Abstract and Applied Analysis

Advances on Integrodifferential Equations and Transforms

Guest Editors: H. M. Srivastava, Xiao-Jun Yang, Dumitru Baleanu, Juan J. Nieto, and Jordan Hristov

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Editorial Board

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Editorial

Advances on Integrodifferential Equations and Transforms

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It is indeed a fairly common practice for scientific research journals and scientific research periodicals to publish special issues as well as conference proceedings. Quite frequently, these special issues are devoted exclusively to specific topics and/or are dedicated respectfully to commemorate the celebrated works of renowned research scientists. This special issue is an outcome of the ongoing importance and popularity of such topics as the theory and applications of various families of differential, integral, and integrodifferential equations as well as their *fractional* counterparts and associated integral and other transformations. We choose here to summarize most (if not all) of the main investigations which are contained in *this* special issue.

To begin with, C. Bianca et al. have investigated the existence problems for a partial integrodifferential equation with thermostat and time delay. Several Krasnoselskii type hybrid fixed point theorems together with their applications involving fractional integral equations are presented in the work by H. M. Srivastava et al. N. Wan et al. have studied the stabilized discretization in spline element method for solutions of some two-dimensional Navier-Stokes problems. Algorithmic investigation for a system of integral equations has been presented by Abdujabar Rasulov, Adem Kilicman, Zainidin Eshkuvatov, and Gulnora Raimova. I. Area et al. have derived fractional derivatives and primitives of several periodic functions. Applications of a local fractional functional method in solving diffusion equations on Cantor sets are discussed by Y. Cao et al. A study of higher-order sequential fractional differential inclusions with nonlocal three-point boundary conditions is presented by B. Ahmad and S. K. Ntouyas. D. Liu et al. have considered the Gerber-Shiu expected penalty function for the risk model with dependence and a constant dividend barrier. Some generalizations of convex functions on fractal sets are given by H. Mo and X. Sui. H. Guo et al. have successfully applied a Jacobi-collocation method for the second kind Volterra integral equations with a smooth kernel. Solutions of initial-boundary value problems for local fractional differential equation by means of local fractional Fourier series method are presented by Y. Zhang. X.-F. Niu et al., on the other hand, have studied some local fractional derivative boundary value problems for the Tricomi equation arising in fractal transonic flow. Existence of solutions for fractional q-integrodifference equations with nonlocal fractional q-integral conditions is discussed by S. Asawasamrit et al. Further generalizations of the celebrated Hölder's inequality and related results on fractal space are presented by G.-S. Chen et al. Q. M. Ul Hassan et al. introduce and study an analytical technique for finding solutions for higher-order nonlinear fractional evolution equations. Applications of some expansion techniques for solving the time-fractional modified Camassa-Holm (MCH) equation are discussed by M. Shakeel et al. N. K. Ashirbayev et al. consider the problem of solvability of an integral equation of Volterra-Wiener-Hopf type. Exact solutions of some nonlinear wave equations by the exp-function method are derived by M. Hu et al. E. Malkawi and D. Baleanu have investigated some fractional Killing-Yano tensors and Killing vectors using the Caputo (or, more accurately, the Liouville-Caputo) derivative in one- and two-dimensional curved space.

Acknowledgments

Finally, we thank all of the participating authors and the referees for their *invaluable* contributions toward the remarkable success of *this* special issue.

H. M. Srivastava
Xiao-Jun Yang
Dumitru Baleanu
Juan J. Nieto
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Corrigendum

Corrigendum to "Krasnosel'skii Type Hybrid Fixed Point Theorems and Their Applications to Fractional Integral Equations"

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In this note we correct some discrepancies that appeared in the paper by rewriting some statements and deleting proof of some theorems which already exist in our previous paper.

After examining different sections in the paper "Krasnosel'skii Type Hybrid Fixed Point Theorems and Their Applications to Fractional Integral Equations," we found some discrepancies. In this note, we slightly modify some of discrepancies by rewriting some statements and deleting proof of some theorems which already exist in our previous paper to achieve our claim.

We rewrite page 1, left side, line 1–line 4 (from Introduction), as follows.

The main result of Nieto and Rodríguez-López [1] is the following hybrid fixed point theorem.

We rewrite page 1, left side, line 1-line 2 (from bottom), as follows.

Another version of the above fixed point theorem can be stated as follows.

We rewrite page 2, left side, line 22–line 39, as follows.

The fixed point result of Heikkilä and Lakshmikantham [3] which originates all the above theoretical results differentiates in the convergence criteria of the sequence of iterations of the monotone mapping is as follows.

We rewrite page 2, left side, line 1–line 7 (from bottom), and right side, line 1–line 13, as follows.

Recently, Dhage [4, 5] and Bedre et al. [6] have obtained the Krasnosel'skii type fixed point theorems for monotone mappings.

We rewrite page 2, right side, line 11-line 12 (from bottom), as follows.

Now we consider the following definitions.

We rewrite page 3, left side, line 7–line 10 (from bottom), as follows.

We now state the basic hybrid fixed point results by Bedre et al. [6] using the argument from algebra, analysis, and geometry. The slight generalization of Theorem 4 and Dhage [8] using M-contraction is stated as follows.

We delete the proof of Theorem 14 and Corollary 15 and rewrite the statements as follows.

Theorem 14 (see Bedre et al. [6]). *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $T : X \to X$ be a monotone function (nondecreasing or nonincreasing) such that there exists an M-function φ_T such that*

$$d\left(T\left(x\right), T\left(y\right)\right) \leqq \varphi_T\left(d\left(x, y\right)\right) \tag{6}$$

for all comparable elements $x, y \in X$ satisfying $\varphi_T(r) < r$ $(r > 0)$. Suppose that either T is continuous or X is such that if $x_n \to x$ is a sequence in X whose consecutive terms *are comparable, then there exists a subsequence* $\{x_{n_k}\}_{k\in\mathbb{N}}$ *of* $\{x_n\}_{n\in\mathbb{N}}$ *such that every term comparable to the limit* x*. If there exists* $x_0 \in X$ *with* $x_0 \leqq T(x_0)$ *or* $x_0 \geqq T(x_0)$*, then* T *has a fixed point which is unique if "every pair of elements in* X *has a lower and an upper bound."*

Corollary 15 (see Bedre et al. [6]). *Let* $(X, \preceq)$ *be a partially ordered set and suppose that there exists a metric* d *in* X *such that* (X, d) *is a complete metric space. Let* $T : X \to X$ *be a monotone function (nondecreasing or nonincreasing) such that there exists an* M*-function* φ_T *and a positive integer* p *such that*

$$d\left(T^{p}\left(x\right), T^{p}\left(y\right)\right) \leqq \varphi_T\left(d\left(x, y\right)\right) \tag{7}$$

for all comparable elements $x, y \in X$ *satisfying* $\varphi_T(r) < r$ $(r > 0)$*. Suppose that either* T *is continuous or* X *is such that if* $x_n \to x$ *is a sequence in* X *whose consecutive terms are comparable, then there exists a subsequence* $\{x_{n_k}\}_{k\in\mathbb{N}}$ *of* $\{x_n\}_{n\in\mathbb{N}}$ *such that every term comparable to the limit* x*. If there exists* $x_0 \in X$ *with* $x_0 \leqq T(x_0)$ *or* $x_0 \geqq T(x_0)$*, then* T *has a fixed point which is unique if "every pair of elements in* X *has a lower and an upper bound."*

We rewrite page 4, left side, line 7–line 15 (from bottom), as follows.

We now consider the following definition.

We rewrite page 4, right side, line 6–line 11 (from bottom), as follows.

The following Krasnosel'skii type fixed point theorem is proved in Dhage [5].

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Research Article

Fractional Cauchy Problem with Caputo Nabla Derivative on Time Scales

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The definition of Caputo fractional derivative is given and some of its properties are discussed in detail. After then, the existence of the solution and the dependency of the solution upon the initial value for Cauchy type problem with fractional Caputo nabla derivative are studied. Also the explicit solutions to homogeneous equations and nonhomogeneous equations are derived by using Laplace transform method.

1. Introduction

Fractional differential equation theory has gained considerable popularity and importance due to their numerous applications in many fields of science and engineering including physics, population dynamics, chemical technology, biotechnology, aerodynamics, electrodynamics of complex medium, polymer rheology, control of dynamical systems, and so on (see, e.g., [1–4], and the references therein). On the other hand, in real applications, it is not always continuous case, but also discrete case. For example, in recent papers [5–8], in order to deeply understand the background of the discrete dynamics behaviors, some interesting results are obtained by applying the discrete fractional calculus to discrete chaos behaviors. In [9–12], the delta type discrete fractional calculus is studied. In [13, 14], the nabla type discrete fractional calculus is studied. In [15], the theory of fractional backward difference equations (i.e., the nabla type fractional difference equations) has been studied in detail. So how to unify continuous fractional calculus and discrete fractional calculus is a natural problem. In order to unify differential equations and difference equations, Hilger [16] proposed firstly the time scale and then some relevant basic theories are studied by some authors (see [17–22]). Recently, some authors studied fractional calculus on time scales (see [23–25]), where Williams [24] gives a definition of fractional integral and derivative on time scales to unify three cases of specific time scales, which improved the results in [23]. Bastos gives definition of fractional Δ-integral and Δ-derivative on time scales in [25]. The delta fractional calculus and Laplace transform on some specific discrete time scales are also discussed in [26–28]. In the light of the above work, we further studied the theory of fractional integral and derivative on general time scales in [29], where ∇-Laplace transform, fractional ∇-power function, ∇-Mittag-Leffler function, fractional ∇-integrals, and fractional ∇-differential on time scales are defined. Some of their properties are discussed in detail. After then, by using Laplace transform method, the existence of the solution and the dependency of the solution upon the initial value for Cauchy type problem with Riemann-Liouville fractional ∇-derivative are studied. Also the explicit solutions to homogeneous equations and nonhomogeneous equations are derived by using Laplace transform method. But there is a shortcoming for Riemann-Liouville fractional ∇-derivative. That is, Cauchy type problem with Riemann-Liouville fractional order derivative and the Laplace transform of Riemann-Liouville fractional order derivative require the initial conditions in terms of non-integer derivatives, which are very difficult to be interpreted from the physical point of view. Thus this paper's focus on defining nabla type Caputo fractional derivative on time scales proves some useful property about Caputo fractional derivative and then studies some Caputo fractional differential equations on time scales.

The structure of this paper is as follows. In Section 2, we give some preliminaries about time scales, generalized ∇-power function, and Riemann-Liouville ∇-integral and ∇-derivative. In Section 3, we present the definitions and the properties of the Caputo nabla derivative on time scales in detail. Then in Section 4, Cauchy type problem with Caputo fractional derivative is discussed. For the Caputo fractional differential initial value problem, we discuss the dependency of the solution upon the initial value. In Section 5, by applying the Laplace transform method, we study the fractional order linear differential equations with Caputo fractional derivative. We derive explicit solutions and fundamental system of solutions to homogeneous equations with constant coefficients and find particular solution and general solutions of the corresponding nonhomogeneous equations.

2. Preliminaries

First, we present some preliminaries about time scales in [17].

Definition 1 (see [17]). A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers.

Definition 2 (see [17]). For $t \in \mathbb{T}$ one defines the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \tag{1}$$

while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}. \tag{2}$$

If $\sigma(t) > t$, we say that t is right-scattered, while if $\rho(t) < t$, we say that t is left-scattered. Points that are right-scattered and left-scattered at the same time are called isolated. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. Finally, the graininess function $\nu : \mathbb{T} \to [0, \infty)$ is defined by

$$\nu(t) := t - \rho(t). \tag{3}$$

Definition 3 (see [17]). If $\mathbb{T}$ has a right-scattered minimum m, then one defines $\mathbb{T}_k = \mathbb{T} - \{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$. Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}_k$. Then one defines $f^{\nabla}(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t (i.e., $U = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$\left|[f(\rho(t)) - f(s)] - f^{\nabla}(t)[\rho(t) - s]\right| \le \varepsilon \left|\rho(t) - s\right| \quad \forall s \in U. \tag{4}$$

We call $f^{\nabla}(t)$ the nabla derivative of f at t.

Definition 4 (see [17]). A function $f : \mathbb{T} \to \mathbb{R}$ is called regulated provided its right-sided limits exist (finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$. A function $f : \mathbb{T} \to \mathbb{R}$ is called ld-continuous provided it is continuous at left-dense points in $\mathbb{T}$ and its right-sided limits exist (finite) at right-dense points in $\mathbb{T}$.

Definition 5 (see [17, page 100]). The generalized nabla type polynomials are the functions $\widehat{h}_k : \mathbb{T}^2 := \mathbb{T} \times \mathbb{T} \to \mathbb{R}$, $k \in \mathbb{N}_0$, defined recursively as follows. The function $\widehat{h}_0$ is

$$\widehat{h}_0(t, s) = 1 \quad \forall s, t \in \mathbb{T}, \tag{5}$$

and given $\widehat{h}_k$ for $k \in \mathbb{N}_0$, the function $\widehat{h}_{k+1}$ is

$$\widehat{h}_{k+1}(t, s) = \int_s^t \widehat{h}_k(\tau, s) \nabla\tau \quad \forall s, t \in \mathbb{T}. \tag{6}$$

Definition 6 (see [18, page 38]). The generalized delta type polynomials are the functions $h_k : \mathbb{T}^2 := \mathbb{T} \times \mathbb{T} \to \mathbb{R}$, $k \in \mathbb{N}_0$, defined recursively as follows. The function h_0 is

$$h_0(t, s) = 1 \quad \forall s, t \in \mathbb{T}, \tag{7}$$

and given h_k for $k \in \mathbb{N}_0$, the function h_{k+1} is

$$h_{k+1}(t, s) = \int_s^t h_k(\tau, s) \Delta\tau \quad \forall s, t \in \mathbb{T}. \tag{8}$$

It is similar to the discussion in the reference [17, (page 103)] for $n \in \mathbb{N}_0$ and ld-continuous functions $p_i : \mathbb{T} \to \mathbb{R}$, $1 \le i \le n$, we consider the nth order linear dynamic equation

$$Ly = 0, \quad \text{where } Ly = y^{\nabla^n} + \sum_{i=1}^{n} p_i y^{\nabla^{n-i}}. \tag{9}$$

Definition 7 (see [17]). One defines the Cauchy function $y : \mathbb{T} \times \mathbb{T}_{k^n} \to \mathbb{R}$ for the linear dynamic equation (9) to be for each fixed $s \in \mathbb{T}_{k^n}$ the solution of the initial value problem

$$\begin{aligned} Ly = 0, \qquad & y^{\nabla^i}(\rho(s), s) = 0, \quad 0 \le i \le n-2, \\ & y^{\nabla^{n-1}}(\rho(s), s) = 1. \end{aligned} \tag{10}$$

Remark 8 (see [17]). Note that

$$y(t, s) := \widehat{h}_{n-1}(t, \rho(s)) \tag{11}$$

is the Cauchy function for y^{∇^n}.

Theorem 9 (see [17] (variation of constants)). *Let $\alpha \in \mathbb{T}_{k^n}$ and $t \in \mathbb{T}$. If $f \in C_{ld}$, then the solution of the initial value problem*

$$\begin{aligned} Ly &= f(t), \\ y^{\nabla^i}(\alpha) &= 0, \quad 0 \le i \le n-1 \end{aligned} \tag{12}$$

is given by

$$y(t) = \int_\alpha^t y(t, \tau) f(\tau) \nabla\tau, \tag{13}$$

where $y(t, \tau)$ is the Cauchy function for (9).

Theorem 10 (see [17] (Taylor's Formula)). *Let $n \in \mathbb{N}$. Suppose the function f is such that $f^{\nabla^{n+1}}$ is ld-continuous on $\mathbb{T}_{k^{n+1}}$. Let $\alpha \in \mathbb{T}_{k^n}$, $t \in \mathbb{T}$. Then one has*

$$f(t) = \sum_{k=0}^{n} \widehat{h}_k(t, \alpha) f^{\nabla^k}(\alpha) + \int_\alpha^t \widehat{h}_n(t, \rho(\tau)) f^{\nabla^{n+1}}(\tau) \nabla\tau. \tag{14}$$

Definition 11 (see [24]). A subset $I \subset \mathbb{T}$ is called a time scale interval, if it is of the form $I = A \cap \mathbb{T}$ for some real interval $A \subset \mathbb{R}$. For a time scale interval I, a function $f : I \to \mathbb{R}$ is said to be left-dense absolutely continuous if for all $\varepsilon > 0$ there exist $\delta > 0$ such that $\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon$ whenever a disjoint finite collection of subtime scale intervals $(a_k, b_k] \cap \mathbb{T} \subset I$ for $1 \le k \le n$ satisfies $\sum_{k=1}^{n} |b_k - a_k| < \delta$. One denotes $f \in AC_\nabla$. If $f^{\nabla^{m-1}} \in AC$, then one denotes $f \in AC_\nabla^m$.

Theorem 12 (see [4]). *Let X be a normed linear space, $\mathscr{C} \subset X$ a convex set, and U open in $\mathscr{C}$ with $\theta \in U$. Let $T : \overline{U} \to \mathscr{C}$ be a continuous and compact mapping. Then either*

(i) *the mapping T has a fixed point in $\overline{U}$, or*

(ii) *there exists $u \in \partial U$ and $\lambda \in (0,1)$ with $u = \lambda T u$.*

The following results can be found in our recent paper [29].

Lemma 13 (see [29]). *Let $E \subset \mathbb{T} - \{\max \mathbb{T}\}$ be a measurable set. If $f : \mathbb{T} \to \mathbb{R}$ is integrable on E, then*

$$\int_E f^\sigma(s)\,\Delta s = \int_E f(s)\,\nabla s. \tag{15}$$

From now on, let $\mathbb{T}$ be a time scale such that $\sup \mathbb{T} = \infty$ and fix $t_0 \in \mathbb{T}$.

Definition 14 (see [29]). Assume that $x : \mathbb{T} \to \mathbb{R}$ is regulated and $t_0 \in \mathbb{T}$. Then the Laplace transform of x is defined by

$$\mathscr{L}_{\nabla,t_0}\{x\}(z) = \int_{t_0}^{\infty} x(t)\,\widehat{e}^{\rho}_{\ominus_\nu z}(t,t_0)\,\nabla t. \tag{16}$$

for $z \in \mathscr{D}\{x\}$, where $\mathscr{D}\{x\}$ consists of all complex numbers $z \in \mathscr{R}_\nu$ for which the improper integral exists.

Theorem 15 (see [29]). *Assume that $x : \mathbb{T} \to \mathbb{C}$ is such that x^{∇^k} is regulated. Then*

$$\mathscr{L}_{\nabla,t_0}\left\{x^{\nabla^k}\right\}(z) = z^k \mathscr{L}_{\nabla,t_0}\{x\}(z) - \sum_{i=0}^{k-1} z^{k-i-1} x^{\nabla^i}(t_0) \tag{17}$$

for those regressive $z \in \mathbb{C}$ satisfying $\lim_{t\to\infty}\{x^{\nabla^i}(t)\widehat{e}_{\ominus_\nu z}(t,t_0)\} = 0$, $i = 0, 1, \dots, k-1$.

Definition 16 (see [29]). One defines fractional generalized ∇-power function on time scales

$$\widehat{h}_\alpha(t,t_0) = \mathscr{L}^{-1}_{\nabla,t_0}\left\{\frac{1}{z^{\alpha+1}}\right\}(t) \quad (\alpha > -1) \tag{18}$$

to those regressive $z \in \mathbb{C}\backslash\{0\}$, $t \ge t_0$; and for $t < t_0$, $\widehat{h}_\alpha(t,t_0) = 0$.

Here we introduce generalized ∇-derivative on time scales:

$$\int f^\nabla g \nabla t = -\int f^\rho g^\nabla \nabla t. \tag{19}$$

Since $\widehat{h}_\alpha(t,t_0)$ $(\alpha > -1)$ is integral, we can consider it as a generalized function, and thus we can define $\widehat{h}_\alpha(t,t_0) = D_\nabla \widehat{h}_{\alpha+1}(t,t_0)$ for $-2 < \alpha \le -1$, where D_∇ here means a generalized derivative. In the same way, we can define $\widehat{h}_\alpha(t,t_0)$ for $\alpha \le -1$.

For $\alpha > 0$, we have

$$\widehat{h}_\alpha(t_0,t_0) = 0. \tag{20}$$

Definition 17 (see [29]). For a given $f : [t_0,\infty)_\mathbb{T} \to \mathbb{C}$, the solution of the shifting problem

$$\begin{gathered} u^{\nabla_t}(t,\rho(s)) = -u^{\nabla_s}(t,s), \quad t,s \in \mathbb{T},\ t \ge s \ge t_0, \\ u(t,t_0) = f(t), \quad t \in \mathbb{T},\ t \ge t_0, \end{gathered} \tag{21}$$

is denoted by $\widetilde{f}$ and is called the shift of f.

Definition 18 (see [29]). For given functions $f, g : \mathbb{T} \to \mathbb{R}$, their convolution $f * g$ is defined by

$$(f * g)(t) = \int_{t_0}^{t} \widetilde{f}(t,\rho(\tau))\,g(\tau)\,\nabla\tau, \quad t \in \mathbb{T}, \tag{22}$$

where $\widetilde{f}$ is the shift of f, which is introduced in Definition 17.

Definition 19 (see [29]). Fractional generalized ∇-power function $\widehat{h}_\alpha(t,s)$ on time scales is defined as the shift of $\widehat{h}_\alpha(t,t_0)$; that is,

$$\widehat{h}_\alpha(t,s) = \widetilde{\widehat{h}_\alpha(\cdot,t_0)}(t,s) \quad (t \ge s \ge t_0). \tag{23}$$

In this paper, we always denote $\Omega := [t_0,t_1]_\mathbb{T}$ a finite interval on a time scale $\mathbb{T}$ $(\sup \mathbb{T} = \infty)$.

Definition 20 (see [29]). Let $t, t_0 \in \Omega$. The Riemann-Liouville fractional ∇-integral $I^\alpha_{\nabla,t_0} f$ of order $\alpha > 0$ is defined by

$$\begin{aligned} I^\alpha_{\nabla,t_0} f(t) &:= \widehat{h}_{\alpha-1}(t,t_0) * f(t) \\ &= \int_{t_0}^{t} \widetilde{\widehat{h}_{\alpha-1}(\cdot,t_0)}(t,\rho(\tau))\,f(\tau)\,\nabla\tau \\ &= \int_{t_0}^{t} \widehat{h}_{\alpha-1}(t,\rho(\tau))\,f(\tau)\,\nabla\tau \quad (t > t_0). \end{aligned} \tag{24}$$

Definition 21 (see [29]). Let $t, t_0 \in \Omega$. The Riemann-Liouville fractional ∇-derivative $D^\alpha_{\nabla,t_0} f$ of order $\alpha \ge 0$ is defined by

$$D^\alpha_{\nabla,t_0} f(t) = D^m_\nabla I^{m-\alpha}_{\nabla,t_0} f(t) \quad (m = [\alpha]+1;\ t > t_0). \tag{25}$$

Throughout this paper, we denote $f^{\nabla^n} = D^n_\nabla f = D^n_{\nabla,t_0} f$, $n \in \mathbb{N}$.

Property 1 (see [29]). Let $\alpha \ge 0$, $m = [\alpha]+1$, $\beta > 0$, $t, t_0 \in \Omega_{k^m}$. Then

$$\begin{gathered} (1)\ I^\alpha_{\nabla,t_0} \widehat{h}_{\beta-1}(t,t_0) = \widehat{h}_{\alpha+\beta-1}(t,t_0), \quad (\alpha > 0); \\ (2)\ D^\alpha_{\nabla,t_0} \widehat{h}_{\beta-1}(t,t_0) = \widehat{h}_{\beta-\alpha-1}(t,t_0), \quad (\alpha \ge 0). \end{gathered} \tag{26}$$

Property 2 (see [29]). If $\alpha > 0$ and $\beta > 0$, then the equation

$$I^{\alpha}_{\nabla,t_0} I^{\beta}_{\nabla,t_0} f(t) = I^{\alpha+\beta}_{\nabla,t_0} f(t) \tag{27}$$

is satisfied at almost every point $t \in \Omega$ for $f(t) \in L_{\nabla,p}(\Omega)$ $(1 \le p \le \infty)$.

Property 3 (see [29]). If $\alpha > 0$ and $f(t) \in L_{\nabla,p}(\Omega)$ $(1 \le p \le \infty)$, then the following equality

$$D^{\alpha}_{\nabla,t_0} I^{\alpha}_{\nabla,t_0} f(t) = f(t) \tag{28}$$

holds almost everywhere on Ω.

Property 4 (see [29]). If $\alpha > \beta > 0$, then, for $f(t) \in L_{\nabla,p}(\Omega)$ $(1 \le p \le \infty)$, the relation

$$D^{\beta}_{\nabla,t_0} I^{\alpha}_{\nabla,t_0} f(t) = I^{\alpha-\beta}_{\nabla,t_0} f(t) \tag{29}$$

holds almost everywhere on Ω. In particular, when $\beta = k \in \mathbb{N}$ and $\alpha > k$, then

$$D^{k}_{\nabla,t_0} I^{\alpha}_{\nabla,t_0} f(t) = I^{\alpha-k}_{\nabla,t_0} f(t). \tag{30}$$

Property 5 (see [29]). Let $\alpha > 0$, $m = [\alpha]+1$ and let $f_{m-\alpha}(t) = I^{m-\alpha}_{\nabla,t_0} f(t)$.

(1) If $1 \le p \le \infty$ and $f(t) \in I^{\alpha}_{\nabla,t_0}(L_{\nabla,p})$, then

$$I^{\alpha}_{\nabla,t_0} D^{\alpha}_{\nabla,t_0} f(t) = f(t). \tag{31}$$

(2) If $f(t) \in L_{\nabla,1}(\Omega)$ and $f_{m-\alpha}(t) \in AC^{m}_{\nabla}(\Omega)$, then the equality

$$I^{\alpha}_{\nabla,t_0} D^{\alpha}_{\nabla,t_0} f(t) = f(t) - \sum_{k=1}^{m} \widehat{h}_{\alpha-k}(t,t_0) D^{\alpha-k}_{\nabla,t_0} f(t_0) \tag{32}$$

holds almost everywhere on Ω, where $D^{\alpha-m}_{\nabla,t_0} y(t_0) = \lim_{t\to t_0^+} I^{m-\alpha}_{\nabla,t_0} y(t)$.

Lemma 22 (see [29]). *Let $\alpha > 0$, $m-1 < \alpha \le m$ $(m \in \mathbb{N})$ and $f : \Omega \to \mathbb{R}$. For $t_0, t \in \Omega_{k^m}$ with $t_0 < t$. Then one has the following.*

(1) *If $f \in L_{\nabla,p}(\Omega)$, then*

$$\mathscr{L}_{\nabla,t_0}\left\{I^{\alpha}_{\nabla,t_0} f(t)\right\}(z) = \frac{1}{z^{\alpha}} \mathscr{L}_{\nabla,t_0}\left\{f(t)\right\}(z). \tag{33}$$

(2) *If $f \in AC^{m}_{\nabla}(\Omega)$, then*

$$\begin{aligned} &\mathscr{L}_{\nabla,t_0}\left\{D^{\alpha}_{\nabla,t_0} f(t)\right\}(z) \\ &\quad = z^{\alpha}\mathscr{L}_{\nabla,t_0}\left\{f(t)\right\}(z) - \sum_{j=1}^{m} z^{j-1} D^{\alpha-j}_{\nabla,t_0} f(t_0), \end{aligned} \tag{34}$$

for those regressive $z \in \mathbb{C}$ satisfying $\lim_{t\to\infty}\{D^{j}_{\nabla} I^{m-\alpha}_{\nabla,t_0} f(t) \widehat{e}_{\ominus_{\nu} z}(t,t_0)\} = 0$, $j = 0,1,\ldots,m-1$.

Definition 23 (see [29]). ∇-Mittag-Leffler function is defined as

$$_{\nabla}F_{\alpha,\beta}(\lambda;t,t_0) = \sum_{j=0}^{\infty} \lambda^{j} \widehat{h}_{\alpha j+\beta-1}(t,t_0) \tag{35}$$

provided the right-hand series is convergent, where $\alpha, \beta > 0$, $\lambda \in \mathbb{R}$.

Theorem 24 (see [29]). *The Laplace transform of ∇-Mittag-Leffler function is*

$$\mathscr{L}_{\nabla,t_0}\left\{_{\nabla}F_{\alpha,\beta}(\lambda;t,t_0)\right\}(z) = \frac{z^{\alpha-\beta}}{z^{\alpha}-\lambda} \left(|\lambda| < |z|^{\alpha}\right). \tag{36}$$

By differentiating k times with respect to λ on both sides of the formula in the theorem above, we get the following result:

$$\mathscr{L}_{\nabla,t_0}\left\{\frac{\partial^k}{\partial\lambda^k}\, _{\nabla}F_{\alpha,\beta}(\lambda;t,t_0)\right\}(z) = \frac{k!\, z^{\alpha-\beta}}{(z^{\alpha}-\lambda)^{k+1}}. \tag{37}$$

3. Definition and Properties of Caputo Fractional Derivative on Time Scales

Definition 25. Let $t, t_0 \in \Omega$. The Caputo fractional derivative of order $\alpha \ge 0$ is defined via Riemann-Liouville fractional derivative by

$$\begin{aligned} {}^{C}D^{\alpha}_{\nabla,t_0} f(t) &:= D^{\alpha}_{\nabla,t_0}\left[f(t) - \sum_{k=0}^{m-1} \widehat{h}_k(t,t_0) f^{\nabla^k}(t_0)\right] \\ &\qquad (t > t_0), \end{aligned} \tag{38}$$

where

$$m = [\alpha]+1 \quad \text{for } \alpha \notin \mathbb{N};\; m = \alpha \text{ for } \alpha \in \mathbb{N}. \tag{39}$$

In particular, when $0 < \alpha < 1$, the relation (38) takes the following forms:

$${}^{C}D^{\alpha}_{\nabla,t_0} f(t) = D^{\alpha}_{\nabla,t_0}\left[f(t) - f(t_0)\right]. \tag{40}$$

If $\alpha \notin \mathbb{N}$, then the Caputo fractional derivative coincides with the Riemann-Liouville fractional derivative in the following case:

$${}^{C}D^{\alpha}_{\nabla,t_0} f(t) = D^{\alpha}_{\nabla,t_0} f(t), \tag{41}$$

if $f^{\nabla^k}(t_0) = 0$ $(k = 0,1,\ldots,m-1, m = [\alpha]+1)$.

In particular, when $0 < \alpha < 1$, we have

$${}^{C}D^{\alpha}_{\nabla,t_0} f(t) = D^{\alpha}_{\nabla,t_0} f(t), \quad \text{when } f(t_0) = 0. \tag{42}$$

If $\alpha = m \in \mathbb{N}$ and the usual nabla derivative $f^{\nabla^m}(t)$ of order m exists, then ${}^{C}D^{m}_{\nabla,t_0} f(t)$ coincides with $f^{\nabla^m}(t)$:

$${}^{C}D^{m}_{\nabla,t_0} f(t) = f^{\nabla^m}(t) \quad (m \in \mathbb{N}). \tag{43}$$

The Caputo fractional derivative ${}^{C}D^{\alpha}_{\nabla,t_0}f(t)$ is defined for functions $f(t)$ for which the Riemann-Liouville fractional derivative of the right-hand sides of (38) exists. In particular, they are defined for $f(t)$ belonging to the space $AC^{m}_{\nabla}(\Omega)$ of absolutely continuous functions defined in Definition 11. Thus the following statement holds.

Property 6. Let $\alpha \geq 0$ and let m be given by (39). If $f(t) \in AC^{m}_{\nabla}(\Omega)$, then the Caputo fractional derivative ${}^{C}D^{\alpha}_{\nabla,t_0}f(t)$ exists almost everywhere on Ω_{k^m}.

(a) If $\alpha \notin \mathbb{N}$, ${}^{C}D^{\alpha}_{\nabla,t_0}f(t)$ is represented by

$$ {}^{C}D^{\alpha}_{\nabla,t_0}f(t) = \widehat{h}_{m-\alpha-1}(t,t_0) * f^{\nabla^m}(t) =: I^{m-\alpha}_{\nabla,t_0}D^{m}_{\nabla}f(t), \tag{44} $$

where $m = [\alpha]+1$. Thus when $\alpha \notin \mathbb{N}$, ${}^{C}D^{\alpha}_{\nabla,t_0}f(t_0) = 0$, where the notation ${}^{C}D^{\alpha}_{\nabla,t_0}f(t_0)$ denote the limit of ${}^{C}D^{\alpha}_{\nabla,t_0}f(t)$ as $t \to t_0^{+}$.

In particular, when $0 < \alpha < 1$ and $f(t) \in AC_{\nabla}(\Omega)$,

$$ {}^{C}D^{\alpha}_{\nabla,t_0}f(t) = \widehat{h}_{-\alpha}(t,t_0) * f^{\nabla}(t) =: I^{1-\alpha}_{\nabla,t_0}f^{\nabla}(t). \tag{45} $$

(b) If $\alpha = m \in \mathbb{N}$, then ${}^{C}D^{\alpha}_{\nabla,t_0}f(t)$ is represented by (43). In particular,

$$ {}^{C}D^{0}_{\nabla,t_0}f(t) = f(t). \tag{46} $$

Proof. (a) By Taylor's formula on time scales

$$ \begin{aligned} f(t) &= \sum_{k=0}^{m-1}\widehat{h}_k(t,t_0)f^{\nabla^k}(t_0) + \int_{t_0}^{t}\widehat{h}_{m-1}(t,\rho(\tau))f^{\nabla^m}(\tau)\nabla\tau \\ &= \sum_{k=0}^{m-1}\widehat{h}_k(t,t_0)f^{\nabla^k}(t_0) + I^{m}_{\nabla,t_0}f^{\nabla^m}(t) \end{aligned} \tag{47} $$

and using (29), we have

$$ \begin{aligned} {}^{C}D^{\alpha}_{\nabla,t_0}f(t) &= D^{\alpha}_{\nabla,t_0}\left[f(t) - \sum_{k=0}^{m-1}\widehat{h}_k(t,t_0)f^{\nabla^k}(t_0)\right] \\ &= D^{\alpha}_{\nabla,t_0}I^{m}_{\nabla,t_0}f^{\nabla^m}(t) \\ &= I^{m-\alpha}_{\nabla,t_0}f^{\nabla^m}(t). \end{aligned} \tag{48} $$

(b) If $\alpha = m \in \mathbb{N}$, then (38) takes the form

$$ {}^{C}D^{m}_{\nabla,t_0}f(t) = D^{m}_{\nabla,t_0}\left[f(t) - \sum_{k=0}^{m-1}\widehat{h}_k(t,t_0)f^{\nabla^k}(t_0)\right], \tag{49} $$

and, from Taylor's formula and (28), we derive ${}^{C}D^{m}_{\nabla,t_0}f(t) = f^{\nabla^m}(t)$. □

Property 7. Let $\alpha > 0$ and let m be given by (39), $\beta > 0$, $t \in \Omega_{k^m}$. Then

$$ {}^{C}D^{\alpha}_{\nabla,t_0}\widehat{h}_{\beta-1}(t,t_0) = \widehat{h}_{\beta-\alpha-1}(t,t_0) \quad (\beta > m), \tag{50} $$

$$ {}^{C}D^{\alpha}_{\nabla,t_0}\widehat{h}_k(t,t_0) = 0 \quad (k = 0,1,\ldots,m-1). \tag{51} $$

In particular,

$$ {}^{C}D^{\alpha}_{\nabla,t_0}1 = 0. \tag{52} $$

Proof. From Property 6 and (26), it is obtained that for $\alpha \notin \mathbb{N}$,

$$ \begin{aligned} {}^{C}D^{\alpha}_{\nabla,t_0}\widehat{h}_{\beta-1}(t,t_0) &= I^{m-\alpha}_{\nabla,t_0}D^{m}_{\nabla}\widehat{h}_{\beta-1}(t,t_0) \\ &= I^{m-\alpha}_{\nabla,t_0}\widehat{h}_{\beta-m-1}(t,t_0) = \widehat{h}_{\beta-\alpha-1}(t,t_0), \\ {}^{C}D^{\alpha}_{\nabla,t_0}\widehat{h}_k(t,t_0) &= I^{m-\alpha}_{\nabla,t_0}D^{m}_{\nabla}\widehat{h}_k(t,t_0) = I^{m-\alpha}_{\nabla,t_0}0 = 0 \\ &\qquad (k = 0,1,\ldots,m-1,\ t > t_0), \end{aligned} \tag{53} $$

while for $\alpha = m \in \mathbb{N}$,

$$ \begin{aligned} {}^{C}D^{m}_{\nabla,t_0}\widehat{h}_{\beta-1}(t,t_0) &= D^{m}_{\nabla}\widehat{h}_{\beta-1}(t,t_0) = \widehat{h}_{\beta-m-1}(t,t_0), \\ {}^{C}D^{m}_{\nabla,t_0}\widehat{h}_k(t,t_0) &= D^{m}_{\nabla}\widehat{h}_k(t,t_0) = 0 \\ &\qquad (k = 0,1,\ldots,m-1,\ t > t_0). \end{aligned} \tag{54} $$

□

Property 8. Let $\alpha > 0$ and let $f(t) \in L_{\nabla,\infty}(\Omega)$ or $f(t) \in AC_{\nabla}(\Omega)$. Then

$$ {}^{C}D^{\alpha}_{\nabla,t_0}I^{\alpha}_{\nabla,t_0}f(t) = f(t). \tag{55} $$

Proof. Let $f(t) \in L_{\nabla,\infty}(\Omega)(f(t) \in AC_{\nabla}(\Omega))$, and let $\alpha > 0$ and $k = 0,1,\ldots,m-1$. Since $f(t) \in L_{\nabla,\infty}(\Omega)(f(t) \in AC_{\nabla}(\Omega))$, then for a.e. (for any) $t \in \Omega_{k^m}$, we get

$$ I^{\alpha-k}_{\nabla,t_0}f(t) = \int_{t_0}^{t}\widehat{h}_{\alpha-k-1}(t,\rho(\tau))f(\tau)\nabla\tau \leq K^2\widehat{h}_1(t,t_0) $$
$$ \left(\text{where } \max_{t,\tau\in\Omega}\left\{\|f\|_{L_{\nabla,\infty}}, \|f\|_{AC_{\nabla}}, \left|\widehat{h}_{\alpha-k-1}(t,\rho(\tau))\right|\right\} \leq K\right) \tag{56} $$

for any $k = 0,1,\ldots,m-1 = [\alpha]$, and hence

$$ D^{k}_{\nabla}I^{\alpha}_{\nabla,t_0}f(t_0) = I^{\alpha-k}_{\nabla,t_0}f(t_0) = 0 \quad (k = 0,1,\ldots,m-1). \tag{57} $$

Thus using (41) for $\alpha \notin \mathbb{N}$ with $f(t)$ replaced by $I^{\alpha}_{\nabla,t_0}f(t)$ and (28), we derive

$$ {}^{C}D^{\alpha}_{\nabla,t_0}I^{\alpha}_{\nabla,t_0}f(t) = f(t). \tag{58} $$

For $\alpha = m \in \mathbb{N}$,

$$ {}^{C}D^{\alpha}_{\nabla,t_0}I^{\alpha}_{\nabla,t_0}f(t) = D^{m}_{\nabla,t_0}I^{m}_{\nabla,t_0}f(t) = f(t). \tag{59} $$

□

Property 9. Let $\alpha > 0$ and let m be given by (39). If $f(t) \in AC_{\nabla}^{m}(\Omega)$, then

$$I_{\nabla,t_0}^{\alpha}\,{}^{C}D_{\nabla,t_0}^{\alpha} f(t) = f(t) - \sum_{k=0}^{m-1} \widehat{h}_k(t,t_0)\, D_{\nabla,t_0}^{k} f(t_0). \tag{60}$$

In particular, if $0 < \alpha \leq 1$ and $f(t) \in AC_{\nabla}(\Omega)$, then

$$I_{\nabla,t_0}^{\alpha}\,{}^{C}D_{\nabla,t_0}^{\alpha} f(t) = f(t) - f(t_0). \tag{61}$$

Proof. Let $\alpha \notin \mathbb{N}$. If $f(t) \in AC_{\nabla}^{m}(\Omega)$, then using Property 6, (27) and (32), we have

$$\begin{aligned} I_{\nabla,t_0}^{\alpha}\,{}^{C}D_{\nabla,t_0}^{\alpha} f(t) &= I_{\nabla,t_0}^{\alpha} I_{\nabla,t_0}^{m-\alpha} D_{\nabla}^{m} f(t) = I_{\nabla,t_0}^{m} D_{\nabla}^{m} f(t) \\ &= f(t) - \sum_{k=0}^{m-1} \widehat{h}_k(t,t_0)\, D_{\nabla,t_0}^{k} f(t_0). \end{aligned} \tag{62}$$

For $\alpha = m \in \mathbb{N}$, the result is obvious from Property 6 and (32). □

Property 10. Assume that $f(t) \in AC_{\nabla}^{m}(\Omega)$ and $m-1 < \beta < \alpha < m$. Then, for all $k \in \{1, \ldots, m-1\}$,

$${}^{C}D_{\nabla,t_0}^{\alpha-m+k} D_{\nabla,t_0}^{m-k} f(t) = {}^{C}D_{\nabla,t_0}^{\alpha} f(t), \tag{63}$$

$${}^{C}D_{\nabla,t_0}^{\alpha-\beta}\,{}^{C}D_{\nabla,t_0}^{\beta} f(t) = {}^{C}D_{\nabla,t_0}^{\alpha} f(t) \tag{64}$$

for all $t \in \Omega_{k^m}$.

Proof. For each $k \in \{1, \ldots, m-1\}$, by Property 6,

$$\begin{aligned} {}^{C}D_{\nabla,t_0}^{\alpha} f(t) &= I_{\nabla,t_0}^{m-\alpha} D_{\nabla}^{m} f(t) \\ &= \int_{t_0}^{t} \widehat{h}_{m-\alpha-1}(t,\rho(s))\, D_{\nabla}^{m} f(s)\, \nabla s \\ &= \int_{t_0}^{t} \widehat{h}_{k-(\alpha-(m-k))-1}(t,\rho(s))\, D_{\nabla}^{k} D_{\nabla}^{m-k} f(s)\, \nabla s. \end{aligned} \tag{65}$$

Noting that $\alpha-(m-k) \in (k-1,k)$ and according to Property 6, we have

$$\begin{aligned} &\int_{t_0}^{t} \widehat{h}_{k-(\alpha-(m-k))-1}(t,\rho(s))\, D_{\nabla}^{k} D_{\nabla}^{m-k} f(s)\, \nabla s \\ &\quad = {}^{C}D_{\nabla,t_0}^{\alpha-m+k} D_{\nabla,t_0}^{m-k} f(t). \end{aligned} \tag{66}$$

Thus (63) holds.

Now, for all $\alpha_0, \beta_0 \in (0,1)$ with $\alpha_0 + \beta_0 < 1$, we have

$${}^{C}D_{\nabla,t_0}^{\alpha_0}\,{}^{C}D_{\nabla,t_0}^{\beta_0} f(t) = {}^{C}D_{\nabla,t_0}^{\alpha_0+\beta_0} f(t) = {}^{C}D_{\nabla,t_0}^{\beta_0}\,{}^{C}D_{\nabla,t_0}^{\alpha_0} f(t). \tag{67}$$

In fact, from Property 6, we can get ${}^{C}D_{\nabla,t_0}^{\beta_0} f(t_0) = 0$. Since $\alpha_0, \beta_0 \in (0,1)$, then by (41), (29), and Property 6

$$\begin{aligned} {}^{C}D_{\nabla,t_0}^{\alpha_0}\,{}^{C}D_{\nabla,t_0}^{\beta_0} f(t) &= D_{\nabla,t_0}^{\alpha_0}\,{}^{C}D_{\nabla,t_0}^{\beta_0} f(t) = D_{\nabla,t_0}^{\alpha_0} I_{\nabla,t_0}^{1-\beta_0} f^{\nabla}(t) \\ &= I_{\nabla,t_0}^{1-\beta_0-\alpha_0} f^{\nabla}(t) = {}^{C}D_{\nabla,t_0}^{\alpha_0+\beta_0} f(t). \end{aligned} \tag{68}$$

Similarly, we have ${}^{C}D_{\nabla,t_0}^{\beta_0}\,{}^{C}D_{\nabla,t_0}^{\alpha_0} f(t) = {}^{C}D_{\nabla,t_0}^{\alpha_0+\beta_0} f(t)$. Thus (67) holds. Then, by using (63) and (67), we have that

$$\begin{aligned} {}^{C}D_{\nabla,t_0}^{\alpha} f(t) &= {}^{C}D_{\nabla,t_0}^{\alpha-m+1} D_{\nabla}^{m-1} f(t) \\ &= {}^{C}D_{\nabla,t_0}^{(\alpha-\beta)+(\beta-m+1)} D_{\nabla}^{m-1} f(t) \\ &= {}^{C}D_{\nabla,t_0}^{\alpha-\beta}\,{}^{C}D_{\nabla,t_0}^{\beta-m+1} D_{\nabla}^{m-1} f(t) \\ &= {}^{C}D_{\nabla,t_0}^{\alpha-\beta}\,{}^{C}D_{\nabla,t_0}^{\beta} f(t). \end{aligned} \tag{69}$$

That is, (64) holds. The results follow. □

The next assertion yields the Laplace transform of the Caputo fractional nabla derivative.

Property 11. Let $\alpha > 0$, $m-1 < \alpha \leq m$ $(m \in \mathbb{N})$ be such that $f(t) \in AC_{\nabla}^{m}(\Omega)$. Then

$$\begin{aligned} &\mathscr{L}_{\nabla,t_0}\left\{{}^{C}D_{\nabla,t_0}^{\alpha} f(t)\right\}(z) \\ &\quad = z^{\alpha}\mathscr{L}_{\nabla,t_0}\{f(t)\}(z) - \sum_{k=0}^{m-1} z^{\alpha-k-1} f^{\nabla^k}(t_0) \end{aligned} \tag{70}$$

for those regressive $z \in \mathbb{C}$ satisfying $\lim_{t\to\infty}\{f^{\nabla k}(t)\widehat{e}_{\ominus_\nu z}(t,t_0)\} = 0$ $(k = 0, 1, \ldots, m-1)$.

In particular, if $0 < \alpha \leq 1$, then

$$\mathscr{L}_{\nabla,t_0}\left\{{}^{C}D_{\nabla,t_0}^{\alpha} f(t)\right\}(z) = z^{\alpha}\mathscr{L}_{\nabla,t_0}\{f(t)\}(z) - z^{\alpha-1} f(t_0) \tag{71}$$

for those regressive $z \in \mathbb{C}$ satisfying $\lim_{t\to\infty}\{f(t)\widehat{e}_{\ominus_\nu z}(t,t_0)\} = 0$.

Proof. By Property 6, (33), and (17), for $\alpha \notin \mathbb{N}$, we have

$$\begin{aligned} &\mathscr{L}_{\nabla,t_0}\left\{{}^{C}D_{\nabla,t_0}^{\alpha} f(t)\right\}(z) \\ &\quad = \mathscr{L}_{\nabla,t_0}\left\{I_{\nabla,t_0}^{m-\alpha} D_{\nabla}^{m} f(t)\right\}(z) \\ &\quad = \frac{1}{z^{m-\alpha}}\mathscr{L}_{\nabla,t_0}\{D_{\nabla}^{m} f(t)\}(z) \\ &\quad = \frac{1}{z^{m-\alpha}}\left[z^{m}\mathscr{L}_{\nabla,t_0}\{f(t)\}(z) - \sum_{k=0}^{m-1} z^{m-k-1} f^{\nabla^k}(t_0)\right] \\ &\quad = z^{\alpha}\mathscr{L}_{\nabla,t_0}\{f(t)\}(z) - \sum_{k=0}^{m-1} z^{\alpha-k-1} f^{\nabla^k}(t_0), \end{aligned} \tag{72}$$

and for $\alpha = m \in \mathbb{N}$, we have

$$\begin{aligned} \mathscr{L}_{\nabla,t_0}\left\{{}^{C}D_{\nabla,t_0}^{m} f(t)\right\}(z) &= \mathscr{L}_{\nabla,t_0}\left\{f^{\nabla^m}(t)\right\}(z) \\ &= z^{m}\mathscr{L}_{\nabla,t_0}\{f(t)\}(z) \\ &\quad - \sum_{k=0}^{m-1} z^{m-k-1} f^{\nabla^k}(t_0). \end{aligned} \tag{73}$$

The result follows. □

Remark 26. (1) For Riemann-Liouville fractional derivative,

$$D^{\alpha}_{\nabla,t_0}1 = \widehat{h}_{-\alpha}(t,t_0) \quad (0<\alpha<1), \tag{74}$$

while for the Caputo fractional derivative,

$${}^{C}D^{\alpha}_{\nabla,t_0}1 = 0, \tag{75}$$

which shows that the Caputo fractional derivative is more near to the usual sense derivative than Riemann-Liouville fractional derivative.

(2) Comparing (34) and (70), we know that the Laplace transform of the Caputo fractional derivative involves only initial value with integer order derivative, such as $f^{\nabla^k}(t_0), k = 0, 1, \ldots, m-1$, while the Laplace transform of the Riemann-Liouville fractional derivative is related to initial value with fractional order derivative which is difficult to understand the physics background, such as $D^{\alpha-k}_{\nabla,t_0}f(t_0), k = 1, \ldots, m$. Thus, the Caputo fractional derivative is used more widely in realistic applications.

4. The Cauchy Problem with Caputo Fractional Derivative

4.1. Existence and Uniqueness of the Solution to the Cauchy Type Problem. In this section we consider the nonlinear differential equation of order $\alpha > 0$:

$${}^{C}D^{\alpha}_{\nabla,t_0}y(t) = f(t,y(t)) \tag{76}$$

involving the Caputo fractional derivative ${}^{C}D^{\alpha}_{\nabla,t_0}y(t)$, defined in (38), with the initial conditions

$$D^{k}_{\nabla}y(t_0) = b_k, \quad b_k \in \mathbb{R} \ \ (k = 0,1,\ldots,m-1;\ m = -[-\alpha]). \tag{77}$$

We give the conditions for a unique solution $y(t)$ to this problem in the space $AC^{m}_{\nabla}(\Omega)$. Our investigations are based on reducing the problem (76)-(77) to the integral equation

$$y(t) = \sum_{j=0}^{m-1}\widehat{h}_j(t,t_0)b_j + \int_{t_0}^{t}\widehat{h}_{\alpha-1}(t,\rho(\tau))f(\tau,y(\tau))\nabla\tau. \tag{78}$$

First we establish an equivalence between the problem (76)-(77) and the integral equation (78).

Theorem 27. *Let $\alpha > 0$ and let m be given by* (39)*. Let G be an open set in $\mathbb{R}$ and let $f : \Omega \times G \to \mathbb{R}$ be a function such that, for any $y \in G$, $f(t,y) \in AC_{\nabla}(\Omega)$. If $y(t) \in AC^{m}_{\nabla}(\Omega)$, then $y(t)$ satisfies the relation* (76)-(77) *if and only if $y(t)$ satisfies the Volterra integral equation* (78).

Proof. First we prove the necessity. Let $y(t)$ be the solution to the Cauchy problem (76)-(77). Applying the operator I^{α}_{∇,t_0} to (76) and taking into account

$$I^{\alpha}_{\nabla,t_0}\,{}^{C}D^{\alpha}_{\nabla,t_0}y(t) = y(t) - \sum_{j=0}^{m-1}\widehat{h}_j(t,t_0)D^{j}_{\nabla,t_0}y(t_0) \tag{79}$$

and (77), we arrive at the integral equation (78) since $y(t) \in AC^{m}_{\nabla}(\Omega)$.

Inversely, if $y(t)$ satisfies (78), for $f(t,y) \in AC_{\nabla}(\Omega)$, applying the operator ${}^{C}D^{\alpha}_{\nabla,t_0}$ to both sides of (78) and taking into account (51) and (55), we have

$$\begin{aligned} {}^{C}D^{\alpha}_{\nabla,t_0}y(t) &= \sum_{j=0}^{m-1}{}^{C}D^{\alpha}_{\nabla,t_0}\widehat{h}_j(t,t_0)b_j + {}^{C}D^{\alpha}_{\nabla,t_0}I^{\alpha}_{\nabla,t_0}f(t,y(t)) \\ &= f(t,y(t)). \end{aligned} \tag{80}$$

In addition, by term-by-term differentiation of (78) and using (51), we have

$$\begin{aligned} D^{k}_{\nabla}y(t) &= \sum_{j=0}^{m-1}D^{k}_{\nabla}\widehat{h}_j(t,t_0)b_j + D^{k}_{\nabla}I^{\alpha}_{\nabla,t_0}f(t,y(t)) \\ &= \sum_{j=0}^{m-1}D^{k}_{\nabla}\widehat{h}_j(t,t_0)b_j \\ &\quad + \int_{t_0}^{t}\widehat{h}_{\alpha-k-1}(t,\rho(\tau))f(\tau,y(\tau))\nabla\tau \\ &= \sum_{j=0}^{k-1}D^{k}_{\nabla}\widehat{h}_j(t,t_0)b_j + \sum_{j=k}^{m-1}D^{k}_{\nabla}\widehat{h}_j(t,t_0)b_j \\ &\quad + \int_{t_0}^{t}\widehat{h}_{\alpha-k-1}(t,\rho(\tau))f(\tau,y(\tau))\nabla\tau \\ &= \sum_{j=k}^{m-1}\widehat{h}_{j-k}(t,t_0)b_j \\ &\quad + \int_{t_0}^{t}\widehat{h}_{\alpha-k-1}(t,\rho(\tau))f(\tau,y(\tau))\nabla\tau \end{aligned} \tag{81}$$

for $k = 0,1,\ldots,m-1$. Thus we obtain relations in (77) by letting $t = t_0$ in (81). □

In the following, we bring into Lipschitzian-type condition:

$$\left|f(t,y_1(t)) - f(t,y_2(t))\right| \le A\left|y_1(t) - y_2(t)\right|, \tag{82}$$

where $A > 0$ does not depend on $t \in \Omega$. We will derive a unique solution to the Cauchy problem (76)-(77).

Theorem 28. *Let $\alpha > 0$ and let m be given by* (39)*. Let G be an open set in $\mathbb{R}$ and $f : \Omega \times G \to \mathbb{R}$ a function such that, for any $y \in G, f(t,y) \in AC_{\nabla}(\Omega), y(t) \in AC^{m}_{\nabla}(\Omega)$. Let $f(t,y)$ satisfies the Lipschitzian condition* (82)*, and $\max_{y\in G,t,s\in\Omega}\{|f(t,y)|, |\widehat{h}_{\alpha-1}(t,s)|\} \le M$. Then there exists a unique solution $y(t)$ to initial value problem* (76)-(77).

Proof. Since the Cauchy type problem (76)-(77) and the nonlinear Volterra integral equation (78) are equivalent, we only need to prove there exists a unique solution to (78).

We define function sequences:

$$y_l(t) = y_0(t) + \int_{t_0}^{t} \widehat{h}_{\alpha-1}(t, \rho(\tau)) f(\tau, y_{l-1}(\tau)) \nabla\tau \quad (l = 1, 2, \ldots), \tag{83}$$

where

$$y_0(t) = \sum_{j=0}^{m-1} \widehat{h}_j(t, t_0) b_j. \tag{84}$$

To simplify our proof, without loss of generality, we assume that G is large enough such that $y_l(t) \in G, \forall t \in \Omega, \forall l$.

We obtain by inductive method that

$$|y_l(t) - y_{l-1}(t)| \le A^{l-1} M^{l+1} \widehat{h}_l(t, t_0). \tag{85}$$

In fact, for $l = 1$, since $\max_{y\in G, t, s\in\Omega}\{|f(t,y)|, |\widehat{h}_{\alpha-1}(t,s)|\} \le M$, we have

$$\begin{aligned} |y_1(t) - y_0(t)| &\le \int_{t_0}^{t} |\widehat{h}_{\alpha-1}(t, \rho(\tau))| |f(\tau, y_0(\tau))| \nabla\tau \\ &\le M^2 \int_{t_0}^{t} \nabla\tau = M^2 \widehat{h}_1(t, t_0). \end{aligned} \tag{86}$$

If

$$|y_{l-1}(t) - y_{l-2}(t)| \le A^{l-2} M^{l} \widehat{h}_{l-1}(t, t_0), \tag{87}$$

then

$$\begin{aligned} &|y_l(t) - y_{l-1}(t)| \\ &\le \int_{t_0}^{t} |\widehat{h}_{\alpha-1}(t, \rho(\tau))| |f(\tau, y_{l-1}(\tau)) - f(\tau, y_{l-2}(\tau))| \nabla\tau \\ &\le AM \int_{t_0}^{t} |y_{l-1}(\tau) - y_{l-2}(\tau)| \nabla\tau \\ &\le AM \int_{t_0}^{t} A^{l-2} M^{l} \widehat{h}_{l-1}(\tau, t_0) \nabla\tau \\ &= A^{l-1} M^{l+1} \widehat{h}_l(t, t_0). \end{aligned} \tag{88}$$

According to

$$\begin{aligned} \sum_{l=1}^{\infty} |y_l(t) - y_{l-1}(t)| &\le \sum_{l=1}^{\infty} A^{l-1} M^{l+1} \widehat{h}_l(t, t_0) \\ &\le \frac{M}{A} \sum_{l=1}^{\infty} (AM)^l h_l(\sigma(t), t_0) \\ &\le \frac{M}{A} \sum_{l=1}^{\infty} (AM)^l \frac{(\sigma(t) - t_0)^l}{l!} \end{aligned} \tag{89}$$

and by Weierstrass discriminance, we obtain $y_l(t)$ convergent uniformly and the limit is the solution. Thus we prove the existence of solution.

Next we will show the uniqueness. Assume $z(t)$ is another solution to (78); that is,

$$z(t) = y_0(t) + \int_{t_0}^{t} \widehat{h}_{\alpha-1}(t, \rho(\tau)) f(\tau, z(\tau)) \nabla\tau. \tag{90}$$

Since

$$\max_{y\in G, t, s\in\Omega} \{|f(t, y)|, |\widehat{h}_{\alpha-1}(t, s)|\} \le M, \tag{91}$$

we have

$$\begin{aligned} |y_0(t) - z(t)| &\le \int_{t_0}^{t} |\widehat{h}_{\alpha-1}(t, \rho(\tau))| |f(\tau, z(\tau))| \nabla\tau \\ &\le M^2 \int_{t_0}^{t} \nabla\tau = M^2 \widehat{h}_1(t, t_0). \end{aligned} \tag{92}$$

If

$$|y_{l-1}(t) - z(t)| \le A^{l-1} M^{l+1} \widehat{h}_l(t, t_0), \tag{93}$$

then

$$\begin{aligned} &|y_l(t) - z(t)| \\ &\le \int_{t_0}^{t} |\widehat{h}_{\alpha-1}(t, \rho(\tau))| |f(\tau, y_{l-1}(\tau)) - f(\tau, z(\tau))| \nabla\tau \\ &\le AM \int_{t_0}^{t} |y_{l-1}(\tau) - z(\tau)| \nabla\tau \\ &\le AM \int_{t_0}^{t} A^{l-1} M^{l+1} \widehat{h}_l(\tau, t_0) \nabla\tau \\ &\le A^l M^{l+2} \widehat{h}_{l+1}(t, t_0). \end{aligned} \tag{94}$$

By mathematical induction, we have

$$|y_l(t) - z(t)| \le A^l M^{l+2} \widehat{h}_{l+1}(t, t_0). \tag{95}$$

and then we get that

$$\begin{aligned} \sum_{l=0}^{\infty} |y_l(t) - z(t)| &\le \sum_{l=0}^{\infty} A^l M^{l+2} \widehat{h}_{l+1}(t, t_0) \\ &\le \frac{M}{A} \sum_{l=0}^{\infty} (AM)^{l+1} h_{l+1}(\sigma(t), t_0) \\ &\le \frac{M}{A} \sum_{l=0}^{\infty} (AM)^{l+1} \frac{(\sigma(t) - t_0)^{l+1}}{(l+1)!}. \end{aligned} \tag{96}$$

Thus, $\lim_{l\to\infty} |y_l(t) - z(t)| = 0$, and then we have $z(t) = y(t)$ owing to the uniqueness of the limit. The result follows. □

In the following, we consider generalized Cauchy type problems:

$$\begin{gathered} {}^{C}D^{\alpha}_{\nabla,t_0} y(t) = f\left(t, y(t), {}^{C}D^{\alpha_1}_{\nabla,t_0} y(t), \ldots, {}^{C}D^{\alpha_l}_{\nabla,t_0} y(t)\right) \\ (0 = \alpha_0 \le \alpha_1 \le \cdots \le \alpha_l \le \alpha), \\ D^{k}_{\nabla,t_0} y(t_0) = b_k \quad (k = 1, \ldots, m,\ \alpha = -[-\alpha]). \end{gathered} \tag{97}$$

Theorem 29. *Let $\alpha > 0, G$ be an open set and let $f : \Omega \times G \to \mathbb{R}$ be a function such that, for any $(y, y_1, \ldots, y_l) \in G$, $f(t, y, y_1, \ldots, y_l) \in AC_\nabla(\Omega)$. If $y(t) \in AC_\nabla^m(\Omega)$, then $y(t)$ satisfies* (97) *if and only if $y(t)$ satisfies the integral equation*

$$y(t) = \sum_{j=0}^{m-1} \widehat{h}_j(t, t_0) b_j + \int_{t_0}^{t} \widehat{h}_{\alpha-1}(t, \rho(\tau)) \times f\left(\tau, y(\tau), {}^{C}D_{\nabla,t_0}^{\alpha_1} y(\tau), \ldots, {}^{C}D_{\nabla,t_0}^{\alpha_l} y(\tau)\right) \nabla\tau. \tag{98}$$

Suppose that f satisfies generalized Lipschitzian condition:

$$\left|f(t, y_0, y_1, \ldots, y_l) - f(t, z_0, z_1, \ldots, z_l)\right| \leq A\left[\sum_{j=0}^{l}\left|y_j - z_j\right|\right] \quad (A > 0). \tag{99}$$

According to the theorem above and the proof of Theorem 28, we have the following theorem.

Theorem 30. *Let the condition of Theorem 29 be valid. If f satisfies Lipschitzian condition* (99) *and $\max_{y\in G,t,s\in\Omega}\{|f(t, y, y_1, \ldots, y_l)|, |\widehat{h}_{\alpha-1}(t, s)|\} \leq M$ holds, then there exists a unique solution $y(t)$ to initial value problem* (97).

4.2. The Dependency of the Solution upon the Initial Value. We consider Caputo fractional differential initial value problem again:

$$\begin{gathered} {}^{C}D_{\nabla,t_0}^{\alpha} y(t) = f(t, y(t)), \\ D_\nabla^k y(t_0) = b_k \quad (k = 0, 1, \ldots, m-1;\ m = -[-\alpha]), \end{gathered} \tag{100}$$

where $\alpha > 0$.

Using Theorem 27, we have

$$y(t) = y_0(t) + \int_{t_0}^{t} \widehat{h}_{\alpha-1}(t, \rho(\tau)) f(\tau, y(\tau)) \nabla\tau, \tag{101}$$

where

$$y_0(t) = \sum_{j=0}^{m-1} \widehat{h}_j(t, t_0) b_j. \tag{102}$$

Suppose $z(t)$ is the solution to the initial value problem:

$$\begin{gathered} {}^{C}D_{\nabla,t_0}^{\alpha} y(t) = f(t, y(t)), \\ D_\nabla^k y(t_0) = c_k \quad (k = 0, 1, \ldots, m-1;\ m = -[-\alpha]). \end{gathered} \tag{103}$$

We denote $\|y\| := \sup_{t\in\Omega} y(t)$. We can derive the dependency of the solution upon the initial value.

Theorem 31. *Let $y(t), z(t)$ be the solutions to* (100) *and* (103), *respectively, and let $t_0, t, s \in \Omega$, $|\widehat{h}_{\alpha-1}(t, s)| \leq M$. Suppose f satisfies the Lipschitz condition; that is,*

$$\left|f(t, z) - f(t, y)\right| \leq A\left|z - y\right| \quad (A > 0). \tag{104}$$

Then we have

$$\left|z(t) - y(t)\right| \leq \left\|z_0 - y_0\right\| \sum_{j=0}^{\infty}(AM)^j \frac{(\sigma(t) - t_0)^j}{j!}, \quad \forall t \in \Omega. \tag{105}$$

Proof. By the proof of Theorem 28, we know that $y(t) = \lim_{m\to\infty} y_m(t)$, $z(t) = \lim_{m\to\infty} z_m(t)$, where

$$\begin{aligned} y_0(t) &= \sum_{j=0}^{m-1} \widehat{h}_j(t, t_0) b_j, \\ y_m(t) &= y_0(t) + \int_{t_0}^{t} \widehat{h}_{\alpha-1}(t, \rho(\tau)) f(\tau, y_{m-1}(\tau)) \nabla\tau, \\ z_0(t) &= \sum_{j=0}^{m-1} \widehat{h}_j(t, t_0) c_j, \\ z_m(t) &= z_0(t) + \int_{t_0}^{t} \widehat{h}_{\alpha-1}(t, \rho(\tau)) f(\tau, z_{m-1}(\tau)) \nabla\tau. \end{aligned} \tag{106}$$

Using the Lipschitz condition, we have

$$\begin{aligned} &\left|z_1(t) - y_1(t)\right| \\ &\quad \leq \left\|z_0 - y_0\right\| + \int_{t_0}^{t}\left|\widehat{h}_{\alpha-1}(t, \rho(\tau))\right|\left|f(\tau, z_0(\tau)) - f(\tau, y_0(\tau))\right| \nabla\tau \\ &\quad \leq \left\|z_0 - y_0\right\| + M\int_{t_0}^{t} A\left|z_0(\tau) - y_0(\tau)\right| \nabla\tau \\ &\quad \leq \left\|z_0 - y_0\right\| + \left\|z_0 - y_0\right\| AM \int_{t_0}^{t} \nabla\tau \\ &\quad = \left\|z_0 - y_0\right\| + \left\|z_0 - y_0\right\| AM\widehat{h}_1(t, t_0) \\ &\quad = \left\|z_0 - y_0\right\|\left[1 + AM\widehat{h}_1(t, t_0)\right]. \end{aligned} \tag{107}$$

Suppose

$$\left|z_{m-1}(t) - y_{m-1}(t)\right| \leq \left\|z_0 - y_0\right\| \sum_{j=0}^{m-1}(AM)^j\widehat{h}_j(t, t_0), \tag{108}$$

then

$$\begin{aligned}
&\left|z_m(t) - y_m(t)\right| \\
&\quad \le \|z_0 - y_0\| \\
&\qquad + \int_{t_0}^{t} \left|\widehat{h}_{\alpha-1}(t, \rho(\tau))\right| \\
&\qquad\qquad \times \left|f(\tau, z_{m-1}(\tau)) - f(\tau, y_{m-1}(\tau))\right| \nabla\tau \\
&\quad \le \|z_0 - y_0\| \\
&\qquad + M \int_{t_0}^{t} A \left|z_{m-1}(\tau) - y_{m-1}(\tau)\right| \nabla\tau \\
&\quad \le \|z_0 - y_0\| \\
&\qquad + M \int_{t_0}^{t} A \|z_0 - y_0\| \sum_{j=0}^{m-1} (AM)^j \widehat{h}_j(\tau, t_0) \nabla\tau \\
&\quad = \|z_0 - y_0\| + \|z_0 - y_0\| \sum_{j=0}^{m-1} (AM)^{j+1} \int_{t_0}^{t} \widehat{h}_j(\tau, t_0) \nabla\tau \\
&\quad = \|z_0 - y_0\| + \|z_0 - y_0\| \sum_{j=0}^{m-1} (AM)^{j+1} \widehat{h}_{j+1}(t, t_0) \\
&\quad = \|z_0 - y_0\| \sum_{j=0}^{m} (AM)^j \widehat{h}_j(t, t_0).
\end{aligned} \tag{109}$$

According to mathematical induction, we have

$$\begin{aligned}
\left|z_m(t) - y_m(t)\right| &\le \|z_0 - y_0\| \sum_{j=0}^{m} (AM)^j \widehat{h}_j(t, t_0) \\
&\le \|z_0 - y_0\| \sum_{j=0}^{m} (AM)^j h_j(\sigma(t), t_0) \\
&\le \|z_0 - y_0\| \sum_{j=0}^{m} (AM)^j \frac{(\sigma(t) - t_0)^j}{j!}.
\end{aligned} \tag{110}$$

Taking the limit $m \to \infty$, we obtain that

$$\left|z(t) - y(t)\right| \le \|z_0 - y_0\| \sum_{j=0}^{\infty} (AM)^j \frac{(\sigma(t) - t_0)^j}{j!}, \tag{111}$$

and the proof is completed. □

4.3. Initial Value Problems for Nonlinear Term Containing Fractional Derivative. In this section, we are interested in the nonlinear differential equation

$${}^{C}D^{\alpha}_{\nabla,t_0} u(t) = f\left(t, {}^{C}D^{\beta}_{\nabla,t_0} u(t)\right) \quad (t \in \Omega, t > t_0), \tag{112}$$

of fractional order $\alpha \in (m-1, m)$, where $\beta \in (n-1, n)$, $m, n \in \mathbb{N}$, and $\alpha > \beta$, with the initial conditions

$$D^{k}_{\nabla,t_0} u(t_0) = \eta_k, \quad k = 0, \dots, m-1. \tag{113}$$

We obtain the existence of at least one solution for integral equations using the Leray-Schauder Nonlinear Alternative for several types of initial value problems and establish sufficient conditions for unique solutions using the Banach contraction principle.

Our objective is to find solutions to the initial value problem (112) and (113) in the space $AC^{m}_{\nabla}(\Omega)$. There are two cases to investigate: $n-1 < \beta < n \le m-1 < \alpha < m$ and $n-1 < \beta < \alpha < n$.

Throughout this section, we suppose that the following are satisfied:

(H_1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a ld-continuously and nabla differentiable function;

(H_2) there exist nonnegative functions $a_1, a_2 \in AC_{\nabla}(\Omega)$ such that $|f(t,z)| \le a_1(t) + a_2(t)|z|$;

(H_3) $f(t_0, 0) = 0$ and $f(t, 0) \ne 0$ on a compact subinterval of $\Omega \setminus \{t_0\}$.

The following shows that the solvability of the initial value problem (112) and (113) is equivalent to that of the Volterra-type integral equation (115) in the space $AC_{\nabla}(\Omega)$.

Lemma 32. *Let $n-1 < \beta < n \le m-1 < \alpha < m$ and assume that (H_1) and (H_3) hold. A function $u(t) \in AC^{m}_{\nabla}(\Omega)$ is a solution of the initial value problem* (112) *and* (113) *if and only if*

$$u(t) = \sum_{k=0}^{n-1} \widehat{h}_k(t, t_0) \eta_k + \int_{t_0}^{t} \widehat{h}_{n-1}(t, \rho(s)) v(s) \nabla s, \quad t \in \Omega, \tag{114}$$

where $v \in AC_{\nabla}(\Omega)$ is a solution of the integral equation

$$\begin{aligned}
v(t) = &\sum_{i=0}^{m-n-1} \widehat{h}_i(t, t_0) \eta_{n+i} \\
&+ \int_{t_0}^{t} \widehat{h}_{\alpha-n-1}(t, \rho(s)) \\
&\qquad \times f\left(s, \int_{t_0}^{s} \widehat{h}_{n-\beta-1}(s, \rho(\tau)) v(\tau) \nabla\tau\right) \nabla s.
\end{aligned} \tag{115}$$

Proof. By (63), we have

$${}^{C}D^{\alpha-n}_{\nabla,t_0} D^{n}_{\nabla} u(t) = {}^{C}D^{\alpha}_{\nabla,t_0} u(t) = f\left(t, {}^{C}D^{\beta}_{\nabla,t_0} u(t)\right). \tag{116}$$

By using Property 6, ${}^{C}D^{\beta}_{\nabla,t_0}u(t) = I^{n-\beta}_{\nabla,t_0}D^{n}_{\nabla}u(t)$, thus we have

$$ {}^{C}D^{\alpha-n}_{\nabla,t_0}D^{n}_{\nabla}u(t) = f\left(t, \int_{t_0}^{t} \widehat{h}_{n-\beta-1}(t,\rho(\tau)) D^{n}_{\nabla}u(\tau)\nabla\tau\right). \tag{117} $$

Let $v(t) = D^{n}_{\nabla}u(t)$, by using Theorem 27, we obtain

$$ \begin{aligned} v(t) = & \sum_{i=0}^{m-n-1} \widehat{h}_i(t,t_0) D^{i}_{\nabla}v(t_0) \\ & + \int_{t_0}^{t} \widehat{h}_{\alpha-n-1}(t,\rho(s)) \\ & \times f\left(s, \int_{t_0}^{s} \widehat{h}_{n-\beta-1}(s,\rho(\tau)) v(\tau)\nabla\tau\right)\nabla s. \end{aligned} \tag{118} $$

As $D^{i}_{\nabla}v(t) = D^{n+i}_{\nabla}u(t)$ and by (113), the above equation transforms into (115). An application of Definition 7 and Theorem 9 yields (114) in view of $v(t) = D^{n}_{\nabla}u(t)$ and (113).

To prove the converse, let $v \in AC_{\nabla}(\Omega)$ be a solution of (115). Since $v \in AC_{\nabla}(\Omega)$, the function

$$ s \longrightarrow \int_{t_0}^{s} \widehat{h}_{n-\beta-1}(s,\rho(\tau)) v(\tau)\nabla\tau \tag{119} $$

is ld-continuous on $\Omega \setminus \{t_0\}$ and so is

$$ s \longrightarrow f\left(s, \int_{t_0}^{s} \widehat{h}_{n-\beta-1}(s,\rho(\tau)) v(\tau)\nabla\tau\right). \tag{120} $$

We have

$$ \begin{aligned} D^{n}_{\nabla}u(t) &= v(t) \\ &= \sum_{i=0}^{m-n-1} \widehat{h}_i(t,t_0)\eta_{n+i} \\ &\quad + \int_{t_0}^{t} \widehat{h}_{\alpha-n-1}(t,\rho(s)) \\ &\quad \times f\left(s, \int_{t_0}^{s} \widehat{h}_{n-\beta-1}(s,\rho(\tau)) D^{n}_{\nabla}u(\tau)\nabla\tau\right)\nabla s \\ &= \sum_{i=0}^{m-n-1} \widehat{h}_i(t,t_0)\eta_{n+i} + I^{\alpha-n}_{\nabla,t_0} f\left(t, {}^{C}D^{\beta}_{\nabla,t_0}u(t)\right). \end{aligned} \tag{121} $$

Since $\alpha - n \in (m-n-1, m-n)$, by

$$ \begin{aligned} {}^{C}D^{\alpha}_{\nabla,t_0}u(t) &= {}^{C}D^{\alpha-n}_{\nabla,t_0}D^{n}_{\nabla}u(t) \\ &= {}^{C}D^{\alpha-n}_{\nabla,t_0}\left(\sum_{i=0}^{m-n-1} \widehat{h}_i(t,t_0)\eta_{n+i}\right) \\ &\quad + {}^{C}D^{\alpha-n}_{\nabla,t_0} I^{\alpha-n}_{\nabla,t_0} f\left(t, {}^{C}D^{\beta}_{\nabla,t_0}u(t)\right) \\ &= f\left(t, {}^{C}D^{\beta}_{\nabla,t_0}u(t)\right), \end{aligned} \tag{122} $$

so u is a solution of (112) in view of (H_1). By absolute continuity of the integral, differentiating (115), we obtain

$$ \begin{aligned} D^{k}_{\nabla}v(t) &= \sum_{i=0}^{m-n-1} D^{k}_{\nabla}\widehat{h}_i(t,t_0)\eta_{n+i} \\ &\quad + D^{k}_{\nabla} I^{\alpha-n}_{\nabla,t_0} f\left(t, {}^{C}D^{\beta}_{\nabla,t_0}u(t)\right) \\ &= \sum_{i=0}^{k-1} D^{k}_{\nabla}\widehat{h}_i(t,t_0)\eta_{n+i} + \sum_{i=k}^{m-n-1} D^{k}_{\nabla}\widehat{h}_i(t,t_0)\eta_{n+i} \\ &\quad + D^{k}_{\nabla} I^{\alpha-n}_{\nabla,t_0} f\left(t, {}^{C}D^{\beta}_{\nabla,t_0}u(t)\right) \\ &= 0 + \sum_{i=k}^{m-n-1} \widehat{h}_{i-k}(t,t_0)\eta_{n+i} + I^{\alpha-n-k}_{\nabla,t_0} f\left(t, {}^{C}D^{\beta}_{\nabla,t_0}u(t)\right) \end{aligned} \tag{123} $$

for each $k = 0,\ldots,m-n-1$. Thus, $D^{n+k}_{\nabla}u(t_0) = D^{k}_{\nabla}v(t_0) = \eta_{n+k}$, $k = 0,\ldots,m-n-1$; that is, $D^{i}_{\nabla}u(t_0) = \eta_i$, $i = n,\ldots,m-1$. On the other hand, from (114),

$$ \begin{aligned} D^{i}_{\nabla}u(t) &= \sum_{k=0}^{n-1} D^{i}_{\nabla}\widehat{h}_k(t,t_0)\eta_k + D^{i}_{\nabla} I^{n}_{\nabla,t_0} v(t) \\ &= \sum_{k=0}^{i-1} D^{i}_{\nabla}\widehat{h}_k(t,t_0)\eta_k + \sum_{k=i}^{n-1} D^{i}_{\nabla}\widehat{h}_k(t,t_0)\eta_k + I^{n-i}_{\nabla,t_0} v(t) \\ &= \sum_{k=i}^{n-1} D^{i}_{\nabla}\widehat{h}_k(t,t_0)\eta_k + I^{n-i}_{\nabla,t_0} v(t) \end{aligned} \tag{124} $$

and thus $D^{i}_{\nabla}u(t_0) = \eta_i$ $(i = 0,\ldots,n-1)$. Also it is easy to see that $D^{m-n}_{\nabla}v(t) = D^{m}_{\nabla}u(t) \in AC_{\nabla}(\Omega)$. □

For the sake of brevity, by ϕ, we denote the first term in the right-hand side of (115).

Theorem 33. *Suppose that (H_1)–(H_3) hold. Then the integral equation (115) has a solution in $AC_{\nabla}(\Omega)$ provided*

$$ \begin{aligned} A = \sup_{t\in\Omega} \int_{t_0}^{t} & \left(\left|\widehat{h}_{\alpha-n-1}(t,\rho(s))\right| \right. \\ & \left. \times \int_{t_0}^{s} \left|\widehat{h}_{n-\beta-1}(s,\rho(\tau))\right| \nabla\tau \right) a_2(s)\nabla s < 1, \\ 0 < B = \sup_{t\in\Omega} & \left(|\phi(t)| + \int_{t_0}^{t} \left|\widehat{h}_{\alpha-n-1}(t,\rho(s))\right| a_1(s)\nabla s \right) < \infty. \end{aligned} \tag{125} $$

Proof. In the normed space $(AC_{\nabla}(\Omega), \|\cdot\|_0)$ with the sup-norm $\|\cdot\|_0$, we define the mapping T by

$$ (Tv)(t) = \phi(t) $$

$$+\int_{t_0}^{t}\widehat{h}_{\alpha-n-1}(t,\rho(s))\times f\left(s,\int_{t_0}^{s}\widehat{h}_{n-\beta-1}(s,\rho(\tau))\,v(\tau)\,\nabla\tau\right)\nabla s \tag{126}$$

for all $t\in\Omega$. Indeed, one can easily verify that the mapping T is well defined and $T:AC_{\nabla}(\Omega)\to AC_{\nabla}(\Omega)$.

Let

$$U=\{v\in AC_{\nabla}(\Omega):\|v\|_0<R\} \tag{127}$$

with

$$R=\frac{B}{1-A}>0. \tag{128}$$

Let $\mathscr{C}\subset AC_{\nabla}(\Omega)$ be defined by $\mathscr{C}=\overline{U}$.

Let $v\in\overline{U}$; that is, $\|v\|_0\le R$. Then

$$\begin{aligned}
&\|Tv\|_0\\
&=\sup_{t\in\Omega}\left|\phi(t)+\int_{t_0}^{t}\widehat{h}_{\alpha-n-1}(t,\rho(s))\times f\left(s,\int_{t_0}^{s}\widehat{h}_{n-\beta-1}(s,\rho(\tau))\,v(\tau)\,\nabla\tau\right)\nabla s\right|\\
&\le\sup_{t\in\Omega}\left(|\phi(t)|+\int_{t_0}^{t}\left|\widehat{h}_{\alpha-n-1}(t,\rho(s))\right|\times\left|f\left(s,\int_{t_0}^{s}\widehat{h}_{n-\beta-1}(s,\rho(\tau))\,v(\tau)\,\nabla\tau\right)\right|\nabla s\right)\\
&\le\sup_{t\in\Omega}\left(|\phi(t)|+\int_{t_0}^{t}\left|\widehat{h}_{\alpha-n-1}(t,\rho(s))\right|\times\left(a_1(s)+a_2(s)\times\int_{t_0}^{s}\left|\widehat{h}_{n-\beta-1}(s,\rho(\tau))\right||v(\tau)|\,\nabla\tau\right)\nabla s\right)\\
&\le\sup_{t\in\Omega}\left(|\phi(t)|+\int_{t_0}^{t}\left|\widehat{h}_{\alpha-n-1}(t,\rho(s))\right|a_1(s)\,\nabla s\right)\\
&\quad+\sup_{t\in\Omega}\int_{t_0}^{t}\left(\left|\widehat{h}_{\alpha-n-1}(t,\rho(s))\right|\times\int_{t_0}^{s}\left|\widehat{h}_{n-\beta-1}(s,\rho(\tau))\right|\nabla\tau\right)a_2(s)\,\nabla s\|v\|_0\\
&=B+A\|v\|_0\\
&\le B+AR\\
&=R,
\end{aligned} \tag{129}$$

which shows that $Tv\in\mathscr{C}$.

In addition,

$$\begin{aligned}
&\left|(Tv)(t_1)-(Tv)(t_2)\right|\\
&\le\left|\phi(t_1)-\phi(t_2)\right|\\
&\quad+\left|\int_{t_0}^{t_1}\widehat{h}_{\alpha-n-1}(t_1,\rho(s))\times f\left(s,\int_{t_0}^{s}\widehat{h}_{n-\beta-1}(s,\rho(\tau))\,v(\tau)\,\nabla\tau\right)\nabla s\right.\\
&\qquad\left.-\int_{t_0}^{t_2}\widehat{h}_{\alpha-n-1}(t_2,\rho(s))\times f\left(s,\int_{t_0}^{s}\widehat{h}_{n-\beta-1}(s,\rho(\tau))\,v(\tau)\,\nabla\tau\right)\nabla s\right|\\
&=\left|\sum_{i=0}^{m-n-1}\widehat{h}_i(t_1,t_0)\,\eta_{n+i}-\sum_{i=0}^{m-n-1}\widehat{h}_i(t_2,t_0)\,\eta_{n+i}\right|\\
&\quad+\left|\int_{t_0}^{t_1}\int_{\rho(s)}^{t_1}\widehat{h}_{\alpha-n-2}(\theta,\rho(s))\,\nabla\theta\times f\left(s,\int_{t_0}^{s}\widehat{h}_{n-\beta-1}(s,\rho(\tau))\,v(\tau)\,\nabla\tau\right)\nabla s\right.\\
&\qquad\left.-\int_{t_0}^{t_2}\int_{\rho(s)}^{t_2}\widehat{h}_{\alpha-n-2}(\theta,\rho(s))\,\nabla\theta\times f\left(s,\int_{t_0}^{s}\widehat{h}_{n-\beta-1}(s,\rho(\tau))\,v(\tau)\,\nabla\tau\right)\nabla s\right|\\
&\le\left|\sum_{i=0}^{m-n-1}\int_{t_0}^{t_1}\widehat{h}_{i-1}(\tau,t_0)\,\nabla\tau\eta_{n+i}-\sum_{i=0}^{m-n-1}\int_{t_0}^{t_2}\widehat{h}_{i-1}(\tau,t_0)\,\nabla\tau\eta_{n+i}\right|\\
&\quad+\left|\int_{t_0}^{t_1}\int_{\rho(s)}^{t_1}\widehat{h}_{\alpha-n-2}(\theta,\rho(s))\,\nabla\theta\right.
\end{aligned}$$

$$\times f\left(s,\int_{t_0}^{s}\widehat{h}_{n-\beta-1}(s,\rho(\tau))v(\tau)\nabla\tau\right)\nabla s$$
$$-\int_{t_0}^{t_1}\int_{\rho(s)}^{t_2}\widehat{h}_{\alpha-n-2}(\theta,\rho(s))\nabla\theta$$
$$\times f\left(s,\int_{t_0}^{s}\widehat{h}_{n-\beta-1}(s,\rho(\tau))v(\tau)\nabla\tau\right)\nabla s\Bigg|$$
$$+\left|\int_{t_0}^{t_1}\int_{\rho(s)}^{t_2}\widehat{h}_{\alpha-n-2}(\theta,\rho(s))\nabla\theta\right.$$
$$\times f\left(s,\int_{t_0}^{s}\widehat{h}_{n-\beta-1}(s,\rho(\tau))v(\tau)\nabla\tau\right)\nabla s$$
$$-\int_{t_0}^{t_2}\int_{\rho(s)}^{t_2}\widehat{h}_{\alpha-n-2}(\theta,\rho(s))\nabla\theta$$
$$\left.\times f\left(s,\int_{t_0}^{s}\widehat{h}_{n-\beta-1}(s,\rho(\tau))v(\tau)\nabla\tau\right)\nabla s\right|$$
$$\leq\sum_{i=0}^{m-n-1}\left|\int_{t_2}^{t_1}\widehat{h}_{i-1}(\tau,t_0)\nabla\tau\eta_{n+i}\right|$$
$$+\int_{t_0}^{t_1}\left|\int_{t_2}^{t_1}\widehat{h}_{\alpha-n-2}(\theta,\rho(s))\nabla\theta\right|$$
$$\times\left|f\left(s,\int_{t_0}^{s}\widehat{h}_{n-\beta-1}(s,\rho(\tau))v(\tau)\nabla\tau\right)\right|\nabla s$$
$$+\int_{t_2}^{t_1}\left|\int_{\rho(s)}^{t_2}\widehat{h}_{\alpha-n-2}(\theta,\rho(s))\nabla\theta\right|$$
$$\times\left|f\left(s,\int_{t_0}^{s}\widehat{h}_{n-\beta-1}(s,\rho(\tau))v(\tau)\nabla\tau\right)\right|\nabla s$$
$$\leq M\sum_{i=0}^{m-n-1}|\eta_{n+i}||t_1-t_2|+M^2|t_1-t_2|+M|t_1-t_2|$$
$$=\left(M\sum_{i=0}^{m-n-1}|\eta_{n+i}|+M^2+M\right)|t_1-t_2|,\tag{130}$$

where $\max_{\tau,\theta,s,t_1,t_2\in\Omega}\{\widehat{h}_{i-1}(\tau,t_0)\ (i=0,\dots,m-n-1),\ \widehat{h}_{\alpha-n-2}(\theta,\rho(s)),\ \int_{t_0}^{t_1}|f(s,\int_{t_0}^{s}\widehat{h}_{n-\beta-1}(s,\rho(\tau))v(\tau)\nabla\tau)|\nabla s,\ \int_{t_2}^{t_1}|f(s,\int_{t_0}^{s}\widehat{h}_{n-\beta-1}(s,\rho(\tau))v(\tau)\nabla\tau)|\nabla s\}\leq M$.

Thus, Tv is equicontinuous on Ω. This shows that T is a compact mapping.

Consider the eigenvalue problem

$$v=\lambda Tv,\quad \lambda\in(0,1).\tag{131}$$

Under the assumption that v is a solution of (131) for a $\lambda\in(0,1)$, one obtains

$$\begin{aligned}\|v\|_0&=\sup_{t\in\Omega}\left|\lambda\phi(t)+\lambda\int_{t_0}^{t}\widehat{h}_{\alpha-n-1}(t,\rho(s))\times f\left(s,\int_{t_0}^{s}\widehat{h}_{n-\beta-1}(s,\rho(\tau))v(\tau)\nabla\tau\right)\nabla s\right|\\&<\sup_{t\in\Omega}\left(|\phi(t)|+\int_{t_0}^{t}\left|\widehat{h}_{\alpha-n-1}(t,\rho(s))\right|\times\left|f\left(s,\int_{t_0}^{s}\widehat{h}_{n-\beta-1}(s,\rho(\tau))v(\tau)\nabla\tau\right)\right|\nabla s\right)\\&\leq B+A\|v\|_0\leq R,\end{aligned}\tag{132}$$

which shows that $v\notin\partial U$. By Theorem 12, T has a fixed point in $\overline{U}$, which we denote by v_0, such that $\|v_0\|_0\leq R$. □

It follows from Lemma 32 that

$$u_0(t)=\sum_{k=0}^{n-1}\widehat{h}_k(t,t_0)\eta_k+\int_{t_0}^{t}\widehat{h}_{n-1}(t,\rho(s))v_0(s)\nabla s,\quad t\in\Omega,\tag{133}$$

is a solution of (112) and (113).

In the following, we will discuss another case: $n-1<\beta<\alpha<n$.

Lemma 34. *Let $n-1<\beta<\alpha<n$ and suppose that (H_1) and (H_3) hold. A function $u\in AC_\nabla^n(\Omega)$ is a solution of the initial value problem* (112) *and* (113) *if and only if*

$$u(t)=\sum_{k=0}^{n-1}\widehat{h}_k(t,t_0)\eta_k+\int_{t_0}^{t}\widehat{h}_{\beta-1}(t,\rho(s))v(s)\nabla s,\quad t\in\Omega,\tag{134}$$

where $v\in AC_\nabla(\Omega)$ is a solution of

$$v(t)=\int_{t_0}^{t}\widehat{h}_{\alpha-\beta-1}(t,\rho(s))f(s,v(s))\nabla s.\tag{135}$$

Proof. Let $u\in AC_\nabla^n(\Omega)$ be a solution of the

$${}^{C}D_{\nabla,t_0}^{\alpha-\beta}{}^{C}D_{\nabla,t_0}^{\beta}u(t)={}^{C}D_{\nabla,t_0}^{\alpha}u(t)=f\left(t,{}^{C}D_{\nabla,t_0}^{\beta}u(t)\right),\tag{136}$$

which, after the substitution $v(t)={}^{C}D_{\nabla,t_0}^{\beta}u(t)$, becomes

$${}^{C}D_{\nabla,t_0}^{\alpha-\beta}v(t)=f(t,v(t)).\tag{137}$$

Next, by Property 9 and (113)

$$u(t) = \sum_{k=0}^{n-1} \widehat{h}_k(t, t_0)\eta_k + \int_{t_0}^{t} \widehat{h}_{\beta-1}(t, \rho(s)) v(s) \nabla s,$$
$$v(t) = v(t_0) + \int_{t_0}^{t} \widehat{h}_{\alpha-\beta-1}(t, \rho(s)) f(s, v(s)) \nabla s. \tag{138}$$

By Property 6, we have that $v(t_0) = {}^{C}D^{\beta}_{\nabla,t_0}u(t_0) = 0$, and thus the above equation becomes (135).

Conversely, let $v \in AC_\nabla(\Omega)$ be a solution of the integral equation (135); that is,

$$v(t) = \int_{t_0}^{t} \widehat{h}_{\alpha-\beta-1}(t, \rho(s)) f(s, v(s)) \nabla s, \tag{139}$$

$$u(t) = \sum_{k=0}^{n-1} \widehat{h}_k(t, t_0)\eta_k + \int_{t_0}^{t} \widehat{h}_{\beta-1}(t, \rho(s)) v(s) \nabla s. \tag{140}$$

Then, by (H_1),

$${}^{C}D^{\alpha-\beta}_{\nabla,t_0} v(t) = f(t, v(t)), \quad t \in \Omega,\ t > t_0, \tag{141}$$

and ${}^{C}D^{\beta}_{\nabla,t_0}u(t) = v(t)$. Hence

$${}^{C}D^{\alpha-\beta}_{\nabla,t_0}{}^{C}D^{\beta}_{\nabla,t_0}u(t) = f\left(t, {}^{C}D^{\beta}_{\nabla,t_0}u(t)\right) \quad (t \in \Omega, t > t_0), \tag{142}$$

and we obtain (112). Also, it follows from (140) that $u \in AC^n_\nabla(\Omega)$ and (113) are satisfied since, for $i = 0, \ldots, n-1$,

$$\begin{aligned} D^i_\nabla u(t) &= \sum_{k=0}^{n-1} D^i_\nabla \widehat{h}_k(t, t_0)\eta_k + D^i_\nabla I^{\beta}_{\nabla,t_0} v(t) \\ &= \sum_{k=0}^{i-1} D^i_\nabla \widehat{h}_k(t, t_0)\eta_k + \sum_{k=i}^{n-1} D^i_\nabla \widehat{h}_k(t, t_0)\eta_k + D^i_\nabla I^{\beta}_{\nabla,t_0} v(t) \\ &= \sum_{k=i}^{n-1} D^i_\nabla \widehat{h}_k(t, t_0)\eta_k + I^{\beta-i}_{\nabla,t_0} v(t). \end{aligned} \tag{143}$$

□

Our next existence result corresponds to the case $n-1 < \beta < \alpha < n$.

Theorem 35. *Suppose that (H_1)–(H_3) are satisfied. Then the integral equation (135) has a solution in $AC_\nabla(\Omega)$ provided*

$$A = \sup_{t\in\Omega} \int_{t_0}^{t} \left|\widehat{h}_{\alpha-\beta-1}(t, \rho(s))\right| a_2(s) \nabla s < 1,$$
$$0 < B = \sup_{t\in\Omega} \int_{t_0}^{t} \left|\widehat{h}_{\alpha-\beta-1}(t, \rho(s))\right| a_1(s) \nabla s < \infty. \tag{144}$$

Proof. We endow $AC_\nabla(\Omega)$ with the sup-norm and define, for $v \in AC_\nabla(\Omega)$, the mapping T by

$$Tv(t) = \int_{t_0}^{t} \widehat{h}_{\alpha-\beta-1}(t, \rho(s)) f(s, v(s)) \nabla s. \tag{145}$$

The mapping T is well defined and $T : AC_\nabla(\Omega) \to AC_\nabla(\Omega)$. Let

$$U = \{v \in AC_\nabla(\Omega) : \|v\|_0 < R\} \tag{146}$$

with

$$R = \frac{B}{1-A} > 0. \tag{147}$$

Let $\mathscr{C} \subset AC_\nabla(\Omega)$ be defined by $\mathscr{C} = \overline{U}$.

If $v \in \overline{U}$, then

$$\begin{aligned} \|Tv\|_0 &= \sup_{t\in\Omega} \left| \int_{t_0}^{t} \widehat{h}_{\alpha-\beta-1}(t, \rho(s)) f(s, v(s)) \nabla s \right| \\ &\le \sup_{t\in\Omega} \int_{t_0}^{t} \left|\widehat{h}_{\alpha-\beta-1}(t, \rho(s))\right| \left|f(s, v(s))\right| \nabla s \\ &\le \sup_{t\in\Omega} \int_{t_0}^{t} \left|\widehat{h}_{\alpha-\beta-1}(t, \rho(s))\right| (a_1(s) + a_2(s)|v(s)|) \nabla s \\ &\le \sup_{t\in\Omega} \int_{t_0}^{t} \left|\widehat{h}_{\alpha-\beta-1}(t, \rho(s))\right| a_1(s) \nabla s \\ &\quad + \sup_{t\in\Omega} \int_{t_0}^{t} \left|\widehat{h}_{\alpha-\beta-1}(t, \rho(s))\right| a_2(s) \nabla s \|v\|_0 \\ &= B + A\|v\|_0 \\ &\le R; \end{aligned} \tag{148}$$

that is, $T : \overline{U} \to \mathscr{C}$. Certainly, $T : \overline{U} \to \mathscr{C}$ is continuous and compact. Consider

$$v = \lambda T v, \quad \lambda \in (0, 1). \tag{149}$$

The rest of the proof is the same as the corresponding part of the proof of Theorem 30. □

The uniqueness results are based on applications of the Banach contraction principle.

The main assumption in the existence theorems below is that

(H_4) for each $R > 0$, there exists a nonnegative function γ such that $|f(t, z_1(t)) - f(t, z_2(t))| \le \gamma(t)|z_1 - z_2|$, $t \in \Omega$, $z_1, z_2 \in \mathbb{R}$.

The first uniqueness result is for the case $n-1 < \beta < \alpha < n$.

Theorem 36. *Suppose that (H_1), (H_3), and (H_4) hold. Assume that*

$$\zeta = \sup_{t\in\Omega} \int_{t_0}^{t} \left|\widehat{h}_{\alpha-\beta-1}(t, \rho(s))\right| \gamma(s) \nabla s < 1,$$
$$0 < \sup_{t\in\Omega} \int_{t_0}^{t} \left|\widehat{h}_{\alpha-\beta-1}(t, \rho(s))\right| \left|f(s, 0)\right| \nabla s < \infty. \tag{150}$$

Then the integral equation (135) has a unique solution.

Proof. In the Banach space $\mathscr{B} = (AC_\nabla(\Omega), \|\cdot\|_0)$ we define $\mathscr{C}$ by

$$\mathscr{C} = \{v \in \mathscr{B} : \|v\|_0 \le R\}, \tag{151}$$

where

$$R = \frac{1}{1-\zeta}\sup_{t\in\Omega}\int_{t_0}^{t}\left|\widehat{h}_{\alpha-\beta-1}(t,\rho(s))\right|\left|f(s,0)\right|\nabla s. \tag{152}$$

We define the mapping $T : AC_\nabla(\Omega) \to AC_\nabla(\Omega)$ as in the proof of Theorem 31.

If $v \in \mathscr{C}$, then

$$\begin{aligned}\|Tv\|_0 &\le \|Tv - T\theta\|_0 + \|T\theta\|_0\\ &\le \zeta\|v\|_0 + \sup_{t\in\Omega}\int_{t_0}^{t}\left|\widehat{h}_{\alpha-\beta-1}(t,\rho(s))\right|\left|f(s,0)\right|\nabla s\\ &= \zeta\|v\|_0 + (1-\zeta)R\\ &\le R;\end{aligned} \tag{153}$$

that is, $T : \mathscr{C} \to \mathscr{C}$.

Let $v_1, v_2 \in \mathscr{C}$. Then

$$\begin{aligned}&\|Tv_1 - Tv_2\|_0\\ &\quad= \sup_{t\in\Omega}\left|Tv_1 - Tv_2\right|\\ &\quad\le \sup_{t\in\Omega}\int_{t_0}^{t}\left|\widehat{h}_{\alpha-\beta-1}(t,\rho(s))\right|\left|f(s,v_1(s)) - f(s,v_2(s))\right|\nabla s\\ &\quad\le \sup_{t\in\Omega}\int_{t_0}^{t}\left|\widehat{h}_{\alpha-\beta-1}(t,\rho(s))\right|\gamma(s)\left|v_1(s) - v_2(s)\right|\nabla s\\ &\quad\le \sup_{t\in\Omega}\int_{t_0}^{t}\left|\widehat{h}_{\alpha-\beta-1}(t,\rho(s))\right|\gamma(s)\nabla s\|v_1 - v_2\|_0\\ &\quad\le \zeta\|v_1 - v_2\|_0;\end{aligned} \tag{154}$$

that is, T is a contraction since $\zeta < 1$.

By the Banach contraction principle, T has a unique fixed point, which is a solution of the integral equation (135). □

For the case $n-1 < \beta < n \le m-1 < \alpha < m$, the uniqueness result is given without proof.

Theorem 37. *Suppose that $(H_1), (H_3)$, and (H_4) hold and assume that*

$$\begin{aligned}\zeta = \sup_{t\in\Omega}\int_{t_0}^{t}&\left|\widehat{h}_{\alpha-\beta-1}(t,\rho(s))\right|\gamma(s)\\ &\times\left(\int_{t_0}^{s}\left|\widehat{h}_{n-\beta-1}(s,\rho(\tau))\right|\nabla\tau\right)\nabla s < 1.\end{aligned} \tag{155}$$

Assume further that

$$0 < \sup_{t\in\Omega}\left(\left|\phi(t)\right| + \int_{t_0}^{t}\left|\widehat{h}_{\alpha-\beta-1}(t,\rho(s))\right|\left|f(s,0)\right|\nabla s\right) < \infty. \tag{156}$$

Then the integral equation (115) *has a unique solution.*

5. Laplace Transform Method for Solving Ordinary Differential Equations with Caputo Fractional Derivatives

5.1. Homogeneous Equations with Constant Coefficients. In this section we apply the Laplace transform method to derive explicit solutions to homogeneous equations of the form

$$\begin{gathered}\sum_{k=1}^{m}A_k\left[{}^{C}D^{\alpha_k}_{\nabla,t_0}y(t)\right] + A_0y(t) = 0\\ (m\in\mathbb{N};\ 0<\alpha_1<\cdots<\alpha_m;\\ l-1<\alpha_m<l,\ l\in\mathbb{N},\ t_0,t\in\Omega_{k^l},\ t>t_0)\end{gathered} \tag{157}$$

involving the Caputo fractional derivatives ${}^{C}D^{\alpha_k}_{\nabla,t_0}y$ $(k = 1,\ldots,m)$, with real constants $A_k \in \mathbb{R}$ $(k = 0,\ldots,m-1)$ and $A_m = 1$.

The Laplace transform method is based on the formula:

$$\begin{aligned}\mathscr{L}_{\nabla,t_0}&\left\{{}^{C}D^{\alpha}_{\nabla,t_0}y(t)\right\}(z)\\ &= z^{\alpha}\mathscr{L}_{\nabla,t_0}\{y(t)\}(z)\\ &\quad-\sum_{j=0}^{l-1}d_jz^{\alpha-j-1}\qquad(l-1<\alpha\le l\in\mathbb{N}),\end{aligned} \tag{158}$$

$$d_j = D^j_\nabla y(t_0)\quad(j = 0,\ldots,l-1). \tag{159}$$

First, we derive explicit solutions to (157) with $m = 1$:

$$\begin{gathered}{}^{C}D^{\alpha}_{\nabla,t_0}y(t) - \lambda y(t) = 0\\ (t>t_0;\ l-1<\alpha\le l;\ l\in\mathbb{N};\ \lambda\in\mathbb{R}).\end{gathered} \tag{160}$$

In order to prove our result, we also need the following definition and lemma.

Definition 38. The function $W(t)$ is defined by

$$W(t) = \det\left(\left(D^k_\nabla y_j\right)(t)\right)^n_{k,j=1}\quad(t\in\Omega_{k^n}). \tag{161}$$

Lemma 39. *The solutions $y_1(t), y_2(t), \ldots, y_n(t)$ are linearly independent if and only if $W(t^*) \ne 0$ at some point $t^* \in \Omega$.*

Proof. We first prove sufficiency. If, to the contrary, $y_j(t)$ $(j = 1,2,\ldots,n)$ are linearly dependent in Ω, then there exist n constants $\{c_j\}_{j=1}^n$, not all zero, such that

$$\left(\left(D^k_{\nabla,t_0}y_j\right)(t)\right)^n_{k,j=1}\begin{pmatrix}c_1\\ c_2\\ \vdots\\ c_n\end{pmatrix}\equiv 0 \tag{162}$$

holds, and thus, $W(t) \equiv 0$ which leads to a contradiction. Therefore, if $W(t^*) \ne 0$ at some point $t^* \in \Omega$, then $y_1(t), y_2(t), \ldots, y_n(t)$ are linearly independent. Now we prove the necessity. Suppose, to the contrary, for any $t \in \Omega$, $W(t) = 0$. Consider the equations

$$\left(\left(D^{k}_{\nabla,t_0} y_j\right)(t^*)\right)^{n}_{k,j=1} C = 0, \tag{163}$$

where $t^* \in \Omega$, $C = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$. As $W(t^*) = 0$, the equations have nontrivial solution c_j $(j = 1, 2, \dots, n)$. Now we construct a function using these constants:

$$y(t) = \sum_{j=1}^{n} c_j y_j(t), \tag{164}$$

and we get $y(t)$ as a solution. From (163), we obtain that $y(t)$ satisfies initial value condition

$$D^{k}_{\nabla,t_0} y(t^*) = 0, \quad k = 1, \dots, n. \tag{165}$$

However, $y(t) = 0$ is also a solution to equation satisfying the initial value condition. By the uniqueness of solution, we have

$$\sum_{j=1}^{n} c_j y_j(t) = 0, \tag{166}$$

and thus, $y_j(t)$ $(j = 1, 2, \dots, n)$ are linearly dependant which leads to a contradiction. Thus, if the solutions $y_1(t), y_2(t), \dots, y_n(t)$ are linearly independent, then $W(t^*) \neq 0$ at some point $t^* \in \Omega$. The result follows. □

The following statements hold.

Theorem 40. *Let $l - 1 < \alpha \le l$ $(l \in \mathbb{N})$ and $\lambda \in \mathbb{R}$. Then the functions*

$$y_j(t) = {}_{\nabla}F_{\alpha,j+1}(\lambda; t, t_0) \quad (j = 0, \dots, l-1) \tag{167}$$

yield the fundamental system of solutions to (160).

Proof. Applying the Laplace transform to (160) and taking (158) into account, we have

$$\mathscr{L}_{\nabla,t_0}\{y(t)\}(z) = \sum_{j=0}^{l-1} d_j \frac{z^{\alpha-j-1}}{z^{\alpha} - \lambda}, \tag{168}$$

where d_j $(j = 0, \dots, l-1)$ are given by (159).

Formula (36) with $\beta = j + 1$ yields

$$\mathscr{L}_{\nabla,t_0}\left\{{}_{\nabla}F_{\alpha,j+1}(\lambda; t, t_0)\right\}(z) = \frac{z^{\alpha-j-1}}{z^{\alpha} - \lambda} \quad \left(|\lambda| < |z|^{\alpha}\right). \tag{169}$$

Thus, from (168), we derive the following solution to (160):

$$y(t) = \sum_{j=0}^{l-1} d_j y_j(t), \qquad y_j(t) = {}_{\nabla}F_{\alpha,j+1}(\lambda; t, t_0). \tag{170}$$

It is easily verified that the functions $y_j(t)$ are solutions to (160):

$$\begin{gathered} {}^{C}D^{\alpha}_{\nabla,t_0}\left[{}_{\nabla}F_{\alpha,j+1}(\lambda; t, t_0)\right] = \lambda {}_{\nabla}F_{\alpha,j+1}(\lambda; t, t_0) \\ (j = 0, \dots, l-1). \end{gathered} \tag{171}$$

In fact,

$$\begin{aligned} &{}^{C}D^{\alpha}_{\nabla,t_0}\left[{}_{\nabla}F_{\alpha,j+1}(\lambda; t, t_0)\right] \\ &\quad = {}^{C}D^{\alpha}_{\nabla,t_0}\left[\sum_{k=0}^{\infty} \lambda^{k} \widehat{h}_{k\alpha+j}(t, t_0)\right] \\ &\quad = {}^{C}D^{\alpha}_{\nabla,t_0} \lambda^{0} \widehat{h}_{j}(t, t_0) + {}^{C}D^{\alpha}_{\nabla,t_0} \sum_{k=1}^{\infty} \lambda^{k} \widehat{h}_{k\alpha+j}(t, t_0) \\ &\quad = 0 + \sum_{k=1}^{\infty} \lambda^{k} \widehat{h}_{(k-1)\alpha+j}(t, t_0) \\ &\quad = \sum_{k=0}^{\infty} \lambda^{k+1} \widehat{h}_{k\alpha+j}(t, t_0) \\ &\quad = \lambda \sum_{k=0}^{\infty} \lambda^{k} \widehat{h}_{k\alpha+j}(t, t_0) \\ &\quad = \lambda\, {}_{\nabla}F_{\alpha,j+1}(\lambda; t, t_0). \end{aligned} \tag{172}$$

Moreover,

$$\begin{aligned} D^{k}_{\nabla} y_j(t) &= D^{k}_{\nabla\nabla}F_{\alpha,j+1}(\lambda; t, t_0) \\ &= D^{k}_{\nabla}\left[\sum_{s=0}^{\infty} \lambda^{s} \widehat{h}_{s\alpha+j}(t, t_0)\right] = \sum_{s=0}^{\infty} \lambda^{s} \widehat{h}_{s\alpha+j-k}(t, t_0). \end{aligned} \tag{173}$$

It follows from (173) and (20) that

$$\begin{gathered} D^{k}_{\nabla} y_j(t_0) = 0 \quad (k, j = 0, \dots, l-1;\ j > k), \\ D^{k}_{\nabla} y_k(t_0) = 1 \quad (k = 0, \dots, l-1). \end{gathered} \tag{174}$$

If $j < k$, then

$$\begin{aligned} D^{k}_{\nabla} y_j(t_0) &= D^{k}_{\nabla} \widehat{h}_{j}(t, t_0) + D^{k}_{\nabla} \sum_{s=1}^{\infty} \lambda^{s} \widehat{h}_{s\alpha+j}(t, t_0) \\ &= 0 + \sum_{s=1}^{\infty} \lambda^{s} \widehat{h}_{s\alpha+j-k}(t, t_0) \\ &= \sum_{s=0}^{\infty} \lambda^{s+1} \widehat{h}_{s\alpha+\alpha+j-k}(t, t_0), \end{aligned} \tag{175}$$

and, since $\alpha + j - k \ge \alpha + 1 - l > 0$ for any $k, j = 0, \dots, l-1$, the following relations hold:

$$D^{k}_{\nabla} y_j(t_0) = 0 \quad (k, j = 0, \dots, l-1;\ j < k). \tag{176}$$

By (174) and (176), the Wronskian function

$$W(t) = \det\left(D^{k}_{\nabla} y_j(t)\right)^{l-1}_{k,j=0} \tag{177}$$

at t_0 is equal to 1: $W(t_0) = 1$. Then $y_j(t)$ $(j = 0, \dots, l-1)$ yield the fundamental system of solutions to (160). □

Corollary 41. *The equation*

$$^{C}D^{\alpha}_{\nabla,t_0}y(t)-\lambda y(t)=0 \quad (t>t_0;\ 0<\alpha\le 1;\ \lambda\in\mathbb{R}) \tag{178}$$

has its solution given by

$$y(t)={}_{\nabla}F_{\alpha,1}(\lambda;t,t_0), \tag{179}$$

while the equation

$$^{C}D^{\alpha}_{\nabla,t_0}y(t)-\lambda y(t)=0 \quad (t>t_0;\ 1<\alpha\le 2;\ \lambda\in\mathbb{R}) \tag{180}$$

has the fundamental system of solutions given by

$$y_0(t)={}_{\nabla}F_{\alpha,1}(\lambda;t,t_0), \qquad y_1(t)={}_{\nabla}F_{\alpha,2}(\lambda;t,t_0). \tag{181}$$

Next we derive the explicit solutions to (157) with $m=2$:

$$\begin{gathered} ^{C}D^{\alpha}_{\nabla,t_0}y(t)-\lambda\, ^{C}D^{\beta}_{\nabla,t_0}y(t)-\mu y(t)=0 \\ (t>t_0;\ l-1<\alpha\le l;\ l\in\mathbb{N};\ 0<\beta<\alpha) \end{gathered} \tag{182}$$

with $\lambda,\mu\in\mathbb{R}$.

Theorem 42. *Let $l-1<\alpha\le l$ $(l\in\mathbb{N})$, $0<\beta<\alpha$, and $\lambda,\mu\in\mathbb{R}$. Then the functions*

$$\begin{aligned} y_j(t)&=\sum_{n=0}^{\infty}\frac{\mu^n}{n!}\frac{\partial^n}{\partial\lambda^n}{}_{\nabla}F_{\alpha-\beta,\beta n+j+1}(\lambda;t,t_0) \\ &\quad-\lambda\sum_{n=0}^{\infty}\frac{\mu^n}{n!}\frac{\partial^n}{\partial\lambda^n}{}_{\nabla}F_{\alpha-\beta,\beta n+j+1+\alpha-\beta}(\lambda;t,t_0), \\ &\qquad j=0,\dots,m-1; \end{aligned} \tag{183}$$

$$\begin{aligned} y_j(t)&=\sum_{n=0}^{\infty}\frac{\mu^n}{n!}\frac{\partial^n}{\partial\lambda^n}{}_{\nabla}F_{\alpha-\beta,\beta n+j+1}(\lambda;t,t_0), \\ &\qquad j=m,\dots,l-1 \end{aligned} \tag{184}$$

yield the fundamental system of solutions to (182), *provided that the series in* (183) *and* (184) *are convergent.*

Proof. Let $m-1<\beta\le m$ $(m\le l;\ m\in\mathbb{N})$. Applying the Laplace transform to (182) and using (158), we obtain

$$\begin{aligned} \mathscr{L}_{\nabla,t_0}\{y(t)\}(z)&=\sum_{j=0}^{l-1}d_j\frac{z^{\alpha-j-1}}{z^{\alpha}-\lambda z^{\beta}-\mu} \\ &\quad-\lambda\sum_{j=0}^{m-1}d_j\frac{z^{\beta-j-1}}{z^{\alpha}-\lambda z^{\beta}-\mu}, \end{aligned} \tag{185}$$

where d_j $(j=0,\dots,l-1)$ are given by (159).

For $z\in\mathbb{C}$ and $|\mu z^{-\beta}/(z^{\alpha-\beta}-\lambda)|<1$, we have

$$\begin{aligned} \frac{1}{z^{\alpha}-\lambda z^{\beta}-\mu}&=\frac{z^{-\beta}}{z^{\alpha-\beta}-\lambda}\cdot\frac{1}{1-(\mu z^{-\beta}/(z^{\alpha-\beta}-\lambda))} \\ &=\sum_{n=0}^{\infty}\mu^n\frac{z^{-\beta-n\beta}}{(z^{\alpha-\beta}-\lambda)^{n+1}}. \end{aligned} \tag{186}$$

In addition, for $z\in\mathbb{C}$ and $|\lambda z^{\beta-\alpha}|<1$, we have

$$\begin{aligned} &\frac{z^{\alpha-j-1-\beta-n\beta}}{(z^{\alpha-\beta}-\lambda)^{n+1}} \\ &\quad=\frac{z^{(\alpha-\beta)-(\beta n+j+1)}}{(z^{\alpha-\beta}-\lambda)^{n+1}} \\ &\quad=\frac{1}{n!}\mathscr{L}_{\nabla,t_0}\left\{\frac{\partial^n}{\partial\lambda^n}{}_{\nabla}F_{\alpha-\beta,\beta n+j+1}(\lambda;t,t_0)\right\}(z), \\ &\frac{z^{\beta-j-1-\beta-n\beta}}{(z^{\alpha-\beta}-\lambda)^{n+1}} \\ &\quad=\frac{z^{(\alpha-\beta)-(\beta n+j+1+\alpha-\beta)}}{(z^{\alpha-\beta}-\lambda)^{n+1}} \\ &\quad=\frac{1}{n!}\mathscr{L}_{\nabla,t_0}\left\{\frac{\partial^n}{\partial\lambda^n}{}_{\nabla}F_{\alpha-\beta,\beta n+j+1+\alpha-\beta}(\lambda;t,t_0)\right\}(z). \end{aligned} \tag{187}$$

From (185) and (187), we derive the solution to (182):

$$y(t)=\sum_{j=0}^{l-1}d_jy_j(t), \tag{188}$$

where $y_j(t)$ $(j=0,\dots,l-1)$ are given by (183) for $j=0,\dots,m-1$ and by (184) for $j=m,\dots,l-1$. For $k=0,\dots,l-1$, the direct evaluation yields

$$\begin{aligned} &D^k_{\nabla}y_j(t) \\ &\quad=D^k_{\nabla}\Bigg[\sum_{n=0}^{\infty}\frac{\mu^n}{n!}\frac{\partial^n}{\partial\lambda^n}{}_{\nabla}F_{\alpha-\beta,\beta n+j+1}(\lambda;t,t_0) \\ &\qquad\quad-\lambda\sum_{n=0}^{\infty}\frac{\mu^n}{n!}\frac{\partial^n}{\partial\lambda^n}{}_{\nabla}F_{\alpha-\beta,\beta n+j+1+\alpha-\beta}(\lambda;t,t_0)\Bigg] \\ &\quad=\sum_{n=0}^{\infty}\frac{\mu^n}{n!}\frac{\partial^n}{\partial\lambda^n}D^k_{\nabla}\left[\sum_{s=0}^{\infty}\lambda^s\widehat{h}_{s(\alpha-\beta)+\beta n+j}(t,t_0)\right] \\ &\qquad-\lambda\sum_{n=0}^{\infty}\frac{\mu^n}{n!}\frac{\partial^n}{\partial\lambda^n}D^k_{\nabla}\left[\sum_{s=0}^{\infty}\lambda^s\widehat{h}_{s(\alpha-\beta)+\beta n+j+\alpha-\beta}(t,t_0)\right] \\ &\quad=\sum_{n=0}^{\infty}\frac{\mu^n}{n!}\frac{\partial^n}{\partial\lambda^n}\sum_{s=0}^{\infty}\lambda^s\widehat{h}_{s(\alpha-\beta)+\beta n+j-k}(t,t_0) \\ &\qquad-\lambda\sum_{n=0}^{\infty}\frac{\mu^n}{n!}\frac{\partial^n}{\partial\lambda^n}\sum_{s=0}^{\infty}\lambda^s\widehat{h}_{s(\alpha-\beta)+\beta n+j+\alpha-\beta-k}(t,t_0) \\ &\qquad\qquad(j=0,\dots,m-1), \\ &D^k_{\nabla}y_j(t)=\sum_{n=0}^{\infty}\frac{\mu^n}{n!}\frac{\partial^n}{\partial\lambda^n}\sum_{s=0}^{\infty}\lambda^s\widehat{h}_{s(\alpha-\beta)+\beta n+j-k}(t,t_0) \\ &\qquad\qquad(j=m,\dots,l-1). \end{aligned} \tag{189}$$

For $j>k$, $D^k_{\nabla}y_j(t_0)=0$, and for $j=k$, $D^k_{\nabla}y_j(t_0)=1$. Thus we have $W(t_0)=1$. Thus the functions $y_j(t)$ $(j=0,\dots,l-1)$

in (183) and (184) are linearly independent solutions to (182). The result follows. □

Corollary 43. *The equation*

$$ {}^{C}D^{\alpha}_{\nabla,t_0} y(t) - \lambda {}^{C}D^{\beta}_{\nabla,t_0} y(t) = 0 $$
$$ (t > t_0;\ l-1 < \alpha \le l;\ l \in \mathbb{N};\ 0 < \beta < \alpha) \tag{190} $$

has its fundamental system of solutions given by

$$ y_j(t) = {}_{\nabla}F_{\alpha-\beta,j+1}(\lambda; t, t_0) - \lambda {}_{\nabla}F_{\alpha-\beta,\alpha-\beta+j+1} \quad (j = 0, \ldots, m-1), $$
$$ y_j(t) = {}_{\nabla}F_{\alpha-\beta,j+1}(\lambda; t, t_0) \quad (j = m, \ldots, l-1). \tag{191} $$

Corollary 44. *The equation*

$$ {}^{C}D^{\alpha}_{\nabla,t_0} y(t) - \lambda {}^{C}D^{\beta}_{\nabla,t_0} y(t) - \mu y(t) = 0 $$
$$ (t > t_0;\ 0 < \beta < \alpha \le 1;\ \lambda, \mu \in \mathbb{R}) \tag{192} $$

has its solution by

$$ y_0(t) = \sum_{n=0}^{\infty} \frac{\mu^n}{n!} \frac{\partial^n}{\partial \lambda^n} {}_{\nabla}F_{\alpha-\beta,\beta n+1}(\lambda; t, t_0) - \lambda \sum_{n=0}^{\infty} \frac{\mu^n}{n!} \frac{\partial^n}{\partial \lambda^n} {}_{\nabla}F_{\alpha-\beta,\beta n+1+\alpha-\beta}(\lambda; t, t_0). \tag{193} $$

In particular,

$$ y_0(t) = {}_{\nabla}F_{\alpha-\beta,1}(\lambda; t, t_0) - \lambda {}_{\nabla}F_{\alpha-\beta,\alpha-\beta+1}(\lambda; t, t_0) \tag{194} $$

is a solution to the equation

$$ {}^{C}D^{\alpha}_{\nabla,t_0} y(t) - \lambda {}^{C}D^{\beta}_{\nabla,t_0} y(t) = 0 $$
$$ (t > t_0;\ 0 < \beta < \alpha \le 1;\ \lambda \in \mathbb{R}). \tag{195} $$

Corollary 45. *The equation*

$$ {}^{C}D^{\alpha}_{\nabla,t_0} y(t) - \lambda {}^{C}D^{\beta}_{\nabla,t_0} y(t) - \mu y(t) = 0, \tag{196} $$

where $t > t_0$; $1 < \alpha \le 2$, $0 < \beta < \alpha$; $\lambda, \mu \in \mathbb{R}$, has one solution $y_0(t)$, given by (193), and a second solution $y_1(t)$ given by

$$ y_1(t) = \sum_{n=0}^{\infty} \frac{\mu^n}{n!} \frac{\partial^n}{\partial \lambda^n} {}_{\nabla}F_{\alpha-\beta,\beta n+2}(\lambda; t, t_0) - \lambda \sum_{n=0}^{\infty} \frac{\mu^n}{n!} \frac{\partial^n}{\partial \lambda^n} {}_{\nabla}F_{\alpha-\beta,\beta n+\alpha-\beta+2}(\lambda; t, t_0) \tag{197} $$

for $1 < \beta < \alpha$, while, for $0 < \beta \le 1$, by

$$ y_1(t) = \sum_{n=0}^{\infty} \frac{\mu^n}{n!} \frac{\partial^n}{\partial \lambda^n} {}_{\nabla}F_{\alpha-\beta,\beta n+2}(\lambda; t, t_0). \tag{198} $$

In particular, the equation

$$ {}^{C}D^{\alpha}_{\nabla,t_0} y(t) - \lambda {}^{C}D^{\beta}_{\nabla,t_0} y(t) = 0 $$
$$ (t > t_0;\ 1 < \alpha \le 2,\ 0 < \beta < \alpha;\ \lambda \in \mathbb{R}) \tag{199} $$

has one solution $y_0(t)$ given by (194), and a second $y_1(t)$ given by

$$ y_1(t) = {}_{\nabla}F_{\alpha-\beta,2}(\lambda; t, t_0) - \lambda {}_{\nabla}F_{\alpha-\beta,\alpha-\beta+2}(\lambda; t, t_0) \tag{200} $$

for $1 < \beta < \alpha$, while for $0 < \beta \le 1$, by

$$ y_1(t) = {}_{\nabla}F_{\alpha-\beta,2}(\lambda; t, t_0). \tag{201} $$

Finally, we find explicit solutions to (157) with any $m \in \mathbb{N} \setminus \{1, 2\}$. It is convenient to rewrite (157) in the form (202)

$$ {}^{C}D^{\alpha}_{\nabla,t_0} y(t) - \lambda {}^{C}D^{\beta}_{\nabla,t_0} y(t) - \sum_{k=0}^{m-2} A_k {}^{C}D^{\alpha_k}_{\nabla,t_0} y(t) = 0 $$
$$ (t > t_0;\ m \in \mathbb{N} \setminus \{1, 2\};\ 0 = \alpha_0 < \alpha_1 < \cdots < \alpha_{m-2} < \beta < \alpha; $$
$$ \lambda, A_0, \ldots, A_{m-2} \in \mathbb{R}). \tag{202} $$

Theorem 46. *Let $\alpha, \beta, \alpha_{m-2}, \ldots, \alpha_0$ and $l, l_{m-1}, \ldots, l_0 \in \mathbb{N}_0$ $(m \in \mathbb{N} \setminus \{1, 2\})$ be such that*

$$ \begin{gathered} 0 = \alpha_0 < \alpha_1 < \cdots < \alpha_{m-2} < \beta < \alpha, \\ 0 = l_0 \le l_1 \le \cdots \le l_{m-1} \le l, \\ l - 1 < \alpha \le l, \\ l_{m-1} - 1 < \beta \le l_{m-1}, \\ l_k - 1 < \alpha_k \le l_k \\ (k = 0, \ldots, m-2), \end{gathered} \tag{203} $$

and let $\lambda, A_0, \ldots, A_{m-2} \in \mathbb{R}$. Then the fundamental system of solutions to (202) is given by the formulas

$$ \begin{aligned} y_j(t) &= \sum_{n=0}^{\infty} \left(\sum_{k_0+\cdots+k_{m-2}=n} \right) \frac{1}{k_0! \cdots k_{m-2}!} \left[\prod_{\nu=0}^{m-2} (A_\nu)^{k_\nu} \right] \\ &\quad \cdot \left\{ \frac{\partial^n}{\partial \lambda^n} {}_{\nabla}F_{\alpha-\beta, j+1+\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu}(\lambda; t, t_0) \right. \\ &\quad - \lambda \frac{\partial^n}{\partial \lambda^n} {}_{\nabla}F_{\alpha-\beta, \alpha-\beta+j+1+\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu}(\lambda; t, t_0) \\ &\quad \left. - \sum_{k=0}^{m-2} A_k \frac{\partial^n}{\partial \lambda^n} {}_{\nabla}F_{\alpha-\beta, \alpha-\alpha_k+j+1+\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu}(\lambda; t, t_0) \right\} \end{aligned} \tag{204} $$

for $j = 0, \dots, l_{m-2} - 1$*; by*

$$y_j(t) = \sum_{n=0}^{\infty} \left(\sum_{k_0+\cdots+k_{m-2}=n} \right) \frac{1}{k_0! \cdots k_{m-2}!} \left[\prod_{\nu=0}^{m-2} (A_\nu)^{k_\nu} \right] \cdot \left\{ \frac{\partial^n}{\partial \lambda^n} {}_\nabla F_{\alpha-\beta, j+1+\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu}(\lambda; t, t_0) - \lambda \frac{\partial^n}{\partial \lambda^n} {}_\nabla F_{\alpha-\beta, \alpha-\beta+j+1+\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu}(\lambda; t, t_0) \right\} \tag{205}$$

for $j = l_{m-2}, \dots, l_{m-1} - 1$*; and by*

$$y_j(t) = \sum_{n=0}^{\infty} \left(\sum_{k_0+\cdots+k_{m-2}=n} \right) \frac{1}{k_0! \cdots k_{m-2}!} \left[\prod_{\nu=0}^{m-2} (A_\nu)^{k_\nu} \right] \times \frac{\partial^n}{\partial \lambda^n} {}_\nabla F_{\alpha-\beta, j+1+\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu}(\lambda; t, t_0) \tag{206}$$

for $j = l_{m-1}, \dots, l - 1$.

