 0.0056946, \\ E(|P_2 - P|) &= 0.0044808, \text{Var}(|P_2 - P|) = 0.0029812, \end{aligned} \quad (22)$$

where P is the actual closing price of put option, P_1 is the value of put option computed by the Black–Scholes model, and P_2 is the value of put option computed by Theorem 3. It is obvious that $E(|P_2 - P|) < E(|P_1 - P|)$. In the following, we will support the statement by the hypothesis test (i.e., t -test function of MATLAB R2018b).

In fact, for the hypothesis test,

$$\begin{aligned} H_0: E(|P_2 - P|) - E(|P_1 - P|) &\geq 0, \\ H_1: E(|P_2 - P|) - E(|P_1 - P|) &< 0, \end{aligned} \quad (23)$$

The t -statistics equals -4.7567 with degree of freedom 61, and its p value is 6.2304×10^{-6} . Thus, we accept H_1 , i.e., $E(|P_2 - P|) < E(|P_1 - P|)$. That is, the prices of put options computed by Theorem 3 are far nearer to their actual prices than those computed by the Black–Scholes model.

Therefore, our pricing formulas in Theorem 2 and Theorem 3 have less absolute errors than those of the Black–Scholes model for both call options and put options. That is, the fitting effect of our pricing formulas is superior to that of the Black–Scholes model.

5. Empirical Analysis of S&P 500 Options

In the section, we will present empirical analysis on S&P 500 options and use the data of April 1 and 2, 2019 to compare the fitting effect of our pricing model and the Black–Scholes pricing model. All used data come from the Chicago Board Options Exchange, which includes actual option price C or P , trading date t , exercise date T , strike price K , and current price of underlying asset S_0 . We consider the out-of-the-money put and call options, which are more liquid and actively traded than in-the-money options. And observations with trading volume below average, prices less than \$0.5 or left expiration time less than 10 days or longer than 360 days are discarded. The annualized historical volatility $\sigma = 0.1174$ based on closing prices of the underlying asset over the last month. According to the Board of Governors of the Federal Reserve System (<https://www.federalreserve.gov/releases/h15/data.htm>), the annualized risk-free interest rate is 2.43% on April 1 and 2.42% on April 2, 2019.

5.1. Parameter Estimation. We consider the daily logarithmic returns of S&P 500 index closing prices from January 2015 to December 2018. The augmented Dickey–Fuller test shows the Dickey–Fuller statistic equals -10.283 with lag order 10 and p value is 0.01, which indicates that the time series of returns is stationary. Furthermore, the autocorrelation shows the coefficients of autocorrelation mostly fall within double standard deviations (see Figure 1). Thus, we accept that the time series of daily logarithmic returns of S&P 500 index closing prices are stationary and independent. We take the annual average return rate based on 252

TABLE 3: Pricing results by our pricing formula and the Black–Scholes model for call options.

T	K	S_0	τ	d_1	d_2	d_3	C_1	C_2	C
2018/09/26	3.00	2.55	0.0603	-3.2949	-3.2457	-3.1965	0.0005	0.0000	0.0005
2018/12/26	2.40	2.55	0.3096	0.5586	0.6702	0.7817	0.2239	0.2087	0.2200
2018/12/26	2.35	2.55	0.3096	0.7473	0.8589	0.9705	0.2601	0.2451	0.2555
2018/09/26	2.90	2.55	0.0603	-2.6062	-2.5570	-2.5077	0.0017	0.0002	0.0007
2019/03/27	2.50	2.55	0.5589	0.1525	0.3024	0.4523	0.2079	0.1883	0.2036
2019/03/27	2.55	2.55	0.5589	0.0204	0.1703	0.3202	0.1816	0.1626	0.1773
2019/03/27	2.65	2.55	0.5589	-0.2362	-0.0863	0.0636	0.1364	0.1188	0.1353
2018/12/26	2.60	2.55	0.3096	-0.1589	-0.0473	0.0643	0.1107	0.0963	0.1067
2018/09/26	2.20	2.55	0.0603	3.0060	3.0552	3.1044	0.3548	0.3520	0.3510
2018/09/26	2.85	2.55	0.0603	-2.2529	-2.2036	-2.1544	0.0030	0.0006	0.0012
2019/03/27	2.20	2.55	0.5589	1.0053	1.1552	1.3051	0.4160	0.3955	0.4070
2018/12/26	2.30	2.55	0.3096	0.9401	1.0517	1.1632	0.2991	0.2846	0.2925
2018/10/24	2.35	2.55	0.1370	1.1108	1.1850	1.2592	0.2264	0.2166	0.2255
2018/10/24	2.30	2.55	0.1370	1.4006	1.4748	1.5490	0.2698	0.2610	0.2653
2018/10/24	2.70	2.55	0.1370	-0.7601	-0.6859	-0.6117	0.0356	0.0258	0.0300
2018/12/26	2.20	2.55	0.3096	1.3386	1.4501	1.5617	0.3840	0.3706	0.3817
2018/09/26	2.50	2.55	0.0603	0.4090	0.4582	0.5074	0.0905	0.0800	0.0879
2018/09/26	2.95	2.55	0.0603	-2.9535	-2.9042	-2.8550	0.0010	0.0001	0.0005
2018/12/26	2.95	2.55	0.3096	-1.2909	-1.1794	-1.0678	0.0229	0.0150	0.0197
2018/09/26	2.80	2.55	0.0603	-1.8933	-1.8441	-1.7948	0.0052	0.0015	0.0019
2018/12/26	2.85	2.55	0.3096	-0.9818	-0.8703	-0.7587	0.0371	0.0272	0.0341
2018/10/24	2.55	2.55	0.1370	0.0101	0.0843	0.1585	0.0901	0.0780	0.0854
2018/10/24	2.40	2.55	0.1370	0.8271	0.9013	0.9755	0.1862	0.1753	0.1837
2018/09/26	3.10	2.55	0.0603	-3.9610	-3.9118	-3.8626	0.0001	0.0000	0.0004
2018/12/26	2.70	2.55	0.3096	-0.4972	-0.3856	-0.2740	0.0733	0.0604	0.0700
2019/03/27	2.35	2.55	0.5589	0.5653	0.7152	0.8651	0.3016	0.2811	0.2985
2019/03/27	2.40	2.55	0.5589	0.4249	0.5748	0.7246	0.2679	0.2476	0.2628
2018/12/26	2.75	2.55	0.3096	-0.6616	-0.5501	-0.4385	0.0588	0.0469	0.0550
2019/03/27	2.25	2.55	0.5589	0.8554	1.0053	1.1552	0.3758	0.3552	0.3679
2019/03/27	2.80	2.55	0.5589	-0.6035	-0.4536	-0.3038	0.0856	0.0707	0.0810
2018/12/26	2.65	2.55	0.3096	-0.3296	-0.2181	-0.1065	0.0905	0.0768	0.0871
2018/09/26	2.30	2.55	0.0603	2.1029	2.1521	2.2014	0.2571	0.2528	0.2538
2018/09/26	3.60	2.55	0.0603	-6.9988	-6.9496	-6.9003	0.0000	0.0000	0.0004
2019/03/27	2.75	2.55	0.5589	-0.4833	-0.3334	-0.1835	0.1004	0.0846	0.0988
2019/03/27	2.30	2.55	0.5589	0.7088	0.8587	1.0086	0.3376	0.3170	0.3316
2018/09/26	2.40	2.55	0.0603	1.2383	1.2875	1.3368	0.1658	0.1583	0.1641
2018/09/26	2.25	2.55	0.0603	2.5494	2.5986	2.6479	0.3055	0.3022	0.3040
2019/03/27	2.60	2.55	0.5589	-0.1091	0.0408	0.1907	0.1578	0.1394	0.1549
2018/10/24	2.45	2.55	0.1370	0.5492	0.6234	0.6976	0.1497	0.1380	0.1461
2018/12/26	2.90	2.55	0.3096	-1.1377	-1.0262	-0.9146	0.0292	0.0203	0.0266
2018/10/24	2.50	2.55	0.1370	0.2770	0.3512	0.4254	0.1175	0.1054	0.1135
2018/10/24	2.75	2.55	0.1370	-1.0074	-0.9332	-0.8590	0.0252	0.0167	0.0195
2018/09/26	3.30	2.55	0.0603	-5.2311	-5.1819	-5.1327	0.0000	0.0000	0.0003
2018/12/26	2.50	2.55	0.3096	0.1927	0.3043	0.4158	0.1607	0.1455	0.1572
2018/12/26	2.45	2.55	0.3096	0.3738	0.4854	0.5969	0.1907	0.1754	0.1870
2018/10/24	2.60	2.55	0.1370	-0.2516	-0.1774	-0.1031	0.0675	0.0559	0.0623
2018/09/26	2.70	2.55	0.0603	-1.1545	-1.1053	-1.0560	0.0153	0.0081	0.0103
2018/12/26	2.80	2.55	0.3096	-0.8232	-0.7116	-0.6000	0.0469	0.0359	0.0445
2018/09/26	2.55	2.55	0.0603	0.0067	0.0559	0.1052	0.0621	0.0512	0.0590
2018/09/26	3.50	2.55	0.0603	-6.4265	-6.3773	-6.3280	0.0000	0.0000	0.0003
2018/09/26	2.65	2.55	0.0603	-0.7747	-0.7255	-0.6763	0.0253	0.0164	0.0206
2018/09/26	2.35	2.55	0.0603	1.6660	1.7152	1.7645	0.2101	0.2045	0.2077
2018/09/26	2.45	2.55	0.0603	0.8194	0.8686	0.9179	0.1254	0.1162	0.1232
2018/12/26	2.25	2.55	0.3096	1.1371	1.2487	1.3602	0.3405	0.3265	0.3330
2018/10/24	2.65	2.55	0.1370	-0.5083	-0.4340	-0.3598	0.0495	0.0387	0.0442
2018/09/26	2.75	2.55	0.0603	-1.5272	-1.4780	-1.4288	0.0090	0.0037	0.0048
2018/09/26	3.40	2.55	0.0603	-5.8376	-5.7884	-5.7392	0.0000	0.0000	0.0003
2019/03/27	2.70	2.55	0.5589	-0.3609	-0.2110	-0.0611	0.1173	0.1006	0.1162
2018/09/26	3.20	2.55	0.0603	-4.6060	-4.5568	-4.5076	0.0000	0.0000	0.0003
2018/12/26	2.55	2.55	0.3096	0.0152	0.1268	0.2383	0.1341	0.1192	0.1311
2018/09/26	2.60	2.55	0.0603	-0.3878	-0.3386	-0.2893	0.0405	0.0303	0.0377
2019/03/27	2.45	2.55	0.5589	0.2873	0.4372	0.5871	0.2367	0.2166	0.2319

TABLE 4: Pricing results by our pricing formula and the Black–Scholes model for put options.

T	K	S_0	τ	d_1	d_2	d_3	P_1	P_2	P
2018/12/26	2.50	2.55	0.3096	0.1927	0.3043	0.4158	0.0963	0.0839	0.0983
2019/03/27	2.30	2.55	0.5589	0.7088	0.8587	1.0086	0.0583	0.0478	0.0635
2019/03/27	2.25	2.55	0.5589	0.8554	1.0053	1.1552	0.0463	0.0364	0.0520
2018/12/26	2.20	2.55	0.3096	1.3386	1.4501	1.5617	0.0186	0.0104	0.0205
2018/12/26	2.90	2.55	0.3096	-1.1377	-1.0262	-0.9146	0.3631	0.3568	0.3618
2018/09/26	2.30	2.55	0.0603	2.1029	2.1521	2.2014	0.0050	0.0007	0.0030
2018/09/26	3.20	2.55	0.0603	-4.6060	-4.5568	-4.5076	0.6473	0.6471	0.6437
2018/10/24	2.70	2.55	0.1370	0.7601	-0.6859	-0.6117	0.1831	0.1703	0.1724
2018/09/26	2.50	2.55	0.0603	0.4090	0.4582	0.5074	0.0440	0.0278	0.0359
2018/09/26	2.65	2.55	0.0603	-0.7747	-0.7255	-0.6763	0.1282	0.1140	0.1191
2019/03/27	2.55	2.55	0.5589	0.0204	0.1703	0.3202	0.1521	0.1413	0.1569
2018/10/24	2.60	2.55	0.1370	-0.2516	-0.1774	-0.1031	0.1162	0.1005	0.1054
2018/09/26	2.20	2.55	0.0603	3.0060	3.0552	3.1044	0.0012	0.0000	0.0010
2019/03/27	2.40	2.55	0.5589	0.4249	0.5748	0.7246	0.0887	0.0775	0.0939
2019/03/27	2.60	2.55	0.5589	-0.1091	0.0408	0.1907	0.1780	0.1677	0.1829
2018/09/26	2.85	2.55	0.0603	-2.2529	-2.2036	-2.1544	0.3010	0.2980	0.2955
2018/12/26	2.40	2.55	0.3096	0.5586	0.6702	0.7817	0.0605	0.0475	0.0607
2019/03/27	2.45	2.55	0.5589	0.2873	0.4372	0.5871	0.1074	0.0962	0.1121
2019/03/27	2.65	2.55	0.5589	-0.2362	-0.0863	0.0636	0.2062	0.1967	0.2094
2019/03/27	2.80	2.55	0.5589	-0.6035	-0.4536	-0.3038	0.3040	0.2973	0.3078
2018/09/26	2.45	2.55	0.0603	0.8194	0.8686	0.9179	0.0277	0.0140	0.0219
2018/12/26	2.95	2.55	0.3096	-1.2909	-1.1794	-1.0678	0.4061	0.4013	0.4056
2018/12/26	2.25	2.55	0.3096	1.1371	1.2487	1.3602	0.0257	0.0161	0.0271
2018/09/26	2.80	2.55	0.0603	-1.8933	-1.8441	-1.7948	0.2541	0.2490	0.2486
2018/10/24	2.65	2.55	0.1370	-0.5083	-0.4340	-0.3598	0.1478	0.1332	0.1339
2018/12/26	2.75	2.55	0.3096	-0.6616	-0.5501	-0.4385	0.2446	0.2341	0.2387
2018/12/26	2.85	2.55	0.3096	-0.9818	-0.8703	-0.7587	0.3217	0.3140	0.3201
2018/09/26	2.70	2.55	0.0603	-1.1545	-1.1053	-1.0560	0.1668	0.1557	0.1585
2018/09/26	3.50	2.55	0.0603	-6.4265	-6.3773	-6.3280	0.9470	0.9468	0.9443
2019/03/27	2.35	2.55	0.5589	0.5653	0.7152	0.8651	0.0724	0.0614	0.0784
2018/10/24	2.55	2.55	0.1370	0.0101	0.0843	0.1585	0.0889	0.0728	0.0788
2018/12/26	2.70	2.55	0.3096	-0.4972	-0.3856	-0.2740	0.2096	0.1979	0.2079
2018/10/24	2.75	2.55	0.1370	-1.0074	-0.9332	-0.8590	0.2219	0.2110	0.2120
2018/12/26	2.45	2.55	0.3096	0.3738	0.4854	0.5969	0.0776	0.0640	0.0782
2018/10/24	2.40	2.55	0.1370	0.8271	0.9013	0.9755	0.0333	0.0204	0.0260
2018/09/26	3.10	2.55	0.0603	-3.9610	-3.9118	-3.8626	0.5474	0.5472	0.5451
2018/10/24	2.50	2.55	0.1370	0.2770	0.3512	0.4254	0.0661	0.0503	0.0569
2018/09/26	3.30	2.55	0.0603	-5.2311	-5.1819	-5.1327	0.7472	0.7470	0.7420
2018/10/24	2.30	2.55	0.1370	1.4006	1.4748	1.5490	0.0149	0.0063	0.0108
2018/12/26	2.35	2.55	0.3096	0.7473	0.8589	0.9705	0.0463	0.0343	0.0467
2018/09/26	2.40	2.55	0.0603	1.2383	1.2875	1.3368	0.0166	0.0062	0.0118
2018/09/26	3.40	2.55	0.0603	-5.8376	-5.7884	-5.7392	0.8471	0.8469	0.8453
2018/09/26	2.60	2.55	0.0603	-0.3878	-0.3386	-0.2893	0.0943	0.0779	0.0844
2018/09/26	2.25	2.55	0.0603	2.5494	2.5986	2.6479	0.0025	0.0002	0.0015
2018/10/24	2.35	2.55	0.1370	1.1108	1.1850	1.2592	0.0226	0.0118	0.0178
2019/03/27	2.70	2.55	0.5589	-0.3609	-0.2110	-0.0611	0.2368	0.2280	0.2402
2018/09/26	2.35	2.55	0.0603	1.6660	1.7152	1.7645	0.0094	0.0023	0.0065
2018/09/26	2.95	2.55	0.0603	-2.9535	-2.9042	-2.8550	0.3982	0.3974	0.3921
2018/10/24	2.45	2.55	0.1370	0.5492	0.6234	0.6976	0.0476	0.0330	0.0387
2018/12/26	2.65	2.55	0.3096	-0.3296	-0.2181	-0.1065	0.1772	0.1645	0.1753
2018/12/26	2.55	2.55	0.3096	0.0152	0.1268	0.2383	0.1212	0.1073	0.1210
2019/03/27	2.75	2.55	0.5589	-0.4833	-0.3334	-0.1835	0.2694	0.2616	0.2738
2018/09/26	2.75	2.55	0.0603	-1.5272	-1.4780	-1.4288	0.2091	0.2012	0.2011
2018/09/26	3.60	2.55	0.0603	-6.9988	-6.9496	-6.9003	1.0469	1.0467	1.0452
2019/03/27	2.20	2.55	0.5589	1.0053	1.1552	1.3051	0.0363	0.0272	0.0423
2018/12/26	2.80	2.55	0.3096	-0.8232	-0.7116	-0.6000	0.2821	0.2729	0.2800
2018/12/26	2.30	2.55	0.3096	0.9401	1.0517	1.1632	0.0348	0.0239	0.0353
2018/09/26	2.90	2.55	0.0603	-2.6062	-2.5570	-2.5077	0.3492	0.3476	0.3430
2018/09/26	2.55	2.55	0.0603	0.0067	0.0559	0.1052	0.0661	0.0489	0.0573
2019/03/27	2.50	2.55	0.5589	0.1525	0.3024	0.4523	0.1285	0.1174	0.1330
2018/12/26	2.60	2.55	0.3096	-0.1589	-0.0473	0.0643	0.1477	0.1342	0.1468
2018/09/26	3.00	2.55	0.0603	-3.2949	-3.2457	-3.1965	0.4477	0.4473	0.4459

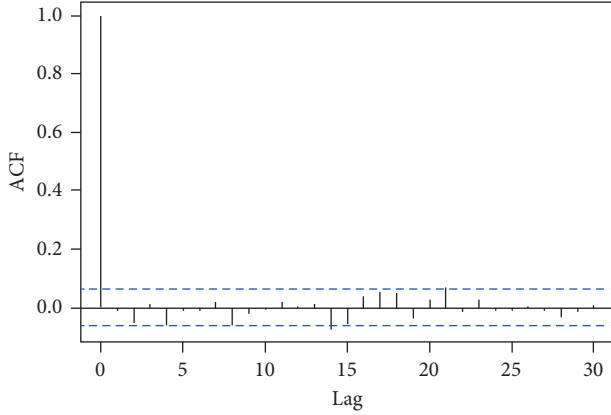


FIGURE 1: The autocorrelation function of the daily logarithmic returns of S&P 500 index closing prices.

effective trading days as the drift coefficient of the underlying asset μ .

There are 41 call options on April 1, 2019, and the current price of underlying asset $S_0 = 2867.19$. According to simple computation, we obtain $\mu = 4.932\%$ and the left expiration time $\tau = (T - t)/365$ (years), where $t = 2019/04/01$, and then we work out d_1, d_2 , and d_3 by Theorem 2; see Table 5 for detailed data, where C is the actual closing price of call option, τ is the left expiration time of call option, K is the strike price of call option, and d_1, d_2 , and d_3 are the parameters in Theorem 2. Furthermore, we obtain the estimated value of the risk-compensation factor $\lambda = -0.0246533$ by the least square method with $R^2 = 0.9532$.

There are 62 put options on April 1, 2019, and the current price of underlying asset $S_0 = 2867.19$. Similarly, we obtain $\mu = 4.932\%$ and the left expiration time $\tau = (T - t)/365$ (years), where $t = 2019/04/01$, and then we work out d_1, d_2 , and d_3 by Theorem 3; see Table 6 for detailed data, where P is the actual closing price of put option, τ is the left expiration time of put option, K is the strike price of put option, and d_1, d_2 , and d_3 are the parameters in Theorem 3. Furthermore, we obtain the estimated value of the risk-compensation factor $\lambda = 0.0269767$ by the least square method with $R^2 = 0.9561$.

5.2. Comparison of Pricing Effect. In the section, we will compare the fitting effect of our pricing formulas with the Black-Scholes model.

5.2.1. Comparison of Pricing Effect on Call Options. There are 25 call options on April 2, 2019, and the current price of underlying asset $S_0 = 2867.24$. Based on the estimated parameters $\mu = 4.932\%$, $\sigma = 0.1174$, $r = 2.42\%$, and $\lambda = -0.0246533$ for call options, the call options of S&P 500 on April 2, 2019, are priced by our obtained pricing formula in Theorem 1 and the Black-Scholes model, respectively; see Table 7 for detailed data, where K is the strike price of call option, τ is the left expiration time of call option, d_1, d_2 , and d_3 are the parameters in Theorem 3, C is the actual closing price of call option, C_1 is the value of call option computed

by the Black-Scholes model, and C_2 is the value of call option computed by Theorem 1.

According to simple computing, the expectation and variance of absolute errors follow as

$$\begin{aligned} E(|\widehat{C_1} - C|) &= 6.902386, \text{Var}(|\widehat{C_1} - C|) = 6.810772, \\ E(|\widehat{C_2} - C|) &= 5.156575, \text{Var}(|\widehat{C_2} - C|) = 9.726556, \end{aligned} \quad (24)$$

where C is the actual closing price of call option, C_1 is the value of call option computed by the Black-Scholes model, and C_2 is the value of call option computed by Theorem 2. It is obvious that $E(|\widehat{C_2} - C|) < E(|\widehat{C_1} - C|)$. In the following, we will support the statement by the hypothesis test (i.e., test function of MATLAB R2018b).

In fact, for the hypothesis test,

$$\begin{aligned} H_0: E(|\widehat{C_2} - C|) - E(|\widehat{C_1} - C|) &\geq 0, \\ H_1: E(|\widehat{C_2} - C|) - E(|\widehat{C_1} - C|) &< 0, \end{aligned} \quad (25)$$

the t -statistics equals -2.1465 with degree of freedom 24, and its p value is 0.01854. Thus, we accept H_1 , i.e., $E(|\widehat{C_2} - C|) < E(|\widehat{C_1} - C|)$. That is, the prices of call options computed by Theorem 2 are far nearer to their actual prices than those computed by the Black-Scholes model.

5.2.2. Comparison of Pricing Effect on Put Options. There are 39 put options on April 2, 2019, and the current price of underlying asset $S_0 = 2867.24$. Based on the estimated parameters $\mu = 4.932\%$, $\sigma = 0.1174$, $r = 2.42\%$, and $\lambda = 0.0269767$ for put options, the put options of S&P 500 on April 2, 2019, are priced by our obtained pricing formula in Theorem 3 and the Black-Scholes model, respectively; see Table 8 for detailed data, where K is the strike price of put option, τ is the left expiration time of put option, d_1, d_2 , and d_3 are the parameters in Theorem 3, P is the actual closing price of put option, P_1 is the value of put option computed by the Black-Scholes model, and P_2 is the value of put option computed by Theorem 3.

