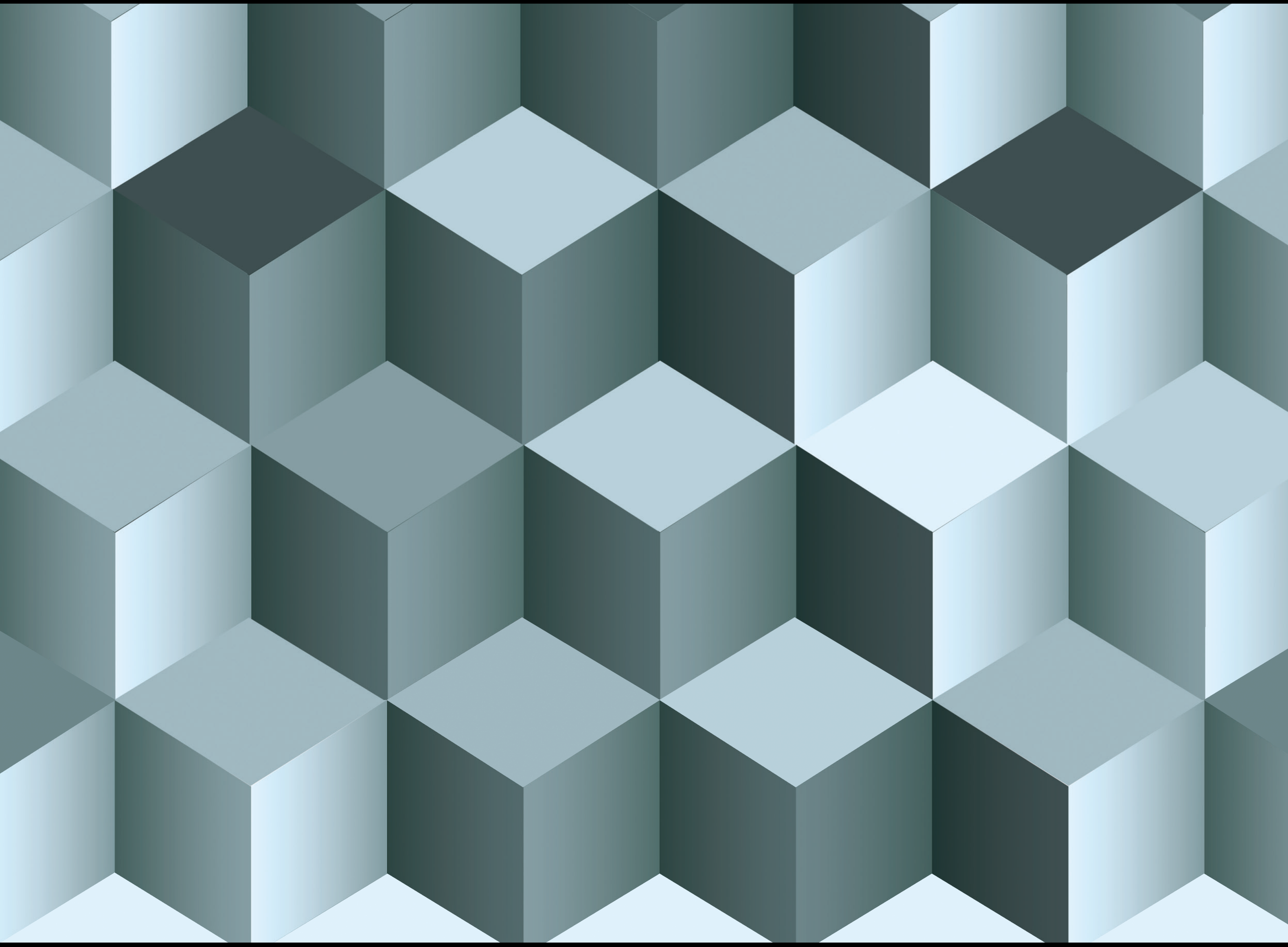
Journal of Function Spaces

Recent Advances in Function Spaces and its Applications in Fractional Differential Equations 2021

Lead Guest Editor: Xinguang Zhang
Guest Editors: Yong Hong Wu, Chuanjun Chen, Fu-Dong Ge, and Chun Lu



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Research Article

Dynamical Behavior of Stochastic Markov Switching Hepatitis B Epidemic Model with Saturated Incidence Rate

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The article researches a stochastic hepatitis B epidemic model with saturated incidence rate, which is perturbed by both white noise and colored noise. Firstly, we obtain a significant criterion R_0^S which relies on environmental noises. By means of Lyapunov function approach, we show that there is a stationary distribution if $R_0^S > 1$. Its condition implies that when white noise is small, in the stochastic model, there exists a stochastic positive equilibrium state without changing the basic properties of its corresponding deterministic model. Secondly, we derive sufficient criteria for extinction of the disease. Finally, we propose a definition of the solution to an impulsive stochastic functional differential equation with Markovian switching (ISFDM).

1. Introduction

Hepatitis B virus is a severe infectious disease that has emerged as one of the greatest threats to human health in the 21st century. An estimated 350 million people worldwide have been infected with hepatitis B virus [1]. The mathematical model to describe hepatitis B virus transmission and its dynamics has been extensively explored, which provides some effective suggestions for further study on the progression and its control [2–5]. Recently, Khan et al. [6] investigated a hepatitis B epidemic model with saturated incidence rate:

$$\begin{cases} \dfrac{dS}{dt} = \Lambda - \dfrac{\alpha SI}{1+\gamma I} - (\mu_0 + \nu)S, \\ \dfrac{dI}{dt} = \dfrac{\alpha SI}{1+\gamma I} - (\mu_0 + \mu_1 + \beta)I, \\ \dfrac{dR}{dt} = \beta I + \nu S - \mu_0 R, \end{cases} \tag{1}$$

with $S(0) > 0, I(0) > 0$, and $R(0) > 0$. In model (1), the birth rate is denoted by Λ. The transmission rate of hepatitis B is given by α, while μ_0 and μ_1, respectively, demonstrated the natural and disease-induced death rates. Recovery rate is denoted by β, while the vaccination and saturation rates are ν and γ, respectively. According to the theory in [6], model (1) always has the disease-free equilibrium $E^0 = (S^0, 0, R^0)$, where the components are defined as $S^0 = \Lambda/(\mu_0 + \nu)$, and $R^0 = \Lambda\nu/(\mu_0(\mu_0 + \nu))$. If $R_0 < 1$, E^0 is globally asymptotically stable. If $R_0 > 1$, E^0 is unstable and there exists an endemic equilibrium $E^* = (S^*, I^*, R^*)$ which is globally asymptotically stable, where $R_0 = \alpha\Lambda/((\mu_0 + \nu)(\mu_0 + \mu_1 + \beta))$.

In fact, epidemic models are inherently subject to a continuous spectrum of disturbances [7–11]. Many authors demonstrated that the white noise and colored noise have a great destabilizing influence on the epidemic transmission. Moreover, considering the effect of environment noise on the epidemic model has become a popular trend in controlling the spread of disease [12–16]. In this respect, some researches on stochastic hepatitis B virus models have been reported [17–19]. Particularly, in the epidemic model, the disease transmission rate α represents an extremely important coefficient [16, 20]. In this paper, by taking into account the effect of continuous-time Markov chain on the transmission rate α, we consider a stochastic analogue of the deterministic model (1):

$$\begin{cases} dS = \left(\Lambda - \dfrac{\alpha(\xi(t))SI}{1+\gamma I} - (\mu_0+\nu)S \right)dt + \sigma_1(\xi(t))SdB_1(t), \\ dI = \left(\dfrac{\alpha(\xi(t))SI}{1+\gamma I} - (\mu_0+\mu_1+\beta)I \right)dt + \sigma_2(\xi(t))IdB_2(t), \\ dR = (\beta I + \nu S - \mu_0 R)dt + \sigma_3(\xi(t))RdB_3(t), \end{cases} \quad (2)$$

where $B_i(t)$ are independent standard Brownian motions and σ_i^2 stand for the intensities of $B_i(t), i=1,2,3$. $\xi(t), t \geq 0$, is a right-continuous Markov chain on the complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with values in a finite space $\mathcal{M} = \{1, 2, \cdots, N\}$ (see [21, 22]).

It is widely known that the stability of biomathematical model has always been a hot issue in recent years [23–26]. Compared with their corresponding deterministic cases, lots of stochastic models have no traditional positive equilibrium state. Consequently, the research of ergodic stationary distribution of s stochastic biomathematical model has been a research highlight. In addition, model (2) incorporates white noise as well as colored noise possessing important practical significance [27]. The main aim of this article is to prove the existence of stationary distribution for model (2). Above all, to guarantee existence and uniqueness of globally positive solution for model (2), we establish the following conclusion. Since the proof is standard, we omit it here.

Lemma 1. *For any initial value $(S(0), I(0), R(0), \xi(0)) \in \mathbb{R}_+^3 \times \mathcal{M}$, there exists a unique positive solution $(S(t), I(t), R(t), \xi(t)) \in \mathbb{R}_+^3 \times \mathcal{M}$ of model (2) on $t \geq 0$ almost surely (a.s.).*

2. Existence of a Unique and Ergodic Stationary Distribution

Theorem 2. *If $R_0^S > 1$, where*

$$R_0^S = \frac{\sum_{k\in\mathcal{M}} \pi_k \alpha(k)\Lambda}{\left(\mu_0+\nu+\sum_{k\in\mathcal{M}}\pi_k\left(\sigma_1^2(k)/2\right)\right)\left(\mu_0+\mu_1+\beta+\sum_{k\in\mathcal{M}}\pi_k\left(\sigma_2^2(k)/2\right)\right)}, \quad (3)$$

then for any initial value $(S(0), I(0), R(0), \xi(0)) \in \mathbb{R}_+^3 \times \mathcal{M}$, model (2) has a unique stationary distribution which is ergodic.

Proof. In order to prove Theorem 2, we need to validate that the feasibility of (A1), (A2), and (A3) in Lemma 7 in the appendix holds. We have assumed (A1) holds in Section 1. To verify (A3), we need to find a nonnegative C^2-function $V(S, I, R, k)$ and a compact set $D_\varepsilon \in \mathbb{R}_+^4$ such that $LV \leq -1$ for all $(S, I, R, k) \in (\mathbb{R}_+^3 \setminus D_\varepsilon) \times \mathcal{M}$. Construct a C^2-function

$$\begin{aligned} V(S,I,R) &= M(-c_1 \ln S - c_2 \ln I) + \rho(k) + (S+I+R)^{\rho+1} \\ &\quad - \ln S - \ln I - \ln R = MV_1 + V_2 + V_3 + V_4 + V_5, \end{aligned} \quad (4)$$

where $V_1 = -c_1 \ln S - c_2 \ln I + \rho(k)$, $V_2 = (S+I+R)^{\rho+1}$, $V_3 = -\ln S$, $V_4 = -\ln I$, $V_5 = -\ln R$, and $0 < \rho < 2\mu_0 / \max_{i=1,2,3}\{\breve{\sigma}_i^2\}$, where $\breve{\sigma}_i = \max_{k\in\mathcal{M}}\{\sigma_i(k)\}$, and constants M, c_1, c_2, compact set D_ε and function $\rho(k)$ will be determined later. Employing Itô's formula [28–34], we can get

$$\begin{aligned} \mathcal{L}V_1 &= -\frac{c_1\Lambda}{S} + \frac{c_1\alpha(k)I}{1+\gamma I} + c_1\left(\mu_0+\nu+\frac{1}{2}\sigma_1^2(k)\right) \\ &\quad - \frac{c_2\alpha(k)S}{1+\gamma I} + c_2\left(\mu_0+\mu_1+\beta+\frac{1}{2}\sigma_2^2(k)\right) \\ &= +\sum_{l\in\mathcal{M}}\zeta_{kl}\rho(l) - \frac{c_1\Lambda}{S} - \frac{c_2\alpha(k)S}{1+\gamma I} - (1+\gamma I) + \frac{c_1\alpha(k)I}{1+\gamma I} \\ &\quad + c_1\left(\mu_0+\nu+\frac{1}{2}\sigma_1^2(k)\right) + c_2\left(\mu_0+\mu_1+\beta+\frac{1}{2}\sigma_2^2(k)\right) \\ &\quad + (1+\gamma I) + \sum_{l\in\mathcal{M}}\zeta_{kl}\rho(l) \leq -3\sqrt[3]{c_1c_2\alpha(k)\Lambda} + 1 \\ &\quad + c_1\left(\mu_0+\nu+\frac{1}{2}\sigma_1^2(k)\right) + c_2\left(\mu_0+\mu_1+\beta+\frac{1}{2}\sigma_2^2(k)\right) \\ &\quad + \gamma I + \frac{c_1\alpha(k)I}{1+\gamma I} + \sum_{l\in\mathcal{M}}\zeta_{kl}\rho(l). \end{aligned} \quad (5)$$

Choose $M_1(k) = -3\sqrt[3]{c_1c_2\alpha(k)\Lambda} + 1 + c_1(\mu_0+\nu+(1/2)\sigma_1^2(k)) + c_2(\mu_0+\mu_1+\beta+(1/2)\sigma_2^2(k))$; on the basis of the irreducibility of generator matrix Γ, one can find that for $\Theta = (\Theta(1), \Theta(2), \cdots, \Theta(N))$, there exists $\rho = (\rho(1), \rho(2), \cdots, \rho(N))^T$ satisfying the following Poisson system $\Gamma\rho = (\sum_{k=1}^N \pi_k\Theta(k))\overleftarrow{1} - \Theta$. Let c_1 and c_2 satisfy

$$\begin{aligned} c_1\left(\mu_0+\nu+\sum_{k\in\mathcal{M}}\pi_k\frac{\sigma_1^2(k)}{2}\right) &= c_2\left(\mu_0+\mu_1+\beta+\sum_{k\in\mathcal{M}}\pi_k\frac{\sigma_2^2(k)}{2}\right) \\ &= \frac{\sum_{k\in\mathcal{M}}\pi_k\alpha(k)\Lambda}{\left(\mu_0+\nu+\sum_{k\in\mathcal{M}}\pi_k\left(\sigma_1^2(k)/2\right)\right)\left(\mu_0+\mu_1+\beta+\sum_{k\in\mathcal{M}}\pi_k\left(\sigma_2^2(k)/2\right)\right)}. \end{aligned} \quad (6)$$

Then,

$$\begin{aligned} \mathcal{L}V_1 &\leq -\frac{\sum_{k\in\mathcal{M}}\pi_k\alpha(k)\Lambda}{\left(\mu_0+\nu+\sum_{k\in\mathcal{M}}\pi_k\left(\sigma_1^2(k)/2\right)\right)\left(\mu_0+\mu_1+\beta+\sum_{k\in\mathcal{M}}\pi_k\left(\sigma_2^2(k)/2\right)\right)} \\ &\quad + 1 + \gamma I + \frac{c_1\alpha(k)I}{1+\gamma I} \leq -\left(R_0^S - 1\right) + \gamma I + c_1\breve{\alpha}I = -\lambda + \varphi(I), \end{aligned} \quad (7)$$

where

$$\begin{aligned}\lambda &= R_0^S - 1,\\ \varphi(I) &= \gamma I + c_1 \breve{\alpha} I,\end{aligned} \tag{8}$$

and set $\breve{\alpha} = \max_{k\in\mathcal{M}}\{\alpha(k)\}$. Applying Itô's formula, one can obtain

$$\begin{aligned}\mathscr{L}V_2 &= (\rho+1)(S+I+R)^{\rho}(\Lambda - \mu_0 S - (\mu_0+\mu_1)I - \mu_0 R)\\ &\quad + \frac{1}{2}\rho(\rho+1)(S+I+R)^{\rho-1}(\sigma_1^2(k)S^2 + \sigma_2^2(k)I^2\\ &\quad + \sigma_3^2(k)R^2) \le (\rho+1)(S+I+R)^{\rho}(\Lambda - \mu_0(S+I+R))\\ &\quad + \max_{i=1,2,3}\{\breve{\sigma}_i{}^2\}\frac{\rho}{2}(\rho+1)(S+I+R)^{\rho+1} = \Lambda(\rho+1)(S+I+R)^{\rho}\\ &\quad - (\rho+1)\left(\mu_0 - \frac{\rho}{2}\max_{i=1,2,3}\{\breve{\sigma}_i{}^2\}\right)(S+I+R)^{\rho+1}\\ &\le B - \frac{1}{2}(\rho+1)\left(\mu_0 - \frac{\rho}{2}\max_{i=1,2,3}\{\breve{\sigma}_i{}^2\}\right)(S+I+R)^{\rho+1}\\ &\le B - \frac{1}{2}(\rho+1)\left(\mu_0 - \frac{\rho}{2}\max_{i=1,2,3}\{\breve{\sigma}_i{}^2\}\right)(S^{\rho+1}+I^{\rho+1}+R^{\rho+1}),\end{aligned} \tag{9}$$

where

$$\begin{aligned}B = \sup_{(S,I,R)\in\mathbb{R}_+^3}&\left\{\Lambda(\rho+1)(S+I+R)^{\rho} - \frac{1}{2}(\rho+1)\right.\\ &\left.\cdot\left(\mu_0 - \frac{\rho}{2}\max_{i=1,2,3}\{\breve{\sigma}_i{}^2\}\right)(S+I+R)^{(\rho+1)}\right\} < \infty.\end{aligned} \tag{10}$$

Denote

$$C = \sup_{(S,I,R)\in\mathbb{R}_+^3}\left\{\theta - \frac{1}{2}(\rho+1)\left(\mu_0 - \frac{\rho}{2}\max_{i=1,2,3}\{\breve{\sigma}_i{}^2\}\right)(S^{\rho+1}+I^{\rho+1}+R^{\rho+1})\right\}, \tag{11}$$

where $\theta = B + (\mu_0 + \nu + (1/2)\breve{\sigma}_1{}^2) + (\mu_0 + \mu_1 + \beta + (1/2)\breve{\sigma}_2{}^2) + (\mu_0 + (1/2)\breve{\sigma}_3{}^2)$. By using Itô's formula, we also have

$$\begin{aligned}\mathscr{L}V_3 &= -\frac{\Lambda}{S} + \frac{\alpha(k)I}{1+\gamma I} + \mu_0 + \nu + \frac{1}{2}\sigma_1^2(k),\\ \mathscr{L}V_4 &= -\frac{\alpha(k)S}{1+\gamma I} + \mu_0 + \mu_1 + \beta + \frac{1}{2}\sigma_2^2(k),\\ \mathscr{L}V_5 &= -\beta\frac{I}{R} - \nu\frac{S}{R} + \mu_0 + \frac{1}{2}\sigma_3^2(k).\end{aligned} \tag{12}$$

Hence, by (7), (9), and (12), we get

$$\begin{aligned}\mathscr{L}V &\le -M\lambda + M\varphi(I) + \breve{\alpha}I - \frac{\Lambda}{S} - \frac{\widehat{\alpha}S}{1+\gamma I} - \beta\frac{I}{R} - \nu\frac{S}{R} + \theta\\ &\quad - \frac{1}{2}(\rho+1)\left(\mu_0 - \frac{\rho}{2}\max_{i=1,2,3}\{\breve{\sigma}_i{}^2\}\right)(S^{\rho+1}+I^{\rho+1}+R^{\rho+1}),\end{aligned} \tag{13}$$

where $\widehat{\alpha} = \min_{k\in\mathcal{M}}\{\alpha(k)\}$. Here, we choose that the positive constant M satisfies the following inequality:

$$-M\lambda + C \le -2. \tag{14}$$

For arbitrary $\varepsilon > 0$, define the following bounded closed set:

$$D_\varepsilon = \left\{\varepsilon \le S \le \frac{1}{\varepsilon}, \varepsilon \le I \le \frac{1}{\varepsilon}, \varepsilon^2 \le R \le \frac{1}{\varepsilon^2}\right\}, \tag{15}$$

where ε satisfies the following conditions:

$$\begin{aligned}&-\frac{\Lambda}{\varepsilon} + K \le -1,\\ &-M\lambda + M\varphi(\varepsilon) + \breve{\alpha}\varepsilon + C \le -1,\\ &-\frac{\beta}{\varepsilon} + K \le -1,\\ &-\frac{1}{2}(\rho+1)\left(\mu_0 - \frac{\rho}{2}\max_{i=1,2,3}\{\breve{\sigma}_i{}^2\}\right)\frac{1}{\varepsilon^{\rho+1}} + D \le -1,\\ &-\frac{1}{2}(\rho+1)\left(\mu_0 - \frac{\rho}{2}\max_{i=1,2,3}\{\breve{\sigma}_i{}^2\}\right)\frac{1}{\varepsilon^{\rho+1}} + E \le -1,\\ &-\frac{1}{2}(\rho+1)\left(\mu_0 - \frac{\rho}{2}\max_{i=1,2,3}\{\breve{\sigma}_i{}^2\}\right)\frac{1}{\varepsilon^{\rho+1}} + F \le -1,\end{aligned} \tag{16}$$

where

$$\begin{aligned}K &= \sup_{(S,I,R)\in\mathbb{R}_+^3}\{M\varphi(I) + \breve{\alpha}I + C\},\\ D &= \sup\left\{M\varphi(I) + \breve{\alpha}I + \theta - \frac{1}{2}(\rho+1)\left(\mu_0 - \frac{\rho}{2}\max_{i=1,2,3}\{\breve{\sigma}_i{}^2\}\right)(I^{\rho+1}+R^{\rho+1})\right\},\\ E &= \sup\left\{M\varphi(I) + \breve{\alpha}I + \theta - \frac{1}{2}(\rho+1)\left(\mu_0 - \frac{\rho}{2}\max_{i=1,2,3}\{\breve{\sigma}_i{}^2\}\right)(S^{\rho+1}+R^{\rho+1})\right\},\\ F &= \sup\left\{M\varphi(I) + \breve{\alpha}I + \theta - \frac{1}{2}(\rho+1)\left(\mu_0 - \frac{\rho}{2}\max_{i=1,2,3}\{\breve{\sigma}_i{}^2\}\right)(S^{\rho+1}+I^{\rho+1})\right\}.\end{aligned} \tag{17}$$

Furthermore,

$$\mathbb{R}_+^3 \setminus D_\varepsilon = D_1 \cup D_2 \cup D_3 \cup D_4 \cup D_5 \cup D_6, \tag{18}$$

where

$$\begin{aligned}
D_1 &= \left\{(S,I,R)\in\mathbb{R}_+^3, 0<S<\varepsilon\right\},\\
D_2 &= \left\{(S,I,R)\in\mathbb{R}_+^3, 0<I<\varepsilon\right\},\\
D_3 &= \left\{(S,I,R)\in\mathbb{R}_+^3, 0<R<\varepsilon^2, S>\varepsilon, I>\varepsilon\right\},\\
D_4 &= \left\{(S,I,R)\in\mathbb{R}_+^3, S>\frac{1}{\varepsilon}\right\},\\
D_5 &= \left\{(S,I,R)\in\mathbb{R}_+^3, I>\frac{1}{\varepsilon}\right\},\\
D_6 &= \left\{(S,I,R)\in\mathbb{R}_+^3, R>\frac{1}{\varepsilon}\right\}.
\end{aligned} \tag{19}$$

Case 1. If $(S,I,R)\in D_1$, we derive that

$$\begin{aligned}
\mathscr{L}V \le & -M\lambda + M\varphi(I) + \check{\alpha}I - \frac{\Lambda}{S} - \frac{\widehat{\alpha}S}{1+\gamma I} - \beta\frac{I}{R} - \nu\frac{S}{R} + \theta \\
& - \frac{1}{2}(\rho+1)\left(\mu_0 - \frac{\rho}{2}\max_{i=1,2,3}\left\{\check{\sigma}_i^{\,2}\right\}\right)\left(S^{\rho+1}+I^{\rho+1}+R^{\rho+1}\right)\\
\le & -\frac{\Lambda}{\varepsilon} + K \le -1.
\end{aligned} \tag{20}$$

Case 2. If $(S,I,R)\in D_2$, we have

$$\begin{aligned}
\mathscr{L}V \le & -M\lambda + M\varphi(I) + \check{\alpha}I - \frac{\Lambda}{S} - \frac{\widehat{\alpha}S}{1+\gamma I} - \beta\frac{I}{R} - \nu\frac{S}{R} + \theta \\
& - \frac{1}{2}(\rho+1)\left(\mu_0 - \frac{\rho}{2}\max_{i=1,2,3}\left\{\check{\sigma}_i^{\,2}\right\}\right)\left(S^{\rho+1}+I^{\rho+1}+R^{\rho+1}\right)\\
\le & -M\lambda + M\varphi(\varepsilon) + \check{\alpha}\varepsilon + C \le -1.
\end{aligned} \tag{21}$$

Case 3. If $(S,I,R)\in D_3$, we compute

$$\begin{aligned}
\mathscr{L}V \le & -M\lambda + M\varphi(I) + \check{\alpha}I - \frac{\Lambda}{S} - \frac{\widehat{\alpha}S}{1+\gamma I} - \beta\frac{I}{R} - \nu\frac{S}{R} + \theta \\
& - \frac{1}{2}(\rho+1)\left(\mu_0 - \frac{\rho}{2}\max_{i=1,2,3}\left\{\check{\sigma}_i^{\,2}\right\}\right)\left(S^{\rho+1}+I^{\rho+1}+R^{\rho+1}\right)\\
\le & -\frac{\beta}{\varepsilon} + K \le -1.
\end{aligned} \tag{22}$$

Case 4. If $(S,I,R)\in D_4$, we derive

$$\begin{aligned}
\mathscr{L}V \le & -M\lambda + M\varphi(I) + \check{\alpha}I - \frac{\Lambda}{S} - \frac{\widehat{\alpha}S}{1+\gamma I} - \beta\frac{I}{R} - \nu\frac{S}{R} + \theta \\
& - \frac{1}{2}(\rho+1)\left(\mu_0 - \frac{\rho}{2}\max_{i=1,2,3}\left\{\check{\sigma}_i^{\,2}\right\}\right)\left(S^{\rho+1}+I^{\rho+1}+R^{\rho+1}\right)\\
\le & -\frac{1}{2}(\rho+1)\left(\mu_0 - \frac{\rho}{2}\max_{i=1,2,3}\left\{\check{\sigma}_i^{\,2}\right\}\right)\frac{1}{\varepsilon^{\rho+1}} + D \le -1.
\end{aligned} \tag{23}$$

Case 5. If $(S,I,R)\in D_5$, we conclude

$$\begin{aligned}
\mathscr{L}V \le & -M\lambda + M\varphi(I) + \check{\alpha}I - \frac{\Lambda}{S} - \frac{\widehat{\alpha}S}{1+\gamma I} - \beta\frac{I}{R} - \nu\frac{S}{R} + \theta \\
& - \frac{1}{2}(\rho+1)\left(\mu_0 - \frac{\rho}{2}\max_{i=1,2,3}\left\{\check{\sigma}_i^{\,2}\right\}\right)\left(S^{\rho+1}+I^{\rho+1}+R^{\rho+1}\right)\\
\le & -\frac{1}{2}(\rho+1)\left(\mu_0 - \frac{\rho}{2}\max_{i=1,2,3}\left\{\check{\sigma}_i^{\,2}\right\}\right)\frac{1}{\varepsilon^{\rho+1}} + E \le -1.
\end{aligned} \tag{24}$$

Case 6. If $(S,I,R)\in D_6$, we have

$$\begin{aligned}
\mathscr{L}V \le & -M\lambda + M\varphi(I) + \check{\alpha}I - \frac{\Lambda}{S} - \frac{\widehat{\alpha}S}{1+\gamma I} - \beta\frac{I}{R} - \nu\frac{S}{R} + \theta \\
& - \frac{1}{2}(\rho+1)\left(\mu_0 - \frac{\rho}{2}\max_{i=1,2,3}\left\{\check{\sigma}_i^{\,2}\right\}\right)\left(S^{\rho+1}+I^{\rho+1}+R^{\rho+1}\right)\\
\le & -\frac{1}{2}(\rho+1)\left(\mu_0 - \frac{\rho}{2}\max_{i=1,2,3}\left\{\check{\sigma}_i^{\,2}\right\}\right)\frac{1}{\varepsilon^{\rho+1}} + F \le -1.
\end{aligned} \tag{25}$$

Then, we can obtain that for a sufficiently small ε, $LV < -1$ for any $(S,I,R)\in\mathbb{R}_+^3\setminus D_\varepsilon$. Therefore, we can verify (A3) in Lemma 7 of the appendix. On the other hand, the diffusion matrix $D(x,k)=\operatorname{diag}\{\sigma_1^2(k)S^2,\sigma_2^2(k)I^2,\sigma_3^2(k)R^2\}$ of model (2) is positive definite, which implies that condition (A2) in Lemma 7 holds. This completes the proof.□ □

Now, consider the corresponding model (2) without Markov switching:

$$\begin{cases}
dS = \left(\Lambda - \dfrac{\alpha SI}{1+\gamma I} - (\mu_0+\nu)S\right)dt + \sigma_1 S dB_1(t),\\
dI = \left(\dfrac{\alpha SI}{1+\gamma I} - (\mu_0+\mu_1+\beta)I\right)dt + \sigma_2 I dB_2(t),\\
dR = (\beta I + \nu S - \mu_0 R)dt + \sigma_3 R dB_3(t).
\end{cases} \tag{26}$$

Define a parameter

$$\widehat{R}_0 = \frac{\alpha\int_0^\infty x\pi(x)dx}{\mu_0+\mu_1+\beta+\left(\sigma_2^2/2\right)}, \tag{27}$$

where

$$\pi(x) = Qx^{-2-(2(\mu_0+\nu))/\sigma_1^2}\sigma_1^{-2+(2(\mu_0+\nu))/\sigma_1^2}e^{-\left(2/\sigma_1^2\right)\left((\Lambda/x)+(\mu_0+\nu)\right)}, \quad x\in(0,+\infty). \tag{28}$$

Similar to Theorem 3.1 in [35], it is easy to obtain the following result.

Theorem 3. *Let $(S(t),I(t),R(t))$ be the solution of model (26). If $\widehat{R}_0<1$, for any initial value $(S(0),I(0),R(0))\in\mathbb{R}^3$,*

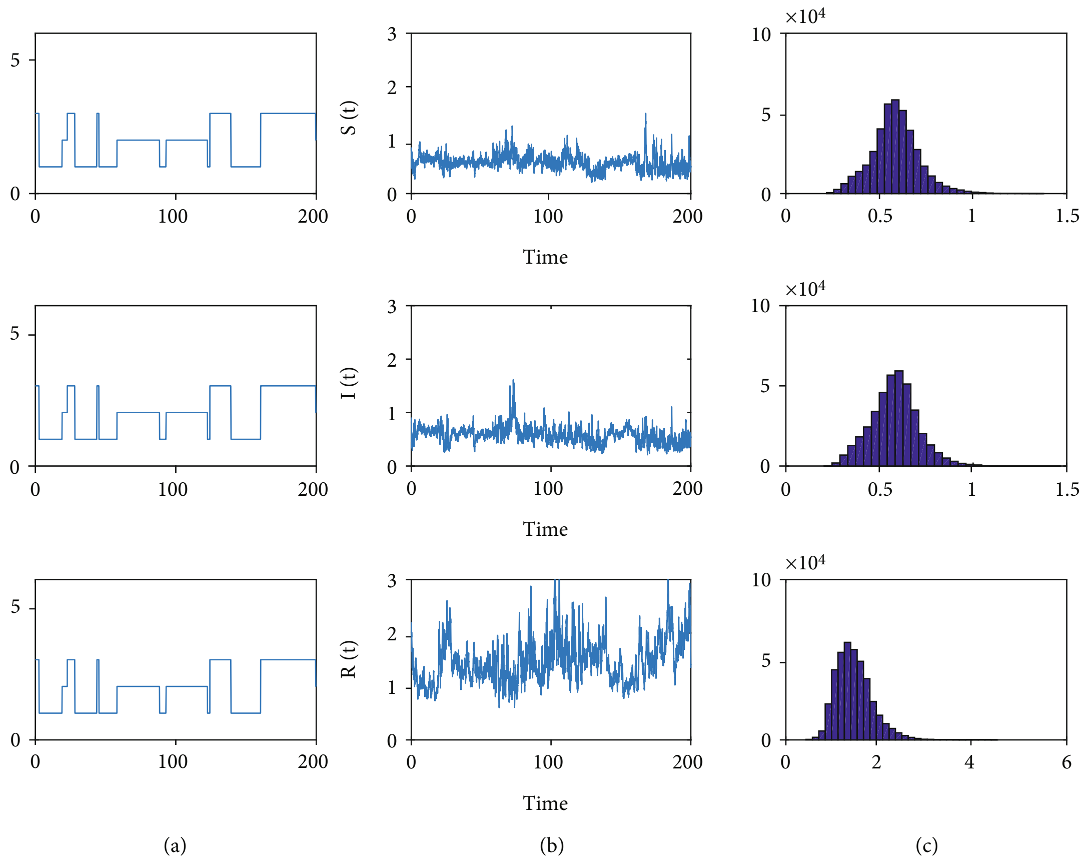

FIGURE 1: $S(t), I(t)$, and $R(t)$ have ergodic property. The pictures in (a) are Markovian chain. The pictures in (c) are the density functions of model (2) for $k \in \mathcal{M} = \{1, 2, 3\}$. The initial value $S(0) = 0.8, I(0) = 0.7$, and $R(0) = 1.1$. Step size $\Delta t = 0.001$.

then the solution $(S(t), I(t), R(t))$ *of model (26) satisfies*

$$\lim_{t \longrightarrow +\infty} I(t) = 0\, a.s., \tag{29}$$

and the distribution of $S(t)$ *converges weakly to the measure which has the density*

$$\pi(x) = Qx^{-2-\left((2(\mu_0+\nu))/\sigma_1^2\right)} \sigma_1^{-2+\left((2(\mu_0+\nu))/\sigma_1^2\right)} e^{-\left(2/\sigma_1^2\right)\left((\Lambda/x)+(\mu_0+\nu)\right)}, \quad x \in (0, +\infty), \tag{30}$$

where Q is a constant such that $\int_0^\infty \pi(x)dx = 1$.

Remark 4. In Theorem 2, we derive $R_0^S = R_0$ when $\alpha(k) \equiv \alpha$ and $\sigma_i(k) \equiv 0$. This conclusion accords with practice.

3. Numerical Examples

In this section, we will test our theory conclusion by Milstein's higher order method in [36].

Example 1. Let the generator of the Markov chain ζ_{ij} be

$$\Gamma = \begin{pmatrix} -\frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{2} \end{pmatrix}, \tag{31}$$

in which ζ_{ij} is a right-continuous Markov chain taking value in $\mathcal{M} = \{1, 2, 3\}$. By solving the linear equation $\pi\Gamma = 0$, we obtain the unique stationary (probability) distribution $\pi = (\pi_1, \pi_2, \pi_3) = (2/7, 3/7, 2/7)$. Choose parameters $\Lambda = 0.232, \gamma = 0.9, \mu_0 = 0.000232, \nu = 0.02, \mu_1 = 0.0000547, \beta = 0.12, \alpha(1) = 0.0013, \alpha(2) = 0.00129, \alpha(3) = 0.00132, \sigma_1(1) = 0.01, \sigma_2(1) = 0.02, \sigma_3(1) = 0.06, \sigma_1(2) = 0.011, \sigma_2(2) = 0.022, \sigma_3(2) = 0.055, \sigma_1(3) = 0.009, \sigma_2(3) = 0.019$, and $\sigma_3(3) = 0.063$. Then, $R_0^S = 1.2226 > 1$. In view of Theorem 2, there is a stationary distribution of model (2), and it is ergodic. Phase portrait of $(S(t), I(t), R(t))$ and histograms of $(S(t), I(t), R(t))$ are plotted in Figure 1.

Example 2. Select parameters $\Lambda = 0.232, \gamma = 0.9, \mu_0 = 0.000232, \nu = 0.02, \mu_1 = 0.0000547, \beta = 0.12, \alpha = 0.04, \sigma_1 = 0.1, \sigma_2 = 0.08$, and $\sigma_3 = 0.05$. By calculation, $R_0 = \alpha\Lambda/((\mu_0$

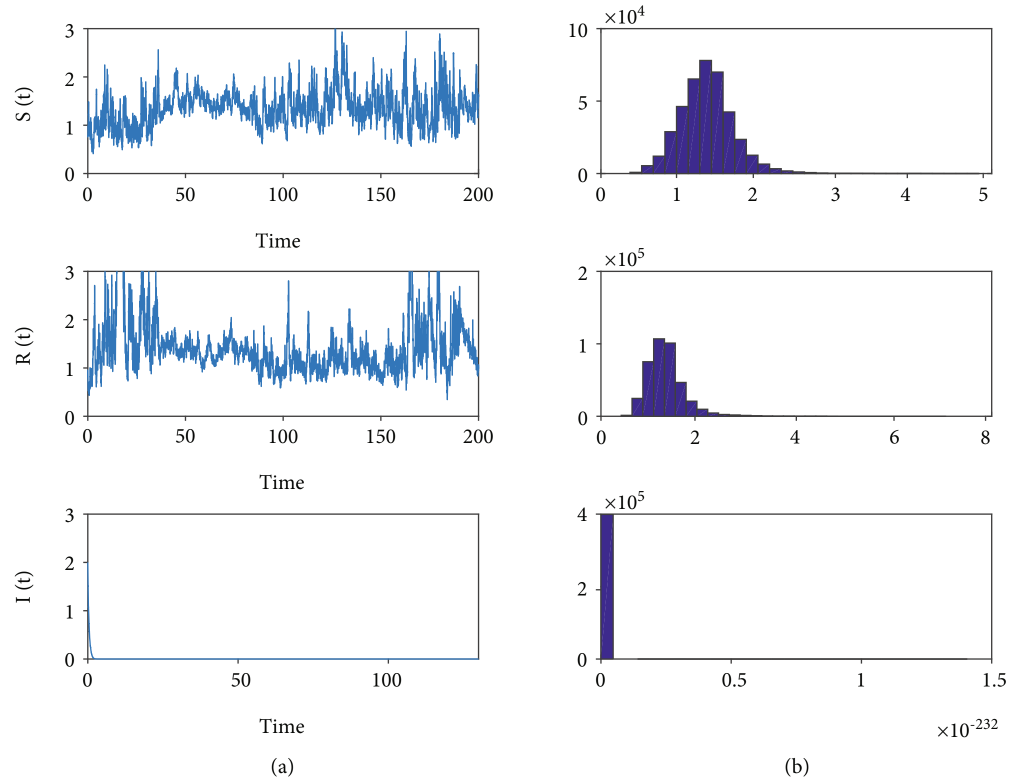

FIGURE 2: The left column reflects the simulation of number variations of $S(t), R(t)$ and $I(t)$ in model (26) with the initial value $S(0) = 0.8, R(0) = 1.1$, and $I(0) = 0.7$ and the noise intensities given in Example 2. The right column reveals the relevant histogram of density functions of the classes $S(t), R(t)$, and $I(t)$. Step size $\Delta t = 0.001$.

$+\nu)(\mu_0 + \mu_1 + \beta)) = 3.87 > 1$, $\int_0^\infty x\pi(x)dx = 1.16$, and $\widehat{R}_0 = 0.377 < 1$. It means that there exists a unique endemic equilibrium of determined model (1), which is globally asymptotically stable. Instead, in view of Theorem 3, we have $\lim_{t\longrightarrow+\infty} I(t) = 0$ a.s. and the distribution of $S(t)$ in model (26) converges weakly to the measure $\pi(x)$ (see Figure 2).

4. Concluding Remarks

The paper successfully investigates extinction and stationary distribution of a stochastic Markov switching hepatitis B epidemic model with saturated incidence rate. Besides the effect of Markovian switching on the deterministic SIRS epidemic models [37–39], pulse vaccination strategy (PVS) has been adopted to control the outbreaks and fastly tackle the spread of disease by wide areas [40]. In order to help future research, we propose the following definition related to SIR model by taking into account Markovian switching, impulse, and infinite delay.

