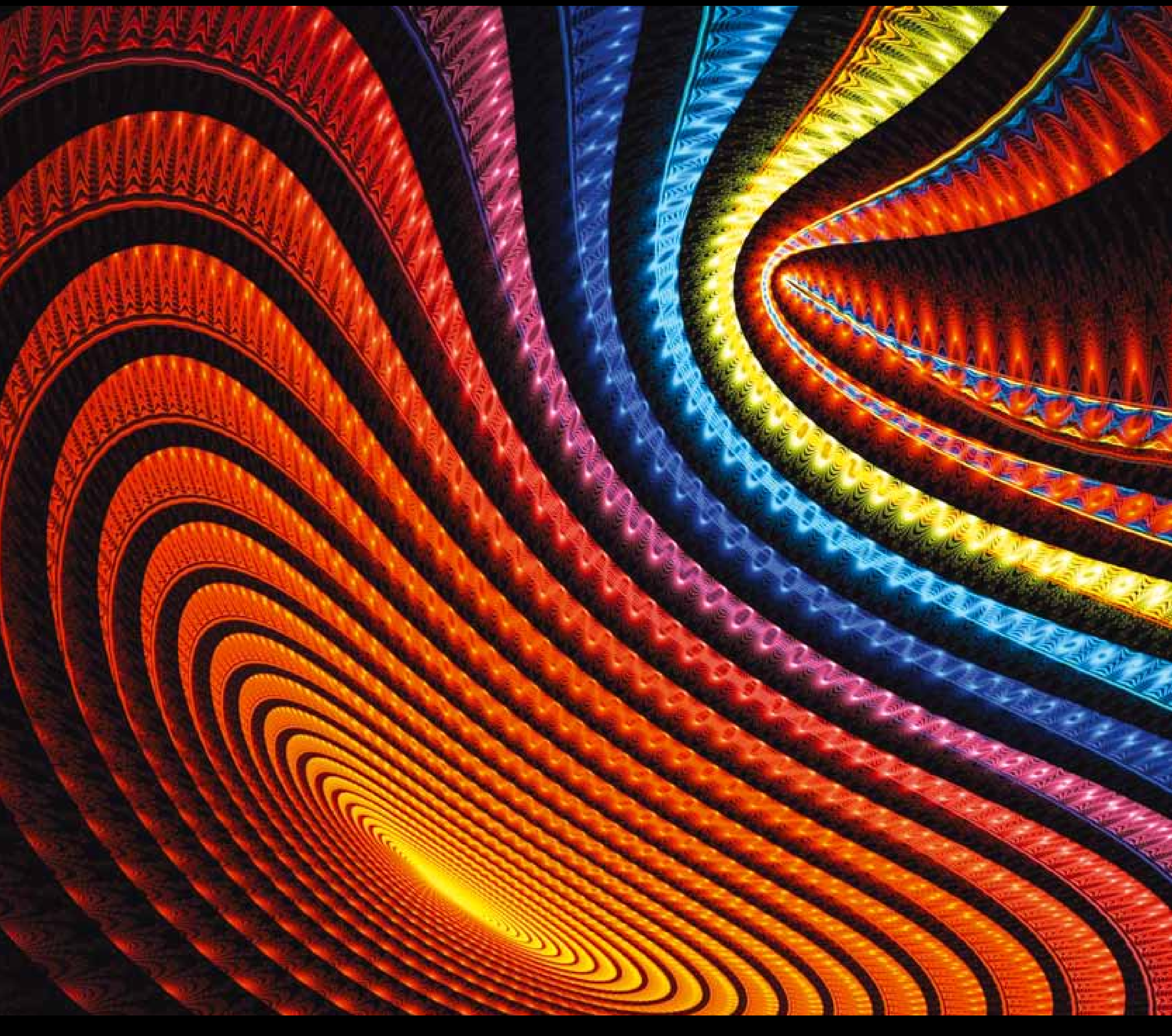
Abstract and Applied Analysis

Neurodynamic System Theory and Applications

Guest Editors: Jinde Cao, Xuerong Mao, and Qi Luo



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Editorial Board

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Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 734536, 1 page
http://dx.doi.org/10.1155/2013/734536

Editorial

Neurodynamic System Theory and Applications

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Received 12 June 2013; Accepted 12 June 2013

Neurodynamical systems have gradually become a popular research topic owing to their broad applications in such fields as optimization, linear and nonlinear programming, associative memory, pattern recognition, and computer vision. In view of some inevitable factors, there have been formed various neurodynamical systems including delayed neural networks, stochastic neural networks, impulsive neural networks, reaction-diffusion neural networks, and fuzzy neural networks. Over the last few decades, considerable attention has been devoted to this research area not only for enriching the theory of differential equations and dynamical systems but also for deeply understanding the dynamic states of neural networks for better modelling the brain.

The current special issue puts its emphasis on the study of neurodynamical system theory and applications. Call for papers has been carefully prepared by the guest editors and posted on the journal's web page, which has received many attentions followed by some submissions among wide topics such as delayed neural systems, stochastic neural systems, impulsive neural systems, reaction-diffusion neural systems, fuzzy neural systems, evolutionary neural systems, mathematical modeling of neural systems, computational neuroscience, neurodynamical optimization and adaptive dynamic programming, cognitive models, pattern recognition, and neural network applications.

All manuscripts submitted to this special issue went through a thorough peer-refereeing process. Based on the reviewers' reports, eleven original research articles are finally accepted. The contents embrace the synchronization of coupled neural networks, the numerical analysis of stochastic delayed partial differential equations, and the stability analysis of delayed impulsive reaction-diffusion neural networks and switched neural networks.

It is certainly impossible to provide in this short editorial a more comprehensive description for all articles in this special issue. However, the team of the guest editors believes that the results included reflect some recent trends in research and outline new ideas for future studies of neurodynamical system theory and applications.

Acknowledgments

We would like to express sincere gratitude to the authors who submitted papers for consideration and the many reviewers whose comments are important for us to make the decisions. All the participants have made it possible to have a very stimulating interchange of ideas. Many thanks are also given to the editorial board members of this journal owing to their great support and help for this special issue.

Jinde Cao
Xuerong Mao
Qi Luo

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 420605, 10 pages
http://dx.doi.org/10.1155/2013/420605

Research Article

Study of the Method of Multi-Frequency Signal Detection Based on the Adaptive Stochastic Resonance

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Received 1 January 2013; Accepted 12 March 2013

Academic Editor: Xuerong Mao

Recently, the stochastic resonance effect has been widely used by the method of discovering and extracting weak periodic signals from strong noise through the stochastic resonance effect. The detection of the single-frequency weak signals by using stochastic resonance effect is widely used. However, the detection methods of the multifrequency weak signals need to be researched. According to the different frequency input signals of a given system, this paper puts forward a detection method of multifrequency signal by using adaptive stochastic resonance, which analyzed the frequency characteristics and the parallel number of the input signals, adjusted system parameters automatically to the low frequency signals in the fixed step size, and then measured the stochastic resonance phenomenon based on the frequency of the periodic signals to select the most appropriate indicators in the middle or high frequency. Finally, the optimized system parameters are founded and the frequency of the given signals is extracted in the frequency domain of the stochastic resonance output signals. Compared with the traditional detection methods, the method in this paper not only improves the work efficiency but also makes it more accurate by using the color noise, the frequency is more accurate being extracted from the measured signal. The consistency between the simulation results and analysis shows that this method is effective and feasible.

1. Introduction

Now, we need to find and extract useful signal through the signal detection in engineering technology and scientific research. The traditional method to detect signal usually uses linear filtering, wavelet analysis [1], and so on to reduce and eliminate noise and finally obtain the useful signal. Although some weak signals are often overwhelmed by strong noise, the weak periodic signal is also reduced in the denoise to a certain extent, which made some weak periodic signal fail to be detected and extracted. In 1981, Benzi et al. proposed the concept of stochastic resonance [2] which provides a new research method for the detection of weak periodic signal. Compared to the traditional signal detection method, stochastic resonance is a kind of nonlinear phenomenon, which adds a certain intensity noise rather than reducs the noise, then uses the synergy among signal frequency, noise intensity, and nonlinear system to drive part of the noise energy into the measuring signal energy, and finally highlights in the output signal.

With the development of the theory of stochastic resonance, the method of finding and extracting weak periodic signals from strong noise by stochastic resonance effect has been widely used in various fields of science such as nerve physiology, intelligence theory, nonlinear optics, signal processing, communication engineering, and sociology [3–11]. Among them, the method of detecting single-frequency weak signals by using stochastic resonance effect has been more mature. Its main method is to analyze the relationship between the characteristics of the measured input signal and the system parameters through the nonlinear bistable system, through adjusting the system parameters [12] or increasing the strength of the noise [13, 14] to realize stochastic resonance. In 1990, Gang et al. [15] put forward the famous idea of adiabatic approximation theory, which proved that stochastic resonance is used to detect small parameter signal.

Then the method of stochastic resonance detection to single-frequency signal is gradually perfect. However, in the actual research, we found that the signal submerged by strong noise is unknown weak periodic signal and even unknown high frequency signal. Then, the research on the detection of multiple frequency signals received the widespread attention rapidly.

It is mainly used to realize stochastic resonance through adjusting system parameters manually or increasing the strength of noise so that we can find and extract the unknown multiple frequency signal. Due to the manual, adjusting has low work efficiency, and cannot achieve continuous search which will omit part of the signal, and it is difficult to find and search the optimal system parameters which will certainly omit part of the signal. This paper combines the theory of stochastic resonance and adaptive algorithm to put forward a kind of adaptive stochastic resonance detection method for multiple-frequency signal, respectively, of the low frequency and high frequency input signals. Based on the traditional single-frequency weak signal detection, selected the SNR to be a measurement index of the generation of stochastic resonance and reducing the range of parameter values by the threshold analysis, this method can find the optimal system parameters effectively and can detect a multiple weak periodic signals. A large number of simulation results show that the output signal of stochastic resonance system will be interfered by some noise which will lead to distortion of waveform slightly. Therefore, this paper makes processing the output signal of stochastic resonance by using the autocorrelation method which only changes the amplitude and phase, without changing the frequency. It can reduce the impact of noise, make the waveform more similar to measured signal, highlight the frequency of the signal cycle component, and enhance the SNR.

The methods to detect the high-frequency signals are subsampled, frequency-shifted and rescaling, wavelet analysis [16, 17], and so forth. Its main idea is transforming the high frequency into the low frequency through scale changes to meet the conditions of stochastic resonance then detect and extract the low-frequency signal, and finally achieve recovery. However, the output signal waveform extracted by these methods often exists with some distortion. In 2008, Mao et al. [18] proposed a method, which adds one cycle modulated signal to the stochastic resonance system, and then adjust the frequency of the modulation signal close to the frequency of the signal to be measured and generate the differential frequency which meets the adiabatic approximation theory. Finally, significant changes of the output signal spectrogram occurred in the approximation process. This characteristic can be taken as the basis for signal detection and extraction. But it used ideal Gaussian white noise during the experiment rather than the nonzero color noise which is often encountered in practical engineering applications such as the mechanical fault detection [8], and its frequency is concentrated in a frequency band and can easily be confused with the frequency of signal to be measured. It is considered that the frequency of the multi-frequency signal to be measured may be odd multiples. This paper contemplated to select the reciprocal of the power spectrum in the autocorrelation function of the output signal as measurement index under the interference of the color noise, which can distinguish the color noise with the signal to be measured and extract the high frequency of multiple parallel input signals effectively. This paper made a large number of numerical simulations by MATLAB, and the simulation results show the effectiveness and feasibility of the method and have a good prospect.

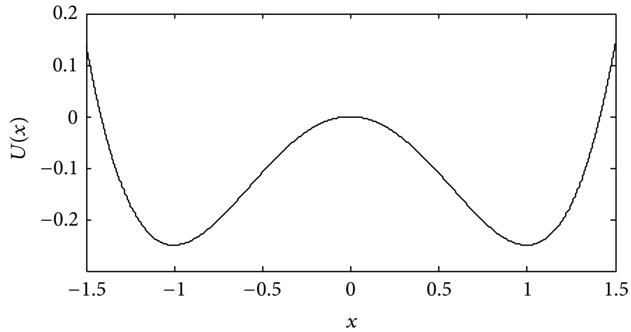


Figure 1: When $A = 0$, $D = 0$, the corresponding potential function curve $U(x)$.

2. Bistable System and Its Performance Analysis

This paper uses the bistable system model: Langevin equation. It is actually an overdamped bistable system model driven by cycle, and its mathematical expression is [19]

$$\frac{dx}{dt} = ax - bx^3 + s(t) + \Gamma(t), \qquad (1)$$

where a, b are the system parameters, $s(t)$ is the system input signal to be measured, $s(t) = A\cos(2\pi f_0 t)$, f_0 is the frequency of the input signal to be measured and $\Gamma(t)$ is the Gaussian white noise with noise intensity D, and it satisfied: $\langle\Gamma(t)\rangle = 0$, $\langle\Gamma(t)\Gamma(t')\rangle = 2D\delta(t-t')$. When the input signal $A = 0$, the noise intensity $D = 0$, the potential function corresponding to the nonlinear bistable system is

$$U(x) = -\frac{1}{2}ax^3 + \frac{1}{4}bx^3. \qquad (2)$$

As shown in Figure 1, the system has two potential wells and a potential barrier. Stochastic resonance is actually shown the phenomenon that the signal has enough energy to transfer between two potential wells under the synergistic effect of the bistable system. At present, the main method is adjusted system parameters and increased a certain intensity of noise to generate stochastic resonance. However, the characteristic of input signal to be measured with noise is usually unknown in the measurement of the practical engineering. It is difficult to meet the actual demand only by adjusting the system parameters manually. Therefore, this paper integrates the adaptive iterative algorithm into the stochastic resonance detection method to study the adaptive stochastic resonance detection method for multi-frequency signals, seeks the optimal system parameters to generate stochastic resonance, and finally finds and extracts the frequency of unknown weak cycle signal in the frequency domain.

3. Adaptive Stochastic Resonance Detection for Low-Frequency Signals

3.1. Measurement Index and Iterative Algorithm. Adaptive stochastic resonance signal detection involves two important factors: measurement index and iterative algorithm.

(1) *Measurement Index.* Selecting the appropriate measurement index to measure the effectiveness of the system output which means whether to generate stochastic resonance. The commonly measurement index in the study of stochastic resonance contains signal-to-noise ratio (SNR), autocorrelation function, cross-correlation function, mutual information, residence time distribution, [20–23] and so on. For the detection of low-frequency signals, this paper is mainly based on the SNR to extract effective signal. SNR is an index of the proportion that the energy of input signal frequency f_0 is contained in the system output signal $y(t) = g(x(t))$, which is defined as

$$\mathrm{SNR} = 10\log\frac{S}{N} = 10\log\frac{S(f_0)}{N(f_0)}dB. \tag{3}$$

This paper uses the fourth-order Runge-Kutta method to solve the nonlinear systems. Set the sample step $h = 1/f_s$, where f_s is the sampling frequency. The output signal is $y(t)$. The power spectrum of the input signal $S(f_0)$ is the energy of the output signal power spectrum $Y(f)$ in the input signal at the frequency f_0. The noise power spectrum $N(f_0)$ is a period of average power spectrum estimate near the input signal frequency f_0.

(2) *Iterative Algorithm.* Choose a suitable iterative algorithm to make the system tends to the optimal state, which generates stochastic resonance. In the measurement of the practical engineering, by the limit of the algorithm accuracy requirements and working conditions, many algorithms cannot be applied to the actual detection because of its high complexity. This paper mainly uses adaptive iterative algorithm: fix the step size and adjust the system parameters linearity. The steps of adaptive stochastic resonance detection of low-frequency signal are as follows.

(a) Firstly, to set the system parameters, to input the signal to be measured with noise, to fix the step size, and to select the appropriate value range of parameter, increase the step size during this interval gradually to adjust the system parameters a.

(b) Secondly, to use the Runge-Kutta algorithm to take numerical simulation to the corresponding system of each parameter, every parameter a has a corresponding system output signal.

(c) Then, to calculate the SNR according to (3), find the optimal parameters a_{best} corresponding to the maximum SNR.

(d) Finally, to reset nonlinear bistable system based on the optimal parameters to drive the signal to be measured with noise, generate stochastic resonance in this system. The output signal can show the signal to be measured to the greatest extent. The frequency corresponding to the spectrum peak in the spectrum diagram of the output signal is the frequency of the signal to be measured.

3.2. Simulation of Single Weak Signal Detection. Let the input signal to be tested is $S(t) = A\sin(2\pi f_0 t)$, in which $A = 0.8$, $f_0 = 0.03$ Hz, the noise intensity $D = 0.6$, the sampling frequency $f_s = 5$ Hz. Figure 2(b) shows that the input signal to be measured has been completely submerged by the noise at this time, the parameter of bistable system $b = 1$ is fixed. But it has a problem which is how to set the range of values about the system parameter a.

Let the input signal be a constant A and the noise intensity $D = 0$ (without considering the noise). The barrier of the bistable system exists with a static threshold condition: $A_c = \sqrt{4a^3/27b}$. Thus we can calculate a system parameter threshold $a = 1.1$ according to the above conditions of the system. Set the adjustment range of system parameters as $[1.1, 5]$ and the step size $h = 1/f_s = 0.2$. According to the adaptive iterative algorithm mentioned above, we can obtain the variation curve of SNR as the system parameter changes in Figure 3. The maximum $\mathrm{SNR}_{\max} = 0.0609$, and the corresponding optimal system parameters $a_{\text{best}} = 1.2$. Reset system parameters and the system obviously generated stochastic resonance effect, as shown in Figure 2(c). Although there is still some noise in the output signal, but the noise energy is significantly weakened, and it has been fully utilized and transformed into the energy of the signal to be measured. Figure 2(d) is a spectrum diagram of the output signal, when $f = 0.03$ Hz there is a very clear and sharp spectral peak.

However, the frequency of low-frequency signal is prominent by the processing of the stochastic resonance system and is easy to be extracted. Although, as the Figure 2(c) shows that the time domain diagram of output signal is still interfered by part of the noise, there are some glitches. In order to solve this problem, this paper uses the autocorrelation techniques on the postprocessing program.

Define the autocorrelation function of the signal $x(t)$ as follows:

$$R_x(\tau) = \lim_{T\to\infty}\frac{1}{T}x(t)\,x(t+\tau)\,dt, \tag{4}$$

where T is the observation time of the signal $x(t)$, and $R_x(\tau)$ describes the correlation between the signal $x(t)$ and $x(t+\tau)$, due to the actual observation time T is limited. Therefore define the autocorrelation function is,

$$\widehat{R}_x(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_0^{T-\tau} x(t)\,x(t+\tau)\,dt. \tag{5}$$

The signals to be measured with noise are as follows:

$$S_n(t) = s(t) + \Gamma(t) = A\cos(2\pi f_0 t) + \Gamma(t). \tag{6}$$

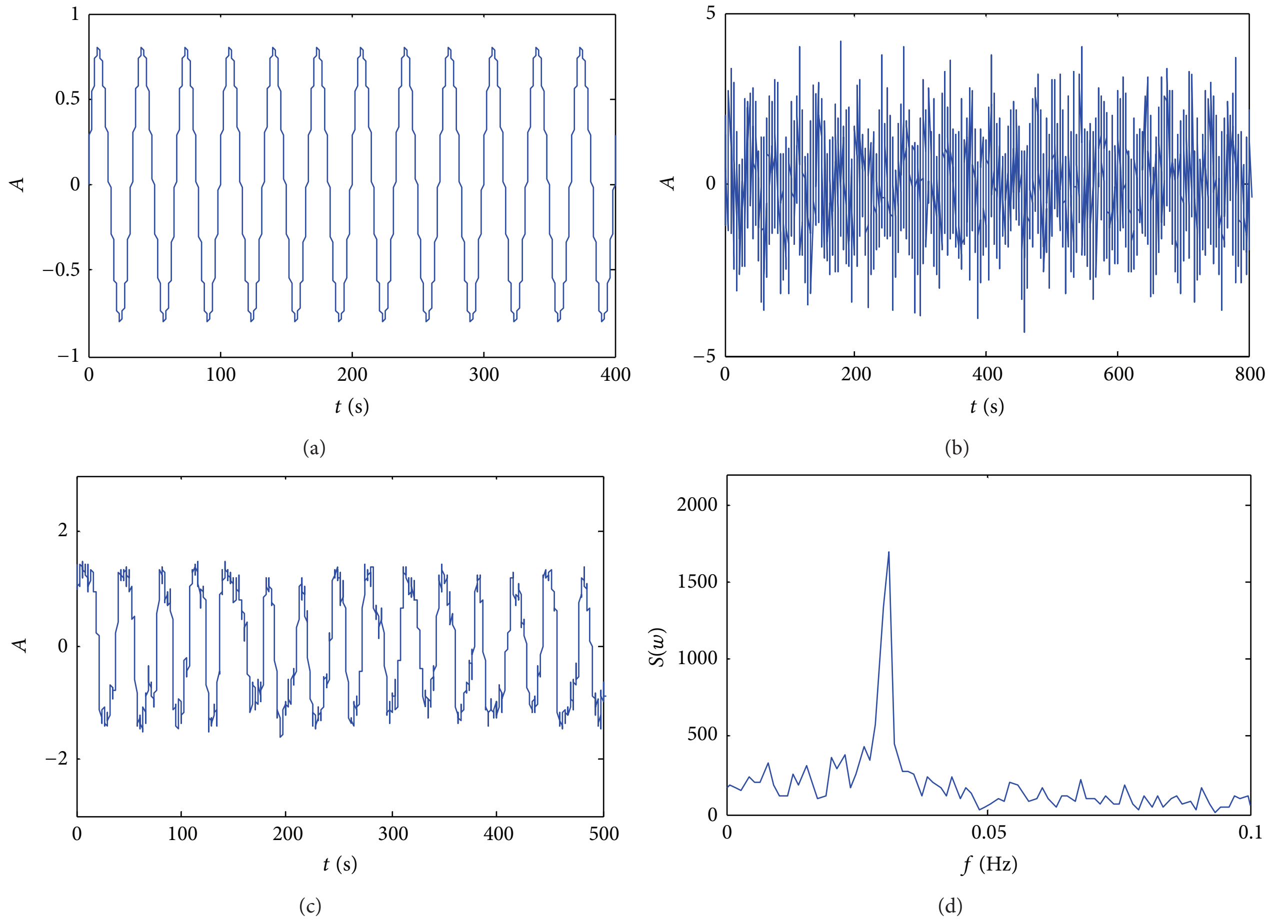


Figure 2: (a) The input signal to be measured. (b) The input signal to be measured contains white Gaussian noise. (c) The stochastic resonance output signal. (d) The spectrum figure of the stochastic resonance output signal.

For the actual engineering signal, the integration time can be approximated by T instead of $T-\tau$, and the signal after the autocorrelation processing is:

$$
\begin{aligned}
R_Y(\tau) = {} & \frac{A^2}{2}\cos(\omega t) + \frac{A^2}{2T}\int_0^T \cos[\omega(2t+\tau)+2\phi]\,dt \\
& + \frac{1}{T}\int_0^T s(t)\,dt \cdot \frac{1}{T}\int_0^T \Gamma(t+\tau)\,dt \\
& + \frac{1}{T}\int_0^T s(t+\tau)\,dt \cdot \frac{1}{T}\int_0^T \Gamma(t)\,dt + R_\Gamma(\tau),
\end{aligned} \tag{7}
$$

in which $R_x(\tau)$ is the autocorrelation function of the noise. The noise cannot be the ideal Gaussian white noise in the measurement of the actual engineering. Therefore, $R_x(\tau)$ is always present and its amplitude is drastically reduced compared with the original noise amplitude, and can be regarded as a new noise.

The output signal by autocorrelation processing can be abbreviated as

$$y_1(t) = A_1\cos(f_1 t + \phi_1) + \Gamma_1(t). \tag{8}$$

Compared to the original noise signal to be measured, the amplitude and phase of the two signals have changed, but the frequency is not changed. It improves the SNR to a certain extent. Therefore, this paper takes advantage of this feature to postprocess the output signal of stochastic resonance (see Figure 7). It not only reduces the influence of the noise but also makes the waveform of the output signal more close to the original signal to be measured in the time domain.

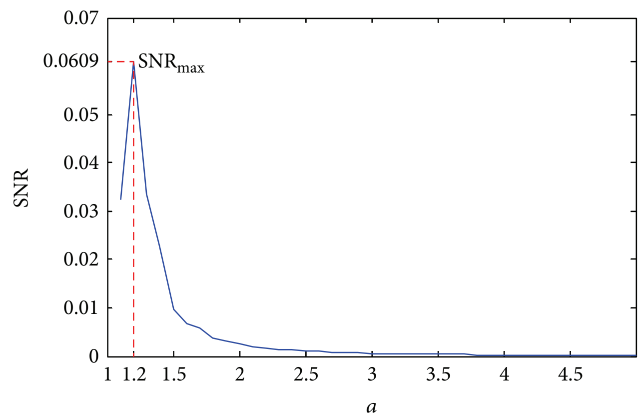
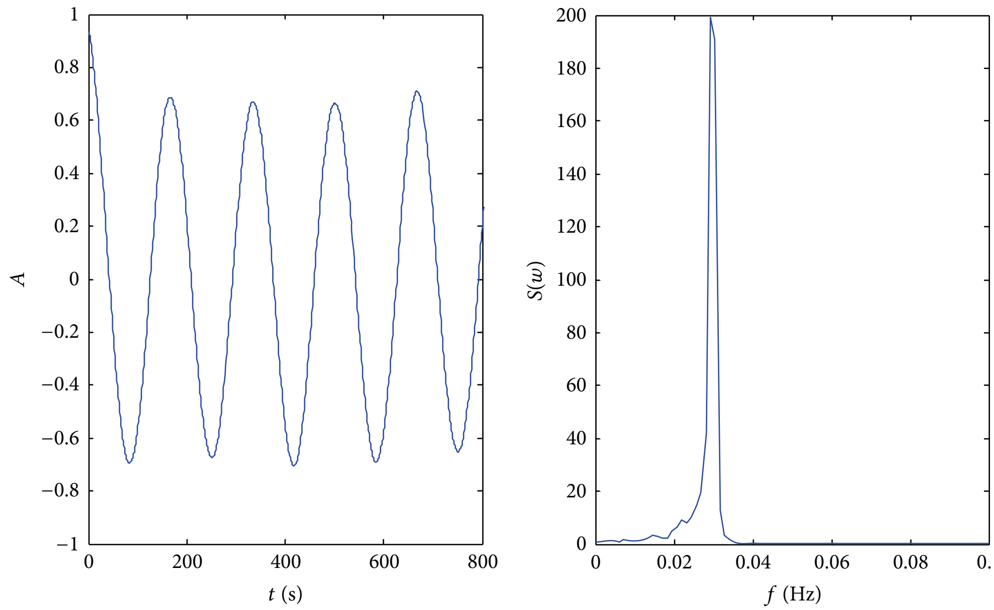


Figure 3: The variation curve of SNR while adjusting the system parameter a.



(a)

(b)

Figure 4: (a) The time-domain diagram of stochastic resonance output signal after correlation processing and (b) the spectrum diagram of stochastic resonance output signal after correlation processing.

With the signal cycle components characteristic frequency is even more pronounced in the spectrogram. We verify the feasibility of this theory through a numerical example. Make autocorrelation processing of the output signal of stochastic resonance as shown in Figure 2(c). As Figure 4 shows that the waveform of the output signal is obviously undistorted in the time-domain diagram, and it is almost unanimous with the waveform of the measured signal. The frequency of the signal to be measured is more prominent under the background of noise.

3.3. Simulation of Multifrequency Weak Superposition Signal Detection. When the input signal to be measured is the multi-frequency weak signal and parallel input, the multi-frequency input signal to be tested is

$$s(t) = \sum_{i=1}^{3} A_i \cos(2\pi f_i t). \tag{9}$$

While $A_1 = 0.6$, $A_2 = 0.8$, $A_3 = 1.0$, $f_1 = 0.02$ Hz, $f_2 = 0.03$ Hz, and $f_3 = 0.05$ Hz, $\Gamma(t)$ is Gaussian white noise with noise intensity $D = 0.6$. Sampling frequency $f_s = 5$ Hz, and let the bistable system parameter $b = 1$. The study has shown that only the frequency, noise intensity, and system parameters of signal must be matched, and the system can generate stochastic resonance effect, so that we define a set of system parameters as a signal path for the system [22]. It generates mixing phenomenon when the signal band is too close, and the spectrum peaks of output signal are not obvious. Therefore, we can define the frequency number as not only the channel capacity of the signal path adapts to this set of parameters to generate a stochastic resonance effect, but also the mixing frequency phenomenon does not occur. Similarly, according to the above adaptive iterative algorithm, we can calculate the optimal parameters $a_{\text{best}} = 1.5$ while SNR is maximum ($\text{SNR}_{\max}$), as shown in Figure 6. As shown in Figure 5(d), the frequency of obviously spectral peak is 0.02 Hz, 0.03 Hz, and 0.05 Hz. The degree of waveform distortion is weakened by autocorrelation processing, and the frequency of the signal to be measured is more prominent which indicates that this algorithm is suitable for the parallel multi-frequency weak input signal detection. Parameter a_{best} matches the frequency of signal to be measured and noise intensity. The channel capacity is $N = 3$ at this time.

4. Adaptive Stochastic Resonance in the High Frequency Signal Detection

According to (1), the power spectrum of the system output signal can be calculated as [23]

$$\begin{aligned} S(f) &= S_1(f) + S_2(f) \\ &= \frac{2a^4A^2\exp\left(\left(-a^2/2D\right)/\pi D^2\right)}{\left(2a^2\exp\left(-a^2/2D\right)/\pi^2\right)^2} \times \delta\left(f_0 - f\right) \\ &\quad + \left[1 - \frac{2a^4A^2\exp\left(\left(-a^2/2D\right)/\pi D^2\right)}{\left(\left(2a^2\exp\left(-a^2/2D\right)/\pi^2\right) + 2\pi f_0\right)^2}\right] \\ &\quad \times \left[\frac{4\sqrt{2}a^4\exp\left(\left(-a^2/4D\right)/\pi\right)}{\left(\left(2a^2\exp\left(-a^2/2D\right)/\pi^2\right) + 2\pi f_0\right)^2}\right]. \end{aligned} \tag{10}$$

Stochastic resonance of the output signal spectrum is caused by the input signal and noise, as $S_1(f)$ and $S_2(f)$,

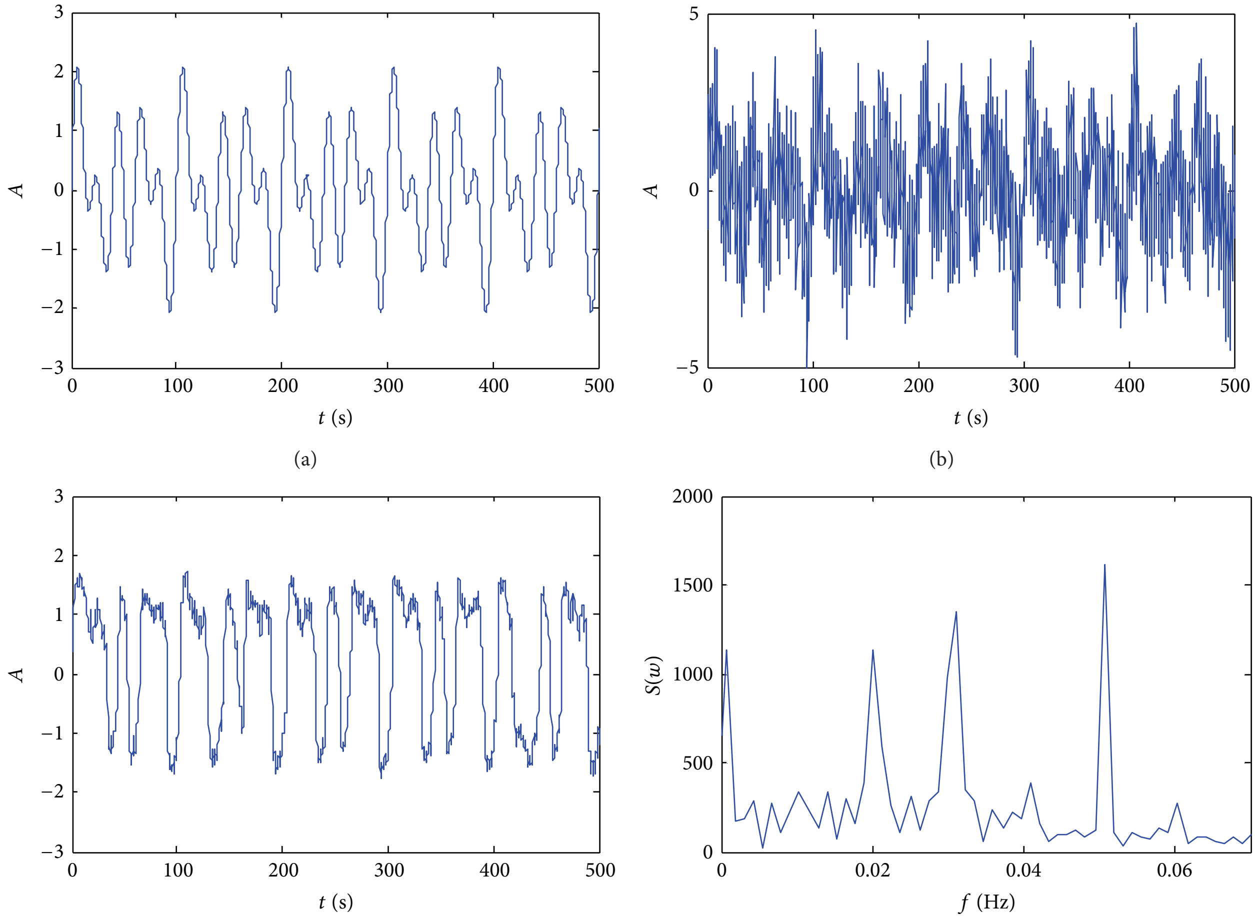


Figure 5: (a) The multi-frequency input signal to be measured. (b) The multi-frequency input signal to be measured contains white Gaussian noise. (c) The stochastic resonance output signal. (d) The spectrum figure of the stochastic resonance output signal.

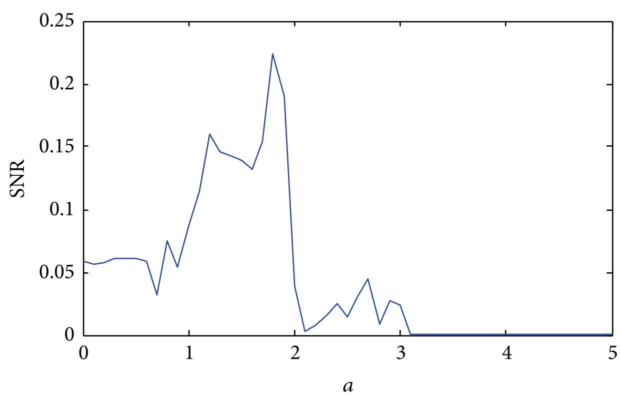


Figure 6: The variation curve of SNR while adjusting the system parameter a.

respectively. Since the output of the noise power spectrum $S_2(f)$ has Lorentz distribution, the subband which can generate stochastic resonance spectrum peak is generally limited to the low frequency band. Therefore, the bistable system of stochastic resonance is generally suitable for small parameters ($f \ll 1$) of weak signal detection. For the detection of high frequency signals, the current methods are: secondary sampling, frequency shift by varying scale and modem [24, 25], and so on. The main idea is transform the high frequency into the low frequency through the scale change to meet the low frequency of the small parameter conditions, so that it is able to generate stochastic resonance effect. Finally, the frequency of the output signal recover its actual measurement scale, which is the frequency of the signal to be measured. These methods have some inevitably problem of the efficiency and practicality.

(i) In the measurement of the actual engineering, such as mechanical failure diagnosis, most of the signal to be measured is the high-frequency signal, and the noise is often colored noise, rather than idealized Gaussian white noise.

(ii) In the field of classical stochastic resonance, most theoretical studies only discuss the linear response of single frequency weak signal, and it can be observed clearly that the output signal of stochastic resonance system has some distortion. Compared to the original

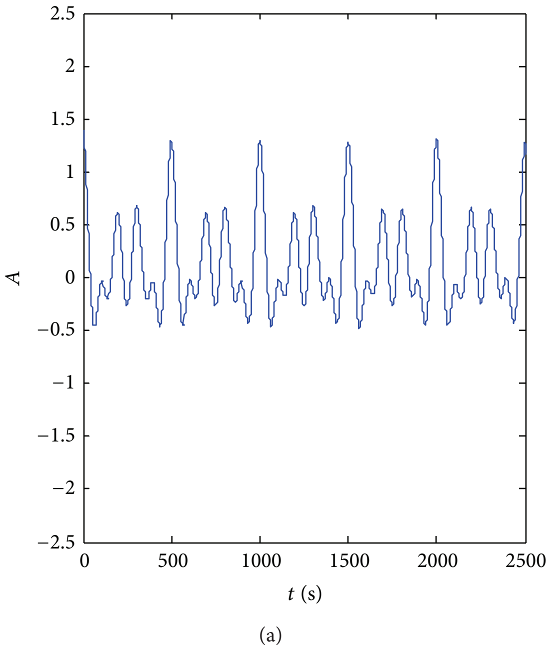


Figure 7: (a) The time-domain diagram of stochastic resonance output signal after correlation processing. (b) The spectrum diagram of stochastic resonance output signal after correlation processing.

sinusoidal signal, the output signal is more similar to a rectangular wave. Depending on the nature of the rectangular wave, the Fourier expansion is

$$x(t) = \frac{4A}{\pi}\left(\sin\omega_0 t + \frac{1}{3}\sin 3\omega_0 t + \frac{1}{5}\sin 5\omega_0 t + \cdots\right). \quad (11)$$

Except for the fact that the ω_0 has peak, its odd multiples of frequency $3\omega_0, 5\omega_0 \ldots$ have peaks in the spectrum diagram of the system output signal. Taking into account the influence of noise, the signal to be measured with noise meet is Lorentz distribution through the stochastic resonance system, and the odd multiples of the output signal frequency are not obvious in the spectrum diagram. However, in the detection of actual signals, the measured signal may exist with multi-frequencies, and satisfy the relationship of odd multiple, and it is difficult to determine the frequency which, corresponding to the peak, is the frequency of the output signal or some other weak signals by nonlinear response. Therefore, the method of low-frequency signal detection is not suitable for it and it needs to make some adjustments. A method is proposed for the above problems in this paper, which is approaching constantly the frequency of the signal to be measured by automatically adjusting the modulation signal frequency f_c of the system externally added, and thereby detecting the frequency of the signal being measured. The main idea is as follows.

Let the input signal be measured as

$$s(t) = \sum_{i=1}^{M} A_i \cos(2\pi f_i t) + \Gamma(t), \quad (12)$$

where f_i $(i = 1, 2, \ldots M)$ is the frequency of the signal to be measured. $\Gamma(t)$ is color noise distinguished from white Gaussian noise, and color noise is nonzero. Let its frequency mainly concentrate in some band of 0.2 Hz–0.5 Hz in this paper. Adding one cycle of the modulation signal to the system, the input signal to be measured is transformed into:

$$\begin{aligned} F(t)\cdot S(t) = {} & \frac{1}{2}\sum_{i=0}^{M} A_i \cos[2\pi(f_i - f_c)] \\ & + \frac{1}{2}\sum_{i=1}^{M} A_i \cos[2\pi(f_i - f_c)] \\ & + \Gamma(t)\cdot\cos(2\pi f_c t). \end{aligned} \quad (13)$$

The signal is composed of two parts: the difference frequency $f_i - f_c$, and the added frequency $f_i + f_c$.

It constantly approachs the frequency of the signal being measured f_i by adjusting the frequency f_c from $f_c < f_i$ via $f_c = f_i$ to $f_c > f_i$, difference frequency $f_i - f_c \ll 1$ which meets the generated conditions of the stochastic resonance in a certain frequency band. The system will generate a random resonance effect at this time, which means that each f_c will exists with a significantly nonzero spectral peak corresponding to the output signal spectrogram. Particularly, while $f_c = f_i$, the stochastic resonance disappears. The maximum spectral peak power is close to 0, and its reciprocal is infinite, which seems like a sharp peak in the diagram. So that we can use this feature to exacte the frequency of the input signals to be measured f_i. This method avoids the problem of odd multiples mentioned above. The frequency of the color noise is often concentrated in some frequency band. So it is difficult to distinguish the color noise and the frequency of the signal to be measured from the frequency domain. It is no longer applicable to use SNR as the index.

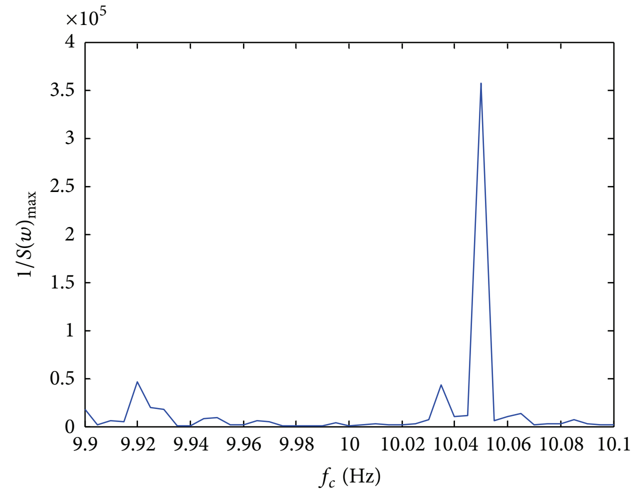


FIGURE 8: The change curve about the reciprocal of the stochastic resonance output signal spectrum peak with the adjustment of f_c in the single high frequency.

This paper selects the reciprocal of the maximum power spectrum peak of the output signal the autocorrelation function as measurement index.

The steps of adaptive stochastic resonance in the high-frequency signal detection are as follows.

(a) Set the system parameters, select the appropriate value interval, and fix the step size $h = 1/f_s$. Increase the step size gradually to adjust f_c, approaching the frequency of the signal to be measured f_i.

(b) Make numerical simulation of each f_c corresponding system by the fourth-order Runge-Kutta algorithm, and get the system output signal corresponding to each parameter points. Plott the curve of the maximum power spectral peak in the output signal with the modulating signal frequency f_c changed.

(c) Sharp peaks will appear in the curve which is drawn above, and each frequency corresponding to the peak is the frequency of the signal to be measured f_i.

The flow chart is shown in Figure 10.

4.1. Simulation of the Single High-Frequency Signal Detector. Let the system parameters $a = 1.4$, $b = 1$, the signal to be measured is $s(t) = A\cos(2\pi f_0 t)$, while $A = 2$, $f_0 = 10.05$ Hz, the color noise is generated by the MATLAB script. The sampling frequency is $f_s = 5000$. The adjustment interval of the modulation frequency f_c is [9.9 10.1]. Adjust the frequency f_c to approach the frequency of the signal being measured f_i. As shown in Figure 8, it occurred a sharp peak while $f_c = 10.05$ Hz, which means that the frequency of the signal being measured is 10.05 Hz. The numerical simulation results comes together with the theoretical analysis, so this method is effective and feasible.

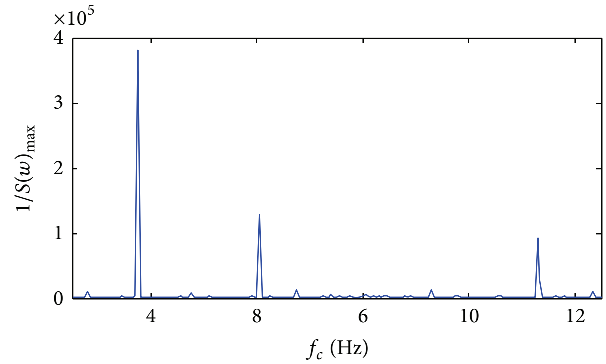


FIGURE 9: The change curve about the reciprocal of the stochastic resonance output signal spectrum peak with the adjustment of f_c in the multiple high frequency.

4.2. Simulation of the Multiple High-Frequency Signal Detector. Let the input signal be detected with multiple high frequency as follows:

$$s(t) = \sum_{i=1}^{3} A_i \cos(2\pi f_i t), \tag{14}$$

where the amplitude $A_1 = 2$, $A_2 = 1.5$, $A_3 = 2.1$, the frequency $f_1 = 3.75$ Hz, $f_2 = 6.05$ Hz, $f_3 = 11.30$ Hz, the bistable system parameters $a = 1.3$, $b = 1$, and the noise intensity $D = 10$. Sampling frequency $f_s = 5000$. The modulation signal frequency range is [2.5, 12.5]. As shown in Figure 9, the frequencies f_1, f_2, and f_3 all appear obvious as sharp peaks, it detected the frequency of the multiple signals submerged by strong noise efficiently. The odd multiples of the frequency $3f_1$ are close to the frequency f_3. The simulation results show that the detected signal frequency is f_3 which is the frequency of the input signal to be measured rather than the odd multiples. It proves that the method is feasible, effective, and suitable for the actual engineering measurement.

5. Conclusions

In order to meet the needs of practical engineering, this paper combined the adaptive algorithm with stochastic resonance theory. According to the frequency characteristics of the input signal to be tested, it proposed a feasible and effective adaptive stochastic resonance signal detection. Considering the actual situation, it improves work efficiency to a certain extent and has great value and development prospects in the measurement of the actual engineering. This paper chooses the SNR and the power spectrum of the autocorrelation function estimates as the index. The characteristics of the signal to be measured contain a lot of complexity in practical applications. In the actual engineering, we can choose a more precise measurement of indicators to measure the generation of stochastic resonance effect. Among the system parameters, noise intensity and the frequency of the signal being measured, which have a close relationship. We can analyze the degree of association by genetic algorithm to

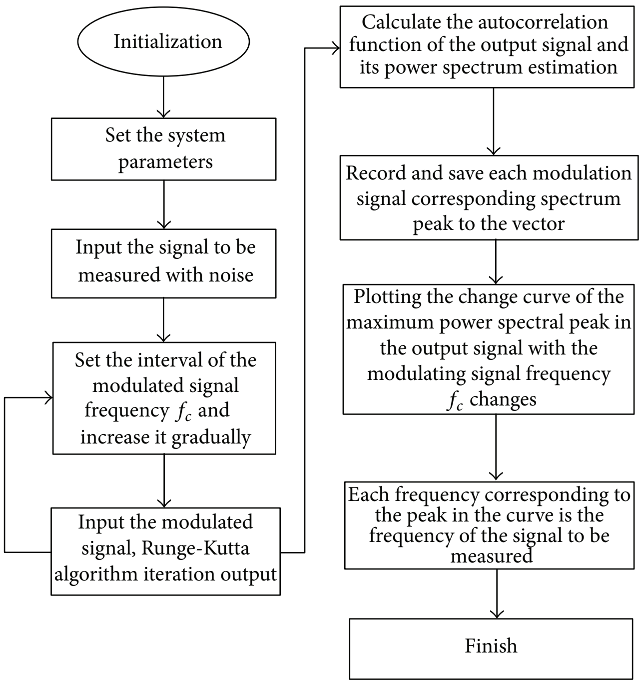


Figure 10: The flow chart.

further expand the system of stochastic resonance signal detection.

Acknowledgments

This work was supported by National Natural Science Foundation of China (nos. 61104062 and 61174077), Jiangsu Qing Lan Project, and PAPD.

References

[1] M. Witzke, "Linear and widely linear filtering applied to iterative detection of generalized MIMO signals," *Annales des Telecommunications*, vol. 60, no. 2, pp. 113–117, 2005.

[2] R. Benzi, A. Sutera, and A. Vulpiani, "The mechanism of stochastic resonance," *Journal of Physics A*, vol. 14, no. 11, pp. L453–L457, 1981.

[3] F. Duan and D. Abbott, "Binary modulated signal detection in a bistable receiver with stochastic resonance," *Physica A*, vol. 376, no. 1-2, pp. 173–190, 2007.

[4] C. Y. Chen, M. H. Ma, B. Zhao, S. F. Xie, and B. R. Xiang, "Stochastic resonance algorithm applied to quantitative analysis for weak liquid chromatographic signal of pyrene in water samples," *International Journal of Environmental Analytical Chemistry*, vol. 91, no. 1, pp. 112–119, 2011.

[5] W. Wei, X. Suyun, and X. Shaofei, "An adaptive single-well stochastic resonance algorithm applied to trace analysis of clenbuterol in human urine," *Molecules*, vol. 17, no. 2, pp. 1929–1938, 2012.

[6] L.-F. Li and J.-Y. Zhu, "Gravitational wave detection: stochastic resonance method with matched filtering," *General Relativity and Gravitation*, vol. 43, no. 11, pp. 2991–3000, 2011.

[7] F. Guo and Y. R. Zhou, "Stochastic resonance in a stochastic bistable system subject to additive white noise and dichotomous noise," *Physica A*, vol. 388, no. 17, pp. 3371–3376, 2009.

[8] H. Yan, W. Tai-yong, W. Jian, and Z. Pan, "Mechanical fault diagnosis based on the cascaded bistable stochastic resonance and multi-fractal," *Journal of Vibration Shock*, vol. 31, no. 8, pp. 181–185, 2012.

[9] P. Ping, Y. Ping, and H. Zhaoxia, "Speaker recognition method based on stochastic resonance," *Telecommunications Science*, vol. 26, no. S2, pp. 74–78, 2010.

[10] J. W. Mo, S. Ouyang, H. L. Xiao, and X. Y. Sun, "High sensitive GPS signal acquisition algorithm based on stochastic resonance," *Systems Engineering and Electronics*, vol. 33, no. 4, pp. 838–841, 2011.

[11] Z. S. Chen and Y. M. Yang, "Stochastic resonance mechanism for wideband and low frequency vibration energy harvesting based on piezoelectric cantilever beams," *Acta Physica Sinica*, vol. 60, no. 7, Article ID 074301, pp. 1–7, 2011.

[12] Y. G. Leng, "Mechanism of parameter-adjusted stochastic resonance based on Kramers rate," *Acta Physica Sinica*, vol. 58, no. 8, pp. 5196–5200, 2009.

[13] G. Q. Zhu, K. Ding, Y. Zhang, and Y. Zhao, "Experimental research of weak signal detection based on the stochastic resonance of nonlinear system," *Acta Physica Sinica*, vol. 59, no. 5, pp. 3001–3006, 2010.

[14] Y. Hasegawa and M. Arita, "Escape process and stochastic resonance under noise intensity fluctuation," *Physics Letters A*, vol. 375, no. 39, pp. 336–372, 2011.

[15] H. Gang, G. Nicolis, and C. Nicolis, "Periodically forced Fokker-Planck equation and stochastic resonance," *Physical Review A*, vol. 42, no. 4, pp. 2030–2041, 1990.

[16] L. Yonggang and W. Tai-yong, "Numerical research of twice sampling stochastic resonance for the detection of a weak signal submerged in a heavy Noise," *Acta Physica Sinica*, vol. 52, no. 10, pp. 2432–2437, 2003.

[17] F. Chunfeng, G. Ke, C. Xin-wu, and G. Jiantao, "Application of frequency-shifted and re-scaling adaptive stochastic resonance in signal detection," *Journal of Xinyang Normal University*, vol. 23, no. 3, pp. 415–419, 2010.

[18] Q. Mao, M. Lin, and Y. Zheng, "Study of weak multi-frequencies signal detection based on stochastic resonance," *Journal of Basic Science and Engineering*, vol. 16, no. 1, pp. 86–91, 2008.

[19] Y. G. Leng, T. Y. Wang, Y. Guo, and Z. Y. Wu, "Study of the property of the parameters of bistable stochastic resonance," *Acta Physica Sinica*, vol. 56, no. 1, pp. 30–35, 2007.

[20] J. F. Wang, F. Liu, J. Y. Wang, G. Chen, and W. Wang, "Frequency characteristics of the input thresholds of stochastic resonant systems," *Acta Physica Sinica*, vol. 46, no. 12, pp. 2311–2312, 1997.

[21] X. Jingsong, *Weak Signal Deteetion Based on the theory of Stoehastic Resonance*, Lanzhou University Of Information and Communication Engineering, Gansu, China, 2008.

[22] G. L. Zhang and F. Z. Wang, "Research of multiple signals in stochastic resonance system," *Journal of System Simulation*, vol. 21, no. 13, pp. 4190–4193, 2009.

[23] Y. G. Leng, T. Y. Wang, X. D. Qin, R. X. Li, and Y. Guo, "Power spectrum research of twice sampling stochastic resonance response in a bistable system," *Acta Physica Sinica*, vol. 53, no. 3, pp. 717–723, 2004.

[24] Y. G. Leng, Y. S. Leng, T. Y. Wang, and Y. Guo, "Numerical analysis and engineering application of large parameter stochastic

resonance," *Journal of Sound and Vibration*, vol. 292, no. 3-5, pp. 788–801, 2006.

[25] Y. Dingxin, H. Zheng, and Y. Yongmin, "The analysis of stochastic resonance of periodic signal with large parameters," *Acta Physica Sinica*, vol. 61, no. 8, Article ID 080501, 2012.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 325310, 10 pages
http://dx.doi.org/10.1155/2013/325310

Research Article

Stability of Impulsive Neural Networks with Time-Varying and Distributed Delays

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Received 26 December 2012; Accepted 21 March 2013

Academic Editor: Xuerong Mao

This work is devoted to investigating the stability of impulsive cellular neural networks with time-varying and distributed delays. We use the new method of fixed point theory to obtain some new and concise sufficient conditions to ensure the existence and uniqueness of solution and the global exponential stability of trivial equilibrium. The presented algebraic criteria are easily checked and do not require the differentiability of delays.

1. Introduction

Since cellular neural networks (CNNs) were proposed by Chua and Yang in 1988 [1, 2], many researchers have put great effort into this subject due to their numerous successful applications in various fields such as optimization, linear and nonlinear programming, associative memory, pattern recognition, and computer vision.

Owing to the finite switching speed of amplifiers, there is no doubt that time delays exist in the communication and response of neurons. Moreover, as neural networks usually have a spatial extent due to the presences of a multitude of parallel pathways with a variety of axon sizes and lengths, there is a distribution of conduction velocities along these pathways and a distribution of propagation designed with discrete delays. Therefore, a more appropriate and ideal way is to incorporate continuously distributed delays with a result that a more effective model of cellular neural networks with time-varying and distributed delays proposed.

In fact, beside delay effects, stochastic and impulsive as well as diffusing effects are also likely to exist in neural networks. So far, there have been many results [3–11] on the study of dynamic behaviors of complex CNNs such as impulsive delayed reaction-diffusion CNNs and stochastic delayed reaction-diffusion CNNs. Summing up the existing researches on the stability of complex CNNs, we see that the primary method is Lyapunov theory. However, there are also lots of difficulties in the applications of corresponding theories to specific problems. It is therefore necessary to seek some new methods to deal with the stability in order to overcome those difficulties.

Recently, it is inspiring that Burton and other authors have applied the fixed point theory to investigate the stability of deterministic systems and obtained some more applicable conclusions, for example, see the monograph [12] and the work in [13–24]. In addition, more recently, there have been a few papers where the fixed point theory is employed to investigate the stability of stochastic (delayed) differential equations, for instance, see [25–31]. Precisely, in [26–28], Luo used the fixed point theory to study the exponential stability of mild solutions for stochastic partial differential equations with bounded delays and with infinite delays. In [29, 30], Sakthivel used the fixed point theory to discuss the asymptotic stability in pth moment of mild solutions to nonlinear impulsive stochastic partial differential equations with bounded delays and with infinite delays. In [31], Luo used the fixed point theory to study the exponential stability of stochastic Volterra-Levin equations. We wonder if we can obtain some new and more applicable stability criteria of complex CNNs by applying the fixed point theory.

With this motivation, in this paper, we aim to discuss the global exponential stability of impulsive CNNs with time-varying and distributed delays. It is worth noting that our research technique is based on the contraction mapping principle rather than the usual method of Lyapunov theory. We deal with, by employing the fixed point theorem, the existence and uniqueness of solution and the global exponential stability of trivial equilibrium at the same time, for which Lyapunov method feels helpless. The obtained stability criteria are easily checked and do not require the differentiability of delays.

2. Preliminaries

Let R^n denote the n-dimensional Euclidean space and $\|\cdot\|$ represent the Euclidean norm $\mathcal{N} \triangleq \{1, 2, \ldots, n\}$ and $R_+ = [0, \infty)$. $C[X, Y]$ corresponds to the space of continuous mappings from the topological space X to the topological space Y.

In this paper, we consider the following impulsive cellular neural networks with time-varying and distributed delays:

$$\begin{aligned}\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = & -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t)) \\ & + \sum_{j=1}^{n} c_{ij} g_j(x_j(t - \tau_{ij}(t))) \\ & + \sum_{j=1}^{n} d_{ij} \int_0^{\rho(t)} \sigma_j(x_j(t-\theta))\, \mathrm{d}\theta \\ & \qquad t \geq 0,\ t \neq t_k,\end{aligned} \tag{1}$$

$$\begin{aligned}\Delta x_i(t_k) = x_i(t_k + 0) - x_i(t_k) = I_{ik}(x_i(t_k)), \\ k = 1, 2, \ldots,\end{aligned} \tag{2}$$

where $i \in \mathcal{N}$ and n is the number of neurons in the neural network. $x_i(t)$ corresponds to the state of the ith neuron at time t. f_j, g_j, and σ_j denote the activation functions, respectively. The constant $a_i > 0$ represents the rate with which the ith neuron will reset its potential to the resting state when disconnected from the network and external inputs. The constants b_{ij}, c_{ij}, and d_{ij} represent the connection weights of the jth neuron to the ith neuron, respectively. $\tau_{ij}(t)$ and $\rho(t)$ correspond to the transmission delays meeting $0 \leq \tau_{ij}(t) \leq \tau$ (τ = constant) and $0 \leq \rho(t) \leq \rho$ (ρ = constant). The fixed impulsive moments t_k ($k = 1, 2, \ldots$) satisfy $0 = t_0 < t_1 < t_2 < \cdots$ and $\lim_{k\to\infty} t_k = \infty$. $x_i(t_k + 0)$ and $x_i(t_k - 0)$ stand for the right-hand and left-hand limits of $x_i(t)$ at time t_k, respectively. $I_{ik}(x_i(t_k))$ shows the impulsive perturbation of the ith neuron at the impulsive moment t_k.

Throughout this paper, we always assume that $f_i(0) = g_i(0) = \sigma_i(0) = I_{ik}(0) = 0$ for $i \in \mathcal{N}$ and $k = 1, 2, \ldots$. Thereby, problems (1) and (2) admit a trivial equilibrium $\mathbf{x} = 0$.

Denote by $\mathbf{x}(t) \triangleq \mathbf{x}(t; s, \varphi) = (x_1(t; s, \varphi_1), \ldots, x_n(t; s, \varphi_n))^T \in R^n$ the solution to (1) and (2) with the initial condition

$$x_i(s) = \varphi_i(s), \quad -m^* \leq s \leq 0,\ i \in \mathcal{N}, \tag{3}$$

where $m^* = \max\{\tau, \rho\}$, $\varphi_i(s) \in C[[-m^*, 0], R]$ and $\varphi(s) = (\varphi_1(s), \ldots, \varphi_n(s))^T \in R^n$.

The solution $\mathbf{x}(t) \triangleq \mathbf{x}(t; s, \varphi) \in R^n$ to (1)–(3) is, for the time variable t, a piecewise continuous vector-valued function with the first-kind discontinuous points t_k ($k = 1, 2, \ldots$), where it is left-continuous; that is, the following relations are true:

$$\begin{aligned}x_i(t_k - 0) = x_i(t_k),\ x_i(t_k + 0) = x_i(t_k) + I_{ik}(x_i(t_k)), \\ i \in \mathcal{N},\ k = 1, 2, \ldots.\end{aligned} \tag{4}$$

Definition 1. The trivial equilibrium $\mathbf{x} = 0$ is said to be globally exponentially stable if for any initial condition $\varphi(s) \in C[[-m^*, 0], R^n]$, there exists a pair of positive constants λ and M such that

$$\|\mathbf{x}(t; s, \varphi)\| \leq M e^{-\lambda t}, \quad \forall t \geq 0. \tag{5}$$

The consideration of this paper is based on the following fixed point theorem.

Theorem 2 (see [32]). *Let Υ be a contraction operator on a complete metric space Θ, then there exists a unique point $\zeta \in \Theta$ for which $\Upsilon(\zeta) = \zeta$.*

3. Main Results

In this section, we will, for (1)–(3), use the contraction mapping principle to prove the existence and uniqueness of the solution and the global exponential stability of trivial equilibrium all at once. Before proceeding, we firstly introduce some assumptions as follows.

(A1) There exist nonnegative constants l_j such that for any η, $\upsilon \in R$,

$$|f_j(\eta) - f_j(\upsilon)| \leq l_j |\eta - \upsilon|, \quad j \in \mathcal{N}. \tag{6}$$

(A2) There exist nonnegative constants k_j such that for any η, $\upsilon \in R$,

$$|g_j(\eta) - g_j(\upsilon)| \leq k_j |\eta - \upsilon|, \quad j \in \mathcal{N}. \tag{7}$$

(A3) There exist nonnegative constants p_{jk} such that for any η, $\upsilon \in R$,

$$|I_{jk}(\eta) - I_{jk}(\upsilon)| \leq p_{jk} |\eta - \upsilon|, \quad j \in \mathcal{N},\ k = 1, 2, \ldots. \tag{8}$$

(A4) There exist nonnegative constants ω_j such that for any η, $\upsilon \in R$,

$$|\sigma_j(\eta) - \sigma_j(\upsilon)| \leq \omega_j |\eta - \upsilon|, \quad j \in \mathcal{N}. \tag{9}$$

Let $\mathcal{H} = \mathcal{H}_1 \times \cdots \times \mathcal{H}_n$, and let $\mathcal{H}_i$ ($i \in \mathcal{N}$) be the space embracing functions $\phi_i(t) : [-m^*, +\infty) \to R$, wherein $\phi_i(t)$ satisfies the following:

(1) $\phi_i(t)$ is continuous on $t \neq t_k$ ($k = 1, 2, \ldots$),

(2) $\lim_{t\to t_k^-} \phi_i(t)$ and $\lim_{t\to t_k^+} \phi_i(t)$ exist; moreover, $\lim_{t\to t_k^-} \phi_i(t) = \phi_i(t_k)$ for $k = 1, 2, \ldots$,

(3) $\phi_i(s) = \varphi_i(s)$ on $s \in [-m^*, 0]$,

(4) $e^{\alpha t}\phi_i(t) \to 0$ as $t \to \infty$, where α = const and $0 < \alpha < \min_{i\in\mathcal{N}}\{a_i\}$,

where t_k and $\varphi_i(s)$ are defined as shown in Section 2. Also $\mathscr{H}$ is a complete metric space when it is equipped with a metric defined by

$$d\left(\overline{\mathbf{q}}(t), \overline{\mathbf{h}}(t)\right) = \sum_{i=1}^{n} \sup_{t\geq -m^*} \left|q_i(t) - h_i(t)\right|, \tag{10}$$

where $\overline{\mathbf{q}}(t) = (q_1(t), \ldots, q_n(t)) \in \mathscr{H}$ and $\overline{\mathbf{h}}(t) = (h_1(t), \ldots, h_n(t)) \in \mathscr{H}$.

Theorem 3. *Assume that conditions (A1)–(A4) hold provided that*

(i) *there exists a constant μ such that $\inf_{k=1,2,\ldots}\{t_k - t_{k-1}\} \geq \mu$,*

(ii) *there exist constants p_i such that $p_{ik} \leq p_i\mu$ for $i \in \mathcal{N}$ and $k = 1, 2, \ldots$,*

(iii) $\sum_{i=1}^{n}\{(1/a_i)\max_{j\in\mathcal{N}}|b_{ij}l_j| + (1/a_i)\max_{j\in\mathcal{N}}|c_{ij}k_j| + (\rho/a_i)\max_{j\in\mathcal{N}}|\omega_j d_{ij}|\} + \max_{i\in\mathcal{N}}\{p_i(\mu + (1/a_i))\} \triangleq \chi < 1$,

and then the trivial equilibrium $\mathbf{x} = 0$ is globally exponentially stable.

Proof. Multiplying both sides of (1) with $e^{a_i t}$ gives, for $t > 0$ and $t \neq t_k$,

$$\begin{aligned} \mathrm{d}e^{a_i t}x_i(t) &= e^{a_i t}\mathrm{d}x_i(t) + a_i x_i(t)e^{a_i t}\mathrm{d}t \\ &= e^{a_i t}\left\{\sum_{j=1}^{n} b_{ij}f_j\left(x_j(t)\right) + \sum_{j=1}^{n} c_{ij}g_j\left(x_j\left(t - \tau_{ij}(t)\right)\right)\right. \\ &\qquad \left. + \sum_{j=1}^{n} d_{ij}\int_0^{\rho(t)} \sigma_j\left(x_j(t-\theta)\right)\mathrm{d}\theta\right\}\mathrm{d}t, \end{aligned} \tag{11}$$

which yields after integrating from $t_{k-1} + \varepsilon$ $(\varepsilon > 0)$ to $t \in (t_{k-1}, t_k)$ $(k = 1, 2, \ldots)$ that

$$\begin{aligned} x_i(t)e^{a_i t} &= x_i\left(t_{k-1} + \varepsilon\right)e^{a_i(t_{k-1}+\varepsilon)} \\ &+ \int_{t_{k-1}+\varepsilon}^{t} e^{a_i s}\left\{\begin{aligned} &\sum_{j=1}^{n} b_{ij}f_j\left(x_j(s)\right) + \sum_{j=1}^{n} c_{ij}g_j\left(x_j\left(s - \tau_{ij}(s)\right)\right) \\ &+ \sum_{j=1}^{n} d_{ij}\int_0^{\rho(s)} \sigma_j\left(x_j(s-\theta)\right)\mathrm{d}\theta \end{aligned}\right\}\mathrm{d}s. \end{aligned} \tag{12}$$

Letting $\varepsilon \to 0$ in (12), we have, for $t \in (t_{k-1}, t_k)$ $(k = 1, 2, \ldots)$,

$$\begin{aligned} x_i(t)e^{a_i t} &= x_i\left(t_{k-1} + 0\right)e^{a_i t_{k-1}} \\ &+ \int_{t_{k-1}}^{t} e^{a_i s}\left\{\begin{aligned} &\sum_{j=1}^{n} b_{ij}f_j\left(x_j(s)\right) + \sum_{j=1}^{n} c_{ij}g_j\left(x_j\left(s - \tau_{ij}(s)\right)\right) \\ &+ \sum_{j=1}^{n} d_{ij}\int_0^{\rho(s)} \sigma_j\left(x_j(s-\theta)\right)\mathrm{d}\theta \end{aligned}\right\}\mathrm{d}s. \end{aligned} \tag{13}$$

Setting $t = t_k - \varepsilon$ $(\varepsilon > 0)$ in (13), we get

$$\begin{aligned} x_i\left(t_k - \varepsilon\right)e^{a_i(t_k-\varepsilon)} &= x_i\left(t_{k-1} + 0\right)e^{a_i t_{k-1}} \\ &+ \int_{t_{k-1}}^{t_k-\varepsilon} e^{a_i s}\left\{\begin{aligned} &\sum_{j=1}^{n} b_{ij}f_j\left(x_j(s)\right) + \sum_{j=1}^{n} c_{ij}g_j\left(x_j\left(s - \tau_{ij}(s)\right)\right) \\ &+ \sum_{j=1}^{n} d_{ij}\int_0^{\rho(s)} \sigma_j\left(x_j(s-\theta)\right)\mathrm{d}\theta \end{aligned}\right\}\mathrm{d}s, \end{aligned} \tag{14}$$

which generates by letting $\varepsilon \to 0$

$$\begin{aligned} x_i\left(t_k - 0\right)e^{a_i t_k} &= x_i\left(t_{k-1} + 0\right)e^{a_i t_{k-1}} \\ &+ \int_{t_{k-1}}^{t_k} e^{a_i s}\left\{\begin{aligned} &\sum_{j=1}^{n} b_{ij}f_j\left(x_j(s)\right) + \sum_{j=1}^{n} c_{ij}g_j\left(x_j\left(s - \tau_{ij}(s)\right)\right) \\ &+ \sum_{j=1}^{n} d_{ij}\int_0^{\rho(s)} \sigma_j\left(x_j(s-\theta)\right)\mathrm{d}\theta \end{aligned}\right\}\mathrm{d}s. \end{aligned} \tag{15}$$

Noting $x_i(t_k - 0) = x_i(t_k)$, (15) can be rearranged as

$$\begin{aligned} x_i\left(t_k\right)e^{a_i t_k} &= x_i\left(t_{k-1} + 0\right)e^{a_i t_{k-1}} \\ &+ \int_{t_{k-1}}^{t_k} e^{a_i s}\left\{\begin{aligned} &\sum_{j=1}^{n} b_{ij}f_j\left(x_j(s)\right) + \sum_{j=1}^{n} c_{ij}g_j\left(x_j\left(s - \tau_{ij}(s)\right)\right) \\ &+ \sum_{j=1}^{n} d_{ij}\int_0^{\rho(s)} \sigma_j\left(x_j(s-\theta)\right)\mathrm{d}\theta \end{aligned}\right\}\mathrm{d}s. \end{aligned} \tag{16}$$

Combining (13) and (16), we derive that

$$\begin{aligned} x_i(t)e^{a_i t} &= x_i\left(t_{k-1} + 0\right)e^{a_i t_{k-1}} \\ &+ \int_{t_{k-1}}^{t} e^{a_i s}\left\{\begin{aligned} &\sum_{j=1}^{n} b_{ij}f_j\left(x_j(s)\right) + \sum_{j=1}^{n} c_{ij}g_j\left(x_j\left(s - \tau_{ij}(s)\right)\right) \\ &+ \sum_{j=1}^{n} d_{ij}\int_0^{\rho(s)} \sigma_j\left(x_j(s-\theta)\right)\mathrm{d}\theta \end{aligned}\right\}\mathrm{d}s \end{aligned} \tag{17}$$

is true for $t \in (t_{k-1}, t_k]$ $(k = 1, 2, \ldots)$. Hence, we get, for $t \in (t_{k-1}, t_k]$ $(k = 1, 2, \ldots)$,

$$\begin{aligned}
& x_i(t)\,\mathrm{e}^{a_i t} \\
&= \{x_i(t_{k-1}) + I_{i(k-1)}(x_i(t_{k-1}))\}\,\mathrm{e}^{a_i t_{k-1}} \\
&\quad + \int_{t_{k-1}}^{t} \mathrm{e}^{a_i s} \left\{ \begin{array}{c} \sum_{j=1}^{n} b_{ij} f_j(x_j(s)) + \sum_{j=1}^{n} c_{ij} g_j(x_j(s - \tau_{ij}(s))) \\ + \sum_{j=1}^{n} d_{ij} \int_0^{\rho(s)} \sigma_j(x_j(s-\theta))\,\mathrm{d}\theta \end{array} \right\} \mathrm{d}s \\
&= x_i(t_{k-1})\,\mathrm{e}^{a_i t_{k-1}} \\
&\quad + \int_{t_{k-1}}^{t} \mathrm{e}^{a_i s} \left\{ \begin{array}{c} \sum_{j=1}^{n} b_{ij} f_j(x_j(s)) + \sum_{j=1}^{n} c_{ij} g_j(x_j(s - \tau_{ij}(s))) \\ + \sum_{j=1}^{n} d_{ij} \int_0^{\rho(s)} \sigma_j(x_j(s-\theta))\,\mathrm{d}\theta \end{array} \right\} \mathrm{d}s \\
&\quad + I_{i(k-1)}(x_i(t_{k-1}))\,\mathrm{e}^{a_i t_{k-1}},
\end{aligned} \tag{18}$$

which results in

$$\begin{aligned}
& x_i(t_{k-1})\,\mathrm{e}^{a_i t_{k-1}} \\
&= x_i(t_{k-2})\,\mathrm{e}^{a_i t_{k-2}} \\
&\quad + \int_{t_{k-2}}^{t_{k-1}} \mathrm{e}^{a_i s} \left\{ \begin{array}{c} \sum_{j=1}^{n} b_{ij} f_j(x_j(s)) + \sum_{j=1}^{n} c_{ij} g_j(x_j(s - \tau_{ij}(s))) \\ + \sum_{j=1}^{n} d_{ij} \int_0^{\rho(s)} \sigma_j(x_j(s-\theta))\,\mathrm{d}\theta \end{array} \right\} \mathrm{d}s \\
&\quad + I_{i(k-2)}(x_i(t_{k-2}))\,\mathrm{e}^{a_i t_{k-2}} \\
&\qquad \vdots \\
& x_i(t_2)\,\mathrm{e}^{a_i t_2} \\
&= x_i(t_1)\,\mathrm{e}^{a_i t_1} \\
&\quad + \int_{t_1}^{t_2} \mathrm{e}^{a_i s} \left\{ \begin{array}{c} \sum_{j=1}^{n} b_{ij} f_j(x_j(s)) + \sum_{j=1}^{n} c_{ij} g_j(x_j(s - \tau_{ij}(s))) \\ + \sum_{j=1}^{n} d_{ij} \int_0^{\rho(s)} \sigma_j(x_j(s-\theta))\,\mathrm{d}\theta \end{array} \right\} \mathrm{d}s \\
&\quad + I_{i1}(x_i(t_1))\,\mathrm{e}^{a_i t_1}, \\
& x_i(t_1)\,\mathrm{e}^{a_i t_1} \\
&= \varphi_i(0) \\
&\quad + \int_{0}^{t_1} \mathrm{e}^{a_i s} \left\{ \begin{array}{c} \sum_{j=1}^{n} b_{ij} f_j(x_j(s)) + \sum_{j=1}^{n} c_{ij} g_j(x_j(s - \tau_{ij}(s))) \\ + \sum_{j=1}^{n} d_{ij} \int_0^{\rho(s)} \sigma_j(x_j(s-\theta))\,\mathrm{d}\theta \end{array} \right\} \mathrm{d}s.
\end{aligned} \tag{19}$$

We therefore conclude, for $t > 0$,

$$\begin{aligned}
& x_i(t) \\
&= \varphi_i(0)\,\mathrm{e}^{-a_i t} \\
&\quad + \mathrm{e}^{-a_i t} \int_{0}^{t} \mathrm{e}^{a_i s} \left\{ \begin{array}{c} \sum_{j=1}^{n} b_{ij} f_j(x_j(s)) + \sum_{j=1}^{n} c_{ij} g_j(x_j(s - \tau_{ij}(s))) \\ + \sum_{j=1}^{n} d_{ij} \int_0^{\rho(s)} \sigma_j(x_j(s-\theta))\,\mathrm{d}\theta \end{array} \right\} \mathrm{d}s \\
&\quad + \mathrm{e}^{-a_i t} \sum_{0<t_k<t} \{I_{ik}(x_i(t_k))\,\mathrm{e}^{a_i t_k}\}.
\end{aligned} \tag{20}$$

Note that $x_i(0) = \varphi_i(0)$ in (20). We then define the following operator π acting on $\mathscr{H}$, for $\overline{\mathbf{y}}(t) = (y_1(t), \ldots, y_n(t)) \in \mathscr{H}$:

$$\pi(\overline{\mathbf{y}})(t) = (\pi(y_1)(t), \ldots, \pi(y_n)(t)), \tag{21}$$

where $\pi(y_i)(t) : [-m^*, +\infty) \to R$ $(i \in \mathscr{N})$ obeys the rule as follows:

$$\begin{aligned}
& \pi(y_i)(t) \\
&= \varphi_i(0)\,\mathrm{e}^{-a_i t} \\
&\quad + \mathrm{e}^{-a_i t} \int_{0}^{t} \mathrm{e}^{a_i s} \left\{ \begin{array}{c} \sum_{j=1}^{n} b_{ij} f_j(y_j(s)) + \sum_{j=1}^{n} c_{ij} g_j(y_j(s - \tau_{ij}(s))) \\ + \sum_{j=1}^{n} d_{ij} \int_0^{\rho(s)} \sigma_j(y_j(s-\theta))\,\mathrm{d}\theta \end{array} \right\} \mathrm{d}s \\
&\quad + \mathrm{e}^{-a_i t} \sum_{0<t_k<t} \{I_{ik}(y_i(t_k))\,\mathrm{e}^{a_i t_k}\}
\end{aligned} \tag{22}$$

on $t \geq 0$ and $\pi(y_i)(s) = \varphi_i(s)$ on $s \in [-m^*, 0]$.

In what follows, we will apply the contraction mapping principle to prove the existence and uniqueness of solution and the global exponential stability of trivial equilibrium at the same time. The subsequent proof can be divided into two steps.

Step 1. We need to prove that $\pi(\mathscr{H}) \subset \mathscr{H}$. For $y_i(t) \in \mathscr{H}_i$ $(i \in \mathscr{N})$, it is necessary to show that $\pi(y_i)(t) \subset \mathscr{H}_i$. As defined above, we see that $\pi(y_i)(s) = \varphi_i(s)$ on $s \in [-m^*, 0]$. Owing to the continuity of $\varphi_i(s)$ on $s \in [-m^*, 0]$, we immediately know that $\pi(y_i)(t)$ is continuous on $t \in [-m^*, 0]$.

Choose a fixed time $t > 0$, and it is then derived from (22) that

$$\pi(y_i)(t + r) - \pi(y_i)(t) = Q_1 + Q_2 + Q_3 + Q_4 + Q_5, \quad t > 0, \tag{23}$$

where,

$$\begin{aligned}
Q_1 &= \varphi_i(0)\,\mathrm{e}^{-a_i(t+r)} - \varphi_i(0)\;\mathrm{e}^{-a_i t}, \\
Q_2 &= \mathrm{e}^{-a_i(t+r)} \int_0^{t+r} \mathrm{e}^{a_i s} \sum_{j=1}^{n} b_{ij} f_j(y_j(s))\,\mathrm{d}s \\
&\quad - \mathrm{e}^{-a_i t} \int_0^{t} \mathrm{e}^{a_i s} \sum_{j=1}^{n} b_{ij} f_j(y_j(s))\,\mathrm{d}s,
\end{aligned}$$

$$Q_3 = e^{-a_i(t+r)} \int_0^{t+r} e^{a_i s} \sum_{j=1}^{n} c_{ij} g_j\left(y_j\left(s-\tau_{ij}(s)\right)\right) ds - e^{-a_i t} \int_0^{t} e^{a_i s} \sum_{j=1}^{n} c_{ij} g_j\left(y_j\left(s-\tau_{ij}(s)\right)\right) ds,$$

$$Q_4 = e^{-a_i(t+r)} \int_0^{t+r} e^{a_i s} \sum_{j=1}^{n} d_{ij} \int_0^{\rho(s)} \sigma_j\left(y_j(s-\theta)\right) d\theta ds - e^{-a_i t} \int_0^{t} e^{a_i s} \sum_{j=1}^{n} d_{ij} \int_0^{\rho(s)} \sigma_j\left(y_j(s-\theta)\right) d\theta ds,$$

$$Q_5 = e^{-a_i(t+r)} \sum_{0<t_k<(t+r)} \left\{I_{ik}\left(y_i(t_k)\right) e^{a_i t_k}\right\} - e^{-a_i t} \sum_{0<t_k<t} \left\{I_{ik}\left(y_i(t_k)\right) e^{a_i t_k}\right\}. \quad (24)$$

Since $y_i(t) \in \mathscr{H}_i$, we know that $y_i(t)$ is continuous on $t \neq t_k$ $(k = 1, 2, \ldots)$; moreover, $\lim_{t\to t_k^-} y_i(t)$ and $\lim_{t\to t_k^+} y_i(t)$ exist, in addition, $\lim_{t\to t_k^-} y_i(t) = y_i(t_k)$.

Letting $t \neq t_k$ $(k = 1, 2, \ldots)$ in (23), it is easy to see that $Q_i \to 0$ as $r \to 0$ for $i = 1, \ldots, 5$. Thus, $\pi(y_i)(t + r) - \pi(y_i)(t) \to 0$ as $r \to 0$ holds on $t > 0$ and $t \neq t_k$ $(k = 1, 2, \ldots)$.

