# Control, Stability, and Qualitative Theory of Dynamical Systems 

Guest Editors: Nazim Idrisoqlu MaHmudov, Mark A. McKibben,
Sakithivel Rathinasamy, and Yong Ren


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## Abstract and Applied Analysis

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## Editorial

# Control, Stability, and Qualitative Theory of Dynamical Systems 

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Dynamical systems play a crucial role in the mathematical modeling of phenomena across disciplines. Understanding issues concerning controllability, stability, and other qualitative aspects of such systems is important in enhancing our understanding of the mathematical models described by these systems. This issue compiles 18 manuscripts covering various aspects of deterministic and stochastic dynamical systems theory and its applications. Elements of the theory and techniques from dynamical systems, control and optimal control theory, stochastic analysis, and stochastic evolution equations are used throughout these papers to establish an impressive collection of results. The results established in all articles in this issue have application in multiple disciplines, and they often contain replicable numerical analysis components that could have broader applicability.

On the more abstract end of the spectrum, two papers explore theoretical issues arising in deterministic and stochastic fractional differential equations. Specifically, H. Aktuglu et al. established existence results for Caputo fractional BVPs and G. Shen et al. studied the Holder regularity in local time of the fractional Ornstein-Uhlenbeck process; both have applications in diffusion, finance, and econophysics. Two papers focus on abstract optimal control problems: A. R. Safari et al. establish a maximum principle for abstract systems with integral boundary conditions arising in the mathematical modeling of heat conduction and plasma physics, while S. Meherrem and R. Akbarov study the role that exhausters and quasidifferentiability play in switching control problems arising in the mathematical study of chemical processes, automotive systems, and circuit theory. One
paper, by X. Qin, focuses on the stabilization of a class of stochastic nonholonomic system, which involved constructing a smooth state-feedback control law that ensures certain stochastic properties of the solution. Also, Y. Li and Y. Zhao study tracking control and synchronization of the hyperchaotic Lorenz-Stenflo system, which contributes to the growing literature in that direction.

Three papers study control problems for very different systems (networks, deterministic systems, and stochastic systems), all with time delays. B. Wang and Y. Sun investigated control problems for a multiagent system with heterogeneous delays in directed networks; A. E. Bashirov and M. Jneid established partial complete controllability results for abstract deterministic systems; and X. Zhou et al. studied the BIBO stabilization in the mean square for discrete-time stochastic systems. All of these results have significant applicability for mathematical models involving the respective type of system.

Qualitative results involving the notion of periodicity were established in two of the papers. D. de C. Braga et al. studied the approximation of periodic orbits for dynamical systems of the type arising in the circuit analysis and mathematical ecology. G. Ge and W. Wang established perioddoubling bifurcation results for feedback control systems of damped linear oscillators.

Finally, on the more applied end of the spectrum, nearly half of the papers in this issue study more concrete systems arising in the mathematical models from population ecology, mathematical biology, and mathematical finance. Two papers focus on different aspects of the SIRS epidemic model: W. Liu studied the SIRS model with random perturbations, while
Y. Cai et al. focused on related stochastic dynamics. Four papers studied population models of various types. C. Xu and Y . Wu studied the Lotka-Volterra model with time-varying delays. $\mathrm{H} . \mathrm{Li}$ et al. investigated the (non)persistence of the food chain model with stochastic perturbation. L. Qi and J. Cui discussed the parasite disease, schistosomiasis, with mating structure. And, Z. Luo and L. Luo established the existence, uniqueness, and global attractivity of positive periodic solutions for multispecies population model with impulses. One paper focused on mathematical finance; specifically, Y. Zhai et al. performed a bifurcation analysis for the Rayleigh price model with time delay.

It is our hope that this compilation of papers will provide our readers and researchers with new ideas to continue similar lines of research in control, stability, and qualitative theory of dynamical systems.

Nazim Idrisoglu Mahmudov<br>Mark A. McKibben<br>Sakthivel Rathinasamy Yong Ren

## Research Article

# Chaotic Control and Generalized Synchronization for a Hyperchaotic Lorenz-Stenflo System 

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#### Abstract

This paper is devoted to investigate the tracking control and generalized synchronization of the hyperchaotic Lorenz-Stenflo system using the tracking model and the feedback control scheme. We suppress the chaos to unstable equilibrium via three feedback methods, and we achieve three globally generalized synchronization controls. Novel tracking controllers with corresponding parameter update laws are designed such that the Lorenz-Stenflo systems can be synchronized asymptotically. Moreover, numerical simulations are presented to demonstrate the effectiveness, through the contrast between the orbits before being stabilized and the ones after being stabilized.


## 1. Introduction

Study of chaotic control and generalized synchronization has received great attention in the past several decades [110]; many hyperchaotic systems have been proposed and studied in the last decade, for example, a new hyperchaotic Rössler system [4], the hyperchaotic L system [5], Chua's circuit [6], the hyperchaotic Chen system [7, 8], and so forth. Hyperchaotic system has been proposed for secure communication and the presence of more than one positive Lyapunov exponent clearly improves the security of the communication scheme [9-12]. Therefore, hyperchaotic system generates more complex dynamics than the low-dimensional chaotic system, which has much wider application than the low-dimensional chaotic system.

Until now, a variety of approaches have been proposed for the synchronization of low-dimensional chaotic systems, including Q-S method [ 13,14 ], active control $[15,16]$, adaptive control [17-24], and time-delay feedback control [25]. Recently, Stenflo [26] presented a new hyperchaotic LorenzStenflo (LS) system:

$$
\begin{gathered}
\dot{x}=a(y-x)+d w, \\
\dot{y}=x(c-z)-y \\
\dot{z}=y x-b z \\
\dot{w}=-x-a w .
\end{gathered}
$$

In (1), $x, y, z$, and $w$ are the state variables of the system and $a, b, c$, and $d$ are real constant parameters. System (1) is generated from the originally three-dimensional Lorenz chaotic system by introducing a new control parameter $b$ and a state variable $w$.

In this paper, we will consider chaos control and generalized synchronization related to hyperchaotic Lorenz-Stenflo system. we found that the feedback control achieved in the low-dimensional system like many other studies of dynamics in low-dimensional systems. We suppress the hyperchaotic Lorenz-Stenflo system to unstabilize equilibrium via three control methods: linear feedback control, speed feedback control, and doubly-periodic function feedback control. By designing a nonlinear controller, we achieve the generalized synchronization of two Lorenz-Stenflo systems up to a scaling factor. Moreover, numerical simulations are applied to verify the effectiveness of the obtained controllers.

## 2. The Hyperchaotic Lorenz-Stenflo System

In the following we would like to consider the hyperchaotic cases of system (1). When $a=1.0, b=0.7, c=26$, and $d=1.5$, system (1) exhibits hyperchaotic behavior. Simulated results are depicted in Figures 1 and 2. Figures 1(a)-1(d) depict the projection of the chaotic attractor in different spaces; Figures 2(a)-2(d) depict the states of system (1) before being


Figure 1: Chaotic attractor in different spaces: (a) $(x, y, z),(\mathrm{b})(x, y, w),(\mathrm{c})(x, w, z)$, and (d) $(y, w, z)$.
stabilized, respectively. The volume of the elements in the phase space $\delta X(t)=\delta x \delta y \delta z \delta w$ and the divergence of flow (1) are defined by

$$
\begin{equation*}
\nabla X=\frac{\partial X}{\partial x}+\frac{\partial X}{\partial y}+\frac{\partial X}{\partial z}+\frac{\partial X}{\partial w}=-(2 a+b+1) \tag{2}
\end{equation*}
$$

where $X=(\dot{x}, \dot{y}, \dot{z}, \dot{w})=[a(y-x)+d w, x(c-z)-y, y x-$ $b z,-x-a w]$.

System (1) is dissipative when $2 a+b+1>0$. Moreover, an exponential contraction rate is given by

$$
\begin{equation*}
\frac{d X(t)}{d t}=-(2 a+b+1) X(t) \tag{3}
\end{equation*}
$$

It is clear that $X(t)=X_{0} e^{-(2 a+b+1) t}$, which implies that the solutions of system (1) are bounded as $t->+\infty$. It is easy to find the three equilibria $E_{1}(0,0,0,0)$,

$$
\begin{gathered}
E_{2}\left(\frac{\sqrt{\left((c-1) a^{2}-d\right) b\left(d+a^{2}\right)}}{d+a^{2}}\right. \\
\frac{\sqrt{\left((c-1) a^{2}-d\right) b\left(d+a^{2}\right)}}{a^{2}}
\end{gathered}
$$

$$
\left.\begin{array}{c}
\frac{(c-1) a^{2}-d}{a^{2}}, \\
\\
\left.-\frac{\sqrt{\left(d+a^{2}\right) b\left(a^{2} c-a^{2}-d\right)}}{\left(d+a^{2}\right) a}\right) \\
E_{3}\left(-\frac{\sqrt{\left((c-1) a^{2}-d\right) b\left(d+a^{2}\right)}}{d+a^{2}},\right. \\
 \tag{4}\\
\\
\\
\\
\\
\\
\\
\frac{(c-1) a^{2}-d}{a^{2}}, \frac{\sqrt{\left((c-1) a^{2}-d\right) b\left(d+a^{2}\right)}}{a^{2}}, \\
\left(d+a^{2}\right) b\left(a^{2} c-a^{2}-d\right) \\
\left(d+a^{2}\right) a
\end{array}\right) .
$$

To determine the stability of the equilibria point $E_{1}(0$, $0,0,0$ ), evaluating the Jacobian matrix of system (1) at $E_{1}$ yields

$$
\left.J\right|_{E_{1}}=\left(\begin{array}{cccc}
-a & a & 0 & d  \tag{5}\\
c & -1 & 0 & 0 \\
0 & 0 & -b & 0 \\
-1 & 0 & 0 & -a
\end{array}\right)
$$



Figure 2: The states of system (1) before being stabilized: (a) $x-t$, (b) $y-t$, (c) $z-t$, and (d) $w-t$.

If $a=1.0, b=0.7, c=26$, and $d=1.5$, the four eigenvalues of the characteristic polynomial of Jacobian matrix (5) are

$$
\begin{equation*}
\lambda_{1}=3.94975, \quad \lambda_{2}=-5.9497, \quad \lambda_{3}=-1, \quad \lambda_{4}=-0.7 \tag{6}
\end{equation*}
$$

Thus, the equilibria $E_{1}$ is a saddle point of the hyperchaotic system (1). The Jacobian matrix of system (1) at $E_{2}$ yields

$$
\left.J\right|_{E_{2}}=\left(\begin{array}{ccc}
-a & a & 0  \tag{7}\\
d \\
c & -1 & \frac{\sqrt{\left((c-1) a^{2}-d\right) b\left(d+a^{2}\right)}}{d+a^{2}} \\
\frac{\sqrt{\sqrt{\left((c-1) a^{2}-d\right) b\left(d+a^{2}\right)}}}{a^{2}} & \frac{\sqrt{\left((c-1) a^{2}-d\right) b\left(d+a^{2}\right)}}{d+a^{2}} & -b \\
-1 & 0 & 0
\end{array}\right.
$$

If $a=1.0, b=0.7, c=26$, and $d=1.5$, the four eigenvalues of the characteristic polynomial of Jacobian matrix (7) are

$$
\lambda_{1}=4.8887, \quad \lambda_{2}=-6.3032
$$

$$
\begin{equation*}
\lambda_{3}=-1.1427+0.5438 i, \quad \lambda_{4}=-1.1427-0.5438 i \tag{8}
\end{equation*}
$$

Thus, the equilibria $E_{2}$ are unstable, and $E_{3}$ is similar.


Figure 3: The states of system (9): (a) $x-t$, (b) $y-t$, (c) $z-t$, (d) $w-t$.

## 3. The Hyperchaotic Control for Lorenz-Stenflo System

In this section, we control the hyperchaotic system (1) such that all trajections converge to the equilibrium point ( $0,0,0,0$ ). The controlled hyperchaotic Lorenz-Stenflo system is given by

$$
\begin{gather*}
\dot{x}=a(y-x)+d w+u_{1}, \\
\dot{y}=x(c-z)-y+u_{2}, \\
\dot{z}=y x-b z+u_{3},  \tag{9}\\
\dot{w}=-x-a w+u_{4},
\end{gather*}
$$

where $u_{1}, u_{2}, u_{3}$, and $u_{4}$ are external control inputs which will be suitably derived from the trajectory of the chaotic system (1), specified by $(x, y, z, w)$ to the equilibrium $(0,0,0)$ of uncontrolled system $u_{i}=0(i=1,2,3,4)$.
3.1. Linear Function Feedback Control. For the modified hyperchaotic Lorenz-Stenflo system (9), if one of the following feedback controllers $u_{i}(i=1,2,3,4)$ is chosen for the
system (9), then $u_{1}=-k_{1} x_{1}, u_{2}=-k_{2} y, u_{3}=-k_{3} z$, $u_{4}=-k_{4} w$, and where $k_{i}$ 's are feedback coefficients. Therefore controlled system (9) is rewritten as

$$
\begin{gather*}
\dot{x}=a(y-x)+d w-k_{1} x, \\
\dot{y}=x(c-z)-y-k_{2} y, \\
\dot{z}=y x-b z-k_{3} z,  \tag{10}\\
\dot{w}=-x-a w-k_{4} w
\end{gather*}
$$

whose Jacobian matrix is

$$
J=\left(\begin{array}{cccc}
-a-k_{1} & a & 0 & d  \tag{11}\\
c & -1-k_{2} & 0 & 0 \\
0 & 0 & -b-k_{3} & 0 \\
-1 & 0 & 0 & -a-k_{4}
\end{array}\right)
$$

The characteristic equation of $J$ is

$$
\begin{aligned}
\lambda^{4} & +(-B-A-D-C) \lambda^{3} \\
& +(C D+A B+B D+B C-c a+A D+A C+d) \lambda^{2}
\end{aligned}
$$



Figure 4: The states of system (19): (a) $x-t$, (b) $y-t$, (c) $z-t$, and (d) $w-t$.

$$
\begin{aligned}
& +(-A B D-A B C-B C D+c a D \\
& \quad+c a C-A C D-d C-d B) \lambda+E=0,
\end{aligned}
$$

where $A=-a-k_{1}, B=-1-k_{2}, C=-b-k_{3}, D=-a-k_{4}$, and $E=A B C D-c a C D+d B C$.

The abbreviated characteristic equation is

$$
\begin{equation*}
\lambda^{4}+R_{1} \lambda^{3}+R_{2} \lambda^{2}+R_{3} \lambda+E=0 \tag{13}
\end{equation*}
$$

where $R_{1}=-B-A-D-C, R_{2}=C D+A B+B D+B C-c a+$ $A D+A C+d$, and $R_{3}=-A B D-A B C-B C D+c a D+c a C-$ $A C D-d C-d B, R_{4}=E$.

According to the Routh-Hurwitz criterion, constraints are imposed as follows:

$$
\begin{gathered}
H_{1}=R_{1}=-B-A-D-C>0, \\
\begin{aligned}
H_{2} & =\left|\begin{array}{cc}
R_{1} & R_{3} \\
1 & R_{2}
\end{array}\right|=R_{1} R_{2}-R_{3} \\
& =(-A-C-B) D^{2}
\end{aligned}
\end{gathered}
$$

$$
\begin{align*}
& +\left((-2 A-2 C) B-B^{2}-A^{2}-C^{2}-d-2 A C\right) D \\
& +(-A-C) B^{2}-A^{2} C+\left(-2 A C+c a-A^{2}-C^{2}\right) B  \tag{12}\\
& +\left(-d-C^{2}+c a\right) A>0 \\
& H_{3}=\left|\begin{array}{ccc}
R_{1} & R_{3} & 0 \\
1 & R_{2} & E \\
0 & R_{1} & R_{3}
\end{array}\right|=R_{1} R_{2} R_{3}-E R_{1}^{2}-R_{3}^{2}>0 \\
& H_{4}=\left|\begin{array}{cccc}
R_{1} & R_{3} & 0 & 0 \\
1 & R_{2} & R_{4} & 0 \\
0 & R_{1} & R_{3} & 0 \\
0 & 1 & R_{2} & R_{4}
\end{array}\right| \\
& \quad=R_{1} R_{2} R_{3} R_{4}-R_{1}^{2} R_{4}^{2}-R_{3}^{2} R_{4}>0 . \tag{14}
\end{align*}
$$

This characteristic polynomial has four roots, all with negative real roots, under the condition of $H_{1}>0, H_{2}>0$, $H_{3}>0$, and $H_{4}>0$. Therefore, the equilibria $(0,0,0,0)$ are the stable manifold $W^{s}$ and the controlled chaotic Lorenz-Stenflo system (9) is asymptotically stable. The concrete dynamics of


Figure 5: The states of system (25): (a) $x-t$, (b) $y-t$, (c) $z-t$, and (d) $w-t$.
(9) can be demonstrated by Proposition 1, while the Maple program is demonstrated in Appendix A.

Proposition 1. If one chooses the control coefficients $k_{1}=8$, $k_{2}=4, k_{3}=3$, and $k_{4}=2$ and the parameters $a=1.0$, $b=0.7, c=26$, and $d=1.5$, the controlled chaotic LorenzStenflo system (9) is asymptotically stable at the equilibrium ( $0,0,0,0$ ).

Proof. When the parameters were selected by the above value, we obtain the Jacobian matrix

$$
J=\left(\begin{array}{cccc}
-9.0 & 1.0 & 0 & 1.5  \tag{15}\\
26 & -5 & 0 & 0 \\
0 & 0 & -3.7 & 0 \\
-1 & 0 & 0 & -3.0
\end{array}\right)
$$

The characteristic equation of $J$ is given by

$$
\begin{equation*}
\lambda^{4}+20 \frac{7}{10} \lambda^{3}+125 \frac{2}{5} \lambda^{2}+295 \frac{3}{4} \lambda+238 \frac{13}{20}=0 \tag{16}
\end{equation*}
$$

According to Appendix A, we easily obtain

$$
\begin{array}{ll}
H_{1}=20 \frac{7}{10}>0, & H_{2}=2300>0 \\
H_{3}=578>0, & H_{4}=\frac{69}{500}>0 \tag{17}
\end{array}
$$

which yields the eigenvalues via the compute simulation

$$
\begin{gather*}
\lambda_{1}=-12.36844706, \quad \lambda_{2}=-1.931149367 \\
\lambda_{3}=-2.700403573, \quad \lambda_{4}=-3.70000 \tag{18}
\end{gather*}
$$

Thus the zero solution of system (9) is exponentially stable, Proposition 1 is proved.

Numerical simulations are used to investigate the controlled chaotic Lorenz-Stenflo system (1) using the fourthorder Runge-Kutta scheme with time step 0.01 . The parameters and the corresponding feedback coefficients are given by the above value. The initial values are taken as $[x(0)=$ $1, y(0)=0.7, z(0)=20, w(0)=0.1]$. The behaviors of the states $(x, y, z, w)$ of the controlled chaotic Lorenz-Stenflo system (1) with time are displayed in Figures 3(a)-3(d).


Figure 6: The solutions of the master and slave systems without active control law. (a) Signals $x_{1}$ (the dashed line) and $x_{2}$ (the solid line). (b) Signals $y_{1}$ (the dashed line) and $y_{2}$ (the solid line). (c) Signals $z_{1}$ (the dashed line) and $z_{2}$ (the solid line). (d) Signals $w_{1}$ (the dashed line) and $w_{2}$ (the solid line).
3.2. Speed Function Feedback Control. Suppose that $u_{1}=u_{3}=$ $0 ; u_{2}$ and $u_{4}$ are of the speed forms $u_{2}=k_{2}(-b z+x y)$ and $u_{4}=-k_{4}[a(y-x)+d w]$, where $k_{2}$ and $k_{4}$ are speed feedback coefficients. Therefore the controlled chaotic system (9) is rewritten as

$$
\begin{gather*}
\dot{x}=a(y-x)+d w \\
\dot{y}=x(c-z)-y+k_{2}(-b z+x y)  \tag{19}\\
\dot{z}=y x-b z \\
\dot{w}=-x-a w-k_{4}[a(y-x)+d w] .
\end{gather*}
$$

The Jacobian matrix is

$$
J=\left(\begin{array}{cccc}
-a & a & 0 & d  \tag{20}\\
c & -1 & -k_{2} b & 0 \\
0 & 0 & -b & 0 \\
-1+k_{4} a & -k_{4} a & 0 & -a-k_{4} d
\end{array}\right)
$$

The characteristic equation of $J$ is

$$
\begin{aligned}
& \lambda^{4}+\left(1-k_{1}+2 a+b\right) \lambda^{3} \\
& \quad+\left(2 a b-k_{1} b-k_{1} a+2 a-k_{1}+b+d+a^{2}-c a\right) \lambda^{2}
\end{aligned}
$$

$$
\begin{align*}
& +\left(-c a^{2}-c a b+2 a b+d b+a^{2} b-k_{1} a\right. \\
& \left.\quad+a^{2}-k_{1} b-k_{1} b a+d\right) \lambda \\
& -c a^{2} b-k_{1} b a+a^{2} b+d b=0 . \tag{21}
\end{align*}
$$

Proposition 2. If one chooses the control coefficients: $k_{2}=9$, $k_{4}=-1$ and the parameters $a=1.0, b=0.7, c=26$, and $d=1.5$, the controlled chaotic Lorenz-Stenflo system (9) is asymptotically stable at the equilibrium ( $0,0,0,0$ ).

Similar to Proposition 1, the proof of Proposition 2 is straightforward and thus is omitted.

In the following, we give the eigenvalues via the compute simulation. When the parameters were selected by the above value, we obtain the Jacobian matrix

$$
J=\left(\begin{array}{cccc}
-1.0 & 1.0 & 0 & 1.5  \tag{22}\\
26 & -1 & -6.3 & 0 \\
0 & 0 & -0.7 & 0 \\
-2.0 & 1.0 & 0 & 0.5
\end{array}\right)
$$



Figure 7: The solutions of the drive and response systems with control law. (a) Signals $x_{1}$ (the dashed line) and $x_{2}$ (the solid line). (b) Signals $y_{1}$ (the dashed line) and $y_{2}$ (the solid line). (c) Signals $z_{1}$ (the dashed line) and $z_{2}$ (the solid line). (d) Signals $w_{1}$ (the dashed line) and $w_{2}$ (the solid line).

The characteristic equation of $J$ changes the following:

$$
\begin{equation*}
\lambda^{4}+2.2 \lambda^{3}-21.95 \lambda^{2}-39.6 \lambda-16.45=0 . \tag{23}
\end{equation*}
$$

The eigenvalues of the above equation are

$$
\begin{gather*}
\lambda_{1}=-5.10412, \quad \lambda_{2}=-1  \tag{24}\\
\lambda_{3}=-0.70000, \quad \lambda_{4}=-4.60412
\end{gather*}
$$

Numerical simulations are used to investigate the controlled chaotic Lorenz-Stenflo system (20) using the fourth-order Runge-Kutta scheme with time step 0.01 . The initial values are taken as $[x(0)=1, y(0)=0.7, z(0)=20, w(0)=0.1]$. The behaviors of the states $(x, y, z, w)$ of the controlled chaotic Lorenz-Stenflo system (20) with time are displayed in Figures 4(a)-4(d).
3.3. The Doubly Periodic Function Feedback Control. Suppose that $u_{2}=0, u_{3}=0$, and $u_{4}=0 ; u_{1}$ is of the doubly periodic function $u_{1}=k_{1} c n(x, m)$, where $k_{1}$ is speed feedback coefficients and $0<m<1$ is the modulus of Jacobi
elliptic function. Therefore the controlled chaotic system (9) is rewritten as

$$
\begin{gather*}
\dot{x}=a(y-x)+d w+k_{1} c n(x, m), \\
\dot{y}=x(c-z)-y, \\
\dot{z}=y x-b z,  \tag{25}\\
\dot{w}=-x-a w .
\end{gather*}
$$

The Jacobian matrix is

$$
J=\left(\begin{array}{cccc}
-a+k_{1} & a & 0 & d  \tag{26}\\
c & -1 & 0 & 0 \\
-1 & 0 & 0 & -a
\end{array}\right)
$$

The characteristic equation of $J$ is

$$
\begin{aligned}
& \lambda^{4}+\left(1-k_{1}+2 a+b\right) \lambda^{3} \\
& \quad+\left(2 a b-k_{1} b-k_{1} a+2 a-k_{1}+b\right. \\
& \left.\quad+d+a^{2}-c a\right) \lambda^{2}
\end{aligned}
$$



Figure 8: The dynamics of synchronization errors. (a) Signal $e_{1}$, (b) signal $e_{2}$, (c) signal $e_{3}$ and (d) signal $e_{4}$.

$$
\begin{align*}
& +\left(-c a^{2}-c a b+2 a b+d b+a^{2} b-k_{1} a\right. \\
& \left.\quad+a^{2}-k_{1} b-k_{1} b a+d\right) \lambda \\
& -c a^{2} b-k_{1} b a+a^{2} b+d b=0 . \tag{27}
\end{align*}
$$

Proposition 3. If one chooses the control coefficients $k_{1}=-30$, $m=0.3$ and the parameters $a=1.0, b=0.7, c=26$, and $d=1.5$, the controlled chaotic Lorenz-Stenflo system (9) is asymptotically stable at the equilibrium ( $0,0,0,0$ ).

The proof of Proposition 3 is the same as Proposition 1, which is straightforward and thus is omitted.

In the following, we give the eigenvalues via the compute simulation. When the parameters were selected by the above value, we obtain the Jacobian matrix

$$
J=\left(\begin{array}{cccc}
-31.0 & 1.0 & 0 & 1.5  \tag{28}\\
26 & -1 & 0 & 0 \\
0 & 0 & -0.7 & 0 \\
-1.0 & 0 & 0 & -1.0
\end{array}\right)
$$

The characteristic equation of $J$ changes the following:

$$
\begin{equation*}
\lambda^{4}+33.7 \lambda^{3}+61.60 \lambda^{2}+33.450 \lambda+4.55=0 \tag{29}
\end{equation*}
$$

The eigenvalues of the above equation are

$$
\begin{gather*}
\lambda_{1}=-31.795569, \quad \lambda_{2}=-0.20443,  \tag{30}\\
\lambda_{3}=-1, \quad \lambda_{4}=-0.70000 .
\end{gather*}
$$

Thus the zero solution of system (9) is exponentially stable; Proposition 3 is proved.

Numerical simulations are used to investigate the controlled chaotic Lorenz-Stenflo system (25) using the fourthorder Runge-Kutta scheme with time step 0.01 . The initial values are taken as $[x(0)=1, y(0)=0.7, z(0)=20, w(0)=$ $0.1]$. The behaviors of the states $(x, y, z, w)$ of the controlled chaotic Lorenz-Stenflo system (25) with time are displayed in Figures 5(a)-5(d).


FIGURE 9: The projection of the synchronized attractors in different spaces: (a) $(x, y, z),(\mathrm{b})(x, y, w),(\mathrm{c})(x, w, z)$, and (d) ( $y, w, z)$. The solid line denotes and the drive system; the dashed line denotes respond system synchronized.

## 4. Globally Exponential Hyperchaotic Projective Synchronization Control

Consider two chaotic systems given by

$$
\begin{gather*}
\dot{x}_{m}=f\left(x_{m}, t\right),  \tag{31}\\
\dot{y}_{s}=g\left(y_{s}, t\right)+u\left(x_{m}, y_{s}, t\right), \tag{32}
\end{gather*}
$$

where $x_{m}=\left(x_{1 m}, x_{2 m}, \ldots x_{n m}\right)^{T}, y_{s}=\left(y_{1 s}, y_{2 s}, \ldots y_{n s}\right)^{T}$, $f, g \in C^{r}\left[R_{+} \times R^{n}, R^{n}\right], u \in C^{r}\left[R_{+} \times R^{n} \times R^{n}, R^{n}\right]$, and $r \geq 1$. $R_{+}$comprises the set of non negative real numbers. Assume that (31) is the master system, (32) is the slave system, and $u\left(x_{m}, y_{s}, t\right)$ is the control vector. Let the error state be

$$
\begin{align*}
e(t)= & {\left[e_{1}(t), e_{2}(t) \ldots, e_{n}(t)\right]^{T} } \\
= & {\left[x_{1 m}-\left(a_{11} y_{1 s}+a_{12}\right) y_{1 s}, \ldots, x_{n m}\right.}  \tag{33}\\
& \left.-\left(a_{n 1} y_{n s}+a_{n 2}\right) y_{1 s}\right] .
\end{align*}
$$

Then the error dynamics of $e(t)$ are defined by

$$
\begin{equation*}
\dot{e}(t)=f\left(x_{m}, t\right)-g\left(y_{s}, t\right)-u\left(x_{m}, y_{s}, t\right) . \tag{34}
\end{equation*}
$$

The slave and master systems are said to be exponential, hyperchaotic, projective and synchronized if, for all $x_{m}\left(t_{0}\right)$, $y_{s}\left(t_{0}\right) \in R^{n}$, and $i \in R^{n},\left\|x_{m}-\left(a_{i 1} y_{s}+a_{i 2}\right) y_{s}\right\| \rightarrow 0$ as $t \rightarrow \infty$.

Lemma 4 (see $[27,28]$ ). The zero solution of the error dynamical system (34) is globally and exponentially stable; the master-slave systems (31) and (32) are globally and exponentially projective, synchronized, if there exists a positive definite quadratic polynomial $V=\left(e_{1}, e_{2}, \ldots e_{n}\right) P\left(e_{1}, e_{2}, \ldots e_{n}\right)^{T}$ such that $d V / d t=\left(e_{1}, e_{2}, \ldots e_{n}\right) Q\left(e_{1}, e_{2}, \ldots e_{n}\right)^{T}$. Moreover, the following negative Lyapunov exponent estimation for the error dynamical system (34) holds:

$$
\begin{equation*}
\sum_{i=1}^{n} e_{i}^{2}(t) \leq \frac{\lambda_{\max }(P)}{\lambda_{\min }(P)} \sum_{i=1}^{n} e_{i}^{2}(0) \exp \left[-\frac{\lambda_{\max }(Q)}{\lambda_{\min }(P)}(t)\right], \tag{35}
\end{equation*}
$$

where $P=P^{T} \in R^{n \times n}$ and $Q=Q^{T} \in R^{n \times n}$ are both positive definite matrices, $\lambda_{\max }(P)$ and $\lambda_{\min }(P)$ stand for the minimum and maximum eigenvalues of the matrix $P$, respectively, and $\lambda_{\text {min }}(Q)$ denotes the minimum eigenvalue of the matrix $Q$.

In the following, we consider the hyperchaotic system (1) as a master system:

$$
\begin{gather*}
\dot{x}_{m}=a\left(y_{m}-x_{m}\right)+d w_{m} \\
\dot{y}_{m}=x_{m}\left(c-z_{m}\right)-y_{m} \\
\dot{z}_{m}=y_{m} x_{m}-b z_{m}  \tag{36}\\
\dot{w}_{m}=-x_{m}-a w_{m}
\end{gather*}
$$



FIGURE 10: Simulated phase portraits of the chaotic system (1), projected on (a) the $x-y$ plane, (b) the $x-z$ plane, (c) the $x-w$ plane, (d) the $y-z$ plane, (e) the $y-w$ plane, and (f) the $z-w$ plane.
and the system related to (36), given by

$$
\begin{gathered}
\dot{x}_{s}=a\left(y_{s}-x_{s}\right)+d w_{s}+u_{1} \\
\dot{y}_{s}=x_{s}\left(c-z_{s}\right)-y_{s}+u_{2} \\
\dot{z}_{s}=y_{s} x_{s}-b z_{s}+u_{3} \\
\dot{w}_{s}=-x_{s}-a w_{s}+u_{4}
\end{gathered}
$$

as a slave system, where the subscripts " $m$ " and " $s$ " stand for the master system and slave system, respectively. Let the error state be

$$
\begin{aligned}
e(t) & =\left(e_{1}, e_{2}, e_{3}, e_{4}\right)^{T} \\
& =\left[x_{m}-\left(a_{11} x_{s}+a_{12}\right) x_{s},\right.
\end{aligned}
$$



FIgURE 11: The solutions of the master and slave systems with control law. (a) Signals $x_{1}$ (the dashed line) and $x_{2}$ (the solid line). (b) Signals $y_{1}$ (the dashed line) and $y_{2}$ (the solid line). (c) Signals $z_{1}$ (the dashed line) and $z_{2}$ (the solid line). (d) Signals $w_{1}$ (the dashed line) and $w_{2}$ (the solid line).

$$
\begin{align*}
& y_{m}-\left(a_{21} y_{s}+a_{22}\right) y_{s} \\
& z_{m}-\left(a_{31} z_{s}+a_{32}\right) z_{s} \\
& \left.w_{m}-\left(a_{41} w_{s}+a_{42}\right) w_{s}\right]^{T} \tag{38}
\end{align*}
$$

Then the derivative of $e(t)$ along the trajectories of (36) and (37), we obtain the error system:

$$
\begin{aligned}
\dot{e}_{1}= & a\left(y_{m}-x_{m}\right)+d w_{m}-2 a_{11} x_{s} \\
& \times\left[a\left(y_{s}-x_{s}\right)+b w_{s}+u_{1}\right] \\
& -\left[a\left(y_{s}-x_{s}\right)+b w_{s}+u_{1}\right] a_{12} \\
\dot{e}_{2}= & \left(c-z_{m}\right) x_{m}-y_{m}-2 a_{21} y_{s} \\
& \times\left[\left(c-z_{s}\right) x_{s}-y_{s}+u_{2}\right] \\
& -\left[\left(c-z_{s}\right) x_{s}-y_{s}+u_{2}\right] a_{22} \\
\dot{e}_{3}= & -b z_{m}+x_{m} y_{m}-2 a_{31} z_{s} \\
& \times\left(-d z_{s}+x_{s} y_{s}+u_{3}\right) \\
& -\left(-d z_{s}+x_{s} y_{s}+u_{3}\right) a_{32}
\end{aligned}
$$

$$
\begin{align*}
\dot{e}_{4}= & -x_{m}-a w_{m}-2 a_{41} w_{s}\left(-x_{s}-a w_{s}+u_{4}\right) \\
& -\left(-x_{s}-a w_{s}+u_{4}\right) a_{42} . \tag{39}
\end{align*}
$$

To demonstrate the synchronization control between systems (36) and (37), we have the following cases based on [29-44].

Case 1 (modified projective synchronization control).
Theorem I. When $\left[a_{11}, a_{12}, a_{21}, a_{22}, a_{31}, a_{32}, a_{41}, a_{42}\right]=[0, m$, $0, m, 0, m, 0, m], a>0$ and $b>0$. For the hyperchaotic system (1), if one of the following families of feedback controllers $u_{i}(i=$ $1,2,3,4)$ is given for the slave system (36):

$$
\begin{gather*}
u_{1}=\frac{(d-b) m w_{2}-v_{1}}{m}, \\
u_{2}=\frac{(1-m) m x_{2} z_{2}-v_{2}}{m},  \tag{40}\\
u_{3}=\frac{\left(m x_{2} y_{2}-b z_{2}+d z_{2}-x_{2} y_{2}\right) m-v_{3}}{m}, \\
u_{4}=\frac{-x_{1}+m x_{2}-v_{4}}{m},
\end{gather*}
$$



Figure 12: The dynamics of synchronization errors. (a) Signal $e_{1}$, (b) signal $e_{2}$, (c) signal $e_{3}$, and (d) signal $e_{4}$.
there exist many possible choices for $v_{1}, v_{2}, v_{3}$, and $v_{4}$. Then the zero solution of the error dynamical system (39) is globally and exponentially stable, and thus globally exponential modified projective synchronization can be achieved.

The concrete proof of Theorem I can be demonstrated by Appendix B. In the following, tracking numerical simulations are used to solve differential equations (36), (37), and (42) with Runge-Kutta integration method. The initial values of the drive system and response system are $x_{1}(0)=0.1, y_{1}(0)=$ $0.1, z_{1}(0)=20$, and $w_{1}(0)=0.1$ and $x_{2}(0)=0.2, y_{2}(0)=0.2$, $z_{2}(0)=21$, and $w_{2}(0)=0.2$ respectively. The parameters are chosen to be $m=-2, a=1.0, b=0.7, d=1.5$, and $c=$ 26 so that the Lorenz-Stenflo hyperchaotic system exhibits a chaotic behavior if no control is applied. The numerical simulation of the master and the slave systems without active synchronization control law is shown in Figures 6(a)-6(d). The diagram of the solutions of the master and the slave systems with feedback control law is presented in Figures 7(a)-7(d). The synchronization errors are shown in Figures $8(a)-8(d)$. Figures $9(a)-9(d)$ depicts the projection of the
synchronized attractors. Figures $10(\mathrm{a})-10(\mathrm{f})$ depicts the phase portraits of the synchronized attractors.

Case 2 (generalized projective synchronization control).
Theorem II. When $\left[a_{11}, a_{12}, a_{21}, a_{22}, a_{31}, a_{32}, a_{41}, a_{42}\right]=[0$, $\left.m_{1}, 0, m_{2}, 0, m_{3}, 0, m_{4}\right], m_{i}$ 's are different, $a>0$ and $b>0$. For the hyperchaotic system (1), if one of the following families of feedback controllers $u_{i}(i=1,2,3,4)$ is given for the slave system (36):

$$
\begin{gather*}
u_{1}=\frac{\left(m_{2}-m_{1}\right) a y_{2}+\left(d m_{4}-m_{1} b\right) w_{2}-v_{1}}{m_{1}} \\
u_{2}=\frac{\left(m_{1}-m_{2}\right) c x_{2}+\left(m_{2}-m_{1} m_{3}\right) c x_{2}-v_{2}}{m_{2}}  \tag{41}\\
u_{3}=\frac{(d-b) m_{3} z_{2}+\left(m_{1} m_{2}-m_{3}\right) c x_{2}-v_{3}}{m_{3}}, \\
u_{4}=\frac{-x_{1}+m_{4} x_{2}-v_{4}}{m_{4}}
\end{gather*}
$$



Figure 13: The function projection of the synchronized attractors in different spaces: (a) $(x, y, z),(b)(x, y, w),(\mathrm{c})(x, w, z)$, and (d) $(y, w, z)$. The solid line denotes the drive system; the dashed line denotes respond system synchronized.
there exist many possible choices for $v_{1}, v_{2}, v_{3}$, and $v_{4}$. Then the zero solution of the error dynamical system (39) is globally and exponentially stable, and thus globally exponential generalized projective synchronization.

The concrete proof of Theorem II can be demonstrated by Appendix B. In the following, tracking numerical simulations are used to solve differential equations (36), (37), and (39) with Runge-Kutta integration method. The parameters are chosen to be $m_{1}=2, m_{2}=3, m_{3}=3$, and $m_{4}=5$. The initial values and others parameters are the same as the above cases so that the Lorenz-Stenflo hyperchaotic system exhibits a chaotic behavior if no control is applied. The diagram of the solutions of the master and the slave systems with feedback control law is presented in Figures 11(a)-11(d). The synchronization errors are shown in Figures 12(a)-12(d).

Case 3 (function synchronization control [45-58]).
Theorem III. When $f_{1}=a_{11} x_{2}+a_{12}, f_{2}=a_{21} y_{2}+a_{22}$, $f_{3}=a_{31} z_{2}+a_{32}, f_{4}=a_{41} w_{2}+a_{42}, a_{i 1} \neq 0, a>0, b>0$. For the hyperchaotic system (1), if one of the following families of feedback controllers $u_{i}(i=1,2,3,4)$ is given for the slave system (36):

$$
\begin{gather*}
u_{1}=\frac{\left(f_{2}-f_{1}\right) a y_{2}+\left(d f_{4}-f_{1} b\right) w_{2}-k_{1} e_{1}}{f_{1}} \\
u_{2}=\frac{\left(f_{1}-f_{2}\right) c x_{2}+\left(f_{2}-f_{1} f_{3}\right) c x_{2}-k_{2} e_{2}}{f_{2}}  \tag{42}\\
u_{3}=\frac{(d-b) f_{3} z_{2}+\left(f_{1} f_{2}-f_{3}\right) c x_{2}-k_{3} e_{3}}{f_{3}} \\
u_{4}=\frac{-x_{1}+f_{4} x_{2}-k_{4} e_{4}}{f_{4}}
\end{gather*}
$$

where $k_{1}>0, k_{2}>0, k_{3}>0$, and $k_{4}>0$, then the zero solution of the error dynamical system (39) is globally stable, and thus global function, projective synchronization occurs between the master systems (36) and (37).

The concrete Proof of Theorem III can be demonstrated by Appendix B. In the following, tracking numerical simulations are used to solve differential equations (36), (37), and (39) with Runge-Kutta integration method. The initial values and the parameters are the same as the above cases so that the Lorenz-Stenflo hyperchaotic system exhibits a chaotic behavior if no control is applied. The diagram of the solutions


FIGURE 14: The dynamics of synchronization errors. (a) Signal $e_{1}$, (b) signal $e_{2}$, (c) signal $e_{3}$ and (d) signal $e_{4}$.
of the master and the slave systems with active control law is presented in Figures 13(a)-13(d). The synchronization errors are shown in Figures 14(a)-14(d).

## 5. Summary and Conclusions

In this paper, we have introduced the tracking control and generalized synchronization of the hyperchaotic system which is different from the Lorenz-Stenflo attractor. We suppress the chaos to unstabilize equilibrium via three feedback methods, and we achieve three globally generalized synchronization controls of two Lorenz-Stenflo systems. As a result, some powerful controllers are obtained. Then, we investigate the hyperchaotic system applying the complex system calculus technique. Moreover, numerical simulations are used to verify the effectiveness of our results, through the contrast between the orbits before being stabilized and the ones after being stabilized.

## Appendices

## A. The Maple Program

>restart: with(student): with(PDEtools): with(linalg): with (LinearAlgebra):
$>J:=$ matrix $(4,4,[[A, a, 0, d],[c, B, 0,0],[0,0, C, 0]$, $[-1,0,0, D]]$ );
$>U:=\operatorname{charmat}(J, \lambda)$;
$>\operatorname{poly} 1:=\operatorname{collect}(\operatorname{det}(U), \lambda)$;
$>a 1:=\operatorname{coeff}($ poly $1, \lambda, 3)$;
$>a 2:=\operatorname{coeff}($ poly1, $\lambda, 2)$;
$>a 3:=\operatorname{coeff}($ poly $1, \lambda, 1)$;
$>a 4:=\operatorname{coeff}($ poly $1, \lambda, 0)$;
$>H:=$ matrix $(4,4,[a 1, a 3,0,0,1, a 2, a 4,0,0, a 1$, a3, $0,0,1, a 2, a 4]$ );
> H11 := submatrix( $H, 1 . .1,1 . .1$ );

$$
\begin{aligned}
& >H 22:=\text { submatrix }(H, 1 . .2,1 . .2) ; \\
& >H 33:=\operatorname{submatrix}(H, 1 . .3,1 . .3) ; \\
& >H 44:=\operatorname{submatrix}(H, 1 . .4,1 . .4) ; \\
& >H 1:=\operatorname{det}(H 11) ; \\
& >H 2:=\operatorname{det}(H 22) ; \\
& >H 3:=\operatorname{det}(H 33) ; \\
& >H 4:=\operatorname{det}(H 44) .
\end{aligned}
$$

## B. The Proof of Theorem

Proof of Theorem I. Consider the controller (40) and choose the following $v_{i}$ :

$$
\begin{gather*}
v_{1}=-a e_{2}-d e_{4}, \\
v_{2}=-c e_{1}+e_{1} e_{3}+e_{1} m z_{2}+m x_{2} e_{3},  \tag{B.1}\\
v_{3}=-e_{2} e_{1}-e_{2} m x_{2}-m y_{2} e_{1}, \\
v_{4}=(a-1) e_{4} .
\end{gather*}
$$

Then the system (39) is reduced into

$$
\begin{gather*}
\dot{e}_{1}=-a e_{1}, \\
\dot{e}_{2}=-e_{2},  \tag{B.2}\\
\dot{e}_{3}=-b e_{3}, \\
\dot{e}_{4}=-e_{4} .
\end{gather*}
$$

Let us consider the Lyapunov function for the system (B.2) as follows:

$$
\begin{equation*}
V\left(e_{1}, e_{2}, e_{3}, e_{4}\right)=\frac{1}{2}\left(e_{1}^{2}+e_{2}^{2}+e_{3}^{2}+e_{4}^{2}\right) \tag{B.3}
\end{equation*}
$$

In addition, the derivative of $V$ has the form

$$
\begin{equation*}
\frac{d V(t)}{d t}=-\left(a e_{1}^{2}+e_{2}^{2}+b e_{3}^{2}+e_{4}^{2}\right) \tag{B.4}
\end{equation*}
$$

which is negatively defined. So (B.2) is asymptotically stable. This implies that the two Lorenz-Stenflo hyperchaotic systems are projective and synchronized.

Proof of Theorem II. Consider the controller (41) and choose $v_{i}$ as follows:

$$
\begin{gather*}
v_{1}=-a e_{2}-d e_{4}, \\
v_{2}=-c e_{1}+e_{1} e_{3}+e_{1} m_{3} z_{2}+m_{1} x_{2} e_{3},  \tag{B.5}\\
v_{3}=-e_{2} e_{1}-e_{2} m_{1} x_{2}-m_{2} y_{2} e_{1}, \\
v_{4}=(a-1) e_{4} .
\end{gather*}
$$

Then the following steps are the same (B.2), (B.3), and (B.4). So we know that the two Lorenz-Stenflo hyperchaotic systems are modified, projective, and synchronized.

Proof of Theorem III. Consider the controller (42) and choose the following positive definite, quadratic form of Lyapunov function:

$$
\begin{equation*}
V(t)=\frac{1}{2}\left[e_{1}^{2}+e_{2}^{2}+e_{3}^{2}+e_{4}^{2}\right] \tag{B.6}
\end{equation*}
$$

We differentiate $V(t)$ and substitute the trajectory of system (39) which yields

$$
\begin{align*}
\left.\frac{d V(t)}{d t}\right|_{(18)} & =e_{1} \dot{e}_{1}+e_{2} \dot{e}_{2}+e_{3} \dot{e}_{3} \\
& =-k_{1} e_{1}^{2}-k_{2} e_{2}^{2}-k_{3} e_{3}^{2}-k_{4} e_{4}^{2}  \tag{B.7}\\
& =-\left[e_{1}, e_{2}, e_{3}, e_{4}\right] P\left[e_{1}, e_{2}, e_{3}, e_{4}\right]^{T}
\end{align*}
$$

where $P=\operatorname{diag}\left(k_{1}, k_{2}, k_{3}, k_{4}\right)$, which implies that the conclusion of Theorem III is true. So (39) is asymptotically stable. This implies that the two Lorenz-Stenflo hyperchaotic systems are projective synchronized functions.

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# Adaptive Exponential Stabilization for a Class of Stochastic Nonholonomic Systems 

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#### Abstract

This paper investigates the adaptive stabilization problem for a class of stochastic nonholonomic systems with strong drifts. By using input-state-scaling technique, backstepping recursive approach, and a parameter separation technique, we design an adaptive state feedback controller. Based on the switching strategy to eliminate the phenomenon of uncontrollability, the proposed controller can guarantee that the states of closed-loop system are global bounded in probability.


## 1. Introduction

The nonholonomic systems cannot be stabilized by stationary continuous state feedback, although it is controllable, due to Brockett's theorem [1]. So the well-developed smooth nonlinear control theory and the method cannot be directly used in these systems. Many researchers have studied the control and stabilization of nonholonomic systems in the nonlinear control field and obtained some success [26]. It should be mentioned that many literatures consider the asymptotic stabilization of nonholonomic systems; the exponential convergence is also an important topic theme, which is demanded in many practical applications. However, the exponential regulation problem, particularly the systems with parameterization, has received less attention. Recently, [3] firstly introduced a class of nonholonomic systems with strong nonlinear uncertainties and obtained global exponential regulation. References [4, 5] studied a class of nonholonomic systems with output feedback control. Reference [6] combined the idea of combined input-statescaling and backstepping technology, achieving the asymptotic stabilization for nonholonomic systems with nonlinear parameterization.

It is well known that when the backstepping designs were firstly introduced, the stochastic nonlinear control had obtained a breakthrough [7]. Based on quartic Lyapunov
functions, the asymptotical stabilization control in the large of the open-loop system was discussed in [8]. Further research was developed by the recent work [9-16]. [17-19] studied a class of nonholonomic systems with stochastic unknown covariance disturbance. Since stochastic signals are very prevalent in practical engineering, the study of nonholonomic systems with stochastic disturbances is very significant. So, there exists a natural problem that is how to design an adaptive exponential stabilization for a class of nonholonomic systems with stochastic drift and diffusion terms. Inspired by these papers, we will study the exponential regulation problem with nonlinear parameterization for a class of stochastic nonholonomic systems. We use the input-state-scaling, the backstepping technique, and the switching scheme to design a dynamic state-feedback controller with $\sum^{T} \sum \neq I$; the closed-loop system is globally exponentially regulated to zero in probability.

This paper is organized as follows. In Section 2, we give the mathematical preliminaries. In Section 3, we construct the new controller and offer the main result. In the last section, we present the conclusions.

## 2. Problem Statement and Preliminaries

In this paper, we consider a class of stochastic nonholonomic systems as follows:

$$
\begin{gather*}
d x_{0}=d_{0}(t) u_{0} d t+f_{0}\left(t, x_{0}\right) d t \\
d x_{i}=d_{i}(t) x_{i+1} u_{0} d t+f_{i}\left(t, x_{0}, \bar{x}_{i}\right) d t+\varphi_{i}\left(\bar{x}_{i}\right) \sum(t) d \omega \\
\quad i=1, \ldots, n-1, \\
d x_{n}=d_{n}(t) u_{1} d t+f_{n}\left(t, x_{0}, x\right) d t+\varphi_{n}(\bar{x}) \sum(t) d \omega, \tag{1}
\end{gather*}
$$

where $x_{0} \in R$ and $x=\left[x_{1}, \ldots, x_{n}\right]^{T} \in R^{n}$ are the system states and $u_{0} \in R$ and $u_{1} \in R$ are the control inputs, respectively. $\bar{x}_{i}=\left[x_{1}, x_{2}, \ldots, x_{i}\right]^{T} \in R^{i},(i=1,2, \ldots, n)$, and $\bar{x}_{n}=x$; $\omega \in R^{r}$ is an $r$-dimensional standard Wiener process defined on the complete probability space $(\Omega, F, P)$ with $\Omega$ being a sample space, $F$ being a filtration, and $P$ being measure. The drift and diffusion terms $f_{i}(\cdot), \varphi_{i}(\cdot)$ are assumed to be smooth, vanishing at the origin $\left(x_{1}, x_{2}, \ldots, x_{i}\right)=(0,0, \ldots, 0)$; $\sum(t): R_{+} \rightarrow R^{r \times r}$ is the Borel bounded measurable functions and is nonnegative definite for each $t \geq 0 . d_{i}(t)$ are disturbed virtual control coefficients, where $i=0,1 \ldots n$.

Next we introduce several technical lemmas which will play an important role in our later control design.

Consider the following stochastic nonlinear system:

$$
\begin{equation*}
d x=f(x, t) d t+g(x, t) d \omega, \quad x(0)=x_{0} \in R^{n} \tag{2}
\end{equation*}
$$

where $x \in R^{n}$ is the state of system (2), the Borel measurable functions: $f: R^{n+1} \rightarrow R^{n}$ and $g: R^{n+1} \rightarrow R^{n \times r}$ are assumed to be $C^{1}$ in their arguments, and $\omega \in R^{r}$ is an $r$-dimensional standard Wiener process defined on the complete probablity space $(\Omega, F, P)$.

Definition 1 (see [8]). Given any $V(x, t) \in C^{1,2}$, for stochastic nonlinear system (2), the differential operator $L$ is defined as follows:

$$
\begin{equation*}
L V(x, t)=\frac{\partial V}{\partial t}+\frac{\partial V}{\partial x} f+\frac{1}{2} \operatorname{tr}\left(g^{T} \frac{\partial^{2} V}{\partial x^{2}} g\right) \tag{3}
\end{equation*}
$$

where $C^{1,2}\left(R^{n} \times R_{+} ; R_{+}\right)$denotes all nonnegative functions $V(x, t)$ on $R^{n} \times R_{+}$, which are $C^{1}$ in $t$ and $C^{2}$ in $x$, and for simplicity, the smooth function $f(\cdot)$ is denoted by $f$.

Lemma 2 (see [8]). Let $x$ and $y$ be real variables. Then, for any positive integers $m, n$, and any real number $\varepsilon>0$, the following inequality holds:

$$
\begin{align*}
\alpha(\cdot) x^{m} y^{n} \leq & \varepsilon|x|^{m+n}+\frac{n}{m+n}\left(\frac{m+n}{m}\right)^{-m / n}  \tag{4}\\
& \times \alpha(\cdot)^{(m+n) / n} \varepsilon^{-m / n}|y|^{m+n}
\end{align*}
$$

Lemma 3 (see [7]). Considering the stochastic nonlinear system (2), if there exist a $C^{1,2}$ function $V(x, t), K_{\infty}$ class functions $\underline{\alpha}$ and $\bar{\alpha}$, constant $\bar{c}$, and a nonnegative functions $W(x, t)$ such that

$$
\begin{equation*}
\underline{\alpha}|(x)| \leq V(x) \leq \bar{\alpha}|(x)|, \quad L V(x) \leq-W(x, t)+\bar{c} \tag{5}
\end{equation*}
$$

then for each $x_{0} \in R^{n}$. (1) For (2), there exists an almost surely unique solution on $[0, \infty]$. (2) When $\bar{c}=0, f(0, t)=0$, $g(0, t)=0$, and $W(x, t)=W(x)$ is continuous, the equilibrium $x=0$ is globally stable in probability, and the solution $x(t)$ satisfies $P\left\{\lim _{t \rightarrow \infty} W(x(t)=0\}=1\right.$. (3) For any given $\varepsilon>0$, there exist a class KL function $\beta_{c}(\cdot, \cdot)$ and $K$ function $\gamma(\cdot)$ such that $P\left\{|(x(t))|<\beta_{c}\left(\left|x_{0}\right|, t\right)+\gamma(c)\right\} \geq 1-\varepsilon$ for any $t \geq 0$, $x_{0} \in R^{n} \backslash\{0\}$.

Lemma 4 (see [20]). For any real-valued continuous function $f(x, y), x \in R^{m}, y \in R^{n}$, there exist smooth scalar-value funcions $a(x) \geq 0, b(y) \geq 0, c(x)>1$, and $d(y) \geq 1$, such that $|f(x, y)| \leq a(x)+b(y)$, and $|f(x, y)| \leq c(x) d(y)$.

## 3. Controller Design and Analysis

The purpose of this paper is to construct a smooth statefeedback control law such that the solution process of system (1) is bounded in probability. For clarity, the case that $x_{0}\left(t_{0}\right) \neq 0$ is firstly considered. Then, the case where the initial $x_{0}\left(t_{0}\right)=0$ is dealt with later. The triangular structure of system (1) suggests that we should design the control inputs $u_{0}$ and $u_{1}$ in two separate stages.

To design the controller for system (1), the following assumptions are needed.

Assumption 5. For $0 \leq i \leq n$, there are some positive constants $\lambda_{i 1}$ and $\lambda_{i 2}$ that satisfy the inequality $\lambda_{i 1} \leq d_{i}(t) \leq$ $\lambda_{i 2}$.

Assumption 6. For $f_{0}\left(t, x_{0}\right)$, there exists a nonnegative smooth function $\gamma_{0}\left(t, x_{0}\right)$, such that $\left|f_{0}\left(t, x_{0}\right)\right| \leq\left|x_{0}\right| \gamma_{0}\left(t, x_{0}\right)$.

For each $f_{i}\left(t, x_{0}, \bar{x}_{i}\right), \varphi_{i}\left(\bar{x}_{i}\right)$, there exist nonnegative smooth functions $\gamma_{i}\left(t, x_{0}, \bar{x}_{i}\right)$ and $\rho_{i}\left(\bar{x}_{i}\right)$, such that $\left|f_{i}\left(t, x_{0}, \bar{x}_{i}\right)\right| \leq\left(\sum_{k=1}^{i}\left|x_{k}\right|\right) \gamma_{i}\left(t, x_{0}, \bar{x}_{i}\right),\left|\varphi_{i}\left(\bar{x}_{i}\right)\right| \leq\left(\sum_{k=1}^{i}\left|x_{k}\right|\right)$ $\rho_{i}\left(\bar{x}_{i}\right)$.
3.1. Designing $u_{0}$ for $x_{0}$-Subsystem. For $x_{0}$-subsystem, the control $u_{0}$ can be chosen as

$$
\begin{equation*}
u_{0}=-\lambda_{0} x_{0} \tag{6}
\end{equation*}
$$

where $\lambda_{0}=\left(k_{0}+\gamma_{0}\right) / \lambda_{01}$ and $k_{0}$ is a positive design parameter.
Consider the Lyapunov function candidate $V_{0}=x_{0}^{2} / 2$. From (6) and Assumptions 5 and 6, we have

$$
\begin{align*}
L V_{0} & =x_{0}\left(d_{0} u_{0}+f_{0}\left(t, x_{0}\right)\right) \\
& \leq d_{0} u_{0} x_{0}+x_{0}^{2} \gamma_{0} \leq-k_{0} x_{0}^{2}=-2 k_{0} V_{0} \tag{7}
\end{align*}
$$

So, we obtain the first result of this paper.
Theorem 7. The $x_{0}$-subsystem, under the control law (6) with an appropriate choice of the parameters $k_{0}, \lambda_{01}, \lambda_{02}$, is globally exponentially stable.

Proof. Clearly, from (7), $L V_{0} \leq 0$, which implies that $\left|x_{0}(t)\right| \leq$ $\left|x_{0}\left(t_{0}\right)\right| e^{-k_{0}\left(t-t_{0}\right)}$. Therefore, $x_{0}$ is globally exponentially convergent. Consequently, $x_{0}$ can be zero only at $t=t_{0}$, when
$x\left(t_{0}\right)=0$ or $t=\infty$. It is concluded that $x_{0}$ does not cross zero for all $t \in\left(t_{0}, \infty\right)$ provided that $x\left(t_{0}\right) \neq 0$.

Remark 8. If $x\left(t_{0}\right) \neq 0, u_{0}$ exists and does not cross zero for all $t \in\left(t_{0}, \infty\right)$ independent of the $x$-subsystem from (6).
3.2. Backstepping Design for $u_{1}$. From the above analysis, the $x_{0}$-state in (1) can be globally exponentially regulated to zero as $t \rightarrow \infty$, obviously. In this subsection, we consider the control law $u_{1}$ for the $x$-subsystem by using backstepping technique. To design a state-feedback controller, one first introduces the following discontinuous input-state-scaling transformation:

$$
\begin{equation*}
\eta_{i}=\frac{e^{\alpha t} x_{i}}{u_{0}^{n-i}}, \quad i=1 \ldots, n, u=e^{\alpha t} u_{1} . \tag{8}
\end{equation*}
$$

Under the new $x$-coordinates, $x$-subsystems is transformed into

$$
\begin{align*}
& d \eta_{i}=d_{i} \eta_{i+1} d t+\bar{f}_{i} d t+\phi_{i}^{T} \sum^{T}(t) d \omega, \quad i=1, \ldots, n-1, \\
& d \eta_{n}=d_{n} u d t+\bar{f}_{n} d t+\phi_{n}^{T} \sum^{T}(t) d \omega \tag{9}
\end{align*}
$$

where

$$
\begin{gather*}
\bar{f}_{i}=\alpha \eta_{i}+\frac{e^{\alpha t} f_{i}}{u_{0}^{n-i}}-\frac{(n-i) \eta_{i}}{u_{0}} \frac{\partial u_{0}}{\partial x_{0}}\left(d_{0} u_{0}+f_{0}\right)  \tag{10}\\
\phi_{i}=\frac{e^{\alpha t} \varphi_{i}}{u_{0}^{n-i}}
\end{gather*}
$$

In order to obtain the estimations for the nonlinear functions $\bar{f}_{i}$ and $\phi_{i}$, the following Lemma can be derived by Assumption 6.

Lemma 9. For $i=1,2 \ldots n$, there exist nonnegative smooth functions $\bar{\gamma}_{i}(\cdot), \bar{\rho}_{i}(\cdot)$, such that

$$
\begin{align*}
\left|\bar{f}_{i}\right| & \leq\left(\sum_{k=1}^{i}\left|\eta_{k}\right|\right) \bar{\gamma}_{i}\left(x_{0}, \bar{x}_{i}\right),  \tag{11}\\
\left|\phi_{i}\right| & \leq\left(\sum_{k=1}^{i}\left|\eta_{k}\right|\right) \bar{\rho}_{i}\left(\bar{x}_{i}\right) . \tag{12}
\end{align*}
$$

Proof. We only prove (11). The proof of (12) is similar to that of (11). In view of (6), (8), (10) and Assumption 6, one obtains

$$
\begin{aligned}
\left|\bar{f}_{i}\right|= & \left|\alpha \eta_{i}+\frac{e^{\alpha t} f_{i}}{u_{0}^{n-i}}-\frac{(n-i) \eta_{i}}{u_{0}} \frac{\partial u_{0}}{\partial x_{0}}\left(d_{0} x_{0}+f_{0}\right)\right| \\
\leq & \left|\alpha \eta_{i}\right|+\left(\sum_{k=1}^{i} \frac{e^{\alpha t}\left|x_{k}\right|}{u_{0}^{n-k}}\left|u_{0}^{i-k}\right|\right) \gamma_{i} \\
& +(n-i)\left(\lambda_{0} \lambda_{02}+\gamma_{0}\right)\left|\eta_{i}\right|
\end{aligned}
$$

$$
\begin{align*}
\leq & |\alpha|\left|\eta_{i}\right|+\left(\sum_{k=1}^{i}\left|\eta_{k}\right| \lambda_{0}^{i-k}\left|x_{0}^{i-k}\right|\right) \gamma_{i} \\
& +(n-i)\left(\lambda_{0} \lambda_{02}+\gamma_{0}\right)\left|\eta_{i}\right| \\
\leq & \left(\sum_{k=1}^{i}\left|\eta_{k}\right|\right)\left(|\alpha|+\left|\lambda_{0}^{i-k}\right|\left|x_{0}^{i-k}\right| \gamma_{i}+(n-i)\left(\lambda_{0} \lambda_{02}+\gamma_{0}\right)\right) \\
\leq & \left(\sum_{k=1}^{i}\left|\eta_{k}\right|\right) \bar{\gamma}_{i}\left(x_{0}, \bar{x}_{i}\right) \tag{13}
\end{align*}
$$

where $\bar{\gamma}_{i}\left(x_{0}, \bar{x}_{i}\right) \geq|\alpha|+\left|\lambda_{0}^{i-k}\right| x_{0}^{i-k} \mid \gamma_{i}+(n-i)\left(\lambda_{0} \lambda_{02}+\gamma_{0}\right)$.
To design a state-feedback controller, one introduces the coordinate transformation

$$
\begin{gather*}
z_{1}=\eta_{1} \\
z_{i}=\eta_{i}-\alpha_{i}\left(\bar{z}_{i-1}\right), \quad i=1,2 \ldots, n \tag{14}
\end{gather*}
$$

where $\alpha_{2}, \ldots, \alpha_{n}$ are smooth virtual control laws and will be designed later and $\alpha_{1}=0 . \hat{\theta}$ denotes the estimate of $\theta$, where

$$
\begin{gather*}
\theta=\sup _{t \geq 0}\left\{\operatorname { m a x } \left\{\left\|\sum(t) \sum^{T}(t)\right\|^{2},\left\|\sum(t) \sum^{T}(t)\right\|^{4 / 3},\right.\right. \\
\left.\left.\left\|\sum(t) \sum^{T}(t)\right\|\right\}\right\} \tag{15}
\end{gather*}
$$

Then using (9), (10), (14) and It $\widehat{o}$ differentiation rule, one has

$$
\begin{align*}
d z_{i}= & d\left(\eta_{i}-\alpha_{i}\right) \\
= & \left(d_{i} \eta_{i+1}+F_{i}\left(\bar{z}_{i}, x_{0}\right)-\frac{\partial \alpha_{i} \dot{\hat{\theta}}}{\partial \widehat{\theta}}\right) d t+G_{i}^{T}\left(\bar{z}_{i}\right) \sum^{T}(t) d \omega \\
& -\frac{1}{2} \sum_{k, m=1}^{i-1} \frac{\partial^{2} \alpha_{i}}{\partial z_{k} \partial z_{m}} \phi_{k}^{T}\left(\bar{z}_{k}\right) \sum^{T}(t) \sum(t) \phi_{m}\left(\bar{z}_{m}\right) d t \\
& i=1,2 \ldots n \tag{16}
\end{align*}
$$

where $\eta_{n+1}=u, F_{i}\left(\bar{z}_{i}, x_{0}\right)=\bar{f}_{i}+\sum_{k=1}^{i-1}\left(\partial \alpha_{i} / \partial z_{k}\right)\left(d_{k} \eta_{k+1}+\bar{f}_{k}\right)$, and $G_{i}\left(\bar{z}_{i}, x_{0}\right)=\phi_{i}+\sum_{k=1}^{i-1}\left(\partial \alpha_{i} / \partial z_{k}\right) \phi_{k}$, where $i=1,2 \ldots n$. Using Lemmas 2, 4, and 9 and (14), we easily obtain the following lemma.

Lemma 10. For $1 \leq i \leq n$, there exist nonnegative smooth functions $\gamma_{i 1}\left(\bar{z}_{i}, x_{0}\right), p_{i 1}\left(\bar{z}_{i}\right)$, and $\bar{p}_{i}\left(\bar{z}_{i}\right)$, such that

$$
\begin{align*}
& \left|F_{i}\right| \leq\left(\sum_{k=1}^{i}\left|z_{k}\right|\right) \gamma_{i 1}\left(\bar{z}_{i}, x_{0}\right), \\
& \left|G_{i}\right| \leq\left(\sum_{k=1}^{i}\left|z_{k}\right|\right) p_{i 1}\left(\bar{z}_{i}\right)  \tag{17}\\
& \left|\Phi_{i}\right| \leq\left(\sum_{k=1}^{i}\left|z_{k}\right|\right) \bar{p}_{i}\left(\bar{z}_{i}\right) .
\end{align*}
$$

The proof of Lemma 10 is similar to that of Lemma 9, so we omitted it.

We now give the design process of the controller.
Step 1. Consider the first Lyapunov function $V_{1}\left(z_{1}, \widehat{\theta}\right)=$ $(1 / 4) z_{1}^{4}+(1 / 2)(\hat{\theta}-\theta)^{2}$. By (14), (15), and (16), we have

$$
\begin{align*}
L V_{1}= & z_{1}^{3}\left(d_{1} \eta_{2}+F_{1}\right)+\frac{3}{2} z_{1}^{2} \operatorname{Tr}\left(G_{1}^{T} \sum^{T}(t) \sum(t) G_{1}\right)  \tag{18}\\
& +(\hat{\theta}-\theta) \dot{\hat{\theta}}
\end{align*}
$$

Using Lemma 10 and Lemma 4, we have

$$
\begin{gather*}
\left|z_{1}^{3} F_{1}\right| \leq z_{1}^{4} \gamma_{11}\left(z_{1}, x_{0}\right) \\
\left|\frac{3}{2} z_{1}^{2} \operatorname{Tr}\left(G_{1}^{T} \sum^{T}(t) \sum(t) G_{1}\right)\right|  \tag{19}\\
\leq z_{1}^{4} p_{11}^{2}\left(z_{1}, x_{0}\right)\left|\sum^{T}(t) \sum(t)\right| \leq z_{1}^{4} p_{11}^{2}\left(z_{1}, x_{0}\right) \theta
\end{gather*}
$$

Substituting (19) into (18) and using (14), we have

$$
\begin{align*}
L V_{1} \leq & d_{1} z_{1}^{3}\left(\eta_{2}-\alpha_{2}\right)+d_{1} z_{1}^{3} \alpha_{2}+z_{1}^{4} p_{11}^{2}\left(z_{1}, x_{0}\right) \theta \\
& +z_{1}^{4} \gamma_{11}\left(z_{1}, x_{0}\right)+(\hat{\theta}-\theta) \dot{\hat{\theta}} \\
\leq & d_{1} z_{1}^{3} z_{2}+d_{1} z_{1}^{3} \alpha_{2}+z_{1}^{4} p_{11}^{2}\left(z_{1}, x_{0}\right) \theta  \tag{20}\\
& +z_{1}^{4} \gamma_{11}\left(z_{1}, x_{0}\right)+(\hat{\theta}-\theta) \dot{\hat{\theta}}
\end{align*}
$$

where $\alpha_{2}=-z_{1} \beta_{1}=-z_{1}\left(\left(c_{1}+\gamma_{11}+p_{11}^{2} \widehat{\theta}\right) / \lambda_{11}\right)$. Substituting $\alpha_{2}$ into (20), we have

$$
\begin{equation*}
L V_{1} \leq d_{1} z_{1}^{3} z_{2}-c_{1} z_{1}^{4}+(\hat{\theta}-\theta)\left(\dot{\hat{\theta}}-\tau_{1}\right) \tag{21}
\end{equation*}
$$

where $\tau_{1}=z_{1}^{4} p_{11}^{2}$.
Step i. $(2 \leq i \leq n)$. Assume that at step $i-1$, there exists a smooth state-feedback virtual control $\alpha_{i}=$ $-z_{i-1} \beta_{i-1}\left(\bar{z}_{i-1}, \widehat{\theta}\right)=-z_{i-1}\left(\left(c_{i-1}+\hat{\theta} \sqrt{1+\left(\psi_{i-12}+\psi_{i-13}\right)^{2}}+b_{i-1}+\right.\right.$ $\left.\left.\psi_{i-11}+\psi_{i-14}\right) / \lambda_{i-11}\right)$, such that

$$
\begin{align*}
L V_{i-1} \leq & -\sum_{j=1}^{i-2}\left(c_{j}-\varepsilon_{j}-e_{j}\right) z_{j}^{4}-c_{i-1} z_{i-1}^{4}+d_{i-1} z_{i-1}^{3} z_{i}  \tag{22}\\
& +\left(\hat{\theta}-\theta-\sum_{k=2}^{i-1} z_{k}^{3} \frac{\partial \alpha_{k}}{\partial \widehat{\theta}}\right)\left(\dot{\hat{\theta}}-\tau_{i-1}\right),
\end{align*}
$$

where $V_{i-1}=\sum_{j=1}^{i-1}(1 / 4) z_{j}^{4}+(1 / 2)(\hat{\theta}-\theta)^{2}, \tau_{i-1}=\tau_{1}+$ $\sum_{k=2}^{i-1} z_{k}^{4}\left(\psi_{i-12}+\psi_{i-13}\right)$, and $\varepsilon_{j}=\sum_{k=1}^{j}\left(\varepsilon_{k 1}+\varepsilon_{k 2}+\varepsilon_{k 3}+\varepsilon_{k 4}\right)$, where $j=1, \ldots, n$.

Then, define the $i$ th Lyapunov candidate function $V_{i}\left(\bar{z}_{i}, \widehat{\theta}\right)=V_{i-1}+(1 / 4) z_{i}^{4}$. From (16) and (22), it follows that

$$
\begin{align*}
L V_{i} \leq & -\sum_{j=1}^{i-2}\left(c_{j}-\varepsilon_{j}-e_{j}\right) z_{j}^{4}-c_{i-1} z_{i-1}^{4}+d_{i-1} z_{i-1}^{3} z_{i} \\
& +z_{i}^{3}\left(d_{i} \eta_{i+1}+F_{i}\left(\bar{z}_{i}, x_{0}\right)-\frac{\partial \alpha_{i} \dot{\hat{\theta}}}{\partial \widehat{\theta}}\right. \\
& \left.-\frac{1}{2} \sum_{k, m=1}^{i-1} \frac{\partial^{2} \alpha_{i}}{\partial z_{k} \partial z_{m}} \phi_{k}^{T}\left(\bar{z}_{k}\right) \sum^{T}(t) \sum(t) \phi_{m}\left(\bar{z}_{m}\right)\right) \\
& +\frac{3}{2} z_{i}^{2} \operatorname{Tr}\left(G_{i}^{T}\left(\bar{z}_{i}\right) \sum^{T}(t) \sum(t) G_{i}\left(\bar{z}_{i}\right)\right) \\
& +\left(\hat{\theta}-\theta-\sum_{k=2}^{i-1} z_{k}^{3} \frac{\partial \alpha_{k}}{\partial \widehat{\theta}}\right)\left(\dot{\hat{\theta}}-\tau_{i-1}\right) . \tag{23}
\end{align*}
$$

Using Lemmas 9 and 4, there are always known nonnegative smooth functions $\psi_{i 1}\left(\bar{z}_{i}\right), \psi_{i 2}\left(\bar{z}_{i}\right), \psi_{i 3}\left(\bar{z}_{i}\right), \psi_{i 4}\left(\bar{z}_{i}\right)$ and constant $\varepsilon_{i}>0, \varepsilon_{i j}>0$, where $i=1, \ldots, n$ and $j=1,2,3,4$.

Consider

$$
\begin{align*}
z_{i}^{3} F_{i} & \leq\left|z_{i}^{3}\right|\left(\sum_{k=1}^{i-1}\left|z_{k}\right|\right) \gamma_{i 1}\left(\bar{z}_{i}, x_{0}\right) \\
& \leq \gamma_{i 1} z_{i}^{4}+\sum_{k=1}^{i-1}\left(\varepsilon_{k 1} z_{k}^{4}+\frac{3}{4}\left(4 \varepsilon_{k 1}\right)^{-1 / 3} \gamma_{i 1}^{4 / 3} z_{i}^{4}\right)  \tag{24}\\
& \leq \sum_{k=1}^{i-1} \varepsilon_{k 1} z_{k}^{4}+\psi_{i 1} z_{i}^{4}
\end{align*}
$$

where $\psi_{i 1} \geq \gamma_{i 1}+\sum_{k=1}^{i-1}(3 / 4)\left(4 \varepsilon_{k 1}\right)^{-1 / 3} \gamma_{i 1}^{4 / 3}$.

$$
\begin{align*}
& -\frac{1}{2} z_{i}^{3} \sum_{k, m=1}^{i-1} \frac{\partial^{2} \alpha_{i}}{\partial z_{k} \partial z_{m}} \phi_{k}^{T} \sum^{T}(t) \sum(t) \phi_{m} \\
& \quad \leq \frac{1}{2} z_{i}^{3} \sum_{k, m=1}^{i-1}\left|\frac{\partial^{2} \alpha_{i}}{\partial z_{k} \partial z_{m}}\right|\left(\sum_{j=1}^{k}\left|z_{j}\right|\right) \bar{p}_{k}\left(\bar{z}_{k}\right) \\
& \quad \times\left(\sum_{j=1}^{m}\left|z_{j}\right|\right) \bar{p}_{m}\left(\bar{z}_{m}\right)\left|\sum^{T}(t) \sum(t)\right|  \tag{25}\\
& \quad \leq z_{i}^{3}\left(\sum_{k=1}^{i-1} z_{k}^{2}\right) \overline{\bar{p}}_{i}\left(\bar{z}_{i}\right)\left|\sum^{T}(t) \sum(t)\right| \\
& \quad \leq z_{i}^{4} \psi_{i 2}\left(\bar{z}_{i}\right) \theta+\sum_{k=1}^{i-1} \varepsilon_{k 2} z_{k}^{4}
\end{align*}
$$

where $\psi_{i 2} \geq \sum_{k=1}^{i-1}(3 / 4)\left(4 \varepsilon_{k 2}\right)^{-1 / 3}\left(\overline{\bar{p}}_{i}\left(\bar{z}_{i}\right)\right)^{3 / 4}$.

$$
\begin{aligned}
& \frac{3}{2} z_{i}^{2} \operatorname{Tr}\left(G_{i}^{T} \sum^{T}(t) \sum(t) G_{i}\right) \\
& \quad \leq \frac{3}{2} z_{i}^{2} p_{i 2}^{2}\left(\bar{z}_{i}\right)\left(\sum_{k=1}^{i}\left|z_{k}\right|\right)^{2}\left|\sum^{T}(t) \sum(t)\right|
\end{aligned}
$$

$$
\begin{align*}
& \leq \frac{3}{2} z_{i}^{2} i p_{i 2}^{2}\left(\bar{z}_{i}\right)\left(\sum_{k=1}^{i} z_{k}^{2}\right)\left|\sum^{T}(t) \sum^{T}(t)\right| \\
& \leq \frac{3}{2} z_{i}^{4} i p_{i 2}^{2}\left(\bar{z}_{i}\right) \theta+\sum_{k=1}^{i-1} \varepsilon_{k 3} z_{k}^{4}+\sum_{k=1}^{i-1} \frac{1}{4 \varepsilon_{k 3}}\left(\frac{3}{2} i p_{i 2}^{2}\right)^{2} z_{i}^{4} \theta \\
& \leq \sum_{k=1}^{i-1} \varepsilon_{k 3} z_{k}^{4}+\psi_{i 3} z_{i}^{4} \theta \tag{26}
\end{align*}
$$

where $\psi_{i 3} \geq(3 / 2) i p_{i 2}^{2}+\sum_{k=1}^{i-1}\left(1 / 4 \varepsilon_{k 3}\right)\left((3 / 2) i p_{i 2}^{2}\right)^{2}$.

$$
\begin{align*}
d_{i-1} z_{i-1}^{3} z_{i} & \leq \lambda_{i-12}\left|z_{i-1}^{3} z_{i}\right| \\
& \leq e_{i-1} z_{i-1}^{4}+\frac{1}{4}\left(\frac{4}{3} e_{i-1}\right)^{-3} z_{i}^{4} \lambda_{i 2}^{4}  \tag{27}\\
& \leq e_{i-1} z_{i-1}^{4}+b_{i} z_{i}^{4}
\end{align*}
$$

where $b_{i} \geq(1 / 4)\left((4 / 3) e_{i-1}\right)^{-3} \lambda_{i 2}^{4}, \tau_{i-1}=z_{1}^{4} p_{11}^{2}+\sum_{k=2}^{i-1} z_{k}^{4}\left(\psi_{k 2}+\right.$ $\left.\psi_{k 3}\right)$, and $\tau_{i}=\tau_{i-1}+\left(\psi_{i 2}+\psi_{i 3}\right) z_{i}^{4}$.

$$
\begin{align*}
&-z_{i}^{3} \frac{\partial \alpha_{i}}{\partial \widehat{\theta}} \tau_{i} \\
& \leq z_{i}^{3}\left|\frac{\partial \alpha_{i}}{\partial \widehat{\theta}}\right|\left(\tau_{i-1}+z_{i}^{4}\left(\psi_{i 2}+\psi_{i 3}\right)\right) \\
& \leq z_{i}^{4} \sqrt{1+\left(z_{i}^{3} \frac{\partial \alpha_{i}}{\partial \widehat{\theta}}\right)^{2}}\left(\psi_{i 2}+\psi_{i 3}\right) \\
&+z_{i}^{3}\left|\frac{\partial \alpha_{i}}{\partial \widehat{\theta}}\right|\left(z_{1}^{4} p_{11}^{2}+\sum_{k=2}^{i-1} z_{k}^{4}\left(\psi_{k 2}+\psi_{k 3}\right)\right) \\
&+\frac{3}{4}(4)^{-1 / 3}\left(\left|\frac{\partial \alpha_{2}}{\partial \widehat{\theta}}\right| z_{1}^{3} p_{11}^{2}\right)^{4 / 3} \varepsilon_{i 4}{ }^{-1 / 3} z_{i}^{4} \\
& \leq \varepsilon_{i 4} z_{1}^{4}+\frac{3}{4}(4)^{-1 / 3} \sqrt{1+\left(\frac{\partial \alpha_{i}}{\partial \widehat{\theta}} z_{1}^{3} P_{11}^{2}\right)^{2} \varepsilon_{i 4}^{-1 / 3} z_{i}^{4}} \\
&+\sum_{k=2}^{i-1} \varepsilon_{k 4} z_{k}^{4} \\
&+\sum_{k=2}^{i-1} \frac{3}{4}(4)^{-1 / 3} \sqrt{1+\left(\frac{\partial \alpha_{i}}{\partial \widehat{\theta}} z_{k}^{3}\left(\psi_{k 2}+\psi_{k 3}\right)\right)^{2} \varepsilon_{k 4}^{-1 / 3} z_{i}^{4}} \\
& \leq \sum_{k=1}^{i-1} \varepsilon_{k 4} z_{k}^{4}+\psi_{i 4} z_{i}^{4}, \tag{28}
\end{align*}
$$

where $\psi_{i 4} \geq(3 / 4)\left(4 \varepsilon_{i 4}\right)^{-1 / 3} \sqrt{1+\left(\left(\partial \alpha_{i} / \partial \hat{\theta}\right) z_{1}^{3} P_{11}^{2}\right)^{2 / 3}}+$ $\sum_{k=2}^{i-1}(3 / 4)\left(4 \varepsilon_{k 4}\right)^{-1 / 3} \sqrt{1+\left(\left(\partial \alpha_{i} / \partial \hat{\theta}\right) z_{k}^{3}\left(\psi_{k 2}+\psi_{k 3}\right)\right)^{2}}$.

$$
\begin{gather*}
\alpha_{i+1}\left(\bar{z}_{i}, \hat{\theta}\right)=-z_{i} \beta_{i}\left(\bar{z}_{i}, \hat{\theta}\right) \\
\beta_{i}\left(\bar{z}_{i}, \widehat{\theta}\right)=\frac{c_{i}+\psi_{i 1}+\psi_{i 4}+b_{i}+\sqrt{1+\left(\psi_{i 2}+\psi_{i 3}\right)^{2}} \widehat{\theta}}{\lambda_{i 1}} \tag{29}
\end{gather*}
$$

where $c_{i}>0$ is a design parameter to be chosen.
With the aid of (24)-(29) and (14), (23) can be simplified as

$$
\begin{align*}
L V_{i} \leq & -\sum_{j=1}^{i-1}\left(c_{j}-\varepsilon_{j}-e_{j}\right) z_{j}^{4}-c_{i} z_{i}^{4} \\
& +d_{i} z_{i}^{3} z_{i+1}+\left(\hat{\theta}-\theta-\sum_{k=2}^{i} \frac{\partial \alpha_{k}}{\partial \widehat{\theta}} z_{k}^{3}\right)\left(\dot{\hat{\theta}}-\tau_{i}\right) \tag{30}
\end{align*}
$$

Finally, when $i=n, z_{n+1}=u$ is the actual control. By choosing the actual control law and the adaptive law,

$$
\begin{gather*}
u\left(\bar{z}_{n}, \widehat{\theta}\right)=-z_{n} \beta_{n}\left(\bar{z}_{n}, \widehat{\theta}\right) \\
\dot{\hat{\theta}}=\tau_{n}=z_{1}^{4} p_{11}^{2}+\sum_{k=2}^{n} z_{k}^{4}\left(\psi_{k 2}+\psi_{k 3}\right) \\
\beta_{n}\left(\bar{z}_{n}, \hat{\theta}\right)=\frac{c_{n}+b_{n}+\psi_{n 1}+\psi_{n 4}+\sqrt{1+\left(\psi_{n 2}+\psi_{n 3}\right)^{2}} \hat{\theta}}{\lambda_{n 1}}, \\
u_{1}=e^{-\alpha t} u \tag{31}
\end{gather*}
$$

where $c_{n}>0$ is a design parameter to be chosen and $\psi_{n i}, i=$ $1, \ldots 4$ are smooth functions; we get

$$
\begin{equation*}
L V_{n} \leq-\sum_{j=1}^{n}\left(c_{j}-\varepsilon_{j}-e_{j}\right) z_{j}^{4}, \tag{32}
\end{equation*}
$$

where $V_{n}(z, \widehat{\theta})=\sum_{k=1}^{n}(1 / 4) z_{k}^{4}+(1 / 2)(\widehat{\theta}-\theta)^{2}, z=$ $\left(z_{1}, \ldots z_{n}\right)$. We have finished the controller design procedure for $x_{0}\left(t_{0}\right) \neq 0$ and the parameter identification. Without loss of generality, we can assume that $t_{0} \neq 0$.
3.3. Switching Control and Main Result. In the preceding subsection, we have given controller design for $x_{0} \neq 0$. Now, we discuss how to choose the control laws $u_{0}$ and $u_{1}$ when $x_{0}=0$. We choose $u_{0}$ as $u_{0}=-\lambda_{0} x_{0}+u_{0}^{*}, u_{0}^{*}>0$. And choose the Lyapunov function $V_{0}=(1 / 2) x_{0}^{2}$. Its time derivative is given by $L V_{0}=-\lambda_{0} x_{0}^{2}+u_{0}^{*}$, which leads to the bounds of $x_{0}$. During the time period $\left[0, t_{s}\right)$, using $u_{0}=-\lambda_{0} x_{0}+u_{0}^{*}$, new control law $u$ can be obtained by the control procedure described above to the original $x$-subsystem in (1). Then, we can conclude that the $x$-state of (1) cannot be blown up during the time period $\left[0, t_{s}\right.$ ). Since at $x\left(t_{s}\right) \neq 0$, we can switch the control inputs $u_{0}$ and $u$ to (6) and (31), respectively.

Now, we state the main results as follows.
Theorem 11. Under Assumption 5, if the proposed adaptive controller (31) together with the above switching control strategy is used in (1), then for any initial contidion $\left(x_{0}, x, \widehat{\theta}\right) \in R^{n}$,
the closed-loop system has an almost surely unique solution on $[0, \infty)$, the solution process is bounded in probability, and $P\left\{\lim _{t \rightarrow \infty} \widehat{\theta}(t)\right.$ exists and is finite $\}=1$.

Proof. According to the above analysis, it suffices to prove in the case $x_{0}(0) \neq 0$. Since we have already proven that $x_{0}$ can be globally exponentially convergent to zero in probability in Section 3.1, we only need prove that $x(t)$ is convergent to zero in probability also. In this case, we choose the Lyapunov function $V=V_{n}$, and $c_{i}>\varepsilon_{i}+e_{i}$; from (32) and Lemma 3, we know that the closed-loop system has an almost surely unique solution on $[0, \infty)$, and the solution process is bounded in probability.

## 4. Conclusions

This paper investigates the globally exponential stabilization problem for a class of stochastic nonholonomic systems in chained form. To deal with the nonlinear parametrization problem, a parameter separation technique is introduced. With the help of backstepping technique, a smooth adaptive controller is constructed which ensures that the closed-loop system is globally asymptotically stable in probability. A further work is how to design the output-feedback tracking control for more high-order stochastic nonholonomic systems.

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## Research Article

# $H_{\infty}$ Consensus for Multiagent Systems with Heterogeneous Time-Varying Delays 

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#### Abstract

We apply the linear matrix inequality method to consensus and $H_{\infty}$ consensus problems of the single integrator multiagent system with heterogeneous delays in directed networks. To overcome the difficulty caused by heterogeneous time-varying delays, we rewrite the multiagent system into a partially reduced-order system and an integral system. As a result, a particular Lyapunov function is constructed to derive sufficient conditions for consensus of multiagent systems with fixed (switched) topologies. We also apply this method to the $H_{\infty}$ consensus of multiagent systems with disturbances and heterogeneous delays. Numerical examples are given to illustrate the theoretical results.


## 1. Introduction

In recent years, decentralized coordination of multiagent systems has received many researchers' attention in the areas of system control theory, biology, communication, applied mathematics, computer science, and so forth. In cooperative control of multiagent systems, a critical problem is to design appropriate protocols such that multiple agents in a group can reach consensus. So far, by using the matrix theory, the graph theory, the frequency-domain analysis method, the Lyapunov direct method, and so forth, consensus problems for various kinds of multiagent systems have been studied extensively [1$3]$.

In the field of systems and control theory, the pioneering work was done by Borkar and Varaiya [4] and Tsitsiklis and Athans [5], where the asynchronous consensus problem with an application in distributed decision-making systems was considered. Later, Vicsek et al. [6] proposed a simple but interesting discrete-time model of multiple agents which can be viewed as a special case of a computer model mimicking animal aggregation. Jadbabaie et al. [7] provided a theoretical explanation of the consensus property of the Vicsek model.

For the case of single integrator multiagent systems, Olfati-Saber and Murray [8] discussed the consensus prob-
lem for networks of dynamic agents by defining a disagreement function. Ren and Beard [9] established some more relaxable consensus conditions under dynamically changing interaction topologies. Hui and Haddad [10], Liu et al. [11], and Bauso et al. [12] investigated the consensus problem for nonlinear multiagent systems. Tan and Liu [13] studied consensus of networked multiagent systems via the networked predictive control. Li and Zhang [14] gave the necessary and sufficient condition of mean square averageconsensus for multiagent systems with noises. Zheng and Wang [15] studied finite-time consensus of heterogenous multiagent systems with and without velocity measurements. The constrained consensus problem for multiagent systems in unbalanced networks was investigated in Lin and Ren [16, 17]. Liu et al. [18] considered the consensus problem for multiagent systems with inherent nonlinear dynamics under directed topologies. A distributed shortest-distance consensus problem under dynamically changing network topologies was studied by Lin and Ren [19]. Li et al. [20] studied the distributed consensus problem of multiagent systems with general continuous-time dynamics for both the case without and with a leader.

When considering communication delays in the feedback, three types of consensus protocols have been analyzed: (i) both the state of the agent and its neighbors are affected by
identical delays [21-27]; (ii) communication delays only affect the state received from neighbors of the agent [28-32]; (iii) the state of the agent and its neighbors are affected by heterogeneous delays. Note that (i) and (ii) are special cases of (iii). Compared to the cases of (i) and (ii), consensus for the case of (iii) receives less attention. Under the restricted assumptions that the graph is undirected and the communication delays are constants, Münz et al. [32,33] investigated the robustness of the third kind of consensus protocols in both identical and nonidentical agent dynamics.

We are here concerned with the third type of consensus protocols, where the heterogeneous delays are time varying, and the involved graph is directed. It seems to us that the frequency-domain analysis method in $[32,33]$ becomes invalid in this case, and it is difficult to employ the Lyapunov method directly. In this paper, we will apply the linear matrix inequality method to this problem. In [21-25], the linear matrix inequality method has been used successfully in the case of (i). When applying this method to the case of (iii), we have to overcome some difficulties caused by heterogeneous time-varying delays. For this sake, we first skillfully rewrite the multiagent system by virtue of a partially-reduced-order system and an integral system. Then, by defining a particular Lyapunov function based on the above partially-reducedorder system and the integral system, sufficient conditions to consensus are derived in both cases of fixed topology and switching topologies by using the linear matrix inequality method.

We also consider the $H_{\infty}$ consensus problem for the single integrator multiagent system with heterogeneous timevarying delays. So far, there are few $H_{\infty}$ consensus results for continuous time multiagent systems. Lin et al. [34] studied distributed robust $H_{\infty}$ consensus problem in directed networks of agents with identical delays. Ugrinovskii [35] considered a problem of design of distributed robust filters with $H_{\infty}$ consensus by using the recent vector dissipativity theory. Sun and Wang [36] investigated the $H_{\infty}$ consensus for the double integrator multiagent system with asymmetric delays.

To the best of our knowledge, little has been known about the $H_{\infty}$ consensus problem for the single integrator multiagent systems with heterogeneous time-varying delays and directed network topologies. Therefore, another purpose of this paper is to establish $H_{\infty}$ consensus criteria in the cases of heterogeneous delays and directed fixed topology (or switching topologies) by using the linear matrix inequality technique.

This paper is structured as follows. The problem statement and the transformation of the multiagent system are summarized in Section 2. In Section 3, sufficient conditions in terms of linear matrix inequalities are established for consensus and $H_{\infty}$ consensus of the single integrator multiagent system with heterogeneous delays. These results are also extended to the case of switching topologies. Numerical examples and simulation results are given in Section 4. The paper is concluded in Section 5.

The notation used throughout this paper is fairly standard. $A^{T}$ means the transpose of the matrix $A . I_{m}$ is an
$m \times m$-dimensional identity matrix. $\mathbf{a}_{m}=(a, a, \ldots, a)^{T}$ is an $m$-dimensional column vector with $a \in \mathbb{R}$. We say $X>Y$ if $X-Y$ is positive definite, where $X$ and $Y$ are symmetric matrices of the same dimensions. We use an asterisk * to represent a term that is induced by symmetry and $\operatorname{diag}\{\cdots\}$ stands for a block-diagonal matrix. $L_{2}[0, \infty)$ denotes the space of square-integrable vector functions over $[0, \infty)$.

## 2. Preliminaries

Throughout this paper, we denote a weighted digraph by $\mathscr{G}=$ $(V, E, A)$, where $V=\left\{v_{1}, v_{2}, \ldots, v_{n}\right\}$ is the set of nodes with $n \geq 2$ and node $v_{i}$ represents the $i$ th agent; $E \subseteq V \times V$ is the set of edges, an edge of $\mathscr{G}$ is denoted by an order pair $(i, j)$, and $(i, j) \in E$ if and only if $a_{j i}>0 ; A=\left[a_{i j}\right]$ is an $n \times n-$ dimensional weighted adjacency matrix with $a_{i i}=0$. If $(i, j)$ is an edge of $\mathscr{G}$, node $i$ is called the parent of node $j$. A directed tree is a directed graph, where every node, except one special node without any parent, which is called the root, has exactly one parent, and the root can be connected to any other nodes through paths. The Laplacian matrix $L=\left[l_{i j}\right]$ of digraph $\mathscr{G}$ is defined by $l_{i i}=-\sum_{j=1}^{n} a_{i j}$ and $l_{i j}=a_{i j}$ for $i \neq j, i, j \in \mathcal{N}=$ $\{1,2, \ldots, n\}$.

Consider the following multiagent system with heterogeneous delays:

$$
\begin{equation*}
\dot{x}_{i}(t)=\sum_{j=1}^{n} a_{i j}\left(x_{j}\left(t-d_{1}(t)\right)-x_{i}\left(t-d_{2}(t)\right)\right), \quad i \in \mathcal{N}, \tag{1}
\end{equation*}
$$

where $t \geq 0, a_{i j} \geq 0, i, j \in \mathcal{N}$, are entries of the weighted adjacency matrix $A, d_{1}(t)$ and $d_{2}(t)$ are different piecewise continuous communication delays satisfying $0 \leq d_{1}(t) \leq h_{1}$ and $0 \leq d_{2}(t) \leq h_{2}$ for $t \geq 0$, and $h_{1}$ and $h_{2}$ are positive constants.

Denote $x=\left(x_{1}, x_{2}, \ldots \text { sd, } x_{n}\right)^{T}$ and $y=\left(y_{1}, y_{2}, \ldots\right.$, $\left.y_{n-1}\right)^{T}$. If we set $y_{i}=x_{i+1}-x_{1}$ for $i=1,2, \ldots, n-1$, by the straightforward computation, we get

$$
\begin{equation*}
y=E x, \quad x=x_{1} \mathbf{1}_{n}+F y \tag{2}
\end{equation*}
$$

where the $(n-1) \times n$-dimensional matrix $E$ and the $n \times(n-$ 1)-dimensional matrix $F$ are defined as follows:

$$
\begin{equation*}
E=\left(-\mathbf{1}_{n-1}, I_{n-1}\right), \quad F=\binom{\mathbf{0}_{n-1}^{T}}{I_{n-1}} \tag{3}
\end{equation*}
$$

Next, we make use of transformation (2) to derive two descriptor systems of the system (1). Rewrite (1) as follows:

$$
\begin{align*}
\dot{x}_{i}(t)= & \sum_{j=1}^{n} a_{i j}\left(x_{j}(t)-x_{i}(t)\right)+\sum_{j=1}^{n} a_{i j}\left(x_{j}\left(t-d_{1}(t)\right)-x_{j}(t)\right) \\
& +\sum_{j=1}^{n} a_{i j}\left(x_{i}(t)-x_{i}\left(t-d_{2}(t)\right)\right), \quad i \in \mathscr{N} . \tag{4}
\end{align*}
$$

Therefore, by the Newton-Leibniz formula, we get a matrix form of (4):

$$
\begin{equation*}
\dot{x}(t)=L x(t)+A \int_{t}^{t-d_{1}(t)} \dot{x}(s) d s+\Delta \int_{t-d_{2}(t)}^{t} \dot{x}(s) d s \tag{5}
\end{equation*}
$$

where $\Delta=A-L$.
Denote $z(t)=\dot{x}(t)$. By transformation (2) and the property of the Laplacian matrix $L$ that $L \mathbf{1}_{n}=\mathbf{0}_{n}$, we get from (5) the following two descriptor systems:

$$
\begin{gather*}
\dot{y}(t)=E L F y(t)+E A \int_{t}^{t-d_{1}(t)} z(s) d s+E \Delta \int_{t-d_{2}(t)}^{t} z(s) d s \\
z(t)=L F y(t)+A \int_{t}^{t-d_{1}(t)} z(s) d s+\Delta \int_{t-d_{2}(t)}^{t} z(s) d s \tag{6}
\end{gather*}
$$

When the system involves disturbance input, we consider the following multiagent system of the form

$$
\begin{array}{r}
\dot{x}_{i}(t)=\sum_{j=1}^{n} a_{i j}\left(x_{j}\left(t-d_{1}(t)\right)-x_{i}\left(t-d_{2}(t)\right)+w_{i j}(t)\right),  \tag{7}\\
i \in \mathcal{N},
\end{array}
$$

where $a_{i j}, d_{1}(t)$, and $d_{2}(t)$ are defined as above and $w_{i j}(t)$ is the disturbance input satisfying $w_{i j}(t) \in L_{2}[0, \infty)$.

Denote the $i$ th row of the matrix $A$ by $\alpha_{i}, \Sigma=$ $\operatorname{diag}\left\{\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}\right\}, w_{i}(t)=\left(w_{i 1}(t), w_{i 2}(t), \ldots, w_{i n}(t)\right]^{T}$, and $w(t)=\left(w_{1}^{T}(t), w_{2}^{T}(t), \ldots, w_{n}^{T}(t)\right]^{T}$. Similar to the above procedure, we can get the following matrix form of (7):

$$
\begin{align*}
\dot{y}(t)= & E L F y(t)+E A \int_{t}^{t-d_{1}(t)} z(s) d s \\
& +E \Delta \int_{t-d_{2}(t)}^{t} z(s) d s+E \Sigma w(t)  \tag{8}\\
z(t)= & L F y(t)+A \int_{t}^{t-d_{1}(t)} z(s) d s \\
& +\Delta \int_{t-d_{2}(t)}^{t} z(s) d s+\Sigma w(t) .
\end{align*}
$$

In this paper, say that system (1) achieves consensus asymptotically if

$$
\begin{equation*}
\lim _{t \rightarrow \infty}\left[x_{i}(t)-x_{j}(t)\right]=0, \quad \forall i \neq j, i, j \in \mathcal{N} \tag{9}
\end{equation*}
$$

Say that system (7) solves $H_{\infty}$ consensus if system (1) achieves consensus asymptotically, and there exists a constant $\gamma>0$ such that

$$
\begin{equation*}
\int_{0}^{\infty} y^{T}(t) y(t) d t \leq \gamma \int_{0}^{\infty} w^{T}(t) w(t) d t \tag{10}
\end{equation*}
$$

for all nonzero $w \in L_{2}[0, \infty)$ under zero initial condition.

## 3. Main Results

The following two lemmas can be concluded from [21, 22].
Lemma 1. The digraph $\mathscr{G}=\{V, E, A\}$ has a spanning tree if and only if the matrix ELF is Hurwitz.

Lemma 2. For any continuous vector $u(t) \in \mathbb{R}^{m}$ on $\left[t-h_{k}, t\right]$ for $k=1,2$ and $t \geq 0$ and any $m \times m$-dimensional matrix $W>0$, the following inequality holds:

$$
\begin{gather*}
\left(\int_{t}^{t-d_{1}(t)} u(\alpha) d \alpha\right)^{T} W \int_{t}^{t-d_{1}(t)} u(\alpha) d \alpha \\
\leq h_{1} \int_{t-h_{1}}^{t} u^{T}(\alpha) W u(\alpha) d \alpha  \tag{11}\\
\left(\int_{t-d_{2}(t)}^{t} u(\alpha) d \alpha\right)^{T} W \int_{t-d_{2}(t)}^{t} u(\alpha) d \alpha \\
\leq h_{2} \int_{t-h_{2}}^{t} u^{T}(\alpha) W u(\alpha) d \alpha
\end{gather*}
$$

Based on Lemmas 1 and 2 and the preliminaries given in Section 2, we have the following consensus result for system (1).

Theorem 3. If the digraph $\mathscr{G}$ has a spanning tree, system (1) achieves consensus asymptotically for appropriate constants $h_{1}>0$ and $h_{2}>0$ which satisfy the following matrix inequality:

$$
\left(\begin{array}{ccccc}
P \widetilde{L}+\widetilde{L}^{T} P & P \widetilde{A} & P \widetilde{\Delta} & h_{1} \bar{L}^{T} Q_{1} & h_{2} \bar{L}^{T} Q_{2}  \tag{12}\\
* & -Q_{1} & 0 & h_{1} A^{T} Q_{1} & h_{2} A^{T} Q_{2} \\
* & * & -Q_{2} & h_{1} \Delta^{T} Q_{1} & h_{2} \Delta^{T} Q_{2} \\
* & * & * & -Q_{1} & 0 \\
* & * & * & * & -Q_{2}
\end{array}\right)<0,
$$

where $\widetilde{L}=E L F, \widetilde{A}=E A, \widetilde{\Delta}=E \Delta, \bar{L}=L F$, and $P, Q_{1}$, and $Q_{2}$ are positive-definite matrices of appropriate dimensions.

Proof. First, we see that (12) is equivalent to

$$
\begin{equation*}
\Omega+h_{1}^{2} \Phi^{T} Q_{1} \Phi+h_{2}^{2} \Phi^{T} Q_{2} \Phi<0 \tag{13}
\end{equation*}
$$

where

$$
\Phi=(\bar{L}, A, \Delta), \quad \Omega=\left(\begin{array}{ccc}
P \widetilde{L}+\widetilde{L}^{T} P & P \widetilde{A} & P \widetilde{\Delta}  \tag{14}\\
* & -Q_{1} & 0 \\
* & * & -Q_{2}
\end{array}\right)
$$

On the other hand, we show that (13) is feasible if the digraph $\mathscr{G}$ has a spanning tree. By Lemma 1, we see that there exists a positive-definite matrix $P$ such that $P \widetilde{L}+\widetilde{L}^{T} P<$ 0 . Once the positive-definite matrix $P$ is determined, it is not difficult to see that $\Omega<0$ for certain positive-definite matrices $Q_{1}$ and $Q_{2}$. Once the matrices $P>0, Q_{1}>0$, and $Q_{2}>0$ are determined, (13) is valid for sufficiently small $h_{1}>0$ and $h_{2}>0$. Therefore, (13) always holds for appropriate matrices $P>0, Q_{1}>0$, and $Q_{2}>$ 0 and constants $h_{1}>0, h_{2}>0$ if $\mathscr{G}$ has a spanning tree.

Next, by the definition of vector $y$, it is sufficient to show that $\lim _{t \rightarrow \infty} y(t)=0$ if (13) holds. We construct the Lyapunov function as follows:

$$
\begin{align*}
V(t)= & y^{T}(t) P y(t) \\
& +h_{1} \int_{t-h_{1}}^{t}\left(s-t+h_{1}\right) z^{T}(s) Q_{1} z(s) d s  \tag{15}\\
& +h_{2} \int_{t-h_{2}}^{t}\left(s-t+h_{2}\right) z^{T}(s) Q_{2} z(s) d s
\end{align*}
$$

where positive constants $h_{1}$ and $h_{2}$ and positive-definite matrices $P, Q_{1}$, and $Q_{2}$ satisfy (13). We now consider the

$$
\left(\begin{array}{ccc}
P \widetilde{L}+\widetilde{L}^{T} P+\alpha I_{n-1} & P \widetilde{A} & P \widetilde{\Delta} \\
* & -Q_{1} & 0 \\
* & * & -Q_{2} \\
* & * & * \\
* & * & * \\
* & * & *
\end{array}\right.
$$

where $\widetilde{\Sigma}=E \Sigma, \widetilde{A}, \widetilde{\Delta}, \widetilde{L}$, and $\bar{L}$ are defined as above, then system (7) solves $H_{\infty}$ consensus with $\gamma=\beta / \alpha$.

Proof. Choose the Lyapunov function defined by (15). First, (19) implies that (12) holds. By Theorem 3, system (1) achieves consensus asymptotically. Next, we show that (10) holds with
derivative of $V(t)$ along the trajectory of system (6). For the sake of convenience, set

$$
\begin{equation*}
\eta_{1}(t)=\int_{t}^{t-d_{1}(t)} z(s) d s, \quad \eta_{2}(t)=\int_{t-d_{2}(t)}^{t} z(s) d s \tag{16}
\end{equation*}
$$

A straightforward computation yields that

$$
\begin{align*}
\dot{V}(t)= & 2 y^{T}(t) P\left[\widetilde{L} y(t)+\widetilde{A} \eta_{1}(t)+\widetilde{\Delta} \eta_{2}(t)\right] \\
& +h_{1}^{2} z^{T}(t) Q_{1} z(t)-h_{1} \int_{t-h_{1}}^{t} z^{T}(s) Q_{1} z(s) d s  \tag{17}\\
& +h_{2}^{2} z^{T}(t) Q_{2} z(t)-h_{2} \int_{t-h_{2}}^{t} z^{T}(s) Q_{2} z(s) d s .
\end{align*}
$$

By Lemma 1, (6), and (17), we have that

$$
\begin{align*}
\dot{V}(t) \leq & 2 y^{T}(t) P\left[\widetilde{L} y(t)+\widetilde{A} \eta_{1}(t)+\widetilde{\Delta} \eta_{2}(t)\right] \\
& +h_{1}^{2} z^{T}(t) Q_{1} z(t)-\eta_{1}^{T}(t) Q_{1} \eta_{1}(t) \\
& +h_{2}^{2} z^{T}(t) Q_{2} z(t)-\eta_{2}^{T}(t) Q_{2} \eta_{2}(t)  \tag{18}\\
= & \xi^{T}(t)\left(\Omega+h_{1}^{2} \Phi^{T} Q_{1} \Phi+h_{2}^{2} \Phi^{T} Q_{2} \Phi\right) \xi(t)
\end{align*}
$$

where $\xi(t)=\left(y^{T}(t), \eta_{1}^{T}(t), \eta_{2}^{T}(t)\right)^{T}$ and $\Phi$ and $\Omega$ are defined by (14). From (13) and (18), we have that $\dot{V}(t)<0$ for $t \geq 0$. Therefore, we eventually conclude that $\lim _{t \rightarrow \infty} y(t)=0$. This completes the proof of Theorem 3.

The method used in Theorem 3 can also be applied to the $H_{\infty}$ consensus problem of system (7).

Theorem 4. For given constants $h_{1}>0$ and $h_{2}>0$, if there exist positive-definite matrices $P, Q_{1}$, and $Q_{2}$ of appropriate dimensions and positive constants $\alpha>0$ and $\beta>0$ such that the following linear matrix inequality holds:

$$
\left.\begin{array}{ccc}
P \widetilde{\Sigma} & h_{1} \bar{L}^{T} Q_{1} & h_{2} \bar{L}^{T} Q_{2}  \tag{19}\\
0 & h_{1} A^{T} Q_{1} & h_{2} A^{T} Q_{2} \\
0 & h_{1} \Delta^{T} Q_{1} & h_{2} \Delta^{T} Q_{2} \\
-\beta I_{n-1} & h_{1} \Sigma^{T} Q_{1} & h_{2} \Sigma^{T} Q_{2} \\
* & -Q_{1} & 0 \\
* & * & -Q_{2}
\end{array}\right)<0
$$

$\gamma=\beta / \alpha$ for all nonzero $w \in L_{2}[0, \infty)$ under zero initial condition. In fact, similar to the computation in Theorem 3, we have

$$
\begin{aligned}
\dot{V}(t) \leq & 2 y^{T}(t) P\left[\widetilde{L} y(t)+\widetilde{A} \eta_{1}(t)+\widetilde{\Delta} \eta_{2}(t)+\widetilde{\Sigma} w(t)\right] \\
& +h_{1}^{2} z^{T}(t) Q_{1} z(t)-\eta_{1}^{T}(t) Q_{1} \eta_{1}(t)
\end{aligned}
$$

$$
\begin{equation*}
+h_{2}^{2} z^{T}(t) Q_{2} z(t)-\eta_{2}^{T}(t) Q_{2} \eta_{2}(t) \tag{20}
\end{equation*}
$$

which implies that

$$
\begin{align*}
& \dot{V}(t)+\alpha y^{T}(t) y(t)-\beta w^{T}(t) w(t) \\
& \quad \leq \widetilde{\xi}^{T}\left(\widetilde{\Omega}+h_{1}^{2} \widetilde{\Phi}^{T} Q_{1} \widetilde{\Phi}+h_{2}^{2} \widetilde{\Phi}^{T} Q_{2} \widetilde{\Phi}\right) \widetilde{\xi} \tag{21}
\end{align*}
$$

where $\tilde{\xi}=\left(y^{T}(t), \eta_{1}^{T}(t), \eta_{2}^{T}(t), w^{T}(t)\right)^{T}$,

$$
\begin{gather*}
\widetilde{\Phi}=(\bar{L}, A, \Delta, \Sigma) \\
\Omega=\left(\begin{array}{cccc}
P \widetilde{L}+\widetilde{L}^{T} P+\alpha I_{n-1} & P \widetilde{A} & P \widetilde{\Delta} & P \widetilde{\Sigma} \\
* & -Q_{1} & 0 & 0 \\
* & * & -Q_{2} & 0 \\
* & * & * & -\beta I_{n-1}
\end{array}\right) \tag{22}
\end{gather*}
$$

By (19), (21), and (22), we have that

$$
\begin{equation*}
\dot{V}(t)+\alpha y^{T}(t) y(t)-\beta w^{T}(t) w(t)<0, \quad t \geq 0 \tag{23}
\end{equation*}
$$

Integrating (23) from 0 to $\infty$ under zero initial condition, we get

$$
\begin{equation*}
\int_{0}^{\infty} y^{T}(t) y(t) d t \leq \frac{\beta}{\alpha} \int_{0}^{\infty} w^{T}(t) w(t) d t \tag{24}
\end{equation*}
$$

This completes the proof of Theorem 4.
Remark 5. Similar to the analysis in the proof of Theorem 3, we see that a necessary condition to Theorem 3 is that the digraph $\mathscr{G}$ has a spanning tree. That is, the linear matrix inequality (19) implies that $\mathscr{G}$ has a spanning tree.