Definition 5. Considering the following impulsive stochastic functional differential equation with Markovian switching(ISFDM),

$$\begin{cases} dY(t) = F_1\left(t, \zeta(t), Y(t), \int_{-\infty}^{0} Y(t+\theta)d\mu_1(\theta)\right)dt + F_2\left(t, \zeta(t), Y(t), \int_{-\infty}^{0} Y(t+\theta)d\mu_2(\theta)\right)dB(t), \\ t \neq t_k, \quad k \in N, \\ Y(t_k^+) - Y(t_k) = H_k Y(t_k), \quad k \in N, \end{cases} \tag{32}$$

where $Y(t+\theta), -\infty<\theta\le 0$, represents C_g-value stochastic process, $C_g = \{\psi \in C((-\infty,0]\,;\mathbb{R}^d): \|\psi\|_{c_g} = \sup_{-\infty<s\le 0} e^{qs}|\psi(s)| < +\infty\}$, $g(s) = e^{-qs}, q > 0, |\psi(s)| = \sqrt{\psi_1^2(s) + \psi_2^2(s) + \cdots + \psi_d^2(s)}$, and $(\psi_1(s), \psi_2(s), \cdots, \psi_d(s)) \in \mathbb{R}^d$. $H_k > -1$, $\zeta(t)$ denotes the regime switching [41, 42]. For $i = 1, 2$, $\mu_i(\theta)$ is a measure on $(-\infty, 0]$, $0 < t_1 < t_2 < \cdots$, $\lim_{k\longrightarrow+\infty} t_k = +\infty$. The initial condition $Y_0 \in C_g$ and $\zeta(0) = 0$, where $Y_0 = \vartheta = \{\vartheta(\theta): -\infty<\theta\le 0\}$

is an $\mathscr{F}_0$-measurable C_g-valued random variable such that $\vartheta \in \mathscr{M}^2((-\infty, 0] ; \mathbb{R}^d)$ which is the family of all $\mathscr{F}_0$-measurable, $\mathbb{R}^d$-valued processes $\psi(t), t \in (-\infty, 0]$ such that $\mathbb{E}\int_{-\infty}^0 |\psi(t)|^2 dt < +\infty$. An $\mathbb{R}^d$-value stochastic process $Y(t)$ defined on $\mathbb{R}$ is called a solution of Equation (32) with initial condition above when $Y(t)$ satisfies the following criterion:

(i) $Y(t)$ is $\mathscr{F}_t$-adapted and continuous on $(0, t_1)$ and $(t_k, t_{k+1}), k \in N$; $F_1(t, \zeta(t), Y(t), \int_{-\infty}^0 Y(t+\theta) d\mu_1(\theta)) \in \mathscr{L}^1(\bar{\mathbb{R}}_+ ; \mathbb{R}^d)$ and $F_2(t, \zeta(t), Y(t), \int_{-\infty}^0 Y(t+\theta) d\mu_2(\theta)) \in \mathscr{L}^2(\bar{\mathbb{R}}_+ ; \mathbb{R}^{d\times m})$. Here, the interpretations of $\mathscr{L}^1(\bar{\mathbb{R}}_+ ; \mathbb{R}^d)$ and $\mathscr{L}^2(\bar{\mathbb{R}}_+ ; \mathbb{R}^{d\times m})$ can be found in [43]. $B(t)$ stands for a m-dimension standard Brownian motion

(ii) For each $t_k, k \in N$, $Y(t_k^+) = \lim_{t\longrightarrow t_k^+} Y(t)$ and $Y(t_k) = Y(t_k^-) = \lim_{t\longrightarrow t_k^-} Y(t)$ a.s.

(iii) $Y(t)$ satisfies the equivalent integral equation of (32) for almost every $t \in [0,\infty) \setminus t_k$ and satisfies the impulsive criterion at each $t = t_k, k \in N$ with probability one

Remark 6. Liu and Wang [44] give a new definition of a solution of an impulsive stochastic differential equation (ISDE). We propose Definition 5, which generalizes the definition of a solution of ISDE to ISFDM, because time memory and Markovian switching are very important in the fields of infectious disease, biological engineering, chemical engineering, etc.

Appendix

Let $(X(t), \xi(t))$ be the diffusion process described by the following equation [(31)]:

$$dX(t) = b(X(t), \xi(t))dt + \sigma(X(t), \xi(t))dB(t), X(0) = x_0, r(0) = \gamma, \tag{A1}$$

where $b(\cdot, \cdot) : \mathbb{R} \times \mathscr{M} \longrightarrow \mathbb{R}^n$, $\sigma(\cdot, \cdot) : \mathbb{R} \times \mathscr{M} \longrightarrow \mathbb{R}^{n\times n}$, and $D(x, k) = \sigma(x, k)\sigma^T(x, k) = (d_{ij}(x, k))$. For each $k \in \mathscr{M}$, let $V(\cdot, k)$ be any twice continuously differentiable function; the operator $\mathscr{L}$ can be defined by

$$\mathscr{L}V(x, k) = \sum_{i=1}^{n} b_i(x, k)\frac{\partial V(x, k)}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n} d(x, k)\frac{\partial^2 V(x, k)}{\partial x_i \partial x_j} + \sum_{l=1}^{N} \vartheta_{kl} V(x, l). \tag{A2}$$

According to theorems in [27], it follows the following lemma which provides a criterion for the ergodic stationary distribution of the solution $(X(t), \xi(t))$ to model (A1).

Lemma 7 ([22]). *If the following conditions are satisfied:*

(A1) $\vartheta_{ij} > 0$ for any $i \neq j$.

(A2) For each $k \in \mathscr{M}$, $D(x, k) = (d_{ij}(x, k))$ is symmetric and satisfies $\lambda|\varpi|^2 \le \langle D(x, k)\varpi, \varpi\rangle \le \lambda^{-1}|\varpi|^2$ for all $\varpi \in \mathbb{R}^n$, with some constant $\lambda \in (0, 1]$ for all $x \in \mathbb{R}^n$.

(A3) There exists a nonempty open set $\mathscr{D}$ with compact closure, satisfying that, for each $k \in \mathscr{M}$, there is a nonnegative function $V(\cdot, k)$: $\mathscr{D}^c \longrightarrow \mathbb{R}$ such that $V(x, k)$ is twice continuously differential and that for some $\alpha > 0$, $\mathscr{L}V(x, k) \le -\alpha$, ($x, k \in \mathscr{D}^c \times \mathscr{M}$), then $(x(t), \xi(t))$ of system (A1) is positive recurrent and ergodic. That is to say, there exists a unique stationary distribution.

Data Availability

No data were used in this study.

Conflicts of Interest

The author declares that there are no competing interests.

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Research Article

Monotonicity and Symmetry of Solutions to Fractional Laplacian in Strips

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In this paper, using the method of moving planes, we study the monotonicity in some directions and symmetry of the Dirichlet problem involving the fractional Laplacian $\begin{cases} (-\Delta)^{\alpha/2}u(x) = f(u(x)), & x \in \Omega, \\ u(x) > 0, & x \in \Omega, \\ u(x) = 0, & x \in \mathbb{R}^n \setminus \Omega, \end{cases}$ in a slab-like domain $\Omega = \mathbb{R}^{n-1} \times (0, h) \subset \mathbb{R}^n$.

1. Introduction

The fractional Laplacian in $\mathbb{R}^n$ is a nonlocal pseudo-differential operator defined by

$$(-\Delta)^{\alpha/2}u(x) = C_{n,\alpha} \lim_{\varepsilon \longrightarrow 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{u(x) - u(z)}{|x - z|^{n+\alpha}} dz, \quad (1)$$

where $C_{n,\alpha}$ is a normalisation constant and α is any real number between 0 and 2. Let

$$L_\alpha = \left\{ u : \mathbb{R}^n \longrightarrow \mathbb{R}^1 \mid \int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+\alpha}} dx < \infty \right\}. \quad (2)$$

Then, it is easy to verify that for $u \in L_\alpha \cap C^{1,1}_{loc}$, the integral on the right-hand side of (1) is well defined. Throughout this paper, we consider the fractional Laplacian in this setting.

Due to applications in physics, chemistry, biology, probability, and finance, differential equations involving the fractional Laplacian $(-\Delta)^{\alpha/2}$ have received growing attention from the mathematical communicity in recent years (see [1–14]). There are many papers devoted to the study of qualitative properties of fractional Laplacian equations in bounded or unbounded domains, but seldom are concerned with slab-like domains. For example, in [15], the authors established the symmetry and monotonicity of positive solutions of the following problem with more general nonlinearity on a bounded domain.

$$\begin{cases} (-\Delta)^{\alpha/2}u(x) = f(u(x)), & x \in B_1(0), \\ u(x) = 0, & x \in \mathbb{R}^n \setminus B_1(0), \end{cases} \quad (3)$$

using a direct method of moving planes. For local elliptic operators, these kinds of approaches were introduced decades ago in the paper [16] and then summarized in the book [17], among which the narrow region principle and the decay at infinity have been applied extensively by many researchers to solve various problems. For more articles concerning the method of moving plans for nonlocal equations, please see [18–20] and the references therein.

However, there are some papers of elliptic second-order boundary value problems concerned with features like monotonicity in some directions and symmetry for positive solutions in slab-like domains. For instance, in [21], using the "sliding method," the authors studied monotonicity in some directions and symmetry of elliptic second-order boundary value problems of the type.

$$\begin{cases} \Delta u + f(u) = 0, & x \in \Omega, \\ u(x) > 0, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{4}$$

in a slab $\Omega = \mathbb{R} \times (0, h) \subset \mathbb{R}^2$. For more articles concerning the "sliding method," please see [22, 23] and the references therein.

Motivated by the above work, in this paper, using the direct method of moving planes, we study the monotonicity in some directions and symmetry of fractional Laplacian boundary value problems of the type.

$$\begin{cases} (-\Delta)^{\alpha/2} u(x) = f(u(x)), & x \in \Omega, \\ u(x) > 0, & x \in \Omega, \\ u(x) = 0, & x \in \mathbb{R}^n \setminus \Omega, \end{cases} \tag{5}$$

in a class of special unbounded domains Ω of $\mathbb{R}^n$: infinite cylinders or more generally, product domains of the form

$$\Omega = \mathbb{R}^{n-j} \times \omega, \tag{6}$$

where ω is a smooth bounded domain in $\mathbb{R}^j$.

We denote the variables in Ω by (x', y), $x' \in \mathbb{R}^{n-j}$, and $y \in \omega \subset \mathbb{R}^j$ with $j \geq 1$. It is not assumed that Ω is bounded. The function f appearing in (5) will always be assumed to be (globally) Lipschitz continuous. We firmly believe that the result introduced here is of great importance, and the ideals and methods can be applied to study a variety of nonlocal problems with more general operators and nonlinearities.

In most of what follows, we consider the case $j = 1$. In this case, the proof of monotonicity and symmetry yields the following statement for $j = 1$.

Theorem 1. *Let*

$$\Sigma = \left\{ \left(x', y\right) \mid x' \in \mathbb{R}^{n-1}, 0 < y < h \right\}. \tag{7}$$

Suppose $u \in L_\alpha \cap C^{1,1}_{loc}(\Sigma)$ satisfies

$$\begin{cases} (-\Delta)^{\alpha/2} u = f(u), & in\, \Sigma, \\ u(x) > 0, & in\, \Sigma, \\ u(x) = 0, & in\, \mathbb{R}^n \setminus \Sigma, \end{cases} \tag{8}$$

with $f(\cdot)$ being Lipschitz continuous. Then, for any positive $l < h/2$,

$$u\left(x', y\right) < u\left(x', 2l - y\right), in \sum_l = \left\{ \left(x', y\right) \mid x' \in \mathbb{R}^{n-1}, 0 < y < l \right\}, \tag{9}$$

and u is symmetric in y about $y = h/2$.

If we further assume that $u \in C^3_{loc}(\bar{\Sigma}_{h/2})$, then

$$\frac{\partial u}{\partial y} > 0, in \sum_{h/2} = \left\{ \left(x', y\right) \mid 0 < y < \frac{h}{2} \right\}. \tag{10}$$

Remark 2. Here, the domain Ω is an infinite cylinder, and it is more general than the usual unbounded domains. For instance, if we let $h \longrightarrow \infty$ in Theorem 1, we can get monotonicity of positive solutions of the Dirichlet problem involving the fractional Laplacian in the half space.

2. Preliminaries and Lemmas

Let T_λ be a hyperplane in $\mathbb{R}^n$. Without loss of generality, we may assume that

$$\begin{aligned} T_\lambda &= \left\{ x = \left(x', y\right) \in \mathbb{R}^{n-1} \times (0, h) \mid y = \lambda \right\}, \\ \sum_\lambda &= \left\{ x = \left(x', y\right) \in \mathbb{R}^{n-1} \times (0, h) \mid 0 < y < \lambda \right\}. \end{aligned} \tag{11}$$

And for $(x', y) \in \Sigma_\lambda$, we let $x^\lambda = (x', 2\lambda - y)$ be the reflection of x about the plane T_λ. Denote $w_\lambda(x) = u(x^\lambda) - u(x)$. For simplicity of notation, in the following, we denote w_λ by w and Σ_λ by Σ.

Lemma 3 (Narrow region principle [15]). *Let Ω be a bounded narrow region in Σ, such that it is contained in $\{x \mid \lambda - l < y < \lambda\}$ with small l. Suppose that $w \in L_\alpha \cap C^{1,1}_{loc}(\Omega)$ and is lower semicontinuous on $\bar{\Omega}$. If $c(x)$ is bounded from below in Ω and*

$$\begin{cases} (-\Delta)^{\alpha/2} w(x) + c(x) w(x) \geq 0 & in\, \Omega, \\ w(x) \geq 0 & in\, \Sigma \setminus \Omega, \\ w\left(x^\lambda\right) = -w(x) & in\, \Sigma, \end{cases} \tag{12}$$

then for sufficiently small l, we have

$$w(x) \geq 0 \, in\, \Omega. \tag{13}$$

Furthermore, if $w = 0$ at some point in Ω, then

$$w(x) = 0 \text{ almost every where in } \mathbb{R}^n. \tag{14}$$

These conclusions hold for unbounded region Ω if we further assume that

$$\varliminf_{|x| \longrightarrow \infty} w(x) \geq 0. \tag{15}$$

Lemma 4 (A Hopf type lemma for antisymmetric functions [24]). *Assume that $w \in C^3_{loc}(\bar{\Sigma})$, $\overline{\lim}_{x \longrightarrow \partial\Sigma} c(x) = o(1/[dist(x, \partial\Sigma)]^2)$, and*

$$\begin{cases} (-\Delta)^{\alpha/2} w(x) + c(x)w(x) = 0 & in\, \Sigma, \\ w(x) \geq 0 & in\, \Sigma, \\ w\left(x^{\lambda}\right) = -w(x) & in\, \Sigma. \end{cases} \quad (16)$$

Then,

$$\frac{\partial w}{\partial \nu} < 0, x \in \partial\Sigma. \quad (17)$$

3. Proof of Theorem 1

Proof of Theorem 1. Now we carry on the method of moving planes on the solution u along y direction. □

Step 1. We show that, for sufficiently small $\lambda > 0$,

$$w_{\lambda}(x) > 0, x \in \sum_{\lambda}, \quad (18)$$

where $w_{\lambda}(x) = u(x^{\lambda}) - u(x)$.

As usual, we can easily verify that w_{λ} satisfies the following linear equation

$$(-\Delta)^{\alpha/2} w_{\lambda} + c_{\lambda}(x) w_{\lambda} = 0, x \in \sum_{\lambda}. \quad (19)$$

Indeed, $u(x^{\lambda})$ satisfies the same equation in (8) as $u(x)$; thus, (19) is obtained by subtracting one from the other and letting

$$c_{\lambda}(x) = \begin{cases} \dfrac{f\left(u\left(x^{\lambda}\right)\right) - f(u(x))}{u(x) - u\left(x^{\lambda}\right)}, & u(x) \neq u\left(x^{\lambda}\right), \\ 0, & u(x) = u\left(x^{\lambda}\right). \end{cases} \quad (20)$$

By the assumption that f is (globally) Lipschitz continuous, with some Lipschitz constant b, we have

$$\|c_{\lambda}\|_{L^{\infty}(\Sigma_{\lambda})} \leq b, \forall \lambda \in \left(0, \frac{h}{2}\right). \quad (21)$$

From the narrow region principle, we can easily know that for sufficiently small $\sigma > 0$,

$$w_{\lambda}(x) \geq 0, \forall x \in \sum_{\lambda}, \lambda \in (0, \sigma). \quad (22)$$

Furthermore, it follows from $w_{\lambda}(x', 0) > 0$ that we have

$$w_{\lambda}(x) > 0, \forall x \in \sum_{\lambda}, \lambda \in (0, \sigma). \quad (23)$$

Step 2. The proof in Step 1 provides a starting point, from which we can now move the plane T_{λ} to the right as long as (18) holds to its limiting position.

Let

$$\lambda_0 = \sup \left\{ \lambda \in \left(0, \frac{h}{2}\right) \mid w_{\mu}(x) > 0, \forall x \in \sum_{\mu}, \mu \leq \lambda \right\}. \quad (24)$$

In this part, we show that

$$\lambda_0 = \frac{h}{2}, \quad w_{\lambda_0}(x) \equiv 0, x \in \sum_{\lambda_0}. \quad (25)$$

Suppose that $\lambda_0 < h/2$, we show that the plane T_{λ} can be moved further. To be more rigorous, we only need to prove that there exists $\varepsilon > 0$, such that for any $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$, we have

$$w_{\lambda_0}(x) > 0, x \in \sum_{\lambda_0}. \quad (26)$$

This is a contradiction with the definition of λ_0. Hence, we have $\lambda_0 = h/2$.

Now we prove (26) by the narrow region principle (Lemma 3). By the definition of λ_0, we can easily have

$$w_{\lambda_0}(x) \geq 0, x \in \sum_{\lambda_0}. \quad (27)$$

In fact, when $\lambda_0 < h/2$, we have

$$(x) > 011 w_{\lambda_0}(x) > 0, x \in \sum_{\lambda_0}. \quad (28)$$

If not, there exists $\widehat{x}$ such that

$$w_{\lambda_0}(\widehat{x}) = \min_{\Sigma_{\lambda_0}} w_{\lambda_0}(x) = 0. \quad (29)$$

Then, we have

$$\begin{aligned} (-\Delta)^{\alpha/2} w_{\lambda_0}(\widehat{x}) &= C_{n,\alpha} PV \int_{\mathbb{R}^n} \frac{-w_{\lambda_0}(z)}{|\widehat{x} - z|^{n+\alpha}} dz \\ &= C_{n,\alpha} PV \int_{\Sigma_{\lambda_0}} \frac{-w_{\lambda_0}(z)}{|\widehat{x} - z|^{n+\alpha}} dz + \int_{\mathbb{R}^n \setminus \Sigma_{\lambda_0}} \frac{-w_{\lambda_0}(z)}{|\widehat{x} - z|^{n+\alpha}} dz \\ &= C_{n,\alpha} PV \int_{\Sigma_{\lambda_0}} \frac{-w_{\lambda_0}(z)}{|\widehat{x} - z|^{n+\alpha}} dz + \int_{\Sigma_{\lambda_0}} \frac{w_{\lambda_0}(z)}{|\widehat{x} - z^{\lambda}|^{n+\alpha}} dz \\ &= C_{n,\alpha} PV \int_{\Sigma_{\lambda_0}} \left(\frac{1}{|\widehat{x} - z^{\lambda}|^{n+\alpha}} dz - \frac{1}{|\widehat{x} - z|^{n+\alpha}} \right) \\ &\quad \cdot w_{\lambda_0}(z) dz < 0. \end{aligned} \quad (30)$$

On the other hand,

$$\begin{aligned}(-\Delta)^{\alpha/2} w_{\lambda_0}(\hat{x}) &= \left(-\Delta^{\alpha/2}\right) u\left(\hat{x}^{\lambda_0}\right) - \left(-\Delta^{\alpha/2}\right) u(\hat{x}) \\ &= f\left(u\left(\hat{x}^{\lambda_0}\right)\right) - f(u(\hat{x})) = 0. \end{aligned} \tag{31}$$

This is a contradiction with (30). Thus, (28) holds.

Then, it follows from (28) that there exists a constant $c_0 > 0$ and $\delta > 0$, such that

$$w_{\lambda_0}(x) \geq c_0, x \in \bar{\Sigma}_{\lambda_0-\delta}. \tag{32}$$

Since w_λ depends on λ continuously, there exists $\varepsilon \in (0, \delta)$, such that for all $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$, we have

$$w_\lambda(x) > 0, x \in \bar{\Sigma}_{\lambda_0-\delta}. \tag{33}$$

Then, from the narrow region principle (Lemma 3), we conclude that for all $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$,

$$w_\lambda(x) > 0, x \in \bar{\Sigma}_\lambda. \tag{34}$$

This is a contradiction with the definition of λ_0. Therefore, we must have $\lambda_0 = h/2$, and

$$w_{\lambda_0}(x) \equiv 0, x \in \sum_{\lambda_0}. \tag{35}$$

Consequently, for all λ: $0 < \lambda < h/2$, we have $w_\lambda > 0$ in Σ_λ. Therefore, (9) holds, and u is symmetric in y about $y = h/2$.

Further, if we assume $u \in C^3_{loc}(\bar{\Sigma}_{h/2})$, we now prove (10) holds. Indeed, w_λ satisfies the following linear equation

$$(-\Delta)^{\alpha/2} w_\lambda + c_\lambda(x) w_\lambda = 0, x \in \sum_\lambda, \tag{36}$$

with $w_\lambda(x', \lambda) = 0$. Also, by the former proof, we know that $w_\lambda > 0$ in Σ_λ. Here, we consider the distance from x to the upper boundary $\{y = \lambda\}$ of Σ_λ, denoted by $\operatorname{dist}(x, \partial\Sigma_\lambda) =: d$. Then, $d(x, \partial\Sigma_\lambda) = \lambda - y$. Thus, by (20) we know that

$$\overline{\lim_{x \longrightarrow \partial\Sigma_\lambda}} c(x) \left[d\left(x, \partial \sum_\lambda \right) \right]^2 = \overline{\lim_{x \longrightarrow \partial\Sigma_\lambda}} c(x) [d(\lambda - x_2)]^2 = 0. \tag{37}$$

Therefore,

$$\overline{\lim_{x \longrightarrow \partial\Sigma_\lambda}} c(x) = o\left(\frac{1}{[d(x, \partial\Sigma_\lambda)]^2} \right). \tag{38}$$

Consequently, the Hopf type lemma for antisymmetric functions (Lemma 4) leads to

$$-2 \frac{\partial u}{\partial y}\left(x', \lambda\right) \equiv \frac{\partial w_\lambda}{\partial y}\left(x', \lambda\right) < 0, \forall x' \in \mathbb{R}^{n-1}, \lambda \in \left(0, \frac{h}{2}\right), \tag{39}$$

which implies that (10) holds. This completes the proof.

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Authors' Contributions

All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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Research Article

Numerical Investigation of Fractional-Order Differential Equations via φ-Haar-Wavelet Method

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In this paper, we propose a novel and efficient numerical technique for solving linear and nonlinear fractional differential equations (FDEs) with the φ-Caputo fractional derivative. Our approach is based on a new operational matrix of integration, namely, the φ-Haar-wavelet operational matrix of fractional integration. In this paper, we derived an explicit formula for the φ-fractional integral of the Haar-wavelet by utilizing the φ-fractional integral operator. We also extended our method to nonlinear φ-FDEs. The nonlinear problems are first linearized by applying the technique of quasilinearization, and then, the proposed method is applied to get a numerical solution of the linearized problems. The current technique is an effective and simple mathematical tool for solving nonlinear φ-FDEs. In the context of error analysis, an exact upper bound of the error for the suggested technique is given, which shows convergence of the proposed method. Finally, some numerical examples that demonstrate the efficiency of our technique are discussed.

1. Introduction

Fractional differential equations are used to describe a wide range of phenomena in natural science, and because of its numerous applications in physical, chemical, and biological sciences, fractional calculus has captivated the scientific community. Several researchers have recently focused their attention on the concept of the fractional derivative. The fractional derivative is introduced in fractional calculus through the fractional integral. Riemann, Liouville, Caputo, Hadamard, Grunwald, and Letinkow are the pioneers in this field, having contributed and published extensively on the subject. The nonlinear fractional Schrodinger equations with the Riesz space and the Caputo time-fractional derivatives are studied using the finite difference/spectral-Galerkin method in [1]. For the Higgs boson equation in the de Sitter spacetime, a finite difference/Galerkin spectral scheme was introduced in [2] which retains the discrete energy dissipation property. For the two-dimensional fractional wave equation with the Weyl space-fractional operators, Ref. [3] proposes a high-order compact difference method with fourth-order precision in space and second order in time. Explicit solutions to differential equations of complex fractional orders with respect to functions and continuous variable coefficients are determined in [4]. Different types of fractional derivatives have appeared in the literature that strengthen and generalize the classical fractional operators defined by the aforementioned authors [5, 6]. Katugampola recently discovered a new type of fractional integral operator which encompasses the Riemann-Liouville and Hadamard operators in a single form [7, 8]. Moreover, several other fractional operators are being introduced to date. Due to a wide range of definitions for fractional-order integrals and derivatives [9–11], the idea of a fractional derivative of one function with respect to another function emerged. This class of fractional operators depends on a kernel function and unifies many definitions of fractional operators. Almeida used the idea of fractional derivatives in the Caputo sense and introduced the φ-Caputo fractional derivative of one function with respect to some other function [12]. The proper choice of a trial function helps in the modeling of physical phenomenon and makes the approach more suitable from the application point of view [13, 14].

Wavelet analysis is a well-known and widely used mathematical method in engineering and other sciences [15, 16]. Wavelets are made up of function expressions that have been extended into a sum of basic functions. A mother wavelet function is translated and compressed to obtain these basic functions. As a result, it inherits properties of locality and smoothness, making it simple to research the properties of integer and locality during the process of expressing functions. Wavelets have sparked a lot of interest in using them to solve classical ordinary and partial differential equations numerically. Researchers have recently succeeded in extending several standard wavelet methods to numerical solutions for fractional differential equations. Numerical integration and numerical solutions of fractional ordinary and fractional partial differential equations are some of the other applications of wavelet methods in applied mathematics. So, for now, wavelets such as the Haar-wavelet, B-spline, Daubechies, and Legendre wavelet are used [17–21]. In Ref. [22], the Genocchi wavelet-like operational matrix was used together with the collocation method to solve nonlinear FDEs. For solving fractional integrodifferential equations, the Jacobi wavelet operational matrix of fractional integration is constructed and utilized in [23]. The Haar-wavelet is a simple form of orthonormal wavelets with compact support and has been used by many researchers. The Haar-wavelet family consisted of rectangular functions. It also includes the lower member of the Daubechies wavelet family, which is suitable for computer implementation. The Haar-wavelets are used to transform a fractional differential equation into an algebraic structure of finite variables [24–27].

For modeling different physical problems, it is difficult to pick the right operator. Therefore, generalized operators of fractional order should be developed for which classical operators are special cases. An effective way to deal with such a variety is to merge these definitions into one by considering fractional derivatives of function f with respect to another function φ. The Riemann-Liouville operators of fractional order are generalized by introducing the fractional-order differentiation and integration of a function by another function [28, 29]. In [12, 30], Almeida defined the φ-Caputo fractional differential and integral operators and discussed its characteristics. The contribution made by Almeida et al. plays a pivotal role in putting together a wide range of fractional operators. Moreover, recent work on the φ-Caputo derivative indicates that φ-Caputo fractional differential-based mathematical models are more flexible and provide felicitous results in many situations. In order to evaluate the growth of the world population, Almeida [12] implemented the φ-Caputo derivative and illustrated that the appropriate selection of a fractional operator determines the model's precision. Using fixed-point theorems, Almeida et al. in [13] investigated the existence and uniqueness of a solution for nonlinear FDEs involving a φ-Caputo derivative. Almeida et al. in [31] introduced the φ-shifted Legendre polynomials for solving fractional oscillation equations containing the φ-Caputo derivative of fractional order. We therefore see the theory of φ-FDEs as a promising field for further study. In this paper, taking motivation by the work cited above, we developed a new numerical method for solving linear and nonlinear boundary value problems in φ-FDEs.

The rest of the paper is organized as follows: We start Section 2 with an overview of the fractional calculus followed by a discussion of the classical Haar-wavelet and an approximation of the functions by the Haar-wavelet. In Section 3, we developed the φ-Haar operational matrix of fractional-order integration of the Haar-wavelet and then utilize it for a numerical solution of the φ-FDEs. In Section 3.1, the error estimate of the developed technique is discussed in depth. Section 4 is devoted to some numerical results and figures that show the precision and effectuality of the developed technique. Finally, a conclusion is given in the last section.

2. Preliminaries

Here, we present some vital definitions of φ-fractional operators and their basic properties which will be used in the subsequent sections of the paper.

Let the function $f : [\alpha, \beta] \longrightarrow \mathbb{R}$ be integrable, ρ a positive real number, n a natural number, and $\varphi \in C^1([\alpha, \beta])$ an increasing function such that $\varphi'(\zeta) \neq 0 \forall \zeta \in [\alpha, \beta]$.

Definition 1. The Caputo fractional derivative of a function f is defined by

$$ {}^{C}D_{\alpha}^{\rho} f(\zeta) = \frac{1}{\Gamma(n-\rho)} \int_{\alpha}^{\zeta} [\zeta - \mathfrak{I}]^{n-\rho-1} \left(\frac{d}{d\mathfrak{I}}\right)^{n} f(\mathfrak{I}) d\mathfrak{I}, \quad (1) $$

where $\zeta \in [\alpha, \beta]$, $\rho \in \mathbb{R}^+$, and $n = \lceil \rho \rceil$.

Definition 2 (see [9, 30, 32]). The φ-Riemann-Liouvile (φ-RL) integration operator of fractional-order ρ of a function $f(\zeta)$ is defined as follows:

$$ \mathcal{J}_{\alpha}^{\rho,\varphi} f(\zeta) = \frac{1}{\Gamma(\rho)} \int_{\alpha}^{\zeta} \varphi'(\mathfrak{I}) [\varphi(\zeta) - \varphi(\mathfrak{I})]^{\rho-1} f(\mathfrak{I}) d\mathfrak{I}. \quad (2) $$

The φ-RL derivative operator of fractional-order ρ of the function $f(\zeta)$ is defined as follows:

$$ \begin{aligned} D_{\alpha}^{\rho,\varphi} f(\zeta) &= \left(\frac{1}{\varphi'(\zeta)} \frac{d}{d\zeta}\right)^{n} \mathcal{J}_{\alpha}^{n-\rho,\varphi} f(\zeta) \\ &= \frac{1}{\Gamma(n-\rho)} \left(\frac{1}{\varphi'(\zeta)} \frac{d}{d\zeta}\right)^{n} \\ &\quad \cdot \int_{\alpha}^{\zeta} \varphi'(\mathfrak{I}) [\varphi(\zeta) - \varphi(\mathfrak{I})]^{n-\rho-1} f(\mathfrak{I}) d\mathfrak{I}, \end{aligned} \quad (3) $$

where $n = \lfloor \rho \rfloor + 1$.

Definition 3 (see [12]). Let ρ be a positive real number, n a natural number, and $f, \varphi \in C^n([\alpha, \beta])$ such that φ is increasing and $\varphi'(\zeta) \neq 0 \forall \zeta \in [\alpha, \beta]$. The φ-Caputo differential operator of fractional-order ρ is defined by

$$ {}^{C}D_{\alpha}^{\rho,\varphi} f(\zeta) = \frac{1}{\Gamma(n-\rho)} \int_{\alpha}^{\zeta} \varphi'(\mathfrak{J})[\varphi(\zeta) - \varphi(\mathfrak{J})]^{n-\rho-1} D^{n,\varphi} f(\mathfrak{J}) d\mathfrak{J}, \tag{4} $$

where $f_{\varphi}^{[n]}(\zeta) = ((1/\varphi'(\zeta))(\mathrm{d}/d\zeta))^n f(\zeta)$, $n = \lfloor \rho \rfloor + 1$ if $\rho \notin \mathbb{N}$, whereas $n = \rho$ if $\rho \in \mathbb{N}$.

Remark 4. For particular choices of $\varphi(\zeta)$, these operators are reduced to the following given operators of the fractional order:

(i) $\varphi(\zeta) = \zeta$ refer to the classical RL and Caputo fractional operators

(ii) $\varphi(\zeta) = \ln(\zeta)$ refer to the classical Hadamard and Caputo-Hadamard fractional operators

2.1. Characteristics of the φ-Fractional Operators. Some fundamental characteristics of the φ-fractional operators are listed below [12, 30].

Let $f(\zeta) = (\varphi(\zeta) - \varphi(\alpha))^{\gamma}$, where $\gamma > n$ and $\rho > 0$. Then,

$$ \begin{aligned} I_{\alpha}^{\rho,\varphi} f(\zeta) &= \frac{\Gamma(\gamma+1)}{\Gamma(\rho+\gamma+1)} (\varphi(\zeta) - \varphi(\alpha))^{\rho+\gamma}, \\ D_{\alpha}^{\rho,\varphi} f(\zeta) &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\rho)} (\varphi(\zeta) - \varphi(\alpha))^{\gamma-\rho}, \\ I_{\alpha}^{\rho,\varphi} D_{\alpha}^{\rho,\varphi} f(\zeta) &= f(\zeta) - \sum_{k=0}^{n-1} \frac{D^{k,\varphi} f(\zeta)}{k!} (\varphi(\zeta) - \varphi(\alpha))^{k}. \end{aligned} \tag{5} $$

Example 5. Let $f(\zeta) = (\zeta - \alpha)^{\gamma}$, with $\gamma > n$ and $\rho > 0$. Then, the Caputo fractional derivative is given by

$$ {}^{C}D_{\alpha}^{\rho} f(\zeta) = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\rho)} (\zeta - \alpha)^{\gamma-\rho}. \tag{6} $$

The Caputo fractional derivatives of $\sin(\zeta)$ and $\cos(\zeta)$ are given by

$$ \begin{aligned} {}^{C}D_{\alpha}^{\rho} \sin(\zeta) &= (\zeta)^{(1-\rho)} E_{2,2-\rho}\left(-\zeta^2\right), \\ {}^{C}D_{\alpha}^{\rho} \cos(\zeta) &= (\zeta)^{-\rho} E_{2,1-\rho}\left(-\zeta^2\right), \end{aligned} \tag{7} $$

where $E_{\alpha,\beta}$ is the two-parameter Mittag-Leffler function defined by

$$ E_{\alpha,\beta} = \sum_{\ell=0}^{\infty} \frac{\zeta^{\ell}}{\Gamma(\alpha\ell + \beta)}. \tag{8} $$

2.2. Existence and Uniqueness of Solution for Nonlinear φ-FDEs. In this section, we provide existence and uniqueness theorems for nonlinear φ-FDEs.

Consider the nonlinear φ-FDE:

$$ \begin{aligned} D_{\alpha}^{\rho,\varphi} y(\zeta) &= f(\zeta, y(\zeta)), \\ t &\in [\alpha, \beta]. \end{aligned} \tag{9} $$

We have the initial conditions, namely, $y(\alpha) = y_{\alpha}$ and $y_{\varphi}^{[\ell]}(\alpha) = y_{\alpha}^{\ell}$, $\ell = 1, \cdots, n-1$, where

(1) $0 < \rho \notin N$ and $n = [\rho] + 1$

(2) y_{α} and y_{α}^{ℓ}, where $\ell = 1, \cdots, n-1$, are fixed reals

(3) $y \in C^{n-1}[\alpha, \beta]$, such that $D_{\alpha}^{\rho,\varphi}$ exists and is continuous in $[\alpha, \beta]$

(4) $f : [\alpha, \beta] \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous

Theorem 6. *A function $y \in C^{n-1}[\alpha, \beta]$ is a solution to problem (9) if and only if y satisfies the following fractional integral equation:*

$$ y(\zeta) = f(\zeta, y(\zeta)) - \sum_{\ell=0}^{n-1} \frac{y_{\alpha}^{\ell}}{\ell!} (\varphi(\zeta) - \varphi(\alpha))^{\ell}. \tag{10} $$

Theorem 7. *Let f be a Lipschitz continuous function with respect to the second variable, that is, ∃ is a positive constant L such that*

$$ |f(\zeta, x_1) - f(\zeta, x_2)| \le L|x_1 - x_2|, \quad \forall \zeta \in [\alpha, \beta], \forall x_1, x_2 \in \mathbb{R}. \tag{11} $$

Then, there is a constant $h \in \mathbb{R}^+$, such that there exists a unique solution to problem (9) on the interval $[\alpha, \alpha + h] \subseteq [\alpha, \beta]$.

Proof of Theorems 6 and 7 can be seen in [13].

2.3. Approximation of Function by the Haar-Wavelet. The ith Haar-wavelet defined on the interval $[\alpha, \beta]$ is given by

$$ h_i(\zeta) = \begin{cases} 1, & \text{for } \zeta \in [\varkappa_1(i), \varkappa_2(i)), \\ -1, & \text{for } \zeta \in [\varkappa_2(i), \varkappa_3(i)), \\ 0, & \text{elsewhere}, \end{cases} \tag{12} $$

where $\varkappa_1(i) = \alpha + (\beta - \alpha)(k/m)$, $\varkappa_2(i) = \alpha + (\beta - \alpha)(2k + 1/m)$, $\varkappa_3(i) = \alpha + (\beta - \alpha)(k + 1/m)$, and $m = 2^j$, where $j = 0, 1, 2, 3, \cdots, J$ and $k = 0, 1, 2, 3, \cdots, m - 1$. Here, j and k are the wavelet's dilation and translation parameters, whereas J is the maximum level of resolution. The relationship $i = m + k + 1$ identifies the wavelet number i. For $i \geq 3$, equation (12) holds true.

The corresponding scaling functions of the Haar-wavelet family for $i = 1$ and $i = 2$ are

$$h_1(\zeta) = \begin{cases} 1, & \text{for } \zeta \in [\alpha, \beta], \\ 0, & \text{elsewhere}, \end{cases}$$
$$h_2(\zeta) = \begin{cases} 1, & \text{if } \zeta \in \left[\alpha, \dfrac{\alpha+\beta}{2}\right), \\ -1, & \text{if } \zeta \in \left[\dfrac{\alpha+\beta}{2}, \beta\right), \\ 0, & \text{elsewhere}. \end{cases} \quad (13)$$

Any function $y(\zeta)$ defined and square integrable over the interval $[0, 1)$ can be expressed in terms of the Haar-wavelet as follows:

$$y(\zeta) = \sum_{i=0}^{\infty} c_i h_i(\zeta), \quad (14)$$

where the coefficients c_i of the Haar-wavelet are defined by

$$c_i = \langle y(\zeta), h_i(\zeta) \rangle = \int_0^1 y(\zeta) h_i(\zeta) d\zeta. \quad (15)$$

In practice, only the first m terms of the series in equation (14) are considered, where m is a power of 2, that is,

$$y(\zeta) \cong y_m(\zeta) = \sum_{i=0}^{m-1} c_i h_i(\zeta), \quad (16)$$

with vector form as

$$y(\zeta) \cong y_m(\zeta) = C_m^T H_m(\zeta), \quad (17)$$

where $C_m^T = [c_0, c_1, c_2, \cdots, c_{m-1}]$ and $H_m(\zeta) = [h_0(\zeta), h_1(\zeta), h_2(\zeta), \cdots, h_{m-1}(\zeta)]^T$.

3. The φ-Haar-Wavelet Operational Matrix

In this section, our endeavor is to construct the φ-Haar-wavelet operational matrix $P^{\rho,\varphi}$ of fractional-order ρ and use it to solve φ-FDEs numerically. The φ-fractional integration of the Haar-wavelet is performed using equation (2). Mathematically, the generalized fractional-order integration of the Haar-wavelet, $H_m = [h_0, h_1, h_2, \cdots, h_{m-1}]$, is given by

$$P_i^{\rho,\varphi}(\zeta) = \frac{1}{\Gamma(\rho)} \int_\alpha^\zeta \varphi'(\mathfrak{I}) [\varphi(\zeta) - \varphi(\mathfrak{I})]^{\rho-1} h_i(\mathfrak{I}) d\mathfrak{I}. \quad (18)$$

Analytically, these generalized φ-fractional integrals can be approximated as follows:

$$P_i^{\rho,\varphi}(\zeta) = \begin{cases} 0, & \text{if } \zeta < \varkappa_1(i), \\ \Phi_1, & \text{if } \zeta \in [\varkappa_1(i), \varkappa_2(i)), \\ \Phi_2, & \text{if } \zeta \in (\varkappa_2(i), \varkappa_3(i)], \\ \Phi_3, & \text{if } \zeta > \varkappa_3(i), \end{cases} \quad (19)$$

where

$$\Phi_1 = \frac{1}{\Gamma(\rho+1)} [\varphi(\zeta) - \varphi(\varkappa_1(i))]^\rho,$$

$$\Phi_2 = \frac{1}{\Gamma(\rho+1)} \left[(\varphi(\zeta) - \varphi(\varkappa_1(i)))^\rho - 2(\varphi(\zeta) - \varphi(\varkappa_2(i)))^\rho \right],$$
$$\Phi_3 = \frac{1}{\Gamma(\rho+1)} \left[(\varphi(\zeta) - \varphi(\varkappa_1(i)))^\rho - 2(\varphi(\zeta) - \varphi(\varkappa_2(i)))^\rho + \mathrm{big}(\varphi(\zeta) - \varphi(\varkappa_3(i)))^\rho \right]. \quad (20)$$

Equation (19) holds for $i > 1$; for $i = 1$, we have

$$P_1^{\rho,\varphi}(\zeta) = \frac{1}{\Gamma(\rho+1)} [\varphi(\zeta) - \varphi(\alpha)]^\rho. \quad (21)$$

The fractional-order φ-Haar-wavelet operational matrix $P^{\rho,\varphi}$ for the function $\varphi(\zeta) = \zeta^2$ and $\rho = 0.75$ is given by

$$P_{m\times m}^{\rho,\varphi} = \begin{bmatrix} 0.4342 & -0.2816 & -0.0998 & -0.1763 & -0.0356 & -0.0623 & -0.0806 & -0.0953 \\ -0.0210 & 0.1735 & -0.0998 & 0.2392 & -0.0356 & -0.0623 & 0.1297 & 0.1153 \\ -0.0739 & 0.0653 & 0.0613 & -0.0204 & -0.0356 & 0.0833 & -0.0173 & -0.0058 \\ 0.0653 & -0.0653 & 0 & 0.1167 & 0 & 0 & -0.1051 & 0.1635 \\ -0.0285 & 0.0022 & 0.0221 & -0.00291 & 0.0211 & -0.0066 & -0.0019 & -0.0010 \\ -0.0094 & 0.0318 & -0.0224 & -0.00901 & 0 & 0.0435 & -0.0088 & -0.0020 \\ 0.0064 & -0.0064 & 0 & 0.06786 & 0 & 0 & 0.0616 & -0.0113 \\ 0.0280 & -0.0280 & 0 & -0.05604 & 0 & 0 & 0 & 0.0779 \end{bmatrix}. \quad (22)$$

Also, the approximate and exact φ-RL fractional integration of $\varphi(\zeta) = \sin(5\zeta)$ for $J = 6$ and various choices of ρ is plotted in Figure 1.