According to simple computing, the expectation and variance of absolute errors follow as

$$\begin{aligned} E(|\widehat{P_1} - P|) &= 7.850124, \text{Var}(|\widehat{P_1} - P|) = 29.945065, \\ E(|\widehat{P_2} - P|) &= 3.492115, \text{Var}(|\widehat{P_2} - P|) = 6.764690, \end{aligned} \quad (26)$$

where P is the actual closing price of put option, P_1 is the value of put option computed by the Black-Scholes model, and P_2 is the value of put option computed by Theorem 3. It is obvious that $E(|\widehat{P_2} - P|) < E(|\widehat{P_1} - P|)$. In the following, we will support the statement by the hypothesis test (i.e., t -test function of MATLAB R2018b).

In fact, for the hypothesis test,

$$\begin{aligned} H_0: E(|\widehat{P_2} - P|) - E(|\widehat{P_1} - P|) &\geq 0, \\ H_1: E(|\widehat{P_2} - P|) - E(|\widehat{P_1} - P|) &< 0, \end{aligned} \quad (27)$$

TABLE 5: Parameter estimation of d_1 , d_2 , and d_3 for call options.

C	τ	K	d_1	d_2	d_3
24.40	0.046575	2870	0.039320	0.064659	0.089997
21.50	0.046575	2875	-0.029375	-0.004037	0.021302
10.60	0.046575	2900	-0.371070	-0.345732	-0.320393
4.50	0.046575	2925	-0.709832	-0.684493	-0.659155
3.85	0.046575	2930	-0.777237	-0.751898	-0.726559
3.14	0.046575	2935	-0.844526	-0.819188	-0.793849
1.67	0.046575	2950	-1.045710	-1.020372	-0.995033
0.65	0.046575	2975	-1.378754	-1.353416	-1.328077
0.60	0.046575	2980	-1.445027	-1.419689	-1.394350
46.15	0.126027	2870	0.104771	0.146452	0.188133
44.00	0.126027	2875	0.063010	0.104691	0.146371
31.00	0.126027	2900	-0.144713	-0.103032	-0.061351
19.35	0.126027	2925	-0.350653	-0.308972	-0.267291
7.00	0.126027	2975	-0.757303	-0.715622	-0.673941
3.73	0.126027	3000	-0.958073	-0.916392	-0.874711
3.30	0.126027	3005	-0.998026	-0.956345	-0.914664
2.50	0.126027	3015	-1.077733	-1.036052	-0.994371
66.70	0.221918	2870	0.152504	0.207814	0.263123
63.80	0.221918	2875	0.121033	0.176343	0.231652
48.60	0.221918	2900	-0.035505	0.019805	0.075114
36.70	0.221918	2925	-0.190699	-0.135390	-0.080080
28.32	0.221918	2940	-0.283181	-0.227871	-0.172561
27.70	0.221918	2950	-0.344573	-0.289263	-0.233954
13.03	0.221918	3000	-0.648446	-0.593137	-0.537827
6.00	0.221918	3050	-0.947297	-0.891987	-0.836678
3.10	0.221918	3100	-1.241288	-1.185978	-1.130669
0.85	0.221918	3200	-1.815306	-1.759996	-1.704686
28.30	0.298630	2975	-0.377841	-0.313680	-0.249519
21.90	0.298630	3000	-0.508267	-0.444106	-0.379945
17.00	0.375342	3050	-0.637911	-0.565980	-0.494048
42.20	0.471233	3000	-0.313762	-0.233165	-0.152567
15.10	0.471233	3100	-0.720596	-0.639998	-0.559401
133.50	0.720548	2875	0.279420	0.379083	0.478746
69.70	0.720548	3000	-0.147614	-0.047950	0.051713
33.36	0.720548	3100	-0.476619	-0.376956	-0.277292
27.50	0.720548	3125	-0.557212	-0.457549	-0.357885
15.70	0.720548	3200	-0.795178	-0.695515	-0.595852
7.00	0.720548	3300	-1.103934	-1.004271	-0.904607
75.30	0.797260	3000	-0.109289	-0.004455	0.100380
18.70	0.797260	3200	-0.724912	-0.620077	-0.515243
52.44	0.969863	3100	-0.319343	-0.203716	-0.088089

TABLE 6: Parameter estimation of d_1 , d_2 , and d_3 for put options.

P	τ	K	d_1	d_2	d_3
0.35	0.046575	2400	7.097458	7.122797	7.148136
0.42	0.046575	2410	6.933361	6.958699	6.984038
0.40	0.046575	2420	6.769942	6.795281	6.820619
0.40	0.046575	2435	6.526076	6.551415	6.576753
0.47	0.046575	2440	6.445121	6.470460	6.495799
0.46	0.046575	2450	6.283708	6.309047	6.334385
0.55	0.046575	2455	6.203248	6.228587	6.253926
0.54	0.046575	2485	5.723905	5.749243	5.774582
0.62	0.046575	2490	5.644577	5.669915	5.695254
0.56	0.046575	2500	5.486398	5.511737	5.537076
0.72	0.046575	2515	5.250313	5.275652	5.300990
0.74	0.046575	2525	5.093704	5.119042	5.144381
0.75	0.046575	2550	4.704878	4.730217	4.755555
0.80	0.046575	2600	3.938534	3.963872	3.989211

TABLE 6: Continued.

P	τ	K	d_1	d_2	d_3
0.93	0.046575	2605	3.862711	3.888050	3.913389
1.38	0.046575	2640	3.335995	3.361334	3.386673
1.50	0.046575	2650	3.186787	3.212126	3.237464
1.80	0.046575	2675	2.816217	2.841555	2.866894
2.44	0.046575	2700	2.449093	2.474432	2.499770
2.55	0.046575	2705	2.376076	2.401415	2.426754
2.65	0.046575	2710	2.303195	2.328533	2.353872
3.03	0.046575	2720	2.157833	2.183172	2.208511
3.25	0.046575	2725	2.085353	2.110692	2.136030
4.50	0.046575	2750	1.724935	1.750274	1.775612
0.55	0.126027	2000	8.769770	8.811451	8.853132
1.15	0.126027	2275	5.678836	5.720517	5.762198
1.50	0.126027	2350	4.900655	4.942336	4.984017
1.95	0.126027	2400	4.395546	4.437227	4.478907
2.20	0.126027	2425	4.146924	4.188605	4.230285
3.20	0.126027	2500	3.416152	3.457833	3.499514
5.10	0.126027	2575	2.706983	2.748664	2.790345
6.05	0.126027	2600	2.475176	2.516857	2.558538
6.38	0.126027	2610	2.383077	2.424758	2.466439
8.55	0.126027	2650	2.018175	2.059856	2.101537
10.31	0.126027	2675	1.792898	1.834579	1.876260
12.35	0.126027	2700	1.569717	1.611398	1.653079
14.35	0.126027	2725	1.348593	1.390274	1.431955
18.07	0.126027	2750	1.129488	1.171169	1.212850
22.54	0.126027	2775	0.912366	0.954047	0.995728
27.70	0.126027	2800	0.697191	0.738872	0.780553
32.85	0.126027	2825	0.483929	0.525610	0.567291
41.25	0.126027	2850	0.272547	0.314227	0.355908
1.30	0.221918	2000	6.682380	6.737690	6.793000
5.39	0.221918	2375	3.575320	3.630630	3.685939
5.80	0.221918	2400	3.385999	3.441308	3.496618
9.26	0.221918	2500	2.647935	2.703245	2.758554
11.74	0.221918	2550	2.289903	2.345212	2.400522
12.90	0.221918	2565	2.183861	2.239171	2.294480
13.55	0.221918	2575	2.113511	2.168820	2.224130
15.32	0.221918	2600	1.938823	1.994133	2.049442
17.70	0.221918	2625	1.765807	1.821117	1.876426
19.60	0.221918	2650	1.594431	1.649740	1.705050
26.23	0.221918	2700	1.256476	1.311786	1.367095
28.70	0.221918	2715	1.156310	1.211619	1.266929
31.25	0.221918	2725	1.089839	1.145148	1.200458
35.95	0.221918	2750	0.924723	0.980033	1.035342
46.70	0.221918	2800	0.598948	0.654257	0.709567
59.32	0.221918	2850	0.278939	0.334248	0.389558
63.55	0.221918	2860	0.215611	0.270920	0.326230
64.70	0.221918	2865	0.184030	0.239340	0.294649
46.60	0.298630	2750	0.847874	0.912035	0.976196
57.50	0.298630	2800	0.567041	0.631203	0.695364

TABLE 7: Pricing results by our pricing formula and the Black-Scholes model for call option.

K	τ	d_1	d_2	d_3	C	C_1	C_2
2870	0.043836	0.036511	0.061093	0.085675	21.05	28.26	30.77
2875	0.043836	-0.034298	-0.009716	0.014866	18.15	25.85	26.25
2880	0.043836	-0.104985	-0.080403	-0.055821	18.20	23.58	21.99
2890	0.043836	-0.245990	-0.221408	-0.196826	13.70	19.45	14.27
2900	0.043836	-0.386509	-0.361927	-0.337345	8.50	15.87	7.65
2870	0.123288	0.103532	0.144758	0.185983	45.00	50.06	54.77
2875	0.123288	0.061310	0.102535	0.143760	40.85	47.55	50.91

TABLE 7: Continued.

K	τ	d_1	d_2	d_3	C	C_1	C_2
2895	0.123288	-0.106850	-0.065625	-0.024399	31.68	38.36	36.73
2900	0.123288	-0.148708	-0.107483	-0.066258	29.50	36.27	33.49
2925	0.123288	-0.356924	-0.315698	-0.274473	18.10	27.00	19.26
2940	0.123288	-0.481000	-0.439775	-0.398549	14.10	22.35	12.25
2950	0.123288	-0.563367	-0.522141	-0.480916	11.50	19.61	8.19
2955	0.123288	-0.604445	-0.563220	-0.521995	10.45	18.34	6.35
2870	0.219178	0.151657	0.206624	0.261591	64.00	69.14	75.39
2875	0.219178	0.119990	0.174957	0.229924	63.80	66.58	71.78
2900	0.219178	-0.037523	0.017444	0.072411	47.50	54.73	54.98
2950	0.219178	-0.348517	-0.293550	-0.238583	26.75	35.58	27.81
3000	0.219178	-0.654284	-0.599317	-0.544350	13.30	21.94	8.99
3025	0.219178	-0.805261	-0.750294	-0.695327	8.75	16.88	2.42
2900	0.295890	0.018662	0.082528	0.146394	62.77	67.32	69.15
2900	0.372603	0.062039	0.133708	0.205376	76.38	78.74	81.78
3100	0.372603	-0.868516	-0.796848	-0.725179	9.40	18.99	3.50
2900	0.468493	0.105947	0.186310	0.266673	91.40	91.87	96.10
3000	0.468493	-0.315908	-0.235545	-0.155182	41.26	52.22	44.38
3100	0.468493	-0.723929	-0.643566	-0.563203	15.00	27.13	12.45

TABLE 8: Pricing results by our pricing formula and the Black-Scholes model for put option.

K	τ	d_1	d_2	d_3	P	P_1	P_2
2550	0.043836	4.845658	4.870240	4.894822	0.59	0.00	0.05
2575	0.043836	4.448776	4.473358	4.497940	0.70	0.00	0.14
2600	0.043836	4.055729	4.080311	4.104893	0.81	0.00	0.35
2650	0.043836	3.280846	3.305428	3.330010	1.20	0.01	1.59
2660	0.043836	3.127625	3.152207	3.176790	1.40	0.02	2.08
2675	0.043836	2.898871	2.923453	2.948035	1.64	0.04	3.02
2700	0.043836	2.520449	2.545031	2.569613	1.95	0.14	5.27
2300	0.123288	5.474086	5.515311	5.556536	1.30	0.00	0.01
2400	0.123288	4.441720	4.482946	4.524171	1.95	0.00	0.14
2500	0.123288	3.451504	3.492730	3.533955	3.26	0.01	1.11
2590	0.123288	2.593606	2.634832	2.676057	5.40	0.20	4.65
2600	0.123288	2.500131	2.541356	2.582581	6.00	0.27	5.32
2615	0.123288	2.360589	2.401814	2.443040	6.20	0.41	6.45
2625	0.123288	2.268005	2.309230	2.350456	7.10	0.53	7.29
2670	0.123288	1.855696	1.896922	1.938147	9.20	1.62	11.98
2675	0.123288	1.810314	1.851539	1.892764	9.40	1.82	12.60
2700	0.123288	1.584666	1.625892	1.667117	11.50	3.13	15.93
2725	0.123288	1.361099	1.402324	1.443549	14.30	5.16	19.69
2775	0.123288	0.920051	0.961277	1.002502	20.50	12.41	28.36
2800	0.123288	0.702499	0.743724	0.784950	25.85	18.18	33.26
2825	0.123288	0.486881	0.528106	0.569331	31.50	25.71	38.61
2850	0.123288	0.273162	0.314387	0.355612	39.00	35.19	44.52
2200	0.219178	4.988269	5.043237	5.098204	2.60	0.00	0.03
2400	0.219178	3.405298	3.460265	3.515232	5.80	0.02	1.16
2500	0.219178	2.662636	2.717603	2.772570	9.00	0.22	4.11
2540	0.219178	2.373857	2.428824	2.483791	11.00	0.54	6.28
2550	0.219178	2.302373	2.357340	2.412307	11.95	0.67	6.94
2575	0.219178	2.124882	2.179849	2.234816	13.20	1.10	8.80
2600	0.219178	1.949106	2.004073	2.059040	14.80	1.76	10.99
2650	0.219178	1.602568	1.657535	1.712502	19.95	4.13	16.46
2725	0.219178	1.094832	1.149799	1.204766	29.92	12.23	27.74
2825	0.219178	0.439168	0.494135	0.549103	51.10	37.78	49.91
2850	0.219178	0.278879	0.333846	0.388814	58.00	47.68	57.10
2865	0.219178	0.183379	0.238346	0.293314	65.20	54.36	61.80
2425	0.295890	2.819513	2.883379	2.947245	10.76	0.16	3.16
2500	0.295890	2.342589	2.406455	2.470321	14.75	0.70	6.54
2675	0.295890	1.283205	1.347071	1.410937	32.50	9.91	24.37
2850	0.295890	0.290978	0.354844	0.418710	69.79	55.32	64.91
2860	0.295890	0.236134	0.300000	0.363866	76.22	59.70	68.22

The t -statistics equals -4.4919 with degree of freedom 38, and its p value is 0.00019. Thus, we accept H_1 , i.e., $E(|P_2 - P|) < E(|P_1 - P|)$. That is, the prices of put options computed by Theorem 3 are far nearer to their actual prices than those computed by the Black–Scholes model.

Therefore, our pricing formulas in Theorems 2 and 3 have less absolute errors than those of the Black–Scholes model for both call options and put options. That is, the fitting effect of our pricing formulas is superior to that of the Black–Scholes model.

6. Conclusion

In this study, we obtain the pricing formula of European options, including European call option and European put option, in a risk-averse market. Corollaries of our obtained results improve the Black–Scholes model owing to its much weaker conditions. It follows from our obtained results that European option value depends on the drift coefficient μ of its underlying security, which does not display in the Black–Scholes model only because $\mu = r$ in a risk-neutral financial market according to the no-arbitrage opportunity principle. Empirical analyses show that the fitting effect of our pricing formulas is superior to that of the Black–Scholes model.

Appendix

A. Proof of Theorem 2

In order to prove Theorem 2, we need the following lemma.

Lemma A.1. Assuming $\ln X \sim N(\mu, \sigma^2)$, then for any real number $m \in \mathbb{R}$ and positive real number $K \in \mathbb{R}^+$, it follows that

$$E[X^m 1_{\{X \geq K\}}] = e^{m\mu + (1/2)m^2\sigma^2} \Phi\left(\frac{1}{\sigma}\left(\ln \frac{1}{K} + \mu + m\sigma^2\right)\right). \quad (\text{A.1})$$

Proof. If $\ln X \sim N(\mu, \sigma^2)$, denote the density function of X by $f(x; \mu, \sigma)$, and then

$$f(x; \mu, \sigma) = \begin{cases} \frac{1}{x\sigma\sqrt{2\pi}} \exp\left\{-\frac{(\ln x - \mu)^2}{2\sigma^2}\right\}, & x > 0, \\ 0, & x \leq 0. \end{cases} \quad (\text{A.2})$$

For any real number $m \in \mathbb{R}$ and positive real number $K \in \mathbb{R}^+$, it follows that

$$\begin{aligned} E[X^m 1_{\{X \geq K\}}] &= \int_{-\infty}^{+\infty} x^m 1_{\{x \geq K\}} f(x; \mu, \sigma^2) dx \\ &= \int_K^{+\infty} x^m \frac{1}{x\sigma\sqrt{2\pi}} \exp\left\{-\frac{(\ln x - \mu)^2}{2\sigma^2}\right\} dx \\ &= \int_{\ln K}^{+\infty} e^{my} \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(y - \mu)^2}{2\sigma^2}\right\} dy \\ &= e^{m\mu + (1/2)m^2\sigma^2} \int_{\ln K}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{\left[y - (\mu + m\sigma^2)\right]^2}{2\sigma^2}\right\} dy \\ &= e^{m\mu + (1/2)m^2\sigma^2} \int_{1/\sigma[\ln K - (\mu + m\sigma^2)]}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2}\right\} dz \\ &= e^{m\mu + (1/2)m^2\sigma^2} \left(1 - \Phi\left(\frac{1}{\sigma}[\ln K - (\mu + m\sigma^2)]\right)\right) \\ &= e^{m\mu + (1/2)m^2\sigma^2} \Phi\left(\frac{1}{\sigma}\left(\ln \frac{1}{K} + \mu + m\sigma^2\right)\right). \end{aligned} \quad (\text{A.3})$$

The proof is complete. \square

Noting that the underlying asset follows a geometric Brownian motion with drift coefficient μ and diffusion coefficient $\sigma > 0$ and the current price of underlying asset is S_0 , it follows from Itô formula that

$$S_\tau = S_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)\tau + \sigma W_\tau\right\}, \quad (\text{A.4})$$

where $W = \{W_t, t \geq 0\}$ is the standard Wiener process, so

$$\ln S_\tau \sim N\left(\ln S_0 + \left(\mu - \frac{1}{2}\sigma^2\right)\tau, \sigma^2\tau\right). \quad (\text{A.5})$$

It yields from Lemma A.1 and (A.5) that

$$E[1_{\{S_\tau \geq K\}}] = \Phi(d_1), \quad (\text{A.6})$$

$$E[S_\tau 1_{\{S_\tau \geq K\}}] = S_0 e^{\mu\tau} \Phi(d_2), \quad (\text{A.7})$$

$$E[S_\tau^2 1_{\{S_\tau \geq K\}}] = S_0^2 e^{(\mu + \sigma^2)\tau} \Phi(d_3). \quad (\text{A.8})$$

It follows from (A.6) to (A.8) that

$$\begin{aligned}
E[(S_\tau - K)^+] &= E[(S_\tau - K)1_{\{S_\tau \geq K\}}], \\
&= E[S_\tau 1_{\{S_\tau \geq K\}}] - KE[1_{\{S_\tau \geq K\}}] \quad (\text{A.9}) \\
&= S_0 e^{\mu\tau} \Phi(d_2) - K\Phi(d_1),
\end{aligned}$$

Furthermore, it obtains from (A.9) and (A.10) that

$$\begin{aligned}
E[(S_\tau - K)^+)^2] &= E[(S_\tau - K)^2 1_{\{S_\tau \geq K\}}], \\
&= E[S_\tau^2 1_{\{S_\tau \geq K\}}] - 2KE[S_\tau 1_{\{S_\tau \geq K\}}] \\
&\quad + K^2 E[1_{\{S_\tau \geq K\}}] \\
&= S_0^2 e^{(\mu+\sigma^2)\tau} \Phi(d_3) - 2KS_0 e^{\mu\tau} \Phi(d_2) \\
&\quad + K^2 \Phi(d_1). \quad (\text{A.10})
\end{aligned}$$

$$\begin{aligned}
\text{std}((S_\tau - K)^+) &= \text{sqrt}\left\{E[(S_\tau - K)^+)^2] - E^2[(S_\tau - K)^+]\right\}, \\
&= \text{sqrt}\left\{S_0^2 e^{2\mu\tau} \left(e^{\sigma^2\tau} \Phi(d_3) - \Phi^2(d_2)\right) - KS_0 e^{\mu\tau} \Phi(d_2) \Phi(-d_1) + K^2 \Phi(d_1) \Phi(-d_1)\right\}. \quad (\text{A.11})
\end{aligned}$$

Thus, it yields from Proposition 1, (A.9), and (A.11) that

$$\begin{aligned}
C(S_0, K, r, \sigma, \tau, \lambda) &= e^{-r\tau} \{E[(S_\tau - K)^+] + \lambda \cdot \text{std}((S_\tau - K)^+)\}, \\
&= S_0 e^{(\mu-r)\tau} \Phi(d_2) - K e^{-r\tau} \Phi(d_1) + \lambda e^{-r\tau} \cdot \text{sqrt}\left\{S_0^2 e^{2\mu\tau} \left(e^{\sigma^2\tau} \Phi(d_3) - \Phi^2(d_2)\right) - KS_0 e^{\mu\tau} \Phi(d_2) \Phi(-d_1) \right. \\
&\quad \left. + K^2 \Phi(d_1) \Phi(-d_1)\right\}. \quad (\text{A.12})
\end{aligned}$$

The proof is complete. \square

$$E[X^m 1_{\{X \leq K\}}] = e^{m\mu + (1/2)m^2\sigma^2} \Phi\left(-\frac{1}{\sigma} \left(\ln \frac{1}{K} + \mu + m\sigma^2\right)\right). \quad (\text{B.1})$$

B. Proof of Theorem 3

In order to prove Theorem 3, we first present another lemma as follows.