Proof. Applying the Laplace transform to (202) and using (158), we obtain

$$\begin{aligned} \mathscr{L}_{\nabla, t_0}\{y(t)\}(z) = &\sum_{j=0}^{l-1} d_j \frac{z^{\alpha-j-1}}{z^\alpha - \lambda z^\beta - \sum_{k=0}^{m-2} A_k z^{\alpha_k}} \\ &- \lambda \sum_{j=0}^{l_{m-1}-1} d_j \frac{z^{\beta-j-1}}{z^\alpha - \lambda z^\beta - \sum_{k=0}^{m-2} A_k z^{\alpha_k}} \\ &- \sum_{k=0}^{m-2} A_k \sum_{j=0}^{l_k-1} d_j \frac{z^{\alpha_k-j-1}}{z^\alpha - \lambda z^\beta - \sum_{k=0}^{m-2} A_k z^{\alpha_k}}, \end{aligned} \tag{207}$$

where d_j $(j = 0, \dots, l-1)$ are given by (159).

For $z \in \mathbb{C}$ and $|\sum_{k=0}^{m-2} A_k z^{\alpha_k-\beta}/(z^{\alpha-\beta} - \lambda)| < 1$, we have

$$\begin{aligned} &\frac{1}{z^\alpha - \lambda z^\beta - \sum_{k=0}^{m-2} A_k z^{\alpha_k}} \\ &= \frac{z^{-\beta}}{z^{\alpha-\beta} - \lambda} \cdot \frac{1}{\left(1 - \left(\sum_{k=0}^{m-2} A_k z^{\alpha_k-\beta} / \left(z^{\alpha-\beta} - \lambda\right)\right)\right)} \\ &= \sum_{n=0}^{\infty} \frac{z^{-\beta}}{\left(z^{\alpha-\beta} - \lambda\right)^{n+1}} \left(\sum_{k=0}^{m-2} A_k z^{\alpha_k-\beta} \right)^n \\ &= \sum_{n=0}^{\infty} \left(\sum_{k_0+\cdots+k_{m-2}=n} \right) \frac{n!}{k_0! \cdots k_{m-2}!} \times \left[\prod_{\nu=0}^{m-2} (A_\nu)^{k_\nu} \right] \frac{z^{-\beta-\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu}}{\left(z^{\alpha-\beta} - \lambda\right)^{n+1}}, \end{aligned} \tag{208}$$

if we also take into account the following relation:

$$(x_0 + \cdots + x_{m-2})^n = \left(\sum_{k_0+\cdots+k_{m-2}=n} \right) \frac{n!}{k_0! \cdots k_{m-2}!} \prod_{\nu=0}^{m-2} x_\nu^{k_\nu}, \tag{209}$$

where the summation is taken over all $k_0, \dots, k_{m-2} \in \mathbb{N}_0$ such that $k_0 + \cdots + k_{m-2} = n$.

In addition, for $z \in \mathbb{C}$ and $|\lambda z^{\beta-\alpha}| < 1$, we have

$$\begin{aligned} &\frac{z^{\alpha-j-1-\beta-\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu}}{\left(z^{\alpha-\beta} - \lambda\right)^{n+1}} \\ &= \frac{z^{(\alpha-\beta)-(j+1+\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu)}}{\left(z^{\alpha-\beta} - \lambda\right)^{n+1}} \\ &= \frac{1}{n!} \mathscr{L}_{\nabla, t_0} \left\{ \frac{\partial^n}{\partial \lambda^n} {}_\nabla F_{\alpha-\beta, j+1+\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu}(\lambda; t, t_0) \right\}(z), \end{aligned} \tag{210}$$

$$\begin{aligned} &\frac{z^{\beta-j-1-\beta-\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu}}{\left(z^{\alpha-\beta} - \lambda\right)^{n+1}} \\ &= \frac{z^{(\alpha-\beta)-(\alpha-\beta+j+1+\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu)}}{\left(z^{\alpha-\beta} - \lambda\right)^{n+1}} \\ &= \frac{1}{n!} \mathscr{L}_{\nabla, t_0} \left\{ \frac{\partial^n}{\partial \lambda^n} {}_\nabla F_{\alpha-\beta, \alpha-\beta+j+1+\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu}(\lambda; t, t_0) \right\}(z), \end{aligned} \tag{211}$$

$$\begin{aligned} &\frac{z^{\alpha_k-j-1-\beta-\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu}}{\left(z^{\alpha-\beta} - \lambda\right)^{n+1}} \\ &= \frac{z^{(\alpha-\beta)-(\alpha-\alpha_k+j+1+\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu)}}{\left(z^{\alpha-\beta} - \lambda\right)^{n+1}} \\ &= \frac{1}{n!} \mathscr{L}_{\nabla, t_0} \left\{ \frac{\partial^n}{\partial \lambda^n} {}_\nabla F_{\alpha-\beta, \alpha-\alpha_k+j+1+\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu}(\lambda; t, t_0) \right\}(z). \end{aligned} \tag{212}$$

From (210) to (212), we derive the solution to (202):

$$y(t) = \sum_{j=0}^{l-1} d_j y_j(t), \tag{213}$$

where $y_j(t)$ $(j = 0, \dots, l-1)$ are given by (204) for $j = 0, \dots, l_{m-2} - 1$, by (205) for $j = l_{m-2}, \dots, l_{m-1} - 1$, and by (206) for $j = l_{m-1}, \ldots, l-1$. For $k = 0, \ldots, l-1$, the direct evaluation yields

$$\begin{aligned}
D_{\nabla}^{k} y_j(t) &= \sum_{n=0}^{\infty} \left(\sum_{k_0+\cdots+k_{m-2}=n} \right) \frac{1}{k_0! \cdots k_{m-2}!} \left[\prod_{\nu=0}^{m-2} (A_\nu)^{k_\nu} \right] \\
&\quad \cdot D_{\nabla}^{k} \left\{ \frac{\partial^n}{\partial \lambda^n} \sum_{s=0}^{\infty} \lambda^s \widehat{h}_{s(\alpha-\beta)+\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu + j} (t, t_0) \right. \\
&\quad - \lambda \frac{\partial^n}{\partial \lambda^n} \sum_{s=0}^{\infty} \lambda^s \widehat{h}_{s(\alpha-\beta)+\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu + j+\alpha-\beta} (t, t_0) \\
&\quad - \sum_{k=0}^{m-2} A_k \frac{\partial^n}{\partial \lambda^n} \\
&\quad \left. \times \sum_{s=0}^{\infty} \lambda^s \widehat{h}_{s(\alpha-\beta)+\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu + j+\alpha-\alpha_k} (t, t_0) \right\} \\
&= \sum_{n=0}^{\infty} \left(\sum_{k_0+\cdots+k_{m-2}=n} \right) \frac{1}{k_0! \cdots k_{m-2}!} \left[\prod_{\nu=0}^{m-2} (A_\nu)^{k_\nu} \right] \\
&\quad \cdot \left\{ \frac{\partial^n}{\partial \lambda^n} \sum_{s=0}^{\infty} \lambda^s \widehat{h}_{s(\alpha-\beta)+\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu + j-k} (t, t_0) \right. \\
&\quad - \lambda \frac{\partial^n}{\partial \lambda^n} \sum_{s=0}^{\infty} \lambda^s \widehat{h}_{s(\alpha-\beta)+\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu + j-k+\alpha-\beta} (t, t_0) \\
&\quad - \sum_{k=0}^{m-2} A_k \frac{\partial^n}{\partial \lambda^n} \\
&\quad \left. \times \sum_{s=0}^{\infty} \lambda^s \widehat{h}_{s(\alpha-\beta)+\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu + j-k+\alpha-\alpha_k} (t, t_0) \right\}
\end{aligned} \tag{214}$$

for $j = 0, \ldots, l_{m-2} - 1$,

$$\begin{aligned}
D_{\nabla}^{k} y_j(t) &= \sum_{n=0}^{\infty} \left(\sum_{k_0+\cdots+k_{m-2}=n} \right) \frac{1}{k_0! \cdots k_{m-2}!} \left[\prod_{\nu=0}^{m-2} (A_\nu)^{k_\nu} \right] \\
&\quad \cdot \left\{ \frac{\partial^n}{\partial \lambda^n} \sum_{s=0}^{\infty} \lambda^s \widehat{h}_{s(\alpha-\beta)+\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu + j-k} (t, t_0) - \lambda \frac{\partial^n}{\partial \lambda^n} \right. \\
&\quad \left. \times \sum_{s=0}^{\infty} \lambda^s \widehat{h}_{s(\alpha-\beta)+\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu + j-k+\alpha-\beta} (t, t_0) \right\}
\end{aligned} \tag{215}$$

for $j = l_{m-2}, \ldots, l_{m-1} - 1$, and

$$\begin{aligned}
D_{\nabla}^{k} y_j(t) &= \sum_{n=0}^{\infty} \left(\sum_{k_0+\cdots+k_{m-2}=n} \right) \frac{1}{k_0! \cdots k_{m-2}!} \left[\prod_{\nu=0}^{m-2} (A_\nu)^{k_\nu} \right] \\
&\quad \times \frac{\partial^n}{\partial \lambda^n} \sum_{s=0}^{\infty} \lambda^s \widehat{h}_{s(\alpha-\beta)+\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu + j-k} (t, t_0)
\end{aligned} \tag{216}$$

for $j = l_{m-1}, \ldots, l-1$. For $j > k$, $D_{\nabla}^{k} y_j(t_0) = 0$, and for $j = k$, $D_{\nabla}^{k} y_j(t_0) = 1$. Thus we have $W(t_0) = 1$. Thus the functions $y_j(t)$ $(j = 0, \ldots, l-1)$ in (204)–(206) are linearly independent solutions to (202). The result follows. □

5.2. Nonhomogeneous Equations with Constant Coefficients. In this section, we still use Laplace transform method to find general solutions to the corresponding nonhomogeneous equations

$$\begin{gathered}
\sum_{k=1}^{m} A_k \left[{}^{C}D_{\nabla, t_0}^{\alpha_k} y(t) \right] + A_0 y(t) = f(t) \\
(m \in \mathbb{N};\ 0 < \alpha_1 < \cdots < \alpha_m; \\
l-1 < \alpha_m < l,\ l \in \mathbb{N},\ t_0, t \in \Omega_{k^l},\ t > t_0)
\end{gathered} \tag{217}$$

with real coefficients $A_k \in \mathbb{R}$ $(k = 0, \ldots, m)$ and a given function $f(t)$. The general solution to (217) is a sum of its particular solution and of the general solution to the corresponding homogeneous equation (157). It is sufficient to construct a particular solution to (217).

By (158) and (159), for suitable functions y, the Laplace transform of ${}^{C}D_{\nabla, t_0}^{\alpha} y$ is given by

$$\mathscr{L}_{\nabla, t_0} \left\{ {}^{C}D_{\nabla, t_0}^{\alpha} y(t) \right\}(z) = z^{\alpha} \mathscr{L}_{\nabla, t_0} \{ y(t) \}(z). \tag{218}$$

Applying the Laplace transform to (217) and taking (218) into account, we have

$$\left[A_0 + \sum_{k=1}^{m} A_k z^{\alpha_k} \right] \mathscr{L}_{\nabla, t_0} \{ y(t) \}(z) = \mathscr{L}_{\nabla, t_0} \{ f(t) \}(z). \tag{219}$$

Using the inverse Laplace transform $\mathscr{L}_{\nabla}^{-1}$ from here we obtain a particular solution to (217) in the form

$$y(t) = \mathscr{L}_{\nabla, t_0}^{-1} \left[\frac{\mathscr{L}_{\nabla, t_0} \{ f(t) \}(z)}{A_0 + \sum_{k=1}^{m} A_k z^{\alpha_k}} \right](t). \tag{220}$$

Using the Laplace convolution formula

$$\mathscr{L}_{\nabla, t_0} \{ f * g \}(z) = \mathscr{L}_{\nabla, t_0} \{ f \}(z) \mathscr{L}_{\nabla, t_0} \{ g \}(z), \tag{221}$$

we can introduce the *Laplace fractional analog* of the Green function as follows:

$$\begin{gathered}
G_{\alpha_1, \ldots, \alpha_m}(t) = \mathscr{L}_{\nabla, t_0}^{-1} \left\{ \frac{1}{P_{\alpha_1, \ldots, \alpha_m}(z)} \right\}(t), \\
P_{\alpha_1, \ldots, \alpha_m}(z) = A_0 + \sum_{k=1}^{m} A_k z^{\alpha_k},
\end{gathered} \tag{222}$$

and express a particular solution of (217) in the form of the Laplace convolution $G_{\alpha_1, \ldots, \alpha_m}(t)$ and $f(t)$

$$y(t) = G_{\alpha_1, \ldots, \alpha_m}(t) * f(t). \tag{223}$$

Theorem 47. *Let $l-1<\alpha\le l$ $(l\in\mathbb{N})$, $\lambda\in\mathbb{R}$, and $f(t)$ be a given function. Then the equation*

$$ {}^{C}D^{\alpha}_{\nabla,t_0}y(t)-\lambda y(t)=f(t) \tag{224}$$

is solvable, and its general solution is given by

$$y(t)={}_{\nabla}F_{\alpha,\alpha}(\lambda;t,t_0)*f(t)+\sum_{j=0}^{l-1}c_j\,{}_{\nabla}F_{\alpha,j+1}(\lambda;t,t_0), \tag{225}$$

where c_j $(j=0,\dots,l-1)$ are arbitrary real constants.

In particular, the general solutions to (224) *with $0<\alpha\le 1$ and $1<\alpha\le 2$ have the forms*

$$y(t)={}_{\nabla}F_{\alpha,\alpha}(\lambda;t,t_0)*f(t)+c_{0\nabla}F_{\alpha,1}(\lambda;t,t_0), \tag{226}$$

$$\begin{aligned}y(t)&={}_{\nabla}F_{\alpha,\alpha}(\lambda;t,t_0)*f(t)+c_{0\nabla}F_{\alpha,1}(\lambda,t,t_0)\\&\quad+c_{1\nabla}F_{\alpha,2}(\lambda;t,t_0),\end{aligned} \tag{227}$$

respectively, where c_0 and c_1 are arbitrary real constants.

Proof. Equation (224) is (217) with $m=1$, $\alpha_1=\alpha$, $A_1=1$, $A_0=-\lambda$ and (222) takes the form

$$G_{\alpha}(t)=\mathscr{L}^{-1}_{\nabla,t_0}\left\{\frac{1}{z^{\alpha}-\lambda}\right\}(t)={}_{\nabla}F_{\alpha,\alpha}(\lambda;t,t_0). \tag{228}$$

Thus (223), with $G_{\alpha_1,\dots,\alpha_m}(t)=G_{\alpha}(t)$, and Theorem 40 yield (225). Theorem is proved. □

Theorem 48. *Let $l-1<\alpha\le l$ $(l\in\mathbb{N})$, $0<\beta<\alpha$, $\lambda,\mu\in\mathbb{R}$, and let $f(x)$ be a given function. Then the equation*

$$ {}^{C}D^{\alpha}_{\nabla,t_0}y(t)-\lambda\,{}^{C}D^{\beta}_{\nabla,t_0}y(t)-\mu y(t)=f(t) \tag{229}$$

is solvable, and its general solution has the form

$$y(t)=\sum_{n=0}^{\infty}\frac{\mu^n}{n!}\frac{\partial^n}{\partial\lambda^n}\,{}_{\nabla}F_{\alpha-\beta,\alpha+n\beta}(\lambda,t,t_0)*f(t)+\sum_{j=0}^{l-1}c_jy_j(t), \tag{230}$$

where $y_j(t)$ $(j=0,\dots,l-1)$ are given by (183) *and* (184) *and c_j $(j=0,\dots,l-1)$ are arbitrary real constants.*

Proof. Equation (229) is the same as (217) with $m=2$, $\alpha_2=\alpha$, $\alpha_1=\beta$, $A_2=1$, $A_1=-\lambda$, $A_0=-\mu$, and (222) is given by

$$G_{\alpha,\beta;\lambda,\mu}(t)=\mathscr{L}^{-1}_{\nabla,t_0}\left\{\frac{1}{z^{\alpha}-\lambda z^{\beta}-\mu}\right\}(t). \tag{231}$$

For $z\in\mathbb{C}$ and $|\mu z^{-\beta}/(z^{\alpha-\beta}-\lambda)|<1$, we have

$$\begin{aligned}\frac{1}{z^{\alpha}-\lambda z^{\beta}-\mu}&=\frac{z^{-\beta}}{z^{\alpha-\beta}-\lambda}\cdot\frac{1}{1-\left(\mu z^{-\beta}/\left(z^{\alpha-\beta}-\lambda\right)\right)}\\&=\sum_{n=0}^{\infty}\frac{\mu^nz^{-\beta-n\beta}}{\left(z^{\alpha-\beta}-\lambda\right)^{n+1}}\end{aligned} \tag{232}$$

and then we get

$$G_{\alpha,\beta;\lambda,\mu}(t)=\mathscr{L}^{-1}_{\nabla,t_0}\left\{\sum_{n=0}^{\infty}\mu^n\frac{z^{-\beta-n\beta}}{\left(z^{\alpha-\beta}-\lambda\right)^{n+1}}\right\}(t). \tag{233}$$

In addition, for $z\in\mathbb{C}$ and $|\lambda z^{\beta-\alpha}|<1$, we have

$$\frac{z^{-\beta-n\beta}}{\left(z^{\alpha-\beta}-\lambda\right)^{n+1}}=\frac{1}{n!}\mathscr{L}_{\nabla,t_0}\left\{\frac{\partial^n}{\partial\lambda^n}\,{}_{\nabla}F_{\alpha-\beta,\alpha+n\beta}(\lambda;t,t_0)\right\}(z) \tag{234}$$

and hence (233) takes the following form:

$$G_{\alpha,\beta;\lambda,\mu}(t)=\sum_{n=0}^{\infty}\frac{\mu^n}{n!}\frac{\partial^n}{\partial\lambda^n}\,{}_{\nabla}F_{\alpha-\beta,\alpha+n\beta}(\lambda;t,t_0). \tag{235}$$

Thus the result in (230) follows from (223) with $G_{\alpha_1,\dots,\alpha_m}(t)=G_{\alpha,\beta;\lambda,\mu}(t)$ and Theorem 42. □

Theorem 49. *Let $m\in\mathbb{N}\setminus\{1,2\}$, $l-1<\alpha\le l$ $(l\in\mathbb{N})$, $\beta,\alpha_1,\dots,\alpha_{m-2}$ be such that $\alpha>\beta>\alpha_{m-2}>\cdots>\alpha_1>\alpha_0=0$, and let $\lambda,A_0,\dots,A_{m-2}\in\mathbb{R}$, and let $f(t)$ be a given function. The equation*

$$\begin{aligned}&{}^{C}D^{\alpha}_{\nabla,t_0}y(t)-\lambda\,{}^{C}D^{\beta}_{\nabla,t_0}y(t)-\sum_{k=0}^{m-2}A_k\,{}^{C}D^{\alpha_k}_{\nabla,t_0}y(t)=f(t)\\&\quad(m\in\mathbb{N}\setminus\{1,2\};\ \alpha>\beta>\alpha_{m-2}>\cdots>\alpha_1>\alpha_0=0;\\&\quad\lambda,A_0,\dots,A_{m-2}\in\mathbb{R})\end{aligned} \tag{236}$$

is solvable, and its general solution is given by

$$\begin{aligned}y(t)&=\sum_{n=0}^{\infty}\left(\sum_{k_0+\cdots+k_{m-2}=n}\right)\frac{1}{k_0!\cdots k_{m-2}!}\\&\quad\times\left[\prod_{\nu=0}^{m-2}(A_\nu)^{k_\nu}\right]\frac{\partial^n}{\partial\lambda^n}\,{}_{\nabla}F_{\alpha-\beta,\alpha+\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu}(\lambda;t,t_0)*f(t)\\&\quad+\sum_{j=0}^{l-1}c_jy_j(t),\end{aligned} \tag{237}$$

where $y_j(t)$ $(j=0,\dots,l-1)$ are given by (204)–(206) *and c_j $(j=0,\dots,l-1)$ are arbitrary real constants.*

Proof. Equation (236) is the same equation as (217) with $\alpha_m=\alpha$, $\alpha_{m-1}=\beta$, $A_m=1$, $A_{m-1}=-\lambda$, and with $-A_k$ instead of A_k for $k=0,\dots,m-2$. Since $\alpha_0=0$, (222) takes the form

$$G_{\alpha_1,\dots,\alpha_{m-2},\beta,\alpha;\lambda}(t)=\mathscr{L}^{-1}_{\nabla,t_0}\left\{\frac{1}{z^{\alpha}-\lambda\alpha^{\beta}-\sum_{k=0}^{m-2}A_kz^{\alpha_k}}\right\}(t). \tag{238}$$

For $z \in \mathbb{C}$ and $|\sum_{k=0}^{m-2} A_k z^{\alpha_k-\beta}/(z^{\alpha-\beta}-\lambda)| < 1$, we have

$$\begin{aligned}
&\frac{1}{z^{\alpha}-\lambda z^{\beta}-\sum_{k=0}^{m-2} A_k z^{\alpha_k}} \\
&\quad = \frac{z^{-\beta}}{z^{\alpha-\beta}-\lambda}\cdot\frac{1}{\left(1-\left(\sum_{k=0}^{m-2} A_k z^{\alpha_k-\beta}/\left(z^{\alpha-\beta}-\lambda\right)\right)\right)} \\
&\quad = \sum_{n=0}^{\infty}\frac{z^{-\beta}}{\left(z^{\alpha-\beta}-\lambda\right)^{n+1}}\left(\sum_{k=0}^{m-2} A_k z^{\alpha_k-\beta}\right)^{n} \\
&\quad = \sum_{n=0}^{\infty}\left(\sum_{k_0+\cdots+k_{m-2}=n}\right)\frac{n!}{k_0!\cdots k_{m-2}!} \\
&\qquad \times\left[\prod_{\nu=0}^{m-2}\left(A_\nu\right)^{k_\nu}\right]\frac{z^{-\beta-\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu}}{\left(z^{\alpha-\beta}-\lambda\right)^{n+1}},
\end{aligned} \tag{239}$$

if we also take into account the following relation:

$$\left(x_0+\cdots+x_{m-2}\right)^{n} = \left(\sum_{k_0+\cdots+k_{m-2}=n}\right)\frac{n!}{k_0!\cdots k_{m-2}!}\prod_{\nu=0}^{m-2} x_\nu^{k_\nu}, \tag{240}$$

where the summation is taken over all $k_0, \ldots, k_{m-2} \in \mathbb{N}_0$ such that $k_0 + \cdots + k_{m-2} = n$, and then we get

$$\begin{aligned}
&G_{\alpha_1,\ldots,\alpha_{m-2},\beta,\alpha;\lambda}(t) \\
&\quad = \mathscr{L}_{\nabla,t_0}^{-1}\left\{\sum_{n=0}^{\infty}\left(\sum_{k_0+\cdots+k_{m-2}=n}\right)\frac{n!}{k_0!\cdots k_{m-2}!}\right. \\
&\qquad\left.\times\left[\prod_{\nu=0}^{m-2}\left(A_\nu\right)^{k_\nu}\right]\frac{z^{-\beta-\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu}}{\left(z^{\alpha-\beta}-\lambda\right)^{n+1}}\right\}(t).
\end{aligned} \tag{241}$$

For $z \in \mathbb{C}$ and $|\lambda z^{\beta-\alpha}| < 1$, we have

$$\begin{aligned}
&\frac{z^{-\beta-\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu}}{\left(z^{\alpha-\beta}-\lambda\right)^{n+1}} \\
&\quad = \frac{1}{n!}\mathscr{L}_{\nabla,t_0}\left\{\frac{\partial^n}{\partial\lambda^n}\,{}_{\nabla}F_{\alpha-\beta,\alpha+\sum_{\nu=0}^{m-2}(\beta-\alpha_\nu)k_\nu}\left(\lambda;t,t_0\right)\right\}(z).
\end{aligned} \tag{242}$$

Thus the result in (237) follows from (223) with $G_{\alpha_1,\ldots,\alpha_m}(t) = G_{\alpha_1,\ldots,\alpha_{m-2},\beta,\alpha;\lambda}(t)$ and Theorem 46. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' Contribution

All authors contributed equally and significantly in writing this paper. All authors read and approved the final paper.

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Research Article

On the Fourier-Transformed Boltzmann Equation with Brownian Motion

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We establish a global existence theorem, and uniqueness and stability of solutions of the Cauchy problem for the Fourier-transformed Fokker-Planck-Boltzmann equation with singular Maxwellian kernel, which may be viewed as a kinetic model for the stochastic time-evolution of characteristic functions governed by Brownian motion and collision dynamics.

1. Introduction

In this paper, we consider the Cauchy problem for the space-homogeneous Fokker-Planck-Boltzmann equation which takes the form

$$\begin{aligned} \partial_t f(v,t) &= Q(f,f)(v,t) + \nu\Delta f(v,t) \\ &\quad \text{for } (v,t) \in \mathbb{R}^3 \times (0,\infty), \\ f(v,0) &= f_0(v) \quad \text{for } v \in \mathbb{R}^3. \end{aligned} \tag{1}$$

Here, the diffusion constant $\nu \geq 0$, f_0 is a nonnegative initial datum and $Q(f)$ stands for the collision term defined as

$$\begin{aligned} &Q(f,f)(v) \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b(\mathbf{k}\cdot\sigma)\left[f(v')f(v'_*) - f(v)f(v_*)\right] d\sigma\, dv_* \end{aligned} \tag{2}$$

for each scalar-valued function f on $\mathbb{R}^3$ where

$$\begin{aligned} v' &= \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \\ v'_* &= \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma, \\ \mathbf{k} &= \frac{v-v_*}{|v-v_*|}, \end{aligned} \tag{3}$$

the collision kernel b is a nonnegative function on $[-1,1]$, and $d\sigma$ denotes the area measure on the unit sphere $\mathbb{S}^2$.

In kinetic theory of a rarefied gas, the Fokker-Planck-Boltzmann equation (1) models the single-particle distribution function f of its molecules which evolve under binary and elastic collision dynamics as well as Brownian motion (see below). Each pair (v', v'_*) represents the postcollision velocities of two molecules colliding with velocities (v, v_*).

The collision kernel b is an implicitly-defined function which represents a specific type of collision dynamics in terms of the deviation angle θ defined by $\cos\theta = \mathbf{k}\cdot\sigma$. In a physically relevant model known as the Maxwellian kernel, it is customary to assume that $b(\cos\theta)$ is supported in $[0,\pi/2]$, bounded away from $\theta = 0$, but develops a singularity at $\theta = 0$ in the form

$$b(\cos\theta)\sin\theta \sim \theta^{-3/2} \quad \text{as } \theta \longrightarrow 0+, \tag{4}$$

which accounts for grazing collisions in the long-range interactions.

The Maxwellian kernel is a special instance of

$$B = \left|v - v_*\right|^\lambda b(\cos\theta) \quad (-3 < \lambda \le 2), \tag{5}$$

known as the collision kernel of inverse-power potential type, and we refer to Villani's review paper [1] for more details. Besides the physically relevant assumption (4) on b, a simplified one is that

$$\|b\|_{L^1(\mathbb{S}^2)} = 2\pi \int_0^{\pi/2} b(\cos\theta)\sin\theta\, d\theta < +\infty, \tag{6}$$

referred to as Grad's angular cutoff assumption.

The inhomogeneous Fokker-Planck-Boltzmann equation reads

$$\partial_t f + v\cdot\nabla_x f = Q(f,f) + \nu\Delta_v f \quad \text{in } \mathbb{R}^3\times\mathbb{R}^3\times(0,\infty) \tag{7}$$

for the unknown density $f = f(x,v,t)$, where the space variable $x \in \mathbb{R}^3$ stands for the position. In the case when the collision kernel B takes form (5) and the angular part b satisfies certain cutoff assumption of type (6), let us mention some of the earlier works on the Cauchy problem for (7). In the small perturbations of the vacuum state, a global existence result is obtained by Hamdache [2]. In the context of renormalized solutions, global existence and stability of solutions with large data are established by DiPerna and Lions [3]. In the linearized context around the global Maxwellian $M(v) = (2\pi)^{-3/2}\exp(-|v|^2/2)$, global existence or asymptotic behavior of solutions is investigated by Li and Matsumura [4], Xiong et al. [5], and Zhong and Li [6]. We also refer to Li [7] for the diffusive property of solutions and further references cited in the aforementioned work.

As for the homogeneous Fokker-Planck-Boltzmann equation, we are aware only of results of Goudon [8] for the global existence of a weak solution in the case when the collision kernel is given by (5) with $-3 < \lambda < -2$ and b satisfies a singular condition of type (4). For the homogeneous Boltzmann equation, however, more extensive results are available. We refer to Arkeryd [9, 10], Goudon [8], and Villani [11] and to the references cited therein.

We recall that the Fourier transform of a complex Borel measure μ on $\mathbb{R}^3$ is defined by

$$\widehat{\mu}(\xi) = \int_{\mathbb{R}^3} e^{-i\xi\cdot v}\, d\mu(v) \quad \left(\xi\in\mathbb{R}^3\right), \tag{8}$$

which extends to any tempered distribution on $\mathbb{R}^3$ via the usual functional pairing relations. If μ is a probability measure, that is, a nonnegative Borel measure with unit mass, $\widehat{\mu}$ is said to be a characteristic function.

From a probability theory point of view, Cauchy problem (1), with an initial probability density f_0, could be considered as a governing equation for the time-evolution of a family of probability densities $\{f(\cdot,t)\}_{t\ge 0}$ and, hence, it is natural to study the problem on the Fourier transform side for it is fundamental in probability theory to investigate a probability distribution through its characteristic function.

In [12], Bobylev discovered a remarkably simple formula for the Fourier transform of the collision term which reads

$$\begin{aligned} &[Q(f,f)]\,\widehat{}\,(\xi) \\ &\quad = \int_{\mathbb{S}^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\left[\widehat{f}(\xi^+)\widehat{f}(\xi^-) - \widehat{f}(\xi)\widehat{f}(0)\right] d\sigma, \\ &\qquad \xi^+ = \frac{\xi + |\xi|\sigma}{2}, \qquad \xi^- = \frac{\xi - |\xi|\sigma}{2} \end{aligned} \tag{9}$$

for each nonzero $\xi \in \mathbb{R}^3$. To simplify, we introduce the Boltzmann-Bobylev operator $\mathscr{B}$ defined by

$$\mathscr{B}(\phi)(\xi) = \int_{\mathbb{S}^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\left[\phi(\xi^+)\phi(\xi^-) - \phi(\xi)\phi(0)\right] d\sigma \tag{10}$$

for each complex-valued function ϕ on $\mathbb{R}^3$. In view of Bobylev's formula, the Fourier-transformed version of (1) takes the form

$$\begin{aligned} &\left(\partial_t + \nu|\xi|^2\right)\phi(\xi,t) = \mathscr{B}(\phi)(\xi,t) \\ &\qquad \text{for } (\xi,t)\in\mathbb{R}^3\times(0,\infty), \\ &\phi(\xi,0) = \phi_0(\xi) \quad \text{for } \xi\in\mathbb{R}^3, \end{aligned} \tag{11}$$

which is equivalent to the integral equation

$$\phi(\xi,t) = e^{-\nu|\xi|^2 t}\phi_0(\xi) + \int_0^t e^{-\nu|\xi|^2(t-\tau)}\mathscr{B}(\phi)(\xi,\tau)\, d\tau, \tag{12}$$

provided that differentiation under the integral sign was permissible.

In the theory of stochastic processes, a Markov process $\{X_t\}_{t\ge 0}$ in any Euclidean space $\mathbb{R}^n$, with stationary independent increments, for which the characteristic functions of its continuous transition probability densities are given by the Gaussian family $\{e^{-|\xi|^2 t}\}_{t\ge 0}$ is known as Brownian motion or the symmetric stable Lévi process of index 2 (see [13]). Hence, Cauchy problem (11) may be viewed as a kinetic model for the stochastic time-evolution of characteristic functions governed by Brownian motion and Maxwellian collision dynamics. For more detailed interpretations and motivations, we refer to the inspiring paper [14] of Bisi et al. which deals with Cauchy problem (11) in the inelastic setting.

In this paper, we are concerned about global existence and uniqueness and stability of solutions of Cauchy problem (11) in the space of characteristic functions. Before proceeding further, let us describe briefly some of the earlier works about the Cauchy problem for the corresponding Fourier-transformed Boltzmann equation:

$$\begin{aligned} &\partial_t\phi(\xi,t) = \mathscr{B}(\phi)(\xi,t) \\ &\quad \text{for } (\xi,t)\in\mathbb{R}^3\times(0,\infty), \\ &\phi(\xi,0) = \phi_0(\xi) \quad \text{for } \xi\in\mathbb{R}^3 \end{aligned} \tag{13}$$

for which the Maxwellian kernel b is assumed to satisfy the singular or noncutoff condition as described in (4).

(a) It is Pulvirenti and Toscani [15] who first established a global existence of solution to (13) on the space of characteristic functions ϕ satisfying

$$\phi(0) = 1, \quad \nabla\phi(0) = 0, \quad \Delta\phi(0) = -3. \tag{14}$$

They also proved uniqueness and stability of solutions in terms of Tanaka's functionals related with probabilistic Wasserstein distance.

(b) In [16], Toscani and Villani proved uniqueness and stability, on the same solution space, with respect to the Fourier-based metric d_2 which is a particular case of

$$d_\alpha(f, g) = \|\phi - \psi\|_\alpha = \sup_{\xi\in\mathbb{R}^3} \frac{|\phi(\xi) - \psi(\xi)|}{|\xi|^\alpha} \tag{15}$$

for each $\alpha \geq 0$ where $\phi = \widehat{f}$ and $\psi = \widehat{g}$ (see also [17] for the properties of Fourier-based metrics and their applications to the Boltzmann and Fokker-Planck-Boltzmann equations in the inelastic setting).

(c) In [18], Bobylev and Cercignani constructed an explicit class of self-similar solutions whose probability densities possess infinite energy for all time. Specifically, they exhibited a class of characteristic functions $\Phi(\xi, t)$ satisfying (13) and $\Delta\Phi(0, t) = -\infty$ for all $t \geq 0$.

(d) In [19], Cannone and Karch established global existence and uniqueness and stability of solutions on the space $\mathcal{K}^\alpha$, to be explained below, which turns out to be larger than the solution space of Pulvirenti and Toscani and closely related with infinite energy solutions. In [20], Morimoto improved their work by weakening the assumptions on the kernel and providing another proof of uniqueness.

As to Cauchy problem (11), our aim is to establish global existence and uniqueness and stability of solutions on the space introduced by Cannone and Karch [19]. Following their notation, let $\mathcal{K}$ be the set of all characteristic functions on $\mathbb{R}^3$. For $0 < \alpha \leq 2$, let

$$\mathcal{K}^\alpha = \left\{\phi \in \mathcal{K} : \|\phi - 1\|_\alpha = \sup_{\xi\in\mathbb{R}^3} \frac{|\phi(\xi) - 1|}{|\xi|^\alpha} < +\infty\right\}. \tag{16}$$

While $\mathcal{K}^\alpha$ is not a vector space, it is a complete metric space with respect to the Fourier-based metric d_α defined in (15) (for the proofs and further properties, see [19]). As a monotonically indexed family, the embedding

$$\{1\} \subset \mathcal{K}^\beta \subset \mathcal{K}^\alpha \subset \mathcal{K} \tag{17}$$

holds for $0 < \alpha \leq \beta \leq 2$. Any characteristic function ϕ satisfying (14) clearly belongs to $\mathcal{K}^2$. More extensively, it can be trivially verified that if μ is a probability measure on $\mathbb{R}^3$ such that

$$\int_{\mathbb{R}^3} |v|^\alpha \mu(dv) < +\infty \tag{18}$$

with the additional assumption that the first-order moments vanish in the case $1 < \alpha \leq 2$, then $\widehat{\mu} \in \mathcal{K}^\alpha$. The reverse implication, however, is false as it can be seen from the Lévi characteristic function $\widehat{\mu}(\xi) = e^{-|\xi|^\alpha}$ with $0 < \alpha < 2$ which belongs to $\mathcal{K}^\alpha$ but

$$\int_{\mathbb{R}^3} |v|^\alpha \mu(dv) = +\infty. \tag{19}$$

As a means of treating singularity, we follow Morimoto to consider weak integrability of the kernel b in the form

$$\int_0^{\pi/2} b(\cos\theta)\sin\theta \sin^{\alpha_0}\left(\frac{\theta}{2}\right) d\theta < +\infty \tag{20}$$

with $0 < \alpha_0 \leq 2$. It is certainly satisfied by the true Maxwellian kernel b which behaves like (4) as long as $\alpha_0 > 1/2$. In addition, we will consider

$$\begin{aligned}\lambda_\alpha &= \int_{\mathbb{S}^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\left(\frac{|\xi^+|^\alpha + |\xi^-|^\alpha}{|\xi|^\alpha} - 1\right) d\sigma \\ &= 2\pi \int_0^{\pi/2} b(\cos\theta)\sin\theta \\ &\quad \times \left[\cos^\alpha\left(\frac{\theta}{2}\right) + \sin^\alpha\left(\frac{\theta}{2}\right) - 1\right] d\theta\end{aligned} \tag{21}$$

for $0 < \alpha \leq 2$, which is independent of $\xi \neq 0$ and finite under condition (20) for all $\alpha_0 \leq \alpha \leq 2$. Introduced by Cannone and Karch, these quantities will serve as the stability exponents.

To state our results, we set down the precise solution spaces. Let $T > 0$ be arbitrary. As it is customary, we denote by $C([0, T]; \mathcal{K}^\alpha)$ the space of all complex-valued functions ϕ on $\mathbb{R}^3 \times [0, T]$ such that $\phi(\cdot, t) \in \mathcal{K}^\alpha(\mathbb{R}^3)$ for each $t \in [0, T]$ and the map $t \mapsto \|\phi(t) - 1\|_\alpha$ is continuous on $[0, T]$. By the Riemann-Lebesgue lemma, each characteristic function is continuous in $\mathbb{R}^3$ and, hence, the space continuity is alluded in the definition of $C([0, T]; \mathcal{K}^\alpha)$.

In consideration of time regularity, we will write $\Omega^\alpha(\mathbb{R}^3 \times [0, T])$ for the space of $\phi \in C([0, T]; \mathcal{K}^\alpha)$ such that $\phi(\xi, \cdot) \in C([0, T])$, $\partial_t\phi(\xi, \cdot) \in C((0, T))$ for each fixed $\xi \in \mathbb{R}^3$. We put

$$\Omega^\alpha\left(\mathbb{R}^3 \times [0, \infty)\right) = \bigcup_{T>0} \Omega^\alpha\left(\mathbb{R}^3 \times [0, T]\right). \tag{22}$$

Our main result for global existence is as follows.

Theorem 1. *Assume that the collision kernel b satisfies a weak integrability condition (20) for some $0 < \alpha_0 \leq 2$ and $\alpha_0 \leq \alpha \leq 2$. Then, for any initial datum $\phi_0 \in \mathcal{K}^\alpha$, there exists a classical solution ϕ to the Cauchy problem (11) in the space $\Omega^\alpha(\mathbb{R}^3 \times [0, \infty))$ satisfying*

$$|\phi(\xi, t)| \leq e^{-\nu|\xi|^2 t} \quad \forall (\xi, t) \in \mathbb{R}^3 \times [0, \infty). \tag{23}$$

A distinctive feature is the existence of a solution satisfying the stated maximum growth estimate which asserts in a sense that the solution stays within Brownian motion for all time.

To state our main result for stability and uniqueness, we put

$$\begin{aligned}
&\Omega_\nu^\alpha\left(\mathbb{R}^3 \times [0,\infty)\right) \\
&\quad = \left\{ \phi \in \Omega^\alpha\left(\mathbb{R}^3 \times [0,\infty)\right) : |\phi(\xi,t)| \le e^{-\nu|\xi|^2 t} \right. \\
&\qquad \left. \forall (\xi,t) \in \mathbb{R}^3 \times [0,\infty) \right\}.
\end{aligned} \tag{24}$$

Theorem 2. *Under the same hypotheses on α, b as in Theorem 1, if ϕ, ψ are solutions to Cauchy problem (11) in the space $\Omega_\nu^\alpha(\mathbb{R}^3\times[0,\infty))$ corresponding to the initial data $\phi_0, \psi_0 \in \mathcal{K}^\alpha$, respectively, then, for all $t \ge 0$,*

$$\begin{aligned}
&\sup_{\xi\in\mathbb{R}^3}\left[e^{\nu|\xi|^2 t}\frac{|\phi(\xi,t)-\psi(\xi,t)|}{|\xi|^\alpha}\right] \\
&\qquad \le e^{\lambda_\alpha t}\sup_{\xi\in\mathbb{R}^3}\frac{|\phi_0(\xi)-\psi_0(\xi)|}{|\xi|^\alpha}.
\end{aligned} \tag{25}$$

In particular, for any initial datum $\phi_0 \in \mathcal{K}^\alpha$, Cauchy problem (11) has at most one solution in the space $\Omega_\nu^\alpha(\mathbb{R}^3 \times [0,\infty))$.

Upon setting $\nu = 0$, both theorems are reduced to those of Cannone and Karch and Morimoto. In fact, due to a special structure of the Boltzmann-Bobylev operator, to be explained below, the existence theorem is an almost instant consequence of their existence theorem except for some technical points. On the other hand, the stability theorem is not so straightforward and our proof will be carried out along Gronwall-type reasonings.

As some functionals or expressions involving the space and time variables are too lengthy to put effectively, we will often abbreviate the space variables for simplicity in the sequel.

2. Preliminaries

A well-known Fourier transform formula states that

$$e^{-|\xi|^2 t} = \frac{1}{(4\pi t)^{3/2}}\int_{\mathbb{R}^3} e^{-i\xi\cdot\nu}e^{-|\nu|^2/4t}d\nu \quad \left(\xi \in \mathbb{R}^3,\ t>0\right) \tag{26}$$

and, hence, it is clear that the Gaussian family $\{e^{-|\xi|^2 t}\}_{t\ge0} \subset \mathcal{K}$ whose probability densities are self-similar Gaussian functions (see, e.g., [21]).

Lemma 3. *If $0 < \alpha \le 2$ and $t > 0$, then*

$$\sup_{\xi\in\mathbb{R}^3}\frac{1-e^{-|\xi|^2 t}}{|\xi|^\alpha} \le t^{\alpha/2}. \tag{27}$$

Proof. Observe that

$$\begin{aligned}
&\sup_{\xi\in\mathbb{R}^3}\frac{1-e^{-|\xi|^2 t}}{|\xi|^\alpha} = t^{\alpha/2}\cdot\sup_{r>0} g_\alpha(r) \\
&\qquad \text{with } g_\alpha(r) = \left(\frac{1-e^{-r}}{r^{\alpha/2}}\right).
\end{aligned} \tag{28}$$

Since g_α is a smooth function on $(0,\infty)$ with

$$\begin{aligned}
&g_\alpha(r) \le \min\left(r^{1-\alpha/2}, 1\right), \\
&\lim_{r\to0+} g_\alpha(r) = \lim_{r\to\infty} g_\alpha(r) = 0,
\end{aligned} \tag{29}$$

the assertion follows. □

The Boltzmann-Bobylev operator defined in (10) takes the form

$$\mathcal{B}(\phi)(\xi) = \int_{\mathbb{S}^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\left[\phi(\xi^+)\phi(\xi^-)-\phi(\xi)\right]d\sigma \tag{30}$$

for each characteristic function ϕ. We set $\mathcal{B}(\phi)(0) = 0$ in the sequel.

For a nonzero $\xi \in \mathbb{R}^3$, by considering a parametrization of the unit sphere in terms of the deviation angle from $\xi/|\xi|$, it is well known that

$$\begin{aligned}
\mathcal{B}(\phi)(\xi) = &\int_0^{\pi/2} b(\cos\theta)\sin\theta \\
&\times\left\{\int_{\mathbb{S}^1(\xi)}\left[\phi(\xi^+)\phi(\xi^-)-\phi(\xi)\right]d\omega\right\}d\theta
\end{aligned} \tag{31}$$

in which $\mathbb{S}^1(\xi) = \mathbb{S}^2\cap\xi^\perp$ and $d\omega$ denotes the area measure on the unit circle $\mathbb{S}^1 \subset \mathbb{R}^3$. As it is defined in (9), the spherical variables ξ^+, ξ^- are expressed in terms of θ, ω via

$$\sigma = \cos\theta\frac{\xi}{|\xi|} + \sin\theta\omega. \tag{32}$$

The following are due to Morimoto [20, page 555]. We put

$$\mu_\alpha = 2\pi\int_0^{\pi/2} b(\cos\theta)\sin\theta\sin^\alpha\left(\frac{\theta}{2}\right)d\theta, \tag{33}$$

which is finite under condition (20) for any $\alpha_0 \le \alpha \le 2$.

Lemma 4. *For $0 < \alpha \le 2$, assume that the kernel b satisfies the condition $\mu_\alpha < +\infty$. Let $\phi \in \mathcal{K}^\alpha$ and $\xi \in \mathbb{R}^3 - \{0\}$. Then,*

$$\begin{aligned}
&\left|\int_{\mathbb{S}^1(\xi)}\left[\phi(\xi^+)\phi(\xi^-)-\phi(\xi)\right]d\omega\right| \\
&\qquad \le 16\pi\|\phi-1\|_\alpha|\xi|^\alpha\sin^\alpha\left(\frac{\theta}{2}\right)
\end{aligned} \tag{34}$$

for each $\theta \in (0,\pi/2]$. Moreover,

$$|\mathcal{B}(\phi)(\xi)| \le 8\mu_\alpha\|\phi-1\|_\alpha|\xi|^\alpha. \tag{35}$$

As an application, we have the following time-continuity property.

Lemma 5. *For $0 < \alpha \le 2$, assume that the kernel b satisfies the condition $\mu_\alpha < +\infty$ and $T > 0$. If $\phi \in C([0,T];\mathscr{K}^\alpha)$ and $\phi(\xi,\cdot) \in C([0,T])$ for each $\xi \in \mathbb{R}^3$, then $\mathscr{B}(\phi)(\xi,\cdot) \in C([0,T])$ for each $\xi \in \mathbb{R}^3$.*

Proof. Fix a nonzero $\xi \in \mathbb{R}^3$ and $t_0 \in [0,T]$. For any sequence $(t_n) \subset [0,T]$ with $t_n \to t_0$, we may write, with the aid of (31),

$$\begin{aligned}\mathscr{B}(\phi)(\xi,t_n) &= \int_{\mathbb{S}^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\left[\phi(\xi^+,t_n)\phi(\xi^-,t_n)-\phi(\xi,t_n)\right]d\sigma \\ &= \int_0^{\pi/2} b(\cos\theta)\sin\theta A_n(\xi,\theta)\,d\theta \quad \text{where} \\ A_n(\xi,\theta) &= \int_{\mathbb{S}^1(\xi)}\left[\phi(\xi^+,t_n)\phi(\xi^-,t_n)-\phi(\xi,t_n)\right]d\omega.\end{aligned} \tag{36}$$

By the estimate (34), we notice that

$$\begin{aligned}\left|A_n(\xi,\theta)\right| &\le 16\pi\left\|\phi(t_n)-1\right\|_\alpha|\xi|^\alpha\sin^\alpha\left(\frac{\theta}{2}\right) \\ &\le 16\pi C_\alpha(T)|\xi|^\alpha\sin^\alpha\left(\frac{\theta}{2}\right) \quad \text{where} \\ C_\alpha(T) &= \max_{t\in[0,T]}\left\|\phi(t)-1\right\|_\alpha.\end{aligned} \tag{37}$$

By the continuity of $t \mapsto \|\phi(t)-1\|_\alpha$, we have $C_\alpha(T) < +\infty$. Since

$$\begin{aligned}& b(\cos\theta)\sin\theta\left|A_n(\xi,\theta)\right| \\ &\quad\le 16\pi C_\alpha(T)|\xi|^\alpha b(\cos\theta)\sin\theta\sin^\alpha\left(\frac{\theta}{2}\right) \\ &\quad\equiv A(\xi,\theta)\end{aligned} \tag{38}$$

uniformly in n and the definition of μ_α gives

$$\int_0^{\pi/2} A(\xi,\theta)\,d\theta = 8\mu_\alpha C_\alpha(T)|\xi|^\alpha < +\infty, \tag{39}$$

we may apply Lebesgue's dominated convergence theorem to evaluate the limit under the integral sign $\lim_{n\to\infty}\mathscr{B}(\phi)(\xi,t_n) = \mathscr{B}(\phi)(\xi,t_0)$, which proves continuity at t_0. □

3. Global Existence

An important feature of the Boltzmann-Bobylev operator $\mathscr{B}$ is that it satisfies the pointwise identity

$$e^{h(t)|\xi|^2}\mathscr{B}(\phi)(\xi,t) = \mathscr{B}\left(e^{h(t)|\xi|^2}\phi\right)(\xi,t) \tag{40}$$

for any scalar-valued function h defined on $[0,\infty)$ and for any scalar-valued function ϕ on $\mathbb{R}^3\times[0,\infty)$, which results from $|\xi^+|^2 + |\xi^-|^2 = |\xi|^2$ for all $\xi \in \mathbb{R}^3$ and $\sigma \in \mathbb{S}^2$. As a consequence, at the formal level, it is straightforward to find that if ϕ is a solution to Cauchy problem (13) of the Fourier-transformed Boltzmann equation, then $e^{-\nu|\xi|^2 t}\phi$ is a solution to Cauchy problem (11) of our consideration.

To be rigorous, we begin with quoting the existence theorem of Cannone and Karch [19] and Morimoto [20] in a combined manner.

Theorem 6. *Assume that b satisfies (20) for some $0 < \alpha_0 \le 2$. Let $\alpha_0 \le \alpha \le 2$ and $\phi_0 \in \mathscr{K}^\alpha$. Then, there exists a unique classical solution ϕ to Cauchy problem (13) in the space $\Omega^\alpha(\mathbb{R}^3\times[0,\infty))$.*

Remark 7. In their work, Cannone and Karch constructed a unique solution on the space $C([0,\infty);\mathscr{K}^\alpha)$ without mentioning time-regularity conditions. Since $\mathscr{K}^\alpha$ is not a Banach space, the meaning of a classical solution to Cauchy problem (20) is not so clear in this space. Inspecting their proof and making use of the time continuity of the Boltzmann-Bobylev operator as stated in Lemma 5, however, it is not hard to find that their solution is indeed a classical solution in the space $\Omega^\alpha(\mathbb{R}^3\times[0,\infty))$ for which the partial derivative in time is taken in the usual pointwise sense.

Let us consider an equivalent formulation of (13):

$$\phi(\xi,t) = \phi_0(\xi) + \int_0^t \mathscr{B}(\phi)(\xi,\tau)\,d\tau, \tag{41}$$

where the time integration is taken in the usual Riemann sense. By Lemma 5, if $\phi \in C([0,T];\mathscr{K}^\alpha)$ and $\phi(\xi,t)$ is continuous in t for each fixed ξ, then this integral is well defined for a kernel b satisfying $\mu_\alpha < +\infty$.

We will need a technical lemma in support of Theorem 6.

Lemma 8. *For $0 < \alpha \le 2$, let $\mu_\alpha < +\infty$ and $\phi_0 \in \mathscr{K}^\alpha$. Assume that $\phi \in C([0,T];\mathscr{K}^\alpha)$ and $\phi(\xi,t)$ is continuous in $t \in [0,T]$ for each fixed ξ. If ϕ is a solution to (41), then, for all $s,t \in [0,T]$,*

$$\begin{aligned}&\text{(i)} \quad \left\|\phi(t)-1\right\|_\alpha \le e^{8\mu_\alpha t}\left\|\phi_0-1\right\|_\alpha, \\ &\text{(ii)} \quad \left\|\phi(t)-\phi(s)\right\|_\alpha \le \left(8\mu_\alpha e^{8\mu_\alpha T}\left\|\phi_0-1\right\|_\alpha\right)|t-s|.\end{aligned} \tag{42}$$

Proof. (i) An application of Lemma 4 yields

$$\left\|\phi(t)-1\right\|_\alpha \le \left\|\phi_0-1\right\|_\alpha + 8\mu_\alpha\int_0^t\left\|\phi(\tau)-1\right\|_\alpha d\tau, \tag{43}$$

which yields the desired estimate in view of Gronwall's lemma.

(ii) Assuming $s < t$, we apply Lemma 4 once again to find

$$\begin{aligned}&\left\|\phi(t)-\phi(s)\right\|_\alpha \\ &\quad\le \int_s^t \sup_{\xi\in\mathbb{R}^3}\frac{\left|\mathscr{B}(\phi)(\xi,\tau)\right|}{|\xi|^\alpha}d\tau \\ &\quad\le 8\mu_\alpha\left(\max_{\tau\in[0,T]}\left\|\phi(\tau)-1\right\|_\alpha\right)|t-s|,\end{aligned} \tag{44}$$

which yields the desired estimate upon combining with (i). □

Proof of Theorem 1. Since the stated assumptions on b and ϕ_0 are the same as those of Theorem 6, there exists a unique solution ϕ to Cauchy problem (13) in the space $\Omega^\alpha(\mathbb{R}^3 \times [0,\infty))$. Put

$$\Phi(\xi,t) = e^{-\nu|\xi|^2 t}\phi(\xi,t). \tag{45}$$

We will verify that Φ is a solution to Cauchy problem (11) satisfying the stated maximum growth estimate.

(i) Clearly $\Phi(\cdot,t) \in \mathscr{K}$ for any fixed $t \geq 0$. Moreover, Lemma 3 gives

$$\begin{aligned} &\frac{|\Phi(\xi,t)-1|}{|\xi|^\alpha} \\ &\quad\leq e^{-\nu|\xi|^2 t}\frac{|\phi(\xi,t)-1|}{|\xi|^\alpha} + \frac{1-e^{-\nu|\xi|^2 t}}{|\xi|^\alpha} \\ &\quad\leq \|\phi(t)-1\|_\alpha + (\nu t)^{\alpha/2}, \end{aligned} \tag{46}$$

which implies $\Phi(\cdot,t) \in \mathscr{K}^\alpha$ for any fixed $t \geq 0$ with

$$\|\Phi(t)-1\|_\alpha \leq e^{8\mu_\alpha t}\|\phi_0 - 1\|_\alpha + (\nu t)^{\alpha/2}. \tag{47}$$

(ii) For $s,t \in [0,T]$, with an arbitrary $T > 0$, writing

$$\begin{aligned} \Phi(t)-\Phi(s) &= -e^{-\nu|\xi|^2 s}\left[1-e^{-\nu|\xi|^2(t-s)}\right]\phi(t) \\ &\quad + e^{-\nu|\xi|^2 s}[\phi(t)-\phi(s)], \end{aligned} \tag{48}$$

we deduce from Lemmas 3 and 8

$$\begin{aligned} &\|\Phi(t)-\Phi(s)\|_\alpha \\ &\quad\leq (\nu|t-s|)^{\alpha/2} + \|\phi(t)-\phi(s)\|_\alpha \\ &\quad\leq C_T|t-s|^{\alpha/2} \quad \text{where} \\ &C_T = \nu^{\alpha/2} + 8\mu_\alpha e^{8\mu_\alpha T}\|\phi_0-1\|_\alpha T^{1-\alpha/2}. \end{aligned} \tag{49}$$

Thus, the map $t \mapsto \|\Phi(t)-1\|_\alpha$ is Lipschitz continuous in $[0,T]$ for

$$\left|\|\Phi(t)-1\|_\alpha - \|\Phi(s)-1\|_\alpha\right| \leq \|\Phi(t)-\Phi(s)\|_\alpha. \tag{50}$$

Therefore, $\Phi \in \Omega^\alpha(\mathbb{R}^3 \times [0,\infty))$ for the time-regularity conditions are obviously valid. In particular, Lemmas 4 and 5 imply that $\mathscr{B}(\Phi)$ is well defined with

$$|\mathscr{B}(\Phi)(\xi,t)| \leq 8\mu_\alpha \|\Phi(t)-1\|_\alpha |\xi|^\alpha \tag{51}$$

for each $(\xi,t) \in \mathbb{R}^3 \times [0,\infty)$ and $\mathscr{B}(\Phi)(\xi,t)$ is continuous in t for each ξ. Clearly, $\Phi(\xi,0) = \phi_0(\xi)$. We calculate

$$\begin{aligned} &\partial_t\Phi(\xi,t) \\ &\quad= -\nu|\xi|^2 e^{-\nu|\xi|^2 t}\phi(\xi,t) + e^{-\nu|\xi|^2 t}\partial_t\phi(\xi,t) \\ &\quad= -\nu|\xi|^2\Phi(\xi,t) + e^{-\nu|\xi|^2 t}\mathscr{B}(\phi)(\xi,t) \\ &\quad= -\nu|\xi|^2\Phi(\xi,t) + \mathscr{B}(\Phi)(\xi,t) \end{aligned} \tag{52}$$

for all $(\xi,t) \in \mathbb{R}^3 \times (0,\infty)$, where we have used (40). Thus, Φ satisfies Cauchy problem (11). Since it is obvious that

$$|\Phi(\xi,t)| \leq e^{-\nu|\xi|^2 t} \quad \forall(\xi,t) \in \mathbb{R}^3 \times [0,\infty), \tag{53}$$

our proof of Theorem 1 is complete. □

Remark 9. In our forthcoming paper [22], we study the Cauchy problem for the Boltzmann equation coupled with fractional Laplacian diffusion terms on the Fourier transform side in which we give direct proofs of global existence.

4. Uniqueness and Stability of Solutions

To proceed our proof for stability of solutions, we begin with estimating the time-growth behavior of $\|\phi(t)-1\|_\alpha$ for each solution ϕ of Cauchy problem (11) or integral equation (12).

Lemma 10. *Under the same hypotheses on α, b, ϕ as stated in Lemma 8, if ϕ is a solution to (12), then*

$$\|\phi(t)-1\|_\alpha \leq e^{8\mu_\alpha t}\left[\|\phi_0-1\|_\alpha + (\nu t)^{\alpha/2}\right] \quad (t \geq 0). \tag{54}$$

Proof. Writing

$$\begin{aligned} \frac{\phi(\xi,t)-1}{|\xi|^\alpha} &= \frac{e^{-\nu|\xi|^2 t}\phi_0(\xi)-1}{|\xi|^\alpha} \\ &\quad + \int_0^t e^{-\nu|\xi|^2(t-\tau)}\frac{\mathscr{B}(\phi)(\xi,\tau)}{|\xi|^\alpha}d\tau \end{aligned} \tag{55}$$

and applying Lemma 4, it is straightforward to obtain

$$\begin{aligned} \|\phi(t)-1\|_\alpha &\leq \|\phi_0-1\|_\alpha + (\nu t)^{\alpha/2} \\ &\quad + 8\mu_\alpha\int_0^t \|\phi(\tau)-1\|_\alpha d\tau. \end{aligned} \tag{56}$$

A Gronwall-type argument yields

$$\begin{aligned} &\|\phi(t)-1\|_\alpha \\ &\quad\leq e^{8\mu_\alpha t}\left\{\|\phi_0-1\|_\alpha + \frac{\alpha\nu^{\alpha/2}}{2}\int_0^t e^{-8\mu_\alpha\tau}\tau^{\alpha/2-1}d\tau\right\} \\ &\quad\leq e^{8\mu_\alpha t}\left[\|\phi_0-1\|_\alpha + (\nu t)^{\alpha/2}\right]. \end{aligned} \tag{57}$$

□

Proof of Theorem 2. We will prove the stated stability inequality for each $t \in [0,T]$ with an arbitrarily fixed $T > 0$.

Let us consider a monotone sequence (b_n) of kernels obtained from b by cutting off the singularity at $\theta = 0$ in the manner

$$b_n(\cos\theta) = b(\cos\theta)\chi_{[1/n,\pi/2]}(\theta), \quad n = 1,2,\ldots. \tag{58}$$

Since b is assumed to be at least bounded away from $\theta = 0$, it is clear that each b_n is integrable on the unit sphere, $b_n \leq b$

and $b_n \to b$ monotonically. Setting $b_n^r = b - b_n$ for each n, we introduce two sequences of operators $(\mathcal{G}_n)$, $(\mathcal{R}_n)$ defined by

$$\mathcal{G}_n(\phi)(\xi) = \int_{\mathbb{S}^2} b_n\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\phi(\xi^+)\phi(\xi^-)\,d\sigma,$$
$$\mathcal{R}_n(\phi)(\xi) = \int_{\mathbb{S}^2} b_n^r\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\left[\phi(\xi^+)\phi(\xi^-) - \phi(\xi)\right]d\sigma. \tag{59}$$

Suppose that $\phi, \psi \in \Omega_\nu^\alpha(\mathbb{R}^3\times[0,\infty))$ are solutions to Cauchy problem (11) with the initial data $\phi_0, \psi_0 \in \mathcal{K}^\alpha$, respectively. Then,

$$\begin{aligned}\partial_t(\phi-\psi) &+ \left(\|b_n\|_1 + \nu|\xi|^2\right)(\phi-\psi)\\ &= \left[\mathcal{G}_n(\phi) - \mathcal{G}_n(\psi)\right] + \left[\mathcal{R}_n(\phi) - \mathcal{R}_n(\psi)\right]\end{aligned} \tag{60}$$

for which we denote

$$\|b_n\|_1 = 2\pi\int_{1/n}^{\pi/2} b(\cos\theta)\sin\theta\,d\theta < +\infty. \tag{61}$$

Upon setting

$$U(\xi,t) = e^{\nu|\xi|^2 t}\left[\frac{\phi(\xi,t)-\psi(\xi,t)}{|\xi|^\alpha}\right] \tag{62}$$

for $\xi \neq 0$ and $U(0,t) = 0$, the above identity implies

$$\begin{aligned}&\left|\partial_t\left[e^{\|b_n\|_1 t}U(\xi,t)\right]\right|\\ &\quad\le e^{(\|b_n\|_1+\nu|\xi|^2)t}\\ &\qquad\times\left\{\left|\frac{\mathcal{G}_n(\phi)-\mathcal{G}_n(\psi)}{|\xi|^\alpha}\right| + \left|\frac{\mathcal{R}_n(\phi)-\mathcal{R}_n(\psi)}{|\xi|^\alpha}\right|\right\}.\end{aligned} \tag{63}$$

Let $\rho > 0$ be arbitrary. Put $U_\rho(t) = \sup_{|\xi|\le\rho}|U(\xi,t)|$ and

$$\begin{aligned}\gamma_{n,\alpha} &= \int_{\mathbb{S}^2} b_n\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\left(\frac{|\xi^+|^\alpha + |\xi^-|^\alpha}{|\xi|^\alpha}\right)d\sigma\\ &= 2\pi\int_{1/n}^{\pi/2} b(\cos\theta)\sin\theta\left[\cos^\alpha\left(\frac{\theta}{2}\right) + \sin^\alpha\left(\frac{\theta}{2}\right)\right]d\theta.\end{aligned} \tag{64}$$

For $|\xi| \le \rho$, we make use of $|\xi^+| \le |\xi|$ to estimate

$$\begin{aligned}&\left|\phi(\xi^+,t) - \psi(\xi^+,t)\right|\\ &\quad\le e^{-\nu|\xi^+|^2 t}\left|\xi^+\right|^\alpha\\ &\qquad\times\left\{e^{\nu|\xi^+|^2 t}\left|\frac{\phi(\xi^+,t)-\psi(\xi^+,t)}{|\xi^+|^\alpha}\right|\right\}\\ &\quad\le e^{-\nu|\xi^+|^2 t}\left|\xi^+\right|^\alpha U_\rho(t).\end{aligned} \tag{65}$$

Likewise, we make use of $|\xi^-| \le |\xi|$ to estimate

$$\left|\phi(\xi^-,t) - \psi(\xi^-,t)\right| \le e^{-\nu|\xi^-|^2 t}\left|\xi^-\right|^\alpha U_\rho(t) \tag{66}$$

for all $|\xi| \le \rho$. Since $|\phi(\xi,t)| \le e^{-\nu|\xi|^2 t}$, $|\psi(\xi,t)| \le e^{-\nu|\xi|^2 t}$, we find

$$\begin{aligned}&\left|\phi(\xi^+,t)\phi(\xi^-,t) - \psi(\xi^+,t)\psi(\xi^-,t)\right|\\ &\quad\le e^{-\nu|\xi|^2 t}\left(\left|\xi^+\right|^\alpha + \left|\xi^-\right|^\alpha\right)U_\rho(t)\end{aligned} \tag{67}$$

for all $|\xi| \le \rho$. Henceforth, it is straightforward to deduce

$$e^{\nu|\xi|^2 t}\left|\frac{\mathcal{G}_n(\phi)-\mathcal{G}_n(\psi)}{|\xi|^\alpha}\right| \le \gamma_{n,\alpha}U_\rho(t) \tag{68}$$

for all $|\xi| \le \rho$. On the other hand, Lemma 4 gives

$$\begin{aligned}\left|\mathcal{R}_n(\phi)(\xi,t)\right| &= 16\pi\left\|\phi(t)-1\right\|_\alpha|\xi|^\alpha\\ &\quad\times\int_0^{1/n} b(\cos\theta)\sin\theta\sin^\alpha\left(\frac{\theta}{2}\right)d\theta.\end{aligned} \tag{69}$$

By considering similar estimate for $\mathcal{R}_n(\psi)$, hence, we note

$$e^{\nu|\xi|^2 t}\left|\frac{\mathcal{R}_n(\phi)-\mathcal{R}_n(\psi)}{|\xi|^\alpha}\right| \le M_n \tag{70}$$

for all $|\xi| \le \rho$ and $t \in [0,T]$, where we put

$$\begin{aligned}M_n &= M(\rho,T)\int_0^{1/n} b(\cos\theta)\sin\theta\sin^\alpha\left(\frac{\theta}{2}\right)d\theta,\\ M(\rho,T) &= 16\pi e^{\nu\rho^2 T}\\ &\quad\times\max_{t\in[0,T]}\left(\left\|\phi(t)-1\right\|_\alpha + \left\|\psi(t)-1\right\|_\alpha\right).\end{aligned} \tag{71}$$

In view of the growth estimate of Lemma 10,

$$\begin{aligned}&M(\rho,T)\\ &\quad\le 16\pi e^{(\nu\rho^2+8\mu_\alpha)T}\\ &\qquad\times\left[\left\|\phi_0-1\right\|_\alpha + \left\|\psi_0-1\right\|_\alpha + 2(\nu T)^{\alpha/2}\right] < +\infty,\end{aligned} \tag{72}$$

and so an application of Lebesgue's dominated convergence theorem shows $M_n \to 0$ as $n \to \infty$ under the assumption $\mu_\alpha < +\infty$.

Now, estimates (68) and (70) imply

$$\left|\partial_t\left[e^{\|b_n\|_1 t}U(\xi,t)\right]\right| \le \gamma_{n,\alpha}e^{\|b_n\|_1 t}U_\rho(t) + M_n e^{\|b_n\|_1 t} \tag{73}$$

for all $|\xi| \le \rho$, $t \in [0,T]$. A standard Gronwall-type argument gives

$$U_\rho(t) \le e^{(\gamma_{n,\alpha}-\|b_n\|_1)t}U_\rho(0) + \frac{M_n}{\gamma_{n,\alpha}-\|b_n\|_1}\left[e^{(\gamma_{n,\alpha}-\|b_n\|_1)t} - 1\right]. \tag{74}$$

Since

$$\gamma_{n,\alpha} - \|b_n\|_1 = 2\pi \int_{1/n}^{\pi/2} b(\cos\theta)\sin\theta \times \left[\cos^\alpha\left(\frac{\theta}{2}\right) + \sin^\alpha\left(\frac{\theta}{2}\right) - 1\right] d\theta, \tag{75}$$

we notice $0 < \gamma_{n,\alpha} - \|b_n\|_1 \to \lambda_\alpha$ increasingly as $n \to \infty$. Passing to the limit, we conclude $U_\rho(t) \le e^{\lambda_\alpha t} U_\rho(0)$ for all $t \in [0, T]$. Letting $\rho \to +\infty$, we finally obtain

$$\sup_{\xi\in\mathbb{R}^3} |U(\xi,t)| \le e^{\lambda_\alpha t} \sup_{\xi\in\mathbb{R}^3} |U(\xi,0)|, \tag{76}$$

which is equivalent to the desired stability estimate on $[0, T]$. □

5. Concluding Remarks

Having established global existence and uniqueness and stability of solutions to the Fourier-transformed version of Fokker-Planck-Boltzmann equation on the space $\mathcal{K}^\alpha$, we end our paper with a few additional remarks.

(a) Concerning Theorem 1, while it asserts that there exists a solution ϕ of Cauchy problem (11) satisfying

$$|\phi(\xi,t)| \le e^{-\nu|\xi|^2 t} \quad \forall (\xi,t) \in \mathbb{R}^3 \times [0,\infty), \tag{77}$$

a natural question is whether this dominating property would hold for any solution of (11). The answer is affirmative in the case when the collision kernel b satisfies Grad's angular cutoff assumption.

Suppose $b \in L^1(\mathbb{S}^2)$ and $\phi_0 \in \mathcal{K}$. If ϕ is a solution to (11) in the space $C([0,\infty); \mathcal{K})$, then necessarily (77) holds.

As it can be proved in an elementary way, we leave its verification to the interested reader. For the singular case of b, however, we were not able to draw any conclusion.

(b) In the cutoff case of b, it is possible to construct an explicit solution of Cauchy problem (11) by using the Wild sum method as developed in [15, 17, 23]. Assuming $\|b\|_{L^1(\mathbb{S}^2)} = 1$, if we follow the same known method, then it is straightforward to derive

$$\phi(\xi,t) = e^{-(1+\nu|\xi|^2)t} \sum_{n=0}^{\infty} u_n(\xi)\left(1 - e^{-t}\right)^n, \tag{78}$$

where $u_0 = \phi_0$ is the initial datum and

$$u_{n+1}(\xi) = \frac{1}{n+1}\sum_{j=0}^{n} \int_{\mathbb{S}^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right) u_j(\xi^+) u_{n-j}(\xi^-)\, d\sigma \tag{79}$$

for $n = 0, 1, \ldots$. It can be shown plainly that if $\phi_0 \in \mathcal{K}^\alpha$, then this explicit solution $\phi \in \mathcal{K}^\alpha$ for $0 < \alpha \le 2$. By uniqueness, this solution coincides with the solution of Theorem 1.

(c) Concerning the asymptotic behavior of a solution ϕ to Cauchy problem (11), an important question common in kinetic theory is whether there exists a steady-state equilibrium ϕ_∞ such that $\phi \to \phi_\infty$ as $t \to \infty$ in some sense. For instance, in the inelastic case, it is shown that there exists such steady-state equilibrium for a solution of the Cauchy problem for the corresponding Fokker-Planck-Boltzmann equation (see [17] and further references therein). In the elastic case, however, it is likely that the answer would be negative in view of the pointwise behavior $\phi(\xi,t) \to 0$ due to growth estimate (77). A seemingly reasonable alternative is to investigate if the solution gets close to the Gaussian $e^{-\nu|\xi|^2 t}$ in an appropriate sense.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

The Existence and Uniqueness of Global Solutions to the Initial Value Problem for the System of Nonlinear Integropartial Differential Equations in Spatial Economics: The Dynamic Continuous Dixit-Stiglitz-Krugman Model in an Urban-Rural Setting

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Assume that economic activities are conducted in a bounded continuous domain where workers move toward regions that offer higher real wages and away from regions that offer below-average real wages. The density of real wages is calculated by solving the nominal wage equation of the continuous Dixit-Stiglitz-Krugman model in an urban-rural setting. The evolution of the density of workers is described by an unknown function of the replicator equation whose growth rate is equal to the difference between the density of real wages and the average real wage. Hence, the evolution of the densities of workers and real wages is described by the system of the nominal wage equation and the replicator equation. This system of equations is an essentially new kind of system of nonlinear integropartial differential equations in the theory of functional equations. The purpose of this paper is to obtain a sufficient condition for the initial value problem for this system to have a unique global solution.

1. Introduction

The *new economic geography* (NEG) is a new branch of spatial economics that was initiated by Krugman in the early 1990s. This new branch has attracted many social scientists and becomes one of the most important major branches of spatial economics at present. In 2008 Krugman received the Nobel Memorial Prize in Economic Sciences (officially Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel) for his great contribution to the NEG (see [1–6]). A large number of mathematical models have been built in the NEG. In particular there are many models described by nonlinear integropartial differential equations that are new and important in the theory of functional equations (see [1–3]). Hence the NEG is regarded as a new frontier of the theory of nonlinear integropartial differential equations.

The Krugman core-periphery model (the CP model) is the origin of the NEG (see [1, Chapter 5]) since various models are constructed as its extension. The CP model is a discrete model. In this model economic activities are conducted at two points. These two points represent a core region and a periphery region, respectively. An extension to the case of a finite set of points has been studied in [1]. This model is called *the Dixit-Stiglitz-Krugman model* (DSK model). Its mathematical foundation is studied in [7–13]. Moreover, in [14], we consider an extension of the CP model to the case of a bounded continuous domain where economic activities are conducted continuously in space. This model is called *the continuous DSK model* (cDSK model).

These models are static models with no population dynamics. It is very important in spatial economics to build population dynamics into them. Hence Krugman constructs *the dynamic DSK model* (dDSK model) by combining the DSK model with *the replicator dynamics* that is one of the most fundamental dynamics in evolutionary game theory (see [15] and [16, Chapter 3]). His dynamic model is very important in spatial economics since it describes economies of agglomeration in the case where workers move from one point to another to seek higher real wages within a finite set of points at which economic activities are conducted (see [1, p. 62, p. 77]). Hence, by following this line, we consider *the dynamic cDSK model* (dcDSK model) in this paper; that is, we combine the cDSK model with the replicator dynamics. This dynamic model is regarded as a continuous version of the dDSK model and explains agglomeration of capital and concentration of workers when workers move in a bounded continuous domain where economic activities are conducted continuously in space.

Let us discuss the dcDSK model from the viewpoint of the theory of functional equations. The dcDSK model is described by the system of *the nominal wage equation* and *the replicator equation*. We refer to this system of equations as *the dcDSK system*. The nominal wage equation is a nonlinear integral equation that contains the density of nominal wages as an unknown function and the density of workers as a known function (see [14, (2.4)]). Hence, if we solve this equation under the condition that the density of workers is a given function, then we can obtain the density of nominal wages. However, the integral kernel of the nominal wage equation contains not only these densities but also *the price index*. We must note that the price index itself is a nonlinear integral operator acting on the density of nominal wages and the density of workers (see [14, (2.7)]). Therefore, we can say that the nominal wage equation is a *double nonlinear* integral equation (see [14, Remark 2.3] for mathematical difficulties caused by the double nonlinearity).

The replicator equation is a nonlinear integropartial differential equation whose unknown function denotes the density of workers. Its coefficient denotes *the growth rate* of worker population (see [1, (5.1), (5.2)] and [16, p. 73]). The coefficient is equal to the difference between *the density of real wages* and *the average real wage*, where the density of real wages is defined as the density of nominal wages deflated by a fractional power of the price index, and the average real wage is defined as the integral of the product of the density of workers and the density of real wages (see [1, (5,1), (5.6)]). Hence, the coefficient is regarded as a double nonlinear integral operator acting on the density of workers and the density of nominal wages.

Moreover, the coefficient of the replicator equation of the dcDSK system contains an unknown function implicitly in the sense that the coefficient is determined by solving the nominal wage equation under the condition that the unknown function is given, in contrast to the replicator equation whose coefficient explicitly contains an unknown function in evolutionary game theory (see [16, (3.3)]). If we can define an operator that maps the density of workers to the density of real wages by solving the nominal wage equation under the condition that the density of workers is a given function, then the replicator equation is regarded as a nonlinear integropartial differential equation whose coefficient contains the operator that acts on an unknown function.

For these reasons we deduce that the dcDSK system is an essentially new kind of system of nonlinear integropartial differential equations. Therefore, it is important to study this system not only in spatial economics but also in the theory of functional equations. In this paper we prove a sufficient condition for the initial value problem for the dcDSK system to have a unique global solution and obtain estimates of the solution. The main result is Theorem 4.

2. The System of Equations

Let us introduce the notations. Let E be a domain of a Euclidean space. By $L^1(E)$ we denote the Banach space of all Lebesgue summable functions of $x \in E$. By $L^\infty(E)$ we denote the Banach space of all essentially bounded functions of $x \in E$. By $C(E)$ we denote the Banach space of all uniformly bounded continuous functions of $x \in E$.

We assume that economic activities are conducted continuously in a bounded domain D of an m-dimensional Euclidean space, where m is a positive integer. If $m \geq 3$, then the model is unrealistic from the viewpoint of economics, but we accept such a case for mathematical generality in this paper. We denote the norms of the Banach spaces $L^1(D)$ and $C(D)$ by $|||\cdot|||$ and $\|\cdot\|$, respectively; that is, we define

$$|||u(\cdot)||| := \int_{y\in D} |u(y)|\, dy, \qquad \|u(\cdot)\| := \sup_{y\in D} |u(y)|. \quad (1)$$

Let $T \geq 0$. By $L^{\infty,1}([0,T]\times D)$ we denote the Banach space of all functions $h = h(t,x)$ such that

$$\operatorname*{ess\,sup}_{0\leq t\leq T} |||h(t,\cdot)||| < +\infty. \quad (2)$$

By $L^1{}_+(E)$, $L^\infty{}_+(E)$, $C_+(E)$, and $L^{\infty,1}{}_+([0,T]\times D)$ we denote the set of all positive-valued functions of $L^1(E)$, $L^\infty(E)$, $C(E)$, and $L^{\infty,1}([0,T]\times D)$, respectively. By $L^1{}_{0+}(E)$, $L^\infty{}_{0+}(E)$, $C_{0+}(E)$, and $L^{\infty,1}{}_{0+}([0,T]\times D)$ we denote the set of all nonnegative-valued functions of $L^1(E)$, $L^\infty(E)$, $C(E)$, and $L^{\infty,1}([0,T]\times D)$, respectively.

Let us introduce the dcDSK system. The nominal wage equation is a nonlinear integral equation of the following form (see [1, (5.5)] and [14, (2.4)]):

$$w(t,x)^\sigma = \int_{y\in D} Y_\mu(\lambda(t,y), w(t,y))\, G_\sigma(\lambda(t,\cdot), w(t,\cdot); y)^{\sigma-1} \cdot e^{-(\sigma-1)c(x,y)} dy, \quad (3)$$

where $w = w(t,x)$ is an unknown function that denotes the density of nominal wages at time $t \geq 0$ and at point $x \in D$. By $G_\sigma = G_\sigma(\lambda(t,\cdot), w(t,\cdot); x)$ we denote the price index, which is a nonlinear integral operator of the following form (see [1, (5.4)] and [14, (2.7)]):

$$G_\sigma(\lambda(t,\cdot), w(t,\cdot); x)^{\sigma-1} := \frac{1}{\int_{y\in D} \lambda(t,y)\,(1/w(t,y))^{\sigma-1} e^{-(\sigma-1)c(x,y)} dy}, \quad (4)$$

where $\lambda = \lambda(t, y)$ represents the density of workers at time $t \geq 0$ and at point $y \in D$. By σ we denote *the elasticity of substitution* among varieties of manufactured goods. We assume that

$$\sigma > 1. \tag{5}$$

When σ increases, varieties of manufactured goods are close to *perfect substitutes*; as σ decreases toward 1, the desire to consume a greater variety of manufactured goods increases (see [1, p. 46] and [2, p. 308]). We denote *the income* at time $t \geq 0$ and at point $y \in D$ by $Y_\mu = Y_\mu(\lambda(t, y), w(t, y))$. The dcDSK model consists of a monopolistically competitive sector (*manufacturing*) and a perfectly competitive sector (*agriculture*) (see [1, p. 61]). Hence the income consists of agricultural income and manufacturing income; that is, it has the following form (see [1, (5.3)]):

$$Y_\mu(\lambda(t, y), w(t, y)) = \mu\lambda(t, y) w(t, y) + (1 - \mu)\phi(y), \tag{6}$$

$$0 < \mu < 1, \tag{7}$$

where μ and $(1-\mu)$ denote the share of manufacturing expenditure and the share of agricultural expenditure, respectively, and we denote the density of farmers at point $y \in D$ by $\phi = \phi(y)$. We assume that $\phi = \phi(y)$ is a given function such that (see [14, (2.12), (2.14)])

$$\begin{gathered} \phi(y) \in {L^1}_{0+}(D), \\ |||\phi(\cdot)||| = 1. \end{gathered} \tag{8}$$

Note that this function is independent of the time variable $t \geq 0$.

The function $c = c(x, y)$ represents *the iceberg form of transport costs* (see [17]). We refer to this function as *the transport cost function*. We reasonably accept the following assumption (see [14, Assumption 2.1]).

Assumption 1. The transport cost function $c = c(x, y)$ is a nonnegative-valued continuous function of $(x, y) \in D \times D$ such that

$$\begin{gathered} c(x, x) = 0 \quad \text{for each } x \in D, \\ c(x, y) = c(y, x) \quad \text{for each } x, y \in D, \\ c(x, y) > 0 \quad \text{if } x \neq y, \\ \mathbf{C} := \sup_{(x,y)\in D\times D} c(x, y) < +\infty. \end{gathered} \tag{9}$$

Considering (6), and noting that the right-hand side of (4) is a nonlinear integral operator acting on the density of workers $\lambda = \lambda(t, x)$ and the density of nominal wages $w = w(t, x)$, we see that the right-hand side of (3) is a double nonlinear integral operator acting on these densities.

We define the density of real wages $\omega = \omega(t, x)$ at time $t \geq 0$ and at point $x \in D$ by deflating the density of nominal wages by a fractional power of the price index as follows (see [1, (5.6)], (4), and (7)):

$$\omega = \omega(t, x) := \frac{w(t, x)}{G_\sigma(\lambda(t, \cdot), w(t, \cdot); x)^\mu}. \tag{10}$$

The density of real wages can be regarded as a nonlinear integral operator acting on $\lambda = \lambda(t, x)$ and $w = w(t, x)$.

The replicator equation is a nonlinear integropartial differential equation of the following form (see [1, (5.2)] and [13, (2.22)]):

$$\left(\frac{\partial}{\partial t}\right)\lambda(t, x) = aM(\lambda(t, \cdot), \omega(t, \cdot); x)\lambda(t, x), \tag{11}$$

where a denotes a positive constant. We define

$$M(\lambda(t, \cdot), \omega(t, \cdot); x) := \omega(t, x) - m(\lambda(t, \cdot), \omega(t, \cdot)), \tag{12}$$

where $m(\lambda(t, \cdot), \omega(t, \cdot))$ is the average real wage defined as follows in the same way as [1, (5.1)] and [13, (2.23)]:

$$m(\lambda(t, \cdot), \omega(t, \cdot)) := \int_{y\in D} \lambda(t, y)\omega(t, y)\, dy. \tag{13}$$

It follows from (11), (12), and (13) that workers move toward regions that offer higher real wages and away from regions that offer below-average real wages (see [1, p. 62]). Considering (4) and (10), we see that (12) is a double nonlinear integral operator acting on $\lambda = \lambda(t, x)$ and $w = w(t, x)$. In this paper for simplicity we assume that

$$a = 1. \tag{14}$$

Hence (12) is the growth rate. The dcDSK system consists of (3), (10), and (11). In Section 4 we define an operator that maps $\lambda = \lambda(t, x)$ to $\omega = \omega(t, x)$ by solving (3) under the condition that $\lambda = \lambda(t, x)$ is a given function. Substituting this operator in (11), we can transform the dcDSK system into the replicator equation whose coefficient contains the operator that maps $\lambda = \lambda(t, x)$ to $\omega = \omega(t, x)$.

3. Result and Discussion

We consider the initial value problem by imposing the following initial condition on the dcDSK system (3), (10), and (11):

$$\lambda(0, x) = \lambda^0(x) \quad \text{for a.e. } x \in D, \tag{15}$$

where $\lambda^0 = \lambda^0(x)$ is a given function of $x \in D$. This function denotes the initial density of workers. The following assumption is imposed on this function in [1, pp. 61–63] and [14, (2.11), (2.13)].

Assumption 2. (i) $\lambda^0 = \lambda^0(x) \in {L^1}_{0+}(D)$.

(ii) $|||\lambda^0(\cdot)||| = 1$.

Hence, we accept this assumption in this paper also. Let $T > 0$ be a constant. If a function

$$(\lambda, \omega, w) = (\lambda(t, x), \omega(t, x), w(t, x)) \tag{16}$$

belongs to

$${L^{\infty,1}}_{0+}([0, T] \times D) \times {L^{\infty}}_{+}([0, T] \times D) \times {L^{\infty}}_{+}([0, T] \times D) \tag{17}$$

and satisfies the dcDSK system for a.e. $(t,x) \in [0,T] \times D$ and the initial condition (15), then we say that the function (16) is a *solution* to the initial value problem in $[0,T]$. If a function (16) is defined for a.e. $(t,x) \in [0,+\infty) \times D$ and is a solution to the initial value problem in $[0,T]$ for each $T > 0$, then we say that the solution is *global*. No boundary condition needs to be imposed on the density of workers, since the evolution of the density of workers can be determined uniquely in (17) by the initial condition (15) as done in Section 5.

We define a function $V = V(\mu,\sigma)$, $(\mu,\sigma) \in (0,1) \times (1,+\infty)$ in order to state a sufficient condition for the initial value problem to have a unique global solution. Consider the quadratic equation

$$I(\mu,\sigma;u) := \left(1 - \frac{1}{\sigma}\right)u^2 + \left(\frac{\mu}{\sigma}\right)u - 1 = 0, \tag{18}$$

where u denotes an unknown quantity. It follows from (5) that this equation has a positive solution and a negative solution. We denote the positive solution by $U = U(\mu,\sigma)$. We see easily that

$$U(\mu,\sigma) = \frac{\left\{-\mu + \left(\mu^2 + 4\sigma(\sigma-1)\right)^{1/2}\right\}}{(2(\sigma-1))}. \tag{19}$$

By making use of this positive solution, we define the following quadratic equation:

$$J(\mu,\sigma;v) := v^2 + \mu\left(U(\mu,\sigma)^{1/\sigma} - 1\right)v - U(\mu,\sigma)^{1/\sigma} = 0, \tag{20}$$

where v denotes an unknown quantity. This quadratic equation has a positive solution and a negative solution, since $U(\mu,\sigma)^{1/\sigma} > 0$. We denote the positive solution by $V = V(\mu,\sigma)$. We see easily that

$$\begin{aligned} V &= V(\mu,\sigma) \\ &:= \left\{-\mu\left(U(\mu,\sigma)^{1/\sigma} - 1\right)\right. \\ &\qquad \left. + \left(\mu^2\left(U(\mu,\sigma)^{1/\sigma} - 1\right)^2 + 4U(\mu,\sigma)^{1/\sigma}\right)^{1/2}\right\} \cdot 2^{-1}. \end{aligned} \tag{21}$$

The following lemma is proved in [12, Lemma 3.2] (see [12, (3.3)–(3.8)]).

Lemma 3. *(i)* $1 < V(\mu,\sigma) < 1/\mu$ *for each* $(\mu,\sigma) \in (0,1) \times (1,+\infty)$.

(ii) $V = V(\mu,\sigma)$ *is a strictly monotone-decreasing smooth function of* $\sigma > 1$ *for each* $\mu \in (0,1)$.

(iii) $V = V(\mu,\sigma)$ *is a strictly monotone-decreasing smooth function of* $\mu \in (0,1)$ *for each* $\sigma > 1$.

(iv) $\lim_{\sigma \to 1+0} V(\mu,\sigma) > 1$, $\lim_{\sigma \to +\infty} V(\mu,\sigma) = 1$, *for each* $\mu \in (0,1)$.

(v) $\lim_{\mu \to 0+0} V(\mu,\sigma) > 1$, $\lim_{\mu \to 1-0} V(\mu,\sigma) = 1$, *for each* $\sigma > 1$.

The following theorem is the main result of this paper, which is proved in Sections 4 and 5 (see Assumption 1).

Theorem 4. *If* μ, σ, *and* $\mathbf{C}$ *satisfy (5), (7), and the following inequality:*

$$\mathbf{C} < \frac{\{\log(V(\mu,\sigma))\}}{(\sigma-1)}, \tag{22}$$

then the initial value problem has a unique global solution $(\lambda,\omega,w) = (\lambda(t,x),\omega(t,x),w(t,x))$, *and this solution satisfies the following:*

$$|||\lambda(r,\cdot) - \lambda(s,\cdot)||| \le \mathbf{a}|r-s| \quad \text{for each } r,s \ge 0, \tag{23}$$

$$\begin{gathered} \lambda = \lambda(t,x) \ \text{ is continuous at } t = 0+0 \\ \text{and partially differentiable for } t > 0 \ \text{ for a.e. } x \in D, \end{gathered} \tag{24}$$

$$\omega = \omega(t,x), \quad w = w(t,x) \in C_+(D) \quad \text{for each } t \ge 0, \tag{25}$$

$$\begin{gathered} \|w(r,\cdot) - w(s,\cdot)\| \le \delta_1(\mu,\sigma,\mathbf{C})|r-s|, \\ \|\omega(r,\cdot) - \omega(s,\cdot)\| \le \delta_2(\mu,\sigma,\mathbf{C})|r-s| \\ \text{for each } r,s \ge 0, \end{gathered} \tag{26}$$

$$\begin{gathered} \lambda^0(x)\exp(-\mathbf{a}t) \le \lambda(t,x) \le \lambda^0(x)\exp(\mathbf{a}t) \\ \text{for each } t \ge 0 \text{ and a.e. } x \in D, \end{gathered} \tag{27}$$

$$|||\lambda(t,\cdot)||| = 1 \quad \text{for each } t \ge 0, \tag{28}$$

$$\frac{1}{\alpha_+} \le w(t,x) \le \frac{1}{\alpha_-} \quad \text{for each } (t,x) \in [0,+\infty) \times D, \tag{29}$$

$$\begin{gathered} \frac{1}{\alpha_+} \le G_\sigma(\lambda(t,\cdot),w(t,\cdot);x) \le \left(\frac{1}{\alpha_-}\right)\exp(\mathbf{C}) \\ \text{for each } (t,x) \in [0,+\infty) \times D, \end{gathered} \tag{30}$$

$$\mathbf{a}_1 \le \omega(t,x) \le \mathbf{a}_2 \quad \text{for each } (t,x) \in [0,+\infty) \times D, \tag{31}$$

$$\begin{gathered} -\mathbf{a} \le M(\lambda(t,\cdot),\omega(t,\cdot);x) \le \mathbf{a} \\ \text{for each } (t,x) \in [0,+\infty) \times D, \end{gathered} \tag{32}$$

where $\delta_i = \delta_i(\mu,\sigma,\mathbf{C})$, $i = 1,2$, *are positive-valued functions of* $(\mu,\sigma,\mathbf{C}) \in (0,1) \times (1,+\infty) \times (0,+\infty)$ *and*

$$\begin{gathered} \mathbf{a}_1 := \frac{(\exp(-\mu\mathbf{C}))\,{\alpha_-}^{\mu}}{\alpha_+}, \\ \mathbf{a}_2 := \frac{{\alpha_+}^{\mu}}{\alpha_-}, \qquad \mathbf{a} := \mathbf{a}_2 - \mathbf{a}_1, \end{gathered} \tag{33}$$

$$\alpha_\pm := \frac{\{\exp(\pm(\sigma-1)\mathbf{C}) - \mu\}}{(1-\mu)}. \tag{34}$$

Let us discuss this theorem. The inequalities (5), (7), and (22) are a sufficient condition for the initial value problem to have a unique global solution. Note that $V = V(\mu, \sigma)$ is independent of $\mathbf{C}$. Making use of Lemma 3, (i), (ii), (iv), we deduce that

$$\text{if } \sigma > 1 \text{ and } \mathbf{C} > 0 \text{ are sufficiently small, then (22) holds.} \tag{35}$$

Hence we can say that (22) holds for a much larger set of $(\mu, \sigma, \mathbf{C}) \in (0,1) \times (1, +\infty) \times (0, +\infty)$. By Lemma 3, (i), we see that (5), (7), and (22) imply the following inequality:

$$\mathbf{C} < \frac{(\log(1/\mu))}{(\sigma - 1)}. \tag{36}$$

Applying this inequality, Assumption 1, (5), and (7) to (33) and (34), we see that

$$0 < \alpha_- < 1 < \alpha_+, \tag{37}$$

$$0 < \mathbf{a}_1 < 1 < \mathbf{a}_2. \tag{38}$$

Making use of (5), (7), (8), and the above theorem, we see easily that the right-hand sides of (3), (4), and (12) belong to $L^{\infty}([0,T] \times D)$ for each $T > 0$ and that the right-hand side of (11) belongs to $L^{\infty,1}([0,T] \times D)$ for each $T > 0$. The conservation law of workers follows from (17) and (28). It follows from (27) that

$$\{x \in D;\ \lambda(t,x) = 0\} = \left\{x \in D;\ \lambda^0(x) = 0\right\} \quad \text{for each } t \geq 0. \tag{39}$$

Hence, no worker moves toward a point where no worker lives. Combining (25) and (26), we see that

$$\omega = \omega(t,x), \qquad w = w(t,x) \in C_+([0,T] \times D) \quad \text{for each } T > 0. \tag{40}$$

The functions $\delta_i = \delta_i(\mu, \sigma, \mathbf{C})$, $i = 1, 2$, are defined in Lemma 17.

Remark 5. We impose (7) on the dcDSK model; that is, we consider the dcDSK model in an *urban-rural setting*. In this paper we cannot treat the case

$$\mu = 1; \tag{41}$$

that is, we cannot consider the dcDSK model in an *urban setting* (see [1, p. 331]) because it follows from Lemma 3, (i), that (41) cannot be substituted in (22). The DSK model with (41) is studied in [13]. The cDSK model with (41) is studied in [14, Theorem 3.2].

4. Solutions of the Nominal Wage Equation

Let us solve the nominal wage equation (3) under the condition that the density of workers is a given function. In this section we do not deal with the replicator equation. Hence, we have no need to consider the time evolution of the density of workers, the density of nominal wages, and the density of real wages. Therefore, for simplicity of symbols, we omit the time variable t from these densities, and we denote them by $\lambda = \lambda(x)$, $w = w(x)$, and $\omega = \omega(x)$ in (3), (4), (6), and (10) in this section. We refer to these equations with the same numbers. No confusion should arise. We assume that $\lambda = \lambda(x)$ is a given function that satisfies the same condition as Assumption 2 as follows:

$$\lambda(x) \in L^1{}_{0+}(D), \tag{42}$$

$$|||\lambda(\cdot)||| = 1. \tag{43}$$

Proposition 6. *If μ, σ, and $\mathbf{C}$ satisfy (5), (7), and (36), then the following statements (i) and (ii) hold.*

(i) *The nominal wage equation (3) has a solution $w = w(x)$ in $L^{\infty}{}_{+}(D)$.*

(ii) *If (3) has a solution $w = w(x)$ in $L^{\infty}{}_{+}(D)$, then the solution satisfies the following:*

$$w = w(x), \qquad \omega = \omega(x) \in C_+(D), \tag{44}$$

$$\frac{1}{\alpha_+} \leq w(x) \leq \frac{1}{\alpha_-} \quad \textit{for each } x \in D, \tag{45}$$

$$\frac{1}{\alpha_+} \leq G_\sigma(\lambda(\cdot), w(\cdot); x) \leq \left(\frac{1}{\alpha_-}\right) \exp(\mathbf{C}) \quad \textit{for each } x \in D, \tag{46}$$

$$\mathbf{a}_1 \leq \omega(x) \leq \mathbf{a}_2 \quad \textit{for each } x \in D, \tag{47}$$

$$\mathbf{a}_1 \leq m(\lambda(\cdot), \omega(\cdot)) \leq \mathbf{a}_2, \tag{48}$$

$$-\mathbf{a} \leq M(\lambda(\cdot), \omega(\cdot); x) \leq \mathbf{a} \quad \textit{for each } x \in D. \tag{49}$$

Proof. In [14, Theorem 3.1, (i), (ii)], from (5), (7), and (36), we prove (i), and we prove that if (3) has a solution in $L^{\infty}{}_{+}(D)$, then the solution satisfies (44)–(47). Applying (42), (43), and (47) to (12) and (13), we obtain (48) and (49). □

Lemma 7. *Let r and s be positive constants. Let $\beta \in \mathbb{R}$ be a constant. If $g_i = g_i(x) \in C(D)$, $i = 1, 2$, satisfy the following inequality:*

$$r \leq g_i(x) \leq s \quad \textit{for each } x \in D,\ i = 1, 2, \tag{50}$$

then

$$\left\| g_1(\cdot)^\beta - g_2(\cdot)^\beta \right\| \leq h(\beta, r, s) \left\| g_1(\cdot) - g_2(\cdot) \right\|, \tag{51}$$

where

$$h(\beta, r, s) := \beta s^{\beta - 1} \quad \textit{if } \beta > 1, \qquad h(\beta, r, s) := |\beta| r^{\beta - 1} \quad \textit{if } \beta \leq 1. \tag{52}$$

Proof. By the mean value theorem we see easily that if $0 < X_1 \leq X_2$, then there exists a constant $\xi \in [X_1, X_2]$ such that

$$X_1{}^\beta - X_2{}^\beta = \beta \xi^{\beta - 1}(X_1 - X_2). \tag{53}$$

Substituting $(X_1, X_2) = (g_1(y), g_2(y))$ or $(g_2(y), g_1(y))$ in this equality and making use of (50), we see that

$$\left|g_1(y)^\beta - g_2(y)^\beta\right| \le |\beta| \max_{r\le\xi\le s} \xi^{\beta-1} \left|g_1(y) - g_2(y)\right|. \quad (54)$$

Applying (52) to the right-hand side of this inequality, we obtain the present lemma. □

Lemma 8. *Assume that μ, σ, and $\mathbf{C}$ satisfy (5), (7), and (22). If $\lambda_i = \lambda_i(x)$, $i = 1,2$, satisfy (42) and (43), and $w_i = w_i(x) \in C_+(D)$ is a solution of (3) with $\lambda(x) = \lambda_i(x)$, $i = 1,2$, respectively, then*

$$\|w_1(\cdot) - w_2(\cdot)\| \le |||\lambda_1(\cdot) - \lambda_2(\cdot)|||\, \gamma_1(\mu, \sigma, \mathbf{C}), \quad (55)$$

where $\gamma_1 = \gamma_1(\mu, \sigma, \mathbf{C})$ is a positive-valued function of $(\mu, \sigma, \mathbf{C}) \in (0,1) \times (1, +\infty) \times (0, +\infty)$.

Proof. Let us transform the nominal wage equation (3). Define the following new unknown function:

$$W = W(x) := w(x)^\sigma. \quad (56)$$

Defining the following nonlinear integral operator:

$$H(\lambda(\cdot), W(\cdot); x) := \int_{y\in D} \lambda(y) \left(\frac{1}{W(y)}\right)^{1-1/\sigma} e^{-(\sigma-1)c(x,y)} dy, \quad (57)$$

we rewrite (4) as follows:

$$G_\sigma(\lambda(\cdot), w(\cdot); x)^{\sigma-1} = \frac{1}{H(\lambda(\cdot), W(\cdot); x)}. \quad (58)$$

Making use of this equality and defining the following nonlinear integral operator (see (6)):

$$F(\lambda(\cdot), W(\cdot); x) := \int_{y\in D} \left\{ \frac{Y_\mu\left(\lambda(y), W(y)^{1/\sigma}\right)}{H(\lambda(\cdot), W(\cdot); y)} \right\} e^{-(\sigma-1)c(x,y)} dy, \quad (59)$$

we rewrite (3) equivalently as follows:

$$W(x) = F(\lambda(\cdot), W(\cdot); x). \quad (60)$$

Hence, we see that

$$W_i(x) = F\left(\lambda_i(\cdot), W_i(\cdot); x\right), \quad i = 1,2, \quad (61)$$

where $W_i = W_i(x)$ is defined by $w_i = w_i(x)$ in the same way as (56), $i = 1,2$. Subtract both sides of (61) with $i = 2$ from those of (61) with $i = 1$. The right-hand side of the equality thus obtained is transformed as follows (see (6)):

$$\begin{aligned} W_1(x) - W_2(x) &= \int_{y\in D} \mu(\lambda_1(y) - \lambda_2(y)) W_1(y)^{1/\sigma} \cdot \left(\frac{1}{H(\lambda_1(\cdot), W_1(\cdot); y)}\right) e^{-(\sigma-1)c(x,y)} dy \\ &+ \int_{y\in D} \mu\lambda_2(y) \left(W_1(y)^{1/\sigma} - W_2(y)^{1/\sigma}\right) \cdot \left(\frac{1}{H(\lambda_1(\cdot), W_1(\cdot); y)}\right) e^{-(\sigma-1)c(x,y)} dy \\ &+ \int_{y\in D} Y_\mu\left(\lambda_2(y), W_2(y)^{1/\sigma}\right) \Delta_1(y) e^{-(\sigma-1)c(x,y)} dy \\ &+ \int_{y\in D} Y_\mu\left(\lambda_2(y), W_2(y)^{1/\sigma}\right) \Delta_2(y) e^{-(\sigma-1)c(x,y)} dy, \end{aligned} \quad (62)$$

where

$$\Delta_i(y) := \frac{1}{H(\lambda_1(\cdot), W_i(\cdot); y)} - \frac{1}{H(\lambda_i(\cdot), W_2(\cdot); y)}, \quad i = 1,2. \quad (63)$$

We denote the jth term of the right-hand side of this equality by I_j, $j = 1, \ldots, 4$.

Substituting (58) in (46), we see that

$$\mathbf{H}_- \le H\left(\lambda_i(\cdot), W_j(\cdot); y\right) \le \mathbf{H}_+, \quad i, j = 1,2, \quad (64)$$

where

$$\mathbf{H}_- := {\alpha_-}^{\sigma-1} \exp(-(\sigma-1)\mathbf{C}), \qquad \mathbf{H}_+ := {\alpha_+}^{\sigma-1}. \quad (65)$$

It follows from (5) and Assumption 1 that

$$\exp(-(\sigma-1)\mathbf{C}) \le \exp(-(\sigma-1)c(x,y)) \le 1 \quad \text{for each } x, y \in D. \quad (66)$$

Applying this inequality, (7), (45), (56), and (64) to I_1, we see that

$$\|I_1\| \le |||\lambda_1(\cdot) - \lambda_2(\cdot)|||\, \theta_1(\mu, \sigma, \mathbf{C}), \quad (67)$$

where

$$\theta_1(\mu, \sigma, \mathbf{C}) := \mu\left(\frac{1}{\alpha_-}\right)\left(\frac{1}{\mathbf{H}_-}\right). \quad (68)$$

Applying (7), (64), and (66) to I_2, we see that

$$\|I_2\| \le |||\lambda_2(\cdot)|||\, \theta_{2.1}(\mu, \sigma, \mathbf{C}) \left\|W_1(\cdot)^{1/\sigma} - W_2(\cdot)^{1/\sigma}\right\|, \quad (69)$$

where

$$\theta_{2.1}(\mu, \sigma, \mathbf{C}) := \mu\left(\frac{1}{\mathbf{H}_-}\right). \quad (70)$$

Making use of (5), (45), (56), and Lemma 7 with $\beta = 1/\sigma$ and $r = 1/{\alpha_+}^{\sigma}$, we obtain the following inequality:

$$\left\| W_1(\cdot)^{1/\sigma} - W_2(\cdot)^{1/\sigma} \right\| \le \left\| W_1(\cdot) - W_2(\cdot) \right\| \theta_{2.2}(\mu, \sigma, \mathbf{C}), \tag{71}$$

where

$$\theta_{2.2}(\mu, \sigma, \mathbf{C}) := \left(\frac{1}{\sigma}\right)\left(\frac{1}{{\alpha_+}^{\sigma}}\right)^{1/\sigma - 1}. \tag{72}$$

Applying this inequality and (43) to (69), we see that

$$\left\| I_2 \right\| \le \left\| W_1(\cdot) - W_2(\cdot) \right\| \theta_2(\mu, \sigma, \mathbf{C}), \tag{73}$$

where

$$\theta_2(\mu, \sigma, \mathbf{C}) := \theta_{2.1}(\mu, \sigma, \mathbf{C})\, \theta_{2.2}(\mu, \sigma, \mathbf{C}). \tag{74}$$

Applying (5), (43), (45), (56), (66), and Lemma 7 with $\beta = -(1 - 1/\sigma)$ and $r = 1/{\alpha_+}^{\sigma}$ to (57) and performing the same calculations as done in proving (73), we see that

$$\begin{aligned} &\left\| H(\lambda_1(\cdot), W_1(\cdot); \cdot) - H(\lambda_1(\cdot), W_2(\cdot); \cdot) \right\| \\ &\quad \le \left\| W_1(\cdot) - W_2(\cdot) \right\| \theta_{3.1}(\mu, \sigma, \mathbf{C}), \end{aligned} \tag{75}$$

where

$$\theta_{3.1}(\mu, \sigma, \mathbf{C}) := \left(1 - \frac{1}{\sigma}\right)\left(\frac{1}{{\alpha_+}^{\sigma}}\right)^{-(2 - 1/\sigma)}. \tag{76}$$

Combining this inequality and (64), we see that

$$\left\| \Delta_1(\cdot) \right\| \le \left\| W_1(\cdot) - W_2(\cdot) \right\| \theta_{3.2}(\mu, \sigma, \mathbf{C}), \tag{77}$$

where

$$\theta_{3.2}(\mu, \sigma, \mathbf{C}) := \frac{\theta_{3.1}(\mu, \sigma, \mathbf{C})}{{\mathbf{H}_-}^2}. \tag{78}$$

Applying (7), (42), (45), and (56) to (6), we see that

$$\begin{aligned} &\frac{\mu \lambda_i(y)}{\alpha_+} + (1 - \mu)\phi(y) \\ &\quad \le Y_\mu\left(\lambda_i(y), W_j(y)^{1/\sigma}\right) \\ &\quad \le \frac{\mu \lambda_i(y)}{\alpha_-} + (1 - \mu)\phi(y), \quad i, j = 1, 2. \end{aligned} \tag{79}$$

Integrating both sides of these inequalities with respect to $y \in D$ and making use of (7), (8), (42), and (43), we see that

$$\frac{\mu}{\alpha_+} + (1 - \mu) \le \left|\left|\left| Y_\mu\left(\lambda_i(\cdot), W_j(\cdot)^{1/\sigma}\right) \right|\right|\right| \le \frac{\mu}{\alpha_-} + (1 - \mu), \quad i, j = 1, 2. \tag{80}$$

Applying (7) and (37) to this inequality, we see that

$$\left|\left|\left| Y_\mu\left(\lambda_i(\cdot), W_j(\cdot)^{1/\sigma}\right) \right|\right|\right| \le \theta_{3.3}(\mu, \sigma, \mathbf{C}), \quad i, j = 1, 2, \tag{81}$$

where

$$\theta_{3.3}(\mu, \sigma, \mathbf{C}) := \frac{1}{\alpha_-}. \tag{82}$$

Applying (66), (77), and (81) to I_3, we obtain the following inequality:

$$\left\| I_3 \right\| \le \left\| W_1(\cdot) - W_2(\cdot) \right\| \theta_3(\mu, \sigma, \mathbf{C}), \tag{83}$$

where

$$\theta_3(\mu, \sigma, \mathbf{C}) := \theta_{3.2}(\mu, \sigma, \mathbf{C})\, \theta_{3.3}(\mu, \sigma, \mathbf{C}). \tag{84}$$

Performing calculations similar to, but easier than, those done in proving (75), we see that

$$\begin{aligned} &\left\| H(\lambda_1(\cdot), W_2(\cdot); \cdot) - H(\lambda_2(\cdot), W_2(\cdot); \cdot) \right\| \\ &\quad \le \left|\left|\left| \lambda_1(\cdot) - \lambda_2(\cdot) \right|\right|\right| \theta_{4.1}(\mu, \sigma, \mathbf{C}), \end{aligned} \tag{85}$$

where

$$\theta_{4.1}(\mu, \sigma, \mathbf{C}) := \left({\alpha_+}^{\sigma}\right)^{1 - 1/\sigma}. \tag{86}$$

We obtain the following inequality by combining this inequality and (64) in the same way as (77):

$$\left\| \Delta_2(\cdot) \right\| \le \left|\left|\left| \lambda_1(\cdot) - \lambda_2(\cdot) \right|\right|\right| \theta_{4.2}(\mu, \sigma, \mathbf{C}), \tag{87}$$

where

$$\theta_{4.2}(\mu, \sigma, \mathbf{C}) := \frac{\theta_{4.1}(\mu, \sigma, \mathbf{C})}{{\mathbf{H}_-}^2}. \tag{88}$$

Applying this inequality, (66), and (81) to I_4, we obtain the following inequality:

$$\left\| I_4 \right\| \le \left|\left|\left| \lambda_1(\cdot) - \lambda_2(\cdot) \right|\right|\right| \theta_4(\mu, \sigma, \mathbf{C}), \tag{89}$$

where

$$\theta_4(\mu, \sigma, \mathbf{C}) := \theta_{3.3}(\mu, \sigma, \mathbf{C})\, \theta_{4.2}(\mu, \sigma, \mathbf{C}). \tag{90}$$

Applying this inequality, (67), (73), and (83) to (62), we obtain the following inequality:

$$\begin{aligned} \left\| W_1(\cdot) - W_2(\cdot) \right\| &\le \left|\left|\left| \lambda_1(\cdot) - \lambda_2(\cdot) \right|\right|\right| \Theta_1(\mu, \sigma, \mathbf{C}) \\ &\quad + \left\| W_1(\cdot) - W_2(\cdot) \right\| \Theta_2(\mu, \sigma, \mathbf{C}), \end{aligned} \tag{91}$$

where

$$\begin{aligned} \Theta_1 &= \Theta_1(\mu, \sigma, \mathbf{C}) := \theta_1(\mu, \sigma, \mathbf{C}) + \theta_4(\mu, \sigma, \mathbf{C}), \\ \Theta_2 &= \Theta_2(\mu, \sigma, \mathbf{C}) := \theta_2(\mu, \sigma, \mathbf{C}) + \theta_3(\mu, \sigma, \mathbf{C}). \end{aligned} \tag{92}$$

Substituting (34) in the definitions of $\theta_i = \theta_i(\mu,\sigma,\mathbf{C})$, $i = 1,\ldots,4$, we rewrite $\Theta_i = \Theta_i(\mu,\sigma,\mathbf{C})$, $i = 1,2$, as follows in the same way as [12, (5.8), (5.9), (7.19), (7.21)–(7.25)]:

$$\begin{aligned}
&\Theta_1(\mu,\sigma,\mathbf{C}) \\
&\quad = \left\{\frac{(1-\mu)}{(\exp(-\alpha)-\mu)}\right\}^{\sigma} \\
&\qquad \cdot \left\{Q(\exp(\alpha))^{\sigma-1}\exp(2\alpha)+\mu\exp(\alpha)\right\}, \\
&\Theta_2(\mu,\sigma,\mathbf{C}) \\
&\quad = \left(\frac{\mu}{\sigma}\right)Q(\exp(\alpha))^{\sigma-1}\exp(\alpha) \\
&\qquad + \left(1-\frac{1}{\sigma}\right)Q(\exp(\alpha))^{2\sigma-1}\exp(2\alpha),
\end{aligned} \tag{93}$$

where

$$Q = Q(\nu) := \frac{(\nu-\mu)}{(\nu^{-1}-\mu)}, \qquad \alpha := (\sigma-1)\mathbf{C}. \tag{94}$$

It is proved in [12, Lemma 7.2, (7.22)] that (5), (7), and (22) imply the following inequality:

$$\Theta_2(\mu,\sigma,\mathbf{C}) < 1. \tag{95}$$

Making use of (5), (7), and (36), we see easily that

$$\Theta_1(\mu,\sigma,\mathbf{C}) > 1. \tag{96}$$

Hence, it follows from (91) and (95) that

$$\|W_1(\cdot)-W_2(\cdot)\| \le |||\lambda_1(\cdot)-\lambda_2(\cdot)|||\,\theta_5(\mu,\sigma,\mathbf{C}), \tag{97}$$

where

$$\theta_5(\mu,\sigma,\mathbf{C}) := \frac{\Theta_1(\mu,\sigma,\mathbf{C})}{(1-\Theta_2(\mu,\sigma,\mathbf{C}))} > 0. \tag{98}$$

Making use of this inequality, (56), and (71) and defining

$$\gamma_1(\mu,\sigma,\mathbf{C}) := \theta_{2.2}(\mu,\sigma,\mathbf{C})\,\theta_5(\mu,\sigma,\mathbf{C}), \tag{99}$$

we obtain Lemma 8. □

Proposition 9. *If μ, σ, and $\mathbf{C}$ satisfy (5), (7), and (22), then the nominal wage equation (3) has a unique solution $w = w(x)$ in $L^{\infty}{}_{+}(D)$.*

Proof. Recalling that (36) follows from (5), (7), and (22) and combining Proposition 6 and Lemma 8 with $\lambda_1(x) = \lambda_2(x)$, we obtain this proposition. □

Remark 10. (i) If D is not a continuous domain but a finite set of points, then the nominal wage equation is not a nonlinear integral equation but a nonlinear equation in an Euclidean space whose dimension is equal to the number of points of D (see [12, (2.5)]). This subject is treated in [12]. In [12] we prove a result similar to Propositions 6 and 9 by analyzing this nonlinear equation in the Euclidean space. However, in this paper, Propositions 6 and 9 are proved in the Banach spaces in contrast to the finite dimensional proof done in [12]. Propositions 6 and 9 are similar to, but essentially different from, the results obtained in [12].

(ii) The inequalities (5), (7), and (36) are a sufficient condition for the nominal wage equation (3) to have a solution in $L^{\infty}{}_{+}(D)$ (see Proposition 6). The inequalities (5), (7), and (22) are a sufficient condition for this solution to be unique. We make use of (22) in order to obtain (95) in the proof of Lemma 8.

(iii) It is proved in [14, Theorem 3.1, (iii)] that if $\sigma > 1$ and $\mathbf{C} > 0$ are sufficiently small, then (3) has a unique solution. However, the condition (22) is not accepted in [14]. Recalling (35), we see that the result of [14] can be regarded as a corollary of Proposition 9.

By Proposition 9 we can define an operator that maps $\lambda = \lambda(x)$ to $w = w(x)$, where $\lambda = \lambda(x)$ satisfies (42) and (43). We denote this operator by

$$P_1 = P_1(\lambda(\cdot);x); \tag{100}$$

that is, $w(x) := P_1(\lambda(\cdot);x)$ satisfies (3). Applying this operator to (10), we can define an operator that maps $\lambda = \lambda(x)$ to $\omega = \omega(x)$. We denote this operator by $P_2 = P_2(\lambda(\cdot);x)$; that is, we define

$$P_2(\lambda(\cdot);x) := \frac{P_1(\lambda(\cdot);x)}{G_\sigma(\lambda(\cdot),P_1(\lambda(\cdot);\cdot);x)^{\mu}}. \tag{101}$$

Proposition 11. *If $\lambda_i = \lambda_i(x)$, $i = 1,2$, satisfy (42) and (43), then*

$$\begin{aligned}
&\left\|P_j(\lambda_1(\cdot);\cdot)-P_j(\lambda_2(\cdot);\cdot)\right\| \\
&\quad \le |||\lambda_1(\cdot)-\lambda_2(\cdot)|||\,\gamma_j(\mu,\sigma,\mathbf{C}), \quad j = 1,2,
\end{aligned} \tag{102}$$

where $\gamma_2 = \gamma_2(\mu,\sigma,\mathbf{C})$ is a positive-valued function of $(\mu,\sigma,\mathbf{C}) \in (0,1)\times(1,+\infty)\times(0,+\infty)$ (see (99) for $\gamma_1 = \gamma_1(\mu,\sigma,\mathbf{C})$).

Proof. For $j = 1$ the result follows form Lemma 8. Next, we prove the case $j = 2$. Let us define

$$w_i(x) := P_1(\lambda_i(\cdot);x), \quad i = 1,2, \tag{103}$$

$$\omega_i(x) := P_2(\lambda_i(\cdot);x), \quad i = 1,2. \tag{104}$$

We define $W_i = W_i(x)$, $i = 1,2$, by (103) in the same way as (56). Making use of (75), (85), and (97), we see that

$$\begin{aligned}
&\|H(\lambda_1(\cdot),W_1(\cdot);\cdot)-H(\lambda_2(\cdot),W_2(\cdot);\cdot)\| \\
&\quad \le |||\lambda_1(\cdot)-\lambda_2(\cdot)|||\,\theta_6(\mu,\sigma,\mathbf{C}),
\end{aligned} \tag{105}$$

where

$$\theta_6(\mu,\sigma,\mathbf{C}) := \theta_{4.1}(\mu,\sigma,\mathbf{C})+\theta_{3.1}(\mu,\sigma,\mathbf{C})\,\theta_5(\mu,\sigma,\mathbf{C}). \tag{106}$$

Substitute (58) and $(\lambda, \omega, w) = (\lambda_i, \omega_i, w_i)$, $i = 1, 2$, in (10). Subtracting both sides of the equalities thus obtained from each other, we obtain

$$\begin{aligned} &\omega_1(x) - \omega_2(x) \\ &\quad = (w_1(x) - w_2(x)) H(\lambda_1(\cdot), W_1(\cdot); x)^{\mu/(\sigma-1)} \\ &\qquad + w_2(x) \left\{ H(\lambda_1(\cdot), W_1(\cdot); x)^{\mu/(\sigma-1)} \right. \\ &\qquad\qquad \left. - H(\lambda_2(\cdot), W_2(\cdot); x)^{\mu/(\sigma-1)} \right\}. \end{aligned} \tag{107}$$

Apply (5), (7), (42), (43), (45), (64), (105), and Lemmas 8 and 7 with $\beta = \mu/(\sigma-1)$ and $g_i(x) = H(\lambda_i(\cdot), W_i(\cdot); x)$, $i = 1, 2$, to the right-hand side of this equality. Defining

$$\begin{aligned} \gamma_2(\mu, \sigma, \mathbf{C}) &:= \gamma_1(\mu, \sigma, \mathbf{C}) \mathbf{H}_+^{\mu/(\sigma-1)} \\ &\quad + \left(\frac{1}{\alpha_-}\right) h\left(\frac{\mu}{(\sigma-1)}, \mathbf{H}_-, \mathbf{H}_+\right) \theta_6(\mu, \sigma, \mathbf{C}), \end{aligned} \tag{108}$$

we obtain the present lemma when $j = 2$. □

Remark 12. (i) We can prove Propositions 6, 9, and 11 with no boundary condition on $\lambda = \lambda(x)$, $w = w(x)$, and $\omega = \omega(x)$. We make use of (42) and (43) only. See [14, Remark 2.3, (i)].