3.1. Convergence Analysis of the φ-Haar-Wavelet Method. In [33], the Caputo-type FDEs were recently analyzed for error. Furthermore, utilizing the Haar wavelet, [34] proves convergence for the solution of the nonlinear Fredholm integral equations. In the present work, the upper limit for the error estimate is calculated using the φ-Caputo fractional differential operator. The φ-Haar-wavelet method for FDEs is shown to be convergent.

Theorem 8. *Let $y^n(\zeta)$ be continuous on interval $[\alpha, \beta]$, and suppose $\exists K > 0$, such that $|y_\varphi^{[n]}(\zeta)| \le K \forall \zeta \in [\alpha, \beta]$, where $\alpha, \beta \in \mathbb{R}^+$, $y_\varphi^{[n]}(\zeta) = ((1/\varphi'(\zeta))(d/d\zeta))^n y(\zeta)$, and $D_\alpha^{\rho,\varphi} y_m(\zeta)$ is the approximation of $D_\alpha^{\rho,\varphi} y(\zeta)$. Then, we have*

$$\|D_\alpha^{\rho,\varphi} y(\zeta) - D_\alpha^{\rho,\varphi} y_m(\zeta)\|_E \le \frac{(\beta-\alpha)K\left(\varphi'(\beta)\right)^{m-\rho}}{\Gamma(m-\rho+1)} \frac{1}{k^{(m-\rho)}} \frac{1}{\left[1-2^{2(\rho-m)}\right]^{(1/2)}}. \quad (23)$$

Proof. $D_\alpha^{\rho,\varphi} y$ can be approximated as follows:

$$D_\alpha^{\rho,\varphi} y(\zeta) = \sum_{i=\alpha}^{\infty} c_i h_i(\zeta), \quad (24)$$

where

$$c_i = \langle D_\alpha^{\rho,\varphi} y(\zeta), h_i(\zeta)\rangle = \int_\alpha^\beta (D_\alpha^{\rho,\varphi} y(\zeta)) h_i(\zeta) d\zeta. \quad (25)$$

Suppose that $D_\alpha^{\rho,\varphi} y_m$ is the following approximation of $D_\alpha^{\rho,\varphi} y$:

$$D_\alpha^{\rho,\varphi} y_m(\zeta) = \sum_{i=0}^{m-1} c_i h_i(\zeta), \quad (26)$$

where $m = 2^{\kappa+1}$, $\kappa = 1, 2, 3, \cdots$. Then,

$$D_\alpha^{\rho,\varphi} y(\zeta) - D_\alpha^{\rho,\varphi} y_m(\zeta) = \sum_{i=m}^{\infty} c_i h_i(\zeta) = \sum_{i=2^{\kappa+1}}^{\infty} c_i h_i(\zeta), \quad (27)$$

which implies that

$$\begin{aligned}\|D_\alpha^{\rho,\varphi} y(\zeta) - D_\alpha^{\rho,\varphi} y_m(\zeta)\|_E^2 &= \int_\alpha^\zeta (D_\alpha^{\rho,\varphi} y(\zeta) - D_\alpha^{\rho,\varphi} y_m(\zeta))^2 d\zeta \\ &= \sum_{i=2^{\kappa+1}}^{\infty} \sum_{i'=2^{\kappa+1}}^{\infty} c_i c_{i'} \int_\alpha^\zeta h_i(\zeta) h_{i'}(\zeta) d\zeta.\end{aligned} \quad (28)$$

By orthogonality of the sequence $\{h_m(\zeta)\}$, we have $\int_\alpha^\beta h_m(\zeta) h_m(\zeta) d\zeta = I_m$, where I_m is the identity matrix of order m. Therefore,

$$\|D_\alpha^{\rho,\varphi} y(\zeta) - D_\alpha^{\rho,\varphi} y_m(\zeta)\|_E^2 = \sum_{i'=2^{\kappa+1}}^{\infty} c_i^2. \quad (29)$$

From equation (25), we have

$$\begin{aligned}c_i &= \int_\alpha^\beta (D_\alpha^{\rho,\varphi} y(\zeta)) h_i(\zeta) d\zeta \\ &= 2^{(j/2)} \left\{ \int_{\alpha+(\beta-\alpha)k2^{-j}}^{\alpha+(\beta-\alpha)(k+(1/2))2^{-j}} D_\alpha^{\rho,\varphi} y(\zeta) d\zeta \right. \\ &\quad \left. - \int_{\alpha+(\beta-\alpha)(k+(1/2))2^{-j}}^{\alpha+(\beta-\alpha)(k+1)2^{-j}} D_\alpha^{\rho,\varphi} y(\zeta) d\zeta \right\}.\end{aligned} \quad (30)$$

By the mean value theorem for integration, we have $\exists \zeta_1, \zeta_2 \in (\alpha, \beta)$, such that

$$\alpha + (\beta-\alpha)k2^{-j} < \zeta_1 < \alpha + (\beta-\alpha)\left(k + \frac{1}{2}\right)2^{-j},$$

$$\alpha + (\beta-\alpha)\left(k + \frac{1}{2}\right)2^{-j} < \zeta_2 < \alpha + (\beta-\alpha)(k+1)2^{-j},$$

$$\begin{aligned}c_i &= 2^{(j/2)}(\beta-\alpha)\left\{\left(\alpha + \left(k+\frac{1}{2}\right)2^{-j} - \left(\alpha + k2^{-j}\right)\right) D_\alpha^{\rho,\varphi} y(\zeta_1) \right. \\ &\quad \left. - \left(\left(\alpha + (k+1)2^{-j}\right) - \left(\alpha + \left(k+\frac{1}{2}\right)2^{-j}\right) D_\alpha^{\rho,\varphi} y(\zeta_2)\right\} \\ &= 2^{(j/2)}(\beta-\alpha)\left\{2^{-j-1}(D_\alpha^{\rho,\varphi} y(\zeta_1) - D_\alpha^{\rho,\varphi} y(\zeta_2))\right\}.\end{aligned} \quad (31)$$

Hence,

$$c_i^2 = 2^{-j-2}(\beta-\alpha)^2 (D_\alpha^{\rho,\varphi} y(\zeta_1) - D_\alpha^{\rho,\varphi} y(\zeta_2))^2. \quad (32)$$

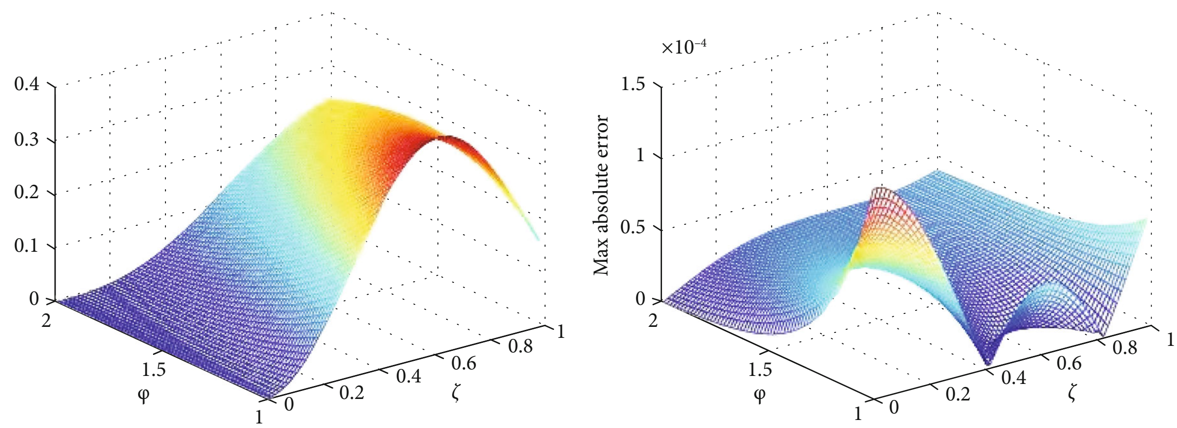

Figure 1: Exact and approximate φ-RL integration of the function $f(\zeta)=\sin(5\zeta)$ for $J=6$ and various choices of ρ and their maximum absolute error.

Employing the definition of the φ-Caputo fractional derivative, the fact that φ is increasing and the condition $|y_\varphi^{[n]}(\zeta)|\le K$, we arrive at

$$\begin{aligned}
&|D_\alpha^{\rho,\varphi}y(\zeta_1)-D_\alpha^{\rho,\varphi}y(\zeta_2)|\\
&=\frac{1}{\Gamma(m-\rho)}\left|\int_\alpha^{\zeta_1}\varphi'(\zeta)(\varphi(\zeta_1)-\varphi(\zeta))^{m-\rho-1}y_\varphi^{[n]}(\zeta)d\zeta\right.\\
&\quad\left.-\int_\alpha^{\zeta_2}\varphi'(\zeta)(\varphi(\zeta_2)-\varphi(\zeta))^{m-\rho-1}y_\varphi^{[n]}(\zeta)d\zeta\right|\\
&\le\frac{1}{\Gamma(m-\rho)}\left|\int_\alpha^{\zeta_1}\varphi'(\zeta)(\varphi(\zeta_1)-\varphi(\zeta))^{m-\rho-1}y_\varphi^{[n]}(\zeta)d\zeta\right.\\
&\quad\left.-\int_\alpha^{\zeta_1}\varphi'(\zeta)(\varphi(\zeta_2)-\varphi(\zeta))^{m-\rho-1}y_\varphi^{[n]}(\zeta)d\zeta\right|\\
&\quad+\left|\int_{\zeta_1}^{\zeta_2}\varphi'(\zeta)(\varphi(\zeta_2)-\varphi(\zeta))^{m-\rho-1}y_\varphi^{[n]}(\zeta)d\zeta\right|\\
&\le\frac{1}{\Gamma(m-\rho)}\left(\int_\alpha^{\zeta_1}\varphi'(\zeta)\left[(\varphi(\zeta_1)-\varphi(\zeta))^{m-\rho-1}\right.\right.\\
&\quad\left.-(\varphi(\zeta_2)-\varphi(\zeta))^{m-\rho-1}\right]\left|y_\varphi^{[n]}(\zeta)\right|d\zeta\\
&\quad\left.+\int_{\zeta_1}^{\zeta_2}\varphi'(\zeta)(\varphi(\zeta_2)-\varphi(\zeta))^{m-\rho-1}\left|y_\varphi^{[n]}(\zeta)\right|d\zeta\right),
\end{aligned}\tag{33}$$

where

$$\begin{aligned}
m-\rho-1>0=\frac{K}{\Gamma(m-\rho+1)}&\left((\varphi(\zeta_1)-\varphi(\alpha))^{m-\rho}-(\varphi(\zeta_2)\right.\\
&\left.-\varphi(\alpha))^{m-\rho}+2(\varphi(\zeta_2)-\varphi(\zeta_1))^{m-\rho}\right).
\end{aligned}\tag{34}$$

Since $\zeta_1>\alpha$, $\zeta_2>\alpha$, and $\zeta_2>\zeta_1$ and $\varphi(\zeta)$ is an increasing function, so

$$(\varphi(\zeta_1)-\varphi(\alpha))^{m-\rho}-(\varphi(\zeta_2)-\varphi(\alpha))^{m-\rho}<0. \tag{35}$$

Table 1: Optimal value of the upper bound of error at different J and $\alpha=0.25$.

J	$\left\|y_{\text{exact}}-y_{\text{approx}}(x)\right\|_E$	Optimality of the upper bound of error
4	3.5102×10^{-4}	0.0714
5	2.8937×10^{-5}	0.0216
6	6.8632×10^{-6}	0.0542
7	3.2381×10^{-6}	0.0139

Table 2: Maximum absolute error for various choices of ρ and J.

J	$\rho=0.50$	$\rho=0.70$	$\rho=0.90$	$\rho=1$
0.5	3.2914×10^{-4}	2.4211×10^{-4}	2.3518×10^{-4}	2.4036×10^{-4}
0.6	1.1220×10^{-4}	6.9659×10^{-5}	5.9464×10^{-5}	6.0560×10^{-5}
0.7	3.8646×10^{-5}	2.0316×10^{-5}	1.5089×10^{-5}	1.5199×10^{-5}
0.8	1.3413×10^{-5}	5.9901×10^{-6}	3.8355×10^{-6}	3.8072×10^{-6}

Therefore,

$$|D_\alpha^{\rho,\varphi}y(\zeta_1)-D_\alpha^{\rho,\varphi}y(\zeta_2)|\le\frac{2K}{\Gamma(m-\rho+1)}(\varphi(\zeta_2)-\varphi(\zeta_1))^{m-\rho}. \tag{36}$$

By mean value theorem, $\exists\varkappa\in[\zeta,\zeta_2]\subseteq[\alpha,\beta]$, such that $\varphi(\zeta_2)-\varphi(\zeta_1)\le(\zeta_2-\zeta_1)\varphi'(\varkappa)$, we get

$$\begin{aligned}
|D_\alpha^{\rho,\varphi}y(\zeta_1)-D_\alpha^{\rho,\varphi}y(\zeta_2)|&\le\frac{2K}{\Gamma(m-\rho+1)}\left((\zeta_2-\zeta_1)\varphi'(\varkappa)\right)^{m-\rho}\\
&\le\frac{2K}{\Gamma(m-\rho+1)2^{j(m-\rho)}}\left(\varphi'(\beta)\right)^{m-\rho},
\end{aligned}\tag{37}$$

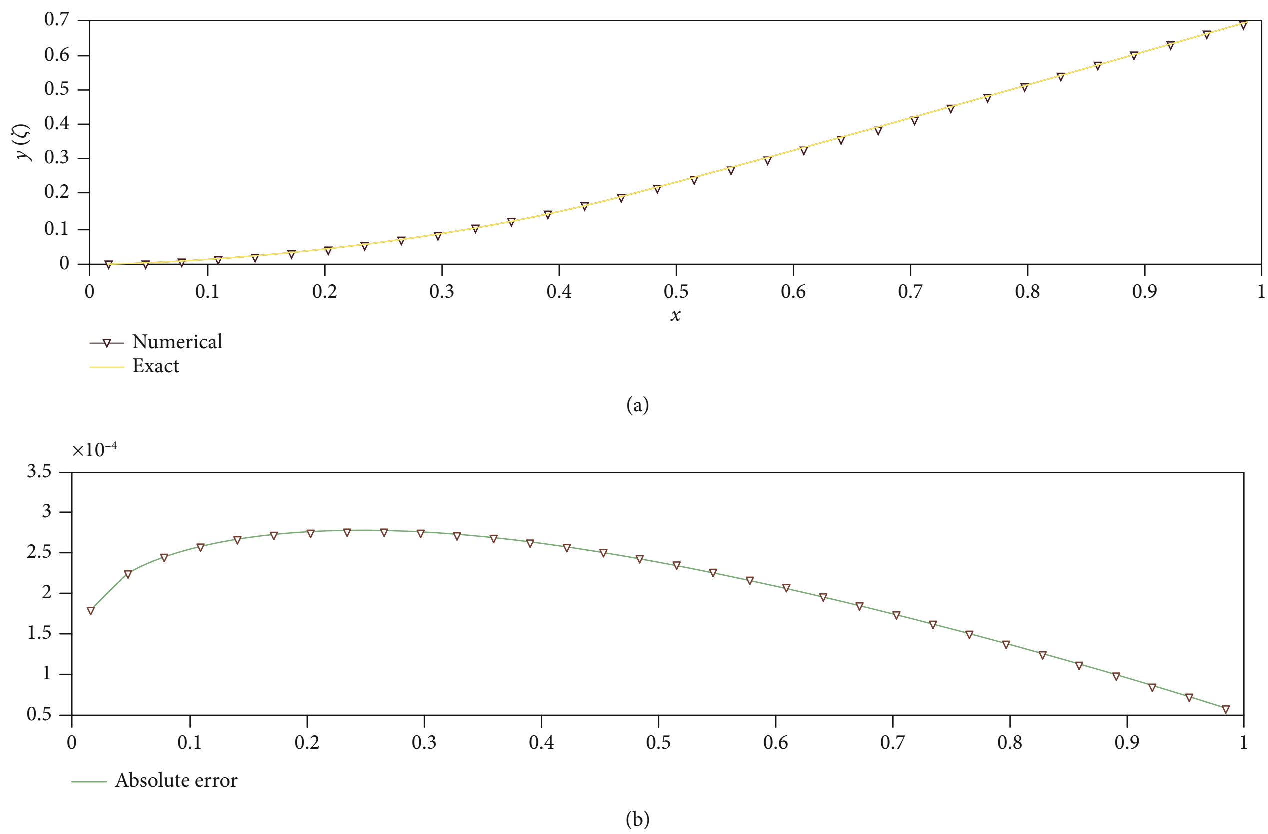

FIGURE 2: For $J = 5$, $\rho = 0.6$, and $\varphi(\zeta) = \sin(\zeta)$: (a) approximate and exact solutions; (b) maximum absolute error.

which gives

$$\left(D_\alpha^{\rho,\varphi}y(\zeta_1) - D_\alpha^{\rho,\varphi}y(\zeta_2)\right)^2 \le \frac{4K^2}{\Gamma^2(m-\rho+1)2^{2j(m-\rho)}}\left(\varphi'(\beta)\right)^{2(m-\rho)}. \tag{38}$$

Putting equation (38) into equation (32), we get

$$c_i^2 \le 2^{-j-2}(\beta-\alpha)^2\frac{4K^2}{\Gamma^2(m-\rho+1)2^{2j(m-\rho)}}\left(\varphi'(\beta)\right)^{2(m-\rho)}. \tag{39}$$

Equations (29) and (39) give

$$\begin{aligned}
&\left\|D_\alpha^{\rho,\varphi}y(\zeta) - D_\alpha^{\rho,\varphi}y_m(\zeta)\right\|_E^2 \\
&\quad = \sum_{i=2^{\kappa+1}}^{\infty} c_i^2 = \sum_{j=\kappa+1}^{\infty}\left(\sum_{i=2^j}^{2^{j+1}-1} c_i^2\right) \\
&\quad \le \sum_{j=\kappa+1}^{\infty}(\beta-\alpha)^2\frac{K^2}{\Gamma^2(m-\rho+1)2^{2j(m-\rho)+j}} \\
&\quad \cdot\left(\varphi'(\beta)\right)^{2(m-\rho)}\left(2^{j+1}-1-2^j+1\right) \\
&\quad = \frac{(\beta-\alpha)^2K^2\left(\varphi'(\beta)\right)^{2(m-\rho)}}{\Gamma^2(m-\rho+1)}\sum_{j=\kappa+1}^{\infty}\frac{1}{2^{2j(m-\rho)}} \\
&\quad = \frac{(\beta-\alpha)^2K^2\left(\varphi'(\beta)\right)^{2(m-\rho)}}{\Gamma^2(m-\rho+1)}\frac{1}{2^{2(\kappa+1)(m-\rho)}}\frac{1}{1-2^{2(\rho-m)}},
\end{aligned} \tag{40}$$

which implies that

$$\begin{aligned}
&\left\|D_\alpha^{\rho,\varphi}y(\zeta) - D_\alpha^{\rho,\varphi}y_m(\zeta)\right\|_E \\
&\quad \le \frac{(\beta-\alpha)K\left(\varphi'(\beta)\right)^{m-\rho}}{\Gamma(m-\rho+1)}\frac{1}{2^{(\kappa+1)(m-\rho)}}\frac{1}{\left[1-2^{2(\rho-m)}\right]^{(1/2)}}.
\end{aligned} \tag{41}$$

Let $k = 2^{\kappa+1}$; (41) can also be written as

$$\begin{aligned}
&\left\|D_\alpha^{\rho,\varphi}y(\zeta) - D_\alpha^{\rho,\varphi}y_m(\zeta)\right\|_E \\
&\quad \le \frac{(\beta-\alpha)K\left(\varphi'(\beta)\right)^{m-\rho}}{\Gamma(m-\rho+1)}\frac{1}{k^{(m-\rho)}}\frac{1}{\left[1-2^{2(\rho-m)}\right]^{(1/2)}}.
\end{aligned} \tag{42}$$

From the value of K, we can get an upper bound for the error.

We first estimate the value of K. Since $y^n(\zeta)$ is continuous and bounded on $[\alpha, \beta]$, so $y_\varphi^{[n]}(\zeta)$ is also continuous and bounded on $[\alpha, \beta]$ and is given by

$$y_\varphi^{[n]}(\zeta) \cong \sum_{i=0}^{m-1} c_i h_i(\zeta) = C_m^T H_m(\zeta), \tag{43}$$

where $C_m = [c_0, c_1, c_2, \cdots, c_{m-1}]^T$ and $H_m(\zeta) = [h_0(\zeta), h_1(\zeta), h_2(\zeta), \cdots, h_{m-1}(\zeta)]^T$.

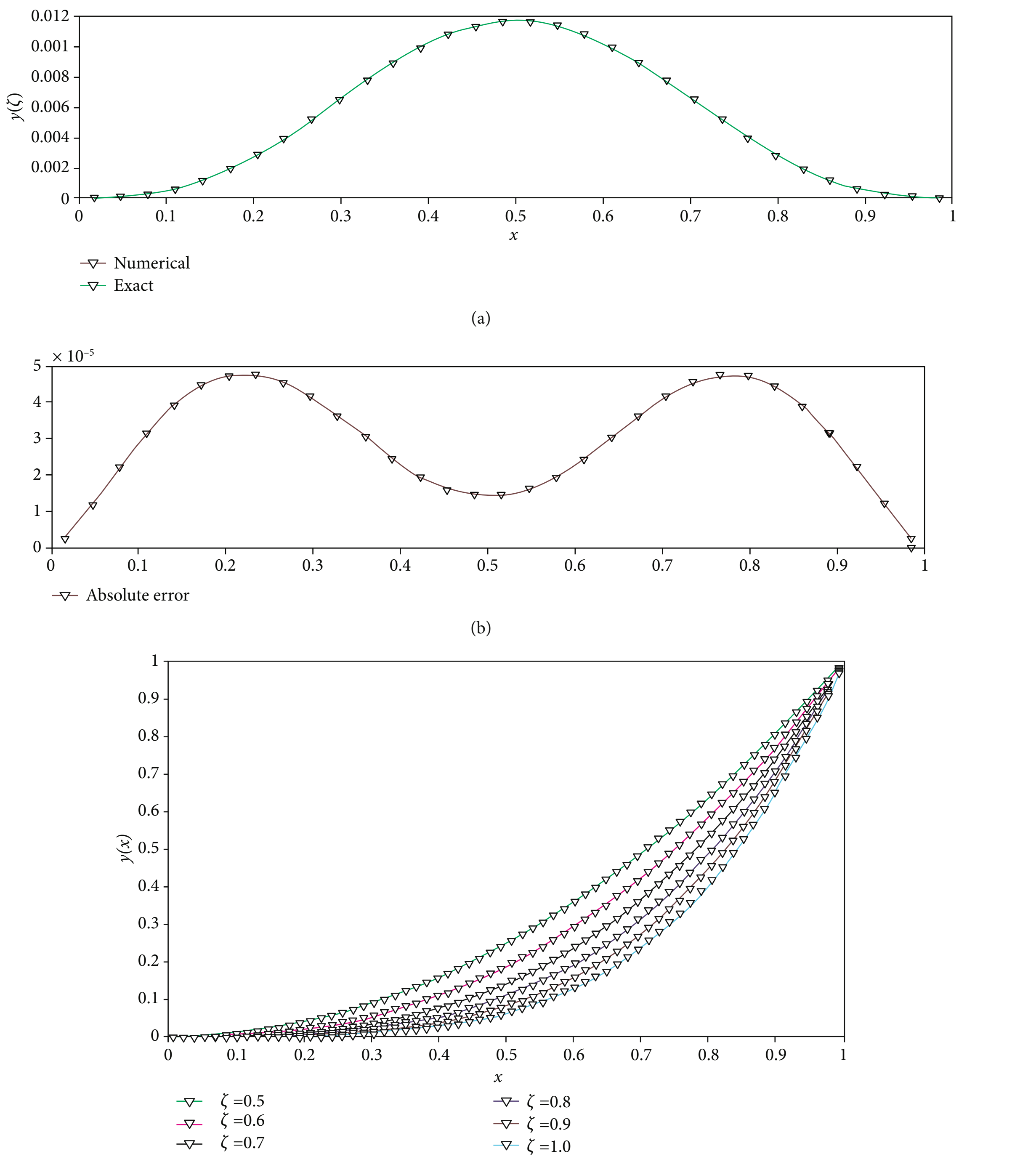

Figure 3: (a) Approximate and exact solutions of equation (57) for $\rho = 0.6$ and $\varphi(\zeta) = \zeta^2 - \zeta$. (b) Maximum absolute error. (c) Approximate solutions of equation (57) for $\varphi(\zeta) = \zeta^2$ and various choices of ρ.

Integrating equation (43), we have

$$\begin{aligned} y_\varphi^{[n-1]}(\zeta) &= \int_\alpha^\zeta y_\varphi^{[n]}(\zeta)d\zeta + y_\varphi^{[n-1]}(\alpha) \\ &= \int_\alpha^\zeta y_\varphi^{[n]}(\zeta)d\zeta \cong C_m^T P_{m\times m}^{1,\varphi} H_m(\zeta). \end{aligned} \tag{44}$$

Similarly,

$$\begin{aligned} y_\varphi^{[n-2]}(\zeta) &= \int_\alpha^\zeta y_\varphi^{[n-1]}(\zeta)d\zeta + y_\varphi^{[n-2]}(\alpha) \\ &= \int_\alpha^\zeta y_\varphi^{[n-1]}(\zeta)d\zeta \cong C_m^T P_{m\times m}^{2,\varphi} H_m(\zeta). \end{aligned} \tag{45}$$

Proceeding in the same way, we get

$$y_\varphi(\zeta) \cong C_m^T P_{m\times m}^{m,\varphi} H_m(\zeta). \tag{46}$$

By defining the points $\zeta_j = ((j - 1/2)/m)$, $j = 0, 1, 2, \cdots, m$. Substituting ζ_j in equation (46), we have

$$y_\varphi(\zeta_j) \cong C_m^T P_{m\times m}^{m,\varphi} H_m(\zeta_j). \tag{47}$$

The matrix form of equation (47) is as follows:

$$y_\varphi \cong C_m^T P_{m\times m}^{m,\varphi} H_m(\zeta_j), \tag{48}$$

where $y_\varphi = [y_\varphi(\zeta_1), y_\varphi(\zeta_2), y_\varphi(\zeta_3), \cdots, y_\varphi(\zeta_m)]$.

By using equation (48), we can investigate C_m^T. From equation (43), we may know the value of $D^{m,\varphi}(\zeta)$ for each $\zeta \in [\alpha, \beta]$.

Suppose $t_i \in [\alpha, \beta]$, for $i = 1, 2, 3, \cdots, l$, $t_i = (t_i = (i - 1/l)/l)$, and we calculate$y_\varphi^{[n]}(t_i)$ for $i = 1, 2, 3, \cdots, l$; then, $\varepsilon + \max | y_\varphi^n(t_i)|$ may be measured as the approximation for K.

Obviously, this approximation would have additional precision if l increases and ε is selected as β. □ □

Theorem 9. *Let $D_\alpha^{\rho,\varphi} y_m$, achieved by applying the φ-Haar-wavelet, be the estimation of $D_\alpha^{\rho,\varphi} y$; then, the actual upper-bound of error is given as follows:*

$$\|y(\zeta) - y_m(\zeta)\|_E \le \frac{KN}{\Gamma(\rho+1)\Gamma(m-\rho+1)} \frac{1}{k^{(m-\rho)}} \frac{1}{\left[1 - 2^{2(\rho-m)}\right]^{(1/2)}}, \tag{49}$$

where $N = \max |(\beta - \alpha)(\varphi(\beta))^{m-\rho}(\varphi(\zeta) - \varphi(0))^\rho|$.

Theorem 9 can be proven simply via Theorem 8. From equation (49), we can understand that $\|y(\zeta) - y_m(\zeta)\|_E$ tends to 0 as m tends to ∞, which shows that the φ-Haar-wavelet technique converges.

Example 10. To demonstrate optimality of the upper bound in equation (49), we consider the following φ-FDE:

$$D_0^{\rho,\varphi} y(\zeta) + y(\zeta) = (\varphi(\zeta))^{2\rho} + \frac{\Gamma(2\rho+1)}{\Gamma(\rho+1)} (\varphi(\zeta))^\rho, \quad 0 < \rho \le 1, \zeta \in [0, 1], \tag{50}$$

with initial condition $y(0) = 0$, having the exact solution $y(\zeta) = (\varphi(\zeta))^2 \rho$.

Table 1 shows the optimal values of the upper bound of error obtained for various options J and $\rho = 0.25$.

4. Numerical Solutions of φ-FDEs

In this section, we provide the solution to some problems in linear and nonlinear φ-FDEs by employing the φ-Haar-wavelet operational matrix technique.

Table 3: Maximum absolute error for various choices of ρ and J.

J	$\rho = 0.60$	$\rho = 0.70$	$\rho = 0.80$	$\rho = 0.90$
0.5	4.7700×10^{-5}	4.4509×10^{-5}	3.833×10^{-5}	3.4673×10^{-5}
0.6	1.4816×10^{-5}	1.2922×10^{-5}	1.0871×10^{-5}	9.0534×10^{-6}
0.7	4.6410×10^{-6}	3.7700×10^{-6}	2.9758×10^{-6}	2.3511×10^{-6}
0.8	1.4674×10^{-6}	1.1071×10^{-6}	8.1696×10^{-7}	6.1074×10^{-7}

4.1. Linear Case. Here, we consider two examples of linear φ-FDEs for the numerical solution by the proposed method.

Example 11. Consider the composite oscillation equation of a fractional order with the φ-Caputo fractional derivative:

$$D_0^{\rho,\varphi} y(\zeta) + y(\zeta) = (\varphi(\zeta))^2 + \frac{2}{\Gamma(3-\rho)} (\varphi(\zeta))^{2-\rho}, \quad 0 < \rho \le 1, \zeta \in [0, 1], \tag{51}$$

with the initial condition $y(0) = 0$. The exact solution of equation (51) is given by $y(\zeta) = (\varphi(\zeta))^2$. For numerical solutions, we approximate $D_0^{\rho,\varphi} y(\zeta)$ as

$$D_0^{\rho,\varphi} y(\zeta) = C_m^T H_m(\zeta). \tag{52}$$

Integrating equation (52) with the φ-Caputo integral operator, we have

$$y(\zeta) = \mathcal{I}_0^{\rho,\varphi} C_m^T H_m(\zeta) + c_1 = C_m^T P_{m\times m}^{\rho,\varphi} H_m(\zeta) + c_1. \tag{53}$$

Substituting the initial conditions in equation (51), we get

$$y(\zeta) = C_m^T P_{m\times m}^{\rho,\varphi} H_m(\zeta) + (\varphi(0))^2. \tag{54}$$

Substituting equations (52) and (54) for equation (51), we have

$$C_m^T (H_m(\zeta) + P_{m\times m}^{\rho,\varphi} H_m(\zeta)) = f(\zeta), \tag{55}$$

where $f(\zeta) = (\varphi(\zeta))^2 + (2/\Gamma(3-\rho))(\varphi(\zeta))^{2-\rho} - (\varphi(0))^2$.

Equation (55) can be expressed in the matrix form as follows:

$$C_m^T (H_m(\zeta) + A P_{m\times m}^{\rho,\varphi} H_m(\zeta)) = F, \tag{56}$$

where $F = f(\zeta)$. The value of C_m^T can be obtained from equation (56). By using C_m^T in equation (54), we can obtain the numerical solutions. Table 2 represents approximate solutions obtained for various choices of ρ and J. The exact and numerical solutions of equation (51) and the maximum absolute error are plotted in Figures 2(a) and 2(b), respectively, for $J = 5$, $\rho = 0.6$, and $\varphi(\zeta) = \sin(\zeta)$.

Table 4: Maximum absolute error for $\varphi(\zeta)=\zeta^3$ and various choices of J and ρ.

J	$\rho=0.60$	$\rho=0.70$	$\rho=0.80$	$\rho=0.90$	$\rho=1$
0.5	2.9805×10^{-4}	2.6189×10^{-4}	9.0556×10^{-5}	8.4937×10^{-5}	4.0843×10^{-5}
0.6	9.3858×10^{-5}	8.2375×10^{-5}	7.5629×10^{-5}	4.5815×10^{-5}	3.0209×10^{-5}
0.7	8.0570×10^{-5}	7.5617×10^{-5}	6.8581×10^{-5}	4.2393×10^{-6}	2.5527×10^{-6}
0.8	4.6741×10^{-5}	3.3469×10^{-5}	5.6175×10^{-6}	3.0351×10^{-6}	6.3818×10^{-7}
0.9	3.2109×10^{-5}	2.3126×10^{-5}	7.8318×10^{-6}	2.6165×10^{-7}	1.5954×10^{-7}

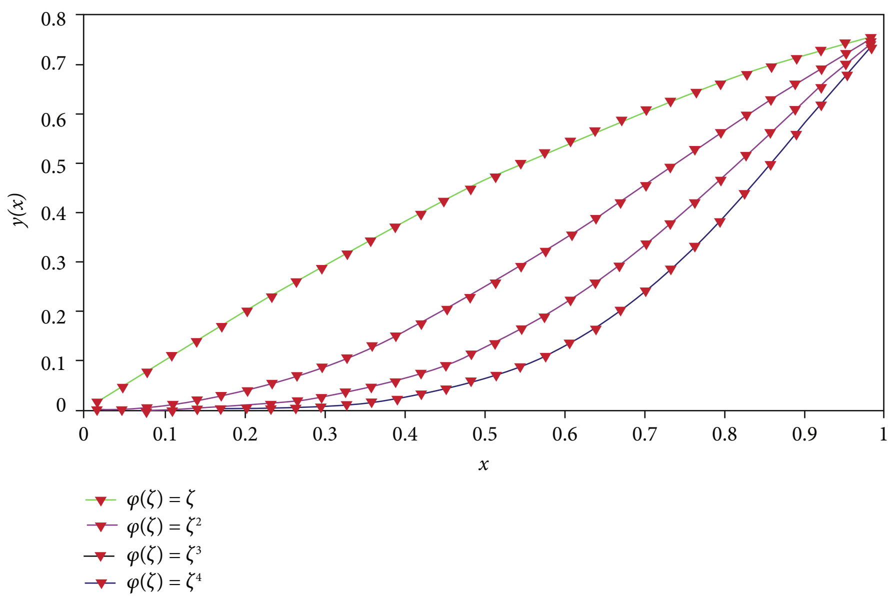

Figure 4: Approximate solutions for $\rho=1$, $J=5$, and different choices of $\varphi(\zeta)$.

Example 12. In this example, consider the FDE involving the φ-Caputo derivative:

$$D_0^{\rho,\varphi}y(\zeta)+y(\zeta)=(\varphi(\zeta))^4-\frac{1}{2}(\varphi(\zeta))^3-\frac{3}{\Gamma(4-\rho)}(\varphi(\zeta))^{3-\rho}+\frac{24}{\Gamma(5-\rho)}(\varphi(\zeta))^{4-\rho}, \quad (57)$$

where $0<\rho\le1$, $\zeta\in[0,1]$, and the initial condition

$$y(0)=0. \quad (58)$$

The exact solution of the problem (57) is given as follows: $y(\zeta)=(\varphi(\zeta))^4-(1/2)(\varphi(\zeta))^3$. To get numerical solutions, the φ-Haar-wavelet method is employed as follows. Let

$$D_0^{\rho,\varphi}y(\zeta)=C_m^T H_m(\zeta). \quad (59)$$

Table 5: Maximum absolute error for various choices of ρ and J.

ρ	$J=0.5$	$J=0.6$	$J=0.7$	$J=0.8$
0.60	3.9081×10^{-4}	2.1491×10^{-4}	1.6723×10^{-4}	3.0569×10^{-5}
0.70	3.6472×10^{-4}	1.7153×10^{-4}	3.2854×10^{-5}	3.7074×10^{-5}
0.80	3.2019×10^{-4}	1.3570×10^{-4}	2.8061×10^{-5}	5.8283×10^{-6}
0.90	2.6195×10^{-4}	9.0689×10^{-5}	1.8584×10^{-5}	4.1523×10^{-6}
0.1	2.2700×10^{-4}	5.8851×10^{-5}	1.4983×10^{-5}	3.7800×10^{-6}

Integrating equation (59) with the φ-Caputo integral operator, we have

$$y(\zeta)=\mathcal{I}_0^{\rho,\varphi}C_m^T H_m(\zeta)+c_1. \quad (60)$$

Substituting initial conditions in equation (60), we get $c_1=y_0$. Equation (60) becomes

$$y(\zeta)=C_m^T P^{\rho,\varphi}H_m(\zeta)+y_0. \quad (61)$$

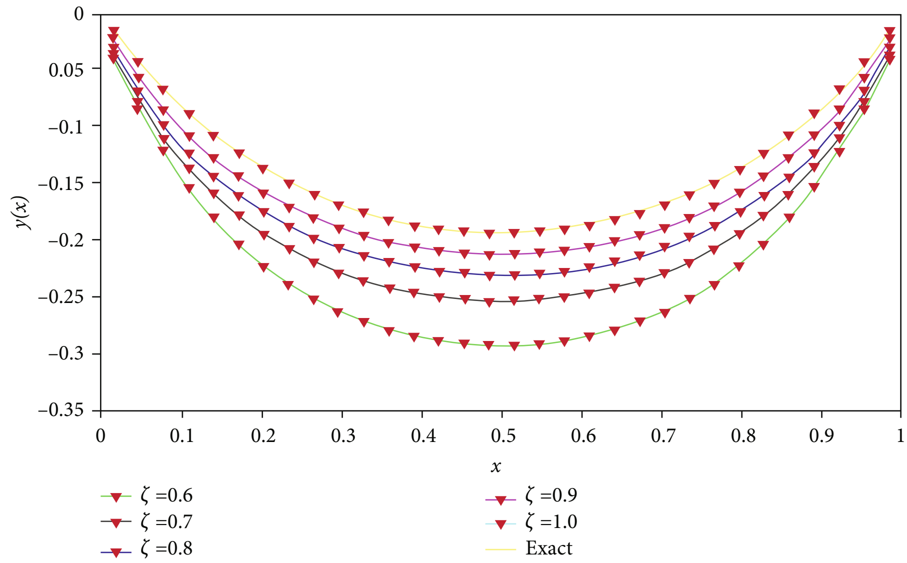

FIGURE 5: Exact solution for $\rho = 1$ and numerical solutions for various choices of ρ.

Substituting equations (59) and (60) for equation (57), we get

$$C_m^T\left(H_m(\zeta) + a(\zeta)P^{\rho-\kappa,\varphi}H_m(\zeta) + b(\zeta)P^{\rho,\varphi}H_m(\zeta)\right) = F(\zeta), \quad (62)$$

where $F(\zeta) = (\varphi(\zeta))^4 - (1/2)(\varphi(\zeta))^3 - (3/\Gamma(4-\rho))(\varphi(\zeta))^{3-\rho} + (24/\Gamma(5-\rho))(\varphi(\zeta))^{4-\rho}$. Approximate solutions are obtained by solving equations (61) and (62). The exact solution, approximate solutions, and the maximum absolute error are plotted in Figure 3 for $J = 6$ and $\rho = 0.6$. Also, the maximum absolute errors obtained for various choices of ρ and J are given in Table 3. We noticed that the maximum absolute error decreases with an increase in J.

4.2. Nonlinear Case

Example 13. Consider the fractional-order Riccati differential equation with the φ-Caputo fractional derivative:

$$D_0^{\rho,\varphi}y(\zeta) = -y^2(\zeta) + 1, 0 < \rho \le 1, \zeta \in [0, 1], \quad (63)$$

subject to the initial condition $y(0) = 0$. For $\rho = 1$, the exact solution of equation (63) is given by $y(\zeta) = (e^{\varphi(2\zeta)} - 1/e^{\varphi(2\zeta)} + 1)$. For numerical solutions, we first utilize the quasilinearization techniques to make the nonlinear terms of equation (63) linear and then solve the linearized problem with the φ-Haar-wavelet method. The linearized form of (63) is

$$D_0^{\rho,\varphi}y_{r+1}(\zeta) + 2y_r(\zeta)y_{r+1}(\zeta) = y_r^2(\zeta) + 1, \quad \zeta > 0,\ 0 < \rho \le 1, \quad (64)$$

with the initial condition $y_{r+1}(0) = 0$.

TABLE 6: Comparison of results obtained in [31] and by our method for $\varphi(\zeta) = (\zeta^2/2) + (\zeta/2)$.