Remark 6. Unlike most of consensus analysis for multiagent systems, it does not require that $a_{i j} \geq 0$ for all $i \neq j$ in the proofs of Theorems 3 and 4. Therefore, even when $a_{i j}<0$ for some $i \neq j$, system (7) can also solve $H_{\infty}$ consensus under appropriate conditions.

The method used in this paper can also be extended to the case of switching topology. Consider the following multiagent system with switched topologies and heterogeneous delays:

$$
\begin{equation*}
\dot{x}_{i}(t)=\sum_{j=1}^{n} a_{i j}^{\sigma}\left(x_{j}\left(t-d_{1}(t)\right)-x_{i}\left(t-d_{2}(t)\right)\right), \quad i \in \mathcal{N}, \tag{25}
\end{equation*}
$$

where $t \geq 0, \sigma(t):[0, \infty) \rightarrow \Gamma=\{1,2, \ldots, p\}$ is a switching signal that determines which subsystem is active at time $t$, and $a_{i j}^{k} \geq 0, i, j \in \mathcal{N}, k \in \Gamma$, are entries of the weighted
adjacency matrix $A_{k}$. When $\sigma(t)=k \in \Gamma$, we denote the involved digraph by $\mathscr{G}_{k}=\left(V, A_{k}, E_{k}\right)$.

Under the reduced-order transformation (2), system (25) can be described by the following two systems:

$$
\begin{align*}
\dot{y}(t)= & E L_{\sigma} F y(t)+E A_{\sigma} \int_{t}^{t-d_{1}(t)} z(s) d s \\
& +E \Delta_{\sigma} \int_{t-d_{2}(t)}^{t} z(s) d s,  \tag{26}\\
z(t)= & L_{\sigma} F y(t)+A_{\sigma} \int_{t}^{t-d_{1}(t)} z(s) d s \\
& +\Delta_{\sigma} \int_{t-d_{2}(t)}^{t} z(s) d s,
\end{align*}
$$

where $y$ and $z$ are defined as above and $L_{k}, A_{k}$, and $\Delta_{k}$ $(k \in \Gamma)$ relative to the digraph $\mathscr{G}_{k}$ are defined as $L, A$, and $\Delta$, respectively. Let the Lyapunov function defined by (15) be the common Lyapunov function for the switched system (26). Then, similar to the proof of Theorem 3, we have the following consensus result in the case of switching topology.

Theorem 7. For given constants $h_{1}>0$ and $h_{2}>$ 0 , system (25) solves consensus under arbitrary switching, if there exist positive-definite matrices $P, Q_{1}$, and $Q_{2}$ of appropriate dimensions such that the following linear matrix inequality

$$
\left(\begin{array}{ccccc}
P \widetilde{L}_{k}+\widetilde{L}_{k}^{T} P & P \widetilde{A}_{k} & P \widetilde{\Delta}_{k} & h_{1} \bar{L}_{k}^{T} Q_{1} & h_{2} \bar{L}_{k}^{T} Q_{2}  \tag{27}\\
* & -Q_{1} & 0 & h_{1} A_{k}^{T} Q_{1} & h_{2} A_{k}^{T} Q_{2} \\
* & * & -Q_{2} & h_{1} \Delta_{k}^{T} Q_{1} & h_{2} \Delta_{k}^{T} Q_{2} \\
* & * & * & -Q_{1} & 0 \\
* & * & * & * & -Q_{2}
\end{array}\right)<0
$$

holds for $k \in \Gamma$, where $\widetilde{L}_{k}=E L_{k} F, \widetilde{A}_{k}=E A_{k}, \widetilde{\Delta}_{k}=E \Delta_{k}$, and $\bar{L}_{k}=L_{k} F$.

Similarly, for the following switched multiagent system with disturbance input and heterogeneous delays

$$
\begin{array}{r}
\dot{x}_{i}(t)=\sum_{j=1}^{n} a_{i j}^{\sigma}\left(x_{j}\left(t-d_{1}(t)\right)-x_{i}\left(t-d_{2}(t)\right)+w_{i j}(t)\right) \\
i \in \mathcal{N} \tag{28}
\end{array}
$$

we have the following $H_{\infty}$ consensus result.
Theorem 8. For given constants $h_{1}>0$ and $h_{2}>0$, if there exist positive-definite matrices $P, Q_{1}$, and $Q_{2}$ of appropriate dimensions and positive constants $\alpha>0$ and $\beta>0$ such that the following linear matrix inequality


Figure 1: Four digraphs: (a) $\mathscr{G}_{a}$, (b) $\mathscr{G}_{b}$, (c) $\mathscr{G}_{c}$, and (d) $\mathscr{G}_{d}$.

$$
\left(\begin{array}{cccccc}
P \widetilde{L}_{k}+\widetilde{L}_{k}^{T} P+\alpha I_{n-1} & P \widetilde{A}_{k} & P \widetilde{\Delta}_{k} & P \widetilde{\Sigma}_{k} & h_{1} \bar{L}_{k}^{T} Q_{1} & h_{2} \bar{L}_{k}^{T} Q_{2}  \tag{29}\\
* & -Q_{1} & 0 & 0 & h_{1} A_{k}^{T} Q_{1} & h_{2} A_{k}^{T} Q_{2} \\
* & * & -Q_{2} & 0 & h_{1} \Delta_{k}^{T} Q_{1} & h_{2} \Delta_{k}^{T} Q_{2} \\
* & * & * & -\beta I_{n-1} & h_{1} \Sigma_{k}^{T} Q_{1} & h_{2} \Sigma_{k}^{T} Q_{2} \\
* & * & * & * & -Q_{1} & 0 \\
* & * & * & * & * & -Q_{2}
\end{array}\right)<0
$$

holds for $k \in \Gamma$, where $\widetilde{L}_{k}, \widetilde{A}_{k}, \widetilde{\Delta}_{k}, \widetilde{\Sigma}_{k}, \Sigma_{k}$, and $\bar{L}_{k}$ relative to the digraph $\mathscr{G}_{k}$ are defined as $\widetilde{L}, \widetilde{A}, \widetilde{\Delta}, \widetilde{\Sigma}, \Sigma$, and $\bar{L}$ in Theorem 4, then system (28) solves $H_{\infty}$ consensus with $\gamma=\beta / \alpha$ under arbitrary switching.

## 4. Numerical Examples

Consider the following four digraphs with six nodes shown in Figure 1, where the weights relative to the edges shown by solid lines and dashed lines are 1 and -0.5 , respectively.

When $\mathscr{G}=\mathscr{G}_{a}$, we have that (12) is feasible for given $h_{1} \leq 0.38$ and $h_{2} \leq 0.24$. By Theorem 3, system
(1) achieves consensus asymptotically. The state trajectory under the stochastic initial condition is shown in Figure 2. In Remark 6, we show that Theorem 3 can also be applied to the extreme case when part of weights $a_{i j}$ is negative. For example, if $\mathscr{G}=\mathscr{G}_{b}$, we have that (12) is still feasible for given $h_{1} \leq 0.26$ and $h_{2} \leq 0.22$. Therefore, system (1) also solves consensus asymptotically even when there exist negative weights $a_{51}=a_{53}=-0.5$. The state trajectory under the stochastic initial condition is shown in Figure 3.

For the case of disturbance input, assume that $\mathscr{G}=\mathscr{G}_{c}$ and $w_{i j}(t)=w_{j}(t)$ for $i, j \in \mathcal{N}$. Therefore, $\Sigma=A$. For given


Figure 2: State trajectory of system (1) with $\mathscr{G}=\mathscr{G}_{a}$.


Figure 3: State trajectory of system (1) with $\mathscr{G}=\mathscr{G}_{b}$.
$h_{1}=0.2$ and $h_{2}=0.1$, we get from (19) that $\gamma=43.0731$. By Theorem 4, we have that system (7) solves $H_{\infty}$ consensus with $\gamma=43.0731$. If we let

$$
w(t)= \begin{cases}(0.1 \sin t) \mathbf{1}, & 0 \leq t \leq 30  \tag{30}\\ 0.0, & \text { otherwise }\end{cases}
$$

the trajectory of system (7) under a stochastic initial condition is shown in Figure 4.

For the case of switching topologies $\left\{\mathscr{G}_{c}, \mathscr{G}_{d}\right\}$, we have that (27) is feasible for given $h_{1} \leq 0.24$ and $h_{2} \leq 0.18$. By Theorem 7, system (25) solves consensus under arbitrary switching.

## 5. Conclusions

In this paper, we first apply the linear matrix inequality method to consensus of the single integrator multiagent


Figure 4: State trajectory of system (7) with $\mathscr{G}=\mathscr{G}_{c}$ and given disturbance.
system with heterogeneous time-varying delays in directed networks. Unlike the case of identical delays, the multiagent system with heterogeneous delays usually cannot be transformed to a reduced-order system. To overcome such difficulty, we introduce a partially-reduced-order system and an integral system. As a result, the linear matrix inequality method becomes useful in the analysis of consensus and $H_{\infty}$ consensus by constructing a particular Lyapunov function. The main results are also extended to the case of switching topologies. Finally, numerical examples and simulation results are given to illustrate the theoretical results.

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## Research Article

# Persistence and Nonpersistence of a Food Chain Model with Stochastic Perturbation 

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#### Abstract

We analyze a three species predator-prey chain model with stochastic perturbation. First, we show that this system has a unique positive solution and its $p$ th moment is bounded. Then, we deduce conditions that the system is persistent in time average. After that, conditions for the system going to be extinction in probability are established. At last, numerical simulations are carried out to support our results.


## 1. Introduction

Recently, the dynamical relationship between predator-prey has been one of the dominant themes in both ecology and mathematical ecology due to its universal importance. Especially, the predator-prey chain model is the typical representative. Thereby it significantly changed the biology, the understanding of the existence, and development of the basic law and has made the model become a research hot spot. One of the most famous models for population dynamics is the Lotka-Volterra predator-prey system which has received plenty of attention and has been studied extensively; see [14]. Specially persistence and extinction of this model are interesting topics.

The three species predator-prey chain model is described as follows:

$$
\begin{align*}
& \dot{x_{1}}(t)=x_{1}(t)\left(a_{1}-b_{11} x_{1}(t)-b_{12} x_{2}(t)\right), \\
& \dot{x_{2}}(t)=x_{2}(t)\left(-a_{2}+b_{21} x_{1}(t)-b_{22} x_{2}(t)-b_{23} x_{3}(t)\right),  \tag{1}\\
& \dot{x}_{3}(t)=x_{3}(t)\left(-a_{3}+b_{32} x_{2}(t)-b_{33} x_{3}(t)\right),
\end{align*}
$$

where $x_{i}(t)(i=1,2,3)$ denotes the population densities of the species at time $t$. The parameters $a_{1}, a_{2}, a_{3}, b_{i i}(i=1,2,3)$ are positive constants that stand for intrinsic growth rate, predator death rate of the second species, predator death
rate of the third species, coefficient of internal competition, respectively. $b_{21}, b_{32}$ represent saturated rate of the second and the third predator and $b_{12}, b_{23}$ represent the decrement rate of predator to prey.

System (1) describes a three species predator-prey chain model in which the latter preys on the former. From a biological viewpoint, we not only require the positive solution of the system but also require its unexploded property in any finite time and stability.

We know that the global asymptotic stability of a positive equilibrium $x^{*}=\left(x_{1}^{*}, x_{2}^{*}, x_{3}^{*}\right)$ holds and is global stability if the following condition holds:

$$
\begin{equation*}
a_{1}-\frac{b_{11}}{b_{21}} a_{2}-\frac{b_{11} b_{22}+b_{12} b_{21}}{b_{21} b_{32}} a_{3}>0 \tag{2}
\end{equation*}
$$

which could refer to [5]. However, population dynamics in the real world is inevitably affected by environmental noise (see, e.g., [6, 7]). Parameters involved in the system are not absolute constants, they always fluctuate around some average values. The deterministic models assume that parameters in the systems are deterministic irrespective of environmental fluctuations which impose some limitations in mathematical modeling of ecological systems. So we cannot omit the influence of the noise on the system. Recently many authors have discussed population systems subject to white
noise (see, e.g., [8-15]). May (see, e.g., [16]) pointed out that due to continuous fluctuation in the environment, the birth rates, death rates, saturated rate, competition coefficients, and all other parameters involved in the model exhibit random fluctuation to some extent, and as a result the equilibrium population distribution never attains a steady value but fluctuates randomly around some average value. Sometimes, large amplitude fluctuation in population will lead to the extinction of certain species, which does not happen in deterministic models.

Therefore, Lotka-Volterra predator-prey chain models in random environments are becoming more and more popular. Ji et al. [14, 15] investigated the asymptotic behavior of the stochastic predator-prey system with perturbation. Liu and Chen introduced periodic constant impulsive immigration of predator into predator-prey system and gave conditions for the system to be extinct and permanence. Polansky [17] and Barra et al. [18] have given some special systems of their invariant distribution. After that, Gard [5] analysed that under some conditions the stochastic food chain model exists an invariant distribution. However, seldom people study the persistent and nonpersistent of the food chain model with stochastic perturbation.

In this paper, we introduce the white noise into the intrinsic growth rate of system (1), and suppose $a_{i} \rightarrow a_{i}+$ $\sigma_{i} \dot{B}_{i}(t)(i=1,2,3)$; then we obtain the following stochastic system:

$$
\begin{align*}
\dot{x}_{1}(t)= & x_{1}(t)\left(a_{1}-b_{11} x_{1}(t)-b_{12} x_{2}(t)\right)+\sigma_{1} x_{1}(t) \dot{B}_{1}(t) \\
\dot{x}_{2}(t)= & x_{2}(t)\left(-a_{2}+b_{21} x_{1}(t)-b_{22} x_{2}(t)-b_{23} x_{3}(t)\right) \\
& -\sigma_{2} x_{2}(t) \dot{B}_{2}(t) \\
\dot{x}_{3}(t)= & x_{3}(t)\left(-a_{3}+b_{32} x_{2}(t)-b_{33} x_{3}(t)\right)-\sigma_{3} x_{3}(t) \dot{B}_{3}(t), \tag{3}
\end{align*}
$$

where $B_{i}(t)(i=1,2,3)$ are independent white noises with $B_{i}(0)=0, \sigma_{i}^{2}>0(i=1,2,3)$ representing the intensities of the noise.

The aim of this paper is to discuss the long time behavior of system (3). We have mentioned that $x^{*}=\left(x_{1}^{*}, x_{2}^{*}, x_{3}^{*}\right)$ is the positive equilibrium of system (1). But, when it suffers stochastic perturbations, there is no positive equilibrium. Hence, it is impossible that the solution of system (3) will tend to a fixed point. In this paper, we show that system (3) is persistent in time average. Furthermore, under certain conditions, we prove that the population of system (3) will die out in probability which will not happen in deterministic system and could reveal that large white noise may lead to extinction.

The rest of this paper is organized as follows. In Section 2, we show that there is a unique nonnegative solution of system (3), and its $p$ th moment is bounded. In Section 3, we show that system (3) is persistent in time average. While in Section 4, we consider three situations when the population of the system will be extinction. In Section 5, numerical simulations are carried out to support our results.

Throughout this paper, unless otherwise specified, let $\left(\Omega,\left\{\mathscr{F}_{t}\right\}_{t \geq 0}, P\right)$ be a complete probability space with a filtration $\left\{\mathscr{F}_{t}\right\}_{t \geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathscr{F}_{0}$ contains all $P$-null sets). Let $R_{+}^{3}$ denote the positive cone of $R^{3}$; namely, $R_{+}^{3}=\left\{x \in R^{3}: x_{i}>0,1 \leq i \leq 3\right\}$, $\bar{R}_{+}^{3}=\left\{x \in R^{3}: x_{i} \geq 0,1 \leq i \leq 3\right\}$.

## 2. Existence and Uniqueness of the Nonnegative Solution

To investigate the dynamical behavior, the first concern thing is whether the solution is global existence. Moreover, for a population model, whether the solution is nonnegative is also considered. Hence, in this section, we show that the solution of system (3) is global and nonnegative. As we have known, in order for a stochastic differential equation to have a unique global (i.e., no explosion at a finite time) solution with any given initial value, the coefficients of the equation are generally required to satisfy the linear growth condition and local Lipschitz condition (see, e.g., [19]). It is easy to see that the coefficients of system (3) are locally Lipschitz continuous, so system (3) has a local solution. By Lyapunov analysis method, we show the global existence of this solution.

Theorem 1. For any given initial value $x(0)=x_{0} \in R_{+}^{3}$, system (3) has a unique global positive solution $x(t)=\left(x_{1}(t), x_{2}(t)\right.$, $\left.x_{3}(t)\right)$ for all $t \geq 0$ with probability one.

Proof. It is clear that the coefficients of system (3) are locally Lipschitz continuous for the given initial value $x(0)=x_{0} \in$ $R_{+}^{3}$. So there is a unique local solution $x(t)$ on $t \in\left[0, \tau_{e}\right)$, where $\tau_{e}$ is the explosion time (see, e.g., [19]). To show that this solution is global, we need to show that $\tau_{e}=\infty$ a.s. Let $m_{0} \geq 1$ be sufficiently large so that each component of $x_{0}$ all lies within the interval [ $\left.1 / m_{0}, m_{0}\right]$. For each integer $m \geq m_{0}$, define the stopping time:

$$
\begin{align*}
\tau_{m}=\inf \left\{t \in\left[0, \tau_{e}\right):\right. & \min \left\{x_{1}(t), x_{2}(t), x_{3}(t)\right\} \leq \frac{1}{m} \\
& \text { or } \left.\max \left\{x_{1}(t), x_{2}(t), x_{3}(t)\right\} \geq m\right\} \tag{4}
\end{align*}
$$

Throughout this paper, we set $\inf \emptyset=\infty$ (as usual $\emptyset$ denotes the empty set). Clearly, $\tau_{m}$ is increasing as $m \rightarrow \infty$. Set $\tau_{\infty}=$ $\lim _{m \rightarrow \infty} \tau_{m}$; then $\tau_{\infty} \leq \tau_{e}$ a.s. If we can show that $\tau_{\infty}=\infty$ a.s., then $\tau_{e}=\infty$ and $\left(x_{1}(t), x_{2}(t), x_{3}(t)\right) \in R_{+}^{3}$ a.s. for all $t \geq 0$. In other words, to complete the proof all we need to show is that $\tau_{\infty}=\infty$ a.s. For if this statement is false, then there is a pair of constants $T>0$ and $\epsilon \in(0,1)$ such that

$$
\begin{equation*}
P\left\{\tau_{\infty} \leq T\right\}>\epsilon \tag{5}
\end{equation*}
$$

Hence there is an integer $m_{1} \geq m_{0}$ such that

$$
\begin{equation*}
P\left\{\tau_{m} \leq T\right\} \geq \epsilon \quad \forall m \geq m_{1} \tag{6}
\end{equation*}
$$

Define a $C^{2}$-function $V: R_{+}^{3} \rightarrow \bar{R}_{+}$by

$$
\begin{align*}
& V\left(x_{1}, x_{2}, x_{3}\right) \\
& =b_{32}\left[b_{21}\left(x_{1}-1-\log x_{1}\right)+b_{12}\left(x_{2}-1-\log x_{2}\right)\right]  \tag{7}\\
& \quad+b_{23} b_{12}\left(x_{3}-1-\log x_{3}\right)
\end{align*}
$$

the nonnegativity of this function can be seen from $u-1-$ $(1 / 2) \log u \geq 0$, for all $u>0$. Using Itô's formula, we get

$$
\begin{align*}
d V:= & L V d t+b_{32} b_{21} \sigma_{1}\left(x_{1}-1\right) d B_{1}(t) \\
& +b_{32} b_{12} \sigma_{2}\left(x_{2}-1\right) d B_{2}(t)  \tag{8}\\
& +b_{23} b_{12} \sigma_{3}\left(x_{3}-1\right) d B_{3}(t),
\end{align*}
$$

where

$$
\begin{align*}
L V= & b_{32} b_{21}\left(x_{1}-1\right)\left(a_{1}-b_{11} x_{1}-b_{12} x_{2}\right) \\
& +b_{32} b_{12}\left(x_{2}-1\right)\left(-a_{2}+b_{21} x_{1}-b_{22} x_{2}-b_{23} x_{3}\right) \\
& +b_{23} b_{12}\left(x_{3}-1\right)\left(-a_{3}+b_{32} x_{2}-b_{33} x_{3}\right) \\
& +b_{32} b_{21} \frac{\sigma_{1}^{2}}{2}+b_{32} b_{12} \frac{\sigma_{2}^{2}}{2}+b_{23} b_{12} \frac{\sigma_{1}^{2}}{2} \\
= & b_{32} b_{21}\left(-a_{1}+\frac{\sigma_{1}^{2}}{2}\right)+b_{32} b_{12}\left(a_{2}+\frac{\sigma_{2}^{2}}{2}\right)  \tag{9}\\
& +b_{23} b_{12}\left(a_{3}+\frac{\sigma_{3}^{2}}{2}\right) \\
& +\left(b_{32} b_{21} a_{1}+b_{32} b_{21} b_{11}-b_{32} b_{12} b_{21}\right) x_{1} \\
& +\left(b_{32} b_{21} b_{12}+b_{32} b_{12} b_{22}-b_{23} b_{12} b_{32}-b_{32} b_{12} a_{2}\right) x_{2} \\
& +\left(b_{32} b_{12} b_{23}+b_{23} b_{12} b_{33}-b_{23} b_{12} a_{3}\right) x_{3} \\
& -b_{32} b_{21} b_{11} x_{1}^{2}-b_{32} b_{12} b_{22} x_{2}^{2}-b_{23} b_{12} b_{33} x_{3}^{2} \leq \widehat{M},
\end{align*}
$$

where $\widehat{M}$ is a constant. Therefore

$$
\begin{align*}
& \int_{0}^{\tau_{m} \wedge T} d V(x(t)) \\
& \quad \leq \int_{0}^{\tau_{m} \wedge T} \widehat{M} d t+\int_{0}^{\tau_{m} \wedge T} b_{32} b_{21} \sigma_{1}\left(x_{1}-1\right) d B_{1}(t)  \tag{10}\\
& \quad+\int_{0}^{\tau_{m} \wedge T} b_{32} b_{12} \sigma_{2}\left(x_{2}-1\right) d B_{2}(t) \\
& \quad+\int_{0}^{\tau_{m} \wedge T} b_{23} b_{12} \sigma_{3}\left(x_{3}-1\right) d B_{3}(t)
\end{align*}
$$

which implies that

$$
\begin{equation*}
E\left[V\left(x\left(\tau_{m} \wedge T\right)\right)\right] \leq V\left(x_{0}\right)+\widehat{M} T . \tag{11}
\end{equation*}
$$

Set $\Omega_{m}=\left\{\tau_{m} \leq T\right\}$ for $m \geq m_{1}$. By (6), we know $P\left(\Omega_{m}\right) \geq \epsilon$. Notice that for every $\omega \in \Omega_{m}$, there is at least one of $x_{i}\left(\tau_{m}, \omega\right)$ equals either $m$ or $1 / m$; then

$$
\begin{equation*}
V\left(x\left(\tau_{m}\right)\right) \geq(m-1-\log m) \wedge\left(m^{-1}-1+\log m\right) \tag{12}
\end{equation*}
$$

It then follows from (11) that

$$
\begin{align*}
& V\left(x_{0}\right)+\widehat{M} T \\
& \quad \geq E\left[1_{\Omega_{m}(\omega)} V\left(x\left(\tau_{m}\right)\right)\right]  \tag{13}\\
& \quad \geq \epsilon(m-1-\log m) \wedge\left(m^{-1}-1+\log m\right)
\end{align*}
$$

where $1_{\Omega_{m}(\omega)}$ is the indicator function of $\Omega_{m}$. Letting $m \rightarrow$ $\infty$ leads to the contradiction that $\infty>V\left(x_{0}\right)+\widehat{M} T=\infty$. So we must have $\tau_{\infty}=\infty$ a.s.

Theorem 2. Let $x(t)=\left(x_{1}(t), x_{2}(t)\right.$, and let $\left.x_{3}(t)\right)$ be the solution of system (3) with any given initial value $x(0)=x_{0} \in$ $R_{+}^{3}$, then there exists a positive constant $K(p)$ such that

$$
\begin{array}{r}
E\left[\left(b_{32} b_{21} x_{1}(t)+b_{32} b_{12} x_{2}(t)+b_{23} b_{12} x_{3}(t)\right)^{p}\right] \leq K(p) \\
\forall t \in(0, \infty), \quad p>1 \tag{14}
\end{array}
$$

Proof. Let $y(t)=b_{32} b_{21} x_{1}(t)+b_{32} b_{12} x_{2}(t)+b_{23} b_{12} x_{3}(t)$; then

$$
\begin{align*}
d y(t)= & \left(a_{1} b_{21} b_{32} x_{1}-a_{2} b_{12} b_{32} x_{2}-a_{3} b_{12} b_{23} x_{3}\right. \\
& \left.-b_{11} b_{21} b_{32} x_{1}^{2}-b_{12} b_{22} b_{32} x_{2}^{2}-b_{12} b_{23} b_{33} x_{3}^{2}\right) d t \\
+ & \sigma_{1} b_{21} b_{32} x_{1} d B_{1}(t)+\sigma_{2} b_{12} b_{32} x_{2} d B_{2}(t) \\
+ & \sigma_{3} b_{12} b_{23} x_{3} d B_{3}(t), \tag{15}
\end{align*}
$$

and so

$$
\begin{align*}
d y^{p}= & p y^{p-1}\left(a_{1} b_{21} b_{32} x_{1}-a_{2} b_{12} b_{32} x_{2}-a_{3} b_{12} b_{23} x_{3}\right. \\
& -b_{11} b_{21} b_{32} x_{1}^{2}-b_{12} b_{22} b_{32} x_{2}^{2} \\
& \left.-b_{12} b_{23} b_{33} x_{3}^{2}\right) d t \\
& +p y^{p-1}\left(\sigma_{1} b_{21} b_{32} x_{1} d B_{1}(t)+\sigma_{2} b_{12} b_{32} x_{2} d B_{2}(t)\right. \\
& \left.+\sigma_{3} b_{12} b_{23} x_{3} d B_{3}(t)\right) \\
& +\frac{1}{2} p(p-1) y^{p-2} \\
& \times\left(\sigma_{1}^{2} b_{21}^{2} b_{32}^{2} x_{1}^{2}+\sigma_{2}^{2} b_{12}^{2} b_{32}^{2} x_{2}^{2}+\sigma_{3}^{2} b_{12}^{2} b_{23}^{2} x_{3}^{2}\right) d t \tag{16}
\end{align*}
$$

Note that

$$
\begin{align*}
& b_{11} b_{21} b_{32} x_{1}^{2}+b_{12} b_{22} b_{32} x_{2}^{2}+b_{12} b_{23} b_{33} x_{3}^{2} \\
& \geq \frac{\min \left\{b_{11}, b_{22}, b_{33}\right\}}{b_{32} b_{21}+b_{32} b_{12}+b_{23} b_{12}} \\
& \quad \times\left(b_{32} b_{21} x_{1}+b_{32} b_{12} x_{2}+b_{23} b_{12} x_{3}\right)^{2}  \tag{17}\\
& =\frac{\min \left\{b_{11}, b_{22}, b_{33}\right\}}{b_{32} b_{21}+b_{32} b_{12}+b_{23} b_{12}} y^{2} .
\end{align*}
$$

Then

$$
\begin{align*}
d y^{p}= & p y^{p-1}\left(a_{1} y-\frac{\min \left\{b_{11}, b_{22}, b_{33}\right\}}{b_{32} b_{21}+b_{32} b_{12}+b_{23} b_{12}} y^{2}\right) d t \\
& +\frac{1}{2} p(p-1) y^{p-2} \max \left\{\sigma_{1}^{2}, \sigma_{2}^{2}, \sigma_{3}^{2}\right\} y^{2} d t \\
& +p y^{p-1}\left(\sigma_{1} b_{21} b_{32} x_{1} d B_{1}(t)+\sigma_{2} b_{12} b_{32} x_{2} d B_{2}(t)\right. \\
& \left.+\sigma_{3} b_{12} b_{23} x_{3} d B_{3}(t)\right) \\
\leq & {\left[p\left(a_{1}+\frac{p}{2} \max \left\{\sigma_{1}^{2}, \sigma_{2}^{2}, \sigma_{3}^{2}\right\}\right) y^{p}\right.}  \tag{18}\\
& \left.\quad-\frac{p \min \left\{b_{11}, b_{22}, b_{33}\right\}}{b_{32} b_{21}+b_{32} b_{12}+b_{23} b_{12}} y^{p+1}\right] d t \\
& +p y^{p-1}\left(\sigma_{1} b_{21} b_{32} x_{1} d B_{1}(t)+\sigma_{2} b_{12} b_{32} x_{2} d B_{2}(t)\right. \\
& \left.+\sigma_{3} b_{12} b_{23} x_{3} d B_{3}(t)\right)
\end{align*}
$$

Hence

$$
\begin{align*}
& \frac{d E\left[y^{p}(t)\right]}{d t} \\
& \quad \leq p\left(a_{1}+p \frac{\max \left\{\sigma_{1}^{2}, \sigma_{2}^{2}, \sigma_{3}^{2}\right\}}{2}\right) E\left[y^{p}(t)\right] \\
& \quad-p \frac{\min \left\{b_{11}, b_{22}, b_{33}\right\}}{b_{32} b_{21}+b_{32} b_{12}+b_{23} b_{12}} E\left[y^{p+1}(t)\right]  \tag{19}\\
& \quad \leq p\left(a_{1}+p \frac{\max \left\{\sigma_{1}^{2}, \sigma_{2}^{2}, \sigma_{3}^{2}\right\}}{2}\right) E\left[y^{p}(t)\right] \\
& \quad-p \frac{\min \left\{b_{11}, b_{22}, b_{33}\right\}}{b_{32} b_{21}+b_{32} b_{12}+b_{23} b_{12}} E\left[y^{p}(t)\right]^{(p+1) / p}
\end{align*}
$$

Therefore, by comparison theorem, we get

$$
\begin{align*}
& \limsup _{t \rightarrow \infty} E\left[y^{p}(t)\right] \\
& \leq\left[\frac{\left(a_{1}+p \max \left\{\sigma_{1}^{2}, \sigma_{2}^{2}, \sigma_{3}^{2}\right\}\right)\left(b_{32} b_{21}+b_{32} b_{12}+b_{23} b_{12}\right)}{\min \left\{b_{11}, b_{22}, b_{33}\right\}}\right]^{p} . \tag{20}
\end{align*}
$$

Besides, note that $E\left[y^{p}(t)\right]$ is continuous; then there is a positive constant $K(p)$ such that

$$
\begin{equation*}
E\left[y^{p}(t)\right] \leq K(p), \quad \forall t \in[0, \infty) \tag{21}
\end{equation*}
$$

## 3. Persistent in Time Average

There is no equilibrium of system (3). Hence we cannot show the permanence of the system by proving the stability of the positive equilibrium as the deterministic system. In this section we first show that this system is persistent in mean. Before we give the result, we should do some prepared work.
L. S. Chen and J. Chen in [20] proposed the definition of persistence in mean for the deterministic system. Here, we also use this definition for the stochastic system.

Definition 3. System (3) is said to be persistent in mean, if

$$
\begin{equation*}
\liminf _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} x_{3}(s) d s>0, \text { a.s. } \tag{22}
\end{equation*}
$$

Lemma 4 (see [21, Lemma 17]). Let $f \in C([0,+\infty) \times$ $\Omega,(0,+\infty))$ and $F \in C([0,+\infty) \times \Omega, R)$. If there exist positive constants $\lambda_{0}, \lambda$, such that

$$
\begin{equation*}
\log f(t) \geq \lambda t-\lambda_{0} \int_{0}^{t} f(s) d s+F(t), \quad t \geq 0 \text { a.s. } \tag{23}
\end{equation*}
$$

and $\lim _{t \rightarrow \infty}(F(t) / t)=0$ a.s., then

$$
\begin{equation*}
\liminf _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} f(s) d s \geq \frac{\lambda}{\lambda_{0}}, \text { a.s. } \tag{24}
\end{equation*}
$$

From Lemma 4, it is easy to see that we could get Lemmas 5 and 6 with the same method.

Lemma 5. Let $f \in C([0,+\infty) \times \Omega,(0,+\infty))$ and $F \in C([0$, $+\infty) \times \Omega, R)$. If there exist positive constants $\lambda_{0}, \lambda$, such that

$$
\begin{equation*}
\log f(t) \leq \lambda t-\lambda_{0} \int_{0}^{t} f(s) d s+F(t), \quad t \geq 0 \text { a.s. } \tag{25}
\end{equation*}
$$

and $\lim _{t \rightarrow \infty}(F(t) / t)=0$ a.s., then

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} f(s) d s \leq \frac{\lambda}{\lambda_{0}}, \text { a.s. } \tag{26}
\end{equation*}
$$

Lemma 6. Let $f \in C([0,+\infty) \times \Omega,(0,+\infty))$ and $F \in C([0$, $+\infty) \times \Omega, R)$. If there exist positive constants $\lambda_{0}, \lambda$, such that

$$
\begin{equation*}
\log f(t)=\lambda t-\lambda_{0} \int_{0}^{t} f(s) d s+F(t), \quad t \geq 0 \text { a.s. } \tag{27}
\end{equation*}
$$

and $\lim _{t \rightarrow \infty}(F(t) / t)=0$ a.s., then

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} f(s) d s=\frac{\lambda}{\lambda_{0}}, \text { a.s. } \tag{28}
\end{equation*}
$$

From the stochastic comparison theorem [11], it is easy to get the following result.

Lemma 7. Let $x(t) \in R_{+}^{3}$ be a solution of system (3) with $x(0)=\left(x_{1}(0), x_{2}(0), x_{3}(0)\right)$. Then one has

$$
\begin{equation*}
x(t) \leq \Phi(t) ; \tag{29}
\end{equation*}
$$

that is,

$$
\begin{equation*}
x_{i}(t) \leq \Phi_{i}(t), \quad i=1,2,3, \tag{30}
\end{equation*}
$$

where

$$
\begin{equation*}
\Phi(t)=\left(\Phi_{1}(t), \Phi_{2}(t), \Phi_{3}(t)\right)^{\top} \tag{31}
\end{equation*}
$$

$\Phi_{i}(t)$ is solutions of the following stochastic differential equations:

$$
\begin{align*}
d \Phi_{1}(t)= & \Phi_{1}(t)\left(a_{1}-b_{11} \Phi_{1}(t)\right) d t \\
& +\sigma_{1} \Phi_{1}(t) d B_{1}(t), \quad \Phi_{1}(0)=x_{1}(0), \\
d \Phi_{2}(t)= & \Phi_{2}(t)\left(-a_{2}+b_{21} \Phi_{1}(t)-b_{22} \Phi_{2}(t)\right) d t \\
& -\sigma_{2} \Phi_{2}(t) d B_{2}(t), \quad \Phi_{2}(0)=x_{2}(0),  \tag{32}\\
d \Phi_{3}(t)= & \Phi_{3}(t)\left(-a_{3}+b_{32} \Phi_{2}(t)-b_{33} \Phi_{3}(t)\right) d t \\
& -\sigma_{3} \Phi_{3}(t) d B_{3}(t), \quad \Phi_{3}(0)=x_{3}(0) .
\end{align*}
$$

Assumption 8. Consider

$$
\begin{gather*}
r_{1}-\frac{b_{11}}{b_{21}} r_{2}-\frac{b_{11} b_{22}+b_{12} b_{21}}{b_{21} b_{32}} r_{3}>0,  \tag{33}\\
r_{1}=a_{1}-\frac{\sigma_{1}^{2}}{2}>0, \quad r_{i}=a_{i}+\frac{\sigma_{i}^{2}}{2} \quad i=2,3 .
\end{gather*}
$$

Lemma 9. If Assumption 8 is satisfied, the solution $\Phi(t)$ of system (32) with any initial value $\Phi(0) \in R_{+}^{3}$ has the following property:

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \frac{\log \Phi_{i}(t)}{t}=0, \quad \lim _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} \Phi_{i}(s) d s=M_{i}, \text { a.s. } \tag{34}
\end{equation*}
$$

where

$$
\begin{align*}
& M_{1}=\frac{r_{1}}{b_{11}}, \quad M_{2}=\frac{r_{1} b_{21}-r_{2} b_{11}}{b_{11}}  \tag{35}\\
& M_{3}=\frac{r_{1} b_{21} b_{32}-r_{2} b_{11} b_{32}-r_{3} b_{11} b_{22}}{b_{11} b_{22} b_{33}}
\end{align*}
$$

Proof. From the result in [14] and Assumption 8 being satisfied, we know

$$
\begin{gather*}
\lim _{t \rightarrow \infty} \frac{\log \Phi_{1}(t)}{t}=0 \\
\liminf _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} \Phi_{1}(s) d s=\frac{a_{1}-\sigma_{1}^{2} / 2}{b_{11}}=\frac{r_{1}}{b_{11}}, \tag{36}
\end{gather*}
$$

Besides, according to Itô's formula, the second population of system (32) is changed into

$$
\begin{equation*}
d \log \Phi_{2}(t)=\left(-r_{2}+b_{21} \Phi_{1}(t)-b_{22} \Phi_{2}(t)\right) d t-\sigma_{2} d B_{2}(t) \tag{37}
\end{equation*}
$$

It then follows

$$
\begin{align*}
\log \Phi_{2}(t)= & \log \Phi_{2}(0)-r_{2} t \\
& +b_{21} \int_{0}^{t} \Phi_{1}(s) d s-b_{22} \int_{0}^{t} \Phi_{2}(s) d s-\sigma_{2} B_{2}(t) \tag{38}
\end{align*}
$$

With Lemma 6 and Assumption 8, we could get

$$
\begin{align*}
\lim _{t \rightarrow \infty} & \frac{1}{t} \int_{0}^{t} \Phi_{2}(s) d s  \tag{39}\\
& =\frac{-r_{2}+b_{21}\left(r_{1} / b_{11}\right)}{b_{22}}=\frac{r_{1} b_{21}-r_{2} b_{11}}{b_{11} b_{22}}>0
\end{align*}
$$

Let (38) divide $t$, and $t \rightarrow \infty$, together with (36) and (39), consequently

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \frac{\log \Phi_{2}(t)}{t}=0 \tag{40}
\end{equation*}
$$

Similarly, according to Itô's formula, the third population of system (25) is changed into

$$
\begin{equation*}
d \log \Phi_{3}(t)=\left(-r_{3}+b_{32} \Phi_{2}(t)-b_{33} \Phi_{3}(t)\right) d t-\sigma_{3} d B_{3}(t) \tag{41}
\end{equation*}
$$

it then follows

$$
\begin{gather*}
\log \Phi_{3}(t)=\log \Phi_{3}(0)-r_{3} t+b_{32} \int_{0}^{t} \Phi_{2}(s) d s \\
\quad-b_{33} \int_{0}^{t} \Phi_{3}(s) d s-\sigma_{3} B_{3}(t) \\
\lim _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} \Phi_{3}(s) d s  \tag{42}\\
=\frac{-r_{3}+b_{32}\left(\left(r_{1} b_{21}-r_{2} b_{11}\right) / b_{11} b_{22}\right)}{b_{33}}>0 \\
\lim _{t \rightarrow \infty} \frac{\log \Phi_{3}(t)}{t}=0
\end{gather*}
$$

From this, together with Lemmas 7 and 9, the following result is obviously true.

Theorem 10. If Assumption 8 is satisfied, the solution $x(t)$ of system (3) with any initial value $x(0) \in R_{+}^{3}$ has the following property:

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \frac{\log x_{i}(t)}{t} \leq 0, \quad i=1,2,3 \tag{43}
\end{equation*}
$$

Above all, we could get.
Theorem 11. If Assumption 8 is satisfied, the the solution $x(t)$ of system (3) with any initial value $x(0) \in R_{+}^{3}$ has the following property:

$$
\begin{equation*}
\liminf _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} x_{3}(s) d s \geq \tilde{x}_{3}^{*}, \text { a.s. } \tag{44}
\end{equation*}
$$

where $\widetilde{x}^{*}=\left(\widetilde{x}_{1}^{*}, \tilde{x}_{2}^{*}, \widetilde{x}_{3}^{*}\right)$ is the only nonnegative solution of the following equation:

$$
\begin{gather*}
r_{1}-b_{11} x_{1}-b_{12} x_{2}=0 \\
-r_{2}+b_{21} x_{1}-b_{22} x_{2}-b_{23} x_{3}=0  \tag{45}\\
-r_{3}+b_{32} x_{2}-b_{33} x_{3}=0
\end{gather*}
$$

Proof. From system (3), such that

$$
\begin{align*}
& d\left(c_{1} \log x_{1}(t)+c_{2} \log x_{2}(t)+c_{3} \log x_{3}(t)\right) \\
& =[ \\
& \quad\left[\left(r_{1} c_{1}-r_{2} c_{2}-r_{3} c_{3}\right)+\left(-b_{11} c_{1}+b_{21} c_{2}\right) x_{1}\right.  \tag{46}\\
& \quad+\left(-b_{12} c_{1}-b_{22} c_{2}+b_{32} c_{3}\right) x_{2} \\
& \left.\quad-\left(b_{23} c_{2}+b_{33} c_{3}\right) x_{3}\right] d t \\
& \quad+c_{1} \sigma_{1} d B_{1}(t)-c_{2} \sigma_{2} d B_{2}(t)-c_{3} \sigma_{3} d B_{3}(t) .
\end{align*}
$$

Let $c_{1}=b_{21}, c_{2}=b_{11}$, and $c_{3}=\left(b_{11} b_{22}+b_{12} b_{21}\right) / b_{32}$, together with Assumption 8, we know

$$
\begin{equation*}
r_{1} c_{1}-r_{2} c_{2}-r_{3} c_{3}>0 ; \tag{47}
\end{equation*}
$$

hence

$$
\begin{align*}
& \left(c_{1}\left(\log x_{1}(t)-\log x_{1}(0)\right)+c_{2}\left(\log x_{2}(t)-\log x_{2}(0)\right)\right. \\
& \left.\quad+c_{3}\left(\log x_{3}(t)-\log x_{3}(0)\right)\right) \times(t)^{-1} \\
& \quad=\left(r_{1} c_{1}-r_{2} c_{2}-r_{3} c_{3}\right) \\
& \quad-\left(c_{2} b_{23}+c_{3} b_{33}\right) \frac{1}{t} \int_{0}^{t} x_{3}(s) d s \\
& \quad+\frac{c_{1} \sigma_{1} B_{1}(t)-c_{2} \sigma_{2} B_{2}(t)-c_{3} \sigma_{3} B_{3}(t)}{t} \tag{48}
\end{align*}
$$

According to Theorem 10, where

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \frac{\log x_{i}(t)}{t} \leq 0, \quad i=1,2,3 \tag{49}
\end{equation*}
$$

and $\lim _{t \rightarrow \infty}\left(B_{i}(t) / t\right)=0, i=1,2,3$,

$$
\begin{equation*}
\liminf _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} x_{3}(s) d s \geq \frac{r_{1} c_{1}-r_{2} c_{2}-r_{3} c_{3}}{c_{2} b_{23}+c_{3} b_{33}}=\tilde{x}_{3}^{*}, \tag{50}
\end{equation*}
$$

where $\tilde{x}^{*}=\left(\widetilde{x}_{1}^{*}, \widetilde{x}_{2}^{*}, \widetilde{x}_{3}^{*}\right)$ is the only nonnegative solution of the following equation when Assumption 8 is satisfied:

$$
\begin{gather*}
r_{1}-b_{11} x_{1}-b_{12} x_{2}=0, \\
-r_{2}+b_{21} x_{1}-b_{22} x_{2}-b_{23} x_{3}=0,  \tag{51}\\
-r_{3}+b_{32} x_{2}-b_{33} x_{3}=0 .
\end{gather*}
$$

## 4. Nonpersistence

In this section, we show the situation when the population of system (3) will be extinction in three cases.

Case $1\left(r_{1}<0\right)$. According to Itô's formula, the first population of system (25) is changed into

$$
\begin{equation*}
d \log \Phi_{1}(t) \leq\left(r_{1}-b_{11} \Phi_{1}(t)\right) d t-\sigma_{1} d B_{1}(t) . \tag{52}
\end{equation*}
$$

If $r_{1}<0$, we could get

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \frac{\log \Phi_{1}(t)}{t} \leq \frac{r_{1}}{b_{11}}<0 \text { a.s. } \tag{53}
\end{equation*}
$$

From the stochastic comparison theorem, we have

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \frac{\log x_{1}(t)}{t} \leq \frac{r_{1}}{b_{11}}<0 \text { a.s., } \tag{54}
\end{equation*}
$$

hence

$$
\begin{equation*}
\lim _{t \rightarrow \infty} x_{1}(t)=0 \text {, a.s. } \tag{55}
\end{equation*}
$$

From the second population of system (25), we have

$$
\begin{align*}
& \limsup _{t \rightarrow \infty} \frac{\log \Phi_{2}(t)}{t} \\
& \quad \leq-a_{2}+b_{21} \limsup _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} \Phi_{1}(s) d s \leq-a_{2} \text { a.s.; } \tag{56}
\end{align*}
$$

similarly

$$
\begin{align*}
& \limsup _{t \rightarrow \infty} \frac{\log \Phi_{3}(t)}{t} \leq-a_{3} \text { a.s., }  \tag{57}\\
& \lim _{t \rightarrow \infty} x_{i}(t)=0 \quad \text { a.s. } \mathrm{i}=2,3 .
\end{align*}
$$

Case $2\left(r_{1}>0, r_{1}-\left(b_{11} / b_{21}\right) r_{2}<0\right)$. It is clear that from the proof section of Case 1, we get

$$
\begin{align*}
& \frac{\log \Phi_{2}(t)-\log \Phi_{2}(0)}{t} \\
& \quad \leq-r_{2}+b_{21} \frac{1}{t} \int_{0}^{t} \Phi_{1}(s) d s-\frac{\sigma_{2} d B_{2}(t)}{t} \text { a.s., } \tag{58}
\end{align*}
$$

hence

$$
\begin{align*}
& \limsup _{t \rightarrow \infty} \frac{\log \Phi_{2}(t)}{t}  \tag{59}\\
& \quad \leq-r_{2}+b_{21} M_{1}=-r_{2}+b_{21} \frac{r_{1}}{b_{11}}<0 \text { a.s. }
\end{align*}
$$

Similarly

$$
\begin{align*}
& \limsup _{t \rightarrow \infty} \frac{\log \Phi_{3}(t)}{t} \\
& \quad \leq-r_{3}+b_{32} \limsup _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} \Phi_{2}(s) d s  \tag{60}\\
& \quad \leq-a_{3}<0 \text { a.s.; }
\end{align*}
$$

thus,

$$
\begin{equation*}
\lim _{t \rightarrow \infty} x_{i}(t)=0 \quad \text { a.s., } \quad \mathrm{i}=2,3 . \tag{61}
\end{equation*}
$$

Above all, and from the conclusion in [22], we could easily know that the distribution of $x_{1}(t)$ converges weekly to the probability measure with density:

$$
\begin{equation*}
f^{*}(\zeta)=C_{0} \zeta^{2 r_{1} / \sigma_{1}^{2}-1} e^{-2 b_{11} \zeta / \sigma_{1}^{2}} \tag{62}
\end{equation*}
$$

where $C_{0}=\left(2 b_{11} / \sigma_{1}^{2}\right)^{2 r_{1} / \sigma_{1}^{2}} / \Gamma\left(2 r_{1} / \sigma_{1}^{2}\right)$ and

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} x_{1}(s) d s=\frac{r_{1}}{b_{11}}, \text { a.s. } \tag{63}
\end{equation*}
$$

Case $3\left(r_{1}-\left(b_{11} / b_{21}\right) r_{2}-\left(\left(b_{11} b_{22}+b_{12} b_{21}\right) / b_{21} b_{32}\right) r_{3}<0\right)$. It is clear that

$$
\begin{align*}
& d\left(c_{1} \log x_{1}(t)+c_{2} \log x_{2}(t)+c_{3} \log x_{3}(t)\right) \\
& =[ \\
& \quad\left[\left(r_{1} c_{1}-r_{2} c_{2}-r_{3} c_{3}\right)+\left(-b_{11} c_{1}+b_{21} c_{2}\right) x_{1}\right.  \tag{64}\\
& \quad+\left(-b_{12} c_{1}-b_{22} c_{2}+b_{32} c_{3}\right) x_{2} \\
& \left.\quad-\left(b_{23} c_{2}+b_{33} c_{3}\right) x_{3}\right] d t \\
& \quad+c_{1} \sigma_{1} d B_{1}(t)-c_{2} \sigma_{2} d B_{2}(t)-c_{3} \sigma_{3} d B_{3}(t)
\end{align*}
$$

Since $c_{1}=b_{21}, c_{2}=b_{11}, c_{3}=\left(b_{11} b_{22}+b_{12} b_{21}\right) / b_{32}$, we get

$$
\begin{align*}
c_{1} \log x_{1} & (t)+c_{2} \log x_{2}(t)+c_{3} \log x_{3}(t) \\
\leq & \left(r_{1} c_{1}-r_{2} c_{2}-r_{3} c_{3}\right) t  \tag{65}\\
& +c_{1} \log x_{1}(0)+c_{2} \log x_{2}(0)+c_{3} \log x_{3}(0) \\
& +c_{1} \sigma_{1} B_{1}(t)-c_{2} \sigma_{2} B_{2}(t)-c_{3} \sigma_{3} B_{3}(t),
\end{align*}
$$

Moreover,

$$
\begin{align*}
& \frac{\log x_{1}^{c_{1}}(t) x_{2}^{c_{2}}(t) x_{3}^{c_{3}}(t)}{t} \\
& \quad \leq\left(r_{1} c_{1}-r_{2} c_{2}-r_{3} c_{3}\right) \\
& \quad+\frac{c_{1} \log x_{1}(0)+c_{2} \log x_{2}(0)+c_{3} \log x_{3}(0)}{t}  \tag{66}\\
& \quad+c_{1} \sigma_{1} \frac{B_{1}(t)}{t}-c_{2} \sigma_{2} \frac{B_{2}(t)}{t}-c_{3} \sigma_{3} \frac{B_{3}(t)}{t} .
\end{align*}
$$

And $\lim _{t \rightarrow \infty}\left(B_{i}(t) / t\right)=0, i=1,2,3$, implies

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \frac{\log x_{1}^{c_{1}}(t) x_{2}^{c_{2}}(t) x_{3}^{c_{3}}(t)}{t} \leq r_{1} c_{1}-r_{2} c_{2}-r_{3} c_{3}<0 \tag{67}
\end{equation*}
$$

then

$$
\begin{equation*}
\lim _{t \rightarrow \infty} x_{1}^{c_{1}}(t) x_{2}^{c_{2}}(t) x_{3}^{c_{3}}(t)=0 \text { a.s. } \tag{68}
\end{equation*}
$$

Therefore, by the above arguments, we get the following conclusion.

Theorem 12. Let $x(t)$ be the solution of system (3) with any initial value $x(0) \in R_{+}^{3}$. Then
(1) if $r_{1}<0$, then

$$
\begin{equation*}
\lim _{t \rightarrow \infty} x_{i}(t)=0 \quad \text { a.s., } \mathrm{i}=1,2,3 \tag{69}
\end{equation*}
$$

(2) if $r_{1}>0, r_{1}-\left(b_{11} / b_{21}\right) r_{2}<0$, then

$$
\begin{equation*}
\lim _{t \rightarrow \infty} x_{i}(t)=0 \quad \text { a.s., } \mathrm{i}=2,3 \tag{70}
\end{equation*}
$$

and the distribution of $x_{1}(t)$ converges weekly to the probability measure with density:

$$
\begin{equation*}
f^{*}(\zeta)=C_{0} \zeta^{2 r_{1} / \sigma_{1}^{2}-1} e^{-2 b_{11} \zeta / \sigma_{1}^{2}} \tag{71}
\end{equation*}
$$

where $C_{0}=\left(2 b_{11} / \sigma_{1}^{2}\right)^{2 r_{1} / \sigma_{1}^{2}} / \Gamma\left(2 r_{1} / \sigma_{1}^{2}\right)$ and

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} x_{1}(s) d s=\frac{r_{1}}{b_{11}}, \text { a.s } \tag{72}
\end{equation*}
$$

(3) if $r_{1}-\left(b_{11} / b_{21}\right) r_{2}-\left(\left(b_{11} b_{22}+b_{12} b_{21}\right) / b_{21} b_{32}\right) r_{3}<0$, then

$$
\begin{equation*}
\lim _{t \rightarrow \infty} x_{1}^{c_{1}}(t) x_{2}^{c_{2}}(t) x_{3}^{c_{3}}(t)=0, \text { a.s. } \tag{73}
\end{equation*}
$$

where $c_{1}=b_{21}, c_{2}=b_{11}$, and $c_{3}=\left(b_{11} b_{22}+b_{12} b_{21}\right) / b_{32}$.

## 5. Numerical Simulation

In this section, we give out the numerical experiment to support our results. Consider

$$
\begin{align*}
\dot{x}_{1}(t)= & x_{1}(t)\left(a_{1}-b_{11} x_{1}(t)-b_{12} x_{2}(t)\right) \\
& +\sigma_{1} x_{1}(t) \dot{B}_{1}(t), \\
\dot{x}_{2}(t)= & x_{2}(t)\left(-a_{2}+b_{21} x_{1}(t)-b_{22} x_{2}(t)-b_{23} x_{3}(t)\right) \\
& -\sigma_{2} x_{2}(t) \dot{B}_{2}(t),  \tag{74}\\
\dot{x}_{3}(t)= & x_{3}(t)\left(-a_{3}+b_{32} x_{2}(t)-b_{33} x_{3}(t)\right) \\
& -\sigma_{3} x_{3}(t) \dot{B}_{3}(t) .
\end{align*}
$$

By the Milstein method in [23], we have the difference equation:

$$
\begin{align*}
& x_{1, k+1}= x_{1, k}+x_{1, k} \\
& \times {\left[\left(a_{1}-b_{11} x_{1, k}-b_{12} x_{2, k}\right) \Delta t\right.} \\
&\left.+\sigma_{1} \epsilon_{1, k} \sqrt{\Delta} t+\frac{\sigma_{1}^{2}}{2}\left(\epsilon_{1, k}^{2} \Delta t-\Delta t\right)\right], \\
& x_{2, k+1}=x_{2, k}+x_{2, k} \\
& \times {\left[\left(-a_{2}+b_{21} x_{1, k}-b_{22} x_{2, k}-b_{23} x_{3, k}\right) \Delta t\right.}  \tag{75}\\
&\left.-\sigma_{2} \epsilon_{2, k} \sqrt{\Delta} t+\frac{\sigma_{2}^{2}}{2}\left(\epsilon_{2, k}^{2} \Delta t-\Delta t\right)\right], \\
& x_{3, k+1}=x_{3, k}+x_{3, k} \\
& \times {\left[\left(-a_{3}+b_{32} x_{2, k}-b_{33} x_{3, k}\right) \Delta t\right.} \\
&\left.-\sigma_{3} \epsilon_{3, k} \sqrt{\Delta} t+\frac{\sigma_{3}^{2}}{2}\left(\epsilon_{3, k}^{2} \Delta t-\Delta t\right)\right],
\end{align*}
$$

where $\epsilon_{1, k}, \epsilon_{2, k}$, and $\epsilon_{3, k}, i=1,2,3$, are the Gaussian random variables $N(0,1), r_{1}=a_{1}-\sigma_{1}^{2} / 2>0$, and $r_{i}=a_{i}+\sigma_{i}^{2} / 2, i=2,3$. Choosing $\left(x_{1}(0), x_{2}(0), x_{3}(0)\right) \in R_{+}^{3}$, and suitable parameters, by Matlab, we get Figures 1, 2, and 3.


Figure 1: The solution of system (1) and system (3) with $\left(x_{1}(0), x_{2}(0), x_{3}(0)\right)=(1,0.8,0.5), a_{1}=0.3, a_{2}=0.4, a_{3}=0.1$, $b_{11}=0.1, b_{12}=0.1, b_{21}=0.6, b_{22}=0.6, b_{23}=0.6, b_{32}=0.8$, and $b_{33}=0.6$. The red lines represent the solution of system (1), while the blue lines represent the solution of system (3) with $\sigma_{1}=0.02$, $\sigma_{2}=0.01$, and $\sigma_{3}=0.01$.


Figure 2: Two of the species will die out in probability. The red lines represent the solution of system (1), while the blue lines represent the solution of system (3) with $\left(x_{1}(0), x_{2}(0), x_{3}(0)\right)=(1,0.8,0.5)$, $a_{1}=0.4, a_{2}=0.4, a_{3}=0.1, b_{11}=0.1, b_{12}=0.1, b_{21}=0.6, b_{22}=0.6$, $b_{23}=0.6, b_{32}=0.8$, and $b_{33}=0.6$. The red lines represent the solution of system (1), while the blue lines represent the solution of system (3) with $\sigma_{1}=0.02, \sigma_{2}=3$, and $\sigma_{3}=0.01$.

In Figure 1, when the noise is small, choosing parameters satisfying the condition of Theorem 10 , the solution of system (3) will persist in time average.

In Figure 2, we observe case (3) in Theorem 12 and choose parameters $r_{1}>0, r_{1}-\left(b_{11} / b_{21}\right) r_{2}<0$. As Theorem 12 indicated that two predators will die out in probability. The prey solution of system (3) will persist in time average.


Figure 3: One of the species or both species will die out in probability. The red lines represent the solution of system (1), while the blue lines represent the solution of system (3) with $\left(x_{1}(0), x_{2}(0), x_{3}(0)\right)=(1,0.8,0.5), a_{1}=-0.1, a_{2}=0.4, a_{3}=0.1$, $b_{11}=0.1, b_{12}=0.1, b_{21}=0.6, b_{22}=0.6, b_{23}=0.6, b_{32}=0.8$, and $b_{33}=0.6$. The red lines represent the solution of system (1), while the blue lines represent the solution of system (3) with $\sigma_{1}=0.02$, $\sigma_{2}=3$, and $\sigma_{3}=0.01$.

In Figure 3, we observe case (1) in Theorem 12 and choose parameters $r_{1}<0$. As Theorem 12 indicated that not only predators but also prey will die out in probability when the noise of the prey is large, and it does not happen in the deterministic system.

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## Research Article

# A SIRS Epidemic Model Incorporating Media Coverage with Random Perturbation 

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#### Abstract

We investigate the complex dynamics of a SIRS epidemic model incorporating media coverage with random perturbation. We first deal with the boundedness and the stability of the disease-free and endemic equilibria of the deterministic model. And for the corresponding stochastic epidemic model, we prove that the endemic equilibrium of the stochastic model is asymptotically stable in the large. Furthermore, we perform some numerical examples to validate the analytical finding, and find that if the conditions of stochastic stability are not satisfied, the solution for the stochastic model will oscillate strongly around the endemic equilibrium.


## 1. Introduction

Epidemiology is the study of the spread of diseases with the objective of tracing factors that are responsible for or contribute to their occurrence. Mathematical modeling has become an important tool in analyzing the epidemiological characteristics of infectious diseases and can provide useful control measures. Various models have been used to study different aspects of diseases spreading [1-11].

Let $S(t)$ be the number of susceptible individuals, $I(t)$ the number of infective individuals, and $R(t)$ the number of removed individuals at time $t$, respectively. A general SIRS epidemic model can be formulated as

$$
\begin{gather*}
\frac{d S}{d t}=b-d S-g(I) S+\gamma R, \\
\frac{d I}{d t}=g(I) S-(d+\mu+\delta) I,  \tag{1}\\
\frac{d R}{d t}=\mu I-(d+\gamma) R,
\end{gather*}
$$

where $b$ is the recruitment rate of the population, $d$ is the natural death rate of the population, $\mu$ is the natural recovery rate of the infective individuals, $\gamma$ is the rate at which recovered individuals lose immunity and return to the susceptible class, and $\delta$ is the disease-induced death
rate. The transmission of the infection is governed by the incidence rate $g(I) S$, and $g(I)$ is called the infection force.

In modelling of communicable diseases, the incidence rate $g(I) S$ has been considered to play a key role in ensuring that the models indeed give reasonable qualitative description of the transmission dynamics of the diseases. Some factors, such as media coverage, density of population, and life style, may affect the incidence rate directly or indirectly [12-18]. It is worthy to note that, during the spreading of severe acute respiratory syndrome (SARS) from 2002 to 2004 and the outbreak of influenza A (H1N1) in 2009, media coverage plays an important role in helping both the government authority make interventions to contain the disease and people response to the disease [12, 18]. And a number of mathematical models have been formulated to describe the impact of media coverage on the transmission dynamics of infectious diseases. Especially, Liu and Cui [15], Tchuenche et al. [17], and Sun et al. [16] incorporated a nonlinear function of the number of infective (2) in their transmission term to investigate the effects of media coverage on the transmission dynamics:

$$
\begin{equation*}
g(I)=\beta_{1}-\frac{\beta_{2} I}{m+I} \tag{2}
\end{equation*}
$$

where $\beta_{1}$ is the contact rate before media alert; the terms $\beta_{2} I /(m+I)$ measure the effect of reduction of the contact rate when infectious individuals are reported in the media.

Because the coverage report cannot prevent disease from spreading completely we have $\beta_{1} \geq \beta_{2}$. The half-saturation constant $m>0$ reflects the impact of media coverage on the contact transmission. The function $I /(m+I)$ is a continuous bounded function which takes into account disease saturation or psychological effects. Then model (1) becomes

$$
\begin{gather*}
\frac{d S}{d t}=b-d S-\left(\beta_{1}-\frac{\beta_{2} I}{m+I}\right) S I+\gamma R \\
\frac{d I}{d t}=\left(\beta_{1}-\frac{\beta_{2} I}{m+I}\right) S I-(d+\mu+\delta) I  \tag{3}\\
\frac{d R}{d t}=\mu I-(d+\gamma) R
\end{gather*}
$$

where all the parameters are nonnegative and have the same definitions as before.

For model (3), the basic reproduction number

$$
\begin{equation*}
R_{0}=\frac{b \beta_{1}}{d(d+\mu+\delta)} \tag{4}
\end{equation*}
$$

is the threshold of the system for an epidemic to occur. Model (3) has a the disease-free equilibrium $P_{0}=(b / d, 0,0)$ which exists for all parameter values. And the endemic equilibrium $E^{*}=\left(S^{*}, I^{*}, R^{*}\right)$ of model (3) satisfies

$$
\begin{gather*}
b-d S-\left(\beta_{1}-\frac{\beta_{2} I}{m+I}\right) S I+\gamma R=0 \\
\left(\beta_{1}-\frac{\beta_{2} I}{m+I}\right) S I-(d+\mu+\delta) I=0  \tag{5}\\
\mu I-(d+\gamma) R=0
\end{gather*}
$$

which yields

$$
\begin{gather*}
S^{*}=\frac{(d+\mu+\gamma)\left(m+I^{*}\right)}{\beta_{1}\left(m+I^{*}\right)-\beta_{2} I^{*}} \\
R^{*}=\frac{\mu I^{*}}{d+\gamma}  \tag{6}\\
H_{1} I^{* 2}+H_{2} I^{*}+H_{3}=0
\end{gather*}
$$

where

$$
\begin{gather*}
H_{1}=-\frac{1}{d+\gamma}\left(\beta_{1}-\beta_{2}\right)(\gamma(d+\delta)+d(d+\mu+\delta)) \\
H_{2}=-\frac{d \beta_{1} m \mu}{d+\gamma}-\beta_{1} m(d+\delta)-b \beta_{2}+b \beta_{1}\left(1-\frac{1}{R_{0}}\right),  \tag{7}\\
H_{3}=d m(d+\mu+\delta)\left(R_{0}-1\right)
\end{gather*}
$$

When $R_{0}>1$, we know that $H_{1}<0, H_{3}>0$; hence, model (3) has a unique endemic equilibrium $E^{*}=\left(S^{*}, I^{*}, R^{*}\right)$. These results of model (3) were studied in [15].

On the other hand, if the environment is randomly varying, the population is subject to a continuous spectrum
of disturbances [19, 20]. That is to say, population systems are often subject to environmental noise; that is, due to environmental fluctuations, parameters involved in epidemic models are not absolute constants, and they may fluctuate around some average values. Therefore, many stochastic models for the populations have been developed and studied [21-40]. But, to our knowledge, the research on the dynamics of SIRS epidemic model incorporating media coverage with random perturbation seems rare.

In this paper, our basic approach is analogous to that of Beretta et al. [24]. They assumed that stochastic perturbations were of white noise type, which were directly proportional to distances $S(t), I(t)$, and $R(t)$ from values of $S^{*}, I^{*}$, and $R^{*}$, influenced the $S(t), I(t)$, and $R(t)$, respectively. By this method, we formulate our stochastic differential equation corresponding to model (3) as follows:

$$
\begin{gather*}
d S=b-d S-\left(\beta_{1}-\frac{\beta_{2} I}{m+I}\right) S I+\gamma R+\sigma_{1}\left(S-S^{*}\right) d B_{1}(t) \\
d I=\left(\beta_{1}-\frac{\beta_{2} I}{m+I}\right) S I-(d+\mu+\delta) I+\sigma_{2}\left(I-I^{*}\right) d B_{2}(t) \\
d R=\mu I-(d+\gamma) R+\sigma_{3}\left(R-R^{*}\right) d B_{3}(t) \tag{8}
\end{gather*}
$$

where $\sigma_{1}, \sigma_{2}$, and $\sigma_{3}$ are real constants and known as the intensity of environmental fluctuations; $B_{1}(t), B_{2}(t)$, and $B_{3}(t)$ are independent standard Brownian motions.

The aim of this paper is to consider the stochastic dynamics of model (8). The paper is organized as follows. In Section 2, we carry out the analysis of the dynamical properties of stochastic model (8). And in Section 3, we give some numerical examples and make a comparative analysis of the stability of the model with deterministic and stochastic environments and have some discussions.

## 2. Mathematical Properties of the Deterministic Model (3)

The following result shows that the solutions for model (3) are bounded and, hence, lie in a compact set and are continuable for all positive time.

Lemma 1. The plane $S+I+R \leq b / d$ is an invariant manifold of model (3), which is attracting in the first octant.

Proof. Summing up the three equations in (3) and denoting $N(t)=S(t)+I(t)+R(t)$, we have

$$
\begin{equation*}
\frac{d N}{d t}=b-d N-\delta I \leq b-d N \tag{9}
\end{equation*}
$$

Hence, by integration, we check

$$
\begin{equation*}
N(t) \leq \frac{b}{d}+\left(N(0)-\frac{b}{d}\right) e^{-d t} \tag{10}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \sup N(t) \leq \frac{b}{d} \tag{11}
\end{equation*}
$$

which implies the conclusion.

Therefore, from biological consideration, we study model (3) in the closed set

$$
\begin{equation*}
\Gamma=\left\{(S, I, R) \in \mathbb{R}_{+}^{3}: 0<S+I+R \leq \frac{b}{d}\right\} \tag{12}
\end{equation*}
$$

Proposition 2 is proved in [15] and is here just recalled.
Proposition 2. (i) The disease-free equilibrium $E_{0}=(b / d, 0$, 0 ) is globally asymptotically stable if $R_{0}<1$ and unstable if $R_{0}>1$ in the set $\Gamma$.
(ii) The endemic equilibrium $E^{*}=\left(S^{*}, I^{*}, R^{*}\right)$ of model (3) is locally asymptotically stable if $R_{0}>1$ in the set $\Gamma$.

Next, we present the following theorem which gives condition for the global asymptotical stability of the endemic equilibrium $E^{*}$ of model (3).

Theorem 3. If $R_{0}>1$, the endemic equilibrium $E^{*}=\left(S^{*}, I^{*}\right.$, $R^{*}$ ) of model (3) is globally asymptotically stable in the set $\Gamma$.

Proof. By summing all the equations of model (3), we find that the total population size verify the following equation:

$$
\begin{equation*}
\frac{d N}{d t}=b-d N-\delta I \tag{13}
\end{equation*}
$$

where $N=S+I+R$.
It is convenient to choose the variable ( $N, I, R$ ) instead of $(S, I, R)$. That is, consider the following model:

$$
\begin{gather*}
\frac{d N}{d t}=b-d N-\delta I \\
\frac{d I}{d t}=\left(\beta_{1}-\frac{\beta_{2} I}{m+I}\right)(N-I-R) I-(d+\mu+\delta) I  \tag{14}\\
\frac{d R}{d t}=\mu I-(d+\gamma) R
\end{gather*}
$$

changing the variables such that $x=N-N^{*}, y=I-I^{*}$, and $z=R-R^{*}$, where $N^{*}=S^{*}+I^{*}+R^{*}$, so model (14) becomes as follows:

$$
\begin{align*}
\frac{d x}{d t}= & -d x-\delta y \\
\frac{d y}{d t}= & \left(\beta_{1}-\frac{\beta_{2} I^{*}}{m+I^{*}}\right) I^{*} \\
& \times\left(x-\left(1+\frac{m \beta_{2}(d+\mu+\delta)}{I^{*}\left(\beta_{1}-\beta_{2}\right)+\beta_{1} m}\right) y-z\right)  \tag{15}\\
\frac{d z}{d t}= & \mu y-(d+\gamma) z
\end{align*}
$$

Consider the function

$$
\begin{equation*}
V(x, y, z)=\frac{1}{2}\left(k_{1} x^{2}+y^{2}+k_{2} z^{2}\right) \tag{16}
\end{equation*}
$$

where $k_{1}$ and $k_{2}$ are positive constants which will be chosen later. Then the derivative of $V$ along the solution for model (15) is given by

$$
\begin{align*}
\frac{d V}{d t}= & k_{1} x x_{t}+y y_{t}+k_{2} z z_{t} \\
= & k_{1} x(-d x-\delta y)+\left(\beta_{1}-\frac{\beta_{2} I^{*}}{m+I^{*}}\right) I^{*} y \\
& \times\left(x-\left(1+\frac{m \beta_{2}(d+\mu+\delta)}{I^{*}\left(\beta_{1}-\beta_{2}\right)+\beta_{1} m}\right) y-z\right) \\
& +k_{2} z(\mu y-(d+\gamma) z) \\
= & -d k_{1} x^{2}-\left(\beta_{1}-\frac{\beta_{2} I^{*}}{m+I^{*}}\right)  \tag{17}\\
& \times\left(1+\frac{m \beta_{2}(d+\mu+\delta)}{I^{*}\left(\beta_{1}-\beta_{2}\right)+\beta_{1} m}\right) \\
& \times I^{*} y^{2}-k_{2}(d+\gamma) z^{2} \\
& +\left(\frac{\left(I^{*}\left(\beta_{1}-\beta_{2}\right)+\beta_{1} m\right) I^{*}}{m+I^{*}}-k_{1} \delta\right) x y \\
& +\left(k_{2} \mu-\frac{\left(I^{*}\left(\beta_{1}-\beta_{2}\right)+\beta_{1} m\right) I^{*}}{m+I^{*}}\right) y z
\end{align*}
$$

Let us choose $k_{1}$ and $k_{2}$ such that

$$
\begin{align*}
& \frac{\left(I^{*}\left(\beta_{1}-\beta_{2}\right)+\beta_{1} m\right) I^{*}}{m+I^{*}}-k_{1} \delta=0 \\
& k_{2} \mu-\frac{\left(I^{*}\left(\beta_{1}-\beta_{2}\right)+\beta_{1} m\right) I^{*}}{m+I^{*}}=0 \tag{18}
\end{align*}
$$

then $k_{1}=\left(I^{*}\left(\beta_{1}-\beta_{2}\right)+\beta_{1} m\right) I^{*} / \delta\left(m+I^{*}\right)$ and $k_{2}=\left(I^{*}\left(\beta_{1}-\right.\right.$ $\left.\left.\beta_{2}\right)+\beta_{1} m\right) I^{*} / \mu\left(m+I^{*}\right)$. Thus, we have

$$
\begin{align*}
\frac{d V}{d t}= & -d k_{1} x^{2}-\left(\beta_{1}-\frac{\beta_{2} I^{*}}{m+I^{*}}\right) \\
& \times\left(1+\frac{m \beta_{2}(d+\mu+\delta)}{I^{*}\left(\beta_{1}-\beta_{2}\right)+\beta_{1} m}\right) I^{*} y^{2}-k_{2}(d+\gamma) z^{2} \leq 0 \tag{19}
\end{align*}
$$

By applying the Lyapunov-LaSalle asymptotic stability theorem [41, 42], the endemic equilibrium $E^{*}$ of model (3) is globally asymptotically stable. This completes the proof.

Example 4. We now use the parameter values

$$
\begin{gather*}
b=5, \quad d=0.02, \quad \beta_{1}=0.002, \quad \beta_{2}=0.0018 \\
m=30, \quad \delta=0.1, \quad \mu=0.05, \quad \gamma=0.01 \tag{20}
\end{gather*}
$$



Figure 1: The global stability of the endemic equilibrium $E^{*}=$ $\left(S^{*}, I^{*}, R^{*}\right)$ for model (21) with initial values $S(0)=85, I(0)=15$, and $R(0)=0$. The parameters are taken as (20).
and show the stability of the endemic equilibrium $E^{*}$ of model (3). Model (3) becomes

$$
\begin{gather*}
\frac{d S}{d t}=5-0.02 S-\left(0.002-\frac{0.0018 I}{30+I}\right) S I+0.01 R \\
\frac{d I}{d t}=\left(0.002-\frac{0.0018 I}{30+I}\right) S I-(0.02+0.05+0.1) I  \tag{21}\\
\frac{d R}{d t}=0.05 I-(0.02+0.01) R
\end{gather*}
$$

Note that

$$
\begin{equation*}
R_{0}=\frac{b \beta_{1}}{d(d+\mu+\delta)}=2.941>1 . \tag{22}
\end{equation*}
$$

From Theorem 3, one can therefore conclude that, for any initial values $(S(0), I(0), R(0))$, the endemic equilibrium $E^{*}=$ (124.564, 16.361, 27.269) of model (21) is globally stable (see Figure 1).

## 3. Stochastic Stability of the Endemic Equilibrium of Model (8)

Throughout this paper, let $(\Omega, \mathscr{F}, \mathscr{P})$ be a complete probability space with a filtration $\left\{\mathscr{F}_{t}\right\}_{t \in \mathbb{R}_{+}}$satisfying the usual conditions (i.e., it is right continuous and increasing while $\mathscr{F}_{0}$ contains all $\mathscr{P}$-null sets).

Considering the general $n$-dimensional stochastic differential equation

$$
\begin{equation*}
d x(t)=f(x(t), t) d t+\varphi(x(t), t) d B(t) \tag{23}
\end{equation*}
$$

on $t \geq 0$ with initial value $x(0)=x_{0}$, the solution is denoted by $x\left(t, x_{0}\right)$. Assume that $f(0, t)=0$ and $\varphi(0, t)=0$ for all $t \geq 0$, so (23) has the solution $x(t)=0$. This solution is called the trivial solution.

Definition 5 ( see [43]). The trivial solution $x(t)=0$ of (23) is said to be as follows:
(i) stable in probability if for all $\varepsilon>0$,

$$
\begin{equation*}
\lim _{x_{0} \rightarrow 0} \mathscr{P}\left(\sup _{t \geq 0}\left|x\left(t, x_{0}\right)\right| \geq \varepsilon\right)=0 \tag{24}
\end{equation*}
$$

(ii) asymptotically stable if it is stable in probability and, moreover,

$$
\begin{equation*}
\lim _{x_{0} \rightarrow 0} \mathscr{P}\left(\lim _{t \rightarrow \infty} x\left(t, x_{0}\right)=0\right)=1 ; \tag{25}
\end{equation*}
$$

(iii) asymptotically stable in the large if it is stable in probability and, moreover, for all $x_{0} \in \mathbb{R}^{n}$

$$
\begin{equation*}
\mathscr{P}\left(\lim _{t \rightarrow \infty} x\left(t, x_{0}\right)=0\right)=1 . \tag{26}
\end{equation*}
$$

Define the differential operator $L$ associated to (23) by

$$
\begin{equation*}
L=\frac{\partial}{\partial t}+\sum_{i=1}^{n} f_{i}(x, t) \frac{\partial}{\partial x_{i}}+\frac{1}{2} \sum_{i, j=1}^{n}\left[\varphi^{T}(x, t) \varphi(x, t)\right]_{i j} \frac{\partial^{2}}{\partial x_{i} \partial x_{j}} . \tag{27}
\end{equation*}
$$

If $L$ acts on a function $V(x, t) \in C^{2,1}\left(\mathbb{R}^{d} \times(0, \infty) ; \mathbb{R}_{+}\right)$, then

$$
\begin{align*}
L V(x, t)= & V_{t}(x, t)+V_{x}(x, t) f(x, t) \\
& +\frac{1}{2} \operatorname{trace}\left[\varphi^{T}(x, t) V_{x x}(x, t) \varphi(x, t)\right] \tag{28}
\end{align*}
$$

where $T$ means transposition. For more definitions of stability we refer to [43].

In the following, we will give the result of the asymptotical stability in the large of the endemic equilibrium $E^{*}$ of model (8).

If $R_{0}>1$, stochastic model (8) can center at its endemic equilibrium $E^{*}$. By the change of variables

$$
\begin{equation*}
u=S-S^{*}, \quad v=I-I^{*}, \quad w=R-R^{*} \tag{29}
\end{equation*}
$$

we obtain the following system:

$$
\begin{equation*}
d z(t)=f_{1}(z(t)) d t+f_{2}(z(t)) d B(t), \tag{30}
\end{equation*}
$$

where

$$
\begin{gather*}
z(t)=(u(t), v(t), w(t))^{T}, \\
f_{1}(z(t))=\left(\begin{array}{c}
-\left(d+\left(\beta_{1}-\frac{\beta_{2} I^{*}}{I^{*}+m}\right) I^{*}\right) u-\left(d+\mu+\delta-\frac{\beta_{2} m S^{*} I^{*}}{\left(I^{*}+m\right)^{2}}\right) v+\gamma w \\
\left(\beta_{1}-\frac{\beta_{2} I^{*}}{I^{*}+m}\right) I^{*} u-\frac{\beta_{2} m S^{*} I^{*}}{\left(I^{*}+m\right)^{2}} v \\
\mu v-(d+\gamma) w
\end{array}\right) \\
f_{2}(z(t))=\left(\begin{array}{ccc}
\sigma_{1} u(t) & 0 & 0 \\
0 & \sigma_{2} v(t) & 0 \\
0 & 0 & \sigma_{3} w(t)
\end{array}\right) . \tag{31}
\end{gather*}
$$

It is easy to see that the stability of the endemic equilibrium $E^{*}$ of model (8) is equivalent to the stability of the trivial solution for model (30).

Before proving the stochastic stability of the trivial solution for model (30), we put forward a Lemma in [44].

Lemma 6 (see [44]). Suppose that there exists a function $V(z, t) \in C^{2}(\Omega)$ satisfying the following inequalities:

$$
\begin{gather*}
K_{1}|z|^{\omega} \leq V(z, t) \leq K_{2}|z|^{\omega} \\
L V(z, t) \leq-K_{3}|z|^{\omega} \tag{32}
\end{gather*}
$$

where $\omega>0$ and $K_{i}(i=1,2,3)$ is positive constant. Then the trivial solution for model (30) is exponentially $\omega$-stable for all time $t \geq 0$. When $\omega=2$, it is usually said to be exponentially stable in mean square and the trivial solution $x=0$ is asymptotically stable in the large.

From the above Lemma, we obtain the following theorem.
Theorem 7. Assume that $R_{0}=b \beta_{1} / d(d+\mu+\delta)>1$. If the following conditions are satisfied:

$$
\begin{gather*}
\sigma_{1}^{2}<2 d, \quad \sigma_{3}^{2}<2(d+\gamma) \\
\frac{2\left(\gamma^{2}+\mu^{2}\right)}{2(d+\gamma)-\sigma_{3}^{2}}<d+\mu+\delta+\frac{\beta_{2} m \theta S^{*} I^{*}}{\left(I^{*}+m\right)^{2}} \\
\sigma_{2}^{2}<\frac{2}{1+\theta}\left(d+\mu+\delta+\frac{\beta_{2} m \theta S^{*} I^{*}}{\left(I^{*}+m\right)^{2}}-\frac{2\left(\gamma^{2}+\mu^{2}\right)}{2(d+\gamma)-\sigma_{3}^{2}}\right), \tag{33}
\end{gather*}
$$

where

$$
\begin{equation*}
\theta=\frac{(2 d+\mu+\delta)\left(I^{*}+m\right)}{\left(\left(\beta_{1}-\beta_{2}\right) I^{*}+\beta_{1} m\right) I^{*}} \tag{34}
\end{equation*}
$$

then the trivial solution of model (30) is asymptotically stable in the large. And the endemic point $E^{*}$ of model (8) is asymptotically stable in the large.

Proof. We define the Lyapunov function $V(u, v, w)$ as follows:

$$
\begin{align*}
V(z(t)) & =\frac{1}{2} c_{1}(u+v)^{2}+\frac{1}{2} c_{2} v^{2}+\frac{1}{2} c_{3} w^{2}  \tag{35}\\
& :=V_{1}(z(t))+V_{2}(z(t))+V_{3}(z(t)),
\end{align*}
$$

where $c_{1}>0, c_{2}>0$ and $c_{3}>0$ are real positive constants to be chosen later. It is easy to check that inequalities (32) are true.

Furthermore, by the Itô formula, we have

$$
\begin{align*}
L V_{1}= & c_{1}(u+v)(-d u-(d+\mu+\delta) v+\gamma w) \\
& +\frac{1}{2} c_{1} \sigma_{1}^{2} u^{2}+\frac{1}{2} c_{1} \sigma_{2}^{2} v^{2} \\
= & -c_{1}\left(d-\frac{1}{2} \sigma_{1}^{2}\right) u^{2}-c_{1}(2 d+\mu+\delta) u v \\
& -c_{1}\left(d+\mu+\delta-\frac{1}{2} \sigma_{2}^{2}\right) v^{2}+c_{1} \gamma u w+c_{1} \gamma v w, \\
L V_{2}= & c_{2} v\left(\left(\beta_{1}-\frac{\beta_{2} I^{*}}{I^{*}+m}\right) I^{*} u-\frac{\beta_{2} m S^{*} I^{*}}{\left(I^{*}+m\right)^{2}} v\right) \\
& +\frac{1}{2} c_{2} \sigma_{2}^{2} v^{2} \\
= & c_{2}\left(\beta_{1}-\frac{\beta_{2} I^{*}}{I^{*}+m}\right) I^{*} u v-c_{2}\left(\frac{\beta_{2} m S^{*} I^{*}}{\left(I^{*}+m\right)^{2}}-\frac{1}{2} \sigma_{2}^{2}\right) v^{2}, \\
L V_{3}= & c_{3} w(\mu v-(d+\gamma) w)+\frac{1}{2} c_{3} \sigma_{3}^{2} w^{2} \\
= & c_{3} \mu v w-c_{3}\left(d+\gamma-\frac{1}{2} \sigma_{3}^{2}\right) w^{2} . \tag{36}
\end{align*}
$$

Then we have

$$
\begin{aligned}
L V & =L V_{1}+L V_{2}+L V_{3} \\
& =-c_{1}\left(d-\frac{1}{2} \sigma_{1}^{2}\right) u^{2}
\end{aligned}
$$

$$
\begin{align*}
& -\left(c_{1}(d+\mu+\delta)+\frac{c_{2} \beta_{2} m S^{*} I^{*}}{\left(I^{*}+m\right)^{2}}\right. \\
& \left.\quad-\frac{1}{2} \sigma_{2}^{2}\left(c_{1}+c_{2}\right)\right) v^{2} \\
& -c_{2}\left(d+\gamma-\frac{1}{2} \sigma_{3}^{2}\right) w^{2} \\
& -\left(c_{1}(2 d+\mu+\delta)-c_{2}\left(\beta_{1}-\frac{\beta_{2} I^{*}}{I^{*}+m}\right) I^{*}\right) u v \\
& +c_{1} \gamma u w+c_{3} \mu v w+c_{1} \gamma v w \tag{37}
\end{align*}
$$

Choose $c_{3}=c_{1}$ and

$$
\begin{equation*}
c_{1}(2 d+\mu+\delta)-c_{2}\left(\beta_{1}-\frac{\beta_{2} I^{*}}{I^{*}+m}\right) I^{*}=0 \tag{38}
\end{equation*}
$$

then

$$
\begin{equation*}
c_{2}=\frac{c_{1}(2 d+\mu+\delta)\left(I^{*}+m\right)}{\left(\left(\beta_{1}-\beta_{2}\right) I^{*}+\beta_{1} m\right) I^{*}}=c_{1} \theta \tag{39}
\end{equation*}
$$

Moreover, using Cauchy inequality to $\gamma u w, \gamma v w$, and $\mu v w$, we can obtain

$$
\begin{align*}
\gamma u w & \leq \frac{\gamma^{2} u^{2}}{d+\gamma-(1 / 2) \sigma_{3}^{2}}+\frac{1}{4}\left(d+\gamma-\frac{1}{2} \sigma_{3}^{2}\right) w^{2} \\
\mu \nu w & \leq \frac{\mu^{2} v^{2}}{d+\gamma-(1 / 2) \sigma_{3}^{2}}+\frac{1}{4}\left(d+\gamma-\frac{1}{2} \sigma_{3}^{2}\right) w^{2}  \tag{40}\\
\gamma \nu w & \leq \frac{\gamma^{2} v^{2}}{d+\gamma-(1 / 2) \sigma_{3}^{2}}+\frac{1}{4}\left(d+\gamma-\frac{1}{2} \sigma_{3}^{2}\right) w^{2}
\end{align*}
$$

Substituting (39) and (40) into (37), yields

$$
\begin{aligned}
L V= & -\left(d-\frac{1}{2} \sigma_{1}^{2}-\frac{c_{1} \gamma^{2}}{d+\gamma-(1 / 2) \sigma_{3}^{2}}\right) u^{2} \\
& -\frac{1}{4}\left(3 c_{3}-2 c_{1}\right)\left(d+\gamma-\frac{1}{2} \sigma_{3}^{2}\right) w^{2} \\
& -\left(c_{1}(d+\mu+\delta)+\frac{c_{2} \beta_{2} m S^{*} I^{*}}{\left(I^{*}+m\right)^{2}}-\frac{1}{2} \sigma_{2}^{2}\left(c_{1}+c_{2}\right)\right. \\
= & -\left(A u^{2}+B v^{2}+C w^{2}\right) \\
& \left.-\frac{c_{1} \gamma^{2}+c_{3} \mu^{2}}{d+\gamma-(1 / 2) \sigma_{3}^{2}}\right) v^{2}
\end{aligned}
$$

where

$$
\begin{gather*}
A=d-\frac{1}{2} \sigma_{1}^{2}-\frac{c_{1} \gamma^{2}}{d+\gamma-(1 / 2) \sigma_{3}^{2}} \\
B=c_{1}\left(d+\mu+\delta+\frac{\beta_{2} m \theta S^{*} I^{*}}{\left(I^{*}+m\right)^{2}}-\frac{1}{2} \sigma_{2}^{2}(1+\theta)\right.  \tag{42}\\
\left.-\frac{\gamma^{2}+\mu^{2}}{d+\gamma-(1 / 2) \sigma_{3}^{2}}\right) \\
C=\frac{1}{4} c_{1}\left(d+\gamma-\frac{1}{2} \sigma_{3}^{2}\right)
\end{gather*}
$$

Let us choose $c_{1}$ such that

$$
\begin{equation*}
0<c_{1}<\frac{1}{\gamma^{2}}\left(d-\frac{1}{2} \sigma_{1}^{2}\right)\left(d+\gamma-\frac{1}{2} \sigma_{3}^{2}\right) \tag{43}
\end{equation*}
$$

On the other hand, the conditions in (33) are satisfied, so $A, B$, and $C$ are positive constants. Let $\lambda=\min \{A, B, C\}$; then $\lambda>0$. From (41), one sees that

$$
\begin{equation*}
L V(z(t)) \leq-\lambda|z(t)|^{2} . \tag{44}
\end{equation*}
$$

According to Lemma 6, we therefore conclude that the trivial solution of model (30) is asymptotically stable in the large. We therefore have the assertion.

Next, for further studying the effects of noise on the dynamics of model (8), we give some numerical examples to illustrate the dynamical behavior of stochastic model (8) by using the Milstein method mentioned in Higham [45]. In this way, model (8) can be rewritten as the following discretization equations:

$$
\begin{align*}
S_{k+1}= & S_{k}+\left(b-d S_{k}-\left(\beta_{1}-\frac{\beta_{2} I_{k}}{m+I_{k}}\right) S_{k} I_{k}+\gamma R_{k}\right) \\
& \times \Delta t+\sigma_{1}\left(S_{k}-S^{*}\right) \sqrt{\Delta t} \xi_{k} \\
I_{k+1}= & I_{k}+\left(\left(\beta_{1}-\frac{\beta_{2} I_{k}}{m+I_{k}}\right) S_{k} I_{k}-(d+\mu+\delta) I_{k}\right) \\
& \times \Delta t+\sigma_{2}\left(I_{k}-I^{*}\right) \sqrt{\Delta t} \eta_{k} \\
R_{k+1}= & R_{k}+\left(\mu I_{k}-(d+\gamma) R_{k}\right) \\
& \times \Delta t+\sigma_{3}\left(R_{k}-R^{*}\right) \sqrt{\Delta t} \zeta_{k} \tag{45}
\end{align*}
$$

where $\xi_{k}, \zeta_{k}$, and $\eta_{k}, k=1,2, \ldots, n$, are the Gaussian random variables $N(0,1)$.


Figure 2: The asymptotic behavior of the solutions to the stochastic model (47) around the endemic equilibrium $E^{*}$ with initial values $S(0)=85, I(0)=15$, and $R(0)=0$. The parameters are taken as (20).

The parameters of model (8) are fixed as (20). Then model (8) has the endemic point $E^{*}=(124.564,16.361,27.269)$. And model (8) becomes

$$
\begin{align*}
d S= & 5-0.02 S-\left(0.002-\frac{0.0018 I}{30+I}\right) S I \\
& +0.01 R+\sigma_{1}(S-124.564) d B_{1}(t) \\
d I= & \left(0.002-\frac{0.0018 I}{30+I}\right) S I-(0.02+0.05+0.1) I  \tag{46}\\
& +\sigma_{2}(I-16.361) d B_{2}(t) \\
d R= & 0.05 I-(0.02+0.01) R \\
& +\sigma_{3}(R-27.269) d B_{3}(t)
\end{align*}
$$

Choosing $\left(\sigma_{1}, \sigma_{2}, \sigma_{3}\right)=(0.025,0.1,0.05)$ and noting that

$$
\begin{gather*}
R_{0}=\frac{b \beta_{1}}{d(d+\mu+\delta)}=2.941>1, \\
\sigma_{1}^{2}=0.025^{2}<2 d=0.04, \\
\sigma_{3}^{2}=0.05^{2}<2(d+\gamma)=0.06, \\
\sigma_{2}^{2}=0.1^{2}<\frac{2}{1+\theta}\left(d+\mu+\delta+\frac{\beta_{2} m \theta S^{*} I^{*}}{\left(I^{*}+m\right)^{2}}-\frac{2\left(\gamma^{2}+\mu^{2}\right)}{2(d+\gamma)-\sigma_{3}^{2}}\right) \\
=0.12739-\frac{0.0011}{0.06-0.05^{2}}=0.108 . \tag{47}
\end{gather*}
$$

It is easy to see that all the conditions of Theorem 7 are satisfied, and we can therefore conclude that the endemic
point $E^{*}$ of model (47) is asymptotically stable in the large. The numerical examples shown in Figure 2(b) clearly support these results. To further illustrate the effect of the noise intensity on model (47), we keep all the parameters in (20) unchanged but increase $\left(\sigma_{1}, \sigma_{2}, \sigma_{3}\right)$ to $(0.18,0.18,0.22)$. In this case,

$$
\begin{gather*}
\sigma_{1}^{2}=0.0324<0.04, \quad \sigma_{3}^{2}=0.0484<0.06 \\
\sigma_{2}^{2}=0.0324<0.12739-\frac{0.0011}{0.06-0.22^{2}}=0.033 \tag{48}
\end{gather*}
$$

we can therefore conclude, by Theorem 7, that for any initial value ( $S(0), I(0), R(0)$ ), the endemic point $E^{*}$ of model (47) is asymptotically stable in the large (see Figure 2(b)).

In the above case, if we adopt $d=0.01$ and keep the other parameters unchanged, in this case, model (47) has the endemic point $E^{*}=(138.935,26.746,66.864)$. And it is easy to compute

$$
\begin{gather*}
R_{0}=\frac{b \beta_{1}}{d(d+\mu+\delta)}=6.25>1 \\
\sigma_{1}^{2}=0.0324>2 d=0.02  \tag{49}\\
\sigma_{3}^{2}=0.0484>2(d+\gamma)=0.04
\end{gather*}
$$

Therefore, the conditions of Theorem 7 are not satisfied, and the solution of model (47) will oscillate strongly around the endemic point $E^{*}=(138.935,26.746,66.864)$, which is not asymptotically stable in the large (see Figure 3).


Figure 3: The global stability of the endemic equilibrium $E^{*}=$ $\left(S^{*}, I^{*}, R^{*}\right)$ for model (47) with initial values $S(0)=85, I(0)=15$, and $R(0)=0$. The parameters are taken as $b=5, d=0.01$, $\beta_{1}=0.002, \beta_{2}=0.0018, m=30, \delta=0.1, \mu=0.05, \gamma=0.01$, $\sigma_{1}=0.18, \sigma_{2}=0.18$, and $\sigma_{3}=0.22$.

## 4. Conclusions and Discussions

In this paper, by using the theory of stochastic differential equation, we investigate the dynamics of a SIRS epidemic model incorporating media coverage with random perturbation. The value of this study lies in two aspects. First, it presents some relevant properties of the deterministic model (3), including boundedness and the stability of the diseasefree and endemic points. Second, it verifies the stochastic stability in the large of the endemic equilibrium for the stochastic model (8).

From the theoretical and numerical results, we can know that, when the noise density is not large, the stochastic model (8) preserves the property of the stability of the deterministic model (3). To a great extent, we can ignore the noise and use the deterministic model (3) to describe the population dynamics. However, when the noise is sufficiently large, it can force the population to become largely fluctuating. In this case, we can not use deterministic model (3) but instead stochastic model (8) to describe the population dynamics. Needless to say, both deterministic and stochastic epidemic models have their important roles.

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## Research Article

# Optimal Control Problem for Switched System with the Nonsmooth Cost Functional 

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#### Abstract

We examine the relationships between lower exhausters, quasidifferentiability (in the Demyanov and Rubinov sense), and optimal control for switching systems. Firstly, we get necessary optimality condition for the optimal control problem for switching system in terms of lower exhausters. Then, by using relationships between lower exhausters and quasidifferentiability, we obtain necessary optimality condition in the case that the minimization functional satisfies quasidifferentiability condition.


## 1. Introduction

A switched system is a particular kind of hybrid system that consists of several subsystems and a switching law specifying the active subsystem at each time instant. There are some articles which are dedicated to switching system [1-8]. Examples of switched systems can be found in chemical processes, automotive systems, and electrical circuit systems, and so forth.

Regarding the necessary optimality conditions for switching system in the smooth cost functional, it can be found in $[1,4,6]$. The more information connection between quasidifferential, exhausters and Hadamard differential are in [8-10]. Concerning the necessary optimality conditions for discrete switching system is in [5], and switching system with Frechet subdifferentiable cost functional is in [3]. This paper addresses the role exhausters and quasi-differentiability in the switching control problem. This paper is also extension of the results in the paper [5] (additional conditions are switching points unknown, and minimization functional is nonsmooth) in the case of first optimality condition. The rest of this paper is organized as follows. Section 2 contains some preliminaries, definitions, and theorems. Section 3 contains problem formulations and necessary optimality conditions for switching optimal control problem in the terms of exhausters. Then, the main theorem in Section 3 is extended to the case in which minimizing function is quasidifferentiable.

## 2. Some Preliminaries of Non-Smooth Analysis

Let us begin with basic constructions of the directional derivative (or its generalization) used in the sequel. Let $f$ : $X \rightarrow R, X \subset R^{n}$ be an open set. The function $f$ is called Hadamard upper (lower) derivative of the function $f$ at the point $x \in X$ in the direction $g \in X$ if there exist limit such that

$$
\begin{align*}
f_{H}^{\uparrow} & :=\lim _{[\alpha, g] \rightarrow[+0, g]} \frac{1}{\alpha}[f(x+\alpha g)-f(x)],  \tag{1}\\
\left(f_{H}^{\downarrow}\right. & \left.:=\liminf _{[\alpha, g] \rightarrow[+0, g]} \frac{1}{\alpha}[f(x+\alpha g)-f(x)]\right),
\end{align*}
$$

where $[\alpha, g] \rightarrow[+0, g]$ means that $\alpha \rightarrow+0$ and $g \rightarrow g$.
Note that limits in (1) always exist, but there are not necessary finite. This derivative is positively homogeneous functions of direction. The Gateaux upper (lower) subdifferential of the function $f$ at a point $x_{0} \in X$ can be defined as follows:

$$
\begin{gather*}
\partial_{G}^{+} f\left(x_{0}\right)=\left\{v \in R^{n} \left\lvert\, \lim _{t \leq 0} \sup \frac{f\left(x_{0}+t g\right)-f\left(x_{0}\right)}{t}\right.\right.  \tag{2}\\
\left.\leq(v, g), \quad \forall g \in R^{n}\right\} .
\end{gather*}
$$

The set

$$
\begin{equation*}
\hat{\partial}^{+} f\left(x_{0}\right)=\left\{v \in R^{n} \left\lvert\, \limsup _{x \rightarrow x_{0}} \frac{f(x)-f\left(x_{0}\right)-\left\langle v, x-x_{0}\right\rangle}{\left\|x-x_{0}\right\|} \leq 0\right.\right\} \tag{3}
\end{equation*}
$$

is called, respectively, the upper (lower) Frechet subdifferential of the function $f$ at the point $x_{0}$.

As observed in $[9,10]$, if $f$ is a quasidifferentiable function then its directional derivative at a point $x$ is represented as

$$
\begin{equation*}
f^{\prime}(x, g)=\max _{v \in \underline{\partial} f(x)}(v, g)+\min _{w \in \bar{\partial} f(x)}(w, g) \tag{4}
\end{equation*}
$$

where $\underline{\partial} f(x), \bar{\partial} f(x) \subset R^{n}$ are convex compact sets. From the last relation, we can easily reduce that

$$
\begin{equation*}
f^{\prime}(x, g)=\min _{w \in \bar{\partial} f(x)} \max _{v \in w+\underline{\partial} f(x)}(v, g)=\max _{v \in \underline{\partial} f(x)} \max _{w \in v+\bar{\partial} f(x)}(v, g) \tag{5}
\end{equation*}
$$

This means that for the function $h(g)=f^{\prime}(x, g)$ the upper and lower exhausters can be described in the following way:

$$
\begin{align*}
& E^{*}=\{C=w+\underline{\partial} f(x) \mid w \in \bar{\partial} f(x)\},  \tag{6}\\
& E_{*}=\{C=v+\bar{\partial} f(x) \mid v \in \underline{\partial} f(x)\}
\end{align*}
$$

It is clear that the Frechet upper subdifferential can be expressed with the Hadamard upper derivative in the following way; see [9, Lemma 3.2]:

$$
\begin{equation*}
\partial_{F}^{+} f\left(x_{0}\right)=\partial_{F}^{+} f_{H}^{\uparrow}\left(x_{0}, 0_{n}\right) \tag{7}
\end{equation*}
$$

Theorem 1. Let $E_{*}$ be lower exhausters of the positively homogeneous function $h: R^{n} \rightarrow R$. Then, $\bigcap_{C \subset E_{*}} C=\widehat{\partial}^{+} h\left(0_{n}\right)$, where $\widehat{\partial}^{+} h$ is the Frechet upper subdifferential of the $h$ at $0_{n}$, and for the positively homogeneous function $h: R^{n} \rightarrow R$ the Frechet superdifferential at the point zero follows

$$
\begin{equation*}
\hat{\partial}^{+} h\left(0_{n}\right)=\left\{v \in R^{n} \mid h(x)-(v, x) \leq 0, x \in R^{n}\right\} . \tag{8}
\end{equation*}
$$

Proof. Take any $v_{0} \in \bigcap_{C \subset E_{*}} C$. Then by using definition an lower exhausters we can write

$$
\begin{equation*}
v_{0}(x) \geq h(x), \quad \forall x \in R^{n} \Longrightarrow \bigcap_{C \subset E_{*}} C \subset \hat{\partial}^{+} h\left(0_{n}\right) \tag{9}
\end{equation*}
$$

Consider now any $v_{0} \in \widehat{\partial}^{+} h\left(0_{n}\right) \Rightarrow$

$$
\begin{equation*}
v_{0}(x) \geq h(x) . \tag{10}
\end{equation*}
$$

Let us consider $v_{0} \notin \bigcap_{C \subset E_{*}} C$. Then, there exists $C_{0} \in E^{*}$ where $v_{0} \notin C_{0}$. Then, by separation theorem, there exists $x_{0} \in$ $R^{n}$ such that

$$
\begin{equation*}
\left(x_{0}, v_{0}\right) \leq \max _{v \in C_{0}}\left(x_{0}, v\right) \leq h(x) . \tag{11}
\end{equation*}
$$

It is conducts (3) and $v_{0} \in C$ for every $C \in E^{*}$ and due to arbitrary. This means that $v_{0} \in \bigcap_{C \subset E_{*}} C$. The proof of the theorem is complete.

Lemma 2. The Frechet upper and Gateaux lower subdifferentials of a positively homogeneous function at zero coincide.

Proof. Let $h: R^{n} \rightarrow R$ be a positively homogenous function. It is not difficult to observe that every $g \in R^{n}$ and every $t>0$ :

$$
\begin{equation*}
\frac{h\left(0_{n}+t g\right)-h\left(0_{n}\right)}{t}=\frac{t h(g)}{t}=h(g) . \tag{12}
\end{equation*}
$$

Hence, the Gateaux lower subdifferential of $h$ at $0_{n}$ takes the forms

$$
\begin{equation*}
\partial_{\mathrm{G}}^{+} h\left(0_{n}\right)=\left\{v \in R^{n} \mid h(g) \leq(v, g), \forall g \in R^{n}\right\} \tag{13}
\end{equation*}
$$

which coincides with the representation of the Frechet upper subdifferential of the positively homogenous function (see [11, Proposition 1.9]).

## 3. Problem Formulation and Necessary Optimality Condition

Let investigating object be described by the differential equation

$$
\begin{array}{r}
\dot{x}_{K}(t)=f_{K}\left(x_{K}(t), u_{K}(t), t\right), \quad t \in\left[t_{k-1}, t_{k}\right],  \tag{14}\\
K=1,2, \ldots, N
\end{array}
$$

with initial condition

$$
\begin{equation*}
x_{1}\left(t_{0}\right)=x_{0} \tag{15}
\end{equation*}
$$

and the phase constraints at the end of the interval

$$
\begin{equation*}
F_{K}\left(x_{N}\left(t_{N}\right), t_{N}\right)=0, \quad K=1,2, \ldots, N \tag{16}
\end{equation*}
$$

and switching conditions on switching points (the conditions which determine that at the switching points the phase trajectories must be connected to each other by some relations):

$$
\begin{equation*}
x_{K+1}\left(t_{K}\right)=M_{K}\left(x_{K}\left(t_{K}\right), t_{k}\right), \quad K=1,2, \ldots, N-1 . \tag{17}
\end{equation*}
$$

The goal of this paper is to minimize the following functional:

$$
\begin{align*}
S\left(u_{1}, \ldots, u_{N}, t_{1}, \ldots, t_{N}\right)= & \sum_{K=1}^{N} \varphi_{K}\left(x_{K}\left(t_{K}\right)\right) \\
& +\sum_{K=1}^{N} \int_{t_{K-1}}^{t_{K}} L\left(x_{K}(t), u_{K}(t), t\right) d t \tag{18}
\end{align*}
$$

with the conditions (14)-(16). Namely, it is required to find the controls $u_{1}, u_{2}, \ldots, u_{N}$, switching points $t_{1}, t_{2}, \ldots, t_{N-1}$, and the end point $t_{N}$ (here $t_{1}, t_{2}, \ldots, t_{N}$ are not fixed) with the corresponding state $x_{1}, x_{2}, \ldots, x_{N}$ satisfying (14)-(16) so that the functional $J\left(u_{1}, \ldots, u_{N}, t_{1}, \ldots, t_{N}\right)$ in (18) is minimized. We will derive necessary conditions for the nonsmooth version of these problems (by using the Frechet superdifferential and exhausters, quasidifferentiable in the Demyanov and Rubinov sense).

Here $f_{K}: \mathbb{R} \times \mathbb{R}^{n} \times \mathbb{R}^{r} \rightarrow \mathbb{R}^{n}, M_{K}$ and $F_{K}$ are continuous, at least continuously partially differentiable vectorvalued functions with respect to their variables, $L: \mathbb{R}^{n} \times \mathbb{R}^{r} \times$ $\mathbb{R} \rightarrow \mathbb{R}$ are continuous and have continuous partial derivative with respect to their variables, $\varphi_{k}(\cdot)$ has Frechet upper subdifferentiable (superdifferentiable) at a point $\bar{x}_{K}\left(t_{K}\right)$ and positively homogeneous functional, and $u_{K}(t): \mathbb{R} \rightarrow$ $U_{K} \subset \mathbb{R}^{r}$ are controls. The sets $U_{K}$ are assumed to be nonempty and open. Here (16) is switching conditions. If we denote this as follows: $\theta=\left(t_{1}, t_{2}, \ldots, t_{N}\right), x(t)=\left(x_{1}(t)\right.$, $\left.x_{2}(t), \ldots, x_{N}(t)\right), u(t)=\left(u_{1}(t), u_{2}(t), \ldots, u_{N}(t)\right)$, then it is convenient to say that the aim of this paper is to find the triple $(x(t), u(t), \theta)$ which solves problem (14)-(18). This triple will be called optimal control for the problem (14)-(18). At first we assume that $\varphi_{k}(\cdot)$ is the Hadamar upper differentiable at the point $\bar{x}_{K}\left(t_{K}\right)$ in the direction of zero. Then, $\varphi_{k}(\cdot)$ is upper semicontinuous, and it has an exhaustive family of lower concave approximations of $\varphi_{k}(\cdot)$.

Theorem 3 (Necessary optimality condition in terms of lower exhauster). Let $\left(\bar{u}_{K}(\cdot), \bar{x}_{K}(\cdot), \bar{\theta}\right)$ be an optimal solution to the control problem (14)-(18). Then, for every element $x_{K}^{*}$ from intersection of the subsets $C_{K}$ of the lower exhauster $E_{*, K}$ of the functional $\varphi_{K}\left(\bar{x}_{K}\left(\bar{t}_{K}\right)\right)$, that is, $x_{K}^{*} \in \bigcap_{C_{K} \in E_{*, K}} C_{K}, K=1$, $2, \ldots, N$, there exist vector functions $\bar{p}_{K}(t), K=1, \ldots, N$ for which the following necessary optimality condition holds:
(i) State equation:

$$
\begin{equation*}
\dot{\bar{x}}_{K}(t)=\frac{\partial H_{K}\left(\bar{x}_{K}(t), \bar{u}_{K}(t), \bar{p}_{K}(t), t\right)}{\partial p_{K}}, \quad t \in\left[\bar{t}_{K-1}, \bar{t}_{K}\right] ; \tag{19}
\end{equation*}
$$

(ii) Costate equation:

$$
\begin{equation*}
\dot{\bar{p}}_{K}(t)=\frac{\partial H_{K}\left(\bar{x}_{K}(t), \bar{u}_{K}(t), \bar{p}_{K}(t), t\right)}{\partial x_{K}}, \quad t \in\left[\bar{t}_{K-1}, \bar{t}_{K}\right] \tag{20}
\end{equation*}
$$

(iii) At the switching points, $\bar{t}_{1}, \bar{t}_{2}, \ldots, \bar{t}_{N-1}$,

$$
\begin{array}{r}
x_{K}^{*}-\bar{p}_{K}\left(\bar{t}_{K}\right)-\bar{p}_{K+1}\left(\bar{t}_{K}\right) \frac{\partial M_{K}\left(\bar{x}_{K}\left(\bar{t}_{K}\right), \bar{t}_{K}\right)}{\partial x_{K}}=0  \tag{21}\\
K=1,2, \ldots, N-1
\end{array}
$$

(iv) Minimality condition:

$$
\begin{gather*}
H_{K}\left(\bar{x}_{K}(t), \bar{p}_{K}(t), \bar{u}_{K}(t)+\delta u_{K}(t), t\right) \\
\geq H_{K}\left(\bar{x}_{K}(t), \bar{p}_{K}(t), \bar{u}_{K}(t), t\right), \tag{22}
\end{gather*}
$$

$$
\text { for all admissible } \delta u_{K}, \quad t \in\left[\bar{t}_{K-1}, \bar{t}_{K}\right] ;
$$

(v) At the end point $\bar{t}_{N}$,

$$
\bar{p}_{N}\left(\bar{t}_{N}\right)=x_{N}^{*}+\sum_{K=1}^{N} \lambda_{K} \frac{\partial F_{K}\left(\bar{x}_{N}\left(\bar{t}_{N}\right), \bar{t}_{N}\right)}{\partial x_{N}}
$$

$$
\begin{align*}
& \left(\sum_{K=1}^{N} \lambda_{K} \frac{\partial F_{K}\left(\bar{x}_{N}\left(\bar{t}_{N}\right), \bar{t}_{N}\right)}{\partial t_{N}}\right) \delta_{L, N} \\
& \quad-\frac{1}{N}\left(\sum_{K=1}^{N-1} \bar{p}_{K+1}\left(\bar{t}_{K}\right) \frac{\partial M_{K}\left(\bar{x}_{K}\left(\bar{t}_{K}\right), \bar{t}_{K}\right)}{\partial t_{K}}\right)\left(1-\delta_{L, N}\right)=0, \tag{23}
\end{align*}
$$

here

$$
\delta_{L, N}=\left\{\begin{array}{ll}
1, & L=N,  \tag{24}\\
0, & L \neq N,
\end{array} \quad L=1,2, \ldots, N\right.
$$

is a Kronecker symbol, $H_{K}\left(x_{K}, u_{K}, p_{K}, t\right)=L_{K}\left(x_{K}\right.$, $\left.u_{K}, p_{K}, t\right)+p_{K}^{T} \cdot f_{K}\left(x_{K}, u_{K}, p_{K}, t\right)$, is a HamiltonPontryagin function, $E_{*, K}$ is lower exhauster of the functional $\varphi_{K}\left(x_{K}\left(t_{K}\right)\right), \lambda_{K}, K=1, \ldots, N$ are the vectors, and $p_{k}(\cdot)$ is defined by the conditions (ii) and (iii) in the process of the proof of the theorem, later.