(ii) For simplicity we omit the time variable t in this section. By replacing $\lambda = \lambda(x)$, $w = w(x)$, and $\omega = \omega(x)$ by $\lambda = \lambda(t, x)$, $w = w(t, x)$, and $\omega = \omega(t, x)$ respectively, we make use of Propositions 6, 9, and 11 in the next section.

5. The Iteration Scheme

The purpose of this section is to prove Theorem 4. Let us construct an iteration scheme to obtain a solution to the initial value problem for the dcDSK system (3), (10), and (11). Let Δt be a constant such that (see (33))

$$0 < \Delta t \le \frac{1}{\mathbf{a}}. \tag{109}$$

Decompose the time interval $[0, +\infty)$ into an infinite number of intervals $[t_n, t_{n+1})$, $n \in \mathbb{N} \cup \{0\}$, where

$$t_n := n\Delta t, \quad n \in \mathbb{N} \cup \{0\}. \tag{110}$$

Let us define $\lambda_{\Delta t} = \lambda_{\Delta t}(t, x)$ by the following iteration scheme (see (15)):

$$\lambda_{\Delta t}(0, x) := \lambda^0(x), \tag{111}$$

$$\begin{aligned} \lambda_{\Delta t}(t, x) &:= \lambda_{\Delta t}(t_n, x) + M_{\Delta t}(t_n, x) \lambda_{\Delta t}(t_n, x)(t - t_n), \\ &\qquad \text{for } t \in [t_n, t_{n+1}), \quad n \in \mathbb{N} \cup \{0\}, \end{aligned} \tag{112}$$

$$\begin{aligned} \lambda_{\Delta t}(t_{n+1}, x) &= \lambda_{\Delta t}(t_{n+1} - 0, x), \\ &\qquad \text{for each } n \in \mathbb{N} \cup \{0\}, \end{aligned} \tag{113}$$

where we define (see (12)–(14))

$$M_{\Delta t}(t, x) := M(\lambda_{\Delta t}(t, \cdot), \omega_{\Delta t}(t, \cdot); x), \quad t \ge 0, \tag{114}$$

$$\begin{aligned} w_{\Delta t}(t, x) &:= P_1(\lambda_{\Delta t}(t, \cdot); x), \\ \omega_{\Delta t}(t, x) &:= P_2(\lambda_{\Delta t}(t, \cdot); x), \\ & t \ge 0. \end{aligned} \tag{115}$$

Lemma 13. *For each $T > 0$, the following (i–viii) statements hold.*

(i) $\lambda_{\Delta t} = \lambda_{\Delta t}(t, x) \in L^{\infty,1}{}_{0+}([0, T] \times D)$.

(ii) $|||\lambda_{\Delta t}(t, \cdot)||| = 1$ *for each* $t \in [0, T]$.

(iii) $\omega_{\Delta t} = \omega_{\Delta t}(t, x)$, $w_{\Delta t} = w_{\Delta t}(t, x) \in C_+(D)$ *for each* $t \in [0, T]$.

(iv) $1/\alpha_+ \le w_{\Delta t}(t, x) \le 1/\alpha_-$ *for each* $(t, x) \in [0, T] \times D$.

(v) $1/\alpha_+ \le G_\sigma(\lambda_{\Delta t}(t, \cdot), w_{\Delta t}(t, \cdot); x) \le (1/\alpha_-) \exp(\mathbf{C})$ *for each* $(t, x) \in [0, T] \times D$.

(vi) $\mathbf{a}_1 \le \omega_{\Delta t}(t, x) \le \mathbf{a}_2$ *for each* $(t, x) \in [0, T] \times D$.

(vii) $\mathbf{a}_1 \le m(\lambda_{\Delta t}(t, \cdot), \omega_{\Delta t}(t, \cdot)) \le \mathbf{a}_2$ *for each* $t \in [0, T]$.

(viii) $-\mathbf{a} \le M_{\Delta t}(t, x) \le \mathbf{a}$ *for each* $(t, x) \in [0, T] \times D$.

Proof. By (111) and Assumption 2, we can consider that (i) and (ii) hold when $T = 0$. Hence, making use of (115) and Propositions 6 and 9, we obtain (iii)–(viii) when $T = 0$. Applying (i), (ii), and (viii) with $T = 0$, (109), and (110) to (112) and (113) when $n = 0$, we obtain (i) with $T = t_1$. Integrating both sides of (112) with respect to $x \in D$, recalling (12) and (13), and making use of (i) with $T = t_1$, we obtain (ii) with $T = t_1$. Making use of (i) and (ii) with $T = t_1$, (115), and Propositions 6 and 9, we obtain (iii)–(viii) when $T = t_1$. Assume that (i)–(viii) hold when $T = t_k$, $k \in \mathbb{N}$. We can prove (i)–(viii) with $T = t_{k+1}$ in the same way as in obtaining (i)–(viii) with $T = t_1$ from (i)–(viii) with $T = 0$. Therefore we prove this lemma. □

Note that Propositions 6 and 9 are proved on the basis of (42) and (43). Hence, we need Lemma 13, (i), (ii), in order to make use of Propositions 6 and 9 in the proof of Lemma 13, (iii)–(viii).

Differentiating both sides of (112) with respect to $t \in (t_n, t_{n+1})$, we obtain the following equality:

$$\begin{aligned} \left(\frac{\partial}{\partial t}\right) \lambda_{\Delta t}(t, x) &= M_{\Delta t}(t, x) \lambda_{\Delta t}(t, x) + R_{\Delta t}(t, x), \\ &\qquad \text{for each } t \in (t_n, t_{n+1}), \quad n \in \mathbb{N} \cup \{0\}, \end{aligned} \tag{116}$$

where

$$\begin{aligned} R_{\Delta t}(t, x) &:= -M_{\Delta t}(t, x)(\lambda_{\Delta t}(t, x) - \lambda_{\Delta t}(t_n, x)) \\ &\quad - (M_{\Delta t}(t, x) - M_{\Delta t}(t_n, x)) \lambda_{\Delta t}(t_n, x). \end{aligned} \tag{117}$$

Lemma 14. $|||R_{\Delta t}(t, \cdot)||| \le \mathbf{b}\Delta t$ *for each* $t \in [t_n, t_{n+1})$, $n \in \mathbb{N} \cup \{0\}$, *where*

$$\mathbf{b} := \mathbf{a}^2 + \mathbf{a}\mathbf{c}, \qquad \mathbf{c} := 2\gamma_2(\mu, \sigma, \mathbf{C}) + \mathbf{a}_2. \tag{118}$$

Proof. Subtract both sides of (114) with $t = t_n$ from those of (114). Applying (12) and (13) to the equality thus obtained, we obtain

$$\begin{aligned} M_{\Delta t}(t,x) - M_{\Delta t}(t_n,x) &= \omega_{\Delta t}(t,x) - \omega_{\Delta t}(t_n,x) \\ &\quad - \left(\int_{y\in D} \lambda_{\Delta t}(t,y)\,\omega_{\Delta t}(t,y)\,dy \right. \\ &\quad \left. - \int_{y\in D} \lambda_{\Delta t}(t_n,y)\,\omega_{\Delta t}(t_n,y)\,dy\right). \end{aligned} \tag{119}$$

Applying Lemma 13, (ii), (vi), and Proposition 11 with $j = 2$ and $(\lambda_1(x), \lambda_2(x)) = (\lambda_{\Delta t}(t,x), \lambda_{\Delta t}(t_n,x))$ to the right-hand side of this equality, we see that

$$\begin{gathered} \left\|M_{\Delta t}(t,\cdot) - M_{\Delta t}(t_n,\cdot)\right\| \le \mathbf{c}\left|\left|\left|\lambda_{\Delta t}(t,\cdot) - \lambda_{\Delta t}(t_n,\cdot)\right|\right|\right| \\ \text{for each } t \in [t_n, t_{n+1}), \quad n \in \mathbb{N}\cup\{0\}. \end{gathered} \tag{120}$$

Applying Lemma 13, (ii), (viii), and (110) to (112), we see that

$$\left|\left|\left|\lambda_{\Delta t}(t,\cdot) - \lambda_{\Delta t}(t_n,\cdot)\right|\right|\right| \le \mathbf{a}\Delta t. \tag{121}$$

Applying this inequality, Lemma 13, (ii), (viii), and (120) to (117), we obtain the present lemma. □

Lemma 15. *The function*

$$(\lambda_{\Delta t}(t,x), \omega_{\Delta t}(t,x), w_{\Delta t}(t,x)) \tag{122}$$

converges in (17) as $\Delta t \to 0+0$ for each $T > 0$.

Proof. Replace Δt by Δs in (116), where Δs is a positive constant that satisfies the same inequality as (109). Subtracting both sides of the equality thus obtained from those of (116), we obtain

$$\left(\frac{\partial}{\partial t}\right)\Lambda_{\Delta t,\Delta s}(t,x) = M_{\Delta t}(t,x)\,\Lambda_{\Delta t,\Delta s}(t,x) + \mathbf{R}_{\Delta t,\Delta s}(t,x), \tag{123}$$

where

$$\Lambda_{\Delta t,\Delta s}(t,x) := \lambda_{\Delta t}(t,x) - \lambda_{\Delta s}(t,x), \tag{124}$$

$$\mathbf{R}_{\Delta t,\Delta s}(t,x) := R_{\Delta t}(t,x) - R_{\Delta s}(t,x) + r_{\Delta t,\Delta s}(t,x), \tag{125}$$

$$r_{\Delta t,\Delta s}(t,x) := \left(M_{\Delta t}(t,x) - M_{\Delta s}(t,x)\right)\lambda_{\Delta s}(t,x). \tag{126}$$

Performing the same calculations as done in proving (120) and making use of Lemma 13, (ii), we obtain

$$\left|\left|\left|r_{\Delta t,\Delta s}(t,\cdot)\right|\right|\right| \le \mathbf{c}\mathbf{k}(t), \tag{127}$$

where

$$\mathbf{k}(t) := \left|\left|\left|\Lambda_{\Delta t,\Delta s}(t,\cdot)\right|\right|\right|. \tag{128}$$

Applying this inequality and Lemma 14 to (125), we deduce that

$$\left|\left|\left|\mathbf{R}_{\Delta t,\Delta s}(t,\cdot)\right|\right|\right| \le \mathbf{b}(\Delta s + \Delta t) + \mathbf{c}\mathbf{k}(t). \tag{129}$$

Let us solve (123) with respect to (124) by considering (125) as a perturbation term. We perform the same calculations as done in proving Gronwall's lemma. Replace t by r in (123), multiply both sides by

$$\exp\left(-\int_0^r M_{\Delta t}(s,x)\,ds\right), \tag{130}$$

and integrate both sides with respect to $r \in [0,t]$. Applying Lemma 13, (viii), and (129) to the equality thus obtained and noting that $\Lambda_{\Delta t,\Delta s}(0,x) = 0$ (see (111)), we can easily obtain an integral inequality for $\mathbf{k}(t)$. Consider this integral inequality as a differential inequality whose unknown function is

$$\mathbf{K} = \mathbf{K}(t) := \int_0^t \mathbf{k}(r)\,e^{-\mathbf{a}r}\,dr. \tag{131}$$

Solve this differential inequality with respect to $\mathbf{K} = \mathbf{K}(t)$. Applying the inequality thus solved to the integral inequality for $\mathbf{k} = \mathbf{k}(t)$, we obtain

$$\left|\left|\left|\lambda_{\Delta t}(t,\cdot) - \lambda_{\Delta s}(t,\cdot)\right|\right|\right| \le \mathbf{r}(t)(\Delta t + \Delta s), \tag{132}$$

where

$$\mathbf{r}(t) := \left(\frac{\mathbf{b}}{(\mathbf{a}+\mathbf{c})}\right)\left\{\exp\left((\mathbf{a}+\mathbf{c})t\right) - 1\right\}. \tag{133}$$

Applying this inequality and Proposition 11 with $(\lambda_1(x), \lambda_2(x)) = (\lambda_{\Delta t}(t,x), \lambda_{\Delta s}(t,x))$ to (115), we deduce that

$$\begin{aligned} \left\|w_{\Delta t}(t,\cdot) - w_{\Delta s}(t,\cdot)\right\| &\le \gamma_1(\mu,\sigma,\mathbf{C})\,\mathbf{r}(t)(\Delta t + \Delta s), \\ \left\|\omega_{\Delta t}(t,\cdot) - \omega_{\Delta s}(t,\cdot)\right\| &\le \gamma_2(\mu,\sigma,\mathbf{C})\,\mathbf{r}(t)(\Delta t + \Delta s). \end{aligned} \tag{134}$$

From (132)–(134), we obtain the present lemma. □

Lemma 16. *The limit*

$$\begin{aligned} &(\lambda(t,x), \omega(t,x), w(t,x)) \\ &\quad := \lim_{\Delta t\to 0+0}(\lambda_{\Delta t}(t,x), \omega_{\Delta t}(t,x), w_{\Delta t}(t,x)) \end{aligned} \tag{135}$$

is a global solution and satisfies (24), (25), and (27)–(32).

Proof. Making use of Lemmas 13 and 15, we deduce that (135) satisfies (25) and (28)–(32). Recalling (115), we can substitute (122) in (3) and (10). Let $\Delta t \to 0+0$ in the equalities thus obtained. By Lemma 15, we see easily that (135) satisfies (3) and (10). Replace t by r in (116). Multiply both sides by (130) and integrate both sides with respect to $r \in [0,t]$. We obtain an integral equation whose unknown function is $\lambda_{\Delta t} = \lambda_{\Delta t}(t,x)$. Apply (111) and Lemmas 14 and 15 to this integral equation, recall (12) and (13), and let $\Delta t \to 0+0$. We deduce that (135) satisfies the following equality:

$$\lambda(t,x) = \lambda^0(x)\exp\left(\int_0^t M(\lambda(s,\cdot), \omega(s,\cdot); x)\,ds\right). \tag{136}$$

Hence, (135) satisfies (11) and (15) (see (14)). Applying (32) to (136), we see that (135) satisfies (24) and (27). □

The following lemma gives necessary conditions for the initial value problem to have a solution that belongs to (17).

Lemma 17. *If the initial value problem has a solution*

$$(\lambda, \omega, w) = (\lambda(t, x), \omega(t, x), w(t, x)) \quad (137)$$

in (17) for some $T > 0$, then this solution satisfies (23), (25), (26), and (28)–(32) in $[0, T] \times D$, where $\delta_i(\mu, \sigma, \mathbf{C}) := \mathbf{a}\gamma_i(\mu, \sigma, \mathbf{C})$, $i = 1, 2$ (see Proposition 11).

Proof. Recalling (12), (13), and the definition of (17), we see easily that

$$\omega(t, x) \in L^\infty([0, T] \times D), \qquad m(t) \in L^\infty([0, T]), \quad (138)$$

$$\lambda(t, x) \in L^{\infty,1}{}_{0+}([0, T] \times D), \quad (139)$$

where

$$m(t) := m(\lambda(t, \cdot), \omega(t, \cdot)). \quad (140)$$

Hence, integrating both sides of (11) with respect to $x \in D$ and recalling (12) and (13), we see easily that

$$\frac{dn(t)}{dt} = m(t)(1 - n(t)), \quad (141)$$

where

$$n(t) := |||\lambda(t, \cdot)|||. \quad (142)$$

Making use of (138) and (139), we can transform (11) into (136). Applying (138) and Assumption 2, (ii), to (136), we obtain

$$n(0 + 0) = 1. \quad (143)$$

Solving (141) with this initial condition, we see that $\lambda = \lambda(t, x)$ satisfies (28) for each $t \in [0, T]$. Making use of this result and (139), we can apply Propositions 6, 9, and 11 to (137). Applying Propositions 6 and 9 to (137), we see that (137) satisfies (25) and (29)–(32) for each $(t, x) \in [0, T] \times D$. Integrate both sides of (11) with respect to $t \in [r, s]$, calculate the absolute values of both sides, and integrate them with respect to $x \in D$. Applying (28) and (32) to the right-hand side of the equality thus obtained, we see that (137) satisfies (23) for each $r, s \in [0, T]$. Combining this result and Proposition 11, we see that (137) satisfies (26) for each $r, s \in [0, T]$. □

Proof of Theorem 4. Let

$$(\lambda_1, \omega_1, w_1) = (\lambda_1(t, x), \omega_1(t, x), w_1(t, x)) \quad (144)$$

be a solution that belongs to (17). Substitute this solution and (135) in (11). Subtracting both sides of the equalities thus obtained from each other, we obtain the following equation in the same way as (123):

$$\left(\frac{\partial}{\partial t}\right)\Lambda(t, x) = M(t, x)\Lambda(t, x) + \mathbf{r}(t, x), \quad (145)$$

where

$$\begin{gathered}\Lambda(t, x) := \lambda(t, x) - \lambda_1(t, x), \\ \mathbf{r}(t, x) := (M(t, x) - M_1(t, x))\lambda_1(t, x), \\ M(t, x) := M(\lambda(t, \cdot), \omega(t, \cdot); x), \\ M_1(t, x) := M(\lambda_1(t, \cdot), \omega_1(t, \cdot); x).\end{gathered} \quad (146)$$

Note that both (135) and (144) satisfy (25) and (28)–(32) (see Lemmas 16 and 17). Making use of this result, we can perform the same calculations as done in proving (127). Hence we obtain

$$|||\mathbf{r}(t, \cdot)||| \le \mathbf{c}\,|||\Lambda(t, \cdot)|||. \quad (147)$$

Making use of this inequality in place of (129) and performing the same calculations as done in proving (132)–(134), we prove that (144) is the same as (135). By Lemmas 16 and 17 we prove Theorem 4. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Existence of Solutions of a Partial Integrodifferential Equation with Thermostat and Time Delay

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This paper deals with the mathematical analysis of a retarded partial integrodifferential equation that belongs to the class of thermostatted kinetic equations with time delay. Specifically, the paper is devoted to the proof of the existence and uniqueness of mild solutions of the related Cauchy problem. The main result is obtained by employing integration along the characteristic curves and successive approximations sequence arguments. Applications and perspective are also discussed within the paper.

1. Introduction

The derivation and analysis of mathematical frameworks have recently gained much attention in the applied sciences and specifically for the modeling of complex phenomena occurring in biology, chemistry, vehicular traffic, crowd/swarm dynamics, economics, and social systems. At the origin of complex dynamics, there are interactions that occur in nonlinear fashion and randomly among the elements (particles, cells, and pedestrians) composing the complex system [1]. Complex systems are also characterized by emergent properties, which are properties of the system as a whole which do not exist at the individual element level.

Different ordinary differential equations and partial differential equations have been derived with the aim of obtaining an accurate description of complex phenomena. Moreover, thermostatted kinetic frameworks have been developed for the mathematical modeling of complex systems in physics and life sciences [2–5]. The goal of the thermostatted kinetic models is the possibility of modeling the interactions among the particles at the microscopic scale. In particular, these frameworks allow modeling the ability of the particles to express strategies.

The present paper deals with further developments of the thermostatted kinetic theory proposed in [6]. Specifically, this paper is devoted to the mathematical analysis of a partial integrodifferential equation with thermostat and time delay that belongs to the class of the thermostatted kinetic theory frameworks. The paper focuses on the proof of the existence and uniqueness of mild solutions of the related Cauchy problem. The main result is obtained by employing integration along the characteristic curves and successive approximations sequence arguments. Applications and perspective are also discussed within the paper.

The time delay is introduced in order to take into account the fact that most of the emerging behaviours occurring in complex systems at a certain time are strictly related to the interactions among the particles of the system at a previous time. In the pertinent literature, mathematical models with time delays have been proposed only in ODE-based models; see, among others, [7–16]. The introduction of the time delay has provoked the onset of fluctuations and Hopf bifurcation; see [17–20].

It is worth stressing that, to the best of our knowledge, this is the first time that time delay is introduced into a thermostatted partial integrodifferential equation (kinetic).

The contents of the present paper are developed through four more sections, which follow this introduction. In detail, Section 2 deals with the retarded partial integrodifferential equation and the related Cauchy problem; Section 3 is concerned with some preliminary results that are needed for the proof of the main result that is outlined in Section 4. Finally, Section 5 is devoted to a critical analysis of the proposed mathematical equation including research perspective and applications.

2. The Retarded Integrodifferential Equation

This paper is devoted to the result about the existence and uniqueness of a solution of the following Cauchy problem:

$$\begin{aligned} &\partial_t f(t,u) + F\partial_u \left(f(t,u) \left(1 - u \int_{D_u} u f(t,u)\, du \right)\right) \\ &\quad = \eta J[f, f_\tau](t,u), \\ &f(0,u) = f_0(u), \end{aligned} \tag{1}$$

where $f = f(t,u) : [0,+\infty) \times D_u \to \mathbb{R}^+$ is the unknown function, $f_\tau = f(t-\tau,u) : [-\tau,+\infty)\times D_u \to \mathbb{R}^+$, $F, \eta \in \mathbb{R}^+$, and $J[f, f_\tau] = J[f, f_\tau](t,u)$ is the following integral operator:

$$\begin{aligned} J[f, f_\tau] = &\int_{D_u\times D_u} \mathscr{A}(u_*, u^*, u)\, f(t, u_*) \\ &\times f(t-\tau, u^*)\, du_* du^* \\ &- f(t,u) \int_{D_u} f(t-\tau, u^*)\, du^*, \end{aligned} \tag{2}$$

with $\mathscr{A}(u_*, u^*, u) : D_u \times D_u \times D_u \to \mathbb{R}^+$, τ the time delay, and f_0 the initial datum.

In the partial integrodifferential equation with time delay

$$\begin{aligned} &\partial_t f(t,u) + F\partial_u \left(f(t,u) \left(1 - u \int_{D_u} u f(t,u)\, du \right)\right) \\ &\quad = \eta J[f, f_\tau](t,u), \end{aligned} \tag{3}$$

$J[f, f_\tau](t,u) = G[f, f_\tau](t,u) - L[f, f_\tau](t,u)$ is the conservative interaction operator, which is splitted into the gain (of particles into state u) operator $G[f, f_\tau] = G[f, f_\tau](t,u)$ and the loss (of particles into state u) operator $L[f, f_\tau] = L[f, f_\tau](t,u)$:

$$\begin{aligned} &G[f, f_\tau] \\ &\quad = \int_{D_u\times D_u} \mathscr{A}(u_*, u^*, u)\, f(t, u_*)\, f(t-\tau, u^*)\, du_* du^*, \end{aligned} \tag{4}$$

$$L[f, f_\tau] = f(t,u) \int_{D_u} f(t-\tau, u^*)\, du^*. \tag{5}$$

Bearing all of the above in mind, and under suitable integrability assumptions on f, the pth-order moment of f is defined as follows:

$$\mathbb{E}_p[f](t) = \int_{D_u} u^p f(t,u)\, du, \quad p \in \mathbb{N}. \tag{6}$$

In particular, the zero-order, first-order, and second-order moments represent the *density* (mass), *mean activation* (linear momentum), and *activation energy* (kinetic energy), respectively. The term

$$\mathscr{T}_F[f] := F\partial_u \left(f(t,u) \left(1 - u \int_{D_u} u f(t,u)\, du \right)\right) \tag{7}$$

is a damping operator that allows the control of the activation energy. This term is based on the Gaussian isokinetic thermostat (the interested reader is referred, among others, to [21–23]).

In what follows, we assume that $f(t,u) = 0$ for $(t,u) \in [-\tau, 0] \times D_u$ (initial function).

3. Preliminary Results

This section is concerned with some preliminary results that are at the basis of the main result of the present paper.

Lemma 1. *The gain operator* (4) *satisfies, for all functions f and g, the following identity:*

$$G[f, f_\tau] - G[g, g_\tau] = G[f-g, f_\tau] + G[g, f_\tau - g_\tau]. \tag{8}$$

Proof. It is obtained by straightforward calculations. □

The main result is based on the following assumptions on the probability density function $\mathscr{A}$.

(A_1) The probability density function $\mathscr{A}$ satisfies, for all $u_*, u^* \in D_u$, the following identity:

$$\int_{D_u} \mathscr{A}(u_*, u^*, u)\, du = 1, \tag{9}$$

which models the conservation of particles.

(A_2) The probability density function $\mathscr{A}$ is an even function with respect to u; then, in particular,

$$\int_{D_u} u\mathscr{A}(u_*, u^*, u)\, du = 0. \tag{10}$$

(A_3) The probability density function $\mathscr{A}$ satisfies, for all $u_*, u^* \in D_u$, the following identity:

$$\int_{D_u} u^2\mathscr{A}(u_*, u^*, u)\, du = u_*^2. \tag{11}$$

Lemma 2. *If the function $\mathscr{A}$ satisfies assumptions* (9), (10), *and* (11), *then*

$$\begin{aligned} &\int_{D_u} G[f, f_\tau](t,u)\, du \\ &\quad = \left(\int_{D_u} f(t,u)\, du \right)\left(\int_{D_u} f(t-\tau,u)\, du \right), \end{aligned} \tag{12}$$

$$\int_{D_u} u^{2p+1} G[f, f_\tau](t,u)\, du = 0, \quad p \in \mathbb{N}, \tag{13}$$

$$\begin{aligned} &\int_{D_u} u^2 G[f, f_\tau](t,u)\, du \\ &\quad = \left(\int_{D_u} u^2 f(t,u)\, du \right)\left(\int_{D_u} f(t-\tau,u)\, du \right). \end{aligned} \tag{14}$$

Proof. Condition (9) implies

$$\begin{aligned}\int_{D_u} G[f,f_\tau](t,u)\,du &= \int_{D_u}\left(\int_{D_u\times D_u}\mathscr{A}(u_*,u^*,u)\,f(t,u_*)\right.\\ &\qquad\left.\times f(t-\tau,u^*)\,du_*du^*\right)du\\ &= \int_{D_u\times D_u}\left(\int_{D_u}\mathscr{A}(u_*,u^*,u)\,du\right)\\ &\qquad\times f(t,u_*)\,f(t-\tau,u^*)\,du_*du^*\\ &= \left(\int_{D_u} f(t,u_*)\,du^*\right)\left(\int_{D_u} f(t-\tau,u^*)\,du^*\right).\end{aligned} \tag{15}$$

Bearing in mind condition (10), we have

$$\begin{aligned}\int_{D_u} u^{2p+1}G[f,f_\tau](t,u)\,du &= \int_{D_u} u^{2p+1}\left(\int_{D_u\times D_u}\mathscr{A}(u_*,u^*,u)\,f(t,u_*)\right.\\ &\qquad\left.\times f(t-\tau,u^*)\,du_*du^*\right)du\\ &= \int_{D_u\times D_u}\left(\int_{D_u} u^{2p+1}\mathscr{A}(u_*,u^*,u)\,du\right)\\ &\qquad\times f(t,u_*)\,f(t-\tau,u^*)\,du_*du^* = 0.\end{aligned} \tag{16}$$

Finally, condition (11) implies

$$\begin{aligned}\int_{D_u} u^2G[f,f_\tau](t,u)\,du &= \int_{D_u} u^2\left(\int_{D_u\times D_u}\mathscr{A}(u_*,u^*,u)\,f(t,u_*)\right.\\ &\qquad\left.\times f(t-\tau,u^*)\,du_*du^*\right)du\\ &= \int_{D_u\times D_u}\left(\int_{D_u} u^2\mathscr{A}(u_*,u^*,u)\,du\right)\\ &\qquad\times f(t,u_*)\,f(t-\tau,u^*)\,du_*du^*\\ &= \int_{D_u\times D_u} u_*^2 f(t,u_*)\,f(t-\tau,u^*)\,du_*du^*\\ &= \left(\int_{D_u} u_*^2 f(t,u_*)\,du_*\right)\left(\int_{D_u} f(t-\tau,u^*)\,du^*\right).\end{aligned} \tag{17}$$

Therefore, the proof of the lemma is concluded. □

Bearing the previous lemma in mind, the evolution equation for the 1st-order moment of f is stated in the following results.

Theorem 3. *Assume that assumptions* (9), (10), *and* (11) *hold. If there exists a nonnegative solution f of the partial integrodifferential equation* (3) *such that $f(t,u)=0$ as $u\in\partial D_u$, then the 1st-order moment $\mathbb{E}_1[f](t)$ is solution of the following delayed first-order nonlinear ordinary differential equation:*

$$\begin{aligned}\frac{d}{dt}\mathbb{E}_1[f](t) &= F\left[\mathbb{E}_0[f](t)-(\mathbb{E}_1[f](t))^2\right]\\ &\quad-\eta\mathbb{E}_1[f](t)\,\mathbb{E}_0[f](t-\tau).\end{aligned} \tag{18}$$

Proof. The integral operator $J[f,f_\tau]$ can be written as follows:

$$\begin{aligned}J[f,f_\tau](t,u) &= G[f,f_\tau](t,u)\\ &\quad- f(t,u)\int_{D_u} f(t-\tau,u^*)\,du^*.\end{aligned} \tag{19}$$

Multiplying both sides of $J[f,f_\tau]$ by u, integrating over D_u, and considering (10), we have

$$\begin{aligned}\int_{D_u} uJ[f,f_\tau](t,u)\,du &= -\left(\int_{D_u} uf(t,u)\,du\right)\left(\int_{D_u} f(t-\tau,u)\,du\right)\\ &= -\mathbb{E}_1[f](t)\,\mathbb{E}_0[f](t-\tau).\end{aligned} \tag{20}$$

Since

$$\begin{aligned}\int_{D_u} u\partial_u\left(f(t,u)\left(1-u\int_{D_u} uf(t,u)\,du\right)\right)du\\ = (\mathbb{E}_1[f](t))^2-\mathbb{E}_0[f](t),\end{aligned} \tag{21}$$

then we have the proof. □

Corollary 4. *Assume that assumptions* (9), (10), *and* (11) *hold. If there exists a nonnegative solution f of the Cauchy problem* (1) *such that*

(i) $\mathbb{E}_0[f](t)=\mathbb{E}_0[f](t-\tau)=1$,

(ii) $f(t,u)=0$ *as* $u\in\partial D_u$,

then the 1st-order moment $\mathbb{E}_1[f](t)$ reads as follows:

$$\begin{aligned}\beta(t) &:= \mathbb{E}_1[f](t)\\ &= \frac{\mathbb{E}_1^+\left(\mathbb{E}_1^- - \mathbb{E}_1^0\right)-\mathbb{E}_1^-\left(\mathbb{E}_1^+ - \mathbb{E}_1^0\right)e^{-(\sqrt{\eta^2+4F^2}/F)t}}{\left(\mathbb{E}_1^- - \mathbb{E}_1^0\right)-\left(\mathbb{E}_1^+ - \mathbb{E}_1^0\right)e^{-(\sqrt{\eta^2+4F^2}/F)t}},\end{aligned} \tag{22}$$

where

$$\begin{aligned}\mathbb{E}_1^\pm &= \frac{-\eta\pm\sqrt{\eta^2+4F^2}}{2F},\\ \mathbb{E}_1^0 &= \mathbb{E}_1[f](0) = \int_{D_u} uf_0\,du.\end{aligned} \tag{23}$$

Proof. The proof is obtained by coupling the delayed differential equation (18) with the initial condition $\mathbb{E}_1[f](0) = \mathbb{E}_1^0$. If $\mathbb{E}_1^0 = \mathbb{E}_1^+$ or $\mathbb{E}_1^0 = \mathbb{E}_1^-$, then we have $\mathbb{E}_1[f](t) = \mathbb{E}_1^0$ for all $t > 0$. Otherwise, the unique solution is function (22). □

Theorem 5. *Let p be an odd number and $t \geq 0$. Assume that assumptions (9), (10), and (11) hold. Then, the pth-order moment of f satisfies the following delayed ordinary differential equation:*

$$\begin{aligned}\frac{d}{dt}\mathbb{E}_p[f](t) &= -\mathbb{E}_p[f](t)\left[pF\mathbb{E}_1[f](t) + \eta\mathbb{E}_0[f](t-\tau)\right] \\ &\quad + pF\mathbb{E}_{p-1}[f](t).\end{aligned} \tag{24}$$

Proof. The proof follows by multiplying both sides of (3) by u^p, taking into account assumptions (9), (10), and (11), and performing integration by parts on the thermostat term. □

According to Corollary 4, (3) can be rewritten as follows:

$$\partial_t f + F(1 - u\beta(t))\partial_u f + (\eta - F\beta(t)) f = G[f, f_\tau](t,u), \tag{25}$$

and, after integrating along the characteristics, (25) reads as follows:

$$\frac{d}{dt} f_U + (\eta - F\beta(t)) f_U = G_U[f, f_\tau](t,u), \tag{26}$$

where

$$\begin{aligned} f_U(t,u) &:= f(t, U(t,u)), \\ G_U[f, f_\tau](t,u) &:= G[f, f_\tau](t, U(t,u)), \end{aligned} \tag{27}$$

being

$$\begin{aligned} U(t,u) = \varphi_t(u) &= ue^{-\lambda(t)} + Fe^{-\lambda(t)}\int_0^t e^{\lambda(s)} ds, \\ \lambda(t) &= F\int_0^t \beta(s)\, ds, \end{aligned} \tag{28}$$

$$u = \varphi_t^{-1}(U) = Ue^{\lambda(t)} - F\int_0^t e^{\lambda(s)} ds. \tag{29}$$

Bearing all of the above in mind, the integral form of (26) is

$$\begin{aligned} f_U(t,u) &= e^{-\Lambda(t)} f_U(0,u) \\ &\quad + e^{-\Lambda(t)}\int_0^t e^{\Lambda(\alpha)} G_U[f, f_\tau](\alpha, u)\, d\alpha, \\ &\qquad \forall t \in [0,T], \end{aligned} \tag{30}$$

where $\Lambda(t) = \int_0^t (\eta - F\beta(s)) ds = \eta t - \lambda(t)$.

4. Existence of Mild Solutions

Definition 6. A function f is said to be a mild solution to the Cauchy problem (1) on the time interval $[0,T]$ if $f(t,\cdot) \in L^1(D_u, du)$ and f is solution to the following integral equation:

$$f(t,u) = \Phi_{f_0}[f, f_\tau](t,u), \tag{31}$$

where

$$\begin{aligned} \Phi_{f_0}[f, f_\tau](t,u) &= e^{-\Lambda(t)} f_0\left(\varphi_t^{-1}(u)\right) \\ &\quad + e^{-\Lambda(t)}\int_0^t e^{\Lambda(\alpha)} G[f, f_\tau]\left(\alpha, \varphi_\alpha \circ \varphi_t^{-1}(u)\right) d\alpha. \end{aligned} \tag{32}$$

Lemma 7. *Let $\{f^{(n)}(t,u)\}_n$ be the following successive approximations sequence:*

$$\begin{aligned} f^{(1)}(t,u) &= 0, \\ f^{(n)}(t,u) &= \Phi_{f_0}\left[f^{(n-1)}, f_\tau^{(n-1)}\right](t,u), \quad n > 1, \end{aligned} \tag{33}$$

where f_0 is a nonnegative function such that $\mathbb{E}_0[f_0](t) = 1$. Then, $\{f^{(n)}(t,u)\}_n$ admits, as n goes to infinity, a nonnegative limit $f(t,\cdot) \in L^1(D_u, du)$ such that $\mathbb{E}_0[f](t) = 1$.

Proof. Since f_0 is a nonnegative function, then $f^{(n)}(t,u) > 0$, $\forall n \geq 1$. Moreover,

$$f^{(2)}(t,u) = e^{-\Lambda(t)} f_0\left(\varphi_t^{-1}(u)\right) \geq 0 = f^{(1)}(t,u), \tag{34}$$

and $\mathbb{E}_0[f^{(2)}](t) = e^{-\eta t} \leq 1$.

We now prove by induction that the sequence $\{f^{(n)}(t,\cdot)\}_n$ is monotone, and specifically

$$f^{(n)}(t,u) \geq f^{(n-1)}(t,u), \quad \forall u \in D_u,\ \forall n \geq 1, \tag{35}$$

and $\mathbb{E}_0[f](t) \leq 1$.

Assume as the induction hypothesis that, for some $n \geq 3$, we have $f^{(n-1)} \geq f^{(n-2)}$ and $\mathbb{E}_0[f^{(n-1)}](t) \leq 1$. Then,

$$\begin{aligned} f^{(n)} - f^{(n-1)} &= e^{-\Lambda(t)}\int_0^t e^{\Lambda(\alpha)}\left(G\left[f^{(n-1)}, f_\tau^{(n-1)}\right] - G\left[f^{(n-2)}, f_\tau^{(n-2)}\right]\right) \\ &\qquad \times \left(\alpha, \varphi_\alpha \circ \varphi_t^{-1}(u)\right) d\alpha. \end{aligned} \tag{36}$$

Taking into account property (8), equation (36) thus reads as follows:

$$\begin{aligned} e^{-\Lambda(t)}\int_0^t e^{\Lambda(\alpha)} &\left(G\left[f^{(n-1)} - f^{(n-2)}, f_\tau^{(n-1)}\right]\right. \\ &\left. + G\left[f^{(n-2)}, f_\tau^{(n-1)} - f_\tau^{(n-2)}\right]\right) d\alpha, \end{aligned} \tag{37}$$

and by using the induction hypothesis we conclude that the sequence $\{f^{(n)}(t,\cdot)\}_n$ is monotone. Moreover,

$$\int_{D_u} f^{(n)}(t,u)\,du = e^{-\eta t} + e^{-\eta t}\int_0^t e^{\eta\alpha}\int_{D_u} G\left[f^{(n-1)}, f_\tau^{(n-1)}\right](\alpha,u)\,d\alpha\,du. \tag{38}$$

Taking into account property (12), we have

$$\mathbb{E}_0\left[f^{(n)}\right](t) = e^{-\eta t} + \eta e^{-\eta t}\int_0^t e^{\eta\alpha}\mathbb{E}_0\left[f^{(n-1)}\right](\alpha) \times \mathbb{E}_0\left[f^{(n-1)}\right](\alpha-\tau)\,d\alpha, \tag{39}$$

and by using the induction hypothesis we have

$$\mathbb{E}_0\left[f^{(n)}\right](t) \le e^{-\eta t} + \eta e^{-\eta t}\int_0^t e^{\eta\alpha}d\alpha = 1. \tag{40}$$

Bearing all of the above in mind, we conclude that the sequence $\{f^{(n)}(t,\cdot)\}_n$ has a nonnegative limit $f(t,\cdot)\in L^1(D_u)$ such that $f^{(n)} \to f$, as $n\to\infty$. Then, the Levi theorem implies that f satisfies the following equation:

$$\mathbb{E}_0[f](t) = e^{-\eta t} + \eta e^{-\eta t}\int_0^t e^{\eta\alpha}\mathbb{E}_0[f](\alpha)\,\mathbb{E}_0[f](\alpha-\tau)\,d\alpha, \tag{41}$$

whose unique solution is $\mathbb{E}_0[f](t)=1$. Therefore, the lemma is completely proved. □

The main result of this paper is the following.

Theorem 8. *Let f_0 be a given nonnegative function such that $\mathbb{E}_0[f_0]=1$. Then, there exists a unique nonnegative mild solution*

$$f\in C\left((0,\infty);L^1(D_u,du)\right) \tag{42}$$

to the Cauchy problem (1).

Proof. Since Lemma 7 states that f solves (31), in order to prove that f is a mild solution of (1), it is enough to show that $\mathbb{E}_1[f](t)=\beta(t)$. In order to prove that $\mathbb{E}_1[f](t)=\beta(t)$, we consider the following successive approximations sequence:

$$\begin{aligned} g^{(1)}(t,u) &= f_0(u), \\ g^{(n)}(t,u) &= \Phi_{f_0}\left[g^{(n-1)}, g_\tau^{(n-1)}\right](t,u), \quad n>1, \end{aligned} \tag{43}$$

or the following equivalent form of (43):

$$e^{\Lambda(t)}g^{(n)}(t,\varphi_t(u)) = f_0(u) + \int_0^t e^{\Lambda(\tau)}G\left[g^{(n-1)}, g_\tau^{(n-1)}\right](\alpha,\varphi_\alpha(u))\,d\alpha. \tag{44}$$

The assumption on f_0 implies that the zero-order moment $\mathbb{E}_0[g^{(1)}](t)=1$. Assume now as induction hypothesis that $\mathbb{E}_0[g^{(n-1)}](t)=1$, for some $n\ge 2$. Integrating both sides of (44) over D_u with respect to u, and using (28), we obtain

$$e^{\eta t}\int_{D_u} g^{(n)}(t,u)\,du = \int_{D_u} f_0(u)\,du + \int_0^t e^{\eta\alpha}\int_{D_u} G\left[g^{(n-1)}, g_\tau^{(n-1)}\right](\alpha,u)\,d\alpha\,du. \tag{45}$$

Taking into account property (12) and by using the induction hypothesis, the right-hand side of (45) thus reads as follows:

$$e^{\eta t}\int_{D_u} g^{(n)}(t,u)\,du = \int_{D_u} f_0(u)\,du + \eta\int_0^t e^{\eta\alpha}d\alpha = e^{\eta t}. \tag{46}$$

Therefore, for all $t\ge 0$, we have $\mathbb{E}_0[g^{(n)}](t)=1$.

Multiplying both sides of (44) by u, and integrating over D_u with respect to u, we have

$$\begin{aligned}\int_{D_u} ug^{(n)}(t,u)\,du &= e^{-\Lambda(t)}\int_{D_u}\int_0^t e^{\Lambda(\alpha)}uG\left[g^{(n-1)}, g_\tau^{(n-1)}\right] \times \left(\alpha,\varphi_\alpha\circ\varphi_t^{-1}(u)\right)d\alpha\,du \\ &\quad + e^{-\Lambda(t)}\int_{D_u} uf_0\left(\varphi_t^{-1}(u)\right)du.\end{aligned} \tag{47}$$

Taking into account (12), (13) and repeating the computations developed in [6], it is easy to prove by induction on n that

$$\mathbb{E}_1\left[g^{(n)}\right](t) = \int_{D_u} ug^{(n)}(t,u)\,du = \beta(t). \tag{48}$$

Multiplying both sides of (44) by u^2 and integrating over D_u with respect to u, we have

$$e^{\Lambda(t)}\int_{D_u} u^2g^{(n)}(t,\varphi_t(u))\,du = \int_{D_u}\int_0^t e^{\Lambda(\alpha)}u^2G\left[g^{(n-1)}, g_\tau^{(n-1)}\right](\alpha,\varphi_\alpha(u))\,d\alpha\,du + \int_{D_u} u^2f_0(u)\,du, \tag{49}$$

and according to [6] it is easy to prove by induction on n that

$$\mathbb{E}_2\left[g^{(n)}\right](t) = \int_{D_u} u^2g^{(n)}(t,u)\,du = 1. \tag{50}$$

Let $\sum = \{u \in D_u : f(t,u) \geq g^{(n)}(t,u)\}$. Then,

$$\int_{D_u} \left| f - g^{(n)} \right| du = 2 \int_{\Sigma} \left(f - g^{(n)} \right) du - \int_{D_u} \left(f - g^{(n)} \right) du. \tag{51}$$

Since $\mathbb{E}_0[f](t) = \mathbb{E}_0[g^{(n)}](t) = 1$ and by construction $f^n \leq g^n$, we have

$$\int_{D_u} \left| f - g^{(n)} \right| du = 2 \int_{\Sigma} \left(f - g^{(n)} \right) du \leq 2 \int_{D_u} \left(f - f^{(n)} \right) du \longrightarrow 0, \quad n \longrightarrow \infty. \tag{52}$$

Therefore, $g^{(n)} \rightarrow f$ in $L^1(D_u, du)$, and since $\mathbb{E}_2[g^{(n)}](t)$ is bounded, then $\mathbb{E}_1[f](t) = \beta(t)$.

We now prove the uniqueness of the solution. Let $\overline{f}(t,u)$ be any solution to (31). The positivity of the operators G and Φ_{f_0} implies that, for all n, $f^{(n)}(t,u) \leq \overline{f}(t,u)$, and then $f(t,u) \leq \overline{f}(t,u)$. Since $\mathbb{E}_0[f](t) = \mathbb{E}_0[\overline{f}](t) = 1$, we thus have $\overline{f} = f$. □

Corollary 9. *Let f_0 be a given nonnegative function such that*

(i) $\mathbb{E}_0[f_0] = 1$;

(ii) $\int_{D_u} |u|^3 f_0(u)\, du < \infty$.

Then, the mild solution f of the Cauchy problem (1) *belongs to $\mathscr{M}(D_u)$ where*

$$\mathscr{M}(D_u) = \{ f = f(t,u) : [0,\infty) \times D_u \longrightarrow \mathbb{R}^+ \text{ such that } \mathbb{E}_0[f](t) = \mathbb{E}_2[f](t) = 1 \}. \tag{53}$$

Proof. It is sufficient to remember (24) and note that assumption (ii) implies that

$$\int_{D_u} |u|^3 g^{(n)}(t,u)\, du < \infty, \tag{54}$$

which allows us to conclude that $E_2[f](t) = 1$. □

5. Applications and Research Perspectives

The main goal of the present paper refers to the proof of the global existence of mild solutions of a thermostatted partial integrodifferential equation with time delay. As already mentioned in Section 2, this equation can be proposed for the modeling of complex systems in nature and society where only the interactions at the microscopic scales are affected by time delay. Specifically, the partial integrodifferential equation with time delay

$$\partial_t f(t,u) = F\partial_u \left(f(t,u) \left(1 - u \int_{D_u} uf(t,u)\, du \right) \right) = \eta J[f, f_r](t,u) \tag{55}$$

can be proposed as a general mathematical framework for the modeling of complex systems composed by a large number of interacting particles and subjected to the external force field F. The overall state of the system is described by the distribution function f (statistical description). Particles are able to express a specific function; this ability of the particles is modeled by the variable $u \in D_u \subseteq \mathbb{R}$. Moreover, η is the encounter rate between particles with states u_* (or u) and u^*. Finally, $\mathscr{A}$ is the probability density that a particle with state u_* ends up into the state u after the interaction with the particle with state u^*. The action of the external force field is controlled by the thermostat term that, as shown in Section 4, maintains constant the first-order and the second-order moments (number density and activation energy of the system).

From the applications viewpoint, we consider a simple model for the evolution of malignancy in tumor cells. Specifically, we assume that the variable u models the magnitude of the malignancy of tumor cells and $\mathscr{A}$ is a delta Dirac function (deterministic output $m(u_*, u^*)$ of a pair interaction) depending on the microscopic state of the interacting particles:

$$\mathscr{A}(u_*, u^*, u) = \delta(u - m(u_*, u^*)), \tag{56}$$

and finally we assume that the malignancy of tumor cells increases when cells interact with each other with rate η. Accordingly, we have

$$m(u_*, u^*) = u_* + \epsilon, \tag{57}$$

where ϵ is a positive parameter. Bearing all of the above in mind, the integral operator $J[f, f_\tau]$ reads as follows:

$$J[f, f_\tau] = [f(t, u - \epsilon) - f(t,u)] \int_{D_u} f(t - \tau, u)\, du, \tag{58}$$

and then delayed equation (55) now reads as follows:

$$\partial_t f(t,u) + F\partial_u \left(f(t,u) \left(1 - u \int_{D_u} uf(t,u)\, du \right) \right) = \eta [f(t, u-\epsilon) - f(t,u)] \int_{D_u} f(t-\tau, u)\, du. \tag{59}$$

However, thermostatted equation (55) does not include the role of the space and velocity variables; then, applications refer to the modeling of complex phenomena that are homogeneous in space and velocity. The mathematical analysis performed in the present paper has to be thus generalized for taking also into account the dynamics described by these variables. Moreover, (55) refers to complex systems where the mutual interactions do not produce modification in the number density (conservative interactions).

Research perspectives include the possibility of performing an asymptotic analysis by parabolic (low-field) and hyperbolic (high-field) scalings (see [24–34]) with the aim of obtaining the dynamics of the system at the macroscopic scale. This is a work in progress and results will be presented in due course.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Krasnosel'skii Type Hybrid Fixed Point Theorems and Their Applications to Fractional Integral Equations

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Some hybrid fixed point theorems of Krasnosel'skii type, which involve product of two operators, are proved in partially ordered normed linear spaces. These hybrid fixed point theorems are then applied to fractional integral equations for proving the existence of solutions under certain monotonicity conditions blending with the existence of the upper or lower solution.

1. Introduction

Recently, Nieto and Rodríguez-López [1] proved the following hybrid fixed point theorem for the monotone mappings in partially ordered metric spaces using the mixed arguments from algebra and geometry.

Theorem 1 (Nieto and Rodríguez-López [1]). *Let $(X, \preceq)$ be a partially ordered set and suppose that there is a metric d in X such that (X, d) is a complete metric space. Let $T : X \to X$ be a monotone non-decreasing mapping such that there exists a constant $k \in (0, 1)$ such that*

$$d(Tx, Ty) \leqq kd(x, y) \tag{1}$$

for all comparable elements $x, y \in X$. Assume that either T is continuous or X is such that if $\{x_n\}$ is a non-decreasing sequence with $x_n \to x$ in X, then

$$x_n \leqq x \quad (n \in \mathbb{N} := \{1, 2, 3, \ldots\}). \tag{2}$$

Further, if there is an element $x_0 \in X$ satisfying $x_0 \preceq Tx_0$, then T has a fixed point which is unique if "every pair of elements in X has a lower and an upper bound."

Another fixed point theorem in the above direction can be stated as follows.

Theorem 2 (Nieto and Rodríguez-López [1]). *Let $(X, \preceq)$ be a partially ordered set and suppose that there is a metric d in X such that (X, d) is a complete metric space. Let $T : X \to X$ be a monotone non-decreasing mapping such that there exists a constant $k \in (0, 1)$ such that (1) satisfies for all comparable elements $x, y \in X$. Assume that either T is continuous or X is such that if $\{x_n\}$ is a non-decreasing sequence with $x_n \to x$ in X, then*

$$x_n \geqq x \quad (n \in \mathbb{N}). \tag{3}$$

Further, if there is an element $x_0 \in X$ satisfying $x_0 \succeq Tx_0$, then T has a fixed point which is unique if "every pair of elements in X has a lower and an upper bound."

Remark 3. If we suppose that $d(a, c) \geqq d(b, c)$ $(a \leqq b \leqq c)$ and $\{x_n\} \to x$ is a sequence in X whose consecutive terms are comparable, then there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that every term comparable to the limit x implies the conditions (2) and (3), since (in the monotone case) the existence of a subsequence whose terms are comparable with the limit is equivalent to saying that all the terms in the sequence are also comparable with the limit.

Taking Remark 3 into account, the results discussed by Nieto and Rodríguez-López and the fact that, in conditions

$\{x_n\} \rightarrow x$, there is a sequence in X whose consecutive terms are comparable, there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that every term comparable to the limit x implies the validity of the conditions (2) and (3). Here the key is that the terms in the sequence (starting at a certain term) are comparable to the limit. Nieto and Rodríguez-López [2] obtained the following results, which improve Theorems 1 and 2.

Theorem 4 (Nieto and Rodríguez-López [2]). *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $T : X \rightarrow X$ be a monotone function (non-decreasing or non-increasing) such that there exists $k \in [0, 1)$ with*

$$d(T(x), T(y)) \leqq kd(x, y) \quad (x \geqq y). \qquad (1')$$

Suppose that either T is continuous or X is such that if $x_n \rightarrow x$ is a sequence in X whose consecutive terms are comparable, then there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that every term comparable to the limit x. If there exists $x_0 \in X$ with $x_0 \leqq T(x_0)$ or $x_0 \geqq T(x_0)$, then T has a fixed point which is unique if "every pair of elements in X has a lower and an upper bound."

After the publication of the above fixed point theorems, there is a huge upsurge in the development of the metric fixed point theory in partially ordered metric spaces. A good number of fixed and common fixed point theorems have been proved in the literature for two, three, and four mappings in metric spaces by suitably modifying the contraction condition (1) as per the requirement of the results. We claim that almost all the results proved so far along this line, though not mentioned here, have their origin in a paper due to Heikkilä and Lakshmikantham [3]. The main difference is the convergence criteria of the sequence of iterations of the monotone mappings under consideration. The convergence of the sequence in Heikkilä and Lakshmikantham [3] is straightforward, whereas the convergence of the sequence in Nieto and Rodríguez-López [1, 2] is due mainly to the metric condition of contraction. The hybrid fixed point theorem of Heikkilä and Lakshmikantham [3] for the monotone mappings in ordered metric spaces is as follows.

Theorem 5 (Heikkillä and Lakshmikantham [3]). *Let $[a, b]$ be an order interval in a subset Y of the ordered metric space X and let $G : [a, b] \rightarrow [a, b]$ be a non-decreasing mapping. If the sequence $\{Gx_n\}$ converges in Y whenever $\{x_n\}$ is a monotone sequence in $[a, b]$, then the well-ordered chain of G-iterations of a has the maximum x^* which is a fixed point of G. Moreover,*

$$x^* = \max\{y \in [a, b] \mid y \leqq Gy\}. \qquad (4)$$

The above hybrid fixed point theorem is applicable in the study of discontinuous nonlinear equations and has been used throughout the research monograph of Heikkillä and Lakshmikantham [3]. We also claim that the convergence of the monotone sequence in Theorem 5 is replaced in Theorem 4 by the Cauchy sequence $\{x_n\}$ and completeness of X. Further, the Cauchy non-decreasing sequence is replaced by the equivalent contraction condition for comparable elements in X. Theorem 4 is the best hybrid fixed point theorem because it is derived for the mixed arguments from algebra and geometry. The main advantage of Theorem 4 is that the uniqueness of the fixed point of the monotone mappings is obtained under certain additional conditions on the domain space such as lattice structure of the partially ordered space under consideration and these fixed point results are useful in establishing the uniqueness of the solution of nonlinear differential and integral equations. Again, some hybrid fixed point theorems of Krasnosel'skii type for monotone mappings are proved in Dhage [4, 5] along the lines of Heikkilä and Lakshmikantham [3].

The main object of this paper is first to establish some hybrid fixed point theorems of Krasnosel'skii type in partially ordered normed linear spaces, which involve product of two operators. We then apply these hybrid fixed point theorems to fractional integral equations for proving the existence of solutions under certain monotonicity conditions blending with the existence of the upper or lower solution.

2. Hybrid Fixed Point Theorems

Let X be a linear space or vector space. We introduce a partial order $\preceq$ in X as follows. A relation $\preceq$ in X is said to be a partial order if it satisfies the following properties:

(1) reflexivity: $a \preceq a$ for all $a \in X$;

(2) antisymmetry: $a \preceq b$ and $b \preceq a$ implies $a = b$;

(3) transitivity: $a \preceq b$ and $b \preceq c$ implies $a \preceq c$;

(4) order linearity: $x_1 \preceq y_1$ and $x_2 \preceq y_2 \Rightarrow x_1 + x_2 \preceq y_1 + y_2$; and $x \preceq y \Rightarrow tx \preceq ty$ for $t \geqq 0$.

The linear space X together with a partial order $\preceq$ becomes a partially ordered linear or vector space. Two elements x and y in a partially ordered linear space X are called comparable if the relation either $x \preceq y$ or $y \preceq x$ holds true. We introduce a norm $\|\cdot\|$ in partially ordered linear space X so that X becomes now a partially ordered normed linear space. If X is complete with respect to the metric d defined through the above norm, then it is called a partially ordered complete normed linear space.

The following definitions are frequently used in our present investigation.

Definition 6. A mapping $T : X \rightarrow X$ is called monotone non-decreasing if $x \preceq y$ implies $Tx \preceq Ty$ for all $x, y \in X$.

Definition 7. A mapping $T : X \rightarrow X$ is called monotone non-increasing if $x \preceq y$ implies $Tx \succeq Ty$ for all $x, y \in X$.

Definition 8. A mapping $T : X \rightarrow X$ is called monotone if it is either monotone non-increasing or monotone non-decreasing.

Definition 9 (see [6, 7]). A mapping $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is called a monotone dominating function or, in short, an M-function if it is an upper or lower semicontinuous and monotonic non-decreasing or non-increasing function satisfying the condition: $\varphi(0) = 0$.

Definition 10 (see [6, 7]). Given a partially ordered normed linear space E, a mapping $Q : E \to E$ is called partially M-Lipschitz or partially nonlinear M-Lipschitz if there is an M-function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\|Qx - Qy\| \leqq \varphi(\|x - y\|) \tag{5}$$

for all comparable elements $x, y \in E$. The function is called an M-function of Q on E. If $\varphi(r) = kr$ $(k > 0)$, then Q is called partially M-Lipschitz with the Lipschitz constant k. In particular, if $k < 1$, then Q is called a partially M-contraction on X with the contraction constant k. Further, if $\varphi(r) < r$, for $r > 0$, then Q is called a partially nonlinear M-contraction with an M-function φ of Q on X.

There do exist M-functions and the commonly used M-functions are $\varphi(r) = kr$ and $\varphi(r) = r/1+r$, *et cetera*. These M-functions can be used in the theory of nonlinear differential and integral equations for proving the existence results via fixed point methods.

Definition 11 (see [8]). An operator Q on a normed linear space E into itself is called compact if $Q(E)$ is a relatively compact subset of E. Q is called totally bounded if, for any bounded subset S of E, $Q(S)$ is a relatively compact subset of E. If Q is continuous and totally bounded, then it is called completely continuous on E.

Definition 12 (see [8]). An operator Q on a normed linear space E into itself is called partially compact if $Q(C)$ is a relatively compact subset of E for all totally ordered set or chain C in E. The operator Q is called partially totally bounded if, for any totally ordered and bounded subset C of E, $Q(C)$ is a relatively compact subset of E. If the operator Q is continuous and partially totally bounded, then it is called partially completely continuous on E.

Remark 13. We note that every compact mapping in a partially metric space is partially compact and every partially compact mapping is partially totally bounded. However, the reverse implication does not hold true. Again, every completely continuous mapping is partially completely continuous and every partially completely continuous mapping is continuous and partially totally bounded, but the converse may not be true.

We now state and prove the basic hybrid fixed point results of this paper by using the argument from algebra, analysis, and geometry. The slight generalization of Theorem 4 and Dhage [8] using M-contraction is stated as follows.

Theorem 14. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $T : X \to X$ be a monotone function (non-decreasing or non-increasing) such that there exists an M-function φ_T such that*

$$d(T(x), T(y)) \leqq \varphi_T(d(x, y)) \tag{6}$$

for all comparable elements $x, y \in X$ and satisfying $\varphi_T(r) < r$ $(r > 0)$. Suppose that either T is continuous or X is such that if $x_n \to x$ is a sequence in X whose consecutive terms are comparable, then there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that every term comparable to the limit x. If there exists $x_0 \in X$ with $x_0 \leqq T(x_0)$ or $x_0 \geqq T(x_0)$, then T has a fixed point which is unique if "every pair of elements in X has a lower and an upper bound."

Proof. The proof is standard. Nevertheless, for the sake of completeness, we give the details involved. Define a sequence $\{x_n\}$ of successive iterations of T by

$$x_{n+1} = Tx_n \quad (n \in \mathbb{N}). \tag{7}$$

By the monotonicity property of T, we obtain

$$x_0 \preceq x_1 \preceq \cdots \preceq x_n \cdots \tag{8}$$

or

$$x_0 \succeq x_1 \succeq \cdots \succeq x_n \cdots. \tag{9}$$

If $x_n = x_{n+1}$, for some $n \in \mathbb{N}$, then $u = x_n$ is a fixed point of T. Therefore, we assume that $x_n = x_{n+1}$ for some $n \in \mathbb{N}$. If $x = x_{n-1}$ and $y = x_n$, then, by the condition (6), we obtain

$$d(x_n, x_{n+1}) \leqq \varphi(d(x_{n-1}, x_n)) \tag{10}$$

for each $n \in \mathbb{N}$.

Let us write $r_n = d(x_n, x_{n+1})$. Since φ is an M-function, $\{r_n\}$ is a monotonic sequence of real numbers which is bounded. Hence $\{r_n\}$ is convergent and there exists a real number r such that

$$\lim_{n\to\infty} r_n = d(x_n, x_{n+1}) = r. \tag{11}$$

We show that $r = 0$. If $r \neq 0$, then

$$\begin{aligned} r &= \lim_{n\to\infty} r_n = \lim_{n\to\infty} d(x_n, x_{n+1}) \\ &\leqq \lim_{n\to\infty} \varphi(d(x_{n-1}, x_n)) \leqq \varphi(r) < r, \end{aligned} \tag{12}$$

which is a contradiction. Hence $r = 0$.

We now show that $\{x_n\}$ is a Cauchy sequence in X. If not, then, for $\epsilon > 0$, there exists a positive integer k such that

$$d(x_{m(k)}, x_{n(k)}) \geqq \epsilon \tag{13}$$

for all positive integers $m(k) \geqq n(k) \geqq k$.

If we write $r_k = d(x_{m(k)}, x_{n(k)})$, then

$$\begin{aligned} \epsilon &\leqq r_k = d(x_{m(k)}, x_{n(k)}) \\ &\leqq d(x_{m(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{n(k)}) \\ &= r_{m(k)-1} + \epsilon, \end{aligned} \tag{14}$$

so that we have

$$\lim_{k\to\infty} r_k = \epsilon. \tag{15}$$

Again, we have

$$\begin{aligned}\epsilon \leqq r_k &= d\left(x_{m(k)}, x_{n(k)}\right) \\ &\leqq d\left(x_{m(k)}, x_{m(k)+1}\right) + d\left(x_{m(k)+1}, x_{n(k)+1}\right) \\ &\quad + d\left(x_{n(k)+1}, x_{n(k)}\right) = r_{m(k)} + \varphi\left(r_k\right) + r_{n(k)}.\end{aligned} \tag{16}$$

Taking the limit as $k \to \infty$, we obtain

$$\epsilon \leqq \varphi(\epsilon) < \epsilon, \tag{17}$$

which is a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence in X. By the metric space (X, d) being complete, there is a point $x^* \in X$ such that $\lim_{n\to 0} x_n = x^*$. The rest of the proof is similar to above fixed point Theorem 4 given in Nieto and Rodríguez-López [2]. Hence we omit the details involved. □

Corollary 15. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $T : X \to X$ be a monotone function (non-decreasing or non-increasing) such that there exists an M-function φ and a positive integer p such that*

$$d\left(T^p(x), T^p(y)\right) \leqq \varphi_T\left(d(x, y)\right) \tag{18}$$

for all comparable elements $x, y \in X$ and satisfying $\varphi_T(r) < r$ $(r > 0)$. Suppose that either T is continuous or X is such that if $x_n \to x$ is a sequence in X whose consecutive terms are comparable, then there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that every term comparable to the limit x. If there exists $x_0 \in X$ with $x_0 \leqq T(x_0)$ or $x_0 \geqq T(x_0)$, then T has a fixed point which is unique if "every pair of elements in X has a lower and an upper bound."

Proof. Let us first set $Q = T^p$. Then $Q : X \to X$ is a continuous monotonic mapping. Also there exists the element $x_0 \in X$ such that $x_0 \preceq Qx_0$. Now, an application of Theorem 14 yields that Q has an unique fixed point; that is, it is a point $u \in X$ such that $Q(u) = T^p(u) = u$. Now $T(u) = T(T^p u) = Q(Tu)$, showing that Tu is again a fixed point of Q. By the uniqueness of u, we get $Tu = u$. The proof is complete. □

Fixed point Theorem 14 and Corollary 15 have some nice applications to various nonlinear problems modelled on nonlinear equations for proving existence as well as uniqueness of the solutions under generalized Lipschitz condition. The following fixed point theorem is presumably new in the literature. The basic principle in formulating this theorem is the same as that of Dhage [5, 8] and Nieto and Rodríguez-López [2]. Before stating these results, we give an useful definition.

Definition 16. The order relation $\preceq$ and the norm $\|\cdot\|$ in a nonempty set X are said to be compatible if $\{x_n\}$ is a monotone sequence in X and if a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x_0 impling that the whole sequence $\{x_n\}$ converges to x_0. Similaraly, given a partially ordered normed linear space $(X, \preceq, \|\cdot\|)$, the ordered relation $\preceq$ and the norm $\|\cdot\|$ are said to be compatible if $\preceq$ and the metric d defined through the norm are compatible.

Clearly, the set $\mathbb{R}$ with the usual order relation $\leqq$ and the norm defined by absolute value function has this property. Similarly, the space $C(J, \mathbb{R})$ with usual order relation defined by $x \leqq y$ if and only if $x(t) \leqq y(t)$ for all $t \in J$ or $x \leqq y$ if and only if $x(t) \geqq y(t)$ for all $t \in J$ and the usual standard supremum norm $\|\cdot\|$ are compatible.

We now state a more basic hybrid fixed point theorem. Since the proof is straightforword, we omit the details involved.

Theorem 17. *Let X be a partially ordered linear space and suppose that there is a norm in X such that X is a normed linear space. Let $T : X \to X$ be a monotonic (non-decreasing or non-increasing), partially compact and continuous mapping. Further, if the order relation $\preceq$ or $\succeq$ and the norm $\|\cdot\|$ in X are compatible and if there is an element $x_0 \in X$ satisfying $x_0 \leqq Tx_0$ or $x_0 \geqq Tx_0$, then T has a fixed point.*

In this paper, we combine Theorems 14 and 17 and Corollary 15 to derive some Krasnosel'skii type fixed point theorems in partially ordered complete normed linear spaces and discuss some of their applications to fractional integral equations of mixed type. We freely use the conventions and notations for fractional integrals as in (for example) [9–11].

3. Krasnosel'skii Type Fixed Point Theorems

We first state the following result.

Theorem 18 (see Krasnosel'skii [12]). *Let S be a closed convex and bounded subset of a Banach space X and let $A : X \to X$ and $B : S \to X$ be two operators satisfying the following conditions:*

(a) *A is nonlinear contraction;*

(b) *B is completely continuous;*

(c) *$Ax + By = x$ for all $y \in S$ implies $x \in S$.*

Then the following operator equation

$$Ax + Bx = x \tag{19}$$

has a solution.

Theorem 18 is very much useful and applied to linear perturbations of differential and integral equations by several authors in the literature for proving the existence of the solutions. The theory of Krasnosel'skii type fixed point theorem is initiated by Dhage [5]. The following Krasosel'skii type fixed point theorem is proved in Dhage [5].

Theorem 19 (see Dhage [5]). *Let S be a nonempty, closed, convex, and bounded subset of the Banach algebra X. Also let $A : X \to X$ and $B : S \to X$ be two operators such that*

(a) *A is D-Lipschitz with the D-function ψ;*

(b) *B is completely continuous;*

(c) $x = AxBy \Rightarrow x \in S$ *for all* $y \in S$;

$M\psi(r) < r$, $r > 0$ *where*

$$M = \|B(S)\| = \sup\{\|B(x)\| : x \in S\}. \tag{20}$$

Then the operator equation $AxBx = x$ *has a solution in* S.

Remark 20. $(I/A)^{-1}B$ is monotone (non-decreasing or non-increasing) if A and B are monotone (non-decreasing or non-increasing), but the converse may not be true.

We now obtain another version of Krasnosel'skii type fixed point theorems in partially ordered complete normed linear spaces under weaker conditions, which improve Theorem 19, and discuss some of their applications to fractional integral equations of mixed type.

Theorem 21. *Let* $(X, \preceq, \|\cdot\|)$ *be a partially ordered complete normed linear space such that the order relation* $\preceq$ *and the norm* $\|\cdot\|$ *in* X *are compatible. Let* $A, B : X \to X$ *be two monotone operators (non-decreasing or non-increasing) such that*

(a) A *is continuous and partially nonlinear* M*-contraction;*

(b) B *is continuous and partially compact;*

(c) *there exists an element* $x_0 \in X$ *such that* $x_0 \preceq Ax_0By$ *or* $x_0 \succeq Ax_0By$ *for all* $y \in X$;

(d) *every pair of elements* $x, y \in X$ *has a lower and an upper bound in* X;

(e) $K\varphi(r) < r$, $r > 0$ *where*

$$K = \|B(X)\| = \sup\{\|Bx\| : x \in X\}. \tag{21}$$

Then the operator equation $AxBx = x$ *has a solution.*

Proof. Define an operator $T : X \to X$ by

$$T(x) = \left(\frac{I}{A}\right)^{-1} B. \tag{22}$$

Clearly, the operator T is well defined. To see this, let $y \in X$ be fixed and define a mapping $A_y : X \to X$ by

$$A_y(x) = AxBy. \tag{23}$$

Now, for any two comparable elements $x_1, x_2 \in X$, we have

$$\begin{aligned} &\left\|A_y(x_1) - A_y(x_2)\right\| \\ &\quad = \left\|Ax_1By - Ax_2By\right\| \leqq \left\|Ax_1 - Ax_2\right\| \cdot \left\|By\right\| \\ &\quad \leqq K\varphi_A\left(\left\|x_1 - x_2\right\|\right), \end{aligned} \tag{24}$$

where A is an M-function of T on X. Hence, by an application of fixed point Theorem 14, A_y has an unique fixed point; say $x^* \in X$. Therefore, we have an unique element $x^* \in X$ such that

$$A_y(x^*) = Ax^*By = x^*, \tag{25}$$

which implies that

$$\left(\frac{I}{A}\right)^{-1} By = x^* \tag{26}$$

or, equivalently, that

$$Ty = x^*. \tag{27}$$

Thus the mapping $T : X \to X$ is well defined.

We now define a sequence $\{x_n\}$ of iterates of T; that is, $x_{n+1} = Tx_n$ for $n \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$. It follows from the hypothesis (c) that $x_0 \leqq T(x_0)$ or $x_0 \geqq T(x_0)$. Again, by Remark 20, we find that the mapping T is monotonic (non-decreasing or non-increasing) on X. So we have

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots x_n \preceq \cdots \tag{28}$$

or

$$x_0 \succeq x_1 \succeq x_2 \succeq \cdots x_n \succeq \cdots. \tag{29}$$

Since B is partially compact and $(I/A)^{-1}$ is continuous, the composition mapping $T = (I/A)^{-1}B$ is partially compact and continuous on X into X. Therefore, the sequence $\{x_n\}$ has a convergent subsequence and, from the compatibility of the order relation and the norm, it follows that the whole sequence converges to a point in X. Hence, an application of Theorem 17 implies that T has a fixed point. This further implies that

$$\left(\frac{I}{A}\right)^{-1} Bx^* = x^* \quad \text{or} \quad Ax^*Bx^* = x^*, \tag{30}$$

which evidently completes the proof of Theorem 21. □

Theorem 22. *Let* $(X, \preceq, \|\cdot\|)$ *be a partially ordered complete normed linear space such that the order relation* $\preceq$ *and the norm* $\|\cdot\|$ *in* X *are compatible. Let* $A, B : X \to X$ *be two monotone mappings (non-decreasing or non-increasing) satisfying the following conditions:*

(a) A *is linear and bounded and* A^p *is partially nonlinear* M*-contraction for some positive integer* p;

(b) B *is continuous and partially compact;*

(c) *there exists an element* $x_0 \in X$ *such that* $x_0 \preceq Ax_0By$ *or* $x_0 \succeq Ax_0By$ *for all* $y \in X$;

(d) *every pair of elements* $x, y \in X$ *has a lower and an upper bound in* X;

(e) $K\varphi(r) < r$, $r > 0$ *where*

$$K = \|B(X)\| = \sup\{\|Bx\| : x \in X\}. \tag{31}$$

Then the operator equation $AxBx = x$ *has a solution.*

Proof. Define an operator T on X by

$$T(x) = \left(\frac{I}{A}\right)^{-1} B. \tag{32}$$

Now the mapping $(I/A)^{-1}$ exists in view of the relation

$$\left(\frac{I}{A}\right)^{-1} = \left(\frac{I}{A^p}\right)^{-1} \prod_{j=1}^{p-1} A^j, \tag{33}$$

where $\prod_{j=1}^{p-1} A^j$ is bounded and $(I/A^p)^{-1}$ exists in view of Corollary 15. Hence, $(I/A)^{-1}$ exists and is continuous on X. Next, the operator T is well defined. To see this, let $y \in X$ be fixed and define a mapping $A_y : X \to X$ by

$$A_y(x) = AxBy. \tag{34}$$

Then, for any two comparable elements $x, y \in X$, we have

$$\begin{aligned} &\left\| A_y^p(x_1) - A_y^p(x_2) \right\| \\ &\quad = \left\| A^p x_1 By - A^p x_2 By \right\| \leqq \left\| A^p x_1 - A^p x_2 \right\| \cdot \|By\| \\ &\quad \leqq K\varphi_A(\|x_1 - x_2\|). \end{aligned} \tag{35}$$

Hence, by Corollary 15 again, there exists an unique element x^* such that

$$A_y^p(x^*) = A_p(x^*) By = x^*. \tag{36}$$

This further implies that $A_y(x^*) = x^*$ and x^* is an unique fixed point of A_y. Thus we have

$$A_y(x^*) = x^* = Ax^* By \quad \text{or} \quad \left(\frac{I}{A}\right)^{-1} By = x^*. \tag{37}$$

Consequently, $Ty = x^*$ and so T is well defined. The rest of the proof is similar to that of Theorem 21 and we omit the details. The proof is complete. □

Remark 23. The hypothesis (d) of Theorems 21 and 22 holds true if the partially ordered set X is a lattice. Furthermore, the space $C(J, \mathbb{R})$ of continuous real-valued functions on the closed and bounded interval $J = [a, b]$ is a lattice, where the order relation $\leqq$ is defined as follows. For any $x, y \in C(J, \mathbb{R})$, $x \leqq y$ if and only if $x(t) \leqq y(t)$ for all $t \in J$. The real-variable operations show that $\min(x, y)$ and $\max(x, y)$ are, respectively, the lower and upper bounds for the pair of elements x and y in X.

4. Fractional Integral Equations of Mixed Type

In this section we apply the hybrid fixed point theorems proved in the preceding sections to some nonlinear fractional integral equations of mixed type.

Given a closed and bounded interval $J = [t_0, t_0 + a]$ in $\mathbb{R}$, $\mathbb{R}$ being the set of real numbers or some real numbers $t_0 \in \mathbb{R}$ and $a \in \mathbb{R}$ with $a > 0$ and given a real number $0 < q < 1$, consider the following nonlinear hybrid fractional integral equation (in short HFIE):

$$x(t) = [f(t, x(t))] \left(\frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1} g(s, x(s))\, ds \right), \tag{38}$$

where $f : J \times \mathbb{R} \to \mathbb{R}$ is continuous and $g : J \times \mathbb{R} \to \mathbb{R}$ is locally Hölder continuous.

We seek the solutions of HFIE (38) in the space $C(J, \mathbb{R})$ of continuous real-valued functions defined on J. We consider the following set of hypotheses in what follows.

(H_1) g is bounded on $J \times \mathbb{R}$ with bound C_g.

(H_2) $g(t, x)$ is non-decreasing in x for each $t \in J$.

(H_3) There exist constants $L > 0$ and $K > 0$ such that

$$0 \leqq (f(t, x) - f(t, y)) \leqq \frac{L(x - y)}{K + (x - y)} \tag{39}$$

for all $x, y \in \mathbb{R}$ with $x \geqq y$. Moreover, $L \leqq K$.

(H_4) There exists an element $u_0 \in X = C(J, \mathbb{R})$ such that

$$u_0(t) \leqq [f(t, u_0(t))] \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1} g(s, y(s))\, ds \tag{40}$$

for all $t \in J$ and $y \in X$ or

$$u_0(t) \geqq [f(t, u_0(t))] \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1} g(s, y(s))\, ds \tag{41}$$

for all $t \in J$ and $y \in X$.

Remark 24. The condition given in the hypothesis (H_4) is a little more restrictive than that of a lower solution of the HFIE (38). It is clear that u_0 is a lower solution of the HFIE (38); however, the converse is not true.

Theorem 25. *Assume that the hypotheses* (H_1) *through* (H_4) *hold true. Then the HFIE* (38) *admits a solution.*

Proof. Define two operators A and B on $X = C(J, \mathbb{R})$, the Banach space of continuous real-valued functions on J with the usual supremum norm $\|\cdot\|$ given by

$$\|x\| = \sup_{t \in J} |x(t)|. \tag{42}$$

We define an order relation $\leqq$ in X with help of a cone $\mathscr{K}$ defined by

$$\mathscr{K} = \{x : x \in C(J, \mathbb{R}), x(t) \geqq 0 \ (\forall t \in J)\}. \tag{43}$$

Clearly, the Banach space X together with this order relation becomes an ordered Banach space. Furthermore, the order relation $\leqq$ and the norm $\|\cdot\|$ in X are compatible. Define two operators $A, B : C(J, \mathbb{R}) \to C(J, \mathbb{R})$ by

$$\begin{aligned} Ax(t) &= f(t, x(t)) \quad (t \in J), \\ Bx(t) &= \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - s)^{q-1} g(s, x(s))\, ds. \end{aligned} \tag{44}$$

Then the given Hybrid fractional integral equation (38) is transformed into an equivalent operator equation as follows:

$$Ax(t) \cdot Bx(t) = x(t) \quad (t \in J). \tag{45}$$

We show that the operators A and B satisfy all the conditions of Theorem 21 on $C(J,\mathbb{R})$. First of all, we show that A is a nonlinear M-contraction on $C(J,\mathbb{R})$. Let $x, y \in X$. Then, by the hypothesis (H_3), we obtain

$$\begin{aligned}|Ax(t)-Ay(t)| &= |f(t,x(t))-f(t,y(t))| \\ &\leqq \frac{L|x(t)-y(t)|}{K+|x(t)-y(t)|} \\ &\leqq \frac{L\|x-y\|}{K+\|x-y\|}.\end{aligned} \tag{46}$$

Taking the supremum over t, we get

$$\|Ax-Ay\| \leqq \frac{L\|x-y\|}{K+\|x-y\|} = \varphi(\|x-y\|), \tag{47}$$

where

$$\varphi(r) = \frac{Lr}{K+r} < r \quad (r>0). \tag{48}$$

Clearly, φ is an M-function for the operator A on X and so A is a partially nonlinear M-contraction on X.

Next, we show that B is a compact continuous operator on X. To this end, we show that $B(X)$ is a uniformly bounded and equicontinuous set in X. Now, for any $x \in X$, we have

$$\begin{aligned}|Bx(t)| &\leqq \frac{1}{\Gamma(q)}\int_{t_0}^{t}|t-s|^{q-1}|g(s,x(s))|\,ds \\ &\leqq \frac{C_g}{\Gamma(q)}\int_{t_0}^{t}|t-s|^{q-1}ds \leqq \frac{a^q C_g}{\Gamma(q+1)},\end{aligned} \tag{49}$$

which shows that B is a uniformly bounded set in X. We now let $t_1, t_2 \in J$. Then

$$\begin{aligned}&|Bx(t_1)-Bx(t_2)| \\ &\leqq \frac{C_g}{\Gamma(q)}\int_{t_0}^{t_2}\left|(t_1-s)^{q-1}-(t_2-s)^{q-1}\right|ds+\frac{C_g}{\Gamma(q+1)}|t_1-t_2|^q \\ &\leqq \frac{C_g}{\Gamma(q)}\int_{t_0}^{t_0+a}\left|(t_1-s)^{q-1}-(t_2-s)^{q-1}\right|ds+\frac{C_g}{\Gamma(q+1)}|t_1-t_2|^q \\ &\longrightarrow 0 \quad \text{as } t_1 \longrightarrow t_2\end{aligned} \tag{50}$$

uniformly for all $x \in X$. Hence $B(X)$ is an equicontinuous set in X. Now we apply the Arzela-Ascoli theorem to show that $B(X)$ is a compact set in X. The continuity of B follows from the continuity of the function g on $J\times\mathbb{R}$.

Finally, since $f(t,x)$ and $g(t,x)$ are non-decreasing in x for each $t \in J$, the operators A and B are non-decreasing on X. Also the hypothesis (H_3) yields $u_0 \leqq Au_0\cdot Bu_0$. Thus, all of the conditions of Theorem 22 are satisfied and we conclude that the fractional integral equation (38) admits a solution. This completes the proof. □

We now consider the following fractional integral equation of mixed type:

$$\begin{aligned}x(t) = &\left[\int_{t_0}^{t} v(t,s)f(s,x(s))\,ds\right] \\ &\times\left(q(t)+\frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}g(s,x(s))\,ds\right)\end{aligned} \tag{51}$$

for all $t \in J$ and $0<q<1$, where the functions $v : J\times J \to \mathbb{R}^+$ and $f, g : J\times\mathbb{R} \to \mathbb{R}$ are continuous.

We consider the following set of hypotheses in what follows.

(H_5) The function $v : J\times J \to \mathbb{R}^+$ is continuous. Moreover, $v = \sup_{t,s\in J}|v(t,s)|$.

(H_6) $f(t,x)$ is linear in x for each $t \in J$.

(H_7) f is bounded on $J\times\mathbb{R}$ and there exists a constant $L>0$ such that $f(t,x) < L|x|$ for all $t \in J$ and $x \in \mathbb{R}$.

(H_8) There exists an element $u_0 \in X = C(J,\mathbb{R})$ such that

$$\begin{aligned}u_0(t) \leqq &\left[\int_{t_0}^{t} v(t,s)f(s,u_0(s))\,ds\right] \\ &\times\left(q(t)+\frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}g(s,y(s))\,ds\right)\end{aligned} \tag{52}$$

or

$$\begin{aligned}u_0(t) \geqq &\left[\int_{t_0}^{t} v(t,s)f(s,u_0(s))\,ds\right] \\ &\times\left(q(t)+\frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}g(s,y(s))\,ds\right)\end{aligned} \tag{53}$$

for all $t \in J$ and $0<q<1$, where the functions $v : J\times J \to \mathbb{R}^+$ and $f, g : J\times\mathbb{R} \to \mathbb{R}$ are continuous.

Remark 26. The condition given in hypothesis (H_7) is a little more restrictive than that of a lower solution for the HFIE (51) defined on J.

Theorem 27. *Assume that the hypotheses (H_1), (H_2), and (H_5) through (H_8) hold true. Then the HFIE* (51) *admits a solution.*

Proof. Set $X = C(J,\mathbb{R})$ and define an order relation $\leqq$ with the help of the cone $\mathscr{K}$ defined by (43). Clearly, $C(J,\mathbb{R})$ is a lattice with respect to the above order relation $\leqq$ in it. Define two operators A and B on X by

$$\begin{aligned}Ax(t) &= \int_{t_0}^{t} v(t,s)f(s,x(s))\,ds \quad (t\in J), \\ Bx(t) &= q(t)+\frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}g(s,x(s))\,ds \quad (t\in J).\end{aligned} \tag{54}$$

Clearly, the operator A is linear and bounded in view of the hypotheses (H_1), (H_6), and (H_7). We only show that the operator A^n is partially M-contraction on X for every positive integer n. Let $x, y \in X$ be such that $x \geqq y$. Then, by (H_6) and (H_7), we have

$$\begin{aligned} &\left|Ax(t) - Ay(t)\right| \\ &\quad \leqq \int_{t_0}^{t} |V| \cdot \left|f(s, x(s)) - f(s, y(s))\right| ds \\ &\quad \leqq V \int_{t_0}^{t_0+a} L\left|x(s) - y(s)\right| ds \leqq LVa\left\|x - y\right\|, \end{aligned} \tag{55}$$

where $|V|$ is the supremum of $v(t,s)$ over t. Thus, by taking the supremum over t, we obtain

$$\left\|Ax - Ay\right\| \leqq LVa\left\|x - y\right\|. \tag{56}$$

Similarly, it can be proved that

$$\begin{aligned} \left\|A^2x - A^2y\right\| &= \left|A(Ax(t)) - A(Ay(t))\right| \\ &\leqq LV\int_{t_0}^{t_0+a}\left(\int_{t_0}^{t}\left|Ax(s) - Ay(s)\right| ds\right) ds \\ &\leqq \frac{L^2V^2a^2}{2!}\left\|x - y\right\|. \end{aligned} \tag{57}$$

In general, proceeding in the same way, for any positive integer n, we have

$$\left\|A^nx - A^ny\right\| \leqq \frac{L^nV^na^n}{n!}\left\|x - y\right\|. \tag{58}$$

Therefore, for large n, A^n is partially a nonlinear M-contraction mapping on X. The rest of the proof is similar to that of Theorem 25. The desired result now follows by an application of Theorem 22. This completes the proof. □

5. An Illustrative Example

Example 1. Consider a distributed-order fractional hybrid differential equation (DOFHDES) involving the Reimann-Liouville derivative operator of order $0 < q < 1$ with respect to the negative density function $b(q) > 0$ as follows:

$$\begin{gathered} \int_0^1 b(q) D^q\left[\frac{x(t)}{f(t, x(t))}\right] dq = g(t, x(t)) \quad (t \in J), \\ \int_0^1 b(q)\, dq = 1, \\ x(0) = 0. \end{gathered} \tag{59}$$

Moreover, the function $t \rightarrow x/f(t,x)$ is continuous for each $x \in \mathbb{R}$, where $J = [0, T]$ is bounded in $\mathbb{R}$ for some $T \in \mathbb{R}$. Also $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g \in C(J, \mathbb{R})$. It is well known that the DOFHDES (59) is equivalent to the following integral equation:

$$\begin{aligned} &x(t) \\ &\quad = \frac{f(t, x(t))}{\pi}\int_0^t L\left\{S\left\{\frac{1}{B\left(re^{-i\pi}\right)}\right\}; t - \tau\right\} g(\tau, x(\tau))\, d\tau \end{aligned} \tag{60}$$

such that $0 \leqq \tau \leqq t \leqq T$ and

$$B(s) = \int_0^1 b(q)\, s^q dq. \tag{61}$$

The integral equation (60) is valid for all $x \in C(J, \mathbb{R})$. Hence, if Theorem 25 holds true then we further have

$$\frac{LM|h|_{L'}}{\pi} < 1 \quad (M > 0), \tag{62}$$

then the above-mentioned DOFHDES (59) has a solution.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Stabilized Discretization in Spline Element Method for Solution of Two-Dimensional Navier-Stokes Problems

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In terms of the poor geometric adaptability of spline element method, a geometric precision spline method, which uses the rational Bezier patches to indicate the solution domain, is proposed for two-dimensional viscous uncompressed Navier-Stokes equation. Besides fewer pending unknowns, higher accuracy, and computation efficiency, it possesses such advantages as accurate representation of isogeometric analysis for object boundary and the unity of geometry and analysis modeling. Meanwhile, the selection of B-spline basis functions and the grid definition is studied and a stable discretization format satisfying inf-sup conditions is proposed. The degree of spline functions approaching the velocity field is one order higher than that approaching pressure field, and these functions are defined on one-time refined grid. The Dirichlet boundary conditions are imposed through the Nitsche variational principle in weak form due to the lack of interpolation properties of the B-splines functions. Finally, the validity of the proposed method is verified with some examples.

1. Introduction

Comparing with traditional finite element method, spline element method (on the basis of Galerkin Principle and Spline Function Theory) involves less calculation, higher precision, and fewer pending unknowns and it is easier to construct high-order coordination unit. Thus, it has attracted much attention, and Chinese scholars have gained much achievement [1, 2]. However, it mainly focuses on structural problems [3], such as elastic beam, shell, vibration, and dynamic response and the research on nonstructural problem like fluid is far from enough. Currently, the two main methods of the application of spline function in fluid mechanics are Collocation Method and Galerkin Method.

Spline Collocation Method is similar to Chebyshev Spectrum Method in its less calculation and higher efficiency. So Botella [4] applied it to calculate the incompressible Navier-Stokes. Aiming at solving false oscillation by suppressing the pressure value, Botella [5] proposed a staggered grid collocation scheme and achieved stable numerical results. Comparing to the collocation method, Galerkin Method has higher numerical precision and maturer error analysis theory. However, an element type (e.g., Taylor-Hood Element) that satisfies inf-sup stability condition [6] needs to be constructed when Galerkin Method is applied for Navier-Stokes Equations. In the field of spline element, Kumar et al. [7] had adopted weighted extended B-splines (WEB-spline) to compute Stokes. Then, they extended to Navier-Stokes equations [8] containing nonlinear convection term and constructed stable grid discrete format. The basic idea was that the degree of spline function approaching velocity field was one order higher than that approaching pressure field while only two kinds of discrete formats, namely, linear-constant and quadratic-linear, are designed. Meanwhile, WEB-spline Method is a meshless method. It avoided cockamamie grid division by replacing unstructured grid of finite element with regular net, but boundary elements require special treatment. B-Spline Element Method was adopted by Kravchenko et al. [9, 10] to analyze turbulent flow problem to decrease the calculation amounts of large eddy simulation and direct numerical method as well as to increase resolution ratio of boundary layer by embedding partitioned grid. In addition, they adopted divergence-free B-spline to expand and eliminate pressure term in governing equations to decrease numerical disturbance of calculating results.

Moreover, two disadvantages of Spline Element Method have been noticed. One is that it is restricted by "low geometric versatility" and is only appropriate for the solving domain of specially simple geometric shape (e.g., rectangular or those that can be converted into rectangular). The other is that B-spline function has no interpolation property, and the function value is in the convex hull. So Dirichlet boundary condition cannot be imposed directly at the junction. Mingquan [13] solved the first problem by converting quadrilateral area into rectangular region through double linear coordinate transformation. Ronglin et al. [14] calculated boundary value of the one with arc boundary through polar mapping. However, these attempts have failed to fully address this issue. Hughes et al. [15] put forward Isogeometric Analysis Method to bridge geometric modeling and finite element analysis. It can be applied in any complex geometric area, but the primary function needs to be rational function and it is less efficient than finite element and spline element.

This paper aims at solving incompressible Navier-Stokes equation and the main idea is as follows. (1) On the basis of Isogeometric Analysis Method, solution domain can be precisely represented by making rational Bezier patches as geometric mapping and the spline element can be ascertained with the geometry that evens the B-spline function approaching physical field. Appropriate function space can be more flexibly chosen by separating the expressions of geometric solving domain and physical field; (2) construct discrete format of stable spline element that meets inf-sup conditions; (3) impose essential boundary condition through Nitsche variational principle for B-spline function's lack of interpolation property.

2. Flow of Navier-Stokes

Assume that the boundary $\partial\Omega$ of a 2D closed connected region $\Omega \in R^2$ satisfies Lipschitz succession. The incompressible Navier-Stokes flow equation in dimensionless form is

$$-\mu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} + \nabla p = \mathbf{f}, \qquad \nabla\mathbf{u} = 0, \quad \text{in } \Omega, \tag{1}$$

$\mathbf{u} = (u, v)$ refers to velocity vector of fluid, p refers to the pressure, and $\mathbf{f} = (f_1, f_2)$ refers to volume force source term. $\mu = 1/\mathrm{Re}$ stands for scale-free viscosity coefficient, in which Reynolds number $\mathrm{Re} = \rho UL/\nu$ is a dimensionless number of representational fluid property. Nonlinear term of convection form $\mathbf{u}\cdot\nabla\mathbf{u} = u_j(\partial u_i/\partial x_j)$ is adopted in this paper, mainly because of its simple format and numerical stability of high Reynolds number. Additional boundary condition and distribution constraint of pressure field should be added to solve the above equation:

$$\begin{gathered}\mathbf{u} = \mathbf{g}, \quad \text{on } \partial\Omega,\\ \int_\Omega p\,\mathrm{d}x\,\mathrm{d}y = 0, \quad \text{in } \Omega,\end{gathered} \tag{2}$$

where $g = (g_1, g_2)$ refers to Dirichlet boundary condition of speed on boundary $\partial\Omega$. The second equation means that the average pressure is zero.

Assume that a function space $W = \{p \in L^2(\Omega) : \int_\Omega p\,\mathrm{d}x\,\mathrm{d}y = 0\}$, and a vector function space $S = \{u \in H^1(\Omega)\times H^1(\Omega),\ u = g\}$, $V = \{u \in H^1(\Omega)\times H^1(\Omega),\ u = 0\}$ exist; then the weak form solution of constant Navier-Stokes equation can be expressed as search function $(\mathbf{u}, p) \in \mathbf{S}\times W$, and it satisfies

$$\begin{gathered}\int_\Omega \mu\nabla\mathbf{w} : \nabla\mathbf{u}\,\mathrm{d}\mathbf{x} + \int_\Omega \mathbf{w}\,(\mathbf{u}\cdot\nabla\mathbf{u})\,\mathrm{d}\mathbf{x}\\ - \int_\Omega (\nabla\cdot\mathbf{w})\,p\,\mathrm{d}\mathbf{x} = \int_\Omega \mathbf{wf}\,\mathrm{d}\mathbf{x}\\ - \int_\Omega \theta\nabla\cdot\mathbf{u}\,\mathrm{d}\mathbf{x} = 0.\end{gathered} \tag{3}$$

In the equation, arbitrary function $(\mathbf{w}, \theta) \in \mathbf{V}\times W$, Galerkin discretization assumes to project physical quantities into a finite dimensional subspace, and the speed and pressure are approximately expressed as $\mathbf{u}_h = \sum_{i=1}^{n_u} N_i^u(\mathbf{x})\left\{\begin{smallmatrix}u_i\\ v_i\end{smallmatrix}\right\}$ and $p_h = \sum_{i=1}^{n_p} N_i^p(\mathbf{x})p_i$, respectively. N_i^u and N_i^p, respectively, stand for the primary functions of finite element space of speed and pressure field. n_u and n_p are numbers of primary functions. This study is different from traditional finite element method in adopting B-spline function as the primary function.

3. Solution of Navier-Stokes Equation

3.1. Nitsche Type Variational Weak Form. In order to simplify the derivation of variational weak form, nonlinear term $\mathbf{u}\cdot\nabla\mathbf{u}$ can be ignored, and the equitation is reduced to Stokes flow equation: $-\mu\Delta\mathbf{u} + \nabla p = \mathbf{f}$. Its weak form is equivalent to variational extremum: search $\mathbf{u} \in \mathbf{S}$ and then obtained as

$$\begin{gathered}J(\mathbf{u}) = \min_{\mathbf{w}\in\mathbf{S}} J(\mathbf{w}),\\ J(\mathbf{w}) = \frac{1}{2}\int_\Omega \mu\nabla\mathbf{w} : \nabla\mathbf{w}\,\mathrm{d}\mathbf{x} - \int_\Omega \mathbf{wf}\,\mathrm{d}\mathbf{x}.\end{gathered} \tag{4}$$

Then the following constraints should be obeyed: (1) incompressible condition: $\nabla\cdot\mathbf{u} = 0$; (2) pressure constraint: $\int_\Omega p\,\mathrm{d}x\,\mathrm{d}y = 0$. It should be noticed that the above conclusion is made on the basis of natural variational principle, so Dirichlet boundary condition $\mathbf{u}|_{\partial\Omega} = \mathbf{g}$ should be met when finite element space is constructed. But it can be known from the above analysis that B-spline has no interpolation property, so the constraints are hard to be directly imposed as to traditional polynomial unit (e.g., interpolation of Lagrangian unit).

This paper obtained unconstrained functional by imposing Dirichlet boundary condition with Nitsche method [16] and incompressibility and pressure condition with Lagrangian multiplier method:

$$\begin{gathered}J_L(\mathbf{w}) = J(\mathbf{w}) - \int_\Omega p\,(\nabla\cdot\mathbf{w})\,\mathrm{d}\mathbf{x} + \gamma\int_\Omega p\,\mathrm{d}\mathbf{x}\\ + \int_{\partial\Omega} \boldsymbol{\lambda}\,(\mathbf{w}-\mathbf{g})\,\mathrm{d}s + \frac{\beta}{2}\int_{\partial\Omega} (\mathbf{w}-\mathbf{g})^2\mathrm{d}s.\end{gathered} \tag{5}$$

In the equation, $p, \gamma, \boldsymbol{\lambda} = \{\lambda_1, \lambda_2\}$ stands for Lagrangian multipliers and constant β is the penalty factor that depends on mesh size h. After taking the stationary value of above unconstrained functional and several transformations, Lagrangian multipliers $\lambda_1 = -(\partial u/\partial \mathbf{n}) + pn_1$ and $\lambda_2 = -(\partial v/\partial \mathbf{n}) + pn_2$ can be identified, which means the normal component of velocity gradient and pressure. Here $n = (n_1, n_2)$ refers to normal vector outside the unit. After substituting it into variational weak form and adding nonlinear convection term, the following equation can be obtained:

$$\begin{gathered}
\int \mu \nabla \mathbf{w} : \nabla \mathbf{u}\, \mathrm{d}\mathbf{x} - \int_{\partial\Omega} \mu \frac{\partial \mathbf{w}}{\partial \mathbf{n}} \mathbf{u}\, \mathrm{d}s - \int_{\partial\Omega} \mu \mathbf{w} \frac{\partial \mathbf{u}}{\partial \mathbf{n}} \mathrm{d}s \\
+ \beta \int_{\partial\Omega} \mathbf{wu}\, \mathrm{d}s + \int_{\Omega} \mathbf{w} (\mathbf{u} \cdot \nabla \mathbf{u})\, \mathrm{d}\mathbf{x} - \int_{\Omega} (\nabla \cdot \mathbf{w})\, p\, \mathrm{d}x \\
+ \int_{\partial\Omega} \mathbf{wn} p\, \mathrm{d}s \\
= \int_{\Omega} \mathbf{wf}\, \mathrm{d}\mathbf{x} - \int_{\partial\Omega} \mu \frac{\partial \mathbf{w}}{\partial \mathbf{n}} \mathbf{g}\, \mathrm{d}s + \beta \int_{\partial\Omega} \mathbf{wg}\, \mathrm{d}s - \int \theta \nabla \cdot \mathbf{u}\, \mathrm{d}x \\
+ \int_{\partial\Omega} \theta \mathbf{nu}\, \mathrm{d}s + \left(\int_{\Omega} \theta\, \mathrm{d}x \right) \gamma = \int_{\partial\Omega} \theta \mathbf{ng}\, \mathrm{d}s, \\
\int_{\Omega} p\, \mathrm{d}x = 0.
\end{gathered} \tag{6}$$

Equation (6) is different from (29) in [17] in that Lagrangian multiplier λ_i contains pressure part n_i, which makes the first two equations of above formula contain boundary pressure term $\int_{\partial\Omega} \mathbf{wn} p\, \mathrm{d}s$, $\int_{\partial\Omega} \theta \mathbf{nu}\, \mathrm{d}s$, and $\int_{\partial\Omega} \theta \mathbf{ng}\, \mathrm{d}s$. It should be noticed that the modified weak form equation can ensure optimal order convergence of numerical solution.

Assume that the speed $\mathbf{u} = (u_1, u_2) = (u, v)$ and pressure field p can be approximately expressed by adopting spline function:

$$\begin{gathered}
u = \sum_{i=1}^{n_u} N_i^{u_1}(x, y)\, u_i, \qquad v = \sum_{i=1}^{n_v} N_i^{u_2}(x, y)\, p_i \\
p = \sum_{i=1}^{n_p} N_i^{p}(x, y)\, p_i.
\end{gathered} \tag{7}$$

In above equation, velocity components u, v and pressure p adopt different primary functions, namely, N_i^u, N_i^v, and N_i^p. Then nonlinear simultaneous equations can be obtained through settlement after substituting them into the formula

$$\begin{bmatrix} \mathbf{A}_1 + \mathbf{C}_1 & 0 & -\mathbf{G}_1 + \mathbf{R}_1 & 0 \\ 0 & \mathbf{A}_2 + \mathbf{C}_2 & -\mathbf{G}_2 + \mathbf{R}_2 & 0 \\ -\mathbf{G}_1^{\mathrm{T}} + \mathbf{R}_1^{\mathrm{T}} & -\mathbf{G}_2^{\mathrm{T}} + \mathbf{R}_2^{\mathrm{T}} & 0 & \mathbf{W} \\ 0 & 0 & \mathbf{W}^{\mathrm{T}} & 0 \end{bmatrix} \begin{pmatrix} \widehat{\mathbf{u}} \\ \widehat{\mathbf{v}} \\ \widehat{\mathbf{p}} \\ \gamma \end{pmatrix} = \begin{pmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \\ \mathbf{Q} \\ 0 \end{pmatrix}. \tag{8}$$

In the equation, $\widehat{\mathbf{u}} = \{u_i\}$, $\widehat{\mathbf{v}} = \{v_i\}$, and $\widehat{\mathbf{p}} = \{p_i\}$ vectors need to be solved. Partitioned matrix: $\mathbf{A}_k = \mathbf{K}_k - [\mathbf{H}_k + \mathbf{H}_k^{\mathrm{T}}] + \beta \mathbf{M}_k$, $k = 1, 2$. Element of matrix $\mathbf{K}_k$: $K_{ij}^k = \int_\Omega \mu \nabla N_i^{u_k} \cdot \nabla N_j^{u_k} \mathrm{d}x\, \mathrm{d}y$; element of matrix $\mathbf{H}_k$: $H_{ij}^k = \int_{\partial\Omega} \mu (\partial N_i^{u_k}/\partial \mathbf{n}) N_j^{u_k} \mathrm{d}s$; element of matrix $\mathbf{M}_k$: $M_{ij}^k = \int_{\partial\Omega} N_i^{u_k} N_j^{u_k} \mathrm{d}s$; element of nonlinearity matrix $\mathbf{C}_k$: $C_{ij}^k = \int_\Omega N_i^{u_k} (u (\partial N_j^{u_k}/\partial x) + v (\partial N_j^{u_k}/\partial y)) \mathrm{d}x\, \mathrm{d}y$. Element of matrix $\mathbf{G}_1$: $G_{ij}^1 = \int_\Omega (\partial N_i^u/\partial x) N_j^p \mathrm{d}x\, \mathrm{d}y$, element of matrix $\mathbf{G}_2$: $G_{ij}^2 = \int_\Omega (\partial N_i^v/\partial y) N_j^p \mathrm{d}x\, \mathrm{d}y$; element of matrix $\mathbf{R_k}$: $R_{ij}^k = \int_{\partial\Omega} N_i^{u_k} n_k N_j^p \mathrm{d}s$. Element of vector $\mathbf{W}$: $W_i = \int_\Omega N_i^p (\mathbf{x}) \mathrm{d}x\, \mathrm{d}y$. Right-hand member of term: $\mathbf{B}_k = \mathbf{F}_k - \mathbf{T}_k + \beta \mathbf{S}_k$; element of vector $\mathbf{F}_k$: $F_i^k = \int_\Omega N_i^{u_k} f_k \mathrm{d}x\, \mathrm{d}y$; element of vector $\mathbf{T}_k$: $T_i^k = \int_{\partial\Omega} \mu (\partial N_i^{u_k}/\partial \mathbf{n}) g_k \mathrm{d}s$; element of vector $\mathbf{S}_k$: $S_i^k = \int_{\partial\Omega} N_i^{u_k} g_k \mathrm{d}s$; element of vector $\mathbf{Q}$: $Q_i = \int_{\partial\Omega} N_i^p (\mathbf{g} \cdot \mathbf{n}) \mathrm{d}s$.

3.2. Solving of Nonlinear Equation. This study adopts Newton-Raphson method to solve the nonlinear equations (9) because matrix $\mathbf{C}_k$ contains nonlinear term of displacement field. First, (9) is expressed as vector form $\mathbf{L}(\mathbf{a}) \equiv 0$, in which $\mathbf{L}(\mathbf{a}) = (\mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3, L_4)$ and vector quantity $\mathbf{a} = (\widehat{\mathbf{u}}, \widehat{\mathbf{v}}, \widehat{\mathbf{p}}, \gamma)$:

$$\begin{gathered}
\mathbf{L}_1 = \mathbf{A}_1 \widehat{\mathbf{u}} + \mathbf{C}_1 (\widehat{\mathbf{u}}) \widehat{\mathbf{u}} - \mathbf{G}_1 \widehat{\mathbf{p}} + \mathbf{R}_1 \widehat{\mathbf{p}} - \mathbf{B}_1, \\
\mathbf{L}_2 = \mathbf{A}_2 \widehat{\mathbf{v}} + \mathbf{C}_2 (\widehat{\mathbf{u}}) \widehat{\mathbf{v}} - \mathbf{G}_2 \widehat{\mathbf{p}} + \mathbf{R}_2 \widehat{\mathbf{p}} - \mathbf{B}_2, \\
\mathbf{L}_3 = \left(-\mathbf{G}_1^{\mathrm{T}} + R_1^{\mathrm{T}} \right) \widehat{\mathbf{u}} + \left(-\mathbf{G}_2^{\mathrm{T}} + R_2^{\mathrm{T}} \right) \widehat{\mathbf{v}} + \gamma \mathbf{W}, \\
L_4 = \mathbf{W}^{\mathrm{T}} \mathbf{p}.
\end{gathered} \tag{9}$$

Conduct Taylor expansion on vector $\mathbf{L}(\mathbf{a})$ and ignore high-order term and then obtain $0 \equiv \mathbf{L}(\mathbf{a}^{(n)} + \Delta \mathbf{a}^{(n)}) \approx \mathbf{L}(\mathbf{u}^{(n)}) + (\partial \mathbf{L}/\partial \mathbf{a})_{\mathbf{a}=\mathbf{a}^{(n)}} \Delta \mathbf{a}^{(n)}$, in which Jacobian matrix is

$$K_T = \frac{\partial \mathbf{L}}{\partial \mathbf{a}} = \begin{bmatrix} \dfrac{\partial \mathbf{L}_1}{\partial \widehat{\mathbf{u}}} & \dfrac{\partial \mathbf{L}_1}{\partial \widehat{\mathbf{v}}} & \dfrac{\partial \mathbf{L}_1}{\partial \widehat{\mathbf{p}}} & \dfrac{\partial \mathbf{L}_1}{\partial \gamma} \\ \dfrac{\partial \mathbf{L}_2}{\partial \widehat{\mathbf{u}}} & \dfrac{\partial \mathbf{L}_2}{\partial \widehat{\mathbf{v}}} & \dfrac{\partial \mathbf{L}_2}{\partial \widehat{\mathbf{p}}} & \dfrac{\partial \mathbf{L}_2}{\partial \gamma} \\ \dfrac{\partial \mathbf{L}_3}{\partial \widehat{\mathbf{u}}} & \dfrac{\partial \mathbf{L}_3}{\partial \widehat{\mathbf{v}}} & \dfrac{\partial \mathbf{L}_3}{\partial \widehat{\mathbf{p}}} & \dfrac{\partial \mathbf{L}_3}{\partial \gamma} \\ \dfrac{\partial L_4}{\partial \widehat{\mathbf{u}}} & \dfrac{\partial L_4}{\partial \widehat{\mathbf{v}}} & \dfrac{\partial L_4}{\partial \widehat{\mathbf{p}}} & \dfrac{\partial L_4}{\partial \gamma} \end{bmatrix}. \tag{10}$$

In the equations, for matrix $\partial \mathbf{L}_1/\partial \widehat{\mathbf{u}} = \mathbf{A}_1 + \mathbf{C}_1 + \mathbf{D}_1$, element of matrix $\mathbf{D}_1$: $D_{ij}^1 = \int_\Omega N_i^u (\partial u/\partial x) N_j^u \mathrm{d}x\, \mathrm{d}y$, matrix $\partial \mathbf{L}_1/\partial \widehat{\mathbf{v}} = \mathbf{E}_1$, its element: $E_{ij}^1 = \int_\Omega N_i^u (\partial u/\partial y) N_j^v \mathrm{d}x\, \mathrm{d}y$, matrix $\partial \mathbf{L}_1/\partial \widehat{\mathbf{p}} = -\mathbf{G}_1 + \mathbf{R}_1$, vector $\partial \mathbf{L}_1/\partial \gamma = 0$. For matrix $\partial \mathbf{L}_2/\partial \widehat{\mathbf{u}} = \mathbf{E}_2$, its element: $E_{ij}^2 = \int_\Omega N_i^v (\partial v/\partial x) N_j^u \mathrm{d}x\, \mathrm{d}y$, matrix $\partial \mathbf{L}_2/\partial \widehat{\mathbf{v}} = \mathbf{A}_2 + \mathbf{C}_2 + \mathbf{D}_2$, in which, element of matrix $\mathbf{D}_2$: $D_{ij}^2 = \int_\Omega N_i^v (\partial v/\partial y) N_j^v \mathrm{d}x\, \mathrm{d}y$, $\partial \mathbf{L}_2/\partial \widehat{\mathbf{p}} = -\mathbf{G}_2 + \mathbf{R}_2$, vector $\partial \mathbf{L}_2/\partial \gamma = 0$. For matrix $\partial \mathbf{L}_3/\partial \widehat{\mathbf{u}} = -\mathbf{G}_1^{\mathrm{T}} + \mathbf{R}_1^{\mathrm{T}}$, matrix $\partial \mathbf{L}_3/\partial \widehat{\mathbf{v}} = -\mathbf{G}_2^{\mathrm{T}} + \mathbf{R}_2^{\mathrm{T}}$, matrix $\partial \mathbf{L}_3/\partial \widehat{\mathbf{p}} = 0$, vector $\partial \mathbf{L}_3/\partial \gamma = \mathbf{W}$. Vector $\partial L_4/\partial \widehat{\mathbf{u}} = 0$, vector $\partial L_4/\partial \widehat{\mathbf{v}} = 0$, vector $\partial L_4/\partial \widehat{\mathbf{p}} = \mathbf{W}^{\mathrm{T}}$, scalar $\partial L_4/\partial \gamma = 0$.

The equation can be solved with Newton-Raphson iteration method:

$$\begin{gathered}
\mathbf{K}_T \Delta \mathbf{a}^{(n)} = -\mathbf{L}\left(\mathbf{a}^{(n)} \right), \\
\mathbf{a}^{(n+1)} = \mathbf{a}^{(n)} + \omega \Delta \mathbf{a}^{(n)}.
\end{gathered} \tag{11}$$

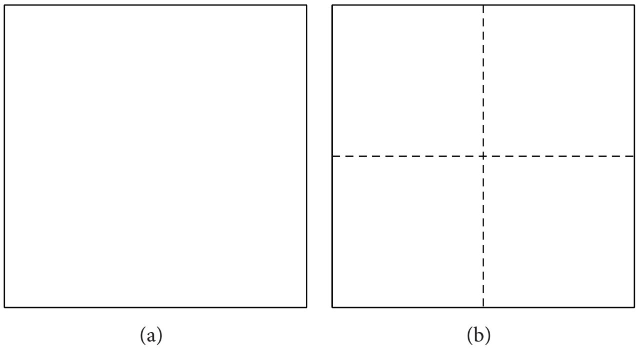


FIGURE 1: Mesh refinement.

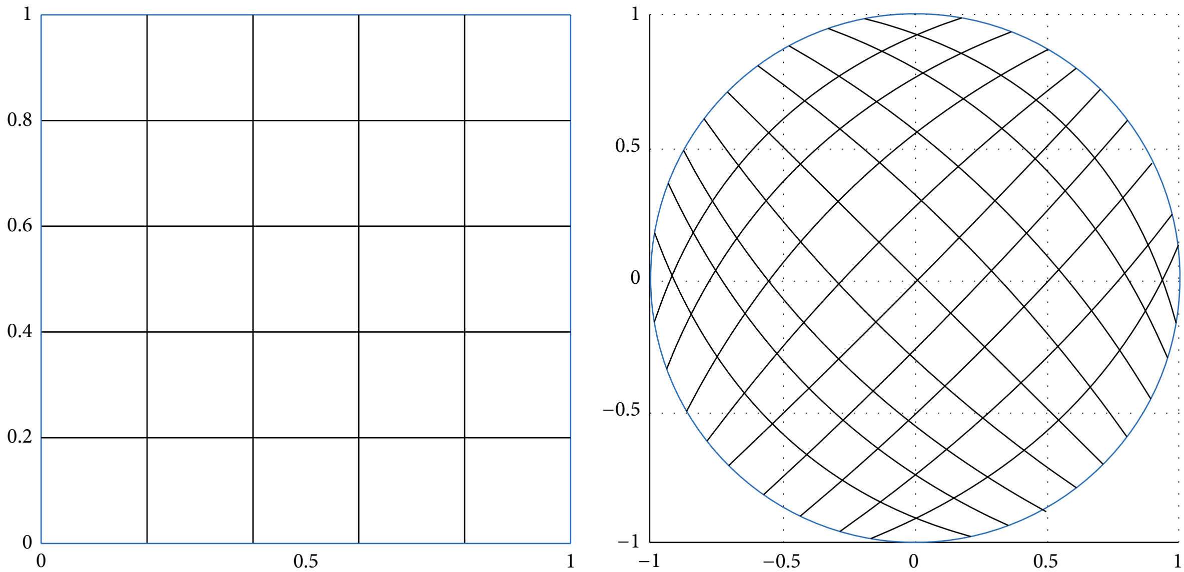


FIGURE 2: Grid division.

In the equations, n refers to iterative times and $0 < \omega \leq 1$ refers to relaxation factor.

4. Stable Grid Discretization

Approximate function space that satisfies LBB condition [6] (or called inf-sup condition) should be constructed when mixed finite element method is applied to solve Navier-Stokes equation

$$\inf_{p\in W} \sup_{\mathbf{u}\in \mathbf{S}} \left(\frac{-\int_{\Omega} p\nabla \cdot \mathbf{u} d\Omega}{|\mathbf{u}|_1 \|p\|_0} \right) \geq \alpha > 0. \tag{12}$$

In the equation, α refers to the constant that is independent of grid discretization. It is theoretically difficult to prove that certain unit format meets above condition. Moreover, perfect error analysis theory on spline element method has not been established and there are only a small amount of literatures for [18]. Therefore, the stability of grid discretization is verified with a kind of numerical test called "inf-sup test," which is similar to the patch test that proves whether the nonconforming finite element is in convergence. It is an effective tool to verify unit quality.

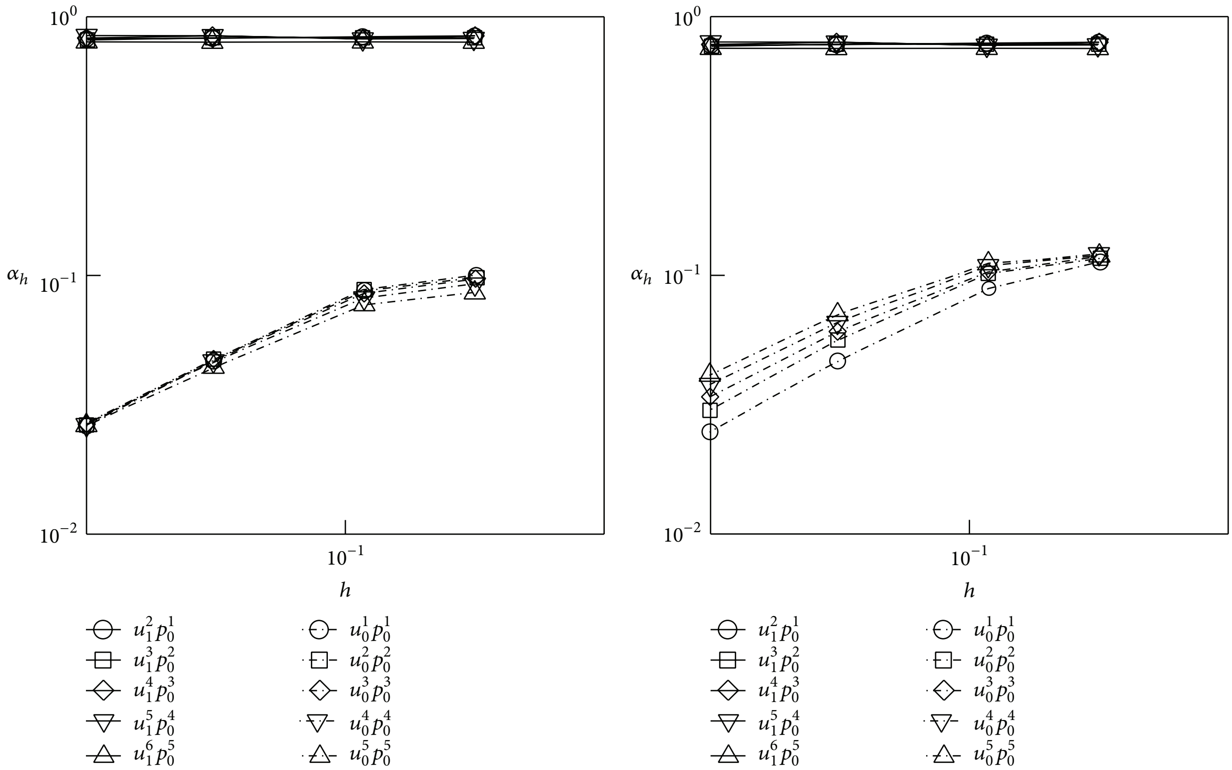
Here the method of inf-sup test mentioned in [19] is briefly introduced. The above LBB condition can be expressed as a discrete version

$$\alpha_h = \inf_{\widehat{\mathbf{u}}_h} \sup_{\widehat{\mathbf{p}}_h} \left(\frac{\widehat{\mathbf{u}}_h^{\mathrm{T}} \mathbf{G} \widehat{\mathbf{p}}_h}{\sqrt{\widehat{\mathbf{u}}_h^{\mathrm{T}} \mathbf{K} \widehat{\mathbf{u}}_h} \sqrt{\mathbf{v}_h^{\mathrm{T}} \mathbf{Q} \mathbf{v}_h}} \right), \tag{13}$$

where element of matrix $\mathbf{Q}$: $\mathbf{Q}_{ij} = \int_{\Omega} N_i^p N_j^p dx$, matrix $\mathbf{K} = \begin{bmatrix} \mathbf{K}_1 & 0 \\ 0 & \mathbf{K}_2 \end{bmatrix}$, and matrix $\mathbf{G} = \begin{bmatrix} \mathbf{G}_1 \\ \mathbf{G}_2 \end{bmatrix}$. Then generalized eigenvalue problem can be approached through a series of transformations:

$$\mathbf{P}\varphi = \lambda \mathbf{Q}\varphi. \tag{14}$$

In the equation, matrix $\mathbf{P} = \mathbf{G}^{\mathrm{T}}\mathbf{K}^{-1}\mathbf{G}$. If the eigenvalue sequence of above problem is $0 = \lambda_1 = \lambda_2 \cdots = \lambda_{k-1} < \lambda_k \leq \lambda_{k+1} \cdots \lambda_n$, then inf-sup constant is $\alpha_h = \sqrt{\lambda_k}$, namely, the



(a) Square area

(b) Circular area

FIGURE 3: Inf-sup constant.

square root of the smallest nonzero eigenvalue. Inf-sup test requires that the inf-sup constant α_h should be independence of mesh size h.

The approximation capability of spline function depends on the function power and grid density, so the approximation precision could be improved through the promotion of power and grid density. Assume that parameter region D is equally divided into n shares along arbitrary coordinate direction; then the total number of units is N^2 and the mesh size is $h = 1/N$, and initial mesh is denoted by $\pi_0(h)$. As shown in Figure 1, take a unit from the grid (Figure 1(a)) and then equally divide them into four small units (Figure 1(b)). After the refinement, the total number of units is $(2N)^2$, the mesh size is $h/2$, and the new grid is denoted by $\pi_1(h/2)$.

This paper adopts such a stable discrete format, namely, the power of spline function closed to velocity field $\mathbf{u}$ is one order higher than that of spline function closed to pressure field p. What is more, velocity field $\mathbf{u}$ adopts one grid refinement $\pi_1(h/2)$ and pressure field p adopts original grid $\pi_0(h)$. As a contrast, in another unstable discrete format, velocity field and pressure field share the grids with same density and same power of primary function. For the convenience of later reference, we called the former as 4/1 format and the later as 1/1 format. What is more, such a mark $u_c^a p_d^b$ is adopted, in which u refers to velocity field and p refers to pressure field. The superscript a (or b) refers to power of spline function and the subscript c (or d) is 0 or 1, in which value 0 refers to origin gird and value 1 refers to refined grid.

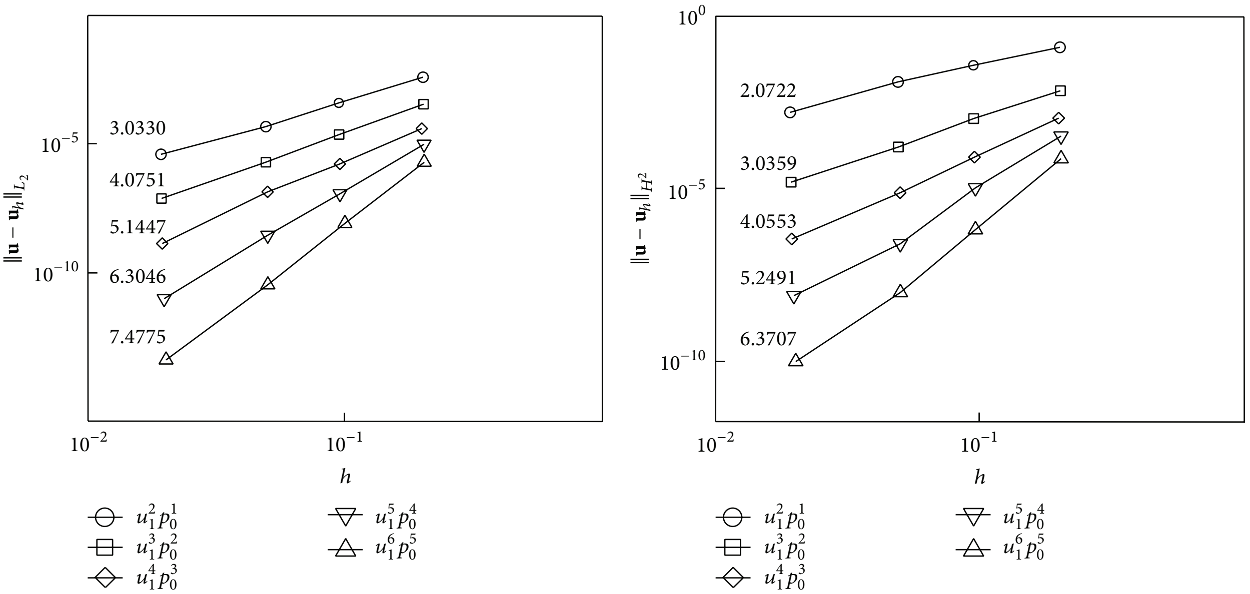
Now, a test on a group of numbers is conducted using two kinds of geometry regions shown on Figure 2. Equally divide them in parameter regions and adopt $n = 1, 2, \ldots, 5$ for the power of spline function closed to pressure field, respectively. For each group of splines, continuously defined grids $N \times N$ will be adopted, in which the number of units in each direction is $N = 5, 10, 20, 40$. The numerical results are shown on Figure 3: horizontal ordinate is mesh size $h = 1/N$ and vertical coordinate is constant α_h of inf-sup. It is shown in the figure that the inf-sup constant of 4/1 format is independent of mesh size h, but the inf-sup constant of 1/1 format is gradually decreased with continuous refinement of grids.