ζ	y-Exact	y-Approximate by [31]	Error by [31]	Error by our method
0.0	0.0000	0.00060	60×10^{-4}	60×10^{-4}
0.1	0.0031	0.0037	70×10^{-4}	60×10^{-4}
0.2	0.0144	0.0151	80×10^{-4}	70×10^{-4}
0.3	0.0380	0.0388	90×10^{-4}	80×10^{-4}
0.4	0.0783	0.0793	10×10^{-4}	10×10^{-4}
0.5	0.1416	0.1427	11×10^{-4}	12×10^{-3}
0.6	0.2313	0.2324	11×10^{-3}	11×10^{-3}
0.7	0.3542	0.3553	12×10^{-4}	11×10^{-4}
0.8	0.5274	0.5284	10×10^{-3}	10×10^{-3}
0.9	0.7411	0.7421	10×10^{-4}	10×10^{-4}
1.0	1.0000	1.0007	70×10^{-4}	70×10^{-4}

Now, we apply the φ-Haar-wavelet method to equation (64). Let

$$D_0^{\rho,\varphi}y_{r+1}(\zeta) = C_m^T H_m(\zeta). \quad (65)$$

Operating the φ-Caputo integral on equation (65), we get

$$y_{r+1}(\zeta) = \mathcal{I}^{\rho,\varphi}C_m^T H_m(\zeta) + c_1 = C_m^T P_{m\times m}^{\rho,\varphi}H_m(\zeta) + c_1. \quad (66)$$

Putting the initial conditions in equation (66) gives

$$y_{r+1}(\zeta) = C_m^T P_{m\times m}^{\rho,\varphi}H_m(\zeta). \quad (67)$$

Substituting equations (65) and (66) for equation (64), we have

$$C_m^T(H_m(\zeta) + 2y_r(\zeta)P_{m\times m}^{\rho,\varphi}H_m(\zeta)) = 1 + y_r^2(\zeta). \quad (68)$$

The matrix form of equation (68) is given by

$$C_m^T(H_m(\zeta) + 2y_rP_{m\times m}^{\rho,\varphi}H_m(\zeta)) = F, \quad (69)$$

where $F = 1 + y_r^2$. By solving the algebraic system given by equation (69) for C_m^T and substituting this value into equation (67), we will have the required numerical solution. In Table 4, the maximum absolute error is given for $\varphi(\zeta) = \zeta^3$. Also, approximate solutions for different choices of the function φ are plotted in Figure 4.

Example 14. Finally, consider the Riccati differential equation of fractional order having the φ-Caputo fractional derivative:

$$D_0^{\rho,\varphi}y(\zeta) = 2y(\zeta) - y(\zeta)^2 + 1, \quad (70)$$

where $0 < \rho \le 1$ and $\zeta \in [0, 1]$.

Then, we subject this to the initial condition:

$$y(0) = 0. \quad (71)$$

When $\rho = 1$, $y(\zeta) = 1 + \sqrt{2}\tanh\left(\sqrt{2}\varphi(\zeta) + (1/2)\log\left((\sqrt{2} - 1)/(\sqrt{2} + 1)\right)\right)$ is the actual solution of problem (70). For numerical solutions, we first utilize the quasilinearization technique to linearize the nonlinear terms in equation (70) and then solve the linearized FDE by the φ-Haar-wavelet method.

Equation (70) in the linearized form is given by

$$D_0^{\rho,\varphi}y_{r+1} - (2 - 2y_r(\zeta))y_{r+1}(\zeta) = y_r^2(\zeta) + 1, \quad \zeta > 0 \text{ and } 0 < \rho \le 1, \quad (72)$$

with the initial condition $y_{r+1}(0) = 0$.

Consider

$$D_0^{\rho,\varphi}y_{r+1} = C_m^TH_m(\zeta). \quad (73)$$

Taking the φ-Caputo integral of (73),

$$y_{r+1} = \mathcal{I}_0^{\rho,\varphi}C_m^TH_m(\zeta) + c_1. \quad (74)$$

Substituting the initial condition in equation (74), we have $c_1 = 0$.

Using $c_1 = 0$ in equation (74), we get

$$y_{r+1} = C_m^TP_{m\times m}^{\rho,\varphi}H_m. \quad (75)$$

Substituting equations (73) and (75) for equation (70), we get

$$C_m^T(H_m(\zeta) - (2 - 2y_r(\zeta)))P_{m\times m}^{\rho,\varphi}H_m(\zeta) = F(\zeta). \quad (76)$$

Required approximate solutions can be obtained by using the value of C_m^T from equation (76) in equation (75). Table 5 shows that the maximum absolute error decreases by increasing the values of J. Also, the approximate solutions are displayed in Figure 5 for various values of ρ.

Example 15. For comparison with another method, we consider the following problem:

$$D_0^{(3/2),\varphi}y(\zeta) + \frac{2}{\Gamma(3/2)}y(\zeta) = \frac{2}{\Gamma(3/2)}\left(1 + (\varphi(\zeta))^{(3/2)}\right), \quad \zeta \in [0, 1], \quad (77)$$

with the initial condition $y(0) = 0$. The exact solution of equation (77) is given by $y(\zeta) = (\varphi(\zeta))^2$. This problem is studied in [31] by using the operational matrix of the φ-shifted Legendre polynomials.

For $\varphi(\zeta) = (\zeta^2/2) + (\zeta/2)$, a comparison of the results obtained in [31] and by the proposed method is given in Table 6.

5. Conclusion

In this article, the φ-FDEs are solved numerically by introducing the φ-Haar-wavelet operational matrix of integration of fractional order. This operational matrix has been used to solve both linear and nonlinear problems with success. In comparison to the other methods, this approach is simple and more convergent. The developed method is used to solve a number of linear and nonlinear problems, demonstrating its efficiency and accuracy. Furthermore, the method's error analysis is thoroughly examined. As a future work, the proposed method may be applied to different wavelets as well as other operators.

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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Research Article

Existence of Positive Solutions for Second-Order Third-Point Semipositive BVP

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In this paper, we study the existence of positive solutions for the following nonlinear second-order third-point semi-positive BVP. We derive an explicit interval of positive parameters, which for any l, μ in this interval, the existence of positive solutions to the boundary value problem is guaranteed under the condition that $a(t,x), b(t,x)$ are all superlinear (sublinear), or one is superlinear, the other is sublinear.

1. Introduction

In the applied mathematical field, three-point BVP can describe many phenomena. Moshinsky [1] introduced the vibrations of a guy wire with a uniform cross-section and composed of N parts of different densities using a multipoint BVP. Timoshenko [2] also revealed that the theory of elastic stability can be used by the method of a three-point BVP. Il'in and Moviseev [3] were the first to study this aspect. Since then, more general nonlinear BVP have been studied by several authors [4–25].

In their paper [7], Ma and Wang obtained the existence of positive solutions for a three-point BVP by Krasnoselskii's fixed theorem:

$$\begin{cases} u''(t) + a(t)u'(t) + b(t)u(t) + h(t)f(u) = 0, & 0 \le t \le 1, \\ u(0) = 0, & u(1) = \alpha u(\eta), \end{cases} \tag{1}$$

where α is a positive constant, $0 < \eta < 1$, $a(t) \in C([0,1], \mathbf{R}^+)$, $b(t) \in C([0,1], \mathbf{R}^-)$, $f \in C(\mathbf{R}^+, \mathbf{R}^+)$,$h \in C([0,1], \mathbf{R}^+)$ and there exists $x_0 \in (0,+\infty)$ such that $h(x_0) > 0$.

In our paper, we study the existence of positive solutions of second-order third-point semipositive BVP:

$$\begin{cases} (Lx)(t) + \lambda a(t,x) + \mu b(t,x), & 0 \le t \le 1, \\ x(0) = 0, & x(1) = \alpha x(\xi), \end{cases} \tag{2}$$

where $(Lu)(t) = u''(t) + f(t)u'(t) + g(t)u(t)$, λ, μ are positive parameters, $0 < \xi < 1$, $f(t) \in C[0,1]$, and $g(t) \in C([0,1], (-\infty,0))$. And our paper also allows that $a(t,x), b(t,x)$ are both semipositive and lower unbounded.

Our main tool is the following fixed point index theory.

Theorem 1 [4]. *We suppose that $K \subset E$ is a cone in E, in which E is a real Banach space, the open bounded set Ω_1, Ω_2 is in E, $\theta \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$, and $T : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \longrightarrow K$. Suppose operator T can be completely continuous and satisfies one of the following conditions:*

(i) $\|Tx\| \le \|x\|, \ \forall x \in K \cap \partial\Omega_1;\ \|Tx\| \ge \|x\|, \ \forall x \in K \cap \partial\Omega_2$

(ii) $\|Tx\| \ge \|x\|, \ \forall x \in K \cap \partial\Omega_1;\ \|Tx\| \le \|x\|, \ \forall x \in K \cap \partial\Omega_2$

Then, operator T has at least one fixed point x^* *in* $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

Theorem 2 [4]. *We suppose that* $P \subset E$ *is a cone in E, in which E is a real Banach space, the open bounded set* $\Omega_1, \Omega_2, \Omega_3$ *is in E,* $\theta \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$, $\bar{\Omega}_2 \subset \Omega_3$, *and* $T : P \cap \Omega_3 \longrightarrow P$. *Suppose operator A is completely continuous and satisfies the following conditions:*

$$\begin{aligned} &\|Tx\| \le \|x\|, \quad \forall x \in P \cap \partial\Omega_1, \\ &\|Tx\| \ge \|x\|, \quad Ax \ne x, \forall x \in P \cap \partial\Omega_2, \\ &\|Tx\| \le \|x\|, \quad \forall x \in P \cap \partial\Omega_3. \end{aligned} \tag{3}$$

Then, operator T has at least two fixed points x^* and x^{**} in $P \cap (\bar{\Omega}_3/\Omega_1)$, and $x^* \in P \cap (\Omega_2/\Omega_1)$ and $x^{**} \in P \cap (\bar{\Omega}_3/\bar{\Omega}_2)$.

2. Preliminaries and Lemmas

We set a Banach space $E = C([0, 1],(-\infty,+\infty))$ with norm $\|x\| = \max_{t\in I}|x(t)|$. We know of the following lemmas from Ref. [6].

Lemma 3. *Setting*$\xi_1(t)$*as the positive solution of the equation, we have:*

$$\begin{cases} (L\xi_1)(t) = 0, & 0 \le t \le 1, \\ \xi_1(0) = 0, & \xi_1(1) = 1. \end{cases} \tag{4}$$

Then, $\xi_1(t) \in [0, 1]$ is strictly increasing on $[0, 1]$, and $\xi_1{}'(0) > 0$.

Lemma 4. *Setting*$\xi_2(t)$*as the positive solution of the equation, we have:*

$$\begin{cases} (L\xi_2)(t) = 0, & 0 \le t \le 1, \\ \xi_2(0) = 1, & \xi_2(1) = 0. \end{cases} \tag{5}$$

Then, $\xi_2(t) \in [0, 1]$ *is strictly decreasing on* $[0, 1]$.

From Lemma 3 and Lemma 4, we know that $0 < \xi_1(t) < 1$, $0 < \xi_2(t) < 1$. In the rest of our paper, the following condition is used:

(C1) $0 < \alpha\xi_1(\eta) < 1$, where$\xi_1(t)$is given by Lemma 3

Throughout this paper, we shall use the following notation:

$$G(t, s) = \frac{1}{\zeta}\begin{cases} \xi_1(t)\xi_2(s), & 0 \le t \le s \le 1, \\ \xi_1(s)\xi_2(t), & 0 \le s \le t \le 1, \end{cases} \tag{6}$$

where $\zeta = \xi_1'(0)\xi_2(0)$.

Obviously, from Ref. [6], we can be assured that when (C1) holds, the BVP

$$\begin{cases} (Lx)(t) + y(t) = 0, & 0 \le t \le 1, \\ x(0) = 0, & x(1) = \alpha u(\xi), \end{cases} \tag{7}$$

is equivalent to the following integral equation:

$$x(t) = \int_0^1 G(t, s)e(s)y(s)ds + \frac{\alpha\xi_1(t)}{1 - \alpha\xi_1(\xi)}\int_0^1 G(\xi, s)e(s)y(s)ds, \tag{8}$$

where $e(t) = \exp\left(\int_0^t f(s)ds\right)$.

Set $z(t) = \min\left((\xi_1(t)/\|\xi_1\|), (\xi_2(t)/\|\xi_2\|)\right)$. From (6), for $t \in [0, 1]$, we know that

$$z(t)G(s, s) \le G(t, s) \le G(s, s). \tag{9}$$

We present some other lemmas that are important to our main results.

Lemma 5 [7]. *Assume that for any* $y \in C([0, 1], (0,+\infty))$, $x(t)$ *is the solution of the following BVP:*

$$\begin{cases} (Lx)(t) + y(t) = 0, & 0 < t < 1, \\ x(0) = 0, & x(1) = \alpha x(\xi). \end{cases} \tag{10}$$

Then, we have

$$x(t) \ge z(t)\|x\|, t \in [0, 1]. \tag{11}$$

Lemma 6. *Assume that* $\bar{w}$ *is a solution of the following BVP:*

$$\begin{cases} (Lx)(t) = -B(t), & 0 < t < 1, \\ x(0) = 0, & x(1) = \alpha x(\xi), \end{cases} \tag{12}$$

where $B \in C(0, 1), M > 0$. *Then, there exists constant* $M > 0$ *and satisfies*

$$\bar{w}(t) \le M\|B\|z(t), \quad t \in [0, 1]. \tag{13}$$

Proof. For $t \in [0, 1]$, we can have

$$\bar{w}(t) = \int_0^1 G(t, s)e(s)B(s)ds + \frac{\alpha\xi_1(t)}{1 - \alpha\xi_1(\xi)}\int_0^1 G(\xi, s)e(s)B(s)ds. \tag{14}$$

Obviously, for $t \in [0,1]$, we have

$$\begin{aligned}
&\int_0^1 G(t,s)e(s)B(s)ds \\
&= \frac{1}{\zeta}\left[\int_0^t \xi_1(s)\xi_2(t)e(s)B(s)ds + \int_t^1 \xi_1(t)\xi_2(s)e(s)M(s)ds\right] \\
&\le \frac{p(1)}{\zeta}\left[\xi_1(t)\xi_2(t)\int_0^t B(s)ds + \xi_1(t)\xi_2(t)\int_t^1 B(s)ds\right] \\
&= \frac{e(1)\|\xi_1\|\|\xi_2\|}{\zeta}\left[\frac{\xi_1(t)}{\|\xi_1\|}\frac{\xi_2(t)}{\|\xi_2\|}\int_0^t M(s)ds \right.\\
&\left. + \frac{\xi_1(t)}{\|\xi_1\|}\frac{\xi_2(t)}{\|\xi_2\|}\int_t^1 B(s)ds\right] \\
&\le \frac{e(1)\|\xi_1\|\|\xi_2\|}{\zeta} z(t)\int_0^1 B(s)ds \le M_1 z(t)\|B\|,
\end{aligned} \tag{15}$$

where $M_1 = (e(1)\|\xi_1\|\|\xi_2\|)/\zeta$.

By the same method, we can know that

$$\frac{\alpha\xi_1(t)}{1-\alpha\xi_1(\xi)}\int_0^1 G(\xi,s)e(s)B(s)ds \le M_2 z(t)\|B\|, \tag{16}$$

where $M_2 = (\alpha e(1)\|\xi_1\|\|\xi_2\|)/(1-\alpha\xi_1(\xi))$.

So, by choosing constant $M \ge M_1 + M_2$, we have

$$\bar{w}(t) \le M\|B\|z(t), \quad 0 \le t \le 1. \tag{17}$$

□

Lemma 7 [7]. *Let $0 \le \overline{\lim}_{x\infty}(b(t,x)/x) \le L_2$, $t \in [0,1]$. Define the following function:*

$$G(\tau) = \max_{0\le t\le 1, 0\le x\le \tau} b(t,\tau). \tag{18}$$

Then

(i) G is a nondecreasing function for τ

(ii) $0 \le \overline{\lim}_{\rho\infty}(G(\tau)/\tau) \le K_2$

For g assumptions:

(C2) $a(t,x), b(t,x) \in C, ([0,1]\times[0,+\infty)R)$

From (C2), there exists a function$B(t) \in C[0,1]$, $B(t) > 0$, which satisfies

$$\begin{aligned}
&a(t,x) \ge -B(t), \\
&b(t,x) \ge -B(t), \\
&\quad \forall t \in (0,1), x \ge 0,
\end{aligned} \tag{19}$$

where $M\|B\| < 1$. M is given by Lemma 6.

(C3) $B_1 \le a_\infty^- \le \infty, B_2 \le b_\infty^- \le \infty$

(C4) $0 \le a_\infty^+ \le b_1, 0 \le b_\infty^+ \le b_2$

(C5) $K_1 \le a_\infty^- \le \infty, 0 \le b_\infty^+ \le K_2$,

where

$$\begin{aligned}
&\min(B_1,B_2) \ge 2\left((\lambda+\mu)\min_{0\le t\le 1}\int_0^1 G(t,s)e(s)z(s)ds\right)^{-1}, \\
&b_1 + b_2 \le \left((\lambda+\mu)p(1)\left[\int_0^1 G(s,s)ds + \frac{\alpha\xi_1(1)}{1-\alpha\xi_1(\xi)}\int_0^1 G(\xi,s)ds\right]\right)^{-1}, \\
&K_1 \ge 2\left(\lambda \min_{0\le t\le 1}\int_0^1 G(t,s)e(s)z(s)ds\right)^{-1}, \\
&a_\infty^- = \underline{\lim}_{x\longrightarrow\infty}\frac{a(t,x)}{x}, \\
&a_\infty^+ = \overline{\lim}_{x\longrightarrow\infty}\frac{a(t,x)}{x}, \\
&b_\infty^- = \underline{\lim}_{x\longrightarrow\infty}\frac{b(t,x)}{x}, \\
&b_\infty^+ = \overline{\lim}_{x\longrightarrow\infty}\frac{b(t,x)}{x}.
\end{aligned} \tag{20}$$

Let $\varepsilon = \min_{0\le t\le 1} z(t)$, and

$$\begin{aligned}
&\bar{H}(t,x) = \begin{cases} H(t,x), & x \ge 0, \\ F(t,0), & x < 0, \end{cases} \\
&\bar{Y}(t,x) = \begin{matrix} Y(t,x), & x \ge 0, \\ G(t,0), & x < 0, \end{matrix}
\end{aligned} \tag{21}$$

where $H(t,x) = a(t,x) + B(t)$, $Y(t,x) = b(t,x) + B(t)$.

For any $l > 0$, we set

$$\begin{aligned}
&H_l = \max_{0\le t\le 1, 0\le x\le l} \bar{H}(t,x), \\
&Y_l = \max_{0\le t\le 1, 0\le x\le l} \bar{Y}(t,x).
\end{aligned} \tag{22}$$

From Lemma 6, letting $w(t) = \bar{w}(t)$, then $x(t)$ is the positive solution of problem (2) if and only if $\tilde{x}(t) = x(t) + w(t)$ is the solution of the following problem:

$$\begin{cases} (Lx)(t) + \lambda\bar{H}(t,x-w) + \mu\bar{Y}(t,x-w) = 0, \\ x(0) = 0, \quad x(1) = \alpha u(\xi), \end{cases} \tag{23}$$

and $\tilde{x}(t) > w(t)$, $0 < t < 1$; here, $\bar{H}$, $\bar{Y}$ is given by (21).

Defining the cone P in E, we have

$$P = \{x \in E : x(t) \ge \|x\|q(t), \quad t \in [0,1]\}. \tag{24}$$

Obviously, problem (18) is equivalent to

$$x(t)=\int_0^1 Y(t,s)e(s)\left[\lambda\bar{H}(s,x-w)+\mu\bar{Y}(s,x-w)\right]ds + \frac{\alpha\xi_1(t)}{1-\alpha\xi_1(\xi)}\int_0^1 G(\xi,s)e(s)\left[\lambda\bar{H}(s,x-w)+\mu\bar{Y}(s,x-w)\right]ds. \tag{25}$$

Defining the operator $T: E \longrightarrow E$, we have

$$(Tx)(t)=\int_0^1 G(t,s)e(s)\left[\lambda\bar{H}(s,x-w)+\mu\bar{Y}(s,x-w)\right]ds + \frac{\alpha\xi_1(t)}{1-\alpha\xi_1(\xi)}\int_0^1 G(\xi,s)e(s) \cdot \left[\lambda\bar{H}(s,x-w)+\mu\bar{Y}(s,x-w)\right]ds. \tag{26}$$

Obviously $T(P)\subset P$ and T is completely continuous.

3. Our Main Three Results

Theorem 8. *Suppose condition (C1), condition (C2), and condition (C3) hold. Then, for the small number λ, μ, problem (2) has at least one positive solution.*

Proof. Firstly, we choose sufficiently small λ, μ which satisfies the following:

$$\lambda+\mu<\left([H_1+Y_1]p(1)\left[\int_0^1 G(s,s)ds+\frac{\alpha\xi_1(1)}{1-\alpha\xi_1(\xi)}\int_0^1 G(\xi,s)ds\right]\right)^{-1}. \tag{27}$$

Letting $\Omega_1=\{x\in E: \|x\|<1\}$, for any $x\in P\cap\partial\Omega_1$, $t\in[0,1]$, by the definition of operator T, we have

$$\begin{aligned}(Tx)(t)&\le\int_0^1 G(s,s)[\lambda H_1+\mu Y_1]ds+\frac{\alpha\xi_1(t)}{1-\alpha\xi_1(\xi)}\cdot\int_0^1 G(\xi,s)e(s)[\lambda H_1+\mu Y_1]ds\\ &\le(\lambda+\mu)[H_1+Y_1]e(1)\cdot\left[\int_0^1 G(s,s)ds+\frac{\alpha\xi_1(1)}{1-\alpha\xi_1(\xi)}\int_0^1 G(\xi,s)ds\right]<1=\|x\|.\end{aligned} \tag{28}$$

Thus, we have

$$\|Tx\|\le\|x\|,\quad \forall x\in P\cap\partial\Omega_1. \tag{29}$$

Secondly, by (C3), we know that there exists constant $l_1>0$ which satisfies

$$a(t,x)\ge B_1x, b(t,x)\ge B_2x, \forall x\ge l_1, t\in[0,1]. \tag{30}$$

Letting $r=\max\{2M\|B\|,(2l_1/\varepsilon),2\}$, then $r>1$. Set $\Omega_2=\{x\in E: \|x\|<r\}$, for any $x\in P\cap\partial\Omega_2$, $t\in[0,1]$, we have

$$x(t)-w(t)\ge x(t)-M\|B\|z(t)\ge x(t)-\frac{M\|B\|_1}{r}x(t)\ge\frac{1}{2}x(t). \tag{31}$$

Therefore, we have $x(t)-w(t)\ge(1/2)x(t)\ge(\|x\|/2)z(t)\ge(\varepsilon r/2)\ge l_1$.

Thus, by the definition of $\bar{H}$, $\bar{Y}$ and (30), we can have

$$\begin{aligned}&\lambda\bar{H}(s,x(s)-w(s))+\mu\bar{Y}(s,x(s)-w(s))\\ &\quad\ge B_1\lambda(x(s)-w(s))+B_2\mu(x(s)-w(s))\\ &\quad\ge\min(B_1,B_2)(\lambda+\mu)(x(s)-w(s)).\end{aligned} \tag{32}$$

We have

$$\begin{aligned}(Tx)(t)&\ge\int_0^1 G(t,s)e(s)\left[\lambda\bar{H}(s,x-w)+\mu\bar{Y}(s,x-w)\right]ds\\ &\ge\min_{0\le t\le 1}\int_0^1 G(t,s)\min(B_1,B_2)(\lambda+\mu)(x(s)-w(s))ds\\ &\ge\frac{1}{2}(\lambda+\mu)\min(B_1,B_2)\min_{0\le t\le 1}\int_0^1 G(t,s)x(s)ds.\end{aligned} \tag{33}$$

Then, by Lemma 5, we have

$$\|(Tx)(t)\|\ge\frac{1}{2}(\lambda+\mu)\min(B_1,B_2)\min_{0\le t\le 1}\int_0^1 G(t,s)e(s)z(s)ds\|x(t)\|. \tag{34}$$

Therefore, by the definition of B_1, B_2, we have

$$\|Tx\|\ge\|x\|,\quad \forall x\in K\cap\partial\Omega_2. \tag{35}$$

Then, by (29), (35) and Theorem 1, operator T has at last one fixed point $\tilde{x}(t)\in P\cap(\bar{\Omega}_2/\Omega_1)$, i.e., $\tilde{x}(t)$ is the solution of problem (2), and it is easy to know $\|\tilde{x}\|\ge 1$.

Finally, by (C2) and Lemma 3, we have

$$\tilde{x}(t)\ge\|\tilde{x}\|z(t)\ge z(t)>M\|B\|z(t)\ge\bar{w}(t)=w(t). \tag{36}$$

Thus, $x=\tilde{x}-w$ is the positive solution of problem (2). ☐

Theorem 9. *We suppose that condition (C1), (C2), and (C4) hold, and the following condition also holds:*

(C6) There exist constant $D>0$, $\rho>0$, and we have

$$\begin{aligned}&a(t,x)\ge\rho,\\ &b(t,x)\ge\rho,\\ &\quad x\in[D,\infty), t\in[0,1].\end{aligned} \tag{37}$$

Then, for the small number λ, μ, problem (2) has at least one positive solution.

Proof. Firstly, let $r = \max\{2M\|B\|, (2D/\varepsilon), 2\}$, and

$$1 = 2r\left(\min_{0\le t\le 1}\int_0^1 Y(t,s)e(s)(\lambda+\mu)\rho ds\right)^{-1}. \tag{38}$$

Set $\Omega_1 = \{x \in E : \|x\| < r\}$, for any $x \in P \cap \partial\Omega_1$, $s \in [0,1]$, we have

$$x(s) - w(s) \ge x(s) - M\|B\|z(s) \ge x(s) - \frac{M\|B\|}{r}x(s) \ge \frac{1}{2}x(s). \tag{39}$$

Thus, $x(s) - w(s) \ge (1/2)x(s) \ge (\|x\|/2)z(s) \ge (\varepsilon r/2) \ge D$. Therefore, by (C6) and the definition of operator T, we have

$$\begin{aligned}(Tx)(t) = &\int_0^1 G(t,s)e(s)\left[\lambda\bar{H}(s,x-w) + \mu\bar{Y}(s,x-w)\right]ds \\ &+ \frac{\alpha\xi_1(t)}{1-\alpha\xi_1(\xi)}\int_0^1 G(\xi,s)e(s) \\ &\cdot \left[\lambda\bar{H}(s,x-w) + \mu\bar{Y}(s,x-w)\right]ds.\end{aligned} \tag{40}$$

For $B(t) > 0$, $t \in (0,1)$, we have

$$\begin{aligned}(Tx)(t) &\ge \int_0^1 G(t,s)e(s)\left[\lambda\bar{H}(s,x-w) + \mu\bar{Y}(s,x-w)\right]ds \\ &\ge \int_0^1 G(t,s)e(s)[\lambda(\rho+B(s)) + \mu(\rho+B(s))]ds \\ &\ge \frac{1}{2}\min_{0\le t\le 1}\int_0^1 G(t,s)e(s)(\lambda+\mu)\rho ds = r = \|x\|.\end{aligned} \tag{41}$$

We can know that by the above discussion, we have

$$\|Tx\| \ge \|x\|, \forall x \in P \cap \partial\Omega_1. \tag{42}$$

Secondly, by (C4), we can have

$$\begin{gathered}0 \le \lim_{x\longrightarrow\infty}\frac{\bar{H}(s,x-w)}{u} \le b_1, \\ 0 \le \lim_{x\longrightarrow\infty}\frac{\bar{Y}(s,x-w)}{u} \le b_2, \\ s \in [0,1].\end{gathered} \tag{43}$$

Then, there exists constant $l_2 > 0$ which satisfies

$$\begin{gathered}\bar{H}(s,x-w) \le b_1 x, \\ \bar{Y}(s,x-w) \le b_2 x, \\ \forall x \ge l_2, s \in [0,1].\end{gathered} \tag{44}$$

Letting $R = \max\{2l_2, 2r\}$, then $r < R$. Set $\Omega_2 = \{x \in E : \|x\| < R\}$, for any $x \in P \cap \partial\Omega_2$, $t \in [0,1]$, we have

$$\begin{aligned}(Tx)(t) \le &\int_0^1 G(s,s)[\lambda b_1 x(s) + \mu b_2 x(s)]ds \\ &+ \frac{\alpha\xi_1(t)}{1-\alpha\xi_1(\xi)}\int_0^1 G(\xi,s)e(s)[\lambda b_1 x(s) + \mu b_2 x(s)]ds.\end{aligned} \tag{45}$$

Thus, we have

$$\begin{aligned}(Tx)(t) \le &(\lambda+\mu)[b_1+b_2]e(1) \\ &\cdot\left[\int_0^1 G(s,s)x(s)ds + \frac{\alpha\xi_1(1)}{1-\alpha\xi_1(\xi)}\int_0^1 G(\xi,s)x(s)ds\right].\end{aligned} \tag{46}$$

So, we have

$$\begin{aligned}\|(Tx)(t)\| \le &(\lambda+\mu)[b_1+b_2]e(1) \\ &\cdot\left[\int_0^1 G(s,s)ds + \frac{\alpha\xi_1(1)}{1-\alpha\xi_1(\xi)}\int_0^1 G(\xi,s)ds\right]\|x\|.\end{aligned} \tag{47}$$

Then, we can have by the definition of b_1, b_2

$$\|Tx\| \le \|x\|, \forall x \in K \cap \partial\Omega_2. \tag{48}$$

□

Then, similar to the proof of heorem 8, we have that result of heorem 9 by Theorem 1.

Theorem 10. *Suppose condition (C1), condition (C2), and condition (C5) hold. Then, for sufficiently small λ, μ, problem (2) has at least two positive solutions.*

Proof. Firstly, by Lemma 7, there exists constant $\tau > 0$ which satisfies

$$G(\tau) \le K_2\tau. \tag{49}$$

Therefore, setting $\Omega_1 = \{x \in E : \|x\| < \tau\}$, for any $x \in P \cap \partial\Omega_1$, $t \in [0,1]$, by the above discussion, for the quite small λ, μ, we have

$$[\lambda H_\tau + \mu Y(\tau)]e(1)\left(\int_0^1 G(s,s)ds + \frac{\alpha\xi_1(1)}{1-\alpha\xi_1(\xi)}\int_0^1 G(\xi,s)ds\right) \le \tau. \tag{50}$$

We have

$$(Tx)(t) \le [\lambda H_\tau + \mu Y(\tau)] \cdot \left[\int_0^1 G(t,s)e(s)ds + \frac{\alpha\xi_1(1)}{1-\alpha\xi_1(\xi)}\int_0^1 G(\xi,s)e(s)ds\right] \le [\lambda H_\tau + \mu Y(\tau)]e(1) \cdot \left(\int_0^1 G(s,s)ds + \frac{\alpha\xi_1(1)}{1-\alpha\xi_1(\xi)}\int_0^1 G(\xi,s)ds\right) \le \tau = \|x\|. \quad (51)$$

Then, we have

$$\|Tx\| \le \|x\|, \quad \forall x \in P \cap \partial\Omega_1. \quad (52)$$

Secondly, by (C5), there exists a constant $l_3 > 1$, which satisfies

$$a(t,x) \ge K_1 x, \quad \forall x \ge l_3. \quad (53)$$

Letting $r = \max\{2M\|B\|, (2l_3/\varepsilon), 2\tau\}$, and $\Omega_2 = \{x \in E : \|x\| < r\}$, for any $x \in P \cap \partial\Omega_2$, $t \in [0,1]$, we have

$$x(t) - w(t) \ge x(t) - M\|B\|z(t) \ge x(t) - \frac{M\|B\|}{r}x(t) \ge \frac{1}{2}x(t). \quad (54)$$

Then, $x(t) - w(t) \ge (1/2)x(t) \ge (\|x\|/2)z(t) \ge (r\varepsilon/2) \ge l_3$.

Therefore, by the definitions of $\bar{H}$, $\bar{Y}$ and the above discussion, we have

$$(Tx)(t) \ge \int_0^1 G(t,s)e(s)\left[\lambda\bar{H}(s,x-w) + \mu\bar{Y}(s,x-w)\right]ds \ge \int_0^1 G(t,s)e(s)\lambda K_1(x-w)ds \ge \frac{K_1}{2}\lambda \min_{0\le t\le 1}\int_0^1 G(t,s)e(s)z(s)ds \ge r = \|x\|. \quad (55)$$

Thus, we have

$$\|Tx\| \ge \|x\|, \forall x \in P \cap \partial\Omega_2. \quad (56)$$

Finally, letting

$$R = \max\left\{[\lambda H_R + \mu Y_R]\left(\int_0^1 G(s,s)e(s)ds + \frac{\alpha\xi_1(1)}{1-\alpha\xi_1(\xi)} \cdot \int_0^1 G(\xi,s)e(s)ds\right], 2r\right\}, \quad (57)$$

then, $\tau < r < R$. Set $\Omega_3 = \{x \in E : \|x\| < R\}$, for any $x \in P \cap \partial\Omega_3$, $t \in [0,1]$, by the definition of operator T, we have

$$(Tx)(t) \le [\lambda H_R + \mu Y_R] \cdot \left(\int_0^1 G(s,s)e(s)ds + \frac{\alpha\xi_1(1)}{1-\alpha\xi_1(\xi)}\int_0^1 G(\xi,s)e(s)ds\right]. \quad (58)$$

Thus, we have

$$\|Tx\| \le \|x\|, \quad \forall x \in P \cap \partial\Omega_3. \quad (59)$$

Then, similar to the proof of heorem 8, we have the result of heorem 10 by Theorem 2. □

Remark 11. The results of these three theorems in our paper also hold under the condition in which nonlinear $a(t,x)$, $b(t,x)$ are both lower semicontinuous.

Remark 12. We can obtain the results of Theorem 10 if we replace condition (C5) with (C6)$K_1 \le b_\infty^- \le \infty$, $0 \le a_\infty^+ \le K_2$.

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Authors' Contributions

All authors contributed equally to the manuscript, and all authors typed, read, and approved the final manuscript.

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Research Article

Stability Analysis Based on Caputo-Type Fractional-Order Quantum Neural Networks

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In this paper, a quantum neural network with multilayer activation function is proposed by using multilayer Sigmoid function superposition and learning algorithm to adjust quantum interval. On this basis, the quasiuniform stability of fractional quantum neural networks with mixed delays is studied. According to the order of two different cases, the conditions of quasi uniform stability of networks are given by using the techniques of linear matrix inequality analysis, and the sufficiency of the conditions is proved. Finally, the feasibility of the conclusion is verified by experiments.

1. Introduction

Fractional calculus is an arbitrary extension of integer calculus in order. It has strong advantages and wide application prospects in the fields of physics, chemistry, biology, economy, control, signal, and image processing. It has attracted extensive attention from scholars at home and abroad and has become one of the current research hotspots. In recent years, due to the continuous development of fractional differential equations, many researchers began to pay attention to the fractional-order theory, and the combination of fractional-order and neural network give full play to the advantages of fractional order. For example, literature [1–4] combined fractional order with neural network and achieved a good effect. Among them, Boroomand and Menhaj [4] presented the fractional-order Hopfield neural network model and studied its stability through the quasienergy function. [5–7] study on different fractional-order neural networks and explore the influence of different factors on fractional-order neural networks. This paper summarizes the synchronization problem of neural network [8–12]. Dominik et al. [13] considered discrete fractional-order artificial neural networks. Chaos and chaotic synchronization of fractional-order neural networks are proposed [14]. Literature [15, 16] explained and analyzed the dynamics of fractional-order neural networks. The fractional-order neural network was applied in different fields [17–21]. In recent years, the stability of fractional-order neural network system has become a research hotspot [22–31]. In reference [22], the stability and passivity of a memristor-based fractional-order competitive neural network (MBFOCNN) are analyzed by using Caputo's fractional derivative. The effectiveness of the proposed results is finally verified by using analysis techniques and other computational tools. In reference [23], the problem of robust dissipation of Hopfield-type complex valued neural network (HTCVNN) model with time-varying delay and linear fractional uncertainty is studied, and many numerical models are designed to verify the results. In reference [24, 25], the global asymptotic stability of fractional quaternion numerical bidirectional associative memory neural networks (FQVBAMNNs) and fractional quaternion numerical memristic neural networks (FOQVMNNs) is studied. The effectiveness of the results is proved by using related methods. In reference [26, 27], the stability of fractional-order continuous time quaternion numerical leaky integral echo state neural network (NN) with multiple time-varying delays is studied, and the feasibility of the method is verified by numerical examples. In reference [28], the uniform stability of a fractional-order leaky integral echo state neural network (FOESN) with multiple delays is studied. The simulation results show the effectiveness of the method. Literature [32, 33] proposed the time-delay correlation study of Caputo

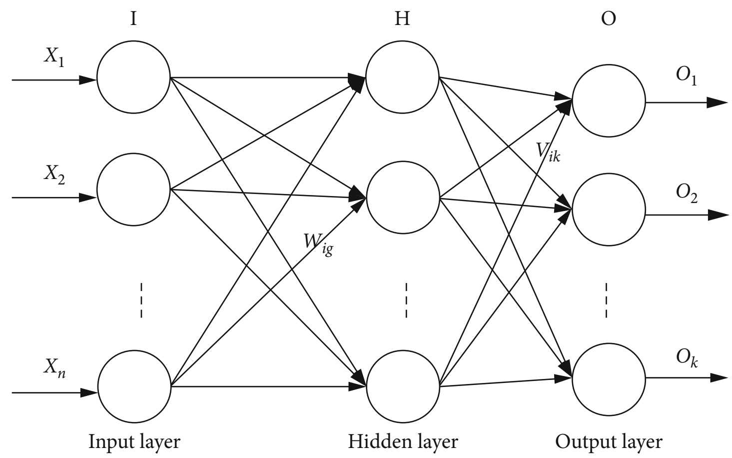

Figure 1: Three-layer feedforward neural network graph structure.

fractional-order neural network. However, there are few studies on the behavior of fractional quantum neural networks with mixed delay. In this paper, a multilayer activation function quantum neural network model is presented, and the quasiuniform stability of fractional quantum neural networks with mixed delay is studied. It is proved by the formula and simulated by the numerical case.

This article is organized as follows. In the second section, we give the structure of the multilayer activation function of the quantum neural network, based on which a fractional quantum neural network model with mixed delay is proposed. In the third section, it is proved that the fractional quantum neural network system with mixed time delay is quasiuniformly stable by corresponding definitions and lemmas. In the fourth section, a concrete example is given to verify the validity and applicability of the given results.

2. Model Composition and Preparation

2.1. Quantum Neural Network. Quantum neural network belongs to the feed-forward type of neural network [34, 35]. Compared with the traditional feed-forward type of neural network, the neurons in the hidden layer of quantum neural network refer to the idea of quantum state superposition in the quantum theory and carry out the linear superposition of several Sigmoid functions, which is called the multilayer activation function. Traditional activation functions can only represent two states and orders of magnitude. When quantized, a hidden layer neuron can represent more states and orders of magnitude.

Each Sigmoid function superimposed has a different quantum interval. By adjusting the quantum interval, the data of different classes can be mapped to different orders of magnitude or steps, so that the classification can have more degrees of freedom. The quantum interval of the quantum neural network can be obtained by training. The uncertainty in the sampled data can be obtained and quantified by a quantum neural network with an appropriate learning algorithm.

Figure 1 shows a traditional three-layer feedforward neural network. Assume that the input layer I has n nodes, the output layer O has k nodes, and the number of nodes in the hidden layer H is m. Adjacent layer nodes are fully interconnected, and nodes of the same layer are not connected. The node output function in the hidden layer is

$$H_r = f\left(\mathbf{W}^{\mathrm{T}}\mathbf{X} - \theta\right) r = 1, 2, \cdots, u. \tag{1}$$

The output function of the node in the output layer is

$$O_i = f\left(\mathbf{V}^{\mathrm{T}}\mathbf{H} - \mathrm{h}\right) j = 1, 2, \cdots, n. \tag{2}$$

In the formula, f adopts Sigmoid function, and W is the connection weight vector between each neuron in the input layer and each neuron in the hidden layer. $\mathbf{V}$ is the connection weight vector between each neuron in the hidden layer and each neuron in the output layer; θ is the threshold of the hidden layer, and h is the threshold of the unit of the output layer.

Quantum neural networks with multiple excitation functions:

$$Hr = \frac{1}{n}\sum_{s=1}^{n} f\left[\mathrm{U}\left(\mathbf{W}^{\mathrm{T}}\mathbf{X} - \theta_s\right)\right]. \tag{3}$$

In the formula: $f(x) = 1/(1 + \exp(-x))$, $\mathbf{W}$ is the network weight vector; $\mathbf{X}$ is the network input vector; U is the slope; $\mathbf{W}^{\mathbf{T}}\mathbf{X}$ is the input excitation of the quantum neuron; θ_s is the quantum interval $(s = 1, 2, \cdots, n)$.

The learning of quantum neural network can be divided into two steps: (1) adjusting the weight to make the input data correspond to different class spaces; (2) adjust the quantum interval of quantum neurons in the hidden layer to reflect the uncertainty of data. The BP algorithm is used to adjust the weight. Once the network weight is obtained, the quantum interval can be adjusted by an appropriate algorithm [36]. The idea of the algorithm is to minimize the output change of the hidden layer neurons in the quantum neural network based on the same kind of sample data.

Assume that for class C_m, the output of the ith hidden layer neuron changes as:

$$e_{i,m}^2 = \sum_{x_k} \sum_{x_k \in C_m} \left(\langle O_{i,m} \rangle - O_{i,k} \right)^2, \quad (4)$$

in the formula: O_i, k represents the output of the i_{th} neuron in the hidden layer when the network input vector is x_k;

$$\langle O_{i,m} \rangle = \frac{1}{|C_m|} \sum_{x_i x_k \in C_m} O_{i,k}. \quad (5)$$

|Cm| in the formula represents the cardinality of class |Cm|. It can be seen that $e_{i,m}^2$ is a function of the quantum interval θ_s. By taking the derivative of $\theta_s (s = 1, \cdots, n)$ on both sides of Equation (4) and finding the minimum value of $e_{i,m}^2$, the variation formula of $\theta_{i,s}$ (i.e., layer S of the ith neuron in the hidden layer) can be obtained.

$$\Delta\theta_{i,s} = Z \frac{U}{nS} \sum_{m=1}^{k_0} \sum_{x_k} \sum_{x_k \in C_m} \left(\langle O_{i,m} \rangle - O_{i,k} \right)^* \left(\langle V_{i,m,s} \rangle - V_{i,k,s} \right), \quad (6)$$

$$\langle V_{i,m,s} \rangle = \frac{1}{|C_m|} \sum_{x_k} \sum_{x_k \in C_m} V_{i,k,s}. \quad (7)$$

In formula (6), Z is the learning rate; k_0 is the number of nodes in the output layer, namely, the total number of classes; k is the number of quantum interval layers; x_k: $x_k \in C_m$ represents all samples belonging to the C_m class.