Proof. Firstly, we will try to reduce optimal control problem (14)-(18) with nonsmooth cost functional to the optimal control problem with smooth minimization functional. In this way, we will use some useful theorems in [12, 13]. Let us note that smooth variational descriptions of Frechet normals theorem in [12, Theorem 1.30] and its subdifferential counterpart [12, Theorem 1.88] provide important variational descriptions of Frechet subgradients of nonsmooth functions in terms of smooth supports. To prove the theorem, take any elements from intersection of the subset of the exhauster, $x_{K}^{*} \in \bigcap C_{K}$, where $C_{K} \in E_{*, K}, K=1,2, \ldots, N$. Then by using Theorem 1 , we can write that $x_{K}^{*} \in \hat{\partial}^{+} \varphi_{K}\left(\bar{x}_{K}\left(t_{K}\right)\right)$. Then, apply the variational description in [12, Theorem 1.88] to the subgradients $-x_{K}^{*} \in \hat{\partial}^{+}\left(-\varphi_{K}\left(\bar{x}_{K}\left(\bar{t}_{K}\right)\right)\right)$. In this way, we find functions $s_{K}$ : $X \rightarrow \mathbb{R}$ for $K=1,2, \ldots, N$ satisfying the relations

$$
\begin{equation*}
s_{K}\left(\bar{x}_{K}\left(t_{K}\right)\right)=\varphi_{K}\left(\bar{x}_{K}\left(t_{K}\right)\right), \quad s_{K}\left(x_{K}(t)\right) \geq \varphi_{K}\left(x_{K}(t)\right) \tag{25}
\end{equation*}
$$

in some neighborhood of $\bar{x}_{K}\left(t_{K}\right)$, and such that each $s_{K}(\cdot)$ is continuously differentiable at $\bar{x}_{K}\left(\bar{t}_{K}\right)$ with $\nabla s_{K}\left(\bar{x}_{K}\left(\bar{t}_{K}\right)\right)=$ $x_{K}^{*}, K=1,2, \ldots, N$. It is easy to check that $\bar{x}_{K}(\cdot)$ is a local solution to the following optimization problem of type (14)(18) but with cost continuously differentiable around $\bar{x}_{K}(\cdot)$. This means that we deduce the optimal control problem (14)(18) with the nonsmooth cost functional to the smooth cost functional data:

$$
\begin{align*}
\min S & \left(u_{1}, \ldots, u_{N}, t_{1}, \ldots, t_{N}\right) \\
& =\sum_{K=1}^{N} s_{K}\left(x_{K}\left(t_{K}\right)\right)+\sum_{K=1}^{N} \int_{t_{K-1}}^{t_{K}} L\left(x_{K}(t), u_{K}(t), t\right) d t, \tag{26}
\end{align*}
$$

taking into account that

$$
\begin{equation*}
\nabla s_{K}\left(\bar{x}_{K}\left(\bar{t}_{K}\right)\right)=x_{K}^{*}, \quad K=1,2, \ldots, N \tag{27}
\end{equation*}
$$

We use multipliers to adjoint to constraints $\dot{x}_{K}(t)-$ $f_{K}\left(x_{K}(t), u_{K}(t), t\right)=0, t \in\left[t_{k-1}, t_{k}\right], K=1,2, \ldots, N$ and $F_{K}\left(x_{N}\left(t_{N}\right), t_{N}\right)=0, K=1,2, \ldots, N$ to $S$ :

$$
\begin{align*}
J^{\prime}= & \sum_{K=1}^{N} s_{K}\left(x_{K}\left(t_{K}\right)\right)+\sum_{K=1}^{N} \lambda_{k} F_{K}\left(x_{N}\left(t_{N}\right), t_{N}\right) \\
& +\sum_{K=1}^{N} \int_{t_{K-1}}^{t_{K}}\left(L\left(x_{K}(t), u_{K}(t), t\right)+p_{K}^{T}(t)\right.  \tag{28}\\
& \left.\quad \times\left(f_{K}\left(x_{K}(t), u_{K}(t), t\right)-\dot{x}_{K}(t)\right)\right) d t
\end{align*}
$$

by introducing the Lagrange multipliers $p_{1}(t), p_{2}(t), \ldots$, $p_{N}(t)$. In the following, we will find it convenient to use the function $H_{K}$, called the Hamiltonian, defined as $H_{K}\left(x_{K}(t), p_{K}(t), u_{K}(t), t\right) \quad=\quad L_{K}\left(x_{K}(t), u_{K}(t), t\right)+$ $p_{K}(t) f_{K}\left(x_{K}, u_{K}, t\right)$ for $t \in\left[t_{K-1}, t_{K}\right]$. Using this notation, we can write the Lagrange functional as

$$
\begin{align*}
J^{\prime}= & \sum_{K=1}^{N} s_{K}\left(x_{K}\left(t_{K}\right)\right)+\sum_{K=1}^{N} \lambda_{k} F_{K}\left(x_{N}\left(t_{N}\right), t_{N}\right) \\
& +\sum_{K=1}^{N} \int_{t_{K-1}}^{t_{K}}\left(H_{K}\left(x_{K}(t), p_{K}(t), u_{K}(t), t\right)-p_{K}^{T} \dot{x}_{K}(t)\right) d t \tag{29}
\end{align*}
$$

Assume $\left\{\bar{x}_{k}, \bar{u}_{k}, \bar{\theta}_{k}\right\}$ is optimal control. To determine the variation $\delta J^{\prime}$, we introduce the variation $\delta x_{K}, \delta u_{K}, \delta p_{K}$, and $\delta t_{K}$. From the calculus of variations, we can obtain that the first variation of $J^{\prime}$ as

$$
\begin{align*}
\delta J^{\prime}= & \sum_{K=1}^{N} \frac{\partial s_{K}\left(\bar{x}_{K}\left(\bar{t}_{K}\right)\right)}{\partial x_{K}} \delta x_{K}\left(t_{K}\right) \\
& +\sum_{K=1}^{N} \lambda_{K} \frac{\partial F_{K}\left(\bar{x}_{N}\left(\bar{t}_{N}\right), \bar{t}_{N}\right)}{\partial x_{N}} \delta x_{N}\left(t_{N}\right) \\
& +\sum_{K=1}^{N} \lambda_{K} \frac{\partial F_{K}\left(\bar{x}_{N}\left(\bar{t}_{N}\right), \bar{t}_{K}\right)}{\partial t_{N}} \delta t_{N} \\
& +\sum_{K=1}^{N} \int_{t_{K-1}}^{t_{K}} \frac{\partial H_{K}\left(\bar{x}_{K}(t), \bar{u}_{K}(t), \bar{p}_{K}(t), t\right)}{\partial x_{K}} \delta x_{K}(t) \\
& +\sum_{K=1}^{N} \int_{t_{K-1}}^{t_{K}} \frac{\partial H_{K}\left(\bar{x}_{K}(t), \bar{u}_{K}(t), \bar{p}_{K}(t), t\right)}{\partial u_{K}} \delta u_{K} \\
& +\sum_{K=1}^{N} \int_{t_{K-1}}^{t_{K}} \frac{\partial H_{K}\left(\bar{x}_{K}(t), \bar{u}_{K}(t), \bar{p}_{K}(t), t\right)}{\partial p_{K}} \delta p_{K} \\
& -\sum_{K=1}^{N} \int_{t_{K-1}}^{t_{K}}\left(\bar{p}_{K}(t) \delta x_{K}(t)+\bar{x}_{K}(t) \delta p_{K}(t)\right) d t \\
& +\operatorname{high} \text { order terms. } \tag{30}
\end{align*}
$$

If we follow the steps in [3, pages 5-7] then, the first variation of the functional takes the following form:

$$
\begin{align*}
& \delta J^{\prime}=\sum_{K=1}^{N-1} \frac{\partial s_{K}\left(\bar{x}_{K}\left(\bar{t}_{K}\right)\right)}{\partial x_{K}} \delta x_{K}\left(t_{K}\right)+\frac{\partial s\left(\bar{x}_{N}\left(\bar{t}_{N}\right)\right)}{\partial x_{N}} \delta x_{N}\left(t_{N}\right) \\
& +\sum_{K=1}^{N} \lambda_{K} \frac{\partial F_{K}\left(\bar{x}_{N}\left(\bar{t}_{N}\right), \bar{t}_{N}\right)}{\partial x_{N}} \delta x_{N}\left(t_{N}\right) \\
& +\sum_{K=1}^{N} \lambda_{K} \frac{\partial F_{K}\left(\bar{x}_{N}\left(\bar{t}_{N}\right), \bar{t}_{N}\right)}{\partial t_{N}} \delta t_{N} \\
& +\sum_{K=1}^{N} \int_{t_{K-1}}^{t_{K}} \frac{\partial H_{K}\left(\bar{u}_{K}, \bar{x}_{K}, \bar{p}_{K}, t\right)}{\partial u_{K}} \delta u_{K} \\
& +\sum_{K=1}^{N} \int_{t_{K-1}}^{t_{K}} \frac{\partial H_{K}\left(\bar{u}_{K}(t), \bar{x}_{K}(t), \bar{p}_{K}(t), t\right)}{\partial \bar{p}_{K}} \delta p_{K} \\
& -\sum_{K=1}^{N-1} \bar{p}_{K}\left(t_{K}\right) \delta x_{K}\left(t_{K}\right)-\bar{p}_{N}\left(t_{N}\right) \delta x_{N}\left(t_{N}\right) \\
& -\sum_{K=1}^{N-1} p_{K+1}\left(\bar{t}_{K}\right) \frac{\partial M_{K}\left(\bar{x}_{K}\left(\bar{t}_{K}\right), \bar{t}_{K}\right)}{\partial x_{K}} \delta x_{K}\left(\bar{t}_{K}\right) \\
& -\sum_{K=1}^{N-1} \bar{p}_{K+1}\left(\bar{t}_{K}\right) \frac{\partial M_{K}\left(\bar{x}_{K}\left(\bar{t}_{K}\right), \bar{t}_{K}\right)}{\partial t_{K}} \delta t_{K} \\
& -\sum_{k=1}^{N} \bar{p}_{K}(t) \delta x_{K}\left(t_{K}\right) \\
& =\sum_{K=1}^{N-1}\left(\frac{\partial \varphi_{K}\left(\bar{x}_{K}\left(\bar{t}_{K}\right)\right)}{\partial x_{K}}-\bar{p}_{K}\left(\bar{t}_{K}\right)\right. \\
& \left.-\bar{p}_{K+1}\left(\bar{t}_{K}\right) \frac{\partial M_{K}\left(\bar{x}_{K}\left(\bar{t}_{K}\right), \bar{t}_{K}\right)}{\partial x_{K}}\right) \delta x_{K}\left(\bar{t}_{K}\right) \\
& +\left(\frac{\partial \varphi_{N}\left(\bar{x}_{N}\left(\bar{t}_{N}\right)\right)}{\partial x_{N}}+\sum_{K=1}^{N} \lambda_{K} \frac{\partial F_{K}\left(\bar{x}_{N}\left(\bar{t}_{N}\right), \bar{t}_{N}\right)}{\partial \bar{x}_{N}}\right. \\
& \left.-\bar{p}_{N}\left(t_{N}\right)\right) \delta x_{N}\left(t_{N}\right) \\
& +\sum_{L=1}^{N}\left[\left(\sum_{K=1}^{N} \lambda_{K} \frac{\partial F_{K}\left(\bar{x}_{N}\left(\bar{t}_{N}\right), \bar{t}_{N}\right)}{\partial \bar{t}_{N}}\right) \delta_{L, N}\right. \\
& -\frac{1}{N}\left(\sum_{K=1}^{N-1} \bar{p}_{K+1}\left(\bar{t}_{K}\right) \frac{\partial M_{K}\left(\bar{x}_{K}\left(\bar{t}_{K}\right), \bar{t}_{K}\right)}{\partial t_{K}}\right) \\
& \left.\times\left(1-\delta_{L, N}\right)\right] \delta t_{L} \\
& +\sum_{K=1}^{N} \int_{\bar{t}_{K-1}}^{\bar{t}_{K}}\left(\frac{\partial H_{K}\left(\bar{x}_{K}(t), \bar{p}_{K}(t), \bar{u}_{K}(t), t\right)}{\partial x_{K}}-\dot{\bar{p}}_{K}(t)\right) \delta x_{K} \\
& +\sum_{K=1}^{N} \int_{\bar{t}_{K-1}}^{\bar{t}_{K}} \frac{\partial H_{K}\left(\bar{x}_{K}(t), \bar{p}_{K}(t), \bar{u}_{K}(t), t\right)}{\partial u_{K}} \delta u_{K} \\
& +\sum_{K=1}^{N} \int_{t_{K-1}}^{t_{K}}\left(\frac{\partial H_{K}\left(\bar{x}_{K}(t), \bar{p}_{K}(t), \bar{u}_{K}(t), t\right)}{\partial p_{K}}-\dot{\bar{x}}_{K}(t)\right) \delta p_{K} . \tag{31}
\end{align*}
$$

The latter sum is known because

$$
\begin{equation*}
\frac{\partial H_{K}\left(\bar{x}_{K}(t), \bar{u}_{K}(t), \bar{p}_{K}(t), t\right)}{\partial p_{K}}=\dot{\bar{x}}_{K}(t), \tag{32}
\end{equation*}
$$

and it is easy to check that

$$
\begin{align*}
& \sum_{K=1}^{N} \lambda_{K} \frac{\partial F_{K}\left(\bar{x}_{N}\left(\bar{t}_{N}\right), \bar{t}_{N}\right)}{\partial t_{N}} \delta t_{N} \\
& \quad-\sum_{K=1}^{N-1} \bar{p}_{K+1}\left(\bar{t}_{K}\right) \frac{\partial M_{K}\left(\bar{x}_{K}\left(\bar{t}_{K}\right), \bar{t}_{K}\right)}{\partial t_{K}} \delta t_{K} \\
& =\sum_{L=1}^{N}\left[\left(\sum_{K=1}^{N} \lambda_{K} \frac{\partial F_{K}\left(\bar{x}_{N}\left(\bar{t}_{N}\right), \bar{t}_{N}\right)}{\partial t_{N}}\right) \delta_{L, N}\right.  \tag{33}\\
& \quad-\frac{1}{N}\left(\sum_{K=1}^{N-1} \bar{p}_{K+1}\left(\bar{t}_{K}\right) \frac{\partial M_{K}\left(\bar{x}_{K}\left(\bar{t}_{K}\right), \bar{t}_{K}\right)}{\partial t_{K}}\right) \\
& \\
& \left.\quad \times\left(1-\delta_{L, N}\right)\right] \delta t_{L} .
\end{align*}
$$

If the state equations (14) are satisfied, $\dot{p}_{k}$ is selected so that coefficient of $\delta x_{k}$ and $\delta t_{N}$ is identically zero. Thus, we have

$$
\begin{align*}
\delta S^{\prime}= & \sum_{K=1}^{N-1} \int_{t_{K-1}}^{t_{K}} \frac{\partial H_{K}\left(\bar{x}_{K}(t), \bar{p}_{K}(t), \bar{u}_{K}(t), t\right)}{\partial u_{K}} \delta u_{K}  \tag{34}\\
& + \text { high order terms. }
\end{align*}
$$

The integrand is the first-order approximation to the change in $H_{K}$ caused by

$$
\begin{align*}
& {\left[\frac{\partial H_{K}\left(\bar{x}_{K}(t), \bar{p}_{K}(t), \bar{u}_{K}(t), t\right)}{\partial u_{K}} \delta u_{K}\right]^{T} \delta u_{K}(t)} \\
& \quad=H_{K}\left(\bar{x}_{K}(t), \bar{p}_{K}(t), \bar{u}_{K}(t)+\delta u_{K}(t), t\right)  \tag{35}\\
& \quad-H_{K}\left(\bar{x}_{K}(t), \bar{p}_{K}(t) \bar{u}_{K}(t), t\right)
\end{align*}
$$

Therefore,

$$
\begin{aligned}
\delta S^{\prime}=\sum_{K=1}^{N-1} \int_{t_{K-1}}^{t_{K}}[ & H_{K}\left(\bar{x}_{K}(t), \bar{p}_{K}(t), \bar{u}_{K}(t)+\delta u_{K}(t), t\right) \\
& \left.-H_{K}\left(\bar{x}_{K}(t), \bar{p}_{K}(t), \bar{u}_{K}(t), t\right)\right] \delta u_{K}(t)
\end{aligned}
$$

+ high order terms.

If $\bar{u}_{K}+\delta u_{K}$ is in a sufficiently small neighborhood of $\bar{u}_{K}$ then the high-order terms are small and the integral in last equation dominates the expression of $\delta S^{\prime}$. Thus, for $\bar{u}_{K}$ to be a minimizing control it is necessary that

$$
\begin{align*}
\sum_{K=1}^{N-1} \int_{t_{K-1}}^{t_{K}}[ & H_{K}\left(\bar{x}_{K}, \bar{p}_{K}, \bar{u}_{K}+\delta u_{K}, t\right)  \tag{37}\\
& \left.\quad-H_{K}\left(\bar{x}_{K}, \bar{p}_{K}, \bar{u}_{K}, t\right)\right] \delta u_{K} \geq 0
\end{align*}
$$

for all admissible $\delta u_{K}$. We assert that in order for the last inequality to be satisfied for all admissible $\delta u_{K}$ in the specified neighborhood, it is necessary that $H_{K}\left(\bar{x}_{K}, \bar{p}_{K}, \bar{u}_{K}+\delta u_{K}, t\right) \geq$ $H_{K}\left(\bar{x}_{K}, \bar{p}_{K}, \bar{u}_{K}, t\right)$ for all admissible $\delta u_{K}$ and for all $t \in$ [ $t_{K-1}, t_{K}$ ]. To show this, consider the control

$$
\Delta u_{K}= \begin{cases}\bar{u}_{K}(t), & t \in\left[t_{K-1}, t_{K}\right],  \tag{38}\\ \bar{u}_{K}(t)+\delta u_{K}(t), & t \in\left[t_{K-1}, t_{K}\right],\end{cases}
$$

where $t \in\left[t_{K-1}, t_{K}\right]$ is an arbitrarily small, but nonzero time interval and $\delta u_{K}$ are admissible control variations. After this, if we consider proof description of the maximum principle in [4], we can come to the last inequality.

According to the fundamental theorem of the calculus of the variation, at the extremal point the first variation of the functional must be zero, that is, $\delta J^{\prime}=0$. Setting to zero, the coefficients of the independent increments $\delta x_{N}\left(t_{N}\right)$, $\delta x_{K}\left(t_{K}\right) \delta x_{K}, \delta u_{K}$ and $\delta p_{K}$, and taking into account that

$$
\begin{equation*}
\nabla s_{K}\left(\bar{x}_{K}\left(t_{K}\right)\right)=x_{K}^{*}, \quad K=1,2, \ldots, N, \tag{39}
\end{equation*}
$$

yield the necessary optimality conditions (i)-(v) in Theorem 3.

This completes the proof of the theorem.
Theorem 4 (Necessary optimality conditions for switching optimal control system in terms of Quasidiffereniability). Let the minimization functional $\varphi_{K}(\cdot)$ be positively homogenous, quasidifferentiable at a point $\bar{x}_{K}(\cdot)$, and let $\left(\bar{u}_{K}(\cdot), \bar{x}_{K}(\cdot), \bar{\theta}\right)$ be an optimal solution to the control problem (14)-(18). Then, there exist vector functions $p_{K}(t), K=1, \ldots, N$, and there exist convex compact and bounded set $M\left(\varphi_{K}(\cdot)\right)$, in which for any elements $x_{K}^{*} \in M\left(\varphi_{K}(\cdot)\right)$, the necessary optimality conditions (i)-(v) in Theorem 3 are satisfied.

Proof. Let minimization functional $\varphi_{K}(\cdot)$ be positively homogenous and quasidifferentiable at a point $\bar{x}_{K}\left(\bar{t}_{K}\right)$. Then, there exist totally bounded lower exhausters $E_{*, K}$ for the $\varphi_{K}(\cdot)$ [ 9 , Theorem 4]. Let us make the substitution $M\left(\varphi_{K}(\cdot)\right)=$ $E_{*, K} ;$ take any element $x_{K}^{*} \in M\left(\varphi_{K}(\cdot)\right)$, then $x_{K}^{*} \in E_{*, K}$ also, and if we follow the proof description and result in Theorem 3 in the current paper, we can prove Theorem 4. If we use the relationship between the Gateaux upper subdifferential and Dini upper derivative [9, Lemma 3.6], substitute $h_{K}(g)=$ $\varphi_{K, H}^{+}\left(\bar{x}_{K}\left(\bar{t}_{K}\right), g\right)$, then we can write the following corollary (here $\varphi_{K, H}^{+}\left(\bar{x}_{K}\left(\bar{t}_{K}\right), g\right.$ is the Hadamard upper derivative of the minimizing functional $\varphi_{K}(\cdot)$ in the direction $\left.g\right)$.

Corollary 5. Let the minimization functional $\varphi_{K}(\cdot)$ be positively homogenous, and let the Dini upper differentiable at a point $x_{K}\left(\bar{t}_{K}\right)$ and $\left(\bar{u}_{K}(\cdot), \bar{x}_{K}(\cdot), \bar{\theta}\right)$ be an optimal solution to the control problem (14)-(18). Then for any elements $x_{K}^{*} \in$ $\partial_{G}^{+} h_{K}\left(0_{n}\right)$, there exist vector functions $p_{K}(t), K=1, \ldots, N$ in which the necessary optimality conditions (i)-(v) in the Theorem 3 hold.

Proof. Let us take any element $x_{K}^{*} \in \partial_{G}^{+} h_{K}\left(0_{n}\right)$. Then by using the lemma in [9, Lemma 3.8] we can write $x_{K}^{*} \in \partial_{F}^{+} h_{K}\left(0_{n}\right)$. Next, if we use the lemma in [9, Lemma 3.2], then we can put
$x_{K}^{*} \in \partial_{F}^{+} \varphi_{K}\left(\bar{x}_{K}\left(\bar{t}_{K}\right)\right)$. At least, if we follow Theorem 1 (relationship between upper Frechet subdifferential and exhausters) and Theorem 3 (necessary optimality condition in terms of exhausters) in the current paper, we can prove the result of Corollary 5.

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## Research Article

# Existence and Stability of Positive Periodic Solutions for a Neutral Multispecies Logarithmic Population Model with Feedback Control and Impulse 

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#### Abstract

We investigate a neutral multispecies logarithmic population model with feedback control and impulse. By applying the contraction mapping principle and some inequality techniques, a set of easily applicable criteria for the existence, uniqueness, and global attractivity of positive periodic solution are established. The conditions we obtained are weaker than the previously known ones and can be easily reduced to several special cases. We also give an example to illustrate the applicability of our results.


## 1. Introduction

As is known to all, ecosystem in the real world is continuously distributed by unpredictable forces which can result in changes in the biological parameters such as survival rates. Of practical interest in ecology is the question of whether or not an ecosystem can withstand those unpredictable disturbances which persist for a finite period of time. In recent years, the qualitative behaviors of the population dynamics with feedback control has attracted the attention of many mathematicians and biologists $[1-5]$. On the other hand, there are some other perturbations in the real world such as fires and floods, which are not suitable to be considered continually. These perturbations bring sudden changes to the system. Systems with such sudden perturbations involving impulsive differential equations have attracted the interest of many researchers in the past twenty years [6-10], since they provide a natural description of several real processes subject to certain perturbations whose duration is negligible in comparison with the duration of the process. Such processes are often investigated in various fields of science and technology such as physics, population dynamics, ecology, biological systems, and optimal control; for details, see [1113]. However, to the best of the author's knowledge, to this day, no scholar considered the neutral multispecies logarithmic population model with feedback control and impulse.

The aim of this paper is to investigate the existence, uniqueness, and global attractivity of the positive periodic solution for the following neutral multispecies logarithmic population system with feedback control and impulse:

$$
\begin{aligned}
& \frac{d N_{i}(t)}{d t} \\
& =N_{i}(t)\left[r_{i}(t)-\sum_{j=1}^{n} a_{i j}(t) \ln N_{j}(t)\right. \\
& \quad-\sum_{j=1}^{n} b_{i j}(t) \ln N_{j}\left(t-\tau_{i j}(t)\right) \\
& \quad-\sum_{j=1}^{n} c_{i j}(t) \int_{-\infty}^{t} K_{i j}(t-s) \ln N_{j}(s) d s \\
& \quad-\sum_{j=1}^{n} d_{i j}(t) \frac{d \ln N_{j}\left(t-\delta_{i j}(t)\right)}{d t} \\
& \left.\quad-e_{i}(t) u_{i}(t)-f_{i}(t) u_{i}\left(t-\sigma_{i}(t)\right)\right], \\
& t \neq t_{k},
\end{aligned}
$$

$$
\begin{gather*}
\frac{d u_{i}(t)}{d t}=-\alpha_{i}(t) u_{i}(t)+\beta_{i}(t) \ln N_{i}(t) \\
\\
\quad+\vartheta_{i}(t) \ln N_{i}\left(t-\gamma_{i}(t)\right), \quad t \geq 0  \tag{1}\\
N_{i}\left(t_{k}^{+}\right)=e^{\left(1+\theta_{i k}\right)} N_{i}\left(t_{k}\right), \quad i=1,2, \ldots, n, \quad k=1,2, \ldots,
\end{gather*}
$$

where $u_{i}(t)$ denote indirect feedback control variables. For the ecological justification of (1) and the similar types, refer to [14-20].

For the sake of generality and convenience, we always make the following fundamental assumptions:
$\left(H_{1}\right) r_{i}(t), a_{i j}(t), b_{i j}(t), c_{i j}(t), d_{i j}(t), e_{i}(t), f_{i}(t), \tau_{i j}(t)$, $\delta_{i j}(t) \in C^{2}(R, R), \sigma_{i}(t), \gamma_{i}(t), \alpha_{i}(t), \beta_{i}(t)$, and $\eta_{i}(t)$ are continuous nonnegative $\omega$-periodic functions with $\int_{0}^{\omega} r_{i}(t)>0, a_{i i}(t)>0, \delta_{i j}^{\prime}(t)<1, \tau=$ $\max _{t \in[0, \omega]}\left\{\tau_{i j}(t), \delta_{i j}(t), \sigma_{i}(t), \gamma_{i}(t)\right\}$, and $\int_{0}^{\infty} K_{i j}(s) d s=$ $1, \int_{0}^{+\infty} s K_{i j}(s) d s<+\infty, i, j=1,2, \ldots, n$;
$\left(H_{2}\right) 0<t_{1}<t_{2}<\cdots<t_{k}<\cdots$ are fixed impulsive points with $\lim _{k \rightarrow \infty} t_{k}=+\infty$;
$\left(H_{3}\right)\left\{\theta_{i k}\right\}$ is a real sequence, $\theta_{i k}+1>0$, and $\prod_{0<t_{k}<t}\left(1+\theta_{i k}\right)$ is an $\omega$-periodic function.

In the following section, some definitions and some useful lemmas are listed. In the third section, by applying the contraction mapping principle, some sufficient conditions which ensure the existence and uniqueness of positive periodic solution of system (1) are established, and then we get a few sufficient conditions ensuring the global attractivity of the positive periodic solution by employing some inequality techniques. Finally, we give an example to show our results.

## 2. Preliminaries

In order to obtain the existence and uniqueness of a periodic solution for system (1), we first give some definitions and lemmas.

Definition 1. A function $N_{i}: R \rightarrow(0, \infty)(i=1,2, \ldots, n)$ is said to be a positive solution of (1), if the following conditions are satisfied:
(a) $N_{i}(t)$ is absolutely continuous on each $\left(t_{k}, t_{k+1}\right)$;
(b) for each $k \in Z_{+}, N_{i}\left(t_{k}^{+}\right)$and $N_{i}\left(t_{k}^{-}\right)$exist, and $N_{i}\left(t_{k}^{-}\right)=$ $N_{i}\left(t_{k}\right)$;
(c) $N_{i}(t)$ satisfies the first equation of (1) for almost everywhere (for short a.e.) in $[0, \infty] \backslash\left\{t_{k}\right\}$ and satisfies $N_{i}\left(t_{k}^{+}\right)=\left(1+\theta_{i k}\right) N_{i}\left(t_{k}\right)$ for $t=t_{k}, k \in Z_{+}=\{1,2, \ldots\}$.

Definition 2. System (1) is said to be globally attractive, if there exists a positive solution $\left(N_{i}(t), u_{i}(t)\right)$ of (1) such that $\lim _{t \rightarrow+\infty}\left|N_{i}(t)-N_{i}^{*}(t)\right|=0, \lim _{t \rightarrow+\infty}\left|u_{i}(t)-u_{i}^{*}(t)\right|=0$, for any other positive solution $\left(N_{i}^{*}(t), u_{i}^{*}(t)\right)$ of the system (1).

Lemma 3. $R_{+}^{2 n}=\left\{\left(N_{i}(t), u_{i}(t)\right): N_{i}(0)>0, u_{i}(0)>0, i=\right.$ $1,2, \ldots, n\}$ is the positive invariable region of the system (1).

Proof. In view of biological population, we obtain $N_{i}(0)>0$, $u_{i}(0)>0$. By the system (1), we have

$$
\begin{align*}
& N_{i}(t)=N_{i}(0) \\
& \times \exp \left\{\int _ { 0 } ^ { t } \left[r_{i}(\eta)-\sum_{j=1}^{n} a_{i j}(\eta) \ln N_{j}(\eta)\right.\right. \\
& -\sum_{j=1}^{n} b_{i j}(\eta) \ln N_{j}\left(\eta-\tau_{i j}(\eta)\right) \\
& -\sum_{j=1}^{n} c_{i j}(\eta) \int_{-\infty}^{t} K_{i j}(\eta-s) \ln N_{j}(s) d s \\
& -\sum_{j=1}^{n} d_{i j}(\eta) \frac{d \ln N_{j}\left(\eta-\delta_{i j}(\eta)\right)}{d t} \\
& -e_{i}(\eta) u_{i}(\eta) \\
& \left.\left.-f_{i}(\eta) u_{i}\left(\eta-\sigma_{i}(\eta)\right)\right] d \eta\right\} \text {, } \\
& t \in\left[0, t_{1}\right], i=1,2, \ldots, n, \\
& N_{i}(t)=N_{i}\left(t_{k}\right) \\
& \times \exp \left\{\int _ { t _ { k } } ^ { t } \left[r_{i}(\eta)\right.\right. \\
& -\sum_{j=1}^{n} a_{i j}(\eta) \ln N_{j}(\eta) \\
& -\sum_{j=1}^{n} b_{i j}(\eta) \ln N_{j}\left(\eta-\tau_{i j}(\eta)\right) \\
& -\sum_{j=1}^{n} c_{i j}(\eta) \int_{-\infty}^{t} K_{i j}(\eta-s) \ln N_{j}(s) d s \\
& -\sum_{j=1}^{n} d_{i j}(\eta) \frac{d \ln N_{j}\left(\eta-\delta_{i j}(\eta)\right)}{d t} \\
& -e_{i}(\eta) u_{i}(\eta) \\
& \left.\left.-f_{i}(\eta) u_{i}\left(\eta-\sigma_{i}(\eta)\right)\right] d \eta\right\} \text {, } \\
& t \in\left(t_{k}, t_{k+1}\right], i=1,2, \ldots, n, \\
& N_{i}\left(t_{k}^{+}\right)=e^{\left(1+p_{i k}\right)} N_{i}\left(t_{k}\right)>0, \quad k \in N, i=1,2, \ldots, n, \\
& u_{i}(t)=\int_{t}^{t+\omega} G(t, s)\left[\beta_{i}(s) \ln N_{i}(s)\right. \\
& \left.+\vartheta_{i}(s) \ln N_{i}\left(s-\gamma_{i}(s)\right)\right] d s \\
& :=\left(\phi_{i} \ln N_{i}\right)(t) \text {, } \tag{2}
\end{align*}
$$

We can easily get the following lemma.
where

$$
\begin{equation*}
G_{i}(t, s)=\frac{\exp \left\{\int_{t}^{s} \alpha_{i}(\xi) d \xi\right\}}{\exp \left\{\int_{t}^{s} \alpha_{i}(\xi) d \xi\right\}-1} \tag{3}
\end{equation*}
$$

Then the solution of the system (1) is positive.
Under the above hypotheses $\left(H_{1}\right)-\left(H_{3}\right)$, we consider the neutral nonimpulsive system:

$$
\begin{aligned}
\frac{d y_{i}(t)}{d t}=y_{i}(t)[ & r_{i}(t)-\sum_{j=1}^{n} A_{i j}(t) \ln y_{j}(t) \\
& -\sum_{j=1}^{n} B_{i j}(t) \ln y_{j}\left(t-\tau_{i j}(t)\right) \\
& -\sum_{j=1}^{n} C_{i j}(t) \int_{-\infty}^{t} K_{i j}(t-s) \ln y_{j}(s) d s \\
& -\sum_{j=1}^{n} D_{i j}(t) \frac{d \ln y_{j}\left(t-\delta_{i j}(t)\right)}{d t} \\
& \left.-e_{i}(t) u_{i}(t)-f_{i}(t) u_{i}\left(t-\sigma_{i}(t)\right)\right], \\
\frac{d u_{i}(t)}{d t}= & -\alpha_{i}(t) u_{i}(t)+\beta_{i}^{*}(t) \ln y_{i}(t) \\
& +\vartheta_{i}^{*}(t) \ln y_{i}\left(t-\gamma_{i}(t)\right),
\end{aligned}
$$

where

$$
\begin{gathered}
A_{i j}(t)=a_{i j}(t) \prod_{0<t_{k}<t}\left(1+\theta_{i k}\right), \\
B_{i j}(t)=b_{i j}(t) \prod_{0<t_{k}<t-\tau_{i j}(t)}\left(1+\theta_{i k}\right), \\
C_{i j}(t)=c_{i j}(t) \prod_{0<t_{k}<t}\left(1+\theta_{i k}\right), \\
D_{i j}(t)=d_{i j}(t) \prod_{0<t_{k}<t-\delta_{i j}(t)}\left(1+\theta_{i k}\right), \\
\beta_{i}^{*}(t)=\beta_{i}(t) \prod_{0<t_{k}<t}\left(1+\theta_{i k}\right), \\
\vartheta_{i}^{*}(t)=\theta_{i}(t) \prod_{0<t_{k}<t-\gamma_{i}(t)}\left(1+\theta_{i k}\right),
\end{gathered}
$$

By a solution $\left(y_{i}(t), u_{i}(t)\right)$ of (4), it means an absolutely continuous function $\left(y_{i}(t), u_{i}(t)\right)$ defined on $[-\tau, 0]$ that satisfies (4) a.e., for $t \geq 0$, and $y(\xi)=\varphi(\xi), y^{\prime}(\xi)=\varphi^{\prime}(\xi)$ on $[-\tau, 0]$.

The following lemma will be used in the proofs of our results, and the proof of the lemma is similar to that of Theorem 1 in [6].

Lemma 4. Suppose that $\left(H_{1}\right)-\left(H_{4}\right)$ hold. Then
(i) if $\left(y_{i}(t), u_{i}(t)\right)$ is a solution of (4) on $[-\tau,+\infty)$, then $\left(N_{i}(t), u_{i}(t)\right)=\left(\prod_{0<t_{k}<t} e^{\left(1+\theta_{i k}\right)} y_{i}(t), u_{i}(t)\right)$ is a solution of (1) on $[-\tau,+\infty)$,
(ii) if $\left(N_{i}(t), u_{i}(t)\right)$ is a solution of (1) on $[-\tau,+\infty)$, then $\left(y_{i}(t), u_{i}(t)\right)=\left(\prod_{0<t_{k}<t}\left(e^{\left(1+\theta_{i k}\right)}\right)^{-1} N_{i}(t), u_{i}(t)\right)$ is a solution of $(4)$ on $[-\tau,+\infty)$.

Proof. (i) It is easy to see that $\left(N_{i}(t), u_{i}(t)\right)=\left(\prod_{0<t_{k}<t} e^{\left(1+\theta_{i k}\right)}\right.$ $\left.y_{i}(t), u_{i}(t)\right)$ is absolutely continuous on every interval $\left(t_{k}\right.$, $\left.t_{k+1}\right], t \neq t_{k}, k=1,2, \ldots$,

$$
\begin{aligned}
& N_{i}^{\prime}(t)-N_{i}(t) \\
& \times\left[r_{i}(t)-\sum_{j=1}^{n} a_{i j}(t) \ln N_{j}(t)-\sum_{j=1}^{n} b_{i j}(t) \ln N_{j}\left(t-\tau_{i j}(t)\right)\right. \\
& -\sum_{j=1}^{n} c_{i j}(t) \int_{-\infty}^{t} K_{i j}(t-s) \ln N_{j}(s) d s \\
& -\sum_{j=1}^{n} d_{i j}(t) \frac{d \ln N_{j}\left(t-\delta_{i j}(t)\right)}{d t} \\
& \left.-e_{i}(t) u_{i}(t)-f_{i}(t) u_{i}\left(t-\sigma_{i}(t)\right)\right] \\
& =\prod_{0<t_{k}<t} e^{\left(1+\theta_{i k}\right)} y_{i}^{\prime}(t)-\prod_{0<t_{k}<t} e^{\left(1+\theta_{i k}\right)} y_{i}(t) \\
& \times\left[r_{i}(t)-\sum_{j=1}^{n} a_{i j}(t) \prod_{0<t_{k}<t}\left(1+\theta_{i k}\right) \ln y_{j}(t)\right. \\
& -\sum_{j=1}^{n} b_{i j}(t) \prod_{0<t_{k}<t-\tau_{i j}(t)}\left(1+\theta_{i k}\right) \ln y_{j}\left(t-\tau_{i j}(t)\right) \\
& -\sum_{j=1}^{n} c_{i j}(t) \int_{-\infty}^{t} K_{j}(t-s) \prod_{0<t_{k}<t}\left(1+\theta_{i k}\right) \ln y_{j}(s) d s \\
& -\sum_{j=1}^{n} d_{i j}(t) \prod_{0<t_{k}<t-\delta_{i j}(t)}\left(1+\theta_{i k}\right) \frac{d y_{j}\left(t-\delta_{i j}(t)\right)}{d t} \\
& \left.-e_{i}(t) u_{i}(t)-f_{i}(t) u_{i}\left(t-\sigma_{i}(t)\right)\right] \\
& =\prod_{0<t_{k}<t} e^{\left(1+\theta_{k}\right)} \\
& \times\left\{y^{\prime}(t)-y(t)\right. \\
& \times\left[r_{i}(t)-\sum_{j=1}^{n} A_{i j}(t) \ln y_{j}(t)\right. \\
& -\sum_{j=1}^{n} B_{i j}(t) \ln y_{j}\left(t-\tau_{i j}(t)\right)
\end{aligned}
$$

$$
\begin{gather*}
-\sum_{j=1}^{n} C_{i j}(t) \times \int_{-\infty}^{t} K_{i j}(t-s) \ln y_{j}(s) d s \\
-\sum_{j=1}^{n} D_{i j}(t) \frac{d \ln y_{j}\left(t-\delta_{i j}(t)\right)}{d t} \\
\left.\left.-e_{i}(t) u_{i}(t)-f_{i}(t) u_{i}\left(t-\sigma_{i}(t)\right)\right]\right\}=0, \\
u_{i}^{\prime}(t)+\alpha_{i}(t) u_{i}(t)-\beta_{i}(t) \ln N_{i}(t)-\vartheta_{i}(t) \ln N_{i}\left(t-\gamma_{i}(t)\right) \\
=u_{i}^{\prime}(t)+\alpha_{i}(t) u_{i}(t)-\beta_{i}^{*}(t) \ln y_{i}(t) \\
-\vartheta_{i}^{*}(t) \ln y_{i}\left(t-\gamma_{i}(t)\right)=0 . \tag{6}
\end{gather*}
$$

On the other hand, for any $t=t_{k}, k=1,2, \ldots$,

$$
\begin{align*}
\begin{aligned}
N_{i}\left(t_{k}^{+}\right) & =\lim _{t \rightarrow t_{k}^{+}} \prod_{0<t_{j}<t} e^{\left(1+\theta_{i k}\right)} y_{i}(t) \\
& =\prod_{0<t_{j} \leq t_{k}} e^{\left(1+\theta_{i k}\right)} y_{i}\left(t_{k}\right), \\
N_{i}\left(t_{k}\right) & =\prod_{0<t_{j}<t_{k}} e^{\left(1+\theta_{i k}\right)} y_{i}\left(t_{k}\right) .
\end{aligned} .
\end{align*}
$$

Thus

$$
\begin{equation*}
N\left(t_{k}^{+}\right)=e^{\left(1+\theta_{i k}\right)} N\left(t_{k}\right), \quad k=1,2, \ldots \tag{8}
\end{equation*}
$$

It follows from (6)-(8) that $\left(N_{i}(t), u_{i}(t)\right)$ is a solution of (1).
(ii) Since $N_{i}(t)=\prod_{0<t_{k}<t} e^{\left(1+\theta_{i k}\right)} y_{i}(t)$ is absolutely continuous on every interval $\left(t_{k}, t_{k+1}\right], t \neq t_{k}, k=1,2, \ldots$, and in view of (8), it follows that for any $k=1,2, \ldots$,

$$
\begin{align*}
y_{i}\left(t_{k}^{+}\right) & =\prod_{0<t_{j} \leq t_{k}}\left(e^{\left(1+\theta_{i k}\right)}\right)^{-1} N_{i}\left(t_{k}^{+}\right) \\
& =\prod_{0<t_{j}<t_{k}}\left(e^{\left(1+\theta_{i k}\right)}\right)^{-1} N_{i}\left(t_{k}\right)=y_{i}\left(t_{k}\right) \\
y_{i}\left(t_{k}^{-}\right) & =\prod_{0<t_{j}<t_{k}}\left(e^{\left(1+\theta_{i k}\right)}\right)^{-1} N_{i}\left(t_{k}^{-}\right)  \tag{9}\\
& =\prod_{0<t_{j} \leq t_{k}^{-}}\left(e^{\left(1+\theta_{i k}\right)}\right)^{-1} N_{i}\left(t_{k}^{-}\right)=y_{i}\left(t_{k}\right),
\end{align*}
$$

which implies that $y_{i}(t)$ is continuous on $[-\tau,+\infty)$. It is easy to prove that $y_{i}(t)$ is absolutely continuous on [ $-\tau$, $+\infty)$. Similar to the proof of (i), we can check that $\left(y_{i}(t)\right.$, $\left.u_{i}(t)\right)=\left(\prod_{0<t_{k}<t}\left(e^{\left(1+\theta_{i k}\right)}\right)^{-1} N_{i}(t), u_{i}(t)\right)$ are solutions of (4) on $[-\tau,+\infty)$. The proof of Lemma 4 is completed.

Lemma 5. $\left(y_{i}(t), u_{i}(t)\right)$ is a $\omega$-periodic solution of (4) if and only if $y_{i}(t)$ is a $\omega$-periodic solution of the following system:

$$
\begin{align*}
& \frac{d y_{i}(t)}{d t} \\
& =y_{i}(t)\left[r_{i}(t)-\sum_{j=1}^{n} A_{i j}(t) \ln y_{j}(t)\right. \\
& \\
& \quad-\sum_{j=1}^{n} B_{i j}(t) \ln y_{j}\left(t-\tau_{i j}(t)\right)  \tag{10}\\
& \\
& \quad-\sum_{j=1}^{n} C_{i j}(t) \int_{-\infty}^{t} K_{i j}(t-s) \ln y_{j}(s) d s \\
& \\
& \quad-\sum_{j=1}^{n} D_{i j}(t) \frac{d \ln y_{j}\left(t-\delta_{i j}(t)\right)}{d t} \\
& \\
& \quad-e_{i}(t)\left(\phi_{i} \ln y_{i}\right)(t) \\
& \\
& \left.\quad-f_{i}(t)\left(\phi_{i} \ln y_{i}\right)(t)\left(t-\sigma_{i}(t)\right)\right]
\end{align*}
$$

where

$$
\left(\phi_{i} \ln y_{i}\right)(t)
$$

$$
\begin{equation*}
:=\int_{t}^{t+\omega} G_{i}(t, s)\left[\beta_{i}^{*}(s) \ln y_{i}(s)+\vartheta_{i}^{*}(s) \ln y_{i}\left(s-\gamma_{i}(s)\right)\right] d s \tag{11}
\end{equation*}
$$

and $G_{i}(t, s)$ is defined by (3).
Proof. The proof of Lemma 5 is similar to that of Lemma 2.2 in [2], and we omit the details here.

Obviously, the existence, uniqueness, and global attractivity of positive periodic solution of system (1) is equivalent to the existence, uniqueness, and global attractivity of periodic solution of system (10).

Lemma 6. Assume that $u(t), \tau(t)$ are all continuously differentiable $\omega$-periodic functions and $a(t)$ is a nonnegative continuous $\omega$-periodic function such that $\int_{0}^{\omega} a(t) d t>0$; then

$$
\begin{align*}
& \int_{-\infty}^{t} \quad e^{-\int_{s}^{t} a(\xi) d \xi} b(s) u^{\prime}(s-\tau(s)) d s \\
& \quad=c(t) u(t-\tau(t)) \\
& \quad-\int_{-\infty}^{t} e^{-\int_{s}^{t} a(\xi) d \xi}\left[a(s) c(s)+c^{\prime}(s)\right] u(s-\tau(s)) d s, \tag{12}
\end{align*}
$$

where $c(t)=b(t) /\left(1-\tau^{\prime}(t)\right)$.

Proof. As

$$
\begin{align*}
\int_{-\infty}^{t} & e^{-\int_{s}^{t} a(\xi) d \xi} b(s) u^{\prime}(s-\tau(s)) d s \\
= & \int_{-\infty}^{t} e^{-\int_{s}^{t} a(\xi) d \xi} c(s) d u(s-\tau(s)) \\
= & \left.e^{-\int_{s}^{t} a(\xi) d \xi} c(s) u(s-\tau(s))\right|_{-\infty} ^{t} \\
& \quad-\int_{-\infty}^{t} u(s-\tau(s)) d\left(e^{-\int_{s}^{t} a(\xi) d \xi} c(s)\right) \\
= & \left.e^{-\int_{s}^{t} a(\xi) d \xi} c(s) u(s-\tau(s))\right|_{-\infty} ^{t} \\
& -\int_{-\infty}^{t} u(s-\tau(s))\left[a(s) c(s)+c^{\prime}(s)\right] e^{-\int_{s}^{t} a(\xi) d \xi} d s . \tag{13}
\end{align*}
$$

Denote $m=e^{-\int_{0}^{\omega} a(t) d t}$; then from $a(t) \geq 0, \int_{0}^{\omega} a(t) d t>0$, it follows that $m<1$. Also, when $t \geq s$ without loss of generality, we may assume that $s+n \omega \leq t \leq s+(n+1) \omega$; thus

$$
\begin{align*}
& \left|e^{-\int_{s}^{t} a(\xi) d \xi} c(s) u(s-\tau(s))\right| \\
& \quad \leq e^{-\int_{s}^{t} a(\xi) d \xi}\|c\|\|u\| \\
& \quad=e^{-\sum_{j=1}^{n-1} \int_{s+j \omega}^{s+(j+1) \omega} a(\xi) d \xi-\int_{s+n \omega}^{t} a(\xi) d \xi}\|c\|\|u\|  \tag{14}\\
& \quad=m^{n} e^{-\int_{s+n \omega}^{t} a(\xi) d \xi}\|c\|\|u\| \\
& \quad \leq m^{n}\|c\|\|u\| .
\end{align*}
$$

Therefore,

$$
\begin{equation*}
\lim _{s \rightarrow-\infty} e^{-\int_{s}^{t} a(\xi) d \xi} c(s) u(s-\tau(s))=0 \tag{15}
\end{equation*}
$$

and so from (13)-(15) it follows that

$$
\begin{align*}
& \int_{-\infty}^{t} \quad e^{-\int_{s}^{t} a(\xi) d \xi} b(s) u^{\prime}(s-\tau(s)) d s \\
& \quad=c(t) u(t-\tau(t)) \\
& \quad-\int_{-\infty}^{t} e^{-\int_{s}^{t} a(\xi) d \xi}\left[a(s) c(s)+c^{\prime}(s)\right] u(s-\tau(s)) d s \tag{16}
\end{align*}
$$

The proof Lemma 6 is complete.

## 3. Main Theorem

In this section, by using contraction principle and some inequality techniques, several conditions on the existence, uniqueness, and global attractivity of periodic solution for system (1) are presented.

Let $y_{i}(t)=e^{h_{i} x_{i}(t)}$; the system (10) can be reduced to

$$
\begin{align*}
\frac{d x_{i}}{d t}= & -A_{i i}(t) x_{i}(t)-h_{i}^{-1} \sum_{j=1, j \neq i}^{n} h_{j} A_{i j}(t) x_{j}(t) \\
& -h_{i}^{-1} \sum_{j=1}^{n} h_{j} B_{i j}(t) x_{j}\left(t-\tau_{i j}(t)\right) \\
& -h_{i}^{-1} \sum_{j=1}^{n} h_{j} C_{i j}(t) \int_{-\infty}^{t} K_{i j}(t-s) x_{j}(s) d s \\
& -h_{i}^{-1} \sum_{j=1}^{n} h_{j} D_{i j}(t)\left(1-\delta_{i j}^{\prime}(t)\right) x_{j}^{\prime}\left(t-\delta_{i j}(t)\right) \\
& -e_{i}(t)\left(\phi_{i} x_{i}\right)(t)-f_{i}(t)\left(\phi_{i} x_{i}\right)\left(t-\sigma_{i}(t)\right)+h_{i}^{-1} r_{i}(t) \tag{17}
\end{align*}
$$

where $h_{i}>0(i=1,2, \ldots, n)$ are $n$ positive real numbers.
Obviously, the existence, uniqueness, and global attractivity of positive periodic solution of system (10) is equivalent to the existence, uniqueness, and global attractivity of periodic solution of system (17).

For the rest of this paper, we will devote ourselves to study the existence, uniqueness, and global attractivity of periodic solution of (17). We denote

$$
\begin{align*}
& \Gamma_{i}^{1}(t):=e_{i}(t)\left(\phi_{i} 1\right)(t), \quad \Gamma_{i}^{2}(t):=f_{i}(t)\left(\phi_{i} 1\right)\left(t-\sigma_{i}(t)\right), \\
& \Delta_{i}(t):= h_{i}^{-1} \sum_{j=1, j \neq i}^{n} h_{j} A_{i j}(t) \\
&+h_{i}^{-1} \sum_{j=1}^{n} h_{j}\left[B_{i j}(t)+C_{i j}(t)\right. \\
&\left.\quad+\left(A_{i i}(s) D_{i j}(t)+\left|D_{i j}^{\prime}(t)\right|\right)\right] \\
&+\Gamma_{i}^{1}(t)+\Gamma_{i}^{2}(t), \quad i=1,2, \ldots, n . \tag{18}
\end{align*}
$$

Our first result on the global existence of a periodic solution of system (1) is stated in the following theorem.

Theorem 7. In addition to $\left(H_{1}\right)-\left(H_{3}\right)$, assume further that there exist positive constants $h_{i}(i=1,2, \ldots, n)$ and a positive constant $M<1$ such that

$$
\begin{aligned}
& \left(H_{4}\right) \sup _{t \in[0, \omega]} \max _{i \in[1, n]}\left\{\sum_{j=1}^{n}\left(h_{j} / h_{i}\right) D_{i j}(t)+\int_{-\infty}^{t} \exp \left\{-\int_{s}^{t} A_{i i}\right.\right. \\
& \left.\quad(\xi) d \xi\} \Delta_{i}(s) d s\right\} \leq M .
\end{aligned}
$$

Then, system (1) has a unique $\omega$-periodic solution with strictly positive components, where $\Delta_{i}(t)$ is defined by (18).

Proof. From the above analysis, to finish the proof of Theorem 7, it is enough to prove under the conditions of Theorem 7 that system (17) has a unique $\omega$-periodic solution. Let

$$
\begin{equation*}
\Omega=\left\{u(t) \mid u \in C\left(R^{n}, R\right), u(t+\omega)=u(t)\right\} ; \tag{19}
\end{equation*}
$$

under the norm $\|u\|=\max _{1 \leq i \leq n} \max _{t \in[0, \omega]}\left\{\left|u_{i}(t)\right|\right\}, \Omega$ is a Banach space. For any $u(t)=\left(u_{1}(t), \ldots, u_{n}(t)\right)^{T} \in \Omega$, we consider the periodic solution $x_{u}(t)$ of periodic differential equation

$$
\begin{align*}
\frac{d x_{i}}{d t}= & -A_{i i}(t) x_{i}(t)-h_{i}^{-1} \sum_{j=1, j \neq i}^{n} h_{j} A_{i j}(t) u_{j}(t) \\
& -h_{i}^{-1} \sum_{j=1}^{n} h_{j} B_{i j}(t) u_{j}\left(t-\tau_{i j}(t)\right) \\
& -h_{i}^{-1} \sum_{j=1}^{n} h_{j} C_{i j}(t) \int_{-\infty}^{t} K_{i j}(t-s) u_{j}(s) d s  \tag{20}\\
& -h_{i}^{-1} \sum_{j=1}^{n} h_{j} D_{i j}(t)\left(1-\delta_{i j}^{\prime}(t)\right) u_{j}^{\prime}\left(t-\delta_{i j}(t)\right) \\
& -e_{i}(t)\left(\phi_{i} u_{i}\right)(t)-f_{i}(t)\left(\phi_{i} u_{i}\right)(t)\left(t-\sigma_{i}(t)\right) \\
& +h_{i}^{-1} r_{i}(t), \quad i=1,2, \ldots, n
\end{align*}
$$

Since $A_{i i}(t)>0$, we know that the linear system of system (20)

$$
\begin{equation*}
\frac{d x_{i}}{d t}=-A_{i i}(t) x_{i}(t), \quad i=1,2, \ldots, n \tag{21}
\end{equation*}
$$

admits exponential dichotomies on $R$, and so system (20) has a unique periodic solution $x_{u}(t)$, which can be expressed as

$$
\begin{align*}
x_{u}(t)= & \left(x_{1 u}(t), \ldots, x_{n u}(t)\right)^{T} \\
= & \left(\int_{-\infty}^{t} \exp \left\{-\int_{s}^{t} A_{11}(\xi) d \xi\right\} F_{1 u}(s) d s, \ldots\right.  \tag{22}\\
& \left.\int_{-\infty}^{t} \exp \left\{-\int_{s}^{t} A_{n n}(\xi) d \xi\right\} F_{n u}(s) d s\right)^{T},
\end{align*}
$$

where

$$
\begin{align*}
F_{i u}(t)= & -h_{i}^{-1} \sum_{j=1, j \neq i}^{n} h_{j} A_{i j}(t) x_{j}(t) \\
& -h_{i}^{-1} \sum_{j=1}^{n} h_{j} B_{i j}(t) x_{j}\left(t-\tau_{i j}(t)\right) \\
& -h_{i}^{-1} \sum_{j=1}^{n} h_{j} C_{i j}(t) \int_{-\infty}^{t} K_{i j}(t-s) x_{j}(s) d s  \tag{23}\\
& -h_{i}^{-1} \sum_{j=1}^{n} h_{j} D_{i j}(t)\left(1-\delta_{i j}^{\prime}(t)\right) x_{j}^{\prime}\left(t-\delta_{i j}(t)\right) \\
& -e_{i}(t)\left(\phi_{i} x_{i}\right)(t)-f_{i}(t)\left(\phi_{i} x_{i}\right)(t)\left(t-\sigma_{i}(t)\right) \\
& +h_{i}^{-1} r_{i}(t) ;
\end{align*}
$$

its proof is similar to that of Theorem 1 in [18]; here we omit it.

Now, by using Lemma 6, $x_{i u}(t)$ can also be expressed as

$$
\begin{align*}
x_{i u}(t)= & -\sum_{j=1}^{n} \frac{h_{j}}{h_{i}} D_{i j}(t) u_{j}\left(t-\delta_{i j}(t)\right) \\
& +\int_{-\infty}^{t} \exp \left\{-\int_{s}^{t} A_{i i}(\xi) d \xi\right\} G_{i u}(s) d s  \tag{24}\\
& i=1,2, \ldots, n
\end{align*}
$$

where

$$
\begin{align*}
G_{i u}(t)= & -h_{i}^{-1} \sum_{j=1, j \neq i}^{n} h_{j} A_{i j}(t) u_{j}(t) \\
& -h_{i}^{-1} \sum_{j=1}^{n} h_{j} B_{i j}(t) u_{j}\left(t-\tau_{i j}(t)\right) \\
& -h_{i}^{-1} \sum_{j=1}^{n} h_{j} \times C_{i j}(t) \int_{-\infty}^{t} K_{i j}(t-s) u_{j}(s) d s \\
& +h_{i}^{-1} \sum_{j=1}^{n} h_{j}\left(A_{i i}(s) D_{i j}(t)+D_{i j}^{\prime}(t)\right) u_{j}\left(t-\delta_{i j}(t)\right) \\
& +e_{i}(t)\left(\phi_{i} u_{i}\right)(t)-f_{i}(t)\left(\phi_{i} u_{i}\right)(t)\left(t-\sigma_{i}(t)\right) \\
& +h_{i}^{-1} r_{i}(t) . \tag{25}
\end{align*}
$$

Now we define mapping $T: \Omega \rightarrow \Omega, T u(t)=x_{u}(t)$. Following this we will prove that $T$ is a contraction mapping; that is, there exists a constant $\beta \in(0,1)$, such that $\| T u-$ $T v\|\leq \beta\| u-v \|$, for all $u, v \in \Omega$. In fact, for any $u(t)=$ $\left(u_{1}(t), \ldots, u_{n}(t)\right)^{T}$ and $v(t)=\left(v_{1}(t), \ldots, v_{n}(t)\right)^{T}$, we have

$$
\begin{align*}
& \left\|G_{i u}(t)-G_{i v}(t)\right\| \\
& \begin{array}{r}
\leq h_{i}^{-1} \sum_{j=1, j \neq i}^{n} h_{j} A_{i j}(t)\left|u_{j}(t)-v_{j}(t)\right| \\
+h_{i}^{-1} \sum_{j=1}^{n} h_{j}\left[B_{i j}(t)\left|u_{j}\left(t-\tau_{i j}(t)\right)-v_{j}\left(t-\tau_{i j}(t)\right)\right|\right. \\
\\
+C_{i j}(t) \int_{-\infty}^{t} K_{i j}(t-s)\left|u_{j}(s)-v_{j}(s)\right| d s \\
\\
+\left(A_{i i}(s) D_{i j}(t)+D_{i j}^{\prime}(t)\right) \\
\\
\left.\quad \times\left|u_{j}\left(t-\delta_{i j}(t)\right)-v_{j}\left(t-\delta_{i j}(t)\right)\right|\right]
\end{array} \\
& \begin{array}{r}
\leq \begin{array}{r}
e_{i}(t)\left(\phi_{i} 1\right)\|u-v\|+f_{i}(t)\left(\phi_{i} 2\right)\left(t-\sigma_{i}(t)\right)\|u-v\| \\
h_{i}^{-1} \sum_{j=1, j \neq i}^{n} h_{j} A_{i j}(t)
\end{array} \\
\quad+h_{i}^{-1} \sum_{j=1}^{n} h_{j}\left[B_{i j}(t)+C_{i j}(t)\right. \\
\left.+\left(A_{i i}(s) D_{i j}(t)+\left|D_{i j}^{\prime}(t)\right|\right)\right]
\end{array} \\
& \left.+\Gamma_{i}^{1}(t)+\Gamma_{i}^{2}(t)\right\}\|u-v\|=\Delta_{i}(t)\|u-v\| .
\end{align*}
$$

Hence,

$$
\begin{align*}
& \|T u-T v\| \\
& =\sup _{t \in[0, \omega]} \max \left\{\left\lvert\, \sum_{j=1}^{n} \frac{h_{j}}{h_{1}} D_{1 j}(t)\right.\right. \\
& \times\left[u_{j}\left(t-\delta_{1 j}(t)\right)-v_{j}\left(t-\delta_{1 j}(t)\right)\right] \mid \\
& +\int_{-\infty}^{t} \exp \left\{-\int_{s}^{t} A_{11}(\xi) d \xi\right\} \\
& \times\left|G_{1 u}(s)-G_{1 v}(s)\right| d s, \ldots, \\
& \left\lvert\, \sum_{j=1}^{n} \frac{h_{j}}{h_{n}} D_{n j}(t)\right. \\
& \times\left[u_{j}\left(t-\delta_{n j}(t)\right)-v_{j}\left(t-\delta_{n j}(t)\right)\right] \mid \\
& +\int_{-\infty}^{t} \exp \left\{-\int_{s}^{t} A_{n n}(\xi) d \xi\right\} \\
& \left.\times\left|G_{n u}(s)-G_{n v}(s)\right| d s\right\} \\
& \leq \sup _{t \in[0, \omega]} \max \left\{\left[\sum_{j=1}^{n} \frac{h_{j}}{h_{1}} D_{1 j}(t)\right.\right. \\
& \left.+\int_{-\infty}^{t} \exp \left\{-\int_{s}^{t} A_{11}(\xi) d \xi\right\} \Delta_{1}(s) d s\right] \\
& \times\|u-v\|, \ldots \text {, } \\
& {\left[\sum_{j=1}^{n} \frac{h_{j}}{h_{n}} D_{n j}(t)\right.} \\
& \left.+\int_{-\infty}^{t} \exp \left\{-\int_{s}^{t} A_{n n}(\xi) d \xi\right\} \Delta_{n}(s) d s\right] \\
& \times\|u-v\|\} \text {. } \tag{27}
\end{align*}
$$

It follows from $\left(H_{4}\right)$ that $\|T u-T v\| \leq\|u-v\|$ for all $u, v \in \Omega$. That is, $T$ is a contraction mapping. Hence, there exists a unique fixed point $x^{*}(t)=\left(x_{1}^{*}(t), \ldots, x_{n}^{*}(t)\right)^{T} \in \Omega$; that is, $T x^{*}(t)=x^{*}(t)$. Therefore, $x^{*}(t)$ is the unique periodic solution of system (17). It follows from (1), (4), (10), and (17) that $\left(N^{*}(t), u^{*}(t)\right)^{T}=\left(N_{1}^{*}(t), \ldots, N_{n}^{*}(t), u_{1}^{*}(t), \ldots, u_{n}^{*}(t)\right)^{T}$ is the unique positive periodic solution of system (1). The proof of Theorem 7 is completed.

Our next theorem is concerned with the global stability of periodic solution for system (1).

Theorem 8. In addition to $\left(H_{1}\right)-\left(H_{4}\right)$, suppose further that the following condition holds:

$$
\left(H_{5}\right) \exp \left\{-\int_{0}^{t} A_{i i}(\xi) d \xi\right\} \rightarrow 0, \text { ast } \rightarrow+\infty, i=1,2, \ldots, n
$$

Then system (1) has a unique periodic solution which is globally attractive.

Proof. Let $N^{*}(t)=\left(N_{1}^{*}(t), N_{2}^{*}(t), \ldots, N_{n}^{*}(t)\right)^{T}$ be the unique positive periodic solution of system (1), whose existence and uniqueness are guaranteed by Theorem 7, and let $N(t)=$ $\left(N_{1}(t), N_{2}(t), \ldots, N_{n}(t)\right)^{T}$ be any other solution of system (1). Let $N_{i}^{*}(t)=\exp \left\{\prod_{0<t_{k}<t} h_{i}\left(1+p_{i k}\right) x_{i}^{*}(t)\right\}, N_{i}(t)=$ $\exp \left\{\prod_{0<t_{k}<t} \times h_{i}\left(1+p_{i k}\right) x_{i}(t)\right\}$; then, similar to (17), we have

$$
\begin{align*}
\frac{d x_{i}^{*}}{d t}= & -A_{i i}(t) x_{i}^{*}(t)-h_{i}^{-1} \sum_{j=1, j \neq i}^{n} h_{j} A_{i j}(t) x_{j}^{*}(t) \\
& -h_{i}^{-1} \sum_{j=1}^{n} h_{j} B_{i j}(t) x_{j}^{*}\left(t-\tau_{i j}(t)\right) \\
& -h_{i}^{-1} \sum_{j=1}^{n} h_{j} C_{i j}(t) \int_{-\infty}^{t} K_{i j}(t-s) x_{j}^{*}(s) d s \\
& -h_{i}^{-1} \sum_{j=1}^{n} h_{j} D_{i j}(t)\left(1-\delta_{i j}^{\prime}(t)\right) x_{j}^{\prime *}\left(t-\delta_{i j}(t)\right) \\
& -e_{i}(t)\left(\phi_{i} x_{i}^{*}\right)(t)-f_{i}(t)\left(\phi_{i} x_{i}^{*}\right)\left(t-\sigma_{i}(t)\right) \\
& +h_{i}^{-1} r_{i}(t), \\
\frac{d x_{i}}{d t}= & -A_{i i}(t) x_{i}(t)-h_{i}^{-1} \sum_{j=1, j \neq i}^{n} h_{j} A_{i j}(t) x_{j}(t) \\
& -h_{i}^{-1} \sum_{j=1}^{n} h_{j} B_{i j}(t) x_{j}\left(t-\tau_{i j}(t)\right) \\
& -h_{i}^{-1} \sum_{j=1}^{n} h_{j} C_{i j}(t) \int_{-\infty}^{t} K_{i j}(t-s) x_{j}(s) d s \\
& -h_{i}^{-1} \sum_{j=1}^{n} h_{j} D_{i j}(t)\left(1-\delta_{i j}^{\prime}(t)\right) x_{j}^{\prime}\left(t-\delta_{i j}(t)\right) \\
& -e_{i}(t)\left(\phi_{i} x_{i}\right)(t)-f_{i}(t)\left(\phi_{i} x_{i}\right)\left(t-\sigma_{i}(t)\right)+h_{i}^{-1} r_{i}(t) . \tag{28}
\end{align*}
$$

Let $x_{i}^{*}(t)-x_{i}(t)=w_{i}(t)$; then

$$
\begin{align*}
\frac{d w_{i}}{d t}= & -A_{i i}(t) w_{i}(t)-h_{i}^{-1} \sum_{j=1, j \neq i}^{n} h_{j} A_{i j}(t) w_{j}(t) \\
& -h_{i}^{-1} \sum_{j=1}^{n} h_{j} B_{i j}(t) w_{j}\left(t-\tau_{i j}(t)\right) \\
& -h_{i}^{-1} \sum_{j=1}^{n} h_{j} C_{i j}(t) \int_{-\infty}^{t} K_{i j}(t-s) w_{j}(s) d s  \tag{29}\\
& -h_{i}^{-1} \sum_{j=1}^{n} h_{j} D_{i j}(t)\left(1-\delta_{i j}^{\prime}(t)\right) w_{j}^{\prime}\left(t-\delta_{i j}(t)\right) \\
& -e_{i}(t)\left(\phi_{i} w_{i}\right)(t)-f_{i}(t)\left(\phi_{i} w_{i}\right)\left(t-\sigma_{i}(t)\right)
\end{align*}
$$

Multiply both sides of (29) with $\exp \left\{\int_{0}^{t} A_{i i}(\xi) d \xi\right\}$, and then integrate from 0 to $t$ to obtain

$$
\begin{align*}
& \int_{0}^{t}\left[w_{i}(u) \exp \left\{\int_{0}^{u} A_{i i}(\xi) d \xi\right\}\right]^{\prime} d u \\
& =-\int_{0}^{t}\left[h_{i}^{-1} \sum_{j=1, j \neq i}^{n} h_{j} A_{i j}(u) w_{j}(u)\right. \\
& \quad+h_{i}^{-1} \sum_{j=1}^{n} h_{j} B_{i j}(u) w_{j}\left(u-\tau_{i j}(u)\right) \\
& \quad+h_{i}^{-1} \sum_{j=1}^{n} h_{j} C_{i j}(u) \int_{-\infty}^{t} K_{i j}(u-s) w_{j}(s) d s \\
& \quad+h_{i}^{-1} \sum_{j=1}^{n} h_{j} D_{i j}(u)\left(1-\delta_{i j}^{\prime}(u)\right) w_{j}^{\prime}\left(u-\delta_{i j}(u)\right) \\
& \left.\quad+e_{i}(u)\left(\phi_{i} w_{i}\right)(u)+f_{i}(u)\left(\phi_{i} w_{i}\right)\left(u-\sigma_{i}(t)\right)\right] \\
& \quad \times \exp \left\{\int_{0}^{u} A_{i i}(\xi) d \xi\right\} d u, \quad i=1,2, \ldots, n ; \tag{30}
\end{align*}
$$

then

$$
\begin{align*}
& w_{i}(t)=w_{i}(0) \exp \left\{-\int_{0}^{t} A_{i i}(\xi) d \xi\right\} \\
&-\int_{0}^{t}[ h_{i}^{-1} \sum_{j=1, j \neq i}^{n} h_{j} A_{i j}(u) w_{j}(u) \\
&+h_{i}^{-1} \sum_{j=1}^{n} h_{j} B_{i j}(u) w_{j}\left(u-\tau_{i j}(u)\right) \\
&+h_{i}^{-1} \sum_{j=1}^{n} h_{j} C_{i j}(u) \\
& \times \int_{-\infty}^{t} K_{i j}(u-s) w_{j}(s) d s \\
&+h_{i}^{-1} \sum_{j=1}^{n} h_{j} D_{i j}(u)\left(1-\delta_{i j}^{\prime}(u)\right) \\
& \quad \times e_{i}(u)\left(\phi_{i} w_{j}\right)(u) \\
&\left.+f_{i}(u)\left(\phi_{i j} w_{i}\right)(u)\right) \\
& \times \exp \left\{-\int_{u}^{t} A_{i i}(\xi) d \xi\right\} d u, \quad i=1,2, \ldots, n .
\end{align*}
$$

Let $D_{0 i j}(t)=D_{i j}(t)\left(1-\delta_{i j}^{\prime}(t)\right)$; we see that

$$
\begin{align*}
& \int_{0}^{t} D_{0 i j}(u) w_{j}^{\prime}\left(u-\delta_{i j}(u)\right) \exp \left\{-\int_{u}^{t} A_{i i}(\xi) d \xi\right\} d u \\
& =\int_{0}^{t} \frac{D_{0 i j}(u) w_{j}^{\prime}\left(u-\delta_{i j}(u)\right)\left(1-\delta_{i j}^{\prime}(u)\right)}{1-\delta_{i j}(u)} \\
& \times \exp \left\{-\int_{u}^{t} A_{i i}(\xi) d \xi\right\} d u \\
& =\int_{0}^{t}\left[\frac{D_{0 i j}(u) \exp \left\{-\int_{u}^{t} A_{i i}(\xi) d \xi\right\}}{1-\delta_{i j}^{\prime}(u)}\right] \\
& \times\left[w_{j}^{\prime}\left(u-\delta_{i j}(u)\right)\left(1-\delta_{i j}^{\prime}(u)\right)\right] d u \\
& =\left[\frac{D_{0 i j}(t)}{1-\delta_{i j}^{\prime}(t)} w_{j}\left(t-\delta_{i j}(t)\right)\right. \\
& \left.-\frac{D_{0 i j}(0)}{1-\delta_{i j}^{\prime}(0)} w_{j}\left(-\delta_{i j}(0)\right) \exp \left\{-\int_{0}^{t} A_{i i}(\xi) d \xi\right\}\right] \\
& -\int_{0}^{t}\left(A_{i i}(u) D_{i j}(u)+D_{i j}^{\prime}(u)\right) \\
& \times \exp \left\{-\int_{u}^{t} A_{i i}(\xi) d \xi\right\} w_{j}\left(u-\delta_{i j}(u)\right) d u \\
& =\left[D_{i j}(t) w_{j}\left(t-\delta_{i j}(t)\right)\right. \\
& \left.-D_{i j}(0) w_{j}\left(-\delta_{i j}(0)\right) \exp \left\{-\int_{0}^{t} A_{i i}(\xi) d \xi\right\}\right] \\
& -\int_{0}^{t}\left(A_{i i}(u) D_{i j}(u)+D_{i j}^{\prime}(u)\right) \\
& \times \exp \left\{-\int_{u}^{t} A_{i i}(\xi) d \xi\right\} w_{j}\left(u-\delta_{i j}(u)\right) d u . \tag{32}
\end{align*}
$$

Substituting (32) into (31), we get

$$
\begin{aligned}
& w_{i}(t) \\
& \qquad=\left[w_{i}(0)+h_{i}^{-1} \sum_{j=1}^{n} h_{j} D_{i j}(0) w_{j}\left(-\delta_{i j}(0)\right)\right] \\
& \quad \times \exp \left\{-\int_{0}^{t} A_{i i}(\xi) d \xi\right\} \\
& \quad+\int_{0}^{t}\left\{h _ { i } ^ { - 1 } \sum _ { j = 1 } ^ { n } h _ { j } \left[\left(A_{i i}(u) D_{i j}(u)\right.\right.\right. \\
& \left.\quad+D_{i j}^{\prime}(u)\right) w_{j}\left(u-\delta_{i j}(u)\right) \\
& \quad-B_{i j}(u) w_{j}\left(u-\tau_{i j}(u)\right) \\
& \left.\quad-C_{i j}(u) \int_{-\infty}^{t} K_{i j}(u-s) w_{j}(s) d s\right]
\end{aligned}
$$

$$
\begin{gather*}
-h_{i}^{-1} \sum_{j=1, j \neq i}^{n} h_{j} A_{i j}(u) w_{j}(u) \\
\left.-e_{i}(u)\left(\phi_{i} w_{i}\right)(u)-f_{i}(u)\left(\phi_{i} w_{i}\right)\left(u-\sigma_{i}(t)\right)\right\} \\
\times \exp \left\{-\int_{u}^{t} A_{i i}(\xi) d \xi\right\} d u \\
+h_{i}^{-1} \sum_{j=1}^{n} h_{j} D_{i j}(t) w_{j}\left(t-\delta_{i j}(t)\right) ; \tag{33}
\end{gather*}
$$

therefore, we have

$$
\begin{align*}
& \|w\| \\
& \leq\left|w_{i}(0)+h_{i}^{-1} \sum_{j=1}^{n} h_{j} D_{i j}(0) w_{j}\left(-\delta_{i j}(0)\right)\right| \\
& \times \exp \left\{-\int_{0}^{t} A_{i i}(\xi) d \xi\right\} \\
& +\left\{\int _ { 0 } ^ { t } \left\{h _ { i } ^ { - 1 } \sum _ { j = 1 } ^ { n } h _ { j } \left[\left(A_{i i}(u) D_{i j}(u)+\left|D_{i j}^{\prime}(u)\right|\right)\right.\right.\right. \\
& \left.\times\left|B_{i j}(u)\right|+\left|C_{i j}(u)\right|\right] \\
& \left.+h_{i}^{-1} \sum_{j=1, j \neq i}^{n} h_{j}\left|A_{i j}(u)\right|+\left|\Gamma_{i}^{1}(u)\right|+\left|\Gamma_{i}^{2}(u)\right|\right\} \\
& \times \exp \left\{-\int_{u}^{t} A_{i i}(\xi) d \xi\right\} d u \\
& \left.+h_{i}^{-1} \sum_{j=1}^{n} h_{j}\left|D_{i j}(t)\right|\right\}\|w\| \\
& =\left|w_{i}(0)+h_{i}^{-1} \sum_{j=1}^{n} h_{j} D_{i j}(0) w_{j}\left(-\delta_{i j}(0)\right)\right| \\
& \times \exp \left\{-\int_{u}^{t} A_{i i}(\xi) d \xi\right\} \\
& +\left[\sum_{j=1}^{n} \frac{h_{j}}{h_{i}} D_{i j}(t)\right. \\
& \left.+\int_{0}^{t} \exp \left\{-\int_{u}^{t} A_{i i}(\xi) d \xi\right\} \Delta_{i}(u) d u\right]\|w\|, \tag{34}
\end{align*}
$$

where $\Delta_{i}(t)$ is defined by (18). From $\left(H_{4}\right)$, we have $\|w\|$

$$
\begin{align*}
& \leq \frac{\left|w_{i}(0)+h_{i}^{-1} \sum_{j=1}^{n} h_{j} D_{i j}(0) w_{j}\left(-\delta_{i j}(0)\right)\right| \exp \left\{-\int_{0}^{t} A_{i i}(\xi) d \xi\right\}}{1-h_{i}^{-1} \sum_{j=1}^{n} h_{j}\left|D_{i j}(t)\right|-\int_{0}^{t} \Delta_{i}(u) \exp \left\{-\int_{u}^{t} A_{i i}(\xi) d \xi\right\} d u} \\
& \leq \frac{\left|w_{i}(0)+h_{i}^{-1} \sum_{j=1}^{n} h_{j} D_{i j}(0) w_{j}\left(-\delta_{i j}(0)\right)\right| \exp \left\{-\int_{0}^{t} A_{i i}(\xi) d \xi\right\}}{1-M} . \tag{35}
\end{align*}
$$

From $\left(H_{5}\right)$, we have

$$
\begin{align*}
\|w\| & =\max _{i \in[0, n]}\left|w_{i}(t)\right|  \tag{36}\\
& =\max _{i \in[0, n]}\left|x_{i}^{*}(t)-x_{i}(t)\right|=0, \quad \text { as } t \longrightarrow+\infty
\end{align*}
$$

thus, $x_{i}(t) \rightarrow x_{i}^{*}(t)$, as $t \rightarrow+\infty, i=1,2, \ldots, n$. Hence, the positive $\omega$-periodic solution of (17) is globally attractive; accordingly, $N_{i}(t)=\exp \left\{\prod_{0<t_{k}<t} h_{i}\left(1+p_{i k}\right) x_{i}(t)\right\} \rightarrow$ $\exp \left\{\prod_{0<t_{k}<t} h_{i}\left(1+p_{i k}\right) x_{i}^{*}(t)\right\}=N_{i}^{*}(t), u_{i}(t)=\left(\phi_{i} \ln N_{i}\right)(t) \rightarrow$ $\left(\phi_{i} \ln N_{i}^{*}\right)(t)=u_{i}^{*}(t)$ as $t \rightarrow+\infty, i=1,2, \ldots, n$, and by Definition 2, the positive $\omega$-periodic solution of (1) is globally attractive. The proof of Theorem 8 is completed.

Remark 9. If $e_{i}(t)=f_{i}(t)=\alpha_{i}(t)=\beta_{i}(t)=\mathcal{\vartheta}_{i}(t)=0, \theta_{i k}+1=$ $0, i=1,2, \ldots, n, k=1,2, \ldots$, then system (1) is studied by [3]. Hence, Theorems 7 and 8 generalize the corresponding results in [3].

Remark 10. If $\theta_{i k}+1=0, i=1,2, \ldots, n, k=1,2, \ldots$, then system (1) is studied by [4]. Hence Theorems 7 and 8 also generalize the corresponding results in [4].

## 4. Example

Consider the following impulsive model:

$$
\begin{align*}
& \frac{d N(t)}{d t} \\
& \begin{aligned}
=N(t)[ & r(t)-a(t) \ln N(t)-b(t) \ln N(t-\tau(t)) \\
& -c(t) \int_{-\infty}^{t} K(t-s) \ln N(s) d s \\
& -d(t) \frac{d \ln N(t-\delta(t))}{d t}-e(t) u(t) \\
& -f(t) u(t-\sigma(t))], \quad t \neq t_{k} \\
\frac{d u(t)}{d t}= & -\alpha(t) u(t)+\beta(t) \ln N(t) \\
& +\theta(t) \ln N(t-\gamma(t)), \quad t \geq 0 \\
N\left(t_{k}^{+}\right) & =e^{\left(1+p_{k}\right)} N\left(t_{k}\right), \quad k=1,2, \ldots,
\end{aligned}
\end{align*}
$$

where $r(t), a(t), b(t), c(t), d(t), e(t), f(t), \delta(t) \in C^{2}(R, R)$, $\sigma(t), \gamma(t), \alpha(t), \beta(t), \theta(t)$ are all nonnegative $\omega$-periodic continuous functions with $\int_{0}^{\omega} r(t)>0, a(t)>0, \delta^{\prime}(t)<1$ and $p_{k}$ is a real sequence, and $\prod_{0<t_{k}<t}\left(1+p_{k}\right)$ is a positive $\omega$-periodic function with $k=1,2, \ldots$. Furthermore, $\int_{0}^{\infty} K(s) d s=1$, $\int_{0}^{+\infty} s K(s) d s<+\infty$.

We denote

$$
\begin{gather*}
A(t)=a(t) \prod_{0<t_{k}<t}\left(1+p_{k}\right), \\
B(t)=b(t) \prod_{0<t_{k}<t-\tau(t)}\left(1+p_{k}\right), \\
C(t)=c(t) \prod_{0<t_{k}<t}\left(1+p_{k}\right), \\
D(t)=d(t) \prod_{\left.0<t_{k}<t-\delta(t)\right)}\left(1+p_{k}\right), \\
D_{0}(t)=D(t)\left(1-\delta^{\prime}(t)\right), \\
\beta^{*}(t)=\beta(t) \prod_{0<t_{k}<t}\left(1+p_{k}\right) \\
\theta^{*}(t)=\theta(t) \prod_{0<t_{k}<t-\gamma(t)}\left(1+p_{k}\right), \\
\Gamma^{1}(t):=e(t)(\phi 1)(t), \\
\Gamma^{2}(t):=f(t)(\phi 1)(t-\sigma(t)), \\
\Delta(t):=B(t)+C(t)+A(t) D(t)+\left|D^{\prime}(t)\right|+\Gamma^{1}(t)+\Gamma^{2}(t) . \tag{38}
\end{gather*}
$$

Similar to Theorems 7 and 8, we can get the following results.
Corollary 11. In addition to conditions $\left(H_{1}\right)-\left(H_{3}\right)$, assume further that there exists a positive constant $M<1$ such that

$$
\left(H_{6}\right) D(t)+\int_{-\infty}^{t} \exp \left\{-\int_{s}^{t} A(\xi) d \xi\right\} \Delta(s) d s \leq M
$$

Then, (37) has a unique $\omega$-periodic solution with strictly positive components, where $\Delta(t)$ is defined by (38).

Corollary 12. In addition to conditions $\left(H_{1}\right)-\left(H_{3}\right)$ and $\left(H_{6}\right)$, suppose further that the following condition holds:

$$
\left(H_{7}\right) \exp \left\{-\int_{0}^{t} A(\xi) d \xi\right\} \rightarrow 0, \text { as } t \rightarrow+\infty
$$

Then, system (37) has a unique periodic solution which is globally attractive.

Remark 13. The results in the work show that by means of appropriate impulsive perturbations and feedback control we can control the dynamics of these equations.

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## Research Article

# A Schistosomiasis Model with Mating Structure 

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#### Abstract

A schistosomiasis model is proposed including single schistosomes, paired schistosomes, snails, and the latent period of infected snails. A reasonable sex ratio of schistosomes and the minimum mating function are considered. A threshold condition determining the stability of the system is given, and the stability of equilibrium for the model is shown. The impact of the latent period of infected snails on schistosomiasis transmission can be found through numerical simulations. Finally, preferable control strategies are obtained by sensitivity analyses. Killing snails may be the preferred control measure. If we choose chemotherapy, we should use some drugs which are sufficient for reducing egg-associated pathology, since paired schistosomes are mostly harmful to definitive hosts.


## 1. Introduction

Schistosomiasis is one of the most prevalent parasite diseases. Its transmission needs two hosts: the definitive hosts and the intermediate snail hosts. In definitive hosts, schistosoma has two distinct sexes. Female and male schistosomes pair up they lay eggs that work their way through the intestinal lining or into the ureters and bladder. The eggs themselves are harmless, but in definitive host the body's immune response results in inflammation and eventual damage, as granulomasfibrous lesions-form around the eggs. These eggs pass in urine or feces into fresh water. In infected water, miracidium hatches from egg and infects the intermediate snail hosts. In snail hosts, parasites asexually reproduce. As a parasite in the body of the snail, the miracidium loses its cilia and undergoes a change into a sac-like primary sporocysts [1]. Many secondary sporocysts are formed in the body of the primary sporocysts. They develop for some time and then rupture the wall of the primary sporocyst and become established in the body of the snail. From the walls of the secondary sporocyst a large number of parthenogenetic eggs are formed, and they develop internally into forms known as cercariae. After about four, weeks cercariae rupture the body-wall of the secondary sporocyst, bore out of the snail's body, and enter the water as very active organisms, characterized by two large suckers
and a forked tail. In this stage, they are ready to enter a mammalian host. When such a free-living cercaria comes in contact with a susceptible mammal, it penetrate, the skin of the definitive host and transforms into single schistosoma. Here, the parasite's life cycle is completed.

During the asexual phase in the host mollusc, several groups have observed that the miracidia are already sexually differentiated [2]. Furthermore, male miracidia will only give rise to male later forms, whilst female miracidia will only give rise to female later forms. However, no apparent differences have been registered between physiological alterations in snails infected by a male miracidium and those in snails infected by a female miracidium. There are no distinctive qualitative and quantitative differences in snail host response to infection with a single miracidium and in snail host reaction to infection with numerous miracidia secured by single or multiple exposures [3]. From the above statement, we know that the mating of male schistosoma and female schistosoma in definitive hosts is very important to the transmission of schistosomiasis. Hence, we differentiate female and male parasites only in definitive hosts in this paper.

In the literature, two-sex problems have been studied by many authors [4-6]. All of these models are established for human population diseases, such as sexually transmitted diseases. They constructed pair-formation models and studied
the existence, uniqueness and the stabilities of exponential solutions. In $[5,6]$ authors put forward three forms of pairformation functions which are also called mating functions: the harmonic mean function, the geometric mean function and the minimum function. All of these three functions can be applied to two-sex models.

As for schistosomiasis, there are few papers considering two-sex problems and mating interactions [4, 7-9]. In [9] Xu et al. have proposed a multi-strain schistosome model with mating structure. Their goal is to infer the impact of drug treatment on the maintenance of schistosome genetic diversity. However, the model of Xu et al. ignores several stages of the parasite's complex life cycle. They consider only the adult parasite populations. Furthermore, in their model they assume that the recruitment rate of single adult parasites at time $t$ depends instantaneously on the total number of parasite pairs at time $t$. Subsequently, Castillo-Chavez et al. [4] have relaxed this assumption on the parasite's life history by introducing a time delay. The time delay accounts for the average time that must elapse between two adult generations. Their model is in the following:

$$
\begin{gather*}
\frac{d m}{d t}=k S_{m}(\tau) p(t-\tau)-\left(\mu_{m}+\frac{\sigma}{\theta}\right) m(t)-\varphi(m(t), f(t)) \\
\frac{d f}{d t}=k S_{f}(\tau) p(t-\tau)-\left(\mu_{f}+\frac{\sigma}{\theta}\right) f(t)-\varphi(m(t), f(t)) \\
\frac{d p}{d t}=\varphi(m(t), f(t))-\left(\mu_{p}+\frac{\sigma}{\theta}\right) p(t) \tag{1}
\end{gather*}
$$

Here $m(t)$ and $f(t)$ represent the densities of single male schistosoma and single female schistosoma, respectively, and $p(t)$ is the density of pairs. The formation of schistosome pairs is described by a mating function $\varphi(m(t), f(t)) . k$ is half of the birth rate of a pair per capita. Note that the ratio of male to female offspring is assumed to be $1: 1$ in their model. $\mu_{m}, \mu_{f}$, and $\mu_{p}$ are the per capita death rates of single male schistosoma, single female schistosoma, and schistosoma in a mated pair, respectively. $\sigma$ is the rate of the chemotherapeutic treatment. $\theta$ denotes the drug resistance of that parasite strain. $S_{m}(\tau)$ and $S_{f}(\tau)$ are functions which keep track of the parasite's survival probabilities in various (nonadult) stages of the life cycle. The dynamical behaviors of their model in [4] are not qualitatively different from those derived from an earlier model in [9], although [9] ignores the impact of time delays associated with the multiple stages in parasite's life cycle. The results of [4] imply that higher treatment rate can allow for coexistence between susceptible and resistant parasite strains. Both in $[4,9]$, the harmonic mean function is chosen as the mating function $\varphi(m(t), f(t))$.

As we know, in the two models of $[4,9]$, the ratio of male to female offspring is assumed to be $1: 1$. In fact, some experiments have shown that the number of male schistosomes is bigger than that of female [10-16]. In [10] authors found the phenomena of the natural male bias by experimental observations on the sex ratio of adult schistosoma mansoni. In [12] the sex ratio of female schistosoma to male schistosoma is $1: 1.81$. In [16], authors reported that the male-biased sex ratio
is 2.36 males to 1 female in Kenya. In two experiments of [14], the sex ratios of female schistosoma to male schistosoma are ( $0.89 \pm 0.03$ ) : 1 and $(0.98 \pm 0.03): 1$, respectively. These results tend to male-biased sex ratio. In [15] the sex ratio of female schistosoma to male schistosoma is $1: 1.38$. In artificial experiments of [13], there are 9543 males and 6597 females in 16140 mature schistosoma japonicum; the ratio of female to male is $1: 1.45$. In [11] authors also support male-biased sex ratio. All of these results imply that the sex ratio is not $1: 1$. Furthermore, Mao [13] also denote that the assumption of the $50: 50$ sex ratio in previous schistosomiasis models does not accord with the result of experiments. Hence, in this paper, we consider reasonably that the number of male offspring is bigger than that of female in our model.

On the other hand, the two models in $[4,9]$ do not include the snail dynamics. In reality, from the life cycle of schistosoma, it is easy to see that the parasite offspring is produced directly by infected snails but not by paired parasites. Nåsell [17] also thought that the number of male and female schistosoma is proportional to the number of infected snails. This implies that the snail dynamics may influence the transmission of schistosomiasis. Hence, it is necessary to add snails to the model. In addition, from the life history of schistosoma, we know that the production of parasite offspring needs about four weeks after the snail hosts are infected by miracidia $[4,9]$. Motivated by $[4,9,13,17]$ we established a new model including a more reasonable sex ratio of schistosoma, snail dynamics, the latent period of infected snails, and the mating structure of schistosoma. Here, we take the minimum function as mating function. In this paper, our purpose is to study the dynamics of single schistosoma, pairs, and snails and to put forward preferable control strategies.

Our paper is organized as follows. In Section 2, we establish a mathematical model with single schistosoma, pairs and snails. For convenience of analysis, we first study the model without time delay and define the basic reproductive number. And then the stabilities of the disease-free and endemic equilibria are obtained in Section 3. In Section 4, the stability of the delayed model is investigated. In Section 5, computational simulations and sensitivity analysis are performed, and we give preferable control strategies.

## 2. Mathematical Model

Considering the mating structure of parasites and snail dynamics, we propose a model with state variables $m(t)$, $f(t), p(t), s(t)$, and $i(t)$, where $m(t), f(t)$, and $p(t)$ represent the densities of single male schistosoma, single female schistosoma and pairs, respectively; $s(t)$ and $i(t)$ represent the number of susceptible and infectious snail hosts, respectively. The basic assumptions are as follows.
(i) Since the single male and female schistosoma are from the cercaria produced by infected snails, in [17, 18] authors consider that the number of male and female schistosoma is proportional to the number of infected snails. According to [17, 18] we also assume that the recruitment rate of single male and female schistosoma is proportional to the number of infected snail hosts, that is, the recruitment rates of single
male and female worms are $k_{m} i(t)$ and $k_{f} i(t)$, respectively. Following papers [10-16], we assume $k_{m} \geq k_{f}$.
(ii) Natural death rates for single male, single female, and pair worms per capita are $\mu_{m}, \mu_{f}$, and $\mu_{p}$, respectively. In reality, pairs of schistosoma may live for a few years while single parasites may only live for a few weeks [9]. Therefore, we assume $\mu_{p}<\mu_{m}, \mu_{f}$. As for the investigation of the death rate of single male and female schistosoma, there are many results. In the experiment of [12], the livability of single male parasites is stronger than that of single female parasites, which implies that single female parasites have a higher natural death rate than single male parasites. In [19], the author found that single male schistosoma is larger more muscular. May and Woolhouse [20] and Tchuente et al. [21] also found that the growth of single female schistosoma is restrained when it is not paired up. Standen [22] deduced that single female schistosoma is incapable of going against the blood stream but single male schistosoma can. These results imply that the livability of single male parasites is stronger than that of single female parasites. Cornford and Huot [23] and Cornford and Fitzpatrick [24, 25] had shown that single female schistosoma has less glucose than single male schistosoma. B. G. Atkinson and K. H. Atkinson [26] and Davis et al. [27] found that single female schistosoma has less actin than single male schistosoma. These indicate that single female parasite is not easy to survive. In addition, Mao [13] also reported that the survival rate of single male schistosoma is not lower than that of single female schistosoma. All of the above results imply $\mu_{m} \leq \mu_{f}$. Hence, we assume $\mu_{p}<\mu_{m} \leq \mu_{f}$.
(iii) The mating function $\varphi(m(t), f(t))$ takes the minimum function $\varphi(m(t), f(t))=\rho \min (m(t), f(t))$ in which $\rho$ represents the effective mating rate $[5,6]$.
(iv) The parameter $\Lambda_{s}$ is the recruitment rate of snail hosts. $\mu_{s}$ is per capita natural death rate of snail hosts. $\alpha_{s}$ is the disease-induced death rate of snail hosts. The transmission rate from pairs to susceptible snails is a constant $\xi$.
(v) $\tau$ is the latent period of infected snails, and $e^{-\left(\mu_{s}+\alpha_{s}\right) \tau}$ represents the survival rate of infected snails in the latent period.
(vi) We suppose that infected snails do not recover from schistosomiasis as their life spans are short.
(vii) All parameters are assumed nonnegative in reality. Then, we have a model with the following form:

$$
\begin{align*}
\frac{d m}{d t} & =k_{m} i(t-\tau) e^{-\left(\mu_{s}+\alpha_{s}\right) \tau}-\mu_{m} m(t)-\varphi(m(t), f(t)) \\
\frac{d f}{d t} & =k_{f} i(t-\tau) e^{-\left(\mu_{s}+\alpha_{s}\right) \tau}-\mu_{f} f(t)-\varphi(m(t), f(t)) \\
\frac{d p}{d t} & =\varphi(m(t), f(t))-\mu_{p} p(t) \\
\frac{d s}{d t} & =\Lambda_{s}-\mu_{s} s(t)-\xi p(t) s(t) \\
\frac{d i}{d t} & =\xi p(t) s(t)-\left(\mu_{s}+\alpha_{s}\right) i(t) \tag{2}
\end{align*}
$$

For convenience of the stability analysis of (2), we first study the case when $\tau=0$, that is,

$$
\begin{align*}
\frac{d m}{d t} & =k_{m} i(t)-\mu_{m} m(t)-\varphi(m(t), f(t)) \\
\frac{d f}{d t} & =k_{f} i(t)-\mu_{f} f(t)-\varphi(m(t), f(t)) \\
\frac{d p}{d t} & =\varphi(m(t), f(t))-\mu_{p} p(t) \\
\frac{d s}{d t} & =\Lambda_{s}-\mu_{s} s(t)-\xi p(t) s(t) \\
\frac{d i}{d t} & =\xi p(t) s(t)-\left(\mu_{s}+\alpha_{s}\right) i(t) \tag{3}
\end{align*}
$$

In the current paper, we take the mating function $\varphi(m(t)$, $f(t))=\rho \min (m(t), f(t))[5,6]$. The function is linear on each of the following two sets:

$$
\begin{align*}
K_{f}= & \left\{(m, f, p, s, i) \in R^{5} \mid f \geq m \geq 0\right. \\
& \left.p \geq 0, s \geq 0, i \geq 0,0 \leq s+i \leq \frac{\Lambda_{s}}{\mu_{s}}\right\} \\
K_{m}= & \left\{(m, f, p, s, i) \in R^{5} \mid m \geq f \geq 0\right.  \tag{4}\\
& \left.p \geq 0, s \geq 0, i \geq 0,0 \leq s+i \leq \frac{\Lambda_{s}}{\mu_{s}}\right\}
\end{align*}
$$

In general, none of these sets is positively invariant for (3) with $\varphi(m(t), f(t))=\rho \min (m(t), f(t))$. But under the conditions

$$
\begin{equation*}
k_{m} \geq k_{f}, \quad \mu_{p}<\mu_{m} \leq \mu_{f} \tag{5}
\end{equation*}
$$

the set $K_{m}$ is positively invariant. Thus, the model (3) with an initial surplus of males becomes the so-called female dominance model on the set $K_{m}$ and the system (3) can be rewritten in the following form:

$$
\begin{align*}
\frac{d m}{d t} & =k_{m} i(t)-\mu_{m} m(t)-\rho f(t) \\
\frac{d f}{d t} & =k_{f} i(t)-\mu_{f} f(t)-\rho f(t) \\
\frac{d p}{d t} & =\rho f(t)-\mu_{p} p(t)  \tag{6}\\
\frac{d s}{d t} & =\Lambda_{s}-\mu_{s} s(t)-\xi p(t) s(t) \\
\frac{d i}{d t} & =\xi p(t) s(t)-\left(\mu_{s}+\alpha_{s}\right) i(t)
\end{align*}
$$

Using the standard method, it is easy to see that the disease-free equilibrium $E_{0}=\left(0,0,0, \Lambda_{s} / \mu_{s}, 0\right)$ always exists. According to the calculational method of the basic
reproductive number [28], the basic reproduction number for model (6) is

$$
\begin{equation*}
R_{0}=\sqrt[3]{\frac{\xi \rho \Lambda_{s} k_{f}}{\mu_{s}\left(\mu_{s}+\alpha_{s}\right) \mu_{p}\left(\mu_{f}+\rho\right)}} \tag{7}
\end{equation*}
$$

The following section shows that the basic reproductive number $R_{0}$ provides a threshold condition for parasite extinction in (6).

## 3. Stability Analysis of the System (6)

In this section, we will analyze the stability of model (6). The stability of the disease-free equilibrium determines whether the disease will be prevalent in an uninfected population. The following result shows that the parasites will go extinct if $R_{0}<$ 1.

Theorem 1. The disease-free equilibrium $E_{0}$ of the system (6) is locally asymptotically stable if $R_{0}<1$ and unstable if $R_{0}>1$.

Proof. The Jacobian matrix for system (6) is given by

$$
J=\left(\begin{array}{ccccc}
-\mu_{m} & -\rho & 0 & 0 & k_{m}  \tag{8}\\
0 & -\left(\mu_{f}+\rho\right) & 0 & 0 & k_{f} \\
0 & \rho & -\mu_{p} & 0 & 0 \\
0 & 0 & -\xi_{s}(t) & -\mu_{s}-\xi_{p}(t) & 0 \\
0 & 0 & \xi_{s}(t) & \xi_{p}(t) & -\left(\mu_{s}+\alpha_{s}\right)
\end{array}\right) .
$$

Thus, the eigenvalues of $E_{0}$ are $-\mu_{s},-\mu_{m}$, and the roots of the equation are

$$
\begin{equation*}
\lambda^{3}+a_{1} \lambda^{2}+a_{2} \lambda+a_{3}=0 \tag{9}
\end{equation*}
$$

where

$$
\begin{align*}
& a_{1}=\left(\mu_{s}+\alpha_{s}\right)+\mu_{p}+\left(\mu_{f}+\rho\right)>0 \\
& a_{2}=\left(\mu_{s}+\alpha_{s}\right) \mu_{p}+\left(\mu_{s}+\alpha_{s}\right)\left(\mu_{f}+\rho\right)+\mu_{p}\left(\mu_{f}+\rho\right)>0, \\
& a_{3}=\left(\mu_{s}+\alpha_{s}\right) \mu_{p}\left(\mu_{f}+\rho\right)-\frac{\xi \rho \Lambda_{s} k_{f}}{\mu_{s}} . \tag{10}
\end{align*}
$$

Note that $a_{3}>0$ if and only if $R_{0}<1$.
By the Routh-Hurwitz criterion, we know

$$
\begin{align*}
& H_{1}=a_{1}>0, \\
& H_{2}=\left|\begin{array}{cc}
a_{1} & a_{3} \\
1 & a_{2}
\end{array}\right|>0,  \tag{11}\\
& H_{3}=\left|\begin{array}{ccc}
a_{1} & a_{3} & 0 \\
1 & a_{2} & 0 \\
0 & a_{1} & a_{3}
\end{array}\right|=a_{3} H_{2},
\end{align*}
$$

which implies that all the eigenvalues of $E_{0}$ have negative real parts if and only if $R_{0}<1$. It follows that the disease-free equilibrium $E_{0}$ is locally asymptotically stable if $R_{0}<1$ and unstable if $R_{0}>1$.

Note that if $R_{0}>1,(6)$ has a unique endemic equilibrium $E^{*}=\left(m^{*}, f^{*}, p^{*}, s^{*}, i^{*}\right)$, where

$$
\begin{align*}
s^{*} & =\frac{\left(\mu_{s}+\alpha_{s}\right) \mu_{p}\left(\mu_{f}+\rho\right)}{\xi \rho k_{f}} \\
p^{*} & =\frac{\Lambda_{s} \rho k_{f}}{\left(\mu_{s}+\alpha_{s}\right) \mu_{p}\left(\mu_{f}+\rho\right)}-\frac{\mu_{s}}{\xi}=\frac{\mu_{s}}{\xi}\left(R_{0}^{3}-1\right) \\
f^{*} & =\frac{\mu_{p}}{\rho} p^{*}  \tag{12}\\
i^{*} & =\frac{\mu_{p}\left(\mu_{f}+\rho\right)}{\rho k_{f}} p^{*} \\
m^{*} & =\frac{k_{m}\left(\mu_{f}+\rho\right)-\rho k_{f}}{k_{f} \mu_{m}} \frac{\mu_{p}}{\rho} p^{*}
\end{align*}
$$

Similarly, using the Routh-Hurtwitz criterion we can obtain the stability of the endemic equilibrium $E^{*}$.

Theorem 2. The endemic equilibrium $E^{*}$ of the system (6) is locally asymptotically stable if $R_{0}>1$.

## 4. Stability Analysis of the System (2)

In this section, we analyze the stability of model (2). We recall it with the following form:

$$
\begin{align*}
\frac{d m}{d t} & =k_{m} i(t-\tau) e^{-\left(\mu_{s}+\alpha_{s}\right) \tau}-\mu_{m} m(t)-\rho f(t) \\
\frac{d f}{d t} & =k_{f} i(t-\tau) e^{-\left(\mu_{s}+\alpha_{s}\right) \tau}-\mu_{f} f(t)-\rho f(t) \\
\frac{d p}{d t} & =\rho f(t)-\mu_{p} p(t)  \tag{13}\\
\frac{d s}{d t} & =\Lambda_{s}-\mu_{s} s(t)-\xi p(t) s(t) \\
\frac{d i}{d t} & =\xi p(t) s(t)-\left(\mu_{s}+\alpha_{s}\right) i(t) .
\end{align*}
$$

Define

$$
\begin{equation*}
\widetilde{R}_{0}=\frac{\xi \rho \Lambda_{s} k_{f} e^{-\left(\mu_{s}+\alpha_{s}\right) \tau}}{\mu_{s}\left(\mu_{s}+\alpha_{s}\right) \mu_{p}\left(\mu_{f}+\rho\right)} . \tag{14}
\end{equation*}
$$

We can obtain that the disease-free equilibrium $\widetilde{E}_{0}=$ $\left(0,0,0, \Lambda_{s} / \mu_{s}, 0\right)$ always exists and if $\widetilde{R}_{0}>1$ there exists the endemic equilibrium $\widetilde{E}^{*}=\left(\widetilde{m}^{*}, \widetilde{f}^{*}, \widetilde{p}^{*}, \widetilde{s}^{*}, \widetilde{i}^{*}\right)$, where

$$
\begin{aligned}
& \widetilde{s}^{*}=\frac{\left(\mu_{s}+\alpha_{s}\right) \mu_{p}\left(\mu_{f}+\rho\right)}{\xi \rho k_{f} e^{-\left(\mu_{s}+\alpha_{s}\right) \tau}} \\
& \widetilde{p}^{*}=\frac{\Lambda_{s} \rho k_{f} e^{-\left(\mu_{s}+\alpha_{s}\right) \tau}}{\left(\mu_{s}+\alpha_{s}\right) \mu_{p}\left(\mu_{f}+\rho\right)}-\frac{\mu_{s}}{\xi}=\frac{\mu_{s}}{\xi}\left(\widetilde{R}_{0}-1\right), \\
& \widetilde{f}^{*}=\frac{\mu_{p}}{\rho} \widetilde{p}^{*}
\end{aligned}
$$

$$
\begin{align*}
\widetilde{i}^{*} & =\frac{\mu_{p}\left(\mu_{f}+\rho\right)}{\rho k_{f} e^{-\left(\mu_{s}+\alpha_{s} \tau\right.}} \widetilde{p}^{*}, \\
\widetilde{m}^{*} & =\frac{k_{m}\left(\mu_{f}+\rho\right)-\rho k_{f}}{k_{f} \mu_{m}} \frac{\mu_{\rho}}{\rho} \widetilde{p}^{*} . \tag{15}
\end{align*}
$$

Theorem 3. If $\widetilde{R}_{0}<1$, the disease-free equilibrium $\widetilde{E}_{0}$ of the system (13) is locally asymptotically stable.

Proof. The Jacobian matrix for system (13) is given by

$$
|J-\lambda I|=\left\lvert\, \begin{array}{cc}
-\left(\mu_{m}+\frac{\sigma}{\theta}\right) & -\rho  \tag{16}\\
0 & -\left(\mu_{f}+\rho+\frac{\sigma}{\theta}\right) \\
0 & \rho \\
0 & 0 \\
0 & 0
\end{array}\right.
$$

$$
\left.\begin{array}{ccc}
0 & 0 & k_{m} e^{-\left(\mu_{s}+\alpha_{s}\right) \tau} e^{-\lambda} \\
0 & 0 & k_{f} e^{-\left(\mu_{s}+\alpha_{s}\right) \tau} e^{-\lambda} \\
-\left(\mu_{p}+\frac{\sigma}{\theta}\right) & 0 & 0 \\
-\xi_{s}(t) & -\mu_{s}-\xi_{p}(t) & 0 \\
\xi_{s}(t) & \xi_{p}(t) & -\left(\mu_{s}+\alpha_{s}\right)
\end{array} \right\rvert\,
$$

Thus, the eigenvalues of $\widetilde{E}_{0}$ are $-\mu_{s},-\mu_{m}$ and the roots of the equation are

$$
\begin{equation*}
\lambda^{3}+\widetilde{a}_{1} \lambda^{2}+\widetilde{a}_{2} \lambda+\widetilde{a}_{3}=0 \tag{17}
\end{equation*}
$$

where

$$
\begin{align*}
& \widetilde{a}_{1}=\left(\mu_{s}+\alpha_{s}\right)+\mu_{p}+\left(\mu_{f}+\rho\right)>0 \\
& \widetilde{a}_{2}=\left(\mu_{s}+\alpha_{s}\right) \mu_{p}+\left(\mu_{s}+\alpha_{s}\right)\left(\mu_{f}+\rho\right)+\mu_{p}\left(\mu_{f}+\rho\right)>0 \\
& \widetilde{a}_{3}=\left(\mu_{s}+\alpha_{s}\right) \mu_{p}\left(\mu_{f}+\rho\right)-\frac{\xi \rho \Lambda_{s} k_{f}}{\mu_{s}} e^{-\left(\mu_{s}+\alpha_{s}\right) \tau} e^{-\lambda} \tag{23}
\end{align*}
$$

If $\lambda \geq 0$ and $\tau>0,0<e^{-\lambda} \leq 1$, and then

$$
\begin{equation*}
\widetilde{a}_{3} \geq\left(\mu_{s}+\alpha_{s}\right) \mu_{p}\left(\mu_{f}+\rho\right)-\frac{\xi \rho \Lambda_{s} k_{f}}{\mu_{s}} e^{-\left(\mu_{s}+\alpha_{s}\right) \tau} \tag{19}
\end{equation*}
$$

Note that $\widetilde{a}_{3}>0$ if $\widetilde{R}_{0}<1$.
Thus, the left-hand side in (17) is positive for all $\lambda \geq 0$ and $\tau>0$ while the right-hand side is zero. This leads to a contradiction. Then, (17) does not have non-negative real solutions. Following the proof of Theorem 1, we know that (17) has eigenvalues with negative real parts when $\tau=0$. Hence, for the case $\tau>0$, if (17) has roots with non-negative real parts they must be complex roots. Moreover, these complex roots should be obtained from a pair of complex conjugate roots crossing the imaginary axis. Thus, (17) must have a pair of purely imaginary roots.