Letting $t = t_k$ $(k = 1, 2, \ldots)$ in (23), it is not difficult to find that $Q_i \to 0$ as $r \to 0$ for $i = 1, \ldots, 4$. Letting $r < 0$ be small enough, we compute

$$\begin{aligned} Q_5 &= e^{-a_i(t_k+r)} \sum_{0<t_m<(t_k+r)} I_{im}\left(y_i(t_m)\right) e^{a_i t_m} - e^{-a_i t_k} \sum_{0<t_m<t_k} I_{im}\left(y_i(t_m)\right) e^{a_i t_m} \\ &= \left\{e^{-a_i(t_k+r)} - e^{-a_i t_k}\right\} \sum_{0<t_m<t_k} \left\{I_{im}\left(y_i(t_m)\right) e^{a_i t_m}\right\} \end{aligned} \quad (25)$$

which implies $\lim_{r\to 0^-} Q_5 = 0$. Letting $r > 0$ be small enough, we have

$$\begin{aligned} Q_5 &= e^{-a_i(t_k+r)} \sum_{0<t_m<(t_k+r)} I_{im}\left(y_i(t_m)\right) e^{a_i t_m} - e^{-a_i t_k} \sum_{0<t_m<t_k} I_{im}\left(y_i(t_m)\right) e^{a_i t_m} \\ &= e^{-a_i(t_k+r)} \left\{\sum_{0<t_m<t_k} \left\{I_{im}\left(y_i(t_m)\right) e^{a_i t_m}\right\} + I_{ik}\left(y_i(t_k)\right) e^{a_i t_k}\right\} - e^{-a_i t_k} \sum_{0<t_m<t_k} \left\{I_{im}\left(y_i(t_m)\right) e^{a_i t_m}\right\} \\ &= \left\{e^{-a_i(t_k+r)} - e^{-a_i t_k}\right\} \sum_{0<t_m<t_k} \left\{I_{im}\left(y_i(t_m)\right) e^{a_i t_m}\right\} + e^{-a_i(t_k+r)} I_{ik}\left(y_i(t_k)\right) e^{a_i t_k}, \end{aligned} \quad (26)$$

which implies $\lim_{r\to 0^+} Q_5 = e^{-a_i t_k} I_{ik}(y_i(t_k)) e^{a_i t_k}$.

According to the above discussion, we see that $\pi(y_i)(t) : [-m^*, +\infty) \to R$ is continuous on $t \neq t_k$ $(k = 1, 2, \ldots)$, while for $t = t_k$ $(k = 1, 2, \ldots)$, $\lim_{t\to t_k^-} \pi(y_i)(t)$ and $\lim_{t\to t_k^+} \pi(y_i)(t)$ exist; moreover, $\lim_{t\to t_k^-} \pi(y_i)(t) = \pi(y_i)(t_k) \neq \lim_{t\to t_k^+} \pi(y_i)(t)$.

Next, we will prove that $e^{\alpha t}\pi(y_i)(t) \to 0$ as $t \to \infty$ for $i \in \mathscr{N}$. To begin with, we give the expression of $e^{\alpha t}\pi(y_i)(t)$ as follows:

$$e^{\alpha t}\pi\left(y_i\right)(t) = W_1 + W_2 + W_3 + W_4 + W_5, \quad t > 0, \quad (27)$$

where

$W_1 = \varphi_i(0) e^{-(a_i-\alpha)t}$,

$W_2 = e^{\alpha t} e^{-a_i t} \int_0^t e^{a_i s} \sum_{j=1}^n b_{ij} f_j(y_j(s)) ds$,

$W_5 = e^{\alpha t} e^{-a_i t} \sum_{0<t_k<t} \{I_{ik}(y_i(t_k)) e^{a_i t_k}\}$,

$W_3 = e^{\alpha t} e^{-a_i t} \int_0^t e^{a_i s} \sum_{j=1}^n c_{ij} g_j(y_j(s-\tau_{ij}(s))) ds$, and

$W_4 = e^{\alpha t} e^{-a_i t} \int_0^t e^{a_i s} \sum_{j=1}^n d_{ij} \int_0^{\rho(s)} \sigma_j(y_j(s-\theta)) d\theta ds$.

First, it is obvious that $\lim_{t\to\infty} W_1 = 0$ as $a_i - \alpha > 0$. Furthermore, for $y_j(t) \in \mathscr{H}_j$ $(j \in \mathscr{N})$, we see $\lim_{t\to\infty} e^{\alpha t} y_j(t) = 0$. Then, for any $\varepsilon > 0$, there exists a $T_j > 0$ such that $s \geq T_j$ implies $|e^{\alpha s} y_j(s)| < \varepsilon$. Choose $T^* = \max_{j\in\mathscr{N}}\{T_j\}$. It is derived form (A1) that

$$\begin{aligned} W_2 &\leq e^{\alpha t} e^{-a_i t} \int_0^t e^{a_i s} \sum_{j=1}^n \left\{\left|b_{ij} l_j\right| \left|y_j(s)\right|\right\} ds \\ &= e^{\alpha t} e^{-a_i t} \int_0^t e^{a_i s} e^{-\alpha s} \sum_{j=1}^n \left\{\left|b_{ij} l_j\right| e^{\alpha s} \left|y_j(s)\right|\right\} ds \\ &= e^{-(a_i-\alpha)t} \int_0^{T^*} e^{(a_i-\alpha)s} \sum_{j=1}^n \left\{\left|b_{ij} l_j\right| e^{\alpha s} \left|y_j(s)\right|\right\} ds \\ &\quad + e^{-(a_i-\alpha)t} \int_{T^*}^t e^{(a_i-\alpha)s} \sum_{j=1}^n \left\{\left|b_{ij} l_j\right| e^{\alpha s} \left|y_j(s)\right|\right\} ds \\ &\leq e^{-(a_i-\alpha)t} \sum_{j=1}^n \left\{\left|b_{ij} l_j\right| \sup_{s\in[0,T^*]} \left|e^{\alpha s} y_j(s)\right|\right\} \left\{\int_0^{T^*} e^{(a_i-\alpha)s} ds\right\} \\ &\quad + \varepsilon \sum_{j=1}^n \left\{\left|b_{ij} l_j\right|\right\} e^{-(a_i-\alpha)t} \int_{T^*}^t e^{(a_i-\alpha)s} ds \\ &\leq e^{-(a_i-\alpha)t} \sum_{j=1}^n \left\{\left|b_{ij} l_j\right| \sup_{s\in[0,T^*]} \left|e^{\alpha s} y_j(s)\right|\right\} \left\{\int_0^{T^*} e^{(a_i-\alpha)s} ds\right\} \\ &\quad + \frac{\varepsilon}{a_i - \alpha} \sum_{j=1}^n \left\{\left|b_{ij} l_j\right|\right\}, \end{aligned} \quad (28)$$

which leads to $W_2 \to 0$ as $t \to \infty$.

Similarly, for the given $\varepsilon > 0$ above, there also exists a $T_j' > 0$ such that $s \geq T_j' - \tau$ implies $|e^{\alpha s} y_j(s)| < \varepsilon$. Select $\widehat{T} = \max_{j \in \mathcal{N}} \{T_j'\}$. It follows from (A2) that

$$\begin{aligned}
W_3 &\leq e^{\alpha t} e^{-a_i t} \int_0^t e^{a_i s} \sum_{j=1}^n \left\{ \left|c_{ij} k_j\right| \left| y_j\left(s - \tau_{ij}(s)\right)\right| \right\} ds \\
&\leq e^{-(a_i-\alpha)t} \\
&\quad \times \int_0^t e^{a_i s} e^{-\alpha\{s-\tau\}} \\
&\qquad \times \sum_{j=1}^n \left\{ \left|c_{ij} k_j\right| e^{\alpha[s-\tau_{ij}(s)]} \left| y_j\left(s - \tau_{ij}(s)\right)\right| \right\} ds \\
&= e^{\alpha\tau} e^{-(a_i-\alpha)t} \\
&\quad \times \int_0^{\widehat{T}} e^{(a_i-\alpha)s} \sum_{j=1}^n \left\{ \left|c_{ij} k_j\right| e^{\alpha[s-\tau_{ij}(s)]} \left| y_j\left(s - \tau_{ij}(s)\right)\right| \right\} ds \\
&\quad + e^{\alpha\tau} e^{-(a_i-\alpha)t} \\
&\quad \times \int_{\widehat{T}}^t e^{(a_i-\alpha)s} \sum_{j=1}^n \left\{ \left|c_{ij} k_j\right| e^{\alpha[s-\tau_{ij}(s)]} \left| y_j\left(s - \tau_{ij}(s)\right)\right| \right\} ds \\
&\leq e^{\alpha\tau} \sum_{j=1}^n \left\{ \left|c_{ij} k_j\right| \sup_{s \in [-\tau, \widehat{T}]} \left| e^{\alpha s} y_j(s)\right| \right\} \\
&\quad \times e^{-(a_i-\alpha)t} \int_0^{\widehat{T}} e^{(a_i-\alpha)s} ds \\
&\quad + e^{\alpha\tau} \varepsilon \sum_{j=1}^n \left\{ \left|c_{ij} k_j\right| \right\} e^{-(a_i-\alpha)t} \int_{\widehat{T}}^t e^{(a_i-\alpha)s} ds \\
&\leq e^{\alpha\tau} \sum_{j=1}^n \left\{ \left|c_{ij} k_j\right| \sup_{s \in [-\tau, \widehat{T}]} \left| e^{\alpha s} y_j(s)\right| \right\} \\
&\quad \times e^{-(a_i-\alpha)t} \int_0^{\widehat{T}} e^{(a_i-\alpha)s} ds + \frac{e^{\alpha\tau}\varepsilon}{a_i - \alpha} \sum_{j=1}^n \left\{ \left|c_{ij} k_j\right| \right\},
\end{aligned} \tag{29}$$

which results in $W_3 \to 0$ as $t \to \infty$. In addition, it is derived from (A4) that

$$\begin{aligned}
W_4 &\leq e^{\alpha t} e^{-a_i t} \int_0^t e^{a_i s} \sum_{j=1}^n \left\{ d_{ij} \int_0^\rho \omega_j \left| y_j(s-\theta)\right| d\theta \right\} ds \\
&= e^{\alpha t} e^{-a_i t} \\
&\quad \times \int_0^t e^{a_i s} e^{-\alpha s} \sum_{j=1}^n \left\{ d_{ij} \int_0^\rho e^{\alpha\theta} \omega_j e^{\alpha(s-\theta)} \left| y_j(s-\theta)\right| d\theta \right\} ds \\
&\leq e^{\alpha t} e^{-a_i t} \\
&\quad \times \int_0^t e^{a_i s} e^{-\alpha s} \\
&\qquad \times \sum_{j=1}^n \left\{ d_{ij} \sup_{\zeta \in [s-\rho, s]} \left\{ e^{\alpha\zeta} \left| y_j(\zeta)\right| \right\} \int_0^\rho e^{\alpha\theta} \omega_j d\theta \right\} ds.
\end{aligned} \tag{30}$$

Since $e^{\alpha\zeta} |y_j(\zeta)| \to 0$ as $\zeta \to \infty$, we know that, for any $\varepsilon > 0$, there exists a $T_j'' > 0$ such that $\zeta > T_j'' - \rho$ implies $e^{\alpha\zeta} |y_j(\zeta)| < \varepsilon$. Selecting $\overline{T} = \max_{j \in \mathcal{N}} \{T_j''\}$, it follows from (30) that

$$\begin{aligned}
W_4 &\leq e^{(\alpha - a_i)t} \\
&\quad \times \int_0^{\overline{T}} e^{(a_i-\alpha)s} \sum_{j=1}^n \left\{ d_{ij} \sup_{\zeta \in [s-\rho, s]} \left\{ e^{\alpha\zeta} \left| y_j(\zeta)\right| \right\} \right. \\
&\qquad \left. \times \int_0^\rho e^{\alpha\theta} \omega_j d\theta \right\} ds \\
&\quad + e^{(\alpha - a_i)t} \\
&\quad \times \int_{\overline{T}}^t e^{(a_i-\alpha)s} \sum_{j=1}^n \left\{ d_{ij} \sup_{\zeta \in [s-\rho, s]} \left\{ e^{\alpha\zeta} \left| y_j(\zeta)\right| \right\} \right. \\
&\qquad \left. \times \int_0^\rho e^{\alpha\theta} \omega_j d\theta \right\} ds \\
&\leq \frac{e^{\alpha\rho}}{\alpha} \left\{ \sum_{j=1}^n d_{ij} \omega_j \sup_{\zeta \in [-\rho, \overline{T}]} \left\{ e^{\alpha\zeta} \left| y_j(\zeta)\right| \right\} \right\} \\
&\quad \times e^{(\alpha - a_i)t} \int_0^{\overline{T}} e^{(a_i-\alpha)s} ds \\
&\quad + \frac{e^{\alpha\rho}}{\alpha} \sum_{j=1}^n \left\{ d_{ij} \omega_j \sup_{\zeta \in [\overline{T}-\rho, t]} \left\{ e^{\alpha\zeta} \left| y_j(\zeta)\right| \right\} \right\} \\
&\quad \times e^{(\alpha - a_i)t} \int_{\overline{T}}^t e^{(a_i-\alpha)s} ds \\
&\leq e^{(\alpha - a_i)t} \frac{e^{\alpha\rho}}{\alpha} \left\{ \sum_{j=1}^n d_{ij} \omega_j \sup_{\zeta \in [-\rho, \overline{T}]} \left\{ e^{\alpha\zeta} \left| y_j(\zeta)\right| \right\} \right\} \\
&\quad \times \int_0^{\overline{T}} e^{(a_i-\alpha)s} ds + \varepsilon \sum_{j=1}^n \left\{ d_{ij} \omega_j \right\} \frac{e^{\alpha\rho}}{\alpha (a_i - \alpha)},
\end{aligned} \tag{31}$$

which yields $W_4 \to 0$ as $t \to \infty$.

Furthermore, from (A3), we see that $|I_{ik}(x_i(t_k))| \leq p_{ik} |y_i(t_k)|$. So,

$$W_5 \leq e^{\alpha t} e^{-a_i t} \sum_{0 < t_k < t} \left\{ p_{ik} \left| y_i(t_k)\right| e^{a_i t_k} \right\}. \tag{32}$$

As $y_i(t) \in \mathcal{H}_i$, we have $\lim_{t \to \infty} e^{\alpha t} y_i(t) = 0$. Then, for any $\varepsilon > 0$, there exists a nonimpulsive point $T_i > 0$ such that

$s \geq T_i$ implies $|e^{\alpha s} y_i(s)| < \varepsilon$. It then follows from conditions (i) and (ii) that

$$\begin{aligned}
W_5 &\leq e^{\alpha t} e^{-a_i t} \left\{ \sum_{0<t_k<T_i} \left\{ p_{ik} \left| y_i(t_k) \right| e^{a_i t_k} \right\} + \sum_{T_i<t_k<t} \left\{ p_{ik} \left| y_i(t_k) \right| e^{\alpha t_k} e^{(a_i-\alpha)t_k} \right\} \right\} \\
&\leq e^{\alpha t} e^{-a_i t} \sum_{0<t_k<T_i} \left\{ p_{ik} \left| y_i(t_k) \right| e^{a_i t_k} \right\} + e^{\alpha t} e^{-a_i t} p_i \varepsilon \sum_{T_i<t_k<t} \left\{ \mu e^{(a_i-\alpha)t_k} \right\} \\
&\leq e^{-(a_i-\alpha)t} \sum_{0<t_k<T_i} \left\{ p_{ik} \left| y_i(t_k) \right| e^{a_i t_k} \right\} + e^{-(a_i-\alpha)t} p_i \varepsilon \left\{ \sum_{T_i<t_r<t_k} \left\{ e^{(a_i-\alpha)t_r} \left(t_{r+1} - t_r \right) \right\} + \mu e^{(a_i-\alpha)t_k} \right\} \\
&\leq e^{-(a_i-\alpha)t} \sum_{0<t_k<T_i} \left\{ p_{ik} \left| y_i(t_k) \right| e^{a_i t_k} \right\} + e^{-(a_i-\alpha)t} p_i \varepsilon \int_{T_i}^{t} e^{(a_i-\alpha)s} ds + e^{-(a_i-\alpha)t} p_i \varepsilon \mu e^{(a_i-\alpha)t} \\
&\leq e^{-(a_i-\alpha)t} \sum_{0<t_k<T_i} \left\{ p_{ik} \left| y_i(t_k) \right| e^{a_i t_k} \right\} + \frac{p_i \varepsilon}{a_i - \alpha} + p_i \varepsilon \mu,
\end{aligned} \tag{33}$$

which means that $W_5 \to 0$ as $t \to \infty$.

Now, we can derive from (27) that $e^{\alpha t} \pi(y_i)(t) \to 0$ as $t \to \infty$ for $i \in \mathcal{N}$. It is therefore concluded that $\pi(y_i)(t) \subset \mathcal{H}_i$ which results in $\pi(\mathcal{H}) \subset \mathcal{H}$.

Step 2. We need to prove that π is contractive. For $\bar{\mathbf{z}} = (z_1(t), \ldots, z_n(t)) \in \mathcal{H}$ and $\bar{\mathbf{y}} = (y_1(t), \ldots, y_n(t)) \in \mathcal{H}$, we estimate

$$\left| \pi(y_i)(t) - \pi(z_i)(t) \right| \leq J_1 + J_2 + J_3 + J_4, \tag{34}$$

where

$$\begin{aligned}
J_1 &= e^{-a_i t} \int_0^t e^{a_i s} \sum_{j=1}^{n} \left[\left| b_{ij} \right| \left| f_j(y_j(s)) - f_j(z_j(s)) \right| \right] ds, \\
J_2 &= e^{-a_i t} \int_0^t e^{a_i s} \sum_{j=1}^{n} \left[\left| c_{ij} \right| \left| g_j\left(y_j\left(s - \tau_{ij}(s) \right) \right) - g_j\left(z_j\left(s - \tau_{ij}(s) \right) \right) \right| \right] ds, \\
J_3 &= e^{-a_i t} \int_0^t e^{a_i s} \sum_{j=1}^{n} \left| d_{ij} \right| \int_0^{\rho(s)} \left| \sigma_j\left(y_j(s-\theta) \right) - \sigma_j\left(z_j(s-\theta) \right) \right| d\theta ds, \\
J_4 &= e^{-a_i t} \sum_{0<t_k<t} \left\{ e^{a_i t_k} \left| I_{ik}\left(y_i(t_k) \right) - I_{ik}\left(z_i(t_k) \right) \right| \right\}.
\end{aligned} \tag{35}$$

Note that

$$\begin{aligned}
J_1 &\leq e^{-a_i t} \int_0^t e^{a_i s} \sum_{j=1}^{n} \left[\left| b_{ij} l_j \right| \left| y_j(s) - z_j(s) \right| \right] ds \\
&\leq \max_{j \in \mathcal{N}} \left| b_{ij} l_j \right| \sum_{j=1}^{n} \left\{ \sup_{s \in [0,t]} \left| y_j(s) - z_j(s) \right| \right\} \times e^{-a_i t} \int_0^t e^{a_i s} ds \\
&\leq \frac{1}{a_i} \max_{j \in \mathcal{N}} \left| b_{ij} l_j \right| \sum_{j=1}^{n} \left\{ \sup_{s \in [0,t]} \left| y_j(s) - z_j(s) \right| \right\}, \\
J_2 &\leq e^{-a_i t} \int_0^t e^{a_i s} \sum_{j=1}^{n} \left[\left| c_{ij} k_j \right| \left| y_j\left(s - \tau_{ij}(s) \right) - z_j\left(s - \tau_{ij}(s) \right) \right| \right] ds \\
&\leq \max_{j \in \mathcal{N}} \left| c_{ij} k_j \right| \sum_{j=1}^{n} \left\{ \sup_{\xi \in [-\tau,t]} \left| y_j(\xi) - z_j(\xi) \right| \right\} \times e^{-a_i t} \int_0^t e^{a_i s} ds \\
&\leq \frac{1}{a_i} \max_{j \in \mathcal{N}} \left| c_{ij} k_j \right| \sum_{j=1}^{n} \left\{ \sup_{\xi \in [-\tau,t]} \left| y_j(\xi) - z_j(\xi) \right| \right\}, \\
J_3 &\leq e^{-a_i t} \int_0^t e^{a_i s} \sum_{j=1}^{n} \left\{ \left| d_{ij} \right| \int_0^{\rho(s)} \omega_j \left| y_j(s-\theta) - z_j(s-\theta) \right| d\theta \right\} ds \\
&\leq e^{-a_i t} \int_0^t e^{a_i s} \sum_{j=1}^{n} \left\{ \left| d_{ij} \right| \sup_{\xi \in [s-\rho, s]} \left| y_j(\xi) - z_j(\xi) \right| \times \int_0^{\rho(s)} \omega_j d\theta \right\} ds \\
&\leq e^{-a_i t} \int_0^t e^{a_i s} \sum_{j=1}^{n} \left\{ \omega_j \left| d_{ij} \right| \sup_{\xi \in [-\rho, t]} \left| y_j(\xi) - z_j(\xi) \right| \rho(s) \right\} ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \max_{j\in\mathcal{N}}\left\{\omega_j\left|d_{ij}\right|\right\}\sum_{j=1}^{n}\left\{\sup_{\xi\in[-\rho,t]}\left|y_j(\xi)-z_j(\xi)\right|\right\}\\
&\quad\times \mathrm{e}^{-a_i t}\int_0^t \mathrm{e}^{a_i s}\rho(s)\,\mathrm{d}s\\
&\leq \frac{\rho}{a_i}\max_{j\in\mathcal{N}}\left\{\omega_j\left|d_{ij}\right|\right\}\sum_{j=1}^{n}\left\{\sup_{\xi\in[-\rho,t]}\left|y_j(\xi)-z_j(\xi)\right|\right\},\\
J_4 &\leq \mathrm{e}^{-a_i t}\sum_{0<t_k<t}\left\{\mathrm{e}^{a_i t_k}p_{ik}\left|y_i(t_k)-z_i(t_k)\right|\right\}\\
&\leq p_i\mathrm{e}^{-a_i t}\sup_{s\in[0,t]}\left|y_i(s)-z_i(s)\right|\sum_{0<t_k<t}\left\{\mathrm{e}^{a_i t_k}\mu\right\}\\
&\leq p_i\mathrm{e}^{-a_i t}\sup_{s\in[0,t]}\left|y_i(s)-z_i(s)\right|\\
&\quad\times\left\{\sum_{0<t_r<t_k}\left\{\mathrm{e}^{a_i t_r}(t_{r+1}-t_r)\right\}+\mathrm{e}^{a_i t_k}\mu\right\}\\
&\leq p_i\sup_{s\in[0,t]}\left|y_i(s)-z_i(s)\right|\mathrm{e}^{-a_i t}\left\{\int_0^t \mathrm{e}^{a_i s}\mathrm{d}s+\mathrm{e}^{a_i t}\mu\right\}\\
&\leq p_i\left(\mu+\frac{1}{a_i}\right)\sup_{s\in[0,t]}\left|y_i(s)-z_i(s)\right|.
\end{aligned}
\tag{36}
$$

It is then derived from (36) that

$$
\begin{aligned}
&\sup_{t\in[-m^*,T]}\left|\pi(y_i)(t)-\pi(z_i)(t)\right|\\
&\leq \frac{1}{a_i}\max_{j\in\mathcal{N}}\left|b_{ij}l_j\right|\sum_{j=1}^{n}\left\{\sup_{s\in[-m^*,T]}\left|y_j(s)-z_j(s)\right|\right\}\\
&\quad+\frac{1}{a_i}\max_{j\in\mathcal{N}}\left|c_{ij}k_j\right|\sum_{j=1}^{n}\left\{\sup_{\xi\in[-m^*,T]}\left|y_j(\xi)-z_j(\xi)\right|\right\}\\
&\quad+\frac{\rho}{a_i}\max_{j\in\mathcal{N}}\left\{\omega_j\left|d_{ij}\right|\right\}\sum_{j=1}^{n}\left\{\sup_{\xi\in[-m^*,T]}\left|y_j(\xi)-z_j(\xi)\right|\right\}\\
&\quad+p_i\left(\mu+\frac{1}{a_i}\right)\sup_{s\in[-m^*,T]}\left|y_i(s)-z_i(s)\right|,
\end{aligned}
\tag{37}
$$

which means that

$$
\begin{aligned}
&\sum_{i=1}^{n}\sup_{t\in[-m^*,T]}\left|\pi(y_i)(t)-\pi(z_i)(t)\right|\\
&\qquad\leq \chi\sum_{j=1}^{n}\left\{\sup_{s\in[-m^*,T]}\left|y_j(s)-z_j(s)\right|\right\},
\end{aligned}
\tag{38}
$$

where

$$
\begin{aligned}
\chi \triangleq \sum_{i=1}^{n}&\left\{\frac{1}{a_i}\max_{j\in\mathcal{N}}\left|b_{ij}l_j\right|+\frac{1}{a_i}\max_{j\in\mathcal{N}}\left|c_{ij}k_j\right|+\frac{\rho}{a_i}\max_{j\in\mathcal{N}}\left|\omega_j d_{ij}\right|\right\}\\
&+\max_{i\in\mathcal{N}}\left\{p_i\left(\mu+\frac{1}{a_i}\right)\right\}.
\end{aligned}
\tag{39}
$$

In view of condition (iii), we know that π is a contraction mapping, and hence, there exists a unique fixed point $\overline{\mathbf{y}}(\cdot)$ of π in $\mathscr{H}$ which means that $\overline{\mathbf{y}}^{\mathrm{T}}(\cdot)$ is the solution to (1)–(3) and $\mathrm{e}^{\alpha t}\|\overline{\mathbf{y}}^{\mathrm{T}}(\cdot)\| \to 0$ as $t \to \infty$. This completes the proof. □

Lemma 4. *Assume conditions (A1)–(A4) hold. Provided that*

(i) $\inf_{k=1,2,\ldots}\{t_k-t_{k-1}\}\geq 1$,

(ii) *there exist constants p_i such that $p_{ik}\leq p_i$ for $i\in\mathcal{N}$ and $k=1,2,\ldots$,*

(iii) $\sum_{i=1}^{n}\{(1/a_i)\max_{j\in\mathcal{N}}|b_{ij}l_j| + (1/a_i)\max_{j\in\mathcal{N}}|c_{ij}k_j| + (\rho/a_i)\max_{j\in\mathcal{N}}|\omega_j d_{ij}|\}+\max_{i\in\mathcal{N}}\{p_i(1+(1/a_i))\}\triangleq\chi<1$,

then the trivial equilibrium $\mathbf{x}=0$ is globally exponentially stable.

Proof. Lemma 4 is a direct conclusion by letting $\mu = 1$ in Theorem 3. □

Remark 5. In Theorem 3, we use the fixed point theorem to prove the existence and uniqueness of solution and the global exponential stability of trivial equilibrium all at once, while Lyapunov method fails to do this.

Remark 6. The presented sufficient conditions in Theorem 3 and Lemma 4 do not require even the differentiability of delays, let alone the monotone decreasing behavior of delays which is necessary in some relevant works.

4. Example

Consider the following two-dimensional impulsive cellular neural network with time-varying and distributed delays.

$$
\begin{aligned}
\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} &= -a_i x_i(t)+\sum_{j=1}^{2}b_{ij}f_j\left(x_j(t)\right)\\
&\quad+\sum_{j=1}^{2}c_{ij}g_j\left(x_j\left(t-\tau_{ij}(t)\right)\right)\\
&\quad+\sum_{j=1}^{2}d_{ij}\int_0^{\rho(t)}\sigma_j\left(x_j(t-\theta)\right)\mathrm{d}\theta,\quad t\geq 0,\ t\neq t_k,\\
\Delta x_i(t_k) &= x_i(t_k+0)-x_i(t_k)=\arctan\left(0.4x_i(t_k)\right),\\
&\qquad k=1,2,\ldots,
\end{aligned}
\tag{40}
$$

with the initial conditions $x_1(s) = \cos(s)$, $x_2(s) = \sin(s)$ on $-m^* \leq s \leq 0$, where $\tau_{ij}(t) = 0.8 + 0.4\cos(t)$, $\rho(t) = 0.5 + 0.3\sin(t)$, m^* is defined as shown in (3), $a_1 = a_2 = 7$, $b_{ij} = 0$, $c_{11} = 0$, $c_{12} = 1/7$, $c_{21} = -1/7$, $c_{22} = -1/7$, $d_{11} = 3/7$, $d_{12} = 2/7$, $d_{21} = 0$, $d_{22} = 1/7$, $f_j(s) = g_j(s) = \sigma_j(s) = (|s+1| - |s-1|)/2$, and $t_k = t_{k-1} + 0.5k$.

It is easily to find that $\mu = 0.5$, $l_j = k_j = \omega_j = 1$, and $p_{ik} = 0.4$. Let $p_i = 0.8$ and compute

$$\sum_{i=1}^{2} \left\{ \frac{1}{a_i} \max_{j \in \mathcal{N}} \left| b_{ij} l_j \right| + \frac{1}{a_i} \max_{j \in \mathcal{N}} \left| c_{ij} k_j \right| + \frac{\rho}{a_i} \max_{j \in \mathcal{N}} \left| \omega_j d_{ij} \right| \right\} + \max_{i \in \mathcal{N}} \left\{ p_i \left(\mu + \frac{1}{a_i} \right) \right\} < 1. \quad (41)$$

From Theorem 3, we conclude that the trivial equilibrium $\mathbf{x} = 0$ of this two-dimensional impulsive cellular neural network with time-varying and distributed delays is globally exponentially stable.

5. Conclusions

This article is a new attempt of applying the fixed point theory to the stability analysis of impulsive neural networks with time-varying and distributed delays, which is different from the existing relevant publications where Lyapunov theory is the main technique. From what have been discussed above, we see that the contraction mapping principle is effective for not only the investigation of the existence and uniqueness of solution but also for the stability analysis of trivial equilibrium. In the future, we will continue to explore the application of other kinds of fixed point theorems to the stability research of complex neural networks.

Acknowledgment

This work is supported by the National Natural Science Foundation of China under Grant no. 61174077 and Natural Science Foundation of Jiangsu of China under Grant no. BK2012682.

References

[1] L. O. Chua and L. Yang, "Cellular neural networks: theory," *IEEE Transactions on Circuits and Systems*, vol. 35, no. 10, pp. 1257–1272, 1988.

[2] L. O. Chua and L. Yang, "Cellular neural networks: applications," *IEEE Transactions on Circuits and Systems*, vol. 35, no. 10, pp. 1273–1290, 1988.

[3] G. T. Stamov and I. M. Stamova, "Almost periodic solutions for impulsive neural networks with delay," *Applied Mathematical Modelling*, vol. 31, no. 7, pp. 1263–1270, 2007.

[4] S. Ahmad and I. M. Stamova, "Global exponential stability for impulsive cellular neural networks with time-varying delays," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 69, no. 3, pp. 786–795, 2008.

[5] K. Li, X. Zhang, and Z. Li, "Global exponential stability of impulsive cellular neural networks with time-varying and distributed delay," *Chaos, Solitons and Fractals*, vol. 41, no. 3, pp. 1427–1434, 2009.

[6] J. Qiu, "Exponential stability of impulsive neural networks with time-varying delays and reaction-diffusion terms," *Neurocomputing*, vol. 70, no. 4-6, pp. 1102–1108, 2007.

[7] X. Wang and D. Xu, "Global exponential stability of impulsive fuzzy cellular neural networks with mixed delays and reaction-diffusion terms," *Chaos, Solitons & Fractals*, vol. 42, no. 5, pp. 2713–2721, 2009.

[8] Y. Zhang and Q. Luo, "Global exponential stability of impulsive delayed reaction-diffusion neural networks via Hardy-Poincare Inequality," *Neurocomputing*, vol. 83, pp. 198–204, 2012.

[9] Y. Zhang and Q. Luo, "Novel stability criteria for impulsive delayed reaction-diffusion Cohen-Grossberg neural networks via Hardy-Poincarè inequality," *Chaos, Solitons & Fractals*, vol. 45, no. 8, pp. 1033–1040, 2012.

[10] W. Zhang, Y. Tang, J.-a. Fang, and X. Wu, "Stability of delayed neural networks with time-varying impulses," *Neural Networks*, vol. 36, pp. 59–63, 2012.

[11] S. Zhu and Y. Shen, "Robustness analysis for connection weight matrices of global exponential stability of stochastic recurrent neural networks," *Neural Networks*, vol. 38, pp. 17–22, 2013.

[12] T. A. Burton, *Stability by Fixed Point Theory for Functional Differential Equations*, Dover, Mineola, NY, USA, 2006.

[13] L. C. Becker and T. A. Burton, "Stability, fixed points and inverses of delays," *Proceedings of the Royal Society of Edinburgh A*, vol. 136, no. 2, pp. 245–275, 2006.

[14] T. A. Burton, "Fixed points, stability, and exact linearization," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 61, no. 5, pp. 857–870, 2005.

[15] T. A. Burton, "Fixed points, Volterra equations, and Becker's resolvent," *Acta Mathematica Hungarica*, vol. 108, no. 3, pp. 261–281, 2005.

[16] T. A. Burton, "Fixed points and stability of a nonconvolution equation," *Proceedings of the American Mathematical Society*, vol. 132, no. 12, pp. 3679–3687, 2004.

[17] T. A. Burton, "Perron-type stability theorems for neutral equations," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 55, no. 3, pp. 285–297, 2003.

[18] T. A. Burton, "Integral equations, implicit functions, and fixed points," *Proceedings of the American Mathematical Society*, vol. 124, no. 8, pp. 2383–2390, 1996.

[19] T. A. Burton and T. Furumochi, "Krasnoselskii's fixed point theorem and stability," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 49, no. 4, pp. 445–454, 2002.

[20] T. A. Burton and B. Zhang, "Fixed points and stability of an integral equation: nonuniqueness," *Applied Mathematics Letters*, vol. 17, no. 7, pp. 839–846, 2004.

[21] T. Furumochi, "Stabilities in FDEs by Schauder's theorem," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 63, no. 5–7, pp. e217–e224, 2005.

[22] C. Jin and J. Luo, "Fixed points and stability in neutral differential equations with variable delays," *Proceedings of the American Mathematical Society*, vol. 136, no. 3, pp. 909–918, 2008.

[23] Y. N. Raffoul, "Stability in neutral nonlinear differential equations with functional delays using fixed-point theory," *Mathematical and Computer Modelling*, vol. 40, no. 7-8, pp. 691–700, 2004.

[24] B. Zhang, "Fixed points and stability in differential equations with variable delays," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 63, no. 5–7, pp. e233–e242, 2005.

[25] J. Luo, "Fixed points and stability of neutral stochastic delay differential equations," *Journal of Mathematical Analysis and Applications*, vol. 334, no. 1, pp. 431–440, 2007.

[26] J. Luo, "Fixed points and exponential stability of mild solutions of stochastic partial differential equations with delays," *Journal of Mathematical Analysis and Applications*, vol. 342, no. 2, pp. 753–760, 2008.

[27] J. Luo, "Stability of stochastic partial differential equations with infinite delays," *Journal of Computational and Applied Mathematics*, vol. 222, no. 2, pp. 364–371, 2008.

[28] J. Luo and T. Taniguchi, "Fixed points and stability of stochastic neutral partial differential equations with infinite delays," *Stochastic Analysis and Applications*, vol. 27, no. 6, pp. 1163–1173, 2009.

[29] R. Sakthivel and J. Luo, "Asymptotic stability of impulsive stochastic partial differential equations with infinite delays," *Journal of Mathematical Analysis and Applications*, vol. 356, no. 1, pp. 1–6, 2009.

[30] R. Sakthivel and J. Luo, "Asymptotic stability of nonlinear impulsive stochastic differential equations," *Statistics & Probability Letters*, vol. 79, no. 9, pp. 1219–1223, 2009.

[31] J. Luo, "Fixed points and exponential stability for stochastic Volterra-Levin equations," *Journal of Computational and Applied Mathematics*, vol. 234, no. 3, pp. 934–940, 2010.

[32] D. R. Smart, *Fixed Point Theorems*, Cambridge University Press, London, UK, 1974.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 128625, 8 pages
http://dx.doi.org/10.1155/2013/128625

Research Article

Numerical Analysis for Stochastic Partial Differential Delay Equations with Jumps

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Received 3 January 2013; Accepted 21 March 2013

Academic Editor: Xuerong Mao

We investigate the convergence rate of Euler-Maruyama method for a class of stochastic partial differential delay equations driven by both Brownian motion and Poisson point processes. We discretize in space by a Galerkin method and in time by using a stochastic exponential integrator. We generalize some results of Bao et al. (2011) and Jacob et al. (2009) in finite dimensions to a class of stochastic partial differential delay equations with jumps in infinite dimensions.

1. Introduction

The theory and application of stochastic differential equations have been widely investigated [1–7]. Liu [2] studied the stability of infinite dimensional stochastic differential equations. For the numerical analysis of stochastic partial differential equations, Gyöngy and Krylov [8] discussed the numerical approximations for linear stochastic partial differential equations in whole space. Jentzen et al. [9] studied the numerical simulations of nonlinear parabolic stochastic partial differential equations with additive noise. Kloeden et al. [10] gave the error analysis for the pathwise approximation of a general semilinear stochastic evolution equations.

By contrast, stochastic partial differential equations with jumps have begun to gain attention [11–15]. Röckner and Zhang [15] considered the existence, uniqueness, and large deviation principles of stochastic evolution equation with jump. In [12], the successive approximation of neutral SPDEs was studied. There are few papers on the convergence rate of numerical solutions for stochastic partial differential equations with jump, although there are some papers on the convergence rate of numerical solutions for stochastic differential equations with jump in finite dimensions [16, 17].

Being motivated by the papers [16, 17], we will discuss the convergence rate of Euler-Maruyama scheme for a class of stochastic partial delay equations with jump, where the numerical scheme is based on spatial discretization by Galerkin method and time discretization by using a stochastic exponential integrator. In consequence, we generalize some results of Bao et al. (2011) and Jacob et al. (2009) in finite dimensions to a class of stochastic partial delay equations with jump in infinite dimensions. The rest of this paper is arranged as follows. We give some preliminary results of Euler-Maruyama scheme in Section 2. The convergence rate is discussed in Section 3.

2. Preliminary Results

Throughout this paper, let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with some filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathscr{F}_0$ contains all $\mathbb{P}$-null sets). Let $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ and $(K, \langle\cdot,\cdot\rangle_K, \|\cdot\|_K)$ be two real separable Hilbert spaces. We denote by $(\mathscr{L}(K, H), \|\cdot\|)$ the family of bounded linear operators. Let $\tau > 0$ and $D([-\tau, 0], H)$ denote the family of right-continuous function and left-hand limits φ from $[-\tau, 0]$ to H with the norm $\|\varphi\|_D = \sup_{-\tau\leq\theta\leq 0}\|\varphi(\theta)\|_H$. $D^b_{\mathscr{F}_0}([-\tau, 0], H)$ denotes the family of almost surely bounded, $\mathscr{F}_0$-measurable, $D([-\tau, 0], H)$-valued random variables. For all $t \geq 0$, $X_t = \{X(t+\theta) : -\tau \leq \theta \leq 0\}$ is regarded as $D([-\tau, 0], H)$-valued stochastic process.

Let T be a positive constant. For given $\tau \geq 0$, consider the following stochastic partial differential delay equations with jumps:

$$\begin{aligned} dX(t) = {} & [AX(t) + f(X(t), X(t-\tau))]\,dt \\ & + g(X(t), X(t-\tau))\,dW(t) \\ & + \int_{\mathbb{Z}} h(X(t), X(t-\tau), u)\,N(dt, du) \end{aligned} \tag{1}$$

on $t \in [0, T]$ with initial datum $X(t) = \xi(t) \in D^b_{\mathcal{F}_0}([-\tau, 0], H)$, $-\tau \leq t \leq 0$. Here $(A, D(A))$ is a self-adjoint operator on H. $\{W(t), t \geq 0\}$ is K-valued $\{\mathcal{F}_t\}_{t\geq 0}$-Wiener process defined on the probability space $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\}$ with covariance operator Q. We assume that $-A$ and the covariance operator Q of the Wiener process have the same eigenbasis $\{e_m\}_{m\geq 1}$ of H; that is,

$$\begin{aligned} -Ae_m &= \lambda_m e_m, \\ Qe_m &= \alpha_m e_m, \quad m = 1, 2, 3, \ldots, \end{aligned} \tag{2}$$

where $\{\lambda_m, m \in \mathbb{N}\}$ are the discrete spectrum of $-A$ and $0 \leq \lambda_1 \leq \lambda_2 \cdots \leq \lim_{m\to\infty}\lambda_m = \infty$, $\{\alpha_m, m \in \mathbb{N}\}$ are the eigenvalues of Q. Then, $W(t)$ is defined by

$$W(t) = \sum_{n=1}^{\infty} \sqrt{\alpha_m}\beta_m(t)\,e_m, \quad t \geq 0, \tag{3}$$

where $\beta_m(t)$ $(m = 1, 2, 3, \ldots)$ is a sequence of real-valued standard Brownian motions mutually independent of the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$.

According to Da Prato and Zabczyk [1], we define stochastic integrals with respect to the Q-Wiener process $W(t)$. Let $K_0 = Q^{1/2}(K)$ be the subspace of K with the inner product $\langle u, v\rangle_{K_0} = \langle Q^{-1/2}u, Q^{-1/2}v\rangle_K$. Obviously, K_0 is a Hilbert space. Denote by $\mathcal{L}^0_2 = \mathcal{L}(K_0, H)$ the family of Hilbert-Schmidt operators from K_0 into H with the norm $\|\Psi\|^2_{\mathcal{L}^0_2} = \mathrm{tr}((\Psi Q^{1/2})(\Psi Q^{1/2})^*)$.

Let $\Phi : (0, \infty) \to \mathcal{L}^0_2$ be a predictable, $\mathcal{F}_t$-adapted process such that

$$\int_0^t \mathbb{E}\|\Phi(s)\|_{\mathcal{L}^0_2} ds \leq \infty, \quad \forall t > 0. \tag{4}$$

Then, the H-valued stochastic integral $\int_0^t \Phi(s)dW(s)$ is a continuous square martingale. Let $N(dt, du)$ be the Poisson measure which is independent of the Q-Wiener process $W(t)$. Denote the compensated or centered Poisson measure as

$$\widetilde{N}(dt, du) = N(dt, du) - \rho dt\pi(du), \tag{5}$$

where $0 < \rho < \infty$ is known as the jump rate and $\pi(\cdot)$ is the jump distribution (a probability measure). Let $\mathbb{Z} \in \mathcal{B}(K - \{0\})$ be the measurable set. Denote by $\mathcal{P}^2([0, T] \times \mathbb{Z}, H)$ the space of all predictable mappings $h : [0, T] \times \mathbb{Z} \to H$ for which

$$\int_0^T \int_{\mathbb{Z}} \mathbb{E}\|h(t, u)\|^2_H dt\pi(du) < \infty. \tag{6}$$

Then, the H-valued stochastic integral

$$\int_0^T \int_{\mathbb{Z}} h(t, u)\,\widetilde{N}(dt, du) \tag{7}$$

is a centred square-integrable martingale.

We recall the definition of the mild solution to (1) as follows.

Definition 1. A stochastic process $\{X(t) : t \in [0, T]\}$ is called a mild solution of (1) if

(i) $X(t)$ is adapted to $\mathcal{F}_t$, $t \geq 0$, and has *càdlàg* path on $t \geq 0$ almost surely,

(ii) for arbitrary $t \in [0, T]$, $\mathbb{P}\{w : \int_0^t \|X(s)\|^2_H ds < \infty\} = 1$, and almost surely

$$\begin{aligned} X(t) = {} & e^{tA}\xi(0) + \int_0^t e^{(t-s)A} f(X(s), X(s-\tau))\,ds \\ & + \int_0^t e^{(t-s)A} g(X(s), X(s-\tau))\,dW(s) \\ & + \int_0^t \int_{\mathbb{Z}} e^{(t-s)A} h(X(s), X(s-\tau), u)\,N(ds, du) \end{aligned} \tag{8}$$

for any $X(t) = \xi(t) \in D^b_{\mathcal{F}_0}([-\tau, 0], H)$, $-\tau \leq t \leq 0$.

For the existence and uniqueness of the mild solution to (1) (see [11]), we always make the following assumptions.

(H1) $(A, D(A))$ is a self-adjoint operator on H such that $-A$ has discrete spectrum $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lim_{m\to\infty}\lambda_m = \infty$ with corresponding eigenbasis $\{e_m\}_{m\geq 1}$ of H. In this case A generates a compact C_0-semigroup e^{tA}, $t \geq 0$, such that $\|e^{tA}\| \leq e^{-\alpha t}$.

(H2) The mappings $f : H \times H \to H$, $g : H \times H \to \mathcal{L}(K, H)$, and $h : H \times H \times \mathbb{Z} \to H$ are Borel measurable and satisfy the following Lipschitz continuity condition for some constant $L_1 > 0$ and arbitrary $x, y, x_1, y_1, x_2, y_2 \in H$ and $u \in \mathbb{Z}$:

$$\begin{aligned} & \|f(x_1, y_1) - f(x_2, y_2)\|^2_H \\ & \qquad \vee \|g(x_1, y_1 - g(x_2, y_2))\|^2_{\mathcal{L}^2_0} \\ & \qquad \leq L_1\left(\|x_1 - x_2\|^2_H + \|y_1 - y_2\|^2_H\right), \\ & \|h(x_1, y_1, u) - h(x_2, y_2, u)\|^2_H \\ & \qquad \leq L_1\left(\|x_1 - x_2\|^2_H + \|y_1 - y_2\|^2_H\right). \end{aligned} \tag{9}$$

This further implies the linear growth condition; that is,

$$\|f(x, y)\|^2_H + \|g(x, y)\|^2_{\mathcal{L}^0_2} \leq L_0\left(1 + \|x\|^2_H + \|y\|^2_H\right), \tag{10}$$

where

$$L_0 := 2\left(L_2 \vee \|f(0, 0)\|^2_H \vee \|g(0, 0)\|^2_{\mathcal{L}^0_2}\right). \tag{11}$$

(H3) There exists $L_2 > 0$ satisfying

$$\|h(x,y,u)\|_H^2 \leq L_2\left(\|x\|_H^2 + \|y\|_H^2\right), \tag{12}$$

for each $x, y \in H$ and $u \in \mathbb{Z}$.

(H4) For $\xi \in D^b_{\mathscr{F}_0}([-\tau,0],H)$, there exists a constant $L_3 > 0$ such that

$$\mathbb{E}\left(|\xi(s)-\xi(t)|^2\right) \leq L_3|t-s|^2, \quad t,s \in [-\tau,0]. \tag{13}$$

We now describe our Euler-Maruyama scheme for the approximation of (1). For any $n \geq 1$, let $\pi_n : H \to H_n = \operatorname{span}\{e_1, e_2, \ldots, e_n\}$ be the orthogonal projection; that is, $\pi_n x = \sum_{i=1}^n \langle x, e_i\rangle_H e_i$, $x \in H$, $A_n = \pi_n A$, $f_n = \pi_n f$, $g_n = \pi_n g$, and $h_n = \pi_n h$.

Consider the following stochastic differential delay equations with jumps on H_n:

$$\begin{aligned} dX^n(t) = {} & \left[A_n X^n(t) + f_n\left(X^n(t), X^n(t-\tau)\right)\right]dt \\ & + g_n\left(X^n(t), X^n(t-\tau)\right)dW(t) \\ & + \int_{\mathbb{Z}} h_n\left(X^n(t), X^n(t-\tau), u\right)N(dt,du), \\ X^n(\theta) = {} & \pi_n\xi(\theta), \quad \theta \in [-\tau,0]. \end{aligned} \tag{14}$$

This spatial approximation (14) is called the Galerkin approximation of (1). Due to the fact that $\pi_n Ax = \pi_n A(\sum_{i=1}^n \langle x, e_i\rangle_H e_i) = -\sum_{i=1}^n \lambda_i \langle x, e_i\rangle_H e_i$, $x \in H_n$, it follows that for $x \in H_n$, $A_n x = Ax$, $e^{tA_n x} = e^{tAx}$.

By (H2) and (H3) and the property of the projection operator, we have that

$$\begin{aligned} & \|A_n x - A_n y\|_H^2 = \|A_n(x-y)\|_H^2 \leq \lambda_n^2 \|x-y\|_H, \\ & \|f_n(x_1,y_1) - f_n(x_2,y_2)\|_H^2 \\ & \quad \vee \|g(x_1,y_1) - g(x_2,y_2)\|_{\mathscr{L}_2^0}^2 \\ & \quad = \|f(x_1,y_1) - f(x_2,y_2)\|_H^2 \\ & \quad \vee \|g(x_1,y_1) - g(x_2,y_2)\|_{\mathscr{L}_2^0}^2 \\ & \quad \leq L_1\left(\|x_1-x_2\|_H^2 + \|y_1-y_2\|_H^2\right), \\ & \|h_n(x_1,y_1,u) - h_n(x_2,y_2,u)\|_H^2 \\ & \quad = \|h(x_1,y_1,u) - h(x_2,y_2,u)\|_H^2 \\ & \quad \leq L_1\left(\|x_1-x_2\|_H^2 + \|y_1-y_2\|_H^2\right), \\ & \|h_n(x,y,u)\|_H^2 = \|h(x,y,u)\|_H^2 \leq L_2\left(\|x\|_H^2 + \|y\|_H^2\right) \end{aligned} \tag{15}$$

for arbitrary $x, y, x_1, x_2, y_1, y_2 \in H_n$ and $u \in \mathbb{Z}$. Hence, (14) admits a unique solution $X^n(t)$ on H_n.

We introduce a time discretization scheme for (14) by using a stochastic exponential integrator. For given $T \geq 0$ and $\tau > 0$, the time-step size $\Delta \in (0,1)$ is defined by $\Delta := \tau/N$, for some sufficiently large integer $N > \tau$. For any integer $k \geq 0$, the time discretization scheme applied to (14) produces approximations $\overline{Y}^n(k\Delta) \approx X^n(k\Delta)$ by forming

$$\begin{aligned} & \overline{Y}^n((k+1)\Delta) \\ & \quad = e^{\Delta A_n}\Big\{\overline{Y}^n(k\Delta) + f_n\left(\overline{Y}^n(k\Delta), \overline{Y}^n(k\Delta-\tau)\right)\Delta \\ & \qquad + g_n\left(\overline{Y}^n(k\Delta), \overline{Y}^n(k\Delta-\tau)\right)\Delta W_k \\ & \qquad + \int_{\mathbb{Z}} h_n\left(\overline{Y}^n(k\Delta), \overline{Y}^n(k\Delta-\tau), u\right)\Delta N_k(u)\Big\}, \\ & \qquad \overline{Y}^n(\theta) = \pi_n\xi(\theta), \quad \theta \in [-\tau,0], \end{aligned} \tag{16}$$

where $\Delta W_k = W((k+1)\Delta) - W(k\Delta)$ and $\Delta N_k(du) = N((0,(k+1)\Delta],du) - N((0,k\Delta],du)$.

The continuous-time version of this scheme associated with (14) is defined by

$$\begin{aligned} & Y^n(t) \\ & \quad = e^{tA_n}Y^n(0) + \int_0^t e^{(t-\lfloor s\rfloor)A_n} f_n\left(Y^n(\lfloor s\rfloor), Y^n(\lfloor s\rfloor - \tau)\right)ds \\ & \qquad + \int_0^t e^{(t-\lfloor s\rfloor)A_n} g_n\left(Y^n(\lfloor s\rfloor), Y^n(\lfloor s\rfloor - \tau)\right)dW(s) \\ & \qquad + \int_0^t \int_{\mathbb{Z}} e^{(t-\lfloor s\rfloor)A_n} h_n\left(Y^n(\lfloor s\rfloor), Y^n(\lfloor s\rfloor - \tau), u\right) \\ & \qquad\quad \times N(ds,du), \\ & \qquad Y^n(\theta) = \pi_n\xi(\theta), \quad \theta \in [-\tau,0], \end{aligned} \tag{17}$$

where $\lfloor t\rfloor = [t/\Delta]\Delta$ with $[t/\Delta]$ denotes the integer of t/Δ.

From (16) and (17), we have $Y^n(k\Delta) = \overline{Y}^n(k\Delta)$ for every $k \geq 0$. That is, the discrete-time and continuous-time schemes coincide at the grid points.

3. Convergence Rate

In this section, we shall investigate the convergence rate of the Euler-Maruyama method. In what follows, $C > 0$ is a generic constant whose values may change from line to line.

Lemma 2. *Let (H1)–(H4) hold; then there is a positive constant $C > 0$ which depends on T, ξ, L_1, L_2, and L_3 but is independent of Δ, such that*

$$\sup_{0\leq t\leq T}\left(\mathbb{E}\|X(t)\|_H^2\right)^{1/2} \vee \sup_{0\leq t\leq T}\left(\mathbb{E}\|Y^n(t)\|_H^2\right)^{1/2} \leq C. \tag{18}$$

Proof. Due to the fact that $(\mathbb{E}\|\cdot\|_H^2)^{1/2}$ is a norm, we have from (8) that

$$\begin{aligned}
&\left(\mathbb{E}\|X(t)\|_H^2\right)^{1/2} \\
&\quad\leq \left(\mathbb{E}\left\|e^{tA_n}\xi(0)\right\|_H^2\right)^{1/2} \\
&\qquad+\left(\mathbb{E}\left\|\int_0^t e^{(t-s)A} f(X(s),X(s-\tau))\,ds\right\|_H^2\right)^{1/2} \\
&\qquad+\left(\mathbb{E}\left\|\int_0^t e^{(t-s)A} g(X(s),X(s-\tau))\,dW(s)\right\|_H^2\right)^{1/2} \\
&\qquad+\left(\mathbb{E}\left\|\int_0^t e^{(t-s)A} h(X(s),X(s-\tau),u)\,N(ds,du)\right\|_H^2\right)^{1/2} \\
&\quad=\sum_{i=1}^4 I_i(t).
\end{aligned} \tag{19}$$

Recall the property of the operator A (see [18]):

$$\begin{gathered}
\left\|(-A)^{\delta_1}e^{At}\right\| \leq Ct^{-\delta_1}, \\
\left\|(-A)^{\delta_2}\left(1-e^{At}\right)\right\| \leq Ct^{\delta_2}, \quad \delta_1\geq 0,\ \delta_2\in[0,1], \\
(-A)^{\alpha+\beta}x=(-A)^{\alpha}(-A)^{\beta}x, \quad x\in D\left((-A)^r\right),
\end{gathered} \tag{20}$$

for $\alpha,\beta\in\mathbb{R}$, where $r=\max\{\alpha,\beta,\alpha+\beta\}$.

By (H1) and (H2), together with the Minkowski integral inequality, we derive that

$$\begin{aligned}
I_2(t) &\leq \int_0^t \left(\mathbb{E}\left\|e^{(t-s)A} f(X(s),X(s-\tau))\right\|_H^2\right)^{1/2} ds \\
&\leq C\int_0^t \Big\{1+\left(\mathbb{E}\|X(s)\|_H^2\right)^{1/2} \\
&\qquad +\left(\mathbb{E}\|X(s-\tau)\|_H^2\right)^{1/2}\Big\}\, ds \\
&\leq C + C\int_0^t \left(\mathbb{E}\|X(s)\|_H^2\right)^{1/2} ds.
\end{aligned} \tag{21}$$

By (H1), (H2), and (H3) and using the Itô isometry, we have

$$\begin{aligned}
&I_3(t)+I_4(t) \\
&\quad\leq \left(\int_0^t \mathbb{E}\left\|e^{(t-s)A} g(X(s),X(s-\tau))\right\|_{\mathscr{L}_2^0}^2 ds\right)^{1/2} \\
&\qquad+\Bigg(\mathbb{E}\Bigg\|\int_0^t\int_Z e^{(t-s)A} h(X(s),X(s-\tau),u)\,\widetilde{N}(ds,du) \\
&\qquad\qquad+\rho\int_0^t\int_{\mathbb{Z}} e^{(t-s)A}h \\
&\qquad\qquad\qquad\times(X(s),X(s-\tau),u)\,\pi(du)\Bigg\|_H^2\Bigg)^{1/2} \\
&\quad\leq \left(\int_0^t L_0\left(1+\mathbb{E}\|X(s)\|_H^2+\mathbb{E}\|X(s-\tau)\|_H^2\right) ds\right)^{1/2} \\
&\qquad+\Bigg(\mathbb{E}\Bigg\|\int_0^t\int_Z e^{(t-s)A}h \\
&\qquad\qquad\times(X(s),X(s-\tau),u)\,\widetilde{N}(ds,du)\Bigg\|_H^2\Bigg)^{1/2} \\
&\qquad+\rho\Bigg(\mathbb{E}\Bigg\|\int_0^t\int_{\mathbb{Z}} e^{(t-s)A}h \\
&\qquad\qquad\times(X(s),X(s-\tau),u)\,\pi(du)\,ds\Bigg\|_H^2\Bigg)^{1/2}.
\end{aligned} \tag{22}$$

Using Hölder inequality and (H3), for the last term of (22), we have

$$\begin{aligned}
&\rho\left(\mathbb{E}\left\|\int_0^t\int_Z e^{(t-s)A}h(X(s),X(s-\tau),u)\,\pi(du)\,ds\right\|_H^2\right)^{1/2} \\
&\quad\leq C\left(\mathbb{E}\int_0^t\int_Z \|h(X(s),X(s-\tau),u)\|_H^2\pi(du)\,ds\right)^{1/2} \\
&\quad\leq C\sqrt{L_2}\left(\int_0^t\left(\mathbb{E}\|X(s)\|_H^2+\mathbb{E}\|X(s-\tau)\|_H^2\right)ds\right)^{1/2} \\
&\quad\leq C\sqrt{L_2}\sqrt{\tau}\mathbb{E}\|\xi\|_H+pC\sqrt{2L_2}\left(\int_0^t\mathbb{E}\|X(s)\|_H^2 ds\right)^{1/2}.
\end{aligned} \tag{23}$$

Moreover, by using the Itô isometry and (H3), we obtain that

$$\begin{aligned}
&\left(\mathbb{E}\left\|\int_0^t\int_{\mathbb{Z}} e^{(t-s)A}h(X(s),X(s-\tau),u)\,\widetilde{N}(ds,du)\right\|_H^2\right)^{1/2} \\
&\quad\leq\left(\int_0^t\int_{\mathbb{Z}}\mathbb{E}\|h(X(s),X(s-\tau),u)\|_H^2\pi(du)\,ds\right)^{1/2} \\
&\quad\leq\sqrt{L_2}\left(\int_0^t\left(\mathbb{E}\|X(s)\|_H^2+\mathbb{E}\|X(s-\tau)\|_H^2\right)ds\right)^{1/2} \\
&\quad\leq\sqrt{L_2}\sqrt{\tau}\mathbb{E}\|\xi\|_H+\sqrt{2L_2}\left(\int_0^t(\mathbb{E}\|X(s)\|_H^2 ds\right)^{1/2}.
\end{aligned} \tag{24}$$

Substituting (23) and (24) into (22), it follows that

$$I_3(t)+I_4(t)\leq C+C\,\mathbb{E}\|\xi\|_H+C\left(\int_0^t\mathbb{E}\|X(s)\|_H^2 ds\right)^{1/2}. \tag{25}$$

Hence,

$$\left(\mathbb{E}\|X(t)\|_H^2\right)^{1/2}\leq C+C\,\mathbb{E}\|\xi\|_H+C\left(\int_0^t\mathbb{E}\|X(s)\|_H^2 ds\right)^{1/2}. \tag{26}$$

Applying the Gronwall inequality, we have

$$\sup_{0\le t\le T}\left(\mathbb{E}\|X(t)\|_H^2\right)^{1/2} \le C. \tag{27}$$

Using the similar argument, the second assertion of (18) follows. □

Lemma 3. *Let (H1)–(H4) hold; for sufficiently small* Δ,

$$\sup_{0\le t\le T}\left(\mathbb{E}\|X(t)-X(\lfloor t\rfloor)\|_H^2\right)^{1/2} \le C\Delta^{1/2}, \tag{28}$$

where $C>0$ *is constant dependent on* T, ξ, L_1, L_2, L_3, *and* L_4, *while being independent of* Δ.

Proof. For any $t\in[0,T]$, we have from (8) that

$$\begin{aligned}
X(t)&-X(\lfloor t\rfloor)\\
&= e^{\lfloor t\rfloor A}\left(e^{(t-\lfloor t\rfloor)A}-\mathbf{1}\right)\xi(0)\\
&\quad+\int_0^{\lfloor t\rfloor}\left(e^{(t-\lfloor t\rfloor)A}-\mathbf{1}\right)e^{(\lfloor t\rfloor-s)A}f(X(s),X(s-\tau))\,ds\\
&\quad+\int_{\lfloor t\rfloor}^{t}e^{(t-s)A}f(X(s),X(s-\tau))\,ds\\
&\quad+\int_0^{\lfloor t\rfloor}\left(e^{(t-\lfloor t\rfloor)A}-\mathbf{1}\right)e^{(\lfloor t\rfloor-s)A}g(X(s),X(s-\tau))\,dW(s)\\
&\quad+\int_0^{\lfloor t\rfloor}\int_{\mathbb{Z}}\left(e^{(t-\lfloor t\rfloor)A}-\mathbf{1}\right)e^{(\lfloor t\rfloor-s)A}\\
&\qquad\qquad\times h(X(s),X(s-\tau),u)\,N(ds,du)\\
&\quad+\int_{\lfloor t\rfloor}^{t}e^{(t-s)A}g(X(s),X(s-\tau))\,dW(s)\\
&\quad+\int_{\lfloor t\rfloor}^{t}\int_{\mathbb{Z}}e^{(t-s)A}h(X(s),X(s-\tau),u)\,N(ds,du)\\
&=\sum_{i=1}^{7}J_i(t).
\end{aligned} \tag{29}$$

Since $(\mathbb{E}\|\cdot\|_H^2)^{1/2}$ is a norm, it follows that

$$\left(\mathbb{E}\|X(t)-X(\lfloor t\rfloor)\|_H^2\right)^{1/2} \le \sum_{i=1}^{7}\left(\mathbb{E}\|J_i(t)\|_H^2\right)^{1/2}. \tag{30}$$

Recalling the fundamental inequality $1-e^{-y}\le y$, $y>0$, we get from (H1) that

$$\begin{aligned}
\left\|\left(e^{(t-\lfloor t\rfloor)A}-\mathbf{1}\right)x\right\|_H^2
&=\left\|\sum_{i=1}^{\infty}\left(e^{-\lambda_i(t-\lfloor t\rfloor)}-1\right)\langle x,e_i\rangle e_i\right\|_H^2\\
&\le\left(1-e^{-\lambda_1(t-\lfloor t\rfloor)}\right)^2\|x\|_H^2\\
&\le\lambda_1^2\Delta^2\|x\|_H^2.
\end{aligned} \tag{31}$$

Therefore,

$$\begin{aligned}
\left(\mathbb{E}\|J_1(t)\|_H^2\right)^{1/2}
&=\left(\mathbb{E}\left\|e^{\lfloor t\rfloor A}\left\{e^{(t-\lfloor t\rfloor)A}-\mathbf{1}\right\}\xi(0)\right\|_H^2\right)^{1/2}\\
&\le\lambda_1\left(\mathbb{E}\|\xi(0)\|_H^2\right)^{1/2}\Delta.
\end{aligned} \tag{32}$$

By (H1), (H2), and the Minkowski integral inequality, we obtain that

$$\begin{aligned}
\sum_{i=2}^{3}\left(\mathbb{E}\|J_i(t)\|_H^2\right)^{1/2}
&\le\int_0^{\lfloor t\rfloor}\left\|e^{(t-\lfloor t\rfloor)A}-\mathbf{1}\right\|\left\|e^{(\lfloor t\rfloor-s)A}\right\|\\
&\qquad\times\left(\mathbb{E}\|f(X(s),X(s-\tau))\|_H^2\right)^{1/2}ds\\
&\quad+\int_{\lfloor t\rfloor}^{t}\left(\mathbb{E}\|f(X(s),X(s-\tau))\|_H^2\right)^{1/2}ds.
\end{aligned} \tag{33}$$

Together with (31), we arrive at

$$\begin{aligned}
\sum_{i=2}^{3}\left(\mathbb{E}\|J_i(t)\|_H^2\right)^{1/2}
&\le\left(\lambda_1\Delta\int_0^{\lfloor t\rfloor}ds+\Delta\right)C\sup_{0\le t\le T}\left(\mathbb{E}\|f(X(t),X(t-\tau))\|_H^2\right)^{1/2}\\
&\le C\left(1+\sup_{0\le t\le T}\left(\mathbb{E}\|X(t)\|_H^2\right)^{1/2}\right)\Delta.
\end{aligned} \tag{34}$$

Following the argument of (22), we derive that

$$\begin{aligned}
\sum_{i=4}^{7}\left(\mathbb{E}\|J_i(t)\|_H^2\right)^{1/2}
&\le\left(\int_0^{\lfloor t\rfloor}\left\|e^{(t-\lfloor t\rfloor)A}-1\right\|^2\left\|e^{(\lfloor t\rfloor-s)A}\right\|^2\right.\\
&\qquad\left.\times\mathbb{E}\|g(X(s),X(s-\tau))\|_{\mathscr{L}_2^0}^2ds\right)^{1/2}\\
&\quad+C\left(\int_0^{\lfloor t\rfloor}\int_{\mathbb{Z}}\left\|e^{(t-\lfloor t\rfloor)A}-1\right\|^2\left\|e^{(\lfloor t\rfloor-s)A}\right\|^2\right.\\
&\qquad\left.\times\mathbb{E}\|h(X(s),X(s-\tau),u)\|_H^2\pi(du)\,ds\right)^{1/2}\\
&\quad+\left(\int_{\lfloor t\rfloor}^{t}\left\|e^{(t-s)A}\right\|^2\mathbb{E}\|g(X(s),X(s-\tau))\|_{\mathscr{L}_2^0}^2ds\right)^{1/2}\\
&\quad+C\left(\int_{\lfloor t\rfloor}^{t}\int_{\mathbb{Z}}\left\|e^{(t-s)A}\right\|^2\right.\\
&\qquad\left.\times\mathbb{E}\|h(X(s),X(s-\tau),u)\|_H^2\pi(du)\,ds\right)^{1/2}\\
&\le C\left(1+\sup_{0\le t\le T}\left(\mathbb{E}\|X(t)\|_H^2\right)^{1/2}\right)\Delta^{1/2}.
\end{aligned} \tag{35}$$

Substituting (32), (34), and (35) into (30), we arrive at

$$\begin{aligned}&\left(\mathbb{E}\|X(t)-X(\lfloor t\rfloor)\|_H^2\right)^{1/2}\\&\quad\leq C\left(1+\sup_{0\leq t\leq T}\left(\mathbb{E}\|X(t)\|_H^2\right)^{1/2}\right)\Delta^{1/2}.\end{aligned}\tag{36}$$

Therefore, by Lemma 2, the required assertion (28) follows. □

Now, we state our main result in this paper as follows.

Theorem 4. *Let (H1)–(H4) hold, and*

$$\sqrt{L_1}\left(2\alpha^{-1}+(\rho+3)(2\alpha)^{-1/2}\right)<1.\tag{37}$$

Then,

$$\sup_{0\leq t\leq T}\left(\mathbb{E}\|X(t)-Y^n(t)\|_H^2\right)^{1/2}\leq C\left\{\lambda_n^{-1/2}+\Delta^{1/2}\right\},\tag{38}$$

where $C>0$ is a constant dependent on T,ξ,L_1,L_2,L_3, and L_4, while being independent of n and Δ.

Proof. By (8) and (17), we obtain

$$\begin{aligned}&X(t)-Y^n(t)\\&\quad=e^{tA}(1-\pi_n)\xi(0)\\&\qquad+\int_0^t e^{(t-s)A}\left(f(X(s),X(s-\tau))\right.\\&\qquad\qquad\left.-f_n(X(s),X(s-\tau))\right)ds\\&\qquad+\int_0^t e^{(t-s)A}\left(f_n(X(s),X(s-\tau))\right.\\&\qquad\qquad\left.-f_n(X(\lfloor s\rfloor),X(\lfloor s\rfloor-\tau))\right)ds\\&\qquad+\int_0^t e^{(t-s)A}\left(g_n(X(s),X(s-\tau))\right.\\&\qquad\qquad\left.-g_n(X(\lfloor s\rfloor),X(\lfloor s\rfloor-\tau))\right)dW(s)\\&\qquad+\int_0^t e^{(t-s)A}\left(f_n(X(\lfloor s\rfloor),X(\lfloor s\rfloor-\tau))\right.\\&\qquad\qquad\left.-f_n(Y^n(\lfloor s\rfloor),Y^n(\lfloor s\rfloor-\tau))\right)ds\\&\qquad+\int_0^t e^{(t-s)A}\left(g_n(X(\lfloor s\rfloor),X(\lfloor s\rfloor-\tau))\right.\\&\qquad\qquad\left.-g_n(Y^n(\lfloor s\rfloor),Y^n(\lfloor s\rfloor-\tau))\right)dW(s)\\&\qquad+\int_0^t e^{(t-s)A}\left(g(X(s),X(s-\tau))\right.\\&\qquad\qquad\left.-g_n(X(s),X(s-\tau))\right)dW(s)\\&\qquad+\int_0^t e^{(t-s)A}\left(1-e^{(s-\lfloor s\rfloor)A}\right)f_n(Y^n(\lfloor s\rfloor),Y^n(\lfloor s\rfloor-\tau))ds\\&\qquad+\int_0^t e^{(t-s)A}\left(1-e^{(s-\lfloor s\rfloor)A}\right)\\&\qquad\qquad\times g_n(Y^n(\lfloor s\rfloor),Y^n(\lfloor s\rfloor-\tau))dW(s)\\&\qquad+\int_0^t\int_{\mathbb{Z}}e^{(t-s)A}\{h(X(s),X(s-\tau),u)\\&\qquad\qquad-h_n(X(s),X(s-\tau),u)\}N(ds,du)\\&\qquad+\int_0^t\int_{\mathbb{Z}}e^{(t-s)A}\{h_n(X(s),X(s-\tau),u)\\&\qquad\qquad-h_n(X(\lfloor s\rfloor),X(\lfloor s\rfloor-\tau),u)\}N(ds,du)\\&\qquad+\int_0^t\int_{\mathbb{Z}}e^{(t-s)A}\{h_n(X(\lfloor s\rfloor),X(\lfloor s\rfloor-\tau),u)-h_n\\&\qquad\qquad\times(Y^n(\lfloor s\rfloor),Y^n(\lfloor s\rfloor-\tau),u)\}N(ds,du)\\&\qquad+\int_0^t\int_Z e^{(t-s)A}\left(1-e^{(s-\lfloor s\rfloor)A}\right)h_n\\&\qquad\qquad\times(Y^n(\lfloor s\rfloor),Y^n(\lfloor s\rfloor-\tau),u)N(ds,du)\\&\quad=\sum_{i=1}^{13}K_i(t).\end{aligned}\tag{39}$$

Noting that $(\mathbb{E}\|\cdot\|_H^2)^{1/2}$ is a norm, we have

$$\left(\mathbb{E}\|X(t)-Y^n(t)\|_H^2\right)^{1/2}\leq\sum_{i=1}^{13}\left(\mathbb{E}\|K_i(t)\|_H^2\right)^{1/2}.\tag{40}$$

By (H1) and the nondecreasing spectrum $\{\lambda_m\}_{m\geq1}$, it easily follows that

$$\begin{aligned}&\mathbb{E}\left\|e^{tA}(1-\pi_n)\xi(0)\right\|_H\\&\quad=\mathbb{E}\left(\sum_{m=n+1}^{\infty}e^{-2\lambda_m t}\langle\xi(0),e_m\rangle_H^2\right)^{1/2}\\&\quad=\mathbb{E}\left(\sum_{m=n+1}^{\infty}\frac{e^{-2\lambda_m t}}{\lambda_m^2}\lambda_m^2\langle\xi(0),e_m\rangle_H^2\right)^{1/2}\\&\quad\leq\frac{1}{\lambda_n}\mathbb{E}\|A\xi(0)\|_H.\end{aligned}\tag{41}$$

By (H2), the Minkowski integral inequality, and Lemma 2, we have

$$\begin{aligned}
&\left(\mathbb{E}\|K_2(t)\|_H^2\right)^{1/2} \\
&\quad\le \int_0^t \left(\mathbb{E}\left\|e^{(t-s)A}(1-\pi_n)f(X(s),X(s-\tau))\right\|_H^2\right)^{1/2} ds \\
&\quad= \int_0^t \left(\mathbb{E}\sum_{m=n+1}^{\infty} e^{-2\lambda_m(t-s)}\left\langle f(X(s),X(s-\tau)),e_m\right\rangle_H^2\right)^{1/2} ds \\
&\quad\le \int_0^t e^{-\lambda_n(t-s)}\left(\mathbb{E}\sum_{m=n+1}^{\infty}\left\langle f(X(s),X(s-\tau)),e_m\right\rangle_H^2\right)^{1/2} ds \\
&\quad\le C\int_0^t e^{-\lambda_n(t-s)} \\
&\qquad\times\left\{1+\left(\mathbb{E}\|X(s)\|_H^2\right)^{1/2}+\left(\mathbb{E}\|X(s-\tau)\|_H^2\right)^{1/2}\right\} ds \\
&\quad\le C\lambda_n^{-1}.
\end{aligned} \tag{42}$$

Applying (H1), (H2), and Lemma 3 and combining the Minkowski integral inequality and the Itô isometry yield

$$\begin{aligned}
&\sum_{i=3}^{6}\left(\mathbb{E}\|K_i(t)\|_H^2\right)^{1/2} \\
&\quad\le \sqrt{L_1}\int_0^t \left\|e^{(t-s)A}\right\|\left(\mathbb{E}\left(\|X(s)-X(\lfloor s\rfloor)\|_H^2\right.\right. \\
&\qquad\qquad\left.\left.+\|X(s-\tau)-X(\lfloor s\rfloor-\tau)\|_H^2\right)\right)^{1/2} ds \\
&\qquad+\sqrt{L_1}\int_0^t \left\|e^{(t-s)A}\right\|\left(\mathbb{E}\left(\|X(\lfloor s\rfloor)-Y^n(\lfloor s\rfloor)\|_H^2\right.\right. \\
&\qquad\qquad\left.\left.+\|X(\lfloor s\rfloor-\tau)-Y^n(\lfloor s\rfloor-\tau)\|_H^2\right)\right)^{1/2} ds \\
&\qquad+\sqrt{L_1}\left(\int_0^t \left\|e^{(t-s)A}\right\|^2\left(\mathbb{E}\left(\|X(s)-X(\lfloor s\rfloor)\|_H^2\right.\right.\right. \\
&\qquad\qquad\left.\left.\left.+\|X(s-\tau)-Y^n(\lfloor s\rfloor-\tau)\|_H^2\right)\right) ds\right)^{1/2} \\
&\qquad+\sqrt{L_1}\left(\int_0^t \left\|e^{(t-s)A}\right\|^2\left(\mathbb{E}\|X(\lfloor s\rfloor)-Y^n(\lfloor s\rfloor)\|_H^2\right.\right. \\
&\qquad\qquad\left.\left.+\|X(\lfloor s\rfloor-\tau)-Y^n(\lfloor s\rfloor-\tau)\|_H^2\right) ds\right)^{1/2} \\
&\quad\le C\Delta^{1/2}+\sqrt{L_1}\sup_{0\le s\le t}\left(\mathbb{E}\|X(s)-Y^n(s)\|_H^2\right)^{1/2} \\
&\qquad\times\int_0^t e^{-\alpha(t-s)} ds \\
&\qquad+\sqrt{L_1}\sup_{0\le s\le t}\left(\mathbb{E}\|X(s)-Y^n(s)\|_H^2\right)^{1/2}\left(\int_0^t e^{-2\alpha(t-s)} ds\right)^{1/2} \\
&\qquad+\sqrt{L_1}\sup_{0\le s\le t}\left(\mathbb{E}\|X(s)-Y^n(s)\|_H^2\right)^{1/2}\int_{-\tau}^{t-\tau} e^{-\alpha(t-s-\tau)} ds \\
&\qquad+\sqrt{L_1}\sup_{0\le s\le t}\left(\mathbb{E}\|X(s)-Y^n(s)\|_H^2\right)^{1/2}\left(\int_{-\tau}^{t-\tau} e^{-2\alpha(t-s-\tau)} ds\right)^{1/2} \\
&\quad\le C\Delta^{1/2}+\sqrt{L_1}\sup_{0\le s\le t}\left(\mathbb{E}\|X(s)-Y^n(s)\|_H^2\right)^{1/2} \\
&\qquad\times\left(2\alpha^{-1}+2(2\alpha)^{-1/2}\right).
\end{aligned} \tag{43}$$

By the Itô isometry and a similar argument to that of (42), we deduce that

$$\begin{aligned}
&\left(\mathbb{E}\|K_7(t)\|_H^2\right)^{1/2} \\
&\quad\le \left(\int_0^t \mathbb{E}\left\|e^{(t-s)A}(1-\pi_n)g(X(s),X(s-\tau))\right\|_{\mathscr{L}_2^0}^2 ds\right)^{1/2} \\
&\quad\le C\left(\int_0^t e^{-2\lambda_n(t-s)}\mathbb{E}\|g(X(s),X(s-\tau))\|_{\mathscr{L}_2^0}^2 ds\right)^{1/2} \\
&\quad\le C\lambda_n^{-1/2}.
\end{aligned} \tag{44}$$

Moreover, by (31), (H2), and Lemma 2 and combining the Minkowski integral inequality and the Itô isometry, we have

$$\begin{aligned}
&\sum_{i=8}^{9}\left(\mathbb{E}\|K_i(t)\|_H^2\right)^{1/2} \\
&\quad\le \int_0^t \left(\mathbb{E}\left\|e^{(t-s)A}\left(1-e^{(s-\lfloor s\rfloor)A}\right)\right\|^2\right. \\
&\qquad\qquad\left.\times\left\|f_n(Y^n(\lfloor s\rfloor),Y^n(\lfloor s\rfloor-\tau))\right\|_H^2\right)^{1/2} ds \\
&\qquad+\left(\int_0^t \mathbb{E}\left\|e^{(t-s)A}\left(1-e^{(s-\lfloor s\rfloor)A}\right)\right\|^2\right. \\
&\qquad\qquad\left.\times\left\|g_n(Y^n(\lfloor s\rfloor),Y^n(\lfloor s\rfloor-\tau))\right\|_H^2 ds\right)^{1/2} \\
&\quad\le C\Delta\int_0^t \left(\mathbb{E}\left\|f_n(Y^n(\lfloor s\rfloor),Y^n(\lfloor s\rfloor-\tau))\right\|_H^2\right)^{1/2} ds \\
&\qquad+C\Delta\left(\int_0^t \mathbb{E}\left\|g_n(Y^n(\lfloor s\rfloor),Y^n(\lfloor s\rfloor-\tau))\right\|_{\mathscr{L}_2^0}^2 ds\right)^{1/2} \\
&\quad\le C\Delta.
\end{aligned} \tag{45}$$

By (31) and the Itô isometry, we obtain that

$$\begin{aligned}
&\left(\mathbb{E}\|K_{10}(t)\|_H^2\right)^{1/2} \\
&\quad\le \left(\mathbb{E}\left\|\int_0^t\int_{\mathbb{Z}} e^{(t-s)A}(1-\pi_n)h\right.\right. \\
&\qquad\qquad\left.\left.\times(X(s),X(s-\tau),u)\widetilde{N}(ds,du)\right\|_H^2\right)^{1/2} \\
&\qquad+\rho\left(\mathbb{E}\left\|\int_0^t\int_{\mathbb{Z}} e^{(t-s)A}(1-\pi_n)h\right.\right. \\
&\qquad\qquad\left.\left.\times(X(s),X(s-\tau),u)\pi(du)ds\right\|_H^2\right)^{1/2} \\
&\quad\le \left(\int_0^t\int_{\mathbb{Z}} \left\|e^{(t-s)A}(1-\pi_n)\right\|^2\mathbb{E}\right. \\
&\qquad\qquad\left.\times\|h(X(s),X(s-\tau),u)\|_H^2\pi(du)ds\right)^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&+\rho\left(\int_0^t\int_{\mathbb{Z}}\left\|e^{(t-s)A}\left(1-\pi_n\right)\right\|^2\right.\\
&\qquad\left.\times\mathbb{E}\|h\left(X\left(s\right),X\left(s-\tau\right),u\right)\|_H^2\pi(du)ds\right)^{1/2}\\
&\le C\left(\int_0^t e^{-2\lambda_n(t-s)}\left(\mathbb{E}\|X\left(s\right)\|_H^2+\mathbb{E}\|X\left(s-\tau\right)\|_H^2\right)ds\right)^{1/2}\\
&\le C\lambda_n^{-1/2}.
\end{aligned}
\tag{46}
$$

Carrying out the similar arguments to those of (43) and (45), we derive that

$$
\begin{aligned}
&\left(\mathbb{E}\left\|K_{11}\left(t\right)\right\|_H^2\right)^{1/2}+\left(\mathbb{E}\left\|K_{12}\left(t\right)\right\|_H^2\right)^{1/2}\\
&\quad\le C\Delta^{1/2}+(2\alpha)^{-1/2}\left(\rho+1\right)\\
&\qquad\times\sqrt{L_1}\sup_{0\le s\le t}\left(\mathbb{E}\left\|X\left(s\right)-Y^n\left(s\right)\right\|_H^2\right)^{1/2},\\
&\qquad\left(\mathbb{E}\left\|K_{13}\left(t\right)\right\|_H^2\right)^{1/2}\le C\Delta.
\end{aligned}
\tag{47}
$$

As a result, putting (41)–(47) into (40) gives that

$$
\begin{aligned}
&\sup_{0\le s\le t}\left(\mathbb{E}\left\|X\left(s\right)-Y^n\left(s\right)\right\|_H^2\right)^{1/2}\\
&\quad\le C\lambda_n^{-1/2}+C\Delta^{1/2}+\sqrt{L_1}\left(2\alpha^{-1}+\left(\rho+3\right)(2\alpha)^{-1/2}\right)\\
&\qquad\times\sup_{0\le s\le t}\left(\mathbb{E}\left\|X\left(s\right)-Y^n\left(s\right)\right\|_H^2\right)^{1/2},
\end{aligned}
\tag{48}
$$

and therefore the desired assertion follows. □

Remark 5. For finite-dimensional Euler-Maruyama method, the condition (37) can be deleted by the Gronwall inequality [16, 17].

