Abstract and Applied Analysis

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Guest Editors: Jifeng Chu, Juntao Sun, Patricia J. Y. Wong, and Yonghui Xia

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Editorial Board

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Editorial

Dynamical Aspects of Initial/Boundary Value Problems for Ordinary Differential Equations

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Dynamical aspects of initial/boundary value problems for ordinary differential equations have become a rapidly growing area of research in the theory of differential equations and dynamical systems and have gathered substantial research interests during the last decades. The attractiveness of this field not only is derived from theoretical interests but also is motivated by the insights that such dynamical aspects could reveal in several phenomena observed in applied sciences.

The current special issue places its emphasis on the study of the dynamical aspects of initial/boundary value problems for ordinary differential equations. Call for papers has been carefully prepared by the guest editors and posted on the journals web page, which has attracted many researchers to submit their contribution on wide topics such as oscillation theory, delay differential equation, impulsive differential equation, multipoint boundary value problems, stochastic mutualism system, chaotic system, homoclinic solutions, Hamiltonian systems, stability and bifurcation, exponential extinction, singular elliptic problem, nonuniform exponential contraction and dichotomy.

All manuscripts submitted to this special issue went through a thorough peer-refereeing process. Based on the reviewers' reports, we collect twenty-five original research articles by more than fifty active international researchers in differential equations and from different countries such as Korea, China, Malaysia, Singapore, Czech Republic, Turkey, Slovenia, India, and USA. Besides, one survey on recent results for the existence of singular periodic problems is also contained.

It is certainly impossible to provide in this short editorial note a more comprehensive description for all articles in this special issue. However, the team of the guest editors believes that the results included reflect some recent trends in research and outline new ideas for future studies of dynamical aspects of initial/boundary value problems for ordinary differential equations.

Acknowledgment

We would like to express our gratitude to the authors who have submitted papers for consideration. Thanks also are given to the many reviewers whose reports are important for us to make the decisions. All the participants have made it possible to have a very stimulating interchange of ideas. We would also like to thank the editorial board members of this journal, for their support and help throughout the preparation of this special issue.

Jifeng Chu
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Yonghui Xia

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Research Article

Poincaré Map and Periodic Solutions of First-Order Impulsive Differential Equations on Moebius Stripe

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This paper is mainly concerned with the existence, stability, and bifurcations of periodic solutions of a certain scalar impulsive differential equations on Moebius stripe. Some sufficient conditions are obtained to ensure the existence and stability of one-side periodic orbit and two-side periodic orbit of impulsive differential equations on Moebius stripe by employing displacement functions. Furthermore, double-periodic bifurcation is also studied by using Poincaré map.

1. Introduction

Many systems in physics, chemistry, biology, and information science have impulsive dynamical behavior due to abrupt jumps at certain instants during the evolving processes. This complex dynamical behavior can be modeled by impulsive differential equations. The theory of impulsive differential systems has been developed by numerous mathematicians (see [1–9]). As to the stability theory and boundary value problems to impulsive differential equations, There have been extensive studies in this area. However, there are very few works on the qualitative theory of impulsive differential equations and impulsive semidynamical systems. Recently, Bonotto and Federson have given a version of the Poincaré-Bendixson Theorem for impulsive semidynamical systems in [10, 11]. As it is known, the method of Poincaré map plays an important role in the research of qualitative theory and is a natural means to study the existence of periodic solutions and its asymptotic stability. However, due to the complexity of the associated impulsive dynamic models, this approach has only been applied successfully to Raibert's one-legged-hopper (see [12–14]) predator-prey models (see [15–18]), and so forth. The bifurcation theory for ordinary differential equations or smooth systems appeared during the last decades (see, e.g., [19]); however, little is known about the bifurcation theory of impulsive differential equations due to its complexity (see [20]). In this paper, we mainly study a certain scalar impulsive differential equations on Moebius stripe undergoing impulsive effects at fixed time:

$$\begin{aligned} \frac{dx}{dt} &= f(t,x), \quad t \neq \tau_k, \\ \Delta x|_{t=\tau_k} &= I_k(x), \quad k \in \mathbb{Z}^+, \end{aligned} \tag{1}$$

where $0 \le \tau_k < \tau_{k+1}$, $k \in \mathbb{Z}^+$ are fixed with $\tau_k \to +\infty$ as, $k \to +\infty$, and $\Delta x|_{t=\tau_k} = x(\tau_k^+) - x(\tau_k)$. Hu and Han (see [20]) investigated the existence of periodic solutions and bifurcations of (1) under the assumptions that $f(t,x)$ and $I_k(x)$ are periodic; that is, the following assumption holds.

(H^*) There exist a constant $T > 0$, a positive integer q, and two mutual coprime positive integers m and n such that

$$\begin{gathered} f(t+T,x) = f(t,x), \quad \forall t \in \mathbb{R}^+, \ x \in \mathbb{R}, \\ I_{k+q}(x) = I_k(x), \quad \tau_{k+q} - \tau_k = T, \quad \forall k \in \mathbb{Z}^+, \ x \in \mathbb{R}, \\ m\left(t_{k+q} - t_k\right) = nT, \quad k \ge 1. \end{gathered} \tag{2}$$

In this paper, we assume that the following conditions hold.

(H1) Assume that both $f(t,x)$ and $f_x(t,x)$ are continuous scalar functions on $\mathbb{R} \times \mathbb{R}$, $I_k(x) : \mathbb{R} \to \mathbb{R}, k \in \mathbb{Z}$ are odd, continuous functions; that is, $I_k(-x) = -I_k(x)$, $k \in \mathbb{Z}^+$.

(H2) There exists a constant $T > 0$, a positive integer q such that

$$f(t+T,-x) = -f(t,x), \quad \forall t \in \mathbb{R},\ x \in \mathbb{R},$$
$$I_{k+q}(x) = I_k(x), \quad \tau_{k+q} - \tau_k = T, \quad \forall k \in \mathbb{Z}^+,\ x \in \mathbb{R}. \tag{3}$$

From (H2), we have $f(t,x)$ is $2T$ periodic and $\tau_{k+q} - \tau_k = 2T$. Hence assumption (H^*) holds naturally. However, we show some new and fruitful results of system (1) with the condition (H1)-(H2). For example, we obtain the existence and stability of $2T$ periodic solutions to system (1) by double-periodic bifurcation.

This paper is organized as follows. In Section 2, for the sake of self-containedness of the paper, we present some basic definitions of impulsive differential equations. In Section 3, we describe the scalar impulsive differential equations on Moebius stripe and define the Poincaré map. Then we prove several essential lemmas and give sufficient conditions to ensure the existence and stability of one-side and two-side orbits of impulsive differential equation on Moebius stripe. In Section 4, we are mainly concerned with the double-periodic bifurcation impulsive differential equations on Moebius stripe.

2. Preliminaries

For the sake of self-containedness of the paper, we present the basic definitions and notations of the theory of impulsive differential equations we need (see [1, 2, 8]). We also include some fundamental results which are necessary for understanding the theory.

Let $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{Z}^+$ be the sets of real numbers, integers, and positive integers, respectively. Denote by $\theta = \{\theta_i\}$ a strictly increasing sequence of real numbers such that the set $\mathfrak{A}$ of indexes i is an interval in $\mathbb{Z}$.

Definition 1. A function $\phi : \mathbb{R} \to \mathbb{R}^n$, $n \in \mathbb{R}$, is from the set $PC(\mathbb{R},\theta)$ if

(i) it is left continuous;

(ii) it is continuous, except, possibly, points of θ, where it has discontinuities of the first kind.

The last definition means that if $\phi(t) \in PC(\mathbb{R},\theta)$, then the right limit $\phi(\theta_i+) = \lim_{t\to\theta_i^+}\phi(t)$ exists and $\phi(\theta_i(-)) = \phi(\theta_i)$, where $\phi(\theta_i-) = \lim_{t\to\theta_i^-}\phi(t)$, for each $\theta_i \in \theta$.

Definition 2. A function $\phi : \mathbb{R} \to \mathbb{R}^n$ is from the set $PC^1(\mathbb{R},\theta)$ if $\phi(t), \phi'(t) \in PC(\mathbb{R},\theta)$, where the derivative at points of θ is assumed to be the left derivative.

In what follows, in this section, $J \in \mathbb{R}$ is an interval in $\mathbb{R}$. For simplicity of notation, θ is not necessary a subset of J.

Definition 3. The solution $\phi(t)$ is stable if to any $\varepsilon > 0$ and $t_0 \in J$ there corresponds $\delta(t_0,\varepsilon) > 0$ such that for any other solution $\psi(t)$ of (1) with $\|\phi(t_0) - \psi(t_0)\| < \delta(t_0,\varepsilon)$ we have $\|\phi(t) - \psi(t)\| < \varepsilon$ for $t \geq t_0$; the solution $\phi(t)$ is uniformly stable, if $\delta(t_0,\varepsilon)$ can be chosen independently of t_0.

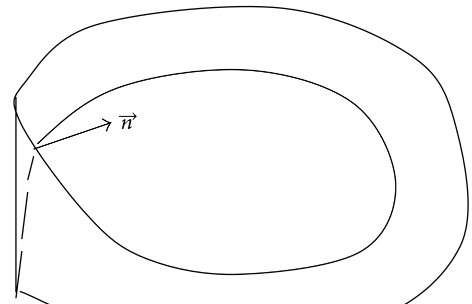

Figure 1: Moebius stripe.

Definition 4. The solution $\phi(t)$ is asymptotically stable if it is stable in the sense of Definition 3 and there exists a positive number $\kappa(t_0)$ such that if $\psi(t)$ is any other solution of (1) with $\|\phi(t_0) - \psi(t_0)\| < \kappa(t_0)$, then $\|\phi(t) - \psi(t)\| \to 0$ as $t \to \infty$; if $\kappa(t_0)$ can be chosen to be independent of t_0 and $\phi(t)$ is uniformly stable, then $\phi(t)$ is said to be uniformly asymptotically stable.

Definition 5. The solution $\phi(t)$ is unstable if there exist numbers $\varepsilon_0 > 0$ and $t_0 \in J$ such that for any $\delta > 0$ there exists a solution $y_\delta(t)$, $\|\phi(t_0) - y_\delta(t_0)\| < \delta$, of (1) such that either it is not continuable to ∞ or there exists a moment t_1, $t_1 > t_0$ such that $\|\phi(t_1) - y_\delta(t_1)\| \geq \delta$.

For any $t_0 \in \mathbb{R}$, we assume that there exists a $k \in \mathbb{Z}^+$, such that $\tau_{k-1} < t_0 \leq \tau_k$; then the initial value problem (IVP) to first-order impulsive differential equations (1) is given as

$$\frac{dx}{dt} = f(t,x), \quad t \neq \tau_k,$$
$$\Delta x|_{t=\tau_k} = I_k(x), \quad k \in \mathbb{Z}^+, \tag{4}$$
$$x(t_0^+) = x_0.$$

In what followed, we use $x(t,t_0,x_0)$ to denote the solution of IVP (4).

In [20], Hu and Han investigated system (1) under the assumption (H^*) and obtained the following stability results for the periodic solutions.

Theorem 6 (see [20]). *Let $x(t,t_0,x_0^*)$ be a periodic solution of system* (1) *with period T. If $0 < |P'(x_0^*)| < 1$ (>1), then it is uniformly asymptotically stable (unstable), where $P(x_0) = x(t_0 + nT^+, t_0, x_0)$ is the Poincaré map of system* (1).

3. Poincaré Map and Periodic Solutions

In this section, we describe the scalar impulsive differential equations on Moebius stripe and define the Poincaré map. Then we prove several essential lemmas and give sufficient conditions to ensure the existence and stability of one-side and two-side orbits (Figure 2) of impulsive differential equation on Moebius stripe.

Lemma 7. *Assume that conditions (H1), (H2) hold. Suppose that $x(t, t_0, x_0)$ is a solution of* (1) *satisfying initial value $x(t_0^+) = x_0$. Then $-x(t + T, t_0, x_0)$ is also a solution of* (1), *and*

$$-x(t+T, t_0, x_0) = x(t, t_0, -x(t_0 + T, t_0, x_0)), \quad t \in \mathbb{R}. \tag{5}$$

Proof. Let $\varphi(t) \equiv -x(t + T, t_0, x_0)$, $\psi(t) \equiv x(t, t_0, -x(t_0 + T, t_0, x_0))$. Then for $t \neq \tau_k$, $k \in \mathbb{Z}$, we have by (H2) that

$$\begin{aligned} \frac{d\varphi(t)}{dt} &= -\frac{dx(t+T, t_0, x_0)}{dt} \\ &= -f(t+T, x(t+T, t_0, x_0)) \\ &= f(t, -x(t+T, t_0, x_0)) = f(t, \varphi(t)). \end{aligned} \tag{6}$$

For $t = \tau_k$, $k \in \mathbb{Z}^+$, it follows from (H1), (H2) that $\tau_k + T = \tau_{k+q}$, $k \in \mathbb{Z}^+$ and

$$\begin{aligned} \Delta\varphi|_{t=\tau_k} &= -x(\tau_k + T^+, t_0, x_0) + x(\tau_k + T, t_0, x_0) \\ &= -x(\tau_{k+q}^+, t_0, x_0) + x(\tau_{k+q}, t_0, x_0) \\ &= -I_{k+q}(x(\tau_{k+q}, t_0, x_0)) \\ &= -I_k(x(\tau_{k+q}, t_0, x_0)) = -I_k(x(\tau_k + T, t_0, x_0)) \\ &= I_k(-x(\tau_k + T, t_0, x_0)) = I_k(\varphi(\tau_k)). \end{aligned} \tag{7}$$

Thus, we proved that $\varphi(t) \equiv -x(t + T, t_0, x_0)$ is a solution of (1). On the other hand, it is obvious that

$$\varphi(t)|_{t=t_0} = -x(t_0 + T, t_0, x_0) = \psi(t)|_{t=t_0}. \tag{8}$$

Hence, by uniqueness theorem we have that $\varphi(t) \equiv \psi(t)$, $t \in \mathbb{R}$. This completes the proof. □

Let D denotes the stripe area on the plain $\{(t, x) \mid (t, x) \in \mathbb{R} \times \mathbb{R}\}$ between two lines $t = t_0$ and $t = t_0 + T$; that is,

$$D = \{(t, x) \mid t_0 \le t \le t_0 + T, -\infty < x < +\infty\}. \tag{9}$$

Assume that $x(t, t_0, x_0)$ exists for all $t \in [t_0, +\infty)$. Define $L_0 = \{(t, x(t, t_0, x_0)) \mid t_0 \le t \le t_0 + T\}$. In general, we denote L_k $(k \ge 1)$ by

$$L_k = \{(t, x(t, t_0, -x_k)) \mid t_0 \le t \le t_0 + T\}, \tag{10}$$

where $x_k = x(t_0 + T^+, t_0, -x_{k-1})$, $t \ge t_0$.

It follows from Lemma 7 that L_k has the form

$$L_k = \{(t, (-1)^k x(t + kT, t_0, x_0)) \mid t_0 \le t \le t_0 + T\}. \tag{11}$$

We now introduce an equivalence relation $\sim$ on D such that for $(t, x), (t', x') \in D$

$$(t, x) \sim (t', x') \quad \text{iff } |t - t'| = T, \ x = -x'. \tag{12}$$

Then we denote the corresponding quotient space by M_2. From geometric point of view, M_2 is obtained by considering two elements (t_0, x) and $(t_0 + T, -x)$ on D as the same point (or sticking (t_0, x) and $(t_0 + T, -x)$ together). Thus M_2 is a surface with only one side or the well-known *Moebius stripe*. Obviously, by Lemma 7 the union

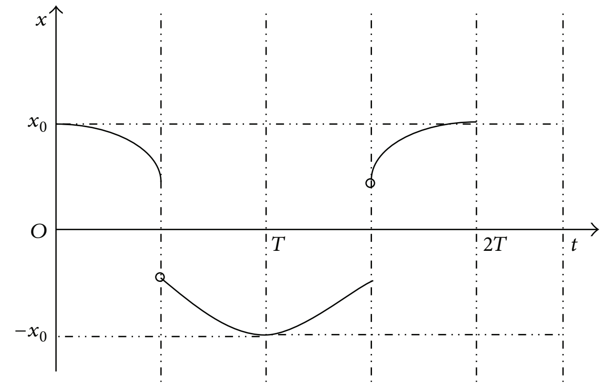

Figure 2: Figure of one-side and two-side orbits.

$$\bigcup_{k\in\mathbb{Z}^+} L_k = \bigcup_{k\in\mathbb{Z}^+} \{(t, (-1)^k x(t + kT, t_0, x_0)) \mid t_0 \le t \le t_0 + T\} \tag{13}$$

define a flow on M_2. From this point of view, we call (1) satisfying (H1) and (H2) an impulsive dynamical system on Moebius stripe (see Figure 1).

Definition 8 (Poincaré Map). Let $x(t, t_0, x_0)$ be the solution of (IVP) (4). Assume that there exists an interval J such that for any $x_0 \in J$, $x(t, t_0, x_0)$ exists on $[t_0, t_0 + T]$. A map $P : J \to \mathbb{R}$ is called a Poincaré map of system (1) if for any $x_0 \in J$

$$P(x_0) = -x(t_0 + T^+, t_0, x_0). \tag{14}$$

Definition 9. A closed curve $\gamma^+(x_0)$ is called a one-side periodic orbit on M_2 if $\gamma^+(x_0) = L_0$. And a closed curve $\gamma^+(x_0)$ is called a two-side periodic orbit on M_2 if $\gamma^+(x_0) = L_0 \cup L_1 \neq L_1$.

From Definitions 8 and 9, we can easily prove the following assertion.

Lemma 10. *One of following alternatives is valid:*

(i) *$\gamma^+(x_0)$ is a one-side periodic orbit;*

(ii) *x_0 is a fixed point of P; that is, $P(x_0) = x_0$;*

(iii) *$x(t + T, t_0, x_0) = -x(t, t_0, x_0)$, $t \in \mathbb{R}$.*

Proof. We prove it from (i) ⇒ (ii) ⇒ (iii) ⇒ (i). Assume (i) is true; that is, $\gamma^+(x_0)$ is a one-side periodic orbit. Then by Definition 9 we have that

$$-x(t_0 + T^+, t_0, x_0) = x(t_0^+, t_0, x_0) = x_0, \tag{15}$$

that is, $P(x_0) = x_0$. Hence (ii) is valid.

Next, we suppose that (ii) is fulfilled; that is, $-x(t_0 + T^+, t_0, x_0) = x_0$. Then by Lemma 7 we know

$$\begin{aligned} x(t+T, t_0, x_0) &= x(t, t_0, -x(t_0+T^+, t_0, x_0)) \\ &= -x(t, t_0, x_0). \end{aligned} \tag{16}$$

Thus, (iii) is proved.

Finally, if (iii) is true, then $x(t_0+T^+, t_0, x_0) = -x_0$. By the uniqueness of solution of IVP (4), we know

$$\begin{aligned} x(t, t_0, -x(t_0+T^+, t_0, x_0)) &= x(t, t_0, x_0), \\ & t \in [t_0, t_0+T]. \end{aligned} \tag{17}$$

Thus we obtain that $\gamma^+(x_0)$ is a one-side periodic orbit. The proof is completed. □

Similarly, as proof of Lemma 10, we have the following lemma.

Lemma 11. *One of following alternatives is valid:*

(i) $\gamma^+(x_0)$ *is a two-side periodic orbit;*

(ii) x_0 *is a 2-periodic point of P; that is,* $P(x_0) \neq x_0$, $P^2(x_0) = x_0$;

(iii) $x(t+2T, t_0, x_0) = x(t, t_0, x_0)$, $t \in \mathbb{R}$. *And there exists a* t_0, *such that* $x(t_0+T^+, t_0, x_0) \neq -x_0$.

Remark 12. From Lemmas 10 and 11, we see that a one-side periodic orbit must be a two-side periodic orbit since

$$P(x_0) = x_0 \quad \text{implies } P^2(x_0) = P(P(x_0)) = P(x_0) = x_0. \tag{18}$$

Nevertheless, the converse is not true.

From Remark 12, we give the definition of stability of the mentioned orbits.

Definition 13. Let $\gamma^+(x_0)$ be a periodic orbit of system (1) (one-side or two-side). Then $\gamma^+(x_0)$ of system (1) is called stable (asymptotically stable or unstable) if $\gamma^+(x_0)$ as a $2T$ periodic solution is stable (asymptotically stable or unstable).

Theorem 14. *Assume* $x(t, t_0 x_0)$ *is the solution of IVP* (4) *and let* $z(t) = \partial x(t, t_0, x_0)/\partial x_0$. *Then* $z(t)$ *is a solution to the following IVP of impulsive differential equations:*

$$\begin{aligned} \frac{dz}{dt} &= f_x(t, x) z, \quad t \neq t_k, \\ \Delta z|_{t=\tau_k} &= I_k'(x), \quad k \in \mathbb{Z}^+, \\ z(t_0) &= 1. \end{aligned} \tag{19}$$

Proof. Let $J = (t_0, +\infty)$ and $J_k = (\tau_{k-1}, \tau_k]$, $k \in \mathbb{Z}^+$. Without losing generality, we assume that $t_0 \in J_j$ for some $j \geq 1$. The solution of IVP

$$\begin{aligned} \frac{dx}{dt} &= f(t, x), \\ x(t_0^+) &= x_0 \end{aligned} \tag{20}$$

can be expressed as

$$x(t, t_0, x_0) = x_0 + \int_{t_0}^{t} f(s, x(s, t_0, x_0))\, ds. \tag{21}$$

Differentiate between both sides of the above equation with respect to x_0, we have

$$\frac{\partial x(t, t_0, x_0)}{\partial x_0} = 1 + \int_{t_0}^{t} f_x(s, x(s, t_0, x_0)) \cdot \frac{\partial x(s, t_0, x_0)}{\partial x_0} ds. \tag{22}$$

Let $z(t) = \partial x(t, t_0, x_0)/\partial x_0$, then for $t \in [t_0, \tau_j)$, $z(t)$ is the solution of IVP to ordinary differential equation

$$\begin{aligned} \frac{dz}{dt} &= f_x(t, x) z, \quad t \neq t_k, \\ z(t_0) &= 1. \end{aligned} \tag{23}$$

Thus

$$z(t) = \exp \int_{t_0}^{t} f_x(s, x(s, t_0, x_0))\, ds. \tag{24}$$

Since $z(t)$ is left continuous on $[t_0, \infty)$, we have

$$z(\tau_j) = \exp \int_{t_0}^{\tau_j} f_x(s, x(s, t_0, x_0))\, ds. \tag{25}$$

For $t \in J_{j+1}$, $x(t, t_0, x_0)$ is a solution of system

$$\begin{aligned} \frac{dx}{dt} &= f(t, x), \\ x(\tau_j) &= x_1, \end{aligned} \tag{26}$$

where $x_1 = x(\tau_{j^+}, t_0, x_0) = x(\tau_j, t_0, x_0) + I_j(x(\tau_j, t_0, x_0))$. Thus, we have

$$\begin{aligned} x(t, t_0, x_0) &\equiv x(t, \tau_j, x_1) \\ &= x_1 + \int_{\tau_j}^{t} f(s, x(s, \tau_j, x_1))\, ds, \quad t \in J_{j+1}. \end{aligned} \tag{27}$$

Similarly, we have $t \in (\tau_j, \tau_{j+1})$,

$$\begin{aligned} \frac{\partial x(t, t_0, x_0)}{\partial x_1} &= \frac{\partial x(t, \tau_j, x_1)}{\partial x_1} \\ &= \exp \int_{\tau_j}^{t} f_x(s, x(s, \tau_j, x_1))\, ds \\ &= \exp \int_{\tau_j}^{t} f_x(s, x(s, t_0, x_0))\, ds. \end{aligned} \tag{28}$$

Note

$$\begin{aligned}\frac{\partial x_1}{\partial x_0} &= \frac{\partial\left[x\left(\tau_j, t_0, x_0\right) + I_j\left(x\left(\tau_j, t_0, x_0\right)\right)\right]}{\partial x\left(\tau_j, t_0, x_0\right)} \\ &\quad \cdot \frac{\partial x\left(\tau_j, t_0, x_0\right)}{\partial x_0} \\ &= \left(1 + I_j'\left(x\left(\tau_j, t_0, x_0\right)\right)\right)\frac{\partial x\left(\tau_j, t_0, x_0\right)}{\partial x_0}.\end{aligned} \tag{29}$$

We obtain for $t \in (\tau_j, \tau_{j+1})$ that

$$\begin{aligned}\frac{\partial x\left(t, t_0, x_0\right)}{\partial x_0} &= \frac{\partial x\left(t, \tau_j, x_1\right)}{\partial x_0} \\ &= \left(1 + I_j'\left(x\left(\tau_j, t_0, x_0\right)\right)\right) \\ &\quad \cdot \exp\int_{t_0}^{t} f_x\left(s, x\left(s, t_0, x_0\right)\right) ds.\end{aligned} \tag{30}$$

Deducing in a similar way, we get

$$\begin{aligned}\frac{\partial x\left(t, t_0, x_0\right)}{\partial x_0} &= \prod_{t_0<\tau_k\le t}\left(1 + I_k'\left(x\left(\tau_j, t_0, x_0\right)\right)\right) \\ &\quad \cdot \exp\int_{t_0}^{t} f_x\left(s, x\left(s, t_0, x_0\right)\right) ds,\end{aligned} \tag{31}$$

where $t \in J$. Then the proof is completed. □

By Definitions 9 and (31), we conclude the following assertion.

Corollary 15. *Assume that conditions (H1), (H2) hold. Then*

$$\begin{aligned}P'\left(x_0\right) &= -\prod_{t_0<\tau_k\le t_0+T}\left(1 + I_k'\left(x\left(\tau_k, t_0, x_0\right)\right)\right) \\ &\quad \cdot \exp\int_{t_0}^{t_0+T} f_x\left(t, x\left(t, t_0, x_0\right)\right) dt.\end{aligned} \tag{32}$$

As usual, one uses the notion $P^2(x_0) = P(P(x_0))$. Then one has

$$\begin{aligned}\left[P^2\left(x_0\right)\right]' &= \prod_{t_0<\tau_k\le t_0+2T}\left(1 + I_k'\left(x\left(\tau_k, t_0, x_0\right)\right)\right) \\ &\quad \cdot \exp\int_{t_0}^{t_0+2T} f_x\left(t, x\left(t, t_0, x_0\right)\right) dt.\end{aligned} \tag{33}$$

Definition 16. x_0 is called a hyperbolic fixed point of P if $x_0 = P(x_0)$ and $P'(x_0) \ne -1$; the corresponding one-side periodic orbit $\gamma^+(x_0)$ is called hyperbolic one-side periodic orbit. If $\gamma^+(x_0)$ is a two-side periodic orbit with $(P^2)'(x_0) \ne 1$, then we call $\gamma^+(x_0)$ a hyperbolic two-side periodic orbit.

Theorem 17. *Assume that the conditions (H1), (H2) hold. Let $\gamma^+(x_0)$ be a periodic orbit of system (1) and $I_k'(x(\tau_k, t_0, x_0)) \ne -1$. Then (i) $\int_{t_0}^{t_0+2T} f_x(t, x(t, t_0, x_0))dt < -\sum_{t_0<\tau_k\le t_0+2T}\ln|1 + I_k'(x(\tau_k, t_0, x_0))|$ implies $\gamma^+(x_0)$ is asymptotically stable,*

(ii) $\int_{t_0}^{t_0+2T} f_x(t, x(t, t_0, x_0))dt > -\sum_{t_0<\tau_k\le t_0+2T}\ln|1 + I_k'(x(\tau_k, t_0, x_0))|$ implies $\gamma^+(x_0)$ is unstable.

Proof. If $\gamma^+(x_0)$ is a two-side periodic orbit; that is, $x(t, t_0, x_0)$ is a $2T$ periodic solution of (1). Since both (H1) and (H2) hold, we know that (1) is a periodic impulsive differential equation. Then by (33) and Theorem 6, the conclusion is straightforward. □

Example 18. Consider the linear periodic impulsive differential equations on Moebius stripe as follows:

$$\begin{aligned}\frac{dx}{dt} &= a(t)x + b(t), \quad t \ne \tau_k, \\ \Delta x|_{t=\tau_k} &= c_k x\left(\tau_k\right), \quad k \in \mathbb{Z}^+,\end{aligned} \tag{34}$$

where $\tau_k < \tau_{k+1}$ $(k \ge 1)$, $\tau_k \to +\infty$, $k \to +\infty$, $c_k \ne -1$ and there exists a constant $T > 0$, a positive integer q, such that the following conditions are satisfied:

($\widetilde{\mathrm{H}}1$) $a(t+T) = a(t)$ and $b(t+T) = -b(t)$ for $t \in \mathbb{R}$;

($\widetilde{\mathrm{H}}2$) $a(t)$ and $b(t)$ are continuous;

($\widetilde{\mathrm{H}}3$) $c_{k+q} = c_k$, for all $k \in \mathbb{Z}^+$;

($\widetilde{\mathrm{H}}4$) $\tau_{k+q} - \tau_k = T$, for all $k \in \mathbb{Z}^+$.

Assume that $x(t, t_0, x_0)$ is a one-side periodic solution of system (34), by the method of variation of constants formula (see [1]), we get

$$\begin{aligned}&x\left(t, t_0, x_0\right) \\ &\quad = \prod_{t_0<\tau_k\le t}\left(1 + c_k\right) \\ &\qquad \cdot \exp\int_{t_0}^{t} a(t) dt \\ &\qquad \times\left[x_0 + \int_{t_0}^{t}\exp\left(-\int_{t_0}^{s} a(u) du\right) b(s) ds\right],\end{aligned} \tag{35}$$

$$\begin{aligned}&P\left(x_0\right) \\ &\quad = -\prod_{t_0<\tau_k\le t_0+T}\left(1 + c_k\right) \\ &\qquad \cdot \exp\int_{t_0}^{t_0+T} a(t) dt \\ &\qquad \times\left[x_0 + \int_{t_0}^{t_0+T}\exp\left(-\int_{t_0}^{t} a(u) du\right) b(t) dt\right],\end{aligned} \tag{36}$$

$$
\begin{aligned}
P^2(x_0) &= \prod_{t_0<\tau_k\le t_0+2T}(1+c_k) \\
&\quad \cdot \exp\int_{t_0}^{t_0+2T} a(t)\,dt \\
&\quad \times\left[x_0+\int_{t_0}^{t_0+2T}\exp\left(-\int_{t_0}^{t} a(u)\,du\right)b(t)\,dt\right].
\end{aligned} \tag{37}
$$

Let $A=\prod_{t_0<\tau_k\le t_0+T}(1+c_k)\cdot\exp\int_{t_0}^{t_0+T}a(t)dt$; therefore, we have the following theorem.

Theorem 19. *Suppose that ($\widetilde{H}1$–$\widetilde{H}4$) are satisfied, then*

(i) *there exists a unique one-side periodic orbit for system* (34) *if* $A\ne-1$, *which is asymptotically stable (unstable) provided* $0<|A|<1$ ($|A|>1$),

(ii) *if* $A^2\ne1$, (34) *has no two-side periodic orbit. If* $A=1$ *all the trajectories are two-side periodic orbits expect for a unique one-side periodic orbit.*

Proof. For the sake of convenience, we denote

$$
\begin{aligned}
B_1&=\int_{t_0}^{t_0+T}\exp\left(-\int_{t_0}^{t}a(u)\,du\right)b(t)\,dt,\\
B_2&=\int_{t_0}^{t_0+2T}\exp\left(-\int_{t_0}^{t}a(u)\,du\right)b(t)\,dt.
\end{aligned} \tag{38}
$$

Then

$$
P(x_0)=-A(x_0+B_1),\qquad P^2(x_0)=A^2(x_0+B_2). \tag{39}
$$

Obviously, $P(x_0)=x_0$ has a unique solution for any $x_0\in\mathbb{R}$ if $A\ne-1$, and $P^2(x_0)=x_0$ has a unique solution for any $x_0\in\mathbb{R}$ if $A^2\ne1$. Observing that any two-side periodic orbit obtained under the assumption $A^2\ne1$ must be a one-side periodic orbit since $A^2\ne1$ implies $A\ne-1$, together with Remark 12, we have (34) has no two-side periodic orbit.

It follows from (36) that $P'(x_0)=-A$. Then by Theorem 6 we have the one-side orbit is asymptotically stable (unstable) provided $0<|A|<1$ ($|A|>1$).

Next, let $A=1$. By taking (36) and (37) into account, we have

$$
\begin{aligned}
P(x_0)&=-x_0-\int_{t_0}^{t_0+T}\exp\left(-\int_{t_0}^{t}a(u)\,du\right)b(t)\,dt\\
&\equiv -x_0-B_1,\\
P^2(x_0)&=x_0+\int_{t_0}^{t_0+2T}\exp\left(-\int_{t_0}^{t}a(u)\,du\right)b(t)\,dt\\
&\equiv x_0+B_2.
\end{aligned} \tag{40}
$$

Suppose that P has a unique fixed point $x_0^*=-B_1/2$, from the above we have $-B_1/2+B_2=-B_1/2$, then $B_2\equiv0$ and $P^2(x_0)=x_0$. So by taking Lemma 11, $\gamma^+(x_0)$ is a two-side periodic orbit if $x_0\ne x_0^*$.

The proof is ended. □

Remark 20. If $c_k\equiv0$, $k\in\mathbb{Z}^+$, in (34); that is, (34) reduces to an ordinary differential equation. We see that $A=\exp\int_{t_0}^{t_0+T}a(t)dt$. Hence $A\ne-1$ holds automatically, and therefore (34) always has a unique one-side periodic orbit.

Corollary 21. *Let ($\widetilde{H}1$–$\widetilde{H}4$) be fulfilled and* $A=1$. *Then*

$$
B_2=\int_{t_0}^{t_0+2T}\exp\left(-\int_{t_0}^{t}a(u)\,du\right)b(t)\,dt=0. \tag{41}
$$

Now we are in position to consider nonlinear impulsive system on Meobius stripe. To explore the uniqueness of one-side periodic orbit, we induce the following condition.

(H3) Operator $B_k:\mathbb{R}\to\mathbb{R}$, $B_k(x)=x+I_k(x)$ is strictly increasing, for all $k\in\mathbb{Z}^+$.

Theorem 22. *Suppose that conditions (H1)–(H3) hold, then*

(i) *system* (1) *has at most one one-side periodic orbit;*

(ii) *if any solution* $x(t,t_0,x_0)$ *of* (1) *with* $|x_0|\le|P(0)|$ *is well defined on* $t\in[t_0,t_0+T]$, *then system* (1) *must has a unique one-side periodic orbit.*

Proof. We first prove that system (1) cannot have two one-side periodic orbits. Suppose $\gamma_1^+(x_0^*): x=x(t,t_0,x_0^*)$, $t\in[t_0,t_0+T]$ and $\gamma_2^+(x_0): x=x(t,t_0,x_0)$, $t\in[t_0,t_0+T]$ are two one-side periodic orbits system (1). Then

$$
x(t_0+T,t_0,x_0^*)=-x_0^*,\qquad x(t_0+T,t_0,x_0)=-x_0. \tag{42}
$$

Without losing generality, we assume $x_0>x_0^*$, then it follows from uniqueness theorem of ordinary differential equations that $x=x(t,t_0,x_0)$ and $x(t,t_0,x_0^*)$ cannot intersect when t is not an impulsive time. Therefore we have

$$
x(t,t_0,x_0)>x(t,t_0,x_0^*),\quad t_0\le t\le\tau_1. \tag{43}
$$

Note $B_k(x)=x+I_k(x)$ is strictly increasing, we get

$$
x(\tau_1^+,t_0,x_0)>x(\tau^+,t_0,x_0^*). \tag{44}
$$

In a similar way, we can prove that $x(t,t_0,x_0)>x(t,t_0,x_0^*)$, $t_0\le t\le t_0+T$. That is, the curve $\{(t,x)\mid x=x(t,t_0,x_0),\ t_0\le t\le t_0+T\}$ always stays above curve $\{(t,x)\mid x=x(t,t_0,x_0^*),\ t_0\le t\le t_0+T\}$. This contradicts (42). We put it in another way that

$$
x_0>x_0^*\Longrightarrow P(x_0)>P(x_0^*). \tag{45}
$$

Thus, (1) has at most a one-side periodic orbit.

Further, let the solution $x(t,t_0,x_0)$ of system (1) be all defined on $t\in[t_0,t_0+T]$. If $P(0)=0$, the conclusion is proved. We assume that $P(0)>0$, then we know $P(x_0)<P(0)$ if $0<x_0\le P(0)$. Note

$$
P^2(0)-P(0)=P(P(0))-P(0)<P(0)-P(0)=0. \tag{46}
$$

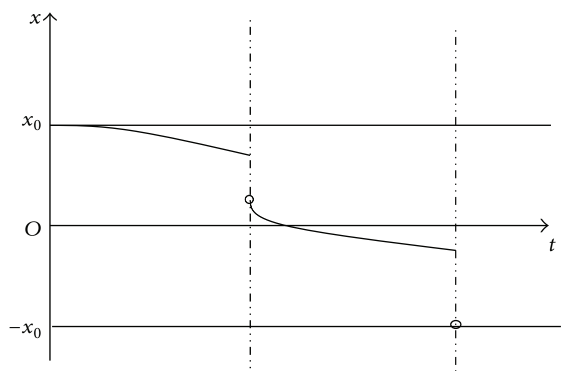

Figure 3: A one-side periodic orbit.

We obtain that $P(x_0)-x_0$ have opposite signs between $x_0 = 0$ and $x_0 = P(0)$, and then it follows from the continuity of P that there exists $x_0^* \in (0, P(0))$ such that $P(x_0^*) = x_0^*$. Similarly, we can prove P has a fixed point in the case of $P(0) < 0$. The proof is completed. □

Theorem 23. *Assume that conditions (H1)–(H3) hold. Furthermore, suppose there exists a positive number N such that*

$$f(t, N) \le 0, \quad f(t, -N) \ge 0, \quad t \in [t_0, t_0 + T], \tag{47}$$

$$-2N \le I_k(N) \le 0, \quad \forall k \in \mathbb{Z}^+. \tag{48}$$

Then (1) *has a unique one-side periodic orbit.*

Proof. From (47) we have that $x(t, t_0, x_0)$ will stay inside $[-N, N]$ for $t \ne \tau_k, k \in \mathbb{Z}^+$. On the other hand, by (H3), we have that $-N + I_k(-N) \le x(\tau_k) + I_k(x(\tau_k)) \le N + I_k(N)$ for $-N \le x(\tau_k) \le N$. Then it follows from (48) that

$$\begin{gathered} -N \le N + I_k(N) \le N \le N, \\ -N \le -N + I_k(-N) = -N - I_k(N) \le N \end{gathered} \tag{49}$$

(see Figure 3). □

Thus,

$$|P(0)| = |-x(t_0 + T, 0)| = |x(t_0 + T, 0)| \le N. \tag{50}$$

This implies that $P(x_0)$ is well defined for $|x_0| \le |P(0)|$. By Theorem 22, we obtain that (1) has a unique one-side periodic orbit.

4. Double-Period Bifurcation

In this section, we mainly discuss the bifurcation on periodic orbits. If system (1) has a one-side periodic orbit, without losing generality, we may assume that $f(t, 0) = 0$; that is, $x = 0$ is the one-side periodic orbit. Actually, if $x(t)$ is a one-side periodic orbit, then we let $y = x - x(t)$; therefore there exists a transformation of system (1) that

$$\begin{gathered} \frac{dy}{dt} = f(t, y + x(t)) - f(t, x(t)) \equiv g(t, y), \quad t \ne \tau_k, \\ \Delta y|_{t=\tau_k} = I_k(y + x(\tau_k)) - I_k(x(\tau_k)) \equiv h_k(y), \quad k \in \mathbb{Z}^+, \end{gathered} \tag{51}$$

By (H2) and Lemma 10, we know $g(t + T, -y) = -g(t, y)$, $g(t, 0) = 0$, $h_{k+q}(y) = h_k(y)$, and $\tau_{k+q} - \tau_k = T$, for all $k \in \mathbb{Z}^+$.

Next, we consider the following perturbed system of system (1):

$$\begin{gathered} \frac{dx}{dt} = F(t, x, \varepsilon), \quad t \ne \tau_k, \\ \Delta x|_{t=\tau_k} = \tilde{I}_k(x(\tau_k), \varepsilon), \quad k \in \mathbb{Z}^+, \end{gathered} \tag{52}$$

where $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is C^3 with respect to x, continuously differentiable with respect to ε. $\tilde{I}_k : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is C^3 $(k \in \mathbb{Z}^+)$ with respect to x. Moreover, we suppose $F(t + T, -x, \varepsilon) = -F(t, x, \varepsilon)$, $\tilde{I}_k(x, \varepsilon) = \tilde{I}_{k+q}(x, \varepsilon)$, $\tilde{I}_k(-x) = -\tilde{I}_k(x)$, for all $k \in \mathbb{Z}^+$, where $\tau_{k+q} - \tau_k = T$. For $\varepsilon = 0$, we have $F(t, x, 0) = f(t, x)$, $\tilde{I}_k(x, 0) = I_k(x)$. These assumptions mean (H1) and (H2) hold for F and $\tilde{I}_k$, for all $k \in \mathbb{Z}^+$. Furthermore, assume that $x + \tilde{I}_k(x, \varepsilon)$ is strictly increasing, then by Theorem 22 we have that system (52) has at most a one-side periodic orbit.

Suppose that (1) has a one-side periodic orbit and $f(t, 0) = 0$. Then by using implicit function theorem in the Poincaré map of system (52), we know that system (52) has a one-side periodic orbit when $|\varepsilon|$ is sufficiently small. Now let $x^*(t, \varepsilon)$ be the solution of system (52) and $y(t, \varepsilon) = x(t, \varepsilon) - x^*(t, \varepsilon)$. Then we can get a transformation of system (52):

$$\begin{aligned} \frac{dy}{dt} &= F(t, y + x^*, \varepsilon) - F(t, x^*, \varepsilon) = G(t, y, \varepsilon), \quad t \ne \tau_k, \\ \Delta y|_{t=\tau_k} &= \tilde{I}_k(y + x^*(\tau_k, \varepsilon), \varepsilon) \\ &\quad - \tilde{I}_k(x^*(\tau_k, \varepsilon), \varepsilon) = H_k(y, \varepsilon), \quad k \in \mathbb{Z}^+. \end{aligned} \tag{53}$$

By Taylor's formula, we have

$$\begin{aligned} G(t, y, \varepsilon) &= A_1(t, \varepsilon) y + A_2(t, \varepsilon) y^2 \\ &\quad + A_3(t, \varepsilon) y^3 + o(y^3), \end{aligned} \tag{54}$$

$$H_k(y, \varepsilon) = B_{k1}(\varepsilon) y + B_{k3}(\varepsilon) y^3 + o(y^3),$$

where

$$\begin{gathered} A_i(t, \varepsilon) = \frac{1}{i!} \frac{\partial^i F}{\partial x^i}(t, x^*, \varepsilon), \\ B_{kj}(\varepsilon) = \frac{1}{j!} \frac{\partial^j \tilde{I}_k}{\partial x^j}(x^*, \varepsilon), \\ A_i(t + T, \varepsilon) = (-1)^{i-1} A_i(t, \varepsilon), \end{gathered} \tag{55}$$

for $k \ge 1$, $i = 1, 2, 3$; $j = 1, 3$.

If $\varepsilon = 0$, then $x^* = 0$. So $A_i(t,0) = (1/i!)(\partial^i f/\partial x^i)(t,0)$ and $B_{kj}(0) = (1/j!)I_k^{(j)}(0)$, for $k \ge 1$, $i = 1,2,3$, $j = 1,3$.

Suppose that $y(t, y_0, \varepsilon)$ $(t \ge t_0)$ is the solution of system (53) with the initial value $y(t_0^+, y_0, \varepsilon) = y_0$, $P(y_0, \varepsilon)$ is the Poincaré map of system (53). Note

$$\widetilde{P}(y_0,\varepsilon) = y(t_0 + 2T^+, y_0, \varepsilon) = P^2(y_0,\varepsilon). \tag{56}$$

Without losing generality, let $x^*(t,\varepsilon) = 0$ is a nonhyperbolic solution. That is, $P(0,0) = 0$ and $(\partial P/\partial y_0)(0,0) = -1$.

Noting that $P(0,\varepsilon) = 0$, then by Taylor's formula, we have

$$P(y_0,\varepsilon) = \overline{A}_1(\varepsilon) y_0 + \overline{A}_2(\varepsilon) y_0^2 + \overline{A}_3(\varepsilon) y_0^3 + o\left(y_0^3\right), \tag{57}$$

where $\overline{A}_1(\varepsilon) = (\partial P/\partial y_0)(0,\varepsilon)$, $\overline{A}_2(\varepsilon) = (1/2)(\partial^2 P/\partial y_0^2)(0,\varepsilon)$, and $\overline{A}_3(\varepsilon) = (1/6)(\partial^3 P/\partial y_0^3)(0,\varepsilon)$.

Theorem 24. *Suppose that $f(t,0) = 0$ and $x = 0$ is a one-side periodic orbit of system $(52)_{\varepsilon=0}$ with $P(0,0) = 0$ and $(\partial P/\partial y_0)(0,0) = -1$. Let $a_3^* = (1/6)\widetilde{P}'''(0,0)$. If $a_3^* \neq 0$, then for $|\varepsilon|$ sufficiently small and $[\overline{A}_1(\varepsilon) + 1]a_3^* > 0$ (≤ 0) implies that system (52) has a unique (no) two-sides periodic orbit near $x = 0$, except for a one-side periodic orbit $x^*(t,\varepsilon)$.*

Proof. As before, we obtain that $\widetilde{P}'(0,\varepsilon) = [P'(0,\varepsilon)]^2 = \overline{A}_1^2(\varepsilon)$:

$$\begin{aligned}
\widetilde{P}(y_0,\varepsilon) &= P(P(y_0,\varepsilon),\varepsilon) \\
&= \overline{A}_1(\varepsilon) P(y_0,\varepsilon) + \overline{A}_2(\varepsilon) P^2(y_0,\varepsilon) \\
&\quad + \overline{A}_3(\varepsilon) P^3(y_0,\varepsilon) + o\left(P^3(y_0,\varepsilon)\right) \\
&= \overline{A}_1^2(\varepsilon) y_0 + \left[\overline{A}_1^2(\varepsilon) + \overline{A}_1(\varepsilon)\right]\overline{A}_2(\varepsilon) y_0^2 \\
&\quad + \overline{A}_1\left[\overline{A}_3(\varepsilon) + 2\overline{A}_2^2(\varepsilon) + \overline{A}_1^2(\varepsilon)\overline{A}_3(\varepsilon)\right] y_0^3 \\
&\quad + o\left(y_0^3\right).
\end{aligned} \tag{58}$$

By our assumption, we have $\overline{A}_1(\varepsilon) = -1 + \overline{A}_1'(0)\varepsilon + o(\varepsilon)$. Therefore,

$$\begin{aligned}
\widetilde{d}(y_0,\varepsilon) &= \frac{P^2(y_0,\varepsilon) - y_0}{y_0} \\
&= d_0(\varepsilon) + d_1(\varepsilon) y_0 + d_2(\varepsilon) y_0^2 + o\left(y_0^2\right),
\end{aligned} \tag{59}$$

where

$$\begin{aligned}
d_0(\varepsilon) &= \overline{A}_1^2(\varepsilon) - 1 = -2\left[\overline{A}_1(\varepsilon) + 1\right] + o\left(\left[\overline{A}_1(\varepsilon) + 1\right]\right), \\
d_1(\varepsilon) &= \left[\overline{A}_1^2(\varepsilon) + \overline{A}_1(\varepsilon)\right]\overline{A}_2(\varepsilon) \\
&= O\left(\overline{A}_1(\varepsilon) + 1\right) = -\overline{A}_1'(0)\overline{A}_2(0)\varepsilon + o(\varepsilon), \\
d_2(\varepsilon) &= \overline{A}_1(\varepsilon)\left[\overline{A}_3(\varepsilon) + 2\overline{A}_2^2(\varepsilon) + \overline{A}_1^2(\varepsilon)\overline{A}_3(\varepsilon)\right] \\
&= -2\left[\overline{A}_2^2(0) + \overline{A}_3(0)\right] + o(1) = a_3^* + o(1).
\end{aligned} \tag{60}$$

By the implicit function theorem, there exists a unique function $y_0 = y_1(\varepsilon)$, $y_1(0) = 0$ such that $(\partial\widetilde{d}/\partial y_0)(y_1(\varepsilon),\varepsilon) = 0$. Therefore, for $|\varepsilon|$ sufficiently small, there is a unique extremal point $y_0 = y_1(\varepsilon)$ near $x = 0$. Moreover, the function $\widetilde{d}(y_0,\varepsilon)$ takes its minimum (maximum) $\overline{\Delta}(\varepsilon) \equiv \widetilde{d}(y_1(\varepsilon),\varepsilon)$ only if $a_3^* > 0$ (<0):

$$\overline{\Delta}(\varepsilon) = d_0(\varepsilon) + o(\varepsilon) = \overline{A}_1^2(\varepsilon) - 1 = -2\left[\overline{A}_1(\varepsilon) + 1\right] + o(\varepsilon). \tag{61}$$

Without loss of generality, we can let $a_3^* = d_2(0) > 0$ and then $y_0 = 0$ is the minimum point of $d(y_0, 0)$. So there exists $\varepsilon_0 > 0$, such that

$$\widetilde{d}(\pm\varepsilon_0, 0) > 0, \qquad \frac{\partial\widetilde{d}}{\partial y_0}(\varepsilon_0, 0) > 0, \qquad \frac{\partial\widetilde{d}}{\partial y_0}(-\varepsilon_0, 0) < 0, \tag{62}$$

and for $|y_0| \le \varepsilon_0$, $(\partial^2\widetilde{d}/\partial y_0^2)(y_0, 0) > 0$ exists. Therefore, there exists a δ_0, such that, for $|\varepsilon| \le \delta_0$, we have

$$\widetilde{d}(\pm\varepsilon_0, \varepsilon) > 0, \qquad \frac{\partial\widetilde{d}}{\partial y_0}(\varepsilon_0, \varepsilon) > 0, \qquad \frac{\partial\widetilde{d}}{\partial y_0}(-\varepsilon_0, \varepsilon) < 0. \tag{63}$$

For $|\varepsilon| \le \delta_0$ and $|y_0| \le \varepsilon_0$, we have

$$\frac{\partial^2\widetilde{d}}{\partial y_0^2}(y_0,\varepsilon) > 0. \tag{64}$$

From (64), for any $|\varepsilon| \le \delta_0$, we have

$$\overline{\Delta}(\varepsilon) = \min_{|y_0| \le \varepsilon_0} \widetilde{d}(y_0,\varepsilon), \quad -\varepsilon_0 < y_1(\varepsilon) < \varepsilon_0. \tag{65}$$

And for $y_0 \in (-\varepsilon_0, y_1(\varepsilon))(\in (y_1(\varepsilon), \varepsilon_0))$,

$$\frac{\partial\widetilde{d}}{\partial y_0}(y_0,\varepsilon) < 0 \ \ (> 0). \tag{66}$$

If $\overline{\Delta}(\varepsilon) > 0$, then for all $|\varepsilon| \le \delta_0$ and $|x_0| < \varepsilon_0$, we have $0 < \overline{\Delta}(\varepsilon) \le \widetilde{d}$.

If $\overline{\Delta}(\varepsilon) = 0$, then $y_0 = y_1(\varepsilon)$ is the unique solution of function $\widetilde{d}$.

If $\overline{\Delta}(\varepsilon) < 0$, then there exist a unique $y_1(\varepsilon)$ and a unique $y_2(\varepsilon)$, such that

$$\widetilde{d}(y_i(\varepsilon),\varepsilon) = 0, \quad \frac{\partial\widetilde{d}}{\partial y_0}(y_i(\varepsilon),\varepsilon) \neq 0, \quad i = 1, 2. \tag{67}$$

Thus system (52) has two (no) two-side periodic orbits if $a_3^*\overline{\Delta}(\varepsilon) < 0$ (≥ 0). The conclusion is completed (see Figures 4, 5, and 6). □

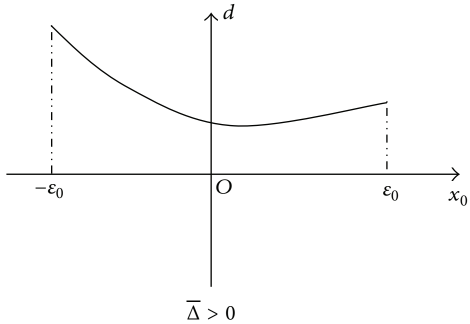

FIGURE 4

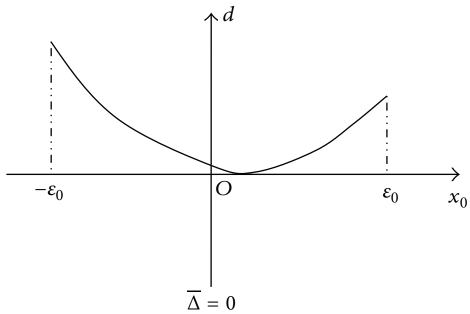

FIGURE 5

Now we shall calculate $\overline{A}_1(0)$ and a_3^* in the simplest case, let $q = 1$. For $q > 1$ we can calculate them in the same way. In this case, $I_k \equiv I$ and $B_{ki}(y_0, \varepsilon) = B_i(y_0, \varepsilon)$, $i = 1, 3$. Suppose $y(t, y_0, \varepsilon)$ $(t \le t_0)$ is the solution to system (53) with initial value $y(t_0+, y_0, \varepsilon) = y_0$. For $y(t, 0, \varepsilon) = 0$, let

$$\begin{aligned} y(t, y_0, \varepsilon) &= \varphi_1(t,\varepsilon) y_0 + \varphi_2(t,\varepsilon) y_0^2 \\ &\quad + \varphi_3(t,\varepsilon) y_0^3 + o\left(y_0^3\right), \quad t \le t_0. \end{aligned} \tag{68}$$

Then for $t \in [t_0, t_0 + T]$, taking $y(t, y_0, \varepsilon)$ into system (53), we can obtain φ_1, φ_2, and φ_3 satisfying the following equations:

$$\begin{aligned} \varphi_1'(t,\varepsilon) &= A_1(t,\varepsilon)\varphi_1(t,\varepsilon), \\ \varphi_2'(t,\varepsilon) &= A_1(t,\varepsilon)\varphi_2(t,\varepsilon) + A_2(t,\varepsilon)\varphi_1^2(t,\varepsilon), \\ \varphi_3'(t,\varepsilon) &= A_1(t,\varepsilon)\varphi_3(t,\varepsilon) + 2A_2(t,\varepsilon)\varphi_1(t,\varepsilon)\varphi_2(t,\varepsilon) \\ &\quad + \varphi_3(t,\varepsilon)\varphi_1^3(t,\varepsilon). \end{aligned} \tag{69}$$

For $y(0, y_0, \varepsilon) = y_0$, we know

$$\varphi_1(0,\varepsilon) = 1, \qquad \varphi_2(0,\varepsilon) = \varphi_3(0,\varepsilon) = 0. \tag{70}$$

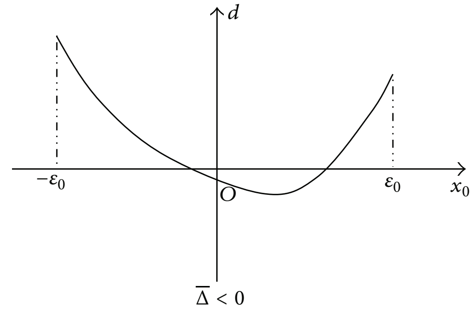

FIGURE 6

From (69) and (70), we have

$$\begin{aligned} \varphi_1(t,\varepsilon) &= \exp\int_{t_0}^{t} A_1(u,\varepsilon)\,du, \\ \varphi_2(t,\varepsilon) &= \varphi_1(t,\varepsilon)\int_{t_0}^{t} A_2(s,\varepsilon)\varphi_1(s,\varepsilon)\,ds, \\ \varphi_3(t,\varepsilon) &= \varphi_1(t,\varepsilon) \\ &\quad \times \int_{t_0}^{t}\left[2A_2(s,\varepsilon)\varphi_2(s,\varepsilon) + A_3(s,\varepsilon)\varphi_1^2(s,\varepsilon)\right]ds. \end{aligned} \tag{71}$$

For $t_0 < t < t_0 + T$, as we know, we get

$$\begin{aligned} y\left(t_0 + T^+, y_0, \varepsilon\right) &= \left[1 + B_1(\varepsilon)\right] y\left(t_0 + T, \varepsilon\right) \\ &\quad + B_2(\varepsilon) y^2\left(t_0 + T, \varepsilon\right) \\ &\quad + B_3(\varepsilon) y^3\left(t_0 + T, \varepsilon\right) \\ &\quad + o\left(y^3\left(t_0 + T, \varepsilon\right)\right) \\ &= \varphi_1\left(t_0 + T^+, \varepsilon\right) y_0 + \varphi_2\left(t_0 + T^+, \varepsilon\right) y_0^2 \\ &\quad + \varphi_3\left(t_0 + T^+, \varepsilon\right) y_0^3 + o\left(y_0^3\right), \end{aligned} \tag{72}$$

where

$$\begin{aligned} \varphi_1\left(t_0 + T^+, \varepsilon\right) &= \left[1 + B_1(\varepsilon)\right]\varphi_1\left(t_0 + T, \varepsilon\right), \\ \varphi_2\left(t_0 + T^+, \varepsilon\right) &= \left[1 + B_1(\varepsilon)\right]\varphi_2\left(t_0 + T, \varepsilon\right) \\ &\quad + B_2(\varepsilon)\varphi_1^2\left(t_0 + T, \varepsilon\right), \\ \varphi_3\left(t_0 + T^+, \varepsilon\right) &= \left[1 + B_1(\varepsilon)\right]\varphi_3\left(t_0 + T, \varepsilon\right) \\ &\quad + 2B_2(\varepsilon)\varphi_1\left(t_0 + T, \varepsilon\right)\varphi_2\left(t_0 + T, \varepsilon\right) \\ &\quad + B_3(\varepsilon)\varphi^3\left(t_0 + T, \varepsilon\right). \end{aligned} \tag{73}$$

Clearly, $\overline{A}_1(\varepsilon) = -\varphi_1(t_0 + T^+, \varepsilon)$, $\overline{A}_2(\varepsilon) = -\varphi_2(t_0 + T^+, \varepsilon)$, and $\overline{A}_3(\varepsilon) = -\varphi_3(t_0 + T^+, \varepsilon)$.

Moreover, we know

$$\begin{aligned}\int_{t_0}^{t_0+T} & A_2(s,\varepsilon)\varphi_2(s,\varepsilon)\,ds \\ &= \int_{t_0}^{t_0+T} A_2(s,\varepsilon)\varphi_1(s,\varepsilon) \\ &\qquad \times \left[\int_{t_0}^{s} A_2(u,\varepsilon)\varphi_1(u,\varepsilon)\,du\right] ds \\ &= \frac{1}{2}\left[\int_{t_0}^{t_0+T} A_2(u,\varepsilon)\varphi_1(u,\varepsilon)\,du\right]^2.\end{aligned} \tag{74}$$

Denote $\Phi(\varepsilon) = \int_{t_0}^{t_0+T} A_2(t,\varepsilon)\varphi_1(t,\varepsilon)dt$ and $\phi_1(\varepsilon) = \varphi_1(t_0 + T,\varepsilon)$. Then,

$$\begin{aligned}\varphi_2(t_0+T,\varepsilon) &= \phi_1(\varepsilon)\Phi(\varepsilon), \\ \varphi_3(t_0+T,\varepsilon) &= \phi_1(\varepsilon)\Phi^2(\varepsilon) + \int_{t_0}^{t_0+T} A_3(s,\varepsilon)\varphi_1^2(s,\varepsilon)\,ds.\end{aligned} \tag{75}$$

Then we can obtain

$$\begin{aligned}\overline{A}_1(\varepsilon) &= -\left[1+B_1(\varepsilon)\right]\exp\left(\int_{t_0}^{t_0+T} A_1(u,\varepsilon)\,du\right), \\ \overline{A}_2(\varepsilon) &= -\left[1+B_1(\varepsilon)\right]\phi_1(\varepsilon)\Phi(\varepsilon), \\ \overline{A}_3(\varepsilon) &= -\left[1+B_1(\varepsilon)\right]\phi_1(\varepsilon) \\ &\quad \times \left[\Phi^2(\varepsilon) + \int_{t_0}^{t_0+T} A_3(s,\varepsilon)\varphi_1^2(s,\varepsilon)\,ds\right] \\ &\quad - B_3(\varepsilon)\phi_1^3(\varepsilon).\end{aligned} \tag{76}$$

For $\overline{A}_1(0) = -1$, we can have $\phi_1(0) = 1/(1 + B_1(0))$. Then

$$\overline{A}_2(0) = -\Phi(0), \tag{77}$$

$$\overline{A}_3(0) = -\Phi^2(0) - \frac{B_3(0)}{\left[1+B_1(0)\right]^3} - \Delta(0), \tag{78}$$

where $\Delta(\varepsilon) = \int_{t_0}^{t_0+T} A_3(s,\varepsilon)\varphi_1^2(s,\varepsilon)ds = \int_{t_0}^{t_0+T} A_3(s,\varepsilon)\exp[2\int_{t_0}^{s} A_1(u,\varepsilon)du]ds$. Therefore,

$$a_3^* = -2\left[\overline{A}_2^2(0) + \overline{A}_3(0)\right] = 2\Delta(0) + \frac{2B_3(0)}{\left[1+B_1(0)\right]^3}. \tag{79}$$

By considering (76)–(79), we can easily have the following theorem when $q = 1$.

Theorem 25. *Suppose that $f(t,0) = 0$ and $x = 0$ is a one-side periodic solution of system $(52)_{\varepsilon=0}$ with $P(0,0) = 0$ and $(\partial P/\partial y_0)(0,0) = -1$. Let*

$$\begin{aligned}\overline{A}_1(\varepsilon) &= -\left(1+\widetilde{I}'(x^*,\varepsilon)\right)\exp\int_{t_0}^{t_0+T} F_x(t,x^*,\varepsilon)\,dt, \\ a_3^* &= \frac{1}{6}\int_{t_0}^{t_0+2T} f_x'''(s,0)\,e^{2\int_{t_0}^{s} f_x(u,0)du}ds + \frac{2\widetilde{I}'''(0)}{\left[1+\widetilde{I}'(0)\right]^3}.\end{aligned} \tag{80}$$

If $a_3^* \neq 0$, then for $|\varepsilon|$ sufficiently small, $[\overline{A}_1(\varepsilon) + 1]a_3^* > 0$ (≤ 0) implies that system (52) has a unique (no) two-side periodic orbit of near $x = 0$, except for a one-side periodic orbit $x^*(t,\varepsilon)$.

By virtue of Theorem 25, we can have the following conclusion.

Corollary 26. *(i) Let $\overline{A}_1(0) = -1$, $a_3^* > 0$ (<0). Then $x = 0$ is a nonhyperbolic one-side periodic orbit of system (48) ($\varepsilon = 0$), which is asymptotically stable (unstable). (ii) Let $\overline{A}_1(0) = -1$, $[\overline{A}_1(\varepsilon) + 1]a_3^* > 0$, $0 < |\varepsilon| \ll 1$. Then $(\overline{A}_1(\varepsilon) + 1) < 0$ (>0), $x^*(t,\varepsilon)$ is a hyperbolic one-side periodic orbit of system (48) ($\varepsilon = 0$), which is asymptotically stable (unstable. Moreover, the two-side periodic orbit is unstable (asymptotically stable) near $x = 0$.*

Finally, we give an example to illustrate it.

Example 27. Consider

$$\begin{aligned}\frac{dx}{dt} &= \varepsilon^2 x + |\varepsilon|(\sin t)x^2 - (1-2\cos 2t)x^3, \quad t \neq k\pi, \\ \Delta x|_{t=k\pi} &= -\varepsilon x(k\pi) + (b_3+\varepsilon)x^3(k\pi), \quad k \in \mathbb{Z}^+,\end{aligned} \tag{81}$$

where $|\varepsilon| > 0$. It is obvious that for $|\varepsilon| > 0$ sufficiently small, $x - \varepsilon x + (b_3 + \varepsilon)x^3$ is strictly increasing, and then $x^*(t,\varepsilon) = 0$ is the unique π-periodic solution. By direct computation, we have $A_1(t,\varepsilon) = \varepsilon^2$, $A_2(t,\varepsilon) = |\varepsilon|(\sin t)$, $A_3(t,\varepsilon) = 2\cos 2t - 1$, $B_1(\varepsilon) = -\varepsilon$, and $B_3(\varepsilon) = b_3 + \varepsilon$. Therefore, $\overline{A}_1(\varepsilon) + 1 = (\varepsilon - 1)e^{\varepsilon^2\pi} + 1 = \varepsilon + o(\varepsilon)$, $a_3^* = 2b_3 - \pi$. It follows from Theorem 25 that system (81) has two (no) 2π-periodic solution of near $x = 0$ if $|\varepsilon|$ is sufficiently small and $a_3^*\varepsilon > 0$ (≤ 0).

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Research Article

Existence Result for Impulsive Differential Equations with Integral Boundary Conditions

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We investigate the following differential equations: $-(y^{[1]}(x))' + q(x)y(x) = \lambda f(x, y(x))$, with impulsive and integral boundary conditions $-\Delta(y^{[1]}(x_i)) = I_i(y(x_i))$, $i = 1, 2, \ldots, m$, $y(0) - ay^{[1]}(0) = \int_0^\omega g_0(s)y(s)ds$, $y(\omega) - by^{[1]}(\omega) = \int_0^\omega g_1(s)y(s)ds$, where $y^{[1]}(x) = p(x)y'(x)$. The expression of Green's function and the existence of positive solution for the system are obtained. Upper and lower bounds for positive solutions are also given. When $p(t)$, $I(\cdot)$, $g_0(s)$, and $g_1(s)$ take different values, the system can be simplified to some forms which has been studied in the works by Guo and LakshmiKantham (1988), Guo et al. (1995), Boucherif (2009), He et al. (2011), and Atici and Guseinov (2001). Our discussion is based on the fixed point index theory in cones.

1. Introduction

The theory of impulsive differential equations in abstract spaces has become a new important branch and has developed rapidly (see [1–4]). As an important aspect, impulsive differential equations with boundary value problems have gained more attention. In recent years, experiments in a variety of different areas (especially in applied mathematics and physics) show that integral boundary conditions can represent the model more accurately. And researchers have obtained many good results in this field.

In this paper, we study the existence of positive solutions for the following system:

$$\begin{gathered}
-\left(y^{[1]}(x)\right)' + q(x)y(x) = \lambda f(x, y(x)), \quad x \neq x_i,\ x \in J^-, \\
-\Delta\left(y^{[1]}(x_i)\right) = I_i(y(x_i)), \quad i = 1, 2, \ldots, m, \\
y(0) - ay^{[1]}(0) = \int_0^\omega g_0(s)y(s)ds, \\
y(\omega) - by^{[1]}(\omega) = \int_0^\omega g_1(s)y(s)ds,
\end{gathered} \tag{1}$$

where $y^{[1]}(x) = p(x)y'(x)$, $J^- = J \setminus \{x_1, x_2, \ldots, x_m\}$, $J = [0, \omega]$, $0 < x_1 < x_2 < \cdots < x_m < \omega$, $f \in C(J \times R^+, R^+)$. $y(x)$, $y^{[1]}(x)$ are left continuous at $x = x_i$, $\Delta(y^{[1]}(x_i)) = y^{[1]}(x_i^+) - y^{[1]}(x_i^-)$. $I_i \in C(R^+, R^+)$. And $a > 0$, $b < 0$, $g_0, g_1 : [0, 1] \to [0, \infty)$ are continuous and positive functions.

When $p(t)$, $I(\cdot)$, $g_0(s)$, and $g_1(s)$ take different values, the system can be simplified to some forms which have been studied. For example, [5–10] discussed the existence of positive solution in case $p(t) = 1$.

Let $p(t) = 1$, $g_0, g_1 = 0$, [11, 12] investigated the system with only one impulse. Reference [13] studied the system when $I(\cdot) = 0$, $g_0, g_1 = 0$. Readers can read the papers in [13] for details.

Throughout the rest of the paper, we assume ω is a fixed positive number, and λ is a parameter. $p(x)$, $q(x)$ are real-valued measurable functions defined on J, and they satisfy the following condition:

(H1) $p(x) > 0$, $q(x) \geq 0$, $q(x) \not\equiv 0$ almost everywhere, and

$$\int_0^\omega \frac{1}{p(x)}dx < \infty, \qquad \int_0^\omega q(x)dx < \infty. \tag{2}$$

This paper aims to obtain the positive solution for (1). In Section 2, we introduce some lemmas and notations. In particular, the expression and some properties of Green's functions are investigated. After the preparatory work, we draw the main results in Section 3.

2. Preliminaries

Theorem 1 (Krasnoselskii's fixed point theorem). *Let E be a Banach space and $C \in E$. Assume Ω_1, Ω_2 are open sets in E with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and $S : C\bigcap(\overline{\Omega}_2 \setminus \Omega_1) \to C$ be a completely continuous operator such that either*

(i) $\|s(y)\| \leq \|y\|$, $y \in C \cap \partial\Omega_1$, *and* $\|s(y)\| \geq \|y\|$, $y \in C \cap \partial\Omega_2$; *or*

(ii) $\|s(y)\| \geq \|y\|$, $y \in C \cap \partial\Omega_1$, *and* $\|s(y)\| \leq \|y\|$, $y \in C \cap \partial\Omega_2$.

Then S has a fixed point in $C\bigcap(\overline{\Omega}_2 \setminus \Omega_1)$.

Definition 2. For two differential functions y and z, we defined their Wronskian by

$$\begin{aligned} W_x(y,z) &= y(x) z^{[1]}(x) - y^{[1]}(x) z(x) \\ &= p(x)\left[y(x) z'(x) - y'(x) z(x)\right]. \end{aligned} \tag{3}$$

Consider the linear nonhomogeneous problem of the form

$$-\left(y^{[1]}(x)\right)' + q(x) y(x) = h(x), \quad x \in J. \tag{4}$$

Its corresponding homogeneous equation is

$$-\left(y^{[1]}(x)\right)' + q(x) y(x) = 0, \quad x \in J. \tag{5}$$

Lemma 3. *Suppose that y_1 and y_2 form a fundamental set of solutions for the homogeneous problem (5). Then the general solution of the nonhomogeneous problem (4) is given by*

$$\begin{aligned} y(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &\quad + \int_0^x \frac{y_1(x) y_2(s) - y_1(s) y_2(x)}{w_s(y_1, y_2)} h(s)\, ds, \end{aligned} \tag{6}$$

where c_1 and c_2 are arbitrary constants.

Proof. We just need to show that the function

$$z(x) = \int_0^x \frac{y_1(x) y_2(s) - y_1(s) y_2(x)}{w_s(y_1, y_2)} h(s)\, ds \tag{7}$$

is a particular solution of (4). From (7), we have for $x \in [0, \omega]$,

$$z'(x) = \int_0^x \frac{y_1'(x) y_2(s) - y_1(s) y_2'(x)}{w_s(y_1, y_2)} h(s)\, ds, \tag{8}$$

$$\left[p(x) z'(x)\right]' = -h(x) + q(x) z(x). \tag{9}$$

Besides, from (7) and (8), we have

$$z(0) = 0, \qquad z^{[1]}(0) = 0. \tag{10}$$

Thus, $z(x)$ satisfies (4). □

Consider the following boundary value problem with integral boundary conditions:

$$\begin{aligned} -\left(y^{[1]}(x)\right)' + q(x) y(x) &= h(x), \quad x \in J, \\ y(0) - a y^{[1]}(0) &= \int_0^\omega g_0(s) \sigma_0(s)\, ds, \\ y(\omega) - b y^{[1]}(\omega) &= \int_0^\omega g_1(s) \sigma_1(s)\, ds. \end{aligned} \tag{11}$$

Denote by $u(x)$ and $v(x)$ the solutions of the homogenous equation (5) satisfying the initial conditions

$$\begin{aligned} u(0) &= a, \qquad u^{[1]}(0) = 1, \\ v(\omega) &= -b, \qquad v^{[1]}(\omega) = -1. \end{aligned} \tag{12}$$

(H2) Let $x, s \in J$, denote a function

$$\phi(x,s) = \frac{u(x)}{u(\omega) - b u^{[1]}(\omega)} g_1(s) + \frac{v(x)}{v(0) - a v^{[1]}(0)} g_0(s) \tag{13}$$

satisfies $0 \leq \phi(x,s) < 1/\omega$.

For convenience, we denote $m := \min\{\phi(x,s); x, s \in J\}$, $M := \max\{\phi(x,s); x, s \in J\}$.

Lemma 4. *Let $K(x,s)$ be a nonnegative continuous function defined for $-\infty < x_1 \leq x, s \leq x_2 < \infty$ and $\psi(x)$ a nonnegative integrable function on $[x_1, x_2]$. Then for arbitrary nonnegative continuous function $\varphi(x)$ defined on $[x_1, x_2]$, the Volterra integral equation*

$$y(x) = \varphi(x) + \int_{x_1}^x K(x,s) \psi(s) y(s)\, ds, \quad x_1 \leq x \leq x_2 \tag{14}$$

has a unique solution $y(x)$. Moreover, this solution is continuous and satisfied the inequality

$$y(x) \geq \varphi(x), \quad x_1 \leq x \leq x_2. \tag{15}$$

Proof. We solve (14) by the method of successive approximations setting

$$\begin{aligned} y_0(x) &= \varphi(x), \\ y_n &= \int_{x_1}^x K(x,s) \psi(s) y_{n-1}(s)\, ds, \quad n = 1, 2, \ldots. \end{aligned} \tag{16}$$

If the series $\sum_{n=0}^{\infty} y_n(x)$ converges uniformly with respect to $x \in [x_1, x_2]$, then its sum will be, obviously, a continuous solution of (14). To prove the uniform convergence of this series, we put

$$\max_{x_1 \leqslant x \leqslant x_2} \varphi(x) = c, \qquad \max_{x_1 \leqslant x, s \leqslant x_2} K(x,s) = c_1. \tag{17}$$

Then it is easy to get from (16) that

$$0 \leqslant y_n(x) \leqslant c \frac{c_1^n}{n!} \left[\int_{x_1}^x \psi(s)\, ds\right]^n, \quad n = 0, 1, 2, \ldots. \tag{18}$$

Hence it follows that (14) has a continuous solution

$$y(x) = \sum_{n=0}^{\infty} y_n(x) \tag{19}$$

and because $y_0 = \varphi(x)$, $y_n \geq 0$, $n = 1, 2, \ldots$, for this solution the inequality (15) holds. Uniqueness of the solution of (14) can be proved in a usual way. The proof is complete. □

Remark 5. Evidently, the statement of Lemma 4 is also valid for the Volterra equation of the form

$$y(x) = \varphi(x) + \int_x^{x_2} K(x,s)\psi(s)y(s)\,ds, \quad x_1 \leq x \leq x_2. \tag{20}$$

Lemma 6. *For the solution $y(x)$ of the BVP* (11), *the formula*

$$y(x) = w(x) + \int_0^{\omega} G(x,s)h(s)\,ds, \quad x \in J \tag{21}$$

holds, where

$$\begin{aligned} w(x) &= \frac{u(x)}{u(\omega) - bu^{[1]}(\omega)} \int_0^{\omega} g_1(s)\sigma_1(s)\,ds \\ &\quad + \frac{v(x)}{v(0) - av^{[1]}(0)} \int_0^{\omega} g_0(s)\sigma_0(s)\,ds, \end{aligned} \tag{22}$$

$$G(x,s) = -\frac{1}{w_s(u,v)} \begin{cases} u(s)v(x), & 0 \leq s \leq x \leq \omega, \\ u(x)v(s), & 0 \leq x \leq s \leq \omega. \end{cases}$$

Proof. By Lemma 3, the general solutions of the nonhomogeneous problem (4) has the form

$$\begin{aligned} y(x) &= c_1 u(x) + c_2 v(x) \\ &\quad + \int_0^x \frac{u(x)v(s) - u(s)v(x)}{W_s(u,v)} h(s)\,ds, \end{aligned} \tag{23}$$

where c_1 and c_2 are arbitrary constants. Now we try to choose the constants c_1 and c_2 so that the function $y(x)$ satisfies the boundary conditions of (11).

From (23), we have

$$\begin{aligned} y^{[1]}(x) &= c_1 u^{[1]}(x) + c_2 v^{[1]}(x) \\ &\quad + \int_0^x \frac{u^{[1]}(x)v(s) - u(s)v^{[1]}(x)}{W_s(u,v)} h(s)\,ds. \end{aligned} \tag{24}$$

Consequently,

$$\begin{aligned} y(0) &= c_1 a + c_2 v(0), \\ y^{[1]}(0) &= c_1 + c_2 v^{[1]}(0). \end{aligned} \tag{25}$$

Substituting these values of $y(0)$ and $y^{[1]}(0)$ into the first boundary condition of (11), we find

$$c_2 = \frac{1}{v(0) - av^{[1]}(0)} \int_0^{\omega} g_0(s)\sigma_0(s)\,ds. \tag{26}$$

Similarly from the second boundary condition of (11), we can find

$$\begin{aligned} c_1 &= \frac{1}{u(\omega) - bu^{[1]}(\omega)} \int_0^{\omega} g_1(s)\sigma_1(s)\,ds \\ &\quad - \int_0^{\omega} \frac{v(s)}{W_s(u,v)} h(s)\,ds. \end{aligned} \tag{27}$$

Putting these values of c_1 and c_2 in (23), we get the formula (21), (22). □

Lemma 7. *Let condition (H1) hold. Then for the Wronskian of solution $u(x)$ and $v(x)$, the inequality $W_x(u,v) < 0$, $x \in J$ holds.*

Proof. Using the initial conditions (12), we can deduce from (5) for $u(x)$ and $v(x)$ the following equations:

$$\begin{aligned} u^{[1]}(x) &= 1 + \int_0^x q(s)u(s)\,ds, \\ u(x) &= a + \int_0^x \frac{1}{p(t)}\,dt \\ &\quad + \int_0^x \left[\int_s^x \frac{dt}{p(t)}\right] q(s)u(s)\,ds, \\ v^{[1]}(x) &= -1 - \int_x^{\omega} q(s)v(s)\,ds, \\ v(x) &= -b + \int_x^{\omega} \frac{1}{p(t)}\,dt \\ &\quad + \int_x^{\omega} \left[\int_x^s \frac{dt}{p(t)}\right] q(s)v(s)\,ds. \end{aligned} \tag{28}$$

From (28), by condition (H1) and Lemma 4, it follows that

$$\begin{aligned} u(x) &\geq a + \int_0^x \frac{dt}{p(t)} > 0, \quad u^{[1]}(x) \geq 1 > 0, \\ v(x) &\geq -b + \int_x^{\omega} \frac{dt}{p(t)} > 0, \quad v^{[1]}(x) \leq -1 < 0. \end{aligned} \tag{29}$$

Now from (3), we get $W_x(u,v) < 0$, $x \in J$. The proof is complete. □

From (21), (22), and Lemma 7, the following lemma follows.

Lemma 8. *Under condition (H1) the Green's function $G(x,s)$ of the BVP* (11) *is positive. That is, $G(x,s) > 0$ for $x, s \in J$.*

Let $C(J)$ denote the Banach of all continuous functions $y : I \to \mathbb{R}$ equipped with the form $\|y\| = \max\{|y(x)|; x \in J\}$, for any $y \in C(J)$. Denote $P = \{y \in C(J); y(x) \geqslant 0, y \in J\}$, then P is a positive cone in $C(J)$.

Let us set $A = \max_{0 \leqslant x,s \leqslant \omega} G(x,s)$, $B = \min_{0 \leqslant x,s \leqslant \omega} G(x,s)$, and by Lemma 8, obviously, $A > B > 0$, $x, s \in J$.

Define a mapping Φ in Banach space $C(J)$ by

$$(\Phi y)(x) = w(x) + \lambda \int_0^\omega G(x,s) f(s, y(s))\, ds + \sum_{i=0}^{m} G(x, x_i) I_i(y(x_i)), \quad x \in J, \tag{30}$$

where

$$w(x) = \frac{u(x)}{u(\omega) - bu^{[1]}(\omega)} \int_0^\omega g_1(s) y(s)\, ds + \frac{v(x)}{v(0) - av^{[1]}(0)} \int_0^\omega g_0(s) y(s)\, ds. \tag{31}$$

Lemma 9. *The fixed point of the mapping Φ is a solution of* (1).

Proof. Clearly, Φy is continuous in x for $x \in J$. For $x \neq x_k$,

$$(\Phi y)'(x) = w'(x) + \lambda \int_0^\omega \frac{\partial G}{\partial x} f(s, y(s))\, ds + \sum_{i=0}^{m} \frac{\partial G(x, x_i)}{\partial x} I_i(y(x_i)), \tag{32}$$

where

$$w'(x) = \frac{u'(x)}{u(\omega) - bu^{[1]}(\omega)} \int_0^\omega g_1(s) y(s)\, ds + \frac{v'(x)}{v(0) - av^{[1]}(0)} \int_0^\omega g_0(s) y(s)\, ds. \tag{33}$$

We have

$$(\Phi y)^{[1]}(x) = w^{[1]}(x) + \lambda \int_0^\omega p(x) \frac{\partial G}{\partial x} f(s, y(s))\, ds + \sum_{i=0}^{m} p(x) \frac{\partial G(x, x_i)}{\partial x} I_i(y(x_i)), \tag{34}$$

where

$$w^{[1]}(x) = \frac{u^{[1]}(x)}{u(\omega) - bu^{[1]}(\omega)} \int_0^\omega g_1(s) y(s)\, ds + \frac{v^{[1]}(x)}{v(0) - av^{[1]}(0)} \int_0^\omega g_0(s) y(s)\, ds. \tag{35}$$

We can easy get that

$$(\Phi y)(0) - a(\Phi y)^{[1]}(0) = \int_0^\omega g_0(s) y(s)\, ds,$$

$$(\Phi y)(\omega) - b(\Phi y)^{[1]}(\omega) = \int_0^\omega g_1(s) y(s)\, ds,$$

$$\begin{aligned}
\Delta(\Phi y)^{[1]}(x_k) &= p(x_k^+)(\Phi y)'(x_k^+) - p(x_k^-)(\Phi y)'(x_k^-) \\
&= p(x_k)\left[-\frac{u(x_k) v'(x_k)}{W_{x_k}(u,v)} + \frac{u'(x_k) v(x_k)}{W_{t_k}(u,v)}\right] \times I_k(y(x_k)) \\
&= -I_k(y(x_k)),
\end{aligned}$$

$$\begin{aligned}
\left(p(x)(\Phi y)'(x)\right)' &= \Bigg[p(x) w'(x) + \lambda \int_0^\omega p(x) \frac{\partial G}{\partial x} f(s, y(s))\, ds + \sum_{i=0}^{m} p(x) \frac{\partial G(x, x_i)}{\partial x} I_i(y(x_i))\Bigg]' \\
&= q(x) w(x) + \lambda q(x) \times \int_0^\omega G(x,s) f(s, y(s))\, ds - \lambda f(x, y(x)) + q(x) \times \sum_{i=0}^{m} G(x, x_i) I_i(y(x_i)) \\
&= q(x)(\Phi y)(x) - \lambda f(x, y(x)),
\end{aligned} \tag{36}$$

which implies that the fixed poind of Φ is a solution of (1). The proof is complete. □

Lemma 10. *Let $P_0 := \{y \in P; \min_{x \in J} y(x) \geq ((1 - M\omega)B/(1 - m\omega)A)\|y\|\}$, then P_0 is a cone.*

Proof. (i) For for all $y_1, y_2 \in P_0$ and for all $\alpha \geq 0$, $\beta \geq 0$, we have

$$\min(\alpha y_1) \geqslant \alpha \cdot \frac{(1 - M\omega) B}{(1 - m\omega) A} \|y_1\|, \qquad \min(\beta y_2) \geqslant \beta \cdot \frac{(1 - M\omega) B}{(1 - m\omega) A} \|y_2\|. \tag{37}$$

Moreover

$$\min(\alpha y_1 + \beta y_2) \geqslant \frac{(1 - M\omega) B}{(1 - m\omega) A} (\alpha \|y_1\| + \beta \|y_2\|) \geqslant \frac{(1 - M\omega) B}{(1 - m\omega) A} \|\alpha y_1 + \beta y_2\|. \tag{38}$$

Thus $\alpha y_1 + \beta y_2 \in P_0$.

(ii) If $y \in P_0$ and $-y \in P_0$, we have

$$\min_{x \in J}(y(x)) \geqslant \frac{(1 - M\omega) B}{(1 - m\omega) A} \|y\|, \qquad \min_{x \in J}(-y(x)) \geqslant \frac{(1 - M\omega) B}{(1 - m\omega) A} \|y\|. \tag{39}$$

It implies that $y = 0$. Hence P_0 is a cone. □

Defined a linear operator $A : C(J) \to C(J)$ by

$$(Ay)(x) = \int_0^\omega \phi(x,s)\, y(s)\, ds. \tag{40}$$

Then we have the following lemma.

Lemma 11. *If (H2) is satisfied, then*

(i) *A is a bounded linear operator, $A(P) \subset P$;*

(ii) *$(I - A)$ is invertible;*

(iii) $\|(I-A)^{-1}\| \le 1/(1 - M\omega)$.

Proof. (i)

$$\begin{aligned} A(\alpha y_1(x) + \beta y_2(x)) &= \int_0^\omega \phi(x,s)\left[\alpha y_1(s) + \beta y_2(s)\right] ds \\ &= \alpha (Ay_1)(x) + \beta (Ay_2)(x), \end{aligned} \tag{41}$$

for all $\alpha, \beta \in \mathbb{R}$, $y_1, y_2 \in C(J)$.

Using $\phi(x,s) \le M$, it is easy to see that $|(Ay)(t)| \le M\omega\|y\|$.

Let $y \in P$. Then $y(s) \ge 0$ for all $s \in J$. Since $\phi(t,s) \ge m \ge 0$, it follows that $(Ay)(x) \ge 0$ for each $x \in J$. So $A(P) \subset P$.

(ii) We want to show that $(I - A)$ is invertible, or equivalently 1 is not an eigenvalue of A.

Since $M < 1/\omega$, it follows from condition (H2) that $\|Ay\| \le M\omega\|y\| < \|y\|$.

So

$$\|A\| = \sup_{y \ne 0} \frac{\|Ay\|}{\|y\|} \leqslant M\omega < 1. \tag{42}$$

On the other hand, we suppose 1 is an eigenvalue of A, then there exists a $y \in C(J)$ such that $Ay = y$. Moreover, we can obtain that $\|Ay\|/\|y\| = 1$. So $\|A\| \geqslant 1$. Thus this assumption is false.

Conversely, 1 is not an eigenvalue of A. Equivalently, $(I - A)$ is invertible.

(iii) We use the theory of Fredholm integral equations to find the expression for $(I - A)^{-1}$.

Obviously, for each $x \in J$, $y(x) = (I-A)^{-1}z(x) \Leftrightarrow y(x) = z(x) + (Ay)(x)$.

By (40), we can get

$$y(x) = z(x) + \int_0^\omega \phi(x,s)\, y(s)\, ds. \tag{43}$$

The condition $M < 1/\omega$ implies that 1 is not an eigenvalue of the kernel $\phi(x,s)$. So (43) has a unique continuous solution y for every continuous function z.

By successive substitutions in (43), we obtain

$$y(x) = z(x) + \int_0^\omega R(x,s)\, z(s)\, ds, \tag{44}$$

where the resolvent kernel $R(x,s)$ is given by

$$R(x,s) = \sum_{j=1}^{\infty} \phi_j(x,s). \tag{45}$$

Here $\phi_j(x,s) = \int_0^\omega \phi(x,\tau)\phi_{j-1}(\tau,s)ds$, $j = 2, \dots$ and $\phi_1(x,s) = \phi(x,s)$.

The series on the right in (45) is convergent because $|\phi(x,s)| \leqslant M < 1/\omega$.

It can be easily verified that $R(x,s) \le M/(1 - M\omega)$.

So we can get

$$(I-A)^{-1} z(x) = z(x) + \int_0^\omega R(x,s)\, z(s)\, ds. \tag{46}$$

Therefore

$$\begin{aligned} (I-A)^{-1} z(x) &\le z(x) + \frac{M}{1 - M\omega} \int_0^\omega z(s)\, ds \\ &\le \|z\| \left(1 + \frac{M\omega}{1 - M\omega}\right) = \frac{1}{1 - M\omega} \|z\|. \end{aligned} \tag{47}$$

So

$$\frac{\left\|(I-A)^{-1} z\right\|}{\|z\|} \le \frac{1}{1 - M\omega}. \tag{48}$$

Thus $\|(I-A)^{-1}\| \le 1/(1 - M\omega)$. This completes the proof of the lemma. □

Remark 12. Since $\phi(x,s) \ge m$ for each $(x,s) \in J$, it is easy to prove that $R(x,s) \ge m/(1 - m\omega)$.

3. Main Results

Consider the following boundary value problem (BVP) with impulses:

$$\begin{gathered} -\left(y^{[1]}(x)\right)' + q(x)\, y(x) = \lambda f(x, y(x)), \\ x \ne x_i, \quad x \in J, \\ -\Delta\left(y^{[1]}(x_i)\right) = I_i\left(y(x_i)\right), \quad i = 1, 2, \dots, m, \\ y(0) - a y^{[1]}(0) = \int_0^\omega g_0(s)\, y(s)\, ds, \\ y(\omega) - b y^{[1]}(\omega) = \int_0^\omega g_1(s)\, y(s)\, ds. \end{gathered} \tag{49}$$

Denote a nonlinear operator $T : PC(J) \to PC(J)$ by

$$\begin{aligned} (Ty)(x) = \lambda &\int_0^\omega G(x,s)\, f(s, y(s))\, ds \\ &+ \sum_{i=0}^{m} G(x, x_i)\, I_i\left(y(x_i)\right). \end{aligned} \tag{50}$$

It is easy to see that solutions of (49) are solutions of the following equation:

$$y(x) = Ty(x) + Ay(x), \quad x \in J^{-1}. \tag{51}$$

According to Lemma 11, y is a solution of (51) if and only if it is a solution of

$$y(x) = (I - A)^{-1} T y(x). \tag{52}$$

It follows from (46) that y is a solution of (52) if and only if

$$y(x) = (Ty)(x) + \int_0^\omega R(x,s)(Ty)(s)\,ds. \tag{53}$$

So, the operator Φ can be written as

$$(\Phi y)(x) = (Ty)(x) + \int_0^\omega R(x,s)(Ty)(s)\,ds. \tag{54}$$

It satisfies the conditions of Theorem 1 with $E = C(J)$ and the cone $C = P_0$.

Let us list some marks and conditions for convenience.

The nonlinearity $f : J \times [0,\infty) \to [0,\infty)$ is continuous and satisfies the following.

(H3) There exist $L_1 > 0$ and $\alpha(x) \in P$, $r_1 \in \mathbb{R}$ with $r_1 \geqslant \sum_{i=0}^m I_i(y(x_i))/\lambda \int_0^\omega \alpha(s)ds$ such that

$$f(x,y) \le \alpha(x)\left[y(1 - M\omega) - r_1\right] \tag{55}$$

for all $y \in (0, L_1]$, $x \in J$.

(H4) There exist $L_2 > L_1$ and $\beta(x) \in P$, $p_1 \in \mathbb{R}$ with $p_1 \leqslant \sum_{i=0}^m I_i(y(x_i))/\lambda \int_0^\omega \beta(s)ds$ such that

$$f(x,y) \ge \beta(x)\left[y(1 - m\omega) - p_1\right] \tag{56}$$

for all $y \in (L_2, \infty]$, $x \in J$.

Then, we can get the following theorem.

Theorem 13. *Assume (H1), (H2), (H3), and (H4) are satisfied. And*

$$(1 - m\omega) A^2 \int_0^\omega \alpha(s)\,ds \leqslant (1 - M\omega) B^2 \int_0^\omega \beta(s)\,ds, \tag{57}$$

then, if λ satisfies

$$\frac{(1 - m\omega) A}{(1 - M\omega) B^2 \int_0^\omega \beta(s)\,ds} \leqslant \lambda \leqslant \frac{1}{A \int_0^\omega \alpha(s)\,ds}. \tag{58}$$

The problem (49) *has at least one positive solution.*

Proof. First of all, we show that operator Φ is defined by (54) maps P_0 into itself. Let $y \in P_0$.

Then $(\Phi y)(x) \ge 0$ for all that $t \in J^{-1}$, and

$$\begin{aligned}(\Phi y)(x) \leqslant{}& \frac{\lambda A}{1 - M\omega} \int_0^\omega f(s, y(s))\,ds \\ &+ \frac{A}{1 - M\omega} \sum_{i=0}^m I_i(y(x_i)).\end{aligned} \tag{59}$$

Because from the formula (54), we have

$$\begin{aligned}(\Phi y)(x) &= (Ty)(x) + \int_0^\omega R(x,s)(Ty)(s)\,ds \\
&= \lambda \int_0^\omega G(x,s) f(s, y(s))\,ds \\
&\quad + \sum_{i=0}^m G(x, x_i) I_i(y(x_i)) \\
&\quad + \lambda \int_0^\omega R(x,s) \int_0^\omega G(x,\tau) f(\tau, y(\tau))\,d\tau\,ds \\
&\quad + \int_0^\omega R(x,s) \sum_{i=0}^m G(x, x_i) I_i(y(x_i))\,ds \\
&\le \lambda \left(1 + \frac{M\omega}{1 - M\omega}\right) \int_0^\omega G(x,s) f(s, y(s))\,ds \\
&\quad + \sum_{i=0}^m G(x, x_i) I_i(y(x_i)) \\
&\quad + \frac{M\omega}{1 - M\omega} \sum_{i=0}^m G(x, x_i) I_i(y(x_i)) \\
&\le \frac{\lambda A}{1 - M\omega} \int_0^\omega f(s, y(s))\,ds \\
&\quad + \frac{A}{1 - M\omega} \sum_{i=0}^m I_i(y(x_i)).\end{aligned} \tag{60}$$

Hence, inequality (59) is established.

This implies that

$$\|\Phi y\| \leqslant \frac{\lambda A}{1 - M\omega} \int_0^\omega f(s, y(s))\,ds + \frac{A}{1 - M\omega} \sum_{i=0}^m I_i(y(x_i)), \tag{61}$$

or equivalently

$$\int_0^\omega f(s, y(s))\,ds \geqslant \frac{1 - M\omega}{\lambda A} \|\Phi y\| - \frac{1}{\lambda} \sum_{i=0}^m I_i(y(x_i)). \tag{62}$$

On the other hand, it follows that

$$\begin{aligned}(\Phi y)(x) \geqslant{}& \frac{\lambda B}{1 - m\omega} \int_0^\omega f(s, y(s))\,ds \\ &+ \frac{B}{1 - m\omega} \sum_{i=0}^m I_i(y(x_i)).\end{aligned} \tag{63}$$

In fact, we have

$$\begin{aligned}(\Phi y)(x) &= \lambda \int_0^\omega G(x,s) f(s,y(s))\,ds \\ &\quad + \sum_{i=0}^m G(x,x_i) I_i(y(x_i)) \\ &\quad + \lambda \int_0^\omega R(x,s) \int_0^\omega G(x,\tau) f(\tau,y(\tau))\,d\tau\,ds \\ &\quad + \int_0^\omega R(x,s) \sum_{i=0}^m G(x,x_i) I_i(y(x_i))\,ds \\ &\geq \lambda \left(1+\frac{m\omega}{1-m\omega}\right) \int_0^\omega G(x,s) f(s,y(s))\,ds \\ &\quad + \sum_{i=0}^m G(x,x_i) I_i(y(x_i)) \\ &\quad + \frac{m\omega}{1-m\omega} \sum_{i=0}^m G(x,x_i) I_i(y(x_i)) \\ &\geq \frac{\lambda B}{1-m\omega} \int_0^\omega f(s,y(s))\,ds + \frac{B}{1-m\omega} \sum_{i=0}^m I_i(y(x_i)).\end{aligned} \tag{64}$$

It follows from (62) that

$$\begin{aligned}(\Phi y)(x) &\geq \frac{\lambda B}{1-m\omega} \cdot \left[\frac{1-M\omega}{\lambda A}\|\Phi y\| - \frac{1}{\lambda}\sum_{i=0}^m I_i(y(x_i))\right] \\ &\quad + \frac{B}{1-m\omega}\sum_{i=0}^m I_i(y(x_i)) \\ &= \frac{(1-M\omega)B}{(1-m\omega)A}\|\Phi y\| - \frac{B}{1-m\omega}\sum_{i=0}^m I_i(y(x_i)) \\ &\quad + \frac{B}{1-m\omega}\sum_{i=0}^m I_i(y(x_i)) \\ &= \frac{(1-M\omega)B}{(1-m\omega)A}\|\Phi y\|.\end{aligned} \tag{65}$$

So, we get

$$(\Phi y)(x) \geqslant \frac{(1-M\omega)B}{(1-m\omega)A}\|\Phi y\|. \tag{66}$$

This show that $\Phi y \in P_0$.

It is easy to see that Φ is the complete continuity.

We now proceed with the construction of the open sets Ω_1 and Ω_2.

First, let $y \in P_0$ with $\|y\| = L_1$. Inequality (59) implies

$$\begin{aligned}(\Phi y)(x) &\leq \frac{\lambda A}{1-M\omega}\int_0^\omega f(s,y(s))\,ds + \frac{A}{1-M\omega}\sum_{i=0}^m I_i(y(x_i)) \\ &\leq \frac{\lambda A}{1-M\omega}\int_0^\omega \alpha(s)\left[y(s)(1-M\omega) - r_1\right]ds \\ &\quad + \frac{A}{1-M\omega}\sum_{i=0}^m I_i(y(x_i)) \\ &= \lambda A\int_0^\omega \alpha(s) y(s)\,ds - \frac{\lambda A}{1-M\omega} r_1 \\ &\quad \times \int_0^\omega \alpha(s)\,ds + \frac{A}{1-M\omega}\sum_{i=0}^m I_i(y(x_i)) \\ &= \lambda A\int_0^\omega \alpha(s) y(s)\,ds + \frac{A}{1-M\omega} \\ &\quad \times \left[\sum_{i=0}^m I_i(y(x_i)) - \lambda r_1 \int_0^\omega \alpha(s)\,ds\right].\end{aligned} \tag{67}$$

By condition (H3) and (58), we obtain

$$\begin{gathered}\sum_{i=0}^m I_i(y(x_i)) - \lambda r_1 \int_0^\omega \alpha(s)\,ds \leqslant 0, \\ \lambda A \int_0^\omega \alpha(s)\,ds \leqslant 1.\end{gathered} \tag{68}$$

So

$$(\Phi y)(x) \leqslant \lambda A \int_0^\omega \alpha(s)\,ds\,\|y\| \leqslant \|y\|. \tag{69}$$

Consequently, $\|\Phi y\| \leqslant \|y\|$.

Let $\Omega_1 := \{y \in C(J); \|y\| < L_1\}$. Then, we have $\|\Phi y\| \leqslant \|y\|$ for $y \in P_0 \cap \partial\Omega_1$.

Next, let $\widetilde{L_2} = \max\{2L_1, ((1-m\omega)A/(1-M\omega)B)L_2\}$ and set $\Omega_2 := \{y \in C(J); \|y\| < \widetilde{L_2}\}$.

For $y \in P_0$ with $\|y\| = \widetilde{L_2}$, we have

$$\begin{aligned}\min_{x\in J} y(x) &\geq \frac{(1-M\omega)B}{(1-m\omega)A}\|y\| = \frac{(1-M\omega)B}{(1-m\omega)A}\widetilde{L_2} \\ &\geq \frac{(1-M\omega)B}{(1-m\omega)A}\cdot\frac{(1-m\omega)A}{(1-M\omega)B}L_2 = L_2.\end{aligned} \tag{70}$$

It follows from (63) that

$$\begin{aligned}(\Phi y)(x) &\geq \frac{\lambda B}{1-m\omega}\int_0^\omega f(s,y(s))\,ds + \frac{B}{1-m\omega}\sum_{i=0}^m I_i(y(x_i)) \\ &\geq \frac{\lambda B}{1-m\omega}\int_0^\omega \beta(s)\left[y(s)(1-m\omega) - p_1\right]ds \\ &\quad + \frac{B}{1-m\omega}\sum_{i=0}^m I_i(y(x_i))\end{aligned}$$

$$= \lambda B \int_0^\omega \beta(s)\, y(s)\, ds + \frac{B}{1-m\omega} \times \left(\sum_{i=0}^m I_i(y(x_i)) - \lambda p_1 \int_0^\omega \beta(s)\, ds \right). \tag{71}$$

By condition (H4) and (58), we obtain

$$\sum_{i=0}^m I_i(y(x_i)) - \lambda p_1 \int_0^\omega \beta(s)\, ds \geqslant 0,$$
$$\lambda B \geqslant \frac{(1-m\omega) A}{(1-M\omega) B \int_0^\omega \beta(s)\, ds}. \tag{72}$$

Since $y \in P_0$ we have $y(x) \geqslant ((1-M\omega)B/(1-m\omega)A)\|y\|$ for all $x \in J$. It follows from the above inequality that

$$(\Phi y)(x) \geqslant \frac{(1-m\omega) A}{(1-M\omega) B \int_0^\omega \beta(s)\, ds} \int_0^\omega \beta(s)\, ds \cdot \frac{(1-M\omega) B}{(1-m\omega) A} \|y\| = \|y\|. \tag{73}$$

Hence $\|\Phi y\| \geqslant \|y\|$ for $y \in P_0 \cap \partial\Omega_2$.

It follows from (i) of Theorem 1 that Φ has a fixed point in $P_0 \cap (\overline{\Omega}_2 \setminus \Omega_1)$, and this fixed point is a solution of (49).

This completes the proof. □

Next, with L_1 and L_2 as above, we assume that f satisfied the following.

(H5) There exist $\alpha^*(x) \in P$, $r_1^* \in \mathbb{R}$ with $r_1^* \leqslant \sum_{i=0}^m I_i(y(x_i))/\lambda \int_0^\omega \alpha^*(s) ds$ such that

$$f(x, y) \geq \alpha^*(x) \left[y(1-m\omega) - r_1^* \right] \tag{74}$$

for all $y \in (0, L_1]$, $x \in J$.

(H6) There exist $\beta^*(x) \in P$, $p_1^* \in \mathbb{R}$ with $p_1^* \geqslant \sum_{i=0}^m I_i(y(x_i))/\lambda \int_0^\omega \beta^*(s) ds$ such that

$$f(x, y) \leq \beta^*(x) \left[y(1-M\omega) - p_1^* \right] \tag{75}$$

for all $y \in (L_2, \infty]$, $x \in J$.

Theorem 14. *Assume (H1), (H2), (H5), and (H6) are satisfied. And*

$$(1-m\omega) A^2 \int_0^\omega \beta^*(s)\, ds \leqslant (1-M\omega) B^2 \int_0^\omega \alpha^*(s)\, ds, \tag{76}$$

then, if λ satisfies

$$\frac{(1-m\omega) A}{(1-M\omega) B^2 \int_0^\omega \alpha^*(s)\, ds} \leqslant \lambda \leqslant \frac{1}{A \int_0^\omega \beta^*(s)\, ds}. \tag{77}$$

The problem (49) *has at least one positive solution.*

Proof. Let Φ be a completely continuous operator defined by (54). Then Φ maps the cone P_0 into itself.

First, let $y \in P_0$ with $\|y\| = L_1$. Inequality (63) implies

$$(\Phi y)(x) \geq \frac{\lambda B}{1-m\omega} \int_0^\omega f(s, y(s))\, ds + \frac{B}{1-m\omega} \sum_{i=0}^m I_i(y(x_i))$$
$$\geq \frac{\lambda B}{1-m\omega} \int_0^\omega \alpha^*(s) \left[y(s)(1-m\omega) - r_1^* \right] ds + \frac{B}{1-m\omega} \sum_{i=0}^m I_i(y(x_i))$$
$$= \lambda B \int_0^\omega \alpha^*(s)\, y(s)\, ds + \frac{B}{1-m\omega} \times \left(\sum_{i=0}^m I_i(y(x_i)) - \lambda r_1^* \int_0^\omega \alpha^*(s)\, ds \right). \tag{78}$$

By condition (H5) and (77), we obtain

$$\sum_{i=0}^m I_i(y(x_i)) - \lambda r_1^* \int_0^\omega \alpha^*(s)\, ds \geqslant 0,$$
$$\lambda B \geqslant \frac{(1-m\omega) A}{(1-M\omega) B \int_0^\omega \alpha^*(s)\, ds}. \tag{79}$$

Hence

$$(\Phi y)(x) \geqslant \frac{(1-m\omega) A}{(1-M\omega) B \int_0^\omega \alpha^*(s)\, ds} \int_0^\omega \alpha^*(s)\, y(s)\, ds. \tag{80}$$

Since $y \in P_0$, we have $y(x) \geqslant ((1-M\omega)B/(1-m\omega)A)\|y\|$ for all $x \in J$. It follows from the above inequality that

$$(\Phi y)(x) \geqslant \frac{(1-m\omega) A}{(1-M\omega) B \int_0^\omega \alpha^*(s)\, ds} \int_0^\omega \alpha^*(s)\, ds \cdot \frac{(1-M\omega) B}{(1-m\omega) A} \|y\| = \|y\|. \tag{81}$$

Let $\Omega_1 := \{y \in C(J); \|y\| < L_1\}$. Then, we have $\|\Phi y\| \geqslant \|y\|$ for $y \in P_0 \cap \partial\Omega_1$.

Next, let $\widetilde{L_2} = \max\{2L_1, ((1-m\omega)A/(1-M\omega)B)L_2\}$ and set $\Omega_2 := \{y \in C(J); \|y\| < \widetilde{L_2}\}$.

Then for $y \in P_0$ with $\|y\| = \widetilde{L_2}$ for all $x \in J$, we have $\min_{x\in J} y(x) \geqslant L_2$. Inequality (59) implies

$$(\Phi y)(x) \leq \frac{\lambda A}{1-M\omega} \int_0^\omega f(s, y(s))\, ds + \frac{A}{1-M\omega} \sum_{i=0}^m I_i(y(x_i))$$

$$\begin{aligned}&\leq \frac{\lambda A}{1-M\omega}\int_0^\omega \beta^*(s)\left[y(s)(1-M\omega)-p_1^*\right]ds\\&\quad+\frac{A}{1-M\omega}\sum_{i=0}^m I_i(y(x_i))\\&=\lambda A\int_0^\omega \beta^*(s)\,y(s)\,ds\\&\quad+\frac{A}{1-M\omega}\left[\sum_{i=0}^m I_i(y(x_i))-\lambda p_1^*\int_0^\omega \beta^*(s)\,ds\right].\end{aligned} \tag{82}$$

By condition (H6) and (77), we obtain

$$\begin{gathered}\sum_{i=0}^m I_i(y(x_i))-\lambda p_1^*\int_0^\omega \beta^*(s)\,ds\leqslant 0,\\ \lambda A\int_0^\omega \beta^*(s)\,ds\leqslant 1.\end{gathered} \tag{83}$$

So

$$(\Phi y)(x)\leqslant \lambda A\int_0^\omega \beta^*(s)\,ds\,\|y\|\leqslant\|y\|\leqslant 1. \tag{84}$$

Therefore $\|\Phi y\|\leqslant\|y\|$ with $\|y\|=\widetilde{L_2}$.

Then, we have $\|\Phi y\|\leqslant\|y\|$ for $y\in P_0\cap\partial\Omega_2$.

We see the case (ii) of Theorem 1 is met. It follows that Φ has a fixed point in $P_0\cap(\overline{\Omega}_2\setminus\Omega_1)$, and this fixed point is a solution of (49).

This completes the proof. □

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Research Article

Interval Oscillation Criteria for Second-Order Nonlinear Forced Dynamic Equations with Damping on Time Scales

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By using the Riccati transformation technique and constructing a class of Philos-type functions on time scales, we establish some new interval oscillation criteria for the second-order damped nonlinear dynamic equations with forced term of the form $(r(t)x^{\Delta}(t))^{\Delta} + p(t)x^{\Delta\sigma}(t) + q(t)(x^{\sigma}(t))^{\alpha} = F(t, x^{\sigma}(t))$ on a time scale $\mathbb{T}$ which is unbounded, where α is a quotient of odd positive integer. Our results in this paper extend and improve some known results. Some examples are given here to illustrate our main results.

1. Introduction

In this paper, we are concerned with the oscillation criteria for the following forced second-order nonlinear dynamic equations with damping:

$$\left(r(t)x^{\Delta}(t)\right)^{\Delta} + p(t)x^{\Delta\sigma}(t) + q(t)\left(x^{\sigma}(t)\right)^{\alpha} = F\left(t, x^{\sigma}(t)\right) \quad (1)$$

on a time scale $\mathbb{T}$, where α is a quotient of odd positive integer. Throughout this paper and without further mention, we assume that the functions $r, p, q \in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}}, \mathbb{R})$, $F \in C(\mathbb{T}\times\mathbb{R}, \mathbb{R})$ with $r(t) > 0$, $p(t) \le 0$, and $p/r^{\sigma} \in \mathscr{R}^{+}$.

The theory of time scales, which has recently received a lot of attention, was originally introduced by Stefan Hilger in his Ph.D. thesis in 1988 (see [1]). Since then a rapidly expanding body of the literature has sought to unify, extend, and generalize ideas from discrete calculus, quantum calculus, and continuous calculus to arbitrary time scale calculus, where a time scale is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$, and the cases when this time scale is equal to the reals or to the integers represent the classical theories of differential and of difference equations. Many other interesting time scales exist, and they give rise to many applications (see [2]). Not only does the new theory of the so-called dynamic equations unify the theories of differential equations and difference equations, but also it extends these classical cases to cases "in between", for example, to the so-called q-difference equations when $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0, q > 1\}$ (which has important applications in quantum theory) and can be applied on different types of time scales like $\mathbb{T} = h\mathbb{N}$, $\mathbb{T} = \mathbb{N}^2$, and $\mathbb{T} = \mathbb{T}_n$, the space of the harmonic numbers. A book on the subject of time scales by Bohner and Peterson [2] summarizes and organizes much of the time scale calculus. For advances of dynamic equations on the time scales we refer the reader to the book [3].

Since we are interested in the oscillatory behavior of solutions near infinity, we make the assumption throughout this paper that the given time scale $\mathbb{T}$ is unbounded above. We assume $t_0 \in \mathbb{T}$ and it is convenient to assume $t_0 > 0$. We define the time scale interval of the form $[t_0,\infty)_{\mathbb{T}}$ by $[t_0,\infty)_{\mathbb{T}} = [t_0,\infty)\bigcap\mathbb{T}$. We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$.

By a solution of (1), we mean a nontrivial real-valued function x satisfying (1) on $[t_x,\infty)_{\mathbb{T}}$. A solution x of (1) is said to be oscillatory on $[t_x,\infty)_{\mathbb{T}}$ in case it is neither eventually positive nor eventually negative; otherwise, it is nonoscillatory. Equation (1) is said to be oscillatory in case all its solutions are oscillatory. Our attention is restricted to those solutions of (1) which exist on some half line $[t_x, \infty)_{\mathbb{T}}$ and satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq t_x$.

In recent years, there has been much research activity concerning the interval oscillation criteria for various second order differential equations; see [4–9]. A great deal of effort has been spent in obtaining criteria for oscillation of dynamic equations on time scales without forcing terms and it is usually assumed that the potential function q is positive. We refer the reader to the papers [10–25] and the references cited therein. On the other hand, there has been an increasing interest in obtaining sufficient conditions for the oscillation and nonoscillation of solutions of dynamic equations with forcing terms on time scales, and we refer the reader to the papers [26–35].

In 2004, by using two inequalities due to Hölder and Hardy and Littlewood and Polya as well as averaging functions, Li [4] established several interval oscillation criteria for the second order damped quasilinear differential equation with forced term of the following form:

$$\left(r(t)\left|y'(t)\right|^{\alpha-1}y'(t)\right)' + p(t)\left|y'(t)\right|^{\alpha-1}y'(t) + q(t)\left|y(t)\right|^{\beta-1}y(t) = e(t), \tag{2}$$

where $r \in C^1([t_0,\infty),\mathbb{R}^+)$, and $\beta > \alpha > 0$ are constants. The obtained results were based on the information only on a sequence of subintervals of $[t_0,\infty)$, rather than on the whole half line, made use of the oscillatory properties of the forcing term, and extended a known result which is obtained by means of a Picone identity.

Erbe et al. [26] studied the forced second-order nonlinear dynamic equation

$$\left(p(t)x^{\Delta}(t)\right)^{\Delta} + q(t)\left|x^{\sigma}(t)\right|^{\gamma}\operatorname{sgn}x^{\sigma}(t) = f(t) \tag{3}$$

on a time scale $\mathbb{T}$, where $\gamma \geq 1$. By using the Riccati substitution, the authors established some new interval oscillation criteria, that is, the criteria given by the behavior of q and f on a sequence of subintervals of $[a,\infty)_{\mathbb{T}}$.

In [31], by constructing a class of Philos-type functions on time scales, Li et al. established some oscillation criteria for the second order nonlinear dynamic equations with the forced term

$$x^{\Delta\Delta}(t) + a(t)f\left(x\left(q(t)\right)\right) = e(t) \tag{4}$$

on a time scale $\mathbb{T}$, where a, q, and e are real-valued rd-continuous functions defined on $\mathbb{T}$, with $q : T \to T, q(t) \to \infty$ as $t \to \infty$, and $f \to C(\mathbb{R},\mathbb{R}), xf(x) > 0$ whenever $x \neq 0$. The obtained results unified the oscillation of the second order forced differential equation and the second order forced difference equation. An example was considered to illustrate the main results in the end.

Erbe et al. [32] were concerned with the oscillatory behavior of the forced second-order functional dynamic equation with mixed nonlinearities

$$\left(a(t)x^{\Delta}(t)\right)^{\Delta} + \sum_{i=0}^{n}p_i(t)\left|x\left(\tau_i(t)\right)\right|^{\alpha_i}\operatorname{sgn}x\left(\tau_i(t)\right) = e(t) \tag{5}$$

on an arbitrary time scale $\mathbb{T}$, where $\alpha_0 = 1$, $\alpha_1 > \alpha_2 > \cdots > \alpha_m > 1 > \alpha_{m+1} > \cdots > \alpha_n$, and $\tau_i : \mathbb{T} \to \mathbb{T}$ are nondecreasing rd-continuous functions on $\mathbb{R}$, $\tau_i(t) \leq \sigma(t)$, and $\lim_{t\to\infty}\tau_i(t) = \infty$, for $i = 0,1,\ldots,n$. Their results in a particular case solved a problem posed by Anderson, and their results in the special cases when the time scale is the set of real numbers and the set of integers involved and improved some oscillation results for second-order differential and difference equations, respectively.

In this paper, we intend to use the Riccati transformation technique to obtain some interval oscillation criteria for (1). Our results do not require that q and f be of definite sign and are based on the information only on a sequence of subintervals of $[t_0,\infty]_{\mathbb{T}}$ rather than the whole half line. To the best of our knowledge, nothing is known regarding the oscillation behavior of (1) on time scales until now, and there are few results regarding the interval oscillation criteria for (1) on time scales without the damping term when $\alpha < 1$, so our results expand the known scope of the study.

The paper is organized as follows. In Section 2, we present some basic definitions and useful results from the theory of calculus on time scales on which we rely in the later section. In Section 3, we intend to use the Riccati transformation technique, integral averaging technique, and inequalities to obtain some sufficient conditions for oscillation of every solution of (1). In Section 4, we give two examples to illustrate Theorems 3 and 7, respectively.

2. Some Preliminaries

On any time scale $\mathbb{T}$, we define the forward and the backward jump operators by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},\qquad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \tag{6}$$

where $\inf\emptyset = \sup\mathbb{T}$ and $\sup\emptyset = \inf\mathbb{T}$. A point $t \in \mathbb{T}$ is said to be left-dense if $\rho(t) = t$, right-dense if $\sigma(t) = t$, left-scattered if $\rho(t) < t$, and right-scattered if $\sigma(t) > t$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) = \sigma(t) - t$.

For a function $f : \mathbb{T} \to \mathbb{R}$, the (delta) derivative is defined by

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}, \tag{7}$$

if f is continuous at t and t is right-scattered. If t is right-dense, then the derivative is defined by

$$f^{\Delta}(t) = \lim_{s\to t^+}\frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} = \lim_{s\to t^+}\frac{f(t) - f(s)}{t - s}, \tag{8}$$

provided this limit exists. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous provided f is continuous at right-dense points and there exists a finite left limit at all left-dense points in $\mathbb{T}$. The set of all such rd-continuous functions is denoted by $C_{\mathrm{rd}}(\mathbb{T})$. The derivative f^{Δ} of f and the forward jump operator σ are related by the formula

$$f^{\sigma}(t) = f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t). \tag{9}$$

Also, we will use $x^{\Delta\sigma}$ which is shorthand for $(x^{\Delta})^{\sigma}$ to denote $x^{\Delta}(t) + \mu(t)x^{\Delta\Delta}(t)$. We will make use of the following product and quotient rules for the derivative of two differentiable functions f and g:

$$\begin{aligned} (fg)^{\Delta}(t) &= f^{\Delta}(t) g(t) + f^{\sigma}(t) g^{\Delta}(t) \\ &= f(t) g^{\Delta}(t) + f^{\Delta}(t) g^{\sigma}(t), \\ \left(\frac{f}{g}\right)^{\Delta}(t) &= \frac{f^{\Delta}(t) g(t) - f(t) g^{\Delta}(t)}{g(t) g^{\sigma}(t)}, \quad \text{if } gg^{\sigma} \neq 0. \end{aligned} \tag{10}$$

The integration by parts formula reads

$$\begin{aligned} \int_b^c f^{\Delta}(t) g(t) \Delta t &= f(c) g(c) - f(b) g(b) \\ &\quad - \int_b^c f^{\sigma}(t) g^{\Delta}(t) \Delta t. \end{aligned} \tag{11}$$

We say that a function $p : \mathbb{T} \to \mathbb{R}$ is regressive provided

$$1 + \mu(t) p(t) \neq 0, \quad \forall t \in \mathbb{T}. \tag{12}$$

The set of all regressive and rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by

$$\mathscr{R} = \mathscr{R}(\mathbb{T}) = \mathscr{R}(\mathbb{T}, \mathbb{R}). \tag{13}$$

If $p \in \mathscr{R}$, then we can define the exponential function by

$$e_p(t,s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau)) \Delta\tau\right) \quad \text{for } s, t \in \mathbb{T}, \tag{14}$$

where $\xi_h(z)$ is the cylinder transformation, which is defined by

$$\xi_h(z) = \begin{cases} \dfrac{\log(1 + hz)}{h}, & h \neq 0, \\ z, & h = 0. \end{cases} \tag{15}$$

Next, we give the following lemmas which will be used in the proof of our main results.

Lemma 1 (see [2, Chapter 2]). *If $g \in \mathscr{R}^{+}$; that is, $g : \mathbb{T} \to \mathbb{R}$ is rd-continuous and such that $1 + \mu(t)g(t) > 0$ for all $t \in [t_0, \infty)_{\mathbb{T}}$, then the initial value problem $y^{\Delta} = g(t)y$, $y(t_0) = y_0 \in \mathbb{R}$ has a unique and positive solution on $[t_0, \infty)_{\mathbb{T}}$, denoted by $e_g(t, t_0)y_0$. This "exponential function" $e_g(\cdot, t_0)$ satisfies the semigroup property $e_g(a, b)e_g(b, c) = e_g(a, c)$.*

Lemma 2 (see [36]). *If $\lambda > 1$ and $\rho > 1$ are conjugate numbers $(1/\lambda + 1/\rho = 1)$, then for any $X, Y \in \mathbb{R}$,*

$$\frac{|X|^{\lambda}}{\lambda} + \frac{|Y|^{\rho}}{\rho} \geq |XY|. \tag{16}$$

3. Main Results

Now, we are in a position to state and prove some new results which guarantee that every solution of (1) oscillates. In the sequel, we say that a function u belongs to a function class

$$\xi(a,b) := \left\{u \in C_{\mathrm{rd}}^{1}[a,b]_{\mathbb{T}} : u(a) = u(b) = 0, u(t) \not\equiv 0\right\}, \tag{17}$$

denoted by $u \in \xi(a,b)$.

Theorem 3. *Assume that $\alpha > 1$ and for any $T \in [t_0, \infty)_{\mathbb{T}}$, there exist constants a_k and $b_k \in [T, \infty)_{\mathbb{T}}$, such that $a_k < b_k$, $k = 1, 2$, with*

$$q(t) \geq 0, \quad \text{for } t \in [a_1, b_1]_{\mathbb{T}} \cup [a_2, b_2]_{\mathbb{T}}, \tag{18}$$

$$(-1)^k F(t, x^{\sigma}(t)) \geq (-1)^k f(t) \geq 0, \quad \text{for } t \in [a_k, b_k]_{\mathbb{T}}, \quad k = 1, 2, \tag{19}$$

where $f \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$. Furthermore, assume that there exist functions $\eta \in C_{rd}^{1}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}^{+})$, $\eta^{\Delta}(t) \geq 0$, and $u \in \xi(a_k, b_k)$, $k = 1, 2$, such that

$$\int_{a_k}^{b_k} \left(\eta(t) r(t) \left(u^{\Delta}(t)\right)^2 - P(t, a_k) \left(u^{\sigma}(t)\right)^2\right) \Delta t \leq 0, \quad k = 1, 2, \tag{20}$$

where

$$\begin{aligned} P(t, a_k) &= \delta_0(t) - \eta^{\Delta}(t) \delta_1(t, a_k) \\ &\quad + \frac{\eta^{\sigma}(t) p(t)}{r^{\sigma}(t)} \delta_1(\sigma(t), a_k), \\ \delta_0(t) &= \alpha^{1/\alpha} \left(\frac{\alpha}{\alpha - 1}\right)^{(\alpha-1)/\alpha} \eta^{\sigma}(t) q^{1/\alpha}(t) \left|f(t)\right|^{(\alpha-1)/\alpha}, \\ \delta_1(t, a_k) &= \frac{1}{e_{p/r^{\sigma}}(t, a_k)} \\ &\quad \times \left(\int_{a_k}^{t} \frac{1}{r(s) e_{p/r^{\sigma}}(s, a_k)} \Delta s\right)^{-1}. \end{aligned} \tag{21}$$

Then (1) is oscillatory on $[t_0, \infty)_{\mathbb{T}}$.

Proof. Assume that x is a nonoscillatory solution of (1) on $[t_0, \infty)_{\mathbb{T}}$. Without loss of generality, we may assume that there exists a $t_1 \in [t_0, \infty)_{\mathbb{T}}$, such that $x(t) > 0$, $x^{\sigma}(t) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$. By assumption, we can choose $b_1 > a_1 > t_1$, then $q(t) \geq 0$ and $F(t, x^{\sigma}(t)) \leq 0$ on the interval $[a_1, b_1]_{\mathbb{T}}$. From (1), we have

$$\left(r(t) x^{\Delta}(t)\right)^{\Delta} + p(t) x^{\Delta\sigma}(t) \leq 0. \tag{22}$$

Using Lemma 1 and the above inequality, we get

$$\left(r(t) x^{\Delta}(t) e_{p/r^{\sigma}}(t, a_1)\right)^{\Delta} \leq 0. \tag{23}$$

Hence $r(t)x^{\Delta}(t)e_{p/r^{\sigma}}(t,a_1)$ is nonincreasing on $[a_1,b_1]_{\mathbb{T}}$. So for $t\in[a_1,b_1]_{\mathbb{T}}$,

$$\begin{aligned}x(t)>x(t)-x(a_1)&=\int_{a_1}^{t}\frac{r(s)x^{\Delta}(s)e_{p/r^{\sigma}}(s,a_1)}{r(s)e_{p/r^{\sigma}}(s,a_1)}\Delta s\\&\geq r(t)x^{\Delta}(t)e_{p/r^{\sigma}}(t,a_1)\int_{a_1}^{t}\frac{1}{r(s)e_{p/r^{\sigma}}(s,a_1)}\Delta s.\end{aligned}\tag{24}$$

Therefore,

$$\begin{aligned}\frac{r(t)x^{\Delta}(t)}{x(t)}&<\frac{1}{e_{p/r^{\sigma}}(t,a_1)}\left(\int_{a_1}^{t}\frac{1}{r(s)e_{p/r^{\sigma}}(s,a_1)}\Delta s\right)^{-1}\\&=\delta_1(t,a_1).\end{aligned}\tag{25}$$

Define the function ω by

$$\omega(t)=\eta(t)\frac{r(t)x^{\Delta}(t)}{x(t)},\quad t\in[a_1,b_1]_{\mathbb{T}}.\tag{26}$$

Using the product rule and the quotient rule, we obtain

$$\begin{aligned}\omega^{\Delta}(t)&=\eta^{\sigma}(t)\frac{\left(r(t)x^{\Delta}(t)\right)^{\Delta}x(t)-r(t)\left(x^{\Delta}(t)\right)^{2}}{x(t)x^{\sigma}(t)}\\&\quad+\eta^{\Delta}(t)\frac{r(t)x^{\Delta}(t)}{x(t)}.\end{aligned}\tag{27}$$

In view of (1), (26), and (27), we have

$$\begin{aligned}\omega^{\Delta}(t)&=-\eta^{\sigma}(t)\frac{p(t)x^{\Delta\sigma}(t)}{x^{\sigma}(t)}-\eta^{\sigma}(t)q(t)\left(x^{\sigma}(t)\right)^{\alpha-1}\\&\quad+\eta^{\sigma}(t)\frac{F(t,x^{\sigma}(t))}{x^{\sigma}(t)}-\eta^{\sigma}(t)\frac{r(t)\left(x^{\Delta}(t)\right)^{2}}{x(t)x^{\sigma}(t)}\\&\quad+\eta^{\Delta}(t)\frac{r(t)x^{\Delta}(t)}{x(t)}\\&=-\frac{\eta^{\sigma}(t)p(t)}{r^{\sigma}(t)}\frac{r^{\sigma}(t)x^{\Delta\sigma}(t)}{x^{\sigma}(t)}-\eta^{\sigma}(t)q(t)\left(x^{\sigma}(t)\right)^{\alpha-1}\\&\quad-\eta^{\sigma}(t)\frac{\left|F(t,x^{\sigma}(t))\right|}{x^{\sigma}(t)}-\frac{\eta^{\sigma}(t)x(t)}{\eta^{2}(t)r(t)x^{\sigma}(t)}\omega^{2}(t)\\&\quad+\eta^{\Delta}(t)\frac{r(t)x^{\Delta}(t)}{x(t)}.\end{aligned}\tag{28}$$

From (19), (25), and (28), we get

$$\begin{aligned}\omega^{\Delta}(t)&\leq-\frac{\eta^{\sigma}(t)p(t)}{r^{\sigma}(t)}\delta_1(\sigma(t),a_1)-\eta^{\sigma}(t)q(t)\left(x^{\sigma}(t)\right)^{\alpha-1}\\&\quad-\eta^{\sigma}(t)\frac{|f(t)|}{x^{\sigma}(t)}-\frac{\eta^{\sigma}(t)x(t)}{\eta^{2}(t)r(t)x^{\sigma}(t)}\omega^{2}(t)\\&\quad+\eta^{\Delta}(t)\delta_1(t,a_1).\end{aligned}\tag{29}$$

Set

$$\begin{gathered}G(x)=\eta^{\sigma}(t)q(t)x^{\alpha-1}+\eta^{\sigma}(t)\frac{|f(t)|}{x},\\\lambda=\alpha,\qquad\rho=\frac{\alpha}{\alpha-1}.\end{gathered}\tag{30}$$

From Lemma 2, it is easy to see that

$$\begin{aligned}G(x^{\sigma})&\geq\alpha^{1/\alpha}\left(\frac{\alpha}{\alpha-1}\right)^{(\alpha-1)/\alpha}\eta^{\sigma}(t)q^{1/\alpha}(t)\left|f(t)\right|^{(\alpha-1)/\alpha}\\&=\delta_0(t).\end{aligned}\tag{31}$$

Since $x(t)>0$, we obtain

$$\begin{aligned}0<\frac{x(t)}{r(t)x^{\sigma}(t)}&=\frac{1}{r(t)+\mu(t)\left(r(t)x^{\Delta}(t)/x(t)\right)}\\&=\frac{\eta(t)}{\eta(t)r(t)+\mu(t)\omega(t)}.\end{aligned}\tag{32}$$

Thus, combining (29)–(32) and noticing that $\eta^{\Delta}(t)\geq0$, we have

$$\omega^{\Delta}(t)\leq-P(t,a_1)-\frac{1}{\eta(t)r(t)+\mu(t)\omega(t)}\omega^{2}(t),\tag{33}$$

where P is defined as in Theorem 3. Multiplying (33) by $(u^{\sigma}(t))^2$ and integrating from a_1 to b_1, we get

$$\begin{aligned}\int_{a_1}^{b_1}\left(u^{\sigma}(t)\right)^{2}\omega^{\Delta}(t)\Delta t&\leq-\int_{a_1}^{b_1}P(t,a_1)\left(u^{\sigma}(t)\right)^{2}\Delta t\\&\quad-\int_{a_1}^{b_1}\frac{\left(u^{\sigma}(t)\right)^{2}\omega^{2}(t)}{\eta(t)r(t)+\mu(t)\omega(t)}\Delta t.\end{aligned}\tag{34}$$

Using integration by parts on the first integral, we obtain

$$\begin{aligned}&u^{2}(t)\omega(t)\Big|_{a_1}^{b_1}-\int_{a_1}^{b_1}\left(u(t)+u^{\sigma}(t)\right)u^{\Delta}(t)\omega(t)\Delta t\\&\quad\leq-\int_{a_1}^{b_1}P(t,a_1)\left(u^{\sigma}(t)\right)^{2}\Delta t-\int_{a_1}^{b_1}\frac{\left(u^{\sigma}(t)\right)^{2}\omega^{2}(t)}{\eta(t)r(t)+\mu(t)\omega(t)}\Delta t.\end{aligned}\tag{35}$$

Rearranging and using $u(a_1) = 0 = u(b_1)$, we have

$$\begin{aligned}0 \geq& \int_{a_1}^{b_1} \frac{(u^\sigma(t))^2 \omega^2(t)}{\eta(t) r(t) + \mu(t)\omega(t)} \Delta t \\ &- \int_{a_1}^{b_1} (u(t) + u^\sigma(t)) u^\Delta(t) \omega(t) \Delta t \\ &+ \int_{a_1}^{b_1} P(t, a_1)(u^\sigma(t))^2 \Delta t \\ =& \int_{a_1}^{b_1} \frac{(u^\sigma(t))^2 \omega^2(t)}{\eta(t) r(t) + \mu(t)\omega(t)} \Delta t \\ &- \int_{a_1}^{b_1} \Big(2u^\sigma(t) u^\Delta(t) \omega(t) \\ &\quad -\mu(t) \left(u^\Delta(t)\right)^2 \omega(t)\Big) \Delta t \\ &+ \int_{a_1}^{b_1} P(t, a_1)(u^\sigma(t))^2 \Delta t.\end{aligned} \tag{36}$$

Adding and subtracting the term $\int_{a_1}^{b_1} \eta(t) r(t)(u^\Delta(t))^2 \Delta t$ and using (20), we get

$$\begin{aligned}0 \geq& \int_{a_1}^{b_1} \left[\frac{(u^\sigma(t))^2 \omega^2(t)}{\eta(t) r(t) + \mu(t)\omega(t)} - 2u^\sigma(t) u^\Delta(t) \omega(t) \right. \\ &\left. + (\eta(t) r(t) + \mu(t)\omega(t)) \left(u^\Delta(t)\right)^2 \right] \Delta t \\ &- \int_{a_1}^{b_1} \left(\eta(t) r(t) \left(u^\Delta(t)\right)^2 - P(t, a_1)(u^\sigma(t))^2\right) \Delta t \\ \geq& \int_{a_1}^{b_1} \left[\frac{u^\sigma(t) \omega(t)}{\sqrt{\eta(t) r(t) + \mu(t)\omega(t)}} \right. \\ &\left. - \sqrt{\eta(t) r(t) + \mu(t)\omega(t)} u^\Delta(t) \right]^2 \Delta t.\end{aligned} \tag{37}$$

It follows that

$$\begin{aligned}\int_{a_1}^{b_1} &\left[\frac{u^\sigma(t) \omega(t)}{\sqrt{\eta(t) r(t) + \mu(t)\omega(t)}} \right. \\ &\left. - \sqrt{\eta(t) r(t) + \mu(t)\omega(t)} u^\Delta(t) \right]^2 \Delta t = 0.\end{aligned} \tag{38}$$

This implies that

$$\begin{aligned}&\frac{u^\sigma(t) \omega(t)}{\sqrt{\eta(t) r(t) + \mu(t)\omega(t)}} \\ &\quad - \sqrt{\eta(t) r(t) + \mu(t)\omega(t)} u^\Delta(t) = 0, \quad t \in [a_1, b_1]_{\mathbb{T}}.\end{aligned} \tag{39}$$

Solving for u^Δ, we get that u solves the IVP

$$\begin{aligned}u^\Delta(t) &= \frac{\omega(t)}{\eta(t) r(t) + \mu(t)\omega(t)} u^\sigma(t), \\ u(a_1) &= 0, \quad \text{for } t \in [a_1, b_1]_{\mathbb{T}}.\end{aligned} \tag{40}$$

Since $-\omega/(\eta r + \mu\omega) \in \mathcal{R}$, we obtain from [2, Theorem 2.7.1] that $u(t) \equiv 0$ on $[a_1, b_1]_{\mathbb{T}}$, which is a contradiction. The proof when x is eventually negative follows the same arguments using the interval $[a_2, b_2]_{\mathbb{T}}$ instead of $[a_1, b_1]_{\mathbb{T}}$, where we use $q(t) \geq 0$, $F(t, x^\sigma(t)) \geq 0$ on $[a_2, b_2]_{\mathbb{T}}$, and $\int_{a_2}^{b_2} (\eta(t) r(t)(u^\Delta(t))^2 - P(t)(u^\sigma(t))^2) \Delta t \leq 0$. The proof is complete. □

Remark 4. When $p(t) = 0$ and $F(t, x^\sigma(t)) = f(t)$, Theorem 3 contains Theorem 3.2 in [26].

Theorem 5. *Assume that $\alpha = 1$ and for any $T \in [t_0, \infty)_{\mathbb{T}}$, there exist constants a_k and $b_k \in [T, \infty)_{\mathbb{T}}$, such that $a_k < b_k$, $k = 1, 2$, with*

$$\begin{aligned}q(t) \geq 0, &\quad \text{for } t \in [a_1, b_1]_{\mathbb{T}} \cup [a_2, b_2]_{\mathbb{T}}, \\ (-1)^k F(t, x^\sigma(t)) \geq 0, &\quad \text{for } t \in [a_k, b_k]_{\mathbb{T}},\ k = 1, 2.\end{aligned} \tag{41}$$

Furthermore, assume that there exist functions $\eta \in C_{rd}^1([t_0, \infty)_{\mathbb{T}}, \mathbb{R}^+)$, $\eta^\Delta(t) \geq 0$, and $u \in \xi(a_k, b_k)$, $k = 1, 2$, such that

$$\begin{aligned}\int_{a_k}^{b_k} \left(\eta(t) r(t) \left(u^\Delta(t)\right)^2 - K(t, a_k)(u^\sigma(t))^2\right) \Delta t \leq 0, \\ k = 1, 2,\end{aligned} \tag{42}$$

where

$$\begin{aligned}K(t, a_k) =& \eta^\sigma(t) q(t) - \eta^\Delta(t) \delta_1(t, a_k) \\ &+ \frac{\eta^\sigma(t) p(t)}{r^\sigma(t)} \delta_1(\sigma(t), a_k),\end{aligned} \tag{43}$$

and δ_1 is defined as in Theorem 3. Then (1) is oscillatory on $[t_0, \infty)_{\mathbb{T}}$.

Proof. Assume that x is a nonoscillatory solution of (1) on $[t_0, \infty)_{\mathbb{T}}$. Without loss of generality, we may assume that there exists a $t_1 \in [t_0, \infty)_{\mathbb{T}}$, such that $x(t) > 0$, $x^\sigma(t) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$. By assumption, we can choose $b_1 > a_1 > t_1$, then $q(t) \geq 0$ and $F(t, x^\sigma(t)) \leq 0$ on the interval $[a_1, b_1]_{\mathbb{T}}$. We define ω as in Theorem 3. Proceeding as in the proof of Theorem 3 and from (25) and (32), we get

$$\begin{aligned}\omega^\Delta(t) =& -\frac{\eta^\sigma(t) p(t)}{r^\sigma(t)} \frac{r^\sigma(t) x^{\Delta\sigma}(t)}{x^\sigma(t)} - \eta^\sigma(t) q(t) \\ &- \eta^\sigma(t) \frac{|F(t, x^\sigma(t))|}{x^\sigma(t)} \\ &- \frac{\eta^\sigma(t) x(t)}{\eta^2(t) r(t) x^\sigma(t)} \omega^2(t) + \eta^\Delta(t) \frac{r(t) x^\Delta(t)}{x(t)} \\ \leq& -K(t, a_1) - \frac{1}{\eta(t) r(t) + \mu(t)\omega(t)} \omega^2(t),\end{aligned} \tag{44}$$

where K is defined as in Theorem 5. Multiplying (44) by $(u^\sigma(t))^2$ and integrating from a_1 to b_1, we get

$$\int_{a_1}^{b_1} (u^\sigma(t))^2 \omega^\Delta(t)\,\Delta t \le -\int_{a_1}^{b_1} K(t,a_1)(u^\sigma(t))^2 \Delta t - \int_{a_1}^{b_1} \frac{(u^\sigma(t))^2 \omega^2(t)}{\eta(t) r(t) + \mu(t)\omega(t)} \Delta t. \tag{45}$$

The rest of the argument proceeds as in Theorem 3 to get a contradiction to (42). The proof is complete. □

Remark 6. When $p(t) = 0$ and $F(t, x^\sigma(t)) = f(t)$, Theorem 5 contains Theorem 2.1 in [26].

Theorem 7. *Assume that $\alpha < 1$ and for any $T \in [t_0,\infty)_{\mathbb{T}}$, there exist constants a_k and $b_k \in [T,\infty)_{\mathbb{T}}$, such that $a_k < b_k$, $k = 1,2$, with*

$$q(t) \ge 0, \quad \text{for } t \in [a_1,b_1]_{\mathbb{T}} \cup [a_2,b_2]_{\mathbb{T}}, \tag{46}$$

$$(-1)^k F(t, x^\sigma(t)) \ge (-1)^k f(t)(x^\sigma(t))^{\alpha+1} \ge 0 \quad \text{for } t \in [a_k,b_k]_{\mathbb{T}}, \quad k = 1,2, \tag{47}$$

where $f \in C_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R})$. Furthermore, assume that there exist functions $\eta \in C^1_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R}^+)$, $\eta^\Delta(t) \ge 0$, and $u \in \xi(a_k,b_k)$, $k = 1,2$, such that

$$\int_{a_k}^{b_k} \left(\eta(t) r(t) (u^\Delta(t))^2 - P_1(t,a_k)(u^\sigma(t))^2\right) \Delta t \le 0, \quad k = 1,2, \tag{48}$$

where

$$P_1(t,a_k) = \delta_2(t) - \eta^\Delta(t)\delta_1(t,a_k) + \frac{\eta^\sigma(t) p(t)}{r^\sigma(t)} \delta_1(\sigma(t), a_k), \tag{49}$$

$$\delta_2(t) = \frac{1}{\alpha^\alpha (1-\alpha)^{1-\alpha}} \eta^\sigma(t) q^\alpha(t) |f(t)|^{1-\alpha},$$

and δ_1 is defined as in Theorem 3. Then (1) *is oscillatory on $[t_0,\infty)_{\mathbb{T}}$.*

Proof. Assume that x is a nonoscillatory solution of (1) on $[t_0,\infty)_{\mathbb{T}}$. Without loss of generality, we may assume that there exists a $t_1 \in [t_0,\infty)_{\mathbb{T}}$, such that $x(t) > 0$, $x^\sigma(t) > 0$ for all $t \in [t_1,\infty)_{\mathbb{T}}$. By assumption, we can choose $b_1 > a_1 > t_1$, then $q(t) \ge 0$ and $F(t, x^\sigma(t)) \le 0$ on the interval $[a_1,b_1]_{\mathbb{T}}$. We define ω as in Theorem 3. Proceeding as in the proof of Theorem 3, we have (28). Hence, from (25), (28), and (47), we get

$$\omega^\Delta(t) \le -\frac{\eta^\sigma(t) p(t)}{r^\sigma(t)} \delta_1(\sigma(t), a_1) - \frac{\eta^\sigma(t) q(t)}{(x^\sigma(t))^{1-\alpha}} - \eta^\sigma(t)|f(t)|(x^\sigma(t))^\alpha - \frac{\eta^\sigma(t) x(t)}{\eta^2(t) r(t) x^\sigma(t)} \omega^2(t) + \eta^\Delta(t)\delta_1(t,a_1). \tag{50}$$

Set

$$G(x) = \frac{\eta^\sigma(t) q(t)}{x^{1-\alpha}} - \eta^\sigma(t)|f(t)| x^\alpha, \quad \lambda = \frac{1}{\alpha}, \quad \rho = \frac{1}{1-\alpha}. \tag{51}$$

From Lemma 2, it is easy to see that

$$G(x^\sigma) \ge \frac{1}{\alpha^\alpha (1-\alpha)^{1-\alpha}} \eta^\sigma(t) q^\alpha(t) |f(t)|^{1-\alpha} = \delta_2(t). \tag{52}$$

Thus, combining (32), (50), and (52) and noticing that $\eta^\Delta(t) \ge 0$, we have

$$\omega^\Delta(t) \le -P_1(t,a_1) - \frac{1}{\eta(t) r(t) + \mu(t)\omega(t)} \omega^2(t), \tag{53}$$

where P_1 is defined as in Theorem 7. Multiplying (53) by $(u^\sigma(t))^2$ and integrating from a_1 to b_1, we get

$$\int_{a_1}^{b_1} (u^\sigma(t))^2 \omega^\Delta(t)\,\Delta t \le -\int_{a_1}^{b_1} P_1(t,a_1)(u^\sigma(t))^2 \Delta t - \int_{a_1}^{b_1} \frac{(u^\sigma(t))^2 \omega^2(t)}{\eta(t) r(t) + \mu(t)\omega(t)} \Delta t. \tag{54}$$

The rest of the argument proceeds as in Theorem 3 to get a contradiction to (47). The proof is complete. □

Next, let us introduce the class of functions Y, which will be extensively used in the sequel.

Let $\mathbb{D}_0 = \{(t,s) \in \mathbb{T}^2 : t > s \ge t_0\}$ and $\mathbb{D} = \{(t,s) \in \mathbb{T}^2 : t \ge s \ge t_0\}$. We say that the function $H \in C_{\mathrm{rd}}(\mathbb{D}, \mathbb{R})$ belongs to the class Y, if

(i) $H(t,t) = 0$, $t \ge t_0$, $H(t,s) > 0$ on $\mathbb{D}_0$;

(ii) H has continuous Δ-partial derivatives $H^{\Delta_t}(t,s)$ and $H^{\Delta_s}(t,s)$ on $\mathbb{D}$ such that

$$H^{\Delta_t}(t,\sigma(s)) = h_1(t,s)\sqrt{H(\sigma(t),\sigma(s))}, \quad H^{\Delta_s}(\sigma(t),s) = -h_2(t,s)\sqrt{H(\sigma(t),\sigma(s))}, \tag{55}$$

where h_1 and $h_2 \in C_{\mathrm{rd}}(\mathbb{D}, \mathbb{R})$.

Theorem 8. *Assume that $\alpha > 1$ and for any $T \in [t_0,\infty)_{\mathbb{T}}$, there exist constants a_k and $b_k \in [T,\infty)_{\mathbb{T}}$, such that $a_k < b_k$, $k = 1,2$, with*

$$q(t) \ge 0, \quad \text{for } t \in [a_1,b_1]_{\mathbb{T}} \cup [a_2,b_2]_{\mathbb{T}}, \quad (-1)^k F(t, x^\sigma(t)) \ge (-1)^k f(t) \ge 0, \quad \text{for } t \in [a_k,b_k]_{\mathbb{T}}, \quad k = 1,2, \tag{56}$$

where $f \in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$. Furthermore, assume that there exists a function $\eta \in C^1_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+)$ such that for some $H \in Y$ and $c_k \in (a_k,b_k)_{\mathbb{T}}$,

$$\begin{aligned}
&\frac{1}{H(\sigma(c_k),\sigma(a_k))}\int_{a_k}^{c_k}\left[H(\sigma(s),\sigma(a_k))Q(s,a_k) - \frac{\eta^2(s)r(s)}{4\eta^\sigma(s)\delta(s,a_k)}\phi_1^2(s,a_k)\right]\Delta s \\
&+\frac{1}{H(\sigma(b_k),\sigma(c_k))}\int_{c_k}^{b_k}\left[H(\sigma(b_k),\sigma(s))Q(s,a_k) - \frac{\eta^2(s)r(s)}{4\eta^\sigma(s)\delta(s,a_k)}\phi_2^2(b_k,s)\right] \\
&\times\Delta s>0,\quad k=1,2,
\end{aligned}\tag{57}$$

where

$$\begin{aligned}
\phi_1(s,a_k) &= h_1(s,a_k)+\sqrt{H(\sigma(s),\sigma(a_k))}\frac{\eta^\Delta(s)}{\eta(s)},\\
\phi_2(b_k,s) &= h_2(b_k,s)-\sqrt{H(\sigma(b_k),\sigma(s))}\frac{\eta^\Delta(s)}{\eta(s)},\\
Q(t,a_k) &= \delta_0(t)+\frac{\eta^\sigma(t)p(t)}{r^\sigma(t)}\delta_1(\sigma(t),a_k),\\
\delta(t,a_k) &= \int_{a_k}^{t}\frac{\Delta s}{r(s)e_{p/r^\sigma}(s,a_k)}\left(\int_{a_k}^{\sigma(t)}\frac{\Delta s}{r(s)e_{p/r^\sigma}(s,a_k)}\right)^{-1},
\end{aligned}\tag{58}$$

and δ_0 and δ_1 are defined as in Theorem 3. Then (1) is oscillatory on $[t_0,\infty)_{\mathbb{T}}$.

Proof. Assume that x is a nonoscillatory solution of (1) on $[t_0,\infty)_{\mathbb{T}}$. Without loss of generality, we may assume that there exists a $t_1 \in [t_0,\infty)_{\mathbb{T}}$, such that $x(t)>0$, $x^\sigma(t)>0$ for all $t\in[t_1,\infty)_{\mathbb{T}}$. By assumption, we can choose $b_1>a_1>t_1$, then $q(t)\ge 0$ and $F(t,x^\sigma(t))\le 0$ on the interval $[a_1,b_1]_{\mathbb{T}}$. We define the function ω as in Theorem 3. Proceeding as in the proof of Theorem 3 and from (25) and (31), we get

$$\omega^\Delta(t)\le -Q(t,a_1)+\frac{\eta^\Delta(t)}{\eta(t)}\omega(t)-\frac{\eta^\sigma(t)x(t)}{\eta^2(t)r(t)x^\sigma(t)}\omega^2(t),\tag{59}$$

where Q is defined as in Theorem 7. Since $r(t)x^\Delta(t)e_{p/r^\sigma}(t,a_1)$ is nonincreasing on $[a_1,b_1]_{\mathbb{T}}$, we obtain

$$\begin{aligned}
x^\sigma(t)-x(t) &= \int_t^{\sigma(t)}\frac{r(s)x^\Delta(s)e_{p/r^\sigma}(s,a_1)}{r(s)e_{p/r^\sigma}(s,a_1)}\Delta s\\
&\le r(t)x^\Delta(t)e_{p/r^\sigma}(t,a_1)\int_t^{\sigma(t)}\frac{\Delta s}{r(s)e_{p/r^\sigma}(s,a_1)},
\end{aligned}\tag{60}$$

hence

$$\frac{x^\sigma(t)}{x(t)}\le 1+\frac{r(t)x^\Delta(t)e_{p/r^\sigma}(t,a_1)}{x(t)}\int_t^{\sigma(t)}\frac{\Delta s}{r(s)e_{p/r^\sigma}(s,a_1)}.\tag{61}$$

From (25), we have

$$\frac{r(t)x^\Delta(t)e_{p/r^\sigma}(t,a_1)}{x(t)}<\left(\int_{a_1}^{t}\frac{\Delta s}{r(s)e_{p/r^\sigma}(s,a_1)}\right)^{-1}.\tag{62}$$

Therefore, from (61) and (62), we get

$$\begin{aligned}
\frac{x^\sigma(t)}{x(t)} &< \int_{a_1}^{\sigma(t)}\frac{\Delta s}{r(s)e_{p/r^\sigma}(s,a_1)}\left(\int_{a_1}^{t}\frac{\Delta s}{r(s)e_{p/r^\sigma}(s,a_1)}\right)^{-1}\\
&=\frac{1}{\delta(t,a_1)}.
\end{aligned}\tag{63}$$

Combining (59) and (63), we obtain

$$\begin{aligned}
\omega^\Delta(t) &\le -Q(t,a_1)+\frac{\eta^\Delta(t)}{\eta(t)}\omega(t)\\
&\quad-\frac{\eta^\sigma(t)\delta(t,a_1)}{\eta^2(t)r(t)}\omega^2(t),\quad t\in[a_1,b_1]_{\mathbb{T}}.
\end{aligned}\tag{64}$$

Multiplying both sides of (64) by $H(\sigma(s),\sigma(t))$ and integrating with respect to s from t to c_1 for $t\in(a_1,c_1]_{\mathbb{T}}$, we have

$$\begin{aligned}
&\int_t^{c_1}H(\sigma(s),\sigma(t))Q(s,a_1)\Delta s\\
&\quad\le -\int_t^{c_1}H(\sigma(s),\sigma(t))\omega^\Delta(s)\Delta s\\
&\qquad+\int_t^{c_1}H(\sigma(s),\sigma(t))\frac{\eta^\Delta(s)}{\eta(s)}\omega(s)\Delta s\\
&\qquad-\int_t^{c_1}H(\sigma(s),\sigma(t))\frac{\eta^\sigma(s)\delta(s,a_1)}{\eta^2(s)r(s)}\omega^2(s)\Delta s.
\end{aligned}\tag{65}$$

In view of (i) and (ii), we see that

$$\begin{aligned}
&\int_t^{c_1}H(\sigma(s),\sigma(t))\omega^\Delta(s)\Delta s\\
&\quad= H(\sigma(c_1),\sigma(t))\omega(c_1)\\
&\qquad-\int_t^{c_1}h_1(s,t)\sqrt{H(\sigma(s),\sigma(t))}\omega(s)\Delta s.
\end{aligned}\tag{66}$$

Using (66) in (65) leads to

$$
\begin{aligned}
&\int_t^{c_1} H(\sigma(s),\sigma(t))\,Q(s,a_1)\,\Delta s\\
&\quad\le -H(\sigma(c_1),\sigma(t))\,\omega(c_1)\\
&\qquad-\int_t^{c_1} H(\sigma(s),\sigma(t))\,\frac{\eta^\sigma(s)\,\delta(s,a_1)}{\eta^2(s)\,r(s)}\omega^2(s)\,\Delta s\\
&\qquad+\int_t^{c_1}\left(h_1(s,t)\sqrt{H(\sigma(s),\sigma(t))}\right.\\
&\qquad\qquad\left.+H(\sigma(s),\sigma(t))\,\frac{\eta^\Delta(s)}{\eta(s)}\right)\omega(s)\,\Delta s\\
&\quad= -H(\sigma(c_1),\sigma(t))\,\omega(c_1)\\
&\qquad+\int_t^{c_1}\frac{\eta^2(s)\,r(s)}{4\eta^\sigma(s)\,\delta(s,a_1)}\phi_1^2(s,t)\,\Delta s\\
&\qquad-\int_t^{c_1}\left(\sqrt{H(\sigma(s),\sigma(t))\,\frac{\eta^\sigma(s)\,\delta(s,a_1)}{r(s)}}\,\frac{\omega(s)}{\eta(s)}\right.\\
&\qquad\qquad\left.-\frac{\eta(s)\sqrt{r(s)}}{2\sqrt{\eta^\sigma(s)\,\delta(s,a_1)}}\phi_1(s,t)\right)^2\Delta s\\
&\quad\le -H(\sigma(c_1),\sigma(t))\,\omega(c_1)\\
&\qquad+\int_t^{c_1}\frac{\eta^2(s)\,r(s)}{4\eta^\sigma(s)\,\delta(s,a_1)}\phi_1^2(s,t)\,\Delta s.
\end{aligned}
\tag{67}
$$

Letting $t \to a_1^+$ in the above inequality, we get

$$
\begin{aligned}
&\frac{1}{H(\sigma(c_1),\sigma(a_1))}\int_{a_1}^{c_1}\left[H(\sigma(s),\sigma(a_1))\,Q(s,a_1)\right.\\
&\qquad\qquad\left.-\frac{\eta^2(s)\,r(s)}{4\eta^\sigma(s)\,\delta(s,a_1)}\phi_1^2(s,a_1)\right]\Delta s\\
&\quad\le -\omega(c_1).
\end{aligned}
\tag{68}
$$

Similarly, multiplying both sides of (64) by $H(\sigma(t),\sigma(s))$ and integrating with respect to s from c_1 to t for $t\in[c_1,b_1)_{\mathbb{T}}$, we obtain

$$
\begin{aligned}
&\int_{c_1}^{t} H(\sigma(t),\sigma(s))\,Q(s,a_1)\,\Delta s\\
&\quad\le -\int_{c_1}^{t} H(\sigma(t),\sigma(s))\,\omega^\Delta(s)\,\Delta s\\
&\qquad+\int_{c_1}^{t} H(\sigma(t),\sigma(s))\,\frac{\eta^\Delta(s)}{\eta(s)}\omega(s)\,\Delta s\\
&\qquad-\int_{c_1}^{t} H(\sigma(t),\sigma(s))\,\frac{\eta^\sigma(s)\,\delta(s,a_1)}{\eta^2(s)\,r(s)}\omega^2(s)\,\Delta s\\
&\quad\le H(\sigma(t),\sigma(c_1))\,\omega(c_1)\\
&\qquad-\int_{c_1}^{t} H(\sigma(t),\sigma(s))\,\frac{\eta^\sigma(s)\,\delta(s,a_1)}{\eta^2(s)\,r(s)}\omega^2(s)\,\Delta s\\
&\qquad-\int_{c_1}^{t}\left(h_2(t,s)\sqrt{H(\sigma(t),\sigma(s))}\right.\\
&\qquad\qquad\left.-H(\sigma(t),\sigma(s))\,\frac{\eta^\Delta(s)}{\eta(s)}\right)\omega(s)\,\Delta s\\
&\quad= H(\sigma(t),\sigma(c_1))\,\omega(c_1)\\
&\qquad+\int_{c_1}^{t}\frac{\eta^2(s)\,r(s)}{4\eta^\sigma(s)\,\delta(s,a_1)}\phi_2^2(t,s)\,\Delta s\\
&\qquad-\int_{c_1}^{t}\left(\sqrt{H(\sigma(t),\sigma(s))\,\frac{\eta^\sigma(s)\,\delta(s,a_1)}{r(s)}}\,\frac{\omega(s)}{\eta(s)}\right.\\
&\qquad\qquad\left.+\frac{\eta(s)\sqrt{r(s)}}{2\sqrt{\eta^\sigma(s)\,\delta(s,a_1)}}\phi_2(t,s)\right)^2\Delta s\\
&\quad\le H(\sigma(t),\sigma(c_1))\,\omega(c_1)\\
&\qquad+\int_{c_1}^{t}\frac{\eta^2(s)\,r(s)}{4\eta^\sigma(s)\,\delta(s,a_1)}\phi_2^2(t,s)\,\Delta s.
\end{aligned}
\tag{69}
$$

Letting $t \to b_1^-$ in the above inequality, we get

$$
\begin{aligned}
&\frac{1}{H(\sigma(b_1),\sigma(c_1))}\int_{c_1}^{b_1}\left[H(\sigma(b_1),\sigma(s))\,Q(s,a_1)\right.\\
&\qquad\qquad\left.-\frac{\eta^2(s)\,r(s)}{4\eta^\sigma(s)\,\delta(s,a_1)}\phi_2^2(b_1,s)\right]\Delta s\\
&\quad\le \omega(c_1).
\end{aligned}
\tag{70}
$$

Adding (68) and (70), we get a contradiction to (57). This completes the proof. □

Theorem 9. *Assume that $\alpha < 1$ and for any $T\in[t_0,\infty)_{\mathbb{T}}$, there exist constants a_k and $b_k\in[T,\infty)_{\mathbb{T}}$, such that $a_k<b_k$, $k=1,2$, with*

$$
\begin{aligned}
&q(t)\ge 0 \quad \text{for } t\in[a_1,b_1]_{\mathbb{T}}\cup[a_2,b_2]_{\mathbb{T}},\\
&(-1)^k F(t,x^\sigma(t)) \ge (-1)^k f(t)\,(x^\sigma(t))^{\alpha+1}\ge 0,\\
&\qquad\qquad \text{for } t\in[a_k,b_k]_{\mathbb{T}},\ k=1,2,
\end{aligned}
\tag{71}
$$

where $f \in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$. Furthermore, assume that there exists a function $\eta \in C^1_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+)$ such that for some $H \in Y$ and $c_k \in (a_k,b_k)_{\mathbb{T}}$,

$$\frac{1}{H(\sigma(c_k),\sigma(a_k))}\int_{a_k}^{c_k}\left[H(\sigma(s),\sigma(a_k))\widetilde{Q}(s,a_k) - \frac{\eta^2(s)r(s)}{4\eta^\sigma(s)\delta(s,a_k)}\phi_1^2(s,a_k)\right]\Delta s + \frac{1}{H(\sigma(b_k),\sigma(c_k))}\int_{c_k}^{b_k}\left[H(\sigma(b_k),\sigma(s))\widetilde{Q}(s,a_k) - \frac{\eta^2(s)r(s)}{4\eta^\sigma(s)\delta(s,a_k)}\phi_2^2(b_k,s)\right]\times\Delta s > 0, \quad k=1,2, \tag{72}$$

where

$$\widetilde{Q}(t,a_k) = \delta_2(t) + \frac{\eta^\sigma(t)p(t)}{r^\sigma(t)}\delta_1(\sigma(t),a_k). \tag{73}$$

δ_1 is defined as in Theorem 3, δ_2 is defined as in Theorem 7, and ϕ_1, ϕ_2, and δ are defined as in Theorem 8. Then (1) is oscillatory on $[t_0,\infty)_{\mathbb{T}}$.

The proof of Theorem 9 is similar to that of Theorem 8, so we omit the proof.

Remark 10. The main results in this paper can also be extended to the following second order damped dynamic equations with mixed nonlinearities:

$$\begin{aligned}&\left(r(t)x^\Delta(t)\right)^\Delta + p(t)x^{\Delta\sigma}(t) + q_0(t)x(\tau_0(t))\\&\quad+q_1(t)\left|x(\tau_1(t))\right|^{\beta-1}x(\tau_1(t))\\&\quad+q_2(t)\left|x(\tau_2(t))\right|^{\gamma-1}x(\tau_2(t)) = F(t,x^\sigma(t)),\end{aligned} \tag{74}$$

where $\gamma > 1 > \beta > 0$, $\tau_i(t) \le \sigma(t)$, $i=0,1,2$, or the more general equation

$$\begin{aligned}&\left(r(t)x^\Delta(t)\right)^\Delta + p(t)x^{\Delta\sigma}(t)\\&\quad+\sum_{i=0}^{n}q_i(t)\left|x(\tau_i(t))\right|^{\alpha_i-1}x(\tau_i(t)) = F(t,x^\sigma(t))\end{aligned} \tag{75}$$

on any arbitrary time scale $\mathbb{T}$, where $\alpha_0 = 1$, $\alpha_1 > \alpha_2 > \cdots > \alpha_m > 1 > \alpha_{m+1} > \cdots > \alpha_n > 0$ and τ_i are nondecreasing rd-continuous functions on $\mathbb{R}$ with $\tau_i(t) \le \sigma(t)$, $i=0,1,\ldots,n$. Due to the limited space, we omit it here and leave it to the readers who are interested in this problem.

4. Examples

In this section, we will show the applications of our interval oscillation criteria in two examples. Firstly, we will give an example to illustrate Theorem 3.

Example 1. Consider the following second order forced difference equations with damping:

$$\begin{aligned}&\Delta\left(t\left(\sin\frac{\pi t}{4}+2\right)\Delta x(t)\right)\\&\quad-\frac{t^2-1}{t^2}\left(\sin\frac{\pi(t+1)}{4}+2\right)\Delta x(t)\\&\quad+\frac{c_0}{(t+1)^2}\left(\sin\frac{\pi t}{4}+2\right)x^2(t) = -\cos\frac{\pi t}{4},\end{aligned} \tag{76}$$

for $t \ge 2$, where c_0 is a positive constant. Here

$$\begin{aligned}r(t) &= t\left(\sin\frac{\pi t}{4}+2\right),\\ p(t) &= -\frac{t^2-1}{t^2}\left(\sin\frac{\pi(t+1)}{4}+2\right),\\ q(t) &= \frac{c_0}{(t+1)^2}\left(\sin\frac{\pi t}{4}+2\right),\\ F(t,x(t)) &= f(t) = -\cos\frac{\pi t}{4}, \qquad \alpha = 2.\end{aligned} \tag{77}$$

Let

$$\begin{aligned}&a_1 = 8h, \qquad b_1 = a_2 = 8h+2,\\&b_2 = 8h+4, \qquad h = 1,2,\ldots,\end{aligned} \tag{78}$$

such that

$$\begin{aligned}&q(t) \ge 0, \quad (-1)^k f(t) \ge 0,\\&t \in [8h, 8h+2) \cup [8h+2, 8h+4), \quad k = 1,2.\end{aligned} \tag{79}$$

For $t \ge 2$, we obtain

$$\begin{aligned}&\delta_1(\sigma(t),a_k)\\&= \frac{1}{(1+\mu(t)(p(t)/r^\sigma(t)))e_{p/r^\sigma}(t,a_k)}\\&\quad\times\left(\int_{a_k}^{\sigma(t)}\frac{1}{(1+\mu(s)(p(s)/r^\sigma(s)))r(s)e_{p/r^\sigma}(s,a_k)}\Delta s\right)^{-1}\\&\le\frac{t}{t-1}\delta_1(t,a_k).\end{aligned} \tag{80}$$

Setting $\eta(t) = 1/t$ and $u(t) = \sin(\pi t/2)$, we have

$$\begin{aligned}
P(t,a_k) &\geq 2(t+1)\left(\frac{c_0}{(t+1)^2}\left(\sin\frac{\pi t}{4}+2\right)\right)^{1/2}\left|-\cos\frac{\pi t}{4}\right|^{1/2} \\
&= 2\left(c_0\left(\sin\frac{\pi t}{4}+2\right)\left|\cos\frac{\pi t}{4}\right|\right)^{1/2}, \\
&\sum_{j=a_1}^{b_1-1}\left(\eta(j)r(j)(\Delta u(j))^2 - P(j,a_1)u^2(j+1)\right) \\
&\leq \sum_{j=8h}^{8h+1}\left(\frac{1}{j}\cdot j\left(\sin\frac{\pi j}{4}+2\right)\left(\sin\frac{\pi(j+1)}{2}-\sin\frac{\pi j}{2}\right)^2\right. \\
&\quad \left. -2\left(c_0\left(\sin\frac{\pi j}{4}+2\right)\cos\frac{\pi j}{4}\right)^{1/2}\sin^2\frac{\pi(j+1)}{2}\right) \\
&= \frac{\sqrt{2}}{2}+4-2(2c_0)^{1/2}.
\end{aligned} \tag{81}$$

Then by Theorem 3, every solution of (76) is oscillatory if

$$c_0 \geq \frac{1}{2}\left(\frac{\sqrt{2}}{4}+2\right)^2. \tag{82}$$

Next, we will give an example to illustrate Theorem 7.

Example 2. Consider the following second order forced differential equations with damping:

$$\begin{aligned}
&\left(t(\sin 2t+2)x'(t)\right)' - (\sin 2t+2)x'(t) + \frac{c_0\cos^2 2t}{t^{1/\alpha}}x^\alpha(t) \\
&\quad = -\sin 2t, \quad t \geq 1,
\end{aligned} \tag{83}$$

where c_0 is a positive constant. Here,

$$\begin{aligned}
&r(t) = t(\sin 2t+2), \qquad p(t) = -\sin 2t - 2, \\
&q(t) = \frac{c_0\cos^2 2t}{t^{1/\alpha}}, \qquad F(t,x(t)) = f(t) = -\sin 2t, \\
&\qquad\qquad \alpha < 1.
\end{aligned} \tag{84}$$

Let

$$\begin{aligned}
&a_1 = 2h\pi, \qquad b_1 = a_2 = 2h\pi + \frac{\pi}{2}, \\
&b_2 = 2h\pi + \pi, \quad h = 1,2,\ldots,
\end{aligned} \tag{85}$$

such that

$$\begin{aligned}
&q(t) \geq 0, \quad (-1)^k f(t) \geq 0, \\
&t \in \left[2h\pi, 2h\pi+\frac{\pi}{2}\right) \cup \left[2h\pi+\frac{\pi}{2}, 2h\pi+\pi\right), \quad k = 1,2.
\end{aligned} \tag{86}$$

Setting $\eta(t) = 1/t$ and $u(t) = \sin 2t$, we obtain

$$\begin{aligned}
P_1(t,a_k) &= \frac{t}{\alpha^\alpha(1-\alpha)^{1-\alpha}}\left(\frac{c_0\cos^2 2t}{t^{1/\alpha}}\right)^\alpha |-\sin 2t|^{1-\alpha} \\
&= \frac{c_0^\alpha}{\alpha^\alpha(1-\alpha)^{1-\alpha}}\cos^{2\alpha}2t|-\sin 2t|^{1-\alpha}, \\
&\int_{a_1}^{b_1}\left(\eta(t)r(t)\left(u'(t)\right)^2 - P_1(t,a_1)u^2(t)\right)\mathrm{d}t \\
&= \int_0^{\pi/2}\left(\frac{1}{t}\cdot t(\sin 2t+2)(2\cos 2t)^2\right. \\
&\qquad \left. -\frac{c_0^\alpha}{\alpha^\alpha(1-\alpha)^{1-\alpha}}\cos^{2\alpha}2t\,\sin^{3-\alpha}2t\right)\mathrm{d}t \\
&= \frac{6}{5}\sqrt{\pi}+\pi-\frac{c_0^\alpha}{4\alpha^\alpha(1-\alpha)^{1-\alpha}} \\
&\quad \times \frac{\Gamma(2-\alpha/2)\,\Gamma(\alpha+1/2)}{\Gamma((\alpha+5)/2)},
\end{aligned} \tag{87}$$

where Γ is the gamma function. Then by Theorem 7, every solution of (83) is oscillatory if

$$\frac{6}{5}\sqrt{\pi}+\pi \leq \frac{c_0^\alpha}{4\alpha^\alpha(1-\alpha)^{1-\alpha}}\frac{\Gamma(2-\alpha/2)\,\Gamma(\alpha+1/2)}{\Gamma((\alpha+5)/2)}. \tag{88}$$

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Review Article

Existence for Singular Periodic Problems: A Survey of Recent Results

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We present a survey on the existence of periodic solutions of singular differential equations. In particular, we pay our attention to singular scalar differential equations, singular damped differential equations, singular impulsive differential equations, and singular differential systems.

1. Introduction

During the last two decades, singular differential equations have attracted many researchers [1–11] because such equations describe many problems in the applied sciences, such as the Brillouin focusing system [12–14], nonlinear elasticity [15], and gravitational forces [3]. Besides these important applications, it has been found that a particular case of singular equations, the Ermakov-Pinney equation, plays an important role in studying the Lyapunov stability of periodic solutions of Lagrangian equations [16–18].

In the literature, two different approaches have been used to establish the existence results for singular equations. The first one is the variational approach [3, 4, 6, 19, 20] and the second one is topological methods [1, 10, 21–28]. In our opinion, the first important result was proved in the pioneering paper of Lazer and Solimini [29]. They proved that a necessary and sufficient condition for the existence of a positive periodic solution for

$$x'' = \frac{1}{x^{\lambda}} + e(t) \tag{1}$$

is that the mean value of e is negative; that is, $\bar{e} < 0$, here $\lambda \geq 1$, which corresponds to a strong force condition, according to a terminology first introduced by Gordon [30]. Moreover, if $0 < \lambda < 1$, which corresponds to a weak force condition, they found examples of functions e with negative mean values and yet no periodic solutions exist. Therefore, there is an essential difference between a strong singularity and a weak singularity. Since the work of Lazer and Solimini, the strong force condition became standard in related work, see, for instance, [8, 15, 18, 27, 28]. Compared with the case of a strong singularity, the study of the existence of periodic solutions under the presence of a weak singularity is more recent, but it has also attracted many researchers [31–39]. In [39], for the first time in this topic, Torres et al. proved an existence result which is valid for a weak singularity, whereas the validity of such results under a strong force assumption remains as an open problem, which was partially solved in [32].

The main aim of this survey is to present some recent existence results for singular differential equations. In particular, we will consider the scalar singular equations, singular damped equations, singular impulsive equations, and singular differential systems. We will also include some examples to illustrate the results presented.

The rest of this paper is organized as follows. In Section 2, we will state some important results for the second-order scalar singular differential equations. Singular damped equations will be considered in Section 3. In Section 4, singular impulsive differential equations will be studied. Finally in Section 5, we will focus on the singular differential systems. Sections 2 and 3 are mainly written by the first author. Section 4 is mainly written by the second author, and Section 5 is mainly completed by the third author.

All the results presented in Sections 3–5 shed some lights on the differences between a strong singularity and a weak singularity.

Finally in this section, we must note that besides the results presented in this survey, many interesting and important results on singular differential equations have been obtained by other researchers, see, for example, [9, 40–45] and the references cited therein.

In this paper, we denote the essential supremum and infimum of p by p^* and p_*, respectively, for a given function $p \in L^1[0,T]$ essentially bounded.

2. Second-Order Scalar Singular Equations

In this section, we recall some results for second-order singular differential equations

$$x'' + a(t)x = f(t,x) + e(t), \tag{2}$$

here $a(t)$, $e(t)$ are continuous, T-periodic functions. The nonlinearity $f(t,x)$ is continuous in (t,x) and T-periodic in t and has a singularity at $x = 0$.

First we need to present some preliminary results on the linear equation

$$x'' + a(t)x = p(t) \tag{3}$$

with periodic boundary conditions

$$x(0) = x(T), \qquad x'(0) = x'(T). \tag{4}$$

We assume the following:

(A) the Green function $G(t,s)$, associated with (3)-(4), is positive for all $(t,s) \in [0,T] \times [0,T]$, or

(B) the Green function $G(t,s)$, associated with (3)-(4), is nonnegative for all $(t,s) \in [0,T] \times [0,T]$.

When $a(t) = k^2$, condition (A) is equivalent to $0 < k^2 < \lambda_1 = (\pi/T)^2$ and condition (B) is equivalent to $0 < k^2 \le \lambda_1$. In this case, we have

$$G(t,s) = \begin{cases} \dfrac{\sin k(t-s) + \sin k(T-t+s)}{2k(1-\cos kT)}, & 0 \le s \le t \le T, \\ \dfrac{\sin k(s-t) + \sin k(T-s+t)}{2k(1-\cos kT)}, & 0 \le t \le s \le T. \end{cases} \tag{5}$$

For a nonconstant function $a(t)$, there is an L^p-criterion proved in [46], which is given in Lemma 1 for the sake of completeness. Let $\mathbf{K}(q)$ denote the best Sobolev constant in the following inequality:

$$C\|u\|_q^2 \le \|u'\|_2^2, \quad \forall u \in H_0^1(0,T). \tag{6}$$

The explicit formula for $\mathbf{K}(q)$ is

$$\mathbf{K}(q) = \begin{cases} \dfrac{2\pi}{qT^{1+2/q}}\left(\dfrac{2}{2+q}\right)^{1-2/q}\left(\dfrac{\Gamma(1/q)}{\Gamma(1/2+1/q)}\right)^2 & \text{if } 1 \le q < \infty, \\ \dfrac{4}{T} & \text{if } q = \infty, \end{cases} \tag{7}$$

where Γ is the gamma function, see [47, 48].

Lemma 1 (see [46, Corollary 2.3]). *Assume that $a(t) \succ 0$ and $a \in L^p[0,T]$ for some $1 \le p \le \infty$. If*

$$\|a\|_p < \mathbf{K}(2\tilde{p}), \tag{8}$$

then the condition (A) holds. Moreover, condition (B) holds if

$$\|a\|_p \le \mathbf{K}(2\tilde{p}). \tag{9}$$

When the hypothesis (A) is satisfied, we denote

$$m = \min_{0\le s,t\le T} G(t,s), \quad M = \max_{0\le s,t\le T} G(t,s), \quad \sigma = \frac{m}{M}. \tag{10}$$

Obviously, $M > m > 0$ and $0 < \sigma < 1$.

The first existence result deals with the case of a strong singularity and the proof is based on the following nonlinear alternative of Leray-Schauder, which can be found in [49] or [50, pages 120–130].

Lemma 2. *Assume Ω is an open subset of a convex set K in a normed linear space X and $p \in \Omega$. Let $T : \overline{\Omega} \to K$ be a compact and continuous map. Then one of the following two conclusions holds.*

(I) *T has at least one fixed point in $\overline{\Omega}$.*

(II) *There exists $x \in \partial\Omega$ and $0 < \lambda < 1$ such that $x = \lambda Tx + (1-\lambda)p$.*

Theorem 3 (see [37, Theorem 4.1]). *Suppose that $a(t)$ satisfies (A) and $f(t,x)$ satisfies the following.*

(H_1) *There exists a nonincreasing positive continuous function $g_0(x)$ on $(0,\infty)$ and a constant $R_0 > 0$ such that $f(t,x) \ge g_0(x)$ for $(t,x) \in [0,T]\times(0,R_0]$, where $g_0(x)$ satisfies*

$$\lim_{x\to 0^+} g_0(x) = +\infty, \qquad \lim_{x\to 0^+}\int_x^{R_0} g_0(u)\,du = +\infty. \tag{11}$$

(H_2) *There exist continuous, nonnegative functions $g(x)$ and $h(x)$ such that*

$$0 \le f(t,x) \le g(x) + h(x) \quad \forall (t,x) \in [0,T]\times(0,\infty), \tag{12}$$

$g(x) > 0$ is nonincreasing and $h(x)/g(x)$ is nondecreasing in $x \in (0,\infty)$.

(H_3) *There exists a positive number r such that $\sigma r + \gamma_* > 0$ and*

$$\frac{r}{g(\sigma r + \gamma_*)\{1 + (h(r+\gamma^*)/g(r+\gamma^*))\}} > \omega^*, \tag{13}$$

here

$$\gamma(t) = \int_0^T G(t,s)e(s)\,ds, \qquad \omega(t) = \int_0^T G(t,s)\,ds. \tag{14}$$

Then for each $e \in \mathbb{C}(\mathbb{R}/T\mathbb{Z},\mathbb{R})$, (2) has at least one positive periodic solution x with $x(t) > \gamma(t)$ for all t and $0 < \|x-\gamma\| < r$.

Note that the study in [37, Theorem 4.1] is slightly different from the above presentation. However, the proof of the above theorem follows from that of [37, Theorem 4.1] with some minor necessary changes. Condition (H_1) corresponds to the classical strong force condition, which was first introduced by Gordon in [30]. In fact, condition (H_1) is only used when we try to obtain a prior lower bound. In Theorem 4, we will show that, for the case $\gamma_* \geq 0$, we can remove the strong force condition (H_1) and replace it by one weak force condition.

Theorem 4 (see [33, Theorem 3.1]). *Assume that (A) and (H_2)-(H_3) are satisfied. Suppose further the following condition.*

(H_4) *For each constant $L > 0$, there exists a continuous function $\phi_L \succ 0$ such that $f(t,x) \geq \phi_L(t)$ for all $(t,x) \in [0,T] \times (0,L]$.*

Then for each $e(t)$ with $\gamma_* \geq 0$, (2) has at least one positive periodic solution x with $x(t) > \gamma(t)$ for all t and $0 < \|x-\gamma\| < r$.

For the superlinear case, we can establish the multiplicity result. The proof is based on a well-known fixed point theorem in cones, which can be found in [51]. Let K be a cone in X and D is a subset of X, we write $D_K = D \cap K$ and $\partial_K D = (\partial D) \cap K$.

Theorem 5 (see [51]). *Let X be a Banach space and K a cone in X. Assume Ω^1, Ω^2 are open bounded subsets of X with $\Omega_K^1 \neq \emptyset$, $\overline{\Omega}_K^1 \subset \Omega_K^2$. Let*

$$T : \overline{\Omega}_K^2 \longrightarrow K \tag{15}$$

be a completely continuous operator such that

(a) *$\|Tx\| \leq \|x\|$ for $x \in \partial_K \Omega^1$,*

(b) *there exists $v \in K \setminus \{0\}$ such that $x \neq Tx + \lambda v$ forall $x \in \partial_K \Omega^2$ and all $\lambda > 0$.*

Then T has a fixed point in $\overline{\Omega}_K^2 \setminus \Omega_K^1$.

Theorem 6 (see [33, Theorem 3.2]). *Suppose that $a(t)$ satisfies (A) and $f(t,x)$ satisfies (H_2)-(H_3). Furthermore, assume the following conditions.*

(H_5) *There exist continuous, nonnegative functions $g_1(x)$, $h_1(x)$ such that*

$$f(t,x) \geq g_1(x) + h_1(x), \quad \forall (t,x) \in [0,T] \times (0,\infty), \tag{16}$$

$g_1(x) > 0$ is nonincreasing and $h_1(x)/g_1(x)$ is nondecreasing in x.

(H_6) *There exists $R > 0$ with $\sigma R > r$ such that*

$$\frac{\sigma R}{g_1(R+\gamma^*)\{1+(h_1(\sigma R+\gamma_*)/g_1(\sigma R+\gamma_*))\}} \leq \omega_*. \tag{17}$$

Then (2) has one positive periodic solution $\tilde{x}$ with $r < \|\tilde{x}-\gamma\| \leq R$.