Lemma B.1. Assuming $\ln X \sim N(\mu, \sigma^2)$, then for any real number $m \in \mathbb{R}$ and positive real number $K \in \mathbb{R}^+$, it follows that

Proof. If $\ln X \sim N(\mu, \sigma^2)$, denote the density function of X by $f(x; \mu, \sigma)$, and then

$$f(x; \mu, \sigma) = \begin{cases} \frac{1}{x\sigma\sqrt{2\pi}} \exp\left\{-\frac{(\ln x - \mu)^2}{2\sigma^2}\right\}, & x > 0, \\ 0, & x \leq 0. \end{cases} \quad (\text{B.2})$$

For any real number $m \in \mathbb{R}$ and positive real number $K \in \mathbb{R}^+$, it follows that

$$\begin{aligned}
 E[X^m 1_{\{X \leq K\}}] &= \int_{-\infty}^{+\infty} x^m 1_{\{x \leq K\}} f(x; \mu, \sigma^2) dx, \\
 &= \int_{-\infty}^K x^m \frac{1}{x\sigma\sqrt{2\pi}} \exp\left\{-\frac{(\ln x - \mu)^2}{2\sigma^2}\right\} dx \\
 &= \int_{-\infty}^{\ln K} e^{my} \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(y - \mu)^2}{2\sigma^2}\right\} dy \\
 &= e^{m\mu + (1/2)m^2\sigma^2} \int_{-\infty}^{\ln K} e^{my} \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{[y - (\mu + m\sigma^2)]^2}{2\sigma^2}\right\} dy \\
 &= e^{m\mu + (1/2)m^2\sigma^2} \int_{-\infty}^{1/\sigma[\ln K - (\mu + m\sigma^2)]} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2}\right\} dz \\
 &= e^{m\mu + (1/2)m^2\sigma^2} \Phi\left(\frac{1}{\sigma}[\ln K - (\mu + m\sigma^2)]\right) \\
 &= e^{m\mu + (1/2)m^2\sigma^2} \Phi\left(-\frac{1}{\sigma}\left(\ln \frac{1}{K} + \mu + m\sigma^2\right)\right).
 \end{aligned} \tag{B.3}$$

The proof is complete. \square

Noting that the underlying asset follows a geometric Brownian motion with drift coefficient μ and diffusion coefficient $\sigma > 0$ and the current price of underlying asset is S_0 , it follows from Itô formula that

$$S_\tau = S_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)\tau + \sigma W_\tau\right\}, \tag{B.4}$$

where $W = \{W_t, t \geq 0\}$ is a standard Wiener process, so

$$\ln S_\tau \sim N\left(\ln S_0 + \left(\mu - \frac{1}{2}\sigma^2\right)\tau, \sigma^2\tau\right). \tag{B.5}$$

It yields from Lemma B.1 and (B.5) that

$$E[1_{\{S_\tau \leq K\}}] = \Phi(-d_1), \tag{B.6}$$

$$E[S_\tau 1_{\{S_\tau \leq K\}}] = S_0 e^{\mu\tau} \Phi(-d_2), \tag{B.7}$$

$$E[S_\tau^2 1_{\{S_\tau \leq K\}}] = S_0^2 e^{(\mu + \sigma^2)\tau} \Phi(-d_3). \tag{B.8}$$

It follows from (B.6) to (B.8) that

$$\begin{aligned}
 E[(K - S_\tau)^+] &= E[(K - S_\tau) 1_{\{S_\tau \leq K\}}], \\
 &= KE[1_{\{S_\tau \leq K\}}] - E[S_\tau 1_{\{S_\tau \leq K\}}] \\
 &= K\Phi(-d_1) - S_0 e^{\mu\tau} \Phi(-d_2),
 \end{aligned} \tag{B.9}$$

$$\begin{aligned}
 E[((K - S_\tau)^+)^2] &= E[(K - S_\tau)^2 1_{\{S_\tau \leq K\}}], \\
 &= E[S_\tau^2 1_{\{S_\tau \leq K\}}] - 2KE[S_\tau 1_{\{S_\tau \leq K\}}] \\
 &\quad + K^2 E[1_{\{S_\tau \leq K\}}] \\
 &= S_0^2 e^{(\mu + \sigma^2)\tau} \Phi(-d_3) - 2KS_0 e^{\mu\tau} \Phi(-d_2) \\
 &\quad + K^2 \Phi(-d_1).
 \end{aligned} \tag{B.10}$$

Furthermore, it obtains from (B.9) and (B.10) that

$$\begin{aligned}
 \text{std}((K - S_\tau)^+) &= \text{sqr}t\left\{E[((K - S_\tau)^+)^2] - E^2[(K - S_\tau)^+]\right\}, \\
 &= \text{sqr}t\left\{S_0^2 e^{2\mu\tau} \left(e^{\sigma^2\tau} \Phi(-d_3) - \Phi^2(-d_2)\right) - KS_0 e^{\mu\tau} \Phi(-d_2) \Phi(d_1) + K^2 \Phi(-d_1) \Phi(d_1)\right\}.
 \end{aligned} \tag{B.11}$$

Thus, it yields from (B.9) and (B.11) that

$$\begin{aligned}
 P(S_0, K, r, \sigma, \tau, \lambda) &= e^{-r\tau} \{E[(K - S_\tau)^+] + \lambda \cdot \text{std}((K - S_\tau)^+)\}, \\
 &= Ke^{-r\tau} \Phi(-d_1) - S_0 e^{(\mu - r)\tau} \Phi(-d_2) + \lambda e^{-r\tau} \cdot \text{sqr}t\left\{S_0^2 e^{2\mu\tau} \left(e^{\sigma^2\tau} \Phi(-d_3) - \Phi^2(-d_2)\right) - KS_0 e^{\mu\tau} \Phi(-d_2) \Phi(d_1) \right. \\
 &\quad \left. + K^2 \Phi(-d_1) \Phi(d_1)\right\}.
 \end{aligned} \tag{B.12}$$

The proof is complete. \square

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

The Optimal Reinsurance Strategy under Conditional Tail Expectation (CTE) and Wang's Premium Principle

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In this study, we take the conditional tail expectation (CTE) as the constraint condition and consider the optimal reinsurance issues under Wang's premium principle in general insurance contracts. With the confidence level and the distortion function in Wang's premium principle given by the insurer in advance, a threshold can be obtained. When the insurer's risk tolerance level is greater than this value, the optimal reinsurance is a proportional reinsurance in which the deductible equals to this value, else the optimal form of reinsurance is a stop-loss reinsurance. Corresponding numerical examples and economic explanations are also given.

1. Introduction

Reinsurance is an important tool for the insurer to manage risks; through reinsurance, the insurer can mitigate the underwriting risk, thereby facilitating more effective risk management. For more than half a century, academics have established lots of optimal reinsurance models from the perspective of insurers and have investigated optimal reinsurance strategies under different conditions. Borch [1] solved that, under the principle of expected premium and fixed reinsurance premiums, a stop-loss reinsurance is the optimal for an insurer if the objective function is to minimize the variance of the insurer's loss. Kenneth [2] also supposed that, under the principle of expected premiums, the optimal reinsurance that a risk-averse insurer desires to maximize the expected utility of his own wealth is a stop-loss reinsurance contract. With the rapid development of financial liberalization, a new risk measure approach VaR (value at risk) is widely employed by banks, insurance companies, other financial institutions, and market regulators. Cai and Tan [3] hypothesized that the form of insurance is a stop-loss reinsurance, and the minimality of gross loss of the insurer measured by VaR and CTE as the objective function, and the optimal deductible is calculated under the principle of expected premium. Hu et al. [4]

researched the calculation of the optimal retention of stop-loss reinsurance under the condition of incomplete information on the aggregate loss function of the insurance company, while minimizing the VaR risk metric, and contrasted with the results of Cai and Tan [3]. Kong et al. [5] studied the optimal reinsurance issues in which both insurers and reinsurers face risks and uncertainties under general premium principles. Chi [6] considered a type of premium principle $P_{(X)} = E[X] + g(\text{var}(X))$ (where g is an increasing function of $g(0) = 0$). This type of premium principle includes the variance premium principle and the standard deviation premium principle. Under this type of premium principle, Chi [6] measured the aggregate loss of the insurer by VaR and CTE and indicated that the stop-loss reinsurance with the upper bounded is optimal. While taking the principle of variance premium and standard deviation premium as an example, the calculation shows how to obtain the optimal deductible and the upper boundary of indemnity. Chi and Tan [7] considered another type of premium principle, including Wang's premium principle and Dutch premium principle, and utilized VaR and CTE to calculate the aggregate loss of the insurer, and the optimal reinsurance is still a stop-loss reinsurance with the upper boundary. Cai et al. [8] studied Pareto optimal problem of reinsurance counterparty under one category of

risk measure and gave a proof for the TVaR measure method. Putri et al. [9] considered the fixed premium under the expected premium standard. In order to minimize the risk of the insurer under VaR, they combined proportional reinsurance with stop loss reinsurance to solve the optimal reinsurance strategy. Liang and Young [10] researched the minimum probability of bankruptcy of insurance companies, and Matteo and Claudia [11] introduced time variables and considered the necessities to pay fixed costs when signing reinsurance contracts. Jiang et al. [12] considered the maximization of the interests of both insurer and reinsurer from the perspectives of expected utility maximization and risk minimization, and they found that the layer reinsurance is optimal.

Before introducing the framework of this paper, we first introduce the following notations. Let the insurer's possible loss X in the future periods of time be a nonnegative random variable, the probability density function is $f(x)$, and the survival function is $S_X(x) = P\{X > x\}$. In order to control the risk effectively, the insurer transfers the loss $I(X)$ to the reinsurer. In exchange for underwriting risks, the reinsurer charges the insurer a reinsurance premium P_X . There are many calculation criteria for reinsurance premiums, and one of the more commonly used premium principle is the expected premium principle. In this paper, the more general Wang's premium principle is applied. It was first proposed by Wang [13]; it comprises the net premium principle, quantile premium principle, and dual risk premium principle. It is defined as $P_X = \int_0^{+\infty} g(S_X(x))dx$, for $g: [0, 1] \rightarrow [0, 1]$ is a nondecreasing concave function which satisfies $g(x) \geq x$, $g(0) = 0$, and $g(1) = 1$. After the loss $I(X)$ ceded by the insurer, its retained loss function is $R(X) = X - I(X)$. Now, the insurer wants to control its retained losses within a certain spectrum because the measurement method only considers the risk of a certain quantile while ignoring the tail risk, but CTE considers the expectation of the entire tail risk. They are defined as follows: for a random variable, given a confidence level $1 - \alpha$ ($0 < \alpha < 1$), VaR is defined as

$\text{VaR}_\alpha(X) = \inf\{x: \Pr\{X > x\} \leq \alpha\}$ and CTE is defined as $\text{CTE}_\alpha(X) = E[X|X \geq \text{VaR}_\alpha(X)]$.

The model constructed in this paper aims to minimize the reinsurance premium purchased by the insurer while controlling the risk of the insurer within a certain spectrum. Among them, the insurer's risk measure is CTE under this risk measure, the maximum risk that the insurer can accept is N , that is, the constraint is $\text{CTE}_\alpha(R(X)) \leq N$. The objective function is $\min P_X = \min \int_0^{+\infty} g(S_{I(X)}(x))dx$. Because the larger the insurer's retention risk, the smaller the corresponding ceded risk, which results in a smaller premium, so there is an optimal solution as $\text{CTE}_\alpha(R(X)) = N$.

2. Optimal Reinsurance

In general insurance contracts, there are stop-loss reinsurance, quota-share reinsurance, layer reinsurance, and proportional reinsurance. The reinsurance used in this paper can include all forms of general insurance contracts. The specific notation is as follows:

$$I(x) = l_1(x-a)_+ - l_2(x-b)_+, \quad (1)$$

$$(x-b)_+ = \begin{cases} 0, & 0 \leq x < a, \\ l_1(x-a), & a \leq x < b, \\ l_1(x-a) - l_2(x-b), & x \geq b, \end{cases}$$

where $(l_1 \geq l_2)$.

Insurer's retention risk is

$$R(x) = x - l_1(x-a)_+ + l_2(x-b)_+, \quad (2)$$

$$(x-b)_+ = \begin{cases} x, & 0 \leq x < a, \\ x - l_1(x-a), & a \leq x < b, \\ x - l_1(x-a) + l_2(x-b), & x \geq b, \end{cases}$$

where $(l_1 \geq l_2)$.

When the insurer's ceded loss function has the form of (1), the corresponding premium expression can be simplified as follows:

$$\begin{aligned} P_X &= \int_0^{+\infty} g(S_{I(X)}(x))dx = \int_0^{+\infty} g(P\{I(X) > x\})dx \\ &= \int_0^{+\infty} g(P\{0 > x, X < a\} + P\{l_1(X-a) > x, a \leq X < b\} + P\{l_1(X-a) - l_2(X-b) > x, X \geq b\})dx \\ &= \int_0^{l_1(b-a)} g\left(P\left\{X > \frac{x}{l_1} + a\right\}\right)dx + \int_{l_1(x-a)}^{+\infty} g\left(P\left\{X > \frac{x + l_1 a - l_2 b}{l_1 - l_2}\right\}\right)dx \\ &= l_1 \int_a^{+\infty} g(S_X(x))dx - l_2 \int_b^{+\infty} g(S_X(x))dx. \end{aligned} \quad (3)$$

Combine the objective function and the constraint conditions together with the Lagrange function:

$$\begin{aligned} L(a, b, l_1, l_2, \lambda) &= \int_0^{+\infty} g(S_{I(X)}(x))dx + \lambda [\text{CTE}_\alpha(R(X)) - N] \\ &= l_1 \int_a^{+\infty} g(S_X(x))dx - l_2 \int_b^{+\infty} g(S_X(x))dx \\ &\quad + \lambda [\text{CTE}_\alpha(R(X)) - N]. \end{aligned} \quad (4)$$

While solving this equation, because of the magnitude relationship between a, b and $\text{VaR}_\alpha(X)$ is unknown, the value of $\text{CTE}_\alpha(R(X))$ is uncertain. Therefore, the discussion should be divided into the following three circumstances.

Case 1. $\text{VaR}_\alpha(X) \geq b$:

$$\begin{aligned} L(a, b, l_1, l_2, \lambda) &= l_1 \int_a^{+\infty} g(S_X(x))dx - l_2 \int_b^{+\infty} g(S_X(x))dx \\ &\quad + \lambda \left[\frac{1}{\alpha} \int_{\text{VaR}_\alpha(X)}^{+\infty} (x - l_1(x - a) + l_2(x - b))f(x)dx - N \right]. \end{aligned} \quad (5)$$

Solving the above equation,
$$\begin{cases} b = a \\ a = a_0 \\ l_1 - l_2 = l \\ \lambda = g(S_X(a_0)) \end{cases}.$$

The ceded loss function is $I(x) = l(x - a_0)_+$, where a_0 and l satisfies:

$$\begin{aligned} \int_{a_0}^{+\infty} g(S_X(x))dx - \frac{g(S_X(a_0))}{\alpha} \int_{\text{VaR}_\alpha(X)}^{+\infty} (x - a_0)f(x)dx &= 0, \\ \int_{\text{VaR}_\alpha(X)}^{+\infty} (x - l(x - a_0))f(x)dx &= \alpha N. \end{aligned} \quad (6)$$

Because $l \in [0, 1]$, the value range of N , $N \in [a_0, \text{CTE}_\alpha(X)]$ can be calculated in equation (6). When $N < a_0$, since there is no stable point for solving the equation, only the boundary $l = 1, a = N$ can be taken. At this time, the ceded loss function is that $I(x) = (x - N)_+$; when

$N \geq \text{CTE}_\alpha(X) \geq \text{CTE}_\alpha(R(X))$, the insurer can bear the risk without purchasing reinsurance, that is, the optimal strategy is no reinsurance, and the reinsurance premium is 0.

Case 2. $a \leq \text{VaR}_\alpha(X) < b$:

$$\begin{aligned} L(a, b, l_1, l_2, \lambda) &= l_1 \int_a^{+\infty} g(S_X(x))dx - l_2 \int_b^{+\infty} g(S_X(x))dx \\ &\quad + \lambda \left[\frac{1}{\alpha} \int_{\text{VaR}_\alpha(X)}^b (x - l_1(x - a))f(x)dx + \frac{1}{\alpha} \int_b^{+\infty} (x - l_1(x - a) + l_2(x - b))f(x)dx - N \right]. \end{aligned} \quad (7)$$

Solving the above equation,
$$\begin{cases} b \rightarrow +\infty \\ a = a_0 \\ l_1 = l \\ l_2 = 0 \\ \lambda = g(S_X(a_0)) \end{cases}.$$
 The ceded

loss function is $I(x) = l(x - a_0)_+$, where a_0 and l satisfies

$$\begin{aligned} \int_{a_0}^{+\infty} g(S_X(x))dx - \frac{g(S_X(a_0))}{\alpha} \int_{\text{VaR}_\alpha(X)}^{+\infty} (x - a_0)f(x)dx &= 0, \\ \int_{\text{VaR}_\alpha(X)}^{+\infty} (x - l(x - a_0))f(x)dx &= \alpha N. \end{aligned} \quad (8)$$

Because $l \in [0, 1]$, the value range of N , $N \in [a_0, \text{CTE}_\alpha(X)]$ can be calculated in equation (8). When $N < a_0$, since there is no stable point for solving the equation, only the boundary $l = 1$ and $a = N$ can be taken. At this time, the ceded loss function is that $I(x) = (x - N)_+$ when $N \geq \text{CTE}_\alpha(X) \geq \text{CTE}_\alpha(R(X))$, the insurer can bear the risk

without purchasing reinsurance, that is, the optimal strategy is no reinsurance, and the reinsurance premium is 0. The result is the same as in the first case.

Case 3. $a > \text{VaR}_\alpha(X)$:

$$L(a, b, l_1, l_2, \lambda) = l_1 \int_a^{+\infty} g(S_X(x))dx - l_2 \int_b^{+\infty} g(S_X(x))dx + \lambda \left[\frac{1}{\alpha} \int_{\text{VaR}_\alpha(X)}^a x f(x)dx + \frac{1}{\alpha} \int_a^b (x - l_1(x - a))f(x)dx + \frac{1}{\alpha} \int_b^{+\infty} (x - l_1(x - a) + l_2(x - b))f(x)dx - N \right]. \quad (9)$$

Solving the above equation, we can get two sets of solutions:

- (i) $a = b = T(N \geq \text{CTE}_\alpha(X))$, the insurer is no need to buy reinsurance
- (ii) $\begin{cases} b = a \text{ or } b \rightarrow +\infty \\ a = a_0 \\ l_1 - l_2 = l \\ \lambda = \alpha \cdot (g(S_X(a_0))/S_X(a_0)) \end{cases}$, where a_0 and l

satisfies $\int_{\text{VaR}_\alpha(X)}^{+\infty} (x - l(x - a_0))f(x)dx = \alpha N$ ($\text{VaR}_\alpha(X) \leq N < \text{CTE}_\alpha(X)$).

In this case, the optimal ceded loss function is $I(x) = l(x - a_0)_+$.

Remark 1. Especially, if the internal constant $g''(x) = 0$ in $(0, \alpha)$ is established and if N and $g(x)$ are given, P_X will be a constant value. a, b, l_1, l_2 only need to satisfy $\text{CTE}_\alpha(R(X)) = N$. At this time, $I(x) = l(x - a_0)_+$ and $I(x) = (x - a)_+$ that meet the condition $\text{CTE}_\alpha(R(X)) = N$ both can be used as the optimal reinsurance strategy, and the value of reinsurance premiums is the same.

Theorem 1. For any given N and $g(x)$, a special point $a_0 \in [0, \text{VaR}_\alpha(X)]$ can be calculated, whereas a_0 satisfies

$$\int_{a_0}^{+\infty} g(S_X(x))dx - \frac{g(S_X(a_0))}{\alpha} \int_{\text{VaR}_\alpha(X)}^{+\infty} (x - a_0)f(x)dx = 0. \quad (10)$$

- (i) $0 \leq N < a_0$: the optimal ceded loss function is $I(x) = (x - N)_+$, that is, the optimal reinsurance is the stop-loss reinsurance with a deductible of N .
- (ii) $a_0 \leq N < \text{CTE}_\alpha(X)$: the optimal ceded loss function is $I(x) = l(x - a_0)_+$, where a_0 and l satisfy the following constraints:

$$\int_{a_0}^{+\infty} g(S_X(x))dx - \frac{g(S_X(a_0))}{\alpha} \int_{\text{VaR}_\alpha(X)}^{+\infty} (x - a_0)f(x)dx = 0, \quad (11)$$

$$\int_{\text{VaR}_\alpha(X)}^{+\infty} (x - l(x - a_0))f(x)dx = \alpha N.$$

If the internal constant $g''(x) = 0$ in $x \in (0, \alpha)$ is established, it can be satisfied. a, b, l_1, l_2 only need to satisfy $\text{CTE}_\alpha(R(X)) = N$. At this time, the ceded loss function $I(x) = l(x - a_0)_+$, $I(x) = (x - a)_+$ that meet the condition $\text{CTE}_\alpha(R(X)) = N$ can be used as the optimal reinsurance strategy.

- (iii) $N \geq \text{CTE}_\alpha(X)$: in this situation, the insurer can bear the risk without purchasing reinsurance contract, so the optimal strategy is not to buy reinsurance, and the reinsurance premium is zero.

Remark 2. Especially, when $g(x) = x$, at that time, Wang's premium principle altered to the net premium principle. Calculating the following formula $\int_{a_0}^{+\infty} g(S_X(x))dx - (g(S_X(a_0))/\alpha) \int_{\text{VaR}_\alpha(X)}^{+\infty} (x - a_0)f(x)dx = 0$ can be solved as $a_0 = \text{VaR}_\alpha(X)$, for a $N \in [0, \text{CTE}_\alpha(X)]$ which is given by the insurer, and $I(x) = (x - a)_+$ can be selected as the optimal reinsurance strategy. This result is consistent with the conclusion in [14].

3. Numerical Examples

Assume that the insurer's loss variable X obeys the Pareto distribution. For any $x \geq 0$, the corresponding survival function

$$S_X(x) = \left(\frac{200}{200 + x} \right)^3, \quad x \geq 0. \quad (12)$$

The mean value of the variable X is 100. We assume that the confidence level of the insurance company to measure the risk

TABLE 1: When $g(x) = 1 - (1 - x)^3$, the ceded functions and premiums correspond to different values of N .

α	$\text{VaR}_\alpha(X)$	$\text{CTE}_\alpha(X)$	N	a	l	P
0.01	728.32	1144.48	400	400	1	32.84
			726.44	726.44	1	13.93
			900	726.44	0.63	8.74
0.025	483.99	825.99	200	200	1	71.35
			480.48	480.48	1	25.65
			700	480.48	0.65	16.78
0.1	230.89	441.53	100	100	1	118.51
			221.82	221.82	1	64.78
			300	221.82	0.65	42.11

TABLE 2: When $g(x) = x$, the ceded functions and premiums correspond to different values of N .

α	$\text{VaR}_\alpha(X)$	$\text{CTE}_\alpha(X)$	N	a	l	P
0.01	728.32	1144.48	400	400	1	11.11
			726.44	726.44	1	4.66
			900	970 (726)	1 (0.63)	2.92
0.025	483.99	825.99	200	200	1	25.00
			480.48	480.48	1	8.64
			700	927 (480)	1 (0.37)	3.15
0.1	230.89	441.53	100	100	1	44.44
			221.84	221.82	1	22.48
			300	323 (222)	1 (0.65)	14.63

is $1 - \alpha$; according to the insurer's different confidence levels, when the retained risk is less than the insurer's maximum risk tolerance, solve the reinsurance strategy when the reinsurance premium is the smallest. Since the different distortion functions of Wang's premium can get the optimal ceded loss function under the corresponding premium principles, this paper considers the following two distortion functions:

- (1) When $g(x) = 1 - (1 - x)^3$, then P_X is the dual risk premium principle
- (2) When $g(x) = x$, then P_X is the principle of net premium

According to the above two premium principles, Theorem 1 and (α, N) under different risk levels, the optimal reinsurance form can be obtained, as shown in Tables 1 and 2.

When $g(x) = x$, the proportional coefficient l can always get 1, so the expression of the optimal ceded loss function is $I(x) = (x - a)_+$, when $0 \leq N \leq \text{VaR}_\alpha(X)$, $a = N$, whereas $\text{VaR}_\alpha(X) \leq N \leq \text{CTE}_\alpha(X)$, and there was $\int_{\text{VaR}_\alpha(X)}^a x f(x) dx + \int_a^{+\infty} a f(x) dx = \alpha N$.