Acknowledgment

This work is partially supported by National Natural Science Foundation of China Under Grants 60904005 and 11271146.

References

[1] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, vol. 44 of *Encyclopedia of Mathematics and Its Applications*, Cambridge University Press, Cambridge, UK, 1992.

[2] K. Liu, *Stability of Infinite Dimensional Stochastic Differential Equations with Applications*, Chapman & Hall/CRC, Boca Raton, Fla, USA, 2004.

[3] Q. Luo, F. Deng, J. Bao, B. Zhao, and Y. Fu, "Stabilization of stochastic Hopfield neural network with distributed parameters," *Science in China F*, vol. 47, no. 6, pp. 752–762, 2004.

[4] Q. Luo, F. Deng, X. Mao, J. Bao, and Y. Zhang, "Theory and application of stability for stochastic reaction diffusion systems," *Science in China F*, vol. 51, no. 2, pp. 158–170, 2008.

[5] X. Mao, *Stochastic Differential Equations and Applications*, Horwood Publishing, Chichester, UK, 2007.

[6] Y. Shen and J. Wang, "An improved algebraic criterion for global exponential stability of recurrent neural networks with time-varying delays," *IEEE Transactions on Neural Networks*, vol. 19, no. 3, pp. 528–531, 2008.

[7] Y. Shen and J. Wang, "Almost sure exponential stability of recurrent neural networks with markovian switching," *IEEE Transactions on Neural Networks*, vol. 20, no. 5, pp. 840–855, 2009.

[8] I. Gyöngy and N. Krylov, "Accelerated finite difference schemes for linear stochastic partial differential equations in the whole space," *SIAM Journal on Mathematical Analysis*, vol. 42, no. 5, pp. 2275–2296, 2010.

[9] A. Jentzen, P. E. Kloeden, and G. Winkel, "Efficient simulation of nonlinear parabolic SPDEs with additive noise," *The Annals of Applied Probability*, vol. 21, no. 3, pp. 908–950, 2011.

[10] P. E. Kloeden, G. J. Lord, A. Neuenkirch, and T. Shardlow, "The exponential integrator scheme for stochastic partial differential equations: pathwise error bounds," *Journal of Computational and Applied Mathematics*, vol. 235, no. 5, pp. 1245–1260, 2011.

[11] J. Bao, A. Truman, and C. Yuan, "Stability in distribution of mild solutions to stochastic partial differential delay equations with jumps," *Proceedings of The Royal Society of London A*, vol. 465, no. 2107, pp. 2111–2134, 2009.

[12] B. Boufoussi and S. Hajji, "Successive approximation of neutral functional stochastic differential equations with jumps," *Statistics and Probability Letters*, vol. 80, no. 5-6, pp. 324–332, 2010.

[13] E. Hausenblas, "Finite element approximation of stochastic partial differential equations driven by Poisson random measures of jump type," *SIAM Journal on Numerical Analysis*, vol. 46, no. 1, pp. 437–471, 2007/08.

[14] S. Peszat and J. Zabczyk, *Stochastic Partial Differential Equations with Lévy Noise: An Evolution Equation Approach*, vol. 113 of *Encyclopedia of Mathematics and Its Applications*, Cambridge University Press, Cambridge, UK, 2007.

[15] M. Röckner and T. Zhang, "Stochastic evolution equations of jump type: existence, uniqueness and large deviation principles," *Potential Analysis*, vol. 26, no. 3, pp. 255–279, 2007.

[16] J. Bao, B. Böttcher, X. Mao, and C. Yuan, "Convergence rate of numerical solutions to SFDEs with jumps," *Journal of Computational and Applied Mathematics*, vol. 236, no. 2, pp. 119–131, 2011.

[17] N. Jacob, Y. Wang, and C. Yuan, "Numerical solutions of stochastic differential delay equations with jumps," *Stochastic Analysis and Applications*, vol. 27, no. 4, pp. 825–853, 2009.

[18] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, vol. 44 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 1983.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 631639, 9 pages
http://dx.doi.org/10.1155/2013/631639

Research Article

A Note on the Observability of Temporal Boolean Control Network

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Received 2 December 2012; Accepted 24 February 2013

Academic Editor: Qi Luo

Temporal Boolean network is a generalization of the Boolean network model that takes into account the time series nature of the data and tries to incorporate into the model the possible existence of delayed regulatory interactions among genes. This paper investigates the observability problem of temporal Boolean control networks. Using the semi tensor product of matrices, the temporal Boolean networks can be converted into discrete time linear dynamic systems with time delays. Then, necessary and sufficient conditions on the observability via two kinds of inputs are obtained. An example is given to illustrate the effectiveness of the obtained results.

1. Introduction

Boolean network (BN) is the simplest logical dynamic system. It was proposed by Kauffman for modeling complex and nonlinear biological systems; see [1–3]. Since then, it has been a powerful tool in describing, analyzing, and simulating the cell networks. In this model, gene state is quantized to only two levels: true and false. Then, the state of each gene is determined by the states of its neighborhood genes, using logical rules.

The control of BN is a challenging problem. So far, there are only few results on it because of the shortage of systematic tools to deal with logical dynamic systems; see [4, 5]. Recently, a new matrix product, which was called the semitensor product (STP) [4], was provided to convert a logical function into an algebraic function, and the logical dynamics of BNs could be converted into standard discrete-time dynamics. Based on this, a new technique has been developed for analyzing and synthesizing Boolean (control) networks (BCNs); see [4, 6–9]. Furthermore, [10] have presented some simple criteria to judge the controllability with respect to input-state incidence matrices of BCNs. A Mayer-type optimal control problem for BCNs with multi-input and single input has been studied in [11, 12].

Systematic analysis of biological systems is an important topic in systems biology, and the observability is a structural property of systems. There have been many results on the controllability and observability of dynamic systems; see [13–18]. When it comes to the observability problem of BNs, Cheng and Qi have obtained necessary and sufficient conditions for the observability of BCNs in [8]. However, simple Boolean method cannot be used to study the kinetic properties of networks because it does not have time components, and time delay behaviors happen frequently in biological and physiological systems. In [19], the observability problem for a class of Boolean control systems with time delay is investigated.

It is well known that time delay phenomenon is very common in the real world [20, 21] and very important in analysis and control for dynamic systems. Since many experiments involve obtaining gene expression data by monitoring the expression of genes involved in some biological process (e.g., neural development) over a period of time, the resulting data is in the form of a time series [22]. It is interesting to understand how the expression of a gene at some stage in the process is influenced by the expression levels of other genes during the stages of the process preceding it. Temporal Boolean networks (TBNs) are developed to help model the temporal dependencies that span several time steps and model regulatory delays, which may come about due to missing intermediary genes and spatial or biochemical delays between transcription and regulation; see [23–25].

It should be noticed that TBCN is similar with higher-order BCN from Chapter 5 of [26] in which the higher-order BCN can be rewritten by a BCN by using the first algebraic form of the network. Hence, the observability analysis for higher-order BCNs can be obtained from [26]. However, if the first algebraic form is used, the dimension of network transition matrix depending on the number of logical variables will be much larger which would make computation cost much higher [27]. Motivated by the above analysis, the purpose of this paper is to use STP developed in [4, 6–9, 28] to analyze the observability problem of TBCN without changing it into BCN, which generalizes the BN model to cope with dependencies that span over more than one unit of time.

The rest of this paper is organized as follows. Section 2 provides a brief review for the STP of matrices and the matrix expression of logical function. In Section 3, we convert TBCN into discrete time delay systems. In Section 4, necessary and sufficient conditions for the observability of the temporal BCNs are obtained. An example is given to illustrate the efficiency of the proposed results in Section 5. Finally, a brief conclusion is presented.

2. Preliminaries

For simplicity, we first give some notations as in [4]. Denote $M_{m\times n}$ as the set of all $m\times n$ matrices. The delta set $\Delta_k := \{\delta_k^i \mid i=1,2,\dots,k\}$, where δ_k^i is the ith column of identity matrix I_k with degree k. A matrix $A\in M_{m\times n}$ is called a logical matrix if the columns set of A, denoted by $\mathrm{Col}(A)$, satisfies $\mathrm{Col}(A)\subset\Delta_m$. The set of all $m\times n$ logical matrices is denoted by $\mathscr{L}_{m\times n}$. Assuming $A=[\delta_m^{i_1},\delta_m^{i_2},\dots,\delta_m^{i_n}]\in\mathscr{L}_{m\times n}$, we denote it as $A=\delta_m[i_1,i_2,\dots,i_n]$.

We recall the concept of STP. Let X be a row vector of dimension np and Y a column vector of dimension p. Then, we split X into equal-sized blocks as $X^1,\dots,X^p$, which are $1\times p$ rows. Define the STP, denoted by $\ltimes$, as

$$X\ltimes Y=\sum_{i=1}^{p}X^i y_i\in R^n,$$
$$Y^T\ltimes X^T=\sum_{i=1}^{p}y_i\left(X^i\right)^T\in R^n. \tag{1}$$

In this paper, "$\ltimes$" is omitted, and throughout this paper the matrix product is assumed to be the semi-tensor product as in [9].

The swap matrix $W_{[m,n]}$ is an $mn\times mn$ matrix. Label its columns by $(11,12,\dots,1n,\dots,m1,m2,\dots,mn)$ and its rows by $(11,21,\dots,m1,\dots,1n,2n,\dots,mn)$. Then, its element in the position $((I,J),(i,j))$ is assigned as

$$w_{(I,J),(i,j)}=\delta_{i,j}^{I,J}=\begin{cases}1, & I=i,J=j,\\ 0, & \text{otherwise}.\end{cases} \tag{2}$$

When $m=n$, we briefly denote $W_{[n]}=W_{[m,n]}$. Furthermore, for $X\in\mathbb{R}^m$ and $Y\in\mathbb{R}^n$, $W_{[m,n]}\ltimes X\ltimes Y=Y\ltimes X$ and $W_{[n,m]}\ltimes Y\ltimes X=X\ltimes Y$.

A logical domain, denoted by $\mathscr{D}$, is defined as $\mathscr{D}:=\{T=1,F=0\}$. To use matrix expression, we identify each element in $\mathscr{D}$ with a vector as $T\sim\delta_2^1$ and $F\sim\delta_2^2$ and denote $\Delta:=\Delta_2=\{\delta_2^1,\delta_2^2\}$. Using STP of matrices, a logical function with n arguments $L:\mathscr{D}^n\to\mathscr{D}$ can be expressed in the algebraic form as follows.

Lemma 1 (see [9]). *Any logical function $L(A_1,\dots,A_n)$ with logical arguments $A_1,\dots,A_n\in\Delta$ can be expressed in a multilinear form as*

$$L\left(A_1,\dots,A_n\right)=M_L A_1\cdots A_n, \tag{3}$$

where $M_L\in\mathscr{L}_{2\times 2^n}$ is unique which is called the structure matrix of L.

Lemma 2 (see [9]). *Assume that $P_k=A_1\cdots A_k$ with logical arguments $A_1,\dots,A_k\in\Delta$, then*

$$P_k^2=\Phi_k P_k, \tag{4}$$

where $\Phi_k=\prod_{i=1}^{k}I_{2^{i-1}}\otimes[(I_2\otimes W_{[2,2^{k-i}]})M_r]$, $M_r=\delta_4[1,4]$.

3. Algebraic Form of Temporal Boolean Networks

We consider the temporal Boolean network [25] of a set of nodes $A_1,\dots,A_n\in\Delta$ as follows:

$$\begin{aligned}A_i(t+1) &= f_i\left(A_1(t),\dots,A_n(t),A_1(t-1),\dots,A_n(t-1),\dots,\right.\\ &\qquad \left.A_1(t-\tau),\dots,A_n(t-\tau)\right),\quad i=1,2,\dots,n,\end{aligned} \tag{5}$$

where f_i, $i=1,2,\dots,n$ are logical functions, $t=0,1,2,\dots$, and τ is a positive integer delay.

Using Lemma 1, for each logical function f_i, $i=1,2,\dots,n$, we can find its structure matrix M_i. Let $x(t)=\ltimes_{i=1}^{n}A_i(t)$. Then, the system (5) can be converted into an algebraic form as

$$\begin{aligned}A_i(t+1) &= M_i\ltimes_{j=1}^{n}A_j(t)\cdots\ltimes_{j=1}^{n}A_j(t-\tau)\\ &= M_i x(t)\cdots x(t-\tau),\quad i=1,\dots,n.\end{aligned} \tag{6}$$

From Lemma 2, multiplying all systems in (6) together yields

$$\begin{aligned} x(t+1) &= \ltimes_{i=1}^{n} A_i(t+1) \\ &= \ltimes_{i=1}^{n} [M_i x(t)\cdots x(t-\tau)] \\ &= M_1 [(I_{2^{n(\tau+1)}} \otimes M_2)\Phi_{n(\tau+1)}] x(t)\cdots \\ &\quad \times x(t-\tau) M_3 \cdots M_n x(t)\cdots x(t-\tau) \\ &= M_1 [\ltimes_{i=2}^{3} I_{2^{n(\tau+1)}} \otimes M_i \Phi_{n(\tau+1)}] x(t)\cdots \\ &\quad \times x(t-\tau) M_4 \cdots M_n x(t)\cdots x(t-\tau) \\ &= \cdots \\ &= M_1 [\ltimes_{i=2}^{n} I_{2^{n(\tau+1)}} \otimes M_i \Phi_{n(\tau+1)}] x(t)\cdots x(t-\tau). \end{aligned} \tag{7}$$

Denote $L_0 := M_1[\ltimes_{i=2}^{n} I_{n(\tau+1)} \otimes M_i \Phi_{n(\tau+1)}]$. Then (7) can be expressed as

$$x(t+1) = L_0 x(t)\cdots x(t-\tau), \tag{8}$$

and L_0 is called the network transition matrix of (5).

Next, we consider temporal Boolean control network with outputs as follows:

$$\begin{aligned} &A_i(t+1) \\ &\quad = f_i(u_1(t),\dots u_m(t), A_1(t),\dots,A_n(t),\dots, \\ &\qquad A_1(t-\tau),\dots,A_n(t-\tau)), \quad i=1,\dots,n, \\ &\qquad y_j(t) = h_j(A_1(t),\dots,A_n(t)), \quad j=1,\dots,p, \end{aligned} \tag{9}$$

where u_i, $i=1,2,\dots,m$ are inputs (or controls); $y_j(t)$, $j=1,\dots,p$ are outputs; f_i, $i=1,\dots,n$; h_j, $j=1,\dots,p$ are logical functions.

In this paper, two kinds of inputs (or controls) are considered for (9).

(A) The controls satisfying certain logical rules are called input networks such as

$$u_j(t+1) = g_j(u_1(t), u_2(t)\cdots u_m(t)), \quad j=1,\dots,m, \tag{10}$$

where g_i, $i=1,2,\dots,m$ are logical functions, and the initial states $u_j(0)$, $j=1,2,\dots,m$, can be arbitrarily given.

(B) The controls are free Boolean sequences, which means that the controls do not satisfy any logical rule.

Let $u(t) = \ltimes_{j=1}^{m} u_j(t)$, $y(t) = \ltimes_{j=1}^{p} y_j(t)$. From Lemma 1, for every logical function f_i, g_j, h_l, we can find its structure matrix M_{1i}, M_{2j}, M_{3l}, $i=1,\dots,n$, $j=1,\dots,m$, $l=1,\dots,p$, respectively. Then from (9) and (10), we can obtain

$$A_i(t+1) = M_{1i} u(t) x(t)\cdots x(t-\tau), \quad i=1,\dots,n, \tag{11}$$

$$u_j(t+1) = M_{2j} u(t), \quad j=1,\dots,m, \tag{12}$$

$$y_l(t) = M_{3l} x(t), \quad l=1,\dots,p. \tag{13}$$

Similar with (7), multiplying (11) yields

$$\begin{aligned} x(t+1) &= \ltimes_{i=1}^{n} [M_{1i} u(t) x(t)\cdots x(t-\tau)] \\ &= M_{11} [(I_{2^{m+n(\tau+1)}} \otimes M_{12})\Phi_{m+n(\tau+1)}] u(t) x(t)\cdots \\ &\quad \times x(t-\tau) M_{13}\cdots \\ &\quad \times M_{1n} u(t) x(t)\cdots x(t-\tau) \\ &= \cdots \\ &= M_{11} [\ltimes_{i=2}^{n} (I_{2^{m+n(\tau+1)}} \otimes M_{1i}\Phi_{m+n(\tau+1)})] u(t) x(t)\cdots \\ &\quad \times x(t-\tau) \\ &\triangleq L u(t) x(t)\cdots x(t-\tau). \end{aligned} \tag{14}$$

And, multiplying (12), it leads to

$$\begin{aligned} u(t+1) &= u_1(t+1) u_2(t+1)\cdots u_m(t+1) \\ &= M_{21} u(t) M_{22} u(t)\cdots M_{2n} u(t) \\ &= M_{21} (I_{2^m} \otimes M_{22}) \Phi_m (I_{2^m} \otimes M_{23}) \Phi_m \cdots \\ &\quad \times (I_{2^m} \otimes M_{2m}) \Phi_m u(t) \\ &\triangleq G u(t). \end{aligned} \tag{15}$$

Multiplying (13) yields $y(t) = Hx(t)$, where $H = M_{31}[\ltimes_{l=2}^{p}(I_{2^n} \otimes M_{3l}\Phi_n)]$. From the above conclusion, in an algebraic form, a BCN (9) and (10) can be expressed as

$$\begin{aligned} x(t+1) &= L u(t) x(t)\cdots x(t-\tau), \\ y(t) &= Hx(t), \end{aligned} \tag{16}$$

$$u(t+1) = Gu(t), \tag{17}$$

where L, H are the network transition matrices of two kinds of equations in (9), respectively, and G is the network transition matrix of (10).

Remark 3. It should be noticed that by using the first algebraic form of the network from Chapter 5 of [26], TBCN can be rewritten by a BCN with no delay. Hence, it can be a good idea to study the observability of TBCNs by using the corresponding BCNs from the results in [10]. However, if the first algebraic form is used, the dimension of network transition matrix of corresponding BCNs will be much bigger which would make computation cost much higher. From (16), it is easy to calculate that the dimension of L is $2^n \times 2^{n(\tau+1)+m}$. However, if the TBCNs are rewritten by BCNs using the first algebraic form, then the dimension of the corresponding network transition matrix of the BCNs would be $2^{n(\tau+1)} \times 2^{n(\tau+1)+m}$, which is much bigger if n or τ is a large number. Furthermore, considering the TBCNs directly, we can find the relationship between the network transition matrix (or the Boolean functions) of the TBCN and the state clearly. However, if the BCN is used, the relationship would not be so clear.

4. Observability of Temporal Boolean Control Networks

In this section, we consider the observability problem of temporal Boolean control network (9), equivalently (16), and the analysis is given via two kinds of controls (A) and (B), respectively.

Definition 4 (see [19]). The temporal Boolean network (16) is observable if for the initial state sequence $x(-i)$, $i \in \{0, 1, \dots, \tau\}$, there exists a finite time $s \in \mathbb{N}$, such that the initial state sequence can be uniquely determined by the input controls $u(0), u(1), \dots, u(s)$ and the outputs $y(0), y(1), \dots, y(s)$.

For simplicity, we denote the vector $\mathscr{X}(i) = \ltimes_{j=0}^{i} x(-j) \in \Delta_{2^{n(\tau+1)}}$, $i \in \{0, 1, \dots, \tau\}$.

Definition 5 (see [19]). For temporal Boolean network (16) and control (17) with fixed G, the input-state transfer matrix $\mathscr{L}_i^G \in \mathscr{L}_{2^n \times 2^{m+n(\tau+1)}}$, $i \in \mathbb{N}^+$, is defined as follows: for any $u(0) \in \Delta_{2^m}$ and any $x(-i) \in \Delta_{2^n}$, $i \in \{0, 1, \dots, \tau\}$, we have

$$x(i) = \mathscr{L}_i^G u(0) \mathscr{X}(\tau), \quad i \in \mathbb{N}^+. \tag{18}$$

Now we need a dummy operator to add some fabricated variables when these variables do not appear. Define

$$\begin{aligned} E_{n,m} &:= [\underbrace{I_{2^n} I_{2^n} \cdots I_{2^n}}_{2^{mn}}] \\ &= \delta_{2^n} [\underbrace{\underbrace{1, 2, \dots, 2^n}, \dots, \underbrace{1, 2, \dots, 2^n}}_{2^{mn}}]. \end{aligned} \tag{19}$$

A straightforward computation shows the following.

Lemma 6. *Consider the temporal Boolean network* (16),

$$x(0) = E_{n,\tau} W_{[2^n, 2^{n\tau}]} \mathscr{X}(\tau). \tag{20}$$

Proof. Since $\ltimes_{i=1}^{\tau} x(-i) \in \Delta_{2^{n\tau}}$, from the definition of $E_{n,m}$, we have

$$E_{n,\tau} \ltimes_{i=1}^{\tau} x(-i) = I_{2^n}. \tag{21}$$

Hence,

$$\begin{aligned} x(0) &= I_{2^n} x(0) = E_{n,\tau} \ltimes_{i=1}^{\tau} x(-i)\, x(0) \\ &= E_{n,\tau} W_{[2^n, 2^{n\tau}]} \mathscr{X}(\tau). \end{aligned} \tag{22}$$

□

4.1. Observability of Input Boolean Networks. We first consider the case that controls satisfy certain logical rules as (17). Define a sequence of matrices $\mathscr{L}_s^G \in \mathscr{L}_{2^n \times 2^{m+n(\tau+1)}}$ as (23):

$$\mathscr{L}_s^G := \begin{cases} L, & s = 1, \\ LG\left[\left(I_{2^m} \otimes \mathscr{L}_1^G\right)\Phi_m\right]\left[I_{2^m} \otimes W_{2^{n\tau}, 2^{n(\tau+1)}} \Phi_{n(\tau)}\right], & s = 2, \\ LG^{s-1}\left[\left(I_{2^m} \otimes \mathscr{L}_{s-1}^G\right)\Phi_m\right]\left[\ltimes_{i=s-2}^{1} \mathscr{M}_i^G\right] \ltimes \left[I_{2^m} \otimes W_{[2^{n(\tau-s+2)}, 2^{n(\tau+1)}]} \Phi_{n(\tau-s+2)}\right], & s = 3, \dots, \tau+1, \\ LG^{s-1}\left[\left(I_{2^m} \otimes \mathscr{L}_{s-1}^G\right)\Phi_m\right]\left[\ltimes_{i=s-2}^{s-\tau-1} \mathscr{M}_i^G\right], & s > \tau + 1, \end{cases} \tag{23}$$

where $\mathscr{M}_i^G = I_{2^{m+n(\tau+1)}} \otimes \mathscr{L}_i^G \Phi_{m+n(\tau+1)}$ and $\mathscr{H}_0^G = HE_{n,\tau} W_{[2^n, 2^{n\tau}]}$, $\mathscr{H}_s^G = H\mathscr{L}_s^G$, $s \in \mathbb{N}^+$, and the transition matrices L, G, and H are defined in (16) and (17). Furthermore, we split $\mathscr{H}_j^G \in \mathscr{L}_{2^p \times 2^{m+n(\tau+1)}}$, $j \in \mathbb{N}^+$, into 2^m equal blocks as $\mathscr{H}_j^G = [\mathscr{H}_{j,1}^G, \mathscr{H}_{j,2}^G, \dots, \mathscr{H}_{j,2^m}^G]$ with each $\mathscr{H}_{j,i}^G \in \mathscr{L}_{2^p \times 2^{n(\tau+1)}}$, $i = 1, 2, \dots, 2^m$, $j \in \mathbb{N}^+$.

Theorem 7. *Consider the temporal Boolean network* (16) *with control* (17). *Assume that* $u(0) = \delta_{2^m}^i$, $i \in \{1, 2, \dots, 2^m\}$. *Then,* (16) *and* (17) *are observable if and only if there exists a finite time* s *such that* $\operatorname{rank}(\mathscr{O}_{1,i,s}) = 2^{n(\tau+1)}$, *where*

$$\mathscr{O}_{1,i,s} = \begin{bmatrix} \mathscr{H}_0^G \\ \mathscr{H}_{1,i}^G \\ \vdots \\ \mathscr{H}_{s,i}^G \end{bmatrix}. \tag{24}$$

Proof. Firstly, from Lemma 6 and (16),

$$y(0) = Hx(0) = HE_{n,\tau} W_{[2^n, 2^{n\tau}]} \mathscr{X}(\tau) \triangleq \mathscr{H}_0^G \mathscr{X}(\tau). \tag{25}$$

Since $u(0) = \delta_{2^m}^i$, we have from (18) that

$$\begin{aligned} y(1) &= Hx(1) = HLu(0)\mathscr{X}(\tau) \\ &\triangleq H\mathscr{L}_1^G u(0) \mathscr{X}(\tau) = \mathscr{H}_{1,i}^G \mathscr{X}(\tau), \\ y(2) &= HLu(1)x(1)\mathscr{X}(\tau-1) \\ &= HLGu(0)\mathscr{L}_1^G u(0)\mathscr{X}(\tau)\mathscr{X}(\tau-1) \\ &= HLG\left[\left(I_{2^m} \otimes \mathscr{L}_1^G\right)\Phi_m\right] u(0)\mathscr{X}(\tau)\mathscr{X}(\tau-1) \\ &= HLG\left[\left(I_{2^m} \otimes \mathscr{L}_1^G\right)\Phi_m\right] u(0) W_{[2^{n\tau}, 2^{n(\tau+1)}]} \Phi_{n\tau} \mathscr{X}(\tau) \end{aligned}$$

$$
\begin{aligned}
&= HLG\left[\left(I_{2^m}\otimes\mathscr{L}_1^G\right)\Phi_m\right]\\
&\quad\times\left[I_{2^m}\otimes W_{[2^{n\tau},2^{n(\tau+1)}]}\Phi_{n\tau}\right]u(0)\mathscr{X}(\tau)\\
&\triangleq H\mathscr{L}_2^G u(0)\mathscr{X}(\tau)=\mathscr{H}_{2,i}^G\mathscr{X}(\tau),\\
y(3)&\\
&= HLu(2)x(2)x(1)\mathscr{X}(\tau-2)\\
&= HLG^2u(0)\mathscr{L}_2^Gu(0)\mathscr{X}(\tau)\mathscr{L}_1^Gu(0)\mathscr{X}(\tau)\mathscr{X}(\tau-2)\\
&= HLG^2\left[\left(I_{2^m}\otimes\mathscr{L}_2^G\right)\Phi_m\right]\\
&\quad\times u(0)\mathscr{X}(\tau)\mathscr{L}_1^Gu(0)\mathscr{X}(\tau)\mathscr{X}(\tau-2)\\
&= HLG^2\left[\left(I_{2^m}\otimes\mathscr{L}_2^G\right)\Phi_m\right]\\
&\quad\times\left[\left(I_{2^{m+n(\tau+1)}}\otimes\mathscr{L}_1^G\right)\Phi_{m+n(\tau+1)}\right]\\
&\quad\times u(0)\mathscr{X}(\tau)\mathscr{X}(\tau-2)\\
&= HLG^2\left[\left(I_{2^m}\otimes\mathscr{L}_2^G\right)\Phi_m\right]\\
&\quad\times\left[\left(I_{2^{m+n(\tau+1)}}\otimes\mathscr{L}_1^G\right)\Phi_{m+n(\tau+1)}\right]\\
&\quad\times\left[I_{2^m}\otimes W_{[2^{n(\tau-1)},2^{n(\tau+1)}]}\Phi_{n(\tau-1)}\right]u(0)\mathscr{X}(\tau)\\
&\triangleq H\mathscr{L}_3^Gu(0)\mathscr{X}(\tau)=\mathscr{H}_{3,i}^G\mathscr{X}(\tau),\\
&\vdots\\
y(\tau+1)&\\
&= HLu(\tau)x(\tau)\cdots x(1)\mathscr{X}(0)\\
&= HLG^{\tau}u(0)\left[\ltimes_{i=\tau}^{1}\mathscr{L}_i^Gu(0)\mathscr{X}(\tau)\right]\mathscr{X}(0)\\
&= HLG^{\tau}\left[\left(I_{2^m}\otimes\mathscr{L}_\tau^G\right)\Phi_m\right]\left[\ltimes_{i=\tau-1}^{1}\mathscr{M}_i^G\right]\\
&\quad\times\left[I_{2^m}\otimes W_{[2^n,2^{n(\tau+1)}]}\Phi_n\right]u(0)\mathscr{X}(\tau)\\
&\triangleq H\mathscr{L}_{\tau+1}^Gu(0)\mathscr{X}(\tau)=\mathscr{H}_{\tau+1,i}^G\mathscr{X}(\tau).
\end{aligned}
\tag{26}
$$

For $s>\tau+1$, we can obtain that

$$
\begin{aligned}
y(\tau+2)&\\
&= HLu(\tau+1)x(\tau+1)\cdots x(1)\\
&= HLG^{\tau+1}u(0)\left[\ltimes_{i=\tau+1}^{1}\mathscr{L}_i^Gu(0)\mathscr{X}(\tau)\right]\\
&= HLG^{\tau+1}\left[\left(I_{2^m}\otimes\mathscr{L}_{\tau+1}^G\right)\Phi_m\right]\\
&\quad\times\left[\ltimes_{i=\tau}^{1}\mathscr{M}_i^G\right]u(0)\mathscr{X}(\tau)\\
&\triangleq H\mathscr{L}_{\tau+2}^Gu(0)\mathscr{X}(\tau)=\mathscr{H}_{\tau+2,i}^G\mathscr{X}(\tau),\\
y(\tau+3)&\\
&= HLu(\tau+2)x(\tau+2)\cdots x(2)\\
&= HLG^{\tau+2}u(0)\left[\ltimes_{i=\tau+2}^{2}\mathscr{L}_i^Gu(0)\mathscr{X}(\tau)\right]\\
&= HLG^{\tau+2}\left[\left(I_{2^m}\otimes\mathscr{L}_{\tau+2}^G\right)\Phi_m\right]\\
&\quad\times\left[\ltimes_{i=\tau+1}^{2}\mathscr{M}_i^G\right]u(0)\mathscr{X}(\tau)\\
&\triangleq H\mathscr{L}_{\tau+3}^Gu(0)\mathscr{X}(\tau)=\mathscr{H}_{\tau+3,i}^G\mathscr{X}(\tau),\\
&\vdots\\
y(s)&\\
&= HLu(s-1)x(s-1)\cdots x(s-\tau-1)\\
&= HLG^{s-1}u(0)\left[\ltimes_{i=s-2}^{s-\tau-1}\mathscr{L}_i^Gu(0)\mathscr{X}(\tau)\right]\\
&= HLG^{s-1}\left[\left(I_{2^m}\otimes\mathscr{L}_{s-1}^G\right)\Phi_m\right]\\
&\quad\times\left[\ltimes_{i=s-2}^{s-\tau-1}\mathscr{M}_i^G\right]u(0)\mathscr{X}(\tau)\\
&\triangleq H\mathscr{L}_s^Gu(0)\mathscr{X}(\tau)=\mathscr{H}_{s,i}^G\mathscr{X}(\tau).
\end{aligned}
\tag{27}
$$

From the above analysis, and definition of $\mathscr{O}_{1,i,s}$ in (24), we can see that

$$
\mathscr{O}_{1,i,s}\mathscr{X}(\tau)=\begin{bmatrix}y(0)\\y(1)\\\vdots\\y(s)\end{bmatrix}. \tag{28}
$$

Since $\mathscr{X}(\tau)\in\Delta_{2^{n(\tau+1)}}$, $\mathscr{O}_{1,i,s}\mathscr{X}(\tau)\in\mathrm{Col}(\mathscr{O}_{1,i,s})$. It implies that $\mathscr{X}(\tau)$ is determined uniquely by the outputs $y(0),\dots,y(s)$ if and only if there exist no similar elements in $\mathrm{Col}(\mathscr{O}_{1,i,s})$, or equivalently, there are no equal columns in $\mathscr{O}_{1,i,s}$, that is, $\mathrm{rank}(\mathscr{O}_{1,i,s})=2^{n(\tau+1)}$. The proof is completed. □

Corollary 8. *Consider the temporal Boolean network* (16) *with control* (17). *Equations* (16) *and* (17) *are observable if and only if there exist a finite time s and* $i\in\{1,2,\dots,2^m\}$ *such that* $\mathrm{rank}(\mathscr{O}_{1,i,s})=2^{n(\tau+1)}$.

Remark 9. When the time delay $\tau=0$, then the temporal Boolean control network (16) and (17) become a Boolean control network. In this case, it can be induced from (23) that

$$
\mathscr{L}_s^G=\begin{cases}L, & s=1,\\ LG^{s-1}\left[\left(I_{2^m}\otimes\mathscr{L}_{s-1}^G\right)\Phi_m\right], & s>1.\end{cases} \tag{29}
$$

Then, the observability of the BCN with input Boolean network controls can be deduced from Theorem 7 and Corollary 8.

4.2. Control via Free Boolean Sequence. In the following, the case where the controls are free Boolean sequences is considered. We split L given in (16) into 2^m equal blocks as

$$L = [L_1, L_2, \dots, L_{2^m}], \tag{30}$$

with each $L_i \in \mathscr{L}_{2^n \times 2^{n(\tau+1)}}$, $i = 1, 2, \dots, 2^m$. Define a sequence of matrices $\widetilde{\mathscr{L}}_{s,i_{s-1},\dots,i_0} \in \mathscr{L}_{2^n \times 2^{n(\tau+1)}}$, $s \in \mathbb{N}^+$, $i_{s-1} \in \{1, 2, \dots, 2^m\}$ as (31):

$$\widetilde{\mathscr{L}}_{s,i_{s-1},\dots,i_0} = \begin{cases} L_{i_0}, & s = 1, \\ L_{i_1} L_{i_0} W_{[2^{n\tau}, 2^{n(\tau+1)}]} \Phi_{n\tau}, & s = 2, \\ L_{i_{s-1}} \widetilde{\mathscr{L}}_{s-1, i_{s-2},\dots,i_0} \left[\ltimes_{j=s-2}^{1} \widetilde{\mathscr{M}}_j \right] \\ \ltimes W_{[2^{n(\tau-s+2)}, 2^{n(\tau+1)}]} \Phi_{n(\tau-s+2)}, & s = 3, \dots, \tau+1, \\ L_{i_{s-1}} \widetilde{\mathscr{L}}_{s-1, i_{s-2},\dots,i_0} \left[\ltimes_{j=s-2}^{s-\tau-1} \widetilde{\mathscr{M}}_j \right], & s > \tau+1, \end{cases} \tag{31}$$

where $\widetilde{\mathscr{M}}_j = I_{2^{n(\tau+1)}} \otimes \widetilde{\mathscr{L}}_{j,i_{j-1},\dots,i_0} \Phi_{n(\tau+1)}$, the transition matrices L, G, and H are defined in (16) and (17).

Theorem 10. *Consider the temporal Boolean network* (16). *Assume that the controls are free Boolean sequences with* $u(l) = \delta_{2^m}^{i_l}$, $l \in \mathbb{N}$, $i_l \in \{1, 2, \dots, 2^m\}$. *Then,* (16) *is observable if and only if there exists a finite time s such that* $\operatorname{rank}(\mathscr{O}_{2,s}) = 2^{n(\tau+1)}$, *where*

$$\mathscr{O}_{2,s} = \begin{bmatrix} \mathscr{H}_0^G \\ H\widetilde{\mathscr{L}}_{1,i_0} \\ H\widetilde{\mathscr{L}}_{2,i_1,i_0} \\ \vdots \\ H\widetilde{\mathscr{L}}_{s,i_{s-1},\dots,i_0} \end{bmatrix}. \tag{32}$$

Proof. Since the controls are free Boolean sequences with $u(l) = \delta_{2^m}^{i_l}$, $l \in \mathbb{N}$, $i_l \in \{1, 2, \dots, 2^m\}$, from (16) we have

$$\begin{aligned}
y(1) &= Hx(1) = HLu(0)\mathscr{X}(\tau) \\
&= HL_{i_0}\mathscr{X}(\tau) \triangleq H\widetilde{\mathscr{L}}_{1,i_0}\mathscr{X}(\tau), \\
y(2) &= Hx(2) = HLu(1)x(1)\mathscr{X}(\tau-1) \\
&= HLu(1)Lu(0)\mathscr{X}(\tau)\mathscr{X}(\tau-1) \\
&= HLu(1)Lu(0)W_{[2^{n\tau}, 2^{n(\tau+1)}]}\Phi_{n\tau}\mathscr{X}(\tau) \\
&= HL_{i_1}L_{i_0}W_{[2^{n\tau}, 2^{n(\tau+1)}]}\Phi_{n\tau}\mathscr{X}(\tau) \\
&\triangleq H\widetilde{\mathscr{L}}_{2,i_1,i_0}\mathscr{X}(\tau), \\
y(3) &= Hx(3) = HLu(2)x(2)x(1)\mathscr{X}(\tau-2) \\
&= HLu(2)\widetilde{\mathscr{L}}_{2,i_1,i_0}\mathscr{X}(\tau)\widetilde{\mathscr{L}}_{1,i_0}\mathscr{X}(\tau)\mathscr{X}(\tau-2) \\
&= HL_{i_2}\widetilde{\mathscr{L}}_{2,i_1,i_0}\left[\left(I_{2^{n(\tau+1)}} \otimes \widetilde{\mathscr{L}}_{1,i_0}\right)\Phi_{n(\tau+1)}\right] \\
&\quad \times \mathscr{X}(\tau)\mathscr{X}(\tau-2) \\
&= HL_{i_2}\widetilde{\mathscr{L}}_{2,i_1,i_0}\left[\left(I_{2^{n(\tau+1)}} \otimes \widetilde{\mathscr{L}}_{1,i_0}\right)\Phi_{n(\tau+1)}\right] \\
&\quad \times \left[W_{[2^{n(\tau-1)}, 2^{n(\tau+1)}]}\Phi_{n(\tau-1)}\right]\mathscr{X}(\tau) \\
&\triangleq H\widetilde{\mathscr{L}}_{3,i_2,i_1,i_0}\mathscr{X}(\tau), \\
&\ \ \vdots \\
y(\tau+1) &= HLu(\tau)x(\tau)\cdots x(1)\mathscr{X}(0) \\
&= HLu(\tau)\left[\ltimes_{j=\tau}^{1}\widetilde{\mathscr{L}}_{j,i_{j-1},\dots,i_0}\mathscr{X}(\tau)\right]\mathscr{X}(0) \\
&= HL_{i_\tau}\widetilde{\mathscr{L}}_{\tau,i_{\tau-1},\dots,i_0}\left[\ltimes_{j=\tau-1}^{1}\widetilde{\mathscr{M}}_j\right] \\
&\quad \times W_{[2^n, 2^{n(\tau+1)}]}\Phi_n\mathscr{X}(\tau) \\
&\triangleq H\widetilde{\mathscr{L}}_{\tau+1,i_\tau,\dots,i_0}\mathscr{X}(\tau).
\end{aligned} \tag{33}$$

For $s > \tau + 1$, we can obtain that

$$\begin{aligned}
y(\tau+2) &= HLu(\tau+1)x(\tau+1)\cdots x(1) \\
&= HLu(\tau+1)\left[\ltimes_{j=\tau+1}^{1}\widetilde{\mathscr{L}}_{j,i_{j-1},\dots,i_0}\mathscr{X}(\tau)\right] \\
&= HL_{i_{\tau+1}}\widetilde{\mathscr{L}}_{\tau+1,i_\tau,\dots,i_0}\left[\ltimes_{i=\tau}^{1}\widetilde{\mathscr{M}}_j\right]\mathscr{X}(\tau) \\
&\triangleq H\widetilde{\mathscr{L}}_{\tau+2,i_{\tau+1},\dots,i_0}\mathscr{X}(\tau),
\end{aligned} \tag{34}$$

$$\begin{aligned}
y(\tau+3) &= HLu(\tau+2)x(\tau+2)\cdots x(2) \\
&= HLu(\tau+2)\left[\ltimes_{j=\tau+2}^{2}\widetilde{\mathscr{L}}_{j,i_{j-1},\dots,i_0}\mathscr{X}(\tau)\right] \\
&= HL_{i_{\tau+2}}\widetilde{\mathscr{L}}_{\tau+2,i_{\tau+1},\dots,i_0}\left[\ltimes_{i=\tau+1}^{2}\widetilde{\mathscr{M}}_j\right]\mathscr{X}(\tau) \\
&\triangleq H\widetilde{\mathscr{L}}_{\tau+3,i_{\tau+2},\dots,i_0}\mathscr{X}(\tau), \\
&\ \ \vdots
\end{aligned} \tag{35}$$

$$\begin{aligned} y(s) &= HLu(s-1)\,x(s-1)\cdots x(s-\tau-1) \\ &= HLu(s-1)\left[\ltimes_{i=s-2}^{s-\tau-1}\widetilde{\mathscr{L}}_{j,i_{j-1},\ldots,i_0}\mathscr{X}(\tau)\right] \\ &= HL_{i_{s-1}}\widetilde{\mathscr{L}}_{s-1,i_{s-2},\ldots,i_0}\left[\ltimes_{i=s-2}^{s-\tau-1}\widetilde{\mathscr{M}}_j\right]\mathscr{X}(\tau) \\ &\triangleq H\widetilde{\mathscr{L}}_{s,i_{s-1},\ldots,i_0}\mathscr{X}(\tau). \end{aligned} \tag{36}$$

Thus, from (25) and the definition of $\mathscr{O}_{2,s}$ in (32), we can see that

$$\mathscr{O}_{2,s}\mathscr{X}(\tau) = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(s) \end{bmatrix}. \tag{37}$$

Similar with the proof of Theorem 7, we conclude that $\mathscr{X}(\tau)$ can be determined uniquely by the outputs $y(0),\ldots,y(s)$ if and only if $\operatorname{rank}(\mathscr{O}_{2,s}) = 2^{n(\tau+1)}$. The proof is completed. □

Corollary 11. *Consider the temporal Boolean network* (16). *The system* (16) *is observable if and only if there exists a finite time s and a sequence $i_0, i_1, \ldots, i_{s-1} \in \{1,2,\ldots,2^m\}$ such that* $\operatorname{rank}(\mathscr{O}_{2,s}) = 2^{n(\tau+1)}$.

Remark 12. As a special case, when $\tau = 0$, then from the proof of Theorem 10, we have $\mathscr{H}_0^G = H$, and

$$\begin{aligned} \widetilde{\mathscr{L}}_{1,i_0} &= L_{i_0}, \\ \widetilde{\mathscr{L}}_{s+1,i_s,\ldots,i_0} &= L_{i_{s+1}}\widetilde{\mathscr{L}}_{s,i_{s-1},\ldots,i_0}, \quad s > 0. \end{aligned} \tag{38}$$

Then, Corollary 11 is equivalent with Theorem 26 in [8] for the observability of BCNs.

Remark 13. For Theorems 7 and 10, when $\tau = 1$, the third explicit expressions of $\mathscr{L}_s^G$ in (23) and $\widetilde{\mathscr{L}}_{s,i_{s-1},\ldots,i_0}$ in (31) for $s = 3,\ldots,\tau+1$ should be omitted.

5. An Example

Given logical arguments $P, Q \in \Delta$, we have the following structure matrices for the fundamental logical functions: $\neg P = M_n P$, $P \vee Q = M_d PQ$, $P \wedge Q = M_c PQ$, $P \rightarrow Q = M_i PQ$, $P \leftrightarrow Q = M_e PQ$, where $M_n = \delta_2[2,1]$, $M_d = \delta_2[1,1,1,2]$, $M_c = \delta_2[1,2,2,2]$, $M_i = \delta_2[1,2,1,1]$, $M_e = \delta_2[1,2,2,1]$.

Example 14. Consider the following temporal Boolean network:

$$\begin{aligned} A(t+1) &= u(t) \vee A(t) \longrightarrow A(t-1) \longleftrightarrow A(t-2), \\ y(t) &= \neg A(t). \end{aligned} \tag{39}$$

Let $x(t) = A(t)$, it is easy to get $H = M_n$, $L = M_e M_i M_d$, and $\tau = 2$.

(A) When the controls satisfy the logical rule

$$u(t+1) = \neg u(t), \tag{40}$$

then the transition matrix $G = M_n$. Now, assume that $u(0) = \delta_2^1$, by calculation, we have

$$\begin{aligned} \mathscr{H}_0^G &= \delta_2[2,2,2,2,1,1,1,1], \\ \mathscr{H}_{1,1}^G &= \delta_2[2,1,1,2,2,1,1,2], \\ \mathscr{H}_{2,1}^G &= \delta_2[2,2,2,2,1,2,2,1], \\ \mathscr{H}_{3,1}^G &= \delta_2[2,1,1,2,2,1,1,2], \\ \mathscr{H}_{4,1}^G &= \delta_2[2,1,1,2,1,1,1,1], \\ \mathscr{H}_{5,1}^G &= \delta_2[2,1,1,2,1,1,1,1], \\ &\vdots \end{aligned}$$

$$\mathscr{O}_{1,1,s} = \begin{bmatrix} \mathscr{H}_0^G \\ \mathscr{H}_{1,1}^G \\ \mathscr{H}_{2,1}^G \\ \mathscr{H}_{3,1}^G \\ \mathscr{H}_{4,1}^G \\ \mathscr{H}_{5,1}^G \\ \vdots \end{bmatrix} = \begin{bmatrix} 2&2&2&2&1&1&1&1 \\ 2&1&1&2&2&1&1&2 \\ 2&2&2&2&1&2&2&1 \\ 2&1&1&2&2&1&1&2 \\ 2&1&1&2&1&1&1&1 \\ 2&1&1&2&1&1&1&1 \\ \vdots &&&&&&& \end{bmatrix}. \tag{41}$$

Hence, for any $s > 0$, there are only 4 linearly independent columns, which means that $\operatorname{rank}(\mathscr{O}_{1,1,s}) < 2^{n(\tau+1)} = 8$ for any $s > 0$, and the system is not observable from Theorem 7. Similarly, if $u(0) = \delta_2^2$, we have the same conclusion.

(B) When controls are free sequences with $u(0) = \delta_2^1$, $u(i) = \delta_2^2$, $i = 1,2,\ldots$. By calculation, it leads to

$$\begin{aligned} \mathscr{H}_0^G &= \delta_2[2,2,2,2,1,1,1,1], \\ H\widetilde{\mathscr{L}}_{1,1} &= \delta_2[2,1,1,2,2,1,1,2], \\ H\widetilde{\mathscr{L}}_{2,2,1} &= \delta_2[2,2,1,1,1,2,1,2], \\ H\widetilde{\mathscr{L}}_{3,2,2,1} &= \delta_2[2,1,2,2,1,2,1,1], \\ H\widetilde{\mathscr{L}}_{4,2,2,2,1} &= \delta_2[2,1,2,1,2,1,1,2], \\ &\vdots \end{aligned} \tag{42}$$

and hence,

$$\mathcal{O}_{2,s}=\begin{bmatrix}\mathscr{H}_0^G\\ H\widetilde{\mathscr{L}}_{1,1}\\ H\widetilde{\mathscr{L}}_{2,2,1}\\ H\widetilde{\mathscr{L}}_{3,2,2,1}\\ H\widetilde{\mathscr{L}}_{4,2,2,2,1}\\ \vdots\end{bmatrix}=\begin{bmatrix}2&2&2&2&1&1&1&1\\2&1&1&2&2&1&1&2\\2&2&1&1&1&2&1&2\\2&1&2&2&1&2&1&1\\2&1&2&1&2&1&1&2\\ \vdots&&&&&&&\end{bmatrix}. \quad (43)$$

When $s=2$, it is enough to see that there are no equal columns in $\mathcal{O}_{2,2}$. So, the system is observable by Theorem 10.

From cases (A) and (B), it is easy to notice that the selection of controls is very important for the observability of the temporal Boolean control network.

6. Conclusion

In this brief paper, necessary and sufficient conditions for the observability of temporal Boolean control networks have been derived. By using semi-tensor product of matrices and the matrix expression of logic, we have converted the temporal Boolean control networks into discrete systems with time delays. Moreover, the observability has been investigated via two different kinds of controls. Finally, an example has been given to show the efficiency of the proposed results.

Acknowledgments

The authors would like to take this opportunity to thank the reviewers for their constructive comments and useful suggestions. This work was partially supported by the NNSF of China (Grants nos. 11101373, 11271333, and 61074011).

References

[1] S. A. Kauffman, "Metabolic stability and epigenesis in randomly constructed genetic nets," *Journal of Theoretical Biology*, vol. 22, no. 3, pp. 437–467, 1969.

[2] S. Kauffman, *The Origins of Order: Self-Organization and Selection in Evolution*, Oxford University Press, New York, NY, USA, 1993.

[3] S. Kauffman, *At Home in the Universe*, Oxford University Press, New York, NY, USA, 1995.

[4] D. Cheng, "Semi-tensor product of matrices and its applications: a survey," in *Proceedings of the 4th International Congress of Chinese Mathematicians*, pp. 641–668, Hangzhou, China, December 2007.

[5] S. Huang and D. E. Ingber, "Shape-dependent control of cell growth, differentiation, and apoptosis: Switching between attractors in cell regulatory networks," *Experimental Cell Research*, vol. 261, no. 1, pp. 91–103, 2000.

[6] D. Cheng, "Input-state approach to Boolean networks," *IEEE Transactions on Neural Networks*, vol. 20, no. 3, pp. 512–521, 2009.

[7] D. Cheng, Z. Li, and H. Qi, "Realization of Boolean control networks," *Automatica*, vol. 46, no. 1, pp. 62–69, 2010.

[8] D. Cheng and H. Qi, "Controllability and observability of Boolean control networks," *Automatica*, vol. 45, no. 7, pp. 1659–1667, 2009.

[9] D. Cheng and H. Qi, "A linear representation of dynamics of Boolean networks," *IEEE Transactions on Automatic Control*, vol. 55, no. 10, pp. 2251–2258, 2010.

[10] D. Laschov and M. Margaliot, "Controllability of Boolean control networks via the Perron-Frobenius theory," *Automatica*, vol. 48, no. 6, pp. 1218–1223, 2012.

[11] D. Laschov and M. Margaliot, "A maximum principle for single-input Boolean control networks," *IEEE Transactions on Automatic Control*, vol. 56, no. 4, pp. 913–917, 2011.

[12] D. Laschov and M. Margaliot, "A pontryagin maximum principle for multi-input boolean control networks," in *Recent Advances in Dynamics and Control of Neural Networks*, Cambridge Scientific Publishers, 2011.

[13] Z.-H. Guan, T.-H. Qian, and X. Yu, "Controllability and observability of linear time-varying impulsive systems," *IEEE Transactions on Circuits and Systems I*, vol. 49, no. 8, pp. 1198–1208, 2002.

[14] Y. Liu and S. Zhao, "Controllability for a class of linear time-varying impulsive systems with time delay in control input," *IEEE Transactions on Automatic Control*, vol. 56, no. 2, pp. 395–399, 2011.

[15] Y. Liu and S. Zhao, "Controllability analysis of linear time-varying systems with multiple time delays and impulsive effects," *Nonlinear Analysis: Real World Applications*, vol. 13, no. 2, pp. 558–568, 2012.

[16] G. Xie and L. Wang, "Necessary and sufficient conditions for controllability and observability of switched impulsive control systems," *IEEE Transactions on Automatic Control*, vol. 49, no. 6, pp. 960–966, 2004.

[17] S. Zhao and J. Sun, "Controllability and observability for time-varying switched impulsive controlled systems," *International Journal of Robust and Nonlinear Control*, vol. 20, no. 12, pp. 1313–1325, 2010.

[18] S. Zhao and J. Sun, "A geometric approach for reachability and observability of linear switched impulsive systems," *Nonlinear Analysis*, vol. 72, no. 11, pp. 4221–4229, 2010.

[19] F. Li, J. Sun, and Q. Wu, "Observability of boolean control networks with state time delays," *IEEE Transactions on Neural Networks*, vol. 22, no. 6, pp. 948–954, 2011.

[20] J. Lu, D. W. C. Ho, and J. Kurths, "Consensus over directed static networks with arbitrary finite communication delays," *Physical Review E*, vol. 80, no. 6, Article ID 066121, 2009.

[21] J. Lu, D. W. C. Ho, and J. Cao, "Synchronization in an array of nonlinearly coupled chaotic neural networks with delay coupling," *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, vol. 18, no. 10, pp. 3101–3111, 2008.

[22] S. Lyu, "Combining boolean method with delay times for determining behaviors of biological networks," in *Proceedings of the 31st Annual International Conference of the IEEE Engineering in Medicine and Biology Society*, pp. 4884–4887, Minneapolis, Minn, USA, September 2009.

[23] C. Cotta, "On the evolutionary inference of temporal boolean networks," in *Computational Methods in Neural Modeling*, vol. 2686 of *Lecture Notes in Computer Science*, pp. 494–501, 2003.

[24] C. Fogelberg and V. Palade, "Machine learning and genetic regulatory networks: a review and a roadmap," in *Foundations of Computational, Intelligence*, vol. 201 of *Studies in Computational Intelligence*, pp. 3–34, 2009.

[25] A. Silvescu and V. Honavar, "Temporal Boolean network models of genetic networks and their inference from gene expression time series," *Complex Systems*, vol. 13, no. 1, pp. 61–78, 2001.

[26] D. Cheng, H. Qi, and Z. Li, *Analysis and Control of Boolean Networks: A Semi-Tensor Product Approach*, Springer, New York, NY, USA, 2011.

[27] Y. Liu, J. Lu, and B. Wu, "Some necessary and sufficient conditions for the outputcontrollability of temporal Boolean control networks," *ESAIM: Control, Optimization and Calculus of Variations*. In press.

[28] Y. Liu, H. Chen, and B. Wu, "Controllability of Boolean control networks with impulsive effects and forbidden states," *Mathematical Methods in the Applied Sciences*, 2013.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 132619, 8 pages
http://dx.doi.org/10.1155/2013/132619

Research Article

New Results on Impulsive Functional Differential Equations with Infinite Delays

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Received 28 January 2013; Accepted 6 April 2013

Academic Editor: Jinde Cao

We investigate the stability for a class of impulsive functional differential equations with infinite delays by using Lyapunov functions and Razumikhin-technique. Some new Razumikhin-type theorems on stability are obtained, which shows that impulses do contribute to the system's stability behavior. An example is also given to illustrate the importance of our results.

1. Introduction

Impulsive differential equations have attracted the interest of many researchers in recent years. It arises naturally from a wide variety of applications such as orbital transfer of satellite, ecosystems management, and threshold theory in biology. There has been a significant development in the theory of impulsive differential equations in the past several years ago, and various interesting results have been reported; see [1–4]. Recently, systems with impulses and time delay have received significant attention [5–16]. In fact, the system stability and convergence properties are strongly affected by time delays, which are often encountered in many industrial and natural processes due to measurement and computational delays, transmission, and transport lags. In [5, 6, 8], the authors considered the stability of impulsive differential equations with finite delay and got some results. In [7], by using Lyapunov functions and Razumikhin technique, some Razumikhin-type theorems on stability are obtained for a class of impulsive functional differential equations with infinite-delay. However, not much has been developed in the direction of the stability theory of impulsive functional differential systems, especially for infinite delays impulsive functional differential systems. As we know, there are a number of difficulties that one must face in developing the corresponding theory of impulsive functional differential systems with infinite-delay; for example, the interval $(-\infty, \sigma]$ is not compact, and the images of a solution map of closed and bounded sets in $C((-\infty, 0], R_n)$ space may not be compact. Therefore, it is an interesting and complicated problem to study the stability theory for impulsive functional differential systems with infinite delays.

In the present paper, we will consider the stability of impulsive infinite-delay differential equations by using Lyapunov functions and the Razumikhin technique, we get some new results. The effect of delay and impulses which do contribute to the equations's stability properties will be shown in this paper.

The rest of this paper is organized as follows. In Section 2, we introduce some notations and definitions. In Section 3, we get some criteria for uniform stability and uniform asymptotic stability for impulsive infinite-delay differential equations, and an example is given to illustrate our results. Finally, concluding remarks are given in Section 4.

2. Preliminaries

Let R denote the set of real numbers, R_+ the set of nonnegative real numbers, and R^n the n-dimensional real space equipped with the Euclidean norm $\|\cdot\|$. For any $t \ge t_0 \ge 0 > \alpha \ge -\infty$, let $f(t, x(s))$ where $s \in [t+\alpha, t]$ or $f(t, x(\cdot))$ be a Volterra-type functional. In the case when $\alpha = -\infty$, the interval $[t+\alpha, t]$ is understood to be replaced by $(-\infty, t]$.

We consider the impulsive functional differential equations

$$\begin{aligned} x'(t) &= f(t, x(\cdot)), \quad t \geq t_0,\ t \neq t_k, \\ \Delta x|_{t=t_k} &= x(t_k) - x(t_k^-) \\ &= I_k(x(t_k^-)), \quad k = 1, 2, \ldots, \end{aligned} \tag{1}$$

where the impulse times t_k satisfy $0 \leq t_0 < t_1 < \cdots < t_k < \cdots$, $\lim_{k\to+\infty} t_k = +\infty$ and x' denotes the right-hand derivative of x. $f \in C([t_{k-1}, t_k) \times C,\ R^n)$, $f(t, 0) = 0$. C is an open set in $PC([\alpha, 0], R^n)$, where $PC([\alpha, 0], R^n) = \{\psi : [\alpha, 0] \to R^n$ is continuous everywhere except at finite number of points t_k, at which $\psi(t_k^+)$ and $\psi(t_k^-)$ exist and $\psi(t_k^+) = \psi(t_k)\}$. For each $k = 1, 2, \ldots$, $I(t, x) \in C([t_0, \infty) \times R^n, R^n)$, $I(t_k, 0) = 0$.

For any $\rho > 0$, there exists a $\rho_1 > 0$ $(0 < \rho_1 < \rho)$ such that $x \in S(\rho_1)$ implies that $x + I(t_k, x) \in S(\rho)$, where $S(\rho) = \{x : \|x\| < \rho,\ x \in R^n\}$.

Define $PCB(t) = \{x \in C : x \text{ is bounded}\}$. For $\psi \in PCB(t)$, the norm of ψ is defined by $\|\psi\| = \sup_{\alpha\leq\theta\leq 0}|\psi(\theta)|$. For any $\sigma \geq 0$, let $PCB_\delta(\sigma) = \{\psi \in PCB(\sigma) : \|\psi\| < \delta\}$.

For any given $\sigma \geq t_0$, the initial condition for system (1) is given by

$$x_\sigma = \phi, \tag{2}$$

where $\phi \in PC([\alpha, 0], R^n)$.

We assume that the solution for the initial problems, (1)-(2) does exist and is unique which will be written in the form $x(t, \sigma, \phi)$; see [4, 10]. Since $f(t, 0) = 0$, $I(t_k, 0) = 0$, $k = 1, 2, \ldots$, then $x(t) = 0$ is a solution of (1)-(2), which is called the trivial solution. In this paper, we always assume that the solution $x(t, \sigma, \phi)$ of (1)-(2) can be continued to ∞ from the right of σ.

For convenience, we also have the following classes in later sections:

$K_1 = \{a \in C(R_+, R_+) \mid a(0) = 0 \text{ and } a(s) > 0 \text{ for } s > 0\}$;

$K_2 = \{a \in C(R_+, R_+) \mid a(0) = 0 \text{ and } a(s) > 0 \text{ for } s > 0$ and a is nondecreasing in $s\}$;

$\Delta V(t_k, \psi(0)) = V(t_k, \psi(0) + I_k(t_k, \psi)) - V(t_k^-, \psi(0))$, $k = 1, 2, \ldots$;

$\Delta t_k = t_k - t_{k-1}$, $k = 1, 2, \ldots$.

We introduce some definitions as follows.

Definition 1 (see [4]). The function $V : [\alpha, \infty) \times C \to R_+$ belongs to class v_0 if

(A_1) V is continuous on each of the sets $[t_{k-1}, t_k) \times C$ and $\lim_{(t,\varphi)\to(t_k^-,\psi)} V(t, \varphi) = V(t_k^-, \psi)$ exists;

(A_2) $V(t, x)$ is locally Lipschitzian in x and $V(t, 0) \equiv 0$.

Definition 2 (see [4]). Let $V \in v_0$, for any $(t, \psi) \in [t_{k-1}, t_k) \times C$, the upper right-hand Dini derivative of $V(t, x)$ along the solution of (1)-(2) is defined by

$$D^+V(t, \psi(0)) = \frac{\limsup_{h\to 0^+}\{V(t + h, \psi(0) + hf(t, \psi)) - V(t, \psi(0))\}}{h}. \tag{3}$$

Similarly, we can define $D^-V(t, \psi(0))$, $D\_V(t, \psi(0)$, $D_+V(t, \psi(0))$. If $V \in C'$, then $DV(t, \psi(0)) = \dot{V}(t, \psi(0))$, where D is any of the four Dini derivatives.

For $V \in v_0$, $(t, \psi) \in [t_{k-1}, t_k) \times C$, the upper right-hand Dini derivative of $\dot{V}(t, x)$ along the solution of (1)-(2) is defined by

$$D^+\dot{V}(t, \psi(0)) = \frac{\limsup_{h\to 0^+}\{\dot{V}(t + h, \psi(0) + hf(t, \psi)) - \dot{V}(t, \psi(0))\}}{h}. \tag{4}$$

Similarly, we can define $D^-\dot{V}(t, \psi(0))$. If $V \in C''$, then these are simply the second derivative of V.

Definition 3 (see [4]). Assume $x(t) = x(t, \sigma, \phi)$ to be the solution of (1)-(2) through (σ, ϕ). Then, the zero solution of (1)-(2) is said to be

(1) uniformly stable, if for any $\varepsilon > 0$ and $\sigma \geq t_0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that $\phi \in PCB_\delta(\sigma)$ implies $\|x(t)\| < \varepsilon$, $t \geq \sigma$.

(2) uniformly asymptotically stable, if it is uniformly stable, and there exists a $\delta > 0$ such that for any $\varepsilon > 0$, $\sigma \geq t_0$, there is a $T = T(\varepsilon) > 0$ such that $\phi \in PCB_\delta(\sigma)$ implies $\|x(t)\| < \varepsilon$, $t \geq \sigma + T$.

3. Main Results

Theorem 4. *Assume that there exist functions $w_i \in K_1$, $g \in K_2$, $c_i, p, q \in C(R_+, R_+)$, $V(t, x) \in v_0$, $i = 1, 2$, and constants $m > 1$, such that the following conditions hold:*

(i) $w_1(\|x\|) \leq V(t, x) \leq w_2(\|x\|)$, $(t, x) \in [\alpha, \infty) \times S(\rho)$;

(ii) *for any $\sigma \geq t_0$ and $\psi \in PC([\alpha, 0], S(\rho))$, if $V(t, \psi(0)) \geq m^{-2}g(V(t + \theta, \psi(\theta)))$, $\max\{\alpha, -q(V(t))\} \leq \theta \leq 0$, $t \neq t_k$, then*

$$D^+V(t, \psi(0)) \leq p(t)c_1(V(t, \psi(0))), \tag{5}$$

where $s/m \leq g(s) < s$ for any $s > 0$;

(iii) *for all $(t, \psi(0)) \in (t_{k-1}, t_k) \times PC([\alpha, 0], S(\rho_1))$,*

$$D^-\dot{V}(t, \psi(0)) \geq 0. \tag{6}$$

Also, for all $(t_k, \psi) \in R_+ \times PC([\alpha, 0], S(\rho_1))$,

$$\Delta t_k\dot{V}(t_k^-, \psi(0)) + \Delta V(t_k, \psi(0)) \leq -\mu_k c_2(V(t_k, \psi(0))), \tag{7}$$

where $c_2(s) \leq sc_2'(s)$, $s > 0$, μ_k satisfies $\liminf_{k\to\infty} \mu_k \geq 2 \cdot \sup_{s>0}(s/c_2(m^{-1} \cdot s))$;

(iv) *there exist constants $M_1, M_2 > 0$ such that the following inequalities hold:*

$$\sup_{t\ge 0}\int_t^{t+\tau} p(s)\,ds = M_1 < \infty,$$
$$\inf_{s>0}\int_{g(s)}^{s}\frac{dt}{c_1(t)} = M_2 > M_1, \tag{8}$$

where $\tau = \max_{k\ge 1}\{t_k - t_{k-1}\} < \infty$.

Then, the zero solution of (1)-(2) *is uniformly asymptotically stable.*

Proof. Condition (i) implies that $w_1(s) \le w_2(s)$ for $s \in [0,\rho]$. So let W_1 and W_2 be continuous, strictly increasing functions satisfying $W_1(s) \le w_1(s) \le w_2(s) \le W_2(s)$ for all $s \in [0,\rho]$. Then

$$W_1(\|x\|) \le V(t,x) \le W_2(\|x\|), \quad (t,x) \in [\alpha,\infty)\times S(\rho). \tag{9}$$

We first show uniform stability.

For any $\varepsilon > 0 (< \rho_1)$, one may choose a $\delta = \delta(\varepsilon) > 0$ such that $W_2(\delta) \le g(W_1(\varepsilon))$. Let $x(t) = x(t,\sigma,\phi)$ be a solution of (1)-(2) through (σ,ϕ), $\sigma \ge t_0$. For any $\phi \in \mathrm{PCB}_\delta(\sigma)$, we will prove that $\|x(t)\| < \varepsilon, t \ge \sigma$.

For convenience, let $V(t) = V(t, x(t))$. Suppose that $\sigma \in [t_{l-1}, t_l)$, $l \in Z_+$. First, for $\sigma + \alpha \le t \le \sigma$, we have

$$W_1(\|x\|) \,\&\, \le V(t) < W_2(\delta) \le g(W_1(\varepsilon)) < W_1(\varepsilon). \tag{10}$$

So, $\|x(t)\| < \varepsilon < \rho_1$, $t \in [\sigma+\alpha, \sigma]$.

Next, we claim that

$$V(t) < W_1(\varepsilon), \quad t \in [\sigma, t_l). \tag{11}$$

Suppose on the contrary that there exists some $t \in [\sigma, t_l)$ such that $V(t) \ge W_1(\varepsilon)$. Since $V(\sigma) < W_1(\varepsilon)$, we can define $\hat{t} = \inf\{t \in [\sigma, t_l) \mid V(t) \ge W_1(\varepsilon)\}$. Thus, $\hat{t} \in (\sigma, t_l)$, $V(\hat{t}) = W_1(\varepsilon)$, and $V(t) < W_1(\varepsilon)$, $t \in [\sigma, \hat{t})$. Also, from (10) we obtain

$$V(t) < W_1(\varepsilon), \quad t \in [\sigma+\alpha, \hat{t}). \tag{12}$$

On the other hand, note that $V(\hat{t}) = W_1(\varepsilon) > g(W_1(\varepsilon))$ and $V(\sigma) < g(W_1(\varepsilon))$ in view of (10), we can define $t^* = \sup\{t \in [\sigma, \hat{t}]V(t) \le g(W_1(\varepsilon))\}$; it is obvious that $t^* \in [\sigma, \hat{t}), V(t^*) = g(W_1(\varepsilon))$ and $V(t) > g(W_1(\varepsilon))$ for $t \in (t^*, \hat{t}]$. Therefore, combining (12), we have for $t \in (t^*, \hat{t})$

$$V(t) > g(W_1(\varepsilon)) > g(V(t+\theta)), \quad \alpha \le \theta \le 0; \tag{13}$$

that is,

$$V(t, \psi(0)) > m^{-2} g(V(t+\theta, \psi(\theta))),$$
$$\max\{\alpha, -q(V(t))\} \le \theta \le 0. \tag{14}$$

By assumption (ii), (iv), we have

$$\int_{V(t^*)}^{V(\hat{t})}\frac{ds}{c_1(s)} = \int_{g(W_1(\varepsilon))}^{W_1(\varepsilon)}\frac{ds}{c_1(s)} \ge M_2 > M_1. \tag{15}$$

However, we also have

$$\int_{V(t^*)}^{V(\hat{t})}\frac{ds}{c_1(s)} \le \int_{t^*}^{\hat{t}} p(s)\,ds < \int_{t^*}^{t^*+\tau} p(s)\,ds \le M_1, \tag{16}$$

which is a contradiction. So, (11) holds.

Hence, $W_1(\|x\|) \le V(t) < W_1(\varepsilon)$, $t \in [\sigma, t_l)$ implies that $\|x(t_l^-)\| < \varepsilon < \rho_1$. Thus, $x(t_l) \in S(\rho)$.

On the other hand, from condition (iii), we note for $k = 1, 2, \ldots$,

$$\begin{aligned} V(t_k) - V(t_{k-1}) &= V(t_k) - V(t_k^-) + V(t_k^-) - V(t_{k-1}) \\ &= \Delta V(t_k) + \int_{t_{k-1}}^{t_k} \dot{V}(t)\,dt \\ &\le \Delta V(t_k) + \Delta t_k \dot{V}(t_k^-) \\ &\le -\mu_k c_2(V(t_k)) \le 0. \end{aligned} \tag{17}$$

Hence, we obtain $V(t_k) \le V(t_{k-1})$, $k = 1, 2, \ldots$. particularly, $V(t_l) \le V(t_{l-1})$. In view of (10), we get

$$V(t_l) \le V(t_{l-1}) < g(W_1(\varepsilon)) < W_1(\varepsilon). \tag{18}$$

Next, we claim that

$$V(t) < W_1(\varepsilon), \quad t \in [t_l, t_{l+1}). \tag{19}$$

Suppose on the contrary that there exists some $t \in [t_l, t_{l+1})$ such that $V(t) \ge W_1(\varepsilon)$. Then applying exactly the same argument as in the proof of (11) yields our desired contradiction.

By induction hypothesis, we may prove, in general, that for $t \in [t_{l+k}, t_{l+k+1})$, $k > 0$,

$$V(t) < W_1(\varepsilon). \tag{20}$$

In view of condition (i), we obtain that $\|x(t)\| < \varepsilon, t \ge \sigma$. Therefore, we have proved that the solutions of (1)-(2) are uniformly stable.

Next, we claim that they are uniformly asymptotically stable. Since the zero solution of (1)-(2) is uniformly stable, for any given constant $H > 0$ $(< \rho_1)$, then there exists $\delta > 0$ such that $\phi \in \mathrm{PCB}_\delta(\sigma)$ implies that $V(t) < W_1(H)$, $\|x(t)\| < \rho_1$, $t \ge \sigma$.

For any $\varepsilon \in (0, H)$, let

$$\begin{aligned} d &< \min\{\hat{d}, W_1(\varepsilon)\}, \\ \hat{d} &= \inf\{s - g(s) \mid m^{-1}W_1(\varepsilon) \le s \le W_1(H)\}, \\ h &= \sup\{q(s) \mid m^{-1}W_1(\varepsilon) \le s \le W_1(H)\}, \\ n_0 &= \frac{W_1(H)}{2\cdot \sup_{s>0}(s/c_2(m^{-1}s))\, c_2(m^{-1}W_1(\varepsilon))} + 1. \end{aligned} \tag{21}$$

From condition (iii), we get that there exists a $n_1 > 0$ such that for $k > n_1$,

$$\mu_k \ge 2\cdot \sup_{s>0}\frac{s}{c_2(m^{-1}\cdot s)}. \tag{22}$$

Choose a positive integer N satisfying

$$W_1(\varepsilon) + (N-1)d < W_1(H) \le W_1(\varepsilon) + Nd, \tag{23}$$

and define $T = N(h + n_0\tau) + n_1$, we will prove that $\phi \in \mathrm{PCB}_\delta(\sigma)$ implies $\|x(t)\| < \varepsilon$, $t \ge \sigma + T$.

First, we prove that there exists $\widehat{t} \in [\sigma+h+n_1, \sigma+h+n_1+n_0\tau]$ such that

$$V(\widehat{t}) < m^{-1}[W_1(\varepsilon) + (N-1)d]. \tag{24}$$

Suppose on the contrary that for all $t \in [\sigma + h + n_1, \sigma + h + n_1 + n_0\tau]$,

$$V(t) \ge m^{-1}[W_1(\varepsilon) + (N-1)d] \ge m^{-1}W_1(\varepsilon). \tag{25}$$

Let $t_{k_1} = \min\{t_k : t_k \ge \sigma + h + n_1\}$, from (17), we get

$$\begin{aligned} V(t_{k_1}) - V(t_{k_1-1}) &\le -\mu_{k_1} c_2(V(t_{k_1})) \\ &\le -\mu_{k_1} c_2(m^{-1}W_1(\varepsilon)), \\ V(t_{k_1+1}) - V(t_{k_1}) &\le -\mu_{k_1+1} c_2(m^{-1}W_1(\varepsilon)), \\ &\vdots \\ V(t_{k_1+n_0}) - V(t_{k_1+n_0-1}) &\le -\mu_{k_1+n_0} c_2(m^{-1}W_1(\varepsilon)), \end{aligned} \tag{26}$$

In general, combining (22), we deduce that

$$\begin{aligned} V(t_{k_1+n_0}) &\le V(t_{k_1-1}) - \sum_{s=0}^{n_0} \mu_{k_1+s} c_2(m^{-1}W_1(\varepsilon)) \\ &\le W_1(H) - 2(n_0+1) \\ &\quad \cdot \sup_{s>0} \frac{s}{c_2(m^{-1}s)} c_2(m^{-1}W_1(\varepsilon)) \\ &= -4 \cdot \sup_{s>0} \frac{s}{c_2(m^{-1}s)} c_2(m^{-1}W_1(\varepsilon)) < 0, \end{aligned} \tag{27}$$

which is a contradiction. So, (24) holds.

Suppose $\widehat{t} \in [t_{l-1}, t_l)$, $l > 1$. Furthermore, we can prove that for $t \in [\widehat{t}, t_l)$

$$V(t) < W_1(\varepsilon) + (N-1)d. \tag{28}$$

Suppose this assertion is false, then there exists some $t \in [\widehat{t}, t_l)$ such that $V(t) \ge W_1(\varepsilon) + (N-1)d$. Since $V(\widehat{t}) < m^{-1}[W_1(\varepsilon) + (N-1)d] < W_1(\varepsilon) + (N-1)d$, so define

$$t^* = \inf\{t \in [\widehat{t}, t_l) \mid V(t) \ge W_1(\varepsilon) + (N-1)d\}; \tag{29}$$

then $t^* \in (\widehat{t}, t_l)$, $V(t^*) = W_1(\varepsilon) + (N-1)d$ and $V(t) < W_1(\varepsilon) + (N-1)d$, $t \in (\widehat{t}, t^*)$. Note that

$$\begin{gathered} V(t^*) = W_1(\varepsilon) + (N-1)d > g(W_1(\varepsilon) + (N-1)d), \\ V(\widehat{t}) < m^{-1}[W_1(\varepsilon) + (N-1)d] < g(W_1(\varepsilon) + (N-1)d); \end{gathered} \tag{30}$$

thus, we can define

$$\bar{t} = \sup\{t \in [\widehat{t}, t^*] \mid V(t) \le g(W_1(\varepsilon) + (N-1)d)\}, \tag{31}$$

then $\bar{t} \in [\widehat{t}, t^*)$, $V(\bar{t}) = g(W_1(\varepsilon) + (N-1)d)$ and $V(t) > g(W_1(\varepsilon) + (N-1)d)$ for $t \in (\bar{t}, t^*]$.

Hence, we get for $t \in (\bar{t}, t^*]$

$$\begin{aligned} V(t) &> g(W_1(\varepsilon) + (N-1)d) \\ &\ge m^{-1}[W_1(\varepsilon) + (N-1)d] \\ &\ge m^{-1}W_1(\varepsilon), \end{aligned} \tag{32}$$

which implies that for $t \in (\bar{t}, t^*]$

$$\begin{aligned} V(t) &\ge g(V(t)) + d \ge m^{-1}V(t) + d \\ &> \frac{mV(t)}{m^2} + \frac{d}{m^2} \ge \frac{W_1(\varepsilon) + Nd}{m^2} \\ &\ge \frac{W_1(H)}{m^2} \ge \frac{V(s)}{m^2} > \frac{g(V(s))}{m^2}, \quad t + \alpha < s \le t. \end{aligned} \tag{33}$$

Thus, $V(t) \ge (1/m^2)g(V(t+\theta, \psi(\theta)))$, $\max\{\alpha, -q(V(t))\} \le \theta \le 0$.

By assumption, (ii), (iv), we have for $t \in (\bar{t}, t^*)$,

$$\int_{V(\bar{t})}^{V(t^*)} \frac{ds}{c_1(s)} = \int_{g(W_1(\varepsilon)+(N-1)d)}^{W_1(\varepsilon)+(N-1)d} \frac{ds}{c_1(s)} \ge M_2 > M_1. \tag{34}$$

However, we also have

$$\int_{V(\bar{t})}^{V(t^*)} \frac{ds}{c_1(s)} < \int_{\bar{t}}^{t^*} p(s)\,ds < \int_{\bar{t}}^{\bar{t}+\tau} p(s)\,ds < M_1, \tag{35}$$

which is a contradiction. So, (28) holds.

On the other hand, it is easy to prove that the functions $s/c_2(m^{-1}s)$ are nonincreasing for $s \in (0, +\infty)$ in view of condition $c_2(s) \le sc_2'(s)$ for any $s > 0$.

Hence, the following inequalities hold: for $k > n_1$,

$$\begin{aligned} \frac{W_1(\varepsilon) + (N-i)d}{c_2(m^{-1}(W_1(\varepsilon) + (N-i-1)d))} &\le \frac{W_1(\varepsilon) + d}{c_2(m^{-1}W_1(\varepsilon))} \\ &< \frac{2W_1(\varepsilon)}{c_2(m^{-1}W_1(\varepsilon))} \\ &\le \mu_k, \quad i = 1, 2, \ldots, N-1. \end{aligned} \tag{36}$$

Next, we claim that

$$V(t_l) < m^{-1}[W_1(\varepsilon) + (N-1)d]. \tag{37}$$

Or else, then $V(t_l) \ge m^{-1}[W_1(\varepsilon) + (N-1)d]$; from (17), we get

$$\begin{aligned} V(t_l) - V(t_{l-1}) &\le -\mu_l c_2(V(t_l)) \\ &\le -\mu_l c_2(m^{-1}[W_1(\varepsilon) + (N-1)d]). \end{aligned} \tag{38}$$

Considering (36), it holds that

$$\begin{aligned} V(t_l) &\le V(t_{l-1}) - \mu_l c_2\left(m^{-1}\left[W_1(\varepsilon) + (N-1)d\right]\right) \\ &\le W_1(H) - \mu_l c_2\left(m^{-1}\left[W_1(\varepsilon) + (N-1)d\right]\right) \\ &\le W_1(\varepsilon) + Nd - \mu_l c_2\left(m^{-1}\left[W_1(\varepsilon) + (N-1)d\right]\right) \\ &\le c_2\left(m^{-1}\left[W_1(\varepsilon) + (N-1)d\right]\right) \\ &\quad \times \left\{\frac{W_1(\varepsilon) + Nd}{c_2\left(m^{-1}\left[W_1(\varepsilon) + (N-1)d\right]\right)} - \mu_l\right\} \\ &< 0, \end{aligned} \tag{39}$$

which is a contradiction and (37) holds.