Combined Theorems 3 and 4 with Theorem 6, we can get the following two multiplicity results.

Theorem 7. *Suppose that $a(t)$ satisfies (A) and $f(t,x)$ satisfies (H_1)–(H_3) and (H_5)-(H_6). Then (2) has two different positive periodic solutions x and $\tilde{x}$ with $0 < \|x-\gamma\| < r < \|\tilde{x}-\gamma\| \leq R$.*

Theorem 8. *Suppose that $a(t)$ satisfies (A) and $f(t,x)$ satisfies (H_2)–(H_6). Then (2) has two different positive periodic solutions x and $\tilde{x}$ with $0 < \|x-\gamma\| < r < \|\tilde{x}-\gamma\| \leq R$.*

To illustrate our results, we have selected the following singular equation:

$$x'' + a(t)x = x^{-\alpha} + \mu x^{\beta} + e(t), \tag{18}$$

here $a, e \in \mathbb{C}[0,T]$, $\alpha, \beta > 0$, and $\mu \in \mathbb{R}$ is a given parameter. The corresponding results are also valid for the general case

$$x'' + a(t)x = \frac{b(t)}{x^{\alpha}} + \mu c(t)x^{\beta} + e(t), \tag{19}$$

with $b, c \in \mathbb{C}[0,T]$.

Corollary 9. *Assume that $a(t)$ satisfies (A) and $\alpha > 0$, $\beta \geq 0$, $\mu > 0$. Then one has the following results.*

(i) *If $\alpha \geq 1, \beta < 1$, then for each $e \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$, (18) has at least one positive periodic solution for all $\mu > 0$.*

(ii) *If $\alpha \geq 1, \beta \geq 1$, then for each $e \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$, (18) has at least one positive periodic solution for each $0 < \mu < \mu_1$; here μ_1 is some positive constant.*

(iii) *If $\alpha \geq 1, \beta > 1$, then for each $e \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$, (18) has at least two positive periodic solutions for each $0 < \mu < \mu_1$.*

(iv) *If $\alpha > 0, \beta < 1$, then for each $e \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$, with $\gamma_* \geq 0$, (18) has at least one positive periodic solution for all $\mu > 0$.*

(v) *If $\alpha > 0, \beta \geq 1$, then for each $e \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$, with $\gamma_* \geq 0$, (18) has at least one positive periodic solution for each $0 < \mu < \mu_1$.*

(vi) *If $\alpha > 0, \beta > 1$, then for each $e \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$, with $\gamma_* \geq 0$, (18) has at least two positive periodic solutions for each $0 < \mu < \mu_1$.*

All the above results require that the linear equation satisfies (A), which cannot cover the critical case. The next few results deal with the case when the condition (B) is satisfied and the proof is based on Schauder's fixed point theorem.

Theorem 10 (see [31, Theorem 3.1]). *Assume that conditions (B) and (H$_2$) and (H$_4$) are satisfied. Furthermore, suppose that*

(H$_7$) *there exists a positive constant $R > 0$ such that $R > \Phi_*, \Phi_* + \gamma_* > 0$ and*

$$R \geq g(\Phi_* + \gamma_*)\left\{1 + \frac{h(R+\gamma^*)}{g(R+\gamma^*)}\right\}\omega^*, \tag{20}$$

here $\Phi_* = \min_t \Phi(t)$, $\Phi(t) = \int_0^T G(t,s)\phi_{R+\gamma^*}(s)ds$.

Then (2) *has at least one positive T-periodic solution.*

As an application of Theorem 10, we consider the case $\gamma_* = 0$. Corollary 11 is a direct result of Theorem 10.

Corollary 11 (see [31, Corollary 3.2]). *Assume that conditions (B) and (H$_2$) and (H$_4$) are satisfied. Furthermore, assume that*

(H$_8$) *there exists a positive constant $R > 0$ such that $R > \Phi_*$ and*

$$R \geq g(\Phi_*)\left\{1 + \frac{h(R+\gamma^*)}{g(R+\gamma^*)}\right\}\omega^*. \tag{21}$$

If $\gamma_* = 0$, then (2) *has at least one positive T-periodic solution.*

Corollary 12 (see [31, Example 3.5]). *Suppose that a satisfies (B) and $0 < \alpha < 1$, $\beta \geq 0$, then for each $e(t) \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$, with $\gamma_* = 0$, one has the following:*

(i) *if $\alpha + \beta < 1 - \alpha^2$, then* (18) *has at least one positive periodic solution for each $\mu \geq 0$,*

(ii) *if $\alpha + \beta \geq 1 - \alpha^2$, then* (18) *has at least one positive T-periodic solution for each $0 \leq \mu < \mu_2$, where μ_2 is some positive constant.*

The next results explore the case when $\gamma_* > 0$.

Theorem 13 (see [31, Theorem 3.6]). *Suppose that $a(t)$ satisfies (B) and $f(t,x)$ satisfies condition (H$_2$). Furthermore, assume that*

(H$_9$) *there exists $R > \gamma^*$ such that*

$$g(\gamma_*)\left\{1 + \frac{h(R+\gamma^*)}{g(R+\gamma^*)}\right\}\omega^* \leq R. \tag{22}$$

If $\gamma_* > 0$, then (2) *has at least one positive T-periodic solution.*

Corollary 14 (see [31, Example 3.8]). *Suppose that $a(t)$ satisfies (B) and $\alpha, \beta \geq 0$, then for each $e \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$, with $\gamma_* > 0$, one has the following:*

(i) *if $\alpha + \beta < 1$, then* (18) *has at least one positive T-periodic solution for each $\mu \geq 0$,*

(ii) *if $\alpha + \beta \geq 1$, then* (18) *has at least one positive T-periodic solution for each $0 \leq \mu < \mu_3$, where μ_3 is some positive constant.*

3. Singular Damped Equations

In this section, we recall some results on second-order singular damped differential equations

$$x'' + h(t)x' + a(t)x = f(t, x, x'), \tag{23}$$

where $h, a \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}, \mathbb{R})$ and the nonlinearity $f \in \mathbb{C}((\mathbb{R}/T\mathbb{Z}) \times (0,\infty) \times \mathbb{R}, \mathbb{R})$. In particular, the nonlinearity may have a repulsive singularity at $x = 0$, which means that

$$\lim_{x \to 0^+} f(t,x,y) = +\infty, \quad \text{uniformly in } (t,y) \in \mathbb{R}^2. \tag{24}$$

First we recall some results on the linear damped equation

$$x'' + h(t)x' + a(t)x = 0, \tag{25}$$

associated to periodic boundary conditions (4). As in the last section, we say that (25)-(4) is nonresonant when its unique T-periodic solution is the trivial one. When (25)-(4) is nonresonant, as a consequence of Fredholm's alternative, the nonhomogeneous equation

$$x'' + h(t)x' + a(t)x = l(t) \tag{26}$$

admits a unique T-periodic solution which can be written as

$$x(t) = \int_0^T G_2(t,s)\, l(s)\, ds, \tag{27}$$

where $G_2(t,s)$ is the Green's function of problem (25)-(4). We also assume that the following standing hypothesis is satisfied.

(C) The Green's function $G_2(t,s)$, associated with (25)-(4), is positive for all $(t,s) \in [0,T] \times [0,T]$.

To guarantee that (C) is satisfied, we require the antimaximum principle for (25)-(4) proved by Hakl and Torres in [52]. To do this, let us define the functions

$$\begin{gathered}\sigma(h)(t) = \exp\left(\int_0^t h(s)\,ds\right),\\ \sigma_1(h)(t) = \sigma(h)(T)\int_0^t \sigma(h)(s)\,ds + \int_t^T \sigma(h)(s)\,ds.\end{gathered} \tag{28}$$

Lemma 15 (see [52, Theorem 2.2]). *Assume that $a \not\equiv 0$ and the following two inequalities are satisfied:*

$$\begin{gathered}\int_0^T a(s)\,\sigma(h)(s)\,\sigma_1(-h)(s)\,ds \geq 0,\\ \sup_{0\leq t\leq T}\left\{\int_t^{t+T}\sigma(-h)(s)\,ds\int_t^{t+T}[a(s)] + \sigma(h)(s)\,ds\right\} \leq 4,\end{gathered} \tag{29}$$

where $[a(s)]_+ = \max\{a(s), 0\}$. Then (C) holds.

For the special case $\int_0^T a(t)\sigma(h)(t)dt > 0$ and $h \in \widetilde{\mathbb{C}}(\mathbb{R}/T\mathbb{Z}) := \{h \in \mathbb{C}(\mathbb{R}/T\mathbb{Z}) : \overline{h} = 0\}$, one criterion has been developed by Cabada and Cid in [40].

Theorem 16 (see [40, Theorem 5.1]). *Assume that* $h \in \widetilde{\mathbb{C}}(\mathbb{R}/T\mathbb{Z})$ *and* $\int_0^T a(t)\sigma(h)(t)dt > 0$. *Suppose further that there exists* $1 \le p \le \infty$ *such that*

$$(B(T))^{1+1/q}\|\mathscr{A}_+\|_{p,T} < \mathbf{M}^2(2q), \quad (30)$$

where

$$B(T) = \int_0^T \sigma(-h)(t)\,dt, \qquad \mathscr{A}_+(t) = a_+(t)(\sigma(h)(t))^{2-1/p}. \quad (31)$$

Then (C) holds.

Theorem 17 (see [35, Theorem 3.2]). *Suppose that* (25) *satisfies (C) and*

$$\int_0^T a(t)\sigma(h)(t)\,dt > 0. \quad (32)$$

Furthermore, assume that there exists a constant $r > 0$ *such that*

(G_1) *there exists a continuous function* $\phi_r \succ 0$ *such that* $f(t,x,y) \ge \phi_r(t)$ *for all* $(t,x,y) \in [0,T]\times(0,r]\times(-\infty,\infty)$,

(G_2) *there exist continuous, nonnegative functions* $g(\cdot)$, $h(\cdot)$, *and* $\varrho(\cdot)$ *such that*

$$0 \le f(t,x,y) \le (g(x)+h(x))\,\varrho(|y|), \qquad \forall (t,x,y) \in [0,T]\times(0,r]\times\mathbb{R}, \quad (33)$$

where $g(\cdot) > 0$ *is nonincreasing,* $h(\cdot)/g(\cdot)$ *is non-decreasing in* $(0,r]$, *and* $\varrho(\cdot)$ *is non-decreasing in* $(0,\infty)$,

(G_3) *the following inequality holds:*

$$\frac{r}{g(\iota r)\{1+(h(r)/g(r))\}\varrho(Lr)} > \omega^*, \quad (34)$$

where

$$\omega(t) = \int_0^T G(t,s)\,ds, \qquad L = \frac{2\int_0^T a(t)\sigma(h)(t)\,dt}{\min_{0\le t\le T}\sigma(h)(t)},$$
$$\iota = \frac{m}{M}, \quad m = \min_{0\le s,t\le T} G(t,s), \quad M = \max_{0\le s,t\le T} G(t,s), \quad (35)$$

then (23) *has at least one positive* T*-periodic solution* x *with* $0 < \|x\| \le r$.

Corollary 18 (see [35, Corollary 3.3]). *Let the nonlinearity in* (23) *be*

$$f(t,x,y) = \left(1+|y|^\gamma\right)\left(x^{-\alpha}+\mu x^\beta\right), \quad (36)$$

where $\alpha > 0$, $\beta, \gamma \ge 0$, $\mu > 0$ *is a positive parameter.*

(i) *If* $\beta+\gamma < 1$, *then* (23) *has at least one positive periodic solution for each* $\mu > 0$.

(ii) *If* $\beta+\gamma \ge 1$, *then* (23) *has at least one positive periodic solution for each* $0 < \mu < \mu_1^*$, *where* μ_1^* *is some positive constant.*

Corollary 19 (see [35, Corollary 3.4]). *Let the nonlinearity in* (23) *be*

$$f(t,x,y) = \left(1+|y|^\gamma\right)\left(\frac{1}{x^\alpha}-\frac{\mu}{x^\beta}\right), \quad (37)$$

where $\alpha > \beta > 0$, $\gamma \ge 0$ *with* $\gamma < \alpha+1$, $\mu > 0$ *is a positive parameter. Then there exists a positive constant* μ_2^* *such that* (23) *has at least one positive* T*-periodic solution for each* $0 \le \mu < \mu_2^*$.

Corollary 19 is interesting because the singularity on the right-hand side combines attractive and repulsive effects. The analysis of such differential equations with mixed singularities is at this moment very incomplete, and few references can be cited [22, 44]. Therefore, the results in Corollary 19 can be regarded as one contribution to the literature trying to fill partially this gap in the study of singularities of mixed type.

As in the last section, if we assume that the linear equation (25)-(4) has a nonnegative Green's function, we can also get some results based on Schauder's fixed point theorem, and the results can cover the critical case.

4. Singular Impulsive Differential Equations

In this section, we will study the existence of periodic solutions for some singular differential equations with impulsive effects by using variational methods.

Firstly, we consider the following second-order nonautonomous singular problem:

$$u'' - \frac{b(t)}{u^\alpha} = e(t), \quad \text{a.e. } t \in (0,T), \qquad u(0)-u(T) = u'(0)-u'(T) = 0, \quad (38)$$

under the impulse conditions

$$\Delta u'(t_j) = I_j(u(t_j)), \quad j = 1,2,\ldots,p-1, \quad (39)$$

where t_j, $j = 1,2,\ldots,p-1$ are the instants where the impulses occur and $0 = t_0 < t_1 < t_2 < \cdots < t_{p-1} < t_p = T$, $I_j : \mathbb{R} \to \mathbb{R}$ $(j = 1,2,\ldots,p-1)$ are continuous.

Our result is presented as follows.

Theorem 20 (see [19, Theorem 1.1]). *Assume that* $\alpha > 1$ *and the following conditions hold.*

(S_1) $b \in \mathbb{C}^1([0,T],(0,\infty))$ *is* T*-periodic and* $b'(t) \ge 0$ *for all* $t \in [0,T]$.

(S_2) $e \in L^2([0,T],\mathbb{R})$ *is* T*-periodic and* $\int_0^T e(t)dt < 0$.

(S_3) *There exist two constants* m, M *such that for any* $t \in \mathbb{R}$,

$$m \le I_j(t) \le M, \quad j = 1,2,\ldots,p-1, \quad (40)$$

where $m < 0$ *and* $0 \le M < (-1/(p-1))\int_0^T e(t)dt$.

(S_4) *For any* $t \in \mathbb{R}$,

$$\int_0^t I_j(s)\,ds \ge 0, \quad j = 1, 2, \ldots, p-1. \tag{41}$$

Then problem (38)-(39) *has at least one solution.*

Remark 21. In fact, it is not difficult to find some functions I_j satisfying (S_3) and (S_4). For example,

$$I_j(t) = \sin t, \quad t \in \mathbb{R}. \tag{42}$$

Let

$$\begin{aligned} H_T^1 = \{&u : [0,T] \longrightarrow \mathbb{R} \mid u \text{ is absolutely continuous}, \\ &u(0) = u(T) \text{ and } u' \in L^2([0,T], \mathbb{R})\}, \end{aligned} \tag{43}$$

with the inner product

$$(u, v) = \int_0^T u(t)\,v(t)\,dt + \int_0^T u'(t)\,v'(t)\,dt, \quad \forall u, v \in H_T^1. \tag{44}$$

The corresponding norm is defined by

$$\|u\|_{H_T^1} = \left(\int_0^T |u(t)|^2 dt + \int_0^T |u'(t)|^2 dt\right)^{1/2}, \quad \forall u \in H_T^1. \tag{45}$$

Then H_T^1 is a Banach space (in fact it is a Hilbert space).

If $u \in H_T^1$, then u is absolutely continuous and $u' \in L^2([0,T], \mathbb{R})$. In this case, $\Delta u'(t) = u'(t^+) - u'(t^-) = 0$ is not necessarily valid for every $t \in (0,T)$ and the derivative u' may exist some discontinuities. It may lead to impulse effects.

Following the ideas of [53], take $v \in H_T^1$ and multiply the two sides of the equality

$$-u'' + \frac{b(t)}{u^\alpha} + e(t) = 0 \tag{46}$$

by v and integrate from 0 to T, so we have

$$\int_0^T \left[-u'' + \frac{b(t)}{u^\alpha} + e(t)\right] v\,dt = 0. \tag{47}$$

Note that since $u'(0) - u'(T) = 0$, one has

$$\begin{aligned} &\int_0^T u''(t)\,v(t)\,dt \\ &= \sum_{j=0}^{p-1} \int_{t_j}^{t_{j+1}} u''(t)\,v(t)\,dt \\ &= \sum_{j=0}^{p-1} \left(u'(t_{j+1}^-)\,v(t_{j+1}^-) - u'(t_j^+)\,v(t_j^+)\right) \\ &\quad - \sum_{j=0}^{p-1} \int_{t_j}^{t_{j+1}} u'(t)\,v'(t)\,dt \\ &= u'(T)\,v(T) - u'(0)\,v(0) - \sum_{j=1}^{p-1} \Delta u'(t_j)\,v(t_j) \\ &\quad - \int_0^T u'(t)\,v'(t)\,dt \\ &= -\sum_{j=1}^{p-1} I_j(u(t_j))\,v(t_j) - \int_0^T u'(t)\,v'(t)\,dt. \end{aligned} \tag{48}$$

Combining with (47), we get

$$\begin{aligned} &\int_0^T u'(t)\,v'(t)\,dt + \sum_{j=1}^{p-1} I_j(u(t_j))\,v(t_j) \\ &\quad + \int_0^T \frac{b(t)}{u^\alpha} v(t)\,dt + \int_0^T e(t)\,v(t)\,dt = 0. \end{aligned} \tag{49}$$

As a result, we introduce the following concept of a weak solution for problem (38)-(39).

Definition 22. One says that a function $u \in H_T^1$ is a weak solution of problem (38)-(39) if

$$\begin{aligned} &\int_0^T u'(t)\,v'(t)\,dt + \sum_{j=1}^{p-1} I_j(u(t_j))\,v(t_j) \\ &\quad + \int_0^T \frac{b(t)}{u^\alpha} v(t)\,dt + \int_0^T e(t)\,v(t)\,dt = 0 \end{aligned} \tag{50}$$

holds for any $v \in H_T^1$.

Define the functional $\Phi : H_T^1 \to \mathbb{R}$ by

$$\begin{aligned} \Phi(u) := &\frac{1}{2} \int_0^T |u'(t)|^2 dt + \sum_{j=1}^{p-1} \int_0^{u(t_j)} I_j(s)\,ds \\ &+ \int_0^T b(t) \left(\int_1^{u(t)} \frac{1}{s^\alpha} ds\right) dt + \int_0^T e(t)\,u(t)\,dt, \end{aligned} \tag{51}$$

for every $u \in H_T^1$. Clearly, Φ_λ is well defined on H_T^1, continuously Gâteaux differentiable functional whose Gâteaux derivative is the functional $\Phi'_\lambda(u)$, given by

$$\begin{aligned} \Phi'_\lambda(u)\,v = &\int_0^T u'(t)\,v'(t)\,dt + \sum_{j=1}^{p-1} I_j(u(t_j))\,v(t_j) \\ &- \int_0^T \frac{b(t)}{u^\alpha} v(t)\,dt + \int_0^T e(t)\,v(t)\,dt, \end{aligned} \tag{52}$$

for any $v \in H_T^1$. Moreover, it is easy to verify that Φ_λ is weakly lower semicontinuous. Indeed, if $\{u_n\} \subset H_T^1$, $u \in H_T^1$, and $u_n \rightharpoonup u$, then $\{u_n\}$ converges uniformly to u on $[0,T]$

and $u_n \to u$ on $L^2([0,T])$, and combining the fact that $\liminf_{n\to\infty}\|u_n\|_{H_T^1} \geq \|u\|_{H_T^1}$, one has

$$\begin{aligned}
&\liminf_{n\to\infty}\Phi_\lambda(u_n)\\
&\quad = \liminf_{n\to\infty}\Bigg(\frac{1}{2}\|u_n\|_{H_T^1}^2 - \frac{1}{2}\int_0^T |u_n(t)|^2 dt\\
&\qquad + \sum_{j=1}^{p-1}\int_0^{u_n(t_j)} I_j(s)\,ds\\
&\qquad - \int_0^T b(t)\left(\int_1^{u_n(t)}\frac{1}{s^\alpha}ds\right)dt\\
&\qquad + \int_0^T e(t)\,u_n(t)\,dt\Bigg)\\
&\quad \geq \frac{1}{2}\int_0^T |u'(t)|^2 dt + \sum_{j=1}^{p-1}\int_0^{u(t_j)} I_j(s)\,ds\\
&\qquad - \int_0^T b(t)\left(\int_1^{u(t)}\frac{1}{s^\alpha}ds\right)dt + \int_0^T e(t)\,u(t)\,dt = \Phi_\lambda(u).
\end{aligned} \tag{53}$$

By the standard discussion, the critical points of Φ_λ are the weak solutions of problem (38)-(39), see [53, 54].

The following version of the mountain pass theorem will be used in our argument.

Theorem 23 (see [55, Theorem 4.10]). *Let X be a Banach space and let $\varphi \in C^1(X,\mathbb{R})$. Assume that there exist $x_0, x_1 \in X$ and an open neighborhood Ω of x_0 such that $x_1 \in X \setminus \overline{\Omega}$ and*

$$\max\{\varphi(x_0),\varphi(x_1)\} < \inf_{x\in\partial\Omega}\varphi(x). \tag{54}$$

Let

$$\begin{gathered}
\Gamma = \{h \in C([0,1],X) : h(0) = x_0, h(1) = x_1\},\\
c = \inf_{h\in\Gamma}\max_{s\in[0,1]}\varphi(h(s)).
\end{gathered} \tag{55}$$

If φ satisfies the (PS)-condition, that is, a sequence $\{u_n\}$ in X satisfying $\varphi(u_n)$ is bounded and $\varphi'(u_n) \to 0$ as $n \to \infty$ has a convergent subsequence, then c is a critical value of φ and $c > \max\{\varphi(x_0),\varphi(x_1)\}$.

Next we consider T-periodic solution for another impulsive singular problem:

$$u''(t) - \frac{1}{u^\alpha(t)} = e(t), \tag{56}$$

under impulsive conditions

$$\Delta u'(t_j) = I_j(u(t_j)), \quad j = 1,2,\ldots,p-1, \tag{57}$$

where $\alpha \geq 1$, $e \in L^1([0,T],\mathbb{R})$ is T-periodic, $\Delta u'(t_j) = u'(t_j^+) - u'(t_j^-)$ with $u'(t_j^\pm) = \lim_{t\to t_j^\pm}u'(t)$; t_j, $j = 1,2,\ldots,p-1$ are the instants where the impulses occur, and $0 = t_0 < t_1 < t_2 < \cdots < t_{p-1} < t_p = T$, $t_{j+p} = t_j + T$; $I_j : \mathbb{R} \to \mathbb{R}$ $(j = 1,2,\ldots,p-1)$ are continuous and $I_{j+p} \equiv I_j$.

In 1987, Lazer and Solimini [29] proved a famous result as follows.

Theorem 24 (see [29]). *Assume that $e \in L^1([0,T],\mathbb{R})$ is T-periodic. Then problem (56) has a positive T-periodic weak solution if and only if $\int_0^T e(t)dt < 0$.*

From Theorem 24, if $\int_0^T e(t)dt \geq 0$, then problem (52) does not have a positive T-periodic weak solution. However, if the impulses happen, for this singular problem may exist a positive T-periodic weak solution. Inspired by the above facts, our aim is to reveal a new existence result on positive T-periodic solution for singular problem (56) when impulsive effects are considered, that is, problem (56)-(57). Indeed, this periodic solution is generated by impulses. Here, we say a solution is generated by impulses if this solution is nontrivial when $I_j \not\equiv 0$ for some $1 < j < p-1$, but it is trivial when $I_j \equiv 0$ for all $1 < j < p-1$. For example, if problem (56)-(57) does not possess positive periodic solution when $I_j \equiv 0$ for all $1 < j < p-1$, then a positive periodic solution u of problem (56)-(57) with $I_j \not\equiv 0$ for some $1 < j < p-1$ is called a positive periodic solution generated by impulses.

Our result is presented as follows.

Theorem 25 (see [35, Theorem 1.2]). *Assume the following:*

(S_1) *$e \in L^1([0,T],\mathbb{R})$ is T-periodic and $\int_0^T e(t)dt \geq 0$;*

(S_2) *there exist two constants m, M such that for any $s \in \mathbb{R}$,*

$$m \leq I_j(s) \leq M, \quad j = 1,2,\ldots,p-1, \tag{58}$$

where $m \leq M < (-1/(p-1))\int_0^T e(t)dt \leq 0$.
Then problem (56)-(57) has at least a positive T-periodic solution.

5. Singular Differential Systems

In this section, we will consider the system of Hill's equations

$$\begin{gathered}
u_i''(t) + a_i(t)\,u_i(t) = F_i(t, u_1(t), u_2(t), \ldots, u_n(t)),\\
1 \leq i \leq n.
\end{gathered} \tag{59}$$

Here, a_i and F_i are T-periodic in the variable t, $a_i \in L^1[0,T]$, and the nonlinearities $F_i(t,x_1,x_2,\ldots,x_n)$ can be *singular* at $x_j = 0$ where $j \in \{1,2,\ldots,n\}$.

Throughout, let $u = (u_1,u_2,\ldots,u_n)$. We are interested in establishing the existence of continuous T-periodic solutions u of the system (59), that is, $u \in (C(\mathbb{R}))^n$ and $u(t) = u(t+T)$ for all $t \in \mathbb{R}$. Moreover, we are concerned with constant-sign solutions u, by which we mean $\theta_i u_i(t) \geq 0$ for all $t \in \mathbb{R}$ and $1 \leq i \leq n$, where $\theta_i \in \{1,-1\}$ is fixed. Note that positive solution, the usual consideration in the literature, is a special case of constant-sign solution when $\theta_i = 1$ for $1 \leq i \leq n$.

We will employ the Schauder's fixed point theorem to establish the existence of solutions. Indeed, in Section 5.1 we will first tackle a particular case of (59) when

$$F_i(t,u(t)) = \partial_2 h_i\left(t, \frac{1}{2}|u(t)|^2\right) u_i(t) + f_i(t). \tag{60}$$

Here, $\partial_2 h_i$ is the partial derivative of h_i with respect to the second variable, and $|\cdot|$ is a norm in $\mathbb{R}^n$. The particular case (60) occurs in the problem [36]

$$\ddot{u}(t) + \nabla_u P(t,u(t)) = f(t), \tag{61}$$

where the potential

$$P(t,u) = \frac{1}{2}a(t)|u|^2 - h\left(t, \frac{1}{2}|u|^2\right), \tag{62}$$

and h presents a singularity of the repulsive type, that is, $\lim_{|x|\to 0} h(t,x) = \infty$ uniformly in t. The general problem (59) will be investigated in Section 5.2; here the singularities are not necessarily generated by a potential as in the case of (60). To illustrate our results, several examples will be presented.

In [45], the authors use a nonlinear alternative of the Leray-Schauder type and a fixed point theorem in cones to establish the existence of two positive periodic solutions for the system

$$\ddot{u}(t) + a(t)u(t) = G(u(t)), \tag{63}$$

where G can be expressed as a sum of two positive functions satisfying certain monotone conditions. Therefore, the results in [45] are not applicable to (59) with F_i as in (60). In [45] it is also shown that the system

$$\begin{aligned} u_1''(t) + a_1(t)u_1(t) &= \left(\sqrt{u_1^2+u_2^2}\right)^{-\beta} + \nu\left(\sqrt{u_1^2+u_2^2}\right)^{\gamma}, \\ u_2''(t) + a_2(t)u_2(t) &= \left(\sqrt{u_1^2+u_2^2}\right)^{-\beta} + \nu\left(\sqrt{u_1^2+u_2^2}\right)^{\gamma} \end{aligned} \tag{64}$$

has a solution when $\beta > 0$, $\gamma \in [0,1)$, and $\nu > 0$. We will generalize the system (64) in Examples 46–48 to allow ν to be zero or negative. The improvement is possible probably due to the fact that we do not need to make a technical truncation to get compactness when we employ the Schauder fixed point theorem as compared to when the Leray-Schauder alternative is used. In fact, the set that we work on excludes the singularities. The results presented in this section not only generalize the papers [36, 39, 45] to systems and existence of constant-sign solutions, but also improve and/or complement the results in these earlier work as well as other research papers [56–60]. This section is based on the work in [61].

5.1. Existence Results for (60). In this section we will consider the system of Hill's equations

$$u_i''(t) + a_i(t)u_i(t) = \partial_2 h_i\left(t, \frac{1}{2}|u(t)|^2\right) u_i(t) + f_i(t), \quad 1 \le i \le n. \tag{65}$$

Here, $\partial_2 h_i(t,s) \equiv (\partial/\partial s)h_i(t,s)$ and $|\cdot|$ is a norm in $\mathbb{R}^n$. Moreover, $a_i(t)$, $\partial_2 h_i(t,s)$, and $f_i(t)$ are T-periodic in t, $a_i \in L^1[0,T]$, $f_i \in L^1[0,T]$, and $\partial_2 h_i(t,s)$ can be *singular* at $s = 0$.

To seek a T-periodic solution $u^T = (u_1^T, u_2^T, \ldots, u_n^T)$ of the system (65), we first obtain a solution $u^* = (u_1^*, u_2^*, \ldots, u_n^*)$ of the following system of boundary value problems:

$$\begin{aligned} &u_i''(t) + a_i(t)u_i(t) \\ &\quad = \partial_2 h_i\left(t, \frac{1}{2}|u(t)|^2\right) u_i(t) + f_i(t), \quad t \in [0,T], \\ &u_i(0) = u_i(T), \quad u_i'(0) = u_i'(T), \quad 1 \le i \le n. \end{aligned} \tag{66}$$

Then, set

$$u^T(t) = u^*(t - mT), \quad t \in [mT, (m+1)T], \; m \in \mathbb{Z}. \tag{67}$$

Our main tool is Schauder's fixed point theorem, which is stated below for completeness.

Theorem 26 (see [62]). *Let Ω be a convex subset of a Banach space B and $S : \Omega \to \Omega$ a continuous and compact map. Then S has a fixed point.*

To begin, let g_i be Green's function of the boundary value problem

$$\begin{aligned} x''(t) + a_i(t)x(t) = 0, \quad t \in [0,T], \\ x(0) = x(T), \qquad x'(0) = x'(T). \end{aligned} \tag{68}$$

Throughout, we will assume that the functions $a_i \in L^1[0,T]$ are such that

(C1) the Hill's equation $x''(t) + a_i(t)x(t) = 0$ is nonresonant (i.e., the unique periodic solution is the trivial solution), and $g_i(t,s) \ge 0$ for all $(t,s) \in [0,T] \times [0,T]$.

Note that Torres [46] has a result on $a_i(t)$ that ensures that condition (C1) is satisfied. In fact, if $a_i(t) = k^2$, then (C1) holds if $k \in (0, \pi/T]$; if $a_i(t)$ is not a constant, then (C1) is valid if the L_p norm of $a_i(t)$ is bounded above by some specific constant.

Let $\theta_i \in \{1, -1\}$, $1 \le i \le n$ be fixed. Define

$$\phi_i(t) = \int_0^T g_i(t,s)\theta_i f_i(s)\,ds, \quad t \in [0,T], \; 1 \le i \le n. \tag{69}$$

We also let

$$\phi_i^{\min} = \min_{t\in[0,T]} \phi_i(t), \qquad \phi_i^{\max} = \max_{t\in[0,T]} \phi_i(t). \tag{70}$$

We now present our main result which tackles (65) when the norm $|\cdot|$ in $\mathbb{R}^n$ is the l_p norm or the l_∞ norm.

Theorem 27. *Assume that the following conditions hold for each $1 \le i \le n$: (C1),*

(C2) $\phi_i^{\min} > 0$;

(C3) *let* $H_i(t,s) = (\partial/\partial s)h_i(t,s)$*; for any numbers* b, b' *with* $b' \geq b > 0$*, the function* $H_i : [0,T] \times [b,b'] \to \mathbb{R}$ *is an* L^1*-Carathéodory function, that is,*

(i) *the map* $s \mapsto H_i(t,s)$ *is continuous for almost all* $t \in [0,T]$*,*

(ii) *the map* $t \mapsto H_i(t,s)$ *is measurable for all* $s \in [b,b']$*,*

(iii) *for any* $r > 0$*, there exists* $\mu_{r,i} \in L^1[0,T]$ *such that* $|s| \leq r(s \in [b,b'])$ *implies* $|H_i(t,s)| \leq \mu_{r,i}(t)$ *for almost all* $t \in [0,T]$*;*

(C4) $(\partial/\partial s)h_i(t,s) \geq 0$ *for* $t \in [0,T]$ *and* $s > 0$*;*

(C5) *there exist* $c_i > 0$ *and* $\alpha_i > 0$ *such that*

$$\frac{\partial}{\partial s} h_i(t,s) \leq c_i s^{-\alpha_i}, \quad t \in [0,T], \ s > 0; \tag{71}$$

(C6) *the norm* $|\cdot|$ *is the* l_p *norm where* $1 \leq p \leq \infty$ *is fixed, and*

$$\int_0^T g_i(t,s)\,ds < A_i^p \left(c_i 2^{\alpha_i}\right)^{-1}, \quad t \in [0,T], \tag{72}$$

where

$$A_i^p = \begin{cases} \left[\sum_{k=1}^n \left(\phi_k^{\min}\right)^p \right]^{2\alpha_i/p}, & 1 \leq p < \infty, \\ \left[\max_{1\leq k\leq n} \phi_k^{\min} \right]^{2\alpha_i}, & p = \infty. \end{cases} \tag{73}$$

Then, (65) *has a* T*-periodic constant-sign solution* $u^T \in (C(\mathbb{R}))^n$ *such that*

$$\phi_i^{\min} \leq \theta_i u_i^T(t) \leq R_i, \quad t \in \mathbb{R}, \ 1 \leq i \leq n, \tag{74}$$

where

$$R_i \geq \phi_i^{\min}, \qquad R_i \geq \phi_i^{\max} \left[1 - \frac{c_i 2^{\alpha_i}}{A_i^p} \max_{t\in[0,T]} \int_0^T g_i(t,s)\,ds \right]^{-1}, \quad 1 \leq i \leq n. \tag{75}$$

Theorem 27 is proved using Theorem 26; in fact we will seek a constant-sign solution of (66) in $(C[0,T])^n$ and then extend it to a T-periodic constant-sign solution of (65) as in (67). Here, let Ω be the closed convex set given by

$$\Omega = \left\{ u \in (C[0,T])^n \mid \phi_i^{\min} \leq \theta_i u_i(t) \leq R_i, \ t \in [0,T], \ 1 \leq i \leq n \mid \phi_i^{\min} \right\}, \tag{76}$$

where R_i ($\geq \phi_i^{\min} > 0$) is chosen as in (75), and define the operator $S : \Omega \to (C[0,T])^n$ as

$$Su(t) = \left(S_1 u(t), S_2 u(t), \ldots, S_n u(t)\right), \quad t \in [0,T], \tag{77}$$

where

$$S_i u(t) = \int_0^T g_i(t,s) \left[\partial_2 h_i \left(s, \frac{1}{2}|u(s)|^2 \right) u_i(s) + f_i(s) \right] ds, \quad t \in [0,T], \ 1 \leq i \leq n. \tag{78}$$

Clearly, a fixed point of $Su = u$ is a solution of (66). We can show that $S(\Omega) \subseteq \Omega$; that is, $S_i(\Omega) \subseteq \Omega$ for each $1 \leq i \leq n$. Further, we can prove that $S : \Omega \to \Omega$ is continuous and compact; that is, $S_i u$ is bounded and is equicontinuous for any $u \in \Omega$ and $1 \leq i \leq n$. By Theorem 26, the system (66) has a constant-sign solution $u^* \in \Omega$. Now, a T-periodic constant-sign solution u^T of (65) can be obtained as in (67).

Remark 28. The constants c_i that appear in (C5) determine the upper bounds R_i of the solution u_i^T, $1 \leq i \leq n$. Noting (75), we see that a smaller (bigger) c_i gives a smaller (bigger) R_i, and hence a smaller (bigger) set Ω where the solution lies.

In the next result, we will relax the condition (C6). The tradeoff is the upper bounds R_i of the solution that may be bigger than those in (75). Also the bounds R_i do not depend on p (p as in l_p norm) and so the information of p is not utilized. This result is obtained by following the main arguments in the derivation of Theorem 27 but modify the proof of $\theta_i S_i u(t) \leq R_i$, $t \in [0,T]$.

Theorem 29. *Assume that (C1)–(C5) hold for each* $1 \leq i \leq n$*. The norm* $|\cdot|$ *is the* l_p *norm where* $1 \leq p \leq \infty$ *is fixed. Then* (65) *has a* T*-periodic constant-sign solution* $u^T \in (C(\mathbb{R}))^n$ *such that*

$$\phi_i^{\min} \leq \theta_i u_i^T(t) \leq R_i, \quad t \in \mathbb{R}, \ 1 \leq i \leq n, \tag{79}$$

where, for $1 \leq i \leq n$ *we have* $R_i \geq \phi_i^{\min}$*,*

$$R_i^{2\alpha_i} > c_i 2^{\alpha_i} \max_{t\in[0,T]} \int_0^T g_i(t,s)\,ds, \quad \text{if } \alpha_i \in \left(0, \frac{1}{2}\right), \tag{80}$$

$$R_i \left[1 - c_i 2^{\alpha_i} R_i^{-2\alpha_i} \max_{t\in[0,T]} \int_0^T g_i(t,s)\,ds \right] \geq \phi_i^{\max}, \quad \text{if } \alpha_i \in \left(0, \frac{1}{2}\right), \tag{81}$$

$$R_i \geq c_i 2^{\alpha_i} \left(\phi_i^{\min}\right)^{1-2\alpha_i} \max_{t\in[0,T]} \int_0^T g_i(t,s)\,ds + \phi_i^{\max}, \quad \text{if } \alpha_i \geq \frac{1}{2}. \tag{82}$$

Remark 30. A similar remark as Remark 28 also holds for Theorem 29. Moreover, we note that the upper bounds R_i that fulfill (80)–(82) are independent of p, thus the information of $|\cdot|$ being a particular l_p norm is not used. On the other hand, in Theorem 27, the upper bounds R_i that satisfy (75) depend on p. The sharpness of the bounds in both theorems cannot be compared in general; however, we will give an example at the end of this section to illustrate the results.

In the next result, we will relax the condition (C2). Here, we allow $\phi_i(t) \le 0$ for some $i \in \{1,2,\ldots,n\}$ and some $t \in [0,T]$.

Theorem 31. *Suppose that*

(C7) *there exists $j \in \{1,2,\ldots,n\}$ such that $\phi_j^{\min} > 0$.*

Let $J = \{j \in \{1,2,\ldots,n\} \mid \phi_j^{\min} > 0\}$ and let $J' = \{1,2,\ldots,n\} \setminus J$. Assume that the following conditions hold for each $1 \le i \le n$: (C1), (C3), (C4), and

(C8) *there exist $c_i > 0$ such that*

$$\frac{\partial}{\partial s} h_i(t,s) \le c_i s^{-\alpha_i}, \quad t \in [0,T],\ s > 0, \tag{83}$$

where $\alpha_j > 0$ for $j \in J$ and $\alpha_k \in (0,1/2)$ for $k \in J'$.

Further, let the following hold for each $j \in J$:

(C9) *the norm $|\cdot|$ is the l_p norm where $1 \le p \le \infty$ is fixed, and*

$$\int_0^T g_j(t,s)\,ds < \overline{A}_j^p \left(c_j 2^{\alpha_j}\right)^{-1}, \quad t \in [0,T], \tag{84}$$

where

$$\overline{A}_j^p = \begin{cases} \left[\sum_{\ell \in J} \left(\phi_\ell^{\min}\right)^p\right]^{2\alpha_j/p}, & 1 \le p < \infty, \\ \left[\max_{\ell \in J} \phi_\ell^{\min}\right]^{2\alpha_j}, & p = \infty. \end{cases} \tag{85}$$

Then, (65) *has a T-periodic solution $u^T \in (C(\mathbb{R}))^n$ such that*

$$\begin{gathered} \phi_j^{\min} \le \theta_j u_j^T(t) \le R_j, \quad t \in \mathbb{R},\ j \in J, \\ \left|u_k^T(t)\right| \le R_k, \quad t \in \mathbb{R},\ k \in J', \end{gathered} \tag{86}$$

where

$$R_j \ge \phi_j^{\min},$$

$$R_j \ge \phi_j^{\max}\left[1 - c_j 2^{\alpha_j}\left(\overline{A}_j^p\right)^{-1} \max_{t \in [0,T]} \int_0^T g_j(t,s)\,ds\right]^{-1}, \quad j \in J, \tag{87}$$

$$R_k^{2\alpha_k} > c_k 2^{\alpha_k} \max_{t \in [0,T]} \int_0^T g_k(t,s)\,ds, \quad k \in J', \tag{88}$$

$$R_k\left[1 - c_k 2^{\alpha_k} R_k^{-2\alpha_k} \max_{t \in [0,T]} \int_0^T g_k(t,s)\,ds\right] \ge \max_{t \in [0,T]} \left|\phi_k(t)\right|, \quad k \in J'. \tag{89}$$

To derive Theorem 31, we let the closed convex set Ω^* be

$$\begin{aligned} \Omega^* = \Big\{ & u \in (C[0,T])^n \mid \phi_j^{\min} \le \theta_j u_j(t) \le R_j,\ t \in [0,T], \\ & j \in J;\ \left|u_k(t)\right| \le R_k,\ t \in [0,T],\ k \in J' \Big\}, \end{aligned} \tag{90}$$

where $R_j\,(\ge \phi_j^{\min} > 0)$ and R_k are chosen as in (87)–(89). Next, we define the operator $S : \Omega^* \to (C[0,T])^n$ as in (78) and show that Theorem 26 is applicable.

Remark 32. From the conclusion of Theorem 29, we see that the solution u^T is "partially" of constant sign, in the sense that $\theta_j u_j^T(t) \ge 0$ for $j \in J$, but may not be so for $j \in J'$. Further, the constants c_i that appear in (C8) determine the upper bounds R_i of the solution u_i^T, $1 \le i \le n$. From (87) and (88), we see that a smaller (bigger) c_i gives a smaller (bigger) R_i, and hence a smaller (bigger) set Ω^* where the solution lies.

Using similar arguments as in the derivation of Theorems 31 and 29 (in getting $S_j u \in \Omega^*$ for $j \in J$ and $u \in \Omega^*$), we obtain the following result.

Theorem 33. *Suppose that (C7) hold. Let $J = \{j \in \{1,2,\ldots,n\} \mid \phi_j^{\min} > 0\}$ and let $J' = \{1,2,\ldots,n\} \setminus J$. Assume the following conditions hold for each $1 \le i \le n$: (C1), (C3), (C4), and (C8). Then,* (65) *has a T-periodic solution $u^T \in (C(\mathbb{R}))^n$ such that*

$$\begin{gathered} \phi_j^{\min} \le \theta_j u_j^T(t) \le R_j, \quad t \in \mathbb{R},\ j \in J, \\ \left|u_k^T(t)\right| \le R_k, \quad t \in \mathbb{R},\ k \in J', \end{gathered} \tag{91}$$

where

$$\begin{gathered} R_j \ge \phi_j^{\min}, \quad j \in J, \\ R_j^{2\alpha_j} > c_j 2^{\alpha_j} \max_{t \in [0,T]} \int_0^T g_j(t,s)\,ds, \quad \text{if } \alpha_j \in \left(0, \frac{1}{2}\right), \quad j \in J, \\ R_j\left[1 - c_j 2^{\alpha_j} R_j^{-2\alpha_j} \max_{t \in [0,T]} \int_0^T g_j(t,s)\,ds\right] \ge \phi_j^{\max}, \\ \text{if } \alpha_j \in \left(0, \frac{1}{2}\right),\ j \in J, \\ R_j \ge c_j 2^{\alpha_j}\left(\phi_j^{\min}\right)^{1-2\alpha_j} \max_{t \in [0,T]} \int_0^T g_j(t,s)\,ds + \phi_j^{\max}, \\ \text{if } \alpha_j \ge \frac{1}{2},\ j \in J, \end{gathered} \tag{92}$$

$$\begin{gathered} R_k^{2\alpha_k} > c_k 2^{\alpha_k} \max_{t \in [0,T]} \int_0^T g_k(t,s)\,ds, \quad k \in J', \\ R_k\left[1 - c_k 2^{\alpha_k} R_k^{-2\alpha_k} \max_{t \in [0,T]} \int_0^T g_k(t,s)\,ds\right] \ge \max_{t \in [0,T]} \left|\phi_k(t)\right|, \\ k \in J'. \end{gathered} \tag{93}$$

Remark 34. A similar remark as Remark 32 holds for Theorem 33. Also, we observe once again that the upper bounds R_j that fulfill (92) are independent of p, thus the information of $|\cdot|$ being a particular l_p norm is not used. On the other hand, in Theorem 31, the upper bounds R_j that satisfy (87) depend on p.

We will now present an example that illustrates Theorems 27 and 29.

Example 35. Consider (65) when

$$T = 2\pi, \qquad n = 2, \qquad a_1(t) = a_2(t) = \frac{1}{4},$$
$$f_1(t) = 1, \qquad f_2(t) = \frac{1}{2}, \qquad h_1(t,s) = \frac{\ln(s+1)}{|\sin t| + 1},$$
$$h_2(t,s) = \frac{\ln(s+1)}{3(|\cos t| + 1)}, \qquad |\cdot| = l_p \text{ norm } (1 \le p \le \infty). \tag{94}$$

Fix $\theta_i = 1$, $1 \le i \le n$, that is, we are seeking positive solutions. The corresponding Green's function has the explicit expression [36]

$$g_1(t,s) = g_2(t,s) = \begin{cases} \cos\frac{1}{2}(t - s - \pi), & 0 \le s \le t \le 2\pi, \\ \cos\frac{1}{2}(s - t - \pi), & 0 \le t \le s \le 2\pi. \end{cases} \tag{95}$$

Condition (C1) is satisfied. By direct computation, we get $\phi_1(t) = 4$ and $\phi_2(t) = 2$ for $t \in [0, 2\pi]$. Thus, (C2) is fulfilled with

$$\phi_1^{\min} = \phi_1^{\max} = 4, \qquad \phi_2^{\min} = \phi_2^{\max} = 2. \tag{96}$$

Moreover, we have

$$\begin{aligned} \frac{\partial}{\partial s} h_1(t,s) &= \frac{1}{|\sin t| + 1}\frac{1}{s+1} \le \frac{1}{s+1} \le \frac{1}{s}, \\ \frac{\partial}{\partial s} h_2(t,s) &= \frac{1}{3(|\cos t| + 1)}\frac{1}{s+1} \le \frac{1}{3(s+1)} \le \frac{1}{3s} \end{aligned} \tag{97}$$

and so it is clear that (C4) and (C5) are satisfied with

$$\alpha_1 = 1, \qquad c_1 = 1, \qquad \alpha_2 = 1, \qquad c_2 = \frac{1}{3}. \tag{98}$$

Finally, we compute

$$\begin{aligned} A_1^p = A_2^p &= (4^p + 2^p)^{2/p}, \quad 1 \le p < \infty, \\ A_1^\infty &= A_2^\infty = 16. \end{aligned} \tag{99}$$

Since $\int_0^{2\pi} g_i(t,s)ds = 4$ for $t \in [0, 2\pi]$ and $i = 1, 2$, we check that (C6) holds for all $1 \le p \le \infty$.

All the conditions of Theorem 27 are satisfied, thus we conclude that the problem (65) with (94) has a positive 2π-periodic solution $u = (u_1, u_2)$ such that

$$\phi_i^{\min} \le u_i(t) \le R_i, \quad t \in \mathbb{R},\ i = 1, 2, \tag{100}$$

where (from (75))

$$R_i \ge \phi_i^{\max}\left[1 - \frac{8c_i}{A_i^p}\right]^{-1} \equiv L_i^p, \quad 1 \le p \le \infty,\ i = 1, 2. \tag{101}$$

We can also apply Theorem 29 to conclude that the problem (65) with (94) has a positive 2π-periodic solution $u = (u_1, u_2)$ satisfying (100) and (from (82))

$$R_i \ge 8c_i\left(\phi_i^{\min}\right)^{-1} + \phi_i^{\max} \equiv M_i, \quad i = 1, 2. \tag{102}$$

As mentioned in Remark 30, in general we cannot compare L_i^p and M_i. In fact, a direct calculation gives

$$\begin{aligned} &p = 1: \\ &L_1^1 = 5.14 < M_1 = 6, \qquad L_2^1 = 2.16 < M_2 = 3.33, \\ &p = 2: \\ &L_1^2 = 6.67 > M_1 = 6, \qquad L_2^2 = 2.31 < M_2 = 3.33, \\ &p = \infty: \\ &L_1^\infty = 8 > M_1 = 6, \qquad L_2^\infty = 2.4 < M_2 = 3.33. \end{aligned} \tag{103}$$

5.2. Existence Results for (59). In this section we will consider the general system of Hill's equations

$$u_i''(t) + a_i(t)u_i(t) = F_i(t, u(t)), \quad 1 \le i \le n. \tag{104}$$

Here, a_i and F_i are T-periodic in the variable t, $a_i \in L^1[0,T]$, and the nonlinearities $F_i(t, x_1, x_2, \ldots, x_n)$ can be singular at $x_j = 0$ where $j \in \{1, 2, \ldots, n\}$.

Once again, to obtain a T-periodic solution $u^T = (u_1^T, u_2^T, \ldots, u_n^T)$ of the system (104), we first seek a solution $u^* = (u_1^*, u_2^*, \ldots, u_n^*)$ of the following system of boundary value problems:

$$\begin{aligned} &u_i''(t) + a_i(t)u_i(t) = F_i(t, u(t)), \quad t \in [0,T], \\ &u_i(0) = u_i(T), \quad u_i'(0) = u_i'(T), \quad 1 \le i \le n. \end{aligned} \tag{105}$$

The periodic solution is then given by

$$u^T(t) = u^*(t - mT), \quad t \in [mT, (m+1)T],\ m \in \mathbb{Z}. \tag{106}$$

With g_i being the Green's function of the boundary value problem (68), throughout we will assume that (C1) is satisfied. Moreover, for fixed $\theta_i \in \{-1, 1\}$ and T-periodic functions $q_i \in L^1[0,T]$, $1 \le i \le n$, we define

$$\eta_i(t) = \int_0^T g_i(t,s)\theta_i q_i(s)\,ds, \quad t \in [0,T],\ 1 \le i \le n \tag{107}$$

and also

$$\eta_i^{\min} = \min_{t\in[0,T]} \eta_i(t), \qquad \eta_i^{\max} = \max_{t\in[0,T]} \eta_i(t). \tag{108}$$

For $b \ge b' \ge 0$ and $1 \le i \le n$, we denote the interval

$$[b, b']_i = \begin{cases} [b, b'], & \text{if } \theta_i = 1, \\ [-b', -b], & \text{if } \theta_i = -1. \end{cases} \tag{109}$$

A similar definition is valid for $(b, b')_i$.

Using Schauder's fixed point theorem, we will establish existence results for the system (104).

Theorem 36. *Assume the following conditions hold for each $1 \le i \le n$: (C1);*

(C10) *for any numbers b_j, b_j', $1 \le j \le n$ with $b_j' \ge b_j > 0$, the function $F_i : [0,T] \times \prod_{j=1}^n [b_j, b_j']_j \to \mathbb{R}$ is a L^1-Carathéodory function, that is,*

(i) *the map $u \mapsto F_i(t,u)$ is continuous for almost all $t \in [0,T]$,*

(ii) *the map $t \mapsto F_i(t,u)$ is measurable for all $u \in \prod_{j=1}^n [b_j, b_j']_j$,*

(iii) *for any $r > 0$, there exists $\mu_{r,i} \in L^1[0,T]$ such that $|u| \le r(u \in \prod_{j=1}^n [b_j, b_j']_j)$ implies $|F_i(t,u)| \le \mu_{r,i}(t)$ for almost all $t \in [0,T]$;*

(C11) *there exist $\beta_i > 0$, $\gamma_i \in [0,1)$, and T-periodic functions w_i, q_i with $w_i \in L^1[0,T]$, $q_i \in L^1[0,T]$ and $w_i(t) > 0$ for a.e. $t \in [0,T]$ such that*

$$\theta_i q_i(t)|u|^{\gamma_i} \le \theta_i F_i(t,u) \le \theta_i q_i(t)|u|^{\gamma_i} + w_i(t)|u|^{-\beta_i},$$
$$t \in [0,T],\ u \in \prod_{k=1}^n (0,\infty)_k \tag{110}$$

(here $|\cdot|$ is the l_p norm where $1 \le p \le \infty$ is fixed);

(C12) $\eta_i^{\min} > 0$.

Then, (104) has a T-periodic constant-sign solution $u^T \in (C(\mathbb{R}))^n$ such that

$$r_i \le \theta_i u_i^T(t) \le R_i, \quad t \in \mathbb{R},\ 1 \le i \le n, \tag{111}$$

where, for $1 \le i \le n$ one has

$$0 < r_i \le R_i, \quad r_i \le \left(\eta_i^{\min}\right)^{1/(1-\gamma_i)}, \tag{112}$$

$$R_i \ge \eta_i^{\max}|R|_p^{\gamma_i} + |r|_p^{-\beta_i} \max_{t\in[0,T]} \int_0^T g_i(t,s)\, w_i(s)\, ds, \tag{113}$$

(here $|R|_p$ is the l_p norm of $(R_1, R_2, \ldots, R_n)$, likewise $|r|_p$ is the l_p norm of $(r_1, r_2, \ldots, r_n)$).

In proving Theorem 36, we actually seek a constant-sign solution of (105) in $(C[0,T])^n$ and then extend it to a T-periodic constant-sign solution of (104) as in (106). Let Ω be the closed convex set given by

$$\Omega = \{u \in (C[0,T])^n \mid r_i \le \theta_i u_i(t) \le R_i,\ t \in [0,T],\ 1 \le i \le n\}, \tag{114}$$

where $R_i \ge r_i > 0$ are chosen as in (112) and (113), and define the operator $S : \Omega \to (C[0,T])^n$ as

$$Su(t) = (S_1u(t), S_2u(t), \ldots, S_nu(t)), \quad t \in [0,T], \tag{115}$$

where

$$S_iu(t) = \int_0^T g_i(t,s)\, F_i(s,u(s))\, ds, \quad t \in [0,T],\ 1 \le i \le n. \tag{116}$$

Clearly, a fixed point of $Su = u$ is a solution of (105). The conditions of Theorem 26 are then shown to be satisfied.

Remark 37. As seen from (112) and (113), the functions w_i and q_i that appear in (C11) determine the lower and upper bounds of the solution u_i^T, $1 \le i \le n$.

Theorem 38. *Assume that the following conditions hold for each $1 \le i \le n$: (C1), (C10), (C11), and (C12). Then, (104) has a T-periodic constant-sign solution $u^T \in (C(\mathbb{R}))^n$ such that*

$$r \le \theta_i u_i^T(t) \le R, \quad t \in \mathbb{R},\ 1 \le i \le n, \tag{117}$$

where $0 < r \le R$, and for all $1 \le i \le n$,

$$r \le \begin{cases} \left(\eta_i^{\min} n^{\gamma_i/p}\right)^{1/(1-\gamma_i)}, & 1 \le p < \infty, \\ \left(\eta_i^{\min}\right)^{1/(1-\gamma_i)}, & p = \infty, \end{cases} \tag{118}$$

$$R \ge \begin{cases} R^{\gamma_i}\eta_i^{\max} n^{\gamma_i/p} + r^{-\beta_i} n^{-\beta_i/p} \left[\max_{t\in[0,T]} \int_0^T g_i(t,s)\, w_i(s)\, ds\right], & 1 \le p < \infty, \\ R^{\gamma_i}\eta_i^{\max} + r^{-\beta_i} \left[\max_{t\in[0,T]} \int_0^T g_i(t,s)\, w_i(s)\, ds\right], & p = \infty. \end{cases} \tag{119}$$

Theorem 38 is obtained by similar arguments used in the derivation of Theorem 36, with a new Ω defined as

$$\Omega = \{u \in (C[0,T])^n \mid r \le \theta_i u_i(t) \le R,\ t \in [0,T],\ 1 \le i \le n\}, \tag{120}$$

where $R \ge r > 0$ are chosen as in (118) and (119).

Remark 39. Remark 37 also holds for Theorem 38. Further, comparing the bounds r_i, R_i, $1 \le i \le n$ in Theorem 36 (see (112), (113)) with the bounds r, R in Theorem 38 (see (118), (119)), we note that r_i and R_i are lower and upper bounds for a particular $\theta_i u_i^T$, whereas r and R are uniform lower and upper bounds for *all* $\theta_i u_i^T$, $1 \le i \le n$. However, the computation of R_i from (113) is more difficult than calculating R from (119).

Our next result tackles the case when $\eta_i^{\min} = 0$.

Theorem 40. *Assume that the following conditions hold for each $1 \le i \le n$: (C1), (C10),*

(C13) *there exist $\beta_i \in (0,1)$, $\gamma_i \in [0,1)$, and T-periodic functions w_i, v_i, q_i with $w_i \in L^1[0,T]$, $v_i \in L^1[0,T]$, $q_i \in L^1[0,T]$, and $w_i(t), v_i(t) > 0$ for a.e. $t \in [0,T]$ such that*

$$\theta_i q_i(t)|u|^{\gamma_i} + v_i(t)|u|^{-\beta_i} \le \theta_i F_i(t,u) \le \theta_i q_i(t)|u|^{\gamma_i} + w_i(t)|u|^{-\beta_i},$$
$$t \in [0,T],\ u \in \prod_{k=1}^n (0,\infty)_k \tag{121}$$

(here $|\cdot|$ is the l_p norm where $1 \le p \le \infty$ is fixed);

(C14) $\eta_i^{\min} = 0$.

Then, (104) *has a T-periodic constant-sign solution* $u^T \in (C(\mathbb{R}))^n$ *such that*

$$\frac{1}{R} \leq \theta_i u_i^T(t) \leq R, \quad t \in \mathbb{R},\ 1 \leq i \leq n, \tag{122}$$

where $R \geq 1$, *and for all* $1 \leq i \leq n$,

$$R \geq \begin{cases} n^{\beta_i/p(1-\beta_i)}\left[\min_{t\in[0,T]}\int_0^T g_i(t,s)v_i(s)ds\right]^{-1/(1-\beta_i)}, & 1 \leq p < \infty, \\ \left[\min_{t\in[0,T]}\int_0^T g_i(t,s)v_i(s)ds\right]^{-1/(1-\beta_i)}, & p = \infty, \end{cases}$$

$$R \geq \begin{cases} R^{\gamma_i}\eta_i^{\max}n^{\gamma_i/p} + R^{\beta_i}n^{-\beta_i/p}\left[\max_{t\in[0,T]}\int_0^T g_i(t,s)w_i(s)ds\right], & 1 \leq p < \infty, \\ R^{\gamma_i}\eta_i^{\max} + R^{\beta_i}\left[\max_{t\in[0,T]}\int_0^T g_i(t,s)w_i(s)ds\right], & p = \infty. \end{cases} \tag{123}$$

The closed convex set used to get Theorem 40 is given by

$$\Omega = \{u \in (C[0,T])^n \mid r \leq \theta_i u_i(t) \leq R,\ t \in [0,T],\ 1 \leq i \leq n\}, \tag{124}$$

where $r = 1/R$ and $R \geq 1$ satisfies (123).

Remark 41. As seen from (123), the functions w_i, v_i, and q_i that appear in (C13) determine the lower and upper bounds of the solution u_i^T, $1 \leq i \leq n$.

Finally, the next result tackles the case when $\eta_i^{\max} < 0$.

Theorem 42. *Assume that the following conditions hold for each* $1 \leq i \leq n$: *(C1), (C10),*

(C15) *there exist* $\beta \in (0,1)$ *and T-periodic functions* w_i, v_i, q_i *with* $w_i \in L^1[0,T]$, $v_i \in L^1[0,T]$, $q_i \in L^1[0,T]$, *and* $w_i(t)$, $v_i(t) > 0$ *for a.e.* $t \in [0,T]$ *such that*

$$\theta_i q_i(t) + v_i(t)|u|^{-\beta} \leq \theta_i F_i(t,u) \leq \theta_i q_i(t) + w_i(t)|u|^{-\beta}, \quad t \in [0,T],\ u \in \prod_{k=1}^{n}(0,\infty)_k \tag{125}$$

(here $|\cdot|$ *is the* l_p *norm where* $1 \leq p \leq \infty$ *is fixed);*

(C16) $\eta_i^{\max} < 0$;

(C17) $\eta_i^{\min} \geq n^{-\beta/(1+\beta)p}W^{-\beta/(1-\beta^2)}(V\beta^2)^{1/(1-\beta^2)}(1 - 1/\beta^2)$ *where*

$$W = \max_{1\leq k\leq n}\left[\max_{t\in[0,T]}\int_0^T g_k(t,s)w_k(s)ds\right],$$
$$V = \min_{1\leq k\leq n}\left[\min_{t\in[0,T]}\int_0^T g_k(t,s)v_k(s)ds\right]. \tag{126}$$

Then, (104) *has a T-periodic constant-sign solution* $u^T \in (C(\mathbb{R}))^n$ *such that*

$$r \leq \theta_i u_i^T(t) \leq R, \quad t \in \mathbb{R},\ 1 \leq i \leq n, \tag{127}$$

where $0 < r \leq R$ *are given by*

$$r = \begin{cases} n^{-\beta/(1+\beta)p}W^{-\beta/(1-\beta^2)}(V\beta^2)^{1/(1-\beta^2)}, & 1 \leq p < \infty \\ W^{-\beta/(1-\beta^2)}(V\beta^2)^{1/(1-\beta^2)}, & p = \infty, \end{cases}$$
$$R = \begin{cases} n^{-\beta/(1+\beta)p}W^{1/(1-\beta^2)}(V\beta^2)^{-\beta/(1-\beta^2)}, & 1 \leq p < \infty \\ W^{1/(1-\beta^2)}(V\beta^2)^{-\beta/(1-\beta^2)}, & p = \infty. \end{cases} \tag{128}$$

Theorem 42 is obtained by considering the closed convex set

$$\Omega = \{u \in (C[0,T])^n \mid r \leq \theta_i u_i(t) \leq R,\ t \in [0,T],\ 1 \leq i \leq n\}, \tag{129}$$

where $R \geq r > 0$ are determined later as those given in (128).

Remark 43. As seen from (128), the functions w_i and v_i that appear in (C15) determine the lower and upper bounds of the solution u_i^T, $1 \leq i \leq n$.

We have so far established the results when (i) $\eta_i^{\min} > 0$, (ii) $\eta_i^{\min} = 0$, and (iii) $\eta_i^{\max} < 0$ for *all* $1 \leq i \leq n$. However, it could be that we only have $\eta_i(t) \geq 0$ for some i and $\eta_j(t) < 0$ for some j, which results in $\eta_i^{\min} \geq 0$ and $\eta_j^{\max} < 0$ for *some* $1 \leq i, j \leq n$. We present two results for such a case as follows. Note that Theorem 44 is obtained by applying Theorems 38–42, while Theorem 45 is obtained by applying Theorems 36, 40, and 42.

Theorem 44. *Let (C1) and (C10) hold for each* $1 \leq i \leq n$. *Assume the following:*

(C18) *conditions (C11) and (C12) hold for some* $i \in I \subseteq \{1,2,\ldots,n\}$;

(C19) *conditions (C13) and (C14) hold for some* $i \in J \subseteq \{1,2,\ldots,n\}$;

(C20) *conditions (C15), (C16), and (C17) hold for some* $i \in K \subseteq \{1,2,\ldots,n\}$;

where $I \cup J \cup K = \{1,2,\ldots,n\}$. *Then,* (104) *has a T-periodic constant-sign solution* $u^T \in (C(\mathbb{R}))^n$ *such that*

$$r \leq \theta_i u_i^T(t) \leq R, \quad t \in \mathbb{R},\ 1 \leq i \leq n, \tag{130}$$

where $0 < r \leq R$ *satisfy*

(a) (118) *and* (119) *for* $i \in I$;

(b) $r = 1/R$, $R \geq 1$, (123) *for* $i \in J$;

(c) (128) *for* $i \in K$.

Theorem 45. *Let (C1) and (C10) hold for each $1 \le i \le n$. Assume that (C18)–(C20) hold with $I \cup J \cup K = \{1, 2, \ldots, n\}$. Then,* (104) *has a T-periodic constant-sign solution $u^T \in (C(\mathbb{R}))^n$ such that*

$$r_i \le \theta_i u_i^T(t) \le R_i, \quad t \in \mathbb{R},\ i \in I, \tag{131}$$

where $0 < r_i \le R_i$ satisfy (112) *and* (113) *for $i \in I$, and*

$$r \le \theta_i u_i^T(t) \le R, \quad t \in \mathbb{R},\ i \in J \cup K, \tag{132}$$

where $0 < r \le R$ satisfy conclusions (b) and (c) of Theorem 44.

We will now apply the results obtained to the following system of Hill's equations, a particular form of it (see (64)) that has been discussed in [45],

$$\begin{aligned} u_1''(t) + a_1(t) u_1(t) &= \left(\sqrt{u_1^2 + u_2^2}\right)^{-\beta_1} + \nu_1 \left(\sqrt{u_1^2 + u_2^2}\right)^{\gamma_1}, \\ u_2''(t) + a_2(t) u_2(t) &= \left(\sqrt{u_1^2 + u_2^2}\right)^{-\beta_2} + \nu_2 \left(\sqrt{u_1^2 + u_2^2}\right)^{\gamma_2}. \end{aligned} \tag{133}$$

Clearly, the system (133) corresponds to (104) where $n = 2$ and

$$F_i(t, u) = \left(\sqrt{u_1^2 + u_2^2}\right)^{-\beta_i} + \nu_i \left(\sqrt{u_1^2 + u_2^2}\right)^{\gamma_i}, \quad i = 1, 2. \tag{134}$$

We will assume that $a_1, a_2 \in L^1[0, T]$ satisfy (C1). Note that condition (C10) is clearly satisfied. Further, let $\theta_1 = \theta_2 = 1$, that is, we are interested in positive periodic solutions of (133).

Example 46. Consider the system (133) with

$$\nu_i > 0, \quad \beta_i > 0, \quad \gamma_i \in [0, 1), \quad i = 1, 2. \tag{135}$$

Clearly, (C11) is satisfied with $p = 2$, $q_i = \nu_i$ and $w_i = 1$, $i = 1, 2$. Thus, (C12) also holds since

$$\begin{aligned} \eta_i^{\min} &= \min_{t \in [0,T]} \int_0^T g_i(t, s) \theta_i q_i(s)\, ds \\ &= \nu_i \min_{t \in [0,T]} \int_0^T g_i(t, s)\, ds > 0. \end{aligned} \tag{136}$$

Theorem 38 (or Theorem 36) is applicable and we conclude that the system (133) with (135) has a T-periodic positive solution $u^T \in (C(\mathbb{R}))^2$ such that

$$r \le u_i^T(t) \le R, \quad t \in \mathbb{R},\ i = 1, 2, \tag{137}$$

where $0 < r \le R$ are such that

$$r \le \min_{i=1,2} \left\{ \left(\eta_i^{\min} 2^{\gamma_i/2}\right)^{1/(1-\gamma_i)} \right\}, \tag{138}$$

$$R \ge \max_{i=1,2} \left\{ R^{\gamma_i} \eta_i^{\max} 2^{\gamma_i/2} + r^{-\beta_i} 2^{-\beta_i/2} \left[\max_{t \in [0,T]} \int_0^T g_i(t, s)\, ds \right] \right\}. \tag{139}$$

To illustrate numerically, suppose

$$\begin{aligned} &a_1(t) = a_2(t) = \frac{1}{4}, \qquad T = 2\pi, \qquad \nu_1 = \frac{1}{4}, \\ &\nu_2 = 1, \qquad \gamma_1 = \gamma_2 = \frac{1}{2}, \qquad \beta_1 = \beta_2 = 1. \end{aligned} \tag{140}$$

Green's function is given in (95) and

$$\eta_i^{\min} = \nu_i \min_{t \in [0,T]} \int_0^T g_i(t, s)\, ds = 4\nu_i. \tag{141}$$

Hence, (138) yields $r \le \sqrt{2}$. Let $r = \sqrt{2}$, then (139) reduces to

$$R \ge \max_{i=1,2} \left\{ R^{1/2} 4\nu_i 2^{1/4} + r^{-1} 2^{-1/2} 4\nu_i \right\} = R^{1/2} 2^{9/4} + 2, \tag{142}$$

which is satisfied by $R \ge 26.48$. Let $R = 26.48$, then from (137) we conclude that the system (133) with (140) has a 2π-periodic positive solution $u \in (C(\mathbb{R}))^2$ such that

$$\sqrt{2} \le u_i(t) \le 26.48, \quad t \in \mathbb{R},\ i = 1, 2. \tag{143}$$

Example 47. Consider the system (133) with

$$\nu_i = 0, \qquad \beta_i \in (0, 1), \qquad \gamma_i \in [0, 1), \quad i = 1, 2. \tag{144}$$

Here, (C13) is satisfied with $p = 2$, $q_i = \nu_i = 0$ and $w_i = \nu_i = 1$, $i = 1, 2$. Subsequently, (C14) also holds since

$$\begin{aligned} \eta_i^{\min} &= \min_{t \in [0,T]} \int_0^T g_i(t, s) \theta_i q_i(s)\, ds \\ &= \nu_i \min_{t \in [0,T]} \int_0^T g_i(t, s)\, ds = 0. \end{aligned} \tag{145}$$

Employing Theorem 40, we conclude that the system (133) with (144) has a T-periodic positive solution $u^T \in (C(\mathbb{R}))^2$ such that

$$\frac{1}{R} \le u_i^T(t) \le R, \quad t \in \mathbb{R},\ i = 1, 2, \tag{146}$$

where $R \ge 1$, and from (123), we have for $i = 1, 2$,

$$\begin{aligned} R &\ge 2^{\beta_i/2(1-\beta_i)} \left[\min_{t \in [0,T]} \int_0^T g_i(t, s)\, ds \right]^{-1/(1-\beta_i)}, \\ R &\ge \left\{ 2^{-\beta_i/2} \left[\max_{t \in [0,T]} \int_0^T g_i(t, s)\, ds \right] \right\}^{1/(1-\beta_i)}. \end{aligned} \tag{147}$$

Combining the inequalities, we see that R should satisfy

$$\begin{aligned} R \ge \max \Bigg\{ 1, &\max_{i=1,2} 2^{\beta_i/2(1-\beta_i)} \\ &\times \left[\min_{t \in [0,T]} \int_0^T g_i(t, s)\, ds \right]^{-1/(1-\beta_i)}, \\ &\max_{i=1,2} \left\{ 2^{-\beta_i/2} \left[\max_{t \in [0,T]} \int_0^T g_i(t, s) ds \right] \right\}^{1/(1-\beta_i)} \Bigg\}. \end{aligned} \tag{148}$$

Example 48. Consider the system (133) with

$$\nu_i < 0, \quad \beta_i = \beta \in (0,1), \quad \gamma_i = 0, \quad i = 1,2, \tag{149}$$

$$\nu_i \min_{t\in[0,T]} \int_0^T g_i(t,s)\,ds \geq 2^{-\beta/2(1+\beta)} W^{-\beta/(1-\beta^2)} \left(V\beta^2\right)^{1/(1-\beta^2)} \times \left(1 - \frac{1}{\beta^2}\right), \quad i = 1,2, \tag{150}$$

where

$$W = \max_{k=1,2}\left[\max_{t\in[0,T]} \int_0^T g_k(t,s)\,ds\right], \qquad V = \min_{k=1,2}\left[\min_{t\in[0,T]} \int_0^T g_k(t,s)\,ds\right]. \tag{151}$$

Obviously, (C15) is satisfied with $p = 2$, $q_i = \nu_i < 0$ and $w_i = v_i = 1$, $i = 1,2$. Then, (C16) also holds since

$$\eta_i^{\max} = \max_{t\in[0,T]} \int_0^T g_i(t,s)\,\theta_i q_i(s)\,ds = \nu_i \max_{t\in[0,T]} \int_0^T g_i(t,s)\,ds < 0. \tag{152}$$

Moreover, condition (C17) is simply (150). Hence, we conclude from Theorem 42 that the system (133) with (149) and (150) has a T-periodic positive solution $u^T \in (C(\mathbb{R}))^2$ such that

$$r \leq u_i^T(t) \leq R, \quad t \in \mathbb{R},\ i = 1,2, \tag{153}$$

where $0 < r \leq R$ are given by

$$r = 2^{-\beta/2(1+\beta)} W^{-\beta/(1-\beta^2)} \left(V\beta^2\right)^{1/(1-\beta^2)}, \qquad R = 2^{-\beta/2(1+\beta)} W^{1/(1-\beta^2)} \left(V\beta^2\right)^{-\beta/(1-\beta^2)}. \tag{154}$$

Remark 49. In [45], it is shown that (64) has a solution when $\beta > 0$, $\gamma \in [0,1)$ and $\nu > 0$. As seen from Examples 46–48, we have generalized the system (64) to allow ν to be zero or negative.

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Research Article

Stability and Hopf Bifurcation Analysis of Coupled Optoelectronic Feedback Loops

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The dynamics of a coupled optoelectronic feedback loops are investigated. Depending on the coupling parameters and the feedback strength, the system exhibits synchronized asymptotically stable equilibrium and Hopf bifurcation. Employing the center manifold theorem and normal form method introduced by Hassard et al. (1981), we give an algorithm for determining the Hopf bifurcation properties.

1. Introduction

In recent research [1–5], it is found that even if several individual systems behave chaotically, in the case where the systems are identical, by proper coupling, the systems can be made to evolve toward a situation of exact isochronal synchronism. Synchronization phenomena are common in coupled semiconductor systems, and they are important examples of oscillators in general, and many works are concerned with coupled semiconductor systems [6–15].

We consider a feedback loop comprises a semiconductor laser that serves as the optical source, a Mach-Zehnder electrooptic modulator, a photoreceiver, an electronic filter, and an amplifier. The dynamics of the feedback loop can be modeled by the delay differential equations [14, 15]:

$$\begin{aligned} \frac{\mathrm{d}x_1(t)}{\mathrm{d}t} &= -(\gamma_1+\gamma_2)\,x_1(t)-\gamma_2 y_1(t) \\ &\quad -\beta\gamma_2\cos^2\left[x_1(t-\tau)+\varphi_0\right], \qquad (1) \\ \frac{\mathrm{d}y_1(t)}{\mathrm{d}t} &= \gamma_1 x_1(t). \end{aligned}$$

Here, $x_1(t)$ is the normalized voltage signal applied to the electrooptic modulator, τ is the feedback time delay, γ_1 and γ_2 are the filter low-pass and high-pass corner frequencies, β is the dimensionless feedback strength, they are all positive constants, and φ_0 is the bias point of the modulator.

Depending on the value of the feedback strength β and delay τ, the loop, which is modeled by system (1), is capable of producing dynamics ranging from periodic oscillations to high-dimensional chaos [1, 14, 15].

We couple two nominally identical optoelectronic feedback loops unidirectionally, that is, the transmitter affects the dynamics of the receiver but not vice versa. Thus, the equations of motion describing the coupled system are given by (1) for the transmitter and

$$\begin{aligned} \frac{\mathrm{d}x_2(t)}{\mathrm{d}t} &= -(\gamma_1+\gamma_2)\,x_2(t)-\gamma_2 y_2(t) \\ &\quad -\beta\gamma_2\cos^2\left[kx_1(t-\tau)+(1-k)\,x_2(t-\tau)+\varphi_0\right], \\ \frac{\mathrm{d}y_2(t)}{\mathrm{d}t} &= \gamma_1 x_2(t), \end{aligned} \qquad (2)$$

for the receiver. In (2), $k>0$ denotes the coupling strength. We will find that with the variety of k, the dynamical behavior of the coupled system can be different, while the feedback strength β keeps the same value.

The paper is organized as follows. In Section 2, using the method presented in [16], we study the stability, and the local Hopf bifurcation of the equilibrium of the coupled system (1) and (2) by analyzing the distribution of the roots of the associated characteristic equation. In Section 3, we use the normal form method and the center manifold theory introduced by Hassard et al. [17] to analyze the direction, stability and the period of the bifurcating periodic solutions at critical values of β. In Section 4, some numerical simulations are carried out to illustrate the results obtained from the analysis. In Section 5, we come to some conclusion about the effect caused by the variety of parameters.

2. Stability Analysis

In this section, we consider the linear stability of the nonlinear coupled system

$$\begin{aligned}\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} &= -(\gamma_1+\gamma_2)x_1(t)-\gamma_2 y_1(t)\\&\quad-\beta\gamma_2\cos^2[x_1(t-\tau)+\varphi_0],\\\frac{\mathrm{d}y_1(t)}{\mathrm{d}t} &= \gamma_1 x_1(t),\\\frac{\mathrm{d}x_2(t)}{\mathrm{d}t} &= -(\gamma_1+\gamma_2)x_2(t)-\gamma_2 y_2(t)\\&\quad-\beta\gamma_2\cos^2[kx_1(t-\tau)+(1-k)x_2(t-\tau)+\varphi_0],\\\frac{\mathrm{d}y_2(t)}{\mathrm{d}t} &= \gamma_1 x_2(t).\end{aligned}\tag{3}$$

It is easy to see that $E(0,-\beta\cos^2\varphi_0,0,-\beta\cos^2\varphi_0)$ is the only equilibrium of system (3). Linearizing system (3) around E and denote $\delta=\sin 2\varphi_0$, we get the linearization system

$$\begin{aligned}\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} &= -(\gamma_1+\gamma_2)x_1(t)-\gamma_2 y_1(t)+\beta\delta\gamma_2 x_1(t-\tau),\\\frac{\mathrm{d}y_1(t)}{\mathrm{d}t} &= \gamma_1 x_1(t),\\\frac{\mathrm{d}x_2(t)}{\mathrm{d}t} &= -(\gamma_1+\gamma_2)x_2(t)-\gamma_2 y_2(t)\\&\quad+k\beta\delta\gamma_2 x_1(t-\tau)+(1-k)\beta\delta\gamma_2 x_2(t-\tau),\\\frac{\mathrm{d}y_2(t)}{\mathrm{d}t} &= \gamma_1 x_2(t),\end{aligned}\tag{4}$$

and the characteristic equation of system (4)

$$\begin{aligned}&[\lambda^2+(\gamma_1+\gamma_2)\lambda+\gamma_1\gamma_2-(1-k)\beta\delta\gamma_2\lambda e^{-\lambda\tau}]\\&\quad\times[\lambda^2+(\gamma_1+\gamma_2)\lambda+\gamma_1\gamma_2-\beta\delta\gamma_2\lambda e^{-\lambda\tau}]=0,\end{aligned}\tag{5}$$

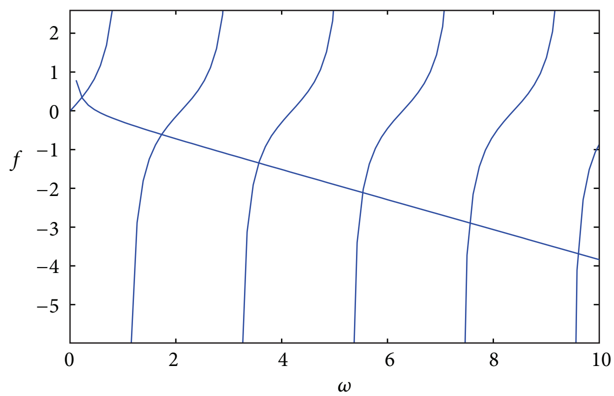


FIGURE 1: The points of intersection of $f_1=\tan\omega\tau$ and $f_2=(-\omega^2+\gamma_1\gamma_2)/\omega(\gamma_1+\gamma_2)$, when $\gamma_1=0.1$, $\gamma_2=2.5$, $\tau=1.5$.

which is equivalent to

$$\lambda^2+(\gamma_1+\gamma_2)\lambda+\gamma_1\gamma_2-\beta\delta\gamma_2\lambda e^{-\lambda\tau}=0,\tag{6}$$

$$\lambda^2+(\gamma_1+\gamma_2)\lambda+\gamma_1\gamma_2-(1-k)\beta\delta\gamma_2\lambda e^{-\lambda\tau}=0.\tag{7}$$

Notice that when $\beta=0$, (5) becomes

$$[\lambda^2+(\gamma_1+\gamma_2)\lambda+\gamma_1\gamma_2]^2=0,\tag{8}$$

whose roots are

$$\lambda_{1,2}=-\gamma_1,\qquad \lambda_{3,4}=-\gamma_2.\tag{9}$$

So, we have the following lemma.

Lemma 1. *The equilibrium $E_0(0,-\beta\cos^2\varphi_0,0,-\beta\cos^2\varphi_0)$ is asymptotically stable when $\beta=0$.*

Next, we regard β as the bifurcation parameter to investigate the distribution of roots of (6) and (7).

Let $\lambda=i\omega$ $(\omega>0)$ be a root of (6) and substituting $\lambda=i\omega$ into (6), separating the real and imaginary parts yields

$$\begin{aligned}-\omega^2+\gamma_1\gamma_2&=\beta\delta\gamma_2\omega\sin\omega\tau,\\\omega(\gamma_1+\gamma_2)&=\beta\delta\gamma_2\omega\cos\omega\tau.\end{aligned}\tag{10}$$

Then, we can get

$$\tan\omega\tau=\frac{-\omega^2+\gamma_1\gamma_2}{\omega(\gamma_1+\gamma_2)}.\tag{11}$$

Hence, (11) has a sequence of roots $\{\omega_j\}_{j\geq 0}$ (see Figure 1), and

$$\omega_j\in\begin{cases}\left(\dfrac{2j\pi}{\tau},\dfrac{2j\pi+\pi/2}{\tau}\right), & \omega_j^2<\gamma_1\gamma_2,\\[2ex]\left(\dfrac{2j\pi+3\pi/2}{\tau},\dfrac{2(j+1)\pi}{\tau}\right), & \omega_j^2>\gamma_1\gamma_2,\end{cases}\qquad j=0,1,2,\ldots.\tag{12}$$

Define

$$\beta_j = \frac{\gamma_1 + \gamma_2}{\delta\gamma_2 \cos\omega_j\tau}. \tag{13}$$

Then, (ω_j, β_j) is the solution of (10).

From (10), we know that

$$\beta^2 = \frac{1}{\delta^2\gamma_2^2}\left[\left(\frac{\gamma_1\gamma_2}{\omega} - \omega\right)^2 + (\gamma_1 + \gamma_2)^2\right], \tag{14}$$

which gives that

$$\frac{d\beta}{d\omega} = \frac{1}{\beta\delta^2\gamma_2^2\omega^3}\left(\omega^2 + \gamma_1\gamma_2\right)\left(\omega^2 - \gamma_1\gamma_2\right). \tag{15}$$

From Figure 1, we know that $\omega_j \to \infty$ when $j \to \infty$, which means that $\omega_j^2 > \gamma_1\gamma_2$; furthermore, β is increasing with respect to ω, when j is sufficiently big.

Reorder the set $\{\beta_j\}$ such that $\beta_0 = \min\{\beta_j\}$ and ω_j is correspondent of β_j $(j = 0, 1, 2, \ldots)$. Then, we have the following lemma.

Lemma 2. *There exists a sequence values of β denoted by*

$$0 < \beta_0 < \beta_1 < \cdots, \tag{16}$$

such that (6) *has a pair of imaginary roots $\pm i\omega_j$ when $\beta = \beta_j$ $(j = 0, 1, 2, \ldots)$, where β_j is defined by* (13), *and ω_j is the root of* (11).

Let

$$\lambda(\tau) = \alpha(\beta) + i\omega(\beta) \tag{17}$$

be the root of (6) satisfying $\alpha(\beta_j) = 0$ and $\omega(\beta_j) = \omega_j$. We have the following conclusion.

Lemma 3. $\alpha'(\beta_j) > 0$.

Proof. Substituting $\lambda(\beta)$ into (6) and taking the derivative with respect to β, it follows that

$$\begin{aligned} &2\lambda\frac{d\lambda}{d\beta} + (\gamma_1 + \gamma_2)\frac{d\lambda}{d\beta} - \delta\gamma_2\lambda e^{-\lambda\tau} - \beta\delta\gamma_2 e^{-\lambda\tau}\frac{d\lambda}{d\beta} \\ &\quad + \tau\beta\delta\gamma_2\lambda e^{-\lambda\tau}\frac{d\lambda}{d\beta} = 0. \end{aligned} \tag{18}$$

Therefore, noting that $\beta\delta\gamma_2\lambda e^{-\lambda\tau} = \lambda^2 + (\gamma_1 + \gamma_2)\lambda + \gamma_1\gamma_2$, we have

$$\frac{d\lambda}{d\beta} = \frac{1}{\beta}\frac{\lambda^3 + (\gamma_1 + \gamma_2)\lambda^2 + \gamma_1\gamma_2\lambda}{\lambda^2 - \gamma_1\gamma_2 + \tau\left[\lambda^3 + (\gamma_1 + \gamma_2)\lambda^2 + \gamma_1\gamma_2\lambda\right]}, \tag{19}$$

and by a straight computation, we get

$$\begin{aligned} \alpha'(\beta_j) = \frac{\omega_j^2}{\beta_j\Delta}&\left[\tau\omega_j^2\left(\gamma_1^2 + \gamma_2^2\right) + \left(\omega_j^2 + \gamma_1\gamma_2\right)(\gamma_1 + \gamma_2)\right. \\ &\left. + \tau\omega_j^4 + \tau\gamma_1^2\gamma_2^2\right] > 0, \end{aligned} \tag{20}$$

where

$$\begin{aligned} \Delta = &\left[\left(\omega_j^2 + \gamma_1\gamma_2\right) + \tau\omega_j^2(\gamma_1 + \gamma_2)\right]^2 \\ &+ \left[\tau\omega_j\left(\omega_j^2 - \gamma_1\gamma_2\right)\right]^2. \end{aligned} \tag{21}$$

□

As to (7), it can be easily found that $-\gamma_1, -\gamma_2$ are two negative roots when $k = 1$, so, next, we only focus on (7) with $k \neq 1$.

Let $\lambda = i\varpi(\varpi) > 0$ be a root of (7). Using the same method above, we get

$$\begin{aligned} -\varpi^2 + \gamma_1\gamma_2 &= (1 - k)\beta\delta\gamma_2\varpi\sin\varpi\tau, \\ (\gamma_1 + \gamma_2)\varpi &= (1 - k)\beta\delta\gamma_2\varpi\cos\varpi\tau, \end{aligned} \tag{22}$$

$$\tan\varpi\tau = \frac{-\varpi^2 + \gamma_1\gamma_2}{\varpi(\gamma_1 + \gamma_2)}. \tag{23}$$

Then, when $0 < k < 1$, (23) has a sequence of roots $\{\varpi_j\}_{j\geq 0}$, which are the same as those of (11).

When $k > 1$, (23) has a sequence of roots $\{\varpi_j\}_{j\geq 0}$, and

$$\varpi_j \in \begin{cases} \left(\dfrac{(2j+1)\pi}{\tau}, \dfrac{(2j+1)\pi + \pi/2}{\tau}\right), & \varpi_j^2 < \gamma_1\gamma_2, \\ \left(\dfrac{2j\pi + \pi/2}{\tau}, \dfrac{(2j+1)\pi}{\tau}\right), & \varpi_j^2 > \gamma_1\gamma_2, \end{cases} \quad j = 0, 1, 2, \ldots. \tag{24}$$

Define

$$\overline{\beta}_j = \frac{\gamma_1 + \gamma_2}{(1 - k)\delta\gamma_2\cos\varpi\tau}. \tag{25}$$

Then, $(\varpi_j, \overline{\beta}_j)$ is the solution of (22).

Repeat the previous process, we have

$$\frac{d\overline{\beta}}{d\varpi} = \frac{1}{(1 - k)^2\beta\delta^2\gamma_2^2\varpi^3}\left(\varpi^2 + \gamma_1\gamma_2\right)\left(\varpi^2 - \gamma_1\gamma_2\right). \tag{26}$$

Reorder the set $\{\overline{\beta}_j\}$ such that $\overline{\beta}_0 = \min\{\overline{\beta}_j\}$ and ϖ_j is correspondent of $\overline{\beta}_j$ $(j = 0, 1, 2, \ldots)$.

Lemma 4. *There exists a sequence values of $\overline{\beta}$ denoted by*

$$0 < \overline{\beta}_0 < \overline{\beta}_1 < \cdots, \tag{27}$$

such that (7) *has a pair of imaginary roots $\pm i\varpi_j$ when $\overline{\beta} = \overline{\beta}_j$ $(j = 0, 1, 2, \ldots)$, where $\overline{\beta}_j$ is defined by* (25), *and ϖ_j is the root of* (23).

Let

$$\lambda(\tau) = \alpha\left(\overline{\beta}\right) + i\varpi\left(\overline{\beta}\right) \tag{28}$$

be the root of (7) satisfying $\alpha(\overline{\beta}_j) = 0$, $\varpi(\overline{\beta}_j) = \varpi_j$. Then, similar to the proof of Lemma 3, we have the following conclusion.

Lemma 5. $\alpha'(\overline{\beta}_j) > 0$.

Compare β_j, $\overline{\beta}_j$ and reorder the set $\{\beta_j\}$ and $\{\overline{\beta}_j\}$ and remove the "−" of $\overline{\beta}_j$, such that

$$0 < \beta_0 < \beta_1 < \cdots, \tag{29}$$

then from previous lemmas and the Hopf bifurcation theorem for functional differential equations [18], we have the following results on stability and bifurcation to system (3).

Theorem 6. *For system*(3)*, the equilibrium E is asymptotically stable when $\beta \in [0, \beta_0)$ and unstable when $\beta \in (\beta_0, +\infty)$; system* (3) *undergoes a Hopf bifurcation at E when $\beta = \beta_j$, $j = 0, 1, 2, \ldots$, where β_j are defined by* (13) *or* (25).

3. The Direction and Stability of the Hopf Bifurcation

In Section 2 we obtained some conditions under which system (3) undergoes the Hopf bifurcation at some critical values of β. In this section, we study the direction, stability, and the period of the bifurcating periodic solutions. The method we used is based on the normal form method and the center manifold theory introduced by Hassard et al. [17].

Move $E(0, -\beta\cos^2\varphi_0, 0, -\beta\cos^2\varphi_0)$ to the origin $O(0, 0, 0, 0)$ and denote $\delta = \sin 2\varphi_0$, $\rho = \cos 2\varphi_0$, then system (3) can be written as the form

$$\begin{aligned}
\frac{\mathrm{d}x_1(t)}{\mathrm{d}t} &= -(\gamma_1+\gamma_2)x_1(t) - \gamma_2 y_1(t) + \beta\delta\gamma_2 x_1(t-\tau) \\
&\quad + \beta\gamma_2\rho\varphi_0 x_1^2(t-\tau) - \frac{2}{3}\beta\delta\gamma_2 x_1^3(t-\tau) + O(4), \\
\frac{\mathrm{d}y_1(t)}{\mathrm{d}t} &= \gamma_1 x_1(t), \\
\frac{\mathrm{d}x_2(t)}{\mathrm{d}t} &= -(\gamma_1+\gamma_2)x_2(t) - \gamma_2 y_2(t) \\
&\quad + \beta\gamma_2\Big[k\delta x_1(t-\tau) + (1-k)\delta x_2(t-\tau) \\
&\qquad + k^2\rho x_1^2(t-\tau) \\
&\qquad + 2k(1-k)\rho x_1(t-\tau)x_2(t-\tau) \\
&\qquad + (1-k)^2\rho x_2^2(t-\tau) - \frac{2}{3}k^3\delta x_1^3(t-\tau) \\
&\qquad - 2k^2(1-k)\delta x_1^2(t-\tau)x_2(t-\tau) \\
&\qquad - 2k(1-k)^2\delta x_1(t-\tau)x_2^2(t-\tau) \\
&\qquad - \frac{2}{3}(1-k)^3\delta x_2^3(t-\tau)\Big] + O(4), \\
\frac{\mathrm{d}y_2(t)}{\mathrm{d}t} &= \gamma_1 x_2(t).
\end{aligned} \tag{30}$$

Clearly, the phase space is $\mathscr{C} = \mathscr{C}([-\tau, 0], \mathbb{R}^4)$. For convenience, let

$$\beta^* \in \{\beta_j\} \cup \{\overline{\beta}_j\}, \tag{31}$$

and $\beta = \beta^* + \mu$, $\mu \in \mathbb{R}$. From the analysis above we know that $\mu = 0$ is the Hopf bifurcation value for system(30). Let $i\omega^*$ be the root of the characteristic equation associate with the linearization of system (30) when $\beta = \beta^*$. For $\phi = (\phi_1, \phi_2, \phi_3, \phi_4) \in \mathscr{C}$, let

$$L_\mu(\phi) = B\phi(0) + C\phi(-\tau), \tag{32}$$

where

$$\begin{gathered}
B = \begin{pmatrix} -(\gamma_1+\gamma_2) & -\gamma_2 & 0 & 0 \\ \gamma_1 & 0 & 0 & 0 \\ 0 & 0 & -(\gamma_1+\gamma_2) & -\gamma_2 \\ 0 & 0 & \gamma_1 & 0 \end{pmatrix}, \\
C = \begin{pmatrix} \beta\delta\gamma_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ k\beta\delta\gamma_2 & 0 & (1-k)\beta\delta\gamma_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\end{gathered} \tag{33}$$

By the Rieze representation theorem, there exists a 4×4 matrix, $\eta(\theta, \mu)$ $(-\tau \le \theta \le 0)$, whose elements are of bounded variation functions such that

$$L_\mu(\phi) = \int_{-\tau}^{0} \mathrm{d}\eta(\theta, \mu)\phi(\theta), \quad \phi \in \mathscr{C}. \tag{34}$$

In fact, we can choose

$$\eta(\theta, \mu) = \begin{cases} B, & \theta = 0, \\ 0, & \theta \in (-\tau, 0) \\ -C, & \theta = -\tau. \end{cases} \tag{35}$$

Then, (30) is satisfied.

For $\phi \in \mathscr{C}$, define the operator $A(\mu)$ as

$$A(\mu)\phi(\theta) = \begin{cases} \dfrac{\mathrm{d}\phi(\theta)}{\mathrm{d}\theta}, & \theta \in [-\tau, 0), \\ \displaystyle\int_{-\tau}^{0} \mathrm{d}\eta(t, \mu)\phi(t), & \theta = 0, \end{cases} \tag{36}$$

and $R(\mu)\phi$ as

$$R(\mu)\phi(\theta) = \begin{cases} 0, & \theta \in [-\tau, 0), \\ f(\mu, \phi), & \theta = 0, \end{cases} \tag{37}$$

where

$$f(\mu,\phi)=\beta^{*}\gamma_2\begin{pmatrix}\rho\phi_1^2(-\tau)-\dfrac{2}{3}\delta\phi_1^3(-\tau)+O(4)\\ 0\\ k^2\rho\phi_1^2(-\tau)+k(1-k)\rho\phi_1(-\tau)\phi_3(-\tau)\\ +(1-k)^2\rho\phi_3^2(-\tau)-\dfrac{2}{3}k^3\delta\phi_1^3(-\tau)-2k^2(1-k)\delta\phi_1^2(-\tau)\phi_3(t-\tau)\\ -2k(1-k)^2\delta\phi_1(-\tau)\phi_3^2(-\tau)-\dfrac{2}{3}(1-k)^3\delta\phi_3^3(-\tau)+O(4)\\ 0\end{pmatrix}. \tag{38}$$

Then, system (30) is equivalent to the following operator equation:

$$\dot{u}_t=A(\mu)u_t+R(\mu)u_t, \tag{39}$$

where $u(t)=(x_1(t),y_1(t),x_2(t),y_2(t))^T$, $u_t=u(t+\theta)$, for $\theta\in[-\tau,0]$.

For $\psi\in\mathscr{C}^1([0,\tau],\mathbb{R}^4)$, define

$$A^{*}\psi(s)=\begin{cases}-\dfrac{\mathrm{d}\psi(s)}{\mathrm{d}s}, & s\in(0,\tau],\\ \displaystyle\int_{-\tau}^{0}\psi(-\xi)\,\mathrm{d}\eta(\xi,0), & s=0.\end{cases} \tag{40}$$

For $\phi\in\mathscr{C}[-\tau,0]$ and $\psi\in\mathscr{C}[0,\tau]$, define the bilinear form

$$\begin{aligned}\langle\psi(s),\phi(\theta)\rangle&=\overline{\psi}(0)\phi(0)\\&\quad-\int_{-\tau}^{0}\int_{0}^{\theta}\overline{\psi}(\xi-\theta)\,\mathrm{d}\eta(\theta)\phi(\xi)\,\mathrm{d}\xi,\end{aligned} \tag{41}$$

where $\eta(\theta)=\eta(\theta,0)$. Then, $A(0)$ and A^{*} are adjoint operators.

Let $q(\theta)$ and $q^{*}(s)$ be eigenvectors of $A(0)$ and A^{*} associated to $i\omega^{*}$ and $-i\omega^{*}$, respectively. It is not difficult with verify that

$$\begin{aligned}q(\theta)&=\left(1,\frac{\gamma_1}{i\omega^{*}},1,\frac{\gamma_1}{i\omega^{*}}\right)^{T}e^{i\omega^{*}\theta},\\ q^{*}(s)&=\frac{1}{D}\left(1,\frac{\gamma_2}{i\omega^{*}},1,\frac{\gamma_2}{i\omega^{*}}\right)e^{i\omega^{*}s},\end{aligned} \tag{42}$$

where

$$D=2+\frac{2\gamma_1\gamma_2}{\omega^{*2}}+2\beta^{*}\delta\gamma_2\tau e^{-i\omega^{*}\tau}. \tag{43}$$

Then, $\langle q^{*}(s),q(\theta)\rangle=1$, $\langle q^{*}(s),\overline{q}(\theta)\rangle=0$.

Let u_t be the solution of (39) and define

$$z(t)=\langle q^{*},u_t\rangle,\qquad W(t,\theta)=u_t(\theta)-2\operatorname{Re}\{z(t)q(\theta)\}. \tag{44}$$

On the center manifold $\mathscr{C}_0$, we have

$$W(t,\theta)=W(z(t),\overline{z}(t),\theta), \tag{45}$$

where

$$W(z,\overline{z},\theta)=W_{20}\frac{z^2}{2}+W_{11}z\overline{z}+W_{02}\frac{\overline{z}^2}{2}+\cdots, \tag{46}$$

z and $\overline{z}$ are local coordinates for center manifold $\mathscr{C}_0$ in the direction of q^{*} and $\overline{q}^{*}$. Note that W is real if u_t is real. We only consider real solutions.

For solution u_t in $\mathscr{C}_0$, since $\mu=0$, we have

$$\begin{aligned}\dot{z}(t)&=i\omega^{*}z+\langle q^{*}(\theta),f(0,W+2\operatorname{Re}\{z(t)q(\theta)\})\rangle\\&=i\omega^{*}z+\overline{q}^{*}(0),f(0,W(z,\overline{z},0)+2\operatorname{Re}\{z(t)q(0)\})\\&=i\omega^{*}z+\overline{q}^{*}(0)f_0(z,\overline{z}).\end{aligned} \tag{47}$$

We rewrite this equation as

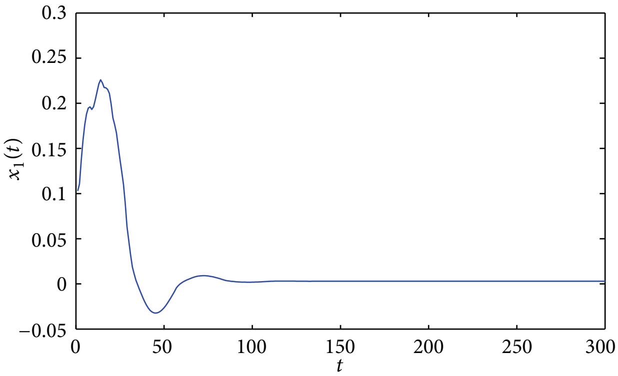
$$\dot{z}(t)=i\omega^{*}z+g(z,\overline{z}), \tag{48}$$

where

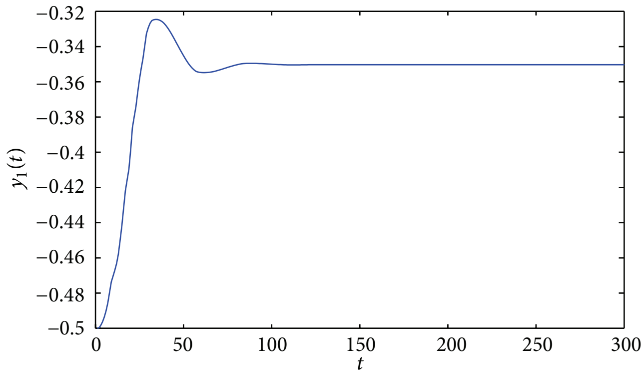
$$g(z,\overline{z})=g_{20}\frac{z^2}{2}+g_{11}z\overline{z}+g_{02}\frac{\overline{z}^2}{2}+g_{21}\frac{z^2\overline{z}}{2}\cdots. \tag{49}$$

By (39) and (48), we have

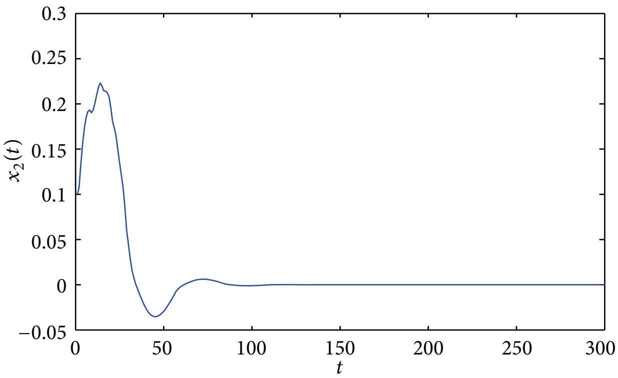
$$\begin{aligned}\dot{W}&=\dot{u}_t-\dot{z}q-\dot{\overline{z}}\overline{q}\\&=\begin{cases}AW-2\operatorname{Re}\{\overline{q}^{*}(0)f_0q(\theta)\}, & \theta\in[-\tau,0),\\ AW-2\operatorname{Re}\{\overline{q}^{*}(0)f_0q(0)\}+f_0, & \theta=0,\end{cases}\\&=AW+H(z,\overline{z},\theta),\end{aligned} \tag{50}$$



(a)



(b)



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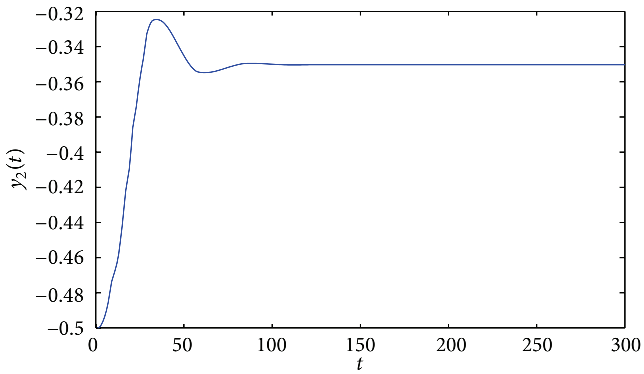

(d)

FIGURE 2: $\gamma_1 = 0.1, \gamma_2 = 2.5, \tau = 1.5, k = 1.9$, which means that condition (H_1) holds, and $\beta = 0.7 < \beta_0$. The initial value is $(0.1, -0.5, 0.1, -0.5)$.

where

$$H(z,\bar{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots. \tag{51}$$

Expanding the above series and comparing the coefficients, we obtain

$$\begin{gathered}(A - 2i\omega^* I)W_{20}(\theta) = -H_{20}(\theta), \\ AW_{11}(\theta) = -H_{11}(\theta), \ldots.\end{gathered} \tag{52}$$

Notice that

$$\begin{gathered}q(\theta) = \left(1, \frac{\gamma_1}{i\omega^*}, 1, \frac{\gamma_1}{i\omega^*}\right)^T e^{i\omega^*\theta}, \\ u_t(\theta) = zq(\theta) + \overline{zq}(\theta) + W(z,\bar{z},\theta),\end{gathered} \tag{53}$$

where

$$\begin{aligned}W^{(i)}(z,\bar{z},\theta) = {} & W_{20}^{(i)}(\theta)\frac{z^2}{2} + W_{11}^{(i)}(\theta)z\bar{z} \\ & + W_{02}^{(i)}(\theta)\frac{\bar{z}^2}{2} + \cdots, \quad i = 1,2,3,4.\end{aligned} \tag{54}$$

Combing (38) and by straightforward computation, we can obtain the coefficients which will be used in determining the important quantities:

$$\begin{aligned}
g_{20} &= \frac{2\beta^*\gamma_2\rho}{D}e^{-2i\omega^*\tau}\left(k^2 - k + 2\right), \\
g_{11} &= \frac{2\beta^*\gamma_2\rho}{D}\left(k^2 - k + 2\right), \\
g_{02} &= \frac{2\beta^*\gamma_2\rho}{D}e^{2i\omega^*\tau}\left(k^2 - k + 2\right), \\
g_{21} &= \frac{2\beta^*\gamma_2}{D}\Bigg\{\rho\left(e^{i\omega^*\tau}W_{20}^{(1)}(-\tau) + 2e^{-i\omega^*\tau}W_{11}^{(1)}(-\tau)\right) \\
&\quad + k^2\rho\left(e^{i\omega^*\tau}W_{20}^{(1)}(-\tau) + 2e^{-i\omega^*\tau}W_{11}^{(1)}(-\tau)\right) \\
&\quad + k(1-k)\rho \\
&\quad \times\left(e^{-i\omega^*\tau}W_{11}^{(3)}(-\tau) + e^{i\omega^*\tau}\frac{W_{20}^{(3)}(-\tau)}{2}\right. \\
&\quad \left. + e^{i\omega^*\tau}\frac{W_{20}^{(1)}(-\tau)}{2} + e^{-i\omega^*\tau}W_{11}^{(1)}(-\tau)\right)
\end{aligned}$$

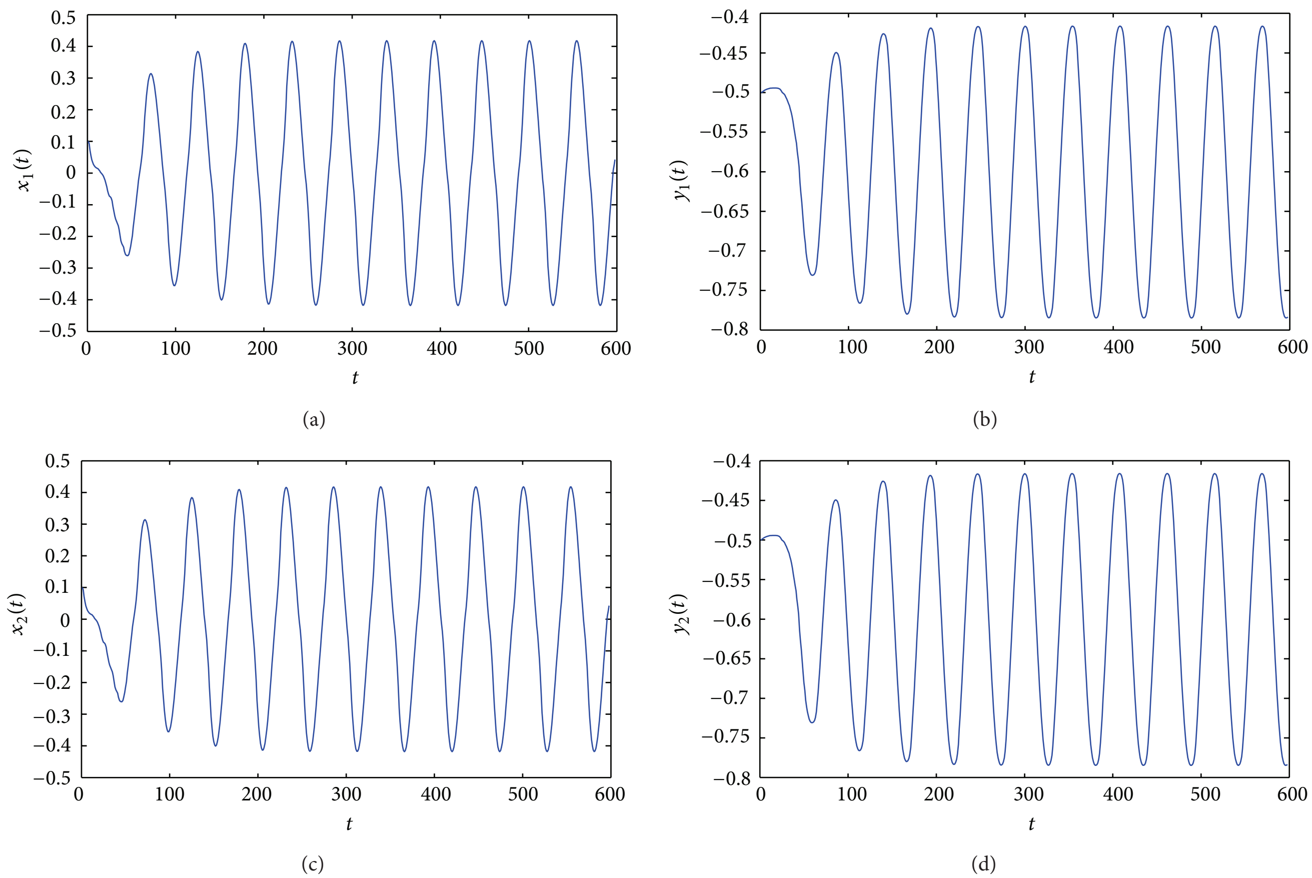

FIGURE 3: $\gamma_1 = 0.1, \gamma_2 = 2.5, \tau = 1.5, k = 1.9$, which means that condition (H_1) holds, and $\beta = 1.2 > \beta_0$. The initial value is $(0.1, -0.5, 0.1, -0.5)$.

$$+(1-k)^2 \rho \left(2e^{-i\omega^* \tau} W_{11}^{(3)}(-\tau) + e^{i\omega^* \tau} W_{20}^{(3)}(-\tau)\right) - 4\delta e^{-i\omega^* \tau} \Bigg\} . \quad (55)$$

We still need to compute $W_{20}(\theta)$ and $W_{11}(\theta)$, for $\theta \in [-\tau, 0)$. We have

$$\begin{aligned} H(z,\overline{z},\theta) &= -\overline{q}^*(0) f_0 q(\theta) - q^*(0) \overline{f}_0 \overline{q}(\theta) \\ &= -g(z,\overline{z}) q(\theta) - \overline{g}(z,\overline{z}) \overline{q}(\theta). \end{aligned} \quad (56)$$

Comparing the coefficients about $H(z,\overline{z},\theta)$ gives that

$$\begin{aligned} H_{20}(\theta) &= -g_{20} q(\theta) - \overline{g}_{02} \overline{q}(\theta), \\ H_{11} &= -g_{11} q(\theta) - \overline{g}_{11} \overline{q}(\theta). \end{aligned} \quad (57)$$

Then, from (52), we get

$$\begin{aligned} \dot{W}_{20}(\theta) &= 2i\omega^* W_{20}(\theta) + g_{20} q(\theta) + \overline{g}_{02} \overline{q}(\theta), \\ \dot{W}_{11}(\theta) &= g_{11} q(\theta) + \overline{g}_{11} \overline{q}(\theta), \end{aligned} \quad (58)$$

which implies that

$$\begin{aligned} W_{20}(\theta) &= \frac{g_{20} q(0)}{-i\omega^*} e^{i\omega^* \theta} + \frac{\overline{g}_{02} \overline{q}(0)}{-3i\omega^*} e^{-i\omega_j^* \theta} + E e^{2i\omega_j^* \theta}, \\ W_{11}(\theta) &= \frac{g_{11} q(0)}{i\omega^*} e^{i\omega^* \theta} + \frac{\overline{g}_{11} \overline{q}(0)}{-i\omega^*} e^{-i\omega^* \theta} + F. \end{aligned} \quad (59)$$

Here, E and F are both four-dimensional vectors and can be determined by setting $\theta = 0$ in $H(z,\overline{z},\theta)$. In fact, from (38) and

$$H(z,\overline{z},0) = -2 \operatorname{Re}\{\overline{q}^*(0) f_0 q(0)\} + f_0, \quad (60)$$

we have

$$\begin{aligned} H_{20}(0) = &- g_{20} q(0) - \overline{g}_{02} \overline{q}(0) \\ &+ 2\beta^* \gamma_2 \rho e^{-2i\omega^* \tau} \left(1, 0, k^2 - k + 1, 0\right)^T, \\ H_{11}(0) = &- g_{11} q(0) - \overline{g}_{11} \overline{q}(0) \\ &+ 2\beta^* \gamma_2 \rho \left(1, 0, k^2 - k + 1, 0\right)^T. \end{aligned} \quad (61)$$

It follows from (52) and the definition of A that

$$\begin{aligned} \beta^* B W_{20}(0) + \beta^* C W_{20}(-\tau) &= 2i\omega^* W_{20}(0) - H_{20}(0), \\ \beta^* B W_{11}(0) + \beta^* C W_{11}(-\tau) &= -H_{11}(0), \end{aligned} \quad (62)$$

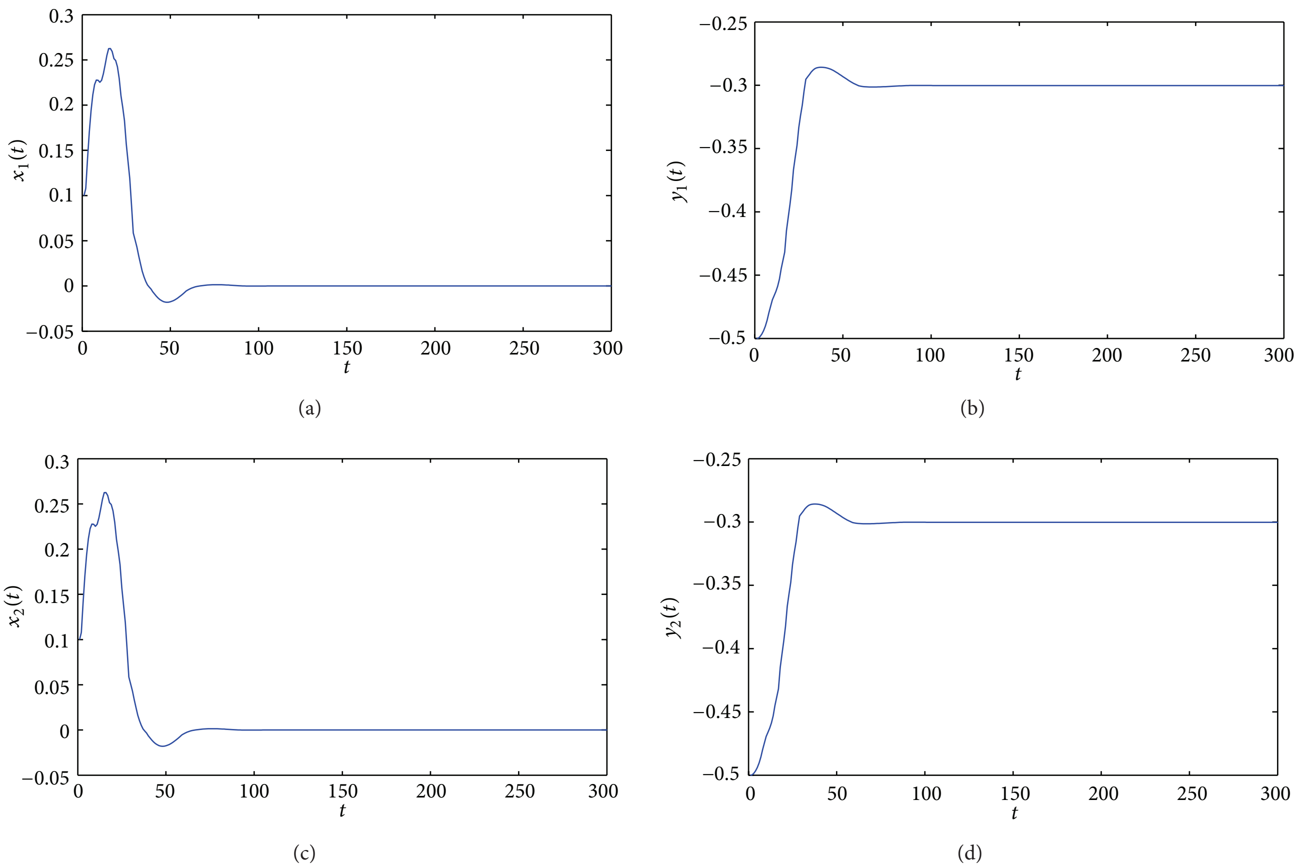

FIGURE 4: $\gamma_1 = 0.1$, $\gamma_2 = 2.5$, $\tau = 1.5$, $k = 3$, which means that condition (H_2) holds, and $\beta = 0.6 < \overline{\beta}_0$. The initial value is $(0.1, -0.5, 0.1, -0.5)$.

which implies that

$$
\begin{aligned}
E = {} & \left(B + e^{-2i\omega^*\tau}C - 2i\omega^* \mathrm{I}\right)^{-1} \\
& \times \left[B\left(\frac{g_{20}q(0)}{i\omega^*} + \frac{\overline{g}_{02}\overline{q}(0)}{3i\omega^*}\right) \right. \\
& \quad + C\left(\frac{g_{20}q(0)}{i\omega^*}e^{-i\omega^*\tau} + \frac{\overline{g}_{02}\overline{q}(0)}{3i\omega^*}e^{i\omega^*\tau}\right) \\
& \quad + \frac{1}{\beta^*}\left(g_{20}q(0) + \overline{g}_{02}\overline{q}(0)\right) \\
& \quad \left. \times 2\beta^*\gamma_2\rho e^{-2i\omega^*\tau}\left(1, 0, k^2 - k + 1, 0\right)^T \right], \\
F = {} & (B + C)^{-1}\left[B\left(\frac{g_{11}q(0)}{-i\omega^*} + \frac{\overline{g}_{11}\overline{q}(0)}{i\omega^*}\right) \right. \\
& \quad + C\left(\frac{g_{11}q(0)}{-i\omega^*}e^{-i\omega^*\tau} + \frac{\overline{g}_{11}\overline{q}(0)}{i\omega^*}e^{i\omega^*\tau}\right) \\
& \quad + \frac{1}{\beta^*}\left(g_{11}q(0) + \overline{g}_{11}\overline{q}(0)\right) \\
& \quad \left. -2\beta^*\gamma_2\rho\left(1, 0, k^2 - k + 1, 0\right)^T \right].
\end{aligned} \tag{63}
$$

Consequently, the above g_{21} can be expressed by the parameters and delay in system (30). Thus, we can compute the following quantities:

$$
\begin{aligned}
c_1(0) &= \frac{i}{2\omega^*}\left(g_{20}g_{11} - 2\left|g_{11}\right|^2 - \frac{1}{3}\left|g_{20}\right|^2\right) + \frac{g_{21}}{2}, \\
\mu_2 &= -\frac{\operatorname{Re} c_1(0)}{\operatorname{Re} \lambda'(\beta^*)}, \\
\beta_2 &= 2\operatorname{Re} c_1(0), \\
T_2 &= -\frac{\operatorname{Im} c_1(0) + \mu_2 \operatorname{Im} \lambda'(\beta^*)}{\omega^*},
\end{aligned} \tag{64}
$$

which determine the properties of bifurcating periodic solutions at the critical value τ_0. The direction and stability of the

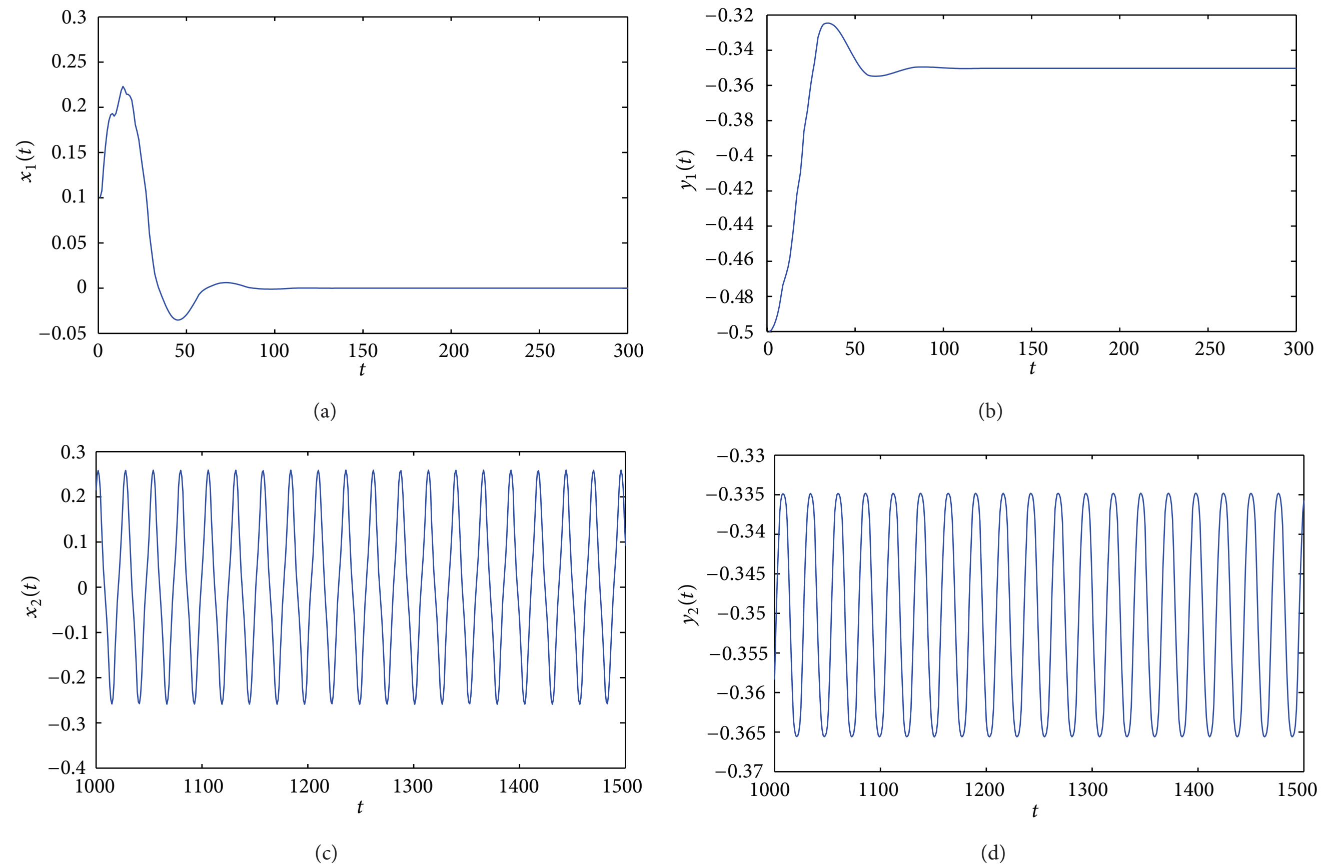

Figure 5: $\gamma_1 = 0.1$, $\gamma_2 = 2.5$, $\tau = 1.5$, $k = 3$, which means that condition (H_2) holds, and $\overline{\beta}_0 < \beta = 0.7 < \beta_0$. The initial value is $(0.1, -0.5, 0.1, -0.5)$.

Hopf bifurcation in the center manifold can be determined by μ_2 and β_2, respectively. In fact, if $\mu_2 > 0$ ($\mu_2 < 0$), then the bifurcating periodic solutions are forward (backward); the bifurcating periodic solutions on the center manifold are stable (unstable) if $\beta_2 < 0$ ($\beta_2 > 0$); and T_2 determines the period of the bifurcating periodic solutions: the period increases (decreases) if $T_2 > 0$ ($T_2 < 0$).

From the discussion in Section 2, we have known that $\operatorname{Re}\lambda'(\beta_j) > 0$; therefore; we have the following result.

Theorem 7. *The direction of the Hopf bifurcation for system* (3) *at the equilibrium* $E(0, -\beta\cos^2\varphi_0, 0, -\beta\cos^2\varphi_0)$ *when* $\beta = \beta^*$ *is forward (backward), and the bifurcating periodic solutions on the center manifold are stable (unstable) if* $\operatorname{Re}(c_1(0)) < 0$ (> 0). *Particularly, the stability of the bifurcation periodic solutions of system* (3) *and the reduced equations on the center manifold are coincident at the first bifurcation value* $\beta = \beta_0$.

4. Numerical Simulations

In this section, we will carry out numerical simulations on system (3) at special values of β. We choose a set of data as follows:

$$\gamma_1 = 0.1, \quad \gamma_2 = 2.5, \quad \varphi_0 = \frac{\pi}{4}, \quad \tau = 1.5, \tag{65}$$

which are the same as those in [1]. Then, $\delta = 1$, $\rho = 0$.

Then, we can obtain

$$\begin{gathered} \omega_0 \doteq 0.2225, \qquad \omega_1 \doteq 3.5677, \ldots, \\ \varpi_0 \doteq 1.7294, \qquad \varpi_1 \doteq 5.5531, \ldots, \\ \beta_0 \doteq 1.1008, \qquad \beta_1 \doteq 1.7434, \ldots, \\ \overline{\beta}_0 \doteq 1.3534, \qquad \overline{\beta}_1 \doteq 2.5235, \ldots, \quad k = 1.9, \\ \overline{\beta}_0 \doteq 0.6093, \qquad \overline{\beta}_1 \doteq 1.1356, \ldots, \quad k = 3. \end{gathered} \tag{66}$$

From the analysis in Section 2, we know that $\beta(\overline{\beta})$ is increasing with respect to $\omega(\varpi)$ when $\omega(\varpi) > \gamma_1\gamma_2$, which means that

$$\beta_0 = \min\{\beta_j\}, \quad \overline{\beta}_0 = \min\{\overline{\beta}_j\}, \quad j = 0, 1, 2, \ldots, \tag{67}$$

that is, $\beta_0(\overline{\beta}_0)$ is the first critical value at which system (3) undergoes a Hopf bifurcation.

When $k = 1.9$, by the previous results, it follows that

$$\begin{gathered} \lambda'(\beta_0) \doteq 0.2440 - 0.0491\mathrm{i}, \qquad c_1(0) \doteq -0.5373 + 0.1082\mathrm{i}, \\ \mu_2 \doteq 2.2020, \qquad \beta_2 \doteq -1.0746, \qquad T_2 = -0.0018. \end{gathered} \tag{68}$$

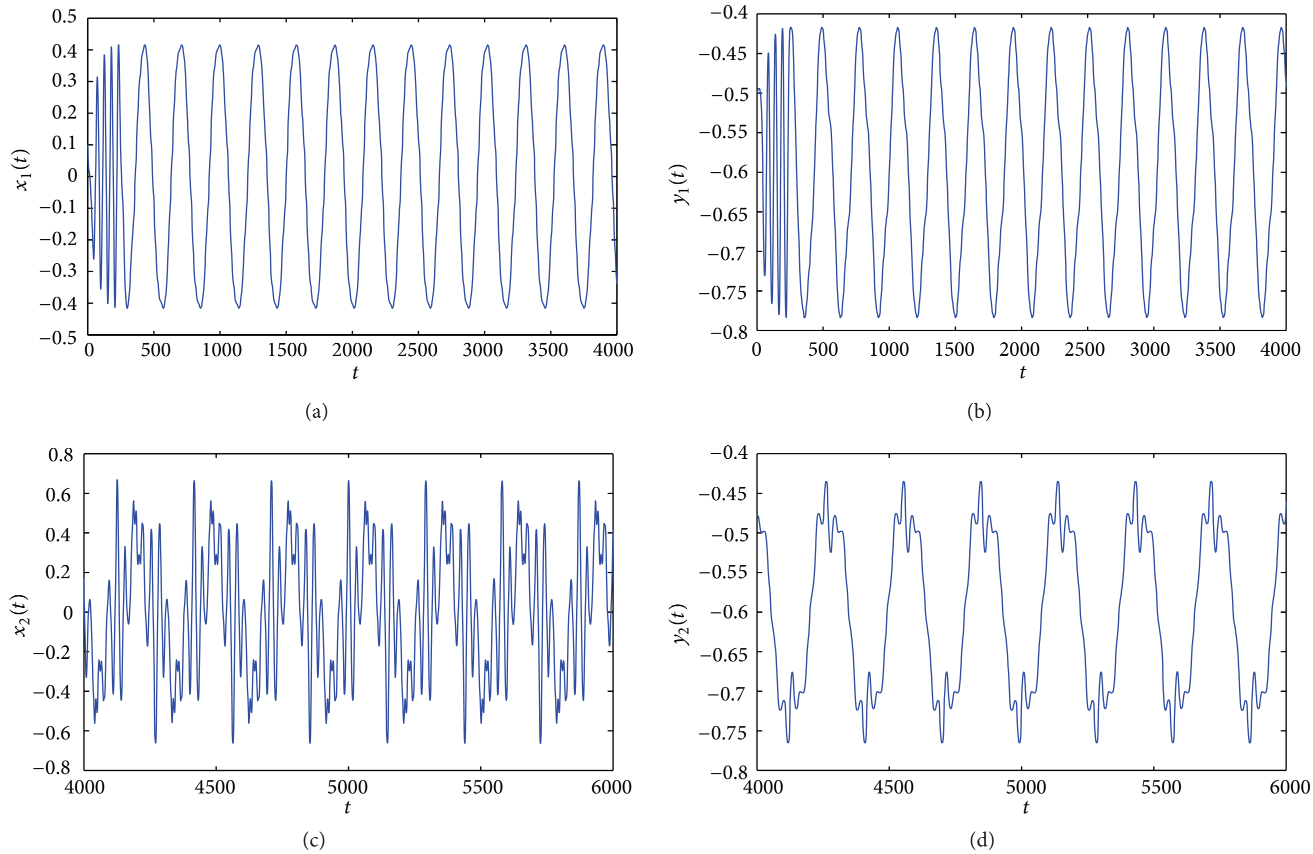

Figure 6: $\gamma_1 = 0.1$, $\gamma_2 = 2.5$, $\tau = 1.5$, $k = 3$, which means that condition (H_2) holds, and $\beta = 1.2 > \beta_0 > \overline{\beta}_0$. The initial value is $(0.1, -0.5, 0.1, -0.5)$.

Hence, we arrive at the following conclusion: the equilibrium E is asymptotically stable when $\beta \in [0, 1.1008)$ and unstable when $\beta \in (1.1008, +\infty)$, and, at the first critical value, the bifurcating periodic solutions are asymptotically stable, and the direction of the bifurcation is forward (see Figures 2 and 3).

When $k = 3$, we can get

$$\lambda'\left(\overline{\beta}_0\right) \doteq 0.9004 + 0.0924i, \qquad c_1(0) \doteq -1.9116 + 0.7930i,$$
$$\mu_2 \doteq 2.1231, \qquad \beta_2 \doteq -3.8232, \qquad T_2 = -0.5720. \tag{69}$$

Then, we have the following: the equilibrium E is asymptotically stable when $\beta \in [0, 0.6093)$, and unstable when $\beta \in (0.6093, +\infty)$, and, at the first critical value, the bifurcating periodic solutions are asymptotically stable, and the direction of the bifurcation is forward (see Figures 4, 5, and 6).

5. Conclusion

Ravoori et al. [1] explored an experimental system of two nominally identical optoelectronic feedback loops coupled unidirectionally, which are described by system (3). In the experiment, they found that depending on the value of the feedback strength β and delay τ, system (1) is capable of producing dynamics ranging from periodic oscillations to high-dimensional chaos [14, 15].

This paper investigates the stability and the existence of periodic solutions. We find that with the variety of the coupling strength k, even if all other parameters keep the same, the dynamical behavior can change greatly. In fact, it is clear that the first two equations, $x_1(t)$ and $y_1(t)$ are uncoupled with equations $x_2(t)$ and $y_2(t)$, so system (1) are independent of (2), which means that coupling strength k does not appear in (1). The characteristic equation of (1) has the same form as (6), so the first critical value β_0 is independent of k. The analysis of characteristic equation (7) shows that the value of k can affect the first critical value $\overline{\beta}_0$ definitely. And we draw a conclusion that when k is in an interval, in which $\beta_0 < \overline{\beta}_0$ holds, solutions of system (1) and (2) keep synchronous; when k belongs to the interval, in which $\overline{\beta}_0 < \beta_0$ holds, solutions of system (1) and (2) can also keep synchronous with $\beta < \overline{\beta}_0$, while they lose their synchronization when $\beta > \overline{\beta}_0$, no matter whether $\beta < \beta_0$ or not.

As a result, the modulation of the coupling strengths k together with the feedback strength β would be an efficient and an easily implementable method to control the behavior of the coupled chaotic oscillators.

Acknowledgment

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Research Article

Oscillation Criteria for Some New Generalized Emden-Fowler Dynamic Equations on Time Scales

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By means of novel analytical techniques, we have established several new oscillation criteria for the generalized Emden-Fowler dynamic equation on a time scale $\mathbb{T}$, that is, $(r(t)|Z^{\Delta}(t)|^{\alpha-1}Z^{\Delta}(t))^{\Delta}+f(t,x(\delta(t)))=0$, with respect to the case $\int_{t_0}^{\infty} r^{-1/\alpha}(s)\Delta s=\infty$ and the case $\int_{t_0}^{\infty} r^{-1/\alpha}(s)\Delta s<\infty$, where $Z(t)=x(t)+p(t)x(\tau(t))$, α is a constant, $|f(t,u)|\geqslant q(t)|u^{\beta}|$, β is a constant satisfying $\alpha\geqslant\beta>0$, and r, p, and q are real valued right-dense continuous nonnegative functions defined on $\mathbb{T}$. Noting the parameter value α probably unequal to β, our equation factually includes the existing models as special cases; our results are more general and have wider adaptive range than others' work in the literature.

1. Introduction

In the past two decades, the theory of time scales proposed by Hilger [1] in 1990 has received extensive attention because of its advantage to unify continuous model and discrete model into one case under the scholars' investigation. Numerous authors have considered many aspects of this new theory. Many of those results focus on oscillation and nonoscillation of some equations on time scales. Reader can refer to articles [2–25] and there references cited therein.

In this paper, we consider the oscillatory behavior of the solutions of second-order generalized Emden-Fowler dynamic equation of the form

$$\left(r(t)\left|Z^{\Delta}(t)\right|^{\alpha-1}Z^{\Delta}(t)\right)^{\Delta}+f(t,x(\delta(t)))=0,\quad t\in\mathbb{T},\ t\geqslant t_0, \tag{1}$$

with $Z(t)=x(t)+p(t)x(\tau(t))$, parameter constant α, and conditions (H_1)–(H_6):

(H_1) $\mathbb{T}$ is a time scale which is unbounded above. $[t_0,\infty)_{\mathbb{T}}:=[t_0,\infty)\cap\mathbb{T}$, where $t_0\in\mathbb{T}$ with $t_0>0$, $C_{\mathrm{rd}}(\mathbb{T},\mathbb{S})$ denotes the collection of all functions $f:\mathbb{T}\to\mathbb{S}$ which are right-dense continuous on $\mathbb{T}$;

(H_2) $r(t)\in C_{\mathrm{rd}}(\mathbb{T},(0,\infty))$, $R(t):=\int_{t_0}^{t} r^{-1/\alpha}(s)\Delta s$;

(H_3) $p(t)\in C_{\mathrm{rd}}(\mathbb{T},[0,1])$;

(H_4) $\tau(t)\in C_{\mathrm{rd}}(\mathbb{T},\mathbb{T})$, $\tau(t)\leqslant t$, for $t\in\mathbb{T}$, $\lim_{t\to\infty}\tau(t)=\infty$, $\delta(t)\in C_{\mathrm{rd}}(\mathbb{T},\mathbb{T})$, $\delta(t)\leqslant t$, for $t\in\mathbb{T}$, $\lim_{t\to\infty}\delta(t)=\infty$;

(H_5) $\delta^{\Delta}(t)>0$ is right-dense continuous on $\mathbb{T}$, and $\delta(\sigma(t))=\sigma(\delta(t))$ for all $t\in\mathbb{T}$, where $\sigma(t)$ is the forward jump operator on $\mathbb{T}$;

(H_6) $f(t,u)\in C(\mathbb{T}\times\mathbb{R},\mathbb{R})$ is a continuous function such that $uf(t,u)>0$, for all $u\neq 0$ and there exists a positive right-dense continuous function $q(t)$ defined on $\mathbb{T}$ such that $|f(t,u)|\geqslant q(t)|u^{\beta}|$ for all $t\in\mathbb{T}$ and for all $u\in\mathbb{R}$, where β is a constant satisfying $\alpha\geqslant\beta>0$.

As a solution of (1), we mean a function $x(t)$ such that $x(t)+p(t)x(\tau(t))\in C^1_{\mathrm{rd}}(t_x,\infty)_{\mathbb{T}}$ and $r(t)|[x(t)+p(t)x(\tau(t))]^{\Delta}|^{\alpha-1}[x(t)+p(t)x(\tau(t))]^{\Delta}\in C^1_{\mathrm{rd}}(t_x,\infty)_{\mathbb{T}}$, $t_x\geqslant t_0$ and satisfying (1) for all $t\geqslant t_x$, where $C^1_{\mathrm{rd}}(t_x,\infty)_{\mathbb{T}}$ denotes the set of right-dense continuously Δ-differentiable functions on $(t_x,\infty)_{\mathbb{T}}$. In the sequel, we restrict our attention to those solutions of (1) which exist on the half-line $[t_x,\infty)_{\mathbb{T}}$ and satisfy $\sup\{|x(t)|:t>\widetilde{T}\}>0$ for any $\widetilde{T}\geqslant t_x$. We say that a nontrivial solution of (1) is oscillatory if it has arbitrary large zeros, otherwise we say that it is nonoscillatory. We say that (1) is oscillatory if all its solutions are oscillatory.

Among researchers in the oscillation of functional equations with time scales, Agarwal et al. [2] studied a special case of (1), which is

$$\left(r(t)\left(\left[y(t)+p(t)\,y(t-\tau_0)\right]^{\Delta}\right)^{\gamma}\right)^{\Delta} + f\left(t, y\left(t-\delta_0\right)\right) = 0, \quad t \in \mathbb{T},\ t \geqslant t_0, \tag{2}$$

where

$$\left|f(t,u)\right| \geqslant q(t)\,|u|^{\gamma}, \qquad \int_{t_0}^{\infty} r^{-1/\gamma}(s)\,\Delta s = \infty, \tag{3}$$

τ_0 and δ_0 are positive constants and $\gamma > 0$ is a quotient of odd positive integers. They got some oscillation criteria of (2) for the case when $\gamma > 0$ under the condition $r^{\Delta}(t) \geqslant 0$, and the case when $\gamma \geqslant 1$ under the condition $\mu(t) > 0$. Subsequently, for the case when $\gamma \geqslant 1$ is an odd positive integer, Saker [7] did not require the conditions $r^{\Delta}(t) \geqslant 0$ and $\mu(t) > 0$ and obtained some new oscillation results for (2) under the conditions (3).

Very Recently, in [10–13], Saker et al. have considered the oscillation of several equations with time scales. For example in paper [13], the author is concerned with the quasilinear equation of the form:

$$\left(p(t)\left(\left[y(t)+r(t)\,y(\tau(t))\right]^{\Delta}\right)^{\gamma}\right)^{\Delta} + f\left(t, y(\delta(t))\right) = 0, \tag{4}$$

where $|f(t,u)| \geqslant q(t)|u^{\beta}|$, $\gamma > 0$, and $\beta > 0$ are ratios of odd positive integers.

However the value range of the equation parameters in our work is wider than those in [2, 7, 10–13] and the equation itself is also different from those in [2, 7, 10–13]. In fact, our approach in constructing the criteria is different from those of Saker and his coauthors' work.

For (2) with $\gamma \geqslant 1$ being a quotient of odd positive integers and without the restrictive conditions $r^{\Delta}(t) \geqslant 0$ and without $\mu(t) > 0$, Wu et al. [21] obtained several oscillation criteria for the equation:

$$\left(r(t)\left(\left[y(t)+p(t)\,y(\tau(t))\right]^{\Delta}\right)^{\gamma}\right)^{\Delta} + f\left(t, y(\delta(t))\right) = 0, \quad t \in \mathbb{T},\ t \geqslant t_0, \tag{5}$$

under the conditions (3).

Chen [25] investigated the following second-order Emden-Fowler neutral delay dynamic equation

$$\left(r(t)\left|x^{\Delta}(t)\right|^{\gamma-1}x^{\Delta}(t)\right)^{\Delta} + f\left(t, y(\delta(t))\right) = 0, \quad t \in \mathbb{T},\ t \geqslant t_0, \tag{6}$$

with $x(t) = y(t) + p(t)y(\tau(t))$, under the conditions (3). He obtained some oscillation criteria when $\gamma > 0$ is a constant and without assuming the conditions $r^{\Delta}(t) \geqslant 0$ and $\mu(t) > 0$.

All the above results cannot apply to our model (1) since our model (1) is more general than (2), (6) and those in [10–13], and the function $f(t,u)$ in (1) satisfies (H_6) which makes our model (1) distinguished from all the existing cases. To the best of our knowledge, nothing is known regarding the necessary and sufficient conditions for the qualitative behavior of (1) with $\alpha \neq \beta$ in (H_6) on time scales.

In this paper, even if $\alpha \neq \beta$ in (H_6) and there is no assumptions $r^{\Delta}(t) \geqslant 0$ and $\mu(t) > 0$, we have established several new oscillation criteria of (1) for the both cases

$$\lim_{t\to\infty}\int_{t_0}^{t} r^{-1/\alpha}(s)\,\Delta s = \infty, \tag{7}$$

$$\lim_{t\to\infty}\int_{t_0}^{t} r^{-1/\alpha}(s)\,\Delta s < \infty. \tag{8}$$

Factually, we have employed new analytical techniques to present and construct our criteria in Section 3 after reciting two useful lemmas in Section 2. Our results have extended and unified a number of other existing results and handled the cases which are not covered by current criteria. Finally, in Section 4 two examples are demonstrated to illustrate the efficiency of our work with relevant remark.

2. Some Lemmas

Lemma 1 (see [25]). *Suppose that (H_5) holds. Let $x : \mathbb{T} \to \mathbb{R}$. If x^{Δ} exists for all sufficiently large $t \in \mathbb{T}$, then $(x(\delta(t)))^{\Delta} = x^{\Delta}(\delta(t))\delta^{\Delta}(t)$ for all sufficiently large $t \in \mathbb{T}$.*

Lemma 2 (Bohner and Peterson [26, Theorem 1.90]). *Assume that $x(t)$ is Δ-differentiable and eventually positive or eventually negative, then*

$$\left(x^{\alpha}(t)\right)^{\Delta} = \alpha\left\{\int_0^1 \left[(1-h)\,x(t) + hx(\sigma(t))\right]^{\alpha-1}dh\right\}x^{\Delta}(t). \tag{9}$$

Lemma 3 (see [27]). *Let $\Psi(u) = au - bu^{(\lambda+1)/\lambda}$, where a, b, λ are constants, $a \geqslant 0$, $b > 0$, $\lambda > 0$, and $u \in [0,\infty)$. Then $\Psi(u)$ attains its maximum value on $[0,\infty)$ at $u = u^* := (a\lambda/b(\lambda+1))^{\lambda}$, and*

$$\max_{u\in[0,\infty)}\Psi(u) = \Psi\left(u^*\right) = \frac{\lambda^{\lambda}}{(\lambda+1)^{\lambda+1}}\frac{a^{\lambda+1}}{b^{\lambda}}. \tag{10}$$

3. Main Results

The case

$$\lim_{t\to\infty}\int_{t_0}^{t} r^{-1/\alpha}(s)\,\Delta s = \infty. \tag{11}$$

Theorem 4. *Assume that* (H_1)–(H_6) *and* (7) *hold. If there exists a function* $\xi(t) \in C^1_{rd}(\mathbb{T}, (0,\infty))$ *such that for any positive number* M,

$$\varlimsup_{t\to\infty} \int_{t_0}^{t} (\xi(s)\,\overline{p}(s) - Q(s))\,\Delta s = \infty, \tag{12}$$

where

$$\overline{p}(s) = q(s)\left[1 - p(\delta(s))\right]^{\beta},$$
$$Q(s) = \frac{\alpha^{\alpha} M (R(\sigma(s)))^{\alpha-\beta} r(\delta(s)) \left(\left(\xi^{\Delta}(s)\right)_{+}\right)^{\alpha+1}}{(\alpha+1)^{\alpha+1}\beta^{\alpha}\xi^{\alpha}(s)\left(\delta^{\Delta}(s)\right)^{\alpha}}, \tag{13}$$
$$\left(\xi^{\Delta}(s)\right)_{+} := \max\left\{\xi^{\Delta}(s), 0\right\},$$

then (1) *is oscillatory.*

Proof. Suppose that (1) has a nonoscillatory solution $x(t)$, then there exists $T_0 \geqslant t_0$ such that $x(t) \neq 0$ for all $t \geqslant T_0$. Without loss of generality, we assume that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\delta(t)) > 0$ for $t \geqslant T_0$, because a similar analysis holds for $x(t) < 0$, $x(\tau(t)) < 0$ and $x(\delta(t)) < 0$. Then the following are deduced from (1), (H_3), and (H_6):

$$Z(t) \geqslant x(t) > 0 \quad \text{for } t \geqslant T_0,$$
$$\left(r(t)\left|Z^{\Delta}(t)\right|^{\alpha-1} Z^{\Delta}(t)\right)^{\Delta} \leqslant 0, \quad t \geqslant T_0. \tag{14}$$

Therefore $r(t)|Z^{\Delta}(t)|^{\alpha-1}Z^{\Delta}(t)$ is a nonincreasing function and $Z^{\Delta}(t)$ is eventually of one sign.

We claim that

$$Z^{\Delta}(t) > 0 \quad \text{or} \quad Z^{\Delta}(t) = 0, \quad t \geqslant T_0. \tag{15}$$

Otherwise, if there exists a $t_1 \geqslant T_0$ such that $Z^{\Delta}(t) < 0$ for $t \geqslant t_1$, then from (14), for some positive constant K, we have

$$-r(t)\left(-Z^{\Delta}(t)\right)^{\alpha} \leqslant -K, \quad t \geqslant t_1, \tag{16}$$

that is,

$$-Z^{\Delta}(t) \geqslant \left(\frac{K}{r(t)}\right)^{1/\alpha}, \quad t \geqslant t_1, \tag{17}$$

integrating the above inequality from t_1 to t, we have

$$Z(t) \leqslant Z(t_1) - K^{1/\alpha}\left(R(t) - R(t_1)\right). \tag{18}$$

Letting $t \to \infty$, from (7), we get $\lim_{t\to\infty} Z(t) = -\infty$, which contradicts (14). Thus, we have proved (15).

We choose some $T_1 \geqslant T_0$ such that $\delta(t) \geqslant T_0$ for $t \geqslant T_1$. Therefore from (14), (15), and the fact $\delta(t) \leqslant \sigma(t)$, we have that

$$r(\sigma(t))\left(Z^{\Delta}(\sigma(t))\right)^{\alpha} \leqslant r(\delta(t))\left(Z^{\Delta}(\delta(t))\right)^{\alpha}, \quad t \geqslant T_1, \tag{19}$$

which follows that

$$Z^{\Delta}(\delta(t)) \geqslant Z^{\Delta}(\sigma(t))\left(\frac{r(\sigma(t))}{r(\delta(t))}\right)^{1/\alpha}, \quad t \geqslant T_1. \tag{20}$$

On the other hand, from (1), (H_6), and (15), we have

$$\left(r(t)\left(Z^{\Delta}(t)\right)^{\alpha}\right)^{\Delta} + q(t)\left(Z(\delta(t)) - p(\delta(t))\,x(\tau(\delta(t)))\right)^{\beta} \leqslant 0, \quad t \geqslant T_1. \tag{21}$$

Noticing (15) and the fact $Z(t) \geqslant x(t)$, we get

$$\left(r(t)\left(Z^{\Delta}(t)\right)^{\alpha}\right)^{\Delta} + \overline{p}(t)\,Z^{\beta}(\delta(t)) \leqslant 0, \quad t \geqslant T_1, \tag{22}$$

where $\overline{p}(t) = q(t)[1 - p(\delta(t))]^{\beta}$.

Define

$$w(t) = \xi(t)\,\frac{r(t)\left(Z^{\Delta}(t)\right)^{\alpha}}{Z^{\beta}(\delta(t))}, \quad \text{for } t \geqslant T_1. \tag{23}$$

Obviously, $w(t) > 0$. By (22), (23) and the product rule and the quotient rule, we obtain

$$\begin{aligned}
w^{\Delta}(t) &= \frac{\xi(t)}{Z^{\beta}(\delta(t))}\left(r(t)\left(Z^{\Delta}(t)\right)^{\alpha}\right)^{\Delta} + r(\sigma(t))\left(Z^{\Delta}(\sigma(t))\right)^{\alpha} \\
&\quad \times \frac{\xi^{\Delta}(t)\,Z^{\beta}(\delta(t)) - \xi(t)\left(Z^{\beta}(\delta(t))\right)^{\Delta}}{Z^{\beta}(\delta(t))\,Z^{\beta}(\delta(\sigma(t)))} \\
&\leqslant -\xi(t)\,\overline{p}(t) + \frac{\xi^{\Delta}(t)}{\xi(\sigma(t))}\,w(\sigma(t)) \\
&\quad - \frac{r(\sigma(t))\left(Z^{\Delta}(\sigma(t))\right)^{\alpha}\xi(t)\left(Z^{\beta}(\delta(t))\right)^{\Delta}}{Z^{\beta}(\delta(t))\,Z^{\beta}(\delta(\sigma(t)))}.
\end{aligned} \tag{24}$$

Now we consider the following two cases.

Case 1. Let $\beta \geqslant 1$. By (15), Lemmas 1 and 2, we have

$$\begin{aligned}
&\left(Z^{\beta}(\delta(t))\right)^{\Delta} \\
&\quad = \beta\left\{\int_0^1 \left[(1-h)\,Z(\delta(t)) + hZ(\delta(\sigma(t)))\right]^{\beta-1} dh\right\} \\
&\qquad \times \left(Z(\delta(t))\right)^{\Delta} \\
&\quad \geqslant \beta (Z(\delta(t)))^{\beta-1} Z^{\Delta}(\delta(t))\,\delta^{\Delta}(t).
\end{aligned} \tag{25}$$

From (H_5), (20), (23)–(25), and the fact that $Z(t)$ is nondecreasing, we obtain

$$\begin{aligned}
&w^{\Delta}(t)\\
&\leqslant -\xi(t)\overline{p}(t)+\frac{\xi^{\Delta}(t)}{\xi(\sigma(t))}w(\sigma(t))\\
&\quad-\frac{r(\sigma(t))\big(Z^{\Delta}(\sigma(t))\big)^{\alpha}\xi(t)\beta(Z(\delta(t)))^{\beta-1}Z^{\Delta}(\delta(t))\delta^{\Delta}(t)}{Z^{\beta}(\delta(t))Z^{\beta}(\delta(\sigma(t)))}\\
&\leqslant -\xi(t)\overline{p}(t)+\frac{\xi^{\Delta}(t)}{\xi(\sigma(t))}w(\sigma(t))\\
&\quad-\frac{r(\sigma(t))\big(Z^{\Delta}(\sigma(t))\big)^{\alpha}\xi(t)\beta Z^{\Delta}(\delta(t))\delta^{\Delta}(t)}{Z^{\beta+1}(\delta(\sigma(t)))}\\
&\leqslant -\xi(t)\overline{p}(t)+\frac{\xi^{\Delta}(t)}{\xi(\sigma(t))}w(\sigma(t))\\
&\quad-\frac{\beta\xi(t)r(\sigma(t))\big(Z^{\Delta}(\sigma(t))\big)^{\alpha+1}\delta^{\Delta}(t)}{Z^{\beta+1}(\delta(\sigma(t)))}\\
&\quad\times\left(\frac{r(\sigma(t))}{r(\delta(t))}\right)^{1/\alpha}\\
&=-\xi(t)\overline{p}(t)+\frac{\xi^{\Delta}(t)}{\xi(\sigma(t))}w(\sigma(t))\\
&\quad-\frac{\beta\xi(t)\delta^{\Delta}(t)}{(\xi(\sigma(t)))^{1+1/\alpha}(Z(\delta(\sigma(t))))^{(\alpha-\beta)/\alpha}(r(\delta(t)))^{1/\alpha}}\\
&\quad\times w^{(\alpha+1)/\alpha}(\sigma(t))\\
&=-\xi(t)\overline{p}(t)+\frac{\xi^{\Delta}(t)}{\xi(\sigma(t))}w(\sigma(t))\\
&\quad-\frac{\beta\xi(t)\delta^{\Delta}(t)}{(\xi(\sigma(t)))^{1+1/\alpha}(Z(\sigma(t)))^{(\alpha-\beta)/\alpha}(r(\delta(t)))^{1/\alpha}}\\
&\quad\times w^{(\alpha+1)/\alpha}(\sigma(t)).
\end{aligned}\tag{26}$$

Case 2. Let $0<\beta<1$. By (15), Lemmas 1 and 2, we get

$$\begin{aligned}
&\big(Z^{\beta}(\delta(t))\big)^{\Delta}\\
&\quad=\beta\left\{\int_0^1[(1-h)Z(\delta(t))+hZ(\delta(\sigma(t)))]^{\beta-1}\mathrm{d}h\right\}\\
&\qquad\times(Z(\delta(t)))^{\Delta}\\
&\quad\geqslant\beta(Z(\delta(\sigma(t))))^{\beta-1}Z^{\Delta}(\delta(t))\delta^{\Delta}(t).
\end{aligned}\tag{27}$$

From (H_4), (H_5), (20), (23)–(25), and the fact that $Z(t)$ is nondecreasing, we have

$$\begin{aligned}
&w^{\Delta}(t)\\
&\leqslant -\xi(t)\overline{p}(t)+\frac{\xi^{\Delta}(t)}{\xi(\sigma(t))}w(\sigma(t))\\
&\quad-\frac{r(\sigma(t))\big(Z^{\Delta}(\sigma(t))\big)^{\alpha}\xi(t)\beta(Z(\delta(\sigma(t))))^{\beta-1}Z^{\Delta}(\delta(t))\delta^{\Delta}(t)}{Z^{\beta}(\delta(t))Z^{\beta}(\delta(\sigma(t)))}\\
&=-\xi(t)\overline{p}(t)+\frac{\xi^{\Delta}(t)}{\xi(\sigma(t))}w(\sigma(t))\\
&\quad-\frac{r(\sigma(t))\big(Z^{\Delta}(\sigma(t))\big)^{\alpha}\xi(t)\beta Z^{\Delta}(\delta(t))\delta^{\Delta}(t)}{Z^{\beta+1}(\delta(\sigma(t)))}\\
&\leqslant -\xi(t)\overline{p}(t)+\frac{\xi^{\Delta}(t)}{\xi(\sigma(t))}w(\sigma(t))\\
&\quad-\frac{\beta\xi(t)r(\sigma(t))\big(Z^{\Delta}(\sigma(t))\big)^{\alpha+1}\delta^{\Delta}(t)}{Z^{\beta+1}(\delta(\sigma(t)))}\left(\frac{r(\sigma(t))}{r(\delta(t))}\right)^{1/\alpha}\\
&=-\xi(t)\overline{p}(t)+\frac{\xi^{\Delta}(t)}{\xi(\sigma(t))}w(\sigma(t))\\
&\quad-\frac{\beta\xi(t)\delta^{\Delta}(t)}{(\xi(\sigma(t)))^{1+1/\alpha}(Z(\delta(\sigma(t))))^{(\alpha-\beta)/\alpha}(r(\delta(t)))^{1/\alpha}}\\
&\quad\times w^{(\alpha+1)/\alpha}(\sigma(t))\\
&=-\xi(t)\overline{p}(t)+\frac{\xi^{\Delta}(t)}{\xi(\sigma(t))}w(\sigma(t))\\
&\quad-\frac{\beta\xi(t)\delta^{\Delta}(t)}{(\xi(\sigma(t)))^{1+1/\alpha}(Z(\sigma(t)))^{(\alpha-\beta)/\alpha}(r(\delta(t)))^{1/\alpha}}\\
&\quad\times w^{(\alpha+1)/\alpha}(\sigma(t)).
\end{aligned}\tag{28}$$

Therefore, for $\beta>0$, from (26) and (28), we get

$$\begin{aligned}
w^{\Delta}(t)\leqslant&-\xi(t)\overline{p}(t)+\frac{\xi^{\Delta}(t)}{\xi(\sigma(t))}w(\sigma(t))\\
&-\frac{\beta\xi(t)\delta^{\Delta}(t)}{(\xi(\sigma(t)))^{1+1/\alpha}(Z(\sigma(t)))^{(\alpha-\beta)/\alpha}(r(\delta(t)))^{1/\alpha}}\\
&\times w^{(\alpha+1)/\alpha}(\sigma(t)).
\end{aligned}\tag{29}$$

From (14) and (15), there exists a constant $M_1>0$ such that

$$r(t)\big(Z^{\Delta}(t)\big)^{\alpha}\leqslant M_1,\quad t\geqslant T_1,\tag{30}$$

that is

$$Z^{\Delta}(t)\leqslant\left(\frac{M_1}{r(t)}\right)^{1/\alpha},\quad t\geqslant T_1,\tag{31}$$

integrating the above inequality from T_1 to t, we have

$$Z(t) \leqslant Z(T_1) + M_1^{1/\alpha}(R(t) - R(T_1)). \tag{32}$$

Thus, there exist a constant $M_2 > 0$, and $T_2 \geqslant T_1$ such that

$$Z(t) \leqslant M_2 R(t), \quad t \geqslant T_2, \tag{33}$$

so we have

$$\begin{aligned} Z^{(\alpha-\beta)/\alpha}(\sigma(t)) &\leqslant M_2^{(\alpha-\beta)/\alpha}(R(\sigma(t)))^{(\alpha-\beta)/\alpha} \\ &= M_3(R(\sigma(t)))^{(\alpha-\beta)/\alpha}, \quad t \geqslant T_2, \end{aligned} \tag{34}$$

where $M_3 = M_2^{(\alpha-\beta)/\alpha}$.

From (29) and (34), we obtain

$$\begin{aligned} w^{\Delta}(t) \leqslant{} & -\xi(t)\overline{p}(t) + \frac{\xi^{\Delta}(t)}{\xi(\sigma(t))}w(\sigma(t)) \\ & - \frac{\beta\xi(t)\delta^{\Delta}(t)}{(\xi(\sigma(t)))^{1+1/\alpha}M_3(R(\sigma(t)))^{(\alpha-\beta)/\alpha}(r(\delta(t)))^{1/\alpha}} \\ & \times w^{(\alpha+1)/\alpha}(\sigma(t)), \quad t \geqslant T_2. \end{aligned} \tag{35}$$

Let

$$\Psi(t) = \frac{\beta\xi(t)\delta^{\Delta}(t)}{(\xi(\sigma(t)))^{1+1/\alpha}M_3(R(\sigma(t)))^{(\alpha-\beta)/\alpha}(r(\delta(t)))^{1/\alpha}}; \tag{36}$$

then $\Psi(t) > 0$. So from (35) and (36) we get

$$\begin{aligned} w^{\Delta}(t) \leqslant{} & -\xi(t)\overline{p}(t) + \frac{\xi^{\Delta}(t)}{\xi(\sigma(t))}w(\sigma(t)) \\ & - \Psi(t)w^{(\alpha+1)/\alpha}(\sigma(t)) \\ \leqslant{} & -\xi(t)\overline{p}(t) + \frac{\left(\xi^{\Delta}(t)\right)_+}{\xi(\sigma(t))}w(\sigma(t)) \\ & - \Psi(t)w^{(\alpha+1)/\alpha}(\sigma(t)), \end{aligned} \tag{37}$$

where $(\xi^{\Delta}(t))_+ := \max\{\xi^{\Delta}(t), 0\}$.

Taking $a = (\xi^{\Delta}(t))_+/\xi(\sigma(t))$, $b = \Psi(t)$, by Lemma 3 and (37), we obtain

$$\begin{aligned} w^{\Delta}(t) \leqslant{} & -\xi(t)\overline{p}(t) + \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}\Psi^{\alpha}(t)}\left(\frac{(\xi^{\Delta}(t))_+}{\xi(\sigma(t))}\right)^{\alpha+1} \\ ={} & -\left[\xi(t)\overline{p}(t) - \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}\Psi^{\alpha}(t)}\left(\frac{(\xi^{\Delta}(t))_+}{\xi(\sigma(t))}\right)^{\alpha+1}\right] \\ ={} & -\left[\xi(t)\overline{p}(t) - \frac{\alpha^{\alpha}M_3^{\alpha}(R(\sigma(t)))^{\alpha-\beta}r(\delta(t))\left(\left(\xi^{\Delta}(t)\right)_+\right)^{\alpha+1}}{(\alpha+1)^{\alpha+1}\beta^{\alpha}\xi^{\alpha}(t)\left(\delta^{\Delta}(t)\right)^{\alpha}}\right] \\ ={} & -\left[\xi(t)\overline{p}(t) - \frac{\alpha^{\alpha}M_4(R(\sigma(t)))^{\alpha-\beta}r(\delta(t))\left(\left(\xi^{\Delta}(t)\right)_+\right)^{\alpha+1}}{(\alpha+1)^{\alpha+1}\beta^{\alpha}\xi^{\alpha}(t)\left(\delta^{\Delta}(t)\right)^{\alpha}}\right], \end{aligned} \tag{38}$$

where $M_4 = M_3^{\alpha}$.

Integrating the above inequality (38) from T_2 to t, we have

$$\begin{aligned} w(t) \leqslant{} & w(T_2) \\ & - \int_{T_2}^{t}\Bigg(\xi(s)\overline{p}(s) - \Big(\alpha^{\alpha}M_4(R(\sigma(s)))^{\alpha-\beta}r(\delta(s)) \\ & \times \left(\left(\xi^{\Delta}(s)\right)_+\right)^{\alpha+1}\Big) \\ & \times \left((\alpha+1)^{\alpha+1}\beta^{\alpha}\xi^{\alpha}(s)\left(\delta^{\Delta}(s)\right)^{\alpha}\right)^{-1}\Bigg)\Delta s \\ \leqslant{} & w(T_2) + \int_{t_0}^{T_2}\xi(s)\overline{p}(s)\Delta s \end{aligned}$$

$$-\int_{t_0}^{t}\left(\xi(s)\overline{p}(s)-\left(\alpha^{\alpha}M_4(R(\sigma(s)))^{\alpha-\beta}r(\delta(s))\times\left(\left(\xi^{\Delta}(s)\right)_+\right)^{\alpha+1}\right)\times\left((\alpha+1)^{\alpha+1}\beta^{\alpha}\xi^{\alpha}(s)\left(\delta^{\Delta}(s)\right)^{\alpha}\right)^{-1}\right)\Delta s. \tag{39}$$

Since $w(t)>0$ for $t>T_2$, we have

$$\begin{aligned}\int_{t_0}^{t}&\left(\xi(s)\overline{p}(s)-\frac{\alpha^{\alpha}M_4(R(\sigma(s)))^{\alpha-\beta}r(\delta(s))\left(\left(\xi^{\Delta}(s)\right)_+\right)^{\alpha+1}}{(\alpha+1)^{\alpha+1}\beta^{\alpha}\xi^{\alpha}(s)\left(\delta^{\Delta}(s)\right)^{\alpha}}\right)\Delta s\\&\leqslant w(T_2)+\int_{t_0}^{T_2}\xi(s)\overline{p}(s)\Delta s-w(t)\\&\leqslant w(T_2)+\int_{t_0}^{T_2}\xi(s)\overline{p}(s)\Delta s,\end{aligned} \tag{40}$$

which contradicts (12). This completes the proof of Theorem 4. □

Next, we use the general weighted functions from the class $\mathcal{F}$ which will be extensively used in the sequel.

Letting $\mathbb{D}\equiv\{(t,s)\in\mathbb{T}\times\mathbb{T}:t\geqslant s\geqslant t_0\}$, we say that a continuous function $H(t,s)\in C_{\mathrm{rd}}(\mathbb{D},\mathbb{R})$ belongs to the class $\mathcal{F}$ if

(i) $H(t,t)=0$ for $t\geqslant t_0$ and $H(t,s)>0$ for $t>s\geqslant t_0$,

(ii) $H(t,s)$ has a nonpositive right-dense continuous Δ-partial derivative $H^{\Delta_s}(t,s)$ with respect to the second variable.

Theorem 5. *Assume that (H_1)–(H_6) and (7) hold. If there exist a function $H(t,s)\in\mathcal{F}$ and a function $\xi(t)\in C^1_{\mathrm{rd}}(\mathbb{T},(0,\infty))$ such that for any positive number M,*

$$\overline{\lim_{t\to\infty}}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)\xi(s)\overline{p}(s)-\widetilde{U}(t,s)\right]\Delta s=\infty, \tag{41}$$

where

$$\overline{p}(s)=q(s)\left[1-p(\delta(s))\right]^{\beta}, \tag{42}$$

$$\widetilde{U}(t,s)=\frac{\alpha^{\alpha}\left(\phi_+(t,s)\right)^{\alpha+1}(\xi(\sigma(s)))^{\alpha+1}M(R(\sigma(s)))^{\alpha-\beta}r(\delta(s))}{(\alpha+1)^{\alpha+1}\beta^{\alpha}(H(t,s))^{\alpha}\xi^{\alpha}(s)\left(\delta^{\Delta}(s)\right)^{\alpha}}, \tag{43}$$

$$\phi_+(t,s):=\max\left\{H^{\Delta_s}(t,s)+\frac{H(t,s)\left(\xi^{\Delta}(s)\right)_+}{\xi(\sigma(s))},0\right\}, \tag{44}$$

$$\left(\xi^{\Delta}(s)\right)_+:=\max\left\{\xi^{\Delta}(s),0\right\}, \tag{45}$$

then (1) is oscillatory.

Proof. We proceed as in the proof of Theorem 4 to have (37). From (37) we obtain

$$\xi(t)\overline{p}(t)\leqslant-w^{\Delta}(t)+\frac{\left(\xi^{\Delta}(t)\right)_+}{\xi(\sigma(t))}w(\sigma(t))-\Psi(t)w^{(\alpha+1)/\alpha}(\sigma(t)),\quad t\geqslant T_2. \tag{46}$$

Multiplying (46) (with t replaced by s) by $H(t,s)$, integrating it with respect to s from T_2 to t for $t>T_2$, using integration by parts and (i)-(ii), we get

$$\begin{aligned}\int_{T_2}^{t}&H(t,s)\xi(s)\overline{p}(s)\Delta s\\&\leqslant-\int_{T_2}^{t}H(t,s)w^{\Delta}(s)\Delta s+\int_{T_2}^{t}\frac{H(t,s)\left(\xi^{\Delta}(s)\right)_+}{\xi(\sigma(s))}w(\sigma(s))\Delta s-\int_{T_2}^{t}H(t,s)\Psi(s)w^{(\alpha+1)/\alpha}(\sigma(s))\Delta s\\&=H(t,T_2)w(T_2)+\int_{T_2}^{t}H^{\Delta_s}(t,s)w(\sigma(s))\Delta s+\int_{T_2}^{t}\frac{H(t,s)\left(\xi^{\Delta}(s)\right)_+}{\xi(\sigma(s))}w(\sigma(s))\Delta s-\int_{T_2}^{t}H(t,s)\Psi(s)w^{(\alpha+1)/\alpha}(\sigma(s))\Delta s\\&=H(t,T_2)w(T_2)+\int_{T_2}^{t}\left(H^{\Delta_s}(t,s)+\frac{H(t,s)\left(\xi^{\Delta}(s)\right)_+}{\xi(\sigma(s))}\right)w(\sigma(s))\Delta s-\int_{T_2}^{t}H(t,s)\Psi(s)w^{(\alpha+1)/\alpha}(\sigma(s))\Delta s\\&=H(t,T_2)w(T_2)+\int_{T_2}^{t}\left[\left(H^{\Delta_s}(t,s)+\frac{H(t,s)\left(\xi^{\Delta}(s)\right)_+}{\xi(\sigma(s))}\right)w(\sigma(s))\right.\end{aligned}$$

$$
\begin{aligned}
&\left. -H(t,s)\Psi(s)w^{(\alpha+1)/\alpha}(\sigma(s)) \right]\Delta s \\
&\leqslant H(t,T_2)w(T_2) \\
&\quad + \int_{T_2}^{t}\left[\phi_{+}(t,s)w(\sigma(s)) \right.\\
&\quad \left. -H(t,s)\Psi(s)w^{(\alpha+1)/\alpha}(\sigma(s)) \right]\Delta s,
\end{aligned}
\tag{47}
$$

where $\phi_{+}(t,s)$ is defined as in (44).