5. Examples of Numerical Calculation

5.1. Rectangular Area. Taking 2D Navier-Stokes flow that is defined within unit rectangular area $\Omega = [0,1] \times [0,1]$ into account, assume the analysis formulas of flow function $\psi(x, y)$ and pressure function $p(x, y)$ are

$$\psi(x, y) = \frac{\sin(\pi x)\sin(\pi y)}{\pi},$$
$$p(x, y) = \frac{\cos(\pi x)^2 + \cos(\pi y)^2}{2} + 2\pi\cos(\pi x)\cos(\pi y). \tag{15}$$

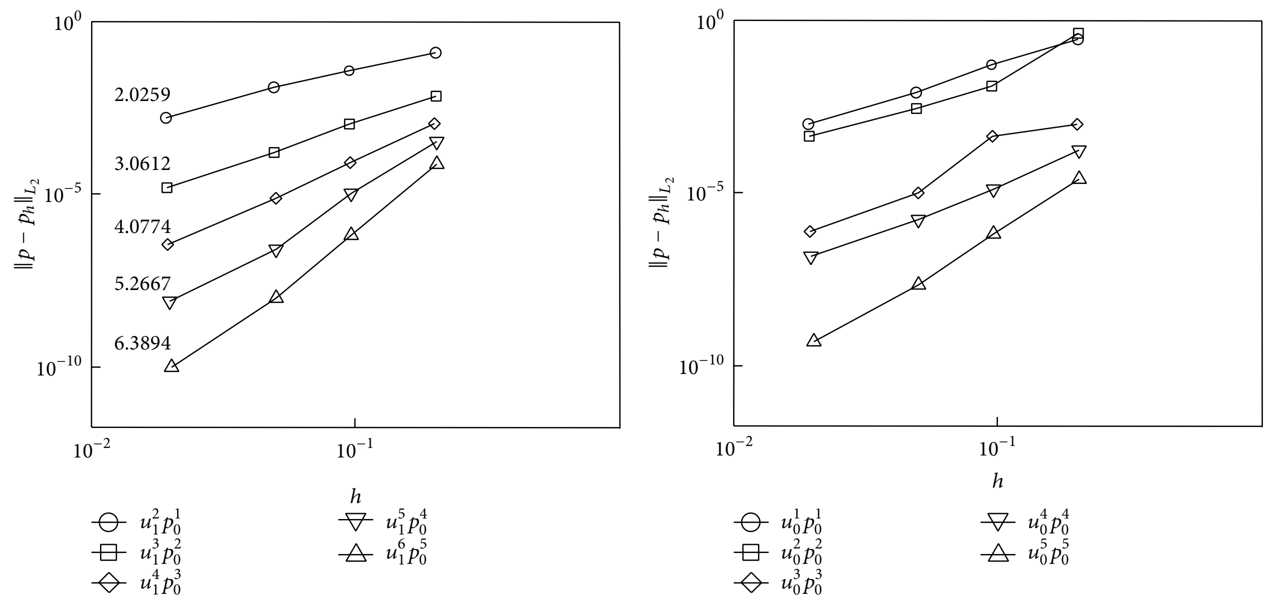
So the components of velocity field are $u = \partial\psi/\partial y = \sin(\pi x)\cos(\pi y)$ and $v = -\partial\psi/\partial x = -\cos(\pi x)\sin(\pi y)$, and



(a) L_2-norm errors

(b) H^2-norm errors

FIGURE 4: Convergence rate of velocity of Section 5.1.



(a) L_2-norm errors of 4/1 mesh format

(b) L_2-norm errors of 1/1 mesh format

FIGURE 5: Convergence rate of pressure of Section 5.1.

the components of volume force source term are $f_1 = 0$ and $f_2 = -4\pi^2\cos(\pi x)\sin(\pi y)$. Adopt grid division shown on Figure 3(a) and add Dirichlet velocity boundary conditions on the assumption of viscosity coefficient being $\mu = 1$.

After a group of tests, it is found that the unit number from each direction is $N = 5, 10, 20, 40$ when the degree of spline function closed to pressure field is $n = 1, 2, \ldots 5$, and t. Figure 4(a) shows the convergence curve of approximate velocity field $\mathbf{u}_h$ under L2-norm, in which horizontal ordinate refers to mesh size h and vertical coordinate refers to error of L_2-norm. And the convergence rate value is marked beside every curve. Similarly, Figure 4(b) and Figure 5(a) show the convergence curves of approximate velocity field $\mathbf{u}_h$ under H^1-norm and approximate solution of pressure field p_h under L2-norm. It should be noticed that if 4/1 discrete format is adopted, then solution of Navier-Stokes equation through spline function can get optimal convergence rate (optimal convergence rate refers to convergence of $p+1$ order under

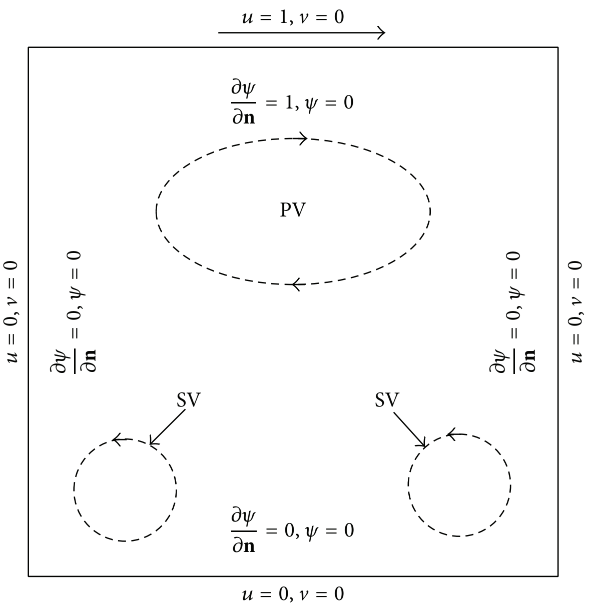


Figure 6: Lid-driven cavity flow.

L_2-norm, convergence of p order under H^1-norm.). But for unstable 1/1 discrete format, false numerical oscillation will exist on pressure field p_h (especially $u_0^2 p_0^2$ and $u_0^3 p_0^3$, as is shown as Figure 5(b)).

5.2. Cavity Flow. Lid-driven flow within unit square area $\Omega = [0,1] \times [0,1]$ is approached in this part. As is shown in Figure 6, side wall and cavity bottom is fix as ($u = 0$, $v = 0$), the lid is moving with constant velocity ($u = 1$, $v = 0$), and the fluid in the cavity flows with the driven of surface viscous force. Under moderate condition of Re number, except primary vortex (PV) in the center of square cavity, there still exists secondary vortex (SV) on the corner of cavity bottom.

Now, the postprocessing is presented. Due to the relationship between flow function and velocity field $\mathbf{u} = (u, v)$

$$-\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \qquad \psi\big|_{\partial\Omega} = h. \tag{16}$$

In the equation, $h(x) - h(x_0) = \int_{(\widehat{\mathbf{x}_0,\mathbf{x}})} \nabla\psi \cdot \boldsymbol{\tau}\, ds$, boundary tangent vector $\boldsymbol{\tau} = (-n_2, n_1)$. After solving such a Dirichlet poison equation, distribution of the stream-function can be obtained through postprocessing.

Distribution of the stream-functions of Re = 100 and Re = 1000 is, respectively, shown in Figure 7, in which power of spline function closed to pressure field is $n = 3$, unit number from each direction is $N = 20$. Values of contour line in Figures 8(a) and 8(b) are −0.10, −0.09, −0.07, −0.05, −0.03, −0.01, −0.001, −0.0001, 1×10^{-7}, 1×10^{-6}, 1×10^{-5}, 0.001 and −0.115, −0.11, −0.10, −0.09, −0.07, −0.05, −0.03, −0.01, −0.001, 1×10^{-6}, 0.001, 0.002, 0.005, 0.001, 0.0015, 0.0017, respectively. After listing minimum stream function value of primary vortex and vortex center position in Table 1 and comparing with the date of literature [11, 12], it is found that the results are basically identical.

Table 1: Comparison on minimum stream function value of primary vortex and position of vortex center.

Case	x	y	$\psi_{\min}$
Re = 100/present	0.6160	0.7360	−0.103523
Re = 100/reference [11]	0.6172	0.7344	−0.103423
Re = 1000/present	0.5320	0.5640	−0.118683
Re = 1000/reference [11]	0.5313	0.5625	−0.117929
Re = 1000/reference [12]	0.5300	0.5650	−0.118885

Table 2: Control vertexes and weight factors of Section 5.3.

i	$d_{i,1}$	$d_{i,2}$	$d_{i,3}$	$w_{i,1}$	$w_{i,2}$	$w_{i,3}$
1	(0, 1)	(−1, 1)	(−1, 0)	1	1	1
2	(1, 1)	(0, 0)	(−1, 1)	$\sqrt{2}/2$	1	$\sqrt{2}/2$
3	(1, 0)	(1, −1)	(0, −1)	1	1	1

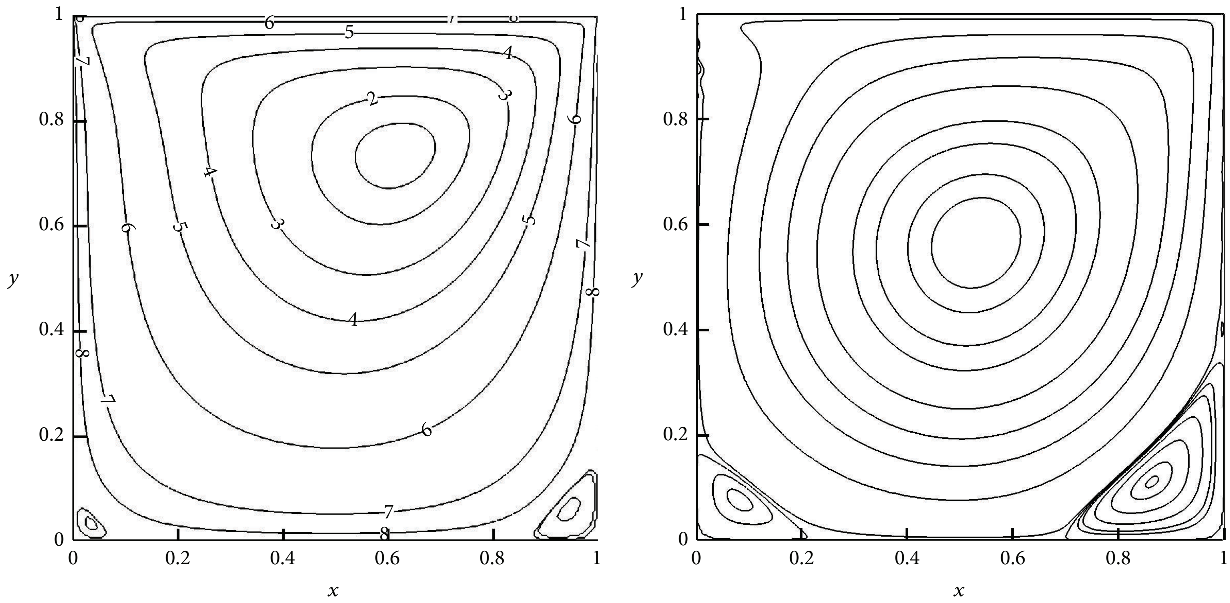
At last, in order to visually compare the results, velocity magnitudes on parallel centerline and vertical centerline are given. In Figure 8, horizontal ordinate refers to x-coordinate and vertical coordinate refers to velocity component v. In Figure 9, horizontal ordinate refers to velocity component u and vertical coordinate refers to y-coordinate vertical centerline. The grid of 20×20 is adopted in the calculation, and power of spline function closed to pressure field is $n = 1, 2, \ldots, 5$. The data chosen as testing benchmark are from Literature [11, 12], which coincides with the results in this paper.

5.3. Unit Circular Area. 2D Navier-Stokes flow in unit circular area (the center is on original point) is defined. Assume the analysis formulas of flow function $\psi(x, y)$ and pressure function $p(x, y)$ are

$$\begin{aligned} \psi(x,y) &= \frac{\left(1 - x^2 - y^2\right)^2}{64}, \\ p(x,y) &= \frac{x^2 y^2 \left(x^2 + y^2 - 2\right)}{512} + \frac{x^2\left(3 - 3x^2 + x^4\right)}{1536} \\ &\quad + \frac{y^2\left(3 - 3y^2 + y^4\right)}{1536} + \frac{xy}{2} - \frac{1}{2048}. \end{aligned} \tag{17}$$

Velocity components are $u = (-1/16)y(1 - x^2 - y^2)$ and $v = (1/16)x(1 - x^2 - y^2)$; components of volume force source term are $f_1 = 0$ and $f_2 = x$. Assume Dirichlet boundary condition and viscosity coefficient $\mu = 1$ are added. Parameterization secondary rational Bezier carve surface should be adopted in unit circular area, and its control vertexes and weight factors are shown on Table 2.

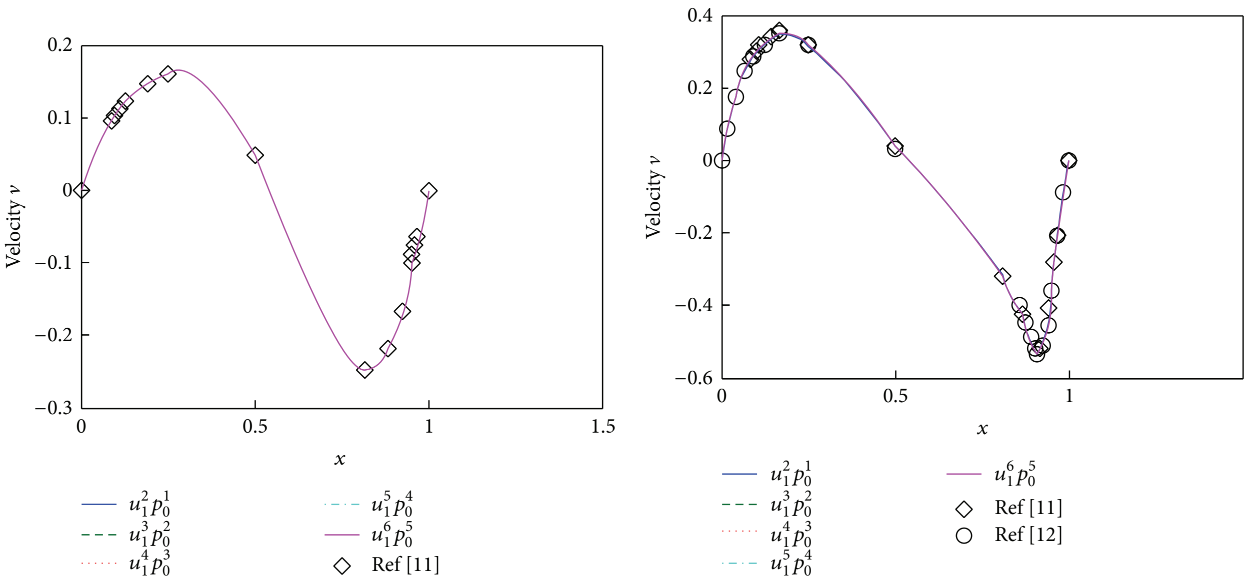
The set of spline function power and parameter grid is the same as that of Section 5.1. Figure 10 shows the convergence curves of approximate velocity field $\mathbf{u}_h$ under L_2-norm and H^1-norm, both of which reached the optimal convergence rate. For comparison, Figure 11 shows the convergence curves of pressure numerical solution p_h under L_2-norm calculated in 4/1 grid and 1/1 grid, respectively. It is obvious that the false numerical oscillation of unstable format causes the degradation of convergence rate (Figure 11(b)).



(a) Re = 100

(b) Re = 1000

FIGURE 7: Streamline distribution of fluid ($p = 3$, $N = 20$).



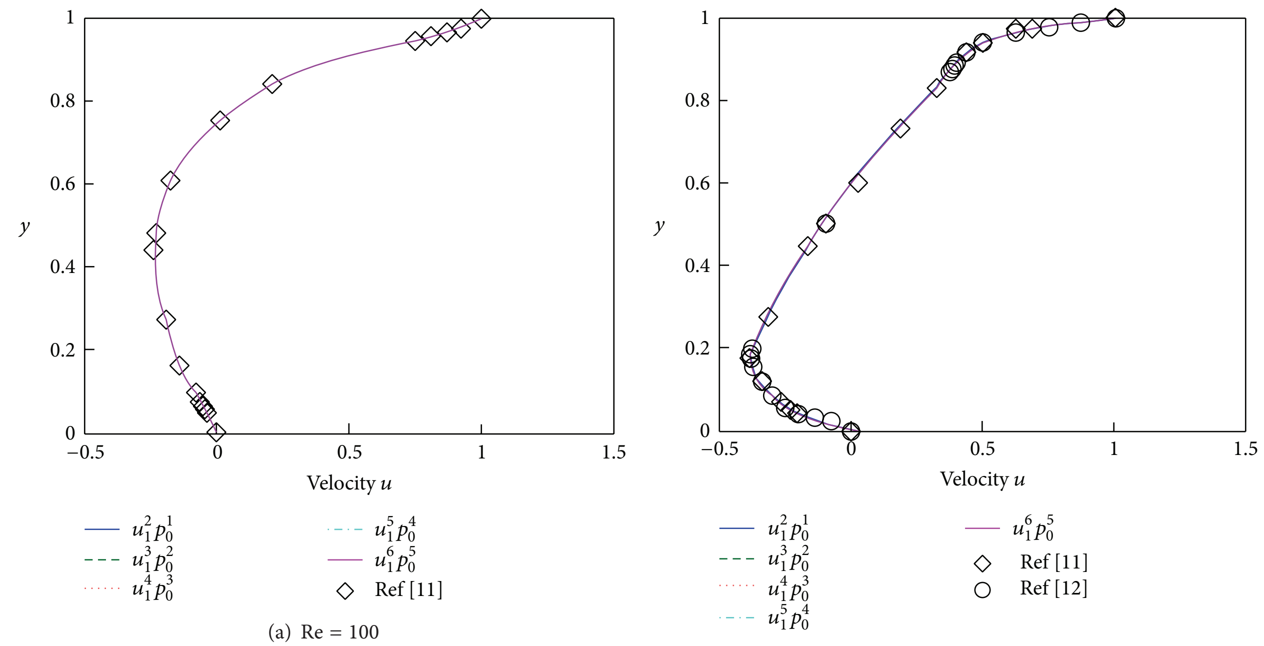
(a) Re = 100

(b) Re = 1000

FIGURE 8: Velocity magnitude along parallel centerline.

5.4. Circular Couette Flow. Lastly, the typical problem of Circular Couette Flow is considered. As is shown in Figure 12, there are viscous incompressible fluids between two infinite-length concentric cylinders. The radiuses of outer cylinder and inner cylinder are R_1 and R_2 and they are rotated with the constant angular velocity of Ω_1 and Ω_2. Assume that the rotating velocity is slower and the fluid is in steady laminar flow phase; then there is an analytical solution of tangential velocity:

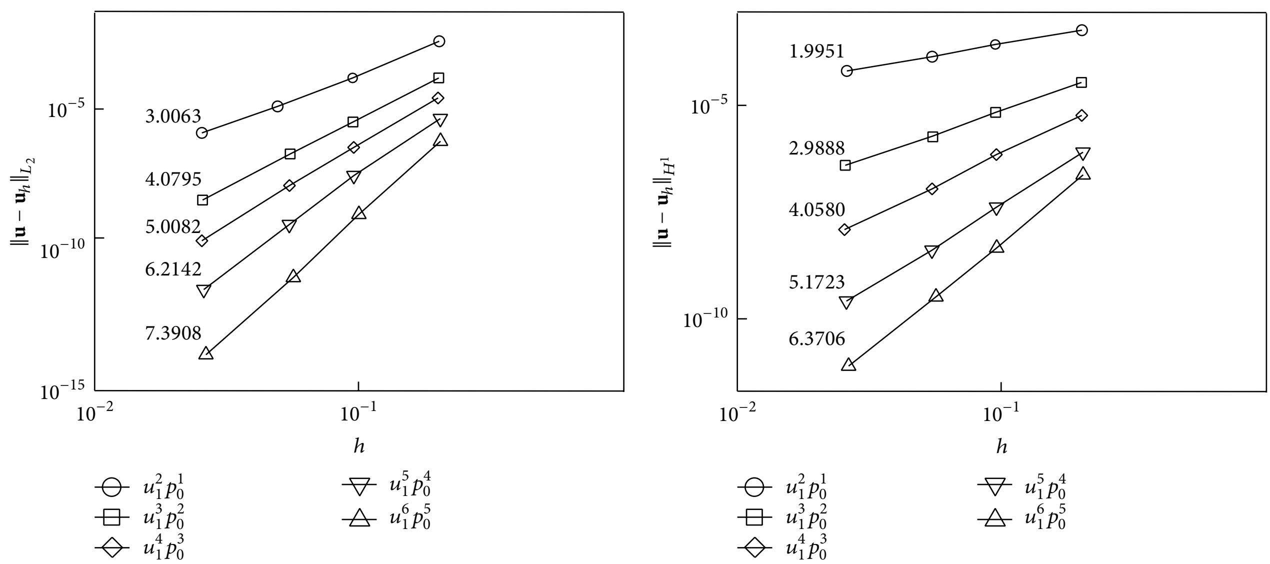
$$u_\theta = Ar + \frac{B}{r},$$
$$A = \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2}, \qquad B = \frac{(\Omega_1 - \Omega_2) R_1^2 R_2^2}{R_2^2 - R_1^2}. \tag{18}$$



(a) Re = 100

(b) Re = 1000

FIGURE 9: Velocity magnitude along vertical centerline.



(a) L_2-norm errors

(b) H^1-norm errors

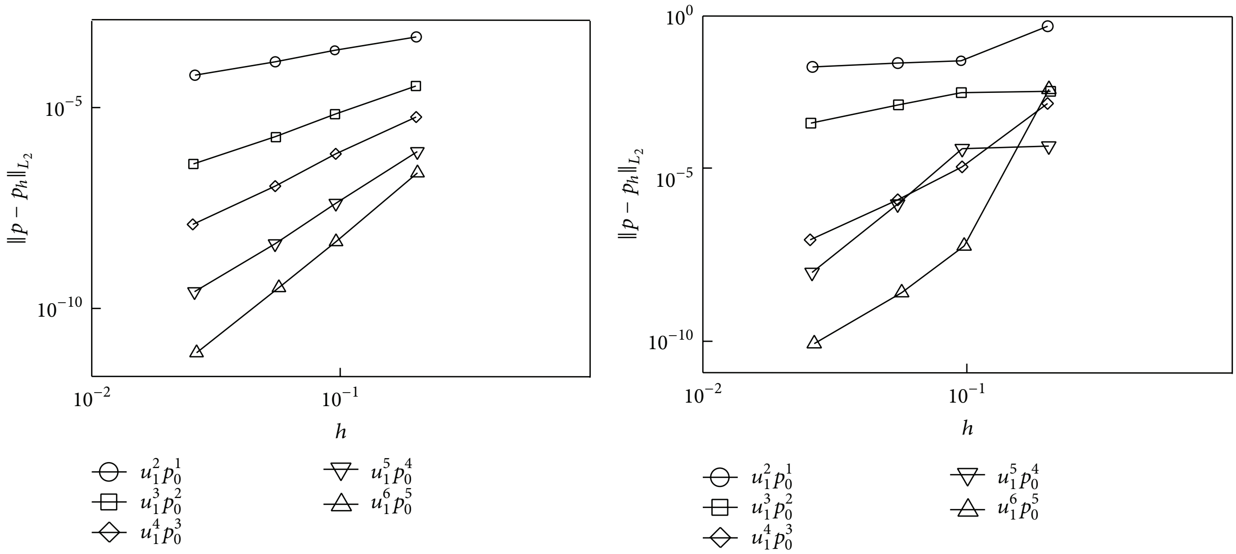
FIGURE 10: Convergence rate of velocity of Section 5.3.

In the equations, $r = \sqrt{x^2 + y^2}$ refers to radial coordinate. This paper assumes that fixation of outer cylinder is $\Omega_2 = 0$ and angular velocity of inner cylinder is $\Omega_1 = 1$. Due to its symmetry, 1/4 is taken for analysis and the settings of geometric definition, grid, and boundary condition are shown on Figure 13. Please refer to Table 3 for control vertex and weight factor within defined geometry area.

Figure 14 shows the distribution of tangential velocity u_θ. The power of spline function approaching pressure field is $n = 3$ and the grid is 20×20. Figure 15 shows the distribution of tangential velocity u_θ on radial coordinate with angle of 45°. It can be noticed that they coincide with the analytical solution.

TABLE 3: Control vertex and weight factor of Section 5.3.

i	$d_{i,1}$	$d_{i,2}$	$d_{i,3}$	$w_{i,1}$	$w_{i,2}$	$w_{i,3}$
1	(1, 0)	(1, 1)	(0, 1)	1	$\sqrt{2}/2$	1
2	(2, 0)	(2, 2)	(0, 2)	1	$\sqrt{2}/2$	1



(a) L_2-norm errors of 4/1 mesh format (b) L_2-norm errors of 1/1 mesh format

FIGURE 11: Convergence rate of pressure of Section 5.3.

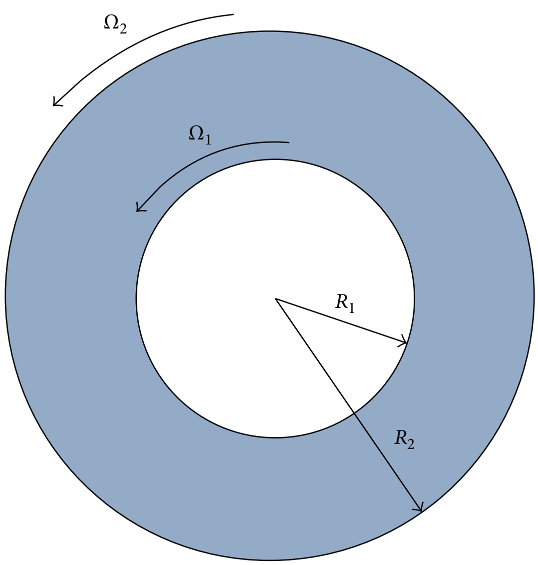


FIGURE 12: Circular Couette flow.

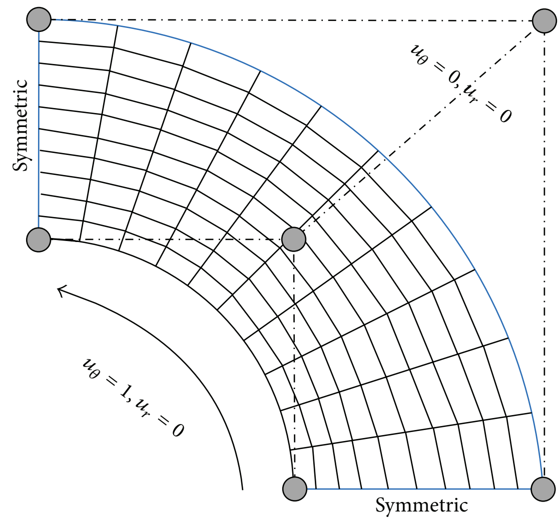


FIGURE 13: Settings of grid and boundary condition.

6. Conclusion

This paper solved the problem of incompressible Navier-Stokes flow through geometrical precise spline element method. (1) This method overcame the poor geometric versatility of spline element method; adoption of rational Bezier surface patch in mapping function can accurately express complex geometry areas. (2) It presents a stable discrete mesh format meeting in-sup condition, which expanded the spline method into fluid.

This paper only discussed 2D fluid problem, but its conclusion can be directly generalized to 3D conditions. Problems to be solved are (1) variation of transient problems with time; (2) multiarea tiling problem, which shall adapt to more complicated computational area of topology; (3) parameterized method of complex solution domain.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

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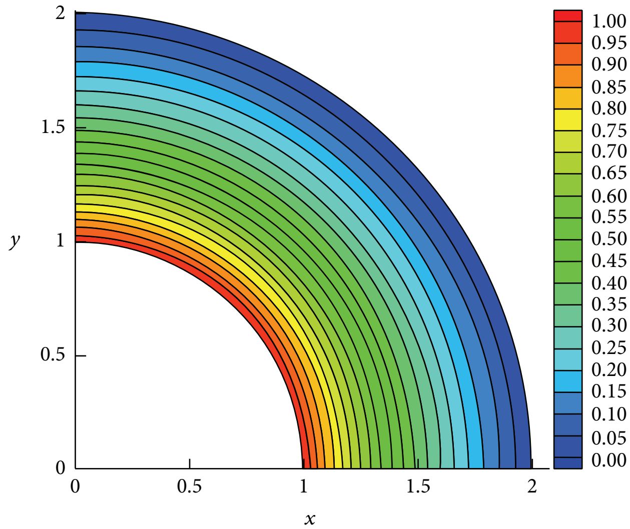

FIGURE 14: Distribution of tangential velocity u_θ.

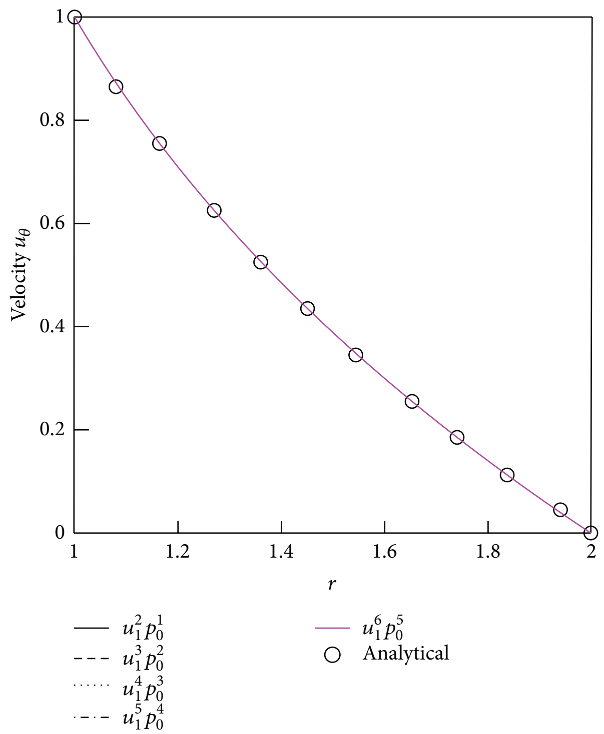

FIGURE 15: Distribution of tangential velocity u_θ on radial coordinate.

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Research Article

A New Algorithm for System of Integral Equations

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We develop a new algorithm to solve the system of integral equations. In this new method no need to use matrix weights. Beacause of it, we reduce computational complexity considerable. Using the new algorithm it is also possible to solve an initial boundary value problem for system of parabolic equations. To verify the efficiency, the results of computational experiments are given.

1. Introduction

The theory and application of integral equations are an important subject within pure and applied mathematics and they appear in various types in many fields of science and engineering. The integral equations can also be represented as convolution integral equations; see Srivastava and Buschman [1]. In the applications, the number of computational problems can be reduced to the solution of a system of integral equations (system of IEs) of the second kind; see [2–4]. However, solving systems of integrodifferential equations are very important and such systems might be difficult analytically, so many researchers have attempted to propose different numerical methods which are accurate and efficient. For example, numerical expansion methods for solving a system of linear IDEs by interpolation and Clenshaw Curtis quadrature rules were presented in [5], where the integral system was transferred into a matrix equation by the interpolation points. Pour in [6] studied an extension of the Tau method to obtain the numerical solution of Fredholm integrodifferential equations systems ad applied Chebyshev basis to solve IDEs. Similarly, Arikoglu and Ozkol [7] obtained solutions of integral and integrodifferential equation systems by using differential transform method where the approch provides very good approximation to the exact solution.

Recently, the solution of the system has been estimated by many different basic functions, such as orthonormal bases and wavelets; see, for example [8, 9], and the hybrid Legendre Block-Pulse functions, that is, a combination of the Block-Pulse functions on $[0, 1]$ and Legendre polynomials was proposed. In addition, the Bessel matrix method was introduced in [10] for solving a system of high order linear Fredholm differential equations with variable coefficients. In the literature there are several methods to solve the different type of integral equations; see [11–16]. One of the novel methods is known as the vector Monte Carlo algorithms to solve the system of IEs. Among the vector Monte Carlo algorithms the following are well known:

(i) an algorithm for solving the system of transfer equations with polarization;

(ii) a vector algorithm for solving multigroup transfer equations;

(iii) a Monte Carlo technique combined with the finite sum method and vector Monte Carlo method for solving metaharmonic equations.

In the use of this method one can easily see that the variance of the vector estimate largely depends on the form of transitional density. Thus appropriate choice of the density leads to the reduction of the complexity calculations, which is defined as the product of the variance and the computational time. To determine the density is difficult as to solve the problem itself, although in some cases it is possible to obtain a minimal criterian of uniform optimality of the method. The transitional density that corresponds to minimum complexity of algoritm is said to be optimal for a given problem.

In Mikhailov [17], vector Monte Carlo algorithms are used to solve system of IEs. The distinguished feature of that vector algorithm is that its "weight" appears in the form of a matrix weight. This matrix weight is multiplied by the kernel matrix of the system of IEs dividing by a transition density function in the Markov chain simulation, so that a number of computational problems can be reduced to the solution of a system of IEs of second kind. By introducing a suitable discrete-continuous measure of the integration, we can write the system of IEs in the form of a single integral equation, and this allows us to use standard algorithms of the Monte Carlo method. However, it is more expedient to make use of the matrix structure of the system and solve the problem by the Monte Carlo method with vector weights. The following vector Monte Carlo algorithms are well known: an algorithm for solving the system of transfer equations with polarization taken into account, a vector algorithm for solving multigroup transfer equations, a Monte Carlo technique combined with the finite sum method, and vector Monte Carlo method for solving metaharmonic equation.

In this study, a new algorithm is proposed for the numerical solution of system of IEs but in this algorithm we do not use matrix weights. The proposed algorithm has usual advantages of ordinary Monte Carlo method. The new algorithm is considerably reduced to computational complexity. Using this new algorithm we have solved an initial boundary value problem for system of parabolic equations. The paper is organized as follows. In Section 2, we present the description of the problem and proposed a new Monte Carlo algorithm for the solution of system of IEs. In Section 3, we discuss the application of the method to the solution of system of parabolic equations. In Section 4, we will construct biased and ε-biased estimators for the solution. In Section 5, the results of computational experiments are given, followed by the conclusion in Section 6.

2. Description of the Problem and a New Approach for the Solution of System of IEs

Let us consider second kind nonhomogeneous system of IEs of the form

$$\varphi_i(x) = \sum_{j=1}^{n} \int_X k_{i,j}(x,y)\,\varphi_j(y)\,dy + h_i(x), \quad i = 1,\dots,n, \tag{1}$$

where $x \in X \subseteq \mathbb{R}^m$, $m \geq 1$ or in vector form

$$\Phi = K\Phi + H, \tag{2}$$

here operator $K : L_\infty \to L_\infty$ where L_∞ is the space of bounded function almost everywhere and

$$\begin{gathered} H = (h_1,\dots,h_n) \in L_\infty, \qquad K = \left(K_{ij}\right) \in L_\infty, \\ \Phi = (\varphi_1,\varphi_2,\dots,\varphi_n) \in L_\infty, \end{gathered} \tag{3}$$

where the norm of H is

$$\|H\|_{L_\infty} = \operatorname*{vrai\ sup}_{1\leq i\leq n,\ x\in X} \left|h_i(x)\right|. \tag{4}$$

Suppose the spectral radius $\rho(K)$ satisfy the inequalities

$$\rho(K) = \lim_{n\to\infty} \left\|K^n\right\|^{1/n} < 1, \qquad \rho(K) \leq \left\|K^n\right\|^{1/2}, \tag{5}$$

where $K^n\varphi = K^{n-1}K\varphi$.

Let Markov chain $\{x_n,\ n = 0,1,\dots,N\}$ with transition density $p(x,y)$ be

$$g(x) = 1 - \int_X p(x,y)\,dy \geq 0, \tag{6}$$

where $g(x)$ is the probability of absorption at the point x_N, where N is the random number of the last moment and in initial moment $x_0 = x$.

A standard vector algorithm of Monte Carlo for $\Phi(x)$ is

$$\begin{gathered} \Phi(x) = \left(M\xi_x, \xi_x\right) = H(x) + \sum_{n=1}^{N} Q_n H(x_n), \\ Q_0 = I, \qquad Q_{n+1} = \frac{Q_n K(x_n, x_{n+1})}{P(x_n, x_{n+1})}, \quad n = 0,1,2,\dots, \end{gathered} \tag{7}$$

where I is a unit matrix, $K(x,y)$ is a kernel matrix $\{k_{ij}(x,y)\}$, and $p(x_n, x_{n+1})$ is the transition density function at the points (x_n, x_{n+1}). The condition for unbiasedness is

$$p(K_1) < 1 \quad \text{or } K_1 = \|K\| < 1. \tag{8}$$

We will assume also that the spectral radius of the operator K_1 obtained from K by the substitution $k_{i,j} \to |k_{i,j}|$ is less than one. Then, by using standard methods of Monte Carlo theory we can show that

$$\begin{gathered} \Phi(x) = E\xi_x, \qquad \xi_x = \sum_{n=0}^{N} Q_n H(x_n), \\ Q_0 = \left\{\delta_{i,j}\right\}_{i,j=(1,\dots,n)}, \qquad Q_n = Q_{n-1}\frac{K(x_{n-1},x_n)}{p(x_{n-1},x_n)}, \end{gathered} \tag{9}$$

where Q_n can be considered as matrix weight and

$$\frac{K(x_{n-1},x_n)}{p(x_{n-1},x_n)} = \left\{\frac{k_{ij}(x_{n-1},x_n)}{p(x_{n-1},x_n)}\right\}, \quad i,j = \{1,2,\dots,n\}. \tag{10}$$

The Monte Carlo method is used to estimate linear functionals of the form

$$(F,\Phi) = \int_X F'(x)\,\Phi(x)\,dx, \tag{11}$$

where $F'(x) = (f_1(x), f_2(x), \dots f_n(x))$ with

$$\|F\|_{L_1} = \sum_{j=1}^{n} \int_X \left|f_j(x)\right| dx < \infty. \tag{12}$$

Let the point x_0 be distributed with initial probability density $\pi(x)$ such that

$$\pi(x) \neq 0 \quad \text{if } F'(x)\Phi(x) \neq 0. \tag{13}$$

Then, obviously (see Mikhailov [17]),

$$\begin{aligned}(F,\Phi) &= E\left[\frac{F'(x_0)}{\pi(x_0)}\xi_{x_0}\right] = E\left[\sum_{n=0}^{N}\frac{F'(x_0)}{\pi(x_0)}Q_n H(x_n)\right] \\ &= E\left[\sum_{n=0}^{N} H'(x_n) Q'_n \frac{F'(x_0)}{\pi(x_0)}\right].\end{aligned} \tag{14}$$

The random vector with weight $Q_n^{(1)} = Q'_n(F'(x_0)/\pi(x_0))$ is computed by the formula

$$Q_n^{(1)} = \frac{K'(x_{n-1},x_n)}{p(x_{n-1},x_n)} Q_{n-1}^{(1)}. \tag{15}$$

Precisely such a vector algorithm, corresponding to the representation $I = (\Phi^*, H)$, has been formulated in the work of Mikhailov [17]. Below on the contrary to vector algorithms we will propose a new algorithm for the solution of system of integral equations. Our method does not use matrix weights.

Suppose we have to find the solution of the inhomogeneous system of IEs (1) of the second kind at the point $x \in X$. We will define two types of Markov chain $\{i_K\}$ and $\{x_K\}$ by the following way.

(a) Definition of the First Homogeneous Markov Chain. Now we simulate the Markov chain $i_0, i_1 \cdots \in N$ with $n+1$ state. Initial state i_0 will simulate according to initial distribution $\pi = (\pi_1, \ldots, \pi_n, 0)$ and the next i_1 with the transition matrix

$$\begin{gathered} A = A(x) = \left\|\alpha_{i,j}(x)\right\|_{i,j=1}^{n+1}, \\ \sum_{j=1}^{n}\alpha_{i,j}(x) = 1 - g_i(x), \\ g_i(x) = \alpha_{i,n+1}(x), \quad i = 1,\ldots,n. \end{gathered} \tag{16}$$

Here $\alpha_{n+1,n+1}(x) = 1$ and

$$A(x) = \begin{pmatrix} \alpha_{11}(x),\ldots,\alpha_{1n}(x), & g_1(x) \\ \alpha_{21}(x),\ldots,\alpha_{2n}(x), & g_2(x) \\ \alpha_{n1}(x),\ldots,\alpha_{nn}(x), & g_n(x) \\ 0,\ldots,0, & 1 \end{pmatrix}. \tag{17}$$

Let N be a random absorption moment with $N = \{\max k,\ i_k \neq n+1\}$, a life time of chain.

(b) A second homogeneous Markov chain $\{x_k\}$ with space phase X is defined by the following way.

Firstly, we define the transition density matrix as

$$P\left(x^1 \longrightarrow x\right) = \left\|P_{i,j}\left(x^1 \longrightarrow x\right)\right\|_{i,j=1}^{n+1},$$

$$P\left(x^1 \longrightarrow x\right) = \begin{pmatrix} P_{11}\left(x^1 \longrightarrow x\right),\ldots,P_{1n}\left(x^1 \longrightarrow x\right), & 0 \\ P_{21}\left(x^1 \longrightarrow x\right),\ldots,P_{2n}\left(x^1 \longrightarrow x\right), & 0 \\ P_{n1}\left(x^1 \longrightarrow x\right),\ldots,P_{nn}\left(x^1 \longrightarrow x\right), & 0 \\ 0,\ldots,0, & 1 \end{pmatrix}. \tag{18}$$

Let an initial point $x_0 = x$; using $\pi(x)$ we will simulate initial moment i_0, then according to the transition matrix $A(x_0)$ we are able to simulate again the next state of chain i_1. It means with the probability $\alpha_{i_0,i_1}(x_0)$, $P(i_1 = k) = \alpha_{0k}(x_0)$.

The next phase coordinates of the chain x_1 simulated according to $p_{i_0,i_1}(x_0,x_1)$. The probability of absorption of the trajectory is $g_{i_0}(x_0)$. Let (i_k, x_k) be known then the next value of i_{k+1} will be defined according to the matrix $A(x_k)$ and next random point x_{k+1} simulated according to the probability density function $P_{i_k,i_{k+1}}(x_k^1 \rightarrow x)$ and so on.

Let $\xi_{x_0} = \xi_N(i_0, i_1, \ldots i_N; x_0, x_1, \ldots x_N)$ be some random variable which is defined by the set of trajectory Markov chains. The mathematical expectations of random variable will be

$$\begin{aligned} E\xi_{x_0} = \sum_{k=0}^{\infty}\sum_{i_0,\ldots i_k=1}^{n} \underbrace{\int\cdots\int}_{k} & \pi_{i_0}\alpha_{i_0,i_1}(x_0)\, p_{i_0,i_1}(x_0 x_1)\cdot\alpha_{i_1,i_2}(x_1) \\ & p_{i_1,i_2}(x_1,x_2)\cdots\alpha_{i_{k-1},i_k}(x_{k-1}) \\ & \times p_{i_{k-1}-1,i_k}(x_{k-1},x_k)\, g_{i_k}(x_k) \\ & \xi_k(i_0,\ldots i_k, x_0,\ldots x_k)\ dx_1\cdots dx_k. \end{aligned} \tag{19}$$

Let us consider calculation of the functional $(\Phi\ , F)$, where $F^T = (f_1(x), \ldots, f_n(x))$ column vector. Let us compute the functional $(\Phi, F) = \sum_{i=1}^{n}\varphi_i(x) f_i(x)$. For doing this task we introduce two well-known estimators according to the Monte Carlo theory. First of them is analog of absorption estimator

$$\begin{aligned}\xi_1(x_0) = {} & \frac{f_{i_0}(x_0)}{\pi_{i_0}(x_0)}\frac{k_{i_0 i_1}(x_0,x_1)}{\alpha_{i_0,i_1}(x_0)\, p_{i_0,i_1}(x_0,x_1)} \\ & \cdots \frac{k_{i_{n-1},i_n}(x_{n-1},x_n)}{\alpha_{i_{n-1},i_n}(x_{n-1})\, p_{i_{n-1},i_n}(x_{n-1},x_n)}\cdot\frac{h_{i_n}(x_n)}{g_{i_n}(x_n)}\end{aligned} \tag{20}$$

and the second one is analog of collision estimator

$$\begin{aligned}\xi_2(x_0) = {} & \sum_{j=1}^{N}\frac{f_{i_0}(x_0)}{\pi_{i_0}(x_0)}\frac{K_{i_0 i_1}(x_0,x_1)}{\alpha_{i_0,i_1}(x_0)\, p_{i_0,i_1}(x_0,x_1)} \\ & \cdots \frac{K_{i_{j-1},i_j}\left(x_{j-1},x_j\right)}{\alpha_{i_{j-1},i_j}\left(x_{j-1}\right) p_{i_{j-1},i_j}\left(x_{j-1},x_j\right)}\cdot h_{i_j}\left(x_j\right).\end{aligned} \tag{21}$$

Theorem 1. *If* $f_{i_0}(x_{i_0}) \neq 0$ *then* $\pi_{i_0}(x_{i_0}) \neq 0$ *and if* $k_{i,i_j}(x_i, x_j) \neq 0$ *then*

$$\alpha_{i,j}(x_i)\, p_{i_i i_j}(x_i, x_j) \neq 0, \quad \text{for any } 1 \le i, j \le n. \tag{22}$$

In this case $E\xi_{1,2} = (\Phi, F)$.

The proof of the theorem is similar to the theorem Ermakov [23], and therefore proof is omitted. Now we will apply the obtained results to the solution system of parabolic equations.

3. Application to System of Parabolic Equations

In this section we consider initial boundary problem for system of parabolic equations. Let D be bounded domain in R^m with enough smooth boundary ∂D of $\Omega = D \times [0,T]$ and Ω is the cylinder in R^{m+1} with parallel spin axis t. The basement is the domain D on the surface $t = 0$ and T is the fixed constant. The functions

$$\begin{gathered} y_{0i}(x) \in C(\overline{D}), \qquad y_i(x,t) \in C(\partial D \times [0,T]), \\ f_i(x,t) \in C(\overline{\Omega}), \end{gathered} \tag{23}$$

where $C(\overline{D})$ stands for a continuous function on the closed domain D.

Now consider the following initial boundary value problem (BVP) for system of parabolic equations:

$$\begin{aligned} &\frac{\partial u_1(x,t)}{\partial t} - a_1 \Delta u_1(x,t) + c_{11} u_1(x,t) \\ &\quad - c_{12} u_2(x,t) - \cdots - c_{1n} u_n(x,t) = f_1(x,t) \\ &\frac{\partial u_2(x,t)}{\partial t} - a_2 \Delta u_2(x,t) + c_{22} u_2(x,t) \\ &\quad - c_{21} u_1(x,t) - \cdots - c_{2n} u_n(x,t) = f_2(x,t) \\ &\qquad \vdots \\ &\frac{\partial u_n(x,t)}{\partial t} - a_n \Delta\, u_n(x,t) + c_{nn} u_n(x,t) - c_{n1} u_1(x,t) \\ &\quad - \cdots - c_{(n-1)n} u_{n-1}(x,t) = f_n(x,t), \end{aligned} \tag{24}$$

where the coefficients $a_i > 0$, $c_{ij} > 0, (i = 1, \ldots, n,\ j = 1, \ldots, n)$, and $(x,t) \in \Omega$ with initial and boundary conditions

$$\begin{aligned} &u_i(x,t) = y_i(x,t), \quad x \in \partial D,\ t \in [0,T],\ i = \overline{1,n} \\ &u_i(x,0) = y_{oi}(x), \quad x \in D,\ i = \overline{1,n}. \end{aligned} \tag{25}$$

Further suppose $f_i(x,t)$, $y_{oi}(x)$, $y_i(x,t)$, and coefficients a_i, c_{ij} $(i,\ j = \overline{1,n})$ are given such that there exists unique solution Ladyzhenskaya et al. [18] and Lions [19] of the initial BVR (24)-(25) and

$$\begin{aligned} &u_i(x,t) \in C(\overline{D} \times [0,T]) \\ &\quad \cap\ C^{2,1}(\overline{D} \times [0,T]) \quad (i = \overline{1,n}), \end{aligned} \tag{26}$$

where $C^{2,1}$ is the set of continuous functions in the given region with continuous derivatives u_x, u_{xx}, and u_t.

Now we construct unbiased estimator for the problem (24)-(25) in the arbitrary point $(x,t) \in \Omega$ on the trajectory some random process. For that we use mean value formula and construct some special system of integral equations for $u_i(x,t)$ in special constructed domains (spheroid or balloid with the center (x,t)).

According to Section 2 below we will propose a new nonstationary Markov chain on which trajectory will construct unbiased estimators for the obtained system of integral equations.

In our algorithm we do not used matrix weight; it means the computational complexity of new algorithm is much better. The basis for the constructing of algorithms will be the formula of parabolic mean for the heat conductivity equations. As we know the fundamental solution $Z(x,t,y,\tau)$ for heat equation $u_t - a\Delta u = 0$ is given by

$$\begin{aligned} &Z(x,t;y,\tau) \\ &\quad = (4\pi a(t-\tau))^{-m/2} \exp\left(-\frac{|x-y|^2}{4a(t-\tau)}\right). \end{aligned} \tag{27}$$

Firstly, we define a special domain using a fundamental solution of the heat equation $Q_r(x,t)$ which depends on $r > 0$ and points $(x,t) \in R^{m+1}$ as

$$Q_r(x,t) = \left\{(y,\tau) : Z(x,t;y,\tau) > (4\pi a r)^{-m/2},\ \tau < t\right\}. \tag{28}$$

The domain $Q_r(x,t)$, we call balloid and $\partial Q_r(x,t)$, spheroid with the center in the point (x,t). From the definition balloid $Q_r(x,t)$, described by following inequality (Kupcov [20]):

$$Q_r(x,t) = \left\{(y,\tau) : |x-y|^2 < 2ma(t-\tau)\ln\frac{r}{t-r}, \tau < t\right\}. \tag{29}$$

Each section with the sectional plain of balloid when $\tau =$ constant will be m-dimensional ball $B(x, R(t-\tau))$ with the center x and with the radius

$$R(t-\tau) = \sqrt{2ma(t-\tau)\ln\frac{r}{t-r}}. \tag{30}$$

Let $(x,t) \in \Omega$ and

$$r = r(x,t) = \min\left\{\frac{R_1^2(x)\, e}{2am}, t\right\}, \tag{31}$$

where $R_1(x)$ is the minimum distance from point x until the boundary; that is,

$$R_1(x) = \inf\left\{\left|x - x'\right|, x \in \partial D, x' \in \overline{D}\right\}. \tag{32}$$

In this case $Q_r(x,t) \subset \overline{\Omega}$. By further using Greens function and fundamental solution we will transfer from the system of differential equations into the system of integral equations. In the book [21] special balance equation analogies were constructed as in [22], which connected the value of function $u(x,t)$ with its integral from the spheroid and balloid with the center in the point (x,t).

Lemma 2 (Kurbanmuradov [22]). *Let the function $u(x,t)$ satisfy the following equation:*

$$\frac{\partial u(x,t)}{\partial t} - a\Delta u(x,t) = f(x,t), \quad (x,t) \in \Omega. \tag{33}$$

Then the following formula of mean is true (mean value formula):

$$u(x,t) = a\iint_{\partial Q_r(x,t)} \left(1 - \frac{t-\tau}{r}\right) \times \left(-\frac{\partial Z(x,t;y,\tau)}{\partial n_y}\right) u(y,\tau)\, ds\, d\tau + \frac{1}{r}\iint_{Q_r(x,t)} Z_r(x,t;y,\tau)\, u(y,\tau)\, dy\, d\tau + F_r(x,t), \tag{34}$$

where

$$F_r(x,t) = \frac{1}{r}\iint_{Q_r(x,t)} (r-(t-\tau))\, Z_r(x,t;y,\tau)\, f(y,\tau)\, dy\, d\tau,$$
$$Z_r(x,t;y,\tau) = Z(x,t;y,\tau) - (4\pi a r)^{-m/2}, \tag{35}$$

here ds is the element of small area of sphere $\partial B(x, R(t-\tau))$. In further using these results we will get special integral representation.

3.1. Transforming a System and Obtaining Integral Representation. Let us define the family of domains $Q_r^{(i)}(x,t)$, which depends on positive parameters $r > 0$ and point $(x,t) \in R^{m+1}$, where

$$Q_r^{(i)}(x,t) = \left\{(y,\tau) : Z_r^i(x,t;y,\tau) > 0, \tau < t\right\}, \tag{36}$$

where $Z_r^i(x,t;y,\tau)$ defined analogous $Z_r(x,t;y,\tau)$ changing a for a_i (see above Lemma 2). The domain $Q_r^i(x,t)$ we will call a balloid with radius r which a center in a point (x,t) and a boundary $\partial Q_r^{(i)}(x,t) = \{(y,\tau) : Z_r^i(x,t;y,\tau) = 0,\ \tau \le t\}$ is spheroid. Here

$$r = r(x,t) = \min\left\{\frac{R_1^2(x)\, e}{2a_1 m}, \ldots, \frac{R_1^2(x)\, e}{2a_n m}, t\right\}. \tag{37}$$

Let $(x,t) \in \Omega$ and $D : R_1(x) = \inf\{|x - x'|,\ x \in \partial D,\ x' \in \overline{D}\}$ where $R_1(x)$ is the distance from the point (x,t) to the boundary of domain. In this case $Q_r^i(x,t) \subset \overline{\Omega}$. Appling the expression (34) to each of the equations we will get the following system of integral equations ($i = 1, 2, \ldots, n$):

$$\begin{aligned}
u_i(x,t) &= a_i\iint_{\partial Q_r^i(x,t)} \left(1-\frac{t-\tau}{r}\right)\left(\frac{\partial Z^{(i)}(x,t;y,\tau)}{\partial n_y}\right) u_i(y,\tau)\, ds\, d\tau \\
&+ \frac{1}{r}\iint_{\partial Q_r^i(x,t)} \left(1 - \left(r - (t-\tau)\, c_{ii}\right)\right) \times Z_r^{(i)}(x,t;y,\tau)\, u_i(y,\tau)\, dy\, d\tau \\
&+ \frac{1}{r}\iint_{\partial Q_r^i(x,t)} (r-(t-\tau))\, Z_r^{(i)}(x,t;y,\tau) \times \sum_{j=1,n;i\neq j} c_{ij} u_j(y,\tau)\, dy\, d\tau \\
&+ \frac{1}{r}\iint_{Q_r^i(x,t)} (r-(t-\tau))\, Z_r^{(i)}(x,t;y,\tau)\, f_i(y,\tau)\, dy\, d\tau, \\
&\qquad i = 1, \ldots, n,
\end{aligned} \tag{38}$$

where

$$Z^{(i)}(x,t;y,\tau) = (4\pi a_i(t-\tau))^{-m/2} \exp\left(-\frac{|x-y|^2}{4a_i(t-\tau)}\right),$$
$$Z_r^i(x,t;y,\tau) = Z^{(i)}(x,t;y,\tau) - (4\pi a_i(t-\tau))^{-m/2}. \tag{39}$$

The derived system (38) is similar to system IEs which was considered in Section 2. That is way we can use the method which was given in Section 2.

3.2. The Probabilistic Representation of the Solution. After some transformation we will get for separate terms of the system (38) as follows:

$$\begin{aligned}
I_1^{(i)}(x,t) &= a_i\iint_{\partial Q_r^i(x,t)} \left(1-\frac{t-\tau}{r}\right)\left(\frac{\partial Z^{(i)}(x,t;y,\tau)}{\partial n_y}\right) u_i(y,\tau)\, ds\, d\tau \\
&= (1-q_m)\int_0^\infty q_1(\rho)\, d\rho \int_{S_1(0)} q_2(\omega)\, u_i\left(y^{(i)}(\rho,\omega), \tau(\rho)\right) ds \\
&= (1-q_m)\, Eu_i\left(y^{(i)}(\xi,\omega), \tau(\xi)\right),
\end{aligned} \tag{40}$$

where

$$q_1(\rho) = \rho^{m/2}\exp(-\rho)\left(1-\exp\left(-\frac{2\rho}{m}\right)\right) \times \left((1-q_m)\,\Gamma\left(1+\frac{m}{2}\right)\right)^{-1},$$
$$q_2(\omega) = \frac{1}{\sigma_m} = \Gamma\left(\frac{m}{2}\right)\left(2\pi^{m/2}\right)^{-1},$$
$$q_m = \left(1+\frac{2}{m}\right)^{-(1+m/2)},$$

$$y^{(i)}(\xi,\omega) = x + \sqrt{4r\xi a_i \exp\left(-\frac{2\xi}{m}\right)}\,\omega,$$
$$\tau(\xi) = t - r\exp\left(-\frac{2\xi}{m}\right), \tag{41}$$

ξ is a random variable with density functions $q_1(\rho)$, ω random point on the surface $S_1(0)$, which has a density function $q_2(w)$, $S_1(0)$ unit sphere, ds element of surface, σ_m square of the surface unit sphere, and $\Gamma(\cdot)$ Gamma function.

Let us consider the second terms of (38)

$$\begin{aligned} I_2^{(i)}(x,t) &= \frac{1}{r}\iint_{Q_r^i(x,t)} (r-(t-\tau))\, Z_r^i(x,t;y,\tau)\, f_i(y,\tau)\,dy\,d\tau \\ &= rq_m\int_0^1 q_3(\nu)\,d\nu \int_0^\infty q_4(z)\,dz \\ &\quad\cdot\int_{S_1(0)}\left(1-\nu^{2/m}\exp\left(\frac{-2z}{m+2}\right)\right) \\ &\qquad\times f_i\left(y_1^{(i)}(z,\nu,w),\tau_1(z,\nu)\right)dS_w \\ &= rq_m E\left\{\left(1-\nu^{2/m}\exp\left(\frac{-2\xi_1}{m+2}\right)\right)\right. \\ &\qquad\left.\times f_i\left(y_1^{(i)}(\xi_1,\nu,\omega),\tau_1(\xi_1,\nu)\right)\right\}, \end{aligned} \tag{42}$$

where $q_3(\nu) = (1-\nu)\nu^{2/m-1}(B(2,\ 2/m))^{-1}$. Then the density of Beta distribution with parameters $(2, 2/m)$

$$q_4(z) = \exp(-z)\,z^{m/2-1}\left(\Gamma\left(\frac{m}{2}\right)\right)^{-1}, \tag{43}$$

and the density of Gamma distribution with parameters $m/2$, ω—unit random vector,

$$y_1^i(\xi_1,\nu,\omega) = x + \left[\frac{4m}{m+2} ra_i\xi_1\nu^{2/m}\exp\left(-\frac{2\xi_1}{m+2}\right)\right]^{1/2}\omega,$$
$$\tau_1(\xi_1,\nu) = t - r\nu^{2/m}\exp\left(-\frac{2\xi_1}{m+2}\right), \tag{44}$$

where ξ_1 is the random variable with density function $q_4(z)$ and ν is another random variable with the density function $q_3(\nu)$.

Let

$$r = r(x,t) = \min\left\{\frac{eR_1^2(x)}{2ma_1},\ldots,\frac{eR_1^2(x)}{2ma_n};\frac{1}{c_{11}},\ldots,\frac{1}{c_{nn}};t\right\}, \tag{45}$$

then $\overline{Q_r^{(i)}}(x,t) \in \Omega$ and the function

$$p_1^{(i)}(x,t;y,\tau) = \frac{[1-(r-(t-\tau))c_i]\,Z_r^{(i)}(x,t;y,\tau)}{rq_m(1-rq_{1m}c_{ii})} \times I_{Q_r^{(i)}(x,t)}(y,\tau) \tag{46}$$

is the transition density in $Q_r^i(x,t)$ with fixed point (x,t), where

$$q_{1m} = 1 - \frac{1}{2}\left(\frac{m+2}{m+4}\right)^{1+m/2}. \tag{47}$$

Let $(y_2^{(i)},\tau_2^{(i)})$ be a random point of balloid $Q_r^i(x,t)$ which has the following density function $p_1^{(i)}(x,t;y,\tau)$ $(i=1,\ldots,n)$ in the fixed point (x,t).

In this case

$$\begin{aligned} &\frac{1}{r}\iint_{Q_r^{(i)}(x,t)}(1-(r-(t-\tau))c_{ii})\,Z_r^{(i)}(x,t;y,\tau)\,u_i(y,\tau)\,dy\,d\tau \\ &\quad = q_m(1-rc_{ii}q_{1m})\,Eu_i\left(y_2^{(i)},\tau_2^{(i)}\right),\quad i=1,\ldots,n. \end{aligned} \tag{48}$$

The obtained results we will put to (38) and we will get the probabilistic representation of problem (24)-(25). It follows there from that we could to following proposition.

Theorem 3. *For the solution of initial BVP* (24)-(25) *the following probabilistic representation is valid:*

$$\begin{aligned} u_i(x,t) &= (1-q_m)\,Eu_i\left(y^{(i)}(\xi,\omega),\tau(\xi)\right) \\ &\quad + q_m(1-rc_{ii}q_{1m})\,Eu_i\left(y_2^{(i)},\tau_2^{(i)}\right) \\ &\quad + q_m rE\left\{\left(1-\nu^{2/m}\exp\left(-\frac{2\xi_1}{m+2}\right)\right)\right. \\ &\qquad\left.\times\sum_{j=1,j\neq i}^n c_{ij}u_j\left(y_1^{(i)}(\xi_1,\nu,\omega,),\tau_1(\xi_1,\nu)\right)\right\} \\ &\quad + q_m rE\left(\left(1-\nu^{2/m}\exp\left(-\frac{2\xi_1}{m+2}\right)\right)\right. \\ &\qquad\left.\times f_i\left(y_1^{(i)}(\xi_1,\nu,\omega)\ ,\tau_1(\xi_1,\nu)\right)\right),\quad \left(i=\overline{1,n}\right), \end{aligned} \tag{49}$$

where $(y^{(i)}(\xi,\omega),\tau(\xi))$ *is defined by* (40) *and* $(y^{(i)}(\xi,\nu,\omega),\tau(\xi,\nu))$ *is determined by* (41).

The proof of Theorem 3 is the consequence of the above mentioned reasoning.

By further using the presentation (44) we will construct a random process in Ω and propose the Monte Carlo algorithm for the solution of system IEs.

3.3. Description Random Process and the Algorithm Simulation. Let

$$r = r(x,t) = \min\left\{\frac{eR_1^2(x)}{2ma_1},\ldots,\frac{eR_1^2(x)}{2ma_n};\frac{1}{c_{11}},\ldots,\frac{1}{c_{nn}};t\right\}, \tag{50}$$

The functions

$$p_0^{(i)}(x,t;y,\tau) = \frac{1}{1-q_m}\left(1-\frac{t-\tau}{r}\right)\left(-\frac{\partial Z^{(i)}(x,t;y,\tau)}{\partial n_y}\right) \times I_{\partial Q_r^{(i)}(x,t)}(y,\tau),$$
$$p_1^{(i)}(x,t;y,\tau) = \frac{(1-(r-(t-\tau))c_{ii})Z_r^{(i)}(x,t;y,\tau)}{rq_m(1-rq_{1m}c_{ii})} \times I_{\partial Q_r^{(i)}(x,t)}(y,\tau),$$
$$p_2^{(i)}(x,t;y,\tau) = \frac{Z_r^{(i)}(x,t;y,\tau)}{rq_m} I_{\partial Q_r^{(i)}(x,t)}(y,\tau), \tag{51}$$

are the transition density functions in $\overline{Q_r^{(i)}}(x,t)$ at a fixed point (x,t) $(i=\overline{1,n})$. We will define in Ω a random process as was proposed in Section 2.

Let us define a transition matrix as

$$A(x,t) = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1(n+1)} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2(n+1)} \\ \dots & \dots & \dots & \dots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{n(n+1)} \\ 0 & 0 & \dots & 1 \end{bmatrix}, \tag{52}$$

where $\alpha_{ii} = 1 - q_m c_{ii} q_{1m} r(x,t)$, and let

$$\beta_i = \frac{q_m c_{ii} q_{1m} r(x,t)(n-1)}{n},$$
$$\alpha_{ij} = \frac{\beta_i c_{ij}}{M_i}, \quad \left(i,j=\overline{1,n}; i\neq j\right),$$
$$M_i = \sum_{j=1,n;\ j\neq i} c_{ij}, \quad \left(i=\overline{1,n}\right), \tag{53}$$
$$\alpha_{i(n+1)} = \frac{q_m q_{1m} r(x,t) c_{ii}}{n}, \quad \left(i=\overline{1,n}\right).$$

Now we will define the density function of transition matrix $P(x,t;y,\tau)$:

$$P(x,t;y,\tau) = \begin{bmatrix} p_{11}(x,t;y,\tau) & p_{12}(x,t;y,\tau) & \dots & p_{1(n+1)}(x,t;y,\tau) \\ p_{21}(x,t;y,\tau) & p_{22}(x,t;y,\tau) & \dots & p_{2(n+1)}(x,t;y,\tau) \\ \dots & \dots & \dots & \dots \\ p_{n1}(x,t;y,\tau) & p_{n2}(x,t;y,\tau) & \dots & p_{n(n+1)}(x,t;y,\tau) \\ 0 & 0 & \dots & 1 \end{bmatrix}, \tag{54}$$

where

$$p_{ij}(x,t;y,\tau) = p_2^{(i)}(x,t;y,\tau),$$
$$\left(i=\overline{1,n};\ j=\overline{1,n+1}; i\neq j\right),$$
$$p_{ii}(x,t;y,\tau) = \frac{(1-q_m)p_0^{(i)}(x,t;y,\tau) + q_m(1-rq_{1m}c_{ii})p_1^{(i)}(x,t;y,\tau)}{1-r(x,t)q_{1m}c_{ii}q_m}. \tag{55}$$

Then we will fix the initial point $(x_0,t_0) = (x,t)$ and the number of equations $i_0 \in \{1,\dots,n\}$. Let an initial moment at the point $(x_0,t_0) = (x,t)$; we will have one particle. For one step a particle $i_k \to i_{k+1}$ moves from its position according to the transition matrix $A(x_k,t_k)$ and moves with probability $\alpha_{i_k,i_{k+1}}(x_k,t_k)$ from the point (x_k,t_k) to the point (x_{k+1},t_{k+1}). The next (x_{k+1},t_{k+1}) point will be simulated using the density function $p_{i_k i_{k+1}}(x_k,t_k;y,\tau)$.

The probability of breaking of trajectory in the point (x_n,t_n) is

$$g(x_n,t_n) = \begin{cases} 1, & (x_n,t_n)\in\Omega; \\ \alpha_{i_n,n+1}(x_{n-1},t_{n-1}), & (x_n,t_n)\in\Omega. \end{cases} \tag{56}$$

The next coordinate of the particle will be defined in the following way.

(1) If the density function of the point (x_{n+1},t_{n+1}) equals $p_0^{(i)}(x_n,t_n;y,\tau)$ in the fixed point (x_n,t_n) then

$$x_{n+1} = x_n + 2(r(x_n,t_n)\xi_n a_i)^{1/2}\exp\left(-\frac{\xi_n}{m}\right)\omega_n,$$
$$t_{n+1} = t_n - r(x_n,t_n)\ \exp\left(-\frac{2\xi_n}{m}\right), \tag{57}$$

where $\{\xi_n\}_{n=0}^{\infty}$, $\{\omega_n\}_{n=0}^{\infty}$ the sequence of independent random variables with the density function $q_1(\rho)$ and independent isotropic vectors. The value $r(x_n,t_n)$ will be defined as (50).

(2) If the density function of the point (x_{n+1},t_{n+1}) is equal to $p_1^{(i)}(x_n,t_n;y,\tau)$ at a fixed (x_n,t_n) then

$$x_{n+1} = x_n + 2\left(\frac{m}{m+2}r(x_n,t_n) \times \xi_n'\left(v_n'\right)^{2/m}a_i\exp\left(-\frac{2\xi_n'}{m+2}\right)\right)^{1/2}\omega_n, \tag{58}$$
$$t_{n+1} = t_n - r(x_n,t_n)\left(v_n'\right)^{2/m}\exp\left(-\frac{2\xi_n'}{m+2}\right),$$

where $\{\xi_n'\}_{n=0}^{\infty}$, $\{v_n'\}_{n=0}^{\infty}$ is a sequence of independent random variables, which will be obtained from the algorithm below (Algorithm 4) (Neumann acceptance rejection method).

Algorithm 4. (a) We firstly simulate ξ, Gamma distributed random variable with the parameters $(m/2)$, secondly simulate γ, uniformly distributed random variable on $(0,1)$, and thirdly simulate ν, Beta distributed random variable with the parameters $(2, 2/m)$.

(b) If $\gamma > 1 - c_{ii} r(1-\nu^{2/m} \exp(-2\xi/(m+2)))$ then we will go to (a) and so on; otherwise $\nu' = \nu$, $\xi' = \xi$.

(3) If the density function at the point (x_{n+1}, t_{n+1}) equals $p_2^{(i)}(x_n, t_n; y, \tau)$ under fixed point (x_n, t_n), then

$$\begin{aligned} x_{n+1} &= x_n + 2\left(\frac{m}{m+2} r(x_n, t_n)\, \xi_n (\nu_n)^{2/m} \right. \\ &\qquad \left. \times a_i \exp\left(-\frac{2\xi_n}{m+2}\right)\right)^{1/2} \omega_n, \\ t_{n+1} &= t_n - r(x_n, t_n)(\nu_n)^{2/m} \exp\left(-\frac{2\xi_n}{m+2}\right), \end{aligned} \tag{59}$$

where $\{\xi_n\}_{n=0}^{\infty}, \{\nu_n\}_{n=0}^{\infty}, \{\omega_n\}_{n=0}^{\infty}$ is sequence of independent Gamma distributed random variables with parameters $(2/m)$, Beta distributed random variables with parameters $(2, 2/m)$, and independent isotropic vectors, respectively.

If at the moment n was held break, then we will put $(x_{n+k}, t_{n+k}) = (x_n, t_n)$, $k = 0, 1, 2, \ldots$ obviously the sequence of coordinate of the particle forms Markov chains. The random process which was described above was considered in Ermakov et al. [23] for the solution of initial BVP for the heat equation and adapted in Kurbanmuradov [22] for the heat equation with variable coefficients.

Now we prove the auxiliary Lemma 5.

Lemma 5. *With the probability one Markov chain $\{x_n, t_n\}_{n=0}^{\infty}$ converges when $n \to \infty$ to the random point of boundary $(x_\infty, t_\infty) \in \partial\Omega$, or it is absorbed inside of the domain.*

Proof. Since $\{t_n\}$ is decreasing sequence and $t_n \geq 0$, it has a limit $t_\infty = \lim_{n\to\infty} t_n$. Let $\mathfrak{R}_n - \sigma$ be algebra, which was generated by random variables

$$\begin{gathered} \{\omega_k\}_{k=0}^{n-1}, \quad \{\nu_k\}_{k=0}^{n-1}, \\ \{\xi_k\}_{k=0}^{n-1}, \quad \{\nu'_k\}_{k=0}^{n-1}, \quad \{\xi'_k\}_{k=0}^{n-1}. \end{gathered} \tag{60}$$

From the definition $\mathfrak{R}_n$ and (57)–(59) it follows that x_n is measurable relatively $\mathfrak{R}_n$. It is obvious that the coordinates of vector process formed limited martingale relatively $\{\mathfrak{R}_m\}_{m=1}^{\infty}$:

$$\begin{aligned} E&\left(\frac{x_{n+1}}{\mathfrak{R}_n}\right) \\ &= E\Bigg\{(1-q_m) \\ &\quad \times \left[x_n + 2\left(r(x_n, t_n)\,\xi_n a_{i_n}\right)^{1/2} \exp\left(-\frac{\xi_n}{m}\right)\omega_n\right] \\ &\quad + q_m\left(1 - q_{1m} r(x_n, t_n)\, c_{i_n i_n}\right) \\ &\quad \times \left[x_n + 2\left(\frac{m}{m+2} r(x_n, t_n)\, a_{i_n} \xi'_n {\nu'_n}^{2/m}\right)^{1/2} \right. \\ &\qquad \left. \times \exp\left(-\frac{\xi'_n}{m+2}\right)\omega_n\right] \\ &\quad + c_{i_n i_n} r(x_n, t_n)\, q_m q_{1m} \frac{n-1}{n} \\ &\quad \times \left[x_n + 2\left(\frac{m}{m+2} r(x_n, t_n)\, a_{i_n} \varsigma_n \nu_n^{2/m}\right)^{1/2} \right. \\ &\qquad \left. \times \exp\left(-\frac{\varsigma}{m+2}\right)\omega_n\right] \\ &\quad + \frac{c_{i_n i_n} r(x_n, t_n)\, q_m q_{1m}}{n} \frac{x_n}{\mathfrak{R}_n}\Bigg\} \\ &= (1-q_m)\left[x_n + 2\left(r(x_n, t_n)\, a_{i_n}\right)^{1/2} \right. \\ &\qquad \left. \times E\left(\xi_n^{1/2} \exp\left(-\frac{\xi_n}{m}\right)\omega_n\right)\right] \\ &\quad + q_m\left(1 - q_{1m} c_{i_n i_n} r(x_n, t_n)\right) \\ &\quad \times \left[x_n + 2\left(\frac{m}{m+2} r(x_n, t_n)\, a_{i_n}\right)^{1/2} \right. \\ &\qquad \left. \times E\left(\left(\xi'_n {\nu'_n}^{2/m}\right) \exp\left(-\frac{\xi_n}{m+2}\right)\omega_n\right)\right] \\ &\quad + c_{i_n i_n} r(x_n, t_n)\, q_m q_{1m} \frac{n-1}{n} \\ &\quad \times \left[x_n + 2\left(\frac{m}{m+2} r(x_n, t_n)\, a_{i_n}\right)^{1/2} \right. \\ &\qquad \left. \times E\left(\left(\xi'_n {\nu'_n}^{2/m}\right) \exp\left(-\frac{\xi_n}{m+2}\right)\omega_n\right)\right] \\ &\quad + c_{i_n i_n} r(x_n, t_n)\, q_m q_{1m} \frac{n-1}{n} \\ &\quad \times \left[x_n + 2\left(\frac{m}{m+2} r(x_n, t_n)\, a_{i_n}\right)^{1/2} \right. \\ &\qquad \left. \times E\left(\left(\varsigma_n {\nu_n}^{2/m}\right)^{1/2} \exp\left(-\frac{\varsigma_n}{m+2}\right)\omega_n\right)\right] \\ &\quad + \frac{c_{i_n i_n} r(x_n, t_n)\, q_m q_{1m}}{n} x_n \\ &= (1-q_m)\, x_n + q_m\left(1 - q_{1m} c_{i_n i_n} r(x_n, t_n)\right) x_n \\ &\quad + c_{i_n i_n} r(x_n, t_n)\, q_m q_{1m} x_n = x_n, \end{aligned} \tag{61}$$

where $\{x_n\}$ are limited martingale; it converges with the probability one Shiryaev [24].

Let $(x_\infty, t_\infty) = \lim_{n\to\infty}(x_n, t_n)$ be the limit vector. We show that $(x_\infty, t_\infty) \in \partial\Omega$. If $t_\infty = 0$ then the process is broken inside the domain. Let $t_\infty > 0$. As far as the process converges, according to the formulas (50)–(58) we have

$$E_{(x_0,t_0)}\left|x_{n+1} - x_n\right| \longrightarrow 0,$$
$$E_{(x_0,t_0)}\left|x_{n+1} - x_n\right| = E_{(x_0,t_0)}\left\{\sqrt{r(x_n,t_n)}h(r(x_n,t_n))\right\}, \quad (62)$$

where $h(r)$ is strictly positive. Applying Lebesque Theorem (about the limited convergence) we get

$$E_{(x_0,t_0)}(r(x_\infty, t_\infty))^{1/2} = 0. \quad (63)$$

It means $r(x_\infty, t_\infty) = 0$. Then from the definition of $r(x,t)$ and using the formulae (49) we obtain

$$R_1(x_\infty) = 0, \quad (x_\infty, t_\infty) \in \partial\Omega. \quad (64)$$

Lemma is proven. □

4. Construction Unbiased and ε-Biased Estimators

Let $(x_k, t_k)_{k=0}^\infty$ be the trajectories of random process which was described above. We will define on it the sequence of the random variables $\{\eta_n(i_0)\}_{n=0}^\infty$. Let

$$\Theta_0 = 1, \quad \Theta_n = \Theta_{n-1} \times V_{i_{n-1}i_n}(x_{n-1}, t_{n-1}; x_n, t_n), \quad (65)$$

where $V_{ij}(x_{n-1}, t_{n-1}; x_n, t_n)$ is defined as follows:

$$\begin{aligned} &V_{ij}(x_{n-1}, t_{n-1}; x_n, t_n) \\ &\quad = \frac{nM_i}{(n-1)c_{ii}q_{1m}}\left(1 - \nu_n^{2/m}\exp\left(-\frac{2\varsigma_n}{m+2}\right)\right), \\ &\qquad\qquad (i, j = \overline{1,n}; i \neq j), \\ &V_{ii}(x_{n-1}, t_{n-1}; x_n, t_n) = 1; \quad (i = \overline{1,n}); \\ &V_{i(n+1)}(x_{n-1}, t_{n-1}; x_n, t_n) \\ &\quad = \frac{n}{c_{ii}q_{1m}}\left(1 - \nu_n^{2/m}\exp\left(-\frac{2\varsigma_n}{m+2}\right)\right); \quad (i = \overline{1,n}). \end{aligned} \quad (66)$$

Here $\{\varsigma_n\}_{n=0}^\infty$, $\{\nu_n\}_{n=0}^\infty$ are the sequences of independent Gamma function with the parameters $(m/2)$ and Beta function with parameters $(2, 2/m)$ distributed random variables, respectively. We will define the sequence

$$\eta_n(i_0) = \Theta_n \times F(x_n, t_n) = \Theta_n \begin{cases} u_j(x_n, t_n), & i_n = j, \ j \neq n+1, \\ f_{i_{n-1}}(x_n, t_n), & i_n = n+1. \end{cases} \quad (67)$$

If at the moment n happen break n, we will put

$$\begin{gathered} \eta_{n+k}(i_0) = \eta_n(i_0), \\ (x_{n+k}, t_{n+k}) = (x_n, t_n), \\ k = 1, 2, \ldots, \end{gathered} \quad (68)$$

where algebra $\mathfrak{R}_n - \sigma$ generated until the moment n.

Theorem 6. *Let the sequence be form martingale $\{\eta_n(i_0)\}_{n=1}^\infty$ with $\mathfrak{R}_n$, respectively. If*

$$\begin{gathered} \sum_{j=1,\ldots,n;\ j\neq i} c_{ij} < \frac{(n-1)c_{ii}q_{1m}}{n}, \quad (i = \overline{1,n}), \\ \max_{(x,t)\in\Omega}\left|f_i(x,t)\right| \le c_0, \quad (c_0 = const, i = \overline{1,n}). \end{gathered} \quad (69)$$

Then the sequence will be $\{\eta_n(i_0)\}$ uniformly integrable martingale.

Proof. From the definition $\eta_n(i_0)$ the $\mathfrak{R}_n$ is measurable. In this case

$$\begin{aligned} &E\left(\eta_{n+1}\frac{(i_0)}{R_n}\right) \\ &= E\left(\Theta_{n+1} \times \frac{F(x_{n+1}, t_{n+1})}{\mathfrak{R}_n}\right) \\ &= E\left(\Theta_n \times V_{i_n i_{n+1}}(x_n, t_n; x_{n+1}, t_{n+1}) \times \frac{F(x_{n+1}, t_{n+1})}{\mathfrak{R}_n}\right) \\ &= \Theta_n E\left(V_{i_n i_{n+1}}(x_n, t_n; x_{n+1}, t_{n+1}) \times F(x_{n+1}, t_{n+1})\right) \\ &= \Theta_n\left[\sum_{j=1,n}\alpha_{i_n j}\iint_{\overline{Q_r^{(i_n)}(x_n,t_n)}} p_{i_n j}(x_n, t_n; y, \tau)\,u_j(y,\tau)\,dy\,d\tau\right. \\ &\qquad\left. +\alpha_{i_n n+1}\iint_{\overline{Q_r^{(i_n)}(x_n,t_n)}} p_{i_n n+1}(x_n, t_n; y, \tau)\,f_{i_n}(y,\tau)\,dy\,d\tau\right] \\ &= \eta_n(i_0). \end{aligned} \quad (70)$$

As far as the sequence is martingale $\{\eta_n(i_0)\}$. We can show the uniformly integrability of $\eta_n(i_0)$. To do that it is enough to show $|\eta_n(i_0)| < \infty$.

Since

$$u_i(x,t) \in C\left(\overline{D} \times [0,T]\right) \cap C^{2,1}\left(\overline{D} \times [0,T]\right), \quad (71)$$

and Ω is bounded domain $|u_i(x,t)| \le$ const, for any $(x,t) \in \Omega$. From the condition of theorem $|\Theta_n| \le 1$, it is followed $|\eta_n(i_0)| \le$ const. It means $\{\eta_n(i_0)\}$ is uniformly integrable.

The theorem is proved. □

Now we will construct computable (realizable) estimator $\eta_n(i_0)$. We will take ε-neighborhoods of the domain $(\partial\Omega)_\varepsilon = \{D \times [0,\varepsilon]\} \cup \{(\partial D)_\varepsilon \times [0,T]\}$.

Let N_1 be a breaking moment of process inside of domain. N_ε is the first passage moment $(\partial\Omega)_\varepsilon$. $N = \min\{N_1, N_\varepsilon\}$, stopping moment of process $\{(x_n, t_n)\}$. In this case the probability of absorbing will be

$$g(x_n, t_n) = \begin{cases} 1, & (x_n, t_n) \in (\partial\Omega)_\varepsilon \\ \alpha_{i_n,(n+1)}(x_{n-1}, t_{n-1}), & (x_n, t_n) \in \overline{\Omega} \setminus (\partial\Omega)_\varepsilon. \end{cases} \tag{72}$$

From Lemma 5 it follows that $N < \infty$. It could be proved that the mathematical expectation of the stopping time $\{(x_n, t_n)\}$ of Markov process is finite.

Theorem 7. *Let the condition of Theorem 6 be satisfied; then the estimator $\eta_n(i_0)$ will be unbiased estimator with finite variance, where $u_{i_0}(x,t)$ is i_0th component of solution vector $u(x,t)$.*

Proof. Since $\eta_n(i_0)$ is uniformly integrable martingale and N is Markov moment, according to the theorem in ([24, 25]) for the martingale $\{\eta_n(i_0)\}$ we obtain

$$E_{(x_0,t_0)}\eta_N(i_0) = E_{(x_0,t_0)}\eta_1(i_0). \tag{73}$$

From the definition $\eta_n(i_0)$ holds $E_{(x_0,t_0)}\eta_1(i_0) = u_{i_0}(x,t)$. From condition of Theorem 6 $E(\eta_N)^2(i_0) < \infty$ is valid accordingly the variance is finite. The theorem is proved. □

Further, from $\eta_N(i_0)$ we could construct biased but computable (realizable) estimator $\eta_N^*(i_0)$. Let, for $x \in \partial D$, $t \in [0,T]$ and $\psi_i(x,t) = y_i(x,t)$ for $x \in D$, $\psi_i(x,0) = y_{0i}(x)$, (x^*, t^*) closed to the point (x,t) of boundary $\partial\Omega$. η_N^* will be obtained with changing

$$u_i(x_N, t_N) \quad \text{in } \eta_N(i_0) \quad \text{to } \psi_i(x_N^*, t_N^*). \tag{74}$$

Let us evaluate bias $\eta_N^*(i_0)$. It is clear that

$$\left|E_{(x,t)}\eta_N(i_0) - u_{i_0}(x,t)\right| \le E_{(x,t)}\left|\eta_N^*(i_0) - \eta_N(i_0)\right|. \tag{75}$$

If $N = N_1$, in this case the process is broken when do not reach the boundary $(\partial\Omega)_\varepsilon$ and $\eta_N^*(i_0) = \eta_N(i_0)$. If $N = N_\varepsilon$ then $(x_N, t_N) \in (\partial\Omega)_\varepsilon$.

Let $A_i(\varepsilon)$ be a module of continuity function $u_i(x,t)$. In this case it is true:

$$\left|\eta_N^*(i_0) - \eta_N(i_0)\right| \le \left|\Theta_N\right| A(\varepsilon), \tag{76}$$

where $A(\varepsilon) = \max_i A_i(\varepsilon)$, since $|\Theta_N| \le 1$ then $E_{(x,t)}|\eta_N^*(i_0) - u_{i_0}(x,t)| \le A(\varepsilon)$. Finiteness of variance followed from $E\eta_N^2(i_0) < \infty$. The proposed algorithm we could generalize for the case with variable coefficients $c_{ij} = c_{ij}(x,t)$ and one could get the same results.

5. Computational Example

Let $D \in R^3$ be bounded domains, $\Omega = D \times [0,T]$. We will consider for some mode linitial boundary value problem

$$\begin{aligned} &\frac{\partial u_i(x,t)}{\partial t} - a_i \Delta u_i(x,t) + c_{ii} u_i(x,t) \\ &\quad - \sum_{j=1,4;\ j\neq i} c_{ij} u_j(x,t) = f_i(x,t), \quad \left(i = \overline{1,4}\right) \end{aligned} \tag{77}$$

for $(x,t) \in \Omega$ with the initial boundary conditions

$$\begin{aligned} u_i(x,t) &= y_i(x,t), \quad x \in \partial D,\ t \in [0,T],\ i = \overline{1,4}, \\ u_i(x,0) &= y_{0i}(x), \quad x \in D,\ i = \overline{1,4}. \end{aligned} \tag{78}$$

As domain is chosen as the simple ball, $D = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 \le R^2\}$.

The coefficients

$$\begin{gathered} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.7 \\ 0.1 \\ 1.0 \end{pmatrix}, \\ \{c_{ij}\}_{i,j=1,\dots,4} = \begin{bmatrix} 2 & 0.4 & 0.5 & 0.2 \\ 0.7 & 3 & 0.4 & 0.6 \\ 0.3 & 0.1 & 1 & 0.1 \\ 0.2 & 0.3 & 0.3 & 1.5 \end{bmatrix} \end{gathered} \tag{79}$$

The initial and boundary conditions

$$\begin{aligned} &y_{01}(x) = x_1^2 + x_2^2 + x_3^2; \\ &y_1(x,t) = R^2 \exp(t); \\ &y_{02}(x) = (x_1 x_2 x_3)^2; \\ &y_2(x,t) = \exp(t)(x_2 x_3)^2 \left(R^2 - x_2^2 - x_3^2\right); \\ &y_{03}(x) = \exp(x_1 + x_2 + x_3); \\ &y_3(x,t) = \exp(t + x_1 + x_2 + x_3); \\ &y_{04}(x) = 1; \qquad y_4(x,t) = \exp(t x_1 x_2 x_3). \end{aligned} \tag{80}$$

Left hand sides

$$\begin{aligned} f_1(x,t) = {} & \exp(t)\left[4\left(x_1^2 + x_2^2 + x_3^2\right) - 0.3(x_1 x_2 x_3)^2 - 3\right] \\ & - 0.2(x_1 + x_2 + x_3 + t) - 0.5 \exp(x_1 x_2 x_3 t); \\ f_2(x,t) = {} & \exp(t)\left[3.5(x_1 x_2 x_3)^2 - 1.4(x_1 x_2)^2\right. \\ & \left. + (x_2 x_3)^2 + (x_1 x_3)^2 - 0.4\left(x_1^2 + x_2^2 + x_3^2\right)\right] \\ & - 0.3(x_1 + x_2 + x_3 + t) - 0.2 \exp(x_1 x_2 x_3 t); \end{aligned}$$

Table 1: The results of computational experiments.

(x_0, t_0)	i_0	N_t	ε	U_e	MC	3σ	err
(0.1; −0.8; 0.4; 1.2)	1	5000	0.005	2.689294	2.881862	0.2821	0.1925
(0.1; −0.8; 0.4; 1.2)	2	5000	0.005	0.003399	0.018739	0.224699	0.015339
(0.1; −0.8; 0.4; 1.2)	3	5000	0.005	2.459603	2.483835	0.21815	0.02423
(0.1; −0.8; 0.4; 1.2)	4	5000	0.005	0.962327	0.95895	0.03377	0.1598
(−0.5; 0.25; −0.4; 1.3)	1	5000	0.005	1.733942	1.554352	0.32319	0.17939
(−0.5; 0.25; −0.4; 1.3)	2	5000	0.005	0.009173	0.009841	0.215628	0.000668
(−0.5; 0.25; −0.4; 1.3)	3	5000	0.005	1.91554	1.980669	0.18057	0.064528
(−0.5; 0.25; −0.4; 1.3)	4	5000	0.005	1.067159	1.084443	0.162095	0.01728

$(x_0; t_0)$ is the point which solved BVP; N_t is the quantity of samples (trajectories); ε is neighborhood area; i_0 is the number of equation; U_e is exact solution at the point $(x_0; t_0)$. MC is Monte Carlo solutions; 3σ is confidence interval; err is the difference between exact ant MC solutions err $= |u_{i_0}(x_0; t_0) - \text{MS}|$.

$$\begin{aligned} f_3(x,t) = {} & 4.7\exp(x_1+x_2+x_3+t) \\ & - \exp(t)\left[0.5\left(x_1^2+x_2^2+x_3^2\right)+0.4(x_1x_2x_3)^2\right] \\ & - 0.7\exp(x_1x_2x_3t); \\ f_4(x,t) = {} & \exp(x_1x_2x_3t) \\ & \times\left[(x_1x_2x_3)+3.5-t^2\left((x_1x_2)^2\right.\right. \\ & \left.\left.+(x_2x_3)^2+(x_1x_3)^2\right)\right] \\ & - \exp(t)\left[0.3\left(x_1^2+x_2^2+x_3^2\right)-0.4(x_1x_2x_3)^2\right] \\ & - 0.6(x_1+x_2+x_3+t). \end{aligned} \tag{81}$$

The exact solutions are known:

$$\begin{aligned} u_1(x,t) &= \exp(t)\left(x_1^2+x_2^2+x_3^2\right), \\ u_2(x,t) &= \exp(t)(x_1x_2x_3)^2, \\ u_3(x,t) &= \exp(x_1+x_2+x_3+t), \\ u_4(x,t) &= \exp(x_1x_2x_3t). \end{aligned} \tag{82}$$

6. Conclusions

It is known that the distinguishing feature of the vector algorithm is that its "weight" appears to be a matrix weight. This matrix weight is multiplied by the kernel matrix of the system of integral equations divided by a transition density function after each transition in the Markov chain simulation. In this case the computational complexity is higher enough than simple Monte Carlo method. On the contrary to the vector algorithms we proposed a new Monte Carlo algorithm for the solution of system of integral equations. This method has the simple structure of the computation algorithm and the errors do not depend on the dimension of domain and smoothness of boundary. One can solve the problem at one point and we do not use matrix weights. Proposed algorithm applied to the solution of system of the parabolic equations. To do so we derived corresponding system of integral equations and construct a special probabilistic representation. This probabilistic representation uses for simulation the random process and construction the unbiased and ε-biased estimator for the solution of systems IEs.

Numerical experiments show that the computational complexity of our algorithm is reduced. In the future the proposed algorithm might be generalized for the case with variable coefficients $c_{ij} = c_{ij}(x,t)$. The results of numerical experiments are shown with the probability almost one; the approximate solution tends to the exact solution of the problem. In the given example the exact solution is known; therefore we can make sure that all the estimators really are in the confidence intervals (see Table 1).

Conflict of Interests

The authors declare that they have no conflict of interests regarding publication of this paper and they do not have direct financial relation that might lead to conflict of interests for any of the authors.

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Research Article

On Fractional Derivatives and Primitives of Periodic Functions

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We prove that the fractional derivative or the fractional primitive of a T-periodic function cannot be a $\widetilde{T}$-periodic function, for any period $\widetilde{T}$, with the exception of the zero function.

1. Introduction

Periodic functions [1, Ch. 3, pp. 58–92] play a central role in mathematics since the seminal works of Fourier [2, 3]. Nowadays, periodic functions appear in applications ranging from electromagnetic radiation to blood flow and of course in control theory in linear time-varying systems driven by periodic input signals [4]. Linear time-varying systems driven by periodic input signals are ubiquitous in control systems, from natural sciences to engineering, economics, physics, and the life science [4, 5]. Periodic functions also appear in automotive engine applications [6], optimal periodic scheduling of sensor networks [7, 8], or cyclic gene regulatory networks [9], to give some applications.

It is an obvious fact that the classical derivative, if it exists, of a periodic function is also a periodic function of the same period. Also the primitive of a periodic function may be periodic (e.g., $\cos t$ as primitive of $\sin t$).

The idea of integral or derivatives of noninteger order goes back to Riemann and Liouville [3, 10]. Probably the first application of fractional calculus was made by Abel in the solution of the integral equation that arises in the formulation of the tautochrone problem [11]. Fractional calculus appears in many different contexts as speech signals, cardiac tissue electrode interface, theory of viscoelasticity, or fluid mechanics. The asymptotic stability of positive fractional-order nonlinear systems has been proved in [12] by using the Lyapunov function. We do not intend to give a full list of applications but to show the wide range of them.

In this paper we prove that periodicity is not transferred by fractional integral or derivative, with the exception of the zero function. Although this property seems to be known [10, 13, 14], in Section 3 we give a different proof by using the Laplace transform. Our approach relies on the classical concepts of fractional calculus and elementary analysis. Moreover, by using a similar argument as in [15], in Section 4 we prove that the fractional derivative or primitive of a T-periodic function cannot be $\widetilde{T}$-periodic for any period $\widetilde{T}$. A particular but nontrivial example is explicitly given. Finally, as a consequence we show in Section 5 that an autonomous fractional differential equation cannot have periodic solutions with the exception of constant functions.

2. Preliminaries

Let $T > 0$. If $f : \mathbb{R} \to \mathbb{R}$ is T periodic and $f \in \mathscr{C}^1(\mathbb{R})$, then the derivative f' is also T-periodic. However, the primitive of f

$$F(t) = \int_0^t f(s)\, ds \tag{1}$$

is not, in general, T-periodic. Just take $f(t) = 1$ so that $F(t) = t$ is not T'-periodic for any $T' > 0$. The necessary and sufficient condition for F to be T-periodic is that

$$\int_0^T f(s)\, ds = 0. \tag{2}$$

The purpose of this note is to show that the fractional derivative or the fractional primitive of a T-periodic function cannot be T-periodic function with the exception, of course, of the zero function. We use the notation

$$F = I^1 f, \qquad f' = D^1 f \tag{3}$$

and note that

$$D^1 \left(I^1 f\right)(t) = D^1 F(t) = f(t) \tag{4}$$

but

$$I^1 \left(D^1 f\right)(t) = f(t) - f(0), \tag{5}$$

and $I^1(D^1 f)$ does not coincide with f unless $f(0) = 0$.

We recall some elements of fractional calculus. Let $\alpha \in (0,1)$ and $f : \mathbb{R} \to \mathbb{R}$. We point out that f is not necessarily continuous. The fractional integral of f of order α is defined by [16]

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds, \tag{6}$$

provided the right-hand side is defined for a.e. $t \in \mathbb{R}$. If, for example, $f \in \mathscr{L}^1(\mathbb{R})$, then the fractional integral (6) is well defined and $I^\alpha f \in \mathscr{L}^1(0,T)$, for any $T > 0$. Moreover, the fractional operator

$$I^\alpha : \mathscr{L}^1(0,T) \longrightarrow \mathscr{L}^1(0,T) \tag{7}$$

is linear and bounded.

The fractional Riemann-Liouville derivative of order α of f is defined as [16, 17]

$$D^\alpha f(t) = D^1 I^{1-\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} f(s)\, ds. \tag{8}$$

This is well defined if, for example, $f \in \mathscr{L}^1_{\text{loc}}(\mathbb{R})$.

There are many more fractional derivatives. We are not giving a complete list but recall the Caputo derivative [16, 17]

$$^cD^\alpha f(t) = I^{1-\alpha} D^1 f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s)\, ds, \tag{9}$$

which is well defined, for example, for absolutely continuous functions.

As in the integer case we have

$$D^\alpha \left(I^\alpha f\right)(t) = f(t), \qquad {}^cD^\alpha \left(I^\alpha f\right)(t) = f(t) \tag{10}$$

but $I^\alpha(D^\alpha f)$ or $I^\alpha({}^cD^\alpha f)$ are not, in general, equal to f. Indeed

$$I^\alpha \left({}^cD^\alpha f\right)(t) = f(t) - f(0), \tag{11}$$

and (see [17, (2.113), p. 71])

$$I^\alpha \left(D^\alpha f\right)(t) = f(t) - \frac{D^{\alpha-1} f(0)}{\Gamma(\alpha)} t^{\alpha-1}. \tag{12}$$

Also [16, (2.4.4), p. 91]

$$^cD^\alpha f(t) = D^\alpha \left(f(t) - f(0)\right). \tag{13}$$

3. The Fractional Derivative or Primitive of a T-Periodic Function Cannot Be T-Periodic

We prove the following result in Section 3.1 below.

Theorem 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a nonzero T-periodic function with $f \in \mathscr{L}^1_{\text{loc}}(\mathbb{R})$. Then $I^\alpha f$ cannot be T-periodic for any $\alpha \in (0,1)$.*

Corollary 2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a nonzero T-periodic function such that $f \in \mathscr{L}^1_{\text{loc}}(\mathbb{R})$. Then the Caputo derivative ${}^cD^\alpha f$ cannot be T-periodic for any $\alpha \in (0,1)$. The same result holds for the fractional derivative $D^\alpha f$.*

Proof. Suppose that ${}^cD^\alpha f$ is T-periodic. Then by Theorem 1, $I^\alpha({}^cD^\alpha f)$ cannot be T-periodic. However,

$$I^\alpha \left({}^cD^\alpha f\right)(t) = f(t) - f(0) \tag{14}$$

is T-periodic. In relation to the fractional Riemann-Liouville derivative, suppose that $D^\alpha f$ is T-periodic and consider the function $\hat{f} = f - f(0)$ which is also T-periodic. Then

$$^cD^\alpha \hat{f} = D^\alpha \hat{f} \tag{15}$$

cannot be T-periodic. □

3.1. Proof of Theorem 1. Let $\alpha \in (0,1)$ and $T > 0$. By reduction to the absurd, in this section we suppose that $I^\alpha f$ is T-periodic. Then

$$I^\alpha f(0) = 0 = I^\alpha f(T); \tag{16}$$

that is,

$$\int_0^T (T-s)^{\alpha-1} f(s)\, ds = 0. \tag{17}$$

Lemma 3. *Assume $f \in \mathscr{L}^1_{\text{loc}}(\mathbb{R})$ is T-periodic. If $I^\alpha f$ is also T-periodic, then*

$$\int_0^T (nT-s)^{\alpha-1} f(s)\, ds = 0, \quad (n \in \mathbb{N} := \{1,2,3,\ldots\}). \tag{18}$$

Proof. For $n = 1$ the latter equality reduces to (17). For $n = 2$,

$$\begin{aligned} 0 &= I^{\alpha} f(2T) \\ &= \frac{1}{\Gamma(\alpha)} \int_0^{2T} (2T-s)^{\alpha-1} f(s)\, ds \\ &= \frac{1}{\Gamma(\alpha)} \int_0^{T} (2T-s)^{\alpha-1} f(s)\, ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_T^{2T} (2T-s)^{\alpha-1} f(s)\, ds \\ &= \frac{1}{\Gamma(\alpha)} \int_0^{T} (2T-s)^{\alpha-1} f(s)\, ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^{T} (T-r)^{\alpha-1} f(r+T)\, dr \\ &= \frac{1}{\Gamma(\alpha)} \int_0^{T} (2T-s)^{\alpha-1} f(s)\, ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^{T} (T-r)^{\alpha-1} f(r)\, dr \\ &= \frac{1}{\Gamma(\alpha)} \int_0^{T} (2T-s)^{\alpha-1} f(s)\, ds. \end{aligned} \tag{19}$$

The proof follows by induction on n. Assume that (18) is valid for some $n \in \mathbb{N}$. Then

$$\begin{aligned} &\int_0^{(n+1)T} ((n+1)T-s)^{\alpha-1} f(s)\, ds \\ &\quad = \sum_{j=0}^{n} \int_{jT}^{(j+1)T} ((n+1)T-s)^{\alpha-1} f(s)\, ds, \end{aligned} \tag{20}$$

and, by periodicity,

$$\int_0^{(n+1)T} ((n+1)T-s)^{\alpha-1} f(s)\, ds = I^{\alpha} f((n+1)T) = 0. \tag{21}$$

Moreover, for $j = 1, 2, \ldots, n$,

$$\begin{aligned} &\sum_{j=1}^{n} \int_{jT}^{(j+1)T} ((n+1)T-s)^{\alpha-1} f(s)\, ds \\ &\quad = \sum_{j=1}^{n} \int_0^{T} ((n+1-j)T-r)^{\alpha-1} f(r)\, dr = 0 \end{aligned} \tag{22}$$

by hypothesis of induction since $1 \le n+1-j \le n$. Hence,

$$\begin{aligned} 0 &= \sum_{j=0}^{n} \int_{jT}^{(j+1)T} ((n+1)T-s)^{\alpha-1} f(s)\, ds \\ &= \int_0^{T} ((n+1)T-s)^{\alpha-1} f(s)\, ds. \end{aligned} \tag{23}$$

□

Lemma 4. *Under the hypothesis of Lemma 3,*

$$\int_0^T f(s)\, ds = 0. \tag{24}$$

Proof. Let f^+ and f^- be the positive and negative parts of f,

$$\begin{aligned} f^+(x) &= \max(f(x), 0), \\ f^-(x) &= -\min(f(x), 0), \\ f &= f^+ - f^-. \end{aligned} \tag{25}$$

Equation (18) implies that

$$\int_0^T (nT-s)^{\alpha-1} f^+(s)\, ds = \int_0^T (nT-s)^{\alpha-1} f^-(s)\, ds. \tag{26}$$

If $\int_0^T f^+(s)ds = 0$ or $\int_0^T f^-(s)ds = 0$, then from (18) we get $f = 0$. We consider the case

$$\int_0^T f^+(s)\, ds > \int_0^T f^-(s)\, ds > 0. \tag{27}$$

For n large

$$\left(\frac{nT}{(n-1)T}\right)^{\alpha-1} > \frac{\int_0^T f^-(s)\, ds}{\int_0^T f^+(s)\, ds} \tag{28}$$

or equivalently

$$(nT)^{\alpha-1} \int_0^T f^+(s)\, ds > ((n-1)T)^{\alpha-1} \int_0^T f^-(s)\, ds. \tag{29}$$

Hence,

$$\begin{aligned} 0 &= \int_0^T (nT-s)^{\alpha-1} f(s)\, ds \\ &\ge (nT)^{\alpha-1} \int_0^T f^+(s)\, ds - ((n-1)T)^{\alpha-1} \int_0^T f^-(s)\, ds > 0, \end{aligned} \tag{30}$$

which is a contradiction.

The case

$$\int_0^T f^-(s)\, ds > \int_0^T f^+(s)\, ds > 0 \tag{31}$$

is analogous.

Therefore,

$$\begin{aligned} \int_0^T f^-(s)\, ds &= \int_0^T f^+(s)\, ds > 0, \\ \int_0^T f(s)\, ds &= 0. \end{aligned} \tag{32}$$

□

Lemma 5. *Under the hypothesis of Lemma 3,*

$$\int_0^T (T+\delta-s)^{\alpha-1} f(s)\,ds = 0, \quad \forall \delta \in [0,T]. \tag{33}$$

Proof. If $\delta = 0$ and $\delta = T$, the equation reduces to (17) and (18), respectively. Let $0 < \delta < T$.

$$\begin{aligned} I^\alpha f(T+\delta) &= \frac{1}{\Gamma(\alpha)} \int_0^{T+\delta} (T+\delta-s)^{\alpha-1} f(s)\,ds \\ &= \frac{1}{\Gamma(\alpha)} \int_0^{T} (T+\delta-s)^{\alpha-1} f(s)\,ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_T^{T+\delta} (T+\delta-s)^{\alpha-1} f(s)\,ds \\ &= \frac{1}{\Gamma(\alpha)} \int_0^{T} (T+\delta-s)^{\alpha-1} f(s)\,ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^{\delta} (\delta-r)^{\alpha-1} f(r+T)\,dr \\ &= \frac{1}{\Gamma(\alpha)} \int_0^{T} (T+\delta-s)^{\alpha-1} f(s)\,ds + I^\alpha f(\delta). \end{aligned} \tag{34}$$

By using the periodicity of $I^\alpha f$ we get (33). □

Lemma 6. *Under the hypothesis of Lemma 3,*

$$\int_0^T (T+t-s)^{\alpha-1} f(s)\,ds = 0, \quad \forall t \in \mathbb{R}. \tag{35}$$

Proof. For $t \in [0,T]$ or $t = nT$, $n = 1,2,\ldots$, relation (35) is true. Let $t = nT+\delta$, so that $T+t = (n+1)T+\delta$. Then

$$\begin{aligned} I^\alpha f(\delta) &= I^\alpha f(T+t) \\ &= \frac{1}{\Gamma(\alpha)} \int_0^{(n+1)T+\delta} ((n+1)T+\delta-s)^{\alpha-1} f(s)\,ds. \end{aligned} \tag{36}$$

Now, using the additive property of the integral, we have

$$\begin{aligned} &\frac{1}{\Gamma(\alpha)} \int_0^{(n+1)T+\delta} ((n+1)T+\delta-s)^{\alpha-1} f(s)\,ds \\ &\quad = \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} \int_{jT}^{(j+1)T} ((n+1)T+\delta-s)^{\alpha-1} f(s)\,ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{(n+1)T}^{(n+1)T+\delta} ((n+1)T+\delta-s)^{\alpha-1} f(s)\,ds. \end{aligned} \tag{37}$$

Let us compute separately the integrals in the right-hand side. In all the integrals depending on j, we use the (linear) change of variable $r = s - jT$ and rename $t' = (n-j)T+\delta$ to obtain

$$\begin{aligned} &\int_{jT}^{(j+1)T} (nT+T+\delta-s)^{\alpha-1} f(s)\,ds \\ &\quad = \int_0^T (T+(n-j)T+\delta-r)^{\alpha-1} f(r+jT)\,dr \\ &\quad = \int_0^T \left(T+t'-s\right)^{\alpha-1} f(s)\,ds. \end{aligned} \tag{38}$$

For the last integral we use the (linear) change of variable $r = s-(n+1)T$ to get

$$\begin{aligned} &\int_{(n+1)T}^{(n+1)T+\delta} ((n+1)T+\delta-s)^{\alpha-1} f(s)\,ds \\ &\quad = \int_0^\delta (\delta-r)^{\alpha-1} f(r+(n+1)T)\,dr = I^\alpha f(\delta). \end{aligned} \tag{39}$$

By induction on n, as in Lemma 3, the proof follows. □

Lemma 7. *Let f be a continuous and T-periodic function, $T > 0$. Let $0 < \alpha < 1$ be fixed. Assuming that*

$$\begin{gathered} \int_0^T (T-s+t)^{\alpha-1} f(s) = 0, \quad \forall t \in \mathbb{R}, \\ \int_0^T f(s)\,ds = 0, \end{gathered} \tag{40}$$

then $f \equiv 0$.

Proof. Since $\int_0^T f(s)ds = 0$ then $0 = \int_0^T f(s)ds = \int_0^T (f^+(s) - f^-(s))ds$ and therefore we can define $c = \int_0^T f^+(s)ds = \int_0^T f^{-s}(s)ds > 0$. If $c = 0$ then $f = 0$.

Let us define

$$\phi(t) = \int_0^T (T-s+t)^{\alpha-1} f(s)\,ds. \tag{41}$$

From the hypothesis, we have that $\phi(t) = 0$ at any $t \in \mathbb{R}$. Therefore, its integral is also zero. Let us integrate with respect to t from a to b for $0 \le a \le b \le T$

$$\begin{aligned} 0 &= \int_a^b \phi(t)\,dt = \int_a^b \left(\int_0^T (T-s+t)^{\alpha-1} f(s)\,ds \right) dt \\ &= \int_0^T \left(\int_a^b (T-s+t)^{\alpha-1} dt \right) f(s)\,ds \\ &= \int_0^T \left(\frac{(b-s+T)^\alpha - (a-s+T)^\alpha}{\alpha} \right) f(s)\,ds, \end{aligned} \tag{42}$$

where we have assumed $0 \le a < b$, $s < T$. Thus,

$$\int_0^T \left[(b-s+T)^\alpha - (a-s+T)^\alpha \right] f(s)\,ds = 0 \tag{43}$$

which implies that

$$\psi(t) = \int_0^T (T-s+t)^\alpha f(s)\,ds \tag{44}$$

is a constant function.

Moreover, since

$$\begin{aligned} t^\alpha c - (T+t)^\alpha c &\le \int_0^T (T-s+t)^\alpha f(s)\,ds \\ &\le (T+t)^\alpha c - t^\alpha c, \end{aligned} \tag{45}$$

where

$$c = \int_0^T f^+(s)\,ds = \int_0^T f^-(s)\,ds \tag{46}$$

in view of (24) and

$$\lim_{t\to+\infty}\left((T+t)^\alpha - t^\alpha\right) = 0, \tag{47}$$

we have that

$$\int_0^T (T-s+t)^\alpha f(s)\,ds = 0, \quad \forall t\in\mathbb{R}. \tag{48}$$

Let

$$\tilde{f} = f\cdot\chi_{[0,T]}, \quad \tilde{f}(t) = \begin{cases} f(t), & t\in[0,T] \\ 0, & t>T. \end{cases} \tag{49}$$

If we define

$$\varphi(t) = (T+t)^\alpha \tag{50}$$

then the convolution of φ and $\tilde{f}$ is given by

$$\begin{aligned} \left(\varphi * \tilde{f}\right) &= \int_0^{+\infty} \varphi(t-s)\,\tilde{f}(s)\,ds \\ &= \int_0^T (T+t-s)^\alpha f(s)\,ds = 0. \end{aligned} \tag{51}$$

Therefore, if we apply the Laplace transform [18, Chapter 17] to the above equality it yields

$$\mathscr{L}\left[\varphi * \tilde{f}\right] = \mathscr{L}[\varphi]\,\mathscr{L}\left[\tilde{f}\right] = \mathscr{L}[0] = 0. \tag{52}$$

Since

$$\mathscr{L}[\varphi] = s^{-\alpha-1}e^{sT}\Gamma(\alpha+1, sT), \tag{53}$$

where $\Gamma(a,z)$ denotes the incomplete gamma function [19, Section 6.5], then $\mathscr{L}[\varphi]\neq 0$ which implies that $\mathscr{L}[\tilde{f}] = 0$ and therefore $\tilde{f} = 0$, that is, $f = 0$, on $[0,T]$. □

4. The Fractional Derivative or Primitive of a T-Periodic Function Cannot Be $\widetilde{T}$-Periodic for any Period $\widetilde{T}$

Let f be a T-periodic function and consider u such that

$${}^cD^\alpha u = f(t), \quad 0<\alpha<1. \tag{54}$$

Then

$$u(t) = u(0) + I^\alpha f(t), \tag{55}$$

and therefore

$$\mathscr{L}[u(t)] = \mathscr{L}u_0 + \mathscr{L}\left[I^\alpha f(t)\right]. \tag{56}$$

Let us assume that u is a $\widetilde{T}$-periodic function. Then by using some basic properties of the Laplace transform it yields

$$\frac{\int_0^{\widetilde{T}} u(t)\exp(-\lambda t)\,dt}{1-\exp\left(-\lambda\widetilde{T}\right)} = \frac{u_0}{\lambda} + \frac{1}{\lambda^\alpha}\frac{\int_0^T f(t)\exp(-\lambda t)\,dt}{1-\exp(-\lambda T)}. \tag{57}$$

Therefore,

$$\begin{aligned} &\lambda\left(1-\exp(-\lambda T)\right)\int_0^{\widetilde{T}} u(t)\exp(-\lambda t)\,dt \\ &\quad= u_0\left(1-\exp(-\lambda T)\right)\left(1-\exp\left(-\lambda\widetilde{T}\right)\right) \\ &\qquad+ \lambda^{1-\alpha}\left(1-\exp\left(-\lambda\widetilde{T}\right)\right)\int_0^T f(t)\exp(-\lambda t)\,dt. \end{aligned} \tag{58}$$

Let us consider $v = u - u_0$ so that v is also $\widetilde{T}$-periodic and $v(0) = 0$. The above equality becomes

$$\begin{aligned} &\lambda\left(1-\exp(-\lambda T)\right)\int_0^{\widetilde{T}} v(t)\exp(-\lambda t)\,dt \\ &\quad= \lambda^{1-\alpha}\left(1-\exp\left(-\lambda\widetilde{T}\right)\right)\int_0^T f(t)\exp(-\lambda t)\,dt \end{aligned} \tag{59}$$

or equivalently

$$\begin{aligned} &\lambda^\alpha\frac{\left(1-\exp(-\lambda T)\right)}{\left(1-\exp\left(-\lambda\widetilde{T}\right)\right)}\int_0^{\widetilde{T}} v(t)\exp(-\lambda t)\,dt \\ &\quad= \int_0^T f(t)\exp(-\lambda t)\,dt. \end{aligned} \tag{60}$$

Thus,

$$\begin{aligned} &\frac{\left(1-\exp(-\lambda T)\right)}{\left(1-\exp\left(-\lambda\widetilde{T}\right)\right)}\sum_{i=0}^{\infty}(-1)^i\frac{\lambda^{\alpha+i}}{i!}\int_0^{\widetilde{T}} v(t)\,t^i\,dt \\ &\quad= \sum_{i=0}^{\infty}(-1)^i\frac{\lambda^i}{i!}\int_0^T f(t)\,t^i\,dt. \end{aligned} \tag{61}$$

Since

$$\lim_{\lambda\to 0^+}\frac{(1-\exp(-\lambda T))}{\left(1-\exp\left(-\lambda\widetilde{T}\right)\right)}=\frac{T}{\widetilde{T}},\qquad \lim_{\lambda\to 0^+}\lambda^{\alpha+i}=0, \tag{62}$$

by using $0<\alpha<1$ and $i\geq 0$, the limit as $\lambda\to 0^+$ of the left-hand side is zero, which implies

$$\int_0^T f(t)\,dt=0. \tag{63}$$

Then

$$\begin{aligned}
&\frac{(1-\exp(-\lambda T))}{\left(1-\exp\left(-\lambda\widetilde{T}\right)\right)}\sum_{i=0}^{\infty}(-1)^i\frac{\lambda^i}{i!}\int_0^{\widetilde{T}}v(t)\,t^i dt\\
&=\lambda^{-\alpha}\sum_{i=1}^{\infty}(-1)^i\frac{\lambda^i}{i!}\int_0^T f(t)\,t^i dt\\
&=\lambda^{1-\alpha}\sum_{i=0}^{\infty}(-1)^{i+1}\frac{\lambda^i}{(i+1)!}\int_0^T f(t)\,t^{i+1}dt.
\end{aligned} \tag{64}$$

If we consider $\lambda\to 0^+$ in the latter expression we get

$$\frac{T}{\widetilde{T}}\int_0^{\widetilde{T}}v(t)\,dt=0, \tag{65}$$

and therefore

$$\int_0^{\widetilde{T}}v(t)\,dt=0. \tag{66}$$

By induction, we obtain that

$$\int_0^T f(t)\,t^i dt=0,\qquad \int_0^{\widetilde{T}}v(t)\,t^i dt=0,\qquad i=0,1,2,\ldots. \tag{67}$$

Therefore, $f=u=0$ and there are no nonzero $\widetilde{T}$-periodic L^∞-solutions of the problem.

Example 8. Let $f(t)=\sin(t)$ and $0<\alpha<1$. The Caputo-fractional derivative of $f(t)$ is given by

$${}^cD^\alpha f(t)=\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\,{}_1F_2\left(1;\frac{3-\alpha}{2},1-\frac{\alpha}{2};-\frac{t^2}{4}\right), \tag{68}$$

where the hypergeometric series ${}_1F_2(a;b,c;d)$ is defined as ([20, 21], Chapter 15)

$${}_1F_2(a;b,c;d)=\sum_{j=0}^{\infty}\frac{(a)_j}{j!(b)_j\,(c)_j}d^j, \tag{69}$$

and the Pochhammer symbol $(A)_j=A(A+1)\cdots(A+j-1)$, with $(A)_0=1$.

Since

$$\begin{aligned}
&\frac{{}^cD^\alpha f(\pi)}{{}^cD^\alpha f\left(\pi+\widetilde{T}\right)}\\
&=\pi^{1-\alpha}\left(\widetilde{T}+\pi\right)^{\alpha-1}\,{}_1F_2\left(1;1-\frac{\alpha}{2},\frac{3}{2}-\frac{\alpha}{2};-\frac{\pi^2}{4}\right)\\
&\quad\times\left({}_1F_2\left(1;1-\frac{\alpha}{2},\frac{3}{2}-\frac{\alpha}{2};-\frac{1}{4}\left(\widetilde{T}+\pi\right)^2\right)\right)^{-1},\\
&\frac{{}^cD^\alpha f(\pi/2)}{{}^cD^\alpha f\left(\pi/2+\widetilde{T}\right)}\\
&=\left(\frac{2T}{\pi}+1\right)^{\alpha-1}\,{}_1F_2\left(1;1-\frac{\alpha}{2},\frac{3}{2}-\frac{\alpha}{2};-\frac{\pi^2}{16}\right)\\
&\quad\times\left({}_1F_2\left(1;1-\frac{\alpha}{2},\frac{3}{2}-\frac{\alpha}{2};-\frac{1}{16}(2T+\pi)^2\right)\right)^{-1},
\end{aligned} \tag{70}$$

we have that ${}^cD^\alpha f(t)$ is not a $\widetilde{T}$-periodic function for any positive $\widetilde{T}$ and $\alpha\in(0,1)$. Plotting both functions $\sin(t)$ and ${}^cD^\alpha\sin(t)$, this last function seems to be periodic but it is not according to our results. Notice that Kaslik and Sivasundaram [10] gave the following alternate representation:

$${}^cD^\alpha\sin(t)=\frac{1}{2}t^{1-\alpha}\left[E_{1,2-\alpha}(it)+E_{1,2-\alpha}(-it)\right], \tag{71}$$

in terms of the two-parameter Mittag-Leffler function ([20, 21], Chapter 10)

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}. \tag{72}$$

5. Periodic Solutions of Fractional Differential Equations

In this section we show how Theorem 1 can be used to give a nonexistence result of periodic solutions for fractional differential equations.

Consider the first order ordinary differential equation

$$D^1u(t)=\varphi(u(t)),\qquad t\in\mathbb{R}, \tag{73}$$

where $\varphi:\mathbb{R}\to\mathbb{R}$ is continuous. An important question is the existence of periodic solutions [22–24].

If $u:\mathbb{R}\to\mathbb{R}$ is a T-periodic solution of (73) then obviously

$$u(0)=u(T). \tag{74}$$

One can find T-periodic solutions of (73) by solving the equation only on the interval $[0,T]$ and then checking the values $u(0)$ and $u(T)$. If (74) holds, then extending by T-periodicity the function $u(t)$, $t\in[0,T]$, to $\mathbb{R}$ we have a T-periodic solution of (73).

However, this is not possible for a fractional differential equation. Consider, for $\alpha\in(0,1)$, the equation

$${}^cD^\alpha u(t)=\varphi(u(t)),\qquad t\in\mathbb{R}. \tag{75}$$

If u is a solution of (75), let $f(t) = \varphi(u(t))$. Then

$$u(t) = u(0) + I^{\alpha} f(t). \tag{76}$$

In the case that u is a T-periodic solution of (75) we have that f is also T-periodic. According to Theorem 1, $I^{\alpha} f$ cannot be T-periodic unless it is the zero function and we have the following relevant result.

Theorem 9. *The fractional equation* (75) *cannot have periodic solutions with the exception of constant functions* $u(t) = u_0$, $t \in \mathbb{R}$, *with* $\varphi(u_0) = 0$.

Remark 10. It is possible to consider the periodic boundary value problem

$$\begin{gathered} {}^{c}D^{\alpha} u(t) = \varphi(u(t)), \quad t \in [0, T], \\ u(0) = u(T), \end{gathered} \tag{77}$$

as in, for example, [25], but one cannot extend the solution of that periodic boundary value problem on $[0, T]$ to a T-periodic solution on $\mathbb{R}$ (unless u is a constant function, as indicated in Theorem 9).

Remark 11. The same applies to the Riemann-Liouville fractional differential equation

$$D^{\alpha} u(t) = \varphi(u(t)), \quad t \in \mathbb{R}, \tag{78}$$

taking into account that

$$\lim_{t \to 0^{+}} t^{1-\alpha} u(t) = \frac{D^{\alpha-1} u(0)}{\Gamma(\alpha)}. \tag{79}$$

Example 12. Considering the fractional equation

$${}^{c}D^{\alpha} u(t) = \psi(t, u(t)), \quad t \in \mathbb{R}, \tag{80}$$

with $\psi : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\begin{aligned} &\psi(t, u) \\ &\quad = u + \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \, {}_{1}F_{2}\left(1; \frac{3-\alpha}{2}, 1 - \frac{\alpha}{2}; -\frac{t^2}{4}\right) - \sin(t), \end{aligned} \tag{81}$$

we have that $u(t) = \sin(t)$ is a 2π-periodic solution of (80). This shows that the result of Theorem 9 is not valid for a nonautonomous fractional differential equation as (80).