Among them:

$$V_{i,k,s} = O_{i,k,s} * (1 - O_{i,k,s}). \quad (8)$$

In the formula: $O_{i,k,s} = \text{sig}(U * (W^{\text{T}} \text{x}_{\text{k}} - \theta_s))$ represents the output of the sth quantum layer of the ith hidden layer neuron when the input vector is $x_k (s = 1, 2, \cdots, n)$.

2.2. Caputo-Type Fractional Derivative Definition. In definition, $f(s)$ is a continuous function over R, for any β>0; the β-order Caputo-type derivative of 0,$f(s)$ is defined as:

$$D_{0,s}^{\beta} f(s) = D_{0,s}^{-(n-\beta)} \frac{d^n}{dt^n} f(s) = \frac{1}{\Gamma(n-\beta)} \int_0^s (s-\tau)^{\beta-1} f^n(\tau) \mathrm{d}\tau. \quad (9)$$

The following corollary can be drawn:

(1) When $0 < \beta < 1$, $D_{0,s}^{\beta} f(s) = (1/(\Gamma(1-\beta))) \int_0^s (s-\tau)^{\beta-1} f(\tau) \mathrm{d}\tau$

(2) When $f(s)$ is a constant function, $D_{0s}^{\beta} f(s) = 0$

(3) $D^{-\beta} D^{\beta} f(s) = f(s) - \sum - f^{(m)}$, $\beta > 0$,especially,when $0 < \beta < 1$, when $f(s)$

(4) is a one-dimensional function, $D^{-\beta} D^{\beta} f(s) = f(s) - f(0)$.

(5) If α and γ are two constants, then $D^{-\beta}(\alpha f(s) + \gamma h(t)) = \alpha D^{-\beta} f(s) + \gamma D^{-\beta} h(t), \beta \geq 0$

2.3. Fractional-Order Quantum Neural Network Model. Suppose the following two conclusions are true:

(1) If vector $x = (x_i)$ and matrix $\bar{A} = (a_{ij})$, we define the Euclidean norm $\|\text{x}\|$ of vector $\mathbf{x}$ to be $\|x\| = \sum |x_i|$. The matrix norm of the matrix $\|\bar{A}\|$ is defined as $\|\bar{A}\| = \max_{1 \leq i \leq n} \sum |a_{ij}|$. In this paper, we set $C = \|\bar{C}\|$, $A = \|\bar{A}\|$, $B = \|\bar{B}\|$, and $M = \|\bar{M}\|$

(2) The excitation functions $F(x)$, $G(x)$, and $H(x)$ of the fractional quantum neural network with mixed delay both satisfy the Lipschiz condition, that is, for any $u, v \in R$, $u \neq v$, there exists a corresponding real number $F, G, H > 0$, such that $\|F(u) - F(v)\| \leq F\|u - v\|$, $\|G(u) - G(v)\| \leq G\|u - v\|$

The fractional quantum neural network model with mixed time delay is shown below:

$$\begin{cases} D^{\beta} x_i(t) = -c_i x_i(t) + \sum a_{ij} f_j(x_j(t)) + \sum b_{ij} g_j(x_j(t-\tau)) + \sum m_{ij} \int_{t-\sigma}^{t} h_i(x_i(\mu)) \mathrm{d}\mu + I_i, \\ x_i(t) = \psi_i(t), \quad t \in [-\gamma, 0), \gamma \in \max\{\tau, 0\}, \end{cases} \quad (10)$$

is converted to:

$$\begin{cases} D^{\beta} x(t) = -\bar{C} x(t) + \bar{A} F(x(t)) + \bar{B} G(x(t-\tau)) + \bar{M} \int_{t-\sigma}^{t} H(\mu) \mathrm{d}\mu + I, \\ x(t) = \psi(t), \quad t \in [-\gamma, 0), \gamma \in \max\{\tau, \sigma\}. \end{cases} \quad (11)$$

Among them, $0 < \beta < 1$, $(i = 1, 2, \cdots, n)$, n represents the number of neurons in a fractional quantum neural network with mixed delay, and $x(t) = (x_1(t), x_2(t), \cdots, x_n(t)) \in R$ is the state vector of the neuron at time t.

$$F(x(t)) = (f_1(x_1(t)), f_2(x_2(t)), \cdots, f_n(x_n(t))), G(x(t)) = (g_1(x_1(t)), g_2(x_2(t)), \cdots, g_n(x_n(t)))$$

and $H(x(t)) = (h_1(x_1(t)), h_2(x_2(t)), \cdots, h_n(x_n(t)))^T$ are the activation function of fractional quantum neural network; $\bar{C} = \text{diag}\,(c_i > 0)$, $\bar{A} = (a_{ii})$, $\bar{B} = (b_{ij})$, and $\bar{M} = (m_{ij})$ are all constant matrices; $c_i > 0$ represents the rate of the isolated resting state of the first neuron in the fractional-order quantum neural network in the state of unconnected and without external additional voltage difference; a_j, b_{ij}, and m_{ij} represent the weight of the connection between the jth neuron and the ith neuron; τ_j and σ_i represent the transmission delay of the jth neuron along the axon; and $I = (I_1(t), I_2(t), \cdots, I_n(t))^T$ represents the external input and deviation of the neuron.

Set the initial conditions of the system, usually assuming $\psi_i(s) \in C([-\gamma, 0]R), i \in N^+$, and the norm on C is defined as $\|\psi\| = \sup \|\psi(s)\|$.

Assume that $x(t)$ and $y(t)$ are two different solutions whose initial values of model (11) are $\psi \in C$ and $\phi \in C$, respectively, where $\psi(0) = \phi(0) = 0$; let $\varphi = \psi - \varphi$, $x(t) = y(t) = e(t) = (e_1(t), e_2(t), \cdots, e_n(t))^T$ can be obtained:

$$\begin{aligned} D^\beta e(t) = & - C(t) \\ & + A(F(x(t)) - F(y(t))) \\ & + B(G(x(t-\tau)) - G(y(t-\tau))) \\ & + M\int_{t-\sigma}^t (H(x(\mu)) - H(y(\mu))) \\ & \cdot d\mu e(t) = \varphi(t), \quad t \in [-\gamma, 0), \gamma \in \max\{\tau, \sigma\}, \end{aligned} \tag{12}$$

where $\varphi \in C, \varphi(0) = 0$ is the initial condition of model (12), $\|\varphi\| = \sup_{s\in[-\gamma,0]}\|\varphi(s)\|$.

3. Main Result

Relevant definition:

Lemma 1. *If $x(s) \in C^n[0,+\infty)$ and $n-1 < \alpha, \beta < n \in Z^+$ then*

$$\begin{aligned} & D^{-\alpha}D^{-\beta}x(s) = D^{-(\alpha+\beta)}x(s), \quad \alpha, \beta > 0, \\ & D^\beta D^{-\beta}x(s) = x(s), \quad \beta \geq 0, \\ & D^{-\beta}D^\beta x(s) = x(s) - \sum_{m=0}^{n-1} \frac{s^m}{m!} x^{(m)}, \quad \beta \geq 0. \end{aligned} \tag{13}$$

Lemma 2 (Hölder inequality). *Suppose that the real number $p, q > 1$, and p, q satisfies $(1/p) + (1/q) = 1$, if $|f(\cdot)|^p$, $|h(\cdot)|^q$ is a measurable function in space, and $f, g : E \longrightarrow R$ satisfies $\int_E |f(x)|dx < \infty$, $\int_E |g(x)|dx < \infty$, then $f(\cdot)h(\cdot)$ is also a measurable function and satisfies*

$$\int_E |f(x)h(x)|dx \leq \left(\int_E |f(x)|^p dx\right)^{1/p} \left(\int_E |h(x)|^q dx\right)^{1/q}. \tag{14}$$

In particular, when $p = q = 2$, it is the inequality that we usually see. That is

$$\int_E |f(x)h(x)|dx \leq \left(\int_E |f(x)|^2 dx\right)^{1/2} \left(\int_E |h(x)|^2 dx\right)^{1/2}. \tag{15}$$

Lemma 3. *Let $k \in N, x_1, x_2, \cdots, x_k$ be a nonnegative real number, then it can be obtained for any*

$$\left(\sum_{i=1}^k x_i\right)^\eta \leq k^{\eta-1} \sum_{i=1}^k (x_i)^\eta. \tag{16}$$

Lemma 4 (Gronwall inequality). *If $x(t), f(t), g(t) \geq 0$ is a continuous function on $[0, T)$, $T < \infty$ and satisfies the following inequality*

$$x(t) \leq f(t) + \int_0^t g(\mu)x(\mu)d\mu, \quad t \in [0, T). \tag{17}$$

Then, we can get

$$x(t) \leq f(t) + \int_0^t g(\mu)f(\mu) \exp\left\{\int_\mu^t g(\nu)d\nu\right\}d\mu, \quad t \in [0, T). \tag{18}$$

In special cases, if $f(t)$ is a nonincreasing function, you can get

$$x(t) \leq f(t) \exp\left\{\int_0^t g(\nu)\mathrm{d}\nu\right\}, \quad t \in [0, T). \tag{19}$$

Definition 5. The initial time of the fractional quantum neural network system (11) with mixed delay is set to t_0. For any $\xi > 0$, there are two constants δ and $T, 0 < \delta < \xi, T > 0$, so that for any $t \in J = [t_0, t_0 + T]$, when $\|e(t_0)\| < \delta$ has $\|e(t)\| < \xi$, then the system (11) is called quasiuniformly stable.

Theorem 6. *When the order $\beta \in [0.5, 1)$ of the fractional quantum neural network system (11) with mixed delay is established, if the assumptions 2.3.1 and 2.3.2 are true and*

$$\begin{aligned} & \sqrt{P + Qe^{2t} + W(t)e^{(w(t)+2)t}\left(P\frac{1 - e^{-(2+W(t))t}}{2 + W(t)} + Q\frac{1 - e^{-W(t)t}}{W(t)}\right)} \\ & < \frac{\xi}{\delta}\ t \in J, \end{aligned} \tag{20}$$

is true, where

$$\begin{aligned} & P = 5 - \frac{5M^2H^2\gamma^2\Gamma(2\beta-1)}{\Gamma^2(\beta)4^\beta}, N = \frac{5M^2H^2}{2\beta\Gamma(\beta)}, \\ & Q = \frac{5M^2H^2\gamma^2\Gamma(2\beta-1)}{\Gamma^2(\beta)4^\beta} + \frac{5B^2G^2\Gamma(2\beta-1)\left(1 - e^{-2\gamma}\right)}{\Gamma^2(\beta)4^\beta}, \\ & L = \frac{10\Gamma(2\beta-1)\left[(C + AF)^2 + B^2G^2e^{-2\gamma}\right]}{\Gamma^2(\beta)4^\beta}, \\ & W(t) = L + Nt^{2\beta}\left(1 - e^{-2t}\right). \end{aligned} \tag{21}$$

Then, the system (11) is quasiuniformly stable.

Proof. We set the initial time $t_0 = 0$ of the error system (12), the initial condition is $e_0 = \varphi(0)$, and the expression of the solution of the error system can be obtained from Lemma 1 as

$$
\begin{aligned}
e(t) &= \varphi(0) + D^{-\beta}\big[-\bar{C}e(t) + \bar{A}(F(x(t)) - F(y(t))) \\
&\quad + \bar{B}(G(x(t-\tau)) - G(y(t-\tau))) + \bar{M}\int_{t-\sigma}^{t}(H(x(\mu)) - H(y(\mu)))\mathrm{d}\mu\big] \\
&= \varphi(0) + \frac{1}{\Gamma(\beta)}\int_0^t (t-\mu)^{\beta-1}\big[-\bar{C}e(\mu) + \bar{A}(F(x(\mu)) - F(y(\mu))) \\
&\quad + \bar{B}(G(x(\mu-\tau)) - G(y(\mu-\tau))) + \bar{M}\int_{t-\delta}^{t}(H(x(s)) - H(y(s)))\mathrm{d}s\big]\mathrm{d}\mu.
\end{aligned} \tag{22}
$$

From the hypotheses 1 and 2 and the basic properties of the norm, we can get

$$
\begin{aligned}
\|e(t)\| &\le \|\varphi(0)\| + \frac{1}{\Gamma(\beta)}\int_0^t (t-\mu)^{\beta-1}[C\|e(\mu)\| + AF\|e(\mu)\| \\
&\quad + BG\|e(\mu-\tau)\| + \int_{\mu-\sigma}^{\mu} MH\|e(s)\|\mathrm{d}s]\mathrm{d}\mu \\
&\le \|\varphi(0)\| + \frac{1}{\Gamma(\beta)}\int_0^t (t-\mu)^{\beta-1}(C+AF)\|e(\mu)\|\mathrm{d}\mu \\
&\quad + \frac{1}{\Gamma(\beta)}\int_0^t (t-\mu)^{\beta-1}BG\|e(\mu-\tau)\|\mathrm{d}\mu \\
&\quad + \frac{1}{\Gamma(\beta)}\int_0^t (t-\mu)^{\beta-1}\left(\int_{-\sigma}^{t} MH\|e(s)\|\mathrm{d}s\right)\mathrm{d}\mu \\
&\le \|\varphi\| + \frac{1}{\Gamma(\beta)}\int_0^t (t-\mu)^{\beta-1}(C+AF)\|e(\mu)\|\mathrm{d}\mu \\
&\quad + \frac{1}{\Gamma(\beta)}\int_0^t (t-\mu)^{\beta-1}BG\|e(\mu-\tau)\|\mathrm{d}\mu \frac{MHt^{\beta}}{\beta\Gamma(\beta)}\int_0^t \|e(\mu)\|\mathrm{d}\mu \\
&\quad + \frac{MH\sigma\|\varphi\|}{\Gamma(\beta)}\int_0^t (t-\mu)^{\beta-1}\mathfrak{d}\mu.
\end{aligned} \tag{23}
$$

According to the Cauchy-Schwartz inequality in Lemma 2, we know

$$
\begin{aligned}
\|e(t)\| &\le \|\varphi\| + \frac{1}{\Gamma(\beta)}\left(\int_0^t (t-\mu)^{2\beta-2}e^{2\mu}\mathrm{d}\mu\right)^{1/2} \\
&\quad \cdot \left(\int_0^t (C+AF)^2\|e(\mu)\|^2 e^{-2\mu}\mathrm{d}\mu\right)^{1/2} \\
&\quad + \frac{1}{\Gamma(\beta)}\left(\int_0^t (t-\mu)^{2\beta-2}e^{2\mu}\mathrm{d}\mu\right)^{1/2} \\
&\quad \cdot \left(\int_0^t B^2G^2\|e(\mu-\tau)\|^2 e^{-2\mu}\mathrm{d}\mu\right)^{1/2} \\
&\quad + \frac{MHt^{\beta}}{\beta\Gamma(\beta)}\left(\int_0^t e^{2\mu}\mathrm{d}\mu\right)^{1/2}\left(\int_0^t \|e(\mu)\|^2 e^{-2\mu}\mathrm{d}\mu\right)^{1/2} \\
&\quad + \frac{MH\sigma\|\varphi\|}{\Gamma(\beta)}\left(\int_0^t e^{2\mu}(t-\mu)^{2\beta-2}\mathrm{d}\mu\right)^{1/2}\left(\int_0^t e^{-2\mu}\mathrm{d}\mu\right)^{1/2}.
\end{aligned} \tag{24}
$$

Bring

$$
\begin{aligned}
\int_0^t (t-\mu)^{2\beta-2}e^{2\mu}\mathrm{d}\mu &= \int_0^t z^{2\beta-2}e^{2(t-z)}\mathrm{d}z = e^{2t}\int_0^t z^{2\beta-2}e^{-2z}\mathrm{d}z \\
&= \frac{e^{2t}}{2^{2\beta-1}}\int_0^{2t}\mu^{2\beta-2}e^{-\mu}\mathrm{d}\mu < \frac{2e^{2t}}{4^{\beta}}\Gamma(2\beta-1),
\end{aligned} \tag{25}
$$

into Equation (24) to get

$$
\begin{aligned}
\|e(t)\| &\le \|\varphi\| + \frac{1}{\Gamma(\beta)}\left(\frac{2\Gamma(2\beta-1)e^{2t}}{4^{\beta}}\right)^{1/2} \\
&\quad \cdot \left(\int_0^t (C+AF)^2\|e(\mu)\|^2 e^{-2\mu}\mathrm{d}\mu\right)^{1/2} \\
&\quad + \frac{1}{\Gamma(\beta)}\left(\frac{2\Gamma(2\beta-1)e^{2t}}{4^{\beta}}\right)^{1/2} \\
&\quad \cdot \left(\int_0^t B^2G^2\|e(\mu-\tau)\|^2 e^{-2\mu}\mathrm{d}\mu\right)^{1/2} \\
&\quad + \frac{MHt^{\beta}}{\beta\Gamma(\beta)}\left(\frac{e^{2t}-1}{2}\right)^{1/2}\left(\int_0^t \|e(\mu)\|^2 e^{-2\mu}\mathrm{d}\mu\right)^{1/2} \\
&\quad + \frac{MH\sigma\|\varphi\|}{\Gamma(\beta)}\left(\frac{2\Gamma(2\beta-1)e^{2t}}{4^{\beta}}\right)^{1/2}\left(\frac{1-e^{-2t}}{2}\right)^{1/2}.
\end{aligned} \tag{26}
$$

In Lemma 3, let $k = 5, \eta = 2$, we can get

$$
\begin{aligned}
\|e(t)\|^2 &\le \Bigg\{5 - \frac{5M^2H^2\gamma^2\Gamma(2\beta-1)}{\Gamma^2(\beta)4^{\beta}} + \Bigg[\frac{5M^2H^2\sigma^2\Gamma(2\beta-1)}{\Gamma^2(\beta)4^{\beta}} \\
&\quad + \frac{5B^2G^2\Gamma(2\beta-1)\left(1-e^{2\gamma}\right)}{\Gamma^2(\beta)4^{\beta}}\Bigg]e^{2t}\Bigg\}\|\varphi\|^2 \\
&\quad + \Bigg\{\frac{10\Gamma(2\beta-1)\left[(C+AF)^2 + B^2G^2e^{-2\gamma}\right]e^{2t}}{\Gamma^2(\beta)4^{\beta}} \\
&\quad + \frac{5M^2H^2t^{2\beta}\left(e^{2t}-1\right)}{2\beta\Gamma(\beta)}\Bigg\}\int_0^t \|e(\mu)\|^2 e^{-2\mu}\mathrm{d}\mu,
\end{aligned}
$$

$$
\begin{aligned}
P &= 5 - \frac{5M^2H^2\gamma^2\Gamma(2\beta-1)}{\Gamma^2(\beta)4^{\beta}}, N = \frac{5M^2H^2}{2\beta\Gamma(\beta)}, \\
Q &= \frac{5M^2H^2\gamma^2\Gamma(2\beta-1)}{\Gamma^2(\beta)4^{\beta}} + \frac{5B^2G^2\Gamma(2\beta-1)\left(1-e^{-2\gamma}\right)}{\Gamma^2(\beta)4^{\beta}}, \\
L &= \frac{10\Gamma(2\beta-1)\left[(C+AF)^2 + B^2G^2e^{-2\gamma}\right]}{\Gamma^2(\beta)4^{\beta}},
\end{aligned} \tag{27}
$$

get

$$
\begin{aligned}
\|e(t)\|^2 e^{-2t} &\le \left(Pe^{-2t} + Q\right)\|\varphi\|^2 + \left(L + Nt^{2\beta}\right) \\
&\quad \cdot \left(1 - e^{-2t}\right)\int_0^t \|e(\mu)\|^2 e^{-2\mu}\mathrm{d}\mu.
\end{aligned} \tag{28}
$$

Using Gronwall inequality and letting $W(t) = L + Nt^{2\beta}(1 - e^{-2t})$, get

$$
\begin{aligned}
\|e(t)\|^2 e^{-2t} &\le \left[Pe^{-2t} + Q + W(t)e^{\mathrm{tw}(t)}\left(P\frac{1-e^{-(2+W(t))t}}{2+W(t)} + Q\frac{1-e^{-W(t)t}}{W(t)}\right)\right] \\
&\quad \cdot \|\varphi\|^2,
\end{aligned} \tag{29}
$$

so

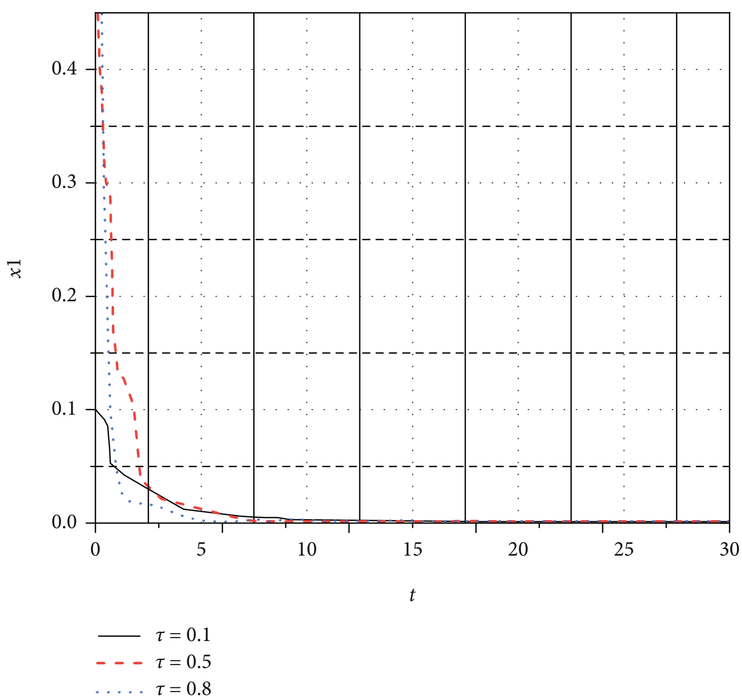

Figure 2: The trajectory graph of x_1 for different τ.

$$\|e(t)\|^2 \le \left[P + Qe^{2t} + W(t)e^{(w(t)+2)t}\left(P\frac{1-e^{-(2+W(t))t}}{2+W(t)} + Q\frac{1-e^{-W(t)t}}{W(t)}\right)\right] \cdot \|\varphi\|^2, \tag{30}$$

that is

$$\|e(t)\| \le \sqrt{P + Qe^{2t} + W(t)e^{(w(t)+2)t}\left(P\frac{1-e^{-(2+W(t))t}}{2+W(t)} + Q\frac{1-e^{-W(t)t}}{W(t)}\right)} \cdot \|\varphi\|. \tag{31}$$

□

It can be seen that when $\|\varphi\| < \delta$, $\|e(t)\| < \xi$ is easy to know from Theorem 6. From Definition 5, it can be concluded that the fractional quantum neural system (11) with mixed time delay is quasiuniformly stable.

Theorem 7. *If the order $\beta \in (0, 0.5)$ of fractional-order quantum neural network system (11) with mixed delay is true, assuming that 1 and 2 are true and*

$$\sqrt[q]{\tilde{P} + \tilde{Q}e^{qt} + \frac{\tilde{W}(t)\tilde{P}\left(e^{(\tilde{w}(t)+q)t} - 1\right)}{q + \tilde{W}(t)} + \frac{\tilde{W}(t)\tilde{Q}e^{(\tilde{w}(t)+q)t}\left(1 - e^{-W(t)t}\right)}{\tilde{W}(t)}} < \frac{\xi}{\delta}\ t \in J, \tag{32}$$

is true, where

$$\begin{aligned}
&\tilde{P} = 5^{q-1}, \tilde{Q} = \frac{5^{q-1}B^q G^q \tilde{E}\left(1 - e^{-q\gamma}\right)}{q} + 5^{q-1}M^q H^q \gamma^q \tilde{E},\\
&\tilde{L} = 5^{q-1}\tilde{E}(C + AF)^q + 5^{q-1}B^q G^q e^{-q\gamma}\tilde{E},\\
&\tilde{E} = \left[\frac{\Gamma(p\beta - p + 1)}{\Gamma^p(\beta)p^{p\beta-p+1}}\right]^{q/p}, \tilde{W}(t) = L + \tilde{N}t^{q\beta}, \tilde{N} = \frac{5^{q-1}M^q H^q}{\beta^q \Gamma^q(\beta)},
\end{aligned} \tag{33}$$

then the system (11) is quasiuniformly stable.

Proof from Theorem 6, we get

$$\begin{aligned}
\|e(t)\| \le \|\varphi\| &+ \frac{1}{\Gamma(\beta)}\int_0^t (t-\mu)^{\beta-1}(C + AF)\|e(\mu)\|\mathrm{d}\mu\\
&+ \frac{1}{\Gamma(\beta)}\int_0^t (t-\mu)^{\beta-1}BG\|e(\mu-\tau)\|\mathrm{d}\mu\\
&+ \frac{1}{\Gamma(\beta)}\int_0^t (t-\mu)^{\beta-1}\left(\int_{\mu-\sigma}^{\mu} MH\|e(s)\|\mathrm{d}s\right)\mathrm{d}\mu,
\end{aligned} \tag{34}$$

let $p = 1 + \beta$ and $p = 1 + (1/\beta)$, obviously $p, q > 1$, from Hölder's inequality we can get

$$\begin{aligned}
\|e(t)\| \le \|\varphi\| &+ \frac{1}{\Gamma(\beta)}\left(\int_0^t (t-\mu)^{p\beta-p}e^{p\mu}\mathrm{d}\mu\right)^{1/p}\\
&\cdot \left(\int_0^t (C + AF)^q\|e(\mu)\|^q e^{-q\mu}\mathrm{d}\mu\right)^{1/q} + \frac{1}{\Gamma(\beta)}\\
&\cdot \left(\int_0^t (t-\mu)^{p\beta-p}e^{p\mu}\mathrm{d}\mu\right)^{1/p}\left(\int_0^t B^q G^q\|e(\mu-\tau)\|^q e^{-q\mu}\mathrm{d}\mu\right)^{1/q}\\
&+ \frac{MHt^\beta e^{pt}}{\beta\Gamma(\beta)}\left(\int_0^t \|e(\mu)\|^q e^{-q\mu}\mathrm{d}\mu\right)^{1/q} + \frac{MH\sigma\|\varphi\|}{\Gamma(\beta)}\\
&\cdot \left(\int_0^t e^{p\mu}(t-\mu)^{p\beta-p}\mathrm{d}\mu\right)^{1/q},
\end{aligned} \tag{35}$$

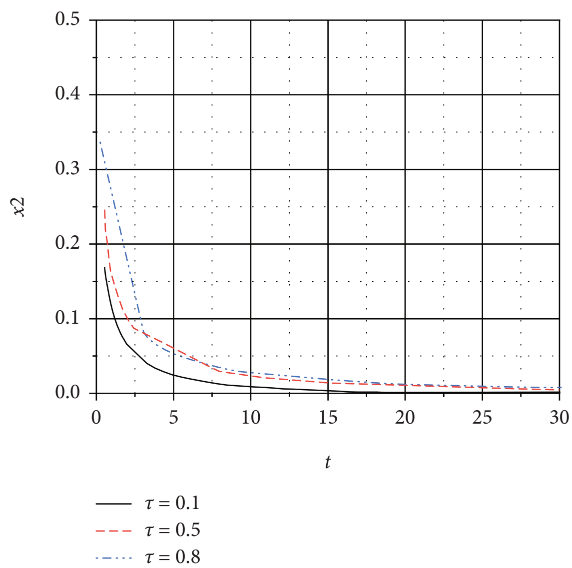

FIGURE 3: The trajectory graph of x_2 for different τ.

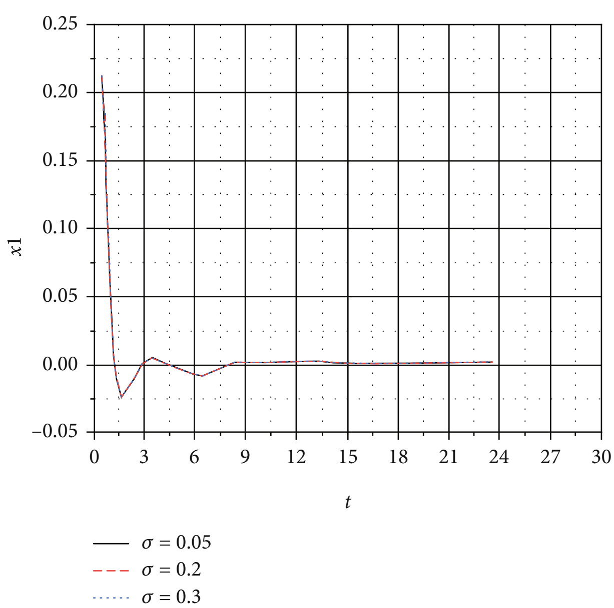

FIGURE 4: The trajectory graph of x_1 for different σ.

because

$$\int_0^t (t-\mu)^{p\beta-p} e^{p\mu}\,\mathrm{d}\mu = \int_0^t z^{p\beta-p} e^{p(t-z)}\,\mathrm{d}z = e^{pt}\int_0^t z^{p\beta-p} e^{-pz}\,\mathrm{d}z = \frac{e^{pt}}{p^{p\beta-p+1}}\int_0^{pt} \mu^{p\beta-p} e^{-\mu}\,\mathrm{d}\mu < \frac{e^{pt}}{p^{p\beta-p+1}}\Gamma(p\beta-p+1), \tag{36}$$

so

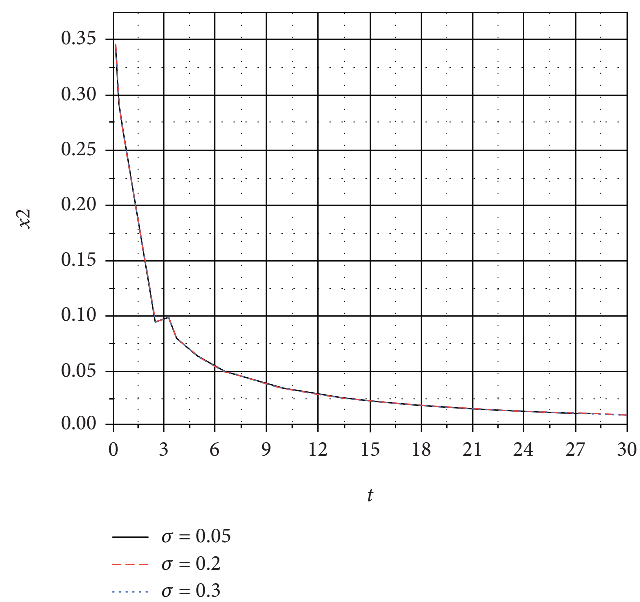

FIGURE 5: The trajectory graph of x_2 for different σ.

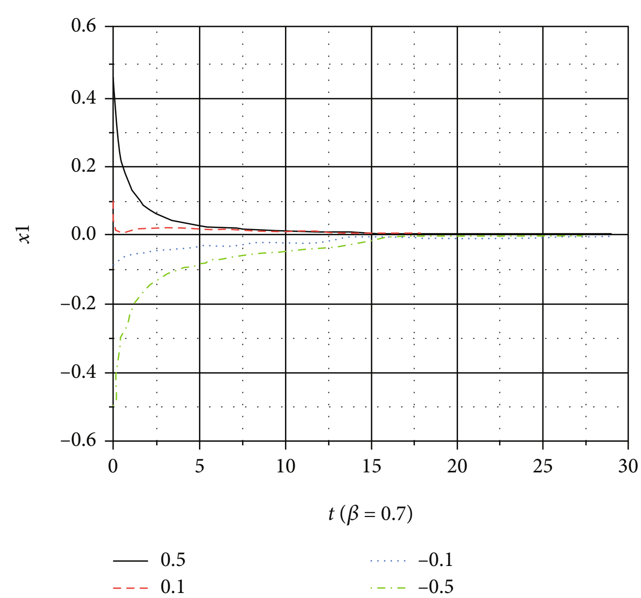

FIGURE 6: For the trajectory graph of different initial values x_1.

$$\begin{aligned} \|e(t)\| \le{} & \|\varphi\| + \left(\frac{e^{pt}\Gamma(p\beta-p+1)}{\Gamma^p(\beta)p^{p\beta-p+1}}\right)^{1/p}\left(\int_0^t (C+AF)^q \|e(\mu)\|^q e^{-q\mu}\,\mathrm{d}\mu\right)^{1/q} \\ & + \left(\frac{e^{pt}\Gamma(p\beta-p+1)}{\Gamma^p(\beta)p^{p\beta-p+1}}\right)^{1/p}\left(\int_0^t B^q G^q \|e(\mu-\tau)\|^q e^{-q\mu}\,\mathrm{d}\mu\right)^{1/q} \\ & + \frac{MHt^\beta e^{pt}}{\beta\Gamma(\beta)}\left(\int_0^t \|e(\mu)\|^q e^{-q\mu}\,\mathrm{d}\mu\right)^{1/q} + MH\sigma\|\varphi\| \\ & \cdot \left(\frac{e^{pt}\Gamma(p\beta-p+1)}{\Gamma^p(\beta)p^{p\beta-p+1}}\right)^{1/p}. \end{aligned} \tag{37}$$

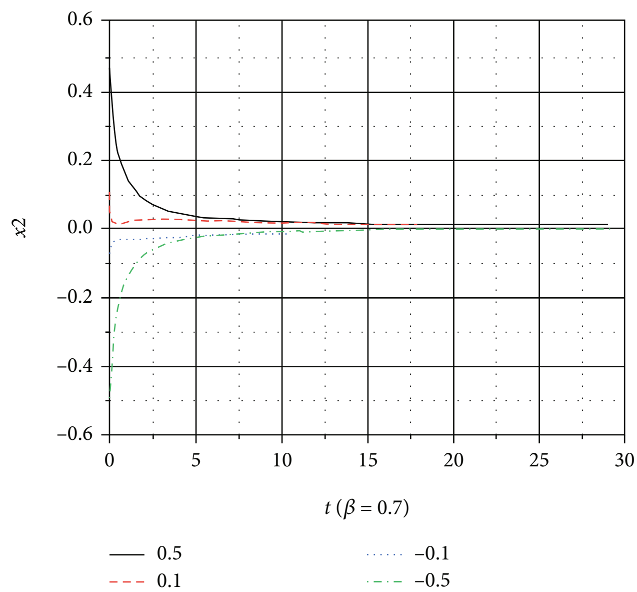

FIGURE 7: For the trajectory graph of different initial values x_2.

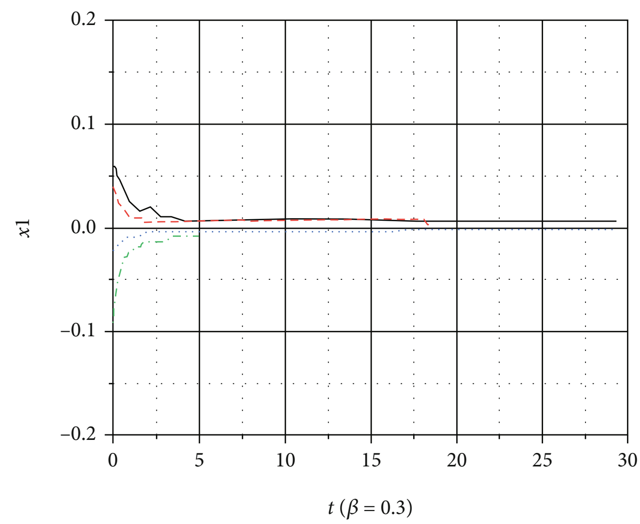

FIGURE 8: For the trajectory graph of different initial values x_1.

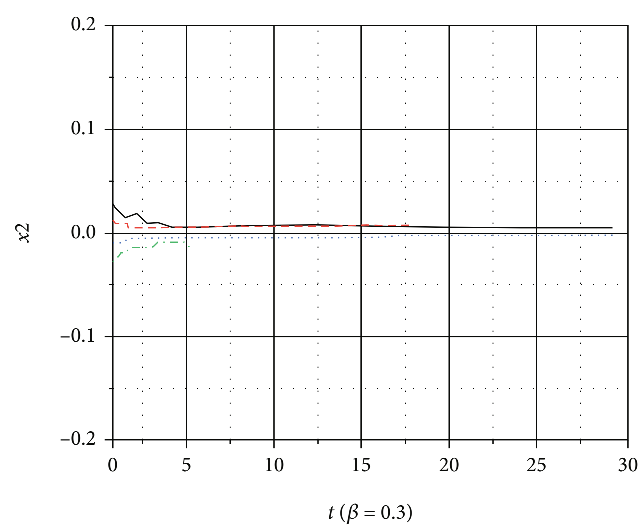

FIGURE 9: For the trajectory graph of different initial values x_2.

In Lemma 3, let $k = 5, \eta = q$, we can get

$$\begin{aligned} \|e(t)\|^q \le{}& 5^{q-1}\|\varphi\|^q + 5^{q-1}\left(\frac{e^{pt}\Gamma(p\beta-p+1)}{\Gamma^p(\beta)p^{p\beta-p+1}}\right)^{q/p} \\ &\cdot \int_0^t (C+AF)^q\|e(\mu)\|^q e^{-q\mu}\mathrm{d}\mu + 5^{q-1}\left(\frac{e^{pt}\Gamma(p\beta-p+1)}{\Gamma^p(\beta)p^{p\beta-p+1}}\right)^{q/p} \\ &\cdot \int_0^t B^qG^q\|e(\mu-\tau)\|^q e^{-q\mu}\mathrm{d}\mu + \frac{5^{q-1}M^qH^qt^{q\beta}e^{pt}}{\beta^q\Gamma^q(\beta)} \\ &\cdot \int_0^t \|e(\mu)\|^q e^{-q\mu}\mathrm{d}\mu + 5^{q-1}M^qH^q\sigma^q\|\varphi\|^q \\ &\cdot \left(\frac{e^{pt}\Gamma(p\beta-p+1)}{\Gamma^p(\beta)p^{p\beta-p+1}}\right)^{q/p}. \end{aligned} \tag{38}$$

Let

$$\tilde{E} = \left[\frac{\Gamma(p\beta-p+1)}{\Gamma^p(\beta)p^{p\beta-p+1}}\right]^{q/p}, \tag{39}$$

get

$$\begin{aligned} \|e(t)\|^q \le{}& \left(5^{q-1} + 5^{q-1}M^qH^q\gamma^q\tilde{E}e^{qt} + \frac{5^{q-1}B^qG^q\tilde{E}(1-e^{-q\gamma})e^{qt}}{q}\right)\|\varphi\|^q \\ &+ \left[5^{q-1}\tilde{E}(C+AF)^q e^{qt} + \frac{5^{q-1}M^qH^qt^{q\beta}e^{qt}}{\beta^q\Gamma^q(\beta)} + 5^{q-1}B^qG^qe^{-q\gamma}\tilde{E}e^{qt}\right] \\ &\cdot \int_0^t \|e(\mu)\|^q e^{-q\mu}\mathrm{d}\mu, \end{aligned} \tag{40}$$

then let

$$\begin{aligned} &\tilde{P} = 5^{q-1}, \tilde{N} = \frac{5^{q-1}M^qH^q}{\beta^q\Gamma^q(\beta)}, \\ &\tilde{Q} = \frac{5^{q-1}B^qG^q\tilde{E}(1-e^{-q\gamma})}{q} + 5^{q-1}M^qH^q\gamma^q\tilde{E}, \\ &\tilde{L} = 5^{q-1}\tilde{E}(C+AF)^q + 5^{q-1}B^qG^qe^{-q\gamma}\tilde{E}, \end{aligned} \tag{41}$$

then

$$\|e(t)\|^q e^{-qt} \le \left(\tilde{P}e^{-qt} + Q\right)\|\varphi\|^q + \left(\tilde{L} + \tilde{N}t^{q\beta}\right)\int_0^t \|e(\mu)\|^q e^{-q\mu}\mathrm{d}\mu. \tag{42}$$

Use Gronwall inequality and make $\tilde{W}(t) = L + \tilde{N}t^{q\beta}$, and get

$$\|e(t)\|^q \le \left[\tilde{P} + \tilde{Q}e^{qt} + \frac{\tilde{W}(t)\tilde{P}\left(e^{(\tilde{w}(t)+q)t} - 1\right)}{q + \tilde{W}(t)} + \frac{\tilde{W}(t)\tilde{Q}e^{(\tilde{w}(t)+q)t}\left(1 - e^{-W(t)t}\right)}{\tilde{W}(t)}\right] \cdot \|\varphi\|^q, \tag{43}$$

which is

$$\|e(t)\| \le \sqrt[q]{\tilde{P} + \tilde{Q}e^{qt} + \frac{\tilde{W}(t)\tilde{P}\left(e^{(\tilde{w}(t)+q)t} - 1\right)}{q + \tilde{W}(t)} + \frac{\tilde{W}(t)\tilde{Q}e^{(\tilde{w}(t)+q)t}\left(1 - e^{-W(t)t}\right)}{\tilde{W}(t)}} \cdot \|\varphi\|. \tag{44}$$

It can be seen that when $\|\varphi\| < \delta$, it is easy to know $\|e(t)\| < \xi$ from Theorem 6. From Definition 5, it can be obtained that the fractional quantum neural system (11) with mixed time delay is quasiuniformly stable.