Taking $a = \phi_{+}(t,s)$, $b = H(t,s)\Psi(s)$, by Lemma 3 and (47), we obtain

$$
\begin{aligned}
&\int_{T_2}^{t} H(t,s)\xi(s)\overline{p}(s)\Delta s \\
&\leqslant H(t,T_2)w(T_2) \\
&\quad + \int_{T_2}^{t}\left[\left(\alpha^{\alpha}(\phi_{+}(t,s))^{\alpha+1}(\xi(\sigma(s)))^{\alpha+1} \times M_3^{\alpha}(R(\sigma(s)))^{\alpha-\beta}r(\delta(s))\right)\right.\\
&\quad \left.\times\left((\alpha+1)^{\alpha+1}\beta^{\alpha}(H(t,s))^{\alpha} \times\xi^{\alpha}(s)\left(\delta^{\Delta}(s)\right)^{\alpha}\right)^{-1}\right]\Delta s \\
&\leqslant H(t,T_2)w(T_2) \\
&\quad + \int_{T_2}^{t}\left[\left(\alpha^{\alpha}(\phi_{+}(t,s))^{\alpha+1}(\xi(\sigma(s)))^{\alpha+1} \times M_4(R(\sigma(s)))^{\alpha-\beta}r(\delta(s))\right)\right.\\
&\quad \left.\times\left((\alpha+1)^{\alpha+1}\beta^{\alpha}(H(t,s))^{\alpha} \times\xi^{\alpha}(s)\left(\delta^{\Delta}(s)\right)^{\alpha}\right)^{-1}\right]\Delta s \\
&\leqslant H(t,t_0)w(T_2) + \int_{T_2}^{t} U(t,s)\Delta s,
\end{aligned}
\tag{48}
$$

where $M_4 = M_3^{\alpha}$,

$$
U(t,s) = \frac{\alpha^{\alpha}(\phi_{+}(t,s))^{\alpha+1}(\xi(\sigma(s)))^{\alpha+1}M_4(R(\sigma(s)))^{\alpha-\beta}r(\delta(s))}{(\alpha+1)^{\alpha+1}\beta^{\alpha}(H(t,s))^{\alpha}\xi^{\alpha}(s)\left(\delta^{\Delta}(s)\right)^{\alpha}}.
\tag{49}
$$

Then it follows that

$$
\frac{1}{H(t,t_0)}\int_{T_2}^{t}\left[H(t,s)\xi(s)\overline{p}(s) - U(t,s)\right]\Delta s \leqslant w(T_2).
\tag{50}
$$

Thus we get

$$
\begin{aligned}
&\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)\xi(s)\overline{p}(s) - U(t,s)\right]\Delta s \\
&= \frac{1}{H(t,t_0)}\left(\int_{t_0}^{T_2} + \int_{T_2}^{t}\right)\left[H(t,s)\xi(s)\overline{p}(s) - U(t,s)\right]\Delta s \\
&\leqslant w(T_2) + \frac{1}{H(t,t_0)}\int_{t_0}^{T_2}\left[H(t,s)\xi(s)\overline{p}(s) - U(t,s)\right]\Delta s \\
&\leqslant w(T_2) + \int_{t_0}^{T_2}\left[\frac{H(t,s)}{H(t,t_0)}\xi(s)\overline{p}(s) - \frac{U(t,s)}{H(t,t_0)}\right]\Delta s \\
&\leqslant w(T_2) + \int_{t_0}^{T_2}\xi(s)\overline{p}(s)\Delta s.
\end{aligned}
\tag{51}
$$

Then

$$
\overline{\lim_{t\to\infty}}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)\xi(s)\overline{p}(s) - U(t,s)\right]\Delta s < \infty,
\tag{52}
$$

which contradicts (41). This completes the proof of Theorem 5. □

Theorem 6. *Assume that* (H_1)–(H_6) *and* (7) *hold and* $\beta \geqslant 1$. *Furthermore, assume that* $r^{\Delta}(t) \geqslant 0$. *If there exists a function* $\xi(t) \in C_{\mathrm{rd}}^{1}(\mathbb{T},(0,\infty))$ *such that for any positive number* M,

$$
\overline{\lim_{t\to\infty}}\int_{t_0}^{t}\left(\xi(s)\overline{p}(s) - Q(s)\right)\Delta s = \infty,
\tag{53}
$$

where

$$
\begin{aligned}
\overline{p}(s) &= q(s)\left[1 - p(\delta(s))\right]^{\beta}, \\
Q(s) &= \frac{\left(\xi^{\Delta}(s)\right)^2(r(\sigma(s)))^{(\alpha-\beta)/\alpha}(r(\delta(s)))^{\beta/\alpha}}{4\beta\xi(s)(\delta(s)/2)^{\beta-1}\delta^{\Delta}(s)M^{\alpha-\beta}},
\end{aligned}
\tag{54}
$$

then (1) *is oscillatory.*

Proof. We proceed as in the proof of Theorem 4 to have (24). On the other hand, from (22) and (H_3), we deduce

$$
\left(r(t)\left(Z^{\Delta}(t)\right)^{\alpha}\right)^{\Delta} \leqslant 0, \quad t \geqslant T_1,
\tag{55}
$$

and from $r^{\Delta}(t) \geqslant 0$ for $t \geqslant t_0$, we can get $Z^{\Delta}(t)$ is nonincreasing. Hence, we have

$$
Z(t) - Z(T_1) = \int_{T_1}^{t} Z^{\Delta}(s)\Delta s \geqslant (t - T_1)Z^{\Delta}(t),
\tag{56}
$$

which implies

$$Z(t) \geqslant \frac{t}{2} Z^{\Delta}(t), \quad \text{for } t \geqslant T_2 > 2T_1. \tag{57}$$

Choosing $T_3 \geqslant T_2$ such that $\delta(t) \geqslant T_2$ for $t \geqslant T_3$, we get

$$Z(\delta(t)) \geqslant \frac{\delta(t)}{2} Z^{\Delta}(\delta(t)), \quad \text{for } t \geqslant T_3. \tag{58}$$

From (H_6), (15), (20), (24), (25), (58), and as $Z^{\Delta}(t)$ is nonincreasing, we obtain

$$\begin{aligned}
w^{\Delta}(t) \leqslant & -\xi(t)\overline{p}(t) + \frac{\xi^{\Delta}(t)}{\xi(\sigma(t))} w(\sigma(t)) \\
& - \Big(r(\sigma(t)) \left(Z^{\Delta}(\sigma(t))\right)^{\alpha} \xi(t) \beta (Z(\delta(t)))^{\beta-1} \\
& \times Z^{\Delta}(\delta(t)) \delta^{\Delta}(t) \Big) \left(Z^{2\beta}(\delta(\sigma(t)))\right)^{-1} \\
\leqslant & -\xi(t)\overline{p}(t) + \frac{\xi^{\Delta}(t)}{\xi(\sigma(t))} w(\sigma(t)) \\
& - \Big(r(\sigma(t)) \left(Z^{\Delta}(\sigma(t))\right)^{\alpha} \xi(t) \\
& \times \beta \left((\delta(t)/2) Z^{\Delta}(\delta(t))\right)^{\beta-1} Z^{\Delta}(\delta(t)) \delta^{\Delta}(t) \Big) \\
& \times \left(Z^{2\beta}(\delta(\sigma(t)))\right)^{-1} \\
\leqslant & -\xi(t)\overline{p}(t) + \frac{\xi^{\Delta}(t)}{\xi(\sigma(t))} w(\sigma(t)) \\
& - \left(\beta \xi(t) r(\sigma(t)) \left(Z^{\Delta}(\sigma(t))\right)^{\alpha+\beta} (\delta(t)/2)^{\beta-1} \delta^{\Delta}(t)\right) \\
& \times \left(Z^{2\beta}(\delta(\sigma(t)))\right)^{-1} \left(\frac{r(\sigma(t))}{r(\delta(t))}\right)^{\beta/\alpha} \\
= & -\xi(t)\overline{p}(t) + \frac{\xi^{\Delta}(t)}{\xi(\sigma(t))} w(\sigma(t)) \\
& - \left(\beta \xi(t) (\delta(t)/2)^{\beta-1} \delta^{\Delta}(t)\right) \\
& \times \Bigg(\xi^{2}(\sigma(t)) (r(\sigma(t)))^{(\alpha-\beta)/\alpha} \left(Z^{\Delta}(\sigma(t))\right)^{\alpha-\beta} \\
& \times (r(\delta(t)))^{\beta/\alpha} \Bigg)^{-1} w^{2}(\sigma(t)) \\
\leqslant & -\xi(t)\overline{p}(t) + \frac{\xi^{\Delta}(t)}{\xi(\sigma(t))} w(\sigma(t)) \\
& - \left(\beta \xi(t) (\delta(t)/2)^{\beta-1} \delta^{\Delta}(t)\right) \\
& \times \Bigg(\xi^{2}(\sigma(t)) (r(\sigma(t)))^{(\alpha-\beta)/\alpha} \left(Z^{\Delta}(t)\right)^{\alpha-\beta} \\
& \times (r(\delta(t)))^{\beta/\alpha} \Bigg)^{-1} w^{2}(\sigma(t)).
\end{aligned} \tag{59}$$

Now, from the fact that $Z^{\Delta}(t)$ is nonnegative and nonincreasing, there exists a $T_4 > T_3$ sufficiently large such that

$$Z^{\Delta}(t) \leqslant \frac{1}{M}, \quad t \geqslant T_4, \tag{60}$$

holds for some positive constant M and therefore

$$\left(Z^{\Delta}(t)\right)^{\alpha-\beta} \leqslant \left(\frac{1}{M}\right)^{\alpha-\beta}, \quad t \geqslant T_4. \tag{61}$$

Combining (59) and (61), we obtain that

$$\begin{aligned}
w^{\Delta}(t) \leqslant & -\xi(t)\overline{p}(t) + \frac{\xi^{\Delta}(t)}{\xi(\sigma(t))} w(\sigma(t)) \\
& - \frac{\beta \xi(t) (\delta(t)/2)^{\beta-1} \delta^{\Delta}(t) M^{\alpha-\beta}}{\xi^{2}(\sigma(t)) (r(\sigma(t)))^{(\alpha-\beta)/\alpha} (r(\delta(t)))^{\beta/\alpha}} \\
& \times w^{2}(\sigma(t)), \quad t \geqslant T_4.
\end{aligned} \tag{62}$$

Letting

$$\Phi(t) = \frac{\beta \xi(t) (\delta(t)/2)^{\beta-1} \delta^{\Delta}(t) M^{\alpha-\beta}}{\xi^{2}(\sigma(t)) (r(\sigma(t)))^{(\alpha-\beta)/\alpha} (r(\delta(t)))^{\beta/\alpha}}, \tag{63}$$

then $\Phi(t) \geqslant 0$. So

$$\begin{aligned}
w^{\Delta}(t) \leqslant & -\xi(t)\overline{p}(t) + \frac{\xi^{\Delta}(t)}{\xi(\sigma(t))} w(\sigma(t)) - \Phi(t) w^{2}(\sigma(t)) \\
= & -\xi(t)\overline{p}(t) + \frac{1}{4\Phi(t)} \frac{\left(\xi^{\Delta}(t)\right)^{2}}{\xi^{2}(\sigma(t))} \\
& - \left[\sqrt{\Phi(t)} w(\sigma(t)) - \frac{1}{2\sqrt{\Phi(t)}} \frac{\xi^{\Delta}(t)}{\xi(\sigma(t))}\right]^{2} \\
\leqslant & -\xi(t)\overline{p}(t) + \frac{1}{4\Phi(t)} \frac{\left(\xi^{\Delta}(t)\right)^{2}}{\xi^{2}(\sigma(t))} \\
= & -\Bigg[\xi(t)\overline{p}(t) \\
& - \frac{\left(\xi^{\Delta}(t)\right)^{2} (r(\sigma(t)))^{(\alpha-\beta)/\alpha} (r(\delta(t)))^{\beta/\alpha}}{4\beta \xi(t) (\delta(t)/2)^{\beta-1} \delta^{\Delta}(t) M^{\alpha-\beta}}\Bigg].
\end{aligned} \tag{64}$$

Integrating the above inequality from T_4 to t, we have

$$\begin{aligned} w(t) \leqslant{} & w(T_4) \\ & - \int_{T_4}^{t} \Bigg(\xi(s)\,\overline{p}(s) \\ & \qquad - \left(\left(\xi^{\Delta}(s)\right)^2 (r(\sigma(s)))^{(\alpha-\beta)/\alpha} (r(\delta(s)))^{\beta/\alpha} \right) \\ & \qquad \times \left(4\beta\xi(s)(\delta(s)/2)^{\beta-1}\delta^{\Delta}(s) M^{\alpha-\beta} \right)^{-1} \Bigg) \Delta s \\ \leqslant{} & w(T_4) + \int_{t_0}^{T_4} \xi(s)\,\overline{p}(s)\,\Delta s \\ & - \int_{t_0}^{t} \Bigg(\xi(s)\,\overline{p}(s) \\ & \qquad - \left(\left(\xi^{\Delta}(s)\right)^2 (r(\sigma(s)))^{(\alpha-\beta)/\alpha} (r(\delta(s)))^{\beta/\alpha} \right) \\ & \qquad \times \left(4\beta\xi(s)(\delta(s)/2)^{\beta-1}\delta^{\Delta}(s) M^{\alpha-\beta} \right)^{-1} \Bigg) \Delta s. \end{aligned} \tag{65}$$

Since $w(t) > 0$ for $t > T_4$, we have

$$\begin{aligned} & \int_{t_0}^{t} \left(\xi(s)\,\overline{p}(s) - \frac{\left(\xi^{\Delta}(s)\right)^2 (r(\sigma(s)))^{(\alpha-\beta)/\alpha} (r(\delta(s)))^{\beta/\alpha}}{4\beta\xi(s)(\delta(s)/2)^{\beta-1}\delta^{\Delta}(s) M^{\alpha-\beta}} \right) \Delta s \\ & \quad \leqslant w(T_4) + \int_{t_0}^{T_4} \xi(s)\,\overline{p}(s)\,\Delta s - w(t) \\ & \qquad < w(T_4) + \int_{t_0}^{T_4} \xi(s)\,\overline{p}(s)\,\Delta s. \end{aligned} \tag{66}$$

which contradicts (53). This completes the proof of Theorem 6. □

Theorem 7. *Assume that (H_1)–(H_6) and (7) hold and $\beta \geqslant 1$. Furthermore, assume that $r^{\Delta}(t) \geqslant 0$. If there exist a function $H(t,s) \in \digamma$ and a function $\xi(t) \in C^1_{\mathrm{rd}}(\mathbb{T},(0,\infty))$ such that*

$$H^{\Delta_s}(t,s) + \frac{H(t,s)\,\xi^{\Delta}(s)}{\xi(\sigma(s))} \leqslant 0, \quad \text{for } t \geqslant s \geqslant t_0, \tag{67}$$

$$\overline{\lim_{t\to\infty}} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\,\xi(s)\,\overline{p}(s)\,\Delta s = \infty, \tag{68}$$

where

$$\overline{p}(s) = q(s)\left[1 - p(\delta(s))\right]^{\beta}, \tag{69}$$

then (1) *is oscillatory.*

Proof. We proceed as in the proof of Theorem 6 to have (64). From (64) we obtain

$$\begin{aligned} \xi(t)\,\overline{p}(t) \leqslant{} & -w^{\Delta}(t) + \frac{\xi^{\Delta}(t)}{\xi(\sigma(t))} w(\sigma(t)) \\ & - \Phi(t)\,w^2(\sigma(t)), \quad t \geqslant T_4. \end{aligned} \tag{70}$$

Multiplying (70) (with t replaced by s) by $H(t,s)$, integrating it with respect to s from T_4 to t for $t > T_4$, using integration by parts and (i)-(ii), we get

$$\begin{aligned} & \int_{T_4}^{t} H(t,s)\,\xi(s)\,\overline{p}(s)\,\Delta s \\ & \quad \leqslant -\int_{T_4}^{t} H(t,s)\,w^{\Delta}(s)\,\Delta s + \int_{T_4}^{t} \frac{H(t,s)\,\xi^{\Delta}(s)}{\xi(\sigma(s))} w(\sigma(s))\,\Delta s \\ & \qquad - \int_{T_4}^{t} H(t,s)\,\Phi(s)\,w^2(\sigma(s))\,\Delta s \\ & \quad = H(t,T_4)\,w(T_4) + \int_{T_4}^{t} H^{\Delta_s}(t,s)\,w(\sigma(s))\,\Delta s \\ & \qquad + \int_{T_4}^{t} \frac{H(t,s)\,\xi^{\Delta}(s)}{\xi(\sigma(s))} w(\sigma(s))\,\Delta s \\ & \qquad - \int_{T_4}^{t} H(t,s)\,\Phi(s)\,w^2(\sigma(s))\,\Delta s \\ & \quad = H(t,T_4)\,w(T_4) \\ & \qquad + \int_{T_4}^{t} \left(H^{\Delta_s}(t,s) + \frac{H(t,s)\,\xi^{\Delta}(s)}{\xi(\sigma(s))} \right) w(\sigma(s))\,\Delta s \\ & \qquad - \int_{T_4}^{t} H(t,s)\,\Phi(s)\,w^2(\sigma(s))\,\Delta s. \end{aligned} \tag{71}$$

Using (67) in the above inequality (71), we get

$$\int_{T_4}^{t} H(t,s)\,\xi(s)\,\overline{p}(s)\,\Delta s \leqslant H(t,t_0)\,w(T_4). \tag{72}$$

Then it follows that

$$\frac{1}{H(t,t_0)} \int_{T_4}^{t} H(t,s)\,\xi(s)\,\overline{p}(s)\,\Delta s \leqslant w(T_4). \tag{73}$$

Thus we get

$$\begin{aligned}
&\frac{1}{H(t,t_0)}\int_{t_0}^{t} H(t,s)\,\xi(s)\,\overline{p}(s)\,\Delta s\\
&\quad=\frac{1}{H(t,t_0)}\left(\int_{t_0}^{T_4}+\int_{T_4}^{t}\right)H(t,s)\,\xi(s)\,\overline{p}(s)\,\Delta s\\
&\quad\leqslant w(T_4)+\frac{1}{H(t,t_0)}\int_{t_0}^{T_4} H(t,s)\,\xi(s)\,\overline{p}(s)\,\Delta s\\
&\quad\leqslant w(T_4)+\int_{t_0}^{T_4}\frac{H(t,s)}{H(t,t_0)}\xi(s)\,\overline{p}(s)\,\Delta s\\
&\quad\leqslant w(T_4)+\int_{t_0}^{T_4}\xi(s)\,\overline{p}(s)\,\Delta s.
\end{aligned}\tag{74}$$

Then

$$\overline{\lim_{t\to\infty}}\frac{1}{H(t,t_0)}\int_{t_0}^{t} H(t,s)\,\xi(s)\,\overline{p}(s)\,\Delta s<\infty,\tag{75}$$

which contradicts (68). This completes the proof of Theorem 7. □

Theorem 8. *Assume that* (H_1)–(H_6) *and* (7) *hold and* $\beta\geqslant 1$. *Furthermore, assume that* $r^{\Delta}(t)\geqslant 0$. *If there exist a function* $H(t,s)\in\mathcal{F}$ *and a function* $\xi(t)\in C^1_{\mathrm{rd}}(\mathbb{T},(0,\infty))$ *such that for any positive number* M,

$$\begin{aligned}
&\overline{\lim_{t\to\infty}}\frac{1}{H(t,t_0)}\\
&\quad\times\int_{t_0}^{t}\Bigg[H(t,s)\,\xi(s)\,\overline{p}(s)\\
&\qquad-\frac{\left(H^{\Delta_s}(t,s)+H(t,s)\,\xi^{\Delta}(s)/\xi(\sigma(s))\right)^2}{4H(t,s)\,\Phi(s)}\Bigg]\Delta s=\infty,
\end{aligned}\tag{76}$$

where

$$\begin{aligned}
\overline{p}(s)&=q(s)\left[1-p(\delta(s))\right]^{\beta},\\
\Phi(s)&=\frac{\beta\xi(s)\,(\delta(s)/2)^{\beta-1}\delta^{\Delta}(s)\,M^{\alpha-\beta}}{\xi^2(\sigma(s))\,(r(\sigma(s)))^{(\alpha-\beta)/\alpha}(r(\delta(s)))^{\beta/\alpha}},
\end{aligned}\tag{77}$$

then (1) *is oscillatory.*

Proof. We proceed as those in the proof of Theorem 7 to have (71), that is,

$$\begin{aligned}
&\int_{T_4}^{t} H(t,s)\,\xi(s)\,\overline{p}(s)\,\Delta s\\
&\quad\leqslant H(t,T_4)\,w(T_4)\\
&\qquad+\int_{T_4}^{t}\left(H^{\Delta_s}(t,s)+\frac{H(t,s)\,\xi^{\Delta}(s)}{\xi(\sigma(s))}\right)w(\sigma(s))\,\Delta s\\
&\qquad-\int_{T_4}^{t} H(t,s)\,\Phi(s)\,w^2(\sigma(s))\,\Delta s\\
&\quad=H(t,T_4)\,w(T_4)\\
&\qquad+\int_{T_4}^{t}\frac{\left(H^{\Delta_s}(t,s)+H(t,s)\,\xi^{\Delta}(s)/\xi(\sigma(s))\right)^2}{4H(t,s)\,\Phi(s)}\Delta s\\
&\qquad-\int_{T_4}^{t}\Bigg[\frac{H^{\Delta_s}(t,s)+H(t,s)\,\xi^{\Delta}(s)/\xi(\sigma(s))}{2\sqrt{H(t,s)\,\Phi(s)}}\\
&\qquad\qquad-\sqrt{H(t,s)\,\Phi(s)}w(\sigma(s))\Bigg]^2\Delta s\\
&\quad\leqslant H(t,T_4)\,w(T_4)\\
&\qquad+\int_{T_4}^{t}\frac{\left(H^{\Delta_s}(t,s)+H(t,s)\,\xi^{\Delta}(s)/\xi(\sigma(s))\right)^2}{4H(t,s)\,\Phi(s)}\Delta s\\
&\quad\leqslant H(t,t_0)\,w(T_4)\\
&\qquad+\int_{T_4}^{t}\frac{\left(H^{\Delta_s}(t,s)+H(t,s)\,\xi^{\Delta}(s)/\xi(\sigma(s))\right)^2}{4H(t,s)\,\Phi(s)}\Delta s.
\end{aligned}\tag{78}$$

Then it follows that

$$\begin{aligned}
&\frac{1}{H(t,t_0)}\\
&\quad\times\int_{T_4}^{t}\Bigg[H(t,s)\,\xi(s)\,\overline{p}(s)\\
&\qquad-\frac{\left(H^{\Delta_s}(t,s)+H(t,s)\,\xi^{\Delta}(s)/\xi(\sigma(s))\right)^2}{4H(t,s)\,\Phi(s)}\Bigg]\Delta s\\
&\quad\leqslant w(T_4).
\end{aligned}\tag{79}$$

Thus we get

$$\begin{aligned}
&\frac{1}{H(t,t_0)}\int_{t_0}^{t}\Bigg[H(t,s)\,\xi(s)\,\overline{p}(s)\\
&\qquad-\left(H^{\Delta_s}(t,s)+\frac{H(t,s)\,\xi^{\Delta}(s)}{\xi(\sigma(s))}\right)^2
\end{aligned}$$

$$\left.\times\left(4H(t,s)\Phi(s)\right)^{-1}\right]\Delta s$$

$$\begin{aligned}
&= \frac{1}{H(t,t_0)} \\
&\quad \times \left\{\int_{t_0}^{T_4} + \int_{T_4}^{t}\right\}\Bigg[H(t,s)\xi(s)\overline{p}(s) \\
&\qquad - \left(H^{\Delta_s}(t,s) + \frac{H(t,s)\xi^{\Delta}(s)}{\xi(\sigma(s))}\right)^2 \\
&\qquad \times \left(4H(t,s)\Phi(s)\right)^{-1}\Bigg]\Delta s \\
&\leqslant w(T_4) + \frac{1}{H(t,t_0)} \\
&\quad \times \int_{t_0}^{T_4}\Bigg[H(t,s)\xi(s)\overline{p}(s) \\
&\qquad - \left(H^{\Delta_s}(t,s) + \frac{H(t,s)\xi^{\Delta}(s)}{\xi(\sigma(s))}\right)^2 \\
&\qquad \times \left(4H(t,s)\Phi(s)\right)^{-1}\Bigg]\Delta s \\
&\leqslant w(T_4) \\
&\quad + \int_{t_0}^{T_4}\Bigg[\frac{H(t,s)}{H(t,t_0)}\xi(s)\overline{p}(s) \\
&\qquad - \left(H^{\Delta_s}(t,s) + \frac{H(t,s)\xi^{\Delta}(s)}{\xi(\sigma(s))}\right)^2 \\
&\qquad \times \left(4H(t,s)H(t,t_0)\Phi(s)\right)^{-1}\Bigg]\Delta s \\
&\leqslant w(T_4) + \int_{t_0}^{T_4}\xi(s)\overline{p}(s)\Delta s.
\end{aligned} \tag{80}$$

Then

$$\begin{aligned}
&\overline{\lim_{t\to\infty}}\frac{1}{H(t,t_0)} \\
&\quad \times \int_{t_0}^{t}\Bigg[H(t,s)\xi(s)\overline{p}(s) \\
&\qquad -\frac{\left(H^{\Delta_s}(t,s) + H(t,s)\xi^{\Delta}(s)/\xi(\sigma(s))\right)^2}{4H(t,s)\Phi(s)}\Bigg]\Delta s \\
&\quad < \infty,
\end{aligned} \tag{81}$$

which contradicts (76). This completes the proof of Theorem 8. □

The case

$$\lim_{t\to\infty}\int_{t_0}^{t} r^{-1/\alpha}(s)\Delta s < \infty. \tag{82}$$

Theorem 9. *Assume that* (H_1)*–*(H_6) *and* (8) *hold and there exists a* $T_* \in [t_0,\infty)_{\mathbb{T}}$ *such that* $p^{\Delta}(t) \geqslant 0$, $\tau^{\Delta}(t) \geqslant 0$ *for* $t \geqslant T_*$*, and suppose that there exists a function* $\xi(t) \in C^1_{\mathrm{rd}}(\mathbb{T},(0,\infty))$ *such that* (12) *holds for any positive number* M*, and there exists a function* $\psi(t) \in C^1_{\mathrm{rd}}(\mathbb{T},(0,\infty))$ *satisfying* $\psi(t) \geqslant t$, $\psi^{\Delta}(t) > 0$, $\delta(t) \leqslant \tau(\psi(t))$ *for* $t \geqslant T_*$ *such that for any positive number* M *and for every* $T_1 \in [T_*,\infty)_{\mathbb{T}}$

$$\overline{\lim_{t\to\infty}}\int_{T_1}^{t}\left[\widetilde{p}(s)V^{\alpha}(\sigma(s)) - G(s)\right]\Delta s = \infty, \tag{83}$$

where

$$\begin{aligned}
\widetilde{p}(s) &= q(s)\left(\frac{1}{1+p(\psi(s))}\right)^{\beta}, \\
V(s) &= \int_{\psi(s)}^{\infty} r^{-1/\alpha}(t)\Delta t, \\
G(s) & \\
&= \begin{cases} \dfrac{\alpha^{2\alpha+1}r^{-1/\alpha}(\psi(s))\psi^{\Delta}(s)}{(\alpha+1)^{\alpha+1}\beta^{\alpha}M^{\alpha-\beta}V(\sigma(s))}, & \text{if } 0<\alpha<1, \\ \dfrac{\alpha^{2\alpha+1}r^{-1/\alpha}(\psi(s))V^{\alpha^2-1}(s)\psi^{\Delta}(s)}{(\alpha+1)^{\alpha+1}\beta^{\alpha}M^{\alpha-\beta}V^{\alpha^2}(\sigma(s))}, & \text{if } \alpha \geqslant 1, \end{cases}
\end{aligned} \tag{84}$$

then (1) *is oscillatory.*

Proof. Suppose to the contrary that $x(t)$ is an eventually positive solution of (1), then there exists a $T_1 \geqslant T_* \geqslant t_0$ such that $x(t) > 0$, $x(\delta(t)) > 0$, $x(\sigma(t)) > 0$ for all $t \geqslant T_1$, (the case of $x(t)$ is negative and can be considered by the same method). It follows form (H_3) that $Z(t) \geqslant x(t) > 0$ for $t \geqslant T_1$. From (14) it is easy to conclude that there exist two possible cases of the sign of $Z^{\Delta}(t)$.

Case 1. Suppose $Z^{\Delta}(t) \geqslant 0$ for sufficiently large t, then we are back to the case of Theorem 4. Thus the proof of Theorem 4 goes through, and we may get contradiction by (12).

Case 2. Suppose $Z^{\Delta}(t) < 0$ for $t \geqslant T_1$. Define

$$w(t) = \frac{r(t)\left(-Z^{\Delta}(t)\right)^{\alpha-1}Z^{\Delta}(t)}{Z^{\beta}(\psi(t))}, \quad t \geqslant T_1. \tag{85}$$

Then $w(t) < 0$ for $t \geqslant T_1$. From the fact that $Z(t)$ is positive and nonincreasing, we get that

$$Z(\psi(t)) \leqslant \frac{1}{M_0}, \quad t \geqslant T_1, \tag{86}$$

holds for some positive constant M_0.

Noting that $(r(t)(-Z^{\Delta}(t))^{\alpha-1}Z^{\Delta}(t))^{\Delta} \leqslant 0,\ \psi(t) \geqslant t$, so we have

$$Z^{\Delta}(\psi(t)) \leqslant \left(\frac{r(t)}{r(\psi(t))}\right)^{1/\alpha} Z^{\Delta}(t), \tag{87}$$

$$Z^{\Delta}(s) \leqslant \frac{r^{1/\alpha}(t)}{r^{1/\alpha}(s)} Z^{\Delta}(t), \quad s \geqslant t. \tag{88}$$

Integrating the above inequality (88) with respect to s from $\psi(t)$ to v, we have

$$Z(v) \leqslant Z(\psi(t)) + r^{1/\alpha}(t) Z^{\Delta}(t) \int_{\psi(t)}^{v} r^{1/\alpha}(s)\Delta s. \tag{89}$$

Letting $v \to \infty$ in the above inequality, we obtain

$$0 \leqslant Z(\psi(t)) + r^{1/\alpha}(t) Z^{\Delta}(t) V(t). \tag{90}$$

From (86) and (90), we have

$$-\frac{1}{M_0^{\alpha-\beta}} \leqslant w(t) V^{\alpha}(t) \leqslant 0, \quad t \geqslant T_1. \tag{91}$$

If $0 < \beta < 1$. From $Z^{\Delta}(t) < 0$, Lemmas 1 and 2, we have

$$\begin{aligned}
&\left(Z^{\beta}(\psi(t))\right)^{\Delta} \\
&\quad = \beta\left\{\int_0^1 [(1-h)Z(\psi(t)) + hZ(\psi(\sigma(t)))]^{\beta-1} dh\right\} \\
&\qquad \times (Z(\psi(t)))^{\Delta} \\
&\quad \leqslant \beta\left[\int_0^1 Z^{\beta-1}(\psi(t))\, dh\right] Z^{\Delta}(\psi(t))\psi^{\Delta}(t) \\
&\quad = \beta Z^{\beta-1}(\psi(t)) Z^{\Delta}(\psi(t))\psi^{\Delta}(t).
\end{aligned} \tag{92}$$

From (1), (H_6), (85), and (92), we get

$$\begin{aligned}
&w^{\Delta}(t) \\
&\quad = \frac{1}{Z^{\beta}(\psi(t))}\left(r(t)\left(-Z^{\Delta}(t)\right)^{\alpha-1} Z^{\Delta}(t)\right)^{\Delta} \\
&\qquad - \left(r(\sigma(t))\left(-Z^{\Delta}(\sigma(t))\right)^{\alpha-1} Z^{\Delta}(\sigma(t))\left(Z^{\beta}(\psi(t))\right)^{\Delta}\right) \\
&\qquad \times \left(Z^{\beta}(\psi(t)) Z^{\beta}(\psi(\sigma(t)))\right)^{-1} \\
&\quad \leqslant -q(t)\frac{x^{\beta}(\delta(t))}{Z^{\beta}(\psi(t))} \\
&\qquad - \left(r(\sigma(t))\left(-Z^{\Delta}(\sigma(t))\right)^{\alpha-1} Z^{\Delta}(\sigma(t))\beta Z^{\beta-1}\right. \\
&\qquad\qquad \left.\times (\psi(t)) Z^{\Delta}(\psi(t))\psi^{\Delta}(t)\right) \\
&\qquad \times \left(Z^{\beta}(\psi(t)) Z^{\beta}(\psi(\sigma(t)))\right)^{-1} \\
&\quad \leqslant -q(t)\frac{x^{\beta}(\delta(t))}{Z^{\beta}(\psi(t))} \\
&\qquad - \left(r(\sigma(t))\left(-Z^{\Delta}(\sigma(t))\right)^{\alpha-1} Z^{\Delta}(\sigma(t))\right. \\
&\qquad\qquad \left.\times \beta Z^{\Delta}(\psi(t))\psi^{\Delta}(t)\right)\left(Z(\psi(t)) Z^{\beta}(\psi(\sigma(t)))\right)^{-1} \\
&\quad \leqslant -q(t)\frac{x^{\beta}(\delta(t))}{Z^{\beta}(\psi(t))} \\
&\qquad - \left(r(\sigma(t))\left(-Z^{\Delta}(\sigma(t))\right)^{\alpha-1} Z^{\Delta}(\sigma(t))\right. \\
&\qquad\qquad \left.\times \beta Z^{\Delta}(\psi(t))\psi^{\Delta}(t)\right)\left(Z^{\beta+1}(\psi(t))\right)^{-1}.
\end{aligned} \tag{93}$$

If $\beta \geqslant 1$. From $Z^{\Delta}(t) < 0$, Lemmas 1 and 2, we have

$$\begin{aligned}
&\left(Z^{\beta}(\psi(t))\right)^{\Delta} \\
&\quad = \beta\left\{\int_0^1 [(1-h)Z(\psi(t)) + hZ(\psi(\sigma(t)))]^{\beta-1} dh\right\} \\
&\qquad \times (Z(\psi(t)))^{\Delta} \\
&\quad \leqslant \beta\left[\int_0^1 Z^{\beta-1}(\psi(\sigma(t)))\, dh\right] Z^{\Delta}(\psi(t))\psi^{\Delta}(t) \\
&\quad = \beta Z^{\beta-1}(\psi(\sigma(t))) Z^{\Delta}(\psi(t))\psi^{\Delta}(t).
\end{aligned} \tag{94}$$

From (1), (H_6), (85) and (94), we get

$$\begin{aligned}
w^{\Delta}(t) = &\frac{1}{Z^{\beta}(\psi(t))}\left(r(t)\left(-Z^{\Delta}(t)\right)^{\alpha-1} Z^{\Delta}(t)\right)^{\Delta} \\
&- \left(r(\sigma(t))\left(-Z^{\Delta}(\sigma(t))\right)^{\alpha-1}\right. \\
&\qquad \left.\times Z^{\Delta}(\sigma(t))\left(Z^{\beta}(\psi(t))\right)^{\Delta}\right)
\end{aligned}$$

$$\begin{aligned}
&\times \left(Z^{\beta}(\psi(t)) Z^{\beta}(\psi(\sigma(t)))\right)^{-1} \\
&\leqslant -q(t) \frac{x^{\beta}(\delta(t))}{Z^{\beta}(\psi(t))} \\
&\quad - \Big(r(\sigma(t)) \left(-Z^{\Delta}(\sigma(t))\right)^{\alpha-1} Z^{\Delta}(\sigma(t)) \\
&\qquad \times \beta Z^{\Delta}(\psi(t)) \psi^{\Delta}(t) \Big) \\
&\quad \times \left(Z^{\beta}(\psi(t)) Z(\psi(\sigma(t)))\right)^{-1} \\
&\leqslant -q(t) \frac{x^{\beta}(\delta(t))}{Z^{\beta}(\psi(t))} \\
&\quad - \Big(r(\sigma(t)) \left(-Z^{\Delta}(\sigma(t))\right)^{\alpha-1} Z^{\Delta}(\sigma(t)) \\
&\qquad \times \beta Z^{\Delta}(\psi(t)) \psi^{\Delta}(t) \Big) \left(Z^{\beta+1}(\psi(t))\right)^{-1}.
\end{aligned} \tag{95}$$

Therefore, for $\beta > 0$, from (93) and (95), we get

$$\begin{aligned}
&w^{\Delta}(t) \\
&\leqslant -q(t) \frac{x^{\beta}(\delta(t))}{Z^{\beta}(\psi(t))} \\
&\quad - \frac{r(\sigma(t)) \left(-Z^{\Delta}(\sigma(t))\right)^{\alpha-1} Z^{\Delta}(\sigma(t)) \beta Z^{\Delta}(\psi(t)) \psi^{\Delta}(t)}{Z^{\beta+1}(\psi(t))}.
\end{aligned} \tag{96}$$

Noticing that $p^{\Delta}(t) \geqslant 0$ and $\tau^{\Delta}(t) \geqslant 0$, from $Z^{\Delta}(t) = x^{\Delta}(t) + p^{\Delta}(t)x(\tau(t)) + p(\sigma(t))x^{\Delta}(\tau(t))\tau^{\Delta}(t)$, we see that $x^{\Delta}(t) \leqslant 0$ for $t \geqslant T_1$, and from $\delta(t) \leqslant \tau(\psi(t)) \leqslant \psi(t)$ we can get

$$\begin{aligned}
\frac{x^{\beta}(\delta(t))}{Z^{\beta}(\psi(t))} &= \left(\left(\frac{x(\psi(t))}{x(\delta(t))} + p(\psi(t)) \frac{x(\tau(\psi(t)))}{x(\delta(t))} \right)^{-1} \right)^{\beta} \\
&\geqslant \left(\frac{1}{1 + p(\psi(t))} \right)^{\beta}.
\end{aligned} \tag{97}$$

Thus from (86), (87), (96), (97) and the fact that $(r(t)(-Z^{\Delta}(t))^{\alpha-1} Z^{\Delta}(t))^{\Delta} \leqslant 0$, we have

$$\begin{aligned}
w^{\Delta}(t) \leqslant & -\tilde{p}(t) \\
& - \frac{r(\sigma(t))\left(-Z^{\Delta}(\sigma(t))\right)^{\alpha-1} Z^{\Delta}(\sigma(t)) \beta Z^{\Delta}(t) \psi^{\Delta}(t)}{Z^{\beta+1}(\psi(t))} \\
& \times \left(\frac{r(t)}{r(\psi(t))} \right)^{1/\alpha} \\
= & -\tilde{p}(t) - \frac{r(t) \left(-Z^{\Delta}(t)\right)^{\alpha-1} Z^{\Delta}(t) \beta Z^{\Delta}(t) \psi^{\Delta}(t)}{Z^{\beta+1}(\psi(t))} \\
& \times \left(\frac{r(t)}{r(\psi(t))} \right)^{1/\alpha} \\
= & -\tilde{p}(t) - \frac{r(t) \left(-Z^{\Delta}(t)\right)^{\alpha-1} Z^{\Delta}(t) \beta Z^{\Delta}(t) \psi^{\Delta}(t)}{Z^{\beta+1}(\psi(t))} \\
& \times \left(\frac{r(t)}{r(\psi(t))} \right)^{1/\alpha} \\
\leqslant & -\tilde{p}(t) - \frac{\beta M_0^{(\alpha-\beta)/\alpha} \psi^{\Delta}(t)}{r^{1/\alpha}(\psi(t))} (-w(t))^{(\alpha+1)/\alpha},
\end{aligned} \tag{98}$$

where $\tilde{p}(t) = q(t)(1/(1 + p(\psi(t))))^{\beta}$.

That is

$$w^{\Delta}(t) + \tilde{p}(t) + \frac{\beta M_0^{(\alpha-\beta)/\alpha} \psi^{\Delta}(t)}{r^{1/\alpha}(\psi(t))} (-w(t))^{(\alpha+1)/\alpha} \leqslant 0, \quad t \geqslant T_1. \tag{99}$$

Multiplying (99) (with t replaced by s) by $V^{\alpha}(\sigma(s))$, integrating it with respect to s from T_1 to t, we have

$$\begin{aligned}
V^{\alpha}(t) w(t) &- V^{\alpha}(T_1) w(T_1) - \int_{T_1}^{t} \left(V^{\alpha}(s)\right)^{\Delta} w(s) \Delta s \\
&+ \int_{T_1}^{t} \tilde{p}(s) V^{\alpha}(\sigma(s)) \Delta s \\
&+ \int_{T_1}^{t} \frac{\beta M_0^{(\alpha-\beta)/\alpha} V^{\alpha}(\sigma(s)) \psi^{\Delta}(s)}{r^{1/\alpha}(\psi(s))} (-w(s))^{(\alpha+1)/\alpha} \Delta s \leqslant 0.
\end{aligned} \tag{100}$$

Next, we consider the following two cases.

Case (i) (let $0 < \alpha < 1$). From Lemma 2 and $V^{\Delta}(t) = -r^{-1/\alpha}(\psi(t))\psi^{\Delta}(t) < 0$, we have

$$\begin{aligned}\left(V^{\alpha}(t)\right)^{\Delta} &= \alpha\left\{\int_0^1 \left[(1-h)V(t)+hV(\sigma(t))\right]^{\alpha-1}dh\right\}V^{\Delta}(t)\\ &\geqslant \alpha\left[\int_0^1 V^{\alpha-1}(\sigma(t))\,dh\right]V^{\Delta}(t)\\ &= \alpha V^{\alpha-1}(\sigma(t))V^{\Delta}(t).\end{aligned}\tag{101}$$

From (100) and (101), we get

$$\begin{aligned}&V^{\alpha}(t)w(t)-V^{\alpha}(T_1)w(T_1)\\ &\quad-\int_{T_1}^{t}\alpha V^{\alpha-1}(\sigma(s))V^{\Delta}(s)w(s)\Delta s\\ &\quad+\int_{T_1}^{t}\widetilde{p}(s)V^{\alpha}(\sigma(s))\Delta s\\ &\quad+\int_{T_1}^{t}\frac{\beta M_0^{(\alpha-\beta)/\alpha}V^{\alpha}(\sigma(s))\psi^{\Delta}(s)}{r^{1/\alpha}(\psi(s))}(-w(s))^{(\alpha+1)/\alpha}\Delta s\leqslant 0.\end{aligned}\tag{102}$$

That is

$$\begin{aligned}&V^{\alpha}(t)w(t)+\int_{T_1}^{t}\widetilde{p}(s)V^{\alpha}(\sigma(s))\Delta s\\ &\quad-\int_{T_1}^{t}\left[\alpha V^{\alpha-1}(\sigma(s))\left(-V^{\Delta}(s)\right)(-w(s))\Delta s\right.\\ &\qquad\left.-\frac{\beta M_0^{(\alpha-\beta)/\alpha}V^{\alpha}(\sigma(s))\psi^{\Delta}(s)}{r^{1/\alpha}(\psi(s))}(-w(s))^{(\alpha+1)/\alpha}\right]\Delta s\\ &\leqslant V^{\alpha}(T_1)w(T_1).\end{aligned}\tag{103}$$

Taking $a=\alpha V^{\alpha-1}(\sigma(s))(-V^{\Delta}(s))$, $b=\beta M_0^{(\alpha-\beta)/\alpha}V^{\alpha}(\sigma(s))\psi^{\Delta}(s)/r^{1/\alpha}(\psi(s))$, by Lemma 3 and (103), we obtain

$$\begin{aligned}&V^{\alpha}(t)w(t)+\int_{T_1}^{t}\widetilde{p}(s)V^{\alpha}(\sigma(s))\Delta s\\ &\quad-\int_{T_1}^{t}\frac{\alpha^{\alpha}r(\psi(s))\left(\alpha V^{\alpha-1}(\sigma(s))\left(-V^{\Delta}(s)\right)\right)^{\alpha+1}}{(\alpha+1)^{\alpha+1}\left(\beta M_0^{(\alpha-\beta)/\alpha}V^{\alpha}(\sigma(s))\psi^{\Delta}(s)\right)^{\alpha}}\Delta s\\ &\leqslant V^{\alpha}(T_1)w(T_1).\end{aligned}\tag{104}$$

That is

$$\begin{aligned}&V^{\alpha}(t)w(t)\leqslant V^{\alpha}(T_1)w(T_1)\\ &\quad-\int_{T_1}^{t}\left[\widetilde{p}(s)V^{\alpha}(\sigma(s))\right.\\ &\qquad\left.-\frac{\alpha^{2\alpha+1}r^{-1/\alpha}(\psi(s))\psi^{\Delta}(s)}{(\alpha+1)^{\alpha+1}\beta^{\alpha}M_0^{\alpha-\beta}V(\sigma(s))}\right]\Delta s.\end{aligned}\tag{105}$$

By (83), we get a contradiction with (91).

Case (ii) (let $\alpha \geqslant 1$). From Lemma 2 and $V^{\Delta}(t) < 0$, we get

$$\begin{aligned}\left(V^{\alpha}(t)\right)^{\Delta} &= \alpha\left\{\int_0^1 \left[(1-h)V(t)+hV(\sigma(t))\right]^{\alpha-1}dh\right\}V^{\Delta}(t)\\ &\geqslant \alpha\left[\int_0^1 V^{\alpha-1}(t)\,dh\right]V^{\Delta}(t)=\alpha V^{\alpha-1}(t)V^{\Delta}(t).\end{aligned}\tag{106}$$

From (100) and (106), we obtain

$$\begin{aligned}&V^{\alpha}(t)w(t)-V^{\alpha}(T_1)w(T_1)-\int_{T_1}^{t}\alpha V^{\alpha-1}(s)V^{\Delta}(s)w(s)\Delta s\\ &\quad+\int_{T_1}^{t}\widetilde{p}(s)V^{\alpha}(\sigma(s))\Delta s\\ &\quad+\int_{T_1}^{t}\frac{\beta M_0^{(\alpha-\beta)/\alpha}V^{\alpha}(\sigma(s))\psi^{\Delta}(s)}{r^{1/\alpha}(\psi(s))}(-w(s))^{(\alpha+1)/\alpha}\Delta s\leqslant 0.\end{aligned}\tag{107}$$

That is

$$\begin{aligned}&V^{\alpha}(t)w(t)+\int_{T_1}^{t}\widetilde{p}(s)V^{\alpha}(\sigma(s))\Delta s\\ &\quad-\int_{T_1}^{t}\left[\alpha V^{\alpha-1}(s)\left(-V^{\Delta}(s)\right)(-w(s))\Delta s\right.\\ &\qquad\left.-\frac{\beta M_0^{(\alpha-\beta)/\alpha}V^{\alpha}(\sigma(s))\psi^{\Delta}(s)}{r^{1/\alpha}(\psi(s))}(-w(s))^{(\alpha+1)/\alpha}\right]\Delta s\\ &\leqslant V^{\alpha}(T_1)w(T_1).\end{aligned}\tag{108}$$

Taking $a=\alpha V^{\alpha-1}(s)(-V^{\Delta}(s))$, $b=\beta M_0^{(\alpha-\beta)/\alpha}V^{\alpha}(\sigma(s))\psi^{\Delta}(s)/r^{1/\alpha}(\psi(s))$, by Lemma 3 and (108), we obtain

$$\begin{aligned}&V^{\alpha}(t)w(t)+\int_{T_1}^{t}\widetilde{p}(s)V^{\alpha}(\sigma(s))\Delta s\\ &\quad-\int_{T_1}^{t}\frac{\alpha^{\alpha}r(\psi(s))\left(\alpha V^{\alpha-1}(s)\left(-V^{\Delta}(s)\right)\right)^{\alpha+1}}{(\alpha+1)^{\alpha+1}\left(\beta M_0^{(\alpha-\beta)/\alpha}V^{\alpha}(\sigma(s))\psi^{\Delta}(s)\right)^{\alpha}}\Delta s\\ &\leqslant V^{\alpha}(T_1)w(T_1).\end{aligned}\tag{109}$$

That is

$$\begin{aligned}V^{\alpha}(t)\,w(t) &\leqslant V^{\alpha}(T_1)\,w(T_1) - \int_{T_1}^{t}\left[\widetilde{p}(s)\,V^{\alpha}(\sigma(s)) - \frac{\alpha^{2\alpha+1}r^{-1/\alpha}(\psi(s))\,V^{\alpha^2-1}(s)\,\psi^{\Delta}(s)}{(\alpha+1)^{\alpha+1}\beta^{\alpha}M_0^{\alpha-\beta}V^{\alpha^2}(\sigma(s))}\right]\Delta s.\end{aligned} \tag{110}$$

By (83), we get a contradiction with (91). This completes the proof of Theorem 9. □

4. Examples

Example 10. Consider the following dynamic equation:

$$\begin{aligned}&\left[\left|\left(x(t)+\frac{1}{1+t^2}x(\delta(t))\right)^{\Delta}\right|^{\alpha-1}\left(x(t)+\frac{1}{1+t^2}x(\delta(t))\right)^{\Delta}\right]^{\Delta}\\&\quad+\frac{1}{t^2}\left(1+\frac{1}{\delta^2(t)}\right)^{\beta}|x(\delta(t))|^{\beta-1}x(\delta(t))=0,\quad t\in\mathbb{T},\end{aligned} \tag{111}$$

where $\alpha > \beta > 1$ are constants. In (111), $r(t) = 1$, $p(t) = 1/(1+t^2)$, $q(t) = (1/t^2)(1+1/\delta^2(t))^{\beta}$.

If $\mathbb{T} = \overline{q_0^{\mathbb{Z}}} = \{q_0^n : n \in \mathbb{Z}\} \cup \{0\}$, and $\delta(t) = t/q_0$, where $q_0 > 1$ and $q_0 \in \mathbb{R}$, then $\delta^{\Delta}(t) = 1/q_0$. It is easy to get that $\overline{p}(t) = q(t)[1-p(\delta(t))]^{\beta} = 1/t^2$. Choosing $\xi(t) = t$, therefore,

$$\begin{aligned}&\overline{\lim_{t\to\infty}}\int_{t_0}^{t}\left(\xi(s)\,\overline{p}(s) - \frac{\left(\xi^{\Delta}(s)\right)^2(r(\sigma(s)))^{(\alpha-\beta)/\alpha}(r(\delta(s)))^{\beta/\alpha}}{4\beta\xi(s)(\delta(s)/2)^{\beta-1}\delta^{\Delta}(s)M^{\alpha-\beta}}\right)\Delta s\\&\quad=\overline{\lim_{t\to\infty}}\int_{t_0}^{t}\left(\frac{1}{s}-\frac{2^{\beta-1}q_0^{\beta}}{4\beta s^{\beta}M^{(\alpha-\beta)/\alpha}}\right)\Delta s=\infty.\end{aligned} \tag{112}$$

Hence, by Theorem 6, (111) is oscillatory.

Example 11. Consider the following dynamic equation:

$$\begin{aligned}&\left[t^{\alpha}\left|\left(x(t)+\left(1-\frac{1}{1+t^2}\right)x(\delta(t))\right)^{\Delta}\right|^{\alpha-1}\times\left(x(t)+\left(1-\frac{1}{1+t^2}\right)x(\delta(t))\right)^{\Delta}\right]^{\Delta}\\&\quad+\frac{1}{t}\left(1+\frac{1}{\delta^2(t)}\right)^{\beta}|x(\delta(t))|^{\beta-1}x(\delta(t))=0,\quad t\in\mathbb{T},\end{aligned} \tag{113}$$

where $\alpha > \beta > 1$. In (113), $r(t) = t^{\alpha}$, $p(t) = 1 - 1/(1+t^2)$, $q(t) = (1/t)(1+\delta^2(t))^{\beta}$.

If $\mathbb{T} = \overline{q_0^{\mathbb{Z}}} = \{q_0^n : n \in \mathbb{Z}\} \cup \{0\}$, and $\delta(t) = t/q_0$, where $q_0 > 1$ and $q_0 \in \mathbb{R}$, then $\delta^{\Delta}(t) = 1/q_0$. It is easy to get that $\overline{p}(t) = q(t)[1-p(\delta(t))]^{\beta} = 1/t$. Choosing $\xi(t) = 1$, $H(t,s) = t - s$, therefore, $(t-s)^{\Delta_s} = -1$,

$$\begin{aligned}&\overline{\lim_{t\to\infty}}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)\,\xi(s)\,\overline{p}(s)\,\Delta s\\&\quad=\overline{\lim_{t\to\infty}}\frac{1}{t-t_0}\int_{t_0}^{t}(t-s)\frac{1}{s}\Delta s\\&\quad=\overline{\lim_{t\to\infty}}\frac{t}{t-t_0}\cdot\frac{1}{t}\int_{t_0}^{t}\frac{t-s}{s}\Delta s\\&\quad=\infty.\end{aligned} \tag{114}$$

Hence, by Theorem 7, (111) is oscillatory.

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Research Article

On a Five-Dimensional Chaotic System Arising from Double-Diffusive Convection in a Fluid Layer

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A chaotic system arising from double-diffusive convection in a fluid layer is investigated in this paper based on the theory of dynamical systems. A five-dimensional model of chaotic system is obtained using the Galerkin truncated approximation. The results showed that the transition from steady convection to chaos via a Hopf bifurcation produced a limit cycle which may be associated with a homoclinic explosion at a slightly subcritical value of the Rayleigh number.

1. Introduction

The concept of sensitivity on initial conditions where a small difference on initial conditions may produce large variations in the long-term behaviour of the system is pivotal in chaos theory. This behaviour is also known as the "*butterfly effect*" related to work done by Lorenz [1] where it is already described by Henri Poincare in 1890 in the literature in a particular case of the three-body problem. Chaotic behaviour has been studied intensively in various dynamical systems; see, for example, [2–14].

The investigation of free convection in the Rayleigh-Bénard problem is receiving much attention due to its wide application in different disciplines such as biotechnology for the description of the convection with the microorganisms diffusion, in astrophysics for simulation of the influence of the helium diffusion on convective motions in the stars, in oceanography for the investigation of the salinity influence on the convective motions in the seas, and in engineering and geology. Research in double-diffusive convection begins after the work done by sea-going oceanographers in order to measure the fluctuation of temperature and salinity as a function of depth as stated in the paper of Huppert and Turner [15]. Then, Knobloch et al. [16] and Bhattacharjee [17] studied the transition to chaos in double-diffusive convection with stress-free boundary conditions where oscillatory solution exists. They showed that the instability of fluid becomes oscillatory when thermal Rayleigh number is raised and the truncated model suggests that the appearance of chaos is associated with heteroclinic bifurcations.

Two-dimensional thermosolutal convection between free boundaries was studied numerically by Veronis [18]. From their observation, they found that when the solutal Rayleigh number is large enough, the oscillations underwent a bifurcation to asymmetry as thermal Rayleigh number increased and, for the larger values of solutal Rayleigh number, the transition from chaos to steady motion occurs.

Sibgatullin et al. [19] studied some properties of two-dimensional stochastic regimes of double-diffusive convection in a plane layer. Using the Bubnov-Galerkin method, they obtained that, with the growth of Rayleigh numbers of heat and salinity, the structure of one-dimensional curve becomes more irregular and sophisticated. Transition to chaos in double-diffusive Marangoni convection was studied by Li et al. [20]. It was found that the supercritical solution branch takes a quasiperiodicity and phase locking route to chaos while the subcritical branch follows the Ruelle-Takens-Newhouse scenario. The transitions from regular to chaotic dynamics and analysis of the hyper, hyper-hyper, and spatial-temporal chaos using the Lyapunov exponents of continuous mechanical systems have been studied in [21–24]; they found the Sharkovskii windows of periodicity in the systems investigated.

The objective of the present paper is to study the weak turbulence and chaos in double-diffusive convection involving temperature and concentration as the thermal Rayleigh number increases with rigid, no-slip horizontal boundary condition. Applying the truncated Galerkin approximation to the governing equations yields an autonomous system with five ordinary differential equations which can be used to understand low-dimensional dynamics before moving to studying more complex systems.

2. Mathematical Formulation

Consider a two-dimensional layer of fluid of depth H subject to gravity and heated from below as shown in Figure 1. A Cartesian coordinate system is used such that the vertical axis z is collinear with gravity, that is, $\widehat{\mathbf{e}}_{\mathbf{g}} = -\widehat{\mathbf{e}}_{\mathbf{z}}$. The two long walls are maintained at temperatures T_H and T_C and solute concentrations S_H and S_C, respectively. A relationship between density, temperature, and solute concentration is assumed linear and can be presented by the following form $\rho = \rho_0[1 - \alpha(T - T_C) + \alpha_s(S - S_C)]$, where α and α_s are volume expansion coefficients due to variations of thermal and solute concentrations. The Boussinesq approximation is applied for the effects of density variations for the gravity term in momentum equation. Therefore, the set of equations governing the conservation of mass, momentum, energy, and concentration for fluid flow is given by the following:

$$\nabla \cdot \mathbf{V} = 0, \tag{1}$$

$$\rho_0 \left[\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \mathbf{V} \right] = -\nabla p + \nabla^2 \mathbf{V} + \rho \widehat{\mathbf{e}}_{\mathbf{z}}, \tag{2}$$

$$\frac{\partial T}{\partial t} + \mathbf{V} \cdot \nabla T = \eta \nabla^2 T, \tag{3}$$

$$\frac{\partial S}{\partial t} + \mathbf{V} \cdot \nabla S = \eta_s \nabla^2 S. \tag{4}$$

We nondimensionalize (1)–(4) using the following transformations:

$$\begin{gathered} \mathbf{V}_* = \frac{H_*}{\eta_*} \mathbf{V}, \qquad p_* = \frac{H_*^2}{\rho_0 \eta_*^2} p, \\ T_* = \frac{(T - T_C)}{\Delta T_c}, \qquad S_* = \frac{(S - S_C)}{\Delta S_c}, \\ (x_*, y_*, z_*) = H_* (x, y, z), \qquad t_* = \frac{t H_*^2}{\eta_*}, \end{gathered} \tag{5}$$

where $\mathbf{V}_* = (u_*, v_*, w_*)$ is the velocity component, p_* is the pressure, $(T - T_C)$ and $(S - S_C)$ are the temperature and solute concentration variations, η_* is the effective thermal diffusivity, and ν_* is fluid's viscosity.

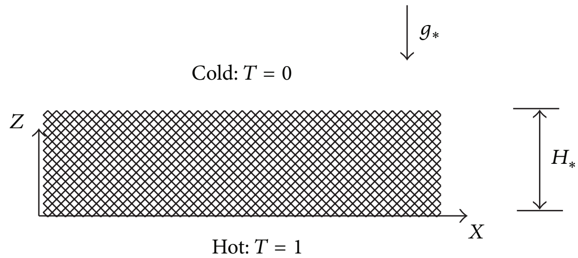


Figure 1: Physical model.

In this model, all the boundaries are rigid and the solution must follow the impermeability conditions there, that is, $\mathbf{V} \cdot \widehat{\mathbf{e}}_{\mathbf{n}} = 0$ on the boundaries, where $\widehat{\mathbf{e}}_{\mathbf{n}}$ is a unit vector normal to the boundary. The temperature and solute concentration boundary conditions are $T = S = 1$ at $z = 0$ and $T = S = 0$ at $z = 1$.

For convective rolls having axes parallel to the shorter dimension (i.e., $y = 0$) $v = 0$, by applying the curl operator on (2) to eliminate the pressure and introducing the stream function defined by $u = \partial\psi/\partial z$ and $w = -\partial\psi/\partial x$, we get

$$\begin{gathered} \left[\frac{1}{\mathrm{Pr}} \left(\frac{\partial}{\partial t} + \frac{\partial \psi}{\partial z} \frac{\partial}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial}{\partial z} \right) - \nabla^2 \right] \nabla^2 \psi = -\mathrm{Ra} \frac{\partial T}{\partial x} + R_s \frac{\partial S}{\partial x}, \\ \frac{\partial T}{\partial t} + \frac{\partial \psi}{\partial z} \frac{\partial T}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial T}{\partial z} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2}, \\ \frac{\partial S}{\partial t} + \frac{\partial \psi}{\partial z} \frac{\partial S}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial S}{\partial z} = \frac{1}{\mathrm{Le}} \left(\frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial z^2} \right), \end{gathered} \tag{6}$$

where

$$\begin{gathered} \mathrm{Pr} = \frac{\nu_*}{\eta_*}, \qquad \mathrm{Ra} = \frac{\alpha_* \Delta T_c g_* H_*^3}{\eta_* \nu_*}, \\ R_s = \frac{\alpha_{s*} \Delta S_c g_* H_*^3}{\eta_* \nu_*}, \qquad \mathrm{Le} = \frac{\eta_*}{\eta_{s*}} \end{gathered} \tag{7}$$

which are, respectively, the Prandtl number, the Rayleigh number, the solutal Rayleigh number, and the Lewis number. The boundary conditions for the stream function are $\psi = 0$ on the horizontal boundaries. Equation (6) forms a nonlinear coupled system which together with the corresponding boundary conditions allows for a basic motionless conduction solution.

3. Diminished Set of Equation

In order to obtain the solution to (6), we represent the stream function, temperature, and solutal distributions in the following form:

$$\begin{gathered} \psi = A_{11} \sin(\kappa x) \sin(\pi z), \\ T = 1 - z + B_{11} \cos(\kappa x) \sin(\pi z) + B_{02} \sin(2\pi z), \\ S = 1 - z + C_{11} \cos(\kappa x) \sin(\pi z) + C_{02} \sin(2\pi z). \end{gathered} \tag{8}$$

Substituting (8) into (6), multiplying the equations by the orthogonal eigenfunctions corresponding to (8), and then integrating them over the spatial domain yield a set of five ordinary differential equations for the time evolution of the following amplitudes:

$$\frac{d\overline{A}_{11}}{d\tau} = \Pr\left[\overline{B}_{11} - \overline{A}_{11} + \overline{C}_{11}\right],$$
$$\frac{d\overline{B}_{11}}{d\tau} = -\overline{B}_{11} + R\overline{A}_{11} - \overline{A}_{11}\overline{B}_{02},$$
$$\frac{d\overline{B}_{02}}{d\tau} = \overline{A}_{11}\overline{B}_{11} - \lambda\overline{B}_{02}, \tag{9}$$
$$\frac{d\overline{C}_{11}}{d\tau} = -\frac{\overline{C}_{11}}{\mathrm{Le}} + R_s\overline{A}_{11} - \overline{A}_{11}\overline{C}_{02},$$
$$\frac{d\overline{C}_{02}}{d\tau} = \overline{A}_{11}\overline{C}_{11} - \frac{\lambda}{\mathrm{Le}}\overline{C}_{02}.$$

In (9), the time, the amplitudes, the Rayleigh number, and the solutal Rayleigh number were rescaled, and the following notations are introduced as follows:

$$\overline{A}_{11} = \frac{(\kappa/\kappa_{\mathrm{cr}})}{\left[(\kappa/\kappa_{\mathrm{cr}})^2 + 2\right]}A_{11}, \qquad \overline{B}_{11} = \kappa_{\mathrm{cr}}RB_{11},$$
$$\overline{B}_{02} = \pi RB_{02}, \qquad \overline{C}_{11} = \kappa_{\mathrm{cr}}R_sC_{11},$$
$$\overline{C}_{02} = \pi R_sC_{02}, \qquad R = \frac{\mathrm{Ra}}{\mathrm{Ra}_c}, \qquad R_s = \frac{R_s}{R_{sc}}, \tag{10}$$
$$\tau = \left(\kappa^2 + \pi^2\right)t, \qquad \lambda = \frac{8}{\left[(\kappa/\kappa_{\mathrm{cr}})^2 + 2\right]},$$
$$\mathrm{Ra}_c = R_{sc} = \frac{\left(\kappa^2 + \pi^2\right)^3}{\kappa^2}, \qquad \kappa_{\mathrm{cr}} = \frac{\pi}{\sqrt{2}}.$$

Rescaling the equation again in the forms

$$X = \frac{\overline{A}_{11}}{\sqrt{\lambda(R-1)}}, \qquad Y = \frac{\overline{B}_{11}}{\sqrt{\lambda(R-1)}}, \qquad Z = \frac{\overline{B}_{02}}{(R-1)},$$
$$U = \frac{\overline{C}_{11}}{\sqrt{\lambda(R-1)}}, \qquad W = \frac{\overline{C}_{02}}{(R-1)}, \tag{11}$$

gives the following set of scaled equations which are equivalent to (9):

$$\dot{X} = \Pr(Y - X - U),$$
$$\dot{Y} = RX - Y - (R-1)XZ,$$
$$\dot{Z} = \lambda(XY - Z), \tag{12}$$
$$\dot{U} = R_sX - \frac{U}{\mathrm{Le}} - (R-1)XW,$$
$$\dot{W} = \lambda\left(XU - \frac{W}{\mathrm{Le}}\right),$$

where the dots $(\cdot)$ denote time derivatives $d()/d\tau$.

4. Linear Stability Analysis

In this paper, we investigate the chaotic behaviour with low Prandtl number in double-diffusive convection. We obtained system (12) that provides a set of nonlinear equations with five parameters. The value of λ has to be consistent with the wave number at the convection threshold, a requirement for the convection cells to fit into the domain and fulfill the boundary conditions. However, the Lorenz equations have been extensively analyzed and solved for parameter values corresponding to convection in pure fluids and, even there, the parameter values most regularly used correspond to $\Pr = 10$ and $\lambda = 8/3$. Therefore, it is of interest to analyze and solve the corresponding equations for parameter values corresponding to the problem under investigation. We employ the MATLAB ODE45 routine for obtaining the numerical solutions.

Before attempting the numerical solution of system (12), it is useful to examine the local stability of equilibrium points. System (12) has the three basic properties which we will discuss in the following: dissipation, fixed points, and stability of fixed points.

4.1. Dissipation. System (12) is dissipative since

$$\begin{aligned}\nabla\cdot\widehat{V} &= \frac{\partial\dot{X}}{\partial X} + \frac{\partial\dot{Y}}{\partial Y} + \frac{\partial\dot{Z}}{\partial Z} + \frac{\partial\dot{U}}{\partial U} + \frac{\partial\dot{W}}{\partial W}\\ &= -\left(1 + \Pr + \frac{1+\lambda}{\mathrm{Le}} + \lambda\right)\\ &< 0.\end{aligned} \tag{13}$$

Therefore, if the set of initial points in the phase space occupies region $\widehat{V}(0)$ at $\tau = 0$, then, after some time, τ, the endpoints of the corresponding trajectories will fill a volume

$$\widehat{V}(\tau) = \widehat{V}(0)\exp\left[-\left(1 + \Pr + \frac{(1+\lambda)}{\mathrm{Le}} + \lambda\right)\right]. \tag{14}$$

The expression indicates that the volume decreases monotonically with time.

4.2. Fixed Points. The fixed points for velocity, temperature, and solute concentration can be obtained by setting the derivatives of system (12) to zero:

$$\Pr(Y - X - U) = 0,$$
$$RX - Y - (R-1)XZ = 0,$$
$$\lambda(XY - Z) = 0, \tag{15}$$
$$R_sX - \frac{U}{\mathrm{Le}} - (R-1)XW = 0,$$
$$\lambda\left(XU - \frac{W}{\mathrm{Le}}\right) = 0.$$

There is one trivial solution, that is, the origin in the phase space

$$X_1 = Y_1 = Z_1 = U_1 = W_1 = 0, \tag{16}$$

which corresponds to the motionless solution. The other nonzero fixed points are given by the following:

$$
\begin{aligned}
X_{2,3} &= \pm\frac{h_2}{\sqrt{2}},\\
Y_{2,3} &= \pm\frac{\left(-1+\mathrm{Le}^2(R+1)-\mathrm{Le}R_s+h_1\right)X_{2,3}}{2\left(\mathrm{Le}^2-1\right)},\\
Z_{2,3} &= \frac{\left(1-2R+\mathrm{Le}^2(R-1)+\mathrm{Le}R_s-h_1\right)}{2\left(\mathrm{Le}^2-1\right)(R-1)},\\
U_{2,3} &= \pm\frac{\left(1+\mathrm{Le}^2(R-1)-\mathrm{Le}R_s+h_1\right)X_{2,3}}{2\left(\mathrm{Le}^2-1\right)},\\
W_{2,3} &= -\frac{\left(1+\mathrm{Le}^2(R-1)+\mathrm{Le}R_s-2\mathrm{Le}^3R_s+h_1\right)}{2\mathrm{Le}\left(\mathrm{Le}^2-1\right)(R-1)},\\
X_{4,5} &= \pm\frac{h_3}{\sqrt{2}},\\
Y_{4,5} &= \pm\frac{\left(-1+\mathrm{Le}^2(R+1)-\mathrm{Le}R_s-h_1\right)X_{4,5}}{2\left(\mathrm{Le}^2-1\right)},\\
Z_{4,5} &= \frac{\left(1-2R+\mathrm{Le}^2(R-1)+\mathrm{Le}R_s+h_1\right)}{2\left(\mathrm{Le}^2-1\right)(R-1)},\\
U_{4,5} &= \pm\frac{\left(1+\mathrm{Le}^2(R-1)-\mathrm{Le}R_s-h_1\right)X_{4,5}}{2\left(\mathrm{Le}^2-1\right)},\\
W_{4,5} &= \frac{\left(-1+\mathrm{Le}^2(1-R)-\mathrm{Le}R_s+2\mathrm{Le}^3R_s+h_1\right)}{2\mathrm{Le}\left(\mathrm{Le}^2-1\right)(R-1)},
\end{aligned}
\tag{17}
$$

where

$$
\begin{aligned}
h_1 &= \sqrt{\left(\left(1+\mathrm{Le}^2(R-1)\right)^2-2\mathrm{Le}R_s\left(\mathrm{Le}^2(R+1)-1\right)+\mathrm{Le}^2R_s^2\right)},\\
h_2 &= \sqrt{-\frac{\left(1+\mathrm{Le}^2(1-R)+\mathrm{Le}R_s+h_1\right)}{\mathrm{Le}^2(R-1)}},\\
h_3 &= \sqrt{\frac{\left(-1+\mathrm{Le}^2(R-1)-\mathrm{Le}R_s+h_1\right)}{\mathrm{Le}^2(R-1)}}.
\end{aligned}
\tag{18}
$$

The system has five fixed points. When $R = 0$, the five fixed points are all real. Thus, when $(R-1) > 0$, h_1 is always real and h_2 and h_3 are always complex; therefore, the three fixed points (X_i, Y_i, U_i) are all complex and the other two fixed points, (Z_i, W_i), are all real for $i = 2, \ldots, 5$. The fixed point $(X_1, Y_1, Z_1, U_1, W_1)$ corresponds to motionless solution and $(X_i, Y_i, Z_i, U_i, W_i)$, where $i = 2, \ldots, 5$ corresponds to the convective solution.

4.3. Stability of the Fixed Points. The Jacobian matrix of (12) can be written as follows:

$$
J = \begin{bmatrix}
-\mathrm{Pr} & \mathrm{Pr} & 0 & -\mathrm{Pr} & 0\\
R-(R-1)Z & -1 & -(R-1)X & 0 & 0\\
\lambda Y & \lambda X & -\lambda & 0 & 0\\
R_s-(R-1)W & 0 & 0 & -\dfrac{1}{\mathrm{Le}} & (R-1)X\\
\lambda U & 0 & 0 & \lambda X & -\dfrac{\lambda}{\mathrm{Le}}
\end{bmatrix}.
\tag{19}
$$

Since the matrix is 5×5, it is hard to obtain the eigenvalues in a closed form. Hence, the numerical calculation can be performed to discuss the stability at the fixed point. The motionless solution loses stability and the convection solution takes over at the fixed point $\{X_1, Y_1, Z_1, U_1, W_1\}$ with the critical value, R_{c1}. Numerical results for the value of R_{c1}, which corresponds to the onset of convection, is obtained for various values of Le and R_s with the value of parameters $\mathrm{Pr} = 10$ and $\lambda = 8/3$ as shown in Figure 2. Increasing the values of Le and R_s increases the value of R_{c1}.

The stability of the fixed points $(X_i, Y_i, Z_i, U_i, W_i)$ $(i = 2, \ldots, 5)$ is associated with the convective solution. The evolution of the complex eigenvalues of J in the case $\mathrm{Pr} = 10$, $\lambda = 8/3$, and $\mathrm{Le} = 0.1$ is plotted as shown in Figure 3(a) for the fixed point $(X_{4,5}, Y_{4,5}, Z_{4,5}, U_{4,5}, W_{4,5})$. These two roots become a complex conjugate at $R \simeq 3.32$, 3.97, 4.63, and 5.28 for the case $R_s = 15$, 20, 25, and 30, respectively. At these points exactly, they still have negative real parts; therefore, the convection fixed points are stable, that is, spiral nodes. Of all the cases, both the imaginary and real parts of these two complex conjugate eigenvalues increase as R increases and they cross the imaginary axis on the complex plane, so as a result their real part becomes nonnegative at a value of $R_{c2} \simeq 46.37$, 57.34, 69.33, 82.18. At these points, the convective fixed points lose their stability and another periodic, quasiperiodic or chaotic solution takes over. Figure 3(b) shows the evolution of the complex eigenvalues of J for the case $\mathrm{Pr} = 10$, $\lambda = 8/3$ and R_s for the fixed point $(X_{4,5}, Y_{4,5}, Z_{4,5}, U_{4,5}, W_{4,5})$. These two roots become a complex conjugate at $R \simeq 3.33$, 5.23, 7.03, 8.73 for the case of $\mathrm{Le} = 0.1$, 0.2, 0.3, 0.4, respectively. As mentioned before, the convection fixed points are stable and become spiral nodes. Their real part becomes nonnegative at a value of $R_{c2} \simeq 31.71$, 34.96, 40.05, 46.37. Therefore, at these points, the convective fixed points lose their stability and another periodic, quasiperiodic, or chaotic solution takes over.

While for the fixed point $(X_{2,3}, Y_{2,3}, Z_{2,3}, U_{2,3}, W_{2,3})$, the evolution of the complex eigenvalues of J with the same parameter values is always on the positive side of $\mathrm{Re}(\Lambda)$, does not cross the zero axis for $\mathrm{Re}(\Lambda)$, and is not of interest in this study.

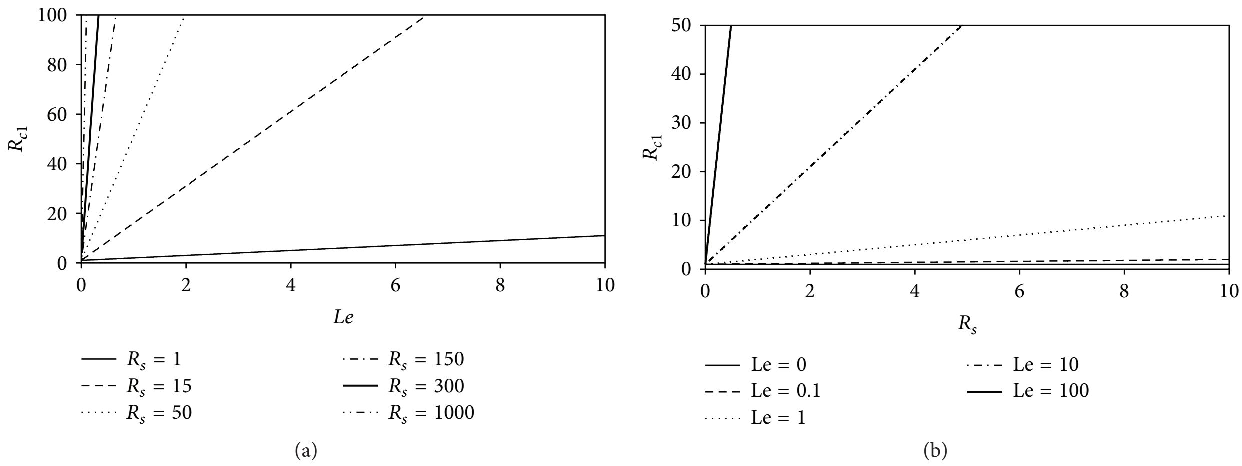


Figure 2: The critical Rayleigh number R_{c1} as function of (a) Le and (b) R_s.

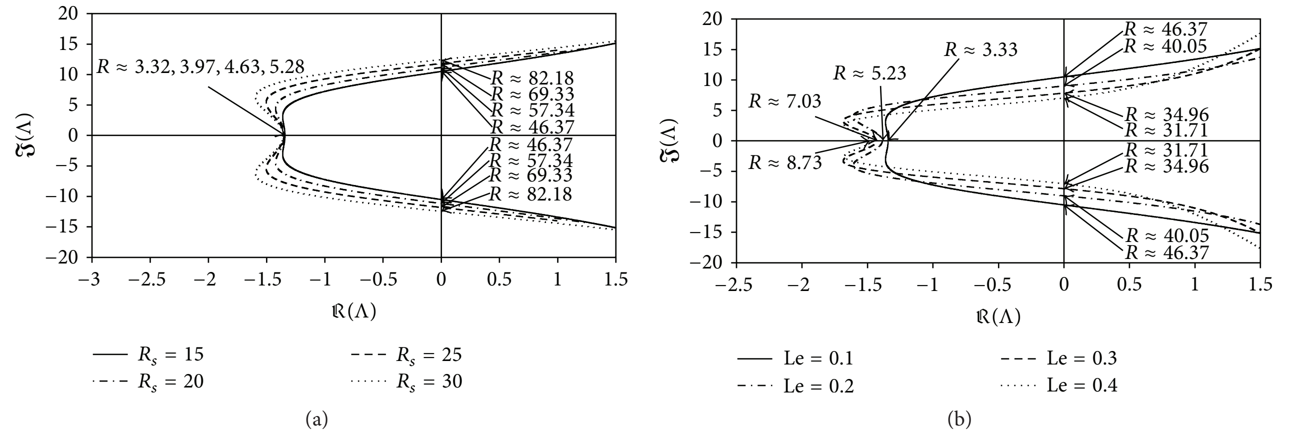


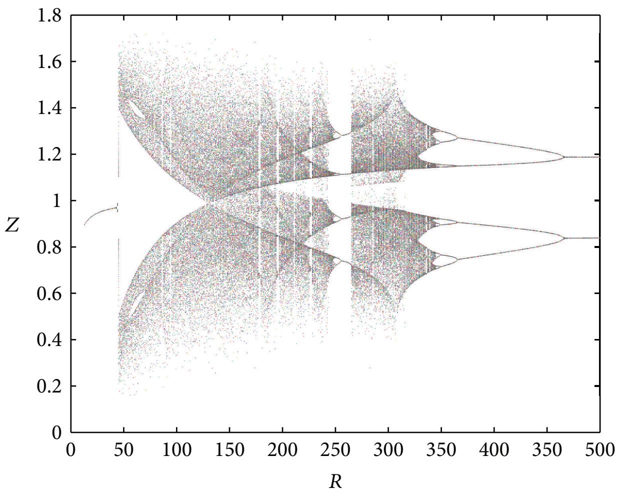
Figure 3: The evolution of the complex eigenvalues with increasing Rayleigh number for Pr = 10, $\lambda = 8/3$, (a) Le = 0.1, and (b) $R_s = 15$.

5. Results and Discussion

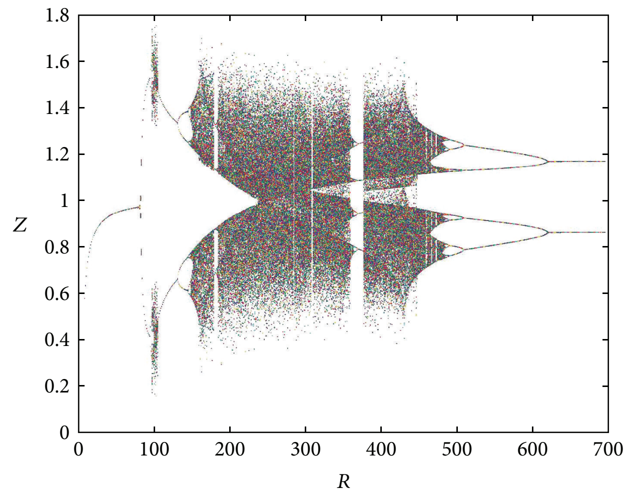
5.1. Bifurcations and Transition to Chaos. In this study, we focused on the dynamic behaviour of thermal convection in double-diffusive fluid layer. The values of Pr and λ used in all computations are 10 and 8/3, respectively, which are consistent with the critical Rayleigh number ($R_c \simeq 24.74$) and the critical wave number at marginal stability in fluid layer convection. All solutions were obtained using the same initial conditions, which were selected to be in the neighborhood of the positive convection fixed point. The initial conditions are at $\tau = 0 : X, Y, Z, U, W = 0.9$. All computations were carried out with the value of maximum time, $\tau_{\max} = 210$, and a step size $\Delta\tau = 0.001$ using the built-in ODE45 method in MATLAB R2010a.

The bifurcation diagrams illustrated in Figure 4 show the peaks and valleys in the posttransient values of Z versus R. In Figure 4(a), for $0 < R < 46.37$ we have one-point attractors, but the "attracted" value of Z increases as R increases, at least to $R \simeq 46.37$. Bifurcation occurs at $R \simeq 46.37, 48$ until just beyond $R = 50$, where the system is chaotic. However, the system is not chaotic for all values of $R > 50$, and we will discuss it using phase-portrait diagram. When we fix Le $= 0.1$ and increase R_s from 15 to 30, the range of one-point attractor changes to $0 < R < 82.31$; this is shown in Figure 4(b), while in Figure 4(c), one-point attractor dropped to $0 < R < 31.86$ in the case of Le = 0.4 and $R_s = 15$. Here we can conclude that increasing the value of the solutal Rayleigh number (with fixed value of Lewis number) will delay the convection process. But increasing the value of Lewis number (with fixed value of solutal Rayleigh number) will enhance the onset of chaos.

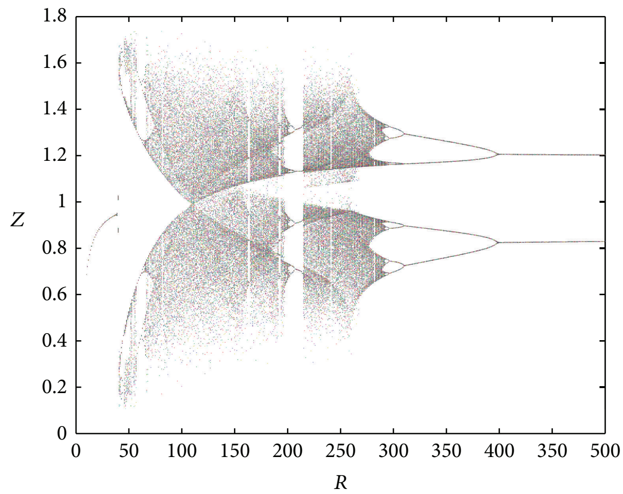
Figure 5 shows the projections of the trajectory's data points on the X-Y-Z plane for Le $= 0.1$ and $R_s = 15$. From Figure 5(a), we obtain a solitary limit cycle signifying the loss of stability of the steady convective fixed points. The subcritical value for this transition is $R_{c2} = 46.37$. Figure 5(b) shows the projections of the trajectory's data points for $R = 48$. At this point, the homoclinic explosion occurs and the chaotic regime with the strange attractor takes over. The homoclinic explosion behaviour giving birth to a stable periodic orbit with period-8 at $R = 250$ is presented



(a)



(b)



(c)

FIGURE 4: Bifurcation diagrams for (a) Le = 0.1, R_s = 15, (b) Le = 0.1, R_s = 30, and (c) Le = 0.4, R_s = 15.

in Figure 5(c). This corresponds to the first wide periodic window within the chaotic regime. In Figure 5(d) we can observe that the data points do align in such a way as to produce an almost clear projection of unconnected points on the projected plane. Increasing the value of R further shows the dynamical behaviour's return to being chaotic again at $R = 300$ as shown in Figure 5(e). At $R = 360$ and $R = 400$, we have a period-8 and period-4 periodic solutions as shown in Figures 5(f) and 5(g). Figure 5(h) shows that a period-2 periodic solution takes over at $R = 500$ and a period-2 periodic type remains when the solutions at higher values of R are obtained. We conclude the observation around these regimes of periodic windows within the broadband of chaotic solutions by pointing out a sequence of period-halving as one increases the Rayleigh number.

Figure 6 shows the projections of the trajectory's data points on the X-Y-Z plane for Le = 0.1 and $R_s = 30$. The subcritical values for limit cycle and homoclinic explosion occur at $R_{c2} = 82.18$ and $R = 98$ as shown in Figures 6(a) and 6(b), respectively, while period-8 is observed at $R = 364$ and 370 as presented in Figures 6(c) and 6(d). The dynamical behaviour returns to being chaotic again as R increases; this happens at $R = 400$ as shown in Figure 6(e). Figures 5(f) and 5(g) show a period-8 and period-4 periodic solutions at $R = 490$ and $R = 500$. Figure 6(h) shows that a period-2 periodic solution takes over at $R = 600$ and remains the same behaviour when the solutions at higher values of R are continued.

5.2. Lyapunov Exponents. The convergence plot of the Lyapunov spectrum for system (12) is shown in Figure 7. The alogrithm as proposed by [24] was employed for this purpose. The values of the Lyapunov exponents for system (12) are tabulated in Table 1. From these results, we can conclude that for eigenvalue λ_1 the system is always unstable and chaotic with the increasing R. For λ_2, the system alternates between being stable and dissipative to unstable and chaotic when the value of R is increased. For eigenvalues λ_3, λ_4, and λ_5, the system is always stable and dissipative with the increasing R.

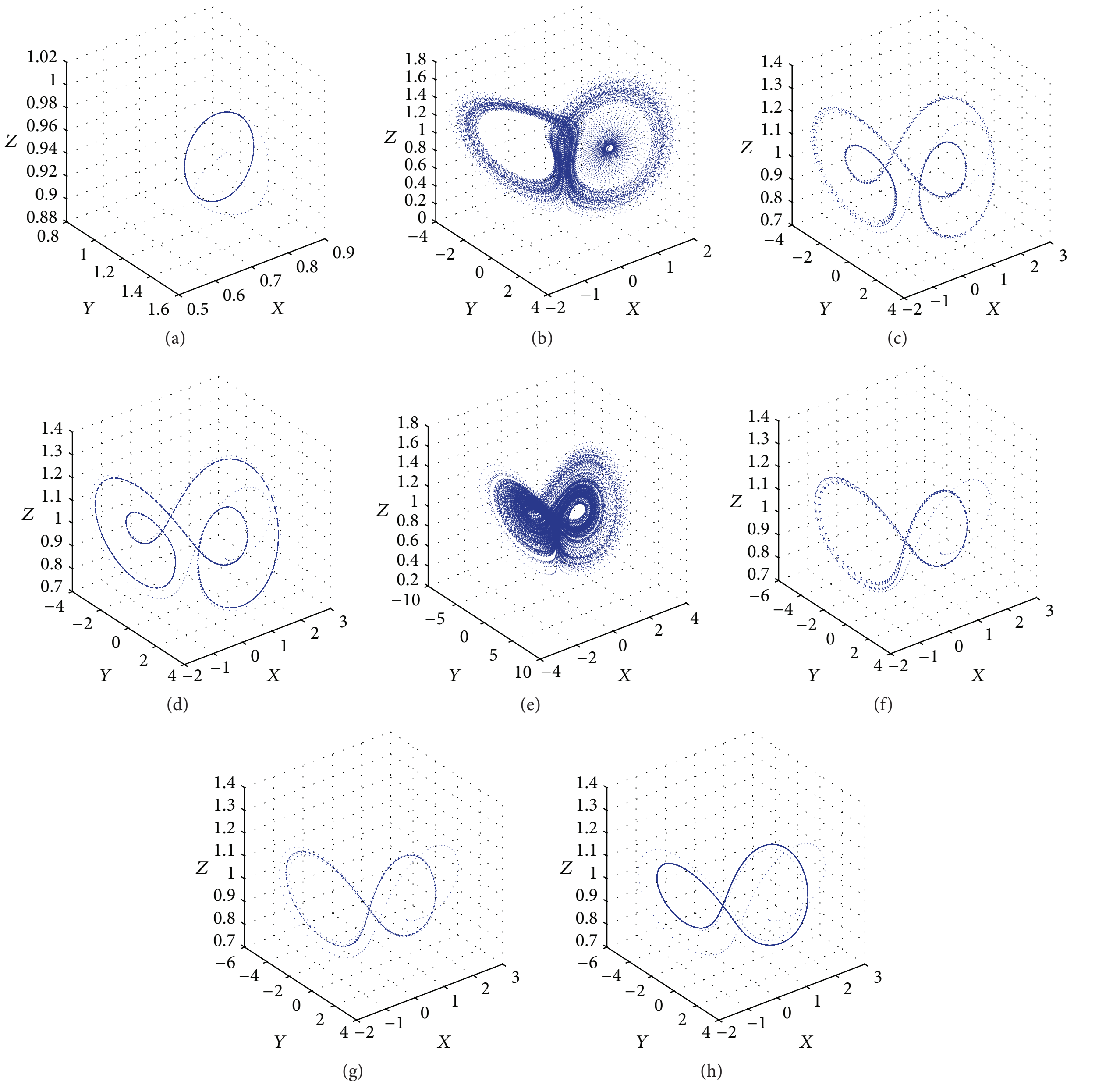


FIGURE 5: Phase portraits for (a) $R \simeq 46.37$, (b) $R = 48$, (c) $R = 250$, (d) $R = 260$, (e) $R = 300$, (f) $R = 360$, (g) $R = 400$, and (h) $R = 500$ for the case where $\mathrm{Le} = 0.1$, $R_s = 15$.

TABLE 1: Lyapunov exponents for system (12) computed from 10,000 data points for the case where $\mathrm{Le} = 0.1$ and $R_s = 15$.

R	λ_1	λ_2	λ_3	λ_4	λ_5
46.37	2.674585	−3.572664	−7.683913	−12.847196	−28.904144
48	2.612254	−3.302684	−7.921422	−12.710783	−29.010697
260	6.350478	0.626358	−4.509057	−13.941471	−38.856433
300	7.469197	−0.458284	−4.653033	−14.610687	−38.076093
360	8.589279	0.219833	−4.754681	−15.043590	−39.338205
400	8.404066	−0.611687	−4.530720	−12.889258	−40.698489
500	4.451812	4.442320	−3.196173	−12.386501	−43.632794

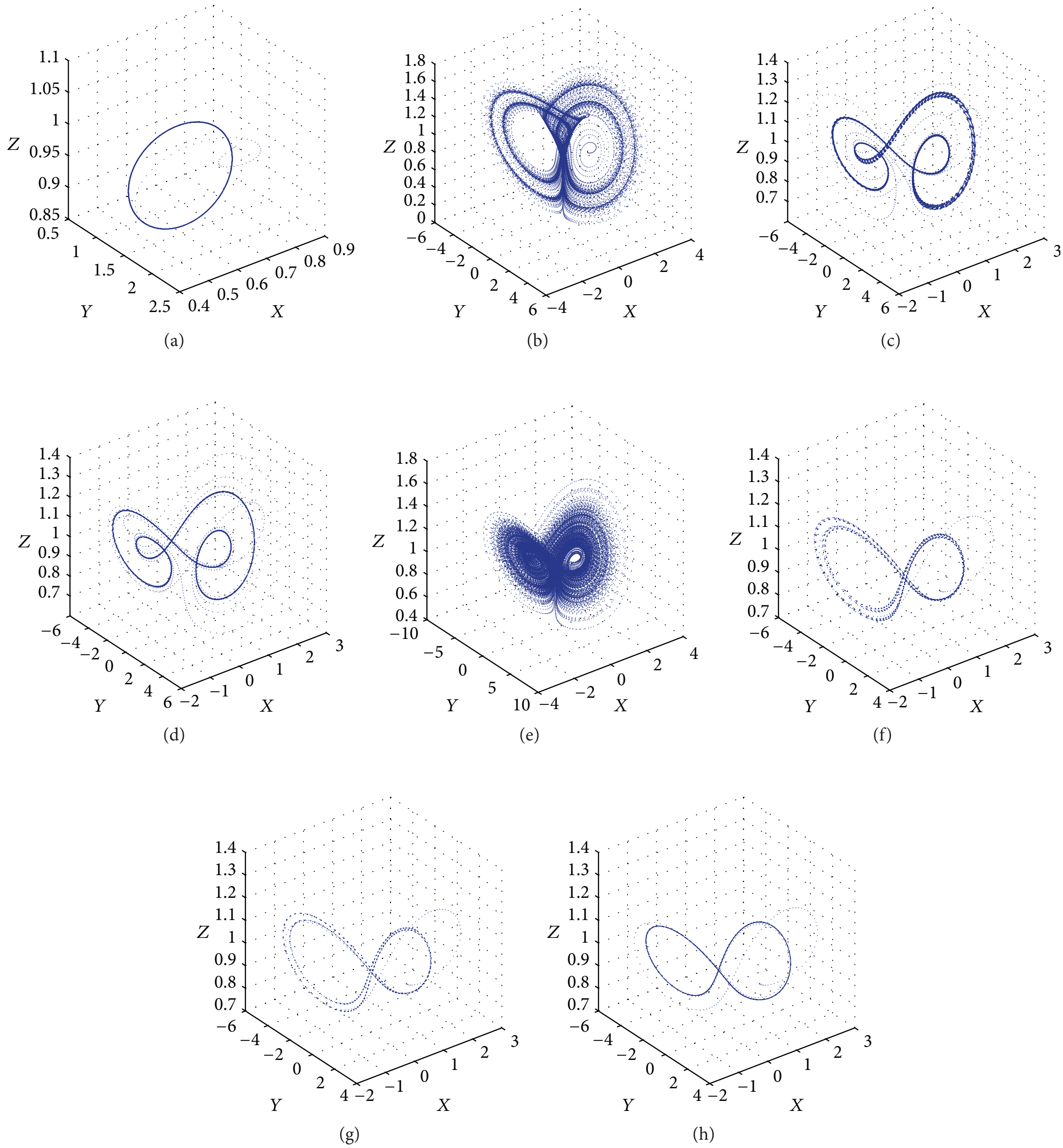


FIGURE 6: Phase portraits for (a) $R = 82$, (b) $R = 98$, (c) $R = 364$, (d) $R = 370$, (e) $R = 400$, (f) $R = 490$, (g) $R = 500$, and (h) $R = 600$ for the case where $\mathrm{Le} = 0.1$, $R_s = 30$.

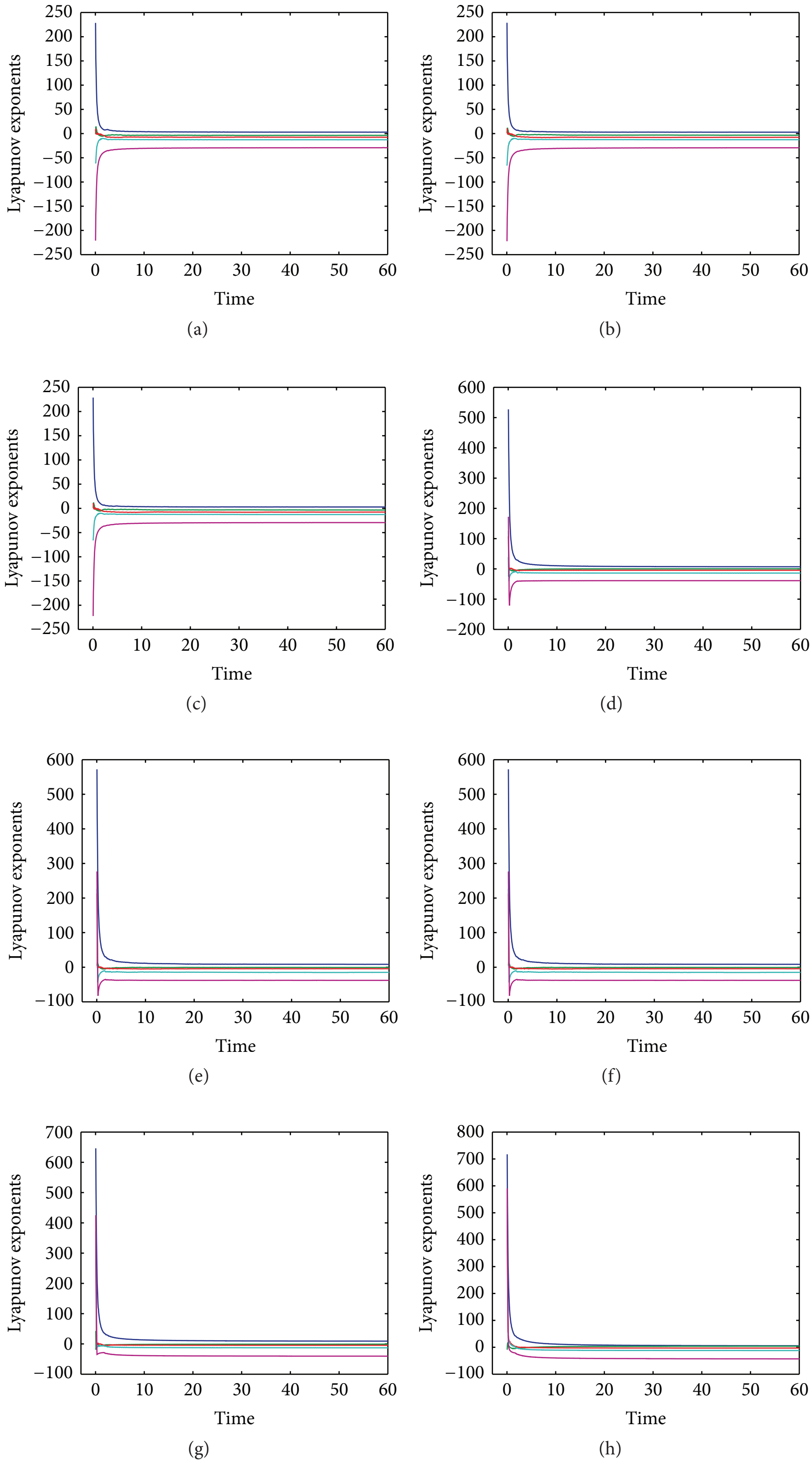


Figure 7: Dynamics of Lyapunov exponents for (a) $R = 46.37$, (b) $R = 48$, (c) $R = 250$, (d) $R = 260$, (e) $R = 300$, (f) $R = 360$, (g) $R = 400$, and (h) $R = 500$ for the case where $\mathrm{Le} = 0.1$, $R_s = 15$.

6. Conclusion

In this work, chaotic behaviour in double-diffusive convection in a fluid layer has been investigated. A five-dimensional model of chaotic system was obtained using the Galerkin truncated approximation. The transition from steady convection to chaos via a Hopf bifurcation produced a limit cycle which may be associated with a homoclinic explosion at a slightly subcritical value of the thermal Rayleigh number. Both the solutal Rayleigh number and Lewis number affect the stability of the system. Increasing the Rayleigh number shows that the trajectory of the data points pointing out a sequence of period-halving and the behaviour remains the same at a higher Rayleigh number. The different transitions of the system, can be implied by the different values of the Lyapunov exponents. Negative eigenvalues lead to a stable and dissipative system and positive eigenvalues show that the system is always unstable and chaotic, while alternate eigenvalues suggest different transitions of the system (i.e., stable and dissipative to unstable and chaotic) as the value of the Rayleigh number is increased.

Acknowledgments

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Research Article

Existence and Iterative Algorithms of Positive Solutions for a Higher Order Nonlinear Neutral Delay Differential Equation

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This paper is concerned with the higher order nonlinear neutral delay differential equation $[a(t)(x(t)+b(t)x(t-\tau))^{(m)}]^{(n-m)}+[h(t,x(h_1(t)),\dots,x(h_l(t)))]^{(i)}+f(t,x(f_1(t)),\dots,x(f_l(t)))=g(t)$, for all $t\ge t_0$. Using the Banach fixed point theorem, we establish the existence results of uncountably many positive solutions for the equation, construct Mann iterative sequences for approximating these positive solutions, and discuss error estimates between the approximate solutions and the positive solutions. Nine examples are included to dwell upon the importance and advantages of our results.

1. Introduction and Preliminaries

In recent years, the existence problems of nonoscillatory solutions for neutral delay differential equations of first, second, third, and higher order have been studied intensively by using fixed point theorems; see, for example, [1–12] and the references therein.

Using the Banach, Schauder, and Krasnoselskii fixed point theorems, Zhang et al. [9] and Liu et al. [7] considered the existence of nonoscillatory solutions for the following first order neutral delay differential equations:

$$\begin{aligned}&[x(t)+P(t)x(t-\tau)]'+Q_1(t)x(t-\tau_1)\\&\quad-Q_2(t)x(t-\tau_2)=0,\quad \forall t\ge t_0,\\&[x(t)+c(t)x(t-\tau)]'\\&\quad+h(t)f(x(t-\sigma_1),x(t-\sigma_2),\dots,x(t-\sigma_k))=g(t),\\&\qquad\forall t\ge t_0,\end{aligned}\tag{1}$$

where $P\in C([t_0,+\infty),\mathbb{R}\setminus\{\pm1\})$ and $c\in C([t_0,+\infty),\mathbb{R})$. Making use of the Banach and Krasnoselskii fixed point theorems, Kulenović and Hadžiomerspahić [2] and Zhou [10] studied the existence of a nonoscillatory solution for the following second order neutral differential equations:

$$\begin{aligned}&[x(t)+cx(t-\tau)]''+Q_1(t)x(t-\sigma_1)\\&\quad-Q_2(t)x(t-\sigma_2)=0,\quad\forall t\ge t_0,\\&\left[r(t)(x(t)+P(t)x(t-\tau))'\right]'\\&\quad+\sum_{i=1}^{m}Q_i(t)f_i(x(t-\sigma_i))=0,\quad\forall t\ge t_0,\end{aligned}\tag{2}$$

where $c\in\mathbb{R}\setminus\{\pm1\}$ and $P\in C([t_0,\infty),\mathbb{R})$. Zhou and Zhang [11], Zhou et al. [12], and Liu et al. [4], respectively, investigated the existence of nonoscillatory solutions for the following higher order neutral delay differential equations:

$$\begin{aligned}&[x(t)+cx(t-\tau)]^{(n)}\\&\quad+(-1)^{n+1}[P(t)x(t-\sigma)-Q(t)x(t-\delta)]=0,\\&\qquad\forall t\ge t_0,\end{aligned}$$

$$
\begin{aligned}
&[x(t)+P(t)x(t-\tau)]^{(n)} \\
&\qquad+\sum_{i=1}^{m}Q_i(t)f_i(x(t-\sigma_i))=g(t),\quad \forall t\ge t_0, \\
&[x(t)+ax(t-\tau)]^{(n)} \\
&\qquad+(-1)^{n+1}f(t,x(t-\sigma_1),x(t-\sigma_2),\dots,x(t-\sigma_k)) \\
&\quad=g(t),\quad \forall t\ge t_0,
\end{aligned}
\tag{3}
$$

where $c \in \mathbb{R}\setminus\{\pm 1\}$, $P \in C([t_0,\infty),\mathbb{R})$ and $a \in \mathbb{R}\setminus\{-1\}$. Candan [1] proved the existence of a bounded nonoscillatory solution for the higher order nonlinear neutral differential equation:

$$
\begin{aligned}
&\left[r(t)(x(t)+P(t)x(t-\tau))^{(n-1)}\right]' \\
&\quad+(-1)^n\left[Q_1(t)g_1(x(t-\sigma_1))\right. \\
&\qquad\left.-Q_2(t)g_2(x(t-\sigma_2))-f(t)\right]=0,\quad \forall t\ge t_0,
\end{aligned}
\tag{4}
$$

where $P \in C([t_0,\infty),\mathbb{R}\setminus\{\pm 1\})$.

Motivated by the results in [1–12], in this paper we consider the following higher order nonlinear neutral delay differential equation:

$$
\begin{aligned}
&\left[a(t)(x(t)+b(t)x(t-\tau))^{(m)}\right]^{(n-m)} \\
&\quad=+\left[h(t,x(h_1(t)),\dots,x(h_l(t)))\right]^{(i)} \\
&\quad=+f(t,x(f_1(t)),\dots,x(f_l(t)))=g(t),\quad \forall t\ge t_0,
\end{aligned}
\tag{5}
$$

where $m,n \in \mathbb{N}$ and $i \in \mathbb{N}_0$ with $i \le n-m-1$, $\tau > 0$, $a \in C([t_0,+\infty),\mathbb{R}\setminus\{0\})$, $b, g, f_j, h_j \in C([t_0,+\infty),\mathbb{R})$, $h \in C^i([t_0,+\infty)\times\mathbb{R}^l,\mathbb{R})$ and $f \in C([t_0,+\infty)\times\mathbb{R}^l,\mathbb{R})$ with

$$
\lim_{t\to+\infty}h_j(t)=\lim_{t\to+\infty}f_j(t)=+\infty,\quad j\in\{1,2,\dots,l\}. \tag{6}
$$

It is clear that (5) includes (1)–(4) as special cases. Utilizing the Banach fixed point theorem, we prove several existence results of uncountably many positive solutions for (5), construct a few Mann iterative schemes, and discuss error estimates between the sequences generated by the Mann iterative schemes and the positive solutions. Nine examples are given to show that the results presented in this paper extend substantially the existing ones in [1, 2, 4, 5, 8, 9, 11].

Throughout this paper, we assume that $\mathbb{R}=(-\infty,+\infty)$, $\mathbb{R}^+=[0,+\infty)$, $\mathbb{N}$ denotes the set of all positive integers, $\mathbb{N}_0=\mathbb{N}\cup\{0\}$,

$$
\begin{aligned}
&H_j=\frac{1}{(m-1)!\,(n-m-j-1)!},\quad j\in\{0,i\}, \\
&\gamma=\min\left\{t_0-\tau,\inf_{t\ge t_0}h_j(t),\inf_{t\ge t_0}f_j(t):j\in\{1,2,\dots,l\}\right\},
\end{aligned}
\tag{7}
$$

$\mathrm{CB}([\gamma,+\infty),\mathbb{R})$ stands for the Banach space of all continuous and bounded functions in $[\gamma,+\infty)$ with norm $\|x\|=\sup_{t\ge\gamma}|x(t)|$, and for any $M>N>0$

$$
\begin{aligned}
\Omega_1(N,M)=&\{x\in \mathrm{CB}([\gamma,+\infty),\mathbb{R}): \\
&\quad N\le x(t)\le M,\ \forall t\ge\gamma\}, \\
\Omega_2(N,M)=&\left\{x\in \mathrm{CB}([\gamma,+\infty),\mathbb{R}):\frac{N}{b(t+\tau)}\le x(t)\right. \\
&\le\frac{M}{b(t+\tau)},\ \forall t\ge T;\ \frac{N}{b(T+\tau)} \\
&\left.\le x(t)\le\frac{M}{b(T+\tau)},\ \forall t\in[\gamma,T)\right\}, \\
\Omega_3(N,M)=&\left\{x\in \mathrm{CB}([\gamma,+\infty),\mathbb{R}):-\frac{N}{b(t+\tau)}\le x(t)\right. \\
&\le-\frac{M}{b(t+\tau)},\ \forall t\ge T;\ -\frac{N}{b(T+\tau)} \\
&\left.\le x(t)\le-\frac{M}{b(T+\tau)},\ \forall t\in[\gamma,T)\right\}.
\end{aligned}
\tag{8}
$$

It is easy to check that $\Omega_1(N,M)$, $\Omega_2(N,M)$ and $\Omega_3(N,M)$ are closed subsets of $\mathrm{CB}([\gamma,+\infty),\mathbb{R})$.

By a solution of (5), we mean a function $x\in C([\gamma,+\infty),\mathbb{R})$ for some $T>1+|t_0|+\tau+|\gamma|$, such that $a(t)(x(t)+b(t)x(t-\tau))^{(m)}$ are $n-m$ times continuously differentiable in $[T,+\infty)$ and such that (5) is satisfied for $t\ge T$.

Lemma 1. *Let $\tau>0$, $c\ge 0$, $F\in C([c,+\infty)^3,\mathbb{R}^+)$ and $G\in C([c,+\infty)^2,\mathbb{R}^+)$. Then*

(a) $\int_c^{+\infty}\int_r^{+\infty}\int_u^{+\infty}rF(s,u,r)ds\,du\,dr<+\infty\Leftrightarrow$ $\sum_{j=0}^{\infty}\int_{c+j\tau}^{+\infty}\int_r^{+\infty}\int_u^{+\infty}F(s,u,r)ds\,du\,dr<+\infty$;

(b) $\int_c^{+\infty}\int_u^{+\infty}uG(s,u)ds\,du<+\infty\Leftrightarrow$ $\sum_{j=0}^{\infty}\int_{c+j\tau}^{+\infty}\int_u^{+\infty}G(s,u)ds\,du<+\infty$;

(c) *if* $\int_c^{+\infty}\int_r^{+\infty}\int_u^{+\infty}rF(s,u,r)ds\,du\,dr<+\infty$, *then*

$$
\begin{aligned}
&\sum_{j=1}^{\infty}\int_{t+j\tau}^{+\infty}\int_r^{+\infty}\int_u^{+\infty}F(s,u,r)\,ds\,du\,dr \\
&\quad\le\frac{1}{\tau}\int_{t+\tau}^{+\infty}\int_r^{+\infty}\int_u^{+\infty}rF(s,u,r)\,ds\,du\,dr \\
&\quad<+\infty,\quad \forall t\ge c;
\end{aligned}
\tag{9}
$$

(d) *if* $\int_c^{+\infty}\int_u^{+\infty} uG(s,u)ds\,du < +\infty$, *then*

$$\begin{aligned}\sum_{j=1}^{\infty}\int_{t+j\tau}^{+\infty}\int_u^{+\infty} G(s,u)\,ds\,du &\leq \frac{1}{\tau}\int_{t+\tau}^{+\infty}\int_u^{+\infty} uG(s,u)\,ds\,du \\ &< +\infty, \quad \forall t \geq c.\end{aligned} \tag{10}$$

Proof. Let $[t]$ denote the largest integral number not exceeding $t \in \mathbb{R}$. Note that

$$\lim_{r\to+\infty}\frac{[(r-c)/\tau]+1}{r} = \frac{1}{\tau}, \tag{11}$$

$$c+n\tau \leq r < c+(n+1)\tau \Longleftrightarrow n \leq \frac{r-c}{\tau} < n+1, \quad \forall n \in \mathbb{N}_0. \tag{12}$$

Clearly (12) means that

$$\begin{aligned}\sum_{j=0}^{\infty}\int_{c+j\tau}^{+\infty}\int_r^{+\infty}\int_u^{+\infty} F(s,u,r)\,ds\,du\,dr &= \int_c^{+\infty}\int_r^{+\infty}\int_u^{+\infty} F(s,u,r)\,ds\,du\,dr \\ &\quad + \int_{c+\tau}^{+\infty}\int_r^{+\infty}\int_u^{+\infty} F(s,u,r)\,ds\,du\,dr \\ &\quad + \int_{c+2\tau}^{+\infty}\int_r^{+\infty}\int_u^{+\infty} F(s,u,r)\,ds\,du\,dr \\ &\quad + \int_{c+3\tau}^{+\infty}\int_r^{+\infty}\int_u^{+\infty} F(s,u,r)\,ds\,du\,dr + \cdots \\ &= \int_c^{c+\tau}\int_r^{+\infty}\int_u^{+\infty} F(s,u,r)\,ds\,du\,dr \\ &\quad + 2\int_{c+\tau}^{c+2\tau}\int_r^{+\infty}\int_u^{+\infty} F(s,u,r)\,ds\,du\,dr \\ &\quad + 3\int_{c+2\tau}^{c+3\tau}\int_r^{+\infty}\int_u^{+\infty} F(s,u,r)\,ds\,du\,dr \\ &\quad + 4\int_{c+3\tau}^{c+4\tau}\int_r^{+\infty}\int_u^{+\infty} F(s,u,r)\,ds\,du\,dr + \cdots \\ &= \sum_{n=0}^{\infty}\int_{c+n\tau}^{c+(n+1)\tau}\int_r^{+\infty}\int_u^{+\infty} (n+1)F(s,u,r)\,ds\,du\,dr \\ &= \sum_{n=0}^{\infty}\int_{c+n\tau}^{c+(n+1)\tau}\int_r^{+\infty}\int_u^{+\infty}\left(\left[\frac{r-c}{\tau}\right]+1\right) \\ &\qquad \times F(s,u,r)\,ds\,du\,dr \\ &= \int_c^{+\infty}\int_r^{+\infty}\int_u^{+\infty}\left(\left[\frac{r-c}{\tau}\right]+1\right)F(s,u,r)\,ds\,du\,dr.\end{aligned} \tag{13}$$

Thus (a) follows from (11) and (13).

Assume that $\int_c^{+\infty}\int_r^{+\infty}\int_u^{+\infty} rF(s,u,r)ds\,du\,dr < +\infty$. As in the proof of (a), we infer that

$$\begin{aligned}\sum_{j=1}^{\infty}\int_{t+j\tau}^{+\infty}\int_r^{+\infty}\int_u^{+\infty} F(s,u,r)\,ds\,du\,dr &= \int_{t+\tau}^{+\infty}\int_r^{+\infty}\int_u^{+\infty}\left[\frac{r-t}{\tau}\right]F(s,u,r)\,ds\,du\,dr \\ &\leq \frac{1}{\tau}\int_{t+\tau}^{+\infty}\int_r^{+\infty}\int_u^{+\infty} rF(s,u,r)\,ds\,du\,dr \\ &\leq \frac{1}{\tau}\int_c^{+\infty}\int_r^{+\infty}\int_u^{+\infty} rF(s,u,r)\,ds\,du\,dr \\ &< +\infty, \quad \forall t \geq c,\end{aligned} \tag{14}$$

that is, (c) holds.

Similar to the proofs of (a) and (c), we conclude that (b) and (d) hold. This completes the proof. □

2. Existence of Uncountably Many Positive Solutions and Mann Iterative Schemes

Now we show the existence of uncountably many positive solutions for (5) and discuss the convergence of the Mann iterative sequences to these positive solutions.

Theorem 2. *Assume that there exist three constants M, N, and b_0 and four functions $P, Q, R, W \in C([t_0,+\infty),\mathbb{R}^+)$ satisfying*

$$0 < N < M, \quad b_0 < \frac{M-N}{2M}, \quad |b(t)| \leq b_0 \text{ eventually}; \tag{15}$$

$$\begin{aligned}&\left|f(t,u_1,\ldots,u_l) - f(t,\overline{u}_1,\ldots,\overline{u}_l)\right| \\ &\qquad \leq P(t)\max\left\{\left|u_j-\overline{u}_j\right| : 1 \leq j \leq l\right\}, \\ &\left|h(t,u_1,\ldots,u_l) - h(t,\overline{u}_1,\ldots,\overline{u}_l)\right| \\ &\qquad \leq R(t)\max\left\{\left|u_j-\overline{u}_j\right| : 1 \leq j \leq l\right\}, \\ &\forall (t,u_1,\ldots,u_l,\overline{u}_1,\ldots,\overline{u}_l) \in [t_0,+\infty)\times[N,M]^{2l};\end{aligned} \tag{16}$$

$$\begin{aligned}&\left|f(t,u_1,\ldots,u_l)\right| \leq Q(t), \qquad \left|h(t,u_1,\ldots,u_l)\right| \leq W(t), \\ &\qquad \forall (t,u_1,\ldots,u_l) \in [t_0,+\infty)\times[N,M]^l;\end{aligned} \tag{17}$$

$$\begin{aligned}\int_{t_0}^{+\infty}\int_u^{+\infty}\frac{|u|^{m-1}}{|a(u)|}&\Big[|s|^{n-m-1}\max\{P(s),Q(s),|g(s)|\} \\ &+|s|^{n-m-i-1}\max\{R(s),W(s)\}\Big]ds\,du < +\infty.\end{aligned} \tag{18}$$

Then

(a) for any $L \in (b_0M+N,(1-b_0)M)$, there exist $\theta \in (0,1)$ and $T > 1+|t_0|+\tau+|\gamma|$ such that for each $x_0 \in \Omega_1(N,M)$,

the Mann iterative sequence $\{x_k\}_{k\in\mathbb{N}_0}$ *generated by the following scheme*

$$x_{k+1}(t) = \begin{cases} (1-\alpha_k)\,x_k(t) \\ \quad +\alpha_k \Big\{ L - b(t)\,x_k(t-\tau) + (-1)^n H_0 \\ \qquad \times \displaystyle\int_t^{+\infty}\int_u^{+\infty} \frac{(s-u)^{n-m-1}(u-t)^{m-1}}{a(u)} \\ \qquad\qquad \times \left[g(s) - f\left(s, x_k(f_1(s)), \ldots, x_k(f_l(s))\right)\right] ds\,du \\ \qquad +(-1)^{n-i-1} H_i \\ \qquad \times \displaystyle\int_t^{+\infty}\int_u^{+\infty} \frac{(s-u)^{n-m-i-1}(u-t)^{m-1}}{a(u)} \\ \qquad\qquad \times h\left(x_k(h_1(s)), \ldots, x_k(h_l(s))\right) ds\,du \Big\}, \\ \qquad\qquad t \ge T,\ k \in \mathbb{N}_0, \\ (1-\alpha_k)\,x_k(T) \\ \quad +\alpha_k \Big\{ L - b(T)\,x_k(T-\tau) \\ \qquad +(-1)^n H_0 \\ \qquad \times \displaystyle\int_T^{+\infty}\int_u^{+\infty} \frac{(s-u)^{n-m-1}(u-T)^{m-1}}{a(u)} \\ \qquad\qquad \times \left[g(s) - f\left(s, x_k(f_1(s)), \ldots, x_k(f_l(s))\right)\right] ds\,du \\ \qquad +(-1)^{n-i-1} H_i \\ \qquad \times \displaystyle\int_T^{+\infty}\int_u^{+\infty} \frac{(s-u)^{n-m-i-1}(u-T)^{m-1}}{a(u)} \\ \qquad\qquad \times h\left(s, x_k(h_1(s)), \ldots, x_k(h_l(s))\right) ds\,du \Big\}, \\ \qquad\qquad t_0 \le t < T,\ k \in \mathbb{N}_0 \end{cases} \tag{19}$$

converges to a positive solution $x \in \Omega_1(N,M)$ *of* (5) *and has the following error estimate:*

$$\left\|x_{k+1} - x\right\| \le e^{-(1-\theta)\sum_{p=0}^{k}\alpha_p} \left\|x_0 - x\right\|, \quad \forall k \in \mathbb{N}_0, \tag{20}$$

where $\{\alpha_k\}_{k\in\mathbb{N}_0}$ *is an arbitrary sequence in* $[0,1]$ *such that*

$$\sum_{k=0}^{\infty} \alpha_k = +\infty; \tag{21}$$

(*b*) *Equation* (5) *has uncountably many positive solutions in* $\Omega_1(N,M)$.

Proof. Firstly, we prove that (a) holds. Set $L \in (b_0 M + N, (1-b_0)M)$. From (15) and (18), we know that there exist $\theta \in (0,1)$ and $T > 1 + |t_0| + \tau + |\gamma|$ satisfying

$$|b(t)| \le b_0, \quad \forall t \ge T; \tag{22}$$

$$\theta = b_0 + \int_T^{+\infty}\int_u^{+\infty} \frac{u^{m-1}}{|a(u)|} \left[H_0 s^{n-m-1} P(s) + H_i s^{n-m-i-1} R(s) \right] ds\,du; \tag{23}$$

$$\int_T^{+\infty}\int_u^{+\infty} \frac{u^{m-1}}{|a(u)|} \left[H_0 s^{n-m-1} \left(|g(s)| + Q(s)\right) + H_i s^{n-m-i-1} W(s) \right] ds\,du < \min\left\{(1-b_0)M - L, L - b_0 M - N\right\}. \tag{24}$$

Define a mapping $S_L : \Omega_1(N,M) \to \mathrm{CB}([\gamma,+\infty),\mathbb{R})$ by

$$S_L x(t) = \begin{cases} L - b(t)\,x(t-\tau) + (-1)^n H_0 \\ \quad \times \displaystyle\int_t^{+\infty}\int_u^{+\infty} \frac{(s-u)^{n-m-1}(u-t)^{m-1}}{a(u)} \\ \quad \times \left[g(s) - f\left(s, x(f_1(s)), \ldots, x(f_l(s))\right)\right] ds\,du \\ \quad +(-1)^{n-i-1} H_i \\ \quad \times \displaystyle\int_t^{+\infty}\int_u^{+\infty} \frac{(s-u)^{n-m-i-1}(u-t)^{m-1}}{a(u)} \\ \quad \times h\left(s, x(h_1(s)), \ldots, x(h_l(s))\right) ds\,du, \\ \qquad t \ge T,\ x \in \Omega_1(N,M), \\ S_L x(T), \qquad \gamma \le t < T,\ x \in \Omega_1(N,M). \end{cases} \tag{25}$$

It is obvious that $S_L x$ is continuous for each $x \in \Omega_1(N,M)$. By means of (16), (22), (23), and (25), we deduce that for any $x, y \in \Omega_1(N,M)$ and $t \ge T$

$$\begin{aligned} &\left|S_L x(t) - S_L y(t)\right| \\ &\quad \le |b(t)| \left|x(t-\tau) - y(t-\tau)\right| \\ &\qquad + H_0 \int_t^{+\infty}\int_u^{+\infty} \frac{(s-u)^{n-m-1}(u-t)^{m-1}}{|a(u)|} \\ &\qquad \times \left| f\left(s, x(f_1(s)), \ldots, x(f_l(s))\right) \right. \\ &\qquad\quad \left. - f\left(s, y(f_1(s)), \ldots, y(f_l(s))\right)\right| ds\,du \\ &\qquad + H_i \int_t^{+\infty}\int_u^{+\infty} \frac{(s-u)^{n-m-i-1}(u-t)^{m-1}}{|a(u)|} \\ &\qquad \times \left| h\left(s, x(h_1(s)), \ldots, x(h_l(s))\right) \right. \\ &\qquad\quad \left. - h\left(s, y(h_1(s)), \ldots, y(h_l(s))\right)\right| ds\,du \end{aligned}$$

$$\begin{aligned}&\leq b_0\|x-y\|+\|x-y\|\\&\quad\times\int_T^{+\infty}\int_u^{+\infty}\frac{u^{m-1}}{|a(u)|}\left[H_0 s^{n-m-1}P(s)\right.\\&\qquad\qquad\left.+H_i s^{n-m-i-1}R(s)\right]ds\,du\\&=\theta\|x-y\|,\end{aligned}\tag{26}$$

which yields that

$$\|S_Lx-S_Ly\|\leq\theta\|x-y\|,\quad\forall x,y\in\Omega_1(N,M).\tag{27}$$

On the basis of (17), (22), (24), and (25), we acquire that for any $x\in\Omega_1(N,M)$ and $t\geq T$

$$\begin{aligned}&S_Lx(t)\\&\quad\leq L+|b(t)|\,x(t-\tau)\\&\qquad+H_0\int_t^{+\infty}\int_u^{+\infty}\frac{s^{n-m-1}u^{m-1}}{|a(u)|}\\&\qquad\qquad\times\left[|g(s)|+|f(s,x(f_1(s)),\ldots,\right.\\&\qquad\qquad\qquad\left.x(f_l(s)))|\right]ds\,du\\&\qquad+H_i\int_t^{+\infty}\int_u^{+\infty}\frac{s^{n-m-i-1}u^{m-1}}{|a(u)|}\\&\qquad\qquad\times|h(s,x(h_1(s)),\ldots,x(h_l(s)))|\,ds\,du\\&\quad\leq L+b_0M\\&\qquad+\int_T^{+\infty}\int_u^{+\infty}\frac{u^{m-1}}{|a(u)|}\\&\qquad\qquad\times\left[H_0s^{n-m-1}(|g(s)|+Q(s))\right.\\&\qquad\qquad\qquad\left.+H_is^{n-m-i-1}W(s)\right]ds\,du\\&\quad<L+b_0M+\min\{(1-b_0)M-L,L-b_0M-N\}\\&\quad\leq M,\\&S_Lx(t)\\&\quad\geq L-|b(t)|\,x(t-\tau)\\&\qquad-H_0\int_t^{+\infty}\int_u^{+\infty}\frac{s^{n-m-1}u^{m-1}}{|a(u)|}\\&\qquad\qquad\times\left[|g(s)|+|f(s,x(f_1(s)),\ldots,\right.\\&\qquad\qquad\qquad\left.x(f_l(s)))|\right]ds\,du\\&\qquad-H_i\int_t^{+\infty}\int_u^{+\infty}\frac{s^{n-m-i-1}u^{m-1}}{|a(u)|}\\&\qquad\qquad\times|h(s,x(h_1(s)),\ldots,x(h_l(s)))|\,ds\,du\\&\quad\geq L-b_0M-\int_T^{+\infty}\int_u^{+\infty}\frac{u^{m-1}}{|a(u)|}\\&\qquad\qquad\times\left[H_0s^{n-m-1}(|g(s)|+Q(s))\right.\\&\qquad\qquad\qquad\left.+H_is^{n-m-i-1}W(s)\right]ds\,du\\&\quad>L-b_0M-\min\{(1-b_0)M-L,L-b_0M-N\}\\&\quad\geq N,\end{aligned}\tag{28}$$

which guarantee that $S_L(\Omega_1(N,M))\subseteq\Omega_1(N,M)$. Consequently, (27) gives that S_L is a contraction mapping in $\Omega_1(N,M)$ and it has a unique fixed point $x\in\Omega_1(N,M)$. It is easy to see that $x\in\Omega_1(N,M)$ is a positive solution of (5).

It follows from (19), (25), and (27) that

$$\begin{aligned}&|x_{k+1}(t)-x(t)|\\&=\left|(1-\alpha_k)x_k(t)\right.\\&\qquad+\alpha_k\left\{L-b(t)x_k(t-\tau)+(-1)^nH_0\right.\\&\qquad\qquad\times\int_t^{+\infty}\int_u^{+\infty}\frac{(s-u)^{n-m-1}(u-t)^{m-1}}{a(u)}\\&\qquad\qquad\times\left[g(s)-f(s,x_k(f_1(s)),\ldots,\right.\\&\qquad\qquad\qquad\left.x_k(f_l(s)))\right]ds\,du\\&\qquad\qquad+(-1)^{n-i-1}H_i\\&\qquad\qquad\times\int_t^{+\infty}\int_u^{+\infty}\frac{(s-u)^{n-m-i-1}(u-t)^{m-1}}{a(u)}\\&\qquad\qquad\left.\left.\times h(s,x_k(h_1(s)),\ldots,x_k(h_l(s)))\,ds\,du\right\}-x(t)\right|\\&\leq(1-\alpha_k)|x_k(t)-x(t)|+\alpha_k|S_Lx_k(t)-S_Lx(t)|\\&\leq(1-\alpha_k)|x_k(t)-x(t)|+\alpha_k\theta|x_k(t)-x(t)|\\&=(1-(1-\theta)\alpha_k)|x_k(t)-x(t)|\\&\leq e^{-(1-\theta)\alpha_k}\|x_k-x\|\\&\leq e^{-(1-\theta)\sum_{p=0}^k\alpha_p}\|x_0-x\|,\quad\forall k\in\mathbb{N}_0,\ t\geq T,\end{aligned}\tag{29}$$

which yields that

$$\|x_{k+1}-x\|\leq e^{-(1-\theta)\sum_{p=0}^k\alpha_p}\|x_0-x\|,\quad\forall k\in\mathbb{N}_0.\tag{30}$$

That is, (20) holds. Thus (20) and (21) ensure that $\lim_{k\to\infty}x_k=x$.

Secondly, we show that (b) holds. Let $L_1,L_2\in(b_0M+N,(1-b_0)M)$ with $L_1\neq L_2$. In light of (15) and (18), we know that for each $p\in\{1,2\}$, there exist $\theta_p\in(0,1)$, T_p and T^*

with $T_p > 1 + |t_0| + \tau + |\gamma|$ and $T^* > \max\{T_1, T_2\}$ satisfying (22)–(24) and

$$\int_{T^*}^{+\infty}\int_u^{+\infty}\frac{u^{m-1}}{|a(u)|}\left[H_0 s^{n-m-1}P(s) + H_i s^{n-m-i-1}R(s)\right]ds\,du < |L_1 - L_2|, \tag{31}$$

where θ and T are replaced by θ_p and T_p, respectively. Let the mapping S_{L_p} be defined by (25) with L and T replaced by L_p and T_p, respectively. As in the proof of (a), we deduce easily that the mapping S_{L_p} possesses a unique fixed point $z_p \in \Omega_1(N, M)$, that is, z_p is a positive solution of (5) in $\Omega_1(N, M)$. In order to prove (b), we need only to show that $z_1 \neq z_2$. In fact, (25) means that for each $t \geq T^*$ and $p \in \{1, 2\}$

$$\begin{aligned}
z_p(t) &= L_p - b(t)z_p(t-\tau) + (-1)^n H_0 \\
&\quad\times\int_t^{+\infty}\int_u^{+\infty}\frac{(s-u)^{n-m-1}(u-t)^{m-1}}{a(u)} \\
&\quad\times\left[g(s) - f\left(s, z_p(f_1(s)), \ldots, z_p(f_l(s))\right)\right]ds\,du \\
&\quad+ (-1)^{n-i-1}H_i\int_t^{+\infty}\int_u^{+\infty}\frac{(s-u)^{n-m-i-1}(u-t)^{m-1}}{a(u)} \\
&\quad\times h\left(s, z_p(h_1(s)), \ldots, z_p(h_l(s))\right)ds\,du.
\end{aligned} \tag{32}$$

It follows from (16), (22), (31), and (32) that for each $t \geq T^*$

$$\begin{aligned}
&|z_1(t) - z_2(t)| \\
&\quad\geq |L_1 - L_2| - |b(t)|\,|z_1(t-\tau) - z_2(t-\tau)| \\
&\quad\quad - H_0\int_t^{+\infty}\int_u^{+\infty}\frac{(s-u)^{n-m-1}(u-t)^{m-1}}{|a(u)|} \\
&\quad\quad\times\left|f(s, z_1(f_1(s)), \ldots, z_1(f_l(s)))\right. \\
&\quad\quad\quad\left.-f(s, z_2(f_1(s)), \ldots, z_2(f_l(s)))\right|ds\,du \\
&\quad\quad - H_i\int_t^{+\infty}\int_u^{+\infty}\frac{(s-u)^{n-m-i-1}(u-t)^{m-1}}{|a(u)|} \\
&\quad\quad\times\left|h(s, z_1(h_1(s)), \ldots, z_1(h_l(s)))\right. \\
&\quad\quad\quad\left.-h(s, z_2(h_1(s)), \ldots, z_2(h_l(s)))\right|ds\,du \\
&\quad\geq |L_1 - L_2| - b_0\|z_1 - z_2\| \\
&\quad\quad - \|z_1 - z_2\|\int_{T^*}^{+\infty}\int_u^{+\infty}\frac{u^{m-1}}{|a(u)|} \\
&\quad\quad\times\left[H_0 s^{n-m-1}P(s) + H_i s^{n-m-i-1}R(s)\right]ds\,du \\
&\quad\geq |L_1 - L_2| - (b_0 + |L_1 - L_2|)\|z_1 - z_2\|,
\end{aligned} \tag{33}$$

which implies that

$$\|z_1 - z_2\| \geq \frac{|L_1 - L_2|}{1 + b_0 + |L_1 - L_2|} > 0, \tag{34}$$

that is, $z_1 \neq z_2$. This completes the proof. □

Theorem 3. *Assume that there exist three constants M, N, and b_0 and four functions $P, Q, R, W \in C([t_0, +\infty), \mathbb{R}^+)$ satisfying* (16)–(18) *and*

$$0 < N < M, \quad b_0 < \frac{M-N}{M}, \quad 0 \leq b(t) \leq b_0 \text{ eventually}. \tag{35}$$

Then

(a) *for any $L \in (b_0 M + N, M)$, there exist $\theta \in (0,1)$ and $T > 1 + |t_0| + \tau + |\gamma|$ such that for each $x_0 \in \Omega_1(N, M)$, the Mann iterative sequence $\{x_k\}_{k\in\mathbb{N}_0}$ generated by* (19) *converges to a positive solution $x \in \Omega_1(N, M)$ of* (5) *and has the error estimate* (20), *where $\{\alpha_k\}_{k\in\mathbb{N}_0}$ is an arbitrary sequence in $[0, 1]$ satisfying* (21)*;*

(b) *Equation* (5) *has uncountably many positive solutions in $\Omega_1(N, M)$.*

Proof. Let $L \in (b_0 M + N, M)$. Equations (18) and (36) ensure that there exist $\theta \in (0, 1)$ and $T > 1 + |t_0| + \tau + |\gamma|$ satisfying (23),

$$0 \leq b(t) \leq b_0, \quad \forall t \geq T; \tag{36}$$

$$\begin{aligned}
\int_T^{+\infty}\int_u^{+\infty}\frac{u^{m-1}}{|a(u)|}&\left[H_0 s^{n-m-1}\left(|g(s)| + Q(s)\right)\right. \\
&\left.+H_i s^{n-m-i-1}W(s)\right]ds\,du \\
&< \min\{M - L, L - b_0 M - N\}.
\end{aligned} \tag{37}$$

Define a mapping $S_L : \Omega_1(N, M) \to \mathrm{CB}([\gamma, +\infty), \mathbb{R})$ by (25). Obviously, $S_L x$ is continuous for every $x \in \Omega_1(N, M)$.

Using (16), (23), (25), and (36), we conclude that for any $x, y \in \Omega_1(N, M)$ and $t \geq T$

$$\begin{aligned}
&\left|S_L x(t) - S_L y(t)\right| \\
&\quad \leq b(t)\left|x(t-\tau) - y(t-\tau)\right| \\
&\qquad + H_0 \int_t^{+\infty}\int_u^{+\infty} \frac{(s-u)^{n-m-1}(u-t)^{m-1}}{|a(u)|} \\
&\qquad \times \left|f(s, x(f_1(s)), \ldots, x(f_l(s)))\right. \\
&\qquad\quad \left.-f(s, y(f_1(s)), \ldots, y(f_l(s)))\right| ds\, du \\
&\qquad + H_i \int_t^{+\infty}\int_u^{+\infty} \frac{(s-u)^{n-m-i-1}(u-t)^{m-1}}{|a(u)|} \\
&\qquad \times \left|h(s, x(h_1(s)), \ldots, x(h_l(s)))\right. \\
&\qquad\quad \left.-h(s, y(h_1(s)), \ldots, y(h_l(s)))\right| ds\, du \\
&\quad \leq b_0 \|x-y\| + \|x-y\| \\
&\qquad \times \int_T^{+\infty}\int_u^{+\infty} \frac{u^{m-1}}{|a(u)|}\left[H_0 s^{n-m-1} P(s)\right. \\
&\qquad\qquad \left.+ H_i s^{n-m-i-1} R(s)\right] ds\, du \\
&\quad = \theta \|x-y\|,
\end{aligned} \tag{38}$$

which implies that (27) holds. In light of (17), (25), (36), and (37), we know that for any $x \in \Omega_1(N, M)$ and $t \geq T$

$$\begin{aligned}
&S_L x(t) \\
&\quad \leq L + H_0 \int_t^{+\infty}\int_u^{+\infty} \frac{s^{n-m-1}u^{m-1}}{|a(u)|} \\
&\qquad\qquad \times \left[\left|g(s)\right| + \left|f(s, x(f_1(s)), \ldots,\right.\right. \\
&\qquad\qquad\qquad \left.\left. x(f_l(s)))\right|\right] ds\, du \\
&\qquad + H_i \int_t^{+\infty}\int_u^{+\infty} \frac{s^{n-m-i-1}u^{m-1}}{|a(u)|} \\
&\qquad\qquad \times \left|h(s, x(h_1(s)), \ldots, x(h_l(s)))\right| ds\, du \\
&\quad \leq L + \int_T^{+\infty}\int_u^{+\infty} \frac{u^{m-1}}{|a(u)|}\left[H_0 s^{n-m-1}\left(\left|g(s)\right| + Q(s)\right)\right. \\
&\qquad\qquad \left.+ H_i s^{n-m-i-1} W(s)\right] ds\, du \\
&\quad < L + \min\{M-L, L-b_0M-N\} \\
&\quad \leq M, \\
&S_L x(t) \\
&\quad \geq L - |b(t)|\, x(t-\tau) \\
&\qquad - H_0 \int_t^{+\infty}\int_u^{+\infty} \frac{s^{n-m-1}u^{m-1}}{|a(u)|} \\
&\qquad\qquad \times \left[\left|g(s)\right| + \left|f(s, x(f_1(s)), \ldots,\right.\right. \\
&\qquad\qquad\qquad \left.\left. x(f_l(s)))\right|\right] ds\, du \\
&\qquad - H_i \int_t^{+\infty}\int_u^{+\infty} \frac{s^{n-m-i-1}u^{m-1}}{|a(u)|} \\
&\qquad\qquad \times \left|h(s, x(h_1(s)), \ldots, x(h_l(s)))\right| ds\, du \\
&\quad \geq L - b_0M - \int_T^{+\infty}\int_u^{+\infty} \frac{u^{m-1}}{|a(u)|} \\
&\qquad\qquad \times \left[H_0 s^{n-m-1}\left(\left|g(s)\right| + Q(s)\right)\right. \\
&\qquad\qquad\quad \left.+ H_i s^{n-m-i-1} W(s)\right] ds\, du \\
&\quad > L - b_0M - \min\{M-L, L-b_0M-N\} \\
&\quad \geq N,
\end{aligned} \tag{39}$$

which mean that $S_L(\Omega_1(N, M)) \subseteq \Omega_1(N, M)$. Equation (27) guarantees that S_L is a contraction mapping in $\Omega_1(N, M)$ and it possesses a unique fixed point $x \in \Omega_1(N, M)$. As in the proof of Theorem 2, we infer that $x \in \Omega_1(N, M)$ is a positive solution of (5). The rest of the proof is similar to that of Theorem 2 and is omitted. This completes the proof. □

Theorem 4. *Assume that there exist three constants M, N, and b_0 and four functions $P, Q, R, W \in C([t_0, +\infty), \mathbb{R}^+)$ satisfying* (16)–(18) *and*

$$0 < N < M, \quad b_0 < \frac{M-N}{M}, \quad -b_0 \leq b(t) \leq 0 \text{ eventually.} \tag{40}$$

Then

(a) *for any $L \in (N, (1-b_0)M)$, there exist $\theta \in (0,1)$ and $T > 1 + |t_0| + \tau + |\gamma|$ such that for each $x_0 \in \Omega_1(N, M)$, the Mann iterative sequence $\{x_k\}_{k\in\mathbb{N}_0}$ generated by* (19) *converges to a positive solution $x \in \Omega_1(N, M)$ of* (5) *and has the error estimate* (20), *where $\{\alpha_k\}_{k\in\mathbb{N}_0}$ is an arbitrary sequence in $[0, 1]$ satisfying* (21)*;*

(b) *Equation* (5) *has uncountably many positive solutions in $\Omega_1(N, M)$.*

Proof. Set $L \in (N, (1-b_0)M)$. It follows from (18) and (40) that there exist $\theta \in (0, 1)$ and $T > 1 + |t_0| + \tau + |\gamma|$ satisfying (23),

$$-b_0 \leq b(t) \leq 0, \quad \forall t \geq T; \tag{41}$$

$$\begin{aligned}
&\int_T^{+\infty}\int_u^{+\infty} \frac{u^{m-1}}{|a(u)|}\left[H_0 s^{n-m-1}\left(\left|g(s)\right| + Q(s)\right)\right. \\
&\qquad\qquad \left.+ H_i s^{n-m-i-1} W(s)\right] ds\, du \\
&\quad < \min\{L-N, (1-b_0)M - L\}.
\end{aligned} \tag{42}$$

Define a mapping $S_L : \Omega_1(N,M) \to \mathrm{CB}([\gamma,+\infty),\mathbb{R})$ by (25). Distinctly, S_Lx is continuous for each $x \in \Omega_1(N,M)$. In terms of (16), (23), (25), and (41), we reason that for any $x, y \in \Omega_1(N,M)$ and $t \ge T$

$$\begin{aligned}
&\left|S_Lx(t) - S_Ly(t)\right| \\
&\quad \le |b(t)|\left|x(t-\tau) - y(t-\tau)\right| \\
&\qquad + H_0 \int_t^{+\infty}\int_u^{+\infty} \frac{(s-u)^{n-m-1}(u-t)^{m-1}}{|a(u)|} \\
&\qquad \times \left| f(s, x(f_1(s)), \ldots, x(f_l(s))) \right. \\
&\qquad \quad \left. - f(s, y(f_1(s)), \ldots, y(f_l(s)))\right| ds\,du \\
&\qquad + H_i \int_t^{+\infty}\int_u^{+\infty} \frac{(s-u)^{n-m-i-1}(u-t)^{m-1}}{|a(u)|} \\
&\qquad \times \left| h(s, x(h_1(s)), \ldots, x(h_l(s))) \right. \\
&\qquad \quad \left. - h(s, y(h_1(s)), \ldots, y(h_l(s)))\right| ds\,du \\
&\quad \le b_0\|x-y\| + \|x-y\| \\
&\qquad \times \int_T^{+\infty}\int_u^{+\infty} \frac{u^{m-1}}{|a(u)|}\left[H_0 s^{n-m-1}P(s)\right. \\
&\qquad\qquad \left. + H_i s^{n-m-i-1}R(s)\right] ds\,du \\
&\quad = \theta\|x-y\|,
\end{aligned} \tag{43}$$

which means that (27) holds. Owing to (17), (25), (41), and (42), we earn that for any $x \in \Omega_1(N,M)$ and $t \ge T$

$$\begin{aligned}
S_Lx(t) &\le L + |b(t)|\, x(t-\tau) \\
&\quad + H_0 \int_t^{+\infty}\int_u^{+\infty} \frac{s^{n-m-1}u^{m-1}}{|a(u)|} \\
&\qquad \times \left[|g(s)| + \left| f(s, x(f_1(s)), \ldots, \right.\right. \\
&\qquad\qquad \left.\left. x(f_l(s)))\right|\right] ds\,du \\
&\quad + H_i \int_t^{+\infty}\int_u^{+\infty} \frac{s^{n-m-i-1}u^{m-1}}{|a(u)|} \\
&\qquad \times \left| h(s, x(h_1(s)), \ldots, \right. \\
&\qquad\qquad \left. x(h_l(s)))\right| ds\,du \\
&\le L + b_0M + \int_T^{+\infty}\int_u^{+\infty} \frac{u^{m-1}}{|a(u)|} \\
&\qquad \times \left[H_0 s^{n-m-1}\left(|g(s)| + Q(s)\right)\right. \\
&\qquad\qquad \left. + H_i s^{n-m-i-1}W(s)\right] ds\,du \\
&< L + b_0M + \min\{L-N, (1-b_0)M - L\} \\
&\le M, \\
S_Lx(t) &\ge L - H_0 \int_t^{+\infty}\int_u^{+\infty} \frac{s^{n-m-1}u^{m-1}}{|a(u)|} \\
&\qquad \times \left[|g(s)| + \left| f(s, x(f_1(s)), \ldots, \right.\right. \\
&\qquad\qquad \left.\left. x(f_l(s)))\right|\right] ds\,du \\
&\quad - H_i \int_t^{+\infty}\int_u^{+\infty} \frac{s^{n-m-i-1}u^{m-1}}{|a(u)|} \\
&\qquad \times \left| h(s, x(h_1(s)), \ldots, \right. \\
&\qquad\qquad \left. x(h_l(s)))\right| ds\,du \\
&\ge L - \int_T^{+\infty}\int_u^{+\infty} \frac{u^{m-1}}{|a(u)|} \\
&\qquad \times \left[H_0 s^{n-m-1}\left(|g(s)| + Q(s)\right)\right. \\
&\qquad\qquad \left. + H_i s^{n-m-i-1}W(s)\right] ds\,du \\
&> L - \min\{L-N, (1-b_0)M - L\} \\
&\ge N,
\end{aligned} \tag{44}$$

which yield that $S_L(\Omega_1(N,M)) \subseteq \Omega_1(N,M)$. Thus (27) ensures that S_L is a contraction mapping in $\Omega_1(N,M)$ and it owns a unique fixed point $x \in \Omega_1(N,M)$. As in the proof of Theorem 2, we infer that $x \in \Omega_1(N,M)$ is a positive solution of (5). The rest of the proof is parallel to that of Theorem 2, and hence is elided. This completes the proof. □

Theorem 5. *Assume that there exist three constants M, N, and b_0 and four functions $P, Q, R, W \in C([t_0,+\infty),\mathbb{R}^+)$ satisfying* (18) *and*

$$M > N > 0, \quad b_0 > \frac{M}{M-N}, \quad b(t) \ge b_0 \text{ eventually}; \tag{45}$$

$$\begin{aligned}
&\left| f(t, u_1, \ldots, u_l) - f(t, \bar{u}_1, \ldots, \bar{u}_l)\right| \\
&\quad \le P(t)\max\left\{\left|u_j - \bar{u}_j\right| : 1 \le j \le l\right\}, \\
&\left| h(t, u_1, \ldots, u_l) - h(t, \bar{u}_1, \ldots, \bar{u}_l)\right| \\
&\quad \le R(t)\max\left\{\left|u_j - \bar{u}_j\right| : 1 \le j \le l\right\}, \\
&\forall (t, u_1, \ldots, u_l, \bar{u}_1, \ldots, \bar{u}_l) \in [t_0,+\infty) \times \left[0, \frac{M}{b_0}\right]^{2l};
\end{aligned} \tag{46}$$

$$\begin{aligned}
&\left| f(t, u_1, \ldots, u_l)\right| \le Q(t), \quad \left| h(t, u_1, \ldots, u_l)\right| \le W(t), \\
&\qquad \forall (t, u_1, \ldots, u_l) \in [t_0,+\infty) \times \left[0, \frac{M}{b_0}\right]^{l}.
\end{aligned} \tag{47}$$

Then

(a) *for any $L \in (N+M/b_0, M)$, there exist $\theta \in (0,1)$ and $T > 1+|t_0|+\tau+|\gamma|$ such that for each $x_0 \in \Omega_2(N,M)$, the Mann iterative sequence $\{x_k\}_{k\in\mathbb{N}_0}$ generated by the following scheme*

$$x_{k+1}(t) = \begin{cases} (1-\alpha_k)x_k(t) + \dfrac{\alpha_k}{b(t+\tau)} \\ \quad \times \Big\{ L - x_k(t+\tau) + (-1)^n H_0 \\ \quad\quad \times \displaystyle\int_{t+\tau}^{+\infty}\int_u^{+\infty} \frac{(s-u)^{n-m-1}(u-t-\tau)^{m-1}}{a(u)} \\ \quad\quad \times [g(s) - f(s, x_k(f_1(s)), \ldots, x_k(f_l(s)))]\,ds\,du \\ \quad\quad + (-1)^{n-i-1}H_i \\ \quad\quad \times \displaystyle\int_{t+\tau}^{+\infty}\int_u^{+\infty} \frac{(s-u)^{n-m-i-1}(u-t-\tau)^{m-1}}{a(u)} \\ \quad\quad \times h(s, x_k(h_1(s)), \ldots, x_k(h_l(s)))\,ds\,du \Big\}, \\ \qquad t \ge T,\ k \in \mathbb{N}_0, \\ (1-\alpha_k)x_k(T) + \dfrac{\alpha_k}{b(t+\tau)} \\ \quad \times \Big\{ L - x_k(T+\tau) + (-1)^n H_0 \\ \quad\quad \times \displaystyle\int_{T+\tau}^{+\infty}\int_u^{+\infty} \frac{(s-u)^{n-m-1}(u-T-\tau)^{m-1}}{a(u)} \\ \quad\quad \times [g(s) - f(s, x_k(f_1(s)), \ldots, x_k(f_l(s)))]\,ds\,du \\ \quad\quad + (-1)^{n-i-1}H_i \\ \quad\quad \times \displaystyle\int_{T+\tau}^{+\infty}\int_u^{+\infty} \frac{(s-u)^{n-m-i-1}(u-T-\tau)^{m-1}}{a(u)} \\ \quad\quad \times h(s, x_k(h_1(s)), \ldots, x_k(h_l(s)))\,ds\,du \Big\}, \\ \qquad \gamma \le t < T,\ k \in \mathbb{N}_0 \end{cases} \tag{48}$$

converges to a positive solution $x \in \Omega_2(N,M)$ of (5) *and has the error estimate* (20), *where $\{\alpha_k\}_{k\in\mathbb{N}_0}$ is an arbitrary sequence in $[0,1]$ with* (21);

(b) *Equation* (5) *has uncountably many positive solutions in $\Omega_2(N,M)$.*

Proof. First of all, we show that (a) holds. Set $L \in (N + M/b_0, M)$. It follows from (18) and (45) that there exist $\theta \in (0,1)$ and $T > 1 + |t_0| + \tau + |\gamma|$ such that

$$b(t) \ge b_0, \quad \forall t \ge T; \tag{49}$$

$$\theta = \frac{1}{b_0} + \frac{1}{b_0}\int_T^{+\infty}\int_u^{+\infty} \frac{u^{m-1}}{|a(u)|} \times \left[H_0 s^{n-m-1}P(s) + H_i s^{n-m-i-1}R(s)\right] ds\,du; \tag{50}$$

$$\int_T^{+\infty}\int_u^{+\infty} \frac{u^{m-1}}{|a(u)|}\left[H_0 s^{n-m-1}(|g(s)| + Q(s)) + H_i s^{n-m-i-1}W(s)\right] ds\,du < \min\left\{M-L, L-\frac{M}{b_0}-N\right\}. \tag{51}$$

Define a mapping $S_L : \Omega_2(N,M) \to CB([\gamma,+\infty),\mathbb{R})$ by

$$S_L x(t) = \begin{cases} \dfrac{L}{b(t+\tau)} - \dfrac{x(t+\tau)}{b(t+\tau)} + \dfrac{(-1)^n H_0}{b(t+\tau)} \\ \quad \times \displaystyle\int_{t+\tau}^{+\infty}\int_u^{+\infty} \frac{(s-u)^{n-m-1}(u-t-\tau)^{m-1}}{a(u)} \\ \quad \times [g(s) - f(s, x(f_1(s)), \ldots, x(f_l(s)))]\,ds\,du \\ \quad + \dfrac{(-1)^{n-i-1}H_i}{b(t+\tau)} \\ \quad \times \displaystyle\int_{t+\tau}^{+\infty}\int_u^{+\infty} \frac{(s-u)^{n-m-i-1}(u-t-\tau)^{m-1}}{a(u)} \\ \quad \times h(s, x(h_1(s)), \ldots, x(h_l(s)))\,ds\,du, \\ \qquad t \ge T,\ x \in \Omega_2(N,M), \\ S_L x(T), \quad \gamma \le t < T,\ x \in \Omega_2(N,M). \end{cases} \tag{52}$$

In light of (46), (49), (50), and (52), we conclude that for $x, y \in \Omega_2(N,M)$ and $t \ge T$

$$\begin{aligned} &|S_L x(t) - S_L y(t)| \\ &\le \frac{1}{b(t+\tau)}|x(t+\tau) - y(t+\tau)| + \frac{H_0}{b(t+\tau)} \\ &\quad \times \int_{t+\tau}^{+\infty}\int_u^{+\infty} \frac{s^{n-m-1}u^{m-1}}{|a(u)|} \\ &\quad \times |f(s, x(f_1(s)), \ldots, x(f_l(s))) - f(s, y(f_1(s)), \ldots, y(f_l(s)))|\,ds\,du \\ &\quad + \frac{H_i}{b(t+\tau)} \\ &\quad \times \int_{t+\tau}^{+\infty}\int_u^{+\infty} \frac{s^{n-m-i-1}u^{m-1}}{a(u)} \\ &\quad \times |h(s, x(h_1(s)), \ldots, x(h_l(s))) - h(s, y(h_1(s)), \ldots, y(h_l(s)))|\,ds\,du \\ &\le \frac{1}{b_0}\|x-y\| + \frac{1}{b_0}\|x-y\| \\ &\quad \times \int_T^{+\infty}\int_u^{+\infty} \frac{u^{m-1}}{|a(u)|}\left[H_0 s^{n-m-1}P(s) + H_i s^{n-m-i-1}R(s)\right] ds\,du \\ &= \theta\|x-y\|, \end{aligned} \tag{53}$$

which yields that

$$\|S_L x - S_L y\| \le \theta \|x - y\|, \quad \forall x, y \in \Omega_2(N, M). \tag{54}$$

In view of (47), (49), (51), and (52), we obtain that for any $x \in \Omega_2(N, M)$ and $t \ge T$

$$\begin{aligned}
S_L x(t) \le{} & \frac{1}{b(t+\tau)} \times \Big\{ L - x(t+\tau) + H_0 \int_{t+\tau}^{+\infty} \int_u^{+\infty} \frac{s^{n-m-1} u^{m-1}}{|a(u)|} \times \big[|g(s)| + |f(s, x(f_1(s)), \dots, x(f_l(s)))| \big]\, ds\, du \\
& + H_i \int_{t+\tau}^{+\infty} \int_u^{+\infty} \frac{s^{n-m-i-1} u^{m-1}}{|a(u)|} \times |h(s, x(h_1(s)), \dots, x(h_l(s)))|\, ds\, du \Big\} \\
\le{} & \frac{1}{b(t+\tau)} \times \Big\{ L - \frac{N}{b(t+\tau)} + \int_T^{+\infty} \int_u^{+\infty} \frac{u^{m-1}}{|a(u)|} \times \big[H_0 s^{n-m-1} (|g(s)| + Q(s)) + H_i s^{n-m-i-1} W(s) \big]\, ds\, du \Big\} \\
<{} & \frac{1}{b(t+\tau)} \times \Big(L - \frac{N}{b(t+\tau)} + \min\Big\{ M - L, L - \frac{M}{b_0} - N \Big\} \Big) \\
\le{} & \frac{M}{b(t+\tau)}, \\
S_L x(t) \ge{} & \frac{1}{b(t+\tau)} \times \Big\{ L - x(t+\tau) - H_0 \int_{t+\tau}^{+\infty} \int_u^{+\infty} \frac{s^{n-m-1} u^{m-1}}{|a(u)|} \times \big[|g(s)| + |f(s, x(f_1(s)), \dots, x(f_l(s)))| \big]\, ds\, du \\
& - H_i \int_{t+\tau}^{+\infty} \int_u^{+\infty} \frac{s^{n-m-i-1} u^{m-1}}{|a(u)|} \times |h(s, x(h_1(s)), \dots, x(h_l(s)))|\, ds\, du \Big\} \\
\ge{} & \frac{1}{b(t+\tau)} \times \Big\{ L - \frac{M}{b(t+\tau)} - \int_T^{+\infty} \int_u^{+\infty} \frac{u^{m-1}}{|a(u)|} \times \big[H_0 s^{n-m-1} (|g(s)| + Q(s)) + H_i s^{n-m-i-1} W(s) \big]\, ds\, du \Big\} \\
>{} & \frac{1}{b(t+\tau)} \times \Big(L - \frac{M}{b(t+\tau)} - \min\Big\{ M - L, L - \frac{M}{b_0} - N \Big\} \Big) \\
\ge{} & \frac{N}{b(t+\tau)},
\end{aligned} \tag{55}$$

which imply that $S_L(\Omega_2(N, M)) \subseteq \Omega_2(N, M)$. It follows from (50) and (54) that S_L is a contraction mapping in $\Omega_2(N, M)$ and it has a unique fixed point $x \in \Omega_2(N, M)$. It is clear that $x \in \Omega_2(N, M)$ is a positive solution of (5).

Note that (48), (52), and (54) undertake that

$$\begin{aligned}
|x_{k+1}(t) - x(t)| ={} & \Big| (1 - \alpha_k) x_k(t) + \frac{\alpha_k}{b(t+\tau)} \times \Big\{ L - x_k(t+\tau) + (-1)^n H_0 \\
& \times \int_{t+\tau}^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-1} (u-t-\tau)^{m-1}}{a(u)} \times [g(s) - f(s, x_k(f_1(s)), \dots, sx_k(f_l(s)))]\, ds\, du \\
& + (-1)^{n-i-1} H_i
\end{aligned}$$

$$
\begin{aligned}
&\times \int_{t+\tau}^{+\infty} \int_{u}^{+\infty} \frac{(s-u)^{n-m-i-1}(u-t-\tau)^{m-1}}{a(u)} \\
&\qquad \times h(s, x_k(h_1(s)), \ldots, \\
&\qquad\qquad x_k(h_l(s)))\, ds\, du \Big\} - x(t) \Big| \\
&\le (1-\alpha_k)\left|x_k(t) - x(t)\right| + \alpha_k \left|S_L x_k(t) - S_L x(t)\right| \\
&\le (1-\alpha_k)\left|x_k(t) - x(t)\right| + \alpha_k \theta \left|x_k(t) - x(t)\right| \\
&= (1-(1-\theta)\alpha_k)\left|x_k(t) - x(t)\right| \\
&\le e^{-(1-\theta)\alpha_k} \left\|x_k - x\right\| \\
&\le e^{-(1-\theta)\sum_{p=0}^{k}\alpha_p} \left\|x_0 - x\right\|, \quad \forall k \in \mathbb{N}_0,\ t \ge T,
\end{aligned} \tag{56}
$$

which indicates that (20) holds. Thus (20) and (21) assure that $\lim_{k\to\infty} x_k = x$.

Next we prove that (b) holds. Let $L_1, L_2 \in (N + M/b_0, M)$ with $L_1 \ne L_2$. As in the proof of (a) we infer that for each $p \in \{1,2\}$ there exist $\theta_p \in (0,1)$, $T_p > 1 + |t_0| + \tau + |\gamma|$ and S_{L_p} satisfying (49)–(52), where L, θ, T, and S_L are replaced by L_p, θ_p, T_p, and S_{L_p}, respectively, and S_{L_p} has a unique fixed point $z_p \in \Omega_2(N, M)$, which is a positive solution of (5) in $\Omega_2(N, M)$. It follows that for each $t \ge T_p$ and $p \in \{1,2\}$

$$
\begin{aligned}
z_p(t) &= \frac{L_p}{b(t+\tau)} - \frac{z_p(t+\tau)}{b(t+\tau)} + \frac{(-1)^n H_0}{b(t+\tau)} \\
&\quad \times \int_{t+\tau}^{+\infty} \int_{u}^{+\infty} \frac{(s-u)^{n-m-1}(u-t-\tau)^{m-1}}{a(u)} \\
&\quad \times \left[g(s) - f\left(s, z_p(f_1(s)), \ldots, z_p(f_l(s))\right)\right] ds\, du \\
&\quad + \frac{(-1)^{n-i-1} H_i}{b(t+\tau)} \\
&\quad \times \int_{t+\tau}^{+\infty} \int_{u}^{+\infty} \frac{(s-u)^{n-m-i-1}(u-t-\tau)^{m-1}}{a(u)} \\
&\quad \times h\left(s, z_p(h_1(s)), \ldots, z_p(h_l(s))\right) ds\, du.
\end{aligned} \tag{57}
$$

On behalf of proving (b), we need only to show that $z_1 \ne z_2$. Notice that (18) guarantees that there exits $T_3 > \max\{T_1, T_2\}$ satisfying

$$
\begin{aligned}
&\int_{T_3}^{+\infty} \int_{u}^{+\infty} \frac{u^{m-1}}{|a(u)|} \left[H_0 s^{n-m-1} P(s) + H_i s^{n-m-i-1} R(s)\right] ds\, du \\
&\qquad < \frac{|L_1 - L_2|}{1 + 2\left\|z_1 - z_2\right\|}.
\end{aligned} \tag{58}
$$

Due to (46), (51), (57), and (58), we conclude that for each $t \ge T_3$

$$
\begin{aligned}
&\left| z_1(t) - z_2(t) + \frac{z_1(t+\tau)}{b(t+\tau)} - \frac{z_2(t+\tau)}{b(t+\tau)} \right| \\
&\quad \ge \frac{1}{b(t+\tau)} \\
&\qquad \times \Bigg(|L_1 - L_2| \\
&\qquad\quad - H_0 \int_{T_3}^{+\infty} \int_{u}^{+\infty} \frac{s^{n-m-1} u^{m-1}}{|a(u)|} \\
&\qquad\quad \times \left| f((s, z_1(f_1(s)), \ldots, z_1(f_l(s))) \right. \\
&\qquad\qquad \left. - f(s, z_2(f_1(s)), \ldots, z_2(f_l(s)))) \right| ds\, du \\
&\qquad\quad - H_i \int_{T_3}^{+\infty} \int_{u}^{+\infty} \frac{s^{n-m-i-1} u^{m-1}}{|a(u)|} \\
&\qquad\quad \times \left| h(s, z_1(f_1(s)), \ldots, z_1(f_l(s))) \right. \\
&\qquad\qquad \left. - h(s, z_2(f_1(s)), \ldots, z_2(f_l(s))) \right| ds\, du \Bigg) \\
&\quad \ge \frac{1}{b(t+\tau)} \\
&\qquad \times \Bigg(|L_1 - L_2| - \left\|z_1 - z_2\right\| \\
&\qquad\quad \times \int_{T_3}^{+\infty} \int_{u}^{+\infty} \frac{u^{m-1}}{|a(u)|} \left[H_0 s^{n-m-1} P(s) \right. \\
&\qquad\qquad \left. + H_i s^{n-m-i-1} R(s)\right] ds\, du \Bigg) \\
&\quad > \frac{1}{b(t+\tau)} \left(|L_1 - L_2| - \left\|z_1 - z_2\right\| \frac{|L_1 - L_2|}{1 + 2\left\|z_1 - z_2\right\|} \right) \\
&\quad > \frac{|L_1 - L_2|}{2b(t+\tau)} \\
&\quad > 0,
\end{aligned} \tag{59}
$$

which yields that $z_1 \ne z_2$. This completes the proof. □

Theorem 6. *Assume that there exist three constants M, N, and b_0 and four functions $P, Q, R, W \in C([t_0, +\infty), \mathbb{R}^+)$ satisfying* (18), (46), (47), *and*

$$
0 < N < M, \qquad \frac{M}{M-N} < b_0, \qquad b(t) \le -b_0 \text{ eventually.} \tag{60}
$$

Then

(*a*) *for any* $L \in (N, (1 - 1/b_0)M)$, *there exist* $\theta \in (0,1)$ *and* $T > 1 + |t_0| + \tau + |\gamma|$ *such that for each* $x_0 \in \Omega_3(N, M)$, *the Mann iterative sequence* $\{x_k\}_{k \in \mathbb{N}_0}$ *generated by the following scheme*

$$x_{k+1}(t) = \begin{cases} (1-\alpha_k) x_k(t) + \dfrac{\alpha_k}{b(t+\tau)} \\ \times \Big\{ -L - x_k(t+\tau) + (-1)^n H_0 \\ \quad \times \displaystyle\int_{t+\tau}^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-1}(u-t-\tau)^{m-1}}{a(u)} \\ \qquad \times [g(s) - f(s, x_k(f_1(s)), \ldots, \\ \qquad\quad x_k(f_l(s)))] \, ds \, du \\ \quad +(-1)^{n-i-1} H_i \\ \quad \times \displaystyle\int_{t+\tau}^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-i-1}(u-t-\tau)^{m-1}}{a(u)} \\ \quad \times h(s, x_k(h_1(s)), \ldots, x_k(h_l(s))) \, ds \, du \Big\}, \\ \qquad\qquad t \geq T, \ k \in \mathbb{N}_0, \\ (1-\alpha_k) x_k(T) + \dfrac{\alpha_k}{b(t+\tau)} \\ \times \Big\{ -L - x_k(T+\tau) + (-1)^n H_0 \\ \quad \times \displaystyle\int_{T+\tau}^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-1}(u-T-\tau)^{m-1}}{a(u)} \\ \qquad \times [g(s) - f(s, x_k(f_1(s)), \ldots, \\ \qquad\quad x_k(f_l(s)))] \, ds \, du \\ \quad +(-1)^{n-i-1} H_i \\ \quad \times \displaystyle\int_{T+\tau}^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-i-1}(u-T-\tau)^{m-1}}{a(u)} \\ \quad \times h(s, x_k(h_1(s)), \ldots, x_k(h_l(s))) \, ds \, du \Big\}, \\ \qquad\qquad \gamma \leq t < T, \ k \in \mathbb{N}_0 \end{cases} \tag{61}$$

converges to a positive solution $x \in \Omega_3(N, M)$ *of* (5) *and has the error estimate* (20), *where* $\{\alpha_k\}_{k \in \mathbb{N}_0}$ *is an arbitrary sequence in* $[0,1]$ *satisfying* (21);

(*b*) *Equation* (5) *has uncountably many positive solutions in* $\Omega_3(N, M)$.

Proof. Put $L \in (N, (1 - 1/b_0)M)$. It follows from (18) and (60) that there exist $\theta \in (0,1)$ and $T > 1 + |t_0| + \tau + |\gamma|$ satisfying (50) and

$$\begin{gathered} b(t) \leq -b_0, \quad \forall t \geq T; \\ \int_T^{+\infty} \int_u^{+\infty} \frac{u^{m-1}}{|a(u)|} \left[H_0 s^{n-m-1} \left(|g(s)| + Q(s) \right) \right. \\ \left. + H_i s^{n-m-i-1} W(s) \right] ds \, du \\ < \min \left\{ M \left(1 - \frac{1}{b_0} \right) - L, L - N \right\}. \end{gathered} \tag{62}$$

Define a mapping $S_L : \Omega_3(N, M) \to \mathrm{CB}([\gamma, +\infty), \mathbb{R})$ by

$$S_L x(t) = \begin{cases} \dfrac{-L}{b(t+\tau)} - \dfrac{x(t+\tau)}{b(t+\tau)} + \dfrac{(-1)^n H_0}{b(t+\tau)} \\ \quad \times \displaystyle\int_{t+\tau}^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-1}(u-t-\tau)^{m-1}}{a(u)} \\ \quad \times [g(s) - f(s, x(f_1(s)), \ldots, \\ \qquad x(f_l(s)))] \, ds \, du \\ \quad + \dfrac{(-1)^{n-i-1} H_i}{b(t+\tau)} \\ \quad \times \displaystyle\int_{t+\tau}^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-i-1}(u-t-\tau)^{m-1}}{a(u)} \\ \quad \times h(s, x(h_1(s)), \ldots, x(h_l(s))) \, ds \, du, \\ \qquad\qquad t \geq T, \ x \in \Omega_3(N, M), \\ S_L x(T), \qquad \gamma \leq t < T, \ x \in \Omega_3(N, M). \end{cases} \tag{63}$$

By virtue of (47), (62), and (63), we know that for any $x \in \Omega_3(N, M)$ and $t \geq T$

$$\begin{aligned} S_L x(t) &\leq \frac{1}{b(t+\tau)} \\ &\quad \times \Bigg(-L - x(t+\tau) - H_0 \\ &\qquad \times \int_{t+\tau}^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-1}(u-t-\tau)^{m-1}}{|a(u)|} \\ &\qquad \times [|g(s)| + |f(s, x(f_1(s)), \ldots, \\ &\qquad\quad x(f_l(s)))|] \, ds \, du \\ &\qquad - H_i \int_{t+\tau}^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-i-1}(u-t-\tau)^{m-1}}{|a(u)|} \\ &\qquad \times |h(s, x(h_1(s)), \ldots, x(h_l(s)))| \, ds \, du \Bigg) \\ &\leq \frac{1}{b(t+\tau)} \\ &\quad \times \Bigg(-L + \frac{M}{b(t+\tau)} \\ &\qquad - \int_T^{+\infty} \int_u^{+\infty} \frac{u^{m-1}}{|a(u)|} \\ &\qquad\quad \times \left[H_0 s^{n-m-1} \left(|g(s)| + Q(s) \right) \right. \\ &\qquad\qquad \left. + H_i s^{n-m-i-1} W(s) \right] ds \, du \Bigg) \end{aligned}$$

$$
\begin{aligned}
&< \frac{1}{b(t+\tau)}\left(-L+\frac{M}{b(t+\tau)}\right.\\
&\qquad\left.-\min\left\{M\left(1-\frac{1}{b_0}\right)-L, L-N\right\}\right)\\
&\le \frac{-M}{b(t+\tau)},\\
&S_L x(t)\\
&\ge \frac{1}{b(t+\tau)}\\
&\quad\times\Bigg(-L-x(t+\tau)+H_0\\
&\qquad\times\int_{t+\tau}^{+\infty}\int_u^{+\infty}\frac{(s-u)^{n-m-1}(u-t-\tau)^{m-1}}{|a(u)|}\\
&\qquad\qquad\times\left[|g(s)|+|f(s,x(f_1(s)),\ldots,\right.\\
&\qquad\qquad\qquad\left.x(f_l(s)))|\right]ds\,du\\
&\qquad+H_i\int_{t+\tau}^{+\infty}\int_u^{+\infty}\frac{(s-u)^{n-m-i-1}(u-t-\tau)^{m-1}}{|a(u)|}\\
&\qquad\times\left|h(s,x(h_1(s)),\ldots,x(h_l(s)))\right|ds\,du\Bigg)\\
&\ge \frac{1}{b(t+\tau)}\\
&\quad\times\Bigg(-L+\frac{N}{b(t+\tau)}\\
&\qquad+\int_T^{+\infty}\int_u^{+\infty}\frac{u^{m-1}}{|a(u)|}\\
&\qquad\qquad\times\left[H_0 s^{n-m-1}(|g(s)|+Q(s))\right.\\
&\qquad\qquad\left.+H_i s^{n-m-i-1}W(s)\right]ds\,du\Bigg)\\
&> \frac{1}{b(t+\tau)}\left(-L+\frac{N}{b(t+\tau)}\right.\\
&\qquad\left.+\min\left\{M\left(1-\frac{1}{b_0}\right)-L, L-N\right\}\right)\\
&\ge \frac{-N}{b(t+\tau)},
\end{aligned}
\tag{64}
$$

which imply that $S_L(\Omega_3(N,M)) \subseteq \Omega_3(N,M)$. The rest of the proof is identical with the proof of Theorem 5 and hence is omitted. This completes the proof. □

Theorem 7. *Let $m \ge 2$. Assume that there exist two constants M, N with $M > N > 0$ and four functions $P, Q, R, W \in C([t_0,+\infty),\mathbb{R}^+)$ satisfying* (16)–(18) *and*

$$
b(t) = 1 \quad \textit{eventually}. \tag{65}
$$

Then

(*a*) *for any $L \in (N,M)$, there exist $\theta \in (0,1)$ and $T > 1+|t_0|+\tau+|\gamma|$ such that for each $x_0 \in \Omega_1(N,M)$, the Mann iterative sequence $\{x_k\}_{k\in\mathbb{N}_0}$ generated by the following scheme*

$$
x_{k+1}(t) =
\begin{cases}
(1-\alpha_k)x_k(t)+\alpha_k \\
\quad\times\Big\{L+(-1)^n(m-1)H_0 \\
\quad\times\displaystyle\sum_{j=1}^{\infty}\int_{t+(2j-1)\tau}^{t+2j\tau}\int_r^{+\infty}\int_u^{+\infty}\Big(\big((s-u)^{n-m-1}(u-r)^{m-2}\big) \\
\qquad\qquad\times(a(u))^{-1}\Big) \\
\quad\times\left[g(s)-f(s,x_k(f_1(s)),\ldots,\right. \\
\qquad\left. x_k(f_l(s)))\right]ds\,du\,dr \\
\quad+(-1)^{n-i-1}(m-1)H_i \\
\quad\times\displaystyle\sum_{j=1}^{\infty}\int_{t+(2j-1)\tau}^{t+2j\tau}\int_r^{+\infty}\int_u^{+\infty}\Big(\big((s-u)^{n-m-i-1}(u-r)^{m-2}\big) \\
\qquad\qquad\times(a(u))^{-1}\Big) \\
\qquad\times h(s,x_k(h_1(s)),\ldots, \\
\qquad\quad x_k(h_l(s)))\,ds\,du\,dr\Big\}, \\
\qquad\qquad t\ge T,\ k\in\mathbb{N}_0, \\
(1-\alpha_k)x_k(T)+\alpha_k \\
\quad\times\Big\{L+(-1)^n(m-1)H_0 \\
\quad\times\displaystyle\sum_{j=1}^{\infty}\int_{T+(2j-1)\tau}^{T+2j\tau}\int_r^{+\infty}\int_u^{+\infty}\Big(\big((s-u)^{n-m-1}(u-r)^{m-2}\big) \\
\qquad\qquad\times(a(u))^{-1}\Big) \\
\quad\times\left[g(s)-f(s,x_k(f_1(s)),\ldots,\right. \\
\qquad\left. x_k(f_l(s)))\right]ds\,du\,dr \\
\quad+(-1)^{n-i-1}(m-1)H_i \\
\quad\times\displaystyle\sum_{j=1}^{\infty}\int_{T+(2j-1)\tau}^{T+2j\tau}\int_r^{+\infty}\int_u^{+\infty}\Big(\big((s-u)^{n-m-i-1}(u-r)^{m-2}\big) \\
\qquad\qquad\times(a(u))^{-1}\Big) \\
\quad\times h(s,x_k(h_1(s)),\ldots, \\
\qquad x_k(h_l(s)))\,ds\,du\,dr\Big\}, \\
\qquad\qquad \gamma\le t<T,\ k\in\mathbb{N}_0
\end{cases}
\tag{66}
$$

converges to a positive solution $x \in \Omega_1(N,M)$ of (5) *and has the error estimate* (20), *where $\{\alpha_k\}_{k\in\mathbb{N}_0}$ is an arbitrary sequence in* $[0,1]$ *with* (21);

(*b*) *Equation* (5) *has uncountably many positive solutions in $\Omega_1(N,M)$.*

Proof. Let $L \in (N, M)$. It follows from (18) and (65) that there exist $\theta \in (0,1)$ and $T > 1 + |t_0| + \tau + |\gamma|$ satisfying

$$b(t) = 1, \quad \forall t \geq T; \tag{67}$$

$$\theta = \int_T^{+\infty} \int_u^{+\infty} \frac{u^{m-1}}{|a(u)|} \left[H_0 s^{n-m-1} P(s) + H_i s^{n-m-i-1} R(s) \right] ds\, du; \tag{68}$$

$$\int_T^{+\infty} \int_u^{+\infty} \frac{u^{m-1}}{|a(u)|} \left[H_0 s^{n-m-1} \left(|g(s)| + Q(s) \right) + H_i s^{n-m-i-1} W(s) \right] ds\, du < \min \{ M - L, L - N \}. \tag{69}$$

Define a mapping $S_L : \Omega_1(N, M) \to \mathrm{CB}([\gamma, +\infty), \mathbb{R})$ by

$$S_L x(t) = \begin{cases} L + (-1)^n (m-1) H_0 \\ \times \sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} \int_r^{+\infty} \int_u^{+\infty} \left(\left((s-u)^{n-m-1} (u-r)^{m-2} \right) \times (a(u))^{-1} \right) \\ \times \left[g(s) - f(s, x(f_1(s)), \ldots, x(f_l(s))) \right] ds\, du\, dr \\ + (-1)^{n-i-1} (m-1) H_i \\ \times \sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} \int_r^{+\infty} \int_u^{+\infty} \left(\left((s-u)^{n-m-i-1} (u-r)^{m-2} \right) \times (a(u))^{-1} \right) \\ \times h(s, x(h_1(s)), \ldots, x(h_l(s))) \, ds\, du\, dr, \\ \qquad t \geq T, \ x \in \Omega_1(N, M), \\ S_L x(T), \qquad \gamma \leq t < T, \ x \in \Omega_1(N, M). \end{cases} \tag{70}$$

With a view to (16), (68), and (70), we derive that for any $x, y \in \Omega_1(N, M)$ and $t \geq T$

$$\begin{aligned} &\left| S_L x(t) - S_L y(t) \right| \\ &\leq (m-1) H_0 \\ &\quad \times \sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} \int_r^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-1} (u-r)^{m-2}}{|a(u)|} \\ &\quad \times \left| f(s, x(f_1(s)), \ldots, x(f_l(s))) \right. \\ &\qquad \left. - f(s, y(f_1(s)), \ldots, y(f_l(s))) \right| ds\, du\, dr \\ &\quad + (m-1) H_i \\ &\quad \times \sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} \int_r^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-i-1} (u-r)^{m-2}}{|a(u)|} \\ &\quad \times \left| h(s, x(h_1(s)), \ldots, x(h_l(s))) \right. \\ &\qquad \left. - h(s, y(h_1(s)), \ldots, y(h_l(s))) \right| ds\, du\, dr \\ &\leq (m-1) H_0 \|x - y\| \\ &\quad \times \int_t^{+\infty} \int_r^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-1} (u-r)^{m-2}}{|a(u)|} P(s)\, ds\, du\, dr \\ &\quad + (m-1) H_i \|x - y\| \\ &\quad \times \int_t^{+\infty} \int_r^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-i-1} (u-r)^{m-2}}{|a(u)|} R(s)\, ds\, du\, dr \\ &= H_0 \|x - y\| \\ &\quad \times \int_t^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-1} (u-t)^{m-1}}{|a(u)|} P(s)\, ds\, du \\ &\quad + H_i \|x - y\| \\ &\quad \times \int_t^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-i-1} (u-t)^{m-1}}{|a(u)|} R(s)\, ds\, du \\ &\leq \|x - y\| \\ &\quad \times \int_T^{+\infty} \int_u^{+\infty} \frac{u^{m-1}}{|a(u)|} \left[H_0 s^{n-m-1} P(s) + H_i s^{n-m-i-1} R(s) \right] ds\, du \\ &= \theta \|x - y\|, \end{aligned} \tag{71}$$

which gives (27). By virtue of (17), (69), and (70), we deduce that for any $x \in \Omega_1(N, M)$ and $t \geq T$

$$\begin{aligned} &S_L x(t) \\ &\leq L + (m-1) H_0 \\ &\quad \times \int_t^{+\infty} \int_r^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-1} (u-r)^{m-2}}{|a(u)|} \\ &\quad \times \left[|g(s)| + \left| f(s, x(f_1(s)), \ldots, x(f_l(s))) \right| \right] ds\, du\, dr \\ &\quad + (m-1) H_i \\ &\quad \times \int_t^{+\infty} \int_r^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-i-1} (u-r)^{m-2}}{|a(u)|} \\ &\quad \times \left| h(s, x(h_1(s)), \ldots, x(h_l(s))) \right| ds\, du\, dr \\ &\leq L + (m-1) H_0 \\ &\quad \times \int_t^{+\infty} \int_r^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-1} (u-r)^{m-2}}{|a(u)|} \\ &\quad \times \left[|g(s)| + Q(s) \right] ds\, du\, dr \\ &\quad + (m-1) H_i \\ &\quad \times \int_t^{+\infty} \int_r^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-i-1} (u-r)^{m-2}}{|a(u)|} W(s)\, ds\, du\, dr \end{aligned}$$

$$
\begin{aligned}
&= L + H_0 \int_t^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-1}(u-t)^{m-1}}{|a(u)|} \\
&\qquad \times \left[|g(s)| + Q(s)\right] ds\, du \\
&\quad + H_i \int_t^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-i-1}(u-t)^{m-1}}{|a(u)|} \\
&\qquad \times W(s)\, ds\, du \\
&\le L + \int_T^{+\infty} \int_u^{+\infty} \frac{u^{m-1}}{|a(u)|} \\
&\qquad \times \left[H_0 s^{n-m-1}\left(|g(s)| + Q(s)\right)\right. \\
&\qquad \left. + H_i s^{n-m-i-1} W(s)\right] ds\, du \\
&< L + \min\{M-L, L-N\} \\
&\le M, \\
&S_L x(t) \\
&\ge L - (m-1) H_0 \\
&\quad \times \int_t^{+\infty} \int_r^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-1}(u-r)^{m-2}}{|a(u)|} \\
&\quad \times \left[\, |g(s)| + \left|f\left(s, x\left(f_1(s)\right), \ldots,\right.\right.\right. \\
&\qquad \left.\left.\left. x\left(f_l(s)\right)\right)\right|\right] ds\, du\, dr \\
&\quad - (m-1) H_i \\
&\quad \times \int_t^{+\infty} \int_r^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-i-1}(u-r)^{m-2}}{|a(u)|} \\
&\quad \times \left|h\left(s, x\left(h_1(s)\right), \ldots, x\left(h_l(s)\right)\right)\right| ds\, du\, dr \\
&\ge L - \int_T^{+\infty} \int_u^{+\infty} \frac{u^{m-1}}{|a(u)|} \\
&\quad \times \left[H_0 s^{n-m-1}\left(|g(s)| + Q(s)\right) + H_i s^{n-m-i-1} W(s)\right] ds\, du \\
&> L - \min\{M-L, L-N\} \\
&\ge N,
\end{aligned}
\tag{72}
$$

which mean that $S_L(\Omega_1(N,M)) \subseteq \Omega_1(N,M)$. Coupled with (27) and (68), we get that S_L is a contraction mapping in $\Omega_1(N,M)$ and it possesses a unique fixed point $x \in \Omega_1(N,M)$. Clearly, $x \in \Omega_1(N,M)$ is a positive solution of (5).

From (27), (66), and (70), we gain that

$$
\begin{aligned}
&\left|x_{k+1}(t) - x(t)\right| \\
&= \Bigg| \left(1-\alpha_k\right) x_k(t) \\
&\quad + \alpha_k \Bigg\{ L + (-1)^n (m-1) H_0 \\
&\qquad \times \sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} \int_r^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-1}(u-r)^{m-2}}{a(u)} \\
&\qquad \times \left[g(s) - f\left(s, x_k\left(f_1(s)\right), \ldots, x_k\left(f_l(s)\right)\right)\right] ds\, du\, dr \\
&\qquad + (-1)^{n-i-1} (m-1) H_i \\
&\qquad \times \sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} \int_r^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-i-1}(u-r)^{m-2}}{a(u)} \\
&\qquad \times h\left(s, x_k\left(h_1(s)\right), \ldots, x_k\left(h_l(s)\right)\right) ds\, du\, dr \Bigg\} - x(t) \Bigg| \\
&\le \left(1-\alpha_k\right)\left|x_k(t) - x(t)\right| + \alpha_k \left|S_L x_k(t) - S_L x(t)\right| \\
&\le \left(1-\alpha_k\right)\left|x_k(t) - x(t)\right| + \alpha_k \theta \left|x_k(t) - x(t)\right| \\
&= \left(1 - (1-\theta)\alpha_k\right)\left|x_k(t) - x(t)\right| \\
&\le e^{-(1-\theta)\alpha_k} \left\|x_k - x\right\| \\
&\le e^{-(1-\theta)\sum_{p=0}^{k} \alpha_p} \left\|x_0 - x\right\|, \quad \forall k \in \mathbb{N}_0,\ t \ge T,
\end{aligned}
\tag{73}
$$

which yields (20). It follows from (20) and (21) that $\lim_{k\to\infty} x_k = x$.