Suppose $\lambda=\omega i(\omega>0)$ is a root of (17). Then, we have

$$
\begin{align*}
-\omega^{3} i & -\widetilde{a}_{1} \omega^{2}+\widetilde{a}_{2} \omega i+\widehat{a}_{3} \\
& =\frac{\xi \rho \Lambda_{s} k_{f}}{\mu_{s}} e^{-\left(\mu_{s}+\alpha_{s}\right) \tau}(\cos (\omega \tau)-i \sin (\omega \tau)) \tag{20}
\end{align*}
$$

where

$$
\begin{equation*}
\widehat{a}_{3}=\left(\mu_{s}+\alpha_{s}\right) \mu_{p}\left(\mu_{f}+\rho\right) \tag{21}
\end{equation*}
$$

Therefore,

$$
\begin{gather*}
-\omega^{3}+\widetilde{a}_{2} \omega=-\frac{\xi \rho \Lambda_{s} k_{f}}{\mu_{s}} e^{-\left(\mu_{s}+\alpha_{s}\right) \tau} \sin (\omega \tau)\left(b_{3}-b_{1} \omega^{2}\right) \\
-\widetilde{a}_{1} \omega^{2}+\widehat{a}_{3}=\frac{\xi \rho \Lambda_{s} k_{f}}{\mu_{s}} e^{-\left(\mu_{s}+\alpha_{s}\right) \tau} \cos (\omega \tau) \tag{22}
\end{gather*}
$$

From (22), we can get

$$
\begin{equation*}
\left(-\omega^{3}+\widetilde{a}_{2} \omega\right)^{2}+\left(-\widetilde{a}_{1} \omega^{2}+\widehat{a}_{3}\right)^{2}=\left(\frac{\xi \rho \Lambda_{s} k_{f}}{\mu_{s}} e^{-\left(\mu_{s}+\alpha_{s}\right) \tau}\right)^{2} \tag{18}
\end{equation*}
$$

that is,

$$
\begin{equation*}
\omega^{6}+c_{1} \omega^{4}+c_{2} \omega^{2}+c_{3}=0 \tag{24}
\end{equation*}
$$

Letting $z=\omega^{2}$, we obtain

$$
\begin{equation*}
z^{3}+c_{1} z^{2}+c_{2} z+c_{3}=0 \tag{25}
\end{equation*}
$$

where

$$
\begin{align*}
c_{1} & =\widetilde{a}_{1}^{2}-2 \widetilde{a}_{2}>0, \\
c_{2} & =\widetilde{a}_{2}^{2}-2 \widetilde{a}_{1} \widehat{a}_{3}>0 \\
c_{3} & =\widehat{a}_{3}^{2}-\left(\frac{\xi \rho \Lambda_{s} k_{f}}{\mu_{s}} e^{-\left(\mu_{s}+\alpha_{s}\right) \tau}\right)^{2}  \tag{26}\\
& =\widehat{a}_{3}\left(\widehat{a}_{3}+\frac{\xi \rho \Lambda_{s} k_{f}}{\mu_{s}} e^{-\left(\mu_{s}+\alpha_{s}\right) \tau}\right)\left(1-\widetilde{R}_{0}\right) .
\end{align*}
$$

If $\widetilde{R}_{0}<1, c_{1}>0, c_{2}>0$, and $c_{3}>0$, which implies that (25) does not have positive solutions, then (24) does not have purely imaginary solutions. Hence, If $\widetilde{R}_{0}<1$, the diseasefree equilibrium $\widetilde{E}_{0}$ of the system (13) is locally asymptotically stable.

Remark 4. If $R_{0}>1$,

$$
\begin{equation*}
\widetilde{R}_{0}<1 \Longleftrightarrow \tau>\frac{3}{\mu_{s}+\alpha_{s}} \ln R_{0} \triangleq \tau_{0} . \tag{27}
\end{equation*}
$$

It is easy to see if $R_{0}<1$, we can get $\widetilde{R}_{0}<1$. But if $R_{0}>1$ to get $\widetilde{R}_{0}<1$ the time delay must satisfy $\tau>\tau_{0}$. This implies that to eliminate the disease the time delay must satisfy $\tau>\tau_{0}$. In fact, the time delay is decreasing as the global warming. If the time delay is decreased to be smaller than $\tau_{0}, \widetilde{R}_{0}$ may be larger than 1 , and then the disease may be prevalent.

Now, we turn to the study of the stability of the endemic equilibrium of model (13).

Theorem 5. If $\widetilde{R}_{0}>1$, the endemic equilibrium $\widetilde{E}^{*}$ of the system (13) is locally asymptotically stable.

Proof. The eigenvalues of $\widetilde{E}^{*}$ are $-\mu_{m}$ and the roots of the equation are

$$
\begin{equation*}
\lambda^{4}+b_{1} \lambda^{3}+b_{2} \lambda^{2}+b_{3} \lambda+b_{4}=0 \tag{28}
\end{equation*}
$$

where

$$
\begin{align*}
b_{1}= & \left(\mu_{s}+\alpha_{s}\right)+\mu_{p}+\left(\mu_{f}+\rho\right)+\left(\mu_{s}+\xi \widetilde{p}^{*}\right)>0, \\
b_{2}= & \left(\mu_{s}+\alpha_{s}\right) \mu_{p}+\left(\mu_{s}+\alpha_{s}\right)\left(\mu_{f}+\rho\right)+\left(\mu_{s}+\alpha_{s}\right)\left(\mu_{s}+\xi \widetilde{p}^{*}\right) \\
& +\mu_{p}\left(\mu_{f}+\rho\right)+\mu_{p}\left(\mu_{s}+\xi \widetilde{p}^{*}\right)+\left(\mu_{f}+\rho\right)\left(\mu_{s}+\xi \widetilde{p}^{*}\right), \\
b_{3}= & \left(\mu_{s}+\alpha_{s}\right) \mu_{p}\left(\mu_{f}+\rho\right)+\left(\mu_{s}+\alpha_{s}\right) \mu_{p}\left(\mu_{s}+\xi \widetilde{p}^{*}\right) \\
& +\mu_{p}\left(\mu_{f}+\rho\right)\left(\mu_{s}+\xi \widetilde{p}^{*}\right)-\xi \rho k_{f} \widetilde{s}^{*} e^{-\left(\mu_{s}+\alpha_{s}\right) \tau} e^{-\lambda}, \\
b_{4}= & \left(\mu_{s}+\alpha_{s}\right) \mu_{p}\left(\mu_{f}+\rho\right)\left(\mu_{s}+\xi \widetilde{p}^{*}\right) \\
& -\xi \rho k_{f} \mu_{s} \widetilde{s}^{*} e^{-\left(\mu_{s}+\alpha_{s}\right) \tau} e^{-\lambda} . \tag{29}
\end{align*}
$$

Assume that $\lambda \geq 0$. Since $\xi \rho k_{f} \widetilde{s}^{*} e^{-\left(\mu_{s}+\alpha_{s}\right) \tau}=\left(\mu_{s}+\right.$ $\left.\alpha_{s}\right) \mu_{p}\left(\mu_{f}+\rho\right)$ and $0<e^{-\lambda} \leq 1$ for $\tau>0$, we have $b_{1}>0, b_{2}>0$, $b_{3}>0$, and $b_{4}>0$ which lead to a contradiction in (28). Then, (28) does not have non-negative real solutions. From Theorem 2, we know that (28) has solutions with negative real parts for $\tau=0$. Hence, for the case $\tau>0$, if (28) has roots with non-negative real parts they must be complex roots. Moreover these complex roots should be obtained from a pair of complex conjugate roots crossing the imaginary axis. Thus, (28) must have a pair of purely imaginary roots.

Suppose $\lambda=\omega i(\omega>0)$ is a root of (28). Then, we have

$$
\begin{align*}
& \omega^{4}-b_{1} \omega^{3} i-b_{2} \omega^{2}+\widehat{b}_{3} \omega i+\widehat{b}_{4} \\
& \quad=\xi \rho k_{f} \widetilde{s}^{*} e^{-\left(\mu_{s}+\alpha_{s}\right) \tau}\left(\mu_{s}+\omega i\right)(\cos (\omega \tau)-i \sin (\omega \tau)), \tag{30}
\end{align*}
$$

where

$$
\begin{align*}
\widehat{b}_{3}= & \left(\mu_{s}+\alpha_{s}\right) \mu_{p}\left(\mu_{f}+\rho\right)+\left(\mu_{s}+\alpha_{s}\right) \mu_{p}\left(\mu_{s}+\xi \widetilde{p}^{*}\right) \\
& +\mu_{p}\left(\mu_{f}+\rho\right)\left(\mu_{s}+\xi \widetilde{p}^{*}\right)  \tag{31}\\
\widehat{b}_{4}= & \left(\mu_{s}+\alpha_{s}\right) \mu_{p}\left(\mu_{f}+\rho\right)\left(\mu_{s}+\xi \widetilde{p}^{*}\right)
\end{align*}
$$

Therefore,

$$
\begin{align*}
& \omega^{4}- b_{2} \omega^{2}+\widehat{b}_{4} \\
&=\xi \rho k_{f} \widetilde{s}^{*} e^{-\left(\mu_{s}+\alpha_{s}\right) \tau}\left(\mu_{s} \cos (\omega \tau)+\omega \sin (\omega \tau)\right),  \tag{32}\\
&-b_{1} \omega^{3}+\widehat{b}_{3} \omega \\
&=\xi \rho k_{f} \widetilde{s}^{*} e^{-\left(\mu_{s}+\alpha_{s}\right) \tau}\left(\omega \cos (\omega \tau)-\mu_{s} \sin (\omega \tau)\right) .
\end{align*}
$$

From (32), we can get

$$
\begin{align*}
\left(\omega^{4}-\right. & \left.b_{2} \omega^{2}+\widehat{b}_{4}\right)^{2}+\left(-b_{1} \omega^{3}+\widehat{b}_{3} \omega\right)^{2}  \tag{33}\\
& =\left(\xi \rho k_{f} \widetilde{s}^{*} e^{-\left(\mu_{s}+\alpha_{s}\right) \tau}\right)^{2}\left(\mu_{s}^{2}+\omega^{2}\right) .
\end{align*}
$$

Letting $z=\omega^{2}$ again, we obtain

$$
\begin{equation*}
z^{4}+c_{1} z^{3}+c_{2} z^{2}+c_{3} z+c_{4}=0 \tag{34}
\end{equation*}
$$

where

$$
\begin{align*}
& c_{1}=b_{1}^{2}-2 b_{2}, \\
& c_{2}=b_{2}^{2}+2 \widehat{b}_{4}-2 b_{1} \widehat{b}_{3}, \\
& c_{3}=\widehat{b}_{3}^{2}-2 b_{2} \widehat{b}_{4}-\left(\xi \rho k_{f} s^{*} e^{-\left(\mu_{s}+\alpha_{s}\right) \tau}\right)^{2},  \tag{35}\\
& c_{4}=\widehat{b}_{4}^{2}-\left(\xi \rho k_{f} s^{*} e^{-\left(\mu_{s}+\alpha_{s}\right) \tau}\right)^{2} \mu_{s}^{2} .
\end{align*}
$$

By similar calculations, we can get $c_{1}>0, c_{2}>0, c_{3}>0$, and $c_{4}>0$. Thus, (34) does not have positive roots and then (28) cannot have purely imaginary solutions. Hence, If $\widetilde{R}_{0}>1$ the endemic equilibrium $\widetilde{E}^{*}$ of the system (13) is locally asymptotically stable.

## 5. Simulation and Sensitivity Analysis

In this section, we perform some computational simulations to observe the impact of the time delay on schistosomiasis dynamics. Through sensitivity analysis of the death rate of snails, the death rate of single parasites, and the death rate of pairs, we give preferable control strategies.

The parameters are chosen with $k_{f}=100$ per year, $k_{m}=$ 145 per year $[13,18], \Lambda_{s}=150$ per year, $\xi=0.000018$ per year [18], $\mu_{s}=0.1$ per year, $\alpha_{s}=0.5$ per year [29], $\mu_{f}=0.2$ per year, $\mu_{m}=0.1$ per year, $\mu_{p}=0.02$ per year, $\rho=0.467$ [9] per year and $\tau=18 / 365 \sim 38 / 365 \approx 0.04 \sim 0.1$ per year [4]. Then $R_{0}=12.55, \widetilde{R}_{0}=149.9$ and $\tau_{0}=8.43$ when we choose $\tau=30 / 365=0.082$.

By Remark 4, the time delay $\tau=0.082<8.43=\tau_{0}$ and $\widetilde{R}_{0}>1$ implies that the disease will be prevalent. From Figure 1 we can find, the number of male schistosomes and infected snails increases as the time delay decreases. These phenomena imply that schistosomiasis infection becomes serious as the time delay decreases. As we know, the temperature of snails changes as the environmental temperature changes. Then the environmental temperature can influence the incubation period in infected snails. Currently, the environmental temperature is increasing because of the global warming. Hence, the global warming may reduce the incubation period of infected snails. Consequently, schistosomiasis may become more serious because of the global warming. Hence, the impact of global warming and the incubation period of infected snails on schistosomiasis dynamics can not be ignored.

As for schistosomiasis control, we can control the disease by killing schistosoma and (or) snails. In reality, there are many drugs for killing schistosoma. When we treat definitive hosts with chemotherapy, different drugs aim at different parasites. For example, male schistosoma is more sensitive to praziquantel (PZQ) [30,31], but pair schistosoma is more sensitive to artesunate (ART) [32]. Hence, sometimes the death rate of pairs and single schistosoma is increased under some treatments, but the extent of increase is different. Under the treatment of some drugs more pairs are killed, but under the treatment of another drugs, more single schistosomes are killed. However, it is a pity that we lack the data about the rate of killing schistosoma to numerical simulation. Hence, we have done some qualitative analyses from the perspective of sensitivity.

Denote by $\varepsilon_{m}, \varepsilon_{f}, \varepsilon_{p}$ and $\varepsilon_{s}$ the killing rates of male schistosoma, female schistosoma, paired schistosoma and snails, respectively. We modify the model by adding these killing rates with the following form:

$$
\begin{align*}
\frac{d m}{d t} & =k_{m} i(t)-\left(\mu_{m}+\varepsilon_{m}\right) m(t)-\rho f(t) \\
\frac{d f}{d t} & =k_{f} i(t)-\left(\mu_{f}+\varepsilon_{f}\right) f(t)-\rho f(t) \\
\frac{d p}{d t} & =\rho f(t)-\left(\mu_{p}+\varepsilon_{p}\right) p(t)  \tag{36}\\
\frac{d s}{d t} & =\Lambda_{s}-\left(\mu_{s}+\varepsilon_{s}\right) s(t)-\xi p(t) s(t) \\
\frac{d i}{d t} & =\xi p(t) s(t)-\left(\mu_{s}+\varepsilon_{s}+\alpha_{s}\right) i(t)
\end{align*}
$$

For convenience of computation, we denote

$$
\begin{align*}
\mu_{m}+\varepsilon_{m}=\mu_{m \varepsilon}, & \mu_{f}+\varepsilon_{f}=\mu_{f \varepsilon}  \tag{37}\\
\mu_{p}+\varepsilon_{p}=\mu_{p \varepsilon}, & \mu_{s}+\varepsilon_{s}=\mu_{s \varepsilon} .
\end{align*}
$$

Similarly, we can obtain that the basic reproduction number for model (36) is as follows

$$
\begin{equation*}
R_{0 \varepsilon}=\sqrt[3]{\frac{\xi \rho \Lambda_{s} k_{f}}{\mu_{s \varepsilon}\left(\mu_{s \varepsilon}+\alpha_{s}\right) \mu_{p \varepsilon}\left(\mu_{f \varepsilon}+\rho\right)}} \tag{38}
\end{equation*}
$$

Note that $R_{0 \varepsilon}$ is independent of the death rate of single male schistosoma. It is easy to see that $R_{0 \varepsilon} \leq R_{0}$, which implies that killing schistosoma and (or) snails can reduce the basic reproduction number and consequently control the disease if $R_{0 \varepsilon}<1$. However, we should choose a rather effective control measure in reality. Thus, we perform some sensitivity analyses by comparing the flexibilities of killing rates. The flexibilities of $\mu_{s \varepsilon}, \mu_{f \varepsilon}$, and $\mu_{p \varepsilon}$ on the basic reproduction number $R_{0 \varepsilon}$, respectively, are

$$
\begin{gather*}
\frac{E R_{0 \varepsilon}}{E \mu_{s \varepsilon}}=\frac{\partial R_{0 \varepsilon}}{\partial \mu_{s \varepsilon}} \times \frac{\mu_{s \varepsilon}}{R_{0 \varepsilon}}=-\frac{1}{3} \frac{2 \mu_{s \varepsilon}+\alpha_{s}}{\mu_{s \varepsilon}+\alpha_{s}} \\
\frac{E R_{0 \varepsilon}}{E \mu_{f \varepsilon}}=\frac{\partial R_{0 \varepsilon}}{\partial \mu_{f \varepsilon}} \times \frac{\mu_{f \varepsilon}}{R_{0 \varepsilon}}=-\frac{1}{3} \frac{\mu_{f \varepsilon}}{\mu_{f \varepsilon}+\rho}  \tag{39}\\
\quad \frac{E R_{0 \varepsilon}}{E \mu_{p \varepsilon}}=\frac{\partial R_{0 \varepsilon}}{\partial \mu_{p \varepsilon}} \times \frac{\mu_{p \varepsilon}}{R_{0 \varepsilon}}=-\frac{1}{3}
\end{gather*}
$$

It is easy to see that

$$
\begin{equation*}
\left|\frac{E R_{0 \varepsilon}}{E \mu_{f \varepsilon}}\right|<\frac{1}{3}=\left|\frac{E R_{0 \varepsilon}}{E \mu_{p \varepsilon}}\right|<\left|\frac{E R_{0 \varepsilon}}{E \mu_{s \varepsilon}}\right| \tag{40}
\end{equation*}
$$

It implies that the death rate of snails is the most sensitive parameter to the reduction of $R_{0 \varepsilon}$. In addition, the death rate of pair schistosoma is more sensitive than that of female single schistosoma. This is reasonable because the offspring is reproduced by pair schistosoma and the pair schistosoma has stronger effect on the life cycle of schistosome. Hence, to kill the sail hosts may be the first reasonable control measure. Furthermore, to kill pair schistosoma by using some medications may be the other better control measure.

## 6. Discussion

In this paper, we established a new schistosomiasis model including a more reasonable sex ratio of schistosoma, snail dynamics, the latent period of infected snails and mating structure. By choosing the minimum function as the mating function, we studied the stabilities of model (6) without time delay and model (13) with time delay. If the basic reproduction number of model (6) is less than 1 , one can prove the stability of the disease-free equilibrium in system (6). When the basic reproductive number $R_{0}$ is greater than 1, the stability of the endemic equilibrium in system (6) can be obtained. But to get the stability of the disease-free equilibrium in system (13) the time delay must be larger than $\tau_{0}$. In reality, the environmental temperature because of the global warming is increasing. And then the incubation period of infected snails is shortened as the environmental temperature increases [13]. From Figure 1 we know that schistosomiasis infection become more serious as the incubation period of infected snails decreases. Hence, the impact of the incubation period of infected snails during global warming on schistosomiasis transmission cannot be ignored.

In recent years, the control of schistosomiasis remains one of the highest priorities in parasitology. There are many strategies to control schistosomiasis, such as schistosome


Figure 1: The dot line represents $\tau=0.041$, and the real line represents $\tau=0.082$.
vaccine, killing snails and chemotherapy with PZQ. In this paper, through comparing the flexibilities of $\mu_{s \varepsilon}, \mu_{f \varepsilon}$, and $\mu_{p \varepsilon}$ on $R_{0 \varepsilon}$, we know that the death rate of snails $\mu_{s \varepsilon}$ is the most sensitive to the reduction of $R_{0 \varepsilon}$. Hence killing the snail hosts may be the most reasonable control measure.

Our sensitivity analysis also deduces that $R_{0 \varepsilon}$ is more sensitive to the death rate of pair parasites than to the death rate of female parasite. This implies that killing paired schistosoma is more advantageous than killing single schistosoma to control disease. From the perspective of medicine, eggs are generally accepted as the major cause of pathogenesis of schistosomiasis [14, 30-32]. Nino Incani et al. [14] suggest that the eggs produced by paired schistosoma are the pivotal factor in nosogenesis and transmission, and sexual maturation is the precondition of producing eggs. This means that paired schistosoma is mostly harmful to definitive hosts. Controlling schistosomiasis must attack the ability of the paired schistosoma to produce eggs. Hence, the drug of choice is possibly sufficient for reducing egg-associated pathology.

As we know, PZQ is currently considered the drug of choice for the treatment of schistosomiasis because of its efficacy against all schistosome species, lack of serious side effects, and low cost. However, PZQ has its limitation. PZQ was found to be more active against single male schistosoma [30, 31], not paired schistosomes. In paired schistosoma, male schistosoma are folded together with female schistosoma. The egg production is owed to the paired schistosomes. Reducing male schistosoma in paired schistosoma can help reduce the egg production and consequently the overall parasite density and disease prevalence. In additional, from our analysis, $R_{0 \varepsilon}$ is sensitive to the death rate of pair schistosoma and independent of the death rate of single male schistosoma. Furthermore, recent epidemiological evidence suggests the emergence of PZQ-resistant schistosoma [33]. Hence, PZQ may not be the best drug for schistosomiasis.

Currently, there are some drugs which are possibly sufficient for reducing egg-associated pathology. For example, ART significantly decreased the survival time of both paired male and female schistosomes. It might be responsible for damaging reproductive organs or killing schistosoma [32]. Oxamniquine-praziquantel composite in the treatment of schistosoma mansoni infection is also a controlled trial [31].

In summary, although PZQ remains the drug of choice to treat schistosomiasis, it does not protect from reinfection (especially in children) and is minimally effective against larval stages of the parasite. Resistance can also develop, although its mode of action is poorly understood. Therefore an important priority in developing new control strategies is to search new drug targets, in combination with selection of viable vaccine candidates. Based on our analysis in this paper, we can obtain the following two results. Killing the snail hosts may be the most reasonable control measure. If we choose chemotherapy, we should choose some drugs which are sufficient for reducing egg-associated pathology because paired schistosoma is mostly harmful to definitive hosts.

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## Research Article

# Dynamics in a Lotka-Volterra Predator-Prey Model with Time-Varying Delays 

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#### Abstract

A Lotka-Volterra predator-prey model with time-varying delays is investigated. By using the differential inequality theory, some sufficient conditions which ensure the permanence and global asymptotic stability of the system are established. The paper ends with some interesting numerical simulations that illustrate our analytical predictions.


## 1. Introduction

In 1992, Berryman [1] pointed out that the dynamical relationship between predators and their prey has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance. Dynamical behavior of predatorprey models has been studied by a lot of papers. It is well known that the investigation on predator-prey models not only focuses on the discussion of stability, periodic oscillatory, bifurcation, and chaos [2-26], but also involves many other dynamical behaviors such as permanence. In many applications, the nature of permanence is of great interest. Recently, Chen [27] investigated the permanence of a discrete $n$-species food-chain system with delays. Fan and Li [28] gave a theoretical study on permanence of a delayed ratiodependent predator-prey model with Holling type functional response. Chen [29] focused on the permanence and global attractivity of Lotka-Volterra competition system with feedback control. Zhao and Jiang [30] analyzed the permanence and extinction for nonautonomous Lotka-Volterra system. Teng et al. [31] addressed the permanence criteria for delayed discrete nonautonomous-species Kolmogorov systems. For more research on the permanence behavior of predator-prey models, one can see [32-40].

In 2010, Lv et al. [41] investigated the existence and global attractivity of periodic solution to the following LotkaVolterra predator-prey system:

$$
\begin{align*}
& \frac{d x_{1}(t)}{d t} \\
& =x_{1}(t)\left[r_{1}(t)-a_{11}(t) x_{1}\left(t-\tau_{11}(t)\right)\right. \\
& \left.\quad-a_{12}(t) x_{2}\left(t-\tau_{12}(t)\right)-a_{13}(t) x_{3}\left(t-\tau_{13}(t)\right)\right], \\
& \begin{aligned}
\frac{d x_{2}(t)}{d t}
\end{aligned} \\
& =x_{2}(t)\left[-r_{2}(t)+a_{21}(t) x_{1}\left(t-\tau_{21}(t)\right)\right. \\
& \left.\quad-a_{22}(t) x_{2}\left(t-\tau_{22}(t)\right)-a_{23}(t) x_{3}\left(t-\tau_{23}(t)\right)\right], \\
& \begin{aligned}
& \frac{d x_{3}(t)}{d t} \\
&= x_{3}(t)
\end{aligned} \quad\left[-r_{3}(t)+a_{31}(t) x_{1}\left(t-\tau_{31}(t)\right)\right. \\
& \left.\quad-a_{32}(t) x_{2}\left(t-\tau_{32}(t)\right)-a_{33}(t) x_{3}\left(t-\tau_{33}(t)\right)\right],
\end{align*}
$$

where $x_{1}(t)$ denotes the density of prey species at time $t, x_{2}(t)$ and $x_{3}(t)$ stand for the density of predator species at time
$t, r_{i}, a_{i j} \in C(\mathbb{R},[0, \infty))$ and $\tau_{i j} \in C(\mathbb{R}, \mathbb{R})$. Using Krasnoselskii's fixed point theorem and constructing Lyapunov function, Lv et al. obtained a set of easily verifiable sufficient conditions which guarantee the permanence and global attractivity of system (1).

For the viewpoint of biology, we shall consider (1) together with the initial conditions $x_{i}(0) \geq 0(i=1,2,3)$. The principle object of this paper is to explore the dynamics of system (1), applying the differential inequality theory to study the permanence of system (1). Using the method of Lyapunov function, we investigated the globally asymptotically stability of system (1).

The remainder of the paper is organized as follows: in Section 2, basic definitions and Lemmas are given, and some sufficient conditions for the permanence of the LotkaVolterra predator-prey model in consideration are established. A series of sufficient conditions for the global stability of the Lotka-Volterra predator-prey model in consideration are included in Section 3. In Section 4, we give an example which shows the feasibility of the main results. Conclusions are presented in Section 5 .

## 2. Permanence

For convenience in the following discussing, we always use the notations:

$$
\begin{equation*}
f^{l}=\inf _{t \in \mathbb{R}} f(t), \quad f^{u}=\sup _{t \in \mathbb{R}} f(t) \tag{2}
\end{equation*}
$$

where $f(t)$ is a continuous function. In order to obtain the main result of this paper, we shall first state the definition of permanence and several lemmas which will be useful in the proving the main result.

Definition 1 (see [41]). We say that system (1) is permanence if there are positive constants $M$ and $m$ such that for each positive solution $\left(x_{1}(t), x_{2}(t), x_{3}(t)\right)$ of system (1) satisfies

$$
\begin{equation*}
m \leq \lim _{t \rightarrow+\infty} \inf x_{i}(t) \leq \lim _{t \rightarrow+\infty} \sup x_{i}(t) \leq M \quad(i=1,2,3) \tag{3}
\end{equation*}
$$

Lemma 2 (see [42]). If $a>0, b>0$, and $\dot{x} \geq x(b-a x)$, when $t \geq 0$ and $x(0)>0$, we have

$$
\begin{equation*}
\lim _{t \rightarrow+\infty} \inf x(t) \geq \frac{b}{a} \tag{4}
\end{equation*}
$$

If $a>0, b>0$, and $\dot{x} \leq x(b-a x)$, when $t \geq 0$ and $x(0)>0$, we have

$$
\begin{equation*}
\lim _{t \rightarrow+\infty} \sup x(t) \leq \frac{b}{a} \tag{5}
\end{equation*}
$$

Now we state our permanence result for system (1).
Theorem 3. Let $M_{1}, M_{2}, M_{3}$, and $m_{1}$ be defined by (11), (18), (24), and (30), respectively. Suppose that the following conditions:
(H1) $a_{22}^{u} M_{1}>r_{2}^{l}, a_{31}^{u} M_{1}>r_{3}^{l}$,

hold, and then system (1) is permanent; that is, there exist positive constants $m_{i}, M_{i}(i=1,2,3)$ which are independent of the solution of system (1), such that, for any positive solution $\left(x_{1}(t), x_{2}(t), x_{3}(t)\right)$ of system (1) with the initial condition $x_{i}(0) \geq 0(i=1,2,3)$, one has

$$
\begin{equation*}
m_{i} \leq \lim _{t \rightarrow+\infty} \inf x_{i}(t) \leq \lim _{t \rightarrow+\infty} \sup x_{i}(t) \leq M_{i} \tag{6}
\end{equation*}
$$

Proof. It is easy to see that system (1) with the initial value condition $\left(x_{1}(0), x_{2}(0), x_{3}(0)\right)$ has positive solution $\left(x_{1}(t), x_{2}(t), x_{3}(t)\right)$ passing through $\left(x_{1}(0), x_{2}(0), x_{3}(0)\right)$. Let $\left(x_{1}(t), x_{2}(t), x_{3}(t)\right)$ be any positive solution of system (1) with the initial condition $\left(x_{1}(0), x_{2}(0), x_{3}(0)\right)$. It follows from the first equation of system (1) that

$$
\begin{equation*}
\frac{d x_{1}(t)}{d t} \leq r_{1}(t) x_{1}(t) \leq r_{1}^{u} x_{1}(t) . \tag{7}
\end{equation*}
$$

Integrating both sides of (7) from $t-\tau_{11}(t)$ to $t$, we get

$$
\begin{equation*}
\ln \left[\frac{x_{1}(t)}{x_{1}\left(t-\tau_{11}(t)\right)}\right] \leq \int_{t-\tau_{11}(t)}^{t} r_{1}^{u} d s \leq r_{1}^{u} \tau_{11}^{u}, \tag{8}
\end{equation*}
$$

which leads to

$$
\begin{equation*}
x_{1}\left(t-\tau_{11}(t)\right) \geq x_{1}(t) \exp \left\{-r_{1}^{u} \tau_{11}^{u}\right\} . \tag{9}
\end{equation*}
$$

Substituting (9) into the first equation of system (1), it follows that

$$
\begin{equation*}
\frac{d x_{1}(t)}{d t} \leq x_{1}(t)\left[r_{1}^{u}-a_{11}^{l} \exp \left\{-r_{1}^{u} \tau_{11}^{u}\right\} x_{1}(t)\right] \tag{10}
\end{equation*}
$$

It follows from (10) and Lemma 2 that

$$
\begin{equation*}
\lim _{t \rightarrow+\infty} \sup x_{1}(t) \leq \frac{r_{1}^{u}}{a_{11}^{l}} \exp \left\{r_{1}^{u} \tau_{11}^{u}\right\}:=M_{1} \tag{11}
\end{equation*}
$$

For any positive constant $\varepsilon>0$, it follows from (11) that there exists a $T_{1}>0$ such that, for all $t>T_{1}$,

$$
\begin{equation*}
x_{1}(t) \leq M_{1}+\varepsilon . \tag{12}
\end{equation*}
$$

For $t \geq T_{1}+\tau_{21}^{u}$, from (12) and the second equation of system (1), we have

$$
\begin{align*}
\frac{d x_{2}(t)}{d t} & \leq x_{1}(t)\left[-r_{2}(t)+a_{21}(t) x_{1}\left(t-\tau_{21}(t)\right)\right]  \tag{13}\\
& \leq x_{1}(t)\left[-r_{2}^{l}+a_{21}^{u}\left(M_{1}+\varepsilon\right)\right]
\end{align*}
$$

Integrating both sides of (13) from $t-\tau_{22}(t)$ to $t$, we get

$$
\begin{align*}
\ln \left[\frac{x_{2}(t)}{x_{2}\left(t-\tau_{22}(t)\right)}\right] & \leq \int_{t-\tau_{22}(t)}^{t}\left[-r_{2}^{l}+a_{21}^{u}\left(M_{1}+\varepsilon\right)\right] d s  \tag{14}\\
& \leq\left[-r_{2}^{l}+a_{21}^{u}\left(M_{1}+\varepsilon\right)\right] \tau_{22}^{u},
\end{align*}
$$

which leads to

$$
\begin{equation*}
x_{2}\left(t-\tau_{22}(t)\right) \geq x_{2}(t) \exp \left\{\left[r_{2}^{l}-a_{21}^{u}\left(M_{1}+\varepsilon\right)\right] \tau_{22}^{u}\right\} . \tag{15}
\end{equation*}
$$

Substituting (15) into the second equation of system (1), it follows that

$$
\begin{align*}
\frac{d x_{2}(t)}{d t} \leq x_{2}(t) & \left\{-r_{2}^{l}+a_{22}^{u}\left(M_{1}+\varepsilon\right)\right. \\
& \left.-a_{22}^{l} \exp \left\{\left[r_{2}^{l}-a_{21}^{u}\left(M_{1}+\varepsilon\right)\right] \tau_{22}^{u}\right\} x_{2}(t)\right\} . \tag{16}
\end{align*}
$$

Thus, as a direct corollary of Lemma 2, according to (16), one has

$$
\begin{equation*}
\lim _{t \rightarrow+\infty} \sup x_{2}(t) \leq \frac{-r_{2}^{l}+a_{22}^{u}\left(M_{1}+\varepsilon\right)}{a_{22}^{l} \exp \left\{\left[r_{2}^{l}-a_{21}^{u}\left(M_{1}+\varepsilon\right)\right] \tau_{22}^{u}\right\}} \tag{17}
\end{equation*}
$$

Setting $\varepsilon \rightarrow 0$, it follows that

$$
\begin{equation*}
\lim _{t \rightarrow+\infty} \sup x_{2}(t) \leq \frac{-r_{2}^{l}+a_{22}^{u} M_{1}}{a_{22}^{l} \exp \left\{\left(r_{2}^{l}-a_{21}^{u} M_{1}\right) \tau_{22}^{u}\right\}}:=M_{2} . \tag{18}
\end{equation*}
$$

For $t \geq T_{1}+\tau_{31}^{u}$, from (12) and the third equation of system (1), we have

$$
\begin{align*}
\frac{d x_{3}(t)}{d t} & \leq x_{3}(t)\left[-r_{3}(t)+a_{31}(t) x_{1}\left(t-\tau_{31}(t)\right)\right]  \tag{19}\\
& \leq x_{3}(t)\left[-r_{3}^{l}+a_{31}^{u}\left(M_{1}+\varepsilon\right)\right]
\end{align*}
$$

Integrating both sides of (19) from $t-\tau_{33}(t)$ to $t$, we get

$$
\begin{align*}
\ln \left[\frac{x_{3}(t)}{x_{3}\left(t-\tau_{33}(t)\right)}\right] & \leq \int_{t-\tau_{33}(t)}^{t}\left[-r_{3}^{l}+a_{31}^{u}\left(M_{1}+\varepsilon\right)\right] d s  \tag{20}\\
& \leq\left[-r_{3}^{l}+a_{31}^{u}\left(M_{1}+\varepsilon\right)\right] \tau_{33}^{u}
\end{align*}
$$

which leads to

$$
\begin{equation*}
x_{3}\left(t-\tau_{33}(t)\right) \geq x_{3}(t) \exp \left\{\left[r_{3}^{l}-a_{31}^{u}\left(M_{1}+\varepsilon\right)\right] \tau_{33}^{u}\right\} \tag{21}
\end{equation*}
$$

Substituting (21) into the third equation of system (1), it follows that

$$
\begin{align*}
\frac{d x_{3}(t)}{d t} \leq x_{3}(t) & \left\{-r_{3}^{l}+a_{31}^{u}\left(M_{1}+\varepsilon\right)\right. \\
& \left.-a_{33}^{l} \exp \left\{\left[r_{3}^{l}-a_{31}^{u}\left(M_{1}+\varepsilon\right)\right] \tau_{33}^{u}\right\} x_{3}(t)\right\} \tag{22}
\end{align*}
$$

Thus, as a direct corollary of Lemma 2, according to (22), one has

$$
\begin{equation*}
\lim _{t \rightarrow+\infty} \sup x_{3}(t) \leq \frac{-r_{3}^{l}+a_{31}^{u}\left(M_{1}+\varepsilon\right)}{a_{33}^{l} \exp \left\{\left[r_{3}^{l}-a_{31}^{u}\left(M_{1}+\varepsilon\right)\right] \tau_{33}^{u}\right\}} \tag{23}
\end{equation*}
$$

Setting $\varepsilon \rightarrow 0$, it follows that

$$
\begin{equation*}
\lim _{t \rightarrow+\infty} \sup x_{3}(t) \leq \frac{-r_{3}^{l}+a_{31}^{u} M_{1}}{a_{33}^{l} \exp \left\{\left(r_{3}^{l}-a_{31}^{u} M_{1}\right) \tau_{33}^{u}\right\}}:=M_{3} . \tag{24}
\end{equation*}
$$

For $t \geq T_{1}+\max \left\{\tau_{21}^{u}, \tau_{31}^{u}, \tau_{11}^{u}, \tau_{12}^{u}, \tau_{13}^{u}\right\}$, it follows from the first equation of system (1) that

$$
\begin{align*}
\frac{d x_{1}(t)}{d t} \geq x_{1}(t) & {\left[r_{1}^{l}-a_{11}^{u}\left(M_{1}+\varepsilon\right)\right.}  \tag{25}\\
& \left.-a_{12}^{u}\left(M_{2}+\varepsilon\right)-a_{13}^{u}\left(M_{3}+\varepsilon\right)\right]
\end{align*}
$$

Integrating both sides of (25) from $t-\tau_{11}(t)$ to $t$, one has

$$
\begin{align*}
& \ln \left[\frac{x_{1}(t)}{x_{1}\left(t-\tau_{11}(t)\right)}\right] \\
& \geq \int_{t-\tau_{11}(t)}^{t}\left[r_{1}^{l}-a_{11}^{u}\left(M_{1}+\varepsilon\right)-a_{12}^{u}\left(M_{2}+\varepsilon\right)-a_{13}^{u}\left(M_{3}+\varepsilon\right)\right] d s \\
& \geq\left[r_{1}^{l}-a_{11}^{u}\left(M_{1}+\varepsilon\right)-a_{12}^{u}\left(M_{2}+\varepsilon\right)-a_{13}^{u}\left(M_{3}+\varepsilon\right)\right] \tau_{22}^{u}, \tag{26}
\end{align*}
$$

which leads to

$$
\begin{align*}
& x_{1}\left(t-\tau_{11}(t)\right) \\
& \qquad \begin{array}{l}
\leq x_{1}(t) \exp \left\{-\left[r_{1}^{l}-a_{11}^{u}\left(M_{1}+\varepsilon\right)\right.\right. \\
\\
\left.\left.\quad-a_{12}^{u}\left(M_{2}+\varepsilon\right)-a_{13}^{u}\left(M_{3}+\varepsilon\right)\right] \tau_{22}^{u}\right\}
\end{array}
\end{align*}
$$

Substituting (27) into the first equation of system (1), it follows that

$$
\begin{align*}
& \frac{d x_{1}(t)}{d t} \\
& \geq x_{1}(t)\left\{r_{1}^{l}-a_{12}^{u}\left(M_{2}+\varepsilon\right)-a_{13}^{u}\left(M_{3}+\varepsilon\right)\right. \\
&  \tag{28}\\
& -a_{11}^{u} \exp \left\{-\left[r_{1}^{l}-a_{11}^{u}\left(M_{1}+\varepsilon\right)\right.\right. \\
& \\
& \left.\quad-a_{12}^{u}\left(M_{2}+\varepsilon\right)-a_{13}^{u}\left(M_{3}+\varepsilon\right)\right] \\
& \\
& \left.\left.\times \tau_{22}^{u}\right\} x_{1}(t)\right\} .
\end{align*}
$$

According to Lemma 2, it follows from (28) that

$$
\begin{align*}
& \lim _{t \rightarrow+\infty} \inf x_{1}(t) \\
& \geq\left(r_{1}^{l}-a_{12}^{u}\left(M_{2}+\varepsilon\right)-a_{13}^{u}\left(M_{3}+\varepsilon\right)\right) \\
& \quad \times\left(a _ { 1 1 } ^ { u } \operatorname { e x p } \left\{\left[-r_{1}^{l}-a_{11}^{u}\left(M_{1}+\varepsilon\right)\right.\right.\right. \\
& \left.\left.\left.\quad-a_{12}^{u}\left(M_{2}+\varepsilon\right)-a_{13}^{u}\left(M_{3}+\varepsilon\right)\right] \tau_{22}^{u}\right\}\right)^{-1} \tag{29}
\end{align*}
$$

Setting $\varepsilon \rightarrow 0$ in (29), we can get

$$
\begin{align*}
& \lim _{t \rightarrow+\infty} \inf x_{1}(t) \\
& \geq \frac{r_{1}^{l}-a_{12}^{u} M_{2}-a_{13}^{u} M_{3}}{a_{11}^{u} \exp \left\{-\left(r_{1}^{l}-a_{11}^{u} M_{1}-a_{12}^{u} M_{2}-a_{13}^{u} M_{3}\right) \tau_{22}^{u}\right\}}:=m_{1} . \tag{30}
\end{align*}
$$

For $t \geq T_{1}+\max \left\{\tau_{21}^{u}, \tau_{22}^{u}, \tau_{23}^{u}, \tau_{31}^{u}, \tau_{11}^{u}, \tau_{12}^{u}, \tau_{13}^{u}\right\}$, from the second equation of system (1), we have

$$
\begin{align*}
& \frac{d x_{2}(t)}{d t} \\
& \geq x_{2}(t)\left[-r_{2}^{u}+a_{21}^{l}\left(m_{1}-\varepsilon\right)-a_{22}^{u}\left(M_{2}+\varepsilon\right)-a_{23}^{u}\left(M_{3}+\varepsilon\right)\right] . \tag{31}
\end{align*}
$$

Integrating both sides of (31) from $t-\tau_{22}(t)$ to $t$ leads to

$$
\begin{align*}
& \ln \left[\frac{x_{2}(t)}{x_{2}\left(t-\tau_{22}(t)\right)}\right] \\
& \geq \int_{t-\tau_{22}(t)}^{t}\left[-r_{2}^{u}+a_{21}^{l}\left(m_{1}-\varepsilon\right)\right. \\
& \left.\quad-a_{22}^{u}\left(M_{2}+\varepsilon\right)-a_{23}^{u}\left(M_{3}+\varepsilon\right)\right] d s \\
& \geq\left[-r_{2}^{u}+a_{21}^{l}\left(m_{1}-\varepsilon\right)-a_{22}^{u}\left(M_{2}+\varepsilon\right)-a_{23}^{u}\left(M_{3}+\varepsilon\right)\right] \tau_{22}^{u}, \tag{32}
\end{align*}
$$

which leads to

$$
\begin{align*}
& x_{2}\left(t-\tau_{22}(t)\right) \\
& \qquad \leq x_{2}(t) \exp \left\{\left[r_{2}^{u}-a_{21}^{l}\left(m_{1}-\varepsilon\right)+a_{22}^{u}\left(M_{2}+\varepsilon\right)\right.\right.  \tag{33}\\
& \\
& \left.\left.\quad+a_{23}^{u}\left(M_{3}+\varepsilon\right)\right] \tau_{22}^{u}\right\}
\end{align*}
$$

Substituting (33) into the second equation of system (1), it follows that

$$
\begin{align*}
& \frac{d x_{2}(t)}{d t} \\
& \geq x_{2}(t)\left\{r_{2}^{u}-a_{22}^{u}\right. \\
& \quad \times \exp \left\{\left[r_{2}^{u}-a_{21}^{l}\left(m_{1}-\varepsilon\right)+a_{22}^{u}\left(M_{2}+\varepsilon\right)\right.\right. \\
& \left.\left.\left.\quad+a_{23}^{u}\left(M_{3}+\varepsilon\right)\right] \tau_{22}^{u}\right\} x_{2}(t)-a_{23}^{u}\left(M_{3}+\varepsilon\right)\right\} \tag{34}
\end{align*}
$$

By Lemma 2 and (34), we can get

$$
\begin{align*}
& \lim _{t \rightarrow+\infty} \inf x_{2}(t) \\
& \geq\left(r_{2}^{u}-a_{23}^{u}\left(M_{3}+\varepsilon\right)\right) \\
& \quad \times\left(a _ { 2 2 } ^ { u } \operatorname { e x p } \left\{\left[r_{2}^{u}-a_{21}^{l}\left(m_{1}-\varepsilon\right)\right.\right.\right.  \tag{35}\\
& \left.\left.\left.\quad+a_{22}^{u}\left(M_{2}+\varepsilon\right)+a_{23}^{u}\left(M_{3}+\varepsilon\right)\right] \tau_{22}^{u}\right\}\right)^{-1}
\end{align*}
$$

Setting $\varepsilon \rightarrow 0$ in the above inequality, it follows that

$$
\begin{align*}
& \lim _{t \rightarrow+\infty} \inf x_{2}(t) \\
& \geq \frac{r_{2}^{u}-a_{23}^{u} M_{3}}{a_{22}^{u} \exp \left\{\left(r_{2}^{u}-a_{21}^{l} m_{1}+a_{22}^{u} M_{2}+a_{23}^{u} M_{3}\right) \tau_{22}^{u}\right\}}:=m_{2} . \tag{36}
\end{align*}
$$

For $t \geq T_{1}+\max \left\{\tau_{31}^{u}, \tau_{32}^{u}, \tau_{33}^{u}, \tau_{21}^{u}, \tau_{22}^{u}, \tau_{23}^{u}, \tau_{31}^{u}, \tau_{11}^{u}, \tau_{12}^{u}, \tau_{13}^{u}\right\}$, it follows from the third equation of system (1) that

$$
\begin{align*}
& \frac{d x_{3}(t)}{d t} \\
& =x_{3}(t)\left[-r_{3}(t)+a_{31}(t) x_{1}\left(t-\tau_{31}(t)\right)\right. \\
& \left.\quad-a_{32}(t) x_{2}\left(t-\tau_{32}(t)\right)-a_{33}(t) x_{3}\left(t-\tau_{33}(t)\right)\right] \\
& \geq x_{3}(t)\left[-r_{3}^{u}+a_{31}^{l}\left(m_{1}-\varepsilon\right)\right. \\
& \left.\quad-a_{32}^{u}\left(M_{2}+\varepsilon\right)-a_{33}^{u}\left(M_{3}+\varepsilon\right)\right] \tag{37}
\end{align*}
$$

Integrating both sides of (37) from $t-\tau_{33}(t)$ to $t$, we get

$$
\begin{align*}
& \ln \left[\frac{x_{3}(t)}{x_{3}\left(t-\tau_{33}(t)\right)}\right] \\
& \begin{array}{l}
\geq \int_{t-\tau_{33}(t)}^{t}\left[-r_{3}^{u}+a_{31}^{l}\left(m_{1}-\varepsilon\right)\right. \\
\left.\quad-a_{32}^{u}\left(M_{2}+\varepsilon\right)-a_{33}^{u}\left(M_{3}+\varepsilon\right)\right] d s
\end{array} \\
& \geq\left[-r_{3}^{u}+a_{31}^{l}\left(m_{1}-\varepsilon\right)-a_{32}^{u}\left(M_{2}+\varepsilon\right)-a_{33}^{u}\left(M_{3}+\varepsilon\right)\right] \tau_{33}^{u}
\end{align*}
$$

Hence

$$
\begin{align*}
& x_{3}\left(t-\tau_{33}(t)\right) \\
& \leq x_{3}(t) \exp \left\{\left[r_{3}^{u}-a_{31}^{l}\left(m_{1}-\varepsilon\right)\right.\right.  \tag{39}\\
& \\
& \left.\left.\quad+a_{32}^{u}\left(M_{2}+\varepsilon\right)+a_{33}^{u}\left(M_{3}+\varepsilon\right)\right] \tau_{33}^{u}\right\}
\end{align*}
$$

Substituting (39) into the third equation of system (1), we derive

$$
\begin{align*}
& \frac{d x_{3}(t)}{d t} \\
& \geq x_{3}(t)\left\{-r_{3}^{u}+a_{31}^{l}\left(m_{1}-\varepsilon\right)-a_{32}\left(M_{2}+\varepsilon\right)\right. \\
& \quad-a_{33} \exp \left\{\left[r_{3}^{u}-a_{31}^{l}\left(m_{1}-\varepsilon\right)+a_{32}^{u}\left(M_{2}+\varepsilon\right)\right.\right. \\
& \left.\left.\left.\quad+a_{33}^{u}\left(M_{3}+\varepsilon\right)\right] \tau_{33}^{u}\right\} x_{3}(t)\right\} . \tag{40}
\end{align*}
$$

In view of Lemma 2 and (40), one has

$$
\begin{align*}
& \lim _{t \rightarrow+\infty} \inf x_{3}(t) \\
& \geq\left(-r_{3}^{u}+a_{31}^{l}\left(m_{1}-\varepsilon\right)-a_{32}^{u}\left(M_{2}+\varepsilon\right)\right)  \tag{41}\\
& \quad \times\left(a _ { 3 3 } \operatorname { e x p } \left\{\left[r_{3}^{u}-a_{31}^{l}\left(m_{1}-\varepsilon\right)+a_{32}^{u}\left(M_{2}+\varepsilon\right)\right.\right.\right. \\
& \left.\left.\left.\quad+a_{33}^{u}\left(M_{3}+\varepsilon\right)\right] \tau_{33}^{u}\right\}\right)^{-1}
\end{align*}
$$

Setting $\varepsilon \rightarrow 0$ in (41) leads to

$$
\begin{align*}
& \lim _{t \rightarrow+\infty} \inf x_{3}(t) \\
& \geq \frac{-r_{3}^{u}+a_{31}^{l} m_{1}-a_{32}^{u} M_{2}}{a_{33}^{u} \exp \left\{\left(r_{3}^{u}-a_{31}^{l} m_{1}+a_{32}^{u} M_{2}+a_{33}^{u} M_{3}\right) \tau_{33}^{u}\right\}}:=m_{3} . \tag{42}
\end{align*}
$$

Equations (11), (18), (24), (30), (36), and (42) show that system (1) is permanent. The proof of Theorem 3 is complete.

## 3. Global Asymptotically Stability of Positive Solutions

In this section, we formulate the global asymptotically stability of positive solutions of system (1).

Definition 4. A bounded positive solution $\left(x_{1}^{*}(t), x_{2}^{*}(t)\right.$, $\left.x_{3}^{*}(t)\right)^{T}$ of system (1) is said to be globally asymptotically stable if, for any other positive bounded solution $\left(x_{1}(t)\right.$, $\left.x_{2}(t), x_{3}(t)\right)^{T}$ of system (1), the following equality holds:

$$
\begin{equation*}
\lim _{t \rightarrow+\infty}\left[\sum_{i=1}^{3}\left|x_{i}(t)-x_{i}^{*}(t)\right|\right]=0 \tag{43}
\end{equation*}
$$

Definition 5 (see [24]). Let $\tilde{h}$ be a real number and $f$ be a nonnegative function defined on $[\widetilde{h},+\infty)$ such that $f$ is integrable on $[\widetilde{h},+\infty)$ and is uniformly continuous on $[\widetilde{h},+\infty)$, then $\lim _{t \rightarrow+\infty} f(t)=0$.

Theorem 6. In addition to (H1)-(H2), assume further that
(H3) $\lim _{t \rightarrow \infty} \inf A_{i}(t)>0$,
where $A_{i}(i=1,2,3)$ are defined by (48), (49), and (50), respectively. Then system (1) has a unique positive solution $\left(x_{1}^{*}(t), x_{2}^{*}(t), x_{3}^{*}(t)\right)^{T}$ which is global attractivity.

Proof. According to the conclusion of Theorem 3, there exists $T>0$ and positive constants $m_{i}, M_{i}(i=1,2,3)$ such that

$$
\begin{equation*}
m_{i}<x_{i}^{*}(t) \leq M_{i} \quad i=1,2,3, t>T \tag{44}
\end{equation*}
$$

Define

$$
\begin{equation*}
V(t)=\sum_{i=1}^{3}\left|\ln x_{i}^{*}(t)-\ln x_{i}(t)\right| \tag{45}
\end{equation*}
$$

Calculating the upper-right derivative of $V(t)$ along the solution of (1), it follows for $t \geq T$ that

$$
\begin{aligned}
D^{+} V(t)= & \sum_{i=1}^{3}\left(\frac{x_{i}^{* \prime}(t)}{x_{i}^{*}(t)}-\frac{x_{i}^{\prime}(t)}{x_{i}(t)}\right) \operatorname{sgn}\left(x_{i}^{*}(t)-x_{i}(t)\right) \\
= & \operatorname{sgn}\left(x_{1}^{*}(t)-x_{1}(t)\right) \\
& \times \sum_{i=1}^{3}-a_{1 i}(t)\left[x_{i}^{*}\left(t-\tau_{1 i}(t)\right)-x_{i}\left(t-\tau_{1 i}(t)\right)\right]
\end{aligned}
$$

$$
\begin{align*}
& +\operatorname{sgn}\left(x_{2}^{*}(t)-x_{2}(t)\right) \\
& \times \sum_{i=1}^{3}-a_{2 i}(t)\left[x_{i}^{*}\left(t-\tau_{2 i}(t)\right)-x_{i}\left(t-\tau_{2 i}(t)\right)\right] \\
& +\operatorname{sgn}\left(x_{3}^{*}(t)-x_{3}(t)\right) \\
& \times \sum_{i=1}^{3}-a_{3 i}(t)\left[x_{i}^{*}\left(t-\tau_{3 i}(t)\right)-x_{i}\left(t-\tau_{3 i}(t)\right)\right] \\
& \leq-a_{11}(t)\left|x_{1}^{*}\left(t-\tau_{11}(t)\right)-x_{1}\left(t-\tau_{11}(t)\right)\right| \\
& +a_{12}(t)\left|x_{2}^{*}\left(t-\tau_{12}(t)\right)-x_{2}\left(t-\tau_{12}(t)\right)\right| \\
& +a_{13}(t)\left|x_{3}^{*}\left(t-\tau_{13}(t)\right)-x_{3}\left(t-\tau_{13}(t)\right)\right| \\
& +a_{21}(t)\left|x_{1}^{*}\left(t-\tau_{21}(t)\right)-x_{1}\left(t-\tau_{21}(t)\right)\right| \\
& -a_{22}(t)\left|x_{2}^{*}\left(t-\tau_{22}(t)\right)-x_{2}\left(t-\tau_{22}(t)\right)\right| \\
& +a_{23}(t)\left|x_{3}^{*}\left(t-\tau_{23}(t)\right)-x_{3}\left(t-\tau_{23}(t)\right)\right| \\
& +a_{31}(t)\left|x_{1}^{*}\left(t-\tau_{31}(t)\right)-x_{1}\left(t-\tau_{31}(t)\right)\right| \\
& +a_{32}(t)\left|x_{2}^{*}\left(t-\tau_{32}(t)\right)-x_{2}\left(t-\tau_{32}(t)\right)\right| \\
& -a_{33}(t)\left|x_{3}^{*}\left(t-\tau_{33}(t)\right)-x_{3}\left(t-\tau_{33}(t)\right)\right| \tag{46}
\end{align*}
$$

It follows that

$$
\begin{aligned}
& D^{+} V(t) \\
& \leq-a_{11}(t)\left|x_{1}^{*}\left(t-\tau_{11}(t)\right)-x_{1}\left(t-\tau_{11}(t)\right)\right| \\
& \quad+a_{12}(t)\left|x_{2}^{*}\left(t-\tau_{12}(t)\right)-x_{2}\left(t-\tau_{12}(t)\right)\right| \\
& \quad+a_{13}(t)\left|x_{3}^{*}\left(t-\tau_{13}(t)\right)-x_{3}\left(t-\tau_{13}(t)\right)\right| \\
& \quad+a_{21}(t)\left|x_{1}^{*}\left(t-\tau_{21}(t)\right)-x_{1}\left(t-\tau_{21}(t)\right)\right| \\
& \quad-a_{22}(t)\left|x_{2}^{*}\left(t-\tau_{22}(t)\right)-x_{2}\left(t-\tau_{22}(t)\right)\right| \\
& \quad+a_{23}(t)\left|x_{3}^{*}\left(t-\tau_{23}(t)\right)-x_{3}\left(t-\tau_{23}(t)\right)\right| \\
& \quad+a_{31}(t)\left|x_{1}^{*}\left(t-\tau_{31}(t)\right)-x_{1}\left(t-\tau_{31}(t)\right)\right| \\
& \quad+a_{32}(t)\left|x_{2}^{*}\left(t-\tau_{32}(t)\right)-x_{2}\left(t-\tau_{32}(t)\right)\right| \\
& \quad-a_{33}(t)\left|x_{3}^{*}\left(t-\tau_{33}(t)\right)-x_{3}\left(t-\tau_{33}(t)\right)\right| \\
& \leq-a_{11}(t)\left\{\operatorname { e x p } \left\{\left[r_{1}^{l}-a_{11}^{u}\left(M_{1}+\varepsilon\right)-a_{12}^{u}\left(M_{2}+\varepsilon\right)\right.\right.\right. \\
& \left.\left.\left.\quad-a_{13}^{u}\left(M_{3}+\varepsilon\right)\right] \tau_{22}^{u}\right\}+\exp \left\{-r^{u} \tau_{11}^{u}\right\}\right\} \\
& \\
& \quad \times\left|x_{1}^{*}(t)-x_{1}(t)\right| \\
& \quad+2 a_{12}(t) \exp \left\{\left[-r_{2}^{u}+a_{21}^{l}\left(m_{1}-\varepsilon\right)-a_{22}^{u}\left(M_{2}+\varepsilon\right)\right.\right. \\
& \left.\left.\quad \quad-a_{23}^{u}\left(M_{3}+\varepsilon\right)\right] \tau_{22}^{u}\right\}
\end{aligned}
$$

$$
\begin{align*}
& +2 a_{13}(t) \exp \left\{\left[-r_{3}^{u}+a_{31}^{l}\left(m_{1}-\varepsilon\right)-a_{32}^{u}\left(M_{2}+\varepsilon\right)\right.\right. \\
& \left.\left.-a_{33}^{u}\left(M_{3}+\varepsilon\right)\right] \tau_{33}^{u}\right\} \\
& \times\left|x_{3}^{*}(t)-x_{3}(t)\right| \\
& +2 a_{21}(t) \exp \left\{-\left[r_{1}^{l}-a_{11}^{u}\left(M_{1}+\varepsilon\right)\right.\right. \\
& \left.\left.-a_{12}^{u}\left(M_{2}+\varepsilon\right)-a_{13}^{u}\left(M_{3}+\varepsilon\right)\right] \tau_{22}^{u}\right\} \\
& \times\left|x_{1}^{*}(t)-x_{1}(t)\right| \\
& -a_{22}(t)\left\{\operatorname { e x p } \left\{\left[-r_{2}^{u}+a_{21}^{l}\left(m_{1}-\varepsilon\right)-a_{22}^{u}\left(M_{2}+\varepsilon\right)\right.\right.\right. \\
& \left.\left.-a_{23}^{u}\left(M_{3}+\varepsilon\right)\right] \tau_{22}^{u}\right\} \\
& \left.+\exp \left\{\left[r_{2}^{l}-a_{21}^{u}\left(M_{1}+\varepsilon\right)\right] \tau_{22}^{u}\right\}\right\} \\
& \times\left|x_{2}^{*}(t)-x_{2}(t)\right| \\
& +2 a_{23}(t) \exp \left\{\left[r_{3}^{u}-a_{31}^{l}\left(m_{1}-\varepsilon\right)\right.\right. \\
& \left.\left.+a_{32}^{u}\left(M_{2}+\varepsilon\right)+a_{33}^{u}\left(M_{3}+\varepsilon\right)\right] \tau_{33}^{u}\right\} \\
& \times\left|x_{3}^{*}(t)-x_{3}(t)\right| \\
& +2 a_{31}(t) \exp \left\{-\left[r_{1}^{l}-a_{11}^{u}\left(M_{1}+\varepsilon\right)\right.\right. \\
& \left.\left.-a_{12}^{u}\left(M_{2}+\varepsilon\right)-a_{13}^{u}\left(M_{3}+\varepsilon\right)\right] \tau_{22}^{u}\right\} \\
& \times\left|x_{1}^{*}(t)-x_{1}(t)\right| \\
& +2 a_{32}(t) \exp \left\{\left[-r_{2}^{u}+a_{21}^{l}\left(m_{1}-\varepsilon\right)\right.\right. \\
& \left.\left.-a_{22}^{u}\left(M_{2}+\varepsilon\right)-a_{23}^{u}\left(M_{3}+\varepsilon\right)\right] \tau_{22}^{u}\right\} \\
& \times\left|x_{2}^{*}(t)-x_{2}(t)\right| \\
& -a_{33}(t)\left\{\operatorname { e x p } \left\{\left[-r_{3}^{u}+a_{31}^{l}\left(m_{1}-\varepsilon\right)\right.\right.\right. \\
& \left.\left.-a_{32}^{u}\left(M_{2}+\varepsilon\right)-a_{33}^{u}\left(M_{3}+\varepsilon\right)\right] \tau_{33}^{u}\right\} \\
& \left.+\exp \left\{\left[r_{3}^{l}-a_{31}^{u}\left(M_{1}+\varepsilon\right)\right] \tau_{33}^{u}\right\}\right\} \\
& \times\left|x_{3}^{*}(t)-x_{3}(t)\right| \\
& \leq\left[-A_{1}(t)\left|x_{1}^{*}(t)-x_{1}(t)\right|+A_{2}(t)\left|x_{2}^{*}(t)-x_{2}(t)\right|\right. \\
& \left.+A_{3}\left|x_{3}^{*}(t)-x_{3}(t)\right|\right], \tag{47}
\end{align*}
$$

where $\varepsilon$ is defined by Theorem 3 and

$$
\left.\left.\begin{array}{l}
A_{1}(t) \\
\begin{array}{rl}
=a_{11}(t)\{\exp \{ & {\left[r_{1}^{l}-a_{11}^{u}\left(M_{1}+\varepsilon\right)-a_{12}^{u}\left(M_{2}+\varepsilon\right)\right.}
\end{array} \\
\left.\left.\left.\qquad-a_{13}^{u}\left(M_{3}+\varepsilon\right)\right] \tau_{22}^{u}\right\}+\exp \left\{-r^{u} \tau_{11}^{u}\right\}\right\}
\end{array}\right\} \begin{array}{l}
-2 a_{21}(t) \exp \left\{-\left[r_{1}^{l}-a_{11}^{u}\left(M_{1}+\varepsilon\right)-a_{12}^{u}\left(M_{2}+\varepsilon\right)\right.\right. \\
\left.\left.\quad-a_{13}^{u}\left(M_{3}+\varepsilon\right)\right] \tau_{22}^{u}\right\}
\end{array}\right\} \begin{aligned}
& -2 a_{31}(t) \exp \left\{\tau_{22}^{u}-\left[r_{1}^{l}-a_{11}^{u}\left(M_{1}+\varepsilon\right)-a_{12}^{u}\left(M_{2}+\varepsilon\right)\right.\right. \\
& \left.\left.\quad-a_{13}^{u}\left(M_{3}+\varepsilon\right)\right] \tau_{22}^{u}\right\},
\end{aligned}
$$

$$
\begin{align*}
& A_{2}(t) \\
& =a_{22}(t)\left\{\operatorname { e x p } \left\{\left[-r_{2}^{u}+a_{21}^{l}\left(m_{1}-\varepsilon\right)-a_{22}^{u}\left(M_{2}+\varepsilon\right)\right.\right.\right. \\
& \left.\left.-a_{23}^{u}\left(M_{3}+\varepsilon\right)\right] \tau_{22}^{u}\right\} \\
& \left.+\exp \left\{\left[r_{2}^{l}-a_{21}^{u}\left(M_{1}+\varepsilon\right)\right] \tau_{22}^{u}\right\}\right\} \\
& -2 a_{12}(t) \exp \left\{\left[-r_{2}^{u}+a_{21}^{l}\left(m_{1}-\varepsilon\right)-a_{22}^{u}\left(M_{2}+\varepsilon\right)\right.\right.  \tag{49}\\
& \left.\left.-a_{23}^{u}\left(M_{3}+\varepsilon\right)\right] \tau_{22}^{u}\right\} \\
& -2 a_{32}(t) \exp \left\{\left[-r_{2}^{u}+a_{21}^{l}\left(m_{1}-\varepsilon\right)-a_{22}^{u}\left(M_{2}+\varepsilon\right)\right.\right. \\
& \left.\left.-a_{23}^{u}\left(M_{3}+\varepsilon\right)\right] \tau_{22}^{u}\right\}, \\
& A_{3}(t) \\
& =a_{33}(t)\left\{\operatorname { e x p } \left\{\left[-r_{3}^{u}+a_{31}^{l}\left(m_{1}-\varepsilon\right)\right.\right.\right. \\
& +\exp \left\{\left[r_{3}^{l}-a_{31}^{u}\left(M_{1}+\varepsilon\right)\right] \tau_{33}^{u}\right\} \\
& \left.\left.-a_{32}^{u}\left(M_{2}+\varepsilon\right)-a_{33}^{u}\left(M_{3}+\varepsilon\right)\right] \tau_{33}^{u}\right\} \\
& +\exp \left\{\left[r_{3}^{l}-a_{31}^{u}\left(M_{1}+\varepsilon\right)\right] \tau_{33}^{u}\right\}  \tag{50}\\
& -2 a_{13}(t) \exp \left\{\left[-r_{3}^{u}+a_{31}^{l}\left(m_{1}-\varepsilon\right)-a_{32}^{u}\left(M_{2}+\varepsilon\right)\right.\right. \\
& \left.\left.-a_{33}^{u}\left(M_{3}+\varepsilon\right)\right] \tau_{33}^{u}\right\} \\
& -2 a_{23}(t) \exp \left\{\left[r_{3}^{u}-a_{31}^{l}\left(m_{1}-\varepsilon\right)+a_{32}^{u}\left(M_{2}+\varepsilon\right)\right.\right. \\
& \left.\left.+a_{33}^{u}\left(M_{3}+\varepsilon\right)\right] \tau_{33}^{u}\right\} .
\end{align*}
$$

By hypothesis (H3), there exist constants $\alpha_{i}(i=1,2,3)$ and $T^{*}>T$ such that

$$
\begin{equation*}
A_{i}(t) \geq \alpha_{i}>0, \quad(i=1,2,3) \text { for } t \geq T^{*} \tag{51}
\end{equation*}
$$

Integrating both sides of (51) on interval $\left[T^{*}, t\right]$ yields

$$
\begin{equation*}
V(t)+\sum_{i=1}^{3} \int_{T^{*}}^{t} A_{i}(t)\left|x_{i}^{*}(t)-x_{i}(t)\right| d s \leq V\left(T^{*}\right) \tag{52}
\end{equation*}
$$

It follows from (51) and (52) that

$$
\begin{equation*}
\sum_{i=1}^{3} \int_{T^{*}}^{t} A_{i}(t)\left|x_{i}^{*}(t)-x_{i}(t)\right| d s \leq V\left(T^{*}\right)<\infty, \quad \text { for } t \geq T^{*} \tag{53}
\end{equation*}
$$

Since $x_{i}^{*}(t)(i=1,2,3)$ are bounded for $t \geq T^{*}$, so $\mid x_{i}^{*}(t)-$ $x_{i}(t) \mid(i, j=1,2,3)$ are uniformly continuous on $\left[T^{*}, \infty\right)$. By Barbalat's Lemma [24], we have

$$
\begin{equation*}
\lim _{t \rightarrow \infty}\left|x_{i}^{*}(t)-x_{i}(t)\right|=0, \quad(i=1,2,3) \tag{54}
\end{equation*}
$$

By Theorems 7.4 and 8.2 in [43], we know that the positive solution $\left(x_{1}^{*}(t), x_{2}^{*}(t), x_{3}^{*}(t)\right)^{T}$ of (1) is uniformly asymptotically stable. The proof of Theorem 6 is complete.

## 4. Numerical Example

To illustrate the theoretical results, we present some numerical simulations. Let us consider the following discrete system:

$$
\begin{align*}
& \frac{d x_{1}(t)}{d t} \\
& =x_{1}(t)\left[5-\frac{\cos \pi t}{2}\right. \\
& -\left(4+\frac{\cos \pi t}{5}\right) x_{1}\left(t-\left(1-\frac{\sin \pi t}{4}\right)\right) \\
& -\left(\frac{1+\sin \pi t}{4}\right) x_{2}\left(t-\left(\frac{0.5-\sin \pi t}{4}\right)\right) \\
& \left.-\left(\frac{1+\cos \pi t}{3}\right) x_{3}\left(t-\left(\frac{0.9-\cos \pi t}{4}\right)\right)\right] \text {, } \\
& \frac{d x_{2}(t)}{d t} \\
& =x_{2}(t)\left[-\left(\frac{48-\cos \pi t}{12}\right)\right. \\
& +\left(2-\frac{\cos \pi t}{4}\right) x_{1}\left(t-\left(\frac{0.7-\cos \pi t}{5}\right)\right) \\
& -\left(4-\frac{\cos \pi t}{12}\right) x_{2}\left(t-\left(\frac{1+\sin \pi t}{4}\right)\right) \\
& \left.-\left(1+\frac{\sin \pi t}{4}\right) x_{3}\left(t-\left(\frac{0.2-\sin \pi t}{12}\right)\right)\right], \\
& \frac{d x_{3}(t)}{d t} \\
& =x_{3}(t)\left[-\left(\frac{1-\cos \pi t}{4}\right)\right. \\
& +\left(8+\frac{\sin \pi t}{4}\right) x_{1}\left(t-\left(\frac{0.8-\sin \pi t}{5}\right)\right) \\
& -\left(\frac{0.6-\sin \pi t}{8}\right) x_{2}\left(t-\left(\frac{0.6-\cos \pi t}{12}\right)\right) \\
& \left.-\left(20+\frac{\sin \pi t}{4}\right) x_{3}\left(t-\left(0.5+\frac{\sin \pi t}{2}\right)\right)\right] . \tag{55}
\end{align*}
$$

Here

$$
\begin{array}{ll}
r_{1}(t)=5-\frac{\cos \pi t}{2}, & r_{2}(t)=\frac{48-\cos \pi t}{12}, \\
r_{3}(t)=\frac{2-\cos \pi t}{4}, & a_{11}(t)=4+\frac{\cos \pi t}{5}, \\
a_{12}(t)=\frac{1+\sin \pi t}{4}, & a_{13}(t)=\frac{1+\cos \pi t}{3}, \\
a_{21}(t)=2-\frac{\cos \pi t}{4}, & a_{22}(t)=4-\frac{\cos \pi t}{12} \\
a_{23}(t)=1+\frac{\sin \pi t}{4}, & a_{31}(t)=8+\frac{\sin \pi t}{4} \\
a_{32}(t)=\frac{0.6-\sin \pi t}{8}, & a_{33}(t)=20+\frac{\sin \pi t}{4} \\
\tau_{11}(t)=1-\frac{\sin \pi t}{4}, & \tau_{12}(t)=\frac{0.5-\sin \pi t}{4} \\
\tau_{13}(t)=\frac{0.9-\cos \pi t}{4}, & \tau_{21}(t)=\frac{0.7-\cos \pi t}{5}
\end{array}
$$



Figure 1: The dynamical behavior of the first component of the solution $\left(x_{1}(t), x_{2}(t), x_{3}(t)\right)$.

$$
\begin{gather*}
\tau_{22}(t)=\frac{1+\sin \pi t}{4}, \\
\tau_{23}(t)=\frac{0.2-\sin \pi t}{12} \\
\tau_{33}(t)=\frac{0.8-\sin \pi t}{5},  \tag{56}\\
\tau_{32}(t)=\frac{0.6-\cos \pi t}{12} \\
\tau_{3}, 5+\frac{\sin \pi t}{2}
\end{gather*}
$$

All the coefficients $r_{i}(t)(i=1,2,3), a_{i j}(t)(i, j=1,2,3)$, $\tau_{i j}(t)(i, j=1,2,3)$ are functions with respect to $t$, and it is easy to see that

$$
\begin{array}{cll}
a_{22}^{u}=\frac{49}{12}, & a_{31}^{u}=\frac{33}{4}, & r_{2}^{l}=\frac{47}{12}, \\
r_{3}^{l}=\frac{1}{4}, & a_{12}^{u}=\frac{1}{2}, & a_{13}^{u}=\frac{2}{3} \\
r_{2}^{u}=\frac{49}{12}, & a_{23}^{u}=\frac{5}{4}, & a_{31}^{l}=\frac{31}{4}, \\
r_{3}^{u}=\frac{3}{4}, & a_{32}^{u}=0.2, & r_{1}^{u}=5.5, \\
a_{11}^{l}=3.8, & \tau_{11}^{u}=1.25, & a_{22}^{l}=\frac{47}{12} \\
a_{21}^{u}=2.25, & \tau_{22}^{u}=0.5, & a_{33}^{l}=19.75 \tag{57}
\end{array}
$$

Then $M_{1}=1.2451, M_{2}=0.7395, M_{3}=2.1093, m_{1}=$ 0.6422 . Thus it is easy to see that all the conditions of Theorem 6 are satisfied. Thus system (55) is permanent which is shown in Figures 1, 2, and 3.

## 5. Conclusions

In this paper, we have investigated the dynamical behavior of a Lotka-Volterra predator-prey model with time-varying


Figure 2: The dynamical behavior of the second component of the solution $\left(x_{1}(t), x_{2}(t), x_{3}(t)\right)$.


Figure 3: The dynamical behavior of the third component of the solution $\left(x_{1}(t), x_{2}(t), x_{3}(t)\right)$.
delays. Sufficient conditions which ensure the permanence of the system are derived. Moreover, we also deal with the global stability of the system. It is shown that delay has influence on the permanence and the global stability of system. Thus delay is an important factor to decide the permanence and global stability of the system. Numerical simulations show the feasibility of our main results.

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# Research Article The Local Time of the Fractional Ornstein-Uhlenbeck Process 

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We investigate the Hölder regularity of the local time of the fractional Ornstein-Uhlenbeck process $X^{H}=\left\{X_{t}^{H}, t \geq 0\right\}$. As a related problem, we study the collision local time of two independent fractional Ornstein-Uhlenbeck $X^{H_{i}}=\left\{X_{t}^{H_{i}}, t \geq 0\right\}, i=1,2$ with respective indices $H_{1}, H_{2} \in(0,1)$.

## 1. Introduction

The Brownian motion and the Ornstein-Uhlenbeck process are the two most well-studied and widely applied stochastic processes. The Einstein-Smoluchowski theory may be seen as an idealized Ornstein-Uhlenbeck theory, and predictions of either cannot be distinguished by the experiment. However, if the Brownian particle is under the influence of an external force, the Einstein-Smoluchowski theory breaks down, while the Ornstein-Uhlenbeck theory remains successful. It is well known that a diffusion process $X=\left(X_{t}\right)_{t \geq 0}$ starting from $x \in$ $\mathbb{R}$ is called Ornstein-Uhlenbeck process with coefficients $v>$ 0 if its infinitesimal generator is

$$
\begin{equation*}
L=\frac{1}{2} v^{2} \frac{d^{2}}{d x^{2}}-x \frac{d}{d x} \tag{1}
\end{equation*}
$$

The Ornstein-Uhlenbeck process (see, e.g., Revuz and Yor [1]) has a remarkable history in physics. It is introduced to model the velocity of the particle diffusion process, and later it has been heavily used in finance, and thus in econophysics. It can be constructed as the unique strong solution of Itô stochastic differential equation

$$
\begin{equation*}
d X_{t}=-X_{t} d t+v d B_{t}, \quad X_{0}=x \tag{2}
\end{equation*}
$$

where $B$ is a standard Brownian motion starting at 0 .
Recently, as an extension of Brownian motion, fractional Brownian motion has become an object of intense study, due to its interesting properties and its applications in various scientific areas including condensed matter physics, biological physics, telecommunications, turbulence, image processing, finance, and econophysics (see, e.g., Gouyet [2], Nualart [3],

Biagini et al. [4], Mishura [5], Willinger et al. [6], and references therein). Recall that fractional Brownian motion $B^{H}$ with Hurst index $H \in(0,1)$ is a central Gaussian process with $B_{0}^{H}=0$ and the covariance function

$$
\begin{equation*}
E\left[B_{t}^{H} B_{s}^{H}\right]=\frac{1}{2}\left[t^{2 H}+s^{2 H}-|t-s|^{2 H}\right] \tag{3}
\end{equation*}
$$

for all $t, s \geqslant 0$. This process was first introduced by Kolmogorov and studied by Mandelbrot and van Ness [7], where a stochastic integral representation in terms of a standard Brownian motion was established. For $H=1 / 2, B^{H}$ coincides with the standard Brownian motion $B . B^{H}$ is neither a semimartingale nor a Markov process unless $H=1 / 2$, and so many of the powerful techniques from stochastic analysis are not available when dealing with $B^{H}$. It has self-similar, longrange dependence, Hölder paths, and it has stationary increments. These properties make $B^{H}$ an interesting tool for many applications.

On the other hand, extensions of the classical OrnsteinUhlenbeck process have been suggested mainly on demand of applications. The fractional Ornstein-Uhlenbeck process is an extension of the Ornstein-Uhlenbeck process, where fractional Brownian motion is used as integrator

$$
\begin{equation*}
d X_{t}=-X_{t} d t+v d B_{t}^{H}, \quad X_{0}=x \tag{4}
\end{equation*}
$$

Then (4) has a unique solution $X_{t}^{H}=\left\{X_{t}^{H}, 0 \leq t \leq T\right\}$, which can be expressed as

$$
\begin{equation*}
X_{t}^{H}=e^{-t}\left(x+v \int_{0}^{t} e^{s} d B_{s}^{H}\right) \tag{5}
\end{equation*}
$$

and the solution is called the fractional Ornstein-Uhlenbeck process. More work for the process can be found in Cheridito et al. [8], Lim and Muniandy [9], Metzler and Klafter [10], and Yan et al. [11, 12]. Clearly, when $H=1 / 2$, the fractional Ornstein-Uhlenbeck process is the classical OrnsteinUhlenbeck process $X$ with parameter $v$ starting at $x \in \mathbb{R}$. An advantage of using fractional Ornstein-Uhlenbeck process is to realize stationary long range dependent processes.

The intuitive idea of local time $L(t, x)$ for a stochastic process $X$ is that $L(t, x)$ measures the amount of time $X$ spends at the level $x$ during the interval $[0, t]$. Moreover, since the work of Varadhan [13], the local time of stochastic processes has become an important subject. Therefore, it seems interesting to study the local time of fractional Ornstein-Uhlenbeck process, a rather special class of Gaussian processes.

In this paper, we focus our attention on the Hölder regularity of the local time of fractional Ornstein-Uhlenbeck process.

The rest of this paper is organized as follows. Section 2 contains a brief review on the local times of Gaussian processes and the approach of chaos expansion of the Gaussian process. In Section 3, we give Hölder regularity of the local time. In Section 4, as a related problem, we study the so-called collision local time of two independent fractional OrnsteinUhlenbeck $X^{H_{i}}=\left\{X_{t}^{H_{i}}, t \geq 0\right\}, i=1,2$ with respective indices $H_{1}, H_{2} \in(0,1)$.

## 2. Preliminaries

2.1. Local Times and Local Nondeterminism. We recall briefly the definition of local time. For a comprehensive survey on local times of both random and nonrandom vector fields, we refer to Alder [14], Geman and Horowitz [15], and Xiao [1618]. Let $X(t)$ be any Borel function on $\mathbb{R}$ with values in $\mathbb{R}$. For any Borel set $B \subset \mathbb{R}$, the occupation measure of $X$ is defined by

$$
\begin{equation*}
\mu_{B}(A)=\lambda_{1}\{t \in B, X(t) \in A\} \tag{6}
\end{equation*}
$$

for all Borel set $A \subseteq \mathbb{R}$, where $\lambda_{1}$ is the one-dimensional Lebesgue measure. If $\mu_{B}$ is absolutely continuous with respect to the Lebesgue measure $\lambda_{1}$ on $\mathbb{R}$, we say that $X(t)$ has a local time on $B$ and define its local time $L(B, x)$ to be the RadonNikodym derivative of $\mu_{B}$. If $B=[0, t]$, we simply write $L(B, x)$ as $L(t, x)$. If $I=[0, T]$ and $L(t, x)$ is continuous as a function of $(t, x) \in I \times \mathbb{R}$, then we say that $X$ has a jointly continuous local time on $I$. In this latter case, the set function $L(\cdot, x)$ can be extended to be a finite Borel measure on the level set (see Adler [14, Theorem 8.6.1])

$$
\begin{equation*}
X_{I}^{-1}(x)=\{t \in I: X(t)=x\} \tag{7}
\end{equation*}
$$

This fact has been used by many authors to study fractal properties of level sets, inverse image, and multiple times of stochastic processes. For example, Xiao [16] and Hu [19] have studied the Hausdorff dimension, and exact Hausdorff and packing measure of the level sets of iterated Brownian motion, respectively.

For a fixed sample function at fixed $t$, the Fourier transform on $x$ of $L(t, x)$ is the function

$$
\begin{equation*}
f(t, u)=\int_{\mathbb{R}} e^{i u x} L(t, x) d x \tag{8}
\end{equation*}
$$

Using the density of occupation formula we have

$$
\begin{equation*}
f(t, u)=\int_{0}^{t} e^{i u X(s)} d s \tag{9}
\end{equation*}
$$

We can express the local times $L(t, x)$ as the inverse Fourier transform of $f(t, u)$, namely,

$$
\begin{equation*}
L(t, x)=\frac{1}{2 \pi} \int_{-\infty}^{+\infty}\left(\int_{0}^{t} e^{i u(X(s)-x)} d s\right) d u \tag{10}
\end{equation*}
$$

It follows from (10) that for any $x, y \in \mathbb{R}, t, t+\omega \in[0, T]$ and any integer $n \geq 2$, we have (see, e.g., Boufoussi et al. [20, 21])

$$
\begin{align*}
& E(L(t+h, x)-L(t, x))^{n} \\
& \begin{aligned}
&=\frac{1}{(2 \pi)^{n}} \int_{[t, t+\omega]^{n}} \int_{\mathbb{R}^{n}} e^{-i x \sum_{j=1}^{n} u_{j}} E\left(e^{i \sum_{j=1}^{n} u_{j} X\left(s_{j}\right)}\right) \\
& \times \prod_{j=1}^{n} d u_{j} \prod_{j=1}^{n} d s_{j}
\end{aligned} \tag{11}
\end{align*}
$$

and for every even integer $n \geq 2$,

$$
\begin{align*}
& E(L(t+h, x)-L(t, x)-L(t+h, y)+L(t, y))^{n} \\
&=\frac{1}{(2 \pi)^{n}} \int_{[t, t+\omega]^{n}} \int_{\mathbb{R}^{n}} \prod_{j=1}^{n}\left[e^{-i x u_{j}}-e^{-i y u_{j}}\right] \\
& \times E\left(e^{i \sum_{j=1}^{n} u_{j} X\left(s_{j}\right)}\right) \prod_{j=1}^{n} d u_{j} \prod_{j=1}^{n} d s_{j} . \tag{12}
\end{align*}
$$

The concept of local nondeterminism was first introduced by Berman [22] to unify and extend his methods for studying local times of real-valued Gaussian processes. Let $X=\left\{X(t), t \in \mathbb{R}_{+}\right\}$be a real-valued, separable Gaussian process with mean 0 and let $T \subset \mathbb{R}_{+}$be an open interval. Assume that $E\left[X(t)^{2}\right]>0$ for all $t \in T$ and there exists $\delta>0$ such that

$$
\begin{equation*}
E\left[(X(s)-X(t))^{2}\right]>0 \tag{13}
\end{equation*}
$$

for $s, t \in T$ with $0<|s-t|<\delta$.
Recall from Berman [22] that $X$ is called locally nondeterministic on $T$ if for every integer $n \geq 2$,

$$
\begin{equation*}
\lim _{\epsilon \rightarrow 0} \inf _{t_{n}-t_{1} \leq \epsilon} V_{n}>0 \tag{14}
\end{equation*}
$$

where $V_{n}$ is the relative prediction error as follows:

$$
\begin{equation*}
V_{n}=\frac{\operatorname{Var}\left(X\left(t_{n}\right)-X\left(t_{n-1}\right) \mid X\left(t_{1}\right), \ldots, X\left(t_{n-1}\right)\right)}{\operatorname{Var}\left(X\left(t_{n}\right)-X\left(t_{n-1}\right)\right)} \tag{15}
\end{equation*}
$$

and the infimum in (14) is taken over all ordered points $t_{1}<$ $t_{2}<\cdots<t_{n}$ in $T$ with $t_{n}-t_{1} \leq \epsilon$. Roughly speaking, (14)
means that a small increment of the process $X$ is not almost relatively predictable based on a finite number of observations from the immediate past.

It follows from Berman [22, Lemma 2.3] that (14) is equivalent to the following property which says that $X$ has locally approximately independent increments: for any positive integer $n \geq 2$, there exist positive constants $C_{n}$ and $\delta$ (both may depend on $n$ ) such that

$$
\begin{align*}
& \operatorname{Var}\left(\sum_{j=1}^{n} u_{j}\left[X\left(t_{j}\right)-X\left(t_{j-1}\right)\right]\right)  \tag{16}\\
& \quad \geq C_{n} \sum_{j=1}^{n} u_{j}^{2} \operatorname{Var}\left[X\left(t_{j}\right)-X\left(t_{j-1}\right)\right]
\end{align*}
$$

for all ordered points $0=t_{0}<t_{1}<t_{2}<\cdots<t_{n}$ in $T$ with $t_{n}-t_{1}<\delta$ and all $u_{j} \in \mathbb{R}(1 \leq j \leq n)$. We refer to Nolan [23, Theorem 2.6] for a proof of the above equivalence in a much more general setting.

For simplicity throughout this paper we let $C_{n}$ stand for a positive constant depending only on the subscripts and its value may be different in different appearances, and this assumption is also adaptable to $C, C_{H}$.
2.2. Chaos Expansion. Let $\Omega$ be the space of continuous $\mathbb{R}^{1}$ valued functions $\omega$ on $[0, T]$. Then $\Omega$ is a Banach space with respect to the supreme norm. Let $\mathscr{F}$ be the $\sigma$-algebra on $\Omega$. Let $P$ be the probability measure on the measurable space $(\Omega, \mathscr{F})$. Let $\mathbb{E}$ denote the expectation on this probability space. The set of all square integrable functionals is denoted by $L^{2}(\Omega, P)$, that is,

$$
\begin{equation*}
E\left(F^{2}\right)=\int_{\Omega}|F(\omega)|^{2} P(d \omega)<\infty \tag{17}
\end{equation*}
$$

We can introduce the chaos expansion, which is an orthogonal decomposition of $L^{2}(\Omega, P)$. We refer to Hu [24], Nualart [3], and the references therein for more details. Let $X:=\left\{X_{t}, t \in[0, T]\right\}$ be a Gaussian process defined on the probality space $(\Omega, \mathscr{F}, P)$. If $p_{n}(x)$ is a polynomial of degree $n$ in $x$, then we call $p_{n}\left(X_{t}\right)$ a polynomial function of $X$ with $t \in$ $[0, T]$. Let $\mathscr{P}_{n}$ be the completion with respect to the $L^{2}(\Omega, P)$ norm of the set $\left\{p_{m}\left(X_{t}\right): 0 \leq m \leq n, t \in[0, T]\right\}$. Clearly, $\mathscr{P}_{n}$ is a subspace of $L^{2}(\Omega, P)$. If $\mathscr{C}_{n}$ denotes the orthogonal complement of $\mathscr{P}_{n-1}$ in $\mathscr{P}_{n}$, then $L^{2}(\Omega, P)$ is actually the direct sum of $\mathscr{C}_{n}$, that is,

$$
\begin{equation*}
L^{2}(\Omega, P)=\bigoplus_{n=0}^{\infty} \mathscr{C}_{n} \tag{18}
\end{equation*}
$$

Namely, for any functional $F \in L^{2}(\Omega, P)$, there are $F_{n}$ in $\mathscr{C}_{n}$, $n=0,1,2, \ldots$, such that

$$
\begin{equation*}
F=\sum_{n=0}^{\infty} F_{n} . \tag{19}
\end{equation*}
$$

The decomposition equation (19) is called the chaos expansion of $F$, and $F_{n}$ is called the $n$th chaos of $F$. Clearly, we have

$$
\begin{equation*}
E\left(|F|^{2}\right)=\sum_{n=0}^{\infty} E\left(\left|F_{n}\right|^{2}\right) \tag{20}
\end{equation*}
$$

Recall that Meyer-Watanabe test function space $\mathscr{U}$ (see Watanabe [25]) is defined as

$$
\begin{equation*}
\mathscr{U}:=\left\{F \in L^{2}(\Omega, P): F=\sum_{n=0}^{\infty} F_{n}, \sum_{n=0}^{\infty} n E\left(\left|F_{n}\right|^{2}\right)<\infty\right\}, \tag{21}
\end{equation*}
$$

and $F \in L^{2}(\Omega, P)$ is said to be smooth if $F \in \mathscr{U}$.
Now, for $F \in L^{2}(\Omega, P)$, we define an operator $\Gamma_{u}$ with $u \in$ $[0,1]$ by

$$
\begin{equation*}
\Gamma_{u} F:=\sum_{n=0}^{\infty} u^{n} F_{n} . \tag{22}
\end{equation*}
$$

Set $\Theta(u):=\Gamma_{\sqrt{u}} F$. Then $\Theta(1)=F$. Define $\Phi_{\Theta}(u):=(d / d u)$ $\left(\|\Theta(u)\|^{2}\right)$, where $\|F\|^{2}:=E\left(|F|^{2}\right)$ for $F \in L^{2}(\Omega, P)$. We have

$$
\begin{equation*}
\Phi_{\Theta}(u)=\sum_{n=1}^{\infty} n u^{n-1} E\left(\left|F_{n}\right|^{2}\right) \tag{23}
\end{equation*}
$$

Note that $\|\Theta(u)\|^{2}=E\left(|\Theta(u)|^{2}\right)=\sum_{n=0}^{\infty} E\left(u^{n}\left|F_{n}\right|^{2}\right)$.
Proposition 1. Let $F \in L^{2}(\Omega, P)$. Then $F \in \mathscr{U}$ if and only if $\Phi_{\Theta}(1)<\infty$.

Consider two independent fractional Ornstein-Uhlenbeck $X^{H_{i}}=\left\{X_{t}^{H_{i}}, \quad t \geq 0\right\}, i=1,2$, with respective indices $H_{i} \in(0,1)$. Let $H_{n}(x)$ and $x \in \mathbb{R}$ be the Hermite polynomials of degree $n$. That is,

$$
\begin{equation*}
H_{n}(x)=(-1)^{n} \frac{1}{n!} e^{x^{2} / 2} \frac{\partial^{n}}{\partial x^{n}} e^{-x^{2} / 2} \tag{24}
\end{equation*}
$$

Then,

$$
\begin{equation*}
e^{t x-t^{2} / 2}=\sum_{n=0}^{\infty} t^{n} H_{n}(x) \tag{25}
\end{equation*}
$$

for all $t \in \mathscr{C}$ and $x \in \mathbb{R}$, this implies that

$$
\begin{align*}
& \exp \left(i u \xi\left(X_{t}^{H_{1}}-X_{t}^{H_{2}}\right)+\frac{1}{2} u^{2} \xi^{2} \operatorname{Var}\left(X_{t}^{H_{1}}-X_{t}^{H_{2}}\right)\right) \\
& \quad=\sum_{n=0}^{\infty}(i u)^{n} \sigma^{n}(t, \xi) H_{n}\left(\frac{\xi\left(X_{t}^{H_{1}}-X_{t}^{H_{2}}\right)}{\sigma(t, \xi)}\right) \tag{26}
\end{align*}
$$

where $i=\sqrt{-1}$ and $\sigma(t, \xi)=\sqrt{\operatorname{Var}\left(X_{t}^{H_{1}}-X_{t}^{H_{2}}\right) \xi^{2}}$ for $\xi \in \mathbb{R}$. Because of the orthogonality of $\left\{H_{n}(x), x \in \mathbb{R}\right\}_{n \in \mathbb{Z}_{+}}$, we will get from (19) that

$$
\begin{equation*}
(i u)^{n} \sigma^{n}(t, \xi) H_{n}\left(\frac{\xi\left(X_{t}^{H_{1}}-X_{t}^{H_{2}}\right)}{\sigma(t, \xi)}\right) \tag{27}
\end{equation*}
$$

is the $n$th chaos of $\exp \left(i u \xi\left(X_{t}^{H_{1}}-X_{t}^{H_{2}}\right)+(1 / 2) u^{2} \xi^{2} \operatorname{Var}\left(X_{t}^{H_{1}}-\right.\right.$ $\left.B_{t}^{H_{2}}\right)$ ) for all $t \geq 0$.

## 3. Local Time of Fractional Ornstein-Uhlenbeck Process

In this section, we offer the Hölder regularity of the local time of fractional Ornstein-Uhlenbeck process.

Theorem 2. Let $\left\{X_{t}^{H}, t \geq 0\right\}$ be the fractional Ornstein-Uhlenbeck process. Then, for every $t \in \mathbb{R}^{+}$and any $x \in \mathbb{R}$, there exist positive and finite constants $C_{1}$ and $C_{2}$ such that

$$
\begin{align*}
& \lim _{h \rightarrow 0} \sup _{x} \sup _{x} \frac{L(t+h, x)-L(t, x)}{h^{1-H}\left(\log \log \left(h^{-1}\right)\right)^{H}} \leq C_{1} \quad \text { a.s. }  \tag{28}\\
& \lim \sup \sup _{x, t} \frac{L(t+h, x)-L(t, x)}{h^{1-H}\left(\log \left(h^{-1}\right)\right)^{H}} \leq C_{2} \quad \text { a.s. } \tag{29}
\end{align*}
$$

Proof. Let $t \geq 0$ be a fixed point. Following the Fourier analytic approach of Berman [26], we have

$$
\begin{align*}
& E[L(t+h, x)-L(t, x)]^{n} \\
& =\frac{1}{(2 \pi)^{n}} \\
& \quad \times \int_{[t, t+h]^{n}} \int_{\mathbb{R}^{n}} E\left(\exp \left(i \sum_{j=1}^{n} u_{j}\left(X_{s_{j}}^{H}-X_{t}^{H}\right)\right)\right)  \tag{30}\\
& \quad \times \prod_{j=1}^{n} d u_{j} \prod_{j=1}^{n} d s_{j}
\end{align*}
$$

Let $\Delta X_{s}^{H}=X_{s}^{H}-X_{t}^{H}, s \geq 0$, and denote by $R\left(s_{1}, s_{2}, \ldots, s_{n}\right)$ the covariance matrix of $\left(\Delta X_{s_{1}}^{H}, \ldots, \Delta X_{s_{n}}^{H}\right)$ for different $s_{1}, \ldots, s_{n}$, then we have

$$
\begin{align*}
& \operatorname{det} R\left(s_{1}, s_{2}, \ldots, s_{n}\right) \\
& =\operatorname{Var}\left(\Delta X_{s_{1}}^{H}\right) \operatorname{Var}\left(\Delta X_{s_{2}}^{H} \mid \Delta X_{s_{1}}^{H}\right) \ldots  \tag{31}\\
& \quad \operatorname{Var}\left(\Delta X_{s_{n}}^{H} \mid \Delta X_{s_{1}}^{H}, \ldots, \Delta X_{s_{n-1}}^{H}\right)
\end{align*}
$$

By Yan et al. [11], one can write the fractional Ornstein-Uhlenbeck process starting from zero as

$$
\begin{equation*}
X_{t}^{H}=v \int_{0}^{t} F(t, u) d B_{u}, \quad 0 \leq t \leq T, \tag{32}
\end{equation*}
$$

where $B$ is a standard Brownian motion with $B_{0}=0$, and for $0<u<t$

$$
\begin{align*}
F(t, u)= & \left(H-\frac{1}{2}\right) \kappa_{H} e^{-t} u^{1 / 2-H} \\
& \times \int_{u}^{t} s^{H-1 / 2}(s-u)^{H-3 / 2} e^{s} d s \tag{33}
\end{align*}
$$

with $1 / 2<H<1, \kappa_{H}=(2 H \Gamma((3 / 2)-H) / \Gamma(H+(1 / 2)) \Gamma(2-$ $2 H))^{1 / 2}$, and

$$
\begin{align*}
& F(t, u)=\kappa_{H} u^{1 / 2-H} \\
& \qquad \begin{aligned}
& \times\left(-e^{-t} \int_{u}^{t}(s-u)^{H-1 / 2} s^{H-1 / 2} e^{s} d s\right. \\
& +t^{H-1 / 2}(t-u)^{H-1 / 2}+\frac{2}{1-2 H} e^{-t} \\
& \left.\times \int_{u}^{t}(s-u)^{H-1 / 2} s^{H-3 / 2} e^{s} d s\right),
\end{aligned}
\end{align*}
$$

with $0<H<1 / 2$.
For any $r, s \in[t, t+h]$ such that $r<s$, we have

$$
\begin{align*}
\operatorname{Var}\left(\Delta X_{s}^{H} \mid \Delta X_{u}^{H}, u \leq r\right) & \geq \operatorname{Var}\left(\Delta X_{s}^{H}-\Delta X_{r}^{H} \mid B_{u}, u \leq r\right) \\
& =\operatorname{Var}\left(X_{s}^{H}-X_{r}^{H} \mid B_{u}, u \leq r\right) \\
& =\operatorname{Var}\left(X_{s}^{H} \mid B_{u}, u \leq r\right), \tag{35}
\end{align*}
$$

where the last equality follows from the fact that $X_{r}^{H}$ is measurable with respect to $\sigma\left(B_{u}, u \leq r\right)$. Moreover, we can write

$$
\begin{equation*}
X_{s}^{H}=v \int_{0}^{s} F(s, u) d B_{u}=v \int_{0}^{r} F(s, u) d B_{u}+v \int_{r}^{s} F(s, u) d B_{u} . \tag{36}
\end{equation*}
$$

Hence, by using the measurability of $\int_{0}^{r} F(s, u) d B_{u}$ with respect to $\sigma\left(B_{u}, u \leq s\right)$, we have

$$
\begin{align*}
\operatorname{Var}\left(X_{s}^{H} \mid B_{u}, u \leq r\right) & =\operatorname{Var}\left(v \int_{r}^{s} F(s, u) d B_{u} \mid B_{u}, u \leq r\right) \\
& =\operatorname{Var}\left(v \int_{r}^{s} F(s, u) d B_{u}\right) \\
& \geq C_{H}(s-r)^{2 H}, \tag{37}
\end{align*}
$$

where, to obtain the second equality, we have used the fact that $\int_{r}^{s} F(s, u) d B_{u}$ is independent of $\sigma\left(B_{u}, u \leq s\right)$ (by the independence of the increments of the Brownian motion). Combining (31), inequation (35), and inequation (37), we have

$$
\begin{equation*}
\operatorname{det} R\left(s_{1}, s_{2}, \ldots, s_{n}\right) \geq C_{H} \prod_{j=1}^{n}\left(s_{j}-s_{j-1}\right)^{2 H}>0 \tag{38}
\end{equation*}
$$

where $s_{0}=0$. Hence, the change of variable $V=R^{1 / 2} U, U=$ $\left(u_{1}, \ldots, u_{n}\right)$ implies that

$$
\begin{array}{rl}
\int_{\mathbb{R}^{n}} & E\left(\exp \left(i \sum_{j=1}^{n} u_{j}\left(X_{s_{j}}^{H}-X_{t}^{H}\right)\right)\right) \prod_{j=1}^{n} d u_{j} \\
\quad=\int_{\mathbb{R}^{n}} E\left(\exp \left(i \sum_{j=1}^{n} u_{j} \Delta X_{s_{j}}^{H}\right)\right) \prod_{j=1}^{n} d u_{j}  \tag{39}\\
= & \frac{(2 \pi)^{n / 2}}{\left(\operatorname{det} R\left(s_{1}, s_{2}, \ldots, s_{n}\right)\right)^{1 / 2}} .
\end{array}
$$

Hence,

$$
\begin{align*}
E[ & L(t+h, x)-L(t, x)]^{n} \\
= & \frac{n!}{(2 \pi)^{n / 2}} \\
& \times \int_{t<s_{1}<\cdots<s_{n}<t+h} \frac{1}{\left(\operatorname{det} R\left(s_{1}, s_{2}, \ldots, s_{n}\right)\right)^{1 / 2}} d s_{1} \cdots d s_{n} \\
\leq & C_{H} \frac{n!}{(2 \pi)^{n / 2}} \int_{t<s_{1}<\cdots<s_{n}<t+h} \prod_{j=1}^{n} \frac{1}{\left(s_{j}-s_{j-1}\right)^{H}} d s_{1} \cdots d s_{n} \\
\leq & C_{H} \frac{n!}{(2 \pi)^{n / 2}} h^{n(1-H)} \frac{(\Gamma(1-H))^{n}}{\Gamma(1+n(1-H))} . \tag{40}
\end{align*}
$$

Following from Stirling's formula, we have $n!/ \Gamma(1+n(1-$ $H)) \leq A^{n} n!^{H}, n \geq 2$, for a suitable finite number $A$. So

$$
\begin{equation*}
E\left(\frac{L(t+h, x)-L(t, x)}{h^{1-H}}\right)^{n} \leq C^{n} n!^{H} \tag{41}
\end{equation*}
$$

Following, we first prove that for any $K>0$, there exists a positive and finite constant $B>0$, depending on $t$, such that for sufficiently small $u$

$$
\begin{equation*}
P\left(L(t+h, x)-L(t, x) \geq \frac{B h^{1-H}}{u^{H}}\right) \leq e^{-(K / u)} \tag{42}
\end{equation*}
$$

First consider $u$ of the form $u=1 / n$. By Chebyshev's inequality and inequation (41), we have

$$
\begin{aligned}
P & \left(L(t+h, x)-L(t, x) \geq B h^{1-H} n^{H}\right) \\
& \leq E\left(\frac{L(t+h, x)-L(t, x)}{B h^{1-H} n^{H}}\right)^{n} \\
& \leq \frac{C^{n}}{B^{n}}\left(\frac{1}{n}\right)^{n H} n!^{H}(\text { by Stirling's formula }) \\
& \leq \frac{C^{n}}{B^{n}}(2 \pi n)^{H / 2} e^{-H n} \\
& =\exp \left(n\left(\log \left(\frac{C}{B}\right)-H\right)+\frac{H}{2}(\log n+\log 2 \pi)\right) .
\end{aligned}
$$

Choose $B>C$ and $n_{0}$ large such that for any $n \geq n_{0}$, to dominate (43) by $e^{-2 K n}$. Moreover, for $u$ sufficiently small, there exists $n \geq n_{0}$ such that $u_{n+1}<u<u_{n}$ and since $n \geq 1$, $n /(m+1) \geq 1 / 2$. This proves inequation (42).

On the other hand, if we take $u(h)=1 / \log \log (1 / h)$ and consider $h_{n}$ of the form $2^{-n}$, then inequation (42) implies

$$
\begin{align*}
& P\left(L\left(t+h_{n}, x\right)-L(t, x) \geq B h_{n}^{1-H}\left(\log \log \left(\frac{1}{h_{n}}\right)\right)^{H}\right) \\
& \quad \leq n^{-2} \tag{44}
\end{align*}
$$

for large $n$. So, following that Borel-Cantelli lemma and monotonicity arguments, we have

$$
\begin{equation*}
\frac{L(t+h, x)-L(t, x)}{h^{1-H}} \leq B\left(\log \log \left(\frac{1}{h_{n}}\right)\right)^{H} \quad \text { a.s. } \tag{45}
\end{equation*}
$$

This completes the proof of inequation (28). we can obtain inequation (29) in the similar manner.

## 4. Existence and Smoothness of Collision Local Time

In this section we will study the so-called collision local time of two independent fractional Ornstein-Uhlenbeck $X^{H_{i}}=$ $\left\{X_{t}^{H_{i}}, t \geq 0\right\}, i=1,2$. It is defined formally by the following expression:

$$
\begin{equation*}
\ell_{T}=\int_{0}^{T} \delta_{0}\left(X_{t}^{H_{1}}-X_{t}^{H_{2}}\right) d t \tag{46}
\end{equation*}
$$

where $\delta_{0}$ is the Dirac delta function. It is a measure of the amount of time for which the trajectories of the two processes, $X_{t}^{H_{1}}$ and $X_{t}^{H_{2}}$, collide on the time interval $[0, T]$. The collision local time for fractional Brownian motion has been studied by Jiang and Wang [27]. We shall show that the random variable $\ell_{T}$ exists in $L^{2}$. We approximate the Dirac delta function by the heat kernel

$$
\begin{equation*}
p_{\varepsilon}(x)=\frac{1}{\sqrt{2 \pi \varepsilon}} e^{-x^{2} / 2 \varepsilon} \equiv \frac{1}{2 \pi} \int_{\mathbb{R}} e^{i x \xi} e^{-\varepsilon\left(\xi^{2} / 2\right)} d \xi \tag{47}
\end{equation*}
$$

For $\varepsilon>0$ we define

$$
\begin{align*}
\ell_{\varepsilon, T} & =\int_{0}^{T} p_{\epsilon}\left(X_{t}^{H_{1}}-X_{t}^{H_{2}}\right) d t \\
& =\frac{1}{2 \pi} \int_{0}^{T} \int_{\mathbb{R}} e^{i \xi\left(X_{t}^{H_{1}}-X_{t}^{H_{2}}\right)} \cdot e^{-\varepsilon\left(\xi^{2} / 2\right)} d \xi d t \tag{48}
\end{align*}
$$

and a natural question to study is that of the behavior of $\ell_{\varepsilon, T}$ as $\varepsilon$ tends to zero.

Theorem 3. For $H_{i} \in(0,1), i=1,2$. Then $\ell_{\varepsilon, T}$ converges in $L^{2}(\Omega, \mathscr{F}, P)$, as $\varepsilon \downarrow 0$. Moreover, the limit is denoted by $\ell_{T}$, then $\ell_{T} \in L^{2}(\Omega, \mathscr{F}, P)$.

Proof. First we claim that $\ell_{\varepsilon, T} \in L^{2}(\Omega, \mathscr{F}, P)$ for every $\varepsilon>0$. By (48) we have

$$
\begin{align*}
& E\left(\ell_{\varepsilon, T}^{2}\right) \\
& \begin{array}{l}
=\frac{1}{4 \pi^{2}} \iint_{0}^{T} \int_{\mathbb{R}^{2}} E e^{i \xi\left(X_{t}^{H_{1}}-X_{t}^{H_{2}}\right)+i \eta\left(X_{s}^{H_{1}}-X_{s}^{H_{2}}\right)} \\
\quad \times e^{-\left(\left(\varepsilon\left(\xi^{2}+\eta^{2}\right)\right) / 2\right)} d \xi d \eta d s d t \\
=\frac{1}{4 \pi^{2}} \iint_{0}^{T} \int_{\mathbb{R}^{2}} e^{-(1 / 2) \sigma^{2}} e^{-\left(\left(\varepsilon\left(\xi^{2}+\eta^{2}\right)\right) / 2\right)} d \xi d \eta d s d t
\end{array}
\end{align*}
$$

where $\sigma^{2}$ denotes the variance of random variable $\xi\left(X_{t}^{H_{1}}-\right.$ $\left.X_{t}^{H_{2}}\right)+\eta\left(X_{s}^{H_{1}}-X_{s}^{H_{2}}\right)$, that is,

$$
\begin{equation*}
\sigma^{2}:=\operatorname{Var}\left(\xi\left(X_{t}^{H_{1}}-X_{t}^{H_{2}}\right)+\eta\left(X_{s}^{H_{1}}-X_{s}^{H_{2}}\right)\right) . \tag{50}
\end{equation*}
$$

According to the property of local nondeterminism (see Theorem 3.1 in [11]), we have

$$
\begin{align*}
& \sigma^{2}=\operatorname{Var}\left(\xi\left(X_{t}^{H_{1}}-X_{s}^{H_{1}}\right)-\xi\left(X_{t}^{H_{2}}-X_{s}^{H_{2}}\right)\right. \\
&\left.+(\xi+\eta)\left(X_{s}^{H_{1}}-X_{s}^{H_{2}}\right)\right) \\
& \geq C {\left[\xi^{2}\left((t-s)^{2 H_{1}}+(t-s)^{2 H_{2}}\right)\right.}  \tag{51}\\
&\left.+(\xi+\eta)^{2}\left(s^{2 H_{1}}+s^{2 H_{2}}\right)\right] .
\end{align*}
$$

Thus, we have

$$
\begin{align*}
& \frac{1}{4 \pi^{2}} \iint_{0}^{T} \int_{\mathbb{R}^{2}} e^{-(1 / 2) \sigma^{2}} e^{-\left(\varepsilon\left(\xi^{2}+\eta^{2}\right)\right) / 2} d \xi d \eta d s d t \\
& \leq \int_{0}^{T} \int_{0}^{t} \int_{\mathbb{R}^{2}} e^{-(C / 2)\left[\xi^{2}\left((t-s)^{2 H_{1}}+(t-s)^{2 H_{2}}\right)+(\xi+\eta)^{2}\left(s^{2 H_{1}}+s^{2 H_{2}}\right)\right]} d \xi d \eta d s d t \\
& =C \int_{0}^{T} \int_{0}^{t}\left[\left((t-s)^{2 H_{1}}+(t-s)^{2 H_{2}}\right)\right. \\
& \left.\quad \times\left(s^{2 H_{1}}+s^{2 H_{2}}\right)\right]^{-1 / 2} d s d t \\
& \leq C \int_{0}^{T} \int_{0}^{t}(t-s)^{-(1 / 2)\left(H_{1}+H_{2}\right)} s^{-(1 / 2)\left(H_{1}+H_{2}\right)} d s d t<\infty, \tag{52}
\end{align*}
$$

because of $H_{i} \in(0,1)$, which yields

$$
\begin{equation*}
E\left(\ell_{\varepsilon, T}^{2}\right)<\infty \tag{53}
\end{equation*}
$$

for all $\varepsilon \in(0,1]$.