4. Illustration

In this part, we give a specific example to verify the validity and applicability of the given results.

$$\begin{cases} D^\beta(x_2(t)) = -0.1x_2(t) + 0.1f_1(x_1(t)) - 0.2f_2(x_2(t)) - 0.2g_1(x_1(t-\tau)) - 0.\mathrm{l}g_2(x_2(t-\tau)) + \int_{t-\sigma}^{t} [0.1h_1(x_1(\mu)) + 0.2h_2(x_2(\mu))]d\mu, \\ D^\beta(x_1(t)) = -0.1x_1(t) + 0.2f_1(x_1(t)) - 0.1f_2(x_2(t)) - 0.5g_1(x_1(t-\tau)) - 0.\mathrm{l}g_2(x_2(t-\tau)) + \int_{t-\sigma}^{t} [0.4h_1(x_1(\mu)) - 0.1h_2(x_2(\mu))]d\mu. \end{cases} \tag{45}$$

The activation function in the above formula is:$f_i(x_i(t)) = g_i(x_i(t)) = h_i(x_i(t)) = \mathrm{sigmoid}(x)$, $i = (1, 2 \cdots)$, $F = H = G = 1$.

By $\bar{B} = \begin{pmatrix} -0.5 & -0.1 \\ -0.2 & -0.1 \end{pmatrix}$, $\bar{A} = \begin{pmatrix} 0.2 & -0.1 \\ 0.1 & -0.2 \end{pmatrix}$, $\bar{C} = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}$, and $\bar{M} = \begin{pmatrix} 0.4 & -0.1 \\ 0.1 & 0.2 \end{pmatrix}$, inferred $B = 0.7$, $A = 0.3$, $C = 0.1$, and $M = 0.5$.

In this experiment, the experimental data show that by controlling the corresponding parameters, we can study the influence of another parameter on the trajectory of x_1 and x_2 with different initial values. We set the parameters $\delta = 0.1$, $\sigma = 0.05$, $t_0 = 0$. Set the parameters when β=0.7, find $L = 2.7994$, $Q = 0.2278$, $N = 0.6878$, $W(t) = 3.0884$, $P = 4.9938$, and find $T = 0.6690$ from the following inequality:

$$\sqrt{P + Qe^{2t} + W(t)e^{(W(t)+2)t}\left(P\frac{1 - e^{-(W(t)+2)t}}{W(t)+2} + Q\frac{1 - e^{-W(t)t}}{W(t)}\right)} < \frac{\varepsilon}{\delta}. \tag{46}$$

When $\beta = 0.3$, find $\tilde{L} = 83.7659$, $\tilde{Q} = 9.2269$, $\tilde{N} = 16.9440$, $\tilde{W}(t) = 84.1261$, $\tilde{P} = 213.7471$, $\tilde{E} = 2.4949$. Find $T = 0.0522$ from the following inequality.

Figure 2 is for $\sigma = 0.1$, $\beta = 0.7$, and $x_1(t) = 0.5$. For different τ values ($\tau = 0.1$,0.5,0.8), the corresponding trajectory of x_1. Figure 3 is for $\sigma = 0.1$, $\beta = 0.7$, and $x_2(t) = 0.5$. For different τ values ($\tau = 0.1$, 0.5, 0.8), the corresponding trajectory of x_2. It can be seen that the state trajectories of x_1 and x_2 converge to the equilibrium point.

Figure 4 shows the trajectory of x_1 for $\tau = 1$, $\beta = 0.7$, and the initial value $x_1(t)$ =0.5, for different values of σ=0.05, 0.2, 0.3. Figure 5 shows the trajectory of x_1 for $\tau = 1$, $\beta = 0.7$, the initial value $x_2(t)$ =0.5, for different values of $\sigma = 0.05$,0.2,0.3 the trajectory of x_2. It can be seen that the state trajectories of x_1 and x_2 converge to the equilibrium point.

Figure 6 shows the trajectory of x_1 when $\beta = 0.7$, $\sigma = 0.05$, and $\tau = 0.1$, and the initial value $x_1(t)$ takes different values. Figure 7 shows the trajectory of x_2 when $\beta = 0.7$, $\sigma = 0.05$, and $\tau = 0.1$, and the initial value $x_2(t)$ takes different values. It can be seen that the state trajectories of x_1 and x_2 converge to the equilibrium point.

Figure 8 shows the trajectory of x_1 when $\beta = 0.3$, $\sigma = 0.05$, $\tau = 0.1$, and the initial value $x_1(t)$ takes different values. Figure 9 shows the trajectory of x_2 when $\beta = 0.3$, $\sigma = 0.05$, and $\tau = 0.1$, and the initial value $x_2(t)$ takes different values. It can be seen that the state trajectories of x_1 and x_2 converge to the equilibrium point.

5. Conclusions

This paper uses the linear superposition of multilayer activation functions, uses learning algorithms to adjust quantum intervals and other operations to quantize the neural network, and proposes a quantum neural network model with multilayer activation functions. On this basis, the quasiuniform stability of fractional quantum neural networks with mixed time delays is studied. When β belongs to different ranges, the sufficient conditions for the quasiuniform stability of the fractional quantum neural network system with mixed time delay are, respectively, discussed. Using the corresponding theorem, the proof of the theoretical result is given. Finally, through numerical simulation, the feasibility of the conclusions obtained in this paper is verified.

Data Availability

We did not use the data set in the research of this article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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Research Article

Multiobjective Programming Strategy of Small- and Medium-Sized Microenterprise Credit Based on Random Factors

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In this paper, we select eight indicators from the aspects of an enterprise's bill transaction information, namely, whether the enterprise's loan is in breach of contract, effective invoice rate, total utilization rate of price and tax, negative invoice rate, strength of enterprise, coefficient of variation, flow efficiency of assets, and influence of upstream and downstream enterprises; then, we construct an evaluation index system. According to different industries, different categories, and the impact of random factors, we divide the types of enterprises into 10 categories. Then, we use three kinds of Poisson random numbers to carry out numerical simulation on the total price and tax of enterprises in different industries under the influence of COVID-19.

1. Background

When banks provide loans to small- and medium-sized and microenterprises (small- and medium-sized and microenterprises are abbreviated as SMMEs), they often judge whether to lend or not through credit risk assessment. Because of the lack of mortgage assets in SMMEs, the bank will make credit risk assessment based on the credit policy, influence, strength, and stability of supply and demand relationship of the enterprise, and then determine whether to lend, loan amount, interest rate and term, and other credit strategies. Some corporate banks have credit records, some have no credit records. However, in the face of the impact of sudden random factors on enterprises, how to give the credit strategy when the annual total credit is fixed.

2. The Selection of Credit Risk Quantitative Index

This paper analyzes the relevant data indicators of enterprises with credit records, takes into account the actual situation affecting the credit problems of SMMEs and refers to the advanced international standards, and selects eight quantitative indicators affecting the credit risk of enterprises according to China's national conditions and the bank's credit policy:

(1) Whether the enterprise loan is in breach of contract is an important indicator for the bank to examine whether the enterprise can bring the money. Default is 0 and nondefault is 1. W_i means whether ith enterprise is in breach of contract. $W_i = 0$ means the enterprise defaults, while $W_i = 1$ means that the enterprise has not breached the contract

(2) Effective invoice rate: it is equal to the ratio of the number of valid invoices to the total number of invoices. B_i is used to denote the effective invoice rate of the ith enterprise, YF_i indicates the number of valid invoices for the ith enterprise, and A_i represents the total invoice number of the ith enterprise. Thus, the corresponding formula of the effective invoice rate of the ith enterprise is as follows:

$$B_i = \frac{YF_i}{A_i}. \tag{1}$$

(3) Utilization rate of total price and tax: it is equal to the ratio of the total price and tax of the valid invoice to

the total price and tax of all invoices. Putting β_i represents the utilization rate of the total price and tax of an effective invoice of the ith enterprise

(4) Negative rate of invoice z_i: it is equal to the ratio of the number of invoices of the ith enterprise whose value of the total invoice price and tax is "-" to the number of total invoices of the ith enterprise

(5) EVA_i: it is equal to the ratio of the difference between the total price and tax of the output and input of the ith enterprise to the total price and tax of input, which indicates the strength of the enterprise. Putting S_i represents the total price and tax of the output (sales revenue) of the ith enterprise, and J_i represents the total price and tax of the ith enterprise's input (purchased products), which uses the following corresponding formula:

$$\text{EVA}_i = \frac{S_i - J_i}{J_i}. \tag{2}$$

(6) Coefficient of variation: it indicates the stability of supply and demand relationship of enterprises. Using c_i represents the coefficient of variation of the ith enterprise. x_{ij} represents the total input price and tax of the ith enterprise in the jth month, s_{ij} is the total output value tax of the ith enterprise in the jth month, and I_{ij} represents the net income of the ith enterprise in the jth month. If $x_{ij} = 0$, let us take directly $\text{EVA}_{ij} = 0$. The corresponding formula is follows:

$$\begin{aligned} I_{ij} &= S_{ij} - x_{ij}, \\ \text{EVA}_{ij} &= \frac{I_{ij}}{x_{ij}}, \\ c_i &= \frac{\sqrt{(1/n)\sum_{j=1}^{12}\left(\text{EVA}_{ij} - \overline{\text{EVA}_{ij}}\right)^2}}{(1/n)\sum_{j=1}^{12}\text{EVA}_{ij}}, \end{aligned} \tag{3}$$

where $\overline{\text{EVA}_{ij}}$ represents the average value of EVA_{ij} in 12 months of the ith enterprise

(7) Liquidity efficiency of assets: it refers to the comparative relationship between current assets and current liabilities of SMMEs in the same period, that is, the short-term solvency of SMMEs

The following table shows the asset flow data of the ith enterprise in 12 months, as shown in Table 1.

The net income of the previous month is transferred to the next month as part of the next month's input, which shows the liquidity of funds. The liquidity efficiency of the ith enterprise asset μ_i can be expressed as follows:

$$\mu_i = \frac{m}{12}, \tag{4}$$

where 12 represents 12 months, and m refers to the number greater than 0 in EVA_{ij} of ith enterprise in 12 months. In fact, μ_i is the proportion of the number of months whose value is greater than 0 to the total number of months. The larger the value indicates that the better the flow efficiency of ith enterprise.

(8) Influence of upstream and downstream enterprises v_i: the influence is expressed by the maximum number of effective cooperation between the ith enterprise and upstream and downstream enterprises. In order to quantitatively describe the influence of upstream and downstream enterprises, the influence function of upstream and downstream enterprises is introduced with reference to the negative exponential function of the psychological curve [1]:

$$v_i = 1 - e^{-(n_i)^{1/3}}, \tag{5}$$

where n_i refers to the largest number of input invoice and output invoice of the ith enterprise in 12 months. Following the increase of n_i, the influence of upstream and downstream enterprises v_i will also increase.

In the quantitative index system affecting the credit risk of SMMEs, the first to fourth indexes reflect the reputation of enterprises, the fifth index reflects the strength of enterprises, the sixth index reflects the stability of the supply and demand relationship of enterprises, the seventh index reflects the size of the credit risk of enterprises, and the eighth index reflects the influence of enterprises and upstream and downstream enterprises.

3. Comprehensive Evaluation of Credit Risk Quantitative Index System

In order to eliminate dimension and the positive and negative effects of index, in this paper, the fuzzy membership method is used to standardize the index. Let y_{tj} be the tth index value of the jth evaluation object; w_{tj} be the standardized value of the tth index of the jth evaluation object and n be the number of objects to be evaluated. Then, the positive index standardization formula (6) and the negative index standardization formula (7) can be used to standardize the index [2]:

$$w_{tj} = \frac{y_{tj} - \min_{1\le j\le n} y_{tj}}{\max_{1\le j\le n} y_{tj} - \min_{1\le j\le n} y_{tj}}, \tag{6}$$

$$w_{tj} = \frac{\max_{1\le j\le n} y_{tj} - y_{tj}}{\max_{1\le j\le n} y_{tj} - \min_{1\le j\le n} y_{tj}}. \tag{7}$$

Among the 8 indicators of the quantitative index system affecting credit risk of SMMEs, the fourth indicator (negative invoice rate) and the sixth indicator (enterprise coefficient of variation) are both negative indicators, which need to be

Table 1: Asset flow data of enterprises.

Month	1	2	⋯	12
Total of input price and tax	x_{i1}	x_{i2}	⋯	$x_{i,12}$
Total of output price and tax	s_{i1}	s_{i2}	⋯	$s_{i,12}$
Net income	$I_{i1} = s_{i1} - x_{i1}$	$I_{i2} = I_{i1} + s_{i2} - x_{i2}$	⋯	$I_{i,12} = I_{i,11} + s_{i,12} - x_{i,12}$
EVA_{ij}	I_{i1}/x_{i1}	I_{i2}/x_{i2}	⋯	I_{i12}/x_{i12}

processed with the help of formula (7), while other indicators are calculated with the help of formula (6).

The following uses the entropy weight TOPSIS method to evaluate the credit risk quantitative index system of SMMEs. On the one hand, the entropy weight method is used to determine the coefficient of the credit risk quantitative index system. On the other hand, the TOPSIS method, that is, the technology of approaching the ideal solution, is used to determine the ranking of the evaluated object n SMMEs. The core idea of the TOPSIS method is to define the positive ideal solution and negative ideal solution of the decision problem, and then compare and evaluate the distance between the solution and the positive ideal solution and negative ideal solution, and finally calculate the relative closeness degree between each solution and the ideal solution, and order the advantages and disadvantages of the solution.

3.1. Entropy Weight Method Being Used to Calculate the Objective Weight of Indexes. Set w_{ij} as the normalized value of the jth indicator in the ith system, where $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, 8$. For a given index j, the larger the difference of w_{ij}, the larger the comparative effect of this index has on the system, that means the more information the index contains and transmits.

The specific steps of the entropy method to determine the index weight are as follows:

(i) Calculating the entropy value of 8 indicators such as effective invoice rate. Set e_j as the entropy value of the$j$th index, the solution process is as follows [3]:

$$p_{ij} = \frac{w_{ij}}{\sum_{i=1}^{n} w_{ij}},$$
$$e_j = -\frac{1}{lnn} \sum_{i=1}^{n} p_{ij} \ln p_{ij}, \tag{8}$$

where p_{ij} is the characteristic proportion of the jth index in the ith system, $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, 8$. $\sum_{i=1}^{n} w_{ij}$ are the sum of all system observation data of the jth indicator

(ii) Calculation of the coefficient of variance g_j of the jth index $g_j = 1 - e_j$

(iii) Determine the weight coefficients of 8 indexes $s_j = (g_j / \sum_{j=1}^{8} g_j)$

3.2. Weighting of Standardized Data. Let y_{ij} be the weighted value of the jth index standardized data of the ith SMMEs, w_{ij} be the normalized value of the jth index observed value of the ith SMMEs, and s_j be the weight coefficient. According to the weighting method, it can be seen that

$$y_{ij} = w_{ij} s_j. \tag{9}$$

3.3. Determining the Positive and Negative Ideals of the Evaluation System. Set y_j^+ and y_j^- as the maximum and minimum value of the jth index observation data, respectively, $j = 1, 2, \cdots, 8$:

$$y_j^+ = \max_{1 \le k \le n} y_{kj},$$
$$y_j^- = \min_{1 \le k \le n} y_{kj}. \tag{10}$$

It is easy to know that the positive and negative ideal solutions of the evaluation system are, respectively, $y_j^+ = (y_1^+, y_2^+, \cdots, y_8^+)$ and $y_j^- = (y_1^-, y_2^-, \cdots, y_8^-)$.

3.4. Calculating the Euclidean Distance between the Evaluation System and the Ideal Solution. Let d_i^+ be the Euclidean distance between the weighted value of the ith enterprise and the positive ideal solution and d_i^- be the Euclidean distance between the weighted value of the ith enterprise and the negative ideal solution. Then

$$d_i^+ = \sqrt{(y_1^+ - y_{i1})^2 + (y_2^+ - y_{i2})^2 + \cdots + (y_8^+ - y_{i8})^2},$$
$$d_i^- = \sqrt{(y_1^- - y_{i1})^2 + (y_2^- - y_{i2})^2 + \cdots + (y_8^- - y_{i8})^2}. \tag{11}$$

3.5. Calculating the Relative Closeness Evaluation Result. Set f_i as the relative closeness of all the indexes and the ideal solution of the ith enterprise, then

$$f_i = \frac{d_i^-}{d_i^- + d_i^+}, \tag{12}$$

where $i = 1, 2, \cdots, n$.

Determine the development status of the evaluated index by calculating the closeness. The greater the relative closeness f_i, the closer the evaluated index is to the ideal solution, and the better the development status.

4. Banks' Credit Strategies for SMMEs under Random Factors

Let x_i be the amount of the bank's loan to the ith SMMEs and l_i be the interest rate of the bank's loan to the ith SMMEs. Whether the bank gives loans to ith SMMEs, we use the 0-1 function

$$c_i = \begin{cases} 1, & \text{bank made a loan to the } i\text{th enterprise}, \\ 0, & \text{bank does not lend to the } i\text{th enterprise}. \end{cases} \quad (13)$$

The production, operation, and economic benefits of enterprises may be affected by some unexpected factors, and the size of the impact is related to different industries and different types of enterprises. For example, when COVID-19 became widespread, the demand for medical services and products produced by healthcare companies increased rapidly. With the help of relevant state policies, the total credit amount of banks to such healthcare companies and health enterprises will increase. At the same time, in order to avoid the rapid transmission of COVID-19, the state often needs to cut off some transmission routes. For example, during the outbreak of COVID-19, the state issued policies to close some self-employed small- and medium-sized enterprises, so as to reduce the movement of people and avoid cross-infection caused by too many people. In this regard, banks will reduce the total amount of credit to such self-employed SMMEs to avoid credit risk.

According to different industries, different categories, and the size of the impact, we classify enterprises as follows: self-employed enterprises, trade and transportation industry, literature and art advertising industry, manufacturing industry, service industry, financial investment industry, medical and health industry, high-tech enterprises, catering industry, and other industries.

In order to visually show the impact of credit risk and possible sudden factors on each enterprise, we carry out the numerical fluctuation of the total input price tax and the total output price tax of 10 types of enterprises. According to the actual impact of COVID-19 on society, the total input price and tax and the total output price and tax of the medical and health industry should be increased, while the total input price and tax and the total output price and tax of the individual business should be reduced. The concrete method is to add random numbers (Poisson random numbers) that are divided into three types for simulation.

The first category is to increase the total input and output tax of the medical and health industry, and the total input and output tax are, respectively, J_i and S_i. By adding random number α_i (0~100%), the total input price and tax and the total sales tax after the influence are, respectively $(1+\alpha_i)J_i$ and $(1+\alpha_i)S_i$. In MATLAB software, the function alpha1 = rand (length(location_a), 1) is used to achieve this [4, 5].

In the second category, for self-employed enterprises, the total input price and tax and the total output price and tax of the catering industry are reduced. The original total input price and tax and the total output price and tax are, respectively, J_i and S_i. By adding random number γ_i (-100%~0), the total input price and tax and the total output price and tax are $(1+\gamma_i)J_i$ and $(1+\gamma_i)S_i$. This is achieved with the help of the functiongamma1 = rand (length(location_g), 1).

In the third category, the influence of other industries is relatively small, and the random number φ_i (-50% ~50%) is added and fluctuates randomly, and the original total of the input price and tax and the total of the output price and tax are J_i and S_i, respectively. The total input price and tax and output price and tax are $(1+\varphi_i)J_i$ and $(1+\varphi_i)S_i$. This is achieved with the help of the function phi1 = rand (length(location_p), 1)/2.

5. Multiobjective Planning Strategy of SMME's Credit under Random Factors

When the COVID-19 outbreak occurred, the demand for services and products provided by medical and health enterprises also increased rapidly, and the resulting enterprise profits also increased, so the ability of enterprises to repay loans increased. It is a pity that the profit of the self-employed enterprise is reduced or stagnated, and the ability to repay the loan is weakened. Due to the impact of unexpected factors, the repayment ability is weakened and the bank's income is affected.

5.1. Determination of Objective Function. From the front, we can see that f_i means the comprehensive evaluation score of the ith enterprise out of n enterprises. Let

$$\begin{aligned} f_{\min} &= \min\{f_i\}, \quad i = 1, 2, \cdots n, \\ f_{\max} &= \max\{f_i\}, \quad i = 1, 2, \cdots n. \end{aligned} \quad (14)$$

Let the ith enterprise repayment for the bank loan ratio be τ_i. Taking it here

$$\tau_i = \frac{f_i - f_{\min}}{f_{\max} - f_{\min}}. \quad (15)$$

Thus, the amount of the loan that the ith enterprise can repay is $\tau_i x_i$. To establish the objective function

$$\max \sum_{i=1}^{n} c_i (1 - f_i) l_i \tau_i x_i. \quad (16)$$

On the other hand, the smaller the bank's lending risk, the better. f_i/J_i indicates the unit capital risk of the ith SMMEs, and $x_i(f_i/J_i)$ represents the investment risk brought by the capital flow x_i of the ith SMMEs, and establishes an objective function for this purpose:

$$\min \sum_{i=1}^{n} c_i x_i \frac{f_i}{J_i}. \quad (17)$$

On the other hand, the bank loan amount should take into account the business strength of the enterprise. This paper uses the sample variance index of loan amount and total input price and tax to describe the balance of credit amount, and establishes the objective function for this purpose:

$$\min\ s^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(\frac{x_i}{J_i} - \frac{1}{n}\sum_{i=1}^{n}\frac{x_i}{J_i}\right)^2. \tag{18}$$

5.2. Determination of Constraints

(i) The loan limit of the established bank to the enterprise determined to be loaned is 10-100 (ten thousand), so

$$10 \le x_{\rm i} \le 100, \quad i = 1, 2, \cdots, n. \tag{19}$$

(ii) The annual loan interest rate of the bank to the enterprise determined to lend is 4%~15%. Thus

$$4\% \le l_{\rm i} \le 15\%, \quad i = 1, 2, \cdots, n. \tag{20}$$

(iii) The balance of a bank's investment in enterprises. It's represented by x_i/J_i. The demand for services and products provided by medical and health enterprises is increasing rapidly. Therefore, the total input price and tax of such enterprises should also be increased, and the amount of bank loans to such enterprises should be increased. When the number of self-employed enterprises decreases or stagnates, the total input value and tax should also be reduced, and the amount of bank loans to such enterprises should be reduced. The upper and lower limits of the total balance of input price and tax for medical and health input are adjusted to 0.8 and 2. Considering that an individual business cannot be given a loan completely, the upper and lower limits of the total balance of input price and tax of an individual business are adjusted to 0.3 and 1, the upper and lower limits of other industries remain at 0.5 and 1.5. Set M = "medical enterprise code"; G = "individual enterprise code"; and Q = "all other enterprise codes". We agreed that

$$\begin{aligned} 0.8 \le \frac{x_i}{J_i} \le 2, &\quad i \in M, \\ 0.3 \le \frac{x_i}{J_i} \le 1, &\quad i \in G, \\ 0.5 \le \frac{x_i}{J_i} \le 1.5, &\quad i \in Q. \end{aligned} \tag{21}$$

(iv) Whether the bank loans to the enterprise and the loan amount is consistent, let δ be a very small positive number and M be a very large positive number. The values of 1 and 0 of c_i, respectively, indicate that the bank loans to the ith enterprise and does not lend to the ith enterprise. In order to ensure the consistency of bank loans to the enterprise and the loan amount, there are constraints

$$\delta c_i \le x_i \le M c_i. \tag{22}$$

(v) Total amount of loan. Assuming that the total amount of loan is 100 million when the bank loans to n enterprises, the unit here takes 10000 yuan. We have

$$\sum_{i=1}^{n} x_i = 10^4. \tag{23}$$

6. Example Checking

This paper verifies the multiobjective planning strategy of SMMEs under the influence of COVID-19 by using the related data. The original data of this paper comes from the data of competition question C for CUMCM-2020 (China University mathematical modeling competition), which can be downloaded publicly [6] (http://www.mcm.edu.cn/html_cn/node/10405905647c52abfd6377c0311632b5.html).

Firstly, the Poisson random number is considered, and with the help of the TOPSIS evaluation method, the scores and ranking comparison table of 302 enterprises before and after the introduction of random distribution are obtained [7–17]. The scores and ranking of the top 20 enterprises with enterprise number before and after the introduction of random distribution are shown in Table 2.

It can be seen from Table 2 that, after the introduction of random distribution, the ranking of enterprises with enterprise labels ranging from 1 to 20 changed correspondingly—-some changed greatly, while some changed less—indicating that our model has good practicability.

After the introduction of random distribution, the changes in scores and rankings.

of the top 20 enterprises among the 302 enterprises are shown in Table 3.

As can be seen from Table 3, after the introduction of random distribution, the number of the top 20 enterprises is basically still in the top 20, indicating that our comprehensive evaluation method is relatively good and the ranking distribution is relatively stable.

From Table 4, we can see the ranking changes of enterprises in the case of occurrence of emergent factors and absence of emergent factors. It can be found that under the influence of COVID-19, the rating and ranking of enterprises in the medical and health industry have increased, indicating that under the influence of COVID-19, such enterprises have a good credit situation and a low credit risk. However, the decline in the score and ranking of self-employed enterprises indicates that under the influence of COVID-19, the credit situation of such enterprises is poor and the credit risk is high, which is in line with the actual situation. It indicates that our TOPSIS evaluation method is effective and can be better applied to the situation when random factors occur.

When the total annual credit of the bank is 100 million yuan, we use the data given in the attached table of question C to establish the multiobjective programming model of 302 enterprises.

Table 2: Comparison of scores and rankings of the top 20 enterprises with enterprise numbers before and after the introduction of random distribution.

The score results of entropy weight method before introducing a random distribution			The score results of entropy weight method after introducing a random distribution		
Enterprise numbers	Score f_i	Ranking	Enterprise numbers	Score f_i	Ranking
1	0.03470546	133	1	0.01051545	258
2	0.03473871	132	2	0.01009714	271
3	0.26589653	77	3	0.2664342	52
4	0.29959602	8	4	0.36079418	3
5	0.26882309	21	5	0.31805003	22
6	0.2663194	61	6	0.04038756	92
7	0.26669382	46	7	0.01158391	189
8	0.26632932	59	8	0.0171994	112
9	0.26681615	40	9	0.01643889	114
10	0.26611135	72	10	0.11524284	72
11	0.26909606	19	11	0.32030139	17
12	0.2667195	44	12	0.31590827	32
13	0.26702299	31	13	0.01160196	187
14	0.26627581	63	14	0.11280042	75
15	0.26699131	33	15	0.31609219	31
16	0.27680573	15	16	0.32686684	12
17	0.2469376	90	17	0.29235498	47
18	0.26674657	42	18	0.31711769	26
19	0.26660041	50	19	0.31528929	40
20	0.24583911	93	20	0.24091041	58

Table 3: Changes in scores and rankings of the top 20 companies after the introduction of random distribution.

The score results of entropy weight method before introducing a random distribution			The score results of entropy weight method after introducing a random distribution		
Enterprise numbers	Score f_i	Ranking	Enterprise numbers	Score f_i	Ranking
206	0.65353762	1	206	0.74570025	1
30	0.51530701	2	30	0.55732306	2
4	0.36079418	3	237	0.33358854	3
107	0.34892084	4	235	0.31445308	4
92	0.34436155	5	89	0.31144136	5
89	0.34363768	6	107	0.30352414	6
76	0.34281575	7	220	0.30110706	7
220	0.34233996	8	4	0.29959602	8
122	0.33108545	9	92	0.2928879	9
26	0.32893502	10	76	0.29120591	10
62	0.32855488	11	122	0.28217424	11
16	0.32686684	12	26	0.28128101	12
38	0.32600357	13	38	0.28008684	13
45	0.32299682	14	62	0.27817557	14
110	0.32166147	15	16	0.27680573	15
33	0.32044747	16	45	0.27234886	16
11	0.32030139	17	33	0.27128483	17
53	0.3194912	18	110	0.27112156	18
111	0.31946501	19	11	0.26909606	19
63	0.31856213	20	63	0.26895159	20

Table 4: Changes in medical and individual business scores and rankings after the introduction of random distribution.

The score results of entropy weight method before introducing a random distribution			The score results of entropy weight method after introducing a random distribution		
Enterprise numbers	Score f_i	Ranking	Enterprise numbers	Score f_i	Ranking
E195 (medical)	0.266610	48	E195 (medical)	0.314380	44
E398 (medical)	0.014558	189	E398 (medical)	0.012037	166
E420 (medical)	0.014021	227	E420 (medical)	0.013773	184
E373 (individual)	0.014911	162	E373 (individual)	0.009712	279
E124 (individual)	0.034705	133	E124 (individual)	0.010516	258
E125 (individual)	0.034739	132	E125 (individual)	0.010516	271

Table 5: Analysis of different results obtained by different scale coefficients of the three objective functions.

Plan	1	2	3	4	5	6	7	8
$u(1)$	0.7	0.6	0.6	0.5	0.6	0.5	0.4	0.3
$u(2)$	0.2	0.2	0.1	0.25	0.15	0.15	0.15	0.15
$u(3)$	0.1	0.2	0.3	0.25	0.25	0.35	0.45	0.55
The number of different values of the loan amount	7	9	9	8	11	17	19	20

The multiobjective function includes the following: max $\sum_{i=1}^{302} c_i(1-f_i)l_i\tau_i x_i$, min $\sum_{i=1}^{302} c_i x_i(f_i/J_i)$, and

$$\min\ s^2 = \frac{1}{301}\sum_{i=1}^{302}\left(\tau_i\frac{x_i}{J_i} - \frac{1}{302}\sum_{i=1}^{302}\tau_i\frac{x_i}{J_i}\right)^2. \tag{24}$$

The constraint conditions are as follows:

$$\begin{cases} 10 \le x_i \le 100, \\ 4\% \le l_i \le 15\%, \\ \tau_i = \dfrac{f_i - f_{\min}}{f_{\max} - f_{\min}}, \\ 0.8 \le \dfrac{x_i}{J_i} \le 2, \quad i \in M \\ \delta c_i \le x_i \le M c_i, \\ \sum_{k=1}^{302} x_i = 10^4. \end{cases}, \tag{25}$$

The software programming of Lingo is used to solve the above multiobjective function programming model [18]. In the three target functions, let $u(j), (j = 1, 2, 3)$ be the scale coefficient of the jth objective function, which satisfies

$$u(1) + u(2) + u(3) = 1. \tag{26}$$

Set three different proportional coefficients and get different results of different loan amounts, which are analyzed in the following list (see Table 5).

According to the analysis in Table 5, the balance of credit amount is important for banks. Finally, the eighth plan is selected to obtain the specific credit plan for 302 SMMEs, as shown in Table 6 below. The loan amount of enterprises not listed in Table 6 is 100,000 yuan.

It can be found that during the COVID-19 epidemic, due to the rapid increase in the demand for the services and products provided by the medical and health enterprises, the investment of the finally obtained bank in this industry also increased, the sales of the self-employed industry decreased or stagnated, and the investment of the finally obtained bank in this industry also decreased or stagnated. The investment of banks in other industries is also adjusted accordingly to maintain the survival and operation of the industry, which is more consistent with the actual situation and demonstrates the effectiveness and practicability of our model.

7. Sensitivity Analysis of the Model

The sensitivity of the model is used to analyze the sensitivity and stability of the model. In order to test the stability and effectiveness of the multiobjective programming strategy model, sensitivity analysis is carried out.

In the case of COVID-19, 3 random Poisson numbers were added to simulate the total value of the total tax and total sales tax in each industry. In order to fully demonstrate the sensitivity of the model, we changed the number of algorithm runs and record the average scores of the 302 comprehensive evaluation scores obtained by each algorithm; then, the average score of the comprehensive evaluation of the 20 algorithms was obtained and drawn. The results are shown in Figure 1.

By checking whether the average value of the comprehensive evaluation score is stable and centralized when the algorithm runs for 20 times, we can verify whether the model is stable. One can find that the average value of the comprehensive evaluation score of 20 times is relatively centralized and stable, floating in a certain range. It can be seen that our model is stable and practical.

Table 6: List of the specific amount of 100 million yuan loan from the bank to 302 SMMEs (yuan).

Enterprise number	Loan amount
133	880344.8
147	548104
149	864340.1
166	532681.5
167	867680.4
171	150709.4
173	726597.9
178	836082.4
179	646818.7
182	425921
187	402967.4
190	414155.8
191	735308.3
193	338920
198	732913.7
205	313454.6
211	798931.5
242	184068.4
1	1000000
2	1000000
212	1000000
296	1000000
39	1000000
58	1000000
100	1000000
101	1000000
103	1000000
109	1000000
124	1000000
125	1000000
126	1000000
127	1000000
128	1000000
129	1000000
130	1000000
131	1000000
132	1000000
134	1000000
135	1000000
136	1000000
137	1000000
138	1000000
229	1000000
3	1000000
139	1000000
140	1000000
141	1000000
142	1000000

Table 6: Continued.

Enterprise number	Loan amount
145	1000000
148	1000000
150	1000000
153	1000000
154	1000000
155	1000000
156	1000000
157	1000000
158	1000000
159	1000000
160	1000000
161	1000000
162	1000000
163	1000000
164	1000000
165	1000000
250	1000000
4	100000
169	1000000
170	1000000
174	1000000
175	1000000
176	1000000
177	1000000
181	1000000
183	1000000
184	1000000
185	1000000
186	1000000
189	1000000
196	1000000
197	1000000
199	1000000
200	1000000
201	1000000
202	1000000
203	1000000
204	1000000
258	1000000
5	100000

In this paper, the TOPSIS evaluation method is firstly used to get the score and ranking comparison of 302 enterprises before and after the introduction of a random number, and the ranking results are analyzed. Then, considering that different industries each have a different ability to repay loans when affected by COVID-19, the ratio factor that can repay bank loans is introduced, and considering the floating

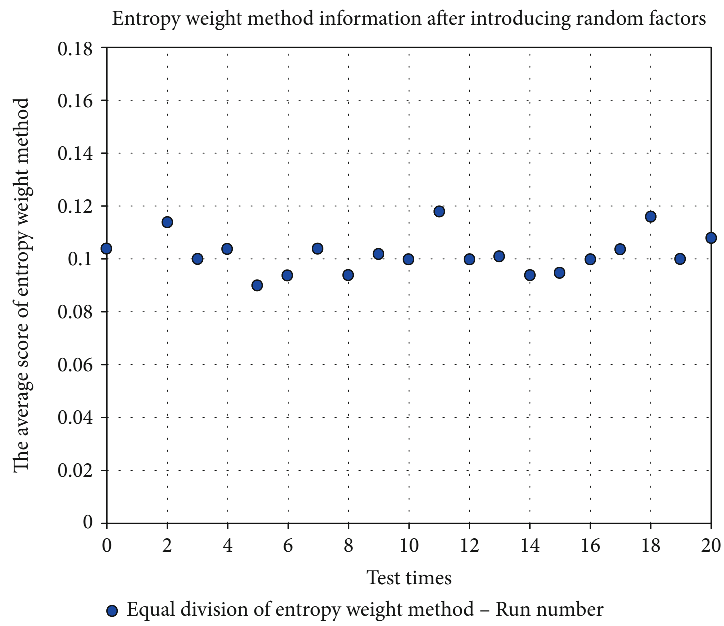

Figure 1: Sensitivity inspection chart.

amount of loans in different industries under national conditions and policies, the objective function and constraints of the multiobjective credit optimization model are modified, and the multiobjective credit optimization model of enterprises influenced by COVID-19 is established. When the total amount is 100 million, the corresponding credit decision is made. Finally, sensitivity analysis is carried out to test the stability and effectiveness of the multiobjective programming strategy model.

Data Availability

The original data of this paper comes from the data of competition question C for CUMCM-2020 (China University mathematical modeling competition), which can be downloaded publicly. Download from the following website: http://www.mcm.edu.cn/html_cn/node/10405905647c52abfd6377c0311632b5.html. The later data used to support the findings of this study are included within the supplementary information file(s).

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Supplementary Materials

We have uploaded three attachments. Annex 1.1: raw data on 302 enterprises without credit records which comes from the data of competition question C for CUMCM-2020. Annex 2: in this paper, we get the relevant data of 302 enterprises before and after the introduction of random distribution. Annex 3: Ligon Program Source Code. *(Supplementary Materials)*

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Research Article

Impulsive Fractional Semilinear Integrodifferential Equations with Nonlocal Conditions

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This paper is devoted to a class of impulsive fractional semilinear integrodifferential equations with nonlocal initial conditions. Based on the semigroup theory and some fixed point theorems, the existence theory of PC-mild solutions is established under the condition of compact resolvent operator. Furthermore, the uniqueness of PC-mild solutions is proved in the case of the noncompact resolvent operator.

1. Introduction

The fractional evolution equation has been applied to many fields, and scholars have obtained abundant research achievements [1–13]. Impulsive fractional integrodifferential equations can describe some phenomena which often occur in physics, geology, and economics, for instance, earthquake, the closing of the switch in the circuit, and so on. Many scholars are committed to this subject and have achieved plentiful results [1–7]. Based on the fact that nonlocal initial conditions are more effective than classical initial conditions in applied physics, the study of differential equations with nonlocal conditions has attracted more and more researchers' attention [8–13].

Ji and Li [14] studied the following impulsive differential evolution equations with nonlocal conditions:

$$\begin{cases} u'(t) = Au(t) + f(t, u(t)), & t \in [0, b], t \neq t_i, \\ \Delta u|_{t=t_i} = I_i(u(t_i)), & i = 1, 2, \cdots, m, \\ u(0) = g(u), \end{cases} \tag{1}$$

where A is the generator of a strongly continuous semigroup $T_\beta(t)$; sufficient conditions for the existence of mild solutions have been established by the Hausdorff measure of noncompactness and fixed point theorems.

Zhu et al. [15] investigated the fractional semilinear integrodifferential equations of mixed type with nonlocal conditions:

$$\begin{cases} {}^cD_t^\beta u(t) = A(t)u(t) + f(t, u(t), \mathcal{G}u(t), \mathcal{S}u(t)), & t \in [0, T_0], \\ u(0) + g(u) = u_0, \end{cases} \tag{2}$$

where $0 < \beta \leq 1$, $A(t)$ is a closed linear operator with domain $D(A)$ defined on a Banach space E; the existence and uniqueness of mild solutions have been established by k-set contraction and β-resolvent family.

Gou and Li [16] studied the fractional impulsive integrodifferential equations in Banach space E; local and global existences of mild solutions have been proved by measure of noncompactness and Sadovskii's fixed point theorem:

$$\begin{cases} {}^cD_t^\beta u(t) + Au(t) = f(t, u(t)) + \int_0^t q(t-s)g(s, u(s))ds, & t \geq 0, t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k^-)), & k = 1, 2, \cdots, m, \\ u(0) = u_0 \in E, \end{cases} \tag{3}$$

where $0 < \beta < 1$, $A : D(A) \subset E \longrightarrow E$ is a closed linear operator and $-A$ generates a uniformly bounded C_0-semigroup $T(t)$.

Inspired by these contributions, we consider the following impulsive fractional semilinear integrodifferential equations with nonlocal initial conditions:

$$\begin{cases} {}^cD_t^\beta x(t) - A(t)x(t) = f(t, x(t), (\mathscr{G}x)(t), (\mathscr{H}x)(t)), & t \in [0, T], t \neq t_k, \\ x(0) + w(x) = x_0, \\ \Delta x(t_k) = I_k(x(t_k)), \quad k = 1, 2, \cdots, m, \end{cases} \tag{4}$$

where ${}^cD_t^\beta$ is the Caputo's fractional derivative of order β, $\beta \in (0, 1]$, $A(t)$ is a closed linear operator with domain $D(A)$ defined on a Banach space E, and two integral operators $\mathscr{G}$ and $\mathscr{H}$ are defined by

$$\begin{aligned} \mathscr{G}x(t) &= \int_0^t g(t, s, x(s))ds, \\ \mathscr{H}x(t) &= \int_0^T h(t, s, x(s))ds, \end{aligned} \tag{5}$$

$g : B \times E \longrightarrow E$, $h : B_0 \times E \longrightarrow E$ are continuous and nonlinear functions, $B = \{(t, s) \mid 0 \le s \le t \le T\}$, $B_0 = \{(t, s) \mid 0 \le t, s \le T\}$, f and ω are to be specified later, $I_k : E \longrightarrow E(k = 1, 2, \cdots, m)$ are continuous impulsive functions, the prefixed numbers $t_k (k = 1, 2, \cdots, m)$ satisfy $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T$, $x(t_k) = x(t_k^-)$, and $x(t_k^-) = \lim_{h \longrightarrow 0^-} x(t_k + h)$ represent the left limit of $x(t)$ at $t = t_k$.

In this paper, we demonstrate the existence of PC-mild solutions for problem (4) via the theory of semigroup and fixed point theorem under the condition of compact resolvent operator. Meanwhile, the uniqueness of PC-mild solutions is proved in the case of noncompact resolvent operator. The kernels g and h of the integral operators $\mathscr{G}$ and $\mathscr{H}$ are nonlinear functions; the function ω of the nonlocal conditions is noncompact. In addition, the closed linear operator $A(t)$ is dependent on t. The rest of this paper is organized as follows. In Section 2, some basic definitions and lemmas are collected that will be needed throughout the remaining sections. The existence and uniqueness of PC-mild solutions are shown in Section 3 via the theories of resolvent operators and various fixed point theorems. Finally, the summary of our results comes in Section 4.