Now we prove that (b) holds. Let $L_1, L_2 \in (N, M)$ and $L_1 \ne L_2$. As in the proof of (a), we conclude that for each $p \in \{1,2\}$, there exist $\theta_p \in (0,1)$, $T_p > 1 + |t_0| + \tau + |\gamma|$ and $S_{L_p} : \Omega_1(N,M) \to \Omega_1(N,M)$ satisfying (69)–(77), where L, θ, T, and S_L are replaced by L_p, θ_p, T_p, and S_{L_p}, respectively, and S_{L_p} has a unique fixed point $z_p \in \Omega_1(N,M)$, which is a positive solution of (5) in $\Omega_1(N,M)$, that is,

$$
\begin{aligned}
z_p(t) &= L_k + (-1)^n (m-1) H_0 \\
&\quad \times \sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} \int_r^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-1}(u-r)^{m-2}}{a(u)} \\
&\quad \times \left[g(s) - f\left(s, z_p\left(f_1(s)\right), \ldots, z_p\left(f_l(s)\right)\right)\right] ds\, du\, dr \\
&\quad + (-1)^{n-i-1} (m-1) H_i \\
&\quad \times \sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} \int_r^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-m-i-1}(u-r)^{m-2}}{a(u)} \\
&\quad \times h\left(s, z_p\left(h_1(s)\right), \ldots, z_p\left(h_l(s)\right)\right) ds\, du\, dr, \\
&\qquad \forall t \ge T_p,\ p \in \{1,2\}.
\end{aligned}
\tag{74}
$$

For purpose of proving (b), we just need to show that $z_1 \neq z_2$. It follows from (16), (27), (68), and (74) that

$$\begin{aligned}
&\left|z_1(t) - z_2(t)\right| \\
&\quad \geq \left|L_1 - L_2\right| - (m-1) H_0 \left\|z_1 - z_2\right\| \\
&\quad\quad \times \int_t^{+\infty} \int_r^{+\infty} \int_u^{+\infty} \left(\left((s-u)^{n-m-1}(u-r)^{m-2}\right)\right. \\
&\quad\quad\quad \left. \times (|a(u)|)^{-1}\right) P(s)\, ds\, du\, dr \\
&\quad\quad - (m-1) H_i \left\|z_1 - z_2\right\| \\
&\quad\quad \times \int_t^{+\infty} \int_r^{+\infty} \int_u^{+\infty} \left(\left((s-u)^{n-m-i-1}(u-r)^{m-2}\right)\right. \\
&\quad\quad\quad \left. \times (|a(u)|)^{-1}\right) R(s)\, ds\, du\, dr \\
&\quad \geq \left|L_1 - L_2\right| - \left\|z_1 - z_2\right\| \\
&\quad\quad \times \int_{\max\{T_1, T_2\}}^{+\infty} \int_u^{+\infty} \frac{u^{m-1}}{|a(u)|} \\
&\quad\quad\quad \times \left[H_0 s^{n-m-1} P(s) + H_i s^{n-m-i-1} R(s)\right] ds\, du \\
&\quad > \left|L_1 - L_2\right| - \max\{\theta_1, \theta_2\} \left\|z_1 - z_2\right\|, \quad \forall t \geq \max\{T_1, T_2\},
\end{aligned} \tag{75}$$

which yields that

$$\left\|z_1 - z_2\right\| \geq \frac{\left|L_1 - L_2\right|}{1 + \max\{\theta_1, \theta_2\}} > 0, \tag{76}$$

that is, $z_1 \neq z_2$. This completes the proof. □

Theorem 8. *Let $m = 1$. Assume that there exist two constants M, N with $M > N > 0$ and four functions $P, Q, R, W \in C([t_0, +\infty), \mathbb{R}^+)$ satisfying* (16), (17), (65), *and*

$$\begin{aligned}
&\int_{t_0}^{+\infty} \int_u^{+\infty} \frac{1}{|a(u)|} \\
&\quad \times \left[|s|^{n-2} \max\{P(s), Q(s), |g(s)|\} + |s|^{n-i-2} \max\{R(s), W(s)\}\right] ds\, du \\
&\quad < +\infty.
\end{aligned} \tag{77}$$

Then

(*a*) *for any $L \in (N, M)$, there exist $\theta \in (0,1)$ and $T > 1 + |t_0| + \tau + |\gamma|$ such that for each $x_0 \in \Omega_1(N, M)$, the Mann iterative sequence $\{x_k\}_{k \in \mathbb{N}_0}$ generated by the following scheme*

$$x_{k+1}(t) = \begin{cases}
(1-\alpha_k) x_k(t) \\
\quad + \alpha_k \left\{ L + \dfrac{(-1)^n}{(n-2)!} \right. \\
\quad \times \displaystyle\sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} \int_u^{+\infty} \frac{(s-u)^{n-2}}{a(u)} \\
\quad \times \left[g(s) - f\left(s, x_k(f_1(s)), \ldots, x_k(f_l(s))\right)\right] ds\, du \\
\quad + \dfrac{(-1)^{n-i-1}}{(n-i-2)!} \\
\quad \times \displaystyle\sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} \int_u^{+\infty} \frac{(s-u)^{n-i-2}}{a(u)} \\
\quad \left. \times h\left(s, x_k(h_1(s)), \ldots, x_k(h_l(s))\right) ds\, du \right\}, \\
\qquad t \geq T,\ k \in \mathbb{N}_0, \\
(1-\alpha_k) x_k(T) \\
\quad + \alpha_k \left\{ L + \dfrac{(-1)^n}{(n-2)!} \right. \\
\quad \times \displaystyle\sum_{j=1}^{\infty} \int_{T+(2j-1)\tau}^{T+2j\tau} \int_u^{+\infty} \frac{(s-u)^{n-2}}{a(u)} \\
\quad \times \left[g(s) - f\left(s, x_k(f_1(s)), \ldots, x_k(f_l(s))\right)\right] ds\, du \\
\quad + \dfrac{(-1)^{n-i-1}}{(n-i-2)!} \\
\quad \times \displaystyle\sum_{j=1}^{\infty} \int_{T+(2j-1)\tau}^{T+2j\tau} \int_u^{+\infty} \frac{(s-u)^{n-i-2}}{a(u)} \\
\quad \left. \times h\left(s, x_k(h_1(s)), \ldots, x_k(h_l(s))\right) ds\, du \right\}, \\
\qquad \gamma \leq t < T,\ k \in \mathbb{N}_0
\end{cases} \tag{78}$$

converges to a positive solution $x \in \Omega_1(N, M)$ of (5) *and has the error estimate* (20), *where $\{\alpha_k\}_{k \in \mathbb{N}_0}$ is an arbitrary sequence in $[0, 1]$ with* (21);

(*b*) *Equation* (5) *has uncountably many positive solutions in $\Omega_1(N, M)$.*

Proof. Let $L \in (N, M)$. It follows from (65) and (77) that there exist $\theta \in (0,1)$ and $T > 1 + |t_0| + \tau + |\gamma|$ satisfying (67),

$$\theta = \int_T^{+\infty} \int_u^{+\infty} \frac{1}{|a(u)|} \times \left[\frac{s^{n-2}}{(n-2)!} P(s) + \frac{s^{n-i-2}}{(n-i-2)!} R(s) \right] ds\, du, \tag{79}$$

$$\begin{aligned} &\int_T^{+\infty} \int_u^{+\infty} \frac{1}{|a(u)|} \left[\frac{s^{n-2}}{(n-2)!} \left(|g(s)| + Q(s) \right) + \frac{s^{n-i-2}}{(n-i-2)!} W(s) \right] ds\, du \\ &\quad < \min \{M - L, L - N\}. \end{aligned} \tag{80}$$

Define a mapping $S_L : \Omega_1(N, M) \to \mathrm{CB}([\gamma, +\infty), \mathbb{R})$ by

$$S_L x(t) = \begin{cases} L + \dfrac{(-1)^n}{(n-2)!} \displaystyle\sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} \int_u^{+\infty} \frac{(s-u)^{n-2}}{a(u)} \\ \quad \times \left[g(s) - f\left(s, x(f_1(s)), \ldots, x(f_l(s))\right) \right] ds\, du \\ \quad + \dfrac{(-1)^{n-i-1}}{(n-i-2)!} \displaystyle\sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} \int_u^{+\infty} \frac{(s-u)^{n-i-2}}{a(u)} \\ \quad \times h\left(s, x(h_1(s)), \ldots, x(h_l(s))\right) ds\, du, \\ \qquad\qquad t \ge T,\ x \in \Omega_1(N, M), \\ S_L x(T), \qquad \gamma \le t < T,\ x \in \Omega_1(N, M). \end{cases} \tag{81}$$

By virtue of (16), (79), and (81), we derive that for any $x, y \in \Omega_1(N, M)$ and $t \ge T$

$$\begin{aligned} &\left| S_L x(t) - S_L y(t) \right| \\ &\quad \le \frac{1}{(n-2)!} \sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} \int_u^{+\infty} \frac{(s-u)^{n-2}}{|a(u)|} \\ &\qquad \times \left| f\left(s, x(f_1(s)), \ldots, x(f_l(s))\right) - f\left(s, y(f_1(s)), \ldots, y(f_l(s))\right) \right| ds\, du \\ &\qquad + \frac{1}{(n-i-2)!} \sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} \int_u^{+\infty} \frac{(s-u)^{n-i-2}}{|a(u)|} \\ &\qquad \times \left| h\left(s, x(h_1(s)), \ldots, x(h_l(s))\right) - h\left(s, y(h_1(s)), \ldots, y(h_l(s))\right) \right| ds\, du \\ &\quad \le \frac{1}{(n-2)!} \|x - y\| \int_T^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-2}}{|a(u)|} P(s)\, ds\, du \\ &\qquad + \frac{1}{(n-i-2)!} \|x - y\| \times \int_T^{+\infty} \int_u^{+\infty} \frac{(s-u)^{n-i-2}}{|a(u)|} R(s)\, ds\, du \\ &\quad \le \|x - y\| \int_T^{+\infty} \int_u^{+\infty} \frac{1}{|a(u)|} \left[\frac{s^{n-2}}{(n-2)!} P(s) + \frac{s^{n-i-2}}{(n-i-2)!} R(s) \right] ds\, du \\ &\quad = \theta \|x - y\|, \end{aligned} \tag{82}$$

which gives (27). It follows from (17), (80), and (81) that for any $x \in \Omega_1(N, M)$ and $t \ge T$

$$\begin{aligned} &S_L x(t) \\ &\quad \le L + \frac{1}{(n-2)!} \sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} \int_u^{+\infty} \frac{(s-u)^{n-2}}{|a(u)|} \\ &\qquad \times \left[|g(s)| + \left| f\left(s, x(f_1(s)), \ldots, x(f_l(s))\right) \right| \right] ds\, du \\ &\qquad + \frac{1}{(n-i-2)!} \sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} \int_u^{+\infty} \frac{(s-u)^{n-i-2}}{|a(u)|} \\ &\qquad \times \left| h\left(s, x(h_1(s)), \ldots, x(h_l(s))\right) \right| ds\, du \\ &\quad \le L + \int_T^{+\infty} \int_u^{+\infty} \frac{1}{|a(u)|} \\ &\qquad \times \left[\frac{s^{n-2}}{(n-2)!} \left(|g(s)| + Q(s) \right) + \frac{s^{n-i-2}}{(n-i-2)!} W(s) \right] ds\, du \\ &\quad < L + \min \{M - L, L - N\} \\ &\quad \le M, \\ &S_L x(t) \\ &\quad \ge L - \frac{1}{(n-2)!} \sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} \int_u^{+\infty} \frac{(s-u)^{n-2}}{|a(u)|} \\ &\qquad \times \left[|g(s)| + \left| f\left(s, x(f_1(s)), \ldots, x(f_l(s))\right) \right| \right] ds\, du \\ &\qquad - \frac{1}{(n-i-2)!} \sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} \int_u^{+\infty} \frac{(s-u)^{n-i-2}}{|a(u)|} \\ &\qquad \times \left| h\left(s, x(h_1(s)), \ldots, x(h_l(s))\right) \right| ds\, du \\ &\quad \ge L - \int_T^{+\infty} \int_u^{+\infty} \frac{1}{|a(u)|} \end{aligned}$$

$$\times \left[\frac{s^{n-2}}{(n-2)!} \left(|g(s)| + Q(s) \right) + \frac{s^{n-i-2}}{(n-i-2)!} W(s) \right] ds\, du$$
$$> L - \min \{M - L, L - N\}$$
$$\geq N, \tag{83}$$

which mean that $S_L(\Omega_1(N, M)) \subseteq \Omega_1(N, M)$. Combined with (27) and (79), we know that S_L is a contraction mapping in $\Omega_1(N, M)$ and it possesses a unique fixed point $x \in \Omega_1(N, M)$. Obviously, $x \in \Omega_1(N, M)$ is a positive solution of (5).

In light of (27), (78), and (81), we gain that

$$\begin{aligned}
&|x_{k+1}(t) - x(t)| \\
&= \Bigg| (1 - \alpha_k) x_k(t) \\
&\quad + \alpha_k \Bigg\{ L + \frac{(-1)^n}{(n-2)!} \sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} \int_u^{+\infty} \frac{(s-u)^{n-2}}{a(u)} \\
&\qquad \times [g(s) - f(s, x_k(f_1(s)), \ldots, x_k(f_l(s)))]\, ds\, du \\
&\qquad + \frac{(-1)^{n-i-1}}{(n-i-2)!} \sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} \int_u^{+\infty} \frac{(s-u)^{n-i-2}}{a(u)} \\
&\qquad \times h(s, x_k(h_1(s)), \ldots, x_k(h_l(s)))\, ds\, du \Bigg\} - x(t) \Bigg| \\
&\leq (1 - \alpha_k) |x_k(t) - x(t)| + \alpha_k |S_L x_k(t) - S_L x(t)| \\
&\leq (1 - \alpha_k) |x_k(t) - x(t)| + \alpha_k \theta |x_k(t) - x(t)| \\
&= (1 - (1 - \theta) \alpha_k) |x_k(t) - x(t)| \\
&\leq e^{-(1-\theta)\alpha_k} \|x_k - x\| \\
&\leq e^{-(1-\theta) \sum_{p=0}^{k} \alpha_p} \|x_0 - x\|, \quad \forall k \in \mathbb{N}_0,\ t \geq T,
\end{aligned} \tag{84}$$

which yields (20). It follows from (20) and (21) that $\lim_{k\to\infty} x_k = x$.

Now we prove that (b) holds. Let $L_1, L_2 \in (N, M)$ and $L_1 \neq L_2$. As in the proof of (a), we conclude that for each $p \in \{1, 2\}$, there exist $\theta_p \in (0, 1)$, $T_p > 1 + |t_0| + \tau + |\gamma|$ and $S_{L_p} : \Omega_1(N, M) \to \Omega_1(N, M)$ satisfying (67) and (79)–(81), where L, θ, T, and S_L are replaced by L_p, θ_p, T_p, and S_{L_p}, respectively, and S_{L_p} has a unique fixed point $z_p \in \Omega_1(N, M)$, which is a positive solution of (5) in $\Omega_1(N, M)$, that is,

$$\begin{aligned}
z_p(t) &= L_p + \frac{(-1)^n}{(n-2)!} \\
&\quad \times \sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} \int_u^{+\infty} \frac{(s-u)^{n-2}}{a(u)} \\
&\quad \times \left[g(s) - f\left(s, z_p(f_1(s)), \ldots, z_p(f_l(s))\right) \right] ds\, du \\
&\quad + \frac{(-1)^{n-i-1}}{(n-i-2)!} \sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} \int_u^{+\infty} \frac{(s-u)^{n-i-2}}{a(u)} \\
&\quad \times h\left(s, z_p(h_1(s)), \ldots, z_p(h_l(s))\right) ds\, du, \\
&\qquad \forall t \geq T_p,\ p \in \{1, 2\}.
\end{aligned} \tag{85}$$

In order to prove (b), we just need to show that $z_1 \neq z_2$. In view of (16), (27), (79), and (85), we get that

$$\begin{aligned}
&|z_1(t) - z_2(t)| \\
&\geq |L_1 - L_2| - \frac{1}{(n-2)!} \sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} \int_u^{+\infty} \frac{(s-u)^{n-2}}{|a(u)|} \\
&\quad \times |f(s, z_2(f_1(s)), \ldots, z_2(f_l(s))) \\
&\qquad - f(s, z_1(f_1(s)), \ldots, z_1(f_l(s)))|\, ds\, du \\
&\quad - \frac{1}{(n-i-2)!} \sum_{j=1}^{\infty} \int_{t+(2j-1)\tau}^{t+2j\tau} \int_u^{+\infty} \frac{(s-u)^{n-i-2}}{|a(u)|} \\
&\quad \times |h(s, z_1(h_1(s)), \ldots, z_1(h_l(s))) \\
&\qquad - h(s, z_2(h_1(s)), \ldots, z_2(h_l(s)))|\, ds\, du \\
&\geq |L_1 - L_2| - \frac{\|z_1 - z_2\|}{(n-2)!} \int_t^{+\infty} \int_u^{+\infty} \frac{s^{n-2}}{|a(u)|} P(s)\, ds\, du \\
&\quad - \frac{\|z_1 - z_2\|}{(n-i-2)!} \int_t^{+\infty} \int_u^{+\infty} \frac{s^{n-i-2}}{|a(u)|} R(s)\, ds\, du \\
&\geq |L_1 - L_2| - \|z_1 - z_2\| \\
&\quad \times \int_{\max\{T_1, T_2\}}^{+\infty} \int_u^{+\infty} \frac{1}{|a(u)|} \\
&\quad \times \left[\frac{s^{n-2}}{(n-2)!} P(s) + \frac{s^{n-i-2}}{(n-i-2)!} R(s) \right] ds\, du \\
&> |L_1 - L_2| - \max\{\theta_1, \theta_2\} \|z_1 - z_2\|, \\
&\qquad \forall t \geq \max\{T_1, T_2\},
\end{aligned} \tag{86}$$

which implies that

$$\|z_1 - z_2\| \geq \frac{|L_1 - L_2|}{1 + \max\{\theta_1, \theta_2\}} > 0, \tag{87}$$

that is, $z_1 \neq z_2$. This completes the proof. □

Theorem 9. *Let $m \geq 2$. Assume that there exist two constants M, N with $M > N > 0$ and four functions $P, Q, R, W \in C([t_0, +\infty), \mathbb{R}^+)$ satisfying* (16), (17),

$$\begin{aligned} &\int_{t_0}^{+\infty}\int_{r}^{+\infty}\int_{u}^{+\infty} \frac{|r|\,|u|^m}{|a(u)|} \\ &\qquad \times \left[|s|^{n-m-1}\max\{P(s), Q(s), |g(s)|\}\right. \\ &\qquad \left. +|s|^{n-m-i-1}\max\{R(s), W(s)\}\right] ds\,du\,dr \\ &< +\infty, \end{aligned} \tag{88}$$

$$b(t) = -1 \text{ eventually.} \tag{89}$$

Then

(*a*) *for any $L \in (N, M)$, there exist $\theta \in (0,1)$ and $T > 1 + |t_0| + \tau + |\gamma|$ such that for each $x_0 \in \Omega_1(N, M)$, the Mann iterative sequence $\{x_k\}_{k\in\mathbb{N}_0}$ generated by the following scheme*

$$x_{k+1}(t) = \begin{cases} (1-\alpha_k)\,x_k(t) \\ +\alpha_k\Big\{L + (-1)^{n-1}(m-1)H_0 \\ \times\sum_{j=1}^{\infty}\int_{t+j\tau}^{+\infty}\int_{r}^{+\infty}\int_{u}^{+\infty}\frac{(s-u)^{n-m-1}(u-r)^{m-2}}{a(u)} \\ \times\left[g(s) - f\left(s, x_k(f_1(s)), \ldots, x_k(f_l(s))\right)\right] ds\,du\,dr \\ +(-1)^{n-i}(m-1)H_i \\ \times\sum_{j=1}^{\infty}\int_{t+j\tau}^{+\infty}\int_{r}^{+\infty}\int_{u}^{+\infty}\frac{(s-u)^{n-m-i-1}(u-r)^{m-2}}{a(u)} \\ \times h\left(s, x_k(h_1(s)), \ldots, x_k(h_l(s))\right) ds\,du\,dr\Big\}, \\ \qquad\qquad t \geq T,\ k \in \mathbb{N}_0, \\ (1-\alpha_k)\,x_k(T) \\ +\alpha_k\Big\{L + (-1)^{n-1}(m-1)H_0 \\ \times\sum_{j=1}^{\infty}\int_{T+j\tau}^{+\infty}\int_{r}^{+\infty}\int_{u}^{+\infty}\frac{(s-u)^{n-m-1}(u-r)^{m-2}}{a(u)} \\ \times\left[g(s) - f\left(s, x_k(f_1(s)), \ldots, x_k(f_l(s))\right)\right] ds\,du\,dr \\ +(-1)^{n-i}(m-1)H_i \\ \times\sum_{j=1}^{\infty}\int_{T+j\tau}^{+\infty}\int_{r}^{+\infty}\int_{u}^{+\infty}\frac{(s-u)^{n-m-i-1}(u-r)^{m-2}}{a(u)} \\ \times h\left(s, x_k(h_1(s)), \ldots, x_k(h_l(s))\right) ds\,du\,dr\Big\}, \\ \qquad\qquad \gamma \leq t < T,\ k \in \mathbb{N}_0 \end{cases} \tag{90}$$

converges to a positive solution $x \in \Omega_1(N, M)$ of (5) *and has the error estimate* (20), *where $\{\alpha_k\}_{k\in\mathbb{N}_0}$ is an arbitrary sequence in $[0,1]$ with* (21);

(*b*) *Equation* (5) *has uncountably many positive solutions in $\Omega_1(N, M)$.*

Proof. Set $L \in (N, M)$. In view of (88) and (89), there exist $\theta \in (0,1)$ and $T > 1 + |t_0| + \tau + |\gamma|$ such that

$$b(t) = -1, \quad \forall t \geq T; \tag{91}$$

$$\begin{aligned}\theta = \frac{m-1}{\tau}&\int_{T}^{+\infty}\int_{r}^{+\infty}\int_{u}^{+\infty}\frac{ru^{m-2}}{|a(u)|} \\ &\times\left[H_0 s^{n-m-1}P(s) + H_i s^{n-m-i-1}R(s)\right] ds\,du\,dr;\end{aligned} \tag{92}$$

$$\begin{aligned}&\frac{m-1}{\tau}\int_{T}^{+\infty}\int_{r}^{+\infty}\int_{u}^{+\infty}\frac{ru^{m-2}}{|a(u)|} \\ &\quad\times\left[H_0 s^{n-m-1}\left(|g(s)| + Q(s)\right) + H_i s^{n-m-i-1}W(s)\right] ds\,du\,dr \\ &< \min\{M - L, L - N\}.\end{aligned} \tag{93}$$

Define a mapping $S_L : \Omega_1(N, M) \to CB([\gamma, +\infty), \mathbb{R})$ by

$$S_L x(t) = \begin{cases} L + (-1)^{n-1}(m-1)H_0 \\ \times\sum_{j=1}^{\infty}\int_{t+j\tau}^{+\infty}\int_{r}^{+\infty}\int_{u}^{+\infty}\frac{(s-u)^{n-m-1}(u-r)^{m-2}}{a(u)} \\ \times\left[g(s) - f\left(s, x(f_1(s)), \ldots, x(f_l(s))\right)\right] ds\,du\,dr \\ +(-1)^{n-i}(m-1)H_i \\ \times\sum_{j=1}^{\infty}\int_{t+j\tau}^{+\infty}\int_{r}^{+\infty}\int_{u}^{+\infty}\frac{(s-u)^{n-m-i-1}(u-r)^{m-2}}{a(u)} \\ \times h\left(s, x(h_1(s)), \ldots, x(h_l(s))\right) ds\,du\,dr, \\ \qquad\qquad t \geq T,\ x \in \Omega_1(N, M), \\ S_L x(T), \qquad \gamma \leq t < T,\ x \in \Omega_1(N, M). \end{cases} \tag{94}$$

By virtue of (16), (92), (94), and Lemma 1, we acquire that for any $x, y \in \Omega_1(N, M)$ and $t \geq T$

$$\begin{aligned}&|S_L x(t) - S_L y(t)| \\ &\leq (m-1)H_0\sum_{j=1}^{\infty}\int_{t+j\tau}^{+\infty}\int_{r}^{+\infty}\int_{u}^{+\infty}\frac{(s-u)^{n-m-1}(u-r)^{m-2}}{|a(u)|} \\ &\quad\times\left|f\left(s, x(f_1(s)), \ldots, x(f_l(s))\right)\right. \\ &\qquad\left.-f\left(s, y(f_1(s)), \ldots, y(f_l(s))\right)\right| ds\,du\,dr\end{aligned}$$

$$+(m-1)H_i\sum_{j=1}^{\infty}\int_{t+j\tau}^{+\infty}\int_r^{+\infty}\int_u^{+\infty}\frac{(s-u)^{n-m-i-1}(u-r)^{m-2}}{|a(u)|}\times\left|h(s,x(h_1(s)),\ldots,x(h_l(s)))-h(s,y(h_1(s)),\ldots,y(h_l(s)))\right|ds\,du\,dr$$
$$\le(m-1)\frac{H_0}{\tau}\|x-y\|\times\int_T^{+\infty}\int_r^{+\infty}\int_u^{+\infty}\frac{(s-u)^{n-m-1}(u-r)^{m-2}r}{|a(u)|}P(s)\,ds\,du\,dr$$
$$+(m-1)\frac{H_i}{\tau}\|x-y\|\times\int_T^{+\infty}\int_r^{+\infty}\int_u^{+\infty}\frac{(s-u)^{n-m-i-1}(u-r)^{m-2}r}{|a(u)|}R(s)\,ds\,du\,dr$$
$$\le\frac{m-1}{\tau}\int_T^{+\infty}\int_r^{+\infty}\int_u^{+\infty}\frac{ru^{m-2}}{|a(u)|}\times\left[H_0s^{n-m-1}P(s)+H_is^{n-m-i-1}R(s)\right]ds\,du\,dr$$
$$=\theta\|x-y\|, \tag{95}$$

which yields that (27) holds. From (17), (94), (98), and Lemma 1, we obtain that for any $x\in\Omega_1(N,M)$ and $t\ge T$

$$\left|S_Lx(t)-L\right|$$
$$\le(m-1)H_0\sum_{j=1}^{\infty}\int_{t+j\tau}^{+\infty}\int_r^{+\infty}\int_u^{+\infty}\frac{(s-u)^{n-m-1}(u-r)^{m-2}}{|a(u)|}\times\left[|g(s)|+|f(s,x(f_1(s)),\ldots,x(f_l(s)))|\right]ds\,du\,dr$$
$$+(m-1)H_i\sum_{j=1}^{\infty}\int_{t+j\tau}^{+\infty}\int_r^{+\infty}\int_u^{+\infty}\frac{(s-u)^{n-m-i-1}(u-r)^{m-2}}{|a(u)|}\times|h(s,x(h_1(s)),\ldots,x(h_l(s)))|\,ds\,du\,dr$$
$$\le(m-1)\frac{H_0}{\tau}\int_T^{+\infty}\int_r^{+\infty}\int_u^{+\infty}\frac{rs^{n-m-1}u^{m-2}}{|a(u)|}\times\left[|g(s)|+Q(s)\right]ds\,du\,dr$$
$$+(m-1)\frac{H_i}{\tau}\int_T^{+\infty}\int_r^{+\infty}\int_u^{+\infty}\frac{rs^{n-m-i-1}u^{m-2}}{|a(u)|}\times W(s)\,ds\,du\,dr$$
$$=\frac{m-1}{\tau}\int_T^{+\infty}\int_r^{+\infty}\int_u^{+\infty}\frac{ru^{m-2}}{|a(u)|}\times\left[H_0s^{n-m-1}(|g(s)|+Q(s))+H_is^{n-m-i-1}W(s)\right]ds\,du\,dr$$
$$<\min\{M-L,L-N\}, \tag{96}$$

which means that $S_L(\Omega_1(N,M))\subseteq\Omega_1(N,M)$. It follows from (27) and (92) that S_L is a contraction mapping and it has a unique fixed point $x\in\Omega_1(N,M)$. It is clear that $x\in\Omega_1(N,M)$ is a positive solution of (5).

On the basis of (27), (90), and (94), we deduce that

$$\left|x_{k+1}(t)-x(t)\right|$$
$$=\Bigg|(1-\alpha_k)x_k(t)+\alpha_k\Bigg\{L+(-1)^{n-1}(m-1)H_0\times\sum_{j=1}^{\infty}\int_{t+j\tau}^{+\infty}\int_r^{+\infty}\int_u^{+\infty}\frac{(s-u)^{n-m-1}(u-r)^{m-2}}{a(u)}\times\left[g(s)-f(s,x_k(f_1(s)),\ldots,x_k(f_l(s)))\right]ds\,du\,dr$$
$$+(-1)^{n-i}(m-1)H_i\times\sum_{j=1}^{\infty}\int_{t+j\tau}^{+\infty}\int_r^{+\infty}\int_u^{+\infty}\frac{(s-u)^{n-m-i-1}(u-r)^{m-2}}{a(u)}\times h(s,x_k(h_1(s)),\ldots,x_k(h_l(s)))\,ds\,du\,dr\Bigg\}-x(t)\Bigg|$$
$$\le(1-\alpha_k)\left|x_k(t)-x(t)\right|+\alpha_k\left|S_Lx_k(t)-S_Lx(t)\right|$$
$$\le(1-\alpha_k)\left|x_k(t)-x(t)\right|+\alpha_k\theta\left|x_k(t)-x(t)\right|$$
$$=(1-(1-\theta)\alpha_k)\left|x_k(t)-x(t)\right|$$
$$\le e^{-(1-\theta)\alpha_k}\|x_k-x\|$$
$$\le e^{-(1-\theta)\sum_{p=0}^{k}\alpha_p}\|x_0-x\|,\quad\forall k\in\mathbb{N}_0,\ t\ge T, \tag{97}$$

which signifies that (20) holds. It follows from (20) and (21) and that $\lim_{k\to\infty}x_k=x$.

Now we show that (b) holds. Let $L_1,L_2\in(N,M)$ and $L_1\ne L_2$. As in the proof of (a), we conclude that for each $p\in\{1,2\}$, there exist $\theta_p\in(0,1)$, $T_p>1+|t_0|+\tau+|\gamma|$ and $S_{L_p}:\Omega_1(N,M)\to\Omega_1(N,M)$ satisfying (91)–(94), where L, θ, T, and S_L are replaced by L_p, θ_p, T_p, and S_{L_p}, respectively, and S_{L_p} has a unique fixed point $z_p \in \Omega_1(N,M)$, which is a positive solution of (5) in $\Omega_1(N,M)$, that is,

$$\begin{aligned} z_p(t) &= L_p + (-1)^{n-1}(m-1)H_0 \\ &\quad\times \sum_{j=1}^{\infty}\int_{t+j\tau}^{+\infty}\int_r^{+\infty}\int_u^{+\infty}\frac{(s-u)^{n-m-1}(u-r)^{m-2}}{a(u)} \\ &\quad\times\left[g(s)-f\left(s,z_p(f_1(s)),\ldots,z_p(f_l(s))\right)\right]ds\,du\,dr \\ &\quad+(-1)^{n-i}(m-1)H_i \\ &\quad\times\sum_{j=1}^{\infty}\int_{t+j\tau}^{+\infty}\int_r^{+\infty}\int_u^{+\infty}\frac{(s-u)^{n-m-i-1}(u-r)^{m-2}}{a(u)} \\ &\quad\times h\left(s,z_p(h_1(s)),\ldots,z_p(h_l(s))\right)ds\,du\,dr, \\ &\qquad\forall t\ge T_p,\ p\in\{1,2\}. \end{aligned} \tag{98}$$

In order to prove (b), it is sufficient to show that $z_1 \neq z_2$. Note that (16), (92), (98), and Lemma 1 lead to

$$\begin{aligned} &\left|z_1(t)-z_2(t)\right| \\ &\ge\left|L_1-L_2\right|-(m-1)\frac{H_0}{\tau}\left\|z_1-z_2\right\| \\ &\quad\times\int_{t+\tau}^{+\infty}\int_r^{+\infty}\int_u^{+\infty}\frac{rs^{n-m-1}u^{m-2}}{|a(u)|}P(s)\,ds\,du\,dr \\ &\quad-(m-1)\frac{H_i}{\tau}\left\|z_1-z_2\right\| \\ &\quad\times\int_{t+\tau}^{+\infty}\int_r^{+\infty}\int_u^{+\infty}\frac{rs^{n-m-i-1}u^{m-2}}{|a(u)|}R(s)\,ds\,du\,dr \\ &\ge\left|L_1-L_2\right|-\frac{(m-1)\left\|z_1-z_2\right\|}{\tau} \\ &\quad\times\int_{\max\{T_1,T_2\}}^{+\infty}\int_r^{+\infty}\int_u^{+\infty}\frac{ru^{m-2}}{|a(u)|} \\ &\quad\times\left[H_0s^{n-m-1}P(s)+H_is^{n-m-i-1}R(s)\right]ds\,du\,dr \\ &>\left|L_1-L_2\right|-\max\{\theta_1,\theta_2\}\left\|z_1-z_2\right\|,\quad\forall t\ge\max\{T_1,T_2\}, \end{aligned} \tag{99}$$

which means that

$$\left\|z_1-z_2\right\|\ge\frac{\left|L_1-L_2\right|}{1+\max\{\theta_1,\theta_2\}}>0, \tag{100}$$

that is, $z_1 \neq z_2$. This completes the proof. □

Theorem 10. *Let $m=1$. Assume that there exist two constants M, N with $M>N>0$ and four functions $P,Q,R,W\in C([t_0,+\infty),\mathbb{R}^+)$ satisfying* (16), (17), (89), *and*

$$\begin{aligned} &\int_{t_0}^{+\infty}\int_u^{+\infty}\frac{|u|}{|a(u)|}\left[|s|^{n-2}\max\{P(s),Q(s),|g(s)|\}\right. \\ &\qquad\left.+|s|^{n-i-2}\max\{R(s),W(s)\}\right]ds\,du \\ &<+\infty. \end{aligned} \tag{101}$$

Then

(*a*) *for any $L\in(N,M)$, there exist $\theta\in(0,1)$ and $T>1+|t_0|+\tau+|\gamma|$ such that for each $x_0\in\Omega_1(N,M)$, the Mann iterative sequence $\{x_k\}_{k\in\mathbb{N}_0}$ generated by the following scheme*

$$x_{k+1}(t)=\begin{cases}(1-\alpha_k)x_k(t) \\ \quad+\alpha_k\left\{L+\dfrac{(-1)^{n-1}}{(n-2)!}\displaystyle\sum_{j=1}^{\infty}\int_{t+j\tau}^{+\infty}\int_u^{+\infty}\frac{(s-u)^{n-2}}{a(u)}\right. \\ \qquad\times\left[g(s)-f\left(s,x_k(f_1(s)),\ldots,x_k(f_l(s))\right)\right]ds\,du \\ \qquad+\dfrac{(-1)^{n-i}}{(n-i-2)!}\displaystyle\sum_{j=1}^{\infty}\int_{t+j\tau}^{+\infty}\int_u^{+\infty}\frac{(s-u)^{n-i-2}}{a(u)} \\ \qquad\left.\times h\left(s,x_k(h_1(s)),\ldots,x_k(h_l(s))\right)ds\,du\right\}, \\ \qquad\qquad t\ge T,\ k\in\mathbb{N}_0, \\ (1-\alpha_k)x_k(T) \\ \quad+\alpha_k\left\{L+\dfrac{(-1)^{n-1}}{(n-2)!}\displaystyle\sum_{j=1}^{\infty}\int_{T+j\tau}^{+\infty}\int_u^{+\infty}\frac{(s-u)^{n-2}}{a(u)}\right. \\ \qquad\times\left[g(s)-f\left(s,x_k(f_1(s)),\ldots,x_k(f_l(s))\right)\right]ds\,du \\ \qquad+\dfrac{(-1)^{n-i}}{(n-i-2)!}\displaystyle\sum_{j=1}^{\infty}\int_{T+j\tau}^{+\infty}\int_u^{+\infty}\frac{(s-u)^{n-i-2}}{a(u)} \\ \qquad\left.\times h\left(s,x_k(h_1(s)),\ldots,x_k(h_l(s))\right)ds\,du\right\}, \\ \qquad\qquad\gamma\le t<T,\ k\in\mathbb{N}_0,\end{cases} \tag{102}$$

converges to a positive solution $x\in\Omega_1(N,M)$ of (5) *and has the error estimate* (20), *where $\{\alpha_k\}_{k\in\mathbb{N}_0}$ is an arbitrary sequence in $[0,1]$ with* (21);

(*b*) *Equation* (5) *has uncountably many positive solutions in $\Omega_1(N,M)$.*

Proof. Set $L \in (N, M)$. Due to (101), there exist $\theta \in (0,1)$ and $T > 1 + |t_0| + \tau + |\gamma|$ satisfying (91),

$$\theta = \frac{1}{\tau}\int_T^{+\infty}\int_u^{+\infty}\frac{u}{|a(u)|}\times\left[\frac{s^{n-2}}{(n-2)!}P(s)+\frac{s^{n-i-2}}{(n-i-2)!}R(s)\right]ds\,du, \tag{103}$$

$$\begin{aligned}&\frac{1}{\tau}\int_T^{+\infty}\int_u^{+\infty}\frac{u}{|a(u)|}\\&\times\left[\frac{s^{n-2}}{(n-2)!}(|g(s)|+Q(s))+\frac{s^{n-i-2}}{(n-i-2)!}W(s)\right]ds\,du\\&<\min\{M-L, L-N\}.\end{aligned} \tag{104}$$

Define a mapping $S_L : \Omega_1(N,M) \to \mathrm{CB}([\gamma,+\infty),\mathbb{R})$ by

$$S_Lx(t) = \begin{cases} L + \dfrac{(-1)^{n-1}}{(n-2)!}\displaystyle\sum_{j=1}^{\infty}\int_{t+j\tau}^{+\infty}\int_u^{+\infty}\frac{(s-u)^{n-2}}{a(u)} \\ \times\left[g(s)-f(s,x(f_1(s)),\ldots,x(f_l(s)))\right]ds\,du \\ +\dfrac{(-1)^{n-i}}{(n-i-2)!}\displaystyle\sum_{j=1}^{\infty}\int_{t+j\tau}^{+\infty}\int_u^{+\infty}\frac{(s-u)^{n-i-2}}{a(u)} \\ \times h(s,x(h_1(s)),\ldots,x(h_l(s)))\,ds\,du, \\ \qquad t\ge T,\ x\in\Omega_1(N,M), \\ S_Lx(T), \quad \gamma\le t<T,\ x\in\Omega_1(N,M). \end{cases} \tag{105}$$

In view of (16), (103), (105), and Lemma 1, we achieve that for any $x, y \in \Omega_1(N,M)$ and $t \ge T$

$$\begin{aligned}&|S_Lx(t)-S_Ly(t)|\\&\le\frac{1}{(n-2)!}\sum_{j=1}^{\infty}\int_{t+j\tau}^{+\infty}\int_u^{+\infty}\frac{(s-u)^{n-2}}{|a(u)|}\\&\quad\times|f(s,x(f_1(s)),\ldots,x(f_l(s)))\\&\quad-f(s,y(f_1(s)),\ldots,y(f_l(s)))|\,ds\,du\\&\quad+\frac{1}{(n-i-2)!}\sum_{j=1}^{\infty}\int_{t+j\tau}^{+\infty}\int_u^{+\infty}\frac{(s-u)^{n-i-2}}{|a(u)|}\\&\quad\times|h(s,x(h_1(s)),\ldots,x(h_l(s)))\\&\quad-h(s,y(h_1(s)),\ldots,y(h_l(s)))|\,ds\,du\\&\le\frac{\|x-y\|}{\tau(n-2)!}\int_{t+\tau}^{+\infty}\int_u^{+\infty}\frac{us^{n-2}}{|a(u)|}P(s)\,ds\,du\\&\quad+\frac{\|x-y\|}{\tau(n-i-2)!}\int_{t+\tau}^{+\infty}\int_u^{+\infty}\frac{us^{n-i-2}}{|a(u)|}R(s)\,ds\,du\\&\le\frac{\|x-y\|}{\tau}\int_T^{+\infty}\int_u^{+\infty}\frac{u}{|a(u)|}\\&\quad\times\left[\frac{s^{n-2}}{(n-2)!}P(s)+\frac{s^{n-i-2}}{(n-i-2)!}R(s)\right]ds\,du\\&=\theta\|x-y\|,\end{aligned} \tag{106}$$

which means that (27) holds. It follows from (17), (104), (105), and Lemma 1 that for any $x \in \Omega_1(N,M)$ and $t \ge T$

$$\begin{aligned}&|S_Lx(t)-L|\\&\le\frac{1}{(n-2)!}\sum_{j=1}^{\infty}\int_{t+j\tau}^{+\infty}\int_u^{+\infty}\frac{(s-u)^{n-2}}{|a(u)|}\\&\quad\times\left[|g(s)|+|f(s,x(f_1(s)),\ldots,x(f_l(s)))|\right]ds\,du\\&\quad+\frac{1}{(n-i-2)!}\sum_{j=1}^{\infty}\int_{t+j\tau}^{+\infty}\int_u^{+\infty}\frac{(s-u)^{n-i-2}}{|a(u)|}\\&\quad\times|h(s,x(h_1(s)),\ldots,x(h_l(s)))|\,ds\,du\\&\le\frac{1}{\tau(n-2)!}\int_{t+\tau}^{+\infty}\int_u^{+\infty}\frac{us^{n-2}}{|a(u)|}(|g(s)|+Q(s))\,ds\,du\\&\quad+\frac{1}{\tau(n-i-2)!}\int_{t+\tau}^{+\infty}\int_u^{+\infty}\frac{us^{n-i-2}}{|a(u)|}W(s)\,ds\,du\\&\le\frac{1}{\tau}\int_T^{+\infty}\int_u^{+\infty}\frac{u}{|a(u)|}\\&\quad\times\left[\frac{s^{n-2}}{(n-2)!}(|g(s)|+Q(s))+\frac{s^{n-i-2}}{(n-i-2)!}W(s)\right]ds\,du\\&<\min\{M-L, L-N\},\end{aligned} \tag{107}$$

which means that $S_L(\Omega_1(N,M)) \subseteq \Omega_1(N,M)$. Coupled with (27), we know that S_L is a contraction mapping and it has a unique fixed point $x \in \Omega_1(N,M)$. It follows that $x \in \Omega_1(N,M)$ is a positive solution of (5).

In view of (27), (102), and (105), we deduce that

$$\begin{aligned}&|x_{k+1}(t)-x(t)|\\&=\Bigg|(1-\alpha_k)x_k(t)\\&\quad+\alpha_k\Bigg\{L+\frac{(-1)^{n-1}}{(n-2)!}\sum_{j=1}^{\infty}\int_{t+j\tau}^{+\infty}\int_u^{+\infty}\frac{(s-u)^{n-2}}{a(u)}\\&\quad\times\left[g(s)-f(s,x_k(f_1(s)),\ldots,x_k(f_l(s)))\right]ds\,du\end{aligned}$$

$$+\frac{(-1)^{n-i}}{(n-i-2)!}\sum_{j=1}^{\infty}\int_{t+j\tau}^{+\infty}\int_{u}^{+\infty}\frac{(s-u)^{n-i-2}}{a(u)}$$
$$\left.\times h\left(s,x_k\left(h_1(s)\right),\ldots,x_k\left(h_l(s)\right)\right)ds\,du\right\}-x(t)\Bigg|$$
$$\le\left(1-\alpha_k\right)\left|x_k(t)-x(t)\right|+\alpha_k\left|S_Lx_k(t)-S_Lx(t)\right|$$
$$\le\left(1-\alpha_k\right)\left|x_k(t)-x(t)\right|+\alpha_k\theta\left|x_k(t)-x(t)\right|$$
$$=\left(1-(1-\theta)\alpha_k\right)\left|x_k(t)-x(t)\right|$$
$$\le e^{-(1-\theta)\alpha_k}\left\|x_k-x\right\|$$
$$\le e^{-(1-\theta)\sum_{p=0}^{k}\alpha_p}\left\|x_0-x\right\|,\quad\forall k\in\mathbb{N}_0,\ t\ge T,\tag{108}$$

which signifies that (20) holds. It follows from (20) and (21) that $\lim_{k\to\infty}x_k=x$.

Now we show that (b) holds. Let $L_1,L_2\in(N,M)$ and $L_1\neq L_2$. As in the proof of (a), we conclude that for each $p\in\{1,2\}$, there exist $\theta_p\in(0,1)$, $T_p>1+|t_0|+\tau+|\gamma|$ and $S_{L_p}:\Omega_1(N,M)\to\Omega_1(N,M)$ satisfying (91) and (103)–(105), where L,θ,T, and S_L are replaced by L_p,θ_p,T_p and S_{L_p}, respectively, and S_{L_p} has a unique fixed point $z_p\in\Omega_1(N,M)$, which is a positive solution of (5) in $\Omega_1(N,M)$. It follows that for any $t\ge T_p$ and $p\in\{1,2\}$

$$z_p(t)=L_p+\frac{(-1)^{n-1}}{(n-2)!}\sum_{j=1}^{\infty}\int_{t+j\tau}^{+\infty}\int_{u}^{+\infty}\frac{(s-u)^{n-2}}{a(u)}$$
$$\times\left[g(s)-f\left(s,z_p\left(f_1(s)\right),\ldots,z_p\left(f_l(s)\right)\right)\right]ds\,du$$
$$+\frac{(-1)^{n-i}}{(n-i-2)!}\sum_{j=1}^{\infty}\int_{t+j\tau}^{+\infty}\int_{u}^{+\infty}\frac{(s-u)^{n-i-2}}{a(u)}$$
$$\times h\left(s,z_p\left(h_1(s)\right),\ldots,z_p\left(h_l(s)\right)\right)ds\,du.\tag{109}$$

In order to prove (b), we just need to show that $z_1\neq z_2$. Notice that (16), (103), (109), and Lemma 1 ensure that

$$\left|z_1(t)-z_2(t)\right|$$
$$\ge\left|L_1-L_2\right|-\frac{1}{(n-2)!}\sum_{j=1}^{\infty}\int_{t+j\tau}^{+\infty}\int_{u}^{+\infty}\frac{(s-u)^{n-2}}{|a(u)|}$$
$$\times\left|f\left(s,z_2\left(f_1(s)\right),\ldots,z_2\left(f_l(s)\right)\right)\right.$$
$$\left.-f\left(s,z_1\left(f_1(s)\right),\ldots,z_1\left(f_l(s)\right)\right)\right|ds\,du$$
$$-\frac{1}{(n-i-2)!}\sum_{j=1}^{\infty}\int_{t+j\tau}^{+\infty}\int_{u}^{+\infty}\frac{(s-u)^{n-i-2}}{|a(u)|}$$
$$\times\left|h\left(s,z_1\left(h_1(s)\right),\ldots,z_1\left(h_l(s)\right)\right)\right.$$
$$\left.-h\left(s,z_2\left(h_1(s)\right),\ldots,z_2\left(h_l(s)\right)\right)\right|ds\,du$$
$$\ge\left|L_1-L_2\right|-\frac{\left\|z_1-z_2\right\|}{\tau(n-2)!}\int_{t+\tau}^{+\infty}\int_{u}^{+\infty}\frac{us^{n-2}}{|a(u)|}P(s)\,ds\,du$$
$$-\frac{\left\|z_1-z_2\right\|}{\tau(n-i-2)!}\int_{t+\tau}^{+\infty}\int_{u}^{+\infty}\frac{us^{n-i-2}}{|a(u)|}R(s)\,ds\,du$$
$$\ge\left|L_1-L_2\right|-\frac{\left\|z_1-z_2\right\|}{\tau}\int_{\max\{T_1,T_2\}}^{+\infty}\int_{u}^{+\infty}\frac{u}{|a(u)|}$$
$$\times\left[\frac{s^{n-2}}{(n-2)!}P(s)+\frac{s^{n-i-2}}{(n-i-2)!}R(s)\right]ds\,du$$
$$>\left|L_1-L_2\right|-\max\left\{\theta_1,\theta_2\right\}\left\|z_1-z_2\right\|,\tag{110}$$

which yields that

$$\left\|z_1-z_2\right\|\ge\frac{\left|L_1-L_2\right|}{1+\max\left\{\theta_1,\theta_2\right\}}>0,\tag{111}$$

that is, $z_1\neq z_2$. This completes the proof. □

3. Remark and Examples

Remark 11. Theorems 2–10 extend, improve, and unifies Theorems 1–4 in [1], the theorem in [2], Theorems 2.1–2.4 in [4], Theorems 2.1–2.5 in [5, 8], Theorems 1–3 in [9], and Theorems 1–4 in [11], respectively. The examples below prove that Theorems 2–10 extend substantially the corresponding results in [1, 2, 4, 5, 8, 9, 11]. Note that none of the known results can be applied to these examples.

Example 12. Consider the higher order nonlinear neutral delay differential equation

$$\left[\left(t^{m+1}+1\right)\left(x(t)+\frac{\sin\left(2t^2\right)-\cos\left(t^5-1\right)}{7+2\sin\left(8t^3+2t-1\right)}x(t-\tau)\right)^{(m)}\right]^{(n-m)}$$
$$+\left(\frac{t^2x(t-3)\,x^2(t-4)}{t^{n-m-i+3}+t^2+1}\right)^{(i)}$$
$$+\frac{t^3x^3\left(t^2-t\right)-x^4(t-1)}{t^{n-m+4}+t+2}$$
$$=\frac{t\ln\left(1+t^2\right)-\cos^2\left(t^2-t+1\right)}{t^{2n-m+3}+1},\quad\forall t\ge 2,\tag{112}$$

where $\tau > 0$ and $i \le n - m - 1$. Let $l = 2$, $t_0 = 2$, $\gamma = \min\{2 - \tau, -2\}$, $M = 10$, $N = 1$, $b_0 = 2/5$ and

$$\begin{aligned}
&h_1(t) = t - 3, \quad h_2(t) = t - 4,\\
&f_1(t) = t^2 - t, \quad f_2(t) = t - 1,\\
&a(t) = t^{m+1} + 1, \quad b(t) = \frac{\sin(2t^2) - \cos(t^5 - 1)}{7 + 2\sin(8t^3 + 2t - 1)},\\
&h(t,u,v) = \frac{t^2uv^2}{t^{n-m-i+3} + t^2 + 1},\\
&f(t,u,v) = \frac{t^3u^3 - v^4}{t^{n-m+4} + t + 2},\\
&g(t) = \frac{t\ln(1 + t^2) - \cos^2(t^2 - t + 1)}{t^{2n-m+3} + 1},\\
&P(t) = \frac{M^2(3t^3 + 4M)}{t^{n-m+4} + t + 2}, \quad Q(t) = \frac{M^3(t^3 + M)}{t^{n-m+4} + t + 2},\\
&R(t) = \frac{3M^2t^2}{t^{n-m-i+3} + t^2 + 1},\\
&W(t) = \frac{M^3t^2}{t^{n-m-i+3} + t^2 + 1},\\
&\forall (t,u,v) \in [t_0, +\infty) \times [N, M]^2.
\end{aligned} \tag{113}$$

It is easy to verify that the conditions of Theorem 2 are satisfied. Thus Theorem 2 ensures that (112) has uncountably many positive solutions in $\Omega_1(1,10)$, and for any $L \in (5,6)$, there exist $\theta \in (0,1)$ and $T > 1 + |t_0| + \tau + |\gamma|$ such that the Mann iterative sequence $\{x_k\}_{k\in\mathbb{N}_0}$ generated by (19) and (21) converges to a positive solution $x \in \Omega_1(1,10)$ of (112) and has the error estimate (20).

Example 13. Consider the higher order nonlinear neutral delay differential equation

$$\begin{aligned}
&\left[(t^n + 1)\left(x(t) + \frac{3t^2}{4t^2 + 3}x(t - \tau)\right)^{(m)}\right]^{(n-m)}\\
&\quad + \left(\frac{t^2x(t\ln t) - (t + 1)x(t\ln t)x(\sqrt{2t})}{t^{n+m-i+3}}\right)^{(i)}\\
&\quad + \frac{(2 - t^2)\arctan t + tx^3(t^3 + t^2)x^2(t^2)}{t^{n+m+3} + x^2(t^2)}\\
&\quad = \frac{\sqrt{1 - 8t^3 + 13t^5 + 5t^6}\cos(t^3 - 1)}{t^{2n+m+4}}, \quad \forall t \ge 1,
\end{aligned} \tag{114}$$

where $\tau > 0$ and $i \le n - m - 1$. Let $l = 2$, $t_0 = 1$, $\gamma = \min\{1 - \tau, 0\}$, $M = 6$, $N = 1$, $b_0 = 3/4$ and

$$\begin{aligned}
&h_1(t) = t\ln t, \quad h_2(t) = \sqrt{2t},\\
&f_1(t) = t^3 + t^2, \quad f_2(t) = t^2,\\
&a(t) = t^n + 1, \quad b(t) = \frac{3t^2}{4t^2 + 3},\\
&h(t,u,v) = \frac{t^2u - (t + 1)uv}{t^{n+m-i+3}},\\
&f(t,u,v) = \frac{(2 - t^2)\arctan t + tu^3v^2}{t^{n+m+3} + v^2},\\
&g(t) = \frac{\sqrt{1 - 8t^3 + 13t^5 + 5t^6}\cos(t^3 - 1)}{t^{2n+m+4}},\\
&P(t) = \frac{5M^4t^{n+m+4} + 2M(2 + t^2)\arctan t + 5M^6t}{(t^{n+m+3} + N^2)^2},\\
&Q(t) = \frac{(2 + t^2)\arctan t + M^5t}{t^{n+m+3} + N^2},\\
&R(t) = \frac{t^2 + 2M(t + 1)}{t^{n+m-i+3}}, \quad W(t) = \frac{Mt^2 + M^2(t + 1)}{t^{n+m-i+3}},\\
&\forall (t,u,v) \in [t_0, +\infty) \times [N, M]^2.
\end{aligned} \tag{115}$$

It is easy to check that the conditions of Theorem 3 are satisfied. Therefore (114) has uncountably many positive solutions in $\Omega_1(1,6)$, and for any $L \in (11/2, 6)$, there exist $\theta \in (0,1)$ and $T > 1 + |t_0| + \tau + |\gamma|$ such that the Mann iterative sequence $\{x_k\}_{k\in\mathbb{N}_0}$ generated by (19) and (21) converges to a positive solution $x \in \Omega_1(1,6)$ of (114) and has the error estimate (20).

Example 14. Consider the higher order nonlinear neutral delay differential equation

$$\begin{aligned}
&\left[(t + 1)\left(x(t) - \frac{\arctan t}{2}x(t - \tau)\right)^{(m)}\right]^{(n-m)}\\
&\quad + \left(\frac{t^2x(t^3 + t) - x^2(t^3 - 1)}{t^{2n+m-i+3} + x^2(t^3 + t)}\right)^{(i)}\\
&\quad + \frac{tx(t - 1)\sin(tx(t - \sin t))}{t^{2n+m+3} + t^3 + 2}\\
&\quad = \frac{t\sqrt{t + 1}\sin^2(t^2 + 2t + 1)}{t^{2n+m+3} + 1}, \quad \forall t \ge 0,
\end{aligned} \tag{116}$$

where $\tau > 0$ and $i \le n-m-1$. Let $l=2$, $t_0=0$, $\gamma=\min\{-\tau,-1\}$, $M=8$, $N=1/2$, $b_0=7/8$ and

$$\begin{gathered} h_1(t)=t^3+t, \quad h_2(t)=t^3-1, \\ f_1(t)=t-1, \quad f_2(t)=t-\sin t, \\ a(t)=t+1, \quad b(t)=-\frac{1}{2}\arctan t, \\ h(t,u,v)=\frac{t^2u-v^2}{t^{2n+m-i+3}+u^2}, \\ f(t,u,v)=\frac{tu\sin(tv)}{t^{2n+m+3}+t^3+2}, \\ g(t)=\frac{t\sqrt{t+1}\sin^2\left(t^2+2t+1\right)}{t^{2n+m+3}+1}, \\ P(t)=\frac{Mt^2+t}{t^{2n+m+3}+t^3+2}, \quad Q(t)=\frac{Mt}{t^{2n+m+3}+t^3+2}, \\ R(t)=\frac{t^{2n+m-i+5}+M^2t^2+2Mt^{2n+m-i+3}+4M^3}{\left(t^{2n+m-i+3}+N^2\right)^2}, \\ W(t)=\frac{Mt^2+M^2}{t^{2n+m-i+3}+N^2}, \quad \forall (t,u,v)\in[t_0,+\infty)\times[N,M]^2. \end{gathered} \tag{117}$$

It is easy to prove that the conditions of Theorem 4 are satisfied. Hence (116) has uncountably many positive solutions in $\Omega_1(1/2,8)$, and for any $L\in(1/2,1)$, there exist $\theta\in(0,1)$ and $T>1+|t_0|+\tau+|\gamma|$ such that the Mann iterative sequence $\{x_k\}_{k\in\mathbb{N}_0}$ generated by (19) and (21) converges to a positive solution $x\in\Omega_1(1/2,8)$ of (116) and has the error estimate (20).

Example 15. Consider the higher order nonlinear neutral delay differential equation

$$\begin{aligned} &\left[\left(t^{2i}+1\right)\left(x(t)+2^{t^2+1}x(t-\tau)\right)^{(m)}\right]^{(n-m)} \\ &\quad+\left(\frac{\sqrt{t+1}x^2(t-1)x\left(t^2\right)}{t^{n+i+1}+t\ln(1+|x(t-2)|)+4}\right)^{(i)} \\ &\quad+\frac{t^2x(t-12)x\left(t^2-9\right)}{t^{2n+3}+t|x(t-3)|+3} \\ &=\frac{t^2\cos(2t)+\arctan t^3}{t^{n+4}+\sin^2\left(1-t^3+t^4\right)+1}, \quad \forall t\ge 3, \end{aligned} \tag{118}$$

where $\tau > 0$ and $i \le n-m-1$. Let $l=3$, $t_0=0$, $\gamma=\min\{3-\tau,-9\}$, $M=12$, $N=5$, $b_0=2$ and

$$\begin{gathered} h_1(t)=t-1, \quad h_2(t)=t^2, \quad h_3(t)=t-2, \\ f_1(t)=t-12, \quad f_2(t)=t^2-9, \quad f_3(t)=t-3, \\ a(t)=t^{2i}+1, \quad b(t)=2^{t^2+1}, \\ h(t,u,v,w)=\frac{\sqrt{t+1}u^2v}{t^{n+i+1}+t\ln(1+|w|)+4}, \\ g(t)=\frac{t^2\cos(2t)+\arctan t^3}{t^{n+4}+\sin^2\left(1-t^3+t^4\right)+1}, \\ f(t,u,v,w)=\frac{t^2uv}{t^{2n+3}+t|w|+3}, \\ P(t)=\frac{Mt^2\left(6b_0+3M^2t+2b_0t^{2n+3}\right)}{b_0^2\left(t^{2n+3}+3\right)^2}, \\ Q(t)=\frac{M^2t^2}{b_0^2\left(t^{2n+3}+3\right)}, \\ R(t)=\frac{M^2\sqrt{t+1}}{b_0^2\left(t^{n+i+1}+4\right)^2} \\ \times\left[3t^{n+i+1}+12+\frac{M}{b_0}t+3t\ln\left(1+\frac{M}{b_0}\right)\right], \\ W(t)=\frac{M^3\sqrt{t+1}}{b_0^3\left(t^{n+i+1}+4\right)}, \\ \forall (t,u,v,w)\in[t_0,+\infty)\times\left[0,\frac{M}{b_0}\right]^3. \end{gathered} \tag{119}$$

It is easy to verify that the conditions of Theorem 5 are satisfied. Hence Theorem 5 ensures that (118) has uncountably many positive solutions in $\Omega_2(5,12)$, and, for any $L\in(11,12)$, there exist $\theta\in(0,1)$ and $T>1+|t_0|+\tau+|\gamma|$ such that the Mann iterative sequence $\{x_k\}_{k\in\mathbb{N}_0}$ generated by (48) and (21) converges to a positive solution $x\in\Omega_2(5,12)$ of (118) and has the error estimate (20).

Example 16. Consider the higher order nonlinear neutral delay differential equation

$$\begin{aligned} &\left[\left(t^{3n}+2t^{n+1}+1\right)\left(x(t)-\left(t^2+2t+4\right)x(t-\tau)\right)^{(m)}\right]^{(n-m)} \\ &\quad+\left(\frac{tx(t-3)x^2(t-4)}{t^{n+10}+x^2(t-3)}\right)^{(i)} \end{aligned}$$

$$+\frac{\sqrt{t+1}x^3(t\ln t)-tx^4(t^2-t)}{(t+1)^{n+3}+tx^2(t\ln t)}$$
$$=\frac{t\ln(1+t^2)-\sqrt{t+3}\cos^3(t^3+1)}{t^{n+5}+4t^3-1-t\sin^5(t^2-3)},\quad \forall t\ge 1, \tag{120}$$

where $\tau>0$, and $i\le n-m-1$. Let $l=2$, $t_0=1$, $\gamma=\min\{1-\tau,-3\}$, $M=6$, $N=2$, $b_0=3$ and

$$\begin{gathered}
h_1(t)=t-3,\quad h_2(t)=t-4,\\
f_1(t)=t\ln t,\quad f_2(t)=t^2-t,\\
a(t)=t^{3n}+2t^{n+1}+1,\quad b(t)=-t^2-2t-4,\\
h(t,u,v)=\frac{tuv^2}{t^{n+10}+u^2},\\
f(t,u,v)=\frac{\sqrt{t+1}u^3-tv^4}{(t+1)^{n+3}+tu^2},\\
g(t)=\frac{t\ln(1+t^2)-\sqrt{t+3}\cos^3(t^3+1)}{t^{n+5}+4t^3-1-t\sin^5(t^2-3)},\\
P(t)=\frac{M^2}{b_0^2(t+1)^{2n+6}}\times\left[3(t+1)^{n+7/2}+\frac{M^2}{b_0^2}t\sqrt{t+1}+4\frac{M}{b_0}t(t+1)^{n+3}+6\frac{M^3}{b_0^3}t^2\right],\\
Q(t)=\frac{M^3}{b_0^3(t+1)^{n+8}}\left(\sqrt{t+1}+\frac{M}{b_0}t\right),\\
R(t)=\frac{3M^2}{b_0^2t^{2n+19}}\left(t^{n+10}+\frac{M^2}{b_0^2}\right),\\
W(t)=\frac{M^3}{b_0^3t^{n+9}},\\
\forall(t,u,v)\in[t_0,+\infty)\times\left[0,\frac{M}{b_0}\right]^2.
\end{gathered}\tag{121}$$

It is easy to check that the conditions of Theorem 6 are satisfied. Thus Theorem 6 ensures that (120) has uncountably many positive solutions in $\Omega_3(2,6)$, and, for any $L\in(2,4)$, there exist $\theta\in(0,1)$ and $T>1+|t_0|+\tau+|\gamma|$ such that the Mann iterative sequence $\{x_k\}_{k\in\mathbb{N}_0}$ generated by (61) and (21) converges to a positive solution $x\in\Omega_3(2,6)$ of (120) and has the error estimate (20).

Example 17. Consider the higher order nonlinear neutral delay differential equation

$$\begin{aligned}
&\left[(t^2\ln t)(x(t)+x(t-\tau))^{(m)}\right]^{(n-m)}\\
&\quad+\left(\frac{x(\sqrt{t}-2)+x(2t-1)}{t^{3n-m-i+2}+x^2(t-\cos t)}\right)^{(i)}\\
&\quad+\frac{x^2(t-4)+x(\sqrt{t-1})x^2(t-\sin(t^9+1))}{t^{2n-m+3}+1}\\
&=\frac{\sin^{13}(t^5-\sqrt{t}+1)}{t^{n+7/2}+1},\quad\forall t\ge 4,
\end{aligned}\tag{122}$$

where $\tau>0$, $m\ge 2$ and $i\le n-m-1$. Let $l=3$, $t_0=4$, $\gamma=\min\{4-\tau,0\}$, $M=100$, $N=1$ and

$$\begin{gathered}
h_1(t)=\sqrt{t}-2,\quad h_2(t)=2t-1,\quad h_3(t)=t-\cos t,\\
f_1(t)=t-4,\quad f_2(t)=\sqrt{t-1},\quad f_3(t)=t-\sin(t^9+1),\\
a(t)=t^2\ln t,\quad b(t)=1,\\
h(t,u,v,w)=\frac{u+v}{t^{3n-m-i+2}+w^2},\\
f(t,u,v,w)=\frac{u^2+vw^2}{t^{2n-m+3}+1},\\
g(t)=\frac{\sin^{13}(t^5-\sqrt{t}+1)}{t^{n+7/2}+1},\quad P(t)=\frac{2M+3M^2}{t^{2n-m+3}+1},\\
Q(t)=\frac{M^2+M^3}{t^{2n-m+3}+1},\quad R(t)=\frac{2t^{3n-m-i+2}+6M^2}{(t^{3n-m-i+2}+N^2)^2},\\
W(t)=\frac{2M}{t^{3n-m-i+2}+N^2},\\
\forall(t,u,v,w)\in[t_0,+\infty)\times[N,M]^3.
\end{gathered}\tag{123}$$

It is easy to check that the conditions of Theorem 7 are satisfied. Thus (122) has uncountably many positive solutions in $\Omega_1(1,100)$, and, for any $L\in(1,100)$, there exist $\theta\in(0,1)$ and $T>1+|t_0|+\tau+|\gamma|$ such that the Mann iterative sequence $\{x_k\}_{k\in\mathbb{N}_0}$ generated by (66) and (21) converges to a positive solution $x\in\Omega_1(1,100)$ of (122) and has the error estimate (20).

Example 18. Consider the higher order nonlinear neutral delay differential equation

$$\begin{aligned}
&\left[\frac{2+\sin\left(t+\sqrt{t}\right)}{t^2}\left(x(t)+x(t-\tau)\right)'\right]^{(n-1)}\\
&\quad+\left(\frac{x^3(t-2)-t^2x^4\left(\sqrt{t+1}-1\right)}{t^{5n+3}+t+1}\right)^{(i)}\\
&\quad+\frac{x^3(t-3)-t}{t^{3n+4}+x^2\left(t-(-1)^n\right)}\\
&=\frac{\sin\left(t^4-\sqrt{t^2+1}\right)}{t^{n+3}+\ln t},\quad \forall t\geq 3,
\end{aligned}\tag{124}$$

where $\tau>0$, $m=1$ and $i\leq n-2$. Let $l=2$, $t_0=3$, $\gamma=\min\{3-\tau,0\}$, $M=10$, $N=9$ and

$$\begin{gathered}
h_1(t)=t-2,\quad h_2(t)=\sqrt{t+1}-1,\\
f_1(t)=t-3,\quad f_2(t)=t-(-1)^n,\\
a(t)=\frac{2+\sin\left(t+\sqrt{t}\right)}{t^2},\quad b(t)=1,\\
h(t,u,v)=\frac{u^3-t^2v^4}{t^{5n+3}+t+1},\\
f(t,u,v)=\frac{u^3-t}{t^{3n+4}+v^2},\\
g(t)=\frac{\sin\left(t^4-\sqrt{t^2+1}\right)}{t^{n+3}+\ln t},\\
P(t)=\frac{M\left(5M^3+2t+2Mt^{3n+4}\right)}{\left(t^{3n+4}+N^2\right)^2},\\
Q(t)=\frac{M^3+t}{t^{3n+4}+N^2},\\
R(t)=\frac{M^2\left(3+4Mt^2\right)}{t^{5n+3}+t+1},\quad W(t)=\frac{M^3\left(1+Mt^2\right)}{t^{5n+3}+t+1},\\
\forall(t,u,v)\in\left[t_0,+\infty\right)\times[N,M]^2.
\end{gathered}\tag{125}$$

It is easy to check that the conditions of Theorem 8 are satisfied. Thus Theorem 8 ensures that (124) has uncountably many positive solutions in $\Omega_1(9,10)$, and, for any $L\in(9,10)$, there exist $\theta\in(0,1)$ and $T>1+|t_0|+\tau+|\gamma|$ such that the Mann iterative sequence $\{x_k\}_{k\in\mathbb{N}_0}$ generated by (78) and (21) converges to a positive solution $x\in\Omega_1(9,10)$ of (124) and has the error estimate (20).

Example 19. Consider the higher order nonlinear neutral delay differential equation

$$\begin{aligned}
&\left[t^{m+1}\ln\left(4+\sin\left(t^2-\sqrt{t}\right)\right)\left(x(t)-x(t-\tau)\right)^{(m)}\right]^{(n-m)}\\
&\quad+\left(\frac{t}{t^{n-m+4}+x^4\left(t-\sqrt{t}\right)}\right)^{(i)}\\
&\quad+\frac{\sin\left(t^3-2t+\sqrt{t^3+1}\right)}{t^{n+m}+|t-x(t-3)|}\\
&=\frac{t\cos^5\left(t^7-t^4+1\right)}{t^n+2t-\cos^3\left(t^2-3\right)},\quad\forall t\geq 4,
\end{aligned}\tag{126}$$

where $\tau>0$, $m\geq 2$ and $i\leq n-m-1$. Let $l=1$, $t_0=4$, $\gamma=\min\{4-\tau,1\}$, $M=7$, $N=5$ and

$$\begin{gathered}
h_1(t)=t-\sqrt{t},\quad f_1(t)=t-3,\\
a(t)=t^{m+1}\ln\left(4+\sin\left(t^2-\sqrt{t}\right)\right),\\
b(t)=-1,\quad h(t,u)=\frac{t}{t^{n-m+4}+u^4},\\
f(t,u)=\frac{\sin t^3\left(-2t+\sqrt{t^3+1}\right)}{t^{n+m}+|t-u|},\\
g(t)=\frac{t\cos^5\left(t^7-t^4+1\right)}{t^n+2t-\cos^3\left(t^2-3\right)},\\
P(t)=\frac{1}{t^{2n+2m}},\quad Q(t)=\frac{1}{t^{n+m}},\\
R(t)=\frac{4M^3}{t^{2n-2m+7}},\quad W(t)=\frac{1}{t^{n-m+3}},\\
\forall(t,u)\in\left[t_0,+\infty\right)\times[N,M]^2.
\end{gathered}\tag{127}$$

It is easy to check that the conditions of Theorem 9 are satisfied. Thus Theorem 9 ensures that (126) has uncountably many positive solutions in $\Omega_1(5,7)$, and, for any $L\in(5,7)$, there exist $\theta\in(0,1)$ and $T>1+|t_0|+\tau+|\gamma|$ such that the Mann iterative sequence $\{x_k\}_{k\in\mathbb{N}_0}$ generated by (90) and (21) converges to a positive solution $x\in\Omega_1(5,7)$ of (126) and has the error estimate (20).

Example 20. Consider the higher order nonlinear neutral delay differential equation

$$\begin{aligned}
&\left[\frac{1}{t^3}\left(x(t)-x(t-\tau)\right)'\right]^{(n-1)}\\
&\quad+\left(\frac{t-\sin\left(t^8-4t^5-1\right)}{t^{n+7}+\left|x(t-1)-x^3(t-2)\right|}\right)^{(i)}
\end{aligned}$$

$$+\frac{\ln\left(1+x^{2}\left(t-\arctan t\right)\right)}{t^{2n+6}+x^{2}\left(t-4\right)}$$
$$=\frac{t\ln\left(t+\cos\left(t^{3}-1\right)\right)}{t^{n+8}+1},\quad \forall t\geq 5, \tag{128}$$

where $\tau > 0$, $m = 1$ and $i \leq n-2$. Let $l = 2$, $t_0 = 5$, $\gamma = \min\{5-\tau, 1\}$, $M = 4$, $N = 2$ and

$$\begin{gathered} h_1(t)=t-1,\quad h_2(t)=t-2,\quad f_1(t)=t-\arctan t,\\ f_2(t)=t-4,\quad a(t)=\frac{1}{t^3},\quad b(t)=-1,\\ h(t,u,v)=\frac{t-\sin\left(t^8-4t^5-1\right)}{t^{n+7}+\left|u-v^3\right|},\\ f(t,u,v)=\frac{\ln\left(1+u^2\right)}{t^{2n+6}+v^2},\quad g(t)=\frac{t\ln\left(t+\cos\left(t^3-1\right)\right)}{t^{n+8}+1},\\ P(t)=\frac{2M\left(2M^2+t^{2n+6}\right)}{t^{4n+12}},\quad Q(t)=\frac{M^2}{t^{2n+6}},\\ R(t)=\frac{2+6M^2}{t^{2n+13}},\quad W(t)=\frac{2}{t^{n+6}},\\ \forall(t,u,v)\in\left[t_0,+\infty\right)\times[N,M]^2. \end{gathered} \tag{129}$$

It is easy to check that the conditions of Theorem 10 are satisfied. Thus Theorem 10 ensures that (128) has uncountably many positive solutions in $\Omega_1(2,4)$, and, for any $L \in (2,4)$, there exist $\theta \in (0,1)$ and $T > 1 + |t_0| + \tau + |\gamma|$ such that the Mann iterative sequence $\{x_k\}_{k\in\mathbb{N}_0}$ generated by (102) and (21) converges to a positive solution $x \in \Omega_1(2,4)$ of (128) and has the error estimate (20).

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Research Article

Bifurcation of Limit Cycles by Perturbing a Piecewise Linear Hamiltonian System

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This paper concerns limit cycle bifurcations by perturbing a piecewise linear Hamiltonian system. We first obtain all phase portraits of the unperturbed system having at least one family of periodic orbits. By using the first-order Melnikov function of the piecewise near-Hamiltonian system, we investigate the maximal number of limit cycles that bifurcate from a global center up to first order of ε.

1. Introduction and Main Results

Recently, piecewise smooth dynamical systems have been well concerned, especially in the scientific problems and engineering applications. For example, see the works of Filippov [1], Kunze [2], di Bernardo et al. [3], and the references therein. Because of the variety of the nonsmoothness, there can appear many complicated phenomena in piecewise smooth dynamical systems such as stability (see [4, 5]), chaos (see [6]), and limit cycle bifurcation (see [7–10]). Here, we are more concerned with bifurcation of limit cycles in a perturbed piecewise linear Hamiltonian system:

$$\begin{aligned} \dot{x} &= y + \varepsilon p(x, y, \delta), \\ \dot{y} &= -g(x) + \varepsilon q(x, y, \delta), \end{aligned} \tag{1}$$

where $\varepsilon > 0$ is a sufficiently small real parameter,

$$g(x) = \begin{cases} a_1 x + a_0, & x \geq 0, \\ b_1 x + b_0, & x < 0 \end{cases} \tag{2}$$

with $a_1, a_0, b_1,$ and b_0 real numbers satisfying $a_1^2 + a_0^2 \neq 0, b_1^2 + b_0^2 \neq 0$,

$$p(x, y, \delta) = \begin{cases} p^+(x, y, \delta) = \sum_{i+j=0}^{n} a_{ij}^+ x^i y^j, & x \geq 0, \\ p^-(x, y, \delta) = \sum_{i+j=0}^{n} a_{ij}^- x^i y^j, & x < 0, \end{cases} \tag{3}$$

$$q(x, y, \delta) = \begin{cases} q^+(x, y, \delta) = \sum_{i+j=0}^{n} b_{ij}^+ x^i y^j, & x \geq 0, \\ q^-(x, y, \delta) = \sum_{i+j=0}^{n} b_{ij}^- x^i y^j, & x < 0, \end{cases} \tag{4}$$

and $\delta = (a_{ij}^+, a_{ij}^-, b_{ij}^+, b_{ij}^-) \in D \subset \mathbb{R}^{2(n+1)(n+2)}$ with D compact. Then system (1) has two subsystems

$$\begin{aligned} \dot{x} &= y + \varepsilon p^+(x, y, \delta), \\ \dot{y} &= -a_1 x - a_0 + \varepsilon q^+(x, y, \delta), \end{aligned} \tag{5a}$$

$$\begin{aligned} \dot{x} &= y + \varepsilon p^-(x, y, \delta), \\ \dot{y} &= -b_1 x - b_0 + \varepsilon q^-(x, y, \delta), \end{aligned} \tag{5b}$$

which are called the right subsystem and the left subsystem, respectively. For $\varepsilon = 0$, systems (5a) and (5b) are Hamiltonian with the Hamiltonian functions, respectively,

$$H^+(x,y) = \frac{1}{2}y^2 + \frac{1}{2}a_1x^2 + a_0x,$$
$$H^-(x,y) = \frac{1}{2}y^2 + \frac{1}{2}b_1x^2 + b_0x. \tag{6}$$

Note that the phase portrait of the linear system

$$\dot{x} = y,$$
$$\dot{y} = -ax - b, \tag{7}$$

with $a^2+b^2 \neq 0$ has possibly the following four different phase portraits on the plane (see Figure 1).

Then, one can find that system (1) $|_{\varepsilon=0}$ can have 13 different phase portraits (see Figure 2) when at least one family of periodic orbits appears.

We remark that in Figure 2,

GC: global center,

Ho: homoclinic,

He: heteroclinic,

C^+: center in the region $\{(x,y) \mid x>0\}$,

C^-: center in the region $\{(x,y) \mid x<0\}$,

S^+: saddle in the region $\{(x,y) \mid x>0\}$,

S^-: saddle in the region $\{(x,y) \mid x<0\}$,

L^+: curvilinear or straightline in the region $\{(x,y) \mid x>0\}$,

L^-: curvilinear or straightline in the region $\{(x,y) \mid x<0\}$.

It is easy to obtain the following Table 1 which shows conditions for each possible phase portrait appearing above. Also, cases (3), (5), (7), (9), and (13) in Figure 2 are equivalent to cases (2), (6), (8), (10), and (12), respectively, by making the transformation

$$(x,y) \longrightarrow (-x,y), \tag{8}$$

together with time rescaling $dt = -d\tau$.

The authors Liu and Han [7] studied system (1) in a subcase of the case (1) of Figure 2 by taking $a_1 = b_1 = 1$, $a_0 = b_0 = 0$. By using the first order Melnikov function, they proved that the maximal number of limit cycles on Poincaré bifurcations is n up to first-order in ε. The authors Liang et al. [8] considered system (1) in the case (5) of Figure 2 by taking $a_1 = -1$, $a_0 = 1$, $b_1 = 1$, and $b_0 = 0$. By using the same method, they gave lower bounds of the maximal number of limit cycles in Hopf, and Homoclinic bifurcations, and derived an upper bound of the maximal number of limit cycles bifurcating from the periodic annulus between the center and the Homoclinic loop up to the first-order in ε. Clearly, the maximal number of limit cycles in the case (7) or (8) of Figure 2 is $[(n-1)/2]$ on Poincaré, Hopf and Homoclinic bifurcations up to first-order in ε, by using the first order Melnikov function.

This paper focuses on studying the limit cycle bifurcations of system (1) in the case (1) of Figure 2 by using the first order Melnikov function. That is, system (1) satisfies

$$a_1 \geq 0, \quad a_0 \geq 0, \quad a_0 + a_1 > 0,$$
$$b_1 \geq 0, \quad b_0 \leq 0, \quad b_0 < b_1. \tag{9}$$

Clearly, system (1) $|_{\varepsilon=0}$ satisfying (9) has a family of periodic orbits

$$\begin{aligned} L_h &= L_h^+ \cup L_h^- \\ &= \{(x,y) \mid H^+(x,y) = h\} \\ &\quad \cup \{(x,y) \mid H^-(x,y) = h\}, \quad h > 0, \end{aligned} \tag{10}$$

such that the limit of L_h as $h \to 0^+$ is the origin. The intersection points of the closed curve L_h with the positive y-axis and the negative y-axis are denoted by $A(h) = (0, \sqrt{2h})$ and $A_1(h) = (0, -\sqrt{2h})$, respectively. Let

$$M^+(h,\delta) = \int_{\widehat{AA_1}} q^+dx - p^+dy,$$
$$M^-(h,\delta) = \int_{\widehat{A_1A}} q^-dx - p^-dy, \quad h > 0. \tag{11}$$

Then, from Liu and Han [7], the first-order Melnikove function corresponding to system (1) is

$$M(h,\delta) = M^+(h,\delta) + M^-(h,\delta), \quad h \in (0,+\infty). \tag{12}$$

Let $Z(n)$ denote the maximal number of zeros of $M(h,\delta)$ for $h>0$ and $N(n)$ the cyclicity of system (1) at the origin. Then, we can obtain the following.

Theorem 1. *Let* (9) *be satisfied. For any given* $n \geq 1$, *one has Table 2.*

This paper is organized as follows. In Section 2, we will provide some preliminary lemmas, which will be used to prove the main results. In Section 3, we present the proof of Theorem 1.

2. Preliminary Lemmas

In this section, we will derive expressions of $M^+(h,\delta)$, $M^-(h,\delta)$ in (11). First, we have the following.

Lemma 2. *Suppose system* (1) *satisfies* (9). *Then,*

(i) $M^+(h,\delta)$ *in* (11) *can be written as*

$$M^+(h,\delta) = M_1^+(h,\delta) + \sum_{2k=0}^{n} \frac{2^{k+1+1/2}}{2k+1} a_{0,2k}^+ h^{k+1/2}, \tag{13}$$

where

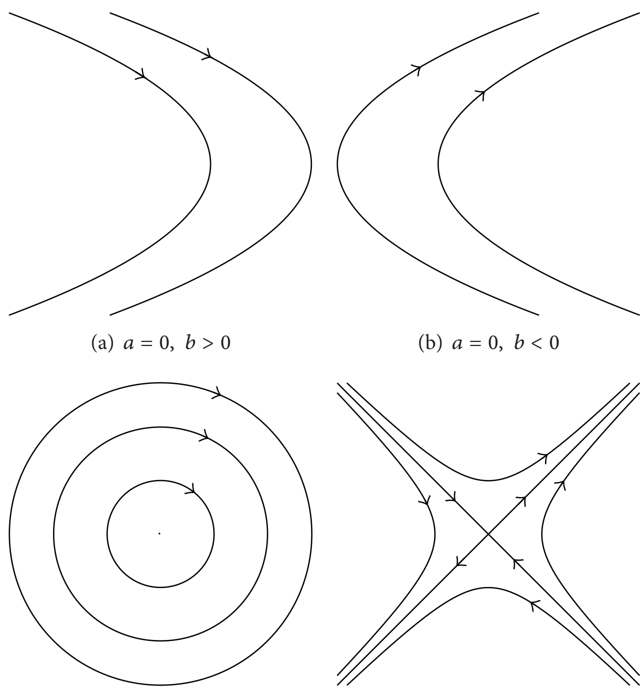
$$M_1^+(h,\delta) = \sum_{i+j=0}^{n-1} \overline{p}_{ij}^+ \int_{\widehat{A_1A}} x^{i+1}y^j dy = \sum_{i+j=0}^{n-1} \overline{q}_{ij}^+ \int_{\widehat{A_1A}} x^i y^{j+1} dx, \tag{14}$$

Table 1: Coefficient conditions for phase portraits (1)–(13).

Coefficient conditions	$a_1 \geq 0$, $a_0 \geq 0$ $a_1 + a_0 > 0$	$a_1 > 0$ $a_0 < 0$	$a_1 \leq 0$, $a_0 \leq 0$ $a_1 + a_0 < 0$	$a_1 < 0$ $a_0 > 0$
$b_1 \geq 0$, $b_0 \leq 0$, $b_1 > b_0$	(1)	(2)		(5)
$b_1 > 0$, $b_0 > 0$	(3)	(4)	(7)	(9)
$b_1 \leq 0$, $b_0 \geq 0$, $b_0 > b_1$		(8)		
$b_1 < 0$, $b_0 < 0$	(6)	(10)		(11) ($a_0^2 b_1 = a_1 b_0^2$), (12) ($a_0^2 b_1 > a_1 b_0^2$), (13) ($a_0^2 b_1 < a_1 b_0^2$)

Table 2

	$a_1 > 0$, $a_0 = 0$	$a_1 = 0$, $a_0 > 0$	$a_1 > 0$, $a_0 > 0$
$b_1 > 0$, $b_0 = 0$	$Z(n) = n$, $N(n) \geq n$	$Z(n) = n + \left[\frac{n+1}{2}\right]$, $N(n) \geq n + \left[\frac{n+1}{2}\right]$	$Z(n) = n + \left[\frac{n+1}{2}\right]$, $N(n) \geq n + \left[\frac{n+1}{2}\right]$
$b_1 = 0$, $b_0 < 0$	$Z(n) = n + \left[\frac{n+1}{2}\right]$, $N(n) \geq n + \left[\frac{n+1}{2}\right]$	$Z(n) = n$, $N(n) \geq n$	$n + \left[\frac{n+1}{2}\right] \leq Z(n) \leq n + 2\left[\frac{n+1}{2}\right]$ $N(n) \geq n + \left[\frac{n+1}{2}\right]$,
$b_1 > 0$, $b_0 < 0$	$Z(n) = n + \left[\frac{n+1}{2}\right]$, $N(n) \geq n + \left[\frac{n+1}{2}\right]$	$n + [\frac{n+1}{2}] \leq Z(n) \leq n + 2\left[\frac{n+1}{2}\right]$ $N(n) \geq n + \left[\frac{n+1}{2}\right]$,	$n \leq Z(n) \leq n + \left[\frac{n+1}{2}\right]$, $\frac{a_0^2}{a_1} = \frac{b_0^2}{b_1}$, $n \leq Z(n) \leq n + 2\left[\frac{n+1}{2}\right]$, $\frac{a_0^2}{a_1} \neq \frac{b_0^2}{b_1}$ $N(n) \geq n$,



(a) $a = 0$, $b > 0$ (b) $a = 0$, $b < 0$

(c) $a > 0$ (d) $a < 0$

Figure 1: The possible phase portraits of system (7).

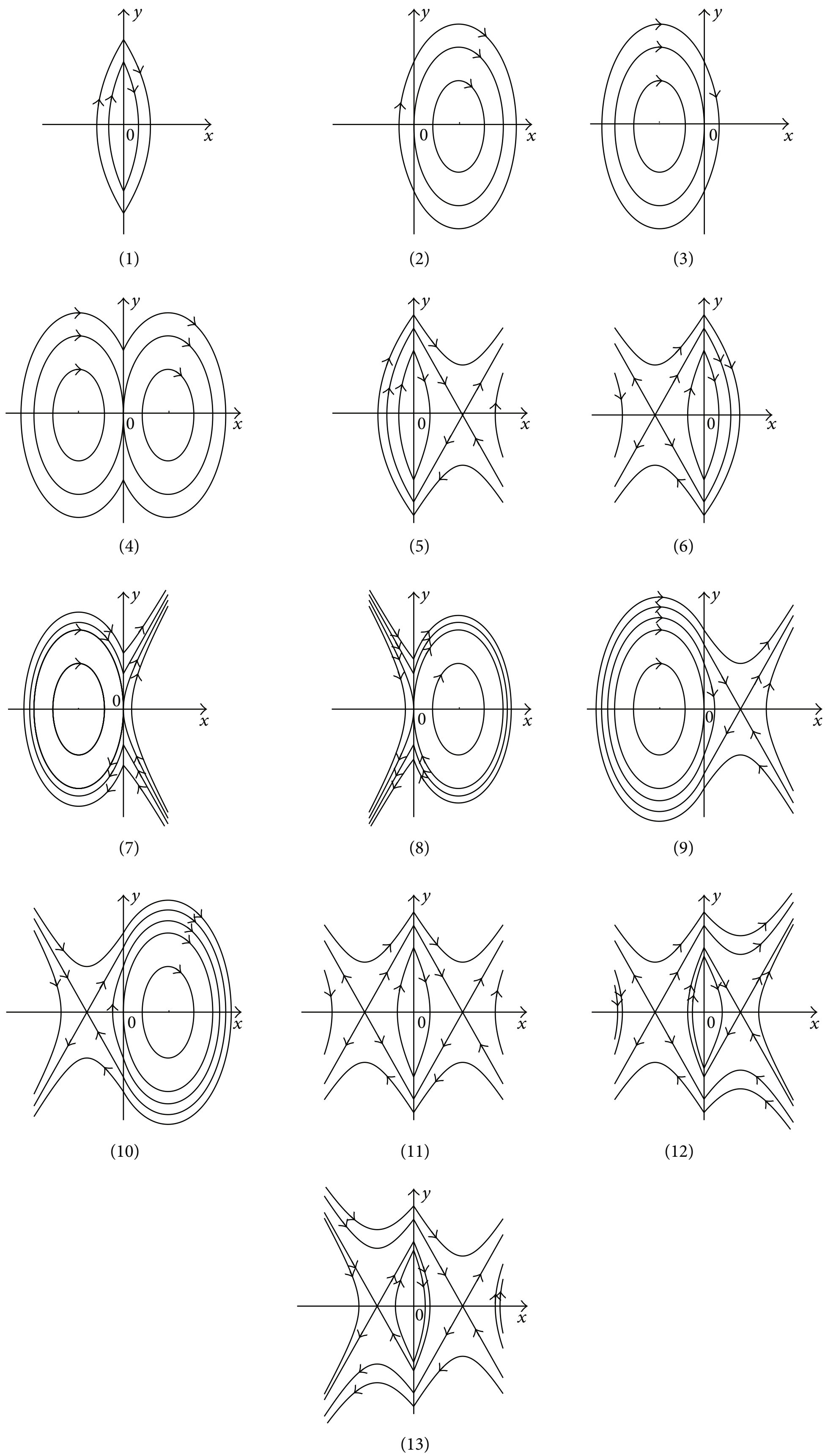


Figure 2: The possible phase portraits of system (1) $|_{\varepsilon=0}$. (1) GC. (2) HoC$^+$. (3) HoC$^-$. (4) HoC$^-$C$^+$. (5) HoS$^+$. (6) HoS$^-$. (7) HoC$^-$L$^+$. (8) HoC$^+$L$^-$. (9) HoC$^-$HoS$^+$. (10) HoC$^+$HoS$^-$. (11) HeS$^-$S$^+$. (12) HeS$^+$. (13) HeS$^-$.

with

$$\overline{p}_{ij}^{+} = a_{i+1,j}^{+} + \frac{j+1}{i+1} b_{i,j+1}^{+}, \qquad \overline{q}_{ij}^{+} = b_{i,j+1}^{+} + \frac{i+1}{j+1} a_{i+1,j}^{+}. \tag{15}$$

(ii) (h,δ) *in* (11) *can be expressed as*

$$M^{-}(h,\delta) = M_1^{-}(h,\delta) - \sum_{2k=0}^{n} \frac{2^{k+1+1/2}}{2k+1} a_{0,2k}^{-} h^{k+1/2}, \tag{16}$$

where

$$M_1^{-}(h,\delta) = \sum_{i+j=0}^{n-1} \overline{p}_{ij}^{-} \int_{\widehat{A_1A}} x^{i+1} y^{j} dy = \sum_{i+j=0}^{n-1} \overline{q}_{ij}^{-} \int_{\widehat{A_1A}} x^{i} y^{j+1} dx, \tag{17}$$

with

$$\overline{p}_{ij}^{-} = a_{i+1,j}^{-} + \frac{j+1}{i+1} b_{i,j+1}^{-}, \qquad \overline{q}_{ij}^{-} = b_{i,j+1}^{-} + \frac{i+1}{j+1} a_{i+1,j}^{-}. \tag{18}$$

Proof. We only prove (i) since (ii) can be verified in a similar way. By (11), we obtain

$$\begin{aligned} M^{+}(h,\delta) = &\int_{\widehat{AA_1}} q^{+}(x,y,\delta)\,dx - p^{+}(x,y,\delta)\,dy \\ &+ \int_{\overrightarrow{A_1A}} q^{+}(x,y,\delta)\,dx - p^{+}(x,y,\delta)\,dy \\ &- \int_{\overrightarrow{A_1A}} q^{+}(x,y,\delta)\,dx - p^{+}(x,y,\delta)\,dy \\ = &\oint_{\widehat{AA_1}\cup\overrightarrow{A_1A}} q^{+}(x,y,\delta)\,dx - p^{+}(x,y,\delta)\,dy \\ &+ \int_{\overrightarrow{A_1A}} p^{+}(0,y,\delta)\,dy, \end{aligned} \tag{19}$$

which follows that by Green formula and (3)

$$M^{+}(h,\delta) = M_1^{+}(h,\delta) + \sum_{2k=0}^{n} \frac{2^{k+1+1/2}}{2k+1} a_{0,2k}^{+} h^{k+1/2}, \tag{20}$$

where

$$M_1^{+}(h,\delta) = \iint_{\operatorname{int}\widehat{AA_1}\cup\overrightarrow{A_1A}} \left(p_x^{+} + q_y^{+}\right) dx\,dy. \tag{21}$$

Then, by Green formula again

$$\begin{aligned} M_1^{+}(h,\delta) = &-\oint_{\widehat{AA_1}\cup\overrightarrow{A_1A}} \overline{p}^{+}(x,y,\delta)\,dy \\ = &\oint_{\widehat{AA_1}\cup\overrightarrow{A_1A}} \overline{q}^{+}(x,y,\delta)\,dx, \end{aligned} \tag{22}$$

where

$$\begin{aligned} \overline{p}^{+}(x,y,\delta) &= p^{+}(x,y,\delta) - p^{+}(0,y,\delta) + \int_0^x q_y^{+}(u,y,\delta)\,du, \\ \overline{q}^{+}(x,y,\delta) &= q^{+}(x,y,\delta) - q^{+}(x,0,\delta) + \int_0^y p_x^{+}(x,v,\delta)\,dv. \end{aligned} \tag{23}$$

By (3), (4), and the above formulas, we have

$$\begin{aligned} \overline{p}^{+}(x,y,\delta) &= \sum_{i+j=0}^{n} a_{ij}^{+} x^i y^j - \sum_{j=0}^{n} a_{0j}^{+} y^j + \sum_{i+j=1}^{n} \frac{j}{i+1} b_{ij}^{+} x^{i+1} y^{j-1} \\ &= x \sum_{i+j=0}^{n-1} \left(a_{i+1,j}^{+} + \frac{j+1}{i+1} b_{i,j+1}^{+}\right) x^i y^j \\ &= x \sum_{i+j=0}^{n-1} \overline{p}_{ij}^{+} x^i y^j, \end{aligned} \tag{24}$$

$$\begin{aligned} \overline{q}^{+}(x,y,\delta) &= \sum_{i+j=0}^{n} b_{ij}^{+} x^i y^j - \sum_{j=0}^{n} b_{i0}^{+} x^i + \sum_{i+j=1}^{n} \frac{i}{j+1} a_{ij}^{+} x^{i-1} y^{j+1} \\ &= y \sum_{i+j=0}^{n-1} \left(b_{i,j+1}^{+} + \frac{i+1}{j+1} a_{i+1,j}^{+}\right) x^i y^j \\ &= y \sum_{i+j=0}^{n-1} \overline{q}_{ij}^{+} x^i y^j. \end{aligned} \tag{25}$$

Combining (20)–(25) gives (13) and (14). Thus, the proof is ended. □

Then, using Lemma 2 and (6) we can obtain the following three lemmas.

Lemma 3. (i) *If* $a_1 = 0$, $a_0 > 0$, *then* $M^{+}(h,\delta)$ *in* (11) *has form*

$$M^{+}(h,\delta) = h^{1/2} \sum_{i+2k=0}^{n} B_{i,2k}^{+} h^{i+k}, \tag{26}$$

where

$$\begin{aligned} B_{0,2k}^{+} &= \frac{2^{k+1+1/2} a_{0,2k}^{+}}{2k+1}, \\ B_{i,2k}^{+} &= \frac{2^{k+1+1/2}}{a_0^{i}} \left(a_{i,2k}^{+} + \frac{2k+1}{i} b_{i-1,2k+1}^{+}\right) \\ &\quad \times \int_0^{\pi/2} \sin^{2k}\theta \cos^{i+1}\theta\, d\theta, \quad 1 \le i \le n. \end{aligned} \tag{27}$$

(ii) *If* $b_1 = 0$, $b_0 < 0$, *then we have*

$$M^{-}(h,\delta) = h^{1/2} \sum_{i+2k=0}^{n} B_{i,2k}^{-} h^{i+k}, \tag{28}$$

where

$$B^-_{0,2k} = -\frac{2^{k+1+1/2}a^-_{0,2k}}{2k+1},$$
$$B^-_{i,2k} = \frac{-2^{k+1+1/2}}{b_0^i}\left(a^-_{i,2k} + \frac{2k+1}{i}b^-_{i-1,2k+1}\right) \times \int_0^{\pi/2} \sin^{2k}\theta\cos^{2i+1}\theta d\theta, \quad 1 \le i \le n. \tag{29}$$

Proof. Note that along $\widehat{AA_1}, x = h/a_0 - (1/2a_0)y^2$. Then, inserting it into (14) follows that

$$M_1^+(h,\delta) = \sum_{i+j=0}^{n-1} \overline{p}^+_{ij} \int_{-\sqrt{2h}}^{\sqrt{2h}} \left(\frac{h}{a_0} - \frac{1}{2a_0}y^2\right)^{i+1} y^j dy = \sum_{i+2k=0}^{n-1} \frac{\overline{p}^+_{i,2k}}{2^i a_0^{i+1}} \int_0^{\sqrt{2h}} \left(2h - y^2\right)^{i+1} y^{2k} dy. \tag{30}$$

Let $y = \sqrt{2h}\sin\theta$. Then we have $dy = \sqrt{2h}\cos\theta\, d\theta$ and the above integral can be carried into

$$\int_0^{\sqrt{2h}} \left(2h - y^2\right)^{i+1} y^{2k} dy = (2h)^{i+1+k+(1/2)} \times \int_0^{\pi/2} \sin^{2k}\theta\cos^{2(i+1)+1}\theta\, \mathrm{d}\theta. \tag{31}$$

Thus, using (30) and the above equation we can write (13) as

$$M^+(h,\delta) = \sum_{i+2k=0}^{n-1} \frac{2^{k+1+1/2}\overline{p}^+_{i,2k}}{a_0^{i+1}} \times \int_0^{\pi/2} \sin^{2k}\theta\cos^{2(i+1)+1}\theta d\theta \times h^{i+1+k+1/2} + \sum_{2k=0}^{n} \frac{2^{k+1+1/2}a^+_{0,2k}}{2k+1} h^{k+1/2} = h^{1/2}\sum_{i+2k=0}^{n} B^+_{i,2k} h^{i+k}, \tag{32}$$

where

$$B^+_{0,2k} = \frac{2^{k+1+1/2}a^+_{0,2k}}{2k+1},$$
$$B^+_{i,2k} = \frac{2^{k+1+1/2}\overline{p}^+_{i-1,2k}}{a_0^i} \int_0^{\pi/2} \sin^{2k}\theta\cos^{2i+1}\theta\, d\theta, \quad 1 \le i \le n, \tag{33}$$

which gives (i) by (15). Thus, (i) holds and we can prove (ii) in the same way by (16)–(18). This ends the proof. □

Lemma 4. *Let system* (5a) *satisfy* (3) *and* (4). *Then*

(i) *If* $a_1 > 0,\ a_0 = 0, M^+(h,\delta)$ *has the expression*

$$M^+(h,\delta) = \sqrt{h}\sum_{i+2k=0}^{n} A^+_{i,2k}\left(\sqrt{h}\right)^{i+2k}, \tag{34}$$

where

$$A^+_{0,2k} = \frac{2\left(\sqrt{2}\right)^{2k+1}a^+_{0,2k}}{2k+1},$$
$$A^+_{i,2k} = \frac{2\left(\sqrt{2}\right)^{2k+1+i}}{\left(\sqrt{a_1}\right)^i}\left(b^+_{i-1,2k+1} + \frac{i}{2k+1}a^+_{i,2k}\right) \times \int_0^{\pi/2} \sin^{i-1}\theta\cos^{2k+2}\theta\, d\theta, \quad i \ge 1. \tag{35}$$

(ii) *If* $a_1 > 0,\ a_0 \ne 0,\ M^+(h,\delta)$ *can be written as*

$$M^+(h,\delta) = \sqrt{h}\left[\sum_{i+2k=0}^{n-1}\left(b^+_{i,2k+1} + \frac{i+1}{2k+1}a^+_{i+1,2k}\right)\phi^+_{ik}(h) + \sum_{2k=0}^{n}\frac{2^{k+1+1/2}a^+_{0,2k}}{2k+1}h^k\right] + \sum_{i+2k=0}^{n-1}\left(b^+_{i,2k+1} + \frac{i+1}{2k+1}a^+_{i+1,2k}\right) \times \sum_{r=0,\ r\ even}^{i} \alpha^+_{irk}\left(2h + \frac{a_0^2}{a_1}\right)^{k+r/2}\overline{I}^+_{00}(h,\delta), \tag{36}$$

or

$$M^+(h,\delta) = \sqrt{h}\psi^+_{[n/2]}(h,\delta) + \left(2h + \frac{a_0^2}{a_1}\right)\varphi^+_{[(n-1)/2]}\left(2h + \frac{a_0^2}{a_1},\delta\right) \times \left(\frac{\pi}{2} - \arcsin\frac{a_0}{\sqrt{2a_1h + a_0^2}}\right), \tag{37}$$

where

$$\overline{I}^+_{00}(h,\delta) = \int_{a_0/\sqrt{a_1}}^{\sqrt{2h + a_0^2/a_1}} \sqrt{2h + \frac{a_0^2}{a_1} - v^2 dv}, \tag{38}$$

each α^+_{irk} *is a nonzero constant and* ϕ^+_{ik}, $\psi^+_{[n/2]}$, $\varphi^+_{[(n-1)/2]}$ *are polynomials of degree* $k + [(i+1)/2]$, $[n/2]$, $[(n-1)/2]$, *respectively.*

Proof. Since $y = \pm\sqrt{2h - a_1x^2 - 2a_0x}$ along the curve $\widehat{AA_1}$, $M_1^+(h,\delta)$ in (14) becomes

$$\begin{aligned} M_1^+(h,\delta) &= \sum_{i+j=0}^{n-1} \overline{q}_{ij}^+ \int_{\widehat{AA_1}} x^i y^{j+1} dx \\ &= \sum_{i+j=0}^{n-1} \overline{q}_{ij}^+ \left[\int_0^{(-a_0+\sqrt{2a_1h+a_0^2})/a_1} x^i \left(2h - a_1x^2 - 2a_0x\right)^{(j+1)/2} dx \right] \\ &\quad + \int_{(-a_0+\sqrt{2a_1h+a_0^2})/a_1}^{0} x^i (-1)^{j+1} \left(2h - a_1x^2 - 2a_0x\right)^{(j+1)/2} dx \\ &= \sum_{i+2k=0}^{n-1} 2\overline{q}_{i,2k}^+ I_{i,2k}^+ (h,\delta), \end{aligned} \tag{39}$$

where

$$I_{i,2k}^+(h,\delta) = \int_0^{(-a_0+\sqrt{2a_1h+a_0^2})/a_1} x^i \left[2h + \frac{a_0^2}{a_1} - a_1\left(x + \frac{a_0}{a_1}\right)^2\right]^{k+1/2} dx. \tag{40}$$

Let $v = \sqrt{a_1}(x + a_0/a_1)$. Then, we have $dv = \sqrt{a_1}dx$ and the above equation becomes

$$I_{i,2k}^+(h,\delta) = \frac{1}{\left(\sqrt{a_1}\right)^{i+1}} \int_{a_0/\sqrt{a_1}}^{\sqrt{2h+a_0^2/a_1}} \left(v - \frac{a_0}{\sqrt{a_1}}\right)^i \times \left(2h + \frac{a_0^2}{a_1} - v^2\right)^{k+(1/2)} dv. \tag{41}$$

For $a_0 = 0$, make the transformation $v = \sqrt{2h}\sin\theta$. Then, we have by (41)

$$I_{i,2k}^+(h,\delta) = \frac{\left(\sqrt{2h}\right)^{2k+1+i+1}}{\left(\sqrt{a_1}\right)^{i+1}} \int_0^{\pi/2} \sin^i\theta \cos^{2k+2}\theta\, d\theta. \tag{42}$$

Substituting the above formula into (37), together with (13), gives that

$$\begin{aligned} M^+(h,\delta) &= \sum_{i+2k=0}^{n-1} \frac{2\overline{q}_{i,2k}^+}{\left(\sqrt{a_1}\right)^{i+1}} \times \int_0^{\pi/2} \sin^i\theta\cos^{2k+2}\theta\, d\theta \times \left(\sqrt{2h}\right)^{2k+1+i+1} \\ &\quad + \sum_{2k=0}^{n} \frac{2a_{0,2k}^+}{2k+1}\left(\sqrt{2h}\right)^{2k+1} \\ &= \sum_{i+2k=0}^{n} A_{i,2k}^+\left(\sqrt{h}\right)^{2k+1+i}, \end{aligned} \tag{43}$$

where

$$\begin{aligned} A_{0,2k}^+ &= \frac{2\left(\sqrt{2}\right)^{2k+1} a_{0,2k}^+}{2k+1}, \\ A_{i,2k}^+ &= \frac{2\left(\sqrt{2}\right)^{2k+1+i} \overline{q}_{i-1,2k}^+}{\left(\sqrt{a_1}\right)^i} \times \int_0^{\pi/2} \sin^{i-1}\theta\cos^{2k+2}\theta\, d\theta, \quad 1 \le i \le n. \end{aligned} \tag{44}$$

Thus, by (15) and the above discussion we know that (i) holds.

For $a_0 \neq 0$, (41) can be represented as

$$I_{i,2k}^+(h,\delta) = \frac{1}{\left(\sqrt{a_1}\right)^{i+1}} \sum_{r=0}^{i} C_i^r \left(-\frac{a_0}{\sqrt{a_1}}\right)^{i-r} \overline{I}_{rk}^+(h,\delta), \tag{45}$$

where

$$\overline{I}_{rk}^+(h,\delta) = \int_{a_0/\sqrt{a_1}}^{\sqrt{2h+a_0^2/a_1}} v^r\left(2h + \frac{a_0^2}{a_1} - v^2\right)^{k+1/2} dv. \tag{46}$$

Recall that

$$\begin{aligned} \int v^r\left(2h + \frac{a_0^2}{a_1} - v^2\right)^{k+1/2} dv &= \frac{v^{r+1}\left(2h + a_0^2/a_1 - v^2\right)^{k+1/2}}{2k+2+r} \\ &\quad + \frac{(2k+1)\left(2h + a_0^2/a_1\right)}{2k+2+r} \times \int v^r\left(2h + \frac{a_0^2}{a_1} - v^2\right)^{k-1/2} dv. \end{aligned} \tag{47}$$

Then, by (46) and the above equation we obtain that

$$\begin{aligned} \overline{I}_{rk}^+(h,\delta) &= -\frac{a_0^{r+1}(2h)^{k+1/2}}{a_1^{(r+1)/2}(2k+2+r)} \\ &\quad + \frac{(2k+1)\left(2h+a_0^2/a_1\right)}{2k+2+r} \overline{I}_{r,k-1}^+(h,\delta), \quad k \ge 1,\ r \ge 0. \end{aligned} \tag{48}$$

It follows that

$$\begin{aligned} \overline{I}_{rk}^+(h,\delta) &= -\frac{a_0^{r+1}}{a_1^{(r+1)/2}}\sqrt{2h}\widetilde{\varphi}_{rk}^+(h) \\ &\quad + \widetilde{\alpha}_{rk}^+\left(2h + \frac{a_0^2}{a_1}\right)^k \overline{I}_{r0}^+(h,\delta), \quad k \ge 1,\ r \ge 0, \end{aligned} \tag{49}$$

where

$$\widetilde{\alpha}_{rk}^{+} = \frac{(2k+1)(2k-1)(2k-3)\times\cdots\times 3}{(2k+2+r)(2k+r)(2k-2+r)\times\cdots\times(4+r)},$$

$$\begin{aligned}\widetilde{\varphi}_{rk}^{+}(h) &= \frac{(2h)^{k}}{2k+2+r} + \frac{2k+1}{(2k+2+r)(2k+r)} \\ &\quad\times\left(2h+\frac{a_0^2}{a_1}\right)(2h)^{k-1} \\ &\quad+\frac{(2k+1)(2k-1)}{(2k+2+r)(2k+r)(2k-2+r)} \\ &\quad\times\left(2h+\frac{a_0^2}{a_1}\right)^{2}(2h)^{k-2}+\cdots \\ &\quad+\frac{(2k+1)(2k-1)\times\cdots\times 5}{(2k+2+r)(2k+r)(2k-2+r)\times\cdots\times(4+r)} \\ &\quad\times\left(2h+\frac{a_0^2}{a_1}\right)^{k-1}2h,\end{aligned} \tag{50}$$

which is a polynomial of degree k in h. For convenience, introduce

$$\overline{\varphi}_{rk}^{+}(h) = \begin{cases}\widetilde{\varphi}_{rk}^{+}(h), & k\geq 1,\\ 0, & k=0,\end{cases}\qquad \overline{\alpha}_{rk}^{+} = \begin{cases}\widetilde{\alpha}_{rk}^{+}, & k\geq 1,\\ 1, & k=0.\end{cases} \tag{51}$$

Then, combining (49) and (51) gives that

$$\begin{aligned}\overline{I}_{rk}^{+}(h,\delta) &= -\frac{a_0^{r+1}}{a_1^{(r+1)/2}}\sqrt{2h}\,\overline{\varphi}_{rk}^{+}(h) \\ &\quad+\overline{\alpha}_{rk}^{+}\left(2h+\frac{a_0^2}{a_1}\right)^{k}\overline{I}_{r0}^{+}(h,\delta), \quad k\geq 0,\ r\geq 0.\end{aligned} \tag{52}$$

Further, by using the formula

$$\begin{aligned}&\int v^{r}\left(2h+\frac{a_0^2}{a_1}-v^2\right)^{1/2}dv \\ &\quad= \frac{-v^{r-1}\left(2h+a_0^2/a_1-v^2\right)^{3/2}}{r+2} \\ &\quad\quad+\frac{(r-1)\left(2h+a_0^2/a_1\right)}{r+2} \\ &\quad\quad\times\int v^{r-2}\left(2h+\frac{a_0^2}{a_1}-v^2\right)^{1/2}dv,\end{aligned} \tag{53}$$

we have that

$$\begin{aligned}\overline{I}_{r0}^{+}(h,\delta) &= \frac{a_0^{r-1}(2h)^{3/2}}{a_1^{(r-1)/2}(r+2)} \\ &\quad+\frac{r-1}{r+2}\left(2h+\frac{a_0^2}{a_1}\right)\overline{I}_{r-2,0}^{+}(h,\delta), \quad r\geq 1.\end{aligned} \tag{54}$$

It follows that

$$\begin{aligned}\overline{I}_{r0}^{+}(h,\delta) &= (2h)^{3/2}\widetilde{\varphi}_{r}^{+}(h) \\ &\quad+\widetilde{\alpha}_{r}^{+}\left(2h+\frac{a_0^2}{a_1}\right)^{r/2}\overline{I}_{00}(h,\delta), \quad r\geq 1,\end{aligned} \tag{55}$$

where

$$\widetilde{\alpha}_{r}^{+} = \begin{cases}0, & r \text{ odd},\\ \dfrac{(r-1)(r-3)\times\cdots\times 3\times 1}{(r+2)r(r-2)\times\cdots\times 6\times 4}, & r \text{ even},\end{cases}$$

$$\begin{aligned}\widetilde{\varphi}_{r}^{+}(h) &= \frac{a_0^{r-1}}{(r+2)a_1^{(r-1)/2}} + \frac{(r-1)a_0^{r-3}}{(r+2)ra_1^{(r-3)/2}}\left(2h+\frac{a_0^2}{a_1}\right) \\ &\quad+\frac{(r-1)(r-3)a_0^{r-5}}{(r+2)r(r-2)a_1^{(r-5)/2}}\left(2h+\frac{a_0^2}{a_1}\right)^{2}+\cdots \\ &\quad+\left((r-1)(r-3)\times\cdots\times\left(r+1-2\left[\frac{(r-1)}{2}\right]\right)\right. \\ &\quad\quad\left.\times a_0^{r-1-2[(r-1)/2]}\right) \\ &\quad\times\left((r+2)r(r-2)\times\cdots\times\left(r+2-2\left[\frac{(r-2)}{2}\right]\right)\right. \\ &\quad\quad\left.\times a_1^{(r-1)/2-[(r-1)/2]}\left[\frac{(r-1)}{2}\right]\right)^{-1} \\ &\quad\times\left(2h+\frac{a_0^2}{a_1}\right)^{[(r-1)/2]},\end{aligned} \tag{56}$$

which is a polynomial of degree $[(r-1)/2]$ in h. Let

$$\overline{\varphi}_{r}^{+}(h) = \begin{cases}\widetilde{\varphi}_{r}^{+}(h), & r\geq 1,\\ 0, & r=0,\end{cases}\qquad \overline{\alpha}_{r}^{+} = \begin{cases}\widetilde{\alpha}_{r}^{+}, & r\geq 1,\\ 1, & r=0.\end{cases} \tag{57}$$

Then, we have that by (55) and the above

$$\begin{aligned}\overline{I}_{r0}^{+}(h,\delta) &= (2h)^{3/2}\overline{\varphi}_{r}^{+}(h) \\ &\quad+\overline{\alpha}_{r}^{+}\left(2h+\frac{a_0^2}{a_1}\right)^{r/2}\overline{I}_{00}^{+}(h,\delta), \quad r\geq 0.\end{aligned} \tag{58}$$

Substituting the above equation into (52), one can find that

$$\begin{aligned}\overline{I}_{rk}^{+}(h,\delta) &= \sqrt{h}\,\overline{\psi}_{rk}^{+}(h)+\overline{\alpha}_{rk}\overline{\alpha}_{r}\left(2h+\frac{a_0^2}{a_1}\right)^{k+r/2} \\ &\quad\times\overline{I}_{00}^{+}(h,\delta), \quad r\geq 0,\ k\geq 0,\end{aligned} \tag{59}$$

where $\overline{\alpha}_r = 0$ for r odd, $\overline{\alpha}_r > 0$ for r even, and

$$\begin{aligned}\overline{\psi}_{rk}^{+}(h,\delta) &= -\frac{\sqrt{2}a_0^{r+1}}{a_1^{(r+1)/2}}\overline{\varphi}_{rk}^{+}(h) \\ &\quad+2\sqrt{2}h\overline{\alpha}_{rk}\left(2h+\frac{a_0^2}{a_1}\right)^{k}\overline{\varphi}_{r}^{+}(h),\end{aligned} \tag{60}$$

which is a polynomial of degree $k+[(r+1)/2]$ in h. Combining (37), (45), and (59) gives that

$$\begin{aligned} M_1^+(h,\delta) = & \sqrt{h}\sum_{i+2k=0}^{n-1}\frac{2\overline{q}_{i,2k}^+}{(\sqrt{a_1})^{i+1}}\sum_{r=0}^{i}C_i^r\left(-\frac{a_0}{\sqrt{a_1}}\right)^{i-r}\overline{\psi}_{rk}^+(h) \\ & +\sum_{i+2k=0}^{n-1}\frac{2\overline{q}_{i,2k}^+}{(\sqrt{a_1})^{i+1}} \\ & \times\sum_{r=0,\ r\text{ even}}^{i}C_i^r\overline{\alpha}_{rk}\overline{\alpha}_r\left(-\frac{a_0}{\sqrt{a_1}}\right)^{i-r} \\ & \times\left(2h+\frac{a_0^2}{a_1}\right)^{k+r/2}\overline{I}_{00}^+(h,\delta), \end{aligned} \tag{61}$$

which implies (35), together with (13) and (15).

Note that

$$\int\sqrt{a^2-x^2}dx=\frac{1}{2}\left[x\sqrt{a^2-x^2}+a^2\arcsin\frac{x}{|a|}\right]. \tag{62}$$

Then, we have

$$\begin{aligned} \overline{I}_{00}^+(h,\delta) = & -\frac{a_0}{2\sqrt{a_1}}\sqrt{2h}+\frac{1}{2}\left(2h+\frac{a_0^2}{a_1}\right) \\ & \times\left(\frac{\pi}{2}-\arcsin\frac{a_0}{\sqrt{2a_1h+a_0^2}}\right). \end{aligned} \tag{63}$$

Inserting the above formula into (35), we can obtain (36). Hence, the proof is finished. □

Similar to Lemma 4, we can obtain the following lemma about $M^-(h,\delta)$.

Lemma 5. *Let system* (5b) *satisfy* (3) *and* (4). *Then*

(i) *If* $b_1>0, b_0=0$, $M^-(h,\delta)$ *in* (11) *has the expression*

$$M^-(h,\delta)=\sqrt{h}\sum_{i+2k=0}^{n}A_{i,2k}^-\left(\sqrt{h}\right)^{i+2k}, \tag{64}$$

where

$$\begin{aligned} A_{0,2k}^- & = -\frac{2\left(\sqrt{2}\right)^{2k+1}a_{0,2k}^-}{2k+1}, \\ A_{i,2k}^- & = \frac{2\left(\sqrt{2}\right)^{2k+1+i}}{\left(\sqrt{b_1}\right)^i}\left(b_{i-1,2k+1}^-+\frac{i}{2k+1}a_{i,2k}^-\right) \\ & \quad\times\int_{-\pi/2}^{0}\sin^{i-1}\theta\cos^{2k+2}\theta\,d\theta, \quad 1\le i\le n. \end{aligned} \tag{65}$$

(ii) *If* $b_1>0$, $b_0\neq 0$, $M^-(h,\delta)$ *in* (11) *has the form*

$$\begin{aligned} M^-(h,\delta) = & \sqrt{h}\left[\sum_{i+2k=0}^{n-1}\left(b_{i,2k+1}^-+\frac{i+1}{2k+1}a_{i+1,2k}^-\right)\phi_{ik}^-(h)\right. \\ & \left.-\sum_{2k=0}^{n}\frac{2^{k+1+1/2}a_{0,2k}^-}{2k+1}h^k\right] \\ & +\sum_{i+2k=0}^{n-1}\left(b_{i,2k+1}^-+\frac{i+1}{2k+1}a_{i+1,2k}^-\right) \\ & \times\sum_{r=0,\ r\ even}^{i}\alpha_{irk}^-\left(2h+\frac{b_0^2}{b_1}\right)^{k+r/2}\overline{I}_{00}^-(h,\delta), \end{aligned} \tag{66}$$

or

$$\begin{aligned} M^-(h,\delta) = & \sqrt{h}\psi_{[n/2]}^-(h,\delta)+\left(2h+\frac{b_0^2}{b_1}\right)\varphi_{[(n-1)/2]}^- \\ & \times\left(2h+\frac{b_0^2}{b_1},\delta\right)\left(\frac{\pi}{2}+\arcsin\frac{b_0}{\sqrt{2b_1h+b_0^2}}\right), \end{aligned} \tag{67}$$

where

$$\overline{I}_{00}^-(h,\delta)=\int_{-\sqrt{2h+b_0^2/b_1}}^{b_0/\sqrt{b_1}}\sqrt{2h+\frac{b_0^2}{b_1}-v^2dv}, \tag{68}$$

each α_{irk}^- *is nonzero constant and* $\phi_{ik}^-,\psi_{[n/2]}^-,\varphi_{[(n-1)/2]}^-$ *are polynomials of degree* $k+[(i+1)/2],[n/2],[(n-1)/2]$, *respectively.*

3. Proof of Theorem 1

In this section, we will prove the main results. Obviously, under (9) there are the following 9 subcases:

(1) $a_1=b_1=0,\ a_0>0,\ b_0<0$,

(2) $a_1>0, b_1>0,\ a_0=b_0=0$,

(3) $a_1>0, b_0<0,\ a_0=b_1=0$,

(4) $a_1>0, a_0>0,\ b_1=0,\ b_0<0$,

(5) $a_1>0, a_0>0,\ b_1>0,\ b_0=0$,

(6) $a_1>0, a_0>0,\ b_1>0,\ b_0<0$,

(7) $a_0>0, b_1>0,\ a_1=b_0=0$,

(8) $a_1=0, a_0>0,\ b_1>0,\ b_0<0$,

(9) $a_1>0, a_0=0,\ b_1>0,\ b_0<0$.

We only give the proof of Subcases 1, 2, 3, 4, 5, and 6. And the Subcases 7, 8, and 9 can be verified, similar to Subcases 3, 4, and 5, respectively.

Subcase 1. $a_1 = b_1 = 0, \quad a_0 > 0, \quad b_0 < 0$. From (12) and Lemma 3, one can obtain that

$$\begin{aligned} M(h,\delta) &= M^{+}(h,\delta) + M^{-}(h,\delta) \\ &= h^{1/2}\sum_{i+2k=0}^{n} B_{i,2k}^{+} h^{i+k} + h^{1/2}\sum_{i+2k=0}^{n} B_{i,2k}^{-} h^{i+k} \\ &= h^{1/2}\sum_{i+2k=0}^{n}\left(B_{i,2k}^{+} + B_{i,2k}^{-}\right) h^{i+k}, \end{aligned} \tag{69}$$

which implies that $M(h,\delta)$ has at most n isolated positive zeros for $h > 0$. To show that this bound can be reached, take $a_{ij}^{-} = b_{ij}^{\pm} = 0, a_{ij}^{+} = 0,\ j \geq 1$. Then, by (27) and (29), (69) has the form

$$M(h,\delta) = h^{1/2}\sum_{i=0}^{n} B_{i0}^{+} h^{i}, \tag{70}$$

where

$$B_{00}^{+} = 2^{1+1/2} a_{00}^{+}, \qquad B_{i0}^{+} = \frac{2^{1+1/2}}{a_0^{i}}\frac{(2i)!!}{(2i+1)!!} a_{i0}^{+}, \quad i \geq 1. \tag{71}$$

Hence, using (70) we can take $a_{i0}^{+}, i = 0,1,\ldots,n$ as free parameters to produce n simple positive zeros of $M(h,\delta)$ near $h = 0$, which gives n limit cycles correspondingly near the origin. Thus, $N(n) \geq n$ in this case. This ends the proof.

Subcase 2. $a_1 > 0,\ b_1 > 0,\ a_0 = b_0 = 0$ Similar to the above and using (32) and (64), $M(h,\delta)$ in (12) has the expression of the form

$$\begin{aligned} M(h,\delta) &= \sum_{i+2k=0}^{n} A_{i,2k}^{+}\left(\sqrt{h}\right)^{2k+1+i} + \sum_{i+2k=0}^{n} A_{i,2k}^{-}\left(\sqrt{h}\right)^{2k+1+i} \\ &= \sqrt{h}\sum_{l=0}^{n} A_{l}\left(\sqrt{h}\right)^{l}, \end{aligned} \tag{72}$$

where $A_l = \sum_{i+2k=l}(A_{i,2k}^{+} + A_{i,2k}^{-})$. Further, taking $a_{ij}^{-} = b_{ij}^{\pm} = 0,\ a_{ij}^{+} = 0,\ j \geq 1$, then, by (34) and (65), $M(h,\delta)$ in (72) becomes

$$M(h,\delta) = h^{1/2}\sum_{i=0}^{n} A_{i0}^{+}\left(\sqrt{h}\right)^{i}, \tag{73}$$

where

$$\begin{aligned} A_{00}^{+} &= \frac{2a_{00}^{+}}{2k+1}, \\ A_{i0}^{+} &= \frac{2ia_{i0}^{+}}{\left(\sqrt{a_1}\right)^{i}}\int_0^{\pi/2}\sin^{i-1}\theta\cos^2\theta\, d\theta, \quad i \geq 1. \end{aligned} \tag{74}$$

Thus, from (72) and (73), we can discuss similar to Subcase 1. This finishes the proof.

Subcase 3. $a_1 > 0,\ b_0 < 0,\ a_0 = b_1 = 0$ By Lemmas 3, and 4 and (12), we can have that

$$\begin{aligned} M(h,\delta) &= \sum_{i+2k=0}^{n} A_{i,2k}^{+}\left(\sqrt{h}\right)^{2k+1+i} + h^{1/2}\sum_{i+2k=0}^{n} B_{i,2k}^{-} h^{i+k} \\ &= h^{1/2}\left(\sum_{i+2k=0}^{n} A_{i,2k}^{+}\left(\sqrt{h}\right)^{i+2k} + \sum_{i+2k=0}^{n} B_{i,2k}^{-} h^{i+k}\right) \\ &= h^{1/2} M^{*}(h,\delta). \end{aligned} \tag{75}$$

Let us prove that $M^{*}(h,\delta)$ has at most $n+[(n+1)/2]$ zeros on the open interval $(0,+\infty)$. For the purpose, let $\sqrt{h} = \lambda$. Then, for $n = 2l,\ l \geq 1$, $M^{*}(h,\delta)$ in (75) has the expression

$$\begin{aligned} M^{*}(h,\delta) &= \sum_{i+2k=0}^{n} A_{i,2k}^{+}\lambda^{i+2k} + \sum_{i+2k=0}^{n} B_{i,2k}^{-}\lambda^{2i+2k} \\ &= \sum_{j=0}^{2l} C_j\lambda^{j} + \sum_{j=1}^{l} C_{2l+j}\lambda^{2j+2l} \triangleq \overline{M}(\lambda,\delta), \end{aligned} \tag{76}$$

where

$$\begin{aligned} C_j &= \sum_{i+2k=j} A_{i,2k}^{+} + \sum_{2i+2k=j} B_{i,2k}^{-}, \quad j = 0,1,2,\ldots,2l, \\ C_{2l+j} &= \sum_{i+k=j+l} B_{i,2k}^{-}, \quad j = 1,2,\ldots,l. \end{aligned} \tag{77}$$

To prove $M^{*}(h,\delta)$ has at most $n+[(n+1)/2]$ zeros, it suffices to prove $\overline{M}(\lambda,\delta)$ has at most $n+[(n+1)/2] = 3l$ zeros for $\lambda > 0$. By Rolles theorem we need only to prove that $d^{2l}\overline{M}(\lambda,\delta)/d\lambda^{2l}$ has at most l zeros for $\lambda \in (0,+\infty)$. From (76), we can have that

$$\frac{d^{2l}\overline{M}(\lambda,\delta)}{d\lambda^{2l}} = C_{2l}(2l)! + \sum_{j=1}^{l} C_{2l+j}A_{2l+2j}^{2l}\lambda^{2j}, \tag{78}$$

which shows that $d^{2l}\overline{M}(\lambda,\delta)/d\lambda^{2l}$ has at most l zeros for $\lambda > 0$. Thus, $M(h,\delta)$ has at most $3l$ zeros for $h > 0$. To prove $3l$ zeros can appear, we only need to prove that $M^{*}(h,\delta)$ in (75) can appear $3l$ zeros for $h > 0$ small. Let $b_{ij}^{\pm} = 0,\ a_{ij}^{\pm} = 0,\ j \geq 1, a_{i0}^{-} = 0,\ 0 \leq i \leq l$, and $a_{2l}^{-} \neq 0$. Then $M^{*}(h,\delta)$ in (75) can be expressed as by (29) and (34)

$$\begin{aligned} M^{*}(h,\delta) &= \sum_{i=0}^{2l} A_{i0}^{+}\left(\sqrt{h}\right)^{i} + \sum_{i=l+1}^{2l} B_{i0}^{-}\left(\sqrt{h}\right)^{2i} \\ &= A_{00}^{+} + A_{10}^{+}\sqrt{h} + A_{20}^{+}\left(\sqrt{h}\right)^{2} + \cdots \\ &\quad + A_{2l-1,0}^{+}\left(\sqrt{h}\right)^{2l-1} + A_{2l,0}^{+}\left(\sqrt{h}\right)^{2l} \\ &\quad + B_{l+1,0}^{-}\left(\sqrt{h}\right)^{2l+2} + B_{l+2,0}^{-}\left(\sqrt{h}\right)^{2l+4} \\ &\quad + \cdots + B_{2l,0}^{-}\left(\sqrt{h}\right)^{4l}, \end{aligned} \tag{79}$$

where

$$\begin{aligned} A_{00}^{+} &= 2\sqrt{2}a_{00}^{+}, \\ A_{i0}^{+} &= \frac{2\left(\sqrt{2}\right)^{i+1}}{\left(\sqrt{a_1}\right)^{i}} i a_{i0}^{+}, \quad i = 1, 2, \ldots, 2l, \\ B_{i0}^{-} &= \frac{-2\sqrt{2}}{b_0^i} a_{i0}^{-} \int_0^{\pi/2} \cos^{2i+1}\theta d\theta \\ &= \frac{-2\sqrt{2}}{b_0^i} \frac{(2i)!!}{(2i+1)!!} a_{i0}^{-}, \quad i = l+1, \ldots, 2l. \end{aligned} \tag{80}$$

Thus, by changing the sign of $a_{2l,0}^{-}, a_{2l-1,0}^{-}, \ldots, a_{l+1,0}^{-}, a_{2l,0}^{+}, a_{2l-1,0}^{+}, \ldots, a_{00}^{+}$ in turn such that

$$\begin{gathered} a_{i-1,0}^{-} a_{i0}^{-} < 0, \qquad i = 2l,\ 2l-1, \ldots, l+2, \\ a_{l+1}^{-} a_{2l,0}^{+} > 0, \qquad a_{i-1,0}^{+} a_{i0}^{+} < 0, \quad i = 2l, 2l-1, \ldots, 1, \\ 0 < \left|a_{00}^{+}\right| \ll \left|a_{10}^{+}\right| \ll \cdots \ll \left|a_{2l,0}^{+}\right| \\ \ll \left|a_{l+1}^{-}\right| \ll \left|a_{l+2}^{-}\right| \ll \left|a_{2l-1}^{-}\right| \ll 1, \end{gathered} \tag{81}$$

we can find $3l$ simply positive zeros $h_1, h_2, \ldots, h_{3l}$ with $0 < h_{3l} < h_{3l-1} < \cdots < h_1 \ll 1$. For $n = 2l+1, l = 0, 1, \ldots$, we can discuss in a similar way. Thus, this bound can be reached and $N(n) \geq n + [(n+1)/2]$. The proof is finished.

Subcase 4. $a_1 > 0,\ a_0 > 0,\ b_1 = 0,\ b_0 < 0$ From (12) and Lemmas 3 and 4, we get that

$$\begin{aligned} M(h,\delta) = &\sqrt{h}\psi_{[n/2]}^{+}(h,\delta) + \left(2h + \frac{a_0^2}{a_1}\right)\varphi_{[(n-1)/2]}^{+} \\ &\times \left(2h + \frac{a_0^2}{a_1}, \delta\right)\left(\pi - 2\arcsin\frac{a_0}{\sqrt{2a_1 h + a_0^2}}\right) \\ &+ \sqrt{h}\sum_{i+2k=0}^{n} B_{i,2k}^{-} h^{i+k} \\ = &\sqrt{h} f_n(h,\delta) + \left(2h + \frac{a_0^2}{a_1}\right)\varphi_{[(n-1)/2]}^{+} \\ &\times \left(2h + \frac{a_0^2}{a_1}, \delta\right)\left(\pi - 2\arcsin\frac{a_0}{\sqrt{2a_1 h + a_0^2}}\right), \end{aligned} \tag{82}$$

where

$$f_n(h,\delta) = \psi_{[n/2]}^{+}(h,\delta) + \sum_{i+2k=0}^{n} B_{i,2k}^{-} h^{i+k}, \tag{83}$$

which is a polynomial of degree n in h. Let $\lambda = \sqrt{h}$. Then $h = \lambda^2,\ \lambda \in (0, +\infty)$, and (82) becomes

$$\begin{aligned} M(h,\delta) = &\lambda f_n\left(\lambda^2\right) \\ &+ v(\lambda,\delta)\left(\frac{\pi}{2} - \arcsin\frac{a_0}{\sqrt{2a_1\lambda^2 + a_0^2}}\right) \\ \triangleq &\widetilde{M}(\lambda,\delta), \end{aligned} \tag{84}$$

where

$$v(\lambda,\delta) = \left(2\lambda^2 + \frac{a_0^2}{a_1}\right)\varphi_{[(n-1)/2]}^{+}\left(2\lambda^2 + \frac{a_0^2}{a_1}, \delta\right). \tag{85}$$

One can see that $(d/d\lambda)(\widetilde{M}(\lambda,\delta)/v(\lambda,\delta)) = u(\lambda,\delta)/v^2(\lambda,\delta)$, where

$$\begin{aligned} u(\lambda,\delta) = &\left(2\lambda^2 + \frac{a_0^2}{a_1}\right)\varphi_{[(n-1)/2]}^{+}\left(2\lambda^2 + \frac{a_0^2}{a_1}, \delta\right) \\ &\times \frac{d}{d\lambda}\left(\lambda f_n\left(\lambda^2,\delta\right)\right) - \lambda f_n\left(\lambda^2,\delta\right) \\ &\times \frac{d}{d\lambda}\left[\left(2\lambda^2 + \frac{a_0^2}{a_1}\right)\varphi_{[(n-1)/2]}^{+}\left(2\lambda^2 + \frac{a_0^2}{a_1}, \delta\right)\right] \\ &+ \frac{\sqrt{2}a_0}{\sqrt{a_1}}\left(2\lambda^2 + \frac{a_0^2}{a_1}\right) \\ &\times \left(\varphi_{[(n-1)/2]}^{+}\left(2\lambda^2 + \frac{a_0^2}{a_1}, \delta\right)\right)^2. \end{aligned} \tag{86}$$

Denote by $\#\{\lambda \in (0,+\infty) \mid f(\lambda) = 0\}$ the number of zeros of the function in the interval $(0, +\infty)$ taking into account their multiplicities. Note that

$$\deg v = 2\left[\frac{n+1}{2}\right], \qquad \deg u = 2\left(n + \left[\frac{n+1}{2}\right]\right), \tag{87}$$

and they are even functions in λ. Therefore,

$$\begin{aligned} \#\{\lambda \in (0,+\infty) \mid v(\lambda,\delta)\} &\leq \left[\frac{n-1}{2}\right], \\ \#\{\lambda \in (0,+\infty) \mid u(\lambda,\delta)\} &\leq n + \left[\frac{n+1}{2}\right]. \end{aligned} \tag{88}$$

Then, from [8], we can obtain that

$$\begin{aligned} \#&\left\{\lambda \in (0,+\infty) \mid \widetilde{M}(\lambda,\delta) = 0\right\} \\ &\leq \#\{\lambda \in (0,+\infty) \mid v(\lambda,\delta)\} \\ &\quad + \#\{\lambda \in (0,+\infty) \mid u(\lambda,\delta)\} + 1 \\ &\leq \left[\frac{n-1}{2}\right] + n + \left[\frac{n+1}{2}\right] + 1 \\ &= n + 2\left[\frac{n+1}{2}\right], \end{aligned} \tag{89}$$

which implies that $Z(n) \le n + 2[(n+1)/2]$. Now, we verify $Z(n) \ge n + [(n+1)/2]$.

Make the transformation $u = \sqrt{v^2/2h - a_0^2/2a_1 h}$. Then $\overline{I}_{00}^{+}(h,\delta)$ in (35) becomes

$$\overline{I}_{00}^{+}(h,\delta) = \frac{(2h)^{3/2}}{\sqrt{a_0^2/a_1}} \int_0^1 \frac{u\left(1-u^2\right)^{1/2}}{\sqrt{1+\left(2ha_1u^2/a_0^2\right)}} du, \tag{90}$$

which follows that as $h > 0$ small

$$\begin{aligned} \overline{I}_{00}^{+}(h,\delta) &= \frac{(2h)^{3/2}}{\sqrt{a_0^2/a_1}} \int_0^1 u\left(1-u^2\right)^{1/2} \\ &\quad \times \left[1 + \sum_{m=1}^{+\infty} \frac{(-1)^m (2m-1)!!}{(2m)!!} \left(\frac{2a_1}{a_0^2}\right)^m h^m u^{2m}\right] du. \end{aligned} \tag{91}$$

Note that

$$\begin{aligned} &\int_0^1 u^{2m+1}\left(1-u^2\right)^{1/2} du \\ &\quad = \int_0^{\pi/2} \sin^{2m+1}\theta \cos^2\theta \, d\theta \quad (\text{Let } u = \sin\theta) \\ &\quad = \frac{(2m)!!}{(2m+3)!!}, \quad m = 0,1,2,\ldots. \end{aligned} \tag{92}$$

Inserting the above formula into (91) gives that

$$\overline{I}_{00}(h,\delta) = 2\sqrt{h}\sqrt{\frac{2a_1}{a_0^2}} \sum_{m=0}^{+\infty} C_m \left(\frac{2a_1}{a_0^2}\right)^m h^{m+1}, \tag{93}$$

where

$$C_m = \frac{(-1)^m}{(2m+1)(2m+3)}, \quad m \ge 0. \tag{94}$$

Take $b_{ij}^- = a_{ij}^+ = 0, a_{ij}^- = 0,\ j \ge 1,\ b_{ij}^+ = 0,\ i \ge 1$. Then, by (28), (35), and (93), we can obtain that for $h > 0$ small

$$\begin{aligned} M(h,\delta) &= \sqrt{h}\left[\sum_{i=0}^{n} \frac{-2\sqrt{2}(2i)!!}{(2i+1)!!b_0^i} a_{i0}^- h^i + \sum_{2k=0}^{n-1} b_{0,2k+1}^+ \psi_{0k}^+(h) \right. \\ &\quad + \sum_{2k=0}^{n-1} 2^{k+1}\alpha_{00k}^+ \sqrt{\frac{2a_1}{a_0^2}} b_{0,2k+1}^+ \left(h + \frac{a_0^2}{2a_1}\right)^k \\ &\quad \left. \times \sum_{m=0}^{+\infty} C_m \left(\frac{2a_1}{a_0^2}\right)^m h^{m+1}\right] \\ &= \sqrt{h}\sum_{i\ge 0} v_i h^i, \end{aligned} \tag{95}$$

where

$$\begin{aligned} v_i &= \frac{-2\sqrt{2}(2i)!!}{(2i+1)!!b_0^i} a_{i0}^- \\ &\quad + L_i\left(b_{01}^+, b_{03}^+, \ldots, b_{0,2[(n-1)/2]+1}^+\right), \quad i = 0,1,2,\ldots,n \\ v_{n+1+i} &= \sum_{2k=0}^{n-1} \alpha_{00k}^+ 2^{k+1} \left(\frac{2a_1}{a_0^2}\right)^{n+i-k+1/2} \\ &\quad \times \sum_{r=0}^{k} C_k^r C_{n+i-r} b_{0,2k+1}^+, \quad i \ge 0, \end{aligned} \tag{96}$$

with $L_i, i = 0,1,\ldots,n$ being linear combination, $L_i(0,0,\ldots,0) = 0$. One can find that

$$\begin{aligned} &\frac{\partial\left(v_0, v_1, \ldots, v_n, v_{n+1}, v_{n+2}, \ldots, v_{n+[(n+1)/2]}\right)}{\partial\left(a_{00}^-, a_{10}^-, \ldots, a_{n0}^-, b_{01}^+, b_{03}^+, \ldots, b_{0,2[(n-1)/2]+1}^+\right)} \\ &\quad = \begin{pmatrix} A_1 & A_2 \\ 0 & 2\sqrt{\dfrac{2a_1}{a_0^2}} A_3 \end{pmatrix} \equiv A, \end{aligned} \tag{97}$$

where A_2 is a $(n+1) \times [(n+1)/2]$ matrix,

$$\begin{aligned} A_1 &= \begin{pmatrix} -2\sqrt{2} & 0 & 0 & \cdots & 0 \\ 0 & \dfrac{-4\sqrt{2}}{b_0} & 0 & \cdots & 0 \\ 0 & 0 & \dfrac{-16\sqrt{2}}{15b_0^2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \dfrac{-2\sqrt{2}(2n)!!}{(2n+1)!!b_0^n} \end{pmatrix}, \\ A_3 &= \left(\alpha_{000}^+\beta_0, 2\alpha_{001}^+\beta_1, 2^2\alpha_{002}^+\beta_2, \ldots, \right. \\ &\quad \left. 2^{[(n-1)/2]}\alpha_{00,[(n-1)/2]}^+\beta_{[(n-1)/2]}\right), \end{aligned} \tag{98}$$

with β_i are $[(n+1)/2] \times 1$ matrix satisfying

$$\beta_i = \begin{pmatrix} \left(\dfrac{2a_1}{a_0^2}\right)^{n-i} \sum_{r=0}^{i} C_i^r C_{n-r} \\ \left(\dfrac{2a_1}{a_0^2}\right)^{n+1-i} \sum_{r=0}^{i} C_i^r C_{n+1-r} \\ \vdots \\ \left(\dfrac{2a_1}{a_0^2}\right)^{n+[(n-1)/2]-i} \sum_{r=0}^{i} C_i^r C_{n+[(n-1)/2]-r} \end{pmatrix}, \quad i = 0,1,2,\ldots,\left[\frac{n-1}{2}\right]. \tag{99}$$

Hence, we can obtain that from (97)

$$|A| = \left(2\sqrt{\frac{2a_1}{a_0^2}}\right)^{[(n+1)/2]} |A_1|\,|A_3| = \left(2\alpha_0^+\sqrt{\frac{2a_1}{a_0^2}}\right)^{[(n+1)/2]} \prod_{i=0}^{[(n-1)/2]} 2^i \alpha_{00i}^+ \prod_{i=0}^{n} \frac{-2\sqrt{2}\,(2i)!!}{(2i+1)!!b_0^i}\,|B|, \tag{100}$$

where

$$B = \left(\beta_0, \beta_1, \ldots, \beta_{[(n-1)/2]}\right), \tag{101}$$

and β_i are given in (99). We claim that $|A| \neq 0$. We only need to prove $|B| \neq 0$ by the above formula. Using elementary transformations to $|B|$ by multiplying ith column by $(2a_1/a_0^2)^{i-1}$, $i = 2, 3, \ldots, [(n+1)/2]$, we can obtain that by (99) and (101)

$$|B| = \left(\frac{2a_1}{a_0^2}\right)^{n[(n+1)/2]} |B_1|, \tag{102}$$

where

$$|B_1| = \begin{vmatrix} C_n & \sum_{r=0}^{1} C_1^r C_{n-r} & \cdots & \sum_{r=0}^{[(n-1)/2]} C_{[(n-1)/2]}^r C_{n-r} \\ C_{n+1} & \sum_{r=0}^{1} C_1^r C_{n+1-r} & \cdots & \sum_{r=0}^{[(n-1)/2]} C_{[(n-1)/2]}^r C_{n+1-r} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n+[(n-1)/2]} & \sum_{r=0}^{1} C_1^r C_{n+[(n-1)/2]-r} & \cdots & \sum_{r=0}^{[(n-1)/2]} C_{[(n-1)/2]}^r C_{n+[(n-1)/2]-r} \end{vmatrix}. \tag{103}$$

Now we will use elementary transformations to B_1 as follows.

(1) Add the first column multiplying by -1 to ith column, $i = 2, 3, \ldots, [(n+1)/2]$.

(2) Add the second column multiplying by $-C_{i-1}^1$ to ith column, $i = 3, 4, \ldots, [(n+1)/2]$.

(3) Add the third column multiplying by $-C_{i-1}^2$ to ith column, $i = 4, 5, \ldots, [(n+1)/2]$

$\vdots$

$[(n-1)/2]$. Add the $[(n-1)/2]$th column multiplying by $-C_{[(n-1)/2]}^{[(n-3)/2]}$ to $[(n+1)/2]$th column,

$[(n+1)/2]$. multiply ith column by $(-1)^{i-1}$, $i = 2, 3, \ldots, [(n+1)/2]$.

Then, $|B_1|$ becomes, together with (94)

$$\begin{aligned} |B_1| &= (-1)^{[(n-1)/2][(n+1)/2]/2} \det\left(\widetilde{\beta}_0, \widetilde{\beta}_1, \ldots, \widetilde{\beta}_{[(n-1)/2]}\right) \\ &= 2^{[(n+1)/2]}(-1)^{n[(n+1)/2]} \det\left(\overline{\beta}_0, \overline{\beta}_1, \ldots, \overline{\beta}_{[(n-1)/2]}\right) \\ &\triangleq 2^{[(n+1)/2]}(-1)^{n[(n+1)/2]} |B_2|, \end{aligned} \tag{104}$$

where

$$\widetilde{\beta}_i = \begin{pmatrix} \dfrac{(-1)^n}{(2(n-i)+1)(2(n-i)+3)} \\ \dfrac{(-1)^{n+1}}{(2(n+1-i)+1)(2(n+1-i)+3)} \\ \vdots \\ \dfrac{(-1)^{n+[(n-1)/2]}}{(2(n+[(n-1)/2]-i)+1)(2(n+[(n-1)/2]-i)+3)} \end{pmatrix},$$

$$\overline{\beta}_i = \begin{pmatrix} \dfrac{1}{2(n-i)+1} - \dfrac{1}{2(n-i)+3} \\ \dfrac{1}{2(n+1-i)+1} - \dfrac{1}{2(n+1-i)+3} \\ \vdots \\ \dfrac{1}{2(n+[(n-1)/2]-i)+1} - \dfrac{1}{2(n+[(n-1)/2]-i)+3} \end{pmatrix}, \tag{105}$$

with $i = 0, 1, \ldots, [(n-1)/2]$. For B_2 in (104) by adding a column on the left and a row on the above, we can obtain that, together with adding ith column to $(i+1)$th column with $i = 1, 2, \ldots, [(n+1)/2]$

$$|B_2| = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ \dfrac{1}{2n+3} & \dfrac{1}{2n+1} & \dfrac{1}{2n-1} & \cdots & \dfrac{1}{2(n-[(n-1)/2])+1} \\ \dfrac{1}{2n+5} & \dfrac{1}{2n+3} & \dfrac{1}{2n+1} & \cdots & \dfrac{1}{2(n-[(n-1)/2])+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \dfrac{1}{2(n+[(n-1)/2])+3} & \dfrac{1}{2(n+[(n-1)/2])+1} & \dfrac{1}{2(n+[(n-1)/2])-1} & \cdots & \dfrac{1}{2n+1} \end{vmatrix}, \tag{106}$$

which implies that $|A_3| \neq 0$ by (102) and (104) if $|B_2| \neq 0$. We claim that $|B_2| \neq 0$ and

$$|B_3| = \begin{vmatrix} \frac{1}{2n+3} & \frac{1}{2n+1} & \frac{1}{2n-1} & \cdots & \frac{1}{2(n-[(n-1)/2])+1} \\ \frac{1}{2n+5} & \frac{1}{2n+3} & \frac{1}{2n+1} & \cdots & \frac{1}{2(n-[(n-1)/2])+3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{2(n+[(n-1)/2])+3} & \frac{1}{2(n+[(n-1)/2])+1} & \frac{1}{2(n+[(n-1)/2])-1} & \cdots & \frac{1}{2n+1} \\ \frac{1}{2(n+[(n+1)/2])+3} & \frac{1}{2(n+[(n+1)/2])+1} & \frac{1}{2(n+[(n+1)/2])-1} & \cdots & \frac{1}{2n+3} \end{vmatrix} \neq 0. \quad (107)$$

Now, we prove them by induction on n. For $n = 1, 2$ we have

$$\begin{gathered} \begin{vmatrix} 1 & 1 \\ \frac{1}{5} & \frac{1}{3} \end{vmatrix} = \frac{2}{15}, \qquad \begin{vmatrix} 1 & 1 \\ \frac{1}{7} & \frac{1}{5} \end{vmatrix} = \frac{2}{35}, \\ \begin{vmatrix} \frac{1}{5} & \frac{1}{3} \\ \frac{1}{7} & \frac{1}{5} \end{vmatrix} = \frac{-4}{525}, \qquad \begin{vmatrix} \frac{1}{7} & \frac{1}{5} \\ \frac{1}{9} & \frac{1}{7} \end{vmatrix} = \frac{-4}{2205}, \end{gathered} \quad (108)$$

which means that (106) and (107) hold for $n = 1, 2$. Suppose (106) and (107) hold for $n = 2l-1,\ 2l, l \geq 1$. That is, we have for $n = 2l$

$$\begin{gathered} |B_2| = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ \frac{1}{4l+3} & \frac{1}{4l+1} & \frac{1}{4l-1} & \cdots & \frac{1}{2l+3} \\ \frac{1}{4l+5} & \frac{1}{4l+3} & \frac{1}{4l+1} & \cdots & \frac{1}{2l+5} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{6l+1} & \frac{1}{6l-1} & \frac{1}{6l-3} & \cdots & \frac{1}{4l+1} \end{vmatrix} \triangleq |C| \neq 0, \\ |B_3| = \begin{vmatrix} \frac{1}{4l+3} & \frac{1}{4l+1} & \frac{1}{4l-1} & \cdots & \frac{1}{2l+3} \\ \frac{1}{4l+5} & \frac{1}{4l+3} & \frac{1}{4l+1} & \cdots & \frac{1}{2l+5} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{6l+1} & \frac{1}{6l-1} & \frac{1}{6l-3} & \cdots & \frac{1}{4l+1} \\ \frac{1}{6l+3} & \frac{1}{6l+1} & \frac{1}{6l-1} & \cdots & \frac{1}{4l+3} \end{vmatrix} \neq 0. \end{gathered} \quad (109)$$

Then for $n = 2l+1$, we have

$$\begin{gathered} |B_2| = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ \frac{1}{4l+5} & \frac{1}{4l+3} & \frac{1}{4l+1} & \cdots & \frac{1}{2l+3} \\ \frac{1}{4l+7} & \frac{1}{4l+5} & \frac{1}{4l+3} & \cdots & \frac{1}{2l+5} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{6l+5} & \frac{1}{6l+3} & \frac{1}{6l+1} & \cdots & \frac{1}{4l+3} \end{vmatrix}, \\ |B_3| = \begin{vmatrix} \frac{1}{4l+5} & \frac{1}{4l+3} & \frac{1}{4l+1} & \cdots & \frac{1}{2l+3} \\ \frac{1}{4l+7} & \frac{1}{4l+5} & \frac{1}{4l+3} & \cdots & \frac{1}{2l+5} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{6l+5} & \frac{1}{6l+3} & \frac{1}{6l+1} & \cdots & \frac{1}{4l+3} \\ \frac{1}{6l+7} & \frac{1}{6l+5} & \frac{1}{6l+3} & \cdots & \frac{1}{4l+5} \end{vmatrix}. \end{gathered} \quad (110)$$

Note that by the first equation of (109) there only exist $\alpha_1, \alpha_2, \ldots, \alpha_{l+1}$ such that

$$\begin{aligned} |B_2| &= \left[\frac{1}{6l+5} - \left(\frac{\alpha_1}{6l+3} + \frac{\alpha_2}{6l+1} + \cdots + \frac{\alpha_{l+1}}{4l+3}\right)\right] |C| \\ &= \frac{-2}{6l+5}\left[\frac{\alpha_1}{6l+3} + \frac{2\alpha_2}{6l+1} + \cdots + \frac{(l+1)\alpha_{l+1}}{4l+3}\right] |C| \end{aligned} \quad (111)$$

since $\alpha_1 + \alpha_2 + \cdots + \alpha_{l+1} = 1$, where C is given in (109). If $|B_2| = 0$, then we can obtain that from (109) and (111)

$$\frac{\alpha_1}{6l+3} + \frac{2\alpha_2}{6l+1} + \cdots + \frac{(l+1)\alpha_{l+1}}{4l+3} = 0. \quad (112)$$

Note that

$$\begin{gathered} \frac{\alpha_1}{4l+3} + \frac{\alpha_2}{4l+1} + \cdots + \frac{\alpha_{l+1}}{2l+3} = \frac{1}{4l+5}, \\ \frac{\alpha_1}{4l+5} + \frac{\alpha_2}{4l+3} + \cdots + \frac{\alpha_{l+1}}{2l+5} = \frac{1}{4l+7}, \\ \vdots \\ \frac{\alpha_1}{6l+1} + \frac{\alpha_2}{6l-1} + \cdots + \frac{\alpha_{l+1}}{4l+1} = \frac{1}{6l+3}, \end{gathered} \quad (113)$$

which follows that, together with (112)

$$\frac{\alpha_1}{4l+3}+\frac{2\alpha_2}{4l+1}+\cdots+\frac{(l+1)\,\alpha_{l+1}}{2l+3}=0,\qquad \frac{\alpha_1}{4l+5}+\frac{2\alpha_2}{4l+3}+\cdots+\frac{(l+1)\,\alpha_{l+1}}{2l+5}=0,$$
$$\vdots$$
$$\frac{\alpha_1}{6l+1}+\frac{2\alpha_2}{6l-1}+\cdots+\frac{(l+1)\,\alpha_{l+1}}{4l+1}=0,\qquad \frac{\alpha_1}{6l+3}+\frac{2\alpha_2}{6l+1}+\cdots+\frac{(l+1)\,\alpha_{l+1}}{4l+3}=0. \tag{114}$$

By the second equation in (109) and the above formula, we have

$$\alpha_1=\alpha_2=\cdots=\alpha_{l+1}=0. \tag{115}$$

This is a contradiction with $\alpha_1+\alpha_2+\cdots+\alpha_{l+1}=1$. Hence $\alpha_1/(6l+3)+2\alpha_2/(6l+1)+\cdots+(l+1)\alpha_{l+1}/(4l+3)\neq 0$, which means that (106) holds for $n=2l+1$. Since $|B_2|\neq 0$ in (110), there only exist $\beta_1,\beta_2,\ldots,\beta_{l+1}$ such that

$$\begin{gathered}\beta_1+\frac{\beta_2}{4l+5}+\frac{\beta_3}{4l+7}+\cdots+\frac{\beta_{l+1}}{6l+5}=\frac{1}{6l+7},\\ \beta_1+\frac{\beta_2}{4l+3}+\frac{\beta_3}{4l+5}+\cdots+\frac{\beta_{l+1}}{6l+3}=\frac{1}{6l+5},\\ \vdots\\ \beta_1+\frac{\beta_2}{2l+3}+\frac{\beta_3}{2l+5}+\cdots+\frac{\beta_{l+1}}{4l+3}=\frac{1}{4l+5},\end{gathered} \tag{116}$$

with $\beta_1\neq 0$ since the last row in the second formula of (110) is linearly independent with all rows in the first formula of (110), which means that (107) holds for $n=2l+1$. In a similar way, we can prove (106) and (107) hold for $n=2l+2$. Hence, the claim holds. So, from (99) we can know that $a_{00}^-,a_{10}^-,\ldots,a_{n0}^-,b_{01}^+,b_{03}^+,\ldots,b_{0,2[(n-1)/2]+1}^+$ can be taken as free parameters. So we can choose these values such that

$$\begin{gathered}v_iv_{i+1}<0,\quad i=0,1,\ldots,n+\left[\frac{(n-1)}{2}\right],\\ 0<|v_0|\ll|v_1|\ll\cdots\ll|v_{n+[(n+1)/2]}|\ll 1,\end{gathered} \tag{117}$$

which yields that by (97) and (96) $M(h,\delta)$ can appear $n+[(n+1)/2]$ positive zeros for $h>0$ small. We also can know that $N(n)\geq n+[(n+1)/2]$. Hence, the conclusion is proved.

Subcase 5. $a_1>0,\ a_0>0,\ b_1>0,\ b_0=0$ By (36) and (67), one can see that

$$\begin{aligned}M(h,\delta)=&\sqrt{h}\psi_{[n/2]}^{+}(h,\delta)+\left(2h+\frac{a_0^2}{a_1}\right)\varphi_{[(n-1)/2]}^{+}\\ &\times\left(2h+\frac{a_0^2}{a_1},\delta\right)\left(\frac{\pi}{2}-\arcsin\frac{a_0}{\sqrt{2a_1h+a_0^2}}\right)\\ &+\sqrt{h}\sum_{i+2k=0}^{n}A_{i,2k}^{-}\left(\sqrt{h}\right)^{i+2k}\\ =&\sqrt{h}g_{[n/2]}(h,\delta)+\overline{g}_{[(n+1)/2]}(h,\delta)\\ &+\left(2h+\frac{a_0^2}{a_1}\right)\varphi_{[(n-1)/2]}^{+}\left(2h+\frac{a_0^2}{a_1},\delta\right)\\ &\times\left(\frac{\pi}{2}-\arcsin\frac{a_0}{\sqrt{2a_1h+a_0^2}}\right),\end{aligned} \tag{118}$$

where

$$\begin{aligned}g_{[n/2]}(h,\delta)&=\psi_{[n/2]}^{+}(h,\delta)+\sum_{l=0}^{[n/2]}\sum_{i+2k=2l}A_{i,2k}^{-}h^{l},\\ \overline{g}_{[(n+1)/2]}(h,\delta)&=\sum_{l=0}^{[(n-1)/2]}\sum_{i+2k=2l+1}A_{i,2k}^{-}h^{l+1},\quad n\geq 1.\end{aligned} \tag{119}$$

For convenience, we denote by g_n any polynomial of degree n although its coefficients may be different when it appears in different place. Then, we claim that for any $2\leq k\leq[(n-1)/2]$

$$\begin{aligned}\frac{d^kM(h,\delta)}{dh^k}=&\sum_{j=0}^{k}h^{1/2-k+j}g_{[n/2]-j}+\frac{d^k\overline{g}_{[(n+1)/2]}}{dh^k}\\ &+\sum_{j=0}^{k-2}g_jh^{-j-1/2}\left(2h+\frac{a_0^2}{a_1}\right)^{-j-1}\\ &+\frac{d^k}{dh^k}\left[\left(2h+\frac{a_0^2}{a_1}\right)\varphi_{[(n-1)/2]}^{+}\left(2h+\frac{a_0^2}{a_1},\delta\right)\right]\\ &\times\left(\frac{\pi}{2}-\arcsin\frac{a_0}{\sqrt{2a_1h+a_0^2}}\right).\end{aligned} \tag{120}$$

Now, we verify this claim by induction on k. For $k = 2$, by (118) we can obtain that

$$\begin{aligned}\frac{dM(h,\delta)}{dh} &= \frac{1}{2}h^{-1/2}g_{[n/2]}(h,\delta) \\ &\quad + h^{1/2}\frac{dg_{[n/2]}(h,\delta)}{dh} + \frac{d\overline{g}_{[(n+1)/2]}}{dh} \\ &\quad + \frac{d}{dh}\left[\left(2h+\frac{a_0^2}{a_1}\right)\varphi^{+}_{[(n-1)/2]}\left(2h+\frac{a_0^2}{a_1},\delta\right)\right] \\ &\quad \times \left(\frac{\pi}{2} - \arcsin\frac{a_0}{\sqrt{2a_1h+a_0^2}}\right) \\ &\quad + \left(2h+\frac{a_0^2}{a_1}\right)\varphi^{+}_{[(n-1)/2]} \\ &\quad \times \left(2h+\frac{a_0^2}{a_1},\delta\right)\frac{a_0}{\sqrt{2a_1}\sqrt{h}\left(2h+a_0^2/a_1\right)} \\ &= h^{-1/2}g_{[n/2]} + h^{1/2}g_{[n/2]-1} + \frac{d\overline{g}_{[(n+1)/2]}}{dh} \\ &\quad + \frac{d}{dh}\left[\left(2h+\frac{a_0^2}{a_1}\right)\varphi^{+}_{[(n-1)/2]}\left(2h+\frac{a_0^2}{a_1},\delta\right)\right] \\ &\quad \times \left(\frac{\pi}{2} - \arcsin\frac{a_0}{\sqrt{2a_1h+a_0^2}}\right),\end{aligned} \tag{121}$$

which follows that

$$\begin{aligned}\frac{d^2M(h,\delta)}{dh^2} &= h^{-3/2}g_{[n/2]} + h^{-1/2}g_{[n/2]-1} \\ &\quad + h^{1/2}g_{[n/2]-2} + \frac{d^2\overline{g}_{[(n+1)/2]}}{dh^2} + \frac{g_0}{\sqrt{h}\left(2h+a_0^2/a_1\right)} \\ &\quad + \frac{d^2}{dh^2}\left[\left(2h+\frac{a_0^2}{a_1}\right)\varphi^{+}_{[(n-1)/2]}\left(2h+\frac{a_0^2}{a_1},\delta\right)\right] \\ &\quad \times \left(\frac{\pi}{2} - \arcsin\frac{a_0}{\sqrt{2a_1h+a_0^2}}\right).\end{aligned} \tag{122}$$

Hence, (120) holds for $k = 2$. Suppose (120) holds for $k, 2 \le k \le [(n-1)/2] - 1$. Then for $k+1$, we have

$$\begin{aligned}&\frac{d^{k+1}M(h,\delta)}{dh^{k+1}} \\ &= \sum_{j=0}^{k}\left(\frac{1}{2}-k+j\right)h^{1/2-(k+1)+j}g_{[n/2]-j} \\ &\quad + \sum_{j=0}^{k}h^{1/2-k+j}g_{[n/2]-1-j} + \frac{d^{k+1}\overline{g}_{[(n+1)/2]}}{dh^{k+1}} \\ &\quad + \frac{d^{k+1}}{dh^{k+1}}\left[\left(2h+\frac{a_0^2}{a_1}\right)\varphi^{+}_{[(n-1)/2]}\left(2h+\frac{a_0^2}{a_1},\delta\right)\right] \\ &\quad \times \left(\frac{\pi}{2} - \arcsin\frac{a_0}{\sqrt{2a_1h+a_0^2}}\right) \\ &\quad + \frac{d^{k}}{dh^{k}}\left[\left(2h+\frac{a_0^2}{a_1}\right)\varphi^{+}_{[(n-1)/2]}\left(2h+\frac{a_0^2}{a_1},\delta\right)\right] \\ &\quad \times \frac{a_0}{\sqrt{2a_1}\sqrt{h}\left(2h+a_0^2/a_1\right)} \\ &\quad + \sum_{j=0}^{k-2}\left(h\left(2h+\frac{a_0^2}{a_1}\right)\frac{d}{dh}g_j\right. \\ &\qquad \left. -g_j\left[2(j+1)h+\left(j+\frac{1}{2}\right)\left(2h+\frac{a_0^2}{a_1}\right)\right]\right) \\ &\quad \times \left(h^{j+3/2}\left(2h+\frac{a_0^2}{a_1}\right)^{j+2}\right)^{-1},\end{aligned} \tag{123}$$

which implies that (120) holds for $k+1$. Thus, the claim is proved. Then, taking $k = [(n-1)/2]$, we can obtain that by differentiating it

$$\begin{aligned}\frac{d^{[(n+1)/2]}M(h,\delta)}{dh^{[(n+1)/2]}} &= \sum_{j=0}^{[n/2]}h^{1/2-[(n+1)/2]+j}g_{[n/2]-j} \\ &\quad + \sum_{j=0}^{[(n+1)/2]-2}g_jh^{-j-1/2}\left(2h+\frac{a_0^2}{a_1}\right)^{-j-1} \\ &\quad + \frac{d^{[(n+1)/2]}\overline{g}_{[(n+1)/2]}}{dh^{[(n+1)/2]}} + \frac{d^{[(n+1)/2]}}{dh^{[(n+1)/2]}} \\ &\quad \times \left[\left(2h+\frac{a_0^2}{a_1}\right)\varphi^{+}_{[(n-1)/2]}\left(2h+\frac{a_0^2}{a_1},\delta\right)\right] \\ &\quad \times \left(\frac{\pi}{2} - \arcsin\frac{a_0}{\sqrt{2a_1h+a_0^2}}\right).\end{aligned} \tag{124}$$

One can find that

$$\begin{aligned}\frac{d^{[(n+1)/2]+1}M(h,\delta)}{dh^{[(n+1)/2]+1}} &= \sum_{j=0}^{[n/2]}h^{1/2-[(n+1)/2]-1+j}g_{[n/2]-j} \\ &\quad + \sum_{j=0}^{[(n+1)/2]-1}g_jh^{-j-1/2}\left(2h+\frac{a_0^2}{a_1}\right)^{-j-1}\end{aligned}$$

$$= h^{-[(n+1)/2]-1/2} \times \left(2h + \frac{a_0^2}{a_1}\right)^{-[(n+1)/2]} F(h,\delta), \tag{125}$$

where

$$F(h,\delta) = \left(2h + \frac{a_0^2}{a_1}\right)^{[(n+1)/2]} \sum_{j=0}^{[n/2]} h^j g_{[n/2]-j} + \sum_{j=0}^{[(n+1)/2]-1} g_j h^{[(n+1)/2]-j} \times \left(2h + \frac{a_0^2}{a_1}\right)^{[(n+1)/2]-j-1}, \tag{126}$$

where F is a polynomial of degree $[n/2] + [(n+1)/2]$. Since $M(0,\delta) = 0$ from (118), it is easy to see that $M(h,\delta)$ has at most $[n/2] + 2[(n+1)/2] = n + [(n+1)/2]$ zeros for $h > 0$ by Rolle theorem. As the above discussion, we only prove $Z(n) \geq n + [(n+1)/2]$ as $h > 0$ small, which implies $N(n) \geq n + [(n+1)/2]$. For the purpose, take $b_{ij}^- = a_{ij}^+ = 0, a_{ij}^- = 0,\ j \geq 1,\ b_{ij}^+ = 0,\ i \geq 1$. Then using (49), (51), we can write $M(h,\delta)$ in (12) as

$$\begin{aligned} M(h,\delta) = \sqrt{h}\Bigg[&-2\sqrt{2}a_{00}^- + \sum_{i=1}^{n} \frac{2(\sqrt{2})^{i+1} i}{(\sqrt{b_1})^i} a_{i0}^- \times \int_{-\pi/2}^{0} \sin^{i-1}\theta\cos^2\theta\, d\theta (\sqrt{h})^i \\ &+ \sum_{2k=0}^{n-1} b_{0,2k+1}^+ \psi_{0k}^+(h) \\ &+ \sum_{2k=0}^{n-1} 2^{k+1}\alpha_{00k}^+ \sqrt{\frac{2a_1}{a_0^2}} b_{0,2k+1}^+ \times \left(h + \frac{a_0^2}{2a_1}\right)^k \sum_{m=0}^{+\infty} C_m \left(\frac{2a_1}{a_0^2}\right)^m h^{m+1}\Bigg]. \end{aligned} \tag{127}$$

For $n = 2l, l \geq 1$, (127) can be written as

$$M(h,\delta) = \sqrt{h}\left[\sum_{i=0}^{2l} \overline{v}_i (\sqrt{h})^i + \sum_{i\geq 0} \overline{v}_{2l+1+i} h^{l+1+i}\right], \tag{128}$$

where

$$\begin{aligned} \overline{v}_0 &= -2\sqrt{2}a_{00}^-, \\ \overline{v}_{2i+1} &= \frac{2(\sqrt{2})^{2i+2}(2i+1)}{(\sqrt{b_1})^{2i+1}} a_{2i+1,0}^- \times \int_{-\pi/2}^{0} \sin^{2i}\theta\cos^2\theta\, d\theta, \quad i = 0,1,\ldots,l-1, \\ \overline{v}_{2i} &= \frac{2(\sqrt{2})^{2i+1} 2i}{(\sqrt{b_1})^{2i}} a_{2i,0}^- \int_{-\pi/2}^{0} \sin^{2i-1}\theta\cos^2\theta\, d\theta + L_{2i}\left(b_{01}^+, b_{03}^+, \ldots, b_{0,2[(n-1)/2]+1}^+\right), \quad i = 1,2,\ldots,l, \\ \overline{v}_{2l+1+i} &= \sum_{k=0}^{l-1} 2^{k+1}\alpha_{00k}^+ \left(\frac{2a_1}{a_0^2}\right)^{l-k+i+1/2} \times b_{0,2k+1}^+ \sum_{r=0}^{k} C_k^r C_{l+i-r}, \quad i = 0,1,\ldots,\left[\frac{n-1}{2}\right], \end{aligned} \tag{129}$$

which implies that $M(h,\delta)$ can appear $n + [(n+1)/2]$ zeros in $h > 0$ small by using the same method with the Subcase 4. For $n = 2l+1,\ l = 0,1,\ldots$, we can discuss by (127) in a similar way. Hence, the conclusion holds.

Subcase 6. $a_1 > 0,\ a_0 > 0,\ b_1 > 0,\ b_0 < 0$. We have as the above

$$\begin{aligned} M(h,\delta) = \sqrt{h}&\left(\psi_{[n/2]}^+(h,\delta) + \psi_{[n/2]}^-(h,\delta)\right) \\ &+ \left(2h + \frac{a_0^2}{a_1}\right)\varphi_{[(n-1)/2]}^+(h,\delta) \times \left(\frac{\pi}{2} - \arcsin\frac{a_0}{\sqrt{2a_1h + a_0^2}}\right) \\ &+ \left(2h + \frac{b_0^2}{b_1}\right)\varphi_{[(n-1)/2]}^-(h,\delta) \times \left(\frac{\pi}{2} + \arcsin\frac{b_0}{\sqrt{2b_1h + b_0^2}}\right). \end{aligned} \tag{130}$$

Similar to the Subcase 5, we can prove that for any $2 \leq k \leq [(n-1)/2]$

$$\frac{d^k M(h,\delta)}{dh^k} = \sum_{j=0}^{k} h^{1/2-k+j} g_{[n/2]-j}$$

$$+\sum_{j=0}^{k-2} g_j h^{-j-1/2}\left(2h+\frac{a_0^2}{a_1}\right)^{-j-1}$$
$$+\sum_{j=0}^{k-2} g_j h^{-j-1/2}\left(2h+\frac{b_0^2}{b_1}\right)^{-j-1}$$
$$+\frac{d^k}{dh^k}\left[\left(2h+\frac{a_0^2}{a_1}\right)\varphi^+_{[(n-1)/2]}\left(2h+\frac{a_0^2}{a_1},\delta\right)\right]$$
$$\times\left(\frac{\pi}{2}-\arcsin\frac{a_0}{\sqrt{2a_1h+a_0^2}}\right)$$
$$+\frac{d^k}{dh^k}\left[\left(2h+\frac{b_0^2}{b_1}\right)\varphi^+_{\left[\frac{(n-1)}{2}\right]}\left(2h+\frac{b_0^2}{a_1},\delta\right)\right]$$
$$\times\left(\frac{\pi}{2}-\arcsin\frac{b_0}{\sqrt{2b_1h+b_0^2}}\right). \tag{131}$$

Taking $k=[(n-1)/2]$ and differentiating the above twice follow that

$$\frac{d^{[(n+1)/2]+1}M(h,\delta)}{dh^{[(n+1)/2]+1}}=\sum_{j=0}^{[n/2]} h^{1/2-[(n+1)/2]-1+j}g_{[n/2]-j}$$
$$+\sum_{j=0}^{[(n+1)/2]-1} g_j h^{-j-1/2}\left(2h+\frac{a_0^2}{a_1}\right)^{-j-1}$$
$$+\sum_{j=0}^{[(n+1)/2]-1} g_j h^{-j-1/2}\left(2h+\frac{b_0^2}{b_1}\right)^{-j-1}. \tag{132}$$

If $a_0^2/a_1=b_0^2/b_1$, then it is easy to see that (132) has the same form with (125). Hence, we can know that $M(h,\delta)$ has at most $[n/2]+2[(n+1)/2]$ zeros for $h\in(0,+\infty)$. If $a_0^2/a_1\neq b_0^2/b_1$, then (132) can be written as

$$\frac{d^{[(n+1)/2]+1}M(h,\delta)}{dh^{[(n+1)/2]+1}}=h^{-[(n+1)/2]-1/2}\left(2h+\frac{a_0^2}{a_1}\right)^{-[(n+1)/2]}$$
$$\times\left(2h+\frac{b_0^2}{b_1}\right)^{-[(n+1)/2]}\overline{F}(h,\delta), \tag{133}$$

where

$$\overline{F}(h,\delta)=\left(2h+\frac{a_0^2}{a_1}\right)^{[(n+1)/2]}\left(2h+\frac{b_0^2}{b_1}\right)^{[(n+1)/2]}$$
$$\times\sum_{j=0}^{[n/2]} h^j g_{[n/2]-j}$$
$$+\left(2h+\frac{b_0^2}{b_1}\right)^{[(n+1)/2]}\sum_{j=0}^{[(n+1)/2]-1} g_j h^{[(n+1)/2]-j}$$
$$\times\left(2h+\frac{a_0^2}{a_1}\right)^{[(n+1)/2]-j-1}$$
$$+\left(2h+\frac{a_0^2}{a_1}\right)^{[(n+1)/2]}\sum_{j=0}^{[(n+1)/2]-1} g_j h^{[(n+1)/2]-j}$$
$$\times\left(2h+\frac{b_0^2}{b_1}\right)^{[(n+1)/2]-j-1}, \tag{134}$$

where $\overline{F}$ is a polynomial of degree $[n/2]+2[(n+1)/2]$ in h. By Rolle theorem, we can obtain that $M(h,\delta)$ has at most $[n/2]+3[(n+1)/2]$ zeros for $h>0$ since $M(0,\delta)=0$. Now, we only need to prove $Z(n)\geq n$. Take $a_{ij}^-=b_{ij}^-=0$, $a_{ij}^+=b_{ij}^+=0$, $i\geq 1$. Then, by Lemmas 4, 5, one can see that for $h>0$ small

$$M(h,\delta)=\sqrt{h}\left[\sum_{2k=0}^{n}\frac{2^{k+1+1/2}a^+_{0,2k}}{2k+1}h^k+\sum_{2k=0}^{n-1}b^+_{0,2k+1}\psi^+_{0k}(h)\right.$$
$$+\sum_{2k=0}^{n-1}2^{k+1}\alpha^+_{00k}\sqrt{\frac{2a_1}{a_0^2}}b^+_{0,2k+1}\left(h+\frac{a_0^2}{2a_1}\right)^k$$
$$\left.\times\sum_{m=0}^{+\infty}C_m\left(\frac{2a_1}{a_0^2}\right)^m h^{m+1}\right]. \tag{135}$$

Similarly, we can discuss the above formula such that such that $M(h,\delta)$ can appear n zeros for $h>0$ small. Hence, the conclusion is proved.

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Research Article

Computation of Positive Solutions for Nonlinear Impulsive Integral Boundary Value Problems with p-Laplacian on Infinite Intervals

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This paper deals with the existence and iteration of positive solutions for nonlinear second-order impulsive integral boundary value problems with p-Laplacian on infinite intervals. Our approach is based on the monotone iterative technique.

1. Introduction

The theory of impulsive differential equations has been emerging as an important area of investigation in recent years. It has been extensively applied to biology, biologic medicine, optimum control in economics, chemical technology, population dynamics, and so on. It is much richer because all the structure of its emergence has deep physical background and realistic mathematical model and coincides with many phenomena in nature. For an introduction of the basic theory of impulsive differential equations in R^n, the reader is referred to see Lakshmikantham et al. [1, 2], Samoĭlenko and Perestyuk [3], and the references therein.

Boundary value problems on infinite intervals arise quite naturally in the study of radially symmetric solutions of nonlinear elliptic equations and models of gas pressure in a semi-infinite porous medium; see [4–7], for example. In a recent paper [8], by means of a fixed-point theorem due to Avery and Peterson, Li and Nieto obtained some new results on the existence of multiple positive solutions for the following multipoint boundary value problem with a finite number of impulsive times on an infinite interval:

$$\begin{gathered}
u''(t) + q(t) f(t, u(t)) = 0, \\
\forall 0 < t < \infty, \quad t \neq t_k, \quad k = 1, 2, \ldots, p, \\
\Delta u(t_k) = I_k(u(t_k)), \quad k = 1, 2, \ldots, p, \\
u(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \quad u'(\infty) = 0,
\end{gathered} \tag{1}$$

where $f \in C([0,+\infty) \times [0,+\infty), [0,+\infty))$, $I_k \in C([0,+\infty), [0,+\infty))$, $u'(\infty) = \lim_{t\to\infty} u'(t)$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < +\infty$, $0 < t_1 < t_2 < \cdots < t_p < +\infty$, and $q \in C([0,+\infty), [0,+\infty))$.

Boundary value problems with integral boundary conditions for ordinary differential equations arise in different fields of applied mathematics and physics such as heat conduction, chemical engineering, underground water flow, thermoelasticity, and plasma physics. Moreover, boundary value problems with Riemann-Stieltjes integral conditions constitute a very interesting and important class of problems. They include two-point, three-point, and multipoint boundary value problems as special cases; see [9–14]. For boundary value problems with other integral boundary conditions and comments on their importance, we refer the reader to the papers [11–20] and the references therein.

There are relatively few papers available for integral boundary value problems for impulsive differential equations on an infinite interval with an infinite number of impulsive times up to now. In [21], Zhang et al. investigated the existence of minimal nonnegative solution for the following second-order impulsive differential equation

$$\begin{aligned} -x''(t) = f\left(t, x(t), x'(t)\right) = 0, &\quad t \in J,\ t \neq t_k, \\ \Delta x|_{t=t_k} = I_k\left(x\left(t_k\right)\right), &\quad k = 1, 2, \ldots, \\ \Delta x'\big|_{t=t_k} = \bar{I}_k\left(x\left(t_k\right)\right), &\quad k = 1, 2, \ldots, \\ x(0) = \int_0^{+\infty} g(t)\,x(t)\,\mathrm{d}t, &\quad x'(\infty) = 0, \end{aligned} \tag{2}$$

where $f \in C(J\times J\times J, J)$, $I_k \in C(J, J)$, $\bar{I}_k \in C(J, J)$, $J = [0, +\infty)$, $0 < t_1 < t_2 < \cdots < t_k < \cdots$, $t_k \to \infty$, and $g(t) \in L[J, J]$ with $\int_0^{+\infty} g(t)\mathrm{d}t < 1$. $\Delta x|_{t=t_k}$ denotes the jump of $x(t)$ at $t = t_k$, that is,

$$\Delta x|_{t=t_k} = x\left(t_k^+\right) - x\left(t_k^-\right), \tag{3}$$

where $x(t_k^+)$ and $x(t_k^-)$ represent the right-hand limit and left-hand limit of $x(t)$ at $t = t_k$, respectively. $\Delta x'|_{t=t_k}$ has a similar meaning to $x'(t)$.

In the past few years, the existence and the multiplicity of bounded or unbounded positive solutions to nonlinear differential equations on infinite intervals have been studied by several different techniques; we refer the reader to [5–8, 21–29] and the references therein. However, most of these papers only considered the existence of positive solutions under various boundary value conditions. Seeing such a fact, a natural question which arises is "how can we find the solutions when they are known to exist?" More recently, Ma et al. [30] and Sun et al. [31, 32] established iterative schemes for approximating the solutions for some boundary value problems defined on finite intervals by virtue of the iterative technique.

However, to the author's knowledge, the corresponding theory for impulsive integral boundary value problems with p-Laplacian operator and infinite impulsive times on infinite intervals has not been considered till now. Motivated by previous papers, the purpose of this paper is to obtain the existence of positive solutions and establish a corresponding iterative scheme for the following impulsive integral boundary value problem of second-order differential equation with p-Laplacian on an infinite interval

$$\begin{aligned} \left(\varphi_p\left(x'(t)\right)\right)' + q(t)\,f\left(t, x(t), x'(t)\right) = 0, &\quad t \in J_+', \\ \Delta x|_{t=t_k} = I_k\left(x\left(t_k\right)\right), &\quad k = 1, 2, \ldots, \\ x(0) = \int_0^{+\infty} g(t)\,x(t)\,\mathrm{d}t, &\quad x'(\infty) = x_\infty, \end{aligned} \tag{4}$$

where $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, $J = [0, +\infty)$, $J_+ = (0, +\infty)$, $J_+' = J_+ \setminus \{t_1, t_2, \ldots, t_k, \ldots\}$, $0 < t_1 < t_2 < \cdots < t_k < \cdots$, $t_k \to \infty$, and $g(t) \in L[J, J]$ with $\int_0^{+\infty} g(t)\mathrm{d}t < 1$, $\int_0^{+\infty} t g(t)\mathrm{d}t < +\infty$, and $0 \le x'(\infty) = \lim_{t\to+\infty} x'(t)$.

It is clear that

$$\varphi_p(s+t) \le \begin{cases} 2^{p-1}\left(\varphi_p(s) + \varphi_p(t)\right), & p \ge 2,\ s, t > 0, \\ \varphi_p(s) + \varphi_p(t), & 1 < p < 2,\ s, t > 0, \end{cases} \tag{5}$$

$$\varphi_p^{-1}(s+t) \le \begin{cases} 2^{1/(p-1)}\left(\varphi_p^{-1}(s) + \varphi_p^{-1}(t)\right), & p \ge 2,\ s, t > 0, \\ \varphi_p^{-1}(s) + \varphi_p^{-1}(t), & 1 < p < 2,\ s, t > 0. \end{cases} \tag{6}$$

Throughout this paper, we adopt the following assumptions.

(H_1) $f(t, u, v) \in C(J \times J \times J, J)$, $f(t, 0, 0) \not\equiv 0$ on any subinterval of J, and when u, v are bounded, $f(t, (1+t)u, v)$ is bounded on J.

(H_2) $q(t)$ is a nonnegative measurable function defined in J_+ and $q(t)$ does not identically vanish on any subinterval of J_+, and

$$\begin{aligned} &0 < \int_0^{+\infty} q(t)\,\mathrm{d}t < +\infty, \\ &0 < \int_0^{+\infty} \varphi_p^{-1}\left(\int_s^{+\infty} q(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s < +\infty. \end{aligned} \tag{7}$$

(H_3) $I_k \in C(J, J)$, and there exist $a_k \ge 0$, $b_k \ge 0$ such that

$$\begin{aligned} &0 \le I_k(x) \le a_k + b_k x, \quad \text{for } x \in J \ (k = 1, 2, 3, \ldots), \\ &a^* = \sum_{k=1}^{\infty} a_k < +\infty, \qquad b^* = \sum_{k=1}^{\infty} b_k\left(1 + t_k\right) < +\infty, \end{aligned} \tag{8}$$

with $b^* < (1/3)(1 - \int_0^{+\infty} g(t)\mathrm{d}t)$.

If $p = 2$, $I_k = 0$ $(k = 1, 2, \ldots)$, $g(t) \equiv 0$, $x'(\infty) = 0$, then BVP (4) reduces to the following two-point boundary value problem:

$$\begin{aligned} -x''(t) = f\left(t, x(t), x'(t)\right) = 0, &\quad t \in J, \\ x(0) = 0, &\quad x'(\infty) = 0, \end{aligned} \tag{9}$$

which has been studied in [23].

Compared with [8, 21], the main features of the present paper are as follows. Firstly, second-order differential operator is replaced by a more general p-Laplacian operator. Secondly, in this paper, x_∞ in boundary value conditions may not be zero which will bring about computational difficulties. Thirdly, by applying monotone iterative techniques, we construct successive iterative schemes starting off with simple known functions. It is worth pointing out that the first terms of our iterative schemes are simple functions. Therefore, the iterative schemes are significant and feasible.

The rest of this paper is organized as follows. In Section 2, we give some preliminaries and establish several lemmas. The main theorems are formulated and proved in Section 3. Then, in Section 4, an example is presented to illustrate the main results.

2. Preliminaries and Several Lemmas

Definition 1. Let E be a real Banach space. A nonempty closed set $P \subset E$ is said to be a cone provided that

(1) $au + bv \in P$ for all $u, v \in P$ and all $a \geq 0$, $b \geq 0$,

(2) $u, -u \in P$ implies that $u = 0$.

Definition 2. A map $\alpha : P \to [0, +\infty)$ is said to be concave on P, if $\alpha(tu + (1-t)v) \geq t\alpha(u) + (1-t)\alpha(v)$ for all $u, v \in P$ and $t \in [0, 1]$.