Since the expected premium principle is similar with the net premium principle, the net premium is merely multiplied by a coefficient. By contrasting different risk levels, it can be discovered that the premium of the dual risk premium principle is significantly higher in various situations.

4. Conclusion

The purpose of this paper is to control the insurer's risk level within his own capacity and choose a reinsurance contract

that minimizes the reinsurance premium. In this paper, a threshold can be obtained based on the distortion function in Wang's premium principle and the insurer's confidence level. When the insurer's risk tolerance level N is less than this value, the insurer will choose a stop-loss reinsurance with a deductible of N , and when the insurer's risk tolerance level N is greater than this threshold, the insurer will choose a proportional stop-loss reinsurance with this threshold as the deductible. With deliberate consideration, this conclusion is reasonable. Since Wang's premium calculation guidelines charge higher premiums for tail risk, when the insurer's risk tolerance is large, the tail risk will not be fully ceded, but a proportional stop-loss reinsurance will be considered. When the insurer's risk tolerance is low, even if the insurer knows that the reinsurance premium for tail risk is high, the insurer still must choose reinsurance to control the risk.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Premium Valuation of the Pension Benefit Guaranty Corporation with Regime Switching

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The Pension Benefit Guaranty Corporation (PBGC) provides insurance coverage for single-employer and multiemployer pension plans in private sector. It has played an important role in protecting the retirement security for over 1.5 million people since it was established about half a decade ago. PBGC collects insurance premiums from employers that sponsor insured pension plans for its coverage and receives funds from pension plans that it takes over. To address the issue of underfunded plans that the PBGC has, this work studies how to evaluate risk-based premiums for the PBGC. Inspired by a couple of existing work in which the premature termination of pension fund and distress termination of sponsor assets are analyzed separately, our work examines the two types of terminations under one framework and considers the occurrence of each termination dynamically. Given that market regime might have a big impact on the dynamics of both pension fund and sponsor's assets, we thus formulate our model using a continuous-time two-state Markov chain in which bull market and bear market are delineated. We thus formulate our model using a continuous-time two-state Markov Chain in which bull market and bear market are delineated. In other words, the pension fund and sponsor assets are market dependent in our work. Given that this additional uncertainty described by regime switching makes the market incomplete, we therefore utilize the Esscher transform to determine an equivalent martingale measure and apply the risk neutral pricing method to obtain the closed-form expressions for premium of PBGC. In addition, we carry out numerical analysis to demonstrate our results and observe that premium increases according to the retirement benefit irrespective of the type of terminations. In comparison to the case of early distress termination of sponsor assets, the premium goes up more quickly when premature termination of pension funds occurs first due to the fact that pension fund is the first venue of retirement security. Furthermore, we look at how the premium changes with respect to other key parameters as well and make some detailed observations in the section of numerical analysis.

1. Introduction

Different from defined contribution pension plans, where employees themselves bear the investment risk and where employees are not sure about the amount of benefit they would receive after retirement, sponsors of defined benefit plans offer their employees a definite amount of benefit by the time of retirement, regardless of the performance of the underlying investment pool. In contrast, a defined benefit plan gives employees a better sense of security, since people under this plan are always eligible to receive benefit as long as they are alive and pensioners know their benefit level

ahead of time. However, as far as the plan's sponsor is concerned, offering defined benefit plan is a huge financial commitment. One immediate problem is that if a private company offering defined benefit pension plan goes bankrupt, it would be difficult for its employees to get protection in this case. Without having insurance protection from elsewhere, employees in this troubled pension plan would suffer a lot. Because of this very reason, a federal agency called the Pension Benefit Guaranty Corporation (PBGC) was created by the Employee Retirement and Income Security Act of 1974 (ERISA) to protect defined benefit pension plans in private sector. Since its creation in 1974,

more than 1.4 million workers have relied on PBGC for their retirement income and most people receive the full benefit that they are expected to earn. For any plans covered by PBGC, sponsor of the plan pays premium to the PBGC. In return, the PBGC steps in and provides coverage to the pension plan if needed. However, the PBGC has most often operated at a deficit and U.S. Accountability Office thus designated the PBGC's single-employer program as high risk in July 2003 and added its multiemployer program as high risk in January 2009 due to its huge amount of deficit. Over the course of time, the problem becomes even more serious, the PBGC had a deficit of 35.6 billion in the year of 2013, and its deficit was 61.8 billion and 76.4 billion, respectively, at the end of fiscal year of 2014 and 2015. Annual report of the PBGC shows that there was a deficit of 79.4 billion in the year of 2016. As far as premium is concerned, the PBGC's premium rates are a key component of its funding and it has drawn attention of many scholars. [1–4] show that it is mispricing PBGC to charge flat premium and the mispricing pension insurance harmfully motivate the company. Stevart [5] gives the economical rationale behind and discusses the resulting consequence when the premium is mispriced. Besides, the level of premiums has not kept pace with the risks that PBGC insures against. There have been proposals that legislative reform should authorize the PBGC board to adjust premiums and redesign a risk-based premium structure, such as consideration of a sponsor's financial health. These recommendations are also the motivations of our work. There have been a series of work on the premium calculation of the PBGC. The pioneering work can be traced back to [6]. In [6], it is assumed that PBGC is the first line of defense for the deficit of pension fund. Marcus [7] used contingent forward to model the PBGC's liability and PBGC is assumed to be allowed to gain surpluses from over funded terminated plans, which potentially means the liability can be negative. Levis and Pennacchi [8] stochastically modeled the PBGC's liability using contingent put option. The major problem of their work is that the maturity of the PBGC's insurance is assumed to be known and the fact that the pension fund will be terminated prematurely due to underfunding has not been taken into consideration. Kalra and Jain [9] firstly considered the premature termination of pension fund. Chen [10] extended their results under the assumption that the PBGC functions as a second line of defense, that is to say, the PBGC covers only the residual deficits of the pension fund that the sponsoring company fails to cover, and in the other related paper, Chen and Uzelac [11] examined the case that the distress termination is triggered due to sponsor's underfunding.

The differences between others' contributions and our work are evident. First, different from [9–11], we include the market regime for the first time into the premium calculations of PBGC's insurance program. We not only assume that the price dynamics of risky assets depend on the states of the economy but also assume that the proportion of pension fund invested in the risky asset also depends on the states of the economy. It is well known that the popularity of regime switching model has been supported by many empirical

evidences for a while. The switching models reflect the changes of macroeconomic environment, such as the adjustment of economic structure, the change of market system, and the business cycle, which are exemplified by a continuous-time Markov chain. The market variable follows one risk model when one state of the economy is specified and transfers to other models when the market scheme transfers between different states. Some applications of regime switching models can be seen in [12, 13]. Second, different from [6, 7], Levis and Pennacchi [8], and others, we consider the risk-based premium that the PBGC should charge. In addition to examining the risk of premature termination for pension fund, we study the financial risk of sponso's assets as well. Finally, explicit solutions of the premiums are provided and numerical analysis is also carried out to demonstrate our results.

The rest of the paper is arranged as follows. Section 2 presents the problem formulation. Section 3 shows the Esscher transform under regime switching. Section 4 introduces the case when pension is underfunded and discusses the case that sponsor asset suffers from distress termination. Section 5 demonstrates how the premiums are calculated based on the two scenario cases. Section 6 provides some numerical examples to illustrate the effect of regime switching on the premium of PBGC. Section 7 concludes the paper with some further remarks.

2. Formulation

This section presents the formulation of the model of our interest. We begin with notation and assumptions. Given a probability space (Ω, \mathcal{F}, P) , we use R as a representative of beneficiary's retirement time. Let ${}_jP_R = P(T_R > j)$ be survival probability in which T_R is the random variable representing the time until death for a representative beneficiary retires at age R and r be long-term interest rate applying to a typical retiree (for example, we can use 30 years' rate of interest without loss of generality based on the fact that most people retire at age of 60 or older and according to data compiled by the Social Security Administration, the average longevity of people is close to 87) and B is the prescribed annual benefit, which depends on employee's years of -service, age, salary before retirement, and other possible factors. Because the focus of this work is on discussing how the PBGC charges the premiums under different scenario cases, we define the benefit that a typical beneficiary expects to receive similar to that in [10] and express it as below:

$$B_R = \sum_{j=R}^{\infty} {}_jP_R e^{-r(j-R)} B. \quad (1)$$

Note that we use a constant interest here to define the expected present value of the whole life annuity due starting at age R for two reasons. (1) It is both mathematically elegant and practically important to obtain a closed-form solution. (2) It is convenient to choose a conservative interest rate in the first place, which, accordingly, contributes to making proactive provisions regarding all the uncertainties.

We further assume that there are three funding sources for the beneficiary's annual benefit and the pension fund is the primary funding resource for the pension benefit; sponsor company provides secondary support; then, the PBGC contributes the rest. These assumptions hold throughout the entire paper. We assume further that the portfolio of pension fund consists of two kinds of assets. One is risk free and the other is risky asset. The growth rate of the risk-free asset is risk-free interest rate and that of the risky asset is its expected rate of return. More often than not, at a given time, the expected rate of return of the risky asset is higher than that of the risk-free asset, which gives a positive risk premium for the risky asset. Different from the classical portfolio selection assumption, we want to take the market regime into consideration to reflect that the dynamics of a given product are different under different market regimes. In this paper, we consider such a case that there are two possible market regimes, which are bull and bear markets, respectively. Mathematically speaking, we assume that there is a continuous-time two-state Markov chain $\alpha(t)$ taking values in $M = \{0, 1\}$ with generator

$$Q_{2 \times 2} = \begin{pmatrix} -\lambda_0 & \lambda_0 \\ \lambda_1 & -\lambda_1 \end{pmatrix}. \quad (2)$$

The Markov chain takes different values when the market is in different regimes. With this practical assumption, the risk-free interest rate and expected rate of return are functions of the market regime $\alpha(t)$. To be specific, the dynamics of the risk-free asset are

$$dx_1(t) = rx_1(t)dt, \quad (3)$$

where $x_1(0) = x_1$ and r is the risk-free interest rate. Moreover, the dynamics of risky asset are given as follows:

$$\begin{aligned} dx_2(t) &= \mu(\alpha(t))x_2(t)dt + \sigma(\alpha(t))x_2(t)dw_1(t), \\ x_2(0) &= x_2, \alpha(0) = i, \quad i = 0, 1, \end{aligned} \quad (4)$$

in which $w_1(t)$, a standard one-dimensional Brownian motion, is independent of $\alpha(t)$. Using $\pi(\alpha(t))$ to denote the proportion of pension fund invested in the risky asset and $1 - \pi(\alpha(t))$ to denote the proportion of pension fund invested in the risk-free asset, we can thus represent the dynamics of the total pension fund $x(t)$ as below:

$$\begin{aligned} dx(t) &= \frac{x(t)(1 - \pi(\alpha(t)))}{x_1(t)} dx_1(t) + \frac{x(t)\pi(\alpha(t))}{x_2(t)} dx_2(t), \\ x(t) &[(1 - \pi(\alpha(t)))r + \pi(\alpha(t))\mu(\alpha(t))dt + \pi(\alpha(t))\sigma(\alpha(t))dw_1(t)], \end{aligned} \quad (5)$$

where $x(0) = x_0$.

The proportion of pension fund invested in the risky asset $\pi(\alpha(t))$ is dependent on the $\alpha(t)$ and it means there will be different proportion of pension fund distributed to risk free as well as risky asset at different times. Note that our interest in this work is about evaluation of premium for the PBGC, so we would assume that there is a given $\pi(\alpha(t))$ at time t for the sake of calculation convenience. As far as the way to find out $\pi(\alpha(t))$, once the objective function is formulated, method of stochastic control can be utilized to find out the representation of $\pi(\alpha(t))$ and this could be one of our further works.

If the pension fund performs well enough, then all the benefit will be paid by the pension fund itself. If the pension fund is underfunded, then company's asset is a potential resource of paying benefit. The dynamics of employer's asset are given by

$$\frac{dy(t)}{y(t)} = \mu_y(\alpha(t))dt + \sigma_y(\alpha(t))\left(\rho dw_1(t) + \sqrt{1 - \rho^2} dw_2(t)\right), \quad (6)$$

where $y(0) = y_0$ and $w_2(t)$ is another standard Brownian motion and is independent of Brownian motion $w_1(t)$. We assume that the sponsoring corporation's assets are correlated to the pension fund's assets with a correlated coefficient

$\rho \in [-1, 1]$ and how the correlation coefficient affects the premiums was analyzed in [10].

Note that typically corporation has corporate debt; we therefore assume that at any time t , the sponsor company always has to pay its debt $\theta y_0 e^{\nu t}$ in which θ is its equity-debt ratio and ν is the predetermined constant to illustrate the growth rate of the corporate debt. It is practically reasonable to assume that company has higher priority to pay back its corporate debt.

The third part of funding resources is the Pension Benefit Guaranty Corporate (PBGC). Let S^p denote the possible contribution that PBGC makes. It is easy to see that S^p depends on the performance of pension fund and sponsor's assets. We will derive the expression of S^p and the premium that PBGC collects from sponsor company at the end of the following sections. To move forward, we will first find the risk neutral probability measure for the dynamic system with the help of Esscher transform in the following section.

3. Esscher Transform under Regime Switching

Since the additional uncertainty described by regime switching makes the market incomplete, there are infinitely many equivalent martingale measures. Here, we will adopt the Esscher transform to determine an equivalent martingale measure for pricing premium of the Pension Benefit

Guaranty Corporation. Esscher transform was first proposed by Gerber and Shiu [14], and it is widely used in the field of finance and insurance. For more details, refer to Bühlmann et al. [15, 16].

We denote $\{\mathcal{F}_t^{x_2}\}_{t \leq R}$ as the filtration generated by $\{x_2(t)\}_{t \leq R}$, $\{\mathcal{F}_t^y\}_{t \leq R}$ the filtration generated by $\{y(t)\}_{t \leq R}$, $\{\mathcal{F}_t^\alpha\}_{t \leq R}$ the filtration generated by $\{\alpha(t)\}_{t \leq R}$, and $\{\mathcal{G}_t\}_{t \leq R}$ the filtration generated by $\mathcal{F}_t^{x_2} \vee \mathcal{F}_t^y \vee \mathcal{F}_t^\alpha$, respectively. The regime-switching Esscher transform $P^* \sim \mathcal{F}$ on \mathcal{G}_t with respect to parameters $\varsigma_1(\alpha(\cdot))$ and $\varsigma_2(\alpha(\cdot))$ is given by

$$\frac{dP^*}{dP}|_{\mathcal{G}_t} = \frac{e^{\int_0^t \varsigma_1(\alpha(s))dw_1(s) + \int_0^t \varsigma_2(\alpha(s))dw_2(s)}}{E_{\mathcal{P}} \left[e^{\int_0^t \varsigma_1(\alpha(s))dw_1(s) + \int_0^t \varsigma_2(\alpha(s))dw_2(s)} | \mathcal{F}_t^\alpha \right]}. \quad (7)$$

Thanks to the well-known result established in [17, 18], the absence of arbitrage opportunities is essentially equivalent to the existence of equivalent martingale, under which the discount price process is a martingale. We know

$\{e^{-rt}x_2(t)\}$ and $\{e^{-rt}y(t)\}$ are martingales under the measure P^* . Thus, we have

$$\begin{aligned} \varsigma_1(\alpha(t)) &= \frac{r - \mu(\alpha(t))}{\sigma(\alpha(t))}, \\ \varsigma_2(\alpha(t)) &= \frac{r - \mu_y(\alpha(t))}{\sigma_y(\alpha(t))\sqrt{1-\rho^2}} - \frac{\rho}{\sqrt{1-\rho^2}} \frac{r - \mu(\alpha(t))}{\sigma(\alpha(t))}. \end{aligned} \quad (8)$$

According to Girsanov's theorem, we know

$$\begin{aligned} w_1^*(t) &= w_1(t) - \int_0^t \frac{r - \mu(\alpha(s))}{\sigma(\alpha(s))} ds, \\ w_2^*(t) &= w_2(t) - \int_0^t \left(\frac{r - \mu_y(\alpha(s))}{\sigma_y(\alpha(s))\sqrt{1-\rho^2}} - \frac{\rho}{\sqrt{1-\rho^2}} \frac{r - \mu(\alpha(s))}{\sigma(\alpha(s))} \right) ds, \end{aligned} \quad (9)$$

are two independent standard Brownian motions under the measure P^* .

Let $z_x(t) = \ln(x(t)/x_0)$; then, we have

$$z_x(t) = \int_0^t \left(r - \frac{\pi(\alpha(s))^2 \sigma^2(\alpha(s))}{2} \right) ds + \int_0^t \pi(\alpha(s)) \sigma(\alpha(s)) dw_1^*(s), \quad (10)$$

$$z_y(t) = \int_0^t \left(r - \frac{\sigma_y^2(\alpha(s))}{2} \right) ds + \int_0^t \sigma_y(\alpha(s)) \rho dw_1^*(s) + \int_0^t \sigma_y(\alpha(s)) \sqrt{1-\rho^2} dw_2^*(s). \quad (11)$$

From (10) and (11), we find that, given \mathcal{F}_t^α under the measure P^* , $z_x(t)$ has normal distribution with mean $\mu_{z_x}(t) = \int_0^t (r - (\pi(\alpha(s))^2 \sigma^2(\alpha(s))/2)) ds$ and variance $\sigma_{z_x}^2(t) = \int_0^t \pi(\alpha(s))^2 \sigma^2(\alpha(s)) ds$. $z_y(t)$ has normal distribution with mean $\mu_{z_y}(t) = \int_0^t (r - (\sigma_y^2(\alpha(s))/2)) ds$ and variance $\sigma_{z_y}^2(t) = \int_0^t \sigma_y^2(\alpha(s)) ds$. Hence, we know that $(z_x(t), z_y(t))$ has bivariate normal distribution $N(\mu_{z_x}(t), \mu_{z_y}(t), \sigma_{z_x}^2(t), \sigma_{z_y}^2(t), \rho_{z_x, z_y}(t))$, with a correlation coefficient $\rho_{z_x, z_y}(t) = (\int_0^t \rho \pi(\alpha(s)) \sigma(\alpha(s)) \sigma_y(\alpha(s)) ds) / (\sigma_{z_x}(t) \sigma_{z_y}(t))$.

4. Scenario Case Analysis

In this section, we will focus on the premium calculations under different scenario cases. The first case is analyzing the pension fund has premature termination. We assume that there is a third-party external regulator (like pension actuary), who is in charge of monitoring the performance of pension fund. The pension fund can thus be forced to close prematurely if a certain threshold value is reached. The other case we will study is from the perspective of plan sponsor. When the performance of the sponsor asset is not good enough, the plan provider is not able to cover its debt and thus remains in business, and the distress termination would happen. We choose to examine a threshold value higher than its liability value to include the other possible expenses for the sponsor company to remain in business.

4.1. Premature Termination of Pension Fund. In this section, we consider the case of premature termination of pension fund. The threshold value at time t is assumed to be $\eta B_R e^{-(R-t)r}$, where η is a positive constant less than 1. Therefore, we can define the first hitting time as

$$\tau = \inf \{t \mid x(t) \leq \eta B_R e^{-(R-t)r}\}. \quad (12)$$

Starting from here, we will consider two cases as below.

The first case is premature termination of pension fund happens prior to retirement time R , i.e., $\tau < R$. When $\tau < R$, pension fund is underfunded, and the possible outcomes for the sponsor assets are

Sponsor company is defaulted, which means $y(\tau) < \theta y_0 e^{y\tau}$. In this case, PBGC takes the whole obligation to pay the part that pension fund fails to cover.

Sponsor company is partially solvent:

$$\theta y_0 e^{y\tau} \leq y(\tau) < \theta y_0 e^{y\tau} + B_R e^{-(R-\tau)r} - x(\tau). \quad (13)$$

In this case, PBGC provides the rest of the part that sponsor company is unable to pay.

Sponsor company is performing very well:

$$y(\tau) \geq \theta y_0 e^{\nu\tau} + B_R e^{-(R-\tau)r} - x(\tau). \quad (14)$$

All the benefit to beneficiary can be paid without the help of PBGC. Note that in our work, we are mainly interested in finding how much premium that PBGC should collect from sponsor company to provide the

corresponding protection. In this case, PBGC does not provide anything, and therefore, we will derive the premium just based on the first two scenarios.

Based on the above analysis, we can model the support from sponsor's company $S^c(\tau)I_{\{\tau < R\}}$ as below:

$$S^c(\tau)I_{\{\tau < R\}} = \begin{cases} 0, & y(\tau) < \theta y_0 e^{\nu\tau}; \\ y(\tau) - \theta y_0 e^{\nu\tau}, & \theta y_0 e^{\nu\tau} \leq y(\tau) < \theta y_0 e^{\nu\tau} + B_R e^{-(R-\tau)r} - x(\tau); \\ B_R e^{-(R-\tau)r} - x(\tau), & y(\tau) \geq \theta y_0 e^{\nu\tau} + B_R e^{-(R-\tau)r} - x(\tau). \end{cases} \quad (15)$$

The second case is that pension fund falls below the threshold after time R , i.e., $\tau \geq R$; this implies $x(R) \geq \eta B_R$. When $\tau > R$, the pension fund is naturally closed at the maturity date R . Note that our assumption is PBGC collects premium up to time R when beneficiary gets retired. Therefore, we just need to consider the case of $\tau = R$. Also, note that if $x(R) \geq B_R$, both

sponsor company and PBGC do not need to provide anything, and therefore, we just focus on the case that $\eta B_R \leq x(R) < B_R$ to proceed and we can represent $S^c(\tau)I_{\{\tau \geq R\}}$ as below:

$$S^c(R)I_{\{\tau \geq R\}} = \begin{cases} 0, & y(R) < \theta y_0 e^{\nu R}, & \eta B_R \leq x(R) < B_R; \\ y(R) - \theta y_0 e^{\nu R}, & \theta y_0 e^{\nu R} \leq y(R) < \theta y_0 e^{\nu R} + B_R - x(R), & \eta B_R \leq x(R) < B_R; \\ B_R - x(R), & y(R) \geq \theta y_0 e^{\nu R} + B_R - x(R), & \eta B_R \leq x(R) < B_R. \end{cases} \quad (16)$$

Therefore, the entire support provided by sponsor company is

$$S^c = S^c(\tau)I_{\{\tau < R\}} + S^c(R)I_{\{\tau \geq R\}}. \quad (17)$$

Accordingly, the entire support provided by PBGC is the sum of the following two parts:

$$\begin{aligned} S^p(\tau)I_{\{\tau < R\}} &= B_R e^{-(R-\tau)r} - x(\tau) - S^c(\tau)I_{\{\tau < R\}}, \\ S^p(R)I_{\{\tau \geq R\}} &= B_R - x(R) - S^c(R)I_{\{\tau \geq R\}}. \end{aligned} \quad (18)$$

Thus, the total contribution supported by PBGC is

$$S^p = S^p(\tau)I_{\{\tau < R\}} + S^p(R)I_{\{\tau \geq R\}}. \quad (19)$$

4.2. Distress Termination of Sponsor Asset. In this section, we consider the case that sponsor asset can be underfunded and it is called "distress termination" in [11]. Therefore, in this case, we use a stopping time τ^p to describe the first time that the sponsor asset falls below or across the threshold $\theta y_0 e^{\nu t}$, given that the sponsor company has corporate debt $\theta y_0 e^{\nu t}$ at time t . We assume $\vartheta \geq \theta$, similar to the assumption used in [11]. Under this framework, the definition of τ^p is given as

$$\tau^p = \inf\{t \mid y(t) \leq \vartheta y_0 e^{\nu t}\}. \quad (20)$$

Similar to what we have discussed previously, we consider the case that $\tau^p < R$ and $\tau^p \geq R$ in the following paragraphs. Similarly, our analysis breaks into two cases as below.