2. Preliminaries

Let $(E, \|\cdot\|)$ be a Banach space, $J = [0, T]$ and $0 < T < \infty$. The collection of all continuous functions from J into E, denoted $C(J, E)$, is a Banach space equipped with the norm $\|x\|_C = \max \{\|x(t)\|, t \in J\}$ for $x \in C(J, E)$. Let $PC(J, E) = \{x \mid x : J \longrightarrow E : x \in C((t_k, t_{k+1}], E)$, and there exist $x(t_k^-)$ and $x(t_k^+)$ with $x(t_k) = x(t_k^-), k = 1, \cdots, m\}$ endowed with the PC-norm $\|x\|_{PC} = \sup \{\|x(t)\|, t \in J\}$, $J_0 = [0, t_1]$, $J_1 = (t_1, t_2]$, $\cdots$, $J_m = (t_m, T]$.

Lemma 1 (nonlinear alternative for single-valued maps). *Let E be a Banach space, $C \subset E$ be a closed convex set, V be an open subset of C, and $0 \in V$. Suppose that $Q : \bar{V} \longrightarrow C$ is completely continuous, then either*

(i) Q has a fixed point in $\bar{V}$ or

(ii) there is a $u \in \partial V$ and $\lambda \in (0, 1)$ with $u = \lambda Q(u)$

Lemma 2 (see [17]). *Let $0 < \eta < 1$, $\gamma > 0$, denote*

$$S_n = \eta^n + C_n^1 \eta^{n-1} \gamma + \frac{C_n^2 \eta^{n-2}}{2!} \gamma^2 + \cdots + \frac{\gamma^n}{n!}, \quad n \in N, \tag{6}$$

where $C_n^k = n!/(k!(n-k)!)$. Then, for any fixed constant $0 < \xi < 1$ and any real number $s > 1$, we get

$$S_n \le O\left(\frac{\xi^n}{\sqrt{n}}\right) + o\left(\frac{1}{n^s}\right) = o\left(\frac{1}{n^s}\right), \text{as } n \longrightarrow \infty. \tag{7}$$

Definition 3 (see [18, 19]). The Caputo fractional derivative of order β of a function $f : (0,\infty) \longrightarrow R$ is defined as

$${}^cD_t^\beta f(t) = \frac{1}{(\Gamma(n-\beta))} \int_0^t (t-s)^{n-\beta-1} f^{(n)}(s)ds, \tag{8}$$

where $n - 1 < \beta < n$, $n \in N$, $\Gamma(\cdot)$ denotes the Gamma function. The Laplace transform of the Caputo fractional derivative of order β is given as

$$\mathscr{L}\left({}^cD_t^\beta f(t)\right)(s) = s^\beta (\mathscr{L}f)(s) - \sum_{j=1}^{n-1} s^{\beta-j-1} x^{(j)}(0), \quad n - 1 < \beta \le n, \tag{9}$$

where $(\mathscr{L}f)(s) = \int_0^\infty e^{-st} f(t)dt$ is the Laplace transform of the function $f(t)$.

Definition 4 (see [20, 21]). Let $A(t)$ be a closed and linear operator with domain $D(A)$ defined on a Banach space E and $\beta > 0$. Let $\rho[A(t)]$ be the resolvent set of $A(t)$; $A(t)$ is called the generator of a β-resolvent family if there exist $\omega \ge 0$ and a strongly continuous function $U_\beta : \mathbb{R}_+^2 \longrightarrow B(E)$ such that $\{\lambda^\beta : \operatorname{Re} \lambda > \omega\} \subset \rho(A)$ and

$$\left(\lambda^\beta I - A(s)\right)^{-1} x = \int_0^\infty e^{-\lambda(t-s)} U_\beta(t, s)x dt, \quad \operatorname{Re}(\lambda) > \omega, x \in E. \tag{10}$$

In this case, $U_\beta(t, s)$ is called the β-resolvent family generated by $A(t)$.

Lemma 5 (see [21, 22]). *$U_\beta(t, s)$ satisfies the following properties:*

(i) $U_\beta(s,s)=I,\ U_\beta(t,s)=U_\beta(t,r)U_\beta(r,s)$, *for* $0\le s\le r\le t\le a$

(ii) $(t,s)\longrightarrow U_\beta(t,s)$ *is strongly continuous for* $0\le s\le t\le a$

(iii) If $U_\beta(t,s)$ *is compact for* $t,s>0$, *then the* $U_\beta(t,s)$ *is continuous in the uniform operator topology*

Definition 6. A function $x\in PC(J,E)$ is said to be a PC-mild solution of problem (4) if $x(t)$ satisfies the integral equation:

$$x(t)=U_\beta(t,0)(x_0-\omega(x))+\int_0^t U_\beta(t,s)f(s,x(s),(\mathcal{G}x)(s),(\mathcal{H}x)(s))ds+\sum_{0<t_k<t}U_\beta(t,t_k)I_k(x(t_k)),\quad t\in J. \tag{11}$$

3. Existence and Uniqueness of Mild Solution

Theorem 7. *Assume that the conditions* (H_1)-(H_3) *hold true and the resolvent operator* $U_\beta(t,s)(t,s>0)$ *is compact.*

(H_1) *The function* $f:J\times E\times E\times E\longrightarrow E$ *is continuous, and there exist nonnegative Lebesgue integrable functions a,* $l_i\in L(J,R_+)(i=1,2,3)$, *for every* $t\in J$, $x_i\in E$, *such that*

$$\|f(t,x_1,x_2,x_3)\|\le a(t)+l_1(t)\|x_1\|+l_2(t)\|x_2\|+l_3(t)\|x_3\|. \tag{12}$$

(H_2) *There exist nonnegative Lebesgue integrable functions b, c,* $l_i\in L(J,\mathbb{R}_+)(i=4,5)$, *for all* $x\in E$, *such that*

$$\begin{aligned}\|g(t,s,x)\|&\le b(t)+l_4(t)\|x\|,\quad (t,s)\in B,\\ \|h(t,s,x)\|&\le c(t)+l_5(t)\|x\|,\quad (t,s)\in B_0.\end{aligned} \tag{13}$$

(H_3) *The functions* $\omega:PC(J,E)\longrightarrow E$ *and* $I_k:E\longrightarrow E$ *are continuous, and there exist constants* d_ω, e_ω, d_k, $e_k>0$, *such that*

$$\begin{aligned}\|\omega(x)\|&\le d_\omega\|x\|_{PC}+e_\omega,\quad x\in PC(J,E),\\ \|I_k(x)\|&\le d_k\|x\|+e_k,\quad x\in E,k=1,2\cdots m.\end{aligned} \tag{14}$$

Then, problem (4) has at least one PC-mild solution in $PC(J,E)$.

Proof. Let us consider the operator $Q:\mathrm{PC}(J,E)\longrightarrow \mathrm{PC}(J,E)$ as follows:

$$(Qx)(t)=U_\beta(t,0)(x_0-\omega(x))+\int_0^t U_\beta(t,s)f(s,x(s),(\mathcal{G}x)(s),(\mathcal{H}x)(s))ds+\sum_{0<t_k<t}U_\beta(t,t_k)I_k(x(t_k)). \tag{15}$$

It is easy to see that the operator Q is well defined in PC (J,E).

At first, we claim that $Q:\mathrm{PC}(J,E)\longrightarrow\mathrm{PC}(J,E)$ is a continuous operator. Let $\{x_n\}_0^\infty\subset\mathrm{PC}(J,E)$ be a sequence such that $x_n\longrightarrow x(n\longrightarrow\infty)$ in $\mathrm{PC}(J,E)$. Since for all $t\in J$,

$$\begin{aligned}\|(Qx_n)(t)-(Qx)(t)\|&\le\left\|U_\beta(t,0)(\omega(x_n)-\omega(x))\right\|\\&+\left\|\int_0^t U_\beta(t,s)(f(s,x_n(s),(\mathcal{G}x_n)(s),(\mathcal{H}x_n)(s))-f(s,x(s),(\mathcal{G}x)(s),(\mathcal{H}x)(s)))ds\right\|\\&+\sum_{k=1}^m\left\|U_\beta(t,t_k)(I_k(x_n(t_k))-I_k(x(t_k)))\right\|\\&\le M\|\omega(x_n)-\omega(x)\|+M\int_0^t\|f(s,x_n(s),(\mathcal{G}x_n)(s),\\&\cdot(\mathcal{H}x_n)(s))-f(s,x(s),(\mathcal{G}x)(s),(\mathcal{H}x)(s))\|ds\\&+\sum_{k=1}^m M\|I_k(x_n(t_k))-I_k(x(t_k))\|,\end{aligned} \tag{16}$$

where $M=\max_{0\le s\le t\le T}\|U_\beta(t,s)\|$. Using the fact that $f:J\times E\times E\times E\longrightarrow E$, $\omega:\mathrm{PC}(J,E)\longrightarrow E$, and $I_k:E\longrightarrow E(k=1,2\cdots m)$ are continuous, we obtain

$$\|Qx_n-Qx\|_{\mathrm{PC}}\longrightarrow 0,\ \text{as}\ n\longrightarrow\infty. \tag{17}$$

Therefore, $Q:\mathrm{PC}(J,E)\longrightarrow\mathrm{PC}(J,E)$ is continuous.

Furthermore, for any $R>0$, we prove that $Q(T_R)$ is equicontinuous in $J_k(k=0,1,2\cdots m)$. For all $x\in T_R=\{x\in\mathrm{PC}(J,E):\|x\|_{\mathrm{PC}}\le R\}$ and $\tau_1,\tau_2\in J_k(\tau_1\le\tau_2)$, by the condition (H_3), we have

$$\begin{aligned}\|(Qx)(\tau_2)-(Qx)(\tau_1)\|&\le\left\|U_\beta(\tau_2,0)-U_\beta(\tau_1,0)\right\|\|x_0-\omega(x)\|\\&+\left\|\int_{\tau_1}^{\tau_2}U_\beta(\tau_2,s)f(s,x(s),(\mathcal{G}x)(s),(\mathcal{H}x)(s))ds\right\|\\&+\left\|\int_0^{\tau_1}(U_\beta(\tau_2,s)-U_\beta(\tau_1,s))f(s,x(s),(\mathcal{G}x)(s),(\mathcal{H}x)(s))ds\right\|\\&+\left\|\sum_{0<t_k<\tau_2}U_\beta(\tau_2,t_k)I_k(x(t_k))-\sum_{0<t_k<\tau_1}U_\beta(\tau_1,t_k)I_k(x(t_k))\right\|\\&\le\left\|U_\beta(\tau_2,0)-U_\beta(\tau_1,0)\right\|(\|x_0\|+d_\omega R+e_\omega)\\&+\int_{\tau_1}^{\tau_2}M\|f(s,x(s),(\mathcal{G}x)(s),(\mathcal{H}x)(s))\|ds\\&+\sup_{s\in J_k}\left\|U_\beta(\tau_2,s)-U_\beta(\tau_1,s)\right\|\\&\cdot\int_0^{\tau_1}\|f(s,x(s),(\mathcal{G}x)(s),(\mathcal{H}x)(s))\|ds\\&+\sum_{k=1}^m\left\|U_\beta(\tau_2,t_k)-U_\beta(\tau_1,t_k)\right\|\|I_k(x(t_k))\|=:I_1+I_2+I_3+I_4.\end{aligned} \tag{18}$$

For all $x \in T_R$, $s \in J$, we get $\|x(s)\| \leq R$, by the condition (H_2),

$$\begin{aligned}\|(\mathscr{G}x)(s)\| &\leq \int_0^s \|g(s,\nu,x(\nu))\|d\nu \leq \int_0^s (b(\nu)+l_4(\nu)\|x(\nu)\|)d\nu \\ &\leq \int_0^T b(\nu)d\nu + R\int_0^T l_4(\nu)d\nu,\end{aligned} \tag{19}$$

meanwhile,

$$\begin{aligned}\|(\mathscr{H}x)(s)\| &\leq \int_0^T \|h(s,\nu,x(\nu))\|d\nu \leq \int_0^T (c(\nu)+l_5(\nu)\|x(\nu)\|)d\nu \\ &\leq \int_0^T c(\nu)d\nu + R\int_0^T l_5(\nu)d\nu.\end{aligned} \tag{20}$$

According to the condition (H_1) and the above inequalities, for all $s \in J$, we get

$$\begin{aligned}&\|f(s,x(s),(\mathscr{G}x)(s),(\mathscr{H}x)(s))\| \\ &\quad\leq a(s)+l_1(s)\|x(s)\|+l_2(s)\|(\mathscr{G}x)(s)\|+l_3(s)\|(\mathscr{H}x)(s)\| \\ &\quad\leq a(s)+l_1(s)R+l_2(s)\left(\int_0^T b(\nu)d\nu+R\int_0^T l_4(\nu)d\nu\right) \\ &\qquad + l_3(s)\left(\int_0^T c(\nu)d\nu + R\int_0^T l_5(\nu)d\nu\right) \leq a_1(s)+b_1(s)R,\end{aligned} \tag{21}$$

where

$$\begin{aligned}a_1(s) &= a(s)+l_2(s)\int_0^T b(\nu)d\nu + l_3(s)\int_0^T c(\nu)d\nu, \\ b_1(s) &= l_1(s)+l_2(s)\int_0^T l_4(\nu)d\nu + l_3(s)\int_0^T l_5(\nu)d\nu.\end{aligned} \tag{22}$$

Obviously, $a_1(s)$ and $b_1(s)$ are nonnegative Lebesgue integrable functions, then

$$\begin{aligned}I_2 &= \int_{\tau_1}^{\tau_2} M\|f(s,x(s),(\mathscr{G}x)(s),(\mathscr{H}x)(s))\|ds \\ &\leq M\int_{\tau_1}^{\tau_2}(a_1(s)+b_1(s)R)ds,\end{aligned}$$

$$\begin{aligned}I_3 &= \sup_{s\in J_k}\left\|U_\beta(\tau_2,s)-U_\beta(\tau_1,s)\right\| \\ &\quad\cdot\int_0^{\tau_1}\|f(s,x(s),(\mathscr{G}x)(s),(\mathscr{H}x)(s))\|ds \\ &\leq \sup_{s\in J_k}\left\|U_\beta(\tau_2,s)-U_\beta(\tau_1,s)\right\|\int_0^{\tau_1}(a_1(s)+b_1(s)R)ds,\end{aligned}$$

$$\begin{aligned}I_4 &= \sum_{k=1}^{m}\left\|U_\beta(\tau_2,t_k)-U_\beta(\tau_1,t_k)\right\|\|I_k(x(t_k))\| \\ &\leq \sum_{k=1}^{m}\left\|U_\beta(\tau_2,t_k)-U_\beta(\tau_1,t_k)\right\|(d_kR+e_k).\end{aligned} \tag{23}$$

In view of Lemma 5, the compactness of the resolvent operator $U_\beta(t,s)(t,s>0)$ implies the continuity in the uniform operator topology. As a result, from the above inequalities, we deduce that $\|(Qx)(\tau_2)-(Qx)(\tau_1)\| \longrightarrow 0$ independently of $x \in T_R$ as $\tau_2-\tau_1 \longrightarrow 0$. That is, $Q(T_R)$ is equicontinuous in $J_k(k=0,1,2\cdots m)$.

In the end, we demonstrate that $Q(T_R) \subset \mathrm{PC}(J,E)$ is precompact.

For any $t(0<t\leq T)$, $0<\varepsilon<t$, and $x \in T_R$, the operator $Q_\varepsilon x$ is defined by

$$\begin{aligned}(Q_\varepsilon x)(t) &= U_\beta(t,0)(x_0-\omega(x)) \\ &\quad + \int_0^{t-\varepsilon} U_\beta(t,s)f(s,x(s),(\mathscr{G}x)(s),(\mathscr{H}x)(s))ds \\ &\quad + \sum_{0<t_k<t} U_\beta(t,t_k)I_k(x(t_k)), \quad t\in J.\end{aligned} \tag{24}$$

Since $U_\beta(t,s)$ is compact resolvent operator, the set $Y_\varepsilon(t)=\{(Q_\varepsilon x)(t)\colon x\in T_R\}$ is relatively compact in E for every ε $(0<\varepsilon<t)$.

Moreover, for any $x \in T_R$, $t \in J$, one can show that

$$\begin{aligned}&\|(Qx)(t)-(Q_\varepsilon x)(t)\| \\ &\quad= \left\|\int_{t-\varepsilon}^t U_\beta(t,s)f(s,x(s),(\mathscr{G}x)(s),(\mathscr{H}x)(s))ds\right\| \\ &\quad\leq M\int_{t-\varepsilon}^t \|f(s,x(s,(\mathscr{G}x)(s),(\mathscr{H}x)(s))\|ds \\ &\quad\leq M\int_{t-\varepsilon}^t (a_1(s)+b_1(s)R)ds.\end{aligned} \tag{25}$$

Thus, $Y(t)=\{(Qx)(t)\colon x\in T_R\}$ is totally bounded. Hence, $Y(t)$ is relatively compact in E, and so, with the help of the Arzelà-Ascoli theorem, $Q:\mathrm{PC}(J,E)\longrightarrow\mathrm{PC}(J,E)$ is completely continuous.

For $0<\lambda<1$, let $x=\lambda(Qx)$, we get

$$\begin{aligned}x(t) &= \lambda U_\beta(t,0)(x_0-\omega(x)) \\ &\quad + \lambda\int_0^t U_\beta(t,s)f(s,x(s),(\mathscr{G}x)(s),(\mathscr{H}x)(s))ds \\ &\quad + \lambda\sum_{0<t_k<t} U_\beta(t,t_k)I_k(x(t_k)).\end{aligned} \tag{26}$$

Then, using the conditions (H_1)-(H_3), it follows that

$$\begin{aligned}\|x(t)\| &\leq M\|x_0-\omega(x)\| + M\int_0^t \|f(s,x(s),\\ &\quad\cdot(\mathscr{G}x)(s),(\mathscr{H}x)(s))\|ds + \sum_{k=1}^m M\|I_k(x(t_k))\|\\ &\leq M(\|x_0\|+d_\omega R+e_\omega)+M\int_0^t(a_1(s)\\ &\quad+b_1(s)R)ds+\sum_{k=1}^m M(d_kR+e_k)\\ &\leq M(\|x_0\|+e_\omega)+M\int_0^T a_1(s)ds+\sum_{k=1}^m Me_k\\ &\quad+\left(Md_\omega+M\int_0^T b_1(s)ds+\sum_{k=1}^m Md_k\right)R=:\rho.\end{aligned} \tag{27}$$

That is, $\|x(t)\|\leq\rho$ for $t\in J$, then there exists a constant $\rho_1>\rho$ such that $\|x\|_{\mathrm{PC}}\neq\rho_1$. Let $V=\{x\in \mathrm{PC}(J,E)\colon \|x\|_{\mathrm{PC}}<\rho_1\}$, obviously, there is no $x\in\partial V$ such that $x=\lambda(Qx)$ for $0<\lambda<1$. Therefore, thanks to Lemma 1, one gets that Q has at least one fixed point x in V, which is a PC-mild solution of problem (4). This completes the proof. □

Remark 8. Theorem 7 is proved under the condition that $U_\beta(t,s)$ is compact for $t,s>0$ and the functions f,g,h meet corresponding conditions; in the case that the resolvent operator $U_\beta(t,s)$ is noncompact, we would obtain Theorem 9 and Theorem 10.

Theorem 9. *Suppose that the conditions (H_4)-(H_6) are satisfied, $M=\max_{0\leq s<t\leq T}\|U_\beta(t,s)\|$, and $M(L_\omega+\int_0^T L_1(s)ds+\int_0^T L_2(s)ds\int_0^T L_4(v)dv+\int_0^T L_3(s)ds\int_0^T L_5(v)dv+\sum_{k=1}^m L_{I_k})<1$.*

(H_4) The function $f: J\times E\times E\times E\longrightarrow E$ is continuous, and there exist nonnegative Lebesgue integrable functions $L_i\in L(J,\mathbb{R}_+)$ $(i=1,2,3)$, for any u_i, $v_i\in E$, $t\in J$, such that

$$\begin{aligned}&\|f(t,u_1,u_2,u_3)-f(t,v_1,v_2,v_3)\|\\ &\quad\leq L_1(t)\|u_1-v_1\|+L_2(t)\|u_2-v_2\|+L_3(t)\|u_3-v_3\|.\end{aligned} \tag{28}$$

(H_5) There exist nonnegative Lebesgue integrable functions $L_4,L_5\in L(J,\mathbb{R}_+)$, for each u, $v\in E$, such that

$$\begin{aligned}&\|g(t,s,u)-g(t,s,v)\|\leq L_4(t)\|u-v\|, \quad (t,s)\in B,\\ &\|h(t,s,u)-h(t,s,v)\|\leq L_5(t)\|u-v\|, \quad (t,s)\in B_0.\end{aligned} \tag{29}$$

(H_6) The functions $I_k: E\longrightarrow E$ and $\omega: PC(J,E)\longrightarrow E$ are continuous, and there exist nonnegative constants $L_\omega, L_{I_k}>0$, such that

$$\begin{aligned}&\|\omega(u)-\omega(v)\|\leq L_\omega\|u-v\|_{PC}, \quad u,v\in PC(J,E),\\ &\|I_k(u)-I_k(v)\|\leq L_{I_k}\|u-v\|, \quad u,v\in E, k=1,2\cdots m.\end{aligned} \tag{30}$$

Then, problem (4) has a unique PC-mild solution x^* in $PC(J,E)$.

Proof. It follows from the conditions (H_4)-(H_6), for any u, $v\in \mathrm{PC}(J,E)$, $t\in J$, one can derive

$$\begin{aligned}&\|(Qu)(t)-(Qv)(t)\|\\ &\quad\leq\left\|U_\beta(t,0)(\omega(u)-\omega(v))\right\|\\ &\qquad+\sum_{0<t_k<t}\left\|U_\beta(t,t_k)(I_k(u(t_k))-I_k(v(t_k)))\right\|\\ &\qquad+\left\|\int_0^t U_\beta(t,s)(f(s,u(s),(\mathscr{G}u)(s),(\mathscr{H}u)(s))\right.\\ &\qquad\left.-f(s,v(s),(\mathscr{G}v)(s),(\mathscr{H}v)(s)))ds\right\|\\ &\quad\leq M\|\omega(u)-\omega(v)\|+\sum_{k=1}^m M\|I_k(u(t_k))-I_k(v(t_k))\|\\ &\qquad+M\int_0^t\|f(s,u(s),(\mathscr{G}u)(s),(\mathscr{H}u)(s))\\ &\qquad-f(s,v(s),(\mathscr{G}v)(s),(\mathscr{H}v)(s))\|ds\\ &\quad\leq ML_\omega\|u-v\|_{\mathrm{PC}}+\sum_{k=1}^m ML_{I_k}\|u(t_k)-v(t_k)\|\\ &\qquad+M\int_0^t(L_1(s)\|u(s)-v(s)\|+L_2(s)\|(\mathscr{G}u)(s)-(\mathscr{G}v)(s)\|\\ &\qquad+L_3(s)\|(\mathscr{H}u)(s)-(\mathscr{H}v)(s)\|)ds\\ &\quad\leq ML_\omega\|u-v\|_{\mathrm{PC}}+\sum_{k=1}^m ML_{I_k}\|u-v\|_{\mathrm{PC}}\\ &\qquad+M\int_0^t L_1(s)ds\|u-v\|_{\mathrm{PC}}+\left(M\int_0^t L_2(s)\int_0^s L_4(v)dvds\right.\\ &\qquad\left.+M\int_0^t L_3(s)\int_0^T L_5(v)dvds\right)\|u-v\|_{\mathrm{PC}}\\ &\quad\leq M\left(L_\omega+\sum_{k=1}^m L_{I_k}+\int_0^T L_1(s)ds+\int_0^T L_2(s)ds\int_0^T L_4(v)dv\right.\\ &\qquad\left.+\int_0^T L_3(s)ds\int_0^T L_5(v)dv\right)\|u-v\|_{\mathrm{PC}}.\end{aligned} \tag{31}$$

Based on the assumption, we have $\|Qu-Qv\|_{\mathrm{PC}}<\|u-v\|_{\mathrm{PC}}$, which means that the operator Q is a contraction mapping. Hence, the operator Q has a unique fixed point $x^*\in \mathrm{PC}(J,E)$, which implies that problem (4) has a unique PC-mild solution. This completes the proof. □

Theorem 10. *Assume that the conditions (H_7) and (H_8) hold, $\omega\equiv 0$, $M=\max_{0\leq s<t\leq T}\|U_\beta(t,s)\|$, $a=\sum_{k=1}^m ML_{I_k}<1$.*

(H_7) *The function* $f : J \times E \times E \times E \longrightarrow E$ *is continuous, and there exist nonnegative Lebesgue integrable functions* $L'_1, L'_2 \in L(J, \mathbb{R}_+)$, *for all* $u_i, v_i \in E$, $t \in J$, *such that*

$$\|f(t, u_1, u_2, u_3) - f(t, v_1, v_2, v_3)\| \le L'_1(t)\|u_1 - v_1\| + L'_2(t)\|u_2 - v_2\|. \quad (32)$$

(H_8) *There exist constants* $L_g, L_{I_k} > 0$, *for all* $u, v \in E$, *satisfying*

$$\|g(t, s, u) - g(t, s, v)\| \le L_g\|u - v\|, \quad (t, s) \in B, \\ \|I_k(u) - I_k(v)\| \le L_{I_k}\|u - v\|, \quad k = 1, 2 \cdots m. \quad (33)$$

Then, problem (4) has a unique PC-mild solution x^* *in* $PC(J, E)$. *For all* $t \in J$, $x_0 \in PC(J, E)$, *iterative sequence* $x_n(t)$ *are defined by*

$$x_n(t) = U_\beta(t, 0)x_0 + \int_0^t U_\beta(t, s)f(s, x_{n-1}(s), \cdot (\mathcal{G}x_{n-1})(s), (\mathcal{H}x_{n-1})(s))ds + \sum_{0<t_k<t} U_\beta(t, t_k)I_k(x_{n-1}(t_k)), \quad n = 1, 2, \cdots, \quad (34)$$

uniformly converge to the unique PC-mild solution $x^*(t)$ *in* $t \in J$, *and for any* $s > 0$,

$$\|x_n - x^*\|_{PC} = o\left(\frac{1}{n^s}\right), \text{ as } n \longrightarrow \infty. \quad (35)$$

Proof. Combining the conditions (H_7) and (H_8), for all $t \in J$, $u, v \in PC(J, E)$, we get

$$\begin{aligned} &\|(Qu)(t) - (Qv)(t)\| \\ &\le M\int_0^t \|f(s, u(s), (\mathcal{G}u)(s), (\mathcal{H}u)(s)) \\ &\quad - f(s, v(s), (\mathcal{G}v)(s), (\mathcal{H}v)(s))\|ds \\ &\quad + \sum_{0<t_k<t} M\|I_k(u(t_k)) - I_k(v(t_k))\| \\ &\le M\int_0^t \left(L'_1(s) + L'_2(s)L_g s\right)\|u(s) - v(s)\|ds \\ &\quad + \sum_{k=1}^m ML_{I_k}\|u(t_k) - v(t_k)\| \le \int_0^t L(s)\|u(s) - v(s)\|ds \\ &\quad + \sum_{k=1}^m ML_{I_k}\|u - v\|_{PC} \le \left(\int_0^t L(s)ds + \sum_{k=1}^m ML_{I_k}\right)\|u - v\|_{PC} \\ &\le \left(\int_0^t L(s)ds + a\right)\|u - v\|_{PC}, \end{aligned} \quad (36)$$

where $L(s) = M(L'_1(s) + L'_2(s)L_g T)$. It is easy to see that $L \in L(J, \mathbb{R}_+)$. Notice that $a = \sum_{k=1}^m ML_{I_k} < 1$, then there is $\varepsilon > 0$ such that $0 < b = \varepsilon + a < 1$. For the above $\varepsilon > 0$, there exists a continuous function $\phi(s)$ such that

$$\int_0^T |L(s) - \phi(s)|ds < \varepsilon. \quad (37)$$

Consequently,

$$\begin{aligned} &\|(Qu)(t) - (Qv)(t)\| \\ &\le \left(\int_0^t |L(s) - \phi(s)|ds + \int_0^t |\phi(s)|ds\right)\|u - v\|_{PC} + a\|u - v\|_{PC} \\ &\le (\varepsilon + \Phi t)\|u - v\|_{PC} + a\|u - v\|_{PC} \le (b + \Phi t)\|u - v\|_{PC} \\ &= \left(C_1^0 b + C_1^1\frac{(\Phi t)}{1!}\right)\|u - v\|_{PC}, \end{aligned} \quad (38)$$

where $\Phi = \max\{|\phi(s)|: s \in J\}$. We next prove the following inequalities, for every positive integer n and $t \in J$,

$$\begin{aligned} &\|(Q^n u)(t) - (Q^n v)(t)\| \\ &\le \left(C_n^0 b^n + C_n^1\frac{b^{n-1}(\Phi t)}{1!} + C_n^2\frac{b^{n-2}(\Phi t)^2}{2!} + \cdots + C_n^n\frac{b^0(\Phi t)^n}{n!}\right)\|u - v\|_{PC}, \end{aligned} \quad (39)$$

where $C_n^m = n!/(m!(n - m)!)$.

Assume that, for any positive integer k, we have

$$\begin{aligned} &\left\|\left(Q^k u\right)(t) - \left(Q^k v\right)(t)\right\| \\ &\le \left(C_k^0 b^k + C_k^1\frac{b^{k-1}(\Phi t)}{1!} + C_k^2\frac{b^{k-2}(\Phi t)^2}{2!} + \cdots + C_k^k\frac{b^0(\Phi t)^k}{k!}\right)\|u - v\|_{PC}. \end{aligned} \quad (40)$$

By the formula $C_{k+1}^m = C_k^m + C_k^{m-1}$, for all $t \in J$,

$$\begin{aligned} &\left\|\left(Q^{k+1}u\right)(t) - \left(Q^{k+1}v\right)(t)\right\| \\ &= \left\|Q\left(Q^k u\right)(t) - Q\left(Q^k v\right)(t)\right\| \\ &\le M\int_0^t \left\|f\left(s, \left(Q^k u\right)(s), \mathcal{G}\left(Q^k u\right)(s), \mathcal{H}\left(Q^k u\right)(s)\right)\right. \\ &\quad \left.- f\left(s, \left(Q^k v\right)(s), \mathcal{G}\left(Q^k v\right)(s), \mathcal{H}\left(Q^k v\right)(s)\right)\right\|ds \\ &\quad + \sum_{0<t_k<t} M\left\|I_k\left(Q^k u\right)(t_k) - I_k\left(Q^k v\right)(t_k)\right\| \\ &\le M\int_0^t \left(L'_1(s) + L'_2(s)L_g s\right)\left\|\left(Q^k u\right)(s) - \left(Q^k v\right)(s)\right\|ds \\ &\quad + \sum_{k=1}^m ML_{I_k}\left\|\left(Q^k u\right)(t_k) - \left(Q^k v\right)(t_k)\right\| \end{aligned}$$

$$
\begin{aligned}
&\le \int_0^t L(s)\left\| \left(Q^k u\right)(s) - \left(Q^k v\right)(s)\right\| ds \\
&\quad + \sum_{k=1}^{m} ML_{I_k}\left\| \left(Q^k u\right)(t_k) - \left(Q^k v\right)(t_k)\right\| \\
&\le \int_0^t |L(s)-\phi(s)| \left(C_k^0 b^k + C_k^1 \frac{b^{k-1}(\Phi s)}{1!} \right. \\
&\quad \left. + C_k^2 \frac{b^{k-2}(\Phi s)^2}{2!} + \cdots + C_k^k \frac{b^0(\Phi s)^k}{k!} \right) ds \|u - \mathrm{v}\|_{PC} \\
&\quad + \int_0^t |\phi(s)| \left(C_k^0 b^k + C_k^1 \frac{b^{k-1}(\Phi s)}{1!} \right. \\
&\quad \left. + C_k^2 \frac{b^{k-2}(\Phi s)^2}{2!} + \cdots + C_k^k \frac{b^0(\Phi s)^k}{k!} \right) ds \|u - v\|_{PC} \\
&\quad + \sum_{k=1}^{m} ML_{I_k} \left(C_k^0 b^k + C_k^1 \frac{b^{k-1}(\Phi t_k)}{1!} \right. \\
&\quad \left. + C_k^2 \frac{b^{k-2}(\Phi t_k)^2}{2!} + \cdots + C_k^k \frac{b^0(\Phi t_k)^k}{k!} \right) \|u - v\|_{PC} \\
&\le \varepsilon \left(C_k^0 b^k + C_k^1 \frac{b^{k-1}(\Phi t)}{1!} + C_k^2 \frac{b^{k-2}(\Phi t)^2}{2!} \right. \\
&\quad \left. + \cdots + C_k^k \frac{b^0(\Phi t)^k}{k!} \right) \|u - v\|_{PC} + \Phi \int_0^t \left(C_k^0 b^k \right. \\
&\quad + C_k^1 \frac{b^{k-1}(\Phi s)}{1!} + C_k^2 \frac{b^{k-2}(\Phi s)^2}{2!} + \cdots \\
&\quad \left. + C_k^k \frac{b^0(\Phi s)^k}{k!} ds \|u - v\|_{PC} + a \left(C_k^0 b^k + C_k^1 \frac{b^{k-1}(\Phi t)}{1!} \right. \right. \\
&\quad \left. + C_k^2 \frac{b^{k-2}(\Phi t)^2}{2!} + \cdots + C_k^k \frac{b^0(\Phi t)^k}{k!} \right) \|u - v\|_{PC} \\
&\le b \left(C_k^0 b^k + C_k^1 \frac{b^{k-1}(\Phi t)}{1!} + C_k^2 \frac{b^{k-2}(\Phi t)^2}{2!} + \cdots \right. \\
&\quad \left. + C_k^k \frac{b^0(\Phi t)^k}{k!} \right) \|u - v\|_{PC} + \Phi \int_0^t \left(C_k^0 b^k + C_k^1 \frac{b^{k-1}(\Phi s)}{1!} \right. \\
&\quad \left. + C_k^2 \frac{b^{k-2}(\Phi s)^2}{2!} + \cdots + C_k^k \frac{b^0(\Phi s)^k}{k!} \right) ds \|u - v\|_{PC} \\
&\le \left(C_{k+1}^0 b^{k+1} + C_{k+1}^1 \frac{b^k(\Phi t)}{1!} + C_{k+1}^2 \frac{b^{k-1}(\Phi t)^2}{2!} \right. \\
&\quad \left. + \cdots + C_{k+1}^{k+1} \frac{b^0(\Phi t)^{k+1}}{(k+1)!} \right) \|u - v\|_{PC}.
\end{aligned}
\tag{41}
$$

By mathematical induction, for every positive integer n, we obtain

$$
\begin{aligned}
\|Q^n u - Q^n v\|_{PC} \le & \left(C_n^0 b^n + C_n^1 \frac{b^{n-1}(\Phi t)}{1!} + C_n^2 \frac{b^{n-2}(\Phi t)^2}{2!} \right. \\
& \left. + \cdots + C_n^n \frac{b^0(\Phi t)^n}{n!} \right) \|u - v\|_{PC}.
\end{aligned}
\tag{42}
$$

Using Lemma 2, it follows that

$$
\|Q^n u - Q^n v\|_{PC} \le o\left(\frac{1}{n^s}\right) \|u - v\|_{PC}, \text{ as } n \longrightarrow \infty. \tag{43}
$$

Thus, for any fixed constant $s > 1$, we can find a positive integer n_0 such that, for any $u, v \in PC(J, E)$ and $n > n_0$, we have

$$
\|Q^n u - Q^n v\|_{PC} \le \frac{1}{n^s} \|u - v\|_{PC}. \tag{44}
$$

Applying the general Banach contraction mapping principle, we deduce that the operator Q has a unique fixed point x^* in $PC(J, E)$, which means that problem (4) has a unique PC-mild solution x^* in $PC(J, E)$. This completes the proof. □

4. Conclusion

In this paper, we demonstrate the existence theory of PC-mild solutions for the impulsive fractional semilinear integrodifferential equations with nonlocal initial conditions (4) via the theory of semigroup and fixed point theorem under the condition of compact resolvent operator. Meanwhile, the uniqueness of PC-mild solutions is proved under the condition of noncompact resolvent operator. The kernels g and h of the integral operators $\mathscr{G}$ and $\mathscr{H}$ are nonlinear functions, and the function ω of the nonlocal initial conditions is noncompact. In addition, the closed linear operator $A(t)$ is dependent on t. As a consequence, our main theorems improve and generalize many existing results on this topic.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Research Article

A New Estimate for the Homogenization Method for Second-Order Elliptic Problem with Rapidly Oscillating Periodic Coefficients

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In this paper, we will investigate a multiscale homogenization theory for a second-order elliptic problem with rapidly oscillating periodic coefficients of the form $(\partial/\partial x_i)(a^{ij}(x/\varepsilon, x)(\partial u^\varepsilon(x)/\partial x_j)) = f(x)$. Noticing the fact that the classic homogenization theory presented by Oleinik has a high demand for the smoothness of the homogenization solution u^0, we present a new estimate for the homogenization method under the weaker smoothness that homogenization solution u^0 satisfies than the classical homogenization theory needs.

1. Introduction

Many people investigated the second-order elliptic problem with a fixed boundary. As far as we know, there is not any work related to the elliptic problem with periodic boundary (see [1–3]). In this article, we will consider the following multiscale elliptic model problem:

$$\begin{cases} L_\varepsilon u^\varepsilon(x) \equiv \dfrac{\partial}{\partial x_i}\left(a^{ij}\left(\dfrac{x}{\varepsilon}, x\right)\dfrac{\partial u^\varepsilon}{\partial x_j}\right) = f(x), & \text{in } \Omega, \\ u^\varepsilon(x) = g(x), & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Here, $\Omega \subset \mathfrak{R}^n (n \geq 1)$ is a bounded domain, and the matrix of coefficients $a^{ij}(\xi, x)$: $\mathfrak{R}^n \longrightarrow \mathfrak{R}^{n\times n}$ is symmetric and satisfies the following conditions:

$$\gamma|\xi|^2 \leq a^{ij}(\xi, x)\xi_i\xi_j \leq \gamma^{-1}|\xi|^2, \xi \in \mathfrak{R}^n, \text{for some } \gamma \in (0, 1]$$
$$a^{ij}\left(\xi + \xi'\right) = a^{ij}(\xi), \xi \in \mathfrak{R}^n, \xi' \in Z^n, 1 \leq i, j \leq n \tag{2}$$

Assume that $Q = [0, 1]^n$. By the homogenization method, Oleinik et al. (see [4, 5]) obtained the 1-order approximation $\tilde{u}(x)$ of u^ε as follows:

$$\tilde{u}(x) = u^0(x) + \varepsilon N^k\left(x, \frac{x}{\varepsilon}\right)\frac{\partial u^0(x)}{\partial x_k}, \tag{3}$$

where $N^k(x, \xi)$ is a 1-periodic function and satisfies the following equations:

$$\begin{cases} \dfrac{\partial}{\partial \xi_i}\left(a^{ij}(x, \xi)\dfrac{\partial N^k(x, \xi)}{\partial \xi_j}\right) = -\dfrac{\partial a^{ik}(x, \xi)}{\partial \xi_i}, & \text{in } \mathfrak{R}^n, \\ \displaystyle\int_Q N^k(\xi, x)d\xi = 0, \end{cases}$$
$$a\wedge^{ij}(x) = \int_Q \left(a^{ij}(x, \xi) + a^{ik}(x, \xi)\frac{\partial N^j(x, \xi)}{\partial \xi_k}\right)d\xi, \tag{4}$$

and the homogenization solution u^0 satisfies the problem as follows:

$$\begin{cases} L_0 u^0(x) \equiv \dfrac{\partial}{\partial x_i}\left(a\wedge^{ij}(x)\dfrac{\partial u^0}{\partial x_j}\right) = f(x), & \text{in } \Omega, \\ u^0(x) = g(x), & \text{on } \partial\Omega. \end{cases} \tag{5}$$

Oleinik et al. (see [5], p. 28) proved the following result.

We end this section with the details of some notations. Throughout this paper, the Einstein summation convention is used: summation is taken over repeated indices, and $\rho(x, \partial\Omega)$ denotes the distance between x and $\partial\Omega$.

2. Some Useful Lemmas

Lemma 1. *Under the assumption that $u^0 \in H^2(\Omega)$, there holds*

$$\|u^\varepsilon - \tilde{u}\|_{H^1(\Omega)} \le c\varepsilon^{1/2}\|u^0\|_{H^2(\Omega)}. \tag{6}$$

There are numerous literatures discussing the homogenization method (see [1, 2, 4–9]). There also are many works (see [3, 10–16]) discussing the numerical methods of the multiscale homogenization problem. We observe that most of them are based on the assumption $u^0 \in H^2(\Omega)$, which is unrealistic for some problems. For example, when $f \notin L^2(\Omega)$. Let $\tilde{u}_i(x) = (\partial u^0(x)/\partial x_i) + (\partial N^k(\xi, x)/\partial \xi_i)(\partial u^0(x)/\partial x_k)$. As far as we know, it is the first time for us to estimate $\tilde{u}_i(x) - (\partial u^\varepsilon(x)/\partial x_i)$ under the assumption that the homogenization solution u^0 belongs to the Sobolev space $H^{1+s}(\Omega)$ for the case that $0 < s < 1$.