Let $PC[J, R] = \{x : x$ is a map from J into R such that $x(t)$ is continuous at $t \neq t_k$, left continous at $t = t_k$ and $x(t_k^+)$ exists for $k = 1, 2, \ldots,\}$, $PC^1[J, R] = \{x \in PC[J, R] : x'(t)$ exists and is continuous at $t \neq t_k$, left continous at $t = t_k$ and $x'(t_k^+)$ exists for $k = 1, 2, \ldots,\}$

$$\begin{aligned} &FPC[J, R] \\ &\quad = \left\{ x \in PC[J, R] : \sup_{t\in J} \frac{|x(t)|}{1+t} < \infty \right\}, \\ &E = DPC^1[J, R] \\ &\quad = \left\{ x \in PC^1[J, R] : \sup_{t\in J} \frac{|x(t)|}{1+t} < \infty, \ \sup_{t\in J} \left| x'(t) \right| < \infty \right\}. \end{aligned} \tag{10}$$

Obviously, $DPC^1[J, R] \subset FPC[J, R]$. It is clear that $FPC[J, R]$ is a Banach space with the norm

$$\|x\|_F = \sup_{t\in J} \frac{|x(t)|}{1+t}, \tag{11}$$

and $DPC^1[J, R]$ is also a Banach space with the norm

$$\|x\|_D = \max\left\{ \|x\|_F, \left\|x'\right\|_B \right\}, \tag{12}$$

where $\|x'\|_B = \sup_{t\in J} |x'(t)|$. Let $J_0 = [0, t_1]$, $J_k = (t_k, t_{k+1}]$ $(k = 1, 2, 3, \ldots)$. Define a cone $P \subset E$ by

$$\begin{aligned} P = \{ & x \in E : x \text{ is concave and nondecreasing on } J, \\ & x(t) \geq 0, \ x'(t) \geq 0, \ t \in J \}. \end{aligned} \tag{13}$$

Remark 3. If x satisfies (4), then $(\varphi_p(x'(t)))' = -q(t)f(t, x(t), x'(t)) \leq 0$, and $t \in [0, +\infty)$ which implies that $\varphi_p(x'(t))$ is nonincreasing on J; that is, $x'(t)$ is also nonincreasing on J. Thus, x is concave on $[0, +\infty)$. Moreover, if $x'(\infty) = x_\infty \geq 0$, then $x'(t) \geq 0$, $t \in [0, +\infty)$, and so x is monotone increasing on $[0, +\infty)$.

Lemma 4. *Let conditions (H_1)–(H_3) hold. Then, $x \in P$ with $(\varphi_p(x'(t)))' \in C(0, +\infty)$ is a solution of BVP (4) if and only if $x \in C[0, +\infty)$ is a fixed point of the following operator equation:*

$$\begin{aligned} (Ax)(t) &= \frac{1}{1 - \int_0^{+\infty} g(t)\, dt} \\ &\quad \times \int_0^{+\infty} g(t) \\ &\quad \times \Bigg[\int_0^t \varphi_p^{-1} \Bigg(\int_s^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right) d\tau \\ &\qquad + \varphi_p(x_\infty) \Bigg) ds \\ &\quad + \sum_{t_k<t} I_k(x(t_k)) \Bigg] dt \\ &\quad + \int_0^t \varphi_p^{-1} \left(\int_s^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right) d\tau + \varphi_p(x_\infty) \right) ds \\ &\quad + \sum_{t_k<t} I_k(x(t_k)). \end{aligned} \tag{14}$$

Proof. Suppose that $x \in P$ with $(\varphi_p(x'(t)))' \in C(0, +\infty)$ is a solution of BVP (4). For $t \in J$, integrating (4) from t to $+\infty$, we have

$$\int_t^{+\infty} \varphi_p\left(x'(\tau)\right)' d\tau = -\int_t^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right) d\tau. \tag{15}$$

That is

$$\varphi_p\left(x'(\infty)\right) - \varphi_p\left(x'(t)\right) = -\int_t^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right) d\tau, \tag{16}$$

which implies that

$$x'(t) = \varphi_p^{-1}\left(\int_t^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right) d\tau + \varphi_p(x_\infty) \right). \tag{17}$$

If $t_1 < t \leq t_2$, integrating (17) from 0 to t_1, we get that

$$x(t_1) - x(0) = \int_0^{t_1} \varphi_p^{-1}\left(\int_s^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right) d\tau + \varphi_p(x_\infty) \right) ds. \tag{18}$$

Integrating (17) from t_1 to t, we obtain

$$x(t) - x(t_1^+) = \int_{t_1}^{t} \varphi_p^{-1}\left(\int_s^{+\infty} q(\tau) f(\tau, x(\tau), x'(\tau))\, d\tau + \varphi_p(x_\infty)\right) ds. \tag{19}$$

Adding (18) and (19) together, we have

$$\begin{aligned} x(t) &= x(0) + \int_0^t \varphi_p^{-1}\left(\int_s^{+\infty} q(\tau) f(\tau, x(\tau), x'(\tau))\, d\tau + \varphi_p(x_\infty)\right) ds \\ &\quad + I_1(x(t_1)), \quad t_1 < t \le t_2. \end{aligned} \tag{20}$$

Repeating previous process, we get that

$$\begin{aligned} x(t) &= x(0) + \int_0^t \varphi_p^{-1}\left(\int_s^{+\infty} q(\tau) f(\tau, x(\tau), x'(\tau))\, d\tau + \varphi_p(x_\infty)\right) ds + \sum_{t_k<t} I_k(x(t_k)) \\ &= \int_0^{+\infty} g(t) x(t)\, dt + \int_0^t \varphi_p^{-1}\left(\int_s^{+\infty} q(\tau) f(\tau, x(\tau), x'(\tau))\, d\tau + \varphi_p(x_\infty)\right) ds + \sum_{t_k<t} I_k(x(t_k)). \end{aligned} \tag{21}$$

It follows that

$$\begin{aligned} \int_0^{+\infty} g(t) x(t)\, dt &= \frac{1}{1 - \int_0^{+\infty} g(t)\, dt} \times \int_0^{+\infty} g(t) \times \left[\int_0^t \varphi_p^{-1} \times \left(\int_s^{+\infty} q(\tau) f(\tau, x(\tau), x'(\tau))\, d\tau + \varphi_p(x_\infty)\right) ds + \sum_{t_k<t} I_k(x(t_k))\right] dt. \end{aligned} \tag{22}$$

Substituting (22) into (21), we get that

$$\begin{aligned} x(t) &= \frac{1}{1 - \int_0^{+\infty} g(t)\, dt} \times \int_0^{+\infty} g(t) \times \left[\int_0^t \varphi_p^{-1} \times \left(\int_s^{+\infty} q(\tau) f(\tau, x(\tau), x'(\tau))\, d\tau + \varphi_p(x_\infty)\right) ds + \sum_{t_k<t} I_k(x(t_k))\right] dt \\ &\quad + \int_0^t \varphi_p^{-1}\left(\int_s^{+\infty} q(\tau) f(\tau, x(\tau), x'(\tau))\, d\tau + \varphi_p(x_\infty)\right) ds + \sum_{t_k<t} I_k(x(t_k)). \end{aligned} \tag{23}$$

For $x \in P$, there exists r_0 such that $\|x\|_D < r_0$. Set $B_{r_0} = \sup\{f(t, (1+t)u, v) \mid (t, u, v) \in J \times [0, r_0] \times [0, r_0]\}$, and we have by (H_1) and (H_3) that

$$\begin{gathered} \int_0^{+\infty} q(s) f(s, x(s), x'(s))\, ds \le \int_0^{+\infty} q(s)\, ds \cdot B_{r_0}, \\ \sum_{t_k<t} I_k(x(t_k)) \le \sum_{k=1}^{\infty} I_k(x(t_k)) \le a^* + b^* r_0 < +\infty. \end{gathered} \tag{24}$$

By (6), (24), we have

$$\begin{aligned} x(t) &= \frac{1}{1 - \int_0^{+\infty} g(t)\, dt} \times \int_0^{+\infty} g(t) \times \left[\int_0^t \varphi_p^{-1} \times \left(\int_s^{+\infty} q(\tau) f(\tau, x(\tau), x'(\tau))\, d\tau + \varphi_p(x_\infty)\right) ds + \sum_{t_k<t} I_k(x(t_k))\right] dt \end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \varphi_p^{-1} \\
&\qquad \times \left(\int_s^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right) \mathrm{d}\tau + \varphi_p\left(x_\infty\right)\right) \mathrm{d}s \\
&+\sum_{t_k<t} I_k\left(x\left(t_k\right)\right) \\
\leq{}& \frac{1}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t} \\
&\times \int_0^{+\infty} t g(t)\,\mathrm{d}t \cdot \varphi_p^{-1} \\
&\qquad \times \left(\int_0^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right) \mathrm{d}\tau + \varphi_p\left(x_\infty\right)\right) \\
&+ t\varphi_p^{-1}\left(\int_0^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right) \mathrm{d}\tau + \varphi_p\left(x_\infty\right)\right) \\
&+ \frac{1}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t}\left(a^* + b^* r_0\right) \\
\leq{}& 2^{1/(p-1)}\left(\frac{1}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t} \cdot \int_0^{+\infty} t g(t)\,\mathrm{d}t + t\right) \\
&\times \left[\varphi_p^{-1}\left(\int_0^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right) \mathrm{d}\tau\right) + x_\infty\right] \\
&+ \frac{1}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t}\left(a^* + b^* r_0\right).
\end{aligned} \tag{25}
$$

It follows from (24) and (25) that the right term in (23) is well defined. Thus, we have proved that x is a fixed point of the operator A defined by (14).

Conversely, suppose that $x \in C[0,+\infty)$ is a fixed point of the operator equation (14). Evidently,

$$
\Delta x|_{t=t_k} = I_k\left(x\left(t_k\right)\right) \quad (k = 1, 2, \ldots). \tag{26}
$$

Direct differentiation of (14) implies that, for $t \neq t_k$,

$$
\begin{gathered}
x'(t) = \varphi_p^{-1}\left(\int_t^{+\infty} q(s) f\left(s, x(s), x'(s)\right) \mathrm{d}s + \varphi_p\left(x_\infty\right)\right), \\
\Delta x'\big|_{t=t_k} = 0 \quad (k = 1, 2, \ldots), \\
\left(\varphi_p\left(x'(t)\right)\right)' = -q(t) f\left(t, x(t), x'(t)\right),
\end{gathered} \tag{27}
$$

which means that $(\varphi_p(x'(t)))' \in C(J')$. It is easy to verify that $x(0) = \int_0^{+\infty} g(t)x(t)\mathrm{d}t$, $x'(\infty) = x_\infty$. The proof of Lemma 4 is complete. □

To obtain the complete continuity of A, the following lemma is still needed.

Lemma 5 (see [33, 34]). *Let W be a bounded subset of P. Then, P is relatively compact in E if $\{W(t)/(1+t)\}$ and $\{W'(t)\}$ are both equicontinuous on any finite subinterval $J_k \cap [0,T]$ $(k = 1, 2, \ldots)$ for any $T > 0$, and for any $\varepsilon > 0$, there exists $N > 0$ such that*

$$
\left|\frac{x\left(t'\right)}{1+t'} - \frac{x\left(t''\right)}{1+t''}\right| < \varepsilon, \quad \left|x'\left(t'\right) - x'\left(t''\right)\right| < \varepsilon, \quad \forall t', t'' \geq N, \tag{28}
$$

uniformly with respect to $x \in W$ as $t', t'' \geq N$, where $W(t) = \{x(t) \mid x \in W\}$, $W'(t) = \{x'(t) \mid x \in W\}$, $t \in [0, +\infty)$.

This lemma is a simple improvement of the Corduneanu theorem in [33, 34].

Lemma 6. *Let (H_1)–(H_3) hold. Then $A : P \to P$ is completely continuous.*

Proof. For any $x \in P$, by (14), we have

$$
\begin{gathered}
\varphi_p\left((Ax)'\right)(t) = \int_t^{+\infty} q(s) f\left(s, x(s), x'(s)\right) \mathrm{d}s + \varphi_p\left(x_\infty\right), \\
\left(\varphi_p(Ax)'(t)\right)' = -q(t) f\left(t, x(t), x'(t)\right).
\end{gathered} \tag{29}
$$

It follows from (14), (29), and (H_1) that $(Ax)(t) \geq 0$, $(Ax)'(t) \geq x_\infty \geq 0$, $(Ax)''(t) \leq 0$, that is, $A(P) \subset P$. Now, we prove that A is continuous and compact respectively. Let $x_n \in P$, $x_n \to x$ as $n \to \infty$, then there exists r_0 such that $\sup_{n \in N\setminus\{0\}} \|x_n\| < r_0$. Let $B_{r_0} = \sup\{f(t, (1+t)u, v) \mid (t, u, v) \in J \times [0, r_0] \times [0, r_0]\}$. By (H_1) and (H_2), we have

$$
\begin{aligned}
&\int_0^{+\infty} q(\tau)\left|f\left(\tau, x_n(\tau), x_n'(\tau)\right) - f\left(\tau, x(\tau), x'(\tau)\right)\right| \mathrm{d}\tau \\
&\qquad \leq 2B_{r_0} \cdot \int_0^{+\infty} q(s)\,\mathrm{d}s < +\infty.
\end{aligned} \tag{30}
$$

It follows from (30) and dominated convergence theorem that

$$
\begin{aligned}
&\int_0^{+\infty} q(\tau)\left|f\left(\tau, x_n(\tau), x_n'(\tau)\right) - f\left(\tau, x(\tau), x'(\tau)\right)\right| \mathrm{d}\tau \\
&\longrightarrow \int_0^{+\infty} q(\tau)\left|f\left(\tau, x(\tau), x'(\tau)\right) - f\left(\tau, x(\tau), x'(\tau)\right)\right| \mathrm{d}\tau,
\end{aligned} \tag{31}
$$

which implies that

$$
\begin{aligned}
&\left|\varphi_p^{-1}\left(\int_0^{+\infty} q(\tau) f\left(\tau, x_n(\tau), x_n'(\tau)\right) \mathrm{d}\tau + \varphi_p\left(x_\infty\right)\right) \mathrm{d}s\right. \\
&\quad \left. -\varphi_p^{-1}\left(\int_0^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right) \mathrm{d}\tau + \varphi_p\left(x_\infty\right)\right) \mathrm{d}s\right| \\
&\longrightarrow 0, \quad n \longrightarrow \infty.
\end{aligned} \tag{32}
$$

By (30)–(32), (H_3) and dominated convergence theorem, we get that

$$\begin{aligned}
&\|Ax_n - Ax\|_F \\
&= \sup_{t\in J}\Bigg\{\frac{1}{1+t} \\
&\quad\times\Bigg|\frac{1}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t} \\
&\quad\times\Bigg[\int_0^{+\infty} g(t) \\
&\quad\times\int_0^t \varphi_p^{-1} \\
&\quad\times\left(\int_s^{+\infty} q(\tau) f\left(\tau, x_n(\tau), x_n'(\tau)\right)\mathrm{d}\tau\right. \\
&\quad\left.+\varphi_p(x_\infty)\right)\mathrm{d}s\,\mathrm{d}t \\
&\quad-\int_0^{+\infty} g(t) \\
&\quad\times\int_0^t \varphi_p^{-1}\left(\int_s^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right)\mathrm{d}\tau\right. \\
&\quad\left.+\varphi_p(x_\infty)\right)\mathrm{d}s\,\mathrm{d}t \\
&\quad+\int_0^{+\infty} g(t) \\
&\quad\cdot\sum_{t_k<t}\left(I_k\left(x_n(t_k)\right) - I_k\left(x(t_k)\right)\right)\Bigg]\mathrm{d}t \\
&\quad+\int_0^t\Bigg[\varphi_p^{-1}\left(\int_s^{+\infty} q(\tau) f\left(\tau, x_n(\tau), x_n'(\tau)\right)\mathrm{d}\tau\right. \\
&\quad\left.+\varphi_p(x_\infty)\right) \\
&\quad-\varphi_p^{-1}\left(\int_s^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right)\mathrm{d}\tau\right. \\
&\quad\left.+\varphi_p(x_\infty)\right)\Bigg]\mathrm{d}s \\
&\quad+\sum_{t_k<t}\left(I_k\left(x_n(t_k)\right) - I_k\left(x(t_k)\right)\right)\Bigg|\Bigg\} \\
&\le \frac{1}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t} \\
&\quad\times\int_0^{+\infty} t g(t)\,\mathrm{d}t \\
&\quad\cdot\left|\varphi_p^{-1}\left(\int_0^{+\infty} q(\tau) f\left(\tau, x_n(\tau), x_n'(\tau)\right)\mathrm{d}\tau + \varphi_p(x_\infty)\right)\right. \\
&\quad\left.-\varphi_p^{-1}\left(\int_0^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right)\mathrm{d}\tau + \varphi_p(x_\infty)\right)\right| \\
&\quad+\left|\varphi_p^{-1}\left(\int_0^{+\infty} q(\tau) f\left(\tau, x_n(\tau), x_n'(\tau)\right)\mathrm{d}\tau + \varphi_p(x_\infty)\right)\right. \\
&\quad\left.-\varphi_p^{-1}\left(\int_0^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right)\mathrm{d}\tau + \varphi_p(x_\infty)\right)\right| \\
&\quad+\frac{1}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t} \\
&\quad\times\int_0^{+\infty} g(t)\cdot\sum_{t_k<t}\left|I_k\left(x_n(t_k)\right) - I_k\left(x(t_k)\right)\right|\mathrm{d}t \\
&\quad+\sum_{t_k<t}\left|I_k\left(x_n(t_k)\right) - I_k\left(x(t_k)\right)\right| \longrightarrow 0 \quad (n\longrightarrow\infty), \\
&\left\|(Ax_n)' - (Ax)'\right\|_B \\
&= \sup_{t\in J}\left\{\left|\varphi_p^{-1}\left(\int_t^{+\infty} q(s) f\left(s, x_n(s), x_n'(s)\right)\mathrm{d}s + \varphi_p(x_\infty)\right)\right.\right. \\
&\quad\left.\left.-\varphi_p^{-1}\left(\int_t^{+\infty} q(s) f\left(s, x(s), x'(s)\right)\mathrm{d}s + \varphi_p(x_\infty)\right)\right|\right\} \\
&\longrightarrow 0 \quad (n\longrightarrow\infty).
\end{aligned}\tag{33}$$

It follows from (33) that A is continuous.

Let $\Omega\subset P$ be any bounded subset. Then, there exists $r>0$ such that $\|x\|_D\le r$ for any $x\in\Omega$. Obviously,

$$\begin{aligned}
&\|Ax\|_F \\
&= \sup_{t\in J}\Bigg\{\frac{1}{1+t} \\
&\quad\times\Bigg|\frac{1}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t} \\
&\quad\times\int_0^{+\infty} g(t) \\
&\quad\times\Bigg[\int_0^t \varphi_p^{-1} \\
&\quad\times\left(\int_s^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right)\mathrm{d}\tau\right. \\
&\quad\left.+\varphi_p(x_\infty)\right)\mathrm{d}s \\
&\quad+\sum_{t_k<t} I_k\left(x(t_k)\right)\Bigg]\mathrm{d}t \\
&\quad+\int_0^t \varphi_p^{-1} \\
&\quad\times\left(\int_s^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right)\mathrm{d}\tau\right. \\
&\quad\left.+\varphi_p(x_\infty)\right)\mathrm{d}s \\
&\quad+\sum_{t_k<t} I_k\left(x(t_k)\right)\Bigg|\Bigg\}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t} \\
&\quad \times \int_0^{+\infty} t g(t)\,\mathrm{d}t \cdot \varphi_p^{-1} \\
&\qquad \times \left(\int_0^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right)\mathrm{d}\tau + \varphi_p\left(x_\infty\right)\right) \\
&\quad + \varphi_p^{-1}\left(\int_0^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right)\mathrm{d}\tau + \varphi_p\left(x_\infty\right)\right) \\
&\quad + \frac{1}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t}\left(a^* + b^* r\right) \\
&\leq 2^{1/(p-1)}\left[\frac{1}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t}\int_0^{+\infty} t g(t)\,\mathrm{d}t + 1\right] \\
&\quad \times \left[\varphi_p^{-1}\left(B_r\right)\cdot \varphi_p^{-1}\left(\int_0^{+\infty} q(\tau)\,\mathrm{d}\tau\right) + x_\infty\right] \\
&\quad + \frac{1}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t}\left(a^* + b^* r\right), \\
&\left\|(Ax)'\right\|_B \\
&\quad = \sup_{t\in J}\left\{\left|\varphi_p^{-1}\left(\int_t^{+\infty} q(s) f\left(s, x(s), x'(s)\right)\mathrm{d}s + \varphi_p\left(x_\infty\right)\right)\right|\right\} \\
&\quad \leq 2^{1/(p-1)}\left[\varphi_p^{-1}\left(\int_0^{+\infty} q(s)\,\mathrm{d}s\right)\cdot \varphi_p^{-1}\left(B_r\right) + x_\infty\right].
\end{aligned} \tag{34}
$$

From (34), (H_2), and (H_3), we know that $A\Omega$ is bounded.

For any $T > 0, x \in \Omega, t', t'' \in J_k \cap [0, T]$ with $t' < t''$, by the absolute continuity of the integral, we have

$$
\begin{aligned}
&\left|\frac{(Ax)\left(t'\right)}{1+t'} - \frac{(Ax)\left(t''\right)}{1+t''}\right| \\
&\quad \leq \frac{1}{\left(1+t''\right)\left(1-\int_0^{+\infty} g(t)\,\mathrm{d}t\right)} \\
&\qquad \cdot \int_0^{+\infty} g(t)\,\mathrm{d}t \\
&\qquad \cdot \int_{t'}^{t''} \varphi_p^{-1} \\
&\qquad\quad \times \left(\int_s^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right)\mathrm{d}\tau + \varphi_p\left(x_\infty\right)\right)\mathrm{d}s \\
&\qquad + \frac{1}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t} \\
&\qquad \times \int_0^{+\infty} g(t)\,\mathrm{d}t \\
&\qquad \cdot \int_0^{t'} \varphi_p^{-1} \\
&\qquad\quad \times \left(\int_s^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right)\mathrm{d}\tau + \varphi_p\left(x_\infty\right)\right)\mathrm{d}s \\
&\qquad \cdot \left|\frac{1}{1+t''} - \frac{1}{1+t'}\right| \\
&\qquad + \frac{1}{1+t''} \\
&\qquad \times \int_{t'}^{t''} \varphi_p^{-1} \\
&\qquad\quad \times \left(\int_s^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right)\mathrm{d}\tau + \varphi_p\left(x_\infty\right)\right)\mathrm{d}s \\
&\qquad + \int_0^{t'} \varphi_p^{-1} \\
&\qquad\quad \times \left(\int_s^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right)\mathrm{d}\tau + \varphi_p\left(x_\infty\right)\right)\mathrm{d}s \\
&\qquad \cdot \left|\frac{1}{1+t''} - \frac{1}{1+t'}\right| \\
&\qquad + \frac{a^* + b^* r}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t}\cdot\left|\frac{1}{1+t''} - \frac{1}{1+t'}\right| \\
&\quad \leq \frac{2^{1/(p-1)}}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t} \\
&\qquad \times \left[\int_{t'}^{t''}\left(\varphi_p^{-1}\left(\int_0^{+\infty} q(\tau)\,\mathrm{d}\tau\right)\cdot \varphi_p^{-1}\left(B_r\right) + x_\infty\right)\mathrm{d}s\right. \\
&\qquad\quad + \int_0^{t'}\left(\varphi_p^{-1}\left(\int_0^{+\infty} q(\tau)\,\mathrm{d}\tau\right)\cdot \varphi_p^{-1}\left(B_r\right) + x_\infty\right)\mathrm{d}s \\
&\qquad\quad \left.\cdot\left|\frac{1}{1+t''} - \frac{1}{1+t'}\right|\right] \\
&\qquad + \frac{a^* + b^* r}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t}\cdot\left|\frac{1}{1+t''} - \frac{1}{1+t'}\right| \\
&\qquad \longrightarrow 0 \quad \text{uniformly as} \quad t' \longrightarrow t'', \\
&\left|\varphi_p\left((Ax)'\left(t'\right)\right) - \varphi_p\left((Ax)'\left(t''\right)\right)\right| \\
&\quad = \left|\int_{t'}^{t''} q(s) f\left(s, x(s), x'(s)\right)\mathrm{d}s\right| \\
&\quad \leq B_r \cdot \left|\int_{t'}^{t''} q(s)\,\mathrm{d}s\right| \\
&\quad \longrightarrow 0 \quad \text{uniformly as} \quad t' \longrightarrow t''.
\end{aligned} \tag{35}
$$

Thus, we have proved that $A\Omega$ is equicontinuous on any $J_k \cap [0, T]$.

Next, we prove that for any $\varepsilon > 0, x \in \Omega$, there exits sufficiently large $N > 0$ such that

$$\left|\frac{(Ax)(t')}{1+t'} - \frac{(Ax)(t'')}{1+t''}\right| < \varepsilon,$$
$$\left|(Ax)'(t') - (Ax)'(t'')\right| < \varepsilon, \quad \forall t', t'' \geq N. \tag{36}$$

For any $x \in \Omega$, we have

$$\begin{aligned}
&\lim_{t\to+\infty}\frac{1}{1+t} \cdot \left[\frac{1}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t} \times \int_0^{+\infty} g(t)\sum_{t_k<t} I_k(x(t_k))\,\mathrm{d}t + \sum_{t_k<t} I_k(x(t_k))\right] \\
&\leq \lim_{t\to+\infty}\frac{1}{1+t}\cdot\frac{a^* + b^* r}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t} = 0, \\
&\lim_{t\to+\infty}\frac{1}{1+t} \cdot \int_0^{+\infty} g(t) \cdot \left[\int_0^t \varphi_p^{-1}\left(\int_s^{+\infty} q(\tau) f(\tau, x(\tau), x'(\tau))\,\mathrm{d}\tau + \varphi_p(x_\infty)\right)\mathrm{d}s\right]\mathrm{d}t \\
&\leq \lim_{t\to+\infty}\frac{1}{1+t}2^{1/(p-1)} \times \left(\frac{1}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t}\int_0^{+\infty} t g(t)\,\mathrm{d}t\right) \times \left[\varphi_p^{-1}(B_r)\cdot\varphi_p^{-1}\left(\int_0^{+\infty} q(\tau)\,\mathrm{d}\tau\right) + x_\infty\right] = 0, \\
&\lim_{t\to+\infty}\frac{1}{1+t} \cdot \int_0^t \varphi_p^{-1}\left(\int_s^{+\infty} q(\tau) f(\tau, x(\tau), x'(\tau))\,\mathrm{d}\tau + \varphi_p(x_\infty)\right)\mathrm{d}s \\
&= \lim_{t\to+\infty}\varphi_p^{-1}\left(\int_t^{+\infty} q(\tau) f(\tau, x(\tau), x'(\tau))\,\mathrm{d}\tau + \varphi_p(x_\infty)\right) \\
&= x_\infty.
\end{aligned} \tag{37}$$

It follows from (37) that

$$\begin{aligned}
&\lim_{t\to\infty}\left|\frac{(Ax)(t)}{1+t}\right| \\
&= \lim_{t\to\infty}\frac{1}{1+t} \times \left\{\frac{1}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t} \times \int_0^{+\infty} g(t) \times \left[\int_0^t \varphi_p^{-1} \times \left(\int_s^{+\infty} q(\tau) f(\tau, x(\tau), x'(\tau))\,\mathrm{d}\tau + \varphi_p(x_\infty)\right)\mathrm{d}s + \sum_{t_k<t} I_k(x(t_k))\right]\mathrm{d}t \right. \\
&\quad \left. + \int_0^t \varphi_p^{-1} \times \left(\int_s^{+\infty} q(\tau) f(\tau, x(\tau), x'(\tau))\,\mathrm{d}\tau + \varphi_p(x_\infty)\right)\mathrm{d}s + \sum_{t_k<t} I_k(x(t_k))\right\} \\
&= x_\infty.
\end{aligned} \tag{38}$$

On the other hand, we arrive at

$$\begin{aligned}
&\lim_{t\to\infty}\left|(Ax)'(t)\right| \\
&= \lim_{t\to\infty}\varphi_p^{-1}\left(\int_t^{+\infty} q(s) f(s, x(s), x'(s))\,\mathrm{d}s + \varphi_p(x_\infty)\right) \\
&= x_\infty.
\end{aligned} \tag{39}$$

Thus, (36) can be easily derived from (38) and (39). So, by Lemma 5, we know that $A\Omega$ is relatively compact. Thus, we have proved that $A : P \to P$ is completely continuous. □

3. Main Results

For notational convenience, we denote that

$$m = 2^{1/(p-1)}\left(\frac{1}{1-\int_0^{+\infty} g(t)\,dt}\int_0^{+\infty} tg(t)\,dt + 1\right) \cdot \varphi_p^{-1}\left(\int_0^{+\infty} q(\tau)\,d\tau\right), \quad (40)$$

$$m' = \left(\frac{1}{1-\int_0^{+\infty} g(t)\,dt}\int_0^{+\infty} tg(t)\,dt + 1\right) \cdot \varphi_p^{-1}\left(\int_0^{+\infty} q(\tau)\,d\tau\right), \quad (41)$$

$$n = 2^{1/(p-1)}\left(\frac{1}{1-\int_0^{+\infty} g(t)\,dt}\int_0^{+\infty} tg(t)\,dt + 1\right) x_\infty, \quad (42)$$

$$n' = \left(\frac{1}{1-\int_0^{+\infty} g(t)\,dt}\int_0^{+\infty} tg(t)\,dt + 1\right) x_\infty, \quad (43)$$

$$\Lambda = \max\left\{\frac{a^*}{1-\int_0^{+\infty} g(t)\,dt - 3b^*}, n\right\}, \quad \Lambda' = \max\left\{\frac{a^*}{1-\int_0^{+\infty} g(t)\,dt - 3b^*}, n'\right\}. \quad (44)$$

Theorem 7. *Assume that (H_1)–(H_3) hold, and there exists*

$$d > \begin{cases} 3\Lambda, & \text{as } p \geq 2, \\ 3\Lambda', & \text{as } 1 < p < 2 \end{cases} \quad (45)$$

such that

(A_1) *$f(t, x_1, y_1) \leq f(t, x_2, y_2)$ for any $0 \leq t < +\infty, 0 \leq x_1 \leq x_2 \leq d, 0 \leq y_1 \leq y_2 \leq d$,*

(A_2)

$$f(t,(1+t)u,v) \leq \begin{cases} \varphi_p\left(\dfrac{d}{3m}\right), & \text{as } p \geq 2, \\ \varphi_p\left(\dfrac{d}{3m'}\right), & \text{as } 1 < p < 2, \end{cases} \quad (46)$$
$$(t,u,v) \in [0,+\infty) \times [0,d] \times [0,d],$$

(A_3) *$I_k(x_1) \leq I_k(x_2)$ $(k = 1, 2, \ldots,)$, for any $0 \leq x_1 \leq x_2$.*

Then, the boundary value problem (4) admits positive, nondecreasing on $[0,+\infty)$ and concave solutions w^* and v^* such that $0 < \|w^*\|_D \leq d$, and $\lim_{n\to\infty} w_n = \lim_{n\to\infty} A^n w_0 = w^*$, where

$$w_0(t) = d + dt, \quad t \in J, \quad (47)$$

and $0 < \|v^*\|_D \leq d$, $\lim_{n\to\infty} v_n = \lim_{n\to\infty} A^n v_0 = v^*$, where $v_0(t) = 0, t \in J$.

Proof. We only prove the case that $p \geq 2$. Another case can be proved in a similar way. By Lemma 6, we know that $A : P \to P$ is completely continuous. From the definition of A, (A_1), and (A_3), we can easily get that $Ax_1 \leq Ax_2$ for any $x_1, x_2 \in P$ with $x_1 \leq x_2, x_1' \leq x_2'$. Denote that

$$\overline{P}_d = \{x \in P \mid \|x\|_D \leq d\}. \quad (48)$$

In what follows, we first prove that $A : \overline{P}_d \to \overline{P}_d$. If $x \in \overline{P}_d$, then $\|x\|_D \leq d$. By (6), (40), (42), (44), (H_3), (A_2), and (A_3), we get that

$$\begin{aligned}
\|Ax\|_F &= \sup_{t\in J}\Bigg\{\frac{1}{1+t}\Bigg|\frac{1}{1-\int_0^{+\infty} g(t)\,dt}\int_0^{+\infty} g(t)\Bigg[\int_0^t \varphi_p^{-1}\left(\int_s^{+\infty} q(\tau) f(\tau, x(\tau), x'(\tau))\,d\tau + \varphi_p(x_\infty)\right) ds + \sum_{t_k<t} I_k(x(t_k))\Bigg] dt \\
&\quad + \int_0^t \varphi_p^{-1}\left(\int_s^{+\infty} q(\tau) f(\tau, x(\tau), x'(\tau))\,d\tau + \varphi_p(x_\infty)\right) ds + \sum_{t_k<t} I_k(x(t_k))\Bigg|\Bigg\} \\
&\leq \frac{1}{1-\int_0^{+\infty} g(t)\,dt}\int_0^{+\infty} tg(t)\,dt \cdot \varphi_p^{-1}\left(\int_0^{+\infty} q(\tau) f(\tau, x(\tau), x'(\tau))\,d\tau + \varphi_p(x_\infty)\right)
\end{aligned}$$

$$
\begin{aligned}
&+ \varphi_p^{-1}\left(\int_0^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right) \mathrm{d}\tau + \varphi_p(x_\infty)\right) \\
&+ \frac{1}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t}\left(a^* + b^* d\right) \\
&\le 2^{1/(p-1)}\left[\frac{1}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t}\int_0^{+\infty} t g(t)\,\mathrm{d}t + 1\right] \\
&\times\left[\varphi_p^{-1}\left(\varphi_p\left(\frac{d}{3m}\right)\right)\cdot\varphi_p^{-1}\left(\int_0^{+\infty} q(\tau)\,\mathrm{d}\tau\right) + x_\infty\right] \\
&+ \frac{1}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t}\left(a^* + b^* d\right) \\
&\le \frac{d}{3} + \frac{d}{3} + \frac{d}{3} = d,
\end{aligned}
\tag{49}
$$

$$
\begin{aligned}
&\left\|(Ax)'\right\|_B \\
&= \sup_{t\in J}\left\{\left|\varphi_p^{-1}\left(\int_t^{+\infty} q(s) f\left(s, x(s), x'(s)\right) \mathrm{d}s + \varphi_p(x_\infty)\right)\right|\right\} \\
&\le 2^{1/(p-1)}\left[\varphi_p^{-1}\left(\int_0^{+\infty} q(s)\,\mathrm{d}s\right)\varphi_p^{-1}\left(\varphi_p\left(\frac{d}{3m}\right)\right) + x_\infty\right] \\
&\le d.
\end{aligned}
\tag{50}
$$

Thus, we get that $\|Ax\|_D \le d$. Hence, we have proved that $A : \overline{P}_d \to \overline{P}_d$.

Let $w_0(t) = d + dt$, $0 \le t < +\infty$, then $w_0(t) \in \overline{P}_d$. Let $w_1 = Aw_0$, $w_2 = A^2 w_0$, then by Lemma 6, we have that $w_1 \in \overline{P}_d$ and $w_2 \in \overline{P}_d$. Denote that

$$
w_{n+1} = Aw_n = A^{n+1} w_0, \quad n = 0, 1, 2, \ldots. \tag{51}
$$

Since $A : \overline{P}_d \to \overline{P}_d$, we have that

$$
w_n \in A\left(\overline{P}_d\right) \subset \overline{P}_d, \; n = 1, 2, 3, \ldots. \tag{52}
$$

It follows from the complete continuity of A that $\{w_n\}_{n=1}^{\infty}$ is a sequentially compact set. We assert that $\{w_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{w_{n_k}\}_{k=1}^{\infty}$, and there exists $w^* \in \overline{P}_d$ such that $w_{n_k} \to w^*$.

By (51), (A_1)–(A_3), we get that

$$
\begin{aligned}
&w_1(t) \\
&= \frac{1}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t} \\
&\times\int_0^{+\infty} g(t) \\
&\times\left[\int_0^{t}\varphi_p^{-1}\right. \\
&\times\left(\int_s^{+\infty} q(\tau) f\left(\tau, w_0(\tau), w_0'(\tau)\right)\mathrm{d}\tau\right. \\
&\left.+\varphi_p(x_\infty)\right)\mathrm{d}s \\
&\left.+\sum_{t_k<t} I_k\left(w_0(t_k)\right)\right]\mathrm{d}t \\
&+\int_0^t \varphi_p^{-1} \\
&\times\left(\int_s^{+\infty} q(\tau) f\left(\tau, w_0(\tau), w_0'(\tau)\right)\mathrm{d}\tau+\varphi_p(x_\infty)\right)\mathrm{d}s \\
&+\sum_{t_k<t} I_k\left(w_0(t_k)\right) \\
&\le \frac{1}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t} \\
&\times\int_0^{+\infty} t g(t)\,\mathrm{d}t\cdot\varphi_p^{-1} \\
&\times\left(\int_0^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right)\mathrm{d}\tau+\varphi_p(x_\infty)\right) \\
&+ t\varphi_p^{-1}\left(\int_0^{+\infty} q(\tau) f\left(\tau, x(\tau), x'(\tau)\right)\mathrm{d}\tau+\varphi_p(x_\infty)\right) \\
&+ \frac{1}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t}\left(a^* + b^* d\right) \\
&\le 2^{1/(p-1)}\left(\frac{1}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t}\cdot\int_0^{+\infty} t g(t)\,\mathrm{d}t\right) \\
&\times\left[\varphi_p^{-1}\left(\int_0^{+\infty} q(\tau)\,\mathrm{d}\tau\right)\varphi_p^{-1}\left(\varphi_p\left(\frac{d}{3m}\right)\right) + x_\infty\right] \\
&+ 2^{1/(p-1)} t\left[\varphi_p^{-1}\left(\int_0^{+\infty} q(\tau)\,\mathrm{d}\tau\right)\varphi_p^{-1}\left(\varphi_p\left(\frac{d}{3m}\right)\right) + x_\infty\right] \\
&+ \frac{1}{1-\int_0^{+\infty} g(t)\,\mathrm{d}t}\left(a^* + b^* d\right) \\
&\le d + dt = w_0(t), \\
&w_1'(t) \\
&= (Aw_0)'(t) \\
&= \varphi_p^{-1}\left(\int_t^{+\infty} q(s) f\left(s, w_0(s), w_0'(s)\right)\mathrm{d}s + \varphi_p(x_\infty)\right) \\
&\le 2^{1/(p-1)}\left[\varphi_p^{-1}\left(\int_0^{+\infty} q(\tau)\,\mathrm{d}\tau\right)\varphi_p^{-1}\left(\varphi_p\left(\frac{d}{3m}\right)\right) + x_\infty\right] \\
&\le d \\
&= w_0'(t), \quad 0 \le t < +\infty.
\end{aligned}
\tag{53}
$$

So, by (53) (A_1)–(A_3) we have

$$
\begin{aligned}
w_2(t) &= (Aw_1)(t) \le (Aw_0)(t) = w_1(t), \quad 0 \le t < +\infty, \\
w_2'(t) &= (Aw_1)'(t) \le (Aw_0)'(t) = (w_1)'(t), \quad 0 \le t < +\infty.
\end{aligned}
\tag{54}
$$

By induction, we get that

$$w_{n+1}(t) \le w_n(t),$$
$$w'_{n+1}(t) \le w'_n(t), \tag{55}$$
$$0 \le t < +\infty, \quad n = 0, 1, 2, \ldots.$$

Hence, we claim that $w_n \to w^*$ as $n \to \infty$. Applying the continuity of A and $w_{n+1} = Aw_n$, we get that $Aw^* = w^*$.

Let $v_0(t) = 0$, $0 \le t < +\infty$, then $v_0(t) \in \overline{P}_d$. Let $v_1 = Av_0, v_2 = A^2 v_0$. By Lemma 6, we have that $v_1 \in \overline{P}_d$ and $v_2 \in \overline{P}_d$. Denote

$$v_{n+1} = Av_n = A^{n+1} v_0, \quad n = 0, 1, 2, \ldots. \tag{56}$$

Since $A : \overline{P}_d \to \overline{P}_d$, we have that $v_n \in A(\overline{P}_d) \subset \overline{P}_d, n = 1, 2, 3, \ldots$. It follows from the complete continuity of A that $\{v_n\}_{n=1}^{\infty}$ is a sequentially compact set. And, we assert that $\{v_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{v_{n_k}\}_{k=1}^{\infty}$ and there exists $v^* \in \overline{P}_d$ such that $v_{n_k} \to v^*$.

Since $v_1 = Av_0 \in \overline{P}_d$, we have

$$v_1(t) = (Av_0)(t) = (A0)(t) \ge 0, \quad 0 \le t < +\infty,$$
$$v'_1(t) = (Av_0)'(t) = (A0)'(t) \ge 0 = v'_0(t), \quad 0 \le t < +\infty. \tag{57}$$

By (A_1)–(A_3), we have

$$v_2(t) = (Av_1)(t) \ge (A0)(t) = v_1(t), \quad 0 \le t < +\infty,$$
$$v'_2(t) = (Av_1)'(t) \ge (A0)'(t) = v'_1(t), \quad 0 \le t < +\infty. \tag{58}$$

By induction, we get that

$$v_{n+1}(t) \ge v_n(t),$$
$$v'_{n+1}(t) \ge v'_n(t), \tag{59}$$
$$0 \le t < +\infty, \quad n = 0, 1, 2, \ldots.$$

Hence, we claim that $v_n \to v^*$ as $n \to \infty$. Applying the continuity of A and $v_{n+1} = Av_n$, we get that $Av^* = v^*$.

Since $f(t, 0, 0) \not\equiv 0$, $0 \le t < \infty$, then the zero function is not the solution of BVP (4). Thus, v^* is a positive solution of BVP (4). By Lemma 4 we know that w^* and v^* are positive, nondecreasing on $[0, +\infty)$ and concave solutions of the BVP (4).

We can easily get that Theorem 7 holds for $1 < p < 2$ in a similar manner. □

Remark 8. The iterative schemes in Theorem 7 are $w_0(t) = d + dt, w_{n+1} = Aw_n = A^{n+1} w_0, n = 0, 1, 2, \ldots$ and $v_0(t) = 0, v_{n+1} = Av_n = A^{n+1} v_0$, $n = 0, 1, 2, \ldots$. They start off with a known simple linear function and the zero function respectively. This is convenient in application.

Theorem 9. *Assume that (H_1)–(H_3) hold, and there exist*

$$d_n > d_{n-1} > \cdots > d_1 > \begin{cases} 3\Lambda, & \text{as } p \ge 2, \\ 3\Lambda', & \text{as } 1 < p < 2 \end{cases} \tag{60}$$

such that

(A'_1) *$f(t, x_1, y_1) \le f(t, x_2, y_2)$ for any $0 \le t < +\infty$, $0 \le x_1 \le x_2$, $0 \le y_1 \le y_2$.*

(A'_2)

$$f(t, (1+t)u, v) \le \begin{cases} \varphi_p\left(\dfrac{d_k}{3m}\right), & \text{as } p \ge 2, \\ \varphi_p\left(\dfrac{d_k}{3m'}\right), & \text{as } 1 < p < 2 \end{cases}$$
$$(t, u, v) \in [0, +\infty) \times [0, d_k] \times [0, d_k], k = 1, 2, \ldots, n. \tag{61}$$

(A'_3) *$I_k(x_1) \le I_k(x_2)$ $(k = 1, 2, \ldots,)$, for any $0 \le x_1 \le x_2$.*

Then, the boundary value problem (4) admits positive nondecreasing on $[0, +\infty)$ and concave solutions w_k^* and v_k^*, such that $0 < \|w_k^*\|_D \le d_k$, and $\lim_{n\to\infty} w_{kn} = \lim_{n\to\infty} A^n w_{k0} = w_k^*$, where

$$w_0(t) = d_k + d_k t, \quad t \in J, \tag{62}$$

and $0 < \|v_k^*\|_D \le d_k$, $\lim_{n\to\infty} v_{kn} = \lim_{n\to\infty} A^n v_{k0} = v_k^*$, where $v_0(t) = 0$, $t \in J$.

Remark 10. It is easy to see that w^* and v^* in Theorem 7 may coincide, and then the boundary value problem (4) has only one solution in P. Similarly, positive solutions w_k^* and v_k^* may also coincide.

4. An Example

Example 11. Consider the following impulsive integral boundary value problem:

$$\left(|x'|\,x'\right)' + e^{-6t} f\left(t, x(t), x'(t)\right) = 0, \quad t \in J_+,$$
$$\Delta x|_{t=k} = \frac{1}{9}\left[\frac{1}{2^{k+2}} x(k) + \frac{1}{2^{k+1}} (1 + x(k))^{1/6}\right], \tag{63}$$
$$x(0) = \int_0^{+\infty} \frac{1}{(1+t)^3} x(t)\,dt, \quad x'(\infty) = \frac{\sqrt{2}}{3},$$

where

$$f(t, u, v) = \begin{cases} \dfrac{1}{64}|\sin(101t + 20)| + \dfrac{1}{72}\left(\dfrac{u}{1+t}\right)^3 + \dfrac{1}{10}\left(\dfrac{v}{20}\right), & u \le 2, \\ \dfrac{1}{64}|\sin(101t + 20)| + \dfrac{1}{72}\left(\dfrac{2}{1+t}\right)^3 + \dfrac{1}{10}\left(\dfrac{v}{20}\right), & u \ge 2. \end{cases} \tag{64}$$

It is clear that conditions (H_1), (A_1), and (A_3) hold for $p = 3$, $q(t) = e^{-6t}$, $g(t) = 1/(1+t)^3$. By direct computation, we obtain that

$$\int_0^{+\infty} q(t)\,dt = \frac{1}{6}, \qquad \int_0^{+\infty} \varphi_p^{-1}\left(\int_s^{+\infty} q(\tau)\,d\tau\right) ds = \frac{\sqrt{3}}{18}, \tag{65}$$

which implies that (H_2) holds.

Obviously, $I_k \in C(J, J)$. Using a simple inequality

$$(1+u)^{\alpha} \leq 1 + \alpha u, \quad \forall u \geq 0,\ 0 < \alpha < 1, \tag{66}$$

we get that

$$\begin{aligned} I_k(x(k)) &\leq \frac{1}{9}\left[\frac{1}{2^{k+2}}x(k) + \frac{1}{2^{k+1}}\left(1 + \frac{1}{6}x(k)\right)\right] \\ &\leq \frac{1}{9}\cdot\frac{1}{2^{k+1}} + \frac{2}{27}\cdot\frac{1}{2^{k+1}}x(k). \end{aligned} \tag{67}$$

Thus, (H_3) holds for $a_k = (1/9)\cdot(1/2^{k+1})$, $b_k = (2/27)\cdot(1/2^{k+1})$. Considering that

$$\begin{gathered} \int_0^{+\infty} g(t)\,dt = \int_0^{+\infty} \frac{1}{(1+t)^3}dt = \frac{1}{2}, \\ \int_0^{+\infty} tg(t)\,dt = \int_0^{+\infty} \frac{t}{(1+t)^3}dt = \frac{1}{2}, \\ \varphi_p^{-1}\left(\int_0^{+\infty} q(\tau)\,d\tau\right) = \varphi_p^{-1}\left(\frac{1}{6}\right) = \frac{\sqrt{6}}{6}, \end{gathered} \tag{68}$$

we can obtain that

$$\begin{gathered} a^* = \sum_{k=1}^{\infty} a_k = \frac{1}{18}, \\ b^* = \frac{2}{27}\sum_{k=1}^{\infty}\frac{1+k}{2^{k+1}} = \frac{2}{27}\left(\sum_{k=1}^{\infty}\frac{1}{2^{k+1}} + \sum_{k=1}^{\infty}\frac{k}{2^{k+1}}\right) = \frac{1}{9}, \\ m = \sqrt{3}, \quad n = 2, \quad \Lambda = 2. \end{gathered} \tag{69}$$

Take $d = 8$. In this case, we have

$$\varphi_p\left(\frac{d}{3m}\right) = \varphi_p\left(\frac{8}{3\sqrt{3}}\right) = \frac{64}{27}. \tag{70}$$

On the other hand, nonlinear term f satisfies

$$\begin{aligned} & f(t, (1+t)u, v) \\ & \leq \frac{1}{64} + \frac{1}{9} + \frac{1}{25} = \frac{2401}{14400}, \quad t \in [0, +\infty),\ u, v \in [0, 8], \end{aligned} \tag{71}$$

which means that (A_2) holds. Thus, we have checked that all the conditions of Theorem 7 are satisfied. Therefore, the conclusion of Theorem 7 holds.

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Research Article

Persistence and Nonpersistence of a Nonautonomous Stochastic Mutualism System

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In this paper, a two-species nonautonomous stochastic mutualism system is investigated. The intrinsic growth rates of the two species at time t are estimated by $r_i(t)+\sigma_i(t)\dot{B}_i(t),\ i=1,2$, respectively. Viewing the different intensities of the noises $\sigma_i(t), i=1,2$ as two parameters at time t, we conclude that there exists a global positive solution and the pth moment of the solution is bounded. We also show that the system is permanent, including stochastic permanence, persistence in mean, and asymptotic boundedness in time average. Besides, we show that the large white noise will make the system nonpersistent. Finally, we establish sufficient criteria for the global attractivity of the system.

1. Introduction

For more than three decades, mutualism of multispecies has attracted the attention of both mathematicians and ecologists. By definition, in a mutualism of multispecies, the interaction is beneficial for the growth of other species. Lotka-Volterra mutualism systems have long been used as standard models to mathematically address questions related to this interaction. Among these, nonautonomous Lotka-Volterra mutualism models are studied by many authors, see [1–7] and references therein. The classical nonautonomous Lotka-Volterra mutualism system can be expressed as follows:

$$\dot{x}_i(t)=x_i(t)\left[r_i(t)-a_{ii}(t)x_i(t)+\sum_{\substack{j=1\\ j\neq i}}^{n}a_{ij}(t)x_j(t)\right],\quad i=1,2,\dots,n, \tag{1}$$

where $x_i(t),\ i=1,2,\dots,n$ is the density of the ith population at time t, $r_i(t)>0,\ i=1,2,\dots,n$ is the intrinsic growth rate of the ith population at time $t, r_i(t)/a_{ii}(t)>0,\ i=1,2,\dots,n$ is the carrying capacity at time t, and coefficient $a_{ij}(t)>0,\ i,j=1,2,\dots,n$ describes the influence of the jth population upon the ith population at time t.

It is shown in [1] that if different conditions hold (see conditions (a)–(e) in [1]), then the solution of system (1) is bounded, permanent, extinct, and global attractive, respectively. However, when the intrinsic growth rate and coefficient $a_{ij}(t)$ are periodic, it is shown in [3] that there exists positive periodic solution and almost periodic solutions are obtained.

From another point of view, environmental noise always exists in real life. It is an interesting problem, both mathematically and biologically, to determine how the structure of the model changes under the effect of a fluctuating environment. Many authors studied the biological models with stochastic perturbation, see [8–12] and references therein. In [8] Ji et al. discussed the following two-species stochastic mutualism system

$$\begin{aligned}dx_1(t)&=x_1(t)\left[\left(r_1-a_{11}x_1(t)+a_{12}x_2(t)\right)dt+\sigma_1dB_1(t)\right],\\ dx_2(t)&=x_2(t)\left[\left(r_2+a_{21}x_1(t)-a_{22}x_2(t)\right)dt+\sigma_2dB_2(t)\right],\end{aligned} \tag{2}$$

where $B_i(t),\ i=1,2$ are mutually independent one dimensional standard Brownian motions with $B_i(0)=0,\ i=1,2,$ and $\sigma_i,\ i=1,2$ are the intensities of white noise. It is shown in [8] that if $a_{11}a_{22}>a_{12}a_{21}$ then there is a unique nonnegative solution of system (2). For small white noise there is a stationary distribution of (2) and it has ergodic property.

Biologically, this implies that with small perturbation of environment, the stability of the two species varies with the intensity of white noise, and both species will survive.

However, almost all known stochastic models assume that the growth rate and the carrying capacity of the population are independent of time t. In contrast, the natural growth rates of many populations vary with t in real situation, for example, due to the seasonality. As a matter of fact, nonautonomous stochastic population systems have recently been studied by many authors, for example, [13–17].

In this paper we consider the system

$$\begin{aligned} dx_1(t) &= x_1(t)\left[\left(r_1(t) - a_{11}(t)x_1(t) + a_{12}(t)x_2(t)\right)dt + \sigma_1(t)\,dB_1(t)\right],\\ dx_2(t) &= x_2(t)\left[\left(r_2(t) + a_{21}(t)x_1(t) - a_{22}(t)x_2(t)\right)dt + \sigma_2(t)\,dB_2(t)\right],\end{aligned} \tag{3}$$

where $r_i(t)$, $a_{ij}(t)$, $\sigma_i(t)$, $i, j = 1, 2$ are all continuous bounded nonnegative functions on $[0, +\infty)$. The objective of our study is to investigate the long-time behavior of system (3). As in [8], we mainly discuss when the system is persistent and when it is not under a fewer conditions. More specifically, we show that there is a positive solution of system (3) and its pth moment bounded in Section 2. In Section 3, we deduce the persistence of the system. If the white noise is not large such that $r_i^l - ((\sigma_i^u)^2/2) > 0$, $i = 1, 2$, we will prove that the solution of system (3) is a stochastic persistence. In addition, we show that every component of the solution is persistent in mean. We further deduce that every component of the solution of system (3) is an asymptotic boundedness in mean. In Section 4, we show that larger white noise will make system (3) nonpersistent. Finally, we study the global attractivity of system (3).

Throughout this paper, unless otherwise specified, let $(\Omega, \{\mathscr{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathscr{F}_0$ contains all P-null sets). Let R_+^2 be the positive cone of R^2, namely, $R_+^2 = \{x \in R^2 : x_i > 0,\ i = 1, 2\}$. If $x \in R^n$, its norm is denoted by $|x| = (\sum_{i=1}^n x_i^2)^{1/2}$. If $f(t)$ is a continuous bounded function on $[0, +\infty)$, we use the notation sup

$$f^u = \sup_{t\in[0,+\infty)} f(t), \qquad f^l = \min_{t\in[0,+\infty)} f(t). \tag{4}$$

2. Existence and Uniqueness of the Positive Solution

In population dynamics, the first concern is that the solution should be nonnegative. In order to do that a stochastic differential equation can have a unique global (i.e., no explosion at any finite time) solution for any given initial value, the coefficients of the equation are generally required to satisfy the linear growth condition and local Lipschitz condition (Mao [18]). However, the coefficients of system (3) do not satisfy the linear growth condition, though they are locally Lipschitz continuous, so the solution of system (3) may explode at a finite time. Following the way developed by Mao et al. [19], we show that there is a unique positive solution of (3).

Theorem 1. *Assume that $a_{11}^l a_{22}^l > a_{12}^u a_{21}^u$. Then, there is a unique positive solution $x(t) = (x_1(t), x_2(t))$ of system (3) on $t \geq 0$ for any given initial value $x(0) \in R_+^2$, and the solution will remain in R_+^2 with probability 1, namely, $x(t) \in R_+^2$ for all $t \geq 0$ almost surely.*

The proof of Theorem 1 is similar to [8]. But it is skilled in taking the value of ϵ. We show it here.

Proof. Since the coefficients of the equation are locally Lipschitz continuous, for any given initial value $x(0) \in R_+^2$ there is an unique local solution $x(t) = (x_1(t), x_2(t))$ on $t \in [0, \tau_e)$, where τ_e is the explosion time. To show that this solution is global, we need to show that $\tau_e = \infty$ a.s. Let $m_0 > 1$ be sufficiently large for every component of $x(0)$ lying within the interval $[1/m_0, m_0]$. For each integer $m \geq m_0$, define the stopping time

$$\tau_m = \inf\left\{t \in [0, \tau_e) : \min\{x_1(t), x_2(t)\} \leq \frac{1}{m} \text{ or } \max\{x_1(t), x_2(t)\} \geq m\right\}, \tag{5}$$

where throughout this paper we set $\inf\emptyset = \infty$ (as usual $\emptyset$ denotes the empty set). Clearly, τ_m is increasing as $m \to \infty$. Set $\tau_\infty = \lim_{m\to\infty}\tau_m$, whence $\tau_\infty \leq \tau_e$ a.s. If we can show that $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ a.s. and $x(t) \in R_+^2$ a.s. for all $t \geq 0$. In other words, to complete the proof, all we need to show is that $\tau_\infty = \infty$ a.s. If this statement is false, there is a pair of constant $T > 0$ and $\varepsilon \in (0, 1)$ such that

$$P\{\tau_\infty \leq T\} > \varepsilon. \tag{6}$$

Hence, there is an integer $m_1 \geq m_0$ such that

$$P\{\tau_m \leq T\} \geq \varepsilon \quad \forall m \geq m_1. \tag{7}$$

We define

$$V(x) = a_{21}^u(x_1 - 1 - \log x_1) + a_{12}^u(x_2 - 1 - \log x_2). \tag{8}$$

By Itô's formula, we have

$$\begin{aligned} dV(x) = \Big\{ & a_{21}^u\left(1 - \frac{1}{x_1}\right)x_1\left[r_1(t) - a_{11}(t)x_1 + a_{12}(t)x_2\right] \\ & + a_{12}^u\left(1 - \frac{1}{x_2}\right)x_2\left[r_2(t) + a_{21}(t)x_1 - a_{22}(t)x_2\right] \\ & + \frac{1}{2}\left[a_{21}^u\sigma_1^2(t) + a_{12}^u\sigma_2^2(t)\right]\Big\}\,dt \end{aligned}$$

$$\begin{aligned} &+ a_{21}^u \sigma_1(t)(x_1 - 1)\, dB_1(t) \\ &+ a_{12}^u \sigma_2(t)(x_2 - 1)\, dB_2(t) \\ := {}& LV dt + a_{21}^u \sigma_1(t)(x_1 - 1)\, dB_1(t) \\ &+ a_{12}^u \sigma_2(t)(x_2 - 1)\, dB_2(t), \end{aligned} \tag{9}$$

where

$$\begin{aligned} LV = {}& a_{21}^u \left(1 - \frac{1}{x_1}\right) x_1 \left[r_1(t) - a_{11}(t) x_1 + a_{12}(t) x_2\right] \\ &+ a_{12}^u \left(1 - \frac{1}{x_2}\right) x_2 \left[r_2(t) + a_{21}(t) x_1 - a_{22}(t) x_2\right] \\ &+ \frac{1}{2}\left[a_{21}^u \sigma_1^2(t) + a_{12}^u \sigma_2^2(t)\right] \\ \leq {}& a_{21}^u \Big[\left(r_1^u + a_{11}^u\right) x_1 - a_{12}^l x_2 - a_{11}^l x_1^2 \\ &\quad + a_{12}^u x_1 x_2 - r_1^l + \frac{1}{2}\left(\sigma_1^u\right)^2 \Big] \\ &+ a_{12}^u \Big[\left(r_2^u + a_{22}^u\right) x_2 - a_{21}^l x_1 - a_{22}^l x_2^2 \\ &\quad + a_{21}^u x_1 x_2 - r_2^l + \frac{1}{2}\left(\sigma_2^u\right)^2 \Big]. \end{aligned} \tag{10}$$

According to Young inequality, note that $x_1 x_2 \leq \epsilon x_1^2 + (1/4\epsilon) x_2^2$, where $a_{21}^u/2a_{22}^l < \epsilon < a_{11}^l/2a_{12}^u$, then,

$$\begin{aligned} LV \leq {}& a_{21}^u \Big[\left(r_1^u + a_{11}^u\right) x_1 - a_{12}^l x_2 - a_{11}^l x_1^2 \\ &\quad + a_{12}^u \left(\epsilon x_1^2 + \frac{1}{4\epsilon} x_2^2\right) - r_1^l + \frac{1}{2}\left(\sigma_1^u\right)^2 \Big] \\ &+ a_{12}^u \Big[\left(r_2^u + a_{22}^u\right) x_2 - a_{21}^l x_1 - a_{22}^l x_2^2 \\ &\quad + a_{21}^u \left(\epsilon x_1^2 + \frac{1}{4\epsilon} x_2^2\right) - r_2^l + \frac{1}{2}\left(\sigma_2^u\right)^2 \Big] \\ = {}& -\left(a_{21}^u a_{11}^l - 2\epsilon a_{21}^u a_{12}^u\right) x_1^2 \\ &+ \left[a_{21}^u \left(r_1^u + a_{11}^u\right) - a_{12}^u a_{21}^l\right] x_1 \\ &- a_{21}^u r_1^l + \frac{1}{2} a_{21}^u \left(\sigma_1^u\right)^2 \\ &- \left(a_{12}^u a_{22}^l - \frac{1}{2\epsilon} a_{21}^u a_{12}^u\right) x_2^2 \\ &+ \left[a_{12}^u \left(r_2^u + a_{22}^u\right) - a_{21}^u a_{12}^l\right] x_2 \\ &- a_{12}^u r_2^l + \frac{1}{2} a_{12}^u \left(\sigma_2^u\right)^2 \\ \leq {}& K. \end{aligned} \tag{11}$$

Since $a_{21}^u/2a_{22}^l < \epsilon < a_{11}^l/2a_{12}^u$, we obtain $-(a_{21}^u a_{11}^l - 2\epsilon a_{12}^u) < 0$ and $-(a_{12}^u a_{22}^l - (1/2\epsilon) a_{21}^u a_{12}^u) < 0$. Hence, K is a positive constant. Integrating both sides of (9) from 0 to $\tau_m \wedge T$, we therefore obtain

$$\begin{aligned} &V\left(x\left(\tau_m \wedge T\right)\right) - V(x(0)) \\ &\quad \leq \int_0^{\tau_m \wedge T} K dt + \int_0^{\tau_m \wedge T} a_{21}^u \sigma_1(t)\left(x_1(t) - 1\right) dB_1(t) \\ &\qquad + \int_0^{\tau_m \wedge T} a_{12}^u \sigma_2(t)\left(x_2(t) - 1\right) dB_2(t). \end{aligned} \tag{12}$$

Whence, taking expectations yields

$$\begin{aligned} E\left[V\left(x\left(\tau_m \wedge T\right)\right)\right] &\leq V(x(0)) + KE\left(\tau_m \wedge T\right) \\ &\leq V(x(0)) + KT. \end{aligned} \tag{13}$$

Set $\Omega_m = \{\tau_m \leq T\}$ for $m \geq m_1$ and by (7), $P(\Omega_m) \geq \varepsilon$. Note that for every $\omega \in \Omega_m$, there is $x_1(\tau_m, \omega)$ or $x_2(\tau_m, \omega)$ equals either m or $1/m$, and therefore

$$\begin{aligned} &V\left(x\left(\tau_m, \omega\right)\right) \\ &\quad \geq \min\left\{a_{21}^u, a_{12}^u\right\}(m - 1 - \log m) \wedge \left(\frac{1}{m} - 1 - \log\frac{1}{m}\right) \\ &\quad := h(m), \end{aligned} \tag{14}$$

where $\lim_{m \to \infty} h(m) = \infty$. It then follows from (13) that

$$E[V(x(0))] + KT \geq E\left[1_{\Omega_m} \cdot V\left(x\left(\tau_m, \omega\right)\right)\right] \geq \epsilon h(m), \tag{15}$$

where 1_{Ω_m} is the indicator function of Ω_m. Letting $m \to \infty$ leads to the contradiction

$$\infty > V(x(0)) + KT = \infty, \tag{16}$$

so we must have $\tau_\infty = \infty$ a.s. This completes the proof of Theorem 1. □

Remark 2. By Theorem 1, we observe that for any given initial value $x(0) \in R_+^2$, there is a unique solution $x(t) = (x_1(t), x_2(t))$ of system (3) on $t \geq 0$ and the solution will remain in R_+^2 with probability 1, no matter how large the intensities of white noise are. So, under the same assumption there is an global unique positive solution of the corresponding deterministic system of system (3).

Next, we show that the pth moment of the solution of system (3) is bounded in time average.

Theorem 3. *Assume that $a_{11}^l a_{22}^l > a_{12}^u a_{21}^u$. Then there exists a positive constant $K(p)$ such that the solution $x(t)$ of system* (3) *has the following property:*

$$E\left[c_1 x_1^p(t) + c_2 x_2^p(t)\right] \le K(p), \quad \forall t \in [0,\infty), \ p > 1, \tag{17}$$

where c_1, c_2 satisfy

$$\frac{(a_{21}^u)^{p+1}}{a_{11}^l (a_{22}^l)^p} < \frac{c_1}{c_2} < \frac{a_{22}^l (a_{11}^l)^p}{(a_{12}^u)^{p+1}}. \tag{18}$$

Proof. By Itô's formula, we have

$$\begin{aligned} dx_1^p(t) &= p x_1^p(t)\left[(r_1(t) - a_{11}(t)x_1(t) + a_{12}(t)x_2(t))\,dt \right.\\ &\qquad \left. + \sigma_1(t)\,dB_1(t)\right] \\ &\quad + \frac{1}{2}p(p-1)x_1^p(t)\sigma_1^2(t)\,dt \\ &= p\left[\left(r_1(t) + \frac{p-1}{2}\sigma_1^2(t)\right)x_1^p(t) - a_{11}(t)x_1^{p+1}(t)\right. \\ &\qquad \left. + a_{12}(t)x_1^p(t)x_2(t)\right]dt \\ &\quad + \sigma_1(t)\,p x_1^p(t)\,dB_1(t) \\ &= p\left[\alpha_1(t)x_1^p(t) - a_{11}(t)x_1^{p+1}(t)\right. \\ &\qquad \left. + a_{12}(t)x_1^p(t)x_2(t)\right]dt \\ &\quad + \sigma_1(t)\,p x_1^p(t)\,dB_1(t) \\ &\le p\left[\alpha_1^u x_1^p(t) - a_{11}^l x_1^{p+1}(t) + a_{12}^u x_1^p(t)x_2(t)\right]dt \\ &\quad + p\sigma_1^u x_1^p(t)\,dB_1(t), \end{aligned} \tag{19}$$

where $\alpha_1(t) = r_1(t) + ((p-1)/2)\sigma_1^2(t)$, and

$$\begin{aligned} dx_2^p(t) &= p x_2^p(t)\left[(r_2(t) - a_{22}(t)x_2(t) + a_{21}(t)x_1(t))\,dt \right.\\ &\qquad \left. + \sigma_2(t)\,dB_2(t)\right] \\ &\quad + \frac{1}{2}p(p-1)x_2^p(t)\sigma_2^2(t)\,dt \\ &= p\left[\left(r_2(t) + \frac{p-1}{2}\sigma_2^2(t)\right)x_2^p(t) - a_{22}(t)x_2^{p+1}(t)\right. \\ &\qquad \left. + a_{21}(t)x_2^p(t)x_1(t)\right]dt \\ &\quad + \sigma_2(t)\,p x_2^p(t)\,dB_2(t) \\ &= p\left[\alpha_2(t)x_2^p(t) - a_{22}(t)x_2^{p+1}(t)\right. \\ &\qquad \left. + a_{21}(t)x_2^p(t)x_1(t)\right]dt + \sigma_2(t)\,p x_2^p(t)\,dB_2(t) \\ &\le p\left[\alpha_2^u x_2^p(t) - a_{22}^l x_2^{p+1}(t) + a_{21}^u x_2^p(t)x_1(t)\right]dt \\ &\quad + p\sigma_2^u x_2^p(t)\,dB_2(t), \end{aligned} \tag{20}$$

where $\alpha_2(t) = r_2(t) + ((p-1)/2)\sigma_2^2(t)$. According to Young inequality, we obtain

$$\begin{aligned} x_1^p(t)x_2(t) &\le \epsilon_1 x_1^{p+1}(t) + \frac{1}{p+1}\left(\frac{p}{p+1}\right)^p\left(\frac{1}{\epsilon_1}\right)^p x_2^{p+1}(t), \\ &\qquad \epsilon_1 = \frac{p a_{11}^l}{(p+1)a_{12}^u}, \\ x_2^p(t)x_1(t) &\le \epsilon_2 x_2^{p+1}(t) + \frac{1}{p+1}\left(\frac{p}{p+1}\right)^p\left(\frac{1}{\epsilon_2}\right)^p x_1^{p+1}(t), \\ &\qquad \epsilon_2 = \frac{p a_{22}^l}{(p+1)a_{21}^u}. \end{aligned} \tag{21}$$

Thus, we have

$$\begin{aligned} dx_1^p(t) &\le p\left[\alpha_1^u x_1^p(t) - a_{11}^l x_1^{p+1}(t) + a_{12}^u \epsilon_1 x_1^{p+1}(t)\right. \\ &\qquad \left. + a_{12}^u \frac{1}{p+1}\left(\frac{p}{p+1}\right)^p\left(\frac{1}{\epsilon_1}\right)^p x_2^{p+1}(t)\right]dt \\ &\quad + p\sigma_1^u x_1^p(t)\,dB_1(t), \\ dx_2^p(t) &\le p\left[\alpha_2^u x_2^p(t) - a_{22}^l x_2^{p+1}(t) + a_{21}^u \epsilon_2 x_2^{p+1}(t)\right. \\ &\qquad \left. + a_{21}^u \frac{1}{p+1}\left(\frac{p}{p+1}\right)^p\left(\frac{1}{\epsilon_2}\right)^p x_1^{p+1}(t)\right]dt \\ &\quad + p\sigma_2^u x_2^p(t)\,dB_2(t). \end{aligned} \tag{22}$$

Since $a_{11}^l a_{22}^l > a_{12}^u a_{21}^u$, there exist two positive constants c_1, c_2 which satisfy

$$\frac{(a_{21}^u)^{p+1}}{a_{11}^l (a_{22}^l)^p} < \frac{c_1}{c_2} < \frac{a_{22}^l (a_{11}^l)^p}{(a_{12}^u)^{p+1}}. \tag{23}$$

Therefore,

$$\begin{aligned} &d\left(c_1 x_1^p(t) + c_2 x_2^p(t)\right) \\ &\quad \le -p\left[\left(c_1 a_{11}^l - c_1 a_{12}^u \epsilon_1 - c_2 a_{21}^u \frac{p^p}{(p+1)^{p+1}\epsilon_2^p}\right)x_1^{p+1}(t)\right. \\ &\qquad + \left(c_2 a_{22}^l - c_1 a_{21}^u \epsilon_2 - c_1 a_{12}^u \frac{p^p}{(p+1)^{p+1}\epsilon_1^p}\right)x_2^{p+1}(t) \\ &\qquad \left. - \sum_{i=1}^{2} c_i \alpha_i^u x_i^p(t)\right]dt + \sum_{i=1}^{2} c_i p \sigma_i^u x_i^p(t)\,dB_i(t). \end{aligned} \tag{24}$$

From (23) and the values of ϵ_1, ϵ_2, we obtain

$$\frac{a_{21}^u \left(p^p/(p+1)^{p+1}\epsilon_2^p\right)}{a_{11}^l - a_{12}^u \epsilon_1} < \frac{c_1}{c_2} < \frac{a_{22}^l - a_{21}^u \epsilon_2}{a_{12}^u \left(p^p/(p+1)^{p+1}\epsilon_1^p\right)}, \tag{25}$$

which implies that $c_1a_{11}^l - c_1a_{12}^u\epsilon_1 - c_2a_{21}^u(p^p/(p+1)^{p+1}\epsilon_2^p) > 0$ and $c_2a_{22}^l - c_1a_{21}^u\epsilon_2 - c_1a_{12}^u(p^p/((p+1)^{p+1}\epsilon_1^p)) > 0$. Let

$$\alpha = \max\{\alpha_1^u, \alpha_2^u\},$$
$$\beta = \min\left\{c_1^{-(p+1)/p}\left[c_1a_{11}^l - c_1a_{12}^u\epsilon_1 - c_2a_{21}^u\frac{p^p}{(p+1)^{p+1}\epsilon_2^p}\right],\right.$$
$$\left. c_2^{-(p+1)/p}\left[c_2a_{22}^l - c_1a_{21}^u\epsilon_2 - c_1a_{12}^u\frac{p^p}{(p+1)^{p+1}\epsilon_1^p}\right]\right\}, \quad (26)$$

then we have

$$\begin{aligned} d\left(c_1x_1^p(t) + c_2x_2^p(t)\right) &\le p\left[\alpha\left(\sum_{i=1}^2 c_ix_i^p(t)\right) - \beta\left(\sum_{i=1}^2 c_i^{1+(1/p)}x_i^{p+1}\right)\right]dt \\ &\quad + \sum_{i=1}^2 c_ip\sigma_i^u x_i^p(t)\,dB_i(t). \end{aligned} \quad (27)$$

Hence, we get

$$\begin{aligned} \frac{dE\left[c_1x_1^p(t) + c_2x_2^p(t)\right]}{dt} &\le p\alpha E\left[c_1x_1^p(t) + c_2x_2^p(t)\right] \\ &\quad - p\beta E\left[c_1^{1+(1/p)}x_1^{p+1}(t) + c_2^{1+(1/p)}x_2^{p+1}(t)\right] \\ &\le p\alpha E\left[c_1x_1^p(t)+c_2x_2^p(t)\right] \\ &\quad - p\beta\left\{\left[E\left(c_1x_1^p(t)\right)\right]^{1+(1/p)} + \left[E\left(c_2x_2^p(t)\right)\right]^{1+(1/p)}\right\} \\ &\le p\alpha E\left[c_1x_1^p(t) + c_2x_2^p(t)\right] \\ &\quad - p\beta\cdot 2^{-1/p}\left[E\left(c_1x_1^p(t) + c_2x_2^p(t)\right)\right]^{1+(1/p)}. \end{aligned} \quad (28)$$

By the comparison theorem, we get

$$\limsup_{t\to\infty} E\left[c_1x_1^p(t) + c_2x_2^p(t)\right] \le 2\left(\frac{\alpha}{\beta}\right)^p := M(p), \quad (29)$$

which implies that there is a $T_0 > 0$, such that

$$E\left[c_1x_1^p(t) + c_2x_2^p(t)\right] \le 2M(p), \quad \forall t > T_0. \quad (30)$$

Besides, note that $E[c_1x_1^p(t) + c_2x_2^p(t)]$ is continuous, then there is a $\widetilde{M}(p) > 0$ such that

$$E\left[c_1x_1^p(t) + c_2x_2^p(t)\right] \le \widetilde{M}(p), \quad \forall t \in [0, T_0]. \quad (31)$$

Let $K(p) = \max\{2M(p), \widetilde{M}(p)\}$, then

$$E\left[c_1x_1^p(t) + c_2x_2^p(t)\right] \le K(p), \quad \forall t \in [0, \infty). \quad (32)$$

□

3. Persistence

Theorem 1 shows that the solution of system (3) will remain in the positive cone R_+^2 if $a_{11}^l a_{22}^l > a_{12}^u a_{21}^u$. Studying a population system, we pay more attention on whether the system is persistent. In this section, we first show that the solution is a stochastic permanence. Next we show that the solution is persistent in time average. Moreover, we show that the solution $x(t)$ of system (3) is an asymptotic boundedness in time average.

3.1. Stochastic Permanence. Let $y(t)$ be the solution of a randomized nonautonomous competitive equation:

$$dy_i(t) = y_i(t)\left[\left(b_i(t) - \sum_{j=1}^n a_{ij}(t)\,y_j(t)\right)dt + \sigma_i(t)\,dB_i(t)\right],$$
$$i = 1, 2, \ldots, n, \quad (33)$$

where $B_i(t)$, $i = 1, 2, \ldots, n$, are independent standard Brownian motions, $y(0) = y_0 > 0$ while y_0 is independent of $B(t)$, and $b_i(t)$, $a_{ij}(t)$, $\sigma_i(t)$ are all continuous bounded nonnegative functions on $[0, +\infty)$.

Lemma 4 (see [15]). *Assume that $b_i^l - ((\sigma_i^u)^2/2) > 0$, $i = 1, 2, \ldots, n$, then for any given initial value $y(0) \in R_+^n$, the solution $y(t)$ of (36) has the properties*

$$\limsup_{t\to\infty} E\left(\frac{1}{|y(t)|^\theta}\right) \le H, \quad (34)$$

where H is a constant, θ is an arbitrary positive constant satisfying

$$\theta \max_{1\le i\le n}(\sigma_i^u)^2 < 2\min_{1\le i\le n}\left(b_i(t) - \frac{\sigma_i^2(t)}{2}\right)^l. \quad (35)$$

Let $N(t)$ be the solution of a randomized nonautonomous logistic equation

$$dN(t) = N(t)\left[(a(t) - b(t)N(t))\,dt + \alpha(t)\,dB(t)\right], \quad (36)$$

where $B(t)$ is a 1-dimensional standard Brownian motion, $N(0) = N_0 > 0$, and N_0 is independent of $B(t)$.

Lemma 5 (see [13]). *Assume that $a(t)$, $b(t)$, and $\alpha(t)$ are bounded continuous functions defined on $[0, \infty)$, $a(t) > 0$ and $b(t) > 0$. Then there exists a unique continuous positive* *solution of* (36) *for any initial value* $N(0) = N_0 > 0$, *which is global and represented by*

$$N(t) = \exp\left\{\int_0^t \left[a(s) - \left(\frac{\alpha^2(s)}{2}\right)\right] ds + \alpha(s)\, dB(s)\right\} \times \left(\left(\frac{1}{N_0}\right) + \int_0^t b(s) \exp\left\{\int_0^s \left[a(\tau) - \left(\frac{\alpha^2(\tau)}{2}\right)\right] d\tau + \alpha(\tau)\, dB(\tau)\right\} ds\right)^{-1}, \quad t \ge 0. \tag{37}$$

From Lemma 4 we have the following.

Lemma 6. *Assume that* $a^l - ((\alpha^u)^2/2) > 0$, *then for any given initial value* $N(0) \in R_+$, *the solution* $N(t)$ *of* (36) *has the properties*

$$\limsup_{t\to\infty} E\left(\frac{1}{N^\theta(t)}\right) \le H, \tag{38}$$

where H *is a constant,* θ *is positive constant satisfying*

$$\theta(\alpha^u)^2 < 2\left[a^l - \frac{(\alpha^u)^2}{2}\right]. \tag{39}$$

Let $\phi(t) = (\phi_1(t), \phi_2(t))^T$ be the solution of

$$d\phi_i(t) = \phi_i(t)\left[(r_i(t) - a_{ii}(t)\phi_i(t))\, dt + \sigma_i(t)\, dB_i(t)\right], \quad i = 1, 2, \tag{40}$$

where $B_i(t)$, $i = 1, 2$, are independent standard Brownian motions, $\phi(0) = \phi_0 > 0$, and $\phi_0 \in R_+^2$, $r_i(t), a_{ii}(t), \sigma_i(t)$, $i = 1, 2$ are all continuous bounded nonnegative functions on $[0, +\infty)$. From Lemma 4 it is easy to know the following.

Lemma 7. *Assume that* $\overline{r}_i^l = r_i^l - ((\sigma_i^u)^2/2) > 0$, $i = 1, 2$, *then for any given initial value* $\phi(0) \in R_+^2$, *the solution* $\phi(t)$ *of* (40) *has the properties*

$$\limsup_{t\to\infty} E\left(\frac{1}{\phi_i^\theta(t)}\right) \le H_i, \quad i = 1, 2, \tag{41}$$

where H_i, $i = 1, 2$ *are two constants,* θ *is positive constant satisfying*

$$\theta(\sigma_i^u)^2 < 2\overline{r}_i^l, \quad i = 1, 2. \tag{42}$$

Lemma 8. *Assume that* $\overline{r}_i^l > 0$, $i = 1, 2$, *then for any given initial value* $x_0 \in R_+^2$, *the solution* $x(t)$ *of system* (3) *has the properties*

$$x_i(t) \ge \phi_i(t), \quad i = 1, 2, \tag{43}$$

$$\limsup_{t\to\infty} E\left(\frac{1}{x_i^\theta(t)}\right) \le H_i, \quad i = 1, 2, \tag{44}$$

where H_i, $i = 1, 2$ *are two constants,* θ *is positive constant satisfying*

$$\theta(\sigma_i^u)^2 < 2\overline{r}_i^l, \quad i = 1, 2. \tag{45}$$

Proof. Equation (43) follows directly from the classical comparison theorem of stochastic differential equations (see [20]). Thus, we obtain

$$\limsup_{t\to\infty} E\left(\frac{1}{x_i^\theta(t)}\right) \le \limsup_{t\to\infty} E\left(\frac{1}{\phi_i^\theta(t)}\right) \le H_i, \quad i = 1, 2. \tag{46}$$

□

Definition 9. System (3) is said to be stochastically permanent if for any $\varepsilon \in (0, 1)$, there exists a pair of positive constants $\delta = \delta(\epsilon)$ and $M = M(\epsilon)$ such that for any initial value $x_0 \in R_+^2$, the solution obeys

$$\liminf_{t\to\infty} P\{x_i(t) \le M(\epsilon)\} \ge 1 - \epsilon, \qquad \liminf_{t\to\infty} P\{x_i(t) \ge \delta(\epsilon)\} \ge 1 - \epsilon, \quad i = 1, 2. \tag{47}$$

Theorem 10. *Assume that* $a_{11}^l a_{22}^l > a_{12}^u a_{21}^u$, $\overline{r}_i^l > 0$, $i = 1, 2$, *then system* (3) *is stochastically permanent.*

The proof is a simple application of the Chebyshev inequality, we omit it.

3.2. Persistence in Time Average. Theorem 10 shows that if the white noise is not large, the solution of system (3) is survive with large probability. In this part, we show $x(t)$ is persistence in mean.

Lemma 11. *Assume that* $\overline{r}_i^l > 0$, $i = 1, 2$, *then for any given initial value* $\phi(0) \in R_+^2$, *the solution* $\phi(t)$ *of* (40) *has the properties*

$$z_i(t)\, e^{-[\max_{0\le s\le t} \int_0^s \sigma_i(\tau)dB_i(\tau) - \int_0^t \sigma_i(\tau)dB_i(\tau)]} \le \phi_i(t) \le z_i(t)\, e^{-[\min_{0\le s\le t} \int_0^s \sigma_i(\tau)dB_i(\tau) - \int_0^t \sigma_i(\tau)dB_i(\tau)]}, \quad i = 1, 2, \tag{48}$$

where $z(t) = (z_1(t), z_2(t))$ *is the solution of*

$$dz_i(t) = z_i(t)\left[r_i(t) - \frac{\sigma_i^2(t)}{2} - a_{ii}^u z_i(t)\right] dt, \qquad z_i(0) = \phi_i(0), \; i = 1, 2. \tag{49}$$

Proof. From Lemma 5, we know

$$\begin{aligned}\frac{1}{\phi_i(t)} &= \frac{1}{\phi_i(0)} e^{-\int_0^t [r_i(s)-(\sigma_i^2(s)/2)]ds+\sigma_i(s)dB_i(s)} \\ &\quad + a_{ii}^u \int_0^t e^{-\int_0^t [r_i(s)-(\sigma_i^2(s)/2)]ds+\sigma_i(s)dB_i(s)} \\ &\quad \times e^{+\int_0^s [r_i(\tau)-(\sigma_i^2(\tau)/2)]d\tau+\sigma_i(\tau)dB_i(\tau)} ds \\ &= e^{-\int_0^t \sigma_i(s)dB_i(s)} \left[\frac{1}{\phi_i(0)} e^{-\int_0^t [r_i(s)-(\sigma_i^2(s)/2)]ds} \right. \\ &\quad + a_{ii}^u \int_0^t e^{-\int_s^t [r_i(\tau)-(\sigma_i^2(\tau)/2)]d\tau} \\ &\quad \left. \times e^{\int_0^s \sigma_i(\tau)dB_i(\tau)} ds \right] \\ &\le e^{-\int_0^t \sigma_i(s)dB_i(s)} \left[\frac{1}{\phi_i(0)} e^{-\int_0^t [r_i(s)-(\sigma_i^2(s)/2)]ds} \right. \\ &\quad + a_{ii}^u e^{\max_{0\le s\le t}(\int_0^s \sigma_i(\tau)dB_i(\tau))} \\ &\quad \left. \times \int_0^t e^{-\int_s^t [r_i(\tau)-(\sigma_i^2(\tau)/2)]d\tau} ds \right] \\ &\le \frac{e^{\max_{0\le s\le t}[\int_0^s \sigma_i(\tau)dB_i(\tau)]-\int_0^t \sigma_i(s)dB_i(s)}}{z_i(t)}.\end{aligned} \tag{50}$$

Similarly, we have

$$\begin{aligned}\frac{1}{\phi_i(t)} &\ge e^{-\int_0^t \sigma_i(s)dB_i(s)} \left[\frac{1}{\phi_i(0)} e^{-\int_0^t [r_i(s)-(\sigma_i^2(s)/2)]ds} \right. \\ &\quad + a_{ii}^u e^{\min_{0\le s\le t}(\int_0^s \sigma_i(\tau)dB_i(\tau))} \\ &\quad \left. \times \int_0^t e^{-\int_s^t [r_i(\tau)-(\sigma_i^2(\tau)/2)]d\tau} ds \right] \\ &\ge \frac{e^{\min_{0\le s\le t}[\int_0^s \sigma_i(\tau)dB_i(\tau)]-\int_0^t \sigma_i(s)dB_i(s)}}{z_i(t)}.\end{aligned} \tag{51}$$

□

Lemma 12. *Assume that $\overline{r}_i^l > 0$, $i = 1,2$, then for any given initial value $z(0) \in R_+^2$, the solution $z(t)$ of (49) has the following properties*

$$\begin{aligned}&\widetilde{z}_i(t) \le z_i(t) \le \widehat{z}_i(t), \\ &\lim_{t\to\infty} \widetilde{z}_i(t) = \frac{\overline{r}_i^l}{a_{ii}^u}, \qquad \lim_{t\to\infty} \widehat{z}_i(t) = \frac{\overline{r}_i^u}{a_{ii}^u},\end{aligned} \tag{52}$$

where $\widetilde{z}(t) = (\widetilde{z}_1(t), \widetilde{z}_2(t))$, $\widehat{z}(t) = (\widehat{z}_1(t), \widehat{z}_2(t))$ are the solutions of the two equations, respectively,

$$d\widetilde{z}_i(t) = \widetilde{z}_i(t)\left[\overline{r}_i^l - a_{ii}^u \widetilde{z}_i(t)\right]dt, \quad \widetilde{z}_i(0) = z_i(0), \ i = 1,2, \tag{53}$$

$$d\widehat{z}_i(t) = \widehat{z}_i(t)\left[\overline{r}_i^u - a_{ii}^u \widehat{z}_i(t)\right]dt, \quad \widehat{z}_i(0) = z_i(0), \ i = 1,2. \tag{54}$$

Proof. Let $\widetilde{z}(t) = (\widetilde{z}_1(t), \widetilde{z}_2(t))$, $\widehat{z}(t) = (\widehat{z}_1(t), \widehat{z}_2(t))$ are the solutions of SDE (53) and (54), respectively, with the positive initial value $z(0)$. By Lemma 5, we know

$$\begin{aligned}\widetilde{z}_i(t) &= \frac{e^{\overline{r}_i^l t}}{1/\widetilde{z}_i(0) + \left(a_{ii}^u/\overline{r}_i^l\right)\left(e^{\overline{r}_i^l t} - 1\right)}, \\ \widehat{z}_i(t) &= \frac{e^{\overline{r}_i^u t}}{1/\widehat{z}_i(0) + \left(a_{ii}^u/\overline{r}_i^u\right)\left(e^{\overline{r}_i^u t} - 1\right)}.\end{aligned} \tag{55}$$

Thus,

$$\lim_{t\to\infty} \widetilde{z}_i(t) = \frac{\overline{r}_i^l}{a_{ii}^u}, \qquad \lim_{t\to\infty} \widehat{z}_i(t) = \frac{\overline{r}_i^u}{a_{ii}^u}. \tag{56}$$

By the classical comparison theorem of ordinary differential equations, we know

$$\widetilde{z}_i(t) \le z_i(t) \le \widehat{z}_i(t). \tag{57}$$

□

Lemma 13. *Assume that $\overline{r}_i^l > 0$, $i = 1,2$, then for any given initial value $\phi(0) \in R_+^2$, the solution $\phi(t)$ of (40) has the properties*

$$\lim_{t\to\infty} \frac{\log \phi_i(t)}{t} = 0, \quad a.s. \tag{58}$$

Proof. By Lemma 12, we know

$$\begin{aligned}&e^{-[\max_{0\le s\le t} \int_0^s \sigma_i(\tau)dB_i(\tau) - \int_0^t \sigma_i(\tau)dB_i(\tau)]} \\ &\quad \le \frac{\phi_i(t)}{z_i(t)} \le e^{-[\min_{0\le s\le t} \int_0^s \sigma_i(\tau)dB_i(\tau) - \int_0^t \sigma_i(\tau)dB_i(\tau)]}.\end{aligned} \tag{59}$$

So, we have

$$\begin{aligned}&\int_0^t \sigma_i(\tau)\,dB_i(\tau) - \max_{0\le s\le t} \int_0^s \sigma_i(\tau)\,dB_i(\tau) \\ &\quad \le \log \phi_i(t) - \log z_i(t) \\ &\quad \le \int_0^t \sigma_i(\tau)\,dB_i(\tau) \\ &\qquad - \min_{0\le s\le t} \int_0^s \sigma_i(\tau)\,dB_i(\tau).\end{aligned} \tag{60}$$

Let $M_i(t) = \int_0^t \sigma_i(\tau)dB_i(\tau)$, then

$$\langle M_i, M_i \rangle_t = \int_0^t \sigma_i^2(\tau)\,d\tau. \tag{61}$$

Since $\sigma_i(t)$, $i = 1, 2$ are bounded, then

$$\lim_{t\to\infty}\frac{\langle M_i, M_i\rangle_t}{t} = \lim_{t\to\infty}\frac{1}{t}\int_0^t \sigma_i^2(\tau)\,d\tau < \infty, \quad \text{a.s.} \tag{62}$$

By the strong law of large numbers, we know

$$\lim_{t\to\infty}\frac{M_i(t)}{t} = \lim_{t\to\infty}\frac{\int_0^t \sigma_i(\tau)\,dB_i(\tau)}{t} = 0, \quad \text{a.s.} \tag{63}$$

Thus,

$$\lim_{t\to\infty}\max_{0\le s\le t}\left|\frac{M_i(s)}{t}\right| = 0, \quad \text{a.s.} \tag{64}$$

Then from (60) we obtain

$$\lim_{t\to\infty}\frac{\log\phi_i(t)}{t} = 0, \quad \text{a.s.} \tag{65}$$

□

Lemma 14. *Assume that $\bar{r}_i^l > 0$, $i = 1, 2$, then for any given initial value $\phi(0) \in R_+^2$, the solution $\phi(t)$ of (40) has the properties*

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t \phi_i(s)\,ds \ge \frac{\bar{r}_i^l}{a_{ii}^u}, \quad a.s. \tag{66}$$

Proof. By Itô's formula, we have

$$d\log\phi_i(t) = \left[r_i(t) - \frac{\sigma_i^2(t)}{2} - a_{ii}^u\phi_i(t)\right]dt + \sigma_i(t)\,dB_i(t). \tag{67}$$

Integrating both sides of this equation from 0 to t yields

$$\begin{aligned}\frac{\log\phi_i(t)}{t} - \frac{\log\phi_i(0)}{t} = {} & \frac{\int_0^t\left[r_i(s) - \left(\sigma_i^2(s)/2\right)\right]ds}{t} \\ & - \frac{a_{ii}^u\int_0^t \phi_i(s)\,ds}{t} + \frac{\int_0^t \sigma_i(s)\,dB_i(s)}{t}.\end{aligned} \tag{68}$$

By Lemma 13, we know that

$$\lim_{t\to\infty}\frac{\int_0^t \sigma_i(s)\,dB_i(s)}{t} = \lim_{t\to\infty}\frac{\log\phi_i(t)}{t} = 0, \quad \text{a.s.} \tag{69}$$

Hence,

$$\begin{aligned}\liminf_{t\to\infty}\frac{1}{t}\int_0^t \phi_i(s)\,ds &= \frac{1}{a_{ii}^u}\liminf_{t\to\infty}\frac{1}{t}\int_0^t\left[r_i(s) - \frac{\sigma_i^2(s)}{2}\right]ds \\ &\ge \frac{\bar{r}_i^l}{a_{ii}^u}, \quad \text{a.s.}\end{aligned} \tag{70}$$

□

Definition 15. System (3) is said to be persistent in time average if

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t x_i(s)\,ds > 0, \quad i = 1, 2. \tag{71}$$

Theorem 16. *Assume that $a_{11}^l a_{22}^l > a_{12}^u a_{21}^u$ and $\bar{r}_i^l > 0$, $i = 1, 2$, then the solution $x(t)$ of system (3) with any initial value $x(0) \in R_+^2$ has the following property:*

$$\begin{aligned}&\liminf_{t\to\infty}\frac{1}{t}\int_0^t x_i(s)\,ds \ge \frac{\bar{r}_i^l}{a_{ii}^u}, \\ &\liminf_{t\to\infty}\frac{\log x_i(t)}{t} \ge 0, \quad a.s.,\end{aligned} \tag{72}$$

and so system (3) is persistent in time average.

Proof. By Lemma 8, we know that

$$x_i(t) \ge \phi_i(t) \quad i = 1, 2, \tag{73}$$

where $\phi(t) = (\phi_1(t), \phi_2(t))$ is the solution of system (40). Moreover, by Lemma 14 we know that

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t x_i(s)\,ds \ge \liminf_{t\to\infty}\frac{1}{t}\int_0^t \phi_i(s)\,ds \ge \frac{\bar{r}_i^l}{a_{ii}^u}, \quad \text{a.s.} \tag{74}$$

Hence, by Lemma 13 we know that

$$\liminf_{t\to\infty}\frac{\log x_i(t)}{t} \ge \liminf_{t\to\infty}\frac{\log\phi_i(t)}{t} = 0, \quad \text{a.s.} \tag{75}$$

□

3.3. Asymptotic Boundedness of Integral Average. Theorem 16 shows that every component of the solution $x(t)$ of system (3) will survive forever in time average, if the white noise is not large. In this part, we further deduce that every component of $x(t)$ of system (3) will be an asymptotic boundedness in time average. Before we give the result, we do some preparation work.

Lemma 17. *Let $f \in C[[0,\infty)\times\Omega, (0,\infty)]$, $F(t) \in ((0,\infty)\times\Omega, R)$. If there exist positive constants λ_0 and λ such that*

$$\log f(t) \ge \lambda t - \lambda_0\int_0^t f(s)\,ds + F(t), \quad t \ge 0, \ a.s., \tag{76}$$

and $\lim_{t\to\infty}(F(t)/t) = 0$ a.s., then

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t f(s)\,ds \ge \frac{\lambda}{\lambda_0}, \quad a.s. \tag{77}$$

Proof. The proof is similar to the proof of Lemma in [21]. Let

$$\varphi(t) = \int_0^t f(s)\,ds. \tag{78}$$

Since $f \in C[[0,\infty)\times\Omega, (0,\infty)]$, $\varphi(t)$ is differentiable on $[0,\infty)$ and

$$\frac{d\varphi(t)}{dt} = f(t) > 0, \quad \text{for } t \ge 0. \tag{79}$$

Substituting $d\varphi(t)/dt$ and $\varphi(t)$ into (76), we obtain the following:

$$\log \frac{d\varphi(t)}{dt} \geq \lambda t - \lambda_0 \varphi(t) + F(t), \tag{80}$$

thus

$$e^{\lambda_0 \varphi(t)} \frac{d\varphi(t)}{dt} \geq e^{\lambda t + F(t)}, \quad \text{for } t \geq 0. \tag{81}$$

Note that $\lim_{t\to\infty}(F(t)/t) = 0$ a.s., then for $0 < \varepsilon < \min\{1, \lambda\}$, $\exists T = T(\omega) > 0$ and $\Omega_\varepsilon \subset \Omega$ such that $P(\Omega_\varepsilon) > 1-\varepsilon$ and $F(t) \geq -\varepsilon t,\ t \geq T,\ \omega \in \Omega_\varepsilon$. Then we have

$$e^{\lambda_0 \varphi(t)} \frac{d\varphi(t)}{dt} \geq e^{(\lambda-\varepsilon)t}, \quad \text{for } t \geq T,\ \omega \in \Omega_\varepsilon. \tag{82}$$

Integrating inequality (82) from 0 to t results in the following:

$$\lambda_0^{-1}\left[e^{\lambda_0 \varphi(t)} - e^{\lambda_0 \varphi(T)}\right] \geq (\lambda - \varepsilon)^{-1}\left[e^{(\lambda-\varepsilon)t} - e^{(\lambda-\varepsilon)T}\right]. \tag{83}$$

This inequality can be rewritten into

$$e^{\lambda_0 \varphi(t)} \geq e^{\lambda_0 \varphi(T)} + \lambda_0 (\lambda - \varepsilon)^{-1}\left[e^{(\lambda-\varepsilon)t} - e^{(\lambda-\varepsilon)T}\right]. \tag{84}$$

Taking the logarithm of both sides and dividing both sides by $t(> 0)$ yields

$$\frac{\varphi(t)}{t} \geq \lambda_0^{-1} \frac{\log\left\{e^{\lambda_0\varphi(T)} + \lambda_0(\lambda-\varepsilon)^{-1}\left[e^{(\lambda-\varepsilon)t - e^{(\lambda-\varepsilon)T}}\right]\right\}}{t}. \tag{85}$$

Then,

$$\liminf_{t\to\infty} \frac{\varphi(t)}{t} \geq \frac{\lambda - \varepsilon}{\lambda_0}, \quad \omega \in \Omega_\varepsilon. \tag{86}$$

Letting $\varepsilon \to \infty$ yields

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t f(s)\,ds \geq \frac{\lambda}{\lambda_0}, \quad \text{a.s.} \tag{87}$$

This finishes the proof of the Lemma. □

Theorem 18. *Assume that* $a_{11}^l a_{22}^l > a_{12}^u a_{21}^u$ *and* $\bar{r}_i^l > 0$, $i = 1, 2$, *then the solution* $x(t)$ *of system* (3) *with any initial value* $x(0) \in R_+^2$ *has the property*

$$\tilde{x}_i^* \leq \lim_{t\to\infty} \frac{1}{t}\int_0^t x_i(s)\,ds \leq \hat{x}_i^*, \quad i = 1, 2, \ a.s., \tag{88}$$

where

$$\begin{aligned} \tilde{x}_1^* &= \frac{a_{22}^u \bar{r}_1^l + a_{12}^l \bar{r}_2^l}{a_{11}^u a_{22}^u - a_{12}^l a_{21}^l}, & \tilde{x}_2^* &= \frac{a_{11}^u \bar{r}_2^l + a_{21}^l \bar{r}_1^l}{a_{11}^u a_{22}^u - a_{12}^l a_{21}^l}, \\ \hat{x}_1^* &= \frac{a_{22}^l \bar{r}_1^u + a_{12}^u \bar{r}_2^u}{a_{11}^l a_{22}^l - a_{12}^u a_{21}^u}, & \hat{x}_2^* &= \frac{a_{11}^l \bar{r}_2^u + a_{21}^u \bar{r}_1^u}{a_{11}^l a_{22}^l - a_{12}^u a_{21}^u}. \end{aligned} \tag{89}$$

Proof. To prove the results, we only need to prove

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t x_i(s)\,ds \geq \tilde{x}_i^*, \quad i = 1, 2, \text{ a.s.} \tag{90}$$

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t x_i(s)\,ds \leq \hat{x}_i^*, \quad i = 1, 2, \text{ a.s.} \tag{91}$$

By Itô's formula, we have

$$\begin{aligned} &d\log x_1(t) \\ &\quad = \left[r_1(t) - \frac{1}{2}\sigma_1^2(t) - a_{11}(t)x_1(t) + a_{12}(t)x_2(t)\right]dt \\ &\qquad + \sigma_1(t)\,dB_1(t), \\ &d\log x_2(t) \\ &\quad = \left[r_2(t) - \frac{1}{2}\sigma_2^2(t) + a_{21}(t)x_1(t) - a_{22}(t)x_2(t)\right]dt \\ &\qquad + \sigma_2(t)\,dB_2(t). \end{aligned} \tag{92}$$

First, we prove (91). Integrating both sides of (92) from 0 to t yields

$$\begin{aligned} \log x_1(t) = {} & \log x_1(0) + \int_0^t \bar{r}_1(s)\,ds - \int_0^t a_{11}(s)x_1(s)\,ds \\ & + \int_0^t a_{12}(s)x_2(s)\,ds + \int_0^t \sigma_1(s)\,dB_1(s), \\ \log x_2(t) = {} & \log x_2(0) + \int_0^t \bar{r}_2(s)\,ds - \int_0^t a_{22}(s)x_2(s)\,ds \\ & + \int_0^t a_{21}(s)x_1(s)\,ds + \int_0^t \sigma_2(s)\,dB_2(s), \end{aligned} \tag{93}$$

where $\bar{r}_i(s) = r_i(s) - (1/2)\sigma_i^2(s),\ i = 1, 2$. Since $x_i(t) > 0,\ i = 1, 2$, hence

$$\begin{aligned} \log x_1(t) \leq {} & \log x_1(0) + \bar{r}_1^u t - a_{11}^l \int_0^t x_1(s)\,ds \\ & + a_{12}^u \int_0^t x_2(s)\,ds + \int_0^t \sigma_1(s)\,dB_1(s), \\ \log x_2(t) \leq {} & \log x_2(0) + \bar{r}_2^u t - a_{22}^l \int_0^t x_2(s)\,ds \\ & + a_{21}^u \int_0^t x_1(s)\,ds + \int_0^t \sigma_2(s)\,dB_2(s). \end{aligned} \tag{94}$$

So we have

$$\begin{aligned} a_{22}^{l}\log x_1(t)+a_{12}^{u}\log x_2(t) &\le a_{22}^{l}\left[\log x_1(0)+\int_0^t\sigma_1(s)\,dB_1(s)\right]+a_{22}^{l}\overline{r}_1^{u}t \\ &\quad +a_{12}^{u}\left[\log x_2(0)+\int_0^t\sigma_2(s)\,dB_2(s)\right]+a_{12}^{u}\overline{r}_2^{u}t \\ &\quad -\left(a_{11}^{l}a_{22}^{l}-a_{21}^{u}a_{12}^{u}\right)\int_0^t x_1(s)\,ds. \end{aligned} \tag{95}$$

By Theorem 16, we know that

$$\liminf_{t\to\infty}\frac{\log x_i(t)}{t}\ge 0,\quad i=1,2,\ \text{a.s.} \tag{96}$$

Obviously,

$$\lim_{t\to\infty}\frac{\log x_i(0)+\int_0^t\sigma_i(s)\,dB_i(s)}{t}=0,\quad i=1,2,\ \text{a.s.} \tag{97}$$

Hence, we have

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t x_1(s)\,ds\le\frac{a_{22}^{l}\overline{r}_1^{u}+a_{12}^{u}\overline{r}_2^{u}}{a_{11}^{l}a_{22}^{l}-a_{21}^{u}a_{12}^{u}}\triangleq\widehat{x}_1^{*},\quad \text{a.s.} \tag{98}$$

Similarly, we have

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t x_2(s)\,ds\le\frac{a_{11}^{l}\overline{r}_2^{u}+a_{12}^{u}\overline{r}_1^{u}}{a_{11}^{l}a_{22}^{l}-a_{21}^{u}a_{12}^{u}}\triangleq\widehat{x}_2^{*},\quad \text{a.s.} \tag{99}$$

Next, we prove that (90) is true. Taking integration both sides of (92) from 0 to t, we have

$$\begin{aligned} \log x_1(t) &\ge \log x_1(0)+\overline{r}_1^{l}t-a_{11}^{u}\int_0^t x_1(s)\,ds \\ &\quad +a_{12}^{l}\int_0^t x_2(s)\,ds+\int_0^t\sigma_1(s)\,dB_1(s), \\ \log x_2(t) &\ge \log x_2(0)+\overline{r}_2^{l}t-a_{22}^{u}\int_0^t x_2(s)\,ds \\ &\quad +a_{21}^{l}\int_0^t x_1(s)\,ds+\int_0^t\sigma_2(s)\,dB_2(s). \end{aligned} \tag{100}$$

By Theorem 16 we know that

$$\begin{aligned} \liminf_{t\to\infty}\frac{1}{t}\int_0^t x_1(s)\,ds &\ge \frac{\overline{r}_1^{l}}{a_{11}^{u}}\triangleq M_1,\quad \text{a.s.}, \\ \liminf_{t\to\infty}\frac{1}{t}\int_0^t x_2(s)\,ds &\ge \frac{\overline{r}_2^{l}}{a_{22}^{u}}\triangleq N_1,\quad \text{a.s.}, \end{aligned} \tag{101}$$

then for any $\varepsilon>0$, there is a $T(\omega)>0$ such that

$$\frac{1}{t}\int_0^t x_2(s)\,ds\ge N_1-\varepsilon, \tag{102}$$

for $t>T(\omega)$. It follows from (100) that, for $t>T(\omega)$,

$$\begin{aligned} \log x_1(t) &\ge \log x_1(0)+\overline{r}_1^{l}t-a_{11}^{u}\int_0^t x_1(s)\,ds \\ &\quad +a_{12}^{l}(N_1-\varepsilon)t+\int_0^t\sigma_1(s)\,dB_1(s) \\ &= \log x_1(0)-a_{11}^{u}\int_0^t x_1(s)\,ds \\ &\quad +\left[\overline{r}_1^{l}+a_{12}^{l}(N_1-\varepsilon)\right]t+\int_0^t\sigma_1(s)\,dB_1(s). \end{aligned} \tag{103}$$

From Lemma 17, we have

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t x_1(s)\,ds\ge\frac{\overline{r}_1^{l}+a_{12}^{l}(N_1-\varepsilon)}{a_{11}^{u}}:=M_2>M_1. \tag{104}$$

Similarly, we have

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t x_2(s)\,ds\ge\frac{\overline{r}_2^{l}+a_{21}^{l}(M_1-\varepsilon)}{a_{22}^{u}}:=N_2>N_1. \tag{105}$$

Continuing this process, we obtain two sequences M_n, N_n $(n=1,2,\ldots)$ such that

$$M_n=\frac{\overline{r}_1^{l}+a_{12}^{l}(N_{n-1}-\varepsilon)}{a_{11}^{u}}, \tag{106}$$

$$N_n=\frac{\overline{r}_2^{l}+a_{21}^{l}(M_{n-1}-\varepsilon)}{a_{22}^{u}}. \tag{107}$$

By induction, we can easily show that $M_{n+1}>M_n$, $N_{n+1}>N_n$, $n=1,2,\ldots$, that is, sequences $\{M_n,n=1,2,\ldots\}$ and $\{N_n,n=1,2,\ldots\}$ are nondecreasing. Moreover, note that (98) and (99), then the sequences $\{M_n,n=1,2,\ldots\}$ and $\{N_n,n=1,2,\ldots\}$, have upper bounds. Therefore, there are two positive M,N such that

$$\begin{gathered} \lim_{n\to\infty}M_n=M,\qquad \lim_{n\to\infty}N_n=N, \\ \liminf_{t\to\infty}\frac{1}{t}\int_0^t x_1(s)\,ds\ge M,\qquad \liminf_{t\to\infty}\frac{1}{t}\int_0^t x_2(s)\,ds\ge N, \end{gathered} \tag{108}$$

which together with (106) implies

$$\begin{aligned} a_{11}^{u}M-a_{12}^{l}N &= \overline{r}_1^{l}-\varepsilon a_{12}^{l}, \\ a_{22}^{u}N-a_{21}^{l}M &= \overline{r}_2^{l}-\varepsilon a_{21}^{l}. \end{aligned} \tag{109}$$

Letting $\varepsilon\to 0$ yields

$$\begin{aligned} M &= \frac{a_{22}^{u}\overline{r}_1^{l}+a_{12}^{l}\overline{r}_2^{l}}{a_{11}^{u}a_{22}^{u}-a_{12}^{l}a_{21}^{l}}\triangleq\widetilde{x}_1^{*}, \\ N &= \frac{a_{11}^{u}\overline{r}_1^{l}+a_{21}^{l}\overline{r}_2^{l}}{a_{11}^{u}a_{22}^{u}-a_{12}^{l}a_{21}^{l}}\triangleq\widetilde{x}_1^{*}. \end{aligned} \tag{110}$$

Hence,

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t x_i(s)\,ds \geq \tilde{x}_i^*, \quad i=1,2, \text{ a.s.,} \tag{111}$$

which is as required. □

4. Nonpersistence

In this section, we discuss the dynamics of system (3) as the white noise is getting larger. We show that system (3) will be nonpersistent if the white noise is large, which does not happen in the deterministic system.

Definition 19. System (3) is said to be nonpersistent, if there are positive constants q_1, q_2 such that

$$\lim_{t\to\infty}\prod_{i=1}^{2}x_i^{q_i}(t) = 0 \quad \text{a.s.} \tag{112}$$

Theorem 20. *Assume that $a_{11}^l a_{22}^l > a_{12}^u a_{21}^u$ and $a_{22}^l\bar{r}_1^u + a_{12}^u\bar{r}_2^u < 0$, then system (3) is nonpersistent, where $\bar{r}_i(s) = r_i(s) - (\sigma_i^2(s)/2)$, $i=1,2$.*

Proof. Since $x_i(t) > 0$, $i=1,2$ and $a_{11}^l a_{22}^l > a_{12}^u a_{21}^u$, from (93) we have

$$\begin{aligned}
&a_{22}^l\log x_1(t) + a_{12}^u\log x_2(t)\\
&\quad\leq \left\{a_{22}^l\bar{r}_1^u + a_{12}^u\bar{r}_2^u\right\}t - \left(a_{11}^l a_{22}^l - a_{21}^u a_{12}^u\right)\int_0^t x_1(s)\,ds\\
&\qquad + a_{22}^l\left[\log x_1(0) + \int_0^t \sigma_1(s)\,dB_1(s)\right]\\
&\qquad + a_{12}^u\left[\log x_2(0) + \int_0^t \sigma_2(s)\,dB_2(s)\right]\\
&\quad\leq K_1 t + a_{22}^l\left[\log x_1(0) + \int_0^t \sigma_1(s)\,dB_1(s)\right]\\
&\qquad + a_{12}^u\left[\log x_2(0) + \int_0^t \sigma_2(s)\,dB_2(s)\right],
\end{aligned} \tag{113}$$

where $K_1 = a_{22}^l\bar{r}_1^u + a_{12}^u\bar{r}_2^u$ which together with

$$\begin{aligned}
&\lim_{t\to\infty}\frac{a_{22}^l\left[\log x_1(0) + \int_0^t \sigma_1(s)\,dB_1(s)\right]}{t}\\
&\quad= \lim_{t\to\infty}\frac{a_{12}^u\left[\log x_2(0) + \int_0^t \sigma_2(s)\,dB_2(s)\right]}{t} = 0, \quad \text{a.s.,}
\end{aligned} \tag{114}$$

implies

$$\lim_{t\to\infty}\frac{1}{t}\left[a_{22}^l\log x_1(t) + a_{12}^u\log x_2(t)\right] \leq K_1, \quad \text{a.s.} \tag{115}$$

If $K_1 < 0$, then there must be

$$\lim_{t\to\infty}x_1^{a_{22}^l}(t)\,x_2^{a_{12}^u}(t) = 0, \quad \text{a.s.} \tag{116}$$

Hence, system (3) is nonpersistent. □

Theorem 21. *Assume that $a_{11}^l a_{22}^l > a_{12}^u a_{21}^u$ and $(a_{21}^u\bar{r}_1^u + a_{11}^l\bar{r}_2^u) < 0$, then system (3) is nonpersistent, where $\bar{r}_i(s) = r_i(s) - (\sigma_i^2(s)/2)$, $i=1,2$.*

Here we omit the proof of Theorem 21 which is similar to the proof of Theorem 20.

Remark 22. If $(\sigma_i^l)^2 > 2r_i^u$, $i=1,2$, then the conditions in Theorems 20 and 21 are obviously satisfied, respectively. That is to say, the large white noise will lead to the population system being non-persistent.

5. Global Attractivity

In this section, we turn to establishing sufficient criteria for the global attractivity of stochastic system (3).

Definition 23. Let $x(t)$, $y(t)$ be two arbitrary solutions of system (3) with initial values $x(0), y(0) \in R_+^2$, respectively. If

$$\lim_{t\to\infty}\left|x(t) - y(t)\right| = 0, \quad \text{a.s.,} \tag{117}$$

then we say system (3) is globally attractive.

Theorem 24. *Assume that $a_{11}^l a_{22}^l > a_{12}^u a_{21}^u$, then system (3) is globally attractive.*

Proof. Let $x(t)$, $y(t)$ be two arbitrary solutions of system (3) with initial values $x(0), y(0) \in R_+^2$. By the Itô's formula, we have

$$\begin{aligned}
d\log x_i(t) &= \left[r_i(t) - \frac{1}{2}\sigma_i^2(t) - a_{ii}(t)x_i(t) + a_{ij}(t)x_j(t)\right]dt\\
&\quad + \sigma_i(t)\,dB_i(t), \quad i,j=1,2,\ j\neq i,\\
d\log y_i(t) &= \left[r_i(t) - \frac{1}{2}\sigma_i^2(t) - a_{ii}(t)y_i(t) + a_{ij}(t)y_j(t)\right]dt\\
&\quad + \sigma_i(t)\,dB_i(t), \quad i,j=1,2,\ j\neq i.
\end{aligned} \tag{118}$$

Then,

$$\begin{aligned}
&d\left(\log x_i(t) - \log y_i(t)\right)\\
&\quad= \left\{-a_{ii}(t)\left[x_i(t) - y_i(t)\right] + a_{ij}(t)\left[x_i(t) - y_i(t)\right]\right\}dt,\\
&\qquad\qquad i,j=1,2,\ j\neq i.
\end{aligned} \tag{119}$$

Since $a_{11}^l a_{22}^l > a_{12}^u a_{21}^u$, there exist two positive constants c_1, c_2 which satisfy

$$\frac{a_{21}^u}{a_{11}^l} < \frac{c_1}{c_2} < \frac{a_{22}^l}{a_{12}^u}. \tag{120}$$

Thus, $c_1 a_{11}^l - c_2 a_{21}^u > 0$, $c_2 a_{22}^l - c_1 a_{12}^u > 0$.

Consider a Lyapunov function $V(t)$ defined by

$$\begin{aligned} V(t) &= c_1 \left|\log x_1(t) - \log y_1(t)\right| \\ &\quad + c_2 \left|\log x_2(t) - \log y_2(t)\right|, \quad t \geq 0. \end{aligned} \tag{121}$$

A direct calculation of the right differential $d^+V(t)$ of $V(t)$ along the ordinary differential equation (119) leads to

$$\begin{aligned} d^+V(t) &= c_1 \operatorname{sgn}(x_1(t) - y_1(t))\, d[\log x_1(t) - \log y_1(t)] \\ &\quad + c_2 \operatorname{sgn}(x_2(t) - y_2(t))\, d[\log x_2(t) - \log y_2(t)] \\ &= c_1 \operatorname{sgn}(x_1(t) - y_1(t)) \\ &\quad \times [-a_{11}(t)(x_1(t) - y_1(t))\, dt \\ &\quad\quad + a_{12}(t)(x_2(t) - y_2(t))\, dt] \\ &\quad + c_2 \operatorname{sgn}(x_2(t) - y_2(t)) \\ &\quad \times [a_{21}(t)(x_1(t) - y_1(t))\, dt \\ &\quad\quad - a_{22}(t)(x_2(t) - y_2(t))\, dt] \\ &\leq -c_1 a_{11}^l |x_1(t) - y_1(t)|\, dt \\ &\quad + c_1 a_{12}^u |x_2(t) - y_2(t)|\, dt \\ &\quad - c_2 a_{22}^l |x_2(t) - y_2(t)|\, dt \\ &\quad + c_2 a_{21}^u |x_1(t) - y_1(t)|\, dt \\ &= -\left(c_1 a_{11}^l - c_2 a_{21}^u\right) |x_1(t) - y_1(t)|\, dt \\ &\quad - \left(c_2 a_{22}^l - c_1 a_{12}^u\right) |x_2(t) - y_2(t)|\, dt \\ &\leq -\gamma \sum_{i=1}^{2} |x_i(t) - y_i(t)|\, dt, \end{aligned} \tag{122}$$

where $\gamma = \min\{c_1 a_{11}^l - c_2 a_{21}^u,\ c_2 a_{22}^l - c_1 a_{12}^u\}$. Integrating both sides of (122) form 0 to t, we have

$$V(t) + \gamma \int_0^t \sum_{i=1}^{2} |x_i(s) - y_i(s)|\, ds \leq V(0) < \infty. \tag{123}$$

Let $t \to \infty$, we obtain

$$\begin{aligned} \int_0^\infty |x(s) - y(s)|\, ds &\leq \int_0^\infty \sum_{i=1}^{2} |x_i(s) - y_i(s)|\, ds \\ &\leq \frac{V(0)}{\gamma} < \infty \quad \text{a.s.} \end{aligned} \tag{124}$$

Note that $u(t) = x(t) - y(t)$. Clearly, $u(t) \in C(R_+, R^2)$ a.s. It is straightforward to see from (124) that

$$\liminf_{t \to \infty} |u(t)| = 0 \quad \text{a.s.} \tag{125}$$

Next, we prove that

$$\lim_{t \to \infty} |u(t)| = 0 \quad \text{a.s.} \tag{126}$$

By Theorem 3 we obtain that the pth moment of the solution of system (3) is bounded, the following proof is similar to the proof of Theorem 6.2 in [15] and hence is omitted. □

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Research Article

Homoclinic Solutions for a Second-Order Nonperiodic Asymptotically Linear Hamiltonian Systems

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We establish a new existence result on homoclinic solutions for a second-order nonperiodic Hamiltonian systems. This homoclinic solution is obtained as a limit of solutions of a certain sequence of nil-boundary value problems which are obtained by the minimax methods. Some recent results in the literature are generalized and extended.

1. Introduction

Consider the following second-order Hamiltonian system:

$$\ddot{u}(t) + \nabla V(t, u(t)) = 0, \quad t \in \mathbb{R}, \tag{HS}$$

where $u = (u_1, u_2, \ldots, u_N) \in \mathbb{R}^N$, $V(t,u) = -K(t,u) + W(t,u)$, $K, W : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are C^1 maps. We will say that a solution $u : \mathbb{R} \to \mathbb{R}^N$ of (HS) is homoclinic (to 0), if $u(t) \to 0$, as $|t| \to \infty$. In addition, if $u \not\equiv 0$, then u is called a nontrivial homoclinic solution.

Inspired by the excellent monographs [1, 2], by now, the existence and multiplicity of homoclinic solutions for Hamiltonian systems have been extensively investigated in many papers via variational methods; see [3–7] for the first order systems and [8–19] for the second systems, and most of them treat the following system:

$$\ddot{u}(t) - L(t)u(t) + \nabla W(t, u(t)) = 0, \quad t \in \mathbb{R}, \tag{1}$$

where $L(t)$ is a symmetric matrix-valued function and $W \in C^1(\mathbb{R}, \mathbb{R}^N)$.

For the periodic case, the periodicity is used to control the lack of compactness due to the fact that (1) is set on all $\mathbb{R}$. In 1990, Rabinowitz [12] first proved that (1) has a $2kT$-periodic solution u_k, which is bounded uniformly for k, and obtained a homoclinic solution for (1) as a limit of $2kT$-periodic solution.

For the nonperiodic case, the problem is quite different from the one described in nature. Rabinowitz and Tanaka [13] introduced a type of coercive condition on the matrix L:

(L_1) $l(t) := \inf_{|x|=1} L(t)x \cdot x \to +\infty$, as $|t| \to \infty$.

They first obtained the existence of homoclinic solution for the nonperiodic system (1) under the well-known (AR) growth condition by using Ekeland's variational principle.

In 1995, Ding [8] strengthened condition (L_1) by

(L_2) there exists a constant $\alpha > 0$ such that

$$l(t)|t|^{-\alpha} \longrightarrow +\infty \quad \text{as } |t| \longrightarrow \infty. \tag{2}$$

Under the condition (L_2) and some subquadratic conditions on $W(t,u)$, Ding proved the existence and multiplicity of homoclinic solutions for the system (1). From then on, the condition (L_1) or (L_2) is extensively used in nonperiodic second-order Hamiltonian systems. However, the assumption (L_1) or (L_2) is a rather restrictive and not very natural condition as it excludes, for example, the case of constant matrices I_N.

In 2005, Izydorek and Janczewska [9] first presented the "pinching" condition (see the following (V_2)) and relaxed the conditions (L_1) and (L_2). They studied the general periodic Hamiltonian system

$$\ddot{u}(t) + \nabla V(t, u(t)) = f(t), \quad t \in \mathbb{R}, \tag{3}$$

where $V(t,u) = -K(t,u) + W(t,u)$ and obtained the following result.

Theorem A (see [9]). *Let the following conditions hold:*

(V_1) *$V(t,u) = -K(t,u) + W(t,u)$, where V is continuous and T periodic with respect to t, $T > 0$;*

(V_2) *there exist $b_1, b_2 > 0$ such that*

$$b_1|u|^2 \le K(t,u) \le b_2|u|^2, \quad \forall (t,u) \in \mathbb{R}\times\mathbb{R}^N; \tag{4}$$

(V_3) *$K(t,u) \le (u, \nabla K(t,u)) \le 2K(t,u)$ for all $(t,u) \in \mathbb{R}\times\mathbb{R}^N$;*

(V_4) *$\nabla W(t,u) = o(|u|)$ as $u \to 0$ uniformly in t;*

(V_5) *there is a constant $\mu > 2$ such that*

$$0 < \mu W(t,u) \le (\nabla W(t,u), u), \quad \forall (t,u) \in \mathbb{R}\times\mathbb{R}^N; \tag{5}$$

(V_6) *$\bar{b}_1 := \min\{1, 2b_1\} > 2M$ and $\|f\|_{L^2} < (\bar{b}_1 - 2M)/2C^*$, where $M = \sup\{W(t,u) : t \in [0,T], |u| = 1\}$ and C^* is a positive constant depending on T.*

Then the system (3) *possesses a nontrivial homoclinic solution $u \in W^{1,2}(\mathbb{R},\mathbb{R}^N)$ such that $\dot{u}(t) \to 0$ as $t \to \pm\infty$.*

From then on, following the idea of [9], some researchers are devoted to relaxing the conditions (L_1) and (L_2) and studying the existence of homoclinic solutions of system (HS) or (3) under the periodicity assumption of the potential, such as [10, 11, 16, 19].

Very recently, Daouas [3] removed the periodicity condition and studied the existence of homoclinic solutions for the nonperiodic system (3), when W is superquadratic at the infinity. Motivated by [3], in this work, we will study the existence of homoclinic solutions of the nonperiodic system (HS), when W satisfies the asymptotically quadratic condition at the infinity. It is worth noticing that there are few works concerning this case for system (HS) or (3) up to now.

Our result is presented as follows.

Theorem 1. *Let $A := \sup\{K(t,u) : t \in \mathbb{R}, |u| \le 1\} < +\infty$ hold. Moreover, assume that the following conditions hold:*

(H_1) *$K(t,0) \equiv 0$, and there exists a constant $a > 0$ such that*

$$K(t,u) \ge a|u|^2, \quad \forall (t,u) \in \mathbb{R}\times\mathbb{R}^N; \tag{6}$$

(H_2) *there exists $\beta \in (1,2]$ such that*

$$K(t,u) \le (u, \nabla K(t,u)) \le \beta K(t,u), \quad \forall (t,u) \in \mathbb{R}\times\mathbb{R}^N; \tag{7}$$

(H_3) *$W(t,0) \equiv 0$ and $\nabla W(t,u) = o(|u|)$ as $u \to 0$ uniformly in t, and there exist, $M_0 > 0$ such that*

$$\frac{|\nabla W(t,u)|}{|u|} \le M_0, \tag{8}$$

for any $t \in \mathbb{R}$ and $u \in \mathbb{R}^N$;

(H_4) *$W(t,u) - w(t)|u|^2 = o(|u|^2)$ as $|u| \to \infty$ uniformly in t, where $w \in L^\infty(\mathbb{R},\mathbb{R})$ with $w_\infty := \inf_{t\in\mathbb{R}} w(t) > 2A$;*

(H_5) *$\widetilde{W}(t,u) := (1/2)(\nabla W(t,u), u) - W(t,u) \to +\infty$ as $|u| \to +\infty$, and*

$$\inf\left\{\frac{\widetilde{W}(t,u)}{|u|^2} : t \in \mathbb{R} \text{ with } c \le |u| < d\right\} > 0, \tag{9}$$

for any $c, d > 0$.

Then the system (HS) possesses a nontrivial homoclinic solution $u \in W^{1,2}(\mathbb{R},\mathbb{R}^N)$ such that $\dot{u}(t) \to 0$ as $t \to \pm\infty$.

Remark 2. Theorem 1 treats the asymptotically quadratic case on W. Consider the functions

$$K(t,u) = \left(1 + e^{-|t|}\right)|u|^2,$$
$$W(t,u) = d(t)|u|^2\left(1 - \frac{1}{\ln(e + |u|)}\right), \tag{10}$$

where $d \in L^\infty(\mathbb{R},\mathbb{R})$ and $\inf_{t\in\mathbb{R}} d(t) > 4 + 32\pi^2$.

A straightforward computation shows that K and W satisfy the assumptions of Theorem 1, but K does not satisfy the conditions (L_1) and (L_2). Hence, Theorem 1 also extends the results in [8, 13].

The remainder of this paper is organized as follows. In Section 2, some preliminary results are presented. In Section 3, we give the proof of Theorem 1.

2. Preliminaries

Following the similar idea of [20], consider the following nil-boundary value problems:

$$\ddot{u}(t) + \nabla V(t, u(t)) = 0, \quad \forall t \in [-T,T],$$
$$u(-T) = u(T) = 0. \tag{11}$$

For each $T > 0$, let $E_T = W^{1,2}([-T,T],\mathbb{R}^N)$, where

$$W^{1,2}\left([-T,T],\mathbb{R}^N\right)$$
$$= \{u : [-T,T] \longrightarrow \mathbb{R}^N \text{ is an absolutely continuous function,}$$
$$u(-T) = u(T) = 0 \text{ and } \dot{u} \in L^2\left([-T,T],\mathbb{R}^N\right)\}, \tag{12}$$

equipped with the norm

$$\|u\| = \left(\int_{-T}^{T}\left[|\dot{u}(t)|^2 + |u(t)|^2\right]dt\right)^{1/2}. \tag{13}$$

Furthermore, for $p > 1$, let $L_T^p = L^p([-T,T],\mathbb{R}^N)$ and $L_T^\infty = L^\infty([-T,T],\mathbb{R}^N)$ under their habitual norms. We need the following result.

Proposition 3 (see [9]). *There is a positive constant C such that for each $T > 0$ and $u \in E_T$ the following inequality holds:*

$$\|u\|_{L_T^\infty} \le C\|u\|. \tag{14}$$

Note that the inequality (14) holds true with constant $C = \sqrt{2}$ if $T \geq 1/2$ (see [9]). Subsequently, we may assume this condition is fulfilled.

Consider a functional $I : E_T \to \mathbb{R}$ defined by

$$\begin{aligned} I(u) &= \int_{-T}^{T} \left[\frac{1}{2} |\dot{u}(t)|^2 - V(t, u(t)) \right] dt \\ &= \frac{1}{2} \int_{-T}^{T} |\dot{u}(t)|^2 dt + \int_{-T}^{T} K(t, u(t))\, dt \\ &\quad - \int_{-T}^{T} W(t, u(t))\, dt. \end{aligned} \tag{15}$$

Then $I \in C^1(E_T, \mathbb{R})$, and it is easy to show that for all $u, v \in E_T$, we have

$$\begin{aligned} I'(u)v &= \int_{-T}^{T} [(\dot{u}(t), \dot{v}(t)) - (W(t,u(t)), \nabla V(t, u(t)), v(t))]\, dt \\ &= \int_{-T}^{T} [(\dot{u}(t), \dot{v}(t)) + (\nabla K(t, u(t)), v(t)) \\ &\quad - (\nabla W(t, u(t)), v(t))]\, dt. \end{aligned} \tag{16}$$

It is well known that critical points of I are classical solutions of the problem (11). We will obtain a critical point of I by using an improved version of the Mountain Pass Theorem. For completeness, we give this theorem.

Recall that a sequence $\{u_j\}$ is a (C)-sequence for the functional φ if $\varphi(u_j)$ is bounded and $(1 + \|u_j\|)\varphi'(u_j) \to 0$. A functional φ satisfies the (C)-condition if and only if any (C)-sequence for φ contains a convergent subsequence.

Theorem 4 (see [21]). *Let E be a real Banach space, and let $\varphi \in C^1(E, \mathbb{R})$ satisfy the (C)-condition and $\varphi(0) = 0$. If φ satisfies the following conditions:*

(A_1) *there exist constants $\rho, \alpha > 0$ such that $\varphi|_{\partial B_\rho(0)} \geq \alpha$;*

(A_2) *there exists $e \in E \setminus \overline{B}_\rho(0)$ such that $\varphi(e) \leq 0$, then φ possesses a critical value $c \geq \alpha$ given by*

$$c = \inf_{g \in \Gamma} \max_{s \in [0,1]} \varphi(f(s)), \tag{17}$$

where $B_\rho(0)$ is an open ball in E of radius ρ at about 0, and

$$\Gamma = \{ f \in C([0,1], E) : f(0) = 0,\ f(1) = e \}. \tag{18}$$

Proof. As shown in Bartolo et al. [22], a deformation lemma can be proved with the (C)-condition replacing the usual (PS)-condition, and it turns out that the standard version Mountain Pass Theorem (see Rabinowitz [21]) holds true under the (C)-condition. □

Lemma 5. *Assume that (H_2) holds, then*

$$K(t, u) \leq K\left(t, \frac{u}{|u|}\right) |u|^\beta, \quad \forall t \in \mathbb{R},\ |u| \geq 1. \tag{19}$$

Proof. From (H_2) it follows that for $u \neq 0$ a map given by

$$(0, \infty) \ni \nu \mapsto W\left(t, \nu^{-1} u\right) \tag{20}$$

is nondecreasing. Similar to the proof in [12], we can get the conclusion. □

Lemma 6 (see [9]). *Let $u : \mathbb{R} \to \mathbb{R}^N$ be a continuous map such that $\dot{u}$ is locally square integrable. Then, for all $t \in \mathbb{R}$, one has*

$$|u(t)| \leq \sqrt{2} \left(\int_{t-1/2}^{t+1/2} \left(|u(s)|^2 + |\dot{u}(s)|^2 \right) ds \right)^{1/2}. \tag{21}$$

3. Proof of Theorem 1

Lemma 7. *Under the assumptions of Theorem 1, the problem (11) possesses a nontrivial solution.*

Proof. It suffices to prove that the functional I satisfies all the assumptions of Theorem 4.

Step 1. We show that the functional I satisfies the (C)-condition. Let

$$I\left(u_j\right) \text{ be bounded and } \left(1 + \left\|u_j\right\|\right) I'\left(u_j\right) \longrightarrow 0. \tag{22}$$

Observe that, for j large, it follows from (H_1) and (H_2) that there exists a constant C_0 such that

$$\begin{aligned} C_0 &\geq I\left(u_j\right) - \frac{1}{2} I'\left(u_j\right) u_j \\ &= \int_{-T}^{T} \left[\frac{1}{2} \left(\nabla W\left(t, u_j\right), u_j \right) - W\left(t, u_j\right) \right] dt \\ &\quad + \int_{-T}^{T} \left[K\left(t, u_j\right) - \frac{1}{2} \left(\nabla K\left(t, u_j\right), u_j \right) \right] dt \\ &\geq \int_{-T}^{T} \widetilde{W}(t, u)\, dt. \end{aligned} \tag{23}$$

Arguing indirectly, assume as a contradiction that $\|u_j\| \to \infty$. Setting $v_j = u_j / \|u_j\|$, then $\|v_j\| = 1$, and by Proposition 3, one has

$$\left\|v_j\right\|_{L_T^\infty} \leq \sqrt{2} \left\|v_j\right\| = \sqrt{2}. \tag{24}$$

Note that

$$\begin{aligned} I'\left(u_j\right) u_j &= \left\|\dot{u}_j\right\|_{L_T^2}^2 + \int_{-T}^{T} \left(\nabla K\left(t, u_j\right), u_j \right) dt \\ &\quad - \int_{-T}^{T} \left(\nabla W\left(t, u_j\right), u_j \right) dt \\ &\geq \left\|\dot{u}_j\right\|_{L_T^2}^2 + \int_{-T}^{T} K\left(t, u_j\right) dt \\ &\quad - \int_{-T}^{T} \left(\nabla W\left(t, u_j\right), u_j \right) dt \\ &\geq C_1 \left\|u_j\right\|^2 - \int_{-T}^{T} \left| \nabla W\left(t, u_j\right) \right| \left| u_j \right| dt, \end{aligned} \tag{25}$$

where $C_1 = \min\{1, a\} > 0$. This implies that

$$\int_{-T}^{T} \frac{\left|\nabla W\left(t, u_j\right)\right|\left|u_j\right|}{\left\|u_j\right\|^2} dt = \int_{-T}^{T} \frac{\left|\nabla W\left(t, u_j\right)\right|\left|v_j\right|^2}{\left|u_j\right|} dt \longrightarrow C_1. \tag{26}$$

Set for $s \geq 0$

$$h(s) := \inf\left\{\widetilde{W}(t, u) : t \in [-T, T],\ u \in \mathbb{R}^N \text{ with } |u| \geq s\right\}. \tag{27}$$

By (H_5), $h(s) \to \infty$ as $s \to \infty$.

For $0 \leq l < m$, let

$$\Omega_j(l, m) = \left\{t \in [-T, T] : l \leq \left|u_j(t)\right| < m\right\},$$
$$C_l^m = \inf\left\{\frac{\widetilde{W}(t, u)}{|u|^2} : t \in [-T, T] \text{ with } l \leq |u(t)| < m\right\}. \tag{28}$$

Then by (H_5), $C_l^m > 0$. One has

$$\widetilde{W}\left(t, u_j\right) \geq C_l^m\left|u_j\right|^2, \quad \forall t \in \Omega_j(l, m). \tag{29}$$

It follows from (23) that

$$\begin{aligned} C_0 \geq & \int_{\Omega_j(0,l)} \widetilde{W}\left(t, u_j\right) dt + \int_{\Omega_j(l,m)} \widetilde{W}\left(t, u_j\right) dt \\ & + \int_{\Omega_j(m,\infty)} \widetilde{W}\left(t, u_j\right) dt \\ \geq & \int_{\Omega_j(0,l)} \widetilde{W}\left(t, u_j\right) dt + C_l^m \int_{\Omega_j(l,m)} \left|u_j\right|^2 dt \\ & + h(m)\left|\Omega_j(m, \infty)\right| \end{aligned} \tag{30}$$

which implies that

$$\left|\Omega_j(m, \infty)\right| \leq \frac{C_0}{h(m)} \longrightarrow 0 \tag{31}$$

as $m \to \infty$ uniformly in j, and for any fixed $0 < l < m$

$$\int_{\Omega_j(l,m)} \left|v_j\right|^2 dt = \frac{1}{\left\|u_j\right\|^2} \int_{\Omega_j(l,m)} \left|u_j\right|^2 dt \leq \frac{C_0}{C_l^m\left\|u_j\right\|^2} \longrightarrow 0 \tag{32}$$

as $j \to \infty$. Using (14) and (31), we have

$$\int_{\Omega_j(m,\infty)} \left|v_j\right|^2 dt \leq \left\|v_j\right\|_{L_T^\infty}^2 \cdot \left|\Omega_j(m, \infty)\right| \leq 2\left|\Omega_j(m, \infty)\right| \longrightarrow 0 \tag{33}$$

as $m \to \infty$ uniformly in j.

Let $0 < \epsilon < C_1/3$. By (H_3) there is $l_\epsilon > 0$ such that

$$|\nabla W(t, u)| < \frac{\epsilon}{4T}|u| \tag{34}$$

for all $|t| \leq l_\epsilon$. Consequently,

$$\begin{aligned} \int_{\Omega_j(0,l_\epsilon)} \frac{\left|\nabla W\left(t, u_j\right)\right|\left|v_j\right|^2}{\left|u_j\right|} dt & \leq \frac{\epsilon}{4T} \int_{\Omega_j(0,l_\epsilon)} \left|v_j\right|^2 dt \\ & \leq \frac{\epsilon}{4T}\left\|v_j\right\|_{L_T^\infty}^2 2T < \epsilon \end{aligned} \tag{35}$$

for all j.

By (31), we can take m_ϵ large such that

$$\int_{\Omega_j(m_\epsilon,\infty)} \left|v_j\right|^2 dt < \frac{\epsilon}{M_0}. \tag{36}$$

Hence, by (H_3) one has

$$\int_{\Omega_j(m_\epsilon,\infty)} \frac{\left|\nabla W\left(t, u_j\right)\right|\left|v_j\right|^2}{\left|u_j\right|} dt \leq M_0 \int_{\Omega_j(m_\epsilon,\infty)} \left|v_j\right|^2 dt < \epsilon \tag{37}$$

for all j. By (32) there exists j_0 such that

$$\int_{\Omega_j(l_\epsilon,m_\epsilon)} \frac{\left|\nabla W\left(t, u_j\right)\right|\left|v_j\right|^2}{\left|u_j\right|} dt \leq M_0 \int_{\Omega_j(l_\epsilon,m_\epsilon)} \left|v_j\right|^2 dt < \epsilon \tag{38}$$

for all $j \geq j_0$. By (35)–(38), one has

$$\limsup_{j\to\infty} \int_{-T}^{T} \frac{\left|\nabla W\left(t, u_j\right)\right|\left|v_j\right|^2}{\left|u_j\right|} dt \leq 3\epsilon < C_1 \tag{39}$$

which contradicts with (26). So $\{u_j\}$ is bounded in E_T. In a similar way to Proposition B. 35 in [21], we can prove that $\{u_j\}$ has a convergent subsequence. Hence I satisfies the (C)-condition.

Step 2. We show that the functional I satisfies the condition (A_1) of Theorem 4.

Observe that, by (H_3) and (H_4), given $0 < \epsilon < a$, there exists some $C_\epsilon > 0$ such that

$$|W(t, u)| \leq \varepsilon|u|^2 + C_\varepsilon|u|^p \tag{40}$$

for all $u \in \mathbb{R}^N$ and $t \in [-T, T]$, where $p > 2$. It follows from (H_1), (40), and Proposition 3 that

$$\begin{aligned} I(u) = & \frac{1}{2}\int_{-T}^{T} |\dot{u}(t)|^2 dt + \int_{-T}^{T} K(t, u(t)) dt \\ & - \int_{-T}^{T} W(t, u(t)) dt \\ \geq & \frac{1}{2}\|\dot{u}\|_{L_T^2}^2 + a\|u\|_{L_T^2}^2 - \epsilon\|u\|_{L_T^2}^2 - C_\epsilon \int_{-T}^{T} |u(t)|^p dt \\ \geq & \frac{1}{2}\|\dot{u}\|_{L_T^2}^2 + a\|u\|_{L_T^2}^2 - \epsilon\|u\|_{L_T^2}^2 - 2TC_\epsilon\|u\|_{L_T^\infty}^p \\ \geq & \min\left\{\frac{1}{2}, a - \epsilon\right\}\|u\|^2 - 2^{p/2+1}TC_\epsilon\|u\|^p. \end{aligned} \tag{41}$$

Hence there exist $\alpha > 0$ and $\rho > 0$ such that $I(u) \geq \alpha$ for all $u \in E_T$ with $\|u\| = \rho$.

Step 3. We show that the functional I satisfies the condition (A_2) of Theorem 4.

By (H_4), there exists $B > 0$ such that

$$W(t,u) \geq w_\infty |u|^2 - B, \quad \forall t \in [-T,T], \ u \in \mathbb{R}^N. \tag{42}$$

Let

$$e(t) = \zeta |\sin(\omega t)| e_0, \quad t \in [-T,T], \tag{43}$$

where $\omega = 2\pi/T$ and $e_0 = (1,0,\ldots,0)$. Clearly, $e \in E_T$. By (15), (42), and Lemma 5, one has

$$\begin{aligned} I(e) &= \frac{\zeta^2\omega^2}{2}\int_{-T}^{T} |\cos(\omega t)|^2 dt + \int_{\{t\in[-T,T];|e(t)|\leq 1\}} K(t,e(t))\,dt \\ &\quad + \int_{\{t\in[-T,T];|e(t)|\geq 1\}} K(t,e(t))\,dt - \int_{-T}^{T} W(t,e(t))\,dt \\ &\leq \frac{T\zeta^2\omega^2}{2} + 2TA + A\int_{\{t\in[-T,T];|e(t)|\geq 1\}} |e(t)|^\beta dt \\ &\quad - w_\infty \zeta^2 \int_{-T}^{T} |\sin(\omega t)|^2 dt + 2TB \\ &= \frac{T\zeta^2\omega^2}{2} + 2TA + A\zeta^2 \int_{-T}^{T} |\sin(\omega t)|^2 dt - T w_\infty \zeta^2 \\ &\quad + 2TB \\ &= T\left(\frac{\omega^2}{2} + A - w_\infty\right)\zeta^2 + 2T(A+B). \end{aligned} \tag{44}$$

Since $w_\infty > 2A + 32\pi^2$ and $T > \sqrt{2/A}\pi$, then $\omega^2/2 + A - w_\infty < 0$. So $I(e) \to -\infty$ as $|\zeta| \to \infty$. So, we can choose large enough $\zeta \in \mathbb{R}$ such that $\|e\| > \rho$ and $I(e) < 0$.

Clearly $I(0) = 0$; then, by application of Theorem 4, there exists a critical point $u_T \in E_T$ of I such that $I(u_T) \geq \alpha$ for all $T > \sqrt{2/A}\pi$. □

Lemma 8. *u_T is bounded uniformly in $T > \sqrt{2/A}\pi$.*

Proof. Define the set of paths

$$\Gamma_T = \{f \in C([0,1], E_T) \mid f(0) = 0, f(1) = e\}. \tag{45}$$

It follows from Lemma 7 that there exists a solution u_T of problem (11) at which

$$\inf_{f\in\Gamma_T} \max_{s\in[0,1]} I(f(s)) \equiv D_T \tag{46}$$

is achieved. Let $\overline{T} > T$. Since any function in E_T can be regarded as belonging to $E_{\overline{T}}$ if one extends it by zero in $[-\overline{T},\overline{T}] \setminus [-T,T]$, then $\Gamma_T \subset \Gamma_{\overline{T}}$. Therefore, for any solution u_T of problem (11), we obtain

$$I(u_T) = D_T \leq D_{1/2} \quad \text{uniformly in} \quad T > \sqrt{\frac{2}{A}}\pi. \tag{47}$$

Notice that $I'(u_T) = 0$, and together with (47), one has

$$I(u_T) \leq D_{1/2}, \qquad (1 + u_T)\left\|I'(u_T)\right\| = 0. \tag{48}$$

The rest of the proof is similar to that of Step 1 in Lemma 7. Hence there exists a constant $M_1 > 0$, independent of T such that

$$\|u_T\| \leq M_1, \quad \forall T > \sqrt{\frac{2}{A}}\pi. \tag{49}$$

The proof is complete. □

Take a sequence $T_n \to \infty$, and consider the problem (11) on the interval $[-T_n, T_n]$. By Lemma 7, there exists a nontrivial solution $u_n := u_{T_n}$ of problem (11).

Lemma 9. *Let $\{u_n\}_{n\in\mathbb{N}}$ be the sequence given above. Then there exists a subsequence $\{u_{n_j}\}_{j\in\mathbb{N}}$ convergent to u_0 in $C^1_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^N)$.*

Proof. First we prove that the sequences $\|u_n\|_{L^\infty_{T_n}}$, $\|\dot{u}_n\|_{L^\infty_{T_n}}$, and $\|\ddot{u}_n\|_{L^\infty_{T_n}}$ are bounded. From (14) and (49), for n large enough, one has

$$\|u_n\|_{L^\infty_{T_n}} \leq CM_1 := M_2. \tag{50}$$

By (11) and (50), for all $t \in [-T_n, T_n]$, there exists $M_3 > 0$ independent of n such that

$$\|\ddot{u}_n\|_{L^\infty_{T_n}} \leq M_3. \tag{51}$$

It follows from the Mean Value Theorem that for every $n \in N$ and $t \in \mathbb{R}$, there exists $\tau_n \in [t-1, t]$ such that

$$\dot{u}_n(\tau_n) = \int_{t-1}^{t} \dot{u}_n(s)\,ds = u_n(t) - u_n(t-1). \tag{52}$$

Combining the above with (50), and (51) we get

$$\begin{aligned} |\dot{u}_n(t)| &= \left|\int_{\tau_n}^{t} \ddot{u}_n(s)\,ds + \dot{u}_n(\tau_n)\right| \\ &\leq \int_{t-1}^{t} |\ddot{u}_n(s)|\,ds + |u_n(t) - u_n(t-1)| \\ &\leq M_3 + 2M_2 := M_4 \end{aligned} \tag{53}$$

and hence for n large enough

$$\|\dot{u}_n\|_{L^\infty_{T_n}} \leq M_4. \tag{54}$$

Second we show that the sequences $\{u_n\}_{n\in\mathbb{N}}$ and $\{\dot{u}_n\}_{n\in\mathbb{N}}$ are equicontinuous. Indeed, for any $n \in \mathbb{N}$ and $t_1, t_2 \in \mathbb{R}$, by (54), we have

$$\begin{aligned} |u_n(t_1) - u_n(t_2)| &= \left|\int_{t_2}^{t_1} \dot{u}_n(s)\,ds\right| \leq \int_{t_2}^{t_1} |\dot{u}_n(s)|\,ds \\ &\leq M_3 |t_1 - t_2|. \end{aligned} \tag{55}$$

Similarly, by (51), one gets

$$\left|\dot{u}_n(t_1) - \dot{u}_n(t_2)\right| \le M_2 \left|t_1 - t_2\right|. \tag{56}$$

By using the Arzelà-Ascoli Theorem, we obtain the existence of a subsequence $\{u_{n_j}\}_{j\in\mathbb{N}}$ and a function u_0 such that

$$u_{n_j} \longrightarrow u_0, \quad \text{as } j \longrightarrow \infty \text{ in } C^1_{\text{loc}}\left(\mathbb{R}, \mathbb{R}^N\right). \tag{57}$$

The proof is complete. □

Lemma 10. *Let $u_0 : \mathbb{R} \to \mathbb{R}^N$ be the function given by* (57). *Then u_0 is the homoclinic solution of* (*HS*).

Proof. First we show that u_0 is a solution of (HS). Let $\{u_{n_j}\}_{j\in\mathbb{N}}$ be the sequence given by Lemma 9, then

$$\ddot{u}_{n_j}(t) + \nabla V\left(t, u_{n_j}(t)\right) = 0 \tag{58}$$

for every $j \in \mathbb{N}$ and $t \in [-T_{n_j}, T_{n_j}]$. Take $b, c \in \mathbb{R}$ with $b < c$. There exists $j_0 \in \mathbb{N}$ such that for all $j > j_0$; we get $[b, c] \subset [-T_{n_j}, T_{n_j}]$ and

$$\ddot{u}_{n_j}(t) = -\nabla V\left(t, u_{n_j}(t)\right), \quad \forall t \in [b, c]. \tag{59}$$

Integrating (59) from b to $t \in [b, c]$, we have

$$\dot{u}_{n_j}(t) - \dot{u}_{n_j}(b) = -\int_b^t \nabla V\left(s, u_{n_j}(s)\right) ds, \quad \forall t \in [b, c]. \tag{60}$$

Since $u_{n_j} \to u_0$ uniformly on $[b, c]$ and $\dot{u}_{n_j} \to \dot{u}_0$ uniformly on $[b, c]$ as $j \to \infty$, then, from (60), we obtain

$$\dot{u}_0(t) - \dot{u}_0(b) = -\int_b^t \nabla V\left(s, u_0(s)\right) ds, \quad \forall t \in [b, c]. \tag{61}$$

Because of the arbitrariness of b and c, we conclude that u_0 satisfies (HS).

Second we prove that $u_0(t) \to 0$, as $|t| \to \infty$. Note that, by (49), for $k \in \mathbb{N}$, there exists $j_0 \in \mathbb{N}$ such that, for all $j > j_0$, one has

$$\int_{-T_{n_j}}^{T_{n_j}} \left[\left|u_{n_j}(t)\right|^2 + \left|\dot{u}_{n_j}(t)\right|^2\right] dt \le \left\|u_{n_j}\right\|^2 \le M_1^2. \tag{62}$$

Letting $j \to \infty$, one gets

$$\int_{-T_{n_j}}^{T_{n_j}} \left[\left|u_0(t)\right|^2 + \left|\dot{u}_0(t)\right|^2\right] dt \le M_1^2 \tag{63}$$

and letting $j \to \infty$, we have

$$\int_{-\infty}^{+\infty} \left[\left|u_0(t)\right|^2 + \left|\dot{u}_0(t)\right|^2\right] dt \le M_1^2 \tag{64}$$

and so

$$\int_{|t|\ge r} \left[\left|u_0(t)\right|^2 + \left|\dot{u}_0(t)\right|^2\right] dt \longrightarrow 0. \tag{65}$$

From Lemma 6 and (65), we obtain $u_0(t) \to 0$ as $|t| \to \infty$.

Next we show that $\dot{u}_0(t) \to 0$ as $|t| \to \infty$. Indeed, applying again Lemma 6 to $\dot{u}_0$, we obtain

$$\left|\dot{u}_0(t)\right| \le \sqrt{2}\left(\int_{t-1/2}^{t+1/2} \left(\left|\dot{u}_0(s)\right|^2 + \left|\ddot{u}_0(s)\right|^2\right) ds\right)^{1/2}. \tag{66}$$

Also, from (65), we get

$$\int_{t-1/2}^{t+1/2} \left|\dot{u}_0(s)\right|^2 ds \longrightarrow 0, \quad \text{as } |t| \longrightarrow \infty. \tag{67}$$

Hence, it is enough to prove that

$$\int_{t-1/2}^{t+1/2} \left|\ddot{u}_0(s)\right|^2 ds \longrightarrow 0, \quad \text{as } |t| \longrightarrow \infty. \tag{68}$$

Since u_0 is a solution of (HS), one has

$$\int_{t-1/2}^{t+1/2} \left|\ddot{u}_0(s)\right|^2 ds = \int_{t-1/2}^{t+1/2} \left|\nabla V\left(s, u_0(s)\right)\right|^2 ds. \tag{69}$$

Since $\nabla V(t, 0) = 0$ for all $t \in \mathbb{R}$ and $u_0(t) \to 0$, as $|t| \to \infty$, (68) follows from (69).

Finally, similar to the proof in [12], we can prove that u_0 is nontrivial, and we omit it here. The proof of Theorem 1 is complete. □

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Research Article

Positive Periodic Solution for Second-Order Singular Semipositone Differential Equations

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We study the existence of a positive periodic solution for second-order singular semipositone differential equation by a nonlinear alternative principle of Leray-Schauder. Truncation plays an important role in the analysis of the uniform positive lower bound for all the solutions of the equation. Recent results in the literature (Chu et al., 2010) are generalized.

1. Introduction

In this paper, we study the existence of positive T-periodic solutions for the following singular semipositone differential equation:

$$x'' + h(t)x' + a(t)x = f(t, x, x'), \tag{1}$$

where $h, a \in C(R/TZ, R)$ and the nonlinearity $f \in C((R/TZ)\times(0,+\infty)\times R, R)$ satisfies $f(t,x,x') \geq -M$ for some $M > 0$. In particular, the nonlinearity may have a repulsive singularity at $x = 0$, which means that

$$\lim_{x\to 0^+} f(t,x,y) = +\infty, \quad \text{uniformly in } (t,y) \in R^2. \tag{2}$$

Electrostatic or gravitational forces are the most important examples of singular interactions.

During the last two decades, the study of the existence of periodic solutions for singular differential equations has attracted the attention of many researchers [1–4]. Some strong force conditions introduced by Gordon [5] are standard in the related earlier works [6, 7]. Compared with the case of a strong singularity, the study of the existence of periodic solutions under the presence of a weak singularity is more recent [2, 8, 9], but has also attracted many researchers. Some classical tools have been used to study singular differential equations in the literature, including the method of upper and lower solutions [10], degree theory [11], some fixed point theorem in cones for completely continuous operators [12], Schauder's fixed point theorem [8, 9, 13], and a nonlinear Leray-Schauder alternative principle [2, 3, 14, 15].

However the singular differential equations, in which there is the damping term, that is, the nonlinearity is dependent on the derivative, has not attracted much attention in the literature. Several existence results can be found in [14, 16, 17].

The aim of this paper is to further show that the nonlinear Leray-Schauder alternative principle can be applied to (1) in the semipositone cases, that is, $f(t,x,x') \geq -M$ for some $M > 0$.

The remainder of the paper is organized as follows. In Section 2, we state some known results. In Section 3, the main results of this paper are stated and proved. To illustrate our result, we select the following system:

$$x'' + h(t)x' + a(t)x = (1+|x|^{\gamma})(x^{-\alpha} + \mu x^{\beta}) + e(t), \tag{3}$$

where $\alpha > 1$, $\beta > 0$, $1 > \gamma \geq 0$, $\mu > 0$ is a positive parameter, $e(t)$ is a T-periodic function.

In this paper, let us fix some notations to be used in the following: given $\varphi \in L^1[0,T]$, we write $\varphi \succ 0$ if $\varphi \geq 0$ for almost everywhere $t \in [0,T]$ and it is positive in a set of positive measure. The usual L^p-norm is denoted by $\|\cdot\|_p$. p^* and p_* the essential supremum and infinum of a given function $p \in L^1[0,T]$, if they exist.

2. Preliminaries

We say that

$$x'' + h(t)x' + a(t)x = 0, \tag{4}$$

associated to the periodic boundary conditions

$$x(0) = x(T), \qquad x'(0) = x'(T), \tag{5}$$

is nonresonant when its unique solutions is the trival one. When (4)-(5) is nonresonant, as a consequence of Fredholm's alternative, the nonhomogeneous equation

$$x'' + h(t)x' + a(t)x = l(t) \tag{6}$$

admits a unique T-periodic solution, which can be written as

$$x(t) = \int_0^T G(t,s)\,l(s)\,ds, \tag{7}$$

where $G(t,s)$ is the Green's function of problem (4)-(5). Throughout this paper, we assume that the following standing hypothesis is satisfied.

(A) The Green function $G(t,s)$, associated with (4)-(5), is positive for all $(t,s) \in [0,T] \times [0,T]$.

In other words, the strict antimaximum principle holds for (4)-(5).

Definition 1. We say that (4) admits the antimaximum principle if (6) has a unique T-periodic solution for any $l \in \mathbb{C}(\mathbb{R}/T\mathbb{Z})$ and the unique T-periodic solution $x_l(t) > 0$ for all t if $l \succ 0$.

Under hypothesis (A), we denote

$$A = \min_{0\le s,t\le T} G(t,s), \qquad B = \max_{0\le s,t\le T} G(t,s), \qquad \imath = \frac{A}{B}. \tag{8}$$

Thus $B > A > 0$ and $0 < \imath < 1$. We also use $w(t)$ to denote the unique periodic solution of (6) with $l(t) = 1$ under condition (5), that is, $w(t) = (\mathscr{L}1)(t)$. In particular, $TA \le w(t) \le TB$.

With the help of [18, 19], the authors give a sufficient condition to ensure that (4) admits the antimaximum principle in [14]. In order to state this result, let us define the functions

$$\begin{gathered}\sigma(h)(t) = \exp\left(\int_0^t h(s)\,ds\right),\\ \sigma_1(h)(t) = \sigma(h)(T)\int_0^t \sigma h(s)\,ds + \int_t^T \sigma(h)(s)\,ds.\end{gathered} \tag{9}$$

Lemma 2 (see [14, Corollary 2.6]). *Assume that $a \not\equiv 0$ and the following two inequalities are satisfied:*

$$\begin{gathered}\int_0^T a(s)\,\sigma(h)(s)\,\sigma_1(-h)(s)\,ds \ge 0,\\ \sup_{0\le t\le T}\left\{\int_t^{t+T}\sigma(-h)(s)\,ds\int_t^{t+T}[a(s)]_+\sigma(h)(s)\,ds\right\} \le 4,\end{gathered} \tag{10}$$

where $[a(s)]_+ = \max\{a(s),0\}$. Then the Green's function $G(t,s)$, associated with (5)*, is positive for all $(t,s) \in [0,T]\times[0,T]$.*

Next, recall a well-known nonlinear alternative principle of Leray-Schauder, which can be found in [20] and has been used by Meehan and O'Regan in [4].

Lemma 3. *Assume Ω is an open subset of a convex set K in a normed linear space X and $p \in \Omega$. Let $T : \overline{\Omega} \to K$ be a compact and continuous map. Then one of the following two conclusions holds:*

(I) *T has at least one fixed point in $\overline{\Omega}$.*

(II) *There exists $x \in \partial\Omega$ and $0 < \lambda < 1$ such that $x = \lambda Tx + (1-\lambda)p$.*

In applications below, we take $K = C_T^1 = \{x : x, x' \in C(R/TZ, R)\} \subset X$ with the norm $\|x\| = \max_{t\in[0,T]}|x(t)|$ and define $\Omega = \{x \in C_T^1 : \|x\| < r\}$.

3. Main Results

In this section, we prove a new existence result of (1).

Theorem 4. *Suppose that* (4) *satisfies (A) and*

$$a(t) \succ 0. \tag{11}$$

Furthermore, assume that there exist three constants $M, R_0, r > Mw^*/\imath$ such that:

(H_1) *$F(t,x,y) = f(t,x,y) + M \ge 0$ for all $(t,x,y) \in [0,T] \times (0,r] \times R$.*

(H_2) *$f(t,x,y) \ge g_0(x)$ for $(t,x,y) \in [0,T] \times (0,R_0] \times R$, where the nonincreasing continuous function $g_0(x) > 0$ satisfies $\lim_{x\to 0^+} g_0(x) = +\infty$ and $\lim_{x\to 0^+}\int_x^{R_0} g_0(u)du = +\infty$.*

(H_3) *$0 \le F(t,x,y) \le (g(x)+h(x))\varrho(|y|)$, for all $(t,x,y) \in [0,T]\times(0,r]\times R$, where $g(\cdot) > 0$ is nonincreasing in $(0,r]$ and $h(\cdot)/g(\cdot) \ge 0$, $\varrho(\cdot) \ge 0$ are nondecreasing in $(0,r]$.*

(H_4)

$$\frac{r}{g(\imath r - Mw^*)\left(1 + h(r)/g(r)\right)\varrho((r+M)L)} > w^*, \tag{12}$$

where

$$L = \frac{2\int_0^T a(t)\,\sigma(h)(t)\,dt}{\min_{0\le t\le T}\sigma(h)(t)}. \tag{13}$$

Then (1) *has at least one positive periodic solution $u(t)$ with $0 < \|u + Mw\| \le r$.*

Proof. For convinence, let us write $Z(t) = x(t) - Mw(t)$, $Z_n(t) = x_n(t) - Mw(t)$, where $w(t) = (\mathscr{L}1)(t)$. Let

$$A_h = \min_{0\le t\le T}\sigma(h)(t), \qquad B_h = \max_{0\le t\le T}\sigma(h)(t), \qquad \imath_h = \frac{B_h}{A_h}, \tag{14}$$

$$\overline{M}_h = (r+M)L\cdot\max_{0\le t\le T}|h(t)|. \tag{15}$$

First we show that

$$x'' + h(t)x' + a(t)x = F\left(t, Z(t), Z'(t)\right) \tag{16}$$

has a solution x satisfying (5), $0 < \|x\| \le r$ and $Z(t) > 0$ for $t \in [0,T]$. If this is true, it is easy to see that $Z(t)$ will be a positive solution of (1)–(5) with $0 < \|Z + Mw\| \le r$.

Choose $n_0 \in \{1,2,\ldots\}$ such that $1/n_0 < r$, and then let $N_0 = \{n_0, n_0+1+\cdots\}$.

Consider the family of equations

$$x'' + h(t)x' + a(t)x = \lambda F_n\left(t, Z(t), Z'(t)\right) + \frac{a(t)}{n}, \tag{17}$$

where $\lambda \in [0,1]$, $n \in N_0$, $x \in B_r = \{x : \|x\| < r\}$ and $F_n(t,x,y) = F(t, \max\{1/n, x\}, y)$.

A T-periodic solution of (17) is just a fixed of the operator equation

$$x = \lambda T_n(x) + (1-\lambda)p, \tag{18}$$

where $p = 1/n$ and T_n is a completely continuous operator defined by

$$(T_n x)(t) = \int_0^T G(t,s) F_n\left(s, Z(s), Z'(s)\right) ds + \frac{1}{n}, \tag{19}$$

where we have used the fact

$$\int_0^T G(t,s)a(s)\,ds \equiv 1. \tag{20}$$

We claim that for any T-periodic solution $x_n(t)$ of (17) satisfies

$$\left\|x_n'\right\| \le Lr. \tag{21}$$

Note that the solution $x_n(t)$ of (17) is also satisfies the following equivalent equation

$$\begin{aligned}&\left(\sigma(h)(t)x_n'\right)' + a(t)\sigma(h)(t)x_n\\ &\quad= \sigma(h)(t)\left(\lambda F_n\left(t, Z_n(t), Z_n'(t)\right) + \frac{a(t)}{n}\right).\end{aligned} \tag{22}$$

Integrating (22) from 0 to T, we obtain

$$\begin{aligned}&\int_0^T a(t)\sigma(h)(t)x_n(t)\,dt\\ &\quad= \int_0^T \sigma(h)(t)\left(\lambda F_n\left(t, Z_n(t), Z_n'(t)\right) + \frac{a(t)}{n}\right)dt.\end{aligned} \tag{23}$$

By the periodic boundary conditions, we have $x'(t_0) = 0$ for some $t_0 \in [0,T]$. Therefore,

$$\begin{aligned}&\left|\sigma(h)(t)x_n'(t)\right|\\ &= \left|\int_{t_0}^t \left(\sigma(h)(s)x_n'(s)\right)' ds\right|\\ &= \left|\int_{t_0}^t \sigma(h)(s)\left(\lambda F_n\left(s, Z_n(s), Z_n'(s)\right) + \frac{a(s)}{n} - a(s)x_n(s)\right)\right| ds\\ &\le \left|\int_0^T \sigma(h)(s)\left(\lambda F_n\left(s, Z_n(s), Z_n'(s)\right) + \frac{a(s)}{n} + a(s)x_n(s)\right)\right| ds\\ &= 2\int_0^T \sigma(h)(s)a(s)x_n(s)\,ds\\ &\le 2r\int_0^T \sigma(h)(s)a(s)\,ds,\end{aligned} \tag{24}$$

where we have used the assumption (11) and $\|x_n\| < r$. Therefore,

$$\left(\min_{0\le t\le T}\sigma(h)(t)\right)\left|x_n'(t)\right| \le 2r\int_0^T \sigma(h)(s)a(s)\,ds, \tag{25}$$

which implies that (21) holds. In particular, let $\lambda F_n(t, Z(t), Z'(t)) + a(t)/n = 1$ in (17), we have

$$\left\|w'(t)\right\| \le L. \tag{26}$$

Choose $n_1 \in N_0$ such that $1/n_1 \le R_1$, and then let $N_1 = \{n_1, n_1+1, \ldots\}$. The following lemma holds. □

Lemma 5. *There exists an integer $n_2 > n_1$ large enough such that, for all $n \in N_2 = \{n_2, n_2+1, \ldots\}$,*

$$Z_n(t) = x_n(t) - Mw(t) \ge \frac{1}{n}. \tag{27}$$

Proof. The lower bound in (27) is established by using the strong force condition of $f(t,x,y)$. By condition (H_2), there exists $R_1 \in (0, R_0)$ and a continuous function $\tilde{g}_0(x)$ such that

$$F(t,x,y) - a(t)x \ge \tilde{g}_0(x) > \max\left\{M + \overline{M}, \iota_h r\|a\|_1\right\} \tag{28}$$

for all $(t,x,y) \in [0,T] \times (0,R_1] \times R$, where $\tilde{g}_0(x)$ satisfies also the strong force condition like in (H_2).

For $n \in N_1$, let $\alpha_n = \min_{0\le t\le T} Z_n(t)$, $\beta_n = \max_{0\le t\le T} Z_n(t)$.

If $\alpha_n \ge R_1$, due to $n \in N_1$, (27) holds.

If $\alpha_n < R_1$, we claim that, for all $n \in N_1$,

$$\beta_n > R_1. \tag{29}$$

Otherwise, suppose that $\beta_n \le R_1$ for some $n \in N_1$. Then it is easy to verify

$$F_n\left(t, Z_n(t), Z_n'(t)\right) > \iota_h r\|a\|_1. \tag{30}$$

In fact, if $1/n \le Z_n(t) \le R_1$, we obtain from (28)

$$\begin{aligned}F_n\left(t, Z_n(t), Z_n'(t)\right) &= F\left(t, Z_n(t), Z_n'(t)\right)\\ &\ge a(t)Z_n(t) + \tilde{g}_0\left(Z_n(t)\right)\\ &\ge \tilde{g}_0\left(Z_n(t)\right)\\ &> \iota_h r\|a\|_1.\end{aligned} \tag{31}$$

and, if $Z_n(t) \le 1/n$, we have

$$F_n\left(t, Z_n(t), Z_n'(t)\right) = F\left(t, \frac{1}{n}, Z_n'(t)\right) \ge \frac{a(t)}{n} + \tilde{g}_0\left(\frac{1}{n}\right) \ge \tilde{g}_0\left(\frac{1}{n}\right) > \iota_h r \|a\|_1. \quad (32)$$

Integrating (22) (with $\lambda = 1$) from 0 to T, we deduce that

$$\begin{aligned} 0 = & \int_0^T \left\{ \left(\sigma(h)(t) x_n'\right)' + a(t)\sigma(h)(t) x_n \right. \\ & \left. -\sigma(h)(t)\left(F_n\left(t, Z_n(t), Z_n'(t)\right) + \frac{a(t)}{n}\right)\right\} dt \\ = & \int_0^T a(t)\sigma(h)(t) x_n dt \\ & - \int_0^T \sigma(h)(t) F_n\left(t, Z_n(t), Z_n'(t)\right) dt \\ & - \int_0^T \sigma(h)(t) \frac{a(t)}{n} dt \\ < & \int_0^T a(t)\sigma(h)(t) x_n dt \\ & - \int_0^T \sigma(h)(t) F_n\left(t, Z_n(t), Z_n'(t)\right) dt \\ < & \ 0, \end{aligned} \quad (33)$$

where estimation (30) and the fact $\|x_n\| < r$ are used. This is a contradiction. Hence (29) holds.

Due to $\alpha_n < R_1$, that is, $\alpha_n = \min_{0\le t\le T}[x_n(t) - Mw(t)] = x_n(a_n) - Mw(a_n) < R_1$ for some $a_n \in [0, T]$. By (29), there exists $c_n \in [0, T]$ (without loss of generality, we assume $a_n < c_n$.) such that $x_n(c_n) = Mw(c_n) + R_1$ and $x_n(t) \le Mw(t) + R_1$ for $a_n \le t \le c_n$.

It can be checked that

$$F_n\left(t, Z_n(t), Z_n'(t)\right) > a(t) Z_n(t) + M + \overline{M}_h, \quad (34)$$

where $\overline{M}_h$ is defined by (15).

In fact, if $t \in [a_n, c_n]$ is such that $1/n \le Z_n(t) \le R_1$, we have

$$\begin{aligned} & F_n\left(t, Z_n(t), Z_n'(t)\right) \\ & \quad = F\left(t, Z_n(t), Z_n'(t)\right) \\ & \quad \ge a(t) Z_n(t) + \tilde{g}_0(x) > \max\left\{M + \overline{M}, \iota_h r \|\alpha\|_1\right\} \\ & \quad \ge a(t) Z_n(t) + M + \overline{M}. \end{aligned} \quad (35)$$

and, if $t \in [a_n, c_n]$ is such that $Z_n(t) \le 1/n$, we have

$$\begin{aligned} F_n\left(t, Z_n(t), Z_n'(t)\right) &= F\left(t, \frac{1}{n}, Z_n'(t)\right) \\ & \ge \frac{a(t)}{n} + \tilde{g}_0\left(\frac{1}{n}\right) > \frac{a(t)}{n} + M + \overline{M}_h \\ & \ge a(t) Z_n(t) + M + \overline{M}_h. \end{aligned} \quad (36)$$

So (34) holds.

Using (17) (with $\lambda = 1$) for $x_n(t)$ and the estimation (34), we have, for $t \in [a_n, c_n]$

$$\begin{aligned} Z_n''(t) = & -h(t) Z_n'(t) - a(t) Z_n(t) \\ & - M + F_n\left(t, Z_n(t), Z_n'(t)\right) + \frac{a(t)}{n} \\ > & -h(t) Z_n'(t) - a(t) Z_n(t) \\ & - M + a(t) Z_n(t) + M + \overline{M}_h + \frac{a(t)}{n} \\ \ge & -\overline{M}_h - a(t) Z_n(t) - M \\ & + a(t) Z_n(t) + M + \overline{M}_h + \frac{a(t)}{n} \\ \ge & \frac{a(t)}{n} \ge 0. \end{aligned} \quad (37)$$

As $Z_n'(a_n) = 0$, $Z_n'(t) > 0$ for all $t \in [a_n, c_n]$, so $Z_n(t)$ is strictly increasing on $[a_n, c_n]$. We use ξ_n to denote the inverse function of Z_n restricted to $[a_n, c_n]$.

Suppose that (27) does not hold, that is, for some $n \in N_1$, $Z_n(t) < 1/n < R_1$. Then there would exist $b_n \in (a_n, c_n)$ such that $Z_n(b_n) = 1/n$ and

$$\begin{aligned} & Z_n(t) \le \frac{1}{n} \quad \text{for } a_n \le t \le b_n, \\ & \frac{1}{n} \le Z_n(t) \le R_1 \quad \text{for } b_n \le t \le c_n. \end{aligned} \quad (38)$$

Multiplying (17) (with $\lambda = 1$) by $Z_n'(t)$ and integrating from b_n to c_n, we obtain

$$\begin{aligned} & \int_{1/n}^{R_1} F\left(\xi_n(Z), Z, Z'\right) dZ \\ & \quad = \int_{b_n}^{c_n} F\left(t, Z_n(t), Z_n'(t)\right) Z_n'(t) dt \\ & \quad = \int_{b_n}^{c_n} F_n\left(t, Z_n(t), Z_n'(t)\right) Z_n'(t) dt \\ & \quad = \int_{b_n}^{c_n} \left(x_n''(t) + h(t) x_n'(t) + a(t) x_n(t) - \frac{a(t)}{n}\right) Z_n'(t) dt \end{aligned}$$

$$
\begin{aligned}
&= \int_{b_n}^{c_n} x_n''(t)\left(x_n'(t) - Mw'(t)\right)dt + \int_{b_n}^{c_n} h(t)\,x_n'(t)\,Z_n'(t)\,dt \\
&\quad + \int_{b_n}^{c_n}\left(a(t)\,x_n(t) - \frac{a(t)}{n}\right)Z_n'(t)\,dt.
\end{aligned}
\tag{39}
$$

By the facts $\|x_n\| < r$, $\|x_n'\| \le Lr$, $\|w'\| \le r$ and the definition of $Z_n(t)$, we can obtain $|Z_n(t)| \le r + TB$, $|Z_n'(t)| \le (r+M)L$, together with $\|x_n\| < r$, implies that the second term and the third term are bounded. The first term is

$$
\begin{aligned}
&\frac{\left(\left[x_n'(c_n)\right]^2 - \left[x_n'(b_n)\right]^2\right)}{2} \\
&\quad - M\left(x_n'(c_n)\,w'(c_n) - x_n'(b_n)\,w'(b_n)\right) \\
&\quad + M\int_{b_n}^{c_n} x_n'(t)\,w''(t)\,dt,
\end{aligned}
\tag{40}
$$

which is also bounded. As a consequence, there exists a $B_1 > 0$ such that

$$
\int_{1/n}^{R_1} F\left(\xi_n(Z), Z, Z'\right)dZ \le B_1. \tag{41}
$$

On the other hand, by (H_2), we can choose $n_2 \in N_1$ large enough such that

$$
\int_{1/n}^{R_1} F\left(\xi_n(Z), Z, Z'\right)dZ \ge \int_{1/n}^{R_1} g_0(Z)\,dZ > B_1 \tag{42}
$$

for all $n \in N_2 = \{n_2, n_2+1, \ldots\}$. So (27) holds. □

Furthermore, we can prove $Z_n(t)$ has a uniform positive lower bound δ.

Lemma 6. *There exist a constant $\delta > 0$ such that, for all $n \in N_2$,*

$$
Z_n(t) \ge \delta. \tag{43}
$$

Proof. Multiplying (17) (with $\lambda = 1$) by $Z_n'(t)$ and integrating from a_n to c_n, we obtain

$$
\begin{aligned}
&\int_{\alpha_n}^{R_1} F\left(\xi_n(Z), Z, Z'\right)dZ \\
&= \int_{a_n}^{c_n} F\left(t, Z_n(t), Z_n'(t)\right)Z_n'(t)\,dt \\
&= \int_{a_n}^{c_n} F_n\left(t, Z_n(t), Z_n'(t)\right)Z_n'(t)\,dt \\
&= \int_{a_n}^{c_n}\left(x_n''(t) + h(t)\,x_n'(t) + a(t)\,x_n(t) - \frac{a(t)}{n}\right)Z_n'(t)\,dt \\
&= \int_{a_n}^{c_n} x_n''(t)\left(x_n'(t) - Mw'(t)\right)dt + \int_{a_n}^{c_n} h(t)\,x_n'(t)\,Z_n'(t)\,dt \\
&\quad + \int_{a_n}^{c_n}\left(a(t)\,x_n(t) - \frac{a(t)}{n}\right)Z_n'(t)\,dt.
\end{aligned}
\tag{44}
$$

In the same way as in the proof of (41), one way readily prove that the right-hand side of the above equality is bounded. On the other hand, if $n \in N_2$, by (H_2),

$$
\begin{aligned}
&\int_{\alpha_n}^{R_1} F\left(\xi_n(Z), Z, Z'\right)dZ \\
&\qquad \ge \int_{\alpha_n}^{R_1} g_0(Z)\,dZ + M\left(R_1 - \alpha_n\right) \longrightarrow +\infty
\end{aligned}
\tag{45}
$$

if $\alpha_n \to 0_+$. Thus we know that there exists a constant $\delta > 0$ such that $\alpha_n \ge \delta$. Hence (43) holds.

Next, we will prove (17) has periodic solution $x_n(t)$.

For $\imath r > 0$, we can choose $n_3 \in N_2$ such that $1/n_3 < \imath r$, which together with (H_4) imply

$$
w^* g\left(\imath r - Mw^*\right)\left(1 + \frac{h(r)}{g(r)}\right)\varrho\left((r+M)L\right) + \frac{1}{n_3} < r. \tag{46}
$$

Let $N_3 = \{n_3, n_3+1, \ldots\}$. For $n \in N_3$, consider (17).

Next we claim that any fixed point x_n of (18) for any $\lambda \in [0,1]$ must satisfy $\|x_n\| \ne r$. So, by using the Leray-Schauder alternative principle, (17) (with $\lambda = 1$) has a periodic solution $x_n(t)$. Otherwise, assume that x_n is a fixed point x_n of (18) for some $\lambda \in [0,1]$ such that $\|x_n\| = r$. Note that

$$
\begin{aligned}
x_n(t) - \frac{1}{n} &= \lambda\int_0^T G(t,s)\,F_n\left(s, Z_n(s), Z_n'(s)\right)ds \\
&\ge \lambda A\int_0^T F_n\left(s, Z_n(s), Z_n'(s)\right)ds \\
&= \imath B\lambda\int_0^T F_n\left(s, Z_n(s), Z_n'(s)\right)ds \\
&\ge \imath \max_{t\in[0,T]}\left\{\lambda\int_0^T G(t,s)\,F_n\left(s, Z_n(s), Z_n'(s)\right)ds\right\} \\
&= \imath\left\|x_n - \frac{1}{n}\right\|.
\end{aligned}
\tag{47}
$$

For $n \in N_3$, we have

$$
x_n(t) \ge \imath\left\|x_n - \frac{1}{n}\right\| + \frac{1}{n} \ge \imath\left(\|x_n\| - \frac{1}{n}\right) + \frac{1}{n} \ge \imath r. \tag{48}
$$

By (27) and assumption (H_3), for all $t \in [0,T]$ and $n \in N_3$, we have

$$
\begin{aligned}
&x_n(t) \\
&\quad = \lambda\int_0^T G(t,s)\,F_n\left(s, Z_n(s), Z_n'(s)\right)ds + \frac{1}{n} \\
&\quad = \lambda\int_0^T G(t,s)\,F\left(s, Z_n(s), Z_n'(s)\right)ds + \frac{1}{n}
\end{aligned}
$$

$$
\begin{aligned}
&\le \int_0^T G(t,s) F\left(s, Z_n(s), Z_n'(s)\right) ds + \frac{1}{n} \\
&\le \int_0^T G(t,s)\left(g\left(Z_n(s)\right) + h\left(Z_n(s)\right)\right) \varrho\left(\left|Z_n'(s)\right|\right) ds \\
&\quad + \frac{1}{n} \\
&\le \int_0^T G(t,s) g\left(Z_n(s)\right)\left(1 + \frac{h\left(Z_n(s)\right)}{g\left(Z_n(s)\right)}\right) \varrho\left(\left|Z_n'(s)\right|\right) ds \\
&\quad + \frac{1}{n} \\
&\le \int_0^T G(t,s) g\left(\iota r - M w^*\right)\left(1 + \frac{h(r)}{g(r)}\right) \varrho\left((r+M)L\right) ds \\
&\quad + \frac{1}{n} \\
&\le g\left(\iota r - M w^*\right)\left(1 + \frac{h(r)}{g(r)}\right) \varrho\left((r+M)L\right) w^* + \frac{1}{n_3}.
\end{aligned}
\tag{49}
$$

Therefore,

$$
r = \|x\| \le g\left(\iota r - M w^*\right)\left(1 + \frac{h(r)}{g(r)}\right) \varrho\left((r+M)L\right) w^* + \frac{1}{n_3}. \tag{50}
$$

This is a contradiction to the choice of n_3 and the claim is proved.