Second, we claim that the sequence $\left\{\ell_{\varepsilon, T}, \varepsilon>0\right\}$ is of Cauchy in $L^{2}(\Omega, \mathscr{F}, P)$. For any $\theta, \varepsilon>0$ we have

$$
\begin{align*}
& E\left(\left|\ell_{\varepsilon, T}-e_{\theta, T}\right|^{2}\right) \\
& =\frac{1}{4 \pi^{2}} \iint_{0}^{T} \int_{\mathbb{R}^{2}} E e^{i \xi\left(X_{t}^{H_{1}}-X_{t}^{H_{2}}\right)+i \eta\left(X_{s}^{H_{1}}-X_{s}^{H_{2}}\right)} \\
& \cdot\left(e^{-(\varepsilon / 2) \xi^{2}}-e^{-(\theta / 2) \xi^{2}}\right) \\
& \times\left(e^{-(\varepsilon / 2) \eta^{2}}-e^{-(\theta / 2) \eta^{2}}\right) d \xi d \eta d s d t \\
& \leq \frac{1}{4 \pi^{2}} \sup _{\xi \in \mathbb{R}}\left(1-e^{-\left(\left(|\varepsilon-\theta|^{2}|\xi|^{2}\right) / 2\right)}\right)^{2} \\
& \quad \times \iint_{0}^{T} \int_{\mathbb{R}^{2}} e^{-(1 / 2) \sigma^{2}} d \xi d \eta d s d t . \tag{54}
\end{align*}
$$

Thus, dominated convergence theorem yields

$$
\begin{equation*}
E\left(\left|\ell_{\varepsilon, T}-\ell_{\theta, T}\right|^{2}\right) \longrightarrow 0 \tag{55}
\end{equation*}
$$

as $\varepsilon \rightarrow 0$ and $\theta \rightarrow 0$, which leads to $\ell_{\varepsilon, T}$ is a Cauchy sequence in $L^{2}(\Omega, \mathscr{F}, P)$. Consequently, $\lim _{\varepsilon \rightarrow 0} \ell_{\varepsilon, T}$ exists in $L^{2}(\Omega, \mathscr{F}, P)$. This completes the proof.

For the increments of collision local time we have the following.

Theorem 4. Let $H_{1}, H_{2} \in(0,1)$ and $\beta=\min \left\{H_{1}, H_{2}\right\}$. Then the collision local time $\ell_{T}$ satisfies the following estimate:

$$
\begin{equation*}
E\left(\left|\ell_{t}-\ell_{s}\right|^{2}\right) \leq C_{H_{1}, H_{2}}(t-s)^{2-2 \beta}, \tag{56}
\end{equation*}
$$

for all $s, t, s<t$.
Proof. For any $0 \leq r, l \leq T$ we denote

$$
\begin{equation*}
\sigma_{r, l}^{2}:=\operatorname{Var}\left(\xi\left(X_{r}^{H_{1}}-X_{r}^{H_{2}}\right)+\eta\left(X_{l}^{H_{1}}-X_{l}^{H_{2}}\right)\right) \tag{57}
\end{equation*}
$$

Then the property of local nondeterminism (see Theorem 3.1 in [11]) yields

$$
\begin{align*}
& \sigma_{r, l}^{2} \geq C\left[\xi^{2}\left((r-l)^{2 H_{1}}+(r-l)^{2 H_{2}}\right)\right. \\
&\left.+(\xi+\eta)^{2}\left(l^{2 H_{1}}+l^{2 H_{2}}\right)\right] \tag{58}
\end{align*}
$$

for a constant $C>0$. It follows from (48) that for $0 \leq s \leq t \leq$ T

$$
\begin{align*}
& E\left(\left|\ell_{\varepsilon, t}-\ell_{\varepsilon, s}\right|^{2}\right) \\
& \quad=\frac{2}{(2 \pi)^{2}} \int_{s}^{t} \int_{s}^{r} d r d l \int_{\mathbb{R}^{2}} e^{-(1 / 2) \sigma_{r, l}^{2}} e^{-(\varepsilon / 2)\left(\xi^{2}+\eta^{2}\right)} d \xi d \eta  \tag{59}\\
& \quad \leq C \int_{s}^{t} d r \int_{s}^{r}(r-l)^{-\beta} l^{-\beta} d l \\
& \quad \leq C(t-s)^{2-2 \beta}
\end{align*}
$$

Thus, Theorem 3 and Fatou's lemma yield

$$
\begin{align*}
& E\left(\left|\ell_{t}-\ell_{s}\right|^{2}\right) \\
& \quad=E\left(\lim _{\varepsilon \rightarrow 0}\left|\ell_{\varepsilon, t}-\ell_{\varepsilon, s}\right|^{2}\right)  \tag{60}\\
& \quad \leq \liminf _{\varepsilon \rightarrow 0} E\left(\left|\ell_{\varepsilon, t}-\ell_{\varepsilon, s}\right|^{2}\right) \leq C(t-s)^{2-2 \beta} .
\end{align*}
$$

This completes the proof.
Let $\lambda_{t}=\operatorname{Var}\left(X_{t}^{H_{1}}-X_{t}^{H_{2}}\right)$ for $t \geq 0$ and

$$
\begin{equation*}
\rho_{s, t}=E\left[\left(X_{t}^{H_{1}}-X_{t}^{H_{2}}\right)\left(X_{s}^{H_{1}}-X_{s}^{H_{2}}\right)\right] \tag{61}
\end{equation*}
$$

for $s, t \geq 0$.
Lemma 5 (An and Yan [28]). For any $x \in[-1,1)$ we have

$$
\begin{equation*}
\sum_{n=1}^{\infty} \frac{(2 n-1)!!}{(2 n-2)!!} x^{n}=x(1-x)^{-3 / 2} \tag{62}
\end{equation*}
$$

where $(2 n-2)!!=1 \cdot 3 \cdot 5 \cdots(2 n-1)$ and $(2 n-1)!!=2 \cdot 4$. $6 \cdots(2 n-2)$.

By Cauchy-Schwartz's inequality, we have $\rho_{s, t}^{2} \leq \lambda_{s} \lambda_{t}$. Hence,

$$
\begin{align*}
\frac{\rho_{s, t}^{2}}{\left(\lambda_{s} \lambda_{t}-\rho_{s, t}^{2}\right)^{3 / 2}} & =\frac{\rho_{s, t}^{2}}{\lambda_{s} \lambda_{t}}\left(1-\frac{\rho_{s, t}^{2}}{\lambda_{s} \lambda_{t}}\right)^{-3 / 2}\left(\frac{1}{\lambda_{s} \lambda_{t}}\right)^{1 / 2}  \tag{63}\\
& =\sum_{n=1}^{\infty} \frac{(2 n-1)!!}{(2 n-2)!!}\left(\frac{\rho_{s, t}^{2}}{\lambda_{s} \lambda_{t}}\right)^{n}\left(\frac{1}{\lambda_{s} \lambda_{t}}\right)^{1 / 2}
\end{align*}
$$

for all $t, s \geq 0$ and $s \neq t$.
Below, we consider the smoothness of the collision local time. Our main object is to explain and prove the following theorem.

Theorem 6. Let $\ell_{T}, T \geq 0$ be the collision local time process of two independent fractional Ornstein-Uhlenbeck $X^{H_{i}}=$ $\left\{X_{t}^{H_{i}}, t \geq 0\right\}, i=1,2$, with respective indices $H_{i} \in(0,1)$. Then $\ell_{T}$ is smooth in the sense of the Meyer-Watanabe if and only if

$$
\begin{equation*}
\min \left\{H_{1}, H_{2}\right\}<\frac{1}{3} \tag{64}
\end{equation*}
$$

Proof. By Yan et al. [11], we have

$$
\begin{align*}
\lambda_{t} \lambda_{s} & -\rho_{s, t}^{2} \\
= & \left(s^{2 H_{1}}+s^{2 H_{2}}\right)\left(t^{2 H_{1}}+t^{2 H_{2}}\right) \\
& \quad-\frac{1}{2}\left(t^{2 H_{1}}+s^{2 H_{1}}-|t-s|^{2 H_{1}}\right.  \tag{65}\\
& \left.\quad+t^{2 H_{2}}+s^{2 H_{2}}-|t-s|^{2 H_{2}}\right) \\
= & \left(s^{2 H_{1}}+s^{2 H_{2}}\right)\left[(t-s)^{2 H_{1}}+(t-s)^{2 H_{2}}\right]
\end{align*}
$$

where the notation $F=G$ means that there are positive constants $c_{1}$ and $c_{2}$ so that

$$
\begin{equation*}
c_{1} G(x) \leq F(x) \leq c_{2} G(x), \tag{66}
\end{equation*}
$$

in the common domain of definition for $F$ and $G$. Hence, following Theorem 2 in An and Yan [28], we have $\iint_{0}^{T}\left(\rho_{s, t}^{2} /\left(\lambda_{s} \lambda_{t}-\rho_{s, t}^{2}\right)^{3 / 2}\right) d s d t<\infty$ if and only if $\min \left\{H_{1}, H_{2}\right\}<1 / 3$. Therefore, in order to prove Theorem 6, it only needs to prove: for $T \geq 0, \ell_{T}$ is smooth in the sense of the Meyer-Watanabe if and only if

$$
\begin{equation*}
\iint_{0}^{T} \rho_{s, t}^{2}\left(\lambda_{t} \lambda_{s}-\rho_{s, t}^{2}\right)^{-3 / 2} d s d t<\infty \tag{67}
\end{equation*}
$$

In fact, for $\varepsilon>0, T \geq 0$ we denote

$$
\begin{equation*}
\Theta_{\varepsilon}\left(u, T, \ell_{\varepsilon, T}\right):=E\left(\left|\Gamma_{\sqrt{u}} \ell_{\varepsilon, T}\right|^{2}\right) \tag{68}
\end{equation*}
$$

and $\Theta\left(u, T, \ell_{T}\right):=E\left(\left|\Gamma_{\sqrt{u}} \ell_{T}\right|^{2}\right)$. Thus, by Proposition 1 to prove that (67) holds if and only if $\Phi_{\Theta}(1)<\infty$. Clearly, we have

$$
\begin{align*}
\ell_{\varepsilon, T}= & \int_{0}^{T} p_{\varepsilon}\left(X_{t}^{H_{1}}-X_{t}^{H_{2}}\right) d t \\
= & \frac{1}{2 \pi} \int_{0}^{T} \int_{\mathbb{R}} e^{i \xi\left(X_{t}^{H_{1}}-X_{t}^{H_{2}}\right)} \cdot e^{-(1 / 2) \varepsilon \xi^{2}} d \xi d t \\
= & \frac{1}{2 \pi} \int_{0}^{T} \int_{\mathbb{R}} e^{-(1 / 2) \xi^{2} \operatorname{Var}\left(X_{t}^{H_{1}}-X_{t}^{H_{2}}\right)} \\
& \quad \times e^{-(1 / 2) \varepsilon \xi^{2}} \sum_{n=0}^{\infty} i^{n} \sigma^{n}(t, \xi) H_{n} \\
& \times\left(\frac{\xi\left(X_{t}^{H_{1}}-X_{t}^{H_{2}}\right)}{\sigma(t, \xi)}\right) d \xi d t \\
\equiv & \sum_{n=0}^{\infty} F_{n} . \tag{69}
\end{align*}
$$

Notice that

$$
\begin{aligned}
& \Phi_{\Theta_{\varepsilon}}(1) \\
& \qquad \begin{array}{l}
=\sum_{n=0}^{\infty} n E\left(\left|F_{n}\right|^{2}\right) \\
=\sum_{n=0}^{\infty} \frac{n}{4 \pi^{2}} \\
\quad \times E\left[\iint_{0}^{T} \int_{\mathbb{R}^{2}}\right.
\end{array} \quad \exp \left(-\frac{1}{2} \varepsilon\left(|\xi|^{2}+|\eta|^{2}\right)\right) \\
& \\
& \quad \times \sigma^{n}(t, \xi) \sigma^{n}(s, \eta) \\
& \\
& \quad \begin{array}{l}
\quad \exp \left(-\frac{1}{2}\left(\xi^{2} \operatorname{Var}\left(X_{t}^{H_{1}}-X_{t}^{H_{2}}\right)\right.\right. \\
\\
\\
\left.\left.\quad+\eta^{2} \operatorname{Var}\left(X_{s}^{H_{1}}-X_{s}^{H_{2}}\right)\right)\right)
\end{array}
\end{aligned}
$$

$$
\begin{gather*}
\cdot H_{n}\left(\frac{\xi\left(X_{t}^{H_{1}}-X_{t}^{H_{2}}\right)}{\sigma(t, \xi)}\right) \\
\left.\times H_{n}\left(\frac{\eta\left(X_{s}^{H_{1}}-X_{s}^{H_{2}}\right)}{\sigma(s, \eta)}\right) d \xi d \eta d s d t\right] \\
=\sum_{n=1}^{\infty} \frac{1}{4 \pi^{2}(2 n-1)!} \\
\times\left[\iint_{0}^{T} \int_{\mathbb{R}^{2}}(\xi \eta)^{2 n}\right. \\
\times\left[E\left(\left(X_{t}^{H_{1}}-X_{t}^{H_{2}}\right)\left(X_{s}^{H_{1}}-X_{s}^{H_{2}}\right)\right)\right]^{2 n} \\
\cdot
\end{gather*}
$$

for all $T \geq 0$, where we have used the following fact: For two random variables $X, Y$ with joint Gaussian distribution such that $E(X)=E(Y)=0$ and $E\left(X^{2}\right)=E\left(Y^{2}\right)=1$ we have (see, for example, Nualart [3])

$$
E\left(H_{n}(X) H_{m}(Y)\right)= \begin{cases}0, & m \neq n  \tag{71}\\ \frac{1}{n!}[E(X Y)]^{n}, & m=n\end{cases}
$$

We obtain

$$
\begin{align*}
\Phi_{\Theta_{\varepsilon}}(1)= & \sum_{n=1}^{\infty} \frac{(\Gamma(n+1 / 2))^{2} 2^{2 n+1}}{4 \pi^{2}(2 n-1)!} \\
& \times \iint_{0}^{T} \frac{\rho_{s, t}^{2 n}}{\left(\left(\lambda_{s}+\varepsilon\right)\left(\lambda_{t}+\varepsilon\right)\right)^{n+(1 / 2)}} d s d t \\
= & \sum_{n=1}^{\infty} \frac{1}{2 \pi} \frac{(2 n-1)!!}{(2 n-2)!!}  \tag{72}\\
& \times \iint_{0}^{T} \frac{\rho_{s, t}^{2 n}}{\left(\left(\lambda_{s}+\varepsilon\right)\left(\lambda_{t}+\varepsilon\right)\right)^{n+(1 / 2)}} d s d t \\
= & \frac{1}{2 \pi} \iint_{0}^{T} \frac{\rho_{s, t}^{2}}{\left(\left(\lambda_{s}+\varepsilon\right)\left(\lambda_{t}+\varepsilon\right)-\rho_{s, t}^{2}\right)^{3 / 2}} d s d t
\end{align*}
$$

where we have used the following equality:

$$
\begin{align*}
& \int_{\mathbb{R}} \xi^{2 n} \exp \left(-\frac{\xi^{2}\left(\operatorname{Var}\left(X_{t}^{H_{1}}-X_{t}^{H_{2}}\right)+\varepsilon\right)}{2}\right) d \xi \\
& \quad=2^{n+(1 / 2)} \Gamma\left(n+\frac{1}{2}\right)\left(\operatorname{Var}\left(X_{t}^{H_{1}}-X_{t}^{H_{2}}\right)+\varepsilon\right)^{-(n+(1 / 2))} \tag{73}
\end{align*}
$$

Hence, we have

$$
\begin{equation*}
\lim _{\varepsilon \rightarrow 0} \Phi_{\Theta_{\varepsilon}}(1)=\frac{1}{2 \pi} \iint_{0}^{T} \frac{\rho_{s, t}^{2}}{\left(\lambda_{s} \lambda_{t}-\rho_{s, t}^{2}\right)^{3 / 2}} d s d t \tag{74}
\end{equation*}
$$

for all $T \geq 0$. This completes the proof.

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# The Application of the Undetermined Fundamental Frequency Method on the Period-Doubling Bifurcation of the 3D Nonlinear System 

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#### Abstract

The analytical method to predict the period-doubling bifurcation of the three-dimensional (3D) system is improved by using the undetermined fundamental frequency method. We compute the stable response of the system subject to the quadratic and cubic nonlinearity by introducing the undetermined fundamental frequency. For the occurrence of the first and second period-doubling bifurcation, the new bifurcation criterion is accomplished. It depends on the stability of the limit cycle on the central manifold. The explicit applications show that the new results coincide with the results of the numerical simulation as compared with the initial methods.


## 1. Introduction

Period-doubling bifurcation can induce complex dynamical behavior in the nonlinear dynamic systems. It is the most classical achievement broadcasted by Feigenbaum [1]. He discovered the ratio of the difference between the values at which such successive period-doubling bifurcation occurs at a constant of around 4.6692 and then showed that the same behavior, with the same mathematical constant, would occur within a wide class of mathematical functions, prior to the onset of chaos. Since then many attempts have been made to study the period-doubling (flip) bifurcation phenomenon in the nonlinear dynamic systems. Wang and Xu [2] developed the relation between two periodic solutions analytically for a general parameter dependent dynamic system. Such relation is further confirmed by one example and shows that the 2T-periodic solution contains all the information of the T-periodic solution near the bifurcation point. From the frequency domain point of view, Floquet multipliers commonly used for the analytical bifurcations of Hopf cycles are the key to detect the appearance of a subharmonic solution. So a quasianalytical
monodromy matrix approach was developed to the perioddoubling bifurcation emerging near a Hopf bifurcation point [3].

As compared with the single freedom system, the dynamical behaviors surrounding the bifurcation point may become more complicated in the 3D system. Rand [4] used the center manifold theory to approximate the newly born limit cycle and then to investigate the stability of the limit cycle corresponding to the flip bifurcation. Later, Belhaq et al. $[5,6]$ improved the approximation of the critical value with a higher-order approximation and further solved the problem of the second period-doubling bifurcation.

In this paper, we use the center manifold theory to reduce a 3D system and then derive the critical values of the first and second period-doubling bifurcation according to the stability of the limit cycle. In terms of the undetermined fundamental frequency method, it produces more accurate results and avoids the computational complexity appending the higherorder approximation at the same time [7, 8]. Finally the whole process is precisely programmed in terms of the computer algebra Mathematica to perform the analysis more efficiently.

## 2. Stable Response with the Undetermined Fundamental Frequency Method

In order to illustrate the analytical process, we refer to the following 3D system:

$$
\begin{gather*}
\dot{x}=\mu x-y-x z, \\
\dot{y}=\mu y+x,  \tag{1}\\
\dot{z}=-z+x^{2} z+y^{2} .
\end{gather*}
$$

This system may be thought of as a feedback control system consisting of a damped linear oscillator in the $x, y$ variables and a control variable $z$. The origin $(x, y, z)=$ $(0,0,0)$ is the equilibrium and may lose its stability at control parameter changing from $\mu<0$ to $\mu>0$. This means that the period-doubling bifurcation appears at the value $\mu=\mu_{c}$ following the limit cycles.

For the value of $\mu_{c}$, the center manifold theory has to be introduced to finish the reduction and obtain the equations on the center manifolds. So we set the second order polynomial of $z$ in terms of $x, y$, and $\mu$

$$
\begin{equation*}
z=a x^{2}+b x y+c y^{2}+d x \mu+e y \mu+f \mu^{2}+o(3) \tag{2}
\end{equation*}
$$

Differentiating (2) with respect to time $t$ and using (1) give

$$
\begin{align*}
-z+x^{2} z+y^{2}= & (2 a x+b y+d \mu)(\mu x-y-x z)  \tag{3}\\
& +(b x+2 c y+e \mu)(\mu y+x)+o(3)
\end{align*}
$$

Equating the same order terms on both sides of (3) produces the coefficients

$$
\begin{gather*}
a=\frac{2}{(1+2 \mu)(5+4 \mu(1+\mu))} \\
b=-\frac{2}{5+4 \mu(1+\mu)} \\
c=\frac{3+4 \mu(1+\mu)}{(1+2 \mu)(5+4 \mu(1+\mu))}  \tag{4}\\
d=0 \\
e=0 \\
f=0
\end{gather*}
$$

That leads to the following approximate flows on the center manifold:

$$
\begin{gather*}
\dot{x}=\mu x-y-a x^{3}-b x^{2} y-c x y^{2}  \tag{5}\\
\dot{y}=\mu y+x .
\end{gather*}
$$

The computational precision of the critical value depends heavily on the stable response, such as the frequency and amplitude of the 3D system. So, in order to perform the limit cycle bifurcation analysis more correctly, Belhaq et al. [6] explored the analysis through a higher-order approximation.

In this paper we introduce the undetermined fundamental frequency method during the course of normal form operation.

In terms of the transformation $x=v-\mu u, y=u$, (5) changes to

$$
\begin{gather*}
\dot{u}=v \\
\dot{v}=-u+2 v \mu-u \mu^{2}-(v-u \mu) \\
\times\left\{-\frac{2 u(v-u \mu)}{5+4 \mu(1+\mu)}+\frac{2(v-u \mu)^{2}}{(1+2 \mu)[5+4 \mu(1+\mu)]}\right.  \tag{6}\\
\left.+\frac{u^{2}(3+4 \mu(1+\mu))}{(1+2 \mu)[5+4 \mu(1+\mu)]}\right\}
\end{gather*}
$$

To obtain the stable response, it demands to transform (6) into a differential equation of the first order with the complex unknown quantities $\xi$. Let

$$
\begin{equation*}
u=\xi+\bar{\xi}, \quad \dot{u}=i \omega_{10}(\xi-\bar{\xi}) \tag{7}
\end{equation*}
$$

where $\omega_{10}$ is the undetermined fundamental frequency. Solving (7) obtains

$$
\begin{equation*}
\xi=\frac{1}{2}\left(u-\frac{i}{\omega_{10}} \dot{u}\right), \quad \bar{\xi}=\frac{1}{2}\left(u+\frac{i}{\omega_{10}} \dot{u}\right) . \tag{8}
\end{equation*}
$$

Differentiating (8) with respect to $t$ and using (6) and (7), give

$$
\begin{align*}
\dot{\xi}=\frac{1}{2 \omega_{10}} i & \left\{\xi+\bar{\xi}+\mu^{2}(\xi+\bar{\xi})-2 i \mu(\xi-\bar{\xi}) \omega_{10}\right. \\
& +(\xi-\bar{\xi}) \omega_{10}^{2}-\frac{1}{5+14 \mu+12 \mu^{2}+8 \mu^{3}} \\
& \times\left[\mu(\xi+\bar{\xi})-i(\xi-\bar{\xi}) \omega_{10}\right]  \tag{9}\\
+ & {\left[\left(3+6 \mu+10 \mu^{2}\right)(\xi+\bar{\xi})^{2}\right.} \\
& -2 i(1+4 \mu)\left(\xi^{2}-\bar{\xi}^{2}\right) \omega_{10} \\
& \left.\left.-2(\xi-\bar{\xi})^{2} \omega_{10}^{2}\right]\right\}
\end{align*}
$$

For the simplification of (9), a third-order nonlinear transformation is considered as

$$
\begin{equation*}
\xi=\eta+h_{1}(\eta, \bar{\eta})+h_{2}(\eta, \bar{\eta})+h_{3}(\eta, \bar{\eta}), \tag{10}
\end{equation*}
$$

where $h_{i}(\eta, \bar{\eta})=\sum_{j=0}^{i} \Gamma_{j, i-j} \eta^{j} \bar{\eta}^{i-j}, i=1,2,3$.

Table 1: Critical value for the first period-doubling bifurcation.

| Method | Numerical simulation | This paper | Reference [4] | Reference [6] |
| :--- | :---: | :---: | :---: | :---: |
| $\mu_{c}$ | 0.439 | 0.443 | 0.45 | 0.446 |

These transformation coefficients $\Gamma_{j, i-j}$ are suitably chosen to eliminate the nonresonance terms [9] in the final expression. So the normal form of (9) is

$$
\begin{align*}
\dot{\eta}= & \frac{1+\mu^{2}+\omega_{10}^{2}}{2\left(\mu-i \omega_{10}\right)} \eta \\
& -\left(\left(3 \mu\left(3+6 \mu+10 \mu^{2}\right)-i\left(3+8 \mu+18 \mu^{2}\right) \omega_{10}\right.\right. \\
& \left.+2(1+5 \mu) \omega_{10}^{2}-6 i \omega_{10}^{3}\right)  \tag{11}\\
& \left.\quad \times\left(2\left(5+14 \mu+12 \mu^{2}+8 \mu^{3}\right)\left(\mu-i \omega_{10}\right)\right)^{-1}\right) \\
& \times \eta^{2} \bar{\eta} .
\end{align*}
$$

Next, $\eta, \bar{\eta}$ require to be expressed in the following polar form $\eta=1 / 2 a e^{i \omega_{10} t}, \bar{\eta}=1 / 2 a e^{-i \omega_{10} t}$. Then separating the real and imaginary parts of the foregoing equation by considering the stationary condition $\dot{a}=0$, we have

$$
\begin{align*}
\mu\{20+ & \mu\left\{56-3 a^{2}[3+2 \mu(3+5 \mu)]\right. \\
& \left.\left.+4 \mu\left[17+2 \mu\left(11+6 \mu+4 \mu^{2}\right)\right]\right\}\right\} \\
+ & \{4 \mu(1+2 \mu)[5+4 \mu(1+\mu)] \\
& \left.\quad-a^{2}[3+2 \mu(5+14 \mu)]\right\} \omega_{10}^{2}-6 a^{2} \omega_{10}^{4}=0,  \tag{12}\\
-10+ & \left(-28+3 a^{2}\right) \mu+\left(-14+5 a^{2}\right) \mu^{2} \\
+ & 6\left(2+a^{2}\right) \mu^{3}+24 \mu^{4}+16 \mu^{5} \\
+ & (1+2 \mu)\left(10+a^{2}+8 \mu+8 \mu^{2}\right) \omega_{10}^{2}=0
\end{align*}
$$

Hence, (12) produces the amplitude and the undetermined fundamental frequency

$$
\begin{aligned}
a=\{ & \frac{1}{3+4(-1+\mu) \mu} \\
& \times\left\{-45+\left\{(1+2 \mu)^{2}[5+4 \mu(1+\mu)]^{2}\right.\right. \\
& \times\{81+8 \mu\{15+\mu[3+2 \mu(6+\mu)]\}\}\}^{1 / 2} \\
& -2 \mu\{73+4 \mu\{23+2 \mu[11+\mu(5+2 \mu)]\}\}\}\}^{1 / 2}, \\
\begin{array}{c}
\omega_{10}=
\end{array} & ((10-\mu\{-14(2+\mu) \\
& \quad+4 \mu^{2}\left(3+6 \mu+4 \mu^{2}\right)
\end{aligned}
$$

$$
\begin{gather*}
\left.\left.+a^{2}[3+\mu(5+6 \mu)]\right\}\right) \\
\left.\times\left((1+2 \mu)\left[10+a^{2}+8 \mu(1+\mu)\right]\right)^{-1}\right)^{1 / 2} \tag{13}
\end{gather*}
$$

## 3. Criterion for the <br> Period-Doubling Bifurcation

Substituting (13) into (7), we obtain the expression of periodic solution in the trigonometric form so that the solution changes into the Cartesian form with the transformation $x_{0}=$ $v-\mu u, y_{0}=u$. Consider

$$
\begin{gather*}
x_{0}=-a \mu \cos \omega_{10} t+a \sin \omega_{10} t  \tag{14}\\
y_{0}=a \cos \omega_{10} t
\end{gather*}
$$

Note that the limit cycle cannot show period-doubling as long as it lies in the center manifold because the latter is two-dimensional and trajectories cannot self-intersect. So it marks $z_{0}$ on the limit cycle for the expression obtained from (2). To investigate the stability, we append disturbance to $z$ in (1), that is,

$$
\begin{equation*}
z=z_{0}+z_{1} \tag{15}
\end{equation*}
$$

and linearize the variation $z_{1}$

$$
\begin{equation*}
\dot{z}_{1}=\left[-1+\left(-a \mu \cos \omega_{10} t+a \sin \omega_{10} t\right)^{2}\right] z_{1} \tag{16}
\end{equation*}
$$

The general solution of (16) is

$$
\begin{equation*}
z_{1}=z_{1}^{*} e^{M} \tag{17}
\end{equation*}
$$

where $M=\int_{0}^{t}\left[-1+\left(-a \mu \cos \omega_{10} t+a \sin \omega_{10} t\right)^{2}\right] d t$. By considering the Floquet theory, the transition from stable to unstable occurs in the condition of $z_{1}(T)=z_{1}(0)$, where $T$ is the period of the limit cycle oscillation. That produces the critical value of period-doubling bifurcation through $M\left(\mu_{c}\right)=0$, and the result is

$$
\begin{equation*}
\mu_{c}=0.443 \tag{18}
\end{equation*}
$$

In order to illustrate the accuracy of the result, the critical values obtained from different methods are presented in Table 1.

Finally, we investigate the stability to find the critical values of the second period doubling bifurcation. That is, from a bifurcation point, the asymmetric 2T-orbit born at the first period-doubling bifurcation point becomes nonstable in a flip bifurcation, where a 4T-orbit emerges. Here, we do not want to give too many details about the second perioddoubling bifurcation because the computational process is

TABLE 2: Critical value for the second period-doubling bifurcation.

| Method | Numerical simulation | This paper | Reference [6] |
| :--- | :---: | :---: | :---: |
| $\mu_{c}$ | 0.476 | 0.481 | 0.486 |



Figure 1: Projections on $(z, x),(x, y)$, and $(y, z)$ plane of the first period-doubling bifurcation.
very similar to the first period doubling bifurcation. We mainly follow the stability analyses of Rand [4] and Belhaq et al. [6]. The main difference exists in finding the solution of the 3D system, where we use the undetermined fundamental frequency method. It produces a better approximation of the asymptotical solution. First of all, we give the general solution in a complex form as follows:

$$
\begin{align*}
u= & \left(\xi_{1}+\bar{\xi}_{1}\right)+\left(1-\delta_{2,1}\right)\left(\xi_{2}+\bar{\xi}_{2}\right) \\
& +\left(1-\delta_{2,1}\right)\left(\xi_{3}+\bar{\xi}_{3}\right) \\
\dot{u}= & i \omega_{10}\left(\xi_{1}-\bar{\xi}_{1}\right)+2 i \omega_{10}\left(1-\delta_{2,1}\right)\left(\xi_{2}-\bar{\xi}_{2}\right)  \tag{19}\\
& +3 i \omega_{10}\left(1-\delta_{2,1}\right)\left(\xi_{3}-\bar{\xi}_{3}\right) .
\end{align*}
$$

Then, we may find the normal form of the reduced system. As for the secular terms, they can be regarded as
the near resonance according to [9]. We use a near identity transformation from $\xi_{1}$ to $\eta_{1}$,

$$
\begin{align*}
\xi_{1}= & \eta_{1}+h_{1}\left(\eta_{1}, \bar{\eta}_{1}, \eta_{2}, \bar{\eta}_{2}, \eta_{3}, \bar{\eta}_{3}\right) \\
& +h_{2}\left(\eta_{1}, \bar{\eta}_{1}, \eta_{2}, \bar{\eta}_{2}, \eta_{3}, \bar{\eta}_{3}\right)  \tag{20}\\
& +h_{3}\left(\eta_{1}, \bar{\eta}_{1}, \eta_{2}, \bar{\eta}_{2}, \eta_{3}, \bar{\eta}_{3}\right)
\end{align*}
$$

to find the normal form of the system which also includes the subharmonic components in its expression. That can be written in different order: first order $\eta_{1}$, second order $\eta_{3} \bar{\eta}_{2}, \eta_{2} \bar{\eta}_{1}$, and third order: $\eta_{2}^{2} \bar{\eta}_{3}, \eta_{3} \bar{\eta}_{1}^{2}, \eta_{1} \eta_{3} \bar{\eta}_{3}, \eta_{1} \eta_{2} \bar{\eta}_{2}, \eta_{1}^{2} \bar{\eta}_{1}$. With these secular terms we find the averaged equation of the system and then amplitude and frequency. Finally, we investigate the stability to find the critical values of the second period-doubling bifurcation. The critical values are presented in Table 2. It exhibits a better approximation than the high order analysis given by Belhaq et al. [6].

In Figure 1, the projections on $(z, x),(x, y)$, and $(y, z)$ plane are plotted at the value of $\mu_{c}=0.439$. Meanwhile the time series of trajectories $x(t), y(t)$, and $z(t)$ appear in


Figure 2: Time series of the first period-doubling trajectories $x(t), y(t)$, and $z(t)$.


Figure 3: 3D phase portrait refers to the first and second period-doubling bifurcation.

Figure 2. A 3D phase portrait refers to the first and second period-doubling bifurcations that are portrayed at the values $\mu_{c}$ equal to 0.439 and 0.476 in Figure 3, respectively. Finally we programme the whole computation process in terms of the computer algebra Mathematica to accelerate the analysis more efficiently.

## 4. Conclusion

The strategy of predicting the period-doubling bifurcation of the 3D system is presented by using the undetermined
fundamental frequency method. It applies the undetermined fundamental frequency to obtain the stable response of the flows on the center manifold and then forms the criterion of period-doubling prediction by considering the stability of the limit cycle. In contrast to the result of numerical simulation, it reveals a good prediction as shown in Tables 1 and 2, compared with the analytical results of the first and second period-doubling bifurcations given by Rand and Belhaq. The whole process is constituted in terms of the computer algebra Mathematica. It enables people to research
the flip bifurcation of the 3D system more accurately and efficiently.

The strategy presented in this work is sufficiently general, so it would be possible to apply the present method to consider other high-dimensional and more complicated systems, which will be the topics for further research.

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## Research Article

# Nonhyperbolic Periodic Orbits of Vector Fields in the Plane Revisited 

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#### Abstract

The main goal of this paper is to present a theory of approximation of periodic orbits of vector fields in the plane. From the theory developed here, it is possible to obtain an approximation to the curve of nonhyperbolic periodic orbits in the bifurcation diagram of a family of differential equations that has a transversal Hopf point of codimension two. Applications of the developed theory are made in Liénard-type equations and in Bazykin's predator-prey system.


## 1. Introduction

The existence of a curve of nonhyperbolic periodic orbits in the bifurcation diagram of a family of differential equations that has a transversal Hopf point of codimension two can be demonstrated with the theories presented in $[1,2]$. However, these theories do not allow us to find or even approximate the curve of nonhyperbolic periodic orbits, except in very special cases as in [3]. On the other hand, good approximations to this curve are essential not only to mathematicians, but primarily for engineers, physicists, and other users of mathematics.

In general, the curve of nonhyperbolic periodic orbits is obtained by numerical methods as in [4] or through specific softwares such as [5], for instance. An analytical alternative proposed in this paper is to generalize the theory of approximation of periodic orbits of [6], using some results and notations of [1,2], in order to obtain an approximation to the curve of nonhyperbolic periodic orbits of a family of differential equations that has transversal Hopf bifurcations of codimension two. Furthermore, the theory developed here does not need normal forms of the vector field in the neighborhood of the Hopf points.

Article [7], among other cases, treats also the generalized Hopf bifurcation in general as $n$-dimensional systems. In particular, it provides quadratic asymptotics for the bifurcation parameter values corresponding to the nonhyperbolic limit cycle, and for this cycle itself. Moreover, these asymptotics are implemented into the standard software MATCONT [5], allowing to automatically initialize the continuation of the cycle-saddle-node curve from the generalized Hopf point. However, the authors believe that the constructions presented here are independent and self-contained. More precisely, both articles give an approximation to the curve of nonhyperbolic periodic orbits of a family of differential equations that has transversal Hopf bifurcations of codimension two. Here we present this theory for 2-dimensional systems without the use of normal forms while in [7], the authors present $n$ dimensional systems using normal forms.

This paper is organized as follows. In Section 2, the theory of approximation of periodic orbits for vector fields in the plane is developed. The stability of the approximate periodic orbits is discussed in Section 3. In Section 4, applications of the theory in Liénard-type differential equations are made, while applications to the Bazykin's predator-prey system are
made in Section 5. Concluding comments about the results obtained here are in Section 6.

## 2. Approximation of Periodic Orbits

Consider a family of the differential equations

$$
\begin{equation*}
\mathbf{x}^{\prime}=f(\mathbf{x}, \xi) \tag{1}
\end{equation*}
$$

where $f: W \times U \rightarrow \mathbb{R}^{2}, W \subset \mathbb{R}^{2}$ is an open set in $\mathbb{R}^{2}$, $f \in \mathscr{C}^{\infty}\left(W \times U, \mathbb{R}^{2}\right)$, and $\xi=(\mu, \nu) \in U \subset \mathbb{R}^{2}$ is the parameter vector. Let $\left(\mathbf{x}_{0}(\xi), \xi\right) \in W \times U$ be an equilibrium point of (1); that is, $f\left(\mathbf{x}_{0}(\xi), \xi\right)=\mathbf{0}$ for $\xi \in U$. Suppose the following assumption:
(H1) the linear part of the vector field $f: W \times U \rightarrow$ $\mathbb{R}^{2}$, evaluated at $\left(\mathbf{x}_{0}(\xi), \xi\right)$ and denoted by $A(\xi)=$ $D f\left(\mathbf{x}_{0}(\xi), \xi\right)$, has eigenvalues $\lambda$ and $\bar{\lambda}$, with $\lambda(\xi)=$ $\gamma(\xi)+i \eta(\xi)$. For $\xi_{0}=\left(\mu_{0}, \nu\right) \in U, \gamma\left(\xi_{0}\right)=0, \partial_{\mu} \gamma\left(\xi_{0}\right) \neq$ 0 , and $\eta\left(\xi_{0}\right)=\omega_{0}(\nu)>0$, where

$$
\begin{equation*}
\partial_{\mu} \gamma\left(\xi_{0}\right)=\left.\frac{\partial}{\partial \mu} \gamma(\xi)\right|_{\xi=\xi_{0}} \tag{2}
\end{equation*}
$$

There is no loss of generality in considering that $\mathbf{x}_{0}(\xi)=\mathbf{0}$ for all $\xi \in U,(0,0) \in U$ and $\mu_{0}=0$. Just make a translation of the equilibrium point and of the critical parameter to their origins and adjust in a convenient way the sets $W \subset \mathbb{R}^{2}$ and $U \subset \mathbb{R}^{2}$. By doing this, (1) can be rewritten as

$$
\begin{equation*}
\mathbf{x}^{\prime}=A(\xi) \mathbf{x}+G(\mathbf{x}, \xi) \tag{3}
\end{equation*}
$$

where $(\mathbf{x}, \xi) \mapsto G(\mathbf{x}, \xi)$ is a smooth vector field with Taylor expansion around $\mathbf{x}=\mathbf{0}$, starting with second-order terms at least, as follows:

$$
\begin{align*}
G(\mathbf{x}, \xi)= & \frac{1}{2} B(\mathbf{x}, \mathbf{x}, \xi)+\frac{1}{6} C(\mathbf{x}, \mathbf{x}, \mathbf{x}, \xi)+\frac{1}{24} D(\mathbf{x}, \mathbf{x}, \mathbf{x}, \mathbf{x}, \xi) \\
& +\frac{1}{120} E(\mathbf{x}, \mathbf{x}, \mathbf{x}, \mathbf{x}, \mathbf{x}, \xi)+O_{G}\left(\|\mathbf{x}\|^{6}, \xi\right) \tag{4}
\end{align*}
$$

where

$$
\begin{gather*}
B_{i}(\mathbf{x}, \mathbf{y}, \xi)=\left.\sum_{j, k=1}^{2} \frac{\partial^{2}}{\partial \eta_{j} \partial \eta_{k}} G_{i}(\eta, \xi)\right|_{\eta=0} x_{j} y_{k}, \\
C_{i}(\mathbf{x}, \mathbf{y}, \mathbf{u}, \xi)=\left.\sum_{j, k, l=1}^{2} \frac{\partial^{3}}{\partial \eta_{j} \partial \eta_{k} \partial \eta_{l}} G_{i}(\eta, \xi)\right|_{\eta=0} x_{j} y_{k} u_{l}, \\
D_{i}(\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}, \xi)=\left.\sum_{j, k, l, r=1}^{2} \frac{\partial^{4}}{\partial \eta_{j} \partial \eta_{k} \partial \eta_{l} \partial \eta_{r}} G_{i}(\eta, \xi)\right|_{\eta=0} x_{j} y_{k} u_{l} v_{r}, \\
E_{i}(\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}, \mathbf{w}, \xi) \\
=\left.\sum_{j, k, l, r, p=1}^{2} \frac{\partial^{5}}{\partial \eta_{j} \partial \eta_{k} \partial \eta_{l} \partial \eta_{r} \partial \eta_{p}} G_{i}(\eta, \xi)\right|_{\eta=0} x_{j} y_{k} u_{l} v_{r} w_{p} \tag{5}
\end{gather*}
$$

are the components of symmetric multilinear functions $B, C$, $D$, and $E$.

Let $q(\xi) \in \mathbb{C}^{2}$ be an eigenvector corresponding to the eigenvalue $\lambda(\xi)$, and let $p(\xi) \in \mathbb{C}^{2}$ be an adjoint eigenvector corresponding to the eigenvalue $\bar{\lambda}(\xi)$ satisfying

$$
\begin{align*}
& A(\xi) q(\xi)=\lambda(\xi) q(\xi),  \tag{6}\\
& A(\xi)^{T} p(\xi)=\bar{\lambda}(\xi) p(\xi), \tag{7}
\end{align*}
$$

and the normalization

$$
\begin{equation*}
\langle p(\xi), q(\xi)\rangle=\sum_{i=1}^{2} \bar{p}_{i}(\xi) q_{i}(\xi)=1 \tag{8}
\end{equation*}
$$

where $\langle\cdot, \cdot\rangle: \mathbb{C}^{2} \times \mathbb{C}^{2} \rightarrow \mathbb{C}$ is the standard inner product in $\mathbb{C}^{2}$ and $A(\xi)^{T}$ is the transpose of the matrix $A(\xi)$. The set $\{q(\xi), \bar{q}(\xi)\}$ is a basis of $\mathbb{C}^{2}$ and the subspace of $\mathbb{C}^{2}$ defined by

$$
\begin{equation*}
\mathbb{R}_{0}^{2}=\left\{(\mathbf{x}, \mathbf{y}) \in \mathbb{C}^{2}: \mathbf{x}, \mathbf{y} \in \mathbb{R}^{2}, \mathbf{y}=\mathbf{0}\right\} \tag{9}
\end{equation*}
$$

is isomorphic to the vector space $\mathbb{R}^{2}$. Taking into account the isomorphism between $\mathbb{R}^{2}$ and $\mathbb{R}_{0}^{2}$, if $(\mathbf{x}, \mathbf{0}) \in \mathbb{R}_{0}^{2}$, then the notation used is $\mathbf{x} \in \mathbb{R}^{2}$. Thus, every vector $\mathbf{x} \in \mathbb{R}^{2}$ can be uniquely represented as a linear combination of elements of $\{q(\xi), \bar{q}(\xi)\}$; that is, there is $z \in \mathbb{C}$ such that

$$
\begin{equation*}
\mathbf{x}=z q(\xi)+\overline{z q}(\xi) \tag{10}
\end{equation*}
$$

It is easy to show that $\langle p(\xi), \bar{q}(\xi)\rangle=0$ and $z=\langle p(\xi), \mathbf{x}\rangle$. So (1) can be written as a complex family of differential equations as follows:

$$
\begin{equation*}
z^{\prime}=g(z, \bar{z}, \xi) \tag{11}
\end{equation*}
$$

for $\left\|\xi-\xi_{0}\right\|$ sufficiently small, where $g \in \mathscr{C}^{\infty}(\mathbb{C} \times \mathbb{C} \times U, \mathbb{C})$ and

$$
\begin{equation*}
g(z, \bar{z}, \xi)=\lambda(\xi) z+\langle p(\xi), G(z q(\xi)+\overline{z q}(\xi), \xi)\rangle \tag{12}
\end{equation*}
$$

The function $(z, \bar{z}, \xi) \mapsto g(z, \bar{z}, \xi)$ has formal Taylor series

$$
\begin{equation*}
g(z, \bar{z}, \xi)=\lambda(\xi) z+\sum_{k=2}^{\infty} \sum_{j=0}^{k} \frac{1}{(k-j)!j!} g_{k-j, j}(\xi) z^{k-j} \bar{z}^{j}, \tag{13}
\end{equation*}
$$

where

$$
\begin{equation*}
g_{k-j, j}(\xi)=\left.\frac{\partial^{k}}{\partial z^{k-j} \partial \bar{z}^{j}}\langle p(\xi), G(z q(\xi)+\bar{z} \bar{q}(\xi), \xi)\rangle\right|_{z=0} \tag{14}
\end{equation*}
$$

for $k=2,3, \ldots$ and $j=0, \ldots, k$.
The coefficients $g_{k-j, j}(\xi)$ for $k=2,3, \ldots$ and $j=0, \ldots, k$ play an important role in the method of approximation of a family of periodic orbits of (1). A simple way to calculate these coefficients, alternative to (14), is through the symmetric multilinear functions. From the symmetric bilinear function $(\mathbf{x}, \mathbf{y}, \xi) \mapsto B(\mathbf{x}, \mathbf{y}, \xi)$ and (10), it follows that

$$
\begin{align*}
& B(z q(\xi)+\overline{z q}(\xi), z q(\xi)+\overline{z q}(\xi), \xi) \\
& \quad=B(q(\xi), q(\xi), \xi) z^{2}+2 B(q(\xi), \bar{q}(\xi), \xi) z \bar{z}  \tag{15}\\
& \quad+B(\bar{q}(\xi), \bar{q}(\xi), \xi) \bar{z}^{2}
\end{align*}
$$

and, therefore,

$$
\begin{align*}
& g_{2,0}(\xi)=\langle p(\xi), B(q(\xi), q(\xi), \xi)\rangle, \\
& g_{1,1}(\xi)=\langle p(\xi), B(q(\xi), \bar{q}(\xi), \xi)\rangle,  \tag{16}\\
& g_{0,2}(\xi)=\langle p(\xi), B(\bar{q}(\xi), \bar{q}(\xi), \xi)\rangle .
\end{align*}
$$

Similarly, for the symmetric trilinear function $(\mathbf{x}, \mathbf{y}, \mathbf{u}, \xi) \mapsto$ $C(\mathbf{x}, \mathbf{y}, \mathbf{u}, \xi)$,

$$
\begin{align*}
& g_{3,0}(\xi)=\langle p(\xi), C(q(\xi), q(\xi), q(\xi), \xi)\rangle, \\
& g_{2,1}(\xi)=\langle p(\xi), C(q(\xi), q(\xi), \bar{q}(\xi), \xi)\rangle, \\
& g_{1,2}(\xi)=\langle p(\xi), C(q(\xi), \bar{q}(\xi), \bar{q}(\xi), \xi)\rangle,  \tag{17}\\
& g_{0,3}(\xi)=\langle p(\xi), C(\bar{q}(\xi), \bar{q}(\xi), \bar{q}(\xi), \xi)\rangle,
\end{align*}
$$

and so on for other symmetric multilinear functions.
The aim of the theory of approximation of periodic orbits in [6] is to build an approximation for a periodic orbit of the complex differential equation (11), from the solution of the linear differential equation

$$
\begin{equation*}
z^{\prime}=\lambda(\xi) z \tag{18}
\end{equation*}
$$

for $\xi=\xi_{0}$. This linear differential equation has the solution

$$
\begin{equation*}
z(t)=z_{0} e^{\lambda(\xi) t} \tag{19}
\end{equation*}
$$

where $z_{0} \in \mathbb{C}$. For $\xi=\xi_{0}$, it follows that

$$
\begin{equation*}
z(t)=z_{0} e^{i \omega_{0}(v) t} \tag{20}
\end{equation*}
$$

and making the change in time $s=\omega_{0}(\nu) t$, this solution is periodic of period $2 \pi$ in the variable $s$. To formalize the method, consider the functions $(\epsilon, \nu) \mapsto \mu=\phi(\epsilon, \nu),(\epsilon, v) \mapsto$ $\omega(\epsilon, \nu)$ and the change of coordinates and time

$$
\begin{equation*}
z(t)=w(s, \epsilon, \nu), \quad s=\omega(\epsilon, \nu) t, \quad \omega\left(\xi_{0}\right)=\omega_{0}(\nu) \tag{21}
\end{equation*}
$$

where

$$
\begin{equation*}
\epsilon=\frac{1}{2 \pi} \int_{0}^{2 \pi} e^{-i s} w(s, \epsilon, v) d s \tag{22}
\end{equation*}
$$

Note that the parameter $\epsilon$, as defined in (22), is a complex number or, more precisely, a complex function whose independent variable is $\nu$. However, it is possible, through a change of variables, to consider the parameter $\epsilon$ as a real number. In fact, as

$$
\begin{equation*}
\epsilon=\varepsilon e^{i \phi}=\frac{1}{2 \pi} \int_{0}^{2 \pi} e^{-i s} w(s, \epsilon, \nu) d s \tag{23}
\end{equation*}
$$

it follows that

$$
\begin{align*}
\varepsilon & =|\epsilon|=\frac{e^{-i \phi}}{2 \pi} \int_{0}^{2 \pi} e^{-i s} w(s, \epsilon, \nu) d s  \tag{24}\\
& =\frac{1}{2 \pi} \int_{0}^{2 \pi} e^{-i(s+\phi)} w(s, \epsilon, \nu) d s
\end{align*}
$$

Thus, making the change of variable $u=s+\phi$ in (24) and setting $(u, \epsilon, v) \mapsto \widetilde{w}(u, \epsilon, v)=w(u-\phi, \epsilon, v)$,

$$
\begin{align*}
\varepsilon & =|\epsilon|=\frac{1}{2 \pi} \int_{\phi}^{2 \pi+\phi} e^{-i u} w(u-\phi, \epsilon, \nu) d u  \tag{25}\\
& =\frac{1}{2 \pi} \int_{0}^{2 \pi} e^{-i u} \widetilde{w}(u, \epsilon, \nu) d u
\end{align*}
$$

since the function $(u, \epsilon, v) \mapsto e^{-i u} \widetilde{w}(u, \epsilon, \nu)$ is periodic of period $2 \pi$ in the variable $s$. Therefore, by (25), the parameter $\epsilon$ as defined in (22) will be considered a real parameter.

The generalization of the theory of approximation of periodic orbits introduced in [6] consists in achieving an approximation to the two-parameter family of periodic orbits

$$
\begin{equation*}
\left\{(s, \epsilon, \nu) \in \mathbb{R} \times U_{\epsilon} \longmapsto w(s, \epsilon, \nu) \in \mathbb{C}:(\epsilon, \nu) \in U_{\epsilon}\right\} \tag{26}
\end{equation*}
$$

where $U_{\epsilon}=\left\{(\epsilon, \nu) \in \mathbb{R}^{2}:(\phi(\epsilon, \nu), \nu) \in U\right\}$.
The change in time $s=\omega(\epsilon, v) t$ is essential, since the period of the family of periodic orbits (26) is unknown and, therefore, the change in time is used only to provide an approximation of the known period $2 \pi$ for the family of periodic orbits (26). If $(\epsilon, \nu) \mapsto T(\epsilon, \nu)$ denotes the period of the family of periodic orbits, then

$$
\begin{equation*}
\omega(\epsilon, v)=\frac{2 \pi}{T(\epsilon, v)} \tag{27}
\end{equation*}
$$

In other words, the knowledge of the function $(\epsilon, \nu) \mapsto \omega(\epsilon, \nu)$ completely determines the period of the family of periodic orbits of (26).

By changing the coordinates and time (21) and applying the chain rule, the complex differential equation (11) is rewritten as

$$
\begin{equation*}
\omega(\epsilon, v) \frac{d}{d s} w(s, \epsilon, \nu)=g(w(s, \epsilon, v), \bar{w}(s, \epsilon, v), \phi(\epsilon, v), \nu) \tag{28}
\end{equation*}
$$

Approximations to the functions $(s, \epsilon, \nu) \mapsto w(s, \epsilon, \nu)$, $(\epsilon, \nu) \mapsto \mu=\phi(\epsilon, \nu)$ and $(\epsilon, \nu) \mapsto \omega(\epsilon, \nu)$ are obtained through (28) and the formal power series

$$
\left(\begin{array}{c}
w(s, \epsilon, v)  \tag{29}\\
\phi(\epsilon, v) \\
\omega(\epsilon, \nu)-\omega_{0}(\nu)
\end{array}\right)=\sum_{k=1}^{\infty} \frac{1}{k!}\left(\begin{array}{c}
w_{k}(s, v) \\
\mu_{k}(v) \\
\omega_{k}(\nu)
\end{array}\right) \epsilon^{k} .
$$

A property of the terms of the sequence $\left\{w_{k}(s, \nu)\right\}_{k \in \mathbb{N}}$, widely used in this theory of approximation of periodic orbits of vector fields in $\mathbb{R}^{2}$, is obtained in Proposition 1.

Proposition 1. Each term of the sequence $\left\{w_{k}(s, v)\right\}_{k \in \mathbb{N}}$ satisfies

$$
\left[w_{k}\right]=\frac{1}{2 \pi} \int_{0}^{2 \pi} e^{-i s} w_{k}(s, v) d s= \begin{cases}1, & k=1  \tag{30}\\ 0, & k=2,3, \ldots\end{cases}
$$

Proof. Setting $\mathscr{W}=\left\{w: \mathbb{R} \times U_{\epsilon} \rightarrow \mathbb{C}: w \in \mathscr{C}^{\infty}\left(\mathbb{R} \times U_{\epsilon}, \mathbb{C}\right)\right\}$, the proof is an immediate consequence of the definition of linear map

$$
\begin{align*}
& {[\quad]: \mathscr{W} \longrightarrow \mathbb{R}} \\
& w \longmapsto[w]=\frac{1}{2 \pi} \int_{0}^{2 \pi} e^{-i s} w(s, \epsilon, \nu) d s \tag{31}
\end{align*}
$$

and the formal power series in the variable $\epsilon$ of the function $(s, \epsilon, \nu) \mapsto w(s, \epsilon, \nu)$, because

$$
\begin{align*}
\epsilon & =\frac{1}{2 \pi} \int_{0}^{2 \pi} e^{-i s}\left(\sum_{k=1}^{\infty} \frac{1}{k!} w_{k}(s, \nu) \epsilon^{k}\right) d s  \tag{32}\\
& =\sum_{k=1}^{\infty} \frac{1}{k!}\left(\frac{1}{2 \pi} \int_{0}^{2 \pi} e^{-i s} w_{k}(s, \nu) d s\right) \epsilon^{k}=\sum_{k=1}^{\infty} \frac{1}{k!}\left[w_{k}\right] \epsilon^{k} .
\end{align*}
$$

The terms of the sequences $\left\{w_{k}(s, \nu)\right\}_{k \in \mathbb{N}},\left\{\mu_{k}(\nu)\right\}_{k \in \mathbb{N}}$ and $\left\{\omega_{k}(\nu)\right\}_{k \in \mathbb{N}}$ are determined through a process that involves analysis of the powers in $\epsilon$, obtained by replacing (29) into the differential equation (28). Note that, for $k=2,3, \ldots$ and $j=0, \ldots, k$, the coefficients of powers in $\epsilon$ are determined by expanding the composition $(\epsilon, \nu) \mapsto g_{k-j, j}(\phi(\epsilon, \nu), \nu)$ in the Taylor series around $\epsilon=0$. Such an expansion, up to the fifth-order terms, is of the following form:

$$
\begin{align*}
& g_{k-j, j}(\phi(\epsilon, \nu), \nu) \\
&=g_{k-j, j}\left(\xi_{0}\right)+\mu_{1}(\nu) \partial_{\mu} g_{k-j, j}\left(\xi_{0}\right) \epsilon \\
&+ \frac{1}{2}\left(\mu_{2}(\nu) \partial_{\mu} g_{k-j, j}\left(\xi_{0}\right)\right. \\
&+\frac{1}{6}\left(\mu_{3}(\nu) \partial_{\mu} g_{k-j, j}\left(\xi_{0}\right)+\mu_{1}(\nu)^{2} \partial_{\mu}^{2} g_{k-j, j}\left(\xi_{0}\right)\right) \epsilon^{2} \\
&+3 \mu_{1}(\nu) \mu_{2}(\nu) \partial_{\mu}^{2} g_{k-j, j}\left(\xi_{0}\right) \\
&\left.+\mu_{1}(\nu)^{3} \partial_{\mu}^{3} g_{k-j, j}\left(\xi_{0}\right)\right) \epsilon^{3} \\
&+\frac{1}{24}\left(\mu_{4}(\nu) \partial_{\mu} g_{k-j, j}\left(\xi_{0}\right)+3 \mu_{2}(\nu)^{2} \partial_{\mu}^{2} g_{k-j, j}\left(\xi_{0}\right)\right. \\
&+4 \mu_{1}(\nu) \mu_{2}(\nu) \partial_{\mu} g_{k-j, j}\left(\xi_{0}\right)+6 \mu_{1}(\nu)^{2} \mu_{2}(\nu) \\
&\left.\quad \times \partial_{\mu}^{3} g_{k-j, j}\left(\xi_{0}\right)+\mu_{1}(\nu)^{4} \partial_{\mu}^{4} g_{k-j, j}\left(\xi_{0}\right)\right) \epsilon^{4} \\
&+\frac{1}{120}( \mu_{5}(\nu) \partial_{\mu} g_{k-j, j}\left(\xi_{0}\right) \\
&+\left(10 \mu_{2}(\nu) \mu_{3}(\nu)+5 \mu_{1}(\nu) \mu_{4}(\nu)\right) \partial_{\mu}^{2} g_{k-j, j}\left(\xi_{0}\right) \\
&+\left(15 \mu_{1}(\nu) \mu_{2}(\nu)^{2}+10 \mu_{1}(\nu)^{2} \mu_{3}(\nu)\right)
\end{align*}
$$

with the same being valid for the composition $(\epsilon, \nu) \mapsto$ $\lambda(\phi(\epsilon, \nu), \nu)=\gamma(\phi(\epsilon, \nu), \nu)+i \eta(\phi(\epsilon, \nu), \nu)$.

The coefficient of the term in $\epsilon$ leads to the following boundary value problem:

$$
\begin{align*}
& w_{1}^{\prime}(s, v)-i w_{1}(s, v)=0 \\
& w_{1}(s, v)=w_{1}(s+2 \pi, v) \tag{34}
\end{align*}
$$

The solution of the differential equation in (34) is

$$
\begin{equation*}
w_{1}(s, v)=C_{1} e^{i s}, \tag{35}
\end{equation*}
$$

and as by Proposition 1, $\left[w_{1}\right]=1$, it follows that

$$
\begin{equation*}
1=\left[w_{1}\right]=\left[C_{1} e^{i s}\right]=C_{1}\left[e^{i s}\right]=C_{1} \tag{36}
\end{equation*}
$$

Thus,

$$
\begin{equation*}
w_{1}(s, \nu)=e^{i s} \tag{37}
\end{equation*}
$$

which is a periodic function of period $2 \pi$ in the variable $s$. In fact, the terms of the sequence $\left\{w_{k}(s, \nu)\right\}_{k \in \mathbb{N}}$ are solutions of certain boundary value problems which appear when (29) is substituted into the differential equation (28). For each $k=$ $1,2, \ldots$, the boundary value problem is of the following form:

$$
\begin{align*}
& w_{k+1}^{\prime}(s, v)-i w_{k+1}(s, v)=H_{k+1}\left(s, \mu_{k}(\nu), \omega_{k}(\nu)\right)  \tag{38}\\
& w_{k+1}(s, v)=w_{k+1}(s+2 \pi, v)
\end{align*}
$$

where $H_{k+1}\left(s, \mu_{k}(\nu), \omega_{k}(\nu)\right)=H_{k+1}\left(s+2 \pi, \mu_{k}(\nu), \omega_{k}(\nu)\right)$.
The following theorem guarantees the existence of the solutions of the boundary value problem (38).

Theorem 2. For each $k=1,2, \ldots$, the boundary value problem (38) admits solution if and only if

$$
\begin{equation*}
\left[H_{k+1}\right]=0 \tag{39}
\end{equation*}
$$

Proof. For fixed $k=1,2, \ldots$, suppose that $(s, \nu) \mapsto \varphi_{k+1}(s, \nu)$ is the solution of (38). Thus,

$$
\begin{align*}
& \frac{1}{2 \pi} \int_{0}^{2 \pi} e^{-i s}\left(\varphi_{k+1}^{\prime}(s, v)-i \varphi_{k+1}(s, v)\right) d s  \tag{40}\\
& \quad=\frac{1}{2 \pi} \int_{0}^{2 \pi} e^{-i s} H_{k+1}\left(s, \mu_{k}(\nu), \omega_{k}(\nu)\right) d s
\end{align*}
$$

and by integrating by parts the left member of (40), it follows that $\left[H_{k+1}\right]=0$. Now suppose that $\left[H_{k+1}\right]=0$ for a fixed $k=1,2, \ldots$. The general solution $(s, v) \mapsto \varphi_{k+1}(s, v)$ of the differential equation in (38) is of the following form:

$$
\begin{equation*}
\varphi_{k+1}(s, \nu)=e^{i s} \varphi_{0}^{k+1}+e^{i s} \int_{0}^{s} e^{-i \zeta} H_{k+1}\left(\zeta, \mu_{k}(\nu), \omega_{k}(\nu)\right) d \zeta \tag{41}
\end{equation*}
$$

where $\varphi_{0}^{k+1}=\varphi_{k+1}(0, \nu)$. This solution will be periodic of period $2 \pi$ if $\varphi_{k+1}(0, \nu)=\varphi_{0}^{k+1}=\varphi_{k+1}(2 \pi, \nu)$; that is, if

$$
\begin{align*}
\varphi_{k+1}(0, \nu) & =\varphi_{k+1}(2 \pi, \nu) \\
& =\varphi_{0}^{k+1}+\int_{0}^{2 \pi} e^{-i \zeta} H_{k+1}\left(\zeta, \mu_{k}(\nu), \omega_{k}(\nu)\right) d \zeta  \tag{42}\\
& =\varphi_{0}^{k+1}+2 \pi\left[H_{k+1}\right]
\end{align*}
$$

Thus, using the hypothesis $\left[H_{k+1}\right]=0$, it follows that $\varphi_{k+1}(s, \nu)=\varphi_{k+1}(s+2 \pi, \nu)$, and, therefore, for each fixed $k=1,2, \ldots$, the function $(s, \nu) \mapsto \varphi_{k+1}(s, \nu)$ is the solution of the boundary value problem (38).

The previous theorem shows that, for $k=1,2, \ldots$, the solution of (38) is obtained by solving the differential equation in (38) with conditions $\left[H_{k+1}\right]=0$ and $\left[w_{k+1}\right]=0$.

Continuing the process and using the result (37), the coefficient of the term in $\epsilon^{2}$ provides the boundary value problem

$$
\begin{align*}
& w_{2}^{\prime}(s, v)-i w_{2}(s, v)=H_{2}\left(s, \mu_{1}(\nu), \omega_{1}(\nu)\right)  \tag{43}\\
& w_{2}(s, v)=w_{2}(s+2 \pi, v)
\end{align*}
$$

where

$$
\begin{align*}
& H_{2}\left(s, \mu_{1}(\nu), \omega_{1}(\nu)\right) \\
& \qquad \begin{array}{l}
=\frac{1}{\omega_{0}(\nu)}\left(e^{2 i s} g_{2,0}\left(\xi_{0}\right)+2 g_{1,1}\left(\xi_{0}\right)+e^{-2 i s} g_{0,2}\left(\xi_{0}\right)\right. \\
\\
\left.\quad+2 e^{i s}\left(\mu_{1}(\nu) \partial_{\mu} \lambda\left(\xi_{0}\right)-i \omega_{1}(\nu)\right)\right)
\end{array} \tag{44}
\end{align*}
$$

By applying Theorem 2 to the function $\left(s, \mu_{1}(\nu), \omega_{1}(\nu)\right) \mapsto$ $H_{2}\left(s, \mu_{1}(\nu), \omega_{1}(\nu)\right)$, it follows that

$$
\begin{equation*}
\left[H_{2}\right]=\mu_{1}(\nu)\left(\partial_{\mu} \gamma\left(\xi_{0}\right)+i \partial_{\mu} \eta\left(\xi_{0}\right)\right)-i \omega_{1}(\nu)=0 \tag{45}
\end{equation*}
$$

and by separating the real and imaginary parts of (45), we have $\mu_{1}(\nu)=0$ and $\omega_{1}(\nu)=0$. Under these conditions, Theorem 2 guarantees the existence of the solution of the boundary value problem (43), which is given by

$$
\begin{align*}
& w_{2}(s, v) \\
& \quad=\frac{1}{3 i \omega_{0}(\nu)}\left(3 e^{2 i s} g_{2,0}\left(\xi_{0}\right)-6 g_{1,1}\left(\xi_{0}\right)-e^{-2 i s} g_{0,2}\left(\xi_{0}\right)\right) . \tag{46}
\end{align*}
$$

For the coefficient of the term in $\epsilon^{3}$, we have the following boundary value problem:

$$
\begin{align*}
& w_{3}^{\prime}(s, v)-i w_{3}(s, v)=H_{3}\left(s, \mu_{2}(v), \omega_{2}(v)\right) \\
& w_{3}(s, v)=w_{3}(s+2 \pi, v) \tag{47}
\end{align*}
$$

with

$$
\begin{align*}
& H_{3}\left(s, \mu_{2}(\nu), \omega_{2}(\nu)\right) \\
& \begin{aligned}
=\frac{3}{\omega_{0}(\nu)} & \left(H_{3}^{3}\left(\xi_{0}\right) e^{3 i s}+H_{3}^{1}\left(\xi_{0}\right) e^{i s}\right. \\
& \left.+H_{3}^{-1}\left(\xi_{0}\right) e^{-i s}+H_{3}^{-3}\left(\xi_{0}\right) e^{-3 i s}\right)
\end{aligned} \tag{48}
\end{align*}
$$

where

$$
\begin{gather*}
H_{3}^{3}\left(\xi_{0}\right)=\frac{1}{3} g_{3,0}\left(\xi_{0}\right)-\frac{i g_{2,0}\left(\xi_{0}\right)^{2}}{\omega_{0}(\nu)}-\frac{i g_{1,1}\left(\xi_{0}\right) \bar{g}_{0,2}\left(\xi_{0}\right)}{\omega_{0}(\nu)}, \\
H_{3}^{1}\left(\xi_{0}\right)=\mu_{2}(\nu) \partial_{\mu} \lambda\left(\xi_{0}\right)-i \omega_{2}(\nu)+G_{2,1}\left(\xi_{0}\right) \\
H_{3}^{-1}\left(\xi_{0}\right)= \\
+\frac{2 i g_{1,1}\left(\xi_{0}\right)^{2}}{\omega_{0}(\nu)}+\frac{i \bar{g}_{2,0}\left(\xi_{0}\right) g_{1,1}\left(\xi_{0}\right)}{\omega_{0}(\nu)}+g_{1,2}\left(\xi_{0}\right) \\
+\frac{i g_{2,0}\left(\xi_{0}\right) g_{0,2}\left(\xi_{0}\right)}{3 \omega_{0}(\nu)}-\frac{2 i g_{0,2}\left(\xi_{0}\right) \bar{g}_{1,1}\left(\xi_{0}\right)}{\omega_{0}(v)}, \\
H_{3}^{-3}\left(\xi_{0}\right)=\frac{1}{3} g_{0,3}\left(\xi_{0}\right)+\frac{i g_{1,1}\left(\xi_{0}\right) g_{0,2}\left(\xi_{0}\right)}{3 \omega_{0}(\nu)}  \tag{49}\\
+\frac{i g_{0,2}\left(\xi_{0}\right) \bar{g}_{2,0}\left(\xi_{0}\right)}{\omega_{0}(\nu)}
\end{gather*}
$$

and the coefficient $G_{2,1}\left(\xi_{0}\right)$ is defined as

$$
\begin{align*}
G_{2,1}\left(\xi_{0}\right)= & \frac{i g_{2,0}\left(\xi_{0}\right) g_{1,1}\left(\xi_{0}\right)+\omega_{0}(\nu) g_{2,1}\left(\xi_{0}\right)}{\omega_{0}(\nu)} \\
& -\frac{2 i\left|g_{1,1}\left(\xi_{0}\right)\right|^{2}}{\omega_{0}(\nu)}-\frac{i\left|g_{0,2}\left(\xi_{0}\right)\right|^{2}}{3 \omega_{0}(\nu)} \tag{50}
\end{align*}
$$

Expression (50) is identical to the one given in [1].
Continuing the process and calculating $\left[\mathrm{H}_{3}\right]$, it follows that

$$
\begin{equation*}
\left[H_{3}\right]=\mu_{2}(\nu)\left(\partial_{\mu} \gamma\left(\xi_{0}\right)+i \partial_{\mu} \eta\left(\xi_{0}\right)\right)-i \omega_{2}(\nu)+G_{2,1}\left(\xi_{0}\right)=0 \tag{51}
\end{equation*}
$$

And by separating the real and imaginary parts,

$$
\begin{gather*}
\mu_{2}(\nu)=-\frac{\operatorname{Re}\left(G_{2,1}\left(\xi_{0}\right)\right)}{\partial_{\mu} \gamma\left(\xi_{0}\right)}  \tag{52}\\
\omega_{2}(\nu)=\operatorname{Im}\left(G_{2,1}\left(\xi_{0}\right)\right)+\mu_{2}(\nu) \partial_{\mu} \eta\left(\xi_{0}\right) \tag{53}
\end{gather*}
$$

Once the coefficients $\mu_{2}(\nu)$ and $\omega_{2}(\nu)$ are determined, the solution of the boundary value problem (47) has the following form:

$$
w_{3}(s, v)
$$

$$
\begin{equation*}
=\frac{1}{4 \omega_{0}(\nu)^{2}}\left(w_{3}^{3}\left(\xi_{0}\right) e^{3 i s}+w_{3}^{-1}\left(\xi_{0}\right) e^{-i s}+w_{3}^{-3}\left(\xi_{0}\right) e^{-3 i s}\right), \tag{54}
\end{equation*}
$$

where

$$
\begin{aligned}
w_{3}^{3}\left(\xi_{0}\right)= & -6 g_{2,0}\left(\xi_{0}\right)^{2}-2 i \omega_{0}(\nu) g_{3,0}\left(\xi_{0}\right) \\
& -2 g_{1,1}\left(\xi_{0}\right) \bar{g}_{0,2}\left(\xi_{0}\right) \\
w_{3}^{-1}\left(\xi_{0}\right)= & -12 g_{1,1}\left(\xi_{0}\right)^{2}-6 \bar{g}_{2,0}\left(\xi_{0}\right) g_{1,1}\left(\xi_{0}\right) \\
& +6 i \omega_{0}(\nu) g_{1,2}\left(\xi_{0}\right)-2 g_{0,2}\left(\xi_{0}\right) g_{2,0}\left(\xi_{0}\right) \\
& +12 g_{0,2}\left(\xi_{0}\right) \bar{g}_{1,1}\left(\xi_{0}\right) \\
w_{3}^{-3}\left(\xi_{0}\right)= & i \omega_{0}(\nu) g_{0,3}\left(\xi_{0}\right)-g_{0,2}\left(\xi_{0}\right) g_{1,1}\left(\xi_{0}\right) \\
& -3 g_{0,2}\left(\xi_{0}\right) \bar{g}_{2,0}\left(\xi_{0}\right)
\end{aligned}
$$

Definition 3. The real number

$$
\begin{align*}
l_{1}\left(\xi_{0}\right) & =\frac{1}{2} \operatorname{Re}\left(G_{2,1}\left(\xi_{0}\right)\right) \\
& =\frac{1}{2 \omega_{0}(\nu)} \operatorname{Re}\left(i g_{2,0}\left(\xi_{0}\right) g_{1,1}\left(\xi_{0}\right)+\omega_{0}(\nu) g_{2,1}\left(\xi_{0}\right)\right) \tag{56}
\end{align*}
$$

is called the first Lyapunov coefficient.
Remark 4. A Hopf point of codimension one for (1) is an equilibrium point $\left(\mathbf{0}, \xi_{0}\right) \in W \times U$, with $\xi_{0}=(0, \nu)$, such that $A\left(\xi_{0}\right)=D f\left(\mathbf{0}, \xi_{0}\right)$ has eigenvalues $\lambda$ and $\bar{\lambda}$, with $\lambda\left(\xi_{0}\right)=$ $\gamma\left(\xi_{0}\right)+i \eta\left(\xi_{0}\right), \gamma\left(\xi_{0}\right)=0, \eta\left(\xi_{0}\right)=\omega_{0}(\nu)>0$, and the first Lyapunov coefficient, $l_{1}\left(\xi_{0}\right) \in \mathbb{R}$, is different from zero. A transversal Hopf point of codimension one is a Hopf point of codimension one such that

$$
\begin{equation*}
\partial_{\mu} \gamma\left(\xi_{0}\right) \neq 0 \tag{57}
\end{equation*}
$$

for $\xi_{0} \in U$. In a neighborhood of a transversal Hopf point of codimension one $\left(\mathbf{0}, \xi_{0}\right) \in W \times U$, with $l_{1}\left(\xi_{0}\right) \neq 0$, the dynamic behavior of differential equation (1) is orbitally topologically equivalent to the following complex normal form:

$$
\begin{equation*}
w^{\prime}=(\alpha+i) w+s w|w|^{2} \tag{58}
\end{equation*}
$$

where $s=\operatorname{sign}\left(l_{1}\left(\xi_{0}\right)\right)$. The sign of the first Lyapunov coefficient determines the stability of the family of periodic orbits that appears (or disappears) from $\left(\mathbf{0}, \xi_{0}\right) \in W \times U$ as will be seen later.

When $l_{1}\left(\xi_{1}\right)=0$, for $\xi_{1}=(0,0) \in U$, there is the possibility of Hopf bifurcations of codimension two. In this case, it is necessary to obtain an expression for $G_{3,2}\left(\xi_{1}\right)$.

Applying Theorem 2 to the boundary value problem for $k=3$, it follows that $\mu_{3}(\nu)=0, \omega_{3}(\nu)=0$ and

$$
\begin{array}{r}
w_{4}(s, v)=\frac{1}{45 \omega_{0}(\nu)^{3}}\left(w_{4}^{4}\left(\xi_{0}\right) e^{4 i s}+w_{4}^{2}\left(\xi_{0}\right) e^{2 i s}+w_{4}^{0}\left(\xi_{0}\right)\right. \\
\left.+w_{4}^{-2}\left(\xi_{0}\right) e^{-2 i s}+w_{4}^{-4}\left(\xi_{0}\right) e^{-4 i s}\right) \tag{59}
\end{array}
$$

where

$$
\begin{aligned}
& w_{4}^{4}\left(\xi_{0}\right)=135 i g_{2,0}\left(\xi_{0}\right)^{3}-120 \omega_{0}(v) g_{3,0}\left(\xi_{0}\right) g_{2,0}\left(\xi_{0}\right) \\
& +{105 i g_{1,1}}\left(\xi_{0}\right) \bar{g}_{0,2}\left(\xi_{0}\right) g_{2,0}\left(\xi_{0}\right) \\
& +5 i g_{0,2}\left(\xi_{0}\right) \bar{g}_{0,2}\left(\xi_{0}\right)^{2}-15 i \omega_{0}(\nu)^{2} g_{4,0}\left(\xi_{0}\right) \\
& -30 \omega_{0}(\nu) g_{2,1}\left(\xi_{0}\right) \bar{g}_{0,2}\left(\xi_{0}\right) \\
& -15 \omega_{0}(\nu) g_{1,1}\left(\xi_{0}\right) \bar{g}_{0,3}\left(\xi_{0}\right) \\
& +15 i g_{1,1}\left(\xi_{0}\right) \bar{g}_{0,2}\left(\xi_{0}\right) \bar{g}_{1,1}\left(\xi_{0}\right), \\
& w_{4}^{2}\left(\xi_{0}\right)=-180 i g_{3,1}\left(\xi_{0}\right) \omega_{0}(\nu)^{2} \\
& -270 i \mu_{2}(\nu) \partial_{\mu} g_{2,0}\left(\xi_{0}\right) \omega_{0}(\nu)^{2} \\
& +450 g_{1,1}\left(\xi_{0}\right) g_{3,0}\left(\xi_{0}\right) \omega_{0}(\nu) \\
& -180 g_{1,2}\left(\xi_{0}\right) \bar{g}_{0,2}\left(\xi_{0}\right) \omega_{0}(\nu) \\
& -45 g_{0,2}\left(\xi_{0}\right) \bar{g}_{0,3}\left(\xi_{0}\right) \omega_{0}(\nu) \\
& -540 g_{2,1}\left(\xi_{0}\right) \bar{g}_{1,1}\left(\xi_{0}\right) \omega_{0}(\nu) \\
& -270 g_{1,1}\left(\xi_{0}\right) \bar{g}_{1,2}\left(\xi_{0}\right) \omega_{0}(\nu) \\
& +270 \mu_{2}(\nu) g_{2,0}\left(\xi_{0}\right) \partial_{\mu} \gamma\left(\xi_{0}\right) \omega_{0}(\nu) \\
& +270 i \mu_{2}(\nu) g_{2,0}\left(\xi_{0}\right) \partial_{\mu} \eta\left(\xi_{0}\right) \omega_{0}(\nu) \\
& +270 i g_{1,1}\left(\xi_{0}\right) g_{2,0}\left(\xi_{0}\right)^{2} \\
& +540 \mathrm{ig}_{1,1}\left(\xi_{0}\right) \bar{g}_{1,1}\left(\xi_{0}\right)^{2} \\
& -630 i g_{1,1}\left(\xi_{0}\right)^{2} \bar{g}_{0,2}\left(\xi_{0}\right) \\
& -45 i g_{0,2}\left(\xi_{0}\right) g_{2,0}\left(\xi_{0}\right) \bar{g}_{0,2}\left(\xi_{0}\right) \\
& -270 \operatorname{ig}_{1,1}\left(\xi_{0}\right) g_{2,0}\left(\xi_{0}\right) \bar{g}_{1,1}\left(\xi_{0}\right) \\
& +225 i g_{0,2}\left(\xi_{0}\right) \bar{g}_{0,2}\left(\xi_{0}\right) \bar{g}_{1,1}\left(\xi_{0}\right) \\
& +90 i g_{1,1}\left(\xi_{0}\right) \bar{g}_{0,2}\left(\xi_{0}\right) \bar{g}_{2,0}\left(\xi_{0}\right) \text {, } \\
& \omega_{4}^{0}\left(\xi_{0}\right)=270 \operatorname{ig}_{2,2}\left(\xi_{0}\right) \omega_{0}(\nu)^{2} \\
& +540 i \mu_{2}(\nu) \partial_{\mu} g_{1,1}\left(\xi_{0}\right) \omega_{0}(\nu)^{2} \\
& -1080 g_{1,1}\left(\xi_{0}\right) g_{2,1}\left(\xi_{0}\right) \omega_{0}(\nu) \\
& -90 g_{0,2}\left(\xi_{0}\right) g_{3,0}\left(\xi_{0}\right) \omega_{0}(\nu) \\
& +90 g_{0,3}\left(\xi_{0}\right) \bar{g}_{0,2}\left(\xi_{0}\right) \omega_{0}(\nu) \\
& +1080 g_{1,2}\left(\xi_{0}\right) \bar{g}_{1,1}\left(\xi_{0}\right) \omega_{0}(\nu) \\
& +270 g_{0,2}\left(\xi_{0}\right) \bar{g}_{1,2}\left(\xi_{0}\right) \omega_{0}(\nu) \\
& -270 g_{2,1}\left(\xi_{0}\right) \bar{g}_{2,0}\left(\xi_{0}\right) \omega_{0}(\nu) \\
& -540 \mu_{2}(\nu) g_{1,1}\left(\xi_{0}\right) \partial_{\mu} \gamma\left(\xi_{0}\right) \omega_{0}(\nu)
\end{aligned}
$$

$$
\begin{aligned}
& -540 i \mu_{2}(\nu) g_{1,1}\left(\xi_{0}\right) \partial_{\mu} \eta\left(\xi_{0}\right) \omega_{0}(\nu) \\
& -1080 i g_{0,2}\left(\xi_{0}\right) \bar{g}_{1,1}\left(\xi_{0}\right)^{2} \\
& -1080 i g_{1,1}\left(\xi_{0}\right)^{2} g_{2,0}\left(\xi_{0}\right) \\
& +570 i g_{0,2}\left(\xi_{0}\right) g_{1,1}\left(\xi_{0}\right) \bar{g}_{0,2}\left(\xi_{0}\right) \\
& +1080 \mathrm{ig}_{1,1}\left(\xi_{0}\right)^{2} \bar{g}_{1,1}\left(\xi_{0}\right) \\
& +270 i g_{0,2}\left(\xi_{0}\right) g_{2,0}\left(\xi_{0}\right) \bar{g}_{1,1}\left(\xi_{0}\right), \\
& w_{4}^{-2}\left(\xi_{0}\right)=-180 i g_{1,1}\left(\xi_{0}\right)^{3}-270 i \bar{g}_{2,0}\left(\xi_{0}\right) g_{1,1}\left(\xi_{0}\right)^{2} \\
& -90 i \bar{g}_{2,0}\left(\xi_{0}\right)^{2} g_{1,1}\left(\xi_{0}\right) \\
& -270 \omega_{0}(\nu) g_{1,2}\left(\xi_{0}\right) g_{1,1}\left(\xi_{0}\right) \\
& -165 i g_{0,2}\left(\xi_{0}\right) g_{2,0}\left(\xi_{0}\right) g_{1,1}\left(\xi_{0}\right) \\
& +330 i g_{0,2}\left(\xi_{0}\right) \bar{g}_{1,1}\left(\xi_{0}\right) g_{1,1}\left(\xi_{0}\right) \\
& -30 \omega_{0}(\nu) \bar{g}_{3,0}\left(\xi_{0}\right) g_{1,1}\left(\xi_{0}\right) \\
& +60 i \omega_{0}(\nu)^{2} g_{1,3}\left(\xi_{0}\right) \\
& -15 \omega_{0}(\nu) g_{0,3}\left(\xi_{0}\right) g_{2,0}\left(\xi_{0}\right) \\
& -120 \omega_{0}(\nu) g_{0,2}\left(\xi_{0}\right) g_{2,1}\left(\xi_{0}\right) \\
& +20 i g_{0,2}\left(\xi_{0}\right)^{2} \bar{g}_{0,2}\left(\xi_{0}\right) \\
& +180 \omega_{0}(\nu) g_{0,3}\left(\xi_{0}\right) \bar{g}_{1,1}\left(\xi_{0}\right) \\
& -180 \omega_{0}(\nu) g_{1,2}\left(\xi_{0}\right) \bar{g}_{2,0}\left(\xi_{0}\right) \\
& -45 i g_{0,2}\left(\xi_{0}\right) g_{2,0}\left(\xi_{0}\right) \bar{g}_{2,0}\left(\xi_{0}\right) \\
& +180 i g_{0,2}\left(\xi_{0}\right) \bar{g}_{1,1}\left(\xi_{0}\right) \bar{g}_{2,0}\left(\xi_{0}\right) \\
& -90 \mu_{2}(\nu) \omega_{0}(\nu) g_{0,2}\left(\xi_{0}\right) \partial_{\mu} \gamma\left(\xi_{0}\right) \\
& -90 i \mu_{2}(\nu) \omega_{0}(\nu) g_{0,2}\left(\xi_{0}\right) \partial_{\mu} \eta\left(\xi_{0}\right) \\
& +90 i \mu_{2}(\nu) \omega_{0}(\nu)^{2} \partial_{\mu} g_{0,2}\left(\xi_{0}\right), \\
& w_{4}^{-4}\left(\xi_{0}\right)=9 i g_{0,4}\left(\xi_{0}\right) \omega_{0}(\nu)^{2}-9 g_{0,3}\left(\xi_{0}\right) g_{1,1}\left(\xi_{0}\right) \omega_{0}(\nu) \\
& -18 g_{0,2}\left(\xi_{0}\right) g_{1,2}\left(\xi_{0}\right) \omega_{0}(\nu) \\
& -9 i g_{0,2}\left(\xi_{0}\right) g_{1,1}\left(\xi_{0}\right)^{2} \\
& -54 g_{0,3}\left(\xi_{0}\right) \bar{g}_{2,0}\left(\xi_{0}\right) \omega_{0}(\nu) \\
& -18 g_{0,2}\left(\xi_{0}\right) \bar{g}_{3,0}\left(\xi_{0}\right) \omega_{0} \\
& -81 i g_{0,2}\left(\xi_{0}\right) \bar{g}_{2,0}\left(\xi_{0}\right)^{2}
\end{aligned}
$$

$$
\begin{align*}
& -3 i g_{0,2}\left(\xi_{0}\right)^{2} g_{2,0}\left(\xi_{0}\right) \\
& -18 i g_{0,2}\left(\xi_{0}\right)^{2} \bar{g}_{1,1}\left(\xi_{0}\right) \\
& -45 i g_{0,2}\left(\xi_{0}\right) g_{1,1}\left(\xi_{0}\right) \bar{g}_{2,0}\left(\xi_{0}\right) \tag{60}
\end{align*}
$$

From the boundary value problem for $k=5$, it follows that

$$
\begin{align*}
\mu_{4}(\nu)= & -\left(2 \operatorname{Re}\left(G_{3,2}\left(\xi_{0}\right)\right)+6 \mu_{2}(\nu) \operatorname{Re}\left(\partial_{\mu} G_{2,1}\left(\xi_{0}\right)\right)\right. \\
& \left.+3 \mu_{2}(\nu)^{2} \partial_{\mu}^{2} \gamma\left(\xi_{0}\right)\right)  \tag{61}\\
& \times\left(\partial_{\mu} \gamma\left(\xi_{0}\right)\right)^{-1} \\
\omega_{4}(\nu)= & 2 \operatorname{Im}\left(G_{3,2}\left(\xi_{0}\right)\right)+\mu_{4}(\nu) \partial_{\mu} \eta\left(\xi_{0}\right) \\
& +6 \mu_{2}(\nu) \operatorname{Im}\left(\partial_{\mu} G_{2,1}\left(\xi_{0}\right)\right)+3 \mu_{2}(\nu)^{2} \partial_{\mu}^{2} \eta\left(\xi_{0}\right) \tag{62}
\end{align*}
$$

where

$$
\begin{align*}
& \partial_{\mu} G_{2,1}\left(\xi_{0}\right)=\frac{3 g_{1,1}\left(\xi_{0}\right) g_{2,0}\left(\xi_{0}\right) \partial_{\mu} \gamma\left(\xi_{0}\right)}{\omega_{0}(\nu)^{2}} \\
& +\frac{g_{0,2}\left(\xi_{0}\right) \bar{g}_{0,2}\left(\xi_{0}\right) \partial_{\mu} \gamma\left(\xi_{0}\right)}{9 \omega_{0}(\nu)^{2}} \\
& +\frac{2 g_{1,1}\left(\xi_{0}\right) \bar{g}_{1,1}\left(\xi_{0}\right) \partial_{\mu} \gamma\left(\xi_{0}\right)}{\omega_{0}(\nu)^{2}} \\
& -\frac{i g_{1,1}\left(\xi_{0}\right) g_{2,0}\left(\xi_{0}\right) \partial_{\mu} \eta\left(\xi_{0}\right)}{\omega_{0}(\nu)^{2}} \\
& +\frac{i g_{0,2}\left(\xi_{0}\right) \bar{g}_{0,2}\left(\xi_{0}\right) \partial_{\mu} \eta\left(\xi_{0}\right)}{3 \omega_{0}(\nu)^{2}} \\
& +\frac{2 i g_{1,1}\left(\xi_{0}\right) \bar{g}_{1,1}\left(\xi_{0}\right) \partial_{\mu} \eta\left(\xi_{0}\right)}{\omega_{0}(\nu)^{2}}  \tag{63}\\
& -\frac{i \bar{g}_{0,2}\left(\xi_{0}\right) \partial_{\mu} g_{0,2}\left(\xi_{0}\right)}{3 \omega_{0}(\nu)} \\
& +\frac{i g_{2,0}\left(\xi_{0}\right) \partial_{\mu} g_{1,1}\left(\xi_{0}\right)}{\omega_{0}(\nu)} \\
& -\frac{2 i \bar{g}_{1,1}\left(\xi_{0}\right) \partial_{\mu} g_{1,1}\left(\xi_{0}\right)}{\omega_{0}(\nu)} \\
& +\frac{i g_{1,1}\left(\xi_{0}\right) \partial_{\mu} g_{2,0}\left(\xi_{0}\right)}{\omega_{0}(\nu)}+\partial_{\mu} g_{2,1}\left(\xi_{0}\right) \\
& -\frac{i g_{0,2}\left(\xi_{0}\right) \partial_{\mu} \bar{g}_{0,2}\left(\xi_{0}\right)}{3 \omega_{0}(\nu)} \\
& -\frac{2 i g_{1,1}\left(\xi_{0}\right) \partial_{\mu} \bar{g}_{1,1}\left(\xi_{0}\right)}{\omega_{0}(\nu)},
\end{align*}
$$

$$
\begin{aligned}
& G_{3,2}\left(\xi_{0}\right)=\frac{12 i \bar{g}_{0,2}\left(\xi_{0}\right) g_{1,1}\left(\xi_{0}\right)^{3}}{\omega_{0}(\nu)^{3}} \\
& -\frac{12 i \bar{g}_{1,1}\left(\xi_{0}\right)^{2} g_{1,1}\left(\xi_{0}\right)^{2}}{\omega_{0}(\nu)^{3}}-\frac{4 g_{3,0}\left(\xi_{0}\right) g_{1,1}\left(\xi_{0}\right)^{2}}{\omega_{0}(\nu)^{2}} \\
& +\frac{12 i g_{2,0}\left(\xi_{0}\right) \bar{g}_{1,1}\left(\xi_{0}\right) g_{1,1}\left(\xi_{0}\right)^{2}}{\omega_{0}(\nu)^{3}} \\
& +\frac{12 \bar{g}_{1,2}\left(\xi_{0}\right) g_{1,1}\left(\xi_{0}\right)^{2}}{\omega_{0}(\nu)^{2}} \\
& -\frac{3 i \bar{g}_{0,2}\left(\xi_{0}\right) \bar{g}_{2,0}\left(\xi_{0}\right) g_{1,1}\left(\xi_{0}\right)^{2}}{\omega_{0}(\nu)^{3}} \\
& +\frac{4 i g_{3,1}\left(\xi_{0}\right) g_{1,1}\left(\xi_{0}\right)}{\omega_{0}(\nu)} \\
& +\frac{9 g_{1,2}\left(\xi_{0}\right) \bar{g}_{0,2}\left(\xi_{0}\right) g_{1,1}\left(\xi_{0}\right)}{\omega_{0}(\nu)^{2}} \\
& +\frac{17 i g_{0,2}\left(\xi_{0}\right) g_{2,0}\left(\xi_{0}\right) \bar{g}_{0,2}\left(\xi_{0}\right) g_{1,1}\left(\xi_{0}\right)}{4 \omega_{0}(\nu)^{3}} \\
& +\frac{31 g_{0,2}\left(\xi_{0}\right) \bar{g}_{0,3}\left(\xi_{0}\right) g_{1,1}\left(\xi_{0}\right)}{12 \omega_{0}(\nu)^{2}} \\
& -\frac{175 i g_{0,2}\left(\xi_{0}\right) \bar{g}_{0,2}\left(\xi_{0}\right) \bar{g}_{1,1}\left(\xi_{0}\right) g_{1,1}\left(\xi_{0}\right)}{12 \omega_{0}(\nu)^{3}} \\
& +\frac{12 g_{2,1}\left(\xi_{0}\right) \bar{g}_{1,1}\left(\xi_{0}\right) g_{1,1}\left(\xi_{0}\right)}{\omega_{0}(\nu)^{2}} \\
& +\frac{6 i \bar{g}_{1,1}\left(\xi_{0}\right)^{2} \bar{g}_{2,0}\left(\xi_{0}\right) g_{1,1}\left(\xi_{0}\right)}{\omega_{0}(\nu)^{3}} \\
& -\frac{g_{3,0}\left(\xi_{0}\right) \bar{g}_{2,0}\left(\xi_{0}\right) g_{1,1}\left(\xi_{0}\right)}{\omega_{0}(v)^{2}} \\
& -\frac{6 i g_{2,0}\left(\xi_{0}\right) \bar{g}_{1,1}\left(\xi_{0}\right) \bar{g}_{2,0}\left(\xi_{0}\right) g_{1,1}\left(\xi_{0}\right)}{\omega_{0}(\nu)^{3}} \\
& +\frac{3 g_{2,0}\left(\xi_{0}\right) \bar{g}_{2,1}\left(\xi_{0}\right) g_{1,1}\left(\xi_{0}\right)}{\omega_{0}(\nu)^{2}} \\
& -\frac{3 i \bar{g}_{2,2}\left(\xi_{0}\right) g_{1,1}\left(\xi_{0}\right)}{\omega_{0}(\nu)} \\
& -\frac{6 \bar{g}_{1,1}\left(\xi_{0}\right) \bar{g}_{2,1}\left(\xi_{0}\right) g_{1,1}\left(\xi_{0}\right)}{\omega_{0}(\nu)^{2}} \\
& -\frac{\bar{g}_{0,2}\left(\xi_{0}\right) \bar{g}_{3,0}\left(\xi_{0}\right) g_{1,1}\left(\xi_{0}\right)}{\omega_{0}(\nu)^{2}} \\
& +\frac{8 i g_{0,2}\left(\xi_{0}\right) \bar{g}_{1,1}\left(\xi_{0}\right)^{3}}{\omega_{0}(\nu)^{3}}
\end{aligned}
$$

$$
\begin{align*}
& -\frac{i g_{0,2}\left(\xi_{0}\right) g_{2,0}\left(\xi_{0}\right) \bar{g}_{0,2}\left(\xi_{0}\right) \bar{g}_{2,0}\left(\xi_{0}\right)}{4 \omega_{0}(\nu)^{3}} \\
& +\frac{g_{0,2}\left(\xi_{0}\right) \bar{g}_{0,3}\left(\xi_{0}\right) \bar{g}_{2,0}\left(\xi_{0}\right)}{12 \omega_{0}(\nu)^{2}} \\
& +\frac{23 i g_{0,2}\left(\xi_{0}\right) \bar{g}_{0,2}\left(\xi_{0}\right) \bar{g}_{1,1}\left(\xi_{0}\right) \bar{g}_{2,0}\left(\xi_{0}\right)}{12 \omega_{0}(\nu)^{3}} \\
& -\frac{2 g_{0,2}\left(\xi_{0}\right) \bar{g}_{0,2}\left(\xi_{0}\right) \bar{g}_{2,1}\left(\xi_{0}\right)}{3 \omega_{0}(\nu)^{2}} . \tag{64}
\end{align*}
$$

Rewriting the coefficient $G_{3,2}\left(\xi_{0}\right)$ in a convenient way, expression (64) is exactly the one that appears in [1].

Definition 5. The real number

$$
\begin{equation*}
l_{2}\left(\xi_{0}\right)=\frac{1}{12} \operatorname{Re}\left(G_{3,2}\left(\xi_{0}\right)\right) \tag{65}
\end{equation*}
$$

where $G_{3,2}\left(\xi_{0}\right)$ is given in (64), is called the second Lyapunov coefficient.

Remark 6. A Hopf point of codimension two for (1) is an equilibrium point $\left(\mathbf{0}, \xi_{1}\right) \in W \times U$, where $\xi_{1}=(0,0)$, that satisfies the definition of a point Hopf of codimension one, except that $l_{1}\left(\xi_{1}\right)=0$. Moreover, it satisfies an additional condition; the second Lyapunov coefficient $l_{2}\left(\xi_{1}\right)$ is nonzero. A Hopf point of codimension two is transversal if

$$
\begin{equation*}
\partial_{\mu} \gamma\left(\xi_{1}\right) \operatorname{Re}\left(\partial_{\nu} G_{2,1}\left(\xi_{1}\right)\right) \neq 0 . \tag{66}
\end{equation*}
$$

In a neighborhood of a transversal Hopf point of codimension two $\left(\mathbf{0}, \xi_{1}\right) \in W \times U$, with $l_{2}\left(\xi_{1}\right) \neq 0$, the dynamic behavior of differential equation (1) is orbitally topologically equivalent to the following complex normal form:

$$
\begin{equation*}
w^{\prime}=(\alpha+i) w+\beta w|w|^{2}+s w|w|^{4} \tag{67}
\end{equation*}
$$

where $s=\operatorname{sign}\left(l_{2}\left(\xi_{1}\right)\right)$. In the bifurcation diagram of (67), there exists a curve of nonhyperbolic periodic orbits that has the exact representations

$$
\begin{equation*}
\Gamma(\epsilon)=\left(s \epsilon^{4},-2 s \epsilon^{2}\right) \tag{68}
\end{equation*}
$$

as a curve parameterized by $\epsilon$ or as a graph of the function

$$
\begin{equation*}
\alpha=\Lambda(\beta)=\frac{1}{4 s} \beta^{2} \tag{69}
\end{equation*}
$$

for $\beta \geq 0$.
The function $(s, \nu) \mapsto w_{5}(s, \nu)$ will not be shown here because it is a long expression and it is not necessary in this work. In many results in this section and, particularly in (63), the following expressions $\partial_{\mu} \gamma\left(\xi_{0}\right), \partial_{\mu} \eta\left(\xi_{0}\right), \partial_{\mu}^{2} \gamma\left(\xi_{0}\right), \partial_{\mu}^{2} \eta\left(\xi_{0}\right)$, $\partial_{\mu} g_{2,0}\left(\xi_{0}\right), \partial_{\mu} g_{1,1}\left(\xi_{0}\right), \partial_{\mu} g_{0,2}\left(\xi_{0}\right)$, and $\partial_{\mu} g_{2,1}\left(\xi_{0}\right)$ appear. These expressions are calculated according to Propositions 7 and 8.

Proposition 7. Consider the differential equation (1) with an equilibrium point $(\mathbf{0}, \xi) \in W \times U$, such that the linear part of the $\operatorname{map}(\mathbf{x}, \xi) \mapsto f(\mathbf{x}, \xi)$, evaluated at $\left(\mathbf{0}, \xi_{0}\right), A\left(\xi_{0}\right)=D f\left(\mathbf{0}, \xi_{0}\right)$, has eigenvalues $\lambda$ and $\bar{\lambda}$, where $\lambda\left(\xi_{0}\right)=\gamma\left(\xi_{0}\right)+i \eta\left(\xi_{0}\right), \gamma\left(\xi_{0}\right)=0$ and $\eta\left(\xi_{0}\right)=\omega_{0}(\nu)>0$. Let also $q(\xi) \in \mathbb{C}^{2}$ be an eigenvector corresponding to the eigenvalue $\lambda(\xi)$, and let $p(\xi) \in \mathbb{C}^{2}$ be an adjoint eigenvector corresponding to the eigenvalue $\bar{\lambda}(\xi)$, satisfying (6), (7), and (8). The following statements hold.
(a) The vector $\partial_{\mu} q\left(\xi_{0}\right) \in \mathbb{C}^{2}$ is the solution of the following nonsingular 3-dimensional system:

$$
\left(\begin{array}{cc}
i \omega_{0}(\nu) I_{2}-A\left(\xi_{0}\right) & q\left(\xi_{0}\right)  \tag{70}\\
\bar{p}\left(\xi_{0}\right) & 0
\end{array}\right)\binom{\partial_{\mu} q\left(\xi_{0}\right)}{s}=\binom{R_{2}\left(\xi_{0}\right)}{0}
$$

with the condition $\left\langle p\left(\xi_{0}\right), \partial_{\mu} q\left(\xi_{0}\right)\right\rangle=0$, where

$$
\begin{equation*}
R_{2}\left(\xi_{0}\right)=\left(\partial_{\mu} A\left(\xi_{0}\right)-\partial_{\mu} \lambda\left(\xi_{0}\right) I_{2}\right) q\left(\xi_{0}\right) \tag{71}
\end{equation*}
$$

(b) The vector $\partial_{\mu} p\left(\xi_{0}\right) \in \mathbb{C}^{2}$ is the solution of the following nonsingular 3-dimensional system:

$$
\left(\begin{array}{cc}
-\left(i \omega_{0}(\nu) I_{2}+A^{T}\left(\xi_{0}\right)\right) & p\left(\xi_{0}\right)  \tag{72}\\
\bar{q}\left(\xi_{0}\right) & 0
\end{array}\right)\binom{\partial_{\mu} p\left(\xi_{0}\right)}{s}=\binom{\bar{R}_{2}\left(\xi_{0}\right)}{0}
$$

with the condition $\left\langle q\left(\xi_{0}\right), \partial_{\mu} p\left(\xi_{0}\right)\right\rangle=0$, where

$$
\begin{equation*}
\bar{R}_{2}\left(\xi_{0}\right)=\left(\partial_{\mu} A^{T}\left(\xi_{0}\right)-\partial_{\mu} \bar{\lambda}\left(\xi_{0}\right) I_{2}\right) p\left(\xi_{0}\right) . \tag{73}
\end{equation*}
$$

(c) The partial derivative with respect to $\mu$ of the real part of the eigenvalue $\lambda(\xi)$, evaluated at $\xi=\xi_{0}$, is given by

$$
\begin{equation*}
\partial_{\mu} \gamma\left(\xi_{0}\right)=\operatorname{Re}\left(\left\langle p\left(\xi_{0}\right), \partial_{\mu} A\left(\xi_{0}\right) q\left(\xi_{0}\right)\right\rangle\right) \tag{74}
\end{equation*}
$$

(d) The partial derivative with respect to $\mu$ of the imaginary part of the eigenvalue $\lambda(\xi)$, evaluated at $\xi=\xi_{0}$, is given by

$$
\begin{equation*}
\partial_{\mu} \eta\left(\xi_{0}\right)=\operatorname{Im}\left(\left\langle p\left(\xi_{0}\right), \partial_{\mu} A\left(\xi_{0}\right) q\left(\xi_{0}\right)\right\rangle\right) . \tag{75}
\end{equation*}
$$

(e) The second-order partial derivative with respect to $\mu$ of the real part of the eigenvalue $\lambda(\xi)$, evaluated at $\xi=\xi_{0}$, is given by

$$
\begin{align*}
\partial_{\mu}^{2} \gamma\left(\xi_{0}\right)=\operatorname{Re}( & \left\langle p\left(\xi_{0}\right), \partial_{\mu}^{2} A\left(\xi_{0}\right) q\left(\xi_{0}\right)\right. \\
& \left.\left.+2\left(\partial_{\mu} A\left(\xi_{0}\right)-\partial_{\mu} \lambda\left(\xi_{0}\right) I_{2}\right) \partial_{\mu} q\left(\xi_{0}\right)\right\rangle\right) \tag{76}
\end{align*}
$$

(f) The second-order partial derivative with respect to $\mu$ of the imaginary part of the eigenvalue $\lambda(\xi)$, evaluated at $\xi=\xi_{0}$, is given by

$$
\begin{align*}
\partial_{\mu}^{2} \eta\left(\xi_{0}\right)=\operatorname{Im}( & \left\langle p\left(\xi_{0}\right), \partial_{\mu}^{2} A\left(\xi_{0}\right) q\left(\xi_{0}\right)\right. \\
& \left.\left.+2\left(\partial_{\mu} A\left(\xi_{0}\right)-\partial_{\mu} \lambda\left(\xi_{0}\right) I_{2}\right) \partial_{\mu} q\left(\xi_{0}\right)\right\rangle\right) \tag{77}
\end{align*}
$$

Proof. Differentiating (6) with respect to the parameter $\mu$ and evaluating at $\xi=\xi_{0}$, we have

$$
\begin{align*}
& \partial_{\mu} A\left(\xi_{0}\right) q\left(\xi_{0}\right)+A\left(\xi_{0}\right) \partial_{\mu} q\left(\xi_{0}\right) \\
& \quad=\partial_{\mu} \lambda\left(\xi_{0}\right) q\left(\xi_{0}\right)+\lambda\left(\xi_{0}\right) \partial_{\mu} q\left(\xi_{0}\right) \tag{78}
\end{align*}
$$

Using the hypotheses, the previous equation is rewritten as

$$
\begin{equation*}
\left(i \omega_{0}(\nu) I_{2}-A\left(\xi_{0}\right)\right) \partial_{\mu} q\left(\xi_{0}\right)=\left(\partial_{\mu} A\left(\xi_{0}\right)-\partial_{\mu} \lambda\left(\xi_{0}\right)\right) q\left(\xi_{0}\right) \tag{79}
\end{equation*}
$$

Taking the inner product of $p\left(\xi_{0}\right) \in \mathbb{C}^{2}$ on both sides of the above equation and using (8), it follows that

$$
\begin{align*}
0 & =\left\langle p\left(\xi_{0}\right),\left(i \omega_{0}(v) I_{2}-A\left(\xi_{0}\right)\right) \partial_{\mu} q\left(\xi_{0}\right)\right\rangle \\
& =\left\langle p\left(\xi_{0}\right), \partial_{\mu} A\left(\xi_{0}\right) q\left(\xi_{0}\right)\right\rangle-\partial_{\mu} \lambda\left(\xi_{0}\right) \tag{80}
\end{align*}
$$

Items (a), (c), and (d) follow from the above equation, the Fredholm alternative (see [1]), and the results of [8]. The proof of part (b) is equal to the previous proof; that is, it is sufficient to differentiate (7) with respect to the parameter $\mu$ and to evaluate at $\xi=\xi_{0}$. The proofs of items (e) and (f) consist of calculating the second-order partial derivative of (6) with respect to the parameter $\mu$, evaluated at $\xi=\xi_{0}$, and to use the Fredholm alternative.

Proposition 8. Consider the coefficients of the formal Taylor series of the map $(z, \bar{z}, \xi) \mapsto g(z, \bar{z}, \xi)$,

$$
\begin{align*}
& g_{2,0}(\xi)=\langle p(\xi), B(q(\xi), q(\xi), \xi)\rangle, \\
& g_{1,1}(\xi)=\langle p(\xi), B(q(\xi), \bar{q}(\xi), \xi)\rangle,  \tag{81}\\
& g_{0,2}(\xi)=\langle p(\xi), B(\bar{q}(\xi), \bar{q}(\xi), \xi)\rangle, \\
& g_{2,1}(\xi)=\langle p(\xi), C(q(\xi), q(\xi), \bar{q}(\xi), \xi)\rangle .
\end{align*}
$$

The following statements hold.
(a) The partial derivative with respect to $\mu$ of the coefficient $g_{2,0}(\xi)$, evaluated at $\xi=\xi_{0}$, is

$$
\begin{align*}
\partial_{\mu} g_{2,0}\left(\xi_{0}\right)= & \left\langle\partial_{\mu} p\left(\xi_{0}\right), B\left(q\left(\xi_{0}\right), q\left(\xi_{0}\right), \xi_{0}\right)\right\rangle \\
+ & \left\langle p\left(\xi_{0}\right), 2 B\left(q\left(\xi_{0}\right), \partial_{\mu} q\left(\xi_{0}\right), \xi_{0}\right)\right.  \tag{82}\\
& \left.+\partial_{\mu} B\left(q\left(\xi_{0}\right), q\left(\xi_{0}\right), \xi_{0}\right)\right\rangle
\end{align*}
$$

(b) The partial derivative with respect to $\mu$ of the coefficient $g_{1,1}(\xi)$, evaluated at $\xi=\xi_{0}$, is given by

$$
\begin{aligned}
\partial_{\mu} g_{1,1}\left(\xi_{0}\right)= & \left\langle\partial_{\mu} p\left(\xi_{0}\right), B\left(q\left(\xi_{0}\right), \bar{q}\left(\xi_{0}\right), \xi_{0}\right)\right\rangle \\
+ & \left\langle p\left(\xi_{0}\right), B\left(\partial_{\mu} q\left(\xi_{0}\right), \bar{q}\left(\xi_{0}\right), \xi_{0}\right)\right. \\
& +B\left(q\left(\xi_{0}\right), \partial_{\mu} \bar{q}\left(\xi_{0}\right), \xi_{0}\right) \\
& \left.+\partial_{\mu} B\left(q\left(\xi_{0}\right), \bar{q}\left(\xi_{0}\right), \xi_{0}\right)\right\rangle .
\end{aligned}
$$

(c) The partial derivative with respect to $\mu$ of the coefficient $g_{0,2}(\xi)$, evaluated at $\xi=\xi_{0}$, is obtained as

$$
\begin{align*}
\partial_{\mu} g_{0,2}\left(\xi_{0}\right)= & \left\langle\partial_{\mu} p\left(\xi_{0}\right), B\left(\bar{q}\left(\xi_{0}\right), \bar{q}\left(\xi_{0}\right), \xi_{0}\right)\right\rangle \\
+ & \left\langle p\left(\xi_{0}\right), 2 B\left(\bar{q}\left(\xi_{0}\right), \partial_{\mu} \bar{q}\left(\xi_{0}\right), \xi_{0}\right)\right.  \tag{84}\\
& \left.+\partial_{\mu} B\left(\bar{q}\left(\xi_{0}\right), \bar{q}\left(\xi_{0}\right), \xi_{0}\right)\right\rangle .
\end{align*}
$$

(d) The partial derivative with respect to $\mu$ of the coefficient $g_{2,1}(\xi)$, evaluated at $\xi=\xi_{0}$, is calculated as

$$
\begin{align*}
\partial_{\mu} g_{2,1}\left(\xi_{0}\right)= & \left\langle\partial_{\mu} p\left(\xi_{0}\right), C\left(q\left(\xi_{0}\right), q\left(\xi_{0}\right), \bar{q}\left(\xi_{0}\right), \xi_{0}\right)\right\rangle \\
& +\left\langle p\left(\xi_{0}\right), 2 C\left(q\left(\xi_{0}\right), \partial_{\mu} q\left(\xi_{0}\right), \bar{q}\left(\xi_{0}\right), \xi_{0}\right)\right. \\
& +C\left(q\left(\xi_{0}\right), q\left(\xi_{0}\right), \partial_{\mu} \bar{q}\left(\xi_{0}\right), \xi_{0}\right) \\
& \left.+\partial_{\mu} C\left(q\left(\xi_{0}\right), q\left(\xi_{0}\right), \bar{q}\left(\xi_{0}\right), \xi_{0}\right)\right\rangle . \tag{85}
\end{align*}
$$

Proof. Observing how the symmetric multilinear functions are defined, the proofs of items (a) to (d) consist in differentiating each expression in (81) with respect to the parameter $\mu$ and evaluating at $\xi=\xi_{0}$.