Next, we can prove that for $t \in [t_l, t_{l+1})$

$$V(t) < W_1(\varepsilon) + (N-1)d. \tag{40}$$

Suppose that this assertion is false, then there exists some $t \in [\hat{t}, t_l)$ such that $V(t) \ge W_1(\varepsilon) + (N-1)d$. Then applying exactly the same argument as in the proof of (24) and (28) yields our desired contradiction. Here, we omit it.

By induction hypothesis, we may prove, for $t \in [t_{l+k}, t_{l+k+1})$, $k = 1, 2, \ldots$,

$$V(t) < W_1(\varepsilon) + (N-1)d; \tag{41}$$

that is,

$$V(t) < W_1(\varepsilon) + (N-1)d, \quad t \ge \hat{t}. \tag{42}$$

Hence, we obtain

$$V(t) < W_1(\varepsilon) + (N-1)d, \quad t \ge \sigma + h + n_1 + n_0\tau. \tag{43}$$

Next, we prove that there exists $\hat{t}_2 \in [\sigma + 2h + n_1 + n_0\tau, \sigma + 2h + n_1 + 2n_0\tau]$ such that

$$V(\hat{t}_2) < m^{-1}\left[W_1(\varepsilon) + (N-2)d\right]. \tag{44}$$

Suppose that for all $t \in [\sigma + 2h + n_1 + n_0\tau, \sigma + 2h + n_1 + 2n_0\tau]$,

$$V(t) \ge m^{-1}\left[W_1(\varepsilon) + (N-2)d\right] \ge m^{-1}W_1(\varepsilon). \tag{45}$$

Using the same argument as in the proof of (24), we get

$$\begin{aligned} V\left(t_{k_2+n_0}\right) &\le V\left(t_{k_2-1}\right) - \sum_{s=0}^{n_0}\mu_{k_2+s}c_2\left(m^{-1}W_1(\varepsilon)\right) \\ &\le W_1(H) - 2(n_0+1) \\ &\quad \cdot \sup_{s>0}\frac{s}{c_2(m^{-1}s)}c_2\left(m^{-1}W_1(\varepsilon)\right) \\ &= -4 \cdot \sup_{s>0}\frac{s}{c_2(m^{-1}s)}c_2\left(m^{-1}W_1(\varepsilon)\right) < 0, \end{aligned} \tag{46}$$

where $t_{k_2} = \min\{t_k : k \ge \sigma + 2h + n_1 + n_0\tau\}$.

This is a contradiction. So, (44) holds.

Suppose $\hat{t}_2 \in [t_{k-1}, t_k)$, $k > l$. Furthermore, we claim that for $t \in [\hat{t}_2, t_k)$

$$V(t) < W_1(\varepsilon) + (N-2)d. \tag{47}$$

Suppose on the contrary, that there exists some $t \in [\hat{t}_2, t_k)$ such that $V(t) \ge W_1(\varepsilon) + (N-2)d$. We define

$$t^\star = \inf\left\{t \in [\hat{t}_2, t_k) \mid V(t) \ge W_1(\varepsilon) + (N-2)d\right\}, \tag{48}$$

since $V(\hat{t}_2) < m^{-1}[W_1(\varepsilon) + (N-2)d] < W_1(\varepsilon) + (N-2)d$ in view of (44). Thus, $t^\star \in (\hat{t}_2, t_k)$, $V(t^\star) = W_1(\varepsilon) + (N-2)d$ and $V(t) < W_1(\varepsilon) + (N-2)d$, $t \in (\hat{t}_2, t^\star)$. Note that

$$\begin{aligned} V(t^\star) &= W_1(\varepsilon) + (N-2)d > g\left(W_1(\varepsilon) + (N-2)d\right), \\ V(\hat{t}_2) &< m^{-1}\left[W_1(\varepsilon) + (N-2)d\right] < g\left(W_1(\varepsilon) + (N-2)d\right); \end{aligned} \tag{49}$$

furthermore, we can define

$$\tilde{t} = \sup\left\{t \in [\hat{t}_2, t^\star] \mid V(t) \le g\left(W_1(\varepsilon) + (N-2)d\right)\right\}, \tag{50}$$

then $\tilde{t} \in [\hat{t}_2, t^\star)$, $V(\tilde{t}) = g(W_1(\varepsilon) + (N-2)d)$ and $V(t) > g(W_1(\varepsilon) + (N-2)d)$ for $t \in (\tilde{t}, t^\star]$.

Hence, we get for $t \in (\tilde{t}, t^\star]$

$$\begin{aligned} V(t) &> g\left(W_1(\varepsilon) + (N-2)d\right) \\ &\ge m^{-1}\left[W_1(\varepsilon) + (N-2)d\right] \\ &\ge m^{-1}W_1(\varepsilon); \end{aligned} \tag{51}$$

considering the definition of d and (43), we get for $t \in (\tilde{t}, t^\star]$

$$\begin{aligned} V(t) &\ge g(V(t)) + d \ge m^{-1}V(t) + d \\ &> \frac{mV(t)}{m^2} + \frac{d}{m^2} \ge \frac{W_1(\varepsilon) + (N-1)d}{m^2} \\ &\ge \frac{V(s)}{m^2} > \frac{g(V(s))}{m^2}, \quad t - h < s \le t. \end{aligned} \tag{52}$$

Thus, $V(t) \ge (1/m^2)g(V(t+\theta, \psi(\theta)))$, $\max\{\alpha, -q(V(t))\} \le \theta \le 0$.

Using assumptions (ii), (iv), we have

$$\int_{V(\tilde{t})}^{V(t^\star)}\frac{ds}{c_1(s)} = \int_{g(W_1(\varepsilon)+(N-2)d)}^{W_1(\varepsilon)+(N-2)d}\frac{ds}{c_1(s)} \ge M_2 > M_1. \tag{53}$$

However,

$$\int_{V(\tilde{t})}^{V(t^\star)}\frac{ds}{c_1(s)} < \int_{\tilde{t}}^{t^\star}p(s)\,ds < \int_{\tilde{t}}^{\tilde{t}+\tau}p(s)\,ds < M_1, \tag{54}$$

giving us a contradiction. So, (47) holds.

Next, we claim that

$$\begin{aligned} V(t_l) &< m^{-1}\left[W_1(\varepsilon) + (N-1)d\right], \\ V(t) &< W_1(\varepsilon) + (N-1)d, \quad t \in [t_l, t_{l+1}), \end{aligned} \tag{55}$$

whose arguments are the same as was employed in the proof of (36), (37). there we omit it.

Repeating this process, it is easy to check that

$$V(t) < W_1(\varepsilon) + (N-2)d, \quad t \ge \sigma + 2h + n_1 + 2n_0\tau. \quad (56)$$

By induction hypothesis, we have

$$V(t) \le W_1(\varepsilon) + (N-i)d, \quad t \ge \sigma + ih + n_1 + in_0\tau. \quad (57)$$

Let $i = N$, then for $t \ge \sigma + N(h + n_0\tau) + n_1$,

$$V(t) < W_1(\varepsilon). \quad (58)$$

Therefore, we arrive at $\|x(t)\| < \varepsilon$, $t \ge T$. The proof of Theorem 4 is complete. □

Corollary 5. *Assume that there exist functions $w_i \in K_1$, $g \in K_2$, $c, p \in C(R_+, R_+)$, $V(t,x) \in \nu_0$, $i = 1,2$, and constants $m > 1$, such that the following conditions hold:*

(i) $w_1(\|x\|) \le V(t,x) \le w_2(\|x\|)$, $(t,x) \in [\alpha,\infty) \times S(\rho)$;

(ii) *for any $\sigma \ge t_0$ and $\psi \in PC([\alpha,0], S(\rho))$, if $V(t,\psi(0)) \ge g(V(t+\theta,\psi(\theta)))$, $\alpha \le \theta \le 0$, $t \ne t_k$, then*

$$D^+V(t,\psi(0)) \le p(t)\,c(V(t,\psi(0))), \quad (59)$$

where $(s/m) \le g(s) < s$ for any $s > 0$;

(iii) *for all $(t,\psi(0)) \in (t_{k-1},t_k) \times PC([\alpha,0], S(\rho_1))$,*

$$D^-\dot V(t,\psi(0)) \ge 0. \quad (60)$$

Also, for all $(t_k,\psi) \in R_+ \times PC([\alpha,0], S(\rho_1))$,

$$\Delta t_k \dot V(t_k^-,\psi(0)) + \Delta V(t_k,\psi(0)) \le 0; \quad (61)$$

(iv) *there exist constants $M_1, M_2 > 0$ such that the following inequalities hold:*

$$\sup_{t\ge 0}\int_t^{t+\tau} p(s)\,ds = M_1 < \infty,$$
$$\inf_{s>0}\int_{g(s)}^{s}\frac{dt}{c(t)} = M_2 > M_1, \quad (62)$$

where $\tau = \max_{k\ge1}\{t_k - t_{k-1}\} < \infty$.

Then the zero solution of (1)-(2) is uniformly stable.

Theorem 4 has a dual result when $\dot V$ is nonincreasing on (t_{k-1},t_k). Here, we only give the results whose proof is very similar to the proof of Theorem 4.

Theorem 6. *Assume that there exist functions $w_i \in K_1$, $g \in K_2$, c_i, p, $q \in C(R_+, R_+)$, $V(t,x) \in \nu_0$, $i = 1,2$, and constants $m > 1$, such that the following conditions hold:*

(i) $w_1(\|x\|) \le V(t,x) \le w_2(\|x\|)$, $(t,x) \in [\alpha,\infty) \times S(\rho)$;

(ii) *for any $\sigma \ge t_0$ and $\psi \in PC([\alpha,0], S(\rho))$, if $V(t,\psi(0)) \ge m^{-2}g(V(t+\theta,\psi(\theta)))$, $\max\{\alpha, -q(V(t))\} \le \theta \le 0$, $t \ne t_k$, then*

$$D^+V(t,\psi(0)) \le p(t)\,c_1(V(t,\psi(0))), \quad (63)$$

where $(s/m) \le g(s) < s$ for any $s > 0$;

(iii) *for all $(t,\psi(0)) \in (t_{k-1},t_k) \times PC([\alpha,0], S(\rho_1))$,*

$$D^-\dot V(t,\psi(0)) \le 0. \quad (64)$$

Also, for all $(t_k,\psi) \in R_+ \times PC([\alpha,0], S(\rho_1))$,

$$\Delta t_k \dot V(t_{k-1}^-,\psi(0)) + \Delta V(t_k,\psi(0)) \le -\mu_k c_2(V(t_k,\psi(0))), \quad (65)$$

where $c_2(s) \le s c_2'(s)$, $s > 0$, μ_k satisfies $\liminf_{k\to\infty}\mu_k \ge 2\cdot\sup_{s>0}(s/c_2(m^{-1}\cdot s))$;

(iv) *there exist constants $M_1, M_2 > 0$ such that the following inequalities hold:*

$$\sup_{t\ge 0}\int_t^{t+\tau} p(s)\,ds = M_1 < \infty, \quad (66)$$

$$\inf_{s>0}\int_{g(s)}^{s}\frac{dt}{c_1(t)} = M_2 > M_1, \quad (67)$$

where $\tau = \max_{k\ge1}\{t_k - t_{k-1}\} < \infty$.

Then the zero solution of (1)-(2) is uniformly asymptotically stable.

Example 7. Consider the impulsive delay differential equations:

$$\begin{gathered} x'(t) = ax(t) - b\int_{-\infty}^{0} e^s x(t+s)\,ds, \quad t \ge 0,\ t \ne t_k, \\ \Delta x|_{t=t_k} = I_k(x), \quad k = 1,2,\ldots, \\ x_0 = \phi > 0, \end{gathered} \quad (68)$$

where $a \in (0,3]$, $b \in (0,2]$, $|x + I_k(x)| \le \sqrt{\lambda}\cdot|x|$, $k = 1,2,\ldots,\lambda \in (0,1)$. For any given $\phi > 0$, we always suppose that (68) has and only has positive solutions, and assume without loss of generality that $x(t) = x(t,0,\phi)$ is a solutions of (68) through $(0,\phi)$. Suppose that there exists $m > 1$ such that the following inequalities hold:

$$\tau < \min\left\{\frac{\ln m}{2(a - b\sqrt{m})}, \frac{1-\lambda-2\lambda m}{2a}\right\}, \quad a > 2b - 1, \quad (69)$$

where $\tau = \max_{k\ge1}\{t_k - t_{k-1}\} < \infty$. Then, the zero solution of (68) is uniformly asymptotically stable.

In fact, let $V(t,x) = x^2/2$, $g(s) = m^{-1}s (m > 1)$, and $c_1(s) = s$ then $V(t,x(t)) > g(V(s,x(s)))$, $-\infty \le s \le t$ implies that $\sqrt{m}\cdot|x(t)| > |x(s)|$, $-\infty \le s \le t$. Thus, for $t \ne t_k$

$$\begin{aligned} D^+V(t,x(\cdot)) &= x(t)x'(t) \\ &= x(t)\left\{ax(t) - b\int_{-\infty}^{0} e^s x(t+s)\,ds\right\} \\ &\le x^2(t)\left\{a - b\sqrt{m}\int_{-\infty}^{0} e^s ds\right\} \\ &= x^2(t)\{a - b\sqrt{m}\} \\ &= p(t)V(t,x(t)), \end{aligned} \quad (70)$$

where $p(t) = 2(a - b\sqrt{m})$.

In view of condition (69), we note

$$\sup_{t\geq 0}\int_t^{t+\tau} p(s)\,ds = 2\tau\left(a-b\sqrt{m}\right) < \ln m = \inf_{s>0}\int_{g(s)}^{s}\frac{dt}{c_1(t)}. \tag{71}$$

So, condition (iv) in Corollary 5 holds.

On the other hand, we have for $t\neq t_k$

$$\begin{aligned}
D^-\dot{V}(t,x(\cdot)) &= \left(x(t)x'(t)\right)' \\
&= x(t)x''(t) + \left(x'(t)\right)^2 \\
&= \left(x'(t)\right)^2 \\
&\quad + x(t)\left(ax(t) - b\int_{-\infty}^{0} e^s x(t+s)\,ds\right)' \\
&= \left(x'(t)\right)^2 + ax(t)x'(t) \\
&\quad - bx(t)\int_{-\infty}^{0} e^s x(t+s)\,ds - bx^2(t) \\
&= \left(x'(t)\right)^2 + ax(t)x'(t) + ax^2(t) \\
&\quad - x(t)x'(t) - bx^2(t) \\
&= \left(x'(t)\right)^2 + (a-1)x(t)x'(t) \\
&\quad + (a-b)x^2(t) \\
&\geq \left(x'(t)\right)^2 - \frac{\left(x'(t)\right)^2 + x^2(t)}{2}(a-1) \\
&\quad + (a-b)x^2(t) \\
&= \frac{3-a}{2}\left(x'(t)\right)^2 + \left(\frac{a+1}{2} - b\right)x^2(t) \\
&\geq 0,
\end{aligned} \tag{72}$$

in view of condition $a > 2b - 1$. Also, considering $x(t)$ to be a positive solution of (68), we get

$$\begin{aligned}
&\Delta t_k \dot{V}\left(t_k^-,\psi(0)\right) + \Delta V\left(t_k,\psi(0)\right) \\
&\quad \leq a\tau x^2\left(t_k^-\right) + (\lambda - 1)\frac{x^2\left(t_k^-\right)}{2} \\
&\quad = (2a\tau + \lambda - 1)x^2\left(t_k^-\right)\frac{x^2\left(t_k^-\right)}{2} \\
&\quad = (2a\tau + \lambda - 1)V\left(t_k^-\right) \\
&\quad \leq -\frac{1 - 2a\tau - \lambda}{\lambda}V\left(t_k\right) \\
&\quad = -\mu_k c_2\left(V\left(t_k,\psi(0)\right)\right),
\end{aligned} \tag{73}$$

where $c_2 = s$, $\mu_k = (1 - 2a\tau - \lambda)/\lambda$.

Note that

$$\sup_{s>0}\frac{2s}{c_2\left(m^{-1}\cdot s\right)} = 2m < \frac{1-2a\tau-\lambda}{\lambda} = \mu_k, \tag{74}$$

in view of (69). So, the zero solution of (68) is uniformly stable by Corollary 5.

Furthermore, choose $q(s) = -\ln(1-1/m)$ (positive constants), which implies that $\int_{-q(s)}^{0} e^s ds = m^{-1}$. On the other hand, since $V(t,x(t)) > m^{-2}g(V(s,x(s)))$, $\max\{\alpha, t - q(V(t))\} \leq s \leq t$, implying that $m^{3/2}|x(t)| > |x(s)|$, $\max\{\alpha, t - q(V(t))\} \leq s \leq t$, then

$$\begin{aligned}
&D^+V\big|_{(68)}(t,x(\cdot)) \\
&\quad \leq ax^2(t) - bx(t)\int_{-\infty}^{0} e^s\left|x(t+s)\right|ds \\
&\quad \leq ax^2(t) - bx(t)\int_{-\infty}^{t} e^{s-t}\left|x(s)\right|ds \\
&\quad \leq ax^2(t) - bx(t)\int_{t-q(V(t,x(\cdot)))}^{t} e^{s-t}\left|x(s)\right|ds \\
&\qquad - x(t)\int_{-\infty}^{t-q(V(t,x(\cdot)))} e^{s-t}\left|x(s)\right|ds \\
&\quad \leq ax^2(t) - bx(t)\int_{t-q(V(t,x(\cdot)))}^{t} e^{s-t}\left|x(s)\right|ds \\
&\quad \leq x^2(t)\left\{a - bm^{3/2}\int_{-q(V(t,x(\cdot)))}^{0} e^s ds\right\} \\
&\quad \leq x^2(t)\left\{a - b\sqrt{m}\right\} \\
&\quad = c\left(V\left(t,x(t)\right)\right)p(t).
\end{aligned} \tag{75}$$

By Theorem 4, we obtain that if (69) holds, then the zero solution of (68) is uniformly asymptotically stable.

Remark 8. In fact, $x(t) = \phi(0)e^t$ is a positive solution of (68) through $(0,\phi)$ in the absence of impulses. It is obvious that the solution is unstable. However, the solution is uniformly asymptotically stable under proper impulses effect, which shows that impulses do contribute to the system's stability behavior.

4. Conclusion

In this work, we have considered the stability of impulsive infinite-delay differential systems. By using Lyapunov functions and the Razumikhin technique, we have obtained some new results. We can see that impulses and delay do contribute to the system's stability behavior.

References

[1] A. A. Soliman, "Stability criteria of impulsive differential systems," *Applied Mathematics and Computation*, vol. 134, no. 2-3, pp. 445–457, 2003.

[2] D. D. Bainov and P. S. Simennov, *Systems with Impulsive Effect Stability*, Theory and Applications, Ellis Horwood, New York, NY, USA, 1989.

[3] T. Yang, *Impulsive Systems and Control*, Theory and Applications, Nova Science, Huntington, NY, USA, 2001.

[4] X. Fu, B. Yan, and Y. Liu, *Introduction of Impulsive Differential Systems*, Science Press, Beijing, China, 2005.

[5] Y. Xing and M. Han, "A new approach to stability of impulsive functional differential equations," *Applied Mathematics and Computation*, vol. 151, no. 3, pp. 835–847, 2004.

[6] J. Shen and J. Yan, "Razumikhin type stability theorems for impulsive functional-differential equations," *Nonlinear Analysis. Theory, Methods & Applications A*, vol. 33, no. 5, pp. 519–531, 1998.

[7] Z. G. Luo and J. H. Shen, "Impulsive stabilization of functional differential equations with infinite delays," *Applied Mathematics Letters*, vol. 16, no. 5, pp. 695–701, 2003.

[8] Z. G. Luo and J. H. Shen, "New Razumikhin type theorems for impulsive functional differential equations," *Applied Mathematics and Computation*, vol. 125, no. 2-3, pp. 375–386, 2002.

[9] Q. Wang and X. Liu, "Exponential stability for impulsive delay differential equations by Razumikhin method," *Journal of Mathematical Analysis and Applications*, vol. 309, no. 2, pp. 462–473, 2005.

[10] J. Shen, "Existence and uniqueness of solutions for a class of infinite delay functional differential equations with applications to impulsive differential equations," *Journal of Huaihua Teachers College*, vol. 15, pp. 45–451, 1996.

[11] Y. Zhang and J. Sun, "Stability of impulsive infinite delay differential equations," *Applied Mathematics Letters*, vol. 19, no. 10, pp. 1100–1106, 2006.

[12] Y. Zhang and J. Sun, "Stability of impulsive functional differential equations," *Nonlinear Analysis. Theory, Methods & Applications A*, vol. 68, no. 12, pp. 3665–3678, 2008.

[13] X. Li, "New results on global exponential stabilization of impulsive functional differential equations with infinite delays or finite delays," *Nonlinear Analysis. Real World Applications*, vol. 11, no. 5, pp. 4194–4201, 2010.

[14] X. Li, "Further analysis on uniform stability of impulsive infinite delay differential equations," *Applied Mathematics Letters*, vol. 25, no. 2, pp. 133–137, 2012.

[15] X. Li, "Uniform asymptotic stability and global stabiliy of impulsive infinite delay differential equations," *Nonlinear Analysis. Theory, Methods & Applications A*, vol. 70, no. 5, pp. 1975–1983, 2009.

[16] X. Li, H. Akca, and X. Fu, "Uniform stability of impulsive infinite delay differential equations with applications to systems with integral impulsive conditions," *Applied Mathematics and Computation*, vol. 219, no. 14, pp. 7329–7337, 2013.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 402031, 12 pages
http://dx.doi.org/10.1155/2013/402031

Research Article

Synchronization in Array of Coupled Neural Networks with Unbounded Distributed Delay and Limited Transmission Efficiency

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Received 25 January 2013; Accepted 11 March 2013

Academic Editor: Qi Luo

This paper investigates global synchronization in an array of coupled neural networks with time-varying delays and unbounded distributed delays. In the coupled neural networks, limited transmission efficiency between coupled nodes, which makes the model more practical, is considered. Based on a novel integral inequality and the Lyapunov functional method, sufficient synchronization criteria are derived. The derived synchronization criteria are formulated by linear matrix inequalities (LMIs) and can be easily verified by using Matlab LMI Toolbox. It is displayed that, when some of the transmission efficiencies are limited, the dynamics of the synchronized state are different from those of the isolated node. Furthermore, the transmission efficiency and inner coupling matrices between nodes play important roles in the final synchronized state. The derivative of the time-varying delay can be any given value, and the time-varying delay can be unbounded. The outer-coupling matrices can be symmetric or asymmetric. Numerical simulations are finally given to demonstrate the effectiveness of the theoretical results.

1. Introduction

In the past few decades, the problem of chaos synchronization and network synchronization has been extensively studied since its potential engineering applications such as communication, biological systems, and information processing (see [1–4] and the references therein). It is found out that neural networks can exhibit chaotic behavior as long as their parameters and delays are properly chosen [5]. Recently, synchronization of coupled chaotic neural networks has received much attention due to its wide applications in many areas [6–12].

An array of coupled neural networks, as a special class of complex networks [12–16], has received increasing attention from researchers of different disciplines. In the literature, synchronization in an array of coupled neural networks has been extensively studied [8, 17–20]. The authors of [8] studied the exponential synchronization problem for coupled neural networks with constant time delay and stochastic noise perturbations. Some novel H_∞ synchronization results have been obtained in [21] for a class of discrete time-varying stochastic networks over a finite horizon. In [18–20, 22], several types of synchronization in dynamical networks with discrete and bounded distributed delays were studied based on LMI approach. However, most of the obtained results concerning synchronization of complex networks including the above-mentioned implicitly assume that the connections among nodes can transmit information from the dispatcher nodes to receiver ones according to the expected effect. In other words, the transmission efficiencies between connected nodes are all perfect. In practical situations, signal transmission efficiency between nodes is limited in general due to either the limited bandwidth of the channels or external causes such as uncertain noisy perturbations and artificial factors. If the transmission efficiency of some connections in a complex network is limited, then most of the existing synchronization criteria are not applicable. Consequently, it is urgent to propose new synchronization criteria for complex networks with arbitrary transmission efficiency.

Time delays usually exist in neural networks. Some papers concerning synchronization of neural networks have considered various time delays. In [6], Cao and Lu investigated the adaptive synchronization of neural networks with or without time-varying delay. In [23], synchronization of neural networks with discrete and bounded distributed time-varying delays was investigated. The authors of [8] studied the exponential synchronization problem for coupled neural networks with constant time delay. Synchronization of coupled neural networks with both discrete and bounded distributed delays was studied in [11, 18–20, 24]. As pointed out in [25], bounded distributed delay means that there is a distribution of propagation delays only over a period of time. At the same time, unbounded distributed delay implies that the distant past has less influence compared to the recent behavior of the state [26]. Note that most existing results on stability or synchronization of neural networks with bounded distributed delays obtained by using LMI approach cannot be directly extended to those with unbounded distributed delays. Although there were some results on stability or synchronization of neural networks with unbounded distributed delays, some of them were obtained by using algebra approach [27–30]. As is well known, compared with LMI result, algebraic one is more conservative, and criteria in terms of LMI can be easily checked by using the powerful Matlab LMI Toolbox. Therefore, in this paper we investigate the synchronization in an array of coupled neural networks with both discrete time-varying delays and unbounded distributed delays based on LMI approach. Results of the present paper are also applicable to synchronization of complex networks with bounded or unbounded distributed time delay.

Motivated by the above analysis, this paper studies the synchronization in an array neural network with both time-varying delays and unbounded distributed delays, under the condition that the transmission efficiencies among nodes are limited. By using a new lemma on infinite integral inequality and the Lyanupov functional method, some synchronization criteria formulated by LMIs are obtained for the considered model. In the obtained synchronization criteria, the time-varying delay studied can be unbounded, and its derivative can be any given value. Especially, when some of the transmission efficiencies are limited (i.e., less than 1), the transmission efficiency and inner coupling matrices between nodes have serious impact on the synchronized state. Results of this paper extend some existing ones. Numerical simulations are finally given to demonstrate the effectiveness of the theoretical results.

The rest of this paper is organized as follows. In Section 2, coupled neural network model with transmission efficiency is presented. Some lemmas and necessary assumptions are also given in this section. Synchronization criteria of the considered model are obtained in Section 3. Then, in Section 4, numerical simulations are given to show the effectiveness of our results. Finally, Section 5 reaches conclusions.

Notations. In the sequel, if not explicitly stated, matrices are assumed to have compatible dimensions. I_q denotes the identity matrix of q-dimension. For vector $x \in \mathbb{R}^n$, the norm is denoted as $\| x \| = \sqrt{x^T x}$, where T denotes transposition. $A = (a_{ij})_{m\times m}$ denotes a matrix of $m \times m$-dimension. $A > 0$ or $A < 0$ denotes that the matrix A is a symmetric and positive or negative definite matrix.

2. Preliminaries

An array of coupled neural networks consisting of N identical nodes with delays and transmission efficiencies is described as follows:

$$\begin{aligned}\dot{x}_i(t) = & -Cx_i(t) + Af(x_i(t)) + Bf(x_i(t-\tau(t))) \\ & + D\int_{-\infty}^{t} K(t-s) f(x_i(s))\,\mathrm{d}s + I(t) \\ & + \sum_{j=1}^{N}\alpha_{ij}u_{ij}\Phi x_j(t) + \sum_{j=1}^{N}\beta_{ij}v_{ij}\Upsilon x_j(t-\tau(t)) \\ & + \sum_{j=1}^{N}\gamma_{ij}w_{ij}\Lambda\int_{-\infty}^{t} K(t-s) x_j(s)\,\mathrm{d}s, \\ & \qquad i = 1,2,\ldots,N,\end{aligned} \tag{1}$$

where $x_i(t) = (x_{i1}(t),\ldots,x_{in}(t))^T \in \mathbb{R}^n$ represents the state vector of the ith node of the network at time t; n corresponds to the number of neurons; $f(x_i(t)) = (f_1(x_{i1}(t)),\ldots, f_n(x_{in}(t)))^T$ is the neuron activation function; $C = \mathrm{diag}(c_1, c_2,\ldots,c_n)$ is a diagonal matrix with $c_i > 0$; $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$, and $D = (d_{ij})_{n\times n}$ are the connection weight matrix, time-delayed weight matrix, and the distributively time-delayed weight matrix, respectively; $I(t) = (I_1(t), I_2(t),\ldots, I_n(t))^T \in \mathbb{R}^n$ is an external input vector; $\tau(t)$ denotes the time-varying delay satisfying $\dot{\tau}(t) \le h$, h is a constant; $K(\cdot)$ is a scalar function describing the delay kernel. The $\Phi = (\phi_{ij})_{n\times n}$, $\Upsilon = (\varepsilon_{ij})_{n\times n}$, and $\Lambda = (\lambda_{ij})_{n\times n}$ are inner coupling matrices of the networks, which describe the individual coupling between two subsystems. Matrices $U = (u_{ij})_{N\times N}$, $V = (v_{ij})_{N\times N}$, and $W = (w_{ij})_{N\times N}$ are outer couplings of the whole networks satisfying the following diffusive conditions:

$$\begin{aligned} & u_{ij} \ge 0\ (i \ne j), \quad u_{ii} = -\sum_{j=1, j\ne i}^{N} u_{ij}, \quad i,j = 1,2,\ldots,N, \\ & v_{ij} \ge 0\ (i \ne j), \quad v_{ii} = -\sum_{j=1, j\ne i}^{N} u_{ij}, \quad i,j = 1,2,\ldots,N, \\ & w_{ij} \ge 0\ (i \ne j), \quad w_{ii} = -\sum_{j=1, j\ne i}^{N} w_{ij}, \quad i,j = 1,2,\ldots,N.\end{aligned} \tag{2}$$

Matrices $\alpha = (\alpha_{ij})_{N\times N}$, $\beta = (\beta_{ij})_{N\times N}$, and $\Gamma = (\gamma_{ij})_{N\times N}$ are transmission efficiency matrices of the coupled network. The constants $0 \le \alpha_{ij}, \beta_{ij}, \gamma_{ij} \le 1$ represent, respectively, signal transmission efficiency from node j to node i through connections u_{ij}, v_{ij}, and w_{ij}. In this paper, we always assume that

$$\alpha_{ii} = \beta_{ii} = \gamma_{ii} = 1, \quad i = 1,2,\ldots,N. \tag{3}$$

The initial condition of (1) is given by $x_i(t) = \phi_i(t) \in C([-\infty, 0], \mathbb{R}^n)$, $i = 1,2,\ldots,N$. In this paper, we assume that at least one matrix of U, V, and W is irreducible in the sense that there is no isolated node in corresponding graph.

Remark 1. Model (1) is general, and some special models can be derived from it. For instance, if

$$K(s)=\begin{cases}0, & s>\theta(t),\\ K(s), & 0\le s\le\theta(t),\end{cases}\tag{4}$$

for any scalar $\theta(t) > 0$, $t \in \mathbb{R}$, then the network (1) becomes the following coupled neural network with bounded distributed delays and transmission efficiencies:

$$\begin{aligned}\dot{x}_i(t)=&-Cx_i(t)+Af(x_i(t))+Bf(x_i(t-\tau(t)))\\&+D\int_{t-\theta(t)}^{t}K(t-s)f(x_i(s))\,\mathrm{d}s+I(t)\\&+\sum_{j=1}^{N}\alpha_{ij}u_{ij}\Phi x_j(t)+\sum_{j=1}^{N}\beta_{ij}v_{ij}\Upsilon x_j(t-\tau(t))\\&+\sum_{j=1}^{N}\gamma_{ij}w_{ij}\Lambda\int_{t-\theta(t)}^{t}K(t-s)x_j(s)\,\mathrm{d}s,\\&\qquad i=1,2,\ldots,N,\end{aligned}\tag{5}$$

which includes the models in [18, 19] as a special case when $\tau(t)=\tau$, $\theta(t)=\theta$, and $\alpha_{ij}=\beta_{ij}=\gamma_{ij}=1$, $1\le i,j\le N$, where τ, θ are nonnegative constants. Furthermore, if $K(s)=1$, $0\le s\le\theta$, then (1) turns out to the model studied in [20].

Remark 2. We introduce transmission efficiencies between nodes in model (1). The two extreme situations are if all the signal channels in the network operate perfectly, then $\alpha_{ij}=\beta_{ij}=\gamma_{ij}=1$, $1\le i,j\le N$; if no signal is transmitted through u_{ij}, v_{ij}, and w_{ij} or $u_{ij}=v_{ij}=w_{ij}=0$, $i\ne j$, then $\alpha_{ij}=\beta_{ij}=\gamma_{ij}=0$, $i\ne j$. Since many practical factors such as limited bandwidth of the channels or external causes and other uncertain perturbations surely exist, the model (1) is more practical than existing models of complex networks including those in [18–20].

Based on (2)-(3), the system (1) can be written as

$$\begin{aligned}\dot{x}_i(t)=&-Cx_i(t)+Af(x_i(t))+Bf(x_i(t-\tau(t)))\\&+D\int_{-\infty}^{t}K(t-s)f(x_i(s))\,\mathrm{d}s+I(t)\\&+\sum_{j=1}^{N}\overline{u}_{ij}\Phi x_j(t)+\sum_{j=1}^{N}\overline{v}_{ij}\Upsilon x_j(t-\tau(t))\\&+\sum_{j=1}^{N}\overline{w}_{ij}\Lambda\int_{-\infty}^{t}K(t-s)x_j(s)\,\mathrm{d}s\\&-\sum_{j=1,j\ne i}^{N}(1-\alpha_{ij})u_{ij}\Phi x_i(t)\\&-\sum_{j=1,j\ne i}^{N}(1-\beta_{ij})v_{ij}\Upsilon x_i(t-\tau(t))\\&-\sum_{j=1,j\ne i}^{N}(1-\gamma_{ij})w_{ij}\Lambda\int_{-\infty}^{t}K(t-s)x_i(s)\,\mathrm{d}s,\end{aligned}\tag{6}$$

where $\overline{u}_{ij}=\alpha_{ij}u_{ij}$, $\overline{u}_{ii}=-\sum_{j=1,j\ne i}\alpha_{ij}u_{ij}$, $\overline{v}_{ij}=\beta_{ij}v_{ij}$, $\overline{v}_{ii}=-\sum_{j=1,j\ne i}\beta_{ij}v_{ij}$, $\overline{w}_{ij}=\gamma_{ij}w_{ij}$, and $\overline{w}_{ii}=-\sum_{j=1,j\ne i}\gamma_{ij}w_{ij}$, $i\ne j$. Obviously, the matrices $\overline{U}=(\overline{u}_{ij})_{N\times N}$, $\overline{V}=(\overline{v}_{ij})_{N\times N}$ and $\overline{W}=(\overline{w}_{ij\infty})_{N\times N}$ are diffusive.

This paper utilizes the following assumptions.

(H_1) The delay kernel $K:[0,+\infty)\to[0,+\infty)$ is a real-valued nonnegative continuous function, and there exists positive number k such that $\int_0^{+\infty}K(s)\mathrm{d}s=k$.

(H_2) There exist constant matrices E_1 and E_2 such that

$$\begin{aligned}&[f(x)-f(y)-E_1(x-y)]^T\\&\quad\times[f(x)-f(y)-E_2(x-y)]\le 0,\quad\forall x,y\in\mathbb{R}^n.\end{aligned}\tag{7}$$

(H_3) There are constants a, b, c such that $\sum_{j=1,j\ne i}^{N}(1-\alpha_{ij})u_{ij}=a$, $\sum_{j=1,j\ne i}^{N}(1-\beta_{ij})v_{ij}=b$, and $\sum_{j=1,j\ne i}^{N}(1-\gamma_{ij})w_{ij}=c$, $i=1,2,\ldots,N$.

Remark 3. The assumption (H_2) was used in [24, 31]. f satisfies the sector condition in the sense that belongs to the sectors $[E_1,E_2]$. Such a sector description is quit general and includes the usual Lipschitz conditions as a special case.

Remark 4. When the transmission efficiencies of all the channels are considered and some of them are limited, the final synchronized state is different from that of a single node without coupling. According to (H_3), the synchronized state can be described as the following:

$$\begin{aligned}\dot{z}(t)=&-(C+a\Phi)z(t)+Af(z(t))+Bf(z(t-\tau(t)))\\&+D\int_{-\infty}^{t}K(t-s)f(z(s))\,\mathrm{d}s+I(t)\\&-b\Upsilon z(t-\tau(t))-c\Lambda\int_{-\infty}^{t}K(t-s)z(s)\,\mathrm{d}s.\end{aligned}\tag{8}$$

In order to derive our main results, some basic definitions and useful lemmas are needed.

Definition 5. The coupled neural network with limited transmission efficiency (1) is said to be globally asymptotically synchronized if

$$\lim_{t\to\infty}\left\|x_i(t)-x_j(t)\right\|=0,\quad i=1,2,\ldots,N,\tag{9}$$

holds for any initial values.

Lemma 6 (see [32]). *Let $\otimes$ denote the Kronecker product, A, B, C, and D are matrices with appropriate dimensions. The following properties are satisfied:*

(1) $(aA)\otimes B=A\otimes(aB)$, *where a is a constant;*

(2) $(A+B)\otimes C=A\otimes C+B\otimes C$;

(3) $(A\otimes B)(C\otimes D)=(AC)\otimes BD$.

Let $T(\epsilon)$ denote the set of matrices of which the sum of the element in each row is equal to the real number ϵ. The set M_1

is defined as follows: if $M = (M_{ij})_{(N-1)\times N} \in M_1$, each row of M contains exactly one element 1 and one element -1, and all other elements are zero. $j_{i1}(j_{i2})$ denotes the column indexes of the first (second) nonzero element in the ith row. The set H is defined by $H = \{\{j_{11}, j_{12}\}, \{j_{21}, j_{22}\}, \ldots, \{j_{p1}, j_{p2}\}\}$. The set M_2 is defined as follows: $M_2 \subset M_1$ and if $M = (m_{ij})_{(N-1)\times N} \in M_2$, for any pair of the column indexes j_s and j_t, there exist indexes $j_1, j_2, \ldots, j_l$ with $j_1 = j_s$ and $j_l = j_t$ such that $\{j_m, j_{m+1}\} \in H$ for $m = 1, 2, \ldots, l-1$.

Lemma 7 (see [33, 34]). *Let $M \in M_2$ be a $(N-1)\times N$ matrix and $G \in T(\epsilon)$ be a $N\times N$ matrix. Then, there exists a $N\times(N-1)$ matrix J such that $MG = \widetilde{G}M$, where $\widetilde{G} = MGJ$. Moreover, let Φ be a constant $n\times n$ matrix and $\mathrm{G} = G\otimes\Phi$, then $\mathrm{MG} = \widetilde{\mathrm{G}}\mathrm{M}$, where $\widetilde{\mathrm{G}} = \widetilde{G}\otimes\Phi$, $\mathrm{M} = M\otimes I_n$. Furthermore, $MJ = I_{N-1}$.*

The following lemma can be easily obtained from [18, 33].

Lemma 8. *Let $x(t) = (x_1^T(t), x_1^T(t), \ldots, x_N^T(t))^T$ and $M \in M_2$, if $\lim_{t\to\infty} \| (M\otimes I_n)x(t) \| = 0$, then $\lim_{t\to\infty} \| x_i(t) - x_j(t) \| = 0$, for all $i, j = 1, 2, \ldots, N$.*

Lemma 9 (see [35]). *Suppose $K(t)$ is a nonnegative bounded scalar function defined on $[0, +\infty)$ and $\int_0^{+\infty} K(u)du = k$. For any constant matrix $D \in \mathbb{R}^{n\times n}$, $D > 0$, and vector function $x : (-\infty, t] \to \mathbb{R}^n$ for $t \geq 0$, one has*

$$k \int_{-\infty}^{t} K(t-s)\, x^T(s)\, Dx(s)\, \mathrm{d}s \geq \left(\int_{-\infty}^{t} K(t-s)\, x(s)\, \mathrm{d}s\right)^T D \int_{-\infty}^{t} K(t-s)\, x(s)\, \mathrm{d}s. \tag{10}$$

Provided that the integrals are all well defined.

3. Synchronization with Limited Transmission Efficiency

In this section, synchronization criteria formulated by LMIs of the general model (1) are derived. When the distributed delays in (1) are bounded, corresponding synchronization criterion is also obtained. In the derived synchronization criteria, the time-varying delays can be unbounded and their derivative can be any given value.

For $M \in M_2$, by Lemma 7, there exists a $N\times(N-1)$ matrix J such that $MJ = I_{N-1}$. Let $\overline{\mathrm{U}} = \overline{U}\otimes\Phi$, $\widetilde{\mathrm{U}} = \widetilde{U}\otimes\Phi$, $\widetilde{U} = M\overline{U}J$, $\overline{\mathrm{V}} = \overline{V}\otimes\Upsilon$, $\widetilde{\mathrm{V}} = \widetilde{V}\otimes\Upsilon$, $\widetilde{V} = M\overline{V}J$, $\overline{\mathrm{W}} = \overline{W}\otimes\Lambda$, $\widetilde{\mathrm{W}} = \widetilde{W}\otimes\Lambda$, $\widetilde{W} = M\overline{W}J$, $\overline{C} = C + a\Phi$, $\overline{\mathrm{C}} = I_N\otimes\overline{C}$, $\overline{\mathrm{C}}_1 = I_{N-1}\otimes\overline{C}$, $\mathrm{A} = I_N\otimes A$, $\mathrm{A}_1 = I_{N-1}\otimes A$, $\mathrm{B} = I_N\otimes B$, $\mathrm{B}_1 = I_{N-1}\otimes B$, $\mathrm{D} = I_N\otimes D$, $\mathrm{D}_1 = I_{N-1}\otimes D$, $\mathrm{K} = I_N\otimes K$, $\mathrm{K}_1 = I_{N-1}\otimes K$, $\mathrm{f}(x(t)) = (f(x_1(t)), f(x_2(t)), \ldots, f(x_N(t)))^T$, $\mathrm{I}(t) = (I(t), I(t), \ldots, I(t))^T$, $\overline{\Upsilon} = I_N\otimes b\Upsilon$, $\overline{\Upsilon}_1 = I_{N-1}\otimes b\Upsilon$, $\overline{\Lambda} = I_N\otimes c\Lambda$, $\overline{\Lambda}_1 = I_{N-1}\otimes c\Lambda$, $x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t))^T$, $x(t) = (x_1(t), x_2(t), \ldots, x_N(t))^T$. Then, the network (1) can be written in the Kronecker product form as

$$\begin{aligned}\dot{x}(t) = & -\overline{\mathrm{C}}x(t) + \mathrm{Af}(x(t)) + \mathrm{Bf}(x(t-\tau(t))) \\ & + \mathrm{D}\int_{-\infty}^{t} \mathrm{K}(t-s)\,\mathrm{f}(x(s))\,\mathrm{d}s + \mathrm{I}(t) \\ & + \overline{\mathrm{U}}x(t) + \overline{\mathrm{V}}x(t-\tau(t)) \\ & + \overline{\mathrm{W}}\int_{-\infty}^{t} \mathrm{K}(t-s)\,x(s)\,\mathrm{d}s - \overline{\Upsilon}x(t-\tau(t)) \\ & - \overline{\Lambda}\int_{-\infty}^{t} \mathrm{K}(t-s)\,x(s)\,\mathrm{d}s.\end{aligned} \tag{11}$$

To obtain synchronization criterion in the array of coupled neural networks (1), we only need to consider the the problem for the system (11). Theorem 10 is our main result.

Theorem 10. *Under assumptions (H_1)–(H_3), if there exist matrices $M \in M_2$ and J satisfying $MJ = I_{N-1}$, positive definite matrices $P, Q, R, G, S \in \mathbb{R}^{(N-1)n\times(N-1)n}$ and two positive diagonal matrices $S_1, S_2 \in \mathbb{R}^{(N-1)n\times(N-1)n}$ such that*

$$\Omega = \begin{pmatrix} \Xi_1 & \mathrm{P}\widetilde{\mathrm{V}} - \mathrm{P}\overline{\Upsilon}_1 & \mathrm{PA}_1 + \mathrm{E}_2 & \mathrm{PB}_1 & \mathrm{P}\widetilde{\mathrm{W}} - \mathrm{P}\overline{\Lambda}_1 & \mathrm{PD}_1 \\ * & (-(1-h)\,\mathrm{S} - \mathrm{E}_1) & 0 & \mathrm{E}_2 & 0 & 0 \\ * & * & \Xi_2 & 0 & 0 & 0 \\ * & * & * & -(1-h)\,\mathrm{G} - I_{(N-1)n} & 0 & 0 \\ * & * & * & * & -\mathrm{R} & 0 \\ * & * & * & * & * & -\mathrm{Q} \end{pmatrix} < 0, \tag{12}$$

where $\Xi_1 = -\overline{\mathrm{C}}_1^{\mathrm{T}}\mathrm{P} - \mathrm{P}\overline{\mathrm{C}}_1 + \mathrm{P}\widetilde{\mathrm{U}} + \widetilde{\mathrm{U}}^T\mathrm{P} + k^2\mathrm{K}^T(0)\mathrm{RK}(0) + \mathrm{S} - \mathrm{E}_1S_1$, $\Xi_2 = k^2\mathrm{K}^T(0)\mathrm{QK}(0) + \mathrm{G} - I_{(N-1)n}S_1$, $\mathrm{E}_1 = I_{N-1}\otimes\widehat{E}_1$, $\mathrm{E}_2 = I_{N-1}\otimes\widehat{E}_2$, $\widehat{E}_1 = (1/2)(E_1^TE_2 + E_2^TE_1)$, and $\widehat{E}_2 = (1/2)(E_1^T + E_2^T)$, then the coupled neural networks (11) is globally asymptotically synchronized.

Proof. Consider the following Lyapunov function:

$$V(t) = \sum_{i=1}^{5} V_i(t), \tag{13}$$

where

$$
\begin{aligned}
V_1(t) &= x^T(t)\mathrm{M}^T\mathrm{PM}x(t),\\
V_2(t) &= k\int_{-\infty}^{0}\int_{t+s}^{t}(\mathrm{MK}(t-\theta)\mathrm{f}(x(\theta)))^T\\
&\qquad\times \mathrm{Q}(\mathrm{MK}(t-\theta)\mathrm{f}(x(\theta)))\,\mathrm{d}\theta\mathrm{d}s,\\
V_3(t) &= k\int_{-\infty}^{0}\int_{t+s}^{t}(\mathrm{MK}(t-\theta)x(\theta))^T\\
&\qquad\times \mathrm{R}(\mathrm{MK}(t-\theta)x(\theta))\,\mathrm{d}\theta\mathrm{d}s,\\
V_4(t) &= \int_{t-\tau(t)}^{t}(\mathrm{Mf}(x(s)))^T\mathrm{G}(\mathrm{Mf}(x(s)))\,\mathrm{d}s,\\
\dot{V}_5(t) &= (\mathrm{M}x(t))^T\mathrm{S}(\mathrm{M}x(t))\\
&\quad-(1-\dot{\tau}(t))(\mathrm{M}x(t-\tau(t)))^T\mathrm{S}(\mathrm{M}x(t-\tau(t))).
\end{aligned}
\tag{14}
$$

Differentiating $V_1(t)$ along the solution of (11) obtains that

$$
\begin{aligned}
\dot{V}_1(t) = &-x^T(t)\left(\overline{\mathrm{C}}^T\mathrm{M}^T\mathrm{PM}+\mathrm{M}^T\mathrm{PM}\overline{\mathrm{C}}\right)x(t)\\
&+2x^T(t)\mathrm{M}^T\mathrm{PM}\Big[\mathrm{Af}(x(t))+\mathrm{Bf}(x(t-\tau(t)))\\
&+\mathrm{D}\int_{-\infty}^{t}\mathrm{K}(t-s)\mathrm{f}(x(s))\,\mathrm{d}s\\
&+\mathrm{I}(t)+\overline{\mathrm{U}}x(t)+\overline{\mathrm{V}}x(t-\tau(t))\\
&+\overline{\mathrm{W}}\int_{-\infty}^{t}\mathrm{K}(t-s)x(s)\,\mathrm{d}s\\
&-\overline{\Upsilon}x(t-\tau(t))\\
&-\overline{\Lambda}\int_{-\infty}^{t}\mathrm{K}(t-s)x(s)\,\mathrm{d}s\Big].
\end{aligned}
\tag{15}
$$

By virtue of Lemma 6, it can be verified that $\mathrm{M}\overline{\mathrm{C}}=\overline{\mathrm{C}}_1\mathrm{M}$, $\mathrm{MA}=\mathrm{A}_1\mathrm{M}$, $\mathrm{MB}=\mathrm{B}_1\mathrm{M}$, $\mathrm{MD}=\mathrm{D}_1\mathrm{M}$, $\mathrm{MK}=\mathrm{K}_1\mathrm{M}$, $\mathrm{M}\overline{\Upsilon}=\overline{\Upsilon}_1\mathrm{M}$, $\mathrm{M}\overline{\Lambda}=\overline{\Lambda}_1\mathrm{M}$, and $\mathrm{MI}(t)=0$. On the other hand, it follows from Lemma 7 that $\mathrm{M}\overline{\mathrm{U}}=\widetilde{\mathrm{U}}\mathrm{M}$, $\mathrm{M}\overline{\mathrm{V}}=\widetilde{\mathrm{V}}\mathrm{M}$, and $\mathrm{M}\overline{\mathrm{W}}=\widetilde{\mathrm{W}}\mathrm{M}$. Therefore,

$$
\begin{aligned}
\dot{V}_1(t) = &-x^T(t)\left(\mathrm{M}^T\overline{\mathrm{C}}_1^T\mathrm{PM}+\mathrm{M}^T\mathrm{P}\overline{\mathrm{C}}_1\mathrm{M}\right)x(t)\\
&+2x^T(t)\mathrm{M}^T\mathrm{P}\Big[\mathrm{A}_1\mathrm{Mf}(x(t))\\
&+\mathrm{B}_1\mathrm{Mf}(x(t-\tau(t)))\\
&+\mathrm{D}_1\mathrm{M}\int_{-\infty}^{t}\mathrm{K}(t-s)\mathrm{f}(x(s))\,\mathrm{d}s\\
&+\widetilde{\mathrm{U}}\mathrm{M}x(t)+\widetilde{\mathrm{V}}\mathrm{M}x(t-\tau(t))\\
&+\widetilde{\mathrm{W}}\mathrm{M}\int_{-\infty}^{t}\mathrm{K}(t-s)x(s)\,\mathrm{d}s\\
&-\overline{\Upsilon}_1\mathrm{M}x(t-\tau(t))\\
&-\overline{\Lambda}_1\mathrm{M}\int_{-\infty}^{t}\mathrm{K}(t-s)x(s)\,\mathrm{d}s\Big].
\end{aligned}
\tag{16}
$$

Moreover, based on Lemma 9, one gets that

$$
\begin{aligned}
\dot{V}_2(t) \le\; & k^2(\mathrm{MK}(0)\mathrm{f}(x(t)))^T\mathrm{Q}(\mathrm{MK}(0)\mathrm{f}(x(t)))\\
&-k\int_{-\infty}^{t}(\mathrm{MK}(t-s)\mathrm{f}(x(s)))^T\\
&\times\mathrm{Q}(\mathrm{MK}(t-s)\mathrm{f}(x(s)))\,\mathrm{d}s\\
\le\; & k^2(\mathrm{Mf}(x(t)))^T\mathrm{K}^T(0)\mathrm{QK}(0)(\mathrm{Mf}(x(t)))\\
&-\left(\int_{-\infty}^{t}\mathrm{MK}(t-s)\mathrm{f}(x(s))\,\mathrm{d}s\right)^T\\
&\times\mathrm{Q}\left(\int_{-\infty}^{t}\mathrm{MK}(t-s)\mathrm{f}(x(s))\,\mathrm{d}s\right).
\end{aligned}
\tag{17}
$$

Similarly,

$$
\begin{aligned}
\dot{V}_3(t) \le\; & k^2(\mathrm{M}x(t))^T\mathrm{K}^T(0)\mathrm{RK}(0)(\mathrm{M}x(t))\\
&-\left(\int_{-\infty}^{t}\mathrm{MK}(t-s)x(s)\,\mathrm{d}s\right)^T\\
&\times\mathrm{R}\left(\int_{-\infty}^{t}\mathrm{MK}(t-s)x(s)\,\mathrm{d}s\right).
\end{aligned}
\tag{18}
$$

By $0\le\dot{\tau}(t)\le h$, it is easy to derive that

$$
\begin{aligned}
\dot{V}_4(t) =\; & (\mathrm{Mf}(x(t)))^T\mathrm{G}(\mathrm{Mf}(x(t)))\\
&-(1-\dot{\tau}(t))(\mathrm{Mf}(x(t-\tau(t))))^T\\
&\times\mathrm{G}(\mathrm{Mf}(x(t-\tau(t))))\\
\le\; & (\mathrm{Mf}(x(t)))^T\mathrm{G}(\mathrm{Mf}(x(t)))\\
&-(1-h)(\mathrm{Mf}(x(t-\tau(t))))^T\\
&\times\mathrm{G}(\mathrm{Mf}(x(t-\tau(t)))),\\
\dot{V}_5(t) \le\; & (\mathrm{M}x(t))^T\mathrm{S}(\mathrm{M}x(t))\\
&-(1-h)(\mathrm{M}x(t-\tau(t)))^T\mathrm{S}(\mathrm{M}x(t-\tau(t))).
\end{aligned}
\tag{19}
$$

In view of assumption (H_2), for any positive diagonal matrices S_1 and S_2, the following two inequalities hold:

$$
\begin{aligned}
&\begin{pmatrix}\mathrm{M}x(t)\\ \mathrm{Mf}(x(t))\end{pmatrix}^T\begin{pmatrix}\mathrm{E}_1S_1 & -\mathrm{E}_2S_1\\ -\mathrm{E}_2^TS_1 & I_{(N-1)n}S_1\end{pmatrix}\begin{pmatrix}\mathrm{M}x(t)\\ \mathrm{Mf}(x(t))\end{pmatrix}\le 0,\\
&\begin{pmatrix}\mathrm{M}x(t-\tau(t))\\ \mathrm{Mf}(x(t-\tau(t)))\end{pmatrix}^T\begin{pmatrix}\mathrm{E}_1S_2 & -\mathrm{E}_2S_2\\ -\mathrm{E}_2^TS_2 & I_{(N-1)n}S_2\end{pmatrix}\\
&\quad\times\begin{pmatrix}\mathrm{M}x(t-\tau(t))\\ \mathrm{Mf}(x(t-\tau(t)))\end{pmatrix}\le 0.
\end{aligned}
\tag{20}
$$

Combining (13)–(20) gives

$$\begin{aligned}
\dot{V}(t) \leq{}& x^T(t)\mathrm{M}^T\Big(-\overline{\mathrm{C}}_1^T\mathrm{P}-\mathrm{P}\overline{\mathrm{C}}_1+\mathrm{P}\widetilde{\mathrm{U}}+\widetilde{\mathrm{U}}^T\mathrm{P} \\
&+k^2\mathrm{K}^T(0)\mathrm{R}\mathrm{K}(0)+\mathrm{S}-\mathrm{E}_1\Big)\mathrm{M}x(t) \\
&+2x^T(t)\mathrm{M}^T(\mathrm{P}\mathrm{A}_1+\mathrm{E}_2)\mathrm{M}\mathrm{f}(x(t)) \\
&+2x^T(t)\mathrm{M}^T\mathrm{P}\mathrm{B}_1\mathrm{M}\mathrm{f}(x(t-\tau(t))) \\
&+2x^T(t)\mathrm{M}^T\mathrm{P}\mathrm{D}_1\mathrm{M}\int_{-\infty}^{t}\mathrm{K}(t-s)\,\mathrm{f}(x(s))\,ds \\
&+2x^T(t)\mathrm{M}^T\big(\mathrm{P}\widetilde{\mathrm{V}}-\mathrm{P}\overline{\Upsilon}_1\big)\mathrm{M}x(t-\tau(t)) \\
&+2x^T(t)\mathrm{M}^T\big(\mathrm{P}\widetilde{\mathrm{W}}-\mathrm{P}\overline{\Lambda}_1\big)\mathrm{M}\int_{-\infty}^{t}\mathrm{K}(t-s)\,x(s)\,ds \\
&+(\mathrm{M}\mathrm{f}(x(t)))^T\big(k^2\mathrm{K}^T(0)\mathrm{Q}\mathrm{K}(0)+\mathrm{G}-I_{(N-1)n}\big) \\
&\times(\mathrm{M}\mathrm{f}(x(t))) \\
&-\left(\int_{-\infty}^{t}\mathrm{M}\mathrm{K}(t-s)\,\mathrm{f}(x(s))\,ds\right)^T \\
&\times\mathrm{Q}\left(\int_{-\infty}^{t}\mathrm{M}\mathrm{K}(t-s)\,\mathrm{f}(x(s))\,ds\right) \\
&-\left(\int_{-\infty}^{t}\mathrm{M}\mathrm{K}(t-s)\,x(s)\,ds\right)^T \\
&\times\mathrm{R}\left(\int_{-\infty}^{t}\mathrm{M}\mathrm{K}(t-s)\,x(s)\,ds\right) \\
&+(\mathrm{M}\mathrm{f}(x(t-\tau(t))))^T\big[-(1-h)\mathrm{G}-I_{(N-1)n}\big] \\
&\times(\mathrm{M}\mathrm{f}(x(t-\tau(t)))) \\
&+(\mathrm{M}x(t-\tau(t)))^T\big[-(1-h)\mathrm{S}-\mathrm{E}_1\big]\mathrm{M}x(t-\tau(t)) \\
&+(\mathrm{M}x(t-\tau(t)))^T\mathrm{E}_2\mathrm{M}f(x(t-\tau(t))) \\
={}& \xi^T\Omega\xi,
\end{aligned} \tag{21}$$

where

$$\begin{aligned}
\xi = \Bigg(&(\mathrm{M}x(t))^T,(\mathrm{M}x(t-\tau(t)))^T, \\
&(\mathrm{M}\mathrm{f}(x(t)))^T,(\mathrm{M}\mathrm{f}(x(t-\tau(t))))^T, \\
&\left(\mathrm{M}\int_{-\infty}^{t}\mathrm{K}(t-s)\,x(s)\,ds\right)^T, \\
&\left(\mathrm{M}\int_{-\infty}^{t}\mathrm{K}(t-s)\,\mathrm{f}(x(s))\,ds\right)^T\Bigg)^T.
\end{aligned} \tag{22}$$

From the given condition (12) and the inequality (21), one derives that $\dot{V}(t) \leq 0$ and $\dot{V}(t) = 0$ if and only if $\xi = 0$. Hence, $\lim_{t\to\infty}\|(M\otimes I_n)x(t)\| = 0$. By virtue of Definition 5 and Lemma 8, the coupled neural network (11) is globally asymptotically synchronize. This completes the proof. □

Corresponding to (5), we now consider the following network with time-varying delays and bounded distributed delays:

$$\begin{aligned}
\dot{x}(t) ={}& -\overline{\mathrm{C}}x(t)+\mathrm{A}\mathrm{f}(x(t))+\mathrm{B}\mathrm{f}(x(t-\tau(t))) \\
&+\mathrm{D}\int_{t-\theta(t)}^{t}\mathrm{K}(t-s)\,\mathrm{f}(x(s))\,ds+\mathrm{I}(t) \\
&+\overline{\mathrm{U}}x(t)+\overline{\mathrm{V}}x(t-\tau(t)) \\
&+\overline{\mathrm{W}}\int_{t-\theta(t)}^{t}\mathrm{K}(t-s)\,x(s)\,ds-\overline{\Upsilon}x(t-\tau(t)) \\
&-\overline{\Lambda}\int_{t-\theta(t)}^{t}\mathrm{K}(t-s)\,x(s)\,ds.
\end{aligned} \tag{23}$$

For the system (23) the following result can be easily derived by similar proof process of Theorem 10.

Corollary 11. *Under assumptions* (H_2) *and* (H_3), *if there is positive constant k such that* $\int_0^{\theta(t)}K(u)du = k(t) \leq k$, *matrices* $M \in M_2$ *and J satisfying* $MJ = I_{N-1}$, *positive definite matrices* $P, Q, R, G, S \in \mathbb{R}^{(N-1)n\times(N-1)n}$, *and two positive diagonal matrices* $S_1, S_2 \in \mathbb{R}^{(N-1)n\times(N-1)n}$ *such that the linear matrix inequality* (12) *holds, then the coupled neural network* (23) *is globally asymptotically synchronized.*

Proof. Consider the following Lyapunov function:

$$V(t) = \sum_{i=1}^{5}V_i(t), \tag{24}$$

where $V_1(t)$, $V_4(t)$, and $V_5(t)$ are the same as those defined in the proof of Theorem 10 and

$$\begin{aligned}
V_2(t) ={}& k\int_{-\theta}^{0}\int_{t+s}^{t}(\mathrm{M}\mathrm{K}(t-\theta)\,\mathrm{f}(x(\theta)))^T \\
&\times\mathrm{Q}(\mathrm{M}\mathrm{K}(t-\theta)\,\mathrm{f}(x(\theta)))\,d\theta\,ds, \\
V_3(t) ={}& k\int_{-\theta}^{0}\int_{t+s}^{t}(\mathrm{M}\mathrm{K}(t-\theta)\,x(\theta))^T \\
&\times\mathrm{R}(\mathrm{M}\mathrm{K}(t-\theta)\,x(\theta))\,d\theta\,ds.
\end{aligned} \tag{25}$$

Based on Lemma 9, one can get that

$$\begin{aligned}
\dot{V}_2(t) \leq{}& k^2(\mathrm{M}\mathrm{K}(0)\,\mathrm{f}(x(t)))^T\mathrm{Q}(\mathrm{M}\mathrm{K}(0)\,\mathrm{f}(x(t))) \\
&-k(t)\int_{t-\theta(t)}^{t}(\mathrm{M}\mathrm{K}(t-s)\,\mathrm{f}(x(s)))^T \\
&\times\mathrm{Q}(\mathrm{M}\mathrm{K}(t-s)\,\mathrm{f}(x(s)))\,ds \\
\leq{}& k^2(\mathrm{M}\mathrm{f}(x(t)))^T\mathrm{K}^T(0)\,\mathrm{Q}\mathrm{K}(0)\,(\mathrm{M}\mathrm{f}(x(t))) \\
&-\left(\int_{t-\theta(t)}^{t}\mathrm{M}\mathrm{K}(t-s)\,\mathrm{f}(x(s))\,ds\right)^T \\
&\times\mathrm{Q}\left(\int_{t-\theta(t)}^{t}\mathrm{M}\mathrm{K}(t-s)\,\mathrm{f}(x(s))\,ds\right),
\end{aligned} \tag{26}$$

$$\begin{aligned}
\dot{V}_3(t) \leq{}& k^2(\mathrm{M}x(t))^T\mathrm{K}^T(0)\,\mathrm{R}\mathrm{K}(0)\,(\mathrm{M}x(t)) \\
&-\left(\int_{t-\theta(t)}^{t}\mathrm{M}\mathrm{K}(t-s)\,x(s)\,ds\right)^T \\
&\times\mathrm{R}\left(\int_{t-\theta(t)}^{t}\mathrm{M}\mathrm{K}(t-s)\,x(s)\,ds\right).
\end{aligned} \tag{27}$$

The rest part of the proof is similar to that of the proof of Theorem 10. This completes the proof. □

Remark 12. In this paper, the least restriction is imposed on the time-varying delay. The derivative of the time-varying delay can be any given value, and the time-varying delay can be unbounded. However, most of former results are based on either that the derivative of the time-varying delay should be less than 1 [16, 17] or that the time-varying delay should be bounded [16] or even both of them [10]. In this sense, results of this paper are less conservative than those of [10, 16, 17].

Remark 13. Synchronization criteria in an array of coupled neural networks with limited transmission efficiency are obtained in Theorem 10 and Corollary 11. One may note that assumption condition (H_3) is strong. Many real-world complex dynamical network models do not satisfy (H_3) and exhibit more complicated dynamical behaviors. How to control complex networks with arbitrary limited transmission efficiency while without (H_3) is our next research topic, which is also a challenging work.

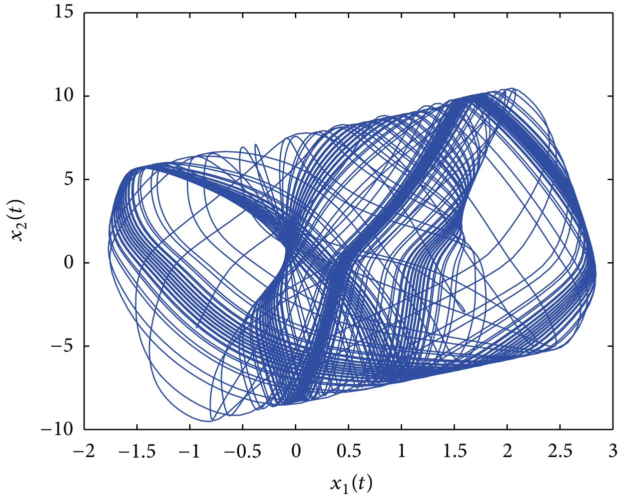

Figure 1: Chaotic-like trajectory of the system (28).

4. Numerical Example

In this section, one example is provided to illustrate the effectiveness of the results obtained above.

Consider a 2-dimensional neural network with both discrete and unbounded distributed delays as follows:

$$\dot{x}(t) = -Cx(t) + Af(x(t)) + Bf(x_i(t-\tau(t))) + D\int_{-\infty}^{t} K(t-s) f(x(s))\,ds + I(t), \tag{28}$$

where $x(t) = (x_1(t), x_2(t))^T$, $f(x(t)) = (\tanh(x_1(t)), \tanh(x_2(t)))^T$, $\tau(t) = 1$, $k(t) = e^{-0.5t}$, and

$$C = \begin{pmatrix} 1.2 & 0 \\ 0 & 1 \end{pmatrix}, \qquad A = \begin{pmatrix} 3 & -0.3 \\ 4 & 5 \end{pmatrix},$$
$$B = \begin{pmatrix} -1.4 & 0.1 \\ 0.3 & -8 \end{pmatrix}, \qquad D = \begin{pmatrix} -1.2 & 0.1 \\ -2.8 & -1 \end{pmatrix}, \tag{29}$$
$$I(t) = \begin{pmatrix} 1 \\ 1.2 \end{pmatrix}.$$

In the case that the initial condition is chosen as $x(t) = (0.4, 0.6)^T$, $\forall t \in [-1, 0]$, and $x(t) = 0$ for $t < -1$, the chaotic-like trajectory of (28) can be seen in Figure 1.

Now we consider a coupled neural network consisting of five identical models (28), which is described as

$$\begin{aligned}\dot{x}_i(t) = {} & -Cx_i(t) + Af(x_i(t)) + Bf(x_i(t-\tau(t))) \\ & + D\int_{-\infty}^{t} K(t-s) f(x_i(s))\,ds + I(t) \\ & + \sum_{j=1}^{N} \alpha_{ij} u_{ij} \Phi\left(x_j(t) - x_i(t)\right) \\ & + \sum_{j=1}^{N} \beta_{ij} v_{ij} \Upsilon\left(x_j(t-\tau(t)) - x_i(t-\tau(t))\right) \\ & + \sum_{j=1}^{N} \gamma_{ij} w_{ij} \Lambda \int_{-\infty}^{t} K(t-s)\left(x_j(s) - x_i(s)\right) ds, \\ & i = 1, 2, \ldots, 5,\end{aligned} \tag{30}$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t))^T$ is the state of the ith neural network, Φ, Υ, and Λ are identity matrices, U, V, and W are asymmetric and zero-row sum matrices as the following:

$$U = 10\begin{pmatrix} -7 & 1 & 3 & 2 & 1 \\ 1 & -4 & 1 & 0 & 2 \\ 1 & 0 & -3 & 1 & 1 \\ 1 & 1 & 1 & -4 & 1 \\ 2 & 0 & 2 & 1 & -5 \end{pmatrix},$$
$$V = W = \begin{pmatrix} -3 & 0 & 1 & 1 & 1 \\ 0 & -2 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & -2 \end{pmatrix}, \tag{31}$$

the transmission efficiency matrices are

$$\alpha = \begin{pmatrix} 1 & 0.99 & 1 & 1 & 0.99 \\ 1 & 1 & 1 & 0 & 0.99 \\ 0.98 & 0 & 1 & 1 & 1 \\ 1 & 0.99 & 0.99 & 1 & 1 \\ 1 & 0 & 1 & 0.98 & 1 \end{pmatrix},$$
$$\beta = \Gamma = \begin{pmatrix} 1 & 0 & 0.9 & 0.9 & 0.9 \\ 0 & 1 & 0.9 & 0 & 0.8 \\ 0 & 0.8 & 1 & 0 & 0.9 \\ 0.7 & 0 & 0 & 1 & 0 \\ 0 & 0.9 & 0 & 0.8 & 1 \end{pmatrix}. \tag{32}$$

It is easy to check that the activation function f satisfies assumption (H_2), and $\widehat{E}_1 = 0$, $\widehat{E}_2 = \mathrm{diag}(0.5, 0.5)$. Moreover,

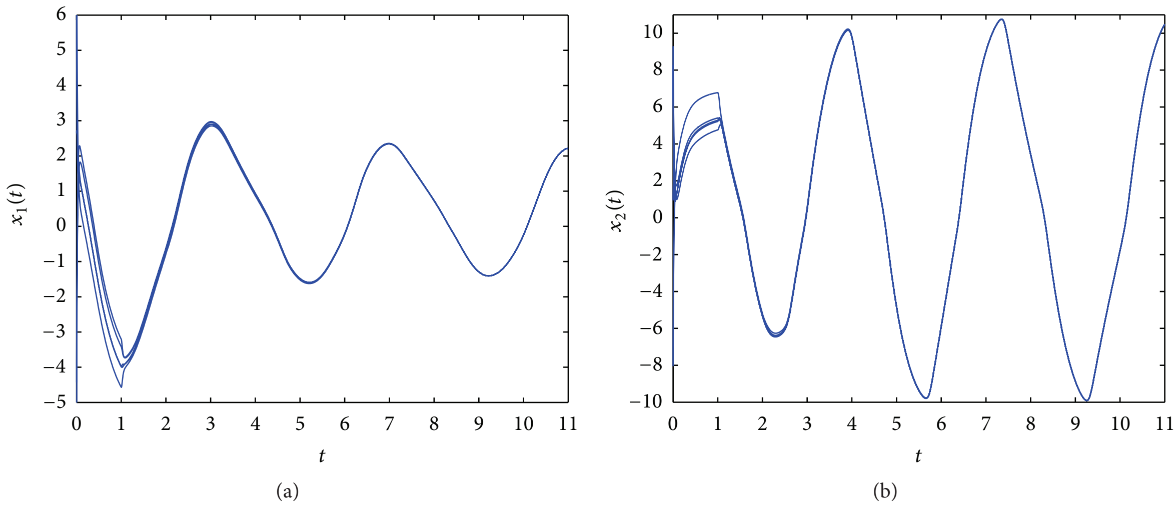

FIGURE 2: Time response of $x_{i1}(t)$ (a) and $x_{i2}(t)$ (b), $i = 1, 2, \ldots, 5$.

(H_1) and (H_3) are satisfied with $k = 2$, $a = 0.2$, and $b = c = 0.3$. Obviously, $h = 0$. Take

$$M = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}, \quad J = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{33}$$

According to Theorem 10, by referring to the MATLAB LMI Toolbox, one can get the feasible solution, see the appendix at the end of this paper. Hence, the system (30) is globally asymptotically synchronized.

In the simulations, the Runge-Kutta numerical scheme is used to simulate by MATLAB. The initial values of (30) are chosen randomly in the real number interval $[-10, 10]$ for $t \in [-1, 0]$ and all the states of the coupled neural networks are zero for $t < 0$. The time step size is $\delta = 0.005$. Figure 2 shows the time response of the states. Figure 3 describes the synchronization errors $e(t) = \sum_{j=1}^{2} \sqrt{\sum_{i=2}^{5} [x_{1j} - x_{ij}]^2}$, which turn to zero quickly as time goes.

Figure 4 presents the synchronized state of (30), which is different from that of Figure 1. Actually, it can be seen from (8) that a, b, and c and Φ, Υ, and Λ have important effects on the synchronized state. Let

$$\Phi = \begin{pmatrix} 1 & 0 \\ 0.5 & 1 \end{pmatrix}, \qquad \Upsilon = \begin{pmatrix} 0.5 & 0 \\ 1 & 1 \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}, \tag{34}$$

in (8). Figure 5 depicts the trajectories of (8) with different a, b, and c, the other parameters are the same as those in (28).

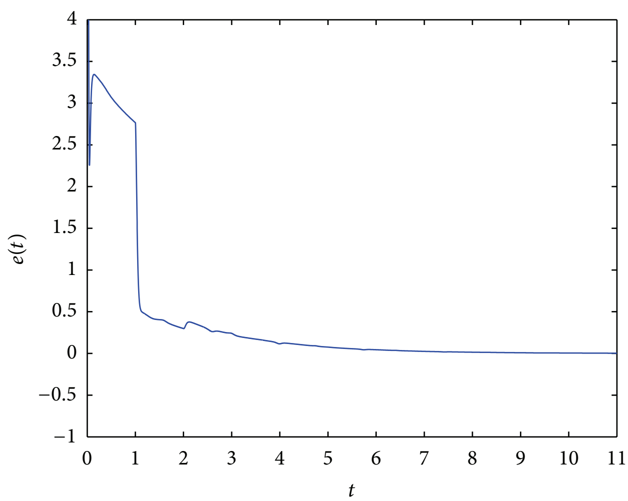

FIGURE 3: Error distance of the coupled network (30).

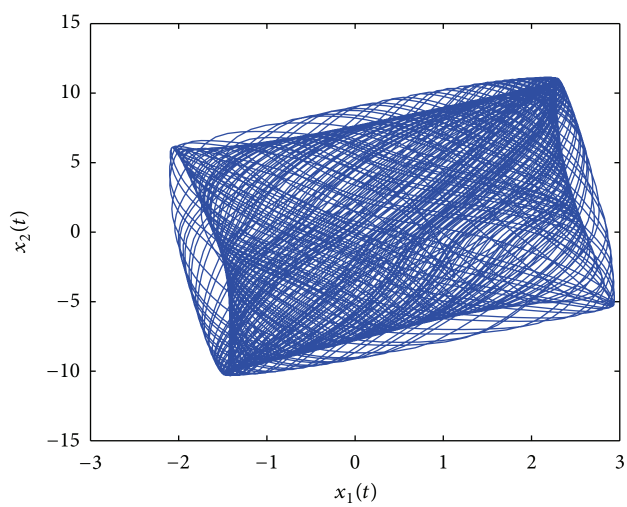

FIGURE 4: Trajectory of the synchronized state of system (30).

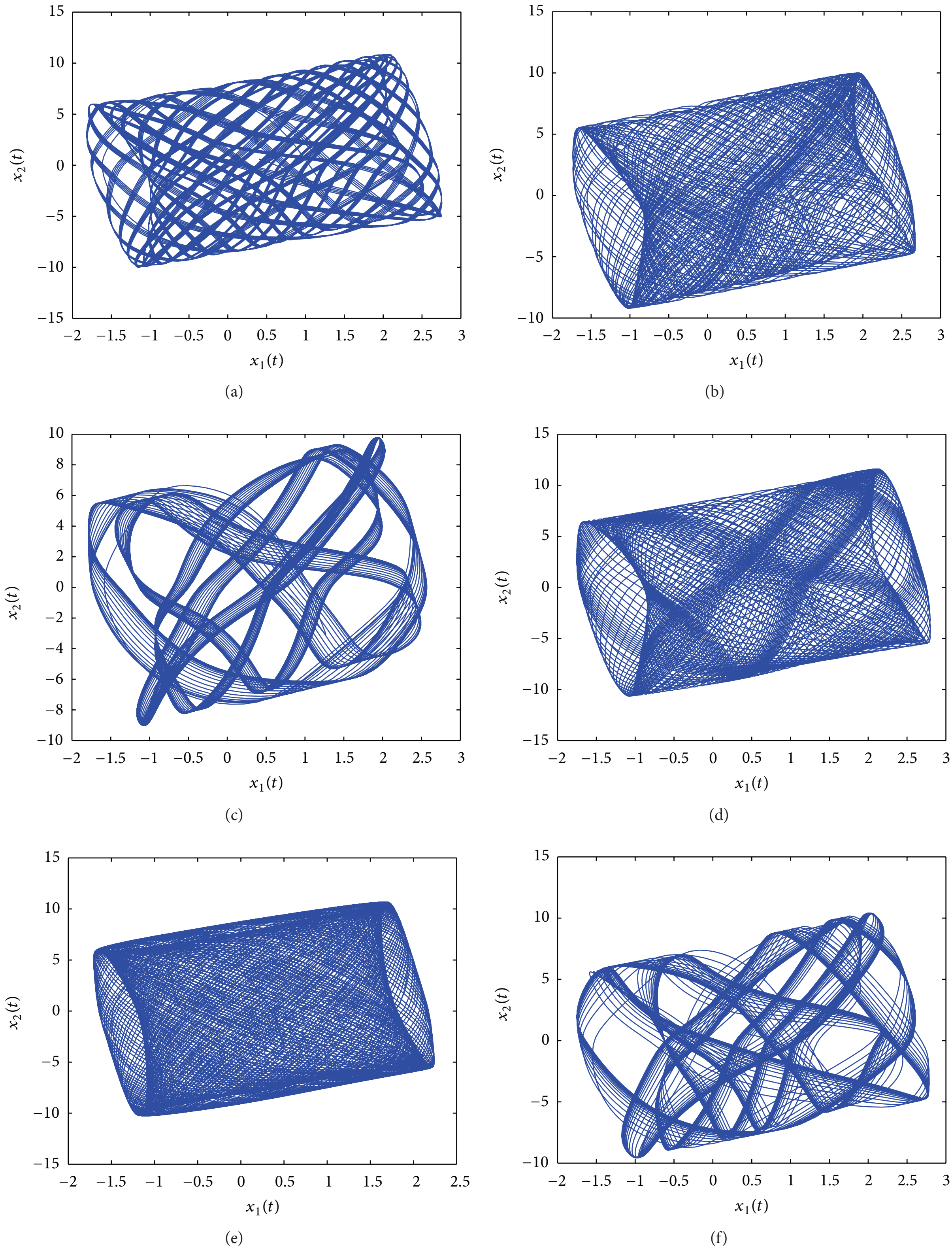

FIGURE 5: Trajectories of system (8) with different a, b, and c: (a) $a = 0.2$, $b = 0.3$, and $c = 0.3$; (b) $a = 0.2$, $b = 0.1$, and $c = 0.3$; (c) $a = 0.2$, $b = 0$, and $c = 0.5$; (d) $a = 0.2$, $b = 0.5$, and $c = 0$; (e) $a = 1$, $b = 1$, and $c = 1$; (f) $a = 0.1$, $b = 0.1$, and $c = 0.1$.

5. Conclusions

In this paper, a general model of coupled neural networks with time-varying delays and unbounded distributed delays is proposed. Limited transmission efficiency between coupled nodes is considered in the dynamical network model. Based on the integral inequality and the Lyapunov functional method, sufficient conditions in terms of LMIs are derived to guarantee the synchronization of the proposed dynamical network with limited transmission efficiency. The restriction on time-varying delay is the least. The derivative of the time-varying delay can be any given value, and the time-varying delay can be unbounded. Numerical examples are given to verify the effectiveness of the theoretical results. Furthermore, numerical simulations show that, when some of the transmission efficiencies are less than 1, the transmission efficiency and inner coupling matrices between nodes play important roles for the final synchronized state. Since many real-world transmission efficiencies between nodes are usually less than 1, the results of this paper are new and extend some of the existing results.