Sponsor asset falls below the threshold before time R , $\tau^p < R$. When $\tau^p < R$, the possible outcomes for the pension fund are

The pension fund performs good enough, that is, the value of pension fund is worth more than the discounted promised pension benefit payment. Both sponsor company and PBGC need to do nothing in the case.

The pension fund is not sufficient to cover all the discounted pension benefit, i.e., $x(\tau^p) < B_R \exp(-(R-\tau^p)r)$.

Hence, we can represent $S^c(\tau^p)I_{\{\tau^p < R\}}$ as below:

$$S^c(\tau^p)I_{\{\tau^p < R\}} = \begin{cases} B_R e^{-(R-\tau^p)r} - x(\tau^p), & 0 < B_R e^{-(R-\tau^p)r} - x(\tau^p) < (\vartheta - \theta)y_0 e^{\nu\tau^p}; \\ (\vartheta - \theta)y_0 e^{\nu\tau^p}, & B_R e^{-(R-\tau^p)r} - x(\tau^p) \geq (\vartheta - \theta)y_0 e^{\nu\tau^p}. \end{cases} \quad (21)$$

Sponsor asset falls below the threshold not earlier than R , $\tau^P \geq R$; this implies $y(R) > \vartheta y_0 e^{yR}$. When $\tau^P \geq R$, the pension fund is naturally closed at the maturity date R . Note that PBGC collects premium up to time R when beneficiary is retired. Therefore, we essentially consider the case of $\tau^P = R$. Also, note that if $x(R) \geq B_R$, both

sponsor company and the PBGC do not need to provide anything, and therefore, we just focus on the case that $x(R) < B_R$ to proceed.

Thus, $S^c(R)I_{\{\tau^P \geq R\}}$ is described as follows:

$$S^c(R)I_{\{\tau^P \geq R\}} = \begin{cases} B_R - x(R), & y(R) \geq \vartheta y_0 e^{yR} + B_R - x(R), \quad B_R > x(R); \\ y(R) - \theta y_0 e^{yR}, & y(R) < \vartheta y_0 e^{yR} + B_R - x(R), \quad B_R > x(R). \end{cases} \quad (22)$$

Combining what we have above together, the entire support provided by the sponsor company is

$$S^c = S^c(\tau^P)I_{\{\tau^P < R\}} + S^c(R)I_{\{\tau^P \geq R\}}. \quad (23)$$

Accordingly, the entire support provided by PBGC is

$$\begin{aligned} S^P(\tau^P)I_{\{\tau^P < R\}} &= (B_R e^{-(R-\tau^P)r} - x(\tau^P) - S^c(\tau^P)I_{\{\tau^P < R\}}) \\ &= (B_R e^{-(R-\tau^P)r} - x(\tau^P) - (\vartheta - \theta)y_0 e^{y\tau^P})I_{\{B_R e^{-(R-\tau^P)r} - x(\tau^P) \geq (\vartheta - \theta)y_0 e^{y\tau^P}\}}, \\ S^P(R)I_{\{\tau^P \geq R\}} &= (B_R - x(R) - S^c(R)I_{\{\tau^P \geq R\}}) \\ &= (B_R - x(R) - y(R) + \theta y_0 e^{yR})I_{\{y(R) < \vartheta y_0 e^{yR} + B_R - x(R), B_R > x(R)\}}. \end{aligned} \quad (24)$$

Thus, the entire support provided by PBGC is summarized as

5. Premium Calculations

Now we can proceed to find the premium for PBGC. Using the no arbitrage idea, the premium paid by plan sponsor to PBGC is the expected discounted insurance payoff under the risk neutral probability measure. Before we discuss the calculation of premium, let us introduce some notations first. Let $J_t = \int_0^t I_{\{\alpha(s)=0\}} ds$ denote the occupation time of Markov chain at state 0 during $[0, t]$. In the calculation below, we will employ the method mentioned in [19] to analyze the case of stopping time being reached before and after retirement time R , $\{\tau^P < R\}$ and $\{\tau^P \geq R\}$ separately due to calculation convenience.

For the sake of simplicity, we first give some symbols. We denote

$$\begin{aligned} m_x(t, R) &= \left(r - \frac{1}{2}\pi(0)^2\sigma^2(0)\right)t + \left(r - \frac{1}{2}\pi(1)^2\sigma^2(1)\right)(R-t) \\ &= \frac{1}{2}(\pi(1)^2\sigma^2(1) - \pi(0)^2\sigma^2(0))t + \left(r - \frac{1}{2}\pi(1)^2\sigma^2(1)\right)R, \\ v_x(t, R) &= \sqrt{(\pi(0)^2\sigma^2(0))t + \pi(1)^2\sigma^2(1)(R-t)} \\ &= \sqrt{(\pi(0)^2\sigma^2(0) - \pi(1)^2\sigma^2(1))t + \pi(1)^2\sigma^2(1)R}, \\ m_y(t, R) &= \left(r - \frac{1}{2}\sigma_y^2(0)\right)t + \left(r - \frac{1}{2}\sigma_y^2(1)\right)(R-t) \\ &= \frac{1}{2}(\sigma_y^2(1) - \sigma_y^2(0))t + \left(r - \frac{1}{2}\sigma_y^2(1)\right)R, \\ v_y(t, R) &= \sqrt{\sigma_y^2(0)t + \sigma_y^2(1)(R-t)} = \sqrt{(\sigma_y^2(0) - \sigma_y^2(1))t + \sigma_y^2(1)R}, \\ v_{xy}(t, R) &= \pi(0)\sigma(0)\sigma_y(0)\rho t + \pi(1)\sigma(1)\sigma_y(1)\rho(R-t). \end{aligned} \quad (26)$$

Then, $(z_x(R), z_y(R) | J_R = t)$ has bivariate normal distribution, with probability density function

$$\phi(z_1, z_2, t) = \frac{1}{2\pi v_x(t, R)v_y(t, R)\sqrt{1-\bar{\rho}^2}} \exp\left(-\frac{z}{2(1-\bar{\rho}^2)}\right), \quad (27)$$

$$z = \frac{(z_1 - m_x(t, R))^2}{v_x^2(t, R)} - 2\bar{\rho} \frac{(z_1 - m_x(t, R))(z_2 - m_y(t, R))}{v_x(t, R)v_y(t, R)} + \frac{(z_2 - m_y(t, R))^2}{v_y^2(t, R)},$$

$$\bar{\rho} = \frac{v_{xy}(t, R)}{v_x(t, R)v_y(t, R)}.$$

From Yoon et al. [20], we can obtain the following lemma.

Lemma 1. Let $f_i(t, u)$ be the probability density of J_t at initial state i , and we have

$$f_0(t, u) = e^{-\lambda_1(t-u)-\lambda_0 u} \left(\left[\frac{u\lambda_1\lambda_0}{t-u} \right]^{1/2} I_1[2(\lambda_0\lambda_1 u(t-u))^{1/2}] + \lambda_0 I_0[2(\lambda_0\lambda_1 u(t-u))^{1/2}] \right),$$

$$f_1(t, u) = e^{-\lambda_1(t-u)-\lambda_0 u} \left(\left[\frac{(t-u)\lambda_1\lambda_0}{u} \right]^{1/2} I_1[2(\lambda_0\lambda_1 u(t-u))^{1/2}] + \lambda_1 I_0[2(\lambda_0\lambda_1 u(t-u))^{1/2}] \right), \quad (29)$$

where $f_0(t, 0) = 0$, $f_0(t, t) = e^{-\lambda_0 t}$, $f_1(t, 0) = e^{-\lambda_1 t}$, $f_1(t, t) = 0$, and $I_\alpha(z)$ is the modified Bessel function ($\alpha = 0, 1$) of the first type such that

$$I_\alpha(z) = \left(\frac{z}{2}\right)^\alpha \sum_{n=0}^{\infty} \frac{(z/2)^{2n}}{n!\Gamma(\alpha+n+1)}. \quad (30)$$

Let $F_\tau(t; J_t) = P^*(\tau \leq t | \mathcal{F}_R^\alpha)$ and $F_{\tau^p}(t; J_t) = P^*(\tau^p \leq t | \mathcal{F}_R^\alpha)$ denote the probability distribution function of τ and τ^p given on \mathcal{F}_R^α . Then, we have the following lemma.

Lemma 2. Let $g(t; J_t)$ and $h(t; J_t)$ denote the condition density functions of τ and τ^p ; $g(t; J_t)$ and $h(t; J_t)$ are then given by (32) and (34).

Proof. It follows from [21, 22] that

$$F_\tau(t; J_t) = N\left(\frac{\ln(\eta B_R e^{-rR}/x_0) + (v_x(J_t; t)^2/2)}{v_x(J_t; t)}\right) + \frac{x_0}{\eta B_R e^{-rR}} N\left(\frac{\ln(\eta B_R e^{-rR}/x_0) - (v_x(J_t; t)^2/2)}{v_x(J_t; t)}\right). \quad (31)$$

By taking derivative with respect to variable t of $F_\tau(t; J_t)$, we have

$$g(t; J_t) = \frac{\pi(1)^2 \sigma(1)^2 (0.5v_x(t; J_t)^2 - \ln(\eta B_R e^{-rR}/x_0))}{2\sqrt{2\pi}v_x(t; J_t)^3} \exp\left\{-0.5\left(\frac{\ln(\eta B_R e^{-rR}/x_0) + (v_x(t; J_t)^2/2)}{v_x(t; J_t)}\right)^2\right\}$$

$$+ \frac{\pi(1)^2 \sigma(1)^2 x_0 (0.5v_x(t; J_t)^2 + \ln(\eta B_R e^{-rR}/x_0))}{2\sqrt{2\pi}\eta B_R e^{-rR} v_x(t; J_t)^3} \exp\left\{-0.5\left(\frac{\ln(\eta B_R e^{-rR}/x_0) - (v_x(t; J_t)^2/2)}{v_x(t; J_t)}\right)^2\right\}. \quad (32)$$

By adopting the same method, we have

$$F_{\tau^p}(t; J_t) = N\left(\frac{\ln \vartheta - (r - \nu)t + (v_y(t; J_t)^2/2)}{v_y(t; J_t)}\right) + \vartheta^{(2(r-\nu)t - v_y(J_t; t)^2)/v_y(J_t; t)^2} N\left(\frac{\ln \vartheta + (r - \nu)t - (v_y(J_t; t)^2/2)}{v_y(J_t; t)}\right). \quad (33)$$

Then,

$$\begin{aligned} h(t; J_t) &= \frac{(\sigma(1)^2 - \sigma(0)^2)[0.5v_y(J_t; t)^2 - \ln \vartheta + (r - \nu)t] - 2(r - \nu)v_y(J_t; t)^2}{2\sqrt{2\pi}v_y(J_t; t)^3} \\ &\times \exp\left\{-0.5\left(\frac{\ln \vartheta - (r - \nu)t + (v_y(J_t; t)^2/2)}{v_y(J_t; t)}\right)^2\right\} \\ &+ \vartheta^{(2(r-\nu)t - v_y(J_t; t)^2)/v_y(J_t; t)^2} \frac{(\sigma(1)^2 - \sigma(0)^2)[0.5v_y(J_t; t)^2 + \ln \vartheta + (r - \nu)t] + 2(r - \nu)v_y(J_t; t)^2}{2\sqrt{2\pi}v_y(J_t; t)^3} \\ &\times \exp\left\{-0.5\left(\frac{\ln \vartheta + (r - \nu)t - (v_y(J_t; t)^2/2)}{v_y(J_t; t)}\right)^2\right\} + \vartheta^{(2(r-\nu)t - v_y(J_t; t)^2)/v_y(J_t; t)^2} \\ &\times N\left(\frac{\ln \vartheta + (r - \nu)t - (v_y(J_t; t)^2/2)}{v_y(J_t; t)}\right) \ln \vartheta \left\{\frac{2(r - \nu)(v_y(J_t; t)^2 + t(\sigma(1)^2 - \sigma(0)^2))}{v_y(J_t; t)^4}\right\}. \end{aligned} \quad (34)$$

Theorem 1. If the Markov chain initial state $\alpha(0) = i$, then the premium received by PBGC with premature closure of pension fund under the risk neutral probability measure P^* is given by (A.14).

Proof. The closed-form solution is given in Appendix A. For details, see Appendix A. \square

Corollary 1. If the Markov chain initial state $\alpha(0) = i$, then the risk-based premium of PBGC with early termination of sponsor assets under the risk neutral probability measure P^* is given by (B.6).

Proof. The closed-form solution is given in Appendix B. For details, see Appendix B. \square

Remark 1.

In reality, sometimes both sponsor company and the PBGC only provide a capped retirement income when the employee gets retired early or when the pension fund is highly underfunded. In this case, we can model the support from PBGC by a constant C with $C < B_R$; by using the similar method as [11], the premium can be calculated.

The premium that sponsor company collects from plan participants can also be calculated by the similar method used before.

6. Numerical Analysis

In this section, we make numerical analysis of the explicit formula derived in the previous section. For numerical demonstration, we take the following parameters:

$$\begin{aligned} r(1) &= 0.05, \sigma(1) = 0.16, x(0) = 600, \eta = 0.8, \theta = 0.6, \\ \pi(0) &= 0.6, \pi(1) = 0.3, \rho = 0.5, y(0) = 800, \\ R &= 15, B_R = 762.12, r = 0.035, \nu = 0.03, \\ r(2) &= 0.02, \sigma(2) = 0.4, \sigma_y(1) = 0.18, \sigma_y(2) = 0.48, \vartheta = 0.63, \\ \mu(1) &= 0.08, \mu(2) = 0.04, \mu_y(1) = 0.07, \mu_y(2) = 0.05. \end{aligned} \quad (35)$$

Figures 1 and 2 demonstrate the effects of B_R on premiums under different situations. To be more specific, Figure 1 illustrates how the premium changes against B_R when premature termination of pension fund occurs and Figure 2 shows the impact of B_R on premium for the case of distress termination of sponsor assets. We make a couple of observations regarding the two graphs. First, the premium is an increasing function of B_R in both graphs. Given that B_R is the retirement benefit that one expects to receive, it makes sense to observe that a higher B_R implies a bigger financial responsibility for the employer to hold. Therefore, it is expected for the employer to pay more premium to the PBGC to transfer the pension risk. Second, we notice that the

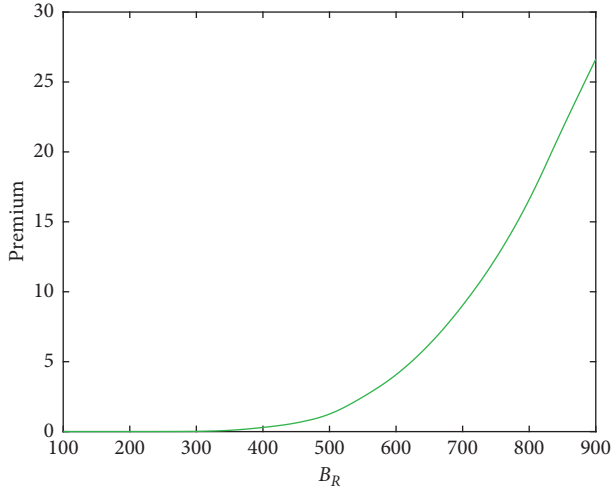


FIGURE 1: The impact of B_R on the premium with premature termination of pension fund.

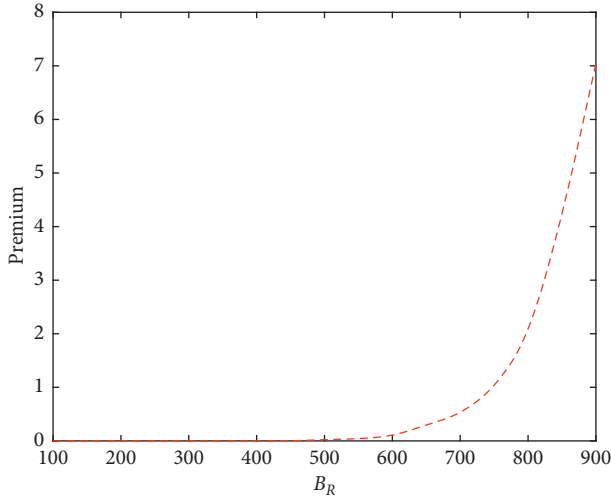


FIGURE 2: The impact of B_R on the premium with distress termination of sponsor asset.

premium increases much more significantly in Figure 1, compared with its trend in Figure 2. Note that pension fund is the first and foremost pool of fund for postretirement payment regardless of financial soundness of the sponsor company, and thus it is reasonable to witness that premium increases more quickly when there is a risk of premature termination of pension fund.

In Figure 3, we demonstrate how the premium changes with respect to η while premature termination of pension is under consideration. Recall that η is the trigger ratio of the pension fund and is also referred to as the regulatory parameter. We know that, on the one hand, termination would never happen if $\eta < 0$ given that the pension fund is non-negative. On the other hand, $\eta > 1$ does not make sense since it implies that more than enough reserve should be set aside in the pension fund pool. Thus, we assume that $\eta \in (0, 1)$. We can see that the dynamics of premium according to η are not monotonic with our choice of parameters. The increase

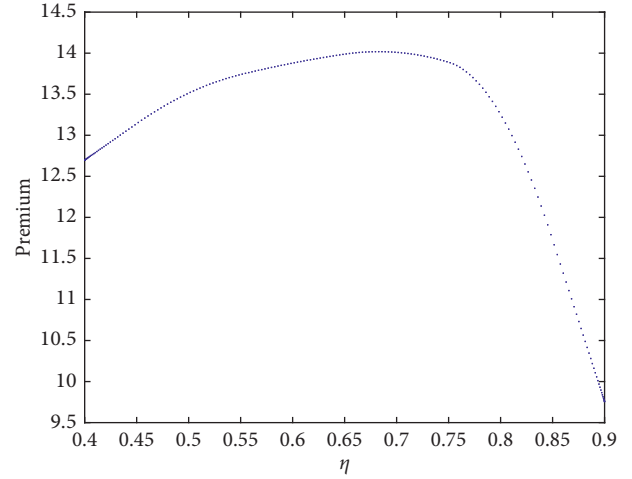


FIGURE 3: The impact of η on the premium with premature termination of pension fund.

of regulatory parameter first leads to a very mild increase of premium and there is a dip of the premium afterwards. The possible reasons behind this interesting pattern are explored as follows. When the regulatory parameter η is small, the pension fund value is small while the threshold value is hit. It is more likely that the PBGC needs to step in when pension fund has premature termination. Our conjecture for the mild increase portion of premium is that it is a reflection of the expected compensation that the PBGC predicts for its higher probability of providing the coverage. With the increase of regulatory parameter, the premium starts to go down to adjust for stronger regulation requirement after an “optimal” value of it being reached first.

Figure 4 is about the relation between premium and threshold value ϑ for the case of distress termination of the sponsor assets. We have the requirements about the ϑ such that it not only satisfies $\vartheta < 1$ but also meets the condition that $\vartheta < \nu$. We need $\vartheta < 1$ to reflect the assumption that the sponsoring company is not in default yet in the beginning. $\vartheta > \nu$ is necessary to take account of the fact that the pension sponsor has the moral obligation to cover some deficits of the claimed pension benefit. Our result shows a decrease of premium in regard to the increase of ϑ in the figure. Note that the three sequential lines of protections are assumed to be the pension fund, the sponsor assets, and the PBGC. Higher threshold value of the sponsor company at distress termination shows that the employer is capable to provide more financial support irrespective of performance of the pension fund, and the PBGC thus charges less premium accordingly with less financial burden.

7. Further Remarks

In this work, we focus on finding the closed-form formula for the risk-based premium of the PBGC with regime switching. The explicit solutions of the premiums are derived.

Further efforts can be directed to the portfolio selection for the pension fund in which stochastic control and Markov

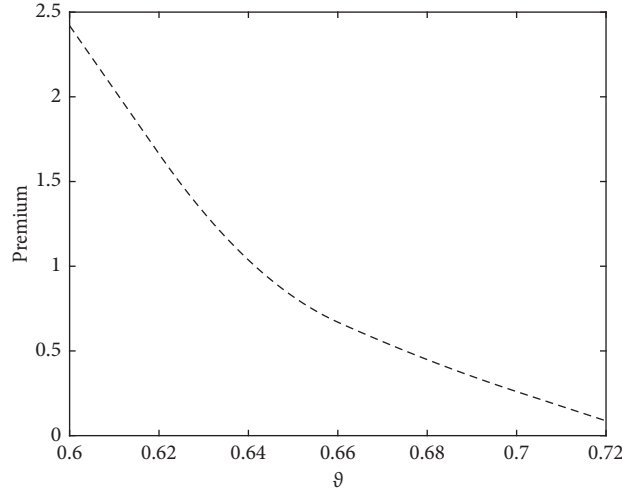


FIGURE 4: The impact of θ on the premium with distress termination of sponsor asset.

chain approximation seem to be reasonable methods to use. Further effort in this direction deserves more thoughts and considerations.

Appendix

A. Proof of Theorem 1

By the risk neutral pricing theory, the premium for PBGC with premature closure of pension fund is the expected discounted insurance payoff under the risk neutral probability measure P^* and is given by

$$S_0^P = E^* \left[e^{-r\tau} S^P(\tau) I_{\{\tau < R\}} \right] + E^* \left[e^{-rR} S^P(R) I_{\{\tau \geq R\}} \right]. \quad (\text{A.1})$$

Hence, the premium for the insurance of the PBGC can be decomposed into two parts:

- (i) $E^* \left[e^{-r\tau} S^P(\tau) I_{\{\tau < R\}} \right]$.
- (ii) $E^* \left[e^{-rR} S^P(R) I_{\{\tau \geq R\}} \right]$.