The theory built up to this point approximates a family of periodic orbits of the complex differential equation (11). In the hypotheses of the Hopf bifurcation, if $(\omega(\epsilon, \nu) t, \epsilon, \nu) \mapsto$ $w(\omega(\epsilon, \nu) t, \epsilon, \nu)$ is a family of periodic orbits of (11), then $(\omega(\epsilon, \nu) t, \epsilon, \nu) \mapsto u(\omega(\epsilon, \nu) t, \epsilon, \nu)$ is a family of periodic orbits associated with the differential equation (1), where

$$
\begin{align*}
u(\omega(\epsilon, \nu) t, \epsilon, \nu)= & w(\omega(\epsilon, v) t, \epsilon, v) q(\phi(\epsilon, \nu), \nu) \\
& +\bar{w}(\omega(\epsilon, v) t, \epsilon, v) \bar{q}(\phi(\epsilon, \nu), \nu), \tag{86}
\end{align*}
$$

or, in a more simple way,

$$
\begin{equation*}
u(s, \epsilon, \nu)=w(s, \epsilon, \nu) q(\phi(\epsilon, \nu), \nu)+\bar{w}(s, \epsilon, \nu) \bar{q}(\phi(\epsilon, \nu), \nu) . \tag{87}
\end{equation*}
$$

The family of periodic orbits $(s, \epsilon, \nu) \mapsto u(s, \epsilon, v)$ has formal Taylor series around $\epsilon=0$ of the following form:

$$
\begin{equation*}
u(s, \epsilon, \nu)=\sum_{k=1}^{\infty} \frac{1}{k!} u_{k}(s, v) \epsilon^{k} \tag{88}
\end{equation*}
$$

and the theory developed previously and the Taylor expansion of (87), around $\epsilon=0$, show that

$$
\begin{aligned}
u_{1}(s, v)= & q\left(\xi_{0}\right) w_{1}(s, v)+\bar{q}\left(\xi_{0}\right) \bar{w}_{1}(s, v) \\
u_{2}(s, v)= & q\left(\xi_{0}\right) w_{2}(s, v)+\bar{q}\left(\xi_{0}\right) \bar{w}_{2}(s, v) \\
u_{3}(s, v)= & q\left(\xi_{0}\right) w_{3}(s, v)+\bar{q}\left(\xi_{0}\right) \bar{w}_{3}(s, v) \\
& +3 \mu_{2}(v)\left(w_{1}(s, v) \partial_{\mu} q\left(\xi_{0}\right)\right. \\
& \left.+\bar{w}_{1}(s, v) \partial_{\mu} \bar{q}\left(\xi_{0}\right)\right)
\end{aligned}
$$

$$
\left.\begin{array}{rl}
u_{4}(s, v)= & q\left(\xi_{0}\right) w_{4}(s, v)
\end{array}\right)+\bar{q}\left(\xi_{0}\right) \bar{w}_{4}(s, v), ~+6 \mu_{2}(\nu)\left(w_{2}(s, v) \partial_{\mu} q\left(\xi_{0}\right) .\right.
$$

The stability of the approximate family of periodic orbits is studied in the next section by means of the Floquet exponent.

## 3. Stability of the Family of Periodic Orbits

According to the Floquet theory (see [9]), the stability of a periodic orbit can be determined through the characteristic exponent that, in this context and for differential equations in $\mathbb{R}^{2}$, is a function $(\epsilon, \nu) \mapsto \chi(\epsilon, \nu)$ such that

$$
\begin{equation*}
\chi(\epsilon, \nu)=\frac{1}{T(\epsilon, \nu)} \int_{0}^{T(\epsilon, v)} \operatorname{Tr}(\mathscr{M}(\omega(\epsilon, \nu) t, \epsilon, \nu)) d t \tag{90}
\end{equation*}
$$

where $\mathscr{M}(\omega(\epsilon, \nu) t, \epsilon, \nu)=\operatorname{Df}(u(\omega(\epsilon, \nu) t, \epsilon, \nu), \phi(\epsilon, \nu), \nu)$. The next proposition provides a simple way to compute (90) in terms of the map $(z, \bar{z}, \xi) \mapsto g(z, \bar{z}, \xi)$.

Proposition 9. Through a change in time $s=\omega(\epsilon, v) t$, the characteristic exponent associated with the differential equation $z^{\prime}=g(z, \bar{z}, \xi)$ is of the following form:

$$
\begin{equation*}
\chi(\epsilon, v)=\frac{1}{2 \pi} \int_{0}^{2 \pi} \mathscr{H}(s, \epsilon, v) d s \tag{91}
\end{equation*}
$$

where

$$
\begin{align*}
\mathscr{H}(s, \epsilon, v)= & \frac{\partial}{\partial w} g(w(s, \epsilon, v), \bar{w}(s, \epsilon, v), \phi(\epsilon, v), v) \\
& +\frac{\partial}{\partial \bar{w}} \bar{g}(w(s, \epsilon, v), \bar{w}(s, \epsilon, v), \phi(\epsilon, v), v) . \tag{92}
\end{align*}
$$

Proof. The differential equation (1) can be written as (11), where $(z, \bar{z}, \xi) \mapsto g(z, \bar{z}, \xi)=g_{1}\left(z_{1}, z_{2}, \xi\right)+i g_{2}\left(z_{1}, z_{2}, \xi\right)$,

$$
\begin{align*}
g_{1}\left(w_{1}, w_{2}, \xi\right) & =\frac{1}{2}(g(w, \bar{w}, \xi)+\bar{g}(w, \bar{w}, \xi)) \\
g_{2}\left(w_{1}, w_{2}, \xi\right) & =-\frac{i}{2}(g(w, \bar{w}, \xi)-\bar{g}(w, \bar{w}, \xi)) \tag{93}
\end{align*}
$$

and $z=z_{1}+i z_{2}$. Thus, through the changes $z(t)=w(s, \epsilon, \nu)$ and $s=\omega(\epsilon, \nu) t$, the characteristic exponent (90) can be rewritten as

$$
\begin{align*}
& \chi(\epsilon, v) \\
& \quad=\frac{1}{2 \pi} \int_{0}^{2 \pi}\left(\frac{\partial}{\partial w_{1}} g_{1}\left(w_{1}, w_{2}, \xi\right)+\frac{\partial}{\partial w_{2}} g_{2}\left(w_{1}, w_{2}, \xi\right)\right) d s \tag{94}
\end{align*}
$$

where

$$
\begin{align*}
& \frac{\partial}{\partial w_{1}} g_{1}\left(w_{1}, w_{2}, \xi\right) \\
& =\frac{1}{2}\left(\frac{\partial}{\partial w} g(w, \bar{w}, \xi) \frac{\partial w}{\partial w_{1}}+\frac{\partial}{\partial \bar{w}} g(w, \bar{w}, \xi) \frac{\partial \bar{w}}{\partial w_{1}}\right)  \tag{95}\\
& \quad+\frac{1}{2}\left(\frac{\partial}{\partial w} \bar{g}(w, \bar{w}, \xi) \frac{\partial w}{\partial w_{1}}+\frac{\partial}{\partial \bar{w}} \bar{g}(w, \bar{w}, \xi) \frac{\partial \bar{w}}{\partial w_{1}}\right), \\
& \frac{\partial}{\partial w_{2}} g_{2}\left(w_{1}, w_{2}, \xi\right) \\
& =-\frac{i}{2}\left(\frac{\partial}{\partial w} g(w, \bar{w}, \xi) \frac{\partial w}{\partial w_{2}}+\frac{\partial}{\partial \bar{w}} g(w, \bar{w}, \xi) \frac{\partial \bar{w}}{\partial w_{2}}\right)  \tag{96}\\
& \quad+\frac{i}{2}\left(\frac{\partial}{\partial w} \bar{g}(w, \bar{w}, \xi) \frac{\partial w}{\partial w_{2}}+\frac{\partial}{\partial \bar{w}} \bar{g}(w, \bar{w}, \xi) \frac{\partial \bar{w}}{\partial w_{2}}\right) .
\end{align*}
$$

Adding equations (95) and (96) and taking into account that $w=w_{1}+i w_{2}$, it follows that

$$
\begin{align*}
& \frac{\partial}{\partial w_{1}} g_{1}\left(w_{1}, w_{2}, \xi\right)+\frac{\partial}{\partial w_{2}} g_{2}\left(w_{1}, w_{2}, \xi\right)  \tag{97}\\
& \quad=\frac{\partial}{\partial w} g(w, \bar{w}, \xi)+\frac{\partial}{\partial \bar{w}} \bar{g}(w, \bar{w}, \xi)
\end{align*}
$$

Therefore, $\mathscr{H}(s, \epsilon, \nu)=K(w, \bar{w}, \xi)$, with

$$
\begin{equation*}
K(w, \bar{w}, \xi)=\frac{\partial}{\partial w} g(w, \bar{w}, \xi)+\frac{\partial}{\partial \bar{w}} \bar{g}(w, \bar{w}, \xi) \tag{98}
\end{equation*}
$$

By the formal Taylor series in the variable $\epsilon$ of the function $(\epsilon, \nu) \mapsto \chi(\epsilon, \nu)$,

$$
\begin{equation*}
\chi(\epsilon, \nu)=\sum_{k=1}^{\infty} \frac{1}{k!} \chi_{k}(\nu) \epsilon^{k} \tag{99}
\end{equation*}
$$

the theory of approximation of a family of periodic orbits developed in the previous section and Proposition 9 allow us to obtain the terms of the sequence $\left\{\chi_{k}(\nu)\right\}_{k \in \mathbb{N}}$. For $k=$ $1, \ldots, 4$, the next theorem provides these terms.

Theorem 10. Let

$$
\begin{equation*}
\chi(\epsilon, v)=\sum_{k=1}^{\infty} \frac{1}{k!} \chi_{k}(v) \epsilon^{k} \tag{100}
\end{equation*}
$$

be the formal Taylor series of the characteristic exponent $(\epsilon, \nu) \mapsto \chi(\epsilon, \nu)$ associated with the differential equation $z^{\prime}=$ $g(z, \bar{z}, \xi)$. Then,

$$
\begin{align*}
& \chi_{1}(\nu)=0 \\
& \chi_{2}(\nu)=2 \operatorname{Re}\left(G_{2,1}\left(\xi_{0}\right)\right), \\
& \chi_{3}(\nu)=0  \tag{101}\\
& \chi_{4}(\nu)=8 \operatorname{Re}\left(G_{3,2}\left(\xi_{0}\right)\right)+12 \mu_{2}(\nu) \operatorname{Re}\left(\partial_{\mu} G_{2,1}\left(\xi_{0}\right)\right), \\
& \chi_{5}(\nu)=0
\end{align*}
$$

where $G_{2,1}\left(\xi_{0}\right), \partial_{\mu} G_{2,1}\left(\xi_{0}\right)$, and $G_{3,2}\left(\xi_{0}\right)$ are given by (50), (63), and (64), respectively.

Proof. From (13) and (14), we have

$$
\begin{align*}
& \frac{\partial}{\partial w} g(w, \bar{w}, \xi)=\lambda(\xi)+\sum_{k=2}^{\infty} \sum_{j=0}^{k} \frac{k-j}{(k-j)!j!} g_{k-j, j}(\xi) w^{k-j-1} \bar{w}^{j}, \\
& \frac{\partial}{\partial \bar{w}} \bar{g}(w, \bar{w}, \xi)=\bar{\lambda}(\xi)+\sum_{k=2}^{\infty} \sum_{j=0}^{k} \frac{k-j}{(k-j)!j!} \bar{g}_{k-j, j}(\xi) \bar{w}^{k-j-1} w^{j} . \tag{102}
\end{align*}
$$

Thus, formally, the map $(w, \bar{w}, \xi) \mapsto K(w, \bar{w}, \xi)$ has the Taylor series

$$
\begin{align*}
& K(w, \bar{w}, \xi)= \lambda(\xi)+ \\
&+\bar{\lambda}(\xi) \\
&+\sum_{k=2}^{\infty} \sum_{j=0}^{k} \frac{k-j}{(k-j)!j!}  \tag{103}\\
& \times\left(g_{k-j, j}(\xi)+\bar{g}_{k-j, j}(\xi)\right) w^{k-j-1} \bar{w}^{j}
\end{align*}
$$

Doing the fourth-order Taylor expansion of the map

$$
\begin{equation*}
(s, \epsilon, \nu) \mapsto \mathscr{H}(s, \epsilon, \nu)=K(w(s, \epsilon, \nu), \bar{w}(s, \epsilon, \nu), \phi(\epsilon, \nu), \nu), \tag{104}
\end{equation*}
$$

around $\epsilon=0$, and taking into account that $\mu_{1}(\nu)=\mu_{3}(\nu)=0$, it results that

$$
\begin{align*}
\mathscr{H}(s, \epsilon, v)= & \mathscr{H}_{1}(s, v) \epsilon+\frac{1}{2} \mathscr{H}_{2}(s, v) \epsilon^{2} \\
& +\frac{1}{6} \mathscr{H}_{3}(s, v) \epsilon^{3}+\frac{1}{24} \mathscr{H}_{4}(s, v) \epsilon^{4}  \tag{105}\\
& +O_{\mathscr{H}}\left(s, \epsilon^{5},|v|\right)
\end{align*}
$$

with

$$
\begin{aligned}
\mathscr{H}_{1}(s, v)= & \bar{w}_{1}(s, v) g_{1,1}\left(\xi_{0}\right)+w_{1}(s, v) g_{2,0}\left(\xi_{0}\right) \\
& +w_{1}(s, v) \bar{g}_{1,1}\left(\xi_{0}\right)+\bar{w}_{1}(s, v) \bar{g}_{2,0}\left(\xi_{0}\right), \\
\mathscr{H}_{2}(s, v)= & w_{1}(s, v)^{2} g_{3,0}\left(\xi_{0}\right)+w_{1}(s, v)^{2} \bar{g}_{1,2}\left(\xi_{0}\right) \\
& +2 w_{1}(s, v) \bar{w}_{1}(s, v) g_{2,1}\left(\xi_{0}\right) \\
& +2 w_{1}(s, v) \bar{w}_{1}(s, v) \bar{g}_{2,1}\left(\xi_{0}\right) \\
& +\bar{w}_{2}(s, v) g_{1,1}\left(\xi_{0}\right)+\bar{w}_{1}(s, v)^{2} g_{1,2}\left(\xi_{0}\right) \\
& +w_{2}(s, v) g_{2,0}\left(\xi_{0}\right)+w_{2}(s, v) \bar{g}_{1,1}\left(\xi_{0}\right) \\
& +\bar{w}_{2}(s, v) \bar{g}_{2,0}\left(\xi_{0}\right) \\
& +\bar{w}_{1}(s, v)^{2} \bar{g}_{3,0}\left(\xi_{0}\right)+2 \mu_{2}(v) \partial_{\mu} \gamma\left(\xi_{0}\right), \\
\mathscr{H}_{3}(s, v)= & w_{1}(s, v)^{3} g_{4,0}\left(\xi_{0}\right)+w_{1}(s, v)^{3} \bar{g}_{1,3}\left(\xi_{0}\right) \\
& +3 w_{1}(s, v)^{2} \bar{w}_{1}(s, v) g_{3,1}\left(\xi_{0}\right) \\
& +3 w_{1}(s, v)^{2} \bar{w}_{1}(s, v) \bar{g}_{2,2}\left(\xi_{0}\right)
\end{aligned}
$$

$$
\begin{aligned}
& +3 w_{1}(s, v) \bar{w}_{2}(s, v) g_{2,1}\left(\xi_{0}\right) \\
& +3 w_{1}(s, \nu) \bar{w}_{1}(s, \nu)^{2} g_{2,2}\left(\xi_{0}\right) \\
& +3 w_{1}(s, v) w_{2}(s, v) g_{3,0}\left(\xi_{0}\right) \\
& +3 w_{1}(s, v) w_{2}(s, v) \bar{g}_{1,2}\left(\xi_{0}\right) \\
& +3 w_{1}(s, v) \bar{w}_{2}(s, v) \bar{g}_{2,1}\left(\xi_{0}\right) \\
& +3 w_{1}(s, \nu) \bar{w}_{1}(s, \nu)^{2} \bar{g}_{3,1}\left(\xi_{0}\right) \\
& +3 \mu_{2}(\nu) w_{1}(s, v) \partial_{\mu} g_{2,0}\left(\xi_{0}\right) \\
& +3 \mu_{2}(\nu) \partial_{\mu} \bar{g}_{1,1}\left(\xi_{0}\right) w_{1}(s, \nu) \\
& +\bar{w}_{3}(s, \nu) g_{1,1}\left(\xi_{0}\right) \\
& +3 \bar{w}_{1}(s, v) \bar{w}_{2}(s, v) g_{1,2}\left(\xi_{0}\right) \\
& +\bar{w}_{1}(s, \nu)^{3} g_{1,3}\left(\xi_{0}\right)+w_{3}(s, \nu) g_{2,0}\left(\xi_{0}\right) \\
& +3 w_{2}(s, v) \bar{w}_{1}(s, v) g_{2,1}\left(\xi_{0}\right) \\
& +w_{3}(s, \nu) \bar{g}_{1,1}\left(\xi_{0}\right)+\bar{w}_{3}(s, \nu) \bar{g}_{2,0}\left(\xi_{0}\right) \\
& +3 w_{2}(s, v) \bar{w}_{1}(s, v) \bar{g}_{2,1}\left(\xi_{0}\right) \\
& +3 \bar{w}_{1}(s, \nu) \bar{w}_{2}(s, \nu) \bar{g}_{3,0}\left(\xi_{0}\right) \\
& +\bar{w}_{1}(s, \nu)^{3} \bar{g}_{4,0}\left(\xi_{0}\right)+3 \mu_{2}(\nu) \bar{w}_{1}(s, \nu) \partial_{\mu} g_{1,1}\left(\xi_{0}\right) \\
& +3 \mu_{2}(\nu) \bar{w}_{1}(s, \nu) \partial_{\mu} \bar{g}_{2,0}\left(\xi_{0}\right), \\
& \mathscr{H}_{4}(s, v)=w_{1}(s, v)^{4} g_{5,0}\left(\xi_{0}\right)+w_{1}(s, v)^{4} \bar{g}_{1,4}\left(\xi_{0}\right) \\
& +4 w_{1}(s, v)^{3} \bar{w}_{1}(s, v) g_{4,1}\left(\xi_{0}\right) \\
& +4 w_{1}(s, v)^{3} \bar{w}_{1}(s, v) \bar{g}_{2,3}\left(\xi_{0}\right) \\
& +6 w_{1}(s, v)^{2} \bar{w}_{2}(s, v) g_{3,1}\left(\xi_{0}\right) \\
& +6 w_{1}(s, v)^{2} \bar{w}_{1}(s, v)^{2} g_{3,2}\left(\xi_{0}\right) \\
& +6 w_{1}(s, v)^{2} w_{2}(s, v) g_{4,0}\left(\xi_{0}\right) \\
& +6 w_{1}(s, v)^{2} w_{2}(s, v) \bar{g}_{1,3}\left(\xi_{0}\right) \\
& +6 w_{1}(s, v)^{2} \bar{w}_{2}(s, v) \bar{g}_{2,2}\left(\xi_{0}\right) \\
& +6 w_{1}(s, v)^{2} \bar{w}_{1}(s, \nu)^{2} \bar{g}_{3,2}\left(\xi_{0}\right) \\
& +6 \mu_{2}(\nu) w_{1}(s, \nu)^{2} \partial_{\mu} g_{3,0}\left(\xi_{0}\right) \\
& +6 \mu_{2}(\nu) w_{1}(s, \nu)^{2} \partial_{\mu} \bar{g}_{1,2}\left(\xi_{0}\right) \\
& +4 w_{1}(s, v) \bar{w}_{3}(s, v) g_{2,1}\left(\xi_{0}\right) \\
& +12 w_{1}(s, \nu) \bar{w}_{1}(s, \nu) \bar{w}_{2}(s, \nu) g_{2,2}\left(\xi_{0}\right) \\
& +4 w_{1}(s, \nu) \bar{w}_{1}(s, \nu)^{3} g_{2,3}\left(\xi_{0}\right) \\
& +12 w_{1}(s, \nu) \bar{w}_{1}(s, \nu) w_{2}(s, \nu) g_{3,1}\left(\xi_{0}\right)
\end{aligned}
$$

$$
\begin{align*}
& +4 w_{1}(s, v) w_{3}(s, \nu) g_{3,0}\left(\xi_{0}\right) \\
& +4 w_{1}(s, v) w_{3}(s, \nu) \bar{g}_{1,2}\left(\xi_{0}\right) \\
& +4 w_{1}(s, v) \bar{w}_{3}(s, v) \bar{g}_{2,1}\left(\xi_{0}\right) \\
& +12 w_{1}(s, \nu) \bar{w}_{1}(s, \nu) w_{2}(s, \nu) \bar{g}_{2,2}\left(\xi_{0}\right)  \tag{106}\\
& +12 w_{1}(s, v) \bar{w}_{1}(s, v) \bar{w}_{2}(s, v) \bar{g}_{3,1}\left(\xi_{0}\right) \\
& +4 w_{1}(s, v) \bar{w}_{1}(s, v)^{3} \bar{g}_{4,1}\left(\xi_{0}\right) \\
& +12 \mu_{2}(\nu) w_{1}(s, v) \bar{w}_{1}(s, v) \partial_{\mu} g_{2,1}\left(\xi_{0}\right) \\
& +12 \mu_{2}(\nu) w_{1}(s, v) \bar{w}_{1}(s, v) \partial_{\mu} \bar{g}_{2,1}\left(\xi_{0}\right) \\
& +\bar{w}_{4}(s, \nu) g_{1,1}\left(\xi_{0}\right)+3 \bar{w}_{2}(s, \nu)^{2} g_{1,2}\left(\xi_{0}\right)  \tag{107}\\
& +4 \bar{w}_{1}(s, v) \bar{w}_{3}(s, v) g_{1,2}\left(\xi_{0}\right) \\
& +6 \bar{w}_{1}(s, \nu)^{2} \bar{w}_{2}(s, \nu) g_{1,3}\left(\xi_{0}\right) \\
& +\bar{w}_{1}(s, v)^{4} g_{1,4}\left(\xi_{0}\right) \\
& +w_{4}(s, \nu) g_{2,0}\left(\xi_{0}\right) \\
& +4 \bar{w}_{1}(s, v) w_{3}(s, v) g_{2,1}\left(\xi_{0}\right)  \tag{108}\\
& +6 w_{2}(s, v) \bar{w}_{2}(s, v) g_{2,1}\left(\xi_{0}\right) \\
& +6 \bar{w}_{1}(s, \nu)^{2} w_{2}(s, \nu) g_{2,2}\left(\xi_{0}\right) \\
& +3 w_{2}(s, \nu)^{2} g_{3,0}\left(\xi_{0}\right) \\
& +w_{4}(s, \nu) \bar{g}_{1,1}\left(\xi_{0}\right) \\
& +3 w_{2}(s, \nu)^{2} \bar{g}_{1,2}\left(\xi_{0}\right) \\
& +\bar{w}_{4}(s, \nu) \bar{g}_{2,0}\left(\xi_{0}\right) \\
& +4 \bar{w}_{1}(s, \nu) w_{3}(s, \nu) \bar{g}_{2,1}\left(\xi_{0}\right)  \tag{109}\\
& +6 w_{2}(s, v) \bar{w}_{2}(s, v) \bar{g}_{2,1}\left(\xi_{0}\right) \\
& +3 \bar{w}_{2}(s, \nu)^{2} \bar{g}_{3,0}\left(\xi_{0}\right) \\
& +4 \bar{w}_{1}(s, v) \bar{w}_{3}(s, v) \bar{g}_{3,0}\left(\xi_{0}\right) \\
& +6 \bar{w}_{1}(s, \nu)^{2} w_{2}(s, \nu) \bar{g}_{3,1}\left(\xi_{0}\right)  \tag{110}\\
& +6 \bar{w}_{1}(s, \nu)^{2} \bar{w}_{2}(s, \nu) \bar{g}_{4,0}\left(\xi_{0}\right) \\
& +\bar{w}_{1}(s, v)^{4} \bar{g}_{5,0}\left(\xi_{0}\right) \\
& +6 \mu_{2}(\nu) \bar{w}_{2}(s, v) \partial_{\mu} g_{1,1}\left(\xi_{0}\right) \\
& +2 \mu_{4}(\nu) \partial_{\mu} \gamma\left(\xi_{0}\right) \\
& +6 \mu_{2}(\nu) \bar{w}_{1}(s, \nu)^{2} \partial_{\mu} g_{1,2}\left(\xi_{0}\right) \\
& +6 \mu_{2}(\nu) w_{2}(s, \nu) \partial_{\mu} g_{2,0}\left(\xi_{0}\right) \\
& +6 \mu_{2}(\nu) w_{2}(s, \nu) \partial_{\mu} \bar{g}_{1,1}\left(\xi_{0}\right) \tag{111}
\end{align*}
$$

$$
\begin{aligned}
& +6 \mu_{2}(\nu) \bar{w}_{2}(s, v) \partial_{\mu} \bar{g}_{2,0}\left(\xi_{0}\right) \\
& +6 \mu_{2}(v) \bar{w}_{1}(s, v)^{2} \partial_{\mu} \bar{g}_{3,0}\left(\xi_{0}\right) \\
& +6 \mu_{2}(v)^{2} \partial_{\mu}^{2} \gamma\left(\xi_{0}\right)
\end{aligned}
$$

where for $k=1,2,3,4$, the functions $(s, v) \mapsto w_{k}(s, v)$ are such as in (37), (46), (54), and (59) and the expressions $\mu_{2}(\nu)$ and $\mu_{4}(\nu)$ are given by (52), and (61), respectively. Thus, from (99) and (105) and by Proposition 9,

$$
\begin{aligned}
\sum_{k=1}^{4} & \frac{1}{k!}\left(\chi_{k}(\epsilon, v)-\frac{1}{2 \pi} \int_{0}^{2 \pi} \mathscr{H}_{k}(s, v) d s\right) \epsilon^{k} \\
& +O_{\mathscr{H}_{I}}\left(\epsilon^{5},|v|\right)=0
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& \frac{1}{2 \pi} \int_{0}^{2 \pi} \mathscr{H}_{1}(s, v) d s=\frac{1}{2 \pi} \int_{0}^{2 \pi} \mathscr{H}_{3}(s, v) d s=0 \\
& \frac{1}{2 \pi} \int_{0}^{2 \pi} \mathscr{H}_{2}(s, v) d s=2 \operatorname{Re}\left(G_{2,1}\left(\xi_{0}\right)\right) \\
& \begin{aligned}
\frac{1}{2 \pi} \int_{0}^{2 \pi} \mathscr{H}_{4}(s, v) d s= & 8 \operatorname{Re}\left(G_{3,2}\left(\xi_{0}\right)\right) \\
& +12 \mu_{2}(v) \operatorname{Re}\left(\partial_{\mu} G_{2,1}\left(\xi_{0}\right)\right)
\end{aligned}
\end{aligned}
$$

which proves the theorem.
It follows from Theorem 10 a corollary that deals with the stability of a family of periodic orbits of the differential equation (1) which exists due to a Hopf bifurcation.

Corollary 11. Let

$$
\chi(\epsilon, \nu)=\frac{1}{2} \chi_{2}(\nu) \epsilon^{2}+\frac{1}{24} \chi_{4}(\nu) \epsilon^{4}+O_{\chi}\left(\epsilon^{5},|\nu|\right)
$$

be the fourth-order Taylor expansion around $\epsilon=0$ of the characteristic exponent $(\epsilon, \nu) \mapsto \chi(\epsilon, \nu)$ associated with the differential equation $z^{\prime}=g(z, \bar{z}, \xi)$, and let

$$
\mu=\phi(\epsilon, \nu)=\frac{1}{2} \mu_{2}(\nu) \epsilon^{2}+\frac{1}{24} \mu_{4}(\nu) \epsilon^{4}+O_{\chi}\left(\epsilon^{5},|\nu|\right)
$$

be the fourth-order Taylor expansion, around $\epsilon=0$, of the function $(\epsilon, \nu) \mapsto \mu=\phi(\epsilon, \nu)$. The following statements hold.
(a) For a fixed $(\epsilon, v) \in U_{\epsilon}, \epsilon \in \mathbb{R}$ sufficiently small, and $\operatorname{Re}\left(G_{2,1}\left(\xi_{0}\right)\right) \neq 0$, the stability of the periodic orbit of the differential equation (1) is given by the sign of $\operatorname{Re}\left(G_{2,1}\left(\xi_{0}\right)\right)$. When $\operatorname{Re}\left(G_{2,1}\left(\xi_{0}\right)\right)<0$ for $\xi_{0} \in U$, the periodic orbit in the phase portrait of differential equation (1) is stable. As for $\in \in \mathbb{R}$, sufficiently small,

$$
\mu=\phi(\epsilon, \nu)=-\frac{1}{2} \frac{\operatorname{Re}\left(G_{2,1}\left(\xi_{0}\right)\right)}{\partial_{\mu} \gamma\left(\xi_{0}\right)} \epsilon^{2}+O_{\mu}\left(\epsilon^{4},|\nu|\right)
$$

if $\partial_{\mu} \gamma\left(\xi_{0}\right)>0$, the periodic orbit in the phase portrait of (1) exists for $\mu>0$, and if $\partial_{\mu} \gamma\left(\xi_{0}\right)<0$, the periodic orbit in the phase portrait exists for $\mu<0$. If $\operatorname{Re}\left(G_{2,1}\left(\xi_{0}\right)\right)>0$, the periodic orbit in the phase portrait of the differential equation (1) is unstable.
(b) Suppose that for $\xi_{1}=(0,0), \operatorname{Re}\left(G_{2,1}\left(\xi_{1}\right)\right)=0$ and $\operatorname{Re}\left(G_{3,2}\left(\xi_{1}\right)\right) \neq 0$. Then, for $\epsilon \in \mathbb{R}$ sufficiently small, the stability is given by the sign of $\operatorname{Re}\left(G_{3,2}\left(\xi_{1}\right)\right)$. When $\operatorname{Re}\left(G_{3,2}\left(\xi_{1}\right)\right)<0$, the periodic orbit in the phase portrait of the differential equation (1) is stable. As, in this case,
$\mu=\phi(\epsilon, \nu)=-\frac{1}{12} \frac{\operatorname{Re}\left(G_{3,2}\left(\xi_{1}\right)\right)}{\partial_{\mu} \gamma\left(\xi_{1}\right)} \epsilon^{4}+O_{\mu}\left(\epsilon^{5},|\nu|\right)$,
if $\partial_{\mu} \gamma\left(\xi_{1}\right)>0$, the periodic orbit in the phase portrait of the differential equation (1) exists for $\mu>0$, and if $\partial_{\mu} \gamma\left(\xi_{1}\right)<0$, the periodic orbit in the phase portrait of the differential equation (1) exists for $\mu<0$. If $\operatorname{Re}\left(G_{3,2}\left(\xi_{1}\right)\right)>0$, the periodic orbit in the phase portrait of the differential equation (1) is unstable.

Proof. As the sign of the Floquet exponent provides the stability of a periodic orbit, by (109), and (110) the proof is immediate.

Corollary 11 does not deal with the case where $\chi(\epsilon, \nu)=0$ for a set of points $(\epsilon, \nu) \in U_{\epsilon}$. The theory developed up to this point enables us to study the curve of nonhyperbolic periodic orbits $C_{\mathrm{NH}}$ in the parameter plane $(\mu, \nu) \in \mathbb{R}^{2}$, associated with a transversal Hopf point of codimension two. This curve is the set

$$
\begin{equation*}
\chi^{-1}(0)=\left\{(\epsilon, \nu) \in U_{\epsilon}: \chi(\epsilon, v)=0\right\} . \tag{113}
\end{equation*}
$$

From the set $\chi^{-1}(0)$ and the Implicit Function Theorem, the parameter $v$ can be obtained as a function of the parameter $\epsilon$. Therefore, the curve $C_{\mathrm{NH}}$ follows from functions $\epsilon \mapsto$ $\nu=\psi(\epsilon)$ and $(\epsilon, \nu) \mapsto \mu=\phi(\epsilon, \nu)$; that is, the curve $C_{\mathrm{NH}}$ can be locally represented as a curve parameterized by $\epsilon$

$$
\begin{equation*}
\Gamma(\epsilon)=(\phi(\epsilon, \psi(\epsilon)), \psi(\epsilon)), \tag{114}
\end{equation*}
$$

or can be locally represented as the graph of a function

$$
\begin{equation*}
\mu=\Lambda(\nu) \tag{115}
\end{equation*}
$$

In fact, the Taylor expansion around $\epsilon=0$ of the exponent characteristic is such as in (99), and, therefore,

$$
\begin{equation*}
\chi(\epsilon, \nu)=\epsilon^{2} \Psi(\epsilon, \nu), \tag{116}
\end{equation*}
$$

where the third-order Taylor expansion around $\epsilon=0$ of the function $(\epsilon, \nu) \mapsto \Psi(\epsilon, \nu)$ is of the following form:

$$
\begin{equation*}
\Psi(\epsilon, \nu)=\frac{1}{2} \chi_{2}(\nu)+\frac{1}{24} \chi_{4}(\nu) \epsilon^{2}+O_{\chi}\left(\epsilon^{3},|\nu|\right) . \tag{117}
\end{equation*}
$$

It is easy to see that $\Psi^{-1}(0)=\left\{(\epsilon, \nu) \in U_{\epsilon}: \Psi(\epsilon, \nu)=0\right\} \subset$ $\chi^{-1}(0)$. Thus, the study of the curve of nonhyperbolic periodic
orbits in the parameter plane $(\mu, \nu) \in \mathbb{R}^{2}$, associated with the differential equation (1), and in the hypotheses of a transversal Hopf bifurcation of codimension two is reduced to the study of the set $\Psi^{-1}(0)$.

The next lemma, whose proof is given in [3], guarantees the existence of the function $\epsilon \mapsto \nu=\psi(\epsilon)$.

Lemma 12. Let

$$
\begin{align*}
D: & \mathbb{R} \times \mathbb{R}^{4} \longrightarrow \mathbb{R} \\
\quad(x, y) & \longmapsto D(x, y) \tag{118}
\end{align*}
$$

be a smooth function, where $y=\left(y_{0}, y_{1}, y_{2}, y_{3}\right)$. Suppose that for $\left(0, y^{0}\right) \in \mathbb{R} \times \mathbb{R}^{4}, y^{0}=\left(y_{0}, 0, y_{2}, y_{3}\right)$, the function in (118) satisfies the following assumptions:
(A1) $D\left(0, y^{0}\right)=0$;
(A2) $\partial_{x} D\left(0, y^{0}\right)=0$;
(A3) $\partial_{y_{1}} D\left(0, y^{0}\right) \neq 0$;
(A4) $\partial_{x}^{2} D\left(0, y^{0}\right) \neq 0$.
Then, there exists a unique smooth function

$$
\begin{equation*}
\left(x, y^{0}\right) \longmapsto y_{1}=\phi\left(x, y^{0}\right) \tag{119}
\end{equation*}
$$

such that $y=\Phi\left(x, y^{0}\right)=y^{0}+\left(0, \phi\left(x, y^{0}\right), 0,0\right)$ and $D\left(x, \Phi\left(x, y^{0}\right)\right) \equiv 0$. Moreover, the function $\left(x, y^{0}\right) \mapsto y_{1}=$ $\phi\left(x, y^{0}\right)$ has the following representation:

$$
\begin{equation*}
\phi\left(x, y^{0}\right)=\frac{1}{2!} \phi_{2}\left(y^{0}\right) x^{2}+O_{\phi}\left(|x|^{3},\left\|y^{0}\right\|\right) \tag{120}
\end{equation*}
$$

where

$$
\begin{equation*}
\phi_{2}\left(y^{0}\right)=-\frac{\partial_{x}^{2} D\left(0, y^{0}\right)}{\partial_{y_{1}} D\left(0, y^{0}\right)} \tag{121}
\end{equation*}
$$

The following theorem can be stated now.
Theorem 13. Let $\left(0, \xi_{1}\right) \in W \times U$ be a transversal Hopf point of codimension two of (1). Then, the curve of nonhyperbolic periodic orbits $C_{N H}$, in the parameter plane $(\mu, \nu) \in \mathbb{R}^{2}$, associated with the differential equation (1), has the following local representations:

$$
\begin{align*}
& \Gamma(\epsilon)=\left(\frac{\operatorname{Re}\left(G_{3,2}\left(\xi_{1}\right)\right)}{12 \partial_{\mu} \gamma\left(\xi_{1}\right)} \epsilon^{4},-\frac{\operatorname{Re}\left(G_{3,2}\left(\xi_{1}\right)\right)}{3 \operatorname{Re}\left(\partial_{\gamma} G_{2,1}\left(\xi_{1}\right)\right)} \epsilon^{2}\right)+O_{\Gamma}(\epsilon),  \tag{122}\\
& \mu=\Lambda(\nu)=\frac{\mu_{2}(\nu)}{\psi_{2}} \nu+\frac{1}{6} \frac{\mu_{4}(\nu)}{\psi_{2}^{2}} \nu^{2}+O_{\Lambda}(|v|), \tag{123}
\end{align*}
$$

where

$$
\begin{equation*}
\psi_{2}=-\frac{\partial_{\epsilon}^{2} \Psi(0,0)}{\partial_{\nu} \Psi(0,0)}=-\frac{2 \operatorname{Re}\left(G_{3,2}\left(\xi_{1}\right)\right)}{3 \partial_{\nu} \operatorname{Re}\left(G_{2,1}\left(\xi_{1}\right)\right)} \tag{124}
\end{equation*}
$$

## Proof. As

$$
\begin{align*}
& \Psi(0,0)=0 \\
& \partial_{\epsilon} \Psi(0,0)=0 \\
& \partial_{\epsilon}^{2} \Psi(0,0)=\frac{1}{12} \chi_{4}(0)=\frac{2}{3} \operatorname{Re}\left(G_{3,2}\left(\xi_{1}\right)\right) \neq 0  \tag{125}\\
& \partial_{\nu} \Psi(0,0)=\frac{1}{2} \chi_{2}^{\prime}(0)=\partial_{\nu} \operatorname{Re}\left(G_{2,1}\left(\xi_{1}\right)\right) \neq 0
\end{align*}
$$

Lemma 12 guarantees the existence of a smooth function $\epsilon \mapsto$ $\nu=\psi(\epsilon)$ such that $\Psi(\epsilon, \psi(\epsilon)) \equiv 0$, or even, $\chi(\epsilon, \psi(\epsilon)) \equiv 0$. Moreover, the function $\epsilon \mapsto \nu=\psi(\epsilon)$ has the second-order Taylor expansion around $\epsilon=0$ of the following form:

$$
\begin{equation*}
\nu=\psi(\epsilon)=\frac{1}{2!} \psi_{2} \epsilon^{2}+O_{\psi}\left(\epsilon^{3}\right) \tag{126}
\end{equation*}
$$

where

$$
\begin{equation*}
\psi_{2}=-\frac{\partial_{\epsilon}^{2} \Psi(0,0)}{\partial_{v} \Psi(0,0)}=-\frac{2 \operatorname{Re}\left(G_{3,2}\left(\xi_{1}\right)\right)}{3 \partial_{v} \operatorname{Re}\left(G_{2,1}\left(\xi_{1}\right)\right)} \tag{127}
\end{equation*}
$$

Thus,

$$
\begin{equation*}
\nu=\psi(\epsilon)=-\frac{\operatorname{Re}\left(G_{3,2}\left(\xi_{1}\right)\right)}{3 \partial_{\nu} \operatorname{Re}\left(G_{2,1}\left(\xi_{1}\right)\right)} \epsilon^{2}+O_{\psi}\left(\epsilon^{3}\right) \tag{128}
\end{equation*}
$$

and substituting (128) into the function $(\epsilon, \nu) \mapsto \mu=\phi(\epsilon, \nu)$ results in the following Taylor expansion:

$$
\begin{equation*}
\mu=\phi(\epsilon, \psi(\epsilon))=\frac{\operatorname{Re}\left(G_{3,2}\left(\xi_{1}\right)\right)}{12 \partial_{\mu} \gamma\left(\xi_{1}\right)} \epsilon^{4}+O_{\phi}\left(\epsilon^{5}\right) \tag{129}
\end{equation*}
$$

So, there is a curve in the parameter plane, $\epsilon \mapsto \Gamma(\epsilon)=$ $(\phi(\epsilon, \psi(\epsilon)), \psi(\epsilon))$, that can be parameterized by $\epsilon$ and represented as in (122). Another representation for this curve is obtained when the Implicit Function Theorem is applied to the following function:

$$
\begin{equation*}
\epsilon^{2}=\frac{2}{\psi_{2}} v+O_{\epsilon}\left(|v|^{2}\right) \tag{130}
\end{equation*}
$$

By substituting (130) into (110), the curve $\epsilon \mapsto \Gamma(\epsilon)$ can also be represented locally as

$$
\begin{align*}
\mu=\Lambda(v) & =\frac{1}{2} \mu_{2}(v)\left(\frac{2}{\psi_{2}} v\right)+\frac{1}{24} \mu_{4}(v)\left(\frac{2}{\psi_{2}} v\right)^{2}+O_{\Lambda}(|v|) \\
& =\frac{\mu_{2}(\nu)}{\psi_{2}} v+\frac{1}{6} \frac{\mu_{4}(v)}{\psi_{2}^{2}} v^{2}+O_{\Lambda}(|v|) . \tag{131}
\end{align*}
$$

Therefore, there exists a curve $\Gamma$ in the parameter plane that locally has the representation (122) or (123). By the hypotheses of the transversal Hopf bifurcation of codimension two, $\operatorname{Re}\left(G_{3,2}\left(\xi_{1}\right)\right) \neq 0$ and equation $z^{\prime}=g(z, \bar{z}, \xi)$ are locally topologically equivalent, around $z=0$, to the complex differential equation (67). Therefore, the curve of nonhyperbolic periodic orbits has the representation (122) or (123).

Example 14. For the complex differential equation (67), we have

$$
\begin{gather*}
\gamma(\alpha, \beta)=\alpha \\
\eta(\alpha, \beta)=1  \tag{132}\\
G_{2,1}(\alpha, \beta)=2 \beta \\
G_{3,2}(\alpha, \beta)=12 s
\end{gather*}
$$

So, by Theorem 13, the curve of nonhyperbolic periodic orbits has the following representations:

$$
\begin{align*}
& \Gamma(\epsilon)=\left(s \epsilon^{4},-2 s \epsilon^{2}\right)+O_{\Gamma}(\epsilon) \\
& \alpha=\Lambda(\beta)=\frac{1}{4 s} \beta^{2}+O_{\Lambda}(\beta) \tag{133}
\end{align*}
$$

which agree with (68) and (69), respectively.
The local representations (122) and (123) in Theorem 13 are valid when the Hopf curve is the set $\{(\mu, \nu): \mu=0\}$. If the Hopf curve is the set $\{(\mu, \nu): \mu=\varphi(\nu)\}$ and the transversal Hopf bifurcation of codimension two occurs for $\xi_{1}=\left(\mu_{1}, \nu_{1}\right) \neq(0,0)$, it is easy to show that the local representations are given by

$$
\begin{align*}
\Gamma(\epsilon)=\left(\mu_{1}+\right. & \frac{1}{2} \varphi_{2} \epsilon^{2}+\frac{\operatorname{Re}\left(G_{3,2}\left(\xi_{1}\right)\right)}{12 \partial_{\mu} \gamma\left(\xi_{1}\right)} \epsilon^{4} \\
& \left.\nu_{1}-\frac{\operatorname{Re}\left(G_{3,2}\left(\xi_{1}\right)\right)}{3 \operatorname{Re}\left(\partial_{\nu} G_{2,1}\left(\xi_{1}\right)\right)} \epsilon^{2}\right)+O_{\Gamma}(\epsilon)  \tag{134}\\
\mu= & \Lambda(\nu)=\varphi(\nu)+\frac{\mu_{2}(\nu)}{\psi_{2}}\left(\nu-v_{1}\right) \\
& +\frac{1}{6} \frac{\mu_{4}(\nu)}{\psi_{2}^{2}}\left(\nu-v_{1}\right)^{2}+O_{\Lambda}(|v|)
\end{align*}
$$

for $v \leq v_{1}$, where

$$
\begin{equation*}
\mu_{1}=\varphi\left(v_{1}\right), \quad \varphi_{2}=\psi_{2} \partial_{\nu} \varphi\left(v_{1}\right) \tag{135}
\end{equation*}
$$

The next two sections present applications of the theory developed here in an extension of the van der Pol equation known as the Liénard equation and in Bazykin's predatorprey system and show how local representations of the the curve $C_{\mathrm{NH}}$ are obtained.

## 4. Liénard Equation

One of the pioneers in nonlinear electrical circuits was, undoubtedly, Balthasar van der Pol, through studies with triodes (vacuum tubes). Balthasar van der Pol showed that in circuits with triodes, the electrical quantities can exhibit nonlinear oscillations under certain conditions. Nowadays, it is known that the model of this circuit with triode presents a Hopf bifurcation. In a simple and theoretical way, the electric circuit of van der Pol consists of a triode, a capacitor of capacitance $C$, and an inductor of inductance $L$, according to the diagram of Figure 1.


Figure 1: van der Pol circuit diagram.

Let $v_{C}, i_{C}$ and $v_{L}, i_{L}$ be the models of voltage and current in the capacitor and inductor, respectively. The triode of van der Pol, by the hypothesis, satisfies the generalized Ohm's law $i_{R} \mapsto v_{R}=X\left(i_{R}\right)$, where $v_{R}$ and $i_{R}$ are the models of voltage and current of the triode of van der Pol, respectively. Applying Kirchhoff's laws to the van der Pol electrical circuit model and using the capacitor and inductor equations, it follows that

$$
\begin{array}{ll}
i_{R}=i_{L}=-i_{C}, & v_{r}+v_{L}-v_{C}=0, \\
v_{L}=L \frac{d}{d t} i_{L}, & i_{C}=C \frac{d}{d t} v_{C} . \tag{136}
\end{array}
$$

Therefore, the van der Pol circuit model is of the following form:

$$
\begin{align*}
& L \frac{d}{d t} i_{L}=v_{c}-X\left(i_{L}\right), \\
& C \frac{d}{d t} v_{c}=-i_{L} . \tag{137}
\end{align*}
$$

The study of differential equation (137) is simplified by the change of coordinates and time

$$
\begin{equation*}
x=\frac{L}{\sqrt{L C}} i_{L}, \quad y=v_{C}, \quad \tau=\frac{1}{\sqrt{L C}} t \tag{138}
\end{equation*}
$$

which leads to the differential equation

$$
\begin{gather*}
x^{\prime}=\frac{d}{d \tau} x=y-\mathscr{X}(x) \\
y^{\prime}=\frac{d}{d \tau} y=-x \tag{139}
\end{gather*}
$$

where $\mathscr{X}(x)=X((C / \sqrt{L C}) x)$. Suppose that

$$
\begin{equation*}
x \longmapsto \mathscr{X}(x)=-\mu x+\nu x^{3}+\frac{1}{5} x^{5} . \tag{140}
\end{equation*}
$$

In the literature, the differential equation (139) satisfying (140) is known as the Liénard-type equation.

The Liénard equation has a unique equilibrium point $(\mathbf{0}, \xi) \in \mathbb{R}^{2} \times \mathbb{R}^{2}$, with $\xi=(\mu, \nu)$, and the linear part of the vector field, evaluated at $(\mathbf{0}, \xi)$,

$$
A(\xi)=\left(\begin{array}{cc}
\mu & 1  \tag{141}\\
-1 & 0
\end{array}\right)
$$

has eigenvalues $\lambda$ and $\bar{\lambda}$, with

$$
\begin{equation*}
\lambda(\xi)=\gamma(\xi)+i \eta(\xi)=\frac{1}{2} \mu+i \frac{1}{2} \sqrt{4-\mu^{2}} \tag{142}
\end{equation*}
$$

for $\mu \in(-2,2)$. When $\xi=\xi_{0}=(0, \nu), \gamma\left(\xi_{0}\right)=0$, and $\eta\left(\xi_{0}\right)=$ 1 , which indicates the occurrence of Hopf bifurcations. The eigenvectors $q\left(\xi_{0}\right) \in \mathbb{C}^{2}$ and $p\left(\xi_{0}\right) \in \mathbb{C}^{2}$, where $q\left(\xi_{0}\right)$ is normalized with respect to $p\left(\xi_{0}\right)$ according to (8), are chosen as

$$
\begin{equation*}
q\left(\xi_{0}\right)=\left(-\frac{i}{2}, \frac{1}{2}\right), \quad p\left(\xi_{0}\right)=(-i, 1) \tag{143}
\end{equation*}
$$

In the case of the Liénard equation, the symmetric multilinear functions are given by

$$
\begin{align*}
& B(\mathbf{x}, \mathbf{y}, \xi)=(0,0) \\
& C(\mathbf{x}, \mathbf{y}, \mathbf{u}, \xi)=\left(6 v x_{1} y_{1} u_{1}, 0\right) \\
& D(\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}, \xi)=(0,0)  \tag{144}\\
& E(\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}, \mathbf{w}, \xi)=\left(24 x_{1} y_{1} u_{1} v_{1} w_{1}, 0\right) .
\end{align*}
$$

Thus, for $k=2,3, \ldots$ and $j=0,1, \ldots, k$, the only nonzero coefficients $g_{k-j, j}(\xi)$ are

$$
\begin{align*}
g_{3,0}\left(\xi_{0}\right) & =g_{1,2}\left(\xi_{0}\right)=-g_{2,1}\left(\xi_{0}\right)=-g_{0,3}\left(\xi_{0}\right)=\frac{3}{4} \nu, \\
g_{4,1}\left(\xi_{0}\right) & =g_{2,3}\left(\xi_{0}\right)=g_{0,5}\left(\xi_{0}\right)=-g_{5,0}\left(\xi_{0}\right)  \tag{145}\\
& =-g_{3,2}\left(\xi_{0}\right)=-g_{1,4}\left(\xi_{0}\right)=\frac{3}{4} .
\end{align*}
$$

The eigenvectors $\partial_{\mu} q\left(\xi_{0}\right)$ and $\partial_{\mu} p\left(\xi_{0}\right)$ and the coefficients $\partial_{\mu} g_{2,0}\left(\xi_{0}\right), \partial_{\mu} g_{1,1}\left(\xi_{0}\right), \partial_{\mu} g_{0,2}\left(\xi_{0}\right)$, and $\partial_{\mu} g_{2,1}\left(\xi_{0}\right)$, computed by Propositions 7 and 8 , are such that

$$
\begin{align*}
& \partial_{\mu} q\left(\xi_{0}\right)=\left(-\frac{1}{8}, \frac{i}{8}\right), \\
& \partial_{\mu} p\left(\xi_{0}\right)=\left(\frac{1}{4},-\frac{i}{4}\right),  \tag{146}\\
& \partial_{\mu} g_{2,0}\left(\xi_{0}\right)=\partial_{\mu} g_{1,2}\left(\xi_{0}\right)=\partial_{\mu} g_{0,2}\left(\xi_{0}\right)=0, \\
& \partial_{\mu} g_{2,1}\left(\xi_{0}\right)=\frac{3}{8} i v .
\end{align*}
$$

Thus, from the previous results and by (50), (63), and (64), it follows that

$$
\begin{gather*}
G_{2,1}\left(\xi_{0}\right)=-\frac{3}{4} \nu, \quad \partial_{\mu} G_{2,1}\left(\xi_{0}\right)=\frac{3}{8} i v, \\
G_{3,2}\left(\xi_{0}\right)=-\frac{3}{4}-\frac{81}{64} i \nu^{2} \tag{147}
\end{gather*}
$$

Therefore, the first Lyapunov coefficient is given by

$$
\begin{equation*}
l_{1}\left(\xi_{0}\right)=\operatorname{Re}\left(G_{2,1}\left(\xi_{0}\right)\right)=-\frac{3}{4} \nu \tag{148}
\end{equation*}
$$

and since

$$
\begin{equation*}
\partial_{\mu} \gamma\left(\xi_{0}\right)=\frac{1}{2}, \tag{149}
\end{equation*}
$$

the Liénard equation presents a transversal Hopf bifurcation of codimension one for $\mu=0$ and $\nu \neq 0$. From Corollary 11,

$$
\begin{gather*}
\mu=\left(\frac{3}{4} \nu\right) \epsilon^{2}+O_{\mu}\left(\epsilon^{4},|\nu|\right) \\
\chi(\epsilon, \nu)=\left(-\frac{3}{4} \nu\right) \epsilon^{2}+O_{\chi}\left(\epsilon^{3},|\nu|\right), \tag{150}
\end{gather*}
$$

and if $v<0$, then there exists a unique unstable periodic orbit in the phase portrait of the Liénard equation when $\mu<0$, and if $\nu>0$, the periodic orbit is stable and there exists for $\mu>0$.

For $v=0$, the Liénard equation has a transversal Hopf point of codimension two, and the second Lyapunov coefficient is given by

$$
\begin{equation*}
l_{2}\left(\xi_{1}\right)=\operatorname{Re}\left(G_{3,2}\left(\xi_{1}\right)\right)=-\frac{1}{16} \tag{151}
\end{equation*}
$$

where $\xi_{1}=(0,0)$. Since

$$
\begin{equation*}
\partial_{\nu} \operatorname{Re}\left(G_{2,1}\left(\xi_{1}\right)\right) \partial_{\mu} \gamma\left(\xi_{1}\right)=\left(-\frac{3}{4}\right)\left(\frac{1}{2}\right)=-\frac{3}{8} \neq 0 \tag{152}
\end{equation*}
$$

by Corollary 11 and Theorem 13, the Liénard equation has a bifurcation diagram as shown in Figure 2.

The curve of nonhyperbolic periodic orbits has the following local representations:

$$
\begin{equation*}
\Gamma(\epsilon)=(\mu(\epsilon, \nu(\epsilon)), \nu(\epsilon))=\left(-\frac{1}{8} \epsilon^{4},-\frac{1}{3} \epsilon^{2}\right)+O_{\Gamma}(\epsilon), \tag{153}
\end{equation*}
$$

as a curve parameterized by $\epsilon$ or as the graph of the function

$$
\begin{equation*}
\mu=\Lambda(\nu)=-\frac{9}{8} v^{2}+O_{v}\left(v^{3}\right) \tag{154}
\end{equation*}
$$

for $v \leq 0$.
Figure 3 emphasizes the comparison between the curve of nonhyperbolic periodic orbits $C_{\mathrm{NH}}$ of (139) obtained numerically with the software MATCONT (see [5]) and the quadratic approximation (154).

## 5. Bazykin's Predator-Prey System

Consider the dynamics of a predator-prey ecosystem, whose model is

$$
\begin{gather*}
x^{\prime}=\frac{x^{2}(1-x)}{\mu+x}-x y  \tag{155}\\
y^{\prime}=-\gamma y(\nu-x)
\end{gather*}
$$



Figure 2: Bifurcation diagram of the Liénard equation (139).


Figure 3: Comparison between the curve of nonhyperbolic periodic orbits $C_{\mathrm{NH}}$ of (139): the dotted curve was obtained numerically with the software MATCONT and the continuous curve from the representation (154).
where $\gamma>0$ (fixed), $\mu \geq 0$, and $0 \leq \nu<1$ are parameters. Model (155) is known in the literature as Bazykin's predatorprey system. See [1] or [10].

Taking $\gamma=1$, the equilibrium point of interest is

$$
\begin{equation*}
\left(\mathbf{x}_{0}(\xi), \xi\right)=\left(\left(\nu, \frac{\nu(1-\nu)}{\mu+\nu}\right), \xi\right) . \tag{156}
\end{equation*}
$$

For

$$
\begin{equation*}
\mu=\varphi(\nu)=\frac{v^{2}}{1-2 v}, \tag{157}
\end{equation*}
$$

the linear part of the vector field, evaluated at $\left(\mathbf{x}_{0}(\xi), \xi\right)$,

$$
A(\xi)=\left(\begin{array}{cc}
\frac{v\left(\mu(1-2 v)-v^{2}\right)}{(\mu+v)^{2}} & -v  \tag{158}\\
\frac{v(1-v)}{\mu+v} & 0
\end{array}\right)
$$

has eigenvalues $\lambda$ and $\bar{\lambda}$, where $\lambda\left(\xi_{0}\right)=i \omega_{0}(\nu), \omega_{0}(\nu)=$ $\sqrt{\nu(1-2 \nu)}$ and $\xi_{0}=(\varphi(\nu), \nu)$.

The eigenvectors $q\left(\xi_{0}\right) \in \mathbb{C}^{2}$ and $p\left(\xi_{0}\right) \in \mathbb{C}^{2}$ are chosen as

$$
\begin{equation*}
q\left(\xi_{0}\right)=\left(\frac{i v}{2 \omega_{0}(\nu)}, \frac{1}{2}\right), \quad p\left(\xi_{0}\right)=\left(\frac{i \omega_{0}(\nu)}{\nu}, 1\right) \tag{159}
\end{equation*}
$$

and by Proposition 7,

$$
\begin{align*}
& \partial_{\mu} \gamma\left(\xi_{0}\right)=\frac{(1-2 \nu)^{3}}{2 \nu(1-\nu)^{2}}, \\
& \partial_{\mu} \eta\left(\xi_{0}\right)=-\frac{(1-2 \nu)^{2}}{2(1-\nu) \omega_{0}(\nu)}, \\
& \partial_{\mu}^{2} \gamma\left(\xi_{0}\right)=-\frac{2(1-2 \nu)^{4}}{\nu^{2}(1-\nu)^{3}}, \\
& \partial_{\mu} q\left(\xi_{0}\right)=\left(\frac{1-(1-\nu)\left(4 \nu-i \omega_{0}(\nu)\right)}{8 \nu(1-\nu)^{2}},\right.  \tag{160}\\
& \left.\frac{\omega_{0}(\nu)^{2}\left(i \omega_{0}(\nu)^{3}-\nu^{2}(1-\nu)\right)}{8 \nu^{4}(1-\nu)^{2}}\right), \\
& \partial_{\mu} p\left(\xi_{0}\right)=\left(\frac{i \omega_{0}(\nu)^{3}\left(i \omega_{0}(\nu)^{3}-\nu^{2}(1-\nu)\right)}{4 \nu^{5}(1-\nu)^{2}},\right. \\
& \left.-\frac{\omega_{0}(\nu)^{2}\left(i \omega_{0}(\nu)^{3}-\nu^{2}(1-\nu)\right)}{4 \nu^{4}(1-\nu)^{2}}\right) .
\end{align*}
$$

The symmetric multilinear functions are given by

$$
\begin{align*}
& B(\mathbf{x}, \mathbf{y}, \xi)=\left(-x_{2} y_{1}-\frac{2\left(v^{3}+3 \mu \nu^{2}+\mu^{2}(3 v-1)\right)}{(\mu+v)^{3}} x_{1} y_{1}\right. \\
& \\
& \left.-x_{1} y_{2}, x_{2} y_{1}+x_{1} y_{2}\right) \\
& \partial_{\mu} B(\mathbf{x}, \mathbf{y}, \xi)=\left(\frac{2 \mu(2 v+\mu(3 v-1))}{(\mu+v)^{4}} x_{1} y_{1}, 0\right), \\
& C(\mathbf{x}, \mathbf{y}, \mathbf{u}, \xi)=\left(-\frac{6 \mu^{2}(\mu+1)}{(\mu+v)^{4}} x_{1} y_{1} z_{1}, 0\right), \\
& \partial_{\mu} C(\mathbf{x}, \mathbf{y}, \mathbf{u}, \xi)=\left(\frac{6 \mu\left(\mu^{2}+(2-3 v) \mu-2 v\right)}{(\mu+v)^{5}} x_{1} y_{1} z_{1}, 0\right), \\
& D(\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}, \xi)=\left(\frac{24 \mu^{2}(\mu+1)}{(\mu+v)^{5}} x_{1} y_{1} z_{1} u_{1}, 0\right),  \tag{161}\\
& E(\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}, \mathbf{w}, \xi)=\left(-\frac{120 \mu^{2}(\mu+1)}{(\mu+v)^{6}} x_{1} y_{1} z_{1} u_{1} v_{1}, 0\right) .
\end{align*}
$$



Figure 4: Comparison between the curve of nonhyperbolic periodic orbits $C_{\mathrm{NH}}$ of (155): the dotted curve was obtained numerically with the software MATCONT and the continuous curve from the representation (164). The dot-dashed curve is the Hopf curve.

Therefore, from the previous results,

$$
\begin{gather*}
\operatorname{Re}\left(G_{2,1}\left(\xi_{0}\right)\right)=\frac{1-4 \nu}{4(1-v)^{2}}, \\
\partial_{\mu} \operatorname{Re}\left(G_{2,1}\left(\xi_{0}\right)\right)=-\frac{5(1-2 \nu)\left(40 \nu^{3}-67 \nu^{2}+27 v-2\right)}{72 v^{2}(1-\nu)^{4}}, \\
\partial_{\nu} \operatorname{Re}\left(G_{2,1}\left(\xi_{0}\right)\right)=-\frac{2 v+1}{2(1-v)^{3}}, \\
\operatorname{Re}\left(G_{3,2}\left(\xi_{0}\right)\right)=\frac{292 \nu^{4}-485 \nu^{3}+286 \nu^{2}-68 v+5}{24 v(1-2 v)^{2}(1-v)^{4}} . \tag{162}
\end{gather*}
$$

When $\nu=\nu_{1}=1 / 4, \mu_{1}=\varphi\left(\nu_{1}\right)=1 / 8$. Thus, for $\xi_{1}=$ $\left(\mu_{1}, \nu_{1}\right)=(1 / 8,1 / 4)$, Bazykin's system (155) has a transversal Hopf point of codimension two, since $\operatorname{Re}\left(G_{2,1}\left(\xi_{1}\right)\right)=0$, $\operatorname{Re}\left(G_{3,2}\left(\xi_{1}\right)\right)=-32 / 27$, and

$$
\begin{equation*}
\partial_{\nu} \operatorname{Re}\left(G_{2,1}\left(\xi_{1}\right)\right) \partial_{\mu} \gamma\left(\xi_{1}\right)=\left(-\frac{16}{9}\right)\left(\frac{4}{9}\right)=-\frac{64}{81} \neq 0 . \tag{163}
\end{equation*}
$$

Using (134), the curve of nonhyperbolic periodic orbits has the following local representations:

$$
\begin{equation*}
\Gamma(\epsilon)=\left(\frac{1}{8}-\frac{\epsilon^{2}}{3}+\frac{14 \epsilon^{4}}{81}, \frac{1}{4}-\frac{2 \epsilon^{2}}{9}\right)+O_{\Gamma}(\epsilon), \tag{164}
\end{equation*}
$$

as a curve parameterized by $\epsilon$ or as a graph of the function

$$
\begin{align*}
\mu= & \Lambda(v)=\frac{v^{2}}{1-2 v}+\frac{9 v(1-4 v)^{2}}{32(2 v-1)^{3}} \\
& +\frac{27(4 v-1)^{2} v\left(120 v^{3}-98 v^{2}+7 v+1\right)}{2048(v-1)^{2}(2 v-1)^{5}}  \tag{165}\\
& +O_{v}\left(v^{3}\right)
\end{align*}
$$

for $v \leq v_{1}=1 / 4$.
Figure 4 emphasizes the comparison between the curve of nonhyperbolic periodic orbits $C_{\mathrm{NH}}$ of (155) obtained


Figure 5: Comparison between the curve of nonhyperbolic periodic orbits $C_{\mathrm{NH}}$ of (155): the dotted curve was obtained numerically with the software MATCONT and the continuous curve from the representation (165). The dot-dashed curve is the Hopf curve.
numerically with the software MATCONT and the quadratic approximation (164).

The comparison between the curve of nonhyperbolic periodic orbits $C_{\mathrm{NH}}$ of (155) obtained numerically with the software MATCONT and the approximation (165) is shown in Figure 5.

## 6. Concluding Comments

This paper shows how to obtain approximations of periodic orbits of a family of differential equations in the plane that has a transversal Hopf point. Moreover, if the family of differential equations has a transversal Hopf point of codimension two, then it is also possible to build an approximation to the curve of nonhyperbolic periodic orbits in the bifurcation diagram. These results are summarized in Corollary 11 and Theorem 13. Example 14, the study of the Liénard equation (139) in Section 4, and Bazykin's predator-prey system in Section 5 demonstrate the applicability of the theory. See also Figures 3, 4, and 5.

Although the theory is formulated for a family of differential equations in the plane, it can be applied to any family of differential equations in $\mathbb{R}^{n}$ that presents a transversal Hopf bifurcation of codimension two. For this, it is necessary to use the Center Manifold Theorem, or more precisely, to apply the proposed theory to the family of differential equations in $\mathbb{R}^{n}$ restricted to the center manifold.

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## Research Article

# Hopf Bifurcation Analysis for the Modified Rayleigh Price Model with Time Delay 

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#### Abstract

This paper mainly modifies and further develops the Reyleigh price model. By modifying the basic Reyleigh model, we can more accurately illustrate the economic phenomena with price varying. First, we research the dynamics of the modified Reyleigh model with time delay. By employing the normal form theory and center manifold theory, we obtain some testable results on these issues. The conclusion confirms that a Hopf bifurcation occurs due to the existence of stability switches when the delay varies. Finally, some numerical simulations are given to illustrate the effectiveness of our results.


## 1. Introduction

With the rapid development of economic society. As one of the many important economic problems, the price oscillation has been widely accepted by many people, especially some researchers. Since the price is a main factor of affecting supply and demand, many researchers have been devoted to study the price model. Different researchers may apply different price models to solve the practical problems. In this paper, our research is based on the traditional Reyleigh price model [1]. As for this model, many researchers have widely studied it by using different methods and proposed some new ideals. However, there is only a limited number of analytical works on the model with time delay. Although in [1] Lv and Liu have studied the Reyleigh model with time delay, they consider the situation that supply depends on the price of the past only. In order to finely interpret economic phenomena, our research is based on the fact that supply depends not only on the price of the past but also on the present price. So, the main purpose of this study is to provide an insight into these unexplored aspects of the Reyleigh price model with time delay.

The traditional Reyleigh price model is described by the following two-dimensional autonomous system:

$$
\begin{gather*}
\dot{x}(t)=-y(t)+l\left(\frac{1}{3} a x^{3}(t)+\frac{1}{2} b x^{2}(t)+c x(t)\right)  \tag{1}\\
\dot{y}(t)=x(t)
\end{gather*}
$$

where $x(t)$ is the price at time $t, y(t)$ is the amount of supply at time $t$, and $a, b, c, l$ are the constants.

By introducing the time delay, the above system (1) can be transformed into the following form:

$$
\begin{gather*}
\dot{x}(t)=-y(t)+l\left(\frac{1}{3} a x^{3}(t)+\frac{1}{2} b x^{2}(t)+c x(t)\right),  \tag{2}\\
\dot{y}(t)=x(t-\tau)
\end{gather*}
$$

where $\tau$ is positive and the other parameters are the same as (1).

In [1], by employing the $\tau-D$ partitioning approach, Lv and Liu have systematically discussed some complex dynamic behaviors of system (2).

Now, based on the economic meaning and the fact discussed at the beginning of the introduction, we modify system (2) as follows:

$$
\begin{gather*}
\dot{x}(t)=-y(t)+l\left(\frac{1}{3} a x^{3}(t)+\frac{1}{2} b x^{2}(t)+c x(t)\right), \\
\dot{y}(t)=\frac{1}{2} d x(t)+\frac{1}{2} k x(t-\tau) \tag{3}
\end{gather*}
$$

where $d, k$ refer to [2].

## 2. The Stability Analysis

In this section, we obtain the domain of the stable equilibrium when time delay $\tau$ varies from small to large. And further, by applying the Hopf bifurcation theorem, we give the condition of the Hopf bifurcation.

It is known that, on the one hand, if the equilibrium of system (3) is stable when $\tau=0$ and the characteristic equation of (3) has no purely imaginary roots for any $\tau>0$, it is also stable for any $\tau>0$. On the other hand, if the equilibrium of system (3) is stable when $\tau=0$ and there exist some positive values $\tau$ such that the characteristic equation of (3) has a pair of purely imaginary roots, there exists a domain concerning $\tau$ such that the equilibrium of system (3) is stable in the domain.

Obviously, system (3) has the only equilibrium ( 0,0 ). And the linearization of system (3) at $(0,0)$ is

$$
\begin{gather*}
\dot{x}(t)=-y(t)+l c x(t), \\
\dot{y}(t)=\frac{1}{2} d x(t)+\frac{1}{2} k x(t-\tau), \tag{4}
\end{gather*}
$$

whose characteristic equation is

$$
\begin{equation*}
\lambda^{2}-l c \lambda+\frac{1}{2} k e^{-\lambda \tau}+\frac{1}{2} d=0 \tag{5}
\end{equation*}
$$

When the case $\tau=0$, we have the following.
Lemma 1. (i) When $c<0$, the equilibrium $(0,0)$ of system (3) is stable.
(ii) When $c>0$, the equilibrium $(0,0)$ of system (3) is unstable.
(iii) When $c=0$ and $b<0(b>0)$, the equilibrium $(0,0)$ of system (3) is stable (unstable).

The proof is straightforward, and we omit it.
Lemma 2. If $d^{2}<k^{2}$ is satisfied, the characteristic equation (5) has a pair of purely imaginary roots $\pm i \omega_{0}$ when $\tau=\tau_{j}$, where

$$
\begin{align*}
& \omega_{0}=\left(\frac{-\left(l^{2} c^{2}-d\right)+\sqrt{\left(l^{2} c^{2}-d\right)^{2}-\left(d^{2}-k^{2}\right)}}{2}\right)^{1 / 2},  \tag{6}\\
& \tau_{j}=\frac{\arccos \left(\left(2 \omega_{0}-d\right) / k\right)+2 j \pi}{\omega_{0}}, \quad j=0,1,2, \ldots
\end{align*}
$$

Proof. Let $i \omega(\omega>0)$ is a root of (5). Then

$$
\begin{equation*}
-\omega^{2}-l c \omega i+\frac{1}{2} d+\frac{1}{2} k(\cos (\omega \tau)-i \sin (\omega \tau))=0 \tag{7}
\end{equation*}
$$

The separation of the real and imaginary parts yields

$$
\begin{gather*}
-\omega^{2}+\frac{1}{2} k \cos (\omega \tau)+\frac{1}{2} d=0 \\
-l c \omega-\frac{1}{2} k \sin (\omega \tau)=0 \tag{8}
\end{gather*}
$$

which lead to

$$
\begin{equation*}
\omega^{4}-d \omega^{2}+l^{2} c^{2} \omega^{2}+\frac{1}{4} d^{2}-\frac{1}{4} k^{2}=0 \tag{9}
\end{equation*}
$$

By solving the second-degree equation (9) concerning $\omega^{2}$, we have

$$
\begin{equation*}
\omega^{2}=\frac{-\left(l^{2} c^{2}-d\right) \pm \sqrt{\left(l^{2} c^{2}-d\right)^{2}-\left(d^{2}-k^{2}\right)}}{2} \tag{10}
\end{equation*}
$$

We take

$$
\begin{equation*}
\omega_{0}=\left(\frac{-\left(l^{2} c^{2}-d\right)+\sqrt{\left(l^{2} c^{2}-d\right)^{2}-\left(d^{2}-k^{2}\right)}}{2}\right)^{1 / 2} \tag{11}
\end{equation*}
$$

And, hence

$$
\begin{equation*}
\tau_{0}=\frac{\arccos \left(\left(2 \omega_{0}^{2}-d\right) / k\right)}{\omega_{0}} \tag{12}
\end{equation*}
$$

By setting $\tau_{j}=\tau_{0}+\left(2 j \pi / \omega_{0}\right), j=0,1,2, \ldots$. Then $\omega_{0}, \tau_{j}$ satisfies the condition of Lemma 2.

The proof is completed.
According to Lemmas 1 and 2 and the assertion at the beginning of this section, we know there must exist some finite interval with regard to $\tau$ in which the equilibrium $(0,0)$ is stable.

Now we investigate how the real part of the roots of (5) varies as $\tau$ varies in a small neighbourhood of $\tau_{j}$.

Assume that $\lambda(\tau)=a(\tau)+i b(\tau)$ is the root of the characteristic equation (5), and it meets $a\left(\tau_{j}\right)=0, b\left(\tau_{j}\right)=$ $\omega_{0}$. By differentiating both sides of (5) with regard to $\tau$ and solving $\lambda^{\prime}(\tau)$, we obtain

$$
\begin{equation*}
\lambda^{\prime}(\tau)=\frac{(1 / 2) k \lambda e^{-\lambda \tau}}{2 \lambda-l c-(1 / 2) k \tau e^{-\lambda \tau}} \tag{13}
\end{equation*}
$$

That is,

$$
\begin{equation*}
\lambda^{\prime}\left(\tau_{j}\right)=\frac{(1 / 2) k \omega_{0} i e^{-\omega_{0} i \tau_{j}}}{2 \omega_{0} i-l c-(1 / 2) k \tau_{j} e^{-\omega_{0} i \tau_{j}}} \tag{14}
\end{equation*}
$$

We substitute (8) into the above equation and separate the real and imaginary parts, and we have

$$
\begin{equation*}
\operatorname{Re}\left\{\lambda^{\prime}\left(\tau_{j}\right)\right\}=\frac{2 \omega_{0}^{4}+(l c-d) \omega_{0}^{2}}{\left((1 / 2) d \tau_{j}-l c-\omega_{0}^{2} \tau_{j}\right)^{2}+\left(2 \omega_{0}-l c \omega_{0} \tau_{j}\right)^{2}} \tag{15}
\end{equation*}
$$

When $l c>d$,

$$
\begin{equation*}
\operatorname{Re}\left\{\lambda^{\prime}\left(\tau_{j}\right)\right\}>0 \tag{*}
\end{equation*}
$$

Here, we know that the root of (5) crosses the imaginary axis from the left to the right as $\tau$ continuously varies from a number less than $\tau_{j}$ to one greater than $\tau_{j}$. Thus, when $\tau>\tau_{0}$,
the characteristic equation (5) has at least one root with positive real part. Further, the equilibrium $(0,0)$ is unstable in the interval $\left(\tau_{0},+\infty\right)$.

By applying Lemmas 1 and 2 and condition (*), we have the following results.

Theorem 3. If $c<0, k^{2}>d^{2}$ and $l c>d$ hold, the equilibrium $(0,0)$ of system (3) is asymptotically stable for $\tau \in\left[0, \tau_{0}\right)$ and unstable for $\tau \in\left(\tau_{0},+\infty\right)$. System (3) exhibits the Hopf bifurcation at the equilibrium $(0,0)$ for $\tau=\tau_{j}, j=0,1,2, \ldots$.

## 3. Hopf Bifurcation Analysis

In Section 2, we obtain the conditions under which family periodic solutions bifurcate from the steady state at the critical value of $\tau$. In this section, by applying the normal form and centre manifold theory, we discuss the direction and stability of the bifurcating periodic solutions. Throughout this section, we always assume that system (3) meets the conditions of the Hopf bifurcation.

By time scaling $t \rightarrow t / \tau$, system (3) is transformed into the following form:

$$
\begin{gather*}
\dot{x}(t)=-\tau y(t)+l \tau\left(\frac{1}{3} a x^{3}(t)+\frac{1}{2} b x^{2}(t)+c x(t)\right), \\
\dot{y}(t)=\frac{1}{2} \tau d x(t)+\frac{1}{2} \tau k x(t-1) . \tag{16}
\end{gather*}
$$

It is not difficult to show that system (16) also exhibits the Hopf bifurcation at the equilibrium $(0,0)$ for $\tau=\tau_{j}$ $(j=0,1,2, \ldots)$. Without loss of generality, we only consider the bifurcation parameter $\tau_{0}$. For convenience, by setting $\tau=$ $\tau_{0}+\mu$, system (16) is rewriten as

$$
\begin{align*}
& \dot{x}(t)=-\left(\tau_{0}+\mu\right) y(t)+l\left(\tau_{0}+\mu\right) \\
& \times\left(\frac{1}{3} a x^{3}(t)+\frac{1}{2} b x^{2}(t)+c x(t)\right),  \tag{17}\\
& \dot{y}(t)=\frac{1}{2}\left(\tau_{0}+\mu\right) d x(t)+\frac{1}{2}\left(\tau_{0}+\mu\right) k x(t-1) .
\end{align*}
$$

Clearly, $\mu=0$ is the Hopf bifurcation value of system (17).
Throughout the following section, $C_{1}=C\left([-1,0] ; R_{+}^{2}\right)$ is a phase space and the superscripts " $T$ " and "*" stand for the transpose and adjoint, respectively.

In $C_{1}$, system (17) can be written as the following equation:

$$
\begin{equation*}
\dot{X}(t)=\left(\tau_{0}+\mu\right) L\left(X_{t}\right)+\left(\tau_{0}+\mu\right) F\left(X_{t}\right) \tag{18}
\end{equation*}
$$

where $X(t)$ is a vector $\left(x_{1}(t), x_{2}(t)\right), X_{t}=X(t+\theta)$ for $\theta \epsilon$ $[-1,0]$, and $L, F$ are given by

$$
\begin{align*}
L(\phi) & =\binom{l c \phi_{1}(0)-\phi_{2}(0)}{\frac{1}{2} d \phi_{1}(0)+\frac{1}{2} k \phi_{1}(-1)},  \tag{19}\\
F(\phi) & =\binom{\frac{1}{3} a l \phi_{1}^{3}(0)+\frac{1}{2} b l \phi_{1}^{2}(0)}{0}
\end{align*}
$$

for $\phi=\left(\phi_{1}, \phi_{2}\right) \in C_{1}$.

Now, we consider the abstract functional differential equation [3]:

$$
\begin{equation*}
\dot{u}_{t}=A(\mu) u_{t}+R(\mu) u_{t} \tag{20}
\end{equation*}
$$

where $u_{t}=u(t+\theta) \in C_{1}$ for $\theta \in[-1,0]$.
The operators $A$ and $R$ are defined as

$$
\begin{gather*}
A(\mu) \phi(\theta)= \begin{cases}\frac{d \phi(\theta)}{d \theta}, & \theta \in[-1,0) \\
\int_{-1}^{0} d(\eta(t, \mu) \phi(t)), & \theta=0\end{cases}  \tag{21}\\
R(\mu) \phi(\theta)= \begin{cases}0, & \theta \in[-1,0) \\
f(\mu, \theta), & \theta=0\end{cases}
\end{gather*}
$$

where $\int_{-1}^{0} d(\eta(t, \mu) \phi(t))=\left(\tau_{0}+\mu\right) L(\phi)$ (here, $\eta(\theta, \mu)$ is a bounded variation function for $\theta \in[-1,0]), f(\mu, \phi)=\left(\tau_{0}+\right.$ $\mu) F(\phi)$.

Consider the adjoint bilinear form $\langle\cdot, \cdot\rangle$ on $C_{1} \times C_{1}^{*}$ :

$$
\begin{equation*}
\langle\psi, \phi\rangle=\bar{\psi}(0) \phi(0)-\int_{-1}^{0} \int_{0}^{\theta} \bar{\psi}(\xi-\theta) d \eta(\theta) \phi(\xi) d \xi \tag{22}
\end{equation*}
$$

where $\eta(\theta)=\eta(\theta, 0)$.
According to the adjoint bilinear form $\langle\cdot, \cdot\rangle$, we can define an adjoint operator $A^{*}(0)$ corresponding to $A(0)$ as the following form:

$$
A^{*} \psi(s)= \begin{cases}-\frac{d \psi(s)}{d s}, & s \in(0,1]  \tag{23}\\ \int_{-1}^{0} d\left(\eta^{T}(t, 0) \psi(-t)\right), & s=0\end{cases}
$$

To determine the normal form of operator $A$, we need to calculate the eigenvectors $q(\theta)$ and $q^{*}(s)$ of $A$ and $A^{*}$ corresponding to $i \tau_{0} \omega_{0}$ and $-i \tau_{0} \omega_{0}$, respectively.

Proposition 4. Assume that $q(\theta)$ and $q^{*}(s)$ are the eigenvector $q(\theta)$ and $q^{*}(s)$ of $A$ and $A^{*}$ corresponding to $i \tau_{0} \omega_{0}$ and $-i \tau_{0} \omega_{0}$, respectively, satisfying $\left\langle q^{*}, q\right\rangle=1$ and $\left\langle q^{*}, \bar{q}\right\rangle=0$.

Then

$$
\begin{align*}
& q(\theta)=(\alpha, \beta)^{T} e^{i \omega_{0} \tau_{0} \theta}=\left(1, l c-\omega_{0} i\right)^{T} e^{i \omega_{0} \tau_{0} \theta},  \tag{24}\\
& q^{*}(s)=D\left(\alpha^{*}, \beta^{*}\right) e^{i \omega_{0} \tau_{0} s}=\left(\omega_{0} i, 1\right) e^{i \omega_{0} \tau_{0} s}
\end{align*}
$$

where $\bar{D}=2 /\left(2 \alpha \alpha^{*}+2 \beta \beta^{*}+k \tau_{0} \alpha \overline{\beta^{*}} e^{-i \omega_{0} \tau_{0}}\right)$.
Proof. Without loss of generality, we just consider the eigenvector $q(\theta)$.

Firstly, when $\theta \in[-1,0)$, by the definition of $A$ and $q(\theta)$, we obtain the form $q(\theta)=(\alpha, \beta)^{T} e^{i \omega_{0} \tau_{0}}$ (here, $\alpha, \beta$ are unknown parameters).

In what follows, notice that $q(0)=(\alpha, \beta)^{T}$ and $A q(0)=$ $\int_{-1}^{0} d(\eta(t, \mu) \phi(t))=i \omega_{0} \tau_{0} q(0)$, and we have $\alpha=1, \beta=l c-\omega_{0} i$.

Finally, by $\left\langle q^{*}, q\right\rangle=1$, we obtain the parameter $\bar{D}$ (refer to $[4,5]$ ).

The proof is completed.

As in [6], the bifurcating periodic solutions $x(t, \mu)$ of system (16) are indexed by a small parameter $\varepsilon$. A solution $x(t, \mu(\varepsilon))$ has amplitude $O(\varepsilon)$, period $T(\varepsilon)$, and nonzero Floquet exponent $\beta(\varepsilon)$ with $\beta(0)=0$. Under the present assumptions, $\mu, T$, and $\beta$ have expansions [7, 8]:

$$
\begin{gather*}
\mu=\mu_{2} \varepsilon^{2}+\mu_{4} \varepsilon^{4}+\cdots, \\
T=\frac{2 \pi}{\omega}\left(1+T_{2} \varepsilon^{2}+T_{4} \varepsilon^{4}+\cdots\right),  \tag{25}\\
\beta=\beta_{2} \varepsilon^{2}+\beta_{4} \varepsilon^{4}+\cdots,
\end{gather*}
$$

where the sign of $\mu_{2}$ determines the directions of the Hopf bifurcations, the sign of $\beta_{2}$ determines the stability of the bifurcation periodic solutions, and $T_{2}$ determines the period of the bifurcating periodic solutions.

The purpose of this section is to compute the coefficients $\mu_{2}, T_{2}, \beta_{2}$ in the above expansions.

Next, we construct the coordinates of the center manifold $C_{0}$ at $\mu=0$. Let

$$
\begin{gather*}
z(t)=\left\langle q^{*}, u_{t}\right\rangle \\
W(t, \theta)=u_{t}(\theta)-2 \operatorname{Re}\{z(t) q(\theta)\} \tag{26}
\end{gather*}
$$

On the center manifold $C_{0}$, we have

$$
\begin{equation*}
W(t, \theta)=W(z(t), \overline{z(t)}, \theta) \tag{27}
\end{equation*}
$$

where

$$
\begin{equation*}
W(z, \bar{z}, \theta)=W_{20}(\theta) \frac{z^{2}}{2}+W_{11}(\theta) z \bar{z}+W_{02} \frac{\bar{z}^{2}}{2}+W_{30} \frac{z^{3}}{6} \cdots \tag{28}
\end{equation*}
$$

$z$ and $\bar{z}$ are local coordinates for the center manifold $C_{0}$ in the direction of $q$ and $q^{*}$, respectively. Since $\mu=0$, we have

$$
\begin{align*}
z^{\prime}(t) & =i \tau_{0} \omega_{0} z(t)+\left\langle q^{*}(\theta), f(W+2 \operatorname{Re}\{z(t) q(\theta)\})\right\rangle \\
& =i \tau_{0} \omega_{0} z(t)+\overline{q^{*}(0)} f(W(z, \bar{z}, 0)+2 \operatorname{Re}\{z(t) q(0)\}) \\
& \triangleq i \tau_{0} \omega_{0} z(t)+\overline{q^{*}(0)} f_{0}(z, \bar{z}), \tag{29}
\end{align*}
$$

where

$$
\begin{equation*}
f_{0}(z, \bar{z})=f_{z^{2}} \frac{z^{2}}{2}+f_{\bar{z}^{2}} \frac{\bar{z}^{2}}{2}+f_{z \bar{z}} z \bar{z}+\cdots \tag{30}
\end{equation*}
$$

We rewrite this as

$$
\begin{equation*}
z^{\prime}(t)=i \tau_{0} \omega_{0} z+g(z, \bar{z}) \tag{31}
\end{equation*}
$$

with

$$
\begin{align*}
g(z, \bar{z}) & =\overline{q^{*}}(0) f_{0}(z, \bar{z}) \\
& =g_{20} \frac{z^{2}}{2}+g_{11} z \bar{z}+g_{02} \frac{\bar{z}^{2}}{2}+g_{21} \frac{z^{2} \bar{z}}{2}+\cdots \tag{32}
\end{align*}
$$

Proposition 5. For (31), one has

$$
\begin{align*}
& \text { (i) } g_{20}=g_{11}=g_{02}=\bar{D} b l \tau_{0} \overline{\alpha^{*}} \alpha^{2} \\
& \text { (ii) } g_{21}=\bar{D} \tau_{0} \alpha^{*}\left(2 a l+2 b l W_{11}^{(1)}(0)+b l W_{20}^{(1)}(0)\right) \tag{33}
\end{align*}
$$

where $W_{11}^{(1)}(0)=\left(-i g_{11} / \omega_{0} \tau_{0}\right) \alpha+\left(i \bar{g}_{11} / \omega_{0} \tau_{0}\right) \bar{\alpha}+E_{2}^{(1)}$, $W_{20}^{(1)}(0)=\left(i g_{20} / \omega_{0} \tau_{0}\right) \alpha+\left(i \overline{g_{02}} / 3 \omega_{0} \tau_{0}\right) \bar{\alpha}+E_{1}^{(1)}, E_{1}^{(1)}=$ $4 l c \alpha^{2} \omega_{0} \tau_{0} i /\left(d+k e^{-i \omega_{0} \tau_{0}}-4 \omega_{0}^{2} \tau_{0}^{2}-2 l c \omega_{0} \tau_{0}\right), E_{2}^{(1)}=2 \alpha \bar{\alpha}$.

Proof. (i) Noticing $x_{t}(\theta)=\left(x_{1 t}(\theta), x_{2 t}(\theta)\right)=W(t, \theta)+z q(\theta)+$ $\bar{z} \overline{q(\theta)}$ and $q(\theta)=(\alpha, \beta)^{T} e^{i \omega_{0} \tau_{0} \theta}$, we have

$$
\begin{align*}
x_{1 t}(0)= & z+\bar{z}+W_{20}^{(1)}(0) \frac{z^{2}}{2}  \tag{34}\\
& +W_{11}^{(1)}(0) z \bar{z}+W_{02}^{(1)}(0) \frac{\bar{z}^{2}}{2}+\cdots .
\end{align*}
$$

By (32), we obtain

$$
\begin{align*}
g(z, \bar{z}) & =\bar{q}^{*}(0) f_{0}(z, \bar{z}) \\
& =\bar{D} \tau_{0}\left(\overline{\alpha^{*}}, \overline{\beta^{*}}\right)\left(\frac{1}{3} a l x_{1 t}^{3}(0)+\frac{1}{2} b l x_{1 t}^{2}(0), 0\right)^{T} . \tag{35}
\end{align*}
$$

Substituting (34) into the above equation and comparing the coefficients with (32), we obtain the results.
(ii) The detail procedure of proof refers to [5] and [9-11].

This completes the proof.
According to Proposition 5, we can compute the following parameters:

$$
\begin{gather*}
C_{1}(0)=\frac{i}{2 \tau_{0} \omega_{0}}\left(g_{20} g_{11}-2\left|g_{11}\right|^{2}-\frac{1}{3}\left|g_{02}\right|^{2}\right)+\frac{g_{21}}{2}, \\
\mu_{2}=-\frac{\operatorname{Re}\left\{C_{1}(0)\right\}}{\operatorname{Re}\left\{\lambda^{\prime}\left(\tau_{0}\right)\right\}}  \tag{36}\\
\beta_{2}=2 \operatorname{Re}\left\{C_{1}(0)\right\} \\
T_{2}=-\frac{\operatorname{Im}\left\{C_{1}(0)\right\}+\mu_{2}\left(\operatorname{Im}\left\{\lambda^{\prime}\left(\tau_{0}\right)\right\}\right)}{\omega_{0}}
\end{gather*}
$$

From the discussion in Section 2 we know that $\operatorname{Re}\left\{\lambda^{\prime}\left(\tau_{0}\right)\right\}>0$. We therefore have the following result.

Theorem 6. If $\operatorname{Re}\left\{C_{1}(0)\right\}<0(>0)$, the direction of the Hopf bifurcation of the system (1) at the equilibrium $(0,0)$ when $\tau=\tau_{0}$ is supercritical (subcritical) and the bifurcating periodic solutions are orbitally asymptotically stable (unstable).

Finally, we give a concrete example to illustrate the dynamics behaviour of the Raleigh model.

We take the coefficients $d=-7, l=2, c=-2, k=9$ in (3). Omitting these complicated expressions, we obtain the numerical results directly by means of the software MatLab: $\omega_{0} \doteq 0.5871, \tau_{0} \doteq 0.935, \operatorname{Re}\left\{\lambda^{\prime}\left(\tau_{0}\right)\right\} \doteq 2.2695 \times 10^{-5}$. So, we directly compete $C_{1}(0) \doteq-0.0431+0.0381 i, \mu_{2} \doteq 12.4372$, $\beta_{2} \doteq-0.862$.


Figure 1: The equilibrium $(0,0)$ of system (3) is stable with $\tau=0.8<$ $\tau_{0}=0.935$.


Figure 2: The equilibrium $(0,0)$ of system (3) is unstable with $\tau=$ $0.97>\tau_{0}=0.935$.

According to Theorem 6, $\operatorname{Re}\left\{C_{1}(0)\right\} \doteq-0.0431<0$. That is, the bifurcating periodic solutions of system (3) with the above coefficients are supercritical and orbitally asymptotically stable at $\tau=\tau_{0}$.

Thus, the conclusion confirms the effectiveness of our research results.

## 4. Conclusion

Firstly, under the condition of $\tau=0$, we discuss the Reyleigh price model. We know that the stability of price varies with the parameters changing. When $c<0, k+d>0$ and $c=0$, $k+d>0, b<0$, the price tends to the stability. The other cases are unstable. morever, we discuss the Reyleigh price model with time delay (3). By adjusting the parameters $d$,


Figure 3: When $\tau_{0}=0.935$, the periodic solutions occur from the equilibrium $(0,0)$.
$k$, we more easily control the price such that the price tends to our expected results. For example, when we take $a=0$, $b=0, d=-7, l=2, c=-2, k=9, \tau=0.8$ in the system (3), the equilibrium $(0,0)$ of system (3) is stable (see Figure 1). By contrast, when we take $a=0, b=0, d=-7$, $l=2, c=-2, k=9, \tau=0.97$ in the system (3), the $(0,0)$ is unstable and there occurs a periodic solution around (0.0) (see Figures 2 and 3). So, by shortening the time delay between the supply and the demand, we can keep the price stable. On the contrary, the price is unstable and undergoes a periodic oscillation. However our analysis indicates that the dynamics of the Reyleigh price model with time delay can be much more complicated than we may have expected. It is still interesting and inspiring to research the price.

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# Maximum Principle in the Optimal Control Problems for Systems with Integral Boundary Conditions and Its Extension 

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#### Abstract

The optimal control problem with integral boundary condition is considered. The sufficient condition is established for existence and uniqueness of the solution for a class of integral boundary value problems for fixed admissible controls. First-order necessary condition for optimality is obtained in the traditional form of the maximum principle. The second-order variations of the functional are calculated. Using the variations of the controls, various optimality conditions of second order are obtained.


## 1. Introduction

Boundary value problems with integral conditions constitute a very interesting and important class of boundary problems. They include two-, three-, and multipoint and nonlocal boundary value problems as special cases, (see [1-3]). The theory of boundary value problems with integral boundary conditions for ordinary differential equations arises in different areas of applied mathematics and physics. For example, heat conduction, thermoelasticity, chemical engineering, plasma physics, and underground water flow can be reduced to the nonlocal problems with integral boundary conditions. For boundary value problems with nonlocal boundary conditions and comments on their importance, we refer the reader to the papers $[4,5]$ and the references therein.

The role of the Pontryagin maximum principle is critical for any research related to optimal processes that have control constraints. The simplicity of the principle's formulation together with its meaningful and beneficial directness has become an extraordinary attraction and one of the major causes for the appearance of new tendencies in mathematical sciences. The maximum principle is by nature a necessary
first-order optimality condition since it was born as an extension of Euler-Lagrange and Weierstrass necessary conditions of variational calculus.

At present, there exists a great amount of works devoted to derivation of necessary optimality conditions of first and second orders for the systems with local conditions (see [612] and the references therein).

Since the systems with nonlocal conditions describe real processes, it is necessary to study the optimal control problems with nonlocal boundary conditions.

The optimal control problems with nonlocal boundary conditions have been investigated in [13-25]. Note that optimal control problems with integral boundary condition are considered and first-order necessary conditions are obtained in [23-25]. In certain cases, the first-order optimality conditions are "degenerated," and are fulfilled trivially on a series of admissible controls. In this case, it is necessary to obtain second-order optimality conditions.

In the present paper, we investigate an optimal control problem in which the state of the system is described by differential equations with integral boundary conditions. Note that this problem is a natural generalization of the

Cauchy problem. The matters of existence and uniqueness of solutions of the boundary value problem are investigated, first and second increments formula of the functional are calculated. Using the variations of the controls, various optimality conditions of first and second order are obtained.

The organization of the present paper is as follows. First, we give the statement of the problem. Second, theorems on existence and uniqueness of a solution for the problem (1)(3) are established under some sufficient conditions on nonlinear terms. Third, the functional increment formula of first order is presented, and Pontryagin's maximum principle is provided. Fourth, variations of the functional of the first and second-order are given. Fifth, Legendre-Clebsh condition is obtained. Finally, a conclusion is given.

Consider the following system of differential equations with integral boundary condition:

$$
\begin{gather*}
\frac{d x}{d t}=f(t, x, u(t)), \quad 0 \leq t \leq T  \tag{1}\\
x(0)+\int_{0}^{T} m(t) x(t) d t=C  \tag{2}\\
u(t) \in U, \quad t \in[0, T] \tag{3}
\end{gather*}
$$

where $x(t) \in R^{n} ; f(t, x, u)$ is $n$-dimensional continuous function and has second-order derivative with respect to $(x, u) ; C \in R^{n}$ is the given constant vector; $m(t) \in R^{n \times n}$ is $n \times n$ matrix function; $u$ is a control parameter; and $U \subset R^{r}$ is an bounded set.

It is required to minimize the functional

$$
\begin{equation*}
J(u)=\varphi(x(0), x(T))+\int_{0}^{T} F(t, x, u) d t \tag{4}
\end{equation*}
$$

subject to (1)-(3).
Here, it is assumed that the scalar functions $\varphi(x, y)$ and $F(t, x, u)$ are continuous by their own arguments and have continuous and bounded partial derivatives with respect to $x, y$, and $u$ to second order, inclusively. Under the solution of boundary value problem (1)-(3) corresponding to the fixed control parameter $u(t)$, we understand the function $x(t):[0, T] \rightarrow R^{n}$ that is absolutely continuous on $[0, T]$. Denote the space of such functions by $A C\left([0, T], R^{n}\right)$. By $C\left([0, T], R^{n}\right)$, we define the space of continuous functions on $[0, T]$ with values from $R^{n}$. It is obvious that this is a Banach space with the norm

$$
\begin{equation*}
\|x\|_{C[0, T]}=\max _{t \in[0, T]}|x(t)|, \tag{5}
\end{equation*}
$$

where $|\cdot|$ is the norm in space $R^{n}$.
Admissible controls are taken from the class of bounded measurable functions with values from the set $U \in R^{r}$. The admissible control together with corresponding solutions of (1), (2) is called an admissible process.

The admissible process $\{u(t), x(t, u)\}$ being the solution of problem (1)-(4), that is, delivering minimum to functional (4) under restrictions (1)-(3) is said to be an optimal process, and is $u(t)$ optimal control.

We suppose the existence of the optimal control in the problem (1)-(4).

## 2. Existence of Solutions of Boundary Value Problem (1)-(3)

Introduce the following conditions:
(1) Let $\|B\|<1$, where $B=\int_{0}^{T} m(t) d t$,
(2) $f:[0, T] \times R^{n} \times R^{r} \rightarrow R^{n}$ is a continuous function, and there exists the constant $K \geq 0$

$$
\begin{array}{r}
|f(t, x, u)-f(t, y, u)| \leq K|x-y| \\
t \in[0, T], x, y \in R^{n}, u \in U \tag{6}
\end{array}
$$

(3) $L=(1-\|B\|)^{-1} K T N<1$, where

$$
N=\max _{0 \leq t, s \leq T}\|N(t, s)\|
$$

$$
N(t, s)= \begin{cases}E+\int_{0}^{s} m(\tau) d \tau, & 0 \leq t \leq s  \tag{7}\\ -\int_{s}^{T} m(\tau) d \tau, & s \leq t \leq T\end{cases}
$$

$E \subset R^{n \times n}$-unit matrix.
Theorem 1. Let condition (1) be satisfied. Then, the function $x(\cdot) \in C\left([0, T], R^{n}\right)$ is an absolutely continuous solution of boundary value problem (1)-(3) if and only if

$$
\begin{equation*}
x(t)=(E+B)^{-1} C+\int_{0}^{T} K(t, \tau) f(\tau, x(\tau), u(\tau)) d \tau \tag{8}
\end{equation*}
$$

where $K(t, \tau)=(E+B)^{-1} N(t, \tau)$.
Proof. Note that under condition (1), the matrix $E+B$ is invertible and the estimation $\left\|(E+B)^{-1}\right\|<(1-\|B\|)^{-1}$ holds [26, page 78]. If $x=x(\cdot)$ is a solution of differential equation (1), then for $t \in(0, T)$

$$
\begin{equation*}
x(t)=x(0)+\int_{0}^{t} f(s, x(s), u(s)) d s \tag{9}
\end{equation*}
$$

where $x(0)$ is still an arbitrary constant. For determining $x(0)$, we require that the function defined by equality (9) satisfies condition (2):

$$
\begin{equation*}
(E+B) x(0)=C-\int_{0}^{T} m(t) \int_{0}^{t} f(\tau, x(\tau), u(\tau)) d \tau d t \tag{10}
\end{equation*}
$$

Since $\operatorname{det}(E+B) \neq 0$, then

$$
\begin{align*}
x(0)= & (E+B)^{-1} C \\
& -(E+B)^{-1} \int_{0}^{T} m(t) \int_{0}^{t} f(\tau, x(\tau), u(\tau)) d \tau d t \tag{11}
\end{align*}
$$

The equality (11) may be written in the following equivalent from

$$
\begin{align*}
x(0)= & (E+B)^{-1} C \\
& -(E+B)^{-1} \int_{0}^{T} \int_{t}^{T} m(\tau) d \tau f(t, x(t), u(t)) d t \tag{12}
\end{align*}
$$

Now, considering the value of $x(0)$ defined by (12) in (9), we get

$$
\begin{align*}
x(t)= & (E+B)^{-1} C \\
& -(E+B)^{-1} \int_{0}^{T} \int_{t}^{T} m(\tau) d \tau f(t, x(t), u(t)) d t  \tag{13}\\
& +\int_{0}^{t} f(\tau, x(\tau), u(\tau)) d \tau .
\end{align*}
$$

It is obvious one can write the last equality as

$$
\begin{align*}
x(t)= & (E+B)^{-1} C \\
+ & \int_{0}^{t}\left(E-(E+B)^{-1} \int_{s}^{T} m(\tau) d \tau\right)  \tag{14}\\
& \times f(s, x(s) u(s)) d s \\
& -(E+B)^{-1} \int_{t}^{T} \int_{s}^{T} m(\tau) d \tau f(s, x(s), u(s)) d s
\end{align*}
$$

Since $B=\int_{0}^{T} m(t) d t$,

$$
\begin{align*}
E & -(E+B)^{-1} \int_{s}^{T} m(\tau) d \tau \\
& =(E+B)^{-1}\left(E+\int_{0}^{T} m(t) d t-\int_{s}^{T} m(\tau) d \tau\right)  \tag{15}\\
& =(E+B)^{-1}\left(E+\int_{0}^{s} m(\tau) d \tau\right)
\end{align*}
$$

Introduce the matrix function

$$
K(t, \tau)= \begin{cases}(E+B)^{-1}\left(E+\int_{0}^{s} m(\tau) d \tau\right), & 0 \leq s \leq t  \tag{16}\\ -(E+B)^{-1} \int_{s}^{T} m(\tau) d \tau, & t<s \leq T\end{cases}
$$

Then, (14) turns to

$$
\begin{equation*}
x(t)=(E+B)^{-1} C+\int_{0}^{T} K(t, \tau) f(\tau, x(\tau), u(\tau)) d \tau \tag{17}
\end{equation*}
$$

Thus, we show that the boundary value problem (1)(3) may be written in the form of integral equation (8). By direct verification, we can show that the solution of integral equation (8) also satisfies to the boundary value problem (1)(3). Theorem 1 is proved.

For every fixed admissible controls, define the operator $P: C\left([0, T], R^{n}\right) \rightarrow C\left([0, T], R^{n}\right)$ by the rule

$$
\begin{equation*}
(P x)(t)=(E+B)^{-1} C+\int_{0}^{T} K(t, \tau) f(\tau, x(\tau), u(\tau)) d \tau \tag{18}
\end{equation*}
$$

Theorem 2. Let conditions (1)-(3) be fulfilled. Then, for any $C \in R^{n}$ and for each fixed admissible control, boundary
value problem (1)-(3) has the unique solution that satisfies the following integral equation:

$$
\begin{equation*}
x(t)=(E+B)^{-1} C+\int_{0}^{T} K(t, \tau) f(\tau, x(\tau), u(\tau)) d \tau \tag{19}
\end{equation*}
$$

Proof. Let $C \in R^{n}$, and let $u(t) \in U, t \in[0, T]$ be fixed. Consider the mapping $P: C\left([0, T], R^{n}\right) \rightarrow C\left([0, T], R^{n}\right)$ defined by equality (18). Clearly, the fixed points of the operator are solutions of the problem (1)-(2). We will use the Banach contraction principle to prove that $P$ defined by (18) has a fixed point. Then, for any $v, w \in C\left([0, T], R^{n}\right)$, we have

$$
\begin{align*}
& |(P v)(t)-(P w)(t)| \\
& \quad \leq \int_{0}^{T}|K(t, s)| \cdot|f(s, v(s), u(s))-f(s, w(s), u(s))| d s \\
& \quad \leq(1-\|B\|)^{-1} K T N\|v(\cdot)-w(\cdot)\|_{C[0, T]}, \quad t \in[0, T] \tag{20}
\end{align*}
$$

or

$$
\begin{equation*}
\|P v-P w\|_{C[0, T]} \leq L\|v-w\|_{C[0, T]} \tag{21}
\end{equation*}
$$

Estimation (21) shows that the operator $P$ is a contraction in the space $C\left([0, T], R^{n}\right)$. Therefore, according to the principle of contraction operators, the operator $P$ defined by equality (18) has a unique fixed point at $C\left([0, T], R^{n}\right)$. So, integral equation (19) or boundary value problem (1)-(3) has a unique solution. Theorem 2 is proved.

## 3. First-Order Optimality Condition

In this section, we assume that $U$ is closed set in $R^{r}$. In order to obtain the necessary conditions for optimality, we will use the standard procedure (see, e.g., [7]). Namely, we should analyze the changing of the objective functional caused by some control impulse. In other words, we must derive the increment formula that originates from Taylor's series expansion. A suitable definition of the conjugate system will facilitate the extraction of the dominant term that is destined to determine the necessary condition for optimality. For the sake of simplicity, it will be reasonable to construct a linearized model of nonlinear system (8), (9) in some small vicinity.
3.1. Increment Formula. Let $\{u, x=x(t, u)\}$ and $\{\tilde{u}=u+$ $\Delta u, \tilde{x}=x+\Delta x=x(t, \widetilde{u})\}$ be two admissible processes. We can determine the boundary value problem for problem (1)-

$$
\begin{align*}
& \Delta \dot{x}=\Delta f(t, x, u), \quad t \in[0, T] \\
& \Delta x(0)+\int_{0}^{T} m(t) \Delta x(t) d t=0 \tag{22}
\end{align*}
$$

where $\Delta f(t, x, u)=f(t, \widetilde{x}, \widetilde{u})-f(t, x, u)$ denotes the total increment of the function $f(t, x, u)$. Then, we can represent the increment of the functional in the form

$$
\begin{align*}
\Delta J(u) & =J(\tilde{u})-J(u) \\
& =\Delta \varphi(x(0), x(T))+\int_{0}^{T} \Delta F(x, u, t) d t \tag{23}
\end{align*}
$$

Let us introduce some nontrivial vector function $\psi(t) \in$ $R^{n}$ and numerical vector $\lambda \in R^{n}$. Then, the increment of functional index (4) may be represented as

$$
\begin{align*}
\Delta J & (u) \\
& =J(\widetilde{u})-J(u) \\
= & \Delta \varphi(x(0), x(T))+\int_{0}^{T} \Delta F(x, u, t) d t  \tag{24}\\
& +\int_{0}^{T}\langle\psi(t), \Delta \dot{x}(t)-\Delta f(t, x, u)\rangle d t \\
& +\left\langle\lambda, \Delta x(0)+\int_{0}^{T} m(t) \Delta x(t) d t\right\rangle .
\end{align*}
$$

After some operations usually used in deriving of the firstorder optimality conditions, for the increment of the functional, we get the formulas

$$
\begin{align*}
\Delta J(u)= & -\int_{0}^{T} \Delta_{\tilde{u}} H(t, \psi, x, u) d t \\
& -\int_{0}^{T}\left\langle\Delta_{\tilde{\mathcal{u}}} \frac{\partial H(t, \psi, x, u)}{\partial x}, \Delta x(t)\right\rangle d t \\
& +\int_{0}^{T}\left\langle\dot{\psi}(t)+\frac{\partial H(t, \psi, x, u)}{\partial x}+m^{\prime}(t) \lambda\right\rangle d t \\
& +\left\langle\frac{\partial \varphi}{\partial x(0)}-\psi(0)+\lambda, \Delta x(0)\right\rangle \\
& +\left\langle\frac{\partial \varphi}{\partial x(T)}+\psi(T), \Delta x(T)\right\rangle+\eta_{\tilde{u}}, \\
\eta_{\tilde{u}}= & o_{\varphi}(\|\Delta x(0)\|,\|\Delta x(T)\|)-\int_{0}^{T} o_{H}(\|x(t)\|) d t, \tag{25}
\end{align*}
$$

where

$$
\begin{equation*}
H(t, \psi, x, u)=\langle\psi, f(t, x, u)\rangle-F(t, x, u) \tag{26}
\end{equation*}
$$

Suppose that the vector function $\psi(t) \in R^{n}$ and vector $\lambda \in R^{n}$ is a solution of the following conjugate problem (the stationary condition of the Lagrangian function by state):

$$
\begin{gather*}
\dot{\psi}(t)=-\frac{\partial H(t, \psi, x, u)}{\partial x}-m^{\prime}(t) \lambda, \quad t \in[0, T]  \tag{27}\\
\frac{\partial \varphi}{\partial x(0)}-\psi(0)+\lambda=0, \quad \frac{\partial \varphi}{\partial x(T)}+\psi(T)=0
\end{gather*}
$$

Then, increment formula (25) takes the form

$$
\begin{align*}
\Delta J(u)= & -\int_{0}^{T} \Delta_{\tilde{u}} H(t, \psi, x, u) d t \\
& -\int_{0}^{T}\left\langle\Delta_{\tilde{u}} \frac{\partial H(t, \psi, x, u)}{\partial x}, \Delta x(t)\right\rangle d t+\eta_{\tilde{u}} . \tag{28}
\end{align*}
$$

3.2. The Maximum Principle. Let us consider the formula for the increment of the functional on the needle-shaped variation of the admissible control. As a parameters, we take the point $\tau \in(0, T]$, number $\varepsilon \in(0, \tau]$, and vector $v \in U$. The variation interval $(\tau-\varepsilon, \tau)$ belongs to $[0, T]$. Needle-shaped variation of the control $u=u(t)$ is given as follows:

$$
\begin{align*}
\widetilde{u}(t) & =u_{\varepsilon}(t) \\
& = \begin{cases}v \in U, t \in(\tau-\varepsilon, \tau] \subset[0, T], & \varepsilon>0, \\
u(t), & t \notin(\tau-\varepsilon, \tau] .\end{cases} \tag{29}
\end{align*}
$$

A traditional form of the necessary optimality condition will follow from the increment formula (28) if we show that on the needle-shaped variation $\widetilde{u}(t)=u_{\varepsilon}(t)$ the state increment $\Delta_{\varepsilon} x(t)$ has the order $\varepsilon$.