Appendix

$$P = \begin{pmatrix} 2.3293 & -0.0507 & -0.4456 & 0.0497 & -0.1796 & 0.0081 & -0.0216 & -0.0071 \\ -0.0507 & 2.0002 & 0.0479 & -0.1233 & 0.0083 & -0.1427 & -0.0066 & -0.0497 \\ -0.4456 & 0.0479 & 4.9740 & -0.1330 & -0.1868 & 0.0201 & -0.4820 & 0.0022 \\ 0.0497 & -0.1233 & -0.1330 & 3.8409 & 0.0208 & -0.0304 & 0.0009 & -0.4124 \\ -0.1796 & 0.0083 & -0.1868 & 0.0208 & 3.8034 & -0.0981 & -0.2548 & 0.0138 \\ 0.0081 & -0.1427 & 0.0201 & -0.0304 & -0.0981 & 3.0951 & 0.0139 & -0.1496 \\ -0.0216 & -0.0066 & -0.4820 & 0.0009 & -0.2548 & 0.0139 & 3.1954 & -0.0674 \\ -0.0071 & -0.0497 & 0.0022 & -0.4124 & 0.0138 & -0.1496 & -0.0674 & 2.7416 \end{pmatrix},$$

$$Q = \begin{pmatrix} 14.1261 & -0.3969 & 0.1779 & 0.0687 & -0.0216 & 0.0133 & 0.0373 & 0.0077 \\ -0.3969 & 12.1352 & 0.0901 & 0.7797 & 0.0090 & 0.0242 & 0.0046 & 0.0984 \\ 0.1779 & 0.0901 & 13.7184 & -0.5675 & 0.1003 & 0.0460 & 0.1842 & 0.0075 \\ 0.0687 & 0.7797 & -0.5675 & 9.3374 & 0.0397 & 0.4060 & 0.0101 & 0.4532 \\ -0.0216 & 0.0090 & 0.1003 & 0.0397 & 14.0715 & -0.5518 & 0.0850 & 0.0348 \\ 0.0133 & 0.0242 & 0.0460 & 0.4060 & -0.5518 & 10.8553 & 0.0308 & 0.3573 \\ 0.0373 & 0.0046 & 0.1842 & 0.0101 & 0.0850 & 0.0308 & 14.1750 & -0.4777 \\ 0.0077 & 0.0984 & 0.0075 & 0.4532 & 0.0348 & 0.3573 & -0.4777 & 11.7793 \end{pmatrix},$$

$$R = \begin{pmatrix} 51.1311 & -2.8226 & 2.1854 & 0.6937 & -0.6251 & 0.2521 & 0.7007 & -0.0726 \\ -2.8226 & 36.5803 & 0.6786 & 7.1163 & 0.2622 & 0.2684 & -0.0770 & 0.6599 \\ 2.1854 & 0.6786 & 44.5831 & -3.7741 & 1.5536 & 0.2002 & 1.6747 & 0.0327 \\ 0.6937 & 7.1163 & -3.7741 & 22.4396 & 0.2177 & 3.3233 & 0.0238 & 1.9847 \\ -0.6251 & 0.2622 & 1.5536 & 0.2177 & 49.0212 & -3.7531 & 1.3611 & 0.2250 \\ 0.2521 & 0.2684 & 0.2002 & 3.3233 & -3.7531 & 28.4807 & 0.2336 & 3.0901 \\ 0.7007 & -0.0770 & 1.6747 & 0.0238 & 1.3611 & 0.2336 & 50.7054 & -3.3712 \\ -0.0726 & 0.6599 & 0.0327 & 1.9847 & 0.2250 & 3.0901 & -3.3712 & 33.8828 \end{pmatrix},$$

$$G = \begin{pmatrix} -31.8299 & -0.6083 & 0.0156 & 0.0565 & 0.2454 & 0.0531 & -0.0814 & 0.0392 \\ -0.6083 & -26.2338 & 0.1191 & -2.0217 & 0.0520 & 0.0315 & 0.0325 & -0.3113 \\ 0.0156 & 0.1191 & -33.6328 & -1.6051 & -0.0230 & 0.0458 & -0.0143 & 0.1821 \\ 0.0565 & -2.0217 & -1.6051 & -18.9430 & 0.0376 & -1.0840 & 0.1756 & -1.5906 \\ 0.2454 & 0.0520 & -0.0230 & 0.0376 & -33.4900 & -1.2137 & 0.0722 & 0.0789 \\ 0.0531 & 0.0315 & 0.0458 & -1.0840 & -1.2137 & -23.4590 & 0.0800 & -0.9480 \\ -0.0814 & 0.0325 & -0.0143 & 0.1756 & 0.0722 & 0.0800 & -33.1772 & -0.9452 \\ 0.0392 & -0.3113 & 0.1821 & -1.5906 & 0.0789 & -0.9480 & -0.9452 & -26.0874 \end{pmatrix},$$

$$S=\begin{pmatrix} 81.6408 & -0.4942 & 0.3979 & 0.5882 & 0.2149 & -0.0492 & 0.0804 & -0.0541 \\ -0.4942 & 75.1089 & 0.5176 & 5.0908 & -0.0392 & 0.5506 & -0.0392 & -0.3170 \\ 0.3979 & 0.5176 & 84.7411 & -1.2587 & 0.4128 & 0.2505 & -0.1328 & 0.0971 \\ 0.5882 & 5.0908 & -1.2587 & 68.4426 & 0.2667 & 2.7250 & 0.0774 & 0.7954 \\ 0.2149 & -0.0392 & 0.4128 & 0.2667 & 83.2899 & -0.5940 & 0.3435 & 0.1248 \\ -0.0492 & 0.5506 & 0.2505 & 2.7250 & -0.5940 & 72.7484 & 0.1203 & 1.8855 \\ 0.0804 & -0.0392 & -0.1328 & 0.0774 & 0.3435 & 0.1203 & 82.3409 & -0.3789 \\ -0.0541 & -0.3170 & 0.0971 & 0.7954 & 0.1248 & 1.8855 & -0.3789 & 74.7768 \end{pmatrix},$$

$$\begin{aligned} S_1 &= \operatorname{diag}(63.3310, 61.7187, 65.1691, 60.1717, 64.2638, 62.0351, 63.7512, 62.1022), \\ S_2 &= \operatorname{diag}(67.5321, 66.8113, 70.6801, 69.0580, 69.6841, 69.7116, 69.0126, 69.7827). \end{aligned} \tag{A.1}$$

Acknowledgments

This work was jointly supported by the National Natural Science Foundation of China (NSFC) under Grants nos. 61263020, 61272530, and 11072059, the Natural Science Foundation of Jiangsu Province of China under Grant no. BK2012741, the Scientific Research Fund of Yunnan Province under Grant no. 2010ZC150, and the Scientific Research Fund of Chongqing Normal University under Grants no. 12XLB031 and no. 940115.

References

[1] S. H. Strogatz and I. Stewart, “Coupled oscillators and biological synchronization,” *Scientific American*, vol. 269, no. 6, pp. 102–109, 1993.

[2] P. Zhou and R. Ding, “Modified function projective synchronization between different dimension fractional-order chaotic systems,” *Abstract and Applied Analysis*, vol. 2012, Article ID 862989, 12 pages, 2012.

[3] X. Yang and J. Cao, “Finite-time stochastic synchronization of complex networks,” *Applied Mathematical Modelling*, vol. 34, no. 11, pp. 3631–3641, 2010.

[4] J. Liang, Z. Wang, Y. Liu, and X. Liu, “Global synchronization control of general delayed discrete-time networks with stochastic coupling and disturbances,” *IEEE Transactions on Systems, Man, and Cybernetics B*, vol. 38, no. 4, pp. 1073–1083, 2008.

[5] X. Yang, Q. Zhu, and C. Huang, “Lag stochastic synchronization of chaotic mixed time-delayed neural networks with uncertain parameters or perturbations,” *Neurocomputing*, vol. 74, no. 10, pp. 1617–1625, 2011.

[6] J. Cao and J. Lu, “Adaptive synchronization of neural networks with or without time-varying delay,” *Chaos*, vol. 16, no. 1, Article ID 013133, 6 pages, 2006.

[7] X. Yang, J. Cao, Y. Long, and W. Rui, “Adaptive lag synchronization for competitive neural networks with mixed delays and uncertain hybrid perturbations,” *IEEE Transactions on Neural Networks*, vol. 21, no. 10, pp. 1656–1667, 2010.

[8] X. Yang and J. Cao, “Stochastic synchronization of coupled neural networks with intermittent control,” *Physics Letters A*, vol. 373, no. 36, pp. 3259–3272, 2009.

[9] W. Yu and J. Cao, “Synchronization control of stochastic delayed neural networks,” *Physica A*, vol. 373, pp. 252–260, 2007.

[10] X. Gao, S. Zhong, and F. Gao, “Exponential synchronization of neural networks with time-varying delays,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 5-6, pp. 2003–2011, 2009.

[11] T. Li, A. Song, S. Fei, and Y. Guo, “Synchronization control of chaotic neural networks with time-varying and distributed delays,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 5-6, pp. 2372–2384, 2009.

[12] J. Lu, D. W. C. Ho, and Z. Wang, “Pinning stabilization of linearly coupled stochastic neural networks via minimum number of controllers,” *IEEE Transactions on Neural Networks*, vol. 20, no. 10, pp. 1617–1629, 2009.

[13] X. Yang, C. Huang, and Z. Yang, “Stochastic synchronization of reaction-diffusion neural networks under general impulsive controller with mixed delays,” *Abstract and Applied Analysis*, vol. 2012, Article ID 603535, 25 pages, 2012.

[14] J. Lu and D. W. C. Ho, “Globally exponential synchronization and synchronizability for general dynamical networks,” *IEEE Transactions on Systems, Man, and Cybernetics B*, vol. 40, no. 2, pp. 350–361, 2010.

[15] J. Lu, D. W. C. Ho, and J. Cao, “A unified synchronization criterion for impulsive dynamical networks,” *Automatica*, vol. 46, no. 7, pp. 1215–1221, 2010.

[16] W. He and J. Cao, “Global synchronization in arrays of coupled networks with one single time-varying delay coupling,” *Physics Letters A*, vol. 373, no. 31, pp. 2682–2694, 2009.

[17] J. Cao, Z. Wang, and Y. Sun, “Synchronization in an array of linearly stochastically coupled networks with time delays,” *Physica A*, vol. 385, no. 2, pp. 718–728, 2007.

[18] W. Yu, J. Cao, G. Chen, J. Lü, J. Han, and W. Wei, “Local synchronization of a complex network model,” *IEEE Transactions on Systems, Man, and Cybernetics B*, vol. 39, no. 1, pp. 230–241, 2009.

[19] J. Cao and L. Li, “Cluster synchronization in an array of hybrid coupled neural networks with delay,” *Neural Networks*, vol. 22, no. 4, pp. 335–342, 2009.

[20] Q. Song, “Synchronization analysis of coupled connected neural networks with mixed time delays,” *Neurocomputing*, vol. 72, no. 16-18, pp. 3907–3914, 2009.

[21] B. Shen, Z. Wang, and X. Liu, “Bounded H_∞ synchronization and state estimation for discrete time-varying stochastic complex networks over a finite horizon,” *IEEE Transactions on Neural Networks*, vol. 22, no. 1, pp. 145–157, 2011.

[22] H. J. Gao, J. Lam, and G. Chen, “New criteria for synchronization stability of general complex dynamical networks with coupling delays,” *Physics Letters A*, vol. 360, no. 2, pp. 263–273, 2006.

[23] Y. Tang, R. Qiu, J. Fang, Q. Miao, and M. Xia, "Adaptive lag synchronization in unknown stochastic chaotic neural networks with discrete and distributed time-varying delays," *Physics Letters A*, vol. 372, no. 24, pp. 4425–4433, 2008.

[24] Y. Liu, Z. Wang, and X. Liu, "Exponential synchronization of complex networks with Markovian jump and mixed delays," *Physics Letters A*, vol. 372, no. 22, pp. 3986–3998, 2008.

[25] T. Li, S. M. Fei, and K. J. Zhang, "Synchronization control of recurrent neural networks with distributed delays," *Physica A*, vol. 387, no. 4, pp. 982–996, 2008.

[26] K. Gopalsamy and X. Z. He, "Stability in asymmetric Hopfield nets with transmission delays," *Physica D*, vol. 76, no. 4, pp. 344–358, 1994.

[27] C. Huang and J. Cao, "Almost sure exponential stability of stochastic cellular neural networks with unbounded distributed delays," *Neurocomputing*, vol. 72, no. 13–15, pp. 3352–3356, 2009.

[28] L. Sheng and H. Yang, "Exponential synchronization of a class of neural networks with mixed time-varying delays and impulsive effects," *Neurocomputing*, vol. 71, no. 16–18, pp. 3666–3674, 2008.

[29] X. Nie and J. Cao, "Multistability of competitive neural networks with time-varying and distributed delays," *Nonlinear Analysis: Real World Applications*, vol. 10, no. 2, pp. 928–942, 2009.

[30] X. Yang, "Existence and global exponential stability of periodic solution for Cohen-Grossberg shunting inhibitory cellular neural networks with delays and impulses," *Neurocomputing*, vol. 72, no. 10–12, pp. 2219–2226, 2009.

[31] Z. Wang, Y. Liu, M. Li, and X. Liu, "Stability analysis for stochastic Cohen-Grossberg neural networks with mixed time delays," *IEEE Transactions on Neural Networks*, vol. 17, no. 3, pp. 814–820, 2006.

[32] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, 1990.

[33] Z. H. Guan, Z. W. Liu, G. Feng, and Y. W. Wang, "Synchronization of complex dynamical networks with time-varying delays via impulsive distributed control," *IEEE Transactions on Circuits and Systems I*, vol. 57, no. 8, pp. 2182–2195, 2010.

[34] Y. Wang, L. Xie, and C. E. de Souza, "Robust control of a class of uncertain nonlinear systems," *Systems & Control Letters*, vol. 19, no. 2, pp. 139–149, 1992.

[35] X. Yang, J. Cao, and J. Lu, "Synchronization of coupled neural networks with random coupling strengths and mixed probabilistic time-varying delays," *International Journal of Robust and Nonlinear Control*, 2012.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 752953, 12 pages
http://dx.doi.org/10.1155/2013/752953

Research Article

Stationary in Distributions of Numerical Solutions for Stochastic Partial Differential Equations with Markovian Switching

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Received 30 December 2012; Accepted 24 February 2013

Academic Editor: Qi Luo

We investigate a class of stochastic partial differential equations with Markovian switching. By using the Euler-Maruyama scheme both in time and in space of mild solutions, we derive sufficient conditions for the existence and uniqueness of the stationary distributions of numerical solutions. Finally, one example is given to illustrate the theory.

1. Introduction

The theory of numerical solutions of stochastic partial differential equations (SPDEs) has been well developed by many authors [1–5]. In [2], Debussche considered the error of the Euler scheme for the nonlinear stochastic partial differential equations by using Malliavin calculus. Gyöngy and Millet [3] discussed the convergence rate of space time approximations for stochastic evolution equations. Shardlow [5] investigated the numerical methods of the mild solutions for stochastic parabolic PDEs derived by space-time white noise by applying finite difference approach.

On the other hand, the parameters of SPDEs may experience abrupt changes caused by phenomena such as component failures or repairs, changing subsystem interconnections, and abrupt environmental disturbances [6–9], and the continuous-time Markov chains have been used to model these parameter jumps. An important equation is a class of SPDEs with Markovian switching

$$\begin{aligned} dX(t) = {} & [AX(t) + f(X(t), r(t))]\,dt \\ & + g(X(t), r(t))\,dW(t), \quad t \geq 0. \end{aligned} \tag{1}$$

Here the state vector has two components $X(t)$ and $r(t)$, the first one is normally referred to as the state while the second one is regarded as the mode. In its operation, the system will switch from one mode to another one in a random way, and the switching among the modes is governed by the Markov chain $r(t)$.

Since only a few SPDEs with Markovian switching have explicit formulae, numerical (approximate) schemes of SPDEs with Markovian switching are becoming more and more popular. In this paper, we will study the stationary distribution of numerical solutions of SPDEs with Markovian switching. Bao et al. [10] investigated the stability in distribution of mild solutions to SPDEs. Bao and Yuan [11] discussed the numerical approximation of stationary distribution for SPDEs. For the stationary distribution of numerical solutions of stochastic differential equations in finite-dimensional space, Mao et al. [12] utilized the Euler-Maruyama scheme with variable step size to obtain the stationary distribution and they also proved that the probability measures induced by the numerical solutions converge weakly to the stationary distribution of the true solution. But since the mild solutions of SPDEs with Markovian switching do not have stochastic differential, a significant consequence of this fact is that the Itô formula cannot be used for mild solutions of SPDEs with Markovian switching directly. Consequently, we generalize the stationary distribution of numerical solutions of the finite dimensional stochastic differential equations with Markovian switching to that of infinite dimensional cases.

Motived by [11–13], we will show in this paper that the mild solutions of SPDE with Markovian switching (1) have a unique stationary distribution for sufficiently small step size.

So this paper is organised as follows: in Section 2, we give necessary notations and define Euler-Maruyama scheme of mild solutions. In Section 3, we give some lemmas and the main result in this paper. Finally, we will give an example to illustrate the theory in Section 4.

2. Statements of Problem

Throughout this paper, unless otherwise specified, we let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ be a real separable Hilbert space and $W(t)$ an H-valued cylindrical Brownian motion (Wiener process) defined on the probability space. Let I_G be the indicator function of a set G. Denote by $(\mathcal{L}(H), \|\cdot\|)$ and $(\mathcal{L}_{\mathrm{HS}}(H), \|\cdot\|_{\mathrm{HS}})$ the family of bounded linear operators and Hilbert-Schmidt operator from H into H, respectively. Let $r(t)$, $t \geq 0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $\mathbb{S} = \{1, 2, \ldots, N\}$ with the generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$\mathbb{P}\{r(t+\delta) = j \mid r(t) = i\} = \begin{cases} \gamma_{ij}\delta + o(\delta) & \text{if } i \neq j, \\ 1 + \gamma_{ij}\delta + o(\delta) & \text{if } i = j, \end{cases} \tag{2}$$

where $\delta > 0$. Here $\gamma_{ij} > 0$ is the transition rate from i to j if $i \neq j$ while

$$\gamma_{ii} = -\sum_{j\neq i}\gamma_{ij}. \tag{3}$$

We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $W(\cdot)$. It is well known that almost every sample path of $r(\cdot)$ is a right-continuous step function with finite number of simple jumps in any finite subinterval of $\mathbb{R}_+ := [0, +\infty)$.

Consider SPDEs with Markovian switching on H

$$\begin{aligned} dX(t) = {} & [AX(t) + f(X(t), r(t))]\,dt \\ & + g(X(t), r(t))\,dW(t), \quad t \geq 0, \end{aligned} \tag{4}$$

with initial value $X(0) = x \in H$ and $r(0) = i \in \mathbb{S}$. Here $f : H \times \mathbb{S} \to H$, $g : H \times \mathbb{S} \to \mathcal{L}_{\mathrm{HS}}(H)$. Throughout the paper, we impose the following assumptions.

(A1) $(A, \mathcal{D}(A))$ is a self-adjoint operator on H generating a C_0-semigroup $\{e^{At}\}_{t\geq 0}$, such that $\| e^{At} \| \leq e^{-\alpha t}$ for some $\alpha > 0$. In this case, $-A$ has discrete spectrum $0 < \rho_1 \leq \rho_2 \leq \cdots \leq \lim_{i\to\infty}\rho_i = \infty$ with corresponding eigenbasis $\{e_i\}_{i\geq 1}$ of H.

(A2) Both f and g are globally Lipschitz continuous. That is, there exists a constant $L > 0$ such that

$$\begin{aligned} & \|f(x,j) - f(y,j)\|_H^2 \vee \|g(x,j) - g(y,j)\|_{\mathrm{HS}}^2 \\ & \quad \leq L\|x - y\|_H^2, \quad \forall x, y \in H,\ j \in \mathbb{S}; \end{aligned} \tag{5}$$

(A3) There exist $\mu > 0$ and $\lambda_j > 0$, $(j = 1, 2, \ldots, N)$ such that

$$\begin{aligned} & 2\lambda_j\langle x - y, f(x,j) - f(y,j)\rangle_H + \lambda_j\|g(x,j) - g(y,j)\|_{\mathrm{HS}}^2 \\ & \quad + \sum_{l=1}^{N}\gamma_{jl}\lambda_l\|x - y\|_H^2 \leq -\mu\|x - y\|_H^2, \quad \forall x, y \in H,\ j \in \mathbb{S}. \end{aligned} \tag{6}$$

It is well known (see [1, 8]) that under (A1)–(A3), (4) has a unique mild solution $X(t)$ on $t \geq 0$. That is, for any $X(0) = x \in H$ and $r(0) = i \in \mathbb{S}$, there exists a unique H-valued adapted process $X(t)$ such that

$$\begin{aligned} X(t) = {} & e^{tA}x + \int_0^t e^{(t-s)A} f(X(s), r(s))\,ds \\ & + \int_0^t e^{(t-s)A} g(X(s), r(s))\,dW(s). \end{aligned} \tag{7}$$

Moreover, the pair $Z(t) = (X(t), r(t))$ is a time-homogeneous Markov process.

Remark 1. We observe that (A2) implies the following linear growth conditions:

$$|f(x,j)|_H^2 \vee \|g(x,j)\|_{\mathrm{HS}}^2 \leq \overline{L}\left(1 + \|x\|_H^2\right), \quad \forall x \in H,\ j \in \mathbb{S}, \tag{8}$$

where $\overline{L} = 2\max_{j\in\mathbb{S}}(L \vee |f(0,j)|_H^2 \vee \| g(0,j)\|_{\mathrm{HS}}^2)$.

Remark 2. We also establish another property from (A3):

$$\begin{aligned} & 2\lambda_j\langle x, f(x,j)\rangle_H + \lambda_j\|g(x,j)\|_{\mathrm{HS}}^2 + \sum_{l=1}^{N}\gamma_{jl}\lambda_l\|x\|_H^2 \\ & \quad \leq 2\lambda_j\langle x, f(x,j) - f(0,j)\rangle_H \\ & \qquad + \lambda_j\|g(x,j) - g(0,j)\|_{\mathrm{HS}}^2 + \sum_{l=1}^{N}\gamma_{jl}\lambda_l\|x\|_H^2 \\ & \qquad + 2\lambda_j\langle x, f(0,j)\rangle_H \\ & \qquad + 2\lambda_j\langle g(x,j) - g(0,j), g(0,j)\rangle_{\mathrm{HS}} + \lambda_j\|g(0,j)\|_{\mathrm{HS}}^2 \\ & \quad \leq -\mu\|x\|_H^2 + \frac{\mu}{4}\|x\|_H^2 + \frac{4\lambda_j^2\|f(0,j)\|_H}{\mu} \\ & \qquad + \frac{\mu}{4L}\|g(x,j) - g(0,j)\|_{\mathrm{HS}}^2 \\ & \qquad + \frac{4L\lambda_j^2}{\mu}\|g(0,j)\|_{\mathrm{HS}}^2 + \lambda_j\|g(0,j)\|_{\mathrm{HS}}^2 \\ & \quad \leq -\mu\|x\|_H^2 + \frac{\mu}{4}\|x\|_H^2 + \frac{\mu}{4}\|x\|_H^2 + \frac{4\lambda_j^2\|f(0,j)\|_H}{\mu} \end{aligned}$$

$$+\frac{4L\lambda_j^2}{\mu}\|g(0,j)\|_{\mathrm{HS}}^2+\lambda_j\|g(0,j)\|_{\mathrm{HS}}^2$$
$$\le -\frac{\mu}{2}\|x\|_H^2+\alpha_1,\quad \forall x\in H,\ j\in\mathbb{S}, \tag{9}$$

where $\alpha_1 := \max_{j\in\mathbb{S}}[(4\lambda_j^2\ \|f(0,j)\|_H^2/\mu)\ +(4L\lambda_j^2/\mu)\ \|g(0,j)\|_{\mathrm{HS}}^2+\lambda_j\ \|\ g(0,j)\|_{\mathrm{HS}}^2]$ and $\langle T,S\rangle_{\mathrm{HS}} := \sum_{i=1}^{\infty}\langle Te_i,Se_j\rangle_H$ for $S,T\in\mathscr{L}_{\mathrm{HS}}(H)$.

Denote by $Z^{x,i}(t)=(X^{x,i}(t),r^i(t))$ the mild solution of (4) starting from $(x,i)\in H\times\mathbb{S}$. For any subset $A\in\mathfrak{B}(H)$, $B\subset\mathbb{S}$, let $\mathbb{P}_t((x,i),A\times B)$ be the probability measure induced by $Z^{x,i}(t)$, $t\ge 0$. Namely,

$$\mathbb{P}_t((x,i),A\times B)=\mathbb{P}\left(Z^{x,i}\in A\times B\right), \tag{10}$$

where $\mathfrak{B}(H)$ is the family of the Borel subset of H.

Denote by $\mathscr{P}(H\times\mathbb{S})$ the family by all probability measures on $H\times\mathbb{S}$. For $P_1,P_2\in\mathscr{P}(H\times\mathbb{S})$, define the metric $d_{\mathbb{L}}$ as follows:

$$d_{\mathbb{L}}(P_1,P_2)=\sup_{\varphi\in\mathbb{L}}\left|\sum_{j=1}^{N}\int_H\varphi(u,j)P_1(du,j)-\sum_{j=1}^{N}\int\varphi(u,j)P_2(du,j)\right|, \tag{11}$$

where $\mathbb{L}=\{\varphi: H\times\mathbb{S}\to\mathbb{R} : |\varphi(u,j)-\varphi(v,l)|\le\|u-v\|_H+|j-l|$, and $|\varphi(u,j)|\le 1$, for $u,v\in K$, $j,l\in\mathbb{S}\}$.

Remark 3. It is known that the weak convergence of probability measures is a metric concept with respect to classes of test function. In other words, a sequence of probability measures $\{P_k\}_{k\ge1}$ of $\mathscr{P}(H\times\mathbb{S})$ converges weakly to a probability measure $P_0\in\mathscr{P}(H\times\mathbb{S})$ if and only if $\lim_{k\to\infty}d_{\mathbb{L}}(P_k,P_0)=0$.

Definition 4. The mild solution $Z(t)=(X(t),r(t))$ of (4) is said to have a stationary distribution $\pi(\cdot\times\cdot)\in\mathscr{P}(H\times\mathbb{S})$ if the probability measure $\mathbb{P}_t((x,i),(\cdot\times\cdot))$ converges weakly to $\pi(\cdot\times\cdot)$ as $t\to\infty$ for every $i\in\mathbb{S}$, and every $x\in U$, a bounded subset of H, that is,

$$\lim_{t\to\infty}d_{\mathbb{L}}\left(\mathbb{P}_t(x,i),\pi(\cdot\times\cdot)\right)=\lim_{t\to\infty}\left(\sup_{\varphi\in\mathbb{L}}\left|\mathbb{E}\varphi\left(Z^{x,i}(t)\right)-\sum_{j=1}^{N}\int_H\varphi(u,j)\pi(du,j)\right|\right)=0. \tag{12}$$

By Theorem 3.1 in [10] and Theorem 3.1 in [14], we have the following.

Theorem 5. *Under (A1)–(A3), the Markov process $Z(t)$ has a unique stationary distribution $\pi(\cdot\times\cdot)\in\mathscr{P}(H\times\mathbb{S})$.*

For any $n\ge1$, let $\pi_n : H\to H_n := \mathrm{Span}\{e_1,e_2,\ldots,e_n\}$ be the orthogonal projection. Consider SPDEs with Markovian switching on H_n,

$$dX^n(t)=\left[A_nX^n(t)+f_n(X^n(t),r(t))\right]dt+g_n(X^n(t),r(t))\,dW(t), \tag{13}$$

with initial data $X^n(0)=\pi_nx=\sum_{i=1}^{n}\langle x,e_i\rangle_He_i$, $x\in H$. Here $A_n=\pi_nA$, $f_n=\pi_nf$, $g_n=\pi_ng$.

Therefore, we can observe that

$$A_nx=Ax,\qquad e^{tA_nx}=e^{tAx},\qquad \langle x,f_n\rangle_H=\langle x,f\rangle_H,$$
$$\langle x,g_n\rangle_H=\langle x,g\rangle_H,\quad\forall x\in H_n, \tag{14}$$

By the property of the projection operator and (A2), we have

$$\begin{aligned}&\|A_n(x-y)\|_H^2\vee\|f_n(x,j)-f_n(y,j)\|_H^2\\&\quad\vee\|g_n(x,j)-g_n(y,j)\|_{\mathrm{HS}}^2\\&\quad\le\lambda_n^2\|(x-y)\|_H^2\vee\|f(x,j)-f(y,j)\|_H^2\\&\quad\vee\|g(x,j)-g(y,j)\|_{\mathrm{HS}}^2\le\left(\lambda_n^2\vee L\right)\|(x-y)\|_H^2,\\&\qquad\forall x,y\in H_n,\ j\in\mathbb{S}.\end{aligned} \tag{15}$$

Hence, (13) admits a unique strong solution $\{X^n(t)\}_{t\ge0}$ on H_n (see [8]).

We now introduce an Euler-Maruyama based computational method. The method makes use of the following lemma (see [15]).

Lemma 6. *Given $\Delta>0$, then $\{r(k\Delta),\ k=0,1,2,\ldots\}$ is a discrete Markov chain with the one-step transition probability matrix*

$$P(\Delta)=\left(P_{i,j}(\Delta)\right)_{N\times N}=e^{\Delta\Gamma}. \tag{16}$$

Given a fixed step size $\Delta>0$ and the one-step transition probability matrix $P(\Delta)$ in (16), the discrete Markov chain $\{r(k\Delta),\ k=0,1,2,\ldots\}$ can be simulated as follows: let $r(0)=i_0$, and compute a pseudorandom number ξ_1 from the uniform $(0,1)$ distribution.

Define

$$r(\Delta)=\begin{cases}i, & i\in\mathbb{S}-\{N\}\ \text{such that}\ \sum_{j=1}^{i-1}P_{r(0),j}(\Delta)\le\xi_1<\sum_{j=1}^{i}P_{r(0),j}(\Delta),\\ N, & \sum_{j=1}^{N-1}P_{r(0),j}(\Delta)\le\xi_1,\end{cases} \tag{17}$$

where we set $\sum_{j=1}^{0} P_{r(0),j}(\Delta) = 0$ as usual. Having computed $r(0), r(\Delta), \ldots, r(k\Delta)$, we can compute $r((k+1)\Delta)$ by drawing a uniform $(0,1)$ pseudorandom number ξ_{k+1} and setting

$$r((k+1)\Delta) = \begin{cases} i, & i \in \mathbb{S} - \{N\} \text{ such that } \sum_{j=1}^{i-1} P_{r(k\Delta),j}(\Delta) \leq \xi_{k+1} < \sum_{j=1}^{i} P_{r(k\Delta),j}(\Delta), \\ N, & \sum_{j=1}^{N-1} P_{r(k\Delta),j}(\Delta) \leq \xi_{k+1}. \end{cases} \tag{18}$$

The procedure can be carried out repeatedly to obtain more trajectories.

We now define the Euler-Maruyama approximation for (13). For a stepsize $\Delta \in (0,1)$, the discrete approximation $\overline{Y}^n(k\Delta) \approx X^n(k\Delta)$, is formed by simulating from $\overline{Y}^n(0) = \pi_n x$, $r(0) = r_0$, and

$$\overline{Y}^n((k+1)\Delta) = e^{\Delta A_n}\left\{\overline{Y}^n(k\Delta) + f_n\left(\overline{Y}^n(k\Delta), r(k\Delta)\right)\Delta + g_n\left(\overline{Y}^n(k\Delta), r(k\Delta)\right)\Delta W_k\right\}, \tag{19}$$

where $\Delta W_k = W((k+1)\Delta) - W(k\Delta))$.

To carry out our analysis conveniently, we give the continuous Euler-Maruyama approximation solution which is defined by

$$\begin{aligned} Y^n(t) &= e^{tA_n}\pi_n x + \int_0^t e^{(t-\lfloor s\rfloor)A_n} f_n(Y^n(\lfloor s\rfloor), r(\lfloor s\rfloor))\, ds + \int_0^t e^{(t-\lfloor s\rfloor)A_n} g_n(Y^n(\lfloor s\rfloor), r(\lfloor s\rfloor))\, dW(s) \\ &= e^{tA_n}\pi_n x + \int_0^t e^{(t-\lfloor s\rfloor)A} f_n(Y^n(\lfloor s\rfloor), r(\lfloor s\rfloor))\, ds + \int_0^t e^{(t-\lfloor s\rfloor)A} g_n(Y^n(\lfloor s\rfloor), r(\lfloor s\rfloor))\, dW(s), \end{aligned} \tag{20}$$

where $\lfloor t\rfloor = [t/\Delta]\Delta$ and $[t/\Delta]$ denotes the integer part of t/Δ and $Y^n(0) = \overline{Y}^n(0) = \pi_n x$, and $Y^n(k\Delta) = \overline{Y}^n(k\Delta)$.

It is obvious that $Y^n(t)$ coincides with the discrete approximation solution at the gridpoints. For any Borel set $A \in \mathfrak{B}(H_n)$, $x \in H_n$, $i, j \in \mathbb{S}$, let $\overline{Z}^n(k\Delta) = (\overline{Y}^n(k\Delta), r(k\Delta))$,

$$\begin{aligned} \mathbb{P}^{n,\Delta}((x,i), A\times\{j\}) &:= \mathbb{P}\left(\overline{Z}^n(\Delta) \in A\times\{j\} \mid \overline{Z}^n(0) = (x,i)\right), \\ \mathbb{P}_k^{n,\Delta}((x,i), A\times\{j\}) &:= \mathbb{P}\left(\overline{Z}^n(k\Delta) \in A\times\{j\} \mid \overline{Z}^n(0) = (x,i)\right). \end{aligned} \tag{21}$$

Following the argument of Theorem 5 in [13], we have the following.

Lemma 7. *$\{\overline{Z}^n(k\Delta)\}_{k\geq 0}$ is a homogeneous Markov process with the transition probability kernel $\mathbb{P}^{n,\Delta}((x,i), A\times\{j\})$.*

To highlight the initial value, we will use notation $\{\overline{Z}^{n,(x,i)}(k\Delta)\}$.

Definition 8. For a given stepsize $\Delta > 0$, $\{\overline{Z}^{n,(x,i)}(k\Delta)\}_{k\geq 0}$ is said to have a stationary distribution $\{\pi^{n,\Delta}(\cdot\times\cdot)\} \in \mathscr{P}(H_n\times\mathbb{S})$ if the k-step transition probability kernel $\mathbb{P}_k^{n,\Delta}((x,i),\cdot\times\cdot)$ converges weakly to $\pi^{n,\Delta}(\cdot\times\cdot)$ as $k\to\infty$, for every $(x,i)\in H_n\times\mathbb{S}$, that is,

$$\lim_{k\to\infty} d_{\mathbb{L}}\left(P_k^{n,\Delta}((x,i),\cdot\times\cdot), \pi^{n,\Delta}(\cdot\times\cdot)\right) = 0. \tag{22}$$

We will establish our result of this paper in Section 3.

Theorem 9. *Under (A1)–(A3), for a given stepsize $\Delta > 0$, and arbitrary $x \in H_n$, $i \in \mathbb{S}$, $\{\overline{Z}^{n,(x,i)}(k\Delta)\}_{k\geq 0}$ has a unique stationary distribution $\pi^{n,\Delta}(\cdot\times\cdot) \in \mathscr{P}(H_n\times\mathbb{S})$.*

3. Stationary in Distribution of Numerical Solutions

In this section, we shall present some useful lemmas and prove Theorem 9. In what follows, $C > 0$ is a generic constant whose values may change from line to line.

For any initial value (x,i), let $Y^{n,x,i}(t)$ be the continuous Euler-Maruyama solution of (20) and starting from $(x,i) \in H\times\mathbb{S}$. Let $X^{x,i}(t)$ be the mild solution of (4) and starting from $(x,i)\in H\times\mathbb{S}$.

Lemma 10. *Under (A1)–(A3), then*

$$\mathbb{E}\left\|Y^{n,x,i}(t) - Y^{n,x,i}(\lfloor t\rfloor)\right\|_H^2 \leq 3\left(\rho_n^2 + 2\overline{L}\right)\Delta\left(1 + \mathbb{E}\left\|Y^n(\lfloor t\rfloor)\right\|_H^2\right). \tag{23}$$

Proof. Write $Y^{n,x,i}(t) = Y^n(t)$, $Y^{n,x,i}(\lfloor t\rfloor) = Y^n(\lfloor t\rfloor)$. From (20), we have

$$Y^n(\lfloor t\rfloor) = e^{\lfloor t\rfloor A}\pi_n x + \int_0^{\lfloor t\rfloor} e^{(\lfloor t\rfloor-\lfloor s\rfloor)A} f_n(Y^n(\lfloor s\rfloor), r(\lfloor s\rfloor))\, ds + \int_0^{\lfloor t\rfloor} e^{(\lfloor t\rfloor-\lfloor s\rfloor)A} g_n(Y^n(\lfloor s\rfloor), r(\lfloor s\rfloor))\, dW(s). \tag{24}$$

Thus,

$$\begin{aligned}
&Y^n(t) - Y^n(\lfloor t\rfloor) \\
&= e^{(t-\lfloor t\rfloor)A} \\
&\quad\times \Big(e^{\lfloor t\rfloor A}\pi_n x \\
&\qquad + \int_0^{\lfloor t\rfloor} e^{(\lfloor t\rfloor-\lfloor s\rfloor)A} \\
&\qquad\quad\times f_n(Y^n(\lfloor s\rfloor), r(\lfloor s\rfloor))\,ds \\
&\qquad + \int_0^{\lfloor t\rfloor} e^{(\lfloor t\rfloor-\lfloor s\rfloor)A} \\
&\qquad\quad\times g_n(Y^n(\lfloor s\rfloor), r(\lfloor s\rfloor))\,dW(s)\Big) \\
&\quad - Y^n(\lfloor t\rfloor) \\
&\quad + \int_{\lfloor t\rfloor}^t e^{(t-\lfloor s\rfloor)A} f_n(Y^n(\lfloor s\rfloor), r(\lfloor s\rfloor))\,ds \\
&\quad + \int_{\lfloor t\rfloor}^t e^{(t-\lfloor s\rfloor)A} g_n(Y^n(\lfloor s\rfloor), r(\lfloor s\rfloor))\,dW(s) \\
&= \left(e^{(t-\lfloor t\rfloor)A} - \mathbf{1}\right) Y^n(\lfloor t\rfloor) \\
&\quad + \int_{\lfloor t\rfloor}^t e^{(t-\lfloor s\rfloor)A} f_n(Y^n(\lfloor s\rfloor), r(\lfloor s\rfloor))\,ds \\
&\quad + \int_{\lfloor t\rfloor}^t e^{(t-\lfloor s\rfloor)A} g_n(Y^n(\lfloor s\rfloor), r(\lfloor s\rfloor))\,dW(s).
\end{aligned} \tag{25}$$

Then, by the Hölder inequality and the Itô isometry, we obtain

$$\begin{aligned}
&\mathbb{E}\left\|Y^n(t) - Y^n(\lfloor t\rfloor)\right\|_H^2 \\
&\quad\leq 3\Big\{\mathbb{E}\left\|\left(e^{(t-\lfloor t\rfloor)A} - \mathbf{1}\right) Y^n(\lfloor t\rfloor)\right\|_H^2 \\
&\qquad + \mathbb{E}\int_{\lfloor t\rfloor}^t \left\|f_n(Y^n(\lfloor s\rfloor), r(\lfloor s\rfloor))\right\|_H^2 ds \\
&\qquad + \mathbb{E}\int_{\lfloor t\rfloor}^t \left\|g_n(Y^n(\lfloor s\rfloor), r(\lfloor s\rfloor))\right\|_{HS}^2 ds\Big\}.
\end{aligned} \tag{26}$$

From (A1), we have

$$\begin{aligned}
&\mathbb{E}\left\|\left(e^{(t-\lfloor t\rfloor)A} - \mathbf{1}\right) Y^n(\lfloor t\rfloor)\right\|_H^2 \\
&\quad= \mathbb{E}\left\|\sum_{i=1}^n \left(e^{-\rho_i(t-\lfloor t\rfloor)} - 1\right)\langle Y^n(\lfloor t\rfloor), e_i\rangle_H e_i\right\|_H^2 \\
&\quad\leq \left(1 - e^{-\rho_n(t-\lfloor t\rfloor)}\right)^2 \mathbb{E}\left\|Y^n(\lfloor t\rfloor)\right\|_H^2 \\
&\quad\leq \rho_n^2\Delta^2\mathbb{E}\left\|Y^n(\lfloor t\rfloor)\right\|_H^2,
\end{aligned} \tag{27}$$

here we use the fundamental inequality $1 - e^{-a} \leq a$, $a > 0$. And, by (8), it follows that

$$\begin{aligned}
&\mathbb{E}\int_{\lfloor t\rfloor}^t \left\|f_n(Y^n(\lfloor s\rfloor), r(\lfloor s\rfloor))\right\|_H^2 ds \\
&\qquad + \mathbb{E}\int_{\lfloor t\rfloor}^t \left\|g_n(Y^n(\lfloor s\rfloor), r(\lfloor s\rfloor))\right\|_{HS}^2 ds \\
&\quad\leq 2\overline{L}\Delta\left(1 + \mathbb{E}\left\|Y^n(\lfloor t\rfloor)\right\|_H^2\right).
\end{aligned} \tag{28}$$

Substituting (27) and (28) into (26), the desired assertion (23) follows. □

Lemma 11. *Under (A1)–(A3), if* $\Delta < \min\{1, 1/3(\rho_n^2 + 2\overline{L}), ((4\alpha p + \mu)/(8q + 4q\rho_n^2\overline{L} + 4q\overline{L} + 24q\hat{r} + 6qL(\rho_n^2 + 2\overline{L})))^2\}$, *then there is a constant* $C > 0$ *that depends on the initial value* x *but is independent of* Δ, *such that the continuous Euler-Maruyama solution of* (20) *has*

$$\sup_{t\geq 0}\mathbb{E}\left\|Y^{n,x,i}(t)\right\|_H \leq C, \tag{29}$$

where $q = \max_{1\leq i\leq N}\lambda_i$, $p = \min_{1\leq i\leq N}\lambda_i$.

Proof. Write $Y^{n,x,i}(t) = Y^n(t)$, $r^i(k\Delta) = r(k\Delta)$. From (20), we have the following differential form:

$$\begin{aligned}
&dY^n(t) \\
&\quad= \left\{AY^n(t) + e^{(t-\lfloor t\rfloor)A} f_n(Y^n(\lfloor t\rfloor), r(\lfloor t\rfloor))\right\} dt \\
&\qquad + e^{(t-\lfloor t\rfloor)A} g_n(Y^n(\lfloor t\rfloor), r(\lfloor t\rfloor))\,dW(t),
\end{aligned} \tag{30}$$

with $Y^n(0) = \pi_n x$.

Let $V(x,i) = \lambda_i \| x\|_H^2$. By the generalised Itô formula, for any $\theta > 0$, we derive from (30) that

$$\begin{aligned}
&e^{\theta t}\mathbb{E}\left(\lambda_{r(t)}\left\|Y^n(t)\right\|_H^2\right) \\
&\quad\leq \lambda_i\|x\|_H^2 + \mathbb{E}\int_0^t e^{\theta s}\lambda_{r(s)} \\
&\qquad\times\Big\{\theta\left\|Y^n(s)\right\|_H^2 + 2\left\langle Y^n(s), AY^n(s)\right\rangle_H \\
&\qquad\quad + 2\left\langle Y^n(s), e^{(s-\lfloor s\rfloor)A} f_n(Y^n(\lfloor s\rfloor), r(\lfloor s\rfloor))\right\rangle_H \\
&\qquad\quad + \left\|g_n(Y^n(\lfloor s\rfloor), r(\lfloor s\rfloor))\right\|_{HS}^2\Big\} ds \\
&\qquad + \mathbb{E}\int_0^t e^{\theta s}\sum_{l=1}^N \gamma_{r(s)l}\lambda_l\left\|Y^n(s)\right\|_H^2 ds
\end{aligned}$$

$$\begin{aligned}
&\leq q\|x\|_H^2 + \theta q \mathbb{E}\int_0^t e^{\theta s}\|Y^n(s)\|_H^2 ds \\
&\quad - 2\alpha p \mathbb{E}\int_0^t e^{\theta s}\|Y^n(s)\|_H^2 ds \\
&\quad + \mathbb{E}\int_0^t e^{\theta s}\lambda_{r(s)} \\
&\qquad \times \left\{2\left\langle Y^n(s), e^{(s-\lfloor s\rfloor)A} f\left(Y^n(\lfloor s\rfloor), r(\lfloor s\rfloor)\right)\right\rangle_H \right. \\
&\qquad\quad \left. + \left\|g\left(Y^n(\lfloor s\rfloor), r(\lfloor s\rfloor)\right)\right\|_{\mathrm{HS}}^2\right\} ds \\
&\quad + \mathbb{E}\int_0^t e^{\theta s}\sum_{l=1}^N \gamma_{r(s)l}\lambda_l\|Y^n(s)\|_H^2 ds.
\end{aligned} \tag{31}$$

By the fundamental transformation, we obtain that

$$\begin{aligned}
&\left\langle Y^n(t), e^{(t-\lfloor t\rfloor)A} f\left(Y^n(\lfloor t\rfloor), r(\lfloor t\rfloor)\right)\right\rangle_H \\
&\quad = \left\langle Y^n(t), f\left(Y^n(t), r(t)\right)\right\rangle_H \\
&\qquad + \left\langle Y^n(t), \left(e^{(t-\lfloor t\rfloor)A} - \mathbf{1}\right) f\left(Y^n(t), r(t)\right)\right\rangle_H \\
&\qquad + \left\langle Y^n(t), e^{(t-\lfloor t\rfloor)A}\left(f\left(Y^n(\lfloor t\rfloor), r(\lfloor t\rfloor)\right) - f\left(Y^n(t), r(t)\right)\right)\right\rangle_H.
\end{aligned} \tag{32}$$

By Höld inequality, we have

$$\begin{aligned}
&\left\|g\left(Y^n(\lfloor t\rfloor), r(\lfloor t\rfloor)\right)\right\|_{\mathrm{HS}}^2 \\
&\quad = \left\|g\left(Y^n(t), r(t)\right) - \left(g\left(Y^n(t), r(t)\right) - g\left(Y^n(\lfloor t\rfloor), r(\lfloor t\rfloor)\right)\right)\right\|_{\mathrm{HS}}^2 \\
&\quad \leq \left(1+\Delta^{1/2}\right)\left\|g\left(Y^n(t), r(t)\right)\right\|_{\mathrm{HS}}^2 + \left(1+\Delta^{-1/2}\right) \\
&\qquad \times \left\|\left(g\left(Y^n(t), r(t)\right) - g\left(Y^n(\lfloor t\rfloor), r(\lfloor t\rfloor)\right)\right)\right\|_{\mathrm{HS}}^2.
\end{aligned} \tag{33}$$

Then, from (31), we have

$$\begin{aligned}
&e^{\theta t}\mathbb{E}\left(\lambda_{r(t)}\|Y^n(t)\|_H^2\right) \\
&\quad \leq q\|x\|_H^2 + \theta q \mathbb{E}\int_0^t e^{\theta s}\|Y^n(s)\|_H^2 ds \\
&\qquad - 2\alpha p \mathbb{E}\int_0^t e^{\theta s}\|Y^n(s)\|_H^2 ds \\
&\qquad + \mathbb{E}\int_0^t e^{\theta s}\lambda_{r(s)}\left\{2\left\langle Y^n(s), f\left(Y^n(s), r(s)\right)\right\rangle_H + \left\|g\left(Y^n(s), r(s)\right)\right\|_{\mathrm{HS}}^2\right\} ds \\
&\qquad + \mathbb{E}\int_0^t e^{\theta s}\sum_{l=1}^N \gamma_{r(s)l}\lambda_l\|Y^n(s)\|_H^2 ds \\
&\qquad + \mathbb{E}\int_0^t e^{\theta s}\lambda_{r(s)} \\
&\qquad\quad \times \left\{2\left\langle Y^n(s), \left(e^{(s-\lfloor s\rfloor)A} - \mathbf{1}\right) f\left(Y^n(s), r(s)\right)\right\rangle_H \right. \\
&\qquad\qquad + 2\left\langle Y^n(s), e^{(s-\lfloor s\rfloor)A} \times \left(f\left(Y^n(\lfloor s\rfloor), r(\lfloor s\rfloor)\right) - f\left(Y^n(s), r(s)\right)\right)\right\rangle_H \\
&\qquad\qquad + \Delta^{1/2}\left\|g\left(Y^n(s), r(s)\right)\right\|_{\mathrm{HS}}^2 + \left(1+\Delta^{-1/2}\right) \\
&\qquad\qquad \left. \times \left\|\left(g\left(Y^n(s), r(s)\right) - g\left(Y^n(\lfloor s\rfloor), r(\lfloor s\rfloor)\right)\right)\right\|_{\mathrm{HS}}^2\right\} ds \\
&\quad \leq q\|x\|_H^2 + \theta q \mathbb{E}\int_0^t e^{\theta s}\|Y^n(s)\|_H^2 ds \\
&\qquad - 2\alpha p \mathbb{E}\int_0^t e^{\theta s}\|Y^n(s)\|_H^2 ds \\
&\qquad - \frac{\mu}{2}\mathbb{E}\int_0^t e^{\theta s}\|Y(s)\|_H^2 ds + \alpha_1\int_0^t e^{\theta s} ds \\
&\qquad + \mathbb{E}\int_0^t e^{\theta s}\lambda_{r(s)} \\
&\qquad\quad \times \left\{2\left\langle Y^n(s), \left(e^{(s-\lfloor s\rfloor)A} - \mathbf{1}\right) f\left(Y^n(s), r(s)\right)\right\rangle_H \right. \\
&\qquad\qquad + 2\left\langle Y^n(s), e^{(s-\lfloor s\rfloor)A}\left(f\left(Y^n(\lfloor s\rfloor), r(\lfloor s\rfloor)\right) - f\left(Y^n(s), r(s)\right)\right)\right\rangle_H \\
&\qquad\qquad + \Delta^{1/2}\left\|g\left(Y^n(s), r(s)\right)\right\|_{\mathrm{HS}}^2 + \left(1+\Delta^{-1/2}\right) \\
&\qquad\qquad \left. \times \left\|\left(g\left(Y^n(s), r(s)\right) - g\left(Y^n(\lfloor s\rfloor), r(\lfloor s\rfloor)\right)\right)\right\|_{\mathrm{HS}}^2\right\} ds \\
&\quad := J_1(t) + J_2(t) + J_3(t) + J_4(t).
\end{aligned} \tag{34}$$

By the elemental inequality: $2ab \leq (a^2/\kappa) + \kappa b^2$, $a, b \in \mathbb{R}$, $\kappa > 0$, and (8), (27), we obtain that, for $\Delta < 1$,

$$\begin{aligned}
J_2(t) \leq \mathbb{E}\int_0^t e^{\theta s}\lambda_{r(s)}&\left\{\Delta^{1/2}\|Y^n(s)\|_H^2 \right. \\
&+ \Delta^{-1/2}\left\|\left(e^{(s-\lfloor s\rfloor)A} - \mathbf{1}\right)\right. \\
&\left.\left. \times\ f\left(Y^n(s), r(s)\right)\right\|_H^2\right\} ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \mathbb{E}\int_0^t e^{\theta s}\lambda_{r(s)}\Delta^{1/2}\|Y^n(s)\|_H^2 ds \\
&\quad + \mathbb{E}\int_0^t e^{\theta s}\lambda_{r(s)}\Delta^{-1/2}\rho_n^2\Delta^2\overline{L}\left(1+\|Y^n(s)\|_H^2\right) ds \\
&\leq \mathbb{E}\int_0^t e^{\theta s}\lambda_{r(s)}\left\{\left(\Delta^{1/2}+\Delta^{1/2}\rho_n^2\overline{L}\right)\right. \\
&\qquad \left.\times\|Y^n(s)\|_H^2+\Delta^{1/2}\rho_n^2\overline{L}\right\} ds \\
&\leq q\mathbb{E}\int_0^t e^{\theta s}\left\{\Delta^{1/2}\left(1+\rho_n^2\overline{L}\right)\|Y^n(s)\|_H^2+\Delta^{1/2}\rho_n^2\overline{L}\right\} ds.
\end{aligned}
\tag{35}
$$

By (A2) and (8), we have

$$
\begin{aligned}
&\|(f(Y^n(\lfloor t\rfloor), r(\lfloor t\rfloor)) - f(Y^n(t), r(t)))\|_H^2 \\
&\quad\leq 2\|(f(Y^n(\lfloor t\rfloor), r(\lfloor t\rfloor)) - f(Y^n(\lfloor t\rfloor), r(t)))\|_H^2 \\
&\qquad + 2\|(f(Y^n(\lfloor t\rfloor), r(t)) - f(Y^n(t), r(t)))\|_H^2 \\
&\quad\leq 8\overline{L}\left(1+\|Y^n(\lfloor t\rfloor)\|_H^2\right) I_{\{r(t)\neq r(\lfloor t\rfloor)\}} \\
&\qquad + 2L\|Y^n(t)-Y^n(\lfloor t\rfloor)\|_H^2.
\end{aligned}
\tag{36}
$$

Similarly, we have

$$
\begin{aligned}
&\|g(Y^n(t), r(t))) - g(Y^n(\lfloor t\rfloor), r(\lfloor t\rfloor))\|_{\mathrm{HS}}^2 \\
&\quad\leq 8\overline{L}\left(1+\|Y^n(\lfloor t\rfloor)\|_H^2\right) I_{\{r(t)\neq r(\lfloor t\rfloor)\}} \\
&\qquad + 2L\|Y^n(t)-Y^n(\lfloor t\rfloor)\|_H^2.
\end{aligned}
\tag{37}
$$

Thus, we obtain from (36) that

$$
\begin{aligned}
&J_3(t) \\
&\leq 2\mathbb{E}\int_0^t e^{\theta s}\lambda_{r(s)}\left\langle Y^n(s), e^{(s-\lfloor s\rfloor)A}\right. \\
&\qquad\times\left(f(Y^n(\lfloor s\rfloor), r(\lfloor s\rfloor))\right. \\
&\qquad\quad\left.\left.-f(Y^n(s), r(s))\right)\right\rangle_H ds \\
&\leq \mathbb{E}\int_0^t e^{\theta s}\lambda_{r(s)}\left\{\Delta^{1/2}\|Y^n(s)\|_H^2 ds+\Delta^{-1/2}\left\|e^{(s-\lfloor s\rfloor)A}\right\|^2\right. \\
&\qquad\times\|f(Y^n(\lfloor s\rfloor), r(\lfloor s\rfloor)) \\
&\qquad\quad\left.-f(Y^n(s), r(s))\|_H^2\right\} ds \\
&\leq q\mathbb{E}\int_0^t e^{\theta s}\left\{\Delta^{1/2}\|Y^n(s)\|_H^2+\Delta^{-1/2}8\overline{L}\right. \\
&\qquad\times\left(1+\|Y^n(\lfloor s\rfloor)\|_H^2\right) I_{\{r(s)\neq r(\lfloor s\rfloor)\}} \\
&\qquad\left.+2\Delta^{-1/2}L\|Y^n(s)-Y^n(\lfloor s\rfloor)\|_H^2\right\} ds.
\end{aligned}
\tag{38}
$$

By Markov property, we compute

$$
\begin{aligned}
&\mathbb{E}\left[\left(1+\|Y(\lfloor t\rfloor)\|_H^2\right) I_{\{r(t)\neq r(\lfloor t\rfloor)\}}\right] \\
&\quad = \mathbb{E}\left(\mathbb{E}\left[\left(1+\|Y(\lfloor t\rfloor)\|_H^2\right) I_{\{r(t)\neq r(\lfloor t\rfloor)\}} \mid r(\lfloor t\rfloor)\right]\right) \\
&\quad = \mathbb{E}\left(\mathbb{E}\left[\left(1+\|Y(\lfloor t\rfloor)\|_H^2\right) \mid r(\lfloor t\rfloor)\right]\right) \\
&\qquad\times\mathbb{E}\left[I_{\{r(t)\neq r(\lfloor t\rfloor)\}} \mid r(\lfloor t\rfloor)\right] \\
&\quad = \mathbb{E}\left(1+\|Y(\lfloor t\rfloor)\|_H^2\right)\sum_{i\in\mathbb{S}} I_{\{r(\lfloor t\rfloor)=i\}}\mathbb{P}(r(t)\neq i \mid r(\lfloor t\rfloor)=i) \\
&\quad = \mathbb{E}\left(1+\|Y(\lfloor t\rfloor)\|_H^2\right)\sum_{i\in\mathbb{S}} I_{\{r(\lfloor t\rfloor)=i\}} \\
&\qquad\times\sum_{j\neq i}\left(\gamma_{ij}(t-\lfloor t\rfloor)+o(t-\lfloor t\rfloor)\right) \\
&\quad = \mathbb{E}\left(1+\|Y(\lfloor t\rfloor)\|_H^2\right)\left(\max_{i\in\mathbb{S}}(-\gamma_{ii})\Delta+o(\Delta)\right)\sum_{i\in\mathbb{S}} I_{\{r(\lfloor t\rfloor)=i\}} \\
&\quad\leq \widehat{\gamma}\Delta\mathbb{E}\left(1+\|Y(\lfloor t\rfloor)\|_H^2\right),
\end{aligned}
\tag{39}
$$

where $\widehat{\gamma} = N[1+\max_{1\leq i\leq N}(-\gamma_{ii})]$. Substituting (39) into (38) gives

$$
\begin{aligned}
&J_3(t) \\
&\quad\leq q\mathbb{E}\int_0^t e^{\theta s}\left\{\Delta^{1/2}\|Y^n(s)\|_H^2\right. \\
&\qquad\left.+2\Delta^{-1/2}L\|Y^n(s)-Y^n(\lfloor s\rfloor)\|_H^2\right\} ds \\
&\qquad + q\int_0^t 8e^{\theta s}\Delta^{1/2}\widehat{\gamma}\overline{L}\mathbb{E}\left(1+\|Y(\lfloor s\rfloor)\|_H^2\right) ds.
\end{aligned}
\tag{40}
$$

Furthermore, due to (37) and (39), we have

$$
\begin{aligned}
&J_4(t) \\
&\quad = \mathbb{E}\int_0^t e^{\theta s}\lambda_{r(s)} \\
&\qquad\times\left\{\Delta^{1/2}\|g(Y^n(s), r(s))\|_{\mathrm{HS}}^2\right. \\
&\qquad\quad + \left(1+\Delta^{-1/2}\right)\|g(Y^n(s), r(s))) \\
&\qquad\qquad\left.-g(Y^n(\lfloor s\rfloor), r(\lfloor s\rfloor))\|_{\mathrm{HS}}^2\right\} ds
\end{aligned}
$$

$$\begin{aligned}
&\leq q\mathbb{E}\int_0^t e^{\theta s}\Delta^{1/2}\overline{L}\left(1+\|Y^n(s)\|_H^2\right)ds+q\left(1+\Delta^{-1/2}\right)\\
&\quad\times\int_0^t e^{\theta s}8\overline{L}\left(1+\|Y^n(\lfloor s\rfloor)\|_H^2\right)I_{\{r(s)\neq r(\lfloor s\rfloor)\}}ds\\
&\quad+2Lq\left(1+\Delta^{-1/2}\right)\int_0^t e^{\theta s}\|Y^n(s)-Y^n(\lfloor s\rfloor)\|_H^2ds\\
&\leq q\Delta^{1/2}\overline{L}\mathbb{E}\int_0^t e^{\theta s}\left(1+\|Y^n(s)\|_H^2\right)ds\\
&\quad+16q\widehat{\gamma}\Delta^{1/2}\overline{L}\int_0^t e^{\theta s}\left(1+\|Y^n(s)\|_H^2\right)ds\\
&\quad+2Lq\left(1+\Delta^{-1/2}\right)\int_0^t e^{\theta s}\|Y^n(s)-Y^n(\lfloor s\rfloor)\|_H^2ds.
\end{aligned}\tag{41}$$

On the other hand, by Lemma 10, when $3(\rho_n^2+2\overline{L})\Delta\leq 1$, we have

$$\begin{aligned}
&\mathbb{E}\|Y^n(t)\|_H^2\\
&\quad\leq 2\mathbb{E}\|Y^n(t)-Y^n(\lfloor t\rfloor)\|_H^2+2\mathbb{E}\|Y^n(\lfloor t\rfloor)\|_H^2\\
&\quad\leq 6\left(\rho_n^2+2\overline{L}\right)\Delta\left(1+\mathbb{E}\|Y^n(\lfloor t\rfloor)\|_H^2\right)\\
&\qquad+2\mathbb{E}\|Y^n(\lfloor t\rfloor)\|_H^2\\
&\quad\leq 4\mathbb{E}\|Y^n(\lfloor t\rfloor)\|_H^2+2.
\end{aligned}\tag{42}$$

Putting (35), (40), and (41) into (34), we have

$$\begin{aligned}
&e^{\theta t}\mathbb{E}\left(\lambda_{r(t)}\|Y^n(t)\|_H^2\right)\\
&\quad\leq q\|x\|_H^2+\int_0^t e^{\theta s}\left[\alpha_1+q\rho_n^2\Delta^{1/2}\overline{L}\right.\\
&\qquad\qquad\left.+24q\Delta^{1/2}\widehat{\gamma}\overline{L}+q\Delta^{1/2}\overline{L}\right]ds\\
&\qquad+\mathbb{E}\int_0^t e^{\theta s}\left[q\theta-2\alpha p-\frac{\mu}{2}+q\Delta^{1/2}\left(2+\rho_n^2\overline{L}\right)+q\Delta^{1/2}\overline{L}\right]\\
&\qquad\quad\times\|Y^n(s)\|_H^2ds+24q\Delta^{1/2}\widehat{\gamma}\overline{L}\mathbb{E}\\
&\qquad\times\int_0^t e^{\theta s}\|Y^n(\lfloor s\rfloor)\|_H^2ds\\
&\qquad+\left(4q\Delta^{-1/2}L+2qL\right)\mathbb{E}\int_0^t e^{\theta s}\|Y^n(s)-Y^n(\lfloor s\rfloor)\|_H^2ds.
\end{aligned}\tag{43}$$

By Lemma 10 and the inequality (42), we obtain that

$$\begin{aligned}
&e^{\theta t}\mathbb{E}\left(\lambda_{r(t)}\|Y^n(t)\|_H^2\right)\\
&\quad\leq q\|x\|_H^2+\int_0^t e^{\theta s}\left[\alpha_1+2q\theta-4\alpha p-\mu\right.\\
&\qquad\qquad+3q\rho_n^2\Delta^{1/2}\overline{L}+24q\Delta^{1/2}\widehat{\gamma}\overline{L}\\
&\qquad\qquad\left.+3q\Delta^{1/2}\overline{L}+4q\Delta^{1/2}\right]ds\\
&\qquad+\int_0^t e^{\theta s}\left[4q\theta-8\alpha p-2\mu+4q\Delta^{1/2}\left(2+\rho_n^2\overline{L}\right)\right.\\
&\qquad\qquad\left.+4q\Delta^{1/2}\overline{L}+24q\Delta^{1/2}\widehat{\gamma}\overline{L}\right]\|Y^n(\lfloor s\rfloor)\|_H^2ds\\
&\qquad+6qL\Delta^{1/2}\left(\rho_n^2+2\overline{L}\right)\mathbb{E}\int_0^t e^{\theta s}\left(1+\|Y^n(\lfloor s\rfloor)\|_H^2\right)ds\\
&\quad\leq q\|x\|_H^2+\int_0^t e^{\theta s}\left[\alpha_1+2q\theta-4\alpha p-\mu+3q\rho_n^2\Delta^{1/2}\overline{L}\right.\\
&\qquad\qquad+24q\Delta^{1/2}\widehat{\gamma}\overline{L}+3q\Delta^{1/2}\overline{L}+4q\Delta^{1/2}\\
&\qquad\qquad\left.+6qL\Delta^{1/2}\left(\rho_n^2+2\overline{L}\right)\right]ds\\
&\qquad+\int_0^t e^{\theta s}\left[4q\theta-8\alpha p-2\mu+4q\Delta^{1/2}\left(2+\rho_n^2\overline{L}\right)\right.\\
&\qquad\qquad+4q\Delta^{1/2}\overline{L}+24q\Delta^{1/2}\widehat{\gamma}\overline{L}\\
&\qquad\qquad\left.+6qL\Delta^{1/2}\left(\rho_n^2+2\overline{L}\right)\right]\|Y^n(\lfloor s\rfloor)\|_H^2ds.
\end{aligned}\tag{44}$$

Let $\theta=(4\alpha p+\mu)/4q$, for $\Delta<((4\alpha p+\mu)/(8q+4q\rho_n^2\overline{L}+4q\overline{L}+24q\widehat{r}+6qL(\rho_n^2+2\overline{L})))^2$, then

$$pe^{\theta t}\mathbb{E}\left(\|Y(t)\|_H^2\right)\leq q\|x\|_H^2+\int_0^t e^{\theta s}\left[\alpha_1+\frac{4\alpha p+\mu}{2}\right]ds.\tag{45}$$

That is,

$$\sup_{t\geq 0}\mathbb{E}\left(\|Y(t)\|_H^2\right)\leq C.\tag{46}$$

□

Lemma 12. *Let (A1)–(A3) hold. If* $\Delta<\min\{1,1/18(\rho_n^2+2L),((2\alpha p+\mu)/(4q+2qL+2q\rho_n^2L+12qL\widehat{r}))^2\}$, *then*

$$\lim_{t\to\infty}\mathbb{E}\left\|Y^{n,x,i}(t)-Y^{n,y,i}(t)\right\|_H^2=0\quad\text{uniformly for } x,y\in U,\tag{47}$$

where U *is a bounded subset of* H_n.

Proof. Write $Y^{n,x,i}(t) = Y^x(t)$, $Y^{n,y,i}(t) = Y^y(t)$, $r^i(k\Delta) = r(k\Delta)$. From (20), it is easy to show that

$$\begin{aligned}(Y^x(t) - Y^y(t)) &- (Y^x(\lfloor t\rfloor) - Y^y(\lfloor t\rfloor))\\ &= (Y^x(t) - Y^x(\lfloor t\rfloor)) - (Y^y(t) - Y^y(\lfloor t\rfloor))\\ &= \left(e^{(t-\lfloor t\rfloor)A} - \mathbf{1}\right)(Y^x(\lfloor t\rfloor) - Y^y(\lfloor t\rfloor))\\ &\quad + \int_{\lfloor t\rfloor}^t e^{(t-\lfloor s\rfloor)A}\left(f_n(Y^x(\lfloor s\rfloor), r(\lfloor s\rfloor)) - f_n(Y^y(\lfloor s\rfloor), r(\lfloor s\rfloor))\right)ds\\ &\quad + \int_{\lfloor t\rfloor}^t e^{(t-\lfloor s\rfloor)A}\left(g_n(Y^x(\lfloor s\rfloor), r(\lfloor s\rfloor)) - g_n(Y^y(\lfloor s\rfloor), r(\lfloor s\rfloor))\right)dW(s).\end{aligned} \tag{48}$$

By using the argument of Lemma 10, we derive that, if $\Delta < 1$,

$$\begin{aligned}\mathbb{E}\|(Y^x(t) - Y^y(t)) &- (Y^x(\lfloor t\rfloor) - Y^y(\lfloor t\rfloor))\|_H^2\\ &\le 3\left(\rho_n^2 + 2L\right)\Delta\mathbb{E}\|Y^x(\lfloor t\rfloor) - Y^y(\lfloor t\rfloor)\|_H^2,\end{aligned} \tag{49}$$

$$\begin{aligned}\mathbb{E}\|(Y^x(t) - Y^y(t))\|_H^2 &= \mathbb{E}\left\|(Y^x(t) - Y^y(t)) - (Y^x(\lfloor t\rfloor) - Y^y(\lfloor t\rfloor)) + (Y^x(\lfloor t\rfloor) - Y^y(\lfloor t\rfloor))\right\|_H^2\\ &\le (1+2)\,\mathbb{E}\left\|(Y^x(t) - Y^y(t)) - (Y^x(\lfloor t\rfloor) - Y^y(\lfloor t\rfloor))\right\|_H^2\\ &\quad + \left(1 + \frac{1}{2}\right)\mathbb{E}\|Y^x(\lfloor t\rfloor) - Y^y(\lfloor t\rfloor)\|_H^2\\ &\le 9\left(\rho_n^2 + 2L\right)\Delta\mathbb{E}\|(Y^x(\lfloor t\rfloor) - Y^y(\lfloor t\rfloor))\|_H^2\\ &\quad + 1.5\mathbb{E}\|(Y^x(\lfloor t\rfloor) - Y^y(\lfloor t\rfloor))\|_H^2.\end{aligned} \tag{50}$$

If $\Delta < 1/18(\rho_n^2 + 2L)$, then

$$\mathbb{E}\|(Y^x(t) - Y^y(t))\|_H^2 \le 2\mathbb{E}\|(Y^x(\lfloor t\rfloor) - Y^y(\lfloor t\rfloor))\|_H^2. \tag{51}$$

Using (30) and the generalised Itô formula, for any $\theta > 0$, we have

$$\begin{aligned}e^{\theta t}\mathbb{E}&\left(\lambda_{r(t)}\|Y^x(t) - Y^y(t)\|_H^2\right)\\ &\le \lambda_i\|x - y\|_H^2\\ &\quad + \mathbb{E}\int_0^t e^{\theta s}\lambda_{r(s)}\Big\{\theta\|Y^x(s) - Y^y(s)\|_H^2 + 2\left\langle Y^x(s) - Y^y(s), AY^x(s) - Y^y(s)\right\rangle_H\\ &\qquad + 2\left\langle Y^x(s) - Y^y(s), e^{(s-\lfloor s\rfloor)A}\times\left(f_n(Y^x(\lfloor s\rfloor), r(\lfloor s\rfloor)) - f_n(Y^y(\lfloor s\rfloor), r(\lfloor s\rfloor))\right)\right\rangle_H\\ &\qquad + \|g_n(Y^x(\lfloor s\rfloor), r(\lfloor s\rfloor)) - g_n(Y^y(\lfloor s\rfloor), r(\lfloor s\rfloor))\|_{\mathrm{HS}}^2\Big\}ds\\ &\quad + \mathbb{E}\int_0^t e^{\theta s}\sum_{l=1}^N \gamma_{r(s)l}\lambda_l\|Y^x(s) - Y^y(s)\|_H^2 ds\\ &\le q\|x - y\|_H^2 + q\theta\mathbb{E}\int_0^t e^{\theta s}\|Y^x(s) - Y^y(s)\|_H^2 ds\\ &\quad - 2\alpha p\mathbb{E}\int_0^t e^{\theta s}\|Y^x(s) - Y^y(s)\|_H^2 ds\\ &\quad + \mathbb{E}\int_0^t e^{\theta s}\lambda_{r(s)}\Big\{2\left\langle Y^x(s) - Y^y(s), e^{(s-\lfloor s\rfloor)A}\times\left(f(Y^x(\lfloor s\rfloor), r(\lfloor s\rfloor)) - f(Y^y(\lfloor s\rfloor), r(\lfloor s\rfloor))\right)\right\rangle_H\\ &\qquad + \|g(Y^x(\lfloor s\rfloor), r(\lfloor s\rfloor)) - g(Y^y(\lfloor s\rfloor), r(\lfloor s\rfloor))\|_{\mathrm{HS}}^2\Big\}ds\\ &\quad + \mathbb{E}\int_0^t e^{\theta s}\sum_{l=1}^N \gamma_{r(s)l}\lambda_l\|Y^x(s) - Y^y(s)\|_H^2 ds.\end{aligned} \tag{52}$$

By the fundamental transformation, we obtain that

$$\begin{aligned}&\left\langle Y^x(t) - Y^y(t), e^{(t-\lfloor t\rfloor)A}\times\left(f(Y^x(\lfloor t\rfloor), r(\lfloor t\rfloor)) - f(Y^y(\lfloor t\rfloor), r(\lfloor t\rfloor))\right)\right\rangle_H\\ &\quad = \left\langle Y^x(t) - Y^y(t), f(Y^x(t), r(t)) - f(Y^y(t), r(t))\right\rangle_H\\ &\qquad + \left\langle Y^x(t) - Y^y(t), \left(e^{(t-\lfloor t\rfloor)A} - \mathbf{1}\right)\times\left(f(Y^x(t), r(t)) - f(Y^y(t), r(t))\right)\right\rangle_H\\ &\qquad + \Big\langle Y^x(t) - Y^y(t), e^{(t-\lfloor t\rfloor)A}\times\left(f(Y^x(\lfloor t\rfloor), r(\lfloor t\rfloor)) - f(Y^y(\lfloor t\rfloor), r(\lfloor t\rfloor))\right)\\ &\qquad\quad - \left(f(Y^x(t), r(t)) - f(Y^y(t), r(t))\right)\Big\rangle_H.\end{aligned} \tag{53}$$

By the Höld inequality, we have

$$\begin{aligned}\|g(Y^x(\lfloor t\rfloor), r(\lfloor t\rfloor)) &- g(Y^y(\lfloor t\rfloor), r(\lfloor t\rfloor))\|_{\mathrm{HS}}^2\\ &\le \left(1 + \Delta^{1/2}\right)\|g(Y^x(t), r(t)) - g(Y^y(t), r(t))\|_{\mathrm{HS}}^2\end{aligned}$$

$$+\left(1+\Delta^{-1/2}\right)\left\|\left(g\left(Y^{x}(t),r(t)\right)-g\left(Y^{y}(t),r(t)\right)\right)-\left(g\left(Y^{x}(\lfloor t\rfloor),r(\lfloor t\rfloor)\right)-g\left(Y^{y}(\lfloor t\rfloor),r(\lfloor t\rfloor)\right)\right)\right\|_{HS}^{2}. \tag{54}$$

Then, from (52) and (A3), we have

$$\begin{aligned}
e^{\theta t}\mathbb{E}\left(\lambda_{r(t)}\|Y^{x}(t)-Y^{y}(t)\|_{H}^{2}\right)
&\le q\|x-y\|_{H}^{2}+(q\theta-2\alpha p-\mu)\,\mathbb{E}\times\int_{0}^{t}e^{\theta s}\|Y^{x}(s)-Y^{y}(s)\|_{H}^{2}ds\\
&\quad+2\mathbb{E}\int_{0}^{t}e^{\theta s}\lambda_{r(s)}\times\left\langle Y^{x}(s)-Y^{y}(s),\left(e^{(s-\lfloor s\rfloor)A}-\mathbf{1}\right)\times\left(f\left(Y^{x}(s),r(s)\right)-f\left(Y^{y}(s),r(s)\right)\right)\right\rangle_{H}ds\\
&\quad+2\mathbb{E}\int_{0}^{t}e^{\theta s}\lambda_{r(s)}\times\left\langle Y^{x}(s)-Y^{y}(s),e^{(s-\lfloor s\rfloor)A}\times\left(f\left(Y^{x}(\lfloor s\rfloor),r(\lfloor s\rfloor)\right)-f\left(Y^{y}(\lfloor s\rfloor),r(\lfloor s\rfloor)\right)\right)-\left(f\left(Y^{x}(s),r(s)\right)-f\left(Y^{y}(s),r(s)\right)\right)\right\rangle_{H}ds\\
&\quad+\mathbb{E}\int_{0}^{t}e^{\theta s}\lambda_{r(s)}\Big\{\Delta^{1/2}\left\|g\left(Y^{x}(s),r(s)\right)-g\left(Y^{y}(s),r(s)\right)\right\|_{HS}^{2}\\
&\qquad+\left(1+\Delta^{-1/2}\right)\times\left\|\left(g\left(Y^{x}(s),r(s)\right)-g\left(Y^{y}(s),r(s)\right)\right)-\left(g\left(Y^{x}(\lfloor s\rfloor),r(\lfloor s\rfloor)\right)-g\left(Y^{y}(\lfloor s\rfloor),r(\lfloor s\rfloor)\right)\right)\right\|_{HS}^{2}\Big\}ds\\
&:=G_{1}(t)+G_{2}(t)+G_{3}(t)+G_{4}(t).
\end{aligned} \tag{55}$$

By (A2) and (27), we have, for $\Delta<1$,

$$\begin{aligned}
G_{2}(t)&\le\mathbb{E}\int_{0}^{t}e^{\theta s}\lambda_{r(s)}\Big\{\Delta^{1/2}\|Y^{x}(s)-Y^{y}(s)\|_{H}^{2}+\Delta^{-1/2}\left\|\left(e^{(s-\lfloor s\rfloor)A}-\mathbf{1}\right)\times\left(f\left(Y^{x}(s),r(s)\right)-Y^{y}(s),r(s)\right)\right\|_{H}^{2}\Big\}ds\\
&\le\mathbb{E}\int_{0}^{t}e^{\theta s}\lambda_{r(s)}\left(\Delta^{1/2}+\Delta^{3/2}\rho_{n}^{2}L\right)\|Y^{x}(s)-Y^{y}(s)\|_{H}^{2}ds\\
&\le q\mathbb{E}\int_{0}^{t}e^{\theta s}\Delta^{1/2}\left(1+\rho_{n}^{2}L\right)\|Y^{x}(s)-Y^{y}(s)\|_{H}^{2}ds.
\end{aligned} \tag{56}$$

It is easy to show that

$$\begin{aligned}
G_{3}(t)&\le q\mathbb{E}\int_{0}^{t}e^{\theta s}\Big\{\Delta^{1/2}\|Y^{x}(s)-Y^{y}(s)\|_{H}^{2}+\Delta^{-1/2}\left\|\left(e^{(s-\lfloor s\rfloor)A}\right)\right\|^{2}\\
&\qquad\times\left\|f\left(Y^{x}(\lfloor s\rfloor),r(\lfloor s\rfloor)\right)-f\left(Y^{y}(\lfloor s\rfloor),r(\lfloor s\rfloor)\right)-\left(f\left(Y^{x}(s),r(s)\right)-Y^{y}(s),r(s)\right)\right\|_{H}^{2}\Big\}ds\\
&\le q\mathbb{E}\int_{0}^{t}e^{\theta s}\Delta^{1/2}\|Y^{x}(s)-Y^{y}(s)\|_{H}^{2}ds+\overline{G}_{3}(t).
\end{aligned} \tag{57}$$

By (39), we have

$$\begin{aligned}
\overline{G}_{3}(t)&\le 2q\Delta^{-1/2}\mathbb{E}\int_{0}^{t}e^{\theta s}\Big[\left\|f\left(Y^{x}(\lfloor s\rfloor),r(\lfloor s\rfloor)\right)-f\left(Y^{y}(\lfloor s\rfloor),r(\lfloor s\rfloor)\right)\right\|_{H}^{2}+\left\|\left(f\left(Y^{x}(s),r(s)\right)-Y^{y}(s),r(s)\right)\right\|_{H}^{2}\Big]\times I_{\{r(s)\neq r(\lfloor s\rfloor)\}}ds\\
&\le 4q\Delta^{-1/2}L\mathbb{E}\int_{0}^{t}e^{\theta s}\|Y^{x}(\lfloor s\rfloor)-Y^{y}(\lfloor s\rfloor)\|_{H}^{2}\times I_{\{r(s)\neq r(\lfloor s\rfloor)\}}ds\\
&\le 4qL\widehat{\gamma}\Delta^{1/2}\mathbb{E}\int_{0}^{t}e^{\theta s}\|Y^{x}(\lfloor s\rfloor)-Y^{y}(\lfloor s\rfloor)\|_{H}^{2}ds.
\end{aligned} \tag{58}$$

Therefore, we obtain that

$$\begin{aligned}
G_{3}(t)&\le q\Delta^{1/2}\mathbb{E}\int_{0}^{t}e^{\theta s}\|Y^{x}(s)-Y^{y}(s)\|_{H}^{2}ds\\
&\quad+4qL\widehat{\gamma}\Delta^{1/2}\mathbb{E}\int_{0}^{t}e^{\theta s}\|Y^{x}(\lfloor s\rfloor)-Y^{y}(\lfloor s\rfloor)\|_{H}^{2}ds.
\end{aligned} \tag{59}$$

On the other hand, using the similar argument of (58), we have

$$\begin{aligned} G_4(t) \le q\mathbb{E}\int_0^t e^{\theta s}\Big\{&\Delta^{1/2}L\|Y^x(s)-Y^y(s)\|_H^2 \\ &+\left(1+\Delta^{-1/2}\right)4L\hat{\gamma}\Delta \\ &\times\|Y^x(\lfloor s\rfloor)-Y^y(\lfloor s\rfloor)\|_H^2\Big\}\,ds. \end{aligned} \tag{60}$$

Hence, we have

$$\begin{aligned} pe^{\theta t}&\mathbb{E}\left(\|Y^x(t)-Y^y(t)\|_H^2\right) \\ &\le q\|x-y\|_H^2 \\ &\quad+\mathbb{E}\int_0^t e^{\theta s}\Big[q\theta-2\alpha p-\mu+q\left(1+\rho_n^2L\right)\Delta^{1/2} \\ &\qquad +q\Delta^{1/2}+qL\Delta^{1/2}\Big]\|Y^x(s)-Y^y(s)\|_H^2ds \\ &\quad+\mathbb{E}\int_0^t e^{\theta s}\left[4qL\hat{\gamma}\Delta^{1/2}+\left(\Delta^{1/2}+\Delta\right)4qL\hat{\gamma}\right] \\ &\qquad\times\|Y^x(\lfloor s\rfloor)-Y^y(\lfloor s\rfloor)\|_H^2ds. \end{aligned} \tag{61}$$

By (50), we obtain that

$$\begin{aligned} pe^{\theta t}&\mathbb{E}\left(\|Y^x(t)-Y^y(t)\|_H^2\right) \\ &\le q\|x-y\|_H^2+\mathbb{E}\int_0^t e^{\theta s}\Big[2q\theta-4\alpha p-2\mu \\ &\qquad+4q\Delta^{1/2}+2qL\Delta^{1/2} \\ &\qquad+2q\rho_n^2L\Delta^{1/2}+12qL\hat{\gamma}\Delta^{1/2}\Big] \\ &\qquad\times\|Y^x(\lfloor s\rfloor)-Y^y(\lfloor s\rfloor)\|_H^2ds. \end{aligned} \tag{62}$$

Let $\theta=(2\alpha p+\mu)/2q$, for $\Delta<((2\alpha p+\mu)/(4q+2qL+2q\rho_n^2L+12qL\hat{r}))^2$, then the desired assertion (47) follows. □

We can now easily prove our main result.