6. Conclusion

By using the classical concepts of fractional calculus and elementary analysis, we have proved that periodicity is not transferred by fractional integral or derivative, with the exception of the zero function. We have also proved that the fractional derivative or primitive of a T-periodic function cannot be $\widetilde{T}$-periodic for any period $\widetilde{T}$. As a consequence we have showed that an autonomous fractional differential equation cannot have periodic solutions with the exception of constant functions.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Local Fractional Functional Method for Solving Diffusion Equations on Cantor Sets

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The analytical solutions for the diffusion equations on Cantor sets with the nondifferentiable terms are discussed by using the local fractional functional method, which is a coupling method for local fractional Fourier series and Laplace transform.

1. Introduction

The local fractional calculus [1, 2], as a new branch of fractional calculus, was successfully applied to describe the fractal problems from science and engineering. For example, the local fractional Fokker-Planck equation [3], the local fractional diffusion equations defined on Cantor sets [4, 5], the local fractional wave equation defined on Cantor sets [6, 7], the local fractional Korteweg-de Vries equation [8], the local fractional Schrödinger equation [9], local fractional Navier-Stokes equations on cantor sets [10], the local fractional Laplace equation [11], the local fractional heat-conduction equation [12–16], the local fractional differential equations arising in the fractal forest gap [17], and others [18–21] were discussed.

In this paper, we consider the local fractional diffusion equations defined on Cantor sets [5] given by

$$u_{xx}^{2\alpha} = \frac{1}{a^{2\alpha}} u_t^{\alpha}, \tag{1}$$

subject to the initial-boundary conditions

$$\frac{\partial^\alpha}{\partial x^\alpha} u(0,t) = g(t), \qquad u(0,t) = f(t),$$
$$u(x,0) = u(x,l) = 0, \tag{2}$$

where the local fractional partial derivatives denote

$$u_t^{\alpha} = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha}, \qquad u_{yx}^{2\alpha} = \frac{\partial^{2\alpha} u(x,t)}{\partial x^\alpha \partial y^\alpha}, \tag{3}$$

and $u(x,t)$, $g(t)$, and $f(t)$ are the local fractional continuous functions. In the high-speed railway healthy monitor system, the problems of diffusion equations with the nondifferentiable terms always exist in fault diagnosing of high-speed trains and their control systems, so we solve this by the local fractional diffusion equations defined on Cantor sets. The local fractional function decomposition method structured in [11, 22], which is a coupling method of the local fractional Fourier series [21, 22] and the Yang-Laplace transform [14, 16, 18, 22], was used to solve the inhomogeneous local fractional wave equations defined on Cantor sets. The main aim of this paper is to discuss the local fractional diffusion equations defined on Cantor sets by the local fractional functional method.

The paper is organized as follows. In Section 2 the basic theory of the local fractional calculus and the Yang-Laplace transform were given. In Section 3, the local fractional functional method is analyzed. Section 4 presents the applications for the local fractional diffusion equations defined on Cantor sets. Finally, the conclusions are given in Section 5.

2. Preliminaries

In this section, we present the basic theory of the local fractional calculus and the local fractional Laplace transform.

Definition 1 (see [1, 5–7]). The local fractional derivative of $f(x)$ at $x = x_0$ is given as follows:

$$\begin{aligned} D_x^{\alpha} f(x_0) &= \left.\frac{d^{\alpha}}{dx^{\alpha}} f(x)\right|_{x=x_0} = f^{(\alpha)}(x) \\ &= \lim_{x \to x_0} \frac{\Delta^{\alpha}(f(x) - f(x_0))}{(x - x_0)^{\alpha}}, \end{aligned} \tag{4}$$

where $\Delta^{\alpha}(f(x) - f(x_0)) \cong \Gamma(\alpha+1)\Delta(f(x) - f(x_0))$.

The local fractional partial derivative of order α is defined as follows [1]:

$$\begin{aligned} \left.\frac{\partial^{\alpha}}{\partial x^{\alpha}} f(x,y)\right|_{x=x_0} &= f^{(\alpha)}(x,y) \\ &= \lim_{x \to x_0} \frac{\Delta^{\alpha}(f(x,y) - f(x_0,y))}{(x - x_0)^{\alpha}}, \end{aligned} \tag{5}$$

and the local fractional partial derivative of high order [1] is

$$\frac{\partial^{k\alpha} f(x,y)}{x^{k\alpha}} = \overbrace{\frac{\partial^{\alpha}}{\partial x^{\alpha}} \frac{\partial^{\alpha}}{\partial x^{\alpha}} \cdots \frac{\partial^{\alpha}}{\partial x^{\alpha}}}^{k \text{ times}} f(x,y), \tag{6}$$

where $\Delta^{\alpha}(f(x) - f(x_0)) \cong \Gamma(\alpha+1)\Delta(f(x) - f(x_0))$.

Definition 2 (see [1, 8–12]). Let us consider a partition of the interval $[a,b]$, which is denoted as (t_j, t_{j+1}), $j = 0, \dots, N-1$, $t_0 = a$ and $t_N = b$ with $\Delta t_j = t_{j+1} - t_j$ and $\Delta t = \max\{\Delta t_0, \Delta t_1, \dots\}$. Local fractional integral of $f(x)$ in the interval $[a,b]$ is defined as follows:

$$\begin{aligned} {}_aI_b^{(\alpha)} f(x) &= \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t)(dt)^{\alpha} \\ &= \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \to 0} \sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^{\alpha}. \end{aligned} \tag{7}$$

Definition 3 (see [1, 5, 11, 16, 21]). The Mittag Leffler, sine and cosine functions defined on Cantor sets are given as follows:

$$\begin{aligned} E_{\alpha}(x^{\alpha}) &= \sum_{k=0}^{\infty} \frac{x^{\alpha k}}{\Gamma(1+k\alpha)}, \\ \sin_{\alpha} x^{\alpha} &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{\alpha(2k+1)}}{\Gamma[1+\alpha(2k+1)]}, \\ \cos_{\alpha} x^{\alpha} &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2\alpha k}}{\Gamma(1+2\alpha k)}, \end{aligned} \tag{8}$$

for $x \in R$, $0 < \alpha < 1$.

Definition 4 (see [11, 20–22]). Let $f(x)$ be $2l$-periodic. For $k \in Z$, local fraction Fourier series of $f(x)$ is given as

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_n \cos_{\alpha} \frac{\pi^{\alpha}(kx)^{\alpha}}{l^{\alpha}} + b_n \sin_{\alpha} \frac{\pi^{\alpha}(kx)^{\alpha}}{l^{\alpha}} \right), \tag{9}$$

where the local fraction Fourier coefficients are as follows:

$$\begin{aligned} a_n &= \frac{2}{l^{\alpha}} \int_0^l f(x) \cos_{\alpha} \frac{\pi^{\alpha}(kx)^{\alpha}}{l^{\alpha}} (dx)^{\alpha}, \\ b_n &= \frac{2}{l^{\alpha}} \int_0^l f(x) \sin_{\alpha} \frac{\pi^{\alpha}(kx)^{\alpha}}{l^{\alpha}} (dx)^{\alpha}. \end{aligned} \tag{10}$$

Definition 5 (see [14, 16, 18, 22]). Let $(1/\Gamma(1+\alpha)) \int_0^{\infty} |f(x)|(dx)^{\alpha} < k < \infty$. The local fractional Laplace transform of $f(x)$ is given as

$$\widetilde{L}_{\alpha}\{f(x)\} = f_s^{L,\alpha}(s) = \frac{1}{\Gamma(1+\alpha)} \int_0^{\infty} E_{\alpha}(-s^{\alpha}x^{\alpha}) f(x)(dx)^{\alpha}, \quad 0 < \alpha \le 1. \tag{11}$$

The inverse formula local fractional Laplace transform of $f(x)$ is given as [14, 16, 18, 22]

$$\begin{aligned} f(x) &= \widetilde{L}_{\alpha}^{-1}\left\{f_s^{L,\alpha}(s)\right\} \\ &= \frac{1}{(2\pi)^{\alpha}} \int_{\beta-i\infty}^{\beta+i\infty} E_{\alpha}(s^{\alpha}x^{\alpha}) f_s^{\widetilde{L},\alpha}(s)(ds)^{\alpha}, \end{aligned} \tag{12}$$

where $f(x)$ is local fractional continuous, $s^{\alpha} = \beta^{\alpha} + i^{\alpha}\infty^{\alpha}$ and $\mathrm{Re}(s) = \beta > 0$.

There is the following formula [14, 16, 18, 22]:

$$\widetilde{L}_{\alpha}\left\{y^{(2\alpha)}(x)\right\} = s^{2\alpha}\widetilde{L}_{\alpha}\{y(x)\} - s^{\alpha}y(0) - f^{(\alpha)}(0). \tag{13}$$

The basic properties of the local fractional calculus and the local fractional Laplace transform were listed in [1, 14, 16, 18, 22].

3. Analysis of the Local Fractional Functional Method

In this section, we introduce the local fractional functional method for the local fractional diffusion equations defined on Cantor sets [11, 22].

Let us consider the nondifferentiable decomposition of the function with the nondifferentiable systems $\{\sin_{\alpha} n^{\alpha}(\pi t/l)^{\alpha}\}$. There are the following functional coefficients of (1) and (2), which are given as follows:

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} A_n(x) \sin_{\alpha} n^{\alpha} \left(\frac{\pi t}{l}\right)^{\alpha}, \\ g(t) &= \sum_{n=1}^{\infty} C_n \sin_{\alpha} n^{\alpha} \left(\frac{\pi t}{l}\right)^{\alpha}, \\ f(t) &= \sum_{n=1}^{\infty} D_n \sin_{\alpha} n^{\alpha} \left(\frac{\pi t}{l}\right)^{\alpha}, \end{aligned} \tag{14}$$

where

$$A_n(x) = \frac{2}{l^\alpha}\int_0^1 u(x,t)\sin_\alpha n^\alpha\left(\frac{\pi t}{l}\right)^\alpha (dt)^\alpha,$$
$$C_n = \frac{2}{l^\alpha}\int_0^1 g(t)\sin_\alpha n^\alpha\left(\frac{\pi t}{l}\right)^\alpha (dt)^\alpha, \tag{15}$$
$$D_n = \frac{2}{l^\alpha}\int_0^1 f(t)\sin_\alpha n^\alpha\left(\frac{\pi t}{l}\right)^\alpha (dt)^\alpha.$$

If we submit (14) into (1) and (2), then we have

$$\frac{d^{2\alpha}}{dx^{2\alpha}}A_n(x) = \frac{1}{a^{2\alpha}}A_n(x)\left(\frac{n\pi}{l}\right)^\alpha,$$
$$A_n^{(\alpha)}(0) = C_n, \tag{16}$$
$$A_n(0) = D_n.$$

Taking the local fractional Laplace transform of (16) gives

$$\widetilde{L}_\alpha\left\{\frac{d^{2\alpha}}{dx^{2\alpha}}A_n(x)\right\} = s^{2\alpha}\widetilde{L}_\alpha\{A_n(x)\} - s^\alpha A_n(0) - A_n^{(\alpha)}(0), \tag{17}$$

which leads to

$$s^{2\alpha}\widetilde{L}_\alpha\{A_n(x)\} - s^\alpha D_n - C_n = \frac{1}{a^{2\alpha}}\left(\frac{n\pi}{l}\right)^\alpha \widetilde{L}_\alpha\{A_n(x)\}. \tag{18}$$

We can rewrite (18) as

$$\begin{aligned}
&A_n(s)\\
&= \widetilde{L}_\alpha\{A_n(x)\}\\
&= \frac{s^\alpha D_n + C_n}{s^{2\alpha} - (1/a^{2\alpha})(n\pi/l)^\alpha}\\
&= \frac{D_n}{s^\alpha - (1/a^\alpha)(n\pi/l)^{\alpha/2}} + \frac{\left(C_n - (1/a^\alpha)(n\pi/l)^{\alpha/2}D_n\right)}{s^{2\alpha} - (1/a^{2\alpha})(n\pi/l)^\alpha}\\
&= \frac{D_n}{s^\alpha - (1/a^\alpha)(n\pi/l)^{\alpha/2}}\\
&\quad + P\left(\frac{1}{s^\alpha - (1/a^\alpha)(n\pi/l)^{\alpha/2}} - \frac{1}{s^\alpha + (1/a^\alpha)(n\pi/l)^{\alpha/2}}\right),
\end{aligned} \tag{19}$$

where

$$P = \frac{C_n - (1/a^\alpha)(n\pi/l)^{\alpha/2}D_n}{(1/a^\alpha)(n\pi/l)^{\alpha/2}}. \tag{20}$$

The inverse formula local fractional Laplace transform of (19) gives

$$\begin{aligned}
&A_n(x)\\
&= \widetilde{L}_\alpha^{-1}\{A_n(s)\}\\
&= \frac{1}{(2\pi)^\alpha}\int_{\beta-i\infty}^{\beta+i\infty} E_\alpha(s^\alpha x^\alpha)A_n(s)(ds)^\alpha\\
&= \frac{1}{(2\pi)^\alpha}\\
&\quad\times\int_{\beta-i\infty}^{\beta+i\infty} E_\alpha(s^\alpha x^\alpha)\left\{\frac{D_n}{s^\alpha - (1/a^\alpha)(n\pi/l)^{\alpha/2}}\right.\\
&\qquad + P\left(\frac{1}{s^\alpha - (1/a^\alpha)(n\pi/l)^{\alpha/2}}\right.\\
&\qquad\left.\left. - \frac{1}{s^\alpha + (1/a^\alpha)(n\pi/l)^{\alpha/2}}\right)\right\}(ds)^\alpha\\
&= D_n E_\alpha\left(\frac{1}{a^\alpha}\left(\frac{n\pi}{l}\right)^{\alpha/2}x^\alpha\right) + PE_\alpha\left(\frac{1}{a^\alpha}\left(\frac{n\pi}{l}\right)^{\alpha/2}x^\alpha\right)\\
&\quad - PE_\alpha\left(-\frac{1}{a^\alpha}\left(\frac{n\pi}{l}\right)^{\alpha/2}x^\alpha\right),
\end{aligned} \tag{21}$$

where

$$P = \frac{C_n - (1/a^\alpha)(n\pi/l)^{\alpha/2}D_n}{(1/a^\alpha)(n\pi/l)^{\alpha/2}}. \tag{22}$$

Hence, the solution of (1) reads as follows:

$$\begin{aligned}
&u(x,t)\\
&= \sum_{n=1}^\infty A_n(x)\sin_\alpha n^\alpha\left(\frac{\pi t}{l}\right)^\alpha\\
&= \sum_{n=1}^\infty D_n E_\alpha\left(\frac{1}{a^\alpha}\left(\frac{n\pi}{l}\right)^{\alpha/2}x^\alpha\right)\sin_\alpha n^\alpha\left(\frac{\pi t}{l}\right)^\alpha\\
&\quad + \sum_{n=1}^\infty PE_\alpha\left(\frac{1}{a^\alpha}\left(\frac{n\pi}{l}\right)^{\alpha/2}x^\alpha\right)\sin_\alpha n^\alpha\left(\frac{\pi t}{l}\right)^\alpha\\
&\quad - \sum_{n=1}^\infty PE_\alpha\left(-\frac{1}{a^\alpha}\left(\frac{n\pi}{l}\right)^{\alpha/2}x^\alpha\right)\sin_\alpha n^\alpha\left(\frac{\pi t}{l}\right)^\alpha,
\end{aligned} \tag{23}$$

where

$$P = \frac{C_n - (1/a^\alpha)(n\pi/l)^{\alpha/2}D_n}{(1/a^\alpha)(n\pi/l)^{\alpha/2}}. \tag{24}$$

4. The Exact Solutions for Local Fractional Diffusion Equations Defined on Cantor Sets

In this section we give two examples for initial boundary problems for local fractional diffusion equations defined on Cantor sets.

Example 6. The initial-boundary values of (1) read as follows:

$$\frac{\partial^\alpha}{\partial x^\alpha} u(0,t) = \sin(t^\alpha), \qquad u(0,t) = 0,$$
$$u(x,0) = u(x,\pi) = 0. \tag{25}$$

Making use of (14), we obtain the following formulas:

$$\sin(t^\alpha) = \sum_{n=1}^{\infty} C_n \sin_\alpha n^\alpha t^\alpha,$$
$$0 = \sum_{n=1}^{\infty} D_n \sin_\alpha n^\alpha t^\alpha, \tag{26}$$

which lead to the following parameters:

$$C_n = 1, \quad n = 1,$$
$$C_n = 0, \quad n > 1, \tag{27}$$
$$D_n = 0, \quad n \geq 1.$$

Therefore, (23) gives the nondifferentiable solution of (1) with initial-boundary values (25)

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} A_n(x) \sin_\alpha n^\alpha t^\alpha \\ &= \sum_{n=1}^{\infty} D_n E_\alpha\left(\frac{1}{a^\alpha} x^\alpha\right) \sin_\alpha n^\alpha t^\alpha + \sum_{n=1}^{\infty} P E_\alpha\left(\frac{1}{a^\alpha} x^\alpha\right) \sin_\alpha n^\alpha t^\alpha \\ &\quad - \sum_{n=1}^{\infty} P E_\alpha\left(-\frac{1}{a^\alpha} x^\alpha\right) \sin_\alpha n^\alpha t^\alpha \\ &= a^\alpha E_\alpha\left(\frac{1}{a^\alpha} x^\alpha\right) \sin_\alpha t^\alpha - a^\alpha E_\alpha\left(-\frac{1}{a^\alpha} x^\alpha\right) \sin_\alpha t^\alpha. \end{aligned} \tag{28}$$

When $a = 1$, we get the nondifferentiable solution

$$u(x,t) = E_\alpha(x^\alpha) \sin_\alpha t^\alpha - E_\alpha(-x^\alpha) \sin_\alpha t^\alpha, \tag{29}$$

and its graph is shown in Figure 1.

Example 7. We present the initial-boundary values of (1) as

$$\frac{\partial^\alpha}{\partial x^\alpha} u(0,t) = \sin(t^\alpha), \qquad u(0,t) = \sin(t^\alpha),$$
$$u(x,0) = u(x,\pi) = 0. \tag{30}$$

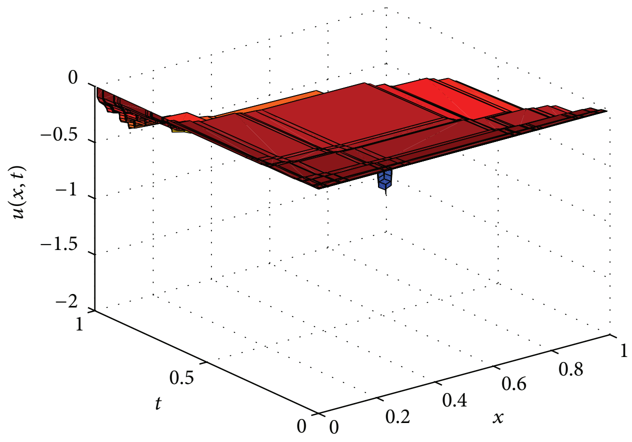

Figure 1: The solution of (1) with initial-boundary value (25) when $a = 1$ and $\alpha = \ln 2/\ln 3$.

Using the relation (14), we get

$$0 = \sum_{n=1}^{\infty} C_n \sin_\alpha n^\alpha t^\alpha,$$
$$\sin(t^\alpha) = \sum_{n=1}^{\infty} D_n \sin_\alpha n^\alpha t^\alpha, \tag{31}$$

which reduce to

$$C_n = 0, \quad n \geq 1,$$
$$D_n = 1, \quad n = 1, \tag{32}$$
$$D_n = 0, \quad n > 1.$$

Using (23), we hence have the nondifferentiable solution of (1) with initial-boundary values (30), which is given as

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} A_n(x) \sin_\alpha n^\alpha t^\alpha \\ &= \sum_{n=1}^{\infty} D_n E_\alpha\left(\frac{1}{a^\alpha} x^\alpha\right) \sin_\alpha n^\alpha t^\alpha + \sum_{n=1}^{\infty} P E_\alpha\left(\frac{1}{a^\alpha} x^\alpha\right) \sin_\alpha n^\alpha t^\alpha \\ &\quad - \sum_{n=1}^{\infty} P E_\alpha\left(-\frac{1}{a^\alpha} x^\alpha\right) \sin_\alpha n^\alpha t^\alpha \\ &= E_\alpha\left(\frac{1}{a^\alpha} x^\alpha\right) \sin_\alpha t^\alpha - a^\alpha E_\alpha\left(\frac{1}{a^\alpha} x^\alpha\right) \sin_\alpha t^\alpha \\ &\quad + a^\alpha E_\alpha\left(-\frac{1}{a^\alpha} x^\alpha\right) \sin_\alpha t^\alpha. \end{aligned} \tag{33}$$

For $a = 1$, the nondifferentiable solution rewrites as follows:

$$u(x,t) = E_\alpha(-x^\alpha) \sin_\alpha t^\alpha, \tag{34}$$

and its graph is illustrated in Figure 2.

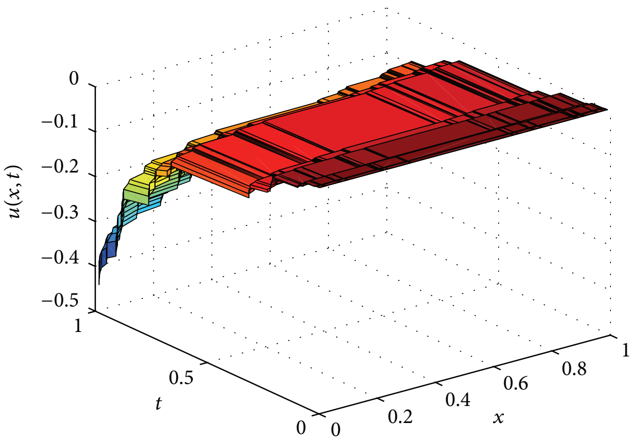

FIGURE 2: The solution of (1) with initial-boundary value (30) when $a = 1$ and $\alpha = \ln 2/\ln 3$.

5. Conclusions

Local fractional calculus was applied to describe the physical problems because of nondifferentiable characteristics. In this work, the initial-boundary value problems for the diffusion equation on Cantor sets within the local fractional derivatives were investigated by using the local fractional functional method, which is a coupling method for local fractional Fourier series and Laplace transform based upon the nondifferentiable decomposition of the function with the nondifferentiable systems. The two examples are given to express the efficiency of the presented method and their graphs are also obtained. The results of this paper could provide the theory support to the problems diffusion equations with the nondifferentiable terms in health monitor of high-speed trains and their control systems.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

On Higher-Order Sequential Fractional Differential Inclusions with Nonlocal Three-Point Boundary Conditions

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We study a nonlinear three-point boundary value problem of sequential fractional differential inclusions of order $\xi + 1$ with $n - 1 < \xi \leq n$, $n \geq 2$. Some new existence results for convex as well as nonconvex multivalued maps are obtained by using standard fixed point theorems. The paper concludes with an example.

1. Introduction

The topic of fractional differential equations has attracted a great attention in the recent years. It is mainly due to the intensive development of the theory and applications of fractional calculus. In fact, the tools of fractional calculus have considerably improved the modeling of several real world phenomena in physics, chemistry, bioengineering, etc. The systematic development of theory, methods, and applications of fractional differential equations can be found in [1–6]. For some recent results on fractional differential equations and inclusions, see [7–23] and the references cited therein.

In this paper, we study the following boundary value problem:

$$\begin{gathered} {}^cD^{\xi}(D+\lambda)x(t) \in F(t, x(t)), \\ 0 < t < 1, \quad n-1 < \xi \leq n, \\ x(0) = 0, \qquad x'(0) = 0, \\ x''(0) = 0, \ldots, x^{(n-1)}(0) = 0, \qquad x(1) = \alpha x(\sigma), \end{gathered} \tag{1}$$

where ${}^cD$ is the Caputo fractional derivative, D is the ordinary derivative, $F : [0,1] \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ is a multivalued map, $\mathscr{P}(\mathbb{R})$ is the family of all subsets of $\mathbb{R}$, $0 < \sigma < 1$, λ is a positive real number, and α is a real number.

The present work is motivated by a recent paper of the authors [14], where the problem (1) was considered for a single-valued case. The existence of solutions for the given multivalued problem is discussed for three cases: (a) convex-valued maps; (b) not necessarily convex-valued maps; (c) nonconvex-valued maps. To establish the existence results, we make use of nonlinear alternative for Kakutani maps, nonlinear alternative of Leray-Schauder type for single-valued maps, selection theorem due to Bressan and Colombo for lower semicontinuous multivalued maps with nonempty closed and decomposable values, and a fixed point theorem for contractive multivalued maps due to Covitz and Nadler. The tools employed in this paper are standard; however, their exposition in the framework of the problem at hand is new.

The paper is organized as follows: in Section 2 we recall some preliminary facts that we used in the sequel. Section 3 contains the main results and an example. In Section 4, we summarize the work obtained in this paper and discuss some special cases.

2. Preliminaries

Let us recall some basic definitions of fractional calculus [2, 4, 6].

Definition 1. For $(n-1)$-times absolutely continuous function $g : [0,\infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$$^{c}D^{q}g(t) = \frac{1}{\Gamma(n-q)}\int_0^t (t-s)^{n-q-1}g^{(n)}(s)\,ds, \qquad n-1<q<n, \quad n=[q]+1, \tag{2}$$

where $[q]$ denotes the integer part of the real number q.

Definition 2. The Riemann-Liouville fractional integral of order q is defined as

$$I^{q}g(t) = \frac{1}{\Gamma(q)}\int_0^t \frac{g(s)}{(t-s)^{1-q}}ds, \quad q>0, \tag{3}$$

provided the integral exists.

Definition 3. A function $x \in AC^{n-1}([0,1],\mathbb{R})$ is called a solution of problem (1) if there exists a function $v \in L^1([0,1],\mathbb{R})$ with $v(t) \in F(t,x(t))$, a.e. $[0,1]$, such that $^{c}D^{\xi}(D+\lambda)x(t) = v(t)$, a.e. $[0,1]$, and $x(0)=0$, $x'(0)=0$, $x''(0)=0,\ldots,x^{(n-1)}(0)=0$, and $x(1)=\alpha x(\sigma)$.

For the forthcoming analysis, we define

$$\begin{aligned} P(t) &= P_o(t) \\ &= \frac{t^{n-1}}{\lambda} - \frac{(n-1)t^{n-2}}{\lambda^2} + \frac{(n-1)(n-2)t^{n-3}}{\lambda^3} \\ &\quad - \cdots - \frac{(n-1)!t}{\lambda^{n-1}} + \frac{(n-1)!}{\lambda^n}\left(1-e^{-\lambda t}\right), \\ &\qquad n \text{ is odd}, \end{aligned} \tag{4}$$

$$\begin{aligned} P(t) &= P_e(t) \\ &= \frac{t^{n-1}}{\lambda} - \frac{(n-1)t^{n-2}}{\lambda^2} + \frac{(n-1)(n-2)t^{n-3}}{\lambda^3} \\ &\quad - \cdots + \frac{(n-1)!t}{\lambda^{n-1}} - \frac{(n-1)!}{\lambda^n}\left(1-e^{-\lambda t}\right), \\ &\qquad n \text{ is even}. \end{aligned} \tag{5}$$

Furthermore, we assume the nonresonance condition, that is, for $P=P_o$ and $P=P_e$, we choose α such that

$$P(1)-\alpha P(\sigma) \neq 0, \quad \text{for } 0<\sigma<1. \tag{6}$$

Lemma 4 (see [14]). *Assume that the nonresonance condition* (6) *holds. Given $y \in C([0,1],\mathbb{R})$, the unique solution of the problem*

$$\begin{gathered} ^{c}D^{\xi}(D+\lambda)x(t) = y(t), \quad 0<t<1, \\ x(0)=0, \qquad x'(0)=0, \\ x''(0)=0,\ldots,x^{(n-1)}(0)=0, \qquad x(1)=\alpha x(\sigma) \end{gathered} \tag{7}$$

is given by

$$\begin{aligned} x(t) &= \int_0^t e^{-\lambda(t-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)}y(u)\,du\right)ds \\ &\quad + \frac{P(t)}{P(1)-\alpha P(\sigma)} \\ &\quad \times \left[\alpha\int_0^{\sigma} e^{-\lambda(\sigma-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)}y(u)\,du\right)ds\right. \\ &\quad \left. -\int_0^1 e^{-\lambda(1-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)}y(u)\,du\right)ds\right], \end{aligned} \tag{8}$$

where $P(t)=P_o(t)$ and $P(t)=P_e(t)$ are given by (4) *and* (5), *respectively.*

3. Existence Results

We begin this section with some preliminary material on multivalued maps [24, 25] that we need in the sequel.

Let $C([0,1])$ denote a Banach space of continuous functions from $[0,1]$ into $\mathbb{R}$ with the norm $\|x\| = \sup_{t\in[0,1]}|x(t)|$. Let $L^1([0,1],\mathbb{R})$ be the Banach space of measurable functions $x:[0,1]\to\mathbb{R}$ which are Lebesgue integrable and normed by $\|x\|_{L^1} = \int_0^1|x(t)|dt$.

Let $(X,\|\cdot\|)$ denote a normed space. Then we define

$$\begin{gathered} \mathscr{P}_{cl}(X) = \{\mathscr{Z}\in\mathscr{P}(X) : \mathscr{Z} \text{ is closed}\}, \\ \mathscr{P}_{b}(X) = \{\mathscr{Z}\in\mathscr{P}(X) : \mathscr{Z} \text{ is bounded}\}, \\ \mathscr{P}_{cp}(X) = \{\mathscr{Z}\in\mathscr{P}(X) : \mathscr{Z} \text{ is compact}\}, \\ \mathscr{P}_{cp,c}(X) = \{\mathscr{Z}\in\mathscr{P}(X) : \mathscr{Z} \text{ is compact and convex}\}. \end{gathered} \tag{9}$$

Definition 5. A multivalued map $\mathfrak{G} : X \to \mathscr{P}(X)$ is convex-(closed-) valued if $\mathfrak{G}(x)$ is convex (closed) for all $x \in X$.

Definition 6. The map $\mathfrak{G}$ is bounded on bounded sets if $\mathfrak{G}(\mathbb{B}) = \cup_{x\in\mathbb{B}}\mathfrak{G}(x)$ is bounded in X for all $\mathbb{B}\in\mathscr{P}_b(X)$ (i.e., $\sup_{x\in\mathbb{B}}\{\sup\{|y| : y\in\mathfrak{G}(x)\}\} < \infty$).

Definition 7. $\mathfrak{G}$ is called upper semicontinuous (u.s.c.) on X if for each $x_0\in X$, the set $\mathfrak{G}(x_0)$ is a nonempty closed subset of X, and if, for each open set N of X containing $\mathfrak{G}(x_0)$, there exists an open neighborhood $\mathscr{N}_0$ of x_0 such that $\mathfrak{G}(\mathscr{N}_0)\subseteq N$.

Definition 8. $\mathfrak{G}$ is said to be completely continuous if $\mathfrak{G}(\mathbb{B})$ is relatively compact for every $\mathbb{B}\in\mathscr{P}_b(X)$.

If the multivalued map $\mathfrak{G}$ is completely continuous with nonempty compact values, then $\mathfrak{G}$ is u.s.c. if and only if $\mathfrak{G}$ has a closed graph; that is, $x_n \to x_*$, $y_n \to y_*$, and $y_n\in\mathfrak{G}(x_n)$ imply that $y_*\in\mathfrak{G}(x_*)$. $\mathfrak{G}$ has a fixed point if there is $x\in X$ such that $x\in\mathfrak{G}(x)$. The fixed point set of the multivalued operator $\mathfrak{G}$ will be denoted by $\operatorname{Fix}\mathfrak{G}$.

Definition 9. A multivalued map $\mathfrak{G} : [0;1] \to \mathscr{P}_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function

$$t \longmapsto d(y, \mathfrak{G}(t)) = \inf\{|y - z| : z \in \mathfrak{G}(t)\} \tag{10}$$

is measurable.

3.1. The Carathéodory Case

Definition 10. A multivalued map $F : [0,1] \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ is said to be Carathéodory if

(i) $t \mapsto F(t,x)$ is measurable for each $x \in \mathbb{R}$;

(ii) $x \mapsto F(t,x)$ is upper semicontinuous for almost all $t \in [0,1]$.

Further a Carathéodory function F is called L^1-Carathéodory if

(iii) for each $\rho > 0$, there exists $\varphi_\rho \in L^1([0,1], \mathbb{R}^+)$ such that

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \le \varphi_\rho(t) \tag{11}$$

for all $\|x\| \le \rho$ and for a.e. $t \in [0,1]$.

For each $y \in C([0,1], \mathbb{R})$, define the set of selections of F by

$$\begin{aligned} S_{F,y} := \Big\{ & v \in L^1([0,1], R) : v(t) \in F(t, y(t)) \\ & \text{for a.e. } t \in [0,1] \Big\}. \end{aligned} \tag{12}$$

For the forthcoming analysis, we need the following lemmas.

Lemma 11 (nonlinear alternative for Kakutanimaps [26]). *Let E be a Banach space, C a closed convex subset of E, U an open subset of C, and $0 \in U$. Suppose that $F : \overline{U} \to \mathscr{P}_{cp,c}(C)$ is an upper semicontinuous compact map. Then either*

(i) *F has a fixed point in $\overline{U}$, or*

(ii) *there is a $u \in \partial U$ and $\lambda \in (0,1)$ with $u \in \lambda F(u)$.*

Lemma 12 (see [27]). *Let X be a Banach space. Let $F : [0,1] \times \mathbb{R} \to \mathscr{P}_{cp,c}(X)$ be an L^1-Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1([0,1], X)$ to $C([0,1], X)$. Then the operator*

$$\begin{aligned} \Theta \circ S_F :\ & C([0,1], X) \longrightarrow \mathscr{P}_{cp,c}(C([0,1], X)), \\ & x \longmapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x}) \end{aligned} \tag{13}$$

is a closed graph operator in $C([0,1], X) \times C([0,1], X)$.

Now we are in a position to prove the existence of the solutions for the boundary value problem (1) when the right-hand side is convex-valued.

Theorem 13. *Assume that the nonresonance condition (6) holds. In addition, we suppose that*

(H_1) *$F : [0,1] \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ is Carathéodory and has nonempty compact and convex values;*

(H_2) *there exist a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and a function $p \in L^1([0,1], \mathbb{R}^+)$ such that*

$$\begin{aligned} \|F(t,x)\|_{\mathscr{P}} &:= \sup\{|y| : y \in F(t,x)\} \\ &\le p(t)\,\psi(\|x\|) \quad \text{for each } (t,x) \in [0,1] \times \mathbb{R}; \end{aligned} \tag{14}$$

(H_3) *there exists a constant $M > 0$ such that*

$$\begin{aligned} M\Bigg(\frac{\psi(M)}{\Gamma(\xi)}\Bigg\{ & (1 + P_1) \int_0^1 e^{-\lambda(1-s)} p(s)\, ds \\ & + \alpha P_1 \int_0^\sigma e^{-\lambda(\sigma - s)} p(s)\, ds \Bigg\}\Bigg)^{-1} > 1, \end{aligned} \tag{15}$$

where $P_1 = \max_{t\in[0,1]} |P(t)/(P(1) - \alpha P(\sigma))| (P(t)$ is defined in (4) and (5)).

Then the boundary value problem (1) has at least one solution on $[0,1]$.

Proof. Define the operator $\Omega_F : C([0,1], \mathbb{R}) \to \mathscr{P}(C([0,1], \mathbb{R}))$ by

$$\begin{aligned} \Omega_F(x) &= \Bigg\{ h \in C([0,1], \mathbb{R}) : h(t) \\ &= \Bigg\{ \int_0^t e^{-\lambda(t-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v(u)\, du \right) ds \\ &\quad + \frac{P(t)}{P(1) - \alpha P(\sigma)} \\ &\quad \times \Bigg[\alpha \int_0^\sigma e^{-\lambda(\sigma-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v(u)\, du \right) ds \\ &\quad - \int_0^1 e^{-\lambda(1-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v(u)\, du \right) ds \Bigg] \Bigg\} \Bigg\}, \end{aligned} \tag{16}$$

for $v \in S_{F,x}$. We will show that Ω_F satisfies the assumptions of the nonlinear alternative of Leray-Schauder type. The proof consists of several steps. As a first step, we show that *Ω_F is convex for each $x \in C([0,1], \mathbb{R})$.* This step is obvious since $S_{F,x}$ is convex (F has convex values), and therefore we omit the proof.

In the second step, we show that Ω_F *maps bounded sets (balls) into bounded sets in* $C([0,1],\mathbb{R})$. For a positive number r, let $B_r = \{x \in C([0,1],\mathbb{R}) : \|x\| \leq r\}$ be a bounded ball in $C([0,1],\mathbb{R})$. Then, for each $h \in \Omega_F(x)$, $x \in B_r$, there exists $v \in S_{F,x}$ such that

$$\begin{aligned} h(t) = {} & \int_0^t e^{-\lambda(t-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v(u)\, du \right) ds \\ & + \frac{P(t)}{P(1) - \alpha P(\sigma)} \\ & \times \left[\alpha \int_0^\sigma e^{-\lambda(\sigma-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v(u)\, du \right) ds \right. \\ & \left. - \int_0^1 e^{-\lambda(1-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v(u)\, du \right) ds \right]. \end{aligned} \tag{17}$$

Then for $t \in [0,1]$, we have

$$\begin{aligned} & |h(t)| \\ & \leq \left| \int_0^t e^{-\lambda(t-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v(u)\, du \right) ds \right. \\ & \quad + \frac{P(t)}{P(1) - \alpha P(\sigma)} \\ & \quad \times \left[\alpha \int_0^\sigma e^{-\lambda(\sigma-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v(u)\, du \right) ds \right. \\ & \quad \left. \left. + \int_0^1 e^{-\lambda(1-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v(u)\, du \right) ds \right] \right| \\ & \leq \int_0^t e^{-\lambda(t-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} p(s) \psi(\|x\|)\, du \right) ds \\ & \quad + \left| \frac{P(t)}{P(1) - \alpha P(\sigma)} \right| \\ & \quad \times \left[\alpha \int_0^\sigma e^{-\lambda(\sigma-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} p(s) \psi(\|x\|)\, du \right) ds \right. \\ & \quad \left. + \int_0^1 e^{-\lambda(1-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} p(s) \psi(\|x\|)\, du \right) ds \right] \\ & \leq \frac{\psi(\|x\|)}{\Gamma(\xi)} \left\{ \int_0^1 e^{-\lambda(1-s)} p(s)\, ds + \alpha P_1 \int_0^\sigma e^{-\lambda(\sigma-s)} p(s)\, ds \right. \\ & \quad \left. + P_1 \int_0^1 e^{-\lambda(1-s)} p(s)\, ds \right\} \\ & = \frac{\psi(\|x\|)}{\Gamma(\xi)} \left\{ (1+P_1) \int_0^1 e^{-\lambda(1-s)} p(s)\, ds \right. \\ & \quad \left. + \alpha P_1 \int_0^\sigma e^{-\lambda(\sigma-s)} p(s)\, ds \right\}. \end{aligned} \tag{18}$$

Consequently,

$$\begin{aligned} \|h\| \leq {} & \frac{\psi(r)}{\Gamma(\xi)} \left\{ (1+P_1) \int_0^1 e^{-\lambda(1-s)} p(s)\, ds \right. \\ & \left. + \alpha P_1 \int_0^\sigma e^{-\lambda(\sigma-s)} p(s)\, ds \right\}. \end{aligned} \tag{19}$$

Now we show that Ω_F maps bounded sets into equicontinuous sets of $C([0,1],\mathbb{R})$. Let $t_1, t_2 \in [0,1]$ and $x \in B_r$. For each $h \in \Omega_F(x)$, we obtain

$$\begin{aligned} & |h(t_1) - h(t_2)| \\ & = \left| \int_0^{t_1} e^{-\lambda(t_1-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v(u)\, du \right) ds \right. \\ & \quad - \int_0^{t_2} e^{-\lambda(t_2-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v(u)\, du \right) ds \\ & \quad + \frac{[P(t_1) - P(t_2)]}{P(1) - \alpha P(\sigma)} \\ & \quad \times \left[\alpha \int_0^\sigma e^{-\lambda(\sigma-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v(u)\, du \right) ds \right. \\ & \quad \left. \left. - \int_0^1 e^{-\lambda(1-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v(u)\, du \right) ds \right] \right| \\ & \leq \int_0^{t_1} e^{-\lambda(t_1-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} \psi(r) p(u)\, du \right) ds \\ & \quad - \int_0^{t_2} e^{-\lambda(t_2-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} \psi(r) p(u)\, du \right) ds \\ & \quad + \left| \frac{P(t_1) - P(t_2)}{P(1) - \alpha P(\sigma)} \right| \\ & \quad \times \left[\alpha \int_0^\sigma e^{-\lambda(\sigma-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} \psi(r) p(u)\, du \right) ds \right. \\ & \quad \left. - \int_0^1 e^{-\lambda(1-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} \psi(r) p(u)\, du \right) ds \right]. \end{aligned} \tag{20}$$

Obviously the right-hand side of the above inequality tends to zero independently of $x \in B_r$ as $t_2 - t_1 \to 0$. As Ω_F satisfies the above three assumptions, therefore it follows from the Ascoli-Arzelá theorem that $\Omega_F : C([0,1],\mathbb{R}) \to \mathscr{P}(C([0,1],\mathbb{R}))$ is completely continuous.

In our next step, we show that Ω_F *has a closed graph*. Let $x_n \to x_*$, $h_n \in \Omega_F(x_n)$, and $h_n \to h_*$. Then we need to show that $h_* \in \Omega_F(x_*)$. Associated with $h_n \in \Omega_F(x_n)$, there exists $v_n \in S_{F,x_n}$ such that, for each $t \in [0,1]$,

$$\begin{aligned} h_n(t) &= \int_0^t e^{-\lambda(t-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v_n(u)\,du\right)ds \\ &+ \frac{P(t)}{P(1)-\alpha P(\sigma)} \\ &\times\left[\alpha\int_0^\sigma e^{-\lambda(\sigma-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v_n(u)\,du\right)ds\right. \\ &\left.-\int_0^1 e^{-\lambda(1-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v_n(u)\,du\right)ds\right]. \end{aligned} \tag{21}$$

Thus, it suffices to show that there exists $v_* \in S_{F,x_*}$ such that, for each $t \in [0,1]$,

$$\begin{aligned} h_*(t) &= \int_0^t e^{-\lambda(t-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v_*(u)\,du\right)ds \\ &+ \frac{P(t)}{P(1)-\alpha P(\sigma)} \\ &\times\left[\alpha\int_0^\sigma e^{-\lambda(\sigma-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v_*(u)\,du\right)ds\right. \\ &\left.-\int_0^1 e^{-\lambda(1-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v_*(u)\,du\right)ds\right]. \end{aligned} \tag{22}$$

Let us consider the linear operator $\Theta : L^1([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ given by

$$\begin{aligned} f &\longmapsto \Theta(v)(t) \\ &= \int_0^t e^{-\lambda(t-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v(u)\,du\right)ds \\ &+ \frac{P(t)}{P(1)-\alpha P(\sigma)} \\ &\times\left[\alpha\int_0^\sigma e^{-\lambda(\sigma-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v(u)\,du\right)ds\right. \\ &\left.-\int_0^1 e^{-\lambda(1-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v(u)\,du\right)ds\right]. \end{aligned} \tag{23}$$

Observe that

$$\begin{aligned} &\|h_n(t)-h_*(t)\| \\ &= \left\|\int_0^t e^{-\lambda(t-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} (v_n(u)-v_*(u))\,du\right)ds\right. \\ &\quad+ \frac{P(t)}{P(1)-\alpha P(\sigma)} \\ &\quad\times\left[\alpha\int_0^\sigma e^{-\lambda(\sigma-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} (v_n(u)-v_*(u))\,du\right)ds\right. \\ &\quad\left.\left.-\int_0^1 e^{-\lambda(1-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} (v_n(u)-v_*(u))\,du\right)ds\right]\right\| \\ &\quad\longrightarrow 0, \end{aligned} \tag{24}$$

as $n \to \infty$.

Thus, it follows from Lemma 12 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, therefore, we have

$$\begin{aligned} h_*(t) &= \int_0^t e^{-\lambda(t-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v_*(u)\,du\right)ds \\ &+ \frac{P(t)}{P(1)-\alpha P(\sigma)} \\ &\times\left[\alpha\int_0^\sigma e^{-\lambda(\sigma-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v_*(u)\,du\right)ds\right. \\ &\left.-\int_0^1 e^{-\lambda(1-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v_*(u)\,du\right)ds\right], \end{aligned} \tag{25}$$

for some $v_* \in S_{F,x_*}$.

Finally, we show that there exists an open set $U \subseteq C([0,1],\mathbb{R})$ with $x \notin \Omega_F(x)$ for any $\lambda \in (0,1)$ and all $x \in \partial U$. Let $\lambda \in (0,1)$ and $x \in \lambda\Omega_F(x)$. Then there exists $v \in L^1([0,1],\mathbb{R})$ with $v \in S_{F,x}$ such that, for $t \in [0,1]$, we have

$$\begin{aligned} x(t) &= \int_0^t e^{-\lambda(t-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v(u)\,du\right)ds \\ &+ \frac{P(t)}{P(1)-\alpha P(\sigma)} \\ &\times\left[\alpha\int_0^\sigma e^{-\lambda(\sigma-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v(u)\,du\right)ds\right. \\ &\left.-\int_0^1 e^{-\lambda(1-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v(u)\,du\right)ds\right]. \end{aligned} \tag{26}$$

Using the computations of the second step above we have

$$|x(t)| \leq \frac{\psi(\|x\|)}{\Gamma(\xi)} \times \left\{ \left(1+P_1\right) \int_0^1 e^{-\lambda(1-s)} p(s)\, ds + \alpha P_1 \int_0^\sigma e^{-\lambda(\sigma-s)} p(s)\, ds \right\}. \tag{27}$$

Consequently, we have

$$\|x\| \left(\frac{\psi(\|x\|)}{\Gamma(\xi)} \left\{ \left(1+P_1\right) \int_0^1 e^{-\lambda(1-s)} p(s)\, ds + \alpha P_1 \int_0^\sigma e^{-\lambda(\sigma-s)} p(s)\, ds \right\} \right)^{-1} \leq 1. \tag{28}$$

In view of (H_3), there exists M such that $\|x\| \neq M$. Let us set

$$U = \{x \in C([0,1], \mathbb{R}) : \|x\| < M\}. \tag{29}$$

Note that the operator $\Omega_F : \overline{U} \to \mathscr{P}(C([0,1],\mathbb{R}))$ is upper semicontinuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x \in \lambda\Omega_F(x)$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 11), we deduce that Ω_F has a fixed point $x \in \overline{U}$ which is a solution of the problem (1). This completes the proof. □

Remark 14. The condition (H_3) in the statement of Theorem 13 may be replaced with the following one.

$(H_3)'$ There exists a constant $M > 0$ such that

$$\frac{M}{(\psi(M)/\Gamma(\xi))\left\{\left(1+(1+\alpha)P_1\right)\|p\|_{L^1}\right\}} > 1, \tag{30}$$

where P_1 is the same as defined in (H_3).

3.2. The Lower Semicontinuous Case. As a next result, we study the case when F is not necessarily convex-valued. Our strategy to deal with this problem is based on the nonlinear alternative of Leray-Schauder type together with the selection theorem of Bressan and Colombo [28] for lower semicontinuous maps with decomposable values.

Let X be a nonempty closed subset of a Banach space E and let $G : X \to \mathscr{P}(E)$ be a multivalued operator with nonempty closed values. G is lower semicontinuous (l.s.c.) if the set $\{y \in X : G(y) \cap B \neq \emptyset\}$ is open for any open set B in E. Let A be a subset of $[0,1] \times \mathbb{R}$. A is $\mathscr{L} \otimes \mathscr{B}$ measurable if A belongs to the σ-algebra generated by all sets of the form $\mathscr{J} \times \mathscr{D}$, where $\mathscr{J}$ is Lebesgue measurable in $[0,1]$ and $\mathscr{D}$ is Borel measurable in $\mathbb{R}$. A subset $\mathscr{A}$ of $L^1([0,1],\mathbb{R})$ is decomposable if, for all $u, v \in \mathscr{A}$ and measurable $\mathscr{J} \subset [0,1] = J$, the function $u\chi_{\mathscr{J}} + v\chi_{J-\mathscr{J}} \in \mathscr{A}$, where $\chi_{\mathscr{J}}$ stands for the characteristic function of $\mathscr{J}$.

Definition 15. Let Y be a separable metric space and let $N : Y \to \mathscr{P}(L^1([0,1],\mathbb{R}))$ be a multivalued operator. We say N has a property (BC) if N is lower semicontinuous (l.s.c.) and has nonempty closed and decomposable values.

Let $F : [0,1] \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ be a multivalued map with nonempty compact values. Define a multivalued operator $\mathscr{F} : C([0,1] \times \mathbb{R}) \to \mathscr{P}(L^1([0,1],\mathbb{R}))$ associated with F as

$$\mathscr{F}(x) = \left\{ w \in L^1([0,1],\mathbb{R}) : w(t) \in F(t, x(t)) \text{ for a.e. } t \in [0,1] \right\}, \tag{31}$$

which is called the Nemytskii operator associated with F.

Definition 16. Let $F : [0,1] \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ be a multivalued function with nonempty compact values. We say F is of lower semicontinuous type (l.s.c. type) if its associated Nemytskii operator $\mathscr{F}$ is lower semicontinuous and has nonempty closed and decomposable values.

Lemma 17 (see [29]). *Let Y be a separable metric space and let $N : Y \to \mathscr{P}(L^1([0,1],\mathbb{R}))$ be a multivalued operator satisfying the property (BC). Then N has a continuous selection; that is, there exists a continuous function (single-valued) $g : Y \to L^1([0,1],\mathbb{R})$ such that $g(x) \in N(x)$ for every $x \in Y$.*

Theorem 18. *Assume that (H_2), (H_3), and the following condition hold:*

(H_4) *$F : [0,1] \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ is a nonempty compact-valued multivalued map such that*

(a) *$(t,x) \mapsto F(t,x)$ is $\mathscr{L} \otimes \mathscr{B}$ measurable,*

(b) *$x \mapsto F(t,x)$ is lower semicontinuous for each $t \in [0,1]$.*

Further the nonresonance condition (6) holds. Then the boundary value problem (1) has at least one solution on $[0,1]$.

Proof. It follows from (H_2) and (H_4) that F is of l.s.c. type. Then from Lemma 17, there exists a continuous function $f : AC^1([0,1],\mathbb{R}) \to L^1([0,1],\mathbb{R})$ such that $f(x) \in \mathscr{F}(x)$ for all $x \in C([0,1],\mathbb{R})$.

Consider the problem

$$\begin{gathered} {}^cD^\xi (D+\lambda)\, x(t) = f(x(t)), \quad 0 < t < 1, \\ x(0) = 0, \qquad x'(0) = 0, \\ x''(0) = 0, \ldots, x^{(n-1)}(0) = 0, \qquad x(1) = \alpha x(\sigma). \end{gathered} \tag{32}$$

Observe that if $x \in AC^1([0,1],\mathbb{R})$ is a solution of (32), then x is a solution to the problem (1). In order to transform the problem (32) into a fixed point problem, we define the operator $\overline{\Omega_F}$ as

$$\begin{aligned}\overline{\Omega_F}x(t) = &\int_0^t e^{-\lambda(t-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} f(x(u))\,du\right)ds\\ &+ \frac{P(t)}{P(1)-\alpha P(\sigma)}\\ &\times\left[\alpha\int_0^\sigma e^{-\lambda(\sigma-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} f(x(u))\,du\right)ds\right.\\ &\left.-\int_0^1 e^{-\lambda(1-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} f(x(u))\,du\right)ds\right].\end{aligned} \tag{33}$$

It can easily be shown that $\overline{\Omega_F}$ is continuous and completely continuous. The remaining part of the proof is similar to that of Theorem 13. So we omit it. This completes the proof. □

3.3. The Lipschitz Case. Now we prove the existence of solutions for the problem (1) with a nonconvex-valued right-hand side by applying a fixed point theorem for multivalued map due to Covitz and Nadler [30].

Let (X, d) be a metric space induced from the normed space $(X, \|\cdot\|)$. Consider $H_d : \mathscr{P}(X)\times\mathscr{P}(X) \to \mathbb{R}\cup\{\infty\}$ given by

$$H_d(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\right\}, \tag{34}$$

where $d(A,b) = \inf_{a\in A} d(a;b)$ and $d(a,B) = \inf_{b\in B} d(a;b)$. Then $(\mathscr{P}_{b,\mathrm{cl}}(X), H_d)$ is a metric space and $(\mathscr{P}_{\mathrm{cl}}(X), H_d)$ is a generalized metric space (see [31]).

Definition 19. A multivalued operator $N : X \to \mathscr{P}_{\mathrm{cl}}(X)$ is called

(a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \le \gamma d(x,y) \quad \text{for each } x, y \in X; \tag{35}$$

(b) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

Lemma 20 (see [30]). *Let (X, d) be a complete metric space. If $N : X \to \mathscr{P}_{cl}(X)$ is a contraction, then* Fix $N \neq \emptyset$.

Theorem 21. *Assume that the nonresonance condition* (6) *holds. In addition, suppose that the following conditions hold:*

(H_5) *$F : [0,1]\times\mathbb{R} \to \mathscr{P}_{cp}(\mathbb{R})$ is such that $F(\cdot, x) : [0,1] \to \mathscr{P}_{cp}(\mathbb{R})$ is measurable for each $x \in \mathbb{R}$;*

(H_6) *$H_d(F(t,x), F(t,\overline{x})) \le m(t)|x-\overline{x}|$ for almost all $t \in [0,1]$ and $x, \overline{x} \in \mathbb{R}$ with $m \in L^1([0,1], \mathbb{R}^+)$ and $d(0, F(t,0)) \le m(t)$ for almost all $t \in [0,1]$.*

Then the boundary value problem (1) *has at least one solution on* $[0,1]$ *if*

$$\begin{aligned}\frac{1}{\Gamma(\xi)}&\left\{(1+P_1)\int_0^1 e^{-\lambda(1-s)} m(s)\,ds\right.\\ &\left.+\alpha P_1\int_0^\sigma e^{-\lambda(\sigma-s)} m(s)\,ds\right\} < 1.\end{aligned} \tag{36}$$

Proof. Observe that the set $S_{F,x}$ is nonempty for each $x \in C([0,1],\mathbb{R})$ by the assumption (H_5), so F has a measurable selection (see Theorem III.6 [32]). Now we show that the operator Ω_F, defined in the beginning of proof of Theorem 13, satisfies the assumptions of Lemma 20. To show that $\Omega_F(x) \in \mathscr{P}_{\mathrm{cl}}((C[0,1],\mathbb{R}))$ for each $x \in C([0,1],\mathbb{R})$, let $\{u_n\}_{n\ge 0} \in \Omega_F(x)$ be such that $u_n \to u$ $(n\to\infty)$ in $C([0,1],\mathbb{R})$. Then $u \in C([0,1],\mathbb{R})$ and there exists $v_n \in S_{F,x_n}$ such that, for each $t \in [0,1]$,

$$\begin{aligned}u_n(t) = &\int_0^t e^{-\lambda(t-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v_n(u)\,du\right)ds\\ &+ \frac{P(t)}{P(1)-\alpha P(\sigma)}\\ &\times\left[\alpha\int_0^\sigma e^{-\lambda(\sigma-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v_n(u)\,du\right)ds\right.\\ &\left.-\int_0^1 e^{-\lambda(1-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v_n(u)\,du\right)ds\right].\end{aligned} \tag{37}$$

As F has compact values, we pass onto a subsequence (if necessary) to obtain that v_n converges to v in $L^1([0,1],\mathbb{R})$. Thus, $v \in S_{F,x}$ and, for each $t \in [0,1]$, we have

$$\begin{aligned}v_n(t) \longrightarrow v(t)&\\ = &\int_0^t e^{-\lambda(t-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v(u)\,du\right)ds\\ &+ \frac{P(t)}{P(1)-\alpha P(\sigma)}\\ &\times\left[\alpha\int_0^\sigma e^{-\lambda(\sigma-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v(u)\,du\right)ds\right.\\ &\left.-\int_0^1 e^{-\lambda(1-s)}\left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v(u)\,du\right)ds\right].\end{aligned} \tag{38}$$

Hence, $u \in \Omega(x)$.

Next we show that there exists $\delta < 1$ such that

$$\begin{aligned}&H_d(\Omega_F(x), \Omega_F(\overline{x}))\\ &\quad\le \delta\|x-\overline{x}\| \quad \text{for each } x, \overline{x} \in AC^1([0,1],\mathbb{R}).\end{aligned} \tag{39}$$

Let $x, \overline{x} \in AC^1([0,1],\mathbb{R})$ and $h_1 \in \Omega_F(x)$. Then there exists $v_1(t) \in F(t, x(t))$ such that, for each $t \in [0,1]$,

$$h_1(t) = \int_0^t e^{-\lambda(t-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v_1(u)\,du \right) ds + \frac{P(t)}{P(1)-\alpha P(\sigma)} \times \left[\alpha \int_0^\sigma e^{-\lambda(\sigma-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v_1(u)\,du \right) ds - \int_0^1 e^{-\lambda(1-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v_1(u)\,du \right) ds \right]. \tag{40}$$

By (H_6), we have

$$H_d(F(t,x), F(t,\overline{x})) \le m(t)\,|x(t) - \overline{x}(t)|. \tag{41}$$

So, there exists $w \in F(t, \overline{x}(t))$ such that

$$|v_1(t) - w| \le m(t)\,|x(t) - \overline{x}(t)|, \quad t \in [0,1]. \tag{42}$$

Define $U : [0,1] \to \mathscr{P}(\mathbb{R})$ by

$$U(t) = \{ w \in \mathbb{R} : |v_1(t) - w| \le m(t)\,|x(t) - \overline{x}(t)| \}. \tag{43}$$

Since the multivalued operator $U(t) \cap F(t, \overline{x}(t))$ is measurable (Proposition III.4 [32]), there exists a function $v_2(t)$ which is a measurable selection for U. So $v_2(t) \in F(t, \overline{x}(t))$ and for each $t \in [0,1]$, we have $|v_1(t) - v_2(t)| \le m(t)|x(t) - \overline{x}(t)|$.

For each $t \in [0,1]$, let us define

$$h_2(t) = \int_0^t e^{-\lambda(t-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v_2(u)\,du \right) ds + \frac{P(t)}{P(1)-\alpha P(\sigma)} \times \left[\alpha \int_0^\sigma e^{-\lambda(\sigma-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v_2(u)\,du \right) ds - \int_0^1 e^{-\lambda(1-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} v_2(u)\,du \right) ds \right]. \tag{44}$$

Thus,

$$\begin{aligned} &|h_1(t) - h_2(t)| \\ &\le \int_0^t e^{-\lambda(t-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} |v_1(u) - v_2(u)|\,du \right) ds \\ &\quad + \left| \frac{P(t)}{P(1)-\alpha P(\sigma)} \right| \\ &\quad \times \left[\alpha \int_0^\sigma e^{-\lambda(\sigma-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} |v_1(u) - v_2(u)|\,du \right) ds \right. \\ &\quad \left. - \int_0^1 e^{-\lambda(1-s)} \left(\int_0^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)} |v_1(u) - v_2(u)|\,du \right) ds \right]. \end{aligned} \tag{45}$$

Hence,

$$\begin{aligned} &\|h_1 - h_2\| \\ &\le \frac{1}{\Gamma(\xi)} \left\{ (1+P_1) \int_0^1 e^{-\lambda(1-s)} m(s)\,ds + \alpha P_1 \int_0^\sigma e^{-\lambda(\sigma-s)} m(s)\,ds \right\} \\ &\quad \times \|x - \overline{x}\|. \end{aligned} \tag{46}$$

Analogously, interchanging the roles of x and $\overline{x}$, we obtain

$$\begin{aligned} &H_d(\Omega_F(x), \Omega_F(\overline{x})) \\ &\le \delta\,\|x - \overline{x}\| \\ &\le \frac{1}{\Gamma(\xi)} \left\{ (1+P_1) \int_0^1 e^{-\lambda(1-s)} m(s)\,ds + \alpha P_1 \int_0^\sigma e^{-\lambda(\sigma-s)} m(s)\,ds \right\} \|x - \overline{x}\|. \end{aligned} \tag{47}$$

Since Ω_F is a contraction, it follows from Lemma 20 that Ω_F has a fixed point x which is a solution of (1). This completes the proof. □

Remark 22. An alternative to the condition (36) in the statement of Theorem 21 may be the following one:

$$\frac{1}{\Gamma(\xi)} \{ (1 + (1+\alpha) P_1) \|m\|_{L^1} \} < 1. \tag{48}$$

Example 23. Consider the problem

$$\begin{aligned} &{}^cD^{7/2}(D+2)\,x(t) \in F(t, x(t)), \quad 0 \le t \le 1, \\ &x(0) = 0, \qquad x'(0) = 0, \\ &x''(0) = 0, \qquad x'''(0) = 0, \qquad x(1) = x\left(\frac{1}{2}\right). \end{aligned} \tag{49}$$

Here, $\xi = 7/2$, $n = 4$, $\lambda = 2$, $\alpha = 1$, $\sigma = 1/2$, and $F : [0,1] \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ is a multivalued map given by

$$x \longrightarrow F(t,x) = \left[\frac{|x|^5}{|x|^5 + 3} + t^3 + t^2 + 4, \frac{|x|^3}{|x|^3 + 1} + t + 2 \right]. \tag{50}$$

For $f \in F$, we have

$$|f| \le \max\left(\frac{|x|^5}{|x|^5 + 3} + t^3 + t^2 + 4, \frac{|x|^3}{|x|^3 + 1} + t + 2 \right) \le 9, \quad x \in \mathbb{R}. \tag{51}$$

Thus,

$$\|F(t,x)\|_{\mathscr{P}} := \sup\{|y| : y \in F(t,x)\} \leq 7 \\ = p(t)\,\psi(\|x\|), \quad x \in \mathbb{R}, \tag{52}$$

with $p(t) = 1, \psi(\|x\|) = 7$. In this case

$$P(t) = P_e(t) = \frac{t^3}{2} - \frac{3t(t-1)}{4} - \frac{3(1-e^{-2t})}{8}, \\ P_1 \approx 6.214821. \tag{53}$$

By the condition (H_3), that is,

$$M\left(\frac{\psi(M)}{\Gamma(\xi)}\left\{(1+P_1)\int_0^1 e^{-\lambda(1-s)}p(s)\,ds \right.\right. \\ \left.\left. +\alpha P_1\int_0^{\sigma} e^{-\lambda(\sigma-s)}p(s)\,ds\right\}\right)^{-1} > 1, \tag{54}$$

we find that $M > M_1$ with $M_1 \approx 10.707326$. Therefore, it follows from Theorem 13 that problem (49) has at least one solution.

4. Conclusions

In this paper, we have solved a three-point boundary value problem of Caputo-type sequential fractional differential inclusions of an arbitrary order $\xi \in (n-1,n)$. The existence of solutions for the given problem with the convex-valued map is obtained by means of nonlinear alternative for Kakutani maps, while the existence result for not necessarily convex-valued map is established by combining nonlinear alternative of Leray-Schauder type for single-valued maps with a selection theorem due to Bressan and Colombo for lower semicontinuous multivalued maps with decomposable values. The nonconvex-valued case relies on a fixed point theorem for contractive multivalued maps due to Covitz and Nadler. Some new existence results follow by fixing the parameters involved in the given problem. For instance, by taking $\alpha = 0$, our results correspond to a two-point Caputo-type multivalued problem of an arbitrary order $\xi \in (n-1,n)$, while the results for sequential differential inclusions of order $(n+1)$ can be obtained by fixing $\xi = n$ in the results of this paper.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

The Gerber-Shiu Expected Penalty Function for the Risk Model with Dependence and a Constant Dividend Barrier

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We consider a compound Poisson risk model with dependence and a constant dividend barrier. A dependence structure between the claim amount and the interclaim time is introduced through a Farlie-Gumbel-Morgenstern copula. An integrodifferential equation for the Gerber-Shiu discounted penalty function is derived. We also solve the integrodifferential equation and show that the solution is a linear combination of the Gerber-Shiu function with no barrier and the solution of an associated homogeneous integrodifferential equation.

1. Introduction

In the classical compound Poisson risk model, the surplus process has the form

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i, \quad t \geq 0, \tag{1}$$

where $u \geq 0$ is the initial surplus, $c \geq 0$ is the premium income rate, and $\{X_i\}_{i=1}^{\infty}$ are i.i.d. random variables representing the individual claim amounts with probability density function (p.d.f.) f_X, cumulative distribution function (c.d.f.) F_X, and Laplace transform (LT) f_X^*. The counting process $\{N(t); t \geq 0\}$ denotes the number of claims up to time t and is defined as $N(t) = \max\{k : W_1 + W_2 + \cdots + W_k \leq t\}$, where the interclaim times $\{W_i, i = 1, 2, \ldots\}$ form a sequence of independent and strictly positive real-valued random variables (r.v.s.). The r.v. $\{W_i, i = 1, 2, \ldots\}$ have common density function $f_W(t) = \lambda e^{-\lambda t}$, $t > 0$, cumulative distribution function F_W, and Laplace transform f_W^*. $\{N(t), t \geq 0\}$ is Poisson process with parameter $\lambda > 0$.

Ruin probability and related problems in the classical risk model have been studied extensively. Gerber and Shiu [1] introduced a discounted penalty function with respect to the time of ruin, the surplus before ruin, and the deficit at ruin. Many quantities can be analyzed through this function in a unified manner.

In ruin theory, the classical compound Poisson risk model is based on the assumption of independence between the claim amount random variable X_i and the interclaim time W_j. However, there exist many real-world situations for which such an assumption is inappropriate. For instance, in modeling natural catastrophic events, we can expect that, on the occurrence of a catastrophe, the total claim amount and the time elapsed since the previous catastrophes are dependent. See, for example, Boudreault [2] and Nikoloulopoulos and Karlis [3] for an application of this type of dependence structure in an earthquake context. And as discussed in Albrecher and Teugels [4], they allow the interclaim time and its subsequent claim size to be dependent according to an arbitrary copula structure, by employing the underlying random walk structure of the risk model; they derive exponential estimates for finite- and infinite-time ruin probabilities in the case of light-tailed claim sizes. In Boudreault et al. [5], a risk model with time-dependent claim sizes (i.e., the distribution of the next claim size depends on the last interarrival time) is analyzed and a defective renewal equation for the Gerber-Shiu discounted penalty function is derived and solved. Marceau [6] has considered the discrete-time renewal risk model with dependence between the claim amount random variable and the interclaim time random variable. Recursive formulas are derived for the probability mass function and the moments of the total claim amount over a fixed period of time. Cossette et al. [7] use the Farlie-Gumbel-Morgenstern (FGM) copula to define the dependence structure between the claim size and the interclaim time; they derive the integrodifferential equation and the Laplace transform (LT) of the Gerber-Shiu discounted penalty function. An explicit expression for the LT of the discounted value of a general function of the deficit at ruin is obtained for claim amounts having an exponential distribution. Zhang and Yang [8] construct the bivariate cumulative distribution function of the claim size and interclaim time by Farlie-Gumbel-Morgenstern copula in a compound Poisson risk model perturbed by a Brownian motion. The integrodifferential equations and the Laplace transforms for the Gerber-Shiu functions are obtained. They also show that the Gerber-Shiu functions satisfy some defective renewal equations.

The FGM copula is given by

$$C_\theta^{\mathrm{FGM}}(u_1,u_2)=u_1u_2+\theta(1-u_1)(1-u_2),\quad 0\le u_1,\quad u_2\le 1, \tag{2}$$

where $-1\le\theta\le 1$. Note that FGM copula allows both negative and positive dependence, and it also includes the independence copula ($\theta=0$).

In this paper, we assume that $\{(X_i,W_j), i\in N^+, j\in N^+\}$ form a sequence of i.i.d. random vectors distributed as the canonical r.v. (X,W). The joint p.d.f. of (X,W) is denoted by $f_{X,W}(x,t)$ with $t\in R^+$ and $x\in R^+$. The joint distribution of (X,W) is defined with a FGM copula; we consider the same dependence risk model with the presence of a constant dividend barrier. We recall that the dividend strategies for insurance risk models were first proposed by De Finetti [9]. Barrier strategies for the compound Poisson risk model have been studied in a number of papers and books, including Landriault [10], Albrecher et al. [11], Yuen et al. [12], Dickson and Waters [13], Lin et al. [14], and Segerdahl [15]. Then, various dividend strategies (threshold dividend strategy, multilayer dividend strategy, etc.) have been studied for different risk models; see, for example, Lin et al. (2006), Chi and Lin [16], D. Liu and Z. Liu [17], Bratiichuk [18], Chadjiconstantinidis and Papaioannou [19], and Wang [20]. As we know, this is the first time to consider the classic risk model with dependence structure based on FGM copula and a constant dividend barrier.

The present paper is organized as follows. In Section 2, the risk model with dependence in the presence of a constant dividend barrier is introduced. And we briefly present some properties of the FGM copula. In Section 3, we derive an integrodifferential equation for the Gerber-Shiu discounted penalty function. Finally, in Section 4, we use a renewal equation to derive an analytical expressions for $m_{b,\delta}(u)$.

2. Dependence Structure and Risk Model

A bivariate copula C is a joint distribution function on $[0,1]\times[0,1]$ with uniform marginal distributions. Assume a bivariate random vector (U,V) with above uniform marginal, which has a dependence structure defined by a copula $F_{U,V}=C(u,v)$ with $(u,v)\in[0,1]\times[0,1]$. Important copulas are the independence copula with $C^{\perp}(u,v)=uv$ and the comonotonic copula with $C^{+}(u,v)=\min(uv)$; the countermonotonic copula with $C^{-}(u,v)=\max(u+v-1;0)$. It is important to mention that all copulas satisfy the inequalities $C^{-}(u,v)\le C(u,v)\le C^{+}(u,v)$, for $(u,v)\in[0,1]\times[0,1]$.

The joint p.d.f. associated to a copula C is defined by

$$c(u_1,u_2)=\frac{\partial^2}{\partial u_1\partial u_2}C(u_1,u_2). \tag{3}$$

Let the bivariate distribution function $F_{X,W}$ of (X,W) with marginals F_X and F_W be defined as $F_{X,W}(x,t)=C(F_X(x),F_W(t))$, for $(x,t)\in R^+\times R^+$. The joint p.d.f. of (X,W) is given by

$$f_{X,W}(x,t)=c(F_X(x),F_W(t))\,f_X(x)\,f_W(t), \tag{4}$$

for $(x,t)\in R^+\times R^+$ (for a survey on copulas we refer the reader to Nelsen [21]).

The FGM copula is given by

$$C_\theta^{\mathrm{FGM}}(u_1,u_2)=u_1u_2+\theta(1-u_1)(1-u_2),\quad(-1\le\theta\le 1), \tag{5}$$

where $C_0^{\mathrm{FGM}}=C^{\perp}$. So we have

$$F_{X,W}(x,t)=F_X(x)F_W(t)+\theta F_X(x)F_W(t)\times(1-F_X(x))(1-F_W(t)), \tag{6}$$

$$f_{X,W}(x,t)=\lambda e^{-\lambda t}f(x)+\theta\left(2\lambda e^{-2\lambda t}-\lambda e^{-\lambda t}\right)h(x), \tag{7}$$

where $h(x)=(1-2F_X(x))f(x)$, with Laplace transform (LT) h_x^*.

In the rest of this paper, we assume that $\{(X_i,W_i), i\in N^+\}$ form a sequence of i.i.d. random vectors distributed like (X,W), which have joint c.d.f. and p.d.f. given by (6) and (7), respectively. In particular, we know from (7) that the conditional p.d.f. of the claim size is given by

$$f_{X\mid W=t}(x)=f(x)+\theta\left(2\lambda e^{-\lambda t}-1\right)h(x). \tag{8}$$

Also, we assume that $\theta\neq 0$; otherwise our model reduces to the constant dividend barrier in the classical risk model.

The total claim amount process $\{S(t),t\ge 0\}$ is defined as $S(t)=\sum_{i=1}^{N(t)}X_i$; let $U_b(0)=u$ and

$$\begin{aligned} dU_b(t)&=c\,dt-dS(t),\quad \text{if } U_b(t)<b,\\ dU_b(t)&=-dS(t),\quad \text{if } U_b(t)=b \end{aligned} \tag{9}$$

be the surplus process in the presence of a constant dividend barrier b ($0<b<\infty$), where $u\ge 0$ is the initial surplus level and c ($c>0$) is the level premium. In other words, we assume that the insurer pays the premium rate c as a dividend whenever the insurer's surplus remains at the threshold level b.

Associated with the risk model, we denote the ruin time by T, which is the first passage time of $U_b(t)$ below zero level; that is,

$$T = \inf \{t \geq 0, U_b(t) < 0\} \quad (10)$$

with $T = \infty$ if $U_b(t) \geq 0$, for all $t \geq 0$. To guarantee that ruin is not a certain event, we assume that the following net profit condition holds:

$$E[cW_i - X_i] > 0, \quad i = 1, 2, \ldots. \quad (11)$$

At the same time, we introduce the Gerber-Shiu function defined by

$$\begin{aligned} & m_{b,\delta}(u) \\ & = E\left[e^{-\delta T}\omega(U(T^-), |U(T)|) I(T < \infty) \mid U(0) = u\right], \end{aligned} \quad (12)$$

where $\delta \geq 0$ is the force of interest, $I(\cdot)$ is the indicator function, and $\omega(U(T^-), |U(T)|)$ defined on $[0,\infty) \times (0,\infty)$ is a nonnegative function of the surplus before ruin $U(T^-)$ and the deficit at ruin $|U(T)|$.

3. Gerber-Shiu Discounted Penalty Function

The main purpose of this section is to derive an integrodifferential equation for the expected discounted penalty function $m_{b,\delta}(u)$, This equation will be useful to derive an explicit solution for $m_{b,\delta}(u)$. Throughout this paper, we denote $\mathbf{I}$ and $\mathbf{D}$ to be the identity and the differential operators, respectively.

Theorem 1. *In the compound Poisson risk model with a dependence structure based on FGM copula defined in* (2) *and a constant dividend b, the expected discounted penalty function $m_{b,\delta}(u)$ satisfies the following integrodifferential equation:*

$$\begin{aligned} & \left(\frac{2\lambda + \delta}{c}\mathbf{I} - \mathbf{D}\right)\left(\frac{\lambda + \delta}{c}\mathbf{I} - \mathbf{D}\right) m_{b,\delta}(u) \\ & \quad = \frac{\lambda}{c}\left(\frac{2\lambda + \delta}{c}\mathbf{I} - \mathbf{D}\right)\sigma_1(u) + \frac{\theta\lambda}{c}\left(\frac{\delta}{c}\mathbf{I} - \mathbf{D}\right)\sigma_2(u) \end{aligned} \quad (13)$$

for $0 \leq u \leq b < \infty$ *with boundary conditions:*

$$m'_{b,\delta}(b) = 0, \quad (14)$$

$$m''_{b,\delta}(b) = -\frac{\lambda}{c}\sigma'_1(b) - \frac{\theta\lambda}{c}\sigma'_2(b), \quad (15)$$

where

$$\sigma_1(u) = \int_0^u m_{b,\delta}(u - x) f(x)\, dx + \omega_1(u), \quad (16)$$

$$\sigma_2(u) = \int_0^u m_{b,\delta}(u - x) h(x)\, dx + \omega_2(u), \quad (17)$$

$$\omega_1(u) = \int_u^\infty \omega(u, x - u) f(x)\, dx, \quad (18)$$

$$\omega_2(u) = \int_u^\infty \omega(u, x - u) h(x)\, dx. \quad (19)$$

Proof. By conditioning on the time and the amount of the first claim, we have

$$\begin{aligned} m_{b,\delta}(u) = & \int_0^{(b-u)/c} \int_0^{u+ct} e^{-\delta t} m_{b,\delta}(u + ct - x) f_{X,W}(x,t)\, dx\, dt \\ & + \int_0^{(b-u)/c} \int_{u+ct}^\infty e^{-\delta t} \times \omega(u + ct, x - u - ct) \\ & \qquad \times f_{X,W}(x,t)\, dx\, dt \\ & + \int_{(b-u)/c}^\infty \int_0^b e^{-\delta t} m_{b,\delta}(b - x) f_{X,W}(x,t)\, dx\, dt \\ & + \int_{(b-u)/c}^\infty \int_b^\infty e^{-\delta t}\omega(b, x - b) f_{X,W}(x,t)\, dx\, dt. \end{aligned} \quad (20)$$

Given from (7), (20) becomes

$$\begin{aligned} m_{b,\delta}(u) = & \lambda \int_0^{(b-u)/c} e^{-(\lambda+\delta)t}\sigma_1(u + ct)\, dt \\ & + 2\theta\lambda \int_0^{(b-u)/c} e^{-(2\lambda+\delta)t}\sigma_2(u + ct)\, dt \\ & - \theta\lambda \times \int_0^{(b-u)/c} e^{-(\lambda+\delta)t}\sigma_2(u + ct)\, dt \\ & + \lambda \int_{(b-u)/c}^\infty e^{-(\lambda+\delta)t}\sigma_1(b)\, dt \\ & + 2\theta\lambda \int_{(b-u)/c}^\infty e^{-(2\lambda+\delta)t}\sigma_2(b)\, dt \\ & - \theta\lambda \int_{(b-u)/c}^\infty e^{-(\lambda+\delta)t}\sigma_2(b)\, dt, \end{aligned} \quad (21)$$

where the functions $\sigma_1(u)$ and $\sigma_2(u)$ are given in (16) and (17), respectively.

Simple modifications of (21) lead to

$$\begin{aligned} m_{b,\delta}(u) = & \frac{\lambda}{c}\int_u^b e^{-(\lambda+\delta)((t-u)/c)}\sigma_1(t)\, dt \\ & + \frac{2\theta\lambda}{c}\int_u^b e^{-(2\lambda+\delta)((t-u)/c)}\sigma_2(t)\, dt \\ & - \frac{\theta\lambda}{c}\int_u^b e^{-(\lambda+\delta)((t-u)/c)} \times \sigma_2(t)\, dt \\ & + \frac{\lambda}{c}\int_b^\infty e^{-(\lambda+\delta)((t-u)/c)}\sigma_1(b)\, dt \\ & + \frac{2\theta\lambda}{c}\int_b^\infty e^{-(2\lambda+\delta)((t-u)/c)}\sigma_2(b)\, dt \\ & - \frac{\theta\lambda}{c}\int_b^\infty e^{-(\lambda+\delta)((t-u)/c)}\sigma_2(b)\, dt. \end{aligned} \quad (22)$$

We can rewrite (22) as

$$\begin{aligned} m_{b,\delta}(u) = {} & \frac{\lambda}{c}\int_u^{\infty} e^{-(\lambda+\delta)((t-u)/c)}\sigma_1(t\wedge b)\,dt \\ & + \frac{2\theta\lambda}{c}\int_u^{\infty} e^{-(2\lambda+\delta)((t-u)/c)}\sigma_2(t\wedge b)\,dt \\ & - \frac{\theta\lambda}{c}\times\int_u^{\infty} e^{-(\lambda+\delta)((t-u)/c)}\sigma_2(t\wedge b)\,dt, \end{aligned} \tag{23}$$

where $t\wedge b = \min(t,b)$.

Now differentiating (23) with respect to u, routine calculations lead to

$$\begin{aligned} m'_{b,\delta}(u) = {} & \frac{\lambda+\delta}{c}\frac{\lambda}{c}\int_u^{\infty} e^{-(\lambda+\delta)((t-u)/c)}\sigma_1(t\wedge b)\,dt - \frac{\lambda}{c}\sigma_1(u) \\ & + \frac{2\lambda+\delta}{c}\frac{2\theta\lambda}{c}\int_u^{\infty} e^{-(2\lambda+\delta)((t-u)/c)}\sigma_2(t\wedge b)\,dt \\ & - \frac{\theta\lambda}{c}\sigma_2(u) - \frac{\lambda+\delta}{c}\frac{\theta\lambda}{c}\int_u^{\infty} e^{-(\lambda+\delta)((t-u)/c)} \\ & \times\sigma_2(t\wedge b)\,dt. \end{aligned} \tag{24}$$

Substituting (23) into (24), we obtain

$$\begin{aligned} & m'_{b,\delta}(u) \\ & = \frac{\lambda+\delta}{c}\left[m_{b,\delta}(u) - \frac{2\theta\lambda}{c}\int_u^{\infty} e^{-(2\lambda+\delta)((t-u)/c)}\sigma_2(t\wedge b)\,dt \right. \\ & \qquad \left. + \frac{\theta\lambda}{c}\int_u^{\infty} e^{-(\lambda+\delta)((t-u)/c)}\sigma_2(t\wedge b)\,dt\right] \\ & \quad + \frac{2\lambda+\delta}{c}\frac{2\theta\lambda}{c}\int_u^{\infty} e^{-(2\lambda+\delta)((t-u)/c)}\sigma_2(t\wedge b)\,dt \\ & \quad - \frac{\lambda+\delta}{c}\frac{\theta\lambda}{c}\int_u^{\infty} e^{-(\lambda+\delta)((t-u)/c)}\sigma_2(t\wedge b)\,dt \\ & \quad - \frac{\lambda}{c}\sigma_1(u) - \frac{\theta\lambda}{c}\sigma_2(u). \end{aligned} \tag{25}$$

That is,

$$\begin{aligned} & m'_{b,\delta}(u) \\ & = \frac{\lambda+\delta}{c}m_{b,\delta}(u) + \frac{\lambda}{c}\frac{2\theta\lambda}{c}\int_u^{\infty} e^{-(2\lambda+\delta)((t-u)/c)}\sigma_2(t\wedge b)\,dt \\ & \quad - \frac{\lambda}{c}\sigma_1(u) - \frac{\theta\lambda}{c}\sigma_2(u). \end{aligned} \tag{26}$$

Differentiating (26) with respect to u, we find

$$\begin{aligned} m''_{b,\delta}(u) = {} & \frac{\lambda+\delta}{c}m'_{b,\delta}(u) + \frac{\delta+2\lambda}{c}\frac{\lambda}{c}\frac{2\theta\lambda}{c} \\ & \times\int_u^{\infty} e^{-(2\lambda+\delta)((t-u)/c)}\sigma_2(t\wedge b)\,dt \\ & - \frac{\lambda}{c}\frac{2\theta\lambda}{c}\sigma_2(u) - \frac{\lambda}{c}\sigma'_1(u) - \frac{\theta\lambda}{c}\sigma'_2(u) \end{aligned} \tag{27}$$

which can be reexpressed as

$$\begin{aligned} & m''_{b,\delta}(u) \\ & = \frac{\lambda+\delta}{c}m'_{b,\delta}(u) + \frac{\delta+2\lambda}{c} \\ & \quad \times\left[m'_{b,\delta}(u) - \frac{\lambda+\delta}{c}m_{b,\delta}(u) + \frac{\lambda}{c}\sigma_1(u) + \frac{\theta\lambda}{c}\sigma_2(u)\right] \\ & \quad - \frac{\lambda}{c}\frac{2\theta\lambda}{c}\sigma_2(u) - \frac{\lambda}{c}\sigma'_1(u) - \frac{\theta\lambda}{c}\sigma'_2(u). \end{aligned} \tag{28}$$

That is,

$$\begin{aligned} m''_{b,\delta}(u) = {} & \frac{3\lambda+2\delta}{c}m'_{b,\delta}(u) - \frac{\lambda+\delta}{c}\frac{\delta+2\lambda}{c}m_{b,\delta}(u) \\ & + \frac{\lambda}{c}\frac{\delta+2\lambda}{c}\sigma_1(u) - \frac{\lambda}{c}\sigma'_1(u) + \frac{\theta\lambda}{c}\frac{\delta}{c}\sigma_2(u) \\ & - \frac{\theta\lambda}{c}\sigma'_2(u). \end{aligned} \tag{29}$$

Using the identical and differential operators, we obtain (13).

Regarding the boundary conditions, (14) is derived from (24) at $u = b$. While (15) can be proven via (27) at $u = b$ and (14).

Note that (13) in itself does not depend on the barrier level b, therefore, one concludes that $m_{\infty,\delta}(u)$, the Gerber-Shiu discounted penalty function in the absence of a barrier, satisfies the second order nonhomogeneous integrodifferential equation:

$$\begin{aligned} & \left(\frac{2\lambda+\delta}{c}\mathbf{I}-\mathbf{D}\right)\left(\frac{\lambda+\delta}{c}\mathbf{I}-\mathbf{D}\right)m_{\infty,\delta}(u) \\ & \quad = \frac{\lambda}{c}\left(\frac{2\lambda+\delta}{c}\mathbf{I}-\mathbf{D}\right)\sigma_3(u) + \frac{\theta\lambda}{c}\left(\frac{\delta}{c}\mathbf{I}-\mathbf{D}\right)\sigma_4(u); \end{aligned} \tag{30}$$

where

$$\begin{aligned} \sigma_3(u) &= \int_0^u m_{\infty,\delta}(u-x)\,f(x)\,dx + \omega_1(u), \\ \sigma_4(u) &= \int_0^u m_{\infty,\delta}(u-x)\,h(x)\,dx + \omega_2(u). \end{aligned} \tag{31}$$

As shown in Cossette et al. [7], it is a solution to a defective renewal equation. □

4. A Representation of the Discounted Penalty Function

In the present section, we derive the defective renewal equation for $m_{b,\delta}(u)$. For that purpose, we use the Dickson-Hipp operator T_s for an integrable real-valued function f (introduced by Dickson and Hipp (2001)) defined by

$$T_s f(x) = \int_x^{\infty} e^{-s(y-x)} f(y)\,dy, \quad s\in C. \tag{32}$$

The operator T_s is commutative; that is, $T_r T_s = T_s T_r$; moreover,

$$T_s T_r f(x) = T_r T_s f(x) = \frac{T_s f(x) - T_r f(x)}{r-s}, \quad s \neq r. \quad (33)$$

From Theorem 1, one concludes that $m_{b,\delta}(u)$ satisfies a nonhomogeneous equation of order 2. From the theory on differential equations, the solution to the second order nonhomogeneous equation (13) for $m_{b,\delta}(u)$ (with boundary conditions (14) and (15)) can be expressed as a particular solution $m_{\infty,\delta}(u)$ and a given combination of two linearly independent solutions to the associated homogeneous integrodifferential equation:

$$\begin{aligned}&\left(\frac{2\lambda+\delta}{c}\mathbf{I}-\mathbf{D}\right)\left(\frac{\lambda+\delta}{c}\mathbf{I}-\mathbf{D}\right)y(u)\\&\quad=\frac{\lambda}{c}\left(\frac{2\lambda+\delta}{c}\mathbf{I}-\mathbf{D}\right)\int_0^u y(u-x)f(x)\,dx+\frac{\theta\lambda}{c}\\&\qquad\times\left(\frac{\delta}{c}\mathbf{I}-\mathbf{D}\right)\int_0^u y(u-x)h(x)\,dx.\end{aligned} \quad (34)$$

By letting $y^*(s) = \int_0^\infty e^{-sx}y(x)dx$, let us take Laplace transform on the both sides of the homogeneous equation (34). We can obtain

$$\begin{aligned}&y^*(s)\\&=\left(\left(s-\frac{3\lambda+2\delta}{c}\right)y(0)+y'(0)\right)\\&\quad\times\left(\left(\frac{2\lambda+\delta}{c}-s\right)\left(\frac{\lambda+\delta}{c}-s\right)-\frac{\lambda}{c}\right.\\&\qquad\left.\times\left[\left(\frac{2\lambda+\delta}{c}-s\right)f^*(s)+\theta\left(\frac{\delta}{c}-s\right)h^*(s)\right]\right)^{-1}.\end{aligned} \quad (35)$$

From (35), it is clear that the solution to (34) can be written as a combination of the two linearly independent solutions $\{y_{1,\delta}(u), u \geq 0\}$ and $\{y_{2,\delta}(u), u \geq 0\}$, where

$$\begin{aligned}&y^*_{1,\delta}(s)\\&=\left(s-\frac{3\lambda+2\delta}{c}\right)\\&\quad\times\left(\left(\frac{2\lambda+\delta}{c}-s\right)\left(\frac{\lambda+\delta}{c}-s\right)-\frac{\lambda}{c}\right.\\&\qquad\left.\times\left[\left(\frac{2\lambda+\delta}{c}-s\right)f^*(s)+\theta\left(\frac{\delta}{c}-s\right)h^*(s)\right]\right)^{-1},\end{aligned} \quad (36)$$

with $y_{1,\delta}(0) = 1$ and $y'_{1,\delta}(0) = 0$, and

$$\begin{aligned}&y^*_{2,\delta}(s) = 1\\&\quad\times\left(\left(\frac{2\lambda+\delta}{c}-s\right)\left(\frac{\lambda+\delta}{c}-s\right)-\frac{\lambda}{c}\right.\\&\qquad\left.\times\left[\left(\frac{2\lambda+\delta}{c}-s\right)f^*(s)+\theta\left(\frac{\delta}{c}-s\right)h^*(s)\right]\right)^{-1},\end{aligned} \quad (37)$$

with $y_{2,\delta}(0) = 0$ and $y'_{2,\delta}(0) = 1$.

Theorem 2. *For the Gerber-Shiu discounted penalty function satisfying* (13), *a closed-form expression for $m_{b,\delta}(u)$ is given by*

$$m_{b,\delta}(u) = m_{\infty,\delta}(u) + \xi_1 y_{1,\delta}(u) + \xi_2 y_{2,\delta}(u), \quad 0 \leq u \leq b, \quad (38)$$

where the constants ξ_1, ξ_2 are the solutions to the following system of linear equations:

$$\xi_1 y'_{1,\delta}(b) + \xi_2 y'_{2,\delta}(b) = -m'_{\infty,\delta}(b), \quad (39)$$

$$\begin{aligned}&\xi_1\left(y''_{1,\delta}(b)+\frac{\lambda}{c}\mathbf{D}\int_0^u y_{1,\delta}(u-x)f(x)\,dx\Big|_{u=b}\right)\\&\quad+\frac{\theta\lambda}{c}\mathbf{D}\int_0^u y_{1,\delta}(u-x)h(x)\,dx\Big|_{u=b}\\&\quad+\xi_2\left(y''_{2,\delta}(b)+\frac{\lambda}{c}\mathbf{D}\int_0^u y_{2,\delta}(u-x)f(x)\,dx\Big|_{u=b}\right)\\&\quad+\frac{\theta\lambda}{c}\mathbf{D}\int_0^u y_{2,\delta}(u-x)h(x)\,dx\Big|_{u=b}\\&=-\left[m''_{\infty,\delta}(b)+\frac{\lambda}{c}\mathbf{D}\int_0^u m_{\infty,\delta}(u-x)f(x)\,dx\Big|_{u=b}\right.\\&\qquad+\frac{\theta\lambda}{c}\mathbf{D}\int_0^u m_{\infty,\delta}(u-x)h(x)\,dx\Big|_{u=b}\\&\qquad\left.+\frac{\lambda}{c}\omega'_1(b)+\frac{\theta\lambda}{c}\omega'_2(b)\right].\end{aligned} \quad (40)$$

Proof. It is immediate that $m_{b,\delta}(u)$ is of the form

$$m_{b,\delta}(u) = m_{\infty,\delta}(u) + \xi_1 y_{1,\delta}(u) + \xi_2 y_{2,\delta}(u). \quad (41)$$

Thus, by (14) and (15), differentiating (41) with respect to u at $u = b$, we obtain

$$m'_{\infty,\delta}(b) + \xi_1 y'_{1,\delta}(b) + \xi_2 y'_{2,\delta}(b) = 0, \quad (42)$$

$$m''_{\infty,\delta}(b) + \xi_1 y''_{1,\delta}(b) + \xi_2 y''_{2,\delta}(b) = -\frac{\lambda}{c}\sigma'_1(b) - \frac{\theta\lambda}{c}\sigma'_2(b). \quad (43)$$

Equation (42) is equivalent to (39).

Using the structural form (38) for $m_{b,\delta}(u)$, differentiation with respect to u of (16) and (17) yields

$$\begin{aligned}\mathbf{D}(\sigma_1(u)) &= \xi_1 \mathbf{D}\int_0^u y_{1,\delta}(u-x) f(x)\,dx \\ &\quad + \xi_2 \mathbf{D}\int_0^u y_{2,\delta}(u-x) f(x)\,dx \\ &\quad + \mathbf{D}\int_0^u m_{\infty,\delta}(u-x) f(x)\,dx + \mathbf{D}\omega_1(u), \\ \mathbf{D}(\sigma_2(u)) &= \xi_1 \mathbf{D}\int_0^u y_{1,\delta}(u-x) h(x)\,dx \\ &\quad + \xi_2 \mathbf{D}\int_0^u y_{2,\delta}(u-x) h(x)\,dx \\ &\quad + \mathbf{D}\int_0^u m_{\infty,\delta}(u-x) h(x)\,dx + \mathbf{D}\omega_2(u).\end{aligned} \tag{44}$$

Substituting (44) into the right-hand side of (43) at $u = b$ leads to (40).

From Propositions 4.1 and 4.2 of Cossette et al. [7], we know that the denominator on the right-hand side of (36) and (37) has only two positive, real, and distinct roots, say, s_1 and s_2.

Using (47) of Cossette et al. [7], (36) and (37) can be expressed as

$$\begin{aligned} y_{1,\delta}^*(s) &= \left(\left(\left(s_1 - \frac{3\lambda+2\delta}{c}\right)\frac{s-s_1}{s_2-s_1} + \left(s_2 - \frac{3\lambda+2\delta}{c}\right)\frac{s-s_2}{s_1-s_2}\right)\right. \\ &\quad \left. \times((s-s_1)(s-s_2))^{-1}\right)\left(1 - T_s T_{s_2} T_{s_1} h_{2,\delta}(0)\right)^{-1}, \end{aligned} \tag{45}$$

$$y_{2,\delta}^*(s) = \frac{1/(s-s_1)(s-s_2)}{1 - T_s T_{s_2} T_{s_1} h_{2,\delta}(0)}, \tag{46}$$

where

$$h_{2,\delta}^*(s) = \frac{\lambda}{c}\left[\left(\frac{2\lambda+\delta}{c} - s\right) f^*(s) + \theta\left(\frac{\delta}{c} - s\right) h^*(s)\right]. \tag{47}$$

Therefore, the Laplace transform (45) and (46) can now be used to find an expression for the two linearly independent solutions $\{y_{1,\delta}(u), u \geq 0\}$ and $\{y_{2,\delta}(u), u \geq 0\}$, respectively. From Proposition 7.2 of Cossette et al. [7], (45) and (46) lead to

$$\begin{aligned} y_{1,\delta}(u) &= k_\delta \int_0^u y_{1,\delta}(u-y)\, g_\delta(y)\,dy + \frac{s_1 - (3\lambda+2\delta)/c}{s_2 - s_1} e^{s_2 u} \\ &\quad + \frac{s_2 - (3\lambda+2\delta)/c}{s_1 - s_2} e^{s_1 u}, \end{aligned} \tag{48}$$

$$y_{2,\delta}(u) = k_\delta \int_0^u y_{2,\delta}(u-y)\, g_\delta(y)\,dy + \frac{e^{s_2 u} - e^{s_1 u}}{s_2 - s_1}, \tag{49}$$

where

$$\begin{aligned} k_\delta &= \frac{\lambda}{c}\left[\left(\frac{\delta+2\lambda}{c} - s_2\right) T_0 T_{s_2} T_{s_1} f(0) + \theta\left(\frac{\delta}{c} - s_2\right)\right. \\ &\quad \left. \times T_0 T_{s_2} T_{s_1} h(0) + T_0 T_{s_1} f(0) + \theta T_0 T_{s_1} h(0)\right], \\ g_\delta(y) &= \frac{T_{s_2} T_{s_1} h_{2,\delta}(u)}{k_\delta}. \end{aligned} \tag{50}$$

The defective renewal equations (48) and (49) may be solved to give an explicit for $y_{1,\delta}(u)$ and $y_{2,\delta}(u)$. By a similar way to the one used in Landriault [10], we choose

$$L_\delta(u) = 1 - \sum_{n=1}^{\infty} (1-k_\delta)(k_\delta)^n \overline{G}_\delta^{*n}(y), \tag{51}$$

where $\overline{G}_\delta^{*n}(y)$ is the survival distribution of the n-fold convolution of the p.d.f. $g_\delta(y)$. □

Theorem 3. *Let $\lambda_{i,\delta}(u) = e^{s_i u} - s_i \int_0^u e^{s_i y} L_\delta(u-y)dy$ for $i = 1, 2$. The solutions to (48) and (49) mat be expressed respectively as follows:*

$$\begin{aligned} y_{1,\delta}(u) &= \left(s_1(\lambda_{1,\delta}(u) - L_\delta(u)) - s_2(\lambda_{2,\delta}(u) - L_\delta(u))\right. \\ &\quad \left. - \frac{3\lambda+2\delta}{c}(\lambda_{1,\delta}(u) - \lambda_{2,\delta}(u))\right) \\ &\quad \times ((1-k_\delta)(s_2-s_1))^{-1}, \end{aligned} \tag{52}$$

$$y_{2,\delta}(u) = \frac{\lambda_{1,\delta}(u) - \lambda_{2,\delta}(u)}{(1-k_\delta)(s_2-s_1)}. \tag{53}$$

Proof. Applying Theorem 9.2 of Willmot and Lin [22] to the defective renewal equation (48) and (49), respectively, we can obtain (52) and (53) immediately.

All above derivations can derive the closed-form expression for $m_{b,\delta}(u)$ by (38). □

Conflict of Interests

The authors declare no conflict of interests regarding the publication for the paper.

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Research Article

Generalized *s*-Convex Functions on Fractal Sets

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We introduce two kinds of generalized s-convex functions on real linear fractal sets $\mathbb{R}^{\alpha}$ $(0 < \alpha < 1)$. And similar to the class situation, we also study the properties of these two kinds of generalized s-convex functions and discuss the relationship between them. Furthermore, some applications are given.

1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$. For any $u, v \in I$ and $t \in [0,1]$, if the following inequality,

$$f(tu + (1-t)v) \leq tf(u) + (1-t)f(v), \tag{1}$$

holds, then f is called a convex function on I.

The convexity of functions plays a significant role in many fields, such as in biological system, economy, and optimization [1, 2]. In [3], Hudzik and Maligranda generalized the definition of convex function and considered, among others, two kinds of functions which are s-convex.

Let $0 < s \leq 1$ and $\mathbb{R}_+ = [0, \infty)$, and then the two kinds of s-convex functions are defined, respectively, in the following way.

Definition 1. A function, $f : \mathbb{R}_+ \to \mathbb{R}$, is said to be s-convex in the first sense if

$$f(\alpha u + \beta v) \leq \alpha^s f(u) + \beta^s f(v), \tag{2}$$

for all $u, v \in \mathbb{R}_+$ and all $\alpha, \beta \geq 0$ with $\alpha^s + \beta^s = 1$. One denotes this by $f \in K_s^1$.

Definition 2. A function, $f : \mathbb{R}_+ \to \mathbb{R}$, is said to be s-convex in the second sense if

$$f(\alpha u + \beta v) \leq \alpha^s f(u) + \beta^s f(v), \tag{3}$$

for all $u, v \in \mathbb{R}_+$ and all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. One denotes this by $f \in K_s^2$.

It is obvious that the s-convexity means just the convexity when $s = 1$, no matter whether it is in the first sense or in the second sense. In [3], some properties of s-convex functions in both senses are considered and various examples and counterexamples are given. There are many research results related to the s-convex functions; see [4–6] and so on.

In recent years, the fractal has received significantly remarkable attention from scientists and engineers. In the sense of Mandelbrot, a fractal set is the one whose Hausdorff dimension strictly exceeds the topological dimension [7–12].

The calculus on fractal set can lead to better comprehension for the various real world models from science and engineering [8]. Researchers have constructed many kinds of fractional calculus on fractal sets by using different approaches. Particularly, in [13], Yang stated the analysis of local fractional functions on fractal space systematically, which includes local fractional calculus. In [14], the authors introduced the generalized convex function on fractal sets and established the generalized Jensen inequality and generalized Hermite-Hadamard inequality related to generalized convex function. And, in [15], Wei et al. established a local fractional integral inequality on fractal space analogous to Anderson's inequality for generalized convex functions. The generalized convex function on fractal sets $\mathbb{R}^{\alpha}$ $(0 < \alpha < 1)$ can be stated as follows.

Let $f : I \subset \mathbb{R} \to \mathbb{R}^{\alpha}$. For any $u, v \in I$ and $t \in [0,1]$, if the following inequality,

$$f(tu + (1-t)v) \leq t^{\alpha} f(u) + (1-t)^{\alpha} f(v), \tag{4}$$

holds, then f is called a generalized convex on I.

Inspired by these investigations, we will introduce the generalized s-convex function in the first or second sense on fractal sets and study the properties of generalized s-convex functions.

The paper is organized as follows. In Section 2, we state the operations with real line number fractal sets and give the definitions of the local fractional calculus. In Section 3, we introduce the definitions of two kinds of generalized s-convex functions and study the properties of these functions. In Section 4, we give some applications for the two kinds of generalized s-convex functions on fractal sets.

2. Preliminaries

Let us recall the operations with real line number on fractal space and use Gao-Yang-Kang's idea to describe the definitions of the local fractional derivative and local fractional integral [13, 16–19].

If a^α, b^α, and c^α belong to the set $\mathbb{R}^\alpha$ $(0 < \alpha \leq 1)$ of real line numbers, then one has the following:

(1) $a^\alpha + b^\alpha$ and $a^\alpha b^\alpha$ belong to the set $\mathbb{R}^\alpha$;

(2) $a^\alpha + b^\alpha = b^\alpha + a^\alpha = (a+b)^\alpha = (b+a)^\alpha$;

(3) $a^\alpha + (b^\alpha + c^\alpha) = (a^\alpha + b^\alpha) + c^\alpha$;

(4) $a^\alpha b^\alpha = b^\alpha a^\alpha = (ab)^\alpha = (ba)^\alpha$;

(5) $a^\alpha (b^\alpha c^\alpha) = (a^\alpha b^\alpha) c^\alpha$;

(6) $a^\alpha (b^\alpha + c^\alpha) = a^\alpha b^\alpha + a^\alpha c^\alpha$;

(7) $a^\alpha + 0^\alpha = 0^\alpha + a^\alpha = a^\alpha$ and $a^\alpha \cdot 1^\alpha = 1^\alpha \cdot a^\alpha = a^\alpha$.

Let us now state some definitions about the local fractional calculus on $\mathbb{R}^\alpha$.

Definition 3 (see [13]). A nondifferentiable function $f : \mathbb{R} \to \mathbb{R}^\alpha$, $x \to f(x)$ is called to be local fractional continuous at x_0, if, for any $\varepsilon > 0$, there exists $\delta > 0$, such that

$$\left|f(x) - f(x_0)\right| < \varepsilon^\alpha \tag{5}$$

holds for $|x - x_0| < \delta$, where $\varepsilon, \delta \in \mathbb{R}$. If f is local fractional continuous on the interval (a, b), one denotes $f \in C_\alpha(a, b)$.

Definition 4 (see [13]). The local fractional derivative of function f of order α at $x = x_0$ is defined by

$$f^{(\alpha)}(x_0) = \left.\frac{d^\alpha f(x)}{dx^\alpha}\right|_{x=x_0} = \lim_{x \to x_0} \frac{\Delta^\alpha (f(x) - f(x_0))}{(x - x_0)^\alpha}, \tag{6}$$

where $\Delta^\alpha(f(x) - f(x_0)) = \Gamma(1+a)(f(x) - f(x_0))$ and the Gamma function is defined by $\Gamma(t) = \int_0^{+\infty} x^{t-1} e^{-x} dx$.

If there exists $f^{((k+1)\alpha)}(x) = \overbrace{D_x^\alpha \cdots D_x^\alpha}^{k+1 \text{ times}} f(x)$ for any $x \in I \subseteq \mathbb{R}$, then one denoted $f \in D_{(k+1)\alpha}(I)$, where $k = 0, 1, 2, \ldots$.

Definition 5 (see [13]). Let $f \in C_\alpha[a, b]$. Then the local fractional integral of the function f of order α is defined by

$$\begin{aligned} {}_aI_b^{(\alpha)} f &= \frac{1}{\Gamma(1+a)} \int_a^b f(t)(dt)^\alpha \\ &= \frac{1}{\Gamma(1+a)} \lim_{\Delta t \to 0} \sum_{j=0}^{N} f(t_j)(\Delta t_j)^\alpha, \end{aligned} \tag{7}$$

with $\Delta t_j = t_{j+1} - t_j$, $\Delta t = \max\{\Delta t_1, \Delta t_2, \Delta t_j, \ldots, \Delta t_N - 1\}$, and $[t_j, t_j + 1]$, $j = 0, \ldots, N-1$, where $t_0 = a < t_1 < \cdots < t_i < \cdots < t_N = b$ is a partition of the interval $[a, b]$.

Lemma 6 (see [13]). *Suppose that $f, g \in C_\alpha[a, b]$ and $f, g \in D_\alpha(a, b)$. If $\lim_{x \to x_0} f(x) = 0^\alpha$, $\lim_{x \to x_0} g(x) = 0^\alpha$ and $g^{(\alpha)}(x) \neq 0^\alpha$. Suppose that $\lim_{x \to x_0}(f^{(\alpha)}(x)/g^{(\alpha)}(x)) = A^\alpha$, and then*

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = A^\alpha. \tag{8}$$

Lemma 7 (see [13]). *Suppose that $f(x) \in C_\alpha[a, b]$; then*

$$\frac{d^\alpha \left({}_aI_x^{(\alpha)} f\right)}{dx^\alpha} = f(x), \quad a < x < b. \tag{9}$$

3. Generalized s-Convexity Functions

The convexity of functions plays a significant role in many fields. In this section, let us introduce two kinds of generalized s-convex functions on fractal sets. And then, we study the properties of the two kinds of generalized s-convex functions.

Definition 8. Let $\mathbb{R}_+ = [0, +\infty)$. A function $f : \mathbb{R}_+ \to \mathbb{R}^\alpha$ is said to be generalized s-convex $(0 < s < 1)$ in the first sense, if

$$f(\lambda_1 u + \lambda_2 v) \leq \lambda_1^{s\alpha} f(u) + \lambda_2^{s\alpha} f(v), \tag{10}$$

for all $u, v \in \mathbb{R}_+$ and all $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1^s + \lambda_2^s = 1$. One denotes this by $f \in GK_s^1$.

Definition 9. A function $f : \mathbb{R}_+ \to \mathbb{R}^\alpha$ is said to be generalized s-convex $(0 < s < 1)$ in the second sense, if

$$f(\lambda_1 u + \lambda_2 v) \leq \lambda_1^{s\alpha} f(u) + \lambda_2^{s\alpha} f(v), \tag{11}$$

for all $u, v \in \mathbb{R}_+$ and all $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 = 1$. One denotes this by $f \in GK_s^2$.

Note that, when $s = 1$, the generalized s-convex functions in both senses are the generalized convex functions; see [14].

Theorem 10. *Let $0 < s < 1$.*

(a) *If $f \in GK_s^1$, then f is nondecreasing on $(0, +\infty)$ and*

$$f(0^+) = \lim_{u \to 0^+} f(u) \leq f(0). \tag{12}$$

(b) *If $f \in GK_s^2$, then f is nonnegative on $[0, +\infty)$.*

Proof. (a) Since $f \in GK_s^1$, we have, for $u > 0$ and $\lambda \in [0,1]$,

$$f\left[\left(\lambda^{1/s} + (1-\lambda)^{1/s}\right)u\right] \leq \lambda^{\alpha} f(u) + (1-\lambda)^{\alpha} f(u) = f(u). \tag{13}$$

The function

$$h(\lambda) = \lambda^{1/s} + (1-\lambda)^{1/s} \tag{14}$$

is continuous on $[0,1]$, decreasing on $[0,1/2]$, and increasing on $[1/2,1]$ and $h([0,1]) = [h(1/2), h(1)] = [2^{1-1/s}, 1]$. This yields that

$$f(tu) \leq f(u), \tag{15}$$

for $u > 0$ and $t \in [2^{1-1/s}, 1]$. If $t \in [2^{2(1-1/s)}, 1]$, then $t^{1/2} \in [2^{1-1/s}, 1]$. Therefore, by the fact that (15) holds, we get

$$f(tu) = f\left(t^{1/2}\left(t^{1/2}u\right)\right) \leq f\left(t^{1/2}u\right) \leq f(u), \tag{16}$$

for all $u > 0$. So we can obtain that

$$f(tu) \leq f(u), \quad \forall u > 0,\ t \in (0,1]. \tag{17}$$

So, taking $0 < u < v$, we get

$$f(u) = f\left(\left(\frac{u}{v}\right)v\right) \leq f(v), \tag{18}$$

which means that f is nondecreasing on $(0, +\infty)$.

As for the second part, for $u > 0$ and $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1^s + \lambda_2^s = 1$, we have

$$f(\lambda_1 u) = f(\lambda_1 u + \lambda_2 0) \leq \lambda_1^{s\alpha} f(u) + \lambda_2^{s\alpha} f(0). \tag{19}$$

And taking $u \to 0^+$, we get

$$\lim_{u \to 0^+} f(u) = \lim_{u \to 0^+} f(\lambda_1 u) \leq \lambda_1^{s\alpha} \lim_{u \to 0^+} f(u) + \lambda_2^{s\alpha} f(0). \tag{20}$$

So,

$$\lim_{u \to 0^+} f(u) \leq f(0). \tag{21}$$

(b) For $f \in GK_s^2$, we can get that, for $u \in \mathbb{R}_+$,

$$f(u) = f\left(\frac{u}{2} + \frac{u}{2}\right) \leq \frac{f(u)}{2^{s\alpha}} + \frac{f(u)}{2^{s\alpha}} = 2^{(1-s)\alpha} f(u). \tag{22}$$

So, $(2^{1-s} - 1)^{\alpha} f(u) \geq 0^{\alpha}$. This means that $f(u) \geq 0^{\alpha}$, since $0 < s < 1$. □

Remark 11. The above results do not hold, in general, in the case of generalized convex functions, that is, when $s = 1$, because a generalized convex function, $f : \mathbb{R}_+ \to \mathbb{R}^{\alpha}$, need not be either nondecreasing or nonnegative.

Remark 12. If $0 < s < 1$, then the function $f \in GK_s^1$ is nondecreasing on $(0, +\infty)$ but not necessarily on $[0, +\infty)$.

Function $F : \mathbb{R}^2 \to \mathbb{R}^{\alpha}$ is called to be generalized convex in each variable, if

$$F(\lambda_1 u + \lambda_2 v, \lambda_1 r + \lambda_2 t) \leq \lambda_1^{\alpha} F(u, r) + \lambda_2^{\alpha} F(v, t). \tag{23}$$

For all (u, r), $(v, t) \in \mathbb{R}^2$ and $\lambda_1, \lambda_2 \in [0,1]$ with $\lambda_1 + \lambda_2 = 1$.

Theorem 13. *Let $0 < s < 1$. If $f, g : \mathbb{R} \to \mathbb{R}$ and $f, g \in K_s^1$ and if $F : \mathbb{R}^2 \to \mathbb{R}^{\alpha}$ is a generalized convex and nondecreasing function in each variable, then the function $h : \mathbb{R}_+ \to \mathbb{R}^{\alpha}$ defined by*

$$h(u) = F(f(u), g(u)) \tag{24}$$

is in GK_s^1. In particular, if $f, g \in K_s^1$, then $f^{\alpha} + g^{\alpha}$, $\max\{f^{\alpha}, g^{\alpha}\} \in GK_s^1$.

Proof. If $u, v \in \mathbb{R}_+$, then for all $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1^s + \lambda_2^s = 1$,

$$\begin{aligned} h(\lambda_1 u + \lambda_2 v) &= F(f(\lambda_1 u + \lambda_2 v), g(\lambda_1 u + \lambda_2 v)) \\ &\leq F(\lambda_1^s f(u) + \lambda_2^s f(v), \lambda_1^s g(u) + \lambda_2^s g(v)) \\ &\leq \lambda_1^{s\alpha} F(f(u), g(u)) + \lambda_2^{s\alpha} F(f(v), g(v)) \\ &= \lambda_1^{s\alpha} h(u) + \lambda_2^{s\alpha} h(v). \end{aligned} \tag{25}$$

Thus, $h \in GK_s^1$.

Moreover, since $F(u, v) = u^{\alpha} + v^{\alpha}$, $F(u, v) = \max\{u^{\alpha}, v^{\alpha}\}$ are nondecreasing generalized convex functions on R^2, so they yield particular cases of our theorem. □

Let us pay attention to the situation when the condition $\lambda_1^s + \lambda_2^s = 1$ $(\lambda_1 + \lambda_2 = 1)$ in the definition of $GK_s^1 (GK_s^2)$ can be equivalently replaced by the condition $\lambda_1^s + \lambda_2^s \leq 1$ $(\lambda_1 + \lambda_2 \leq 1)$.

Theorem 14. *(a) Let $f \in GK_s^1$. Then inequality* (10) *holds for all $u, v \in R_+$ and all $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1^s + \lambda_2^s < 1$ if and only if $f(0) \leq 0^{\alpha}$.*

(b) Let $f \in GK_s^2$. Then inequality (11) *holds for all $u, v \in R_+$ and all $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 < 1$ if and only if $f(0) = 0^{\alpha}$.*

Proof. (a) Necessity is obvious by taking $u = v = 0$ and $\lambda_1 = \lambda_2 = 0$. Let us show the sufficiency.

Assume that $u, v \in \mathbb{R}_+$ and $\lambda_1, \lambda_2 \geq 0$ with $0 < \lambda_3 = \lambda_1^s + \lambda_2^s < 1$. Put $a = \lambda_1 \lambda_3^{-1/s}$ and $b = \lambda_2 \lambda_3^{-1/s}$. Then $a^s + b^s = \lambda_1^s/\lambda_3 + \lambda_2^s/\lambda_3 = 1$ and

$$\begin{aligned} f(\lambda_1 u + \lambda_2 v) &= f\left(a\lambda_3^{1/s} u + b\lambda_3^{1/s} v\right) \\ &\leq a^{s\alpha} f\left(\lambda_3^{1/s} u\right) + b^{s\alpha} f\left(\lambda_3^{1/s} v\right) \end{aligned}$$

$$\begin{aligned}
&= a^{s\alpha} f\left[\lambda_3^{1/s} u + (1-\lambda_3)^{1/s} 0\right] \\
&\quad + b^{s\alpha} f\left[\lambda_3^{1/s} v + (1-\lambda_3)^{1/s} 0\right] \\
&\leq a^{s\alpha}\left[\lambda_3^{\alpha} f(u) + (1-\lambda_3)^{\alpha} f(0)\right] \\
&\quad + b^{s\alpha}\left[\lambda_3^{\alpha} f(v) + (1-\lambda_3)^{\alpha} f(0)\right] \\
&= a^{s\alpha}\lambda_3^{\alpha} f(u) + b^{s\alpha}\lambda_3^{\alpha} f(v) + (1-\lambda_3)^{\alpha} f(0) \\
&\leq \lambda_1^{s\alpha} f(u) + \lambda_2^{s\alpha} f(v).
\end{aligned} \tag{26}$$

(b) Necessity. Taking $u = v = \lambda_1 = \lambda_2 = 0$, we obtain $f(0) \leq 0^{\alpha}$. And using Theorem 10(b), we get $f(0) \geq 0^{\alpha}$. Therefore $f(0) = 0^{\alpha}$.

Sufficiency. Let $u, v \in \mathbb{R}_+$ and $\lambda_1, \lambda_2 \geq 0$ with $0 < \lambda_3 = \lambda_1 + \lambda_2 < 1$. Put $a = \lambda_1/\lambda_3$ and $b = \lambda_2/\lambda_3$, and then $a + b = 1$.

So,

$$\begin{aligned}
& f(\lambda_1 u + \lambda_2 v) \\
&\quad = f(a\lambda_3 u + b\lambda_3 v) \\
&\quad \leq a^{s\alpha} f(\lambda_3 u) + b^{s\alpha} f(\lambda_3 v) \\
&\quad = a^{s\alpha} f[\lambda_3 u + (1-\lambda_3) 0] \\
&\qquad + b^{s\alpha} f[\lambda_3 v + (1-\lambda_3) 0] \\
&\quad \leq a^{s\alpha}\left[\lambda_3^{s\alpha} f(u) + (1-\lambda_3)^{s\alpha} f(0)\right] \\
&\qquad + b^{s\alpha}\left[\lambda_3^{s\alpha} f(v) + (1-\lambda_3)^{s\alpha} f(0)\right] \\
&\quad = a^{s\alpha}\lambda_3^{s\alpha} f(u) + b^{s\alpha}\lambda_3^{s\alpha} f(v) \\
&\qquad + (1-\lambda_3)^{s\alpha} f(0) \\
&\quad = \lambda_1^{s\alpha} f(u) + \lambda_2^{s\alpha} f(v).
\end{aligned} \tag{27}$$

□

Theorem 15. *(a) Let $0 < s \leq 1$. If $f \in GK_s^2$ and $f(0) = 0^{\alpha}$, then $f \in GK_s^1$.*

(b) Let $0 < s_1 \leq s_2 \leq 1$. If $f \in GK_{s_2}^2$ and $f(0) = 0^{\alpha}$, then $f \in GK_{s_1}^2$.

(c) Let $0 < s_1 \leq s_2 \leq 1$. If $f \in GK_{s_2}^1$ and $f(0) \leq 0^{\alpha}$, then $f \in GK_{s_1}^1$.

Proof. (a) Assume that $f \in GK_s^2$ and $f(0) = 0^{\alpha}$. Let $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1^s + \lambda_2^s = 1$, and we have $\lambda_1 + \lambda_2 \leq \lambda_1^s + \lambda_2^s = 1$. From Theorem 14(b), we can get

$$f(\lambda_1 u + \lambda_2 v) \leq \lambda_1^{s\alpha} f(u) + \lambda_2^{s\alpha} f(v), \tag{28}$$

for $u, v \in \mathbb{R}_+$, and then $f \in GK_s^1$.

(b) Assume that $f \in GK_{s_2}^2$, $u, v \in \mathbb{R}_+$, and $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 = 1$. Then we have

$$\begin{aligned}
f(\lambda_1 u + \lambda_2 v) &\leq \lambda_1^{s_2\alpha} f(u) + \lambda_2^{s_2\alpha} f(v) \\
&\leq \lambda_1^{s_1\alpha} f(u) + \lambda_2^{s_1\alpha} f(v).
\end{aligned} \tag{29}$$

So $f \in GK_{s_1}^2$.

(c) Assume that $f \in GK_{s_2}^1$, $u, v \in R_+$, and $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1^{s_1} + \lambda_2^{s_1} = 1$. Then $\lambda_1^{s_2} + \lambda_2^{s_2} \leq \lambda_1^{s_1} + \lambda_2^{s_1} = 1$. Thus, according to Theorem 14(a), we have

$$\begin{aligned}
f(\lambda_1 u + \lambda_2 v) &\leq \lambda_1^{s_2\alpha} f(u) + \lambda_2^{s_2\alpha} f(v) \\
&\leq \lambda_1^{s_1\alpha} f(u) + \lambda_2^{s_1\alpha} f(v).
\end{aligned} \tag{30}$$

So, $f \in GK_{s_1}^1$. □

Theorem 16. *Let $0 < s < 1$ and $p : \mathbb{R}_+ \to \mathbb{R}_+^{\alpha}$ be a nondecreasing function. Then the function f defined for $u \in \mathbb{R}_+$ by*

$$f(u) = u^{(s/(1-s))\alpha} p(u) \tag{31}$$

belongs to GK_s^1.

Proof. Let $v \geq u \geq 0$ and $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1^s + \lambda_2^s = 1$. We consider two cases.

Case I. It is easy to see that f is a nondecreasing function. Let $\lambda_1 u + \lambda_2 v \leq u$, and then

$$\begin{aligned}
f(\lambda_1 u + \lambda_2 v) \leq f(u) &= \left(\lambda_1^{s\alpha} + \lambda_2^{s\alpha}\right) f(u) \\
&\leq \lambda_1^{s\alpha} f(u) + \lambda_2^{s\alpha} f(v).
\end{aligned} \tag{32}$$

Case II. Let $\lambda_1 u + \lambda_2 v > u$, and then $\lambda_2 v > (1-\lambda_1) u$. So, $\lambda_2 > 0$ and $\lambda_1 \leq \lambda_1^s$. Thus,

$$\lambda_1 - \lambda_1^{s+1} \leq \lambda_1^s - \lambda_1^{s+1}; \tag{33}$$

that is,

$$\begin{gathered}
\frac{\lambda_1}{(1-\lambda_1)} \leq \frac{\lambda_1^s}{(1-\lambda_1^s)} = \frac{(1-\lambda_2^s)}{\lambda_2^s}, \\
\frac{\lambda_1\lambda_2}{(1-\lambda_1)} \leq \lambda_2^{1-s} - \lambda_2.
\end{gathered} \tag{34}$$

Thus, we can get that

$$\begin{aligned}
\lambda_1 u + \lambda_2 v &\leq (\lambda_1 + \lambda_2) v \leq (\lambda_1^s + \lambda_2^s) v = v, \\
\lambda_1 u + \lambda_2 v &\leq \frac{\lambda_1\lambda_2 v}{(1-\lambda_1)} + \lambda_2 v \\
&\leq \left(\lambda_2^{1-s} - \lambda_2\right) v + \lambda_2 v = \lambda_2^{1-s} v.
\end{aligned} \tag{35}$$

Then,

$$(\lambda_1 u + \lambda_2 v)^{s/(1-s)} \leq \lambda_2^s v^{s/(1-s)}. \tag{36}$$

We obtain

$$
\begin{aligned}
f(\lambda_1 u+\lambda_2 v) &= (\lambda_1 u+\lambda_2 v)^{(s/(1-s))\alpha} p(\lambda_1 u+\lambda_2 v) \\
&\le \lambda_2^{s\alpha} v^{(s/(1-s))\alpha} p(v) \\
&= \lambda_2^{s\alpha} f(v) \le \lambda_1^{s\alpha} f(u)+\lambda_2^{s\alpha} f(v).
\end{aligned}
\tag{37}
$$

□

Theorem 17. *(a) Let $f \in GK^1_{s_1}$ and $g \in K^1_{s_2}$, where $0 < s_1, s_2 \le 1$. If f is a nondecreasing function and g is a nonnegative function such that $f(0) \le 0^\alpha$ and $g(0) = 0$, then the composition $f \circ g$ of f with g belongs to GK^1_s, where $s = s_1 s_2$.*

(b) Let $f \in GK^1_{s_1}$ and $g \in GK^1_{s_2}$, where $0 < s_1, s_2 \le 1$. Assume that $0 < s_1, s_2 < 1$. If f and g are nonnegative functions such that either $f(0) = 0^\alpha$ and $g(0^+) = g(0)$, or $g(0) = 0^\alpha$ and $f(0^+) = f(0)$, then the product fg of f and g belongs to GK^1_s, where $s = \min\{s_1, s_2\}$.