(1) Regarding (i), we calculate it by iterated expectation formula as follows:

$$\begin{aligned} & E^* \left[e^{-r\tau} S^P(\tau) I_{\{\tau < R\}} \right] \\ &= E^* \left[e^{-r\tau} \left((B_R e^{-r(R-\tau)} - x(\tau)) I_{\{y(\tau) < \theta y_0 e^{y\tau}\}} \right. \right. \\ &\quad \left. \left. + (B_R e^{-r(R-\tau)} - x(\tau) - y(\tau) + \theta y_0 e^{y\tau}) I_{\{\theta y_0 e^{y\tau} \leq y(\tau) < \theta y_0 e^{y\tau} + (1-\eta) B_R e^{-r(R-\tau)}\}} \right) I_{\{\tau < R\}} \right] \\ &= E^* \left[e^{-r\tau} (B_R e^{-r(R-\tau)}) (1-\eta) I_{\{y(\tau) \leq \theta y_0 e^{y\tau}\}} I_{\{\tau < R\}} \right] \\ &\quad + E^* \left[\left(e^{-r\tau} ((1-\eta) B_R e^{-r(R-\tau)} - (y(\tau) - \theta y_0 e^{y\tau})) \right) \times I_{\{\theta y_0 e^{y\tau} \leq y(\tau) < \theta y_0 e^{y\tau} + (1-\eta) B_R e^{-r(R-\tau)}\}} I_{\{\tau < R\}} \right]. \end{aligned} \quad (\text{A.2})$$

Note that here we used the fact that $x(\tau) = \eta B_R e^{-r(R-\tau)}$ in the above. On the one hand,

$$\begin{aligned} & E^* \left[e^{-r\tau} (B_R e^{-r(R-\tau)}) (1-\eta) I_{\{y(\tau) \leq \theta y_0 e^{y\tau}\}} I_{\{\tau < R\}} \right] \\ &= \int_0^R \int_0^u \int_{-\infty}^{\ln \theta + yu} e^{-rR} B_R (1-\eta) g(u; t) f_i(u, t) \phi_2(z_2; u, t) dz_2 dt du \\ &\quad + \Gamma_1(0, u; \theta, R, y, B_R, \lambda) + \Gamma_1(1, 0; \theta, R, y, B_R, \lambda), \end{aligned} \quad (\text{A.3})$$

in which

$$\begin{aligned}\Gamma_1(j, t; \theta, R, \nu, B_R, \lambda) &= \int_0^R \int_{-\infty}^{\ln \theta + \nu u} \delta_j(i) e^{-\lambda_j u} e^{-rR} B_R (1 - \eta) g(u; t) \phi_2(z_2; u, t) dz_2 du, \\ \phi_2(z_2; u, t) &= \frac{1}{\sqrt{2\pi v_y(t, u)}} \exp \left\{ \frac{(z_2 - m_y(t, u))^2}{2v_y(t, u)^2} \right\}, \\ \delta_j(i) &= \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}\end{aligned}\tag{A.4}$$

On the other hand,

$$\begin{aligned}E^* & \left[\left[e^{-r\tau} \left((1 - \eta) B_R e^{-r(R-\tau)} - (y(\tau) - \theta y_0 e^{\nu\tau}) \right) \right] \right. \\ & \quad \times I_{\{\theta y_0 e^{\nu\tau} \leq y(\tau) < \theta y_0 e^{\nu\tau} + (1 - \eta) B_R e^{-r(R-\tau)}\}} I_{\{\tau < R\}} \\ & = \int_0^R \int_0^u \int_{\ln \theta + \nu u}^{\ln (\theta y_0 e^{\nu u} + (1 - \eta) B_R e^{-r(R-u)}) / y_0} \left(e^{-rR} B_R (1 - \eta) - e^{-ru} (y_0 e^{z_2} - \theta y_0 e^{\nu u}) \right) g(u; t) f_i(u, t) \times \phi_2(z_2; u, t) dz_2 dt du \\ & \quad \left. + \Gamma_2(0, u; \theta, R, \nu, B_R, \lambda, y_0) + \Gamma_2(1, 0; \theta, R, \nu, B_R, \lambda, y_0), \right]\end{aligned}\tag{A.5}$$

in which

$$\Gamma_2(j, t; \theta, R, \nu, B_R, \lambda, y_0) = \int_0^R \int_{\ln \theta + \nu u}^{\ln (\theta y_0 e^{\nu u} + (1 - \eta) B_R e^{-r(R-u)}) / y_0} \delta_j(i) e^{-\lambda_j u} \left(e^{-rR} B_R (1 - \eta) - e^{-ru} (y_0 e^{z_2} - \theta y_0 e^{\nu u}) \right) g(u; t) \phi_2(z_2; u, t) dz_2 du.\tag{A.6}$$

According to relationships (A.2), (A.3), and (A.5), we can obtain

$$\begin{aligned}E^* & \left[e^{-r\tau} S^p(\tau) I_{\{\tau < R\}} \right] \\ & = \int_0^R \int_0^u \int_{-\infty}^{\ln \theta + \nu u} e^{-rR} B_R (1 - \eta) g(u; t) f_i(u, t) \phi_2(z_2; u, t) dz_2 dt du \\ & \quad + \Gamma_1(0, u; \theta, R, \nu, B_R, \lambda) + \Gamma_1(1, 0; \theta, R, \nu, B_R, \lambda) \\ & \quad + \int_0^R \int_0^u \int_{\ln \theta + \nu u}^{\ln (\theta y_0 e^{\nu u} + (1 - \eta) B_R e^{-r(R-u)}) / y_0} \left(e^{-rR} B_R (1 - \eta) - e^{-ru} (y_0 e^{z_2} - \theta y_0 e^{\nu u}) \right) g(u; t) f_i(u, t) \times \phi_2(z_2; u, t) dz_2 dt du \\ & \quad + \Gamma_2(0, u; \theta, R, \nu, B_R, \lambda, y_0) + \Gamma_2(1, 0; \theta, R, \nu, B_R, \lambda, y_0).\end{aligned}\tag{A.7}$$

(2) As to (ii), we have

$$\begin{aligned}E^* & \left[e^{-rR} S^p(R) I_{\{\tau \geq R\}} \right] \\ & = E^* \left[E^* \left[e^{-rR} S^p(R) I_{\{\tau \geq R\}} | \tau \geq R, \mathcal{F}_R^\alpha \right] \right] \\ & = E^* \left[I_{\{\tau \geq R\}} E^* \left[e^{-rR} \left((B_R - x(R)) I_{\{y(R) < \theta y_0 e^{\nu R}\}} I_{\{\eta B_R < x(R) < B_R\}} \right. \right. \right. \\ & \quad \left. \left. + (B_R - x(R) - (y(R) - \theta y_0 e^{\nu R})) I_{\{\theta y_0 e^{\nu R} \leq y(R) < \theta y_0 e^{\nu R} + B_R - x(R)\}} \right. \right. \\ & \quad \left. \left. \times I_{\{\eta B_R < x(R) < B_R\}} \right) | \tau \geq R, \mathcal{F}_R^\alpha \right] \right].\end{aligned}\tag{A.8}$$

Therefore, according to Lemma 1, for $0 < t < R$ and $i = 1, 2$, we get

$$\begin{aligned} & E^* \left[I_{\{\tau \geq R\}} E^* \left[e^{-rR} (B_R - x(R)) I_{\{y(R) \leq \theta y_0 e^{vR}\}} I_{\{\eta B_R < x(R) < B_R\}} | \tau \geq R, \mathcal{F}_R^\alpha \right] \right] \\ &= \int_0^R \int_{\ln(\eta B_R/x_0)}^{\ln(B_R/x_0)} \int_{-\infty}^{\nu R + \ln \theta} (1 - F_\tau(R; t)) e^{-rR} (B_R - x_0 e^{z_1}) \phi(z_1, z_2, t) f_i(R, t) dz_2 dz_1 dt \\ &+ \Gamma_3(0, R; \theta, R, \nu, B_R, \lambda, x_0) + \Gamma_3(1, 0; \theta, R, \nu, B_R, \lambda, x_0), \end{aligned} \quad (\text{A.9})$$

in which

$$\begin{aligned} \Gamma_3(j, t; \theta, R, \nu, B_R, \lambda, x_0) &= \delta_j(i) e^{-\lambda_j R - rR} \\ &\times \int_{\ln(\eta B_R/x_0)}^{\ln(B_R/x_0)} \int_{-\infty}^{\nu R + \ln \theta} (1 - F_\tau(R; t)) (B_R - x_0 e^{z_1}) \phi(z_1, z_2, t) \cdot dz_2 dz_1. \end{aligned} \quad (\text{A.10})$$

On the other hand,

$$\begin{aligned} & E^* \left[I_{\{\tau \geq R\}} E^* \left(e^{-rR} (B_R - x(R) - t(y(R) - \theta y_0 e^{vR})) \right) \times I_{\{\theta y_0 e^{vR} \leq y(R) < \theta y_0 e^{vR} + B_R - x(R)\}} I_{\{\eta B_R < x(R) < B_R\}} | \tau \geq R, \mathcal{F}_R^\alpha \right] \\ &= \int_0^R \int_{\ln(\eta B_R/x_0)}^{\ln(B_R/x_0)} \int_{\nu R + \ln \theta}^{\ln((\theta y_0 e^{vR} + B_R - x_0 e^{z_1})/y_0)} (1 - F_\tau(R; t)) e^{-rR} \times (B_R - x_0 e^{z_1} - y_0 e^{z_2} + \theta y_0 e^{vR}) \phi(z_1, z_2, t) f_i(R, t) dz_2 dz_1 dt \\ &+ \Gamma_4(0, R; \theta, R, \nu, B_R, \lambda, x_0, y_0) + \Gamma_4(1, 0; \theta, R, \nu, B_R, \lambda, x_0, y_0), \end{aligned} \quad (\text{A.11})$$

in which

$$\begin{aligned} \Gamma_4(j, t; \theta, R, \nu, B_R, \lambda, x_0, y_0) &= \delta_j(i) e^{-\lambda_j R - rR} \int_{\ln(\eta B_R/x_0)}^{\ln(B_R/x_0)} \int_{\nu R + \ln \theta}^{\ln((\theta y_0 e^{vR} + B_R - x_0 e^{z_1})/y_0)} (1 - F_\tau(R; t)) \\ &\times (B_R - x_0 e^{z_1} - y_0 e^{z_2} + \theta y_0 e^{vR}) \phi(z_1, z_2, t) dz_2 dz_1. \end{aligned} \quad (\text{A.12})$$

Combining (A.9) with (A.11), we have

$$\begin{aligned} E^* \left[e^{-rR} S^p(R) I_{\{\tau \geq R\}} \right] &= \int_0^R \int_{\ln(\eta B_R/x_0)}^{\ln(B_R/x_0)} \int_{-\infty}^{\nu R + \ln \theta} (1 - F_\tau(R; t)) e^{-rR} (B_R - x_0 e^{z_1}) \phi(z_1, z_2, t) f_i(R, t) dz_2 dz_1 dt \\ &+ \Gamma_3(0, R; \theta, R, \nu, B_R, \lambda, x_0) + \Gamma_3(1, 0; \theta, R, \nu, B_R, \lambda, x_0) \\ &+ \int_0^R \int_{\ln(\eta B_R/x_0)}^{\ln(B_R/x_0)} \int_{\nu R + \ln \theta}^{\ln((\theta y_0 e^{vR} + B_R - x_0 e^{z_1})/y_0)} (1 - F_\tau(R; t)) e^{-rR} (B_R - x_0 e^{z_1} - y_0 e^{z_2} + \theta y_0 e^{vR}) \times \phi(z_1, z_2, t) f_i(R, t) dz_2 dz_1 dt \\ &+ \Gamma_4(0, R; \theta, R, \nu, B_R, \lambda, x_0, y_0) + \Gamma_4(1, 0; \theta, R, \nu, B_R, \lambda, x_0, y_0). \end{aligned} \quad (\text{A.13})$$

From (A.7) and (A.13), the pricing formula of the premium of the PBGC with premature closure is given as follows:

$$\begin{aligned}
S_0^P = & \int_0^R \int_0^u \int_{-\infty}^{\ln \theta + \nu u} e^{-rR} B_R (1 - \eta) g(u; t) f_i(u, t) \phi_2(z_2; u, t) dz_2 dt du + \Gamma_1(0, u; \theta, R, \nu, B_R, \lambda) \\
& + \Gamma_1(1, 0; \theta, R, \nu, B_R, \lambda) + \int_0^R \int_0^u \int_{\ln \theta + \nu u}^{\ln((\theta y_0 e^{\nu u} + (1-\eta)B_R e^{-r(R-u)})/y_0)} (e^{-rR} B_R (1 - \eta) \\
& - e^{-ru} (y_0 e^{z_2} - \theta y_0 e^{\nu u})) g(u; t) f_i(u, t) \times \phi_2(z_2; u, t) dz_2 dt du \\
& + \Gamma_2(0, u; \theta, R, \nu, B_R, \lambda, y_0) + \Gamma_2(1, 0; \theta, R, \nu, B_R, \lambda, y_0) \\
& + \int_0^R \int_{\ln(\eta B_R/x_0)}^{\ln(B_R/x_0)} \int_{-\infty}^{\nu R + \ln \theta} (1 - F_\tau(R; t)) e^{-rR} (B_R - x_0 e^{z_1}) \phi(z_1, z_2, t) f_i(R, t) dz_2 dz_1 dt \\
& + \Gamma_3(0, R; \theta, R, \nu, B_R, \lambda, x_0) + \Gamma_3(1, 0; \theta, R, \nu, B_R, \lambda, x_0) \\
& + \int_0^R \int_{\ln(\eta B_R/x_0)}^{\ln(B_R/x_0)} \int_{\nu R + \ln \theta}^{\ln(\theta y_0 e^{\nu R} + B_R - x_0 e^{z_1}/y_0)} (1 - F_\tau(R; t)) e^{-rR} (B_R - x_0 e^{z_1} - y_0 e^{z_2} + \theta y_0 e^{\nu R}) \times \phi(z_1, z_2, t) f_i(R, t) dz_2 dz_1 dt \\
& + \Gamma_4(0, R; \theta, R, \nu, B_R, \lambda, x_0, y_0) + \Gamma_4(1, 0; \theta, R, \nu, B_R, \lambda, x_0, y_0).
\end{aligned} \tag{A.14}$$

B. Proof of Corollary 1

Similarly, we get S_0^P by the risk neutral pricing theory as follows:

$$\begin{aligned}
S_0^P = & E^* \left[e^{-r\tau^P} S^P(\tau^P) I_{\{\tau^P < R\}} \right] + E^* \left[e^{-rR} S^P(R) I_{\{\tau^P \geq R\}} \right] \\
= & E^* \left[e^{-r\tau^P} \left(B_R e^{-(R-\tau^P)r} - x(\tau^P) - (\vartheta - \theta) y_0 e^{\nu \tau^P} \right) I_{\{B_R e^{-(R-\tau^P)r} - x(\tau^P) \geq (\vartheta - \theta) y_0 e^{\nu \tau^P}\}} I_{\{\tau^P < R\}} \right] \\
& + E^* \left[e^{-rR} \left(B_R - x(R) - y(R) + \theta y_0 e^{\nu R} \right) I_{\{y(R) < \theta y_0 e^{\nu R} + B_R - x(R), \quad B_R > x(R)\}} I_{\{\tau^P \geq R\}} \right].
\end{aligned} \tag{B.1}$$

We will deal with the two parts similarly as in the previous section. The details are presented with the detailed calculation omitted.

$$\begin{aligned}
& E^* \left[e^{-r\tau^P} S^P(\tau^P) I_{\{\tau^P < R\}} \right] \\
= & \int_0^R \int_0^u \int_{-\infty}^{\ln(B_R e^{-rR-u} - \vartheta - \theta y_0 e^{\nu u})/x_0} e^{-rR} B_R - e^{-ru} x_0 e^{z_1} - e^{-ru} \vartheta - \theta y_0 e^{\nu u} h u; t \times f_i(u, t) \phi_1(z_1, u, t) dz_1 dt du \\
& + \Gamma_5(0, u; \theta, \vartheta, R, \nu, B_R, \lambda, x_0) + \Gamma_5(1, 0; \theta, \vartheta, R, \nu, B_R, \lambda, x_0),
\end{aligned} \tag{B.2}$$

in which

$$\Gamma_5(j, t; \theta, \vartheta, R, \nu, B_R, \lambda, x_0) = \int_0^R \int_{-\infty}^{\ln((B_R e^{-r(R-u)} - (\vartheta - \theta) y_0 e^{\nu u})/x_0)} \delta_j(i) e^{-\lambda_j u} \times (e^{-rR} B_R - e^{-ru} x_0 e^{z_1} - e^{-ru} ((\vartheta - \theta) y_0 e^{\nu u})) h(u; t) \phi_1(z_1, u, t) dz_1 du, \quad (B.3)$$

$$\phi_1(z_2; u, t) = \frac{1}{\sqrt{2\pi\nu_x(t, u)}} \exp\left\{-\frac{(z_1 - m_x(t, u))^2}{2\nu_x(t, u)^2}\right\},$$

$$\begin{aligned} & E^* \left[e^{-rR} S^P(R) I_{\{\tau^P \geq R\}} \right] \\ &= \int_0^R \int_{-\infty}^{\ln(B_R/x_0)} \int_{-\infty}^{\ln((\theta y_0 e^{\nu R} + B_R - x_0 e^{z_1})/y_0)} (1 - F_{\tau^P}(R; t)) e^{-rR} (B_R - x_0 e^{z_1} - y_0 e^{z_2} + \theta y_0 e^{\nu R}) \times \phi(z_1, z_2, t) f_i(R, t) dz_2 dz_1 dt \\ &+ \Gamma_6(0, R; \theta, \vartheta, R, \nu, B_R, \lambda, x_0, y_0) + \Gamma_6(1, 0; \theta, \vartheta, R, \nu, B_R, \lambda, x_0, y_0), \end{aligned} \quad (B.4)$$

in which

$$\begin{aligned} \Gamma_6(j, t; \theta, \vartheta, R, \nu, B_R, \lambda, x_0, y_0) &= \delta_j(i) e^{-\lambda_j R} \int_{-\infty}^{\ln(B_R/x_0)} \int_{-\infty}^{\ln((\theta y_0 e^{\nu R} + B_R - x_0 e^{z_1})/y_0)} (1 - F_{\tau^P}(R; t)) \\ &\times (B_R - x_0 e^{z_1} - y_0 e^{z_2} + \theta y_0 e^{\nu R}) \phi(z_1, z_2, t) dz_2 dz_1. \end{aligned} \quad (B.5)$$

By combining the results of (B.1), (B.2), and (B.4), we thus get

$$\begin{aligned} S_0^P &= \int_0^R \int_0^u \int_{-\infty}^{\ln((B_R e^{-r(R-u)} - (\vartheta - \theta) y_0 e^{\nu u})/x_0)} (e^{-rR} B_R - e^{-ru} x_0 e^{z_1} - e^{-ru} ((\vartheta - \theta) y_0 e^{\nu u})) h(u; t) \times f_i(u, t) \phi_1(z_1, u, t) dz_1 dt du \\ &+ \Gamma_5(0, u; \theta, \vartheta, R, \nu, B_R, \lambda, x_0) + \Gamma_5(1, 0; \theta, \vartheta, R, \nu, B_R, \lambda, x_0) \\ &+ \int_0^R \int_{-\infty}^{\ln(B_R/x_0)} \int_{-\infty}^{\ln((\theta y_0 e^{\nu R} + B_R - x_0 e^{z_1})/y_0)} (1 - F_{\tau^P}(R; t)) e^{-rR} (B_R - x_0 e^{z_1} - y_0 e^{z_2} + \theta y_0 e^{\nu R}) \times \phi(z_1, z_2, t) f_i(R, t) dz_2 dz_1 dt \\ &+ \Gamma_6(0, R; \theta, \vartheta, R, \nu, B_R, \lambda, x_0, y_0) + \Gamma_6(1, 0; \theta, \vartheta, R, \nu, B_R, \lambda, x_0, y_0). \end{aligned} \quad (B.6)$$

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Optimal Execution considering Trading Signal and Execution Risk Simultaneously

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In this paper, we study the optimal execution problem by considering the trading signal and the transaction risk simultaneously. We propose an optimal execution problem by taking into account the trading signal and the execution risk with the associated decay kernel function and the transient price impact function being of generalized forms. In particular, we solve the stochastic optimal control problems under the assumptions that the decay kernel function is the Dirac function and the transient price function is a linear function. We give the optimal executing strategies in state-feedback form and the Hamilton-Jacobi-Bellman equations that the corresponding value functions satisfy in the cases of a constant execution risk and a linear execution risk. We also demonstrate that our results can recover previous results when the process of the trading signal degenerates.

1. Introduction

It is known that when traders execute a large order in a short time, it will cause severe effect on the stock price in the stock market. This effect is called the price impact or the market impact in academia. The price impact is often adverse for traders because they liquidate or build a large position in a short time with worse average price compared to the initial price. Hence, traders or financial institutions often bear an extra cost due to the price impact except for some fixed cost charged by exchange. Consequently, the topic on how to reduce the cost caused by the price impact has received much attention.

The problem about reducing the cost caused by the price impact is always expressed as an optimal execution problem in the literature, i.e., looking for an optimal executing strategy to minimize the expected cost due to the price impact. Plenty of works have been done on this topic. Bertsimas and Lo [1] studied a discrete time model of price impact with linear impact function and derived dynamic optimal trading strategies to minimize the expected cost. Almgren and Chriss [2] considered the continuous time case of Bertsimas and Lo [1]. They choose the trade-off

between the expectation and the variance of the impact cost as the optimization objective and then gave the explicit solution by the variation method. In addition, they proposed the concept of L-VaR. Almgren [3] further considered nonlinear impact functions and added risk terms in the temporary impact process on the basis of Almgren and Chriss [2]. Obizhaeva and Wang [4] (an early work published later) studied the optimal executing strategy given the dynamic structure of the demand and supply of the equity. Alfonsi et al. [5] extended the model of Obizhaeva and Wang [4] by allowing for a time-dependent resilience rate with more generalized equilibrium dynamics for bid and ask price. Alfonsi et al. [6] considered more general shape of the LOB on the basis of Obizhaeva and Wang [4] and gave the explicit form of optimal executing strategies. They also illustrated the robustness of the optimal strategies with respect to the shape function and resilience type. Gatheral and Schied [7] assumed the asset price followed a geometric Brownian motion and gave the explicit optimal executing strategy with risk aversion. Almgren [8] assumed the market liquidity and volatility were stochastic and time varying, then proposed the HJB equation of the optimal execution problem, and tried to solve it numerically.

Gatheral et al. [9] studied the optimal execution problem in the frame of transient price impact model. Under the assumption of linear impact function, they characterized the optimal executing strategy as the solution of a generalized Fredholm integral equation of the first kind. They also studied the existence problem. Cheridito and Sepin [10] studied the discrete time case of the price impact model with stochastic volatility and stochastic market liquidity. Schoneborn [11] discussed three approaches that remedied the flaw of the optimal executing strategy under the mean-variance framework that big order and small order have the same executing pattern. Cartea and Jaimungal [12] studied the optimal execution problem by taking into account the order flow of all other agents and gave the explicit solution with linear impact function. Cheng et al. [13] considered the execution risk under the framework of Almgren and Chriss and solved the optimal execution problem with different risk aversions. Jin [14] studied the optimal execution problem with an optimization objective of loss probability and talked about the liquidity adjusted VaR. Curato et al. [15] studied the transient impact model with nonlinear impact function and solved the optimal executing problem numerically.

The previous works on the optimal execution problem are mainly based on the framework of Almgren and Chriss [2] or Gatheral et al. [9]. In the framework of Almgren and Chriss [2], the price impact has two kinds of definitions, i.e., the permanent impact and the temporary impact. Orders to be executed are thought to contain some fundamental information about the stock. This information is absorbed into the stock price in the trading and leads to a permanent impact on the intrinsic price of the stock. This is described by a permanent impact function of trading rate in the intrinsic price process of the stock. In addition, the price that we can observe in the market is affected by the trading at the moment and is described by the intrinsic price plus a temporary impact function of trading rate. In the framework of Gatheral et al. [9], the intrinsic stock price is set to be a martingale, i.e., the orders executed have no impact on the intrinsic price. Besides, the observed price is not only affected by instant trading but also affected by historical trading through a decay kernel function and a transient impact function of trading rate. In this paper, we combine the frameworks of Almgren and Chriss [2] and Gatheral et al. [9]. More specifically, we suppose trading gives rise to a permanent impact on the intrinsic price of the stock, and the observed price is affected by historical trading through a decay kernel function and a price impact function of trading rate.