Proof of Theorem 9. Since H_n is finite-dimensional, by Lemma 3.1 in [12], we have

$$\lim_{k\to\infty}d_{\mathbb{L}}\left(\mathbb{P}_k^{n,\Delta}((x,i),\cdot\times\cdot),\mathbb{P}_k^{n,\Delta}((y,i),\cdot\times\cdot)\right)=0, \tag{63}$$

uniformly in $x,y\in H_n,i,j\in\mathbb{S}$.

By Lemma 7, there exists $\pi^{n,\Delta}(\cdot\times\cdot)\in\mathscr{P}(H_n\times\mathbb{S})$, such that

$$\lim_{k\to\infty}d_{\mathbb{L}}\left(\mathbb{P}_k^{n,\Delta}((0,1),\cdot\times\cdot),\pi^{n,\Delta}(\cdot\times\cdot)\right)=0. \tag{64}$$

By the triangle inequality (63) and (64), we have

$$\begin{aligned} \lim_{k\to\infty}&d_{\mathbb{L}}\left(\mathbb{P}_k^{n,\Delta}((x,i),\cdot\times\cdot),\pi^{n,\Delta}(\cdot\times\cdot)\right) \\ &\le\lim_{k\to\infty}d_{\mathbb{L}}\left(\mathbb{P}_k^{n,\Delta}((x,i),\cdot\times\cdot),\mathbb{P}_k^{n,\Delta}((0,1),\cdot\times\cdot)\right) \\ &\quad+\lim_{k\to\infty}d_{\mathbb{L}}\left(\mathbb{P}_k^{n,\Delta}((0,1),\cdot\times\cdot),\pi^{n,\Delta}(\cdot\times\cdot)\right)=0. \end{aligned} \tag{65}$$

□

4. Corollary and Example

In this section, we give a criterion based M-matrices which can be verified easily in applications.

(A4) For each $j\in\mathbb{S}$, there exists a pair of constants β_j and δ_j such that, for $x,y\in H$,

$$\begin{aligned} \langle x-y,f(x,j)-f(y,j)\rangle_H&\le\beta_j\|x-y\|_H^2, \\ \|g(x,j)-g(y,j)\|_{\mathrm{HS}}^2&\le\delta_j\|x-y\|_H^2. \end{aligned} \tag{66}$$

Moreover, $\mathscr{A}:=-\operatorname{diag}(2\beta_1+\delta_1,\ldots,2\beta_N+\delta_N)-\Gamma$ is a nonsingular M-matrix [8].

Corollary 13. *Under (A1), (A2), and (A4), for a given stepsize $\Delta>0$, and arbitrary $x\in H_n,i\in\mathbb{S}$, $\{\overline{Z}^{n,(x,i)}(k\Delta)\}_{k\ge0}$ has a unique stationary distribution $\pi^{n,\Delta}(\cdot\times\cdot)\in\mathscr{P}(H_n\times\mathbb{S})$.*

Proof. In fact, we only need to prove that (A3) holds. By (A4), there exists $(\lambda_1,\lambda_2,\ldots,\lambda_N)^T>0$, such that $(q_1,q_2,\ldots,q_N)^T=\mathscr{A}(\lambda_1,\lambda_2,\ldots,\lambda_N)^T>0$.

Set $\mu=\min_{1\le j\le N}q_j$, by (66), we have

$$\begin{aligned} 2\lambda_j&\langle x-y,f(x,j)-f(y,j)\rangle_H \\ &\quad+\lambda_j\|g(x,j)-g(y,j)\|_H^2+\sum_{l=1}^N\gamma_{jl}\lambda_l\|x-y\|_H^2 \\ &\le2\lambda_j\beta_j\|x-y\|_H^2+\delta_j\lambda_j\|x-y\|_H^2+\sum_{l=1}^N\gamma_{jl}\lambda_l\|x-y\|_H^2 \\ &=\left(2\lambda_j\beta_j+\delta_j\lambda_j+\sum_{l=1}^N\gamma_{jl}\lambda_l\right)\|x-y\|_H^2 \\ &=-q_j\|x-y\|_H^2\le-\mu\|x-y\|_H^2 \end{aligned} \tag{67}$$

□

In the following, we give an example to illustrate the Corollary 13.

Example 14. Consider

$$\begin{aligned} dX(t,\xi)=&\left[\frac{\partial^2}{\partial\xi^2}X(t,\xi)+B(r(t))X(t,\xi)\right]dt \\ &+g(X(t,\xi),r(t))\,dW(t),\quad 0<\xi<\pi,\ t\ge0. \end{aligned} \tag{68}$$

We take $H = L^2(0,\pi)$ and $A = \partial^2/\partial\xi^2$ with domain $\mathscr{D}(A) = H^2(0,\pi) \cap H_0^1(0,\pi)$, then A is a self-adjoint negative operator. For the eigenbasis $e_k(\xi) = (2/\pi)^{1/2}\sin(k\xi)$, $\xi \in [0,\pi]$, $Ae_k = -k^2 e_k$, $k \in \mathbb{N}$. It is easy to show that

$$\left\|e^{tA}x\right\|_H^2 = \sum_{i=1}^{\infty} e^{-2k^2 t}\langle x, e_i\rangle_H^2 \le e^{-2t}\sum_{i=1}^{\infty}\langle x, e_i\rangle_H^2. \tag{69}$$

This further gives that

$$\left\|e^{tA}\right\| \le e^{-t}, \tag{70}$$

where $\alpha = 1$, thus (A1) holds.

Let $W(t)$ be a scalar Brownian motion, let $r(t)$ be a continuous-time Markov chain values in $\mathbb{S} = 1, 2$, with the generator

$$\begin{gathered}\Gamma = \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}, \\ B(1) = B_1 = \begin{pmatrix} -0.3 & -0.1 \\ -0.2 & -0.2 \end{pmatrix}, \\ B(2) = B_2 = \begin{pmatrix} -0.4 & -0.2 \\ -0.3 & -0.2 \end{pmatrix}.\end{gathered} \tag{71}$$

Then $\lambda_{\max}(B_1^T B_1) = 0.1706$, $\lambda_{\max}(B_2^T B_2) = 0.3286$.

Moreover, g satisfies

$$\|g(x,j) - g(y,j)\|_{\mathrm{HS}}^2 \le \delta_j \|x - y\|_H^2, \tag{72}$$

where $\delta_1 = 0.1$, $\delta_2 = 0.06$.

Defining $f(x,j) = B(j)x$, then

$$\begin{aligned}&\|f(x,j) - f(y,j)\|_{\mathrm{HS}}^2 \vee \|g(x,j) - g(y,j)\|_{\mathrm{HS}}^2 \\ &\quad \le \left(\lambda_{\max}\left(B_j^T B_j\right) \vee \delta_j\right)\|x - y\|_H^2 < 0.33\|x - y\|_H^2, \\ &\quad \langle x - y, f(x,j) - f(y,j)\rangle_H \\ &\quad \le \frac{1}{2}\left\langle x - y, \left(B_j^T + B_j\right)(x - y)\right\rangle_H \\ &\quad \le \frac{1}{2}\lambda_{\max}\left(B_j^T + B_j\right)\|x - y\|_H^2.\end{aligned} \tag{73}$$

It is easy to compute

$$\begin{aligned}\beta_1 &= \frac{1}{2}\lambda_{\max}\left(B_1^T + B_1\right) = -0.0919, \\ \beta_2 &= \frac{1}{2}\lambda_{\max}\left(B_2^T + B_2\right) = -0.03075.\end{aligned} \tag{74}$$

So the matrix $\mathscr{A}$ becomes

$$\mathscr{A} = \operatorname{diag}(0.0838, 0.0015) - \Gamma = \begin{pmatrix} 2.0838 & -2 \\ -1 & 1.0015 \end{pmatrix}. \tag{75}$$

It is easy to see that $\mathscr{A}$ is a nonsingular M-matrix. Thus, (A4) holds. By Corollary 13, we can conclude that (68) has a unique stationary distribution $\pi^{n,\Delta}(\cdot \times \cdot)$.

Acknowledgment

This work is partially supported by the National Natural Science Foundation of China under Grants 61134012 and 11271146.

References

[1] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite-Dimensions*, Cambridge University Press, Cambridge, UK, 1992.

[2] A. Debussche, "Weak approximation of stochastic partial differential equations: the nonlinear case," *Mathematics of Computation*, vol. 80, no. 273, pp. 89–117, 2011.

[3] I. Gyöngy and A. Millet, "Rate of convergence of space time approximations for stochastic evolution equations," *Potential Analysis*, vol. 30, no. 1, pp. 29–64, 2009.

[4] A. Jentzen and P. E. Kloeden, *Taylor Approximations for Stochastic Partial Differential Equations*, CBMS-NSF Regional Conference Series in Applied Mathematics, 2011.

[5] T. Shardlow, "Numerical methods for stochastic parabolic PDEs," *Numerical Functional Analysis and Optimization*, vol. 20, no. 1-2, pp. 121–145, 1999.

[6] M. Athans, "Command and control (C2) theory: a challenge to control science," *IEEE Transactions on Automatic Control*, vol. 32, no. 4, pp. 286–293, 1987.

[7] D. J. Higham, X. Mao, and C. Yuan, "Preserving exponential mean-square stability in the simulation of hybrid stochastic differential equations," *Numerische Mathematik*, vol. 108, no. 2, pp. 295–325, 2007.

[8] X. Mao and C. Yuan, *Stochastic Differential Equations with Markovian Switching*, Imperial College, London, UK, 2006.

[9] M. Mariton, *Jump Linear Systems in Automatic Control*, Marcel Dekker, 1990.

[10] J. Bao, Z. Hou, and C. Yuan, "Stability in distribution of mild solutions to stochastic partial differential equations," *Proceedings of the American Mathematical Society*, vol. 138, no. 6, pp. 2169–2180, 2010.

[11] J. Bao and C. Yuan, "Numerical approximation of stationary distribution for SPDEs," http://arxiv.org/abs/1303.1600.

[12] X. Mao, C. Yuan, and G. Yin, "Numerical method for stationary distribution of stochastic differential equations with Markovian switching," *Journal of Computational and Applied Mathematics*, vol. 174, no. 1, pp. 1–27, 2005.

[13] C. Yuan and X. Mao, "Stationary distributions of Euler-Maruyama-type stochastic difference equations with Markovian switching and their convergence," *Journal of Difference Equations and Applications*, vol. 11, no. 1, pp. 29–48, 2005.

[14] C. Yuan and X. Mao, "Asymptotic stability in distribution of stochastic differential equations with Markovian switching," *Stochastic Processes and their Applications*, vol. 103, no. 2, pp. 277–291, 2003.

[15] W. J. Anderson, *Continuous-Time Markov Chains*, Springer, New York, NY, USA, 1991.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 870486, 8 pages
http://dx.doi.org/10.1155/2013/870486

Research Article

An LMI Approach for Dynamics of Switched Cellular Neural Networks with Mixed Delays

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Received 3 January 2013; Revised 1 March 2013; Accepted 1 March 2013

Academic Editor: Jinde Cao

This paper considers the dynamics of switched cellular neural networks (CNNs) with mixed delays. With the help of the Lyapnnov function combined with the average dwell time method and linear matrix inequalities (LMIs) technique, some novel sufficient conditions on the issue of the uniformly ultimate boundedness, the existence of an attractor, and the globally exponential stability for CNN are given. The provided conditions are expressed in terms of LMI, which can be easily checked by the effective LMI toolbox in Matlab in practice.

1. Introduction

Cellular neural networks (CNNs) introduced by Chua and Yang in [1, 2] have attracted increasing interest due to the potential applications in classification, signal processing, associative memory, parallel computation, and optimization problems. In these applications, it is essential to investigate the dynamical behavior [3–5]. Both in biological and artificial neural networks, the interactions between neurons are generally asynchronous. As a result, time delay is inevitably encountered in neural networks, which may lead to an oscillation and furthermore to instability of networks. Since Roska et al. [6, 7] first introduced the delayed cellular neural networks (DCNNs), DCNN has been extensively investigated [8–10]. The model can be described by the following differential equation:

$$\dot{x}_i(t) = -d_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j\left(x_j(t)\right) + \sum_{j=1}^{n} b_{ij} f_j\left(x_j\left(t-\tau_j\right)\right) + J_i, \quad i = 1, \dots, n, \tag{1}$$

where $t \geq 0$, $n(\geq 2)$ corresponds to the number of units in a neural network; $x_i(t)$ denotes the potential (or voltage) of cell i at time t; $f_j(\cdot)$ denotes a nonlinear output function; J_i denotes the ith component of an external input source introduced from outside the network to the cell i at time t; $d_i(> 0)$ denotes the rate with which the cell i resets its potential to the resting state when isolated from other cells and external inputs; a_{ij} denotes the strength of the jth unit on the ith unit at time t; b_{ij} denotes the strength of the jth unit on the ith unit at time $t-\tau_j$; $\tau_j(\geq 0)$ corresponds to the time delay required in processing and transmitting a signal from the jth cell to the ith cell at time t.

Although the use of constant fixed delays in models of delayed feedback provides of a good approximation in simple circuits consisting a small number of cells, recently, it has been well recognized that neural networks usually have a spatial extent due to the presence of a multitude of parallel pathways with a variety of axon sizes and lengths. Therefore, there will be a distribution of conduction velocities along these pathways and a distribution of propagation delays. As the fact that delays in artificial neural networks are usually time varying and sometimes vary violently with time, system (1) can be generalized as follow:

$$\dot{x}(t) = -Dx(t) + AF(x(t)) + BF(x(t-\tau(t))) + C\int_{t-h(t)}^{t} F(x(s))\,ds + J, \tag{2}$$

where, $x(t) = (x_1(t), \ldots, x_n(t))^T$, $D = \mathrm{diag}(d_1, \ldots, d_n)$, $F(\cdot) = (f_1(\cdot), \ldots, f_n(\cdot))^T$, $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$, $C = (c_{ij})_{n\times n}$, $\tau(t) = (\tau_1(t), \ldots, \tau_n(t))^T$, $h(t) = (h_1(t), \ldots, h_n(t))^T$, $J = (J_1, \ldots, J_n)^T$.

On the other hand, neural networks are complex and large-scale nonlinear dynamics; during hardware implementation, the connection topology of networks may change very quickly and link failures or new creation in networks often bring about switching connection topology [11, 12]. To obtain a deep and clear understanding of the dynamics of this complex system, one of the usual ways is to investigate the switched neural network. As a special class of hybrid systems, switched neural network systems are composed of a family of continuous-time or discrete-time subsystems and a rule that orchestrates the switching between the subsystems [13]. A switched DCNN can be characterized by the following differential equation:

$$\begin{aligned} \dot{x}(t) = & -Dx(t) + A_{\sigma(t)}F(x(t)) + B_{\sigma(t)}F(x(t-\tau(t))) \\ & + C_{\sigma(t)}\int_{t-h(t)}^{t} F(x(s))\,\mathrm{d}s + J, \end{aligned} \tag{3}$$

where $\sigma(t) : [0, +\infty) \to \Sigma = \{1, 2, \ldots, m\}$ is the switching signal, which is a piecewise constant function of time.

Corresponding to the switching signal $\sigma(t)$, we have the switching sequence $\{x_{t_0}; (i_0, t_0), \ldots, (i_k, t_k), \ldots, \mid i_k \in \Sigma, k = 0, 1, \ldots\}$, which means that the i_kth subsystem is activated when $t \in [t_k, t_{k-1})$.

Over the past decades, the stability of the unique equilibrium point for switched neural networks has been intensively investigated. There are three basic problems in dealing with the stability of switched systems: (1) find conditions that guarantee that the switched system (3) is asymptotically stable for any switching signal; (2) identify those classes of switching signals for which the switched system (3) is asymptotically stable; (3) construct a switching signal that makes the switched system (3) asymptotically stable [14]. Recently, some novel results on the stability of switched systems have been reported; see for examples [14–22] and references therein.

Just as pointed out in [23], when the activation functions are typically assumed to be continuous, bounded, differentiable, and monotonically increasing, such as the functions of sigmoid type, the existence of an equilibrium point can be guaranteed. However, in some special applications, one is required to use unbounded activation functions. For example, when neural networks are designed for solving optimization problems in the presence of constraints (linear, quadratic, or more general programming problems), unbounded activations modeled by diode-like exponential-type functions are needed to impose constraints satisfaction. Different from the bounded case where the existence of an equilibrium point is always guaranteed, for unbounded activations it may happen that there is no equilibrium point. In this case, it is difficult to deal with the issue of the stability of the equilibrium point for switched neural networks.

In fact, studies on neural dynamical systems involve not only the discussion of stability property but also other dynamics behaviors such as the ultimate boundedness and attractor [24, 25]. To the best of our knowledge, so far there are no published results on the ultimate boundedness and attractor for the switched system (3).

Motivated by the above discussions, in the following, the objective of this paper is to establish a set of sufficient criteria on the attractor and ultimate boundedness for the switched system. The rest of this paper is organized as follows. Section 2 presents model formulation and some preliminary works. In Section 3, ultimate boundedness and attractor for the considered model are studied. In Section 4, a numerical example is given to show the effectiveness of our results. Finally, in Section 5, conclusions are given.

2. Problem Formulation

For the sake of convenience, throughout this paper, two of the standing assumptions are formulated below:

(H_1) Assume the functions $\tau(t)$ and $h(t)$ are bounded:

$$0 \le \tau_i(t) \le \tau, \quad 0 \le h(t) \le h, \quad \tau^* = \max_{1\le i\le n}\{\tau, h\}, \tag{4}$$

where τ, h are scalars.

(H_2) Assume there exist constants l_j and L_j, $i = 1, 2, \ldots, n$, such that

$$l_j \le \frac{f_j(x) - f_j(y)}{x - y} \le L_j, \quad \forall x, y \in R,\ x \ne y. \tag{5}$$

Remark 1. We shall point out that the constants l_j and L_j can be positive, negative, or zero, and the boundedness on $f_j(\cdot)$ is no longer needed in this paper. Therefore, the activation function $f_j(\cdot)$ may be unbounded, which is also more general than the form $|f_j(u)| \le K_j|u|, K_j > 0, j = 1, 2, \ldots, n$. Different from the bounded case where the existence of an equilibrium point is always guaranteed, under the condition (H_2), in the switched system (3) it may happen that there is no equilibrium point. Thus it is of great interest to investigate the ultimate boundedness solutions and the existence of an attractor by replacing the usual stability property for system (3).

Without loss of generality, let $C([-\tau^*, 0], R^n)$ denote the Banach space of continuous mapping from $[-\tau^*, 0]$ to R^n equipped with the supremum norm $\|\ \varphi(t)\ \| = \max_{1\le i\le n}\sup_{t-\tau^*<s\le t}|\varphi_i(s)|$. Throughout this paper, we give some notations: A^T denotes the transpose of any square matrix A, $A > 0$ (< 0) denotes a positive (negative) definite matrix A, the symbol "$*$" within the matrix represents the symmetric term of the matrix, $\lambda_{\min}(A)$ represents the minimum eigenvalue of matrix A, and $\lambda_{\max}(A)$ represents the maximum eigenvalue of matrix A.

System (3) is supplemented with initial values of the type

$$x(t) = \varphi, \quad \varphi \in C([-\tau^*, 0], R^n). \tag{6}$$

Now, we briefly summarize some needed definitions and lemmas as below.

Definition 2 (see [24]). System (3) is uniformly ultimately bounded; if there is $\widetilde{B} > 0$, for any constant $\varrho > 0$, there is $t' = t'(\varrho) > 0$, such that $\| x(t, t_0, \varphi) \| < \widetilde{B}$ for all $t \geq t_0 + t'$, $t_0 > 0, \| \varphi \| < \varrho$.

Definition 3. The nonempty closed set $\mathbb{A} \subset R^n$ is called an attractor for the solution $x(t; \varphi)$ of system (3) if the following formula holds:

$$\lim_{t \to \infty} d\left(x\left(t; \varphi\right), \mathbb{A}\right) = 0, \tag{7}$$

where $d(x, \mathbb{A}) = \inf_{y \in \mathbb{A}} \| x - y \|$.

Definition 4 (see [26]). For any switching signal $\sigma(t)$ and any finite constants T_1, T_2 satisfying $T_2 > T_1 \geq 0$, denote the number of discontinuity of a switching signal $\sigma(t)$ over the time interval (T_1, T_2) by $N_\sigma(T_1, T_2)$. If $N_\sigma(T_1, T_2) \leq N_0 + (T_2 - T_1)/T_\alpha$ holds for $T_\alpha > 0$, $N_0 > 0$, then $T_\alpha > 0$ is called the average dwell time.

3. Main Results

Theorem 5. *Assume there is a constant μ, such that $\dot{\tau}(t) \leq \mu$, and denote $g(\mu)$ as*

$$g\left(\mu\right) = \begin{cases} \left(1 - \mu\right) e^{-a\tau}, & \mu \leq 1; \\ 1 - \mu, & \mu \geq 1. \end{cases} \tag{8}$$

For a given constant $a > 0$, if there exist positive-definite matrixes $P = \operatorname{diag}(p_1, p_2, \ldots, p_n)$, $Y_i = \operatorname{diag}(y_{i1}, y_{i2}, \ldots, y_{in})$, $i = 1, 2$, such that the following condition holds:

$$\Delta_1 = \begin{bmatrix} \Phi_{11} & 0 & \Phi_{13} & \Phi_{14} & \Phi_{15} \\ * & \Phi_{22} & 0 & \Phi_{24} & 0 \\ * & * & \Phi_{33} & 0 & 0 \\ * & * & * & \Phi_{44} & 0 \\ * & * & * & * & 0 \end{bmatrix} < 0, \tag{9}$$

where

$$\begin{gathered} Q = \begin{pmatrix} Q_{11} & Q_{12} \\ * & Q_{22} \end{pmatrix} \geq 0, \qquad Y_i \geq 0, \ i = 1, 2, \\ \Phi_{11} = aP - 2DP + Q_{11} - \Omega_1 Y_1 + P + aI, \\ \Phi_{13} = PA + Q_{12} + \Omega_2 Y_1, \\ \Phi_{14} = PB, \qquad \Phi_{15} = PC, \\ \Phi_{22} = -g\left(\mu\right) Q_{11} - \Omega_1 Y_2 + aI, \\ \Phi_{24} = -g\left(\mu\right) Q_{12} + \Omega_2 Y_2, \qquad \Phi_{33} = Q_{22} - 2Y_1 + aI, \\ \Phi_{44} = -g\left(\mu\right) Q_{22} - 2Y_2 + aI, \\ \Omega_1 = \operatorname{diag}\left\{l_1 L_1, l_2 L_2, \ldots, l_n L_n\right\}, \\ \Omega_2 = \operatorname{diag}\left\{l_1 + L_1, l_2 + L_2, \ldots, l_n + L_n\right\}, \\ L = \max_{1 \leq j \leq n} \left\{\left|l_j\right|^2, \left|L_j\right|^2\right\} + 1, \end{gathered} \tag{10}$$

and then system (2) *is uniformly ultimately bounded.*

Proof. Choose the following Lyapunov functional:

$$V\left(t\right) = V_1\left(t\right) + V_2\left(t\right), \tag{11}$$

where

$$\begin{gathered} V_1\left(t\right) = e^{at} x^T\left(t\right) P x\left(t\right), \\ V_2\left(t\right) = \int_{t-\tau(t)}^{t} e^{as} \xi^T\left(s\right) Q \xi\left(s\right) ds, \\ \xi\left(t\right) = \left[x^T\left(t\right), F^T\left(x\left(t\right)\right)\right]^T. \end{gathered} \tag{12}$$

Computing the derivative of $V_1(t)$ along the trajectory of system (2), one can get

$$\begin{aligned} \dot{V}_1\left(t\right) \leq e^{at} & \left[a x^T\left(t\right) P x\left(t\right) - 2x^T\left(t\right) PD x\left(t\right) \right. \\ & + 2x^T\left(t\right) PAF\left(x\left(t\right)\right) \\ & + 2x^T\left(t\right) PBF\left(x\left(t - \tau\left(t\right)\right)\right) \\ & + 2x^T\left(t\right) \ PC \int_{t-h(t)}^{t} F\left(x\left(s\right)\right) ds \\ & \left. + x^T\left(t\right) P x\left(t\right) + J^T P J\right]. \end{aligned} \tag{13}$$

Similarly, computing the derivative of $V_2(t)$ along the trajectory of system (2), one can get

$$\begin{aligned} & \dot{V}_2\left(t\right) \\ & = e^{at}\left[x^T\left(t\right), F^T x\left(t\right)\right] Q \left[x^T\left(t\right), F^T x\left(t\right)\right]^T \\ & \quad - \left(1 - \dot{\tau}\left(t\right)\right) e^{a(t-\tau(t))} \\ & \quad \times \left[x^T\left(t - \tau\left(t\right)\right), F^T\left(x\left(t - \tau\left(t\right)\right)\right)\right] \\ & \quad \times Q\left[x^T\left(t - \tau\left(t\right)\right), F^T\left(x\left(t - \tau\left(t\right)\right)\right)\right]^T \\ & \leq e^{at}\left[x^T\left(t\right), F^T\left(x\left(t\right)\right)\right] Q\left[x^T\left(t\right), F^T\left(x\left(t\right)\right)\right]^T \\ & \quad - g\left(\mu\right) e^{at}\left[x^T\left(t - \tau\left(t\right)\right), F^T\left(x\left(t - \tau\left(t\right)\right)\right)\right] \\ & \quad \times Q\left[x^T\left(t - \tau\left(t\right)\right), F^T\left(x\left(t - \tau\left(t\right)\right)\right)\right]^T \\ & = e^{at}\left[x^T\left(t\right) Q_{11} x\left(t\right) + F^T\left(x\left(t\right)\right) Q_{12}^T x\left(t\right) \right. \\ & \qquad + x^T\left(t\right) Q_{12} F\left(x\left(t\right)\right) \\ & \qquad \left. + F^T\left(x\left(t\right)\right) Q_{22} F\left(x\left(t\right)\right)\right] - g\left(\mu\right) e^{at} \\ & \quad \times \left[x^T\left(t - \tau\left(t\right)\right) Q_{11} x\left(t - \tau\left(t\right)\right) \right. \\ & \qquad + F^T\left(x\left(t - \tau\left(t\right)\right)\right) Q_{12}^T x\left(t - \tau\left(t\right)\right) \\ & \qquad + x^T\left(t - \tau\left(t\right)\right) Q_{12} F\left(x\left(t - \tau\left(t\right)\right)\right) \\ & \qquad \left. + F^T\left(x\left(t - \tau\left(t\right)\right)\right) Q_{22} F\left(x\left(t - \tau\left(t\right)\right)\right)\right]. \end{aligned} \tag{14}$$

From assumption (H_2), we have

$$\begin{aligned}&[f_i(x_i(t)) - L_i x_i(t) - f_i(0)]\\&\quad\times [f_i(x_i(t)) - l_i x_i(t) - f_i(0)] \le 0,\\&[f_i(x_i(t-\tau(t))) - L_i x_i(t-\tau(t)) - f_i(0)]\\&\quad\times [f_i(x_i(t-\tau(t))) - l_i x_i(t-\tau(t)) - f_i(0)]\\&\quad\le 0, \quad i = 1, 2, \dots, n.\end{aligned} \tag{15}$$

Then we have

$$\begin{aligned}0 &\le e^{at}\Bigg\{-2\sum_{i=1}^{n} y_{1i}[f_i(x_i(t)) - L_i x_i(t) - f_i(0)]\\&\qquad\times [f_i(x_i(t)) - l_i x_i(t) - f_i(0)]\\&\qquad - 2\sum_{i=1}^{n} y_{2i}[f_i(x_i(t-\tau(t)))\\&\qquad\qquad - L_i x_i(t-\tau(t)) - f_i(0)]\\&\qquad\times [f_i(x_i(t-\tau(t))) - l_i x_i(t-\tau(t)) - f_i(0)]\Bigg\}\\&= e^{at}\Bigg\{-2\sum_{i=1}^{n} y_{1i}[f_i(x_i(t)) - L_i x_i(t)]\\&\qquad\times [f_i(x_i(t)) - l_i x_i(t) - f_i(0)]\\&\qquad - 2\sum_{i=1}^{n} y_{2i}[f_i(x_i(t-\tau(t))) - L_i x_i(t-\tau(t))]\\&\qquad\times [f_i(x_i(t-\tau(t))) - l_i x_i(t-\tau(t))]\\&\qquad - 2\sum_{i=1}^{n} y_{1i} f_i^2(0)\\&\qquad + 2\sum_{i=1}^{n} y_{1i} f_i(0)[2f_i(x_i(t)) - (L_i + l_i)x_i(t)]\\&\qquad - 2\sum_{i=1}^{n} y_{2i} f_i^2(0)\\&\qquad + 2\sum_{i=1}^{n} y_{2i} f_i(0)[2f_i(x_i(t-\tau(t)))\\&\qquad\qquad - (L_i + l_i)x_i(t-\tau(t))]\Bigg\}\\&\le e^{at}\Bigg\{-2\sum_{i=1}^{n} y_{1i}[f_i(x_i(t)) - L_i x_i(t)]\\&\qquad\times [f_i(x_i(t)) - l_i x_i(t)]\\&\qquad - 2\sum_{i=1}^{n} y_{1i}[f_i(x_i(t-\tau(t))) - L_i x_i(t-\tau(t))]\\&\qquad\times [f_i(x_i(t-\tau(t))) - l_i x_i(t-\tau(t))]\\&\qquad + \sum_{i=1}^{n}[|4y_{1i} f_i(0) f_i(x_i(t))|\\&\qquad\qquad + |2y_{1i} f_i(0)(L_i + l_i)x_i(t)|]\\&\qquad + \sum_{i=1}^{n}[|4y_{2i} f_i(0) f_i(x_i(t-\tau(t)))|\\&\qquad\qquad + |2y_{1i} f_i(0)(L_i + l_i)x_i(t-\tau(t))|]\Bigg\}\\&\le e^{at}\Bigg\{-2\sum_{i=1}^{n} y_{1i}[f_i(x_i(t)) - L_i x_i(t)]\\&\qquad\times [f_i(x_i(t)) - l_i x_i(t)]\\&\qquad - 2\sum_{i=1}^{n} y_{1i}[f_i(x_i(t-\tau(t))) - L_i x_i(t-\tau(t))]\\&\qquad\times [f_i(x_i(t-\tau(t))) - l_i x_i(t-\tau(t))]\\&\qquad + \sum_{i=1}^{n}\left[a f_i^2(x_i(t)) + 4a^{-1} f_i^2(0) y_{1i}^2\right.\\&\qquad\qquad \left.+ a x_i^2(t) + a^{-1} f_i^2(0) y_{1i}^2 (L_i + l_i)^2\right]\\&\qquad + \sum_{i=1}^{n}\left[a f_i^2(x_i(t-\tau(t))) + 4a^{-1} f_i^2(0) y_{2i}^2\right.\\&\qquad\qquad + a x_i^2(t-\tau(t)) + a^{-1} f_i^2(0)\\&\qquad\qquad \left.\times y_{2i}^2 (L_i + l_i)^2\right]\Bigg\}.\end{aligned} \tag{16}$$

Denote $M^T(t) = (x^T(t), x^T(t-\tau), F^T(x(t)), F^T(x(t-\tau)), (\int_{t-h}^{t} F(x(s))\mathrm{d}s)^T)^T$; combing with (11)–(16), we have

$$\begin{aligned}\dot V(t) &\le e^{at}\left[a x^T(t) P x(t) - 2x^T(t) P D x(t)\right.\\&\qquad + 2x^T(t) P A F(x(t)) + 2x^T(t) P B F(x(t-\tau(t)))\\&\qquad + 2x^T(t) P C \int_{t-h(t)}^{t} F(x(s))\,\mathrm{d}s\\&\qquad \left.+ x^T(t) P x(t) + J^T P J\right]\end{aligned}$$

$$+ e^{at}\left[x^T(t)Q_{11}x(t) + F^T(x(t))Q_{12}^Tx(t) + x^T(t)Q_{12}F(x(t)) + F^T(x(t))Q_{22}F(x(t))\right]$$
$$- g_1(\mu)e^{at}\times\left[x^T(t-\tau(t))Q_{11}x(t-\tau(t)) + F^T(x(t-\tau(t)))Q_{12}^Tx(t-\tau(t)) + x^T(t-\tau(t))Q_{12}F(x(t-\tau(t))) + F^T(x(t-\tau(t)))Q_{22}F(x(t-\tau(t)))\right]$$
$$+ e^{at}\left\{-2\sum_{i=1}^{n}y_{1i}\left[f_i(x_i(t)) - L_ix_i(t)\right]\times\left[f_i(x_i(t)) - l_ix_i(t)\right] - 2\sum_{i=1}^{n}y_{1i}\left[f_i(x_i(t-\tau(t))) - L_ix_i(t-\tau(t))\right]\times\left[f_i(x_i(t-\tau(t))) - l_ix_i(t-\tau(t))\right] + \sum_{i=1}^{n}\left[af_i^2(x_i(t)) + ax_i^2(t) + a^{-1}f_i^2(0)y_{1i}^2\times(L_i+l_i)^2 + 4a^{-1}f_i^2(0)y_{1i}^2\right] + \sum_{i=1}^{n}\left[af_i^2(x_i(t-\tau(t))) + 4a^{-1}f_i^2(0)y_{2i}^2 + ax_i^2(t-\tau(t)) + a^{-1}f_i^2(0)y_{2i}^2(L_i+l_i)^2\right]\right\}, \quad (17)$$

and then we have

$$\dot{V}(t) \le e^{at}M^T(t)\Delta_1M(t) + e^{at}R_1, \quad (18)$$

where $R_1 = \sum_{i=1}^{n}[4a^{-1}f_i^2(0)y_{2i}^2 + a^{-1}f_i^2(0)y_{1i}^2(L_i+l_i)^2 + 4a^{-1}f_i^2(0)y_{2i}^2 + a^{-1}f_i^2(0)y_{2i}^2(L_i+l_i)^2] + J^TPJ$.

Therefore, we obtain

$$Ke^{at}\|x(t)\|^2 \le V(x(t)) \le V(x(t_0)) + a^{-1}e^{at}R_1, \quad (19)$$

where $K = \lambda_{\min}(P)$, which implies

$$\|x(t)\|^2 \le \frac{e^{-at}V(x(0)) + a^{-1}R_1}{K}. \quad (20)$$

If one chooses $\widetilde{B} = \sqrt{(1+a^{-1}R_1)/K} > 0$, then for any constant $\varrho > 0$ and $\|\varphi\| < \varrho$, there is $t' = t'(\varrho) > 0$, such that $e^{-at}V(x(0)) < 1$ for all $t \ge t'$. According to Definition 2, we have $\|x(t,0,\varphi)\| < \widetilde{B}$ for all $t \ge t'$. That is to say, system (2) is uniformly ultimately bounded. This completes the proof. □

Theorem 6. *If all of the conditions of Theorem 5 hold, then there exists an attractor $\mathbb{A}_{\widetilde{B}}$ for the solutions of system (2), where $\mathbb{A}_{\widetilde{B}} = \{x(t) :\|x(t)\| \le \widetilde{B}, t \ge t_0\}$.*

Proof. If one chooses $\widetilde{B} = \sqrt{(1+a^{-1}R_1)/K} > 0$, Theorem 5 shows that for any ϕ there is $t' > 0$, such that $\|x(t,0,\phi)\| < \widetilde{B}$ for all $t \ge t'$. Let $\mathbb{A}_{\widetilde{B}}$ be denoted by $\mathbb{A}_{\widetilde{B}} = \{x(t) :\|x(t)\| \le \widetilde{B}, t \ge t_0\}$. Clearly, $\mathbb{A}_{\widetilde{B}}$ is closed, bounded, and invariant. Furthermore, $\lim_{t\to\infty}\sup\inf_{y\in\mathbb{A}_{\widetilde{B}}}\|x(t;0,\phi) - y\| = 0$. Therefore, $\mathbb{A}_{\widetilde{B}}$ is an attractor for the solutions of system (2). This completes the proof. □

Corollary 7. *In addition to all of the conditions of Theorem 5 holding, if $J = 0$ and $f_i(0) = 0$ for all $i = 1,2,\ldots,n$, then system (2) has a trivial solution $x(t) \equiv 0$, and the trivial solution of system (2) is globally exponentially stable.*

Proof. If $J = 0$ and $f_i(0) = 0$ for all $i = 1,2,\ldots,n$, then $R_1 = 0$, and it is obvious that system (2) has a trivial solution $x(t) \equiv 0$. From Theorem 5, one has

$$\|x(t;0,\phi)\|^2 \le K^*e^{-at}, \quad \forall\phi, \quad (21)$$

where $K^* = V(x(0))/K$. Therefore, the trivial solution of system (2) is globally exponentially stable. This completes the proof. □

By (11) and (19), there is a positive constant C_0, such that

$$\|x(t)\|^2 \le \frac{C_0\|x(t_0)\|^2e^{-a(t-t_0)}}{K} + \frac{\Lambda}{K}, \quad (22)$$

where $\Lambda = a^{-1}R_1$.

We now consider the switched cellular neural networks without uncertainties as system (3). When $t \in [t_k, t_{k+1}]$, the i_kth subsystem is activated; from (22) and Theorem 5, there is a positive constant C_{i_k}, such that

$$\|x(t)\|^2 \le \frac{C_{i_k}\|x(t_k)\|^2e^{-a(t-t_k)}}{K_{i_k}} + \frac{\Lambda}{K_{i_k}}, \quad (23)$$

where $K_{i_k} = \lambda_{\min}(P_i)$.

Theorem 8. *For a given constant $a > 0$, if there exist positive-definite matrixes $P_i = \mathrm{diag}(p_{i1}, p_{i2}, \ldots, p_{in})$, $Y_i = \mathrm{diag}(y_{i1}, y_{i2}, \ldots, y_{in})$, $i = 1,2$, such that the following condition holds:*

$$\Delta_{i1} = \begin{bmatrix} \Phi_{i11} & 0 & \Phi_{i13} & \Phi_{i14} & \Phi_{i15} \\ * & \Phi_{i22} & 0 & \Phi_{i24} & 0 \\ * & * & \Phi_{i33} & 0 & 0 \\ * & * & * & \Phi_{i44} & 0 \\ * & * & * & * & 0 \end{bmatrix} < 0, \quad (24)$$

where

$$Q_i = \begin{pmatrix} Q_{i11} & Q_{i12} \\ * & Q_{i22} \end{pmatrix} \geq 0, \quad Y_i \geq 0, \quad i = 1, 2, \quad \dot{\tau}(t) \leq \mu,$$
$$\Phi_{i11} = aP_i - 2DP_i + Q_{i11} - \Omega_1 Y_1 + P_i + aI,$$
$$\Phi_{i13} = P_i A_i + Q_{i12} + \Omega_2 Y_1,$$
$$\Phi_{i14} = P_i B_i, \qquad \Phi_{15} = P_i C_i,$$
$$\Phi_{i22} = -g(\mu) Q_{i11} - \Omega_1 Y_2 + aI,$$
$$\Phi_{i24} = -g(\mu) Q_{i12} + \Omega_2 Y_2, \qquad \Phi_{i33} = Q_{i22} - 2Y_1 + aI,$$
$$\Phi_{i44} = -g(\mu) Q_{i22} - 2Y_2 + aI. \tag{25}$$

Then system (3) is uniformly ultimately bounded for any switched signal with average dwell time satisfying

$$T_\alpha > T_\alpha^* = \frac{\ln C_{\max}}{a}, \tag{26}$$

where $C_{\max} = \max_{i_k}\{C_{i_k}\}$.

Proof. Define the Lyapunov functional candidate

$$V_{\sigma(t)} = e^{at} x^T(t) P_{\sigma(t)} x(t) + \int_{t-\tau}^{t} e^{as} \xi^T(s) Q_{\sigma(t)} \xi(s)\, ds. \tag{27}$$

Since the system state is continuous, it follows from (23) that

$$\begin{aligned}
\|x(t)\|^2 &\leq \frac{C_{i_k}\|x(t_k)\|^2 e^{-a(t-t_k)}}{K_{i_k}} + \frac{\Lambda}{K_{i_k}} \leq \cdots \\
&\leq \frac{e^{\sum_{v=0}^{k}\ln C_{i_v} - a(t-t_0)}\|x(t_0)\|^2}{K_{\min}^{k+1}} \\
&\quad + \left[C_{i_1}^{k} e^{-a(t-t_1)} \frac{\Lambda}{K_{i_1}^{k+1}} + C_{i_2}^{k-1} e^{-a(t-t_2)} \frac{\Lambda}{K_{i_2}^{k}} \right. \\
&\quad + C_{i_3}^{k-2} e^{-a(t-t_3)} \frac{\Lambda}{K_{i_3}^{k-1}} \\
&\quad + \cdots + C_{i_{k-1}}^{2} e^{-a(t-t_{k-1})} \frac{\Lambda}{K_{i_{k-1}}^{3}} \\
&\quad \left. + C_{i_k} e^{-a(t-t_k)} \frac{\Lambda}{K_{i_k}^{2}} + \frac{\Lambda}{K_{i_{k+1}}} \right] \\
&\leq \frac{e^{(k+1)\ln C_{\max} - a(t-t_0)}}{K_{\min}^{k+1}} \|x(t_0)\|^2 \\
&\quad + \left[C_{\max}^{k} \frac{\Lambda}{K_{\min}^{k+1}} + C_{\max}^{k-1} \frac{\Lambda}{K_{\min}^{k}} \right. \\
&\quad + C_{\max}^{k-2} \frac{\Lambda}{K_{\min}^{k-1}} + \dots + C_{\max}^{2} \frac{\Lambda}{K_{\min}^{3}} \\
&\quad \left. + C_{\max} \frac{\Lambda}{K_{\min}^{2}} + \frac{\Lambda}{K_{\min}} \right] \\
&\leq \frac{C_{\max} e^{k \ln C_{\max} - a(t-t_0)}}{K_{\min}^{k+1}} \|x(t_0)\|^2 \\
&\quad + \frac{\Lambda / K_{\min}}{(C_{\max}/K_{\min}) - 1} \left[\frac{C_{\max}^{n+1}}{K_{\min}^{n+1}} - 1 \right] \\
&\leq \frac{C_{\max} e^{\ln C_{\max} N_\sigma(t_0,t) - a(t-t_0)}}{K_{\min}^{k+1}} \|x(t_0)\|^2 \\
&\quad + \frac{\Lambda}{C_{\max} - K_{\min}} \left[\left(C_{\max}^{n+1} / K_{\min}^{n+1} \right) - 1 \right] \\
&\leq \frac{C_{\max} e^{N_0 \ln C_{\max} - (a - (\ln C_{\max}/T_\alpha))(t-t_0)}}{K_{\min}^{k+1}} \|x(t_0)\|^2 \\
&\quad + \frac{\Lambda\left(\left(C_{\max}^{n+1} / K_{\min}^{n+1} \right) - 1 \right)}{C_{\max} - K_{\min}}.
\end{aligned} \tag{28}$$

If one chooses $\widetilde{B} = \sqrt{1/K_{\min} + \Lambda(C_{\max}^{n+1}/K_{\min}^{n+1} - 1)/(C_{\max} - K_{\min})} > 0$, then for any constant $\varrho > 0$ and $\| \varphi \| < \varrho$, there is $t' = t'(\varrho) > 0$, such that $C_{\max} e^{N_0 \ln C_{\max} - (a - \ln C_{\max}/T_\alpha)(t-t_0)} \| x(t_0)\|^2 < 1$ for all $t \geq t'$. According to Definition 2, we have $\| x(t, 0, \varphi) \| < \widetilde{B}$ for all $t \geq t'$. That is to say, system (3) is uniformly ultimately bounded, and the proof is completed. □

Theorem 9. *If all of the conditions of Theorem 8 hold, then there exists an attractor* $\mathbb{A}'_{\widetilde{B}}$ *for the solutions of system (3), where* $\mathbb{A}'_{\widetilde{B}} = \{x(t) : \| x(t) \| \leq \widetilde{B}, t \geq t_0\}$.

Proof. If one chooses $\widetilde{B} = \sqrt{1/K_{\min} + \Lambda(\overline{C}_{\max}^{n+1}/K_{\min}^{n+1} - 1)/(\overline{C}_{\max} - K_{\min})} > 0$, Theorem 8 shows that for any ϕ there is $t' > 0$, such that $\| x(t, 0, \phi) \| < \widetilde{B}$ for all $t \geq t'$. Let $\mathbb{A}'_{\widetilde{B}}$ be denoted by $\mathbb{A}'_{\widetilde{B}} = \{x(t) : \| x(t) \| \leq \widetilde{B}, t \geq t_0\}$. Clearly, $\mathbb{A}'_{\widetilde{B}}$ is closed, bounded, and invariant. Furthermore, $\lim_{t\to\infty} \sup \inf_{y \in \mathbb{A}'_{\widetilde{B}}} \| x(t; 0, \phi) - y \| = 0$. Therefore, $\mathbb{A}'_{\widetilde{B}}$ is an attractor for the solutions of system (3). This completes the proof. □

Corollary 10. *In addition to all of the conditions of Theorem 8 holding, if $J = 0$ and $f_i(0) = 0$ for all i, then system (2) has a trivial solution $x(t) \equiv 0$, and the trivial solution of system (3) is globally exponentially stable.*

Proof. If $J = 0$ and $f_i(0) = 0$ for all i, then it is obvious that system (3) has a trivial solution $x(t) \equiv 0$. From Theorem 8, one has

$$\|x(t; 0, \phi)\|^2 \leq K_2 e^{-at}, \quad \forall \phi, \tag{29}$$

where

$$K_2 = \frac{C_{\max} e^{N_0 \ln C_{\max} + at_0 + (\ln C_{\max}/T_\alpha)(t-t_0)}}{K_{\min}^{k+1}} \|x(t_0)\|^2. \tag{30}$$

Therefore, the trivial solution of system (3) is globally exponentially stable. This completes the proof. □

Remark 11. Up to now, various dynamical results have been proposed for switched neural networks in the literature. For example, in [15], synchronization control of switched linearly coupled delayed neural networks is investigated; in [16–20], the authors investigated the stability of switched neural networks; in [21, 22], stability and L2-gain analysis for switched delay system have been investigated. To the best of our knowledge, there are few works about the uniformly ultimate boundedness and the existence of an attractor for switched neural networks. Therefore, results of this paper are new.

Remark 12. We notice that Lian and Zhang developed an LMI approach to study the stability of switched Cohen-Grossberg neural networks and obtained some novel results in a very recent paper [20], where the considered model includes both discrete and bounded distributed delays. In [20], the following fundamental assumptions are required: (i) the delay functions $\tau(t), h(t)$ are bounded, and $\dot{\tau}(t) \leq \tau$, $\dot{h}(t) \leq d < 1$; (ii) $f_i(0) = 0$, $l_j \leq (f_j(x) - f_j(y))/(x - y) \leq L_j$, for all $i = 1, 2, \ldots, n$; (iii) the switched system has only one equilibrium point. However, as a defect appearing in [20], just checking the inequality (13) in [20], it is easy to see that the assumed condition on $\dot{\tau}(t) \leq \tau$ is not correct, which should be revised as $\dot{\tau}(t) \leq \tau \leq 1$. On the other hand, just as described by Remark 1 in this paper, for a neural network with unbounded activation functions, the considered system in [20] may have no equilibrium point or have multiple equilibrium points. In this case, it is difficult to deal with the issue of the stability of equilibrium point for switched neural networks. In order to modify this imperfection, after relaxing the conditions $\dot{\tau}(t) \leq \tau \leq 1$, $\dot{h}(t) \leq d < 1$, and $f_i(0) = 0$, replacing (i), (ii), and (iii) with assumptions (H_1) and (H_2), we drop out the assumption of the existence of a unique equilibrium point and investigate the issue of the ultimate boundedness and attractor; this modification seems more natural and reasonable.

Remark 13. When investigating the stability, although the adopted Lyapunov function in this paper is similar to those used in [20]; just from Corollaries 7 and 10, the conservatism of the conditions of the delay function in this paper has been further reduced. Hence, the obtained results on stability in this paper are complementary to the corresponding results in [20].

Remark 14. When the uncertainties appear in the system (3), employing the Lyapunov function as (27) in this paper and applying a similar method to the one used in [20], we can get the corresponding dynamical results. Due to the limitation of space, we choose not to give the straightforward but the tedious computations here for the formulas that determine the uniformly ultimate boundedness, the existence of an attractor, and stability.

4. Illustrative Example

In this section, we present an example to illustrate the effectiveness of the proposed results. Consider the switched cellular neural networks with two subsystems.

Example 15. Consider the switched cellular neural networks system (3) with $d_i = 1$, $f_i(x_i(t)) = 0.5\tanh(x_i(t))(i = 1, 2)$, $\tau(t) = 0.5\sin^2(t)$, $h(t) = 0.3\sin^2(t)$, and the connection weight matrices where

$$A_1 = \begin{pmatrix} 3.1 & 0.4 \\ 0.2 & 0.5 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} 2.1 & -1 \\ -1.4 & 0.4 \end{pmatrix},$$
$$C_1 = \begin{pmatrix} 1.2 & -1.1 \\ -0.5 & 0.1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 2.5 & 0.3 \\ 0.2 & 0.6 \end{pmatrix}, \tag{31}$$
$$B_2 = \begin{pmatrix} 1.4 & -1.2 \\ -2.4 & 0.3 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} 2.4 & -0.1 \\ 0.7 & 0.4 \end{pmatrix}.$$

From assumptions (H_1) and (H_2), we can obtain $d = 1$, $l_i = 0$, $L_i = 0.5$, $i = 1, 2$, $\tau = 0.5$, $h = 0.3$, $\mu = 1$.

Choosing $a = 2$ and solving LMIs (23), we get

$$P_1 = \begin{pmatrix} 0.0324 & 0 \\ 0 & 0.0776 \end{pmatrix},$$
$$P_2 = \begin{pmatrix} 0.0168 & 0 \\ 0 & 0.0295 \end{pmatrix},$$
$$Q_1 = \begin{pmatrix} -1.9748 & 0.3440 & -0.3551 & 0.0168 \\ * & -1.9458 & 0.0168 & -0.3438 \\ * & * & 2.9120 & -0.2371 \\ * & * & * & 2.8760 \end{pmatrix}, \tag{32}$$
$$Q_2 = \begin{pmatrix} -1.9996 & 0.2120 & -0.1452 & 0.0119 \\ * & -1.9918 & 0.0119 & -0.1464 \\ * & * & 2.9029 & -0.1083 \\ * & * & * & 2.8927 \end{pmatrix}.$$

Using (26), we can get the average dwell time $T_a^* = 0.3590$.

5. Conclusions

In this paper, the dynamics of switched cellular neural networks with mixed delays (interval time-varying delays and distributed-time varying delays) are investigated. Novel multiple Lyapunov-Krasovkii functional methods are designed to establish new sufficient conditions guaranteeing the uniformly ultimate boundedness, the existence of an attractor, and the globally exponential stability. The derived conditions are expressed in terms of LMIs, which are more relaxed than algebraic formulation and can be easily checked by the effective LMI toolbox in Matlab in practice.

Acknowledgments

The authors are extremely grateful to Professor Jinde Cao and the anonymous reviewers for their constructive and valuable comments, which have contributed much to the improvement of this paper. This work was jointly supported by the National Natural Science Foundation of China under Grants nos. 11101053, 70921001, and 71171024, the Key Project of Chinese Ministry of Education under Grant no. 211118, and the Excellent Youth Foundation of Educational Committee of Hunan Provincial no. 10B002, the Scientific Research Funds of Hunan Provincial Science and Technology Department of China.

References

[1] L. O. Chua and L. Yang, "Cellular neural networks: theory," *IEEE Transactions on Circuits and Systems*, vol. 35, no. 10, pp. 1257–1272, 1988.

[2] L. O. Chua and L. Yang, "Cellular neural networks: applications," *IEEE Transactions on Circuits and Systems*, vol. 35, no. 10, pp. 1273–1290, 1988.

[3] T. Chen, "Global exponential stability of delayed Hopfield neural networks," *Neural Networks*, vol. 14, no. 8, pp. 977–980, 2001.

[4] K. Lu, D. Xu, and Z. Yang, "Global attraction and stability for Cohen-Grossberg neural networks with delays," *Neural Networks*, vol. 19, no. 10, pp. 1538–1549, 2006.

[5] J. Cao and L. Li, "Cluster synchronization in an array of hybrid coupled neural networks with delay," *Neural Networks*, vol. 22, no. 4, pp. 335–342, 2009.

[6] T. Roska, T. Boros, P. Thiran, and L. Chua, "Detecting simple motion using cellular neural networks," in *Proceedings of the International Workshop Cellular neural networks Application*, pp. 127–138, 1990.

[7] T. Roska and L. O. Chua, "Cellular neural networks with non-linear and delay-type template elements and non-uniform grids," *International Journal of Circuit Theory and Applications*, vol. 20, no. 5, pp. 469–481, 1992.

[8] J. Cao, "A set of stability criteria for delayed cellular neural networks," *IEEE Transactions on Circuits and Systems. I.*, vol. 48, no. 4, pp. 494–498, 2001.

[9] J. Cao, G. Feng, and Y. Wang, "Multistability and multiperiodicity of delayed Cohen-Grossberg neural networks with a general class of activation functions," *Physica D*, vol. 237, no. 13, pp. 1734–1749, 2008.

[10] H. Jiang and Z. Teng, "Global eponential stability of cellular neural networks with time-varying coefficients and delays," *Neural Networks*, vol. 17, no. 10, pp. 1415–1425, 2004.

[11] J. Zhao, D. J. Hill, and T. Liu, "Synchronization of complex dynamical networks with switching topology: a switched system point of view," *Automatica*, vol. 45, no. 11, pp. 2502–2511, 2009.

[12] J. Lu, D. W. C. Ho, and L. Wu, "Exponential stabilization of switched stochastic dynamical networks," *Nonlinearity*, vol. 22, no. 4, pp. 889–911, 2009.

[13] H. Huang, Y. Qu, and H. X. Li, "Robust stability analysis of switched Hopfield neural networks with time-varying delay under uncertainty," *Physics Letters A*, vol. 345, no. 4–6, pp. 345–354, 2005.

[14] D. Liberzon and A. S. Morse, "Basic problems in stability and design of switched systems," *IEEE Control Systems Magazine*, vol. 19, no. 5, pp. 59–70, 1999.

[15] W. Yu, J. Cao, and W. Lu, "Synchronization control of switched linearly coupled neural networks with delay," *Neurocomputing*, vol. 73, no. 4–6, pp. 858–866, 2010.

[16] W.-A. Zhang and L. Yu, "Stability analysis for discrete-time switched time-delay systems," *Automatica*, vol. 45, no. 10, pp. 2265–2271, 2009.

[17] H. Huang, Y. Qu, and H. Li, "Robust stability analysis of switched hop?eld neural networks with time-varying delay under uncertainty," *Physics Letters A*, vol. 345, pp. 345–354, 2005.

[18] P. Li and J. Cao, "Global stability in switched recurrent neural networks with time-varying delay via nonlinear measure," *Nonlinear Dynamics*, vol. 49, no. 1-2, pp. 295–305, 2007.

[19] X. Lou and B. Cui, "Delay-dependent criteria for global robust periodicity of uncertain switched recurrent neural networks with time-varying delay," *IEEE Transactions on Neural Networks*, vol. 19, no. 4, pp. 549–557, 2008.

[20] J. Lian and K. Zhang, "Exponential stability for switched Cohen-Grossberg neural networks with average dwell time," *Nonlinear Dynamics*, vol. 63, no. 3, pp. 331–343, 2011.

[21] D. Xie, Q. Wang, and Y. Wu, "Average dwell-time approach to L_2 gain control synthesis of switched linear systems with time delay in detection of switching signal," *IET Control Theory & Applications*, vol. 3, no. 6, pp. 763–771, 2009.

[22] X.-M. Sun, J. Zhao, and D. J. Hill, "Stability and L_2-gain analysis for switched delay systems: a delay-dependent method," *Automatica*, vol. 42, no. 10, pp. 1769–1774, 2006.

[23] C. Huang and J. Cao, "Convergence dynamics of stochastic Cohen-Grossberg neural networks with unbounded distributed delays," *IEEE Transactions on Neural Networks*, vol. 22, no. 4, pp. 561–572, 2011.

[24] J. Cao and J. Liang, "Boundedness and stability for Cohen-Grossberg neural network with time-varying delays," *Journal of Mathematical Analysis and Applications*, vol. 296, no. 2, pp. 665–685, 2004.

[25] H. Zhang, Z. Yi, and L. Zhang, "Continuous attractors of a class of recurrent neural networks," *Computers & Mathematics with Applications*, vol. 56, no. 12, pp. 3130–3137, 2008.

[26] J. P. Hespanha and A. S. Morse, "Stability of switched systems with average dwell-time," in *Proceedings of the 38th IEEE Conference on Decision and Control (CDC '99)*, pp. 2655–2660, December 1999.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 131836, 10 pages
http://dx.doi.org/10.1155/2013/131836

Research Article

Exponential Stability of Impulsive Delayed Reaction-Diffusion Cellular Neural Networks via Poincaré Integral Inequality

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Received 15 November 2012; Accepted 8 February 2013

Academic Editor: Qi Luo

This work is devoted to the stability study of impulsive cellular neural networks with time-varying delays and reaction-diffusion terms. By means of new Poincaré integral inequality and Gronwall-Bellman-type impulsive integral inequality, we summarize some novel and concise sufficient conditions ensuring the global exponential stability of equilibrium point. The provided stability criteria are applicable to Dirichlet boundary condition and show that not only the reaction-diffusion coefficients but also the regional features including the boundary and dimension of spatial variable can influence the stability. Two examples are finally illustrated to demonstrate the effectiveness of our obtained results.

1. Introduction

Cellular neural networks (CNNs), proposed by Chua and Yang in 1988 [1, 2], have been the focus of a number of investigations due to their potential applications in various fields such as optimization, linear and nonlinear programming, associative memory, pattern recognition, and computer vision [3–7]. As the switching speed of neurons and amplifiers is finite in the implementation of neural networks, time delays are inevitable and therefore a type of more effective models is afterwards introduced, called delayed cellular neural networks (DCNNs). Actually, DCNNs have been found to be helpful in solving some dynamic image processing and pattern recognition problems.

As we all know, all the applications of CNNs and DCNNs depend heavily on the dynamic behaviors such as stability, convergence, and oscillatory [8, 9], wherein stability analysis is a major concern in the designs and applications. Correspondingly, the stability of CNNs and DCNNs is a subject of current interest and considerable theoretical efforts have been put into this topic with many good results reported (see, e.g., [10–13]).

With reference to neural networks, however, it is noteworthy that the state of electronic networks is often subject to instantaneous perturbations which may be caused by a switching phenomenon, frequency change, or other sudden noise. On this account, neural networks will experience abrupt change at certain instants, exhibiting impulse effects [14, 15]. For instance, according to Arbib [16] and Haykin [17], when a stimulus from the body or the external environment is received by receptors, the electrical impulses will be conveyed to the neural net and impulse effects arise naturally in the net. In view of this discovery, many scientists have shown growing interests in the influence that the impulses may have on CNNs or DCNNs with a result that a large number of relevant results have been achieved (see, e.g., [18–24]).

Besides impulsive effects, diffusing effects are also nonignorable in reality since the diffusion is unavoidable when the electrons are moving in asymmetric electromagnetic fields. Therefore, the model of impulsive delayed reaction-diffusion neural networks appears as a natural description of the observed evolution phenomena of several real world problems. This one acknowledgement poses a new challenge to the stability research of neural networks.

So far, there have been some theoretical achievements [25–33] on the stability of impulsive delayed reaction-diffusion neural networks. Previously, authors of [27–32] studied the stability of impulsive delayed reaction-diffusion neural networks and put forward several stability criteria by impulsive differential inequality and Green formula, wherein the reaction-diffusion term is evaluated to be less than zero by means of Green formula and thereby the presented stability criteria are shown to be wholly independent of diffusion. According to this result, we fail to see the influence of diffusion on stability.

Recently, it is encouraging that, for impulsive delayed reaction-diffusion neural network, some new stability criteria involving diffusion are obtained in [25, 26, 33–36]. Meanwhile the estimation of reaction-diffusion term is not merely less than zero, instead a more accurate one is given; that is, the reaction-diffusion term is verified to be less than a negative definite term by using some inequalities together with Green formula. It is thereby testified that the diffusion does contribute to the stability of impulsive neural networks.

In [25], the authors quoted the following inequality to deal with the reaction-diffusion terms:

$$\int_{\Omega^*} \left|\frac{\partial v(x)}{\partial x_j}\right|^2 \mathrm{d}x \geq \frac{1}{l_j^2} \int_{\Omega^*} v^2(x)\,\mathrm{d}x, \quad (1)$$

where Ω^* is a cube $|x_j| < l_j$ $(j = 1, 2, \ldots, m)$ and $v(x)$ is a real-valued function belonging to $C_0^1(\Omega^*)$. We can easily derive from this inequality that

$$\int_{\Omega^*} |\nabla v|^2 \mathrm{d}x \geq \left(\int_{\Omega^*} v^2(x)\,\mathrm{d}x\right)\left(\sum_{j=1}^{m} \frac{1}{l_j^2}\right). \quad (2)$$

For better exploring the influence of diffusion on stability, we wonder if we can get a more accurate estimate of reaction-diffusion term. Fortunately, we find the following new Poincaré integral inequality supporting this idea:

$$\int_{\mathcal{S}} |\nabla v(x)|^2 \mathrm{d}x \geq \frac{4n}{B^2} \int_{\mathcal{S}} v^2(x)\,\mathrm{d}x. \quad (3)$$

One can refer to Lemma 3 in Section 2 for the details of this inequality.

On the other hand, it is well known that the theory of differential and integral inequalities plays an important role in the qualitative and quantitative study of solution to differential equations. Up till now, there have been many applications of impulsive differential inequalities to impulsive dynamic systems, followed by lots of stability criteria provided. However, these stability criteria appear a bit complicated and we wonder if we can deduce relatively concise stability criteria by using impulsive integral inequalities

Motivated by these, we attempt to, for impulsive delayed neural networks, employ new Poincaré integral inequality to further investigate the influence of diffusion on the stability and combine Gronwall-Bellman-type impulsive integral inequality so as to provide some new and concise stability criteria. The rest of this paper is organized as follows. In Section 2, the model of impulsive cellular neural networks with time-varying delays and reaction-diffusion terms as well as Dirichlet boundary condition is outlined; in addition, some facts and lemmas are introduced for later reference. In Section 3, we provide a new estimate on the reaction-diffusion term by the agency of new Poincaré integral inequality and then discuss the global exponential stability of equilibrium point by utilizing Gronwall-Bellman-type impulsive integral inequality with a result of some novel and concise stability criteria presented. To conclude, two illustrative examples are given in Section 4 to verify the effectiveness of our obtained results.

2. Preliminaries

Let $R_+ = [0, \infty)$ and $t_0 \in R_+$. Let R^n denote the n-dimensional Euclidean space, and let $\Omega = \prod_{i=1}^{m} [d_i, k_i]$ be a fixed rectangular region in R^m and $M := \max\{k_i - d_i : i = 1, \ldots, m\}$. As usual, denote

$$C_0^1(\Omega) = \left\{ v \mid v \text{ and } D_j v = \frac{\partial v}{\partial x_j} \text{ are continuous on } \Omega, \right.$$
$$\left. v|_{\partial\Omega} = 0,\ 1 \leq j \leq m \right\}. \quad (4)$$

Consider the following impulsive cellular neural network with time-varying delays and reaction-diffusion terms:

$$\frac{\partial u_i(t,x)}{\partial t} = \sum_{s=1}^{m} \frac{\partial}{\partial x_s}\left(D_{is}\frac{\partial u_i(t,x)}{\partial x_s}\right) - a_i u_i(t,x) + \sum_{j=1}^{n} b_{ij} f_j(u_j(t,x)) + \sum_{j=1}^{n} c_{ij} f_j(u_j(t-\tau_j(t),x)),$$
$$t \geq t_0,\ t \neq t_k,\ x \in \Omega,$$
$$i = 1, 2, \ldots, n,\ k = 1, 2, \ldots, \quad (5)$$

$$u_i(t_k + 0, x) = u_i(t_k, x) + P_{ik}(u_i(t_k, x)),$$
$$x \in \Omega,\ i = 1, 2, \ldots, n,\ k = 1, 2, \ldots, \quad (6)$$

where n corresponds to the numbers of units in a neural network; $x = (x_1, \ldots, x_m)^{\mathrm{T}} \in \Omega$, $u_i(t,x)$ denotes the state of the ith neuron at time t and in space x; $D_{is} = \text{const} > 0$ represents transmission diffusion of the ith unit; activation function $f_j(u_j(t,x))$ stands for the output of the jth unit at time t and in space x; b_{ij}, c_{ij}, and a_i are constants: b_{ij} indicates the connection strength of the jth unit on the ith unit at time t and in space x, c_{ij} denotes the connection weight of the jth unit on the ith unit at time $t-\tau_j(t)$ and in space x, where $\tau_j(t)$ corresponds to the transmission delay along the axon of the jth unit, satisfying $0 \leq \tau_j(t) \leq \tau(\tau = \text{const})$ and $\dot{\tau}_j(t) < (1 - (1/h))$ $(h > 0)$, and $a_i > 0$ represents the rate with which the ith unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs at time t and in space x. The fixed moments t_k $(k = 1, 2, \ldots)$ are called impulsive moments meeting $0 \leq t_0 < t_1 < t_2 < \cdots$ and $\lim_{k\to\infty} t_k = \infty$; $u_i(t_k + 0, x)$ and $u_i(t_k - 0, x)$ represent the right-hand and left-hand limit of $u_i(t,x)$ at time t_k and in space x, respectively. $P_{ik}(u_i(t_k, x))$ stands for the abrupt change of $u_i(t,x)$ at the impulsive moment t_k and in space x.

Denote by $u(t,x) = u(t,x;t_0,\varphi)$, $u \in R^n$, the solution of system (5)-(6), satisfying the initial condition

$$u(s,x;t_0,\varphi) = \varphi(s,x), \quad t_0 - \tau \le s \le t_0, \ x \in \Omega, \tag{7}$$

and Dirichlet boundary condition

$$u(t,x;t_0,\varphi) = 0, \quad t \ge t_0, \ x \in \partial\Omega, \tag{8}$$

where the vector-valued function $\varphi(s,x) = (\varphi_1(s,x),\ldots,\varphi_n(s,x))^{\mathrm{T}}$ is such that $\int_\Omega \sum_{i=1}^n \varphi_i^2(s,x)\mathrm{d}x$ is bounded on $[t_0 - \tau, t_0]$.

The solution $u(t,x) = u(t,x;t_0,\varphi) = (u_1(t,x;t_0,\varphi),\ldots,u_n(t,x;t_0,\varphi))^{\mathrm{T}}$ of problem (5)–(8) is, for the time variable t, a piecewise continuous function with the first kind discontinuity at the points t_k $(k = 1,2,\ldots)$, where it is continuous from the left; that is, the following relations are true:

$$\begin{aligned} u_i(t_k - 0, x) &= u_i(t_k, x), \\ u_i(t_k + 0, x) &= u_i(t_k, x) + P_{ik}(u_i(t_k, x)). \end{aligned} \tag{9}$$

Throughout this paper, the norm of $u(t,x;t_0,\varphi)$ is defined by

$$\|u(t,x;t_0,\varphi)\|_\Omega^2 = \sum_{i=1}^n \int_\Omega u_i^2(t,x;t_0,\varphi)\,\mathrm{d}x. \tag{10}$$

Before proceeding, we introduce two hypotheses as follows:

(H1) $f_i(\bullet) : R \to R$ satisfies $f_i(0) = 0$, and there exists a constant $l_i > 0$ such that $|f_i(y_1) - f_i(y_2)| \le l_i|y_1 - y_2|$ for all $y_1, y_2 \in R$ and $i = 1,2,\ldots,n$.

(H2) $P_{ik}(\bullet) : R \to R$ is continuous and $P_{ik}(0) = 0$, $i = 1,2,\ldots,n$, $k = 1,2,\ldots$.

According to (H1) and (H2), it is easy to see that problem (5)–(8) admits an equilibrium point $u = 0$.

Definition 1 (see [25]). The equilibrium point $u = 0$ of problem (5)–(8) is said to be globally exponentially stable if there exist constants $\kappa > 0$ and $\varpi \ge 1$ such that

$$\|u(t,x;t_0,\varphi)\|_\Omega \le \varpi\, \overline{\|\varphi\|}_\Omega \mathrm{e}^{-\kappa(t-t_0)}, \quad t \ge t_0, \tag{11}$$

where $\overline{\|\varphi\|}_\Omega^2 = \sup_{t_0-\tau\le s\le t_0} \sum_{i=1}^n \int_\Omega \varphi_i^2(s,x)\mathrm{d}x$.

Lemma 2 (see [37] Gronwall-Bellman-type Impulsive Integral Inequality). *Assume that*

(A1) *the sequence* $\{t_k\}$ *satisfies* $0 \le t_0 < t_1 < t_2 < \cdots$, *with* $\lim_{k\to\infty} t_k = \infty$,

(A2) $q \in PC^1[R_+, R]$ *and* $q(t)$ *is left-continuous at* t_k, $k = 1,2,\ldots$,

(A3) $p \in C[R_+, R_+]$ *and for* $k = 1,2,\ldots$,

$$q(t) \le c + \int_{t_0}^t p(s)q(s)\,\mathrm{d}s + \sum_{t_0<t_k<t} \eta_k q(t_k), \quad t \ge t_0, \tag{12}$$

where $\eta_k \ge 0$ *and* $c = const$. *Then,*

$$q(t) \le c \prod_{t_0<t_k<t} (1+\eta_k) \exp\left(\int_{t_0}^t p(s)\,\mathrm{d}s\right), \quad t \ge t_0. \tag{13}$$

Lemma 3 (see [38] Poincaré integral inequality). *Let* $\mathcal{S} = \prod_{i=1}^n [a_i, b_i]$ *be a fixed rectangular region in* R^n *and* $B := \max\{b_i - a_i : i = 1,\ldots,n\}$. *For any* $v(x) \in C_0^1(\mathcal{S})$,

$$\int_{\mathcal{S}} v^2(x)\,\mathrm{d}x \le \frac{B^2}{4n}\int_{\mathcal{S}} |\nabla v(x)|^2\,\mathrm{d}x. \tag{14}$$

Remark 4. According to Lemma 2.1 in [25], we know if $\mathcal{S}$ is a cube $|x_j| < l_j$ $(j = 1,2,\ldots,m)$ and $v(x)$ is a real-valued function belonging to $C_0^1(\mathcal{S})$, then

$$\int_{\mathcal{S}} \left|\frac{\partial v(x)}{\partial x_j}\right|^2 \mathrm{d}x \ge \frac{1}{l_j^2}\int_{\mathcal{S}} v^2(x)\,\mathrm{d}x, \tag{15}$$

which yields

$$\int_{\mathcal{S}} |\nabla v|^2 \mathrm{d}x \ge \left(\int_{\mathcal{S}} v^2(x)\,\mathrm{d}x\right)\left(\sum_{j=1}^m \frac{1}{l_j^2}\right). \tag{16}$$

Through the simple example as follows, we can find that in some cases the estimate $\int_{\mathcal{S}} |\nabla v(x)|^2\mathrm{d}x \ge (4n/B^2)\int_{\mathcal{S}} v^2(x)\mathrm{d}x$ shown in Lemma 3 can do better. Let $\mathcal{S} = [0,1]\times[0,2]$, we derive from Lemma 2.1 in [25] that

$$\int_{\mathcal{S}} |\nabla v|^2 \mathrm{d}x \ge \left(\int_{\mathcal{S}} v^2(x)\,\mathrm{d}x\right)\left(\sum_{j=1}^m \frac{1}{l_j^2}\right) = \frac{5}{4}\int_{\mathcal{S}} v^2(x)\,\mathrm{d}x, \tag{17}$$

whereas the application of Lemma 3 of this paper will give

$$\int_{\mathcal{S}} |\nabla v(x)|^2 \mathrm{d}x \ge \frac{4n}{B^2}\int_{\mathcal{S}} v^2(x)\,\mathrm{d}x = 2\int_{\mathcal{S}} v^2(x)\,\mathrm{d}x, \tag{18}$$

which is obviously superior to $\int_{\mathcal{S}} |\nabla v|^2\mathrm{d}x \ge (5/4)(\int_{\mathcal{S}} v^2(x)\mathrm{d}x)$.

3. Main Results

Theorem 5. *Provided that one has the following:*

(1) *let* $\underline{D} = \min\{D_{is} : i = 1,\ldots,n; s = 1,\ldots,m\} > 0$ *and denote* $8m\underline{D}/M^2 = \chi$;

(2) $P_{ik}(u_i(t_k,x)) = -\theta_{ik}u_i(t_k,x)$, $0 \le \theta_{ik} \le 2$;

(3) *there exists a constant* $\gamma > 0$ *satisfying* $\gamma + \lambda + h\rho e^{\gamma\tau} > 0$ *as well as* $\lambda + h\rho e^{\gamma\tau} < 0$, *where* $\lambda = \max_{i=1,\ldots,n}(-\chi - 2a_i + \sum_{j=1}^n (b_{ij}^2 + c_{ij}^2)) + \rho$, $\rho = n\max_{i=1,\ldots,n}(l_i^2)$;

then, the equilibrium point $u = 0$ *of problem* (5)–(8) *is globally exponentially stable with convergence rate* $-(\lambda + h\rho e^{\gamma\tau})/2$.

Proof. Multiplying both sides of (5) by $u_i(t,x)$ and integrating with respect to spatial variable x on Ω, we get

$$\begin{aligned}\frac{\mathrm{d}\left(\int_\Omega u_i^2(t,x)\,\mathrm{d}x\right)}{\mathrm{d}t} &= 2\sum_{s=1}^{m}\int_\Omega u_i(t,x)\frac{\partial}{\partial x_s}\left(D_{is}\frac{\partial u_i(t,x)}{\partial x_s}\right)\mathrm{d}x \\ &\quad - 2a_i\int_\Omega u_i^2(t,x)\,\mathrm{d}x \\ &\quad + 2\sum_{j=1}^{n} b_{ij}\int_\Omega u_i(t,x) f_j\left(u_j(t,x)\right)\mathrm{d}x \\ &\quad + 2\sum_{j=1}^{n} c_{ij}\int_\Omega u_i(t,x) f_j\left(u_j\left(t-\tau_j(t),x\right)\right)\mathrm{d}x \\ &\qquad t\ge t_0,\ t\neq t_k,\ i=1,\dots,n,\ k=1,2,\dots.\end{aligned} \tag{19}$$

Regarding the right-hand part of (19), the first term becomes by using Green formula, Dirichlet boundary condition, Lemma 3, and condition (1) of Theorem 5

$$\begin{aligned} 2\sum_{s=1}^{m}\int_\Omega u_i(t,x)\frac{\partial}{\partial x_s}\left(D_{is}\frac{\partial u_i(t,x)}{\partial x_s}\right)\mathrm{d}x &= -2\sum_{s=1}^{m}\int_\Omega D_{is}\left(\frac{\partial u_i(t,x)}{\partial x_s}\right)^2\mathrm{d}x \\ &\le \frac{-8m\underline{D}}{M^2}\int_\Omega u_i^2(t,x)\,\mathrm{d}x \triangleq -\chi\int_\Omega u_i^2(t,x)\,\mathrm{d}x.\end{aligned} \tag{20}$$

Moreover, From (H1), we have

$$\begin{aligned} 2\sum_{j=1}^{n} b_{ij}\int_\Omega u_i(t,x) f_j\left(u_j(t,x)\right)\mathrm{d}x &\le 2\sum_{j=1}^{n}\left|b_{ij}\right|\int_\Omega\left|u_i(t,x)\right|\left|f_j\left(u_j(t,x)\right)\right|\mathrm{d}x \\ &\le 2\sum_{j=1}^{n}\int_\Omega l_j\left|b_{ij}\right|\left|u_i(t,x)\right|\left|u_j(t,x)\right|\mathrm{d}x \\ &\le \sum_{j=1}^{n}\int_\Omega\left(b_{ij}^2u_i^2(t,x)+l_j^2u_j^2(t,x)\right)\mathrm{d}x,\end{aligned} \tag{21}$$

$$\begin{aligned} 2\sum_{j=1}^{n} c_{ij}\int_\Omega u_i(t,x) f_j\left(u_j\left(t-\tau_j(t),x\right)\right)\mathrm{d}x &\le 2\sum_{j=1}^{n}\left|c_{ij}\right|\int_\Omega\left|u_i(t,x)\right|\left|f_j\left(u_j\left(t-\tau_j(t),x\right)\right)\right|\mathrm{d}x \\ &\le 2\sum_{j=1}^{n}\int_\Omega l_j\left|c_{ij}\right|\left|u_i(t,x)\right|\left|u_j\left(t-\tau_j(t),x\right)\right|\mathrm{d}x \\ &\le \sum_{j=1}^{n}\int_\Omega\left(c_{ij}^2{u_i}^2(t,x)+l_j^2u_j^2\left(t-\tau_j(t),x\right)\right)\mathrm{d}x.\end{aligned} \tag{22}$$

Consequently, substituting (20)–(22) into (19) produces

$$\begin{aligned}\frac{\mathrm{d}\left(\int_\Omega u_i^2(t,x)\,\mathrm{d}x\right)}{\mathrm{d}t} &\le -\chi\int_\Omega u_i^2(t,x)\,\mathrm{d}x - 2a_i\int_\Omega u_i^2(t,x)\,\mathrm{d}x \\ &\quad + \sum_{j=1}^{n}\int_\Omega\left(b_{ij}^2u_i^2(t,x)+l_j^2u_j^2(t,x)\right)\mathrm{d}x \\ &\quad + \sum_{j=1}^{n}\int_\Omega\left(c_{ij}^2u_i^2(t,x)+l_j^2u_j^2\left(t-\tau_j(t),x\right)\right)\mathrm{d}x\end{aligned} \tag{23}$$

for $t\ge t_0$, $t\neq t_k$, $i=1,\dots,n$, $k=1,2,\dots$.