Proof. (a) Let $u, v \in \mathbb{R}_+, \lambda_1, \lambda_2 \ge 0$ with $\lambda_1^s + \lambda_2^s = 1$, where $s = s_1 s_2$. Since $\lambda_1^{s_i} + \lambda_2^{s_i} \le \lambda_1^{s_1 s_2} + \lambda_2^{s_1 s_2} = 1$ for $i = 1, 2$, then according to Theorem 3(a) in [3] and Theorem 14(a) in the paper, we have

$$
\begin{aligned}
f \circ g&(\lambda_1 u+\lambda_2 v) \\
&= f(g(\lambda_1 u+\lambda_2 v)) \\
&\le f(\lambda_1^{s_2} g(u)+\lambda_2^{s_2} g(v)) \\
&\le \lambda_1^{s_1 s_2 \alpha} f(g(u))+\lambda_2^{s_1 s_2 \alpha} f(g(v)) \\
&= \lambda_1^{s\alpha} f\circ g(u)+\lambda_2^{s\alpha} f\circ g(v)
\end{aligned}
\tag{38}
$$

which means that $f \circ g \in GK^1_s$.

(b) According to Theorem 10(a), f, g are nondecreasing on $(0, +\infty)$.

So,

$$
(f(u)-f(v))(g(v)-g(u)) \le 0^\alpha,
\tag{39}
$$

or, equivalently,

$$
f(u)g(v)+f(v)g(u) \le f(u)g(u)+f(v)g(v),
\tag{40}
$$

for all $v > u > 0$.

If $v > u = 0$, then the inequality is still true because f, g are nonnegative and either $f(0) = 0^\alpha$ and $g(0^+) = g(0)$ or $g(0) = 0^\alpha$ and $f(0^+) = f(0)$.

Now let $u, v \in \mathbb{R}_+$ and $\lambda_1, \lambda_2 \ge 0$ with $\lambda_1^s + \lambda_2^s = 1$, where $s = \min\{s_1, s_2\}$. Then $\lambda_1^{s_i} + \lambda_2^{s_i} \le \lambda_1^s + \lambda_2^s = 1$ for $i = 1, 2$. And by Theorem 14(a), we have

$$
\begin{aligned}
f&(\lambda_1 u+\lambda_2 v)\, g(\lambda_1 u+\lambda_2 v) \\
&\le (\lambda_1^{s_1\alpha} f(u)+\lambda_2^{s_1\alpha} f(v)) \\
&\quad \times (\lambda_1^{s_2\alpha} g(u)+\lambda_2^{s_2\alpha} g(v)) \\
&= \lambda_1^{(s_1+s_2)\alpha} f(u)g(u)+\lambda_1^{s_1\alpha}\lambda_2^{s_2\alpha} f(u)g(v) \\
&\quad + \lambda_1^{s_2\alpha}\lambda_2^{s_1\alpha} f(v)g(u)+\lambda_2^{(s_1+s_2)\alpha} f(v)g(v) \\
&\le \lambda_1^{2s\alpha} f(u)g(u) \\
&\quad + \lambda_1^{s\alpha}\lambda_2^{s\alpha}(f(u)g(v)+f(v)g(u)) \\
&\quad + \lambda_2^{2s\alpha} f(v)g(v) \\
&\le \lambda_1^{2s\alpha} f(u)g(u) \\
&\quad + \lambda_1^{s\alpha}\lambda_2^{s\alpha}(f(u)g(u)+f(v)g(v)) \\
&\quad + \lambda_2^{2s\alpha} f(v)g(v) \\
&= \lambda_1^{s\alpha} f(u)g(u)+\lambda_2^{s\alpha} f(v)g(v),
\end{aligned}
\tag{41}
$$

which means that $fg \in GK^1_s$. □

Remark 18. From the above proof, we can get that if f is a nondecreasing function in GK^2_s and g is a nonnegative convex function on $[0, +\infty)$, then the composition $f \circ g$ of f with g belongs to GK^2_s.

Remark 19. Generalized convex functions on $[0, +\infty)$ need not be monotonic. However, if f and g are nonnegative, generalized convex and either both are nondecreasing or both are nonincreasing on $[0, +\infty)$, then the product fg is also a generalized convex function.

Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function. Then f is said to be a φ-function if $f(0) = 0$ and f is nondecreasing on $\mathbb{R}_+$. Similarly, we can define the φ-type function on fractal sets as follows. A function $f : \mathbb{R}_+ \to \mathbb{R}^\alpha_+$ is said to be a φ-type function if $f(0) = 0^\alpha$ and $f \in C_\alpha(\mathbb{R}_+)$ is nondecreasing.

Corollary 20. *If Φ is a generalized convex φ-type function and $g \in K^1_s$ is a φ-function, then the composition $\Phi \circ g$ belongs to GK^1_s. In particular, the φ-type function $h(u) = \Phi(u^s)$ belongs to GK^1_s.*

Corollary 21. *If Φ is a convex φ-function and $f \in GK^2_s$ is a φ-type function, then the composition $f \circ \Phi$ belongs to GK^2_s. In particular, the φ-type function $h(u) = [\Phi(u)]^{s\alpha}$ belongs to GK^2_s.*

Theorem 22. *If $0 < s < 1$ and $f \in GK^1_s$ is a φ-type function, then there exists a generalized convex φ-type function Φ such that*

$$
f(2^{-1/s}u) \le \Phi(u^s) \le f(u),
\tag{42}
$$

for all $u \ge 0$.

Proof. By the generalized s-convexity of the function f and by $f(0) = 0^\alpha$, we obtain $f(\lambda_1 u) \le \lambda_1^{s\alpha} f(u)$ for all $u \ge 0$ and all $\lambda_1 \in [0, 1]$.

Assume now that $v > u > 0$. Then

$$f\left(u^{1/s}\right) \leq f\left(\left(\frac{u}{v}\right)^{1/s} v^{1/s}\right) \leq \left(\frac{u^{\alpha}}{v^{\alpha}}\right) f\left(v^{1/s}\right); \tag{43}$$

that is,

$$\frac{f\left(u^{1/s}\right)}{u^{\alpha}} \leq \frac{f\left(v^{1/s}\right)}{v^{\alpha}}. \tag{44}$$

Inequality (44) means that the function $f(u^{1/s})/u^{\alpha}$ is a nondecreasing function on $(0, +\infty)$. And, since f is a φ-type function, thus f is local fractional continuous $[0, +\infty)$.

Define

$$\Phi(u) = \begin{cases} 0^{\alpha}, & u = 0, \\ \Gamma(1+\alpha)_0 I_u^{(\alpha)}\left(\dfrac{f\left(t^{1/s}\right)}{t^{\alpha}}\right), & u > 0. \end{cases} \tag{45}$$

From Lemmas 6 and 7, it is easy to see that Φ is a generalized convex φ-type function and

$$\begin{aligned} \Phi\left(u^{s}\right) &= \Gamma(1+\alpha)_0 I_{u^s}^{(\alpha)}\left(\frac{f\left(t^{1/s}\right)}{t^{\alpha}}\right) \\ &\leq \left(\frac{f\left(\left(u^{s}\right)^{1/s}\right)}{u^{s\alpha}}\right) u^{s\alpha} = f(u). \end{aligned} \tag{46}$$

Moreover,

$$\begin{aligned} \Phi\left(u^{s}\right) &\geq \Gamma(1+\alpha)_{(u^s/2)} I_{u^s}^{(\alpha)}\left(\frac{f\left(t^{1/s}\right)}{t^{\alpha}}\right) \\ &\geq \frac{\left(f\left(\left(u^{s}/2\right)^{1/s}\right) 2^{\alpha} u^{-s\alpha}\right) u^{s\alpha}}{2^{\alpha}} = f\left(2^{-1/s} u\right). \end{aligned} \tag{47}$$

Therefore,

$$f\left(2^{-1/s} u\right) \leq \Phi\left(u^{s}\right) \leq f(u), \tag{48}$$

for all $u \geq 0$. □

4. Applications

Based on the properties of the two kinds of generalized s-convex functions in the above section, some applications are given.

Example 1. Let $0 < s < 1$, and $a^{\alpha}, b^{\alpha}, c^{\alpha} \in \mathbb{R}^{\alpha}$. For $u \in \mathbb{R}_{+}$, define

$$f(u) = \begin{cases} a^{\alpha}, & u = 0, \\ b^{\alpha} u^{s\alpha} + c^{\alpha}, & u > 0. \end{cases} \tag{49}$$

We have the following conclusions.

(i) If $b^{\alpha} \geq 0^{\alpha}$ and $c^{\alpha} \leq a^{\alpha}$, then $f \in GK_s^1$.

(ii) If $b^{\alpha} \geq 0^{\alpha}$ and $c^{\alpha} < a^{\alpha}$, then f is nondecreasing on $(0, +\infty)$ but not on $[0, +\infty)$.

(iii) If $b^{\alpha} \geq 0^{\alpha}$ and $0^{\alpha} \leq c^{\alpha} \leq a^{\alpha}$, then $f \in GK_s^2$.

(iv) If $b^{\alpha} > 0^{\alpha}$ and $c^{\alpha} < 0^{\alpha}$, then $f \notin GK_s^2$.

Proof. (i) Let $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1^s + \lambda_2^s = 1$. Then, there are two nontrivial cases.

Case I. Let $u, v > 0$. Then $\lambda_1 u + \lambda_2 v > 0$.

Thus,

$$\begin{aligned} f\left(\lambda_1 u + \lambda_2 v\right) &= b^{\alpha}\left(\lambda_1 u + \lambda_2 v\right)^{s\alpha} + c^{\alpha} \\ &\leq b^{\alpha}\left(\lambda_1^{s\alpha} u^{s\alpha} + \lambda_2^{s\alpha} v^{s\alpha}\right) + c^{\alpha} \\ &= b^{\alpha}\left(\lambda_1^{s\alpha} u^{s\alpha} + \lambda_2^{s\alpha} v^{s\alpha}\right) + c^{\alpha}\left(\lambda_1^{s\alpha} + \lambda_2^{s\alpha}\right) \\ &= \lambda_1^{s\alpha}\left(b^{\alpha} u^{s\alpha} + c^{\alpha}\right) + \lambda_2^{s\alpha}\left(b^{\alpha} v^{s\alpha} + c^{\alpha}\right) \\ &= \lambda_1^{s\alpha} f(u) + \lambda_2^{s\alpha} f(v). \end{aligned} \tag{50}$$

Case II. Let $v > u = 0$. We need only to consider $\lambda_2 > 0$.

Thus, we have

$$\begin{aligned} f\left(\lambda_1 0 + \lambda_2 v\right) &= f\left(\lambda_2 v\right) \\ &= b^{\alpha} \lambda_2^{s\alpha} v^{s\alpha} + c^{\alpha} \\ &= b^{\alpha} \lambda_2^{s\alpha} v^{s\alpha} + c^{\alpha}\left(\lambda_1^{s\alpha} + \lambda_2^{s\alpha}\right) \\ &= \lambda_1^{s\alpha} c^{\alpha} + \lambda_2^{s\alpha}\left(b^{\alpha} v^{s\alpha} + c^{\alpha}\right) \\ &= \lambda_1^{s\alpha} c^{\alpha} + \lambda_2^{s\alpha} f(v) \\ &\leq \lambda_1^{s\alpha} a^{\alpha} + \lambda_2^{s\alpha} f(v) \\ &= \lambda_1^{s\alpha} f(0) + \lambda_2^{s\alpha} f(v). \end{aligned} \tag{51}$$

So, $f \in GK_s^1$.

(ii) From Theorem 10, we can see that property (ii) is true.

(iii) Let $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 = 1$. Similar to the estimate of (i), there are also two cases.

Let $v, v > 0$. Then $\lambda_1 u + \lambda_2 v > 0$,

Thus,

$$\begin{aligned} &f\left(\lambda_1 u + \lambda_2 v\right) \\ &\quad = b^{\alpha}\left(\lambda_1 u + \lambda_2 v\right)^{s\alpha} + c^{\alpha} \\ &\quad < b^{\alpha}\left(\lambda_1^{s\alpha} u^{s\alpha} + \lambda_2^{s\alpha} v^{s\alpha}\right) + c^{\alpha}\left(\lambda_1^{\alpha} + \lambda_2^{\alpha}\right) \\ &\quad \leq b^{\alpha}\left(\lambda_1^{s\alpha} u^{s\alpha} + \lambda_2^{s\alpha} v^{s\alpha}\right) + c^{\alpha}\left(\lambda_1^{s\alpha} + \lambda_2^{s\alpha}\right) \\ &\quad = \lambda_1^{s\alpha}\left(b^{\alpha} u^{s\alpha} + c^{\alpha}\right) + \lambda_2^{s\alpha}\left(b^{\alpha} v^{s\alpha} + c^{\alpha}\right) \\ &\quad \leq \lambda_1^{s\alpha} f(u) + \lambda_2^{s\alpha} f(v). \end{aligned} \tag{52}$$

Let $v > u = 0$. We need only to consider $\lambda_2 > 0$.

Thus, we have

$$\begin{aligned} f(\lambda_1 0 + \lambda_2 v) &= f(\lambda_2 v) \\ &= b^\alpha \lambda_2^{s\alpha} v^{s\alpha} + c^\alpha (\lambda_1^\alpha + \lambda_2^\alpha) \\ &< b^\alpha \lambda_2^{s\alpha} v^{s\alpha} + c^\alpha (\lambda_1^{s\alpha} + \lambda_2^{s\alpha}) \\ &= \lambda_2^{s\alpha} (b^\alpha v^{s\alpha} + c^\alpha) + c^\alpha \lambda_1^{s\alpha} \\ &\le \lambda_2^{s\alpha} (b^\alpha v^{s\alpha} + c^\alpha) + a^\alpha \lambda_1^{s\alpha} \\ &= \lambda_2^{s\alpha} f(v) + \lambda_1^{s\alpha} f(0). \end{aligned} \tag{53}$$

So, $f \in GK_s^2$.

(iv) Assume that $f \in GK_s^2$, and then f is nonnegative on $(0, \infty)$. On the other hand, we can take $u_1 > 0$, $c_1 < 0$ such that $f(u_1) = b^\alpha u_1^{s\alpha} + c_1^\alpha < 0^\alpha$, which contradict the assumption. □

Example 2. Let $0 < s < 1$ and $k > 1$. For $u \in R_+$, define

$$f(u) = \begin{cases} u^{(s/(1-s))\alpha}, & 0 \le u \le 1, \\ k^\alpha u^{(s/(1-s))\alpha}, & u > 1. \end{cases} \tag{54}$$

The function f is nonnegative, not local fractional continuous at $u = 1$ and belongs to GK_s^1 but not to GK_s^2.

Proof. From Theorem 16, we have that $f \in GK_s^1$. In the following, let us show that f is not in GK_s^2.

Take an arbitrary $a > 1$ and put $u = 1$. Consider all $v > 1$ such that $\lambda_1 u + \lambda_2 v = \lambda_1 + \lambda_2 v = a$, where $\lambda_1, \lambda_2 \ge 0$ and $\lambda_1 + \lambda_2 = 1$.

If $f \in GK_s^2$, it must be

$$\begin{aligned} & k^\alpha a^{(s/(1-s))\alpha} \\ & \quad \le \lambda_1^{s\alpha} + k^\alpha (1-\lambda_1)^{s\alpha} \left[\frac{(a-\lambda_1)}{(1-\lambda_1)} \right]^{(s/(1-s))\alpha}, \end{aligned} \tag{55}$$

for all $a > 1$ and all $0 \le \lambda_1 \le 1$.

Define the function

$$\begin{aligned} f_{\lambda_1}(a) = {} & \lambda_1^{s\alpha} + k^\alpha (1-\lambda_1)^{s\alpha} \left[\frac{(a-\lambda_1)}{(1-\lambda_1)} \right]^{(s/(1-s))\alpha} \\ & - k^\alpha a^{(s/(1-s))\alpha}. \end{aligned} \tag{56}$$

Then the function is local fractional continuous on the (λ_1, ∞) and

$$g(\lambda_1) = f_{\lambda_1}(1) = \lambda_1^{s\alpha} + k^\alpha (1-\lambda_1)^{s\alpha} - k^\alpha. \tag{57}$$

It is easy to see that g is local fractional continuous on $[0, 1]$ and $g(1) = 1^\alpha - k^\alpha < 0^\alpha$. So there is a number λ_{1_0}, $0 < \lambda_{1_0} < 1$, such that $g(\lambda_{1_0}) = f_{\lambda_{1_0}}(1) < 0^\alpha$. The local fractional continuity of $f_{\lambda_{1_0}}$ yields that $f_{\lambda_{1_0}}(a) < 0^\alpha$ for a certain $a > 1$, that is, inequality (55) does not hold, which means that $f \notin GK_s^2$. □

5. Conclusion

In the paper, we introduce the definitions of two kinds of generalized s-convex function on fractal sets and study the properties of these generalized s-convex functions. When $\alpha = 1$, these results are the classical situation.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

A Jacobi-Collocation Method for Second Kind Volterra Integral Equations with a Smooth Kernel

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The purpose of this paper is to provide a Jacobi-collocation method for solving second kind Volterra integral equations with a smooth kernel. This method leads to a fully discrete integral operator. First, it is shown that the fully discrete integral operator is stable in both L^∞ and weighted L^2 norms. Then, the proposed approach is proved to arrive at an optimal (the most possible) convergent order in both norms. One numerical example demonstrates the efficiency and accuracy of the proposed method.

1. Introduction

In this paper, we provide a Jacobi-collocation approach for solving the second kind Volterra integral equation of the form

$$u(x) + \int_{-1}^{x} k(x,t)\,u(t)\,dt = f(x), \quad x \in I := [-1,1], \quad (1)$$

where the kernel function k and the input function f are given smooth functions about their variables and u is the unknown function to be determined.

For ease of analysis, we will write (1) into an operator form. By introducing the integral operator $\mathscr{K}$ by

$$(\mathscr{K}v)(x) := \int_{-1}^{x} k(x,t)\,v(t)\,dt, \quad x \in I, \quad (2)$$

(1) is reformulated as

$$(\mathscr{I} + \mathscr{K})\,u = f. \quad (3)$$

It is well known that there are many numerical methods for solving second kind Volterra integral equations such as the Runge-Kutta method and the collocation method based on piecewise polynomials; see, for example, Brunner [1] and references therein. For more information of the progress on the study of the problem, we refer the readers to [2–8]. Recently, a few works touched the spectral approximation to Volterra integral equations. In [9], Elnagar and Kazemi provided a novel Chebyshev spectral method for solving nonlinear Volterra-Hammerstein integral equations. Then, this method was investigated by Fujiwara in [10] for solving the first kind Fredholm integral equation under multiple-precision arithmetic. Nevertheless, no theoretical results were provided to justify the high accuracy. In [11], Tang et al. developed a novel Legendre-collocation method for solving (3). Inspired by the work of [11], Chen and Tang in [5, 12] obtained the spectral Jacobi-collocation method for solving the second kind Volterra integral equations with general weakly singular kernels $k(x,t)(x-t)^{-\mu}$ for $-1 < \mu < 0$. In [13], a spectral and pseudospectral Jacobi-Galerkin approach was presented for solving (3). In [14], Wei and Chen considered a spectral Jacobi-collocation method for solving Volterra type integrodifferential equation. In [15], Cai considered a Jacobi-collocation method for solving Fredholm integral equations of second kind with weakly singular kernels.

Unfortunately, all these papers [5, 11–14] give the convergence analysis but suffer from the stability analysis. Because of lack of the stability analysis, the approximate solutiondoes not attain the most possible convergence order. Moreover, all of those papers do not answer that the approximate equation has a unique solution. Hence, in this paper, we will provide a Jacobi-collocation method for solving (3), which extends the Legendre spectral method developed in [11]. This spectral method leads to a fully discrete linear system. We are going to show that the fully discrete integral operator is stabile; that is, the approximate equation has a unique solution, and then, present the optimal (the most possible) convergent order of the approximate solution based on the stability analysis. We organize this paper as follows. In Section 2, as demonstrated in [13], we review a spectral Jacobi-collocation method for solving (3). In Section 3, a few important results are presented to analyze the Jacobi-collocation approach. In Sections 4 and 5, we analyze the Jacobi-collocation method, including the stability of the approximate equation and the convergent order of the approximate solution, in both L^{∞} and weighted L^2 norms, respectively. In Section 6, one numerical example is presented to show the efficiency and accuracy of this method.

The problem under study deserves more investigations in future works. Moreover, we believe that the semianalytical approaches are useful to investigate the problem. For related terminologies and applications of semianalytical approaches, please refer to [16–18].

2. A Spectral Jacobi-Collocation Method

In this section, we are going to review the spectral Jacobi-collocation method for solving (3). To this end, we introduce several index sets: $\mathbb{N} := \{1, 2, \ldots, n, \ldots\}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{Z}_n := \{0, 1, 2, \ldots, n\}$. We let $w^{\alpha,\beta}(x) := (1-x)^{\alpha}(1+x)^{\beta}$ for $\alpha, \beta > -1$ be a weight function and then use the notation $L^2_{w^{\alpha,\beta}}(I)$ to be the set of all square integrable functions associated with the weight function $w^{\alpha,\beta}$, equipped with the norm

$$\|v\|_{w^{\alpha,\beta}} := \left(\int_I w^{\alpha,\beta}(t) v^2(t) dt\right)^{1/2}. \quad (4)$$

For $n \in \mathbb{N}$, we denote the points by $x_i^{\alpha,\beta}$, $i \in \mathbb{Z}_n$ to be the set of $n+1$ Jacobi-Gauss points corresponding to the Jacobi weight function $w^{\alpha,\beta}$. By introducing

$$\pi(x) := \left(x - x_0^{\alpha,\beta}\right)\left(x - x_1^{\alpha,\beta}\right)\cdots\left(x - x_n^{\alpha,\beta}\right), \quad (5)$$

we define the Lagrange fundamental interpolation polynomial $L_i^{\alpha,\beta}$, $i \in \mathbb{Z}_n$ by

$$L_i^{\alpha,\beta}(x) := \frac{\pi(x)}{\left(x - x_i^{\alpha,\beta}\right)\pi'\left(x_i^{\alpha,\beta}\right)}, \quad x \in I. \quad (6)$$

Let P_n be the set of all polynomials of degree not more than n; clearly,

$$P_n = \operatorname{span}\left\{L_i^{\alpha,\beta} : i \in \mathbb{Z}_n\right\}. \quad (7)$$

We use the notion $C(I)$ to denote the set of all continuous functions on I, equipped with the norm

$$\|v\|_{\infty} := \max_{x \in I} |v(x)|. \quad (8)$$

For $s \in I$, we define a linear functional δ_s on $C(I)$ such that, for any $v \in C(I)$,

$$\langle \delta_s, v \rangle := v(s). \quad (9)$$

The collocation method for solving (3) is to seek a vector $\mathbf{u} := [a_i : i \in \mathbb{Z}_n]^T$ such that

$$u_n(x) := \sum_{i \in \mathbb{Z}_n} a_i L_i^{\alpha,\beta}(x), \quad x \in I \quad (10)$$

satisfies

$$\left\langle \delta_{x_j^{\alpha,\beta}}, (\mathcal{I} + \mathcal{K}) u_n \right\rangle = \left\langle \delta_{x_j^{\alpha,\beta}}, f \right\rangle, \quad j \in \mathbb{Z}_n. \quad (11)$$

The above equation can be rewritten as

$$a_i + \sum_{j \in \mathbb{Z}_n} a_j \int_{-1}^{x_i^{\alpha,\beta}} k\left(x_i^{\alpha,\beta}, t\right) L_j^{\alpha,\beta}(t)\, dt = f\left(x_i^{\alpha,\beta}\right), \quad i \in \mathbb{Z}_n. \quad (12)$$

For $v \in C(I)$, we define the interpolating operator $\mathcal{L}_n^{\alpha,\beta} : C(I) \to P_n$ by

$$\left(\mathcal{L}_n^{\alpha,\beta} v\right)\left(x_i^{\alpha,\beta}\right) = v\left(x_i^{\alpha,\beta}\right), \quad i \in \mathbb{Z}_n. \quad (13)$$

It it well known that $\mathcal{L}_n^{\alpha,\beta} v$ is written as the form

$$\left(\mathcal{L}_n^{\alpha,\beta} v\right)(x) = \sum_{i \in \mathbb{Z}_n} v(x_i) L_i^{\alpha,\beta}(x), \quad x \in I. \quad (14)$$

Using these notations we can reformulate (12) into an operator form

$$\left(\mathcal{I} + \mathcal{L}_n^{\alpha,\beta} \mathcal{K}\right) u_n = \mathcal{L}_n^{\alpha,\beta} f. \quad (15)$$

The difficulty in solving the linear system (12) is to compute the integral term in (12), accurately. In this paper, we adopt the numerical integration rule proposed in [11] to overcome this difficulty. For this purpose, we introduce a simple linear transformation

$$t = g(x, \tau) := \frac{x+1}{2}\tau + \frac{x-1}{2}, \quad (16)$$

which transfers the integral operator $\mathcal{K}$ into the following form:

$$(\mathcal{K} v)(x) = \frac{x+1}{2} \int_I k(x, g(x, \tau))\, v(g(x, \tau))\, d\tau. \quad (17)$$

Then, by using $N+1$-point Legendre-Gauss quadrature formula relative to the Legendre weight $w_i, i \in \mathbb{Z}_N$, we can obtain the discrete integral operator $\mathcal{K}_N$ as follows:

$$(\mathcal{K}_N v)(x) := \frac{x+1}{2} \sum_{i \in \mathbb{Z}_N} w_i k\left(x, g\left(x, x_i^{0,0}\right)\right) v\left(g\left(x, x_i^{0,0}\right)\right). \quad (18)$$

Thus, using those notations, a fully discrete spectral Jacobi-collocation method for solving (3) is to seek a vector $\widetilde{\mathbf{u}} := [\widetilde{a}_i : i \in \mathbb{Z}_n]^T$ such that

$$\widetilde{u}_n(x) := \sum_{i \in \mathbb{Z}_n} \widetilde{a}_i L_i^{\alpha,\beta}(x), \quad x \in I, \quad (19)$$

satisfying

$$\left(\mathcal{I}+\mathcal{L}_n^{\alpha,\beta}\mathcal{K}_N\right)\widetilde{u}_n=\mathcal{L}_n^{\alpha,\beta}f. \tag{20}$$

It is easy to show that the operator equation (20) has the following form:

$$\begin{aligned}&\widetilde{a}_i+\frac{x_i^{\alpha,\beta}+1}{2}\\&\quad\times\sum_{j\in\mathbb{Z}_n}\widetilde{a}_j\sum_{l\in\mathbb{Z}_N}w_l k\left(x_i^{\alpha,\beta},g\left(x_i^{\alpha,\beta},x_l^{0,0}\right)\right)L_j^{\alpha,\beta}\left(g\left(x_i^{\alpha,\beta},x_l^{0,0}\right)\right)\\&=f\left(x_i^{\alpha,\beta}\right),\quad i\in\mathbb{Z}_n.\end{aligned} \tag{21}$$

In [11], for the case $N = n$, based on the Gronwall' inequality, Tang et al. analyze the convergence of a spectral Jacobi-collocation method for solving (3) in both $C(I)$ and weighted $L^2_{w^{0,0}}(I)$ spaces. However, the stability analysis of the spectral method is not given. Moreover, we observe that the convergence order of the approximate solution in the space $L^2_{w^{0,0}}(I)$ is not optimal. Hence, the purpose of this paper is to illustrate that for sufficiently large n and N, the operator $\mathcal{I}+\mathcal{L}_n^{\alpha,\beta}\mathcal{K}_N : P_n \to P_n$ has a uniformly bounded inversion in both $C(I)$ and $L^2_{w^{\alpha,\beta}}(I)$ spaces, respectively. Moreover, we also show that the approximate solution $\widetilde{u}_n$ attains at the most possible convergent order.

3. Some Preliminaries and Useful Results

In this section, we will introduce some technical results, which contribute to analyze the stability and convergence on the spectral Jacobi-collocation method for solving (3). To this end, for $i \in \mathbb{N}_0$, we use the notation $\mathcal{D}_x^i$ to denote the ith differential operator on the variable x. For $r \in \mathbb{N}$, we introduce the nonuniformly weighted Sobolev space $H^r_{w^{\alpha,\beta}}(I)$ by

$$H^r_{w^{\alpha,\beta}}(I):=\left\{v:\mathcal{D}_x^i v\in L^2_{w^{\alpha+i,\beta+i}}(I),i\in\mathbb{Z}_r\right\}. \tag{22}$$

It follows from [19] that there exists a positive constant γ_1 independent of n such that, for $v\in H^r_{w^{\alpha,\beta}}(I)$ and $i\in\mathbb{Z}_r$,

$$\left\|\mathcal{D}_x^i\left(v-\mathcal{L}_n^{\alpha,\beta}v\right)\right\|_{w^{\alpha+i,\beta+i}}\le\gamma_1\left\|\mathcal{D}_x^r v\right\|_{w^{\alpha+r,\beta+r}}n^{i-r}, \tag{23}$$

which implies that

$$\left\|\mathcal{D}_x^i\left(\mathcal{L}_n^{\alpha,\beta}v\right)\right\|_{w^{\alpha+i,\beta+i}}\le\gamma_1\left\|\mathcal{D}_x^r v\right\|_{w^{\alpha+r,\beta+r}}+\left\|\mathcal{D}_x^i v\right\|_{w^{\alpha+i,\beta+i}}. \tag{24}$$

Moreover, we have the following.

Lemma 1. *Suppose that $-1<\alpha_1,\beta_1<1$. If the parameters α,β satisfy the next conditions:*

$$-\frac{1}{2}+\alpha_1<\alpha<\frac{3}{2}+\alpha_1,\qquad -\frac{1}{2}+\beta_1<\beta<\frac{3}{2}+\beta_1, \tag{25}$$

then there exists a positive constant γ_2 independent of n such that, for $v\in H^r_{w^{\alpha,\beta}}(I)\cap C(I)$,

$$\left\|v-\mathcal{L}_n^{\alpha_1,\beta_1}v\right\|_{w^{\alpha,\beta}}\le\gamma_2\left\|\mathcal{D}_x^r v\right\|_{w^{\alpha+r,\beta+r}}n^{-r}. \tag{26}$$

Proof. This is a consequence of Theorem 3.4 and (3.13)-(3.14) in [20]. □

For $r\in\mathbb{N}$, $i\in\mathbb{Z}_r$, the binomial coefficients are given by

$$C_r^i:=r(r-1)\cdots(r-i+1). \tag{27}$$

We use the notation $C^r(I)$ to denote the set of all functions whose rth derivative is continuous on I, endowed with the usual norm

$$\|v\|_r:=\sum_{i\in\mathbb{Z}_r}\left\|\mathcal{D}_x^i v\right\|_\infty. \tag{28}$$

For $r_1,r_2\in\mathbb{N}_0$, the notation $C^{r_1,r_2}(I^2)$ is used to denote the set of all functions such that, for $v\in C^{r_1,r_2}(I^2)$, $\mathcal{D}_x^{r_1}\mathcal{D}_y^{r_2}v$ is continuous on I^2. Let

$$\left\|\mathcal{D}_x^{r_1}\mathcal{D}_y^{r_2}v\right\|_\infty:=\max_{(x,y)\in I^2}\left|\mathcal{D}_x^{r_1}\mathcal{D}_y^{r_2}v(x,y)\right|. \tag{29}$$

Next we consider the difference between $\mathcal{K}v$ and $\mathcal{K}_N v$.

Lemma 2. *Assume that the kernel function $k\in C^{0,m}(I^2)$ for $m\in\mathbb{N}$. If two parameters α and β satisfy the conditons*

$$-\frac{1}{2}<\alpha,\beta<\frac{3}{2},\quad \alpha+\beta\le 1, \tag{30}$$

then there exists a positive constant γ_3 independent of N such that when $v\in C^m(I)$,

$$\left\|\mathcal{K}v-\mathcal{K}_N v\right\|_\infty\le\gamma_3\left(\sum_{i\in\mathbb{Z}_m}\left\|\mathcal{D}_x^i v\right\|_{w^{\alpha+i,\beta+i}}\right)N^{-m}. \tag{31}$$

Proof. First of all, by setting

$$\begin{aligned}&b(x,\tau):=k(x,g(x,\tau))\,v(g(x,\tau)),\\&\left(\mathcal{L}_N^{0,0}b\right)(x,\tau):=\sum_{i\in\mathbb{Z}_N}b\left(x,x_i^{0,0}\right)L_i^{0,0}(\tau),\quad x,\tau\in I,\end{aligned} \tag{32}$$

the integral operator $\mathcal{K}_N$ is written as

$$\left(\mathcal{K}_N v\right)(x)=\frac{x+1}{2}\int_I\left(\mathcal{L}_N^{0,0}b\right)(x,\tau)\,d\tau,\quad x\in I. \tag{33}$$

In addition, using the hypothesis that $k\in C^{0,m}(I^2)$ and $v\in C^m(I)$ implies that $b\in C^{0,m}(I^2)$. Thus, we write the difference between $\mathcal{K}v$ and $\mathcal{K}_N v$ as follows:

$$\begin{aligned}&(\mathcal{K}v)(x)-\left(\mathcal{K}_N v\right)(x)\\&=\frac{x+1}{2}\int_I\left(w^{-\alpha/2,-\beta/2}(\tau)\right)\\&\qquad\times\left(w^{\alpha/2,\beta/2}(\tau)\left(b(x,\tau)-\left(\mathcal{L}_N^{0,0}b\right)(x,\tau)\right)\right)d\tau.\end{aligned} \tag{34}$$

Employing Cauchy-Schwartz inequality to the right hand side of the above equation and then using the result (26) with

$\alpha_1 := 0, \beta_1 := 0, n := N$ and $r := m$ produce that there exists a positive constant ξ independent of N,

$$\left|(\mathscr{K}v)(x) - (\mathscr{K}_N v)(x)\right|^2 \le \xi G(x) \|1\|_{w^{-\alpha,-\beta}} N^{-2m}, \tag{35}$$

where $G(x)$ is given by

$$G(x) := \left(\frac{1+x}{2}\right)^2 \int_I w^{\alpha+m,\beta+m}(\tau)\left(\left(\mathscr{D}_\tau^m b\right)(x,\tau)\right)^2 d\tau, \quad x \in I. \tag{36}$$

It remains to estimate $G(x)$. A direct computation leads to

$$G(x) = \left(\frac{1+x}{2}\right)^{2m+2} \times \int_I w^{\alpha+m,\beta+m}(\tau) \times \left(\sum_{i\in\mathbb{Z}_m} \mathrm{C}_m^i (\mathscr{D}_g^i v)(g(x,\tau))(\mathscr{D}_g^{m-i} k)(x, g(x,\tau))\right)^2 d\tau. \tag{37}$$

Making use of a linear transform $t := g(x,\tau)$ to the right hand side of the above equation produces

$$G(x) = \left(\frac{1+x}{2}\right)^{1-\alpha-\beta} \times \int_{-1}^{x} (1+t)^{\alpha+m}(x-t)^{\beta+m} \times \left(\sum_{i\in\mathbb{Z}_m} \mathrm{C}_m^i (\mathscr{D}_t^i v)(t)(\mathscr{D}_t^{m-i} k)(x,t)\right)^2 dt. \tag{38}$$

Using the discrete Cauchy Schwartz inequality into the right hand side of the above equation obtains

$$G(x) \le \max_{i\in\mathbb{Z}_m} \left\|\mathscr{D}_t^i k\right\|_\infty^2 \left(\sum_{i\in\mathbb{Z}_m} \left(\mathrm{C}_n^i\right)^2\right)\left(\frac{1+x}{2}\right)^{1-\alpha-\beta} \times \left(\sum_{i\in\mathbb{Z}_m} \int_{-1}^{x} (1+t)^{\alpha+m}(x-t)^{\beta+m}\left(\mathscr{D}_t^i v\right)^2(t)\, dt\right), \tag{39}$$

where combining the fact that $\alpha + \beta \le 1$ leads to

$$G(x) \le \max_{i\in\mathbb{Z}_m} \left\|\mathscr{D}_t^i k\right\|_\infty^2 \left(\sum_{i\in\mathbb{Z}_m} \left(\mathrm{C}_n^i\right)^2\right)\left(\sum_{i\in\mathbb{Z}_m} \left\|\mathscr{D}_t^i v\right\|_{w^{\alpha+i,\beta+i}}^2\right). \tag{40}$$

Substituting the above estimate on $G(x)$ into the right hand side of (35) yields the desired conclusion (31) with γ_3 being given by

$$\gamma_3 := \max_{i\in\mathbb{Z}_m} \left\|\mathscr{D}_t^i k\right\|_\infty \left(\xi \|1\|_{w^{-\alpha,-\beta}} \sum_{i\in\mathbb{Z}_m} \left(\mathrm{C}_m^i\right)^2\right)^{1/2}. \tag{41}$$

□

Using Lemma 2, we can obtain the following.

Corollary 3. *Suppose that the conditions of Lemma 2 hold, then for $v \in P_n$, the following two estimates hold:*

$$\left\|\mathscr{K}v - \mathscr{K}_N v\right\|_\infty \le 2\gamma_3 \|v\|_{w^{\alpha,\beta}} \left(\frac{n}{N}\right)^m, \tag{42}$$

$$\left\|\mathscr{K}v - \mathscr{K}_N v\right\|_\infty \le 2\gamma_3 \|1\|_{w^{\alpha,\beta}} \|v\|_\infty \left(\frac{n}{N}\right)^m. \tag{43}$$

Proof. We observe that if (42) holds, then by using the fact

$$\|v\|_{w^{\alpha,\beta}} \le \|1\|_{w^{\alpha,\beta}} \|v\|_\infty, \quad v \in P_n, \tag{44}$$

we can easily obtain the result (43). Thus, we only require to estimate (42). In fact, by using the inverse inequality relative to two norms weighted with different Jacobi weight functions in Theorem 3.31 in [19], there exists a positive constant ξ independent of n such that, for $v \in P_n$ and $i \in \mathbb{Z}_m$,

$$\left\|\mathscr{D}_t^i v\right\|_{w^{\alpha+i,\beta+i}} \le \xi n \left\|\mathscr{D}_t^{i-1} v\right\|_{w^{\alpha+i-1,\beta+i-1}}. \tag{45}$$

By the above inequality, we can obtain that

$$\sum_{i\in\mathbb{Z}_m} \left\|\mathscr{D}_t^i v\right\|_{w^{\alpha+i,\beta+i}} \le \sum_{i\in\mathbb{Z}_m} (\xi n)^i \|v\|_{w^{\alpha,\beta}}, \tag{46}$$

where combining (31) yields the desired conclusion (42). □

4. The Stability and Convergence Analysis under the L^∞ Norm

In this section, we will establish that, for sufficiently large n and N, the operator $\mathscr{I} + \mathscr{L}_n^{\alpha,\beta}\mathscr{K}_N : P_n \to P_n$ has a uniformly bounded inversion in the space $C(I)$ and then show that the approximate solution $\tilde{u}_n$ arrives at the most possible convergent order under the L^∞ norm. To this end, we first give some notations. For $r \in \mathbb{N}_0$ and $\nu \in (0,1]$, the notation $H^{r,\nu}(I)$ is used to denote the space of functions whose rth derivative is Hölder continuous on I with exponent ν. The norm of the space is defined by

$$\|v\|_{r,\nu} := \|v\|_r + \sup_{x,y\in I, x\ne y} \frac{\left|\left(\mathscr{D}_x^r v\right)(x) - \left(\mathscr{D}_y^r v\right)(y)\right|}{|x-y|^\nu}. \tag{47}$$

Lemma 4. *Suppose that the kernel function $k \in C^{1,0}(I^2)$; then the operator $\mathscr{K}$ is a bounded linear operator from $C(I)$ to $H^{0,1}(I)$; that is, for $v \in C(I)$,*

$$\|\mathscr{K}v\|_{0,1} \le \left(4\|k\|_\infty + 2\left\|\mathscr{D}_x^1 k\right\|_\infty\right)\|v\|_\infty. \tag{48}$$

Moreover, for $-1 < \alpha, \beta < 1$, the operator $\mathscr{K} : L^2_{w^{\alpha,\beta}}(I) \to C(I)$ is also a linear bounded operator; that is, for $v \in L^2_{w^{\alpha,\beta}}(I)$,

$$\|\mathscr{K}v\|_\infty \leq \|k\|_\infty \|1\|^{1/2}_{w^{-\alpha,-\beta}} \|v\|_{w^{\alpha,\beta}}. \tag{49}$$

Proof. It is easily proved that the operator $\mathscr{K}$ is a linear operator from the space $C(I)$ to the space $H^{0,1}(I)$ or from $L^2_{w^{\alpha,\beta}}(I)$ to $C(I)$.

Next we illustrate that (48) holds. By the definition of the norm,

$$\|\mathscr{K}v\|_\infty \leq \|v\|_\infty \max_{x\in I} \int_{-1}^{x} |k(x,t)|\, dt, \tag{50}$$

which implies that

$$\|\mathscr{K}v\|_\infty \leq 2\|k\|_\infty \|v\|_\infty. \tag{51}$$

On the other hand, for all $x_1, x_2 \in I$, by introducing

$$\begin{aligned} I_1 &:= \int_{-1}^{x_1} \left(k(x_1,t) - k(x_2,t)\right) v(t)\, dt, \\ I_2 &:= \int_{x_2}^{x_1} k(x_2,t)\, v(t)\, dt, \end{aligned} \tag{52}$$

we can obtain that

$$(\mathscr{K}v)(x_1) - (\mathscr{K}v)(x_2) = I_1 + I_2, \tag{53}$$

where using the triangle inequality yields that

$$\left|(\mathscr{K}v)(x_1) - (\mathscr{K}v)(x_2)\right| \leq |I_1| + |I_2|. \tag{54}$$

The left thing is to give an estimation of I_1 and I_2. First, employing Lagrange midvalue differential theorem to I_1 yields that

$$|I_1| \leq 2\left\|\mathscr{D}^1_x k\right\|_\infty \|v\|_\infty |x_1 - x_2|. \tag{55}$$

A direct estimation for I_2 produces that

$$|I_2| \leq 2\|k\|_\infty \|v\|_\infty |x_1 - x_2|. \tag{56}$$

Thus, substituting the estimates (55)-(56) into the right hand side of (54) leads that

$$\begin{aligned} &\left|(\mathscr{K}v)(x_1) - (\mathscr{K}v)(x_2)\right| \\ &\quad \leq 2\left(\|k\|_\infty + \left\|\mathscr{D}^1_x k\right\|_\infty\right) \|v\|_\infty |x_1 - x_2|, \end{aligned} \tag{57}$$

where (51) yields the desired conclusion (48).

In the following we show that the result (49) holds. Noticing,

$$\begin{aligned} &\|\mathscr{K}v\|_\infty \\ &\quad = \max_{x\in I} \left|\int_{-1}^{x} k(x,t)\, v(t)\, dt\right| \\ &\quad = \max_{x\in I} \left|\int_{-1}^{x} \left(w^{-\alpha/2,-\beta/2}(t)\, k(x,t)\right)\left(w^{\alpha/2,\beta/2}(t)\, v(t)\right) dt\right|. \end{aligned} \tag{58}$$

Using Cauchy-Schwartz inequality to the right hand side of the equation above yields

$$\|\mathscr{K}v\|^2_\infty \leq \max_{x\in I} \int_{-1}^{x} \left(w^{-\alpha,-\beta}(t)\, k^2(x,t)\right) \|v\|^2_{w^{\alpha,\beta}}, \tag{59}$$

which implies that

$$\|\mathscr{K}v\|^2_\infty \leq \|k\|^2_\infty \|1\|_{w^{-\alpha,-\beta}} \|v\|^2_{w^{\alpha,\beta}}. \tag{60}$$

This complete the proof of (49). □

The next result concerns on the bound of the norm $\|\mathscr{K}v - \mathscr{L}^{\alpha,\beta}_n \mathscr{K}v\|_\infty$ for $v \in C(I)$. For this purpose, we introduce the result on the Lebesgue constant corresponding to the Lagrange interpolation polynomials associated with the zeros of the Jacobi polynomials, which comes from Lemma 3.4 in [5]:

$$\begin{aligned} \left\|\mathscr{L}^{\alpha,\beta}_n\right\|_\infty &= \max_{\|v\|_\infty=1} \left\|\mathscr{L}^{\alpha,\beta}_n v\right\|_\infty \\ &= \begin{cases} \mathscr{O}(\log n), & -1 < \alpha, \beta \leq -\dfrac{1}{2}, \\ \mathscr{O}\left(n^{(1/2)+\max\{\alpha,\beta\}}\right), & \text{otherwise}. \end{cases} \end{aligned} \tag{61}$$

Further, we also require to make use of another result of Ragozin, coming from [21, 22], which states that, for any $v \in H^{r,\nu}(I)$, there exist a polynomial $q \in P_n$ and a positive constant ς_1 such that

$$\|v - q\|_\infty \leq \varsigma_1 n^{-r-\nu} \|v\|_{r,\nu}. \tag{62}$$

A combination of (61) and (62) leads to that there exists a positive constant ς_2 such that

$$\begin{aligned} &\left\|v - \mathscr{L}^{\alpha,\beta}_n v\right\|_\infty \\ &\quad \leq \varsigma_2 \|v\|_{r,\nu} \begin{cases} n^{-r-\nu} \log n, & -1 < \alpha, \beta \leq -\dfrac{1}{2}, \\ n^{(1/2)+\max\{\alpha,\beta\}-r-\nu}, & \text{otherwise}. \end{cases} \end{aligned} \tag{63}$$

Lemma 5. *Suppose that the kernel function $k \in C^{1,0}(I^2)$. Then there exists a positive constant ς_3 independent of n such that when $v \in C(I)$,*

$$\begin{aligned} &\left\|\mathscr{K}v - \mathscr{L}^{\alpha,\beta}_n \mathscr{K}v\right\|_\infty \\ &\quad \leq \varsigma_3 \|v\|_\infty \begin{cases} n^{-1} \log n, & -1 < \alpha, \beta \leq -\dfrac{1}{2}, \\ n^{-(1/2)+\max\{\alpha,\beta\}}, & \textit{otherwise}. \end{cases} \end{aligned} \tag{64}$$

Proof. It follows from Lemma 4 that $\mathscr{K}v \in H^{0,1}(I)$ for $v \in C(I)$, where combining (63) obtains that there exists a positive constant ξ independent of n:

$$\begin{aligned} &\left\|\mathscr{K}v - \mathscr{L}^{\alpha,\beta}_n \mathscr{K}v\right\|_\infty \\ &\quad \leq \xi \|\mathscr{K}v\|_{0,1} \begin{cases} n^{-1} \log n, & -1 < \alpha, \beta \leq -\dfrac{1}{2}, \\ n^{-(1/2)+\max\{\alpha,\beta\}}, & \text{otherwise}. \end{cases} \end{aligned} \tag{65}$$

Substituting the estimate (48) into the right hand side of the above equation yields the desired conclusion with ς_3 being given by

$$\varsigma_3 := \xi \left(4\|k\|_\infty + 2\left\|\mathscr{D}_x^1 k\right\|_\infty\right). \tag{66}$$

□

We use the notation $[x]$ to denote the largest integer not more than x. Moreover, by Theorem 3.10 in [23], if the kernel function k is a smooth function, the operator $\mathscr{I}+\mathscr{K} : C(I) \to C(I)$ has a bounded inversion; that is, for any $v \in P_n$, there exists a positive constant ρ such that

$$\|(\mathscr{I} + \mathscr{K})\, v\|_\infty \geq \rho\|v\|_\infty. \tag{67}$$

Theorem 6. *Suppose that $k \in C^{1,m}(I^2)$, $-1/2 < \alpha, \beta < 1/2$. If we choose N as follows:*

$$N \geq N_{\min} := \left[n^{1+(1/2m)+(\min\{\alpha,\beta\}/m)}\log^{1/m} n\right] + 1, \tag{68}$$

then there exists a positive integer n_0 such that when $n \geq n_0$ and for $v \in P_n$,

$$\left\|(\mathscr{I} + \mathscr{L}_n^{\alpha,\beta}\mathscr{K}_N)v\right\|_\infty \geq \frac{\rho}{2}\|v\|_\infty, \tag{69}$$

where ρ appears in (67).

Proof. It follows from the hypothesis that $-1/2 < \alpha, \beta < 1/2$ that $n^{-(1/2)+\max\{\alpha,\beta\}}$ tends to zero as n tends to ∞. Hence, using (64) there exists a positive integer n_1 such that $n \geq n_1$,

$$\left\|\mathscr{K}v - \mathscr{L}_n^{\alpha,\beta}v\right\|_\infty \leq \frac{\rho}{4}\|v\|_\infty. \tag{70}$$

On the other hand, using (61) with the hypothesis that $-1/2 < \alpha, \beta < 1/2$ yields that there exists a positive constant ξ_1 such that, for $v \in P_n$,

$$\left\|\mathscr{L}_n^{\alpha,\beta}\mathscr{K}v - \mathscr{L}_n^{\alpha,\beta}\mathscr{K}_N v\right\|_\infty \leq \xi_1\left\|\mathscr{K}v - \mathscr{K}_N v\right\|_\infty n^{(1/2)+\max\{\alpha,\beta\}}, \tag{71}$$

where combining (43) and (68) produces that there exists a positive constant ξ_2,

$$\left\|\mathscr{L}_n^{\alpha,\beta}\mathscr{K}v - \mathscr{L}_n^{\alpha,\beta}\mathscr{K}_N v\right\|_\infty \leq \xi_2\|v\|_\infty \log^{-1} n. \tag{72}$$

Similarly as before, by the fact that $\log^{-1} n$ tends to 0 as n tends to ∞, there exists a positive integer n_2 such that, for $n \geq n_2$,

$$\left\|\mathscr{L}_n^{\alpha,\beta}\mathscr{K}v - \mathscr{L}_n^{\alpha,\beta}\mathscr{K}_N v\right\|_\infty \leq \frac{\rho}{4}\|v\|_\infty. \tag{73}$$

Hence, when $n \geq n_0 := \max\{n_1, n_2\}$, combining these three estimates (67), (70), and (73) yields that

$$\begin{aligned}
&\left\|(\mathscr{I} + \mathscr{L}_n^{\alpha,\beta}\mathscr{K}_N)v\right\|_\infty \\
&\quad\geq \|v + \mathscr{K}v\|_\infty - \left\|\mathscr{K}v - \mathscr{L}_n^{\alpha,\beta}\mathscr{K}v\right\|_\infty \\
&\quad\quad - \left\|\mathscr{L}_n^{\alpha,\beta}\mathscr{K}v - \mathscr{L}_n^{\alpha,\beta}\mathscr{K}_N v\right\|_\infty \\
&\quad\geq \frac{\rho}{2}\|v\|_\infty,
\end{aligned} \tag{74}$$

proving the desired conclusion (69). □

Theorem 6 ensures that, for sufficient large n, the operator equation (20) has a unique solution $\tilde{u}_n$. The next result considers the convergent order of the approximate solution $\tilde{u}_n$ in L^∞ norm.

Theorem 7. *Suppose that the kernel function $k \in C^{m,m}(I^2)$, $f \in C^m(I)$, and $-1/2 < \alpha, \beta < 1/2$. If we choose N as in (68), then there exist a positive constant η and a positive integer n_0 such that, for $n \geq n_0$,*

$$\left\|u - \tilde{u}_n\right\|_\infty \leq \eta\|u\|_m n^{(1/2)+\max\{\alpha,\beta\}-m}. \tag{75}$$

Proof. We first notice that it follows from the hypothesis that $k \in C^{m,m}(I^2)$ and $f \in C^m(I)$ that (3) has a unique solution $u \in C^m(I)$, which implies that $u \in H^{m-1,1}(I)$. By using the triangle inequality,

$$\left\|u - \tilde{u}_n\right\|_\infty \leq \left\|u - \mathscr{L}_n^{\alpha,\beta}u\right\|_\infty + \left\|\mathscr{L}_n^{\alpha,\beta}u - \tilde{u}_n\right\|_\infty. \tag{76}$$

Upon the estimation (63) with $v := u$, we only require to estimate the second term in the right hand side of the above equation. In fact, employing $\mathscr{L}_n^{\alpha,\beta}$ to both sides of (3) yields that

$$\mathscr{L}_n^{\alpha,\beta}u + \mathscr{L}_n^{\alpha,\beta}\mathscr{K}u = \mathscr{L}_n^{\alpha,\beta}f. \tag{77}$$

A direct computation of the above equation and (20) confirms that

$$\begin{aligned}
&\left(\mathscr{I} + \mathscr{L}_n^{\alpha,\beta}\mathscr{K}_N\right)\left(\tilde{u}_n - \mathscr{L}_n^{\alpha,\beta}u\right) \\
&\quad= \mathscr{L}_n^{\alpha,\beta}\mathscr{K}u - \mathscr{L}_n^{\alpha,\beta}\mathscr{K}_N\mathscr{L}_n^{\alpha,\beta}u.
\end{aligned} \tag{78}$$

By Theorem 6, there exists a positive integer n_0 such that $n \geq n_0$,

$$\left\|\tilde{u}_n - \mathscr{L}_n^{\alpha,\beta}u\right\|_\infty \leq \frac{2}{\rho}\left\|\mathscr{L}_n^{\alpha,\beta}\mathscr{K}u - \mathscr{L}_n^{\alpha,\beta}\mathscr{K}_N\mathscr{L}_n^{\alpha,\beta}u\right\|_\infty, \tag{79}$$

where combining (61) leads that there exists a positive constant ξ_1 such that

$$\left\|\tilde{u}_n - \mathscr{L}_n^{\alpha,\beta}u\right\|_\infty \leq \xi_1\left\|\mathscr{K}u - \mathscr{K}_N\mathscr{L}_n^{\alpha,\beta}u\right\|_\infty n^{(1/2)+\max\{\alpha,\beta\}}. \tag{80}$$

To obtain the estimation of the right hand side of equation (80), we let

$$\begin{aligned}
I_1 &:= \left\|\mathscr{K}u - \mathscr{K}\mathscr{L}_n^{\alpha,\beta}u\right\|_\infty, \\
I_2 &:= \left\|\mathscr{K}\mathscr{L}_n^{\alpha,\beta}u - \mathscr{K}_N\mathscr{L}_n^{\alpha,\beta}u\right\|_\infty.
\end{aligned} \tag{81}$$

Clearly,

$$\left\|\mathscr{K}u - \mathscr{K}_N\mathscr{L}_n^{\alpha,\beta}u\right\|_\infty \leq I_1 + I_2. \tag{82}$$

It remains to estimate I_1 and I_2, respectively. First, using the hypothesis that $-1/2 < \alpha, \beta < 1/2$ and the result (49) with $v := u - \mathscr{L}_n^{\alpha,\beta}u$ produces that there exists a positive constant ξ_2 independent of n such that

$$I_1 \leq \xi_2\left\|u - \mathscr{L}_n^{\alpha,\beta}u\right\|_{w^{\alpha,\beta}}, \tag{83}$$

where combining the result (23) with $i := 0$, $r := m$, $v := u$ yields that there exists a positive constant ξ_3 independent of n such that

$$I_1 \le \xi_3 \left\| \mathscr{D}_x^m u \right\|_{w^{\alpha+m,\beta+m}} n^{-m}. \quad (84)$$

Hence, a combination of (84) and the following inequality

$$\left\| \mathscr{D}_x^l u \right\|_{w^{\alpha+l,\beta+l}} \le \|1\|_{w^{\alpha,\beta}} \left\| \mathscr{D}_x^l u \right\|_\infty, \quad l \in \mathbb{N}, \quad (85)$$

produces that there exists a positive constant ξ_4 such that

$$I_1 \le \xi_4 \|u\|_m n^{-m}. \quad (86)$$

On the other hand, using the results (24) and (31) leads that there exists a positive constant ξ_5 such that

$$I_2 \le \xi_5 \left(\sum_{i \in \mathbb{Z}_m} \left\| \mathscr{D}_x^i u \right\|_{w^{\alpha+i,\beta+i}} \right) N^{-m}, \quad (87)$$

where combining (68) and (85) yields that there exists a positive constant ξ_6,

$$I_2 \le \xi_6 \|u\|_m n^{-m-(1/2)-\max\{\alpha,\beta\}} \log^{-1} n. \quad (88)$$

A combination of the above estimation and (80), (82), and (86) yields the desired result. □

Theorem 7 illustrates that the approximate solution obtained by the proposed method arrives at the most possible convergent order.

5. The Stability and Convergence Analysis under the $L^2_{w^{\alpha,\beta}}$ Norm

As demonstrated in the previous section, in this section we are going to prove that, for sufficiently large n and N, the operator $\mathscr{I} + \mathscr{L}_n^{\alpha,\beta}\mathscr{K}_N : P_n \to P_n$ has a uniformly bounded inversion in the $L^2_{w^{\alpha,\beta}}(I)$ space and then show that the approximate solution arrives at the optimal convergent order. To this end, we first give a few results.

Lemma 8. *Suppose that $-1 < \lambda_1, \lambda_2 < 1$ and $h \in [0,1]$. If $x, x+h \in I$, then one has that*

$$\int_x^{x+h} w^{\lambda_1,\lambda_2}(t)\,dt \le \frac{2\max\left\{h, h^{1+\min\{\lambda_1,\lambda_2\}}\right\}}{1+\min\{\lambda_1,\lambda_2\}}. \quad (89)$$

Proof. We will prove that the result (89) holds in the following four cases: (1) $\lambda_1 \ge 0$ and $\lambda_2 \ge 0$; (2) $\lambda_1 \ge 0$ but $\lambda_2 < 0$; (3) $\lambda_1 < 0$ while $\lambda_2 \ge 0$; (4) $\lambda_1 < 0$ and $\lambda_2 < 0$.

Firstly, we notice that, for $\lambda_1 \ge 0$, $\lambda_2 \ge 0$,

$$(1-t)^{\lambda_1}(1+t)^{\lambda_2} \le 2, \quad (90)$$

which confirms the desired conclusion.

If the conditions $\lambda_1 < 0$ and $\lambda_2 \ge 0$ hold, then using

$$(1+t)^{\lambda_2} \le 2 \quad (91)$$

produces

$$\int_x^{x+h} w^{\lambda_1,\lambda_2}(t)\,dt \le 2\int_x^{x+h} (1-t)^{\lambda_1}\,dt \le \frac{2h^{1+\lambda_1}}{1+\lambda_1}, \quad (92)$$

which ensures the desired conclusion.

In a similar approach as the above case, clearly, the result (89) holds for the case that $\lambda_1 \ge 0$ while $\lambda_2 < 0$.

At last, when the conditions $\lambda_1 < 0$ and $\lambda_2 < 0$ hold, using the next equation

$$(1-t)^{\lambda_1}(1+t)^{\lambda_2} = \frac{(1-t)^{\lambda_1} + (1+t)^{\lambda_2}}{(1-t)^{-\lambda_1} + (1+t)^{-\lambda_2}} \quad (93)$$

can produce

$$(1-t)^{\lambda_1}(1+t)^{\lambda_2} \le (1-t)^{\lambda_1} + (1+t)^{\lambda_2}. \quad (94)$$

Thus, again using the same method as before yields the desired result. □

Next we ensure that the operator $\mathscr{K} : L^2_{w^{\alpha,\beta}}(I) \to H^{0,\kappa}(I)$ is a bounded linear operator with certain positive constant κ.

Lemma 9. *Suppose that $-1 < \alpha, \beta < 1$, $k \in C^{1,0}(I^2)$; then $\mathscr{K}$ is a bounded linear operator from $L^2_{w^{\alpha,\beta}}(I)$ into $H^{0,\kappa}(I)$ with $\kappa := \min\{1/2, (1/2) + \min\{-\alpha/2, -\beta/2\}\}$; that is, there exists a positive constant ζ_1 such that, for $v \in L^2_{w^{\alpha,\beta}}(I)$,*

$$\|\mathscr{K}v\|_{0,\kappa} \le \zeta_1 \|v\|_{w^{\alpha,\beta}}. \quad (95)$$

Proof. By the estimation (49) in Lemma 5, there exists a positive constant ξ_1 such that, for $v \in L^2_{w^{\alpha,\beta}}(I)$,

$$\|\mathscr{K}v\|_\infty \le \xi_1 \|v\|_{w^{\alpha,\beta}}. \quad (96)$$

On the other hand, for $x_1, x_2 \in I$, without loss of generality, we assume that $x_1 \le x_2$. By introducing

$$I_1 := \int_{-1}^{x_2} \left(k(x_1,t) - k(x_2,t)\right) v(t)\,dt,$$
$$I_2 := \int_{x_1}^{x_2} k(x_1,t)\, v(t)\,dt, \quad (97)$$

we have

$$(\mathscr{K}v)(x_1) - (\mathscr{K}v)(x_2) = I_1 + I_2. \quad (98)$$

Hence, it remains to estimate I_1 and I_2, respectively. For this purpose, by reformulating I_1 and I_2 as follows

$$I_1 := \int_{-1}^{x_2} \left(w^{-\alpha/2,-\beta/2}(t)\left(k(x_1,t) - k(x_2,t)\right)\right) \times \left(w^{\alpha/2,\beta/2}(t)\, v(t)\right) dt, \quad (99)$$

$$I_2 := \int_{x_1}^{x_2} \left(w^{-\alpha/2,-\beta/2}(t)\, k(x_1,t)\right)\left(w^{\alpha/2,\beta/2}(t)\, v(t)\right) dt$$

and then employing Cauchy-Schwarz inequality to I_1 and I_2, respectively, we can obtain that

$$I_1^2 \leq \left(\int_{-1}^{x_2} w^{-\alpha,-\beta}(t) \left(k(x_1,t) - k(x_2,t)\right)^2 dt \right) \times \left(\int_{-1}^{x_2} w^{\alpha,\beta}(t)\, v^2(t)\, dt \right),$$
$$I_2^2 \leq \left(\int_{x_1}^{x_2} w^{-\alpha,-\beta}(t)\, k^2(x_1,t)\, dt \right) \left(\int_{x_1}^{x_2} w^{\alpha,\beta}(t)\, v^2(t)\, dt \right). \tag{100}$$

Using the hypothesis that $k \in C^{1,0}(I^2)$ and the Lagrange midvalue differential theorem yields that

$$I_1^2 \leq \left\| \mathscr{D}_x^1 k \right\|_\infty^2 \|1\|_{w^{-\alpha,-\beta}} \|v\|_{w^{\alpha,\beta}}^2 \left| x_1 - x_2 \right|^2. \tag{101}$$

A direct estimation for I_2 produces that

$$I_2^2 \leq \|k\|_\infty^2 \|v\|_{w^{\alpha,\beta}}^2 \int_{x_1}^{x_2} w^{-\alpha,-\beta}(t)\, dt. \tag{102}$$

If the condition $x_2 - x_1 > 1$ holds, then we have

$$I_2^2 \leq \|k\|_\infty^2 \|v\|_{w^{\alpha,\beta}}^2 \|1\|_{w^{-\alpha,-\beta}} \left| x_1 - x_2 \right|^\kappa, \tag{103}$$

otherwise, using (102), where combining (89) with $\lambda_1 := -\alpha$, $\lambda_2 := -\beta$, $x := x_1$ and $x + h := x_2$ leads to that there exists a positive constant ξ_2 such that

$$I_2^2 \leq \xi_2 \|v\|_{w^{\alpha,\beta}}^2 \max \left\{ \left| x_1 - x_2 \right|, \left| x_1 - x_2 \right|^{1+\min\{-\alpha,-\beta\}} \right\}. \tag{104}$$

A combination of (98)–(104) and the triangle inequality yields that there exists a positive constant ξ_3 such that

$$\left| (\mathscr{K} v)(x_1) - (\mathscr{K} v)(x_2) \right| \leq \xi_3 \|v\|_{w^{\alpha,\beta}} \left| x_1 - x_2 \right|^\kappa, \tag{105}$$

where (96) draws the desired conclusion. □

The next result concerns on difference between $\mathscr{K} v$ and $\mathscr{L}_n^{\alpha,\beta} \mathscr{K} v$ for $v \in L^2_{w^{\alpha,\beta}}(I)$. For this purpose, we will make use of the next result proposed in [5]. For any $v \in C(I)$, there exists a positive constant ζ_2 independent of n:

$$\left\| \mathscr{L}_n^{\alpha,\beta} v \right\|_{w^{\alpha,\beta}} \leq \zeta_2 \|v\|_\infty. \tag{106}$$

A combination of (61) and (106) leads to that there exists a positive constant ζ_3 such that, for $v \in H^{r,\nu}(I)$,

$$\left\| v - \mathscr{L}_n^{\alpha,\beta} v \right\|_{w^{\alpha,\beta}} \leq \zeta_3 \|v\|_{r,\nu} n^{-r-\nu}. \tag{107}$$

Again using Theorem 3.10 in [23], we know that 0 is the unique eigenvalue of Volterra integral operator $\mathscr{K}$; consequently, the operator $\mathscr{I} + \mathscr{K} : L^2_{w^{\alpha,\beta}}(I) \to L^2_{w^{\alpha,\beta}}(I)$ has a bounded inversion; that is, for any $v \in P_n$, there exists a positive constant ϱ such that

$$\|(\mathscr{I} + \mathscr{K}) v\|_{w^{\alpha,\beta}} \geq \varrho \|v\|_{w^{\alpha,\beta}}. \tag{108}$$

Theorem 10. *Suppose that $k \in C^{1,m}(I^2)$ and $-1/2 < \alpha, \beta < 1, \alpha + \beta \leq 1$. If one chooses N as follows:*

$$N \geq N_{\min} := \left[\mathrm{n} \log^{1/\mathrm{m}} \mathrm{n} \right] + 1, \tag{109}$$

then there exists a positive integer n_0 such that $n \geq n_0$ and for $v \in P_n$,

$$\left\| (\mathscr{I} + \mathscr{L}_n^{\alpha,\beta} \mathscr{K}_N) v \right\|_{w^{\alpha,\beta}} \geq \frac{\varrho}{2} \|v\|_{w^{\alpha,\beta}}, \tag{110}$$

where ϱ appears in (108)*.*

Proof. This proof is similar to that of Theorem 6. By Lemma 9, for $v \in P_n$, we have $\mathscr{K} v \in H^{0,\kappa}(I)$, where combining (95) and (107) obtains that there exists a positive constant ξ_1,

$$\left\| \mathscr{K} v - \mathscr{L}_n^{\alpha,\beta} \mathscr{K} v \right\|_{w^{\alpha,\beta}} \leq \xi_1 \|v\|_{w^{\alpha,\beta}} n^{-\kappa}. \tag{111}$$

Hence, by the fact that $\lim_{n\to\infty} n^{-\kappa} = 0$, there exists a positive integer n_1 such that, for $n \geq n_1$,

$$\left\| \mathscr{K} v - \mathscr{L}_n^{\alpha,\beta} \mathscr{K} v \right\|_{w^{\alpha,\beta}} \leq \frac{\varrho}{4} \|v\|_{w^{\alpha,\beta}}. \tag{112}$$

On the other hand, using (106) obtains that there exists a positive constant ξ_2 such that

$$\left\| \mathscr{L}_n^{\alpha,\beta} \mathscr{K} v - \mathscr{L}_n^{\alpha,\beta} \mathscr{K}_N v \right\|_{w^{\alpha,\beta}} \leq \xi_2 \left\| \mathscr{K} v - \mathscr{K}_N v \right\|_\infty. \tag{113}$$

By the hypothesis that $-1/2 < \alpha, \beta \leq 1$, and $\alpha + \beta \leq 1$, a combination of (42) and (109) yields that there exists a positive constant ξ_3 such that

$$\left\| \mathscr{K} v - \mathscr{K}_N v \right\|_\infty \leq \xi_3 \|v\|_{w^{\alpha,\beta}} \log^{-1} n. \tag{114}$$

Substituting the above estimation into the right hand side of (113) produces that

$$\left\| \mathscr{L}_n^{\alpha,\beta} \mathscr{K} v - \mathscr{L}_n^{\alpha,\beta} \mathscr{K}_N v \right\|_{w^{\alpha,\beta}} \leq \xi_2 \xi_3 \|v\|_{w^{\alpha,\beta}} \log^{-1} n. \tag{115}$$

Again using the same fact that $\lim_{n\to\infty} \log^{-1} n = 0$, there exists a positive integer n_2 such that for $n \geq n_2$,

$$\left\| \mathscr{L}_n^{\alpha,\beta} \mathscr{K} v - \mathscr{L}_n^{\alpha,\beta} \mathscr{K}_N v \right\|_{w^{\alpha,\beta}} \leq \frac{\varrho}{4} \|v\|_{w^{\alpha,\beta}}. \tag{116}$$

When $n \geq n_0 := \max\{n_1, n_2\}$, these three estimates (108), (112), and (116) yield that

$$\left\| (\mathscr{I} + \mathscr{L}_n^{\alpha,\beta} \mathscr{K}_N) v \right\|_{w^{\alpha,\beta}} \geq \frac{\varrho}{2} \|w\|_{w^{\alpha,\beta}}, \tag{117}$$

which infers our result. □

This above result shows that (20) has a unique solution $\tilde{u}_n$ in the space $L^2_{w^{\alpha,\beta}}(I)$. Next result considers the approximate order of the solution $\tilde{u}_n$.

Theorem 11. *Suppose that the kernel function $k \in C^{m,m}(I^2)$, $f \in C^m(I)$, and $-1/2 < \alpha, \beta < 1, \alpha + \beta \leq 1$. If one chooses N as in* (109)*, then there exist a positive constant θ and a positive integer n_0 such that, for $n \geq n_0$,*

$$\|u - \tilde{u}_n\|_{w^{\alpha,\beta}} \leq \theta \left(\sum_{i \in \mathbb{Z}_m} \left\| \mathscr{D}_x^i u \right\|_{w^{\alpha+i,\beta+i}} \right) n^{-m}. \tag{118}$$

TABLE 1: The numerical result based on the collocation nodes $x_i^{0,0}, i \in \mathbb{Z}_n$.

n	4	6	8	10	12	14
$\|u-\tilde{u}_n\|_\infty$	$1.40e-2$	$3.16e-4$	$4.28e-6$	$3.83e-8$	$2.60e-10$	$1.20e-12$
$\|u-\tilde{u}_n\|_{w^{0,0}}$	$1.85e-2$	$4.10e-4$	$5.48e-6$	$4.87e-8$	$3.07e-10$	$1.45e-12$

TABLE 2: The numerical result based on the collocation nodes $x_i^{1/4,1/3}, i \in \mathbb{Z}_n$.

n	4	6	8	10	12	14
$\|u-\tilde{u}_n\|_\infty$	$1.16e-2$	$2.61e-4$	$3.54e-6$	$4.16e-8$	$2.14e-10$	$9.92e-13$
$\|u-\tilde{u}_n\|_{w^{1/4,1/3}}$	$1.52e-2$	$3.38e-4$	$4.52e-6$	$4.01e-8$	$2.53e-10$	$1.19e-12$

Proof. The proof of Theorem 11 is similar as that of Theorem 7. It follows from Theorem 7 that $u \in C^m(I)$, which implies that $u \in H^m_{w^{\alpha,\beta}}(I)$. By using the triangle inequality,

$$\|u-\tilde{u}_n\|_{w^{\alpha,\beta}} \le \left\|u-\mathscr{L}_n^{\alpha,\beta}u\right\|_{w^{\alpha,\beta}} + \left\|\mathscr{L}_n^{\alpha,\beta}u-\tilde{u}_n\right\|_{w^{\alpha,\beta}}. \quad (119)$$

Upon the estimation in (23) with $i := 0$, $r := m$ and $v := u$, we only need to estimate $\|\mathscr{L}_n^{\alpha,\beta}u-\tilde{u}_n\|_{w^{\alpha,\beta}}$. Employing the result (78), (106), and Theorem 7, there exist a positive constant ξ_1 and a positive integer n_0 such that $n \ge n_0$,

$$\left\|\tilde{u}_n-\mathscr{L}_n^{\alpha,\beta}u\right\|_{w^{\alpha,\beta}} \le \xi_1 \left\|\mathscr{K}u-\mathscr{K}_N\mathscr{L}_n^{\alpha,\beta}u\right\|_\infty \quad (120)$$

To obtain the estimation of the right hand side of (120), we let

$$I_1 := \left\|\mathscr{K}u-\mathscr{K}\mathscr{L}_n^{\alpha,\beta}u\right\|_\infty,$$
$$I_2 := \left\|\mathscr{K}\mathscr{L}_n^{\alpha,\beta}u-\mathscr{K}_N\mathscr{L}_n^{\alpha,\beta}u\right\|_\infty. \quad (121)$$

Clearly,

$$\left\|\mathscr{K}u-\mathscr{K}_N\mathscr{L}_n^{\alpha,\beta}u\right\|_\infty \le I_1 + I_2. \quad (122)$$

Upon the estimation (84), we only require to estimate I_2. In fact, a combination of (87) and (109) yields that there exists a positive constant ξ_2:

$$I_2 \le \xi_2 \left(\sum_{i\in\mathbb{Z}_m} \left\|\mathscr{D}_x^i u\right\|_{w^{\alpha+i,\beta+i}}\right) n^{-m}\log^{-1} n. \quad (123)$$

A combination of (84) and (120)–(123) yields the desired result. □

Theorem 11 illustrates that the proposed method preserves the optimal order of convergence.

6. One Numerical Example

In this section, we are going to present one numerical example to demonstrate the efficiency of the spectral Jacobi-collocation method for solving (3). In each example, we use two spectral collocation approaches associated with the weight function $w^{0,0}$ and $w^{1/4,1/3}$, respectively. Here, we compute the Gauss-Jacobi quadrature rule nodes and weights by Theorems 3.4 and 3.6 discussed in [19]. All computer programs are compiled by Matlab language.

Example. Consider the second kind Volterra integral equation (1) with

$$k(x,t) = e^{xt}, \qquad f(x) = e^{2x} + \frac{e^{x(x+2)} - e^{-(x+2)}}{x+2}. \quad (124)$$

The corresponding exact solution is given by $u(x) = e^{2x}$. As expected, the errors show an exponential decay, since in this semilog representation the error variations are essentially linear versus the degrees of the polynomial.

From the theoretical results we observe that the numerical errors should decay with an exponential rate, and we also find that the errors show an exponential decay (Tables 1 and 2).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Solving Initial-Boundary Value Problems for Local Fractional Differential Equation by Local Fractional Fourier Series Method

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The initial-boundary value problems for the local fractional differential equation are investigated in this paper. The local fractional Fourier series solutions with the nondifferential terms are obtained. Two illustrative examples are given to show efficiency and accuracy of the presented method to process the local fractional differential equations.

1. Introduction

In various fields of physics, mathematics, and engineering, because of the different operators, there are classical differential equations [1], fractional differential equation [2–4], and local fractional differential equations [5, 6]. There are more techniques to achieve analytical approximations to the solutions to differential equations in mathematical physics, such as the decomposition method [7], the variational iteration method [8], the homotopy perturbation method [9], the heat-balance integral method [10], the Fourier transform [11], the Laplace transform [11], and the references therein.

Recently, a new Fourier series (local fractional Fourier series) via local fractional operator was proposed [6] and had various applications in the applied fields such as fractal wave problems in fractal string [12, 13] and the heat-conduction problems arising in fractal heat transfer [14, 15]. For a detailed description of the local fractional Fourier series method, we refer the readers to the recent works [14–16]. This is the main advantage of local fractional differential equations in comparison with classical integer-order and fractional-order models.

In the present paper we consider the local fractional differential equation:

$$\frac{\partial^{\alpha} u(x,y)}{\partial y^{\alpha}} - \frac{\partial^{2\alpha} u(x,y)}{\partial x^{2\alpha}} = 0, \tag{1}$$

subject to the initial-boundary value conditions:

$$\frac{\partial^{\alpha} u(0,t)}{\partial x^{\alpha}} = 0, \qquad \frac{\partial^{\alpha} u(L,t)}{\partial x^{\alpha}} = 0, \qquad u(x,0) = g(x), \tag{2}$$

where the operators are described by the local fractional differential operators [5, 6, 12–15]. The paper is organized as follows. In Section 2, the basic theory of the local fractional calculus and local fractional Fourier series is presented. In Section 3, we discuss the initial-boundary problems for local fractional differential equation. Finally, Section 4 is devoted to the conclusions.

2. Analysis of the Method

In this section, we present the basic theory of the local fractional calculus and analyze the local fractional Fourier series method.

Definition 1. Let F be a subset of the real line and be a fractal. The mass function $\gamma^{\alpha}[F,a,b]$ can be written as [5]

$$\gamma^{\alpha}[F,a,b] = \lim_{\max_{0<i<n-1}(x_{i+1}-x_i)\to 0} \sum_{i=0}^{n-1} \frac{(x_{i+1}-x_i)^{\alpha}}{\Gamma(1+\alpha)}. \tag{3}$$

The following properties are present as follows [5].

(i) If $F \cap (a,b) = \varnothing$, then $\gamma^{\alpha}[F,a,b] = 0$.

(ii) If $a < b < c$ and $\gamma^{\alpha}[F,a,b] < 0$, then $\gamma^{\alpha}[F,a,b] + \gamma^{\alpha}[F,b,c] = \gamma^{\alpha}[F,a,c]$.

If $f : (F,d) \to (\Omega', d')$ is a bi-Lipschitz mapping, then we have [5, 12]

$$\rho^{s} H^{s}(F) \le H^{s}(f(F)) \le \tau^{s} H^{s}(F) \tag{4}$$

such that

$$\rho^{\alpha}|x_1 - x_2|^{\alpha} \le |f(x_1) - f(x_2)| \le \tau^{\alpha}|x_1 - x_2|^{\alpha}. \tag{5}$$

In view of (5), we have

$$|f(x_1) - f(x_2)| \le \tau^{\alpha}|x_1 - x_2|^{\alpha} \tag{6}$$

such that

$$|f(x_1) - f(x_2)| < \varepsilon^{\alpha}, \tag{7}$$

where α is the fractal dimension of F. This result is directly deduced from fractal geometry and relates to the fractal coarse-grained mass function $\gamma^{\alpha}[F,a,b]$, which reads [5, 13]

$$\gamma^{\alpha}[F,a,b] = \frac{H^{\alpha}(F \cap (a,b))}{\Gamma(1+\alpha)} \tag{8}$$

with

$$H^{\alpha}(F \cap (a,b)) = (b-a)^{\alpha}, \tag{9}$$

where H^{α} is α dimensional Hausdorff measure.

Definition 2. If there is [5, 6, 12–15]

$$|f(x) - f(x_0)| < \varepsilon^{\alpha} \tag{10}$$

with $|x - x_0| < \delta$, for $\varepsilon, \delta > 0$ and $\varepsilon, \delta \in R$, then $f(x)$ is called local fractional continuous at $x = x_0$.

If $f(x)$ is local fractional continuous on the interval (a,b), then we can write it in the form [5, 6, 12]

$$f(x) \in C_{\alpha}(a,b). \tag{11}$$

Definition 3. Local fractional derivative of $f(x)$ of order α at $x = x_0$ is defined as follows [5, 6, 12–15]:

$$f^{(\alpha)}(x_0) = \frac{d^{\alpha} f(x)}{dx^{\alpha}}\Big|_{x=x_0} = \lim_{x \to x_0} \frac{\Delta^{\alpha}(f(x) - f(x_0))}{(x-x_0)^{\alpha}}, \tag{12}$$

where $\Delta^{\alpha}(f(x) - f(x_0)) \cong \Gamma(1+\alpha)\Delta(f(x) - f(x_0))$.

From (12) we can rewrite the local fractional derivative as

$$f^{(\alpha)}(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{\gamma^{\alpha}[F, x_0, x]}, \tag{13}$$

where

$$\gamma^{\alpha}[F, x_0, x] = \frac{H^{\alpha}(F \cap (a,b))}{\Gamma(1+\alpha)}. \tag{14}$$

Definition 4. The partition of the interval $[a,b]$ is (t_j, t_{j+1}), $j = 0, \ldots, N-1$, $t_0 = a$ and $t_N = b$ with $\Delta t_j = t_{j+1} - t_j$ and $\Delta t = \max\{\Delta t_1, \Delta t_2, \Delta t_j, \ldots\}$. Local fractional integral of $f(x)$ of order α in the interval $[a,b]$ is given by [5, 6, 12–15]

$$\begin{aligned} {}_{a}I_{b}^{(\alpha)} f(x) &= \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t)(dt)^{\alpha} \\ &= \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \to 0} \sum_{j=0}^{j=N-1} f(t_j)(\Delta t_j)^{\alpha}. \end{aligned} \tag{15}$$

Following (14), we have

$${}_{a}I_{b}^{(\alpha)} f(x) = \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \to 0} \sum_{j=0}^{j=N-1} f(t_j)\gamma^{\alpha}[F, t_j, t_{j+1}], \tag{16}$$

where

$$\gamma^{\alpha}[F,a,b] = \lim_{\max_{0<i<n-1}(x_{i+1}-x_i) \to 0} \sum_{i=0}^{n-1} \frac{(x_{i+1}-x_i)^{\alpha}}{\Gamma(1+\alpha)}. \tag{17}$$

If F are Cantor sets, we can get the derivative and integral on Cantor sets.

Some properties of local fractional integrals are listed as follows:

$$\begin{aligned}
{}_{0}I_{x}^{(\alpha)} E_{\alpha}(x^{\alpha}) &= E_{\alpha}(x^{\alpha}) - 1, \\
{}_{0}I_{x}^{(\alpha)} \frac{x^{n\alpha}}{\Gamma(1+n\alpha)} &= \frac{x^{(n+1)\alpha}}{\Gamma(1+(n+1)\alpha)}, \\
{}_{0}I_{x}^{(\alpha)} \sin_{\alpha}(a^{\alpha}x^{\alpha}) &= \frac{1}{a^{\alpha}}\left[\cos_{\alpha}(a^{\alpha}x^{\alpha}) - 1\right], \\
{}_{0}I_{x}^{(\alpha)} \cos_{\alpha}(a^{\alpha}x^{\alpha}) &= \frac{1}{a^{\alpha}} \sin_{\alpha}(a^{\alpha}x^{\alpha}), \\
{}_{0}I_{x}^{(\alpha)} \frac{x^{\alpha}}{\Gamma(1+\alpha)} \sin_{\alpha}(a^{\alpha}x^{\alpha}) &= -\frac{1}{a^{\alpha}}\left[\frac{x^{\alpha}}{\Gamma(1+\alpha)}\cos_{\alpha}(a^{\alpha}x^{\alpha}) - \frac{1}{a^{\alpha}}\sin_{\alpha}(a^{\alpha}x^{\alpha})\right], \\
{}_{0}I_{x}^{(\alpha)} \frac{x^{\alpha}}{\Gamma(1+\alpha)} \cos_{\alpha}(a^{\alpha}x^{\alpha}) &= \frac{1}{a^{\alpha}}\left\{\frac{x^{\alpha}}{\Gamma(1+\alpha)}\sin_{\alpha}(a^{\alpha}x^{\alpha}) - \frac{1}{a^{\alpha}}\left[\cos_{\alpha}(a^{\alpha}x^{\alpha}) - 1\right]\right\}, \\
{}_{0}I_{x}^{(\alpha)} \left\{E_{\alpha}(x^{\alpha}) \sin_{\alpha}(a^{\alpha}x^{\alpha})\right\} &= \frac{E_{\alpha}(x^{\alpha})\left[\sin_{\alpha}(a^{\alpha}x^{\alpha}) - a^{\alpha}\cos_{\alpha}(a^{\alpha}x^{\alpha})\right] + a^{\alpha}}{1 + a^{2\alpha}}, \\
{}_{0}I_{x}^{(\alpha)} \left\{E_{\alpha}(x^{\alpha}) \cos_{\alpha}(a^{\alpha}x^{\alpha})\right\} &= \frac{E_{\alpha}(x^{\alpha})\left[\cos_{\alpha}(a^{\alpha}x^{\alpha}) + a^{\alpha}\sin(a^{\alpha}x^{\alpha})\right] - 1}{1 + a^{2\alpha}}.
\end{aligned} \tag{18}$$

Definition 5. Local fractional trigonometric Fourier series of $f(t)$ is given by [6, 12–16]

$$f(t) = a_0 + \sum_{i=1}^{\infty} a_k \sin_\alpha \left(k^\alpha {\omega_0}^\alpha t^\alpha\right) + \sum_{i=1}^{\infty} b_k \cos_\alpha \left(k^\alpha {\omega_0}^\alpha t^\alpha\right) \quad (19)$$

for $x \in R$ and $0 < \alpha \leq 1$.

The local fractional Fourier coefficients of (19) can be computed by

$$\begin{aligned} a_0 &= \frac{1}{T^\alpha} \Gamma(1+\alpha)\ {}_0 I_T f(t), \\ a_k &= \left(\frac{2}{T}\right)^\alpha \Gamma(1+\alpha)\ {}_0 I_T \left\{f(t) \sin_\alpha \left(k^\alpha {\omega_0}^\alpha t^\alpha\right)\right\}, \\ b_k &= \left(\frac{2}{T}\right)^\alpha \Gamma(1+\alpha)\ {}_0 I_T \left\{f(t) \cos_\alpha \left(k^\alpha {\omega_0}^\alpha t^\alpha\right)\right\}. \end{aligned} \quad (20)$$

If $\omega_0 = 1$, then we get

$$f(t) = a_0 + \sum_{i=1}^{\infty} a_k \sin_\alpha \left(k^\alpha t^\alpha\right) + \sum_{i=1}^{\infty} b_k \cos_\alpha \left(k^\alpha t^\alpha\right), \quad (21)$$

where the local fractional Fourier coefficients can be computed by

$$\begin{aligned} a_0 &= \frac{1}{T^\alpha} \Gamma(1+\alpha)\ {}_0 I_T f(t), \\ a_k &= \left(\frac{2}{T}\right)^\alpha \Gamma(1+\alpha)\ {}_0 I_T \left\{f(t) \sin_\alpha \left(k^\alpha t^\alpha\right)\right\}, \\ b_k &= \left(\frac{2}{T}\right)^\alpha \Gamma(1+\alpha)\ {}_0 I_T \left\{f(t) \cos_\alpha \left(k^\alpha t^\alpha\right)\right\}. \end{aligned} \quad (22)$$

3. The Initial-Boundary Problems for the Local Fractional Differential Equation

In this section, we consider (1) with the various initial-boundary conditions.

Example 6. The initial-boundary condition (2) becomes

$$\frac{\partial^\alpha u(0,t)}{\partial x^\alpha} = 0, \qquad \frac{\partial^\alpha u(L,t)}{\partial x^\alpha} = 0, \qquad u(x,0) = E_\alpha \left(x^\alpha\right). \quad (23)$$

Let $u = XY$ in (1). Separation of the variables yields

$$XY^{(\alpha)} = YX^{(2\alpha)}. \quad (24)$$

Setting

$$\frac{Y^{(\alpha)}}{Y} = \frac{X^{(2\alpha)}}{X} = -\lambda^{2\alpha}, \quad (25)$$

we obtain

$$\begin{aligned} X^{(2\alpha)} + \lambda^{2\alpha} X &= 0, \\ Y^{(\alpha)} + \lambda^{2\alpha} Y &= 0. \end{aligned} \quad (26)$$

Hence, we have their solutions, which read

$$\begin{aligned} X &= a \cos_\alpha \left(\lambda^\alpha x^\alpha\right) + b \sin_\alpha \left(\lambda^\alpha x^\alpha\right), \\ Y &= c E_\alpha \left(-\lambda^{2\alpha} y^\alpha\right). \end{aligned} \quad (27)$$

Therefore, a solution is written in the form

$$\begin{aligned} u(x,y) &= XY \\ &= E_\alpha \left(-\lambda^{2\alpha} y^\alpha\right) \left(A \cos_\alpha \left(\lambda^\alpha x^\alpha\right) + B \sin_\alpha \left(\lambda^\alpha x^\alpha\right)\right), \end{aligned} \quad (28)$$

where $A = ac$, $B = bc$.

For the given condition

$$\frac{\partial^\alpha u(0,t)}{\partial x^\alpha} = 0, \quad (29)$$

there is $B = 0$, so that

$$u(x,y) = A E_\alpha \left(-\lambda^{2\alpha} y^\alpha\right) \cos_\alpha \left(\lambda^\alpha x^\alpha\right). \quad (30)$$

For the given condition

$$\frac{\partial^\alpha u(L,t)}{\partial x^\alpha} = 0, \quad (31)$$

we obtain

$$\sin_\alpha \left(\lambda^\alpha x^\alpha\right) = 0, \quad (32)$$

$$\lambda^\alpha = \left(\frac{m\pi}{L}\right)^\alpha, \quad m \in Z^+ \cup 0. \quad (33)$$

Thus, from (33) we deduce that

$$u(x,y) = A E_\alpha \left(-\left(\frac{m\pi}{L}\right)^{2\alpha} y^\alpha\right) \cos_\alpha \left(\left(\frac{m\pi x}{L}\right)^\alpha\right), \quad m \in Z^+ \cup 0. \quad (34)$$

To satisfy the condition (23), (34) is written in the form

$$\begin{aligned} &u(x,y) \\ &= A_0 + \sum_{m=1}^{\infty} A_m E_\alpha \left(-\left(\frac{m\pi}{L}\right)^{2\alpha} y^\alpha\right) \cos_\alpha \left(\left(\frac{m\pi x}{L}\right)^\alpha\right). \end{aligned} \quad (35)$$

Then, we derive

$$A_0 = \frac{\Gamma(1+\alpha)}{L^\alpha}\left(E_\alpha\left(L^\alpha\right)-1\right),$$

$$\begin{aligned}
A_m &= \left(\frac{2}{L}\right)^\alpha \Gamma(1+\alpha)\, E_\alpha\left(x^\alpha\right) \\
&\quad\times \frac{\left[\cos_\alpha\left((m\pi/L)^\alpha x^\alpha\right)+(m\pi/L)^\alpha \sin\left((m\pi/L)^\alpha x^\alpha\right)\right]-1}{1+(m\pi/L)^{2\alpha}},
\end{aligned}$$

$$\begin{aligned}
u(x,y) &= \frac{\Gamma(1+\alpha)}{2L^\alpha}\left(E_\alpha\left(L^\alpha\right)-1\right) \\
&\quad+\sum_{m=1}^{\infty}\frac{\left[\cos_\alpha\left((m\pi/L)^\alpha x^\alpha\right)+(m\pi/L)^\alpha \sin\left((m\pi/L)^\alpha x^\alpha\right)\right]-1}{1+(m\pi/L)^{2\alpha}} \\
&\quad\times E_\alpha\left(-\left(\frac{m\pi}{L}\right)^{2\alpha} y^\alpha\right)\cos_\alpha\left(\left(\frac{m\pi x}{L}\right)^\alpha\right) \\
&\quad\times E_\alpha\left(x^\alpha\right)\left(\frac{2}{L}\right)^\alpha \Gamma(1+\alpha).
\end{aligned} \tag{36}$$

Example 7. Let us consider (1) with the initial-boundary value condition, which becomes

$$\frac{\partial^\alpha u(0,t)}{\partial x^\alpha}=0, \qquad \frac{\partial^\alpha u(L,t)}{\partial x^\alpha}=0, \qquad u(x,0)=\frac{x^\alpha}{\Gamma(1+\alpha)}. \tag{37}$$

Following (35), we have

$$u(x,y) = A_0+\sum_{m=1}^{\infty}A_m E_\alpha\left(-\left(\frac{m\pi}{L}\right)^{2\alpha} y^\alpha\right)\cos_\alpha\left(\left(\frac{m\pi x}{L}\right)^\alpha\right), \tag{38}$$

where

$$\begin{aligned}
A_0 &= \frac{1}{L^\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}x^{2\alpha}, \\
A_m &= \frac{1}{(2m\pi)^\alpha}\left\{\frac{x^\alpha}{\Gamma(1+\alpha)}\sin_\alpha\left(\left(\frac{m\pi x}{L}\right)^\alpha\right)\right. \\
&\qquad\left.-\left(\frac{L}{m\pi}\right)^\alpha\left[\cos_\alpha\left(\left(\frac{m\pi x}{L}\right)^\alpha\right)-1\right]\right\}.
\end{aligned} \tag{39}$$

Hence, we get

$$\begin{aligned}
u(x,y) &= \frac{1}{L^\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}x^{2\alpha} \\
&\quad+\sum_{m=1}^{\infty}\frac{1}{(2m\pi)^\alpha}\left\{\frac{x^\alpha}{\Gamma(1+\alpha)}\sin_\alpha\left(\left(\frac{m\pi x}{L}\right)^\alpha\right)\right. \\
&\qquad\left.-\left(\frac{L}{m\pi}\right)^\alpha\left[\cos_\alpha\left(\left(\frac{m\pi x}{L}\right)^\alpha\right)-1\right]\right\} \\
&\quad\times E_\alpha\left(-\left(\frac{m\pi}{L}\right)^{2\alpha} y^\alpha\right)\cos_\alpha\left(\left(\frac{m\pi x}{L}\right)^\alpha\right).
\end{aligned} \tag{40}$$

4. Conclusions

In this work, the initial-boundary value problems for the local fractional differential equation are discussed by using the local fractional Fourier series method. Analytical solutions for the local fractional differential equation with the nondifferentiable conditions are obtained.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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Research Article

Local Fractional Derivative Boundary Value Problems for Tricomi Equation Arising in Fractal Transonic Flow

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The local fractional decomposition method is applied to obtain the nondifferentiable numerical solutions for the local fractional Tricomi equation arising in fractal transonic flow with the local fractional derivative boundary value conditions.

1. Introduction

The Tricomi equation [1] is the second-order linear partial differential equations of mixed type, which had been applied to describe the theory of plane transonic flow [2–7]. The Tricomi equation was used to describe the differentiable problems for the theory of plane transonic flow. However, for the fractal theory of plane transonic flow with nondifferentiable terms, the Tricomi equation is not applied to describe them. Recently, the local fractional calculus [8] was applied to describe the nondifferentiable problems, such as the fractal heat conduction [8, 9], the damped and dissipative wave equations in fractal strings [10], the local fractional Schrödinger equation [11], the wave equation on Cantor sets [12], the Navier-Stokes equations on Cantor sets [13], and others [14–19]. Recently, the local fractional Tricomi equation arising in fractal transonic flow was suggested in the form [19]

$$\frac{y^{\alpha}}{\Gamma(1+\alpha)}\frac{\partial^{2\alpha}u(x,y)}{\partial x^{2\alpha}}+\frac{\partial^{2\alpha}u(x,y)}{\partial y^{2\alpha}}=0, \quad (1)$$

where the quantity $u(x,y)$ is the nondifferentiable function, and the local fractional operator denotes [8]

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}=\frac{\Delta^{\alpha}(u(x,t)-u(x,t_0))}{(t-t_0)^{\alpha}}, \quad (2)$$

where

$$\Delta^{\alpha}(u(x,t)-u(x,t_0))\cong\Gamma(1+\alpha)[u(x,t)-u(x,t_0)]. \quad (3)$$

The local fractional decomposition method [12] was used to solve the diffusion equation on Cantor time-space. The aim of this paper is to use the local fractional decomposition method to solve the local fractional Tricomi equation arising in fractal transonic flow with the local fractional derivative boundary value conditions. The structure of this paper is as follows. In Section 2, the local fractional integrals and derivatives are introduced. In Section 3, the local fractional decomposition method is suggested. In Section 4, the nondifferentiable numerical solutions for local fractional Tricomi equation with the local fractional derivative boundary value conditions are given. Finally, the conclusions are shown in Section 5.

2. Local Fractional Integrals and Derivatives

In this section, we introduce the basic theory of the local fractional integrals and derivatives [8–19], which are applied in the paper.

Definition 1 (see [8–19]). For $|x - x_0| < \delta$, for $\varepsilon, \delta > 0$ and $\varepsilon \in R$, we give the function $f(x) \in C_\alpha(a,b)$, when

$$\left| f(x) - f(x_0) \right| < \varepsilon^\alpha, \quad 0 < \alpha \le 1, \tag{4}$$

is valid.

Definition 2 (see [8–19]). Let (t_j, t_{j+1}), $j = 0, \ldots, N-1$, $t_0 = a$, and $t_N = b$ with $\Delta t_j = t_{j+1} - t_j$ and $\Delta t = \max\{\Delta t_0, \Delta t_1, \ldots\}$, be a partition of the interval $[a,b]$. The local fractional integral of $f(x)$ in the interval $[a,b]$ is defined as

$$\begin{aligned} {}_aI_b^{(\alpha)} f(x) &= \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t)(dt)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \to 0} \sum_{j=0}^{j=N-1} f(t_j)(\Delta t_j)^\alpha. \end{aligned} \tag{5}$$

As the inverse operator of (6), local fractional derivative of $f(x)$ of the order α in the interval (a,b) is presented as [8–19]

$$\frac{d^\alpha f(x_0)}{dx^\alpha} = D_x^{(\alpha)} f(x_0) = \frac{\Delta^\alpha (f(x) - f(x_0))}{(x - x_0)^\alpha}, \tag{6}$$

where

$$\Delta^\alpha (f(x) - f(x_0)) \cong \Gamma(1+\alpha)[f(x) - f(x_0)]. \tag{7}$$

The formulas of local fractional derivative and integral, which appear in the paper, are valid [8]:

$$\begin{aligned} &\frac{d^\alpha}{dx^\alpha} \frac{x^{n\alpha}}{\Gamma(1+n\alpha)} = \frac{x^{(n-1)\alpha}}{\Gamma(1+(n-1)\alpha)}, \quad n \in N, \\ &\frac{d^\alpha}{dx^\alpha} E_\alpha(x^\alpha) = E_\alpha(x^\alpha), \\ &\frac{d^\alpha}{dx^\alpha} \sin_\alpha(x^\alpha) = \cos_\alpha(x^\alpha), \\ &\frac{d^\alpha}{dx^\alpha} \cos_\alpha(x^\alpha) = -\sin_\alpha(x^\alpha), \\ &{}_0I_x^{(\alpha)} \frac{x^{(n-1)\alpha}}{\Gamma(1+(n-1)\alpha)} = \frac{x^{n\alpha}}{\Gamma(1+n\alpha)}, \quad n \in N, \\ &{}_0I_x^{(\alpha)} \cos_\alpha(x^\alpha) = \sin_\alpha(x^\alpha). \end{aligned} \tag{8}$$

3. Analysis of the Method

In this section, we give the local fractional decomposition method [12]. We consider the following local fractional operator equation in the form

$$L_\alpha^{(2)} u + R_\alpha u = 0, \tag{9}$$

where $L_\alpha^{(2)}$ is linear local fractional operators of the order 2α with respect to x and R_α is the linear local fractional operators of order less than 2α. We write (9) as

$$L_{xx}^{(2\alpha)} u + R_\alpha u = 0, \tag{10}$$

where the 2α-th local fractional differential operator denotes

$$L_\alpha^{(n)} = L_{xx}^{(2\alpha)} = \frac{\partial^{2\alpha}}{\partial x^{2\alpha}}, \tag{11}$$

and the linear local fractional operators of order less than 2α denote

$$R_\alpha = \frac{\Gamma(1+\alpha)}{y^\alpha} \frac{\partial^{2\alpha} u(x,y)}{\partial y^{2\alpha}}. \tag{12}$$

Define the 2α-fold local fractional integral operator

$$L_\alpha^{(-2\alpha)} m(s) = {}_0I_x^{(\alpha)} {}_0I_x^{(\alpha)} m(s) \tag{13}$$

so that we obtain the local fractional iterative formula as follows:

$$L_\alpha^{(-2\alpha)} L_{xx}^{(2\alpha)} u + L_\alpha^{(-2\alpha)} L_\alpha^{(-2\alpha)} R_\alpha u = 0, \tag{14}$$

which leads to

$$u(x) = u_0(x) + L_\alpha^{(-2\alpha)} L_\alpha^{(-2\alpha)} R_\alpha u. \tag{15}$$

Therefore, for $n \ge 0$, we obtain the recurrence formula in the form

$$\begin{aligned} u_{n+1}(x) &= L_\alpha^{(-2)} R_\alpha u_n(x), \\ u_0(x) &= r(x). \end{aligned} \tag{16}$$

Finally, the solution of (9) reads

$$u(x) = \lim_{n\to\infty} \phi_n(x) = \lim_{n\to\infty} \sum_{n=0}^{\infty} u_n(x). \tag{17}$$

4. The Nondifferentiable Numerical Solutions

In this section, we discuss the nondifferentiable numerical solutions for the local fractional Tricomi equation arising in fractal transonic flow with the local fractional derivative boundary value conditions.

Example 1. We consider the initial-boundary value conditions for the local fractional Tricomi equation in the form [19]

$$u(0,y) = 0, \tag{18}$$

$$u(l,y) = 0, \tag{19}$$

$$u(x,0) = \frac{x^\alpha}{\Gamma(1+\alpha)}, \tag{20}$$

$$\frac{\partial^\alpha u(x,0)}{\partial x^\alpha} = \frac{x^\alpha}{\Gamma(1+\alpha)}. \tag{21}$$

Using (20)-(21), we structure the recurrence formula in the form

$$\begin{aligned} u_{n+1}(x,y) &= L_\alpha^{(-2)} \left[\frac{\Gamma(1+\alpha)}{y^\alpha} \frac{\partial^{2\alpha} u_n(x,y)}{\partial y^{2\alpha}} \right], \\ u_0(x,y) &= \frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{x^\alpha}{\Gamma(1+\alpha)} \frac{y^\alpha}{\Gamma(1+\alpha)}. \end{aligned} \tag{22}$$

Hence, for $n = 0$, the first term of (22) reads

$$u_1(x, y) = 0. \tag{23}$$

For $n = 1$ the second term of (22) is given as

$$u_2(x, y) = 0. \tag{24}$$

Hence, we obtain

$$u_0(x, y) = u_1(x, y) = \cdots = u_n(x, y) = 0. \tag{25}$$

Finally, the solution of (9) with the local fractional derivative boundary value conditions (19)–(21) can be written as

$$\begin{aligned} u(x, y) &= \lim_{n\to\infty} \phi_n(x, y) \\ &= \lim_{n\to\infty} \sum_{n=0}^{\infty} u_n(x, y) \\ &= \frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{x^\alpha}{\Gamma(1+\alpha)} \frac{y^\alpha}{\Gamma(1+\alpha)} \end{aligned} \tag{26}$$

which is in accordance with the result from the local fractional variational iteration method [19].

Example 2. Let us consider the initial-boundary value conditions for the local fractional Tricomi equation in the form

$$\begin{aligned} u(0, y) &= 0, \\ u(l, y) &= 0, \\ u(x, 0) &= 0, \\ \frac{\partial^\alpha u(x, 0)}{\partial x^\alpha} &= \cos_\alpha(x^\alpha). \end{aligned} \tag{27}$$

In view of (27), we set up the recurrence formula in the form

$$\begin{aligned} u_{n+1}(x, y) &= L_\alpha^{(-2)}\left[\frac{\Gamma(1+\alpha)}{y^\alpha} \frac{\partial^{2\alpha} u_n(x, y)}{\partial y^{2\alpha}}\right], \\ u_0(x, y) &= \cos_\alpha(x^\alpha) \frac{y^\alpha}{\Gamma(1+\alpha)}. \end{aligned} \tag{28}$$

Hence, from (28) we get the following equations:

$$\begin{aligned} u_1(x, y) &= L_\alpha^{(-2)}\left[\frac{\Gamma(1+\alpha)}{y^\alpha} \frac{\partial^{2\alpha} u_0(x, y)}{\partial y^{2\alpha}}\right] \\ &= L_\alpha^{(-2)}\left[\frac{\Gamma(1+\alpha)}{y^\alpha} \frac{\partial^{2\alpha}}{\partial y^{2\alpha}}\left(\cos_\alpha(x^\alpha) \frac{y^\alpha}{\Gamma(1+\alpha)}\right)\right] \\ &= 0, \\ u_2(x, y) &= L_\alpha^{(-2)}\left[\frac{\Gamma(1+\alpha)}{y^\alpha} \frac{\partial^{2\alpha} u_1(x, y)}{\partial y^{2\alpha}}\right] \\ &= L_\alpha^{(-2)}\left[\frac{\Gamma(1+\alpha)}{y^\alpha} \frac{\partial^{2\alpha}}{\partial y^{2\alpha}}(0)\right] \\ &= 0, \\ u_3(x, y) &= L_\alpha^{(-2)}\left[\frac{\Gamma(1+\alpha)}{y^\alpha} \frac{\partial^{2\alpha} u_2(x, y)}{\partial y^{2\alpha}}\right] \\ &= L_\alpha^{(-2)}\left[\frac{\Gamma(1+\alpha)}{y^\alpha} \frac{\partial^{2\alpha}}{\partial y^{2\alpha}}(0)\right] \\ &= 0, \\ u_4(x, y) &= L_\alpha^{(-2)}\left[\frac{\Gamma(1+\alpha)}{y^\alpha} \frac{\partial^{2\alpha} u_3(x, y)}{\partial y^{2\alpha}}\right] \\ &= L_\alpha^{(-2)}\left[\frac{\Gamma(1+\alpha)}{y^\alpha} \frac{\partial^{2\alpha}}{\partial y^{2\alpha}}(0)\right] \\ &= 0, \\ &\vdots \\ u_n(x, y) &= 0. \end{aligned} \tag{29}$$

Finally, we obtain the solution of (9) with the local fractional derivative boundary value conditions (27), namely,

$$\begin{aligned} u(x, y) &= \lim_{n\to\infty} \phi_n(x, y) \\ &= \lim_{n\to\infty} \sum_{n=0}^{\infty} u_n(x, y) \\ &= \cos_\alpha(x^\alpha) \frac{y^\alpha}{\Gamma(1+\alpha)}, \end{aligned} \tag{30}$$

whose graph is shown in Figure 1.

Example 3. Let us consider the initial-boundary value conditions for the local fractional Tricomi equation in the form

$$\begin{aligned} u(0, y) &= 0, \\ u(l, y) &= 0, \\ u(x, 0) &= \frac{x^\alpha}{\Gamma(1+\alpha)}, \\ \frac{\partial^\alpha u(x, 0)}{\partial x^\alpha} &= \sin_\alpha(x^\alpha). \end{aligned} \tag{31}$$

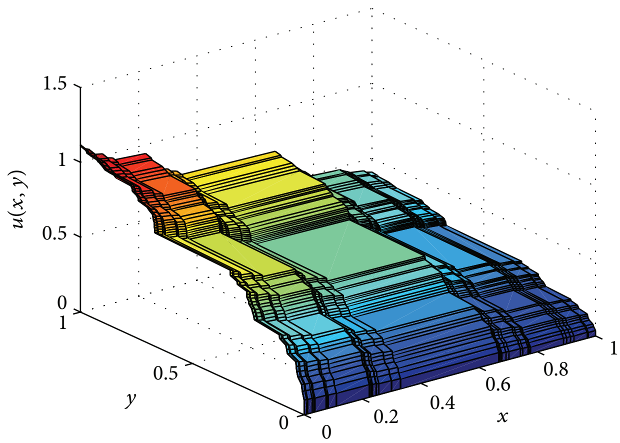

Figure 1: The plot of the solution of (9) with the local fractional derivative boundary value conditions (27) when $\alpha = \ln 2/\ln 3$.

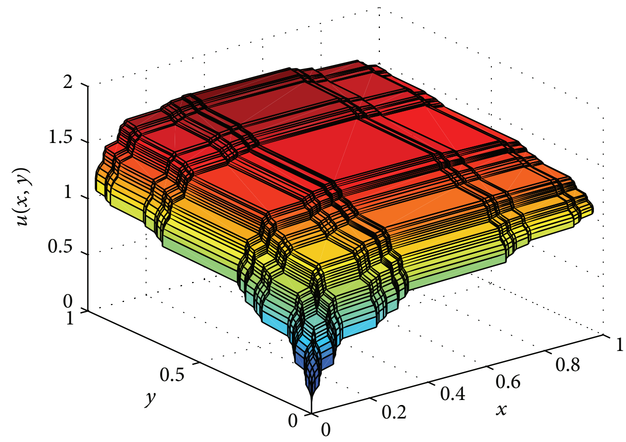

Figure 2: The plot of the solution of (9) with the local fractional derivative boundary value conditions (31) when $\alpha = \ln 2/\ln 3$.

Making use of (31), the recurrence formula can be written as

$$\begin{aligned} u_{n+1}(x, y) &= L_\alpha^{(-2)}\left[\frac{\Gamma(1+\alpha)}{y^\alpha}\frac{\partial^{2\alpha}u_n(x, y)}{\partial y^{2\alpha}}\right], \\ u_0(x, y) &= \frac{x^\alpha}{\Gamma(1+\alpha)} + \sin_\alpha(x^\alpha)\frac{y^\alpha}{\Gamma(1+\alpha)}. \end{aligned} \tag{32}$$

Appling (32) gives the following equations:

$$\begin{aligned} u_1(x, y) &= L_\alpha^{(-2)}\left[\frac{\Gamma(1+\alpha)}{y^\alpha}\frac{\partial^{2\alpha}u_0(x, y)}{\partial y^{2\alpha}}\right] \\ &= L_\alpha^{(-2)}\left[\frac{\Gamma(1+\alpha)}{y^\alpha}\frac{\partial^{2\alpha}}{\partial y^{2\alpha}}\right. \\ &\qquad \left.\times\left(\frac{x^\alpha}{\Gamma(1+\alpha)} + \sin_\alpha(x^\alpha)\frac{y^\alpha}{\Gamma(1+\alpha)}\right)\right] \\ &= 0, \\ u_2(x, y) &= L_\alpha^{(-2)}\left[\frac{\Gamma(1+\alpha)}{y^\alpha}\frac{\partial^{2\alpha}u_1(x, y)}{\partial y^{2\alpha}}\right] \\ &= L_\alpha^{(-2)}\left[\frac{\Gamma(1+\alpha)}{y^\alpha}\frac{\partial^{2\alpha}}{\partial y^{2\alpha}}(0)\right] \\ &= 0, \\ u_3(x, y) &= L_\alpha^{(-2)}\left[\frac{\Gamma(1+\alpha)}{y^\alpha}\frac{\partial^{2\alpha}u_2(x, y)}{\partial y^{2\alpha}}\right] \\ &= L_\alpha^{(-2)}\left[\frac{\Gamma(1+\alpha)}{y^\alpha}\frac{\partial^{2\alpha}}{\partial y^{2\alpha}}(0)\right] \\ &= 0, \\ u_4(x, y) &= L_\alpha^{(-2)}\left[\frac{\Gamma(1+\alpha)}{y^\alpha}\frac{\partial^{2\alpha}u_3(x, y)}{\partial y^{2\alpha}}\right] \\ &= L_\alpha^{(-2)}\left[\frac{\Gamma(1+\alpha)}{y^\alpha}\frac{\partial^{2\alpha}}{\partial y^{2\alpha}}(0)\right] \\ &= 0, \\ &\vdots \\ u_n(x, y) &= 0. \end{aligned} \tag{33}$$

Finally, the solution of (9) with the local fractional derivative boundary value conditions (31) reads

$$\begin{aligned} u(x, y) &= \lim_{n\to\infty}\phi_n(x, y) \\ &= \lim_{n\to\infty}\sum_{n=0}^{\infty}u_n(x, y) \\ &= \frac{x^\alpha}{\Gamma(1+\alpha)} + \sin_\alpha(x^\alpha)\frac{y^\alpha}{\Gamma(1+\alpha)}, \end{aligned} \tag{34}$$

and its graph is shown in Figure 2.

5. Conclusions

In this work we discussed the nondifferentiable numerical solutions for the local fractional Tricomi equation arising in fractal transonic flow with the local fractional derivative boundary value conditions by using the local fractional decomposition method and their plots were also shown in the MatLab software.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Existence of Solutions for Fractional q-Integrodifference Equations with Nonlocal Fractional q-Integral Conditions

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We study a class of fractional q-integrodifference equations with nonlocal fractional q-integral boundary conditions which have different quantum numbers. By applying the Banach contraction principle, Krasnoselskii's fixed point theorem, and Leray-Schauder nonlinear alternative, the existence and uniqueness of solutions are obtained. In addition, some examples to illustrate our results are given.

1. Introduction

In this paper, we deal with the following nonlocal fractional q-integral boundary value problem of nonlinear fractional q-integrodifference equation:

$$\begin{gathered} D_q^{\alpha} x(t) = f\left(t, x(t), I_z^{\delta} x(t)\right), \quad t \in (0,T), \\ x(0) = 0, \qquad \lambda I_p^{\beta} x(\eta) = I_r^{\gamma} x(\xi), \end{gathered} \tag{1}$$

where $0 < p,q,r,z < 1$, $1 < \alpha \le 2$, $\beta,\gamma,\delta > 0$, $\lambda \in \mathbb{R}$ are given constants, D_q^{α} is the fractional q-derivative of Riemann-Liouville type of order α, I_{ϕ}^{ψ} is the fractional ϕ-integral of order ψ with $\phi = p,r,z$, and $\psi = \beta,\gamma,\delta$, $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

The early work on q-difference calculus or *quantum calculus* dates back to Jackson's paper [1]. Basic definitions and properties of quantum calculus can be found in the book [2]. The fractional q-difference calculus had its origin in the works by Al-Salam [3] and Agarwal [4]. Motivated by recent interest in the study of fractional-order differential equations, the topic of q-fractional equations has attracted the attention of many researchers. The details of some recent development of the subject can be found in ([5–17]) and the references cited therein, whereas the background material on q-fractional calculus can be found in a recent book [18].

In this paper, we will study the existence and uniqueness of solutions of a class of boundary value problems for fractional q-integrodifference equations with nonlocal fractional q-integral conditions which have different quantum numbers. So, the novelty of this paper lies in the fact that there are *four different quantum numbers*. In addition, the boundary condition of (1) does not contain the value of unknown function x at the right side of boundary point $t = T$. *One may interpret the q-integral boundary condition in* (1) *as the q-integrals with different quantum numbers are related through a real number λ.*

The paper is organized as follows. In Section 2, for the convenience of the reader, we cite some definitions and fundamental results on q-calculus as well as the fractional q-calculus. Some auxiliary lemmas, needed in the proofs of our main results, are presented in Section 3. Section 4 contains the existence and uniqueness results for problem (1) which are shown by applying Banach's contraction principle, Krasnoselskii's fixed point theorem, and Leray-Schauder's nonlinear alternative. Finally, some examples illustrating the applicability of our results are presented in Section 5.

2. Preliminaries

To make this paper self-contained, below we recall some known facts on fractional q-calculus. The presentation here can be found in, for example, [6, 18, 19].

For $q \in (0, 1)$, define

$$[a]_q = \frac{1-q^a}{1-q}, \quad a \in \mathbb{R}. \tag{2}$$

The q-analogue of the power function $(1-b)^k$ with $k \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$ is

$$(1-b)^{(0)} = 1, \qquad (1-b)^{(k)} = \prod_{i=0}^{k-1} \left(1 - bq^i\right), \quad k \in \mathbb{N}, \quad b \in \mathbb{R}. \tag{3}$$

More generally, if $\gamma \in \mathbb{R}$, then

$$(1-b)^{(\gamma)} = \prod_{i=0}^{\infty} \frac{1-bq^i}{1-bq^{\gamma+i}}. \tag{4}$$

We use the notation $0^{(\gamma)} = 0$ for $\gamma > 0$. The q-gamma function is defined by

$$\Gamma_q(x) = \frac{(1-q)^{(x-1)}}{(1-q)^{x-1}}, \quad x \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}. \tag{5}$$

Obviously, $\Gamma_q(x+1) = [x]_q \Gamma_q(x)$.

The q-derivative of a function h is defined by

$$\left(D_q h\right)(x) = \frac{h(x) - h(qx)}{(1-q)x} \quad \text{for } x \neq 0, \qquad \left(D_q h\right)(0) = \lim_{x \to 0} \left(D_q h\right)(x), \tag{6}$$

and q-derivatives of higher order are given by

$$\left(D_q^0 h\right)(x) = h(x), \qquad \left(D_q^k h\right)(x) = D_q \left(D_q^{k-1} h\right)(x), \quad k \in \mathbb{N}. \tag{7}$$

The q-integral of a function h defined on the interval $[0, b]$ is given by

$$\left(I_q h\right)(x) = \int_0^x h(s)\, d_q s = x(1-q) \sum_{i=0}^{\infty} h\left(xq^i\right) q^i, \quad x \in [0, b]. \tag{8}$$

If $a \in [0, b]$ and h is defined in the interval $[0, b]$, then its integral from a to b is defined by

$$\int_a^b h(s)\, d_q s = \int_0^b h(s)\, d_q s - \int_0^a h(s)\, d_q s. \tag{9}$$

Similar to derivatives, an operator I_q^k is given by

$$\left(I_q^0 h\right)(x) = h(x), \qquad \left(I_q^k h\right)(x) = I_q \left(I_q^{k-1} h\right)(x), \quad k \in \mathbb{N}. \tag{10}$$

The fundamental theorem of calculus applies to operators D_q and I_q; that is,

$$\left(D_q I_q h\right)(x) = h(x), \tag{11}$$

and if h is continuous at $x = 0$. Then

$$\left(I_q D_q h\right)(x) = h(x) - h(0). \tag{12}$$

Definition 1. Let $\nu \geq 0$ and h be a function defined on $[0, T]$. The fractional q-integral of Riemann-Liouville type is given by $(I_q^0 h)(x) = h(x)$ and

$$\left(I_q^\nu h\right)(x) = \frac{1}{\Gamma_q(\nu)} \int_0^x (x - qs)^{(\nu-1)} h(s)\, d_q s, \quad \nu > 0, \quad x \in [0, T]. \tag{13}$$

Definition 2. The fractional q-derivative of Riemann-Liouville type of order $\nu \geq 0$ is defined by $(D_q^0 h)(x) = h(x)$ and

$$\left(D_q^\nu h\right)(x) = \left(D_q^l I_q^{l-\nu} h\right)(x), \quad \nu > 0, \tag{14}$$

where l is the smallest integer greater than or equal to ν.

Definition 3. For any $m, n > 0$,

$$B_q(m, n) = \int_0^1 u^{(m-1)} (1 - qu)^{(n-1)} d_q u \tag{15}$$

is called the q-beta function.

The expression of q-beta function in terms of the q-gamma function can be written as

$$B_q(m, n) = \frac{\Gamma_q(m) \Gamma_q(n)}{\Gamma_q(m+n)}. \tag{16}$$

Lemma 4 (see [4]). *Let $\alpha, \beta \geq 0$, and f be a function defined in $[0, T]$. Then, the following formulas hold:*

(1) $(I_q^\beta I_q^\alpha f)(x) = (I_q^{\alpha+\beta} f)(x)$;
(2) $(D_q^\alpha I_q^\alpha f)(x) = f(x)$.