That follows from conditions (1)-(3) and equalities (19) and (22)

$$
\begin{align*}
\Delta x(t)= & \int_{0}^{T} K(t, \tau)[f(\tau, x+\Delta x, \tilde{u})-f(\tau, x, \tilde{u})] d \tau  \tag{30}\\
& +\int_{0}^{T} K(t, \tau) \Delta_{\tilde{u}} f(\tau, x, u) d \tau
\end{align*}
$$

From this, we obtain

$$
\begin{equation*}
\|\Delta x(t)\| \leq(1-L)^{-1}(1-\|B\|)^{-1} N \int_{0}^{T}\left\|\Delta_{\tilde{u}} f(t, x, u)\right\| d t \tag{31}
\end{equation*}
$$

which proves our hypothesis on response of the state increment caused by the needle-shaped variation given by (29)

$$
\begin{equation*}
\left\|\Delta_{\varepsilon} x(t)\right\| \leq \widetilde{L} \cdot \varepsilon, \quad t \in[0, T], \widetilde{L}=\text { const }>0 . \tag{32}
\end{equation*}
$$

This also implies that for $\widetilde{u}(t)=u_{\varepsilon}(t)$,

$$
\begin{align*}
\int_{\tau-\varepsilon}^{\tau} & \left\langle\Delta_{v} \frac{\partial H(t, \psi, x, u)}{\partial x}, \Delta_{\varepsilon} x(t)\right\rangle d t  \tag{33}\\
& +\eta_{u_{\varepsilon}}\left(\left\|\Delta_{\varepsilon} x(t)\right\|\right)=o(\varepsilon)
\end{align*}
$$

where

$$
\begin{equation*}
\Delta_{\varepsilon} x(t)=x\left(t, u_{\varepsilon}\right)-x(t, u) \sim \varepsilon \tag{34}
\end{equation*}
$$

Therefore, the changing of objective functional caused by the needle-shaped variation (29) can be represented according to (28) as

$$
\begin{align*}
\Delta_{\varepsilon} J(u) & =J\left(u_{\varepsilon}\right)-J(u) \\
& =-\Delta_{v} H(\tau, \psi, x, u) \cdot \varepsilon+o(\varepsilon), \quad v \in U, \tau \in[0, T] \tag{35}
\end{align*}
$$

It should be noted that in the last expression, we used the mean value theorem.

For the needle-shaped variation of optimal process $\left\{u^{0}, x^{0}=x\left(t, u^{0}\right)\right\}$, the increment formula (35) with regard to the estimate (32) implies the necessary optimality condition in the form of the maximum principle.

Theorem 3 (maximum principle). Suppose that the admissible process $\left\{u^{0}, x^{0}=x\left(t, u^{0}\right)\right\}$ is optimal for problem (1)-(4) and $\psi^{0}(t)$ is the solution to conjugate boundary value problem (27) calculated on the optimal process. Then, for all $\tau \in[0, T]$, the following inequality holds:

$$
\begin{equation*}
\Delta_{v} H\left(\tau, \psi^{0}, x^{0}, u^{0}\right) \leq 0, \quad \text { for every } v \in U \tag{36}
\end{equation*}
$$

Remark 4. If the function $f$ is linear with respect to $(x, u)$ and functions $\varphi, F$ are convex with respect to $x(0), x(T)$, and $x(t)$, respectively, then maximum principle (36) is both necessary and sufficient optimality condition. This fact follows from the increment formula

$$
\begin{align*}
\Delta J(u)= & -\int_{0}^{T} \Delta_{\tilde{u}} H(t, \psi, x, u) d t \\
& +o_{\varphi}(\|x(0)\|,\|x(T)\|)+\int_{0}^{T} o_{F}(\|x(t)\|) d t \tag{37}
\end{align*}
$$

where $o_{\varphi} \geq 0, o_{F} \geq 0$.

## 4. Variations of the Functional and Derivation of Legendre-Clebsh Conditions

Let the set $U \subset R^{r}$ be open. Since the functions $\varphi(x, y)$, $F(t, x, u)$, and $f(t, x, u)$ are continuous by their own arguments and have continuous and bounded partial derivatives with respect to $x, y$, and $u$ up to second order, inclusively, then increment formula (28) takes the form

$$
\begin{aligned}
\Delta J(u)= & -\int_{0}^{T}\left\langle\frac{\partial H(t, \psi, x, u)}{\partial u}, \Delta u(t)\right\rangle d t \\
& -\frac{1}{2} \int_{0}^{T}\left\langle\Delta u(t)^{\prime} \frac{\partial^{2} H(t, \psi, x, u)}{\partial u^{2}}, \Delta u(t)\right\rangle d t \\
- & \int_{0}^{T}\left\langle\Delta u(t)^{\prime} \frac{\partial H^{2}(t, \psi, x, u)}{\partial x \partial u}\right. \\
& \left.\quad+\frac{1}{2} \Delta x^{\prime}(t) \frac{\partial^{2} H(t, \psi, x, u)}{\partial x^{2}}, \Delta x(t)\right\rangle d t \\
+ & \frac{1}{2}\left\langle\Delta x(0)^{\prime} \frac{\partial^{2} \varphi}{\partial x(0)^{2}}\right. \\
& \left.+\Delta x(T)^{\prime} \frac{\partial^{2} \varphi}{\partial x(0) \partial x(T)}, \Delta x(0)\right\rangle
\end{aligned}
$$

$$
\begin{align*}
& +\frac{1}{2}\left\langle\Delta x(0)^{\prime} \frac{\partial^{2} \varphi}{\partial x(T) \partial x(0)}\right. \\
& \left.\quad+\Delta x(T)^{\prime} \frac{\partial^{2} \varphi}{\partial x(T)^{2}}, \Delta x(T)\right\rangle+\xi_{\tilde{u}} \tag{38}
\end{align*}
$$

where

$$
\begin{align*}
\xi_{\tilde{u}}= & -\int_{0}^{T} o_{H}\left(\|\Delta x(t)\|^{2}+\|\Delta u(t)\|^{2}\right) d t  \tag{39}\\
& +o_{\varphi}\left(\left\|\Delta x\left(t_{0}\right)\right\|^{2},\left\|\Delta x\left(t_{1}\right)\right\|^{2}\right) .
\end{align*}
$$

Let now $\Delta u(t)=\varepsilon \delta u(t)$, where $\varepsilon>0$ is a rather small number and $\delta u(t)$ is some piecewise continuous function. Then, the increment of the functional $\Delta J(u)=J(\widetilde{u})-J(u)$ for the fixed functions $u(t), \Delta u(t)$ is the function of the parameter $\varepsilon$. If the representation

$$
\begin{equation*}
\Delta J(u)=\varepsilon \delta J(u)+\frac{1}{2} \varepsilon^{2} \delta^{2} J(u)+o\left(\varepsilon^{2}\right) \tag{40}
\end{equation*}
$$

is valid, then $\delta J(u)$ is called the first, and $\delta^{2} J(u)$ is the second variation of the functional. Further, we get an explicit expression for the first and second variations. To achieve the object, we have to select in $\Delta x(t)$ the principal term with respect to $\varepsilon$.

Assume that

$$
\begin{equation*}
\Delta x(t)=\varepsilon \delta x(t)+o(\varepsilon, t) \tag{41}
\end{equation*}
$$

where $\delta x(t)$ is the variation of the trajectory. Such a representation exists, and for the function $\delta x(t)$, one can obtain an equation in variations. Indeed, by definition of $\Delta x(t)$, we have:

$$
\begin{equation*}
\Delta x(t)=\int_{0}^{T} K(t, \tau) \Delta f(\tau, x(\tau), u(\tau)) d \tau \tag{42}
\end{equation*}
$$

Applying the Taylor formula to the integrand expression, we get

$$
\begin{align*}
& \varepsilon \delta x(t)+o(\varepsilon, t) \\
& =\int_{0}^{T} K(t, \tau)\left\{\frac{\partial f(\tau, x, u)}{\partial x}[\varepsilon \delta x(\tau)+o(\varepsilon, \tau)]\right.  \tag{43}\\
& \\
& \left.\quad+\varepsilon \frac{\partial f(\tau, x, u)}{\partial u} \delta u+o_{1}(\varepsilon, \tau)\right\} d \tau .
\end{align*}
$$

Since this formula is true for any $\varepsilon$, then

$$
\begin{align*}
\delta x(t)= & \int_{0}^{T} K(t, \tau) \\
& \times\left\{\frac{\partial f(\tau, x, u)}{\partial x} \delta x(\tau)+\frac{\partial f(\tau, x, u)}{\partial u} \delta u(t)\right\} d \tau . \tag{44}
\end{align*}
$$

Equation (44) is said to be an equation in variations. Obviously, integral equation (44) is equivalent to the following nonlocal boundary value problem:

$$
\begin{gather*}
\delta \dot{x}(t)=\frac{\partial f(t, x, u)}{\partial x} \delta x(t)+\frac{\partial f(t, x, u)}{\partial u} \delta u(t),  \tag{45}\\
\delta x(0)+\int_{0}^{T} m(t) \delta x(t) d t=0 . \tag{46}
\end{gather*}
$$

By [6, page 527], any solution of differential equation (45) may be represented in the form

$$
\begin{equation*}
\delta x(t)=\Phi(t) \delta x(0)+\Phi(t) \int_{0}^{t} \Phi^{-1}(\tau) \frac{\partial f(\tau, x, u)}{\partial u} \delta u(\tau) d \tau \tag{47}
\end{equation*}
$$

where $\Phi(t)$ is a solution of the following differential equation:

$$
\begin{equation*}
\frac{d \Phi(t)}{d t}=\frac{\partial f(t, x, u)}{\partial x} \Phi(t), \quad \Phi(0)=E \tag{48}
\end{equation*}
$$

Assume that the solution of differential equation (45) determined by equality (47) satisfies boundary condition (46). Then, for the solutions of problems (45), (46), we get the following explicit formula:

$$
\begin{equation*}
\delta x(t)=\int_{0}^{T} G(t, \tau) \frac{\partial f(\tau, x, u)}{\partial u} \delta u(\tau) d \tau \tag{49}
\end{equation*}
$$

where

$$
\begin{align*}
& G(t, \tau) \\
& =\left\{\begin{array}{l}
\Phi(t)\left[E+B_{1}\right]^{-1}\left[E+\int_{0}^{s} m(\tau) \Phi(\tau) d \tau\right] \Phi^{-1}(\tau) \\
0 \leq \tau \leq t \\
-\Phi(t)\left[E+B_{1}\right]^{-1} \int_{s}^{T} m(\tau) \Phi(\tau) d \tau \Phi^{-1}(\tau) \\
t \leq \tau \leq T
\end{array}\right. \\
& B_{1}=\int_{0}^{T} m(t) \Phi(t) d t \tag{50}
\end{align*}
$$

Now, substituting (41) into (38), one may get

$$
\begin{aligned}
& \Delta J(u)=-\varepsilon \int_{0}^{T}\left\langle\frac{\partial H(t, \psi, x, u)}{\partial u}, \delta u(t)\right\rangle d t \\
&-\frac{\varepsilon^{2}}{2}\left\{\int_{0}^{T}[ \right. {\left[\left\langle x^{\prime}(t) \frac{\partial^{2} H(t, \psi, x, u)}{\partial x^{2}}, \delta x(t)\right\rangle\right.} \\
&+2\left\langle\delta u^{\prime}(t) \frac{\partial^{2} H(t, \psi, x, u)}{\partial x \partial u}, \delta x(t)\right\rangle \\
&\left.+\left\langle\delta u^{\prime}(t) \frac{\partial^{2} H(t, \psi, x, u)}{\partial u^{2}}, \delta u(t)\right\rangle\right] d t
\end{aligned}
$$

$$
\begin{align*}
& -\left\langle\delta x^{\prime}(0) \frac{\partial^{2} \varphi}{\partial x(0)^{2}}\right. \\
& \left.\quad+\Delta x^{\prime}(T) \frac{\partial^{2} \varphi}{\partial x(0) \partial x(T)}, \delta x(0)\right\rangle \\
& -\left\langle\delta x^{\prime}(0) \frac{\partial^{2} \varphi}{\partial x(T) \partial x(0)}\right. \\
& \left.\left.\quad+\delta x^{\prime}(T) \frac{\partial^{2} \varphi}{\partial x(T)^{2}}, \delta x(T)\right\rangle\right\}+o\left(\varepsilon^{2}\right) \tag{51}
\end{align*}
$$

Considering definition (40), we finally obtain

$$
\begin{align*}
& \delta J(u)=-\int_{0}^{T}\left\langle\frac{\partial H(t, \psi, x, u)}{\partial u}, \delta u(t)\right\rangle d t,  \tag{52}\\
& \delta^{2} J(u)=-\int_{0}^{T}\left[\left\langle\delta x^{\prime}(t) \frac{\partial^{2} H(t, \psi, x, u)}{\partial x^{2}}, \delta x(t)\right\rangle\right. \\
& +2\left\langle\delta u^{\prime}(t) \frac{\partial^{2} H(t, \psi, x, u)}{\partial x \partial u}, \delta x(t)\right\rangle \\
& \left.+\left\langle\delta u^{\prime}(t) \frac{\partial^{2} H(t, \psi, x, u)}{\partial u^{2}}, \delta u(t)\right\rangle\right] d t \\
& +\left\langle\delta x^{\prime}(0) \frac{\partial^{2} \varphi}{\partial x(0)^{2}}\right. \\
& \left.+\Delta x^{\prime}(T) \frac{\partial^{2} \varphi}{\partial x(0) \partial x(T)}, \delta x(0)\right\rangle \\
& +\left\langle\delta x^{\prime}(0) \frac{\partial^{2} \varphi}{\partial x(T) \partial x(0)}\right. \\
& \left.+\delta x^{\prime}(T) \frac{\partial^{2} \varphi}{\partial x(T)^{2}}, \delta x(T)\right\rangle . \tag{53}
\end{align*}
$$

It follows from (40) that the conditions

$$
\begin{equation*}
\delta J\left(u^{0}\right)=0, \quad \delta^{2} J\left(u^{0}\right) \geq 0 \tag{54}
\end{equation*}
$$

are fulfilled for the optimal control $u^{0}(t)$. From the first condition in (54), it follows that

$$
\begin{equation*}
\int_{0}^{T}\left\langle\frac{\partial H\left(t, \psi^{0}, x^{0}, u^{0}\right)}{\partial u}, \delta u(t)\right\rangle d t=0 \tag{55}
\end{equation*}
$$

Hence, we can prove that the following equality is fulfilled along the optimal control (see [11, p. 54]):

$$
\begin{equation*}
\frac{\partial H\left(t, \psi^{0}, x^{0}, u^{0}\right)}{\partial u}=0, \quad t \in[0, T] \tag{56}
\end{equation*}
$$

and it is called the Euler equation. From the second condition in (54), it follows that the following inequality is fulfilled along the optimal control:

$$
\begin{align*}
\delta^{2} J(u) & \\
=-\int_{0}^{T}[ & {\left[\left\langle\delta x^{\prime}(t) \frac{\partial^{2} H(t, \psi, x, u)}{\partial x^{2}}, \delta x(t)\right\rangle\right.} \\
& +2\left\langle\delta u^{\prime}(t) \frac{\partial^{2} H(t, \psi, x, u)}{\partial x \partial u}, \delta x(t)\right\rangle \\
& \left.+\left\langle\delta u^{\prime}(t) \frac{\partial^{2} H(t, \psi, x, u)}{\partial u^{2}}, \delta u(t)\right\rangle\right] d t \\
+ & \left\langle\delta x^{\prime}(0) \frac{\partial^{2} \varphi}{\partial x(0)^{2}}+\Delta x^{\prime}(T) \frac{\partial^{2} \varphi}{\partial x(0) \partial x(T)}, \delta x(0)\right\rangle \\
& +\left\langle\delta x^{\prime}(0) \frac{\partial^{2} \varphi}{\partial x(T) \partial x(0)}+\delta x^{\prime}(T) \frac{\partial^{2} \varphi}{\partial x(T)^{2}}, \delta x(T)\right\rangle \\
\geq & 0 . \tag{57}
\end{align*}
$$

Inequality (57) is an implicit necessary optimality condition of first order. However, the practical value of such conditions is not great, since it requires very complicated calculations.

For obtaining effectively verifiable optimality conditions of second order, following [12, p. 16], we take into account (49) in (57) and introduce the matrix function

$$
\begin{align*}
R(\tau, s)= & -G^{\prime}(0, \tau) \frac{\partial^{2} \varphi}{\partial x(0)^{2}} G(0, s) \\
& -G^{\prime}(T, \tau) \frac{\partial^{2} \varphi}{\partial x(T) \partial x(0)} G(0, s) \\
& -G^{\prime}(0, \tau) \frac{\partial^{2} \varphi}{\partial x(0) \partial x(T)} G(T, s)  \tag{58}\\
& -G^{\prime}(T, \tau) \frac{\partial^{2} \varphi}{\partial x(T)^{2}} G(T, s) \\
& +\int_{0}^{T} G^{\prime}(t, \tau) \frac{\partial^{2} H}{\partial x^{2}} G(t, s) d t .
\end{align*}
$$

Then, for the second variation of the functional, we get the terminal formula

$$
\begin{aligned}
\delta^{2} J(u)=- & \left\{\int _ { 0 } ^ { T } \int _ { 0 } ^ { T } \left\langle\delta^{\prime} u(\tau) \frac{\partial^{\prime} f(\tau, x, u)}{\partial u}\right.\right. \\
& \left.\quad \times R(\tau, s) \frac{\partial f(s, x, u)}{\partial u}, \delta u(s)\right\rangle d t d s \\
& +\int_{0}^{T}\left\langle\delta^{\prime} u(\tau) \frac{\partial^{2} H(t, \psi, x, u)}{\partial u^{2}}, \delta u(t)\right\rangle d t
\end{aligned}
$$

$$
\begin{align*}
+2 \int_{0}^{T} \int_{0}^{T}\langle & \delta u^{\prime}(t) \frac{\partial^{2} H(t, \psi, x, u)}{\partial x \partial u} G(t, s) \\
& \left.\left.\times \frac{\partial f(s, x, u)}{\partial u}, \delta u(s)\right\rangle d t d s\right\} \tag{59}
\end{align*}
$$

Theorem 5. If the admissible control $u(t)$ satisfies condition (56), then for its optimality in problem (1)-(4), the inequality

$$
\begin{align*}
\delta^{2} J(u)=- & \left\{\int _ { 0 } ^ { T } \int _ { 0 } ^ { T } \left\langle\delta^{\prime} u(\tau) \frac{\partial^{\prime} f(\tau, x, u)}{\partial u}\right.\right. \\
& \left.\times R(\tau, s) \frac{\partial f(s, x, u)}{\partial u}, \delta u(s)\right\rangle d \tau d s \\
+ & \int_{0}^{T}\left\langle\delta^{\prime} u(\tau) \frac{\partial^{2} H(t, \psi, x, u)}{\partial u^{2}}, \delta u(t)\right\rangle d t \\
+ & 2 \int_{0}^{T} \int_{0}^{T}\left\langle\delta u^{\prime}(t) \frac{\partial^{2} H(t, \psi, x, u)}{\partial x \partial u} G(t, s)\right. \\
& \left.\left.\times \frac{\partial f(s, x, u)}{\partial u}, \delta u(s)\right\rangle d t d s\right\} \geq 0 \tag{60}
\end{align*}
$$

should be fulfilled for all $\delta u(t) \in L_{\infty}[0, T]$.
The analogy of the Legandre-Klebsh condition for the considered problem follows from condition (60).

Theorem 6. Along the optimal process $(u(t), x(t))$ for all $v \in$ $R^{r}$ and $\theta \in[0, T]$

$$
\begin{equation*}
v^{\prime} \frac{\partial^{2} H(\theta, \psi(\theta), x(\theta), u(\theta))}{\partial u^{2}} v \leq 0 \tag{61}
\end{equation*}
$$

To prove (61), one constructs the variation of the control

$$
\delta u(t)= \begin{cases}v & t \in[\theta, \theta+\varepsilon)  \tag{62}\\ 0 & t \notin[\theta, \theta+\varepsilon),\end{cases}
$$

where $\varepsilon>0$ and $v$ is some $r$-dimensional vector.
By virtue of (62) the corresponding variation of the trajectory indeed is

$$
\begin{equation*}
\delta x(t)=a(t) \varepsilon+o(\varepsilon, t), \quad t \in[0, T], \tag{63}
\end{equation*}
$$

where $a(t)$ is a continuous bounded function.
Substituting variation (62) into (60) and selecting the principal term with respect to $\varepsilon$, one may obtain

$$
\begin{align*}
\delta^{2} J(u) & =-\int_{\theta}^{\theta+\varepsilon} v^{\prime} \frac{\partial^{2} H(t, \psi(t), x(t), u(t))}{\partial u^{2}} v d t+o(\varepsilon) \\
& =-\varepsilon v^{\prime} \frac{\partial^{2} H(\theta, \psi(\theta), x(\theta), u(\theta))}{\partial u^{2}} v+o_{1}(\varepsilon) . \tag{64}
\end{align*}
$$

Thus, considering the second condition of (54), one obtains the Legandre-Klebsh criterion (61).

Condition (61) is the second-order optimality condition. It is obvious that when the right-hand side of system (1) is linear with respect to control parameters, then condition (61) also degenerates fulfills trivially. Following [11, p. 27], [12, p. 40], if for all $\theta \in(0, T), v \in R^{r}$

$$
\begin{gather*}
\frac{\partial H(\theta, \psi(\theta), x(\theta), u(\theta))}{\partial u}=0,  \tag{65}\\
v^{\prime} \frac{\partial^{2} H(\theta, \psi(\theta), x(\theta), u(\theta))}{\partial u^{2}} v=0,
\end{gather*}
$$

then the admissible control $u(t)$ is said be a singular control in the classic sense.

Theorem 7. For singular optimality of the control $u(t)$ in the classic sense, the inequality

$$
\begin{gather*}
v^{\prime}\left\{\int_{0}^{T} \int_{0}^{T}\left\langle\frac{\partial f(t, x, u)}{\partial u} R(t, s), \frac{\partial f(s, x, u)}{\partial u}\right\rangle d t d s\right. \\
+2 \int_{0}^{T} \int_{0}^{T}\left\langle\frac{\partial^{2} H(t, \psi, x, u)}{\partial x \partial u} G(t, s)\right.  \tag{66}\\
\left.\left.\frac{\partial f(s, x, u)}{\partial u}\right\rangle d t d s\right\} v \leq 0
\end{gather*}
$$

should be fulfilled for all $v \in R^{n}$.
Condition (66) is an integral necessary condition of optimality for singular controls in the classic sense. Selecting special variation in different way in formula (60), we can get various necessary optimality conditions.

## 5. Conclusion

In this work, the optimal control problem is considered when the state of the system is described by the differential equations with integral boundary conditions. Applying the Banach contraction principle, the existence and uniqueness of the solution are proved for the corresponding boundary problem by fixed admissible control. The first and second order increment formulas for the functional are calculated. Various necessary conditions of optimality of the first and second order are obtained by the help of the variation of the controls. Of course, such type, the existence and uniqueness results and necessary conditions of optimality hold under the same sufficient conditions on nonlinear terms for the system of nonlinear differential equations (1), subject to multipoint nonlocal and integral boundary conditions

$$
\begin{equation*}
E x(0)+\int_{0}^{T} m(t) x(t) d t+\sum_{j=1}^{J} B_{j} x\left(t_{j}\right)=C \tag{67}
\end{equation*}
$$

where $B_{j} \in R^{n \times n}$ are given matrices and

$$
\begin{equation*}
\|B\|+\sum_{j=1}^{J}\left\|B_{j}\right\|<1 . \tag{68}
\end{equation*}
$$

Here, $0<t_{1}<\cdots<t_{j} \leq T$. Moreover, the method given in [27, 28] and the method presented in the paper may allow one to investigate optimal control for infinite-dimensional systems with integral boundary conditions.

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# BIBO Stabilization of Discrete-Time Stochastic Control Systems with Mixed Delays and Nonlinear Perturbations 

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#### Abstract

The problem of bounded-input bounded-output (BIBO) stabilization in mean square for a class of discrete-time stochastic control systems with mixed time-varying delays and nonlinear perturbations is investigated. Some novel delay-dependent stability conditions for the previously mentioned system are established by constructing a novel Lyapunov-Krasovskii function. These conditions are expressed in the forms of linear matrix inequalities (LMIs), whose feasibility can be easily checked by using MATLAB LMI Toolbox. Finally, a numerical example is given to illustrate the validity of the obtained results.


## 1. Introduction

Many dynamical systems not only depend on the present states but also involve the past ones, generally called the timedelay systems. Generally, as a source of poor or significantly deteriorated performance and instability for the concerned closed-loop system, the time delays are unavoidable in technology and nature. Many works have been done on the stability of time-delay systems; one can see [1-23] and the references therein. The dynamics analysis of continuoustime systems with distributed delay has been well studied in [9-12, 20]. The aspect of simulation and application in control systems, whereas, discrete-time control systems play a more important role than their continuous-time counterparts in the practical digital world. If one wants to simulate or compute the continuous-time systems, it is essential to formulate the discrete-time analogue so as to investigate the dynamical characteristics. It is necessary to take continuous distributed delays into account for modeling realistic systems, for example, neural networks; due to the presence of an amount of parallel pathways of a variety of axon sizes and lengths, a neural network usually has a spatial nature. Very recently, Liu et al. introduced the infinite distributed delay and distributed delay in the form of constant delay into the delay neural networks. See [17-19].

In order to track out the reference input signal in real world, the bounded-input bounded-output stabilization has been investigated by many researchers, one can see [2032] and the references therein. In [22, 23], the sufficient conditions for BIBO stabilization of control systems with no delays were proposed by the Bihari type inequality. In [9, 10], by employing the parameters technique and the Gronwall inequality, the authors investigated the BIBO stability of the systems without distributed time delays. In [20, 27, 29], based on Riccati equations and by constructing appropriate Lyapunov functions, some BIBO stabilization conditions for a class of delayed control systems with nonlinear perturbations were established. In [30], the BIBO stabilization problem of a class of piecewise switched linear systems was further investigated. It should be pointed out that almost all results concerning the BIBO stability for control systems mainly concentrate on continuous-time models. Seldom works have been done for discrete-time control systems one can see [21, 28]. In addition, the previously mentioned works just considered the deterministic systems (see, e.g., $[31,32]$ ). The deterministic systems often fluctuate due to noise, which is random or at least appears to be so. Therefore, we must move from deterministic problems to stochastic ones. So, the BIBO stabilization for stochastic control systems case is necessary and interesting. To the best of our knowledge, there is no
work reported on the mean square BIBO stabilization for the discrete-time stochastic control systems with mixed timevarying delays.

It is well known that the classical technique applied in the study of stability is based on the Lyapunov direct method. However, the Lyapunov direct method has some difficulties with the theory and application to specific problems while discussing the stability of solutions in stochastic systems with time delay. In [33], the midpoint in the time delay's variation interval is introduced, and the variation interval is divided into two subintervals with equal length, by constructing the Lyapunov functional which involved midpoint to reduce the conservatism of stability conditions. This method was first proposed to study the stability and stabilization problems for linear continuous-time systems, and then many successful applications were found in [13-15]. In this paper, we will reconsider this method by introducing a new piecewiselike delay method, given that the point of the time delay's variation interval is arbitrary point rather than midpoint.

Motivated by the aforementioned works, in this paper, we investigate BIBO stabilization in mean square for a class of discrete-time stochastic control systems with mixed timevarying delays and nonlinear perturbations. Some novel delay-dependent stability conditions for the previously mentioned system are derived by constructing a novel LyapunovKrasovskii function. These conditions are expressed in the forms of linear matrix inequalities, whose feasibility can be easily checked by using MATLAB LMI Toolbox. Finally, a numerical example is given to illustrate the validity of the obtained results.

The paper is organized as follows. In Section 2, some notations and the problem formulation are proposed. The main results are given in Section 3. In Section 4, a numerical example is given to illustrate the validity of the obtained theory results. The conclusion is proposed in Section 5.

## 2. Notations and Problem Formulation

Firstly, we propose some notations which will be needed in the sequel. The notations are quite standard. Let $R^{n}$ and $R^{n \times m}$ denote, respectively, the $n$-dimensioned Euclidean space and the set of all $n \times m$ real matrices. The superscript " $T$ " denotes the transpose and the notation $X \geq Y$ (respective $X>Y$ ) means that $X$ and $Y$ are symmetric matrices and that $X-Y$ is positive semidefinitive (respective positive definite). Let $\|\cdot\|$ denote the Euclidean norm in $R^{n}$, let $N^{+}$denote the positive integer set, and let $I$ be the identity matrix with compatible dimension. If $A$ is a matrix, denote by $\|A\|$ its operator norm; that is, $\|A\|=\sup \{\|A x\|:\|x\|=1\}=\sqrt{\lambda_{\text {max }}\left(A^{T} A\right)}$, where $\lambda_{\text {max }}(A)$ (resp., $\left.\lambda_{\text {min }}(A)\right)$ means the largest (resp., smallest) value of $A$. Moreover, let $\left(\Omega, \mathscr{F},\left\{\mathscr{F}_{t}\right\}_{t \geq 0}, P\right)$ be a complete probability space with a filtration $\left\{\mathscr{F}_{t}\right\}_{t \geq 0}$ satisfying the usual conditions (i.e., the filtration contains all $P$-null sets and is right continuous). $E\{\cdot\}$ stands for the mathematical expectation operator with respect to the given probability measure $P$. The asterisk $*$ in a matrix is used to denote term that is induced by its symmetry. Matrices, if not explicitly state are assumed to have compatible dimensions. Denote
$N[a, b]:=\{a, a+1, \ldots, b\}$. Sometimes, the arguments of the functions will be omitted in the analysis without confusions.

In this paper, we consider the discrete-time stochastic control system with mixed time-varying delays and nonlinear perturbations with the following form:

$$
\begin{align*}
& x(k+1)= A x(k)+B x(k-\tau(k))+C u(k) \\
&+D \sum_{i=-d(k)}^{-1} h(x(k+i))+f(k, x(k), x(k-\tau(k))) \\
&+(G x(k)+H x(k-d(k))) \omega(k), \\
& y(k)=M x(k) \tag{1}
\end{align*}
$$

where $x(k)=\left[x_{1}(k), x_{2}(k), \ldots, x_{n}(k)\right]^{T} \in R^{n}$ denotes the state vector, $u(k)=\left[u_{1}(k), u_{2}(k), \ldots, u_{m}(k)\right]^{T} \in R^{m}$ is the control input vector, $y(k)=\left[y_{1}(k), y_{2}(k), \ldots, y_{n}(k)\right]^{T} \in R^{n}$ is the control output vector, $h(x(k))=\left[h_{1}(x(k)), h_{2}(x(k)), \ldots\right.$, $\left.h_{n}(x(k))\right]^{T}, A, B, D, G, C, H$, and $M$ represent the weighting matrices with appropriate dimension, and the positive integers $\tau(k)$ and $d(k)$ are the discrete-time-varying delay, distributed time-varying delay and respectively, satisfying that

$$
\begin{equation*}
\tau_{1} \leq \tau(k) \leq \tau_{2}, \quad d_{1} \leq d(k) \leq d_{2}, \quad k \in N^{+}, \tag{2}
\end{equation*}
$$

with $\tau_{1}, \tau_{2}, d_{1}$, and $d_{2}$ being four known positive integers. For any given $\tau^{*} \in\left(\tau_{1}, \tau_{2}\right), d^{*} \in\left(d_{1}, d_{2}\right)$. The initial conditions of the system (1) are given by

$$
\begin{equation*}
x(k)=\phi(k), \quad k \in\left[-\max \left\{\tau_{2}, d_{2}\right\}, 0\right] . \tag{3}
\end{equation*}
$$

The nonlinear vector-valued perturbation $f(k, x(k), x(k-$ $\tau(k))$ ) satisfies that

$$
\begin{align*}
& \|f(k, x(k), x(k-\tau(k)))\|^{2}  \tag{4}\\
& \quad \leq \alpha_{1}\|x(k)\|^{2}+\alpha_{2}\|x(k-\tau(k))\|^{2}
\end{align*}
$$

where $\alpha_{1}$ and $\alpha_{2}$ are two positive constants. $\omega(k)$ is a scalar Wiener process defined on $\left(\Omega, \mathscr{F},\left\{\mathscr{F}_{t}\right\}_{t \geq 0}, P\right)$ with

$$
\begin{align*}
& E(\omega(k))=0, \quad E\left(\omega(k)^{2}\right)=1, \\
& E(\omega(i) \omega(j))=0, \quad i \neq j \tag{5}
\end{align*}
$$

Remark 1. The $\tau^{*}$ divides the discrete-time delay's variation interval into two subintervals, that is, $\left[\tau_{1}, \tau^{*}\right]$ and $\left(\tau^{*}, \tau_{2}\right]$, and $d^{*}$ divides the distributed time delay's variation interval into two subintervals, that is, $\left[d_{1}, d^{*}\right]$ and $\left(d^{*}, d_{2}\right]$. We will discuss the variation of differences of the Lyapunov-Krasovskii functional $V(t, x(t))$ for each subinterval. Compared with the previous results in the works of [27-32], the BIBO stability conditions are derived in this paper by checking the variation of $V(t, x(t))$ in subintervals rather than in the whole variation interval of the delays.

In what follows, we describe the controller with the form

$$
\begin{equation*}
u(k)=K x(k)+r(k) \tag{6}
\end{equation*}
$$

where $K$ is the feedback gain matrix and $r(k)$ is the reference input.

Assumption 2. For any $\xi_{1}, \xi_{2} \in R, \xi_{1} \neq \xi_{2}$, let

$$
\begin{equation*}
\gamma_{i}^{-} \leq \frac{h_{i}\left(\xi_{1}\right)-h_{i}\left(\xi_{2}\right)}{\xi_{1}-\xi_{2}} \leq \gamma_{i}^{+} \tag{7}
\end{equation*}
$$

where $\gamma_{i}^{-}$and $\gamma_{i}^{+}$are known constant scalars.
Remark 3. The constants $\gamma_{i}^{-}, \gamma_{i}^{+}$in Assumption 2 are allowed to be positive, negative, or zero. Hence, the function $h(x(k))$ could be nonmonotonic and is more general than the usual sigmoid functions and the recently commonly used Lipschitz conditions.

At the end of this section, let us introduce some important definitions and lemmas as which will be used in the sequel.

Definition 4 (see $[28,31])$. A vector function $r(k)=\left(r_{1}(k)\right.$, $\left.r_{2}(k), \ldots, r_{n}(k)\right)^{T}$ is said to be an element of $L_{\infty}^{n}$, if $\|r\|_{\infty}=$ $\sup _{k \in N[0, \infty)}\|r(k)\|<+\infty$, where $\|\cdot\|$ denotes the Euclid norm in $R^{n}$, or the norm of a matrix.

Definition 5 (see [28, 31]). The nonlinear stochastic control system (1) is said to be BIBO stability in mean square, if we can construct a controller (6) such that the output $y(k)$ satisfies that

$$
\begin{equation*}
\mathbb{E}\|y(k)\|^{2} \leq N_{1}+N_{2}\|r\|_{\infty}^{2}, \tag{8}
\end{equation*}
$$

where $N_{1}$ and $N_{2}$ are two positive constants.
Lemma 6 (see [28]). For any given vectors $v_{i} \in R^{n}, i=1$, $2, \ldots, n$, the following inequality holds:

$$
\begin{equation*}
\left[\sum_{i=1}^{n} v_{i}\right]^{T}\left[\sum_{i=1}^{n} v_{i}\right] \leq n \sum_{i=1}^{n} v_{i}^{T} v_{i} \tag{9}
\end{equation*}
$$

Lemma 7 (see [28]). Let $x, y \in R^{n}$ and any $n \times n$ positivedefinite matrix $\mathrm{Q}>0$. Then, one has

$$
\begin{equation*}
2 x^{T} y \leq x^{T} Q^{-1} x+y^{T} Q y \tag{10}
\end{equation*}
$$

Lemma 8 (see [28]). Given the constant matrices $\Omega_{1}, \Omega_{2}$, and $\Omega_{3}$ with appropriate dimensions, where $\Omega_{1}=\Omega_{1}^{T}$ and $\Omega_{2}=$ $\Omega_{2}^{T}>0$, then $\Omega_{1}+\Omega_{3}^{T} \Omega_{2}^{-1} \Omega_{3}<0$ if and only if

$$
\left(\begin{array}{cc}
\Omega_{1} & \Omega_{3}^{T}  \tag{11}\\
* & -\Omega_{2}
\end{array}\right)<0 \quad \text { or } \quad\left(\begin{array}{cc}
-\Omega_{2} & \Omega_{3} \\
* & \Omega_{1}
\end{array}\right)<0
$$

## 3. BIBO Stabilization for the System (1)

In this section, we aim to establish our main results based on the LMI approach. For the conveniences, we denote

$$
\begin{align*}
& \Gamma_{1}=\operatorname{diag}\left\{\gamma_{1}^{-} \gamma_{1}^{+}, \gamma_{2}^{-} \gamma_{2}^{+}, \ldots, \gamma_{n}^{-} \gamma_{n}^{+}\right\}, \\
& \Gamma_{3}=\operatorname{diag}\left\{\gamma_{1}^{+}, \gamma_{2}^{+}, \ldots, \gamma_{n}^{+}\right\}, \\
& \Gamma_{2}=\operatorname{diag}\left\{\frac{\gamma_{1}^{-}+\gamma_{1}^{+}}{2}, \frac{\gamma_{2}^{-}+\gamma_{2}^{+}}{2}, \ldots, \frac{\gamma_{n}^{-}+\gamma_{n}^{+}}{2}\right\}, \\
& a=\tau_{2}-\tau_{1}+1, \\
& b=\left\{\begin{array}{ll}
\frac{\left(d^{*}+d_{1}-1\right)\left(d^{*}-d_{1}\right)+2 d^{*}}{2}, \quad d_{1} \leq d(k) \leq d^{*} \\
\frac{\left(d^{*}+d_{2}-1\right)\left(d_{2}-d^{*}\right)+2 d_{2}}{2}, \quad d^{*}<d(k) \leq d_{2}, \\
c & = \begin{cases}d^{*}, & d_{1} \leq d(k) \leq d^{*} \\
d_{2}, & d^{*}<d(k) \leq d_{2},\end{cases} \\
\theta(k)= \begin{cases}x\left(k-\tau_{1}\right), & \tau_{1} \leq \tau(k) \leq \tau^{*} \\
x\left(k-\tau_{2}\right), & \tau^{*}<\tau(k) \leq \tau_{2},\end{cases} \\
\tilde{\tau}= \begin{cases}\tau^{*}-\tau_{1}, & \tau_{1} \leq \tau(k) \leq \tau^{*} \\
\tau_{2}-\tau^{*}, & \tau^{*}<\tau(k) \leq \tau_{2},\end{cases} \\
\beta= \begin{cases}\frac{\tau(k)-\tau_{1}}{\tau^{*}-\tau_{1}}, & \tau_{1} \leq \tau(k) \leq \tau^{*} \\
\frac{\tau_{2}-\tau(k)}{\tau_{2}-\tau^{*}}, & \tau^{*}<\tau(k) \leq \tau_{2} .\end{cases}
\end{array} .\right.
\end{align*}
$$

Theorem 9. For given positive integers $\tau_{1}, \tau_{2}, d_{1}$, and $d_{2}$, under Assumption 2, the nonlinear discrete-time stochastic control system (1) with the controller (6) is BIBO stabilization in mean square, if there exist symmetric positive-definite matrix $P, R, Q_{i}, i=1,2, \ldots, 5, Z_{1}, Z_{2}$, and $X$ with appropriate dimensional, positive-definite diagonal matrices $\Lambda$ and some positive constants $\zeta$ and $\lambda^{*}$ such that the following two LMIs hold:

$$
\begin{equation*}
P+2\left(\tau^{* 2} Z_{1}+\tilde{\tau}^{2} Z_{2}\right) \leq \lambda^{*} I, \tag{13}
\end{equation*}
$$

$$
\begin{aligned}
& \Xi \\
& =\left(\begin{array}{ccccccccc}
\Xi_{11} & \Xi_{12} & Z_{1} & 0 & 0 & \Gamma_{2} \Lambda & \Xi_{17} & \Xi_{18} & \sqrt{2} X \\
* & \Xi_{22} & \Xi_{23} & 0 & \Xi_{25} & 0 & B^{T} D & B^{T} & 0 \\
* & * & \Xi_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & -Q_{2} & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & \Xi_{55} & 0 & 0 & 0 & 0 \\
* & * & * & * & * & \Xi_{66} & 0 & 0 & 0 \\
* & * & * & * & * & * & \Xi_{77} & \lambda^{*} D^{T} & 0 \\
* & * & * & * & * & * & * & -\zeta I & 0 \\
* & * & * & * & * & * & * & * & -\lambda^{*} I
\end{array}\right)
\end{aligned}
$$

$$
\begin{equation*}
\leq 0 \tag{14}
\end{equation*}
$$

where

$$
\begin{align*}
\Xi_{11}= & Q_{1}+Q_{2}+Q_{3}+a\left(Q_{3}+Q_{4}\right)-Z_{1} \\
& +Q_{5}-\Gamma_{1} \Lambda+G^{T} \lambda^{*} G+2 \tau^{* 2} Z_{1}+2 \widetilde{\tau}^{2} Z_{2} \\
& +2 \lambda^{*} \alpha_{1} I+\alpha_{1} \zeta I-P+2 \lambda^{*} A^{T} A-2 A^{T} X-2 X^{T} A, \\
\Xi_{12}= & \lambda^{*} A^{T} B-X^{T} B+G^{T} \lambda^{*} H, \\
\Xi_{17}= & \lambda^{*} A^{T} D-X^{T} D, \\
\Xi_{18}= & \lambda^{*} A^{T}-X^{T}, \\
\Xi_{22}= & H^{T} \lambda^{*} H+2 \lambda^{*} B^{T} B-Q_{4}-2 Z_{2}-\beta Z_{2} \\
& +2 \lambda^{*} \alpha_{2} I+\alpha_{2} \zeta I-(1-\beta) Z_{2}, \\
\Xi_{23}= & Z_{2}+\beta Z_{2}, \\
\Xi_{25}= & Z_{2}+(1-\beta) Z_{2}, \\
\Xi_{33}= & -Q_{1}-Z_{1}-Z_{2}-\beta Z_{2}, \\
\Xi_{55}= & -Q_{5}-Z_{2}-(1-\beta) Z_{2}, \\
\Xi_{66}= & b R-\Lambda, \\
\Xi_{77}= & 2 \lambda^{*} D^{T} D-\frac{1}{c} R, \\
X= & -\lambda^{*} C K . \tag{15}
\end{align*}
$$

Proof. We construct the following Lyapunov-Krasovskii function for the system (1):

$$
\begin{align*}
V(k, x(k))= & V_{1}(k, x(k))+V_{2}(k)+V_{3}(k)  \tag{16}\\
& +V_{4}(k)+V_{5}(k)+V_{6}(k),
\end{align*}
$$

where

$$
\begin{aligned}
& V_{1}(k, x(k))=x^{T}(k) P x(k), \\
& V_{2}(k)= \sum_{i=k-\tau^{*}}^{k-1} x^{T}(i) Q_{1} x(i) \\
&+\sum_{i=k-d^{*}}^{k-1} x^{T}(i) Q_{2} x(i), \\
& V_{3}(k)= \sum_{i=k-\tau(k)}^{k-1} x^{T}(i) Q_{3} x(i)+\sum_{i=\tau_{1}}^{\tau_{2}-1} \sum_{j=k-i}^{k-1} x^{T}(j) Q_{3} x(j) \\
&+\sum_{i=k-\tau_{1}}^{k-1} x^{T}(i) Q_{3} x(i) \\
&+\sum_{i=-\tau_{2}+1}^{-\tau_{1}+1} \sum_{j=k-1+i}^{k-1} x^{T}(j) Q_{4} x(j), \\
& V_{4}(k)= \tau^{*} \sum_{i=-\tau^{*}}^{-1} \sum_{j=k+i}^{k-1} \eta^{T}(j) Z_{1} \eta(j)
\end{aligned}
$$

$$
\begin{align*}
& \eta(k)=x(k+1)-x(k), \\
& \int \sum_{i=k-\tau_{1}}^{k-1} x^{T}(i) Q_{5} x(i)+\left(\tau^{*}-\tau_{1}\right) \\
& \times \sum_{i=-\tau^{*}}^{-\tau_{1}-1} \sum_{j=k+i}^{k-1} \eta^{T}(j) Z_{2} \eta(j), \\
& \tau_{1} \leq \tau(k) \leq \tau^{*} \\
& V_{5}(k)= \\
& \sum_{i=k-\tau_{2}}^{k-1} x^{T}(i) Q_{5} x(i)+\left(\tau_{2}-\tau^{*}\right) \\
& \times \sum_{i=-\tau_{2}}^{-\tau^{*}-1} \sum_{j=k+i}^{k-1} \eta^{T}(j) Z_{2} \eta(j), \\
& \tau^{*}<\tau(k) \leq \tau_{2}, \\
& V_{6}(k)=\left\{\begin{array}{l}
\sum_{i=-d(k)}^{-1} \sum_{j=k+i}^{k-1} h^{T}(x(j)) R h(x(j)) \\
+\sum_{i=-d^{*}}^{-d_{1}-1} \sum_{j=i+1}^{-1} \sum_{l=k+j}^{k-1} h^{T}(x(l)) R h(x(l)), \\
d_{1} \leq d(k) \leq d^{*} \\
\sum_{i=-d(k)}^{-1} \sum_{j=k+i}^{k-1} h^{T}(x(j)) R h(x(j)) \\
+\sum_{i=-d_{2}}^{-d^{*}-1} \sum_{j=i+1}^{-1} \sum_{l=k+j}^{k-1} h^{T}(x(l)) R h(x(l)), \\
d^{*}<d(k) \leq d_{2} .
\end{array}\right. \tag{17}
\end{align*}
$$

Calculating the difference of $V(k, x(k))$ and taking the mathematical expectation, by Lemma 6, we have

$$
\begin{aligned}
E \Delta & V_{1}(k, x(k)) \\
& =E\left[x^{T}(k+1) P x(k+1)-x^{T}(k) P x(k)\right] \\
& =E\left[\eta^{T}(k) P \eta(k)+2 \eta^{T}(k) P x(k)\right],
\end{aligned}
$$

$E \Delta V_{2}(k)$

$$
\begin{aligned}
=E & {\left[x^{T}(k) Q_{1} x(k)-x^{T}\left(k-\tau^{*}\right) Q_{1} x\left(k-\tau^{*}\right)\right.} \\
& \left.+x^{T}(k) Q_{2} x(k)-x^{T}\left(k-d^{*}\right) Q_{2} x\left(k-d^{*}\right)\right]
\end{aligned}
$$

$E \Delta V_{3}(k)$

$$
\begin{aligned}
=E[ & \left(\sum_{i=k+1-\tau(k+1)}^{k}-\sum_{i=k-\tau(k)}^{k-1}\right) x^{T}(i) Q_{3} x(i) \\
& +\sum_{i=\tau_{1}}^{\tau_{2}-1}\left(\sum_{j=k-i+1}^{k}-\sum_{j=k-i}^{k-1}\right) x^{T}(j) Q_{3} x(j) \\
& +\left(\sum_{i=k-\tau_{1}+1}^{k}-\sum_{i=k-\tau_{1}}^{k-1}\right) x^{T}(i) Q_{3} x(i) \\
& +\sum_{i=-\tau_{2}+1}^{-\tau_{1}+1}\left(\sum_{j=k+i}^{k}-\sum_{j=k+j-1}^{k-1}\right) \\
& \left.\times x^{T}(j) Q_{4} x(j)\right]
\end{aligned}
$$

$$
\begin{aligned}
\leq E[ & \left(\sum_{i=k+1-\tau_{2}}^{k} x^{T}(i) Q_{3} x(i)-\sum_{i=k-\tau_{1}}^{k-1} x^{T}(i) Q_{3} x(i)\right) \\
& +\sum_{i=\tau_{1}}^{\tau_{2}-1}\left(x^{T}(k) Q_{3} x(k)-x^{T}(k-i) Q_{3} x(k-i)\right) \\
& +\left(x^{T}(k) Q_{3} x(k)-x^{T}\left(k-\tau_{1}\right) Q_{3} x\left(k-\tau_{1}\right)\right) \\
& +\sum_{i=-\tau_{2}+1}^{-\tau_{1}+1}\left(x^{T}(k) Q_{4} x(k)-x^{T}\right.
\end{aligned}
$$

$$
\left.\left.\times(k+i-1) Q_{4} x(k+i-1)\right)\right]
$$

$$
=E\left[x^{T}(k)\left[a\left(Q_{3}+Q_{4}\right)+Q_{3}\right] x(k)\right.
$$

$$
-2 x^{T}\left(k-\tau_{1}\right) Q_{3} x\left(k-\tau_{1}\right)
$$

$$
\left.-\sum_{i=k-\tau_{2}}^{k-\tau_{1}} x^{T}(i) Q_{4} x(i)\right]
$$

$$
\leq E\left[x^{T}(k)\left[a\left(Q_{3}+Q_{4}\right)+Q_{3}\right] x(k)\right.
$$

$$
\left.-x^{T}(k-\tau(k)) Q_{4} x(k-\tau(k))\right]
$$

$E \Delta V_{4}(k)$

$$
=E\left[\tau^{*} \sum_{i=-\tau^{*}}^{-1}\left(\sum_{j=k+i+1}^{k}-\sum_{j=k+i}^{k-1}\right) \eta^{T}(j) Z_{1} \eta(j)\right]
$$

$$
=E\left[\tau^{* 2} \eta^{T}(k) Z_{1} \eta(k)-\tau^{*} \sum_{i=k-\tau^{*}}^{k-1} \eta^{T}(i) Z_{1} \eta(i)\right]
$$

$$
\leq E\left[\tau^{* 2} \eta^{T}(k) Z_{1} \eta(k)\right.
$$

$$
\begin{equation*}
\left.-\sum_{i=k-\tau^{*}}^{k-1} \eta^{T}(i) Z_{1} \sum_{i=k-\tau^{*}}^{k-1} \eta(i)\right] \tag{18}
\end{equation*}
$$

Note that

$$
\begin{align*}
& -\sum_{i=k-\tau^{*}}^{k-1} \eta^{T}(i) Z_{1} \sum_{i=k-\tau^{*}}^{k-1} \eta(i) \\
& \quad=\binom{x(k)}{x\left(k-\tau^{*}\right)}^{T}\left(\begin{array}{cc}
-Z_{1} & Z_{1} \\
* & -Z_{1}
\end{array}\right)\binom{x(k)}{x\left(k-\tau^{*}\right)}, \tag{19}
\end{align*}
$$

$$
E \Delta V_{5}(k)=E\left\{x^{T}(k) Q_{5} x(k)-\theta^{T}(k) Q_{5} \theta(k)\right.
$$

$$
\begin{equation*}
\left.+\widetilde{\tau}^{2} \eta^{T}(k) Z_{2} \eta(k)-\widetilde{\tau} \psi(k)\right\} \tag{20}
\end{equation*}
$$

where

$$
\psi(k)= \begin{cases}\sum_{i=k-\tau^{*}}^{k-\tau_{1}-1} \eta^{T}(i) Z_{2} \eta(i), & \tau_{1} \leq \tau(k) \leq \tau^{*}  \tag{21}\\ \sum_{i=k-\tau_{2}}^{k-\tau^{*}-1} \eta^{T}(i) Z_{2} \eta(i), & \tau^{*}<\tau(k) \leq \tau_{2}\end{cases}
$$

When $\tau^{*}<\tau(k) \leq \tau_{2}$, it is easy to compute that

$$
\begin{align*}
-\tilde{\tau} \psi & (k) \\
= & -\left[\left(\tau_{2}-\tau(k)\right)+\left(\tau(k)-\tau^{*}\right)\right] \\
& \times \sum_{i=k-\tau(k)}^{k-\tau^{*}-1} \eta^{T}(i) Z_{2} \eta(i) \\
& -\left[\left(\left(\tau_{2}-\tau(k)\right)+\left(\tau(k)-\tau^{*}\right)\right)\right] \\
& \times \sum_{i=k-\tau_{2}}^{k-\tau(k)-1} \eta^{T}(i) Z_{2} \eta(i) \\
\leq & -\beta \sum_{i=k-\tau(k)}^{k-\tau^{*}-1} \eta^{T}(i) Z_{2} \sum_{i=k-\tau(k)}^{k-\tau^{*}-1} \eta(i)  \tag{22}\\
& -\sum_{i=k-\tau(k)}^{k-\tau^{*}-1} \eta^{T}(i) Z_{2} \sum_{i=k-\tau(k)}^{k-\tau^{*}-1} \eta(i) \\
& -\sum_{i=k-\tau_{2}}^{k-\tau(k)-1} \eta^{T}(i) Z_{2} \sum_{i=k-\tau_{2}}^{k-\tau(k)-1} \eta(i)-(1-\beta) \\
& \times \sum_{i=k-\tau_{2}}^{k-\tau(k)-1} \eta^{T}(i) Z_{2} \sum_{i=k-\tau_{2}}^{k-\tau(k)-1} \eta(i) .
\end{align*}
$$

When $\tau_{1} \leq \tau(k) \leq \tau^{*}$, similarly we can have

$$
\begin{align*}
-\tilde{\tau} \psi(k) \leq & -(1-\beta) \sum_{i=k-\tau(k)}^{k-\tau_{1}-1} \eta^{T}(i) Z_{2} \sum_{i=k-\tau(k)}^{k-\tau_{1}-1} \eta(i) \\
& -\sum_{i=k-\tau(k)}^{k-\tau_{1}-1} \eta^{T}(i) Z_{2} \sum_{i=k-\tau(k)}^{k-\tau-1} \eta(i)  \tag{23}\\
& -\sum_{i=k-\tau^{*}}^{k-\tau(k)-1} \eta^{T}(i) Z_{2} \sum_{i=k-\tau^{*}}^{k-\tau(k)-1} \eta(i) \\
& -\beta \sum_{i=k-\tau^{*}}^{k-\tau(k)-1} \eta^{T}(i) Z_{2} \sum_{i=k-\tau^{*}}^{k-\tau(k)-1} \eta(i) .
\end{align*}
$$

From (20), (22), and (23), we have

$$
\left.\begin{array}{rl}
E \Delta V_{5}(k)=E[ & x^{T}(k) Q_{5} x(k)-\theta^{T}(k) Q_{5} \theta(k) \\
& +\tilde{\tau}^{2} \eta^{T}(k) Z_{2} \eta(k)+\left(\begin{array}{c}
x(k-\tau(k)) \\
x\left(k-\tau^{*}\right) \\
\theta(k)
\end{array}\right)^{T} \\
& \times\left(\begin{array}{ccc}
-2 Z_{2} & Z_{2} & Z_{2} \\
* & -Z_{2} & 0 \\
* & * & -Z_{2}
\end{array}\right)\left(\begin{array}{c}
x(k-\tau(k)) \\
x\left(k-\tau^{*}\right) \\
\theta(k)
\end{array}\right) \\
& +\beta\binom{x(k-\tau(k))}{x\left(k-\tau^{*}\right)}^{T}\left(\begin{array}{cc}
-Z_{2} & Z_{2} \\
* & -Z_{2}
\end{array}\right) \\
& \times\binom{ x(k-\tau(k))}{x\left(k-\tau^{*}\right)}+\left(\begin{array}{l}
1-\beta) \\
\\
\end{array}\right. \\
& \times\binom{ x(k-\tau(k))}{\theta(k)}^{T}\left(\begin{array}{cc}
-Z_{2} & Z_{2} \\
* & -Z_{2}
\end{array}\right) \\
\theta(k-\tau(k)) \tag{24}
\end{array}\right) . ~ \$
$$

When $d_{1} \leq d(k) \leq d^{*}$, by Lemma 6 , it is easy to get

$$
\left.\begin{array}{rl}
E \Delta V_{6}(k)=E[ & \left(\sum_{i=-d(k+1)}^{-1} \sum_{j=k+i+1}^{k}-\sum_{i=-d(k)}^{-1} \sum_{j=k+i}^{k-1}\right) \\
& \times h^{T}(x(j)) R h(x(j)) \\
& +\sum_{i=-d^{*}}^{-d_{1}-1} \sum_{j=i+1}^{-1}\left(\sum_{l=k+j+1}^{k}-\sum_{l=k+j}^{k-1}\right) \\
& \left.\times h^{T}(x(l)) R h(x(l))\right] \\
\leq E\left[\sum_{i=-d^{*}}^{-1} \sum_{j=k+i+1}^{k-1} h^{T}(x(j)) R h(x(j))\right. \\
& \quad-\sum_{i=-d(k)}^{-1} \sum_{j=k+i+1}^{k-1} h^{T}(x(j)) R h(x(j)) \\
& \quad-\sum_{i=-d^{*}}^{-1} h^{T}(x(k)) R h(x(k)) \\
& +\sum_{i=-d(k)}^{-1} h^{T}(x(k+i)) R h(x(k+i)) \\
& \quad-\sum_{i=-d^{*}} \sum_{j=i+1}^{-1} h^{T}(x(k)) R h(x(k)) \\
& -d_{1}-1 \\
i=-1 \\
j=i+1
\end{array} h^{T}(x(k+j)) R h(x(k+j))\right]
$$

$$
\begin{align*}
\leq E[ & \frac{\left(d^{*}+d_{1}-1\right)\left(d^{*}-d_{1}\right)+2 d^{*}}{2} \\
& \times h^{T}(x(k)) R h(x(k)) \\
& -\frac{1}{d^{*}}\left(\sum_{i=-d(k)}^{-1} h(x(k+i))\right)^{T} \\
& \left.\times R\left(\sum_{i=-d(k)}^{-1} h(x(k+i))\right)\right] . \tag{25}
\end{align*}
$$

When $d^{*}<d(k) \leq d_{2}$, similarly we can have

$$
\begin{align*}
& E \Delta V_{6}(k) \leq E\left[\frac{\left(d^{*}+d_{2}-1\right)\left(d_{1}-d^{*}\right)+2 d_{2}}{2}\right. \\
& \times h^{T}(x(k)) R h(x(k)) \\
&-\frac{1}{d_{2}}\left(\sum_{i=-d(k)}^{-1} h(x(k+i))\right)^{T}  \tag{26}\\
&\left.\times R\left(\sum_{i=-d(k)}^{-1} h(x(k+i))\right)\right]
\end{align*}
$$

From (25) and (26), we have

$$
\begin{align*}
E \Delta V_{6}(k) \leq E[ & b h^{T}(x(k)) R h(x(k)) \\
& -\frac{1}{c}\left(\sum_{i=-d(k)}^{-1} h(x(k+i))\right)^{T}  \tag{27}\\
& \left.\times R\left(\sum_{i=-d(k)}^{-1} h(x(k+i))\right)\right] .
\end{align*}
$$

From (7), it follows that

$$
\begin{align*}
& \left(h_{i}(x(k))-\gamma_{i}^{+} x_{i}(k)\right)  \tag{28}\\
& \quad \times\left(h_{i}(x(k))-\gamma_{i}^{-} x_{i}(k)\right) \leq 0, \quad i=1,2, \ldots, n
\end{align*}
$$

which are equivalent to

$$
\begin{align*}
& \binom{x(k)}{h(x(k))}^{T}\left(\begin{array}{cc}
\gamma_{i}^{-} \gamma_{i}^{+} e_{i} e_{i}^{T} & -\frac{\gamma_{i}^{-}+\gamma_{i}^{+}}{2} e_{i} e_{i}^{T} \\
*
\end{array}\right)  \tag{29}\\
& \times\binom{ x(k)}{h(x(k))} \leq 0,
\end{align*}
$$

where $e_{i}$ denotes the unit column vector having one element on its $i$ th row and zeros elsewhere.

Then from (29), for any matrices $\Lambda=\operatorname{diag}\left\{\lambda_{1}, \lambda_{2}, \ldots\right.$, $\left.\lambda_{n}\right\}>0$, it follows that

$$
\binom{x(k)}{h(x(k))}^{T}\left(\begin{array}{cc}
-\Gamma_{1} \Lambda & \Gamma_{2} \Lambda  \tag{30}\\
* & -\Lambda
\end{array}\right)\binom{x(k)}{h(x(k))} \geq 0 .
$$

## Note that, by Lemma 7, we get

$$
+D \sum_{i=-d(k)}^{-1} h(x(k+i))
$$

$$
+f(k, x(k), x(k-\tau(k)))+C r(k)]
$$

$$
+[G x(k)+H x(k-\tau(k))]^{T}
$$

$$
\times \lambda^{*} I[G x(k)+H x(k-\tau(k))]
$$

$$
\left.+x^{T}(k)\left(2 \tau^{* 2} Z_{1}+2 \tilde{\tau} Z_{2}-P\right) x(k)\right\}
$$

$$
\leq E\{[(A+C K) x(k)+B x(k-\tau(k))
$$

$$
+D \sum_{i=-d(k)}^{-1} h(x(k+i))
$$

$$
+f(k, x(k), x(k-\tau(k)))]^{T} \lambda^{*} I
$$

$$
\times[(A+C K) x(k)+B x(k-\tau(k))
$$

$$
\left.+D \sum_{i=-d(k)}^{-1} h(x(k+i))+f(k, x(k), x(k-\tau(k)))\right]
$$

$$
\begin{aligned}
& E\left[\eta^{T}(k) P \eta(k)+2 \eta^{T}(k) P x(k)+\tau^{* 2} \eta^{T}(k) Z_{1} \eta(k)\right. \\
& \left.+\widetilde{\tau} \eta^{T}(k) Z_{2} \eta(k)\right] \\
& =E\left[\eta^{T}(k)\left(P+\tau^{* 2} Z_{1}+\tilde{\tau} Z_{2}\right) \eta(k)+2 \eta^{T}(k) P x(k)\right] \\
& =E\left[x^{T}(k+1)\left(P+\tau^{* 2} Z_{1}+\tilde{\tau} Z_{2}\right) x(k+1)\right. \\
& -2 x^{T}(k+1)\left(\tau^{* 2} Z_{1}+\widetilde{\tau} Z_{2}\right) x(k) \\
& \left.+x^{T}(k)\left(\tau^{* 2} Z_{1}+\widetilde{\tau} Z_{2}-P\right) x(k)\right] \\
& \leq E\left[x^{T}(k+1)\left(P+2 \tau^{* 2} Z_{1}+2 \tilde{\tau} Z_{2}\right) x(k+1)\right. \\
& \left.+x^{T}(k)\left(2 \tau^{* 2} Z_{1}+2 \tilde{\tau} Z_{2}-P\right) x(k)\right] \\
& \leq E\left[x^{T}(k+1) \lambda^{*} I x(k+1)+x^{T}(k)\right. \\
& \left.\times\left(2 \tau^{* 2} Z_{1}+2 \widetilde{\tau} Z_{2}-P\right) x(k)\right] \\
& =E\{[(A+C K) x(k)+B x(k-\tau(k)) \\
& +D \sum_{i=-d(k)}^{-1} h(x(k+i)) \\
& +f(k, x(k), x(k-\tau(k)))+C r(k)]^{T} \\
& \times \lambda^{*} I[(A+C K) x(k)+B x(k-\tau(k))
\end{aligned}
$$

$$
\begin{align*}
& +[G x(k)+H x(k-\tau(k))]^{T} \lambda^{*} I \\
& \times[G x(k)+H x(k-\tau(k))] \\
& +x^{T}(k)\left(2 \tau^{* 2} Z_{1}+2 \widetilde{\tau} Z_{2}-P\right) x(k) \\
& +\lambda^{*} x^{T}(A+C K)^{T}(A+C K) x(k) \\
& +\lambda^{*} x^{T}(k-\tau(k)) B^{T} B x(k-\tau(k)) \\
& +\lambda^{*}\left(\sum_{i=-d(k)}^{-1} h(x(k+i))\right)^{T} D^{T} D \\
& \times\left(\sum_{i=-d(k)}^{-1} h(x(k+i))\right) \\
& \left.+5 \lambda^{*}\|C\|^{2}\|r\|_{\infty}^{2}\right\} \tag{31}
\end{align*}
$$

Then from (18) to (31), we have

$$
\begin{align*}
E \Delta V(k) \leq & E\left\{\xi ^ { T } ( k ) \left[\Xi^{\prime}+(\sqrt{2} X, 0,0,0,0,0,0,0)^{T} \frac{1}{\lambda^{*}}\right.\right. \\
& \times(\sqrt{2} X, 0,0,0,0,0,0,0)] \xi(k)\} \\
+ & \rho\|r\|_{\infty}^{2} \tag{32}
\end{align*}
$$

where

$$
\begin{gather*}
\Xi_{i j}^{\prime}=\Xi_{i j}, \quad i, j=1,2, \ldots, 8, \quad \rho=5 \lambda^{*}\|D\|^{2}, \\
\xi^{T}(k)=\left[x^{T}(k), x^{T}(k-\tau(k)), x\left(k-\tau^{*}\right), x\left(k-d^{*}\right),\right. \\
\left.\theta^{T}(k), h^{T}(x(k)), \sum_{-d(k)}^{-1} h^{T}(x(k+i)), f^{T}\right] . \tag{33}
\end{gather*}
$$

If the LMI (14) holds, by using Lemma 8, it follows that there exists a sufficient small positive $\varepsilon>0$, such that

$$
\begin{equation*}
E \Delta V(k) \leq-\varepsilon E\|x(k)\|^{2}+\rho\|r\|_{\infty}^{2} \tag{34}
\end{equation*}
$$

It is easy to derive that

$$
\begin{align*}
E V(k) \leq & \mu_{1} E\|x(k)\|^{2}+\mu_{2} \sum_{i=k-\tau_{2}}^{k-1} E\|x(i)\|^{2} \\
& +\mu_{3} \sum_{i=k-d_{2}}^{k-1} E\|x(i)\|^{2} \tag{35}
\end{align*}
$$

with

$$
\begin{aligned}
\mu_{1}= & \lambda_{\max }(P), \\
\mu_{2}= & \lambda_{\max }\left(Q_{1}\right)+(a+1) \lambda_{\max }\left(Q_{3}\right) \\
& +2 \lambda_{\max }\left(Q_{4}\right)+4 \tau^{* 2} \lambda_{\max }\left(Z_{1}\right) \\
& +4\left(\tau_{2}-\tau_{1}\right)^{2} \lambda_{\max }\left(Z_{2}\right), \\
\mu_{3}= & \lambda_{\max }\left(Q_{2}\right)+\left[d_{2}+\left(d_{2}-d_{1}\right)\right. \\
& \left.\times\left(d_{2}-d_{1}\right)\right] \\
& \times\|\Gamma\|^{2} \lambda_{\max }(R) .
\end{aligned}
$$

For any $\theta>1$, it follows from (34) and (35) that

$$
\begin{align*}
& E\left[\theta^{j+1} V(j+1)-\theta^{j} V(j)\right] \\
& \quad=\theta^{j+1} E \Delta V(j)+\theta^{j}(\theta-1) E V(j) \\
& \quad \leq \theta^{j}\left[\left(-\varepsilon \theta+(\theta-1) \mu_{1}\right) E\|x(j)\|^{2}\right.  \tag{37}\\
& \quad+(\theta-1) \mu_{2} \sum_{i=j-\tau_{2}}^{j-1} E\|x(i)\|^{2}+\rho \theta\|r\|_{\infty}^{2} \\
& \left.\quad+(\theta-1) \mu_{3} \sum_{i=j-d_{2}}^{j-1} E\|x(i)\|^{2}\right]
\end{align*}
$$

Summing up both sides of (37) from 0 to $k-1$, we can obtain

$$
\begin{align*}
& \theta^{k} E V(k)-E V(0) \\
& \leq\left(\mu_{1}(\theta-1)-\varepsilon \theta\right) \sum_{j=0}^{k-1} \theta^{j} E\|x(j)\|^{2} \\
& \quad+\mu_{2}(\theta-1) \sum_{j=0}^{k-1} \sum_{i=j-\tau_{2}}^{j-1} \theta^{j} E\|x(i)\|^{2}  \tag{38}\\
& \quad+\mu_{3}(\theta-1) \sum_{j=0}^{k-1} \sum_{i=j-d_{2}}^{j-1} \theta^{j} E\|x(i)\|^{2} \\
& \quad+\rho \sum_{j=0}^{k-1} \theta^{j+1}\|r\|_{\infty}^{2} .
\end{align*}
$$

Also it is easy to compute that

$$
\begin{aligned}
& \sum_{j=0}^{k-1} \sum_{i=j-\tau_{2}}^{j-1} \theta^{j} E\|x(i)\|^{2} \\
& \leq\left(\sum_{i=-\tau_{2}}^{-1} \sum_{j=0}^{i+\tau_{2}}+\sum_{i=0}^{k-1-\tau_{2}} \sum_{j=i+1}^{i+\tau_{2}}+\sum_{i=k-\tau_{2}}^{k-1} \sum_{j=i+1}^{k-1}\right) \\
& \quad \times \theta^{j} E\|x(i)\|^{2}
\end{aligned}
$$

$$
\begin{align*}
& \leq \tau_{2} \theta^{\tau_{2}} \sup _{s \in\left[-\tau_{2}, 0\right]} E\|x(s)\|^{2}+\tau_{2} \theta^{\tau_{2}} \sum_{i=0}^{k-1} \theta^{i} E\|x(i)\|^{2}, \\
& \sum_{j=0}^{k-1} \sum_{i=j-d_{2}}^{j-1} \theta^{j} E\|x(i)\|^{2} \\
& \leq\left(\sum_{i=-d_{2}}^{-1} \sum_{j=0}^{i+d_{2}}+\sum_{i=0}^{k-1-d_{2}} \sum_{j=i+1}^{i+d_{2}}+\sum_{i=k-d_{2}}^{k-1} \sum_{j=i+1}^{k-1}\right) \\
& \quad \times \theta^{j} E\|x(i)\|^{2} \\
& \leq d_{2} \theta^{d_{2}} \sup _{s \in\left[-d_{2}, 0\right]} E\|x(s)\|^{2}+d_{2} \theta^{d_{2}} \\
& \quad \times \sum_{i=0}^{k-1} \theta^{i} E\|x(i)\|^{2} . \tag{39}
\end{align*}
$$

Substituting (39) into (38) leads to

$$
\begin{align*}
& \theta^{k} E V(k)-E V(0) \\
& \leq \eta_{1}(\theta) \sup _{s \in\left[-\tau_{2}, 0\right]} E\|x(s)\|^{2}+\rho \sum_{j=0}^{k-1} \theta^{j+1}\|r\|_{\infty}^{2}  \tag{40}\\
& \quad+\eta_{2}(\theta) \sum_{i=0}^{k-1} \theta^{i} E\|x(i)\|^{2}+\eta_{3}(\theta) \sum_{i=0}^{k-1} \theta^{i} E\|x(i)\|^{2}
\end{align*}
$$

where $\eta_{1}(\theta)=\mu_{2}(\theta-1) \tau_{2} \theta^{\tau_{2}}+\mu_{3}(\theta-1) d_{2} \theta^{d_{2}}, \eta_{2}(\theta)=\mu_{2}(\theta-$ 1) $\tau_{2} \theta^{\tau_{2}}+\mu_{1}(\theta-1)-\varepsilon \theta, \eta_{3}(\theta)=\mu_{3}(\theta-1) d_{2} \theta^{d_{2}}+\mu_{1}(\theta-1)-\varepsilon \theta$.

Since $\eta_{2}(1)<0, \eta_{3}(1)<0$, there must exist a positive $\theta_{0}>1$ such that $\eta_{2}\left(\theta_{0}\right)<0, \eta_{3}\left(\theta_{0}\right)<0$. Then we have

EV (k)

$$
\begin{align*}
& \leq \eta_{1}\left(\theta_{0}\right)\left(\frac{1}{\theta_{0}}\right)^{k} \sup _{s \in\left[-\tau_{2}, 0\right]} E\|x(s)\|^{2} \\
& \quad+\left(\frac{1}{\theta_{0}}\right)^{k} E V(0)+\rho \sum_{j=0}^{k-1} \frac{1}{\theta_{0}^{k-j-1}}\|r\|_{\infty}^{2} \\
& \quad+\eta_{2}\left(\theta_{0}\right) \sum_{i=0}^{k-1} \frac{1}{\theta_{0}^{k-i}} E\|x(i)\|^{2}+\eta_{3}\left(\theta_{0}\right) \sum_{i=0}^{k-1} \frac{1}{\theta_{0}^{k-i}} E\|x(i)\|^{2} \\
& \leq\left(\eta_{1}\left(\theta_{0}\right)+\mu_{1}+\mu_{2} \tau_{2}+\mu_{3} d_{2}\right) \\
& \quad \times \sup _{s \in\left[-\max \left\{\tau_{2}, d_{2}\right\}, 0\right]} E\|x(s)\|^{2}+\frac{\rho}{\theta_{0}-1}\|r\|_{\infty}^{2} . \tag{41}
\end{align*}
$$

On the other hand, by (16) we can get

$$
\begin{equation*}
E V(k) \geq \lambda_{\min }(P) E\|x(k)\|^{2} \tag{42}
\end{equation*}
$$

Combining (41) with (42), we have

$$
\begin{align*}
E\|x(k)\|^{2} \leq & \frac{\eta_{1}\left(\theta_{0}\right)+\mu_{1}+\mu_{2} \tau_{2}+\mu_{3} d_{2}}{\lambda_{\min }(P)} \\
& \times \sup _{s \in\left[-\max \left\{\tau_{2}, d_{2}\right\}, 0\right]} E\|x(s)\|^{2}  \tag{43}\\
& +\frac{1}{\lambda_{\text {min }}(P)} \frac{\rho}{\theta_{0}-1}\|r\|_{\infty}^{2} .
\end{align*}
$$

Thus,

$$
\begin{equation*}
\mathbb{E}\|y(k)\|^{2} \leq\|M\|^{2} E\|x(k)\|^{2} \leq N_{1}+N_{2}\|r\|_{\infty}^{2}, \tag{44}
\end{equation*}
$$

where $N_{1}=\|M\|^{2}\left(\left(\eta_{1}\left(\theta_{0}\right)+\mu_{1}+\mu_{2} \tau_{2}+\mu_{3} d_{2}\right) / \lambda_{\text {min }}(P)\right)$ $\sup _{s \in\left[-\max \left\{\tau_{2}, d_{2}\right\}, 0\right]} E\|x(s)\|^{2}, \quad N_{2}=\left(1 / \lambda_{\min }(P)\right)\left(\rho /\left(\theta_{0}-\right.\right.$ 1)) $\|M\|^{2}$. By Definition 5, the nonlinear discrete-time stochastic control system (1) is BIBO stability in mean square. This completes the proof.

If the stochastic term $\omega(K)$ is removed in (1), then the following results can be obtained.

Theorem 10. For given positive integers $\tau_{1}, \tau_{2}, d_{1}$, and $d_{2}$, under Assumption 2, the nonlinear discrete-time stochastic control system (1) with the controller (6) is BIBO stabilization in mean square, if there exist symmetric positive-definite matrix $P, R, Q_{i}, i=1,2, \ldots, 5, Z_{1}, Z_{2}$, and $X$ with appropriate dimensional positive-definite diagonal matrices $\Lambda$ and two positive constants $\zeta$ and $\lambda^{*}$ such that the following two LMIs hold:

$$
P+2\left(\tau^{* 2} Z_{1}+\tilde{\tau}^{2} Z_{2}\right) \leq \lambda^{*} I,
$$

$\Xi$

$$
=\left(\begin{array}{ccccccccc}
\Xi_{11} & \Xi_{12} & Z_{1} & 0 & 0 & \Gamma_{2} \Lambda & \Xi_{17} & \Xi_{18} & \sqrt{2} X \\
* & \Xi_{22} & \Xi_{23} & 0 & \Xi_{25} & 0 & B^{T} D & B^{T} & 0 \\
* & * & \Xi_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & -Q_{2} & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & \Xi_{55} & 0 & 0 & 0 & 0 \\
* & * & * & * & * & \Xi_{66} & 0 & 0 & 0 \\
* & * & * & * & * & * & \Xi_{77} & \lambda^{*} D^{T} & 0 \\
* & * & * & * & * & * & * & -\zeta I & 0 \\
* & * & * & * & * & * & * & * & -\lambda^{*} I
\end{array}\right)
$$

$$
\begin{equation*}
\leq 0 \tag{45}
\end{equation*}
$$

where

$$
\begin{aligned}
\Xi_{11}= & Q_{1}+Q_{2}+Q_{3}+a\left(Q_{3}+Q_{4}\right)-Z_{1} \\
& +Q_{5}-\Gamma_{1} \Lambda+\zeta I+2 \tau^{* 2} Z_{1}+2 \widetilde{\tau}^{2} Z_{2} \\
& +2 \lambda^{*} \alpha_{1} I+\alpha_{1} \zeta I-P+2 \lambda^{*} A^{T} A \\
& -2 A^{T} X-2 X^{T} A, \\
\Xi_{22}= & 2 \lambda^{*} B^{T} B-Q_{4}-2 Z_{2}-\beta Z_{2} \\
& +2 \lambda^{*} \alpha_{2} I+\alpha_{2} \zeta I-(1-\beta) Z_{2},
\end{aligned}
$$

$$
\begin{align*}
& \Xi_{12}=\lambda^{*} A^{T} B-X^{T} B \\
& \Xi_{17}=\lambda^{*} A^{T} D-X^{T} D \\
& \Xi_{18}=\lambda^{*} A^{T}-X^{T} \\
& \Xi_{25}=Z_{2}+(1-\beta) Z_{2}, \\
& \Xi_{23}=Z_{2}+\beta Z_{2} \\
& \Xi_{33}=-Q_{1}-Z_{1}-Z_{2}-\beta Z_{2}, \\
& \Xi_{55}=-Q_{5}-Z_{2}-(1-\beta) Z_{2} \\
& \Xi_{66}=b R-\Lambda \\
& \Xi_{77}=2 \lambda^{*} D^{T} D-\frac{1}{c} R \\
& X=-\lambda^{*} C K . \tag{46}
\end{align*}
$$

Proof. The proof is straightforward and hence omitted.
Corollary 11. System (1) is also stabilization in mean square when all the conditions in Theorems 9 and 10 are satisfied, if the bounded input $r(t)=0$ in (6).

Remark 12. In this paper, a novel BIBO stability criterion for system (1) is derived by checking the variation of derivatives of the Lyapunov-Krasovskii functionals for each subinterval. It is different from [27-32], which checked the variation of the Lyapunov functional in the whole variation interval of the delay.

Remark 13. The BIBO stabilization criteria for discrete-time systems have been investigated in the recently reported paper [28]. However, the stochastic disturbances and nonlinear perturbations have not been taken into account in the control systems. In [28], the time delay is constant time, which is a special case of this paper when $\tau_{1}=\tau_{2}$.

Remark 14. The mean square stabilization conditions in Theorem 9 in this paper depend on the time-delays upper bounds and the lower bounds, time-delays interval, and timedelay interval segmentation point and relate to the delays themselves.

Remark 15. In [33], the time-delay interval is divided into two equal subintervals; the interval segmentation point is midpoint. In this paper, the time-delay interval is divided into two any subintervals; the interval segmentation point is any point in the time-delay interval.

## 4. An Example

In this section, a numerical example will be presented to show the validity of the main results derived in Section 3.

Table 1: For $\tau(k)=1,2,3,4,5, \beta$.

| $\tau(k)$ | 1 | 2 | 3 | 4 | 5 |
| :--- | :--- | :--- | :---: | :---: | :---: |
| $\beta$ | 0 | 1 | $1 / 2$ | $1 / 2$ | 0 |

Example 1. As a simple application of Theorem 9, consider the stochastic control system (1) with the control law (6); the parameters are given by

$$
\begin{array}{lc}
A=\left(\begin{array}{cc}
-0.1 & 0 \\
0.1 & -0.2
\end{array}\right), & B=\left(\begin{array}{cc}
-0.1 & 0.1 \\
-0.1 & 0.1
\end{array}\right) \\
C=\left(\begin{array}{cc}
0.1 & 0.1 \\
0.5 & 0.3
\end{array}\right), & D=\left(\begin{array}{cc}
0.1 & 0.1 \\
0 & 0.2
\end{array}\right) \tag{47}
\end{array}
$$

$G=0.001 I, H=0.02 I, f=[0.1 x(k), \sqrt{0.2} x(k-\tau(k))]^{T}$, $h_{1}(s)=\sin (0.2 s)-0.6 \cos (s), h_{2}(s)=\tanh (-0.4 s), \tau_{1}=1$, $\tau_{2}=5, d_{1}=2, d_{2}=7$.

It is easy to verify that $a=5, \tau^{*}=3, d^{*}=4, \tilde{\tau}=2$, and

$$
\begin{align*}
& \Gamma_{1}=\left(\begin{array}{cc}
-0.64 & 0 \\
0 & 0
\end{array}\right), \quad \Gamma_{2}=\left(\begin{array}{cc}
0 & 0 \\
0 & -0.2
\end{array}\right) \\
& \tau^{*}=\frac{\tau_{1}+\tau_{2}}{2}-\frac{\min \left\{(-1)^{\tau_{1}+\tau_{2}}, 0\right\}}{2}, \\
& d^{*}=\frac{\tau_{1}+\tau_{2}}{2}+\frac{\min \left\{(-1)^{\tau_{1}+\tau_{2}}, 0\right\}}{2},  \tag{48}\\
& b= \begin{cases}9, & d_{1} \leq d(k) \leq d^{*} \\
20, & d^{*}<d(k) \leq d_{2}\end{cases} \\
& c= \begin{cases}4, & d_{1} \leq d(k) \leq d^{*} \\
7, & d^{*}<d(k) \leq d_{2}\end{cases}
\end{align*}
$$

Meanwhile, the corresponding values of $\beta$ for various $\tau(k)$ are listed in Table 1.

By using the MATLAB LMI Toolbox, we solve LMIs (13), (14) and obtain six groups of feasible solutions; we list one case as follows.

When $\beta=1, b=9, c=4$,

$$
\begin{array}{ll}
P=\left(\begin{array}{cc}
145.0684 & -3.2651 \\
-3.2651 & 147.7799
\end{array}\right), & Z_{1}=\left(\begin{array}{cc}
0.0478 & 0.0298 \\
0.0298 & 0.0230
\end{array}\right), \\
Z_{2}=\left(\begin{array}{ll}
0.3443 & 0.2142 \\
0.2142 & 0.1663
\end{array}\right), & Q_{1}=\left(\begin{array}{cc}
2.8401 & 1.7808 \\
1.7808 & 1.3624
\end{array}\right), \\
Q_{2}=\left(\begin{array}{ll}
3.7464 & 2.3462 \\
2.3462 & 1.7985
\end{array}\right), & Q_{3}=\left(\begin{array}{cc}
0.2731 & 0.1713 \\
0.1713 & 0.1309
\end{array}\right), \\
Q_{4}=\left(\begin{array}{ll}
15.3027 & -6.8645 \\
-6.8645 & 15.5746
\end{array}\right), & Q_{5}=\left(\begin{array}{ll}
3.1500 & 1.9740 \\
1.9740 & 1.5118
\end{array}\right), \\
R=\left(\begin{array}{cc}
128.2646 & -92.1100 \\
-92.1100 & 239.0910
\end{array}\right), & N=\left(\begin{array}{cc}
0.4512 & 0 \\
0 & 34.3341
\end{array}\right), \\
X=\left(\begin{array}{cc}
13.3495 & 3.1121 \\
3.1121 & 0.7692
\end{array}\right), & K=\left(\begin{array}{cc}
-1.4212 & -0.3303 \\
0.5268 & 0.1218
\end{array}\right), \tag{49}
\end{array}
$$

and $\zeta=82.0259, \lambda^{*}=150.2996$.
The system (1) exhibits stabilization in mean square behavior as shown in Figure 1.


## 5. Conclusions

In this paper, we have derived some conditions for the BIBO stabilization in mean square for a class of discretetime stochastic control systems with mixed time-varying delays. The results have been obtained by constructing a novel Lyapunov-Krasovskii function. The conditions are expressed in the forms of linear matrix inequalities, which can be easily checked by using MATLAB LMI Toolbox. A numerical example is given to illustrate the validity of the obtained results.

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## Research Article

# On the Solvability of Caputo $q$-Fractional Boundary Value Problem Involving $p$-Laplacian Operator 

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#### Abstract

We consider the model of a Caputo $q$-fractional boundary value problem involving $p$-Laplacian operator. By using the Banach contraction mapping principle, we prove that, under some conditions, the suggested model of the Caputo $q$-fractional boundary value problem involving $p$-Laplacian operator has a unique solution for both cases of $0<p<1$ and $p>2$. It is interesting that in both cases solvability conditions obtained here depend on $q, p$, and the order of the Caputo $q$-fractional differential equation. Finally, we illustrate our results with some examples.


## 1. Introduction

In this section we will give some basic definitions and results that will be needed in the sequel. For more details about the theory of $q$-calculus, fractional calculus, and $q$-fractional calculus, we refer readers to [1-10].

Let $q \in(0,1)$ be a fixed real number. Then for any $\alpha \in \mathbb{R}$,

$$
\begin{equation*}
[\alpha]_{q}:=\frac{1-q^{\alpha}}{1-q} \tag{1}
\end{equation*}
$$

The $q$-binomial function is defined for all $n \in \mathbb{N}$ as

$$
\begin{gather*}
(t-s)_{q}^{n}=\prod_{k=0}^{n-1}\left(t-q^{k} s\right)  \tag{2}\\
(t-s)_{q}^{\beta}=t^{\beta} \prod_{i=0}^{\infty}\left(\frac{1-(s / t) q^{i}}{1-(s / t) q^{i+\alpha}}\right)
\end{gather*}
$$

where $\beta$ is not a positive integer. It is easy to see that

$$
\begin{equation*}
(a t-a s)_{q}^{\beta}=a^{\beta}(t-s)_{q}^{\beta} . \tag{3}
\end{equation*}
$$

The $q$-analog of Euler's gamma function is denoted by $\Gamma_{q}(t)$ and defined as

$$
\begin{equation*}
\Gamma_{q}(t)=\frac{(1-q)_{q}^{t-1}}{(1-q)^{t-1}}, \quad t>0 \tag{4}
\end{equation*}
$$

The following theorem will be used to compare values of $\Gamma(t)$, the usual gamma function, with values of $\Gamma_{q}(t)$ for a fixed $q \in$ $(0,1)$.

Theorem 1 (see [11]). For $0<r<q<1$, one has

$$
\begin{gather*}
\Gamma_{r}(t) \leq \Gamma_{q}(t) \leq \Gamma(t), \quad \text { for } 0<t \leq 1 \text { or } t \geq 2,  \tag{5}\\
\Gamma(t) \leq \Gamma_{q}(t) \leq \Gamma_{r}(t), \quad \text { for } 1 \leq t \leq 2 .
\end{gather*}
$$

It is known that for $0<q<1$,

$$
\mathbb{T}_{q}=\left\{q^{n} ; n \in \mathbb{Z}\right\} \cup\{0\}
$$

$$
\begin{equation*}
\mathbb{T}_{q}^{\alpha}=\left\{q^{n+\alpha} ; n \in \mathbb{Z}\right\} \cup\{0\}, \quad \alpha \in \mathbb{R}^{+} \cup\{0\} \tag{6}
\end{equation*}
$$

The nabla $q$-derivative of the function $f: \mathbb{T}_{q} \rightarrow \mathbb{R}$ is defined by

$$
\begin{equation*}
\nabla_{q} f(s)=\frac{f(s)-f(q s)}{(1-q) s}, \quad s \in \mathbb{T}_{q}-\{0\} \tag{7}
\end{equation*}
$$

The nabla $q$-integral of $f$ is defined by

$$
\begin{equation*}
\int_{0}^{s} f(t) \nabla_{q} t=(1-q) s \sum_{k=0}^{\infty} q^{k} f\left(s q^{k}\right) . \tag{8}
\end{equation*}
$$

Jackson in [12] and Thomae in [13] showed that the $q$-beta function, which is defined by

$$
\begin{equation*}
B_{q}(t, s)=\frac{\Gamma_{q}(t) \Gamma_{q}(s)}{\Gamma_{q}(t+s)}, \tag{9}
\end{equation*}
$$

has the following $q$-integral representation:

$$
\begin{equation*}
B_{q}(t, s)=\int_{0}^{1} \tau^{t-1}(1-q \tau)_{q}^{s-1} \nabla_{q} \tau, \quad t, s>0 \tag{10}
\end{equation*}
$$

The fundamental theorem of $q$-calculus states that

$$
\begin{equation*}
\nabla_{q} \int_{0}^{s} f(t) \nabla_{q} t=f(s) \tag{11}
\end{equation*}
$$

and if $f$ is continuous at 0 , then

$$
\begin{equation*}
\int_{0}^{s} \nabla_{q} f(t) \nabla_{q} t=f(s)-f(0) . \tag{12}
\end{equation*}
$$

Moreover,

$$
\begin{equation*}
\nabla_{q} \int_{0}^{t} f(t, s) \nabla_{q} s=\int_{0}^{t} \nabla_{q} f(t, s) \nabla_{q} s+f(q t, t) \tag{13}
\end{equation*}
$$

where the derivative is applied with respect to $t$.
The nabla $q$-fractional derivative of $(t-s)_{q}^{\alpha}$ with respect to $t$ and for all $\alpha \in \mathbb{R}$ is given by

$$
\begin{equation*}
\nabla_{q}(t-s)_{q}^{\alpha}=\frac{1-q^{\alpha}}{1-q}(t-s)_{q}^{\alpha-1} \tag{14}
\end{equation*}
$$

Moreover, the $q$-fractional integral of order $\alpha \neq 0,-1,-2, \ldots$ is defined by

$$
\begin{equation*}
{ }_{q} I_{0}^{\alpha} f(t)=\frac{1}{\Gamma_{q}(\alpha)} \int_{0}^{t}(t-q s)_{q}^{\alpha-1} f(s) \nabla_{q} s \tag{15}
\end{equation*}
$$

The $\alpha$-order Caputo $q$-fractional derivative of a function $f$ is defined by

$$
\begin{equation*}
{ }_{q} C_{0}^{\alpha} f(t)=\frac{1}{\Gamma_{q}(n-\alpha)} \int_{0}^{t}(t-q s)^{n-\alpha-1} \nabla_{q}^{n} f(s) \nabla_{q} s \tag{16}
\end{equation*}
$$

where $n=[\alpha]+1$ and $[\alpha]$ denotes the greatest integer less than or equal to $\alpha$.

The following lemma enables us to transfer Caputo $q$-fractional differential equations into an equivalent $q$-fractional integral equation.

Lemma 2 (see [3]). Assume that $\alpha>0$ and $f$ is defined on a suitable domain. Then

$$
\begin{equation*}
{ }_{q} I_{0 q}^{\alpha} C_{0}^{\alpha} f(t)=f(t)-\sum_{i=0}^{n-1} \frac{t^{k}}{\Gamma_{q}(k+1)} \nabla_{q}^{k} f(0) \tag{17}
\end{equation*}
$$

and if $0<\alpha \leq 1$, then

$$
\begin{equation*}
{ }_{q} I_{0 q}^{\alpha} C_{0}^{\alpha} f(t)=f(t)-f(0) . \tag{18}
\end{equation*}
$$

On the other hand the operator $\varphi_{p}(s)=|s|^{p-2} s$, where $p>1$ is called the $p$-Laplacian operator. It is easy to see that $\varphi_{p}^{-1}=\varphi_{r}$, where $(1 / p)+(1 / r)=1$. The following properties of $p$-Laplacian operator will be used in the rest of the paper.
(P1) if $1<p<2, x y>0$, and $|x|,|y| \geq m>0$, then $\left|\varphi_{p}(x)-\varphi_{p}(y)\right| \leq(p-1) m^{p-2}|x-y| ;$
(P2) if $p \geq 2$ and $|x|,|y| \leq M$ then, $\left|\varphi_{p}(x)-\varphi_{p}(y)\right| \leq$ $(p-1) M^{p-2}|x-y|$.

## 2. A Model of Caputo $q$-Fractional Boundary Value Problem Involving p-Laplacian Operator

In this paper, our main aim is to prove the existence and uniqueness of the solution for the following Caputo $q$-fractional boundary value problem involving the $p$-Laplacian operator:

$$
\begin{gather*}
\nabla_{q}\left(\varphi_{p}\left({ }_{q} C_{0}^{\alpha} x(t)\right)\right)=f(t, x(t)), \\
\nabla_{q}^{k} x(0)=0, \quad \text { for } k=2,3, \ldots, n-1,  \tag{19}\\
x(0)=a_{0} x(1), \\
\nabla_{q} x(0)=a_{1} \nabla_{q} x(1),
\end{gather*}
$$

where $a_{0}, a_{1} \neq 1,1<\alpha \in \mathbb{R}$, and $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$.
Note that, the boundary value problem given in (19) is antiperiodic for $a_{0}, a_{1}=-1$.

In the following lemma we obtain a $q$-integral equation which is equivalent to the Caputo $q$-fractional boundary value problem given in (19).

Lemma 3. Assume that $\alpha>1, a_{0}, a_{1} \neq 1$, and $h \in C([0,1])$. Then

$$
\begin{gather*}
\nabla_{q}\left(\varphi_{p}\left({ }_{q} C_{0}^{\alpha} x(t)\right)\right)=h(t), \\
\nabla_{q}^{k} x(0)=0, \quad \text { for } k=2,3, \ldots, n-1,  \tag{20}\\
x(0)=a_{0} x(1), \\
\nabla_{q} x(0)=a_{1} \nabla_{q} x(1)
\end{gather*}
$$

are equivalent to the following $q$-integral equation:

$$
\begin{align*}
x(t)= & b_{0}^{q} \int_{0}^{t}(t-q \tau)_{q}^{\alpha-1} \varphi_{r}\left(\int_{0}^{\tau} h(s) \nabla_{q} s\right) \nabla_{q} \tau \\
& +b_{1}^{q} \int_{0}^{1}(1-q \tau)_{q}^{\alpha-1} \varphi_{r}\left(\int_{0}^{\tau} h(s) \nabla_{q} s\right) \nabla_{q} \tau  \tag{21}\\
& +b_{2}^{q}(t) \int_{0}^{1}(1-q \tau)_{q}^{\alpha-2} \varphi_{r}\left(\int_{0}^{\tau} h(s) \nabla_{q} s\right) \nabla_{q} \tau
\end{align*}
$$

where $b_{0}^{q}=1 /\left(\Gamma_{q}(\alpha)\right), b_{1}^{q}=a_{0} /\left(\Gamma_{q}(\alpha)\left(1-a_{0}\right)\right)$, and $b_{2}^{q}(t)=$ $\left(a_{1}\left(t+a_{0}(1-t)\right)\right) /\left(\Gamma_{q}(\alpha-1)\left(1-a_{0}\right)\left(1-a_{1}\right)\right)$.

Proof. Using (20) and the fact that $\varphi_{p}\left({ }_{q} C_{0}^{\alpha} x(0)\right)=0$, we have

$$
\begin{equation*}
\varphi_{p}\left({ }_{q} C_{0}^{\alpha} x(t)\right)=\int_{0}^{t} h(s) \nabla_{q} s \tag{22}
\end{equation*}
$$

or equivalently,

$$
\begin{equation*}
{ }_{q} C_{0}^{\alpha} x(t)=\varphi_{r}\left(\int_{0}^{t} h(s) \nabla_{q} s\right) . \tag{23}
\end{equation*}
$$

Applying $q$-fractional integral operator ${ }_{q} I_{0}^{\alpha}$ to both sides and using Lemma 2, we get

$$
\begin{align*}
x(t) & -\sum_{k=0}^{n-1} \frac{t^{k}}{\Gamma_{q}(k+1)} \nabla_{q}^{k} x(0)  \tag{24}\\
& =\frac{1}{\Gamma_{q}(\alpha)} \int_{0}^{t}(t-q \tau)_{q}^{\alpha-1} \varphi_{r}\left(\int_{0}^{\tau} h(s) \nabla_{q} s\right) \nabla_{q} \tau .
\end{align*}
$$

Using $\nabla_{q}^{k} x(0)=0$, for $k=2,3, \ldots,[\alpha]-1$ in (24), we obtain

$$
\begin{align*}
x(t)= & \frac{1}{\Gamma_{q}(\alpha)} \int_{0}^{t}(t-q \tau)_{q}^{\alpha-1} \varphi_{r}\left(\int_{0}^{\tau} h(s) \nabla_{q} s\right) \nabla_{q} \tau  \tag{25}\\
& +x(0)+t \nabla_{q} x(0)
\end{align*}
$$

According to (13) and (14), we have

$$
\begin{align*}
\nabla_{q} x(t)= & \frac{1}{\Gamma_{q}(\alpha-1)} \int_{0}^{t}(t-q \tau)_{q}^{\alpha-2} \varphi_{r}\left(\int_{0}^{\tau} h(s) \nabla_{q} s\right) \nabla_{q} \tau \\
& +\nabla_{q} x(0) \tag{26}
\end{align*}
$$

Taking $t=1$ in both sides of (25) and (26), we get

$$
\begin{align*}
x(1)= & \frac{1}{\Gamma_{q}(\alpha)} \int_{0}^{1}(1-q \tau)_{q}^{\alpha-1} \varphi_{r}\left(\int_{0}^{\tau} h(s) \nabla_{q} s\right) \nabla_{q} \tau \\
& +x(0)+\nabla_{q} x(0), \\
\nabla_{q} x(1)= & \frac{1}{\Gamma_{q}(\alpha-1)} \int_{0}^{1}(1-q \tau)_{q}^{\alpha-2} \varphi_{r}\left(\int_{0}^{\tau} h(s) \nabla_{q} s\right) \nabla_{q} \tau \\
& +\nabla_{q} x(0) . \tag{27}
\end{align*}
$$

Solving equations obtained by the given boundary value conditions $x(0)=a_{0} x(1)$ and $\nabla_{q} x(0)=a_{1} \nabla_{q} x(1)$, it follows that

$$
\begin{aligned}
\nabla_{q} x(0)= & \frac{a_{1}}{\Gamma_{q}(\alpha-1)\left(1-a_{1}\right)} \\
& \times \int_{0}^{1}(1-q \tau)_{q}^{\alpha-2} \varphi_{r}\left(\int_{0}^{\tau} h(s) \nabla_{q} s\right) \nabla_{q} \tau \\
x(0)= & \frac{a_{0}}{\Gamma_{q}(\alpha)\left(1-a_{0}\right)} \int_{0}^{1}(1-q \tau)_{q}^{\alpha-1} \varphi_{r}\left(\int_{0}^{\tau} h(s) \nabla_{q} s\right) \nabla_{q} \tau
\end{aligned}
$$

$$
\begin{align*}
& +\frac{a_{0}}{\Gamma_{q}(\alpha-1)\left(1-a_{0}\right)\left(1-a_{1}\right)} \\
& \times \int_{0}^{1}(1-q \tau)_{q}^{\alpha-2} \varphi_{r}\left(\int_{0}^{\tau} h(s) \nabla_{q} s\right) \nabla_{q} \tau \tag{28}
\end{align*}
$$

Substituting (28) into (25) gives (21) which completes the proof.

## 3. Solvability of the Caputo $q$-Fractional Boundary Value Problem

This section is devoted to the solvability of the Caputo $q$ fractional boundary value problem given in (19). In the first part we shall prove the existence and uniqueness of the solution, and then we shall illustrate our main results with some examples.

Recall that $C[0,1]$ is a Banach space with the norm $\|x\|=$ $\max _{t \in[0,1]}|x(t)|$. Now consider $T_{i}: C[0,1] \rightarrow C[0,1], i=$ 0,1 with

$$
\begin{align*}
T_{0} x(t) & :=\varphi_{r}\left(\int_{0}^{t} f(s, x(s)) \nabla_{q} s\right) \\
T_{1} x(t)= & b_{0}^{q} \int_{0}^{t}(t-q \tau)_{q}^{\alpha-1} x(\tau) \nabla_{q} \tau  \tag{29}\\
& +b_{1}^{q} \int_{0}^{1}(1-q \tau)_{q}^{\alpha-1} x(\tau) \nabla_{q} \tau \\
& +b_{2}^{q}(t) \int_{0}^{1}(1-q \tau)_{q}^{\alpha-2} x(\tau) \nabla_{q} \tau
\end{align*}
$$

Then $T=T_{1} \circ T_{0}$ is a continuous and compact operator.
Theorem 4. Suppose that $1<r<2, a_{0}, a_{1} \neq 1, q \in(0,1)$ is fixed, and the following conditions hold: $\exists \lambda>0,0<\delta<$ $2 /(2-r)$ and $d$ with

$$
\begin{aligned}
0 & <d \\
< & \lambda^{2-r} \frac{\Gamma_{q}(\delta(r-2)+2+\alpha)}{(r-1) \Gamma_{q}(\delta(r-2)+2)} \\
& \times\left[\frac{\left|1-a_{0}\right|\left|1-a_{1}\right|}{\left(\left|1-a_{0}\right|+\left|a_{0}\right|\right)\left(\left|1-a_{1}\right|+\left|a_{1}\right|[\delta(r-2)+\alpha+1]_{q}\right)}\right]
\end{aligned}
$$

such that

$$
\begin{align*}
& {[\delta]_{q} \lambda t^{\delta-1} \leq f(t, x), \quad \text { for any }(t, x) \in(0,1] \times \mathbb{R},}  \tag{31}\\
& |f(t, x)-f(t, y)| \leq d|x-y|, \quad \text { for } t \in[0,1], x, y \in \mathbb{R} \tag{32}
\end{align*}
$$

Then the boundary value problem (19) has a unique solution.

Proof. Inequality given in (31) implies that

$$
\begin{equation*}
\lambda t^{\delta} \leq \int_{0}^{t} f(s, x) \nabla_{q} s, \quad \text { for any }(t, x) \in[0,1] \times \mathbb{R} \tag{33}
\end{equation*}
$$

On the other hand using (P1) and (32), we have

$$
\begin{array}{rl}
\mid T_{0} & x(t)-T_{0} y(t) \mid \\
& =\left|\varphi_{r}\left(\int_{0}^{t} f(s, x(s)) \nabla_{q} s\right)-\varphi_{r}\left(\int_{0}^{t} f(s, y(s)) \nabla_{q} s\right)\right| \\
& \leq(r-1)\left(\lambda t^{\delta}\right)^{r-2}\left|\int_{0}^{t} f(s, x(s)) \nabla_{q} s-\int_{0}^{t} f(s, y(s)) \nabla_{q} s\right| \\
& \leq(r-1)\left(\lambda t^{\delta}\right)^{r-2} \int_{0}^{t}|f(s, x(s))-f(s, y(s))| \nabla_{q} s \\
& \leq d(r-1)\left(\lambda t^{\delta}\right)^{r-2} \int_{0}^{t}|x(s)-y(s)| \nabla_{q} s \\
& \leq d(r-1) \lambda^{r-2} t^{\delta(r-2)+1}\|x-y\| . \tag{34}
\end{array}
$$

Similarly,

$$
\begin{align*}
& |T x(t)-T y(t)| \\
& =\left|T_{1}\left(T_{0} x(t)\right)-T_{1}\left(T_{0} y(t)\right)\right| \\
& =\mid b_{0}^{q} \int_{0}^{t}(t-q \tau)_{q}^{\alpha-1}\left(\left(T_{0} x\right)(\tau)-\left(T_{0} y\right)(\tau)\right) \nabla_{q} \tau \\
& \quad+b_{1}^{q} \int_{0}^{1}(1-q \tau)_{q}^{\alpha-1}\left(\left(T_{0} x\right)(\tau)-\left(T_{0} y\right)(\tau)\right) \nabla_{q} \\
& \quad+b_{2}^{q}(t) \int_{0}^{1}(1-q \tau)_{q}^{\alpha-2}\left(\left(T_{0} x\right)(\tau)-\left(T_{0} y\right)(\tau)\right) \nabla_{q} \tau \mid \tag{35}
\end{align*}
$$

Finally using (34) in (35), we get

$$
\begin{align*}
& |T x(t)-T y(t)| \\
& \qquad \begin{array}{l}
\leq d(r-1) \lambda^{r-2}\|x-y\| \\
\quad \times\left[b_{0}^{q} \int_{0}^{t}(t-q \tau)_{q}^{\alpha-1} \tau^{\delta(r-2)+1} \nabla_{q} \tau\right. \\
\quad+\left|b_{1}^{q}\right| \int_{0}^{1}(1-q \tau)_{q}^{\alpha-1} \tau^{\delta(r-2)+1} \nabla_{q} \tau \\
\quad
\end{array} \quad\left|\left|b_{2}^{q}(t)\right| \int_{0}^{1}(1-q \tau)_{q}^{\alpha-2} \tau^{\delta(r-2)+1} \nabla_{q} \tau\right] .
\end{align*}
$$

Since

$$
\begin{align*}
& \int_{0}^{t}(t-q \tau)_{q}^{\alpha-1} \tau^{\delta(r-2)+1} \nabla_{q} \tau  \tag{37}\\
& \quad=\int_{0}^{1}(1-q \tau)_{q}^{\alpha-1} t^{\delta(r-2)+\alpha+1} \tau^{\delta(r-2)+1} \nabla_{q} \tau,
\end{align*}
$$

we have

$$
\begin{align*}
& |T x(t)-T y(t)| \\
& \leq d(r-1) \lambda^{r-2}\|x-y\| \\
& \times\left[b_{0}^{q} \int_{0}^{1}(1-q \tau)_{q}^{\alpha-1} t^{\delta(r-2)+\alpha+1} \tau^{\delta(r-2)+1} \nabla_{q} \tau\right. \\
& +\left|b_{1}^{q}\right| \int_{0}^{1}(1-q \tau)_{q}^{\alpha-1} \tau^{\delta(r-2)+1} \nabla_{q} \tau \\
& \left.+\left|b_{2}^{q}(t)\right| \int_{0}^{1}(1-q \tau)_{q}^{\alpha-2} \tau^{\delta(r-2)+1} \nabla_{q} \tau\right] \\
& =d(r-1) \lambda^{r-2}\|x-y\| \\
& \times\left[b_{0}^{q} t^{\delta(r-2)+\alpha+1} B_{q}(\delta(r-2)+2, \alpha)\right. \\
& +\left|b_{1}^{q}\right| B_{q}(\delta(r-2)+2, \alpha) \\
& \left.+\left|b_{2}^{q}(t)\right| B_{q}(\delta(r-2)+2, \alpha-1)\right] \\
& =d(r-1) \lambda^{r-2}\|x-y\| \\
& \times\left[b_{0}^{q} t^{\delta(r-2)+\alpha+1} B_{q}(\delta(r-2)+2, \alpha)\right. \\
& +\left|b_{1}^{q}\right| B_{q}(\delta(r-2)+2, \alpha)+\left|b_{2}^{q}(t)\right| \\
& \left.\times \frac{[\delta(r-2)+\alpha+1]_{q}}{[\alpha-1]_{q}} B_{q}(\delta(r-2)+2, \alpha)\right] \\
& \leq d(r-1) \lambda^{r-2}\|x-y\| B_{q}(\delta(r-2)+2, \alpha) \\
& \times\left[b_{0}^{q} t^{\delta(r-2)+\alpha+1}+\left|b_{1}^{q}\right|+\left|b_{2}^{q}(t)\right| \frac{[\delta(r-2)+\alpha+1]_{q}}{[\alpha-1]_{q}}\right] . \tag{38}
\end{align*}
$$

In other words,

$$
\begin{aligned}
& |T x(t)-T y(t)| \\
& \leq d(r-1) \lambda^{r-2}\|x-y\| \frac{\Gamma_{q}(\delta(r-2)+2) \Gamma_{q}(\alpha)}{\Gamma_{q}(\delta(r-2)+2+\alpha)} \\
& \times
\end{aligned} \begin{aligned}
& {\left[\frac{1}{\Gamma_{q}(\alpha)} t^{\delta(r-2)+\alpha+1}+\left|\frac{a_{0}}{\Gamma_{q}(\alpha)\left(1-a_{0}\right)}\right|\right.} \\
&\left.+\left|\frac{a_{1}\left(t+a_{0}(1-t)\right)}{\Gamma_{q}(\alpha-1)\left(1-a_{0}\right)\left(1-a_{1}\right)}\right| \frac{[\delta(r-2)+\alpha+1]_{q}}{[\alpha-1]_{q}}\right] \\
& \leq d(r-1) \lambda^{r-2}\|x-y\| \frac{\Gamma_{q}(\delta(r-2)+2) \Gamma_{q}(\alpha)}{\Gamma_{q}(\delta(r-2)+2+\alpha)} \\
& \times {\left[\frac{1}{\Gamma_{q}(\alpha)}+\frac{\left|a_{0}\right|}{\Gamma_{q}(\alpha)\left|1-a_{0}\right|}\right.} \\
&\left.\quad+\frac{\left|a_{1}\right|\left(\left|a_{0}\right|+\left|1-a_{0}\right|\right)[\delta(r-2)+\alpha+1]_{q}}{\Gamma_{q}(\alpha)\left|1-a_{0}\right|\left|1-a_{1}\right|}\right]
\end{aligned}
$$

$$
\begin{align*}
& \leq d(r-1) \lambda^{r-2}\|x-y\| \frac{\Gamma_{q}(\delta(r-2)+2) \Gamma_{q}(\alpha)}{\Gamma_{q}(\delta(r-2)+2+\alpha)} \\
& \times \\
& \quad+\left[\left(\left|1-a_{0}\right|\left|1-a_{1}\right|+\left|a_{0}\right|\left|1-a_{1}\right|\right.\right. \\
& \left.\quad \times\left(\left|a_{0}\right|+\left|1-a_{0}\right|\right)[\delta(r-2)+\alpha+1]_{q}\right) \\
& = \\
& \quad d(r-1) \lambda^{r-2} \frac{\Gamma_{q}(\delta(r-2)+2)}{\Gamma_{q}(\delta(r-2)+2+\alpha)} \\
& \times  \tag{39}\\
& \times\left[\frac{\left(\left|1-a_{0}\right|+\left|a_{0}\right|\right)\left(\left|1-a_{1}\right|+\left|a_{1}\right|[\delta(r-2)+\alpha+1]_{q}\right)}{\left|1-a_{0}\right|\left|1-a_{1}\right|}\right] \\
& \quad \times\|x-y\|=K\|x-y\|,
\end{align*}
$$

where $K=d(r-1) \lambda^{r-2}\left(\Gamma_{q}(\delta(r-2)+2) / \Gamma_{q}(\delta(r-2)+2+\alpha)\right)[((\mid 1-$ $\left.\left.a_{0}\left|+\left|a_{0}\right|\right)\left(\left|1-a_{1}\right|+\left|a_{1}\right|[\delta(r-2)+\alpha+1]_{q}\right)\right) /\left(\left|1-a_{0}\right|\left|1-a_{1}\right|\right)\right]$.

By condition (30), we get $0<K<1$, which implies that $T$ is a contraction. As a consequence of the Banach contraction mapping theorem and Lemma 3, the boundary value problem given in (19) has a unique solution.