In the view of some practitioners in trading, the market may not always be so efficient. There exist signals in the stock market with which the traders can predict the future return of the stock to some extent. Indeed, some hedge funds make profit with trading signals founded by technical analysis or other ways. Cartea and Jaimungal [12] studied the optimal executing strategy by incorporating order flow. In this work, the trading rate of all other traders can be treated as a signal of the intrinsic stock return. Motivated by these, we propose a trading signal term in the intrinsic price process. In addition,

the experiences from practitioners indicate that a trading signal usually has the properties of stationarity and mean reversion. Therefore, we assume that the trading signal follows an Ornstein–Uhlenbeck process. Besides, Cheng et al. [13] suggested that the order delivered by traders may not be filled fully, i.e., the traders can face the execution risk; therefore, we investigate the optimal execution problem by taking into account the trading signal and the execution risk simultaneously. More specifically, we propose an optimal execution problem with a generalized kernel function and a generalized transient impact function. To solve this optimal execution problem, we set the kernel function to be the Dirac function, which is compatible with the framework suggested in Almgren and Chriss [2], and the transient impact function to be a linear function. In this setting, we give analytical solutions to the optimal execution problems with a constant execution risk and a linear execution risk, respectively. Moreover, we prove that our results can recover the results in Cheng et al. [13] if the trading signal process degenerates, i.e., the mean reversion speed of the trading signal degenerates to 0. Our results can provide some insights for the hedge funds that possess some trading signals to design their trading scheme.

The rest of this paper is organized as follows. In Section 2, we describe our model. In Section 3, we propose our optimal execution problem. In Section 4, we solve the optimal execution problem with a constant execution risk and a linear execution risk, respectively, and discuss the solutions. In Section 5, we conclude this paper and point out some directions for further work.

2. Model Settings

Suppose that we have a scheme of liquidating X shares of stock in time interval $[t_0, T]$, and $t_0 = 0$. At time t , the amount of stock remaining to be liquidated is denoted as x_t , and thus $x_{t_0} = X$.

We suppose the intrinsic stock price, which cannot be observed directly in the market, follows the stochastic process below:

$$dS_t = \theta dx_t + \rho L_t dt + \sigma_1 dB_t, \quad (1)$$

where L_t is trading signal and is defined in (4), B_t is a standard Brownian motion, and θ , ρ , and σ_1 are constant parameters with $\rho > 0$, $\sigma_1 > 0$.

In the setting of (1), we suppose that trading has a permanent impact on the intrinsic stock price, and the permanent impact function is set to be a linear function. This setting follows the existing works based on the framework of Almgren and Chriss [2], such as Gatheral and Schied [7], Cartea and Jaimungal [12], Cheng et al. [13], Jin [14], and so on. Besides, we further suppose that the intrinsic price of the stock is also affected by other factors, such as trading signals and the trading rate of other traders.

In addition, we suppose the observed price of the stock can be expressed as

$$\tilde{S}_t = S_t + \int_{t_0}^t G(t-s)g(v_s)ds, \quad (2)$$

where $G(\cdot)$ is the decay kernel function, $g(\cdot)$ is the transient impact function, and v_s is the trading rate.

In the setting of (2), we follow the framework of Gatheral et al. [9]. The form of (2) indicates that the trading before time t has a decayed effect on the observed price at time t . We note that when the decay kernel function is set to be the Dirac function $\delta_0(\cdot)$, this model of observed price degenerates to the framework of Almgren and Chriss [2]. Indeed, we have

$$\int_0^t \delta_0(t-s)g(v_s)ds = g(v_t), \quad (3)$$

which is just the form in Almgren and Chriss [2].

The experiences from practitioners indicate that a trading signal usually has the properties of stationarity and mean reversion. Motivated by this, we propose a signal process L_t in (1) and suppose L_t follows an Ornstein–Uhlenbeck process as below:

$$dL_t = -\gamma L_t dt + \sigma_2 dW_t, \quad (4)$$

where γ is the speed of mean reversion such that $\gamma > 0$ and W_t is a standard Brownian motion.

In practice, an order may not be executed fully due to the shortage of liquidity in the market or some other technical reasons. So, traders may face an execution risk. Some previous works have talked about this topic, and we follow the setting in Cheng et al. [13].

We suppose the executing process follows the stochastic process below:

$$dx_t = -v_t dt + m(v_t)dZ_t, \quad (5)$$

where x_t is the amount of stock that remains to be liquidated, v_t is the trading rate, $m(\cdot)$ is a function that affects the diffusion of this process, and Z_t is a standard Brownian motion.

For mathematical tractability, we suppose the standard Brownian motions B_t , W_t , and Z_t are independent.

3. Optimal Execution Problem

With the setting in previous parts, we propose our optimal execution problem in this section.

Following the setting in Cheng et al. [13], we define our PnL as the difference between the realized value by trading and the initial intrinsic value of our position. Indeed, a selling order always pushes the stock price down, so we obtain lower price than the initial price, and thus the PnL is always negative. Naturally, we wish the PnL to be larger. Specifically, we define

$$\text{PnL}_t = \int_{t_0}^t (S_{t_0} - \tilde{S}_u) dx_u + x_t (S_t - S_{t_0}). \quad (6)$$

So, at time T , the PnL is

$$\text{PnL}_T = \int_{t_0}^T (S_{t_0} - \tilde{S}_u) dx_u + x_T (S_T - S_{t_0}). \quad (7)$$

Note that at time T , we may have $x_T > 0$ due to the execution risk. In this case, we need to liquidate the remaining shares immediately, so we put a punishment on the remaining shares. We denote the punishment as $\lambda(x_T)$. In the setting of Cheng et al. [13], the punishment function is quadratic, i.e., $\lambda(x) = -\alpha x^2$ with $\alpha > 0$, which is compatible with the results of Almgren and Chriss [2]. In our work, we also follow this treatment. Hence, we define the adjusted PnL as

$$\text{PnL}_{\text{adj}} = \text{PnL}_T + \lambda(x_T). \quad (8)$$

With all the settings above, we can get the specific form of the adjusted PnL, which is illustrated in Proposition 1.

Proposition 1. *With the settings of (1), (2), (4), (5), (7), and (8) and the assumption that the Brownian motions B_t , W_t , and Z_t are independent, the adjusted PnL defined in (8) has the following expression:*

$$\begin{aligned} \text{PnL}_{\text{adj}} = & \lambda(x_T) + \frac{\theta}{2}(x_T^2 - x_{t_0}^2) + \int_0^T \left[v_t \int_{t_0}^t G(t-s)g(v_s)ds + \frac{\theta}{2}m^2(v_t) + \rho L_t x_t \right] dt \\ & - \int_{t_0}^T m(v_t) \int_{t_0}^t G(t-s)g(v_s)ds dZ_t + \int_{t_0}^T \sigma_1 x_t dB_t. \end{aligned} \quad (9)$$

Proof. Applying Itô's formula and with (1) and (5), we have

$$\begin{aligned} d(x_t S_t) &= x_t dS_t + S_t dx_t + dS_t dx_t, \\ dS_t dx_t &= \theta(dx_t)^2 = \theta m^2(v_t)dt. \end{aligned} \quad (10)$$

By integrating, we get

$$x_T S_T = \int_{t_0}^T x_t dS_t + \int_{t_0}^T S_t dx_t + \int_{t_0}^T \theta m^2(v_t)dt + x_{t_0} S_{t_0}. \quad (11)$$

On the other hand, with (2), we have

$$\int_{t_0}^T (S_{t_0} - \tilde{S}_u) dx_u = S_{t_0}(x_T - x_{t_0}) - \int_{t_0}^T S_t dx_t - \int_{t_0}^T \int_{t_0}^t G(t-s)g(v_s)ds dx_t. \quad (12)$$

Combining (7), (8), (11), and (12), we get

$$\begin{aligned} \text{PnL}_{\text{adj}} = & \lambda(x_T) + \int_{t_0}^T x_t dS_t + \int_{t_0}^T \theta m^2(v_t) dt \\ & - \int_{t_0}^T \int_{t_0}^t G(t-s) g(v_s) ds dx_t. \end{aligned} \quad (13)$$

Applying Itô's formula to (5), we have

$$dx_t^2 = 2x_t dx_t + m^2(v_t) dt. \quad (14)$$

By integrating, we get

$$\int_{t_0}^T x_t dx_t = \frac{1}{2}(x_T^2 - x_{t_0}^2) - \frac{1}{2} \int_{t_0}^T m^2(v_t) dt. \quad (15)$$

Substituting (1), (5), and (15) into (13), we get expression (9). \square

The expression in (9) indicates that the randomness of the adjusted PnL comes from three stochastic sources, i.e., Brownian motions B_t , W_t , and Z_t . In addition, it is worth noting that the last two terms of (9) are Itô integrations and thus are martingales. Hence, the expectation of the adjusted PnL can be expressed as

$$E_{t_0}(\text{PnL}_{\text{adj}}) = E_{t_0} \left\{ \lambda(x_T) + \frac{\theta}{2}(x_T^2 - x_{t_0}^2) + \int_0^T \left[v_t \int_{t_0}^t G(t-s) g(v_s) ds + \frac{\theta}{2} m^2(v_t) + \rho L_t x_t \right] dt \right\}. \quad (16)$$

Note that $E_t(\cdot)$ represents $E(\cdot | x_t = x, L_t = l)$ in here and the other parts of the following context.

With the specific form of the adjusted PnL, it is natural for us to propose an optimal execution problem. More specifically, we look for an optimal trading rate process v_t to maximize the expected utility of the adjusted PnL with a utility function. Here we choose the identity utility function and formulate our optimal execution problem as follows:

$$\begin{cases} \max_{v_t, t_0 \leq t \leq T} E_{t_0}(\text{PnL}_{\text{adj}}) \\ \text{s.t.} \begin{cases} dx_t = -v_t dt + m(v_t) dZ_t, \\ x_{t_0} = X, \\ dL_t = -\gamma L_t dt + \sigma_2 dW_t, \\ L_{t_0} = l_0, \end{cases} \end{cases} \quad (17)$$

where $E_{t_0}(\text{PnL}_{\text{adj}})$ satisfies (16).

So far, we have proposed our optimal execution problem described in (17). In this optimal execution problem, we assume the decay kernel function $G(\cdot)$ and the transient price impact function $g(\cdot)$ are of generalized forms. The topic about the form of these two functions has been widely discussed in the literature, such as Gatheral [16]. We note that the choices of this two functions may lead to different

types of optimal execution problems, thus requiring different techniques to solve the corresponding optimal execution problems. In the next section, we choose some special decay kernel functions and transient price impact functions so that the optimal execution problem becomes a standard stochastic optimal control problem, and we solve it under different cases of execution risk.

4. Optimal Executing Strategy

In this section, we appropriately choose the decay kernel function and the transient price impact function to solve the optimal execution problem (17).

To make the problem more tractable, we choose the Dirac function as the kernel decay function. As mentioned in Section 2, this case conforms to the setting of Almgren and Chriss [2]. In addition, we set price transient impact function $g(\cdot)$ as a linear function; more specifically,

$$g(v_t) = -\eta v_t, \quad (18)$$

where η is a constant and $\eta > 0$. Besides, we follow the treatment about the punishment function $\lambda(\cdot)$ in Cheng et al. [13], i.e., $\lambda(x) = -\alpha x^2$ with $\alpha > 0$. Hence, the expectation of the adjusted PnL in (16) becomes

$$E_{t_0}(\text{PnL}_{\text{adj}}) = E_{t_0} \left\{ -\alpha x_T^2 + \frac{\theta}{2}(x_T^2 - x_{t_0}^2) + \int_{t_0}^T \left[-\eta v_t^2 + \frac{\theta}{2} m^2(v_t) + \rho L_t x_t \right] dt \right\}. \quad (19)$$

Note that under the conditions above, the optimal execution problem (17) has become a standard stochastic optimal control problem. We define the value function as

$$V(t, x, l) = \max_{v_s, t \leq s \leq T} E_t \left\{ -\alpha x_T^2 + \frac{\theta}{2}(x_T^2 - x_t^2) + \int_t^T \left[-\eta v_s^2 + \frac{\theta}{2} m^2(v_s) + \rho L_s x_s \right] ds \right\}. \quad (20)$$

According to (5) and (15), we have

$$x_T^2 = x_t^2 + \int_t^T (m^2(v_s) - 2v_s x_s) ds + 2 \int_t^T x_s m(v_s) dZ_s. \quad (21)$$

$$V(t, x, l) = \max_{v_s, t \leq s \leq T} E_t \left\{ \int_t^T [-\eta v_s^2 + (\theta - \alpha) m^2(v_s) + (2\alpha - \theta) v_s x_s + \rho L_s x_s] ds \right\} - \alpha x^2. \quad (22)$$

So far, the only undefined term in our optimal execution problem is the function $m(\cdot)$. Note that the form of $m(\cdot)$ determines the execution risk, and we follow the setting of Cheng et al. [13]. More specifically, we solve the optimal execution problem (17) with a constant execution risk and a linear execution risk, respectively.

Substituting (21) into (20), we finally get

4.1. Constant Execution Risk. Note that for the constant execution risk, the function $m(\cdot)$ in (5) is a constant, i.e.,

$$m(v_t) \equiv m_0, \quad (23)$$

where $m_0 > 0$.

Under this circumstance, the value function (22) becomes

$$V(t, x, l) = \max_{v_s, t \leq s \leq T} E_t \left\{ \int_t^T [-\eta v_s^2 + (2\alpha - \theta) v_s x_s + \rho L_s x_s] ds \right\} - \alpha x^2 + (\theta - \alpha) m_0^2 (T - t). \quad (24)$$

Under the conditions above, we solve the optimal execution problem (17) and get the following result in Theorem 1.

Theorem 1. Let $G(\cdot) = \delta_0(\cdot)$, $g(x) = -\eta x$, $\lambda(x) = -\alpha x^2$, $m(\cdot) \equiv m_0$, and $\eta > 0$, $\alpha \geq 0$, $\alpha - (\theta/2) > 0$. The optimal execution problem (17) has a unique solution with state-feedback form:

$$v_t^* = \frac{1}{T - t + \beta} x_t - \frac{\rho [(1 - e^{-\gamma(T-t)}) (\gamma\beta - 1) + \gamma(T-t)]}{2\eta\gamma^2 (T - t + \beta)} L_t, \quad (25)$$

where $\beta = (2\eta/2\alpha - \theta)$ and $\beta > 0$.

In addition, the value function (24) satisfies the following HJB equation:

$$\begin{aligned} V_t + \frac{1}{2} m_0^2 V_{xx} + \frac{1}{2} \sigma_2^2 V_{ll} - \gamma l V_l + \rho l x + \theta m_0^2 \\ + \max_v \{-\eta v^2 - (V_x + \theta x) v\} = 0, \end{aligned} \quad (26)$$

with terminal condition $V(T, x, l) = -\alpha x^2$.

Moreover, $V(t, x, l)$ can be expressed as

$$\begin{aligned} V(t, x, l) = (F_1(t) - \alpha) x^2 + G_1(t) l^2 + H(t) x l \\ + m_0^2 \int_t^T F_1(s) ds + \sigma_2^2 \int_t^T G_1(s) ds \\ + (\theta - \alpha) m_0^2 (T - t), \end{aligned} \quad (27)$$

where $F_1(t)$, $G_1(t)$, and $H(t)$ are defined as (38), (43), and (41).

Remark 1. We remark that we assume $\alpha > (\theta/2)$ to make sure $\beta > 0$.

Proof. Instead of taking the standard method to solve the stochastic control problem (17), we use the method of completing the square.

Suppose $F_1(t)$, $F_2(t)$, $G_1(t)$, $G_2(t)$, and $H(t)$ are bounded differentiable functions with $F_1(T) = 0$, $F_2(T) = 0$, $G_1(T) = 0$, $G_2(T) = 0$, and $H(T) = 0$.

Applying Itô's formula, we have

$$\begin{aligned} F_1(T) x_T^2 = F_1(t) x_t^2 + \int_t^T F_1'(s) x_s^2 ds + \int_t^T 2F_1(s) x_s dx_s \\ + \int_t^T F_1(s) (dx_s)^2. \end{aligned} \quad (28)$$

With equation (5), we have

$$\begin{aligned} F_1(T) x_T^2 = F_1(t) x_t^2 \\ + \int_t^T [F_1'(s) x_s^2 - 2F_1(s) x_s v_s + m_0^2 F_1(s)] ds \\ + \int_t^T 2m_0 F_1(s) x_s dZ_s. \end{aligned} \quad (29)$$

Taking similar procedures, we have

$$\begin{aligned}
F_2(T)x_T &= F_2(t)x_t + \int_t^T [F_2'(s)x_s - F_2(s)v_s]ds + \int_t^T m_0 F_2(s)dZ_s, \\
G_1(T)L_T^2 &= G_1(t)L_t^2 + \int_t^T [G_1'(s)L_s^2 - 2\gamma G_1(s)L_s^2 + \sigma_2^2 G_1(s)]ds + \int_t^T 2\sigma_2 G_1(s)L_s dW_s, \\
G_2(T)L_T &= G_2(t)L_t + \int_t^T [G_2''(s)L_s - \gamma G_2(s)L_s]ds + \int_t^T \sigma_2^2 G_2(s)dW_s, \\
H(T)L_T x_T &= H(t)L_t x_t + \int_t^T [H'(s)L_s x_s - H(s)L_s v_s - \gamma H(s)x_s L_s]ds + \int_t^T m_0 H(s)L_s dZ_s + \int_t^T \sigma_2 H(s)x_s dW_s.
\end{aligned} \tag{30}$$

For convenience, we define $J(t, x, l)$ as

$$J(t, x, l) = E_t \left\{ \int_t^T [-\eta v_s^2 + (2\alpha - \theta)v_s x_s + \rho L_s x_s] ds \right\}, \tag{31}$$

and hence

$$V(t, x, l) = \max_{v_s, t \leq s \leq T} J(t, x, l) - \alpha x^2 + (\theta - \alpha)m_0^2(T - t). \tag{32}$$

Note that $F_1(T) = 0$, $F_2(T) = 0$, $G_1(T) = 0$, $G_2(T) = 0$, and $H(T) = 0$; then, we have

$$J(t, x, l) = J(t, x, l) + E_t \{ F_1(T)x_T^2 + F_2(T)x_T + G_1(T)L_T^2 + G_2(T)L_T + H(T)L_T x_T \}. \tag{33}$$

With the conclusions above, we have

$$\begin{aligned}
J(t, x, l) &= E_t \left\{ \int_t^T [-\eta v_s^2 + F_1'(s)x_s^2 + [G_1'(s) - 2\gamma G_1(s)]L_s^2 + [2\alpha - \theta - 2F_1(s)]x_s v_s \right. \\
&\quad \left. - H(s)L_s v_s + [H'(s) - \gamma H(s) + \rho]x_s L_s - F_2(s)v_s + F_2'(s)x_s \right. \\
&\quad \left. + [G_2'(s) - \gamma G_2(s)]L_s] ds \right\} + F_1(t)x^2 + G_1(t)l^2 + H(t)xl + \int_t^T m_0^2 F_1(s) + \int_t^T \sigma_2^2 G_1(s) ds.
\end{aligned} \tag{34}$$

To use the method of completing square, we compare the first integration term above with

$$\int_t^T -\eta \{v_s - g_1(s)x_s - g_2(s)L_s\}^2 ds, \tag{35}$$

where $g_1(s)$, $g_2(s)$ are functions of s .

Matching coefficients, we can get the following ODE system:

$$\begin{cases} F_1'(s) = -\eta g_1^2(s), \\ 2\alpha - \theta - 2F_1(s) = 2\eta g_1(s), \\ G_1'(s) - 2\gamma G_1(s) = -\eta g_2^2(s), \\ -H(s) = 2\eta g_2(s), \\ H'(s) - \gamma H(s) + \rho = -2\eta g_1(s)g_2(s), \\ F_2(s) = 0, \\ F_2'(s) = 0, \\ G_2'(s) - \gamma G_2(s) = 0, \end{cases} \tag{36}$$

with terminal conditions $F_1(T) = 0$, $F_2(T) = 0$, $G_1(T) = 0$, $G_2(T) = 0$, and $H(T) = 0$.

To solve $F_1(s)$, we eliminate $g_1(s)$ in the first row of (36) and get the ODE below:

$$F_1'(s) + \frac{1}{\eta} \left(\alpha - \frac{1}{2}\theta - F_1(s) \right)^2 = 0. \tag{37}$$

Solving this special Riccati equation with the terminal condition, we have

$$F_1(t) = \frac{\eta}{\beta} - \frac{\eta}{T - t + \beta}, \tag{38}$$

where $\beta = (2\eta/2\alpha - \theta)$. Consequently, we have

$$g_1(t) = \frac{1}{T - t + \beta}. \tag{39}$$

To solve $H(s)$, we eliminate $g_2(s)$ in the third row of (36) and get the ODE below:

$$H'(s) + [-\gamma - g_1(s)]H(s) + \rho = 0. \tag{40}$$

Substituting (39) into this linear ODE, we solve it to have

$$H(t) = \frac{\rho \left[(1 - e^{-\gamma(T-t)}) (\gamma\beta - 1) + \gamma(T-t) \right]}{\gamma^2 (T - t + \beta)}. \tag{41}$$

Consequently, we have

$$g_2(t) = -\frac{\rho \left[(1 - e^{-\gamma(T-t)}) (\gamma\beta - 1) + \gamma(T-t) \right]}{2\eta\gamma^2 (T - t + \beta)}. \tag{42}$$

Substituting (42) into the ODE in the second row of (36), we solve the ODE to get

$$G_1(t) = \eta e^{2\gamma t} \int_t^T e^{-2\gamma s} g_2^2(s) ds. \quad (43)$$

In addition, it is obvious that $F_2(t) \equiv 0$ and $G_2(t) \equiv 0$ from (36).