The fact $\|x_n\| < r$ and $\|x_n'(t)\| < Lr$ show that $\{x_n\}_{n\in N_3}$ is a bounded and equicontinuous family on $[0,T]$. Now Arzela-Ascoli Theorem guarantees that $\{x_n\}_{n\in N_3}$ has a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$, converging uniformly on $[0,T]$ to a function $x \in C[0,T]$. From the fact $\|x_n\| < r$ and $x_n(t) > \delta$, x satisfies $\delta \le x(t) \le r$ for all t. Moreover, $\{x_{n_k}\}$ satisfies the integral equation

$$
x_{n_k}(t) = \int_0^T G(t,s) F\left(s, Z_{n_k}(s), Z_{n_k}'(s)\right) ds + \frac{1}{n_k}. \tag{51}
$$

Letting $k \to \infty$, we arrive at

$$
x(t) = \int_0^T G(t,s) F\left(s, x(s) - M w(s), x'(s) - M w'(s)\right) ds, \tag{52}
$$

where the uniform continuity of $F(t,x,y)$ on $[0,T]\times[\delta,r]\times[-(r+M)L,(r+M)L]$ is used. Therefore, x is a positive periodic solution of (16) and $Z(t) = x(t) - Mw(t) \ge \delta$. Thus we complete the prove of Theorem 4. □

Corollary 7. *Let the nonlinearity in* (1) *be*

$$
f(t,x,y) = \left(1 + |y|^\gamma\right)\left(x^{-\alpha} + \mu x^\beta\right) + e(t), \tag{53}
$$

where $\alpha > 1$, $\beta > 0$, $1 > \gamma \ge 0$, $\mu > 0$ is a positive parameter, $e(t)$ is a T-periodic function.

(i) *If $\beta + \gamma < 1$, then* (1) *has at least one positive periodic solution for each $\mu > 0$.*

(ii) *If $\beta + \gamma \ge 1$, then* (1) *has at least one positive periodic solution for each $0 < \mu < \mu_1$, where μ_1 is some positive constant.*

Proof. We will apply Theorem 4 with $M = \max_{0\le t\le T}|e(t)|$ and $g(x) = x^{-\alpha}$, $h(x) = \mu x^\beta + 2M$, $\varrho(y) = 1 + |y|^\gamma$. Then condition (H_1)–(H_3) are satisfied and existence condition (H_4) becomes

$$
\mu < \frac{r\left(\iota r - M w^*\right)^\alpha - w^*\left(1 + (r+M)^\gamma L^\gamma\right)\left(1 + 2M r^\alpha\right)}{w^*\left(1 + (r+M)^\gamma L^\gamma\right) r^{\alpha+\beta}}. \tag{54}
$$

So (1) has at least one positive periodic solution for

$$
\begin{aligned}
0 < \mu < \mu_1 &= \sup_{r > M w^*/\iota} \left(r\left(\iota r - M w^*\right)^\alpha - w^*\left(1 + (r+M)^\gamma L^\gamma\right)\right. \\
&\qquad \left. \times\left(1 + 2M r^\alpha\right)\right) \\
&\qquad \times\left(w^*\left(1 + (r+M)^\gamma L^\gamma\right) r^{\alpha+\beta}\right)^{-1}.
\end{aligned}
\tag{55}
$$

Note that $\mu_1 = \infty$ if $\beta + \gamma < 1$ and $\mu_1 < \infty$ if $\beta + \gamma \ge 1$. We have the desired results. □

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Research Article

Exponential Extinction of Nicholson's Blowflies System with Nonlinear Density-Dependent Mortality Terms

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This paper presents a new generalized Nicholson's blowflies system with patch structure and nonlinear density-dependent mortality terms. Under appropriate conditions, we establish some criteria to guarantee the exponential extinction of this system. Moreover, we give two examples and numerical simulations to demonstrate our main results.

1. Introduction

To describe the population of the Australian sheep blowfly and agree well with the experimental date of Nicholson [1], Gurney et al. [2] proposed the following Nicholson's blowflies equation:

$$N'(t) = -\delta N(t) + pN(t-\tau)e^{-aN(t-\tau)}. \tag{1.1}$$

Here, $N(t)$ is the size of the population at time t, p is the maximum per capita daily egg production, $(1/a)$ is the size at which the population reproduces at its maximum rate, δ is the per capita daily adult death rate, and τ is the generation time. There have been a large number of results on this model and its modifications (see, e.g., [3–8]). Recently, Berezansky et al. [9] pointed out that a new study indicates that a linear model of density-dependent mortality will be most accurate for populations at low densities and marine ecologists are currently in the process of constructing new fishery models with nonlinear density-dependent mortality rates. Consequently Berezansky et al. [9] presented the following Nicholson's blowflies model with a nonlinear density-dependent mortality term

$$N'(t) = -D(N(t)) + PN(t-\tau)e^{-aN(t-\tau)}, \tag{1.2}$$

where P is a positive constant and function D might have one of the following forms: $D(N) = aN/(N+b)$ or $D(N) = a - be^{-N}$ with positive constants $a, b > 0$.

Wang [10] studied the existence of positive periodic solutions for the model (1.2) with $D(N) = a - be^{-N}$. Hou et al. [11] investigated the permanence and periodic solutions for the model (1.2) with $D(N) = aN/(N+b)$. Furthermore, Liu and Gong [12] considered the permanence for a Nicholson-type delay systems with nonlinear density-dependent mortality terms as follows:

$$\begin{aligned} N_1'(t) &= -D_{11}(t, N_1(t)) + D_{12}(t, N_2(t)) + c_1(t)N_1(t-\tau_1(t))e^{-\gamma_1(t)N_1(t-\tau_1(t))} \\ N_2'(t) &= -D_{22}(t, N_2(t)) + D_{21}(t, N_1(t)) + c_2(t)N_2(t-\tau_2(t))e^{-\gamma_2(t)N_2(t-\tau_2(t))}, \end{aligned} \tag{1.3}$$

where

$$D_{ij}(t, N) = \frac{a_{ij}(t)N}{b_{ij}(t) + N} \quad \text{or} \quad D_{ij}(t, N) = a_{ij}(t) - b_{ij}(t)e^{-N}, \tag{1.4}$$

$a_{ij}, b_{ij}, c_i, \gamma_i : R \to (0, +\infty)$ are all continuous functions bounded above and below by positive constants, and $\tau_j(t) \geq 0$ are bounded continuous functions, $r_i = \sup_{t\in R}\tau_i(t) > 0$, and $i, j = 1, 2$.

On the other hand, since the biological species compete and cooperate with each other in real world, the growth models given by patch structure systems of delay differential equation have been provided by several authors to analyze the dynamics of multiple species (see, e.g., [13–16] and the reference therein). Moreover, the extinction phenomenon often appears in the biology, economy, and physics field and the main focus of Nicholson's blowflies model is on the scalar equation and results on patch structure of this model are gained rarely [14, 16], so it is worth studying the extinction of Nicholson's blowflies system with patch structure and nonlinear density-dependent mortality terms. Motivated by the above discussion, we shall derive the conditions to guarantee the extinction of the following Nicholson-type delay system with patch structure and nonlinear density-dependent mortality terms:

$$\begin{aligned} N_i'(t) = &- D_{ii}(t, N_i(t)) + \sum_{j=1, j\neq i}^{n} D_{ij}\left(t, N_j(t)\right) \\ &+ \sum_{j=1}^{l} c_{ij}(t) N_i\left(t - \tau_{ij}(t)\right) e^{-\gamma_{ij}(t)N_i(t-\tau_{ij}(t))}, \end{aligned} \tag{1.5}$$

where

$$D_{ij}(t, N) = \frac{a_{ij}(t)N}{b_{ij}(t) + N} \quad \text{or} \quad D_{ij}(t, N) = a_{ij}(t) - b_{ij}(t)e^{-N}, \tag{1.6}$$

$a_{ij}, b_{ij}, c_{ik}, \gamma_{ik} : R \to (0, +\infty)$ are all continuous functions bounded above and below by positive constants, and $\tau_{ik}(t) \geq 0$ are bounded continuous functions, $r_i = \max_{1 \leq j \leq l}\{\sup_{t \in R} \tau_{ij}(t)\} > 0$, and $i, j = 1, 2, \ldots, n,\ k = 1, 2, \ldots, l$. Furthermore, in the case $D_{ij}(t, N) = a_{ij}(t) - b_{ij}(t)e^{-N}$, to guarantee the meaning of mortality terms we assume that $a_{ij}(t) > b_{ij}(t)$ for $t \in R$ and $i, j = 1, 2, \ldots, n$. The main purpose of this paper is to establish the conditions ensuring the exponential extinction of system (1.5).

For convenience, we introduce some notations. Throughout this paper, given a bounded continuous function g defined on R, let g^+ and g^- be defined as

$$g^- = \inf_{t \in R} g(t), \qquad g^+ = \sup_{t \in R} g(t). \tag{1.7}$$

Let $R^n(R^n_+)$ be the set of all (nonnegative) real vectors, we will use $x = (x_1, \ldots, x_n)^T \in R^n$ to denote a column vector, in which the symbol $(^T)$ denotes the transpose of a vector. We let $|x|$ denote the absolute-value vector given by $|x| = (|x_1|, \ldots, |x_n|)^T$ and define $\|x\| = \max_{1 \leq i \leq n}|x_i|$. Denote $C = \prod_{i=1}^n C([-r_i, 0], R)$ and $C_+ = \prod_{i=1}^n C([-r_i, 0], R_+)$ as Banach spaces equipped with the supremum norm defined by $\|\varphi\| = \sup_{-r_i \leq t \leq 0} \max_{1 \leq i \leq n}|\varphi_i(t)|$ for all $\varphi(t) = (\varphi_1(t), \ldots, \varphi_n(t))^T \in C$ (or $\in C_+$). If $x_i(t)$ is defined on $[t_0 - r_i, \nu)$ with $t_0, \nu \in R$ and $i = 1, \ldots, n$, then we define $x_t \in C$ as $x_t = (x_t^1, \ldots x_t^n)^T$ where $x_t^i(\theta) = x_i(t + \theta)$ for all $\theta \in [-r_i, 0]$ and $i = 1, \ldots, n$.

The initial conditions associated with system (1.5) are of the form:

$$N_{t_0} = \varphi, \quad \varphi = (\varphi_1, \ldots, \varphi_n)^T \in C_+, \quad \varphi_i(0) > 0,\ i = 1, \ldots, n. \tag{1.8}$$

We write $N_t(t_0, \varphi)(N(t; t_0, \varphi))$ for a solution of the initial value problem (1.5) and (1.8). Also, let $[t_0, \eta(\varphi))$ be the maximal right-interval of existence of $N_t(t_0, \varphi)$.

Definition 1.1. The system (1.5) with initial conditions (1.8) is said to be exponentially extinct, if there are positive constants M and κ such that $|N_i(t; t_0, \varphi)| \leq Me^{-\kappa(t-t_0)},\ i = 1, 2 \ldots, n$. Denote it as $N_i(t; t_0, \varphi) = O(e^{-\kappa(t-t_0)}),\ i = 1, 2, \ldots, n$.

The remaining part of this paper is organized as follows. In Sections 2 and 3, we shall derive some sufficient conditions for checking the extinction of system (1.5). In Section 4, we shall give two examples and numerical simulations to illustrate our results obtained in the previous sections.

2. Extinction of Nicholson's Blowflies System with $D_{ij}(t, N) = a_{ij}(t)N/(b_{ij}(t) + N)(i, j = 1, 2, \ldots, n)$

Theorem 2.1. *Suppose that there exists positive constant K_1 such that*

$$\frac{a_{ii}^-}{b_{ii}^+ + K_1} > \sum_{j=1, j \neq i}^n \frac{a_{ij}^+}{b_{ij}^-} + \sum_{j=1}^l \frac{c_{ij}^+}{\gamma_{ij}^- e K_1}, \quad i = 1, 2, \ldots, n. \tag{2.1}$$

Let

$$E^1 = \{\varphi \mid \varphi \in C_+, \varphi(0) > 0,\ 0 \leq \varphi_i(t) < K_1, \forall t \in [-r_i, 0],\ i = 1, 2, \ldots, n\}. \tag{2.2}$$

Moreover, assume $N(t; t_0, \varphi)$ *is the solution of* (1.5) *with* $\varphi \in E^1$ *and* $D_{ij}(t, N) = (a_{ij}(t)N/(b_{ij}(t) + N))$ $(i, j = 1, 2, \ldots, n)$*. Then,*

$$0 \leq N_i(t; t_0, \varphi) < K_1, \quad \forall t \in [t_0, \eta(\varphi)),\ i = 1, 2, \ldots, n, \qquad \eta(\varphi) = +\infty. \tag{2.3}$$

Proof. Set $N(t) = N(t; t_0, \varphi)$ for all $t \in [t_0, \eta(\varphi))$. In view of $\varphi \in C_+$, using Theorem 5.2.1 in [17, p. 81], we have $N_t(t_0, \varphi) \in C_+$ for all $t \in [t_0, \eta(\varphi))$. Assume, by way of contradiction, that (2.3) does not hold. Then, there exist $t_1 \in [t_0, \eta(\varphi))$ and $i \in \{1, 2, \ldots, n\}$ such that

$$N_i(t_1) = K_1, \quad 0 \leq N_j(t) < K_1 \quad \forall t \in [t_0 - r_j, t_1),\ j = 1, 2, \ldots, n. \tag{2.4}$$

Calculating the derivative of $N_i(t)$, together with (2.1) and the fact that $\sup_{u \geq 0} ue^{-u} = 1/e$ and $a(t)N/(b(t) + N) \leq a(t)N/b(t)$ for all $t \in R, N \geq 0$, (1.5) and (2.4) imply that

$$\begin{aligned}
0 &\leq N_i'(t_1) \\
&= -D_{ii}(t_1, N_i(t_1)) + \sum_{j=1, j \neq i}^{n} D_{ij}(t_1, N_j(t_1)) \\
&\quad + \sum_{j=1}^{l} c_{ij}(t_1) N_i(t_1 - \tau_{ij}(t_1)) e^{-\gamma_{ij}(t_1) N_i(t_1 - \tau_{ij}(t_1))} \\
&\leq -\frac{a_{ii}(t_1) N_i(t_1)}{b_{ii}(t_1) + N_i(t_1)} + \sum_{j=1, j \neq i}^{n} \frac{a_{ij}(t_1) N_j(t_1)}{b_{ij}(t_1)} + \sum_{j=1}^{l} \frac{c_{ij}(t_1)}{\gamma_{ij}(t_1)} \frac{1}{e} \\
&\leq \left(-\frac{a_{ii}^-}{b_{ii}^+ + K_1} + \sum_{j=1, j \neq i}^{n} \frac{a_{ij}^+}{b_{ij}^-} + \sum_{j=1}^{l} \frac{c_{ij}^+}{\gamma_{ij}^- e K_1} \right) K_1 \\
&< 0,
\end{aligned} \tag{2.5}$$

which is a contradiction and implies that (2.3) holds. From Theorem 2.3.1 in [18], we easily obtain $\eta(\varphi) = +\infty$. This ends the proof of Theorem 2.1. □

Theorem 2.2. *Suppose that there exists positive constant* K_1 *satisfying* (2.1) *and*

$$\frac{a_{ii}^-}{b_{ii}^+ + K_1} > \sum_{j=1, j \neq i}^{n} \frac{a_{ij}^+}{b_{ij}^-} + \sum_{j=1}^{l} c_{ij}^+, \quad i = 1, 2, \ldots, n. \tag{2.6}$$

Then the solution $N(t; t_0, \varphi)$ *of* (1.5) *with* $\varphi \in E^1$ *and* $D_{ij}(t, N) = (a_{ij}(t)N/(b_{ij}(t) + N))$ $(i, j = 1, 2, \ldots, n)$ *is exponentially extinct as* $t \to +\infty$*.*

Proof. Define continuous functions $\Gamma_i(\omega)$ by setting

$$\Gamma_i(\omega) = \omega - \frac{a_{ii}^-}{b_{ii}^+ + K_1} + \sum_{j=1, j\neq i}^{n} \frac{a_{ij}^+}{b_{ij}^-} + \sum_{j=1}^{l} c_{ij}^+ e^{\omega r_i}, \quad i = 1, 2, \ldots, n. \tag{2.7}$$

Then, from (2.6), we obtain

$$\Gamma_i(0) = -\frac{a_{ii}^-}{b_{ii}^+ + K_1} + \sum_{j=1, j\neq i}^{n} \frac{a_{ij}^+}{b_{ij}^-} + \sum_{j=1}^{l} c_{ij}^+ < 0, \quad i = 1, 2, \ldots, n. \tag{2.8}$$

The continuity of $\Gamma_i(\omega)$ implies that there exists $\lambda > 0$ such that

$$\Gamma_i(\lambda) = \lambda - \frac{a_{ii}^-}{b_{ii}^+ + K_1} + \sum_{j=1, j\neq i}^{n} \frac{a_{ij}^+}{b_{ij}^-} + \sum_{j=1}^{l} c_{ij}^+ e^{\lambda r_i} < 0, \quad i = 1, 2, \ldots, n. \tag{2.9}$$

Let

$$y_i(t) = N_i(t) e^{\lambda(t-t_0)}, \quad i = 1, 2, \ldots, n. \tag{2.10}$$

Calculating the derivative of $y(t)$ along the solution $N(t)$ of system (1.5) with $\varphi \in E^1$, we have

$$\begin{aligned} y_i'(t) &= \lambda y_i(t) + e^{\lambda(t-t_0)} N_i'(t) \\ &= \lambda y_i(t) - \frac{a_{ii}(t) y_i(t)}{b_{ii}(t) + N_i(t)} + \sum_{j=1, j\neq i}^{n} \frac{a_{ij}(t) y_j(t)}{b_{ij}(t) + N_j(t)} \\ &\quad + \sum_{j=1}^{l} c_{ij}(t) e^{\lambda \tau_{ij}(t)} y_i(t - \tau_{ij}(t)) e^{-\gamma_{ij}(t) N_i(t - \tau_{ij}(t))}, \quad i = 1, 2, \ldots, n. \end{aligned} \tag{2.11}$$

Let M_1 denote an arbitrary positive number and set

$$M_1 > y_i(t), \quad \forall t \in [t_0 - r_i, t_0],\ i = 1, 2, \ldots, n. \tag{2.12}$$

We claim that

$$y_i(t) < M_1, \quad \forall t \in [t_0, +\infty),\ i = 1, 2, \ldots, n. \tag{2.13}$$

If this is not valid, there must exist $t_2 \in (t_0, +\infty)$ and $i \in \{1, 2, \ldots, n\}$ such that

$$y_i(t_2) = M_1, \quad y_j(t) < M_1, \quad \forall t < t_2,\ j = 1, 2, \ldots, n. \tag{2.14}$$

Then, from (2.3) and (2.11), we have

$$\begin{aligned}
0 &\le y_i'(t_2) \\
&= \lambda y_i(t_2) - \frac{a_{ii}(t_2) y_i(t_2)}{b_{ii}(t_2) + N_i(t_2)} + \sum_{j=1, j\neq i}^{n} \frac{a_{ij}(t_2) y_j(t_2)}{b_{ij}(t_2) + N_j(t_2)} \\
&\quad + \sum_{j=1}^{l} c_{ij}(t_2) e^{\lambda \tau_{ij}(t_2)} y_i\big(t_2 - \tau_{ij}(t_2)\big) e^{-\gamma_{ij}(t_2) N_i(t_2 - \tau_{ij}(t_2))} \\
&\le \lambda M_1 - \frac{a_{ii}(t_2) M_1}{b_{ii}(t_2) + K_1} + \sum_{j=1, j\neq i}^{n} \frac{a_{ij}(t_2) M_1}{b_{ij}(t_2)} + \sum_{j=1}^{l} c_{ij}(t_2) e^{\lambda r_i} M_1 \\
&\le \left(\lambda - \frac{a_{ii}^-}{b_{ii}^+ + K_1} + \sum_{j=1, j\neq i}^{n} \frac{a_{ij}^+}{b_{ij}^-} + \sum_{j=1}^{l} c_{ij}^+ e^{\lambda r_i}\right) M_1 \\
&< 0.
\end{aligned} \tag{2.15}$$

This contradiction implies that (2.13) holds. Thus,

$$N_i(t) = y_i(t) e^{-\lambda(t-t_0)} \le M_1 e^{-\lambda(t-t_0)} \quad \forall t \in [t_0 - r_i, +\infty),\ i = 1, 2, \ldots, n. \tag{2.16}$$

This completes the proof. □

3. Extinction of Nicholson's Blowflies System with $D_{ij}(t, N) = a_{ij}(t) - b_{ij}(t) e^{-N} (i, j = 1, 2, \ldots, n)$

Theorem 3.1. *Suppose that there exists positive constant K_2 such that*

$$a_{ii}^- > \sum_{j=1, j\neq i}^{n} a_{ij}^+ + \left(b_{ii}^+ - \sum_{j=1, j\neq i}^{n} b_{ij}^-\right) e^{-K_2} + \sum_{j=1}^{l} \frac{c_{ij}^+}{\gamma_{ij}^- e}, \quad i = 1, 2, \ldots, n, \tag{3.1}$$

$$-a_{ii}(t) + b_{ii}(t) + \sum_{j=1, j\neq i}^{n} \big(a_{ij}(t) - b_{ij}(t)\big) \ge 0, \quad i = 1, 2, \ldots, n. \tag{3.2}$$

Let

$$E^2 = \{\varphi \mid \varphi \in C_+,\ \varphi(0) > 0,\ 0 \le \varphi_i(t) < K_2, \forall t \in [-r_i, 0],\ i = 1, 2, \ldots, n\}. \tag{3.3}$$

Moreover, assume $N(t;t_0,\varphi)$ is the solution of (1.5) *with $\varphi \in E^2$ and $D_{ij}(t,N) = a_{ij}(t) - b_{ij}(t) e^{-N}$ $(i,j = 1,2,\ldots,n)$. Then,*

$$0 \le N_i(t;t_0,\varphi) < K_2, \quad \forall t \in \left[t_0,\eta(\varphi)\right),\ i = 1,2,\ldots,n, \tag{3.4}$$

$$\eta(\varphi) = +\infty. \tag{3.5}$$

Proof. Set $N(t) = N(t;t_0,\varphi)$ for all $t \in [t_0,\eta(\varphi))$. Rewrite the system (1.5) as

$$N'(t) = f(t,N_t), \tag{3.6}$$

where $f(t,\phi) = (f_1(t,\phi), f_2(t,\phi),\ldots,f_n(t,\phi))^T$ and

$$\begin{aligned} f_i(t,\phi) = &- a_{ii}(t) + b_{ii}(t)e^{-\phi_i(0)} + \sum_{j=1,j\neq i}^{n}\left(a_{ij}(t) - b_{ij}(t)e^{-\phi_j(0)}\right) \\ &+ \sum_{j=1}^{l} c_{ij}(t)\phi_i(-\tau_{ij}(t))e^{-\gamma_{ij}(t)\phi_i(-\tau_{ij}(t))}, \quad i = 1,2,\ldots,n,\ \phi \in C. \end{aligned} \tag{3.7}$$

In view of (3.2), whenever $\phi \in C$ satisfies $\phi \ge 0$, $\phi_i(0) = 0$ for some i and $t \in R$, then

$$\begin{aligned} f_i(t,\phi) &= - a_{ii}(t) + b_{ii}(t) + \sum_{j=1,j\neq i}^{n}\left(a_{ij}(t) - b_{ij}(t)e^{-\phi_j(0)}\right) \\ &\quad + \sum_{j=1}^{l} c_{ij}(t)\phi_i(-\tau_{ij}(t))e^{-\gamma_{ij}(t)\phi_i(-\tau_{ij}(t))} \\ &\ge - a_{ii}(t) + b_{ii}(t) + \sum_{j=1,j\neq i}^{n}\left(a_{ij}(t) - b_{ij}(t)\right) \\ &\ge 0. \end{aligned} \tag{3.8}$$

Thus, using Theorem 5.2.1 in [17, p. 81], we have $N_t(t_0,\varphi) \in C_+$ for all $t \in [t_0,\eta(\varphi))$ and $\varphi \in E^2 \subset C_+$. Assume, by way of contradiction, that (3.4) does not hold. Then, there exist $t_3 \in [t_0,\eta(\varphi))$ and $i \in \{1,2,\ldots,n\}$ such that

$$N_i(t_3) = K_2, \quad 0 \le N_j(t) < K_2 \quad \forall t \in \left[t_0 - r_j, t_3\right),\ j = 1,2,\ldots,n. \tag{3.9}$$

Calculating the derivative of $N_i(t)$, together with (3.1) and the fact that $\sup_{u\geq 0} ue^{-u} = 1/e$, (1.5) and (3.9) imply that

$$\begin{aligned}
0 &\leq N_i'(t_3) \\
&= - D_{ii}(t_3, N_i(t_3)) + \sum_{j=1, j\neq i}^{n} D_{ij}\big(t_3, N_j(t_3)\big) + \sum_{j=1}^{l} c_{ij}(t_3) N_i\big(t_3 - \tau_{ij}(t_3)\big) \\
&\quad \times e^{-\gamma_{ij}(t_3)N_i(t_3-\tau_{ij}(t_3))} \\
&\leq - a_{ii}(t_3) + b_{ii}(t_3)e^{-K_2} + \sum_{j=1, j\neq i}^{n} \Big(a_{ij}(t_3) - b_{ij}(t_3)e^{-K_2}\Big) + \sum_{j=1}^{l} \frac{c_{ij}(t_3)}{\gamma_{ij}(t_3)} \frac{1}{e} \\
&\leq - a_{ii}^{-} + \sum_{j=1, j\neq i}^{n} a_{ij}^{+} + \left(b_{ii}^{+} - \sum_{j=1, j\neq i}^{n} b_{ij}^{-}\right)e^{-K_2} + \sum_{j=1}^{l} \frac{c_{ij}^{+}}{\gamma_{ij}^{-} e} \\
&< 0,
\end{aligned} \tag{3.10}$$

which is a contradiction and implies that (3.4) holds. From Theorem 2.3.1 in [18], we easily obtain $\eta(\varphi) = +\infty$. This ends the proof of Theorem 3.1. □

Theorem 3.2. *Let* (3.1) *and* (3.2) *hold. Moreover, suppose that there exist two positive constants* $\widetilde{\lambda}$ *and* $\widetilde{M}$ *such that*

$$-a_{ii}(t) + b_{ii}(t) + \sum_{j=1, j\neq i}^{n} \big(a_{ij}(t) - b_{ij}(t)\big) \leq \widetilde{M} e^{-\widetilde{\lambda}(t-t_0)}, \quad t \in R,\ i = 1, 2, \ldots, n, \tag{3.11}$$

$$b_{ii}^{-} > 1 + \frac{K_2}{2} b_{ii}^{+} + \sum_{j=1, j\neq i}^{n} b_{ij}^{+} + \sum_{j=1}^{l} c_{ij}^{+}, \quad i = 1, 2, \ldots, n. \tag{3.12}$$

Then the solution $N(t; t_0, \varphi)$ *of* (1.5) *with* $\varphi \in E^2$ *and* $D_{ij}(t, N) = a_{ij}(t) - b_{ij}(t)e^{-N}$ $(i, j = 1, 2, \ldots, n)$, *is exponentially extinct as* $t \to +\infty$.

Proof. Define continuous functions $\Gamma_i(\omega)$ by setting

$$\Gamma_i(\omega) = \omega - b_{ii}^{-} + 1 + \frac{K_2}{2} b_{ii}^{+} + \sum_{j=1, j\neq i}^{n} b_{ij}^{+} + \sum_{j=1}^{l} c_{ij}^{+} e^{\omega r_i}, \quad i = 1, 2, \ldots, n. \tag{3.13}$$

Then, from (3.12), we obtain

$$\Gamma_i(0) = -b_{ii}^{-} + 1 + \frac{K_2}{2} b_{ii}^{+} + \sum_{j=1, j\neq i}^{n} b_{ij}^{+} + \sum_{j=1}^{l} c_{ij}^{+} < 0, \quad i = 1, 2, \ldots, n. \tag{3.14}$$

The continuity of $\Gamma_i(\omega)$ implies that there exists $0 < \mu < \tilde{\lambda}$ such that

$$\Gamma_i(\mu) = \mu - b_{ii}^- + 1 + \frac{K_2}{2} b_{ii}^+ + \sum_{j=1, j \neq i}^{n} b_{ij}^+ + \sum_{j=1}^{l} c_{ij}^+ e^{\mu r_i} < 0, \quad i = 1, 2, \ldots, n. \tag{3.15}$$

Let

$$x_i(t) = N_i(t) e^{\mu(t-t_0)}, \quad i = 1, 2, \ldots, n. \tag{3.16}$$

Calculating the derivative of $x(t)$ along the solution $N(t)$ of system (1.5) with $\varphi \in E^2$, in view of (3.4) and (3.11), we have

$$\begin{aligned}
x_i'(t) &= \mu x_i(t) + e^{\mu(t-t_0)} N_i'(t) \\
&= \mu x_i(t) + e^{\mu(t-t_0)} \left[-a_{ii}(t) + b_{ii}(t) e^{-N_i(t)} + \sum_{j=1, j \neq i}^{n} \left(a_{ij}(t) - b_{ij}(t) e^{-N_j(t)} \right) \right] \\
&\quad + \sum_{j=1}^{l} c_{ij}(t) e^{\mu \tau_{ij}(t)} x_i\left(t - \tau_{ij}(t)\right) e^{-\gamma_{ij}(t) N_i(t - \tau_{ij}(t))} \\
&\leq \mu x_i(t) + e^{\mu(t-t_0)} \left[- a_{ii}(t) + b_{ii}(t) \left(1 - N_i(t) + \frac{1}{2} N_i^2(t) \right) \right. \\
&\qquad \left. + \sum_{j=1, j \neq i}^{n} \left(a_{ij}(t) - b_{ij}(t) \left(1 - N_j(t) \right) \right) \right] + \sum_{j=1}^{l} c_{ij}^+ e^{\mu r_i} x_i\left(t - \tau_{ij}(t)\right) \\
&= \mu x_i(t) + e^{\mu(t-t_0)} \left[-a_{ii}(t) + b_{ii}(t) + \sum_{j=1, j \neq i}^{n} \left(a_{ij}(t) - b_{ij}(t) \right) \right] - b_{ii}(t) x_i(t) \\
&\quad + \frac{1}{2} b_{ii}(t) N_i(t) x_i(t) + \sum_{j=1, j \neq i}^{n} b_{ij}(t) x_j(t) + \sum_{j=1}^{l} c_{ij}^+ e^{\mu r_i} x_i\left(t - \tau_{ij}(t)\right) \\
&\leq \mu x_i(t) + \widetilde{M} e^{(\mu - \tilde{\lambda})(t-t_0)} - b_{ii}^- x_i(t) + \frac{K_2}{2} b_{ii}^+ x_i(t) \\
&\quad + \sum_{j=1, j \neq i}^{n} b_{ij}^+ x_j(t) + \sum_{j=1}^{l} c_{ij}^+ e^{\mu r_i} x_i\left(t - \tau_{ij}(t)\right).
\end{aligned} \tag{3.17}$$

Let M_2 denote an arbitrary positive number and set

$$M_2 > \max\left\{ x_i(t), \widetilde{M} \right\} \quad \forall t \in [t_0 - r_i, t_0], \ i = 1, 2, \ldots, n. \tag{3.18}$$

We claim that

$$x_i(t) < M_2, \quad \forall t \in [t_0, +\infty), \ i = 1, 2, \ldots, n. \tag{3.19}$$

If this is not valid, there must exist $t_4 \in (t_0, +\infty)$ and $i \in \{1, 2, \ldots, n\}$ such that

$$x_i(t_4) = M_2, \quad x_j(t) < M_2, \quad \forall t < t_4, \ j = 1, 2, \ldots, n. \tag{3.20}$$

Then, from (3.15) and (3.17), we have

$$\begin{aligned}
0 &\leq x_i'(t_4) \\
&\leq \mu x_i(t_4) + \widetilde{M} e^{(\mu-\widetilde{\lambda})(t_4-t_0)} - b_{ii}^- x_i(t_4) + \frac{K_2}{2} b_{ii}^+ x_i(t_4) \\
&\quad + \sum_{j=1, j\neq i}^{n} b_{ij}^+ x_j(t_4) + \sum_{j=1}^{l} c_{ij}^+ e^{\mu r_i} x_i\big(t_4 - \tau_{ij}(t_4)\big) \\
&\leq \left[\mu + 1 - b_{ii}^- + \frac{K_2}{2} b_{ii}^+ + \sum_{j=1, j\neq i}^{n} b_{ij}^+ + \sum_{j=1}^{l} c_{ij}^+ e^{\mu r_i}\right] M_2 \\
&< 0.
\end{aligned} \tag{3.21}$$

This contradiction implies that (3.19) holds. Thus,

$$N_i(t) = x_i(t) e^{-\mu(t-t_0)} \leq M_2 e^{-\mu(t-t_0)} \quad \forall t \in [t_0 - r_i, +\infty), \ i = 1, 2, \ldots, n. \tag{3.22}$$

This completes the proof. □

4. Numerical Examples

In this section, we give two examples and numerical simulations to demonstrate the results obtained in previous sections.

Example 4.1. Consider the following Nicholson's blowflies system with patch structure and nonlinear density-dependent mortality terms:

$$\begin{aligned}
N_1'(t) = {} & -\frac{(25+|\cos 3t|)N_1(t)}{5+|\sin 2t|+N_1(t)} + \frac{(1+|\sin 2t|)N_2(t)}{3+|\cos 3t|+N_2(t)} + \frac{(1+|\cos 2t|)N_3(t)}{3+|\sin 3t|+N_3(t)} \\
& + \frac{1}{4}\left(1+\cos^2 t\right)N_1(t-2|\sin t|)e^{-4N_1(t-2|\sin t|)} \\
& + \frac{1}{4}\left(1+\sin^2 t\right)N_1(t-2|\cos t|)e^{-4N_1(t-2|\cos t|)} \\
N_2'(t) = {} & -\frac{(25+|\sin 3t|)N_2(t)}{5+|\cos 2t|+N_2(t)} + \frac{(1+|\cos 2t|)N_1(t)}{3+|\sin 3t|+N_1(t)} + \frac{(1+|\sin 2t|)N_3(t)}{3+|\cos 3t|+N_3(t)} \\
& + \frac{1}{4}\left(1+\sin^2 t\right)N_2(t-2|\cos t|)e^{-4N_2(t-2|\cos t|)} \\
& + \frac{1}{4}\left(1+\cos^2 t\right)N_2(t-2|\sin t|)e^{-4N_2(t-2|\sin t|)} \\
N_3'(t) = {} & -\frac{(25+|\sin 5t|)N_3(t)}{5+|\cos 6t|+N_3(t)} + \frac{(1+|\cos 3t|)N_1(t)}{3+|\sin 2t|+N_1(t)} + \frac{(1+|\sin 3t|)N_2(t)}{3+|\cos 2t|+N_2(t)} \\
& + \frac{1}{4}\left(1+\sin^2 2t\right)N_3(t-2|\cos 2t|)e^{-4N_3(t-2|\cos 2t|)} \\
& + \frac{1}{4}\left(1+\cos^2 2t\right)N_3(t-2|\sin 2t|)e^{-4N_3(t-2|\sin 2t|)}.
\end{aligned} \tag{4.1}$$

Obviously, $a_{ii}^- = 25,\ b_{ii}^+ = 6,\ (i = 1,2,3),\ a_{ij}^+ = 2,\ b_{ij}^- = 3,\ (i,j = 1,2,3, i \neq j),\ c_{ij}^+ = 1/2,\ \gamma_{ij}^- = 4, (i = 1,2,3,\ j = 1,2)$. Let $K_1 = e$, then we have

$$\begin{aligned}
\frac{25}{6+\mathrm{e}} &= \frac{a_{ii}^-}{b_{ii}^+ + K_1} > \sum_{j=1, j\neq i}^{3} \frac{a_{ij}^+}{b_{ij}^-} + \sum_{j=1}^{2} \frac{c_{ij}^+}{\gamma_{ij}^- e K_1} = \frac{4}{3} + \frac{1}{4e^2}, \\
\frac{25}{6+e} &= \frac{a_{ii}^-}{b_{ii}^+ + K_1} > \sum_{j=1, j\neq i}^{3} \frac{a_{ij}^+}{b_{ij}^-} + \sum_{j=1}^{2} c_{ij}^+ = \frac{7}{3}.
\end{aligned} \tag{4.2}$$

Then (4.2) imply that the system (4.1) satisfies (2.1) and (2.6). Hence, from Theorems 2.1 and 2.2, the solution $N(t)$ of system (4.1) with $D_{ij}(t,N) = a_{ij}(t)N/(b_{ij}(t)+N)(i,j = 1,2,3)$ and $\varphi \in E^1 = \{\varphi \mid \varphi \in C_+,\ \varphi(0) > 0 \text{ and } 0 \le \varphi_i(t) < e,\ \text{for all}, t \in [-2,0],\ i = 1,2,3\}$ is exponentially extinct as $t \to +\infty$ and $N(t) = N(t,0,\varphi) = O(e^{-\kappa t}),\ \kappa \approx 0.0001$. The fact is verified by the numerical simulation in Figure 1.

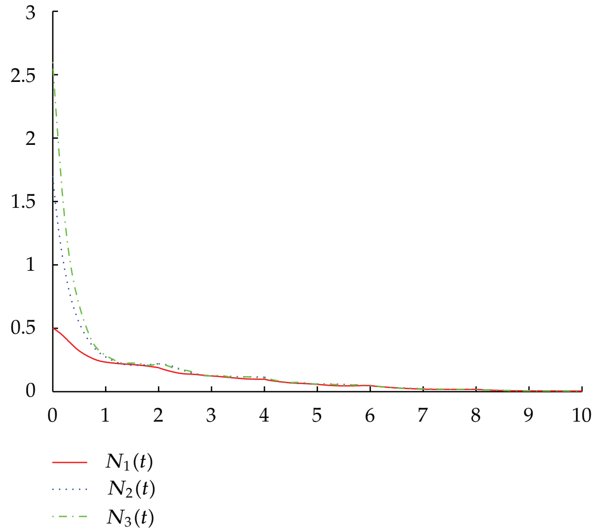

Figure 1: Numerical solution $N(t) = (N_1(t), N_2(t), N_3(t))^T$ of system (4.1) for initial value $\varphi(t) \equiv (0.5, 1.7, 2.6)^T$.

Example 4.2. Consider the following Nicholson's blowflies system with patch structure and nonlinear density-dependent mortality terms:

$$\begin{aligned}
N_1'(t) =& -(12 + |\sin t|) + (11 + |\cos t|)e^{-N_1(t)} + \left(1 + \frac{1}{2}|\sin t|\right) - \left(\frac{1}{2} + \frac{1}{2}|\cos t|\right)e^{-N_2(t)} \\
&+ \left(1 + \frac{1}{2}|\sin t|\right) - \left(\frac{1}{2} + \frac{1}{2}|\cos t|\right)e^{-N_3(t)} + \frac{1}{4}\left(1 + \cos^2 t\right)N_1(t - 2|\sin t|)e^{-N_1(t-2|\sin t|)} \\
&+ \frac{1}{4}\left(1 + \sin^2 t\right)N_1(t - 2|\cos t|)e^{-N_1(t-2|\cos t|)} \\
N_2'(t) =& -(12 + |\cos t|) + (11 + |\sin t|)e^{-N_2(t)} + \left(1 + \frac{1}{2}|\cos t|\right) - \left(\frac{1}{2} + \frac{1}{2}|\sin t|\right)e^{-N_1(t)} \\
&+ \left(1 + \frac{1}{2}|\cos t|\right) - \left(\frac{1}{2} + \frac{1}{2}|\sin t|\right)e^{-N_3(t)} + \frac{1}{4}\left(1 + \sin^2 t\right)N_2(t - 2|\cos t|)e^{-N_2(t-2|\cos t|)} \\
&+ \frac{1}{4}\left(1 + \cos^2 t\right)N_2(t - 2|\sin t|)e^{-N_2(t-2|\sin t|)} \\
N_3'(t) =& -(12 + |\sin 2t|) + (11 + |\cos 2t|)e^{-N_3(t)} + \left(1 + \frac{1}{2}|\sin 2t|\right) - \left(\frac{1}{2} + \frac{1}{2}|\cos 2t|\right)e^{-N_1(t)} \\
&+ \left(1 + \frac{1}{2}|\sin 2t|\right) - \left(\frac{1}{2} + \frac{1}{2}|\cos 2t|\right)e^{-N_2(t)} \\
&+ \frac{1}{4}\left(1 + \cos^2 2t\right)N_3(t - 2|\sin 2t|)e^{-N_3(t-2|\sin 2t|)} \\
&+ \frac{1}{4}\left(1 + \sin^2 2t\right)N_3(t - 2|\cos 2t|)e^{-N_3(t-2|\cos 2t|)}.
\end{aligned} \tag{4.3}$$

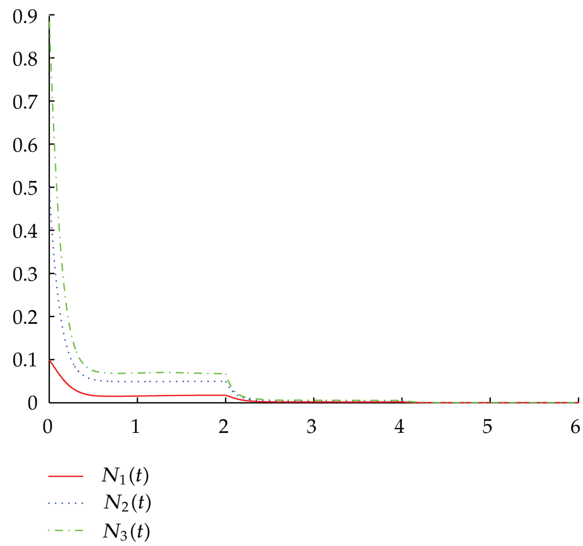

Figure 2: Numerical solution $N(t) = (N_1(t), N_2(t), N_3(t))^T$ of system (4.3) for initial value $\varphi(t) \equiv (0.1, 0.5, 0.9)^T$.

Obviously, $a_{ii}^- = 12$, $b_{ii}^- = 11$, $b_{ii}^+ = 12$, $(i = 1,2,3)$, $a_{ij}^+ = 3/2$, $b_{ij}^- = 1/2$, $b_{ij}^+ = 1$, $(i, j = 1,2,3,\ i \neq j)$, $c_{ij}^+ = 1/2$, $\gamma_{ij}^- = 1$, $(i = 1,2,3,\ j = 1,2)$. Let $K_2 = 1$, then we have

$$
\begin{gathered}
12 = a_{ii}^- > \sum_{j=1, j\neq i}^{3} a_{ij}^+ + \left(b_{ii}^+ - \sum_{j=1, j\neq i}^{n} b_{ij}^- \right) e^{-K_2} + \sum_{j=1}^{2} \frac{c_{ij}^+}{\gamma_{ij}^- e} = 3 + \frac{12}{e}, \quad i = 1,2,3, \\
-a_{ii}(t) + b_{ii}(t) + \sum_{j=1, j\neq i}^{3} \left(a_{ij}(t) - b_{ij}(t) \right) = 0, \quad i = 1,2,3, \qquad (4.4) \\
11 = b_{ii}^- > 1 + \frac{K_2}{2} b_{ii}^+ + \sum_{j=1, j\neq i}^{3} b_{ij}^+ + \sum_{j=1}^{2} c_{ij}^+ = 10, \quad i = 1,2,3.
\end{gathered}
$$

Then (4.4) imply that the system (4.3) satisfies (3.1), (3.2), (3.11), and (3.12). Hence, from Theorems 3.1 and 3.2, the solution $N(t)$ of system (4.1) with $D_{ij}(t, N) = a_{ij}(t) - b_{ij}(t)e^{-N} (i, j = 1,2,3)$ and $\varphi \in E^2 = \{\varphi \mid \varphi \in C_+,\ \varphi(0) > 0 \text{ and } 0 \leq \varphi_i(t) < 1, \text{ for all}, t \in [-2, 0],\ i = 1,2,3\}$ is exponentially extinct as $t \to +\infty$ and $N(t) = N(t, 0, \varphi) = O(e^{-\kappa t})$, $\kappa \approx 0.0001$. The fact is verified by the numerical simulation in Figure 2.

Remark 4.3. To the best of our knowledge, few authors have considered the problems of the extinction of Nicholson's blowflies model with patch structure and nonlinear density-dependent mortality terms. Wang [10] and Hou et al. [11] have researched the permanence and periodic solution for scalar Nicholson's blowflies equation with a nonlinear density-dependent mortality term. Liu and Gong [12] have considered the permanence for Nicholson-type delay systems with nonlinear density-dependent mortality terms and Takeuchi et al. [13] have investigated the global stability of population model with patch structure. Faria [14], Liu [15], and Berzansky et al. [16] have, respectively, studied the local and global stability of positive equilibrium for constant coefficients of Nicholson's blowflies model with patch structure. It is clear that all the results in [10–16] and the references therein cannot be applicable to prove the extinction of (4.1) and (4.3). This implies that the results of this paper are new.

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Research Article

A Generalized Nonuniform Contraction and Lyapunov Function

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For nonautonomous linear equations $x' = A(t)x$, we give a complete characterization of general nonuniform contractions in terms of Lyapunov functions. We consider the general case of nonuniform contractions, which corresponds to the existence of what we call nonuniform (D, μ)-contractions. As an application, we establish the robustness of the nonuniform contraction under sufficiently small linear perturbations. Moreover, we show that the stability of a nonuniform contraction persists under sufficiently small nonlinear perturbations.

1. Introduction

We consider nonautonomous linear equations

$$x' = A(t)x, \tag{1.1}$$

where $A : \mathbb{R}_0^+ \to \mathcal{B}(X)$ is a continuous function with values in the space of bounded linear operators in a Banach space X. Our main aim is to characterize the existence of a general nonuniform contraction for (1.1) in terms of Lyapunov functions.

We assume that each solution of (1.1) is global, and we denote the corresponding evolution operator by $T(t, s)$, which is the linear operator such that

$$T(t, s)x(s) = x(t), \quad t, s \in \mathbb{R}_0^+, \tag{1.2}$$

for any solution $x(t)$ of (1.1). Clearly, $T(t,t) = \mathrm{Id}$ and

$$T(t,\tau)T(\tau,s) = T(t,s), \quad t,\tau,s \in \mathbb{R}_0^+. \tag{1.3}$$

We shall say that an increasing function $\mu : \mathbb{R}_0^+ \to [1,+\infty)$ is a *growth rate* if

$$\mu(0) = 1, \qquad \lim_{t\to+\infty} \mu(t) = +\infty. \tag{1.4}$$

Given two growth rates μ,ν, we say that (1.1) admits a *nonuniform* (μ,ν)*-contraction* if there exist constants $K,\alpha > 0$ and $\varepsilon \geq 0$ such that

$$\|T(t,s)\| \leq K\left(\frac{\mu(t)}{\mu(s)}\right)^{-\alpha} \nu^{\varepsilon}(s), \quad t \geq s \geq 0. \tag{1.5}$$

We emphasize that the notion of nonuniform (μ,ν)-contraction often occurs under reasonably weak assumptions. We refer the reader to [1] for details.

In this work, we mainly consider more general nonuniform contractions (see (2.1) below) and we give a complete characterization of such contractions in terms of Lyapunov functions, especially in terms of quadratic Lyapunov functions, which are Lyapunov functions defined in terms of quadratic forms. The importance of Lyapunov functions is well established, particularly in the study of the stability of trajectories both under linear and nonlinear perturbations. This study goes back to the seminal work of Lyapunov in his 1892 thesis [2]. For more results, we refer the reader to [3–6] for the classical exponential contractions and dichotomies, [7–9] for the nonuniform exponential contractions and nonuniform exponential dichotomies.

The proof of this paper follows from the ideas in [9, 10]. As an application, we provide a very direct proof of the robustness of the nonuniform contraction, that is, of the persistence of the nonuniform contraction in the equation

$$x' = [A(t) + B(t)]x \tag{1.6}$$

for any sufficiently small linear perturbation $B(t)$. We remark that the so-called robustness problem also has a long history. In particular, the problem was discussed by Massera and Schäffer [11], Perron [12], Coppel [3] and in the case of Banach spaces by Daletskiĭ and Kreĭn [13]. For more recent work we refer to [14–16] and the references therein.

Furthermore, for a large class of nonlinear perturbations $f(t,x)$ with $f(t,0) = 0$ for every t, we show that if (1.1) admits a nonuniform contraction, then the zero solution of the equation

$$x' = A(t)x + f(t,x) \tag{1.7}$$

is stable. The proof uses the corresponding characterization between the nonuniform contractions and quadratic Lyapunov functions.

2. Lyapunov Functions and Nonuniform Contractions

Given a growth rate μ and a function $D : \mathbb{R}_0^+ \to (0, +\infty)$, we say that (1.1) admits a *nonuniform* (D, μ)*-contraction* if there exists a constant $\alpha > 0$ such that

$$\|T(t,s)\| \leq D(s)\left(\frac{\mu(t)}{\mu(s)}\right)^{-\alpha}, \quad t \geq s \geq 0. \tag{2.1}$$

The nonuniform (μ, ν)-contraction is a special case of nonuniform (D, μ)-contraction with $D(s) = K\nu^{\varepsilon}(s)$.

Now we introduce the notion of Lyapunov functions. We say that a continuous function $V : (0, +\infty) \times X \to \mathbb{R}_0^-$ is a *strict Lyapunov function* to (1.1) if

(1) for every $t > 0$ and $x \in X$,

$$\|x\| \leq |V(t,x)| \leq D(t)\|x\|, \tag{2.2}$$

(2) for every $t \geq s > 0$ and $x \in X$,

$$V(s,x) \leq V(t, T(t,s)x), \tag{2.3}$$

(3) there exists a constant $\gamma > 0$ such that for every $t \geq s > 0$ and $x \in X$,

$$|V(t, T(t,s)x)| \leq \left(\frac{\mu(t)}{\mu(s)}\right)^{-\gamma} |V(s,x)|. \tag{2.4}$$

The following result gives an optimal characterization of nonuniform (D, μ)-contractions in terms of strict Lyapunov functions.

Theorem 2.1. (1.1) *admits a nonuniform* (D, μ)*-contraction if and only if there exists a strict Lyapunov function for* (1.1)*.*

Proof. We assume that there exists a strict Lyapunov function for (1.1). By (1) and (3), for every $t \geq s > 0$ and $x \in X$, we have

$$\begin{aligned} \|T(t,s)x\| &\leq |V(t, T(t,s)x)| \\ &\leq \left(\frac{\mu(t)}{\mu(s)}\right)^{-\gamma} |V(s,x)| \\ &\leq D(s)\left(\frac{\mu(t)}{\mu(s)}\right)^{-\gamma} \|x\|. \end{aligned} \tag{2.5}$$

Therefore, (1.1) admits a nonuniform (D, μ)-contraction with $\alpha = \gamma$.

Next we assume that (1.1) admits a nonuniform (D,μ)-contraction. For $t>0$ and $x\in X$, we set

$$V(t,x)=-\sup\left\{\|T(\tau,t)x\|\left(\frac{\mu(\tau)}{\mu(t)}\right)^{\alpha}:\tau\geq t\right\}. \tag{2.6}$$

By (2.1), we have $|V(t,x)|\leq D(t)\|x\|$. Moreover, setting $\tau=t$, we obtain $|V(t,x)|\geq\|x\|$. This establishes (1). Furthermore, for $t\geq s$, we have

$$\begin{aligned}
|V(t,T(t,s)x)|&=\sup\left\{\|T(\tau,t)T(t,s)x\|\left(\frac{\mu(\tau)}{\mu(t)}\right)^{\alpha}:\tau\geq t\right\}\\
&=\left(\frac{\mu(s)}{\mu(t)}\right)^{\alpha}\sup\left\{\|T(\tau,s)x\|\left(\frac{\mu(\tau)}{\mu(s)}\right)^{\alpha}:\tau\geq t\right\}\\
&\leq\left(\frac{\mu(s)}{\mu(t)}\right)^{\alpha}\sup\left\{\|T(\tau,s)x\|\left(\frac{\mu(\tau)}{\mu(s)}\right)^{\alpha}:\tau\geq s\right\}\\
&=\left(\frac{\mu(t)}{\mu(s)}\right)^{-\alpha}|V(s,x)|.
\end{aligned} \tag{2.7}$$

Therefore, V is a strict Lyapunov function for (1.1). □

Next we consider another class of Lyapunov functions, namely, those defined in terms of quadratic forms.

Let $S(t)\in\mathcal{B}(X)$ be a symmetric positive-definite operator for each $t>0$. A *quadratic Lyapunov function* V is given as

$$H(t,x)=\langle S(t)x,x\rangle,\qquad V(t,x)=-\sqrt{H(t,x)}. \tag{2.8}$$

Given linear operators M,N, we write $M\leq N$ if $\langle Mx,x\rangle\leq\langle Nx,x\rangle$ for $x\in X$.

Theorem 2.2. *Assume that there exist constants $c>0$ and $d\geq 1$ such that*

$$\|T(t,s)\|\leq c\quad\text{whenever }\mu(t)\leq d\mu(s),\ t\geq s>0. \tag{2.9}$$

Then (1.1) admits a nonuniform (D,μ)-contraction (up to a multiplicative constant) if and only if there exist symmetric positive definite operators $S(t)$ and constants $C,K>0$ such that $S(t)$ is of class C^1 in $t>0$ and

$$\|S(t)\|\leq CD(t)^2, \tag{2.10}$$

$$S'(t)+A^*(t)S(t)+S(t)A(t)\leq-(\mathrm{Id}+KS(t))\frac{\mu'(t)}{\mu(t)}. \tag{2.11}$$

Proof. We first assume that (1.1) admits a nonuniform (D,μ)-contraction. Consider the linear operators

$$S(t) = \int_t^\infty T(\tau,t)^* T(\tau,t)\left(\frac{\mu(\tau)}{\mu(t)}\right)^{2(\alpha-\rho)}\frac{\mu'(\tau)}{\mu(\tau)}d\tau, \tag{2.12}$$

for some constant $\rho \in (0,\alpha)$. Clearly, $S(t)$ is symmetric for each $t > 0$. Moreover, by (2.8), we have

$$\begin{aligned}\|H(t,x)\| &= \int_t^\infty \|T(\tau,t)x\|^2\left(\frac{\mu(\tau)}{\mu(t)}\right)^{2(\alpha-\rho)}\frac{\mu'(\tau)}{\mu(\tau)}d\tau \\ &\le D(t)^2\|x\|^2\int_t^\infty \left(\frac{\mu(\tau)}{\mu(t)}\right)^{-2\rho}\frac{\mu'(\tau)}{\mu(\tau)}d\tau \\ &= \frac{D(t)^2}{2\rho}\|x\|^2.\end{aligned} \tag{2.13}$$

Since $S(t)$ is symmetric, we obtain

$$\|S(t)\| = \sup_{x\neq 0}\frac{|H(t,x)|}{\|x\|^2} \le \frac{D(t)^2}{2\rho} \tag{2.14}$$

and therefore (2.10) holds. Since

$$\frac{\partial}{\partial t}T(\tau,t) = -T(\tau,t)A(t), \qquad \frac{\partial}{\partial t}T(\tau,t)^* = -A(t)^*T(\tau,t)^*, \tag{2.15}$$

we find that $S(t)$ is of class C^1 in t with derivative

$$\begin{aligned}S'(t) = &-\frac{\mu'(t)}{\mu(t)} - \int_t^\infty A(t)^*T(\tau,t)^*T(\tau,t)\left(\frac{\mu(\tau)}{\mu(t)}\right)^{2(\alpha-\rho)}\frac{\mu'(\tau)}{\mu(\tau)}d\tau \\ &- \int_t^\infty T(\tau,t)^*T(\tau,t)A(t)\left(\frac{\mu(\tau)}{\mu(t)}\right)^{2(\alpha-\rho)}\frac{\mu'(\tau)}{\mu(\tau)}d\tau \\ &- 2(\alpha-\rho)\frac{\mu'(t)}{\mu(t)}\int_t^\infty T(\tau,t)^*T(\tau,t)\left(\frac{\mu(\tau)}{\mu(t)}\right)^{2(\alpha-\rho)}\frac{\mu'(\tau)}{\mu(\tau)}d\tau,\end{aligned} \tag{2.16}$$

which implies that

$$S'(t) = -\frac{\mu'(t)}{\mu(t)} - A(t)^*S(t) - S(t)A(t) - 2(\alpha-\rho)\frac{\mu'(t)}{\mu(t)}S(t). \tag{2.17}$$

Therefore,

$$S'(t) + A(t)^* S(t) + S(t)A(t) = -\frac{\mu'(t)}{\mu(t)}\big(\mathrm{Id} + 2(\alpha - \rho)S(t)\big), \tag{2.18}$$

which establishes (2.11) with $K = 2(\alpha - \rho)$.

Now we assume that conditions (2.9) and (2.10)-(2.11) hold. Set $x(t) = T(t,\tau)x(\tau)$. By (2.10), we have

$$\|H(t,x(t))\| \le \|S(t)\| \cdot \|x(t)\|^2 \le CD(t)^2 \|x(t)\|^2. \tag{2.19}$$

Lemma 2.3. *There exists a constant $\eta > 0$ such that*

$$H(t,x(t)) \ge \eta \|x(t)\|^2. \tag{2.20}$$

Proof of Lemma 2.3. Note that

$$\begin{aligned} \frac{d}{dt} H(t,x(t)) &= \langle S'(t)x(t), x(t)\rangle + \langle S(t)A(t)x(t), x(t)\rangle + \langle S(t)x(t), A(t)x(t)\rangle \\ &= \big\langle \big(S'(t) + S(t)A(t) + A(t)^* S(t)\big)x(t), x(t)\big\rangle. \end{aligned} \tag{2.21}$$

Hence, by condition (2.11), and the fact that $K > 0$ we obtain

$$\frac{d}{dt} H(t,x(t)) \le -\frac{\mu'(t)}{\mu(t)} \|x(t)\|^2. \tag{2.22}$$

Now given $\tau > 0$, take $t > \tau$ such that $\mu(t) = d\mu(\tau)$ with d as in (2.9). Then

$$\begin{aligned} H(t,x(t)) - H(\tau,x(\tau)) &= \int_\tau^t \frac{d}{dv} H(v,x(v))dv \\ &\le -\int_\tau^t \frac{\mu'(v)}{\mu(v)} \|x(v)\|^2 dv \\ &= -\int_\tau^t \frac{\mu'(v)}{\mu(v)} \|T(v,\tau)x(\tau)\|^2 dv \\ &\le -\|x(\tau)\|^2 \int_\tau^t \frac{\mu'(v)}{\mu(v)} \frac{1}{\|T(\tau,v)\|^2} dv. \end{aligned} \tag{2.23}$$

It follows from (2.9) that

$$\begin{aligned} H(t,x(t)) - H(\tau,x(\tau)) &\le -\frac{1}{c^2}\|x(\tau)\|^2 \int_\tau^t \frac{\mu'(v)}{\mu(v)}dv \\ &= -\frac{\log d}{c^2}\|x(\tau)\|^2. \end{aligned} \tag{2.24}$$

Since $H(t,x(t)) \ge 0$, we have

$$H(\tau,x(\tau)) \ge H(\tau,x(\tau)) - H(t,x(t)) \ge \frac{\log d}{c^2}\|x(\tau)\|^2 \tag{2.25}$$

which yields (2.20) with $\eta = (\log d)/c^2 > 0$.

Lemma 2.4. *For $t \ge \tau$, one has*

$$H(t,x(t)) \le \left(\frac{\mu(t)}{\mu(\tau)}\right)^{-K} H(\tau,x(\tau)). \tag{2.26}$$

Proof of Lemma 2.4. By conditions (2.11) and (2.21), we have

$$\frac{d}{dt}H(t,x(t)) \le -K\frac{\mu'(t)}{\mu(t)}H(t,x(t)). \tag{2.27}$$

Therefore,

$$\begin{aligned} H(t,x(t)) - H(\tau,x(\tau)) &= \int_\tau^t \frac{d}{dv}H(v,x(v))dv \\ &\le -K\int_\tau^t \frac{\mu'(v)}{\mu(v)}H(v,x(v))dv. \end{aligned} \tag{2.28}$$

It follows from Gronwall's lemma that

$$H(t,x(t)) \le \left(\frac{\mu(t)}{\mu(\tau)}\right)^{-K} H(\tau,x(\tau)), \tag{2.29}$$

which yields the desired result.

By Lemmas 2.3 and 2.4 together with (2.19), we obtain

$$\begin{aligned}
\|T(t,\tau)x(\tau)\|^2 &= \|x(t)\|^2 \\
&\leq \eta^{-1}H(t,x(t)) \\
&\leq \eta^{-1}\left(\frac{\mu(t)}{\mu(\tau)}\right)^{-K} H(\tau,x(\tau)) \\
&\leq \eta^{-1}CD(\tau)^2\left(\frac{\mu(t)}{\mu(\tau)}\right)^{-K} \|x(\tau)\|^2,
\end{aligned} \tag{2.30}$$

and therefore,

$$\|T(t,\tau)\|^2 \leq \eta^{-1}CD(\tau)^2\left(\frac{\mu(t)}{\mu(\tau)}\right)^{-K}, \tag{2.31}$$

which implies that (1.1) admits a nonuniform (D,μ)-contraction. □

As an application of Theorem 2.2, we establish the robustness of nonuniform (D,μ)-contractions. Roughly speaking, a nonuniform contraction for (1.1) is said to be *robust* if (1.6) still admits a nonuniform contraction for any sufficiently small perturbation $B(t)$.

Theorem 2.5. *Let $A,B : \mathbb{R}_0^+ \to \mathcal{B}(X)$ be continuous functions such that* (1.1) *admits a nonuniform (D,μ)-contraction with condition* (2.9)*. Suppose further that $D(t) \geq 1$ for every $t > 0$ and*

$$\|B(t)\| \leq \delta D^{-2}(t)\frac{\mu'(t)}{\mu(t)}, \quad t > 0 \tag{2.32}$$

for some $\delta > 0$ sufficiently small. Then (1.6) *admits a nonuniform (D,μ)-contraction.*

Proof. Let $U(t,s)$ be the evolution operator associated to (1.6). It is easy to verify that

$$U(t,s) = T(t,s) + \int_s^t T(t,\tau)B(\tau)U(\tau,s)d\tau. \tag{2.33}$$

For every $t \geq s > 0$ with $\mu(t) \leq d\mu(s)$, we have

$$\begin{aligned}
\|U(t,s)\| &\leq c + \int_s^t c\delta D^{-2}(\tau)\frac{\mu'(\tau)}{\mu(\tau)}\|U(\tau,s)\|d\tau \\
&\leq c + c\delta\int_s^t \frac{\mu'(\tau)}{\mu(\tau)}\|U(\tau,s)\|d\tau.
\end{aligned} \tag{2.34}$$

Using Gronwall's inequality, we obtain

$$\|U(t,s)\| \leq c\exp\left(c\delta\int_s^t \frac{\mu'(\tau)}{\mu(\tau)}d\tau\right) \leq c\exp(c\delta\log d) \tag{2.35}$$

for every $t \geq s > 0$ with $\mu(t) \leq d\mu(s)$. Therefore condition (2.9) also holds for the perturbed equation (1.6).

Now we consider the matrices $S(t)$ in (2.12). Condition (2.10) can be obtained as in the proof of Theorem 2.2. For condition (2.11), it is sufficient to show that

$$S(t)B(t) + B(t)^* S(t) \leq \vartheta\frac{\mu'(t)}{\mu(t)}\mathrm{Id} \tag{2.36}$$

for some constant $\vartheta < 1$. Using (2.10) and (2.32), we have

$$\begin{aligned} S(t)B(t) + B(t)^* S(t) &\leq 2\|S(t)\| \cdot \|B(t)\| \\ &\leq 2CD(t)^2\delta D(t)^{-2}\frac{\mu'(t)}{\mu(t)} \\ &= 2C\delta\frac{\mu'(t)}{\mu(t)}, \end{aligned} \tag{2.37}$$

and taking δ sufficiently small, we find that (2.36) holds with some $\vartheta < 1$. □

3. Stability of Nonlinear Perturbations

Before stating the result, we fist prove an equivalent characterization of property (3). Given matrices $S(t) \in \mathcal{B}(X)$ for each $t \in \mathbb{R}_0^+$, we consider the functions

$$\begin{aligned} \dot{H}(t,x) &= \frac{d}{dh}H(t+h, T(t+h,h)x)|_{h=0}, \\ \dot{V}(t,x) &= \frac{d}{dh}V(t+h, T(t+h,h)x)|_{h=0}, \end{aligned} \tag{3.1}$$

whenever the derivatives are well defined and H, V are given as (2.8).

Lemma 3.1. *Let V, μ be C^1 functions. Then property* (3) *is equivalent to*

$$\dot{V}(t, T(t,\tau)x) \geq -\gamma V(t, T(t,\tau)x)\frac{\mu'(t)}{\mu(t)}, \quad t > \tau. \tag{3.2}$$

Proof. Now we assume that property (3) holds. If $t > \tau$ and $h > 0$, then

$$\begin{aligned}
V(t+h, T(t+h,\tau)x) &= V(t+h, T(t+h,t)T(t,\tau)x) \\
&\geq \left(\frac{\mu(t+h)}{\mu(t)}\right)^{-\gamma} V(t, T(t,\tau)x), \\
\lim_{h\to 0^+} \frac{V(t+h, T(t+h,\tau)x) - V(t, T(t,\tau)x)}{h} &\geq V(t, T(t,\tau)x) \lim_{h\to 0^+} \frac{(\mu(t+h)/\mu(t))^{-\gamma} - 1}{h} \\
&= -\gamma V(t, T(t,\tau)x) \frac{\mu'(t)}{\mu(t)}.
\end{aligned} \tag{3.3}$$

Similarly, if $h < 0$ is such that $t + h > \tau$, then

$$\begin{aligned}
&V(t+h, T(t+h,\tau)x) \leq \left(\frac{\mu(t+h)}{\mu(t)}\right)^{-\gamma} V(t, T(t,\tau)x), \\
&\lim_{h\to 0^-} \frac{V(t+h, T(t+h,\tau)x) - V(t, T(t,\tau)x)}{h} \\
&\quad\geq V(t, T(t,\tau)x) \lim_{h\to 0^-} \frac{(\mu(t+h)/\mu(t))^{-\gamma} - 1}{h} \\
&\quad= -\gamma V(t, T(t,\tau)x) \frac{\mu'(t)}{\mu(t)}.
\end{aligned} \tag{3.4}$$

This establishes (3.2).

Next we assume that (3.2) holds. We rewrite (3.2) in the form

$$\frac{\dot{V}(t, T(t,\tau)x)}{V(t, T(t,\tau)x)} \geq -\gamma \frac{\mu'(t)}{\mu(t)}, \quad t > \tau, \tag{3.5}$$

which implies that

$$\begin{aligned}
\log\left(\frac{V(t, T(t,\tau)x)}{V(\tau, x)}\right) &= \int_\tau^t \frac{\dot{V}(v, T(v,\tau)x)}{V(v, T(v,\tau)x)} dv \\
&\geq -\gamma \int_\tau^t \frac{\mu'(v)}{\mu(v)} dv \\
&= \log\left(\frac{\mu(t)}{\mu(\tau)}\right)^{-\gamma},
\end{aligned} \tag{3.6}$$

and hence property (3) holds. □

Theorem 3.2. *Assume that* (1.1) *admits a nonuniform* (D,μ)*-contraction satisfying* (2.9). *Suppose further that there exists a constant* $l > 0$ *such that* $l < \alpha$ *and*

$$\|f(t,x)\| \le l\frac{\mu'(t)}{\mu(t)}\|x\|, \quad t > 0,\ x \in X. \tag{3.7}$$

Then for each $k > -\alpha + l$, *there exists* $C > 0$ *such that*

$$\|y(t)\| \le CD(s)\left(\frac{\mu(t)}{\mu(s)}\right)^k \|y(s)\|, \quad t \ge s \tag{3.8}$$

for every solution $y(t)$ *of* (1.7).

Proof. For $S(t)$ as in (2.12) and $H(t,x(t))$ as in (2.8), we have, for every $t \ge s$,

$$\begin{aligned}
H(t,T(t,s)x(s)) &= \int_t^\infty \|T(v,s)x(s)\|^2\left(\frac{\mu(v)}{\mu(t)}\right)^{2(\alpha-\rho)}\frac{\mu'(v)}{\mu(v)}dv \\
&= \left(\frac{\mu(t)}{\mu(s)}\right)^{-2(\alpha-\rho)}\int_t^\infty \|T(v,s)x(s)\|^2\left(\frac{\mu(v)}{\mu(s)}\right)^{2(\alpha-\rho)}\frac{\mu'(v)}{\mu(v)}dv \\
&\le \left(\frac{\mu(t)}{\mu(s)}\right)^{-2(\alpha-\rho)}\int_s^\infty \|T(v,s)x(s)\|^2\left(\frac{\mu(v)}{\mu(s)}\right)^{2(\alpha-\rho)}\frac{\mu'(v)}{\mu(v)}dv \\
&= \left(\frac{\mu(t)}{\mu(s)}\right)^{-2(\alpha-\rho)}H(s,x(s)).
\end{aligned} \tag{3.9}$$

Since $V(t,x) = -\sqrt{H(t,x)}$, we have

$$V(t,T(t,s)x(s)) \ge \left(\frac{\mu(t)}{\mu(s)}\right)^{-(\alpha-\rho)}V(s,x(s)), \quad t \ge s. \tag{3.10}$$

Applying Lemma 3.1, we obtain

$$\dot V(t,T(t,s)x(s)) \ge -(\alpha-\rho)\frac{\mu'(t)}{\mu(t)}V(t,T(t,s)x(s)), \quad t \ge s. \tag{3.11}$$

In particular, for $t = s$,

$$\dot V(s,x(s)) \ge -(\alpha-\rho)\frac{\mu'(s)}{\mu(s)}V(s,x(s)). \tag{3.12}$$

From the identity $\dot H = 2V\dot V$ that for every $s > 0$ and $x \in X$, we have

$$\dot H(s,x) \le -2(\alpha-\rho)\frac{\mu'(s)}{\mu(s)}H(s,x). \tag{3.13}$$

On the other hand,

$$\dot{H}(s,x) = \langle (S'(s) + S(s)A(s) + A(s)^* S(s))x, x\rangle. \tag{3.14}$$

Therefore,

$$\begin{aligned} 0 &\geq \dot{H}(s,x) + 2(\alpha-\rho)\frac{\mu'(s)}{\mu(s)}H(s,x) \\ &= \left\langle \left(S'(s) + S(s)A(s) + A(s)^* S(s) + 2(\alpha-\rho)\frac{\mu'(s)}{\mu(s)}S(s) \right)x, x \right\rangle, \end{aligned} \tag{3.15}$$

and hence

$$S'(t) + S(t)A(t) + A(t)^* S(t) + 2(\alpha-\rho)\frac{\mu'(t)}{\mu(t)}S(t) \leq 0. \tag{3.16}$$

Therefore, if $y(t)$ is a solution of (1.7), then

$$\begin{aligned} \frac{d}{dt}H(t,y(t)) &= \langle S'(t)y(t), y(t)\rangle + \langle S(t)A(t)y(t), y(t)\rangle + \langle S(t)y(t), A(t)y(t)\rangle \\ &\quad + \langle S(t)f(t,y(t)), y(t)\rangle + \langle S(t)y(t), f(t,y(t))\rangle \\ &= \langle (S'(t) + S(t)A(t) + A(t)^* S(t))y(t), y(t)\rangle \\ &\quad + \langle (S(t) + S(t)^*)y(t), f(t,y(t))\rangle \\ &\leq -2(\alpha-\rho)\frac{\mu'(t)}{\mu(t)}\|S(t)\| \cdot \|y(t)\|^2 + \langle (S(t) + S(t)^*)y(t), f(t,y(t))\rangle \\ &\leq -2(\alpha-\rho)\frac{\mu'(t)}{\mu(t)}\|S(t)\| \cdot \|y(t)\|^2 + 2\|S(t)\| \cdot \|f(t,y(t))\| \cdot \|y(t)\| \\ &\leq -2(\alpha-\rho)\frac{\mu'(t)}{\mu(t)}\|S(t)\| \cdot \|y(t)\|^2 + 2l\frac{\mu'(t)}{\mu(t)}\|S(t)\| \cdot \|y(t)\|^2 \\ &= -2(\alpha-\rho-l)\frac{\mu'(t)}{\mu(t)}\|S(t)\| \cdot \|y(t)\|^2. \end{aligned} \tag{3.17}$$

If ρ is small enough such that $\alpha-\rho-l > 0$, then

$$\frac{d}{dt}H(t,y(t)) \leq -2(\alpha-\rho-l)\frac{\mu'(t)}{\mu(t)}H(t,y(t)), \tag{3.18}$$

and hence

$$H(t,y(t)) - H(s,y(s)) \leq -2(\alpha-\rho-l)\int_s^t \frac{\mu'(\tau)}{\mu(\tau)}H(\tau,y(\tau))d\tau. \tag{3.19}$$

It follows from Gronwall's inequality that

$$H(t, y(t)) \leq H(s, y(s)) \left(\frac{\mu(t)}{\mu(s)}\right)^{-2(\alpha-\rho-l)}, \quad t \geq s. \tag{3.20}$$

Now given $s > 0$, take $t > s$ such that $\mu(t) = d\mu(s)$ with d as in (2.9). Then

$$\begin{aligned} H(s, y(s)) &= \int_s^\infty \|T(\tau, s)y(s)\|^2 \left(\frac{\mu(\tau)}{\mu(s)}\right)^{2(\alpha-\rho)} \frac{\mu'(\tau)}{\mu(\tau)} d\tau \\ &\geq \int_s^t \|T(\tau, s)y(s)\|^2 \left(\frac{\mu(\tau)}{\mu(s)}\right)^{2(\alpha-\rho)} \frac{\mu'(\tau)}{\mu(\tau)} d\tau \\ &\geq \frac{1}{c^2}\|y(s)\|^2 \int_s^t \left(\frac{\mu(\tau)}{\mu(s)}\right)^{2(\alpha-\rho)} \frac{\mu'(\tau)}{\mu(\tau)} d\tau \\ &= \frac{1}{2c^2(\alpha-\rho)}\|y(s)\|^2 \left\{\left(\frac{\mu(t)}{\mu(s)}\right)^{2(\alpha-\rho)} - 1\right\} \\ &= \frac{1}{2c^2(\alpha-\rho)}\|y(s)\|^2 \left\{d^{2(\alpha-\rho)} - 1\right\}. \end{aligned} \tag{3.21}$$

Taking

$$\kappa = \frac{1}{2c^2(\alpha-\rho)} \left\{d^{2(\alpha-\rho)} - 1\right\} > 0, \tag{3.22}$$

then

$$H(s, y(s)) \geq \kappa \|y(s)\|^2. \tag{3.23}$$

It follows from (2.13) and (3.20) that

$$\begin{aligned} \|y(t)\| &\leq \kappa^{1/2} \sqrt{H(t, y(t))} \\ &\leq \kappa^{1/2} \sqrt{H(s, y(s))} \left(\frac{\mu(t)}{\mu(s)}\right)^{-(\alpha-\rho-l)} \\ &\leq \kappa^{1/2} \sqrt{\frac{1}{2\rho} D(s)} \left(\frac{\mu(t)}{\mu(s)}\right)^{-(\alpha-\rho-l)} \|y(s)\|. \end{aligned} \tag{3.24}$$

Now the proof is finished. □

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Research Article

Admissibility for Nonuniform (μ, ν) Contraction and Dichotomy

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The relation between the notions of nonuniform asymptotic stability and admissibility is considered. Using appropriate Lyapunov norms, it is showed that if any of their associated $\mathcal{L}^p$ spaces, with $p \in (1, \infty]$, is admissible for a given evolution process, then this process is a nonuniform (μ, ν) contraction and dichotomy. A collection of admissible Banach spaces for any given nonuniform (μ, ν) contraction and dichotomy is provided.

1. Introduction

The study of the admissibility property has a fairly long history, and it goes back to the pioneering work of Perron [1] in 1930. Perron concerned originally the existence of bounded solutions of the equation

$$x' = A(t)x + f(t) \tag{1.1}$$

in $\mathbb{R}_n$ for any bounded continuous perturbation $f : \mathbb{R}_0^+ \to \mathbb{R}_n$. This property can be used to deduce the stability or the conditional stability under sufficiently small perturbations of a given linear equation:

$$x' = A(t)x. \tag{1.2}$$

His result served as a starting point for many works on the qualitative theory of the solutions of differential equations. Moreover, a simple consequence of one of the main results in that paper stated explicitly in [2, Theorem 1] is probably the first step in the literature concerning the study of the relation between admissibility and the notions of stability and conditional stability. We refer the reader to [2] for details. Relevant results concerning the extension of Perron's problem in the more general framework of the infinite-dimensional Banach spaces with bounded $A(t)$ were obtained by Daleckij and Krein [3], Massera and Schäffer [4], and the work of Levitan and Zhikov [5] for certain cases of unbounded $A(t)$.

Over the last decades an increasing interest can be seen in the study of the asymptotic behavior of evolution equations in abstract spaces. In [6, 7], Latushkin et al. studied the dichotomy of linear skew-product semiflows defined on compact spaces. Using the so-called evolution semigroup, they expressed its dichotomy in terms of hyperbolicity of a family of weighted shift operators. In [8–10], Preda et al. considered related problems in the particular case of uniform exponential behavior. A large class of Schäffer spaces, which were introduced by Schäffer in [11] (see also [4]), acted as admissible spaces for the case of uniform exponential dichotomies. It is worth noting here the works by Huy [12–16] in the study of the existence of an exponential dichotomy for evolution equations.

In the case of nonuniform exponential dichotomies, Preda and Megan [17] obtained related results also for the class of Schäffer spaces, but using a notion of dichotomy which is different from the original one motivated by ergodic theory and the nonuniform hyperbolicity theory, as detailed, for example, in [18, 19]. In the more recent work [20], the authors consider the same weaker notion of exponential dichotomy and obtain sharper relations between admissibility and stability for perturbations and solutions in C_0. Important contributions in this aspect have been made by Barreira et al. [2, 18, 19, 21–25]. Particularly, in [22], Barreira and Valls showed an equivalence between the admissibility of their associated $\mathcal{L}^p$ spaces ($p \in (1, \infty]$) and the nonuniform exponential stability of certain evolution families by using appropriate adapted norms. They also establish a collection of admissible Banach spaces for any given nonuniform exponential dichotomy in [2]. Recently, Preda et al. [26] studied the connection between the (non)uniform exponential dichotomy of a non(uniform) exponentially bounded, strongly continuous evolution family and the admissibility of some function spaces, which extended those results established in [2, 22].

In the present paper, inspired by Barreira and Valls [2, 22], we give a characterization of nonuniform asymptotic stability in terms of admissibility property. We consider a more general type of dichotomy which is called (μ, ν) dichotomy in [21], also proposed in [27]. In this dichotomy, not only the usual exponential behavior is replaced by an arbitrary, which may correspond, for example, to situations when the Lyapunov exponents are all infinity or are all zero, but also different growth rates for the uniform and nonuniform parts of the dichotomy are considered. It extended exponential dichotomy in various ways. In [21], it has also been showed that there is a large class of equations exhibiting this behavior. We emphasize that the characterization in our paper is a very general one; it includes as particular cases many interesting situations among them we can mention some results in previous references. To some extent, our results have a certain significance to study the theory of nonuniform hyperbolicity.

2. Admissibility for Nonuniform (μ, ν) Contractions

We first concentrate on the simpler case of admissibility for nonuniform (μ, ν) contractions, leaving the more elaborate case of admissibility for nonuniform (μ, ν) dichotomies for the second part of the paper. This allows us to present the results and their proofs without some accessory technicalities. After the introduction of some basic notions, using appropriate adapted Lyapunov norms, we show that the admissibility with respect to some space $\mathcal{L}^p$ with $p \in (1, \infty]$ is sufficient for an evolution process to be a nonuniform (μ, ν) contraction.

2.1. Basic Notions

We say that an increasing function $\mu : \mathbb{R}^+ \to [1, +\infty)$ is a growth rate if

$$\mu(0) = 1, \qquad \lim_{t \to +\infty} \mu(t) = +\infty. \tag{2.1}$$

We say that a family of linear operators $T(t, s)$, $t \geq s \geq 0$ in a Banach space X is an evolution process if:

(1) $T(t, t) = Id$ and $T(t, \tau)T(\tau, s) = T(t, s)$, $t, \tau, s > 0$;

(2) $(t, s, x) \mapsto T(t, s)x$ is continuous for $t \geq s \geq 0$ and $x \in X$.

In this section, we also assume that

(3) there exist $\omega \geq 0$, $D > 0$ and two growth rates $\mu(t)$, $\nu(t)$ such that

$$\|T(t, s)\| \leq D\left(\frac{\mu(t)}{\mu(s)}\right)^{\omega} \nu^{\varepsilon}(s), \quad t \geq s \geq 0. \tag{2.2}$$

We consider the new norms

$$\|x\|'_t = \sup\left\{\|T(\sigma, t)x\|\left(\frac{\mu(\sigma)}{\mu(t)}\right)^{-\omega}, \sigma \geq t\right\}, \quad x \in X,\ t \in \mathbb{R}^+_0. \tag{2.3}$$

These satisfy

$$\|x\| \leq \|x\|'_t \leq D\nu^{\varepsilon}(t)\|x\|, \quad x \in X,\ t \in \mathbb{R}^+_0. \tag{2.4}$$

Moreover, with respect to these norms the evolution process has the following bounded growth property.

Proposition 2.1. *If T is an evolution process, then*

$$\|T(t, s)x\|'_t \leq \left(\frac{\mu(t)}{\mu(s)}\right)^{\omega} \|x\|'_s \tag{2.5}$$

for every $t \geq s \geq 0$ and $x \in X$.

Proof. We have

$$\begin{aligned} \|T(t, s)x\|'_t &= \sup\left\{\|T(\sigma, t)x\|\left(\frac{\mu(\sigma)}{\mu(t)}\right)^{-\omega}, \sigma \geq t\right\} \\ &\leq \left(\frac{\mu(t)}{\mu(s)}\right)^{\omega} \sup\left\{\|T(\sigma, t)x\|\left(\frac{\mu(\sigma)}{\mu(s)}\right)^{-\omega}, \sigma \geq s\right\} \\ &= \left(\frac{\mu(t)}{\mu(s)}\right)^{\omega} \|x\|'_s \end{aligned} \tag{2.6}$$

which yields the desired inequality. □

Definition 2.2. We say that an evolution process T is a nonuniform (μ, ν) contraction in $\mathbb{R}_0^+$ if there exist some constants α, $D > 0$, $\varepsilon \geq 0$ and two growth rates $\mu(t)$, $\nu(t)$ such that

$$\|T(t,s)\| \leq D\left(\frac{\mu(t)}{\mu(s)}\right)^{-\alpha} \nu^{\varepsilon}(s), \quad t \geq s \geq 0. \tag{2.7}$$

When $\varepsilon = 0$, we say that (1.2) has a uniform (μ, ν) contraction or simply a (μ, ν) contraction.

In the following, we introduce several Banach spaces that are used throughout the paper. We first set

$$L^p = \left\{ f : \mathbb{R}_0^+ \longrightarrow \mathbb{R} \text{ Lebesgue-measurable} : \|f\|_p < \infty \right\} \tag{2.8}$$

for each $p \in [1, \infty)$, and

$$L^\infty = \left\{ f : \mathbb{R}_0^+ \longrightarrow \mathbb{R} \text{ Lebesgue-measurable} : \|f\|_\infty < \infty \right\} \tag{2.9}$$

Respectively, with the norms

$$\|f\|_p = \left(\int_0^\infty |f(t)|^p\right)^{1/p}, \qquad \|f\|_\infty = \operatorname*{ess\,sup}_{t\in\mathbb{R}_0^+} |f(t)|. \tag{2.10}$$

Then for each $p \in [1, \infty]$ the set $\mathcal{L}^p$ of the equivalence classes $[f]$ of functions $g \in L^p$ such that $g = f$ Lebesgue-almost everywhere is a Banach space (again with the norms in (2.10)).

For each Banach space $E = \mathcal{L}^p$, with $p \in [1, \infty]$, we set

$$E(X) = \left\{ f : \mathbb{R}_0^+ \longrightarrow X \text{ Bochner-measurable} : t \longmapsto \|f(t)\|_t' \in E \right\} \tag{2.11}$$

using the norms $\|\cdot\|_t'$ in (2.3), and we endow $E(X) = \mathcal{L}^p(X)$ with the norm

$$\|f\|_p' = \|F\|_p, \quad \text{where } F(t) = \|f(t)\|_t'. \tag{2.12}$$

Repeating arguments in the proof of Theorem 3 in [22], we obtain the following statement.

Lemma 2.3. *For each $p \in [1, \infty]$ and $E = \mathcal{L}^p$, the set $E(X)$ is a Banach space with the norm in* (2.12)*, and the convergence in $E(X)$ implies the pointwise convergence Lebesgue-almost everywhere.*

Definition 2.4. We say that a Banach space E is admissible for the evolution process T if for each $f \in E(X)$ the function $x_f : \mathbb{R}_0^+ \to X$ defined by

$$x_f(t) = \int_0^\infty T(t,\tau) f(\tau) d\tau \tag{2.13}$$

is in $\mathcal{L}^\infty$ (see (2.11)).

By Lemma 2.3 we know that $\mathcal{L}^\infty$ is a Banach space with the norm

$$\|g\|'_\infty = \operatorname*{ess\,sup}_{t\in\mathbb{R}_0^+}\|g(t)\|'_t. \tag{2.14}$$

Lemma 2.5. *There exists $K > 0$ such that*

$$\|x_f\|'_\infty \le K\|f\|'_p \quad \textit{for every } f \in E(X). \tag{2.15}$$

Proof. We define a linear operator $G : E(X) \to \mathcal{L}^\infty(X)$ by $Gf = x_f$. We use the closed graph theorem to show that G is bounded. For this, let us take a sequence $(f_n)_{n\in\mathbb{N}} \subset E(X)$ and $f \in E(X)$ such that $f_n \to f$ in $E(X)$ when $n \to \infty$ and also $h \in \mathcal{L}^\infty(X)$ such that $Gf_n \to h$ in $\mathcal{L}^\infty(X)$ when $n \to \infty$. We need to show that $Gf = h$ Lebesgue-almost everywhere. For each $t \ge 0$ and $n \in N$ we have

$$\begin{aligned}
\|(Gf_n)(t) - (Gf)(t)\|'_t &= \sup\left\{\left\|\int_0^t T(\sigma,t)T(t,\tau)(f_n(\tau)-f(\tau))d\tau\right\|\left(\frac{\mu(\sigma)}{\mu(t)}\right)^{-\omega} : \sigma \ge t\right\} \\
&= \sup\left\{\left\|\int_0^t T(\sigma,\tau)(f_n(\tau)-f(\tau))d\tau\right\|\left(\frac{\mu(\sigma)}{\mu(t)}\right)^{-\omega} : \sigma \ge t\right\} \\
&\le \sup\left\{\int_0^t \|T(\sigma,\tau)(f_n(\tau)-f(\tau))\|\left(\frac{\mu(\sigma)}{\mu(t)}\right)^{-\omega} d\tau : \sigma \ge t\right\} \\
&= \sup\left\{\int_0^t \|T(\sigma,\tau)(f_n(\tau)-f(\tau))\|\left(\frac{\mu(\sigma)}{\mu(\tau)}\right)^{-\omega}\left(\frac{\mu(t)}{\mu(\tau)}\right)^{\omega} d\tau : \sigma \ge t\right\} \\
&= \int_0^t \|f_n(\tau)-f(\tau)\|'_\tau\left(\frac{\mu(t)}{\mu(\tau)}\right)^{\omega} d\tau \\
&\le \mu(t)^\omega \int_0^t \|f_n(\tau)-f(\tau)\|'_\tau d\tau.
\end{aligned} \tag{2.16}$$

According to Hölder's inequality, there exists $\alpha = \alpha([0,t])$ such that

$$\|(Gf_n)(t) - (Gf)(t)\|'_t \le \mu(t)^\omega \int_0^t \|f_n(\tau)-f(\tau)\|'_\tau d\tau \le \mu(t)^\omega \alpha\|(f_n(\tau)-f(\tau))\|'_p. \tag{2.17}$$

Therefore, for each $t \ge 0$, letting $n \to \infty$ we find that $(Gf_n)(t) \to (Gf)(t)$. This shows that $Gf = h$ Lebesgue-almost everywhere, and by the closed graph theorem, we conclude that G is a bounded operator. This completes the proof of the lemma. □

2.2. Criterion for Nonuniform (μ,ν) Contraction

Theorem 2.6. *If for some $p \in (1,\infty]$ the space $E = \mathcal{L}^p$ is admissible for the evolution process T, then T is a nonuniform (μ,ν) contraction.*

Proof. We follow arguments in [22]. Given $x \in X$ and $t_0 \geq 0$, we define a function $f : \mathbb{R}_0^+ \to X$ by

$$f(t) = \begin{cases} T(t,t_0)x, & t \in [t_0, t_0+1] \\ 0, & t \in \mathbb{R}_0^+ \setminus [t_0, t_0+1]. \end{cases} \tag{2.18}$$

We note that

$$\|f(t)\|_t' \leq \|T(t,t_0)x\|_t' \chi_{[t_0,t_0+1]}(t). \tag{2.19}$$

Then, for each $t \in [t_0, t_0+1]$ and $x \in X$, we have

$$\begin{aligned} \|T(t,t_0)x\|_t' &= \sup\left\{ \|T(\sigma,t)T(t,t_0)x\| \left(\frac{\mu(\sigma)}{\mu(t)}\right)^{-\omega}, \sigma \geq t \right\} \\ &\leq \left(\frac{\mu(t)}{\mu(t_0)}\right)^{\omega} \sup\left\{ \|T(\sigma,t_0)x\| \left(\frac{\mu(\sigma)}{\mu(t_0)}\right)^{-\omega}, \sigma \geq t_0 \right\} \\ &= \left(\frac{\mu(t)}{\mu(t_0)}\right)^{\omega} \|x\|_{t_0}' \\ &\leq \left(\frac{\mu(t_0+1)}{\mu(t_0)}\right)^{\omega} \|x\|_{t_0}'. \end{aligned} \tag{2.20}$$

Therefore,

$$\|f(t)\|_p' \leq \left(\frac{\mu(t_0+1)}{\mu(t_0)}\right)^{\omega} \|x\|_{t_0}' \left\|\chi_{[t_0,t_0+1]}(t)\right\|_p = \left(\frac{\mu(t_0+1)}{\mu(t_0)}\right)^{\omega} \|x\|_{t_0}' \tag{2.21}$$

and in particular $f \in E(X)$. On the other hand, according to (2.13) and (2.18), we have

$$x_f(t) = \int_{t_0}^{t_0+1} T(t,\tau)T(\tau,t_0)x\,d\tau = T(t,t_0)x \tag{2.22}$$

for all $t \geq t_0+1$, which implies that

$$\|T(t,t_0)x\|_t' = \|x_f\|_t' \leq \|x_f\|_\infty'. \tag{2.23}$$

By Lemma 2.5 and (2.21)–(2.23), we obtain

$$\|T(t,t_0)x\|_t' \leq \|x_f\|_\infty' \leq K\|f\|_p' \leq K\left(\frac{\mu(t_0+1)}{\mu(t_0)}\right)^{\omega} \|x\|_{t_0}' \tag{2.24}$$

for all $t \geq t_0 + 1$, $t_0 \geq 0$, and $x \in X$. We claim that

$$\|T(t,t_0)\|' := \sup_{x \neq 0} \frac{\|T(t,t_0)x\|'_t}{\|x\|'_{t_0}} \leq L, \quad L = \left(\frac{\mu(t_0+1)}{\mu(t_0)}\right)^{\omega} \max\{K, 1\} \tag{2.25}$$

for all $t \geq t_0$. Indeed, for $t \geq t_0 + 1$ inequality (2.25) follows from (2.24), and for $t \in [t_0, t_0 + 1]$ the inequality follows from (2.20).

Now given $x \in X$, $t_0 \geq 0$, and $\delta > 0$, we define a function $g : \mathbb{R}_0^+ \to X$ by

$$g(t) = \begin{cases} T(t,t_0)x, & t \in [t_0, t_0 + \delta] \\ 0, & t \in \mathbb{R}_0^+ \setminus [t_0, t_0 + \delta]. \end{cases} \tag{2.26}$$

It follows from (2.25) that

$$\|g(t)\|'_t \leq \|T(t,t_0)x\|'_t \leq L\|x\|'_{t_0} \tag{2.27}$$

and thus,

$$g \in E(X), \qquad \|g\|'_p \leq L\delta^{1/p}\|x\|'_{t_0}. \tag{2.28}$$

On the other hand, writing $y = T(t_0 + \delta, t_0)x$,

$$\begin{aligned}
\frac{\delta^2}{2}\|y\|'_{t_0+\delta} &= \left\|\int_{t_0}^{t_0+\delta} (\tau - t_0) y \, d\tau\right\|'_{t_0+\delta} \\
&= \sup\left\{\left\|T(\sigma, t_0 + \delta)\int_{t_0}^{t_0+\delta} (\tau - t_0) y \, d\tau\right\|\left(\frac{\mu(\sigma)}{\mu(t_0+\delta)}\right)^{-\omega} : \sigma \geq t_0 + \delta\right\} \\
&= \sup\left\{\left\|\int_{t_0}^{t_0+\delta} (\tau - t_0) T(\sigma, t_0)x \, d\tau\right\|\left(\frac{\mu(\sigma)}{\mu(t_0+\delta)}\right)^{-\omega} : \sigma \geq t_0 + \delta\right\} \\
&\leq \sup\left\{\int_{t_0}^{t_0+\delta} (\tau - t_0)\|T(\sigma, t_0)x\|\left(\frac{\mu(\sigma)}{\mu(t_0+\delta)}\right)^{-\omega} d\tau : \sigma \geq t_0 + \delta\right\} \\
&= \int_{t_0}^{t_0+\delta} (\tau - t_0) \sup\left\{\|T(\sigma, t_0 + \delta)y\|\left(\frac{\mu(\sigma)}{\mu(t_0+\delta)}\right)^{-\omega} : \sigma \geq t_0 + \delta\right\} d\tau \\
&= \int_{t_0}^{t_0+\delta} (\tau - t_0)\|y\|'_{t_0+\delta} \, d\tau \\
&= \int_{t_0}^{t_0+\delta} (\tau - t_0)\|T(t_0 + \delta, \tau)T(\tau, t_0)x\|'_{t_0} \, d\tau
\end{aligned} \tag{2.29}$$

Since

$$x_g(t) = \int_0^t T(t,\tau)g(\tau)d\tau = \begin{cases} 0, & t \in [0,t_0] \\ (t,t_0)T(t,t_0)x, & t \in [t_0, t_0+\delta], \\ \delta T(t,t_0)x, & t \in [t_0+\delta,\infty), \end{cases} \tag{2.30}$$

it follows from Lemma 2.5, (2.25), and (2.28) that

$$\begin{aligned} \frac{\delta^2}{2}\|T(t_0+\delta,t_0)x\|'_{t_0+\delta} &\le L\int_{t_0}^{t_0+\delta}(\tau-t_0)\|T(\tau,t_0)x\|'_\tau d\tau \\ &= L\int_{t_0}^{t_0+\delta}\|x_g(\tau)\|'_\tau d\tau \le L\delta\|x_g\|'_\infty \\ &\le KL\delta\|g\|'_p \le KL^2\delta^{(p+1)/p}\|x\|'_{t_0} \end{aligned} \tag{2.31}$$

for all $t_0 \ge 0$, $\delta > 0$, and $x \in X$; we thus obtain

$$\frac{\delta^2}{2}\|T(t_0+\delta,t_0)x\|'_{t_0+\delta} \le KL^2\delta^{(p+1)/p}\|x\|'_{t_0} \tag{2.32}$$

so

$$\|T(t_0+\delta,t_0)\|' \le 2KL^2\delta^{(1-p)/p} \tag{2.33}$$

for all $t_0 \ge 0$ and $\delta > 0$. Since $(1-p)/p < 0$ for $p \in (1,\infty]$, there exists $\delta_0 > 0$ sufficiently large such that

$$K_0 := 2KL^2\delta^{(1-p)/p} < 1. \tag{2.34}$$

Setting $n = [(\ln\mu(t) - \ln\mu(t_0))/\delta_0]$ for each $t \ge t_0$, we have

$$T(t,t_0) = T(t,t_0+n\delta_0)T(t_0+n\delta_0,t_0). \tag{2.35}$$

By (2.25) and (2.33) we obtain

$$\begin{aligned} \|T(t,t_0)\|' &\le L\|T(t_0+n\delta_0,t_0)\|' \\ &\le L\prod_{i=0}^{n-1}\|T(t_0+(i+1)\delta_0,t_0+i\delta_0)\|' \le LK_0^n \end{aligned} \tag{2.36}$$

for $t \ge t_0$. By (2.34) and

$$n = \left[\frac{\ln\mu(t)-\ln\mu(t_0)}{\delta_0}\right] \ge \frac{\ln\mu(t)-\ln\mu(t_0)}{\delta_0} - 1 \tag{2.37}$$

this implies that

$$\|T(t,t_0)\|' \leq d\left(\frac{\mu(t)}{\mu(t_0)}\right)^{-\alpha}, \tag{2.38}$$

where

$$d = \frac{L}{K_0}, \qquad \alpha = -\frac{1}{\delta_0}\ln K_0. \tag{2.39}$$

We note that $d, \alpha > 0$. Since

$$\|T(t,t_0)x\|_t' \geq \|T(t,t_0)x\|, \tag{2.40}$$

and by (2.4),

$$\|x\|_{t_0}' = \sup\left\{\|T(\sigma,t_0)x\|\left(\frac{\mu(\sigma)}{\mu(t_0)}\right)^{-\omega} : \sigma \geq t_0\right\} \leq D\nu^{\varepsilon}(t_0)\|x\|. \tag{2.41}$$

It follows from (2.38) that

$$\|T(t,t_0)\| = \sup_{x\neq 0}\frac{\|T(t,t_0)x\|}{\|x\|} \leq D\nu^{\varepsilon}(t_0)\sup_{x\neq 0}\frac{\|T(t,t_0)x\|_t'}{\|x\|_{t_0}'} \leq dD\nu^{\varepsilon}(t_0)\left(\frac{\mu(t)}{\mu(t_0)}\right)^{-\alpha} \tag{2.42}$$

for any $t \geq t_0$. Therefore, the evolution process T is a nonuniform (μ,ν) contraction with α and $\widetilde{D} = dD$. This concludes the proof of Theorem 2.6. □

2.3. Admissible Spaces for Nonuniform (μ,ν) *Contractions*

We consider the spaces

$$L_D^p = \left\{f : \mathbb{R}_0^+ \longrightarrow \mathbb{R} \ \text{ Lebesgue-measurable} : \|f\|_{p,D} < \infty\right\} \tag{2.43}$$

for each $p \in [1,\infty)$, and

$$L_D^\infty = \left\{f : \mathbb{R}_0^+ \longrightarrow \mathbb{R} \ \text{ Lebesgue-measurable} : \|f\|_{\infty,D} < \infty\right\}, \tag{2.44}$$

respectively, with the norms

$$\begin{gathered}\|f\|_{p,D} = \left(\int_0^\infty |f(t)|^p\left(D\nu^{\varepsilon}(t)\left(\frac{\mu(t)}{\mu'(t)}\right)^{1/q}\right)^p\right)^{1/p}, \\ \|f\|_{\infty,D} = \operatorname*{ess\,sup}_{t\in\mathbb{R}_0^+}\left(|f(t)|D\nu^{\varepsilon}(t)\frac{\mu(t)}{\mu'(t)}\right).\end{gathered} \tag{2.45}$$

In a similar manner to that in Lemma 2.3 these normed spaces induce Banach spaces $\mathcal{L}_D^p$ and $\mathcal{L}_D^p(X)$ for each $p \in [1,\infty]$, the last one with norm

$$\|f\|'_{p,D} = \|F\|_{p,D}, \quad \text{where } F(t) = \|f(t)\|'_t. \tag{2.46}$$

Theorem 2.7. *If the evolution process T is a nonuniform (μ,ν) contraction, then for any $p \in [1,\infty]$ the space $\mathcal{L}_D^p$ is admissible for T.*

Proof. We first take $f \in \mathcal{L}_D^\infty$. Then

$$\begin{aligned}
\|x_f(t)\|'_t &= \sup\left\{\left\|\int_0^t T(\sigma,t)T(t,\tau)f(\tau)d\tau\right\|\left(\frac{\mu(\sigma)}{\mu(t)}\right)^{-\omega} : \sigma \ge t\right\} \\
&= \sup\left\{\left\|\int_0^t T(\sigma,\tau)f(\tau)d\tau\right\|\left(\frac{\mu(\sigma)}{\mu(t)}\right)^{-\omega} : \sigma \ge t\right\} \\
&\le \sup\left\{\int_0^t \|T(\sigma,\tau)\| \cdot \|f(\tau)\|d\tau : \sigma \ge t\right\} \\
&\le \sup\left\{\int_0^t D\nu^\tau(\tau)\left(\frac{\mu(\sigma)}{\mu(\tau)}\right)^{-\alpha} \|f(\tau)\|'_\tau d\tau : \sigma \ge t\right\} \\
&\le \sup\left\{\int_0^t D\nu^\tau(\tau)\left(\frac{\mu(t)}{\mu(\tau)}\right)^{-\alpha} \|f(\tau)\|'_\tau d\tau : \sigma \ge t\right\} \\
&\le \|f\|'_{\infty,D}\int_0^t \left(\frac{\mu(t)}{\mu(\tau)}\right)^{-\alpha}\frac{\mu'(\tau)}{\mu(\tau)}d\tau \\
&\le \|f\|'_{\infty,D}\frac{1-\mu(t)^{-\alpha}}{\alpha} \le \frac{1}{\alpha}\|f\|'_{\infty,D}.
\end{aligned} \tag{2.47}$$

Therefore,

$$\|x_f\|'_\infty = \sup_{t\ge0}\|x_f(t)\|'_t \le \sup_{t\ge0}\frac{1}{\alpha}\|f\|'_{\infty,D} < \infty \tag{2.48}$$

and $\mathcal{L}_D^\infty$ is admissible for T.

Now we take $f \in \mathcal{L}_D^p(X)$ for some $p \in [1, \infty)$. Using Hölder's inequality we obtain

$$\begin{aligned}
\|x_f(t)\|_t' &= \sup\left\{ \left\| \int_0^t T(\sigma,t)T(t,\tau)f(\tau)d\tau \right\| \left(\frac{\mu(\sigma)}{\mu(t)}\right)^{-\omega} : \sigma \geq t \right\} \\
&\leq \sup\left\{ \int_0^t D\nu^{\tau}(t)\left(\frac{\mu(t)}{\mu(\tau)}\right)^{-\alpha} \|f(\tau)\|_\tau' d\tau : \sigma \geq t \right\} \\
&\leq \|f\|_{p,D}' \left(\int_0^t \left(\frac{\mu(t)}{\mu(\tau)}\right)^{-\alpha q} \frac{\mu'(\tau)}{\mu(\tau)} d\tau \right)^{1/q} \\
&\leq \|f\|_{p,D}' \left(\frac{1-\mu(t)^{-\alpha q}}{\alpha q} \right)^{1/q} \leq \frac{1}{(\alpha q)^{1/q}} \|f\|_{p,D}',
\end{aligned} \tag{2.49}$$

where $1/p + 1/q = 1$. We conclude that $\mathcal{L}_D^p$ is also admissible for T. □

3. Admissibility for Nonuniform (μ, ν) Dichotomies

We consider in this second part admissibility for nonuniform (μ, ν) dichotomies. It generalizes the usual notion of exponential dichotomy in several ways: besides introducing a nonuniform term, causing that any conditional stability may be nonuniform, we consider rates that may not be exponential as well as different rates in the uniform and nonuniform parts. After introducing some basic notions, we show that the admissibility with respect to some space $\mathcal{L}^p$ with $p \in (1, \infty]$ is sufficient for an evolution process to be a nonuniform (μ, ν) dichotomy. When compared to the case of contractions, this creates substantial complications. We also provide a collection of admissible Banach spaces for any given nonuniform (μ, ν) dichotomy.

3.1. Basic Notions

We consider an evolution process $T(t, s)$, $t \geq s \geq 0$ satisfied 1, 2 in Section 2.

We also consider a function $P : \mathbb{R}_0^+ \to B(X)$, where $B(X)$ is the set of bounded linear operators in X, such that

(1) $P(t)^2 = P(t)$, for every $t \geq 0$;

(2) $(t, x) \mapsto P(t)x$ is continuous in $\mathbb{R}_0^+ \times X$.

We will refer to P as a projection function. Given an evolution process T, we say that a projection function P is compatible with T if:

(1) $T(t, s)p(s) = P(t)T(t, s)$, for every $t, \tau, s > 0$;

(2) the map

$$T(t, \sigma) \mid \ker P(\sigma) : \ker P(\sigma) \longrightarrow \ker P(t) \tag{3.1}$$

is invertible for every $t \geq s \geq 0$.

We also assume that

(3) there exist $D > 0$, $\omega \geq 0$ and two growth rates $\mu(t)$, $\nu(t)$ such that

$$\|T(t,s)P(s)\| \leq D\left(\frac{\mu(t)}{\mu(s)}\right)^{\omega}\nu^{\varepsilon}(s), \quad t \geq s \geq 0. \tag{3.2}$$

$$\|T(t,s)Q(s)\| \leq D\left(\frac{\mu(s)}{\mu(t)}\right)^{\omega}\nu^{\varepsilon}(s), \quad s \geq t \geq 0. \tag{3.3}$$

We note that due to the invertibility assumption in condition (1.2), condition (3.3) is simply a version of (3.2) when time goes backwards.

We always consider in the paper an evolution process T together with a projection function P which is compatible with T (and which satisfies (3.2) and (3.3)). We write

$$U(t,s) = T(t,s)P(s), \qquad V(t,s) = T(t,s)Q(s), \tag{3.4}$$

where $Q(t) = Id - P(t)$ for each $t \geq 0$. we consider the new norms

$$\begin{aligned} \|x\|_t' = \; & \sup\left\{ \|U(\sigma,t)x\|\left(\frac{\mu(\sigma)}{\mu(t)}\right)^{-\omega}, \sigma \geq t\right\} \\ & + \sup\left\{ \|V(\sigma,t)x\|\left(\frac{\mu(t)}{\mu(\sigma)}\right)^{-\omega}, 0 \leq \sigma \leq t\right\}. \end{aligned} \tag{3.5}$$

for each $x \in X$ and $t \in \mathbb{R}_0^+$, where $V(\sigma,t)$ denotes the inverse of the map in (3.1). We have

$$\|x\|_t' \geq \|P(t)x\| + \|Q(t)x\| \geq \|P(t)x + Q(t)x\| = \|x\|. \tag{3.6}$$

Moreover, by (3.2) and (3.3),

$$\|x\|_t' \leq 2D\nu^{\varepsilon}(t)\|x\|, \quad x \in X, \; t \in \mathbb{R}_0^+. \tag{3.7}$$

Definition 3.1. We say that an evolution process T is a nonuniform (μ,ν) dichotomy in $\mathbb{R}^+$ if there exist a projection function $P : \mathbb{R}_0^+ \to B(X)$ compatible with T, some constants α, $D > 0$, $\varepsilon \geq 0$ and two growth rates $\mu(t)$, $\nu(t)$ such that

$$\begin{aligned} \|T(t,s)P(s)\| &\leq D\left(\frac{\mu(t)}{\mu(s)}\right)^{-\alpha}\nu^{\varepsilon}(s), \quad t \geq s \geq 0. \\ \|T(t,s)Q(s)\| &\leq D\left(\frac{\mu(s)}{\mu(t)}\right)^{-\beta}\nu^{\varepsilon}(s), \quad s \geq t \geq 0, \end{aligned} \tag{3.8}$$

When $\varepsilon = 0$, we say that (1.2) has a uniform (μ,ν) dichotomy or simply a (μ,ν) dichotomy.

In the following, we still consider several spaces L^p, L^∞, respectively, with the norms (2.10), which induce Banach spaces $\mathcal{L}^p$ for each $p \in [1, \infty]$. We also set $E(X)$ as in (2.11) but using the norms $\|\cdot\|_t'$ in (3.5), we endow $E(X) = \mathcal{L}^p(X)$ with the norm in (2.12).

We also obtain easily the same statement in Lemma 2.3 for the set $E(X)$ using the norms $\|\cdot\|_t'$ in (3.5).

Definition 3.2. We say that a Banach space E is admissible for the evolution process T if for each $f \in E(X)$

(1) the function

$$\mathbb{R}_0^+ \ni \tau \longmapsto \begin{cases} V(t,\tau)f(\tau), & \tau \geq t, \\ 0, & 0 \leq \tau < t \end{cases} \tag{3.9}$$

is in $\mathcal{L}^1(X)$ for each $t \geq 0$;

(2) the function $x_f : \mathbb{R}_0^+ \to X$ defined by

$$x_f(t) = \int_0^t U(t,\tau)f(\tau)d\tau - \int_t^\infty V(t,\tau)f(\tau)d\tau \tag{3.10}$$

is in $\mathcal{L}^\infty(X)$.

We note that since $\|Q(t)x\| \leq \|Q(t)x\|_t'$ for every $x \in X$, and $t \geq 0$, any function in $\mathcal{L}^1(X)$ is also integrable in R_0^+, and thus the first condition ensures that the function x_f is well defined. By Lemma 2.3 we know that $\mathcal{L}^\infty(X)$ is a Banach space with the norm

$$\|g\|_\infty' = \operatorname*{ess\,sup}_{t\in\mathbb{R}_0^+} \|g(t)\|_t'. \tag{3.11}$$

Lemma 3.3. *If for some $p \in [1, \infty]$ the space $E = \mathcal{L}^p$ is admissible for the evolution process T, then there exists $K > 0$ such that*

$$\|x_f\|_\infty' \leq K\|f\|_p' \quad \text{for every } f \in E(X). \tag{3.12}$$

Proof. We follow arguments in [2]. For each $t \geq 0$, we define a map $H_t : E(X) \to \mathcal{L}^1(X)$ by

$$(H_t f)(\tau) = \begin{cases} V(t,\tau)f(\tau), & \tau \geq t, \\ 0, & 0 \leq \tau < t. \end{cases} \tag{3.13}$$

Clearly, H_t is linear. We use the closed graph theorem to show that H_t is bounded. For this, let us take a sequence $(f_n)_{n\in N} \subset E(X)$ and $f \in E(X)$ such that $f_n \to f$ in $E(X)$ when $n \to \infty$, and also $g \in \mathcal{L}^1(X)$ such that $H_t f_n \to g$ in $\mathcal{L}^1(X)$ when $n \to \infty$. We need to show that

$H_t f = g$ Lebesgue-almost everywhere. By Lemma 2.3, the sequence f_n converges pointwise Lebesgue-almost everywhere. Therefore,

$$(H_t f_n)(\tau) = V(t,\tau) f_n(\tau) \longrightarrow V(t,\tau) f(\tau) = (H_t f)(\tau) \tag{3.14}$$

when $n \to \infty$, for Lebesgue-almost every $\tau \in [t, +\infty)$. On the other hand, since $H_t f_n \to g$ in $\mathcal{L}^1(X)$ when $n \to \infty$, we also have $(H_t f_n)(\tau) \to g(\tau)$ when $n \to \infty$, for Lebesgue-almost every $\tau \in [t, +\infty)$. This shows that $H_t f = g$ Lebesgue-almost everywhere, and H_t is bounded for each $t \geq 0$.

We define a linear operator $G : E(X) \to \mathcal{L}^\infty(X)$ by $Gf = x_f$. We use again the closed graph theorem to show that G is bounded. For this, let us take a sequence $(f_n)_{n \in N} \subset E(X)$ and $f \in E(X)$ such that $f_n \to f$ in $E(X)$ when $n \to \infty$, and also $h \in \mathcal{L}^\infty(X)$ such that $Gf_n \to h$ in $\mathcal{L}^\infty(X)$ when $n \to \infty$. We write

$$\begin{aligned} (G_1 f)(t) &= P(t)(Gf)(t) = \int_0^t U(t,\tau) f(\tau) d\tau, \\ (G_2 f)(t) &= Q(t)(Gf)(t) = -\int_t^\infty V(t,\tau) f(\tau) d\tau. \end{aligned} \tag{3.15}$$

Using the similar proof of Lemma 2.5, for each $t \geq 0$ and $n \in N$ we have

$$\begin{aligned} \|(G_1 f_n)(t) - (G_1 f)(t)\|_t' &= \sup\left\{ \left\| \int_0^t U(\sigma,t) U(t,\tau)(f_n(\tau) - f(\tau)) d\tau \right\| \left(\frac{\mu(\sigma)}{\mu(t)} \right)^{-\omega} : \sigma \geq t \right\} \\ &\leq \mu(t)^\omega \int_0^t \|f_n(\tau) - f(\tau)\|_\tau' d\tau. \end{aligned} \tag{3.16}$$

According to Hölder's inequality, there exists $\alpha = \alpha([0,t])$ such that

$$\|(G_1 f_n)(t) - (G_1 f)(t)\|_t' \leq \mu(t)^\omega \alpha \|(f_n(\tau) - f(\tau))\|_p'. \tag{3.17}$$

Furthermore, we have

$$\begin{aligned} \|(G_2 f_n)(t) - (G_2 f)(t)\|_t' &\leq \sup\left\{ \int_t^\infty \|V(\sigma,t) V(t,\tau)(f_n(\tau) - f(\tau))\| \left(\frac{\mu(t)}{\mu(\sigma)} \right)^{-\omega} d\tau : 0 \leq \sigma \leq t \right\} \\ &= \int_t^\infty \sup\left\{ \|V(\sigma,t) H_t(f_n - f)(\tau)\| \left(\frac{\mu(t)}{\mu(\sigma)} \right)^{-\omega} : 0 \leq \sigma \leq t \right\} d\tau \\ &= \int_t^\infty \|H_t(f_n - f)(\tau)\|_t' d\tau = \|H_t(f_n - f)\|_t^1. \end{aligned} \tag{3.18}$$

It follows from (3.17) and (3.18) that

$$\|(Gf_n)(t)-(Gf)(t)\|_t' \le \mu(t)^{\omega}\alpha\|f_n-f\|_p' + \|H_t(f_n-f)\|_t^1. \tag{3.19}$$

Therefore, for each $t \ge 0$, letting $n \to \infty$ we find that $(Gf_n)(t) \to (Gf)(t)$. This shows that $Gf = h$ Lebesgue-almost everywhere, and by the closed graph theorem, we conclude that G is a bounded operator. This completes the proof of the lemma. □

3.2. Criterion for Nonuniform (μ, ν) *Dichotomy*

Theorem 3.4. *If for some $p \in (1,\infty]$ the space $E = \mathcal{L}^p$ is admissible for the evolution process T, then T is a nonuniform (μ,ν) dichotomy.*

Proof. We first consider the space $P_{t_0} = \operatorname{Im} P(t_0)$. Given $x \in P_{t_0}$ and $t_0 \ge 0$, repeating argument of the proof in Theorem 2.6, except limiting $T(t,t_0)$ on P_{t_0}, we obtain

$$\|T(t,t_0) \mid P_{t_0}\|' \le d\left(\frac{\mu(t)}{\mu(t_0)}\right)^{-\alpha}, \tag{3.20}$$

where

$$d = \frac{L}{K_0}, \qquad \alpha = -\frac{1}{\delta_0}\ln K_0. \tag{3.21}$$

We note that $d, \alpha > 0$. For each $x \in P_{t_0}$, we have

$$\|T(t,t_0)x\|_t' \ge \|T(t,t_0)x\|, \tag{3.22}$$

and by (3.2),

$$\|x\|_{t_0}' = \sup\left\{\|U(\sigma,t_0)x\|\left(\frac{\mu(\sigma)}{\mu(t_0)}\right)^{-\omega} : \sigma \ge t_0\right\} \le D\nu^{\varepsilon}(t_0)\|x\|. \tag{3.23}$$

It follows from (3.5) and (3.20) that

$$\begin{aligned}
\|T(t,t_0) \mid P_{t_0}\| &= \sup_{x\in P_{t_0}\setminus\{0\}} \frac{\|T(t,t_0)x\|}{\|x\|} \\
&\le D\nu^{\varepsilon}(t_0) \sup_{x\in P_{t_0}\setminus\{0\}} \frac{\|T(t,t_0)x\|_t'}{\|x\|_{t_0}'} \\
&\le dD\nu^{\varepsilon}(t_0)\left(\frac{\mu(t)}{\mu(t_0)}\right)^{-\alpha}
\end{aligned} \tag{3.24}$$

for any $t \ge t_0$.

Now we consider the space $Q_{t_0} = \operatorname{Ker} P(t_0)$. Given $x \in Q_{t_0}$ and $t_0 \geq 0$, we define a function $f : \mathbb{R}_0^+ \to X$ by

$$f(t) = \begin{cases} V(t,t_0)x, & t \in \mathbb{R}_0^+ \bigcap [t_0 - 1, t_0] \\ 0, & t \in \mathbb{R}_0^+ \setminus [t_0 - 1, t_0]. \end{cases} \tag{3.25}$$

Clearly, $f(t) \in Q_t$ for every $t \geq 0$. Moreover, for each $t \in [0, t_0 - 1]$ (this interval may be empty), we have

$$\begin{aligned} x_f(t) &= -\int_{t_0-1}^{t_0} V(t,\tau)V(\tau,t_0)x d\tau \\ &= -\int_{t_0-1}^{t_0} V(t,t_0)x d\tau = -V(t,t_0)x, \end{aligned} \tag{3.26}$$

and it follows from Lemma 3.3 that

$$\|V(t,t_0)x\|_t' = \left\|x_f\right\|_\infty' \leq K\left\|f\right\|_p' \tag{3.27}$$

for $t \in [0, t_0 - 1]$.

On the other hand, for each $t \in [t_0 - 1, t_0]$, we have

$$\begin{aligned} \left\|f(t)\right\|_t' &= \sup\left\{ \|V(\sigma,t)f(t)\| \left(\frac{\mu(t)}{\mu(\sigma)}\right)^{-\omega}, 0 \leq \sigma \leq t \right\} \\ &\leq \sup\left\{ \|V(\sigma,t)V(t,t_0)x\| \left(\frac{\mu(t)}{\mu(\sigma)}\right)^{-\omega}, 0 \leq \sigma \leq t \right\} \\ &\leq \left(\frac{\mu(t_0)}{\mu(t)}\right)^{\omega} \sup\left\{ \|V(\sigma,t_0)x\| \left(\frac{\mu(t_0)}{\mu(\sigma)}\right)^{-\omega}, 0 \leq \sigma \leq t_0 \right\} \\ &\leq \left(\frac{\mu(t_0)}{\mu(t_0 - 1)}\right)^{\omega} \|x\|_{t_0}'. \end{aligned} \tag{3.28}$$

So

$$\left\|f\right\|_p' = \int_{\mathbb{R}_0^+ \bigcap [t_0-1,t_0]} \left\|f(t)\right\|_t' dt \leq \left(\frac{\mu(t_0)}{\mu(t_0 - 1)}\right)^{\omega} \|x\|_{t_0}' \tag{3.29}$$

and in particular $f \in E(X)$. We thus have

$$\|V(t,t_0)x\|_t' \leq K\left\|f\right\|_p' \leq K\left(\frac{\mu(t_0)}{\mu(t_0 - 1)}\right)^{\omega} \|x\|_{t_0}' \tag{3.30}$$

for every $t \in [0, t_0 - 1]$, and $x \in Q_{t_0}$. This implies that

$$\|V(t,t_0)\|' := \sup_{x \in Q_{t_0} \setminus \{0\}} \frac{\|V(t,t_0)x\|'_t}{\|x\|'_{t_0}} \le L, \quad L = \left(\frac{\mu(t_0)}{\mu(t_0-1)}\right)^{\omega} \max\{K, 1\} \tag{3.31}$$

for all $t \le t_0$. Indeed, for $t \in [0, t_0 - 1]$ inequality (3.31) follows from (3.30), and for $t \in \mathbb{R}_0^+ \bigcap [t_0 - 1, t_0]$ the inequality follows from (3.28).

Now given $x \in Q_{t_0}$, $t_0 \ge 0$, and $\delta > 0$, we define a function $g : \mathbb{R}_0^+ \to X$ by

$$g(t) = \begin{cases} V(t,t_0)x, & t \in \mathbb{R}_0^+ \bigcap [t_0 - \delta, t_0] \\ 0, & t \in \mathbb{R}_0^+ [t_0 - \delta, t_0]. \end{cases} \tag{3.32}$$

It follows from (3.31) that

$$\|g(t)\|'_t \le \|V(t,t_0)x\|'_t \le L\|x\|'_{t_0}, \tag{3.33}$$

and thus,

$$g \in E(X), \qquad \|g\|'_p \le L\delta^{1/p}\|x\|'_{t_0}. \tag{3.34}$$

On the other hand, in a similar manner to that in (2.29),

$$\begin{aligned} \frac{\delta^2}{2}\|V(t_0-\delta,t_0)x\|'_{t_0-\delta} &= \left\|\int_{t_0-\delta}^{t_0} (\tau - t_0)V(t_0-\delta,t_0)x\,d\tau\right\|'_{t_0-\delta} \\ &\le \int_{t_0-\delta}^{t_0} (t_0-\tau)\sup\left\{\|V(t_0-\delta,t_0)x\|'_{t_0-\delta} : 0 \le \sigma \le t_0-\delta\right\}d\tau \\ &= \int_{t_0-\delta}^{t_0} (t_0-\tau)\|V(t_0-\delta,t_0)x\|'_{t_0}d\tau \\ &= \int_{t_0-\delta}^{t_0} (t_0-\tau)\|V(t_0-\delta,\tau)V(\tau,t_0)x\|'_{t_0}d\tau. \end{aligned} \tag{3.35}$$

Since

$$x_g(t) = -\int_t^{\infty} V(t,\tau)g(\tau)d\tau = \begin{cases} 0, & t \in [t_0, \infty) \\ (t-t_0)V(t,t_0)x, & t \in [t_0-\delta, t_0]. \\ -\delta V(t,t_0)x, & t \in [0, t_0-\delta]. \end{cases} \tag{3.36}$$

It follows from Lemma 3.3, (3.31), and (3.34) that

$$\begin{aligned}\frac{\delta^2}{2}\|V(t_0-\delta,t_0)x\|'_{t_0-\delta} &\le L\int_{t_0-\delta}^{t_0}(t_0-\tau)\|V(\tau,t_0)x\|'_{\tau}d\tau \\ &= L\int_{t_0-\delta}^{t_0}\left\|x_g(\tau)\right\|'_{\tau}d\tau \le L\delta\left\|x_g\right\|'_{\infty} \\ &\le KL\delta\left\|g\right\|'_{p} \le KL^2\delta^{(p+1)/p}\|x\|'_{t_0}\end{aligned} \tag{3.37}$$

for all $t_0 \ge 0$, $\delta > 0$, and $x \in Q_{t_0}$; we thus obtain

$$\frac{\delta^2}{2}\|V(t_0-\delta,t_0)x\|'_{t_0-\delta} \le KL^2\delta^{(p+1)/p}\|x\|'_{t_0} \tag{3.38}$$

so

$$\|V(t_0-\delta,t_0)\|' \le 2KL^2\delta^{(1-p)/p} \tag{3.39}$$

for all $t_0 \ge 0$ and $\delta > 0$. Taking the same δ_0 as before, and setting $n = [(\ln\mu(t_0) - \ln\mu(t))/\delta_0]$ for each $t \le t_0$, we have

$$V(t,t_0) = V(t,t_0-n\delta_0)V(t_0-n\delta_0,t_0). \tag{3.40}$$

By (3.31) and (3.39) we obtain

$$\begin{aligned}\|V(t,t_0)\|' &\le L\|V(t_0-n\delta_0,t_0)\|' \\ &\le L\prod_{i=0}^{n-1}\|V(t_0-(i+1)\delta_0,t_0-i\delta_0)\|' \le LK_0^n\end{aligned} \tag{3.41}$$

for $t \le t_0$, where $K_0 := 2KL^2\delta_0^{(1-p)/p} < 1$. Since

$$n = \left[\frac{\ln\mu(t_0)-\ln\mu(t)}{\delta_0}\right] \ge \frac{\ln\mu(t_0)-\ln\mu(t)}{\delta_0} - 1, \tag{3.42}$$

this implies that

$$\|V(t,t_0)\|' \le d\left(\frac{\mu(t)}{\mu(t_0)}\right)^{\alpha}, \tag{3.43}$$

where

$$d = \frac{L}{K_0}, \qquad \alpha = -\frac{1}{\delta_0}\ln K_0. \tag{3.44}$$

We note that $d, \alpha > 0$. By (3.6)

$$\|V(t,t_0)x\|'_t \geq \|V(t,t_0)x\|, \tag{3.45}$$

and by (3.3),

$$\|x\|'_{t_0} = \sup\left\{\|V(\sigma,t_0)x\|\left(\frac{\mu(t_0)}{\mu(\sigma)}\right)^{-\omega} : 0 \leq \sigma \leq t_0\right\} \leq D\nu^{\varepsilon}(t_0)\|x\| \tag{3.46}$$

for $x \in Q_{t_0}$. It follows from (3.43) that

$$\begin{aligned}
\|V(t,t_0) \mid Q_{t_0}\| &= \sup_{x \in Q_{t_0}\setminus\{0\}} \frac{\|V(t,t_0)x\|}{\|x\|} \\
&\leq D\nu^{\varepsilon}(t_0) \sup_{x \in Q_{t_0}\setminus\{0\}} \frac{\|V(t,t_0)x\|'_t}{\|x\|'_{t_0}} \\
&\leq dD\nu^{\varepsilon}(t_0)\left(\frac{\mu(t)}{\mu(t_0)}\right)^{\alpha}
\end{aligned} \tag{3.47}$$

for any $t \leq t_0$. To show that T is a nonuniform exponential dichotomy, we note that setting $t = s$ in (3.2) and (3.3) yields

$$\|P(s)\| \leq D\nu^{\varepsilon}(s), \qquad \|Q(s)\| \leq D\nu^{\varepsilon}(s) \tag{3.48}$$

for every $s \geq 0$. Together with (3.24) and (3.47) this implies that

$$\begin{aligned}
\|T(t,s)P(s)\| &\leq \|T(t,s) \mid P_s\|\|P(s)\| \\
&\leq dD^2\left(\frac{\mu(t)}{\mu(s)}\right)^{-\alpha}\nu^{2\varepsilon}(s), \quad t \geq s \geq 0. \\
\|T(t,s)Q(s)\| &\leq \|T(t,s) \mid Q_s\|\|P(s)\| \\
&\leq dD^2\left(\frac{\mu(s)}{\mu(t)}\right)^{-\alpha}\nu^{2\varepsilon}(s), \quad s \geq t \geq 0.
\end{aligned} \tag{3.49}$$

This shows that T is a nonuniform (μ, ν) dichotomy with α, α, dD^2 and 2ε. □

3.3. *Admissible Spaces for a Nonuniform* (μ, ν) *Dichotomy*

We consider the spaces

$$L^p_D = \left\{f : \mathbb{R}^+_0 \longrightarrow \mathbb{R} \;\; \text{Lebesgue-measurable} : \|f\|_{p,D} < \infty\right\} \tag{3.50}$$

for each $p \in [1, \infty)$, and

$$L_D^\infty = \left\{ f : \mathbb{R}_0^+ \longrightarrow \mathbb{R} \ \text{ Lebesgue-measurable} : \|f\|_{\infty,D} < \infty \right\}, \tag{3.51}$$

respectively, with the norms

$$\begin{gathered} \|f\|_{p,D} = \left(\int_0^\infty |f(t)|^p \left(D\nu^\varepsilon(t) \left(\frac{\mu(t)}{\mu'(t)} \right)^{1/q} \right)^p \right)^{1/p}, \\ \|f\|_{\infty,D} = \operatorname*{ess\,sup}_{t \in \mathbb{R}_0^+} \left(|f(t)| D\nu^\varepsilon(t) \frac{\mu(t)}{\mu'(t)} \right). \end{gathered} \tag{3.52}$$

In a similar manner to Lemma 2.3 these normed spaces induce Banach spaces $\mathcal{L}_D^p$ and $\mathcal{L}_D^p(X)$ for each $p \in [1, \infty]$, the last one with norm

$$\|f\|'_{p,D} = \|F\|_{p,D}, \quad \text{where } F(t) = \|f(t)\|'_t. \tag{3.53}$$

Theorem 3.5. *If the evolution process T is a nonuniform (μ, ν) dichotomy, then for any $p \in [1, \infty]$ the space $\mathcal{L}_D^p$ is admissible for T.*

Proof. We first take $f \in L_D^\infty$. Then

$$\begin{aligned} \|x_f(t)\|'_t = & \sup\left\{ \left\| \int_0^t U(\sigma,t)U(t,\tau)f(\tau)d\tau \right\| \left(\frac{\mu(\sigma)}{\mu(t)} \right)^{-\omega} : \sigma \geq t \right\} \\ & + \sup\left\{ \left\| \int_t^\infty V(\sigma,t)V(t,\tau)f(\tau)d\tau \right\| \left(\frac{\mu(t)}{\mu(\sigma)} \right)^{-\omega} : 0 \leq \sigma \leq t \right\} \\ \leq & \sup\left\{ \int_0^t \|U(\sigma,\tau)\| \cdot \|f(\tau)\| \left(\frac{\mu(\sigma)}{\mu(t)} \right)^{-\omega} d\tau : \sigma \geq t \right\} \\ & + \sup\left\{ \int_t^\infty \|V(\sigma,\tau)\| \cdot \|f(\tau)\| \left(\frac{\mu(t)}{\mu(\sigma)} \right)^{-\omega} d\tau : 0 \leq \sigma \leq t \right\} \\ \leq & \sup\left\{ \int_0^t D\nu^\varepsilon(\tau) \left(\frac{\mu(\sigma)}{\mu(\tau)} \right)^{-\alpha} \|f(\tau)\| d\tau : \sigma \geq t \right\} \\ & + \sup\left\{ \int_t^\infty D\nu^\varepsilon(\tau) \left(\frac{\mu(\sigma)}{\mu(\tau)} \right)^{\beta} \|f(\tau)\| d\tau : 0 \leq \sigma \leq t \right\} \\ \leq & \sup\left\{ \int_0^t D\nu^\varepsilon(\tau) \left(\frac{\mu(t)}{\mu(\tau)} \right)^{-\alpha} \|f(\tau)\|'_\tau d\tau : \sigma \geq t \right\} \\ & + \sup\left\{ \int_t^\infty D\nu^\varepsilon(\tau) \left(\frac{\mu(t)}{\mu(\tau)} \right)^{\beta} \|f(\tau)\|'_\tau d\tau : 0 \leq \sigma \leq t \right\} \end{aligned}$$

$$\begin{aligned}
&\leq \|f\|'_{\infty,D}\int_0^t\left(\frac{\mu(t)}{\mu(\tau)}\right)^{-\alpha}\frac{\mu'(\tau)}{\mu(\tau)}d\tau+\|f\|'_{\infty,D}\int_t^{\infty}\left(\frac{\mu(t)}{\mu(\tau)}\right)^{\beta}\frac{\mu'(\tau)}{\mu(\tau)}d\tau\\
&=\frac{1}{\alpha}\|f\|'_{\infty,D}\left(1-\mu(t)^{-\alpha}\right)+\frac{1}{\beta}\|f\|'_{\infty,D}\\
&\leq\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)\|f\|'_{\infty,D}.
\end{aligned}\tag{3.54}$$

Therefore,

$$\|x_f\|'_{\infty}=\sup_{t\geq 0}\|x_f(t)\|'_t\leq\sup_{t\geq 0}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)\|f\|'_{\infty,D}<\infty\tag{3.55}$$

and $\mathcal{L}_D^{\infty}$ is admissible for T.

Now we take $f\in\mathcal{L}_D^p(X)$ for some $p\in[1,\infty)$. Using Hölder's inequality we obtain

$$\begin{aligned}
\|x_f(t)\|'_t=&\sup\left\{\left\|\int_0^t U(\sigma,t)U(t,\tau)f(\tau)d\tau\right\|\left(\frac{\mu(\sigma)}{\mu(t)}\right)^{-\omega}:\sigma\geq t\right\}\\
&+\sup\left\{\left\|\int_t^{\infty}V(\sigma,t)V(t,\tau)f(\tau)d\tau\right\|\left(\frac{\mu(t)}{\mu(\sigma)}\right)^{-\omega}:0\leq\sigma\leq t\right\}\\
\leq&\sup\left\{\int_0^t D\nu^{\varepsilon}(\tau)\left(\frac{\mu(t)}{\mu(\tau)}\right)^{-\alpha}\|f(\tau)\|'_{\tau}d\tau:\sigma\geq t\right\}\\
&+\sup\left\{\int_t^{\infty}D\nu^{\varepsilon}(\tau)\left(\frac{\mu(t)}{\mu(\tau)}\right)^{\beta}\|f(\tau)\|'_{\tau}d\tau:0\leq\sigma\leq t\right\}\\
\leq&\|f\|'_{p,D}\left(\int_0^t\left(\frac{\mu(t)}{\mu(\tau)}\right)^{-\alpha q}\frac{\mu'(\tau)}{\mu(\tau)}d\tau\right)^{1/q}\\
&+\|f\|'_{p,D}\left(\int_t^{\infty}\left(\frac{\mu(t)}{\mu(\tau)}\right)^{\beta q}\frac{\mu'(\tau)}{\mu(\tau)}d\tau\right)^{1/q}\\
=&\|f\|'_{p,D}\left(\frac{1-\mu(t)^{-\alpha q}}{\alpha q}\right)^{1/q}+\left(\frac{1}{\beta q}\right)^{1/q}\|f\|'_{p,D}\\
\leq&\left(\frac{1}{(\alpha q)^{1/q}}+\frac{1}{(\beta q)^{1/q}}\right)\|f\|'_{p,D},
\end{aligned}\tag{3.56}$$

where $1/p+1/q=1$. We conclude that $\mathcal{L}_D^p$ is also admissible for T. □

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Research Article

Singular Initial Value Problem for a System of Integro-Differential Equations

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Analytical properties like existence, uniqueness, and asymptotic behavior of solutions are studied for the following singular initial value problem: $g_i(t)y_i'(t) = a_i y_i(t)(1 + f_i(t, \mathbf{y}(t), \int_{0^+}^t K_i(t, s, \mathbf{y}(t), \mathbf{y}(s))ds))$, $y_i(0^+) = 0$, $t \in (0, t_0]$, where $\mathbf{y} = (y_1, \dots, y_n)$, $a_i > 0$, $i = 1, \dots, n$ are constants and $t_0 > 0$. An approach which combines topological method of T. Ważewski and Schauder's fixed point theorem is used. Particular attention is paid to construction of asymptotic expansions of solutions for certain classes of systems of integrodifferential equations in a right-hand neighbourhood of a singular point.

1. Introduction and Preliminaries

Singular initial value problem for ordinary differential and integro-differential equations is fairly well studied (see, e.g., [1–16]), but the asymptotic properties of the solutions of such equations are only partially understood. Although the singular initial value problems were widely considered using various methods (see, e.g., [1–13, 16]), our approach to this problem is essentially different from others known in the literature. In particular, we use a combination of the topological method of T. Ważewski [8] and Schauder's fixed point theorem [11]. Our technique leads to the existence and uniqueness of solutions with asymptotic estimates in the right-hand neighbourhood of a singular point. Asymptotic expansions of solutions are constructed for certain classes of systems of integrodifferential equations as well.

Consider the following problem:

$$g_i(t)y_i'(t) = a_i y_i(t)\left(1 + f_i\left(t, \mathbf{y}(t), \int_{0^+}^t K_i(t, s, \mathbf{y}(t), \mathbf{y}(s))ds\right)\right), \tag{1.1}$$

$$y_i(0^+) = 0, \quad t \in (0, t_0], \tag{1.2}$$

where $\mathbf{y} = (y_1, \ldots, y_n)$, $a_i > 0$ are constants, $f_i \in C^0(J \times \mathbb{R}^n \times \mathbb{R}, \mathbb{R})$, $K_i \in C^0(J \times J \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$, $J = (0, t_0]$, $t_0 > 0$, $i = 1, \ldots, n$.

Denote

(i) $f(t) = O(g(t))$ as $t \to 0^+$ if there is a right-hand neighbourhood $\mathcal{U}(0)$ and a constant $K > 0$ such that $(f(t)/g(t)) \leq K$ for $t \in \mathcal{U}(0)$.

(ii) $f(t) = o(g(t))$ as $t \to 0^+$ if there is valid $\lim_{t \to 0^+} f(t)/g(t) = 0$.

(iii) $f(t) \sim g(t)$ as $t \to 0^+$ if there is valid $\lim_{t \to 0^+} f(t)/g(t) = 1$.

Definition 1.1. The sequence of functions $(\phi_n(t))$ is called an asymptotic sequence as $t \to 0^+$ if

$$\phi_{n+1}(t) = o(\phi_n(t)) \quad \text{as } t \to 0^+ \tag{1.3}$$

for all n.

Definition 1.2. The series $\sum c_n \phi_n(t)$, $c_n \in \mathbb{R}$, is called an asymptotic expansion of the function $f(t)$ up to Nth term as $t \to 0^+$ if

(a) $(\phi_n(t))$ is an asymptotic sequence,

(b)

$$\left[f(t) - \sum_{n=1}^{N} c_n \phi_n(t) \right] = o(\phi_N(t)), \quad \text{as } t \to 0^+. \tag{1.4}$$

The functions g_i, f_i, and K_i will be assumed to satisfy the following:

(i) $g_i(t) \in C^1(J)$, $g_i(t) > 0$, $g_i(0^+) = 0$, $g_i'(t) \sim \varphi_i(t) g_i^{\lambda_i}(t)$ as $t \to 0^+$, $\lambda_i > 0$, $\varphi_i(t) g_i^{\tau}(t) = o(1)$ as $t \to 0^+$ for each $\tau > 0$, $i = 1, \ldots, n$,

(ii) $|f_i(t, u, v)| \leq |u| + |v|$, $|\int_{0^+}^{t} K_i(t, s, \mathbf{y}(t), \mathbf{y}(s)) ds| \leq r_i(t)|\mathbf{y}|$, $0 < r_i(t) \in C(J)$, $r_i(t) = \varphi_i(t, C_i) o(1)$ as $t \to 0^+$ where $\varphi_i(t, C_i) = C_i \exp(\int_{t_0}^{t} (a_i / g_i(s)) ds)$ is the general solution of the equation $g_i(t) y_i'(t) = a_i y_i(t)$.

In the text, we will apply topological method of Ważewski and Schauder's theorem. Therefore we give a short summary of them.

Let $f(t, \mathbf{y})$ be a continuous function defined on an open $(t, \mathbf{y})$ set $\Omega \subset \mathbb{R} \times \mathbb{R}^n$, Ω^0 an open set of Ω, $\partial \Omega^0$ the boundary of Ω^0, and $\overline{\Omega}^0$ the closure of Ω^0. Consider the following system of ordinary differential equations:

$$\mathbf{y}' = f(t, \mathbf{y}). \tag{1.5}$$

Definition 1.3 (see [17]). The point $(t_0, \mathbf{y}_0) \in \Omega \cap \partial \Omega^0$ is called an egress (or an ingress point) of Ω^0 with respect to system (1.5) if for every fixed solution of the problem $\mathbf{y}(t_0) = \mathbf{y}_0$, there exists an $\epsilon > 0$ such that $(t, \mathbf{y}(t)) \in \Omega^0$ for $t_0 - \epsilon \leq t < t_0$ $(t_0 < t \leq t_0 + \epsilon)$. An egress point (ingress point) $(t_0, \mathbf{y}_0)$ of Ω^0 is called a strict egress point (strict ingress point) of Ω^0 if $(t, \mathbf{y}(t)) \notin \overline{\Omega}^0$ on interval $t_0 < t \leq t_0 + \epsilon_1$ $(t_0 - \epsilon_1 \leq t < t_0)$ for an ϵ_1.

Definition 1.4 (see [18]). An open subset Ω^0 of the set Ω is called an (u, v) subset of Ω with respect to system (1.5) if the following conditions are satisfied.

(1) There exist functions $u_i(t, \mathbf{y}) \in C^1(\Omega, \mathbb{R})$, $i = 1, \ldots, m$ and $v_j(t, \mathbf{y}) \in C[\Omega, \mathbb{R}]$ $j = 1, \ldots, n, m + n > 0$ such that

$$\Omega_0 = \{(t, \mathbf{y}) \in \Omega : u_i(t, \mathbf{y}) < 0, v_j(t, \mathbf{y}) < 0 \ \forall i, j\}. \tag{1.6}$$

(2) $\dot{u}_\alpha(t, \mathbf{y}) < 0$ holds for the derivatives of the functions $u_\alpha(t, \mathbf{y})$, $\alpha = 1, \ldots, m$ along trajectories of system (1.5) on the set

$$U_\alpha = \{(t, \mathbf{y}) \in \Omega : u_\alpha(t, \mathbf{y}) = 0, u_i(t, \mathbf{y}) \leq 0, v_j(t, \mathbf{y}) \leq 0, \forall j \text{ and } i \neq \alpha\}. \tag{1.7}$$

(3) $\dot{v}_\beta(t, \mathbf{y}) > 0$ holds for the derivatives of the functions $v_\beta(t, \mathbf{y})$, $\beta = 1, \ldots, n$ along trajectories of system (1.5) on the set

$$V_\beta = \{(t, \mathbf{y}) \in \Omega : u_\beta(t, \mathbf{y}) = 0, u_i(t, \mathbf{y}) \leq 0, v_j(t, y) \leq 0, \forall i \text{ and} j \neq \beta\}. \tag{1.8}$$

The set of all points of egress (strict egress) is denoted by Ω_e^0 (Ω_{se}^0).

Lemma 1.5 (see [18]). *Let the set Ω_0 be a (u, v) subset of the set Ω with respect to system* (1.5)*. Then*

$$\Omega_{\text{se}}^0 = \Omega_e^0 = \bigcup_{\alpha=1}^{m} U_\alpha \setminus \bigcup_{\beta=1}^{n} V_\beta. \tag{1.9}$$

Definition 1.6 (see [18]). Let X be a topological space and $B \subset X$.

Let $A \subset B$. A function $r \in C(B, A)$ such that $r(a) = a$ for all $a \in A$ is a retraction from B to A in X.

The set $A \subset B$ is a retract of B in X if there exists a retraction from B to A in X.

Theorem 1.7 (Ważewski's theorem [18]). *Let Ω^0 be some (u, v) subset of Ω with respect to system* (1.5)*. Let S be a nonempty compact subset of $\Omega^0 \cup \Omega_e^0$ such that the set $S \cap \Omega_e^0$ is not a retract of S but is a retract Ω_e^0. Then there is at least one point $(t_0, \mathbf{y}_0) \in S \cap \Omega_0$ such that the graph of a solution $\mathbf{y}(t)$ of the Cauchy problem $\mathbf{y}(t_0) = \mathbf{y}_0$ for* (1.5) *lies on its right-hand maximal interval of existence.*

Theorem 1.8 (Schauder's theorem [19]). *Let E be a Banach space and S its nonempty convex and closed subset. If P is a continuous mapping of S into itself and PS is relatively compact then the mapping P has at least one fixed point.*

2. Main Results

Theorem 2.1. *Let assumptions (i) and (ii) hold, then for each* $C_i \neq 0$ *there is one solution* $\mathbf{y}(t,\mathbf{C}) = (y_1(t,C_1), y_2(t,C_2),\ldots,y_n(t,C_n))$, $\mathbf{C} = (C_1,\ldots,C_n)$ *of initial problem* (1.1) *and* (1.2) *such that*

$$\left|y_i^{(j)}(t,C_i) - \varphi_i^{(j)}(t,C_i)\right| \leq \delta\left(\varphi_i^2(t,C_i)\right)^{(j)}, \quad j = 0,1, \tag{2.1}$$

for $t \in (0,t^{\Delta}]$, *where* $0 < t^{\Delta} \leq t_0$, $\delta > 1$ *is a constant, and* t^{Δ} *depends on* δ, C_i, $i = 1,\ldots,n$.

Proof. (1) Denote E the Banach space of vector-valued continuous functions $\mathbf{h}(t)$ on the interval $[0,t_0]$ with the norm

$$\|\mathbf{h}(t)\| = \max_{t\in[0,t_0]}|h_i(t)|, \quad i = 1,\ldots,n. \tag{2.2}$$

The subset S of Banach space E will be the set of all functions $h(t)$ from E satisfying the inequality

$$\left|h_i(t) - \varphi_i(t,C_i)\right| \leq \delta\varphi_i^2(t,C_i). \tag{2.3}$$

The set S is nonempty, convex, and closed.

(2) Now we will construct the mapping P. Let $\mathbf{h}_0(t) \in S$ be an arbitrary function. Substituting $\mathbf{h}_0(t)$, $\mathbf{h}_0(s)$ instead of $\mathbf{y}(t)$, $\mathbf{y}(s)$ into (1.1), we obtain the following differential equation:

$$g_i(t)y_i'(t) = a_i y_i(t)\left(1 + f_i\left(t,\mathbf{y}(t),\int_{0^+}^{t} K_i(t,s,\mathbf{h}_0(t),\mathbf{h}_0(s))ds\right)\right), \quad i = 1,\ldots,n. \tag{2.4}$$

Put

$$y_i(t) = \varphi_i(t,C_i) + \varphi_i^{(1-\mu)}(t,C_i)Y_{0i}(t), \tag{2.5}$$

$$y_i'(t) = \varphi_i'(t,C) + \frac{1}{g_i(t)}\varphi_i^{(1-\mu)}(t,C_i)Y_{1i}(t), \tag{2.6}$$

where $0 < \mu < 1$ is a constant and new functions $Y_{0i}(t)$, $Y_{1i}(t)$ satisfy the differential equations as

$$g_i(t)Y_{0i}'(t) = (\mu - 1)a_i Y_{0i}(t) + Y_{1i}(t), \quad i = 1,\ldots,n. \tag{2.7}$$

From (2.3), it follows

$$h_{0i}(t) = \varphi_i(t,C_i) + H_{0i}(t), \quad |H_{0i}(t)| \leq \delta\varphi_i^2(t,C_i). \tag{2.8}$$

Substituting (2.5), (2.6), and (2.8) into (2.4), we get

$$\begin{aligned}
Y_{1i}(t) = {}& a_i Y_{0i}(t) + \left(a_i \varphi_i^{\mu}(t, C_i) + a_i Y_{0i}(t)\right) \\
& \times f_i\Bigg(t, \varphi_1(t, C_1) + \varphi_1^{(1-\mu)}(t, C_1) Y_{01}(t), \dots, \varphi_n(t, C_n) + \varphi_n^{(1-\mu)}(t, C_n) Y_{0n}(t) \ , \\
& \int_{0^+}^{t} K_i\big(t, s, \varphi_1(t, C_1) + H_{01}(t), \dots, \varphi_n(t, C_n) + H_{0n}(t), \varphi_1(s, C_1) \\
& + H_{01}(s), \dots \varphi_n(s, C_n) + H_{0n}(s)\big) ds \Bigg).
\end{aligned} \tag{2.9}$$

Substituting (2.9) into (2.7), we get

$$\begin{aligned}
g_i(t) Y_{0i}'(t) = {}& \mu a_i Y_{0i}(t) + \left(a_i \varphi_i^{\mu}(t, C_i) + a_i Y_{0i}(t)\right) \\
& \times f_i\Bigg(t, \varphi_1(t, C_1) + \varphi_1^{(1-\mu)}(t, C_1) Y_{01}(t), \dots, \varphi_n(t, C_n) + \varphi_n^{(1-\mu)}(t, C_n) Y_{0n}(t), \\
& \int_{0^+}^{t} K_i\big(t, s, \varphi_1(t, C_1) + H_{01}(t), \dots, \varphi_n(t, C_n) + H_{0n}(t), \varphi_1(s, C_1) \\
& + H_{01}(s), \dots \varphi_n(s, C_n) + H_{0n}(s)\big) ds \Bigg).
\end{aligned} \tag{2.10}$$

In view of (2.5) and (2.6), it is obvious that a solution of (2.10) determines a solution of (2.4).

Now we use Ważewski's topological method. Consider an open set $\Omega \subset \mathbb{R}^+ \times \mathbb{R}^n$. Denote $\mathbf{Y}_0 = (Y_{01}, \dots, Y_{0n})$. Define an open subset $\Omega_0 \subset \Omega$ as follows:

$$\begin{aligned}
\Omega_0 &= \{(t, \mathbf{Y}_0) : u_i(t, \mathbf{Y}_0) < 0, v(t, \mathbf{Y}_0) < 0, i = 1, \dots, n\}, \\
U_\alpha &= \{(t, \mathbf{Y}_0) : u_\alpha(t, \mathbf{Y}_0) = 0, u_i(t, \mathbf{Y}_0) \le 0, v(t, \mathbf{Y}_0) \le 0, i = 1, \dots, n, i \neq \alpha\}, \\
V_\beta &= V = \left\{(t, \mathbf{Y}_0) : v(t, \mathbf{Y}_0) = 0, u_j(t, \mathbf{Y}_0) \le 0, i = 1, \dots, n\right\},
\end{aligned} \tag{2.11}$$

where

$$u_i(t, \mathbf{Y}_0) = Y_{0i}^2 - \left(\delta \varphi_i^{(1+\mu)}(t, C_i)\right)^2, \quad v(t, \mathbf{Y}_0) = t - t_0, \quad i = 1, \dots, n. \tag{2.12}$$

Calculating the derivatives $\dot{u}_\alpha(t,\mathbf{Y}_0)$, $\dot{v}(t,\mathbf{Y}_0)$ along the trajectories of (2.10) on the set U_α, V, $\alpha=1,\ldots,n$ we obtain

$$\begin{aligned}\dot{u}_\alpha(t,\mathbf{Y}_0)=\frac{2a_\alpha}{g_\alpha(t)}\Bigg[&\mu Y_{0\alpha}^2(t)+\left(Y_{0\alpha}(t)\varphi_\alpha^\mu(t,C_\alpha)+Y_{0\alpha}^2(t)\right)\\
&\times f_\alpha\Bigg(t,\varphi_1(t,C_1)+\varphi_1^{(1-\mu)}(t,C_1)Y_{01}(t),\ldots,\varphi_n(t,C_n)+\varphi_n^{(1-\mu)}(t,C_n)Y_{0n}(t),\\
&\int_{0^+}^t K_i(t,s,\varphi_1(t,C_1)+H_{01}(t),\ldots,\varphi_n(t,C_n)\\
&+H_{0n}(t),\varphi_1(s,C_1)+H_{01}(s),\ldots\varphi_n(s,C_n)+H_{0n}(s))ds\Bigg).\\
&-\delta^2(1+\mu)\varphi_\alpha^{2(1+\mu)}(t,C_\alpha)\Bigg].\end{aligned}\tag{2.13}$$

Since

$$\begin{gathered}\lim_{t\to+0}\varphi_i(t)g_i^\tau(t)=0\quad\text{for any }\tau>0,\ i=1,\ldots,n\\
g_i'(t)\sim\varphi_i(t)g_i^{\lambda_i}(t)\quad\text{as }t\to0^+,\ \lambda_i>0,\ i=1,\ldots,n,\end{gathered}\tag{2.14}$$

then there exists a positive constant M_i such that

$$g_i'(t)<M_i,\quad t\in(0,t_0],\ i=1,\ldots,n.\tag{2.15}$$

Consequently,

$$\int_{t_0}^t\frac{ds}{g_i(s)}<\frac{1}{M_i}\int_{t_0}^t\frac{g_i'(s)dt}{g_i(s)}=\frac{1}{M_i}\ln\frac{g_i(t)}{g_i(t_0)}\longrightarrow-\infty\quad\text{as}\quad t\longrightarrow0^+,\ i=1,\ldots,n.\tag{2.16}$$

From here $\lim_{t\to0^+}\varphi_i(t,C_i)=0$ and by L'Hospital's rule $\varphi_i^\tau(t,C_i)g_i^\sigma(t)=o(1)$, for $t\to0^+$, $i=1,\ldots,n,\sigma$ is an arbitrary real number. These both identities imply that the powers of $\varphi_i(t,C_i)$ affect the convergence to zero of the terms in (2.13), in a decisive way.

Using the assumptions of Theorem 2.1 and the definition of $Y_0(t)$, $\varphi_i(t,C_i)$, $i=1,\ldots,n$, we get that the first term $\mu Y_{0\alpha}^2(t,C_\alpha)$ in (2.13) has the following form:

$$\mu Y_{0\alpha}^2(t)=\mu\delta^2\varphi_\alpha^{2(1+\mu)}(t,C_\alpha),\tag{2.17}$$

and the second term

$$\begin{aligned}
&\left(Y_{0\alpha}(t)\varphi_\alpha^\mu(t,C_\alpha)+Y_{0\alpha}^2(t)\right)\\
&\quad\times f_\alpha\Big(t,\varphi_1(t,C_1)+\varphi_1^{(1-\mu)}(t,C_1)Y_{01}(t),\dots,\varphi_n(t,C_n)\\
&\qquad+\varphi_n^{(1-\mu)}(t,C_n)Y_{0n}(t),\int_{0^+}^t K_\alpha(t,s,\varphi_1(t,C_1)+H_{01}(t),\dots,\varphi_n(t,C_n)\\
&\qquad\qquad+H_{0n}(t),\varphi_1(s,C_1)+H_{01}(s),\dots\varphi_n(s,C_n)\\
&\qquad\qquad+H_{0n}(s))ds\Big).
\end{aligned}\tag{2.18}$$

is bounded by terms with exponents which are greater than $\varphi_\alpha^{2(1+\mu)}(t,C_\alpha)$, $\alpha=1,\dots,n$. From here, we obtain

$$\operatorname{sgn}\dot{u}_\alpha(t,\mathbf{Y}_0)=-\delta^2(1+\mu)\varphi_\alpha^{2(1+\mu)}(t,C_\alpha)=-1\tag{2.19}$$

for sufficiently small t^*, depending on C_α $\alpha=1,\dots,n$, δ, $0<t^*\le t_0$.

It is obvious that $\operatorname{sgn}\dot{v}(t,\mathbf{Y}_0)=1$.

Change the orientation of the axis t into opposite. Then, with respect to the new system of coordinates, the set Ω_0 is the (u,v) subset with respect to system (2.10). By Ważewski's topological method, we state that there exists at least one integral curve of (2.10) lying in Ω_0 for $t\in(0,t^*)$. It is obvious that this assertion remains true for an arbitrary function $\mathbf{h}_0(t)\in S$.

Now we prove the uniqueness of a solution of (2.10). Let $\overline{\mathbf{Y}}_0(t)=(\overline{Y}_{01}(t),\dots,\overline{Y}_{0n}(t))$ be also the solution of (2.10). Putting

$$Z_{0i}=Y_{0i}-\overline{Y}_{0i},\quad i=1,\dots,n\tag{2.20}$$

and substituting into (2.10), we obtain

$$\begin{aligned}
g_i(t)Z_{0i}'(t)&=\mu a_iY_{0i}(t)+\left(a_i\varphi_i^\mu(t,C_i)+a_iZ_{0i}(t)\right)\\
&\quad\times f_i\bigg(t,\varphi_1(t,C_1)+\varphi_1^{(1-\mu)}(t,C_1)\left(Z_{0i}(t)+\overline{Y}_{01}(t)\right),\dots,\varphi_n(t,C_n)\\
&\qquad+\varphi_n^{(1-\mu)}(t,C_n)\left(Z_{0n}(t)+\overline{Y}_{0n}(t)\right),\\
&\qquad\int_{0^+}^t K_i(t,s,\varphi_1(t,C_1)+H_{01}(t),\dots,\varphi_n(t,C_n)\\
&\qquad\qquad+H_{0n}(t),\varphi_1(s,C_1)+H_{01}(s),\dots\varphi_n(s,C_n)+H_{0n}(s))ds\bigg).
\end{aligned}\tag{2.21}$$

Define

$$\begin{aligned}\Omega_1(\delta) &= \{(t, \mathbf{Z}_0) : 0 < t < t^*, u_{1i}(t, \mathbf{Z}_0) < 0, v_1(t, \mathbf{Z}_0) < 0, 0 < t < t^*, i = 1, \dots, n\} \\ U_{1\alpha} &= \{(t, \mathbf{Z}_0) : u_{1\alpha}(t, \mathbf{Z}_0) = 0, u_{1i}(t, \mathbf{Z}_0) \le 0, v_1(t, \mathbf{Z}_0) \le 0, i = 1, \dots, n, i \ne \alpha\}, \\ V_{1\beta} = V &= \{(t, \mathbf{Z}_0) : v_1(t, \mathbf{Z}_0) = 0, u_j(t, \mathbf{Z}_0) \le 0, i = 1, \dots, n\},\end{aligned} \tag{2.22}$$

where

$$u_{1i}(t, \mathbf{Z}_0) = Z_{0i}^2 - \left(\delta\varphi_i^{(1+\mu-\gamma)}\right)^2, \quad 0 < \gamma < \mu, \; v_1(t, \mathbf{Z}_0) = t - t^*. \tag{2.23}$$

Using the same method as above, we have

$$\operatorname{sgn} \dot{u}_{1i}(t, \mathbf{Z}_0) = -1, \qquad \operatorname{sgn} \dot{v}_1(t, \mathbf{Z}_0) = 1, \quad i = 1, \dots, n \tag{2.24}$$

for sufficiently small $t^\diamond$, $0 < t^\diamond \le t^*$. It is obvious that $\Omega_0 \subset \Omega_1(\delta)$ for $t \in (0, t^\diamond)$. Let $\overline{\mathbf{Z}}_0(t) = (\overline{Z}_{01}(t), \dots, \overline{Z}_{0n}(t))$ be any nonzero solution of (2.10) such that $(t_1, \overline{\mathbf{Z}_0}(t_1)) \in \Omega_1$ for $0 < t_1 < t^\diamond$. Let $\overline{\delta} \in (0, \delta)$ be such a constant that $(t_1, \overline{\mathbf{Z}_0}(t_1)) \in \partial\Omega_1(\overline{\delta})$. If the curve $\overline{\mathbf{Z}_0}(t)$ lay in $\Omega_1(\overline{\delta})$ for $0 < t < t_1$, then $(t_1, \overline{\mathbf{Z}_0}(t_1))$ would have to be a strict egress point of $\partial\Omega_1(\overline{\delta})$ with respect to the original system of coordinates. This contradicts the relation (2.24). Therefore there exists only the trivial solution $\mathbf{Z}_0(t) \equiv 0$ of (2.21), so $\mathbf{Y}_0 = \overline{\mathbf{Y}_0}(t)$ is the unique solution of (2.10).

From (2.5) we obtain

$$\left| y_i(t, C_i) - \varphi_i(t, C_i) \right| \le \delta\varphi_i^2(t, C_i), \quad i = 1, \dots, n, \tag{2.25}$$

where $(y_1(t, C_1), \dots, y_n(t, C_n))$ is the solution of (2.4) for $t \in (0, t^\diamond]$. Similarly, from (2.6) and (2.9), we have

$$\begin{aligned}\left| y_i'(t, C_i) - \varphi_i'(t, C_i) \right| &= \left| \frac{1}{g_i(t)} \varphi_i^{(1-\mu)}(t, C_i) Y_{1i}(t) \right| \\ &\le \left| \frac{1}{g_i(t)} \varphi_i^{(1-\mu)}(t, C_i) 2a_i \delta\varphi_i^{(1-\mu)}(t, C_i) \right| = \delta\left(\varphi_i^2(t, C_i)\right)'.\end{aligned} \tag{2.26}$$

It is obvious (after a continuous extension of $\mathbf{y}(t, \mathbf{C})$ for $t = 0$, $\mathbf{y}(0^+) = 0$) that $P : \mathbf{h}_0 \to \mathbf{y}$ maps S into itself and $PS \subset S$.

(3) We will prove that PS is relatively compact and P is a continuous mapping.

It is easy to see, by (2.25) and (2.26), that PS is the set of uniformly bounded and equicontinuous functions for $t \in [0, t^\diamond]$. By Ascoli's theorem, PS is relatively compact.

Let $\{\mathbf{h}_k(t)\}$ be an arbitrary sequence vector-valued functions in S such that

$$\|\mathbf{h}_k(t) - \mathbf{h}_0(t)\| = \epsilon_k, \qquad \lim_{k\to\infty} \epsilon_k = 0, \quad \mathbf{h}_0(t) \in S. \tag{2.27}$$

The solution $\overline{\mathbf{Y}}_k(t) = (\overline{Y}_{k1}, \ldots, \overline{Y}_{kn})$ of the following equation:

$$
\begin{aligned}
g_i(t)Y'_{0i}(t) = \mu a_i Y_{0i}(t) + \Big(a_i\varphi_i^{\mu}(t,C_i) + a_i Y_{0i}(t)\Big) \\
\times f_i\Bigg(t, \varphi_1(t,C_1) + \varphi_1^{(1-\mu)}(t,C_1)Y_{01}(t), \ldots, \varphi_n(t,C_n) + \varphi_n^{(1-\mu)}(t,C_n)Y_{0n}(t), \\
\int_{0^+}^{t} K_i(t,s,\varphi_1(t,C_1) + H_{01}(t), \ldots, \varphi_n(t,C_n) \\
+H_{0n}(t), \varphi_1(s,C_1) + H_{k1}(s), \ldots \varphi_n(s,C_n) + H_{kn}(s))ds \Bigg).
\end{aligned}
\tag{2.28}
$$

corresponds to the function $\mathbf{h}_k(t)$ and $\overline{\mathbf{Y}_k}(t) \in \Omega_0$ for $t \in (0, t^{\diamond})$. Similarly, the solution $\overline{\mathbf{Y}}_0(t)$ of (2.10) corresponds to the function $h_0(t)$. We will show that $|\overline{\mathbf{Y}_k}(t) - \overline{\mathbf{Y}}_0(t)| \to 0$ uniformly on $[0, t^{\Delta}]$, where $0 < t^{\Delta} \leq t^{\diamond}$, t^{Δ} is a sufficiently small constant which will be specified later. Consider the following region:

$$
\Omega_{0k} = \Big\{(t, \mathbf{Y}_0) : 0 < t < t^{\diamond}, u_{0k_i}(t, \mathbf{Y}_0) < 0, v_0(t, \mathbf{Y}_0) < 0, i = 1, \ldots, n\Big\},
\tag{2.29}
$$

where

$$
\begin{gathered}
u_{0k_i}(t, \mathbf{Y}_0) = \Big(Y_{0i}(t) - \overline{Y}_{0i}(t)\Big)^2 - \Big(\epsilon_k \varphi_i^{(1+\mu-\nu)}(t,C_i)\Big)^2, \quad 0 < \nu < \alpha,\ i = 1, \ldots, n,\ k \geq 1, \\
v_0(t, \mathbf{Y}_0) = t - t^{\diamond}.
\end{gathered}
\tag{2.30}
$$

There exists sufficiently small constant $t^{\Delta} \leq t^{\diamond}$ such that $\Omega_0 \subset \Omega_{0k}$ for any $k, t \in (0, t^{\Delta})$. Investigate the behaviour of integral curves of (2.28) with respect to the boundary $\partial\Omega_{0k}$, $t \in (0, t^{\Delta}]$. Using the same method as above, we obtain the following trajectory derivatives:

$$
\operatorname{sgn} \dot{u}_{0k}(t, \mathbf{Y}_0) = -1, \qquad \operatorname{sgn} \dot{v}_0(t, \mathbf{Y}_0) = 1
\tag{2.31}
$$

for $t \in (0, t^{\Delta}]$ and any k. By Ważewski's topological method, there exists at least one solution $\overline{\mathbf{Y}}_k(t)$ lying in Ω_{0k}, $0 < t < t^{\Delta}$. Hence, it follows that

$$
\Big|\overline{Y}_{ki}(t) - \overline{Y}_{0i}(t)\Big| \leq \epsilon_k \varphi_i^{1+\mu-\nu} \leq N_i \epsilon_k,
\tag{2.32}
$$

$N_i > 0$, $i = 1, \ldots, n$ are constants depending on C_i, t^{Δ}. From (2.5), we obtain

$$
\big|y_{ki}(t) - y_{0i}(t)\big| = \varphi_i^{(1-\mu)}(t,C_i)\Big|\overline{Y}_{ki}(t) - \overline{Y}_{0i}(t)\Big| \leq n_i \epsilon_k,
\tag{2.33}
$$

where $n_i > 0$, $i = 1, \ldots, n$ are constants depending on t^{Δ}, C_i, N_i. This estimate implies that P is continuous.

We have thus proved that the mapping P satisfies the assumptions of Schauder's fixed point theorem and hence there exists a function $\mathbf{h}(t) \in S$ with $\mathbf{h}(t) = P(\mathbf{h}(t))$. The proof of existence of a solution of (1.1) is complete.

Now we will prove the uniqueness of a solution of (1.1). Substituting (2.5) and (2.6) into (1.1), we get

$$\begin{aligned} Y_{1i}(t) = {} & a_i Y_{0i}(t) + \left(a_i \varphi_i^{\mu}(t, C_i) + a_i Y_{0i}(t)\right) \\ & \times f_i\Bigg(t, \varphi_1(t, C_1) + \varphi_1^{(1-\mu)}(t, C_1) Y_{01}(t), \dots, \varphi_n(t, C_n) + \varphi_n^{(1-\mu)}(t, C_n) Y_{0n}(t), \\ & \int_{0^+}^{t} K_i\Big(t, s, \varphi_1(t, C_1) + \varphi_1^{(1-\mu)}(t, C_1) Y_{01}(t), \dots, \varphi_n(t, C_n) \\ & + \varphi_n^{(1-\mu)}(t, C_n) Y_{0n}(t), \varphi_1(s, C_1) + \varphi_1^{(1-\mu)}(s, C_1) Y_{01}(s), \dots \varphi_n(s, C_n) \\ & + \varphi_n^{(1-\mu)}(s, C_n) Y_{0n}(s)\Big) ds \Bigg). \end{aligned} \tag{2.34}$$

Equation (2.7) may be written in the following form:

$$\begin{aligned} g_i(t) Y_{0i}'(t) = {} & a_i Y_{0i}(t) + \left(a_i \varphi_i^{\mu}(t, C_i) + a_i Y_{0i}(t)\right) \\ & \times f_i\Bigg(t, \varphi_1(t, C_1) + \varphi_1^{(1-\mu)}(t, C_1) Y_{01}(t), \dots, \varphi_n(t, C_n) + \varphi_n^{(1-\mu)}(t, C_n) Y_{0n}(t), \\ & \int_{0^+}^{t} K_i\Big(t, s, \varphi_1(t, C_1) + \varphi_1^{(1-\mu)}(t, C_1) Y_{01}(t), \dots, \varphi_n(t, C_n) + \varphi_n^{(1-\mu)}(t, C_n) \\ & \times Y_{0n}(t), \varphi_1(s, C_1) + \varphi_1^{(1-\mu)}(s, C_1) Y_{01}(s), \dots \varphi_n(s, C_n) \\ & + \varphi_n^{(1-\mu)}(s, C_n) Y_{0n}(s)\Big) ds \Bigg). \end{aligned} \tag{2.35}$$

Now we know that there exists the solution $\mathbf{y}_0(t) = (y_{01}(t, C_1), \dots, y_{0n}(t, C_n))$ of (1.1) satisfying (1.2) such that

$$y_{0i}(t, C_i) = \varphi_i(t, C_i) + \varphi_i^{(1-\mu)}(t, C_i) T_{0i}(t), \quad i = 1, \dots, n, \tag{2.36}$$

where $\mathbf{T}_0(t) = (T_{01}(t), \dots, T_{0n}(t))$ is the solution of (2.35).

Denote $W_{i0}(t) = Y_{0i}(t) - T_{0i}(t)$, $i = 1, \dots, n$. Substituting $W_{i0}(t)$ into (2.35), we obtain

$$\begin{aligned}
g_i(t)W'_{0i}(t) = a_i W_{0i}(t) &+ \left(a_i \varphi_i^{\mu}(t, C_i) + a_i W_{0i}(t)\right) \\
&\times \Bigg[f_i\Bigg(t, \varphi_1(t, C_1) + \varphi_1^{(1-\mu)}(t, C_1)(W_{01}(t) + T_{01}(t)), \dots, \varphi_n(t, C_n) \\
&\quad + \varphi_n^{(1-\mu)}(t, C_n) \times (W_{0n}(t) + T_{0n}(t)), \\
&\quad \int_{0^+}^{t} K_i\Big(t, s, \varphi_1(t, C_1) + \varphi_1^{(1-\mu)}(t, C_1)(W_{01}(t) + T_{01}(t)), \dots, \varphi_n(t, C_n) \\
&\quad + \varphi_n^{(1-\mu)}(t, C_n) \times (W_{0n}(t) + T_{0n}(t)), \varphi_1(s, C_1) + \varphi_1^{(1-\mu)}(s, C_1) \\
&\quad \times (W_{01}(s) + T_{01}(s)), \dots \varphi_n(s, C_n) \\
&\quad + \varphi_n^{(1-\mu)}(s, C_n)(W_{0n}(s) + T_{0n}(s))\Big) ds \Bigg) \\
&- f_i\Bigg(t, \varphi_1(t, C_1) + \varphi_1^{(1-\mu)}(t, C_1) T_{01}(t), \dots, \varphi_n(t, C_n) + \varphi_n^{(1-\mu)}(t, C_n), \\
&\quad \int_{0^+}^{t} K_i\Big(t, s, \varphi_1(t, C_1) + \varphi_1^{(1-\mu)}(t, C_1) T_{01}(t), \dots, \varphi_n(t, C_n) T_{0n}(t) \\
&\quad + \varphi_n^{(1-\mu)}(t, C_n) T_{0n}(t), \varphi_1(s, C_1) + \varphi_1^{(1-\mu)}(s, C_1) \\
&\quad \times T_{01}(s), \dots \varphi_n(s, C_n) + \varphi_n^{(1-\mu)}(s, C_n) T_{0n}(s)\Big) ds \Bigg) \Bigg].
\end{aligned} \tag{2.37}$$

Let

$$^1\Omega_0 = \left\{ (t, \mathbf{W}_0) : 0 < t < t^{\Delta}, u_{1i}(t, \mathbf{W}_0) < 0, v_1(t, \mathbf{W}_0) < 0 \right\}, \tag{2.38}$$

where

$$u_{1i}(t, \mathbf{W}_0) = W_{1i}^2 - \left(\delta \varphi_i^{(1+\mu-\rho)}(t, C_i)\right)^2, \quad 0 < \rho < \mu, \quad v_1(t, \mathbf{W}_0) = t - t^{\Delta}, \quad i = 1, \dots, n. \tag{2.39}$$

If (2.37) had only the trivial solution lying in $^1\Omega_0$, then $\mathbf{Y}_0(t) = \mathbf{T}_0(t)$ would be only one solution of (2.37) and from here, by (2.35), $\mathbf{y}_0(t)$ would be only one solution of (1.1) satisfying (1.2) for $t \in (0, t^{\Delta}]$.

We will suppose that there exists nontrivial solution $\overline{\mathbf{W}}_0(t)$ of (2.37) lying in ${}^1\Omega_0$. Substituting $\overline{W}_{0i}(s)$ instead of $W_{0i}(s)$, $i = 1, \dots, n$ into (2.37), we obtain the following differential equation:

$$
\begin{aligned}
g_i(t)W'_{0i}(t) = a_i W_{0i}(t) &+ \left(a_i\varphi_i^{\mu}(t,C_i) + a_i W_{0i}(t)\right) \\
&\times \Bigg[f_i\Bigg(t, \varphi_1(t,C_1) + \varphi_1^{(1-\mu)}(t,C_1)(W_{01}(t) + T_{01}(t)), \dots, \varphi_n(t,C_n) \\
&\qquad + \varphi_n^{(1-\mu)}(t,C_n) \times (W_{0n}(t) + T_{0n}(t)), \\
&\qquad \int_{0^+}^{t} K_i\Big(t,s,\varphi_1(t,C_1) + \varphi_1^{(1-\mu)}(t,C_1)(W_{01}(t) + T_{01}(t)), \dots, \varphi_n(t,C_n) \\
&\qquad\qquad + \varphi_n^{(1-\mu)}(t,C_n) \times (W_{0n}(t) + T_{0n}(t)), \varphi_1(s,C_1) + \varphi_1^{(1-\mu)}(s,C_1) \\
&\qquad\qquad \times \left(\overline{W}_{01}(s) + T_{01}(s)\right), \dots \varphi_n(s,C_n) \\
&\qquad\qquad + \varphi_n^{(1-\mu)}(s,C_n)\left(\overline{W}_{0n}(s) + T_{0n}(s)\right)\Big)ds \Bigg) \\
&\quad - f_i\Bigg(t, \varphi_1(t,C_1) + \varphi_1^{(1-\mu)}(t,C_1)T_{01}(t), \dots, \varphi_n(t,C_n) + \varphi_n^{(1-\mu)}(t,C_n), \\
&\qquad \int_{0^+}^{t} K_i\Big(t,s,\varphi_1(t,C_1) + \varphi_1^{(1-\mu)}(t,C_1)T_{01}(t), \dots, \varphi_n(t,C_n)T_{0n}(t) \\
&\qquad\qquad + \varphi_n^{(1-\mu)}(t,C_n)T_{0n}(t), \varphi_1(s,C_1) + \varphi_1^{(1-\mu)}(s,C_1) \\
&\qquad\qquad \times T_{01}(s), \dots \varphi_n(s,C_n) + \varphi_n^{(1-\mu)}(s,C_n)T_{0n}(s)\Big)ds \Bigg) \Bigg].
\end{aligned}
\tag{2.40}
$$

Calculating the derivative $\dot{u}_{1i}(t,\mathbf{W}_0)$ along the trajectories of (2.40) on the set $\partial^1\Omega_0$, we get $\operatorname{sgn}\ \dot{u}_{1i}(t,\mathbf{W}_0) = -1$ for $t \in (0, t^{\Delta}]$, $i = 1, \dots, n$.

By the same method as in the case of the existence of a solution of (1.1), we obtain that in ${}^1\Omega_0$ there is only the trivial solution of (2.40). The proof is complete. □

3. Asymptotic Expansions of Solutions

Diblík [3] investigated a singular initial problem for implicit ordinary differential equations and constructed asymptotic expansions of solutions in a right-hand neighbourhood of a singular point. Some results about asymptotic expansions of solutions for integrodifferential equations with separable kernels are given in [3, 10, 12].

The aim of this section is to show that results of paper [2] for ordinary differential equations are possible to extend on certain classes systems integrodifferential equations with a separable kernel in the following form:

$$g(t)y_i' = y_i + \int_{0^+}^{t} \left(\sum_{|\sigma_i|+|\omega_i|=2}^{N_i} u_{\sigma_i\omega_i}(t) v_{\sigma_i\omega_i}(s) \mathbf{y}^{\sigma_i}(t) \mathbf{y}^{\omega_i}(s) \right) ds, \tag{3.1}$$

where $N_i \in \mathbb{N}$, $\sigma_i = (l_{i1}, \dots, l_{in})$, $\omega_i = (j_{i1}, \dots, j_{in})$, $l_{ik}, j_{ik} \in \mathbb{N} \cup \{0\}$, $k = 1, \dots, n$,

$$\begin{gathered} |\sigma_i| = \sum_{k=1}^{n} l_{ik}, \qquad |\omega_i| = \sum_{k=1}^{n} j_{ik}, \qquad \mathbf{y}^{\sigma_i}(t) = \prod_{k=1}^{n} y_k^{l_{ik}}(t), \qquad \mathbf{y}^{\omega_i}(s) = \prod_{k=1}^{n} y_k^{j_{ik}}(s), \\ u_{\sigma_i\omega_i}(t), v_{\sigma_i\omega_i}(t) \in C^0(J), \quad J = (0, t_0], \ i = 1, \dots, n. \end{gathered} \tag{3.2}$$

We will construct the solution of (3.1) in the form of one parametric asymptotic expansions as

$$y_i(t, C) = \sum_{h=1}^{\infty} f_{ih}(t) \phi^h(t, C), \quad i = 1, \dots, n, \tag{3.3}$$

where $\phi(t, C)$ is the general solution of the differential equation $g(t)y' = y$ so that

$$\phi(t, C) = C \exp\left[\int_{t_0}^{t} \frac{d\tau}{g(\tau)} \right], \tag{3.4}$$

$f_{i1}(t) \equiv 1$, $f_{ih}(t)$, $h \geq 2$, $i = 1, \dots, n$ are unknown functions, $C \neq 0$ is a constant.

Consider the following differential equation:

$$g(t)y' = qy + p(t). \tag{3.5}$$

Diblík [3] proved asymptotic estimates of the solution of (3.5) which we can be formulated as follows.

Theorem 3.1. *Assume that*

(I) *Let q be a constant, $q < 0$, $g(t) \in C^1(J)$, $g(t) > 0$, $\lim_{t \to t_0^+} g(t) = 0$, $g'(t) \sim \psi_1(t) g^{\lambda_1}(t)$ as $t \to t_0^+$, $\lambda_1 > 0$, $\lim_{t \to t_0^+} \psi_1(t) g^{\tau}(t) = 0$, τ is any positive number.*

(II) $p(t) \in C(J)$, $p(t) = b_0(t) g^{\lambda}(t) + O(b_1(t) g^{\lambda+\epsilon}(t))$, $\epsilon > 0$, $\lim_{t \to t_0^+} b_m(t) g'(t) = 0$, $m = 0, 1$, $b_0(t) \in C(J)$, $b_0(t) \neq 0$, $b_0'(t) \sim \psi_2(t) g^{\lambda_2}(t)$ *as* $t \to t_0^+$, $\lambda_2 + 1 > 0$, $\lim_{t \to t_0^+} \psi_2(t) g^{\tau}(t) = 0$, $\lim_{t \to t_0^+} g^{\tau}(t) (b_0(t))^{-1} = 0$.

Then (3.5) *has a unique solution on* $(0,\bar{t}]$, $0<\bar{t}\le t_0$, *satisfying asymptotic estimates*

$$y(x)=\frac{-1}{q}b_0(x)g^{\lambda}(x)+O(g^{\nu}(x)),\qquad y'(x)=O\left(g^{\nu-1}(x)\right), \tag{3.6}$$

where $\nu\in(\lambda,\lambda+\min\{\lambda_1,\lambda_2+1,\epsilon\})$.

Now we will show the results of Theorem 3.1. regarding only differential equation (3.5) we can apply to system of integrodifferential equations (3.1).