Theorem 5. Suppose that $1<r<2, a_{0}, a_{1} \neq 1$, and the following conditions hold for a fixed $q \in(0,1), \exists \lambda>0,0<\delta<$ $2 /(2-r)$, and $d$ with

$$
\begin{align*}
0< & d \\
< & \lambda^{2-r} \frac{\Gamma_{q}(\delta(r-2)+2+\alpha)}{(r-1) \Gamma_{q}(\delta(r-2)+2)} \\
& \times\left[\frac{\left|1-a_{0}\right|\left|1-a_{1}\right|}{\left(\left|1-a_{0}\right|+\left|a_{0}\right|\right)\left(\left|1-a_{1}\right|+\left|a_{1}\right|[\delta(r-2)+\alpha+1]_{q}\right)}\right] \tag{40}
\end{align*}
$$

such that

$$
\begin{gather*}
f(t, x) \leq-[\delta]_{q} \lambda t^{\delta-1}, \quad \text { for any }(t, x) \in(0,1] \times \mathbb{R}, \\
|f(t, x)-f(t, y)| \leq d|x-y|, \quad \text { for } t \in[0,1], x, y \in \mathbb{R} . \tag{41}
\end{gather*}
$$

Then the boundary value problem (19) has a unique solution.
Remark 6. When $q \rightarrow 1$, Theorems 4 and 5 reduce to Theorems 3.1 and 3.2 of [14].

Theorem 7. Suppose that $r>2, a_{0}, a_{1} \neq 1$, and the following conditions hold for a fixed $q \in(0,1)$. There exists a nonnegative function $g(x) \in L[0,1]$ with $M:=\int_{0}^{1} g(\tau) \nabla_{q} \tau \geq 0$ such that

$$
\begin{equation*}
|f(t, x)| \leq g(t), \quad \text { for any }(t, x) \in[0,1] \times \mathbb{R} \tag{42}
\end{equation*}
$$

and there exists a constant $d$ with

$$
\begin{align*}
& 0<d \\
&<\frac{\Gamma_{q}(\alpha+2)}{(r-1) M^{r-2}}  \tag{43}\\
& \times\left[\frac{\left|1-a_{0}\right|\left|1-a_{1}\right|}{\left(\left|1-a_{0}\right|+\left|a_{0}\right|\right)\left(\left|1-a_{1}\right|+\left|a_{1}\right|[\alpha+1]_{q}\right)}\right] \\
&|f(t, x)-f(t, y)| \leq d|x-y|, \quad \text { for } t \in[0,1], x, y \in \mathbb{R} . \tag{44}
\end{align*}
$$

Then the boundary value problem (19) has a unique solution.
Proof. By (42), we can get that

$$
\begin{equation*}
\int_{0}^{t}|f(\tau, x(\tau))| \nabla_{q} \tau \leq \int_{0}^{1} g(\tau) \nabla_{q} \tau=M \tag{45}
\end{equation*}
$$

for all $t \in[0,1]$. By the definition of $T_{0}$, we have

$$
\begin{align*}
& \left|T_{0} x(t)-T_{0} y(t)\right| \\
& \quad=\left|\varphi_{r}\left(\int_{0}^{t} f(s, x(s)) \nabla_{q} s\right)-\varphi_{r}\left(\int_{0}^{t} f(s, y(s)) \nabla_{q} s\right)\right| \tag{46}
\end{align*}
$$

Using (P2) and (45) gives

$$
\begin{align*}
\mid T_{0} x & (t)-T_{0} y(t) \mid \\
\leq & (r-1) M^{r-2} \\
& \times\left|\int_{0}^{t} f(s, x(s)) \nabla_{q} s-\int_{0}^{t} f(s, y(s)) \nabla_{q} s\right| \\
\leq & (r-1) M^{r-2}  \tag{47}\\
& \times \int_{0}^{t}|f(s, x(s))-f(s, y(s))| \nabla_{q} s \\
\leq & d(r-1) M^{r-2} \int_{0}^{t}|x(s)-y(s)| \nabla_{q} s \\
\leq & d(r-1) M^{r-2} t\|x-y\| .
\end{align*}
$$

Therefore,

$$
\begin{aligned}
& |T x(t)-T y(t)| \\
& =\left|T_{1}\left(T_{0} x(t)\right)-T_{1}\left(T_{0} y(t)\right)\right| \\
& =\mid b_{0}^{q} \int_{0}^{t}(t-q \tau)_{q}^{\alpha-1}\left(\left(T_{0} x\right)(\tau)-\left(T_{0} y\right)(\tau)\right) \nabla_{q} \tau \\
& \quad+b_{1}^{q} \int_{0}^{1}(1-q \tau)_{q}^{\alpha-1}\left(\left(T_{0} x\right)(\tau)-\left(T_{0} y\right)(\tau)\right) \nabla_{q} \\
& \quad+b_{2}^{q}(t) \int_{0}^{1}(1-q \tau)_{q}^{\alpha-2}\left(\left(T_{0} x\right)(\tau)-\left(T_{0} y\right)(\tau)\right) \nabla_{q} \tau \mid
\end{aligned}
$$

$$
\begin{align*}
& \leq d(r-1) M^{r-2}\|x-y\| \\
& \quad \times\left[\frac{1}{\Gamma_{q}(\alpha)} \int_{0}^{t}(t-q \tau)_{q}^{\alpha-1} \tau \nabla_{q} \tau\right. \\
& \quad+\frac{\left|a_{0}\right|}{\Gamma_{q}(\alpha)\left|1-a_{0}\right|} \int_{0}^{1}(1-q \tau)_{q}^{\alpha-1} \tau \nabla_{q} \tau \\
& \left.\quad+\frac{\left|a_{1}\left(t+a_{0}(1-t)\right)\right|}{\Gamma_{q}(\alpha-1)\left|1-a_{0}\right|\left|1-a_{1}\right|} \int_{0}^{1}(1-q \tau)_{q}^{\alpha-2} \tau \nabla_{q} \tau\right] . \tag{48}
\end{align*}
$$

Since

$$
\begin{equation*}
\int_{0}^{t}(t-q \tau)_{\mathrm{q}}^{\alpha-1} \tau \nabla_{q} \tau=\int_{0}^{1} t^{\alpha+1}(1-q \tau)_{q}^{\alpha-1} \tau \nabla_{q} \tau \tag{49}
\end{equation*}
$$

we have

$$
\begin{align*}
& |T x(t)-T y(t)| \\
& \qquad \begin{array}{l}
\leq d(r-1) M^{r-2}\|x-y\| \\
\quad \times\left[\frac{1}{\Gamma_{q}(\alpha)} \int_{0}^{1} t^{\alpha+1}(1-q \tau)_{q}^{\alpha-1} \tau \nabla_{q} \tau\right. \\
\quad+\frac{\left|a_{0}\right|}{\Gamma_{q}(\alpha)\left|1-a_{0}\right|} \int_{0}^{1}(1-q \tau)_{q}^{\alpha-1} \tau \nabla_{q} \tau \\
\left.\quad+\frac{\left|a_{1}\right|\left(\left|a_{0}\right|+t\left|1-a_{0}\right|\right)}{\Gamma_{q}(\alpha-1)\left|1-a_{0}\right|\left|1-a_{1}\right|} \int_{0}^{1}(1-q \tau)_{q}^{\alpha-2} \tau \nabla_{q} \tau\right] .
\end{array}
\end{align*}
$$

Using $q$-Beta function and the fact that $t \in[0,1]$, we get

$$
\begin{aligned}
& |T x(t)-T y(t)| \\
& \leq d(r-1) M^{r-2}\|x-y\| \\
& \quad \times\left[\frac{1}{\Gamma_{q}(\alpha)} B_{q}(2, \alpha)+\frac{\left|a_{0}\right|}{\Gamma_{q}(\alpha)\left|1-a_{0}\right|} B_{q}(2, \alpha)\right. \\
& \left.\quad+\frac{\left|a_{1}\right|\left(\left|a_{0}\right|+\left|1-a_{0}\right|\right)}{\Gamma_{q}(\alpha-1)\left|1-a_{0}\right|\left|1-a_{1}\right|} \frac{[\alpha+1]_{q}}{[\alpha-1]_{q}} B_{q}(2, \alpha)\right] \\
& \leq d(r-1) M^{r-2} B_{q}(2, \alpha)\|x-y\| \\
& \quad \times\left[\frac{1}{\Gamma_{q}(\alpha)}+\frac{\left|a_{0}\right|}{\Gamma_{q}(\alpha)\left|1-a_{0}\right|}\right. \\
& \left.\quad+\frac{\left|a_{1}\right|\left(\left|a_{0}\right|+\left|1-a_{0}\right|\right)[\alpha+1]_{q}}{\Gamma_{q}(\alpha)\left|1-a_{0}\right|\left|1-a_{1}\right|}\right]
\end{aligned}
$$

$$
\begin{align*}
& \leq d(r-1) M^{r-2} B_{q}(2, \alpha)\|x-y\| \\
& \times \\
& \quad+\left[\left(\left|1-a_{0}\right|\left|1-a_{1}\right|+\left|a_{0}\right|\left|1-a_{1}\right|\right.\right. \\
& \left.\quad+\left|a_{1}\right|\left(\left|a_{0}\right|+\left|1-a_{0}\right|\right)[\alpha+1]_{q}\right) \\
& \left.\quad \times\left(\Gamma_{q}(\alpha)\left|1-a_{0}\right|\left|1-a_{1}\right|\right)^{-1}\right] \\
& \leq \frac{d(r-1) M^{r-2}}{\Gamma_{q}(\alpha+2)} \\
& \times\left[\frac{\left(\left|1-a_{0}\right|+\left|a_{0}\right|\right)\left(\left|1-a_{1}\right|+\left|a_{1}\right|[\alpha+1]_{q}\right)}{\left|1-a_{0}\right|\left|1-a_{1}\right|}\right]  \tag{51}\\
& \quad \times\|x-y\| \leq K\|x-y\|,
\end{align*}
$$

where

$$
\begin{align*}
K= & \frac{d(r-1) M^{r-2}}{\Gamma_{q}(\alpha+2)} \\
& \times\left[\frac{\left(\left|1-a_{0}\right|+\left|a_{0}\right|\right)\left(\left|1-a_{1}\right|+\left|a_{1}\right|[\alpha+1]_{q}\right)}{\left|1-a_{0}\right|\left|1-a_{1}\right|}\right] . \tag{52}
\end{align*}
$$

By condition (43), we get $K<1$ which implies that $T$ is a contraction; therefore boundary value problem given in (19) has a unique solution.

Next, we give some examples to illustrate our results.
Example 8. Consider the following Boundary value problem

$$
\begin{gathered}
\nabla_{q}\left(\varphi_{7 / 3}\left({ }_{q} C_{0}^{3 / 2} x(t)\right)\right) \\
=4 t^{2}\left(2+\cos \left(\frac{\sqrt{\pi} x}{24}+\omega\right)\right), \quad t \in(0,1), \\
\nabla_{q}^{k} x(0)=0, \quad \text { for } k=2,3, \ldots, n-1, \\
x(0)=\frac{1}{2} x(1), \\
\nabla_{q} x(0)=\frac{1}{2} \nabla_{q} x(1)
\end{gathered}
$$

where

$$
\begin{gather*}
p=\frac{7}{3}, \quad \alpha=\frac{3}{2}, \quad \delta=4, \\
a_{0}=\frac{1}{2}, \quad a_{1}=\frac{1}{2} . \tag{54}
\end{gather*}
$$

Then $r=7 / 4$, and take $\delta=4, \lambda=1$, and $d=\sqrt{\pi} / 6$. Using Theorem 1 and the fact that $[3 / 2]_{q}>1$ for any fixed $q \in(0,1)$,
we have

$$
\begin{gather*}
\lambda^{2-r} \frac{\Gamma_{q}(\delta(r-2)+2+\alpha)}{(r-1) \Gamma_{q}(\delta(r-2)+2)} \\
\times\left[\left(\left|1-a_{0}\right|\left|1-a_{1}\right|\right)\right. \\
\times\left(\left(\left|1-a_{0}\right|+\left|a_{0}\right|\right)\right. \\
\left.\left.\times\left(\left|1-a_{1}\right|+\left|a_{1}\right|[\delta(r-2)+\alpha+1]_{q}\right)\right)^{-1}\right] \\
=\frac{2}{3}\left[\frac{[3 / 2]_{q} \Gamma_{q}(3 / 2)}{\left(1+[3 / 2]_{q}\right)}\right]>\frac{2}{3}\left[\frac{[3 / 2]_{q} \Gamma_{q}(3 / 2)}{\left([3 / 2]_{q}+[3 / 2]_{q}\right)}\right] \\
>\frac{1}{3} \Gamma\left(\frac{3}{2}\right)=\frac{\sqrt{\pi}}{6}=d>0, \\
=d(r-1) \lambda^{r-2} \frac{\Gamma_{q}(\delta(r-2)+2)}{\Gamma_{q}(\delta(r-2)+2+\alpha)} \\
\times\left[\frac{\mid\left(1-a_{0}\left|+\left|a_{0}\right|\right)\left(\left|1-a_{1}\right|+\left|a_{1}\right|[\delta(r-2)+\alpha+1]_{q}\right)\right.}{\left|1-a_{0}\right|\left|1-a_{1}\right|}\right] \\
=\frac{\sqrt{\pi}}{4 \Gamma_{q}(5 / 2)}\left[\left(1+\left[\frac{3}{2}\right]_{q}\right)\right] \\
\left.<\frac{\sqrt{\pi}}{4[3 / 2]_{q} \Gamma_{q}(3 / 2)}\left[\left(\left[\frac{3}{2}\right]+\left[\frac{3}{2}\right]\right]_{q}\right)\right] \leq \frac{\sqrt{\pi}}{2 \Gamma(3 / 2)}=1 . \tag{55}
\end{gather*}
$$

Moreover, it can be easily seen that

$$
\begin{align*}
{[\delta]_{q} \lambda t^{\delta-1} } & =[4]_{q} t^{3} \leq 4 t^{2}\left(2+\cos \left(\frac{\sqrt{\pi} x}{24}+\omega\right)\right)  \tag{56}\\
& =f(t, x)
\end{align*}
$$

Finally,

$$
\begin{align*}
&|f(t, x)-f(t, y)| \\
&= \left\lvert\, 4 t^{2}\left(2+\cos \left(\frac{\sqrt{\pi} x}{24}+\omega\right)\right)\right. \\
& \left.-4 t^{2}\left(2+\cos \left(\frac{\sqrt{\pi} y}{24}+\omega\right)\right) \right\rvert\, \\
&= 4 t^{2}\left|\cos \left(\frac{\sqrt{\pi} x}{24}+\omega\right)-\cos \left(\frac{\sqrt{\pi} y}{24}+\omega\right)\right|  \tag{57}\\
& \leq 4\left|\left(\frac{\sqrt{\pi} x}{24}+\omega\right)-\left(\frac{\sqrt{\pi} y}{24}+\omega\right)\right| \\
&= \frac{\sqrt{\pi}}{6}|x-y| .
\end{align*}
$$

Therefore as a consequence of Theorem 4, boundary value problem given in (53) has a unique solution.

Example 9. Consider the following boundary value problem:

$$
\begin{gather*}
\nabla_{q}\left(\varphi_{9 / 4}\left(\varphi_{31 / 15}\left({ }_{q} C_{0}^{3 / 2} x(t)\right)\right)\right) \\
=4 t^{2}\left(2+\cos \left(\frac{\sqrt{\pi} x}{24}+\omega\right)\right), \quad t \in(0,1) \\
\nabla_{q}^{k} x(0)=0, \quad \text { for } k=2,3, \ldots, n-1  \tag{58}\\
x(0)=\frac{1}{2} x(1) \\
\nabla_{q} x(0)=\frac{1}{2} \nabla_{q} x(1)
\end{gather*}
$$

Then

$$
\begin{aligned}
\varphi_{9 / 4}\left(\varphi_{31 / 15}(s)\right) & =\varphi_{9 / 4}\left(|s|^{1 / 15} s\right)=\left||s|^{1 / 15} s\right|^{(9 / 4)-2}|s|^{1 / 15} s \\
& =|s|^{(1 / 15)(1 / 4)}|s|^{1 / 4}|s|^{1 / 15} s \\
& =|s|^{(1 / 60)+(1 / 4)+(1 / 15)} s=|s|^{1 / 3} s \\
& =|s|^{(7 / 3)-2} s=\varphi_{7 / 3}(s) .
\end{aligned}
$$

Therefore boundary value problem given in (58) reduces to the boundary value problem given in (53), and it has a unique solution.

Example 10. Now consider the following antiperiodic boundary value problem:

$$
\begin{gather*}
\nabla_{q}\left(\varphi_{7 / 4}\left({ }_{q} C_{0}^{3 / 2} x(t)\right)\right) \\
=\left(\sin ^{2}\left(\frac{\sqrt{\pi} x}{40}+\omega\right)\right), \quad t \in(0,1) \\
\nabla_{q}^{k} x(0)=0, \quad \text { for } k=2,3, \ldots, n-1,  \tag{60}\\
x(0)=-x(1) \\
\nabla_{q} x(0)=-\nabla_{q} x(1)
\end{gather*}
$$

where

$$
\begin{equation*}
p=\frac{7}{4}, \quad \alpha=\frac{3}{2}, \quad a_{0}=-1, \quad a_{1}=-1 \tag{61}
\end{equation*}
$$

Then $r=7 / 3$, and take $d=\sqrt{\pi} / 20$. Using Theorem 1 and taking $g(t)=1$, we get that

$$
M=1,
$$

$$
\begin{align*}
& \frac{\Gamma_{q}(\alpha+2)}{(r-1) M^{r-2}} \\
& \quad \times\left[\frac{\left|1-a_{0}\right|\left|1-a_{1}\right|}{\left(\left|1-a_{0}\right|+\left|a_{0}\right|\right)\left(\left|1-a_{1}\right|+\left|a_{1}\right|[\alpha+1]_{q}\right)}\right] \\
& \quad=\left[\frac{\Gamma_{q}(7 / 2)}{\left(2+[5 / 2]_{q}\right)}\right]>\left[\frac{[5 / 2]_{q}[3 / 2]_{q} \Gamma_{q}(3 / 2)}{\left(2[5 / 2]_{q}+[5 / 2]_{q}\right)}\right] \\
& \quad=\frac{[3 / 2]_{q} \Gamma_{q}(3 / 2)}{3}>\frac{\Gamma(3 / 2)}{3}=\frac{\sqrt{\pi}}{6}>\frac{\sqrt{\pi}}{20}=d . \tag{62}
\end{align*}
$$

On the other hand,

$$
\begin{align*}
|f(t, x)-f(t, y)| \leq & \left|\sin ^{2}\left(\frac{\sqrt{\pi} x}{40}+\omega\right)-\sin ^{2}\left(\frac{\sqrt{\pi} y}{40}+\omega\right)\right| \\
\leq & \frac{\sqrt{\pi}}{20}|x-y|, \quad \text { for } t \in[0,1], x, y \in \mathbb{R}, \\
K= & \frac{d(r-1) M^{r-2}}{\Gamma_{q}(\alpha+2)} \\
& \times\left[\frac{\left(\left|1-a_{0}\right|+\left|a_{0}\right|\right)\left(\left|1-a_{1}\right|+\left|a_{1}\right|[\alpha+1]_{q}\right)}{\left|1-a_{0}\right|\left|1-a_{1}\right|}\right] \\
= & \frac{\sqrt{\pi}}{20 \Gamma_{q}(7 / 2)}\left[2+\left[\frac{5}{2}\right]_{q}\right] \\
< & \frac{\sqrt{\pi}}{20[5 / 2]_{q}[3 / 2]_{q} \Gamma_{q}(3 / 2)}\left[3\left[\frac{5}{2}\right]_{q}\right] \\
= & \frac{3 \sqrt{\pi}}{20[3 / 2]_{q} \Gamma_{q}(3 / 2)}<\frac{3 \sqrt{\pi}}{20 \Gamma(3 / 2)}=\frac{3}{10}<1 . \tag{63}
\end{align*}
$$

Therefore by Theorem 7, the antiperiodic boundary value problem given in (60) has a unique solution.

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## Research Article

# On Partial Complete Controllability of Semilinear Systems 

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#### Abstract

Many control systems can be written as a first-order differential equation if the state space enlarged. Therefore, general conditions on controllability, stated for the first-order differential equations, are too strong for these systems. For such systems partial controllability concepts, which assume the original state space, are more suitable. In this paper, a sufficient condition for the partial complete controllability of semilinear control system is proved. The result is demonstrated through examples.


## 1. Introduction

A concept of controllability, defined by Kalman [1] in 1960 for finite dimensional control systems, is a property of attaining every point in the state space from every initial state point for a finite time. Further studies on this concept in infinite dimensional spaces demonstrated that it is suitable to consider its two versions: a stronger version of complete controllability and a weaker version of approximate controllability. The reason for these versions was the fact that many infinite dimensional control systems are not completely controllable while they are approximately controllable (see Fattorini [2] and Russell [3]). The necessary and sufficient conditions for complete and approximate controllability concepts are almost completely studied and presented in, for example, Curtain and Zwart [4], Bensoussan [5], Bensoussan et al. [6], Zabczyk [7], Bashirov [8], Klamka [9], and so forth for linear systems; Balachandran and Dauer [10, 11], Klamka [12], Mahmudov [13], Li and Yong [14], and so forth for nonlinear systems; Sakthivel et al. [15-17], Yan [18], and so forth for fractional differential systems; and Ren et al. [19] for differential inclusions.

Recently, in Bashirov et al. [20, 21] the partial controllability concepts were initiated. The idea of these concepts is that some control systems, including higher order differential equations, wave equations, and delay equations, can be written as a first-order differential equation only by enlarging the dimension of the state space. Therefore, the theorems on
controllability, which are formulated for control systems in the form of first-order differential equation, are too strong for them because they involve the enlarged state space. In such cases the partial controllability concepts became preferable, which assume the original state space. The basic controllability conditions for linear systems, including resolvent conditions from Bashirov and Mahmudov [22] and Bashirov and Kerimov [23] (see also [24-26]), are extended to partial controllability concepts by just a replacement of the controllability operator by its partial version.

In this paper our aim is to study the partial complete controllability of semilinear systems. The controllability concepts for semilinear systems are intensively discussed in the literature (see Balachandran and Dauer [10, 11], Klamka [12], Mahmudov [13], Sakthivel et al. [15, 17], and references therein). A basic tool of study in these works is fixed point theorems. In this paper, we also use one of the fixed point theorems, a contraction mapping theorem, and find a sufficient condition for the partial complete controllability of a semilinear control system.

The rest of this paper is organised in the following way. In Section 2 we set the problem, give basic definitions, and motivate the partial controllability concepts by considering a higher order differential equation, a wave equation, and a delay equation. Section 3 contains the proof of the main result. In Section 4, we demonstrate the main result in the examples. Finally, Section 5 contains directions of further research regarding partial controllability concepts.

## 2. Setting the Problem and Motivation

Consider the basic semilinear control system

$$
\begin{equation*}
x_{t}^{\prime}=A x_{t}+B u_{t}+f\left(t, x_{t}, u_{t}\right), \quad 0<t \leq T, x_{0} \in X \tag{1}
\end{equation*}
$$

on the interval $[0, T]$ with $T>0$, where $x$ and $u$ are state and control processes. We assume that the following conditions hold.
(A) $X$ and $U$ are separable Hilbert spaces, $H$ is a closed subspace of $X$, and $L$ is a projection operator from $X$ to $H$;
(B) $A$ is a densely defined closed linear operator on $X$, generating a strongly continuous semigroup $e^{A t}, t \geq$ 0 ;
(C) $B$ is a bounded linear operator from $U$ to $X$;
(D) $f$ is a nonlinear function from $[0, T] \times X \times U$ to $X$, satisfying that
(i) $f$ is continuous on $[0, T] \times X \times U$;
(ii) $f$ is Lipschitz continuous with respect to $x$ and $u$ that is, for all $t \in[0, T], u, v \in U$ and $x, y \in X$,

$$
\begin{equation*}
\|f(t, x, u)-f(t, y, v)\| \leq K(\|x-y\|+\|u-v\|) \tag{2}
\end{equation*}
$$

for some $K \geq 0$;
(E) $U_{\text {ad }}=C(0, T ; U)$ is the space of all continuous functions from $[0, T]$ to $U$.
Define the controllability and $L$-partial controllability operators $Q_{t}$ and $\widetilde{Q}_{t}$ by

$$
\begin{equation*}
Q_{t}=\int_{0}^{t} e^{A s} B B^{*} e^{A^{*} s} d s, \quad \widetilde{Q}_{t}=L Q_{t} L^{*} \tag{3}
\end{equation*}
$$

where $L^{*}$ is the adjoint of $L$. The $L$-partial controllability operator becomes the controllability operator if $L=I$ (the identity operator). We will also assume the following condition;
(F) $\widetilde{Q}_{T}$ is coercive; that is, there is $\gamma>0$ such that $\left\langle\widetilde{Q}_{T} h, h\right\rangle \geq \gamma\|h\|^{2}$ for all $h \in H$.
Note that this condition implies the existence of $\widetilde{Q}_{T}^{-1}$ as a bounded linear operator and $\left\|\widetilde{Q}_{T}^{-1}\right\| \leq 1 / \gamma$. Respectively, the linear system associated with (1) (the case when $f=0$ ) is $L$-partially complete controllable on the interval $[0, T]$ (see, Bashirov et al. [20, 21]).

The above conditions imply the existence of a unique continuous solution of (1) in the mild sense for every $u \in U_{\text {ad }}$ and $x_{0} \in X$ (see Li and Yong [14] and Byszewski [27]); that is, there is a unique continuous function $x$ from $[0, T]$ to $X$ such that

$$
\begin{equation*}
x_{t}=e^{A t} x_{0}+\int_{0}^{t} e^{A(t-s)}\left(B u_{s}+f\left(s, x_{s}, u_{s}\right)\right) d s \tag{4}
\end{equation*}
$$

Let

$$
\begin{equation*}
D_{T}^{x_{0}}=\left\{h \in H: \exists u \in U_{\text {ad }} \text { such that } h=L x_{T}\right\} . \tag{5}
\end{equation*}
$$

Following Bashirov et al. [21], the semilinear control system (1) is said to be $L$-partially complete controllable on $U_{\text {ad }}$ if $D_{T}^{x_{0}}=H$ for all $x_{0} \in X$. Similarly, the semilinear system in (1) is said to be $L$-partially approximate controllable on $U_{\mathrm{ad}}$ if $\overline{D_{T}^{x_{0}}}=H$ for all $x_{0} \in X$, where $\overline{D_{T}^{x_{0}}}$ is the closure of $D_{T}^{x_{0}}$. If $H=X$, these are just well-known complete and approximate controllability concepts, respectively. In this paper, we study the concept of $L$-partial complete controllability.

The reason for studying $L$-partial controllability concepts is that many systems can be written in the form of (1) if the original state space is enlarged. Therefore, suitable controllability concepts for such systems are the $L$-partial controllability concepts with the operator $L$ projecting the enlarged state space to the original one. Here are some examples of such systems, which are discussed in Bashirov [8], Section 3.1.1, in more details.

Example 1. Consider the system

$$
\begin{equation*}
x_{t}^{(n)}=f\left(t, x_{t}, x_{t}^{\prime}, \ldots, x_{t}^{(n-1)}, u_{t}\right), \tag{6}
\end{equation*}
$$

assuming that its state space is the one-dimensional space $\mathbb{R}$. The ordinary controllability concepts for this system are the equality to or denseness in $\mathbb{R}$ of the respective attainable set. We can write this system as the first-order differential equation

$$
\begin{equation*}
y_{t}^{\prime}=A y_{t}+F\left(t, y_{t}, u_{t}\right) \tag{7}
\end{equation*}
$$

if

$$
\begin{gather*}
y_{t}=\left[\begin{array}{c}
x_{t} \\
x_{t}^{\prime} \\
\vdots \\
x_{t}^{(n-2)} \\
x_{t}^{(n-1)}
\end{array}\right], \quad A=\left[\begin{array}{ccccc}
0 & 1 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 0 & 0
\end{array}\right], \\
F(t, y, u)=\left[\begin{array}{c}
0 \\
0 \\
\vdots \\
0 \\
f\left(t, x, x^{\prime}, \ldots, x^{(n-1)}, u\right)
\end{array}\right] \tag{8}
\end{gather*}
$$

The state space of this system is the $n$-dimensional Euclidean space $\mathbb{R}^{n}$ and, respectively, its attainable set is a subset of $\mathbb{R}^{n}$. Therefore, the controllability concepts of the system for $y$ are stronger than those of the system for $x$. But if we define the projection operator $L$ by

$$
L=\left[\begin{array}{lllll}
1 & 0 & \cdots & 0 & 0 \tag{9}
\end{array}\right]: \mathbb{R}^{n} \longrightarrow \mathbb{R}
$$

then the $L$-partial controllability concepts of the system for $y$ become the same as the ordinary controllability concepts of the system for $x$.

Example 2. Consider the nonlinear wave equation

$$
\begin{equation*}
\frac{\partial^{2} x_{t, \theta}}{\partial t^{2}}=\frac{\partial^{2} x_{t, \theta}}{\partial \theta^{2}}+f\left(t, x_{t, \theta}, \frac{\partial x_{t, \theta}}{\partial t}, u_{t}\right) \tag{10}
\end{equation*}
$$

where $x$ is a real-valued function of two variables $t \geq 0$ and $0 \leq \theta \leq 1$. The state space of this system is $L_{2}(0,1)$ (the space of square integrable functions on $[0,1]$ ). This system can be written as the first-order abstract differential equation

$$
\begin{equation*}
y_{t}^{\prime}=A y_{t}+F\left(t, y_{t}, u_{t}\right) \tag{11}
\end{equation*}
$$

if

$$
\begin{gather*}
y_{t}=\left[\begin{array}{c}
x_{t, \theta} \\
\frac{\partial x_{t, \theta}}{\partial t}
\end{array}\right], \quad A=\left[\begin{array}{cc}
0 & I \\
\frac{d^{2}}{d \theta^{2}} & 0
\end{array}\right]  \tag{12}\\
F(t, y, u)=\left[\begin{array}{c}
0 \\
f\left(t, y_{1}, y_{2}, u\right)
\end{array}\right]
\end{gather*}
$$

where $y \in L_{2}(0,1) \times L_{2}(0,1)$. The state space $L_{2}(0,1) \times$ $L_{2}(0,1)$ of the system for $y$ is the enlargement of the state space $L_{2}(0,1)$ of the system for $x$. This is a cost that is paid to bring the wave equation to the form of first-order differential equation. The ordinary controllability concepts for the system (11) are too strong for the system (10). If

$$
L=\left[\begin{array}{ll}
I & 0 \tag{13}
\end{array}\right]: L_{2}(0,1) \times L_{2}(0,1) \longrightarrow L_{2}(0,1)
$$

then $L$-partially controllability concepts of the system for $y$ become ordinary controllability concepts of the system for $x$.

Example 3. Consider the system

$$
\begin{equation*}
x_{t}^{\prime}=f\left(t, x_{t}, \int_{-\varepsilon}^{0} x_{t+\theta} d \theta, u_{t}\right) \tag{14}
\end{equation*}
$$

which contains a simple distributed delay in the nonlinear term, assuming that $x$ is a real-valued function. Then the state space is $\mathbb{R}$. To bring this system to a system without delay, enlarge $\mathbb{R}$ to $\mathbb{R} \times L_{2}(-\varepsilon, 0)$ and define $L_{2}(-\varepsilon, 0)$-valued function

$$
\begin{equation*}
\left[\bar{x}_{t}\right]_{\theta}=x_{t+\theta}, \quad t \geq 0,-\varepsilon \leq \theta \leq 0 \tag{15}
\end{equation*}
$$

Then for

$$
\begin{gather*}
y_{t}=\left[\begin{array}{l}
x_{t} \\
\bar{x}_{t}
\end{array}\right], \quad A=\left[\begin{array}{ll}
0 & 0 \\
0 & \frac{d}{d \theta}
\end{array}\right],  \tag{16}\\
F(t, y, u)=\left[\begin{array}{c}
f(t, x, \bar{x}, u) \\
0
\end{array}\right],
\end{gather*}
$$

the above system can be written as the abstract system

$$
\begin{equation*}
y_{t}^{\prime}=A y_{t}+f\left(t, y_{t}, u_{t}\right) \tag{17}
\end{equation*}
$$

Similar to the previous examples, one can easily observe that the ordinary controllability concepts for the system (17) are too strong for the system (14), but the $L$-partial controllability concepts of the system for $y$ with

$$
L=\left[\begin{array}{ll}
I & 0 \tag{18}
\end{array}\right]: \mathbb{R} \times L_{2}(0,1) \longrightarrow \mathbb{R}
$$

are exactly the ordinary controllability concepts of the system for $x$.

These examples motivate a study of the partial controllability concepts. In this paper it is proved that under the conditions (A)-(F), the system in (1) is $L$-partially complete controllable.

## 3. Main Result

Denote $\widetilde{X}=C(0, T ; X)$. Then $\widetilde{X} \times U_{\text {ad }}$ is a Banach space with the norm

$$
\begin{equation*}
\|(\cdot, \cdot)\|_{\widetilde{X} \times U_{a d}}=\|\cdot\|_{\widetilde{X}}+\|\cdot\|_{U_{a d}} \tag{19}
\end{equation*}
$$

Lemma 4. Under the conditions ( $A$ ), (B), and ( $C$ ),

$$
\begin{equation*}
\left\|Q_{t}\right\| \leq\left\|Q_{T}\right\|, \quad\left\|\widetilde{Q}_{t}\right\| \leq\left\|\widetilde{Q}_{T}\right\|, \quad 0 \leq t \leq T \tag{20}
\end{equation*}
$$

Proof. It is easy to see that $Q_{t}=Q_{t}^{*}$ and $\left\langle Q_{t} x, x\right\rangle \geq 0$ for all $x \in X$. Hence,

$$
\begin{equation*}
\left\|Q_{t}\right\|=\sup _{\|x\|=1}\left\langle Q_{t} x, x\right\rangle \tag{21}
\end{equation*}
$$

Then

$$
\begin{align*}
\left\langle Q_{T} x, x\right\rangle & =\int_{0}^{T}\left\langle e^{A s} B B^{*} e^{A^{*} s} x, x\right\rangle d s \\
& =\left\langle Q_{t} x, x\right\rangle+\int_{t}^{T}\left\langle e^{A s} B B^{*} e^{A^{*} s} x, x\right\rangle d s  \tag{22}\\
& =\left\langle Q_{t} x, x\right\rangle+\int_{t}^{T}\left\|B^{*} e^{A^{*} s} x\right\|^{2} d s \\
& \geq\left\langle Q_{t} x, x\right\rangle
\end{align*}
$$

This implies $\left\|Q_{t}\right\| \leq\left\|Q_{T}\right\|$. The conclusion of the lemma regarding $\widetilde{Q}_{T}$ follows from $\left\langle\widetilde{Q}_{t} x, x\right\rangle=\left\langle Q_{t} L^{*} x, L^{*} x\right\rangle$.

The proof of the following lemma appears in different forms in several papers, for example, Mahmudov [13]. Our proof is a minor modification of them.

Lemma 5. Assume that the conditions $(A)-(F)$ hold and take arbitrary $h \in H$. Then for the operator $G: \widetilde{X} \times U_{a d} \rightarrow \widetilde{X} \times$ $U_{a d}$, defined by

$$
\begin{equation*}
G(y, v)(t)=(Y(t), V(t)), \quad 0 \leq t \leq T \tag{23}
\end{equation*}
$$

where

$$
\begin{align*}
Y(t)= & -Q_{t} e^{A^{*}(T-t)} L^{*} \widetilde{Q}_{T}^{-1} L \int_{0}^{T} e^{A(T-s)} f\left(s, y_{s}, v_{s}\right) d s \\
& +\int_{0}^{t} e^{A(t-s)} f\left(s, y_{s}, v_{s}\right) d s  \tag{24}\\
V(t)= & B^{*} e^{A^{*}(T-t)} L^{*} \widetilde{Q}_{T}^{-1} \\
& \times\left(h-L e^{A T} x_{0}-L \int_{0}^{T} e^{A(T-s)} f\left(s, y_{s}, v_{s}\right) d s\right)
\end{align*}
$$

the following inequality holds:

$$
\begin{align*}
\|G(y, v)-G(z, w)\| \leq & \left(1+\frac{\left\|Q_{T}\right\| M}{\gamma}+\frac{\|B\| M}{\gamma}\right)  \tag{25}\\
& \times M K T(\|y-z\|+\|v-w\|)
\end{align*}
$$

where

$$
\begin{equation*}
M=\sup _{0 \leq t \leq T}\left\|e^{A t}\right\| \tag{26}
\end{equation*}
$$

Proof. Let $(y, v)$ and $(z, w)$ be two functions in $\widetilde{X} \times U_{\text {ad }}$ such that $G(y, v)=(Y, V)$ and $G(z, w)=(Z, W)$. Then,

$$
\begin{equation*}
\|G(y, v)-G(z, w)\|_{\widetilde{X} \times U_{\mathrm{ad}}}=\|Y-Z\|_{\widetilde{X}}+\|V-W\|_{U_{\mathrm{ad}}} . \tag{27}
\end{equation*}
$$

Here, $\|Y-Z\|_{\widetilde{X}}$ can be estimated as follows:

$$
\begin{align*}
\|Y-Z\|= & \max _{t \in[0, T]} \|
\end{align*} \int_{0}^{t} e^{A(t-s)}\left(f\left(s, y_{s}, v_{s}\right)-f\left(s, z_{s}, w_{s}\right)\right) d s
$$

Similarly, for $\|V-W\|_{U_{a d}}$, we have

$$
\begin{align*}
\|V-W\|= & \max _{t \in[0, T]} \| B^{*} e^{A^{*}(T-t)} L^{*} \widetilde{Q}_{T}^{-1} L \\
& \times \int_{0}^{T} e^{A(T-s)}\left(f\left(s, y_{s}, v_{s}\right)-f\left(s, z_{s}, w_{s}\right)\right) d s \| \\
\leq & M^{2}\|B\|\left\|\widetilde{Q}_{T}^{-1}\right\| \int_{0}^{T}\left\|f\left(s, y_{s}, v_{s}\right)-f\left(s, z_{s}, w_{s}\right)\right\| d s \\
\leq & M^{2}\|B\| \frac{1}{\gamma} K \int_{0}^{T}\left(\left\|y_{s}-z_{s}\right\|+\left\|v_{s}-w_{s}\right\|\right) d s \\
\leq & \frac{\|B\| M}{\gamma} M K T(\|y-z\|+\|v-w\|) . \tag{29}
\end{align*}
$$

Combining (28) and (29), we obtain the demanded inequality.

Lemma 6. Under the conditions ( $A$ )-(F), if

$$
\begin{equation*}
\left(1+\frac{\left\|Q_{T}\right\| M}{\gamma}+\frac{\|B\| M}{\gamma}\right) M K T<1 \tag{30}
\end{equation*}
$$

then the operator $G$, mapping $\widetilde{X} \times U_{\text {ad }}$ into $\widetilde{X} \times U_{a d}$, has a unique fixed point $(x, u) \in \widetilde{X} \times U_{a d}$.

Proof. By Lemma 5, $G$ is a contraction mapping. Also, the space $\widetilde{X} \times U_{\mathrm{ad}}$ is complete. Hence, $G$ has a fixed point.

Theorem 7. Under the conditions $(A)-(F)$ and (30), the semilinear system (1) is L-partially complete controllable on $[0, T]$.

Proof. Take any $x_{0} \in X$ and $h \in H$. Show that there is $u \in U_{\text {ad }}$ such that $h=L x_{T}$. To this end, consider $u$, defined as follows:

$$
\begin{align*}
u_{t}= & B^{*} e^{A^{*}(T-t)} L^{*} \widetilde{Q}_{T}^{-1} \\
& \times\left(h-L e^{A T} x_{0}-L \int_{0}^{T} e^{A(T-s)} f\left(s, x_{s}, u_{s}\right) d s\right) \tag{31}
\end{align*}
$$

Substituting (31) in (4) and applying Fubini's theorem (see Bashirov [8], p. 45), we obtain

$$
\begin{aligned}
x_{t}= & e^{A t} x_{0} \\
& +\int_{0}^{t} e^{A(t-s)} B B^{*} e^{A^{*}(t-s)} e^{A^{*}(T-t)} L^{*} \widetilde{Q}_{T}^{-1}\left(h-L e^{A T} x_{0}\right) d s \\
& -\int_{0}^{t} \int_{0}^{T} e^{A(t-s)} B B^{*} e^{A^{*}(t-s)} e^{A^{*}(T-t)} L^{*} \widetilde{Q}_{T}^{-1} L e^{A(T-r)} \\
& \quad \times f\left(r, x_{r}, u_{r}\right) d r d s \\
& +\int_{0}^{t} e^{A(t-s)} f\left(s, x_{s}, u_{s}\right) d s \\
= & e^{A t} x_{0}+Q_{t} e^{A^{*}(T-t)} L^{*} \widetilde{Q}_{T}^{-1}\left(h-L e^{A T} x_{0}\right) \\
& +\int_{0}^{t} e^{A(t-s)} f\left(s, x_{s}, u_{s}\right) d s \\
& -\int_{0}^{T} \int_{0}^{t} e^{A(t-s)} B B^{*} e^{A^{*}(t-s)} e^{A^{*}(T-t)} L^{*} \widetilde{Q}_{T}^{-1} L e^{A(T-r)} \\
& \times f\left(r, x_{r}, u_{r}\right) d s d r
\end{aligned}
$$

$$
\begin{align*}
= & e^{A t} x_{0}+Q_{t} e^{A^{*}(T-t)} L^{*} \widetilde{Q}_{T}^{-1}\left(h-L e^{A T} x_{0}\right) \\
& +\int_{0}^{t} e^{A(t-s)} f\left(s, x_{s}, u_{s}\right) d s \\
& -\int_{0}^{T} Q_{t} e^{A^{*}(T-t)} L^{*} \widetilde{Q}_{T}^{-1} L e^{A(T-r)} f\left(r, x_{r}, u_{r}\right) d r \\
= & e^{A t} x_{0}+Q_{t} e^{A^{*}(T-t)} L^{*} \widetilde{Q}_{T}^{-1}\left(h-L e^{A T} x_{0}\right) \\
& +\int_{0}^{t} e^{A(t-s)} f\left(s, x_{s}, u_{s}\right) d s \\
& -Q_{t} e^{A^{*}(T-t)} L^{*} \widetilde{Q}_{T}^{-1} L \int_{0}^{T} e^{A(T-s)} f\left(s, x_{s}, u_{s}\right) d s . \tag{32}
\end{align*}
$$

According to Lemma 6 , there is a unique pair $(x, u) \in \widetilde{X} \times$ $U_{\mathrm{ad}}$, satisfying (31) and (32). So, $u \in U_{\mathrm{ad}}$. Furthermore, we have

$$
\begin{align*}
L x_{T}= & L\left(e^{A T} x_{0}+Q_{T} L^{*} \widetilde{Q}_{T}^{-1}\left(h-L e^{A T} x_{0}\right)\right. \\
& +\int_{0}^{T} e^{A(T-s)} f\left(s, x_{s}, u_{s}\right) d s \\
& \left.\quad-Q_{T} L^{*} \widetilde{Q}_{T}^{-1} L \int_{0}^{T} e^{A(T-s)} f\left(s, x_{s}, u_{s}\right) d s\right)  \tag{33}\\
= & L e^{A T} x_{0}+L Q_{T} L^{*} \widetilde{Q}_{T}^{-1}\left(h-L e^{A T} x_{0}\right) \\
& +L \int_{0}^{T} e^{A(T-s)} f\left(s, x_{s}, u_{s}\right) d s \\
& -L Q_{T} L^{*} \widetilde{Q}_{T}^{-1} L \int_{0}^{T} e^{A(T-s)} f\left(s, x_{s}, u_{s}\right) d s=h
\end{align*}
$$

Thus, there is $u \in U_{\text {ad }}$ which steers $x_{0}$ to $x_{T}$ with $L x_{T}=h$. This means that the semilinear system (1) is $L$-partially complete controllable on $[0, T]$ as desired.

Remark 8. Decomposing $Q_{T}$ in the form

$$
\mathrm{Q}_{T}=\left[\begin{array}{ll}
\widetilde{\mathrm{Q}}_{T} & R_{T}  \tag{34}\\
R_{R}^{*} & P_{T}
\end{array}\right],
$$

where $R_{T}: H^{\perp} \rightarrow H$ and $P_{T}: H^{\perp} \rightarrow H^{\perp}$ are other components of $Q_{T}$ besides $\widetilde{Q}_{T}$ and $H^{\perp}$ is an orthogonal complement of $H$ in $X$, one can calculate

$$
\begin{equation*}
\left\langle Q_{T} h, h\right\rangle=\left\langle\widetilde{Q}_{T} h_{1}, h_{1}\right\rangle+2\left\langle R_{T} h_{2}, h_{1}\right\rangle+\left\langle P_{T} h_{2}, h_{2}\right\rangle, \tag{35}
\end{equation*}
$$

where $h_{1}=L h \in H$ and $h_{2}=h-L h \in H^{\perp}$. Therefore, the coercivity of $Q_{T}$ implies the same of $\widetilde{Q}_{T}$. But, the converse is not true. Theorem 7 is powerful in the cases when $\widetilde{Q}_{T}$ is coercive, but $P_{T}$ is not.

Example 9. Theorem 7 establishes just sufficient condition of $L$-partial complete controllability. In this example we will
demonstrate that this is not a necessary condition. We will consider a simple case of $L=I$ when $L$-partial complete controllability reduces to complete controllability. Consider the one-dimensional control system

$$
\begin{equation*}
x_{t}^{\prime}=2 x_{t}+2 u_{t}, \quad x_{0} \in \mathbb{R} \tag{36}
\end{equation*}
$$

This is a linear system, and the controllability operator of this system is equal to

$$
\begin{equation*}
\int_{0}^{T} 4 e^{4 t} d t=e^{4 T}-1>0 \quad \text { for every } T>0 \tag{37}
\end{equation*}
$$

According to the theory of controllability for linear systems, this system is controllable (completely) for every $T>0$.

Have another look at this system by writing it as

$$
\begin{equation*}
x_{t}^{\prime}=x_{t}+u_{t}+f\left(x_{t}, u_{t}\right), \quad x_{0} \in \mathbb{R} \tag{38}
\end{equation*}
$$

where

$$
\begin{equation*}
f(x, u)=x+u . \tag{39}
\end{equation*}
$$

Here, $f$ satisfies the Lipschitz condition with $K=1$. Also, $A=B=1$, implying $\|B\|=1$ and $M=\sup _{[0, T]}\left\|e^{A t}\right\|=e^{T}$. Furthermore,

$$
\begin{equation*}
Q_{T}=\int_{0}^{T} e^{2 t} d t=\frac{e^{2 T}-1}{2} \tag{40}
\end{equation*}
$$

So, $\left\|Q_{T}\right\|=\gamma=\left(e^{2 T}-1\right) / 2$. Then the inequality (30) becomes

$$
\begin{equation*}
\left(1+e^{T}+\frac{2 e^{T}}{e^{T}-1}\right) e^{T} T<1 \tag{41}
\end{equation*}
$$

The limit of the left-hand side in this inequality when $T \rightarrow$ $\infty$ is equal to $\infty$. This means that there is a sufficiently large $T$ such that the conditions of Theorem 7 do not hold for this $T$, although the system under consideration is completely controllable. Thus, Theorem 7 states a sufficient condition which is not a necessary condition.

## 4. Examples

We demonstrate the features of $L$-partial complete controllability in the following examples of control systems.

Example 1. Consider the system of differential equations

$$
\begin{gather*}
x_{t}^{\prime}=y_{t}+b u_{t}, \quad x_{0} \in \mathbb{R},  \tag{42}\\
y_{t}^{\prime}=f\left(t, x_{t}, y_{t}, u_{t}\right), \quad y_{0} \in \mathbb{R}
\end{gather*}
$$

on $[0, T]$, where $u \in U_{\mathrm{ad}}=C(0, T ; \mathbb{R})$. Besides the complete controllability property, that is,

$$
\begin{equation*}
\left\{(x, y) \in \mathbb{R}^{2}: \exists u \in U_{\text {ad }} \text { such that }\left(x_{T}, y_{T}\right)=(x, y)\right\}=\mathbb{R}^{2} \tag{43}
\end{equation*}
$$

we can investigate the partial complete controllability property, that is,

$$
\begin{equation*}
\left\{x \in \mathbb{R}: \exists u \in U_{\text {ad }} \text { such that } x_{T}=x\right\}=\mathbb{R} . \tag{44}
\end{equation*}
$$

We can write this system in $\mathbb{R}^{2}$ as the following semilinear system:

$$
\begin{equation*}
z_{t}^{\prime}=A z_{t}+F\left(t, z_{t}, u_{t}\right)+B u_{t}, \tag{45}
\end{equation*}
$$

where

$$
\begin{gather*}
z_{t}=\left[\begin{array}{l}
x_{t} \\
y_{t}
\end{array}\right], \quad A=\left[\begin{array}{ll}
0 & 1 \\
0 & 0
\end{array}\right] \quad B=\left[\begin{array}{l}
b \\
0
\end{array}\right], \\
F(t, z, u)=\left[\begin{array}{c}
0 \\
f(t, x, y, u)
\end{array}\right], \tag{46}
\end{gather*}
$$

assuming that

$$
z=\left[\begin{array}{l}
x  \tag{47}\\
y
\end{array}\right]
$$

It can be calculated that

$$
e^{A t}=\left[\begin{array}{ll}
1 & t  \tag{48}\\
0 & 1
\end{array}\right]=\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right]+\left[\begin{array}{ll}
0 & t \\
0 & 0
\end{array}\right]
$$

Hence,

$$
\begin{equation*}
\left\|e^{A t}\right\| \leq 1+t \leq 1+T, \quad 0 \leq t \leq T . \tag{49}
\end{equation*}
$$

The controllability operator is

$$
Q_{T}=\int_{0}^{T} e^{A t} B B^{*} e^{A^{*} t} d t=b^{2} T\left[\begin{array}{ll}
1 & 0  \tag{50}\\
0 & 0
\end{array}\right]
$$

Hence, $Q_{T}$ is not coercive, and the conditions for complete controllability, based on coercivity of $Q_{T}$, fail for this example. Although system (42) can still be complete controllable for properly selected functions $f$, we can investigate the partial complete controllability for this system being interested in just the first component $x_{t}$ of $z_{t}$.

Let $L=\left[\begin{array}{ll}1 & 0\end{array}\right]$. Then

$$
\begin{equation*}
\widetilde{Q}_{T}=L Q_{T} L^{*}=b^{2} T>0 \tag{51}
\end{equation*}
$$

This means that the linear system associated with the semilinear system (45) is $L$-partially complete controllable. Furthermore, the inequality (30) becomes

$$
\begin{equation*}
\left(1+\frac{b^{2} T(1+T)}{b^{2} T}+\frac{b(1+T)}{b^{2} T}\right)(1+T) T K<1 \tag{52}
\end{equation*}
$$

or, simplifying,

$$
\begin{equation*}
K<\frac{b}{(1+T)\left(1+T+2 b T+b T^{2}\right)} \tag{53}
\end{equation*}
$$

This establishes a relation between Lipschitz coefficient $K$ and terminal time moment $T$. Depending on $K, T$ must be taken sufficiently large to satisfy (53). So, the system (42) is $L$ partially complete controllable for the time $T$ if the Lipschitz coefficient $K$, related to $f$, satisfies (53).

Example 2. Delay equations are typical for application of partial controllability concepts. Consider a nonlinear delay equation

$$
\begin{gather*}
x_{t}^{\prime}=a x_{t}+b u_{t}+f\left(t, x_{t}, \int_{-\varepsilon}^{0} x_{t+\theta} d \theta, u_{t}\right),  \tag{54}\\
x_{0}=\xi, \quad x_{\theta}=\eta_{\theta}, \quad-\varepsilon \leq \theta \leq 0
\end{gather*}
$$

on $[0, T]$, where $0<\varepsilon<T, \eta \in L_{2}(-\varepsilon, 0 ; \mathbb{R})$, and $u \in U_{\mathrm{ad}}=$ $C(0, T ; \mathbb{R})$.

Similar to Example 3, introduce the function $\bar{x}:[0, T] \rightarrow$ $L_{2}(-\varepsilon, 0 ; \mathbb{R})$ by

$$
\begin{equation*}
\left[\bar{x}_{t}\right]_{\theta}=x_{t+\theta}, \quad 0 \leq t \leq T,-\varepsilon \leq \theta \leq 0 \tag{55}
\end{equation*}
$$

This function satisfies

$$
\begin{equation*}
\bar{x}_{t}^{\prime}=\left(\frac{d}{d \theta}\right) \bar{x}_{t}, \quad \bar{x}_{0}=\eta, \quad 0<t \leq T \tag{56}
\end{equation*}
$$

Denote by $\mathscr{T}_{t}, t \geq 0$, the semigroup generated by the differential operator $d / d \theta$ and let $\Gamma$ be the integral operator from $L_{2}(-\varepsilon, 0 ; \mathbb{R})$ to $\mathbb{R}$, defined by

$$
\begin{equation*}
\Gamma h=\int_{-\varepsilon}^{0} h_{\theta} d \theta, \quad h \in L_{2}(-\varepsilon, 0 ; \mathbb{R}) \tag{57}
\end{equation*}
$$

noticing that $\|\Gamma\| \leq \sqrt{\varepsilon}$. Then for

$$
y_{t}=\left[\begin{array}{c}
x_{t}  \tag{58}\\
\bar{x}_{t}
\end{array}\right], \quad \zeta=\left[\begin{array}{c}
\xi \\
\eta
\end{array}\right] \in \mathbb{R} \times L_{2}(-\varepsilon, 0 ; \mathbb{R})
$$

we can write system (54) as

$$
\begin{equation*}
y_{t}^{\prime}=A y_{t}+F\left(t, y_{t}, u_{t}\right)+B u_{t}, \quad y_{0}=\zeta \tag{59}
\end{equation*}
$$

where

$$
\begin{array}{cc}
A=\left[\begin{array}{cc}
a & 0 \\
0 & \frac{d}{d \theta}
\end{array}\right], \quad F(t, y, u)=\left[\begin{array}{c}
f(t, x, \Gamma \bar{x}, u) \\
0
\end{array}\right],  \tag{60}\\
B=\left[\begin{array}{l}
b \\
0
\end{array}\right]
\end{array}
$$

assuming that

$$
\begin{equation*}
y=\left[\frac{x}{x}\right] \in \mathbb{R} \times L_{2}(-\varepsilon, 0 ; \mathbb{R}) \tag{61}
\end{equation*}
$$

The semigroup, generated by $A$, has the form

$$
e^{A t}=\left[\begin{array}{cc}
e^{a t} & 0  \tag{62}\\
0 & \mathscr{T}_{t}
\end{array}\right], \quad t \geq 0
$$

Therefore, the controllability operator for system (54) can be calculated as

$$
\begin{align*}
Q_{T} & =\int_{0}^{T} e^{A t} B^{*} B e^{A^{*} t} d t=\int_{0}^{T}\left[\begin{array}{cc}
b^{2} e^{2 a t} & 0 \\
0 & 0
\end{array}\right] d t \\
& =\left[\begin{array}{cc}
\frac{b^{2}\left(e^{2 a T}-1\right)}{2 a} & 0 \\
0 & 0
\end{array}\right] \tag{63}
\end{align*}
$$

This is definitely not a coercive operator.

Taking into account that the original system is given by (54) and (59) is just representation of (54) in the standard form, which enlarges the original state space $\mathbb{R}$ to $\mathbb{R} \times$ $L_{2}(-\varepsilon, 0 ; \mathbb{R})$, we observe that the complete controllability for system (54) is in fact $L$-partial complete controllability for system (59) if

$$
L=\left[\begin{array}{ll}
1 & 0 \tag{64}
\end{array}\right]: \mathbb{R} \times L_{2}(-\varepsilon, 0 ; \mathbb{R}) \longrightarrow \mathbb{R} .
$$

Calculating partial controllability operator, we obtain

$$
\begin{equation*}
\widetilde{Q}_{T}=L Q_{T} L^{*}=\frac{b^{2}\left(e^{2 a T}-1\right)}{2 a}>0 \tag{65}
\end{equation*}
$$

which is coercive.
Furthermore, using

$$
\begin{equation*}
\left\|e^{A t}\right\| \leq 1+e^{a T}, \quad\left\|Q_{T}\right\|=\gamma=\frac{b^{2}\left(e^{2 a T}-1\right)}{2 a} \tag{66}
\end{equation*}
$$

we write the inequality (30) in the form

$$
\begin{equation*}
\left(1+1+e^{a T}+\frac{2 a}{b\left(e^{a T}-1\right)}\right)\left(1+e^{a T}\right) T K<1 \tag{67}
\end{equation*}
$$

If the Lipschitz coefficient $K$ of the function $F$ and terminal time moment $T$ satisfy this inequality, then system (54) is completely controllable, which in turn means that system (59) is $L$-partially complete controllable.

## 5. Conclusion

In this paper a sufficient condition for partial complete controllability of a semilinear control system is proved. This is a continuation of the pioneering research that has been done by Bashirov et al. [20, 21] about partial controllability concepts. A research in this way, concerning partial complete and approximate controllability for semilinear deterministic and stochastic systems, has already been done and awaiting for publication. There are other kinds of systems which besides semilinearity include other features, for example, impulsiveness, fractional derivative issue, and so forth. The result of this paper can be extended to these systems as well.

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## Research Article

# Stochastic Dynamics of an SIRS Epidemic Model with Ratio-Dependent Incidence Rate 

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#### Abstract

We investigate the complex dynamics of an epidemic model with nonlinear incidence rate of saturated mass action which depends on the ratio of the number of infectious individuals to that of susceptible individuals. We first deal with the boundedness, dissipation, persistence, and the stability of the disease-free and endemic points of the deterministic model. And then we prove the existence and uniqueness of the global positive solutions, stochastic boundedness, and permanence for the stochastic epidemic model. Furthermore, we perform some numerical examples to validate the analytical findings. Needless to say, both deterministic and stochastic epidemic models have their important roles.


## 1. Introduction

Since the pioneer work of Kermack and McKendrick [1], mathematical models are used extensively in analyzing the spread, and control of infectious diseases qualitatively and quantitatively. The research results are helpful for predicting the developing tendencies of the infectious disease, for determining the key factors of the disease spreading, and for seeking the optimum strategies for preventing and controlling the spread of infectious diseases [2]. And in modeling communicable diseases, the incidence function has been considered to play a key role in ensuring that the models indeed give reasonable qualitative description of the transmission dynamics of the diseases [3-7].

Let $S(t)$ be the number of susceptible individuals, $I(t)$ the number of infective individuals, and $R(t)$ the number of removed individuals at time $t$, respectively. We consider the general SIRS epidemic model:

$$
\begin{gathered}
\frac{d S}{d t}=b-d S-H(I, S)+\gamma R \\
\frac{d I}{d t}=H(I, S)-(d+\mu+\delta) I
\end{gathered}
$$

$$
\begin{equation*}
\frac{d R}{d t}=\mu I-(d+\gamma) R \tag{1}
\end{equation*}
$$

where $b$ is the recruitment rate of the population, $d$ is the natural death rate of the population, $\mu$ is the natural recovery rate of the infective individuals, $\gamma$ is the rate at which recovered individuals lose immunity and return to the susceptible class, and $\delta$ is the disease-induced death rate. And the transmission of the infection is governed by an incidence rate $H(I, S)$.

In [8], Liu et al. proposed the general saturated nonlinear incidence rate:

$$
\begin{equation*}
H(I, S)=S g(I), \quad g(I)=\frac{k I^{l}}{1+\alpha I^{h}} \tag{2}
\end{equation*}
$$

where the parameters $l$ and $h$ are positive constants, $k$ the proportionality constant, and $\alpha$ is a nonnegative constant, which measures the psychological or inhibitory effect. $k I^{l}$ measures the infection force of the disease, and $1 /\left(1+\alpha I^{h}\right)$ measures the inhibition effect from the behavioral change of the susceptible individuals when their number increases or from the crowding effect of the infective individuals. And the other nonlinear incidence rates are considered in [6, 9-19].

Note that the infectious force $g(I)$ of classical disease transmission models typically is only a function of infective individuals. But in the transmission of communicable diseases, it involves both infective individuals and susceptible individuals. Thus, Yuan et al. [18, 19] studied the infections force function with a ratio-dependent nonlinear incident rate which takes the following form:

$$
\begin{equation*}
g\left(\frac{I}{S}\right)=\frac{k(I / S)^{l}}{1+\alpha(I / S)^{h}} \tag{3}
\end{equation*}
$$

And in [19], Li et al. focus on an epidemic disease of SIRS type, in which they assume that the infectious force takes the form of (3) with $l=1$ and $h=1$, and the model is as follows:

$$
\begin{gather*}
\frac{d S}{d t}=b-d S-\frac{k I S}{S+\alpha I}+\gamma R \\
\frac{d I}{d t}=\frac{k I S}{S+\alpha I}-(d+\mu) I  \tag{4}\\
\frac{d R}{d t}=\mu I-(d+\gamma) R
\end{gather*}
$$

where all the parameters are nonnegative and have the same definitions as in model (1).

From the standpoint of epidemiology, we are only interested in the dynamics of model (4) in the closed first quadrant $\mathbb{R}_{+}^{3}=\{(S, I, R): S \geq 0, I \geq 0, R \geq 0\}$. Thus, we consider only the epidemiological meaningful initial conditions $S(0)>0$, $I(0)>0, R(0)>0$. Straightforward computation shows that model (4) is continuous and Lipschizian in $\mathbb{R}_{+}^{3}$ if we redefine that when $(S, I, R)=(0,0,0), d S / d t=b, d I / d t=0$, $d R / d t=0$. Hence, the solution of model (4) with positive initial conditions exists and is unique.

It is clear that the limit set of model (4) is on the plane $S+I+R=b / d$, and the model can be reduced to the following:

$$
\begin{gather*}
\frac{d S}{d t}=\left(b+\frac{\gamma b}{d}\right)-(d+\gamma) S-\gamma I-\frac{k I S}{S+\alpha I} \\
\frac{d I}{d t}=\frac{k I S}{S+\alpha I}-(d+\mu) I \tag{5}
\end{gather*}
$$

when $(S, I)=(0,0), d S / d t=b+(\gamma b / d), d I / d t=0$.
For mathematical simplicity, let us nondimensionalize model (5) as in [19] with the following scaling:

$$
\begin{equation*}
x=\frac{d(d+\mu)}{b(d+\gamma)} S, \quad y=\frac{d \gamma}{b(d+\gamma)} I, \quad \tau=(d+\mu) t . \tag{6}
\end{equation*}
$$

We still use variable $t$ instead of $\tau$, and model (5) takes the following form:

$$
\begin{gather*}
\frac{d x}{d t}=1-q x-y-\frac{a x y}{x+p y} \\
\frac{d y}{d t}=\left(\frac{R_{0} x}{x+p y}-1\right) y \tag{7}
\end{gather*}
$$

where $q=(d+\gamma) /(d+\mu), p=\alpha(d+\mu) / \gamma, a=k / \gamma$ are positive constants. $R_{0}=k /(d+\mu)$ is the basic reproduction number. And when $(S, I)=(0,0), d x / d t=1, d y / d t=0$.

On the other hand, if the environment is randomly varying, the population is subject to a continuous spectrum of disturbances [20,21]. That is to say, population systems are often subject to environmental noise; that is, due to environmental fluctuations, parameters involved in epidemic models are not absolute constants, and they may fluctuate around some average values. Based on these factors, more and more people began to be concerned about stochastic epidemic models describing the randomness and stochasticity [22-34], and the stochastic epidemic models can provide an additional degree of realism if compared to their deterministic counterparts [10, 35-47]. In Particular, Mao et al. [26] obtained the interesting and surprising conclusion: even a sufficiently small noise can suppress explosions in population dynamics. Beretta et al. [35] obtained the stability of epidemic model with stochastic time delays influenced by probability under certain conditions. Carletti [36] studied the stable properties of a stochastic model for phage-bacteria interaction in open marine environment analytically and numerically. In [37], establishing some stochastic models and studying of several endemic infections with demography, Nåsell found that some deterministic models are unacceptable approximations of the stochastic models for a large range of realistic parameter values. Dalal et al. [39, 40] showed that stochastic models had nonnegative solutions and carried out analysis on the asymptotic stability of models. In [41], Yu et al. presented stochastic asymptotic stability of the epidemic point of the two-group SIR model with random perturbation. It is shown in [45] that the SIR model has a unique global positive and asymptotic solution. But to our knowledge, the research on the stochastic dynamics of the epidemic model with ratiodependent nonlinear incidence rate seems rare.

There are different possible approaches to including random effects in the model, both from a biological and from a mathematical perspectives [48]. Our basic approach is analogous to that of Beddington and May [20], which is pursued in [48], and also, for example, in [45, 47] to epidemic models, in which they considered that the environmental noise was proportional to the variables. Following them, in this paper, we assume that stochastic perturbations are of a white noise type which is directly proportional to $x(t), y(t)$, influenced on the $d x(t) / d t$ and $d y(t) / d t$ in model (4). In this way, we introduce stochastic perturbation terms into the growth equations of susceptible and infected individuals to incorporate the effect of randomly fluctuating environment, and the following stochastic differential equation is corresponding to model (7):

$$
\begin{gather*}
d x=\left(1-q x-y-\frac{a x y}{x+p y}\right) d t+\sigma_{1} x d B_{1}(t),  \tag{8}\\
d y=\left(\frac{R_{0} x y}{x+p y}-y\right) d t+\sigma_{2} y d B_{2}(t),
\end{gather*}
$$

where $\sigma_{1}, \sigma_{2}$ are real constants and known as the intensity of environmental fluctuations, and $B_{1}(t), B_{2}(t)$ are independent standard Brownian motions.

The aim of this paper is to consider the dynamics of the epidemic models (7) and (8). The paper is organized as follows. In Section 2, we give some properties about deterministic model (7). In Section 3, we carry out the analysis of the dynamical properties of stochastic model (8). And in Section 4, we give some numerical examples and make a comparative analysis of the stability of the model with deterministic and stochastic environments and have some discussions.

## 2. Dynamics of the Deterministic Model

Let us begin to determine the location and number of the equilibria of model (7). It is easy to see that if $R_{0}<1$, the disease-free point $E_{0}=(1 / q, 0)$ is the unique equilibrium, corresponding to the extinction of the disease; if $R_{0}>1$, in addition to the disease-free point $E_{0}$, there is a unique endemic point $E^{*}=\left(x^{*}, y^{*}\right)$, corresponding to the survival of the disease, described by the following expressions:

$$
\begin{equation*}
x^{*}=\frac{p R_{0}}{p q R_{0}+\left(R_{0}+a\right)\left(R_{0}-1\right)}, \quad y^{*}=\frac{R_{0}-1}{p} x^{*} \tag{9}
\end{equation*}
$$

The Jacobian matrix of model (7) at $E_{0}$ is as follows:

$$
\left(\begin{array}{cc}
-q & -1-a  \tag{10}\\
0 & R_{0}-1
\end{array}\right)
$$

It follows that $E_{0}$ is asymptotically stable if $R_{0}<1$ and unstable if $R_{0}>1$.

The Jacobian matrix of model (7) at $E^{*}$ is as follows:

$$
J^{*}=\left(\begin{array}{ll}
J_{11} & J_{12}  \tag{11}\\
J_{21} & J_{22}
\end{array}\right)
$$

where

$$
\begin{gather*}
J_{11}=-\frac{p q R_{0}^{2}+a\left(R_{0}-1\right)^{2}}{p R_{0}^{2}}, \quad J_{12}=-\frac{R_{0}^{2}+a}{R_{0}^{2}}  \tag{12}\\
J_{21}=\frac{\left(R_{0}-1\right)^{2}}{p R_{0}}, \quad J_{22}=-\frac{R_{0}-1}{R_{0}}
\end{gather*}
$$

It is easy, by simple computations, to see that

$$
\begin{gather*}
\operatorname{tr}\left(J^{*}\right)=J_{11}+J_{22}<0 \\
\operatorname{det}\left(J^{*}\right)=\frac{p q R_{0}+\left(a+R_{0}\right)\left(R_{0}-1\right)}{p R_{0}^{2}}>0 . \tag{13}
\end{gather*}
$$

Summarizing the above, we have the following results on the dynamics of model (7).

Theorem 1. (i) If $R_{0}<1$, then model (7) has a unique diseasefree equilibrium $E_{0}$ which is asymptotically stable.
(ii) If $R_{0}>1$, then model (7) has two equilibria, a diseasefree equilibrium $E_{0}$ which is an unstable saddle and an endemic equilibrium $E^{*}$ which is asymptotically stable.


Figure 1: The dynamics of model (7). The parameters are taken as $(14) \cdot E_{0}=(0.5,0)$ is a saddle point. $E^{*}=(0.11,0.74)$ is globally asymptotically stable.

As a matter of fact, we can prove that the endemic point $E^{*}=\left(x^{*}, y^{*}\right)$ is also global asymptotically stable. For more details, see [19].

In Figure 1, we show the dynamics of the deterministic model (7) with the following parameters:

$$
\begin{equation*}
a=0.3, \quad p=0.5, \quad q=2, \quad R_{0}=4.5 \tag{14}
\end{equation*}
$$

In this case, $E_{0}=(0.5,0)$ is a saddle point. $E^{*}=(0.10563$, 0.73944 ) is globally asymptotically stable.

In the following, we will focus on the boundedness, dissipation, and persistence of mode (7).

Theorem 2. All the solutions of model (7) with the positive initial condition $(x(0), y(0))$ are uniformly bounded within a region $\Gamma$, where

$$
\begin{equation*}
\Gamma=\left\{(x, y) \in \mathbb{R}_{+}^{2}: x+\frac{a}{R_{0}} y \leq \min \left\{\frac{1}{q}, \frac{R_{0}}{R_{0}+a}\right\}\right\} \tag{15}
\end{equation*}
$$

Proof. Define function

$$
\begin{equation*}
N(t)=x(t)+\frac{a}{R_{0}} y(t) \tag{16}
\end{equation*}
$$

Differentiating $N(t)$ with respect to time $t$ along the solutions of model (7), we can get the following:

$$
\begin{equation*}
\frac{d N(t)}{d t}=\frac{d x}{d t}+\frac{a}{R_{0}} \frac{d y}{d t}=1-q x-\left(1+\frac{a}{R_{0}}\right) y \tag{17}
\end{equation*}
$$

Thus, we obtain the following:

$$
\begin{equation*}
\frac{d N(t)}{d t}+\eta N(t)=1-(q-\eta) x-\left(1+\frac{a}{R_{0}}-\eta\right) y<1 \tag{18}
\end{equation*}
$$

where $\eta<\min \left\{q, 1+\left(a / R_{0}\right)\right\}$. And we obtain the following:

$$
\begin{equation*}
0<N(x, y) \leq \frac{1}{\eta}+N(x(0), y(0)) e^{-\eta t} \tag{19}
\end{equation*}
$$

As $t \rightarrow \infty, 0<N \leq 1 / \eta$. Therefore, all solutions of model (7) enter into the region $\Gamma$. This completes the proof.

Theorem 3. If $R_{0}>1$, model (7) is dissipative.
Proof. Since all solutions of model (7) are positive, by the first equation of (7), we have the following:

$$
\begin{equation*}
\frac{d x}{d t} \leq 1-q x \tag{20}
\end{equation*}
$$

A standard comparison theorem shows that

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} x(t) \leq \frac{1}{q} \tag{21}
\end{equation*}
$$

Hence, for any $0<\varepsilon \ll 1$ and large $t, x \leq(1 / q)+\varepsilon$. It then follows that $y$ satisfies the following:

$$
\begin{equation*}
\frac{d y}{d t} \leq \frac{y\left(\left(\left(R_{0}-1\right) / q\right)+\varepsilon\left(R_{0}-1\right)-p y\right)}{(1 / q)+\varepsilon+p y} \tag{22}
\end{equation*}
$$

The arbitrariness of $\varepsilon$ then implies that

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} y(t) \leq \frac{R_{0}-1}{p q} . \tag{23}
\end{equation*}
$$

Theorem 4. If $R_{0}>1$ and $p q<(1+a)\left(R_{0}-1\right)$, then model (7) is permanent; that is, there exists $\varepsilon>0$ (independent of initial conditions), such that $\liminf _{t \rightarrow \infty} x(t)>\varepsilon, \liminf _{t \rightarrow \infty} y(t)>$ $\varepsilon$.

Proof. By the first equation in (7), we have the following:

$$
\begin{align*}
\frac{d x}{d t} & =1-q x-(1+a) y+\frac{a p y^{2}}{x+p y}  \tag{24}\\
& >1-q x-(1+a) y
\end{align*}
$$

If $R_{0}>1$ and $p q<(1+a)\left(R_{0}-1\right)$, from the proof of Theorem 3, we see that $\lim \sup _{t \rightarrow \infty} y(t) \leq\left(R_{0}-1\right) / p q$. Thus, for any $0<\varepsilon \leq\left(R_{0}-1\right) / p q$ and large $t, y(t)>\left(\left(R_{0}-1\right) / p q\right)-\varepsilon$. As a result, we have the following:

$$
\begin{equation*}
\frac{d x}{d t}>1-(1+a)\left(\frac{R_{0}-1}{p q}-\varepsilon\right)-q x \tag{25}
\end{equation*}
$$

With the comparison principle, the arbitrariness of $\varepsilon$ implies that

$$
\begin{equation*}
\liminf _{t \rightarrow \infty} x(t) \geq \frac{p q-(1+a)\left(R_{0}-1\right)}{p q^{2}} \triangleq \underline{x} . \tag{26}
\end{equation*}
$$

Hence, for any $0<\varepsilon<\left(p q-(1+a)\left(R_{0}-1\right)\right) / p q^{2}$ and large $t, x(t)>\underline{x}-\varepsilon$.

And for large $t$, we have the following:

$$
\begin{equation*}
\frac{d y}{d t}>\frac{y\left((\underline{x}-\varepsilon)\left(R_{0}-1\right)-p y\right)}{\underline{x}-\varepsilon+p y} . \tag{27}
\end{equation*}
$$

Therefore,

$$
\begin{equation*}
\liminf _{t \rightarrow \infty} x(t) \geq \frac{(\underline{x}-\varepsilon)\left(R_{0}-1\right)}{p} \tag{28}
\end{equation*}
$$

The arbitrariness of $\varepsilon$ then implies that

$$
\begin{equation*}
\liminf _{t \rightarrow \infty} x(t) \geq \frac{\underline{x}\left(R_{0}-1\right)}{p} \triangleq \underline{y} . \tag{29}
\end{equation*}
$$

Choosing a positive number $\epsilon$ such that $\epsilon<\min \{\underline{x} / 2, \underline{y} / 2\}$, we see that

$$
\begin{equation*}
\liminf _{t \rightarrow \infty} x(t)>\epsilon, \quad \liminf _{t \rightarrow \infty} y(t)>\varepsilon \tag{30}
\end{equation*}
$$

This ends the proof.
Noting that if the parameters of model (7) are fixed as (14), we can obtain the following:

$$
\begin{equation*}
R_{0}>1, \quad p q=0.6<(1+a)\left(R_{0}-1\right)=4.55 \tag{31}
\end{equation*}
$$

and from Theorems 3 and 4 , we can conclude that model (7) is dissipation and persistence.

## 3. Dynamics of the Stochastic Model

In this subsection, we investigate the dynamical behavior of the stochastic model (8). Throughout this paper, let $(\Omega, \mathscr{F}, \mathscr{P})$ be a complete probability space with a filtration $\left\{\mathscr{F}_{t}\right\}_{t \in \mathbb{R}_{+}}$ satisfying the usual conditions (i.e., it is right continuous and increasing while $\mathscr{F}_{0}$ contains all $\mathscr{P}$-null sets). $B_{1}(t), B_{2}(t)$ are the Brownian motions defined on this probability space. We denote by $X(t)=\left((x(t), y(t))\right.$ and $|X(t)|=\left(x^{2}(t)+y^{2}(t)\right)^{1 / 2}$. Denote $\Lambda=\left\{(x, y) \in \mathbb{R}_{+}^{2}: x \geq a / R_{0}, y>0\right\}$.

Denote by $C^{2,1}\left(\mathbb{R}^{d} \times(0, \infty) ; \mathbb{R}_{+}\right)$the family of all nonnegative functions $V(x, t)$ defined on $\mathbb{R}^{d} \times(0, \infty)$ such that they are continuously twice differentiable in $x$ and once in $t$. Define the differential operator $L$ associated with $d$ dimensional stochastic differential equation:

$$
\begin{equation*}
d x(t)=f(x(t), t) d t+h(x(t), t) d B(t) \tag{32}
\end{equation*}
$$

by

$$
\begin{align*}
L= & \frac{\partial}{\partial t}+\sum_{i=1}^{d} f_{i}(x, t) \frac{\partial}{\partial x_{i}} \\
& +\frac{1}{2} \sum_{i, j=1}^{d}\left[h^{T}(x, t) h(x, t)\right]_{i j} \frac{\partial^{2}}{\partial x_{i} \partial x_{j}} . \tag{33}
\end{align*}
$$

If $L$ acts in a function $V \in C^{2,1}\left(\mathbb{R}^{d} \times(0, \infty)\right.$; $\left.\mathbb{R}_{+}\right)$, then

$$
\begin{align*}
L V(x, t)= & V_{t}(x, t)+V_{x}(x, t) f(x, t) \\
& +\frac{1}{2} \operatorname{trace}\left[h^{T}(x, t) V_{x x}(x, t) h(x, t)\right] \tag{34}
\end{align*}
$$

where $T$ means transposition.
3.1. Existence and Uniqueness of Global Positive Solutions. To investigate the dynamical behavior of model (8), the first thing concerned is whether the solution is global existent. In this section, using the Lyapunov analysis method (mentioned in [24]), we will show the solution of model (8) is global and nonnegative.

Lemma 5. There is a unique local positive solution $(x(t), y(t))$ for $t \in\left[0, \tau_{e}\right)$ to model (8) almost surely (a.s.) for the initial value $(x(0), y(0)) \in \Lambda$, where $\tau_{e}$ is the explosion time.

Proof. Set

$$
\begin{equation*}
u(t)=\ln x(t), \quad v(t)=\ln y(t) \tag{35}
\end{equation*}
$$

by Itô formula, we have the following:

$$
\begin{gather*}
d u=\left(\frac{1}{e^{u}}-q-\frac{e^{v}}{e^{u}}-\frac{a e^{v}}{e^{u}+p e^{v}}-\frac{\sigma_{1}^{2}}{2}\right) d t+\sigma_{1} d B_{1}(t), \\
d v=\left(\frac{R_{0} e^{u}}{e^{u}+p e^{v}}-1-\frac{\sigma_{2}^{2}}{2}\right) d t+\sigma_{2} d B_{2}(t) \tag{36}
\end{gather*}
$$

at $t \geq 0$ with initial value $u(0)=\ln x(0), v(0)=\ln y(0)$.
It is easy to see that the coefficients of model (36) satisfy the local Lipschitz condition, and there is a unique local solution $u(t), v(t)$ on $\left[0, \tau_{e}\right)$ [24]. Therefore, $x(t)=e^{u(t)}$, $y(t)=e^{\nu(t)}$ are the unique positive local solutions to model (36) with the initial value $(x(0), y(0)) \in \Lambda$.

Lemma 5 only tells us that there exists a unique local positive solution to model (8). In the following, we show this solution is global; that is, $\tau_{e}=\infty$, which is motived by the work of Luo and Mao [29].

Theorem 6. Consider model (8), for any given initial value $(x(0), y(0)) \in \Lambda$, there is a unique solution $(x(t), y(t))$ on $t \geq 0$ and the solution will remain in $\Lambda$ with probability 1.

Proof. Let $n_{0}>0$ be sufficiently large for $x(0)$ and $y(0)$ lying within the interval $\left[1 / n_{0}, n_{0}\right]$. For each integer $n>n_{0}$, define the stopping times:

$$
\begin{equation*}
\tau_{n}=\inf \left\{t \in\left[0, \tau_{e}\right]: x(t) \notin\left(\frac{1}{n}, n\right) \text { or } y(t) \notin\left(\frac{1}{n}, n\right)\right\} . \tag{37}
\end{equation*}
$$

We set $\inf \emptyset=\infty$ ( $\emptyset$ represents the empty set) in this paper. $\tau_{n}$ is increasing as $n \rightarrow \infty$. Let $\tau_{\infty}=\lim _{n \rightarrow \infty} \tau_{n}$; then $\tau_{\infty} \leq \tau_{e}$ a.s..

In the following, we need to show $\tau_{\infty}=\infty$ a.s. If this statement is violated, there exist constants $T>0$ and $\varepsilon \in(0,1)$ such that $\mathscr{P}\left\{\tau_{\infty} \leq T\right\}>\varepsilon$. As a consequence, there exists an integer $n_{1} \geq n_{0}$ such that

$$
\begin{equation*}
\mathscr{P}\left\{\tau_{n} \leq T\right\} \geq \varepsilon, \quad n \geq n_{1} . \tag{38}
\end{equation*}
$$

Define a function $V_{1}: \Lambda \rightarrow \mathbb{R}_{+}$by the following:

$$
\begin{equation*}
V_{1}(x, y)=\left(\frac{R_{0}}{a} x-1-\ln \frac{R_{0}}{a} x\right)+(y-1-\ln y) \tag{39}
\end{equation*}
$$

which is a non-negativity function.
If $(x(t), y(t)) \in \Lambda$, by the Ito formula, we compute the following:

$$
\begin{align*}
d V_{1}= & {\left[\left(\frac{R_{0}}{a}-\frac{1}{x}\right)\left(1-q x-y-\frac{a x y}{x+p y}\right)+\frac{\sigma_{1}^{2}}{2}\right] d t } \\
& +\sigma_{1}\left(\frac{R_{0}}{a} x-1\right) d B_{1}(t) \\
& +\left[\left(1-\frac{1}{y}\right)\left(\frac{R_{0} x}{x+p y}-1\right) y+\frac{\sigma_{2}^{2}}{2}\right] d t  \tag{40}\\
& +\sigma_{2}(y-1) d B_{2}(t) \\
= & L V_{1} d t+\sigma_{1}\left(\frac{R_{0}}{a} x-1\right) d B_{1}(t) \\
& +\sigma_{2}(y-1) d B_{2}(t)
\end{align*}
$$

where

$$
\begin{align*}
L V_{1}= & q+\frac{a y-R_{0} x}{x+p y}-\frac{R_{0} q}{a} x \\
& +\left(\frac{R_{0}}{a}-\frac{1}{x}\right)(1-y)+1-y+\frac{\sigma_{1}^{2}}{2}+\frac{\sigma_{2}^{2}}{2} \\
\leq & q+\frac{a}{p}+\left(\frac{R_{0}}{a}-\frac{1}{x}\right)(1-y)  \tag{41}\\
& +1-y+\frac{\sigma_{1}^{2}}{2}+\frac{\sigma_{2}^{2}}{2} .
\end{align*}
$$

Case 1 (assume $a \geq R_{0}$ ). In this case, we have $x \geq 1$. It follows that

$$
\begin{align*}
\left(\frac{R_{0}}{a}\right. & \left.-\frac{1}{x}\right)(1-y)+1-y \\
& \leq \frac{R_{0}}{a}+y\left(\frac{1}{x}-1\right)-\frac{1}{x}+1 \leq 1+\frac{R_{0}}{a} \tag{42}
\end{align*}
$$

Case 2 (assume $a<R_{0}$ ). If $a / R_{0} \leq x<1$, one has the following:
(a) $\left(\left(R_{0} / a\right)-(1 / x)\right)(1-y)+1-y \leq\left(R_{0} / a\right)+((y-1) / x)+$ $1-y \leq 1+\left(R_{0} / a\right)$ provided that $0<y \leq 1$;
(b) $\left(\left(R_{0} / a\right)-(1 / x)\right)(1-y)+1-y \leq 1$ provided that $y>1$.

Hence, there exists a positive number $M$ independent on $x, y$ and $t$ such that $L V_{1} \leq M$. Substituting this inequality into (40), we can get the following:

$$
\begin{align*}
d V_{1} \leq & M d t+\sigma_{1}\left(\frac{R_{0}}{a} x-1\right) d B_{1}(t)  \tag{43}\\
& +\sigma_{2}(y-1) d B_{2}(t)
\end{align*}
$$

Integrating both sides of the above inequality from 0 to $\tau_{n} \wedge T$ and taking expectations leads to the following:
$E V_{1}\left(x\left(\tau_{n} \wedge T\right), y\left(\tau_{n} \wedge T\right)\right) \leq V_{1}(x(0), y(0))+M T$.

Set $\Omega_{n}=\left\{\tau_{n} \leq T\right\}$, for $n \geq n_{1}$ and consider inequality (38), we can get $\mathscr{P}\left(\Omega_{n}\right) \geq \varepsilon$. Note that for every $\omega \in \Omega_{n}$, there exists some $i$ such that $x_{i}\left(\tau_{n}, \omega\right)$ equals either $n$ or $1 / n$ for $i=1,2$; hence,

$$
\begin{align*}
& V_{1}\left(x\left(\tau_{n}, \omega\right), y\left(\tau_{n}, \omega\right)\right) \\
& \quad \geq \min \left\{(n-1-\ln n),\left(\frac{1}{n}-1-\ln \frac{1}{n}\right)\right\} . \tag{45}
\end{align*}
$$

It then follows from (44) that

$$
\begin{align*}
& V_{1}(x(0), y(0))+M T \\
& \quad \geq E\left[I_{\Omega_{n}(\omega)} V_{1}\left(x\left(\tau_{n}\right), y\left(\tau_{n}\right)\right)\right]  \tag{46}\\
& \quad \geq \epsilon \min \left\{(n-1-\ln n),\left(\frac{1}{n}-1-\ln \frac{1}{n}\right)\right\},
\end{align*}
$$

where $I_{\Omega_{n}}$ is the indicator function of $\Omega_{n}$.
As $n \rightarrow \infty$ we have the following:

$$
\begin{equation*}
\infty>V_{1}(x(0), y(0))+M T=\infty \quad \text { a.s. } \tag{47}
\end{equation*}
$$

which leads to the contradiction. This completes the proof.
3.2. Stochastic Boundedness and Permanence. Theorem 6 shows that the solutions to model (8) will remain in $\Lambda$. Generally speaking, the nonexplosion property, the existence, and the uniqueness of the solution are not enough but the property of boundedness and permanence are more desirable since they mean the long-time survival in the population dynamics. Now, we present the definition of stochastic ultimate boundedness and stochastic permanence [31].

Definition 7. The solutions $X(t)=(x(t), y(t))$ of model (8) are said to be stochastically ultimately bounded, if for any $\varepsilon \in$ $(0,1)$, there is a positive constant $\delta=\delta(\varepsilon)$, such that for any initial value $(x(0), y(0)) \in \Lambda$, the solution $X(t)$ of model (8) has the property that

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} P\{|X(t)|>\delta\}<\varepsilon \tag{48}
\end{equation*}
$$

Definition 8. The solutions $X(t)=(x(t), y(t))$ of model (8) are said to be stochastically permanent if for any $\varepsilon \in(0,1)$, there exists a pair of positive constants $\delta=\delta(\varepsilon)$ and $\chi=\chi(\varepsilon)$, such that for any initial value $(x(0), y(0)) \in \Lambda$, the solution $X(t)$ of model (8) has the property that

$$
\begin{align*}
& \liminf _{t \rightarrow \infty} P\{|X(t)| \geq \delta\} \geq 1-\varepsilon \\
& \liminf _{t \rightarrow \infty} P\{|X(t)| \leq \chi\} \geq 1-\varepsilon . \tag{49}
\end{align*}
$$

Theorem 9. The solutions of model (8) are stochastically ultimately bounded for any initial value $(x(0), y(0)) \in \Lambda$.

Proof. Denote functions

$$
\begin{equation*}
V_{2}=e^{t} x^{\theta}, \quad V_{3}=e^{t} y^{\theta} \tag{50}
\end{equation*}
$$

for $(x, y) \in \Lambda$ and $0<\theta<1$.

Applying the Itô formula leads to the following:

$$
\begin{align*}
& d V_{2}=L V_{2} d t+\sigma_{1} \theta e^{t} x^{\theta} d B_{1}(t)  \tag{51}\\
& d V_{3}=L V_{3} d t+\sigma_{2} \theta e^{t} y^{\theta} d B_{2}(t)
\end{align*}
$$

where

$$
\begin{gather*}
L V_{2}=e^{t} x^{\theta}\left(1+\theta\left(\frac{1}{x}-q-\frac{y}{x}-\frac{a y}{x+p y}\right)+\frac{\sigma_{1}^{2} \theta(\theta-1)}{2}\right) \\
L V_{3}=e^{t} y^{\theta}\left(1+\theta\left(\frac{R_{0} x}{x+p y}-1\right)+\frac{\sigma_{2}^{2} \theta(\theta-1)}{2}\right) \tag{52}
\end{gather*}
$$

Thus, there exists the positive constants $M_{1}$ and $M_{2}$ such that we have $L V_{2}<M_{1} e^{t}$ and $L V_{3}<M_{2} e^{t}$. It follows that $e^{t} E x^{\theta}-$ $E x(0)^{\theta} \leq M_{1} e^{t}$ and $e^{t} E y^{\theta}-E y(0)^{\theta} \leq M_{2} e^{t}$. Then we get the following:

$$
\begin{align*}
& \limsup _{t \rightarrow \infty} E x^{\theta} \leq M_{1}<+\infty  \tag{53}\\
& \limsup _{t \rightarrow \infty} E y^{\theta} \leq M_{2}<+\infty
\end{align*}
$$

Note that

$$
\begin{align*}
|X(t)|^{\theta} & =\left(x^{2}(t)+y^{2}(t)\right)^{\theta / 2} \\
& \leq 2^{\theta / 2} \max \left\{x^{\theta}(t), y^{\theta}(t)\right\}  \tag{54}\\
& \leq 2^{\theta / 2}\left(x^{\theta}+y^{\theta}\right)
\end{align*}
$$

Therefore, we obtain the following:

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} E|X(t)|^{\theta} \leq 2^{\theta / 2}\left(M_{1}+M_{2}\right)<+\infty \tag{55}
\end{equation*}
$$

As a result, there exists a positive constant $\delta_{1}$ such that

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} E(\sqrt{|X(t)|})<\delta_{1} \tag{56}
\end{equation*}
$$

Now, for any $\varepsilon>0$, let $\delta=\delta_{1}^{2} / \varepsilon^{2}$; then by Chebyshev's inequality,

$$
\begin{equation*}
\mathscr{P}\{|X(t)|>\delta\} \leq \frac{E(\sqrt{|X(t)|})}{\sqrt{\delta}} . \tag{57}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \mathscr{P}\{|X(t)|>\delta\} \leq \frac{\delta_{1}}{\sqrt{\delta}}=\varepsilon \tag{58}
\end{equation*}
$$

which yields the required assertion.
We are now in the position to show the stochastic permanence. Let us present some hypothesis and a useful lemma.

Lemma 10. Assume $R_{0}>a+\max \{4,2 p q\}$. For any initial value $(x(0), y(0)) \in \Lambda$, the solution $(x(t), y(t))$ satisfies that

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} E\left(\frac{1}{|X(t)|^{\rho}}\right) \leq H \tag{59}
\end{equation*}
$$

where $\rho$ is an arbitrary positive constant satisfying

$$
\begin{equation*}
\frac{\rho+1}{2}\left(\max \left\{\sigma_{1}, \sigma_{2}\right\}\right)^{2}<1+\min \left\{\frac{R_{0}-a}{2 p}-q, \frac{R_{0}-a}{2}-2\right\}, \tag{60}
\end{equation*}
$$

$$
H=\frac{2^{\rho}\left(C_{2}+4 k C_{1}\right)}{4 k C_{1}}
$$

$$
\begin{equation*}
\times \max \left\{1,\left(\frac{2 C_{1}+C_{2}+\sqrt{C_{2}^{2}+4 C_{1} C_{2}}}{2 C_{1}}\right)^{\rho-2}\right\} \tag{61}
\end{equation*}
$$

in which $k$ is an arbitrary positive constant satisfying

$$
\begin{align*}
& \frac{\rho(\rho+1)}{2}\left(\max \left\{\sigma_{1}, \sigma_{2}\right\}\right)^{2}+k \\
& \quad<\rho+\rho \min \left\{\frac{R_{0}-a}{2 p}-q, \frac{R_{0}-a}{2}-2\right\} \tag{62}
\end{align*}
$$

with

$$
\begin{align*}
C_{1}= & \rho+\rho \min \left\{\frac{R_{0}-a}{2 p}-q, \frac{R_{0}-a}{2}-2\right\} \\
& -\frac{\rho(\rho+1)}{2}\left(\max \left\{\sigma_{1}, \sigma_{2}\right\}\right)^{2}-k>0,  \tag{63}\\
C_{2}= & \rho \max \{q, 2+a\}+\frac{\rho R_{0}\left(R_{0}-1\right) \max \left\{1, p^{2}\right\}}{2 a p} \\
& +\rho\left(\max \left\{\sigma_{1}, \sigma_{2}\right\}\right)^{2}+2 k>0 .
\end{align*}
$$

Proof. Set $U(x, y)=1 /(x+y)$ for $(x(t), y(y)) \in \Lambda$, by the Itô formula, we have the following:

$$
\begin{align*}
d U= & -U^{2}\left[1-q x-y-\frac{a x y}{x+p y}+\frac{R_{0} x y}{x+p y}-y\right] d t \\
& +U^{3}\left(\sigma_{1}^{2} x^{2}+\sigma_{2}^{2} y^{2}\right) d t  \tag{64}\\
& -U^{2}\left(\sigma_{1} x d B_{1}(t)+\sigma_{2} y d B_{2}(t)\right) \\
= & L U d t-U^{2}\left(\sigma_{1} x d B_{1}(t)+\sigma_{2} y d B_{2}(t)\right)
\end{align*}
$$

where

$$
\begin{align*}
L U= & -U^{2}\left(1-q x-2 y+\frac{\left(R_{0}-a\right) x y}{x+p y}\right)  \tag{65}\\
& +U^{3}\left(\sigma_{1}^{2} x^{2}+\sigma_{2}^{2} y^{2}\right)
\end{align*}
$$

Choose a positive constant $\rho$ such that it satisfies (60). Applying the Itô formula again, we can get the following:

$$
\begin{align*}
L[ & \left.(1+U)^{\rho}\right] \\
= & \rho(1+U)^{\rho-1} L U \\
& +\frac{\rho(\rho-1)}{2} U^{4}(1+U)^{\rho-2}\left(\sigma_{1}^{2} x^{2}+\sigma_{2}^{2} y^{2}\right)  \tag{66}\\
& =(1+U)^{\rho-2} \Phi,
\end{align*}
$$

where

$$
\begin{align*}
\Phi= & -\rho U^{2}\left(1-q x-2 y+\frac{\left(R_{0}-a\right) x y}{x+p y}\right) \\
& -\rho U^{3}\left(1-q x-2 y+\frac{\left(R_{0}-a\right) x y}{x+p y}\right) \\
& +\rho U^{3}\left(\sigma_{1}^{2} x^{2}+\sigma_{2}^{2} y^{2}\right) \\
& +\frac{\rho(1+\rho) U^{4}}{2}\left(\sigma_{1}^{2} x^{2}+\sigma_{2}^{2} y^{2}\right) \\
\leq & -\rho U^{2}+\rho U^{2}(q x+(2+a) y)  \tag{67}\\
& -\rho U^{3}\left(\left(\frac{R_{0}-a}{2 p}-q\right) x+\left(\frac{R_{0}-a}{2}-2\right) y\right) \\
& +\rho U^{3}\left(\frac{\left(R_{0}-a\right)\left(x^{2}+p^{2} y^{2}\right)}{2 p(x+p y)}\right) \\
& +\rho U^{3}\left(\sigma_{1}^{2} x^{2}+\sigma_{2}^{2} y^{2}\right) \\
& +\frac{\rho(1+\rho) U^{4}}{2}\left(\sigma_{1}^{2} x^{2}+\sigma_{2}^{2} y^{2}\right)
\end{align*}
$$

Using the facts that

$$
\begin{gather*}
U^{3}\left(\sigma_{1}^{2} x^{2}+\sigma_{2}^{2} y^{2}\right)<\left(\max \left\{\sigma_{1}, \sigma_{2}\right\}\right)^{2} U \\
U^{4}\left(\sigma_{1}^{2} x^{2}+\sigma_{2}^{2} y^{2}\right)<\left(\max \left\{\sigma_{1}, \sigma_{2}\right\}\right)^{2} U^{2} \tag{68}
\end{gather*}
$$

so,

$$
\left.\left.\begin{array}{rl}
\Phi \leq & -U^{2}(\rho
\end{array}\right)+\rho \min \left\{\frac{R_{0}-a}{2 p}-q, \frac{R_{0}-a}{2}-2\right\}, ~\left(\max \left\{\sigma_{1}, \sigma_{2}\right\}\right)^{2}\right) .
$$

Now, let $k>0$ sufficiently small such that it satisfies (62), by the Itô formula; then

$$
\begin{align*}
& L\left[e^{k t}(1+U)^{\rho}\right] \\
& \quad=k e^{k t}(1+U)^{\rho}+e^{k t} L(1+U)^{\rho} \\
& \quad=e^{k t}(1+U)^{\rho-2}\left(k(1+U)^{2}+\Phi\right)  \tag{70}\\
& \quad \leq e^{k t}(1+U)^{\rho-2}\left(-C_{1} U^{2}+C_{2} u+k\right) \\
& \quad \leq H_{1} e^{k t},
\end{align*}
$$

where $H_{1}=\left(\left(C_{2}+4 k C_{1}\right) / 4 C_{1}\right) \max \left\{1,\left(\left(2 C_{1}+C_{2}+\right.\right.\right.$ $\left.\left.\left.\sqrt{C_{2}^{2}+4 C_{1} C_{2}}\right) / 2 C_{1}\right)^{\rho-2}\right\}$ and $C_{1}, C_{2}$ have been defined in the statement of the theorem. Thus,

$$
\begin{equation*}
E\left[e^{k t}(1+U)^{\rho}\right] \leq(1+U(0))^{\rho}+\frac{H_{1}}{k} e^{k t} \tag{71}
\end{equation*}
$$

So we can have the following:

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} E\left[U(t)^{\rho}\right] \leq \limsup _{t \rightarrow \infty} E(1+U)^{\rho} \leq \frac{H_{1}}{k} . \tag{72}
\end{equation*}
$$

In addition, we know that $(x+y)^{\rho} \leq 2^{\rho}\left(x^{2}+y^{2}\right)^{\rho / 2}=$ $2^{\rho}|X(t)|^{\rho}$; consequently,

$$
\begin{align*}
\limsup _{t \rightarrow \infty} E\left[\frac{1}{|X(t)|^{\rho}}\right] & \leq 2^{\rho} \limsup _{t \rightarrow \infty} E\left[U(t)^{\rho}\right]  \tag{73}\\
& \leq \frac{2^{\rho} H_{1}}{k}=H
\end{align*}
$$

which complets the proof.
Consider Chebyshev inequality, Theorem 9, and Lemma 10 together, we immediately obtain the following result.

Theorem 11. If the following conditions are satisfied
(i) $a+\max \{2 p q, 4\}<R_{0}$;
(ii) $(1 / 2)\left(\max \left\{\sigma_{1}, \sigma_{2}\right\}\right)^{2}<1+\min \left\{\left(\left(R_{0}-a\right) / 2 p\right)-q\right.$, $\left(\left(R_{0}-\right.\right.$ a) $/ 2)-2\}$,
then the solutions of model (8) is stochastically permanent.

## 4. Conclusions and Discussions

In this paper, by using the theory of stochastic differential equation, we investigate the dynamics of an SIRS epidemic model with a ratio-dependent incidence rate. The value of this study lies in two aspects. First, it presents some relevant properties of the deterministic model (7), including boundedness, dissipation, persistence, and the stability of the disease-free and endemic points. Second, it verifies the existence of global positive solutions, stochastic boundedness, and permanence for the stochastic model (8).

As an example, we give some numerical examples to illustrate the dynamical behavior of stochastic model (8) by
using the Milstein method mentioned in Higham [49]. In this way, model (8) can be rewritten as the following discretization equations:

$$
\begin{align*}
x_{k+1}= & x_{k}+\left(1-q x_{k}-y_{k}-\frac{a x_{k} y_{k}}{x_{k}+p y_{k}}\right) \Delta t \\
& +\sigma_{1} x_{k} \sqrt{\Delta t} \xi_{k}+\frac{\sigma_{1}^{2}}{2} x_{k}^{2}\left(\xi_{k}^{2}-1\right) \Delta t  \tag{74}\\
y_{k+1}= & y_{k}+\left(\frac{R_{0} x_{k} y_{k}}{x_{k}+p y_{k}}-y_{k}\right) \Delta t \\
& +\sigma_{2} y_{k} \sqrt{\Delta t} \eta_{k}+\frac{\sigma_{2}^{2}}{2} y_{k}^{2}\left(\eta_{k}^{2}-1\right) \Delta t
\end{align*}
$$

where $\xi_{k}$ and $\eta_{k}, k=1,2, \ldots, n$, are the Gaussian random variables $N(0,1)$.

The parameters of model (8) are fixed as (14). In this case, model (7) has the endemic point $E^{*}=(0.11,0.74)$. And model (8) becomes as follows:

$$
\begin{gather*}
d x=\left(1-2 x-y-\frac{0.3 x y}{x+0.5 y}\right) d t+\sigma_{1} x d B_{1}(t)  \tag{75}\\
d y=\left(\frac{4.5 x y}{x+0.5 y}-y\right) d t+\sigma_{2} y d B_{2}(t)
\end{gather*}
$$

Simple computations show that

$$
\begin{gather*}
a+\max \{2 p q, 4\}=4.3<4.5=R_{0} \\
\frac{0.03^{2}}{2}=\frac{1}{2}\left(\max \left\{\sigma_{1}, \sigma_{2}\right\}\right)^{2} \\
<1+\min \left\{\frac{R_{0}-a}{2 p}-q, \frac{R_{0}-a}{2}-2\right\}=1.1 \\
\quad \text { if }\left(\sigma_{1}, \sigma_{2}\right)=(0.03,0.01) \\
\frac{0.5^{2}}{2}= \\
\frac{1}{2}\left(\max \left\{\sigma_{1}, \sigma_{2}\right\}\right)^{2} \\
< \tag{76}
\end{gather*}
$$

It is easy to see that, all the conditions of Theorem 11 are satisfied, and we can therefore conclude that, with $\left(\sigma_{1}, \sigma_{2}\right)=$ $(0.03,0.01)$ and $\left(\sigma_{1}, \sigma_{2}\right)=(0.5,0.3)$, the solutions of model (8) is stochastically permanent. The numerical examples shown in Figures 2 and 3 clearly support these results. In Figure 2, with $\left(\sigma_{1}, \sigma_{2}\right)=(0.03,0.01)$, the solutions of model (8) will be oscillating slightly around the endemic point $E^{*}=$ $(0.11,0.74)$ of model (7). And in Figure 3, with $\left(\sigma_{1}, \sigma_{2}\right)=$ $(0.5,0.3)$, the solutions of model (8) will be oscillating strongly around the endemic point $E^{*}=(0.11,0.74)$ of model (7).

It is worthy to note that, throughout this paper, the parameters for model (7), also for model (8), are fixed as the set


Figure 2: The solution of the stochastic model (8) with initial values $x(0)=0.2, y(0)=0.15$. The parameters are taken as (14), $\sigma_{1}=0.03$, $\sigma_{2}=0.01$.


Figure 3: The solution of the stochastic model (8) with initial values $x(0)=0.2, y(0)=0.15$. The parameters are taken as (14), $\sigma_{1}=0.5$, $\sigma_{2}=0.3$.
(14). The reason is that with this parameter set, the conditions of our theoretical results hold. Of course, one can adopt other parameters set to show the numerical results.

From the theoretical and numerical results, we can know that, when the noise density is not large, the stochastic model (8) preserves the property of the stability of the deterministic model (7). To a great extent, we can ignore the noise and use the deterministic model (7) to describe the population dynamics. However, when the noise is sufficiently large, it can force the population to become largely fluctuating. In this
case, we cannot use deterministic model (7) but stochastic model (8) to describe the population dynamics. Needless to say, both deterministic and stochastic epidemic models have their important roles.

Furthermore, from the numerical results in Figure 2, one can see that model (8) is stochastically stable. But we cannot prove the stochastic stability because of the complexity of model (8). This can be further investigated.

On the other hand, we know that there are different possible approaches to including random effects in the epidemic models affected by environmental white noise, here we consider another method to introduce random effects in the epidemic model (7). The martingale approach was initiated by Beretta et al. [35] and applied in [27, 30, 45, 47]. They introduced stochastic perturbation terms into the growth equations to incorporate the effect of a randomly fluctuating environment. In detail, assume that the stochastic perturbations of the state variables around their steady-state $E^{*}$ are of a white noise type which is proportional to the distances of $x, y$ from their steady-state values $x^{*}$ and $y^{*}$, respectively. In this way, model (7) will be reduced to the following form:

$$
\begin{align*}
& d x=\left(1-q x-y-\frac{a x y}{x+p y}\right) d t+\sigma_{1}\left(x-x^{*}\right) d B_{1}(t), \\
& d y=\left(\frac{R_{0} x y}{x+p y}-y\right) d t+\sigma_{2}\left(y-y^{*}\right) d B_{2}(t), \tag{77}
\end{align*}
$$

where the definitions of $\sigma_{1}, \sigma_{2}$ and $B_{1}(t), B_{2}(t)$ are the same as in (8).

If $R_{0}>1$, stochastic model (77) can center at its endemic point $E^{*}$, with the change of variables $u=x-x^{*}, v=y-y^{*}$. The linearized version of model (77) is as follows:

$$
\begin{equation*}
d z(t)=f_{1}(z(t)) d t+f_{2}(z(t)) d B(t) \tag{78}
\end{equation*}
$$

where

$$
\begin{gather*}
z(t)=\binom{u(t)}{v(t)}, \quad f_{1}=\binom{J_{11} u(t)+J_{12} v(t)}{J_{21} u(t)+J_{22} v(t)},  \tag{79}\\
f_{2}=\left(\begin{array}{cc}
\sigma_{1} u(t) & 0 \\
0 & \sigma_{2} v(t)
\end{array}\right),
\end{gather*}
$$

where $J_{11}, J_{12}, J_{21}, J_{22}$ are defined as (12).
It is easy to see that the stability of the endemic point $E^{*}$ of model (77) is equivalent to the stability of zero solution of model (78).

Before proving the stochastic stability of the zero solution of model (78), we put forward a lemma in [50].

Lemma 12. Suppose there exists a function $V(z, t) \in C^{2}(\Omega)$ satisfying the following inequalities:

$$
\begin{gather*}
K_{1}|z|^{\omega} \leq V(z, t) \leq K_{2}|z|^{\omega},  \tag{80}\\
L V(z, t) \leq-K_{3}|z|^{\omega} \tag{81}
\end{gather*}
$$

where $\omega>0$ and $K_{i}(i=1,2,3)$ is positive constant. Then the zero solution of mode (78) is exponentially $\omega$-stable for all time $t \geq 0$.

From the lemma above, note that if $\omega=2$ in (80) and (81), then the zero solution of model (78) is stochastically asymptotically stable in probability. Thus, we obtain the following theorem.

Theorem 13. Assume that $\sigma_{1}^{2}<2\left(p q R_{0}^{2}+a\left(R_{0}-1\right)^{2}\right) / p R_{0}^{2}$, $\sigma_{2}^{2}<2\left(R_{0}-1\right) / R_{0}$ hold; then the zero solution of model (78) is asymptotically mean square stable. And the endemic point $E^{*}$ of model (77) is asymptotically mean square stable.

The details of the proof are shown in the Appendix.
We should point out that the results obtained in this paper are only for the simple case when $l=h=1$ of the incidence rate (3). The dynamical behaviors of the stochastic epidemic model with general ratio-dependent incidence rate (3) are desirable in future studies.

## Appendix

## The proof of Theorem 13

Proof. Let us consider the Lyapunov function

$$
\begin{equation*}
V_{5}(z(t))=\frac{1}{2}\left(u^{2}+\kappa v^{2}\right) \tag{A.1}
\end{equation*}
$$

where $\kappa=\left(R_{0}^{2}+a\right) / R_{0}\left(R_{0}-1\right)^{2}$.
It is easy to check that inequality (80) holds with $\omega=2$. Moreover,

$$
\begin{align*}
L V_{5}(z(t))= & u\left(J_{11} u+J_{12} v\right)+\kappa v\left(J_{21} u+J_{22} v\right) \\
& +\frac{1}{2}\left(\sigma_{1}^{2} u^{2}+\kappa \sigma_{2}^{2} v^{2}\right) \\
= & \left(J_{11}+\frac{\sigma_{1}^{2}}{2}\right) u^{2}+\kappa\left(J_{22}+\frac{\sigma_{2}^{2}}{2}\right) v^{2}  \tag{A.2}\\
= & -z^{T} Q z,
\end{align*}
$$

where

$$
Q=\left(\begin{array}{cc}
J_{11}+\frac{\sigma_{1}^{2}}{2} & 0  \tag{A.3}\\
0 & \kappa\left(J_{22}+\frac{\sigma_{2}^{2}}{2}\right)
\end{array}\right)
$$

When $\sigma_{1}^{2}<2\left(p q R_{0}^{2}+a\left(R_{0}-1\right)^{2}\right) / p R_{0}^{2}, \sigma_{2}^{2}<2\left(R_{0}-1\right) / R_{0}$, the two eigenvalues $\lambda_{1}, \lambda_{2}$ of the matrix $Q$ will be positive. Set $\lambda_{\text {min }}=\min \left\{\lambda_{1}, \lambda_{2}\right\}$, it follows from (A.2) immediately that

$$
\begin{equation*}
L V_{5}(z(t)) \leq-\lambda_{\min }|z(t)|^{2} . \tag{A.4}
\end{equation*}
$$

We therefore have the assertion.

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