With the results above, we have

$$\begin{aligned} J(t, x, l) &= E_t \left\{ \int_t^T -\eta \{v_s - g_1(s)x_s - g_2(s)L_s\}^2 ds \right\} + F_1(t)x^2 + G_1(t)l^2 \\ &\quad + H(t)xl + \int_t^T m_0^2 F_1(s) + \int_t^T \sigma_2^2 G_1(s) ds \\ &\leq F_1(t)x^2 + G_1(t)l^2 + H(t)xl + \int_t^T m_0^2 F_1(s) + \int_t^T \sigma_2^2 G_1(s) ds. \end{aligned} \quad (44)$$

The inequality above indicates that $v_s = g_1(s)x_s + g_2(s)L_s$ is the unique solution to maximize $J(t, x, l)$ and thus the unique solution to the optimal execution problem (17). As a consequence, the value function $V(t, x, l)$ has the following expression:

$$\begin{aligned} V(t, x, l) &= (F_1(t) - \alpha)x^2 + G_1(t)l^2 + H(t)xl \\ &\quad + m_0^2 \int_t^T F_1(s) ds + \sigma_2^2 \int_t^T G_1(s) ds \\ &\quad + (\theta - \alpha)m_0^2(T - t). \end{aligned} \quad (45)$$

Taking the partial derivatives of $V(t, x, l)$, we have

$$\begin{cases} V_t = F_1'(t)x^2 + G_1'(t)l^2 + H'(t)xl - m_0^2 F_1(t) - \sigma_2^2 G_1(t) - (\theta - \alpha)m_0^2, \\ V_{xx} = 2F_1(t) - 2\alpha, \\ V_{ll} = 2G_1(t), \\ V_x = 2F_1(t)x + H(t)l - 2\alpha x, \\ V_l = 2G_1(t)l + H(t)x. \end{cases} \quad (46)$$

Then, it is straightforward to verify that $V(t, x, l)$ satisfies the HJB equation below:

$$\begin{aligned} V_t + \frac{1}{2}m_0^2 V_{xx} + \frac{1}{2}\sigma_2^2 V_{ll} - \gamma l V_l + \rho l x + \theta m_0^2 \\ + \max_v \{-\eta v^2 - (V_x + \theta x)v\} = 0. \end{aligned} \quad (47)$$

□

This theorem indicates that the optimal executing strategy v_t is a linear combination of the remaining position x_t and the trading signal L_t and thus a dynamic executing strategy. Note that when $\rho = 0$ and letting $\alpha \rightarrow +\infty$, we have $v_t^* = (x_t/T - t)$, which is an adaptive VWAP strategy. Furthermore, we denote the weight of L_t in (20) as $w(\gamma)$, i.e.,

$$w(\gamma) = \frac{\rho[(1 - e^{-\gamma(T-t)})(\gamma\beta - 1) + \gamma(T-t)]}{2\eta\gamma^2(T-t+\beta)}, \quad (48)$$

and provide the following results of Corollary 1.

Corollary 1. *With the assumptions in Theorem 1, the weight $w(\gamma)$ of the trading signal L_t in the expression of the optimal executing strategy (25) is monotonic increasing with respect to the mean reversion speed γ of the trading signal for $\gamma \in (0, +\infty)$. In addition, when limiting γ to 0, the limitation of $w(\gamma)$ exists and can be expressed as*

$$\lim_{\gamma \rightarrow 0} w(\gamma) = -\frac{\rho}{4\eta} \left(T - t + \beta - \frac{\beta^2}{T - t + \beta} \right). \quad (49)$$

Proof. We prove the monotonicity first. For convenience, we define $\tilde{w}(\gamma)$ as

$$\tilde{w}(\gamma) = \frac{(1 - e^{-\gamma(T-t)})(\gamma\beta - 1) + \gamma(T-t)}{\gamma^2}, \quad (50)$$

and hence $w(\gamma) = -(\rho/2\eta(T-t+\beta))\tilde{w}(\gamma)$.

Taking the derivative of \tilde{w} , we have

$$\begin{aligned} \bar{w}'(\gamma) &= \frac{e^{-\gamma(T-t)}}{\gamma^3} \left\{ \beta(T-t)\gamma^2 - (T-t-\beta)\gamma - 2 \right. \\ &\quad \left. - [(T-t+\beta)-2]e^{\gamma(T-t)} \right\}. \end{aligned} \quad (51)$$

Further, we define $a(\gamma)$ as

$$a(\gamma) = \beta(T-t)\gamma^2 - (T-t-\beta)\gamma - 2 - [(T-t+\beta)-2]e^{\gamma(T-t)}, \quad (52)$$

and hence $\bar{w}'(\gamma) = (e^{-\gamma(T-t)}/\gamma^3)a(\gamma)$ and $a(0) = 0$.

Taking the derivative of $a(t)$, we have

$$\begin{aligned} a'(\gamma) &= 2\beta(T-t)\gamma - (T-t-\beta) - e^{\gamma(T-t)} \\ &\quad \cdot [(T-t)(T-t+\beta)\gamma + \beta - (T-t)], \end{aligned} \quad (53)$$

with $a'(0) = 0$.

Again, taking the derivative of $a'(\gamma)$, we have

$$a''(\gamma) = 2\beta(T-t) - e^{\gamma(T-t)} [(T-t)^2(T-t+\beta)\gamma + 2\beta(T-t)]. \quad (54)$$

So, it is straightforward to verify that $a''(\gamma) < 0$ for $\gamma \in (0, +\infty)$ with $\beta > 0$, $T-t > 0$, and $e^{\gamma(T-t)} > 1$. Since $a'(0) = 0$, we conclude that $a'(\gamma) < 0$ for $\gamma \in (0, +\infty)$. Furthermore, with $a(0) = 0$, we conclude that $a(\gamma) < 0$ for $\gamma \in (0, +\infty)$, and thus $\bar{w}'(\gamma) < 0$ for $\gamma \in (0, +\infty)$. Finally, the definition of $\bar{w}(\gamma)$ indicates that $w'(\gamma) > 0$ for $\gamma \in (0, +\infty)$, which means $w(\gamma)$ is monotonic increasing with respect to γ for $\gamma \in (0, +\infty)$.

In addition, given the Taylor expansion of $e^{-\gamma(T-t)}$ as

$$e^{-\gamma(T-t)} = 1 - (T-t)\gamma + \frac{1}{2}(T-t)^2\gamma^2 + o(\gamma^2), \quad (55)$$

we then have

$$\begin{aligned} &(1 - e^{-\gamma(T-t)})(\gamma\beta - 1) + \gamma(T-t) \\ &= \beta(T-t)\gamma^2 + \frac{1}{2}(T-t)^2\gamma^2 + o(\gamma^2). \end{aligned} \quad (56)$$

Substituting this to (48) and taking the limit, we get (49). \square

We remark that the corollary above indicates our result can recover the result of Cheng et al. [13] with the mean reversion speed γ of the trading signal L_t degenerating to 0.

4.2. Linear Execution Risk. For the linear execution risk, the function $m(\cdot)$ in (5) is a linear function, i.e.,

$$m(v_t) = m_0 v_t, \quad (57)$$

where $m_0 > 0$.

In this case, the execution risk is related to the trading rate. Specifically, the faster we trade, the bigger the probability that our orders cannot be fully filled. This is in line with our intuition and reality in the market. Indeed, the liquidity of market is limited. If we trade very fast, our orders may merely be filled partially.

Now the value function (22) is of the following form:

$$V(t, x, l) = \max_{v_s, t \leq s \leq T} E_t \left\{ \int_t^T [(\theta - \alpha)m_0^2 - \eta] v_s^2 + (2\alpha - \theta)v_s x_s + \rho L_s x_s \right\} - \alpha x^2. \quad (58)$$

Then, we solve the optimal execution problem (17) and get the following theorem.

Theorem 2. Let $G(\cdot) = \delta_0(\cdot)$, $g(x) = -\eta x$, $\lambda(x) = -\alpha x^2$, $m(v_t) = m_0 v_t$, and $\eta > 0$, $\alpha > 0$, $m_0 > 0$. In addition, we assume $\alpha - (\theta/2) > 0$ and the inequality below holds:

$$\log \left| \frac{(\theta/2) - (\eta/m_0^2)}{\alpha - (\theta/2)} \right| + 1 - \frac{(\theta/2) - (\eta/m_0^2)}{\alpha - (\theta/2)} + \frac{T}{m_0^2} < 0. \quad (59)$$

Then, the optimal execution problem (17) has a unique solution with state-feedback form:

$$v_t^* = \frac{A_1(t) - \alpha + (\theta/2)}{(A_1(t) + \theta - \alpha)m_0^2 - \eta} \left[x_t - \frac{\rho e^{-\gamma(T-t)}}{2} \int_t^T \frac{e^{\gamma(T-s)}}{\alpha - (\theta/2) - A_1(s)} ds \cdot L_t \right], \quad (60)$$

where $A_1(t) = E(t) + \alpha - (\theta/2)$ and $E(t)$ is defined as

$$E(t) = \inf \left\{ E | q(E) = \log \left| \alpha - \frac{\theta}{2} \right| + \frac{(\theta/2) - (\eta/m_0^2)}{\alpha - (\theta/2)} - \frac{T-t}{m_0^2} \right\}, \quad (61)$$

where $q(x) = \log|x| - (k/x)$, $k = (\theta/2) - (\eta/m_0^2)$.

In addition, the value function $V(t, x, l)$ defined as (58) satisfies the HJB equation below:

$$\begin{aligned} &V_t + \frac{1}{2}\sigma^2 V_{ll} - \gamma l V_l + \rho l x + \max_v \left\{ \left(\frac{m_0^2}{2} V_{xx} + \theta m_0^2 - \eta \right) v^2 \right. \\ &\quad \left. - (\theta x + V_x) v \right\} = 0, \end{aligned} \quad (62)$$

with terminal condition $V(T, x, l) = -\alpha x^2$.

Moreover, $V(t, x, l)$ can be expressed as follows:

$$V(t, x, l) = (A_1(t) - \alpha)x^2 + B_1(t)l^2 + C(t)xl + \sigma_2^2 \int_t^T B_1(s)ds, \quad (63)$$

where $B_1(t)$ and $C(t)$ are defined as (79) and (77).

Remark 2. We remark that the condition $\alpha > (\theta/2)$ makes sure that $\log(\alpha - (\theta/2))$ and $(1/\alpha - (\theta/2))$ are well defined.

Condition (59) guarantees that the optimal solution makes the optimal problem achieve the maximum.

Proof. We use the method of completing the square to solve the optimal execution problem again.

Suppose $A_1(t)$, $A_2(t)$, $B_1(t)$, and $C(t)$ are bounded differentiable functions with $A_1(T) = 0$, $A_2(T) = 0$, $B_1(T) = 0$, $B_2(T) = 0$, and $C(T) = 0$.

Applying Itô's formula and with (5), we have

$$\begin{aligned} A_1(T)x_T^2 &= A_1(t)x_t^2 + \int_t^T [A_1'(s)x_s^2 - 2A_1(s)x_s v_s + m_0^2 A_1(s)v_s^2]ds + \int_t^T 2m_0 A_1(s)x_s v_s dZ_s, \\ A_2(T)x_T &= A_2(t)x_t + \int_t^T [A_2'(s)x_s - A_2(s)v_s]ds + \int_t^T m_0 A_2(s)v_s dZ_s, \\ B_1(T)L_T^2 &= B_1(t)L_t^2 + \int_t^T [B_1'(s)L_s^2 - 2\gamma B_1(s)L_s^2 + \sigma_2^2 B_1(s)]ds + \int_t^T 2\sigma_2 B_1(s)L_s dW_s, \\ B_2(T)L_T &= B_2(t)L_t + \int_t^T [B_2'(s)L_s - \gamma B_2(s)L_s]ds + \int_t^T \sigma_2^2 B_2(s)dW_s, \\ C(T)L_T x_T &= C(t)L_t x_t + \int_t^T [C'(s)L_s x_s - C(s)L_s v_s - \gamma C(s)x_s L_s]ds + \int_t^T m_0 C(s)L_s v_s dZ_s + \int_t^T \sigma_2 C(s)x_s dW_s. \end{aligned} \quad (64)$$

Again, we define $J(t, x, l)$ as

$$J(t, x, l) = E_t \left\{ \int_t^T [(\theta - \alpha)m_0^2 - \eta]v_s^2 + (2\alpha - \theta)v_s x_s + \rho L_s x_s \right\} ds. \quad (65)$$

Note that $A_1(T) = 0$, $A_2(T) = 0$, $B_1(T) = 0$, $B_2(T) = 0$, and $C(T) = 0$, and we have

$$\begin{aligned} J(t, x, l) &= J(t, x, l) + E_t \{ A_1(T)x_T^2 + A_2(T)x_T \\ &\quad + B_1(T)L_T^2 + B_2(T)L_T + C(T)L_T x_T \}. \end{aligned} \quad (66)$$

With the conclusions above, we have

$$\begin{aligned} J(t, x, l) &= E_t \int_t^T \{ [(A_1(s) + \theta - \alpha)m_0^2 - \eta]v_s^2 + A_1'(s)x_s^2 + [B_1'(s) - 2\gamma B_1(s)]L_s^2 \\ &\quad + [2\alpha - \theta - 2A_1(s)]x_s v_s - C(s)L_s v_s + [C'(s) - \gamma C(s) + \rho]x_s L_s \\ &\quad - A_2(s)v_s + A_2'(s)x_s + [B_2'(s) - \gamma B_2(s)]L_s \} ds + A_1(t)x^2 \\ &\quad + B_1(t)l^2 + C(t)xl + \int_t^T \sigma_2^2 B_1(s)ds. \end{aligned} \quad (67)$$

We define $D(s) = [A_1(s) + \theta - \alpha]m_0^2 - \eta$. Note that under the assumptions in the theorem we have $D(s) < 0$, which will be verified later. We compare the first integration term above with

$$\int_t^T D(s) \{ v_s - p_1(s)x_s - p_2(s)L_s \}^2 ds, \quad (68)$$

where $p_1(t)$ and $p_2(t)$ are deterministic functions of t .

Matching coefficients, we get the ODE system as follows:

$$\begin{cases} A_1'(s) = D(s)p_1^2(s), \\ 2\alpha - \theta - 2A_1(s) = -2D(s)p_1(s), \\ B_1'(s) - 2\gamma B_1(s) = D(s)p_2^2(s), \\ -C(s) = -2D(s)p_2(s), \\ C'(s) - \gamma C(s) + \rho = 2D(s)p_1(s)p_2(s), \\ A_2(s) = 0, \\ A_2'(s) = 0, \\ B_2'(s) - \gamma B_2(s) = 0, \end{cases} \quad (69)$$

with terminal conditions $A_1(T) = 0$, $A_2(T) = 0$, $B_1(T) = 0$, $B_2(T) = 0$, and $C(T) = 0$.

To solve $A_1(t)$, we eliminate $p_1(s)$ in the first row of (69), and then we have

$$4A_1'(s) \left[(A_1(s) + \theta - \alpha)m_0^2 - \eta \right] - [2\alpha - \theta - 2A_1(s)]^2 = 0, \quad (70)$$

with $A_1(T) = 0$.

Solving this ODE, we have

$$\begin{aligned} \log \left| \alpha - \frac{\theta}{2} - A_1(t) \right| + \frac{(\theta/2) - (\eta/m_0^2)}{\alpha - (\theta/2) - A_1(t)} \\ = \log \left| \alpha - \frac{\theta}{2} \right| + \frac{(\theta/2) - (\eta/m_0^2)}{\alpha - (\theta/2)} - \frac{T-t}{m_0^2}. \end{aligned} \quad (71)$$

We denote $E(t) = A_1(t) - \alpha + (\theta/2)$ and $q(x) = \log|x| - (k/x)$, $k = (\theta/2) - (\eta/m_0^2)$, and then we have

$$q(E(t)) = \log \left| \alpha - \frac{\theta}{2} \right| + \frac{(\theta/2) - (\eta/m_0^2)}{\alpha - (\theta/2)} - \frac{T-t}{m_0^2}. \quad (72)$$

Hence, we define $E(t)$ as

$$\begin{aligned} E(t) = \inf \left\{ E | q(E) = \log \left| \alpha - \frac{\theta}{2} \right| \right. \\ \left. + \frac{(\theta/2) - (\eta/m_0^2)}{\alpha - (\theta/2)} - \frac{T-t}{m_0^2} \right\}, \quad t \in [0, T]. \end{aligned} \quad (73)$$

Thus,

$$A_1(t) = E(t) + \alpha - \frac{\theta}{2}. \quad (74)$$

Now we verify that $D(t) < 0$. Note that $D(t) < 0$ is equivalent to $E(t) < -k$ according to (74). When $k \leq 0$, $q(x)$ is monotonic decreasing on $(-\infty, 0)$, and its value domain on $(-\infty, 0)$ is $(-\infty, \infty)$. According to the definition of $E(t)$, we conclude that $E(t) < 0$, and thus $E(t) < -k$. When $k > 0$,

$q(x)$ is monotonic decreasing on $(-\infty, -k)$ and monotonic increasing on $(-k, 0)$. Besides, the value domain of $q(x)$ on interval $(-\infty, -k)$ and interval $(-k, 0)$ is $(q(-k), +\infty)$. So if $q(E(t)) > q(-k)$ for any t in $[0, T]$, according to the definition of $E(t)$, we conclude that $E(t) < -k$. Note that $q(E(t))$ is monotonic increasing on $[0, T]$. So, it suffices to make $q(E(0)) > q(-k)$ hold, which is just guaranteed by assumption (59). So, we conclude that $D(t) < 0$.

With $A_1(t)$ and according to (69), we have

$$p_1(t) = -\frac{\alpha - (\theta/2) - A_1(t)}{(A_1(t) + \theta - \alpha)m_0^2 - \eta}. \quad (75)$$

To solve $C(t)$, we eliminate $p_2(t)$ in the third row of (69) to get

$$C'(t) + [-\gamma + p_1(t)]C(t) + \rho = 0. \quad (76)$$

Solving this ODE with (75), we have

$$C(t) = \rho e^{-\gamma(T-t)} \frac{\alpha - (\theta/2) - A_1(t)}{\alpha - (\theta/2)} \int_t^T \frac{(\alpha - (\theta/2))e^{\gamma(T-s)}}{\alpha - (\theta/2) - A_1(s)} ds. \quad (77)$$

Consequently,

$$p_2(t) = -\frac{\rho e^{-\gamma(T-t)} [A_1(t) - \alpha + (\theta/2)]}{2[(A_1(t) + \theta - \alpha)m_0^2 - \eta]} \int_t^T \frac{e^{\gamma(T-s)}}{\alpha - (\theta/2) - A_1(s)} ds. \quad (78)$$

Moreover, solving the ODE of $B_1(t)$ in the second row of (69) with $p_2(t)$, we have

$$B_1(t) = e^{2\gamma t} \int_t^T e^{-2\gamma s} [(A_1(s) + \theta - \alpha)m_0^2 - \eta] p_2^2(s) ds. \quad (79)$$

In addition, it is obvious that $A_2(t) \equiv 0$ and $B_2(t) \equiv 0$ from (69).

With the results above, we have

$$\begin{aligned} J(t, x, l) = E_t \left\{ \int_t^T D(s) \{v_s - p_1(s)x_s - p_2(s)L_s\}^2 ds \right\} + A_1(t)x^2 + B_1(t)l^2 + C(t)xl \\ + \int_t^T \sigma_2^2 G_1(s) ds \leq A_1(t)x^2 + B_1(t)l^2 + C(t)xl + \int_t^T \sigma_2^2 G_1(s) ds. \end{aligned} \quad (80)$$

The inequality above indicates $v_t = p_1(t)x_t + p_2(t)L_t$ is the unique solution to maximize $J(t, x, l)$ and thus the unique solution to the optimal execution problem (60). Therefore, the value function (58) can be expressed as

$$V(t, x, l) = (A_1(t) - \alpha)x^2 + B_1(t)l^2 + C(t)xl + \sigma_2^2 \int_t^T B_1(s) ds. \quad (81)$$

To calculate the partial derivatives on $V(t, x, l)$, we have

$$\begin{cases} V_t = A_1'(t)x^2 + B_1'(t)l^2 + C'(t)xl - \sigma_2^2 B_1(t), \\ V_{xx} = 2A_1(t) - 2\alpha, \\ V_{ll} = 2B_1(t), \\ V_x = 2A_1(t)x + C(t)l - 2\alpha x, \\ V_l = 2B_1(t)l + C(t)x. \end{cases} \quad (82)$$

Then, it is straightforward to verify that $V(t, x, l)$ satisfies the HJB equation below:

$$V_t + \frac{1}{2}\sigma_2^2 V_{ll} - \gamma l V_l + \rho l x + \max_v \left\{ \left(\frac{m_0^2}{2} V_{xx} + \theta m_0^2 - \eta \right) v^2 - (\theta x + V_x) v \right\} = 0. \quad (83)$$

□

Note that the optimal strategy (60) is also a linear combination of the remaining position x_t and the trading signal L_t . This means that the trading signal can affect the optimal execution strategy. In addition, the weights of these two terms are affected by the parameter m_0 of the execution risk, and thus the optimal executing strategy is also affected by the execution risk.

We remark that in the case of linear execution risk, the optimal executing strategy (60) can also recover the result in Cheng et al. [13] with mean reversion speed γ degenerating to 0. To illustrate this conclusion, we note that definition (73) implies $E(t)$ is bounded for $t \in [0, T]$ and the verification process of $H(t) < 0$ in the proof of Theorem 2 implies that the value domain of $E(t)$ for $t \in [0, T]$ does not include 0.

Hence, the function $(e^{\gamma(T-t)}/E(t))$ is bounded for $t \in [0, T]$. With $E(t) = A_1(t) - \alpha + (\theta/2)$, we have

$$\lim_{\gamma \rightarrow 0} \int_t^T \frac{e^{\gamma(T-s)}}{\alpha - (\theta/2) - A_1(s)} ds = \int_t^T \frac{1}{\alpha - (\theta/2) - A_1(s)} ds. \quad (84)$$

ODE (70) indicates

$$\frac{ds}{\alpha - (\theta/2) - A_1(s)} = \frac{m_0^2 [A_1(s) + \theta - \alpha - (\eta/m_0^2)]}{[\alpha - (\theta/2) - A_1(s)]^3} dA_1(s). \quad (85)$$

By integration, we have

$$\int_t^T \frac{1}{\alpha - (\theta/2) - A_1(s)} ds = m_0^2 \left\{ \frac{(1/2)((\theta/2) - (\eta/m_0^2)) - (\alpha - (\theta/2))}{(\alpha - (\theta/2))^2} + \frac{1}{\alpha - (\theta/2) - A_1(t)} - \frac{(1/2)((\theta/2) - (\eta/m_0^2))}{[\alpha - (\theta/2) - A_1(t)]^2} \right\}. \quad (86)$$

Finally, we have

$$\lim_{\gamma \rightarrow 0} p_2(t) = \frac{\rho}{2[A_1(t) + \theta - \alpha - (\eta/m_0^2)]} \left\{ \frac{(1/2)((\theta/2) - (\eta/m_0^2)) - (\alpha - (\theta/2))}{(\alpha - (\theta/2))^2} \left[\alpha - \left(\frac{\theta}{2} \right) - A_1(t) \right] + 1 + \frac{(1/2)((\theta/2) - (\eta/m_0^2))}{\alpha - (\theta/2) - A_1(t)} \right\}, \quad (87)$$

which is just the form in Cheng et al. [13].

5. Conclusion

In this paper, we study the optimal execution problem by taking into account the trading signal and the execution risk simultaneously. More specifically, we combine the frameworks of Almgren and Chriss [2] and Gatheral et al. [9] and propose a trading signal term, which follows an Ornstein–Uhlenbeck process, in the intrinsic price process of the stock. In addition, the execution process is affected by execution risk. Under these settings, we propose an optimal executing problem with the decay kernel function and transient impact function being of generalized form. Then, we solve the optimal execution problem with the decay kernel being the Dirac function and the transient impact function being a linear function in the cases of the constant execution risk and the linear execution risk, respectively. We give analytical solutions to the optimal execution problems and prove that our result can recover previous work when the mean reversion speed of the trading signal process degenerates to 0.

Further work can try other types of decay kernel functions and nonlinear transient impact functions. Besides,

other utility functions of the adjusted PnL can be taken into account. Empirical work can also be conducted to validate and calibrate the theoretical model.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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