Substituting (3.3) into (3.1) and comparing the terms with the same powers of $\phi(t,C)$, we obtain the following system of recurrence equations:

$$g(t)f'_{ih}=(1-h)f_{ih}+\phi^{-h}(t,C)\int_{0^+}^{t}R_{ih}(t,s)ds, \tag{3.7}$$

$h\ge 2$, $i=1,\dots,n$ and

$$\begin{aligned}R_{ih}(t,s)=R_{ih}\big[&f_{11}(t),\dots,f_{ih-1}(t),\dots,f_{n1}(t),\dots,f_{nh-1}(t),\\&f_{11}(s),\dots,f_{1h-1}(s),\dots,f_{n1}(s),\dots,f_{nh-1}(s)\big].\end{aligned} \tag{3.8}$$

Denote

$$p_{ih}(t)=\phi^{-h}(t,C)\int_{0^+}^{t}R_{ih}(t,s)ds, \tag{3.9}$$

then it is obvious that the recurrence equations

$$g(t)f'_{ih}=(1-h)f_{ih}+p_{ih}(t) \tag{3.10}$$

$h\ge 2$, $i=1,\dots,n$ have the same form as (3.5) with the constant $q=1-h$. Hence we can apply Theorem 3.1, after the modification of assumption (II) of Theorem 3.1 for indices $h\ge 2$, $i=1,\dots,n$, to recurrence (3.10) which we will demonstrate with the following example.

Example 3.2. Consider the following system of integrodifferential equations:

$$\begin{aligned}t^2y_1'&=y_1+\int_{0^+}^{t}\frac{1}{t^3}y_1(s)y_2(s)ds,\\t^2y_2'&=y_2+\int_{0^+}^{t}\sqrt{t}y_1(t)y_2(s)ds.\end{aligned} \tag{3.11}$$

System (3.11) has the form of system (3.1) for

$$\begin{aligned}&\sigma_1\omega_1=(0,0,1,1),\qquad u_{\sigma_1\omega_1}(t)=\frac{1}{t^3},\qquad v_{\sigma_1\omega_1}(s)=1,\qquad N_1=2\\&\sigma_2\omega_2=(1,0,0,1),\qquad u_{\sigma_2\omega_2}(t)=\sqrt{t},\qquad v_{\sigma_2\omega_2}(s)=1,\qquad N_2=2.\end{aligned} \tag{3.12}$$

We will construct a solution of system (3.11) in the following form:

$$y_1 = \sum_{k=1}^{\infty} f_{1k}(t)\phi^k(t,C), \qquad y_2 = \sum_{k=1}^{\infty} f_{2k}(t)\phi^k(t,C), \tag{3.13}$$

where $\phi(t,C)$ is the general solution of the equation $t^2y' = y$. We will demonstrate the calculation of coefficients f_{ih} for $h = 3$. Substituting (3.13) in (3.11) and comparing the terms with the same powers of $\phi(t,C)$, we obtain the following system of recurrence equations:

$$\begin{aligned} \phi^1(t,C)\colon 1 &= 1, \\ 1 &= 1. \end{aligned} \tag{3.14}$$

$$\begin{aligned} \phi^2(t,C)\colon t^2 f_{12}' &= -f_{12} + \phi^{-2}(t,C)\int_{0^+}^{t} \frac{1}{t^3}\phi^2(s,C)ds, \\ t^2 f_{22}' &= -f_{22} + \phi^{-2}(t,C)\int_{0^+}^{t} \sqrt{t}\;\phi(t,C)\phi(s,C)ds. \end{aligned} \tag{3.15}$$

$$\begin{aligned} \phi^3(t,C)\colon t^2 f_{13}' &= -2f_{13} + \phi^{-3}(t,C)\int_{0^+}^{t} \frac{1}{t^3}\left[f_{12}(s) + f_{22}(s)\right]\phi^3(s,C)ds, \\ t^2 f_{23}' &= -2f_{23} + \phi^{-3}(t,C)\int_{0^+}^{t} \sqrt{t}\Big[f_{12}(t)\phi^2(t,C)\phi(s,C) \\ &\qquad + f_{22}(s)\phi^2(s,C)\phi(t,C)\Big]ds. \end{aligned} \tag{3.16}$$

Put

$$u_1 = \phi^{-2}(t,C)\int_{0^+}^{t} \phi^2(s,C)ds, \qquad u_2 = \phi^{-1}(t,C)\int_{0^+}^{t} \phi(s,C)ds. \tag{3.17}$$

Differentiating both equations (3.17), we obtain the following differential equations:

$$t^2u_1' = -2u_1 + t^2, \tag{3.18}$$

$$t^2u_2' = -u_2 + t^2. \tag{3.19}$$

Equation (3.18) satisfies assumptions of Theorem 3.1. with following functions and coefficients:

$$\begin{aligned} &a = -2, \quad b_0(t) = 1, \quad g^{\lambda}(t) = \left(t^2\right)^1 \Rightarrow \lambda = 1, \quad b_1(t) = 0, \\ &g'(t) = \left(t^2\right)' = 2\left(g(t)\right)^{1/2} \Rightarrow \lambda_1 = \frac{1}{2}, \quad b_0'(t) = 0\cdot g^{\lambda_2}(t). \end{aligned} \tag{3.20}$$

Hence we can choose a constant $\lambda_2 + 1 > 1/2$ and similarly $\epsilon > 1/2$. By Theorem 3.1., we have

$$u_1 = \frac{1}{2}t^2 + O\left(t^{2\nu_1}\right), \qquad \nu_1 \in \left(1, \frac{3}{2}\right). \tag{3.21}$$

Second equation (3.19) is different from (3.18) only in the constant $a = -1$. Thus

$$u_2 = t^2 + O\left(t^{2\nu_2}\right), \qquad \nu_2 \in \left(1, \frac{3}{2}\right). \tag{3.22}$$

Substituting solutions (3.21) and (3.22) into (3.15) instead of integral terms, we obtain for unknown coefficients f_{12}, f_{22} the following differential equations:

$$t^2 f'_{12} = -f_{12} + \frac{1}{2t} + O\left(t^{2\nu_1 - 3}\right), \tag{3.23}$$

$$t^2 f'_{22} = -f_{22} + t^{5/2} + O\left(t^{2\nu_2 + 1/2}\right). \tag{3.24}$$

For (3.23), we can put

$$\begin{gathered} a = -1, \qquad b_0(t) = \frac{1}{2}, \quad g^{\lambda}(t) = \left(t^2\right)^{-1/2}, \qquad \lambda = -\frac{1}{2}, \quad b_1(t) = 1, \\ \epsilon = \nu_1 - 1, \qquad g'(t) = \left(t^2\right)' = 2\left(g(t)\right)^{1/2} \Rightarrow \lambda_1 = \frac{1}{2}, \qquad b'_0(t) = 0 \cdot g^{\lambda_2}(t). \end{gathered} \tag{3.25}$$

Then we can choose a constant $\lambda_2 + 1 > 1/2$. By Theorem 3.1., we get

$$f_{12}(t) = \frac{1}{2t} + O\left(t^{2\nu_{12}}\right), \qquad f'_{12}(t) = O\left(t^{2\nu_{12} - 2}\right), \qquad \nu_{12} \in \left(-\frac{1}{2}, 0\right). \tag{3.26}$$

Similarly for (3.24), we can put $a = -1$, $b_0(t) = 1$, $g^{\lambda}(t) = (t^2)^{5/4}$, $\lambda = 5/4$, $b_1(t) = 1$, $\epsilon = \nu_2 - 1$,

$$g'(t) = \left(t^2\right)' = 2\left(g(t)\right)^{1/2} \Longrightarrow \lambda_1 = \frac{1}{2}, \qquad b'_0(t) = 0 \cdot g^{\lambda_2}(t). \tag{3.27}$$

Then we can choose a constant $\lambda_2 + 1 > 1/2$. By Theorem 3.1., we have

$$f_{22}(t) = t^{5/2} + O\left(t^{2\nu_{22}}\right), \qquad f'_{22}(t) = O\left(t^{2\nu_{22} - 2}\right), \qquad \nu_{22} \in \left(\frac{5}{4}, \frac{7}{4}\right). \tag{3.28}$$

Substituting coefficients f_{12}, f_{22} into (3.16) and using the same method as in the calculation of coefficients f_{12}, f_{22}, we have

$$\begin{aligned} f_{13}(t) &= \frac{1}{12t^2} + O\left(t^{2\nu_{13}}\right), \quad f'_{13}(t) = O\left(t^{2\nu_{13}-1}\right), \quad \nu_{13} \in \left(-1, -\frac{1}{2}\right), \\ f_{23}(t) &= \frac{1}{4}t^{3/2} + O\left(t^{2\nu_{23}}\right), \quad f'_{23}(t) = O\left(t^{2\nu_{23}-1}\right), \quad \nu_{23} \in \left(\frac{3}{4}, \frac{5}{4}\right). \end{aligned} \tag{3.29}$$

Thus the solution of system (3.11) has for $h = 3$ the following asymptotic expansions:

$$\begin{aligned} y_1 &\approx \phi(t,C) + \left[\frac{1}{2t} + O\left(t^{2\nu_{12}}\right)\right]\phi^2(t,C) + \left[\frac{1}{12t^2} + O\left(t^{2\nu_{13}}\right)\right]\phi^3(t,C), \\ y_2 &\approx \phi(t,C) + \left[t^{5/2} + O\left(t^{2\nu_{22}}\right)\right]\phi^2(t,C) + \left[\frac{1}{4}t^{3/2} + O\left(t^{2\nu_{23}}\right)\right]\phi^3(t,C). \end{aligned} \tag{3.30}$$

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Research Article

Solvability of Three-Point Boundary Value Problems at Resonance with a p-Laplacian on Finite and Infinite Intervals

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Three-point boundary value problems of second-order differential equation with a p-Laplacian on finite and infinite intervals are investigated in this paper. By using a new continuation theorem, sufficient conditions are given, under the resonance conditions, to guarantee the existence of solutions to such boundary value problems with the nonlinear term involving in the first-order derivative explicitly.

1. Introduction

This paper deals with the three-point boundary value problem of differential equation with a p-Laplacian

$$\begin{gathered}(\Phi_p(x'))' + f(t, x, x') = 0, \quad 0 < t < T, \\ x(0) = x(\eta), \qquad x'(T) = 0,\end{gathered} \tag{1.1}$$

where $\Phi_p(s) = |s|^{p-2}s$, $p > 1$, $\eta \in (0, T)$ is a constant, $T \in (0, +\infty]$, and $x'(T) = \lim_{t\to T^-} x'(t)$.

Boundary value problems (BVPs) with a p-Laplacian have received much attention mainly due to their important applications in the study of non-Newtonian fluid theory, the turbulent flow of a gas in a porous medium, and so on [1–10]. Many works have been done to discuss the existence of solutions, positive solutions subject to Dirichlet, Sturm-Liouville, or nonlinear boundary value conditions.

In recent years, many authors discussed, solvability of boundary value problems at resonance, especially the multipoint case [3, 11–15]. A boundary value problem of differential equation is said to be at resonance if its corresponding homogeneous one has nontrivial solutions. For (1.1), it is easy to see that the following BVP

$$\begin{gathered}(\Phi_p(x'))' = 0, \quad 0 < t < T, \\ x(0) = x(\eta), \qquad x'(T) = 0\end{gathered} \tag{1.2}$$

has solutions $\{x \mid x = a,\ a \in R\}$. When $a \neq 0$, they are nontrivial solutions. So, the problem in this paper is a BVP at resonance. In other words, the operator L defined by $Lx = (\Phi_p(x'))'$ is not invertible, even if the boundary value conditions are added.

For multi-point BVP at resonance without p-Laplacians, there have been many existence results available in the references [3, 11–15]. The methods mainly depend on the coincidence theory, especially Mawhin continuation theorem. At most linearly increasing condition is usually adopted to guarantee the existence of solutions, together with other suitable conditions imposed on the nonlinear term.

On the other hand, for BVP at resonance with a p-Laplacian, very little work has been done. In fact, when $p \neq 2$, $\Phi_p(x)$ is not linear with respect to x, so Mawhin continuation theorem is not valid for some boundary conditions. In 2004, Ge and Ren [3, 4] established a new continuation theorem to deal with the solvability of abstract equation $Mx = Nx$, where M, N are nonlinear maps; this theorem extends Mawhin continuation theorem. As an application, the authors discussed the following three-point BVP at resonance

$$\begin{gathered}(\Phi_p(u'))' + f(t,u) = 0, \quad 0 < t < 1, \\ u(0) = 0 = G(u(\eta), u(1)),\end{gathered} \tag{1.3}$$

where $\eta \in (0,1)$ is a constant and G is a nonlinear operator. Through some special direct-sum-spaces, they proved that (1.3) has at least one solution under the following condition.

There exists a constant $D > 0$ such that $f(t,D) < 0 < f(t,-D)$ for $t \in [0,1]$ and $G(x,D) < 0 < G(x,-D)$ or $G(x,D) > 0 > G(x,-D)$ for $|x| \leqslant D$.

The above result naturally prompts one to ponder if it is possible to establish similar existence results for BVP at resonance with a p-Laplacian under at most linearly increasing condition and other suitable conditions imposed on the nonlinear term.

Motivated by the works mentioned above, we aim to study the existence of solutions for the three-point BVP (1.1). The methods used in this paper depend on the new Ge-Mawhin's continuation theorem [3] and some inequality techniques. To generalize at most linearly increasing condition to BVP at resonance with a p-Laplacian, a small modification is added to the new Ge-Mawhin's continuation theorem. What we obtained in this paper is applicable to BVP of differential equations with nonlinear term involving in the first-order derivative explicitly. Here we note that the techniques used in [3] are not applicable to such case. An existence result is also established for the BVP at resonance on a half-line, which is new for multi-point BVPs on infinite intervals [16, 17].

The paper is organized as follows. In Section 2, we present some preliminaries. In Section 3, we discuss the existence of solutions for BVP (1.1) when T is a real constant, which we call the finite case. In Section 4, we establish an existence result for the bounded solutions to BVP (1.1) when $T = +\infty$, which we call the infinite case. Some explicit examples are also given in the last section to illustrate our main results.

2. Preliminaries

For the convenience of the readers, we provide here some definitions and lemmas which are important in the proof of our main results. Ge-Mawhin's continuation theorem and the modified one are also stated in this section.

Lemma 2.1. *Let $\Phi_p(s) = |s|^{p-2}s$, $p > 1$. Then Φ_p satisfies the properties.*

(1) *Φ_p is continuous, monotonically increasing, and invertible. Moreover $\Phi_p^{-1} = \Phi_q$ with $q > 1$ a real number satisfying $1/p + 1/q = 1$;*

(2) *for any $u, v \geqslant 0$,*

$$\begin{gathered} \Phi_p(u+v) \leqslant \Phi_p(u) + \Phi_p(v), \quad \text{if } p < 2, \\ \Phi_p(u+v) \leqslant 2^{p-2}\left(\Phi_p(u) + \Phi_p(v)\right), \quad \text{if } p \geqslant 2. \end{gathered} \tag{2.1}$$

Definition 2.2. Let R^2 be an 2-dimensional Euclidean space with an appropriate norm $|\cdot|$. A function $f : [0,T] \times R^2 \to R$ is called Φ_q-Carathéodory if and only if

(1) for each $x \in R^2$, $t \mapsto f(t,x)$ is measurable on $[0,T]$;

(2) for a.e. $t \in [0,T]$, $x \mapsto f(t,x)$ is continuous on R^2;

(3) for each $r > 0$, there exists a nonnegative function $\varphi_r \in L^1[0,T]$ with $\varphi_{r,q}(t) := \Phi_q(\int_t^T \varphi_r(\tau)d\tau) \in L^1[0,T]$ such that

$$|x| \leqslant r \text{ implies } \left|f(t,x)\right|\varphi_r(t), \quad \text{a.e. } t \in [0,T]. \tag{2.2}$$

Next we state Ge-Mawhin's continuation theorem [3, 4].

Definition 2.3. Let X, Z be two Banach spaces. A continuous opeartor $M : X \cap \operatorname{dom} M \to Z$ is called quasi-linear if and only if $\operatorname{Im} M$ is a closed subset of Z and $\operatorname{Ker} M$ is linearly homeomorphic to R^n, where n is an integer.

Let X_2 be the complement space of $\operatorname{Ker} M$ in X, that is, $X = \operatorname{Ker} M \oplus X_2$. $\Omega \subset X$ an open and bounded set with the origin $0 \in \Omega$.

Definition 2.4. A continuous operator $N_\lambda : \overline{\Omega} \to Z$, $\lambda \in [0,1]$ is said to be M-compact in $\overline{\Omega}$ if there is a vector subspace $Z_1 \subset Z$ with $\dim Z_1 = \dim \operatorname{Ker} M$ and an operator $R : \overline{\Omega} \times [0,1] \to X_2$ continuous and compact such that for $\lambda \in [0,1]$,

$$(I-Q)N_\lambda\left(\overline{\Omega}\right) \subset \operatorname{Im} M \subset (I-Q)Z, \tag{2.3}$$

$$QN_\lambda x = 0, \quad \lambda \in (0,1) \Longleftrightarrow QNx = 0, \quad \forall x \in \Omega, \tag{2.4}$$

$$R(\cdot,0) \text{ is the zero operator}, \ \ R(\cdot,\lambda)|_{\Sigma_\lambda} = (I-P)|_{\Sigma_\lambda}, \tag{2.5}$$

$$M[P + R(\cdot,\lambda)] = (I-Q)N_\lambda, \tag{2.6}$$

where P, Q are projectors such that $\operatorname{Im} P = \operatorname{Ker} M$ and $\operatorname{Im} Q = Z_1$, $N = N_1$, $\Sigma_\lambda = \{x \in \overline{\Omega}, Mx = N_\lambda x\}$.

Theorem 2.5 (Ge-Mawhin's continuation theorem). *Let* $(X, \|\cdot\|_X)$ *and* $(Z, \|\cdot\|_Z)$ *be two Banach spaces,* $\Omega \subset X$ *an open and bounded set. Suppose* $M : X \cap \operatorname{dom} M \to Z$ *is a quasi-linear operator and* $N_\lambda : \overline{\Omega} \to Z$, $\lambda \in [0,1]$ *is M-compact. In addition, if*

(i) $Mx \neq N_\lambda x$, *for* $x \in \operatorname{dom} M \cap \partial\Omega$, $\lambda \in (0,1)$,

(ii) $QNx \neq 0$, *for* $x \in \operatorname{Ker} M \cap \partial\Omega$,

(iii) $\deg(JQN, \Omega \cap \operatorname{Ker} M, 0) \neq 0$,

where $N = N_1$. *Then the abstract equation* $Mx = Nx$ *has at least one solution in* $\operatorname{dom} M \cap \overline{\Omega}$.

According to the usual direct-sum spaces such as those in [3, 5, 7, 11–13], it is difficult (maybe impossible) to define the projector Q under the at most linearly increasing conditions. We have to weaken the conditions of Ge-Mawhin continuation theorem to resolve such problem.

Definition 2.6. Let Y_1 be finite dimensional subspace of Y. $Q : Y \to Y_1$ is called a semiprojector if and only if Q is semilinear and idempotent, where Q is called semilinear provided $Q(\lambda x) = \lambda Q(x)$ for all $\lambda \in R$ and $x \in Y$.

Remark 2.7. Using similar arguments to those in [3], we can prove that when Q is a semiprojector, Ge-Mawhin's continuation theorem still holds.

3. Existence Results for the Finite Case

Consider the Banach spaces $X = C^1[0,T]$ endowed with the norm $\|x\|_X = \max\{\|x\|_\infty, \|x'\|_\infty\}$, where $\|x\|_\infty = \max_{0 \leqslant t \leqslant T}|x(t)|$ and $Z = L^1[0,T]$ with the usual Lebesgue norm denoted by $\|\cdot\|_z$. Define the operator M by

$$M : \operatorname{dom} M \cap X \longrightarrow Z, \qquad (Mx)(t) = \left(\Phi_p(x'(t))\right)', \quad t \in [0,T], \tag{3.1}$$

where $\operatorname{dom} M = \{x \in C^1[0,T], \Phi_p(x') \in C^1[0,T], x(0) = x(\eta), x'(T) = 0\}$. Then by direct calculations, one has

$$\begin{gathered}\operatorname{Ker} M = \{x \in \operatorname{dom} M \cap X : x(t) = c \in R,\ t \in [0,T]\}, \\ \operatorname{Im} M = \left\{ y \in Z : \int_0^\eta \Phi_q\left(\int_s^T y(\tau)d\tau\right)ds = 0\right\}.\end{gathered} \tag{3.2}$$

Obviously, $\operatorname{Ker} M \simeq R$ and $\operatorname{Im} M$ is close. So the following result holds.

Lemma 3.1. *Let* M *be defined as* (3.1), *then* M *is a quasi-linear operator.*

Set the projector P and semiprojector Q by

$$P : X \longrightarrow X, \qquad (Px)(t) = x(0), \quad t \in [0,T], \tag{3.3}$$

$$Q : Z \longrightarrow Z, \qquad (Qy)(t) = \frac{1}{\rho}\Phi_p\left(\int_0^{\eta}\Phi_q\left(\int_s^T y(\tau)d\tau\right)ds\right), \quad t \in [0,T], \tag{3.4}$$

where $\rho = ((1/q)(T^q - (T-\eta)^q))^{p-1}$. Define the operator $N_\lambda : X \to Z$, $\lambda \in [0,1]$ by

$$(N_\lambda x)(t) = -\lambda f(t, x(t), x\prime(t)), \quad t \in [0,T]. \tag{3.5}$$

Lemma 3.2. *Let $\Omega \subset X$ be an open and bounded set. If f is a Carathéodory function, N_λ is M-compact in $\overline{\Omega}$.*

Proof. Choose $Z_1 = \operatorname{Im} Q$ and define the operator $R : \overline{\Omega} \times [0,1] \to \operatorname{Ker} P$ by

$$R(x,\lambda)(t) = \int_0^t \Phi_q\left(\int_s^T \lambda(f(\tau, x(\tau), x\prime(\tau)) - (Qf)(\tau))d\tau\right)ds, \quad t \in [0,T]. \tag{3.6}$$

Obviously, $\dim Z_1 = \dim \operatorname{Ker} M = 1$. Since f is a Carathéodory function, we can prove that $R(\cdot,\lambda)$ is continuous and compact for any $\lambda \in [0,1]$ by the standard theories.

It is easy to verify that (2.3)–(2.5) in Definition 2.3 hold. Besides, for any $x \in \operatorname{dom} M \cap \overline{\Omega}$,

$$\begin{aligned} M[Px + R(x,\lambda)](t) &= \left(\Phi_p\left[x(0) + \int_0^t \Phi_q\left(\int_s^T \lambda(f(\tau, x(\tau), x'(\tau))d\tau - (Qf)(\tau))d\tau\right)ds\right]'\right)' \\ &= ((I-Q)N_\lambda x)(t), \quad t \in [0,T]. \end{aligned} \tag{3.7}$$

So N_λ is M-compact in $\overline{\Omega}$. □

Theorem 3.3. *Let $f : [0,T] \times R^2 \to R$ be a Carathéodory function. Suppose that*

(H1) *there exist $e(t) \in L^1[0,T]$ and Carathéodory functions g_1, g_2 such that*

$$\begin{gathered} |f(t,u,v)| \leqslant g_1(t,u) + g_2(t,v) + e(t) \quad \textit{for a.e. } t \in [0,T] \textit{ and all } (u,v) \in R^2, \\ \lim_{x\to\infty}\frac{\int_0^T g_i(\tau,x)d\tau}{\Phi_p(|x|)} = r_i \in [0,+\infty), \quad i = 1,2; \end{gathered} \tag{3.8}$$

(H2) *there exists $B_1 > 0$ such that for all $t_\eta \in [0,\eta]$ and $x \in C^1[0,T]$ with $\|x\|_\infty > B_1$,*

$$\int_{t_\eta}^T f(\tau, x(\tau), x'(\tau))\, d\tau \neq 0; \tag{3.9}$$

(H3) *there exists* $B_2 > 0$ *such that for each* $t \in [0,T]$ *and* $u \in R$ *with* $|u| > B_2$ *either* $uf(t,u,0) \leqslant 0$ *or* $uf(t,u,0) \geqslant 0$. *Then BVP* (1.1) *has at least one solution provided*

$$\alpha_1 := 2^{q-2}\left(T^{p-1}r_1 + r_2\right)^{q-1} < 1, \quad \text{if } p < 2,$$
$$\alpha_2 := \left(2^{p-2}T^{p-1}r_1 + r_2\right)^{q-1} < 1, \quad \text{if } p \geqslant 2. \tag{3.10}$$

Proof. Let X, Z, M, N_λ, P, and Q be defined as above. Then the solutions of BVPs (1.1) coincide with those of $Mx = Nx$, where $N = N_1$. So it is enough to prove that $Mx = Nx$ has at least one solution.

Let $\Omega_1 = \{x \in \operatorname{dom} M : Mx = N_\lambda x,\ \lambda \in (0,1)\}$. If $x \in \Omega_1$, then $QN_\lambda x = 0$. Thus,

$$\Phi_p\left(\int_0^\eta \Phi_q\left(\int_s^T f(\tau, x(\tau), x'(\tau))d\tau\right)ds\right) = 0. \tag{3.11}$$

The continuity of Φ_p and Φ_q together with condition (H2) implies that there exists $\xi \in [0,T]$ such that $|x(\xi)| \leqslant B_1$. So

$$|x(t)| \leqslant |x(\xi)| + \int_\xi^t \left|x'(s)\right|ds \leqslant B_1 + T\|x\prime\|_\infty, \quad t \in [0,T]. \tag{3.12}$$

Noting that $Mx = N_\lambda x$, we have

$$x'(t) = \Phi_q\left(\int_t^T \lambda f(\tau, x(\tau), x'(\tau))d\tau\right),$$
$$x(t) = x(0) + \int_0^t \Phi_q\left(\int_s^T \lambda f(\tau, x(\tau), x'(\tau))d\tau\right)ds. \tag{3.13}$$

If $p < 2$, choose $\epsilon > 0$ such that

$$\alpha_{1,\epsilon} := 2^{q-2}\left(T^{p-1}(r_1+\epsilon) + (r_2+\epsilon)\right)^{q-1} < 1. \tag{3.14}$$

For this $\epsilon > 0$, there exists $\delta > 0$ such that

$$\int_0^T g_i(\tau, x)d\tau \leqslant (r_i + \epsilon)\Phi_p(|x|) \quad \forall |x| > \delta,\ i = 1,2. \tag{3.15}$$

Set

$$g_{i,\delta} = \int_0^T \left(\max_{|x|\leqslant\delta} g_i(\tau, x)\right)d\tau, \quad i = 1,2. \tag{3.16}$$

Noting (3.12)-(3.13), we have

$$\begin{aligned}\left|x'(t)\right| &= \left|\Phi_q\left(\int_t^T \lambda f(\tau,x(\tau),x'(\tau))d\tau\right)\right| \leqslant \Phi_q\left(\int_0^T \left|f(\tau,x(\tau),x'(\tau))\right|d\tau\right)\\ &\leqslant \Phi_q\left(\int_0^T (g_1(\tau,x)+g_2(\tau,x')+e(\tau))d\tau\right)\\ &\leqslant \Phi_q\left((r_1+\epsilon)\Phi_p(|x|)+(r_2+\epsilon)\Phi_p\left(\left|x'\right|\right)+g_{1,\delta}+g_{2,\delta}+\|e\|_{L^1}\right)\\ &\leqslant \alpha_{1,\epsilon}\left\|x'\right\|_\infty + B_\delta,\end{aligned} \tag{3.17}$$

where $B_\delta = 2^{q-2}\left((r_1+\epsilon)B_1^{p-1}+g_{1,\delta}+g_{2,\delta}+\|e\|_{L^1}\right)^{q-1}$. So

$$\left\|x'\right\|_\infty \leqslant \frac{B_\delta}{1-\alpha_{1,\epsilon}} := B'. \tag{3.18}$$

And then $\|x\|_X \leqslant \max\{B_1+TB', B'\} := B$.

Similarly, if $p \geqslant 2$, we can obtain $\|x\|_X \leqslant \max\{B_1+T\widetilde{B}', \widetilde{B}'\} := \widetilde{B}$, where

$$\begin{gathered}\widetilde{B}' = \frac{\left(2^{p-2}(r_1+\epsilon)B_1^{p-1}+g_{1,\delta}+g_{2,\delta}+\|e\|_{L^1}\right)^{q-1}}{1-\alpha_{2,\epsilon}},\\ \alpha_{2,\epsilon} = \left(2^{p-2}T^{p-1}(r_1+\epsilon)+(r_2+\epsilon)\right)^{q-1}.\end{gathered} \tag{3.19}$$

Above all, Ω_1 is bounded.

Set $\Omega_{2,i} := \{x \in \operatorname{Ker} M : (-1)^i\mu x + (1-\mu)JQNx = 0, \mu \in [0,1]\}$, $i = 1,2$, where $J : \operatorname{Im} Q \to \operatorname{Ker} M$ is a homeomorphism defined by $Ja = a$ for any $a \in R$. Next we show that $\Omega_{2,1}$ is bounded if the first part of condition (H3) holds. Let $x \in \Omega_{2,1}$, then $x = a$ for some $a \in R$ and

$$\mu a = (1-\mu)\frac{1}{\rho}\Phi_p\left(\int_0^\eta \Phi_q\left(\int_s^T f(\tau,a,0)d\tau\right)ds\right). \tag{3.20}$$

If $\mu = 0$, we can obtain that $|a| \leqslant B_1$. If $\mu \neq 0$, then $|a| \leqslant B_2$. Otherwise,

$$\begin{aligned}\mu a^2 &= a(1-\mu)\frac{1}{\rho}\Phi_p\left(\int_0^\eta \Phi_q\left(\int_s^T f(\tau,a,0)d\tau\right)ds\right)\\ &= (1-\mu)\frac{1}{\rho}\Phi_p\left(\int_0^\eta \Phi_q\left(\int_s^T af(\tau,a,0)d\tau\right)ds\right) \leqslant 0,\end{aligned} \tag{3.21}$$

which is a contraction. So $\|x\|_X = |a| \leqslant \max\{B_1, B_2\}$ and $\Omega_{2,1}$ is bounded. Similarly, we can obtain that $\Omega_{2,2}$ is bounded if the other part of condition (H3) holds.

Let $\Omega = \{x \in X : \|x\|_X < \max\{B(\widetilde{B}), B_1, B_2\} + 1\}$. Then $\Omega_1 \cup \Omega_{2,1}$ $(\cup\, \Omega_{2,2}) \subset \Omega$. It is obvious that $Mx \neq N_\lambda x$ for each $(x, \lambda) \in (\operatorname{dom} M \cap \partial\Omega) \times (0,1)$.

Take the homotopy $H_i : (\operatorname{Ker} M \cap \overline{\Omega}) \times [0,1] \to X$ by

$$H_i(x,\mu) = (-1)^i \mu x + (1-\mu) JQNx, \quad i = 1 \text{ or } 2. \tag{3.22}$$

Then for each $x \in \operatorname{Ker} M \cap \partial\Omega$ and $\mu \in [0,1]$, $H_i(x,\mu) \neq 0$, so by the degree theory

$$\deg = \{JQN, \operatorname{Ker} M \cap \Omega, 0\} = \deg\left\{(-1)^i I, \operatorname{Ker} M \cap \Omega, 0\right\} \neq 0. \tag{3.23}$$

Applying Theorem 2.5 together with Remark 2.7, we obtain that $Mx = Nx$ has a solution in $\operatorname{dom} M \cap \overline{\Omega}$. So (1.1) is solvable. □

Corollary 3.4. *Let $f : [0,T] \times R^2 \to R$ be a Carathéodory function. Suppose that (H2), (H3) in Theorem 3.3 hold. Suppose further that*

(H1′) *there exist nonnegative functions $g_i \in L^1[0,T]$, $i = 0,1,2$ such that*

$$\left|f(t,u,v)\right| \leqslant g_1(t)|u|^{p-1} + g_2(t)|v|^{p-1} + g_0(t) \quad \text{for a.e. } t \in [0,T] \text{ and all } (u,v) \in R^2. \tag{3.24}$$

Then BVP (1.1) *has at least one solution provided*

$$\begin{aligned} &2^{q-2}\left(T^{p-1}\|g_1\|_{L^1} + \|g_2\|_{L^1}\right)^{q-1} < 1, \quad \text{if } p < 2,\\ &\left(2^{p-2}T^{p-1}\|g_1\|_{L^1} + \|g_2\|_{L^1}\right)^{q-1} < 1, \quad \text{if } p \geqslant 2. \end{aligned} \tag{3.25}$$

If f is a continuous function, we can establish the following existence result.

Theorem 3.5. *Let $f : [0,T] \times R^2 \to R$ be a continuous function. Suppose that (H1), (H3) in Theorem 3.3 hold. Suppose further that*

(H2′) *there exist B_3, $a > 0$, $b, c \geqslant 0$ such that for all $u \in R$ with $|u| > B_3$, it holds that*

$$\left|f(t,u,v)\right| \geqslant a|u| - b|v| - c \quad \forall t \in [0,T] \text{ and all } v \in R. \tag{3.26}$$

Then BVP (1.1) *has at least one solution provided*

$$\begin{aligned} &2^{q-2}\left(\left(\frac{b}{a} + T\right)^{p-1} r_1 + r_2\right)^{q-1} < 1, \quad \text{if } p < 2,\\ &\left(2^{p-2}\left(\frac{b}{a} + T\right)^{p-1} r_1 + r_2\right)^{q-1} < 1, \quad \text{if } p \geqslant 2. \end{aligned} \tag{3.27}$$

Proof. If $x \in \operatorname{dom} M$ such that $Mx = N_\lambda x$ for some $\lambda \in (0,1)$, we have $QN_\lambda x = 0$. The continuity of f and Φ_q imply that there exists $\xi \in [0,T]$ such that $f(\xi, x(\xi), x'(\xi)) = 0$. From (H2$'$), it holds

$$|x(\xi)| \leqslant \max\left\{B_3, \frac{b}{a}\|x'\|_\infty + \frac{c}{a}\right\}. \tag{3.28}$$

Therefore,

$$|x(t)| \leqslant |x(\xi)| + \int_\xi^t |x'(s)| ds \leqslant \left(\frac{b}{a} + T\right)\|x'\|_\infty + \frac{c}{a} + B_1, \quad t \in [0,T]. \tag{3.29}$$

With a similar way to those in Theorem 3.3, we can prove that (1.1) has at least one solution. □

Corollary 3.6. *Let $f : [0,T] \times R^2 \to R$ be a continuous function. Suppose that conditions in Corollary 3.4 hold except (H2) changed with* (H2$'$). *Then BVP* (1.1) *is also solvable.*

4. Existence Results for the Infinite Case

In this section, we consider the BVP (1.1) on a half line. Since the half line is noncompact, the discussions are more complicated than those on finite intervals.

Consider the spaces X and Z defined by

$$\begin{aligned} X &= \left\{x \in C^1[0,+\infty), \lim_{t\to+\infty} x(t) \text{ exists}, \lim_{t\to+\infty} x'(t) \text{ exists}\right\}, \\ Z &= \left\{y \in L^1[0,+\infty), \int_0^{+\infty} \Phi_q\left(\int_s^{+\infty} |y(\tau)| d\tau\right) ds < +\infty\right\}, \end{aligned} \tag{4.1}$$

with the norms $\|x\|_X = \max\{\|x\|_\infty, \|x'\|_\infty\}$ and $\|y\|_Z = \|y\|_{L^1}$, respectively, where $\|x\|_\infty = \sup_{0\leqslant t<+\infty}|x(t)|$. By the standard arguments, we can prove that $(X, \|\cdot\|_X)$ and $(Z, \|\cdot\|_Z)$ are both Banach spaces.

Let the operators M, N_λ, and P be defined as (3.1), (3.3), and (3.5), respectively, expect T replaced by $+\infty$. Set $\omega(t) = ((1 - e^{-(q-1)\eta})/(q-1))^{1-p} e^{-t}$, $t \in [0,+\infty)$ and define the semiprojector $Q : Y \to Y$ by

$$(Qy)(t) = w(t)\Phi_p\left(\int_0^\eta \Phi_q\left(\int_s^{+\infty} y(\tau) d\tau\right) ds\right), \quad t \in [0,+\infty). \tag{4.2}$$

Similarly, we can show that M is a quasi-linear operator. In order to prove that N_λ is M-compact in $\overline{\Omega}$, the following criterion is needed.

Theorem 4.1 (see [16]). *Let $M \subset C_\infty = \{x \in C[0,+\infty), \lim_{t\to+\infty} x(t)$ exists$\}$. Then M is relatively compact if the following conditions hold:*

(a) *all functions from M are uniformly bounded;*

(b) *all functions from M are equicontinuous on any compact interval of* $[0,+\infty)$;

(c) *all functions from M are equiconvergent at infinity, that is, for any given $\epsilon > 0$, there exists a $T = T(\epsilon) > 0$ such that $|f(t) - f(+\infty)| < \epsilon$, for all $t > T$, $f \in M$.*

Lemma 4.2. *Let $\Omega \subset X$ an open and bounded set with $0 \in \Omega$. If f is a Φ_q-Carathéodory function, N_λ is M-compact in $\overline{\Omega}$.*

Proof. Let $Z_1 = \operatorname{Im} Q$ and define the operator $R : \overline{\Omega} \times [0,1] \longrightarrow \operatorname{Ker} P$ by

$$R(x,\lambda)(t) = \int_0^t \Phi_q\left(\int_s^{+\infty} \lambda\left(f(\tau, x(\tau), x'(\tau)) - (Qf)(\tau)\right)d\tau\right)ds, \quad t \in [0,+\infty). \tag{4.3}$$

We just prove that $R(\cdot,\lambda) : \overline{\Omega} \times [0,1] \longrightarrow X$ is what we need. The others are similar and are omitted here.

Firstly, we show that R is well defined. Let $x \in \Omega$, $\lambda \in [0,1]$. Because Ω is bounded, there exists $r > 0$ such that for any $x \in \Omega$, $\|x\|_X \leqslant r$. Noting that f is a Φ_q-Carathéodory function, there exists $\varphi_r \in L^1[0,+\infty)$ with $\varphi_{r,q} \in L^1[0,+\infty)$ such that

$$\left|f(t, x(t), x'(t))\right| \leqslant \left|\varphi_r(t), \quad \text{a.e. } t \in [0,+\infty).\right. \tag{4.4}$$

Therefore

$$\begin{aligned} |R(x,\lambda)(t)| &= \left|\int_0^t \Phi_q\left(\int_s^{+\infty} \lambda\left(f(\tau, x(\tau), x'(\tau)) - (Qf)(\tau)\right)d\tau\right)ds\right| \\ &\leqslant \int_0^{+\infty} \Phi_q\left(\int_s^{+\infty} \left(\varphi_r(\tau) + \Upsilon_r \omega(\tau)\right)d\tau\right)ds < +\infty, \quad \forall t \in [0,+\infty), \end{aligned} \tag{4.5}$$

where $\Upsilon_r = \Phi_p(\int_0^\eta \Phi_q(\int_s^{+\infty} \varphi_r(\tau)d\tau)ds)$. Meanwhile, for any $t_1, t_2 \in [0,+\infty)$, we have

$$\begin{aligned} |R(x,\lambda)(t_1) - R(x,\lambda)(t_2)| &\leqslant \int_{t_1}^{t_2} \Phi_q\left(\int_s^{+\infty} \lambda\left|f(\tau, x(\tau), x'(\tau)) - (Qf)(\tau)\right|d\tau\right)ds \\ &\leqslant \int_{t_1}^{t_2} \Phi_q\left(\int_s^{+\infty} \left(\varphi_r(\tau) + \Upsilon_r \omega(\tau)\right)d\tau\right)ds \\ &\longrightarrow 0, \quad \text{as } t_1 \longrightarrow t_2, \end{aligned} \tag{4.6}$$

$$\left|\int_{t_1}^{t_2} \lambda\left(f(\tau, x(\tau), x'(\tau)) - (Qf)(\tau)\right)d\tau\right| \leqslant \int_{t_1}^{t_2} \left(\varphi_r(\tau) + \Upsilon_r \omega(\tau)\right)d\tau \longrightarrow 0, \quad \text{as } t_1 \longrightarrow t_2. \tag{4.7}$$

The continuity of Φ_q concludes that

$$\left|R(x,\lambda)'(t_1) - R(x,\lambda)'(t_2)\right| \longrightarrow 0, \quad \text{as } t_1 \longrightarrow t_2. \tag{4.8}$$

It is easy to see that $\lim_{t\to+\infty} R(x,\lambda)(t)$ exists and $\lim_{t\to+\infty} R(x,\lambda)\prime(t) = 0$. So $R(x,\lambda) \in X$.

Next, we verify that $R(\cdot,\lambda)$ is continuous. Obviously $R(x,\lambda)$ is continuous in λ for any $x\in\Omega$. Let $\lambda\in[0,1]$, $x_n\to x$ in Ω as $n\to+\infty$. In fact,

$$\begin{gathered}\left|\int_0^{+\infty}\left(f(\tau,x_n,x_n')-f(\tau,x,x')\right)d\tau\right|\leqslant 2\|\varphi_r\|_{L^1},\\ \left|\int_0^t\left[\Phi_q\left(\int_s^{+\infty}f(\tau,x_n,x_n')d\tau\right)-\Phi_q\left(\int_s^{+\infty}f(\tau,x,x')d\tau\right)\right]ds\right|\leqslant 2\|\varphi_{r,q}\|_{L^1}.\end{gathered}\tag{4.9}$$

So by Lebesgue Dominated Convergence theorem and the continuity of Φ_q, we can obtain

$$\|R(x_n,\lambda)-R(x,\lambda)\|_X\longrightarrow 0,\quad \text{as } n\longrightarrow+\infty.\tag{4.10}$$

Finally, $R(\cdot,\lambda)$ is compact for any $\lambda\in[0,1]$. Let $U\subset X$ be a bounded set and $\lambda\in[0,1]$, then there exists $r_0>0$ such that $\|x\|_X\leqslant r_0$ for any $x\in U$. Thus we have

$$\begin{aligned}\|R(x,\lambda)\|_X&=\max\left\{\|R(x,\lambda)\|_\infty,\left\|R'(x,\lambda)\right\|_\infty\right\}\\ &\leqslant\max\left\{\int_0^{+\infty}\Phi_q\left(\int_s^{+\infty}\left(\varphi_{r_0}(\tau)+\Upsilon_{r_0}\omega(\tau)\right)d\tau\right)ds,\right.\\ &\qquad\left.\Phi_q\left(\int_0^{+\infty}\left(\varphi_{r_0}(\tau)+\Upsilon\omega(\tau)\right)d\tau\right)\right\},\\ |R(x,\lambda)(t)-R(x,\lambda)(+\infty)|&=\left|\int_t^{+\infty}\Phi_q\left(\int_s^{+\infty}\lambda\left(f(\tau,x(\tau),x'(\tau))-(Qf)(\tau)\right)d\tau\right)ds\right|\\ &\leqslant\int_t^{+\infty}\Phi_q\left(\int_s^{+\infty}\left(\varphi_{r_0}(\tau)+\Upsilon_{r_0}\omega(\tau)\right)d\tau\right)ds\longrightarrow 0,\\ &\qquad\text{uniformly as } t\longrightarrow+\infty,\\ \left|R(x,\lambda)'(t)-R(x,\lambda)'(+\infty)\right|&=\left|\Phi_q\left(\int_t^{+\infty}\lambda\left(f(\tau,x(\tau),x'(\tau))-(Qf)(\tau)\right)d\tau\right)\right|\\ &\leqslant\Phi_q\left(\int_t^{+\infty}\left(\varphi_{r_0}(\tau)+\Upsilon\omega(\tau)\right)d\tau\right)\longrightarrow 0,\\ &\qquad\text{uniformly as } t\to+\infty.\end{aligned}\tag{4.11}$$

Those mean that $R(\cdot,\lambda)$ is uniformly bounded and equiconvergent at infinity. Similarly to the proof of (4.3) and (4.6), we can show that $R(\cdot,\lambda)$ is equicontinuous. Through Lemma 4.2, $R(\cdot,\lambda)U$ is relatively compact. The proof is complete. □

Theorem 4.3. *Let $f:[0,+\infty)\times R^2\to R$ be a continuous and Φ_q-Carathéodory function. Suppose that*

(H4) *there exist functions* $g_0, g_1, g_2 \in L^1[0,+\infty)$ *such that*

$$\begin{gathered}
\left|f(t,u,v)\right| \leqslant g_1(t)|u|^{p-1} + g_2(t)|v|^{p-1} + g_0(t) \quad \text{for a.e. } t \in [0,+\infty) \text{ and all } (u,v) \in R^2, \\
\left\|g_{i,q}\right\|_{L^1} := \int_0^{+\infty} \Phi_q\left(\int_s^{+\infty} \left|g_i(\tau)\right| d\tau\right) ds < +\infty, \quad i = 0,1,2, \\
\left\|g_1\right\|_1 := \int_0^{+\infty} t^{p-1}\left|g_1(\tau)\right| d\tau < +\infty;
\end{gathered} \tag{4.12}$$

(H5) *there exists* $\gamma > 0$ *such that for all* ζ *satisfying*

$$f(\zeta,u,v) = 0, \qquad f(t,u,v) \neq 0, \quad t \in [0,\zeta), \ (u,v) \in R^2, \tag{4.13}$$

it holds $\zeta \leqslant \gamma$;

(H6) *there exist* B_4, $a > 0$, $b, c \geqslant 0$ *such that for all* $u \in R$ *with* $|u| > B_4$, *it holds*

$$\left|f(t,u,v)\right| \geqslant a|u| - b|v| - c \quad \forall t \in [0,\gamma], \ v \in R; \tag{4.14}$$

(H7) *there exists* $B_5 > 0$ *such that for all* $t \in [0,+\infty)$ *and* $u \in R$ *with* $|u| > B_5$ *either* $uf(t,u,0) \leqslant 0$ *or* $uf(t,u,0) \geqslant 0$. *Then BVP* (1.1) *has at least one solution provided*

$$\begin{gathered}
\max\left\{2^{q-2}\left\|g_{1,q}\right\|_{L^1}, \beta_1\right\} < 1, \quad \text{if } p < 2, \\
\max\left\{\left\|g_{1,q}\right\|_{L^1}, \beta_2\right\} < 1, \quad \text{if } p \geqslant 2,
\end{gathered} \tag{4.15}$$

where

$$\begin{gathered}
\beta_1 := 2^{q-2}\left(\left(\frac{b}{a} + \gamma\right)^{p-1} \left\|g_1\right\|_{L^1} + \left\|g_1\right\|_1 + \left\|g_2\right\|_{L^1}\right)^{q-1}, \\
\beta_2 := \left(2^{2(p-2)}\left(\frac{b}{a} + \gamma\right)^{p-1} \left\|g_1\right\|_{L^1} + 2^{2(q-2)}\left\|g_1\right\|_1 + \left\|g_2\right\|_{L^1}\right)^{q-1}.
\end{gathered} \tag{4.16}$$

Proof. Let X, Z, M, N_λ, P, and Q be defined as above. Let $\Omega_1 = \{x \in \operatorname{dom} M : Mx = N_\lambda x, \lambda \in (0,1)\}$. We will prove that Ω_1 is bounded. In fact, for any $x \in \Omega_1$, $QN_\lambda x = 0$, that is,

$$\omega(t)\ \Phi_p\left(\int_0^\eta \Phi_q\left(\int_s^{+\infty} \lambda f(\tau, x(\tau), x'(\tau))d\tau\right)ds\right) = 0. \tag{4.17}$$

The continuity of Φ_p and Φ_q together with conditions (H5) and (H6) implies that there exists $\xi \leqslant \gamma$ such that

$$|x(\xi)| \leqslant \max\left\{B_4, \frac{b}{a}\|x\prime\|_\infty + \frac{c}{a}\right\}. \tag{4.18}$$

So, we have

$$|x(t)| \leqslant |x(\xi)| + \left|\int_\xi^t x'(s)ds\right| \leqslant \max\left\{B_4, \frac{b}{a}\|x\prime\|_\infty + \frac{c}{a}\right\} + (t+\gamma)\|x'\|_\infty, \quad t \in [0, +\infty). \tag{4.19}$$

If $p < 2$, it holds

$$|x(t)|^{p-1} \leqslant \left(\left(\frac{b}{a} + \gamma\right)^{p-1} + t^{p-1}\right)\|x'\|_\infty^{p-1} + \left(\frac{c}{a} + B_4\right)^{p-1}, \quad t \in [0, +\infty). \tag{4.20}$$

Therefore

$$\begin{aligned} |x'(t)| &= \left|\Phi_q\left(\int_t^{+\infty} \lambda f(\tau, x(\tau), x'(\tau))d\tau\right)\right| \\ &\leqslant \Phi_q\left(\int_0^{+\infty} \left(g_1(\tau)|x(\tau)|^{p-1} + g_2(\tau)|x'(\tau)|^{p-1} + g_0(\tau)\right)d\tau\right) \\ &\leqslant \beta_1\|x'\|_\infty + 2^{q-2}\left((c/a + B_4)^{p-1}\|g_1\|_{L^1} + \|g_0\|_{L^1}\right)^{q-1}, \quad t \in [0, +\infty) \end{aligned} \tag{4.21}$$

concludes that

$$\|x'\|_\infty \leqslant \frac{2^{q-2}\left((c/a + B_4)^{p-1}\|g_1\|_{L^1} + \|g_0\|_{L^1}\right)^{q-1}}{1 - \beta_1} := C. \tag{4.22}$$

Meanwhile

$$\begin{aligned} |x(t)| &= \left|x(0) + \int_0^t \Phi_q\left(\int_s^{+\infty} \lambda f(\tau, x(\tau), x'(\tau))d\tau\right)ds\right| \\ &\leqslant |x(0)| + \int_0^{+\infty} \Phi_q\left(\int_s^{+\infty} \left(g_1|x|^{p-1} + g_2|x'|^{p-1} + g_0\right)d\tau\right)ds \\ &\leqslant 2^{q-2}\|g_{1,q}\|_{L^1}\|x\|_\infty + C_0 \end{aligned} \tag{4.23}$$

implies that

$$\|x\|_\infty \leqslant \frac{C_0}{1 - 2^{q-2}\|g_{1,q}\|_{L^1}}, \tag{4.24}$$

where $C_0 = (b/a + \gamma + 2^{2(q-2)}\|g_{2,q}\|_{L^1})C + B_4 + c/a + 2^{2(q-2)}\|g_{0,q}\|_{L^1}$.

If $p \geqslant 2$, we can prove that

$$\|x'\|_\infty \leqslant \frac{\left(2^{p-2}(B_4 + c/a)^{p-1}\|g_1\|_{L^1} + \|g_0\|_{L^1}\right)^{q-1}}{1-\beta_2} := \widetilde{C},$$
$$\|x\|_\infty \leqslant \frac{\left(b/a + \gamma + \|g_{2,q}\|_{L^1}\right)\widetilde{C} + B_4 + c/a + \|g_{0,q}\|_{L^1}}{1 - \|g_{1,q}\|_{L^1}}. \tag{4.25}$$

So Ω_1 is bounded. With the similar arguments to those in Theorem 3.3, we can complete the proof. □

5. Examples

Example 5.1. Consider the three-point BVPs for second-order differential equations

$$\left(x'(t)\left|x'(t)\right|\right)' = a_2(t)x'(t) + a_1(t)x^2(t)\operatorname{sgn} x(t) + a_0(t), \quad 0 < t < 1,$$
$$x(0) = x(\eta), \qquad x'(1) = 0, \tag{5.1}$$

where $a_i(t) \in C^1[0,1]$, $i = 0,1,2$ with $a_1 = \min|a_1(t)| > 0$.

Take

$$f(t,u,v) = a_1(t)u^2 \operatorname{sgn} u + a_2(t)v + a_0(t),$$
$$g_1(t,u) = |a_1(t)|u^2,$$
$$g_2(t,v) = |a_2(t)||v|, \tag{5.2}$$

and $e(t) = |a_0(t)|$. Then, we have

$$\left|f(t,u,v)\right| \leqslant g_1(t,u) + g_2(t,v) + e(t), \quad \text{for } (t,u,v) \in [0,1] \times R^2$$
$$\max_{0\leqslant t\leqslant 1} \frac{g_1(t,x)}{|x|} = \|a_1\|_{L^1} \in [0,+\infty),$$
$$\max_{0\leqslant t\leqslant 1} \frac{g_1(t,x)}{|x|} = 0, \tag{5.3}$$
$$\left|f(t,u,v)\right| \geqslant a_1|u| - \|a_2\|_\infty|v| - \|a_0\|_\infty, \quad \text{for } (t,|u|,v) \in [0,T] \times [1,+\infty) \times R,$$
$$uf(t,u,0) = a_1(t)|u|^3 + a_0(t)u \geqslant 0, \quad \text{for } (t,|u|) \in [0,1] \times \left[\sqrt{\frac{\|a_0\|_\infty}{a_1}}, +\infty\right).$$

By using Theorem 3.5, we can concluded that BVP (5.1) has at least one solution if

$$\left(\frac{\|a_2\|_\infty}{a_1} + 1\right)^2 \|a_1\|_\infty < \frac{1}{2}. \tag{5.4}$$

Example 5.2. Consider the three-point BVPs for second-order differential equations on a half line

$$x''(t) + e^{-\alpha t} p(t) x(t) + q(t) = 0, \quad 0 < t < +\infty,$$
$$x(0) = x(\eta), \qquad \lim_{t \to +\infty} x'(t) = 0, \tag{5.5}$$

where $\alpha > (1+\sqrt{5})/2$, $p(t) = \max\{\sin \beta t, 1/2\}$ and $q(t)$ continuous on $[0, +\infty)$ with $q(t) > 0$ (or $q(t) < 0$) on $[0, 1)$ and $q \equiv 0$ on $[1, +\infty)$.

Denote $f(t, u) = e^{-\alpha t} p(t) u + q(t)$. Set $g_1(t) = e^{-\alpha t}$, $g_0(t) = q(t)$. By direct calculations, we obtain that $\|g_1\|_{L^1} = 1/\alpha$, $\|g_{1,q}\|_{L^1} = \|g_1\|_1 = 1/\alpha^2$ and $\|g_{0,q}\|_{L^1} \leqslant \|g_0\|_{L^1} \leqslant \|q\|_\infty$. Furthermore,

$$\left|f(t, u)\right| \leqslant \left|g_1(t)\right| |u| + \left|g_0(t)\right|,$$
$$\left|f(t, u)\right| \geqslant \frac{1}{2} e^{-\alpha} |u| - \|q\|_\infty. \tag{5.6}$$

If there exists $\xi \in [0, +\infty)$ such that $f(\xi, u) = 0$, then $\xi \leqslant 1$. Otherwise

$$u f(\xi, u) = e^{-\alpha \xi} p(\xi) u^2 \geqslant \frac{1}{2} e^{-\alpha \xi} u^2 > 0, \quad \forall u \in R \setminus \{0\} \tag{5.7}$$

which is a contraction.

Obviously $\max\{1/\alpha, 1/\alpha + 1/\alpha^2\} < 1$. Meanwhile, it is easy to verify that condition (H7) holds. So Theorem 4.3 guarantees that (5.5) has at least one solution.

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Research Article

Existence of Multiple Solutions for a Singular Elliptic Problem with Critical Sobolev Exponent

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We consider the existence of multiple solutions of the singular elliptic problem $-\mathrm{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u)+|u|^{p-2}u/|x|^{(a+1)p} = f|u|^{r-2}u+h|u|^{s-2}u+|x|^{-bp^*}|u|^{p^*-2}u$, $u(x) \to 0$ as $|x| \to +\infty$, where $x \in \mathbb{R}^N$, $1<p<N$, $a<(N-p)/p$, $a \le b \le a+1$, $r, s > 1$, $p^* = Np/(N-pd)$, $d = a+1-b$. By the variational method and the theory of genus, we prove that the above-mentioned problem has infinitely many solutions when some conditions are satisfied.

1. Introduction and Main Results

In this paper, we consider the existence of multiple solutions for the singular elliptic problem

$$-\mathrm{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right)+\frac{|u|^{p-2}u}{|x|^{(a+1)p}} = f|u|^{r-2}u+h|u|^{s-2}u+|x|^{-bp^*}|u|^{p^*-2}u, \quad x \in \mathbb{R}^N,$$
$$u(x) \longrightarrow 0 \quad \text{as } |x| \longrightarrow +\infty, \tag{1.1}$$

where $1<p<N$, $a<(N-p)/p$, $a \le b \le a+1$, $r>1$, $p^* = Np/(N-pd)$, $d = a+1-b$. $f(x)$ and $h(x)$ are nonnegative functions in $\mathbb{R}^N$.

In recent years, the existence of multiple solutions on elliptic equations has been considered by many authors. In [1], Assunção et al. considered the following quasilinear degenerate elliptic equation:

$$-\mathrm{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right)+\lambda|x|^{-(a+1)p}|u|^{p-2}u = |x|^{-bq}|u|^{q-2}u+f, \tag{1.2}$$

where $x \in \mathbb{R}^N$, $1 < p < N$, $q = Np/[N - p(a+1-b)]$. When $\lambda = 0$, $f = \varepsilon g$, where $0 < \varepsilon \leq \varepsilon_0$ and $0 \leq g \in (L_b^q(\mathbb{R}^N))^*$; the authors proved that problem (1.2) has at least two positive solutions. Rodrigues in [2] studied the following critical problem on bounded domain $\Omega \in \mathbb{R}^N$:

$$- \operatorname{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) = |x|^{-bp^*}|u|^{p^*-2}u + |x|^{-\beta} f|u|^{r-2}u, \quad x \in \Omega,$$
$$u(x) = 0, \quad \text{on } \partial\Omega. \tag{1.3}$$

By the variational method on Nehari manifolds [3, 4], the author proved the existence of at least two positive solutions and the nonexistence of solutions when some certain conditions are satisfied. When $p = 2$ and $a = -1$, Miotto and Miyagaki in [5] considered the semilinear Dirichlet problem in infinite strip domains

$$-\Delta u + u = \lambda f(x)|u|^{q-1} + h(x)|u|^{p-1}, \quad x \in \Omega,$$
$$u(x) = 0, \quad \text{on } \partial\Omega. \tag{1.4}$$

The authors also proved that problem (1.4) has at least two positive solutions by the methods of Nehari manifold. For other references, we refer to [6–11] and the reference therein. In fact, motivated by [1, 2, 5], we consider the problem (1.1). Since our problem is singular and is studied in the whole space $\mathbb{R}^N$, the loss of compactness of the Sobolev embedding renders a variational technique that is more delicate. By the variational method and the theory of genus, we prove that problem (1.1) has infinitely many solutions when some suitable conditions are satisfied.

In order to state our result, we introduce some weighted Sobolev spaces. For $r, s \geq 1$ and $g = g(x) > 0$ in R^N, we define the spaces $L^r(\mathbb{R}^N, g)$ and $L^s(\mathbb{R}^N, g)$ as being the set of Lebesgue measurable functions $u : \mathbb{R}^N \to \mathbb{R}^1$, which satisfy

$$\|u\|_{r,g} = \|u\|_{L^r(\mathbb{R}^N,g)} = \left(\int_{\mathbb{R}^N} g(x)|u|^r dx\right)^{1/r} < \infty,$$
$$\|u\|_{s,g} = \|u\|_{L^s(\mathbb{R}^N,g)} = \left(\int_{\mathbb{R}^N} g(x)|u|^s dx\right)^{1/s} < \infty. \tag{1.5}$$

Particularly, when $g(x) \equiv 1$, we have

$$\|u\|_r = \|u\|_{L^r(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |u|^r dx\right)^{1/r} < \infty. \tag{1.6}$$

We denote the completion of $C_0^\infty(\mathbb{R}^N)$ by $X = W_a^{1,p}(\mathbb{R}^N)$ with the norm of

$$\|u\|_X = \left(\int_{\mathbb{R}^N} |x|^{-ap}|u|^{\mathrm{p}} dx\right)^{1/p}, \tag{1.7}$$

where $1 < p < N$ and $a < (N-p)/p$. It is easy to find that X is a reflexive and separable Banach space with the norm $\|u\|_X$.

The following Hardy-Sobolev inequality is due to Caffarelli et al. [12], which is called Caffarelli-Kohn-Nirenberg inequality. There exist constants $S_1,\ S_2>0$ such that

$$\left(\int_{\mathbb{R}^N}|x|^{-bp^*}|u|^{p^*}dx\right)^{p/p^*}\le S_1\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u|^p dx,\quad \forall u\in C_0^\infty\left(\mathbb{R}^N\right),\tag{1.8}$$

$$\int_{\mathbb{R}^N}|x|^{-(a+1)p}|u|^p dx\le S_2\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u|^p dx,\quad \forall u\in C_0^\infty\left(\mathbb{R}^N\right),\tag{1.9}$$

where $p^*=Np/(N-pd)$ is called the Sobolev critical exponent.

In the present paper, we make the following assumptions:

(A_1) $f(x)\in L^{\sigma_1}(\mathbb{R}^N,g_1)\bigcap L^\infty_{\text{loc}}(\mathbb{R}^N\setminus\{0\})$ for $1<r<p$, where $g_1=|x|^{(a+1)r\sigma_1}$, $\sigma_1=p/(p-r)$;

(A_2) $f(x)\in L^{\sigma_2}(\mathbb{R}^N,g_2)\bigcap L^\infty_{\text{loc}}(\mathbb{R}^N\setminus\{0\})$ for $p<r<p^*$, where $g_2=|x|^{br\sigma_2}$, $\sigma_2=p^*/(p^*-r)$.

(A_3) $h(x)\in L^{\mu}(\mathbb{R}^N,g_3)\bigcap L^\infty_{\text{loc}}(\mathbb{R}^N\setminus\{0\})$ for $p<s<p^*$, where $g_3=|x|^{\mu bp^*}$, $\mu=p^*/(p^*-s)$.

Then, we give some basic definitions.

Definition 1.1. $u\in X$ is said to be a weak solution of (1.1) if for any $\varphi\in C_0^\infty(\mathbb{R}^N)$ there holds

$$\begin{aligned}\int_{\mathbb{R}^N}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi+\frac{|u|^{p-2}u\varphi}{|x|^{(a+1)p}}\right)dx&=\int_{\mathbb{R}^N}f|u|^{r-2}u\varphi\,dx+\int_{\mathbb{R}^N}h|u|^{s-2}u\varphi\,dx\\&\quad+\int_{\mathbb{R}^N}|x|^{-bp^*}|u|^{p^*-2}u\varphi\,dx.\end{aligned}\tag{1.10}$$

Let $I(u):X\to\mathbb{R}^1$ be the energy functional corresponding to problem (1.1), which is defined as

$$I(u)=\frac{1}{p}\int_{\mathbb{R}^N}\left(|x|^{-ap}|\nabla u|^p+\frac{|u|^p}{|x|^{(a+1)p}}\right)dx-\frac{1}{r}\int_{\mathbb{R}^N}f|u|^r dx-\frac{1}{s}\int_{\mathbb{R}^N}h|u|^s dx-\frac{1}{p^*}\int_{\mathbb{R}^N}\frac{|u|^{p^*}}{|x|^{bp^*}}dx,\tag{1.11}$$

for all $u\in X$. Then the functional $I\in C^1(X,R^1)$ and for all $\varphi\in X$, there holds

$$\begin{aligned}\langle I'(u),\varphi\rangle&=\int_{\mathbb{R}^N}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\nabla\varphi+\frac{|u|^{p-2}u\varphi}{|x|^{(a+1)p}}\right)dx-\int_{\mathbb{R}^N}f(x)|u|^{r-2}u\varphi dx\\&\quad-\int_{\mathbb{R}^N}h(x)|u|^{s-2}u\varphi\,dx-\int_{\mathbb{R}^N}|x|^{-bp^*}|u|^{p^*-2}u\varphi\,dx.\end{aligned}\tag{1.12}$$

It is well known that the weak solutions of problem (1.1) are the critical points of the functional $I(u)$, see [13]. Thus, to prove the existence of weak solutions of (1.1), it is sufficient to show that $I(u)$ admits a sequence of critical points in X.

Our main result in this paper is the following.

Theorem 1.2. *Let $1 < p < N$, $a < (N-p)/p$, $a \leq b \leq a+1$, $r > 1$, $p^* = Np/(N-pd)$, $d = a+1-b$, $\max\{r,p\} < s < p^*$. Assume (A_1)–(A_3) are fulfilled. Then problem* (1.1) *has infinitely many solutions in X.*

2. Preliminary Results

Our proof is based on variational method. One important aspect of applying this method is to show that the functional $I(u)$ satisfies $(PS)_c$ condition which is introduced in the following definition.

Definition 2.1. Let $c \in R^1$ and X be a Banach space. The functional $I(u) \in C^1(X,R)$ satisfies the $(PS)_c$ condition if for any $\{u_n\} \subset X$ such that

$$I(u_n) \longrightarrow c, \quad I'(u_n) \longrightarrow 0 \quad \text{in } X^* \text{ as } n \longrightarrow \infty \tag{2.1}$$

contains a convergent subsequence in X.

The following embedding theorem is an extension of the classical Rellich-Kondrachov compactness theorem, see [14].

Lemma 2.2. *Suppose $\Omega \subset \mathbb{R}^N$ is an open bounded domain with C^1 boundary and $0 \in \Omega$. $N \geq 3$, $a < (N-p)/p$. Then the embedding $W_0^{1,p}(\Omega, |x|^{-ap}) \hookrightarrow L^r(\Omega, |x|^{-\alpha})$ is continuous if $1 \leq r \leq Np/(N-p)$ and $0 \leq \alpha \leq (1+a)r + N(1-r/p)$, and is compact if $1 \leq r < Np/(N-p)$ and $0 \leq \alpha < (1+a)r + N(1-r/p)$.*

Now we prove an embedding theorem, which is important in our paper.

Lemma 2.3. *Assume (A_1)-(A_2) and $1 < r < p^*$. Then the embedding $X \hookrightarrow L^r(\mathbb{R}^N, f)$ is compact.*

Proof. We split our proof into two cases.

(i) Consider $1 < r < p$.

By the Hölder inequality and (1.9) we have that

$$\begin{aligned}
\|u\|^r_{L^r(\mathbb{R}^N,f)} &= \int_{\mathbb{R}^N} f(x)|u|^r dx \leq \left(\int_{\mathbb{R}^N} |u|^p |x|^{-(a+1)p} dx\right)^{r/p} \left(\int_{\mathbb{R}^N} f^{\sigma_1} |x|^{(a+1)r\sigma_1} dx\right)^{1/\sigma_1} \\
&= \left(\int_{\mathbb{R}^N} |u|^p |x|^{-(a+1)p} dx\right)^{r/p} \|f\|_{L^{\sigma_1}(\mathbb{R}^N,g_1)} \\
&\leq S_2^{r/p} \|u\|^r_X \|f\|_{L^{\sigma_1}(\mathbb{R}^N,g_1)},
\end{aligned} \tag{2.2}$$

where $g_1 = |x|^{(a+1)r\sigma_1}$, $\sigma_1 = p/(p-r)$. Then the embedding is continuous. Next, we will prove that the embedding is compact.

Let B_R be a ball center at origin with the radius $R > 0$. For the convenience, we denote $L^r(\mathbb{R}^N, f)$ by Z, that is, $Z = L^r(\mathbb{R}^N, f)$. Assume $\{u_n\}$ is a bounded sequence in X. Then $\{u_n\}$ is bounded in $X(B_R)$. We choose $\alpha = 0$ in Lemma 2.2, then there exist $u \in Z(B_R)$ and a subsequence, still denoted by $\{u_n\}$, such that $\|u_n - u\|_{L^r(B_R)} \to 0$ as $n \to \infty$. We want to prove that

$$\lim_{R\to\infty} \sup_{u\in X\setminus\{0\}} \frac{\|u\|_{Z(B_R^c)}}{\|u\|_X} = 0, \tag{2.3}$$

where $B_R^c = \mathbb{R}^N \setminus B_R$. In fact, we obtain from (2.2) that

$$\|u\|_{Z(B_R^c)}^r \le S_2^{r/p} \|u\|_X^r \|f\|_{L^{\sigma_1}(B_R^c, g_1)}. \tag{2.4}$$

The fact $f \in L^{\sigma_1}(\mathbb{R}^N, g_1)$ shows that

$$\lim_{R\to\infty} \int_{B_R^c} f^{\sigma_1} g_1 dx = 0. \tag{2.5}$$

Then (2.4) and (2.5) imply that

$$\frac{\|u\|_{Z(B_R^c)}}{\|u\|_X} \le S_2^{1/p} \|f\|_{L^{\sigma_1}(B_R^c, g_1)}^{1/r}, \tag{2.6}$$

which gives (2.3).

In the following, we will prove that $u_n \to u$ strongly in $Z(R^N)$.

Since X is a reflexive Banach space and $\{u_n\}$ is bounded in X. Then we may assume, up to a subsequence, that

$$u_n \rightharpoonup u \quad \text{in X}. \tag{2.7}$$

In view of (2.3), we get that for any $\varepsilon > 0$ there exists $R_\varepsilon > 0$ large enough such that

$$\|u_n\|_{Z(B_{R_\varepsilon}^c)} \le \varepsilon \|u_n\|_X \quad (n = 1, 2, \ldots). \tag{2.8}$$

On the other hand, due to the compact embedding $X(B_{R_\varepsilon}) \hookrightarrow Z(R_\varepsilon)$ in Lemma 2.2, we have that

$$\lim_{n\to\infty} \|u_n - u\|_{Z(B_{R_\varepsilon})} = 0. \tag{2.9}$$

Therefore, there is $N_0 > 0$ such that

$$\|u_n - u\|_{Z(B_{R_\varepsilon})} < \varepsilon, \tag{2.10}$$

for $n > N_0$. Thus, the inequalities (2.8) and (2.10) show that

$$\begin{aligned} \|u_n - u\|_Z &\le \|u_n - u\|_{Z(B_{R_\varepsilon})} + \|u_n - u\|_{Z(B^c_{R_\varepsilon})} \\ &\le \|u_n - u\|_{Z(B_{R_\varepsilon})} + \|u_n\|_{Z(B^c_{R_\varepsilon})} + \|u\|_{Z(B^c_{R_\varepsilon})} \\ &\le (1 + \|u_n\|_X + \|u\|_X)\varepsilon. \end{aligned} \tag{2.11}$$

This shows that $\{u_n\}$ is convergent in $Z = L^r(\mathbb{R}^N, f)$.

(ii) Consider $p \le r < p^*$.

It follows from (1.8) and the Hölder inequality that

$$\begin{aligned} \|u\|^r_{L^r(\mathbb{R}^N,f)} = \int_{\mathbb{R}^N} f(x)|u|^r dx &\le \left(\int_{\mathbb{R}^N} |x|^{-bp^*}|u|^{p^*} dx\right)^{r/p^*} \left(\int_{\mathbb{R}^N} f^{\sigma_2}|x|^{br\sigma_2} dx\right)^{1/\sigma_2} \\ &\le S_1^{r/p}\left(\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p dx\right)^{r/p} \left(\int_{\mathbb{R}^N} f^{\sigma_2}|x|^{br\sigma_2} dx\right)^{1/\sigma_2} \\ &\le S_1^{r/p}\|u\|^r_X \|f\|_{L^{\sigma_2}(\mathbb{R}^N, g_2)}, \end{aligned} \tag{2.12}$$

where $g_2 = |x|^{br\sigma_2}$, $\sigma_2 = p^*/(p^* - r)$. Thus, the fact of $f \in L^{\sigma_2}(\mathbb{R}^N, g_2)$ and (2.12) imply that the embedding is continuous. Similar to the proof of (i) we can also prove that the embedding $X \hookrightarrow L^r(\mathbb{R}^N, f)$ is compact for $p \le r < p^*$. □

Similarly, we have the following result of compact embedding.

Lemma 2.4. *Assume $1 < p < s < p^*$ and (A_3), then the embedding $X \hookrightarrow L^s(\mathbb{R}^N, h)$ is compact.*

The following concentration compactness principle is a weighted version of the Concentration Compactness Principle II due to Lions [15–18], see also [19, 20].

Lemma 2.5. *Let $1 < p < N$, $-\infty < a < (N-p)/p$, $a \le b \le a+1$, $p^* = Np/(N-pd)$, $d = a+1-b$. Suppose that $\{u_n\} \subset W^{1,p}_a(\mathbb{R}^N)$ is a sequence such that*

$$\begin{gathered} u_n \rightharpoonup u \quad in\ W^{1,p}_a\left(\mathbb{R}^N\right), \\ |x|^{-ap}|\nabla u_n|^p \rightharpoonup \mu \quad in\ \mathcal{M}\left(\mathbb{R}^N\right), \\ |x|^{-bp^*}|u_n|^{p^*} \rightharpoonup \eta \quad in\ \mathcal{M}\left(\mathbb{R}^N\right), \\ u_n \longrightarrow u \quad a.e.\ on\ \mathbb{R}^N, \end{gathered} \tag{2.13}$$

where μ, η are measures supported on Ω and $\mathcal{M}(\mathbb{R}^N)$ is the space of bounded measures in $\mathbb{R}^N$. Then there are the following results.

(1) *There exists some at most countable set* J*, a family* $\{x_j \in \Omega \mid j \in J\}$ *of distinct points in* $\mathbb{R}^N$*, and a family* $\{\eta_j \mid j \in J\}$ *of positive numbers such that*

$$\eta = |x|^{-bp^*}|u|^{p^*} + \sum_{j\in J}\eta_j\delta_{x_j}, \tag{2.14}$$

where δ_{x_j} *is the Dirac measure at* x_j*.*

(2) *The following equality holds*

$$\mu \geq |x|^{-ap}|\nabla u|^p + \sum_{j\in J}\mu_j\delta_{x_j}, \tag{2.15}$$

for some family $\{\mu_j > 0 \mid j \in J\}$ *satisfying*

$$S_1(\eta_j)^{p/p^*} \leq \mu_j \quad \forall j \in J, \qquad \sum_{j\in J}(\eta_j)^{p/p^*} \leq \infty. \tag{2.16}$$

(3) *There hold*

$$\begin{aligned} \lim_{n\to+\infty}\sup\int_\Omega |x|^{-ap}|\nabla u_n|^p dx &= \int_\Omega d\mu + \mu_\infty, \\ \lim_{n\to+\infty}\sup\int_\Omega |x|^{-bp^*}|\nabla u_n|^{p^*} dx &= \int_\Omega d\eta + \eta_\infty, \end{aligned} \tag{2.17}$$

where

$$\begin{aligned} \mu_\infty &:= \lim_{R\to\infty}\lim_{n\to+\infty}\sup\int_{\Omega\bigcap B_R^c} |x|^{-ap}|\nabla u_n|^p dx, \\ \eta_\infty &:= \lim_{R\to\infty}\lim_{n\to+\infty}\sup\int_{\Omega\bigcap B_R^c} |x|^{-bp^*}|\nabla u_n|^{p^*} dx. \end{aligned} \tag{2.18}$$

Lemma 2.6. *Let* $1 < p < r < s < p^*$*. Then* $I(u)$ *satisfies the* $(PS)_c$ *condition with* $c \leq (1/r - 1/p^*)S_1^{p^*/(p^*-p)}$*, where* S_1 *is as in* (1.8).

Proof. We will split the proof into three steps.
Step 1. $\{u_n\}$ is bounded in X.

Let $\{u_n\}$ be a $(PS)_c$ sequence of $I(u)$ in X, that is,

$$I(u_n) \longrightarrow c, \quad I'(u_n) \longrightarrow 0 \quad \text{in } X^* \text{ as } n \longrightarrow \infty. \tag{2.19}$$

Then, we have

$$
\begin{aligned}
1+c+\|u_n\|_X &\geq I(u_n)-\frac{1}{r}\langle I'(u_n),u_n\rangle \\
&=\left(\frac{1}{p}-\frac{1}{r}\right)\|u_n\|_X^p+\left(\frac{1}{r}-\frac{1}{s}\right)\|u_n\|_{L^s(\mathbb{R}^N,h)}^s+\left(\frac{1}{r}-\frac{1}{p^*}\right)\|u_n\|_{L^{p^*}(\mathbb{R}^N)}^{p^*} \\
&\geq\left(\frac{1}{p}-\frac{1}{r}\right)\|u_n\|_X^p.
\end{aligned}
\tag{2.20}
$$

Since $p>1$, (2.20) shows that $\{u_n\}$ is bounded in X.
Step 2. There exists $\{u_n\}$ in X such that $u_n \to u$ in $L^{p^*}(\mathbb{R}^N)$.

The inequality (1.8) shows that $\{u_n\}$ is bounded in $L^{p^*}(\mathbb{R}^N,|x|^{-bp^*})$. Then the above argument and the compactness embedding in Lemma 2.2 mean that the following convergence hold:

$$
\begin{gathered}
u_n \rightharpoonup u \quad \text{in } W_0^{1,p}\left(\mathbb{R}^N\right), \\
u_n \rightharpoonup u \quad \text{in } L^{p^*}\left(\mathbb{R}^N,|x|^{-bp^*}\right), \\
u_n \longrightarrow u \quad \text{a.e. in } \mathbb{R}^N.
\end{gathered}
\tag{2.21}
$$

It follows from Lemma 2.5 that there exist nonnegative measures μ and η such that

$$
|x|^{-bp^*}|u_n|^{p^*} \rightharpoonup \eta=|x|^{-bp^*}|u|^{p^*}+\sum_{j\in J}\eta_j\delta_{x_j}, \tag{2.22}
$$

$$
|x|^{-ap}|\nabla u_n|^p \geq |x|^{-ap}|\nabla u|^p+\sum_{j\in J}\mu_j\delta_{x_j}. \tag{2.23}
$$

Thus, in order to prove $u_n \to u$ in $L^{p^*}(\mathbb{R}^N)$ it is sufficient to prove that $\eta_j=\eta_\infty=0$.

For the proof of $\eta_j=0$, we define the functional $\psi\in C_0^\infty(R^{\mathbb{N}})$ such that

$$
\psi\equiv 1,\quad \text{in } B\left(x_j,\varepsilon\right),\quad \psi\equiv 0,\quad \text{in } B\left(x_j,2\varepsilon\right)^c,\quad \left|\nabla\psi\right|\leq\frac{2}{\varepsilon}, \tag{2.24}
$$

where x_j belongs to the support of $d\eta$. It follows from (2.1) that

$$
\lim_{n\to\infty}\langle I'(u_n),u_n\psi\rangle=0. \tag{2.25}
$$

Since $\|u_n\|_X$ is bounded, we can get from (1.8)-(1.9), Lemmas 2.3 and 2.5 that

$$\begin{aligned}
\lim_{n\to\infty}\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n\nabla\psi u_n dx &= \lim_{n\to\infty}\bigg(\int_{\mathbb{R}^N}|x|^{-bp^*}|u_n|^{p^*}\psi\,dx-\int_{\mathbb{R}^N}|x|^{-(a+1)p}|u_n|^p\psi\,dx\\
&\quad-\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u_n|^p\psi\,dx+\int_{\mathbb{R}^N}f(x)|u_n|^r\psi\,dx\\
&\quad+\int_{\mathbb{R}^N}h(x)|u_n|^s\psi\,dx\bigg)\\
&\longrightarrow\int_{\mathbb{R}^N}\psi d\eta-\int_{\mathbb{R}^N}\psi d\mu=\eta_j-\mu_j\quad(\text{as }\varepsilon\longrightarrow 0).
\end{aligned}\tag{2.26}$$

On the other hand,

$$\begin{aligned}
&\lim_{n\to\infty}\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n\nabla\psi u_n dx\\
&\quad\le\lim_{n\to\infty}\left(\int_{\mathbb{R}^N}|x|^{-ap}|\nabla u_n|^p dx\right)^{(p-1)/p}\left(\int_{\mathbb{R}^N}|x|^{-ap}|u_n|^p\left|\nabla\psi\right|^p dx\right)^{1/p}\\
&\quad\le c\left(\int_{B_{2\varepsilon}}\left|\nabla\psi\right|^N dx\right)^{1/N}\left(\int_{B_{2\varepsilon}}|x|^{(-aNp)/(N-p)}|u|^{Np/(N-p)}dx\right)^{(N-p)/Np}\\
&\quad\le c\left(\int_{B_{2\varepsilon}}|x|^{(-aNp)/(N-p)}|u|^{Np/(N-p)}dx\right)^{(N-p)/Np}\longrightarrow 0\quad(\varepsilon\longrightarrow 0),
\end{aligned}\tag{2.27}$$

where $B_{2\varepsilon}\triangleq B(x_j,2\varepsilon)$. Then $\mu_j=\eta_j$; furthermore, (2.16) implies that $\mu_j=\eta_j=0$ or $\eta_j>S_1^{p^*/(p^*-p)}$. We will prove that the later does not hold. Suppose otherwise, there exists some $j_0\in J$ such that $\eta_{j_0}>S_1^{p^*/(p^*-p)}$. Then (2.19) and Lemma 2.4 show that

$$\begin{aligned}
c+o(1)&=I(u_n)-\frac{1}{r}\left\langle I'(u_n),u_n\right\rangle\\
&=\left(\frac{1}{p}-\frac{1}{r}\right)\|u_n\|_X^p+\left(\frac{1}{r}-\frac{1}{p^*}\right)\int_\Omega|x|^{-bp^*}|u_n|^{p^*}dx\\
&\ge\left(\frac{1}{r}-\frac{1}{p^*}\right)\eta_{j_0}>\left(\frac{1}{r}-\frac{1}{p^*}\right)S_1^{p^*/(p^*-p)},
\end{aligned}\tag{2.28}$$

which contradicts the hypothesis of c. Then $\mu_j=\eta_j=0$.

Similarly, we define the functional $\psi_1\in C_0^\infty(R^{\mathbb{N}})$ as

$$\psi_1\equiv 0,\quad |x|<R,\quad \psi_1\equiv 1,\quad |x|>2R,\quad \left|\nabla\psi_1\right|\le\frac{2}{R}.\tag{2.29}$$

Then, the similar proof as above shows that $\eta_\infty = \mu_\infty = 0$. Thus, we can deduce from (2.22) that

$$\int_{R^N} |x|^{-bp^*}|u_n|^{p^*} dx \longrightarrow \int_{R^N} |x|^{-bp^*}|u|^{p^*} dx \quad \text{as } n \longrightarrow \infty, \tag{2.30}$$

which implies that $u_n \to u$ in $L^{p^*}(R^N, |x|^{-bp^*})$.

Step 3. $\{u_n\}$ converges strongly in X.

The following inequalities [21] play an important role in our proof:

$$|\xi - \zeta|^p \leq \begin{cases} c\left\langle |\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta, \xi - \zeta \right\rangle & \text{for } p \geq 2, \\ c\left\langle |\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta, \xi - \zeta \right\rangle^{p/2} \left(|\xi|^p + |\zeta|^p\right)^{(2-p)/2} & \text{for } 1 < p < 2. \end{cases} \tag{2.31}$$

Our aim is to prove that $\{u_n\}$ is a Cauchy sequence of X. In fact, let $\varphi = u_n - u_m$ in (1.12), it follows from (2.19) that

$$\begin{aligned} A_{mn} &+ \int_{R^N} |x|^{-(a+1)p}\left(|u_n|^{p-2}u_n - |u_m|^{p-2}u_m\right)(u_n - u_m)dx \\ &= \langle I'(u_n) - I'(u_m), u_n - u_m \rangle \\ &\quad + \int_{R^N} f(x)\left(|u_n|^{r-2}u_n - |u_m|^{r-2}u_m\right)(u_n - u_m)dx \\ &\quad + \int_{R^N} h(x)\left(|u_n|^{s-2}u_n - |u_m|^{s-2}u_m\right)(u_n - u_m)dx \\ &\quad + \int_{R^N} |x|^{-bp^*}\left(|u_n|^{p^*-2}u_n - |u_m|^{p^*-2}u_m\right)(u_n - u_m)dx, \end{aligned} \tag{2.32}$$

where

$$A_{mn} = \int_{R^N} |x|^{-ap}\left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^{p-2}\nabla u_m\right) \cdot \nabla(u_n - u_m)dx. \tag{2.33}$$

Using the inequalities (2.31), we can get by direct computation that

$$A_{mn} \geq \begin{cases} c\displaystyle\int_{R^N} |x|^{-ap}|\nabla(u_n - u_m)|^p dx, & p \geq 2 \\ c\left(\displaystyle\int_{R^N} |x|^{-ap}|\nabla(u_n - u_m)|^p dx\right)^{2/p}, & 1 < p < 2, \end{cases} \tag{2.34}$$

with some constant $c > 0$, independent of n and m.

Then the Hölder inequality together with (1.8) and (2.30) yield that

$$\int_{R^N} |x|^{-bp^*}\left(|u_n|^{p^*-2}u_n - |u_m|^{p^*-2}u_m\right)(u_n - u_m)dx \longrightarrow 0 \quad (\text{as } n, m \longrightarrow \infty). \tag{2.35}$$

Similarly, we have from the Hölder inequality, Lemmas 2.3 and 2.4 that

$$\int_{R^N} f(x)\Big(|u_n|^{r-2}u_n - |u_m|^{r-2}u_m\Big)(u_n - u_m)dx \longrightarrow 0 \quad (\text{as } n, m \longrightarrow \infty),$$
$$\int_{R^N} h(x)\Big(|u_n|^{s-2}u_n - |u_m|^{s-2}u_m\Big)(u_n - u_m)dx \longrightarrow 0 \quad (\text{as } n, m \to \infty). \tag{2.36}$$

Therefore, the above estimates imply that $\|u_n - u_m\|_X \to 0$ $(n, m \to \infty)$, that is, $\{u_n\}$ is a Cauchy sequence of X. Then $\{u_n\}$ converges strongly in X and we complete the proof. □

Similarly, we have the following lemma.

Lemma 2.7. *Let $1 < r < p < s < p^*$. Then $I(u)$ satisfies the $(PS)_c$ condition with $c \le (1/s - 1/p^*)S_1^{p^*/(p^*-p)} + (((s-r)/(s-p))S_2)^{r/(p-r)}((r-p)(s-r)/prs)\|f\|^{p/(p-r)}_{L^{\sigma_1}(\mathbb{R}^N, g_1)}$, where S_1, S_2 are as in* (1.8), *and* (1.9) *respectively.*

Proof. Step 1. $\{u_n\}$ is bounded in X.

Let $\{u_n\}$ be a $(PS)_c$ sequence of $I(u)$ in X. Then we have from Lemma 2.3 that

$$\begin{aligned} c + 1 + \|u_n\|_X &\ge I(u_n) - \frac{1}{s}\langle I'(u_n), u_n\rangle \\ &= \left(\frac{1}{p} - \frac{1}{s}\right)\|u_n\|_X^p - \left(\frac{1}{r} - \frac{1}{s}\right)\|u_n\|^r_{L^r(\mathbb{R}^N, f)} + \left(\frac{1}{s} - \frac{1}{p^*}\right)\|u_n\|^{p^*}_{L^{p^*}(\mathbb{R}^N)} \\ &\ge \left(\frac{1}{p} - \frac{1}{s}\right)\|u_n\|_X^p - \left(\frac{1}{r} - \frac{1}{s}\right)S_2^{r/p}\|u_n\|_X^r\|f\|_{L^{\sigma_1}(\mathbb{R}^N, g_1)}. \end{aligned} \tag{2.37}$$

Since $1 < r < p < s$, (2.37) shows that $\|u_n\|$ is bounded in X.

Step 2. There exists $\{u_n\}$ in X such that $u_n \to u$ in $L^{p^*}(\mathbb{R}^N)$.

Similar to the proof of Lemma 2.5, we can get that $\mu_j = \eta_j = 0$ or $\eta_j > S_1^{p^*/(p^*-p)}$ by applying the functional ψ. Now we prove that there is no $j_0 \in J$ such that $\eta_{j_0} > S_1^{p^*/(p^*-p)}$. Suppose otherwise, then

$$\begin{aligned} c + o(1) &= I(u_n) - \frac{1}{s}\langle I'(u), u_n\rangle \\ &= \left(\frac{1}{p} - \frac{1}{s}\right)\|u_n\|_X^p - \left(\frac{1}{r} - \frac{1}{s}\right)\|u_n\|^r_{L^r(\mathbb{R}^N, f)} + \left(\frac{1}{s} - \frac{1}{p^*}\right)\|u_n\|^{p^*}_{L^{p^*}(\mathbb{R}^N)} \\ &\ge \left(\frac{1}{p} - \frac{1}{s}\right)\|u_n\|_X^p - \left(\frac{1}{r} - \frac{1}{s}\right)S_2^{r/p}\|u_n\|_X^r\|f\|_{L^{\sigma_1}(\mathbb{R}^N, g_1)} + \left(\frac{1}{s} - \frac{1}{p^*}\right)S_1^{p^*/(p^*-p)}. \end{aligned} \tag{2.38}$$

Let

$$q(t) = \left(\frac{1}{p} - \frac{1}{s}\right)t^p - \left(\frac{1}{r} - \frac{1}{s}\right)S_2^{r/p}\|f\|_{L^{\sigma_1}(\mathbb{R}^N, g_1)}t^r, \quad t \ge 0. \tag{2.39}$$

Then $q(t)$ has the unique minimum point at

$$t_0 = \left[\frac{s-r}{s-p} S_2^{r/p} \|f\|_{L^{\sigma_1}(\mathbb{R}^N, g_1)}\right]^{1/(p-r)},$$
$$q(t_0) = \left(\frac{s-r}{s-p} S_2\right)^{r/(p-r)} \frac{(r-p)(s-r)}{prs} \|f\|^{p/(p-r)}_{L^{\sigma_1}(\mathbb{R}^N, g_1)}. \tag{2.40}$$

Then it follows from (2.38) that

$$c + o(1) \geq \left(\frac{1}{s} - \frac{1}{p^*}\right) S_1^{p^*/(p^*-p)} + \left(\frac{s-r}{s-p} S_2\right)^{(r/p-r)} \frac{(r-p)(s-r)}{prs} \|f\|^{p/(p-r)}_{L^{\sigma_1}(\mathbb{R}^N, g_1)}, \tag{2.41}$$

which contradicts the hypothesis of c.

Step 3. $\{u_n\}$ converges strongly in X.

By Lemma 2.4, this result can be similarly obtained by the method in Lemma 2.6, so we omit the proof. □

3. Existence of Infinitely Solutions

In this section, we will use the minimax procedure to prove the existence of infinity many solutions of problem (1.1). Let $\mathcal{A}$ denotes the class of $A \subset X \setminus \{0\}$ such that A is closed in X and symmetric with respect to the origin. For $A \in \mathcal{A}$, we recall the genus $\gamma(A)$ which is defined by

$$\gamma(A) := \min\{m \in N : \exists \phi \in C(A, \mathbb{R}^m \setminus \{0\}), \phi(x) = -\phi(-x)\}. \tag{3.1}$$

If there is no mapping ϕ as above for any $m \in N$, then $\gamma(A) = +\infty$, and $\gamma(\emptyset) = 0$. The following proposition gives some main properties of the genus, see [13, 22].

Proposition 3.1. *Let $A, B \in \mathcal{A}$. Then*

(1) *if there exists an odd map $g \in C(A, B)$, then $\gamma(A) \leq \gamma(B)$,*

(2) *if $A \subset B$, then $\gamma(A) \leq \gamma(B)$,*

(3) $\gamma(A \bigcup B) \leq \gamma(A) + \gamma(B)$.

(4) *if S is a sphere centered at the origin in $\mathbb{R}^N$, then $\gamma(S) = N$,*

(5) *if A is compact, then $\gamma(A) < \infty$ and there exists $\delta > 0$ such that $N_\delta(A) \in \mathcal{A}$ and $\gamma(N_\delta(A)) = \gamma(A)$, where $N_\delta(A) = \{x \in X : \|x - A\| \leq \delta\}$.*

Lemma 3.2. *Assume (A_1)–(A_3). Then for any $m \in N$, there exists $\varepsilon = \varepsilon(m) > 0$ such that*

$$\gamma(\{u \in X : I(u) \leq -\varepsilon\}) \geq m. \tag{3.2}$$

Proof. For given $m \in \mathbb{N}^+$, let X_m be a m-dimensional subspace of X. If $p < r < s < p^*$, then for $u \in X_m$ we have

$$I(u) = \frac{1}{p}\|u\|_X^p - \frac{1}{r}\|u\|_{L^r(\mathbb{R}^N,f)}^r - \frac{1}{s}\|u\|_{L^s(\mathbb{R}^N,h)}^s - \frac{1}{p^*}\|u\|_{L^{p^*}}^{p^*} \le \frac{1}{p}\|u\|_X^p - \frac{1}{r}\|u\|_{L^r(\mathbb{R}^N,f)}^r. \tag{3.3}$$

The fact that all the norms on finite dimensional space are equivalent implies that for all $u \in X_m$

$$I(u) \le \frac{1}{p}\|u\|_X^p - c\|u\|_X^r, \tag{3.4}$$

for some constant $c > 0$. Then there exist large $\rho > 0$ and small $\varepsilon > 0$ such that

$$I(u) \le -\varepsilon, \qquad \|u\|_{X_m} = \rho. \tag{3.5}$$

Denote

$$S_\rho = \left\{u \in X_m : \|u\|_{X_m} = \rho\right\}. \tag{3.6}$$

Then S_ρ is a sphere centered at the origin with radius of ρ and

$$S_\rho \subset \{u \in X : I(u) \le -\varepsilon\} \triangleq I^{-\varepsilon}. \tag{3.7}$$

Therefore, Proposition 3.1 shows that $\gamma(I^{-\varepsilon}) \ge \gamma(S_\rho) = m$.

If $r < p < s < p^*$, we have

$$I(u) = \frac{1}{p}\|u\|_X^p - \frac{1}{r}\|u\|_{L^r(\mathbb{R}^N,f)}^r - \frac{1}{s}\|u\|_{L^s(\mathbb{R}^N,h)}^s - \frac{1}{p^*}\|u\|_{L^{p^*}}^{p^*} \le \frac{1}{p}\|u\|_X^p - \frac{1}{s}\|u\|_{L^s(\mathbb{R}^N,h)}^s. \tag{3.8}$$

Since $\|u\|_{L^s(\mathbb{R}^N,h)}^s$ is also a norm and all norms on the finite dimensional space X_m are equivalent, we have

$$I(u) \le \frac{1}{p}\|u\|_X^p - c\|u\|_X^s. \tag{3.9}$$

Then there exist large $\sigma > 0$ and small $\varepsilon > 0$ such that

$$I(u) \le -\varepsilon, \qquad \|u\|_{X_m} = \sigma. \tag{3.10}$$

Denote

$$S_\sigma = \left\{u \in X_m : \|u\|_{X_m} = \sigma\right\}. \tag{3.11}$$

Then S_σ is a sphere centered at the origin with radius of σ and

$$S_\sigma \subset \{u \in X : I(u) \leq -\varepsilon\} \triangleq I^{-\varepsilon}. \tag{3.12}$$

Therefore, Proposition 3.1 shows that $\gamma(I^{-\varepsilon}) \geq \gamma(S_\sigma) = m$. □

Let $\mathcal{A}_m = \{A \in \mathcal{A} : \gamma(A) \geq m\}$. It is easy to check that $\mathcal{A}_{m+1} \subset \mathcal{A}_m (m = 1, 2, \ldots)$. We define

$$c_m = \inf_{A \in \mathcal{A}_m} \sup_{u \in A} I(u). \tag{3.13}$$

It is not difficult to find that

$$c_1 \leq c_2 \leq \cdots \leq c_m \leq \cdots . \tag{3.14}$$

and $c_m > -\infty$ for any $m \in \mathbb{N}$ since $I(u)$ is coercive and bounded below. Furthermore, we define the set

$$K_c = \{u \in X : I(u) = c, I'(u) = 0\}. \tag{3.15}$$

Then, K_c is compact and we have the following important lemma, see [22].

Lemma 3.3. *All the c_m are critical values of $I(u)$. Moreover, if $c = c_m = c_{m+1} = \cdots = c_{m+\tau}$, then $\gamma(K_c) \geq 1 + \tau$.*

Proof of Theorem 1.2. In view of Lemmas 2.6 and 2.7, $I(u)$ satisfies the $(PS)_c$ condition in X. Furthermore, as the standard argument of [13, 22, 23], Lemma 3.3 gives that $I(u)$ has infinity many critical points with negative values. Thus, problem (1.1) has infinitely many solutions in X, and we complete the proof. □

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Research Article

Periodic Solutions of Some Impulsive Hamiltonian Systems with Convexity Potentials

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We study the existence of periodic solutions of some second-order Hamiltonian systems with impulses. We obtain some new existence theorems by variational methods.

1. Introduction

Consider the following systems:

$$\begin{gathered} \ddot{u}(t) = f(t, u(t)), \quad \text{a.e. } t \in [0, T], \\ \Delta \dot{u}(t_k) = g_k\left(u\left(t_k^-\right)\right), \quad k = 1, 2, \ldots, m, \\ u(T) - u(0) = \dot{u}(T) - \dot{u}(0) = 0, \end{gathered} \tag{1.1}$$

where $k \in \mathbb{Z}$, $u \in \mathbb{R}^n$, $\Delta \dot{u}(t_k) = \dot{u}(t_k^+) - \dot{u}(t_k^-)$ with $\dot{u}(t_k^\pm) = \lim_{t \to t_k^\pm} \dot{u}(t)$, $g_k(u) = \operatorname{grad}_u G_k(u)$, $G_k \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ for each $k \in \mathbb{Z}$, there exists an $m \in \mathbb{Z}$ such that $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, and we suppose that $f(t, u) = \operatorname{grad}_u F(t, u)$ satisfies the following assumption.

(A) $F(t, x)$ is measurable in t for $x \in \mathbb{R}^n$ and continuously differentiable in x for a.e. $t \in [0, T]$, and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1(0, T; \mathbb{R}^+)$ such that

$$|F(t, x)| + \left|f(t, x)\right| \le a(|x|) b(t), \tag{1.2}$$

for all $x \in \mathbb{R}^n$ and $t \in [0, T]$.

Many solvability conditions for problem (1.1) without impulsive effect are obtained, such as, the coercivity condition, the convexity conditions (see [1–4] and their references), the sublinear nonlinearity conditions, and the superlinear potential conditions. Recently, by using variational methods, many authors studied the existence of solutions of some second-order differential equations with impulses. More precisely, Nieto in [5, 6] considers linear conditions, [7–10] the sublinear conditions, and [11–16] the sublinear conditions and the other conditions. But to the best of our knowledge, except [7] there is no result about convexity conditions with impulsive effects. By using different techniques, we obtain different results from [7].

We recall some basic facts which will be used in the proofs of our main results. Let

$$H_T^1 = \left\{ u : [0,T] \longrightarrow \mathbb{R}^n \text{ absolutely continuous; } u(0) = u(T),\ \dot{u}(t) \in L^2(0,T;\mathbb{R}^n) \right\}, \tag{1.3}$$

with the inner product

$$\langle u, v \rangle = \int_0^T (u(t), v(t))dt + \int_0^T (\dot{u}(t), \dot{v}(t))dt, \quad \forall u, v \in H_T^1, \tag{1.4}$$

where $(\cdot,\cdot)$ denotes the inner product in $\mathbb{R}^n$. The corresponding norm is defined by

$$\|u\| = \left(\int_0^T (u(t), u(t))dt + \int_0^T (\dot{u}(t), \dot{u}(t))dt \right)^{1/2}, \quad \forall u \in H_T^1. \tag{1.5}$$

The space H_T^1 has some important properties. For $u \in H_T^1$, let $\overline{u} = (1/2T)\int_0^T u(t)dt$, and $\widetilde{u} = u(t) - \overline{u}$. Then one has Sobolev's inequality (see Proposition 1.3 in [1]):

$$\|\widetilde{u}\|_\infty^2 \le \frac{T}{12}\int_0^T |\dot{u}(t)|^2 dt. \tag{1.6}$$

Consider the corresponding functional φ on H_T^1 given by

$$\varphi(u) = \frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt + \int_0^T F(t, u(t))dt + \sum_{k=1}^m G_k(u(t_k)). \tag{1.7}$$

It follows from assumption (A) and the continuity of g_k one has that φ is continuously differentiable and weakly lower semicontinuous on H_T^1. Moreover, we have

$$\langle \varphi'(u), v \rangle = \int_0^T (\dot{u}(t), \dot{v}(t))dt + \int_0^T \big(f(t, u(t)), v(t)\big)dt + \sum_{k=1}^m \big(g_k(u(t_k)), v(t_k)\big), \tag{1.8}$$

for $u, v \in H_T^1$ and φ' is weakly continuous and the weak solutions of problem (1.1) correspond to the critical points of φ (see [8]).

Theorem 1.1 ([2, Theorem 1.1]). *Suppose that V and W are reflexive Banach spaces, $\varphi \in C^1(V \times W, R)$, $\varphi(v, \cdot)$ is weakly upper semi-continuous for all $v \in V$, and $\varphi(\cdot, w) : V \to R$ is convex for all $w \in W$ and φ' is weakly continuous. Assume that*

$$\varphi(0, w) \longrightarrow -\infty \tag{1.9}$$

as $\|w\| \to \infty$ and for every $M > 0$,

$$\varphi(v, w) \longrightarrow +\infty, \tag{1.10}$$

as $\|v\| \to \infty$ uniformly for $\|w\| \le M$. Then φ has at least one critical point.

2. Main Results

Theorem 2.1. *Assume that assumption (A) holds. If further*

- (H_1) *$F(t, \cdot)$ is convex for a.e. $t \in [0, T]$, and*
- (H_2) *there exist η, $\theta > 0$ such that $G_k(x) \ge \eta|x| + \theta$, for all $x \in \mathbb{R}^n$, then* (1.1) *possesses at least one solution in H_T^1.*

Remark 2.2. (H_1) implies there exists a point $\overline{x}$ for which

$$\int_0^T \nabla F(t, \overline{x}) dt = 0. \tag{2.1}$$

Proof of Theorem 2.1. It follows Remark 2.2, (1.6), and (H_2) that

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt + \int_0^T (F(t, u(t)) - F(t, \overline{x})) dt + \int_0^T F(t, \overline{x}) dt + \sum_{k=1}^m G_k(u(t_k)) \\
&= \frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt + \int_0^T F(t, \overline{x}) dt + \int_0^T (f(t, \overline{x}), u(t) - \overline{x}) dt + \sum_{k=1}^m G_k(u(t_k)) \\
&\ge \frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt + \int_0^T F(t, \overline{x}) dt + \int_0^T (f(t, \overline{x}), \widetilde{u}) dt + \sum_{k=1}^m \eta|\widetilde{u} + \overline{u}| + m\theta \\
&\ge \frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt - \left(\int_0^T \big|f(t, \overline{x})\big| dt\right) \|\widetilde{u}\|_\infty + m\eta|\overline{u}| - m\eta\|\widetilde{u}\|_\infty + m\theta \\
&\ge \frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt - C_0\left(\int_0^T |\dot{u}(t)|^2 dt\right)^{1/2} + m\eta|\overline{u}| + m\theta,
\end{aligned} \tag{2.2}$$

for all $u \in H_T^1$ and some positive constant C_0. As $\|u\| \to \infty$ if and only if $(|u|^2 + \|\dot{u}\|_2^2)^{1/2} \to \infty$, we have $\varphi(u) \to +\infty$ as $\|u\| \to \infty$. By Theorem 1.1 and Corollary 1.1 in [1], φ has a minimum point in H_T^1, which is a critical point of φ. Hence, problem (1.1) has at least one weak solution. □

Theorem 2.3. *Assume that assumption* (A) *and* (H_1) *hold. If further*

(H_3) *there exist* $\eta, \theta > 0$ *and* $\alpha \in (0,2)$ *such that* $G_k(x) \le \eta|x|^\alpha + \theta$ *for all* $x \in \mathbb{R}^n$ *and*

(H_4) *there exist some* $\beta > \alpha$ *and* $\gamma > 0$ *such that*

$$|x|^{-\beta}\int_0^T F(t,x)dt \le -\gamma, \tag{2.3}$$

for $|x| \ge M$ *and* $t \in [0,T]$, *where* M *is a constant, then* (1.1) *possesses at least one solution in* H_T^1.

Remark 2.4. We can find that our condition (H_4) is very different from condition (vii) in [7] since we prove this by the saddle point theorem substituted for the least action principle.

Proof of Theorem 2.3. We prove φ satisfies the (PS) condition at first. Suppose $\{u_n\}$ is such an sequence that $\{\varphi(u_n)\}$ is bounded and $\lim_{n\to\infty}\varphi'(u_n) = 0$. We will prove it has a convergent subsequence. By (H_3) and (1.6), we have

$$\begin{aligned}\sum_{k=1}^m G_k(u(t_k)) &\le \sum_{k=1}^m \eta|\tilde{u}(t_k) + \overline{u}|^\alpha + m\theta \\ &\le 4m\eta\left(|\tilde{u}(t_k)|^\alpha + |\overline{u}|^\alpha\right) + m\theta \\ &\le C_1\|\dot{u}\|_2^\alpha + C_2|\overline{u}|^\alpha + C_3,\end{aligned} \tag{2.4}$$

for some positive constants C_1, C_2, C_3. By Remark 2.2, (1.6), and (2.4), we have

$$\begin{aligned}\varphi(u_n) &= \frac{1}{2}\int_0^T |\dot{u}_n(t)|^2dt + \int_0^T (F(t,u_n(t)) - F(t,\overline{x}))dt + \int_0^T F(t,\overline{x})dt + \sum_{k=1}^m G_k(u_n(t_k)) \\ &= \frac{1}{2}\int_0^T |\dot{u}_n(t)|^2dt + \int_0^T F(t,\overline{x})dt + \int_0^T \left(f(t,\overline{x}), u_n(t) - \overline{x}\right)dt + \sum_{k=1}^m G_k(u_n(t_k)) \\ &= \frac{1}{2}\int_0^T |\dot{u}_n(t)|^2dt + \int_0^T F(t,\overline{x})dt + \int_0^T \left(f(t,\overline{x}), \tilde{u}_n\right)dt + \sum_{k=1}^m G_k(u_n(t_k)) \\ &\ge \frac{1}{2}\int_0^T |\dot{u}_n(t)|^2dt - \left(\int_0^T \left|f(t,\overline{x})\right|dt\right)\|\tilde{u}_n\|_\infty - C_1\|\dot{u}_n\|_2^\alpha - C_2|\overline{u}_n|^\alpha - C_4 \\ &\ge \frac{1}{2}\int_0^T |\dot{u}_n(t)|^2dt - C_5\left(\int_0^T |\dot{u}_n(t)|^2dt\right)^{1/2} - C_1\|\dot{u}_n\|_2^\alpha - C_2|\overline{u}_n|^\alpha - C_4,\end{aligned} \tag{2.5}$$

for some positive constants C_4, C_5, which implies that

$$C|\overline{u}_n|^{\alpha/2} \ge \left(\int_0^T |\dot{u}_n(t)|^2dt\right)^{1/2} - C_6, \tag{2.6}$$

for some positive constants C, C_6. By (1.6), the above inequality implies that

$$\|\tilde{u}_n\|_\infty \leq C_7\left(|\overline{u}_n|^{\alpha/2}+1\right), \tag{2.7}$$

for the positive constant C_7. The one has

$$|u_n(t)| \geq |\overline{u}_n| - |\tilde{u}_n| \geq |\overline{u}_n| - \|\tilde{u}_n\|_\infty \geq |\overline{u}_n| - C_7\left(|\overline{u}_n|^{\alpha/2}+1\right), \quad \forall t \in [0,T]. \tag{2.8}$$

If $\{|\overline{u}_n|\}$ is unbounded, we may assume that, going to a subsequence if necessary,

$$|\overline{u}_n| \longrightarrow \infty \quad \text{as } n \longrightarrow \infty. \tag{2.9}$$

By (2.8) and (2.9), we have

$$|u_n(t)| \geq \frac{1}{2}|\overline{u}_n|, \tag{2.10}$$

for all large n and every $t \in [0,T]$. By (2.10) and (H_4), one has $|u_n(t)| \geq M$ for all large n. It follows from (H_4), (2.4), (2.6), (2.7), and above inequality that

$$\begin{aligned}
\varphi(u_n) &\leq \left(C|\overline{u}_n|^{\alpha/2}+C_6\right)^2 - \int_0^T \gamma|u_n(t)|^\beta dt + C_2\|\tilde{u}\|_\infty^\alpha + C_2|\overline{u}|^\alpha + C_3 \\
&\leq \left(C|\overline{u}_n|^{\alpha/2}+C_6\right)^2 - 2^{-\beta}|\overline{u}_n|^\beta T\gamma + C_8\left(|\overline{u}_n|^{\alpha/2}+1\right)^\alpha + C_2|\overline{u}|^\alpha + C_3,
\end{aligned} \tag{2.11}$$

for large n and the positive constant C_8, which contradicts the boundedness of $\varphi(u_n)$ since $\beta > \alpha$. Hence $(|\overline{u}_n|)$ is bounded. Furthermore, (u_n) is bounded by (2.6). A similar calculation to Lemma 3.1 in [9] shows that φ satisfies the (PS) condition. We now prove that φ satisfies the other conditions of the saddle point theorem. Assume that $\widetilde{H}_T^1 = \{u \in H_T^1 : \overline{u} = 0\}$, then $H_T^1 = \widetilde{H}_T^1 \oplus \mathbb{R}^n$. From above calculation, one has

$$\varphi(u) \geq \frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt - C_5\left(\int_0^T |\dot{u}(t)|^2 dt\right)^{1/2} - C_1\|\dot{u}\|_2^\alpha - C_4, \tag{2.12}$$

for all $u \in \widetilde{H}_T^1$, which implies that

$$\varphi(u) \longrightarrow +\infty, \tag{2.13}$$

as $\|u\| \to \infty$ in $\widetilde{H}_T^1$. Moreover, by (H_3) and (H_4) we have

$$\begin{aligned}\varphi(x) &= \int_0^T F(t,x)dt + \sum_{k=1}^m G_k(x)\\ &\le -T\gamma|x|^\beta + m\eta|x|^\alpha + m\theta,\end{aligned} \tag{2.14}$$

for $|x| > M$, which implies that

$$\varphi(x) \longrightarrow -\infty, \tag{2.15}$$

as $|x| \to \infty$ in $\mathbb{R}^n$ since $\beta > \alpha$. Now Theorem 2.3 is proved by (2.13), (2.15), and the saddle point theorem. □

Theorem 2.5. *Assume that assumption (A) holds. Suppose that $F(t,\cdot), G_k(x)$ are concave and satisfy*

(H_5) $G_k(x) \le -\eta|x| + \theta$ *for some positive constant* $\eta, \theta > 0$, *then* (1.1) *possesses at least one solution in* H_T^1.

Proof of Theorem 2.5. Consider the corresponding functional φ on $\mathbb{R}^n \times \widetilde{H}_T^1$ given by

$$\varphi(u) = -\frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt - \int_0^T F(t,u(t))dt - \sum_{k=1}^m G_k(u(t_k)), \tag{2.16}$$

which is continuously differentiable, bounded, and weakly upper semi-continuous on H_T^1. Similar to the proof of Lemma 3.1 in [2], one has that $\varphi(x+w)$ is convex in $x \in \mathbb{R}^n$ for every $w \in \widetilde{H}_T^1$. By the condition, we have $-G_k(x+w) \ge -2G_k((1/2)x) + G_k(-w)$. Similar to the proof of Theorem 3.1, we have

$$\begin{aligned}\varphi(x+w) &= -\frac{1}{2}\int_0^T |\dot{w}|^2 dt - \int_0^T F(t,x+w)dt - \sum_{k=1}^m G_k(x+w)\\ &\ge -\frac{1}{2}\int_0^T |\dot{w}|^2 dt - \left(\int_0^T \left|f(t,\overline{x})\right| dt\right)\|w\|_\infty - \sum_{k=1}^m G_k(x+w) + C_9\\ &\ge -\frac{1}{2}\int_0^T |\dot{w}|^2 dt - C_0\left(\int_0^T |\dot{w}|^2 dt\right)^{1/2} - 2G_k\left(\frac{1}{2}x\right) + G_k(-w) + C_9,\end{aligned} \tag{2.17}$$

which means $\varphi(x+w) \to +\infty$ as $|x| \to \infty$, uniformly for $w \in \widetilde{H}_T^1$ with $\|w\| \le M$ by (H_5) and (1.6). On the other hand,

$$\begin{aligned}\varphi(w) &= -\frac{1}{2}\int_0^T |\dot{w}|^2 dt - \int_0^T F(t,w)dt - \sum_{k=1}^{m} G_k(w) \\ &\le -\frac{1}{2}\int_0^T |\dot{w}|^2 dt + C_0\left(\int_0^T |\dot{w}|^2 dt\right)^{1/2} + m\eta\|w\|_\infty + C_9,\end{aligned} \tag{2.18}$$

which implies that $\varphi(w) \to -\infty$ as $\|w\| \to \infty \in \widetilde{H}_T^1$ by (H_5) and (1.6). We complete our proof by Theorem 1.1. □

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Research Article

Existence and Multiplicity Results for Nonlinear Differential Equations Depending on a Parameter in Semipositone Case

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The existence and multiplicity of solutions for second-order differential equations with a parameter are discussed in this paper. We are mainly concerned with the semipositone case. The analysis relies on the nonlinear alternative principle of Leray-Schauder and Krasnosel'skii's fixed point theorem in cones.

1. Introduction

In this paper, we consider the problem of existence, multiplicity, and nonexistence of positive solutions for the following boundary value problem (BVP):

$$\begin{gathered} -\left(a(t)x'\right)' + b(t)x = \lambda f(t,x), \quad t \in I, \\ x(0) = x(2\pi), \qquad a(0)x'(0) = a(2\pi)x'(2\pi), \end{gathered} \tag{E_λ}$$

where $I := [0, 2\pi]$, λ is a positive parameter, $f(t,x) \in \mathrm{Car}(I \times \mathbb{R}^+, \mathbb{R})$, and $a(t)$, $b(t)$ are real-valued measurable functions defined on $[0, 2\pi]$ and satisfy the following condition:

$$a(t) > 0, \quad b(t) \geq 0, \quad b(t) \not\equiv 0, \quad \int_0^{2\pi} \frac{dt}{a(t)} < \infty, \quad \int_0^{2\pi} b(t)dt < \infty. \tag{H1}$$

Here, the symbol $\mathrm{Car}(I\times\mathbb{R}^+,\mathbb{R})$ denotes the set of functions satisfying the Carathédory conditions on $I\times\mathbb{R}^+$; that is,

(i) $f(\cdot,x) : I \to \mathbb{R}$ is Lebesgue integrable for each fixed $x\in\mathbb{R}^+$, and

(ii) $f(t,\cdot) : \mathbb{R}^+ \to \mathbb{R}$ is continuous for a.e. $t\in I$.

Due to a wide range of applications in physics and engineering, second-order boundary value problems have been extensively investigated by numerous researchers in recent years. For a small sample of such work, we refer the reader to [1–18] and the references therein. When $a(t) = 1$, $b(t) = m^2$, $\lambda = 1$ of (E_λ), in [11, 18], by using Krasnosel'skii's fixed point theorem, the existence and multiplicity of positive solutions are established to the periodic boundary value problem:

$$\begin{gathered} -x'' + m^2 x = f(t,x), \quad t\in I, \\ x(0) = x(2\pi), \qquad x'(0) = x'(2\pi), \end{gathered} \tag{1.1}$$

where $f(t,x)\in\mathrm{Car}(I\times\mathbb{R}^+,\mathbb{R}^+)$.

In [8], Graef et al. consider the second-order periodic boundary value problem:

$$\begin{gathered} -x'' + m^2 x = \lambda g(t) f(x), \quad t\in I, \\ x(0) = x(2\pi), \qquad x'(0) = x'(2\pi), \end{gathered} \tag{1.2}$$

where $g : I \to \mathbb{R}^+$ is continuous and $f : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and $f(x) > 0$ for $x > 0$. Under different combinations of superlinearity and sublinearity of the function f, various existence, multiplicity, and nonexistence results for positive solutions are derived in terms of different value of λ via Krasnosel'skii's fixed point theorem.

Hao et al. [9] use the Global continuation theorem, fixed point index theory, and approximate method to study the following periodic boundary value problems:

$$\begin{gathered} -x'' + a(t)x = \lambda f(t,x), \quad t\in I, \\ x(0) = x(2\pi), \qquad x'(0) = x'(2\pi), \end{gathered} \tag{1.3}$$

where $a\in L^1(0,2\pi)$ and $f(t,x)\in\mathrm{Car}(I\times\mathbb{R}^+,\mathbb{R}^+)$.

In [10], by using the fixed point index theory, He et al. study the existence and multiplicity of positive solutions to BVP (E_λ). Motivated by the above works, we establish the results of existence, multiplicity, and nonexistence of positive solutions for BVP (E_λ) via Leray-Schauder alternative principle and Krasnosel'skii's fixed point in the semipositone case, that is, $f(t,x) + M > 0$ for some $M > 0$. Notice that we do not need $f(t,x) > 0$ for any $t\in[0,2\pi]$ and $x > 0$, which is an essential condition of [9, 10].

The main result of the present paper is summarized as follows.

Theorem 1.1. *Assume that*

$$f^0 := \lim_{x\to 0^+} \max_{t\in[0,2\pi]} \frac{f(t,x)}{x} < \infty, \qquad f^\infty := \lim_{x\to+\infty} \max_{t\in[0,2\pi]} \frac{f(t,x)}{x} < \infty. \tag{1.4}$$

Then, there exist $0 < \underline{\lambda} < \overline{\lambda}$ *such that* (E_λ) *has no positive solution for* $\lambda < \underline{\lambda}$ *and at least two positive solutions for* $\lambda \geq \overline{\lambda}$.

Remark 1.2. The main result above is a generalization of [9, Theorem 1.2] and [10, Theorem 1.2] and some other known results, in which f^0 and f^∞ must be zero, besides $f(t,x)$ is positive.

The remaining part of the paper is organized as follows. Some preliminary results will be given in Section 2. In Section 3, existence results are obtained using a nonlinear alternative of Leray-Schauder and fixed point theorem in cones when λ is large enough; the proof of Theorem 1.1 is also given.

2. Preliminaries and Lemmas

In this section, we present some preliminary results which will be needed in subsequent sections. Denote by $u(x)$ and $v(x)$ the solutions of the corresponding homogeneous equation:

$$-\left(a(t)x'\right)' + b(t)x = 0, \quad t \in I, \tag{2.1}$$

under the initial conditions

$$u(0) = 1, \quad a(0)u(0) = 0, \quad v(0) = 0, \quad a(0)v(0) = 1. \tag{2.2}$$

Lemma 2.1 (see [2, Theorem 2.4], [10, Lemma 2.1]). *Assume that* $(H1)$ *holds and* $h \in C(I, \mathbb{R}^+)$. *Then for the solution* $x(t)$ *of the BVP*

$$\begin{gathered} -\left(a(t)x'\right)' + b(t)x = h(t), \quad t \in I, \\ x(0) = x(2\pi), \qquad a(0)x'(0) = a(2\pi)x'(2\pi), \end{gathered} \tag{2.3}$$

the formula

$$x(t) = (\mathcal{L}h)(t) := \int_0^{2\pi} G(t,s)h(s)ds, \quad t \in I \tag{2.4}$$

holds, where

$$\begin{aligned} G(t,s) = {} & \frac{v(2\pi)}{D}u(t)u(s) - \frac{a(2\pi)u'(2\pi)}{D}v(t)v(s) \\ & + \begin{cases} \dfrac{a(2\pi)v'(2\pi) - 1}{D}u(t)v(s) - \dfrac{u(2\pi) - 1}{D}u(s)v(t), & 0 \leq s \leq t \leq 2\pi, \\ \dfrac{a(2\pi)v'(2\pi) - 1}{D}u(s)v(t) - \dfrac{u(2\pi) - 1}{D}u(t)v(s), & 0 \leq t \leq s \leq 2\pi, \end{cases} \end{aligned} \tag{2.5}$$

and $D = u(2\pi) + a(2\pi)v'(2\pi) - 1 > 0$.

Lemma 2.2 (see [2, Theorem 2.5], [10, Lemma 2.2]). *Under condition* $(H1)$, *the Green's function of the BVP* (2.3) *is positive, that is,* $G(t,s) > 0$ *for* $t, s \in I$.

Remark 2.3. We denote

$$A = \min_{0 \le s,t \le 2\pi} G(t,s), \qquad B = \max_{0 \le s,t \le 2\pi} G(t,s), \qquad \sigma = \frac{A}{B}. \tag{2.6}$$

Thus, $B > A > 0$ and $0 < \sigma < 1$. In this paper, we use $\omega(t)$ to denote the unique periodic solution of (2.3) with $h(t) = 1$, that is, $\omega(t) = (\mathcal{L}1)(t)$. Obviously, $A \le \|\omega\|_\infty / 2\pi \le B$.

Remark 2.4. If $a(t) = 1, b(t) = m^2 > 0$, then the Green's function $G(t,s)$ of the boundary value problem (2.3) has the form

$$G(t,s) = G(|t,s|) = \begin{cases} \dfrac{\exp(m(t-s)) + \exp(m(2\pi - t + s))}{2m(\exp(2m\pi) - 1)}, & 0 \le s \le t \le 2\pi, \\ \dfrac{\exp(m(s-t)) + \exp(m(2\pi - s + t))}{2m(\exp(2m\pi) - 1)}, & 0 \le t \le s \le 2\pi. \end{cases} \tag{2.7}$$

It is obvious that $G(t,s) > 0$ for $0 \le s, t \le 2\pi$, and a direct calculation shows that

$$A = \frac{e^{m\pi}}{m(e^{2m\pi} - 1)}, \qquad B = \frac{1 + e^{2m\pi}}{2m(e^{2m\pi} - 1)}, \qquad \sigma = \frac{2e^{m\pi}}{1 + e^{2m\pi}} < 1. \tag{2.8}$$

In the obtention of the second periodic solution of (E_λ), we need the following well-known fixed point theorem of compression and expansion of cones [19].

Lemma 2.5 (see Krasnosel'skii [19]). *Let* X *be a Banach space and* $K (\subset X)$ *a cone. Assume that* Ω_1, Ω_2 *are open subsets of* X *with* $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, *and let*

$$T : K \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right) \longrightarrow K \tag{2.9}$$

be a continuous and compact operator such that either

(i) $\|Tu\| \ge \|u\|$, $u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \le \|u\|$, $u \in K \cap \partial\Omega_2$, *or*

(ii) $\|Tu\| \le \|u\|$, $u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \ge \|u\|$, $u \in K \cap \partial\Omega_2$.

Then T *has a fixed point in* $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

In the applications below, we take $X = C[0, 2\pi]$ with the supremum norm $\|\cdot\|$ and define

$$K = \left\{ x \in X : x(t) \ge 0 \ \forall t, \min_{0 \le t \le 2\pi} x(t) \ge \sigma \|x\| \right\}, \tag{2.10}$$

where $\|x(t)\| = \max_{0 \le t \le 2\pi} |x(t)|$.

One may readily verify that K is a cone in X. Finally, we define an operator $T : X \to K$ by

$$(Tx)(t) = \int_0^{2\pi} G(t,s)F(s,x(s))ds, \tag{2.11}$$

for $x \in X$ and $t \in [0,2\pi]$, where $F : [0,2\pi] \times \mathbb{R} \to [0,\infty)$ is continuous and $G(t,s)$ is the Green function defined above.

Lemma 2.6 (see [12, Lemmas 2.2, 2.3], [13, Lemma 2.4]). *T is well defined and maps X into K. Moreover, $T : X \to K$ is continuous and completely continuous.*

3. Proof of Theorem 1.1

In this section we establish the existence, multiplicity, and nonexistence of positive solutions to the periodic boundary problem (E_λ). The first existence result is based on the following nonlinear alternative of Leray-Schauder, which can be found in [15].

Lemma 3.1. *Assume Ω is a relatively compact subset of a convex set K in a normed space X. Let $T : \overline{\Omega} \to K$ be a compact map with $0 \in \Omega$. Then one of the following two conclusions holds:*

(I) *T has at least one fixed point in $\overline{\Omega}$.*

(II) *There exist $x \in \partial\Omega$ and $0 < \lambda < 1$ such that $x = \lambda Tx$.*

Since we are mainly interested in the semipositone case, without loss of generality, we may assume that $f(t,x)$ satisfies the following.

(F1) There is a constant $M > 0$ such that $f(t,x)+M > 0$ for all $(t,x) \in [0,2\pi]\times(0,\infty)$ and let $F(t,x) := \lambda(f(t,x)+M) > 0$. Besides, we introduce the following assumption on $f(t,x)$.

(F2) there exists a continuous, nonnegative function $g(x)$ on $(0,\infty)$ such that

$$f(t,x) \le g(x), \quad \forall (t,x) \in [0,2\pi] \times (0,\infty), \tag{3.1}$$

that is,

$$F(t,x) \le \lambda\big(g(x)+M\big), \quad \forall (t,x) \in [0,2\pi] \times (0,\infty), \tag{3.2}$$

and $g(x) > 0$ is nondecreasing in $x \in (0,\infty)$.

Theorem 3.2. *Suppose $f(t,x)$ satisfies (F1) and (F2). Suppose further that*

(F3) *there exists $r > M\|\omega\|/\sigma$ such that*

$$\frac{r}{\lambda(g(r)+M)} > \|\omega\|, \tag{3.3}$$

where σ and ω are as in Section 2.

Then (E_λ) *has at least one positive periodic solution with* $0 < \|x + M\omega\| < r$.

Proof. The existence is proved using the Leray-Schauder alternative principle. Consider the following equation:

$$\begin{aligned} -\left(a(t)x'\right)' + b(t)x &= \mu F(t, x(t) - M\omega(t)), \quad t \in I, \\ x(0) = x(2\pi), \quad & a(0)x'(0) = a(2\pi)x'(2\pi), \end{aligned} \tag{3.4}$$

where $\mu \in [0,1]$. Problem (3.4) is equivalent to the following fixed point problem in $C[0,2\pi]$:

$$x = \mu T x, \tag{3.5}$$

where T denotes the operator defined by (2.11), with $F(t,x)$ replaced by $F(t, x - M\omega)$.

We claim that any fixed point x of (3.5) for any $\mu \in [0,1]$ must satisfy $\|x\| \neq r$.

Then we have from condition (F2), for all $t \in I$,

$$\begin{aligned} x(t) &= \mu T x(t) \\ &= \mu \int_0^{2\pi} G(t,s) F(s, x(s) - M\omega(s)) ds \\ &\leq \int_0^{2\pi} G(t,s) F(s, x(s) - M\omega(s)) ds \\ &\leq \int_0^{2\pi} G(t,s) \left(\lambda\left(g(x - M\omega) + M\right)\right) ds \\ &\leq \lambda\left(g(r) + M\right)\|\omega\|. \end{aligned} \tag{3.6}$$

Therefore,

$$r = \|x\| \leq \lambda\left(g(r) + M\right)\|\omega\|. \tag{3.7}$$

This is a contradiction to the condition (F3). From this claim, the nonlinear alternative of Leray-Schauder guarantees that (3.5) (with $\mu = 1$) has a fixed point, denoted by $\widehat{x}_1(t)$, that is,

$$\begin{aligned} -\left(a(t)\widehat{x}_1'\right)' + b(t)\widehat{x}_1 &= \lambda\left(f(t, \widehat{x}_1(t) - M\omega(t)) + M\right), \quad t \in I, \\ \widehat{x}_1(0) = \widehat{x}_1(2\pi), \qquad & a(0)\widehat{x}_1'(0) = a(2\pi)\widehat{x}_1'(2\pi). \end{aligned} \tag{3.8}$$

Using Lemma 2.5 and condition (F3), for all $t \in I$, we have

$$\widehat{x}_1(t) \geq \sigma\|\widehat{x}_1\| = \sigma r > \sigma \cdot \frac{M\|\omega\|}{\sigma} = M\|\omega\| > 0, \tag{3.9}$$

that is,

$$\widehat{x}_1(t) - M\|\omega\| > 0. \tag{3.10}$$

Let

$$x_1^*(t) = \widehat{x}_1(t) - M\omega. \tag{3.11}$$

It is easy to see that $x_1^*(t)$ is a solution of (E_λ) which satisfies $0 < \|x_1^* + M\omega\| < r$. Thus, the proof of Theorem 3.2 is completed. □

Theorem 3.3. *Suppose that conditions (F1)–(F3) hold. In addition, it is assumed that the following two conditions are satisfied.*

(F4) *There exists a continuous, nonnegative function $h(x)$ on $(0, \infty)$ such that*

$$f(t, x) + M \geq h(x), \quad \forall(t, x) \in [0, 2\pi] \times (0, \infty), \tag{3.12}$$

that is,

$$F(t, x) \geq \lambda h(x), \quad \forall(t, x) \in [0, 2\pi] \times (0, \infty), \tag{3.13}$$

and $h(x) > 0$ is nondecreasing in $x \in (0, \infty)$.

(F5) *There exists a positive number $R > r$ such that*

$$\frac{R}{\lambda h(\sigma R - M\|\omega\|)} \leq \|\omega\|. \tag{3.14}$$

Then, besides the periodic solution x constructed in Theorem 3.2, (E_λ) has another positive periodic solution $\widetilde{x}$ with $r < \|\widetilde{x} + M\omega\| < R$.

Proof. As in the proof of Theorem 3.2, we only need to show that (3.8) has a periodic solution with $\widehat{x}_2 \in C[0, 2\pi]$ with $\widehat{x}_2 > M\omega$ and $r < \|\widehat{x}_2\| < R$.

Let $X = C[0, 2\pi]$ and K the cone in X in Section 2. Let $\Omega_1 = B_r$ and $\Omega_2 = B_R$ be balls in X. The operator $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ is defined by (2.11), with $F(t, x)$ replaced by $F(t, x - M\omega)$. Note that any $x \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$ satisfies $0 < \sigma r \leq x(t) \leq R$, thus T is well defined.

First we have $\|Tx\| \leq \|x\|$ for $x \in K \cap \partial\Omega_1$. In fact, if $x \in K \cap \partial\Omega_1$, then $\|x\| = r$. Now the estimate $\|Tx\| \leq r$ can be obtained almost following the same ideas in proving (3.7). We omit the details here.

Next we show that $\|Tx\| \geq \|x\|$ for $x \in K \cap \partial\Omega_2$. To see this, let $x \in K \cap \partial\Omega_2$, then $\|x\| = R$ and $x \geq \sigma R$; it follows from conditions (F4) and (F5) that, for $0 \geq t \geq 2\pi$,

$$\begin{aligned} Tx(t) &= \int_0^{2\pi} G(t,s)F(s,x(s) - M\omega(s))ds \\ &\geq \int_0^{2\pi} G(t,s)(\lambda(h(x - M\omega)))ds \\ &\geq \lambda h(\sigma R - M\|\omega\|)\|\omega\| \geq R = \|x\|. \end{aligned} \tag{3.15}$$

Now Lemma 2.5 guarantees that T has a fixed point $\widehat{x}_2 \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$, thus $r \leq \|\widehat{x}_2(t)\| \leq R$.

Finally, $x_2^*(t) = \widehat{x}_2(t) - M\omega$ will be the another desired positive periodic solution of (E_λ). We omit the details because they are much similar to that in the proof of Theorem 3.2. □

Now we are in a position to present the proof of Theorem 1.1.

Proof of Theorem 1.1. Consider $v(x) > 0$ be an eigenfunction satisfying

$$\begin{aligned} -(a(t)v')' + b(t)v = \lambda_1 v, \quad t \in I, \\ v(0) = v(2\pi), \qquad a(0)v'(0) = a(2\pi)v'(2\pi), \end{aligned} \tag{3.16}$$

corresponding to the principal eigenvalue λ_1. Let x be a positive solution of (E_λ). Multiplying (3.16) by x and (E_λ) by v, and subtracting we obtain

$$\int_0^{2\pi} (\lambda f(t,x) - \lambda_1 x)v \,\mathrm{d}x = 0. \tag{3.17}$$

Since $f^0 < \infty$ and $f^\infty < \infty$, there exist positive numbers η_1, η_2, ϵ_1, and ϵ_2 such that $\epsilon_1 < \epsilon_2$ and

$$\begin{aligned} |f(t,x)| \leq \eta_1 x \quad \text{for } x \in [0,\epsilon_1], \\ |f(t,x)| \leq \eta_2 x \quad \text{for } x \in [\epsilon_2,\infty), \end{aligned} \tag{3.18}$$

with $t \in I$. Let the positive number η_3 be defined by

$$\eta_3 = \max\left\{\eta_1, \eta_2, \max_{\epsilon_1<x<\epsilon_2}\left\{\left|\frac{f(t,x)}{x}\right|\right\}\right\}. \tag{3.19}$$

Then

$$|f(t,x)| \leq \eta_3 x \quad \text{for } x \in [0,\infty). \tag{3.20}$$

Thus, there exists a $\underline{\lambda} > 0$, for $0 < \lambda < \underline{\lambda}$ satisfying $|\lambda_1/\lambda| > \eta_3$. (3.17) cannot hold, and hence (E_λ) has no positive solution for $\lambda < \underline{\lambda}$.

Note that the sublinearity of $f(t,x)$ near $x = \infty$, we can construct a suitable $g(x)$ in (F2) which satisfies $\lim_{r\to\infty} g(r)/r < \infty$. This means that there exists $\lambda > \overline{\lambda}_1$ satisfying (3.3) with r being large enough. There also exists $\lambda > \overline{\lambda}_2 = R/\|\omega\|(h(\sigma R - M\|\omega\|))$ satisfying (3.14). Let $\overline{\lambda} = \max(\overline{\lambda}_1, \overline{\lambda}_2)$. Thus, with the help of Theorems 3.2 and 3.3, (E_λ) has at least two positive solution for $\lambda > \overline{\lambda}$. This completes the proof of the theorem. □

Example 3.4. Let the nonlinearity in (E_λ) be

$$f(t,x) = \alpha(t) g(x) \exp(-x^\gamma), \tag{3.21}$$

with $\gamma > 0$, $\alpha(t)$ is a continuous function for all $t \in I$ and $g(x)$ is a real coefficient polynomial function which has zero constant term. Then Theorem 1.1 is valid.

Proof. In this case, with the function $f(t,x) = \alpha(t) g(x) \exp(-x^\gamma)$, it is easy to verify

$$\begin{aligned} f^0 &:= \lim_{x\to 0^+} \max_{t\in[0,2\pi]} \frac{f(t,x)}{x} = \lim_{x\to 0^+} \max_{t\in[0,2\pi]} \frac{\alpha(t) g(x)\exp(-x^\gamma)}{x} < \infty, \\ f^\infty &:= \lim_{x\to +\infty} \max_{t\in[0,2\pi]} \frac{f(t,x)}{x} = \lim_{x\to +\infty} \max_{t\in[0,2\pi]} \frac{\alpha(t) g(x)\exp(-x^\gamma)}{x} = 0 < \infty. \end{aligned} \tag{3.22}$$

Then the conclusion follows from Theorem 1.1 that there exists $0 < \underline{\lambda} < \overline{\lambda}$ such that (E_λ) has no positive solution for $\lambda < \underline{\lambda}$ and at least two positive solutions for $\lambda \geq \overline{\lambda}$. □

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Research Article

Principal Functions of Nonselfadjoint Discrete Dirac Equations with Spectral Parameter in Boundary Conditions

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Let L denote the operator generated in $\ell_2(\mathbb{N},\mathbb{C}^2)$ by $a_{n+1}y_{n+1}^{(2)} + b_n y_n^{(2)} + p_n y_n^{(1)} = \lambda y_n^{(1)}$, $a_{n-1}y_{n-1}^{(1)} + b_n y_n^{(1)} + q_n y_n^{(2)} = \lambda y_n^{(2)}$, $n \in \mathbb{N}$, and the boundary condition $(\gamma_0 + \gamma_1\lambda)y_1^{(2)} + (\beta_0 + \beta_1\lambda)y_0^{(1)} = \mathbf{0}$, where $(a_n), (b_n), (p_n)$, and (q_n), $n \in \mathbb{N}$ are complex sequences, $\gamma_i, \beta_i \in \mathbb{C}$, $i = 0, 1$, and λ is an eigenparameter. In this paper we investigated the principal functions corresponding to the eigenvalues and the spectral singularities of L.

1. Introduction

Consider the boundary value problem (BVP)

$$\begin{gathered} -y'' + q(x)y = \lambda^2 y, \quad 0 \le x < \infty, \\ y(0) = 0, \end{gathered} \tag{1.1}$$

in $L^2(\mathbb{R}_+)$, where q is a complex-valued function and $\lambda \in \mathbb{C}$ is a spectral parameter. The spectral theory of the above BVP with continuous and point spectrum was investigated by Naĭmark [1]. He showed the existence of the spectral singularities in the continuous spectrum of (1.1). Note that the eigen and associated functions corresponding to the spectral singularities are not the elements of $L^2(\mathbb{R}_+)$.

In [2, 3] the effect of the spectral singularities in the spectral expansion in terms of the principal vectors was considered. Some problems related to the spectral analysis of difference equations with spectral singularities were discussed in [4–7]. The spectral analysis of eigenparameter dependent nonselfadjoint difference equation was studied in [8, 9].

Let us consider the nonselfadjoint BVP for the discrete Dirac equations

$$\begin{aligned} a_{n+1}y_{n+1}^{(2)}+b_ny_n^{(2)}+p_ny_n^{(1)}&=\lambda y_n^{(1)}, \quad n\in\mathbb{N},\\ a_{n-1}y_{n-1}^{(1)}+b_ny_n^{(1)}+q_ny_n^{(2)}&=\lambda y_n^{(2)}, \quad n\in\mathbb{N}, \end{aligned} \tag{1.2}$$

$$(\gamma_0+\gamma_1\lambda)y_1^{(2)}+(\beta_0+\beta_1\lambda)y_0^{(1)}=0, \quad \gamma_0\beta_1-\gamma_1\beta_0\neq 0, \quad \gamma_1\neq a_0^{-1}\beta_0, \tag{1.3}$$

where $\begin{pmatrix} y_n^{(1)} \\ y_n^{(2)} \end{pmatrix}$, $n\in\mathbb{N}$ are vector sequences, $a_n\neq 0$, $b_n\neq 0$ for all $n\in\mathbb{N}$, $\gamma_i,\beta_i\in\mathbb{C}$, and $i=0,1$ and, λ is a spectral parameter.

In [10] the authors proved that eigenvalues and spectral singularities of (1.2)-(1.3) have a finite number with finite multiplicities, if the condition,

$$\sum_{n=1}^{\infty}\exp\left(\varepsilon n^{\delta}\right)\left(|1-a_n|+|1+b_n|+|p_n|+|q_n|\right)<\infty \tag{1.4}$$

holds, for some $\varepsilon>0$ and $1/2\leq\delta<1$.

In this paper, we aim to investigate the principal functions corresponding to the eigenvalues and the spectral singularities of the BVP (1.2)-(1.3).

2. Discrete Spectrum of (1.2)-(1.3)

Let for some $\varepsilon>0$ and $1/2\leq\delta<1$,

$$\sum_{n=1}^{\infty}\exp\left(\varepsilon n^{\delta}\right)\left(|1-a_n|+|1+b_n|+|p_n|+|q_n|\right)<\infty, \tag{2.1}$$

be satisfied. It has been shown that [10] under the condition (2.1), (1.2) has the solution

$$f_n(z)=\begin{pmatrix} f_n^{(1)}(z) \\ f_n^{(2)}(z) \end{pmatrix}=\alpha_n\left(I+\sum_{m=1}^{\infty}A_{nm}e^{imz}\right)\begin{pmatrix} e^{iz/2} \\ -i \end{pmatrix}e^{inz}, \quad n=1,2,\ldots, \tag{2.2}$$

$$f_0^{(1)}(z)=\alpha_0^{11}\left\{e^{iz/2}\left[1+\sum_{m=1}^{\infty}A_{0m}^{11}e^{imz}\right]-i\sum_{m=1}^{\infty}A_{0m}^{12}e^{imz}\right\}, \tag{2.3}$$

for $\lambda=2\sin(z/2)$ and $z\in\overline{\mathbb{C}}_+:=\{z\in\mathbb{C}:\operatorname{Im}z\geq 0\}$, where

$$\alpha_n=\begin{pmatrix} \alpha_n^{11} & \alpha_n^{12} \\ \alpha_n^{21} & \alpha_n^{22} \end{pmatrix}, \qquad I=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad A_{nm}=\begin{pmatrix} A_{nm}^{11} & A_{nm}^{12} \\ A_{nm}^{21} & A_{nm}^{22} \end{pmatrix}. \tag{2.4}$$

Note that α_n^{ij} and A_{nm}^{ij} $(i,j = 1,2)$ are uniquely expressed in terms of (a_n), (b_n), (p_n), and $(q_n), n \in \mathbb{N}$ as follows

$$
\begin{aligned}
\alpha_n^{11} &= \left[\prod_{k=n+1}^{\infty}(-1)^{n-k} b_k a_{k-1}\right]^{-1}, \\
\alpha_n^{12} &= 0, \\
\alpha_n^{22} &= \left[b_n \prod_{k=n+1}^{\infty}(-1)^{n-k+1} b_k a_{k-1}\right]^{-1}, \\
\alpha_n^{21} &= \alpha_n^{22}\left[p_n + \sum_{k=n+1}^{\infty}(p_k + q_k)\right], \\
A_{n1}^{12} &= -\sum_{k=n+1}^{\infty}(p_k + q_k), \\
A_{n1}^{11} &= \sum_{k=n+1}^{\infty}\left[a_{k+1}a_k + b_k^2 - p_k q_k + (p_k + q_k) A_{k1}^{12} - 2\right], \\
A_{n1}^{22} &= -1 + a_{n+1}a_n + \left(A_{n1}^{12}\right)^2 + A_{n1}^{11}, \\
A_{n1}^{21} &= -\sum_{k=n}^{\infty}\Big\{\left(q_{k+1} + A_{k1}^{12}\right)\Big[a_{k+1}a_k + q_{k+1}(p_{k+1} + q_{k+1}) + q_{k+1}A_{k1}^{12} \\
&\qquad + b_{k+1}^2 + A_{k+1,1}^{11} - 1\Big] - A_{k1}^{12}\left(1 + A_{k1}^{11}\right)\Big\} + \sum_{k=n+1}^{\infty}\left(q_k A_{k1}^{22} - b_k^2 p_k\right), \\
A_{n2}^{12} &= -a_{n+1}a_n\left(q_{n+1} + A_{n1}^{12}\right) + A_{n1}^{12}A_{n1}^{11} + A_{n1}^{12} - A_{n1}^{21}, \\
A_{n2}^{11} &= \sum_{k=n+1}^{\infty}\Big\{\left(b_k^2 - 1\right)A_{k1}^{11} - a_{k+1}a_k\left[\left(q_{k+1} + A_{k1}^{12}\right)A_{k+1,1}^{12} - A_{k+1,1}^{22}\right] \\
&\qquad - \left(p_k - A_{k1}^{12}\right)\left[q_k A_{k1}^{11} + A_{k1}^{12} - A_{k2}^{12}\right] - q_k A_{k1}^{21} + A_{k1}^{12}A_{k2}^{12} - A_{k1}^{22}\Big\}, \\
A_{n2}^{22} &= -a_{n+1}a_n\left(q_{n+1} + A_{n1}^{12}\right)A_{n+1,1}^{12} + a_{n+1}a_n A_{n+1,1}^{22} + A_{n1}^{12}A_{n2}^{12} - A_{n1}^{11} + A_{n2}^{11}, \\
A_{n2}^{21} &= \sum_{k=n}^{\infty}\Big\{A_{k1}^{12}A_{k2}^{11} + A_{k2}^{21} - a_{k+1}a_k\left[\left(q_{k+1} + A_{k1}^{12}\right)A_{k+1,1}^{11} - A_{k+1,1}^{21}\right]\Big\} \\
&\quad - \sum_{k=n+1}^{\infty}\left[\left(q_k + A_{k-1,1}^{12}\right)\left(q_k A_{k2}^{12} - A_{k1}^{11} + A_{k2}^{11}\right) + b_k^2 A_{k2}^{21} - p_k A_{k2}^{22} + A_{k1}^{21}\right],
\end{aligned}
\tag{2.5}
$$

and for $m \geq 3$

$$
\begin{aligned}
A_{nm}^{12} = {} & - a_{n+1} a_n \left[\left(q_{n+1} + A_{n1}^{12} \right) A_{n+1,m-2}^{11} + A_{n+1,m-2}^{21} \right] \\
& + A_{n1}^{12} A_{n,m-1}^{11} + A_{n,m-1}^{12} - A_{n,m-1}^{21}, \\
A_{nm}^{11} = {} & - \sum_{k=n+1}^{\infty} a_{k+1} a_k \left[\left(q_{k+1} + A_{k1}^{12} \right) A_{k+1,m-1}^{12} - A_{k+1,m-1}^{22} \right] \\
& - \sum_{k=n+1}^{\infty} \left(p_k - A_{k1}^{12} \right) \left(q_k A_{k,m-1}^{11} + A_{k,m-1}^{12} - A_{km}^{12} \right) + \sum_{k=n+1}^{\infty} \left(b_k^2 - 1 \right) A_{k,m-1}^{11} \\
& - \sum_{k=n+1}^{\infty} q_k A_{k,m-1}^{21} + \sum_{k=n+1}^{\infty} A_{k1}^{12} A_{km}^{12} - \sum_{k=n+1}^{\infty} A_{k,m-1}^{22}, \\
A_{nm}^{22} = {} & - a_{n+1} a_n \left[\left(q_{n+1} + A_{n1}^{12} \right) A_{n+1,m-1}^{11} - A_{n+1,m-1}^{22} \right] + A_{n1}^{12} A_{nm}^{12} + A_{nm}^{11} - A_{n,m-1}^{11}, \\
A_{nm}^{21} = {} & - \sum_{k=n}^{\infty} a_{k+1} a_k \left[\left(q_{k+1} + A_{k1}^{12} \right) A_{k+1,m-1}^{11} - A_{k+1,m-1}^{21} \right] \\
& - \sum_{k=n+1}^{\infty} \left(q_k - A_{k-1,1}^{12} \right) \left(q_k A_{km}^{21} + A_{k,m-1}^{11} - A_{km}^{22} \right) - \sum_{k=n+1}^{\infty} \left(b_k^2 - 1 \right) A_{km}^{12} \\
& + \sum_{k=n}^{\infty} A_{k1}^{12} A_{km}^{22} + \sum_{k=n+1}^{\infty} q_k A_{km}^{22} + \sum_{k=n}^{\infty} A_{km}^{12} - \sum_{k=n+1}^{\infty} A_{k,m-1}^{21}.
\end{aligned}
\tag{2.6}
$$

Moreover

$$
\left| A_{nm}^{ij} \right| \leq C \sum_{k=n+[|m/2|]}^{\infty} \left(|1 - a_k| + |1 + b_k| + |p_k| + |q_k| \right) \tag{2.7}
$$

holds, where $[|m/2|]$ is the integer part of $m/2$ and $C > 0$ is a constant. Therefore f_n is vector-valued analytic function with respect to z in $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ and continuous in $\overline{\mathbb{C}}_+$ [10]. The solution $f(z) = (f_n(z)) = \begin{pmatrix} f_n^{(1)}(z) \\ f_n^{(2)}(z) \end{pmatrix}$ is called Jost solution of (1.2).

Let us define

$$
F(z) = \left(\gamma_0 + 2\gamma_1 \sin \frac{z}{2} \right) f_1^{(2)}(z) + \left(\beta_0 + 2\beta_1 \sin \frac{z}{2} \right) f_0^{(1)}(z). \tag{2.8}
$$

It follows (2.2) and (2.3) that the function F is analytic in $\mathbb{C}_+$, continuous up to the real axis, and

$$
F(z + 4\pi) = F(z). \tag{2.9}
$$

We denote the set of eigenvalues and spectral singularities of L by $\sigma_d(L)$ and $\sigma_{ss}(L)$, respectively. From the definition of the eigenvalues and spectral singularities we have [10]

$$\begin{aligned} \sigma_d &= \left\{\lambda : \lambda = 2\sin\frac{z}{2},\ z \in P_0,\ F(z) = 0\right\}, \\ \sigma_{ss} &= \left\{\lambda : \lambda = 2\sin\frac{z}{2},\ z \in [0, 4\pi],\ F(z) = 0\right\}, \end{aligned} \tag{2.10}$$

where $P_0 := \{z : z \in \mathbb{C},\ z = x + iy,\ 0 \le x \le 4\pi,\ y > 0\}$. The finiteness of the multiplicities of eigenvalues and spectral singularities has been proven in [10]. Using (2.2), (2.3), and (2.8) we obtain

$$\begin{aligned} F(z) &= \left\{\gamma_0 + \gamma_1\left[(-i)\left(e^{i(z/2)} - e^{-i(z/2)}\right)\right]\right\} f_1^{(2)}(z) \\ &\quad + \left\{\beta_0 + \beta_1\left[(-i)\left(e^{i(z/2)} - e^{-i(z/2)}\right)\right]\right\} f_0^{(1)}(z) \\ &= i\alpha_0^{11}\beta_1 + \left(\gamma_1\alpha_1^{22} + \alpha_0^{11}\beta_0\right)e^{i(z/2)} + i\left(-\gamma_0\alpha_1^{22} + \gamma_1\alpha_1^{22} - \alpha_0^{11}\beta_1\right)e^{iz} \\ &\quad + \left(\gamma_0\alpha_1^{21} - \gamma_1\alpha_1^{22}\right)e^{i(3z/2)} - i\gamma_1\alpha_1^{21}e^{2iz} \\ &\quad + \sum_{m=1}^{\infty}\alpha_0^{11}\beta_1 A_{0m}^{12}e^{i(m-(1/2))z} + i\sum_{m=1}^{\infty}\left(-\alpha_0^{11}\beta_0 A_{0m}^{12} + \alpha_0^{11}\beta_1 A_{0m}^{11}\right)e^{imz} \\ &\quad + \sum_{m=1}^{\infty}\left(\gamma_1\alpha_1^{21}A_{1m}^{12} + \gamma_1\alpha_1^{22}A_{1m}^{22} + \alpha_0^{11}\beta_0 A_{0m}^{11} - \alpha_0^{11}\beta_1 A_{0m}^{12}\right)e^{i(m+(1/2))z} \\ &\quad + i\sum_{m=1}^{\infty}\left(-\gamma_0\alpha_1^{21}A_{1m}^{12} - \gamma_0\alpha_1^{22}A_{1m}^{22} + \gamma_1\alpha_1^{21}A_{1m}^{11} + \gamma_1\alpha_1^{22}A_{1m}^{21}\right. \\ &\qquad\qquad \left. -\alpha_0^{11}\beta_1 A_{0m}^{11}\right)e^{i(m+1)z} \\ &\quad + \sum_{m=1}^{\infty}\left(\gamma_0\alpha_1^{21}A_{1m}^{11} + \gamma_0\alpha_1^{22}A_{1m}^{21} - \gamma_1\alpha_1^{21}A_{1m}^{12} - \gamma_1\alpha_1^{22}A_{1m}^{22}\right)e^{i(m+(3/2))z} \\ &\quad + i\sum_{m=1}^{\infty}\left(-\gamma_1\alpha_1^{21}A_{1m}^{11} - \gamma_1\alpha_1^{22}A_{1m}^{21}\right)e^{i(m+2)z}. \\ &\quad + i\sum_{m=1}^{\infty}\left(-\gamma_1\alpha_1^{21}A_{1m}^{11} - \gamma_1\alpha_1^{22}A_{1m}^{21}\right)e^{i(m+2)z}. \end{aligned} \tag{2.11}$$

Definition 2.1. The multiplicity of a zero of F in $P := P_0 \cup [0, 4\pi]$ is called the multiplicity of the corresponding eigenvalue or spectral singularity of the BVP (1.2), (1.3).

3. Principal Functions

In this section we also assume that (2.1) holds.

Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ and $\lambda_{k+1}, \lambda_{k+2}, \ldots, \lambda_\nu$ denote the zeros of F in P_0 and $[0, 4\pi]$ with multiplicities $m_1, m_2, \ldots, m_k$ and $m_{k+1}, m_{k+2}, \ldots, m_\nu$, respectively.

Let us define $\ell := \binom{\tilde{\ell}}{\hat{\ell}}$ where

$$\begin{aligned}\left(\tilde{\ell}y\right)_n &= a_{n+1}y_{n+1}^{(2)} + b_n y_n^{(2)} + p_n y_n^{(1)}, \quad n \in \mathbb{N},\\ \left(\hat{\ell}y\right)_n &= a_{n-1}y_{n-1}^{(1)} + b_n y_n^{(1)} + q_n y_n^{(2)}, \quad n \in \mathbb{N}.\end{aligned} \tag{3.1}$$

Definition 3.1. Let $\lambda = \lambda_0$ be an eigenvalue of L. If the vectors y_n, $d/(d\lambda y_n), d^2/d\lambda^2 y_n, \ldots, d^\nu/d\lambda^\nu y_n$,

$$\frac{d^j}{d\lambda^j}y := \left\{\frac{d^j}{d\lambda^j}y_n\right\}_{n\in\mathbb{N}}, \quad (j = 0, 1, \ldots, \nu;\ n \in \mathbb{N}), \tag{3.2}$$

satisfy the equations

$$\begin{gathered}(\ell y)_n - \lambda_0 y_n = 0,\\ \left(\ell\left(\frac{d^j}{d\lambda^j}y\right)\right)_n - \lambda_0\frac{d^j}{d\lambda^j}y_n - \frac{d^{j-1}}{d\lambda^{j-1}}y_n = 0, \quad j = 1, 2, \ldots, \nu,\ n \in \mathbb{N},\end{gathered} \tag{3.3}$$

then the vector y_n is called the eigenvector corresponding to the eigenvalue $\lambda = \lambda_0$ of L. The vectors $(d/d\lambda)y_n, (d^2/d\lambda^2)y_n, \ldots, (d^\nu/d\lambda^\nu)y_n$ are called the associated vectors corresponding to $\lambda = \lambda_0$. The eigenvector and the associated vectors corresponding to $\lambda = \lambda_0$ are called the principal vectors of the eigenvalue $\lambda = \lambda_0$. The principal vectors of the spectral singularities of L are defined similarly.

We define the vectors

$$\begin{gathered}\frac{d^j}{d\lambda^j}V_n(\lambda_i) = \begin{pmatrix}\dfrac{1}{j!}\left\{\dfrac{d^j}{d\lambda^j}E_n^{(1)}(\lambda)\right\}_{\lambda=\lambda_i}\\ \dfrac{1}{j!}\left\{\dfrac{d^j}{d\lambda^j}E_n^{(2)}(\lambda)\right\}_{\lambda=\lambda_i}\end{pmatrix},\\ n \in \mathbb{N}, \quad j = 0, 1, \ldots, m_i - 1, \quad i = 1, 2, \ldots, k, k+1, \ldots, \nu,\end{gathered} \tag{3.4}$$

where $\lambda = 2\sin(z/2)$ and

$$E_n(\lambda) = \begin{pmatrix}E_n^{(1)}(\lambda)\\ E_n^{(2)}(\lambda)\end{pmatrix} := f_n\left(2\arcsin\frac{\lambda}{2}\right) = \begin{pmatrix}f_n^{(1)}\left(2\arcsin\dfrac{\lambda}{2}\right)\\ f_n^{(2)}\left(2\arcsin\dfrac{\lambda}{2}\right)\end{pmatrix}. \tag{3.5}$$

If

$$y(\lambda) = \{y_n(\lambda)\} := \begin{pmatrix} y_n^{(1)}(\lambda) \\ y_n^{(2)}(\lambda) \end{pmatrix}_{n\in\mathbb{N}} \tag{3.6}$$

is a solution of (1.2), then

$$\frac{d^j}{d\lambda^j} y(\lambda) = \left\{ \left(\frac{d^j}{d\lambda^j} \right) y_n(\lambda) \right\}_{n\in\mathbb{N}} := \begin{Bmatrix} \left(\dfrac{d^j}{d\lambda^j} \right) y_n^{(1)}(\lambda) \\ \left(\dfrac{d^j}{d\lambda^j} \right) y_n^{(2)}(\lambda) \end{Bmatrix} \tag{3.7}$$

satisfies

$$\begin{aligned} &\begin{pmatrix} a_{n-1} \dfrac{d^j}{d\lambda^j} y_{n+1}^{(2)}(\lambda) + b_n \dfrac{d^j}{d\lambda^j} y_n^{(2)}(\lambda) + p_n \dfrac{d^j}{d\lambda^j} y_n^{(1)}(\lambda) \\ a_{n-1} \dfrac{d^j}{d\lambda^j} y_{n-1}^{(1)}(\lambda) + b_n \dfrac{d^j}{d\lambda^j} y_n^{(1)}(\lambda) + q_n \dfrac{d^j}{d\lambda^j} y_n^{(2)}(\lambda) \end{pmatrix} \\ &\quad = \begin{pmatrix} \lambda \dfrac{d^j}{d\lambda^j} y_n^{(1)}(\lambda) + j \dfrac{d^{j-1}}{d\lambda^{j-1}} y_n^{(1)}(\lambda) \\ \lambda \dfrac{d^j}{d\lambda^j} y_n^{(2)}(\lambda) + j \dfrac{d^{j-1}}{d\lambda^{j-1}} y_n^{(2)}(\lambda) \end{pmatrix}. \end{aligned} \tag{3.8}$$

From (3.4) and (3.8) we get that

$$\begin{gathered} (\ell V(\lambda_i))_n - \lambda_0 V_n(\lambda_i) = 0, \\ \left(\ell \left(\frac{d^j}{d\lambda^j} V(\lambda_i) \right) \right)_n - \lambda_0 \frac{d^j}{d\lambda^j} V_n(\lambda_i) - \frac{d^{j-1}}{d\lambda^{j-1}} V_n(\lambda_i) = 0, \\ n \in \mathbb{N}, \quad j = 1, 2, \ldots, m_i - 1, \quad i = 1, 2, \ldots, \nu. \end{gathered} \tag{3.9}$$

The vectors $d^j/d\lambda^j V_n(\lambda_i)$, $j = 0, 1, 2, \ldots, m_i - 1$, $i = 1, 2, \ldots, k$ and $d^j/d\lambda^j V_n(\lambda_i)$, $j = 0, 1, 2, \ldots, m_i - 1$, $i = k+1, k+2, \ldots, \nu$ are the principal vectors of eigenvalues and spectral singularities of L, respectively.

Theorem 3.2.

$$\begin{gathered} \frac{d^j}{d\lambda^j} V_n(\lambda_i) \in \ell_2\left(\mathbb{N}, \mathbb{C}^2\right), \quad j = 0, 1, 2, \ldots, m_i - 1, \quad i = 1, 2, \ldots, k, \\ \frac{d^j}{d\lambda^j} V_n(\lambda_i) \notin \ell_2\left(\mathbb{N}, \mathbb{C}^2\right), \quad j = 0, 1, 2, \ldots, m_i - 1, \quad i = k+1, k+2, \ldots, \nu. \end{gathered} \tag{3.10}$$

Proof. Using (3.5) we get that

$$\begin{aligned}\left\{\frac{d^j}{d\lambda^j}E_n^{(1)}(\lambda)\right\}_{\lambda=\lambda_i} &= \sum_{t=0}^{j} C_t\left\{\frac{d^t}{d\lambda^t}f_n^{(1)}(z)\right\}_{z=z_i}, \quad n\in\mathbb{N},\\ \left\{\frac{d^j}{d\lambda^j}E_n^{(2)}(\lambda)\right\}_{\lambda=\lambda_i} &= \sum_{t=0}^{j} D_t\left\{\frac{d^t}{d\lambda^t}f_n^{(2)}(z)\right\}_{z=z_i}, \quad n\in\mathbb{N},\end{aligned} \tag{3.11}$$

where $\lambda_i = 2\sin z_i/2$, $z_i \in P = P_0 \cup [0,4]$, $i = 1,2,\ldots,k$ and C_t, D_t are constant depending on λ. From (2.2) we obtain that

$$\begin{aligned}\left\{\frac{d^t}{d\lambda^t}f_n^{(1)}(z)\right\}_{z=z_i} &= \alpha_n^{11} i^t\left(n+\frac{1}{2}\right)^t e^{iz_i(n+(1/2))}\\ &\quad+\sum_{m=1}^{\infty}\alpha_n^{11}\left\{A_{nm}^{11} i^t\left(m+n+\frac{1}{2}\right)^t e^{i(m+n+(1/2))z_i}\right.\\ &\quad\left.-A_{nm}^{12} i^{t+1}(m+n)^t e^{i(m+n)z_i}\right\},\end{aligned} \tag{3.12}$$

$$\begin{aligned}\left\{\frac{d^t}{d\lambda^t}f_n^{(2)}(z)\right\}_{z=z_i} &= \alpha_n^{21} i^t\left(n+\frac{1}{2}\right)^t e^{iz_i(n+(1/2))} - i(in)^t\alpha_n^{22}e^{inz_i}\\ &\quad+\sum_{m=1}^{\infty}\alpha_n^{21}\left\{A_{nm}^{11} i^t\left(m+n+\frac{1}{2}\right)^t e^{i(m+n+(1/2))z_i}\right.\\ &\quad\left.-A_{nm}^{12} i^{t+1}(m+n)^t e^{i(m+n)z_i}\right\}\\ &\quad+\sum_{m=1}^{\infty}\alpha_n^{22}\left\{A_{nm}^{21} i^t\left(m+n+\frac{1}{2}\right)^t e^{i(m+n+(1/2))z_i}\right.\\ &\quad\left.-A_{nm}^{22} i^{t+1}(m+n)^t e^{i(m+n)z_i}\right\}.\end{aligned} \tag{3.13}$$

For the principal vectors $(d^j/d\lambda^j)V_n(\lambda_i) = \{(d^j/d\lambda^j)V_n(\lambda_i)\}_{n\in\mathbb{N}}$, $j = 0,1,\ldots,m_i-1$, $i = 1,2,\ldots,k$ corresponding to the eigenvalues of L we get

$$\begin{aligned}&\frac{1}{j!}\left\{\frac{d^j}{d\lambda^j}E_n^{(1)}(\lambda)\right\}_{\lambda=\lambda_i} = \frac{1^j}{j!}\sum_{t=0}^{j}C_t\left\{\frac{d^t}{d\lambda^t}f_n^{(1)}(z_i)\right\},\\ &\qquad j=0,1,\ldots,m_i-1,\quad i=1,2,\ldots,k,\\ &\frac{1}{j!}\left\{\frac{d^j}{d\lambda^j}E_n^{(2)}(\lambda)\right\}_{\lambda=\lambda_i} = \frac{1^j}{j!}\sum_{t=0}^{j}D_t\left\{\frac{d^t}{d\lambda^t}f_n^{(2)}(z_i)\right\},\\ &\qquad j=0,1,\ldots,m_i-1,\quad i=1,2,\ldots,k.\end{aligned} \tag{3.14}$$

Since $\operatorname{Im}\lambda_i > 0$ for $i = 1, 2, \ldots, k$ from (3.14) we obtain that

$$\begin{aligned}
\left\| \frac{d^j}{d\lambda^j} V_n \right\|^2 &= \sum_{n=1}^{\infty} \left(\left| \frac{1}{j!} \left\{ \frac{d^j}{d\lambda^j} E_n^{(1)}(\lambda) \right\}_{\lambda=\lambda_i} \right|^2 + \left| \frac{1}{j!} \left\{ \frac{d^j}{d\lambda^j} E_n^{(2)}(\lambda) \right\}_{\lambda=\lambda_i} \right|^2 \right) \\
&= \left(\frac{1}{j!} \right)^2 \sum_{n=1}^{\infty} \left(\left| \sum_{t=0}^{j} C_t \left\{ \frac{d^t}{d\lambda^t} f_n^{(1)}(z_i) \right\} \right|^2 + \left| \sum_{t=0}^{j} D_t \left\{ \frac{d^t}{d\lambda^t} f_n^{(2)}(z_i) \right\} \right|^2 \right) \\
&\leq \left(\frac{1}{j!} \right)^2 \sum_{n=1}^{\infty} \left\{ \left(\sum_{t=0}^{j} \left| C_t \left\{ \frac{d^t}{d\lambda^t} f_n^{(1)}(z_i) \right\} \right|^2 \right) + \left(\sum_{t=0}^{j} \left| D_t \left\{ \frac{d^t}{d\lambda^t} f_n^{(2)}(z_i) \right\} \right|^2 \right) \right\} \\
&\leq \left(\frac{1}{j!} \right)^2 \left(\sum_{n=1}^{\infty} \sum_{t=0}^{j} \max\{|C_t|, |D_t|\} \left(\left| \left\{ \frac{d^t}{d\lambda^t} f_n^{(1)}(z_i) \right\} \right| + \left| \left\{ \frac{d^t}{d\lambda^t} f_n^{(2)}(z_i) \right\} \right| \right) \right)^2,
\end{aligned} \tag{3.15}$$

or

$$\begin{aligned}
&\left\| \frac{d^j}{d\lambda^j} V_n \right\|^2 \\
&\leq \left(\frac{1}{j!} \right)^2 \Bigg\{ \sum_{n=1}^{\infty} \Bigg[\sum_{t=0}^{j} \max\{|C_t|, |D_t|\} \\
&\qquad \times \left\{ \left(\left| \alpha_n^{11} \right| + \left| \alpha_n^{21} \right| \right) \left(\left| n + \frac{1}{2} \right|^t e^{-(n+(1/2)) \operatorname{Im} z_i} \right) + \left| \alpha_n^{22} \right| |n|^t e^{-n \operatorname{Im} z_i} \right\} \\
&\qquad + \sum_{t=0}^{j} \max\{|C_t|, |D_t|\} \Bigg\{ \sum_{m=1}^{\infty} \left(\left| \alpha_n^{11} \right| + \left| \alpha_n^{21} \right| \right) \Bigg(\left| A_{nm}^{11} \right| \left| m + n + \frac{1}{2} \right|^t e^{-(m+n+(1/2)) \operatorname{Im} z_i} \\
&\qquad\qquad + \left| A_{nm}^{12} \right| |m+n|^t e^{-(m+n) \operatorname{Im} z_i} \Bigg) \Bigg\} \\
&\qquad + \sum_{t=0}^{j} |D_t| \Bigg\{ \sum_{m=1}^{\infty} \left| \alpha_n^{22} \right| \Bigg(\left| A_{nm}^{21} \right| \left| m + n + \frac{1}{2} \right|^t e^{-(m+n+(1/2)) \operatorname{Im} z_i} + \left| A_{nm}^{22} \right| |m+n|^t \\
&\qquad\qquad \times e^{-(m+n) \operatorname{Im} z_i} \Bigg) \Bigg\} \Bigg] \Bigg\}^2.
\end{aligned} \tag{3.16}$$

From (3.16),

$$\begin{aligned}
&\left(\frac{1}{j!} \right)^2 \sum_{n=1}^{\infty} \sum_{t=0}^{j} \max\{|C_t|, |D_t|\} \left\{ \begin{array}{c} \left(\left| \alpha_n^{11} \right| + \left| \alpha_n^{21} \right| \right) \left(\left| n + \dfrac{1}{2} \right|^t e^{-(n+(1/2)) \operatorname{Im} z_i} \right) \\ + \left| \alpha_n^{22} \right| |n|^t e^{-n \operatorname{Im} z_i} \end{array} \right\} \\
&\quad \leq \frac{A}{(j!)^2} \sum_{n=1}^{\infty} \left(1 + \left(n + \frac{1}{2} \right) + \left(n + \frac{1}{2} \right)^2 + \cdots + \left(n + \frac{1}{2} \right)^j \right) e^{-(n+(1/2)) \operatorname{Im} z_i}
\end{aligned}$$

$$
\begin{aligned}
&+\left(1+n+n^2+\cdots+n^j\right)e^{-n\operatorname{Im} z_i}\\
&\le \frac{A(j+1)^2}{(j!)^2}\sum_{n=1}^{\infty}\left[\left(n+\frac{1}{2}\right)^j e^{-(n+(1/2))\operatorname{Im} z_i}+n^j e^{-n\operatorname{Im} z_i}\right]<\infty.
\end{aligned}
\tag{3.17}
$$

Holds, where

$$
A=\max\{|C_t|,|D_t|\}\max\left\{\left(\left|\alpha_n^{11}\right|+\left|\alpha_n^{21}\right|\right),\left|\alpha_n^{22}\right|\right\}. \tag{3.18}
$$

Now we define the function

$$
\begin{aligned}
g_n(z)={}&\sum_{t=0}^{j}\max\{|C_t|,|D_t|\}\left\{\sum_{m=1}^{\infty}\left(\left|\alpha_n^{11}\right|+\left|\alpha_n^{21}\right|\right)\left(\left|A_{nm}^{11}\right|\left|m+n+\frac{1}{2}\right|^t e^{-(m+n+(1/2))\operatorname{Im} z_i}\right.\right.\\
&\left.\left.+\left|A_{nm}^{12}\right||m+n|^t e^{-(m+n)\operatorname{Im} z_i}\right)\right\}\\
&+\sum_{t=0}^{j}|D_t|\left\{\sum_{m=1}^{\infty}\left|\alpha_n^{22}\right|\left(\left|A_{nm}^{21}\right|\left|m+n+\frac{1}{2}\right|^t e^{-(m+n+(1/2))\operatorname{Im} z_i}\right.\right.\\
&\left.\left.+\left|A_{nm}^{22}\right||m+n|^t e^{-(m+n)\operatorname{Im} z_i}\right)\right\}.
\end{aligned}
\tag{3.19}
$$

So we get

$$
\begin{aligned}
&\left(\frac{1}{j!}\right)^2\sum_{n=1}^{\infty}\left[\sum_{t=0}^{j}\max\{|C_t|,|D_t|\}\left\{\sum_{m=1}^{\infty}\left(\left|\alpha_n^{11}\right|+\left|\alpha_n^{21}\right|\right)\left(\left|A_{nm}^{11}\right|\left|m+n+\frac{1}{2}\right|^t e^{-(m+n+(1/2))\operatorname{Im} z_i}\right.\right.\right.\\
&\left.\left.+\left|A_{nm}^{12}\right||m+n|^t e^{-(m+n)\operatorname{Im} z_i}\right)\right\}\\
&+\sum_{t=0}^{j}|D_t|\left\{\sum_{m=1}^{\infty}\left|\alpha_n^{22}\right|\left(\left|A_{nm}^{21}\right|\left|m+n+\frac{1}{2}\right|^t e^{-(m+n+(1/2))\operatorname{Im} z_i}\right.\right.\\
&\left.\left.\left.+\left|A_{nm}^{22}\right||m+n|^t e^{-(m+n)\operatorname{Im} z_i}\right)\right\}\right]\\
&=\left(\frac{1}{j!}\right)^2\sum_{n=1}^{\infty}g_n(z).
\end{aligned}
\tag{3.20}
$$

Using the boundness of A_{nm}^{ij} and α_n^{ij}, $i, j = 1, 2$ we obtain that

$$g_n(z) \leq \max\{|C_t|, |D_t|\} M \sum_{t=0}^{j} \sum_{m=1}^{\infty} \left\{ \left| m + n + \frac{1}{2} \right|^t e^{-(m+n+(1/2)) \operatorname{Im} z_i} \right. \\ \left. + |m+n|^t e^{-(m+n) \operatorname{Im} z_i} \right\}, \tag{3.21}$$

where

$$M = \max\left\{ \left(\left|\alpha_n^{11}\right| + \left|\alpha_n^{21}\right| \right) \left|A_{nm}^{11}\right|, \left|\alpha_n^{22}\right| \left|A_{nm}^{21}\right|, \right. \\ \left. \left(\left|\alpha_n^{11}\right| + \left|\alpha_n^{21}\right| \right) \left|A_{nm}^{12}\right|, \left|\alpha_n^{22}\right| \left|A_{nm}^{22}\right| \right\}. \tag{3.22}$$

If we take $\max\{|C_t|, |D_t|\} M = N$, we can write

$$\begin{aligned} g_n(z) &\leq N \sum_{t=0}^{j} e^{-n \operatorname{Im} z_i} \sum_{m=1}^{\infty} \left\{ \left(m + n + \frac{1}{2} \right)^t e^{-m \operatorname{Im} z_i} + (m+n)^t e^{-m \operatorname{Im} z_i} \right\} \\ &= N e^{-n \operatorname{Im} z_i} \left\{ \sum_{m=1}^{\infty} 2 e^{-m \operatorname{Im} z_i} + \sum_{m=1}^{\infty} e^{-m \operatorname{Im} z_i} \left(\left(m + n + \frac{1}{2} \right) + (m+n) \right) + \cdots \right. \\ &\qquad \left. + \sum_{m=1}^{\infty} e^{-m \operatorname{Im} z_i} \left(\left(m + n + \frac{1}{2} \right)^j + (m+n)^j \right) \right\} \\ &\leq N e^{-n \operatorname{Im} z_i} \sum_{m=1}^{\infty} \sum_{t=0}^{j} e^{-m \operatorname{Im} z_i} \left(\left(m + n + \frac{1}{2} \right)^t + (m+n)^t \right) \\ &\leq B e^{-n \operatorname{Im} z_i}, \end{aligned} \tag{3.23}$$

where

$$B = A \sum_{m=1}^{\infty} \sum_{t=0}^{j} e^{-m \operatorname{Im} z_i} \left(\left(m + n + \frac{1}{2} \right)^t + (m+n)^t \right). \tag{3.24}$$

Therefore we have

$$\left(\frac{1}{j!} \right)^2 \sum_{n=1}^{\infty} g_n(z) \leq \left(\frac{1}{j!} \right)^2 \sum_{n=1}^{\infty} B e^{-n \operatorname{Im} z_i} < \infty. \tag{3.25}$$

From (3.17) and (3.25) $d^j / d\lambda^j V_n(\lambda_i) \in \ell_2(\mathbb{N}, \mathbb{C}^2)$, $j = 0, 1, \ldots, m_i - 1$, $i = 1, 2, \ldots, k$.