Lemma 5 (see [6]). *Let $\alpha > 0$ and ν be a positive integer. Then, the following equality holds:*

$$\left(I_q^\alpha D_q^\nu f\right)(x) = \left(D_q^\nu I_q^\alpha f\right)(x) - \sum_{k=0}^{\nu-1} \frac{x^{\alpha-\nu+k}}{\Gamma_q(\alpha + k - \nu + 1)} \left(D_q^k f\right)(0). \tag{17}$$

3. Some Auxiliary Lemmas

Lemma 6. *Let $\alpha, \beta > 0$, and $0 < q < 1$. Then one has*

$$\int_0^\eta (\eta - qs)^{(\alpha-1)} s^\beta d_q s = \eta^{\alpha+\beta} B_q(\alpha, \beta + 1). \tag{18}$$

Proof. Using the definitions of q-analogue of power function and q-beta function, we have

$$\begin{aligned}
&\int_0^\eta (\eta - qs)^{(\alpha-1)} s^\beta d_q s \\
&= (1-q)\eta \sum_{n=0}^{\infty} q^n (\eta - q\eta q^n)^{(\alpha-1)} (\eta q^n)^\beta \\
&= (1-q)\eta \sum_{n=0}^{\infty} q^n \eta^{\alpha-1} (1 - qq^n)^{(\alpha-1)} \eta^\beta q^{n\beta} \\
&= (1-q)\eta^{\alpha+\beta} \sum_{n=0}^{\infty} q^n (1 - qq^n)^{(\alpha-1)} q^{n\beta} \\
&= \eta^{\alpha+\beta} \int_0^1 (1-qs)^{(\alpha-1)} s^{(\beta)} d_q s \\
&= \eta^{\alpha+\beta} B_q(\alpha, \beta+1).
\end{aligned} \tag{19}$$

The proof is complete. □

Lemma 7. *Let* $\alpha, \beta, \gamma > 0$*, and* $0 < p, q, r < 1$*. Then one has*

$$\begin{aligned}
&\int_0^\eta \int_0^x \int_0^y (\eta - px)^{(\alpha-1)} (x - qy)^{(\beta-1)} (y - rz)^{(\gamma-1)} d_r z \, d_q y \, d_p x \\
&= \frac{1}{[\gamma]_r} B_p(\alpha, \beta+\gamma+1) B_q(\beta, \gamma+1) \eta^{\alpha+\beta+\gamma}.
\end{aligned} \tag{20}$$

Proof. Taking into account Lemma 6, we have

$$\begin{aligned}
&\int_0^\eta \int_0^x \int_0^y (\eta - px)^{(\alpha-1)} (x - qy)^{(\beta-1)} (y - rz)^{(\gamma-1)} d_r z \, d_q y \, d_p x \\
&= \frac{1}{[\gamma]_r} \int_0^\eta \int_0^x (\eta - px)^{(\alpha-1)} (x - qy)^{(\beta-1)} y^{(\gamma)} d_q y \, d_p x \\
&= \frac{1}{[\gamma]_r} \int_0^\eta (\eta - px)^{(\alpha-1)} \int_0^x (x - qy)^{(\beta-1)} y^{(\gamma)} d_q y \, d_p x \\
&= \frac{1}{[\gamma]_r} B_q(\beta, \gamma+1) \int_0^\eta (\eta - px)^{(\alpha-1)} x^{(\beta+\gamma)} d_p x \\
&= \frac{1}{[\gamma]_r} B_p(\alpha, \beta+\gamma+1) B_q(\beta, \gamma+1) \eta^{\alpha+\beta+\gamma}.
\end{aligned} \tag{21}$$

This completes the proof. □

Lemma 8. *Let* $\beta, \gamma > 0$*,* $\lambda \in \mathbb{R}$*, and* $0 < p, q, r < 1$*. Then, for* $y \in C([0,T], \mathbb{R})$*, the unique solution of boundary value problem,*

$$D_q^\alpha x(t) = y(t), \quad t \in (0,T), \ 1 < \alpha \leq 2, \tag{22}$$

subject to the nonlocal fractional condition,

$$x(0) = 0, \qquad \lambda I_p^\beta x(\eta) = I_r^\gamma x(\xi), \tag{23}$$

is given by

$$\begin{aligned}
x(t) = {} & \frac{\lambda t^{\alpha-1}}{\Omega \Gamma_p(\beta) \Gamma_q(\alpha)} \\
& \times \int_0^\eta \int_0^s (\eta - ps)^{(\beta-1)} (s - qu)^{(\alpha-1)} y(u) \, d_q u \, d_p s \\
& - \frac{t^{\alpha-1}}{\Omega \Gamma_r(\gamma) \Gamma_q(\alpha)} \\
& \times \int_0^\xi \int_0^s (\xi - rs)^{(\gamma-1)} (s - qu)^{(\alpha-1)} y(u) \, d_q u \, d_r s \\
& + \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} y(s) \, d_q s,
\end{aligned} \tag{24}$$

where

$$\Omega = \frac{\Gamma_r(\alpha)}{\Gamma_r(\alpha+\gamma)} \xi^{\alpha+\gamma-1} - \lambda \frac{\Gamma_p(\alpha)}{\Gamma_p(\alpha+\beta)} \eta^{\alpha+\beta-1} \neq 0. \tag{25}$$

Proof. From $1 < \alpha \leq 2$, we let $n = 2$. Using the Definition 2 and Lemma 4, (22) can be expressed as

$$\left(I_q^\alpha D_q^2 I_q^{2-\alpha} x \right)(t) = \left(I_q^\alpha y \right)(t). \tag{26}$$

From Lemma 5, we have

$$x(t) = k_1 t^{\alpha-1} + k_2 t^{\alpha-2} + \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} y(s) \, d_q s \tag{27}$$

for some constants $k_1, k_2 \in \mathbb{R}$. It follows from the first condition of (23) that $k_2 = 0$. Applying the Riemann-Liouville fractional p-integral of order $\beta > 0$ for (27) with $k_2 = 0$ and taking into account of Lemma 6, we have

$$\begin{aligned}
I_p^\beta x(t) = {} & \int_0^t \frac{(t - ps)^{(\beta-1)}}{\Gamma_p(\beta)} \\
& \times \left(k_1 s^{\alpha-1} + \int_0^s \frac{(s - qu)^{(\alpha-1)}}{\Gamma_q(\alpha)} y(u) \, d_q u \right) d_p s \\
= {} & \frac{k_1}{\Gamma_p(\beta)} \int_0^t (t - ps)^{(\beta-1)} s^{\alpha-1} d_p s + \frac{1}{\Gamma_p(\beta) \Gamma_q(\alpha)} \\
& \times \int_0^t \int_0^s (t - ps)^{(\beta-1)} (s - qu)^{(\alpha-1)} y(u) \, d_q u \, d_p s \\
= {} & k_1 \frac{\Gamma_p(\alpha)}{\Gamma_p(\alpha+\beta)} t^{\alpha+\beta-1} + \frac{1}{\Gamma_p(\beta) \Gamma_q(\alpha)} \\
& \times \int_0^t \int_0^s (t - ps)^{(\beta-1)} (s - qu)^{(\alpha-1)} y(u) \, d_q u \, d_p s.
\end{aligned} \tag{28}$$

In particular, we have

$$I_p^{\beta} x(\eta) = k_1 \frac{\Gamma_p(\alpha)}{\Gamma_p(\alpha+\beta)} \eta^{\alpha+\beta-1} + \frac{1}{\Gamma_p(\beta)\Gamma_q(\alpha)} \times \int_0^{\eta}\int_0^{s} (\eta - ps)^{(\beta-1)} (s-qu)^{(\alpha-1)} y(u)\, d_q u\, d_p s. \tag{29}$$

Using the Riemann-Liouville fractional r-integral of order $\gamma > 0$ and repeating the above process, we get

$$I_r^{\gamma} x(\xi) = k_1 \frac{\Gamma_r(\alpha)}{\Gamma_r(\alpha+\gamma)} \xi^{\alpha+\gamma-1} + \frac{1}{\Gamma_r(\gamma)\Gamma_q(\alpha)} \times \int_0^{\xi}\int_0^{s} (\xi - rs)^{(\gamma-1)} (s-qu)^{(\alpha-1)} y(u)\, d_q u\, d_r s. \tag{30}$$

The second nonlocal condition of (23) implies

$$\begin{aligned} k_1 = {} & \frac{\lambda}{\Omega\Gamma_p(\beta)\Gamma_q(\alpha)} \times \int_0^{\eta}\int_0^{s} (\eta - ps)^{(\beta-1)} (s-qu)^{(\alpha-1)} y(u)\, d_q u\, d_p s \\ & - \frac{1}{\Omega\Gamma_r(\gamma)\Gamma_q(\alpha)} \times \int_0^{\xi}\int_0^{s} (\xi - rs)^{(\gamma-1)} (s-qu)^{(\alpha-1)} y(u)\, d_q u\, d_r s. \end{aligned} \tag{31}$$

Substituting the values of k_1 and k_2 in (27), we get the desired result in (24). □

4. Main Results

In this section, we denote $\mathscr{C} = C([0,T],\mathbb{R})$ as the Banach space of all continuous functions from $[0,T]$ to $\mathbb{R}$ endowed with the norm defined by $\|x\| = \sup_{t\in[0,T]}|x(t)|$. In view of Lemma 8, we define an operator $\mathcal{Q} : \mathscr{C} \to \mathscr{C}$ by

$$\begin{aligned} (\mathcal{Q}x)(t) = {} & \frac{\lambda t^{\alpha-1}}{\Omega\Gamma_p(\beta)\Gamma_q(\alpha)} \times \int_0^{\eta}\int_0^{s} (\eta - ps)^{(\beta-1)} (s-qu)^{(\alpha-1)} \times f\left(u, x(u), I_z^{\delta} x(u)\right) d_q u\, d_p s \\ & - \frac{t^{\alpha-1}}{\Omega\Gamma_r(\gamma)\Gamma_q(\alpha)} \times \int_0^{\xi}\int_0^{s} (\xi - rs)^{(\gamma-1)} (s-qu)^{(\alpha-1)} \times f\left(u, x(u), I_z^{\delta} x(u)\right) d_q u\, d_r s \\ & + \int_0^{t} \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f\left(s, x(s), I_z^{\delta} x(s)\right) d_q s, \end{aligned} \tag{32}$$

where $\Omega \neq 0$ is defined by (25). It should be noticed that problem (1) has solutions if and only if the operator $\mathcal{Q}$ has fixed points.

For the sake of convenience of proving the results, we set

$$\begin{aligned} \Lambda = {} & \frac{|\lambda| T^{\alpha-1}}{|\Omega| \Gamma_p(\beta)} \times \left[\frac{\eta^{\alpha+\beta} B_p(\beta,\alpha+1) L_1}{\Gamma_q(\alpha+1)} + \frac{\eta^{\alpha+\beta+\delta} B_q(\alpha,\delta+1) B_p(\beta,\alpha+\delta+1) L_2}{\Gamma_q(\alpha)\Gamma_z(\delta+1)} \right] \\ & + \frac{T^{\alpha-1}}{|\Omega| \Gamma_r(\gamma)} \times \left[\frac{\xi^{\alpha+\gamma} B_r(\gamma,\alpha+1) L_1}{\Gamma_q(\alpha+1)} + \frac{\xi^{\alpha+\gamma+\delta} B_q(\alpha,\delta+1) B_r(\gamma,\alpha+\delta+1) L_2}{\Gamma_q(\alpha)\Gamma_z(\delta+1)} \right] \\ & + \frac{T^{\alpha}}{\Gamma_q(\alpha+1)} L_1 + \frac{T^{\alpha+\delta} B_q(\alpha,\delta+1)}{\Gamma_q(\alpha)\Gamma_z(\delta+1)} L_2, \end{aligned} \tag{33}$$

$$\Psi = \frac{|\lambda| T^{\alpha-1} \eta^{\alpha+\beta} B_p(\beta,\alpha+1)}{|\Omega| \Gamma_p(\beta)\Gamma_q(\alpha+1)} + \frac{T^{\alpha-1} \xi^{\alpha+\gamma} B_r(\gamma,\alpha+1)}{|\Omega| \Gamma_r(\gamma)\Gamma_q(\alpha+1)} + \frac{T^{\alpha}}{\Gamma_q(\alpha+1)}. \tag{34}$$

The first result on the existence and uniqueness of solutions is based on the Banach contraction mapping principle.

Theorem 9. *Let $f : [0,T]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the assumption:*

(H_1) there exist constants $L_1, L_2 > 0$ such that

$$\left| f(t, w_1, w_2) - f(t, \overline{w}_1, \overline{w}_2) \right| \le L_1 \left| w_1 - \overline{w}_1 \right| + L_2 \left| w_2 - \overline{w}_2 \right|, \tag{35}$$

for each $t \in [0,T]$ and $w_1, w_2, \overline{w}_1, \overline{w}_2 \in \mathbb{R}$.

If

$$\Lambda \le \theta < 1, \tag{36}$$

where Λ is given by (33)*, then the boundary value problem* (1) *has a unique solution on* $[0,T]$.

Proof. We transform problem (1) into a fixed point problem, $x = \mathcal{Q}x$, where the operator $\mathcal{Q}$ is defined by (32). By applying the Banach contraction mapping principle, we will show that $\mathcal{Q}$ has a fixed point which is the unique solution of problem (1).

Setting $\sup_{t\in[0,T]}|f(t,0,0)| = M < \infty$ and choosing

$$r \geq \frac{\Psi M}{1-\varepsilon}, \tag{37}$$

where $\theta \leq \varepsilon < 1$, and the constant Ψ defined by (34), we will show that $\mathcal{Q}B_r \subset B_r$, where $B_r = \{x \in \mathcal{C} : \|x\| \leq r\}$. For any $x \in B_r$, we have

$$\begin{aligned}
&|\mathcal{Q}x(t)| \\
&\leq \sup_{t\in[0,T]} \left\{ \frac{|\lambda| t^{\alpha-1}}{|\Omega| \Gamma_p(\beta) \Gamma_q(\alpha)} \right. \\
&\qquad \times \int_0^\eta \int_0^s (\eta - ps)^{(\beta-1)} (s-qu)^{(\alpha-1)} \\
&\qquad\qquad \times \left| f\left(u, x(u), I_z^\delta x(u)\right)\right| d_q u\, d_p s \\
&\qquad + \frac{t^{\alpha-1}}{|\Omega| \Gamma_r(\gamma) \Gamma_q(\alpha)} \\
&\qquad \times \int_0^\xi \int_0^s (\xi - rs)^{(\gamma-1)} (s-qu)^{(\alpha-1)} \\
&\qquad\qquad \times \left| f\left(u, x(u), I_z^\delta x(u)\right)\right| d_q u\, d_r s \\
&\qquad \left. + \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \left| f\left(s, x(s), I_z^\delta x(s)\right)\right| d_q s \right\}.
\end{aligned} \tag{38}$$

The assumption (H_1) implies that

$$\begin{aligned}
\left| f(t, w_1, w_2)\right| &\leq \left| f(t, w_1, w_2) - f(t,0,0)\right| + \left| f(t,0,0)\right| \\
&\leq L_1 |w_1| + L_2 |w_2| + M,
\end{aligned} \tag{39}$$

for all $t \in [0,T]$ and $w_1, w_2 \in \mathbb{R}$.

Then, by using Lemmas 6 and 7, we have

$$\begin{aligned}
&|\mathcal{Q}x(t)| \\
&\leq \frac{|\lambda| T^{\alpha-1}}{|\Omega| \Gamma_p(\beta) \Gamma_q(\alpha)} \\
&\quad \times \int_0^\eta \int_0^s (\eta - ps)^{(\beta-1)} (s-qu)^{(\alpha-1)} \\
&\qquad \times \left| f\left(u, x(u), I_z^\delta x(u)\right)\right| d_q u\, d_p s \\
&\quad + \frac{T^{\alpha-1}}{|\Omega| \Gamma_r(\gamma) \Gamma_q(\alpha)} \\
&\quad \times \int_0^\xi \int_0^s (\xi - rs)^{(\gamma-1)} (s-qu)^{(\alpha-1)} \\
&\qquad \times \left| f\left(u, x(u), I_z^\delta x(u)\right)\right| d_q u\, d_r s \\
&\quad + \int_0^T \frac{(T-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \left| f\left(s, x(s), I_z^\delta x(s)\right)\right| d_q s \\
&\leq \frac{|\lambda| T^{\alpha-1}}{|\Omega| \Gamma_p(\beta) \Gamma_q(\alpha)} \\
&\quad \times \int_0^\eta \int_0^s (\eta - ps)^{(\beta-1)} (s-qu)^{(\alpha-1)} \\
&\qquad \times \left(L_1 r + L_2 r \int_0^u \frac{(u-zv)^{(\delta-1)}}{\Gamma_z(\delta)} d_z v + M \right) d_q u\, d_p s \\
&\quad + \frac{T^{\alpha-1}}{|\Omega| \Gamma_r(\gamma) \Gamma_q(\alpha)} \\
&\quad \times \int_0^\xi \int_0^s (\xi - rs)^{(\gamma-1)} (s-qu)^{(\alpha-1)} \\
&\qquad \times \left(L_1 r + L_2 r \int_0^u \frac{(u-zv)^{(\delta-1)}}{\Gamma_z(\delta)} d_z v + M \right) d_q u\, d_r s \\
&\quad + \int_0^T \frac{(T-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \\
&\qquad \times \left(L_1 r + L_2 r \int_0^s \frac{(s-zv)^{(\delta-1)}}{\Gamma_z(\delta)} d_z v + M \right) d_q s \\
&= \frac{|\lambda| T^{\alpha-1}}{|\Omega| \Gamma_p(\beta) \Gamma_q(\alpha)} \\
&\quad \times \left(\frac{\eta^{\alpha+\beta}}{[\alpha]_q} (L_1 r + M) B_p(\beta, \alpha+1) \right. \\
&\qquad \left. + \frac{L_2 r \eta^{\alpha+\beta+\delta}}{\Gamma_z(\delta) [\delta]_z} B_q(\alpha, \delta+1) B_p(\beta, \alpha+\delta+1) \right) \\
&\quad + \frac{T^{\alpha-1}}{|\Omega| \Gamma_r(\gamma) \Gamma_q(\alpha)} \\
&\quad \times \left(\frac{\xi^{\alpha+\gamma}}{[\alpha]_q} (L_1 r + M) B_r(\gamma, \alpha+1) \right. \\
&\qquad \left. + \frac{L_2 r \xi^{\alpha+\gamma+\delta}}{\Gamma_z(\delta) [\delta]_z} B_q(\alpha, \delta+1) B_r(\gamma, \alpha+\delta+1) \right) \\
&\quad + \frac{T^\alpha}{\Gamma_q(\alpha) [\alpha]_q} (L_1 r + M) + \frac{T^{\alpha+\delta} B_q(\alpha, \delta+1)}{\Gamma_q(\alpha) \Gamma_z(\delta) [\delta]_z} L_2 r \\
&= \Lambda r + \Psi M \leq r.
\end{aligned} \tag{40}$$

Then, we have $\|\mathcal{Q}x\| \leq r$ which yields $\mathcal{Q}B_r \subset B_r$.

Next, for any $x, y \in \mathscr{C}$ and for each $t \in [0, T]$, we have

$$\begin{aligned}
&\left|\mathscr{Q}x(t) - \mathscr{Q}y(t)\right| \\
&\leq \frac{|\lambda| T^{\alpha-1}}{|\Omega| \Gamma_p(\beta) \Gamma_q(\alpha)} \\
&\quad \times \int_0^\eta \int_0^s (\eta - ps)^{(\beta-1)} (s - qu)^{(\alpha-1)} \\
&\qquad \times \left(\left| f\left(u, x(u), I_z^\delta x(u)\right) \right.\right. \\
&\qquad\quad \left.\left. - f\left(u, y(u), I_z^\delta y(u)\right)\right|\right) d_q u \, d_p s \\
&\quad + \frac{T^{\alpha-1}}{|\Omega| \Gamma_r(\gamma) \Gamma_q(\alpha)} \\
&\quad \times \int_0^\xi \int_0^s (\xi - rs)^{(\gamma-1)} (s - qu)^{(\alpha-1)} \\
&\qquad \times \left(\left| f\left(u, x(u), I_z^\delta x(u)\right) \right.\right. \\
&\qquad\quad \left.\left. - f\left(u, y(u), I_z^\delta y(u)\right)\right|\right) d_q u \, d_r s \\
&\quad + \int_0^T \frac{(T - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \\
&\qquad \times \left(\left| f\left(s, x(s), I_z^\delta x(s)\right) - f\left(s, y(s), I_z^\delta y(s)\right)\right|\right) d_q s \\
&\leq \frac{|\lambda| T^{\alpha-1}}{|\Omega| \Gamma_p(\beta) \Gamma_q(\alpha)} \\
&\quad \times \int_0^\eta \int_0^s (\eta - ps)^{(\beta-1)} (s - qu)^{(\alpha-1)} \\
&\qquad \times \left(L_1 \|x - y\| \right. \\
&\qquad\quad \left. + L_2 \|x - y\| \int_0^u \frac{(u - zv)^{(\delta-1)}}{\Gamma_z(\delta)} d_z v \right) d_q u \, d_p s \\
&\quad + \frac{T^{\alpha-1}}{|\Omega| \Gamma_r(\gamma) \Gamma_q(\alpha)} \\
&\quad \times \int_0^\xi \int_0^s (\xi - rs)^{(\gamma-1)} (s - qu)^{(\alpha-1)} \\
&\qquad \times \left(L_1 \|x - y\| + L_2 \|x - y\| \right. \\
&\qquad\quad \left. \times \int_0^u \frac{(u - zv)^{(\delta-1)}}{\Gamma_z(\delta)} d_z v \right) d_q u \, d_r s \\
&\quad + \int_0^T \frac{(T - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \\
&\qquad \times \left(L_1 \|x - y\| + L_2 \|x - y\| \right. \\
&\qquad\quad \left. \times \int_0^s \frac{(s - zv)^{(\delta-1)}}{\Gamma_z(\delta)} d_z v \right) d_q s \\
&= \Lambda \|x - y\|.
\end{aligned} \tag{41}$$

The above result implies that $\|\mathscr{Q}x - \mathscr{Q}y\| \leq \Lambda \|x - y\|$. As $\Lambda < 1$, $\mathscr{Q}$ is a contraction. Hence, by the Banach contraction mapping principle, we deduce that $\mathscr{Q}$ has a fixed point which is the unique solution of problem (1). □

The second existence result is based on Krasnoselskii's fixed point theorem.

Lemma 10 (Krasnoselskii's fixed point theorem [20]). *Let M be a closed, bounded, convex, and nonempty subset of a Banach space X. Let A, B be the operators such that* (a) *$Ax + By \in M$ whenever $x, y \in M$;* (b) *A is compact and continuous;* (c) *B is a contraction mapping. Then there exists $z \in M$ such that $z = Az + Bz$.*

Theorem 11. *Assume that $f : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the assumption (H_1). In addition one supposes that*

(H_2) $|f(t, w_1, w_2)| \leq \kappa(t)$, for all $(t, w_1, w_2) \in [0, T] \times \mathbb{R} \times \mathbb{R}$ and $\kappa \in C([0, T], \mathbb{R}^+)$.

If

$$\frac{T^\alpha}{\Gamma_q(\alpha + 1)} L_1 + \frac{T^{\alpha+\delta} B_q(\alpha, \delta + 1)}{\Gamma_q(\alpha) \Gamma_z(\delta + 1)} L_2 < 1, \tag{42}$$

then the boundary value problem (1) *has at least one solution on* $[0, T]$.

Proof. Let us set $\sup_{t \in [0,T]} |\kappa(t)| = \|\kappa\|$ and choose a suitable constant ρ as

$$\rho \geq \|\kappa\| \Psi, \tag{43}$$

where Ψ is defined by (34). Now, we define the operators $\mathscr{Q}_1$ and $\mathscr{Q}_2$ on the set $B_\rho = \{x \in \mathscr{C} : \|x\| \leq \rho\}$ as

$$\begin{aligned}
&(\mathscr{Q}_1 x)(t) \\
&\quad = \frac{\lambda t^{\alpha-1}}{\Omega \Gamma_p(\beta) \Gamma_q(\alpha)} \\
&\qquad \times \int_0^\eta \int_0^s (\eta - ps)^{(\beta-1)} (s - qu)^{(\alpha-1)} \\
&\qquad\quad \times f\left(u, x(u), I_z^\delta x(u)\right) d_q u \, d_p s \\
&\qquad - \frac{t^{\alpha-1}}{\Omega \Gamma_r(\gamma) \Gamma_q(\alpha)}
\end{aligned}$$

$$\times \int_0^{\xi}\int_0^{s}(\xi - rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)} \times f\left(u, x(u), I_z^{\delta}x(u)\right) d_q u\, d_r s$$

$$(\mathcal{Q}_2 x)(t) = \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f\left(s, x(s), I_z^{\delta}x(s)\right) d_q s. \tag{44}$$

Firstly, we will show that the operators $\mathcal{Q}_1$ and $\mathcal{Q}_2$ satisfy condition (a) of Lemma 10. For $x, y \in B_\rho$, we have

$$\begin{aligned}
&\left\|\mathcal{Q}_1 x + \mathcal{Q}_2 y\right\| \\
&\quad \le \|\kappa\| \frac{|\lambda| T^{\alpha-1}}{|\Omega|\Gamma_p(\beta)\Gamma_q(\alpha)} \times \int_0^{\eta}\int_0^{s}(\eta - ps)^{(\beta-1)}(s-qu)^{(\alpha-1)} d_q u\, d_p s \\
&\quad + \|\kappa\| \frac{T^{\alpha-1}}{|\Omega|\Gamma_r(\gamma)\Gamma_q(\alpha)} \times \int_0^{\xi}\int_0^{s}(\xi - rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)} d_q u\, d_r s \\
&\quad + \|\kappa\| \int_0^T \frac{(T-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} d_q s \\
&\quad = \Psi \|\kappa\| \le \rho.
\end{aligned} \tag{45}$$

Therefore $(\mathcal{Q}_1 x) + (\mathcal{Q}_2 y) \in B_\rho$. Further, condition (H_1) coupled with (42) implies that $\mathcal{Q}_2$ is contraction mapping. Therefore, condition (c) of Lemma 10 is satisfied.

Finally, we will show that $\mathcal{Q}_1$ is compact and continuous. Using the continuity of f and (H_2), we deduce that the operator $\mathcal{Q}_1$ is continuous and uniformly bounded on B_ρ. We define $\sup_{(t,w_1,w_2)\in[0,T]\times B_\rho^2}|f(t,w_1,w_2)| = N < \infty$. For $t_1, t_2 \in [0,T]$ with $t_1 < t_2$ and $x \in B_\rho$, we have

$$\begin{aligned}
&\left|(\mathcal{Q}_1 x)(t_2) - (\mathcal{Q}_1 x)(t_1)\right| \\
&\quad \le \frac{|\lambda|\left|t_2^{\alpha-1} - t_1^{\alpha-1}\right|}{|\Omega|\Gamma_p(\beta)\Gamma_q(\alpha)} \times \int_0^{\eta}\int_0^{s}(\eta - ps)^{(\beta-1)}(s-qu)^{(\alpha-1)} \times \left|f\left(u, x(u), I_z^{\delta}x(u)\right)\right| d_q u\, d_p s \\
&\quad + \frac{\left|t_2^{\alpha-1} - t_1^{\alpha-1}\right|}{|\Omega|\Gamma_r(\gamma)\Gamma_q(\alpha)} \times \int_0^{\xi}\int_0^{s}(\xi - rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)} \times \left|f\left(u, x(u), I_z^{\delta}x(u)\right)\right| d_q u\, d_r s \\
&\quad \le \frac{|\lambda|\left|t_2^{\alpha-1} - t_1^{\alpha-1}\right| N}{|\Omega|\Gamma_p(\beta)\Gamma_q(\alpha)} \times \int_0^{\eta}\int_0^{s}(\eta - ps)^{(\beta-1)}(s-qu)^{(\alpha-1)} d_q u\, d_p s \\
&\quad + \frac{\left|t_2^{\alpha-1} - t_1^{\alpha-1}\right| N}{|\Omega|\Gamma_r(\gamma)\Gamma_q(\alpha)} \times \int_0^{\xi}\int_0^{s}(\xi - rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)} d_q u\, d_r s \\
&\quad \le \frac{|\lambda|\left|t_2^{\alpha-1} - t_1^{\alpha-1}\right| N}{|\Omega|\Gamma_p(\beta)} \left(\frac{\eta^{\alpha+\beta}B_p(\beta,\alpha+1)}{\Gamma_q(\alpha+1)}\right) \\
&\quad + \frac{\left|t_2^{\alpha-1} - t_1^{\alpha-1}\right| N}{|\Omega|\Gamma_r(\gamma)} \left(\frac{\xi^{\alpha+\gamma}B_r(\gamma,\alpha+1)}{\Gamma_q(\alpha+1)}\right).
\end{aligned} \tag{46}$$

Actually, as $t_1 - t_2 \to 0$ the right-hand side of the above inequality tends to zero independently of $x \in B_\rho$. Therefore, $\mathcal{Q}_1$ is relatively compact on B_ρ. Applying the Arzelá-Ascoli theorem, we get that $\mathcal{Q}_1$ is compact on B_ρ. Thus all assumptions of Lemma 10 are satisfied. Therefore, the boundary value problem (1) has at least one solution on $[0, T]$. The proof is complete. □

Using the Leray-Schauder nonlinear alternative, we give the third result.

Lemma 12 (nonlinear alternative for single-valued maps [21]). *Let E be a Banach space, let C be a closed, convex subset of E, let U be an open subset of C, and let $0 \in U$. Suppose that $F : \overline{U} \to C$ is a continuous, compact (i.e., $F(\overline{U})$ is a relatively compact subset of C) map. Then either*

(i) *F has a fixed point in $\overline{U}$, or*

(ii) *there is a $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0,1)$ with $u = \lambda F(u)$.*

For the sake of convenience of proving the last result, we set

$$\Phi_1 = \frac{|\lambda| T^{\alpha-1}\eta^{\alpha+\beta}B_p(\beta,\alpha+1)}{|\Omega|\Gamma_p(\beta)\Gamma_q(\alpha+1)} + \frac{T^{\alpha-1}\xi^{\alpha+\gamma}B_r(\gamma,\alpha+1)}{|\Omega|\Gamma_r(\gamma)\Gamma_q(\alpha+1)} + \frac{T^{\alpha}}{\Gamma_q(\alpha+1)}, \tag{47}$$

$$\begin{aligned}
\Phi_2 &= \frac{|\lambda| T^{\alpha-1}\eta^{\alpha+\beta+\delta}B_q(\alpha,\delta+1)B_p(\beta,\alpha+\delta+1)}{|\Omega|\Gamma_p(\beta)\Gamma_q(\alpha)\Gamma_z(\delta+1)} \\
&+ \frac{T^{\alpha+\delta}B_q(\alpha,\delta+1)}{\Gamma_q(\alpha)\Gamma_z(\delta+1)} \\
&+ \frac{T^{\alpha-1}\xi^{\alpha+\gamma+\delta}B_q(\alpha,\delta+1)B_r(\gamma,\alpha+\delta+1)}{|\Omega|\Gamma_r(\gamma)\Gamma_q(\alpha)\Gamma_z(\delta+1)}.
\end{aligned} \tag{48}$$

Theorem 13. *Assume that $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function. In addition one supposes that*

(H_3) *there exist a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and a function $p \in C([0,T],\mathbb{R}^+)$ such that*

$$\left|f\left(t,w_1,w_2\right)\right| \le p(t)\,\psi\left(\left|w_1\right|\right) + \left|w_2\right| \quad \text{for each } \left(t,w_1,w_2\right) \in [0,T]\times\mathbb{R}^2; \tag{49}$$

(H_4) *there exists a constant $K > 0$ such that*

$$\frac{\left(1-\Phi_2\right)K}{\|p\|\,\psi(K)\,\Phi_1} > 1, \tag{50}$$

where Φ_1 and Φ_2 are defined by (47) *and* (48), *respectively, and*

$$\Phi_2 < 1. \tag{51}$$

Then the boundary value problem (1) *has at least one solution on* $[0,T]$.

Proof. Firstly, we will show that the operator $\mathcal{Q}$, defined by (32), *maps bounded sets (balls) into bounded sets in* $\mathcal{C}$. For a positive number R, we set a bounded ball in $\mathcal{C}$ as $B_R = \{x \in \mathcal{C} : \|x\| \le R\}$. Then, for $t \in [0,T]$, we have

$$\begin{aligned}
&|\mathcal{Q}x(t)| \\
&\le \frac{|\lambda|\,T^{\alpha-1}}{|\Omega|\,\Gamma_p(\beta)\,\Gamma_q(\alpha)} \int_0^\eta\int_0^s (\eta-ps)^{(\beta-1)}(s-qu)^{(\alpha-1)} \times \left|f\left(u,x(u),I_z^\delta x(u)\right)\right| d_qu\,d_ps \\
&\quad + \frac{T^{\alpha-1}}{|\Omega|\,\Gamma_r(\gamma)\,\Gamma_q(\alpha)} \int_0^\xi\int_0^s (\xi-rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)} \times \left|f\left(u,x(u),I_z^\delta x(u)\right)\right| d_qu\,d_rs \\
&\quad + \int_0^T \frac{(T-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \left|f\left(s,x(s),I_z^\delta x(s)\right)\right| d_qs \\
&\le \frac{|\lambda|\,T^{\alpha-1}}{|\Omega|\,\Gamma_p(\beta)\,\Gamma_q(\alpha)} \int_0^\eta\int_0^s (\eta-ps)^{(\beta-1)}(s-qu)^{(\alpha-1)} \times \left(p(u)\,\psi(\|x\|) + \|x\|\int_0^u \frac{(u-zv)^{(\delta-1)}}{\Gamma_z(\delta)}d_zv\right) d_qu\,d_ps \\
&\quad + \frac{T^{\alpha-1}}{|\Omega|\,\Gamma_r(\gamma)\,\Gamma_q(\alpha)} \int_0^\xi\int_0^s (\xi-rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)} \times \left(p(u)\,\psi(\|x\|) + \|x\|\int_0^u \frac{(u-zv)^{(\delta-1)}}{\Gamma_z(\delta)}d_zv\right) d_qu\,d_rs \\
&\quad + \int_0^T \frac{(T-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \times \left(p(s)\,\psi(\|x\|) + \|x\|\int_0^s \frac{(s-zv)^{(\delta-1)}}{\Gamma_z(\delta)}d_zv\right) d_qs \\
&\le \frac{|\lambda|\,T^{\alpha-1}}{|\Omega|\,\Gamma_p(\beta)} \times \left(\frac{\eta^{\alpha+\beta}B_p(\beta,\alpha+1)\,\|p\|\,\psi(R)}{\Gamma_q(\alpha+1)} + \frac{R\eta^{\alpha+\beta+\delta}B_q(\alpha,\delta+1)\,B_p(\beta,\alpha+\delta+1)}{\Gamma_q(\alpha)\,\Gamma_z(\delta+1)}\right) \\
&\quad + \frac{T^{\alpha-1}}{|\Omega|\,\Gamma_r(\gamma)} \times \left(\frac{\xi^{\alpha+\gamma}B_r(\gamma,\alpha+1)\,\|p\|\,\psi(R)}{\Gamma_q(\alpha+1)} + \frac{R\xi^{\alpha+\gamma+\delta}B_q(\alpha,\delta+1)\,B_r(\gamma,\alpha+\delta+1)}{\Gamma_q(\alpha)\,\Gamma_z(\delta+1)}\right) \\
&\quad + \frac{\|p\|\,\psi(R)\,T^\alpha}{\Gamma_q(\alpha+1)} + \frac{RT^{\alpha+\delta}B_q(\alpha,\delta+1)}{\Gamma_q(\alpha)\,\Gamma_z(\delta+1)} \\
&:= G.
\end{aligned} \tag{52}$$

Therefore, we conclude that $\|\mathcal{Q}x\| \le G$.

Secondly, we will show that *$\mathscr{Q}$ maps bounded sets into equicontinuous sets of $\mathscr{C}$*. Let $t_1, t_2 \in [0,T]$ with $t_1 < t_2$ and B_R be a bounded set of $C([0,T],\mathbb{R})$ as in the previous step, and let $x \in B_R$. Then we have

$$\begin{aligned}
&\left|(\mathscr{Q}x)(t_2) - (\mathscr{Q}x)(t_1)\right| \\
&\quad \le \frac{|\lambda|\left|t_2^{\alpha-1} - t_1^{\alpha-1}\right|}{|\Omega|\Gamma_p(\beta)\Gamma_q(\alpha)} \\
&\quad\quad \times \int_0^{\eta}\int_0^{s}(\eta - ps)^{(\beta-1)}(s-qu)^{(\alpha-1)} \\
&\quad\quad\quad \times \left|f\left(u, x(u), I_z^{\delta}x(u)\right)\right| d_q u\, d_p s \\
&\quad\quad + \frac{\left|t_2^{\alpha-1} - t_1^{\alpha-1}\right|}{|\Omega|\Gamma_r(\gamma)\Gamma_q(\alpha)} \\
&\quad\quad \times \int_0^{\xi}\int_0^{s}(\xi - rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)} \\
&\quad\quad\quad \times \left|f\left(u, x(u), I_z^{\delta}x(u)\right)\right| d_q u\, d_r s \\
&\quad\quad + \left|\int_0^{t_2}\frac{(t_2 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\left|f\left(s, x(s), I_z^{\delta}x(s)\right)\right| d_q s\right. \\
&\quad\quad\quad \left. - \int_0^{t_1}\frac{(t_1 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\left|f\left(s, x(s), I_z^{\delta}x(s)\right)\right| d_q s\right| \\
&\quad \le \frac{|\lambda|\left|t_2^{\alpha-1} - t_1^{\alpha-1}\right|}{|\Omega|\Gamma_p(\beta)\Gamma_q(\alpha)} \\
&\quad\quad \times \int_0^{\eta}\int_0^{s}(\eta - ps)^{(\beta-1)}(s-qu)^{(\alpha-1)} \\
&\quad\quad\quad \times \left(p(u)\psi(\|x\|)\right. \\
&\quad\quad\quad\quad \left. + \|x\|\int_0^{u}\frac{(u - zv)^{(\delta-1)}}{\Gamma_z(\delta)}d_z v\right) d_q u\, d_p s \\
&\quad\quad + \frac{\left|t_2^{\alpha-1} - t_1^{\alpha-1}\right|}{|\Omega|\Gamma_r(\gamma)\Gamma_q(\alpha)} \\
&\quad\quad \times \int_0^{\xi}\int_0^{s}(\xi - rs)^{(\gamma-1)}(s-qu)^{(\alpha-1)} \\
&\quad\quad\quad \times \left(p(u)\psi(\|x\|)\right. \\
&\quad\quad\quad\quad \left. + \|x\|\int_0^{u}\frac{(u - zv)^{(\delta-1)}}{\Gamma_z(\delta)}d_z v\right) d_q u\, d_r s \\
&\quad\quad + \left|\int_0^{t_2}\frac{(t_2 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\right. \\
&\quad\quad\quad \times \left(p(s)\psi(\|x\|)\right. \\
&\quad\quad\quad\quad \left. + \|x\|\int_0^{s}\frac{(s - zv)^{(\delta-1)}}{\Gamma_z(\delta)}d_z v\right) d_q s \\
&\quad\quad\quad - \int_0^{t_1}\frac{(t_1 - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \\
&\quad\quad\quad\quad \left. \times \left(p(s)\psi(\|x\|) + \|x\|\int_0^{s}\frac{(s - zv)^{(\delta-1)}}{\Gamma_z(\delta)}d_z v\right) d_q s\right| \\
&\quad \le \frac{|\lambda|\left|t_2^{\alpha-1} - t_1^{\alpha-1}\right|}{|\Omega|\Gamma_p(\beta)} \\
&\quad\quad \times \left(\frac{\eta^{\alpha+\beta}B_p(\beta,\alpha+1)\|p\|\psi(R)}{\Gamma_q(\alpha+1)}\right. \\
&\quad\quad\quad \left. + \frac{R\eta^{\alpha+\beta+\delta}B_q(\alpha,\delta+1)B_p(\beta,\alpha+\delta+1)}{\Gamma_q(\alpha)\Gamma_z(\delta+1)}\right) \\
&\quad\quad + \frac{\left|t_2^{\alpha-1} - t_1^{\alpha-1}\right|}{|\Omega|\Gamma_r(\gamma)} \\
&\quad\quad \times \left(\frac{\xi^{\alpha+\gamma}B_r(\gamma,\alpha+1)\|p\|\psi(R)}{\Gamma_q(\alpha+1)}\right. \\
&\quad\quad\quad \left. + \frac{R\xi^{\alpha+\gamma+\delta}B_q(\alpha,\delta+1)B_r(\gamma,\alpha+\delta+1)}{\Gamma_q(\alpha)\Gamma_z(\delta+1)}\right) \\
&\quad\quad + \frac{\left|t_2^{\alpha-1} - t_1^{\alpha-1}\right|\|p\|\psi(R)}{\Gamma_q(\alpha+1)} + \frac{\left|t_2^{\alpha-1} - t_1^{\alpha-1}\right| R B_q(\alpha,\delta+1)}{\Gamma_q(\alpha)\Gamma_z(\delta+1)}.
\end{aligned} \tag{53}$$

Obviously the right-hand side of the above inequality tends to zero independently of $x \in B_R$ as $t_1 \to t_2$. Therefore, by applying the Arzelá-Ascoli theorem, we deduce that $\mathscr{Q} : \mathscr{C} \to \mathscr{C}$ is completely continuous.

The result will follow from the Leray-Schauder nonlinear alternative once we have proved the boundedness of the set of all solutions to the equation $x(t) = \omega(\mathscr{Q}x)(t)$ for some $0 < \omega < 1$. Let x be a solution. Then, for $t \in [0,T]$, we have

$$\begin{aligned}
(\mathscr{Q}x)(t) = {}& \frac{\omega\lambda t^{\alpha-1}}{\Omega\Gamma_p(\beta)\Gamma_q(\alpha)} \\
&\times \int_0^{\eta}\int_0^{s}(\eta - ps)^{(\beta-1)}(s-qu)^{(\alpha-1)} \\
&\quad \times f\left(u, x(u), I_z^{\delta}x(u)\right) d_q u\, d_p s \\
&- \frac{\omega t^{\alpha-1}}{\Omega\Gamma_r(\gamma)\Gamma_q(\alpha)}
\end{aligned}$$

$$\times \int_0^{\xi} \int_0^{s} (\xi - rs)^{(\gamma-1)} (s - qu)^{(\alpha-1)} \times f\left(u, x(u), I_z^{\delta} x(u)\right) d_q u\, d_r s + \omega \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f\left(s, x(s), I_z^{\delta} x(s)\right) d_q s. \tag{54}$$

As before, one can easily find that

$$\|x\| = \sup_{t\in[0,T]} |\omega(\mathcal{Q}x)(t)| \le \|p\| \psi(\|x\|) \Phi_1 + \|x\| \Phi_2, \tag{55}$$

which can alternatively be written as

$$\frac{(1 - \Phi_2)\|x\|}{\|p\| \psi(\|x\|) \Phi_1} \le 1. \tag{56}$$

In view of (H_4), there exists K such that $\|x\| \ne K$. Let us set

$$\mathcal{U} = \{x \in C([0, T], \mathbb{R}) : \|x\| < K\}. \tag{57}$$

Note that the operator $\mathcal{Q} : \overline{\mathcal{U}} \to C(0, T, \mathbb{R})$ is continuous and completely continuous. From the choice of $\mathcal{U}$, there is no $x \in \partial\mathcal{U}$ such that $x = \omega\mathcal{Q}x$ for some $\omega \in (0, 1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 12), we deduce that $\mathcal{Q}$ has a fixed point $x \in \overline{\mathcal{U}}$ which is a solution of problem (1). This completes the proof. □

5. Examples

In this section, we present some examples to illustrate our results.

Example 1. Consider the following nonlocal fractional q-integral boundary value problem:

$$D_{1/2}^{3/2} x(t) = \frac{2\sin \pi t}{(e^t + 4)^2} \cdot \frac{|x(t)|}{2 + |x(t)|} + \frac{e^{-t^2}}{(6 + t)^2} I_{3/4}^{7/5} x(t) + \frac{1}{2}, \quad 0 < t < 3,$$
$$x(0) = 0, \qquad \frac{1}{5} I_{3/5}^{1/2} x\left(\frac{5}{2}\right) = I_{2/3}^{5/2} x\left(\frac{3}{2}\right). \tag{58}$$

Here $\alpha = 3/2$, $q = 1/2$, $\delta = 7/5$, $z = 3/4$, $\lambda = 1/5$, $\beta = 1/2$, $p = 3/5$, $\eta = 5/2$, $\gamma = 5/2$, $r = 2/3$, $\xi = 3/2$, $T = 3$, and $f(t, x, I_z^{\delta} x) = (2\sin \pi t/(e^t + 4)^2)(|x|/(2 + |x|)) + (e^{-t^2}/((6 + t)^2)) I_{3/4}^{7/5} x + 1/2$.

Since $|f(t, w_1, w_2) - f(t, \overline{w}_1, \overline{w}_2)| \le (1/25)|w_1 - \overline{w}_1| + (1/36)|w_2 - \overline{w}_2|$, then (H_1) is satisfied with $L_1 = 1/25$ and $L_2 = 1/36$. By using the Maple program, we find that

$$\Omega = \frac{\Gamma_r(\alpha)}{\Gamma_r(\alpha + \gamma)} \xi^{\alpha+\gamma-1} - \lambda \frac{\Gamma_p(\alpha)}{\Gamma_p(\alpha + \beta)} \eta^{\alpha+\beta-1} \approx 0.4141558,$$

$$\Lambda = \frac{|\lambda| T^{\alpha-1}}{|\Omega| \Gamma_p(\beta)} \times \left[\frac{\eta^{\alpha+\beta} B_p(\beta, \alpha + 1) L_1}{\Gamma_q(\alpha + 1)} + \frac{\eta^{\alpha+\beta+\delta} B_q(\alpha, \delta + 1) B_p(\beta, \alpha + \delta + 1) L_2}{\Gamma_q(\alpha) \Gamma_z(\delta + 1)} \right] + \frac{T^{\alpha-1}}{|\Omega| \Gamma_r(\gamma)} \times \left[\frac{\xi^{\alpha+\gamma} B_r(\gamma, \alpha + 1) L_1}{\Gamma_q(\alpha + 1)} + \frac{\xi^{\alpha+\gamma+\delta} B_q(\alpha, \delta + 1) B_r(\gamma, \alpha + \delta + 1) L_2}{\Gamma_q(\alpha) \Gamma_z(\delta + 1)} \right] + \frac{T^{\alpha}}{\Gamma_q(\alpha + 1)} L_1 + \frac{T^{\alpha+\delta} B_q(\alpha, \delta + 1)}{\Gamma_q(\alpha) \Gamma_z(\delta + 1)} L_2 \approx 0.8514717 < 1. \tag{59}$$

Hence, by Theorem 9, the nonlocal boundary value problem (58) has a unique solution on $[0, 3]$.

Example 2. Consider the following nonlocal fractional q-integral boundary value problem:

$$D_{2/3}^{9/5} x(t) = \frac{1}{4\pi^2 + t^2} \tan^{-1}\left(\frac{\pi x}{2}\right) + \frac{1}{30\pi} (1 + \sin(\pi t)) + I_{1/10}^{3/5} x(t), \quad 0 < t < 1,$$
$$x(0) = 0, \quad \frac{1}{50} I_{1/5}^{1/10} x\left(\frac{2}{3}\right) = I_{1/8}^{2/9} x\left(\frac{1}{2}\right). \tag{60}$$

Here $\alpha = 9/5$, $q = 2/3$, $\delta = 3/5$, $z = 1/10$, $\lambda = 1/50$, $\beta = 1/10$, $p = 1/5$, $\eta = 2/3$, $\gamma = 2/9$, $r = 1/8$, $\xi = 1/2$, $T = 1$, and $f(t, x, I_z^{\delta} x) = (\tan^{-1}(\pi x/2))/(4\pi^2 + t^2) + (1 + \sin(\pi t))/(30\pi) + I_{1/10}^{3/5} x$.

By using the Maple program, we find that

$$\Omega = \frac{\Gamma_r(\alpha)}{\Gamma_r(\alpha+\gamma)}\xi^{\alpha+\gamma-1} - \lambda\frac{\Gamma_p(\alpha)}{\Gamma_p(\alpha+\beta)}\eta^{\alpha+\beta-1} \approx 0.4691329,$$

$$\Phi_1 = \frac{|\lambda| T^{\alpha-1}\eta^{\alpha+\beta}B_p(\beta,\alpha+1)}{|\Omega|\Gamma_p(\beta)\Gamma_q(\alpha+1)} + \frac{T^{\alpha-1}\xi^{\alpha+\gamma}B_r(\gamma,\alpha+1)}{|\Omega|\Gamma_r(\gamma)\Gamma_q(\alpha+1)} + \frac{T^{\alpha}}{\Gamma_q(\alpha+1)} \approx 1.0408909,$$

$$\Phi_2 = \frac{|\lambda| T^{\alpha-1}\eta^{\alpha+\beta+\delta}B_q(\alpha,\delta+1)B_p(\beta,\alpha+\delta+1)}{|\Omega|\Gamma_p(\beta)\Gamma_q(\alpha)\Gamma_z(\delta+1)} + \frac{T^{\alpha+\delta}B_q(\alpha,\delta+1)}{\Gamma_q(\alpha)\Gamma_z(\delta+1)} + \frac{T^{\alpha-1}\xi^{\alpha+\gamma+\delta}B_q(\alpha,\delta+1)B_r(\gamma,\alpha+\delta+1)}{|\Omega|\Gamma_r(\gamma)\Gamma_q(\alpha)\Gamma_z(\delta+1)} \approx 0.5751429 < 1. \tag{61}$$

Clearly,

$$\begin{aligned}|f(t,w_1,w_2)| &= \left|\frac{1}{4\pi^2+t^2}\tan^{-1}\left(\frac{\pi w_1}{2}\right) + \frac{1}{30\pi}(1+\sin(\pi t)) + w_2\right| \\ &\le \frac{1}{120\pi}(1+\sin\pi t)(15|w_1|+4) + |w_2|.\end{aligned} \tag{62}$$

Choosing $p(t) = 1 + \sin \pi t$ and $\psi(|w_1|) = (1/120\pi)(15|w_1| + 4)$, we can show that

$$\frac{(1-\Phi_2)K}{\|p\|\psi(K)\Phi_1} > 1 \tag{63}$$

which implies that $K > 0.0645811$. Hence, by Theorem 13, the nonlocal boundary value problem (60) has at least one solution on $[0,1]$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Some Further Generalizations of Hölder's Inequality and Related Results on Fractal Space

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We establish some new generalizations and refinements of the local fractional integral Hölder's inequality and some related results on fractal space. We also show that many existing inequalities related to the local fractional integral Hölder's inequality are special cases of the main inequalities which are presented here.

1. Introduction

Let p_j $(j = 1, 2, \ldots)$ be constrained by

$$\sum_{j=1}^{m} \frac{1}{p_j} = 1. \tag{1}$$

Suppose also that $f_j(x) > 0$ and f_j $(j = 1, 2, \ldots, m)$ are continuous real-valued functions on $[a, b]$. Then each of the following assertions holds true.

(1) For $p_j > 0$ $(j = 1, 2, \ldots, m)$, we have the following inequality known as the Hölder inequality (see [1]):

$$\int_a^b \prod_{j=1}^{m} f_j(x)\,dx \leqq \prod_{j=1}^{m} \left(\int_a^b f_j^{p_j}(x) dx \right)^{1/p_j}. \tag{2}$$

(2) For $0 < p_m < 1$ and $p_j < 0$ $(j = 1, 2, \ldots, m-1)$, we have the following *reverse* Hölder inequality (see [2]):

$$\int_a^b \prod_{j=1}^{m} f_j(x)\,dx \geqq \prod_{j=1}^{m} \left(\int_a^b f_j^{p_j}(x) dx \right)^{1/p_j}. \tag{3}$$

In the special case when $m = 2$ and $p_1 = p_2$, inequality (2) reduces to the celebrated Cauchy inequality (see [3]). Both the Cauchy inequality and the Hölder inequality play significant roles in many different branches of modern pure and applied mathematics. A great number of generalizations, refinements, variations, and applications of each of these inequalities have been studied in the literature (see [3–13] and the references cited therein). Recently, Yang [14] established the following local fractional integral Hölder's inequality on fractal space.

Let $f(x), g(x) \in C_\alpha(a, b)$, $p > 1$, and $1/p + 1/q = 1$. Then

$$\begin{aligned} &\frac{1}{\Gamma(1+\alpha)} \int_a^b \left| f(x)\, g(x) \right| (dx)^\alpha \\ &\quad \leqq \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b \left| f(x) \right|^p (dx)^\alpha \right)^{1/p} \\ &\qquad \times \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b \left| g(x) \right|^q (dx)^\alpha \right)^{1/q}. \end{aligned} \tag{4}$$

More recently, Chen [15] gave a generalization of inequality (4) and its corresponding reverse form as follows.

Let $f_j(x) \in C_\alpha(a, b)$, $p_j \in R (j = 1, 2, \ldots, m)$, and

$$\sum_{j=1}^{m} \frac{1}{p_j} = 1. \tag{5}$$

Then each of the following assertions holds true. (1) For $p_j > 1$ $(j = 1, 2, \ldots, m)$, we have

$$\frac{1}{\Gamma(1+\alpha)} \int_a^b \prod_{j=1}^m \left| f_j(x) \right| (dx)^\alpha \leqq \prod_{j=1}^m \left(\int_a^b \frac{1}{\Gamma(1+\alpha)} \left| f_j(x) \right|^{p_j} (dx)^\alpha \right)^{1/p_j}. \tag{6}$$

(2) For $0 < p_1 < 1$ and $p_j < 0$ $(j = 2, \ldots, m)$, we have

$$\frac{1}{\Gamma(1+\alpha)} \int_a^b \prod_{j=1}^m \left| f_j(x) \right| (dx)^\alpha \geqq \prod_{j=1}^m \left(\int_a^b \frac{1}{\Gamma(1+\alpha)} \left| f_j(x) \right|^{p_j} (dx)^\alpha \right)^{1/p_j}. \tag{7}$$

The study of local fractional calculus has been an interesting topic (see [14–25]). In fact, local fractional calculus [14, 16, 17] has turned out to be a very useful tool to deal with the continuously nondifferentiable functions and fractals. This formalism has had a great variety of applications in describing physical phenomena, for example, elasticity [17, 26, 27], continuum mechanics [26], quantum mechanics [28, 29], wave phenomena and heat-diffusion analysis [30–34], and other branches of pure and applied mathematics [15, 35–37] and nonlinear dynamics [38, 39]. For more details and other applications of local fractional calculus, the interested reader may refer to the recent works [14–42] (see also the monograph [43] dealing extensively with fractional differential equations).

The purpose of this paper is to give some new generalizations and refinements of inequalities (6) and (7). Some related inequalities are also considered. This paper is structured as follows. In Section 2, we introduce some basic facts about local fractional calculus. In Section 3, we establish some new generalizations and refinements of the local fractional integral Hölder inequality and their corresponding reverse forms. Finally, we give our concluding remarks and observations in Section 4.

2. Preliminaries

In this section, we recall some known results of local fractional calculus (see [14, 16, 17]). Throughout this section we will always assume that F is a subset of the real line and is a fractal.

Lemma 1 (see [17]). *Assume that $f : (F, d) \to (\Omega', d')$ is a bi-Lipschitz mapping; then there are two positive constants ρ, τ, and $F \subset R$,*

$$\rho^s H^s(F) \leqq H^s(f(F)) \leqq \tau^s H^s(F), \tag{8}$$

such that

$$\rho^\alpha \left| x_1 - x_2 \right|^\alpha \leqq \left| f(x_1) - f(x_2) \right| \leqq \tau^\alpha \left| x_1 - x_2 \right|^\alpha \tag{9}$$

holds true for all $x_1, x_2 \in F$.

Based on Lemma 1, it is easy to show that [14]

$$\left| f(x_1) - f(x_2) \right| \leqq \tau^\alpha \left| x_1 - x_2 \right|^\alpha, \tag{10}$$

such that the following inequality holds true [14]:

$$\left| f(x_1) - f(x_2) \right| \leqq \varepsilon^\alpha, \tag{11}$$

where α is fractal dimension of F.

Definition 2 (see [14, 17]). Assume that $\varepsilon, \delta > 0$, $|x - x_0|^\alpha \leqq \delta$, and $\varepsilon, \delta \in R$; if

$$\left| f(x) - f(x_0) \right| \leqq \varepsilon^\alpha, \tag{12}$$

then $f(x)$ is called local fractional continuous at $x = x_0$, denoted by $\lim_{x \to x_0} f(x) = f(x_0)$. If $f(x)$ is local fractional continuous on the interval (a, b), then we write (see, e.g., [14])

$$f(x) \in C_\alpha(a, b), \tag{13}$$

where $C_\alpha(a, b)$ denotes the space of local fractional continuous functions on (a, b).

Definition 3 (see [16, 17]). Suppose that $f(x)$ is a nondifferentiable function of exponent α $(0 < \alpha \leqq 1)$. If the following inequality holds true

$$\left| f(x) - f(y) \right| \leqq C \left| x - y \right|^\alpha, \tag{14}$$

then $f(x)$ is a Hölder function of exponent α for $x, y \in F$.

Definition 4 (see [16, 17]). If $f(x)$ satisfies the following inequality

$$\left| f(x) - f(x_0) \right| \leqq o\left((x - x_0)^\alpha \right), \tag{15}$$

then $f(x)$ is continuous of order α $(0 < \alpha \leqq 1)$ or, briefly, α-continuous.

Definition 5 (see [14, 16–18]). Suppose that $f(x)$ is local fractional continuous on the interval (a, b); then the local fractional derivative of $f(x)$ of order α at $x = x_0$ is given by

$$f^{(\alpha)}(x_0) = \left. \frac{d^\alpha f(x)}{dx^\alpha} \right|_{x = x_0} = \lim_{x \to x_0} \frac{\Gamma(1+\alpha) \Delta (f(x) - f(x_0))}{(x - x_0)^\alpha}, \tag{16}$$

provided this limit exists.

From Definition 5, we have the following conclusion (see [14]):

$$f^{(\alpha)}(x) = D_x^{(\alpha)} f(x), \tag{17}$$

which is denoted by (see [14])

$$f(x) \in D_x^{(\alpha)}(a, b), \tag{18}$$

where $D_x^{(\alpha)}(a, b)$ denotes the space of local fractional derivable functions on (a, b).

Definition 6 (see [14, 16–18]). Suppose that $f(x)$ is local fractional continuous on the interval (a,b); then the local fractional integral of the function $f(x)$ in the interval $[a,b]$ is defined by

$$\begin{aligned} {}_aI_b^{(\alpha)} f(x) &= \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t)(dt)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \to 0} \sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^\alpha, \end{aligned} \tag{19}$$

where $\Delta t_j = t_{j+1} - t_j$, $\Delta t = \max\{\Delta t_1, \Delta t_2, \ldots, \Delta t_j, \ldots\}$, and $[t_j, t_{j+1}]$ $(j = 1, 2, \ldots, N-1;\ t_0 = a;\ t_N = b)$ are a partition of the interval $[a,b]$.

Let ${}_aI_x^{(\alpha)}(a,b)$ denote the space of local fractional integrable functions on (a,b); from Definition 6, we can obtain the following result (see, for details, [14]):

$$f(x) \in {}_aI_x^{(\alpha)}(a,b), \tag{20}$$

if there exists (see [14])

$${}_aI_x^{(\alpha)} f(x). \tag{21}$$

Remark 7 (see [14]). If we suppose that $f(x) \in D_x^{(\alpha)}(a,b)$ or $C_\alpha(a,b)$, then we have

$$f(x) \in {}_\alpha I_x^{(\alpha)}(a,b). \tag{22}$$

3. Main Results

In this section, we state and prove our main results.

Theorem 8. *Assume that $\alpha_{kj} \in \mathbb{R}$ $(j = 1,2,\ldots,m;\ k = 1,2,\ldots,s)$,*

$$\sum_k^s \frac{1}{p_k} = 1, \qquad \sum_{k=1}^s \alpha_{kj} = 0. \tag{23}$$

If $f_j(x) > 0$ and $f_j \in C_\alpha(a,b)$ $(j = 1,2,\ldots,m)$, then each of the following assertions holds true.

(1) *For $p_k > 0$ $(k = 1,2,\ldots,s)$, one has*

$$\begin{aligned} &\frac{1}{\Gamma(1+\alpha)} \int_a^b \prod_{j=1}^m f_j(x)(dx)^\alpha \\ &\leqq \prod_{k=1}^s \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b \prod_{j=1}^m f_j^{1+p_k\alpha_{kj}}(x)(dx)^\alpha \right)^{1/p_k}. \end{aligned} \tag{24}$$

(2) *For $0 < p_s < 1$ and $p_k < 0$ $(k = 1,2,\ldots,s-1)$, one has*

$$\begin{aligned} &\frac{1}{\Gamma(1+\alpha)} \int_a^b \prod_{j=1}^m f_j(x)(dx)^\alpha \\ &\geqq \prod_{k=1}^s \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b \prod_{j=1}^m f_j^{1+p_k\alpha_{kj}}(x)(dx)^\alpha \right)^{1/p_k}. \end{aligned} \tag{25}$$

Proof. (1) Let

$$g_k(x) = \left(\prod_{j=1}^m f_j^{1+p_k\alpha_{kj}}(x) \right)^{1/p_k}. \tag{26}$$

Applying the assumptions $\sum_k^s (1/p_k) = 1$ and $\sum_{k=1}^s \alpha_{kj} = 0$, a direct computation shows that

$$\begin{aligned} \prod_{k=1}^s g_k(x) &= g_1 g_2 \cdots g_s \\ &= \left(\prod_{j=1}^m f_j^{1+a_1\alpha_{1j}}(x) \right)^{1/a_1} \left(\prod_{j=1}^m f_j^{1+a_2\alpha_{2j}}(x) \right)^{1/a_2} \\ &\quad \cdots \left(\prod_{j=1}^m f_j^{1+a_s\alpha_{sj}}(x) \right)^{1/a_s} \\ &= \prod_{j=1}^m f_j^{1/a_1+\alpha_{1j}}(x) \prod_{j=1}^m f_j^{1/a_2+\alpha_{2j}}(x) \cdots \prod_{j=1}^m f_j^{1/a_s+\alpha_{sj}}(x) \\ &= \prod_{j=1}^m f_j^{1/a_1+1/a_2+\cdots 1/a_s+\alpha_{1j}+\alpha_{2j}+\cdots+\alpha_{sj}}(x) \\ &= \prod_{j=1}^m f_j(x); \end{aligned} \tag{27}$$

that is,

$$\prod_{k=1}^s g_k(x) = \prod_{j=1}^m f_j(x). \tag{28}$$

It is easy to see that

$$\frac{1}{\Gamma(1+\alpha)} \int_a^b \prod_{j=1}^m f_j(x)(dx)^\alpha = \frac{1}{\Gamma(1+\alpha)} \int_a^b \prod_{k=1}^s g_k(x)(dx)^\alpha. \tag{29}$$

It follows from the Hölder inequality (6) that

$$\begin{aligned} &\frac{1}{\Gamma(1+\alpha)} \int_a^b \prod_{k=1}^s g_k(x)(dx)^\alpha \\ &\leqq \prod_{k=1}^s \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b g_k^{p_k}(x)(dx)^\alpha \right)^{1/p_k}. \end{aligned} \tag{30}$$

Substitution of $g_k(x)$ into (30) leads us immediately to inequality (24). This proves inequality (24).

(2) The proof of inequality (25) is similar to the proof of inequality (24). Indeed, by using (26), (29), and (7), we have

$$\begin{aligned} &\frac{1}{\Gamma(1+\alpha)} \int_a^b \prod_{k=1}^s g_k(x)(dx)^\alpha \\ &\geqq \prod_{k=1}^s \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b g_k^{p_k}(x)(dx)^\alpha \right)^{1/p_k}. \end{aligned} \tag{31}$$

Substitution of $g_k(x)$ into (31) leads to inequality (25) immediately. □

Remark 9. Upon setting $s = m$, $\alpha_{kj} = -1/p_k$, for $j \neq k$, and $\alpha_{kk} = 1 - 1/p_k$, inequalities (24) and (25) are reduced to inequalities (6) and (7), respectively.

As we remarked earlier, many existing inequalities related to the local fractional integral Hölder's inequality are special cases of inequalities (24) and (25). For example, we have the following corollary.

Corollary 10. *Under the assumptions of Theorem 8 with $s = m$, $\alpha_{kj} = -t/p_k$, for $j \neq k$, and $\alpha_{kk} = t(1-1/p_k)$ $(t \in \mathbb{R})$, each of the following assertions holds true.*

(1) *For $p_k > 0$ $(k = 1, 2, \ldots, s)$, one has*

$$\frac{1}{\Gamma(1+\alpha)} \int_a^b \prod_{j=1}^m f_j(x)(dx)^\alpha \leqq \prod_{k=1}^m \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b \left(\prod_{j=1}^m f_j(x) \right)^{1-t} \left(f_k^{p_k} \right)^t (x)(dx)^\alpha \right)^{1/p_k}. \tag{32}$$

(2) *For $0 < p_m < 1$ and $p_k < 0$ $(k = 1, 2, \ldots, m-1)$, one has*

$$\frac{1}{\Gamma(1+\alpha)} \int_a^b \prod_{j=1}^m f_j(x)(dx)^\alpha \geqq \prod_{k=1}^m \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b \left(\prod_{j=1}^m f_j(x) \right)^{1-t} \left(f_k^{p_k} \right)^t (x)(dx)^\alpha \right)^{1/p_k}. \tag{33}$$

Theorem 11. *Assume that $r \in \mathbb{R}$, $\alpha_{kj} \in \mathbb{R}$ $(j = 1, 2, \ldots, m;\ k = 1, 2, \ldots, s)$,*

$$\sum_k^s \frac{1}{p_k} = r, \qquad \sum_{k=1}^s \alpha_{kj} = 0. \tag{34}$$

If $f_j(x) > 0$ and $f_j \in C_\alpha(a, b)$ $(j = 1, 2, \ldots, m)$, then each of the following assertions holds true.

(1) *For $rp_k > 0$ $(k = 1, 2, \ldots, s)$, one has*

$$\frac{1}{\Gamma(1+\alpha)} \int_a^b \prod_{j=1}^m f_j(x)(dx)^\alpha \leqq \prod_{k=1}^s \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b \prod_{j=1}^m f_j^{1+rp_k\alpha_{kj}}(x)(dx)^\alpha \right)^{1/rp_k}. \tag{35}$$

(2) *For $0 < rp_s < 1$ and $rp_k < 0$ $(k = 1, 2, \ldots, s-1)$, one has*

$$\frac{1}{\Gamma(1+\alpha)} \int_a^b \prod_{j=1}^m f_j(x)(dx)^\alpha \geqq \prod_{k=1}^s \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b \prod_{j=1}^m f_j^{1+rp_k\alpha_{kj}}(x)(dx)^\alpha \right)^{1/rp_k}. \tag{36}$$

Proof. (1) Since $rp_k > 0$ and $\sum_k^s (1/p_k) = r$, we get $\sum_k^s (1/rp_k) = 1$. Then, by applying (24), we immediately obtain inequality (35).

(2) Since $0 < rp_s < 1$, $rp_k < 0$, and $\sum_k^s (1/p_k) = r$, we have $\sum_k^s (1/rp_k) = 1$. Thus, by applying (25), we immediately have inequality (36). This completes the proof of Theorem 11. □

From Theorem 11, we obtain Corollary 12, which is a generalization of Theorem 11.

Corollary 12. *Under the assumptions of Theorem 11, let $s = 2$, $p_1 = p$, $p_2 = q$, and $\alpha_{1j} = -\alpha_{2j} = \alpha_j$. Then each of the following assertions holds true.*

(1) *For $rp > 0$, one has*

$$\frac{1}{\Gamma(1+\alpha)} \int_a^b \prod_{j=1}^m f_j(x)(dx)^\alpha \leqq \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b \prod_{j=1}^m f_j^{1+rp\alpha_j}(x)(dx)^\alpha \right)^{1/rp} \cdot \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b \prod_{j=1}^m f_j^{1-rq\alpha_j}(x)(dx)^\alpha \right)^{1/rq}. \tag{37}$$

(2) *For $0 < rp < 1$, one has*

$$\frac{1}{\Gamma(1+\alpha)} \int_a^b \prod_{j=1}^m f_j(x)(dx)^\alpha \geqq \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b \prod_{j=1}^m f_j^{1+rp\alpha_j}(x)(dx)^\alpha \right)^{1/rp} \cdot \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b \prod_{j=1}^m f_j^{1-rq\alpha_j}(x)(dx)^\alpha \right)^{1/rq}. \tag{38}$$

Next we present a refinement of each of inequalities (35) and (36).

Theorem 13. *Under the assumptions of Theorem 11, each of the following assertions holds true.*

(1) *For $rp_k > 0$ $(k = 1, 2, \ldots, s)$, one has*

$$\frac{1}{\Gamma(1+\alpha)} \int_a^b \prod_{j=1}^m f_j(x)(dx)^\alpha \leqq \varphi(c) \leqq \prod_{k=1}^s \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b \prod_{j=1}^m f_j^{1+rp_k\alpha_{kj}}(x)(dx)^\alpha \right)^{1/rp_k}, \tag{39}$$

where

$$\varphi(c) \equiv \frac{1}{\Gamma(1+\alpha)} \int_a^c \prod_{j=1}^m f_j(x)(dx)^\alpha + \prod_{k=1}^s \left(\frac{1}{\Gamma(1+\alpha)} \int_c^b \prod_{j=1}^m f_j^{1+rp_k\alpha_{kj}}(x)(dx)^\alpha \right)^{1/rp_k} \tag{40}$$

is a nonincreasing function with $a \leqq c \leqq b$.

(2) *For* $0 < rp_s < 1$ *and* $rp_k < 0(k = 1, 2, \ldots, s-1)$, *one has*

$$\frac{1}{\Gamma(1+\alpha)}\int_a^b \prod_{j=1}^m f_j(x)(dx)^\alpha \geqq \phi(c) \geqq \prod_{k=1}^s \left(\frac{1}{\Gamma(1+\alpha)}\int_a^b \prod_{j=1}^m f_j^{1+rp_k\alpha_{kj}}(x)(dx)^\alpha\right)^{1/rp_k}, \tag{41}$$

where

$$\phi(c) \equiv \frac{1}{\Gamma(1+\alpha)}\int_a^c \prod_{j=1}^m f_j(x)(dx)^\alpha + \prod_{k=1}^s \left(\frac{1}{\Gamma(1+\alpha)}\int_c^b \prod_{j=1}^m f_j^{1+rp_k\alpha_{kj}}(x)(dx)^\alpha\right)^{1/rp_k} \tag{42}$$

is a nondecreasing function with $a \leqq c \leqq b$.

Proof. (1) Let

$$g_k(x) = \left(\prod_{j=1}^m f_j^{1+rp_k\alpha_{kj}}(x)\right)^{1/rp_k}. \tag{43}$$

By rearrangement, it follows from the assumptions of Theorem 11 that

$$\prod_{j=1}^m f_j(x) = \prod_{k=1}^s g_k(x). \tag{44}$$

Then, by Hölder's inequality (6), we obtain

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)}\int_a^b \prod_{j=1}^m f_j(x)(dx)^\alpha \\
&= \frac{1}{\Gamma(1+\alpha)}\int_a^b \prod_{k=1}^s g_k(x)(dx)^\alpha \\
&= \frac{1}{\Gamma(1+\alpha)}\int_a^c \prod_{k=1}^s g_k(x)(dx)^\alpha + \frac{1}{\Gamma(1+\alpha)}\int_c^b \prod_{k=1}^s g_k(x)(dx)^\alpha \\
&\leqq \frac{1}{\Gamma(1+\alpha)}\int_a^c \prod_{k=1}^s g_k(x)(dx)^\alpha \\
&\quad + \prod_{k=1}^s \left(\frac{1}{\Gamma(1+\alpha)}\int_c^b g_k^{rp_k}(x)(dx)^\alpha\right)^{1/rp_k} \\
&\leqq \prod_{k=1}^s \left(\frac{1}{\Gamma(1+\alpha)}\int_a^c g_k^{rp_k}(x)(dx)^\alpha \right.\\
&\quad \left. + \frac{1}{\Gamma(1+\alpha)}\int_c^b g_k^{rp_k}(x)(dx)^\alpha\right)^{1/rp_k} \\
&= \prod_{k=1}^s \left(\frac{1}{\Gamma(1+\alpha)}\int_a^b g_k^{rp_k}(x)(dx)^\alpha\right)^{1/rp_k} \\
&= \prod_{k=1}^s \left(\frac{1}{\Gamma(1+\alpha)}\int_a^b \prod_{j=1}^m f_j^{1+rp_k\alpha_{kj}}(x)(dx)^\alpha\right)^{1/rp_k}.
\end{aligned} \tag{45}$$

Hence, the desired result is obtained.

(2) The proof of inequality (41) is similar to the proof of inequality (39), so we omit the details involved. □

4. Concluding Remarks and Observations

Integral inequalities play a major role in the development of local fractional calculus. In this work, we considered some new generalizations and refinements of the local fractional integral Hölder's inequality and some related results on fractal space. Hölder's inequality was obtained by Yang [14] using local fractional integral. Moreover, the reverse local fractional integral Hölder's inequality was established by Chen [15]. In our present investigation, we have offered further generalizations and refinements of these inequalities by using the local fractional integral which was introduced and investigated by Yang [14, 16, 17]. Special cases of the various results derived in this paper are shown to be related to a number of known results.

For the relevant details about the mathematical, physical, and engineering applications and interpretations of the operators of fractional calculus and local fractional calculus in dealing with the intermediate processes and the intermediate phenomena, the interested reader may be referred to the monographs by Yang [17] and Kilbas et al. [43] (and indeed also to some of the other recent investigations which are cited in this paper).

Conflict of Interests

The authors declare that they have no conflict of interests.

Authors' Contribution

This paper is the result of joint work of all authors who contributed equally to the final version of this paper. All authors read and approved the final paper.

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Research Article

A Novel Analytical Technique to Obtain Kink Solutions for Higher Order Nonlinear Fractional Evolution Equations

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We use the fractional derivatives in Caputo's sense to construct exact solutions to fractional fifth order nonlinear evolution equations. A generalized fractional complex transform is appropriately used to convert this equation to ordinary differential equation which subsequently resulted in a number of exact solutions.

1. Introduction

The concept of differentiation and integration to noninteger order is not new in any case. The notion of fractional calculus emerged when the ideas of classical calculus were proposed by Leibniz, who mentioned it in a letter to L'Hospital in 1695. The foundation of the earliest, more or less, systematic studies can be traced back to the beginning and middle of the 19th century by Liouville in 1832, Riemann in 1853, and Holmgren in 1864, although Euler in 1730, Lagrange in 1772, and others also made contributions. Recently, it has turned out those differential equations involving derivatives of noninteger [1, 2]. For example, the nonlinear oscillation of earthquakes can be modeled with fractional derivatives [3]. There has been some attempt to solve linear problems with multiple fractional derivatives [3, 4]. Not much work has been done on nonlinear problems and only a few numerical schemes have been proposed for solving nonlinear fractional differential equations [5, 6]. More recently, applications have included classes of nonlinear equation with multiorder fractional derivatives. The generalized fractional complex transform was applied in [7–13] to convert fractional order differential equation to ordinary differential equation. Finally, by using Exp-function method [14–25] we obtain generalized solitary solutions and periodic solutions. Recently the theory of local fractional integrals and derivatives [26–28] is one of useful tools to handle the fractal and continuously nondifferentiable functions. It is to be tinted that that $c = d$ and $p = q$ are the only relations that can be obtained by applying Exp-function method [29] to any nonlinear ordinary differential equation. Most scientific problems and phenomena in different fields of sciences and engineering occur nonlinearly. Except in a limited number of these problems are linear, this method has been effectively and accurately shown to solve a large class of nonlinear problems. The solution procedure of this method, with the aid of Maple, is of utter simplicity and this method can easily extend to other kinds of nonlinear evolution equations. In engineering and science, scientific phenomena give a variety of solutions that are characterized by distinct features. Traveling waves appear in many distinct physical structures in solitary wave theory [30] such as solitons, kinks, peakons, cuspons, compactons, and many others. Solitons are localized traveling waves which are asymptotically zero at large distances. In other words, solitons are localized wave packets with exponential wings or tails. Solitons are generated from robust balance between nonlinearity and dispersion. Solitons exhibit properties typically associated with particles. Kink waves [30, 31] are solitons that rise or descend from one asymptotic state to another and hence another type of traveling waves as in the case of the Burgers hierarchy. Peakons, that are peaked solitary wave solutions, are another type of travelling waves as in the case of Camassa-Holm equation. For peakons, the traveling wave solutions are smooth except for a peak at a corner of its crest. Peakons are the points at which spatial derivative changes sign so that peakons have a finite jump in 1st derivative of the solution. Cuspons are other forms of solitons where solution exhibits cusps at their crests. Unlike peakons where the derivatives at the peak differ only by a sign, the derivatives at the jump of a cuspon diverge. The compactons are solitons with compact spatial support such that each compacton is a soliton confined to a finite core or a soliton without exponential tails or wings. Other types of travelling waves arise in science such as negatons, positons, and complexitons. In this research, we use the Exp-function method along with generalized fractional complex transform to obtain new Kink waves' solutions for [30–32].

2. Preliminaries and Notation [1, 2]

In this section, we give some basic definitions and properties of the fractional calculus theory [1, 2] which will be used further in this work. For the finite derivative in $[a, b]$ we define the following fractional integral and derivatives.

Definition 1. A real function $f(x)$, $x > 0$, is said to be in the space $C\mu$, $\mu \in R$ if there exists a real number $(p > \mu)$ such that $f(x) = x^p f_1(x)$, where $f_1(x) = C(0, \infty)$, and it is said to be in the space $C_\mu^m \mu$ if $f^m \in C\mu$, $m \in N$.

Definition 2. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f \in C\mu$, $\mu \geq -1$, is defined as

$$\begin{gathered} J^\alpha (x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt, \\ \alpha > 0, \quad x > 0, \quad J^0(x) = f(x). \end{gathered} \tag{1}$$

Properties of the operator J^a can be found in [1]; we mention only the following.

For $f \in C\mu$, $\mu \geq -1$, $\alpha, \beta \geq 0$, and $\gamma \geq -1$

$$\begin{gathered} J^\alpha J^\beta f(x) = J^{\alpha+B} f(x), \\ J^\alpha J^B f(x) = J^B J^\alpha f(x), \\ J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}. \end{gathered} \tag{2}$$

The Riemann-Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we will introduce a modified fractional differential operator proposed by M. Caputo in his work on the theory of viscoelasticity [2].

Definition 3. For m to be the smallest integer that exceeds α, the Caputo time fractional derivative operator of order $\alpha > 0$ defined as

$$\begin{aligned} {}_a^C D_t^\alpha f(t) &= J^{m-\alpha} D^m f(t) \\ &= \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f^m(t)\, dt, \end{aligned} \tag{3}$$

for $m - 1 < \alpha \leq 1m$, $m \in N$, $t > 0$, $f \in C_{-1}^m$.

3. Chain Rule for Fractional Calculus and Fractional Complex Transform

In [7], the authors used the following chain rule $\partial^\alpha u/\partial t^\alpha = (\partial u/\partial s)(\partial^\alpha s/\partial t^\alpha)$ to convert a fractional differential equation with Jumarie's modification of Riemann-Liouville derivative into its classical differential partner. In [10], the authors showed that this chain rule is invalid and showed following relation [8]:

$$D_t^a u = \sigma_t' \frac{du}{d\eta} D_t^a \eta, \qquad D_x^a u = \sigma_x' \frac{du}{d\eta} D_x^a \eta. \tag{4}$$

To determine σ_s, consider a special case as follows:

$$s = t^\alpha, \qquad u = s^m \tag{5}$$

and we have

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\Gamma(1+m\alpha)\, t^{m\alpha-\alpha}}{\Gamma(1+m\alpha-\alpha)} = \sigma \cdot \frac{\partial u}{\partial s} = \sigma m t^{m\alpha-\alpha}. \tag{6}$$

Thus one can calculate σ_s as

$$\sigma_s = \frac{\Gamma(1+m\alpha)}{\Gamma(1+m\alpha-\alpha)}. \tag{7}$$

Other fractional indexes $(\sigma_x', \sigma_y', \sigma_z')$ can determine in similar way. Li and He [3, 7–9] proposed the following fractional complex transform for converting fractional differential equations into ordinary differential equations, so that all analytical methods for advanced calculus can be easily applied to fractional calculus:

$$u(x,t) = u(\eta), \quad \eta = \frac{kx^\beta}{\Gamma(1+\beta)} + \frac{\omega t^\alpha}{\Gamma(1+\alpha)} + \frac{Mx^\gamma}{\Gamma(1+\gamma)}, \tag{8}$$

where k, ω, and M are constants.

4. Exp-Function Method [33–36]

Consider the general nonlinear partial differential equation of fractional order:

$$\begin{gathered} P(u, u_t, u_x, u_{xx}, u_{xxx}, \ldots, D_t^\alpha u, D_x^\alpha u, D_{xx}^\alpha u, \ldots) = 0, \\ 0 < \alpha \leq 1, \end{gathered} \tag{9}$$

where $D_t^\alpha u$, $D_x^\alpha u$, and $D_{xx}^\alpha u$ are the fractional derivative of u with respect to t, x, xx, respectively.

Use

$$u(x,t) = u(\eta), \quad \eta = \frac{kx^\beta}{\Gamma(1+\beta)} + \frac{\omega t^\alpha}{\Gamma(1+\alpha)} + \frac{Mx^\gamma}{\Gamma(1+\gamma)}, \tag{10}$$

where k, ω, and M are constants.

Then (9) becomes

$$Q\left(u, u', u'', u''', u^{iv}\right) = 0, \tag{11}$$

where the prime denotes derivative with respect to η. In accordance with Exp-function method, we assume that the wave solution can be expressed in the following form:

$$u(\eta) = \frac{\sum_{n-c}^{d} a_n \exp[n\eta]}{\sum_{m-p}^{q} b_m \exp[m\eta]}, \tag{12}$$

where p, q, c, and d are positive integers which are known to be further determined and a_n and b_m are unknown constants. Equation (8) can be rewritten as

$$u(\eta) = \frac{a_c \exp(c\eta) + \cdots + a_{-d} \exp(-d\eta)}{b_p \exp(p\eta) + \cdots + b_{-q} \exp(-q\eta)}. \tag{13}$$

This equivalent formulation plays an important and fundamental for finding the analytic solution of problems. c and p can be determined by [29].

5. Solution Procedure

Consider the following new fifth order nonlinear (2 + 1)-dimensional evolution equations of fractional order:

$$D_t^{3\alpha} u - \left(D_t^\alpha u\right)_{xxxx} - \left(D_t^\alpha u\right)_{xx} - 4\left(u_x \left(D_t^\alpha u\right)_x\right)_x = 0. \tag{14}$$

Using (8) in (14) then it can be converted to an ordinary differential equation. Consider

$$-\omega^3 \ddot{u} + \omega k^4 u^{(v)} + k^2 \omega \ddot{u} + 4\omega k^3 \dot{u}\ddot{u} = 0, \tag{15}$$

where the prime denotes the derivative with respect to η. The solution of (15) can be expressed in form (13). To determine the value of c and p, by using [26],

$$p = c, \qquad q = d. \tag{16}$$

Case 1. We can freely choose the values of c and d, but we will illustrate that the final solution does not strongly depend upon the choice of values of c and d. For simplicity, we set $p = c = 1$ and $q = d = 1$ (15) reduces to

$$u(\eta) = \frac{a_1 \exp[\eta] + a_0 + a_{-1} \exp[-\eta]}{b_1 \exp[\eta] + a_0 + b_{-1} \exp[-\eta]}. \tag{17}$$

Substituting (17) into (15), we have

$$\begin{aligned}\frac{1}{A}\,[c_4 a_1 \exp[4\eta] &= c_3 \exp[3\eta] + c_2 \exp[2\eta] + c_1 \exp[\eta] \\ &+ c_0 + c_{-1} \exp[-\eta] + c_{-2} \exp[-2\eta] \\ &+ c_{-3} \exp[-3\eta] + c_{-4} \exp[-4\eta]] = 0,\end{aligned} \tag{18}$$

where $A = (b_1 \exp(\eta) + b_0 + b_{-1} \exp(-\eta))^4$ and c_i are constants obtained by Maple software 16. Equating the coefficients of $\exp(n\eta)$ to be zero, we obtain

$$\begin{gathered} c_{-4} = 0,\ c_{-3} = 0,\ c_{-2} = 0,\ c_{-1} = 0, \\ c_0 = 0,\ c_1 = 0,\ c_2 = 0,\ c_3 = 0,\ c_4 = 0. \end{gathered} \tag{19}$$

For solution of (19) we have five solution sets satisfying the given (15).

1st Solution Set. Consider

$$\begin{gathered} \omega = -\sqrt{k^2 + 1}k,\ a_{-1} = \frac{b_{-1}(3kb_0 + a_0)}{b_0}, \\ a_0 = a_0,\ a_1 = 0,\ b_{-1} = b_{-1},\ b_0 = b_0,\ b_1 = 0. \end{gathered} \tag{20}$$

We, therefore, obtained the following generalized solitary solution $u(x,t)$ of (14) (Figure 1):

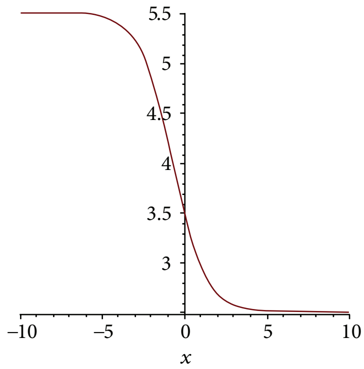
$$u(x,t) = \frac{b_{-1}(3kb_0 + a_0)\, e^{-kx - (\sigma\sqrt{k^2+1}kt^\alpha/\Gamma(1+\alpha))}/b_0 + a_0}{b_{-1} e^{-kx - (\sigma\sqrt{k^2+1}kt^\alpha/\Gamma(1+\alpha))} + b_0}. \tag{21}$$

2nd Solution Set. Consider

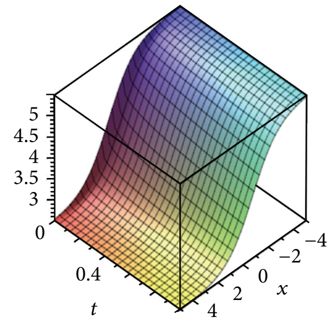
$$\begin{gathered} \omega = \sqrt{4k^2 + 1}k,\ a_{-1} = \frac{b_{-1}(6kb_1 + a_1)}{b_1}, \\ a_0 = a_0,\ a_1 = a_1,\ b_{-1} = b_{-1},\ b_0 = 0,\ b_1 = b_1. \end{gathered} \tag{22}$$

We, therefore, obtained the following generalized solitary solution $u(x,t)$ of (14) (Figure 2):

$$\begin{aligned} u(x,t) = &\Big(b_{-1}(6kb_1 + a_1)\, e^{-kx + (\sigma\sqrt{4k^2+1}kt^\alpha/\Gamma(1+\alpha))}/b_1 \\ &+ a_1 e^{kx - (\sigma\sqrt{4k^2+1}kt^\alpha/\Gamma(1+\alpha))}\Big) \\ &\times \Big(b_{-1} e^{-kx + (\sigma\sqrt{4k^2+1}kt^\alpha/\Gamma(1+\alpha))} \\ &+ b_1 e^{kx - (\sigma\sqrt{4k^2+1}kt^\alpha/\Gamma(1+\alpha))}\Big)^{-1}. \end{aligned} \tag{23}$$



$\alpha = 0.25, t = 1$



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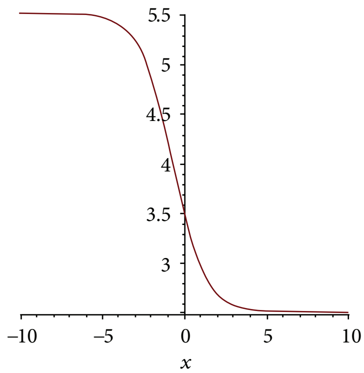
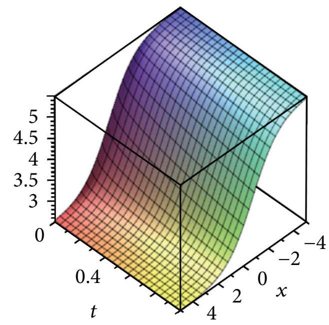
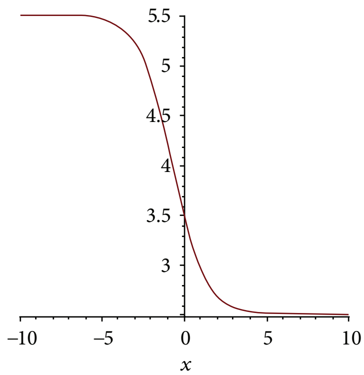


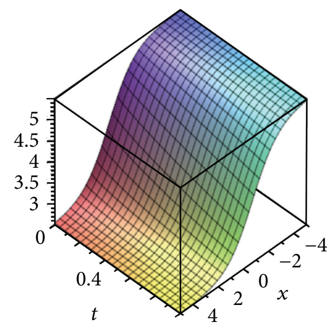
$\alpha = 0.50, t = 1$



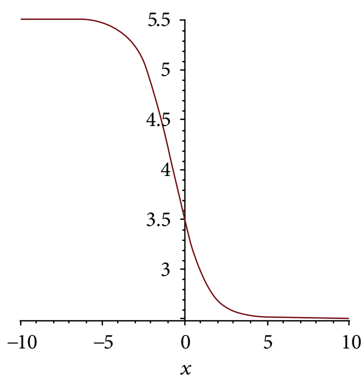
$\alpha = 0.50$



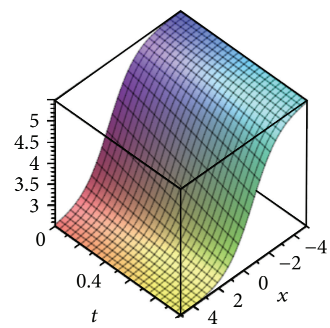
$\alpha = 0.75, t = 1$



$\alpha = 0.75$



$\alpha = 1, t = 1$



$\alpha = 1$

FIGURE 1: Kink waves' solutions of (14) for 1st solution set.

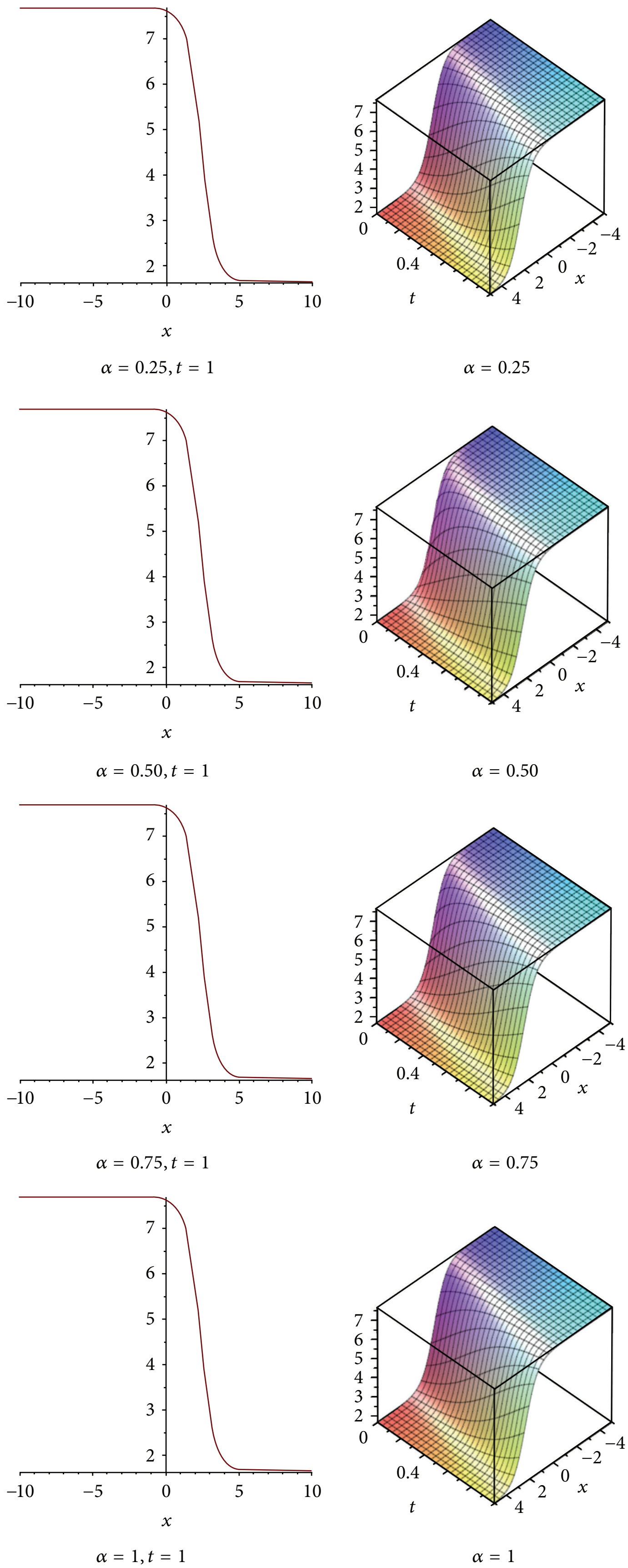


FIGURE 2: Kink waves' solutions of (14) for 2nd solution set.

3rd Solution Set. Consider

$$\omega = \sqrt{k^2+1}k, \quad a_{-1} = \frac{b_{-1}(3kb_0+a_0)}{b_0},$$
$$a_0 = a_0, \quad a_1 = 0, \quad b_{-1} = b_{-1}, \quad b_0 = b_0, \quad b_1 = 0. \tag{24}$$

We, therefore, obtained the following generalized solitary solution $u(x,t)$ of (14) (Figure 3):

$$u(x,t) = \frac{b_{-1}(3kb_0+a_0)e^{-kx+(\sigma\sqrt{k^2+1}kt^{\alpha}/\Gamma(1+\alpha))}/b_0 + a_0}{b_{-1}e^{-kx+(\sigma\sqrt{k^2+1}kt^{\alpha}/\Gamma(1+\alpha))} + b_0}. \tag{25}$$

4th Solution Set. Consider

$$\begin{gathered}\omega = \sqrt{k^2+1}k,\\
a_{-1} = \frac{1}{9}\frac{1}{k^2b_1^4}\left(9k^2a_0b_0b_1^3 - 9k^2a_1b_0^2b_1^2 - 3ka_0^2b_1^3 + 9ka_0a_1b_0b_1^2\right.\\
\left.-6ka_1^2b_0^2b_1 - a_0^2a_1b_1^2 + 2a_0a_1^2b_0b_1 - a_1^3b_0^2\right),\\
b_1 = b_1, \quad b_0 = b_0,\\
a_0 = a_0, \quad a_1 = a_1,\\
b_{-1} = \frac{1}{9}\frac{3ka_0b_0b_1^2 - 3ka_1b_0^2b_1 - a_0^2b_1^2 + 2a_0a_1b_0b_1 - a_1^2b_0^2}{k^2b_1^3}.\end{gathered} \tag{26}$$

We, therefore, obtained the following generalized solitary solution $u(x,t)$ of (14) (Figure 4):

$$\begin{aligned}u(x,t) = {} & \left(\frac{1}{9}\frac{1}{k^2b_1^4}\left(\left(9k^2a_0b_0b_1^3 - 9k^2a_1b_0^2b_1^2\right.\right.\right.\\
& - 3ka_0^2b_1^3 + 9ka_0a_1b_0b_1^2\\
& - 6ka_1^2b_0^2b_1 - a_0^2a_1b_1^2\\
& \left. + 2a_0a_1^2b_0b_1 - a_1^3b_0^2\right)\\
& \left.\times e^{-kx+(\sigma\sqrt{k^2+1}kt^{\alpha}/\Gamma(1+\alpha))}\right)\\
& \left. + a_0 + a_1e^{kx-(\sigma\sqrt{k^2+1}kt^{\alpha}/\Gamma(1+\alpha))}\right)\\
& \times\left(\frac{1}{9}\left(\left(3ka_0b_0b_1^2 - 3ka_1b_0^2b_1 - a_0^2b_1^2\right.\right.\right.\\
& \left. + 2a_0a_1b_0b_1 - a_1^2b_0^2\right)\\
& \left.\times e^{-kx+(\sigma\sqrt{k^2+1}kt^{\alpha}/\Gamma(1+\alpha))}\right)\left(k^2b_1^3\right)^{-1}\\
& \left. + b_0 + b_1e^{kx-(\sigma\sqrt{k^2+1}kt^{\alpha}/\Gamma(1+\alpha))}\right)^{-1}.\end{aligned} \tag{27}$$

5th Solution Set. Consider

$$\omega = \sqrt{k^2+1}k, \quad a_{-1} = \frac{b_{-1}(6kb_1+a_1)}{b_1},$$
$$a_0 = 0, \quad a_1 = a_1, \quad b_{-1} = b_{-1}, \quad b_0 = 0, \quad b_1 = b_1. \tag{28}$$

We, therefore, obtained the following generalized solitary solution $u(x,t)$ of (14) (Figure 5):

$$u(x,t) = \frac{b_{-1}e^{-kx+(\sigma\sqrt{k^2+1}kt^{\alpha}/\Gamma(1+\alpha))} + a_1e^{kx-(\sigma\sqrt{k^2+1}kt^{\alpha}/\Gamma(1+\alpha))}}{b_{-1}e^{-kx+(\sigma\sqrt{k^2+1}kt^{\alpha}/\Gamma(1+\alpha))} + b_1e^{kx-(\sigma\sqrt{k^2+1}kt^{\alpha}/\Gamma(1+\alpha))}}. \tag{29}$$

Case 2. If $p = c = 2$ and $q = d = 1$ then trial solution (14) reduces to

$$u(\eta) = \frac{a_2\exp[2\eta] + a_1\exp[\eta] + a_0 + a_{-1}\exp[-\eta]}{b_2\exp[2\eta] + b_1\exp[\eta] + b_0 + b_{-1}\exp[-\eta]}. \tag{30}$$

Proceeding as before, we obtain the following.

1st Solution. Consider

$$a_{-1} = a_{-1}, \quad a_0 = \frac{a_{-1}b_0}{b_{-1}}, \quad a_1 = \frac{a_{-1}b_1}{b_{-1}}, \quad a_2 = \frac{a_{-1}b_2}{b_{-1}},$$
$$b_{-1} = b_{-1}, \quad b_0 = b_0, \quad b_1 = b_1, \quad b_2 = b_2. \tag{31}$$

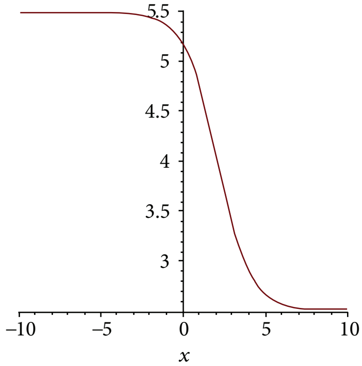
Hence we get the generalized solitary wave solution $u(x,t)$ of (14) (Figure 6):

$$\begin{aligned}u(x,t) = {} & \left(a_{-1}e^{-kx+(\sigma\omega kt^{\alpha}/\Gamma(1+\alpha))} + \frac{a_{-1}b_0}{b_{-1}}\right.\\
& + \frac{a_{-1}b_0}{b_{-1}}e^{kx-(\sigma\omega kt^{\alpha}/\Gamma(1+\alpha))}\\
& \left. + \frac{a_{-1}b_2}{b_{-1}}e^{2kx-2(\sigma\omega kt^{\alpha}/\Gamma(1+\alpha))}\right)\\
& \times\left(b_{-1}e^{-kx+(\sigma\omega kt^{\alpha}/\Gamma(1+\alpha))} + b_0\right.\\
& + b_1e^{kx-(\sigma\omega kt^{\alpha}/\Gamma(1+\alpha))}\\
& \left. + b_2e^{2kx-2(\sigma\omega kt^{\alpha}/\Gamma(1+\alpha))}\right)^{-1}.\end{aligned} \tag{32}$$

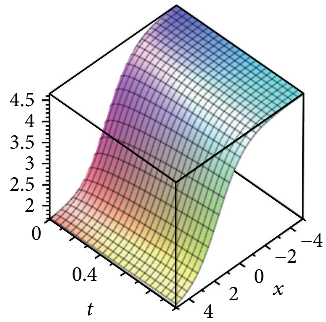
In both cases, for different choices of c, p, d, and q we get the same soliton solutions which clearly illustrate that final solution does not strongly depend on these parameters.

6. Conclusions

Exp-function method is applied to construct solitary solutions of the nonlinear new fifth order evolution equations of fractional orders. The reliability of proposed algorithm is fully supported by the computational work, the subsequent results, and graphical representations. It is observed that Exp-function method is very convenient to apply and is very useful for finding solutions of a wide class of nonlinear problems of fractional orders.



$\alpha = 0.25, t = 1$



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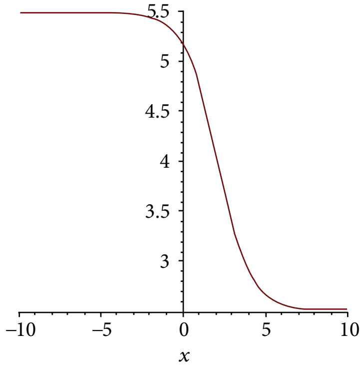
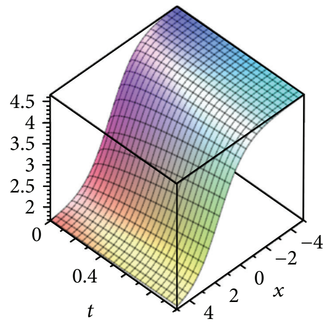
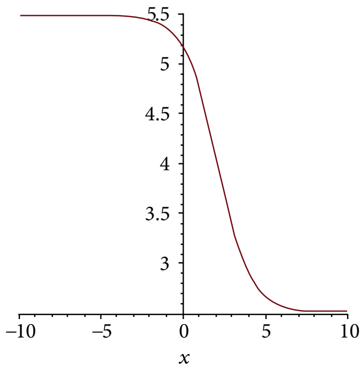


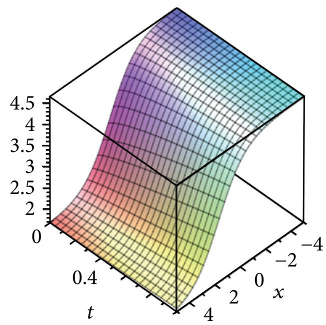
$\alpha = 0.50, t = 1$



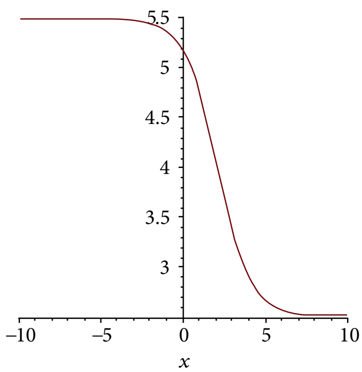
$\alpha = 0.50$



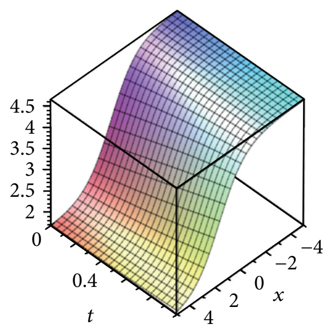
$\alpha = 0.75, t = 1$



$\alpha = 0.75$



$\alpha = 1, t = 1$



$\alpha = 1$

Figure 3: Kink waves' solutions of (14) for 3rd solution set.

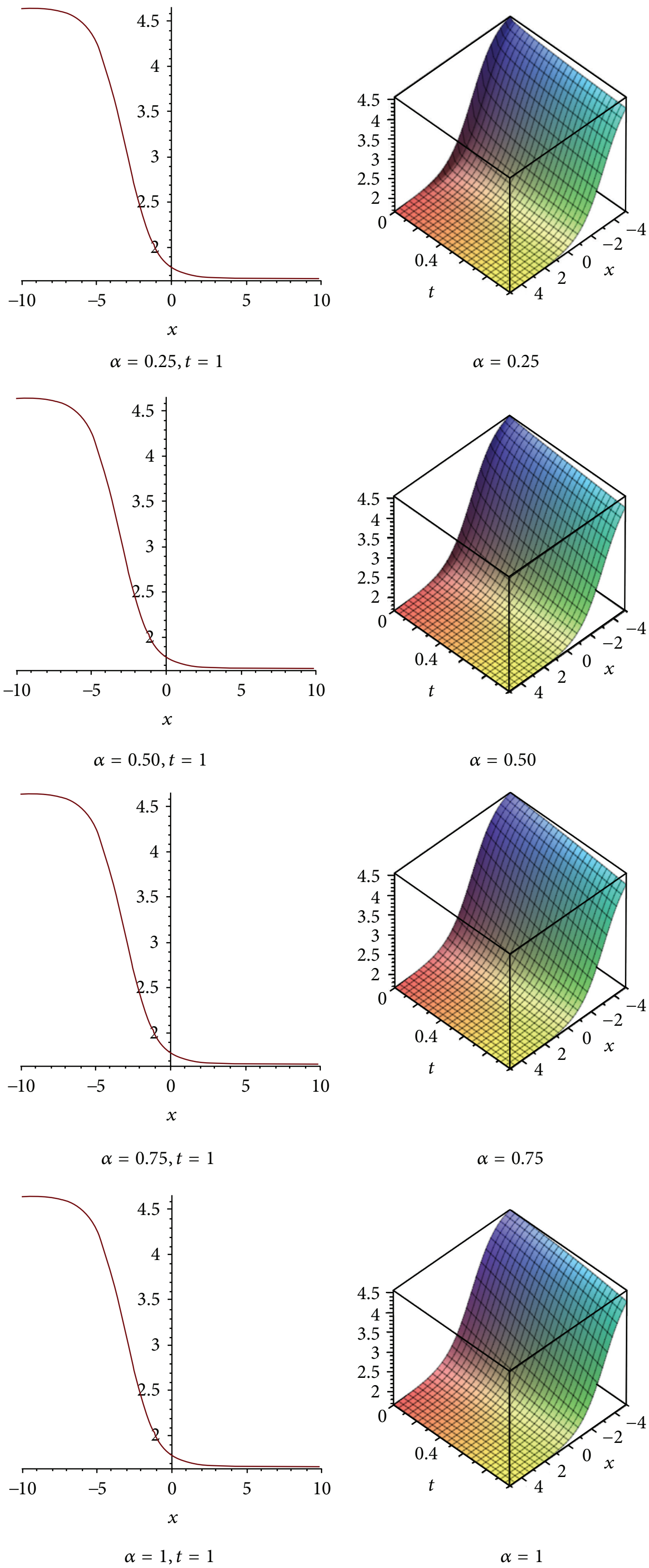


Figure 4: Kink waves' solutions of (14) for 4th solution set.

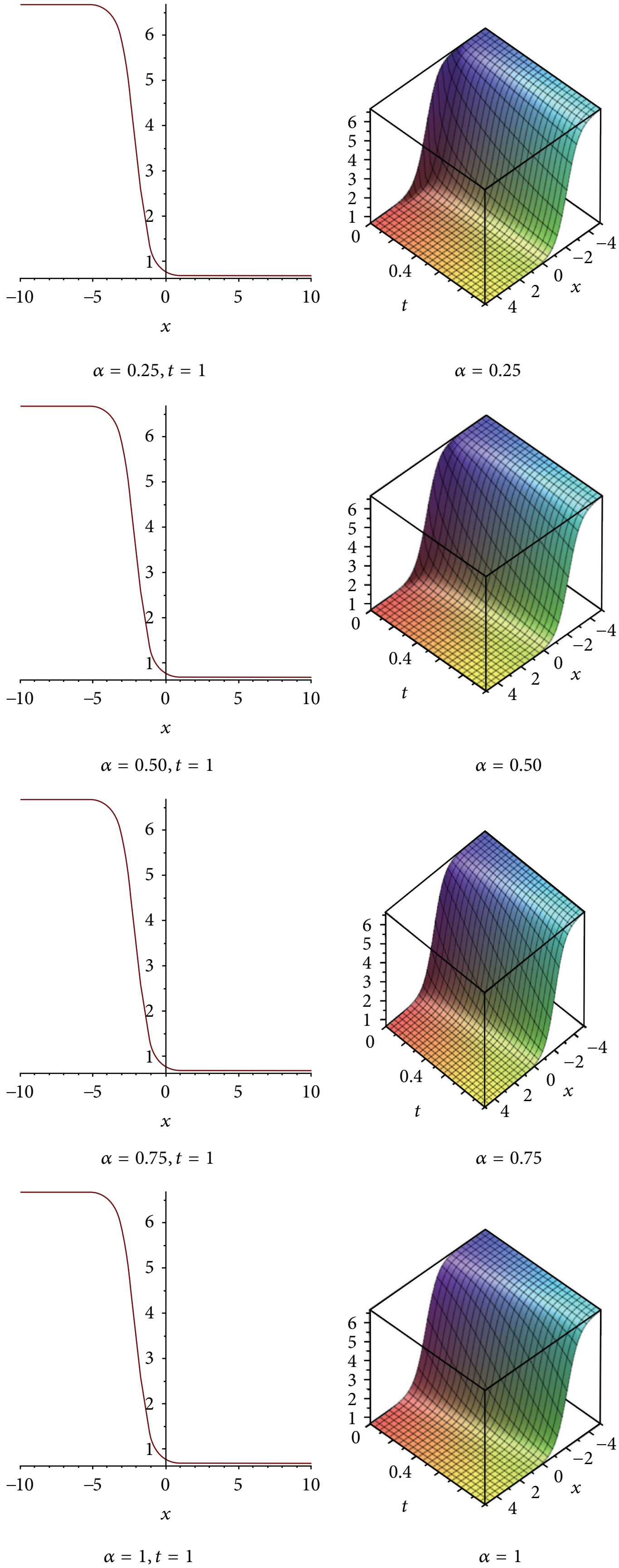


FIGURE 5: Kink waves' solutions of (14) for 5th solution set.

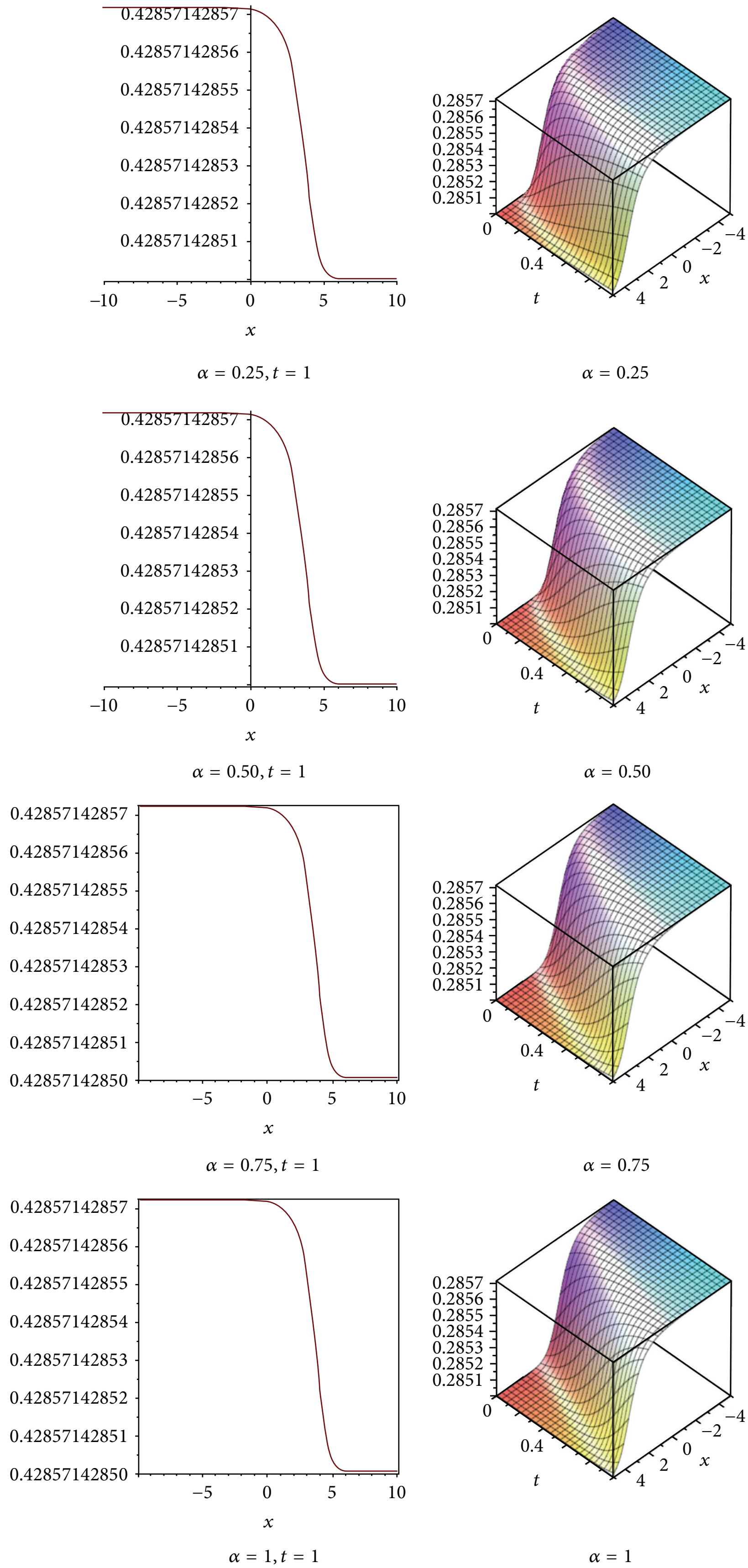


FIGURE 6: Kink waves' solutions of (14) for 1st solution set of Case 2.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Applications of the Novel (G'/G)-Expansion Method for a Time Fractional Simplified Modified Camassa-Holm (MCH) Equation

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We use the fractional derivatives in modified Riemann-Liouville derivative sense to construct exact solutions of time fractional simplified modified Camassa-Holm (MCH) equation. A generalized fractional complex transform is properly used to convert this equation to ordinary differential equation and, as a result, many exact analytical solutions are obtained with more free parameters. When these free parameters are taken as particular values, the traveling wave solutions are expressed by the hyperbolic functions, the trigonometric functions, and the rational functions. Moreover, the numerical presentations of some of the solutions have been demonstrated with the aid of commercial software Maple. The recital of the method is trustworthy and useful and gives more new general exact solutions.

1. Introduction

The class of fractional calculus is one of the most convenient classes of fractional differential equations which were viewed as generalized differential equations [1]. In the sense that much of the theory and, hence, applications of differential equations can be extended smoothly to fractional differential equations with the same flavor and spirit of the realm of differential equation, the seeds of fractional calculus were planted over three hundred years ago from a gracious idea of L'Hopital, who wrote a letter to Leibniz on 1695, asking about a rigorous description of the derivative of order $n = 0.5$. Fractional calculus is the theory of differentiation and integration of noninteger order and embodies the generality of the conventional differential and integral calculus. Therefore, some of the properties of the fractional integral and derivatives differ from the conventional ones in order to allow its implementation in a broader assortment of cases, which cannot be appropriately illustrated by the conventional integer-order calculus. Fractional calculus is painstaking to be a very authoritative tool to help scientists to unearth the concealed properties of the dynamics of multifaceted systems in all fields of sciences and engineering. In recent years, fractional calculus played an imperative role of a proficient, expedient, and elementary theoretical structure for more adequate modeling of multifaceted dynamic processes. Therefore, mounting applications of fractional calculus can be seen in modeling, signal processing, electromagnetism, mechanics, physics, biology, medicine, chemistry, bioengineering, biological systems, and in many other areas [2, 3]. Recently, it has turned out that those differential equations are involving derivatives of noninteger [4]. For example, the nonlinear oscillation of earthquakes can be modeled with fractional derivatives [5]. More recently, applications have included classes of nonlinear equation with multiorder fractional derivatives. We apply a generalized fractional complex transform [6–9] to convert fractional order differential equation to ordinary differential equation. Many important phenomena in electromagnetic, viscoelasticity, electrochemistry, and material science are well described by differential equations of fractional order [10–14]. A physical interpretation of the fractional calculus was given in [15–19]. With the development of symbolic computation software, like Maple, many numerical and analytical methods to search for exact solutions of NLEEs have attracted more attention. As a result, the researchers developed and established many methods, for example, the Cole-Hopf transformation [20], the Tanh-function method [21–24], the inverse scattering transform method [25], the variational iteration method [26, 27], Exp-function method [28–31], and F-expansion method [32, 33] that are used for searching the exact solutions.

Recently, a straightforward and concise method, called (G'/G)-expansion method, was introduced by Wang et al. [34] and demonstrated that it is a powerful method for seeking analytic solutions of NLEEs. (G'/G)-expansion is a reliable technique, which gives various types of the solitary wave solutions including the hyperbolic functions, the trigonometric functions, and the rational functions. It is also evident from the literature that such solutions always satisfy the given nonlinear differential equations. For additional references, see the articles [35–40]. In order to establish the efficiency and assiduousness of (G'/G)-expansion method and to extend the range of applicability, further research has been carried out by several researchers. For instance, Zhang et al. [41] made a generalization of (G'/G)-expansion method for the evolution equations with variable coefficients. Zhang et al. [42] also presented an improved (G'/G)-expansion method to seek more general traveling wave solutions. Zayed [43] presented a new approach of (G'/G)-expansion method where $G(\xi)$ satisfies the Jacobi elliptic equation, $[G'(\xi)]^2 = e_2G^4(\xi) + e_1G^2(\xi) + e_0$, where e_2, e_1, e_0 are arbitrary constants and obtained new exact solutions. Zayed [44] again presented an alternative approach of this method in which $G(\xi)$ satisfies the Riccati equation $G'(\xi) = AG(\xi) + BG^2(\xi)$, where A and B are arbitrary constants.

In this paper, we will apply novel (G'/G)-expansion method introduced by Alam et al. [45] to solve the time fractional simplified modified Camassa-Holm (MCH) equation in the sense of modified Riemann-Liouville derivative by Jumarie [46] and abundant new families of exact solutions are found. The Jumarie modified Riemann-Liouville derivative of order α is defined by the following expression:

$$D_t^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \times \dfrac{d}{dt}\displaystyle\int_0^t (t-\xi)^{-\alpha}\left(f(\xi)-f(0)\right)d\xi, & 0<\alpha<1, \\ \left(f^{(n)}(t)\right)^{(\alpha-n)}, & n \le \alpha < n+1, \\ & n \ge 1. \end{cases} \quad (1)$$

Some important properties of Jumarie's derivative are

$$D_t^\alpha f(t) = \frac{\Gamma(1+\tau)}{\Gamma(1+\tau-\alpha)} t^{\tau-\alpha}, \quad (2)$$

$$D_t^\alpha \left(f(t)\,g(t)\right) = g(t)\,D_t^\alpha f(t) + f(t)\,D_t^\alpha g(t), \quad (3)$$

$$D_t^\alpha f[g(t)] = f_g'[g(t)]\,D_t^\alpha g(t) = D_g^\alpha f[g(t)]\left(g'(t)\right)^\alpha. \quad (4)$$

2. Description of the Method

Suppose that a fractional partial differential equation in the independent variables, say t, is given by

$$S\left(u, u_x, u_t, D_t^\alpha u, \ldots\right) = 0, \quad 0 < \alpha \le 1, \quad (5)$$

where $D_t^\alpha u$ is Jumarie's modified Riemann-Liouville derivatives of $u, u(x,t)$ is an unknown function, S is a polynomial in u, and its various partial derivatives including fractional derivatives in which the highest order derivatives and nonlinear terms are involved.

The main steps of the method are as follows.

Step 1. Li and He [7] proposed a fractional complex transformation to convert fractional partial differential equations into ordinary differential equations (ODE), so all analytical methods devoted to the advanced calculus can be easily applied to the fractional calculus. The traveling wave variable

$$u(x,t) = u(\xi), \qquad \xi = Lx + V\frac{t^\alpha}{\Gamma(1+\alpha)}, \quad (6)$$

where L, V are arbitrary constants with $L, V \ne 0$, permits us to convert (5) into an ordinary differential equation of integer order in the form

$$P\left(u, u', u'', u''', \ldots\right) = 0, \quad (7)$$

where the superscripts stand for the ordinary derivatives with respect to ξ.

Step 2. Integrating (7) term by term one or more times if possible yields constant(s) of integration which can be calculated later on.

Step 3. Assume that the solution of (7) can be represented as

$$u(\xi) = \sum_{i=-m}^{m} \alpha_i (k + \Phi(\xi))^i, \quad (8)$$

where

$$\Phi(\xi) = \frac{G'(\xi)}{G(\xi)}, \quad (9)$$

where both α_{-m} and α_m cannot be zero simultaneously. α_i $(i = 0, \pm1, \pm2, \ldots, \pm m)$ and k are constants to be determined later and $G = G(\xi)$ satisfies the second order nonlinear ordinary differential equation as an auxiliary equation

$$GG'' = AGG' + BG^2 + C\left(G'\right)^2, \quad (10)$$

where A, B, and C are real constants.

Equation (10) can be reduced to the following Riccati equation by making use of the Cole-Hopf transformation $\Phi(\xi) = \ln(G(\xi))_\xi = G'(\xi)/G(\xi)$ as

$$\Phi'(\xi) = B + A\Phi(\xi) + (C-1)\Phi^2(\xi). \quad (11)$$

Equation (11) has twenty five solutions [47].

Step 4. The positive integer m can be determined by balancing the highest order linear term with the nonlinear term of the highest order come out in (7).

Step 5. Substituting (8) together with (9) and (10) into (7), we obtain polynomials in $(k+(G'/G))^i$ and $(k+(G'/G))^{-i}$ $(i = 0,1,2,\ldots,m)$. Collecting each coefficient of the resulted polynomials to zero yields an overdetermined set of algebraic equations for α_i $(i=0,\pm1,\pm2,\ldots,\pm m)$, k, L, and V.

Step 6. The values of the arbitrary constants can be obtained by solving the algebraic equations obtained in Step 4. The obtained values of the arbitrary constants and the solutions of (10) yield abundant exact traveling wave solutions of the nonlinear evolution equation (5).

3. Application of the Method to the Time Fractional Simplified (MCH) Equation

Now, consider the following time fractional simplified modified Camassa-Holm (MCH) equation:

$$D_t^{\alpha}u + 2\delta u_x - u_{xxt} + \gamma u^2 u_x = 0, \quad \text{where } \delta \in \mathfrak{R}, \quad \gamma > 0, \quad 0 < \alpha \le 1, \tag{12}$$

which is the variation of the equation

$$u_t + 2\delta u_x - u_{xxt} + \gamma u^2 u_x = 0, \quad \text{where } \delta \in \mathfrak{R}, \quad \gamma > 0. \tag{13}$$

Many researchers investigated the simplified MCH equation by using different methods to establish exact solutions. For example, Liu et al. [48] were concerned about the (G'/G)-expansion method to solve the simplified MCH equation, whereas the second order linear ordinary differential equation (LODE) is considered as an auxiliary equation. Wazwaz [49] studied this equation by using the sine-cosine algorithm. Zaman and Sultana [50] used the (G'/G)-expansion method together with the generalized Riccati equation to MCH equation to find the exact solutions. Alam and Akbar [51] applied the generalized (G'/G)-expansion method to look for the exact solutions via the simplified MCH equation. Further details of MCH equation can be found in references [52, 53].

By the use of (4), (12) is converted into an ordinary differential equation of integer order and after integrating once, we obtain

$$(V+2\delta L)u - VL^2u'' + \gamma L\frac{u^3}{3} + C_1 = 0, \tag{14}$$

where C_1 is an integral constant which is to be determined later.

Considering the homogeneous balance between u'' and u^3 in (14), we obtain $3m = m+2$; that is, $m = 2$. Therefore, the trial solution formula (8) becomes

$$u(\xi) = \alpha_{-1}(k+\Phi(\xi))^{-1} + \alpha_0 + \alpha_1(k+\Phi(\xi)). \tag{15}$$

Using (15) into (14), left hand side is converted into polynomials in $(k+(G'/G))^i$ and $(k+(G'/G))^{-i}$ $(i = 0,1,2,\ldots,m)$. Equating the coefficients of same power of the resulted polynomials to zero, we obtain a system of algebraic equations for $\alpha_0, \alpha_1, \alpha_{-1}, k, C_1, L$, and V (which are omitted for the sake of simplicity). Solving the overdetermined set of algebraic equations by using the symbolic computation software, such as Maple 13, we obtain the following four solution sets.

Set 1. Consider

$$\begin{aligned}
\alpha_0 &= \pm i\frac{\sqrt{6}\delta L(A+2k-2Ck)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}},\\
\alpha_1 &= \pm i\frac{2\sqrt{6}\delta L(C-1)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}},\\
V &= -\frac{4\delta L}{L^2(A^2-4BC+4B)+2},\\
L &= L, \quad k = k, \quad \alpha_{-1} = 0, \quad C_1 = 0,
\end{aligned} \tag{16}$$

where k, L, A, B, and C are arbitrary constants.

Set 2. Consider

$$\begin{aligned}
\alpha_0 &= \mp i\frac{\sqrt{6}\delta L(A+2k-2Ck)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}},\\
\alpha_{-1} &= \pm i\frac{2\sqrt{6}\delta L(kA+k^2-Ck^2-B)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}},\\
V &= -\frac{4\delta L}{L^2(A^2-4BC+4B)+2},\\
L &= L, \quad k = k, \quad \alpha_1 = 0, \quad C_1 = 0,
\end{aligned} \tag{17}$$

where k, L, A, B, and C are arbitrary constants.