Define a Lyapunov function $V_i(t)$ as $V_i(t)=\int_\Omega u_i^2(t,x)\mathrm{d}x$. It is easy to find that $V_i(t)$ is a piecewise continuous function with the first kind discontinuous points t_k $(k=1,2,\dots)$, where it is continuous from the left, that is, $V_i(t_k-0)=V_i(t_k)$ $(k=1,2,\dots)$. In addition, we also see

$$V_i\left(t_k+0\right)\le V_i\left(t_k\right),\quad k=0,1,2,\dots, \tag{24}$$

as $V_i(t_0+0)\le V_i(t_0)$ and the following estimate derived from condition (2) of Theorem 5:

$$\begin{aligned} u_i^2\left(t_k+0,x\right) &= \left(-\theta_{ik}u_i\left(t_k,x\right)+u_i\left(t_k,x\right)\right)^2 \\ &= \left(1-\theta_{ik}\right)^2u_i^2\left(t_k,x\right)\le u_i^2\left(t_k,x\right), \\ &\qquad k=1,2,\dots.\end{aligned} \tag{25}$$

Put $t\in(t_k,t_{k+1})$, $k=0,1,2,\dots$. It then results from (23) that

$$\begin{aligned}\frac{\mathrm{d}V_i(t)}{\mathrm{d}t} &\le -\chi\int_\Omega u_i^2(t,x)\,\mathrm{d}x - 2a_i\int_\Omega u_i^2(t,x)\,\mathrm{d}x \\ &\quad + \sum_{j=1}^{n}\int_\Omega\left(b_{ij}^2u_i^2(t,x)+l_j^2u_j^2(t,x)\right)\mathrm{d}x \\ &\quad + \sum_{j=1}^{n}\int_\Omega\left(c_{ij}^2u_i^2(t,x)+l_j^2u_j^2\left(t-\tau_j(t),x\right)\right)\mathrm{d}x \\ &\le \left(-\chi-2a_i+\sum_{j=1}^{n}b_{ij}^2+\sum_{j=1}^{n}c_{ij}^2\right)V_i(t) \\ &\quad + \max_{i=1,\dots,n}\left(l_i^2\right)\sum_{j=1}^{n}V_j(t) \\ &\quad + \max_{i=1,\dots,n}\left(l_i^2\right)\sum_{j=1}^{n}V_j\left(t-\tau_j(t)\right) \\ &\qquad t\in\left(t_k,t_{k+1}\right),\ k=0,1,2,\cdots.\end{aligned} \tag{26}$$

Choose $V(t)$ of the form $V(t) = \sum_{i=1}^{n} V_i(t)$. From (26), one reads

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} \le \lambda V(t) + \rho \sum_{j=1}^{n} V_j\left(t - \tau_j(t)\right), \quad t \in \left(t_k, t_{k+1}\right), \; k = 0, 1, 2, \ldots, \tag{27}$$

where $\lambda = \max_{i=1,\ldots,n}(-\chi - 2a_i + \sum_{j=1}^{n}(b_{ij}^2 + c_{ij}^2)) + \rho$ and $\rho = n\max_{i=1,\ldots,n}(l_i^2)$.

Now construct $V^*(t) = \mathrm{e}^{\gamma(t-t_0)}V(t)$ again, where $\gamma > 0$ satisfies $\gamma + \lambda + h\rho\mathrm{e}^{\gamma\tau} > 0$ and $\lambda + h\rho\mathrm{e}^{\gamma\tau} < 0$. Evidently, $V^*(t)$ is also a piecewise continuous function with the first kind discontinuous points t_k $(k = 1, 2, \ldots)$, where it is continuous from the left, that is, $V^*(t_k - 0) = V^*(t_k)$ $(k = 1, 2, \ldots)$. Moreover, at $t = t_k$ $(k = 0, 1, 2, \ldots)$, we find by use of (24)

$$V^*\left(t_k + 0\right) \le V^*\left(t_k\right), \quad k = 0, 1, 2, \ldots. \tag{28}$$

Set $t \in (t_k, t_{k+1})$, $k = 0, 1, 2, \ldots$. By virtue of (27), one has

$$\begin{aligned} \frac{\mathrm{d}V^*(t)}{\mathrm{d}t} &= \gamma\mathrm{e}^{\gamma(t-t_0)}V(t) + \mathrm{e}^{\gamma(t-t_0)}\frac{\mathrm{d}V(t)}{\mathrm{d}t} \\ &\le \gamma\mathrm{e}^{\gamma(t-t_0)}V(t) \\ &\quad + \left(\lambda V(t) + \rho\sum_{j=1}^{n}V_j\left(t - \tau_j(t)\right)\right)\mathrm{e}^{\gamma(t-t_0)} \\ &= \left(\gamma + \lambda\right)V^*(t) + \rho\mathrm{e}^{\gamma(t-t_0)}\sum_{j=1}^{n}V_j\left(t - \tau_j(t)\right) \\ &\qquad t \in \left(t_k, t_{k+1}\right), \; k = 0, 1, 2, \ldots. \end{aligned} \tag{29}$$

Choose small enough $\varepsilon > 0$. Integrating (29) from $t_k + \varepsilon$ to t gives

$$\begin{aligned} V^*(t) &\le V^*\left(t_k + \varepsilon\right) + \left(\gamma + \lambda\right)\int_{t_k+\varepsilon}^{t} V^*(s)\,\mathrm{d}s \\ &\quad + \int_{t_k+\varepsilon}^{t} \rho\mathrm{e}^{\gamma(s-t_0)}\sum_{j=1}^{n}V_j\left(s - \tau_j(s)\right)\mathrm{d}s \\ &\qquad t \in \left(t_k, t_{k+1}\right), \; k = 0, 1, 2, \ldots, \end{aligned} \tag{30}$$

which yields, after letting $\varepsilon \to 0$ in (30),

$$\begin{aligned} V^*(t) &\le V^*\left(t_k + 0\right) + \left(\gamma + \lambda\right)\int_{t_k}^{t} V^*(s)\,\mathrm{d}s \\ &\quad + \int_{t_k}^{t} \rho\mathrm{e}^{\gamma(s-t_0)}\sum_{j=1}^{n}V_j\left(s - \tau_j(s)\right)\mathrm{d}s \\ &\qquad t \in \left(t_k, t_{k+1}\right), \; k = 0, 1, 2, \ldots. \end{aligned} \tag{31}$$

Next we will estimate the value of $V^*(t)$ at $t = t_{k+1}$, $k = 0, 1, 2, \ldots$. For small enough $\varepsilon > 0$, we put $t = t_{k+1} - \varepsilon$. An application of (31) leads to, for $k = 0, 1, 2, \ldots$,

$$\begin{aligned} V^*\left(t_{k+1} - \varepsilon\right) &\le V^*\left(t_k + 0\right) + \left(\gamma + \lambda\right)\int_{t_k}^{t_{k+1}-\varepsilon} V^*(s)\,\mathrm{d}s \\ &\quad + \int_{t_k}^{t_{k+1}-\varepsilon} \rho\mathrm{e}^{\gamma(s-t_0)}\sum_{j=1}^{n}V_j\left(s - \tau_j(s)\right)\mathrm{d}s. \end{aligned} \tag{32}$$

If we let $\varepsilon \to 0$ in (32), there results

$$\begin{aligned} V^*\left(t_{k+1} - 0\right) &\le V^*\left(t_k + 0\right) + \left(\gamma + \lambda\right)\int_{t_k}^{t_{k+1}} V^*(s)\,\mathrm{d}s \\ &\quad + \int_{t_k}^{t_{k+1}} \rho\mathrm{e}^{\gamma(s-t_0)}\sum_{j=1}^{n}V_j\left(s - \tau_j(s)\right)\mathrm{d}s, \\ &\qquad k = 0, 1, 2, \ldots. \end{aligned} \tag{33}$$

Note that $V^*(t_{k+1} - 0) = V^*(t_{k+1})$ is applicable for $k = 0, 1, 2, \ldots$. Thus,

$$\begin{aligned} V^*\left(t_{k+1}\right) &\le V^*\left(t_k + 0\right) + \left(\gamma + \lambda\right)\int_{t_k}^{t_{k+1}} V^*(s)\,\mathrm{d}s \\ &\quad + \int_{t_k}^{t_{k+1}} \rho\mathrm{e}^{\gamma(s-t_0)}\sum_{j=1}^{n}V_j\left(s - \tau_j(s)\right)\mathrm{d}s \end{aligned} \tag{34}$$

holds for $k = 0, 1, 2, \ldots$. By synthesizing (31) and (34), we then arrive at

$$\begin{aligned} V^*(t) &\le V^*\left(t_k + 0\right) + \left(\gamma + \lambda\right)\int_{t_k}^{t} V^*(s)\,\mathrm{d}s \\ &\quad + \int_{t_k}^{t} \rho\mathrm{e}^{\gamma(s-t_0)}\sum_{j=1}^{n}V_j\left(s - \tau_j(s)\right)\mathrm{d}s \\ &\qquad t \in \left(t_k, t_{k+1}\right], \; k = 0, 1, 2, \ldots. \end{aligned} \tag{35}$$

This, together with (28), results in

$$\begin{aligned} V^*(t) &\le V^*\left(t_k\right) + \left(\gamma + \lambda\right)\int_{t_k}^{t} V^*(s)\,\mathrm{d}s \\ &\quad + \int_{t_k}^{t} \rho\mathrm{e}^{\gamma(s-t_0)}\sum_{j=1}^{n}V_j\left(s - \tau_j(s)\right)\mathrm{d}s \end{aligned} \tag{36}$$

for $t \in (t_k, t_{k+1}]$, $k = 0, 1, 2, \ldots$.

Recalling assumptions that $0 \le \tau_j(t) \le \tau$ and $\dot{\tau}_j(t) < (1 - (1/h))(h > 0)$, we obtain

$$\begin{aligned} &\int_{t_k}^{t} \rho\mathrm{e}^{\gamma(s-t_0)}\sum_{j=1}^{n}V_j\left(s - \tau_j(s)\right)\mathrm{d}s \\ &\quad = \sum_{j=1}^{n}\int_{t_k-\tau_j(t_k)}^{t-\tau_j(t)} \rho\mathrm{e}^{\gamma(\theta+\tau_j(s)-t_0)}V_j(\theta)\frac{1}{1 - \dot{\tau}_j(s)}\mathrm{d}\theta \\ &\quad \le h\rho\mathrm{e}^{\gamma\tau}\sum_{j=1}^{n}\int_{t_k-\tau_j(t_k)}^{t-\tau_j(t)} \mathrm{e}^{\gamma(\theta-t_0)}V_j(\theta)\,\mathrm{d}\theta. \end{aligned} \tag{37}$$

Hence,

$$V^*(t) \le V^*(t_k) + (\gamma+\lambda)\int_{t_k}^{t} V^*(s)\,\mathrm{d}s + h\rho e^{\gamma\tau}\sum_{j=1}^{n}\int_{t_k-\tau_j(t_k)}^{t-\tau_j(t)} e^{\gamma(s-t_0)} V_j(s)\,\mathrm{d}s \qquad t\in(t_k,t_{k+1}],\ k=0,1,2,\ldots. \tag{38}$$

By induction argument, we reach

$$\begin{aligned}
V^*(t_k) &\le V^*(t_{k-1}) + (\gamma+\lambda)\int_{t_{k-1}}^{t_k} V^*(s)\,\mathrm{d}s + h\rho e^{\gamma\tau}\sum_{j=1}^{n}\int_{t_{k-1}-\tau_j(t_{k-1})}^{t_k-\tau_j(t_k)} e^{\gamma(s-t_0)} V_j(s)\,\mathrm{d}s,\\
&\vdots\\
V^*(t_2) &\le V^*(t_1) + (\gamma+\lambda)\int_{t_1}^{t_2} V^*(s)\,\mathrm{d}s + h\rho e^{\gamma\tau}\sum_{j=1}^{n}\int_{t_1-\tau_j(t_1)}^{t_2-\tau_j(t_2)} e^{\gamma(s-t_0)} V_j(s)\,\mathrm{d}s,\\
V^*(t_1) &\le V^*(t_0) + (\gamma+\lambda)\int_{t_0}^{t_1} V^*(s)\,\mathrm{d}s + h\rho e^{\gamma\tau}\sum_{j=1}^{n}\int_{t_0-\tau_j(t_0)}^{t_1-\tau_j(t_1)} e^{\gamma(s-t_0)} V_j(s)\,\mathrm{d}s.
\end{aligned} \tag{39}$$

Therefore,

$$\begin{aligned}
V^*(t) &\le V^*(t_0) + (\gamma+\lambda)\int_{t_0}^{t} V^*(s)\,\mathrm{d}s + h\rho e^{\gamma\tau}\sum_{j=1}^{n}\int_{t_0-\tau_j(t_0)}^{t-\tau_j(t)} e^{\gamma(s-t_0)} V_j(s)\,\mathrm{d}s\\
&\le V^*(t_0) + (\gamma+\lambda)\int_{t_0}^{t} V^*(s)\,\mathrm{d}s + h\rho e^{\gamma\tau}\sum_{j=1}^{n}\int_{t_0-\tau_j(t_0)}^{t} e^{\gamma(s-t_0)} V_j(s)\,\mathrm{d}s\\
&= V^*(t_0) + (\gamma+\lambda+h\rho e^{\gamma\tau})\int_{t_0}^{t} V^*(s)\,\mathrm{d}s + h\rho e^{\gamma\tau}\sum_{j=1}^{n}\int_{t_0-\tau_j(t_0)}^{t_0} e^{\gamma(s-t_0)} V_j(s)\,\mathrm{d}s\\
&\qquad t\in(t_k,t_{k+1}],\ k=0,1,2,\ldots.
\end{aligned} \tag{40}$$

Since

$$\begin{aligned}
h\rho e^{\gamma\tau}\sum_{j=1}^{n}\int_{t_0-\tau_j(t_0)}^{t_0} e^{\gamma(s-t_0)} V_j(s)\,\mathrm{d}s
&\le h\rho e^{\gamma\tau}\sum_{j=1}^{n}\int_{t_0-\tau}^{t_0} V_j(s)\,\mathrm{d}s\\
&= h\rho e^{\gamma\tau}\int_{t_0-\tau}^{t_0}\left(\sum_{j=1}^{n}\int_{\Omega}\varphi_j^2(s,x)\,\mathrm{d}x\right)\mathrm{d}s\\
&\le \tau h\rho e^{\gamma\tau}\overline{\|\varphi\|}_{\Omega}^{2},
\end{aligned} \tag{41}$$

we claim

$$V^*(t) \le V^*(t_0) + \tau h\rho e^{\gamma\tau}\overline{\|\varphi\|}_{\Omega}^{2} + (\gamma+\lambda+h\rho e^{\gamma\tau})\int_{t_0}^{t} V^*(s)\,\mathrm{d}s \qquad t\in(t_k,t_{k+1}],\ k=0,1,2\ldots. \tag{42}$$

According to Lemma 2, we know

$$V^*(t) \le \left(V^*(t_0) + \tau h\rho e^{\gamma\tau}\overline{\|\varphi\|}_{\Omega}^{2}\right) \times \exp\left\{(\gamma+\lambda+h\rho e^{\gamma\tau})(t-t_0)\right\}, \qquad t\ge t_0 \tag{43}$$

which reduces to

$$\|u(t,x;t_0,\varphi)\|_{\Omega} \le \sqrt{1+\tau h\rho e^{\gamma\tau}}\ \overline{\|\varphi\|}_{\Omega}\exp\left\{\left(\frac{\lambda+h\rho e^{\gamma\tau}}{2}\right)(t-t_0)\right\}, \qquad t\ge t_0. \tag{44}$$

This completes the proof. □

Remark 6. According to Theorem 5, we see that the diffusion can really influence the stability of equilibrium point $u=0$ of problem (5)–(8), wherein the factors embrace not only the reaction-diffusion coefficients but also the regional features including the dimension and boundary of spatial variable. Owing to the employ of new Poincaré integral inequality, in this paper, the estimation of reaction-diffusion terms is superior to that in [25] in some cases, and this will be helpful to further know the influence of diffusion on stability. What is more, from condition (1) of Theorem 5, we also see that the dimension of spatial variable has an impact on the stability while this is not mentioned in [25].

Remark 7. Among the three conditions of Theorem 5, condition (3) is critical and therefore we must ensure the existence of constant $\gamma > 0$. Fortunately, it is not difficult to find that there must exist a constant $\gamma > 0$ satisfying condition (3) if $\lambda < -h\rho$ which is easily checked.

Theorem 8. *Providing that one has the following:*

(1) *let* $\underline{D} = \min\{D_{is} : i = 1, \ldots, n; s = 1, \ldots, m\} > 0$ *and denote* $8m\underline{D}/M^2 = \chi$;

(2) $P_{ik}(u_i(t_k, x)) = -\theta_{ik}u_i(t_k, x)$, $1 - \sqrt{1+\alpha} \le \theta_{ik} \le 1 + \sqrt{1+\alpha}$, $\alpha \ge 0$;

(3) $\inf_{k=1,2,\ldots}(t_k - t_{k-1}) \ge \mu$;

(4) *there exists a constant* $\gamma > 0$ *which satisfies* $\gamma + \lambda + h\rho e^{\gamma\tau} > 0$ *and* $\lambda + h\rho e^{\gamma\tau} + (\ln(1+\alpha)/\mu) < 0$, *where* $\lambda = \max_{i=1,\ldots,n}(-\chi - 2a_i + \sum_{j=1}^{n}(b_{ij}^2 + c_{ij}^2)) + \rho$ *and* $\rho = n\max_{i=1,\ldots,n}(l_i^2)$;

then, the equilibrium point $u = 0$ *of problem* (5)–(8) *is globally exponentially stable with convergence rate* $-(1/2)(\lambda + h\rho e^{\gamma\tau} + (\ln(1+\alpha)/\mu))$.

Proof. Define Lyapunov function V of the form $V(t) = \sum_{i=1}^{n} V_i(t)$, where $V_i(t) = \int_\Omega u_i^2(t,x)dx$. Obviously, $V(t)$ is a piecewise continuous function with the first kind discontinuous points t_k, $k = 1, 2, \ldots$, where it is continuous from the left, that is, $V(t_k - 0) = V(t_k)$ $(k = 1, 2, \ldots)$. Furthermore, when $t = t_k$ $(k = 0, 1, 2, \ldots)$, it follows from condition (2) of Theorem 8 that

$$u_i^2(t_k+0,x) - u_i^2(t_k,x) = (1-\theta_{ik})^2 u_i^2(t_k,x) - u_i^2(t_k,x) \le \alpha u_i^2(t_k,x). \tag{45}$$

Thereby,

$$V(t_k+0) \le \alpha V(t_k) + V(t_k), \quad k = 0, 1, 2, \ldots. \tag{46}$$

Construct another Lyapunov function $V^*(t) = e^{\gamma(t-t_0)} \times V(t)$, where $\gamma > 0$ satisfies $\gamma + \lambda + h\rho e^{\gamma\tau} > 0$ and $\lambda + h\rho e^{\gamma\tau} + (\ln(1+\alpha)/\mu) < 0$. Then, $V^*(t)$ is also a piecewise continuous function with the first kind discontinuous points t_k, $k = 1, 2, \ldots$, where it is continuous from the left, and for $t = t_k$ $(k = 0, 1, 2, \ldots)$, it results from (46) that

$$V^*(t_k+0) \le \alpha V^*(t_k) + V^*(t_k), \quad k = 0, 1, 2, \ldots. \tag{47}$$

Set $t \in (t_k, t_{k+1}]$, $k = 0, 1, 2, \ldots$. Following the same procedure as in Theorem 5, we get

$$\begin{aligned} V^*(t) &\le V^*(t_k+0) + (\gamma+\lambda)\int_{t_k}^{t} V^*(s)\,ds \\ &\quad + h\rho e^{\gamma\tau}\sum_{j=1}^{n}\int_{t_k-\tau_j(t_k)}^{t-\tau_j(t)} e^{\gamma(\theta-t_0)}V_j(\theta)\,d\theta \\ &\quad t \in (t_k, t_{k+1}], \ k = 0, 1, 2, \ldots. \end{aligned} \tag{48}$$

The relations (47) and (48) yield

$$\begin{aligned} &V^*(t) - V^*(t_k) \\ &\quad \le \alpha V^*(t_k) + (\gamma+\lambda)\int_{t_k}^{t} V^*(s)\,ds \\ &\qquad + h\rho e^{\gamma\tau}\sum_{j=1}^{n}\int_{t_k-\tau_j(t_k)}^{t-\tau_j(t)} e^{\gamma(\theta-t_0)}V_j(\theta)\,d\theta \\ &\qquad t \in (t_k, t_{k+1}], \ k = 0, 1, 2, \ldots. \end{aligned} \tag{49}$$

By induction argument, we reach

$$\begin{aligned} &V^*(t_k) - V^*(t_{k-1}) \\ &\quad \le \alpha V^*(t_{k-1}) + (\gamma+\lambda)\int_{t_{k-1}}^{t_k} V^*(s)\,ds \\ &\qquad + h\rho e^{\gamma\tau}\sum_{j=1}^{n}\int_{t_{k-1}-\tau_j(t_{k-1})}^{t_k-\tau_j(t_k)} e^{\gamma(\theta-t_0)}V_j(\theta)\,d\theta, \\ &\qquad \vdots \\ &V^*(t_2) - V^*(t_1) \\ &\quad \le \alpha V^*(t_1) + (\gamma+\lambda)\int_{t_1}^{t_2} V^*(s)\,ds \\ &\qquad + h\rho e^{\gamma\tau}\sum_{j=1}^{n}\int_{t_1-\tau_j(t_1)}^{t_2-\tau_j(t_2)} e^{\gamma(\theta-t_0)}V_j(\theta)\,d\theta, \\ &V^*(t_1) - V^*(t_0) \\ &\quad \le \alpha V^*(t_0) + (\gamma+\lambda)\int_{t_0}^{t_1} V^*(s)\,ds \\ &\qquad + h\rho e^{\gamma\tau}\sum_{j=1}^{n}\int_{t_0-\tau_j(t_0)}^{t_1-\tau_j(t_1)} e^{\gamma(\theta-t_0)}V_j(\theta)\,d\theta. \end{aligned} \tag{50}$$

Hence,

$$\begin{aligned} &V^*(t) - V^*(t_0) \\ &\quad \le \alpha V^*(t_0) + (\gamma+\lambda)\int_{t_0}^{t} V^*(s)\,ds + h\rho e^{\gamma\tau} \\ &\qquad \times \sum_{j=1}^{n}\int_{t_0-\tau_j(t_0)}^{t-\tau_j(t)} e^{\gamma(\theta-t_0)}V_j(\theta)\,d\theta + \alpha\sum_{t_0<t_k<t} V(t_k) \\ &\quad \le \alpha V^*(t_0) + (\gamma+\lambda+h\rho e^{\gamma\tau})\int_{t_0}^{t} V^*(s)\,ds \\ &\qquad + h\rho e^{\gamma\tau}\sum_{j=1}^{n}\int_{t_0-\tau_j(t_0)}^{t_0} e^{\gamma(\theta-t_0)}V_j(\theta)\,d\theta + \alpha\sum_{t_0<t_k<t} V(t_k) \\ &\qquad t \in (t_k, t_{k+1}], \ k = 0, 1, 2, \ldots. \end{aligned} \tag{51}$$

Introducing $h\rho e^{\gamma\tau}\sum_{j=1}^{n}\int_{t_0-\tau_j(t_0)}^{t_0} e^{\gamma(\theta-t_0)}V_j(\theta)d\theta \le \tau h\rho e^{\gamma\tau} \times \overline{\|\varphi\|}_\Omega^2$ as shown in the proof of Theorem 5 into (51), (51) becomes

$$\begin{aligned} &V^*(t) - V^*(t_0) \\ &\quad \le \alpha V^*(t_0) + \tau h\rho e^{\gamma\tau}\overline{\|\varphi\|}_\Omega^2 \end{aligned}$$

$$+\left(\gamma+\lambda+h\rho e^{\gamma\tau}\right)\int_{t_0}^{t}V^*(s)\,\mathrm{d}s+\alpha\sum_{t_0<t_k<t}V(t_k)$$
$$t\in\left(t_k,t_{k+1}\right],\ k=0,1,2\ldots. \tag{52}$$

It then results from Lemma 2 that, for $t \geq t_0$,

$$\begin{aligned}V^*(t)&\leq\left((\alpha+1)V^*(t_0)+\tau h\rho e^{\gamma\tau}\overline{\|\varphi\|}_\Omega^2\right)\\&\quad\times\prod_{t_0<t_k<t}(1+\alpha)\exp\left(\left(\gamma+\lambda+h\rho e^{\gamma\tau}\right)(t-t_0)\right)\\&=\left((\alpha+1)V^*(t_0)+\tau h\rho e^{\gamma\tau}\overline{\|\varphi\|}_\Omega^2\right)\\&\quad\times(1+\alpha)^k\exp\left(\left(\gamma+\lambda+h\rho e^{\gamma\tau}\right)(t-t_0)\right).\end{aligned} \tag{53}$$

On the other hand, since $\inf_{k=1,2,\ldots}(t_k-t_{k-1})\geq\mu$, one has $k\leq(t_k-t_0)/\mu$. Thereby,

$$\begin{aligned}(1+\alpha)^k&\leq\exp\left\{\frac{\ln(1+\alpha)}{\mu}(t_k-t_0)\right\}\\&\leq\exp\left\{\frac{\ln(1+\alpha)}{\mu}(t-t_0)\right\}\end{aligned} \tag{54}$$

and (53) can be rewritten as

$$\begin{aligned}V^*(t)\leq&\left((\alpha+1)V^*(t_0)+\tau h\rho e^{\gamma\tau}\overline{\|\varphi\|}_\Omega^2\right)\\&\times\exp\left(\left(\gamma+\lambda+h\rho e^{\gamma\tau}+\frac{\ln(1+\alpha)}{\mu}\right)(t-t_0)\right)\end{aligned} \tag{55}$$

which implies

$$\begin{aligned}&\|u(t,x;t_0,\varphi)\|_\Omega\\&\quad\leq\sqrt{(\alpha+1+\tau h\rho e^{\gamma\tau})}\,\overline{\|\varphi\|}_\Omega\\&\quad\times\exp\left(\frac{1}{2}\left(\lambda+h\rho e^{\gamma\tau}+\frac{\ln(1+\alpha)}{\mu}\right)(t-t_0)\right),\\&\qquad t\geq t_0.\end{aligned} \tag{56}$$

The proof is completed. □

As

$$\begin{aligned}&2\sum_{j=1}^{n}b_{ij}\int_\Omega u_i(t,x)f\left(u_j(t,x)\right)\mathrm{d}x\\&\quad\leq\sum_{j=1}^{n}\int_\Omega\left(\varepsilon_1 b_{ij}^2u_i^2(t,x)+\frac{l_j^2}{\varepsilon_1}u_j^2(t,x)\right)\mathrm{d}x,\\&2\sum_{j=1}^{n}c_{ij}\int_\Omega u_i(t,x)f\left(u_j\left(t-\tau_j(t),x\right)\right)\mathrm{d}x\\&\quad\leq\sum_{j=1}^{n}\int_\Omega\left(\varepsilon_2 c_{ij}^2u_i^2(t,x)+\frac{l_j^2}{\varepsilon_2}u_j^2\left(t-\tau_j(t),x\right)\right)\mathrm{d}x\end{aligned} \tag{57}$$

hold for any $\varepsilon_1,\varepsilon_2>0$. In the sequel, analogous to the proofs of Theorems 5 and 8 we arrive at the following.

Theorem 9. *Provided that one has the following:*

(1) *let* $\underline{D}=\min\{D_{is}:i=1,\ldots,n;s=1,\ldots,m\}>0$ *and denote* $8m\underline{D}/M^2=\chi$*;*

(2) $P_{ik}(u_i(t_k,x))=-\theta_{ik}u_i(t_k,x)$, $0\leq\theta_{ik}\leq 2$*;*

(3) *there exist constants* $\gamma>0$ *and* $\varepsilon_1,\varepsilon_2>0$ *such that* $\gamma+\lambda+h\rho e^{\gamma\tau}>0$ *and* $\lambda+h\rho e^{\gamma\tau}<0$, *where* $\lambda=\max_{i=1,\ldots,n}(-\chi-2a_i+\sum_{j=1}^{n}(\varepsilon_1 b_{ij}^2+\varepsilon_2 c_{ij}^2))+(n/\varepsilon_1)\max_{i=1,\ldots,n}(l_i^2)$ *and* $\rho=(n/\varepsilon_2)\max_{i=1,\ldots,n}(l_i^2)$*;*

then, the equilibrium point $u=0$ *of problem* (5)–(8) *is globally exponentially stable with convergence rate* $-(\lambda+h\rho e^{\gamma\tau})/2$.

Remark 10. According to Theorem 5, we know that there must exist constant $\gamma>0$ satisfying condition (3) of Theorem 9 if there are constants $\varepsilon_1,\varepsilon_2>0$ such that $\lambda<-h\rho$.

Theorem 11. *Assume that one has the following:*

(1) *let* $\underline{D}=\min\{D_{is}:i=1,\ldots,n;s=1,\ldots,m\}>0$ *and denote* $8m\underline{D}/M^2=\chi$*;*

(2) $P_{ik}(u_i(t_k,x))=-\theta_{ik}u_i(t_k,x)$, $1-\sqrt{1+\alpha}\leq\theta_{ik}\leq 1+\sqrt{1+\alpha}$, $\alpha\geq 0$*;*

(3) $\inf_{k=1,2,\ldots}(t_k-t_{k-1})\geq\mu$*;*

(4) *there exist constants* $\gamma>0$ *and* $\varepsilon_1,\varepsilon_2>0$ *such that* $\gamma+\lambda+h\rho e^{\gamma\tau}>0$ *and* $\lambda+h\rho e^{\gamma\tau}+\ln(1+\alpha)/\mu<0$, *where* $\lambda=\max_{i=1,\cdots,n}(-\chi-2a_i+\sum_{j=1}^{n}(\varepsilon_1 b_{ij}^2+\varepsilon_2 c_{ij}^2))+(n/\varepsilon_1)\max_{i=1,\ldots,n}(l_i^2)$ *and* $\rho=(n/\varepsilon_2)\max_{i=1,\ldots,n}(l_i^2)$*;*

then, the equilibrium point $u=0$ *of problem* (5)–(8) *is globally exponentially stable with convergence rate* $-(1/2)(\lambda+h\rho e^{\gamma\tau}+(\ln(1+\alpha)/\mu))$.

Further, on the condition that $|P_{ik}(u_i(t_k,x))|\leq\theta_{ik}|u_i(t_k,x)|$, where $\theta_{ik}^2\leq(\alpha-1)/2$ and $\alpha\geq 1$, we obtain, for $t=t_k$ $(k=1,2,\ldots)$,

$$\begin{aligned}&u_i^2(t_k+0,x)-u_i^2(t_k,x)\\&\quad=\left(P_{ik}(u_i(t_k,x))+u_i(t_k,x)\right)^2-u_i^2(t_k,x)\\&\quad\leq 2\left(u_i(t_k,x)\right)^2+2\left(P_{ik}(u_i(t_k,x))\right)^2-u_i^2(t_k,x)\\&\quad\leq\left(2+2\theta_{ik}^2\right)\left(u_i(t_k,x)\right)^2-u_i^2(t_k,x)\\&\quad\leq\alpha u_i^2(t_k,x).\end{aligned} \tag{58}$$

Identical with the proof of Theorem 8, we reach the following.

Theorem 12. *Assume that one has the following:*

(1) *let* $\underline{D}=\min\{D_{is}:i=1,\ldots,n;s=1,\ldots,m\}>0$ *and denote* $8m\underline{D}/M^2=\chi$*;*

(2) $|P_{ik}(u_i(t_k,x))|\leq\theta_{ik}|u_i(t_k,x)|$, *where* $\theta_{ik}^2\leq(\alpha-1)/2$ *and* $\alpha\geq 1$*;*

(3) $\inf_{k=1,2,\ldots}(t_k - t_{k-1}) \geq \mu$;

(4) *there exist constants* $\gamma > 0$ *and* $\varepsilon_1, \varepsilon_2 > 0$ *such that* $\gamma + \lambda + h\rho e^{\gamma\tau} > 0$ *and* $\lambda + h\rho e^{\gamma\tau} + (\ln(1+\alpha)/\mu) < 0$, *where* $\lambda = \max_{i=1,\ldots,n}(-\chi - 2a_i + \sum_{j=1}^{n}(\varepsilon_1 b_{ij}^2 + \varepsilon_2 c_{ij}^2)) + (n/\varepsilon_1)\max_{i=1,\ldots,n}(l_i^2)$ *and* $\rho = (n/\varepsilon_2)\max_{i=1,\ldots,n}(l_i^2)$;

then, the equilibrium point $u = 0$ *of problem* (5)–(8) *is globally exponentially stable with convergence rate* $-(1/2)(\lambda + h\rho e^{\gamma\tau} + (\ln(1+\alpha)/\mu))$.

Remark 13. Different from Theorems 5–11, the impulsive part in Theorem 12 could be nonlinear and this will be of more applicability. Actually, Theorems 5–11 can be regarded as the special cases of Theorem 12.

4. Examples

Example 14. Consider system (5)–(8) equipped with $P_{ik}(u_i(t_k, x)) = 1.343u_i(t_k, x)$. Let $n = 2, m = 2, \Omega = [0, 1.5] \times [0, 2]$, $\tau_j(t) = (3/4)\arctan(t)$, $a_1 = a_2 = 6.5$, $(D_{is})_{2\times 2} = \left(\begin{smallmatrix} 1.2 & 2.3 \\ 2.2 & 1.5 \end{smallmatrix}\right)$, $(b_{ij})_{2\times 2} = \left(\begin{smallmatrix} -0.23 & 1.3 \\ -0.14 & 3.2 \end{smallmatrix}\right)$, $(c_{ij})_{2\times 2} = \left(\begin{smallmatrix} -0.1 & -0.2 \\ 0.25 & -0.13 \end{smallmatrix}\right)$, and $f_j(u_j) = (\sqrt{2}/4)(|u_j + 1| - |u_j - 1|)$.

For $M = 2$ and $\underline{D} = 1.2$, we compute $\chi = 4.8$. This, together with $l_i = \sqrt{2}/2$, yields

$$\rho = n \max_{i=1,\ldots,n}\left(l_i^2\right) = 1, \tag{59}$$

$$\lambda = \max_{i=1,\ldots,n}\left(-\chi - 2a_i + \sum_{j=1}^{n}\left(b_{ij}^2 + c_{ij}^2\right)\right) + \rho = -6.461. \tag{60}$$

Let $h = 4$. Since $\lambda = -6.461 < -4 = -h\rho$, we conclude from Theorem 5 that the equilibrium point $u = 0$ of this system is globally exponentially stable.

Example 15. Consider system (5)–(8) equipped with $P_{ik}(u_i(t_k, x)) = \arctan(0.5u_i(t_k, x))$. Let $n = 2$, $m = 2$, $\tau_j(t) = (1/\pi)\arctan(t)$, $\Omega = [0, 1.5] \times [0, 2]$, $a_i = 6.5$, $(D_{is})_{2\times 2} = \left(\begin{smallmatrix} 1.2 & 2.3 \\ 2.2 & 3.5 \end{smallmatrix}\right)$, $(b_{ij})_{2\times 2} = \left(\begin{smallmatrix} -0.23 & 1.3 \\ -0.14 & 3.2 \end{smallmatrix}\right)$, $(c_{ij})_{2\times 2} = \left(\begin{smallmatrix} -0.1 & -0.2 \\ 0.25 & -0.13 \end{smallmatrix}\right)$, $f_j(u_j) = (\sqrt{2}/4)(|u_j + 1| - |u_j - 1|)$, and $t_k = t_{k-1} + 2k$.

For $M = 2$ and $\underline{D} = 1.2$, we compute $\chi = 4.8$. This, together with $l_i = \sqrt{2}/2$ and $\varepsilon_1 = \varepsilon_2 = 1$, yields

$$\rho = \frac{n}{\varepsilon_2}\max_{i=1,\ldots,n}\left(l_i^2\right) = 1,$$

$$\begin{aligned}\lambda = {} & \max_{i=1,\ldots,n}\left(-\chi - 2a_i + \sum_{j=1}^{n}\left(\varepsilon_1 b_{ij}^2 + \varepsilon_2 c_{ij}^2\right)\right) \\ & + \frac{n}{\varepsilon_1}\max_{i=1,\ldots,n}\left(l_i^2\right) = -6.461.\end{aligned} \tag{61}$$

Letting $\tau = 0.5$, $h = 4$, $\mu = 2$, and $\alpha = 1.5$, we can find $\gamma = 0.78$ satisfying

$$\begin{gathered}\gamma + \lambda + h\rho e^{\gamma\tau} = 0.2269 > 0, \\ \lambda + h\rho e^{\gamma\tau} + \frac{\ln(1+\alpha)}{\mu} = -0.0949 < 0.\end{gathered} \tag{62}$$

It is then concluded from Theorem 12 that this system is globally exponentially stable.

Acknowledgment

The work is supported by National Natural Science Foundation of China under Grant nos. 60904028, 71271118, and 71171116.

References

[1] L. O. Chua and L. Yang, "Cellular neural networks: theory," *IEEE Transactions on Circuits and Systems*, vol. 35, no. 10, pp. 1257–1272, 1988.

[2] L. O. Chua and L. Yang, "Cellular neural networks: applications," *IEEE Transactions on Circuits and Systems*, vol. 35, no. 10, pp. 1273–1290, 1988.

[3] J. Cao, "New results concerning exponential stability and periodic solutions of delayed cellular neural networks," *Physics Letters A*, vol. 307, no. 2-3, pp. 136–147, 2003.

[4] J. Cao, "On stability of cellular neural networks with delay," *IEEE Transactions on Circuits and Systems I*, vol. 40, pp. 157–165, 1993.

[5] P. P. Civalleri and M. Gilli, "A set of stability criteria for delayed cellular neural networks," *IEEE Transactions on Circuits and Systems. I. Fundamental Theory and Applications*, vol. 48, no. 4, pp. 494–498, 2001.

[6] J. J. Hopfield, "Neurons with graded response have collective computational properties like those of two-state neurons," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 81, pp. 3088–3092, 1984.

[7] J. Yan and J. Shen, "Impulsive stabilization of functional-differential equations by Lyapunov-Razumikhin functions," *Nonlinear Analysis. Theory, Methods & Applicationss*, vol. 37, no. 2, pp. 245–255, 1999.

[8] X. Z. Liu and Q. Wang, "Impulsive stabilization of high-order Hopfield-type neural networks with time-varying delays," *IEEE Transactions on Neural Networks*, vol. 19, pp. 71–79, 2008.

[9] X. Z. Liu, "Stability results for impulsive differential systems with applications to population growth models," *Dynamics and Stability of Systems*, vol. 9, no. 2, pp. 163–174, 1994.

[10] S. Arik and V. Tavsanoglu, "On the global asymptotic stability of delayed cellular neural networks," *IEEE Transactions on Circuits and Systems. I. Fundamental Theory and Applications*, vol. 47, no. 4, pp. 571–574, 2000.

[11] L. O. Chua and T. Roska, "Stability of a class of nonreciprocal cellular neural networks," *IEEE Transactions on Circuits and Systems I*, vol. 37, pp. 1520–1527, 1990.

[12] Z. H. Guan and G. Chen, "On delayed impulsive Hopfield neural networks," *Neural Network*, vol. 12, pp. 273–280, 1999.

[13] Q. Zhang, X. Wei, and J. Xu, "On global exponential stability of delayed cellular neural networks with time-varying delays," *Applied Mathematics and Computation*, vol. 162, no. 2, pp. 679–686, 2005.

[14] D. D. Baĭnov and P. S. Simeonov, *Systems with Impulse Effect*, Ellis Horwood, Chichester, UK, 1989.

[15] I. M. Stamova, *Stability Analysis of Impulsive Functional Differential Equations*, Walter de Gruyter, Berlin, Germany, 2009.

[16] M. A. Arbib, *Brains, Machines, and Mathematics*, Springer, New York, NY, USA, 1987.

[17] S. Haykin, *Neural Networks: A Comprehensive Foundation*, Prentice-Hall, Englewood Cliffs, NJ, USA, 1998.

[18] H. Akça, R. Alassar, V. Covachev, Z. Covacheva, and E. Al-Zahrani, "Continuous-time additive Hopfield-type neural networks with impulses," *Journal of Mathematical Analysis and Applications*, vol. 290, no. 2, pp. 436–451, 2004.

[19] G. T. Stamov, "Almost periodic models of impulsive Hopfield neural networks," *Journal of Mathematics of Kyoto University*, vol. 49, no. 1, pp. 57–67, 2009.

[20] G. T. Stamov and I. M. Stamova, "Almost periodic solutions for impulsive neural networks with delay," *Applied Mathematical Modelling*, vol. 31, pp. 1263–1270, 2007.

[21] S. Ahmad and I. M. Stamova, "Global exponential stability for impulsive cellular neural networks with time-varying delays," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 69, no. 3, pp. 786–795, 2008.

[22] X. Liu and K. L. Teo, "Exponential stability of impulsive high-order Hopfield-type neural networks with time-varying delays," *IEEE Transactions on Neural Networks*, vol. 16, pp. 1329–1339, 2005.

[23] Y. Zhang and Q. Luo, "Global exponential stability of impulsive cellular neural networks with time-varying delays via fixed point theory," *Advances in Difference Equations*, vol. 2013, article 23, 2013.

[24] Y. Zhang and M. Zhang, "Stability analysis for impulsive reaction-diffusion Cohen-Grossberg neural networks with time-varying delays," *Journal of Nanjing University of Information Science and Technology*, vol. 4, no. 3, pp. 213–219, 2012.

[25] X. Zhang, S. Wu, and K. Li, "Delay-dependent exponential stability for impulsive Cohen-Grossberg neural networks with time-varying delays and reaction-diffusion terms," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 3, pp. 1524–1532, 2011.

[26] J. Pan and S. Zhong, "Dynamical behaviors of impulsive reaction-diffusion Cohen-Grossberg neural network with delay," *Neurocomputing*, vol. 73, pp. 1344–1351, 2010.

[27] K. Li and Q. Song, "Exponential stability of impulsive Cohen-Grossberg neural networks with time-varying delays and reaction-diffusion terms," *Neurocomputing*, vol. 72, pp. 231–240, 2008.

[28] J. Qiu, "Exponential stability of impulsive neural networks with time-varying delays and reaction-diffusion terms," *Neurocomputing*, vol. 70, pp. 1102–1108, 2007.

[29] X. Wang and D. Xu, "Global exponential stability of impulsive fuzzy cellular neural networks with mixed delays and reaction-diffusion terms," *Chaos, Solitons & Fractals*, vol. 42, no. 5, pp. 2713–2721, 2009.

[30] W. Zhu, "Global exponential stability of impulsive reaction-diffusion equation with variable delays," *Applied Mathematics and Computation*, vol. 205, no. 1, pp. 362–369, 2008.

[31] Z. Li and K. Li, "Stability analysis of impulsive Cohen-Grossberg neural networks with distributed delays and reaction-diffusion terms," *Applied Mathematical Modelling*, vol. 33, no. 3, pp. 1337–1348, 2009.

[32] Z. Li and K. Li, "Stability analysis of impulsive fuzzy cellular neural networks with distributed delays and reaction-diffusion terms," *Chaos, Solitons and Fractals*, vol. 42, no. 1, pp. 492–499, 2009.

[33] J. Pan, X. Liu, and S. Zhong, "Stability criteria for impulsive reaction-diffusion Cohen-Grossberg neural networks with time-varying delays," *Mathematical and Computer Modelling*, vol. 51, no. 9-10, pp. 1037–1050, 2010.

[34] Y. Zhang and Q. Luo, "Novel stability criteria for impulsive delayed reaction-diffusion Cohen-Grossberg neural networks via Hardy-Poincarè inequality," *Chaos, Solitons & Fractals*, vol. 45, no. 8, pp. 1033–1040, 2012.

[35] Y. Zhang and Q. Luo, "Global exponential stability of impulsive delayed reaction-diffusion neural networks via Hardy-Poincarè Inequality," *Neurocomputing*, vol. 83, pp. 198–204, 2012.

[36] Y. Zhang, "Asymptotic stability of impulsive reaction-diffusion cellular neural networks with time-varying delays," *Journal of Applied Mathematics*, vol. 2012, Article ID 501891, 17 pages, 2012.

[37] V. Lakshmikantham, D. D. Baĭnov, and P. S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.

[38] W.-S. Cheung, "Some new Poincaré-type inequalities," *Bulletin of the Australian Mathematical Society*, vol. 63, no. 2, pp. 321–327, 2001.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 409758, 10 pages
http://dx.doi.org/10.1155/2013/409758

Research Article

Stability of Impulsive Cohen-Grossberg Neural Networks with Time-Varying Delays and Reaction-Diffusion Terms

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Received 29 December 2012; Accepted 23 February 2013

Academic Editor: Qi Luo

This work concerns the stability of impulsive Cohen-Grossberg neural networks with time-varying delays and reaction-diffusion terms as well as Dirichlet boundary condition. By means of Poincaré inequality and Gronwall-Bellman-type impulsive integral inequality, we summarize some new and concise sufficient conditions ensuring the global exponential stability of equilibrium point. The proposed criteria are relevant to the diffusion coefficients and the smallest positive eigenvalue of corresponding Dirichlet Laplacian. In conclusion, two examples are illustrated to demonstrate the effectiveness of our obtained results.

1. Introduction

Cohen-Grossberg neural networks (CGNNs) were introduced by Cohen and Grossberg in 1983 [1] and have been a hot topic due to their important applications in various fields such as parallel computation, associative memory, image processing, and optimization problems.

By reason that time delays are unavoidably encountered for the finite switching speed of neurons and amplifiers in the implementation of neural networks, a more powerful model of delayed Cohen-Grossberg neural networks (DCGNNs) is afterwards proposed. This kind of mathematical models is widely applied in dynamic image processing and pattern recognition problems. It is worth noting that all these applications depend heavily on the dynamical behaviors such as stability, convergence, and oscillatory [2–6]. Meanwhile, stability is an important consideration in the designs and applications of neural networks. The stability of delayed neural networks is a subject of current interest, and therefore considerable theoretical efforts have been put into this topic followed by a large number of stability criteria reported; for example, see [7–12] and the references therein.

In real world, however, many evolutionary processes are characterized by abrupt changes at certain instants which may be caused by switching phenomena, frequency changes, or other sudden noises. As such, in the past few years, scientists have become gradually interested in the influence that impulses may have on the CGNNs and DCGNNs, thus obtaining some related results; for example, see [13–18] and the references therein.

Actually, besides impulsive effects, we have to recognize that diffusion effects are also nonignorable in reality as diffusion is unavoidable when electrons are moving in asymmetric electromagnetic fields. On this account, the model of neural networks with both impulses and diffusion should be more effective for describing the evolutionary process of practical systems. Based on this consideration, we wonder what the influence of diffusion on the stability of CGNNs and DCGNNs is.

So far there have appeared a few theoretical achievements [19–29] on the stability of impulsive reaction-diffusion neural networks with or without delays. Particularly, in [21–26], the main research technique is the impulsive differential inequality whereby the authors discussed the stability of equilibrium point and provided a series of sufficient conditions independent of diffusion. From these results, we fail to see the influence of diffusion on the stability of CGNNs and DCGNNs.

Encouragingly, recently there were reported some new results on the stability of CGNNs and DCGNNs in [19, 20, 27]; thereinto, the presented stability criteria derived from the impulsive differential inequality are related to the diffusion terms, and thereby we know the diffusion do contribute to the stability of impulsive neural networks.

In this paper, different from [20, 27], we shall consider the case where the boundary condition is Dirichlet boundary condition rather than Neumann boundary condition. Moreover, unlike [19], we shall utilize the new method of Poincarè inequality to deal with the reaction-diffusion terms, and Gronwall-Bellman-type impulsive integral inequality is also introduced for stability analysis. The obtained results show that not only the reaction-diffusion coefficients but also the first eigenvalue of corresponding Dirichlet Laplacian can affect the stability.

The rest of this paper is structured as follows. In Section 2, the model of impulsive delayed Cohen-Grossberg neural networks with reaction-diffusion terms as well as Dirichlet boundary condition is outlined and some facts and lemmas are introduced for later reference. By the new agencies of Gronwall-Bellman-type impulsive integral inequality and Poincaré inequality, we discuss the global exponential stability of equilibrium point and develop some new and concise algebraic criteria in Section 3. To conclude, two illustrative examples are given in Section 4 to verify the effectiveness of our results.

2. Preliminaries

Let R^n denote the n-dimensional Euclidean space, and let $\Omega \subset R^m$ be an open bounded domain with smooth boundary $\partial\Omega$ and mes$\Omega > 0$. Let $R_+ = [0, \infty)$ and $t_0 \in R_+$.

Consider the following impulsive CGNN with time-varying delays and reaction-diffusion terms:

$$\begin{aligned}\frac{\partial u_i(t,x)}{\partial t} &= \sum_{s=1}^{m} \frac{\partial}{\partial x_s}\left(D_{is}\frac{\partial u_i(t,x)}{\partial x_s}\right) \\ &\quad - a_i(u_i(t,x))\left[\omega_i(u_i(t,x)) - \sum_{j=1}^{n} b_{ij} f_j(u_j(t,x)) \right. \\ &\quad \left. - \sum_{j=1}^{n} c_{ij} f_j(u_j(t-\tau_j(t),x))\right], \\ &\quad t \ge t_0,\ t \ne t_k,\ x \in \Omega,\ i = 1,2,\dots,n,\ k = 1,2,\dots,\end{aligned} \tag{1}$$

$$\begin{aligned} u_i(t_k+0,x) &= u_i(t_k,x) + P_{ik}(u_i(t_k,x)), \\ & x \in \Omega,\ k = 1,2,\dots,\ i = 1,2,\dots,n,\end{aligned} \tag{2}$$

where n corresponds to the numbers of units in a neural network, $x = (x_1,\dots,x_m)^T \in \Omega$, $u_i(t,x)$ denotes the state of the ith neuron at time t and in space x, $D_{is} =$ const > 0 represents transmission diffusion of the ith unit, $a_i(u_i(t,x))$ represents the amplification function, $\omega_i(u_i(t,x))$ is the appropriate behavior function, activation function $f_j(u_j(t,x))$ stands for the output of the jth unit at time i and in space x and b_{ij} and c_{ij} are constants: b_{ij} indicates the connection strength of the jth unit on the ith unit at time t and in space x, while c_{ij} denotes the connection weight of the jth unit on the ith unit at time $t-\tau_j(t)$ and in space x, where $\tau_j(t)$ corresponds to the transmission delay along the axon of the jth unit satisfying $0 \le \tau_j(t) \le \tau$ ($\tau =$ const) and $\dot{\tau}_j(t) < 1-(1/h)(h>0)$. $\{t_k\}$ $(k = 1,2,\dots)$ is the sequence of impulsive moments meeting $0 \le t_0 < t_1 < t_2 < \cdots$ and $\lim_{k\to\infty} t_k = \infty$; $u_i(t_k+0,x)$ and $u_i(t_k-0,x)$ represent the right-hand and left-hand limit of $u_i(t,x)$ at time t_k and in space x, respectively. $P_{ik}(u_i(t_k,x))$ stands for the abrupt change of $u_i(t,x)$ at impulsive moment t_k and in space x.

Denote by $u(t,x) = u(t,x;t_0,\varphi)$, $u \in R^n$, the solution of systems (1)-(2), satisfying the initial condition

$$u(s,x;t_0,\varphi) = \varphi(s,x), \quad t_0-\tau \le s \le t_0,\ x \in \Omega, \tag{3}$$

and Dirichlet boundary condition

$$u(t,x;t_0,\varphi) = 0, \quad t \ge t_0,\ x \in \partial\Omega, \tag{4}$$

where the vector-valued function $\varphi(s,x) = (\varphi_1(s,x),\dots,\varphi_n(s,x))^T$ is such that $\int_\Omega \sum_{i=1}^{n} \varphi_i^2(s,x)dx$ is bounded on $[t_0-\tau, t_0]$.

The solution $u(t,x) = u(t,x;t_0,\varphi) = (u_1(t,x;t_0,\varphi),\dots,u_n(t,x;t_0,\varphi))^T$ of problems (1)–(4) is, for the time variable t, a piecewise continuous function with the first kind discontinuity at the points t_k $(k = 1,2,\dots)$, where it is left-continuous; that is, the following relations are valid:

$$\begin{aligned} u_i(t_k-0,x) &= u_i(t_k,x), \\ u_i(t_k+0,x) &= u_i(t_k,x) + P_{ik}(u_i(t_k,x)).\end{aligned} \tag{5}$$

Throughout this paper, we define the norm of $u(t,x;t_0,\varphi)$ as

$$\|u(t,x;t_0,\varphi)\|_\Omega = \sqrt{\sum_{i=1}^{n}\int_\Omega u_i^2(t,x;t_0,\varphi)\,dx} \tag{6}$$

and make the following assumptions for convenience.

(H1) $a_i(\cdot) : R \to R^+$ is continuous and bounded; that is, there exist constants $\underline{a}_i$ and $\overline{a}_i$ such that

$$0 < \underline{a}_i \le a_i(\zeta) \le \overline{a}_i < \infty, \quad \text{for } i = 1,\dots,n. \tag{7}$$

(H2) $\omega_i(\cdot) : R \to R$ is continuous and $\omega_i(0) = 0$; moreover, there exists constant $p_i > 0$ such that

$$\frac{\omega_i(\zeta_1) - \omega_i(\zeta_2)}{\zeta_1 - \zeta_2} \ge p_i > 0, \quad \text{for } \zeta_1 \ne \zeta_2,\ i = 1,\dots,n. \tag{8}$$

(H3) $f_i(\cdot) : R \to R$ is continuous and $f_i(0) = 0$; furthermore, there exists constant $l_i > 0$ such that

$$l_i = \sup_{\zeta_1 \ne \zeta_2} \frac{f_i(\zeta_1) - f_i(\zeta_2)}{\zeta_1 - \zeta_2} \quad \text{for } \zeta_1 \ne \zeta_2,\ i = 1,2,\dots,n. \tag{9}$$

(H4) $P_{ik}(\cdot) : R \to R$ is continuous and $P_{ik}(0) = 0$ for $i = 1,2,\dots,n$ and $k = 1,2,\dots$.

In the light of (H1)–(H4), it is easy to see that problems (1)-(2) admit an equilibrium point $u = 0$.

Definition 1. The equilibrium point $u = 0$ of problems (1)-(2) is said to be globally exponentially stable if there exist constants $\kappa > 0$ and $M \geq 1$ such that

$$\|u(t, x; t_0, \varphi)\|_\Omega \leq M\overline{\|\varphi\|}_\Omega e^{-\kappa(t-t_0)}, \quad t \geq t_0, \tag{10}$$

where $\overline{\|\varphi\|}_\Omega^2 = \sup_{t_0-\tau\leq s\leq t_0} \sum_{i=1}^n \int_\Omega \varphi_i^2(s, x)dx$.

Lemma 2 (see [30] (Gronwall-Bellman-type impulsive integral inequality)). *Assume the following.*

(A1) *The sequence $\{t_k\}$ satisfies $0 \leq t_0 < t_1 < t_2 < \cdots$, with $\lim_{k\to\infty} t_k = \infty$.*

(A2) *$q \in PC^1[R_+, R]$ and $q(t)$ is left-continuous at t_k, $k = 1, 2, \ldots$.*

(A3) *$p \in C[R_+, R_+]$ and for $k = 1, 2, \ldots$,*

$$q(t) \leq c + \int_{t_0}^t p(s) q(s) ds + \sum_{t_0<t_k<t} \eta_k q(t_k), \quad t \geq t_0, \tag{11}$$

where $\eta_k \geq 0$ and $c =$ const. Then,

$$q(t) \leq c \prod_{t_0<t_k<t} (1 + \eta_k) \exp\left(\int_{t_0}^t p(s) ds\right), \quad t \geq t_0. \tag{12}$$

Lemma 3 (see [31] (Poincaré inequality)). *Let $\mathcal{S}$ be a bounded region in R^n, $v(x) \in C^1(\mathcal{S})$, and $v = 0$ on the boundary of $\mathcal{S}$; then*

$$\lambda_1 \int_{\mathcal{S}} v^2(x) dx \leq \int_{\mathcal{S}} |\nabla v(x)|^2 dx, \tag{13}$$

where λ_1 is the smallest positive eigenvalue of the following problem:

$$\Delta\Psi(x) + \lambda\Psi(x) = 0, \quad x \in \mathcal{S}, \qquad \Psi(x) = 0, \quad x \in \partial\mathcal{S}. \tag{14}$$

Lemma 4. *If $a > 0$ and $b > 0$, then $ab \leq (1/\varepsilon)a^2 + \varepsilon b^2$ holds for any $\varepsilon > 0$.*

3. Main Results

Theorem 5. *Assume the following.*

(1) *$\underline{D} = \min\{D_{is} : i = 1, \ldots, n; s = 1, \ldots, m\} > 0$ and denote $2\underline{D}\lambda_1 = \chi$.*

(2) *$P_{ik}(u_i(t_k, x)) = -\theta_{ik}u_i(t_k, x)$, $0 \leq \theta_{ik} \leq 2$.*

(3) *There exists a constant $\gamma > 0$ satisfying $\gamma + \lambda + h\rho e^{\gamma\tau} > 0$ and $\lambda + h\rho e^{\gamma\tau} < 0$, where*

$$\lambda = \max_{i=1,\ldots,n} \left(-\chi - 2\underline{a}_i p_i + \overline{a}_i \sum_{j=1}^n b_{ij}^2 + \overline{a}_i \sum_{j=1}^n c_{ij}^2 \right) + \rho,$$
$$\rho = \max_{i=1,\ldots,n} \left(l_i^2\right) \sum_{i=1}^n \overline{a}_i. \tag{15}$$

Then, the equilibrium point $u = 0$ of systems (1)-(2) is globally exponentially stable with convergence rate $-(\lambda + h\rho e^{\gamma\tau})/2$.

Proof. Multiplying both sides of (1) by $u_i(t, x)$, we get

$$\begin{aligned}\frac{\partial u_i^2(t, x)}{\partial t} &= 2\sum_{s=1}^m u_i(t, x) \frac{\partial}{\partial x_s}\left(D_{is}\frac{\partial u_i(t, x)}{\partial x_s}\right) \\ &\quad - 2u_i(t, x) a_i(u_i(t, x)) \\ &\quad \times \left[\omega_i(u_i(t, x)) - \sum_{j=1}^n b_{ij} f_j(u_j(t, x)) \right.\\ &\quad \left. - \sum_{j=1}^n c_{ij} f_j(u_j(t - \tau_j(t), x))\right], \\ &\quad t \geq t_0, \ t \neq t_k, \ x \in \Omega, \ k = 1, 2, \ldots,\end{aligned} \tag{16}$$

which yields, after integrating with respect to spatial variable x on Ω,

$$\begin{aligned}\frac{d\left(\int_\Omega u_i^2(t, x) dx\right)}{dt} &= J_1 + J_2, \\ t \geq t_0, \ t \neq t_k, \ k &= 1, 2, \ldots,\end{aligned} \tag{17}$$

where $J_1 = 2\int_\Omega \sum_{s=1}^m (u_i(t, x)(\partial/\partial x_s)(D_{is}(\partial u_i(t, x))/\partial x_s))dx$,

$$\begin{aligned}J_2 = -2\int_\Omega & u_i(t, x) a_i(u_i(t, x)) \\ &\times \left[\omega_i(u_i(t, x)) - \sum_{j=1}^n b_{ij} f_j(u_j(t, x)) \right.\\ &\left. - \sum_{j=1}^n c_{ij} f_j(u_j(t - \tau_j(t), x))\right] dx.\end{aligned} \tag{18}$$

By combining Green formula, Dirichlet boundary condition, Lemma 3, and condition (1) of Theorem 5, we obtain

$$\begin{aligned}J_1 &= -2\sum_{s=1}^m \int_\Omega D_{is}\left(\frac{\partial u_i(t, x)}{\partial x_s}\right)^2 dx \\ &\leq -2\underline{D}\lambda_1 \int_\Omega u_i^2(t, x) dx \triangleq -\chi \int_\Omega u_i^2(t, x) dx.\end{aligned} \tag{19}$$

Moreover, it follows from assumptions (H1), (H2), and (H3) that

$$2\int_\Omega u_i(t,x)\,a_i(u_i(t,x))\,\omega_i(u_i(t,x))\,dx \geq 2\underline{a}_i p_i \int_\Omega |u_i(t,x)|^2 dx, \tag{20}$$

$$\begin{aligned} 2\int_\Omega u_i(t,x)\,a_i(u_i(t,x)) \sum_{j=1}^n b_{ij} f_j(u_j(t,x))\,dx &\leq \bar{a}_i \sum_{j=1}^n \int_\Omega \left(b_{ij}^2 u_i^2(t,x) + f_j^2(u_j(t,x))\right) dx \\ &\leq \bar{a}_i \sum_{j=1}^n \int_\Omega 2\left|b_{ij}\|u_i(t,x)\| f_j(u_j(t,x))\right| dx \\ &\leq \bar{a}_i \sum_{j=1}^n \int_\Omega \left(b_{ij}^2 u_i^2(t,x) + l_j^2 u_j^2(t,x)\right) dx, \end{aligned} \tag{21}$$

$$\begin{aligned} 2\int_\Omega u_i(t,x)\,a_i(u_i(t,x)) \sum_{j=1}^n c_{ij} f_j(u_j(t-\tau_j(t),x))\,dx \\ \leq \bar{a}_i \sum_{j=1}^n \int_\Omega \left(c_{ij}^2 u_i^2(t,x) + l_j^2 u_j^2(t-\tau_j(t),x)\right) dx. \end{aligned} \tag{22}$$

Consequently, substituting (19)–(22) into (17) produces

$$\begin{aligned} \frac{d\left(\int_\Omega u_i^2(t,x)\,dx\right)}{dt} &\leq -\chi \int_\Omega u_i^2(t,x)\,dx - 2\underline{a}_i p_i \int_\Omega u_i^2(t,x)\,dx \\ &\quad + \bar{a}_i \sum_{j=1}^n \int_\Omega \left(b_{ij}^2 {u_i}^2(t,x) + l_j^2 u_j^2(t,x)\right) dx \\ &\quad + \bar{a}_i \sum_{j=1}^n \int_\Omega \left(c_{ij}^2 {u_i}^2(t,x) + l_j^2 u_j^2(t-\tau_j(t),x)\right) dx \end{aligned} \tag{23}$$

for $t \geq t_0$, $t \neq t_k$, $k = 1, 2, \ldots$.

Now define Lyapunov function $V_i(t)$ as $V_i(t) = \int_\Omega u_j^2(t,x)dx$. It is not difficult to see that $V_i(t)$ is a piecewise continuous function with points of discontinuity of the first kind t_k $(k = 1, 2, \ldots)$, where it is continuous from the left; that is, $V_i(t_k - 0) = V_i(t_k)$ $(k = 1, 2, \ldots)$. In addition, for $t = t_k$ $(k = 0, 1, 2, \ldots)$, we know

$$V_i(t_k + 0) \leq V_i(t_k), \quad k = 0, 1, 2, \ldots, \tag{24}$$

as $V_i(t_0 + 0) \leq V_i(t_0)$ and $u_i^2(t_k + 0, x) = (1 - \theta_{ik})^2 u_i^2(t_k, x) \leq u_i^2(t_k, x)$ $(k = 1, 2, \ldots)$, supported by condition 2 of Theorem 5.

Put $t \in (t_k, t_{k+1})$, $k = 0, 1, 2, \ldots$. It is derived from (23) that

$$\begin{aligned} \frac{dV_i(t)}{dt} &\leq \left(-\chi - 2\underline{a}_i p_i + \bar{a}_i \sum_{j=1}^n b_{ij}^2 + \bar{a}_i \sum_{j=1}^n c_{ij}^2\right) V_i(t) \\ &\quad + \bar{a}_i \max_{i=1,\ldots,n} (l_i^2) \sum_{j=1}^n V_j(t) \\ &\quad + \bar{a}_i \max_{i=1,\ldots,n} (l_i^2) \sum_{j=1}^n V_j(t - \tau_j(t)), \\ &\qquad t \in (t_k, t_{k+1}), \quad k = 0, 1, 2, \ldots. \end{aligned} \tag{25}$$

Define function $V(t)$ of the form $V(t) = \sum_{i=1}^n V_i(t)$ again. From (25), one then reads

$$\begin{aligned} \frac{dV(t)}{dt} &\leq \lambda V(t) \\ &\quad + \rho \sum_{j=1}^n V_j(t - \tau_j(t)), \quad t \in (t_k, t_{k+1}), \ k = 0, 1, 2, \ldots, \end{aligned} \tag{26}$$

where $\rho = \max_{i=1,\ldots,n}(l_i^2) \sum_{i=1}^n \bar{a}_i$ and $\lambda = \max_{i=1,\ldots,n}(-\chi - 2\underline{a}_i p_i + \bar{a}_i \sum_{j=1}^n b_{ij}^2 + \bar{a}_i \sum_{j=1}^n c_{ij}^2) + \rho$.

Construct $V^*(t) = e^{\gamma(t-t_0)}V(t)$, where $\gamma > 0$ satisfies $\gamma + \lambda + h\rho e^{\gamma\tau} > 0$ and $\lambda + h\rho e^{\gamma\tau} < 0$. Evidently, $V^*(t)$ is also a piecewise continuous function with the first kind discontinuous points t_k $(k = 1, 2, \ldots)$, in which it is continuous from the left; that is, $V^*(t_k - 0) = V^*(t_k)$ $(k = 1, 2, \ldots)$. Moreover, at $t = t_k$ $(k = 0, 1, 2, \ldots)$, we find by the use of (24)

$$V^*(t_k + 0) \leq V^*(t_k), \quad k = 0, 1, 2, \ldots. \tag{27}$$

Set $t \in (t_k, t_{k+1})$, $k = 0, 1, 2, \ldots$. By virtue of (26), one has

$$\begin{aligned} \frac{dV^*(t)}{dt} &= \gamma e^{\gamma(t-t_0)} V(t) + e^{\gamma(t-t_0)} \frac{dV(t)}{dt} \\ &\leq \gamma e^{\gamma(t-t_0)} V(t) \\ &\quad + \left(\lambda V(t) + \rho \sum_{j=1}^n V_j(t - \tau_j(t))\right) e^{\gamma(t-t_0)} \\ &= (\gamma + \lambda) V^*(t) + \rho e^{\gamma(t-t_0)} \sum_{j=1}^n V_j(t - \tau_j(t)), \\ &\qquad t \in (t_k, t_{k+1}), \quad k = 0, 1, 2, \ldots. \end{aligned} \tag{28}$$

Choose small enough $\varepsilon > 0$. Integrating (28) from $t_k + \varepsilon$ to t gives

$$\begin{aligned} V^*(t) &\leq V^*(t_k + \varepsilon) + (\gamma + \lambda) \int_{t_k+\varepsilon}^t V^*(s)\,ds \\ &\quad + \int_{t_k+\varepsilon}^t \rho e^{\gamma(s-t_0)} \sum_{j=1}^n V_j(s - \tau_j(s))\,ds, \\ &\qquad t \in (t_k, t_{k+1}), \quad k = 0, 1, 2, \ldots \end{aligned} \tag{29}$$

which yields, after letting $\varepsilon \to 0$ in (29),

$$V^*(t) \le V^*(t_k+0) + (\gamma+\lambda)\int_{t_k}^{t} V^*(s)\,ds + \int_{t_k}^{t} \rho e^{\gamma(s-t_0)} \sum_{j=1}^{n} V_j(s-\tau_j(s))\,ds, \quad t \in (t_k, t_{k+1}),\ k = 0,1,2,\ldots. \tag{30}$$

Next, we estimate the value of $V^*(t)$ at $t = t_{k+1}$, $k = 0, 1, 2, \ldots$. For small enough $\varepsilon > 0$, we put $t = t_{k+1} - \varepsilon$. Now an application of (30) leads to, for $k = 0, 1, 2, \ldots$,

$$V^*(t_{k+1}-\varepsilon) \le V^*(t_k+0) + (\gamma+\lambda)\int_{t_k}^{t_{k+1}-\varepsilon} V^*(s)\,ds + \int_{t_k}^{t_{k+1}-\varepsilon} \rho e^{\gamma(s-t_0)} \sum_{j=1}^{n} V_j(s-\tau_j(s))\,ds. \tag{31}$$

If we let $\varepsilon \to 0$ in (31), there results

$$V^*(t_{k+1}-0) \le V^*(t_k+0) + (\gamma+\lambda)\int_{t_k}^{t_{k+1}} V^*(s)\,ds + \int_{t_k}^{t_{k+1}} \rho e^{\gamma(s-t_0)} \sum_{j=1}^{n} V_j(s-\tau_j(s))\,ds, \quad k = 0,1,2,\ldots. \tag{32}$$

Note that $V^*(t_{k+1}-0) = V^*(t_{k+1})$ is applicable for $k = 0, 1, 2, \ldots$. Thus,

$$V^*(t_{k+1}) \le V^*(t_k+0) + (\gamma+\lambda)\int_{t_k}^{t_{k+1}} V^*(s)\,ds + \int_{t_k}^{t_{k+1}} \rho e^{\gamma(s-t_0)} \sum_{j=1}^{n} V_j(s-\tau_j(s))\,ds \tag{33}$$

holds for $k = 0, 1, 2, \ldots$. By synthesizing (30) and (33), we then arrive at

$$V^*(t) \le V^*(t_k+0) + (\gamma+\lambda)\int_{t_k}^{t} V^*(s)\,ds + \int_{t_k}^{t} \rho e^{\gamma(s-t_0)} \sum_{j=1}^{n} V_j(s-\tau_j(s))\,ds, \quad t \in (t_k, t_{k+1}],\ k = 0,1,2,\ldots. \tag{34}$$

This, together with (27), results in

$$V^*(t) \le V^*(t_k) + (\gamma+\lambda)\int_{t_k}^{t} V^*(s)\,ds + \int_{t_k}^{t} \rho e^{\gamma(s-t_0)} \sum_{j=1}^{n} V_j(s-\tau_j(s))\,ds \tag{35}$$

for $t \in (t_k, t_{k+1}]$, $k = 0, 1, 2, \ldots$.

Recalling the assumptions that $0 \le \tau_j(t) \le \tau$ and $\dot{\tau}_j(t) < 1 - (1/h)(h > 0)$, we therefore obtain

$$\begin{aligned}\int_{t_k}^{t} \rho e^{\gamma(s-t_0)} \sum_{j=1}^{n} V_j(s-\tau_j(s))\,ds &= \sum_{j=1}^{n} \int_{t_k-\tau_j(t_k)}^{t-\tau_j(t)} \rho e^{\gamma(\theta+\tau_j(s)-t_0)} V_j(\theta) \frac{1}{1-\dot{\tau}_j(s)}\,d\theta \\ &\le h\rho e^{\gamma\tau} \sum_{j=1}^{n} \int_{t_k-\tau_j(t_k)}^{t-\tau_j(t)} e^{\gamma(\theta-t_0)} V_j(\theta)\,d\theta.\end{aligned} \tag{36}$$

Hence,

$$V^*(t) \le V^*(t_k) + (\gamma+\lambda)\int_{t_k}^{t} V^*(s)\,ds + h\rho e^{\gamma\tau} \sum_{j=1}^{n} \int_{t_k-\tau_j(t_k)}^{t-\tau_j(t)} e^{\gamma(s-t_0)} V_j(s)\,ds, \quad t \in (t_k, t_{k+1}],\ k = 0,1,2,\ldots. \tag{37}$$

By induction argument, we reach

$$\begin{aligned}V^*(t_k) &\le V^*(t_{k-1}) + (\gamma+\lambda)\int_{t_{k-1}}^{t_k} V^*(s)\,ds + h\rho e^{\gamma\tau} \sum_{j=1}^{n} \int_{t_{k-1}-\tau_j(t_{k-1})}^{t_k-\tau_j(t_k)} e^{\gamma(s-t_0)} V_j(s)\,ds, \\ &\vdots \\ V^*(t_2) &\le V^*(t_1) + (\gamma+\lambda)\int_{t_1}^{t_2} V^*(s)\,ds + h\rho e^{\gamma\tau} \sum_{j=1}^{n} \int_{t_1-\tau_j(t_1)}^{t_2-\tau_j(t_2)} e^{\gamma(s-t_0)} V_j(s)\,ds, \\ V^*(t_1) &\le V^*(t_0) + (\gamma+\lambda)\int_{t_0}^{t_1} V^*(s)\,ds + h\rho e^{\gamma\tau} \sum_{j=1}^{n} \int_{t_0-\tau_j(t_0)}^{t_1-\tau_j(t_1)} e^{\gamma(s-t_0)} V_j(s)\,ds.\end{aligned} \tag{38}$$

Thus,

$$\begin{aligned} V^*(t) &\leq V^*(t_0) + (\gamma+\lambda)\int_{t_0}^{t} V^*(s)\,ds \\ &\quad + h\rho e^{\gamma\tau}\sum_{j=1}^{n}\int_{t_0-\tau_j(t_0)}^{t-\tau_j(t)} e^{\gamma(s-t_0)}V_j(s)\,ds \\ &\leq V^*(t_0) + (\gamma+\lambda)\int_{t_0}^{t} V^*(s)\,ds \\ &\quad + h\rho e^{\gamma\tau}\sum_{j=1}^{n}\int_{t_0-\tau_j(t_0)}^{t} e^{\gamma(s-t_0)}V_j(s)\,ds \\ &= V^*(t_0) + (\gamma+\lambda+h\rho e^{\gamma\tau})\int_{t_0}^{t} V^*(s)\,ds \\ &\quad + h\rho e^{\gamma\tau}\sum_{j=1}^{n}\int_{t_0-\tau_j(t_0)}^{t_0} e^{\gamma(s-t_0)}V_j(s)\,ds, \\ &\qquad t\in(t_k,t_{k+1}],\ k=0,1,2,\ldots. \end{aligned} \tag{39}$$

Since

$$\begin{aligned} &h\rho e^{\gamma\tau}\sum_{j=1}^{n}\int_{t_0-\tau_j(t_0)}^{t_0} e^{\gamma(s-t_0)}V_j(s)\,ds \\ &\quad\leq h\rho e^{\gamma\tau}\sum_{j=1}^{n}\int_{t_0-\tau}^{t_0} V_j(s)\,ds \\ &\quad= h\rho e^{\gamma\tau}\int_{t_0-\tau}^{t_0}\left(\sum_{j=1}^{n}\int_{\Omega}\varphi_j{}^2(s,x)\,dx\right)ds \\ &\quad\leq \tau h\rho e^{\gamma\tau}\overline{\|\varphi\|}_{\Omega}^2, \end{aligned} \tag{40}$$

we claim

$$\begin{aligned} V^*(t) &\leq V^*(t_0) + \tau h\rho e^{\gamma\tau}\overline{\|\varphi\|}_{\Omega}^2 \\ &\quad + (\gamma+\lambda+h\rho e^{\gamma\tau})\int_{t_0}^{t} V^*(s)\,ds, \\ &\qquad t\in(t_k,t_{k+1}],\ k=0,1,2\ldots. \end{aligned} \tag{41}$$

According to Lemma 2, we assert that

$$\begin{aligned} V^*(t) &\leq \left(V^*(t_0) + \tau h\rho e^{\gamma\tau}\overline{\|\varphi\|}_{\Omega}^2\right) \\ &\quad\times\exp\{(\gamma+\lambda+h\rho e^{\gamma\tau})(t-t_0)\},\quad t\geq t_0, \end{aligned} \tag{42}$$

which reduces to

$$\begin{aligned} &\|u(t,x;t_0,\varphi)\|_{\Omega} \\ &\quad\leq\sqrt{1+\tau h\rho e^{\gamma\tau}}\,\overline{\|\varphi\|}_{\Omega} \\ &\quad\times\exp\left\{\left(\frac{\lambda+h\rho e^{\gamma\tau}}{2}\right)(t-t_0)\right\},\quad t\geq t_0. \end{aligned} \tag{43}$$

This completes the proof. □

Remark 6. According to the conditions of Theorem 5, we see that the reaction-diffusion terms can influence the stability of equilibrium point $u = 0$. Specifically, the acting factors include the reaction-diffusion coefficients and the first eigenvalue of corresponding Dirichlet Laplacian.

Remark 7. It is not difficult to see that there must exist constant $\gamma > 0$ satisfying condition 3 of Theorem 5 if $\lambda < -h\rho$.

Theorem 8. *Assume the following.*

(1) $\underline{D} = \min\{D_{is} : i = 1,\ldots,n; s = 1,\ldots,m\} > 0$ *and denote* $2\underline{D}\lambda_1 = \chi$.

(2) $P_{ik}(u_i(t_k,x)) = -\theta_{ik}u_i(t_k,x)$, $1-\sqrt{1+\alpha}\leq\theta_{ik}\leq 1+\sqrt{1+\alpha}$, $\alpha\geq 0$.

(3) $\inf_{k=1,2\ldots}(t_k - t_{k-1})\geq\mu$.

(4) *There exists a constant* $\gamma > 0$ *satisfying* $\gamma+\lambda+h\rho e^{\gamma\tau} > 0$ *and* $\lambda + h\rho e^{\gamma\tau} + \ln(1+\alpha)/\mu < 0$, *where* $\lambda = \max_{i=1,\ldots,n}(-\chi - 2\underline{a}_i p_i + \overline{a}_i\sum_{j=1}^{n} b_{ij}^2 + \overline{a}_i\sum_{j=1}^{n} c_{ij}^2) + \rho$ *and* $\rho = \max_{i=1,\ldots,n}(l_i^2)\sum_{i=1}^{n}\overline{a}_i$.

Then, the equilibrium point $u = 0$ of systems (1)-(2) is globally exponentially stable with convergence rate $-(1/2)(\lambda + h\rho e^{\gamma\tau} + \ln(1+\alpha)/\mu)$.