On the other hand, since $\operatorname{Im} z_i = 0$ for $j = 0, 1, \ldots, m_i - 1$; $i = k+1, k+2, \ldots, \nu$ using (3.12) we find that

$$\sum_{n=1}^{\infty} \left| \alpha_n^{11} i^t \left(n + \frac{1}{2} \right)^t e^{iz_i(n+(1/2))} \right|^2 = \infty, \tag{3.26}$$

but the other terms in (3.12) belongs $\ell_2(\mathbb{N}, \mathbb{C}^2)$, so $d^j/(d\lambda^j) E_n^{(1)}(\lambda) \notin \ell_2(\mathbb{N}, \mathbb{C}^2)$. Similarly from (3.13) we get $d^j/(d\lambda^j) E_n^{(2)}(\lambda) \notin \ell_2(\mathbb{N}, \mathbb{C}^2)$, then we obtain that $d^j/(d\lambda^j) V_n(\lambda_i) \notin \ell_2(\mathbb{N}, \mathbb{C}^2)$, $j = 0, 1, \ldots, m_i - 1; i = k+1, k+2, \ldots, \nu$. □

Let us introduce Hilbert space $j = 0, 1, 2, \ldots$

$$H_{-j}(\mathbb{N}) = \left\{ y = \begin{pmatrix} y_n^{(1)} \\ y_n^{(2)} \end{pmatrix} : \sum_{n \in \mathbb{N}} (1 + |n|)^{-2j} \left(\left| y_n^{(1)} \right|^2 + \left| y_n^{(2)} \right|^2 \right) < \infty \right\}, \tag{3.27}$$

with

$$\|y\|_{-j}^2 = \sum_{n \in \mathbb{N}} (1 + |n|)^{-2j} \left(\left| y_n^{(1)} \right|^2 + \left| y_n^{(2)} \right|^2 \right). \tag{3.28}$$

Theorem 3.3. *$d^j/(d\lambda^j) V_n(\lambda_i) \in H_{-(j+1)}(\mathbb{N})$, $j = 0,\ 1,\ 2, \ldots, m_i - 1$, $i = k+1,\ k+2, \ldots, \nu$.*

Proof. Using (3.4), (3.14) we have

$$\begin{aligned}
&\sum_{n \in \mathbb{N}} (1 + |n|)^{-2(j+1)} \left(\left| \frac{1}{j!} \left\{ \frac{d^j}{d\lambda^j} E_n^{(1)}(\lambda) \right\}_{\lambda = \lambda_i} \right|^2 + \left| \frac{1}{j!} \left\{ \frac{d^j}{d\lambda^j} E_n^{(2)}(\lambda) \right\}_{\lambda = \lambda_i} \right|^2 \right) \\
&\quad = \sum_{n \in \mathbb{N}} \frac{(1 + |n|)^{-2(j+1)}}{(j!)^2} \left\{ \left| \sum_{t=0}^{j} C_t \left\{ \frac{d^t}{d\lambda^t} f_n^{(1)}(z_i) \right\} \right|^2 + \left| \sum_{t=0}^{j} D_t \left\{ \frac{d^t}{d\lambda^t} f_n^{(2)}(z_i) \right\} \right|^2 \right\} \\
&\quad \le \frac{1}{(j!)^2} \sum_{n=1}^{\infty} (1 + |n|)^{-2(j+1)} \left\{ \left(\sum_{t=0}^{j} \left| C_t \left\{ \frac{d^t}{d\lambda^t} f_n^{(1)}(z_i) \right\} \right| \right)^2 + \left(\sum_{t=0}^{j} \left| D_t \left\{ \frac{d^t}{d\lambda^t} f_n^{(2)}(z_i) \right\} \right| \right)^2 \right\},
\end{aligned} \tag{3.29}$$

for $j = 0, 1, 2, \ldots, m_i - 1$, $i = k+1, k+2, \nu$. Since $\operatorname{Im} z_i = 0$, using (3.29) we obtain

$$\begin{aligned}
&\sum_{n \in \mathbb{N}} (1 + |n|)^{-2(j+1)} \frac{1}{(j!)^2} \left(\sum_{t=0}^{j} \left| C_t \left\{ \frac{d^t}{d\lambda^t} f_n^{(1)}(z_i) \right\} \right| \right)^2 \\
&\quad = \frac{1}{(j!)^2} \sum_{n=1}^{\infty} \left\{ \sum_{t=0}^{j} (1 + |n|)^{-(j+1)} \left(n + \frac{1}{2} \right)^t \left| \alpha_n^{11} \right| |C_t| \right. \\
&\qquad\qquad \left. + \sum_{t=0}^{j} |C_t| \left| \alpha_n^{11} \right| (1 + |n|)^{-(j+1)} \sum_{m=1}^{\infty} \left| A_{nm}^{11} \right| \left(m + n + \frac{1}{2} \right)^t + \left| A_{nm}^{12} \right| (m + n)^t \right\}^2
\end{aligned}$$

$$= \frac{1}{(j!)^2}\sum_{n=1}^{\infty}\left\{\left(\sum_{t=0}^{j}(1+|n|)^{-(j+1)}\left(n+\frac{1}{2}\right)^t\left|\alpha_n^{11}\right||C_t|\right)^2 \right.$$

$$+2(1+|n|)^{-2(j+1)}\left|\alpha_n^{11}\right|^2\left[\sum_{t=0}^{j}\left(n+\frac{1}{2}\right)^t|C_t|\right]$$

$$\times\left[\sum_{t=0}^{j}|C_t|\sum_{m=1}^{\infty}\left|A_{nm}^{11}\right|\left(m+n+\frac{1}{2}\right)^t+\left|A_{nm}^{12}\right|(m+n)^t\right]$$

$$\left.+\left(\sum_{t=0}^{j}|C_t|(1+|n|)^{-(j+1)}\left|\alpha_n^{11}\right|\sum_{m=1}^{\infty}\left|A_{nm}^{11}\right|\left(m+n+\frac{1}{2}\right)^t+\left|A_{nm}^{12}\right|(m+n)^t\right)^2\right\}. \tag{3.30}$$

Using (3.30), (2.1), and (2.7) we first obtain that

$$\left(\sum_{t=0}^{j}|C_t|\left|\alpha_n^{11}\right|(1+|n|)^{-(j+1)}\sum_{m=1}^{\infty}\left(\left|A_{nm}^{11}\right|\left(m+\mathrm{n}+\frac{1}{2}\right)^t+\left|A_{nm}^{12}\right|(m+n)^t\right)\right)^2$$

$$\leq 4\left(\sum_{t=0}^{j}|C_t|\left|\alpha_n^{11}\right|_{m=1}^{\infty}(1+|n|)^{-(j+1)}\left(m+n+\frac{1}{2}\right)^t C\right.$$

$$\left.\times\sum_{j=n+[m/2]}^{\infty}\left(\left|1-a_j\right|+\left|1+b_j\right|+\left|p_j\right|+\left|q_j\right|\right)e^{-\varepsilon j^{\delta}}e^{\varepsilon j^{\delta}}\right)^2$$

$$\leq 4\left\{\sum_{t=0}^{j}\left|\alpha_n^{11}\right|\sum_{m=1}^{\infty}(1+|n|)^{-(j+1)}\left(m+n+\frac{1}{2}\right)^t C\exp\left(-\varepsilon\left(\frac{n+m}{4}\right)^{\delta}\right)\right.$$

$$\left.\times\sum_{j=n+[m/2]}^{\infty}e^{\varepsilon j^{\delta}}\left(\left|1-a_j\right|+\left|1+b_j\right|+\left|p_j\right|+\left|q_j\right|\right)\right\}^2$$

$$\leq C_1\left(\sum_{t=0}^{j}(1+|n|)^{-(j+1)}\sum_{m=1}^{\infty}\left(m+n+\frac{1}{2}\right)^t\exp\left(-\varepsilon\left(\frac{n+m}{4}\right)^{\delta}\right)\right)^2$$

$$\leq C_1\left(\sum_{t=0}^{j}(1+|n|)^{-(j+1)}\sum_{m=1}^{\infty}\left(m+n+\frac{1}{2}\right)^t\exp\left(-\varepsilon\left(\frac{n+m}{4}\right)^{1/2}\right)\right)^2$$

$$\leq C_1\left(\sum_{t=0}^{j}(1+|n|)^{-(j+1)}\sum_{m=1}^{\infty}\left(m+n+\frac{1}{2}\right)^t\exp\left(-\varepsilon\frac{\sqrt{2}}{4}\left(n^{\frac{1}{2}}+m^{\frac{1}{2}}\right)\right)\right)^2$$

$$
\begin{aligned}
&= C_1(1+|n|)^{-2(j+1)} \exp\left(-\varepsilon\frac{\sqrt{2}}{4}n^{1/2}\right)\left(\sum_{t=0}^{j}\sum_{m=1}^{\infty}\left(m+n+\frac{1}{2}\right)^t \exp\left(-\varepsilon\frac{\sqrt{2}}{4}m^{1/2}\right)\right)^2 \\
&= G\exp\left(-\varepsilon\frac{\sqrt{2}}{4}n\frac{1}{2}\right)(1+|n|)^{-2(j+1)},
\end{aligned}
\tag{3.31}
$$

where

$$
\begin{aligned}
C_1 &= \left(2C\left|\alpha_n^{11}\right|\sum_{j=n+[m/2]}^{\infty} e^{\varepsilon j^{\delta}}\left(\left|1-a_j\right|+\left|1+b_j\right|+\left|p_j\right|+\left|q_j\right|\right)\right)^2, \\
G &= C_1\left(\sum_{t=0}^{j}\sum_{m=1}^{\infty}\left(m+n+\frac{1}{2}\right)^t \exp\left(-\varepsilon\frac{\sqrt{2}}{4}m^{1/2}\right)\right)^2.
\end{aligned}
\tag{3.32}
$$

So we get from (3.31)

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\left(\sum_{t=0}^{j}|C_t|(1+|n|)^{-(j+1)}\left|\alpha_n^{11}\right|\sum_{m=1}^{\infty}\left|A_{nm}^{11}\right|\left(m+n+\frac{1}{2}\right)^t+\left|A_{nm}^{12}\right|(m+n)^t\right)^2 \\
&\quad\le G\sum_{n=1}^{\infty}\exp\left(-\varepsilon\frac{\sqrt{2}}{4}n^{1/2}\right)(1+|n|)^{-2(j+1)}<\infty.
\end{aligned}
\tag{3.33}
$$

Secondly, using (3.30) and (3.31) we obtain that

$$
\begin{aligned}
&\sum_{n=1}^{\infty}2\left\{\left[\sum_{t=0}^{j}\left|\alpha_n^{11}\right||C_t|(1+|n|)^{-(j+1)}\left(n+\frac{1}{2}\right)^t\right]\right. \\
&\qquad\left.\times\left[\sum_{t=0}^{j}|C_t|\left|\alpha_n^{11}\right|\sum_{m=1}^{\infty}(1+|n|)^{-(j+1)}\left(\left(m+n+\frac{1}{2}\right)^t\left|A_{nm}^{11}\right|+(m+n)^t\left|A_{nm}^{12}\right|\right)\right]\right\} \\
&\quad\le\sum_{n=1}^{\infty}\left[\sum_{t=0}^{j}(1+|n|)^{-2(j+1)}\left(n+\frac{1}{2}\right)^t\exp\left(-\varepsilon\frac{\sqrt{2}}{4}n^{1/2}\right)G^{1/2}\right]<\infty,
\end{aligned}
\tag{3.34}
$$

and also the first part of the (3.31) obviously convergent so, we get from (3.33) and (3.34)

$$
\sum_{n\in\mathbb{N}}(1+|n|)^{-2(j+1)}\frac{1}{(j!)^2}\left(\sum_{t=0}^{j}\left|C_t\left\{\frac{d^t}{d\lambda^t}f_n^{(1)}(z_i)\right\}\right|\right)^2<\infty,
\tag{3.35}
$$

and similarly

$$\sum_{n\in\mathbb{N}}(1+|n|)^{-2(j+1)}\frac{1}{(j!)^2}\left(\sum_{t=0}^{j}\left|D_t\left\{\frac{d^t}{d\lambda^t}f_n^{(2)}(z_i)\right\}\right|\right)^2<\infty. \tag{3.36}$$

Finally $d^j/(d\lambda^j)V_n(\lambda_i)\in H_{-(j+1)}(\mathbb{N})$, $j=0,1,2,\dots,m_i-1$, $i=k+1,k+2,\dots,\nu$. □

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Research Article

Existence, Stationary Distribution, and Extinction of Predator-Prey System of Prey Dispersal with Stochastic Perturbation

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We consider a predator-prey model in which the preys disperse among n patches ($n \geq 2$) with stochastic perturbation. We show that there is a unique positive solution and find out the sufficient conditions for the extinction to the system with any given positive initial value. In addition, we investigate that there exists a stationary distribution for the system and it has ergodic property. Finally, we illustrate the dynamic behavior of the system with $n = 2$ via numerical simulation.

1. Introduction

Interest has been growing in the study of the dynamic relationship between predators and their preys due to its universal existence and importance. However, due to the spatial heterogeneity and the increasing spread of human activities, the habitats of many biological species have been separated into isolated patches. In some of these patches, without the contribution from other patches, the species will go to extinction. Recently, the effect of dispersion on the species survival has been an important subject in population biology (see [1–10] and the references cited therein). Particularly, two species predator-prey systems with dispersal have received great attention from both theoretical and mathematical biologists and many good results have been achieved (see [1, 2, 7, 9–11]). The analysis of these papers has been centered around the coexistence of populations, stability (local and global), and permanence of equilibria. Zhang and Teng [11] established the sufficient conditions on the boundedness, permanence, and existence of positive periodic solution for two species predator-prey model. Kuang and Takeuchi [1] studied a predator-prey system with prey dispersal in a two-patch environment; they obtained the existence, local and global stability of the positive steady state and analyzed both the stabilizing and destabilizing effects of dispersion by introducing examples.

Li and Shuai [2] considered the model

$$\begin{aligned}
\dot{x}_i &= x_i\left(r_i - b_i x_i - e_i y_i\right) + \sum_{j=1}^{n} d_{ij}\left(x_j - \alpha_{ij} x_i\right), \\
\dot{y}_i &= y_i\left(-\gamma_i - \delta_i y_i + \varepsilon_i x_i\right), \quad i = 1, 2, \ldots, n,
\end{aligned} \tag{1.1}$$

where x_i, y_i denote the densities of preys and predators on the patch i, respectively. The parameters in the model are nonnegative constants and e_i, ε_i are positive. The constants d_{ij} are the dispersal rate from patch j to i, and the constants α_{ij} can be selected to represent different boundary conditions in the continuous diffusion case [12]. Let (d_{ij}) denote $n \times n$ dispersal matrix. By constructing a Lyapunov function and using graph theory, Li and Shuai proved the uniqueness and globally asymptotically stability of a positive equilibrium, whenever it exists, if (d_{ij}) is irreducible and there exists i such that $b_i > 0$ or $\delta_i > 0$.

The model mentioned above is a deterministic model which assumes that the parameters in the model are all deterministic irrespective environmental fluctuations. In fact, population dynamics is inevitably affected by environmental white noise, such as weather and epidemic disease. Therefore, the deterministic models are often subject to stochastic perturbation, and it is useful to reveal how the noise affects the population system. There are some authors who have studied the dynamics of predator-prey models with stochastic perturbations (see [13–15]). Ji et al. [13] studied a predator-prey with modified Leslie-Gower and Holling-type II schemes with stochastic perturbation; they got some good results about existence, uniqueness, and extinction of positive solution. Cai and Lin [15] investigated a predator-prey stochastic system with competition among predators and obtained the probability distribution of the system state variables at the state of statistical stationarity. But, until now, few people study the dynamical behavior of the predator-prey system with diffusion under the influence of white noise. However, the diffusion phenomenon and environmental white noise are universal existence in nature. Therefore, we want to study the effect of random perturbations on the predator-prey system on the basis of the existing diffusion model and the contents of this paper are of great significance.

In this paper, we take into account the effect of randomly fluctuating and stochastically perturb intrinsic growth rate in each equations of (1.1):

$$\begin{aligned}
r_i &\longrightarrow r_i + \sigma_{1i}\dot{B}_{1i}(t), \\
-\gamma_i &\longrightarrow -\gamma_i + \sigma_{2i}\dot{B}_{2i}(t), \quad i = 1, 2, \ldots, n,
\end{aligned} \tag{1.2}$$

where $B_{1i}(t), B_{2i}(t)$ are mutually independent Brownian motions and σ_{1i}, σ_{2i} are positiveconstants representing the intensity of the white noises, respectively. Then the stochastic system takes the following form:

$$dx_i = \left[x_i(r_i - b_i x_i - e_i y_i) + \sum_{j=1}^{n} d_{ij}(x_j - \alpha_{ij} x_i)\right] dt + \sigma_{1i} x_i dB_{1i}(t),$$
$$dy_i = y_i(-\gamma_i - \delta_i y_i + \varepsilon_i x_i) dt + \sigma_{2i} y_i dB_{2i}(t), \quad i = 1, 2, \ldots, n. \tag{1.3}$$

Throughout this paper, we assume d_{ij} are nonnegative constants, (d_{ij}) is irreducible, and the parameters r_i, γ_i, b_i, e_i, δ_i, ε_i are positive constants.

In order to obtain better dynamic properties of the SDE (1.3), we will show that there exists a unique positive global solution with any initial positive value, and its pth moment is bounded in Section 2. In the study of a population dynamics, permanence is a very important and interesting topic regarding the survival of populations in ecological system. In a deterministic system, it is usually solved by showing the global attractivity of the positive equilibrium of the system. But, as mentioned above, it is impossible to expect stochastic system (1.3) to tend to a steady state. So we attempt to investigate the stationary distribution of this system by Lyapunov functional technique. The stationary can be considered a weak stability, which appears as the solution is fluctuating in a neighborhood of the equilibrium point of the corresponding deterministic model. In Section 3, we will show if the white noise is small, there is a stationary distribution of SDE (1.3) and it has ergodic property. Existing results on dynamics in a patchy environment have largely been restricted to extinction analysis which means that the population system will survive or die out in the future due to the increased complexity of global analysis. In Section 4, we give the sufficient conditions for extinction. In Section 5, we make numerical simulation to conform our analytical results. Finally, for the completeness of the paper, we give an appendix containing some theories which will be used in previous sections.

The key method used in this paper is the analysis of Lyapunov functions [6, 13, 14, 16]. We will also use the graph theory in Section 3 and some graph definitions can be found in the appendix.

Throughout this paper, unless otherwise specified, let $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all P-null sets). Let R_+^{2n} denote the positive cone of R^{2n}, namely, $R_+^{2n} = \{(x_1, y_1, \ldots, x_n, y_n) \in R^{2n} : x_i > 0, y_i > 0, i = 1, 2, \ldots, n\}$. For convenience and simplicity in the following discussion, denote $X(t) = (x_1(t), y_1(t), x_2(t), y_2(t), \ldots, x_n(t), y_n(t))$. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\text{trace}(A^T A)}$ whilst its operator norm is denoted by $\|A\| = \sup\{|Ax| : |x| = 1\}$.

2. Positive and Global Solutions

In order for a stochastic differential equation to have a unique global (i.e., no explosion at any finite time) solution, the coefficients of the equation are generally required to satisfy the linear growth condition and local Lipschitz condition (see [17]). However, the coefficients of SDE (1.3) do not satisfy the linear growth condition, though they are locally Lipschitz continuous, so the solution of SDE (1.3) may explode at a finite time. In this section, we will prove the solution of stochastic system (1.3) with any positive initial value is not only positive but also not exploive in infinity at any finite time.

Theorem 2.1. *For any given initial value $X(0) \in R_+^{2n}$, there is a unique positive solution $X(t)$ of SDE* (1.3)*, and the solution will remain in R_+^{2n} with probability 1.*

Proof. We define a C^2-function $V : R_+^{2n} \to R_+$:

$$V(X(t)) = \sum_{i=1}^{n} \left[\varepsilon_i (x_i - 1 - \log x_i) + e_i (y_i - 1 - \log y_i) \right]. \tag{2.1}$$

Applying Itô's formula, we have

$$\begin{aligned}
LV &= \sum_{i=1}^{n} \left(\varepsilon_i \left\{ (x_i - 1) \left[r_i - b_i x_i - e_i y_i + \sum_{j=1}^{n} d_{ij} \left(\frac{x_j}{x_i} - \alpha_{ij} \right) \right] + \frac{\sigma_{1i}^2}{2} \right\} \right. \\
&\qquad \left. + e_i \left[(y_i - 1)(-\gamma_i - \delta_i y_i + \varepsilon_i x_i) + \frac{\sigma_{2i}^2}{2} \right] dt \right) \\
&= \sum_{i=1}^{n} \left[-\varepsilon_i b_i x_i^2 - e_i \delta_i y_i^2 + \varepsilon_i \left(r_i - \sum_{j=1}^{n} d_{ij} \alpha_{ij} + b_i - e_i \right) x_i + \varepsilon_i \sum_{j=1}^{n} d_{ij} x_j \right. \\
&\qquad \left. - \varepsilon_i \sum_{j=1}^{n} d_{ij} \frac{x_j}{x_i} + e_i (-\gamma_i + \delta_i + \varepsilon_i) y_i + \varepsilon_i \left(\sum_{j=1}^{n} d_{ij} \alpha_{ij} - r_i + \frac{\sigma_{1i}^2}{2} \right) + e_i \left(\gamma_i + \frac{\sigma_{2i}^2}{2} \right) \right] dt \\
&\le \sum_{i=1}^{n} \left[-\varepsilon_i b_i x_i^2 - e_i \delta_i y_i^2 + \varepsilon_i \left(r_i - \sum_{j=1}^{n} d_{ij} \alpha_{ij} + b_i - e_i \right) x_i + \varepsilon_i \sum_{j=1}^{n} d_{ij} x_j \right. \\
&\qquad \left. + e_i (-\gamma_i + \delta_i + \varepsilon_i) y_i + \varepsilon_i \left(\sum_{j=1}^{n} d_{ij} \alpha_{ij} - r_i + \frac{\sigma_{1i}^2}{2} \right) + e_i \left(\gamma_i + \frac{\sigma_{2i}^2}{2} \right) \right].
\end{aligned} \tag{2.2}$$

It is clear that the coefficient of quadratic term is negative, so we must be able to find a positive constant number K that satisfies

$$LV \le K, \tag{2.3}$$

and K is independent of x_i, y_i, and t. By the similar proof of Li and Mao [18, Theorem 2.1], we can obtain the desired assertion. □

Theorem 2.1 shows that the solution of SDE (1.3) will remain in the positive cone R_+^2 with probability 1. This nice property provides us with a great opportunity to discuss the pth moment and stochastically ultimately boundedness of the solution.

Definition 2.2 (see [18]). The solution $X(t)$ of SDE (1.3) is said to be stochastically ultimately bounded, if for any $\epsilon \in (0,1)$, there exists a positive constant $\chi(=\chi(\epsilon))$, such that for any initial value $X(0) \in R_+^{2n}$, the solution $X(t)$ to (1.3) has the property that

$$\limsup_{t\to\infty} P\{|x(t)| > \chi\} < \epsilon. \tag{2.4}$$

Lemma 2.3. *For any given initial value $X(0) \in R_+^{2n}$, there exist positive constants $\kappa(p)$, p_i, and q_i $(i = 1, 2, \dots, n)$, such that the solution $X(t)$ of SDE* (1.3) *has the following property:*

$$E\left[\sum_{i=1}^{n}\left(p_i x_i^p(t) + q_i y_i^p(t)\right)\right] \le \kappa(p), \quad t \ge 0,\ p > 1. \tag{2.5}$$

Proof. By Itô's formula and Young inequality, we compute

$$\begin{aligned}
d\left(\frac{1}{p}x_i^p\right) &= x_i^{p-1}dx_i + \frac{p-1}{2}x_i^{p-2}(dx_i)^2 \\
&= \left[-b_i x_i^{p+1} + \left(r_i + \frac{p-1}{2}\sigma_{1i}^2 - \sum_{j=1}^{n} d_{ij}\alpha_{ij}\right)x_i^p - e_i x_i^p y_i + \sum_{j=1}^{n} d_{ij} x_i^{p-1} x_j\right]dt \\
&\quad + \sigma_{1i} x_i^p dB_{1i}(t) \\
&\le \left[-b_i x_i^{p+1} + \left(r_i + \frac{p-1}{2}\sigma_{1i}^2 + \sum_{j=1}^{n} d_{ij}\alpha_{ij}\right)x_i^p + \sum_{j=1}^{n} d_{ij}\left(\frac{p-1}{p}x_i^p + \frac{1}{p}x_j^p\right)\right]dt \\
&\quad + \sigma_{1i} x_i^p dB_{1i}(t) \\
&= \left[-b_i x_i^{p+1} + \left(r_i + \frac{p-1}{2}\sigma_{1i}^2 + \sum_{j=1}^{n} d_{ij}\alpha_{ij} + \frac{p-1}{p}\sum_{j=1}^{n} d_{ij}\right)x_i^p + \frac{1}{p}\sum_{j=1}^{n} d_{ij} x_j^p\right]dt \\
&\quad + \sigma_{1i} x_i^p dB_{1i}(t), \\
d\left(\frac{1}{p}y_i^p\right) &= y_i^{p-1}dy_i + \frac{p-1}{2}y_i^{p-2}(dy_i)^2 \\
&= \left[-\delta_i y_i^{p+1} + \left(-\gamma_i + \frac{p-1}{2}\sigma_{2i}^2\right)y_i^p + \varepsilon_i x_i y_i^p\right]dt + \sigma_{2i} y_i^p dB_{2i}(t) \\
&\le \left[-\delta_i y_i^{p+1} + \left(-\gamma_i + \frac{p-1}{2}\sigma_{2i}^2\right)y_i^p + \varepsilon_i\left(\frac{1}{p+1}k_i^{-p}x_i^{p+1} + \frac{p}{p+1}k_i y_i^{p+1}\right)\right]dt \\
&\quad + \sigma_{2i} y_i^p dB_{2i}(t) \\
&= \left[\left(-\delta_i + \frac{p\varepsilon_i k_i}{p+1}\right)y_i^{p+1} + \left(-\gamma_i + \frac{p-1}{2}\sigma_{2i}^2\right)y_i^p + \frac{\varepsilon_i k_i^{-p}}{p+1}x_i^{p+1}\right]dt + \sigma_{2i} y_i^p dB_{2i}(t),
\end{aligned} \tag{2.6}$$

where k_i $(i = 1, 2, \dots, n)$ are positive constants to be determined. Hence, for positive constants p_i, q_i, we have

$$\begin{aligned}
&d\left[\sum_{i=1}^{n}\left(p_i x_i^p + q_i y_i^p\right)\right] \\
&\quad\le \sum_{i=1}^{n}\left\{-pp_i b_i x_i^{p+1} + p_i\left[pr_i + \frac{p(p-1)}{2}\sigma_{1i}^2 + \sum_{j=1}^{n} d_{ij}(p-1+p\alpha_{ij}) + \sum_{j=1}^{n} d_{ji}\frac{p_j}{p_i}\right]x_i^p \right.\\
&\qquad\qquad \left. + pq_i\left(-\delta_i + \frac{p\varepsilon_i k_i}{p+1}\right)y_i^{p+1} + pq_i\left(-\gamma_i + \frac{p-1}{2}\sigma_{2i}^2\right)y_i^p + \frac{pq_i\varepsilon_i k_i^{-p}}{p+1}x_i^{p+1}\right\}dt \\
&\quad + p\sum_{i=1}^{n} p_i\sigma_{1i}x_i^p\,dB_{1i}(t) + p\sum_{i=1}^{n} q_i\sigma_{2i}y_i^p\,dB_{2i}(t) \\
&\quad = \sum_{i=1}^{n}\left\{-p\left(p_i b_i - \frac{q_i\varepsilon_i k_i^{-p}}{p+1}\right)x_i^{p+1} - pq_i\left(\delta_i - \frac{p\varepsilon_i k_i}{p+1}\right)y_i^{p+1}\right.\\
&\qquad\qquad + p_i\left[pr_i + \frac{p(p-1)}{2}\sigma_{1i}^2 + \sum_{j=1}^{n} d_{ij}(p-1+p\alpha_{ij}) + \sum_{j=1}^{n} d_{ji}\frac{p_j}{p_i}\right]x_i^p \\
&\qquad\qquad \left. + pq_i\left(-\gamma_i + \frac{p-1}{2}\sigma_{2i}^2\right)y_i^p\right\}dt \\
&\quad + p\sum_{i=1}^{n} p_i\sigma_{1i}x_i^p\,dB_{1i}(t) + p\sum_{i=1}^{n} q_i\sigma_{2i}y_i^p\,dB_{2i}(t).
\end{aligned} \tag{2.7}$$

Next, we claim that there exist $p_i > 0, q_i > 0$, and $k_i > 0$ such that

$$p_i b_i - \frac{q_i\varepsilon_i k_i^{-p}}{p+1} > 0, \qquad \delta_i - \frac{p\varepsilon_i k_i}{p+1} > 0. \tag{2.8}$$

In fact, we only need $0 < k_i < (p+1)\delta_i/p\varepsilon_i$ and $0 < q_i = p_i/m$, where m is a sufficiently large positive integer. Let

$$\alpha_i =: pr_i + \frac{p(p-1)}{2}\sigma_{1i}^2 + \sum_{j=1}^{n} d_{ij}\left(p-1+p\alpha_{ij}\right) + \sum_{j=1}^{n} d_{ji}\frac{p_j}{p_i},$$
$$\alpha_i' =: p\left(-\gamma_i + \frac{p-1}{2}\sigma_{2i}^2\right),$$
$$\beta_i =: pp_i^{-1/p}\left(b_i - \frac{\varepsilon_i k_i^{-p}}{m(p+1)}\right), \tag{2.9}$$
$$\beta_i' =: pq_i^{-1/p}\left(\delta_i - \frac{p\varepsilon_i k_i}{p+1}\right),$$
$$\breve{\alpha} =: \max\{\alpha_1, \alpha_1', \dots, \alpha_n, \alpha_n'\},$$
$$\widehat{\beta} =: \min\{\beta_1, \beta_1', \dots, \beta_n, \beta_n'\}.$$

It is clear that $\breve{\alpha} > 0, \widehat{\beta} > 0$. Then

$$\begin{aligned} d&\left[\sum_{i=1}^{n}\left(p_i x_i^p(t) + q_i y_i^p(t)\right)\right] \\ &\le \left[\breve{\alpha}\sum_{i=1}^{n}\left(p_i x_i^p(t) + q_i y_i^p(t)\right) - \widehat{\beta}\sum_{i=1}^{n}\left(p_i^{(p+1)/p} x_i^{p+1}(t) + q_i^{(p+1)/p} y_i^{p+1}(t)\right)\right]dt \\ &\quad + p\sum_{i=1}^{n} p_i\sigma_{1i} x_i^p(t)\,dB_{1i}(t) + p\sum_{i=1}^{n} q_i\sigma_{2i} y_i^p(t)\,dB_{2i}(t). \end{aligned} \tag{2.10}$$

Integrating it from 0 to t and taking expectations of both sides, we obtain that

$$\begin{aligned} &\frac{dE\left[\sum_{i=1}^{n}\left(p_i x_i^p(t) + q_i y_i^p(t)\right)\right]}{dt} \\ &\le \left\{\breve{\alpha}E\left[\sum_{i=1}^{n}\left(p_i x_i^p(t) + q_i y_i^p(t)\right)\right] - \widehat{\beta}E\left[\sum_{i=1}^{n}\left(p_i^{(p+1)/p} x_i^{p+1}(t) + q_i^{(p+1)/p} y_i^{p+1}(t)\right)\right]\right\} \\ &\le \left\{\breve{\alpha}E\left[\sum_{i=1}^{n}\left(p_i x_i^p(t) + q_i y_i^p(t)\right)\right] - (2n)^{-1/p}\widehat{\beta}E\left[\sum_{i=1}^{n}\left(p_i x_i^p(t) + q_i y_i^p(t)\right)\right]^{(p+1)/p}\right\} \\ &= E\left[\sum_{i=1}^{n}\left(p_i x_i^p(t) + q_i y_i^p(t)\right)\right]\left\{\breve{\alpha} - (2n)^{-1/p}\widehat{\beta}E\left[\sum_{i=1}^{n}\left(p_i x_i^p(t) + q_i y_i^p(t)\right)\right]^{1/p}\right\}. \end{aligned} \tag{2.11}$$

Therefore, letting $z(t) = E[\sum_{i=1}^{n}(p_i x_i^p(t) + q_i y_i^p(t))]$, we have

$$\frac{dz(t)}{dt} \le z(t)\left(\breve{\alpha} - (2n)^{-1/p}\widehat{\beta} z^{1/p}(t)\right). \tag{2.12}$$

Notice that the solution of equation

$$\frac{d\overline{z}(t)}{dt} = \overline{z}(t)\left(\check{\alpha} - (2n)^{-1/p}\widehat{\beta}\overline{z}^{1/p}(t)\right) \tag{2.13}$$

obeys

$$\overline{z}(t) \longrightarrow 2n \cdot \left(\frac{\check{\alpha}}{\widehat{\beta}}\right)^{p}, \quad \text{as } t \longrightarrow \infty. \tag{2.14}$$

By comparison theorem, we can get

$$\limsup_{t\to\infty} E\left[\sum_{i=1}^{n}\left(p_i x_i^p(t) + q_i y_i^p(t)\right)\right] \le 2n \cdot \left(\frac{\check{\alpha}}{\widehat{\beta}}\right)^{p} := L(p), \tag{2.15}$$

which implies that there is a $T > 0$, such that

$$E\left[\sum_{i=1}^{n}\left(p_i x_i^p(t) + q_i y_i^p(t)\right)\right] \le 2L(p), \quad t > T. \tag{2.16}$$

In addition $E[\sum_{i=1}^{n}(p_i x_i^p(t) + q_i y_i^p(t))]$ is continuous, so we have

$$E\left[\sum_{i=1}^{n}\left(p_i x_i^p(t) + q_i y_i^p(t)\right)\right] \le C(p), \quad t \in [0, T]. \tag{2.17}$$

Let $\kappa(p) = \max\{2L(p), C(p)\}$, then

$$E\left[\sum_{i=1}^{n}\left(p_i x_i^p(t) + q_i y_i^p(t)\right)\right] \le \kappa(p), \quad t \ge 0,\ p > 1. \tag{2.18}$$

This completes the proof. □

Theorem 2.4. *The solutions of system* (1.3) *are stochastically ultimately bounded for any initial value* $X(0) \in R_+^{2n}$.

Proof. From Theorem 2.1, the solution $X(t)$ will remain in R_+^{2n} with probability 1. Let $Q = \min\{p_1, q_1, p_2, q_2, \ldots, p_n, q_n\}$. Note that $|X(t)| = [\sum_{i=1}^{n}(x_i^2 + y_i^2)]^{1/2}$ and $|X(t)|^p \le (2n)^{p/2}[\sum_{i=1}^{n}(x_i^p + y_i^p)]$. Therefore, we get

$$E\left[\sum_{i=1}^{n}\left(p_i x_i^p(t) + q_i y_i^p(t)\right)\right] \ge QE\left[\sum_{i=1}^{n}\left(x_i^p(t) + y_i^p(t)\right)\right] \ge (2n)^{-p/2}QE|X(t)|^p, \tag{2.19}$$

and by (2.5) we have

$$E|X(t)|^p \leq \frac{(2n)^{p/2}}{Q}\kappa(p) =: \tilde{\kappa}(p) < +\infty. \tag{2.20}$$

Applying the Chebyshev inequality yields the required assertion. □

3. Stationary Distribution

In this section, we investigate there is a stationary distribution for SDE (1.3) instead of asymptotically stable equilibria. Before giving the main theorem, we first give a lemma (see [2]).

Lemma 3.1 (see [2]). *Assume (d_{ij}) is irreducible. If there exists i such that $b_i > 0$ or $\delta_i > 0$, then, whenever a positive equilibrium E^* exists for system* (1.1), *it is unique and globally asymptotically stable in the positive cone R_+^{2n}.*

In the section, we assume system (1.1) exists and the positive equilibrium $E^* = (x_1^*, y_1^*, \ldots, x_n^*, y_n^*)$ satisfies the equation

$$\begin{aligned} &x_i^*(r_i - b_i x_i^* - e_i y_i^*) + \sum_{j=1}^n d_{ij}\left(x_j^* - \alpha_{ij} x_i^*\right) = 0, \\ &y_i^*(-\gamma_i - \delta_i y_i^* + \varepsilon_i x_i^*) = 0, \quad i = 1, 2, \ldots, n, \end{aligned} \tag{3.1}$$

where $x_i^* > 0, y_i^* > 0$.

We now state a theorem in which the graph theory will be used. Assume $c_i\,(i = 1, 2, \ldots, n)$ defined as in Lemma A.1 are nonnegative constants and $A = (d_{ij}\varepsilon_i x_j^*)_{n\times n}\,(i = 1, 2, \ldots, n)$.

Theorem 3.2. *Assume $\delta < \min\{\min_{1\leq i\leq n}\{c_i\varepsilon_i b_i (x_i^*)^2\}, \min_{1\leq i\leq n}\{c_i e_i \delta_i (y_i^*)^2\}\}$. Then there is a stationary distribution $\mu(\cdot)$ for SDE* (1.3) *and it has ergodic property. Here $(x_1^*, y_1^*, \ldots, x_n^*, y_n^*)$ is the solution of* (3.1), *and $\delta = (1/2)\sum_{i=1}^n (c_i\varepsilon_i b_i x_i^* \sigma_{1i}^2 + c_i e_i \delta_i y_i^* \sigma_{2i}^2)$.*

Proof. Define $V : E_l = R_+^{2n} \rightarrow R_+$:

$$V(X(t)) = \sum_{i=1}^n \left[\varepsilon_i\left(x_i - x_i^* - x_i^* \log\frac{x_i}{x_i^*}\right) + e_i\left(y_i - y_i^* - y_i^* \log\frac{y_i}{y_i^*}\right)\right]. \tag{3.2}$$

Then

$$LV = \sum_{i=1}^{n} c_i \Bigg\{ \varepsilon_i (x_i - x_i^*) \left[(r_i - b_i x_i - e_i y_i) + \sum_{j=1}^{n} d_{ij} \left(\frac{x_j}{x_i} - \alpha_{ij} \right) \right] + e_i (y_i - y_i^*)(-\gamma_i - \delta_i y_i + \varepsilon_i x_i) + \frac{1}{2} \varepsilon_i x_i^* \sigma_{1i}^2 + \frac{1}{2} e_i y_i^* \sigma_{2i}^2 \Bigg\}. \tag{3.3}$$

By (3.1), we have

$$\begin{aligned} r_i &= b_i x_i^* + e_i y_i^* - \sum_{j=1}^{n} d_{ij} \left(\frac{x_j^*}{x_i^*} - \alpha_{ij} \right), \\ -\gamma_i &= \delta_i y_i^* - \varepsilon_i x_i^*, \quad i = 1, 2, \ldots, n. \end{aligned} \tag{3.4}$$

Substituting this into (3.3) gives

$$\begin{aligned} LV &= \sum_{i=1}^{n} c_i \Bigg[-\varepsilon_i b_i (x_i - x_i^*)^2 - e_i \delta_i (y_i - y_i^*)^2 + \varepsilon_i (x_i - x_i^*) \sum_{j=1}^{n} d_{ij} \left(\frac{x_j}{x_i} - \frac{x_j^*}{x_i^*} \right) \\ &\qquad + \frac{1}{2} \varepsilon_i x_i^* \sigma_{1i}^2 + \frac{1}{2} e_i y_i^* \sigma_{2i}^2 \Bigg] \\ &= \sum_{i=1}^{n} c_i \Bigg[-\varepsilon_i b_i (x_i - x_i^*)^2 - e_i \delta_i (y_i - y_i^*)^2 + \frac{1}{2} \varepsilon_i x_i^* \sigma_{1i}^2 + \frac{1}{2} e_i y_i^* \sigma_{2i}^2 \\ &\qquad + \varepsilon_i \sum_{j=1}^{n} d_{ij} x_j^* \left(\frac{x_j}{x_j^*} - \frac{x_i}{x_i^*} - \frac{x_i^* x_j}{x_i x_j^*} + 1 \right) \Bigg] \\ &= \sum_{i=1}^{n} c_i \Bigg\{ -\varepsilon_i b_i (x_i - x_i^*)^2 - e_i \delta_i (y_i - y_i^*)^2 + \frac{1}{2} \varepsilon_i x_i^* \sigma_{1i}^2 + \frac{1}{2} e_i y_i^* \sigma_{2i}^2 \\ &\qquad + \varepsilon_i \sum_{j=1}^{n} d_{ij} x_j^* \left[\left(-\frac{x_i}{x_i^*} + \log \frac{x_i}{x_i^*} \right) - \left(-\frac{x_j}{x_j^*} + \log \frac{x_j}{x_j^*} \right) + \left(1 - \frac{x_i^* x_j}{x_i x_j^*} + \log \frac{x_i^* x_j}{x_i x_j^*} \right) \right] \Bigg\} \\ &\le \sum_{i=1}^{n} c_i \Bigg\{ -\varepsilon_i b_i (x_i - x_i^*)^2 - e_i \delta_i (y_i - y_i^*)^2 + \frac{1}{2} \varepsilon_i x_i^* \sigma_{1i}^2 + \frac{1}{2} e_i y_i^* \sigma_{2i}^2 \\ &\qquad + \varepsilon_i \sum_{j=1}^{n} d_{ij} x_j^* \left[\left(-\frac{x_i}{x_i^*} + \log \frac{x_i}{x_i^*} \right) - \left(-\frac{x_j}{x_j^*} + \log \frac{x_j}{x_j^*} \right) \right] \Bigg\}, \end{aligned} \tag{3.5}$$

Here we use the fact: $1 - a + \log a \leq 0$, for $a > 0$ with equality holding if and only if $a = 1$. Since (d_{ij}) is irreducible, $\varepsilon_i > 0$, $x_j^* > 0$, we know matrix $A = (d_{ij}\varepsilon_i x_j^*)$ is irreducible. Let $G_i(x_i) = -x_i/x_i^* + \log(x_i/x_i^*), G_j(x_j) = -x_j/x_j^* + \log(x_j/x_j^*)$, and by Lemma A.1, we have

$$\sum_{i,j=1}^{n} c_i\left(d_{ij}\varepsilon_i x_j^*\right)\left(-\frac{x_i}{x_i^*} + \log\frac{x_i}{x_i^*}\right) = \sum_{i,j=1}^{n} c_i\left(d_{ij}\varepsilon_i x_j^*\right)\left(-\frac{x_j}{x_j^*} + \log\frac{x_j}{x_j^*}\right) \tag{3.6}$$

which implies that

$$LV \leq \sum_{i=1}^{n}\left[-c_i\varepsilon_i b_i\left(x_i - x_i^*\right)^2 - c_i e_i \delta_i\left(y_i - y_i^*\right)^2\right] + \frac{1}{2}\sum_{i=1}^{n}\left(c_i\varepsilon_i x_i^*\sigma_{1i}^2 + c_i e_i y_i^*\sigma_{2i}^2\right). \tag{3.7}$$

The following proof of ergodicity is similar to Theorem 3.2 in [19]. Assume $\delta = (1/2)\sum_{i=1}^{n}(c_i\varepsilon_i x_i^*\sigma_{1i}^2 + c_i e_i y_i^*\sigma_{2i}^2)$, then

$$LV \leq \sum_{i=1}^{n}\left[-c_i\varepsilon_i b_i\left(x_i - x_i^*\right)^2 - c_i e_i \delta_i\left(y_i - y_i^*\right)^2\right] + \delta. \tag{3.8}$$

Note that $\delta < \min\{\min_{1\leq i\leq n}\{c_i\varepsilon_i b_i(x_i^*)^2\}, \min_{1\leq i\leq n}\{c_i e_i\delta_i(y_i^*)^2\}\}$, then the ellipsoid

$$\sum_{i=1}^{n}\left[-c_i\varepsilon_i b_i\left(x_i - x_i^*\right)^2 - c_i e_i \delta_i\left(y_i - y_i^*\right)^2\right] + \delta = 0 \tag{3.9}$$

lies entirely in R_+^{2n}. We can take U to be a neighborhood of the ellipsoid with $\overline{\mathrm{U}} \subset E_l = R_+^{2n}$, so for $X \in E_l \setminus U$, $LV \leq -N$ (N is a positive constant), which implies the condition (B.2) in Lemma A.2 is satisfied. Therefore, the solution $X(t)$ is recurrent in the domain U, which together with Remark A.3 and Lemma A.4 implies $X(t)$ is recurrent in any bounded domain $D \subset R_+^{2n}$. Thus, for any D, there is

$$M = \min\left\{\min_{1\leq i\leq n}\left\{\sigma_{1i}^2 x_i^2\right\}, \min_{1\leq i\leq n}\left\{\sigma_{2i}^2 y_i^2\right\}, \left(x_1, y_1, \ldots, x_n, y_n\right) \in \overline{D}\right\} > 0 \tag{3.10}$$

such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j = \sum_{i=1}^{n}\left(\sigma_{1i}^2 x_i^2\xi_i^2 + \sigma_{2i}^2 y_i^2\xi_i^2\right) \geq M\|\xi\|^2, \tag{3.11}$$

for all $(x_1, y_1, \ldots, x_n, y_n) \in \overline{\mathrm{U}}$, $\xi \in R^n$, which implies that condition (B.1) is also satisfied. Therefore, the stochastic system (1.3) has a stable stationary distribution $\mu(\cdot)$ and it is ergodic. $\square$

4. Extinction

In this section, we will show that if the noise is sufficiently large, the solution to the associated SDE (1.3) will become extinct with probability 1, although the solution to the original equation (1.1) may be persistent. For example, recall a simple case, namely, the scalar logistic equation

$$dN(t) = N(t)(a - bN(t))dt, \quad t \geq 0, \tag{4.1}$$

with initial value $N_0 > 0$. It is well known that, when $a > 0, b > 0$, the solution $N(t)$ is persistent, because

$$\lim_{t\to\infty} N(t) = \frac{b}{a}. \tag{4.2}$$

However, consider its associated stochastic equation

$$dN(t) = N(t)[(a - bN(t))dt + \sigma dB(t)], \quad t \geq 0, \tag{4.3}$$

where $\sigma > 0$, then the solution to this stochastic equation will become extinct with probability 1, that is to say, if $\sigma^2 > 2a$,

$$\lim_{t\to\infty} N(t) = 0 \quad \text{a.s.} \tag{4.4}$$

The following theorem reveals the important fact that the environmental noise may make the population extinct.

Theorem 4.1. *For any given initial value $X(0) \in R_+^{2n}$, the solution of the SDE* (1.3) *has the property that*

$$\limsup_{t\to\infty} \frac{\log\left(\sum_{i=1}^{n} x_i\right)}{t} \leq \check{l}_1 - \frac{\widehat{\sigma}_1^2}{2} \quad a.s. \tag{4.5}$$

Here

$$\check{l}_1 = \max\left\{\max_{1\leq i\leq n}\left\{r_i + \sum_{j=1}^{n} d_{ji} - \sum_{j=1}^{n} d_{ij}\alpha_{ij}\right\}, 0\right\}, \qquad \frac{\widehat{\sigma}_1^2}{2} = \frac{1}{2\sum_{i=1}^{n}\left(1/\sigma_{1i}^2\right)}. \tag{4.6}$$

Particularly, if $\check{l}_1 - \widehat{\sigma}_1^2/2 < 0$, then $\lim_{t\to\infty}(x_1(t), x_2(t), \dots, x_n(t)) = 0$ a.s.

Proof. Define

$$V = \sum_{i=1}^{n} x_i. \tag{4.7}$$

By Itô's formula, we have

$$\begin{aligned} dV &= \sum_{i=1}^{n}\left[r_i x_i - b_i x_i^2 - e_i x_i y_i + \sum_{j=1}^{n} d_{ij}\left(x_j - \alpha_{ij} x_i\right)\right] dt + \sum_{i=1}^{n} \sigma_{1i} x_i dB_{1i}(t) \\ &= \sum_{i=1}^{n}\left[-b_i x_i^2 - e_i x_i y_i + \left(r_i + \sum_{j=1}^{n} d_{ji} - \sum_{j=1}^{n} d_{ij}\alpha_{ij}\right) x_i\right] dt + \sum_{i=1}^{n} \sigma_{1i} x_i dB_{1i}(t). \end{aligned} \tag{4.8}$$

Thus we compute

$$\begin{aligned} d\log V &= \frac{1}{V} dV - \frac{1}{2V^2}(dV)^2 \\ &= \frac{1}{V}\sum_{i=1}^{n}\left[-b_i x_i^2 - e_i x_i y_i + \left(r_i + \sum_{j=1}^{n} d_{ji} - \sum_{j=1}^{n} d_{ij}\alpha_{ij}\right) x_i\right] dt \\ &\quad + \frac{1}{V}\sum_{i=1}^{n}\sigma_{1i} x_i dB_{1i}(t) - \frac{1}{2V^2}\sum_{j=1}^{n}\sigma_{1i}^2 x_i^2 dt, \end{aligned} \tag{4.9}$$

Letting $\check{l}_1 = \max\{\max_{1\le i\le n}\{r_i + \sum_{j=1}^{n} d_{ji} - \sum_{j=1}^{n} d_{ij}\alpha_{ij}\}, 0\}$, we compute

$$\frac{1}{V}\sum_{i=1}^{n}\left[-b_i x_i^2 - e_i x_i y_i + \left(r_i + \sum_{j=1}^{n} d_{ji} - \sum_{j=1}^{n} d_{ij}\alpha_{ij}\right) x_i\right] \le \frac{1}{V}\sum_{i=1}^{n}\left(r_i + \sum_{j=1}^{n} d_{ji} - \sum_{j=1}^{n} d_{ij}\alpha_{ij}\right) x_i \le \check{l}_1. \tag{4.10}$$

By Cauchy inequality, we compute also

$$-\frac{1}{2V^2}\sum_{i=1}^{n}\sigma_{1i}^2 x_i^2 = -\frac{\sum_{i=1}^{n}\left(\sigma_{1i}^2 x_i^2\right)}{2\left(\sum_{i=1}^{n} x_i\right)^2} \le -\frac{\left(\sum_{i=1}^{n} x_i\right)^2}{2\sum_{i=1}^{n}\left(1/\sigma_{1i}^2\right)\cdot\left(\sum_{i=1}^{n} x_i\right)^2} = -\frac{1}{2\sum_{i=1}^{n}\left(1/\sigma_{1i}^2\right)} =: -\frac{\widehat{\sigma}_1^2}{2}, \tag{4.11}$$

where $\widehat{\sigma}_1^2/2 = 1/2\sum_{i=1}^{n}(1/\sigma_{1i}^2)$. Substituting these two inequalities into (4.9) yields

$$d\log V \le \left(\check{l}_1 - \frac{\widehat{\sigma}_1^2}{2}\right) dt + \frac{1}{V}\sum_{i=1}^{n}\sigma_{1i} x_i dB_{1i}(t). \tag{4.12}$$

This implies

$$\log V(X(t)) \le \log V(X(0)) + \int_0^t \left(\check{l}_1 - \frac{\widehat{\sigma}_1^2}{2}\right) dt + M(t), \tag{4.13}$$

where $M(t)$ is a martingale defined by

$$M(t) = \int_0^t \frac{1}{V}\sum_{i=1}^{n}\sigma_{1i}x_i(s)dB_{1i}(s) = \int_0^t \frac{\sum_{i=1}^{n}\sigma_{1i}x_i(s)dB_{1i}(s)}{\sum_{i=1}^{n}x_i(s)} \tag{4.14}$$

with $M(0) = 0$. The quadratic variation of this martingale is

$$\langle M, M\rangle_t = \int_0^t \frac{\sum_{i=1}^{n}\sigma_{1i}^2 x_i^2(s)}{\left(\sum_{i=1}^{n}x_i(s)\right)^2}ds \le \breve{\sigma}_1^2 \int_0^t \frac{\sum_{i=1}^{n}x_i^2(s)}{\sum_{i=1}^{n}x_i^2(s)}ds = \breve{\sigma}_1^2 t, \tag{4.15}$$

where $\breve{\sigma}_1 = \max_{1\le i\le n}\{\sigma_{1i}\}$. By the strong law of large numbers for martingale (see [17, 20]), we have

$$\lim_{t\to\infty}\frac{M(t)}{t} = 0 \quad \text{a.s.} \tag{4.16}$$

It finally follows from (4.13) by dividing t on the both sides and then letting $t \to \infty$ that

$$\limsup_{t\to\infty}\frac{\log V}{t} \le \limsup_{t\to\infty}\frac{1}{t}\int_0^t\left(\check{l}_1 - \frac{\widehat{\sigma}_1^2}{2}\right)dt = \check{l}_1 - \frac{\widehat{\sigma}_1^2}{2} \quad \text{a.s.} \tag{4.17}$$

which implies the required assertion. □

Remark 4.2. Theorem 4.1 shows that if the condition $\check{l}_1 - \widehat{\sigma}_1^2/2 < 0$ holds, that is, when the prey population is disturbed by large white noise, the species of prey will extinct.

Now we give the following theorem which describes the entire extinction.

Theorem 4.3. *For any given initial value $X(0) \in R_+^{2n}$, the solution of the SDE* (1.3) *has the property that*

$$\limsup_{t\to\infty}\frac{\log\left[\sum_{i=1}^{n}(\varepsilon_i x_i + e_i y_i)\right]}{t} \le \check{l} - \frac{\widehat{\sigma}^2}{2} \quad a.s. \tag{4.18}$$

Here

$$\check{l} = \max\left\{\max_{1\le i\le n}\left\{r_i + \sum_{j=1}^{n}\left(d_{ji}\frac{\varepsilon_j}{\varepsilon_i} - d_{ij}\alpha_{ij}\right)\right\}, 0\right\}, \qquad \frac{\widehat{\sigma}^2}{2} = \frac{1}{2\sum_{i=1}^{n}\left(1/\sigma_{1i}^2 + 1/\sigma_{2i}^2\right)}. \tag{4.19}$$

Particularly, if $\check{l} - \widehat{\sigma}^2/2 < 0$, then $\lim_{t\to\infty}X(t) = 0$ *a.s.*

Proof. The proof of the theorem is similar to Theorem 4.1, we only give the main proof procedure. Define

$$V(X(t)) = \sum_{i=1}^{n} (\varepsilon_i x_i + e_i y_i). \tag{4.20}$$

Let $\check{l} =: \max\{\max_{1\le i\le n}\{r_i + \sum_{j=1}^{n} d_{ji}(\varepsilon_j/\varepsilon_i) - \sum_{j=1}^{n} d_{ij}\alpha_{ij}\}, 0\}$ and $\widehat{\sigma}^2/2 = 1/2\sum_{i=1}^{n}(1/\sigma_{1i}^2 + 1/\sigma_{2i}^2)$. By Itô's formula, we compute

$$\begin{aligned} d\log V \le{}& \frac{1}{V}\sum_{i=1}^{n}\varepsilon_i\left(r_i + \sum_{j=1}^{n} d_{ji}\frac{\varepsilon_j}{\varepsilon_i} - \sum_{j=1}^{n} d_{ij}\alpha_{ij}\right)x_i dt - \frac{1}{2V^2}\sum_{i=1}^{n}\left(\varepsilon_i^2\sigma_{1i}^2 x_i^2 + e_i^2\sigma_{2i}^2 y_i^2\right)dt \\ &+ \frac{1}{V}\left(\sum_{i=1}^{n}\varepsilon_i\sigma_{1i}x_i dB_{1i}(t) + \sum_{i=1}^{n} e_i\sigma_{2i}y_i dB_{2i}(t)\right) \\ \le{}& \left(\check{l} - \frac{\widehat{\sigma}^2}{2}\right)dt + \frac{1}{V}\sum_{i=1}^{n}\left(\varepsilon_i\sigma_{1i}x_i dB_{1i}(t) + e_i\sigma_{2i}y_i dB_{2i}(t)\right). \end{aligned} \tag{4.21}$$

The rest of the proof is similar to Theorem 4.1. □

Remark 4.4. Theorem 4.3 states that when the prey and predator population are all disturbed by large white noise and the condition $\check{l} - \widehat{\sigma}^2/2 < 0$ holds, the two species will be extinct.

5. Numerical Simulation

In this section, in order to better study the effect of white noise in diffusion system, we assume $\alpha_{ij} = 1$, d_{ij} are nonnegative constants, (d_{ij}) is irreducible $(i, j = 1, 2)$, and $d_{11} = d_{22} = 1$. Consider the predator-prey system with $n = 2$, that is,

$$\begin{aligned} dx_1 &= \left[x_1(r_1 - b_1x_1 - e_1y_1) + d_{12}(x_2 - x_1)\right]dt + \sigma_{11}x_1 dB_{11}(t), \\ dy_1 &= y_1(-\gamma_1 - \delta_1 y_1 + \varepsilon_1 x_1)dt + \sigma_{21}y_1 dB_{21}(t), \\ dx_2 &= \left[x_2(r_2 - b_2x_2 - e_2y_2) + d_{21}(x_1 - x_2)\right]dt + \sigma_{12}x_2 dB_{12}(t), \\ dy_2 &= y_2(-\gamma_2 - \delta_2 y_2 + \varepsilon_2 x_2)dt + \sigma_{22}y_2 dB_{22}(t). \end{aligned} \tag{5.1}$$

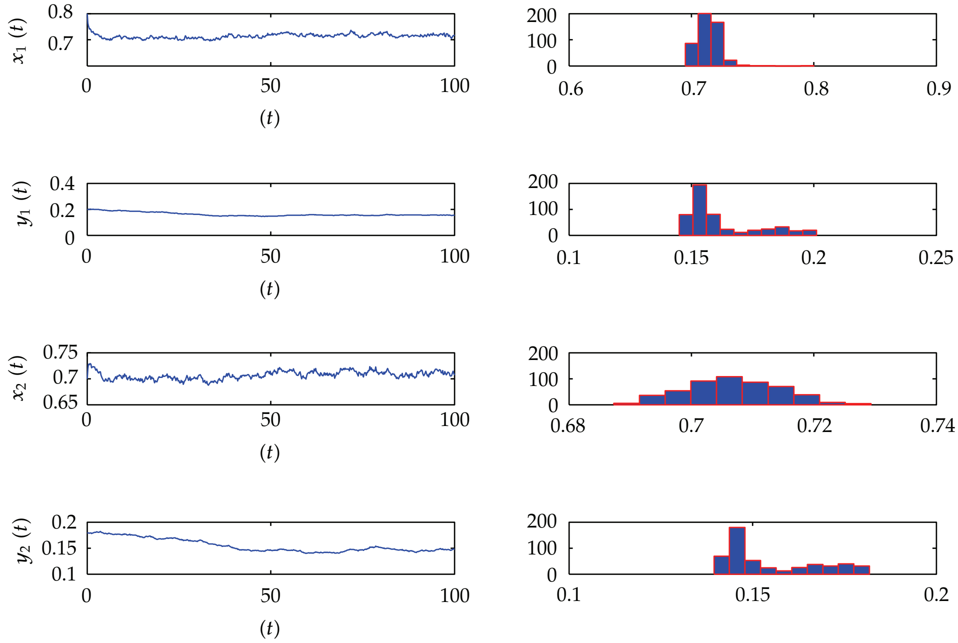

Figure 1: The solution of stochastic system (5.1) with small white noise and its histogram: $\sigma_{11} = \sigma_{12} = \sigma_{21} = \sigma_{22} = 0.01$.

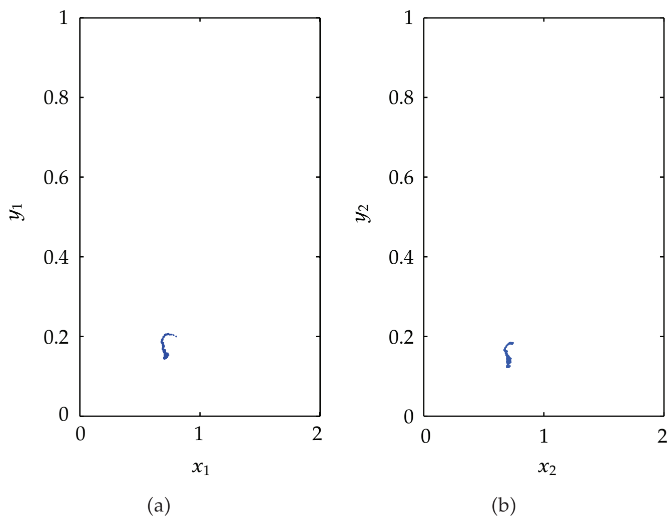

Figure 2: Population distribution scatter corresponding to Figure 1. The scatter points around $(x_1^*, y_1^*) \doteq (0.7174, 0.1545)$ and $(x_2^*, y_2^*) \doteq (0.7107, 0.1356)$, respectively. The system has a stationary distribution.

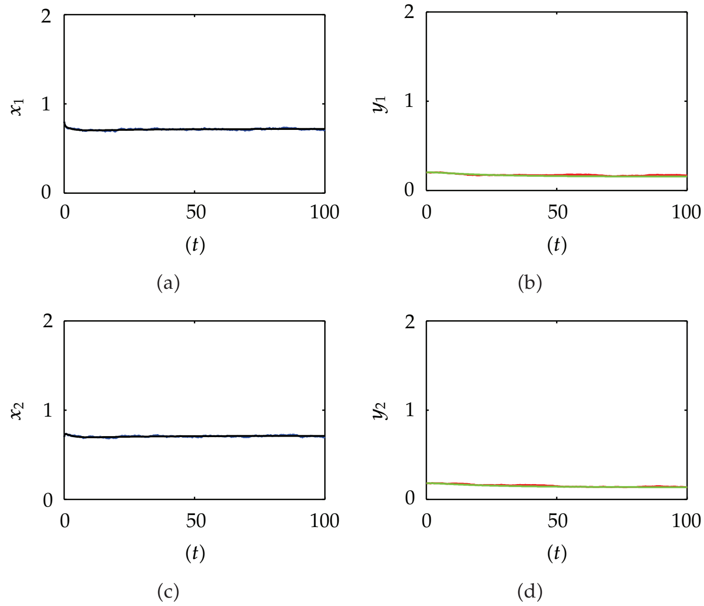

Figure 3: The solution of stochastic system compared to the deterministic system: $\sigma_{11} = \sigma_{12} = \sigma_{21} = \sigma_{22} = 0.01$. The blue and red lines represent the solutions of system (5.1), and the black and green lines represent the solutions of corresponding undisturbed system. These lines are almost coincident.

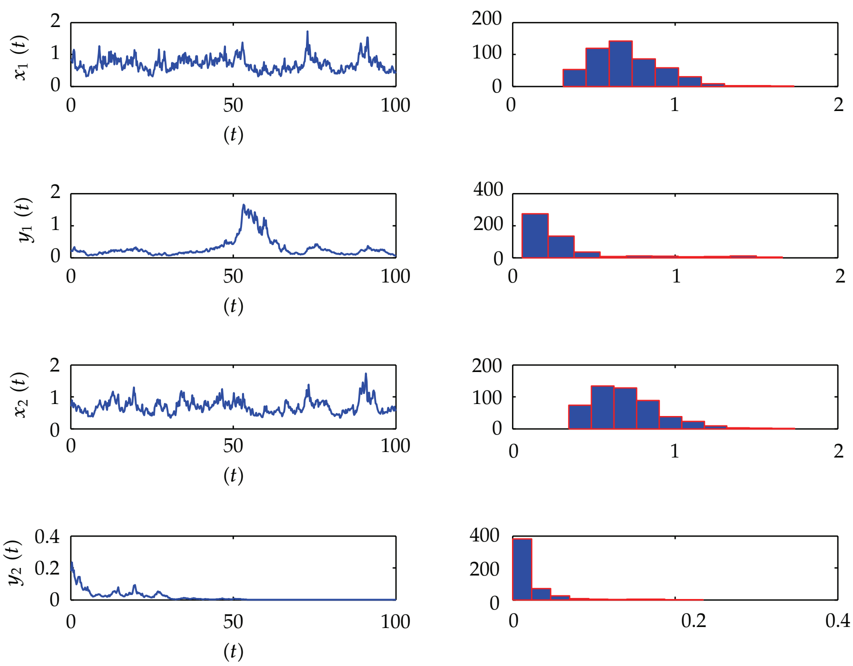

Figure 4: The solution of stochastic system with large white noise and its histogram: $\sigma_{11} = 0.4, \sigma_{12} = 0.3, \sigma_{21} = 0.3, \sigma_{22} = 0.4$. The fluctuations on the left figures are more intense and histogram distribution is not concentrated comparing with Figure 1.

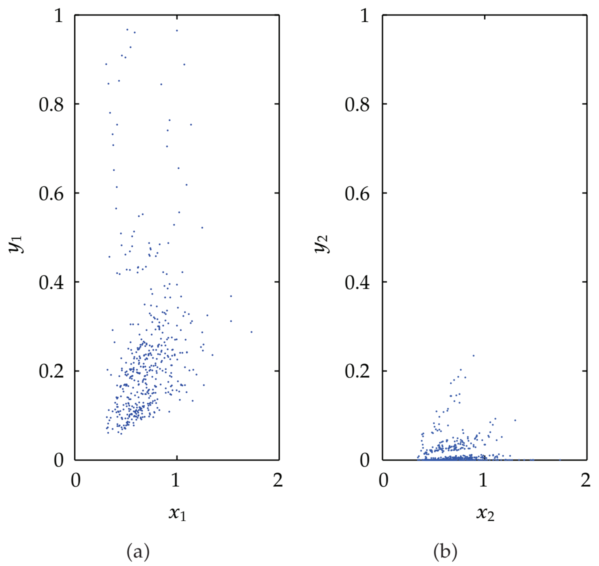

Figure 5: Population distribution scatter corresponding to **Figure 4**. The scatter is not around the equilibrium points of the corresponding deterministic system. There is not a stationary distribution in system (5.1).

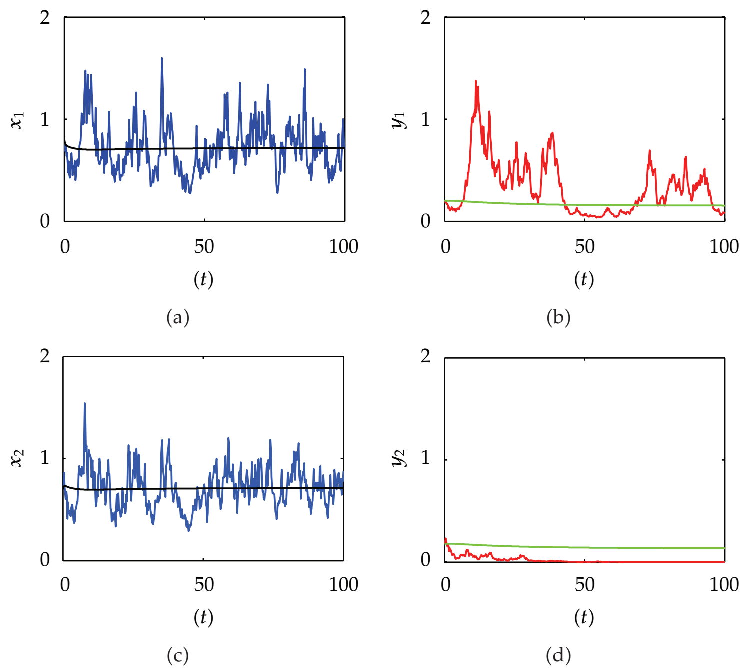

Figure 6: The solution of stochastic system compared to the deterministic system: $\sigma_{11} = 0.4, \sigma_{12} = 0.3, \sigma_{21} = 0.3, \sigma_{22} = 0.4$. The predator population y_2 will die out although its corresponding deterministic system is globally stable. The blue and red lines represent the solutions of system (5.1), and the black and green lines represent the solutions of corresponding undisturb system.

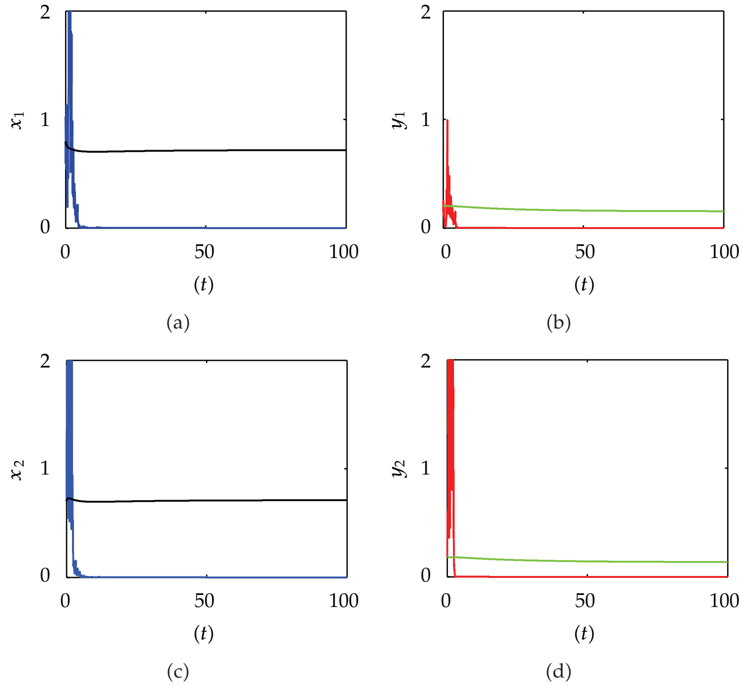

Figure 7: The solution of stochastic system compared to the deterministic system: $\sigma_{11} = 1.4, \sigma_{12} = 1.5, \sigma_{21} = 0.01, \sigma_{22} = 0.01$. The prey population suffer the large white noise which leads to the prey and predator extinction. The blue and red lines represent the solutions of system (5.1), and the black and green lines represent the solutions of corresponding undisturb system.

In order to better study the previous results, we will numerically simulate the solution of (5.1). By the method mentioned in [21], we consider the following discretized equation:

$$\begin{aligned}
x_{1,k+1} &= x_{1,k} + \left[x_{1,k}\left(r_1 - b_1 x_{1,k} - e_1 y_{1,k}\right) + d_{12}\left(x_{2,k} - x_{1,k}\right)\right]h \\
&\quad + \sigma_{11} x_{1,k}\sqrt{h}\xi_{1,k} + \frac{1}{2}\sigma_{11}^2 x_{1,k}\left(h\xi_{1,k}^2 - h\right), \\
y_{1,k+1} &= y_{1,k} + y_{1,k}\left(-\gamma_1 - \delta_1 y_{1,k} + \varepsilon_1 x_{1,k}\right)h + \sigma_{21} y_{1,k}\sqrt{h}\eta_{1,k} + \frac{1}{2}\sigma_{21}^2 y_{1,k}\left(h\eta_{1,k}^2 - h\right), \\
x_{2,k+1} &= x_{2,k} + \left[x_{2,k}\left(r_2 - b_2 x_{2,k} - e_2 y_{2,k}\right) + d_{21}\left(x_{1,k} - x_{2,k}\right)\right]h \\
&\quad + \sigma_{12} x_{2,k}\sqrt{h}\xi_{2,k} + \frac{1}{2}\sigma_{12}^2 x_{2,k}\left(h\xi_{2,k}^2 - h\right), \\
y_{2,k+1} &= y_{2,k} + y_{2,k}\left(-\gamma_2 - \delta_2 y_{2,k} + \varepsilon_2 x_{2,k}\right)h + \sigma_{22} y_{2,k}\sqrt{h}\eta_{2,k} + \frac{1}{2}\sigma_{22}^2 y_{2,k}\left(h\eta_{2,k}^2 - h\right).
\end{aligned} \tag{5.2}$$

Using the discretized equation and the help of Matlab software, we choose the appropriate parameters $r_1 = 0.4, r_2 = 0.3, b_1 = 0.5, b_2 = 0.4, e_1 = e_2 = 0.2, d_{11} = d_{22} = 1, d_{12} = 1.1, d_{21} = 1.2, \gamma_1 = \gamma_2 = 0.2, \delta_1 = \delta_2 = 0.1, \varepsilon_1 = \varepsilon_2 = 0.3$, the initial value $(x_1(0), y_1(0), x_2(0), y_2(0)) = (0.8, 0.2, 0.7, 0.18)$, and time step $h = 0.01$; then $E^* =$

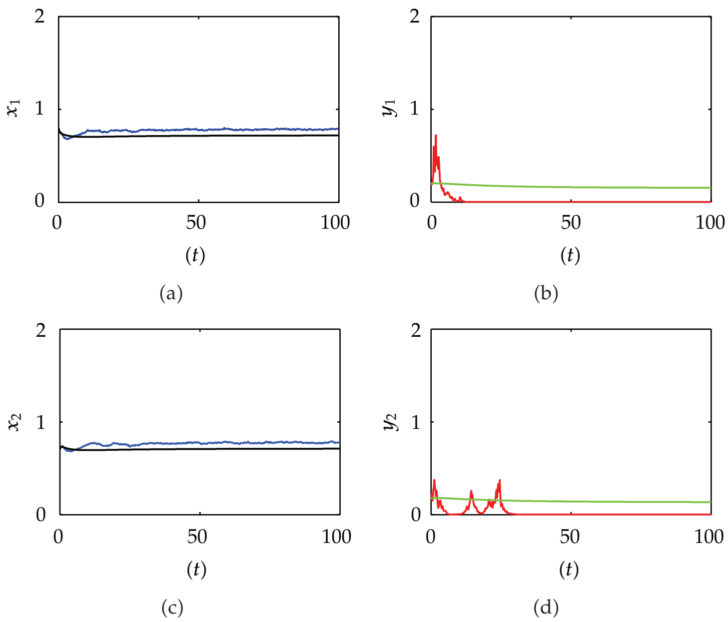

Figure 8: The solution of stochastic system compared to the deterministic system: $\sigma_{11} = \sigma_{12} = 0.01, \sigma_{21} = 0.8, \sigma_{22} = 0.7$. The predator population suffer the large white noise and then extinction. The blue and red lines represent the solutions of system (5.1), and the black and green lines represent the solutions of corresponding undisturb system.

$(x_1^*, y_1^*, x_2^*, y_2^*) \doteq (0.7174, 0.1545, 0.7107, 0.1356)$. In order to better investigate the white noise, we divide its intensity into small, medium, and large three cases to study.

In Figures 1, 2, and 3, we choose $\sigma_{11} = \sigma_{12} = \sigma_{21} = \sigma_{22} = 0.01$ and $c_1 \doteq 0.86088, c_2 \doteq 0.78177$ (c_1, c_2 satisfy Lemma A.1 and Theorem 3.2); then $\delta = (1/2)\sum_{i=1}^{2}(c_i\varepsilon_i b_i x_i^* \sigma_{1i}^2 + c_i e_i \delta_i y_i^* \sigma_{2i}^2) \doteq 9.9 \times 10^{-6}$, and so the condition $\delta < \min\{\min_{1\le i\le 2}\{c_i\varepsilon_i b_i (x_i^*)^2\}, \min_{1\le i\le 2}\{c_i e_i \delta_i (y_i^*)^2\}\} \doteq 4.11 \times 10^{-4}$ is also satisfied. Therefore, by Theorem 3.2, there is a stationary distribution (see the histogram on the right in Figure 1). From the left picture in Figure 1, we can see that the solution of system (5.1) is fluctuating in a small neighborhood. Moreover, from Figure 2, we find that almost all population distribution lies in the neighborhood, which can be imagined by a circular or elliptic region centered at $(x_1^*, y_1^*), (x_2^*, y_2^*)$ (see the scatter picture in Figure 2). Figure 3 shows that when the white noise is small, stochastic system imitates deterministic system and their curves are almost coincident. Hence, the solution of (5.1) is ergodicity, although there is no equilibrium of the stochastic system as the deterministic system. All of these imply system (5.1) is a stochastic stability.

Comparing with small white noise as in Figures 1, 2, and 3, we select the relatively large white noise $\sigma_{11} = 0.4, \sigma_{12} = 0.3, \sigma_{21} = 0.3, \sigma_{22} = 0.4, c_1 \doteq 0.86088, c_2 \doteq 0.78177$ in Figures 4, 5, and 6. We find that $\delta = 0.0107 > \min\{\min_{1\le i\le 2}\{c_i\varepsilon_i b_i (x_i^*)^2\}, \min_{1\le i\le 2}\{c_i e_i \delta_i (y_i^*)^2\}\} = 2.875 \times 10^{-4}$, so the conditions of Theorem 3.2 are not satisfied; therefore, there is not a stationary distribution although the deterministic system is global asymptotic stability. The condition $\check{l} - \hat{\sigma}^2/2 > 0$, that is, all extinction condition does not hold by Theorem 4.3 and the predator y_2 will die. From Figure 4, we see that the fluctuations on the left figures are more intense and histogram distribution is not concentrated comparing with Figure 1, $y_2(t)$ close to the point 0.

In Figure 7, we assume both the predator and the prey population suffered large white noise; we choose $\sigma_{11} = 1.4, \sigma_{12} = 1.5, \sigma_{21} = 0.01, \sigma_{22} = 0.01$, which satisfy the cases said in Theorem 4.1, that is $\check{l}_1 - \widehat{\sigma}_1^2/2 \doteq -0.0239 < 0$. As the case in Theorem 4.1 expected, the large white noise leads to the extinction of the prey which also leads to the extinction of the predator, so the solution of system (5.1) tends to zero.

In Figure 8, we choose the same parameters as in Figure 1, but change the value of σ_{21}, σ_{22} to $\sigma_{21} = 0.8, \sigma_{22} = 0.7$, which do not meet the conditions for extinction of the two species as in Theorem 4.3. The predator population suffer the large white noises and then die out, but the prey will survive.

From these figures, we can get the following conclusions: when the white noise is small, system (5.1) imitates its deterministic system (see Figures 1, 2, and 3). But when the white noise is relatively large, it will bring more big deviation (see Figures 4, 5, and 6) even the extinction of the species (see Figures 6, 7, and 8), which will not happen in the deterministic system. However, when the intensity of the white noise on prey species is large, the predator and prey population will be extinct (see Figure 7). In real world, the large white noise may be bad weather, serious epidemic, which can be considered as the decisive factors responsible for the extinction of populations. Therefore, the study of stochastic model has great practical significance.

Appendix

In this section, we list some theories used in the previous sections.

(1) *Some Graph Theories [2, 22]*

A directed graph or digraph $\mathcal{G}=(V,E)$ contains a set $V = 1,2,\ldots,n$ of vertices and a set E of arcs (i,j) leading from initial vertex i to terminal vertex j. In our convention, $a_{ij} > 0$ if and only if there exists an arc from vertex j to vertex i in $\mathcal{G}$.

A digraph $\mathcal{G}$ is weighted if each arc (j,i) is assigned a positive weight a_{ij}. Given a weighted digraph $(\mathcal{G}, A)$ with n vertices, where $A = (a_{ij})_{n\times n}$ is weight matrix, whose entry a_{ij} equals the weight of arc (j,i) if it exists, and 0 otherwise.

A digraph $(\mathcal{G}, A)$ is strongly connected if for any pair of distinct vertices, there exists a directed path from one to the other. A weighted digraph $(\mathcal{G}, A)$ is strongly connected iff the weight matrix A is irreducible.

The Laplacian matrix of $(\mathcal{G}, A)$ is defined as

$$L_A = \begin{bmatrix} \sum_{i\neq 1} a_{1i} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \sum_{i\neq 2} a_{2i} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \cdots \\ -a_{n1} & a_{n2} & \cdots & \sum_{i\neq n} a_{ni} \end{bmatrix}. \tag{A.1}$$

Let c_i denote the cofactor of the ith diagonal element of L_A, which has the following property.

Lemma A.1 (see [2]). *Assume $n \geq 2$. If $(\mathcal{G}, A)$ is strongly connected. Then*

(1) $c_i > 0$ *for* $1 \leq i \leq n$,

(2) *the following identity holds:*

$$\sum_{i,j=1}^{n} c_i a_{ij} G_i(x_i) = \sum_{i,j=1}^{n} c_i a_{ij} G_j(x_j), \tag{A.2}$$

where $G_i(x_i)$ $(1 \leq i \leq n)$ are arbitrary functions.

(2) *Some Theories about the Stationary Distribution [23]*

Let $X(t)$ be a homogeneous Markov Process in E_l (E_l denotes l-space) described by the stochastic equation

$$dX(t) = b(X)dt + \sum_{r=1}^{k} g_r(X) dB_r(t). \tag{A.3}$$

The diffusion matrix is

$$A(x) = \left(a_{ij}(x)\right), \qquad a_{ij}(x) = \sum_{r=1}^{k} g_r^i(x) g_r^j(x). \tag{A.4}$$

Assumption B. There exists a bounded domain $U \subset E_l$ with regular boundary Γ, having the following properties:

(B.1) In the domain U and some neighborhood thereof, the smallest eigenvalue of the diffusion matrix $A(x)$ is bounded away from zero.

(B.2) If $x \in E_l \setminus U$, the mean time τ at which a path issuing from x reaches the set U is finite and $\sup_{x \in K} E_x \tau < \infty$ for every compact subset $K \subset E_l$.

Lemma A.2 (see [23]). *If Assumption B holds, then the Markov process $X(t)$ has a stationary distribution $\mu(A)$. Let $f(\cdot)$ be a function integrable with respect to the measure μ. Then*

$$P_x \left\{ \lim_{T \to \infty} \frac{1}{T} \int_0^T f(X(t)) dt = \int_{E_l} f(x) \mu(dx) \right\} = 1 \tag{A.5}$$

for all $x \in E_l$.

Remark A.3. Theorem 2.1 shows that there exists a unique positive solution $X(t) = (x_1(t), y_1(t), \ldots x_n(t), y_n(t))$ of SDE (1.3). Besides, from the proof of Theorem 2.1, we obtain

$$LV \leq K. \tag{A.6}$$

Now define $\overline{V} = V + K$; then

$$L\overline{V} \le \overline{V} \tag{A.7}$$

and we can get

$$\overline{V}_R = \inf_{X \in R_+^{2n} \setminus D_m} \overline{V}(X) \longrightarrow \infty \quad \text{as } m \longrightarrow \infty, \tag{A.8}$$

where $D_m = (1/m, m) \times (1/m, m) \times \cdots \times (1/m, m)$. By [23] (Remark 2 of Theorem 4.1), we obtain the solution $X(t)$ is a homogeneous Markov process in R_+^{2n}.

Lemma A.4 (see [23]). *Let $X(t)$ be a regular temporally homogeneous Markov process in E_l. If $X(t)$ is recurrent relative to some bounded domain U, then it is recurrent relative to any nonempty domain in E_l.*

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Research Article

Periodic Solutions of a Class of Fourth-Order Superlinear Differential Equations

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This paper deals with the periodic solutions of a class of fourth-order superlinear differential equations. By using the classical variational techniques and symmetric mountain pass lemma, the periodic solutions of a single equation in literature are extended to that of equations, and also, the cubic growth of nonlinear term is extended to a general form of superlinear growth.

1. Introduction

The existence of periodic solutions of fourth-order differential equations has been studied by more and more researchers [1–6]. The application methods contain mainly Clark theorem [2–4], Cone theory [6], and so on.

For a single equation, Tersian and Chaparova [2] study the existence of infinitely many unbounded solutions, using symmetric mountain pass lemma:

$$\begin{gathered} u^{iv} - pu'' + a(x)u - b(x)u^3 = 0, \quad x \in \mathbf{R}, \\ u(0) = u(L) = 0, \qquad u''(0) = u''(L) = 0. \end{gathered} \tag{1.1}$$

It is a natural problem to wonder whether symmetric mountain pass lemma method may be applied not only to single equations but also to systems of differential equations.

In this paper we study the existence of periodic solutions of the fourth-order equations, by making use of the classical variational techniques and symmetric mountain pass lemma

$$
\begin{gathered}
u^{(4)} - cu'' + a(x)u - \frac{\partial F(x,u,v)}{\partial u} = 0, \quad 0 < x < L, \\
v^{(4)} - dv'' + b(x)v - \frac{\partial F(x,u,v)}{\partial v} = 0, \quad 0 < x < L, \\
u(0) = u''(0) = u(L) = u''(L) = 0, \\
v(0) = v''(0) = v(L) = v''(L) = 0.
\end{gathered}
\tag{1.2}
$$

Through studying System (1.2), (1.1) of the corresponding conclusions are extended.

The paper is organized as follows. In Section 2, we consider the result of System (1.2) under certain conditions. In Section 3, we prove the main result of this paper and give an example.

2. Main Result

In this paper, we state our main result. First we give the following list of assumptions on the parameters in System (1.2):

(**A**) $a(x) > 0$, $b(x) > 0$, $c > -\pi^2/L^2$, $d > -\pi^2/L^2$.

($\mathbf{F_1}$) F is an even functional about (u,v). That is, $F(x,-u,-v) = F(x,u,v)$ for every $(u,v) \in \mathbf{R}^2$.

($\mathbf{F_2}$) There exists $\beta > 2$, as $u^2 + v^2 \neq 0$, we have

$$
u \cdot \frac{\partial F(x,u,v)}{\partial u} + v \cdot \frac{\partial F(x,u,v)}{\partial v} \geq \beta F(x,u,v) > 0 \quad \text{for every } x \in \mathbf{R}. \tag{2.1}
$$

($\mathbf{F_3}$) $F(x,u,v) = o(u^2 + v^2)$ with respect to x consistently, as $u^2 + v^2 \to 0$.

Denote $a_1 = \min_{x\in[0,L]} a(x)$, $a_2 = \max_{x\in[0,L]} a(x)$, $b_1 = \min_{x\in[0,L]} b(x)$, $b_2 = \max_{x\in[0,L]} b(x)$.

From condition (**A**), we obtain $a_i > 0$, $b_i > 0$, when $i = 1, 2$.

Remark 2.1. Let $z = (u,v) \in \mathbf{R}^2$, then condition ($\mathbf{F_2}$) is transformed to

$$
(\nabla F(x,z), z) > \beta F(x,z) > 0 \quad \text{for every } z \neq 0, \tag{2.2}
$$

where $(\cdot,\cdot)$ represents the usual inner product in $\mathbf{R}^2$.

Remark 2.2. From ($\mathbf{F_3}$), we obtain $\lim_{|z|\to 0} F(x,z)/|z|^2 = 0$, where $|\cdot|$ represents normal norm in $\mathbf{R}^2$. Besides, from the continuity of F, we obtain $F(x,0,0) = 0$.

Our main result is as follows.

Theorem 2.3. *Suppose $a(x)$, $b(x)$, and F satisfy* $(\mathbf{A})$, $(\mathbf{F_1})$–$(\mathbf{F_3})$. *Then System* (1.2) *has infinitely many distinct pairs of solutions $z_n = (u_n, v_n)$, which are critical points of the functional $I : X \to \mathbf{R}$, and $I(z_n) \to \infty$ as $n \to \infty$.*

In this paper, the existence of periodic solutions of a single equation in System (1.1) are extended to the case of equations, and also the cubic growth of nonlinear term is extended to a general form of superlinear growth.

3. Variational Structure and the Proof of Result

In this section, we prove the main result stated in Section 2.

3.1. Variational Structure

Denote

$$X(L) = \left(H^2(0,L) \cap H_0^1(0,L)\right)^2. \tag{3.1}$$

Then $X(L)$ is a Hilbert space. The norm is

$$\|z\|^2 = \|u\|_c^2 + \|v\|_d^2, \tag{3.2}$$

where

$$\begin{aligned} \|u\|_c &= \left\{\int_0^L \left[\left|u''(x)\right|^2 + c\left|u'(x)\right|^2 + a(x)|u(x)|^2\right]dx\right\}^{1/2}, \\ \|v\|_d &= \left\{\int_0^L \left[\left|v''(x)\right|^2 + d\left|v'(x)\right|^2 + b(x)|v(x)|^2\right]dx\right\}^{1/2}, \end{aligned} \tag{3.3}$$

$z = (u, v) \in X(L)$. The corresponding inner product are

$$\begin{aligned} \langle z_1, z_2 \rangle| &= \int_0^L \left[\left(z_1'', z_2''\right) + c\left(u_1', u_2'\right) + d\left(v_1', v_2'\right) + a(x)(u_1, u_2) + b(x)(v_1, v_2)\right]dx, \\ \langle u_1, u_2 \rangle|_c &= \int_0^L \left[\left(u_1'', u_2''\right) + c\left(u_1', u_2'\right) + a(x)(u_1, u_2)\right]dx, \\ \langle v_1, v_2 \rangle|_d &= \int_0^L \left[\left(v_1'', v_2''\right) + d\left(v_1', v_2'\right) + b(x)(v_1, v_2)\right]dx. \end{aligned} \tag{3.4}$$

For every $z = (u, v) \in X(L)$, using Poincaré inequality [7], we obtain

$$\int_0^L u^2 dx \leq \frac{L_2}{\pi^2} \int_0^L u'^2 dx, \qquad \int_0^L u'^2 dx \leq \frac{L_2}{\pi^2} \int_0^L u''^2 dx. \tag{3.5}$$

Thus, we can define another norm $\|\cdot\|_1$ in $X(L)$. That is, for every $z \in X(L)$,

$$\|z\|_1 = \left\{ \int_0^L \left| z''(x) \right|^2 dx \right\}^{1/2}. \tag{3.6}$$

The inner product in $X(L)$ as follows:

$$\langle z_1, z_2 \rangle|_1 = \int_0^L \left(z_1''(x), z_2''(x) \right) dx, \quad z_1, z_2 \in X(L). \tag{3.7}$$

The two different norms (3.2) and (3.6) are equivalent in $X(L)$.

In this section we consider System (1.2). The Fréchet derivative of I is given by the following:

$$I(u, v) = \frac{1}{2} \int_0^L \left[u''^2 + cu'^2 + a(x)u^2 + v''^2 + dv'^2 + b(x)v^2 \right] dx - \int_0^L F(x, u, v) dx, \tag{3.8}$$

where $z = (u, v) \in X(L)$.

Remark 3.1. In general, the growth of F is limited by the differentiability of functional I, but we apply truncation techniques in [8]. First, introduce auxiliary functional and the auxiliary functional is Fréchet differentiable. Second, we use critical point theory to prove the existence of critical point of auxiliary functional, then prove the existence of the original equation. However, in order to avoid technical complexity, we assume directly functional I is Fréchet differentiable.

In fact, for every $z = (u, v) \in X(L)$, $\overline{z} = (\overline{u}, \overline{v}) \in X(L)$, we obtain

$$\left\langle I'(z), \overline{z} \right\rangle = \left\langle I_u'(u, v), \overline{u} \right\rangle + \left\langle I_v'(u, v), \overline{v} \right\rangle, \tag{3.9}$$

where

$$\begin{aligned} \left\langle I_u'(u, v), \overline{u} \right\rangle &= \int_0^L \left[u''\overline{u}'' + cu'\overline{u}' + a(x)u\overline{u} - \frac{\partial F(x, u, v)}{\partial u} \overline{u} \right] dx, \\ \left\langle I_v'(u, v), \overline{v} \right\rangle &= \int_0^L \left[v''\overline{v}'' + cv'\overline{v}' + a(x)v\overline{v} - \frac{\partial F(x, u, v)}{\partial v} \overline{v} \right] dx. \end{aligned} \tag{3.10}$$

and $I_u'(u, v), I_v'(u, v) \in [H^2(0, L) \cap H_0^1(0, L)]^*$, $I'(z) \in X(L)^*$.

It is similar to the discussion of [8], the solutions of System (1.2) corresponds to the critical point of the functional I, so we need to discuss the critical point of functional I. In order to prove Theorem 2.3, we introduce below definition and lemma.

Definition 3.2 (see [9]). Let X be a real Banach space, $I \in C^1(X, R)$, I is a Fréchet continuously differentiable functional in $X(L)$. I is said to be satisfying *Palais-Smale* (PS) condition if any sequence $\{u_n\} \subset X$ for which $\{I(u_n)\}$ is bounded and $\{I'(u_n)\} \to 0$ as $j \to \infty$, possesses a convergent subsequence.

Lemma 3.3 (see [8]). *Let X be an infinite dimensional Banach space and $(X_n)_n$ be a sequence of finite dimensional subspaces of X such that* $\dim X_n = n$,

$$X_1 \subset X_2 \subset \cdots \subset X_n \subset X, \qquad \overline{\bigcup_{n=1}^{\infty} X_n} = X. \tag{3.11}$$

Let $I \in C^1(X, R)$ be an even functional, $I(0) = 0$, and I satisfy (PS) condition. Suppose that

- $(\mathbf{A_1})$ *there are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho} \geq \alpha$, and*
- $(\mathbf{A_2})$ *for every n there is an $R_n > 0$ such that $I \leq 0$ on $X_n \setminus B_{R_n}$.*

Then I possesses infinitely many pairs of critical points with unbounded sequence of critical values.

3.2. *The Proof of Result*

Step 1 (Functional I satisfies (PS) condition). Let $\{z_n\} = \{(u_n, v_n)\}$ be a (PS) sequence in X, that is, $\{I(z_n)\}$ is bounded and $I'(z_n) \to 0$, as $n \to \infty$. Suppose that $\{z_n\}$ is unbounded in X, that is, $\|z_n\| \to \infty$ as $n \to \infty$. Since

$$I(z_n) + \frac{1}{\gamma}\left\|I'(z_n)\right\|\|z_n\| \geq I(z_n) - \frac{1}{\gamma}\left\langle I'(z_n), z_n\right\rangle = \frac{1}{\gamma}\|z_n\|^2, \tag{3.12}$$

it follows that

$$\frac{I(z_n)}{\|z_n\|^2} + \frac{\|I'(z_n)\|}{\gamma\|z_n\|} \geq \frac{1}{\gamma}, \tag{3.13}$$

where $\gamma \geq 4$. Letting $n \to \infty$ in (3.13), we have a contradiction with $\|z_n\| \to \infty$ as $n \to \infty$.

Therefore $\{z_n\}$ is a bounded sequence in $X(L)$. Passing if necessary to a subsequence we may assume that $\{z_n\}$ is weakly convergent to a function $z \in X(L)$, $z_n \rightharpoonup z$ in $X(L)$, and $z_n \to z$ in $C[(0, L)]$.

From the Lebesgue theorem, $z \in X(L)$, $z_n \rightharpoonup z$ in $X(L)$, and $z_n \to z$ in $C[(0,L)]$, letting $n \to \infty$ in (3.9)

$$\begin{aligned}\left\langle I'(z_n, z_n)\right\rangle &= \|z_n\|^2 - \int_0^L \frac{\partial F(x,u_n,v_n)}{\partial u} u_n dx - \frac{\partial F(x,u_n,v_n)}{\partial v} v_n dx,\\ \left\langle I'(z_n, z)\right\rangle &= \langle z_n, z\rangle - \int_0^L \frac{\partial F(x,u_n,v_n)}{\partial u} u\, dx - \frac{\partial F(x,u_n,v_n)}{\partial v} v\, dx,\end{aligned} \tag{3.14}$$

we obtain

$$\lim_{n\to\infty} \|z_n\|^2 = \int_0^L \frac{\partial F(x,u,v)}{\partial u} u\, dx + \frac{\partial F(x,u,v)}{\partial v} v\, dx = \|z\|^2. \tag{3.15}$$

From (3.15) and $z \in X(L)$, $z_n \rightharpoonup z$ in $X(L)$, we have $\|z_n - z\| \to 0$ as $n \to \infty$.

Remark 3.4. γ is the largest sum of the order of u and v.

Step 2 (Geometric conditions). Let $e_1 = (1,0)$, $e_2 = (0,1)$, then $\{e_1, e_2\}$ constitutes a pair of standard orthogonal base in $\mathbf{R}^2$. Let us define X_{2m} to be the subspace of $X(L)$

$$X_{2m} = \operatorname{span}\left\{\sin\frac{k\pi x}{L} e_i,\ i = 1,2,\ k = 1,2,\dots,m\right\}, \tag{3.16}$$

for every $m \in \mathbf{N}$. We have $\dim X_{2m} = 2m$, $X_1 \subset X_2 \cdots \subset X_{2m} \subset X$, $\overline{\bigcup_{n=1}^{\infty} X_n} = X$.

For a given constant $\rho > 0$, define a bounded closed set $K \subset X_{2m}$

$$K = \left\{z = (u,v) \in X_{2m} \mid z = \sum_{k=1}^{m}\left[\alpha_k \sin\frac{k\pi x}{L} e_1 + \beta_k \sin\frac{k\pi x}{L} e_2\right], \sum_{k=1}^{m}\left(\alpha_k^2 + \beta_k^2\right) = \rho^2\right\}. \tag{3.17}$$

Define mapping $H : X_{2m} \to \mathbf{R}^{2m}$. For any $z \in X_{2m}$, we obtain

$$H(z) = \frac{(\alpha_1, \beta_1, \alpha_2, \beta_2, \dots, \alpha_m, \beta_m)}{\rho}. \tag{3.18}$$

It is clear that H is a linear odd mapping. For every $z \in X_{2m}$, we have

$$\begin{aligned}\|z\|_1^2 &= \int_0^L \left[\left|u''(x)\right|^2 + \left|v(x)''\right|^2\right] dx\\ &= \frac{\pi^4}{2L^3}\sum_{k=1}^{m} k^4\left(\alpha_k^2 + \beta_k^2\right).\end{aligned} \tag{3.19}$$

So

$$\frac{\rho^2\pi^4}{2L^3}|H(z)|^2 \leq |z|_1^2 \leq \frac{\rho^2(m\pi)^4}{2L^3}|H(z)|^2. \tag{3.20}$$

From (3.20), we obtain H is an odd homeomorphism from X_{2m} to $\mathbf{R}^{2m}$. Then H is an odd homeomorphism from K to S^{2m-1}, since $H(K) = S^{2m-1}$.

On one hand, from functional (3.8) and using Sobolev's embedding theorem, we obtain

$$\begin{aligned} I(z) &\geq \frac{1}{2}\|z\|^2 - \varepsilon|z|^2 L \\ &\geq \frac{1}{2}\|z\|^2 - \varepsilon\frac{\pi^2}{L}\|z\|^2. \end{aligned} \tag{3.21}$$

Thus condition $(\mathbf{A_1})$ is fulfilled if $\varepsilon = L/4\pi^2$, $\rho = \|z\|/2$.

On the other hand, as $-F(x,u,v) < 0$, then there exists σ, such that $-F(x,u,v) < -(1/4)\sigma\|z\|_1^4$.

Denote $A(n) = (n\pi/L)^4 + p(n\pi/L)^2 + a$. From functional (3.8), we obtain

$$\begin{aligned} I(z) &\leq \frac{1}{2}\int_0^L \left(z''^2 + pz'^2 + az^2\right)dx - \int_0^L F(x,z)dx \\ &\leq \frac{L}{4}A(n)\|z\|_1^2 - \int_0^L F(x,z)dx \\ &\leq \frac{L}{4}A(n)\|z\|_1^2 - \frac{L}{4}\sigma\|z\|_1^4, \end{aligned} \tag{3.22}$$

where $p = \max\{c,d\}$, $a = \max\{a_2,b_2\}$. Here choosing $R_n = \|z\|_1 \geq \sqrt{A(n)/\sigma}$, we obtain

$$I(z) \leq 0. \tag{3.23}$$

So $(\mathbf{A_2})$ holds. The proof of Theorem 2.3 is completed.

Example 3.5. In System (1.2), consider the problem:

$$F(x,u,v) = p_0(x)u^n + p_1(x)u^{n-1}v + \cdots + p_i(x)u^{n-i}v^i + \cdots + p_{n-1}(x)uv^{n-1} + p_n(x)v^n, \tag{3.24}$$

where $p_i(x) \geq 0$, but there exists at least one $p_i(x) \neq 0$, n is an even and $n \geq 4$, $i = 0,1,2,\ldots,n$.

It is obvious that $F(x,-u,-v) = F(x,u,v)$ and $F(x,u,v) = o(u^2+v^2)$ as $u^2+v^2 \to 0$.

For the superlinear property, we calculate that

$$
\begin{aligned}
u \cdot \frac{\partial F(x,u,v)}{\partial u} &+ v \cdot \frac{\partial F(x,u,v)}{\partial v} \\
&= np_0(x)u^n + (n-1)p_1(x)u^{n-1}v + \cdots + (n-i)p_i(x)u^{n-i}v^i + \cdots + p_{n-1}(x)uv^{n-1} \\
&\quad + p_1(x)u^{n-1}v + \cdots + ip_i(x)u^{n-i}v^i + \cdots + (n-1)p_{n-1}(x)uv^{n-1} + np_n(x)v^n \\
&= nF(x,u,v) \\
&\geq 4F(x,u,v).
\end{aligned} \tag{3.25}
$$

Therefore, there exists $\beta = 4 > 2$, as $u^2 + v^2 \neq 0$, we have

$$
u \cdot \frac{\partial F(x,u,v)}{\partial u} + v \cdot \frac{\partial F(x,u,v)}{\partial v} \geq 4F(x,u,v) > 0 \quad \text{for every } x \in \mathbf{R}. \tag{3.26}
$$

So F satisfies the conditions $(\mathbf{F_1})$–$(\mathbf{F_3})$. We only choose $a(x) > 0$, $b(x) > 0$, $c > -\pi^2/L^2$, $d > -\pi^2/L^2$, then the condition $(\mathbf{A})$ is satisfied. Therefore, System (1.2) has infinitely many distinct pairs of solutions by using Theorem 2.3.

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Research Article

Multiplicity of Solutions for Perturbed Nonhomogeneous Neumann Problem through Orlicz-Sobolev Spaces

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We investigate the existence of multiple solutions for a class of nonhomogeneous Neumann problem with a perturbed term. By using variational methods and three critical point theorems of B. Ricceri, we establish some new sufficient conditions under which such a problem possesses three solutions in an appropriate Orlicz-Sobolev space.

1. Introduction

Consider the following nonhomogeneous Neumann problem with a perturbed term:

$$\begin{aligned} -\operatorname{div}(\alpha(|\nabla u|)\nabla u) + \alpha(|u|)u &= \lambda f(x,u) + \mu g(x,u), \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{$P_{\lambda,\mu}$}$$

where Ω is a bounded domain in $\mathbb{R}^N (N \geq 3)$ with smooth boundary $\partial\Omega$, ν is the outer normal to $\partial\Omega$, $f, g : \Omega\times\mathbb{R} \to \mathbb{R}$ are two Carathéodory functions, $\lambda > 0, \mu \geq 0$ are two parameters, and the function $\alpha : (0,\infty) \to \mathbb{R}$ is such that $\varphi(t) : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi(t) = \begin{cases} \alpha(|t|)t, & t \neq 0, \\ 0, & t = 0, \end{cases} \tag{1.1}$$

is an odd, strictly increasing homeomorphism from $\mathbb{R}$ to $\mathbb{R}$.

It is well known that these kinds of problems are important in applications in many fields, such as elasticity, fluid dynamics, and image processing (see [1–4]). Since the operator in the divergence form is nonhomogeneous, we introduce Orlicz-Sobolev space which is an appropriate setting for these problems. Such space originated with Nakano [5] and was developed by Musielak and Orlicz [6]. Many properties of Sobolev spaces have been extended to Orlicz-Sobolev space (see [7–10]). Several authors have widely studied the existence of solutions for the relevant problem by means of variational techniques, monotone operator methods, fixed point, and degree theory (see [11–15]). To the best of our knowledge, for the perturbed nonhomogeneous Neumann problem, there has so far been few papers concerning its multiple solutions. Motivated by the above facts, in this paper, we establish some new sufficient conditions under which such a problem possesses three weak solutions in Orlicz-Sobolev space.

This paper is organized as follows. In Section 2, some preliminaries are presented. In Section 3, we discuss the existence of three weak solutions for problem $(P_{\lambda,\mu})$.

2. Preliminaries

We start by recalling some basic facts about Orlicz-Sobolev space. Let φ be as in Introduction and $\Phi(t) : \mathbb{R} \to \mathbb{R}$,

$$\Phi(t) = \int_0^t \varphi(s)ds. \tag{2.1}$$

We observe that Φ is, a Young function, that is $\Phi(0) = 0$, Φ is convex and $\lim_{t\to\infty}\Phi(t) = +\infty$. Furthermore, since $\Phi(t) = 0$ if and only if $t = 0$, $\lim_{t\to 0}(\Phi(t)/t) = 0$, and $\lim_{t\to\infty}(\Phi(t)/t) = +\infty$, then Φ is called an N-function. The function Φ^* is called the complementary function of Φ and it satisfies

$$\Phi^*(t) = \sup\{st - \Phi(s); s \geq 0\}, \quad \forall t \geq 0. \tag{2.2}$$

Assume that Φ satisfies the following structural hypotheses

(Φ_1) $1 < \liminf_{t\to\infty}(t\varphi(t)/\Phi(t)) \leq p^0 := \sup_{t>0}(t\varphi(t)/\Phi(t)) < \infty$;

(Φ_2) $N < p_0 := \inf_{t>0}(t\varphi(t)/\Phi(t)) < \liminf_{t\to\infty}(\log(\Phi(t))/\log(t))$.

Further, we also assume that the function

(Φ_3) $[0,\infty) \ni t \to \Phi(\sqrt{t})$ is convex.

The Orlicz space $L_\Phi(\Omega)$ defined by Φ is the space of measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\|u\|_{L_\Phi} := \sup\left\{\int_\Omega u(x)v(x)dx; \int_\Omega \Phi^*(|v(x)|)dx \leq 1\right\} < \infty. \tag{2.3}$$

Then $(L_\Phi(\Omega), \|\cdot\|_{L_\Phi})$ is a Banach space whose norm is equivalent to the Luxemburg norm

$$\|u\|_\Phi := \inf\left\{k > 0; \int_\Omega \Phi\left(\frac{|u(x)|}{k}\right)dx \leq 1\right\}. \tag{2.4}$$

We denote by $W^1L_\Phi(\Omega)$ the Orlicz-Sobolev space, defined by

$$W^1L_\Phi(\Omega) = \left\{u \in L_\Phi; \frac{\partial u}{\partial x_i} \in L_\Phi, i = 1, 2, \dots, N\right\}. \tag{2.5}$$

This is a Banach space with respect to the norm

$$\|u\|_{1,\Phi} = \||\nabla u|\|_\Phi + \|u\|_\Phi. \tag{2.6}$$

Lemma 2.1 (see [13]). *On $W^1L_\Phi(\Omega)$ the norms*

$$\begin{aligned} \|u\|_{1,\Phi} &= \||\nabla u|\|_\Phi + \|u\|_\Phi, \\ \|u\|_{2,\Phi} &= \max\{\||\nabla u|\|_\Phi, \|u\|_\Phi\}, \\ \|u\| &:= \inf\left\{\mu > 0; \int_\Omega \left[\Phi\left(\frac{|u(x)|}{\mu}\right) + \Phi\left(\frac{|\nabla u(x)|}{\mu}\right)\right] dx \le 1\right\}, \end{aligned} \tag{2.7}$$

are equivalent. Moreover, for every $u \in W^1L_\Phi(\Omega)$, one has

$$\|u\| \le 2\|u\|_{2,\Phi} \le 2\|u\|_{1,\Phi} \le 4\|u\|. \tag{2.8}$$

Lemma 2.2. *Let $u \in W^1L_\Phi(\Omega)$, then*

$$\begin{aligned} \|u\|^{p_0} &\le \int_\Omega [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)]dx \le \|u\|^{p^0}, \quad \textit{if } \|u\| > 1, \\ \|u\|^{p^0} &\le \int_\Omega [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)]dx \le \|u\|^{p_0}, \quad \textit{if } \|u\| < 1. \end{aligned} \tag{2.9}$$

Proof. For the proof of

$$\begin{aligned} \|u\|^{p_0} &\le \int_\Omega [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)]dx, \quad \text{if } \|u\| > 1, \\ \|u\|^{p^0} &\le \int_\Omega [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)]dx, \quad \text{if } \|u\| < 1, \end{aligned} \tag{2.10}$$

we can see Lemma 2.2 of the paper [13]. Since $p^0 \ge (t\varphi(t))/\Phi(t)$ for all $t \ge 0$, it follows that letting $\sigma > 1$, we have

$$\log(\Phi(\sigma t)) - \log(\Phi(t)) = \int_t^{\sigma t} \frac{\varphi(s)}{\Phi(s)} ds \le \int_t^{\sigma t} \frac{p^0}{s} ds = \log\left(\sigma^{p^0}\right). \tag{2.11}$$

Thus, one has

$$\Phi(\sigma t) \le \sigma^{p^0}\Phi(t), \quad t > 0, \ \sigma > 1. \tag{2.12}$$

Moreover, by the definition of the norm, we remark that

$$\int_\Omega \left[\Phi\left(\frac{|u(x)|}{\|u\|}\right) + \Phi\left(\frac{|\nabla u(x)|}{\|u\|}\right)\right] dx \le 1. \tag{2.13}$$

Therefore, we have

$$\begin{aligned}\int_\Omega [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)]dx &= \int_\Omega \left[\Phi\left(\|u\|\frac{|u(x)|}{\|u\|}\right) + \Phi\left(\|u\|\frac{|\nabla u(x)|}{\|u\|}\right)\right] dx \\ &\le \|u\|^{p^0} \int_\Omega \left[\Phi\left(\frac{|u(x)|}{\|u\|}\right) + \Phi\left(\frac{|\nabla u(x)|}{\|u\|}\right)\right] dx \\ &\le \|u\|^{p^0},\end{aligned} \tag{2.14}$$

for all $\|u\| > 1$.

Similar techniques as those used in the proof of (2.12), we have

$$\Phi(t) \le \tau^{p_0}\Phi\left(\frac{t}{\tau}\right), \quad t > 0,\ 0 < \tau < 1. \tag{2.15}$$

Therefore, we can obtain

$$\begin{aligned}\int_\Omega [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)]dx &\le \|u\|^{p_0} \int_\Omega \left[\Phi\left(\frac{|u(x)|}{\|u\|}\right) + \Phi\left(\frac{|\nabla u(x)|}{\|u\|}\right)\right] dx \\ &\le \|u\|^{p_0},\end{aligned} \tag{2.16}$$

for all $\|u\| < 1$. □

Lemma 2.3 (see [13]). *Let $u \in W^1L_\Phi(\Omega)$ and assume that*

$$\int_\Omega [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)]dx \le r \tag{2.17}$$

for some $0 < r < 1$, then one has $\|u\| < 1$.

Lemma 2.4 (see [13]). *If $p_0 > N$, then $W^1L_\Phi(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$ and there exists a constant $c > 0$ such that*

$$\|u\|_\infty \le c\|u\|_{1,\Phi}, \quad \forall u \in W^1L_\Phi(\Omega), \tag{2.18}$$

where $\|u\|_\infty := \sup_{x\in\overline{\Omega}}|u(x)|$.

Now, *one recall*, a three critical theorem of B. Ricceri. If X is a real Banach space, denote by $\mathcal{W}_X$ (see [16]) the class of all functionals $\Phi : X \to \mathbb{R}$ possessing the following property: if $\{u_n\}$ is a sequence in X converging weakly to u and $\liminf_{n\to\infty}\Phi(u_n) \le \Phi(u)$, then $\{u_n\}$

has a subsequence converging strongly to u. For example, if X is uniformly convex and $g : [0,+\infty) \to \mathbb{R}$ is a continuous, strictly increasing function, then, by a classical results, the functional $u \to g(\|u\|)$ belongs to the class $\mathcal{W}_X$.

Lemma 2.5 (see [16]). *Let X be a separable and reflexive real Banach space; let $I : X \to \mathbb{R}$ be a coercive, sequentially weakly lower semicontinuous C^1 functional, belonging to $\mathcal{W}_X$, bounded on each bounded subset of X and whose derivative admits a continuous inverse on X^*; $J : X \to \mathbb{R}$ a C^1 functional with compact derivative. Assume that I has a strict local minimum u_0 with $I(u_0) = J(u_0) = 0$. Finally, setting*

$$\alpha' = \max\left\{0, \limsup_{\|u\|\to+\infty}\frac{J(u)}{I(u)}, \limsup_{u\to u_0}\frac{J(u)}{I(u)}\right\},$$
$$\beta' = \sup_{u\in I^{-1}(0,+\infty)}\frac{J(u)}{I(u)}, \tag{2.19}$$

assume that $\alpha' < \beta'$. Then for each compact interval $[a,b] \subset (1/\beta', 1/\alpha')$ (with the conventions $(1/0) = +\infty, (1/+\infty) = 0$), there exists $B > 0$ with the following property: for every $\lambda \in [a,b]$ and every C^1 functional $\Psi : X \to \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that, for each $\mu \in [0,\delta]$, the equation

$$I'(x) = \lambda J'(x) + \mu\Psi'(x) \tag{2.20}$$

has at least three solutions in X whose norms are less than B.

Lemma 2.6 (see [17]). *Let X be a reflexive real Banach space; $S \subset \mathbb{R}$ an interval, let $I : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous C^1 functional, bounded on each bounded subset of X and whose derivative admits a continuous inverse on X^*; $J : X \to \mathbb{R}$ a C^1 functional with compact derivative. Assume that*

$$\lim_{\|u\|\to\infty}[I(u) - \lambda J(u)] = +\infty \tag{2.21}$$

for all $\lambda \in S$, and that there exists $\rho \in \mathbb{R}$ such that

$$\sup_{\lambda\in S}\inf_{u\in X}\left[I(u) + \lambda(\rho - J(u))\right] < \inf_{u\in X}\sup_{\lambda\in S}\left[I(u) + \lambda(\rho - J(u))\right]. \tag{2.22}$$

Then there exist a nonempty open set $A \subset S$ and a positive number B, with the following property: for every $\lambda \in A$ and every C^1 functional $\Psi : X \to \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that, for each $\mu \in [0,\delta]$, the equation

$$I'(x) = \lambda J'(x) + \mu\Psi'(x) \tag{2.23}$$

has at least three solutions in X whose norms are less than B.

Lemma 2.7 (see [18]). *Let X be a nonempty set and I, J two real functions on X. Assume that there are $r > 0$ and $x_0, x_1 \in X$ such that*

$$I(x_0) = J(x_0) = 0, \qquad I(x_1) > r, \qquad \sup_{x \in I^{-1}(]-\infty, r])} J(x) < r\frac{J(x_1)}{I(x_1)}. \tag{2.24}$$

Then for each ρ satisfying

$$\sup_{x \in I^{-1}(]-\infty, r])} J(x) < \rho < r\frac{J(x_1)}{I(x_1)}, \tag{2.25}$$

one has

$$\sup_{\lambda \geq 0} \inf_{u \in X} [I(u) + \lambda(\rho - J(u))] < \inf_{u \in X} \sup_{\lambda \geq 0} [I(u) + \lambda(\rho - J(u))]. \tag{2.26}$$

3. Proof of the Main Results

Set $\gamma = \inf\{(\int_\Omega (\Phi(|\nabla u(x)|) + \Phi(|u(x)|))dx / \int_\Omega F(x, u(x))dx) : u \in X, \int_\Omega F(x, u(x))dx > 0\}$.

Theorem 3.1. *Let Φ be a function satisfying the structural hypotheses (Φ_1–Φ_3) and the following conditions hold*

(H_1) $\max\{\limsup_{\xi \to 0}(\sup_{x \in \Omega} F(x, \xi)/|\xi|^{p^0}), \limsup_{|\xi| \to \infty}(\sup_{x \in \Omega} F(x, \xi)/|\xi|^{p_0})\} \leq 0$,

(H_2) $\sup_{u \in X} \int_\Omega F(x, u(x))dx > 0$.

Then, for each compact interval $[a, b] \subset (\gamma, \infty)$, there exists $B > 0$ with the following property: for every $\lambda \in [a, b]$ and g, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the problem $(P_{\lambda,\mu})$ has at least three weak solutions whose norms in X are less than B.

Proof of Theorem 3.1. In order to apply Lemma 2.5, we let

$$\begin{aligned} I(u) &= \int_\Omega [\Phi(|\nabla u(x)|) + \Phi(|u(x)|)]dx, \\ J(u) &= \int_\Omega F(x, u(x))dx, \qquad \Psi(u) = \int_\Omega G(x, u(x))dx. \end{aligned} \tag{3.1}$$

We divide our proof into two steps as follows.

Step 1. We show that some fundamental assumptions are satisfied.

$X := W^1L_\Phi(\Omega)$. Obviously, X is a separable and reflexive real Banach space (see [13]). By Lemma 2.2, it is easy to see that $I(u)$ is a coercive, bounded on each bounded subset of X. On the other hand, $I, J, \Psi \in C^1(X, \mathbb{R})$ with the derivatives given by

$$\begin{aligned} \langle I'(u), v \rangle &= \int_\Omega [\alpha(|\nabla u(x)|)\nabla u(x) \cdot \nabla v(x) + \alpha(|u(x)|)u(x)v(x)]dx, \\ \langle J'(u), v \rangle &= \int_\Omega f(x, u(x))v(x)dx, \qquad \langle \Psi'(u), v \rangle = \int_\Omega g(x, u(x))v(x)dx, \end{aligned} \tag{3.2}$$

for any $u, v \in X$. Hence, the critical points of the functional $I - \lambda J - \mu\Psi$ are exactly the weak solutions for problem $(P_{\lambda,\mu})$. Moreover, owing that Φ is convex, it follows that I is convex. Hence, one has that I is sequentially weakly lower semicontinuous. The fact X is compactly embedded into $C^0(\overline{\Omega})$ implies that operators J', Ψ' is compact. As the proof of Lemma 3.2 in [15], we know that I' has a continuous inverse.

Moreover, if $\{u_n\}$ is a sequence in X converging weakly to u and $\liminf_{n\to\infty} I(u_n) \leq I(u)$, remark that I is sequentially weakly lower semicontinuous, one has

$$I(u) \leq \liminf_{n\to\infty} I(u_n) \leq I(u). \tag{3.3}$$

Then, up to a subsequence, we deduce that $I(u_n) \to I(u) = d$. Taking into account that $\{(u_n + u)/2\}$ converges weakly to u and I is sequentially weakly lower semicontinuous, we have

$$d = I(u) \leq \liminf_{n\to\infty} I\left(\frac{u_n + u}{2}\right). \tag{3.4}$$

We assume by contradiction that u_n does not converge to u in X. Hence, there exist $\varepsilon_0 > 0$ and a subsequence $\{u_{n_m}\}$ of (u_n) such that

$$\left\|\frac{u_{n_m} - u}{2}\right\| > \varepsilon_0, \quad \forall m. \tag{3.5}$$

Then there exists $\varepsilon_1 > 0$ such that

$$I\left(\frac{u_{n_m} - u}{2}\right) > \varepsilon_1, \quad \forall m. \tag{3.6}$$

On the other hand,

$$\frac{1}{2}I(u) + \frac{1}{2}I(u_{n_m}) - I\left(\frac{u_{n_m} + u}{2}\right) \geq I\left(\frac{u_{n_m} - u}{2}\right) > \varepsilon_1, \tag{3.7}$$

(see [19]). Letting $m \to \infty$ in the above inequality we obtain

$$d - \varepsilon_1 \geq \limsup_{m\to\infty} I\left(\frac{u_{n_m} + u}{2}\right) \tag{3.8}$$

and that is a contradiction with (3.4). It follows that u_n converges strongly to u and $I \in \mathcal{W}_X$. In addition, $I(0) = J(0) = 0$.

Step 2. We show that $\alpha' = 0, \beta' > 0$.

In view of (H_1), for all $\varepsilon > 0$, there exists $\tau_1 > 0$ such that

$$F(x, \xi) \leq \varepsilon|\xi|^{p^0}, \tag{3.9}$$

for any $|\xi| \in [0, \tau_1]$. For $\|u\| < \min\{1, (\tau_1/2c)\}$, we have

$$|u(x)| \leq \|u\|_\infty \leq c\|u\|_{1,\Phi} \leq 2c\|u\| \leq \tau_1,$$
$$\limsup_{u\to 0} \frac{J(u)}{I(u)} \leq \limsup_{u\to 0} \frac{\varepsilon \int_\Omega |u|^{p^0} dx}{\|u\|^{p^0}} \leq \limsup_{u\to 0} \frac{\varepsilon |\Omega| (2c\|u\|)^{p^0}}{\|u\|^{p^0}} \leq 0. \tag{3.10}$$

By (H_1), for all $\varepsilon > 0$, there exists $0 < \tau_1 < \tau_2$ such that

$$F(x,\xi) \leq \varepsilon |\xi|^{p_0}, \tag{3.11}$$

for any $|\xi| > \tau_2$. Further, for each $\|u\| > 1$, we have

$$\begin{aligned}\frac{J(u)}{I(u)} &\leq \frac{\int_{\Omega(|u|\leq\tau_2)} F(x,u(x))dx}{\|u\|^{p_0}} + \frac{\int_{\Omega(|u|>\tau_2)} F(x,u(x))dx}{\|u\|^{p_0}} \\ &\leq \frac{\int_{\Omega(|u|\leq\tau_2)} F(x,u(x))dx}{\|u\|^{p_0}} + \frac{\varepsilon|\Omega|(2c\|u\|)^{p_0}}{\|u\|^{p_0}}.\end{aligned} \tag{3.12}$$

So we get

$$\limsup_{\|u\|\to\infty} \frac{J(u)}{I(u)} \leq 0. \tag{3.13}$$

Then, with the notation of Lemma 2.5, we have $\alpha' = 0$. By assumption (H_2), we have $\beta' > 0$. Thus, all the hypotheses of Lemma 2.5 are satisfied. Clearly, $\gamma = 1/\beta'$. Finally, by Lemma 2.5, we can obtain the Theorem 3.1. □

Example 3.2. Let $p > N + 1$. Define

$$\varphi(t) = \frac{|t|^{p-2}t}{\log(1+|t|)}, \quad t \neq 0, \tag{3.14}$$

and $\varphi(0) = 0$. By [13], one has

$$p_0 = p - 1 < p^0 = p. \tag{3.15}$$

Let $F(x,t) = |t|^{p+1} - |t|^{p+2}$. Since $F \leq 0$ if $|t|$ is large enough and $F > 0$ if $|t|$ is small enough, moreover, it is easy to see $\limsup_{\xi\to 0}(\sup_{x\in\Omega} F(x,\xi)/|\xi|^{p^0}) = 0$, the conditions of Theorem 3.1 can be satisfied.

Remark 3.3. Since F in [13] is p_0-sublinear, the results of [13] do not fit to the problem treated in the previous Example 3.2 even if $\mu = 0$, that is, there is no perturbed nonlinear term. In addition, for nonhomogeneous Neumann problem with a perturbed term, we can have the following result when F is p_0-sublinear, which extends the results of [13].

Theorem 3.4. *Let Φ be a function satisfying the structural hypotheses (Φ_1–Φ_3) and the following conditions.*

(H_3) There exist two constants γ, δ with $\gamma < 2c$ such that $\Phi(\delta) > (\gamma^{p^0}/(2c)^{p^0}|\Omega|)$ and

$$\frac{\int_\Omega \max_{|\xi|\le\gamma} F(x,\xi)dx}{\gamma^{p^0}} < \frac{\int_\Omega F(x,\delta)dx}{(2c)^{p^0}|\Omega|\Phi(\delta)}, \tag{3.16}$$

where $|\Omega|$ denotes the Lebesgue measure of the set Ω.

(H_4) There exist $h(x), k(x) \in L^1(\Omega;\mathbb{R}^+)$ and $0 < s < p_0$ such that $|F(x,t)| \le h(x) + k(x)|t|^s$ for every $(x,t) \in \Omega \times \mathbb{R}$.

Then, there exist a nonempty open set $A \subset [0,\infty)$ and a positive number B, for each $\lambda \in A$ and for every g, there exists $\delta > 0$ such that, for each $\mu \in [0,\delta]$, the problem $(P_{\lambda,\mu})$ has at least three weak solutions whose norms in X are less than B.

Proof of Theorem 3.4. Let us consider I, J, Ψ as the proof of Theorem 3.1. For any $\lambda > 0$, $u \in X$, by (H_4) we have

$$\begin{aligned} I(u) - \lambda J(u) &\ge \int_\Omega [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)]dx - \lambda\|u\|_\infty^s \int_\Omega k(x)dx - \lambda\int_\Omega h(x)dx \\ &\ge \|u\|^{p_0} - \lambda(2c)^s\|u\|^s \int_\Omega k(x)dx - \lambda\int_\Omega h(x)dx, \end{aligned} \tag{3.17}$$

for $\|u\| > 1$. Since $0 < s < p_0$, one has $\lim_{\|u\|\to\infty}[I(u) - \lambda J(u)] = +\infty$ for all $\lambda \ge 0$.

Let $r := (\gamma^{p^0}/(2c)^{p^0})$, $x_1 = \delta$. For $r > 0, I(u) \le r$, we have

$$|u(x)| \le \|u\|_\infty \le c\|u\|_{1,\Phi} \le 2c\|u\| \le \gamma, \quad \forall x \in \Omega. \tag{3.18}$$

Hence, one has

$$\frac{\sup_{u\in I^{-1}(]-\infty,r])} J(u)}{r} \le \frac{(2c)^{p^0}\int_\Omega \max_{|\xi|\le\gamma} F(x,\xi)dx}{\gamma^{p^0}}. \tag{3.19}$$

From (H_3), it follows that

$$\frac{\sup_{I(u)\le r} J(u)}{r} < \frac{J(\delta)}{I(\delta)}. \tag{3.20}$$

Since all the assumptions of Lemmas 2.7 and 2.6 are satisfied, then, there is a nonempty open set $A \subset [0,\infty)$ and a positive number B, for each $\lambda \in A$ and for every g there exists $\delta > 0$ such that, for each $\mu \in [0,\delta]$, the problem has at least three weak solutions whose norms in X are less than B. □

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Research Article

Application of Mawhin's Coincidence Degree and Matrix Spectral Theory to a Delayed System

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This paper gives an application of Mawhin's coincidence degree and matrix spectral theory to a predator-prey model with M-predators and N-preys. The method is different from that used in the previous work. Some new sufficient conditions are obtained for the existence and global asymptotic stability of the periodic solution. The existence and stability conditions are given in terms of spectral radius of explicit matrices which are much different from the conditions given by the algebraic inequalities. Finally, an example is given to show the feasibility of our results.

1. Introduction and Motivation

1.1. History and Motivations

Mawhin's coincidence degree theory has been applied extensively to study the existence of periodic solutions for nonlinear differential systems (e.g. see [1–16] and references therein). The most important step of applying Mawhin's degree theory to nonlinear differential equations is to obtain the priori bounds of unknown solutions to the operator equation $Lx = \lambda Nx$. However, different estimation techniques for the priori bounds of unknown solutions to the equation $Lx = \lambda Nx$ may lead to different results. Most of papers obtained the priori bounds by employing the inequalities:

$$\begin{aligned} x(t) \le x(\xi) + \int_0^\omega |\dot{x}(t)|dt, &\qquad x(t) \ge x(\eta) - \int_0^\omega |\dot{x}(t)|dt, \\ x(\xi) = \min_{t\in[0,\omega]} x(t), &\qquad x(\eta) = \max_{t\in[0,\omega]} x(t). \end{aligned} \tag{1.1}$$

These inequalities lead to a relatively strong condition given in terms of algebraic inequality or classic norms (see e.g., [3–16]). Different from standard consideration, in this paper, we employ matrix spectral theory to obtain the priori bounds, *not* the above inequalities. So in this paper, the existence and stability of periodic solution for a multispecies predator-prey model is studied by jointly employing Mawhin's coincidence degree and matrix spectral theory.

1.2. Model Formulation

One of classical Lotka-Vlterra system is predator-prey models which have been investigated extensively by mathematicians and ecologist. Many good results have been obtained for stability, bifurcations, chaos, uniform persistence, periodic solution, almost periodic solutions. It has been observed that most of works focus on either two or three species model. There are few paper considering the multispecies model. To model the dynamic behavior of multispecie predator-prey system, Yang and Rui [17] proposed a predator-prey model with M-predators and N-preys of the form:

$$\begin{aligned} \dot{x}_i(t) &= x_i(t)\left[b_i(t) - \sum_{k=1}^{N} a_{ik}(t)x_k(t) - \sum_{l=1}^{M} c_{il}(t)y_l(t)\right], \quad i = 1,2,\ldots,N, \\ \dot{y}_j(t) &= y_j(t)\left[-r_j(t) + \sum_{k=1}^{N} d_{jk}(t)x_k(t) - \sum_{l=1}^{M} e_{jl}(t)y_l(t)\right], \quad j = 1,2,\ldots,M, \end{aligned} \tag{1.2}$$

where $x_i(t)$ denotes the density of prey species X_i at time t, $y_j(t)$ denotes the density of predator species Y_j at time t. The coefficients $b_i(t)$, $r_j(t)$, $a_{ik}(t)$, $c_{il}(t)$, $d_{jk}(t)$, and $e_{jl}(t)$, $(i,k = 1,\ldots,N;\ j,l = 1,\ldots,M)$ are nonnegative continuous periodic functions defined on $t \in (-\infty,+\infty)$. The coefficient b_i is the intrinsic growth rate of prey species X_i, r_j is the death rate of the predator species Y_j, a_{ik} measures the amount of competition between the prey species X_i and X_k $(k \neq i,\ i,k = 1,\ldots,N)$, e_{jl} measures the amount of competition between the predator species Y_j and Y_k $(k \neq j, j,k = 1,\ldots,M)$, and the constant $\widetilde{k}_{ij} \triangleq d_{ij}/c_{ij}$ denotes the coefficient in conversing prey species X_i into new individual of predator species Y_j $(i = 1,\ldots,N;\ j = 1,\ldots,M)$. By using the differential inequality, Zhao and Chen [18] improved the results of Yang and Rui [17]. Recently, Xia et al. [19] obtained some sufficient conditions for the existence and global attractivity of a unique almost periodic solution of the system (1.2).

It is more natural to consider the delay model because most of the species start interacting after reaching a maturity period. Hence many scholars think that the delayed models are more realistic and appropriate to be studied than ordinary model. Delayed system is important also because sometimes time delays may lead to oscillation, bifurcation, chaos, instability which may be harmful to a system. Inspired by the above argument, Wen [20] considered a periodic delayed multispecies predator-prey system as follows:

$$\begin{aligned} \dot{x}_i(t) &= x_i(t)\left[b_i(t) - a_{ii}(t)x_i(t) - \sum_{k=1,k\neq i}^{N} a_{ik}(t)x_k(t-\tau_{ik}) - \sum_{l=1}^{M} c_{il}(t)y_l(t-\eta_{il})\right], \\ \dot{y}_j(t) &= y_j(t)\left[-r_j(t) + \sum_{k=1}^{N} d_{jk}(t)x_k(t-\delta_{jk}) - e_{jj}(t)y_j(t) - \sum_{l=1,l\neq j}^{M} e_{jl}(t)y_l(t-\xi_{jl})\right], \end{aligned} \tag{1.3}$$

where $b_i(t)$, $r_j(t)$, $a_{ik}(t)$, $c_{il}(t)$, $d_{jk}(t)$, $e_{jl}(t)$, $e_{jl}(t)$ $(i,k = 1,2,\ldots,N;\ j,l = 1,2,\ldots,M)$ are assumed to be continuous ω-periodic functions and the delays τ_{ik}, δ_{jk}, η_{il}, ξ_{jl} are assumed to be positive constants. The system (1.3) is supplemented with the initial condition:

$$x_i(\theta) = \phi_i(\theta), \quad y_j(\theta) = \psi_i(\theta), \quad \theta \in [-\tau, 0],\ \phi_i(0) > 0,\ \psi_i(0) > 0, \tag{1.4}$$

where

$$\tau = \max\left\{ \max_{1\le i,k\le n} \tau_{ik},\ \max_{1\le i\le N, 1\le l\le M} \eta_{il},\ \max_{1\le j,l\le M} \xi_{jl},\ \max_{1\le j\le M, 1\le k\le N} \delta_{jk} \right\} > 0. \tag{1.5}$$

It is easy to see that for such given initial conditions, the corresponding solution of the system (1.3) remains positive for all $t \ge 0$. The purpose of this paper is to obtain some new and interesting criteria for the existence and global asymptotic stability of periodic solution of the system (1.3).

1.3. Comparison with Previous Work

To obtain the periodic solutions of the system (1.3), the method used in [20] is based on employing the differential inequality and Brower fixed point theorem. Different from consideration taken by [20], our method is based on combining matrix spectral theory with Mawhin's degree theory. In our method, we study the global asymptotic stability by combining matrix's spectral theory with Lyapunov functional method. The existence and stability conditions are given in terms of spectral radius of explicit matrices. These conditions are much different from the sufficient conditions obtained in [20].

1.4. Outline of This Work

The structure of this paper is as follows. In Section 2, some new and interesting sufficient conditions for the existence of periodic solution of system (1.3) are obtained. Section 3 is devoted to examining the stability of the periodic solution obtained in the previous section. In Section 4, some corollaries are presented to show the effectiveness of our results. Finally, an example is given to show the feasibility of our results.

2. Existence of Periodic Solutions

In this section, we will obtain some sufficient conditions for the existence of periodic solution of the system (1.3).

2.1. Preliminaries on the Matrix Theory and Degree Theory

For convenience, we introduce some notations, definitions, and lemmas. Throughout this paper, we use the following notations.

(i) We always use $i, k = 1, \ldots, N;\ \ j,\ l = 1, \ldots, M$, unless otherwise stated.

(ii) If $f(t)$ is a continuous ω-periodic function defined on R, then we denote

$$\underline{f} = \min_{t\in[0,\omega]} |f(t)|, \qquad \overline{f} = \max_{t\in[0,\omega]} |f(t)|, \qquad m(f) = \frac{1}{\omega}\int_0^{\omega} f(t)dt. \tag{2.1}$$

We use $x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$ to denote a column vector, $\mathfrak{D} = (d_{ij})_{n\times n}$ is an $n \times n$ matrix, $\mathfrak{D}^T$ denotes the transpose of $\mathfrak{D}$, and E_n is the identity matrix of size n. A matrix or vector $\mathfrak{D} > 0$ (resp., $\mathfrak{D} \geq 0$) means that all entries of $\mathfrak{D}$ are positive (resp., nonnegative). For matrices or vectors $\mathfrak{D}$ and E, $\mathfrak{D} > E$ (resp., $\mathfrak{D} \geq E$) means that $\mathfrak{D} - E > 0$ (resp., $\mathfrak{D} - E \geq 0$). We denote the spectral radius of the matrix $\mathfrak{D}$ by $\rho(\mathfrak{D})$.

If $v = (v_1, v_2, \dots, v_n)^T \in \mathbb{R}^n$, then we have a choice of vector norms in $\mathbb{R}^n$, for instance $\|v\|_1$, $\|v\|_2$, and $\|v\|_\infty$ are the commonly used norms, where

$$\|v\|_1 = \sum_{j=1}^{n} |v_i|, \qquad \|v\|_2 = \left\{ \sum_{j=1}^{n} |v_i|^2 \right\}^{1/2}, \qquad \|v\|_\infty = \max_{1\leq i\leq n} |v_i|. \tag{2.2}$$

We recall the following norms of matrices induced by respective vector norms. For instance if $\mathcal{A} = (a_{ij})_{n\times n}$, the norm of the matrix $\|\mathcal{A}\|$ induced by a vector norm $\|\cdot\|$ is defined by

$$\|\mathcal{A}\|_p = \sup_{v\in\mathbb{R}^n, v\neq 0} \frac{\|\mathcal{A}v\|_p}{\|v\|_p} = \sup_{\|v\|_p=1} \|\mathcal{A}v\|_p = \sup_{\|v\|_p\leq 1} \|\mathcal{A}v\|_p. \tag{2.3}$$

In particular one can show that $\|\mathcal{A}\|_1 = \max_{1\leq j\leq n} \sum_{i=1}^n |a_{ij}|$ (column norm), $\|\mathcal{A}\|_2 = [\lambda_{\max}(\mathcal{A}^T\mathcal{A})]^{1/2} = [\text{max. eigenvalue of } (\mathcal{A}^T\mathcal{A})]^{1/2}$ and $\|\mathcal{A}\|_\infty = \max_{1\leq i\leq n} \sum_{j=1}^n |a_{ij}|$ (row norm).

Definition 2.1 (see [1, 21]). Let X, Z be normed real Banach spaces, let $L : \operatorname{Dom} L \subset X \to Z$ be a linear mapping, and $N : X \to Z$ be a continuous mapping. The mapping L is called a Fredholm mapping of index zero, if $\dim\operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < +\infty$ and $\operatorname{Im} L$ is closed in Z. If L is a Fredholm mapping of index zero and there exist continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I - Q)$, then $L \mid \operatorname{dom} L \cap \operatorname{Ker} P : (I - P)X \to \operatorname{Im} L$ is invertible. We denote the inverse of that map by K_P. If Ω is an open bounded subset of X, the mapping N will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I - Q)N : \overline{\Omega} \to X$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exists an isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$.

Definition 2.2 (see [1, 22]). Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $f \in C^1(\Omega, \mathbb{R}^n) \cap C(\overline{\Omega}, \mathbb{R}^n)$ and $y \in \mathbb{R}^n / f(\partial\Omega \cup N_f)$, that is, y is a regular value of f. Here, $N_f = \{x \in \Omega : J_f(x) = 0\}$, the critical set of f and J_f is the Jacobian of f at x. Then the *degree* $\deg\{f, \Omega, y\}$ is defined by

$$\deg\{f, \Omega, y\} = \sum_{x\in f^{-1}(y)} \operatorname{sgn} J_f(x), \tag{2.4}$$

with the agreement that $\sum \phi = 0$. For more details about Degree Theory, the reader may consult Deimling [22].

Lemma 2.3 (Continuation Theorem [1]). *Let $\Omega \subset X$ be an open and bounded set and L be a Fredholm mapping of index zero and N be L-compact on $\overline{\Omega}$ (i.e., $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)N : \overline{\Omega} \to X$ is compact). Assume*

(i) *for each $\lambda \in (0,1)$, $x \in \partial\Omega \cap \operatorname{Dom} L$, $Lx \neq \lambda Nx$;*

(ii) *for each $x \in \partial\Omega \cap \operatorname{Ker} L$, $QNx \neq 0$ and $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$.*

Then $Lx = Nx$ has at least one solution in $\overline{\Omega} \cap \operatorname{Dom} L$.

Definition 2.4 (see [23, 24]). A real $n \times n$ matrix $\mathcal{A} = (a_{ij})$ is said to be an M-matrix if $a_{ij} \leq 0$, $i, j = 1, 2, \ldots, n$, $i \neq j$, and $\mathcal{A}^{-1} \geq 0$.

Lemma 2.5 (see [23, 24]). *Let $\mathcal{A} \geq 0$ be an $n \times n$ matrix and $\rho(\mathcal{A}) < 1$, then $(E_n - \mathcal{A})^{-1} \geq 0$, where E_n denotes the identity matrix of size n.*

Now we introduce some function spaces and their norms, which will be valid throughout this paper. Denote

$$X = \left\{U(t) = (u(t), v(t))^T \in C^1\left(\mathbb{R}, \mathbb{R}^{N+M}\right) \mid U(t+\omega) = U(t) \ \forall t \in \mathbb{R}\right\},$$
$$Z = \left\{U(t) = (u(t), v(t))^T \in C\left(\mathbb{R}, \mathbb{R}^{N+M}\right) \mid U(t+\omega) = U(t) \ \forall t \in \mathbb{R}\right\}. \tag{2.5}$$

The norms are given by

$$|U_n(t)|_0 = \max_{t\in[0,\omega]} |U_n(t)|, \quad |U_n(t)|_1 = |U_n(t)|_0 + \left|\dot{U}_n(t)\right|_0, \quad i = 1, 2, \ldots, N+M,$$
$$\|U_n(t)\|_0 = \max_{1\leq n\leq N+M} \{|U_n(t)|_0\}, \qquad \|U_n(t)\|_1 = \max_{1\leq n\leq N+M} \{|U_n(t)|_1\}. \tag{2.6}$$

Obviously, X and Z, respectively, endowed with the norms $\|\cdot\|_1$ and $\|\cdot\|_0$ are Banach spaces.

2.2. Result on the Existence of Periodic Solutions

Theorem 2.6. *Assume that the following conditions hold:*

(H_1): *the system of algebraic equations:*

$$m(b_i) - m(a_{ii})u_i - \sum_{k=1, k\neq i}^{N} m(a_{ik})u_k - \sum_{l=1}^{M} m(c_{il})w_l = 0, \quad i = 1, 2, \ldots, N,$$
$$m(-r_j) + \sum_{k=1}^{N} m(d_{jk})u_k - m(e_{jj})w_j - \sum_{l=1, l\neq j}^{M} m(e_{jl})w_l = 0, \quad j = 1, 2, \ldots, M, \tag{2.7}$$

has finite solution $(u_1^*, \ldots, u_N^*, w_1^*, \ldots, w_M^*)^T \in \mathbb{R}_+^{N+M}$ with $u^* > 0$, $w^* > 0$;

(H_2): *$\rho(\mathcal{K}) < 1$, where* $\mathcal{K} = \begin{pmatrix} \mathcal{P}_{N\times N} & \mathcal{Q}_{N\times M} \\ \mathcal{M}_{M\times N} & \mathcal{N}_{M\times M} \end{pmatrix}_{(N+M)\times(N+M)}$,

$$\begin{aligned}
\mathcal{P}_{N\times N} &= (p_{ik})_{N\times N}, \qquad p_{ik} = \begin{cases} 0, & i = k, \\ \overline{a}_{ik}\underline{a}_{kk}^{-1}, & i \neq k. \end{cases} \\
\mathcal{Q}_{N\times M} &= (q_{il})_{N\times M}, \qquad q_{il} = \overline{c}_{il}\underline{e}_{ll}^{-1}, \\
\mathcal{M}_{M\times N} &= (m_{jk})_{M\times N}, \qquad m_{jk} = \overline{d}_{jk}\underline{a}_{kk}^{-1}, \\
\mathcal{N}_{M\times M} &= (n_{jl})_{M\times M}, \qquad n_{jl} = \begin{cases} 0, & j = l, \\ \overline{e}_{jl}\underline{e}_{ll}^{-1}, & j \neq l. \end{cases}
\end{aligned} \tag{2.8}$$

Then system (1.3) *has at least one positive ω-periodic solution.*

Proof. Note that every solution

$$U(t) = (u(t), v(t))^T = (u_1(t), \ldots, u_N(t), v_1(t), \ldots, v_M(t))^T \in X \tag{2.9}$$

of the system (1.3) with the initial condition is positive. By using the following changes of variables:

$$u_i(t) = \ln x_i(t), \quad v_j(t) = \ln y_j(t), \quad i = 1, 2, \ldots, N, \; j = 1, 2, \ldots, M, \tag{2.10}$$

the system (1.3) can be rewritten as

$$\begin{aligned}
\dot{u}_i(t) &= b_i(t) - a_{ii}(t)e^{u_i(t)} - \sum_{k=1, k\neq i}^{N} a_{ik}(t)e^{u_k(t-\tau_{ik})} - \sum_{l=1}^{M} c_{il}(t)e^{v_l(t-\eta_{il})}, \quad i = 1, 2, \ldots, N, \\
\dot{v}_j(t) &= -r_j(t) + \sum_{k=1}^{N} d_{jk}(t)e^{u_k(t-\delta_{jk})} - e_{jj}(t)e^{v_j(t)} - \sum_{l=1, l\neq j}^{M} e_{jl}(t)e^{v_l(t-\xi_{jl})}, \quad j = 1, 2, \ldots, M.
\end{aligned} \tag{2.11}$$

Obviously, system (1.3) has at least one ω-periodic solution which is equivalent to the system (2.11) having at least one ω-periodic solution. To prove Theorem 2.6, our main tasks are to construct the operators (i.e., L, N, P, and Q) appearing in Lemma 2.3 and to find an appropriate open set Ω satisfying conditions (i), (ii) in Lemma 2.3.

For any $U(t) \in X$, in view of the periodicity, it is easy to check that

$$\begin{aligned}
\Delta_i(U, t) &= b_i(t) - a_{ii}(t)e^{u_i(t)} - \sum_{k=1, k\neq i}^{N} a_{ik}(t)e^{u_k(t-\tau_{ik})} - \sum_{l=1}^{M} c_{il}(t)e^{v_l(t-\eta_{il})} \in Z, \\
\Delta_j(U, t) &= -r_j(t) + \sum_{k=1}^{N} d_{jk}(t)e^{u_k(t-\delta_{jk})} - e_{jj}(t)e^{v_j(t)} - \sum_{l=1, l\neq j}^{M} e_{jl}(t)e^{v_l(t-\xi_{jl})} \in Z.
\end{aligned} \tag{2.12}$$

Now, we define the operators L, N as follows:

$$L : \operatorname{Dom} L \subset X \longrightarrow Z, \quad L(u(t), v(t)) = \left(\frac{du_i(t)}{dt}, \frac{dv_j(t)}{dt} \right) \in Z,$$
$$N : X \longrightarrow Z \text{ is defined by } NU = \begin{pmatrix} \Delta_i(U,t) \\ \Delta_j(U,t) \end{pmatrix}. \tag{2.13}$$

Define, respectively, the projectors $P : X \to X$ and $Q : Z \to Z$ by

$$PU = \frac{1}{\omega} \int_0^{\omega} U(t)dt, \quad U \in X,$$
$$QU = \frac{1}{\omega} \int_0^{\omega} U(t)dt, \quad U \in Z. \tag{2.14}$$

It is obvious that the domain of L in X is actually the whole space, and

$$\operatorname{Ker} L = \{x(t) \in X \mid Lx(t) = 0, \text{ i.e. } \dot{x}(t) = 0\} = \mathbb{R}^{N+M},$$
$$\operatorname{Im} L = \left\{ z(t) \in Z \mid \int_0^{\omega} z(t)dt = 0 \right\} \text{ is closed in } Z. \tag{2.15}$$

Moreover, P, Q are continuous operators such that

$$\operatorname{Im} P = \mathbb{R}^N = \operatorname{Ker} L, \qquad \operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I - Q),$$
$$\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L = N + M < +\infty. \tag{2.16}$$

It follows that L is a Fredholm mapping of index zero. Furthermore, the generalized inverse (to L) $K_P : \operatorname{Im} L \to \operatorname{Dom} L \cap \operatorname{Ker} P$ exists, which is given by

$$K_P(y) = \int_0^t y(s)ds - \frac{1}{\omega} \int_0^{\omega} \int_0^t y(s)dsdt. \tag{2.17}$$

Then $QN : X \to Z$ and $K_P(I - Q)N : X \to X$ are defined by

$$QNU = \begin{pmatrix} \dfrac{1}{\omega} \displaystyle\int_0^{\omega} \Delta_i(U,t)dt \\ \dfrac{1}{\omega} \displaystyle\int_0^{\omega} \Delta_j(U,t)dt \end{pmatrix},$$
$$K_P(I - Q)Nx = \int_0^t NU(s)ds - \frac{1}{\omega} \int_0^{\omega} \int_0^t NU(s)dsdt - \left(\frac{t}{\omega} - \frac{1}{2} \right) \int_0^{\omega} NU(s)ds. \tag{2.18}$$

Clearly, QN and $K_P(I-Q)N$ are continuous. By using the generalized Arzela-Ascoli theorem, it is not difficult to prove that $(K_P(I-Q)N)(\overline{\Omega})$ is relatively compact in the space $(X, \|\cdot\|_1)$. The proof of this step is complete.

Then, in order to apply condition (i) of Lemma 2.3, we need to search for an appropriate open bounded subset Ω, denoted by

$$\Omega = U_n(t) \in X \mid |U_n(t)|_1 = |U_n(t)|_0 + \left|\dot{U}_n(t)\right|_0 < h_n. \tag{2.19}$$

Specifically, our aim is to find an appropriate h_n. Corresponding to the operator equation $Lx = \lambda Nx$ for each $\lambda \in (0,1)$, we have

$$\begin{aligned} \dot{u}_i(t) &= \lambda\left[b_i(t) - a_{ii}(t)e^{u_i(t)} - \sum_{k=1,k\neq i}^{N} a_{ik}(t)e^{u_k(t-\tau_{ik})} - \sum_{l=1}^{M} c_{il}(t)e^{v_l(t-\eta_{il})}\right], \\ \dot{v}_j(t) &= \lambda\left[-r_j(t) + \sum_{k=1}^{N} d_{jk}(t)e^{u_k(t-\delta_{jk})} - e_{jj}(t)e^{v_j(t)} - \sum_{l=1,l\neq j}^{M} e_{jl}(t)e^{v_l(t-\xi_{jl})}\right]. \end{aligned} \tag{2.20}$$

Since $U(t) \in X$, each $U_n(t), n = 1,2,\ldots,N+M$, as components of $U(t)$, is continuously differentiable and ω-periodic. In view of continuity and periodicity, there exists $t_i \in [0,\omega]$ such that $u_i(t_i) = \max_{t\in[0,\omega]}|u_i(t)|$, $i = 1,2,\ldots,N$, and there also exists $t_{N+j} \in [0,\omega]$ such that $v_j(t_{N+j}) = \max_{t\in[0,\omega]}|v_j(t)|$, $j = 1,2,\ldots,M$. Accordingly, $\dot{u}_i(t_i) = 0$, $\dot{v}_j(t_{N+j}) = 0$, and we get

$$\begin{gathered} b_i(t_i) - a_{ii}(t_i)e^{u_i(t_i)} - \sum_{k=1,k\neq i}^{N} a_{ik}(t_i)e^{u_k(t_i-\tau_{ik})} - \sum_{l=1}^{M} c_{il}(t_i)e^{v_l(t_{N+j}-\eta_{il})} = 0, \\ -r_j(t_{N+j}) + \sum_{k=1}^{N} d_{jk}(t_{N+j})e^{u_k(t_i-\delta_{jk})} - e_{jj}(t_{N+j})e^{v_j(t_{N+j})} - \sum_{l=1,l\neq j}^{M} e_{jl}(t_{N+j})e^{v_l(t_{N+j}-\xi_{jl})} = 0. \end{gathered} \tag{2.21}$$

That is,

$$\begin{aligned} a_{ii}(t_i)e^{u_i(t_i)} &= b_i(t_i) - \sum_{k=1,k\neq i}^{N} a_{ik}(t_i)e^{u_k(t_i-\tau_{ik})} - \sum_{l=1}^{M} c_{il}(t_i)e^{v_l(t_{N+j}-\eta_{il})}, \\ e_{jj}(t_{N+j})e^{v_j(t_{N+j})} &= -r_j(t_{N+j}) + \sum_{k=1}^{N} d_{jk}(t_{N+j})e^{u_k(t_i-\delta_{jk})} - \sum_{l=1,l\neq j}^{M} e_{jl}(t_{N+j})e^{v_l(t_{N+j}-\xi_{jl})}. \end{aligned} \tag{2.22}$$

Note that $u_k(t_k) = \max_{t\in[0,\omega]}|u_k(t)|$ and $v_l(t_{N+l}) = \max_{t\in[0,\omega]}|v_l(t)|$, which implies

$$\begin{gathered} u_k(t_i) \leq u_k(t_k), \qquad u_k(t_i - \tau_{ik}) \leq u_k(t_k), \qquad u_k(t_i - \delta_{jk}) \leq u_k(t_k); \\ v_l(t_{N+j}) \leq v_l(t_{N+l}), \qquad v_l(t_{N+j} - \eta_{il}) \leq v_l(t_{N+l}), \qquad v_l(t_{N+j} - \xi_{jl}) \leq v_l(t_{N+l}). \end{gathered} \tag{2.23}$$

It follows that

$$
\begin{aligned}
\underline{a}_{ii}e^{u_i(t_i)} &\leq \left|a_{ii}(t_i)e^{u_i(t_i)}\right| \\
&= \left|b_i(t_i) - \sum_{k=1,k\neq i}^{N} a_{ik}(t_i)e^{u_k(t_i-\tau_{ik})} - \sum_{l=1}^{M} c_{il}(t_i)e^{v_l(t_{N+j}-\eta_{il})}\right| \\
&\leq \overline{b}_i + \sum_{k=1,k\neq i}^{N} \overline{a}_{ik}e^{u_k(t_i-\tau_{ik})} + \sum_{l=1}^{M}\overline{c}_{il}e^{v_l(t_{N+j}-\eta_{il})} \\
&\leq \overline{b}_i + \sum_{k=1,k\neq i}^{N} \overline{a}_{ik}e^{u_k(t_k)} + \sum_{l=1}^{M}\overline{c}_{il}e^{v_l(t_{N+l})}, \\
\underline{e}_{jj}e^{v_j(t_{N+j})} &\leq \left|e_{jj}\left(t_{N+j}\right)e^{v_j(t_{N+j})}\right| \\
&= \left|-r_j\left(t_{N+j}\right) + \sum_{k=1}^{N} d_{jk}\left(t_{N+j}\right)e^{u_k(t_i-\delta_{jk})} - \sum_{l=1,l\neq j}^{M} e_{jl}\left(t_{N+j}\right)e^{v_l(t_{N+j}-\xi_{jl})}\right| \\
&\leq \overline{r}_j + \sum_{k=1}^{N}\overline{d}_{jk}e^{u_k(t_i-\delta_{jk})} + \sum_{l=1,l\neq j}^{M} \overline{e}_{jl}e^{v_l(t_{N+j}-\xi_{jl})} \\
&\leq \overline{r}_j + \sum_{k=1}^{N}\overline{d}_{jk}e^{u_k(t_k)} + \sum_{l=1,l\neq j}^{M} \overline{e}_{jl}e^{v_l(t_{N+l})}.
\end{aligned}
\tag{2.24}
$$

Let

$$
\underline{a}_{ii}e^{u_i(t_i)} = z_i(t_i), \qquad \underline{e}_{jj}e^{v_j(t_{N+j})} = \widetilde{z}_j\left(t_{N+j}\right). \tag{2.25}
$$

Using (2.25), the inequalities (2.24) become

$$
\begin{aligned}
z_i(t_i) &\leq \overline{b}_i + \sum_{k=1,k\neq i}^{N} \overline{a}_{ik}\underline{a}_{kk}^{-1}z_k(t_k) + \sum_{l=1}^{M}\overline{c}_{il}\underline{e}_{ll}^{-1}\widetilde{z}_l(t_{N+l}), \\
\widetilde{z}_j\left(t_{N+j}\right) &\leq \overline{r}_j + \sum_{k=1}^{N}\overline{d}_{jk}\underline{a}_{kk}^{-1}z_k(t_k) + \sum_{l=1,l\neq j}^{M} \overline{e}_{jl}\underline{e}_{ll}^{-1}\widetilde{z}_l(t_{N+l}),
\end{aligned}
\tag{2.26}
$$

or

$$
\begin{aligned}
z_i(t_i) - \sum_{k=1,k\neq i}^{N} \overline{a}_{ik}\underline{a}_{kk}^{-1}z_k(t_k) - \sum_{l=1}^{M}\overline{c}_{il}\underline{e}_{ll}^{-1}\widetilde{z}_l(t_{N+l}) &\leq \overline{b}_i, \\
\widetilde{z}_j\left(t_{N+j}\right) - \sum_{k=1}^{N}\overline{d}_{jk}\underline{a}_{kk}^{-1}z_k(t_k) - \sum_{l=1,l\neq j}^{M} \overline{e}_{jl}\underline{e}_{ll}^{-1}\widetilde{z}_l(t_{N+l}) &\leq \overline{r}_j,
\end{aligned}
\tag{2.27}
$$

which implies

$$\begin{pmatrix} E_{N\times N}-\mathcal{P}_{N\times N} & -\mathcal{Q}_{N\times M} \\ -\mathcal{M}_{M\times N} & E_{M\times M}-\mathcal{N}_{M\times M} \end{pmatrix}_{(N+M)\times(N+M)} \times \begin{pmatrix} z_1(t_1), \\ \dots, \\ z_N(t_N), \\ \tilde{z}_1(t_{N+1}), \\ \dots, \\ \tilde{z}_M(t_{N+M}) \end{pmatrix} \le \begin{pmatrix} \overline{b}_1, \\ \dots, \\ \overline{b}_N, \\ \overline{r}_1, \\ \dots, \\ \overline{r}_M \end{pmatrix}, \tag{2.28}$$

where

$$\begin{gathered} \mathcal{P}_{N\times N}=(p_{ik})_{N\times N}, \qquad p_{ik}=\begin{cases} 0, & i=k, \\ \overline{a}_{ik}\underline{a}_{kk}^{-1}, & i\neq k, \end{cases} \\ \mathcal{Q}_{N\times M}=(q_{il})_{N\times M}, \qquad q_{il}=\overline{c}_{il}\underline{e}_{ll}^{-1}, \\ \mathcal{M}_{M\times N}=(m_{jk})_{M\times N}, \qquad m_{jk}=\overline{d}_{jk}\underline{a}_{kk}^{-1}, \\ \mathcal{N}_{M\times M}=(n_{jl})_{M\times M}, \qquad n_{jl}=\begin{cases} 0, & j=l, \\ \overline{e}_{jl}\underline{e}_{ll}^{-1}, & j\neq l. \end{cases} \end{gathered} \tag{2.29}$$

Set $D=(\overline{b}_1,\dots,\overline{b}_N,\overline{r}_1,\dots,\overline{r}_M)^T$. It follows from (2.28) and (H_2) that

$$(E-\mathcal{K})(z_1(t_1),\dots,z_N(t_N),\tilde{z}_1(t_{N+1}),\dots,\tilde{z}_M(t_{N+M}))^T \le D. \tag{2.30}$$

In view of $\rho(\mathcal{K})<1$ and Lemma 2.5, we get $(E_{N+M}-\mathcal{K})^{-1}\ge 0$. Let

$$H=\left(\tilde{h}_1,\dots,\tilde{h}_N,\tilde{h}_{N+1},\dots,\tilde{h}_{N+M}\right)^T := (E-\mathcal{K})^{-1}D \ge 0. \tag{2.31}$$

Using (2.30) and (2.31), we get

$$(z_1(t_1),\dots,z_N(t_N),\tilde{z}_1(t_{N+1}),\dots,\tilde{z}_M(t_{N+M}))^T \le H, \tag{2.32}$$

or

$$z_i(t_i)\le \tilde{h}_i, \quad i=1,2,\dots,N, \qquad \tilde{z}_j(t_{N+j})\le \tilde{h}_{N+j}, \quad j=1,2,\dots,M. \tag{2.33}$$

Then

$$u_i(t_i)\le \ln\frac{\tilde{h}_i}{\underline{a}_{ii}}, \qquad v_j(t_{N+j})\le \ln\frac{\tilde{h}_{N+j}}{\underline{e}_{jj}}, \tag{2.34}$$

which implies

$$|u_n(t)|_0 = \max_{t\in[0,\omega]} |u_n(t)| = \max_{t\in[0,\omega]} \{u_i(t_i), v_j(t_{N+j})\} = \max\left\{\ln \frac{\widetilde{h}_i}{\underline{a}_{ii}}, \ln \frac{\widetilde{h}_{N+j}}{\underline{e}_{jj}}\right\}. \tag{2.35}$$

On the other hand, it follows from (2.31) that

$$(E - \mathcal{K})H = D, \quad \text{or} \quad H = \mathcal{K}H + D, \tag{2.36}$$

that is

$$\begin{aligned}
\widetilde{h}_i &= \sum_{k=1,k\neq i}^{N} p_{ik}\widetilde{h}_k + \sum_{l=1}^{M} q_{il}\widetilde{h}_{N+l} + \overline{b}_i, \\
\widetilde{h}_{N+j} &= \sum_{k=1}^{N} m_{jk}\widetilde{h}_k + \sum_{l=1,l\neq j}^{M} n_{jl}\widetilde{h}_{N+l} + \overline{r}_j.
\end{aligned} \tag{2.37}$$

Estimating (2.20), by using (2.25), (2.33), and (2.37), we have

$$\begin{aligned}
|\dot{u}_i(t)|_0 &= \lambda \left| b_i(t) - a_{ii}(t)e^{u_i(t)} - \sum_{k=1,k\neq i}^{N} a_{ik}(t)e^{u_k(t-\tau_{ik})} - \sum_{l=1}^{M} c_{il}(t)e^{v_l(t-\eta_{il})} \right|_0 \\
&\leq \overline{b}_i + \overline{a}_{ii}\left|e^{u_i(t)}\right|_0 + \sum_{k=1,k\neq i}^{N} \overline{a}_{ik}\left|e^{u_k(t-\tau_{ik})}\right|_0 + \sum_{l=1}^{M} \overline{c}_{il}\left|e^{v_l(t-\eta_{il})}\right|_0 \\
&= \overline{b}_i + \overline{a}_{ii}e^{u_i(t_i)} + \sum_{k=1,k\neq i}^{N} \overline{a}_{ik}e^{u_k(t_k)} + \sum_{l=1}^{M} \overline{c}_{il}e^{v_l(t_{N+l})} \\
&= \overline{b}_i + \overline{a}_{ii}\underline{a}_{ii}^{-1}z_i(t_i) + \sum_{k=1,k\neq i}^{N} \overline{a}_{ik}\underline{a}_{kk}^{-1}z_k(t_k) + \sum_{l=1}^{M} \overline{c}_{il}\underline{e}_{ll}^{-1}\widetilde{z}_l(t_{N+l}) \\
&= \overline{b}_i + z_i(t_i) + \sum_{k=1,k\neq i}^{N} \overline{a}_{ik}\underline{a}_{kk}^{-1}z_k(t_k) + \sum_{l=1}^{M} \overline{c}_{il}\underline{e}_{ll}^{-1}\widetilde{z}_l(t_{N+l}) \\
&\leq \overline{b}_i + \widetilde{h}_i + \sum_{k=1,k\neq i}^{N} p_{ik}\widetilde{h}_k + \sum_{l=1}^{M} q_{il}\widetilde{h}_{N+l} \\
&= \widetilde{h}_i + \widetilde{h}_i = 2\widetilde{h}_i,
\end{aligned}$$

$$
\begin{aligned}
\left|\dot{v}_j(t)\right|_0 &= \lambda\left|-r_j(t)+\sum_{k=1}^{N}d_{jk}(t)e^{u_k(t-\delta_{jk})}-e_{jj}(t)e^{v_j(t)}-\sum_{l=1,l\neq j}^{M}e_{jl}(t)e^{v_l(t-\xi_{jl})}\right|_0\\
&\le \overline{r}_j+\sum_{k=1}^{N}\overline{d}_{jk}\left|e^{u_k(t-\delta_{jk})}\right|_0+\sum_{l=1,l\neq j}^{M}\overline{e}_{jl}\left|e^{v_l(t-\xi_{jl})}\right|_0+\overline{e}_{jj}\left|e^{v_j(t)}\right|_0\\
&= \overline{r}_j+\sum_{k=1}^{N}\overline{d}_{jk}e^{u_k(t_k)}+\sum_{l=1,l\neq j}^{M}\overline{e}_{jl}e^{v_l(t_{N+l})}+\overline{e}_{jj}e^{v_j(t_{N+j})}\\
&= \overline{r}_j+\sum_{k=1}^{N}\overline{d}_{jk}\underline{a}_{kk}^{-1}z_k(t_k)+\sum_{l=1,l\neq j}^{M}\overline{e}_{jl}\underline{e}_{ll}^{-1}\widetilde{z}_l(t_{N+l})+\overline{e}_{jj}\underline{e}_{jj}^{-1}z_j\left(t_{N+j}\right)\\
&= \overline{r}_j+\sum_{k=1}^{N}\overline{d}_{jk}\underline{a}_{kk}^{-1}z_k(t_k)+\sum_{l=1,l\neq j}^{M}\overline{e}_{jl}\underline{e}_{ll}^{-1}\widetilde{z}_l(t_{N+l})+\widetilde{z}_j\left(t_{N+j}\right)\\
&\le \overline{r}_j+\sum_{k=1}^{N}m_{jk}\widetilde{h}_k+\sum_{l=1,l\neq j}^{M}n_{jl}\widetilde{h}_{N+l}+\widetilde{h}_{N+j}\\
&= \widetilde{h}_{N+j}+\widetilde{h}_{N+j}=2\widetilde{h}_{N+j}.
\end{aligned}
\tag{2.38}
$$

The above relations imply

$$
|\dot{u}_n(t)|_0=\max_{t\in[0,\omega]}|\dot{u}_n(t)|=\max_{t\in[0,\omega]}\left\{\dot{u}_i(t),\dot{v}_j(t)\right\}=\max\left\{2\widetilde{h}_i,2\widetilde{h}_{N+j}\right\}. \tag{2.39}
$$

Further, it follows form the definition of norm that

$$
|u_n(t)|_1=|u_n(t)|_0+|\dot{u}_n(t)|_0=\max\left\{\ln\frac{\widetilde{h}_i}{\underline{a}_{ii}},\ln\frac{\widetilde{h}_{N+j}}{\underline{e}_{jj}}\right\}+\max\left\{2\widetilde{h}_i,2\widetilde{h}_{N+j}\right\}. \tag{2.40}
$$

Let us set the following:

$$
h_n=\max\left\{\ln\frac{\widetilde{h}_i}{\underline{a}_{ii}},\ln\frac{\widetilde{h}_{N+j}}{\underline{e}_{jj}}\right\}+\max\left\{2\widetilde{h}_i,2\widetilde{h}_{N+j}\right\}+d, \tag{2.41}
$$

where d is any positive constant.

Then for any solution of $Lx=\lambda Nx$, we have $|u_n(t)|_1=|u_n(t)|_0+|\dot{u}_n(t)|_0<h_n$ for all $n=1,2,\ldots,N+M$. Obviously, h_n are independent of λ and the choice of $U(t)$. Consequently, by taking this h_n, the open subset Ω satisfies that $\Omega\cap\operatorname{Dom}L$, that is, the open subset Ω satisfies the assumption (i) of Lemma 2.3.

Now in the last step of the proof, we need to verify that for the given open bounded set Ω obtained in Step 2, the assumption (ii) of Lemma 2.3 also holds. That is, for each $U\in\partial\Omega\cap\operatorname{Ker}L$, $QNU\neq 0$ and $\deg\{JQN,\Omega\cap\operatorname{Ker}L,0\}\neq 0$.

Take $U \in \partial\Omega \cap \operatorname{Ker} L$. Then, in view of $\operatorname{Ker} L = \mathbb{R}^{N+M}$, U is a constant vector in $\mathbb{R}^{N+M}$, denoted by $U = (u_1, \ldots, u_N, v_1, \ldots, v_M)^T$ and with the property

$$|U_n| = |U_n|_0 = |U_n|_1 = h_n. \tag{2.42}$$

By operating U by QN gives

$$(QNU)_n = \begin{pmatrix} m(b_i) - m(a_{ii})e^{u_i} - \sum\limits_{k=1,k\neq i}^{N} m(a_{ik})e^{u_k} - \sum\limits_{l=1}^{M} m(c_{il})e^{v_l} \\ m(-r_j) + \sum\limits_{k=1}^{N} m(d_{jk})e^{u_k} - m(e_{jj})e^{v_j} - \sum\limits_{l=1,l\neq j}^{M} m(e_{jl})e^{v_l} \end{pmatrix}. \tag{2.43}$$

It is easy to obtain that $(QNU)_n$ and $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$, where $\deg(\cdot)$ is the Brouwer degree and J is the identity mapping since $\operatorname{Im} Q = \operatorname{Ker} L$. We have shown that the open subset $\Omega \subset X$ satisfies all the assumptions of Lemma 2.3. Hence, by Lemma 2.3, the system (2.11) has at least one positive ω-periodic solution in $\operatorname{Dom} L \cap \overline{\Omega}$. By (2.10), the system (1.3) has at least one positive ω-periodic solution. This completes the proof of Theorem 2.6. □

3. Globally Asymptotic Stability

Under the assumption of Theorem 2.6, we know that system (1.3) has at least one positive ω-periodic solution, denoted by $X^*(t) = (x_1^*(t), \ldots, x_N^*(t), y_1^*(t), \ldots, y_M^*(t))^T$. The aim of this section is to derive a set of sufficient conditions which guarantee the existence and global asymptotic stability of the positive ω-periodic solution $X^*(t)$.

Before the formal analysis, we recall some facts which will be used in the proof.

Lemma 3.1 (see [25]). *Let f be a nonnegative function defined on $[0, +\infty]$ such that f is integrable on $[0, +\infty]$ and is uniformly continuous on $[0, +\infty]$. Then $\lim_{t\to+\infty} f(t) = 0$.*

Lemma 3.2 (see [23, 24]). *Let $\mathcal{K} = (\Gamma_{ij})_{n\times n}$ be a matrix with nonpositive off-diagonal elements. $\mathcal{K}$ is an M-matrix if and only if there exists a positive diagonal matrix $\xi = \operatorname{diag}(\xi_1, \xi_2, \ldots, \xi_n)$ such that*

$$\xi_i \underline{a}_{ii} > \sum_{j\neq i} \xi_j \overline{a}_{ij}, \quad i = 1, 2, \ldots, n. \tag{3.1}$$

Theorem 3.3. *Assume that all the assumptions in Theorem 2.6 hold. Then system (1.3) has a unique positive ω-periodic solution $X^*(t)$ which is globally asymptotically stable.*

Proof. Let $X(t) = (x(t), y(t))^T = (x_1(t), \ldots, x_N(t), y_1(t), \ldots, y_M(t))^T$ be any positive solution of system (1.3). It is easy to see that $\rho(\mathcal{K}^T) = \rho(\mathcal{K}) < 1$. Thus, in view of Lemma 2.5 and Definition 2.4, $(E - \mathcal{K}^T)$ is an M-matrix, where E denotes an identity matrix of size $N + M$. Therefore, by Lemma 3.2, there exists a diagonal matrix $L = \operatorname{diag}(\alpha_1, \ldots, \alpha_N, \beta_1, \ldots, \beta_M)$ with positive diagonal elements such that the product $(E - \mathcal{K}^T)L$ is strictly diagonally dominant with positive diagonal entries, namely,

$$\alpha_i \underline{a}_{ii} > \sum_{k=1,k\neq i}^{N} \alpha_k \overline{a}_{ki} + \sum_{l=1}^{M} \beta_l \overline{d}_{li}, \quad i = 1,\ldots,N,$$
$$\beta_j \underline{e}_{jj} > \sum_{k=1}^{N} \alpha_k \overline{c}_{kj} + \sum_{l=1,l\neq j}^{M} \beta_l \overline{e}_{lj}, \quad j = 1,\ldots,M. \tag{3.2}$$

Now, we define a Lyapunov function $V(t)$ as follows:

$$\begin{aligned} V(t) = {} & \sum_{i=1}^{N} \alpha_i \Bigg[\left| \ln x_i(t) - \ln x_i^*(t) \right| + \sum_{K=1,K\neq i}^{N} \int_{t-\tau_{ik}}^{t} a_{ik}(s+\tau_{ik}) \left| x_k(t) - x_k^*(t) \right| ds \\ & + \sum_{l=1}^{M} \int_{t-\eta_{il}}^{t} c_{il}(s+\eta_{il}) \left| y_l(t) - y_l^*(t) \right| ds \Bigg] \\ & + \sum_{j=1}^{M} \beta_j \Bigg[\left| \ln y_j(t) - \ln y_j^*(t) \right| + \sum_{K=1}^{N} \int_{t-\delta_{jk}}^{t} d_{jk}(s+\delta_{jk}) \left| x_k(t) - x_k^*(t) \right| ds \\ & + \sum_{l=1,l\neq j}^{M} \int_{t-\xi_{jl}}^{t} e_{jl}(s+\xi_{jl}) \left| y_l(t) - y_l^*(t) \right| ds \Bigg], \quad t \geq t_0. \end{aligned} \tag{3.3}$$

Calculating the upper right derivative of $V(t)$ and using (3.2), we get

$$\begin{aligned} D^+V(t) \leq {} & \sum_{i=1}^{N} \alpha_i \Bigg[-a_{ii}(t) \left| x_i(t) - x_i^*(t) \right| + \sum_{k=1,k\neq i}^{N} a_{ik}(t+\tau_{ik}) \left| x_k(t) - x_k^*(t) \right| \\ & + \sum_{l=1}^{M} c_{il}(t+\eta_{il}) \left| y_l(t) - y_l^*(t) \right| \Bigg] \\ & + \sum_{j=1}^{M} \beta_j \Bigg[-e_{jj}(t) \left| y_j(t) - y_j^*(t) \right| + \sum_{k=1}^{N} d_{jk}(t+\delta_{jk}) \left| x_k(t) - x_k^*(t) \right| \\ & + \sum_{l=1,l\neq j}^{M} e_{jl}(t+\xi_{jl}) \left| y_l(t) - y_l^*(t) \right| \Bigg] \end{aligned}$$

$$
\begin{aligned}
&\leq -\sum_{i=1}^{N}\alpha_i\Bigg[-\underline{a}_{ii}\left|x_i(t)-x_i^*(t)\right|+\sum_{k=1,k\neq i}^{N}\overline{a}_{ik}\left|x_k(t)-x_k^*(t)\right| \\
&\qquad\qquad +\sum_{l=1}^{M}\overline{c}_{il}\left|y_l(t)-y_l^*(t)\right|\Bigg] \\
&\quad +\sum_{j=1}^{M}\beta_j\Bigg[-\underline{e}_{jj}\left|y_j(t)-y_j^*(t)\right|+\sum_{k=1}^{N}\overline{d}_{jk}\left|x_k(t)-x_k^*(t)\right| \\
&\qquad\qquad +\sum_{l=1,l\neq j}^{M}\overline{e}_{jl}\left|y_l(t)-y_l^*(t)\right|\Bigg] \\
&= -\sum_{i=1}^{N}\left(\alpha_i\underline{a}_{ii}-\sum_{k=1,k\neq i}^{N}\alpha_k\overline{a}_{ki}-\sum_{l=1}^{M}\beta_l\overline{d}_{li}\right)\left|x_i(t)-x_i^*(t)\right| \\
&\quad -\sum_{j=1}^{M}\left(\beta_j\underline{e}_{jj}-\sum_{k=1}^{N}\alpha_k\overline{c}_{kj}-\sum_{l=1,l\neq j}^{M}\beta_l\overline{e}_{lj}\right)\left|y_j(t)-y_j^*(t)\right| \\
&= -c\left\{\sum_{i=1}^{N}\left|x_i(t)-x_i^*(t)\right|+\sum_{j=1}^{M}\left|y_j(t)-y_j^*(t)\right|\right\},
\end{aligned}
\tag{3.4}
$$

where

$$
c=\min\left\{\alpha_i\underline{a}_{ii}-\sum_{k=1,k\neq i}^{N}\alpha_k\overline{a}_{ki}-\sum_{l=1}^{M}\beta_l\overline{d}_{li},\beta_j\underline{e}_{jj}-\sum_{k=1}^{N}\alpha_k\overline{c}_{kj}-\sum_{l=1,l\neq j}^{M}\beta_l\overline{e}_{lj}\right\}>0. \tag{3.5}
$$

It follows from (3.4) that $D^+V(t)\leq 0$. Obviously, the zero solution of (1.3) is Lyapunov stable. On the other hand, integrating (3.4) over $[t_0,t]$ leads to

$$
V(t)-V(t_0)\leqslant -c\int_{t_0}^{t}\left[\sum_{i=1}^{N}\left|x_i(s)-x_i^*(s)\right|+\sum_{j=1}^{M}\left|y_j(s)-y_j^*(s)\right|\right]ds,\quad t\geqslant 0, \tag{3.6}
$$

or

$$
V(t)+c\int_{t_0}^{t}\left[\sum_{i=1}^{N}\left|x_i(s)-x_i^*(s)\right|+\sum_{j=1}^{M}\left|y_j(s)-y_j^*(s)\right|\right]ds\leqslant V(t_0)<+\infty,\quad t\geqslant t_0. \tag{3.7}
$$

Noting that $V(t)\geqslant 0$, it follows that

$$
\int_{t_0}^{t}\left[\sum_{i=1}^{N}\left|x_i(s)-x_i^*(s)\right|+\sum_{j=1}^{M}\left|y_j(s)-y_j^*(s)\right|\right]ds\leqslant\frac{V(t_0)}{c}<+\infty,\quad t\geqslant t_0. \tag{3.8}
$$

Therefore, by Lemma 3.1, it is not difficult to conclude that

$$\lim_{t\to+\infty}\left|X_i(t)-X_i^*(t)\right|=0. \tag{3.9}$$

Theorem 3.3 follows. □

4. Corollaries and Remarks

In order to illustrate some features of our main results, we will present some corollaries and remarks in this section. From the proofs of Theorems 2.6 and 3.3, one can easily deduce the following corollary.

Corollary 4.1. *In addition to* (H_1)*, further suppose that* $E-\mathcal{K}$ *or* $E-\mathcal{K}^T$ *is an* M*-matrix. Then system* (1.3) *has a unique positive* ω*-periodic solution which is globally asymptotically stable.*

Now recall that for a given matrix $\mathcal{K}$, its spectral radius $\rho(\mathcal{K})$ is equal to the minimum of all matrix norms of $\mathcal{K}$, that is, for any matrix norm $\|\cdot\|$, $\rho(\mathcal{K})\le\|\mathcal{K}\|$. Therefore, we have the following corollary.

Corollary 4.2. *In addition to* (H_1)*, if one further supposes that there exist positive constants* $\xi_i,\ i=1,2,\dots,n,\ \eta_j,\ j=1,2,\dots,m$ *such that one of the following inequalities holds.*

(1) $\max\{\max_{1\le k\le n}\{\underline{a}_{kk}^{-1}\xi_k^{-1}[\sum_{i=1,i\ne k}^{n}\xi_i\overline{a}_{ik}+\sum_{l=1}^{m}\eta_j\overline{d}_{jk}]\},\max_{1\le l\le m}\{\underline{e}_{ll}^{-1}\eta_l^{-1}[\sum_{i=1}^{n}\xi_i\overline{c}_{li}+\sum_{j=1,j\ne l}^{m}\eta_j\overline{e}_{jl}]\}\}<1$*, or equivalently, for all* $k=1,\dots,n,\ l=1,2,\dots,n,$

$$\begin{aligned}\xi_k\underline{a}_{kk}&>\sum_{i=1,i\ne k}^{n}\xi_i\overline{a}_{ik}+\sum_{l=1}^{m}\eta_j\overline{d}_{jk},\\ \eta_l\underline{e}_{ll}&>\sum_{i=1}^{n}\xi_i\overline{c}_{li}+\sum_{j=1,j\ne l}^{m}\eta_j\overline{e}_{jl}.\end{aligned} \tag{4.1}$$

(2) $\sum_{i=1}^{n+m}\sum_{j=1}^{n+m}(\xi_i^{-1}\xi_jk_{ij})^2<1$*, where* $\mathcal{K}=(k_{ij})_{(n+m)\times(n+m)}$ *has been defined in Theorem 2.6.*

(3) $\max\{\max_{1\le i\le n}\{\underline{a}_{ii}^{-1}\xi_i^{-1}[\sum_{k=1,k\ne i}^{n}\xi_k\overline{a}_{ki}+\sum_{l=1}^{m}\eta_l\overline{d}_{li}]\},\max_{1\le j\le m}\{\underline{e}_{jj}^{-1}\eta_j^{-1}[\sum_{k=1}^{n}\xi_k\overline{c}_{kj}+\sum_{l=1,l\ne j}^{m}\eta_l\overline{e}_{lj}]\}\}<1$*, or equivalently, for all* $i=1,\dots,n,\ j=1,2,\dots,n,$

$$\begin{aligned}\xi_i a_{ii}&>\sum_{k=1,k\ne i}^{n}\xi_k|a_{ki}|+\sum_{l=1}^{m}\eta_l|d_{li}|,\\ \eta_j e_{jj}&>\sum_{k=1}^{n}\xi_k\left|c_{kj}\right|+\sum_{l=1,l\ne j}^{m}\eta_l\left|e_{lj}\right|.\end{aligned} \tag{4.2}$$

Then system (1.3) *has a unique positive* ω*-periodic solution which is globally asymptotically stable.*

Proof. For any matrix norm $\|\cdot\|$ and any nonsingular matrix S, $\|\mathcal{K}\|_S=\|S^{-1}\mathcal{K}S\|$ also defines a matrix norm. Let us denote $D=\operatorname{diag}(\xi_1,\xi_2,\dots,\xi_n)$, then the conditions (1.2) and (1.3) correspond to the column norms and Frobenius norm of matrix $D\mathcal{K}D^{-1}$, respectively.

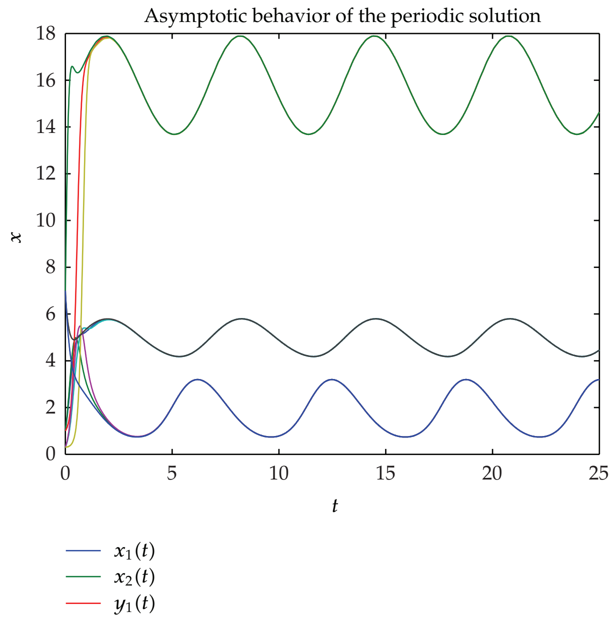

Figure 1: Asymptotic behavior of system (5.1) with initial values $(x_1(0), x_2(0), y_1(0)) = (1,1,1)$, $(0.3, 0.3, 0.3)$, $(7,7,7)$, respectively, $t \in [0,25]$.

Condition (2.10) corresponds to the row norms of $D\mathcal{K}^T D^{-1}$ and note that $\rho(D\mathcal{K}^T D^{-1}) = \rho(D\mathcal{K}D^{-1})$. Now Corollary 4.2 follows immediately. □

5. Example

In this section, an example and its simulations are presented to illustrate the feasibility and effectiveness of our results.

Example 5.1. Consider the following periodic predator-prey model with 2-predators and 1-prey:

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)\left[7 + \sin t - x_1(t) - \frac{1}{4}x_2(t) - \frac{1}{10}y_1(t-1)\right],\\ \dot{x}_2(t) &= x_2(t)\left[7 + \cos t - \frac{1}{4}x_1(t) - x_2(t) - \frac{1}{4}y_1(t)\right],\\ \dot{y}_1(t) &= y_1(t)\left[-\frac{1}{20}(1+\cos t) + \frac{3}{2}x_1(t-1) + \frac{1}{4}x_2(t) - \frac{1}{2}y_1(t)\right].\end{aligned} \tag{5.1}$$

Simple computation leads to

$$\mathcal{K} = \begin{pmatrix} 0 & a_{22}^{-1}a_{12} & e_{11}^{-1}c_{11} \\ a_{11}^{-1}a_{21} & 0 & e_{11}^{-1}c_{21} \\ a_{11}^{-1}d_{11} & a_{22}^{-1}d_{12} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & 0 & 0 \\ \frac{3}{2} & \frac{1}{4} & 0 \end{pmatrix}. \tag{5.2}$$

By using mathematica, we see that $\rho(\mathcal{K}) = 0.633982 < 1$. Thus, the system (5.1) has a periodic solution which is globally asymptotically stable. Figure 1 shows the asymptotic behavior of the periodic solution.

Remark 5.2. In this example, one can observe that though the spectral $\rho(\mathcal{K}) < 1$, the matrix norms of the matrix $\mathcal{K}$ are all bigger than 1. For instance, the column norm: is

$$\|\mathcal{K}\|_1 = \max_{1\le j\le 3}\left\{a_{jj}^{-1}\sum_{i=1,i\ne j}^{3} a_{ij}\right\} = 0 + \frac{3}{2} + \frac{1}{4} > 1. \tag{5.3}$$

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