Set 3. Consider

$$\begin{aligned}
\alpha_1 &= \pm 2i\frac{\sqrt{3}\delta L(C-1)}{\sqrt{\gamma(2L^2(A^2-4BC+4B)+1)}},\\
\alpha_{-1} &= \pm i\frac{\sqrt{3}\delta L(A^2-4BC+4B)}{2\sqrt{\gamma(2L^2(A^2-4BC+4B)+1)}(C-1)},\\
V &= -\frac{2\delta L}{2L^2(A^2-4BC+4B)+1},\\
k &= \frac{A}{2(C-1)}, \quad L = L, \quad \alpha_0 = 0, \quad C_1 = 0,
\end{aligned} \tag{18}$$

where L, A, B, and C are arbitrary constants.

Set 4. Consider

$$\alpha_{-1} = \pm i\frac{\sqrt{6}\delta L\left(A^2 - 4BC + 4B\right)}{2\sqrt{\gamma\left(L^2\left(A^2 - 4BC + 4B\right) + 2\right)}(C-1)},$$
$$V = -\frac{4\delta L}{L^2\left(A^2 - 4BC + 4B\right) + 2}, \tag{19}$$
$$k = \frac{A}{2(C-1)}, \quad L = L, \quad \alpha_0 = 0, \quad \alpha_1 = 0, \quad C_1 = 0,$$

where L, A, B, and C are arbitrary constants.

Substituting (16)–(19) into (15), we obtain

$$u_1(\xi) = \pm i\frac{\sqrt{6}\delta L(A + 2k - 2Ck)}{\sqrt{\gamma\left(L^2\left(A^2 - 4BC + 4B\right) + 2\right)}} \pm i\frac{2\sqrt{6}\delta L(C-1)}{\sqrt{\gamma\left(L^2\left(A^2 - 4BC + 4B\right) + 2\right)}} \times \left(k + \left(\frac{G'}{G}\right)\right), \tag{20}$$

where

$$\xi = Lx - \left(\frac{4\delta L}{L^2\left(A^2 - 4BC + 4B\right) + 2}\right)\frac{t^\alpha}{\Gamma(1+\alpha)},$$
$$u_2(\xi) = \mp i\frac{\sqrt{6}\delta L(A + 2k - 2Ck)}{\sqrt{\gamma\left(L^2\left(A^2 - 4BC + 4B\right) + 2\right)}} \pm i\frac{2\sqrt{6}\delta L\left(kA + k^2 - Ck^2 - B\right)}{\sqrt{\gamma\left(L^2\left(A^2 - 4BC + 4B\right) + 2\right)}} \times \left(k + \left(\frac{G'}{G}\right)\right)^{-1}, \tag{21}$$

where

$$\xi = Lx - \left(\frac{4\delta L}{L^2\left(A^2 - 4BC + 4B\right) + 2}\right)\frac{t^\alpha}{\Gamma(1+\alpha)},$$
$$u_3(\xi) = \pm 2i\frac{\sqrt{3}\delta L(C-1)}{\sqrt{\gamma\left(2L^2\left(A^2 - 4BC + 4B\right) + 1\right)}} \times \left(\frac{A}{2(C-1)} + \left(\frac{G'}{G}\right)\right) \pm i\frac{\sqrt{3}\delta L\left(A^2 - 4BC + 4B\right)}{2\sqrt{\gamma\left(2L^2\left(A^2 - 4BC + 4B\right) + 1\right)}(C-1)} \times \left(\frac{A}{2(C-1)} + \left(\frac{G'}{G}\right)\right)^{-1}, \tag{22}$$

where

$$\xi = Lx - \left(\frac{2\delta L}{2L^2\left(A^2 - 4BC + 4B\right) + 1}\right)\frac{t^\alpha}{\Gamma(1+\alpha)},$$
$$u_4(\xi) = \pm i\frac{\sqrt{6}\delta L\left(A^2 - 4BC + 4B\right)}{2\sqrt{\gamma\left(L^2\left(A^2 - 4BC + 4B\right) + 2\right)}(C-1)} \times \left(\frac{A}{2(C-1)} + \left(\frac{G'}{G}\right)\right)^{-1}, \tag{23}$$

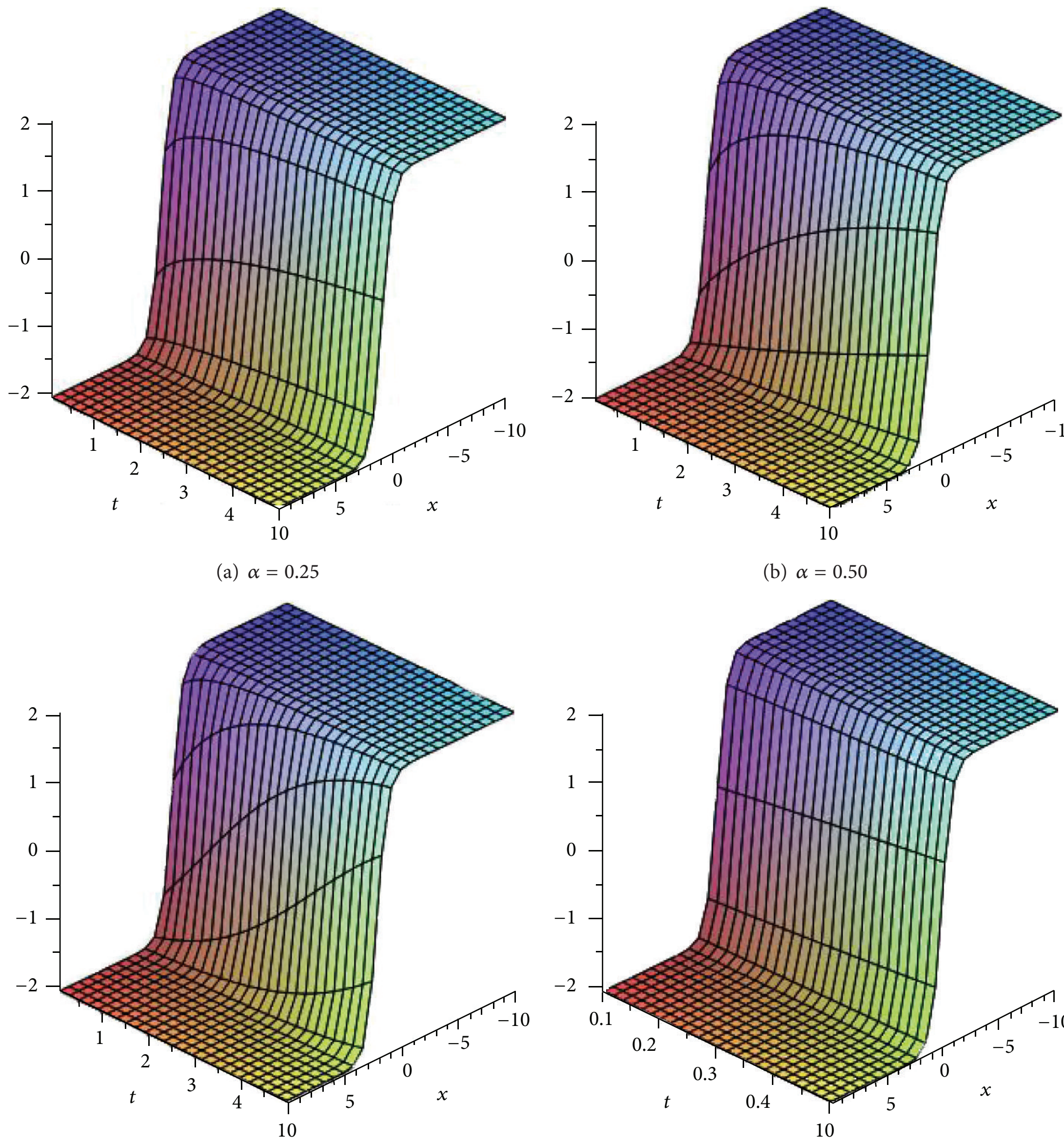
where

$$\xi = Lx - \left(\frac{4\delta L}{L^2\left(A^2 - 4BC + 4B\right) + 2}\right)\frac{t^\alpha}{\Gamma(1+\alpha)}. \tag{24}$$

Substituting the solutions $G(\xi)$ of (10) into (20) and simplifying, we obtain the following solutions.

When $\Delta = A^2 - 4BC + 4B > 0$ and $A(C-1) \neq 0$ (or $B(C-1) \neq 0$) (Figure 1),

$$u_1^1(\xi) = \pm i\frac{\sqrt{6}\delta L(A + 2k - 2Ck)}{\sqrt{\gamma\left(L^2\left(A^2 - 4BC + 4B\right) + 2\right)}} \pm i\frac{2\sqrt{6}\delta L(C-1)}{\sqrt{\gamma\left(L^2\left(A^2 - 4BC + 4B\right) + 2\right)}} \times \left\{k - \frac{1}{2(C-1)}\left(A + \sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)\right\},$$
$$u_1^2(\xi) = \pm i\frac{\sqrt{6}\delta L(A + 2k - 2Ck)}{\sqrt{\gamma\left(L^2\left(A^2 - 4BC + 4B\right) + 2\right)}} \pm i\frac{2\sqrt{6}\delta L(C-1)}{\sqrt{\gamma\left(L^2\left(A^2 - 4BC + 4B\right) + 2\right)}} \times \left\{k - \frac{1}{2(C-1)}\left(A + \sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)\right\},$$
$$u_1^3(\xi) = \pm i\frac{\sqrt{6}\delta L(A + 2k - 2Ck)}{\sqrt{\gamma\left(L^2\left(A^2 - 4BC + 4B\right) + 2\right)}} \pm i\frac{2\sqrt{6}\delta L(C-1)}{\sqrt{\gamma\left(L^2\left(A^2 - 4BC + 4B\right) + 2\right)}} \times \left\{k - \frac{1}{2(C-1)} \times \left(A + \sqrt{\Delta}\left(\tanh\left(\sqrt{\Delta}\xi\right) \pm i\operatorname{sech}\left(\sqrt{\Delta}\xi\right)\right)\right)\right\},$$



(a) $\alpha = 0.25$ (b) $\alpha = 0.50$ (c) $\alpha = 0.75$ (d) $\alpha = 1$

FIGURE 1: (a)–(d) show the kink solution for u_1^1 for different values of parameters.

$$u_1^4(\xi) = \pm i \frac{\sqrt{6\delta} L (A + 2k - 2Ck)}{\sqrt{\gamma (L^2 (A^2 - 4BC + 4B) + 2)}} \pm i \frac{2\sqrt{6\delta} L (C - 1)}{\sqrt{\gamma (L^2 (A^2 - 4BC + 4B) + 2)}} \times \left\{ k - \frac{1}{2(C-1)} \times \left(A + \sqrt{\Delta} \left(\coth\left(\sqrt{\Delta}\xi\right) \pm \operatorname{csch}\left(\sqrt{\Delta}\xi\right)\right)\right)\right\},$$

$$u_1^5(\xi) = \pm i \frac{\sqrt{6\delta} L (A + 2k - 2Ck)}{\sqrt{\gamma (L^2 (A^2 - 4BC + 4B) + 2)}} \pm i \frac{2\sqrt{6\delta} L (C - 1)}{\sqrt{\gamma (L^2 (A^2 - 4BC + 4B) + 2)}} \times \left\{ k - \frac{1}{4(C-1)} \times \left(2A + \sqrt{\Delta} \right.\right.$$

$$\times\left(\tanh\left(\frac{\sqrt{\Delta}\xi}{4}\right)+\coth\left(\frac{\sqrt{\Delta}\xi}{4}\right)\right)\right)\Bigg\},$$

$$u_1^6(\xi)=\pm i\frac{\sqrt{6}\delta L(A+2k-2Ck)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}}\pm i\frac{2\sqrt{6}\delta L(C-1)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}}\times\left[k+\frac{1}{2(C-1)}\times\left\{-A+\left(\pm\sqrt{\Delta(F^2+H^2)}-F\sqrt{\Delta}\cosh\left(\sqrt{\Delta}\xi\right)\right)\times\left(F\sinh\left(\sqrt{\Delta}\xi\right)+B\right)^{-1}\right\}\right],$$

$$u_1^7(\xi)=\pm i\frac{\sqrt{6}\delta L(A+2k-2Ck)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}}\pm i\frac{2\sqrt{6}\delta L(C-1)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}}\times\left[k+\frac{1}{2(C-1)}\times\left\{-A+\left(\pm\sqrt{\Delta(F^2+H^2)}+F\sqrt{\Delta}\cosh\left(\sqrt{\Delta}\xi\right)\right)\times\left(F\sinh\left(\sqrt{\Delta}\xi\right)+B\right)^{-1}\right\}\right], \tag{25}$$

where F and H are real constants (Figure 2). Consider

$$u_1^8(\xi)=\pm i\frac{\sqrt{6}\delta L(A+2k-2Ck)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}}\pm i\frac{2\sqrt{6}\delta L(C-1)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}}\times\left\{k+\frac{2B\cosh\left(\sqrt{\Delta}\xi/2\right)}{\sqrt{\Delta}\sinh\left(\sqrt{\Delta}\xi/2\right)-A\cosh\left(\sqrt{\Delta}\xi/2\right)}\right\},$$

$$u_1^9(\xi)=\pm i\frac{\sqrt{6}\delta L(A+2k-2Ck)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}}\pm i\frac{2\sqrt{6}\delta L(C-1)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}}\times\left\{k+\frac{2B\sinh\left(\sqrt{\Delta}\xi/2\right)}{\sqrt{\Delta}\cosh\left(\sqrt{\Delta}\xi/2\right)-A\sinh\left(\sqrt{\Delta}\xi/2\right)}\right\},$$

$$u_1^{10}(\xi)=\pm i\frac{\sqrt{6}\delta L(A+2k-2Ck)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}}\pm i\frac{2\sqrt{6}\delta L(C-1)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}}\times\left\{k+\frac{2B\cosh\left(\sqrt{\Delta}\xi\right)}{\sqrt{\Delta}\sinh\left(\sqrt{\Delta}\xi\right)-A\cosh\left(\sqrt{\Delta}\xi\right)\pm i\sqrt{\Delta}}\right\},$$

$$u_1^{11}(\xi)=\pm i\frac{\sqrt{6}\delta L(A+2k-2Ck)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}}\pm i\frac{2\sqrt{6}\delta L(C-1)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}}\times\left\{k+\frac{2B\sinh\left(\sqrt{\Delta}\xi\right)}{\sqrt{\Delta}\cosh\left(\sqrt{\Delta}\xi\right)-A\sinh\left(\sqrt{\Delta}\xi\right)\pm\sqrt{\Delta}}\right\}. \tag{26}$$

When $\Delta = A^2-4BC+4B < 0$ and $A(C-1)\neq 0$ (or $B(C-1)\neq 0$),

$$u_1^{12}(\xi)=\pm i\frac{\sqrt{6}\delta L(A+2k-2Ck)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}}\pm i\frac{2\sqrt{6}\delta L(C-1)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}}\times\left\{k+\frac{1}{2(C-1)}\times\left(-A+\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)\right\},$$

$$u_1^{13}(\xi)=\pm i\frac{\sqrt{6}\delta L(A+2k-2Ck)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}}\pm i\frac{2\sqrt{6}\delta L(C-1)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}}\times\left\{k-\frac{1}{2(C-1)}\times\left(A+\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)\right\},$$

$$u_1^{14}(\xi) = \pm i\frac{\sqrt{6\delta}L(A+2k-2Ck)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}} \pm i\frac{2\sqrt{6\delta}L(C-1)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}} \times \left\{k-\frac{1}{2(C-1)} \times \left(-A+\sqrt{-\Delta} \times \left(\tan\left(\sqrt{-\Delta}\xi\right) \pm \sec\left(\sqrt{-\Delta}\xi\right)\right)\right)\right\},$$

$$u_1^{15}(\xi) = \pm i\frac{\sqrt{6\delta}L(A+2k-2Ck)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}} \pm i\frac{2\sqrt{6\delta}L(C-1)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}} \times \left\{k-\frac{1}{2(C-1)} \times \left(A+\sqrt{-\Delta} \times \left(\cot\left(\sqrt{-\Delta}\xi\right) \pm \operatorname{csch}\left(\sqrt{-\Delta}\xi\right)\right)\right)\right\},$$

$$u_1^{16}(\xi) = \pm i\frac{\sqrt{6\delta}L(A+2k-2Ck)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}} \pm i\frac{2\sqrt{6\delta}L(C-1)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}} \times \left[k+\frac{1}{2(C-1)} \times \left\{-A+\left(\pm\sqrt{-\Delta(F^2-H^2)} - F\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi\right)\right) \times \left(F\sin\left(\sqrt{-\Delta}\xi\right)+B\right)^{-1}\right\}\right],$$

$$u_1^{17}(\xi) = \pm i\frac{\sqrt{6\delta}L(A+2k-2Ck)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}} \pm i\frac{2\sqrt{6\delta}L(C-1)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}} \times \left[k+\frac{1}{2(C-1)} \times \left\{-A+\left(\pm\sqrt{-\Delta(F^2-H^2)} - F\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi\right)\right) \times \left(F\sin\left(\sqrt{-\Delta}\xi\right)+B\right)^{-1}\right\}\right],$$

$$u_1^{18}(\xi) = \pm i\frac{\sqrt{6\delta}L(A+2k-2Ck)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}} \pm i\frac{2\sqrt{6\delta}L(C-1)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}} \times \left[k+\frac{1}{2(C-1)} \times \left\{-A+\left(\pm\sqrt{-\Delta(F^2-H^2)} + F\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi\right)\right) \times \left(F\sin\left(\sqrt{-\Delta}\xi\right)+B\right)^{-1}\right\}\right], \tag{27}$$

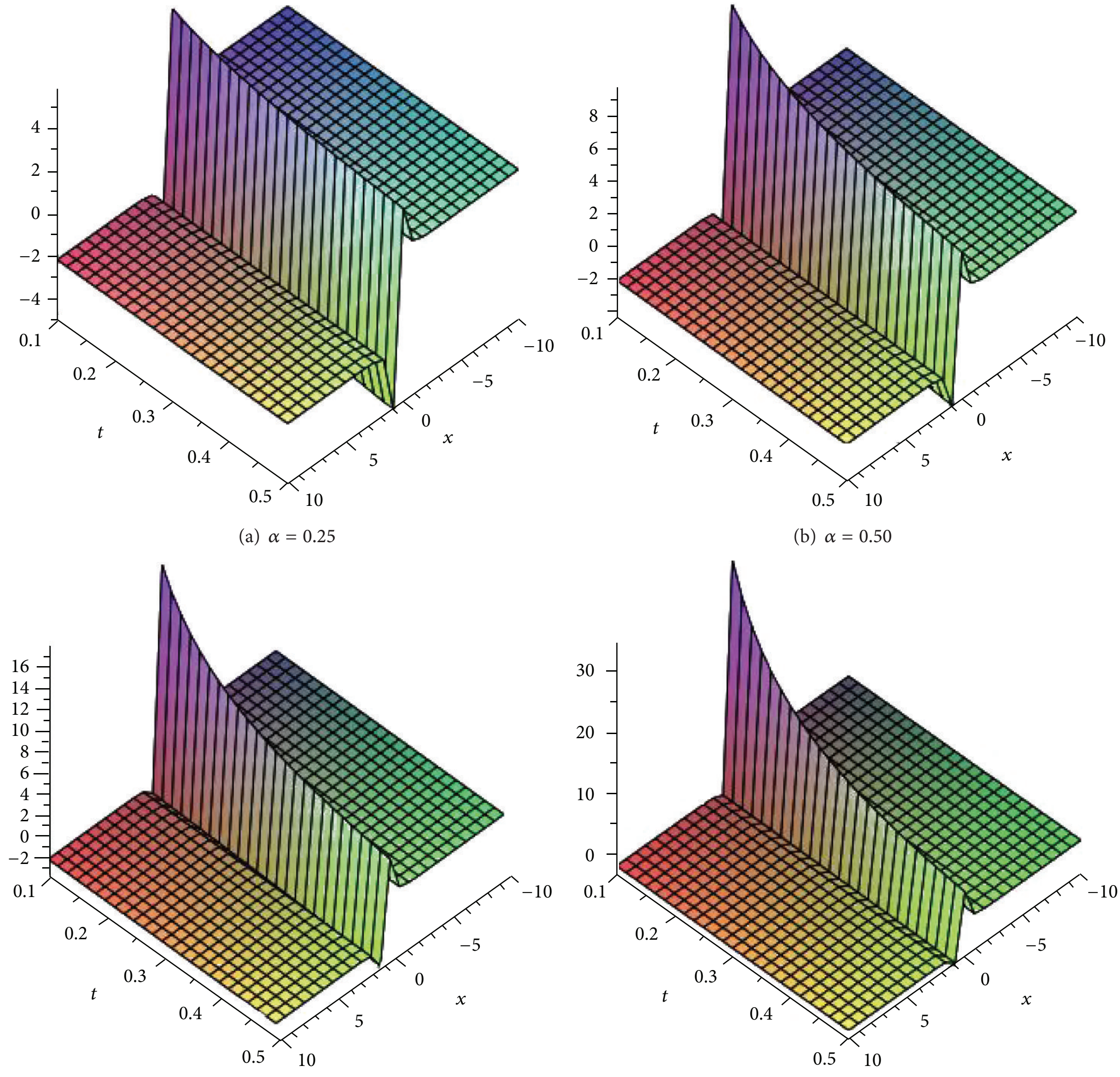
where F and H are real constants such that $F^2 - H^2 > 0$.

Consider

$$u_1^{19}(\xi) = \pm i\frac{\sqrt{6\delta}L(A+2k-2Ck)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}} \pm i\frac{2\sqrt{6\delta}L(C-1)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}} \times \left\{k-\frac{2B\cos\left(\sqrt{-\Delta}\xi/2\right)}{\sqrt{-\Delta}\sin\left(\sqrt{-\Delta}\xi/2\right)+A\cos\left(\sqrt{-\Delta}\xi/2\right)}\right\},$$

$$u_1^{20}(\xi) = \pm i\frac{\sqrt{6\delta}L(A+2k-2Ck)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}} \pm i\frac{2\sqrt{6\delta}L(C-1)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}} \times \left\{k+\frac{2B\sin\left(\sqrt{-\Delta}\xi/2\right)}{\sqrt{-\Delta}\cos\left(\sqrt{-\Delta}\xi/2\right)-A\sin\left(\sqrt{-\Delta}\xi/2\right)}\right\},$$

$$u_1^{21}(\xi) = \pm i\frac{\sqrt{6\delta}L(A+2k-2Ck)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}} \pm i\frac{2\sqrt{6\delta}L(C-1)}{\sqrt{\gamma(L^2(A^2-4BC+4B)+2)}}$$



(a) $\alpha = 0.25$

(b) $\alpha = 0.50$

(c) $\alpha = 0.75$

(d) $\alpha = 1$

FIGURE 2: (a)–(d) show the singular solution for u_1^2 for different values of parameters.

$$\times \left\{ k - \left(2B\cos\left(\sqrt{-\Delta}\xi\right)\right) \times \left(\sqrt{-\Delta}\sin\left(\sqrt{-\Delta}\xi\right) + A\cos\left(\sqrt{-\Delta}\xi\right) \pm \sqrt{-\Delta}\right)^{-1} \right\},$$

$$u_1^{22}(\xi) = \pm i \frac{\sqrt{6}\delta L (A + 2k - 2Ck)}{\sqrt{\gamma\left(L^2\left(A^2 - 4BC + 4B\right) + 2\right)}} \pm i \frac{2\sqrt{6}\delta L (C-1)}{\sqrt{\gamma\left(L^2\left(A^2 - 4BC + 4B\right) + 2\right)}} \times \left\{ k + \left(2B\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right) \times \left(\sqrt{-\Delta}\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right) - A\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right) \pm \sqrt{-\Delta}\right)^{-1} \right\}. \tag{28}$$

When $B = 0$ and $A(C-1) \neq 0$,

$$u_1^{23}(\xi) = \pm i \frac{\sqrt{6}\delta L (A + 2k - 2Ck)}{\sqrt{\gamma\left(L^2\left(A^2 - 4BC + 4B\right) + 2\right)}}$$

$$\pm i\frac{2\sqrt{6}\delta L(C-1)}{\sqrt{\gamma\left(L^{2}\left(A^{2}-4BC+4B\right)+2\right)}}\times\left\{k-\frac{Ac_{1}}{(C-1)\{c_{1}+\cosh(A\xi)-\sinh(A\xi)\}}\right\},$$

$$u_{1}^{24}(\xi)=\pm i\frac{\sqrt{6}\delta L(A+2k-2Ck)}{\sqrt{\gamma\left(L^{2}\left(A^{2}-4BC+4B\right)+2\right)}}\pm i\frac{2\sqrt{6}\delta L(C-1)}{\sqrt{\gamma\left(L^{2}\left(A^{2}-4BC+4B\right)+2\right)}}\times\left\{k-\frac{A(\cosh(A\xi)+\sinh(A\xi))}{(C-1)\{c_{1}+\cosh(A\xi)+\sinh(A\xi)\}}\right\},\tag{29}$$

where c_1 is an arbitrary constant.

When $A=B=0$ and $(C-1)\neq 0$, the solution of (12) is

$$u_{1}^{25}(\xi)=\pm i\frac{\sqrt{6}\delta L(A+2k-2Ck)}{\sqrt{\gamma\left(L^{2}\left(A^{2}-4BC+4B\right)+2\right)}}\pm i\frac{2\sqrt{6}\delta L(C-1)}{\sqrt{\gamma\left(L^{2}\left(A^{2}-4BC+4B\right)+2\right)}}\times\left\{k-\frac{1}{(C-1)\xi+c_{2}}\right\},\tag{30}$$

where c_2 is an arbitrary constant.

Substituting the solutions $G(\xi)$ of (10) in (21) and simplifying, we obtain the following solutions.

When $\Delta = A^2-4BC+4B>0$ and $A(C-1)\neq 0$ (or $B(C-1)\neq 0$),

$$u_{2}^{1}(\xi)=\mp i\frac{\sqrt{6}\delta L(A+2k-2Ck)}{\sqrt{\gamma\left(L^{2}\left(A^{2}-4BC+4B\right)+2\right)}}\pm i\frac{2\sqrt{6}\delta L\left(kA+k^{2}-Ck^{2}-B\right)}{\sqrt{\gamma\left(L^{2}\left(A^{2}-4BC+4B\right)+2\right)}}\times\left\{k-\frac{1}{2(C-1)}\left(A+\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)\right\}^{-1},$$

$$u_{2}^{2}(\xi)=\mp i\frac{\sqrt{6}\delta L(A+2k-2Ck)}{\sqrt{\gamma\left(L^{2}\left(A^{2}-4BC+4B\right)+2\right)}}\pm i\frac{2\sqrt{6}\delta L\left(kA+k^{2}-Ck^{2}-B\right)}{\sqrt{\gamma\left(L^{2}\left(A^{2}-4BC+4B\right)+2\right)}}\times\left\{k-\frac{1}{2(C-1)}\left(A+\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)\right\}^{-1},$$

$$u_{2}^{3}(\xi)=\mp i\frac{\sqrt{6}\delta L(A+2k-2Ck)}{\sqrt{\gamma\left(L^{2}\left(A^{2}-4BC+4B\right)+2\right)}}\pm i\frac{2\sqrt{6}\delta L\left(kA+k^{2}-Ck^{2}-B\right)}{\sqrt{\gamma\left(L^{2}\left(A^{2}-4BC+4B\right)+2\right)}}\times\left\{k-\frac{1}{2(C-1)}\times\left(A+\sqrt{\Delta}\left(\tanh\left(\sqrt{\Delta}\xi\right)\pm i\,\mathrm{sech}\left(\sqrt{\Delta}\xi\right)\right)\right)\right\}^{-1}.\tag{31}$$

The other families of exact solutions of (12) are omitted for convenience.

When $\Delta = A^2-4BC+4B<0$ and $A(C-1)\neq 0$ (or $B(C-1)\neq 0$) (Figure 3),

$$u_{2}^{12}(\xi)=\mp i\frac{\sqrt{6}\delta L(A+2k-2Ck)}{\sqrt{\gamma\left(L^{2}\left(A^{2}-4BC+4B\right)+2\right)}}\pm i\frac{2\sqrt{6}\delta L\left(kA+k^{2}-Ck^{2}-B\right)}{\sqrt{\gamma\left(L^{2}\left(A^{2}-4BC+4B\right)+2\right)}}\times\left\{k+\frac{1}{2(C-1)}\times\left(-A+\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)\right\}^{-1},$$

$$u_{2}^{13}(\xi)=\mp i\frac{\sqrt{6}\delta L(A+2k-2Ck)}{\sqrt{\gamma\left(L^{2}\left(A^{2}-4BC+4B\right)+2\right)}}\pm i\frac{2\sqrt{6}\delta L\left(kA+k^{2}-Ck^{2}-B\right)}{\sqrt{\gamma\left(L^{2}\left(A^{2}-4BC+4B\right)+2\right)}}\times\left\{k-\frac{1}{2(C-1)}\times\left(A+\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)\right\}^{-1},$$

$$u_{2}^{14}(\xi)=\mp i\frac{\sqrt{6}\delta L(A+2k-2Ck)}{\sqrt{\gamma\left(L^{2}\left(A^{2}-4BC+4B\right)+2\right)}}\pm i\frac{2\sqrt{6}\delta L\left(kA+k^{2}-Ck^{2}-B\right)}{\sqrt{\gamma\left(L^{2}\left(A^{2}-4BC+4B\right)+2\right)}}$$

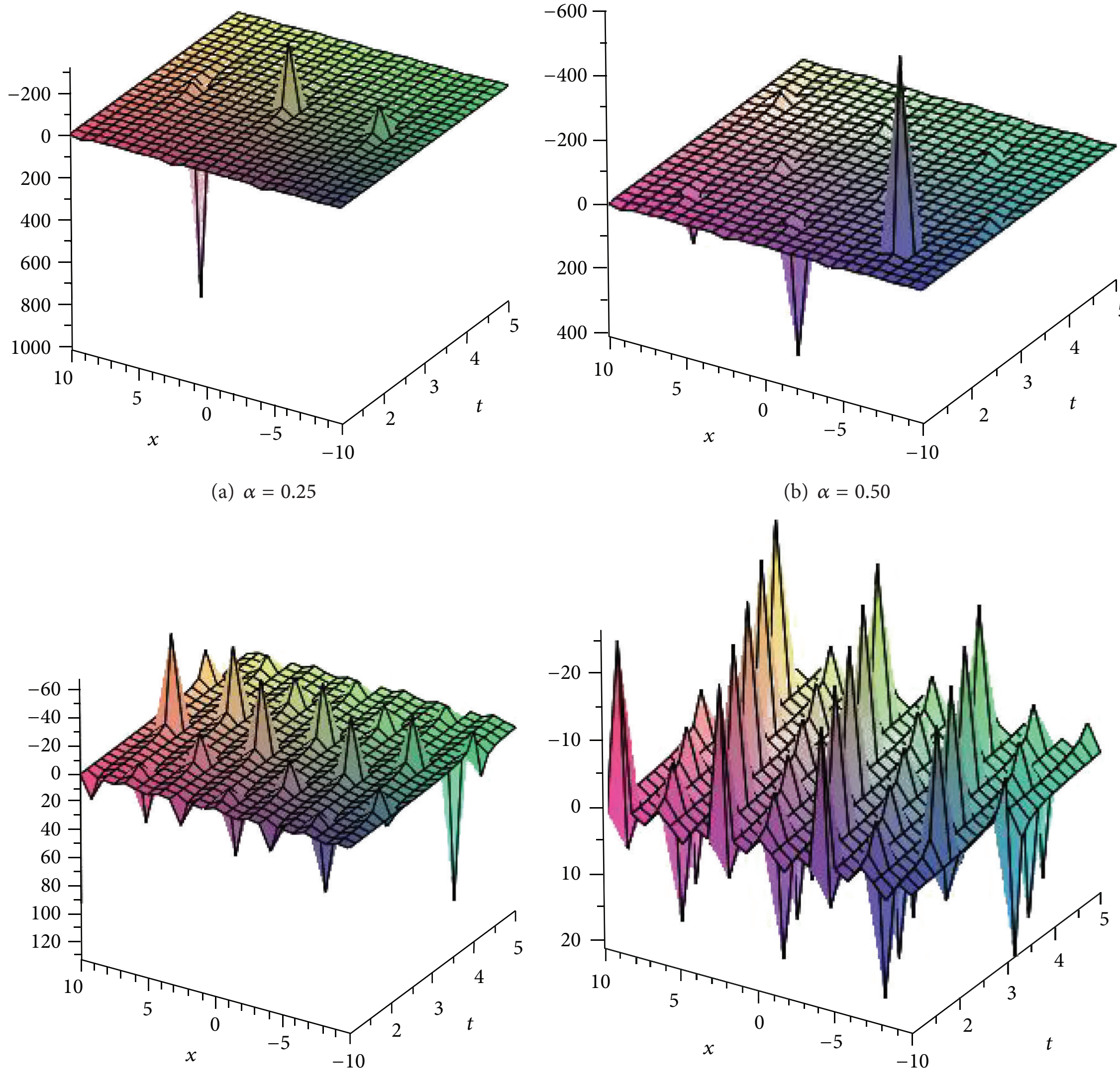


FIGURE 3: (a)–(d) show the periodic solution for u_2^{12} for different values of parameters.

$$\times \left\{k + \frac{1}{2(C-1)} \times \left(-A + \sqrt{-\Delta} \times \left(\tan\left(\sqrt{-\Delta}\xi\right) \pm \sec\left(\sqrt{-\Delta}\xi\right)\right)\right)\right\}^{-1}. \tag{32}$$

When $A = B = 0$ and $(C-1) \neq 0$, the solution of (12) is

$$u_2^{25}(\xi) = u_2(\xi) = \mp i \frac{\sqrt{6}\delta L (A + 2k - 2Ck)}{\sqrt{\gamma\left(L^2\left(A^2 - 4BC + 4B\right) + 2\right)}} \pm i \frac{2\sqrt{6}\delta L\left(kA + k^2 - Ck^2 - B\right)}{\sqrt{\gamma\left(L^2\left(A^2 - 4BC + 4B\right) + 2\right)}} \times \left\{k - \frac{1}{(C-1)\xi + c_2}\right\}^{-1}, \tag{33}$$

where c_2 is an arbitrary constant.

We can write down the other families of exact solutions of (12) which are omitted for practicality.

Similarly, by substituting the solutions $G(\xi)$ of (10) into (22) and simplifying, we obtain the following solutions.

When $\Delta = A^2 - 4BC + 4B > 0$ and $A(C-1) \neq 0$ (or $B(C-1) \neq 0$),

$$u_3^1(\xi) = \pm 2i\frac{\sqrt{3\delta}L(C-1)}{\sqrt{\gamma(2L^2(A^2-4BC+4B)+1)}} \times \left(\frac{1}{2(C-1)}\left(\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)\right) \pm i\frac{\sqrt{3\delta}L\left(A^2-4BC+4B\right)}{2\sqrt{\gamma(2L^2(A^2-4BC+4B)+1)}(C-1)} \times \left(\frac{1}{2(C-1)}\left(\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)\right)^{-1},$$

$$u_3^2(\xi) = \pm 2i\frac{\sqrt{3\delta}L(C-1)}{\sqrt{\gamma(2L^2(A^2-4BC+4B)+1)}} \times \left(\frac{1}{2(C-1)}\left(\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)\right) \pm i\frac{\sqrt{3\delta}L\left(A^2-4BC+4B\right)}{2\sqrt{\gamma(2L^2(A^2-4BC+4B)+1)}(C-1)} \times \left(\frac{1}{2(C-1)}\left(\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)\right)^{-1},$$

$$u_3^3(\xi) = \pm 2i\frac{\sqrt{3\delta}L(C-1)}{\sqrt{\gamma(2L^2(A^2-4BC+4B)+1)}} \times \left(\frac{1}{2(C-1)} \times \left\{\sqrt{\Delta}\tanh\left(\sqrt{\Delta}\xi\right) \pm i\,\mathrm{sech}\left(\sqrt{\Delta}\xi\right)\right\}\right) \pm i\frac{\sqrt{3\delta}L\left(A^2-4BC+4B\right)}{2\sqrt{\gamma(2L^2(A^2-4BC+4B)+1)}(C-1)} \times \left(\frac{1}{2(C-1)} \times \left\{\sqrt{\Delta}\tanh\left(\sqrt{\Delta}\xi\right) \pm i\,\mathrm{sech}\left(\sqrt{\Delta}\xi\right)\right\}\right)^{-1}. \tag{34}$$

Others families of exact solutions are omitted for the sake of simplicity.

When $\Delta = A^2 - 4BC + 4B < 0$ and $A(C-1) \neq 0$ (or $B(C-1) \neq 0$) (Figure 4),

$$u_3^{12}(\xi) = \pm 2i\frac{\sqrt{3\delta}L(C-1)}{\sqrt{\gamma(2L^2(A^2-4BC+4B)+1)}} \times \left(\frac{1}{2(C-1)}\left(\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)\right) \pm i\frac{\sqrt{3\delta}L\left(A^2-4BC+4B\right)}{2\sqrt{\gamma(2L^2(A^2-4BC+4B)+1)}(C-1)} \times \left(\frac{1}{2(C-1)}\left(\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)\right)^{-1},$$

$$u_3^{13}(\xi) = \pm 2i\frac{\sqrt{3\delta}L(C-1)}{\sqrt{\gamma(2L^2(A^2-4BC+4B)+1)}} \times \left(\frac{1}{2(C-1)}\left(\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)\right) \pm i\frac{\sqrt{3\delta}L\left(A^2-4BC+4B\right)}{2\sqrt{\gamma(2L^2(A^2-4BC+4B)+1)}(C-1)} \times \left(\frac{1}{2(C-1)}\left(\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)\right)^{-1},$$

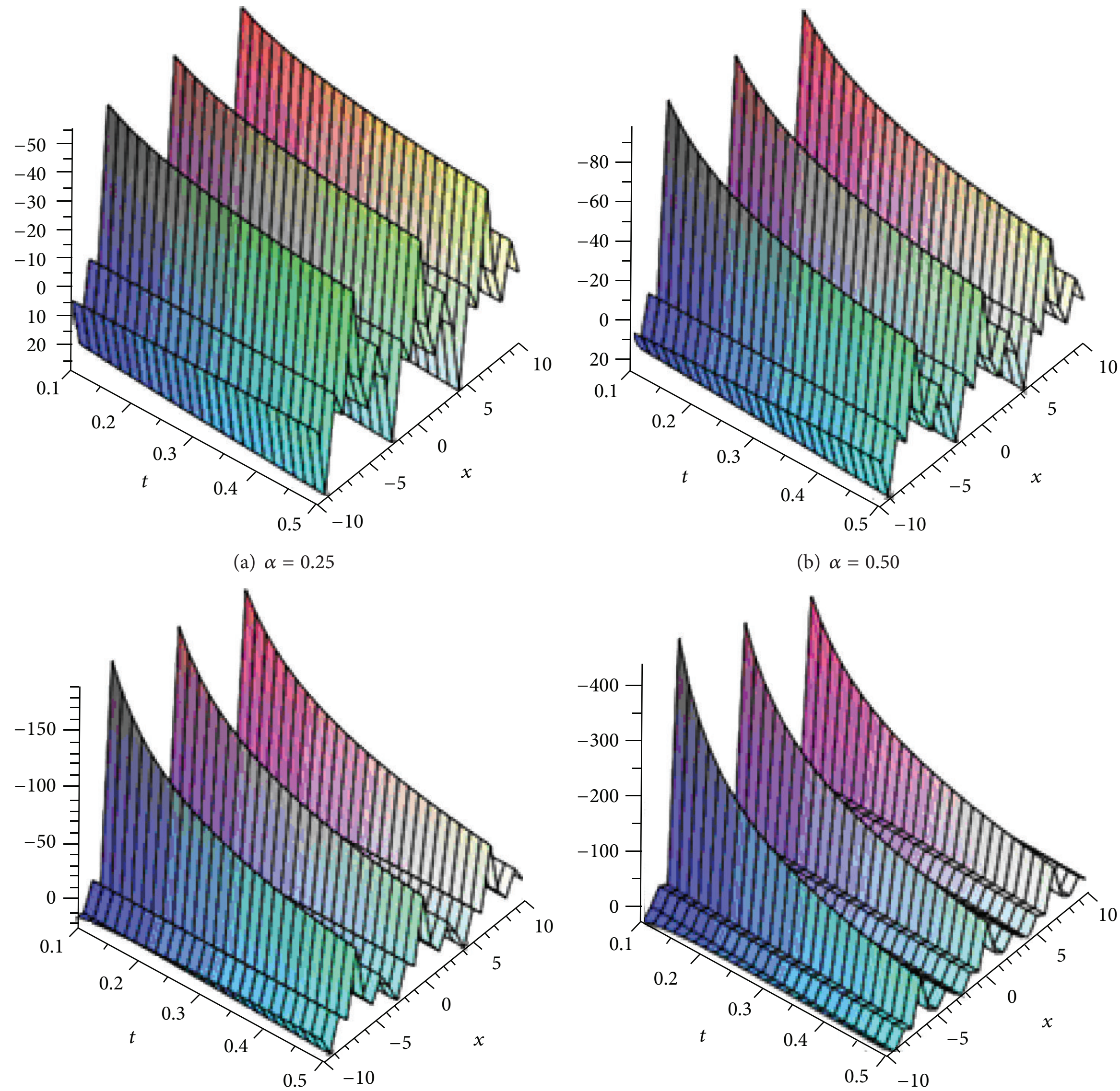
$$u_3^{14}(\xi) = \pm 2i\frac{\sqrt{3\delta}L(C-1)}{\sqrt{\gamma(2L^2(A^2-4BC+4B)+1)}} \times \left(\frac{1}{2(C-1)} \times \left\{\sqrt{-\Delta}\tan\left(\sqrt{-\Delta}\xi\right) \pm \sec\left(\sqrt{-\Delta}\xi\right)\right\}\right) \pm i\frac{\sqrt{3\delta}L\left(A^2-4BC+4B\right)}{2\sqrt{\gamma(2L^2(A^2-4BC+4B)+1)}(C-1)} \times \left(\frac{1}{2(C-1)} \times \left\{\sqrt{-\Delta}\tan\left(\sqrt{-\Delta}\xi\right) \pm \sec\left(\sqrt{-\Delta}\xi\right)\right\}\right)^{-1}. \tag{35}$$

When $(C-1) \neq 0$ and $A = B = 0$, the solution of (12) is

$$u_3^{25}(\xi) = \pm 2i\frac{\sqrt{3\delta}L(C-1)}{\sqrt{\gamma(2L^2(A^2-4BC+4B)+1)}} \times \left(\frac{A}{2(C-1)} - \frac{1}{(C-1)\xi + c_2}\right) \pm i\frac{\sqrt{3\delta}L\left(A^2-4BC+4B\right)}{2\sqrt{\gamma(2L^2(A^2-4BC+4B)+1)}(C-1)} \times \left(\frac{A}{2(C-1)} - \frac{1}{(C-1)\xi + c_2}\right)^{-1}, \tag{36}$$

where c_2 is an arbitrary constant.

Other exact solutions of (12) are omitted here for convenience.



(a) $\alpha = 0.25$ (b) $\alpha = 0.50$

(c) $\alpha = 0.75$ (d) $\alpha = 1$

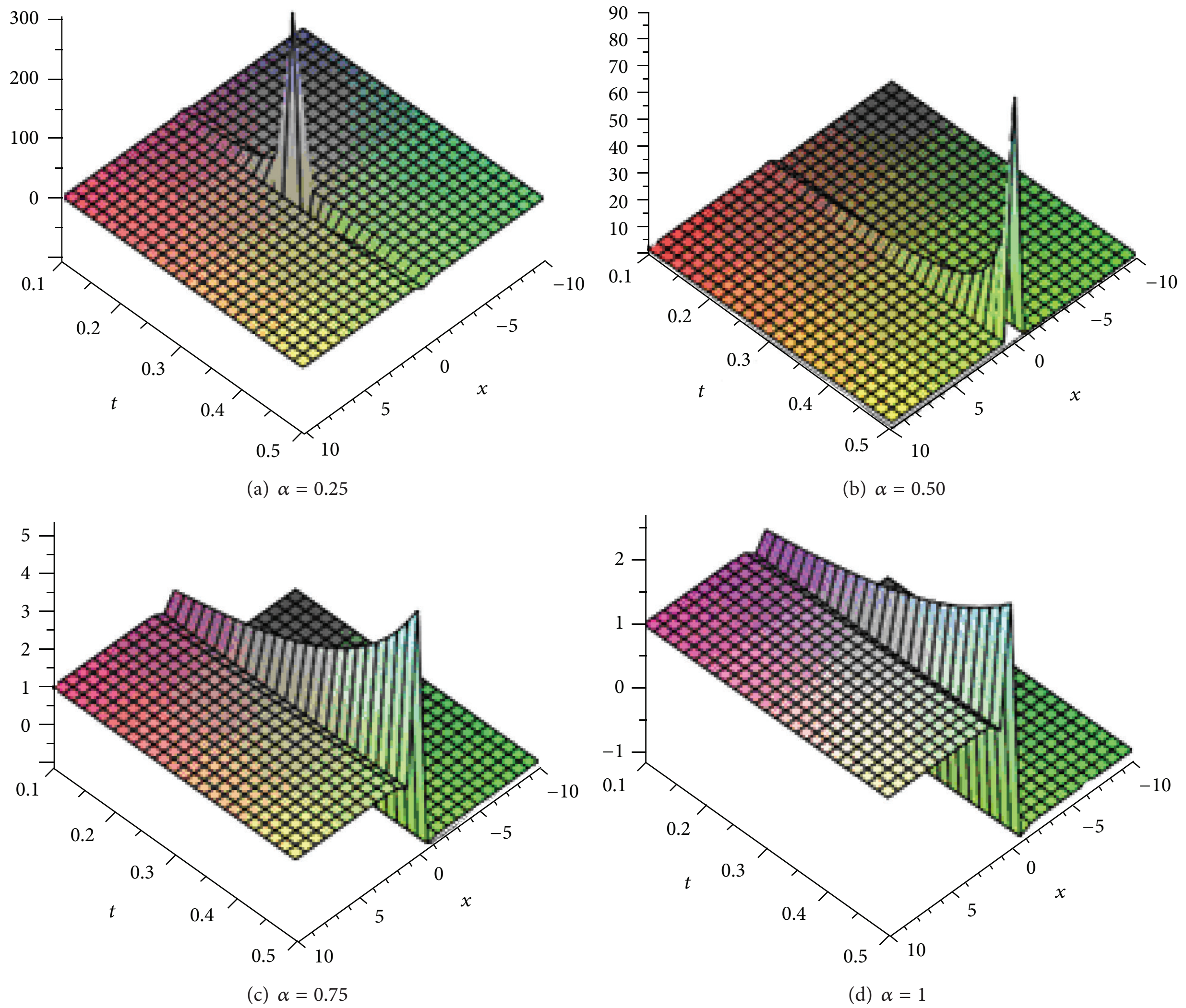
FIGURE 4: (a)–(d) show singular kink solution for u_3^{12} for different values of parameters.

Finally, by substituting the solutions $G(\xi)$ of (10) into (23) and simplifying, we obtain the following solutions.

When $\Delta = A^2 - 4BC + 4B > 0$ and $A(C-1) \neq 0$ (or $B(C-1) \neq 0$) (Figure 5),

$$
\begin{aligned}
u_4^1(\xi) &= \pm i \frac{\sqrt{6}\delta L\left(A^2 - 4BC + 4B\right)}{2\sqrt{\gamma\left(L^2\left(A^2 - 4BC + 4B\right) + 2\right)}(C-1)} \\
&\quad \times \left(\frac{1}{2(C-1)}\left(\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)\right)^{-1}, \\
u_4^2(\xi) &= \pm i \frac{\sqrt{6}\delta L\left(A^2 - 4BC + 4B\right)}{2\sqrt{\gamma\left(L^2\left(A^2 - 4BC + 4B\right) + 2\right)}(C-1)} \\
&\quad \times \left(\frac{1}{2(C-1)}\left(\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)\right)^{-1}, \\
u_4^3(\xi) &= \pm i \frac{\sqrt{6}\delta L\left(A^2 - 4BC + 4B\right)}{2\sqrt{\gamma\left(L^2\left(A^2 - 4BC + 4B\right) + 2\right)}(C-1)} \\
&\quad \times \left(\frac{1}{2(C-1)}\right. \\
&\qquad \left. \times \left\{\sqrt{\Delta}\tanh\left(\sqrt{\Delta}\xi\right) \pm i\,\mathrm{sech}\left(\sqrt{\Delta}\xi\right)\right\}\right)^{-1}.
\end{aligned} \tag{37}
$$

Others families of exact solutions are omitted for the sake of ease.



(a) $\alpha = 0.25$

(b) $\alpha = 0.50$

(c) $\alpha = 0.75$

(d) $\alpha = 1$

FIGURE 5: (a)–(d) show traveling wave solution for u_4^3 for different values of parameters.

When $\Delta = A^2 - 4BC + 4B < 0$ and $A(C-1) \neq 0$ (or $B(C-1) \neq 0$),

$$u_4^{12}(\xi) = \pm i \frac{\sqrt{6}\delta L\left(A^2 - 4BC + 4B\right)}{2\sqrt{\gamma\left(L^2\left(A^2 - 4BC + 4B\right) + 2\right)}(C-1)} \times \left(\frac{1}{2(C-1)}\left(\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)\right)^{-1},$$

$$u_4^{13}(\xi) = \pm i \frac{\sqrt{6}\delta L\left(A^2 - 4BC + 4B\right)}{2\sqrt{\gamma\left(L^2\left(A^2 - 4BC + 4B\right) + 2\right)}(C-1)} \times \left(\frac{1}{2(C-1)}\left(\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)\right)^{-1},$$

$$u_4^{14}(\xi) = \pm i \frac{\sqrt{6}\delta L\left(A^2 - 4BC + 4B\right)}{2\sqrt{\gamma\left(L^2\left(A^2 - 4BC + 4B\right) + 2\right)}(C-1)} \times \left(\frac{1}{2(C-1)} \times \left\{\sqrt{-\Delta}\tan\left(\sqrt{-\Delta}\xi\right) \pm \sec\left(\sqrt{-\Delta}\xi\right)\right\}\right)^{-1}. \tag{38}$$

When $(C-1) \neq 0$ and $A = B = 0$, the solution of (12) is

$$u_4^{25}(\xi) = \pm i \frac{\sqrt{6}\delta L\left(A^2 - 4BC + 4B\right)}{2\sqrt{\gamma\left(L^2\left(A^2 - 4BC + 4B\right) + 2\right)}(C-1)} \times \left(\frac{A}{2(C-1)} - \frac{1}{(C-1)\xi + c_2}\right)^{-1}, \tag{39}$$

where c_2 is an arbitrary constant.

Table 1: Comparison between our solutions and Liu et al. [48] solutions.

Obtained solutions	Liu et al. [48] solutions
(i) If $L = 1, A = 2, B = 0, C = 2, \delta = -1, \gamma = 1, k = 0, \alpha = 1$, and $u_1^1(\xi) = u_{1,2}(x,t)$, then the solution is $u_{1,2}(x,t) = \pm 2\tanh\left(x + \frac{2}{3}t\right)$.	(i) If $C_1 = 1, C_2 = 0, \lambda = 2, \mu = 0, a = 1$, and $k = 1$, then the solution is $u_{1,2}(x,t) = \pm 2\tanh\left(x + \frac{2}{3}t\right)$.
(ii) If $L = 1, A = 2, B = 1, C = 3, \delta = -1, \gamma = 1, k = 0, \alpha = 1$, and $u_1^{12}(\xi) = u_{3,4}(x,t)$, then the solution is $u_{3,4}(x,t) = \pm 2\sqrt{3}\tan(x + 2\ t)$.	(ii) If $C_1 = 1, C_2 = 0, \lambda^2 - 4\ \mu = -4, a = 1$, and $k = 1$, then the solution is $u_{3,4}(x,t) = \pm 2\sqrt{3}\tan(x + 2\ t)$.
(iii) If $L = 1, A = 0, B = 0, C = 2, \delta = -1, \gamma = 1, k = 0$, $\alpha = 1, c_2 = 0$, and $u_1^{25}(\xi) = u_{3,4}(x,t)$, then the solution is $u_{3,4}(x,t) = \pm 2\sqrt{3}\ \frac{1}{x + 2\ t}$.	(iii) If $C_1 = 1, C_2 = 1, \lambda = 2, \mu = 1, a = 1$, and $k = -1$, then the solution is $u_{3,4}(x,t) = \pm\ 2\sqrt{3}\ \frac{1}{x + 2\ t}$.
(iv) If $L = 1, A = 2, B = 0, C = 2, \delta = 1, \gamma = 1, k = 0\ , \alpha = 1$, and $u_1^1(\xi) = u_{1,2}(x,t)$, then the solution is $u_{3,4}(x,t) = \pm 2i\tanh\left(x - \frac{2}{3}t\right)$.	(iv) If $C_1 = 1, C_2 = 0, \lambda = 2, \mu = 0, a = 1$, and $k = 1$, then the solution is $u_{3,4}(x,t) = \pm 2i\tanh\left(x - \frac{2}{3}t\right)$.
(v) If $L = 1, A = 1, B = \frac{1}{2}, C = 3, \delta = -1, \gamma = 1, k = 0, \alpha = 1$, and $u_1^{12}(\xi) = u_{3,4}(x,t)$, then the solution is $u_{3,4}(x,t) = \pm\ \sqrt{6}i\tan\frac{1}{2}(x - 4\ t)$.	(v) If $C_1 = 1, C_2 = 0, \lambda = 0, \mu = \frac{1}{4}, a = 1$, and $k = 1$, then the solution is $u_{3,4}(x,t) = \pm\ \sqrt{6}\ i\tan\frac{1}{2}(x - 4\ t)$.
(vi) If $L = 1, A = 0, B = 0, C = 2, \delta = 1, \gamma = 1, k = 0$, $\alpha = 1, c_2 = 0$, and $u_1^{25}(\xi) = u_{3,4}(x,t)$, then the solution is $u_{3,4}(x,t) = \pm i2\sqrt{3}\ \frac{1}{x - 2\ t}$.	(vi) If $C_1 = 1, C_2 = 1, \lambda = 2, \mu = 1, a = 1$, and $k = 1$, then the solution is $u_{3,4}(x,t) = \pm i2\sqrt{3}\ \frac{1}{x - 2\ t}$.

Other exact solutions of (12) are omitted here for expediency.

4. Conclusions

A novel (G'/G)-expansion method is applied to fractional partial differential equation successfully. As applications, abundant new exact solutions for the time fractional simplified modified Camassa-Holm (MCH) equation have been successfully obtained. The nonlinear fractional complex transformation for ξ is very important, which ensures that a certain fractional partial differential equation can be converted into another ordinary differential equation of integer order. The obtained solutions are more general with more parameters. Also comparison has been made in the form of table (Table 1), which shows that some of our solutions are in full agreement with the results obtained previously. Thus, novel (G'/G)-expansion method would be a powerful mathematical tool for solving nonlinear evolution equations.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Solvability of an Integral Equation of Volterra-Wiener-Hopf Type

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The paper presents results concerning the solvability of a nonlinear integral equation of Volterra-Stieltjes type. We show that under some assumptions that equation has a continuous and bounded solution defined on the interval $[0, \infty)$ and having a finite limit at infinity. As a special case of the mentioned integral equation we obtain an integral equation of Volterra-Wiener-Hopf type. That fact enables us to formulate convenient and handy conditions ensuring the solvability of the equation in question in the class of functions defined and continuous on the interval $[0, \infty)$ and having finite limits at infinity.

1. Introduction

Integral equations play very important and significant role in the description of numerous events appearing in real world. Almost all branches of physics, mathematical physics, engineering, astronomy, economics, biology, and so forth utilize the theory of integral equations, both linear and nonlinear (cf. [1–5], e.g.).

Integral equations of Wiener-Hopf type create very important branch of the theory of integral equations [5]. Integral equations of such a type belong to the part of the theory of integral equations which are often called as integral equations depending on the difference of arguments [5]. It is worthwhile mentioning that integral equations of Wiener-Hopf type find numerous applications. For example, they are applied to describe some problems of radiative equilibria [6] and in the theory of diffraction [7]. Moreover, the reflection of an electromagnetic plane wave by an infinite sets of plates is also investigated with help of Wiener-Hopf integral equations [8]. Other possible applications of the theory of Wiener-Hopf integral equations are associated with dynamic elasticity [9], diffraction of plane waves by circular cone [10], and so forth (cf. also [5]).

Let us recall that the classical Wiener-Hopf integral equation has the form

$$x(t) = a(t) + \int_a^b k(t-s)\, f(s, x(s))\, ds, \tag{1}$$

where $t \in [a, b]$ and $k : \mathbb{R} \to \mathbb{R}$ is a given function which is continuous and integrable on the set of real numbers $\mathbb{R}$; that is, there exists a finite improper integral:

$$\int_{-\infty}^{+\infty} k(u)\, du. \tag{2}$$

Obviously, instead of (1) we may consider its "unbounded domain" counterpart having the form

$$x(t) = a(t) + \int_0^{\infty} k(t-s)\, f(s, x(s))\, ds \tag{3}$$

or even more general equations [5].

In this paper we will investigate the Volterra counterpart of the Wiener-Hopf integral equations (1) and (3), which has the form

$$x(t) = a(t) + \int_0^t k(t-s) f(s, x(s))\, ds, \tag{4}$$

where $t \in \mathbb{R}_+$ or $t \in [0, T]$ with $T > 0$.

Let us pay attention to the fact that Volterra-Wiener-Hopf integral equation (4) appears quite naturally as a special case of (1) and (3). In fact, if we require that

$$k(u) = 0 \quad \text{for } u \leqslant 0, \tag{5}$$

then (3) reduces to (4). This observation justifies the interest in the study of the Volterra-Wiener-Hopf integral equations.

To make our investigations more general and more convenient, we will study the so-called Volterra-Stieltjes integral equation having the form

$$x(t) = a(t) + \int_0^t f(s, x(s))\, d_s K(t, s), \tag{6}$$

where the involved integral is understood in the Riemann-Stieltjes sense.

The details explaining such an approach as well as suitable definitions will be presented in our further considerations.

2. Notation, Definitions, and Auxiliary Facts

In this sectio,n we present notation, definitions, and all auxiliary facts which will be utilized further on. Similar to Section 1, we will denote by $\mathbb{R}$ the set of real numbers. We put also $\mathbb{R}_+ = [0, \infty)$.

The investigation of the paper will be conducted in the Banach function space $BC(\mathbb{R}_+)$ consisting of all real functions defined, continuous, and bounded on the interval $\mathbb{R}_+$. This space is endowed by the classical supremum norm

$$\|x\| = \sup\{|x(t)| : t \geqslant 0\}. \tag{7}$$

Let us notice that in the space $BC(\mathbb{R}_+)$ the classical Ascoli-Arzela criterion for relative compactness fails to work and we know only a few sufficient conditions guaranteeing the relative compactness (cf. [11, 12]). Keeping in mind our further purposes we provide below a sufficient condition of such a type [12].

Theorem 1. *Let X be a nonempty and bounded subset of the space $BC(\mathbb{R}_+)$. Assume that X is locally equicontinuous; that is, for any $T > 0$, the functions from X are equicontinuous on the interval $[0, T]$. Moreover assume that the following condition is satisfied.*

For any $\varepsilon > 0$ there exists a number $T > 0$ such that for any function $x \in X$ and for all $t, s \in [T, \infty)$ the inequality $|x(t) - x(s)| \leqslant \varepsilon$ is satisfied.

Then the set X is relatively compact in the space $BC(\mathbb{R}_+)$.

Remark 2. Let us notice that in the case when a set X satisfies conditions imposed in Theorem 1 all functions from X tend to finite limits at infinity uniformly with respect to the set X (cf. [11, 12]).

In the sequel we will use the concept of the modulus of continuity of a function from the space $BC(\mathbb{R}_+)$. Thus, fix arbitrarily $T > 0$ and take a function $x \in BC(\mathbb{R}_+)$.

Consider the quantity

$$\omega^T(x, \varepsilon) = \sup\{|x(t) - x(s)| : t, s \in [0, T], |t - s| \leqslant \varepsilon\} \tag{8}$$

defined for $\varepsilon > 0$. This quantity is called the modulus of continuity of a function x on the interval $[0, T]$. Obviously $\lim_{\varepsilon \to 0} \omega^T(x, \varepsilon) = 0$ in view of the uniform continuity of x on the interval $[0, T]$.

Now, we provide needed facts concerning functions of bounded variation [13].

At the beginning assume that x is a real function defined on a fixed interval $[a, b]$. Then the symbol $\bigvee_a^b x$ will denote the variation of the function x on the interval $[a, b]$. In the case when $\bigvee_a^b x$ is finite we say that x is of bounded variation on $[a, b]$. If we have a function $u(t, s) = u : [a, b] \times [c, d] \to \mathbb{R}$, then we denote by $\bigvee_{t=p}^q u(t, s)$ the variation of the function $t \to u(t, s)$ on the interval $[p, q] \subset [a, b]$, where s is a fixed number in the interval $[c, d]$. Similarly we define the quantity $\bigvee_{s=p}^q u(t, s)$.

For the properties of functions of bounded variation we refer to [13].

If x and φ are two real functions defined on the interval $[a, b]$ then under some additional conditions [13] we can define the Stieltjes integral (in the Riemann-Stieltjes sense)

$$\int_a^b x(t)\, d\varphi(t) \tag{9}$$

of the function x with respect to the function φ. In such a case we say that x is Stieltjes integrable on the interval $[a, b]$ with respect to φ.

Let us mention that several conditions are known which guarantee the Stieltjes integrability [3, 13, 14]. One of the most frequently used conditions requires that x is continuous and φ is of bounded variation on $[a, b]$.

In what follows we will utilize a few properties of the Stieltjes integral contained in the following given lemmas [13].

Lemma 3. *If x is Stieltjes integrable on the interval $[a, b]$ with respect to a function φ of bounded variation, then*

$$\left| \int_a^b x(t)\, d\varphi(t) \right| \leqslant \int_a^b |x(t)|\, d\left(\bigvee_a^t \varphi \right). \tag{10}$$

Lemma 4. *Let x_1 and x_2 be Stieltjes integrable functions on the interval $[a, b]$ with respect to a nondecreasing function φ such that $x_1(t) \leqslant x_2(t)$ for $t \in [a, b]$. Then*

$$\int_a^b x_1(t)\, d\varphi(t) \leqslant \int_a^b x_2(t)\, d\varphi(t). \tag{11}$$

Further on, we will also consider Stieltjes integrals having the form

$$\int_a^b x(s)\, d_s g(t, s), \tag{12}$$

where $g : [a,b] \times [a,b] \to \mathbb{R}$ and the symbol d_s indicates the integration with respect to s. The details concerning the integral of this type will be given later. Let us only mention that integral (12) allows us to represent the Volterra-Wiener-Hopf integral equation (4) as a particular case of the Volterra-Stieltjes integral equation (6).

3. Main Results

The investigations of this section will be located in the Banach function space $\mathrm{BC}(\mathbb{R}_+)$ described previously in Section 2. Firstly, we will consider the solvability of the Volterra-Stieltjes integral equation having form (6). This equation will be studied under the following formulated assumptions.

(i) The function $a = a(t)$ belongs to the space $\mathrm{BC}(\mathbb{R}_+)$ and is such that there exists the limit $\lim_{t\to\infty} a(t)$ (obviously, this limit is finite).

(ii) $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ which is nondecreasing, $\phi(0) = 0$, $\lim_{t\to 0}\phi(t) = 0$, and such that

$$\left| f(t,x) - f(t,y) \right| \leqslant \phi\left(|x-y|\right) \tag{13}$$

for all $t \in \mathbb{R}_+$ and $x, y \in \mathbb{R}$.

(iii) The function $t \to f(t,0)$ is a member of $\mathrm{BC}(\mathbb{R}_+)$.

(iv) $K(t,s) = K : \Delta \to \mathbb{R}$ is a uniformly continuous function on the triangle

$$\Delta = \{(t,s) : 0 \leqslant s \leqslant t\}. \tag{14}$$

(v) The function $s \to K(t,s)$ is of bounded variation on the interval $[0,t]$ for each fixed $t \in \mathbb{R}_+$.

(vi) For any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $t_1, t_2 \in \mathbb{R}_+$ with $t_1 < t_2$, $t_2 - t_1 \leqslant \delta$, the following inequality holds:

$$\bigvee_{s=0}^{t_1} \left[K(t_2,s) - K(t_1,s) \right] \leqslant \varepsilon. \tag{15}$$

(vii) $K(t,0) = 0$ for all $t \geqslant 0$.

(viii) The function $t \to \bigvee_{s=0}^{t} K(t,s)$ is bounded on $\mathbb{R}_+$.

Before formulating the last assumption let us denote by F_1 and $\overline{K}$ the following constants:

$$\begin{gathered} F_1 = \sup\left\{ \left| f(t,0) \right| : t \in \mathbb{R}_+ \right\}, \\ \overline{K} = \sup\left\{ \bigvee_{s=0}^{t} K(t,s) : t \in \mathbb{R}_+ \right\}. \end{gathered} \tag{16}$$

Obviously $F_1 < \infty$ in view of assumption (iii), while the inequality $\overline{K} < \infty$ is a consequence of assumption (viii).

Now, we can formulate our last assumption.

(ix) There exists a positive solution r_0 of the inequality

$$\|a\| + \left(\phi(r) + F_1\right)\overline{K} \leqslant r. \tag{17}$$

Now, we are prepared to present our first main result.

Theorem 5. *Under assumptions (i)–(ix),* (6) *has at least one solution $x = x(t)$ in the space $BC(\mathbb{R}_+)$ which belongs to the ball $B_{r_0} = \{x \in BC(\mathbb{R}_+) : \|x\| \leqslant r_0\}$ and has a finite limit at infinity.*

In the proof of the above theorem we will need a few auxiliary facts contained in the following given lemmas.

Lemma 6. *The function*

$$p \longrightarrow \bigvee_{s=0}^{p} K(t,s) \tag{18}$$

is continuous on the interval $[0,t]$ for any fixed $t \in \mathbb{R}_+$.

This lemma is an easy consequence of assumptions (iv) and (v) and the properties of the variation of functions (cf. [13], p. 60).

Lemma 7. *Let assumptions (iv)–(vi) be satisfied. Then, for arbitrarily fixed number $t_2 > 0$ and for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $t_1 < t_2$ and $t_2 - t_1 \leqslant \delta$ then*

$$\bigvee_{s=t_1}^{t_2} K(t_2,s) \leqslant \varepsilon. \tag{19}$$

Proof. Fix $t_2 \in (0,\infty)$ and $\varepsilon > 0$. Next, consider the function H defined on the interval $[0,t_2]$ by the formula

$$H(p) = \bigvee_{s=0}^{p} K(t_2,s). \tag{20}$$

Then, in view of Lemma 6, the function H is continuous at the point t_2. Hence we infer that there exists $\delta > 0$ such that for $t_1 \geqslant 0$, $t_1 < t_2$, and $t_2 - t_1 \leqslant \delta$ we have that $|H(t_2) - H(t_1)| \leqslant \varepsilon$. On the other hand, we get

$$\begin{aligned} \left| H(t_2) - H(t_1) \right| &= \left| \bigvee_{s=0}^{t_2} K(t_2,s) - \bigvee_{s=0}^{t_1} K(t_2,s) \right| \\ &= \left| \bigvee_{s=0}^{t_1} K(t_2,s) + \bigvee_{s=t_1}^{t_2} K(t_2,s) - \bigvee_{s=0}^{t_1} K(t_2,s) \right| \\ &= \bigvee_{s=t_1}^{t_2} K(t_2,s) \leqslant \varepsilon. \end{aligned} \tag{21}$$

The proof is complete. □

Proof of Theorem 5. Let us consider the operator F defined on the space $\mathrm{BC}(\mathbb{R}_+)$ in the following way:

$$(Fx)(t) = a(t) + \int_0^t f(s,x(s))\, d_s K(t,s), \tag{22}$$

for $x \in \mathrm{BC}(\mathbb{R}_+)$ and for arbitrarily fixed $t \in \mathbb{R}_+$. Then, keeping in mind the imposed assumptions, we deduce that the function Fx is well defined.

Further, fix arbitrarily $T > 0$ and take $s, t \in [0, T]$. Without loss of generality we may assume that $s < t$. Then, in view of Lemmas 3 and 4, we obtain

$$
\begin{aligned}
&|(Fx)(t) - (Fx)(s)| \\
&\quad \leqslant |a(t) - a(s)| \\
&\qquad + \left| \int_0^t f(\tau, x(\tau))\, d_\tau K(t, \tau) - \int_0^s f(\tau, x(\tau))\, d_\tau K(s, \tau) \right| \\
&\quad \leqslant \omega^T(a, \varepsilon) + \left| \int_0^t f(\tau, x(\tau))\, d_\tau K(t, \tau) - \int_0^s f(\tau, x(\tau))\, d_\tau K(t, \tau) \right| \\
&\qquad + \left| \int_0^s f(\tau, x(\tau))\, d_\tau K(t, \tau) - \int_0^s f(\tau, x(\tau))\, d_\tau K(s, \tau) \right| \\
&\quad \leqslant \omega^T(a, \varepsilon) + \left| \int_s^t f(\tau, x(\tau))\, d_\tau K(t, \tau) \right| \\
&\qquad + \left| \int_0^s f(\tau, x(\tau))\, d_\tau [K(t, \tau) - K(s, \tau)] \right| \\
&\quad \leqslant \omega^T(a, \varepsilon) \\
&\qquad + \int_s^t |f(\tau, x(\tau))|\, d_\tau \left(\bigvee_{p=0}^{\tau} K(t, p) \right) \\
&\qquad + \int_0^s |f(\tau, x(\tau))|\, d_\tau \times \left(\bigvee_{q=0}^{\tau} [K(t, q) - K(s, q)] \right) \\
&\quad \leqslant \omega^T(a, \varepsilon) \\
&\qquad + \int_s^t [|f(\tau, x(\tau)) . - f(\tau, 0)| + |f(\tau, 0)|]\, d_\tau \left(\bigvee_{p=0}^{\tau} K(t, p) \right) \\
&\qquad + \int_0^s [|f(\tau, x(\tau)) . - f(\tau, 0)| + |f(\tau, 0)|]\, d_\tau \left(\bigvee_{q=0}^{\tau} [K(t, q) - K(s, q)] \right) \\
&\quad \leqslant \omega^T(a, \varepsilon) + \int_s^t \{\phi(|x(\tau)|) + F_1\}\, d_\tau \times \left(\bigvee_{p=0}^{\tau} K(t, p) \right) \\
&\qquad + \int_0^s \{\phi(|x(\tau)|) + F_1\}\, d_\tau \times \left(\bigvee_{q=0}^{\tau} [K(t, q) - K(s, q)] \right) \\
&\quad \leqslant \omega^T(a, \varepsilon) + \{\phi(\|x\|) + F_1\} \times \int_s^t d_\tau \left(\bigvee_{p=0}^{\tau} K(t, p) \right) \\
&\qquad + \{\phi(\|x\|) + F_1\} \times \int_0^s d_\tau \left(\bigvee_{q=0}^{\tau} [K(t, q) - K(s, q)] \right) \\
&\quad \leqslant \omega^T(a, \varepsilon) + \{\phi(\|x\|) + F_1\} \bigvee_{p=s}^{t} K(t, p) \\
&\qquad + \{\phi(\|x\|) + F_1\} \bigvee_{q=0}^{s} [K(t, q) - K(s, q)] .
\end{aligned} \tag{23}
$$

Hence, in view of assumption (vi) and Lemma 7, we conclude that the function Fx is continuous on the interval $[0, T]$. Since T was chosen arbitrarily this allows us to infer that Fx is continuous on $\mathbb{R}_+$.

Next, we show that the function Fx is bounded on $\mathbb{R}_+$. To this end, fix arbitrarily $x \in BC(\mathbb{R}_+)$ and $t \geqslant 0$. Then, in virtue of the imposed assumptions and Lemmas 3 and 4, we get

$$
\begin{aligned}
|(Fx)(t)| &\leqslant \|a\| + \int_0^t |f(s, x(s))|\, d_s \left(\bigvee_{p=0}^{s} K(t, p) \right) \\
&\leqslant \|a\| + \int_0^t [|f(s, x(s)) - f(s, 0)| + |f(s, 0)|]\, d_s \times \left(\bigvee_{p=0}^{s} K(t, p) \right) \\
&\leqslant \|a\| + \int_0^t [\phi(|x(s)|) + F_1]\, d_s \left(\bigvee_{p=0}^{s} K(t, p) \right) \\
&\leqslant \|a\| + \{\phi(\|x\|) + F_1\} \int_0^t d_s \left(\bigvee_{p=0}^{s} K(t, p) \right) \\
&\leqslant \|a\| + \{F_1 + \phi(\|x\|)\} \bigvee_{s=0}^{t} K(t, s) .
\end{aligned} \tag{24}
$$

Now, in view of assumption (viii), we conclude that the following inequality holds:

$$\|Fx\| \leqslant \|a\| + \{F_1 + \phi(\|x\|)\} \overline{K}. \tag{25}$$

The above inequality shows that the function Fx is bounded on $\mathbb{R}_+$. This fact in connection with the continuity of the function Fx established above shows that $Fx \in BC(\mathbb{R}_+)$. In other words, the operator F is a self-mapping of the space $BC(\mathbb{R}_+)$.

Moreover, on the basis of inequality (25) and assumption (ix), we conclude that there exists a positive number r_0 such that the operator F transforms the ball B_{r_0} (see assumption (ix)) into itself.

Now we show that the operator F is continuous on the ball B_{r_0}. To this end, fix $\varepsilon > 0$. Next, fix arbitrarily $x, y \in B_{r_0}$ such that $\|x-y\| \leqslant \varepsilon$. Then, taking into account the imposed assumptions, for an arbitrary fixed number $t \in \mathbb{R}_+$, we get

$$\begin{aligned} &|(Fx)(t) - (Fy)(t)| \\ &= \left| \int_0^t [f(s, x(s)) - f(s, y(s))] \, d_s K(t, s) \right| \\ &\leqslant \int_0^t |f(s, x(s)). - f(s, y(s))| \, d_s \left(\bigvee_{p=0}^{s} K(t, p) \right) \\ &\leqslant \int_0^t \phi(|x(s) - y(s)|) \, d_s \left(\bigvee_{p=0}^{s} K(t, p) \right) \\ &\leqslant \int_0^t \phi(\varepsilon) \, d_s \left(\bigvee_{p=0}^{s} K(t, p) \right) \\ &\leqslant \phi(\varepsilon) \bigvee_{s=0}^{t} K(t, s) \leqslant \overline{K} \phi(\varepsilon). \end{aligned} \tag{26}$$

The above obtained estimate (26) shows that the operator F is continuous even on the whole space $BC(\mathbb{R}_+)$.

Now we show that the set $F(B_{r_0})$ is relatively compact in the space $BC(\mathbb{R}_+)$. To show this fact we introduce two auxiliary functions $M = M(\varepsilon)$ and $N = N(\varepsilon)$ defined in the following way:

$$\begin{aligned} M(\varepsilon) = \sup &\left\{ \bigvee_{s=0}^{t_1} [K(t_2, s) - K(t_1, s)] : t_1, t_2 \in \mathbb{R}_+, \right. \\ &\left. t_1 < t_2, \ t_2 - t_1 \leqslant \varepsilon \right\}, \\ N(\varepsilon) = \sup &\left\{ \bigvee_{s=t_1}^{t_2} K(t_2, s) : t_1, t_2 \in \mathbb{R}_+, t_1 < t_2, \ t_2 - t_1 \leqslant \varepsilon \right\}. \end{aligned} \tag{27}$$

Observe that in view of assumption (iv) and Lemma 7 we have that $M(\varepsilon) \to 0$ and $N(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Further, fix arbitrarily $\varepsilon > 0$ and $T > 0$ and choose a function $x \in B_{r_0}$. Next, take $t, s \in [0, T]$ such that $|t - s| \leqslant \varepsilon$. Without loss of generality we may assume that $s < t$. Then, in view of estimate (23), we get

$$\begin{aligned} |(Fx)(t) - (Fx)(s)| \leqslant \omega^T(a, \varepsilon) + [\phi(r_0) + F_1] N(\varepsilon) \\ + [\phi(r_0) + F_1] M(\varepsilon). \end{aligned} \tag{28}$$

This estimate shows that functions from the set $F(B_{r_0})$ are equicontinuous on the interval $[0, T]$.

Next, taking arbitrarily $t, s \in [T, \infty)$ with $s < t$ and arguing in the same way as we done in order to obtain estimate (23), we get

$$\begin{aligned} |(Fx)(t) - (Fx)(s)| \leqslant |a(t) - a(s)| \\ + \{\phi(r_0) + F_1\} [M(\varepsilon) + N(\varepsilon)]. \end{aligned} \tag{29}$$

Hence, keeping in mind assumption (i), we can choose $T > 0$ so big that the term $|a(t) - a(s)|$ is suitably small for $s, t > T$. This assertion in conjunction with the fact that $M(\varepsilon) \to 0$ and $N(\varepsilon) \to 0$ as $\varepsilon \to 0$, in view of Theorem 1, allows us to deduce that the set $F(B_{r_0})$ is relatively compact in the space $BC(\mathbb{R}_+)$.

Now, taking into account the continuity of the operator F and applying the classical Schauder fixed point principle, we conclude that there exists at least one fixed point x of the operator F which belongs to the ball B_{r_0}. Obviously, the function $x = x(t)$ is a solution of the Volterra-Stieltjes integral equation (6). Moreover, let us notice that any fixed point $x = x(t)$ of the operator F from the ball B_{r_0} must belong to the set $F(B_{r_0})$ being relatively compact in the sense of Theorem 1. In the light of Remark 2 this fact allows us to infer that the function $x = x(t)$ being a solution of (6) has a finite limit at infinity.

The proof is complete. □

Now, we pay our attention to assumption (vi) playing a key role in our investigations. It turns out that we can formulate a condition being handy in applications and ensuring that the function $K = K(t, s)$ satisfies assumption (vi).

To formulate that condition assume, as previously, that $K(t, s) = K : \Delta \to \mathbb{R}$, where $\Delta = \{(t, s) : 0 \leqslant s \leqslant t\}$. Then, the announced condition may be formulated as follows.

(vi$'$) For arbitrary $t_1, t_2 \in \mathbb{R}_+$ such that $t_1 < t_2$ the function $s \to K(t_2, s) - K(t_1, s)$ is nonincreasing on the interval $[0, t_1]$.

Remark 8. The above condition and its consequences were discussed in [15] (cf. also [16]) under the assumption that $K : \Delta_1 \to \mathbb{R}$, where $\Delta_1 = \{(t, s) : 0 \leqslant s \leqslant t \leqslant 1\}$. Moreover, instead of $t_1, t_2 \in \mathbb{R}_+$ it was assumed that $t_1, t_2 \in [0, 1]$.

Further, we prove a few consequences of condition (vi$'$).

Lemma 9. *Under assumptions (vi$'$) and (vii), for arbitrarily fixed $s \in \mathbb{R}_+$, the function $t \to K(t, s)$ is nonincreasing on the interval $[s, \infty)$.*

Proof. Fix a number $s \in \mathbb{R}_+$ and take arbitrarily $t_1, t_2 \in [s, \infty)$ with $t_1 < t_2$. Then, in virtue of (vi$'$), we obtain

$$K(t_2, s) - K(t_1, s) \leqslant K(t_2, 0) - K(t_1, 0). \tag{30}$$

Hence, in view of assumption (vii), we have

$$K(t_2, s) - K(t_1, s) \leqslant 0 \tag{31}$$

and the proof is complete. □

The next result indicates the utility of assumption (vi$'$).

Theorem 10. *Suppose that the function $K = K(t,s)$ satisfies assumptions (iv), (vi$'$), and (vii). Then K satisfies assumption (vi).*

Proof. Fix an arbitrary number $\varepsilon > 0$. In view of assumption (iv) we deduce that there exists $\delta > 0$ such that if $t_1, t_2 \in \mathbb{R}_+$, $t_1 < t_2$ and $t_2 - t_1 < \delta$ then

$$\left|K(t_2,t_1) - K(t_1,t_1)\right| \leqslant \varepsilon. \tag{32}$$

In the light of Lemma 9 the above inequality can be written equivalently in the form

$$0 \leqslant K(t_1,t_1) - K(t_2,t_1) \leqslant \varepsilon. \tag{33}$$

Further, assume that t_1, t_2 are fixed. Take a partition $0 = s_0 < s_1 < \cdots < s_n = t_1$ of the interval $[0, t_1]$. Then, in view of assumptions (vi$'$) and (vii) and Lemma 9, we obtain

$$\begin{aligned}
&\sum_{i=1}^{n} \left|\left[K(t_2,s_i) - K(t_1,s_i)\right] - \left[K(t_2,s_{i-1}) - K(t_1,s_{i-1})\right]\right| \\
&\quad = \sum_{i=1}^{n} \left\{\left[K(t_2,s_{i-1}) - K(t_1,s_{i-1})\right] - \left[K(t_2,s_i) - K(t_1,s_i)\right]\right\} \\
&\quad = K(t_1,t_1) - K(t_2,t_1).
\end{aligned} \tag{34}$$

Hence we deduce that

$$\bigvee_{s=0}^{t_1} \left[K(t_2,s) - K(t_1,s)\right] = K(t_1,t_1) - K(t_2,t_1). \tag{35}$$

Finally, combining the above equality with (33), we complete the proof. □

In what follows we show how the result contained in Theorem 5 can be applied to the Volterra-Wiener-Hopf integral equation (4). First of all let us recall that (4) is a special case of the Volterra-Stieltjes integral equation (6) if we put

$$K(t,s) = \int_0^s k(t-z)\,dz \tag{36}$$

for $(t,s) \in \Delta$. Obviously such a substitution has a sense under suitable assumptions concerning the function $k = k(u)$, which will be formulated later.

To adapt the assumptions of Theorem 5 to our situation let us observe that assumption (vii) is then automatically satisfied since $K(t,0) = 0$.

Let us observe that in order to ensure the well definiteness of the function $K = K(t,s)$ we have to assume that the function $k = k(u)$ is locally integrable over $\mathbb{R}_+$ (in Lebesgue sense). Moreover, to adapt assumption (vi), let us notice that taking $t_1, t_2 \in \mathbb{R}_+$, $t_1 < t_2$, we have

$$\begin{aligned}
&\bigvee_{s=0}^{t_1} \left[K(t_2,s) - K(t_1,s)\right] \\
&\quad = \bigvee_{s=0}^{t_1} \left[\int_0^s k(t_2-z)\,dz - \int_0^s k(t_1-z)\,dz\right] \\
&\quad = \bigvee_{s=0}^{t_1} \int_0^s \left[k(t_2-z) - k(t_1-z)\right] dz \\
&\quad = \int_0^{t_1} \left[k(t_2-z) - k(t_1-z)\right] dz.
\end{aligned} \tag{37}$$

In view of the above equality we can reformulate assumption (vi) in the following way.

(vi$_1$) For any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $t_1, t_2 \in \mathbb{R}_+$ with $t_1 < t_2$, $t_2 - t_1 \leqslant \delta$, the following inequality holds:

$$\int_0^{t_1} \left[k(t_2-s) - k(t_1-s)\right] ds \leqslant \varepsilon. \tag{38}$$

In a similar way, assumption (viii) can be translated to the following form.

(viii$_1$) The function $t \to \int_0^t k(t-s)ds$ is bounded on $\mathbb{R}_+$.

In order to present the last assumption in a more transparent form, let us substitute $u = t - s$ in the integral appearing in assumption (viii$_1$). Then we get

$$\int_0^t k(t-s)\,ds = \int_0^t k(u)\,du. \tag{39}$$

Thus, the above condition concerning the local Lebesgue integrability of the function $k = k(u)$ in conjunction with the above observation implies that we should put the following assumption in place of (viii$_1$).

(viii$_2$) The function $k = k(u)$ is Lebesgue integrable over $\mathbb{R}_+$.

It is well-known that the Lebesgue integrability of the function $k = k(u)$ on the interval $\mathbb{R}_+$ implies that the function

$$t \longrightarrow \int_0^t k(u)\,du \tag{40}$$

(the indefinite integral of k) is absolutely continuous on $\mathbb{R}_+$ (cf. [13, 17, 18]). This immediately implies that the function defined by (40) is uniformly continuous on $\mathbb{R}_+$.

Now, let us observe that the above formulated assumption (vi$_1$) connected with the Volterra-Wiener-Hopf integral equation (4) has rather inconvenient form and is not easy to verify in practice. Therefore, in our further investigations, we will utilize assumption (vi$'$) instead of assumption (vi). Obviously, assumption (vi$'$) will be adapted to the case of (4).

To this end choose arbitrarily $t_1, t_2 \in \mathbb{R}_+$ with $t_1 < t_2$. According to assumption (vi$'$) the function

$$s \longrightarrow K(t_2, s) - K(t_1, s) \tag{41}$$

should be nonincreasing on the interval $[0, t_1]$. Taking into account that

$$\begin{aligned} K(t_2, s) - K(t_1, s) &= \int_0^s k(t_2 - z)\, dz - \int_0^s k(t_1 - z)\, dz \\ &= \int_0^s [k(t_2 - z) - k(t_1 - z)]\, dz, \end{aligned} \tag{42}$$

we have to impose the condition requiring that the function

$$s \longrightarrow \int_0^s [k(t_2 - z) - k(t_1 - z)]\, dz \tag{43}$$

is nonincreasing on the interval $[0, t_1]$.

Since the function (42) is absolutely continuous on the interval $[0, t_1]$, by the well-known facts from the theory of real functions [18], this requirement can be expressed equivalently in the following form:

$$k(t_2 - s) - k(t_1 - s) \leqslant 0 \tag{44}$$

for $s \in [0, t_1]$. Obviously this means that the function $k = k(u)$ is nonincreasing on $\mathbb{R}_+$.

On the other hand any monotone function is Riemann integrable. Thus, assuming additionally that $k : \mathbb{R}_+ \to \mathbb{R}_+$ we conclude that k is nonincreasing and bounded on $\mathbb{R}_+$. It is known [13] that in this case the function

$$t \longrightarrow \int_0^t k(u)\, du \tag{45}$$

is Lipschitz continuous on $\mathbb{R}_+$. In other words, in such a situation, we have that the function

$$s \longrightarrow K(t, s) = \int_0^s k(t - z)\, dz \tag{46}$$

is uniformly continuous on the interval $[0, t]$.

Keeping in mind the above conducted considerations we can formulate the following result concerning the Volterra-Wiener-Hopf integral equation (4).

Theorem 11. *Assume that there are satisfied assumptions (i), (ii), and (iii) of Theorem 5. Moreover, we assume that the following conditions are satisfied.*

(x) *The function $k(u) = k : \mathbb{R}_+ \to \mathbb{R}_+$ is nonincreasing and integrable on $\mathbb{R}_+$.*

(xi) *There exists a positive solution r_0 of the inequality*

$$\|a\| + (\phi(r) + F_1)\, \bar{k} \leqslant r, \tag{47}$$

where

$$\bar{k} = \int_0^\infty |k(u)|\, du. \tag{48}$$

Then there exists at least one solution $x = x(t)$ of (4) in the space $BC(\mathbb{R}_+)$ which has a limit at infinity.

Finally, let us mention that the result concerning the nonlinear integral equation (4) obtained in this section generalizes several ones which can be encountered in the literature (cf. [5, 19, 20], e.g.).

4. Further Discussions and Examples

At the beginning of this section we intend to discuss some assumptions imposed on the terms of the integral equation of Volterra-Wiener-Hopf type (4) considered in the previous section.

Let us start with the requirement that the function $k = k(u)$ transforms $\mathbb{R}_+$ into itself and is nonincreasing on $\mathbb{R}_+$. Observe that in that case we allow the function k to take negative values; that is, if we would assume that $k(u) \leqslant k(u_0) < 0$ for $u > u_0$, then we infer that k is not integrable on $\mathbb{R}_+$ and we obtain a contradiction with assumption (x).

Further on, let us notice that in our considerations connected with (6), assumption (vi$'$) can be replaced by the following one.

(vi$''$) For arbitrary $t_1, t_2 \in \mathbb{R}_+$ such that $t_1 < t_2$ the function $s \to K(t_2, s) - K(t_1, s)$ is nondecreasing on the interval $[0, t_1]$.

Indeed, in such a case, arguing similarly to the proof of Lemma 9 and Theorem 10, we can prove the following analogous results.

Lemma 12. *Under assumptions (vi$''$) and (vii), for arbitrarily fixed $s \in \mathbb{R}_+$, the function $t \to K(t, s)$ is nondecreasing on the interval $[s, \infty)$.*

Theorem 13. *Suppose that the function $K = K(t, s)$ satisfies assumptions (iv), (vi$''$), and (vii). Then K satisfies assumption (vi).*

Further, performing similar reasonings as at the end of Section 3, we can easily conclude that, in the case of Volterra-Wiener-Hopf equation (4), assumption (vi$''$) is equivalent to the requirement that the function $k = k(u)$ is nondecreasing on $\mathbb{R}_+$. This immediately yields that in order to ensure the integrability of the function k over the interval $\mathbb{R}_+$ we are forced to assume that $k : \mathbb{R}_+ \to \mathbb{R}_- = (-\infty, 0]$.

Now, we are prepared to formulate other (nondecreasing) versions of Theorem 11.

Theorem 14. *Assume that there are satisfied assumptions (i), (ii), and (iii) of Theorem 5 and assumption (xi) of Theorem 11. Moreover, we assume that the following condition is satisfied.*

(x$'$) *The function $k(u) = k : \mathbb{R}_+ \to \mathbb{R}_-$ is nondecreasing and integrable on $\mathbb{R}_+$.*

Then there exists at least one solution $x = x(t)$ of (4) in the space $BC(\mathbb{R}_+)$ which has a limit at infinity.

Obviously, the proof of Theorem 14 runs in a similar way as the proof of Theorem 11.

In what follows let us pay our attention to the case of Volterra-Wiener-Hopf integral equation (4) considered on a bounded interval $[0, T]$. This means that we consider the following integral equation of Volterra-Wiener-Hopf type:

$$x(t) = a(t) + \int_0^t k(t-s) f(s, x(s))\, ds \tag{49}$$

for $t \in [0, T]$, where $T > 0$ is a given number.

Observe that in this case we can replace assumption (i) by the assumption requiring that $a \in C[0, T]$ and we can delete assumption (iii). Similarly we can modify and adapt suitable assumptions (iv) and (vii). Summing up, we can formulate the following result concerning equation (49) for $t \in [0, T]$.

Theorem 15. *Assume that the following hypotheses are satisfied:*

(1) *$a \in C[0, T]$;*

(2) *$f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ which is nondecreasing, $\phi(0) = 0$, $\lim_{t \to 0} \phi(t) = 0$, and such that*

$$|f(t, x) - f(t, y)| \leqslant \phi(|x - y|) \tag{50}$$

for all $t \in [0, T]$ and $x, y \in \mathbb{R}$;

(3) *the function $k(u) = k : [0, T] \to \mathbb{R}$ is monotone on $[0, T]$;*

(4) *there exists a positive solution r_0 of the inequality*

$$\|a\|_{C[0,T]} + (\phi(r) + F_1)\, \overline{k} \leqslant r, \tag{51}$$

where $\overline{k} = \int_0^T |k(u)| du$ and $F_1 = \max\{|f(t, 0)| : t \in [0, T]\}$.

Then there exists at least one solution $x = x(t)$ of (49) in the space $C[0, T]$.

Let us remark that the space $C[0, T]$ denotes the classical Banach space consisting of real functions being continuous on the interval $[0, T]$ and endowed by the classical maximum norm.

In the remainder of this section we provide a few examples associated with the Volterra-Wiener-Hopf integral equation (4) considered on the real half-axis $\mathbb{R}_+$. At the beginning we present examples of functions $k = k(u)$ satisfying requirement of Theorems 11 and 14.

Example 1. Let us take the function k having the form

$$k(u) = \frac{1}{u^2 + 1}. \tag{52}$$

Obviously $k : \mathbb{R}_+ \to \mathbb{R}_+$ and the function k is nonincreasing on the interval $\mathbb{R}_+$. Moreover, the function k is integrable over $\mathbb{R}_+$ and

$$\int_0^t k(u)\, du = \arctan t. \tag{53}$$

Thus

$$\overline{k} = \int_0^\infty k(u)\, du = \frac{\pi}{2}. \tag{54}$$

Example 2. Consider the function $k(u) = e^{-u}$. Observe that this function satisfies assumption (x) since it is decreasing and integrable on $\mathbb{R}_+$. Moreover, we have that

$$\overline{k} = \int_0^\infty k(u)\, du = \int_0^\infty e^{-u} du = 1, \tag{55}$$

where $\overline{k}$ is the constant defined by (48).

Example 3. Now, consider the function $k = k(u)$ of the form

$$k(u) = (u + 1)\, e^{-u}. \tag{56}$$

It is easy to check that k is decreasing and integrable on the interval $\mathbb{R}_+$. Moreover, we have that

$$\overline{k} = \int_0^\infty (u + 1)\, e^{-u} du = 2. \tag{57}$$

Example 4. Let us take into account the function k defined by the formula

$$k(u) = \frac{-1}{1 + e^u}. \tag{58}$$

Obviously, we can easily verify that $k : \mathbb{R}_+ \to \mathbb{R}_-$ and k is increasing on $\mathbb{R}_+$. Moreover, we have

$$\overline{k} = \int_0^\infty \frac{1}{1 + e^u} du = \ln 2. \tag{59}$$

In what follows we provide an example illustrating Theorem 11.

Example 5. Let us consider the Volterra-Wiener-Hopf integral equation having the form

$$x(t) = \frac{t^2 + 1}{t^2 + 2} + \int_0^t \frac{1}{(t - s)^2 + 1} \sqrt[3]{x^2(s) + \arctan\left(\frac{s}{s^2 + 4}\right)} ds. \tag{60}$$

Observe that (60) is a special case of (4) if we put

$$a(t) = \frac{t^2 + 1}{t^2 + 2}, \tag{61}$$

$$k(u) = \frac{1}{u^2 + 1}, \tag{62}$$

$$f(t, x) = \sqrt[3]{x^2 + \arctan\left(\frac{t}{t^2 + 4}\right)}. \tag{63}$$

Let us verify that the terms involved in (60) satisfy the assumption of Theorem 11.

Indeed, the function $a = a(u)$ satisfies assumption (i) and we have that $\|a\| = 1$. Obviously, the function $f = f(t, x)$ defined by (63) is continuous on the set $\mathbb{R}_+ \times \mathbb{R}$. To prove the second part of assumption (ii) we will use the following inequality:

$$\left| \sqrt[3]{x^2 + a} - \sqrt[3]{y^2 + a} \right| \leqslant \sqrt[3]{(x - y)^2}, \tag{64}$$

(cf. [21]). Thus, in view of (64), we get

$$\begin{aligned}
&\left|f(t,x)-f(t,y)\right| \\
&\quad \leqslant \left|\sqrt[3]{x^2+\arctan\left(\frac{t}{t^2+4}\right)}-\sqrt[3]{y^2+\arctan\left(\frac{t}{t^2+4}\right)}\right| \\
&\quad \leqslant \sqrt[3]{(x-y)^2}.
\end{aligned} \tag{65}$$

Hence we infer that the function $f = f(t,x)$ satisfies assumption (ii) with the function $\phi(r) = r^{2/3}$. Obviously $\phi(0) = 0$, ϕ is nondecreasing, and $\lim_{r\to 0}\phi(r) = 0$.

In order to check that the function $f = f(t,x)$ satisfies assumption (iii) observe that

$$f(t,0) = \sqrt[3]{\arctan\left(\frac{t}{t^2+4}\right)}. \tag{66}$$

Applying the standard methods of differential calculus, we obtain

$$\begin{aligned}
F_1 = \sup\left\{\left|f(t,0)\right| : t \geqslant 0\right\} &= \sqrt[3]{\arctan\left(\frac{1}{4}\right)} \\
&= \sqrt[3]{0.2449\ldots} = 0.62564\ldots.
\end{aligned} \tag{67}$$

Next, in view of Example 1, we derive that the function k given by (52) satisfies assumption (x) and $\overline{k} = \pi/2$.

Finally, let us consider inequality (47) which now has the following form:

$$1 + \left(r^{2/3} + 0.62564\ldots\right)\frac{\pi}{2} \leqslant r. \tag{68}$$

Using the standard methods of mathematical analysis we can show that there exists a number $\overline{r}$ belonging to the interval $(8,9)$ which satisfies the equation

$$1 + \left(r^{2/3} + 0.62564\ldots\right)\frac{\pi}{2} = r. \tag{69}$$

Thus, this allows us to deduce that for any number $r_0 \geqslant \overline{r}$ there is satisfied inequality (68). For example, we can accept that $r_0 = 9$.

Now, invoking Theorem 11, we infer that there exists at least one solution $x = x(t)$ of (60) in the space $BC(\mathbb{R}_+)$ which belongs to the ball B_9 and has a finite limit at infinity. Obviously the limit $\lim_{t\to\infty}x(t)$ belongs to the interval $[-9,9]$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Exact Solutions for Nonlinear Wave Equations by the Exp-Function Method

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This paper elucidates the main advantages of the exp-function method in finding exact solutions of nonlinear wave equations. By the aid of some mathematical software, the solution process becomes extremely simple and accessible.

1. Introduction

One of the most important aspects in nonlinear science is how to solve an exact solution of a nonlinear equation. Recently many different methods have appeared, among which the homotopy perturbation method [1–4], the tanh-method [5], the sinh-method [6, 7], and the F-expansion method [8–11] have caught much attention; however, all these methods are valid for some special kinds of nonlinear equations. It is therefore very much needed to find a universal approach to nonlinear equations; this is very challenging indeed, and the exp-function method [12–15] meets this requirement. The exp-function method itself is mathematically beautiful and extremely accessible to nonmathematicians. The use of the method requires no special knowledge of advanced calculus, and it is especially effective for solitary solutions.

2. Exp-Function Method

The exp-function method was first proposed by He and Wu [16], and we consider a general partial differential equation (PED) in the form

$$P\left(u, u_t, u_x, u_y, u_{xx}, u_{tt}, u_{yy}\right) = 0 \tag{1}$$

to pick out the main solution process and its advantages.

Use a transformation [16]

$$\xi = kx + \omega t + ly, \tag{2}$$

where k, ω, and l are unknown constants and should be determined later. By (2), we can convert (1) to the following nonlinear ordinary differential equation:

$$G\left(u, u', u'', u''', \dots\right) = 0. \tag{3}$$

According to the exp-function method, we assume that its solution can be expressed in the following form [16, 17]:

$$u(\xi) = \frac{\sum_{n=-c}^{d} a_n \exp(n\xi)}{\sum_{m=-p}^{q} b_n \exp(m\xi)}, \tag{4}$$

where c, d, p, q are positive integers that could be freely chosen. To determine the value of c and p, we balance the linear term of highest order of (3) with the highest order of the nonlinear term. Similarly for determining the value of d and q, we balance the lowest orders of linear and nonlinear terms in (3). By substituting (4) into (3), collecting terms of the same term of $\exp(i\xi)$, and equating the coefficient of each power of exp to zero, we can get a set of algebraic equations for determining unknown constants.

3. Exact Solution for Nonlinear Wave Equation

In order to illustrate the basic solution process of the exp-function method, we use the Burgers-Huxley equation as an example, which can be expressed as [18]

$$u_t + u_{xx} + \frac{3}{k} u u_x + cu + u^2 + u^3 = 0, \tag{5}$$

where $u = u(x,t)$ is an unknown function, u_t, u_x are the partial derivatives of $u(x,t)$ with respect to t and x, respectively, and k and c are arbitrary constants.

According to the exp-function method [15–17], we introduce a complex variation ξ defined as

$$\xi = kx + \omega t. \tag{6}$$

Equation (5) thus becomes an ordinary differential equation as

$$k^2 u'' + (3u + \omega) u' + cu + u^2 + u^3 = 0. \tag{7}$$

We suppose that the solution of (7) can be expressed as

$$u(\xi) = \frac{a_c \exp(c\xi) + \cdots + a_{-d} \exp(-d\xi)}{b_p \exp(p\xi) + \cdots + b_{-q} \exp(-q\xi)}. \tag{8}$$

Thus we have

$$\begin{aligned} u'' &= \frac{\gamma_1 \exp((c+3p)\xi)}{\gamma_2 \exp(4\xi)}, \\ u^3 &= \frac{c_3 \exp(3c\xi) + \cdots}{c_4 \exp(3p\xi) + \cdots} = \frac{c_3 \exp((3c+p)\xi)}{c_4 \exp(4p\xi)}. \end{aligned} \tag{9}$$

Balancing highest order of exp-function in (9), we have $3c + p = c + 3p$, which leads to the result $p = c$. Similarly we balance the lowest orders of linear and nonlinear terms in (5) to determine values of d and q, and we can get $d = q$. For simplicity, we set $p = c = 1$ and $q = d = 1$; then (8) reduces to

$$u(\xi) = \frac{a_1 \exp(\xi) + a_0 + a_{-1} \exp(-\xi)}{\exp(\xi) + b_0 + b_{-1} \exp(-\xi)}. \tag{10}$$

Substituting (10) in to (5), we have

$$\begin{aligned} &\frac{1}{A} [E_3 \exp(3\xi) + E_2 \exp(2\xi) + E_1 \exp(\xi) + E_0 \\ &\quad + E_{-1} \exp(-\xi) + E_{-2} \exp(-2\xi) + E_{-3} \exp(-3\xi)] = 0, \end{aligned} \tag{11}$$

where $= [b_0 + \exp(\xi) + b_{-1} \exp(-\xi)]^3$,

$$\begin{aligned} E_3 &= a_1^3 + a_1^2 + ca_1, \\ E_2 &= ca_0 - a_0 a_1 - \omega a_0 + 3a_0 a_1^2 + 4a_1^2 b_0 \\ &\quad + k^2 a_0 + 2a_1 b_0 c + \omega a_1 b_0 - k^2 a_1 b_0, \\ E_0 &= a_0^3 + a_0^2 b_0 - 7a_0 a_{-1} + 6a_0 a_1 a_{-1} \\ &\quad + 11 a_0 a_1 b_{-1} + 2b_0 a_1 a_{-1} + 2ca_0 b_{-1} + 2cb_0 a_{-1} \\ &\quad - 3\omega b_0 a_{-1} + ca_0 b_0^2 - 6k^2 a_0 b_{-1} + 3k^2 a_{-1} b_0 \\ &\quad + 3k^2 a_{-1} b_{-1} b_0 + 2ca_1 b_{-1} b_0 + 3\omega a_1 b_{-1} b_0, \\ E_1 &= ca_{-1} - 4a_1 a_{-1} - 2\omega a_{-1} + 3a_1 a_0^2 + 3a_1^2 a_{-1} \\ &\quad + 7a_1^2 b_{-1} + 4k^2 a_{-1} - 2a_0^2 + \left(k^2 + c\right) a_1 b_0^2 + 5a_0 a_1 b_0 \\ &\quad + 2ca_1 b_{-1} - \omega a_0 b_0 + 2\omega a_1 b_{-1} + 2ca_0 b_0, \\ E_{-1} &= 3a_0^2 a_{-1} + 3a_1 a_{-1}^2 + 4a_0^2 b_{-1} - 5a_{-1}^2 + a_{-1} b_0^2 k^2 \\ &\quad + 4a_1 b_{-1}^2 k^2 - a_0 a_{-1} b_0 + 8a_1 a_{-1} b_{-1} + (2c - 2\omega) a_{-1} b_{-1} \\ &\quad + ca_{-1} b_0^2 + ca_1 b_{-1}^2 - 4a_{-1} b_{-1}^2 k^2 - \omega a_{-1} b_0^2 + 2\omega a_1 b_{-1}^2 \\ &\quad + \left(2c + \omega - k^2\right) a_0 b_{-1} b_0, \\ E_{-2} &= 3a_0 a_{-1}^2 - 2a_{-1}^2 b_0 + k^2 a_0 b_{-1}^2 + 5a_0 a_{-1} b_{-1} \\ &\quad + (c + \omega) a_0 b_{-1}^2 + \left(2c - k^2 - \omega\right) a_{-1} b_{-1} b_0, \\ E_{-3} &= a_{-1}^3 + a_{-1}^2 b_{-1} + ca_{-1} b_{-1}^2. \end{aligned} \tag{12}$$

Setting the coefficients of $\exp(i\xi)$, $(i = 0, \pm 1, \pm 2, \pm 3)$ to zero, we have

$$\begin{gathered} E_3 = 0, \qquad E_2 = 0, \qquad E_1 = 0 \\ E_0 = 0, \\ E_{-3} = 0, \qquad E_{-2} = 0, \qquad E_{-1} = 0. \end{gathered} \tag{13}$$

With the help of some mathematical software, we can solve the solutions of the algebraic equations.

Case 1. Consider

$$\begin{aligned} a_1 &= \frac{\sqrt{1-4c} - 1}{2}, \\ b_0 &= \frac{-a_0 \left(3a_1^2 - a_1 + k^2 + c - \omega\right)}{a_1 \left(-k^2 + 4a_1 + 2c + \omega\right)}, \\ \omega &= \frac{ca_0 - a_0 a_1 + 3a_0 a_1^2 + 4a_1^2 b_0 + a_0 k^2 + 2a_1 b_0 c - a_1 b_0 k^2}{a_0 - a_1 b_0}, \\ a_{-1} &= \frac{b_{-1} \left(\sqrt{1-4c} - 1\right)}{2}. \end{aligned} \tag{14}$$

This implies the following exact solution:

$$u(\xi) = \left(\frac{\sqrt{1-4c}-1}{2} \exp(\xi) + a_0 + \frac{b_2\left(\sqrt{1-4c}-1\right)}{2} \exp(-\xi) \right) \times \left(\exp(\xi) - \frac{\left(3a_0a_1^2 - a_0a_1 + a_0k^2 + ca_0 - \omega a_0\right)}{2ca_1 + \omega c - a_1k^2 + 4a_1^2} + b_2 \exp(-\xi) \right)^{-1}, \tag{15}$$

where

$$\xi = kx + \frac{ca_0 - a_0a_1 + 3a_0a_1^2 + 4a_1^2b_0 + a_0k^2 + 2a_1b_0c - a_1b_0k^2}{a_0 - a_1b_0}t, \tag{16}$$

a_0, b_2 are parameters, $b_2 \neq 0$, and k is a free real number.

Case 2. Consider

$$\begin{gathered} a_1 = \frac{1}{2}, \qquad b_0 = 0, \\ a_2 = -\frac{b_2}{2}, \qquad b_2 = b_2, \\ a_0 = \frac{\sqrt{2}}{2} + \sqrt{-16b_2k^2 + \frac{19}{2}b_2 + 2cb_2 + 8b_2\omega}, \\ \omega = k^2 + c + \frac{1}{4}. \end{gathered} \tag{17}$$

This case gives another exact solution as follows:

$$u(x,t) = \left(\frac{1}{2} \exp\left(kx + \left(k^2 + c + \frac{1}{4}\right)t\right) + \frac{\sqrt{2}}{2}\sqrt{10b_2c - 8b_2k^2 + \frac{19}{2}b_2 + 2b_2} - b_2 \exp\left(-kx - \left(k^2 + c + \frac{1}{4}\right)t\right)\right) \times \left(\exp\left(kx + \left(k^2 + c + \frac{1}{4}\right)t\right) + b_2 \exp\left(-kx - \left(k^2 + c + \frac{1}{4}\right)t\right)\right)^{-1}, \tag{18}$$

where b_2, k, are nonzero free parameters.

Case 3. Consider

$$\begin{gathered} a_1 = -\frac{\sqrt{1-4c}+1}{2}, \qquad a_0 = 0, \\ b_0 = 0, \qquad b_2 = b_2, \qquad a_2 = -\frac{\left(1+\sqrt{1-4c}\right)}{2}b_2. \end{gathered} \tag{19}$$

This results in the following exact solution:

$$u(\xi) = \left(-\frac{\sqrt{1-4c}+1}{2} \exp(\xi) - \frac{\left(1+\sqrt{1-4c}\right)}{2} b_2 \exp(-\xi) \right) \times \left(\exp(\xi) + b_2 \exp(-\xi)\right)^{-1}, \tag{20}$$

where b_2 is nonzero free parameter.

4. Conclusion

By some mathematical software, the solution process is extremely simple and abundant solutions are predicted [19–21]. The exp-function method is a universal tool for nonlinear equations and can be easily extended to fractional calculus [22–27].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Fractional Killing-Yano Tensors and Killing Vectors Using the Caputo Derivative in Some One- and Two-Dimensional Curved Space

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The classical free Lagrangian admitting a constant of motion, in one- and two-dimensional space, is generalized using the Caputo derivative of fractional calculus. The corresponding metric is obtained and the fractional Christoffel symbols, Killing vectors, and Killing-Yano tensors are derived. Some exact solutions of these quantities are reported.

1. Introduction

The tool of the fractional calculus started to be successfully applied in many fields of science and engineering (see, e.g., [1–12] and the references therein). Fractals and its connection to local fractional vector calculus represents another interesting field of application (see, e.g., [13, 14] and the references therein). Several definitions of the fractional differentiation and integration exist in the literature. The most commonly used are the Riemann-Liouville and the Caputo derivatives. The Riemann-Liouville derivative of a constant is not zero while Caputo's derivative of a constant is zero. This property makes the Caputo definition more suitable in all problems involving the fractional differential geometry [15, 16]. The Caputo differential operator of fractional calculus is defined as [1–8]

$$ {}_aD_x^{\alpha} f(x) \equiv \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \\ \times \displaystyle\int_a^x (x-u)^{n-\alpha-1} \frac{d^n f(u)}{du^n} du, & n-1<\alpha<n \\ \dfrac{d^n}{dx^n} f(x), & \alpha = n, \end{cases} \tag{1} $$

where $\Gamma(\cdot)$ is the Gamma function and $x > a$. In this work, we consider the case $a = 0, n-1 < \alpha \leq n$. For the power function x^p, $p \in R$, the Caputo fractional derivative satisfies

$$ D_x^{\alpha} x^p = \begin{cases} \dfrac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} x^{p-\alpha} \\ 0, & p = 0, 1, 2, \ldots, n-1. \end{cases} \tag{2} $$

The role played by Killing and Killing-Yano tensors for the geodesic motion of the particle and the superparticle in a curved background was a topic subjected to an intense debate during the last decades [17–26]. In [27] a generalization of exterior calculus was presented. Besides, the quadratic Lagrangians are introduced by adding surface terms to a free-particle Lagrangian in [28].

Motivated by the above mentioned results in differential geometry, we discuss in this paper the hidden symmetries corresponding to the fractional Killing vectors and Killing-Yano tensors on curved spaces deeply related to physical systems.

The Caputo partial differential operator of fractional order α is defined as

$$
{}_{a}\partial_{x}^{\alpha} f(x,y) \equiv \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \times \displaystyle\int_{a}^{x} (x-u)^{n-\alpha-1} \frac{\partial^{n} f(u,y)}{\partial u^{n}} du, & n-1<\alpha<n \\ \dfrac{\partial^{n}}{\partial x^{n}} f(x,y). & \alpha = n \end{cases} \tag{3}
$$

Again in this work we consider the case $a=0$, $n-1<\alpha\leq n$, and we drop the term a in the notation.

2. The Main Results

In the following, we present the Killing vectors and Killing-Yano tensors corresponding to some curved spaces with some physical significance.

2.1. One-Dimensional Case. Consider the one-dimensional free Lagrangian, admitting a constant of motion; that is, momentum [28]

$$L = \frac{1}{2}\dot{x}^{2} + \dot{\lambda}_{2}\dot{x}. \tag{4}$$

The Lagrangian can be rewritten as

$$L = \frac{1}{2} g_{ij} \dot{u}^{i} \dot{u}^{j}, \tag{5}$$

where $g_{ij} = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. The fractional Lagrangian of order q is given by

$$L = \frac{1}{2} g_{ij} D_{t}^{q} u^{i} D_{t}^{q} u^{j}, \tag{6}$$

where we consider the Caputo fractional derivative.

We generalize the Christoffel symbols in the fractional case, of order $n-1<q<n$, as

$${}^{q}\Gamma_{\beta\mu}^{\gamma} = \frac{1}{2} g^{\alpha\gamma} \left(\partial_{\mu}^{q} g_{\alpha\beta} + \partial_{\beta}^{q} g_{\alpha\mu} - \partial_{\gamma}^{q} g_{\beta\mu} \right), \tag{7}$$

where the partial derivatives of order q are defined in the fractional case.

We notice that because the metric is constant, all the Christoffel symbols vanish,

$${}^{q}\Gamma_{\mu\nu}^{\gamma} = 0. \tag{8}$$

2.1.1. Fractional Killing Vectors and Killing-Yano Tensors. The Killing vectors can be calculated from the generalized equations, namely,

$$V_{\alpha;\beta}^{q} + V_{\beta;\alpha}^{q} = 0, \tag{9}$$

where $V_{\alpha;\beta}^{q}$ is the fractional covariant derivative defined as

$$V_{\alpha;\beta}^{q} = \partial_{\beta}^{q} V_{\alpha} + g_{\alpha\mu}\, {}^{q}\Gamma_{\delta\beta}^{\mu} g^{\delta\lambda} V_{\lambda}. \tag{10}$$

Because all the Christoffel symbols vanish, it is easy to show that

$$
\begin{aligned}
V_{1;1}^{q} &= \partial_{1}^{q} V_{1} = 0, \\
V_{2;2}^{q} &= \partial_{2}^{q} V_{2} = 0, \\
V_{1;2}^{q} + V_{2;1}^{q} &= V_{1,2}^{q} + V_{2,1}^{q} = \partial_{2}^{q} V_{1} + \partial_{1}^{q} V_{2} = 0,
\end{aligned} \tag{11}
$$

For $0<q\leq 1$, a solution of the above equations is $V_{1} = -cy^{q}$, $V_{2} = cx^{q}$, where c is a constant. While for $q>1$, we have the general solution

$$
\begin{aligned}
V_{1} &= -cy^{q} + \sum_{k=0}^{n-1} \left(a_{k} x^{k} + b_{k} y^{k} \right), \\
V_{2} &= cx^{q} + \sum_{k=0}^{n-1} \left(a_{k}' x^{k} + b_{k}' y^{k} \right),
\end{aligned} \tag{12}
$$

where c, a_k, b_k, a_k', b_k' are constants.

The fractional Killing-Yano antisymmetric tensor ${}^{q}f_{\mu\nu}$ can be calculated using the condition

$${}^{q}f_{\mu\nu;\lambda} + {}^{q}f_{\lambda\nu;\mu} = 0, \tag{13}$$

where ${}^{q}f_{\mu\nu;\lambda}$ is the fractional covariant derivative of the Killing-Yano tensor ${}^{q}f_{\mu\nu}$ defined as

$${}^{q}f_{\mu\nu;\lambda} = \partial_{\lambda}^{q} f_{\mu\nu} - f_{\alpha\nu}\, {}^{q}\Gamma_{\lambda\mu}^{\alpha} - f_{\mu\alpha}\, {}^{q}\Gamma_{\lambda\nu}^{\alpha}. \tag{14}$$

We find that

$$\partial_{\lambda}^{q} f_{\mu\nu} = 0 \tag{15}$$

for all values of λ, ν, μ. A solution is $f_{11} = f_{22} = 0$ and $f_{12} = c = -f_{21}$, where c is a constant and for $0<q\leq 1$. While for $q>1$, that is, $n\geq 2$, we have the general solution

$$f_{12} = -f_{21} = \sum_{k=0}^{n-1} \left(a_{k} x^{k} + b_{k} y^{k} \right), \tag{16}$$

where a_k, b_k are constants.

2.2. Two-Dimensional Case. Below we consider the classical free Lagrangian, in two dimensions, admitting a constant of motion; that is, angular momentum [28]

$$L = \frac{1}{2}\left(\dot{x}^{2} + \dot{y}^{2} \right) + \dot{\lambda}_{3} \left(x\dot{y} - y\dot{x} \right). \tag{17}$$

The fractional Lagrangian is given by

$$L = \frac{1}{2} g_{ij} D^{\alpha} q^{i} D^{\alpha} q^{j}, \tag{18}$$

where g_{ij} is given by

$$g_{ij} = \begin{bmatrix} 1 & 0 & -y \\ 0 & 1 & x \\ -y & x & 0 \end{bmatrix}. \tag{19}$$

The inverse matrix of the metric is

$$g^{ij} = \frac{1}{x^2 + y^2}\begin{bmatrix} x^2 & xy & -y \\ xy & y^2 & x \\ -y & x & -1 \end{bmatrix}. \tag{20}$$

We generalize the Christoffel symbols in the fractional case, of order $n - 1 < q < n$, as

$${}^{q}\Gamma^{\gamma}_{\beta\mu} = \frac{1}{2} g^{\alpha\gamma}\left(\partial^{q}_{\mu} g_{\alpha\beta} + \partial^{q}_{\beta} g_{\alpha\mu} - \partial^{q}_{\gamma} g_{\beta\mu}\right). \tag{21}$$

One can show that

$${}^{q}\Gamma^{\gamma}_{\mu\mu} = 0 \tag{22}$$

for $\gamma, \mu = 1, 2, 3$, while

$$\begin{aligned} {}^{q}\Gamma^{\gamma}_{12} &= \frac{g^{3\gamma}}{2}\left(\partial^{q}_{1} g_{32} + \partial^{q}_{2} g_{31}\right), \\ {}^{q}\Gamma^{\gamma}_{13} &= \frac{g^{2\gamma}}{2}\left(\partial^{q}_{1} g_{32} - \partial^{q}_{2} g_{31}\right), \\ {}^{q}\Gamma^{\gamma}_{23} &= \frac{g^{1\gamma}}{2}\left(\partial^{q}_{2} g_{13} + \partial^{q}_{1} g_{23}\right). \end{aligned} \tag{23}$$

2.2.1. Fractional Killing Vectors. The Killing vectors can be calculated from the generalized equations

$$V^{q}_{\alpha;\beta} + V^{q}_{\beta;\alpha} = 0, \tag{24}$$

where $V^{q}_{\alpha;\beta}$ is the fractional covariant derivative defined as

$$V^{q}_{\alpha;\beta} = \partial^{q}_{\beta} V_{\alpha} + g_{\alpha\mu}\, {}^{q}\Gamma^{\mu}_{\delta\beta} g^{\delta\lambda} V_{\lambda}. \tag{25}$$

It is easy to show that

$$\begin{aligned} V^{q}_{1;1} &= \partial^{q}_{1} V_1 = 0, \\ V^{q}_{2;2} &= \partial^{q}_{2} V_2 = 0, \\ V^{q}_{3;3} &= \partial^{q}_{3} V_3 = 0, \\ V^{q}_{1;2} + V^{q}_{2;1} &= V^{q}_{1,2} + V^{q}_{2,1} = \partial^{q}_{2} V_1 + \partial^{q}_{1} V_2 = 0, \\ V^{q}_{1;3} + V^{q}_{3;1} &= \partial^{q}_{3} V_1 + \partial^{q}_{1} V_3 + g^{2\lambda} V_{\lambda} \partial^{q}_{2} g_{13} = 0, \\ V^{q}_{2;3} + V^{q}_{3;2} &= \partial^{q}_{3} V_2 + \partial^{q}_{2} V_3 + g^{1\lambda} V_{\lambda} \partial^{q}_{1} g_{23} = 0. \end{aligned} \tag{26}$$

A solution for V_1 and V_2 can be easily found for any fractional order q, that is, $n - 1 < q < n$, namely,

$$\begin{aligned} V_1 &= c y^{q} + \sum_{k=0}^{n-1}\left(a_k x^k + b_k y^k\right), \\ V_2 &= -c x^{q} + \sum_{k=0}^{n-1}\left(c_k x^k + d_k y^k\right), \end{aligned} \tag{27}$$

where c, a_k, b_k, c_k, d_k are constants. The solution to V_3 is not easy to find for $0 < q \leq 1$. However, for $n \geq 2$, that is, $1 < q$, the equations simplify because

$$\partial^{q}_{2} g_{13} = \partial^{q}_{1} g_{23} = 0. \tag{28}$$

In this case a general solution is obtained as

$$V_3 = \sum_{k=0}^{n-1}\left(a'_k x^k + b'_k y^k\right), \tag{29}$$

where a'_k, b'_k are constants.

2.2.2. Fractional Killing-Yano Tensors. The fractional antisymmetric Killing-Yano tensors can be derived using the condition that

$${}^{q}f_{\mu\nu;\lambda} + {}^{q}f_{\lambda\nu;\mu} = 0, \tag{30}$$

where ${}^{q}f_{\mu\nu;\lambda}$ is the fractional covariant derivative of the Killing-Yano tensor ${}^{q}f_{\mu\nu}$ defined as

$${}^{q}f_{\mu\nu;\lambda} = \partial^{q}_{\lambda} f_{\mu\nu} - f_{\alpha\nu}\, {}^{q}\Gamma^{\alpha}_{\lambda\mu} - f_{\mu\alpha}\, {}^{q}\Gamma^{\alpha}_{\lambda\nu}\,. \tag{31}$$

For the fractional order $0 < q < 1$, it is difficult to find an analytic solution. However, for the order $q > 1$, the Christoffel symbols vanish; we find that

$$\partial^{q}_{\lambda} f_{\mu\nu} = 0 \tag{32}$$

for all values of λ, ν, μ. A solution is that $f_{11} = f_{22} = f_{33} = 0$ and f_{12}, f_{13}, f_{23} are a linear combination of x^k, y^k where $k = 0, 1, 2, \ldots, n - 1$, namely,

$$\begin{aligned} f_{12} = -f_{21} &= \sum_{k=0}^{n-1}\left(a_k x^k + b_k y^k\right), \\ f_{13} = -f_{31} &= \sum_{k=0}^{n-1}\left(a'_k x^k + b'_k y^k\right), \\ f_{23} = -f_{32} &= \sum_{k=0}^{n-1}\left(c_k x^k + d_k y^k\right), \end{aligned} \tag{33}$$

where a_k, b_k, a'_k, b'_k, c_k, d_k are constants.

3. Conclusion

In this work, we investigate the existence of fractional Killing vectors and Killing-Yano tensors for the geometry induced by fractionalizing the classical free Lagrangian admitting a constant of motion. We discuss the cases of one-dimensional and two-dimensional curved space. We use the Caputo definition of the fractional derivative to calculate the fractional Christoffel symbols and consequently we provide explicit solution to the fractional Killing vectors and Killing-Yano tensors.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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