Proof. Define Lyapunov function V of the form $V(t) = \sum_{i=1}^{n} V_i(t)$, where $V_i(t) = \int_{\Omega} u_i^2(t,x)dx$. Obviously, $V(t)$ is a piecewise continuous function with the first kind discontinuous points t_k, $k = 1,2,\ldots$, where it is continuous from the left; that is, $V(t_k - 0) = V(t_k)$ $(k = 1,2,\ldots)$. Furthermore, for $t = t_k$ $(k = 0,1,2,\ldots)$, we derive from condition 2 of Theorem 8 that

$$\begin{aligned} &u_i^2(t_k+0,x) - u_i^2(t_k,x) \\ &\quad= (1-\theta_{ik})^2 u_i^2(t_k,x) - u_i^2(t_k,x)\leq\alpha u_i^2(t_k,x). \end{aligned} \tag{44}$$

Thereby,

$$V(t_k+0)\leq\alpha V(t_k) + V(t_k),\quad k=0,1,2,\ldots. \tag{45}$$

Construct function $V^*(t) = e^{\gamma(t-t_0)}V(t)$ again, where $\gamma > 0$ satisfies $\gamma+\lambda+h\rho e^{\gamma\tau} > 0$ and $\lambda + h\rho e^{\gamma\tau} + \ln(1+\alpha)/\mu < 0$. Then, $V^*(t)$ is also a piecewise continuous function with the first kind discontinuous points t_k, $k = 1,2,\ldots$, where it is continuous from the left; that is, $V^*(t_k - 0) = V^*(t_k)$ $(k = 1,2,\ldots)$. And for $t = t_k$ $(k = 0,1,2,\ldots)$, it follows from (45) that

$$V^*(t_k+0)\leq\alpha V^*(t_k) + V^*(t_k),\qquad k=0,1,2,\ldots \tag{46}$$

Set $t\in(t_k,t_{k+1}]$, $k = 0,1,2,\ldots$. Following the same procedure as shown in the proof of Theorem 5, we get

$$\begin{aligned} V^*(t) &\leq V^*(t_k+0) + (\gamma+\lambda)\int_{t_k}^{t} V^*(s)\,ds \\ &\quad + h\rho e^{\gamma\tau}\sum_{j=1}^{n}\int_{t_k-\tau_j(t_k)}^{t-\tau_j(t)} e^{\gamma(\theta-t_0)}V_j(\theta)\,d\theta, \\ &\qquad t\in(t_k,t_{k+1}],\ k=0,1,2,\ldots. \end{aligned} \tag{47}$$

The relations (46) and (47) yield

$$\begin{aligned} & V^*(t) - V^*(t_k) \\ & \quad \le \alpha V^*(t_k) + (\gamma+\lambda)\int_{t_k}^{t} V^*(s)\,ds \\ & \qquad + h\rho e^{\gamma\tau}\sum_{j=1}^{n}\int_{t_k-\tau_j(t_k)}^{t-\tau_j(t)} e^{\gamma(\theta-t_0)}V_j(\theta)\,d\theta, \\ & \qquad\qquad t\in(t_k,t_{k+1}],\ k=0,1,2,\ldots. \end{aligned} \tag{48}$$

By induction argument, we obtain

$$\begin{aligned} & V^*(t_k) - V^*(t_{k-1}) \\ & \quad \le \alpha V^*(t_{k-1}) + (\gamma+\lambda)\int_{t_{k-1}}^{t_k} V^*(s)\,ds \\ & \qquad + h\rho e^{\gamma\tau}\sum_{j=1}^{n}\int_{t_{k-1}-\tau_j(t_{k-1})}^{t_k-\tau_j(t_k)} e^{\gamma(\theta-t_0)}V_j(\theta)\,d\theta, \\ & \quad \vdots \\ & V^*(t_2) - V^*(t_1) \\ & \quad \le \alpha V^*(t_1) + (\gamma+\lambda)\int_{t_1}^{t_2} V^*(s)\,ds \\ & \qquad + h\rho e^{\gamma\tau}\sum_{j=1}^{n}\int_{t_1-\tau_j(t_1)}^{t_2-\tau_j(t_2)} e^{\gamma(\theta-t_0)}V_j(\theta)\,d\theta, \\ & V^*(t_1) - V^*(t_0) \\ & \quad \le \alpha V^*(t_0) + (\gamma+\lambda)\int_{t_0}^{t_1} V^*(s)\,ds \\ & \qquad + h\rho e^{\gamma\tau}\sum_{j=1}^{n}\int_{t_0-\tau_j(t_0)}^{t_1-\tau_j(t_1)} e^{\gamma(\theta-t_0)}V_j(\theta)\,d\theta. \end{aligned} \tag{49}$$

Hence,

$$\begin{aligned} & V^*(t) - V^*(t_0) \\ & \quad \le \alpha V^*(t_0) + (\gamma+\lambda)\int_{t_0}^{t} V^*(s)\,ds \\ & \qquad + h\rho e^{\gamma\tau}\sum_{j=1}^{n}\int_{t_0-\tau_j(t_0)}^{t-\tau_j(t)} e^{\gamma(\theta-t_0)}V_j(\theta)\,d\theta \\ & \qquad + \alpha\sum_{t_0<t_k<t} V(t_k) \\ & \quad \le \alpha V^*(t_0) + (\gamma+\lambda+h\rho e^{\gamma\tau})\int_{t_0}^{t} V^*(s)\,ds \\ & \qquad + h\rho e^{\gamma\tau}\sum_{j=1}^{n}\int_{t_0-\tau_j(t_0)}^{t_0} e^{\gamma(\theta-t_0)}V_j(\theta)\,d\theta + \alpha\sum_{t_0<t_k<t} V(t_k), \\ & \qquad\qquad t\in(t_k,t_{k+1}],\ k=0,1,2,\ldots. \end{aligned} \tag{50}$$

Introducing $h\rho e^{\gamma\tau}\sum_{j=1}^{n}\int_{t_0-\tau_j(t_0)}^{t_0} e^{\gamma(\theta-t_0)}V_j(\theta)d\theta \le \tau h\rho e^{\gamma\tau}\overline{\|\varphi\|}_{\Omega}^{2}$ as shown in the proof of Theorem 5 into (50), (50) becomes, for $t\in(t_k,t_{k+1}]$, $k=0,1,2,\ldots$,

$$\begin{aligned} & V^*(t) - V^*(t_0) \\ & \quad \le \alpha V^*(t_0) + \tau h\rho e^{\gamma\tau}\overline{\|\varphi\|}_{\Omega}^{2} \\ & \qquad + (\gamma+\lambda+h\rho e^{\gamma\tau})\int_{t_0}^{t} V^*(s)\,ds + \alpha\sum_{t_0<t_k<t} V(t_k). \end{aligned} \tag{51}$$

It then results from Lemma 2 that, for $t\ge t_0$,

$$\begin{aligned} V^*(t) & \le \left((\alpha+1)V^*(t_0) + \tau h\rho e^{\gamma\tau}\overline{\|\varphi\|}_{\Omega}^{2}\right) \\ & \qquad \times\prod_{t_0<t_k<t}(1+\alpha)\exp\left((\gamma+\lambda+h\rho e^{\gamma\tau})(t-t_0)\right) \\ & = \left((\alpha+1)V^*(t_0) + \tau h\rho e^{\gamma\tau}\overline{\|\varphi\|}_{\Omega}^{2}\right) \\ & \qquad \times(1+\alpha)^k\exp\left((\gamma+\lambda+h\rho e^{\gamma\tau})(t-t_0)\right). \end{aligned} \tag{52}$$

On the other hand, since $\inf_{k=1,2,\ldots}(t_k-t_{k-1})\ge\mu$, one has $k\le(t_k-t_0)/\mu$. Thereby,

$$\begin{aligned} (1+\alpha)^k & \le \exp\left\{\frac{\ln(1+\alpha)}{\mu}(t_k-t_0)\right\} \\ & \le \exp\left\{\frac{\ln(1+\alpha)}{\mu}(t-t_0)\right\} \end{aligned} \tag{53}$$

and (52) can be rewritten as

$$\begin{aligned} V^*(t) & \le \left((\alpha+1)V^*(t_0) + \tau h\rho e^{\gamma\tau}\overline{\|\varphi\|}_{\Omega}^{2}\right) \\ & \qquad \times\exp\left(\left(\gamma+\lambda+h\rho e^{\gamma\tau}+\frac{\ln(1+\alpha)}{\mu}\right)(t-t_0)\right) \end{aligned} \tag{54}$$

which implies

$$\begin{aligned} & \|u(t,x;t_0,\varphi)\|_{\Omega} \le \sqrt{(\alpha+1+\tau h\rho e^{\gamma\tau})}\,\overline{\|\varphi\|}_{\Omega} \\ & \quad \times\exp\left(\frac{1}{2}\left(\lambda+h\rho e^{\gamma\tau}+\frac{ln(1+\alpha)}{\mu}\right)(t-t_0)\right), \quad t\ge t_0. \end{aligned} \tag{55}$$

The proof is completed. □

Due to Lemma 4, we know that the following inequalities:

$$2\sum_{j=1}^{n} b_{ij}\int_{\Omega} u_i(t,x) f(u_j(t,x))\,dx \le \sum_{j=1}^{n}\int_{\Omega}\left(\varepsilon_1 b_{ij}^2 u_i^2(t,x)+\frac{l_j^2}{\varepsilon_1}u_j^2(t,x)\right)dx,$$
$$2\sum_{j=1}^{n} c_{ij}\int_{\Omega} u_i(t,x) f(u_j(t-\tau_j,x))\,dx \le \sum_{j=1}^{n}\int_{\Omega}\left(\varepsilon_2 c_{ij}^2 u_i^2(t,x)+\frac{l_j^2}{\varepsilon_2}u_i^2(t-\tau_j,x)\right)dx \tag{56}$$

hold for any $\varepsilon_1,\varepsilon_2>0$. Thus, in a similar way to the proofs of Theorems 5–8, we can prove the following theorems.

Theorem 9. *Assume the following.*

(1) $\underline{D}=\min\{D_{is}: i=1,\ldots,n; s=1,\ldots,m\}>0$ *and denote* $2\underline{D}\lambda_1=\chi$.

(2) $P_{ik}(u_i(t_k,x))=-\theta_{ik}u_i(t_k,x)$, $0\le\theta_{ik}\le 2$.

(3) *There exist constants* $\gamma>0$ *and* $\varepsilon_1,\varepsilon_2>0$ *such that* $\gamma+\lambda+h\rho e^{\gamma\tau}>0$ *and* $\lambda+h\rho e^{\gamma\tau}<0$, *where* $\lambda=\max_{i=1,\ldots,n}(-\chi-2\underline{a}_i p_i+\overline{a}_i\sum_{j=1}^{n}(\varepsilon_1 b_{ij}^2+\varepsilon_2 c_{ij}^2))+(\max_{i=1,\ldots,n}(l_i^2)/\varepsilon_1)\sum_{i=1}^{n}\overline{a}_i$, *and* $\rho=(\max_{i=1,\ldots,n}(l_j^2)/\varepsilon_2)\sum_{i=1}^{n}\overline{a}_i$.

Then, the equilibrium point $u=0$ *of systems* (1)-(2) *is globally exponentially stable with convergence rate* $-(\lambda+h\rho e^{\gamma\tau})/2$.

Remark 10. There must exist constant $\gamma>0$ satisfying condition 3 of Theorem 9 if there are constants $\varepsilon_1,\varepsilon_2>0$ such that $\lambda<-h\rho$.

Theorem 11. *Assume the following.*

(1) $\underline{D}=\min\{D_{is}: i=1,\ldots,n; s=1,\ldots,m\}>0$ *and denote* $2\underline{D}\lambda_1=\chi$.

(2) $P_{ik}(u_i(t_k,x))=-\theta_{ik}u_i(t_k,x)$, $1-\sqrt{1+\alpha}\le\theta_{ik}\le 1+\sqrt{1+\alpha}$, $\alpha\ge 0$.

(3) $\inf_{k=1,2,\ldots}(t_k-t_{k-1})\ge\mu$.

(4) *There exist constants* $\gamma>0$ *and* $\varepsilon_1,\varepsilon_2>0$ *satisfying* $\gamma+\lambda+h\rho e^{\gamma\tau}>0$ *and* $\lambda+h\rho e^{\gamma\tau}+\ln(1+\alpha)/\mu<0$, *where*

$$\lambda=\max_{i=1,\ldots,n}\left(-\chi-2\underline{a}_i p_i+\overline{a}_i\sum_{j=1}^{n}\left(\varepsilon_1 b_{ij}^2+\varepsilon_2 c_{ij}^2\right)\right)+\frac{\max_{i=1,\ldots,n}\left(l_i^2\right)}{\varepsilon_1}\sum_{i=1}^{n}\overline{a}_i,$$
$$\rho=\frac{\max_{i=1,\ldots,n}\left(l_i^2\right)}{\varepsilon_2}\sum_{i=1}^{n}\overline{a}_i. \tag{57}$$

Then, the equilibrium point $u=0$ *of systems* (1)-(2) *is globally exponentially stable with convergence rate* $-(1/2)(\lambda+h\rho e^{\gamma\tau}+\ln(1+\alpha)/\mu)$.

Further, on the condition that $|P_{ik}(u_i(t_k,x))|\le\theta_{ik}|u_i(t_k,x)|$, *where* $\theta_{ik}^2<(\alpha-1)/2$ *and* $\alpha\ge 1$, *we obtain*

$$\begin{aligned} u_i^2(t_k+0,x)-u_i^2(t_k,x) &\le 2(u_i(t_k,x))^2+2(P_{ik}(u_i(t_k,x)))^2-u_i^2(t_k,x)\\ &\le\left(2+2\theta_{ik}^2\right)(u_i(t_k,x))^2-u_i^2(t_k,x)\le\alpha u_i^2(t_k,x)\end{aligned} \tag{58}$$

for $t=t_k$ $(k=1,2,\ldots)$. *In an identical way with the proof of Theorem 8, we can present the following.*

Theorem 12. *Assume the following.*

(1) *Let* $\underline{D}=\min\{D_{is}: i=1,\ldots,n; s=1,\ldots,m\}>0$ *and denote* $2\underline{D}\lambda_1=\chi$.

(2) $|P_{ik}(u_i(t_k,x))|\le\theta_{ik}|u_i(t_k,x)|$, *where* $\theta_{ik}^2\le(\alpha-1)/2$ *and* $\alpha\ge 1$.

(3) $\inf_{k=1,2,\ldots}(t_k-t_{k-1})\ge\mu$.

(4) *There exist constants* $\gamma>0$ *and* $\varepsilon_1,\varepsilon_2>0$ *such that* $\gamma+\lambda+h\rho e^{\gamma\tau}>0$ *and* $\lambda+h\rho e^{\gamma\tau}+\ln(1+\alpha)/\mu<0$, *where*

$$\lambda=\max_{i=1,\ldots,n}\left(-\chi-2\underline{a}_i p_i+\overline{a}_i\sum_{j=1}^{n}\left(\varepsilon_1 b_{ij}^2+\varepsilon_2 c_{ij}^2\right)\right)+\frac{\max_{i=1,\ldots,n}\left(l_i^2\right)}{\varepsilon_1}\sum_{i=1}^{n}\overline{a}_i,$$
$$\rho=\frac{\max_{i=1,\ldots,n}\left(l_i^2\right)}{\varepsilon_2}\sum_{i=1}^{n}\overline{a}_i. \tag{59}$$

Then, the equilibrium point $u=0$ *of systems* (1)-(2) *is globally exponentially stable with convergence rate* $-(1/2)(\lambda+h\rho e^{\gamma\tau}+\ln(1+\alpha)/\mu)$.

Remark 13. Different from Theorems 5–11, the impulsive part in Theorem 12 could be nonlinear, and this will be of more applicability. Actually, Theorems 5–11 can be regarded as the special cases of Theorem 12.

4. Examples

Example 14. Consider problems (1)–(4) with $P_{ik}(u_i(t_k,x))=1.343u_i(t_k,x)$; moreover, $n=2$, $m=2$, $\Omega=\{(x_1,x_2)^T \mid x_1^2+x_2^2<1\}$, $a_i(u_1(t,x))=1$, $\omega_1(u_1(t,x))=6.5u_1(t,x)$, $\omega_2(u_2(t,x))=8.5u_2(t,x)$, $(D_{is})=\begin{pmatrix}1.2 & 2.3\\ 2.2 & 1.5\end{pmatrix}$, $(b_{ij})=\begin{pmatrix}-0.23 & 1.3\\ -0.14 & 3.2\end{pmatrix}$, $(c_{ij})=\begin{pmatrix}-0.1 & -0.2\\ 0.25 & -0.13\end{pmatrix}$, $f_j(u_j)=(\sqrt{2}/4)(|u_j+1|-|u_j-1|)$, and $\tau_j(t)=(3/4)\arctan(t)$.

As $\lambda_1 = 5.783$ and $\underline{D} = 1.2$, we know $\chi = 13.8792$. Further, for $l_i = \sqrt{2}/2$, $\underline{a}_i = \overline{a}_i = 1$, $p_1 = 6.5$, and $p_2 = 8.5$, we compute

$$\rho = \max_{i=1,\ldots,n} \left(l_i^2\right) \sum_{i=1}^{n} \overline{a}_i = 1,$$

$$\lambda = \max_{i=1,\ldots,n} \left(-\chi - 2\underline{a}_i p_i + \overline{a}_i \sum_{j=1}^{n} b_{ij}^2 + \overline{a}_i \sum_{j=1}^{n} c_{ij}^2 \right) + \rho = -15.5402. \tag{60}$$

Let $h = 4$. Since $\lambda = -15.5402 < -4 = -h\rho$, we therefore conclude from Theorem 5 that the zero solution of this system is globally exponential stable.

Example 15. Consider problems (1)–(4) with $P_{ik}(u_i(t_k, x)) = \arctan(0.5u_i(t_k, x))$; moreover, $n = 2$, $m = 2$, $\Omega = \{(x_1, x_2)^T \mid x_1^2 + x_1^2 < 1\}$, $a_i(u_1(t, x)) = 1$, $\omega_1(u_1(t, x)) = 6.5u_1(t, x)$, $\omega_2(u_2(t, x)) = 8.5u_2(t, x)$, $(D_{is}) = \left(\begin{smallmatrix} 1.2 & 2.3 \\ 2.2 & 1.5 \end{smallmatrix}\right)$, $(b_{ij}) = \left(\begin{smallmatrix} -0.23 & 1.3 \\ -0.14 & 3.2 \end{smallmatrix}\right)$, $(c_{ij}) = \left(\begin{smallmatrix} -0.1 & -0.2 \\ 0.25 & -0.13 \end{smallmatrix}\right)$, $f_j(u_j) = (\sqrt{2}/4)(|u_j + 1| - |u_j - 1|)$, $\tau_j(t) = (1/\pi)\arctan(t)$, and $t_k = t_{k-1} + k$.

As $\lambda_1 = 5.783$ and $\underline{D} = 1.2$, we know $\chi = 13.8792$. Further, for $l_i = \sqrt{2}/2$, $\underline{a}_i = \overline{a}_i = 1$, $p_1 = 6.5$, $p_2 = 8.5$, and $\varepsilon_i = 1$, we compute

$$\rho = \frac{\max_{i=1,\ldots,n}}{\varepsilon_2} \sum_{i=1}^{n} \overline{a}_i = 1,$$

$$\lambda = \max_{i=1,\ldots,n} \left(-\chi - 2\underline{a}_i p_i + \overline{a}_i \sum_{j=1}^{n} \left(\varepsilon_1 b_{ij}^2 + \varepsilon_2 c_{ij}^2\right) \right) + \frac{\max_{i=1,\ldots,n} \left(l_i^2\right)}{\varepsilon_1} \sum_{i=1}^{n} \overline{a}_i = -15.5402. \tag{61}$$

Let $\tau = 0.5$, $h = 4$, $\mu = 1$, $\theta_{ik} = 0.5$, and $\alpha = 1.5$; we can find $\gamma = 2.4$ such that

$$\gamma + \lambda + h\rho e^{\gamma\tau} = 0.1403 > 0,$$
$$\lambda + h\rho e^{\gamma\tau} + \frac{\ln(1+\alpha)}{\mu} = -1.3434 < 0. \tag{62}$$

Therefore it is concluded from Theorem 12 that the zero solution of this system is globally exponential stable.

References

[1] M. A. Cohen and S. Grossberg, "Absolute stability of global pattern formation and parallel memory storage by competitive neural networks," *IEEE Transactions on Systems, Man, and Cybernetics*, vol. 13, no. 5, pp. 815–826, 1983.

[2] J. Cao and J. Liang, "Boundedness and stability for Cohen-Grossberg neural network with time-varying delays," *Journal of Mathematical Analysis and Applications*, vol. 296, no. 2, pp. 665–685, 2004.

[3] J. Cao and J. Wang, "Global exponential stability and periodicity of recurrent neural networks with time delays," *IEEE Transactions on Circuits and Systems. I.*, vol. 52, no. 5, pp. 920–931, 2005.

[4] X. Liu, "Stability results for impulsive differential systems with applications to population growth models," *Dynamics and Stability of Systems*, vol. 9, no. 2, pp. 163–174, 1994.

[5] X. Liu and Q. Wang, "Impulsive stabilization of high-order Hopfield-type neural networks with time-varying delays," *IEEE Transactions on Neural Networks*, vol. 19, no. 1, pp. 71–79, 2008.

[6] Z. Yang and D. Xu, "Impulsive effects on stability of Cohen-Grossberg neural networks with variable delays," *Applied Mathematics and Computation*, vol. 177, no. 1, pp. 63–78, 2006.

[7] S. Arik and Z. Orman, "Global stability analysis of Cohen-Grossberg neural networks with time varying delays," *Physics Letters A*, vol. 341, no. 5-6, pp. 410–421, 2005.

[8] T. Huang, C. Li, and G. Chen, "Stability of Cohen-Grossberg neural networks with unbounded distributed delays," *Chaos, Solitons & Fractals*, vol. 34, no. 3, pp. 992–996, 2007.

[9] X. Liao, C. Li, and K. W. Wong, "Criteria for exponential stability of Cohen-Grossberg neural networks," *Neural Networks*, vol. 17, no. 10, pp. 1401–1414, 2004.

[10] J. Zhang, Y. Suda, and H. Komine, "Global exponential stability of Cohen-Grossberg neural networks with variable delays," *Physics Letters A*, vol. 338, no. 1, pp. 44–50, 2005.

[11] W. Zhang, Y. Tang, J. A. Fang, and X. Wu, "Stability of delayed neural networks with time-varying impulses," *Neural Networks*, vol. 36, pp. 59–63, 2012.

[12] X. Lai and Y. Zhang, "Fixed point and asymptotic analysis of cellular neural networks," *Journal of Applied Mathematics*, vol. 2012, Article ID 689845, 12 pages, 2012.

[13] Z. Chen and J. Ruan, "Global stability analysis of impulsive Cohen' Grossberg neural networks with delay," *Physics Letters A*, vol. 345, pp. 101–111, 2005.

[14] Z. Chen and J. Ruan, "Global dynamic analysis of general Cohen-Grossberg neural networks with impulse," *Chaos, Solitons and Fractals*, vol. 32, no. 5, pp. 1830–1837, 2007.

[15] Z. Yang and D. Xu, "Impulsive effects on stability of Cohen-Grossberg neural networks with variable delays," *Applied Mathematics and Computation*, vol. 177, no. 1, pp. 63–78, 2006.

[16] Q. Song and J. Zhang, "Global exponential stability of impulsive Cohen-Grossberg neural network with time-varying delays," *Nonlinear Analysis: Real World Applications*, vol. 9, no. 2, pp. 500–510, 2008.

[17] Z. Chen and J. Ruan, "Global dynamic analysis of general Cohen-Grossberg neural networks with impulse," *Chaos, Solitons and Fractals*, vol. 32, no. 5, pp. 1830–1837, 2007.

[18] Y. Zhang and Q. Luo, "Global exponential stability of impulsive cellular neural networks with time-varying delays via fixed point theory," *Advances in Difference Equations*, vol. 2013, article 23, 2013.

[19] X. Zhang, S. Wu, and K. Li, "Delay-dependent exponential stability for impulsive Cohen-Grossberg neural networks with time-varying delays and reaction-diffusion terms," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 3, pp. 1524–1532, 2011.

[20] J. Pan and S. Zhong, "Dynamical behaviors of impulsive reaction-diffusion Cohen-Grossberg neural network with delays," *Neurocomputing*, vol. 73, no. 7–9, pp. 1344–1351, 2010.

[21] K. Li and Q. Song, "Exponential stability of impulsive Cohen-Grossberg neural networks with time-varying delays and reaction-diffusion terms," *Neurocomputing*, vol. 72, no. 1–3, pp. 231–240, 2008.

[22] J. Qiu, "Exponential stability of impulsive neural networks with time-varying delays and reaction-diffusion terms," *Neurocomputing*, vol. 70, no. 4–6, pp. 1102–1108, 2007.

[23] X. Wang and D. Xu, "Global exponential stability of impulsive fuzzy cellular neural networks with mixed delays and reaction-diffusion terms," *Chaos, Solitons & Fractals*, vol. 42, no. 5, pp. 2713–2721, 2009.

[24] W. Zhu, "Global exponential stability of impulsive reaction-diffusion equation with variable delays," *Applied Mathematics and Computation*, vol. 205, no. 1, pp. 362–369, 2008.

[25] Z. Li and K. Li, "Stability analysis of impulsive Cohen-Grossberg neural networks with distributed delays and reaction-diffusion terms," *Applied Mathematical Modelling*, vol. 33, no. 3, pp. 1337–1348, 2009.

[26] Z. Li and K. Li, "Stability analysis of impulsive fuzzy cellular neural networks with distributed delays and reaction-diffusion terms," *Chaos, Solitons and Fractals*, vol. 42, no. 1, pp. 492–499, 2009.

[27] J. Pan, X. Liu, and S. Zhong, "Stability criteria for impulsive reaction-diffusion Cohen-Grossberg neural networks with time-varying delays," *Mathematical and Computer Modelling*, vol. 51, no. 9-10, pp. 1037–1050, 2010.

[28] Y. Zhang and Q. Luo, "Novel stability criteria for impulsive delayed reaction-diffusion Cohen-Grossberg neural networks via Hardy-Poincarè inequality," *Chaos, Solitons & Fractals*, vol. 45, no. 8, pp. 1033–1040, 2012.

[29] Z. Yutian and Z. Minhui, "Stability analysis for impulsive reaction-diffusion Cohen-Grossberg neural networks with time-varying delays," *Journal of Nanjing University of Information Science and Technology*, vol. 4, no. 3, pp. 213–219, 2012.

[30] V. Lakshmikantham, D. D. Baĭnov, and P. S. Simeonov, *Theory of Impulsive Differential Equations*, vol. 6 of *Series in Modern Applied Mathematics*, World Scientific Publishing, Singapore, 1989.

[31] D. S. Mitrinovic, *Analytic Inequalities*, Springer, New York, NY, USA, 1970.

Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 427827, 10 pages
http://dx.doi.org/10.1155/2013/427827

Research Article

Asymptotic Stability of Impulsive Cellular Neural Networks with Infinite Delays via Fixed Point Theory

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Received 1 November 2012; Accepted 8 February 2013

Academic Editor: Qi Luo

We employ the new method of fixed point theory to study the stability of a class of impulsive cellular neural networks with infinite delays. Some novel and concise sufficient conditions are presented ensuring the existence and uniqueness of solution and the asymptotic stability of trivial equilibrium at the same time. These conditions are easily checked and do not require the boundedness and differentiability of delays.

1. Introduction

Cellular neural networks (CNNs), proposed by Chua and Yang in 1988 [1, 2], have become a hot topic for their numerous successful applications in various fields such as optimization, linear and nonlinear programming, associative memory, pattern recognition, and computer vision.

Due to the finite switching speed of neurons and amplifiers in the implementation of neural networks, it turns out that the time delays should not be neglected, and therefore, the model of delayed cellular neural networks (DCNNs) is put forward, which is naturally of better realistic significances. In fact, besides delay effects, stochastic and impulsive as well as diffusing effects are also likely to exist in neural networks. Accordingly many experts are showing a growing interest in the research on the dynamic behaviors of complex CNNs such as impulsive delayed reaction-diffusion CNNs and stochastic delayed reaction-diffusion CNNs, with a result of many achievements [3–9] obtained.

Synthesizing the reported results about complex CNNs, we find that the existing research methods for dealing with stability are mainly based on Lyapunov theory. However, we also notice that there are still lots of difficulties in the applications of corresponding results to specific problems; correspondingly it is necessary to seek some new techniques to overcome those difficulties.

Encouragingly, in recent few years, Burton and other authors have applied the fixed point theory to investigate the stability of deterministic systems and obtained some more applicable results; for example, see the monograph [10] and papers [11–22]. In addition, more recently, there have been a few publications where the fixed point theory is employed to deal with the stability of stochastic (delayed) differential equations; see [23–29]. Particularly, in [24–26], Luo used the fixed point theory to study the exponential stability of mild solutions to stochastic partial differential equations with bounded delays and with infinite delays. In [27, 28], Sakthivel used the fixed point theory to investigate the asymptotic stability in pth moment of mild solutions to nonlinear impulsive stochastic partial differential equations with bounded delays and with infinite delays. In [29], Luo used the fixed point theory to study the exponential stability of stochastic Volterra-Levin equations.

Naturally, for complex CNNs which have high application values, we wonder if we can utilize the fixed point theory to investigate their stability, not just the existence and uniqueness of solution. With this motivation, in the present paper, we aim to discuss the stability of impulsive CNNs with infinite delays via the fixed point theory. It is worth noting that our research skill is the contraction mapping theory which is different from the usual method of Lyapunov theory. We employ the fixed point theorem to prove the existence and uniqueness of solution and the asymptotic stability of trivial equilibrium all at once. Some new and concise algebraic criteria are provided, and these conditions are easy to verify and, moreover, do not require the boundedness and differentiability of delays.

2. Preliminaries

Let R^n denote the n-dimensional Euclidean space and let $\|\cdot\|$ represent the Euclidean norm. $\mathcal{N} \triangleq \{1,2,\ldots,n\}$. $R_+ = [0,\infty)$. $C[X,Y]$ corresponds to the space of continuous mappings from the topological space X to the topological space Y.

In this paper, we consider the following impulsive cellular neural network with infinite delays:

$$\begin{aligned}\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} &= -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j\left(x_j(t)\right) \\ &\quad + \sum_{j=1}^{n} c_{ij} g_j\left(x_j\left(t-\tau_j(t)\right)\right), \\ &\qquad t \geq 0,\ t \neq t_k,\end{aligned} \tag{1}$$

$$\begin{aligned}\Delta x_i(t_k) &= x_i(t_k+0) - x_i(t_k) \\ &= I_{ik}\left(x_i(t_k)\right), \quad k = 1,2,\ldots,\end{aligned} \tag{2}$$

where $i \in \mathcal{N}$ and n is the number of neurons in the neural network. $x_i(t)$ corresponds to the state of the ith neuron at time t. $f_j(\cdot), g_j(\cdot) \in C[R,R]$ denote the activation functions, respectively. $\tau_j(t) \in C[R_+,R_+]$ corresponds to the known transmission delay satisfying $\tau_j(t) \to \infty$ and $t-\tau_j(t) \to \infty$ as $t \to \infty$. Denote $\vartheta = \inf\{t-\tau_j(t),\ t \geq 0, j \in \mathcal{N}\}$. The constant b_{ij} represents the connection weight of the jth neuron on the ith neuron at time t. The constant c_{ij} denotes the connection strength of the jth neuron on the ith neuron at time $t-\tau_j(t)$. The constant $a_i > 0$ represents the rate with which the ith neuron will reset its potential to the resting state when disconnected from the network and external inputs. The fixed impulsive moments t_k $(k=1,2,\ldots)$ satisfy $0 = t_0 < t_1 < t_2 < \cdots$ and $\lim_{k\to\infty} t_k = \infty$. $x_i(t_k+0)$ and $x_i(t_k-0)$ stand for the right-hand and left-hand limits of $x_i(t)$ at time t_k, respectively. $I_{ik}(x_i(t_k))$ shows the abrupt change of $x_i(t)$ at the impulsive moment t_k and $I_{ik}(\cdot) \in C[R,R]$.

Throughout this paper, we always assume that $f_i(0) = g_i(0) = I_{ik}(0) = 0$ for $i \in \mathcal{N}$ and $k = 1,2,\ldots$. Thereby, problem (1) and (2) admits a trivial equilibrium $\mathbf{x} = 0$.

Denote by $\mathbf{x}(t) \triangleq \mathbf{x}(t;s,\varphi) = (x_1(t;s,\varphi_1),\ldots, x_n(t;s,\varphi_n))^T \in R^n$ the solution to (1) and (2) with the initial condition

$$x_i(s) = \varphi_i(s), \quad \vartheta \leq s \leq 0,\ i \in \mathcal{N}, \tag{3}$$

where $\varphi(s) = (\varphi_1(s),\ldots,\varphi_n(s))^T \in R^n$ and $\varphi_i(s) \in C[[\vartheta,0], R]$. Denote $|\varphi| = \sup_{s\in[\vartheta,0]}\|\varphi(s)\|$.

The solution $\mathbf{x}(t) \triangleq \mathbf{x}(t;\ s,\varphi) \in R^n$ of (1)–(3) is, for the time variable t, a piecewise continuous vector-valued function with the first kind discontinuity at the points t_k $(k=1,2,\ldots)$, where it is left continuous; that is, the following relations are valid:

$$\begin{aligned}&x_i(t_k-0) = x_i(t_k), \\ &x_i(t_k+0) = x_i(t_k) + I_{ik}\left(x_i(t_k)\right), \\ &\qquad i \in \mathcal{N},\ k = 1,2,\ldots.\end{aligned} \tag{4}$$

Definition 1. The trivial equilibrium $\mathbf{x} = 0$ is said to be stable, if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any initial condition $\varphi(s) \in C[[\vartheta,0],R^n]$ satisfying $|\varphi| < \delta$:

$$\left\|\mathbf{x}(t;s,\varphi)\right\| < \varepsilon, \quad t \geq 0. \tag{5}$$

Definition 2. The trivial equilibrium $\mathbf{x} = 0$ is said to be asymptotically stable if the trivial equilibrium $\mathbf{x} = 0$ is stable, and for any initial condition $\varphi(s) \in C[[\vartheta,0],R^n]$, $\lim_{t\to\infty}\|\mathbf{x}(t;s,\varphi)\| = 0$ holds.

The consideration of this paper is based on the following fixed point theorem.

Theorem 3 (see [30]). *Let Υ be a contraction operator on a complete metric space Θ, then there exists a unique point $\zeta \in \Theta$ for which $\Upsilon(\zeta) = \zeta$.*

3. Main Results

In this section, we will consider the existence and uniqueness of solution and the asymptotic stability of trivial equilibrium by means of the contraction mapping principle. Before proceeding, we introduce some assumptions listed as follows.

(A1) There exist nonnegative constants l_j such that, for any $\eta, \upsilon \in R$,

$$\left|f_j(\eta) - f_j(\upsilon)\right| \leq l_j\left|\eta - \upsilon\right|, \quad j \in \mathcal{N}. \tag{6}$$

(A2) There exist nonnegative constants k_j such that, for any $\eta, \upsilon \in R$,

$$\left|g_j(\eta) - g_j(\upsilon)\right| \leq k_j\left|\eta - \upsilon\right|, \quad j \in \mathcal{N}. \tag{7}$$

(A3) There exist nonnegative constants p_{jk} such that, for any $\eta, \upsilon \in R$,

$$\left|I_{jk}(\eta) - I_{jk}(\upsilon)\right| \leq p_{jk}\left|\eta - \upsilon\right|, \quad j \in \mathcal{N},\ k = 1,2,\ldots. \tag{8}$$

Let $\mathcal{H} = \mathcal{H}_1 \times \cdots \times \mathcal{H}_n$, and let $\mathcal{H}_i$ $(i \in \mathcal{N})$ be the space consisting of functions $\phi_i(t) : [\vartheta,\infty) \to R$, where $\phi_i(t)$ satisfies the following:

(1) $\phi_i(t)$ is continuous on $t \neq t_k$ $(k = 1,2,\ldots)$;

(2) $\lim_{t\to t_k^-}\phi_i(t)$ and $\lim_{t\to t_k^+}\phi_i(t)$ exist; furthermore, $\lim_{t\to t_k^-}\phi_i(t) = \phi_i(t_k)$ for $k = 1,2,\ldots$;

(3) $\phi_i(s) = \varphi_i(s)$ on $s \in [\vartheta,0]$;

(4) $\phi_i(t) \to 0$ as $t \to \infty$;

here t_k $(k = 1, 2, \ldots)$ and $\varphi_i(s)$ $(s \in [\vartheta, 0])$ are defined as shown in Section 2. Also $\mathscr{H}$ is a complete metric space when it is equipped with the following metric:

$$d\left(\overline{\mathbf{q}}(t), \overline{\mathbf{h}}(t)\right) = \sum_{i=1}^{n} \sup_{t \geq \vartheta} \left|q_i(t) - h_i(t)\right|, \tag{9}$$

where $\overline{\mathbf{q}}(t) = (q_1(t), \ldots, q_n(t)) \in \mathscr{H}$ and $\overline{\mathbf{h}}(t) = (h_1(t), \ldots, h_n(t)) \in \mathscr{H}$.

In what follows, we will give the main result of this paper.

Theorem 4. *Assume that conditions (A1)–(A3) hold. Provided that*

(i) *there exists a constant* μ *such that* $\inf_{k=1,2,\ldots}\{t_k - t_{k-1}\} \geq \mu$,

(ii) *there exist constants* p_i *such that* $p_{ik} \leq p_i\mu$ *for* $i \in \mathscr{N}$ *and* $k = 1, 2, \ldots$,

(iii) $\lambda^* \triangleq \sum_{i=1}^{n}\{(1/a_i)\max_{j\in\mathscr{N}}|b_{ij}l_j| + (1/a_i)\max_{j\in\mathscr{N}}|c_{ij}k_j|\} + \max_{i\in\mathscr{N}}\{p_i(\mu + (1/a_i))\} < 1$,

(iv) $\max_{i\in\mathscr{N}}\{\lambda_i\} < 1/\sqrt{n}$, *where* $\lambda_i = (1/a_i)\sum_{j=1}^{n}|b_{ij}l_j| + (1/a_i)\sum_{j=1}^{n}|c_{ij}k_j| + p_i(\mu + (1/a_i))$,

then the trivial equilibrium $\mathbf{x} = 0$ *is asymptotically stable.*

Proof. Multiplying both sides of (1) with $e^{a_i t}$ gives, for $t > 0$ and $t \neq t_k$,

$$\begin{aligned} de^{a_i t} x_i(t) &= e^{a_i t} dx_i(t) + a_i x_i(t) e^{a_i t} dt \\ &= e^{a_i t} \left\{ \sum_{j=1}^{n} b_{ij} f_j\left(x_j(t)\right) + \sum_{j=1}^{n} c_{ij} g_j\left(x_j\left(t - \tau_j(t)\right)\right) \right\} dt, \end{aligned} \tag{10}$$

which yields after integrating from $t_{k-1} + \varepsilon$ $(\varepsilon > 0)$ to $t \in (t_{k-1}, t_k)$ $(k = 1, 2, \ldots)$

$$\begin{aligned} x_i(t) e^{a_i t} = {} & x_i\left(t_{k-1} + \varepsilon\right) e^{a_i(t_{k-1}+\varepsilon)} \\ & + \int_{t_{k-1}+\varepsilon}^{t} e^{a_i s} \left\{ \sum_{j=1}^{n} b_{ij} f_j\left(x_j(s)\right) + \sum_{j=1}^{n} c_{ij} g_j\left(x_j\left(s - \tau_j(s)\right)\right) \right\} ds. \end{aligned} \tag{11}$$

Letting $\varepsilon \to 0$ in (11), we have

$$\begin{aligned} x_i(t) e^{a_i t} = {} & x_i\left(t_{k-1} + 0\right) e^{a_i t_{k-1}} \\ & + \int_{t_{k-1}}^{t} e^{a_i s} \times \left\{ \sum_{j=1}^{n} b_{ij} f_j\left(x_j(s)\right) + \sum_{j=1}^{n} c_{ij} g_j\left(x_j\left(s - \tau_j(s)\right)\right) \right\} ds, \end{aligned} \tag{12}$$

for $t \in (t_{k-1}, t_k)$ $(k = 1, 2, \ldots)$. Setting $t = t_k - \varepsilon$ $(\varepsilon > 0)$ in (12), we get

$$\begin{aligned} x_i\left(t_k - \varepsilon\right) e^{a_i(t_k - \varepsilon)} &= x_i\left(t_{k-1} + 0\right) e^{a_i t_{k-1}} \\ & + \int_{t_{k-1}}^{t_k - \varepsilon} e^{a_i s} \left\{ \sum_{j=1}^{n} b_{ij} f_j\left(x_j(s)\right) + \sum_{j=1}^{n} c_{ij} g_j\left(x_j\left(s - \tau_j(s)\right)\right) \right\} ds, \end{aligned} \tag{13}$$

which generates by letting $\varepsilon \to 0$

$$\begin{aligned} x_i\left(t_k - 0\right) e^{a_i t_k} = {} & x_i\left(t_{k-1} + 0\right) e^{a_i t_{k-1}} \\ & + \int_{t_{k-1}}^{t_k} e^{a_i s} \left\{ \sum_{j=1}^{n} b_{ij} f_j\left(x_j(s)\right) + \sum_{j=1}^{n} c_{ij} g_j\left(x_j\left(s - \tau_j(s)\right)\right) \right\} ds. \end{aligned} \tag{14}$$

Noting $x_i(t_k - 0) = x_i(t_k)$, (14) can be rearranged as

$$\begin{aligned} x_i\left(t_k\right) e^{a_i t_k} = {} & x_i\left(t_{k-1} + 0\right) e^{a_i t_{k-1}} \\ & + \int_{t_{k-1}}^{t_k} e^{a_i s} \left\{ \sum_{j=1}^{n} b_{ij} f_j\left(x_j(s)\right) + \sum_{j=1}^{n} c_{ij} g_j\left(x_j\left(s - \tau_j(s)\right)\right) \right\} ds. \end{aligned} \tag{15}$$

Combining (12) and (15), we reach that

$$x_i(t)e^{a_it} = x_i(t_{k-1}+0)e^{a_it_{k-1}} + \int_{t_{k-1}}^{t} e^{a_is}\left\{\sum_{j=1}^{n} b_{ij}f_j(x_j(s)) + \sum_{j=1}^{n} c_{ij}g_j(x_j(s-\tau_j(s)))\right\}ds \tag{16}$$

is true for $t \in (t_{k-1}, t_k]$ $(k = 1, 2, \ldots)$. Further,

$$\begin{aligned} x_i(t)e^{a_it} = {} & x_i(t_{k-1})e^{a_it_{k-1}} + \int_{t_{k-1}}^{t} e^{a_is}\left\{\sum_{j=1}^{n} b_{ij}f_j(x_j(s)) + \sum_{j=1}^{n} c_{ij}g_j(x_j(s-\tau_j(s)))\right\}ds \\ & + I_{i(k-1)}(x_i(t_{k-1}))e^{a_it_{k-1}} \end{aligned} \tag{17}$$

holds for $t \in (t_{k-1}, t_k]$ $(k = 1, 2, \cdots)$. Hence,

$$\begin{aligned} x_i(t_{k-1})e^{a_it_{k-1}} = {} & x_i(t_{k-2})e^{a_it_{k-2}} + \int_{t_{k-2}}^{t_{k-1}} e^{a_is}\left\{\sum_{j=1}^{n} b_{ij}f_j(x_j(s)) + \sum_{j=1}^{n} c_{ij}g_j(x_j(s-\tau_j(s)))\right\}ds \\ & + I_{i(k-2)}(x_i(t_{k-2}))e^{a_it_{k-2}}, \\ & \vdots \\ x_i(t_2)e^{a_it_2} = {} & x_i(t_1)e^{a_it_1} + \int_{t_1}^{t_2} e^{a_is}\left\{\sum_{j=1}^{n} b_{ij}f_j(x_j(s)) + \sum_{j=1}^{n} c_{ij}g_j(x_j(s-\tau_j(s)))\right\}ds \\ & + I_{i1}(x_i(t_1))e^{a_it_1}, \\ x_i(t_1)e^{a_it_1} = {} & \varphi_i(0) + \int_{0}^{t_1} e^{a_is}\left\{\sum_{j=1}^{n} b_{ij}f_j(x_j(s)) + \sum_{j=1}^{n} c_{ij}g_j(x_j(s-\tau_j(s)))\right\}ds, \end{aligned} \tag{18}$$

which produces, for $t > 0$,

$$\begin{aligned} x_i(t) = {} & \varphi_i(0)e^{-a_it} + e^{-a_it}\int_{0}^{t} e^{a_is}\left\{\sum_{j=1}^{n} b_{ij}f_j(x_j(s)) + \sum_{j=1}^{n} c_{ij}g_j(x_j(s-\tau_j(s)))\right\}ds \\ & + e^{-a_it}\sum_{0<t_k<t}\left\{I_{ik}(x_i(t_k))e^{a_it_k}\right\}. \end{aligned} \tag{19}$$

Note $x_i(0) = \varphi_i(0)$ in (19). We then define the following operator π acting on $\mathscr{H}$, for $\overline{\mathbf{y}}(t) = (y_1(t), \ldots, y_n(t)) \in \mathscr{H}$:

$$\pi(\overline{\mathbf{y}})(t) = (\pi(y_1)(t), \ldots, \pi(y_n)(t)), \tag{20}$$

where $\pi(y_i)(t) : [\vartheta, \infty) \to R\,(i \in \mathscr{N})$ obeys the rules as follows:

$$\begin{aligned} \pi(y_i)(t) = {} & \varphi_i(0)e^{-a_it} + e^{-a_it}\int_{0}^{t} e^{a_is}\left\{\sum_{j=1}^{n} b_{ij}f_j(y_j(s)) + \sum_{j=1}^{n} c_{ij}g_j(y_j(s-\tau_j(s)))\right\}ds \\ & + e^{-a_it}\sum_{0<t_k<t}\left\{I_{ik}(y_i(t_k))e^{a_it_k}\right\}, \end{aligned} \tag{21}$$

on $t \geq 0$ and $\pi(y_i)(s) = \varphi_i(s)$ on $s \in [\vartheta, 0]$.

The subsequent part is the application of the contraction mapping principle, which can be divided into two steps.

Step 1. We need to prove $\pi(\mathscr{H}) \subset \mathscr{H}$. Choosing $y_i(t) \in \mathscr{H}_i$ $(i \in \mathscr{N})$, it is necessary to testify $\pi(y_i)(t) \subset \mathscr{H}_i$.

First, since $\pi(y_i)(s) = \varphi_i(s)$ on $s \in [\vartheta, 0]$ and $\varphi_i(s) \in C[[\vartheta, 0], R]$, we know $\pi(y_i)(s)$ is continuous on $s \in [\vartheta, 0]$. For a fixed time $t > 0$, it follows from (21) that

$$\pi(y_i)(t+r) - \pi(y_i)(t) = Q_1 + Q_2 + Q_3 + Q_4, \tag{22}$$

where

$$Q_1 = \varphi_i(0)\, e^{-a_i(t+r)} - \varphi_i(0) \quad e^{-a_i t}, \tag{23}$$

$$\begin{aligned} Q_2 &= e^{-a_i(t+r)} \int_0^{t+r} e^{a_i s} \sum_{j=1}^{n} b_{ij} f_j\left(y_j(s)\right) \mathrm{d}s \\ &\quad - e^{-a_i t} \int_0^{t} e^{a_i s} \sum_{j=1}^{n} b_{ij} f_j\left(y_j(s)\right) \mathrm{d}s, \\ Q_3 &= e^{-a_i(t+r)} \int_0^{t+r} e^{a_i s} \sum_{j=1}^{n} c_{ij} g_j\left(y_j\left(s-\tau_j(s)\right)\right) \mathrm{d}s \\ &\quad - e^{-a_i t} \int_0^{t} e^{a_i s} \sum_{j=1}^{n} c_{ij} g_j\left(y_j\left(s-\tau_j(s)\right)\right) \mathrm{d}s, \\ Q_4 &= e^{-a_i(t+r)} \sum_{0<t_k<(t+r)} \left\{ I_{ik}\left(y_i(t_k)\right) e^{a_i t_k} \right\} \\ &\quad - e^{-a_i t} \sum_{0<t_k<t} \left\{ I_{ik}\left(y_i(t_k)\right) e^{a_i t_k} \right\}. \end{aligned} \tag{24}$$

Owing to $y_i(t) \in \mathscr{H}_i$, we see that $y_i(t)$ is continuous on $t \neq t_k$ $(k = 1, 2, \ldots)$; moreover, $\lim_{t\to t_k^-} y_i(t)$ and $\lim_{t\to t_k^+} y_i(t)$ exist, and $\lim_{t\to t_k^-} y_i(t) = y_i(t_k)$.

Consequently, when $t \neq t_k$ $(k = 1, 2, \ldots)$ in (22), it is easy to find that $Q_i \to 0$ as $r \to 0$ for $i = 1, \ldots, 4$, and so $\pi(y_i)(t)$ is continuous on the fixed time $t \neq t_k$ $(k = 1, 2, \ldots)$.

On the other hand, as $t = t_k$ $(k = 1, 2, \ldots)$ in (22), it is not difficult to find that $Q_i \to 0$ as $r \to 0$ for $i = 1, 2, 3$. Furthermore, if letting $r < 0$ be small enough, we derive

$$\begin{aligned} Q_4 &= e^{-a_i(t_k+r)} \sum_{0<t_m<(t_k+r)} I_{im}\left(y_i(t_m)\right) e^{a_i t_m} \\ &\quad - e^{-a_i t_k} \sum_{0<t_m<t_k} I_{im}\left(y_i(t_m)\right) e^{a_i t_m} \\ &= \left\{ e^{-a_i(t_k+r)} - e^{-a_i t_k} \right\} \\ &\quad \times \sum_{0<t_m<t_k} \left\{ I_{im}\left(y_i(t_m)\right) e^{a_i t_m} \right\}, \end{aligned} \tag{25}$$

which implies $\lim_{r\to 0^-} Q_4 = 0$ as $t = t_k$. While letting $r > 0$ tend to zero gives

$$\begin{aligned} Q_4 &= e^{-a_i(t_k+r)} \sum_{0<t_m<(t_k+r)} I_{im}\left(y_i(t_m)\right) e^{a_i t_m} \\ &\quad - e^{-a_i t_k} \sum_{0<t_m<t_k} I_{im}\left(y_i(t_m)\right) e^{a_i t_m} \\ &= e^{-a_i(t_k+r)} \left\{ \sum_{0<t_m<t_k} \left\{ I_{im}\left(y_i(t_m)\right) e^{a_i t_m} \right\} \right. \\ &\quad \left. + I_{ik}\left(y_i(t_k)\right) e^{a_i t_k} \right\} \\ &\quad - e^{-a_i t_k} \sum_{0<t_m<t_k} \left\{ I_{im}\left(y_i(t_m)\right) e^{a_i t_m} \right\} \\ &= \left\{ e^{-a_i(t_k+r)} - e^{-a_i t_k} \right\} \\ &\quad \times \sum_{0<t_m<t_k} \left\{ I_{im}\left(y_i(t_m)\right) e^{a_i t_m} \right\} \\ &\quad + e^{-a_i(t_k+r)} I_{ik}\left(y_i(t_k)\right) e^{a_i t_k}, \end{aligned} \tag{26}$$

which yields $\lim_{r\to 0^+} Q_4 = e^{-a_i t_k} I_{ik}(y_i(t_k)) e^{a_i t_k}$ as $t = t_k$.

According to the above discussion, we find that $\pi(y_i)(t) : [\vartheta, \infty) \to R$ is continuous on $t \neq t_k$ $(k = 1, 2, \ldots)$; moreover, $\lim_{t\to t_k^-} \pi(y_i)(t)$ and $\lim_{t\to t_k^+} \pi(y_i)(t)$ exist; in addition, $\lim_{t\to t_k^-} \pi(y_i)(t) = \pi(y_i)(t_k) \neq \lim_{t\to t_k^+} \pi(y_i)(t)$.

Next, we will prove $\pi(y_i)(t) \to 0$ as $t \to \infty$. For convenience, denote

$$\pi\left(y_i\right)(t) = J_1 + J_2 + J_3 + J_4, \quad t > 0, \tag{27}$$

where $J_1 = \varphi_i(0) e^{-a_i t}$, $J_2 = e^{-a_i t} \int_0^t e^{a_i s} \sum_{j=1}^n b_{ij} f_j(y_j(s)) \mathrm{d}s$, $J_4 = e^{-a_i t} \sum_{0<t_k<t} \{I_{ik}(y_i(t_k)) e^{a_i t_k}\}$, and $J_3 = e^{-a_i t} \int_0^t e^{a_i s} \sum_{j=1}^n c_{ij} g_j(y_j(s-\tau_j(s))) \mathrm{d}s$.

Due to $y_j(t) \in \mathscr{H}_j$ $(j \in \mathscr{N})$, we know $\lim_{t\to\infty} y_j(t) = 0$. Then for any $\varepsilon > 0$, there exists a $T_j > 0$ such that $t \geq T_j$ implies $|y_j(t)| < \varepsilon$. Choose $T^* = \max_{j\in\mathscr{N}}\{T_j\}$. It is derived from (A1) that, for $t \geq T^*$,

$$\begin{aligned} J_2 &\leq e^{-a_i t} \int_0^t e^{a_i s} \sum_{j=1}^{n} \left\{ \left| b_{ij} l_j \right| \left| y_j(s) \right| \right\} \mathrm{d}s \\ &= e^{-a_i t} \int_0^{T^*} e^{a_i s} \sum_{j=1}^{n} \left\{ \left| b_{ij} l_j \right| \left| y_j(s) \right| \right\} \mathrm{d}s \\ &\quad + e^{-a_i t} \int_{T^*}^{t} e^{a_i s} \sum_{j=1}^{n} \left\{ \left| b_{ij} l_j \right| \left| y_j(s) \right| \right\} \mathrm{d}s \\ &\leq e^{-a_i t} \sum_{j=1}^{n} \left\{ \left| b_{ij} l_j \right| \sup_{s\in[0,T^*]} \left| y_j(s) \right| \right\} \left\{ \int_0^{T^*} e^{a_i s} \mathrm{d}s \right\} \\ &\quad + \varepsilon \sum_{j=1}^{n} \left\{ \left| b_{ij} l_j \right| \right\} e^{-a_i t} \int_{T^*}^{t} e^{a_i s} \mathrm{d}s \end{aligned}$$

$$\le e^{-a_i t}\sum_{j=1}^{n}\left\{\left|b_{ij}l_j\right|\sup_{s\in[0,T^*]}\left|y_j(s)\right|\right\}\times\left\{\int_0^{T^*}e^{a_i s}\mathrm{d}s\right\}+\frac{\varepsilon}{a_i}\sum_{j=1}^{n}\left\{\left|b_{ij}l_j\right|\right\}. \tag{28}$$

Moreover, as $\lim_{t\to\infty}e^{-a_i t}=0$, we can find a $\overline{\overline{T}}>0$ for the given ε such that $t\ge\overline{\overline{T}}$ implies $e^{-a_i t}<\varepsilon$, which leads to

$$J_2\le\varepsilon\left\{\sum_{j=1}^{n}\left\{\left|b_{ij}l_j\right|\sup_{s\in[0,T^*]}\left|y_j(s)\right|\right\}\times\left\{\int_0^{T^*}e^{a_i s}ds\right\}+\frac{1}{a_i}\sum_{j=1}^{n}\left\{\left|b_{ij}l_j\right|\right\}\right\},\quad t\ge\max\left\{T^*,\overline{\overline{T}}\right\}; \tag{29}$$

namely,

$$J_2\longrightarrow 0\quad\text{as } t\longrightarrow\infty. \tag{30}$$

On the other hand, since $t-\tau_j(t)\to\infty$ as $t\to\infty$, we get $\lim_{t\to\infty}y_j(t-\tau_j(t))=0$. Then for any $\varepsilon>0$, there also exists a $T_j'>0$ such that $s\ge T_j'$ implies $|y_j(s-\tau_j(s))|<\varepsilon$. Select $\overline{T}=\max_{j\in\mathcal{N}}\{T_j'\}$. It follows from (A2) that

$$\begin{aligned}J_3&\le e^{-a_i t}\int_0^t e^{a_i s}\sum_{j=1}^{n}\left\{\left|c_{ij}k_j\right|\left|y_j\left(s-\tau_j(s)\right)\right|\right\}\mathrm{d}s\\&=e^{-a_i t}\int_0^{\overline{T}} e^{a_i s}\sum_{j=1}^{n}\left\{\left|c_{ij}k_j\right|\left|y_j\left(s-\tau_j(s)\right)\right|\right\}\mathrm{d}s\\&\quad+e^{-a_i t}\int_{\overline{T}}^t e^{a_i s}\sum_{j=1}^{n}\left\{\left|c_{ij}k_j\right|\left|y_j\left(s-\tau_j(s)\right)\right|\right\}\mathrm{d}s\\&\le\sum_{j=1}^{n}\left\{\left|c_{ij}k_j\right|\sup_{s\in[\vartheta,\overline{T}]}\left|y_j(s)\right|\right\}e^{-a_i t}\int_0^{\overline{T}}e^{a_i s}\mathrm{d}s\\&\quad+\varepsilon\sum_{j=1}^{n}\left\{\left|c_{ij}k_j\right|\right\}e^{-a_i t}\int_{\overline{T}}^t e^{a_i s}\mathrm{d}s\\&\le e^{-a_i t}\sum_{j=1}^{n}\left\{\left|c_{ij}k_j\right|\sup_{s\in[\vartheta,\overline{T}]}\left|y_j(s)\right|\right\}\int_0^{\overline{T}}e^{a_i s}\mathrm{d}s\\&\quad+\frac{\varepsilon}{a_i}\sum_{j=1}^{n}\left\{\left|c_{ij}k_j\right|\right\},\end{aligned} \tag{31}$$

which results in

$$J_3\longrightarrow 0\quad\text{as } t\longrightarrow\infty. \tag{32}$$

Furthermore, from (A3), we know that $|I_{ik}(y_i(t_k))|\le p_{ik}|y_i(t_k)|$. So

$$J_4\le e^{-a_i t}\sum_{0<t_k<t}\left\{p_{ik}\left|y_i(t_k)\right|e^{a_i t_k}\right\}. \tag{33}$$

As $y_i(t)\in\mathscr{H}_i$, we have $\lim_{t\to\infty}y_i(t)=0$. Then for any $\varepsilon>0$, there exists a nonimpulsive point $T_i>0$ such that $s\ge T_i$ implies $|y_i(s)|<\varepsilon$. It then follows from conditions (i) and (ii) that

$$\begin{aligned}J_4&\le e^{-a_i t}\left\{\sum_{0<t_k<T_i}\left\{p_{ik}\left|y_i(t_k)\right|e^{a_i t_k}\right\}+\sum_{T_i<t_k<t}\left\{p_{ik}\left|y_i(t_k)\right|e^{a_i t_k}\right\}\right\}\\&\le e^{-a_i t}\sum_{0<t_k<T_i}\left\{p_{ik}\left|y_i(t_k)\right|e^{a_i t_k}\right\}+e^{-a_i t}p_i\varepsilon\sum_{T_i<t_k<t}\left\{\mu e^{a_i t_k}\right\}\\&\le e^{-a_i t}\sum_{0<t_k<T_i}\left\{p_{ik}\left|y_i(t_k)\right|e^{a_i t_k}\right\}+e^{-a_i t}p_i\varepsilon\left\{\sum_{T_i<t_r<t_k}\left\{e^{a_i t_r}\left(t_{r+1}-t_r\right)\right\}+\mu e^{a_i t_k}\right\}\\&\le e^{-a_i t}\sum_{0<t_k<T_i}\left\{p_{ik}\left|y_i(t_k)\right|e^{a_i t_k}\right\}+e^{-a_i t}p_i\varepsilon\left(\int_{T_i}^t e^{a_i s}\mathrm{d}s+\mu e^{a_i t}\right)\\&\le e^{-a_i t}\sum_{0<t_k<T_i}\left\{p_{ik}\left|y_i(t_k)\right|e^{a_i t_k}\right\}+\frac{\varepsilon p_i}{a_i}+p_i\varepsilon\mu,\end{aligned} \tag{34}$$

which produces

$$J_4\longrightarrow 0\quad\text{as } t\longrightarrow\infty. \tag{35}$$

From (30), (32), and (35), we deduce $\pi(y_i)(t)\to 0$ as $t\to\infty$ for $i\in\mathcal{N}$. We therefore conclude that $\pi(y_i)(t)\subset\mathscr{H}_i$ ($i\in\mathcal{N}$) which means $\pi(\mathscr{H})\subset\mathscr{H}$.

Step 2. We need to prove π is contractive. For $\overline{\mathbf{y}}=(y_1(t),\dots,y_n(t))\in\mathscr{H}$ and $\overline{z}=(z_1(t),\dots,z_n(t))\in\mathscr{H}$, we estimate

$$\left|\pi(y_i)(t)-\pi(z_i)(t)\right|\le I_1+I_2+I_3, \tag{36}$$

where $I_1 = e^{-a_i t}\int_0^t e^{a_i s}\sum_{j=1}^n[|b_{ij}||f_j(y_j(s)) - f_j(z_j(s))|]\,\mathrm{d}s$, $I_3 = e^{-a_i t}\sum_{0<t_k<t}\{e^{a_i t_k}|I_{ik}(y_i(t_k)) - I_{ik}(z_i(t_k))|\}$, and $I_2 = e^{-a_i t}\int_0^t e^{a_i s}\sum_{j=1}^n[|c_{ij}|\ |g_j(y_j(s-\tau_j(s))) - g_j(z_j(s-\tau_j(s)))|]\mathrm{d}s$.

Note

$$\begin{aligned}
I_1 &\leq e^{-a_i t}\int_0^t e^{a_i s}\sum_{j=1}^n\left[\left|b_{ij}l_j\right|\left|y_j(s)-z_j(s)\right|\right]\mathrm{d}s\\
&\leq \max_{j\in\mathcal{N}}\left|b_{ij}l_j\right|\sum_{j=1}^n\left\{\sup_{s\in[0,t]}\left|y_j(s)-z_j(s)\right|\right\}e^{-a_i t}\int_0^t e^{a_i s}\mathrm{d}s\\
&\leq \frac{1}{a_i}\max_{j\in\mathcal{N}}\left|b_{ij}l_j\right|\sum_{j=1}^n\left\{\sup_{s\in[0,t]}\left|y_j(s)-z_j(s)\right|\right\},\\
I_2 &\leq e^{-a_i t}\int_0^t e^{a_i s}\\
&\quad\times\sum_{j=1}^n\left[\left|c_{ij}k_j\right|\left|y_j\left(s-\tau_j(s)\right)\right.\right.\\
&\qquad\qquad\left.\left.-z_j\left(s-\tau_j(s)\right)\right|\right]\mathrm{d}s\\
&\leq \max_{j\in\mathcal{N}}\left|c_{ij}k_j\right|\sum_{j=1}^n\left\{\sup_{s\in[\vartheta,t]}\left|y_j(s)-z_j(s)\right|\right\}e^{-a_i t}\int_0^t e^{a_i s}\mathrm{d}s\\
&\leq \frac{1}{a_i}\max_{j\in\mathcal{N}}\left|c_{ij}k_j\right|\sum_{j=1}^n\left\{\sup_{s\in[\vartheta,t]}\left|y_j(s)-z_j(s)\right|\right\},\\
I_3 &\leq e^{-a_i t}\sum_{0<t_k<t}\left\{e^{a_i t_k}p_{ik}\left|y_i(t_k)-z_i(t_k)\right|\right\}\\
&\leq p_i e^{-a_i t}\sup_{s\in[0,t]}\left|y_i(s)-z_i(s)\right|\sum_{0<t_k<t}\left\{e^{a_i t_k}\mu\right\}\\
&\leq p_i e^{-a_i t}\sup_{s\in[0,t]}\left|y_i(s)-z_i(s)\right|\\
&\quad\times\left\{\sum_{0<t_r<t_k}\left\{e^{a_i t_r}\left(t_{r+1}-t_r\right)\right\}+e^{a_i t_k}\mu\right\}\\
&\leq p_i\sup_{s\in[0,t]}\left|y_i(s)-z_i(s)\right|\ e^{-a_i t}\\
&\quad\times\left\{\int_0^t e^{a_i s}\mathrm{d}s+e^{a_i t}\mu\right\}\\
&\leq p_i\left(\mu+\frac{1}{a_i}\right)\sup_{s\in[0,t]}\left|y_i(s)-z_i(s)\right|.
\end{aligned}\tag{37}$$

It hence follows from (37) that

$$\begin{aligned}
&\left|\pi\left(y_i\right)(t)-\pi\left(z_i\right)(t)\right|\\
&\quad\leq\frac{1}{a_i}\max_{j\in\mathcal{N}}\left|b_{ij}l_j\right|\\
&\quad\quad\times\sum_{j=1}^n\left\{\sup_{s\in[0,t]}\left|y_j(s)-z_j(s)\right|\right\}\\
&\quad\quad+\frac{1}{a_i}\max_{j\in\mathcal{N}}\left|c_{ij}k_j\right|\sum_{j=1}^n\left\{\sup_{s\in[\vartheta,t]}\left|y_j(s)-z_j(s)\right|\right\}\\
&\quad\quad+p_i\left(\mu+\frac{1}{a_i}\right)\sup_{s\in[0,t]}\left|y_i(s)-z_i(s)\right|,
\end{aligned}\tag{38}$$

which implies

$$\begin{aligned}
&\sup_{t\in[\vartheta,T]}\left|\pi\left(y_i\right)(t)-\pi\left(z_i\right)(t)\right|\\
&\quad\leq\frac{1}{a_i}\max_{j\in\mathcal{N}}\left|b_{ij}l_j\right|\sum_{j=1}^n\left\{\sup_{s\in[\vartheta,T]}\left|y_j(s)-z_j(s)\right|\right\}\\
&\quad\quad+\frac{1}{a_i}\max_{j\in\mathcal{N}}\left|c_{ij}k_j\right|\sum_{j=1}^n\left\{\sup_{s\in[\vartheta,T]}\left|y_j(s)-z_j(s)\right|\right\}\\
&\quad\quad+p_i\left(\mu+\frac{1}{a_i}\right)\sup_{s\in[\vartheta,T]}\left|y_i(s)-z_i(s)\right|.
\end{aligned}\tag{39}$$

Therefore,

$$\begin{aligned}
&\sum_{i=1}^n\sup_{t\in[-\tau,T]}\left|\pi\left(y_i\right)(t)-\pi\left(z_i\right)(t)\right|\\
&\quad\leq\lambda^*\sum_{j=1}^n\left\{\sup_{s\in[\vartheta,T]}\left|y_j(s)-z_j(s)\right|\right\}.
\end{aligned}\tag{40}$$

In view of condition (iii), we see π is a contraction mapping, and, thus there exists a unique fixed point $\overline{\mathbf{y}}^*(\cdot)$ of π in $\mathcal{H}$ which means the transposition of $\overline{\mathbf{y}}^*(\cdot)$ is the vector-valued solution to (1)–(3) and its norm tends to zero as $t\to\infty$.

To obtain the asymptotic stability, we still need to prove that the trivial equilibrium $\mathbf{x}=0$ is stable. For any $\varepsilon>0$, from condition (iv), we can find δ satisfying $0<\delta<\varepsilon$ such that $\delta+\max_{i\in\mathcal{N}}\{\lambda_i\}\varepsilon\leq\varepsilon/\sqrt{n}$. Let $|\varphi|<\delta$. According to what has been discussed above, we know that there exists a unique solution $\mathbf{x}(t;s,\varphi)=(x_1(t;s,\varphi_1),\ldots,x_n(t;s,\varphi_n))^T$ to (1)–(3); moreover,

$$x_i(t)=\pi\left(x_i\right)(t)=J_1+J_2+J_3+J_4,\quad t\geq 0;\tag{41}$$

here $J_1=\varphi_i(0)e^{-a_i t}$, $J_2=e^{-a_i t}\int_0^t e^{a_i s}\sum_{j=1}^n b_{ij}f_j(x_j(s))\mathrm{d}s$, $J_3=e^{-a_i t}\int_0^t e^{a_i s}\sum_{j=1}^n c_{ij}g_j(x_j(s-\tau_j(s)))\mathrm{d}s$, and $J_4=e^{-a_i t}\sum_{0<t_k<t}\{I_{ik}(x_i(t_k))e^{a_i t_k}\}$.

Suppose there exists $t^*>0$ such that $\|\mathbf{x}(t^*;s,\varphi)\|=\varepsilon$ and $\|\mathbf{x}(t;s,\varphi)\|<\varepsilon$ as $0\leq t<t^*$. It follows from (41) that

$$\left|x_i\left(t^*\right)\right|\leq\left|J_1\left(t^*\right)\right|+\left|J_2\left(t^*\right)\right|+\left|J_3\left(t^*\right)\right|+\left|J_4\ \left(t^*\right)\right|.\tag{42}$$

As

$$\begin{aligned}
|J_1(t^*)| &= \left|\varphi_i(0)\, e^{-a_i t^*}\right| \le \delta, \\
|J_2(t^*)| &\le e^{-a_i t^*} \int_0^{t^*} e^{a_i s} \sum_{j=1}^{n} \left|b_{ij} l_j x_j(s)\right| \mathrm{d}s \\
&< \frac{\varepsilon}{a_i} \sum_{j=1}^{n} \left|b_{ij} l_j\right|, \\
|J_3(t^*)| &\le e^{-a_i t^*} \int_0^{t^*} e^{a_i s} \\
&\quad \times \sum_{j=1}^{n} \left|c_{ij} k_j x_j\left(s - \tau_j(s)\right)\right| \mathrm{d}s \\
&< \frac{\varepsilon}{a_i} \sum_{j=1}^{n} \left|c_{ij} k_j\right|, \\
|J_4(t^*)| &\le p_i e^{-a_i t^*} \sum_{0<t_k<t^*} \left\{\mu \left|x_i(t_k)\right| e^{a_i t_k}\right\} \\
&< \varepsilon p_i e^{-a_i t^*} \left\{\int_0^{t^*} e^{a_i s} ds + \mu e^{a_i t^*}\right\} \\
&\le \varepsilon p_i \left(\mu + \frac{1}{a_i}\right),
\end{aligned} \tag{43}$$

we obtain $|x_i(t^*)| < \delta + \lambda_i \varepsilon$.

So $\|\mathbf{x}(t^*; s, \varphi)\|^2 = \sum_{i=1}^{n}\{|x_i(t^*)|^2\} < \sum_{i=1}^{n}\{|\delta + \lambda_i \varepsilon|^2\} \le n|\delta + \max_{i\in\mathcal{N}}\{\lambda_i\}\varepsilon|^2 \le \varepsilon^2$. This contradicts the assumption of $\|\mathbf{x}(t^*; s, \varphi)\| = \varepsilon$. Therefore, $\|\mathbf{x}(t; s, \varphi)\| < \varepsilon$ holds for all $t \ge 0$. This completes the proof. □

Corollary 5. *Assume that conditions (A1)–(A3) hold. Provided that*

(i) $\inf_{k=1,2,\ldots}\{t_k - t_{k-1}\} \ge 1$,

(ii) *there exist constants p_i such that $p_{ik} \le p_i$ for $i \in \mathcal{N}$ and $k = 1, 2, \ldots$,*

(iii) $\sum_{i=1}^{n}\{(1/a_i)\max_{j\in\mathcal{N}}|b_{ij} l_j| + (1/a_i)\max_{j\in\mathcal{N}}|c_{ij} k_j|\} + \max_{i\in\mathcal{N}}\{p_i(1 + (1/a_i))\} < 1$,

(iv) $\max_{i\in\mathcal{N}}\{\lambda_i'\} < 1/\sqrt{n}$, *where* $\lambda_i' = (1/a_i)\sum_{j=1}^{n}|b_{ij} l_j| + (1/a_i)\sum_{j=1}^{n}|c_{ij} k_j| + p_i(1 + (1/a_i))$,

then the trivial equilibrium $\mathbf{x} = 0$ *is asymptotically stable.*

Proof. Corollary 5 is a direct conclusion by letting $\mu = 1$ in Theorem 4. □

Remark 6. In Theorem 4, we can see it is the fixed point theory that deals with the existence and uniqueness of solution and the asymptotic analysis of trivial equilibrium at the same time, while Lyapunov method fails to do this.

Remark 7. The presented sufficient conditions in Theorems 4 and Corollary 5 do not require even the boundedness and differentiability of delays, let alone the monotone decreasing behavior of delays which is necessary in some relevant works.

Provided that $I_{ik}(\cdot) \equiv 0$, (1) and (2) will become the following cellular neural network with infinite delays and without impulsive effects:

$$\begin{aligned}
\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} &= -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j\left(x_j(t)\right) \\
&\quad + \sum_{j=1}^{n} c_{ij} g_j\left(x_j\left(t - \tau_j(t)\right)\right), \\
&\qquad i \in \mathcal{N},\ t \ge 0,
\end{aligned} \tag{44}$$

where a_i, b_{ij}, c_{ij}, $f_j(\cdot)$, $g_j(\cdot)$, $\tau_j(t)$, and $x_i(t)$ are the same as defined in Section 2. Obviously, (44) also admits a trivial equilibrium $\mathbf{x} = 0$. From Theorem 4, we reach the following.

Theorem 8. *Assume that conditions (A1)-(A2) hold. Provided that*

(i) $\sum_{i=1}^{n}\{(1/a_i)\max_{j\in\mathcal{N}}|b_{ij} l_j| + (1/a_i)\max_{j\in\mathcal{N}}|c_{ij} k_j|\} < 1$,

(ii) $\max_{i\in\mathcal{N}}\{\lambda_i''\} < 1/\sqrt{n}$, *where* $\lambda_i'' = (1/a_i)\sum_{j=1}^{n}|b_{ij} l_j| + (1/a_i)\sum_{j=1}^{n}|c_{ij} k_j|$,

then the trivial equilibrium $\mathbf{x} = 0$ *is asymptotically stable.*

4. Example

Consider the following two-dimensional impulsive cellular neural network with infinite delays:

$$\begin{aligned}
\frac{\mathrm{d}x_i(t)}{\mathrm{d}t} &= -a_i x_i(t) + \sum_{j=1}^{2} b_{ij} f_j\left(x_j(t)\right) \\
&\quad + \sum_{j=1}^{2} c_{ij} g_j\left(x_j\left(t - \tau_j(t)\right)\right), \\
&\qquad t \ge 0,\ t \ne t_k, \\
\Delta x_i(t_k) &= x_i(t_k + 0) - x_i(t_k) \\
&= \arctan\left(0.4 x_i(t_k)\right), \quad k = 1, 2, \ldots,
\end{aligned} \tag{45}$$

with the initial conditions $x_1(s) = \cos(s)$, $x_2(s) = \sin(s)$ on $-1 \le s \le 0$, where $\tau_j(t) = 0.4t + 1$, $a_1 = a_2 = 7$, $b_{ij} = 0$, $c_{11} = 3/7$, $c_{12} = 2/7$, $c_{21} = 0$, $c_{22} = 1/7$, $f_j(s) = g_j(s) = (|s + 1| - |s - 1|)/2$, and $t_k = t_{k-1} + 0.5k$.

It is easy to see that $\mu = 0.5$, $l_j = k_j = 1$, and $p_{ik} = 0.4$. Let $p_i = 0.8$ and compute

$$\begin{aligned}
&\sum_{i=1}^{2} \left\{\frac{1}{a_i}\max_{j=1,2}\left|c_{ij} k_j\right|\right\} + \max_{i=1,2}\left\{p_i\left(\mu + \frac{1}{a_i}\right)\right\} < 1, \\
&\max_{i\in\mathcal{N}}\{\lambda_i\} < \frac{1}{\sqrt{2}},
\end{aligned} \tag{46}$$

where $\lambda_i = (1/a_i)\sum_{j=1}^{n}|c_{ij}k_j|+p_i(\mu+(1/a_i))$. From Theorem 4, we conclude that the trivial equilibrium $\mathbf{x} = 0$ of this two-dimensional impulsive cellular neural network with infinite delays is asymptotically stable.

5. Conclusions

This work is devoted to seeking new methods to investigate the stability of complex neural networks. From what has been discussed above, we find that the fixed point theory is feasible. With regard to a class of impulsive cellular neural networks with infinite delays, we utilize the contraction mapping principle to deal with the existence and uniqueness of solution and the asymptotic analysis of trivial equilibrium at the same time, for which Lyapunov method feels helpless. Now that there are different kinds of fixed point theorems and complex neural networks, our future work is to continue the study on the application of fixed point theory to the stability analysis of complex neural networks.

Acknowledgment

This work is supported by the National Natural Science Foundation of China under Grants 60904028, 61174077, and 41105057.

References

[1] L. O. Chua and L. Yang, "Cellular neural networks: theory," *IEEE Transactions on Circuits and Systems*, vol. 35, no. 10, pp. 1257–1272, 1988.

[2] L. O. Chua and L. Yang, "Cellular neural networks: applications," *IEEE Transactions on Circuits and Systems*, vol. 35, no. 10, pp. 1273–1290, 1988.

[3] G. T. Stamov and I. M. Stamova, "Almost periodic solutions for impulsive neural networks with delay," *Applied Mathematical Modelling*, vol. 31, no. 7, pp. 1263–1270, 2007.

[4] S. Ahmad and I. M. Stamova, "Global exponential stability for impulsive cellular neural networks with time-varying delays," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 3, pp. 786–795, 2008.

[5] K. Li, X. Zhang, and Z. Li, "Global exponential stability of impulsive cellular neural networks with time-varying and distributed delay," *Chaos, Solitons and Fractals*, vol. 41, no. 3, pp. 1427–1434, 2009.

[6] J. Qiu, "Exponential stability of impulsive neural networks with time-varying delays and reaction-diffusion terms," *Neurocomputing*, vol. 70, no. 4–6, pp. 1102–1108, 2007.

[7] X. Wang and D. Xu, "Global exponential stability of impulsive fuzzy cellular neural networks with mixed delays and reaction-diffusion terms," *Chaos, Solitons and Fractals*, vol. 42, no. 5, pp. 2713–2721, 2009.

[8] Y. Zhang and Q. Luo, "Global exponential stability of impulsive delayed reaction-diffusion neural networks via Hardy-Poincarè inequality," *Neurocomputing*, vol. 83, pp. 198–204, 2012.

[9] Y. Zhang and Q. Luo, "Novel stability criteria for impulsive delayed reaction-diffusion Cohen-Grossberg neural networks via Hardy-Poincarè inequality," *Chaos, Solitons and Fractals*, vol. 45, no. 8, pp. 1033–1040, 2012.

[10] T. A. Burton, *Stability by Fixed Point Theory for Functional Differential Equations*, Dover, New York, NY, USA, 2006.

[11] L. C. Becker and T. A. Burton, "Stability, fixed points and inverses of delays," *Proceedings of the Royal Society of Edinburgh A*, vol. 136, no. 2, pp. 245–275, 2006.

[12] T. A. Burton, "Fixed points, stability, and exact linearization," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 61, no. 5, pp. 857–870, 2005.

[13] T. A. Burton, "Fixed points, Volterra equations, and Becker's resolvent," *Acta Mathematica Hungarica*, vol. 108, no. 3, pp. 261–281, 2005.

[14] T. A. Burton, "Fixed points and stability of a nonconvolution equation," *Proceedings of the American Mathematical Society*, vol. 132, no. 12, pp. 3679–3687, 2004.

[15] T. A. Burton, "Perron-type stability theorems for neutral equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 55, no. 3, pp. 285–297, 2003.

[16] T. A. Burton, "Integral equations, implicit functions, and fixed points," *Proceedings of the American Mathematical Society*, vol. 124, no. 8, pp. 2383–2390, 1996.

[17] T. A. Burton and T. Furumochi, "Krasnoselskii's fixed point theorem and stability," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 49, no. 4, pp. 445–454, 2002.

[18] T. A. Burton and B. Zhang, "Fixed points and stability of an integral equation: nonuniqueness," *Applied Mathematics Letters*, vol. 17, no. 7, pp. 839–846, 2004.

[19] T. Furumochi, "Stabilities in FDEs by Schauder's theorem," *Nonlinear Analysis, Theory, Methods & Applications*, vol. 63, no. 5–7, pp. e217–e224, 2005.

[20] C. Jin and J. Luo, "Fixed points and stability in neutral differential equations with variable delays," *Proceedings of the American Mathematical Society*, vol. 136, no. 3, pp. 909–918, 2008.

[21] Y. N. Raffoul, "Stability in neutral nonlinear differential equations with functional delays using fixed-point theory," *Mathematical and Computer Modelling*, vol. 40, no. 7-8, pp. 691–700, 2004.

[22] B. Zhang, "Fixed points and stability in differential equations with variable delays," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 63, no. 5–7, pp. e233–e242, 2005.

[23] J. Luo, "Fixed points and stability of neutral stochastic delay differential equations," *Journal of Mathematical Analysis & Applications*, vol. 334, no. 1, pp. 431–440, 2007.

[24] J. Luo, "Fixed points and exponential stability of mild solutions of stochastic partial differential equations with delays," *Journal of Mathematical Analysis and Applications*, vol. 342, no. 2, pp. 753–760, 2008.

[25] J. Luo, "Stability of stochastic partial differential equations with infinite delays," *Journal of Computational and Applied Mathematics*, vol. 222, no. 2, pp. 364–371, 2008.

[26] J. Luo and T. Taniguchi, "Fixed points and stability of stochastic neutral partial differential equations with infinite delays," *Stochastic Analysis and Applications*, vol. 27, no. 6, pp. 1163–1173, 2009.

[27] R. Sakthivel and J. Luo, "Asymptotic stability of impulsive stochastic partial differential equations with infinite delays," *Journal of Mathematical Analysis and Applications*, vol. 356, no. 1, pp. 1–6, 2009.

[28] R. Sakthivel and J. Luo, "Asymptotic stability of nonlinear impulsive stochastic differential equations," *Statistics & Probability Letters*, vol. 79, no. 9, pp. 1219–1223, 2009.

[29] J. Luo, "Fixed points and exponential stability for stochastic Volterra-Levin equations," *Journal of Computational and Applied Mathematics*, vol. 234, no. 3, pp. 934–940, 2010.

[30] D. R. Smart, *Fixed Point Theorems*, Cambridge University Press, Cambridge, UK, 1980.