Lemma 2. *Assume that $u \in H^{1+s}(\Omega) \cap W^{1,\infty}(K_{2r})$. Then,*

$$\|\nabla(u - u_r)\|_{L^2(\Omega)} \le c\left(r^s\|u\|_{H^{1+s}(\Omega)} + r^{1/2}\|\nabla u\|_{L^\infty(K_{2r})}\right), \tag{7}$$

$$\|\nabla^2 u_r\|_{L^2(\Omega)} \le c\left(r^{s-1}\|u\|_{H^{1+s}(\Omega)} + r^{-1/2}\|\nabla u\|_{L^\infty(K_{2r})}\right). \tag{8}$$

Proof. One observes that $\|\nabla(u-u_r)\|^2_{L^2(\Omega)}$ can be split into

$$\|\nabla(u-u_r)\|^2_{L^2(\Omega)} = \|\nabla(u-u_r)\|^2_{L^2(\Omega\setminus K_r)} + \|\nabla(u-u_r)\|^2_{L^2(K_r)}. \tag{9}$$

We first estimate $\|\nabla(u-u_r)\|^2_{L^2(\Omega\setminus K_r)}$. Assume that $x \in \Omega \setminus K_r$ and $B(x,r) = \{y \in \Omega : |x-y| \le r\}$. Note that the definition of $\omega_r(z)$ implies $\int_\Omega \omega_r(x-y)dy = 1$. By the definitions of $\omega_r(z)$ and $u_r(x)$, we have, for any $1 \le i \le n$,

$$\begin{aligned} \frac{\partial u_r(x)}{\partial x_i} &= \int_\Omega \frac{\partial \omega_r(x-y)}{\partial x_i}u(y)dy = -\int_\Omega \frac{\partial \omega_r(x-y)}{\partial y_i}u(y)dy \\ &= \int_\Omega \omega_r(x-y)\frac{\partial u(y)}{\partial y_i}dy = \int_{B(x,r)} \omega_r(x-y)\frac{\partial u(y)}{\partial y_i}dy. \end{aligned} \tag{10}$$

Using (10), we obtain

$$\frac{\partial u_r(x)}{\partial x_i} - \frac{\partial u(x)}{\partial x_i} = \int_{B(x,r)} \omega_r(x-y)\left(\frac{\partial u(y)}{\partial y_i} - \frac{\partial u(x)}{\partial x_i}\right)dy. \tag{11}$$

Furthermore, from the definition of $\omega_r(z)$ and (11), it follows that

$$\left\|\frac{\partial(u-u_r)}{\partial x_i}\right\|^2_{L^2(\Omega\setminus K_r)} = \int_{\Omega\setminus K_r}\left[\int_{B(x,r)} \omega_r(x-y)\left(\frac{\partial u(y)}{\partial y_i} - \frac{\partial u(x)}{\partial x_i}\right)dy\right]^2 dx, \tag{12}$$

$$\le cr^{-2n}\int_{\Omega\setminus K_r}\left[\int_{B(x,r)}\left|\frac{\partial u(y)}{\partial y_i} - \frac{\partial u(x)}{\partial x_i}\right|dy\right]^2 dx. \tag{13}$$

Note that

$$\int_{B(x,r)}\left|\frac{\partial u(y)}{\partial y_i} - \frac{\partial u(x)}{\partial x_i}\right|dy \le cr^{s+n/2}\int_{B(x,r)}\left|\frac{\partial u(y)}{\partial y_i} - \frac{\partial u(x)}{\partial x_i}\right||x-y|^{-s-n/2}dy. \tag{14}$$

This, together with (13), gives

$$\begin{aligned} \left\|\frac{\partial(u-u_r)}{\partial x_i}\right\|^2_{L^2(\Omega\setminus K_r)} &\le cr^{2s-n}\int_{\Omega\setminus K_r}\left[\int_{B(x,r)}\left|\frac{\partial u(y)}{\partial y_i} - \frac{\partial u(x)}{\partial x_i}\right||x-y|^{-s-n/2}dy\right]^2 dx \\ &\le cr^{2s}\int_{\Omega\setminus K_r}\int_{B(x,r)}\left|\frac{\partial u(y)}{\partial y_i} - \frac{\partial u(x)}{\partial x_i}\right|^2|x-y|^{-2s-n}dydx \\ &\le cr^{2s}\|u\|^2_{H^{1+s}(\Omega)}. \end{aligned} \tag{15}$$

Next we estimate $\|\nabla(u-u_r)\|^2_{L^2(K_r)}$. Assume that $x \in K_r$. Set $\bar{\omega}_r(x-y) = \omega_r(x-y)/\int_{B(x,r)}\omega_r(x-y)dy$. Let $\bar{x} = x$ or $x + \Delta x$. We have

$$\begin{aligned} u_r(\bar{x}) &= \int_{B(\bar{x},r)} \bar{\omega}_r(\bar{x}-y)u(y)dy \\ &= u(\bar{x}) + \int_{B(\bar{x},r)} \bar{\omega}_r(\bar{x}-y)(u(y)-u(x))dy. \end{aligned} \tag{16}$$

Let $|\Delta x| \le r$. Note that $\bar{\omega}_r(z) = 0$ whenever $|z| \ge r$. By (16), one observes that $u_r(x+\Delta x) - u_r(x)$ can be decomposed into

$$\begin{aligned}u_r(x+\Delta x)-u_r(x) &= \int_{B(x+\Delta x,r)} \bar{\omega}_r(x+\Delta x-y)(u(y)-u(x))dy \\ &\quad - \int_{B(x,r)} \bar{\omega}_r(x-y)(u(y)-u(x))dy \\ &= \int_{B(x+\Delta x,r)} - \int_{B(x,r)} \bar{\omega}_r(x+\Delta x-y)(u(y) \\ &\quad - u(x))dy + \int_{B(x,r)} [\bar{\omega}_r(x+\Delta x-y) \\ &\quad - \bar{\omega}_r(x-y)](u(y)-u(x))dy = I_1 + I_2.\end{aligned} \tag{17}$$

We need estimates I_1 and I_2. Assume that $y \in B(x+\Delta x, r) \setminus B(x,r)$. Note that $x+\Delta x \in \Omega$. One observes that $\int_{B(x+\Delta x,r/2)} \square dy \geq cr^n$. Then, we have

$$\int_{B(x+\Delta x,r/2)} \omega_r(x-y)dy \geq c. \tag{18}$$

By the definition of $\bar{\omega}_r(x)$ and (18), we have

$$|\bar{\omega}_r(x+\Delta x-y)| \leq cr^{-n}. \tag{19}$$

Note that

$$|u(y)-u(x)| \leq cr\|u\|_{W^{1,\infty}(K_{2r+|\Delta x|})}. \tag{20}$$

Inserting (19) and (20) into (17), we have

$$|I_1| \leq cr^{n-1}|\Delta x| r^{-n} r \|u\|_{W^{1,\infty}(K_{2r+|\Delta x|})} \leq c|\Delta x| \|u\|_{W^{1,\infty}(K_{2r+|\Delta x|})}. \tag{21}$$

We turn now to the estimation of I_2. We split $\bar{\omega}_r(x+\Delta x-y) - \bar{\omega}_r(x-y)$ into

$$\begin{aligned}\bar{\omega}_r(x+\Delta x-y)-\bar{\omega}_r(x-y) &= \frac{\omega_r(x+\Delta x-y)}{\int_{B(x+\Delta x,r)}\omega_r(x+\Delta x-y)dy} - \frac{\omega_r(x-y)}{\int_{B(x,r)}\omega_r(x-y)dy} \\ &= \omega_r(x+\Delta x-y)\left[\left(\int_{B(x+\Delta x,r)}\omega_r(x+\Delta x-y)dy\right)^{-1} \right. \\ &\quad \left. - \left(\int_{B(x,r)}\omega_r(x-y)dy\right)^{-1}\right] \\ &\quad + \frac{[\omega_r(x+\Delta x-y)-\omega_r(x-y)}{\int_{B(x,r)}\omega_r(x-y)dy} = J_1 + J_2.\end{aligned} \tag{22}$$

We need to estimate the two items of the right-hand side of (22). Note that

$$\begin{aligned}&\left|\int_{B(x+\Delta x,r)} \omega_r(x+\Delta x-y)dy - \int_{B(x,r)} \omega_r(x-y)dy\right| \\ &\quad \leq \left|\int_{B(x+\Delta x,r)} - \int_{B(x,r)} \omega_r(x+\Delta x-y)dy\right| \\ &\quad + \int_{B(x,r)} [\omega_r(x+\Delta x-y)-\omega_r(x-y)]dy| \leq cr^{n-1}|\Delta x| r^{-n} \\ &\quad + cr^n r^{-n-1}|\Delta x| \leq cr^{-1}|\Delta x|.\end{aligned} \tag{23}$$

By (22) and (23), we have

$$|J_1| \leq cr^{-n}cr^{-1}|\Delta x| \leq cr^{-n-1}|\Delta x|. \tag{24}$$

To estimate J_2, we have

$$|J_2| \leq cr^{-1}|\Delta x| r^{-n} \leq cr^{-n-1}|\Delta x|. \tag{25}$$

Plugging the above two estimates into (22), we obtain

$$|\bar{\omega}_r(x+\Delta x-y) - \bar{\omega}_r(x-y)| \leq cr^{-n-1}|\Delta x|. \tag{26}$$

This, together with (17), gives

$$|I_2| \leq cr^n r^{-n-1}|\Delta x| r \|u\|_{W^{1,\infty}(K_{2r})} \leq c|\Delta x| \|u\|_{W^{1,\infty}(K_{2r})}. \tag{27}$$

Inserting (21) and (27) into (17), we have

$$|u_r(x+\Delta x) - u_r(x)| \leq c|\Delta x| \|u\|_{W^{1,\infty}(K_{2r+|\Delta x|})}. \tag{28}$$

Furthermore, let $\Delta x \longrightarrow 0$, we have

$$\|u_r\|_{W^{1,\infty}(K_r)} \leq c\|u\|_{W^{1,\infty}(K_{2r})}, \tag{29}$$

where we have used (28). Then, (7) follows by combining (15) and (29). We turn now to the estimation of $\|\nabla^2 u_r\|_{L^2(\Omega)}$. We decompose $\|\nabla^2 u_r\|_{L^2(\Omega)}$ into

$$\|\nabla^2 u_r\|^2_{L^2(\Omega)} = \|\nabla^2 u_r\|^2_{L^2(\Omega\setminus K_r)} + \|\nabla^2 u_r\|^2_{L^2(K_r)}. \tag{30}$$

We first estimate $\|\nabla^2 u_r\|^2_{L^2(\Omega\setminus K_r)}$. Assume that $x \in \Omega \setminus K_r$. By (10), we have, for any $1 \leq i, j \leq n$,

$$\begin{aligned}\frac{\partial^2 u_r(x)}{\partial x_i \partial x_j} &= \frac{\partial\left(\int_\Omega \omega_r(x-y)(\partial u(y)/\partial y_i)dy\right)}{\partial x_j} \\ &= \int_\Omega \frac{\partial \omega_r(x-y)}{\partial x_j}\frac{\partial u(y)}{\partial y_i}dy.\end{aligned} \tag{31}$$

Note that $x \in \Omega \setminus K_r$. By the definition of $\omega_r(x-y)$, we have $\int_\Omega (\partial \omega_r(x-y)/\partial x_j)dy = 0$. Then, by (28) and (31), we have

$$\frac{\partial^2 u_r(x)}{\partial x_i \partial x_j} = \frac{\partial\left(\int_\Omega \omega_r(x-y)(\partial u(y)/\partial y_i)dy\right)}{\partial x_j} = \int_\Omega \frac{\partial \omega_r(x-y)}{\partial x_j}\left(\frac{\partial u(y)}{\partial y_i} - \frac{\partial u(x)}{\partial x_i}\right)dy. \quad (32)$$

Finally, similarly to (15), by (32), we have

$$\left\|\nabla^2 u_r\right\|_{L^2(\Omega\setminus K_r)} \le cr^{s-1}\|u\|_{H^{1+s}(\Omega)}. \quad (33)$$

We turn now to the estimation of $\|\nabla^2 u_r\|_{L^2(K_r)}$. Similarly to (17), we have

$$u_r(x+2\Delta x) - 2u_r(x+\Delta x) + u_r(x) = \int_{B(x,2r)} [\bar{\omega}_r(x+2\Delta x - y) - 2\bar{\omega}_r(x+\Delta x - y) + \bar{\omega}_r(x-y)](u(y)-u(x))dy. \quad (34)$$

Note that the definition of $\omega_r(z)$ implies $\|\bar{\omega}_r\|_{W^{2,1}(\mathfrak{R}^n)} \le c r^{-2}$. Therefore, let $\Delta x \longrightarrow 0$, from (34), it follows that

$$\|u_r\|_{W^{2,\infty}(K_r)} \le c\|\bar{\omega}_r\|_{W^{2,1}(\mathfrak{R}^n)} cr\|u\|_{W^{1,\infty}(K_{2r})} \le cr^{-2}r\|u\|_{W^{1,\infty}(K_{2r})} \le cr^{-1}\|u\|_{W^{1,\infty}(K_{2r})}. \quad (35)$$

The desired result (8) follows by combining (33) and (35). □

3. A New Estimate for Multiscale Homogenization Method

In this section, we give the main results as follows.

Theorem 3. *Assume that $K_r = \{x \in \Omega \mid \rho(x, \partial\Omega) \le r\}$ and $Q = [0,1]^n$. Assume also that $N^k \in W^{1,\infty}(Q)$ and $u^0 \in H^{1+s}(\Omega) \cap W^{1,\infty}(K_\varepsilon)$ for some $0 < s < 1$. Then,*

$$\left\|\frac{\partial u^\varepsilon}{\partial x_i} - \tilde{u}_i\right\|_{L^2(\Omega)} \le c\left(\varepsilon^{1/2}\left\|u^0\right\|_{W^{1,\infty}(K_\varepsilon)} + \varepsilon^s\left\|u^0\right\|_{H^{1+s}(\Omega)}\right). \quad (36)$$

Assume that $\chi(z) \in C^\infty(\mathfrak{R}^n)$ is the cutoff function satisfying $0 \le \chi(z) \le 1$, and $\chi(z) = 1$ if $|z| \le 1/2$, and $\chi(z) = 0$ if $|z| \ge 1$. Let $\omega_r(z) = \chi(z/r)/\int_{\mathfrak{R}^n}\chi(y/r)dy$. One observes that $\int_{B(0,r)}\omega_r(z)dz = 1$ and $\|\omega_r\|_{W^{k,\infty}(\mathfrak{R}^n)} \le cr^{-k-n}$ for all $k \ge 0$. Set $u_r(x) = \int_\Omega \omega_r(x-y)u(y)dy/\int_\Omega \omega_r(x-y)dy$. In the process of proving Theorem 3, we need the above Lemma 2.

Based on Lemma 2, we can prove Theorem 3 as follows:

Proof. Assume that $\omega_r(z)$ is defined as in Lemma 2. Set

$$\bar{u}_r^0(x) = \frac{\int_\Omega \omega_r(x-y)u^0(y)dy}{\int_\Omega \omega_r(x-y)dy}, \qquad f_r(x) = \frac{\partial}{\partial x_j}\left(\hat{a}_{ij}(x)\frac{\partial \bar{u}_r^0(x)}{\partial x_i}\right). \quad (37)$$

We introduce $u_r^\varepsilon(x)$ by the following problem:

$$\begin{cases} L_\varepsilon u_r^\varepsilon(x) = f_r(x), & \text{in } \Omega, \\ u_r^\varepsilon(x) = \bar{u}_r^0(x), & \text{on } \partial\Omega. \end{cases} \quad (38)$$

One observes that $u_r^0(x)$ and $\tilde{u}_r(x)$ are the homogenization solution of (38) and the 1-order approximation of $u_r^\varepsilon(x)$, respectively. We decompose $\partial u^\varepsilon(x)/\partial x_i - \tilde{u}_i(x)$ into

$$\frac{\partial u^\varepsilon(x)}{\partial x_i} - \tilde{u}_i(x) = \frac{\partial(u^\varepsilon - u_r^\varepsilon)(x)}{\partial x_i} + \frac{\partial(u_r^\varepsilon - \tilde{u}_r)(x)}{\partial x_i} + \left(\frac{\partial \tilde{u}_r(x)}{\partial x_i} - \tilde{u}_i(x)\right). \quad (39)$$

We first estimate $\nabla(u^\varepsilon - u_r^\varepsilon)(x)$. Let $B_1^\varepsilon(x) = (u^\varepsilon - u_r^\varepsilon)(x)$. Note that $B_1^\varepsilon(x)$ satisfies the following problem:

$$\begin{cases} L_\varepsilon B_1^\varepsilon(x) = f(x) - f_r(x), & x \in \Omega, \\ B_1^\varepsilon(x) = g(x) - \bar{u}_r^0(x), & x \in \partial\Omega. \end{cases} \quad (40)$$

One observes that $B_1^\varepsilon(x)$ can be split into

$$B_1^\varepsilon(x) = e_1(x) + e_2^\varepsilon(x), \quad (41)$$

where $e_1(x) = (u^0 - \bar{u}_r^0)(x)$ and $e_2^\varepsilon(x)$ satisfies the following problem:

$$\begin{cases} L_\varepsilon e_2^\varepsilon(x) = (f - f_r)(x) - \dfrac{\partial}{\partial x_i}\left(a_{ij}\left(x, \dfrac{x}{\varepsilon}\right)\dfrac{\partial e_1(x)}{\partial x_j}\right), & \text{in } \Omega, \\ e_2^\varepsilon(x) = 0, & \text{on } \partial\Omega. \end{cases} \quad (42)$$

From the combination of the definition of $\bar{u}_r^0$ and (7), it follows that

$$\left\|\nabla\left(\bar{u}_r^0 - u^0\right)\right\|_{L^2(\Omega)} \le c\left(r^s\left\|u^0\right\|_{H^{1+s}(\Omega)} + r^{1/2}\left\|u^0\right\|_{W^{1,\infty}(K_{2r})}\right). \quad (43)$$

To estimate $e_2^\varepsilon(x)$, by (8) and the definitions of $\bar{u}_r^0(x)$ and $f_r(x)$, one observes that

$$\|\nabla e_2^\varepsilon\|_{L^2(\Omega)} \le \|f - f_r\|_{H^{-1}(\Omega)} + \left\|\nabla\left(\bar{u}_r^0 - u^0\right)\right\|_{L^2(\Omega)} \le c\left\|\nabla\left(\bar{u}_r^0 - u^0\right)\right\|_{L^2(\Omega)} \le cr^s\left\|u^0\right\|_{H^{1+s}(\Omega)} + cr^{1/2}\left\|u^0\right\|_{W^{1,\infty}(K_{2r})}. \quad (44)$$

Combining (39), (41), (43), and (44), we have

$$\|\nabla(u^\varepsilon - u_r^\varepsilon)\|_{L^2(\Omega)} = \|\nabla B_1^\varepsilon\|_{L^2(\Omega)} \le c\left(r^s\|u^0\|_{H^{1+s}(\Omega)} + r^{1/2}\|u^0\|_{W^{1,\infty}(K_{2r})}\right). \quad (45)$$

Next, we estimate $(u_r^\varepsilon - \tilde{u}_r)(x)$. Set $B_2^\varepsilon(x) = (u_r^\varepsilon - \tilde{u}_r)(x)$. By the method of asymptotic expansion (see [7], p. 27), one finds that $B_2^\varepsilon(x)$ can be split into $B_2^\varepsilon(x) = w_r^\varepsilon(x) + \theta_r^\varepsilon(x)$, where $w_r^\varepsilon(x)$ and $\theta_r^\varepsilon(x)$ are defined by

$$\begin{cases} L_\varepsilon w_r^\varepsilon(x) \equiv \dfrac{\partial}{\partial x_i}\left(a_{ij}\left(x, \dfrac{x}{\varepsilon}\right)\dfrac{\partial w_r^\varepsilon(x)}{\partial x_j}\right) = \dfrac{\partial F_{r,i}(x)}{\partial x_i}, & \text{in } \Omega, \\ w_r^\varepsilon(x) = 0, & \text{on } \partial\Omega, \end{cases}$$
$$\begin{cases} L_\varepsilon \theta_r^\varepsilon(x) = 0, & \text{in } \Omega, \\ \theta_r^\varepsilon(x) = -\varepsilon N^k\left(\dfrac{x}{\varepsilon}, x\right)\dfrac{\partial \bar{u}_r^0(x)}{\partial x_k}, & \text{on } \partial\Omega, \end{cases} \quad (46)$$

respectively, where

$$F_{r,i}(x) = -\left(a_{ij}\left(x, \frac{x}{\varepsilon}\right) + a_{ik}\left(x, \frac{x}{\varepsilon}\right)\frac{\partial N^j(\xi)}{\partial \xi_k} - \widehat{a}_{ij}\right)\frac{\partial \bar{u}_r^0(x)}{\partial x_j} + \varepsilon a_{ij}\left(x, \frac{x}{\varepsilon}\right)N^k\left(\frac{x}{\varepsilon}\right)\frac{\partial^2 \bar{u}_r^0(x)}{\partial x_j \partial x_k}. \quad (47)$$

We first estimate $w_r^\varepsilon(x)$. Note that (8) implies

$$\|\bar{u}_r^0\|_{H^2(\Omega)} \le c\left(r^{-1+s}\|u^0\|_{H^{1+s}(\Omega)} + r^{-1/2}\|u^0\|_{W^{1,\infty}(K_{2r})}\right). \quad (48)$$

By the method of asymptotic expansion (see ([7], p. 27), from (48), it follows that

$$\|w_r^\varepsilon\|_{H^1(\Omega)} \le c\varepsilon\|\bar{u}_r^0\|_{H^2(\Omega)} \le c\left(\varepsilon r^{-1+s}\|u^0\|_{H^{1+s}(\Omega)} + \varepsilon r^{-1/2}\|u^0\|_{W^{1,\infty}(K_{2r})}\right). \quad (49)$$

Assume that $\phi(x) \in C^\infty(\Omega)$ is a cutoff function satisfying $\phi(x) = 1$ if $\rho(x, \partial\Omega) \le \varepsilon$, and $\phi(x) = 0$ if $\rho(x, \partial\Omega) \ge 2\varepsilon$, and $\|\nabla\phi\|_{L^\infty(\Omega)} \le c_2\varepsilon^{-1}$. We split $\theta_r^\varepsilon(x)$ into

$$\theta_r^\varepsilon(x) = \psi_r^\varepsilon(x) + \widehat{\psi}_r^\varepsilon(x), \quad (50)$$

where $\psi_r^\varepsilon(x) = -\varepsilon N^k(x/\varepsilon, x)(\partial\bar{u}_r^0(x)/\partial x_k)\phi(x)$ and $\widehat{\psi}_r^\varepsilon(x)$ satisfies the following problem:

$$\begin{cases} L_\varepsilon \widehat{\psi}_r^\varepsilon(x) = -\dfrac{\partial}{\partial x_i}\left(a_{ij}\left(x, \dfrac{x}{\varepsilon}\right)\dfrac{\partial \psi_r(x)}{\partial x_j}\right), & x \in \Omega, \\ \widehat{\psi}_r^\varepsilon(x) = 0, & x \in \partial\Omega. \end{cases} \quad (51)$$

To estimate $\psi_r^\varepsilon(x)$, one has

$$\|\psi_r^\varepsilon\|_{H^1(\Omega)} \le c\varepsilon r^{-1}\|\bar{u}_r^0\|_{H^1(K_\varepsilon)} + c\varepsilon\|\bar{u}_r^0\|_{H^2(K_\varepsilon)}. \quad (52)$$

We now estimate $\|\bar{u}_r^0\|_{H^1(K_\varepsilon)}$. Assume that $r = \varepsilon$ and V_{K_ε} denotes the volume of K_ε if $n = 3$, or the area of K_ε if $n = 2$. One observes that

$$\|\bar{u}_r^0\|_{H^1(K_\varepsilon)} \le c\sqrt{V_{K_\varepsilon}}\|\bar{u}_r^0\|_{W^{1,\infty}(K_\varepsilon)} \le c\varepsilon^{1/2}\|\bar{u}_r^0\|_{W^{1,\infty}(K_\varepsilon)} \le c\varepsilon^{1/2}\|u^0\|_{W^{1,\infty}(K_\varepsilon)}. \quad (53)$$

The combination of (8), (52), and (53) gives

$$\|\psi_r^\varepsilon\|_{H^1(\Omega)} \le c\left(\varepsilon^{1/2}\|u^0\|_{W^{1,\infty}(K_{2\varepsilon})} + \varepsilon^s\|u^0\|_{H^{1+s}(\Omega)}\right). \quad (54)$$

Using (51) and (54), we derive

$$\|\widehat{\psi}_r^\varepsilon\|_{H^1(\Omega)} \le c\|\psi_r^\varepsilon\|_{H^1(\Omega)} \le c\left(\varepsilon^{1/2}\|u^0\|_{W^{1,\infty}(K_{2\varepsilon})} + \varepsilon^s\|u^0\|_{H^{1+s}(\Omega)}\right). \quad (55)$$

The above two estimates, together with (50), imply

$$\|\theta_r^\varepsilon\|_{H^1(\Omega)} \le c\left(\varepsilon^{1/2} + \varepsilon^{3/2}r^{-1}\right)\|u^0\|_{W^{1,\infty}(K_{2\varepsilon})}. \quad (56)$$

Furthermore, by (49) and (56), we have

$$\|u_r^\varepsilon - \tilde{u}_r\|_{H^1(\Omega)} = \|B_2^\varepsilon\|_{H^1(\Omega)} \le c\left[\left(\varepsilon^{1/2} + \varepsilon^{3/2}\varepsilon^{-1}\right)\|u^0\|_{W^{1,\infty}(K_{2\varepsilon})} + \varepsilon\varepsilon^{-1+s}\|u^0\|_{H^{1+s}(\Omega)}\right] \le c\left(\varepsilon^{1/2}\|u^0\|_{W^{1,\infty}(K_{2\varepsilon})} + \varepsilon^s\|u^0\|_{H^{1+s}(\Omega)}\right), \quad (57)$$

We next estimate $(\partial\tilde{u}_r(x)/\partial x_i) - \tilde{u}_i(x)$. Assume that $r = \varepsilon$. Note that the definitions of u_r^0 and u_r^ε imply $u_r^0(x) = \bar{u}_r^0(x)$. By (7) and (47), we have

$$\left\|\frac{\partial\tilde{u}_r}{\partial x_i} - \tilde{u}_i\right\|_{L^2(\Omega)} \le c\left\|\nabla\left(u^0 - \bar{u}_r^0\right)\right\|_{L^2(\Omega)} \le c\left(\varepsilon^s\|u^0\|_{H^{1+s}(\Omega)} + \varepsilon^{1/2}\|u^0\|_{W^{1,\infty}(K_{2\varepsilon})}\right). \quad (58)$$

Assume that $r = \varepsilon$. This, together with (39), (45), and (57), gives the desired result (36). □

Data Availability

The paper's data available through the email liuxiong980211@163.com or from the author's ORCID: 0000-0002-5452-653X, and other data is given to the journal of functional spaces https://orcid.org/0000-0002-5452-653X.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

A Conservative Crank-Nicolson Fourier Spectral Method for the Space Fractional Schrödinger Equation with Wave Operators

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In this paper, the Crank-Nicolson Fourier spectral method is proposed for solving the space fractional Schrödinger equation with wave operators. The equation is treated with the conserved Crank-Nicolson Fourier Galerkin method and the conserved Crank-Nicolson Fourier collocation method, respectively. In addition, the ability of the constructed numerical method to maintain the conservation of mass and energy is studied in detail. Meanwhile, the convergence with spectral accuracy in space and second-order accuracy in time is verified for both Galerkin and collocation approximations. Finally, the numerical experiments verify the properties of the conservative difference scheme and demonstrate the correctness of theoretical results.

1. Introduction

The Schrödinger equation is one of the most basic equations in quantum mechanics, which was proposed by Austrian physicist Schrödinger in 1926. The equation can correctly describe the quantum behaviors of wave function, which has made great contributions to the study of quantum mechanics. Since then, the Schrödinger system has attracted a large number of mathematicians and physicists to explore the characteristics of its solution and physical applications. The study of conservative methods for the Schrödinger equation is one of the most popular research fields.

Over the past decades, most of the researches on the conservative method of the Schrödinger equation focus on the integer-order Schrödinger equation (e.g., see Refs. [1–7]). As models of science and engineering are needed to be more realistic, the fractional-order Schrödinger equation becomes one of the most important models in the fields of Bose-Einstein condensation, plasma, nonlinear optics, fluid dynamics [8, 9], etc. However, few studies have been investigated on conservative methods for the fractional Schrödinger equation. Besides that, most of the existing fractional-order conservative methods are finite element and finite difference methods [10, 11].

From the viewpoint of mathematics, the solution of the Schrödinger system has important geometric structures such as energy conservation and multisymplectic structure. Therefore, these properties should be maintained as much as possible in the construction of numerical methods. In this paper, we consider the following nonlinear fractional Schrödinger equation:

$$\begin{aligned}\phi_{tt}(y,t) + (-\Delta)^{\alpha/2}\phi(y,t) + i\kappa\phi_t(y,t)\\ + \beta|\phi(y,t)|^2\phi(y,t) = 0, \quad y \in (a,b), 0 < t \le T,\end{aligned} \tag{1}$$

subject to the boundary condition

$$\phi(a,t) = \phi(b,t), \quad 0 < t \le T, \tag{2}$$

and the initial conditions

$$\phi(y,0) = \phi_0(y), \phi_t(y,0) = \phi_1(y), \quad y \in (a,b), \tag{3}$$

where β and κ are positive real constants, $1 < \alpha \le 2$, and $i^2 = -1$. $\phi_0(y)$ and $\phi_1(y)$ are given real functions. The fractional

Laplacian operator $(-\Delta)^{\alpha/2}$ can be defined as a pseudo-differential operator with the symbol $-|\xi|^{\alpha}$:

$$-(-\Delta)^{\alpha/2}u(x,t) := -\mathscr{F}^{-1}\left(|\xi|^{\alpha}\hat{u}(\xi,t)\right), \tag{4}$$

where $\mathscr{F}$ is the Fourier transform and $\hat{u}$ is the Fourier transform of u.

The spectral method is a generalization of a standard separation variable method, for which Chebyshev polynomials and Legendre polynomials are generally used as the basic functions of approximate expansions. And the Fourier series is convenient to deal with the periodic boundary conditions. Bridges and Reich [12] first put forward the Hamiltonian system using the Fourier spectrum discrete method in 2001. Based on their theoretical ideas, Chen and Qin [13] in the same year proposed the Fourier pseudo-spectral method for the Hamiltonian partial differential equation and used it to integrate the nonlinear Schrödinger equation with periodic boundary conditions. For more comprehensive work on the different conservative Fourier pseudo-spectral methods, refer to [2, 14–16] and their references.

Since the equation is calculated on a finite interval $[a,b]$, it is converted into periodic boundary conditions in this paper and studied on $\Omega=[0,2\pi]$ and $I=[0,T]$ below. Let

$$x = \frac{(2y-a-b)\pi}{b-a} + \pi, \quad x \in [0,2\pi]. \tag{5}$$

Denote $u(x,t)=\phi(y,t)$, $u(0,t)=u(2\pi,t)$, $u_0(x)=\phi_0(y)$, and $u_1(x)=\phi_1(y)$. Thus, (1)–(3) become

$$\begin{aligned} &u_{tt}(x,t) + M^{\alpha}(-\Delta)^{\alpha/2}u(x,t) + i\kappa u_t(x,t) \\ &\quad + \beta|u(x,t)|^2u(x,t) = 0, \quad x \in (0,2\pi), 0 < t \le T, \end{aligned} \tag{6}$$

$$u(0,t) = u(2\pi,t), \tag{7}$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), \quad x \in (0,2\pi), \tag{8}$$

where $M = 2\pi/(b-a)$.

In fact, the nonlinear fractional Schrödinger equation ((6)–(8)) has two conserved quantities:

$$Q(t) = Q(0), E(t) = E(0), \quad 0 \le t \le T, \tag{9}$$

where

$$\begin{aligned} Q(t) &= \frac{\kappa}{2}\|u(\cdot,t)\|_{L^2}^2 + \mathrm{Im}\,(u_t,u), \\ E(t) &= \|u_t(\cdot,t)\|_{L^2}^2 + M^{\alpha}\left\|(-\Delta)^{\alpha/4}u(\cdot,t)\right\|_{L^2}^2 + \frac{\beta}{2}\|u(\cdot,t)\|_{L^4}^4, \end{aligned} \tag{10}$$

with

$$\|u(\cdot,t)\|_{L^p}^p = \int_{\Omega}|u(x,t)|^p dx, \quad p = 2,4. \tag{11}$$

The outline of the remainder of this paper is as follows. In Section 2, a conserved Crank-Nicolson Fourier Galerkin method and a conserved Crank-Nicolson Fourier collocation method are constructed to discrete time variables and spatial variables. Energy-preserving and mass-preserving properties of the new method are investigated, and the error estimate is derived in Section 3. In Section 4, numerical experiments are presented to illustrate the theoretical results. Finally, the conclusions are given in Section 5.

2. Crank-Nicolson Fourier Spectral Method and Conservation Laws

Let $C_{\text{per}}^{\infty}(\Omega)$ be the set of all complex-valued and 2π-periodic C^{∞}-functions on Ω. Denote $(\cdot,\cdot)$ as the inner product on the space $L_{\text{per}}^2(\Omega)$ with the L^2 norm $\|\cdot\|_{L_{\text{per}}^2(\Omega)}$ (abbreviated as $\|\cdot\|$):

$$\begin{aligned} (u(x),v(x)) &= \int_{\Omega} u(x)\bar{v}(x)dx, \\ \|u(x)\|_{L_{\text{per}}^2(\Omega)} &= \sqrt{(u(x),u(x))}. \end{aligned} \tag{12}$$

For μ as a nonnegative real number, let $H_{\text{per}}^{\mu}(\Omega)$ be the closure of $C_{\text{per}}^{\infty}(\Omega)$. Note that $H_{\text{per}}^{0}(\Omega) = L_{\text{per}}^2(\Omega)$. For any function $u(x) \in L_{\text{per}}^2(\Omega)$, the following equations [17] can be developed easily:

$$u(x) = \sum_{\omega\in\mathbb{Z}} \hat{u}_{\omega}e^{i\omega x}, \tag{13}$$

where the Fourier coefficients are arranged as

$$\hat{u}_{\omega} = \left(u, e^{i\omega x}\right) = \frac{1}{2\pi}\int_{\Omega} ue^{-i\omega x}dx. \tag{14}$$

For the Fourier transform of fractional Laplacian $-(-\Delta)^{\alpha/2}$, we have

$$\mathscr{F}\left\{-(-\Delta)^{\alpha/2}u(x,t)\right\}(\omega) = -|\omega|^{\alpha}\mathscr{F}\{u(x,t)\}(\omega). \tag{15}$$

In order to discretize the equation in the temporal direction, the time step is defined by $\tau = T/N_t$. Denote difference operator

$$\begin{aligned} \delta_t^2u^n &= \frac{u^{n+1} - 2u^n + u^{n-1}}{\tau^2}, \delta_t u^n = \frac{u^{n+1} - u^n}{\tau}, \delta_{\hat{t}}u^n \\ &= \frac{u^{n+1} - u^{n-1}}{2\tau}, \tilde{u}^n = \frac{u^{n-1} + u^{n+1}}{2}, \end{aligned} \tag{16}$$

where n is a positive integer $(0 \le n \le N_t)$. Therefore, the Crank-Nicolson method was used to discretize equation (6) in the time axis.

$$i\kappa\delta_{\hat{t}}u^n = -\delta_t{}^2u^n - M^\alpha(-\Delta)^{\alpha/2}\tilde{u}^n - \frac{\beta}{2}\left(\left|u^{n-1}\right|^2 + \left|u^{n+1}\right|^2\right)\tilde{u}^n + R^n, \tag{17}$$

where $R^n = O(\tau^2)$.

2.1. Crank-Nicolson Fourier Galerkin Method. For positive even number N, the basis function space can be constructed as

$$S_N = \text{span}\left\{ e^{i\omega x} \middle| -\frac{N}{2} \le \omega \le \frac{N}{2} - 1 \right\}, \tag{18}$$

where the norm and seminorm of $H^\alpha_{\text{per}}(\Omega)$ are characterized by

$$\|u\|_\alpha \triangleq \left(\sum_{\omega=-N/2}^{(N/2)-1} \left(1 + |\omega|^{2\alpha}\right) |\widehat{u}_\omega|^2 \right)^{1/2}, \quad |u|_\alpha \triangleq \left(\sum_{\omega=-N/2}^{(N/2)-1} |\omega|^{2\alpha} |\widehat{u}_\omega|^2 \right)^{1/2}. \tag{19}$$

Let

$$u_N(t) = \sum_{\omega=-N/2}^{(N/2)-1} \widehat{u}_\omega(t) e^{i\omega x}. \tag{20}$$

The orthogonal operators $P_N : L^2_{\text{per}}(\Omega) \to S_N$ are defined as follows:

$$(P_N u - u, v) = 0, \quad \forall v \in S_N. \tag{21}$$

Lemma 1 [18, 19]. *Suppose that $u \in H^s_{per}(\Omega)$ for all $0 \le \mu \le s$; it holds that*

$$\|u - P_N u\|_\mu \le C N^{\mu - s} \|u\|_s. \tag{22}$$

Denote

$$u^n_N = \sum_{\omega=-N/2}^{(N/2)-1} \widehat{u}^n_\omega e^{i\omega x}, \quad n = 0, 1, \cdots, N_t. \tag{23}$$

The time variables of equation (6) are discretized by the Crank-Nicolson method. And the discrete Fourier Galerkin approximation for equation (6) has a modified scheme as follows:

$$\begin{aligned} &\left(\tau\kappa u^{n+1}_N - \tau^2 M^\alpha i(-\Delta)^{\alpha/2} u^{n+1}_N - 2iu^{n+1}_N, v\right) \\ &\quad = \left(\tau\kappa u^{n-1}_N + \tau^2 M^\alpha i(-\Delta)^{\alpha/2} u^{n-1}_N + 2iu^{n-1}_N, v\right) \\ &\quad - (4iu^n_N, v) + \frac{\beta\tau^2 i}{2}\left(\left(\left|u^{n+1}_N\right|^2 u^{n+1}_N + \left|u^{n-1}_N\right|^2 u^{n-1}_N \right.\right. \\ &\quad \left.\left. + \left|u^{n+1}_N\right|^2 u^{n-1}_N + \left|u^{n-1}_N\right|^2 u^{n+1}_N\right), v\right), \end{aligned} \tag{24}$$

$$u^0_N = P_N u_0, \delta_{\hat{t}} u^0_N = P_N u_1, \tag{25}$$

where $u^{n+1}_N \in S_N$, $\forall v \in S_N$.

2.2. Crank-Nicolson Fourier Collocation Method. For positive even number N, consider the points $x_j = 2\pi j/N$, $j = 0, 1, \cdots, N-1$, as collocation nodes. The discrete Fourier coefficients [18] of a function u on $[0, 2\pi]$ with respect to the collocation points are the following form:

$$\widehat{u}_\omega = \frac{1}{N}\sum_{j=0}^{N-1} u(x_j) e^{-i\omega x_j}, \quad \omega = -\frac{N}{2}, \cdots, \frac{N}{2} - 1. \tag{26}$$

Using the inversion formula, we have

$$u(x_j) = \sum_{\omega=-N/2}^{(N/2)-1} \widehat{u}_\omega e^{i\omega x_j}, \quad j = 0, 1, \cdots, N-1. \tag{27}$$

Define the interpolation operator I_N [18] at the collocation points:

$$I_N u(x_j) = u(x_j), \quad j = 0, 1, \cdots, N-1. \tag{28}$$

According to (27),

$$(I_N u)(x) = \sum_{\omega=-N/2}^{(N/2)-1} \widehat{u}_\omega e^{i\omega x}. \tag{29}$$

Lemma 2 [18, 19]. *For any $u \in H^s_{per}(\Omega)$, $s \ge 1$, the estimate*

$$\|u - I_N u\| \le C N^{-s} \|u\|_s \tag{30}$$

in the sense of the Sobolev norm.

Combining Lemma 2 and the triangle inequality, Corollary 3 is drawn.

Corollary 3. *For any $u \in H^s_{per}(\Omega)$, $s \ge 1$, there exists a constant C independent of u and N, such that*

$$\|I_N u\| \le C N^{-s} \|u\|_s + \|u\|. \tag{31}$$

Using the Fourier collocation method to discrete the spatial variables of the equation, we get the fully discrete scheme for equations (6)–(8) as the following forms:

$$\begin{aligned} &\tau\kappa u^{n+1}_N(x_j) - \tau^2 M^\alpha i(-\Delta)^{\alpha/2} u^{n+1}_N(x_j) - 2iu^{n+1}_N(x_j) \\ &= \tau\kappa u^{n-1}_N(x_j) + \tau^2 M^\alpha i(-\Delta)^{\alpha/2} u^{n-1}_N(x_j) \\ &\quad + 2iu^{n-1}_N(x_j) - 4iu^n_N(x_j) + \frac{\beta\tau^2 i}{2} I_N \\ &\quad \cdot \left(\left|u^{n+1}_N(x_j)\right|^2 u^{n+1}_N(x_j) + \left|u^{n-1}_N(x_j)\right|^2 u^{n-1}_N(x_j)\right) \\ &\quad + \frac{\beta\tau^2 i}{2} I_N\left(\left|u^{n+1}_N(x_j)\right|^2 u^{n-1}_N(x_j) + \left|u^{n-1}_N(x_j)\right|^2 u^{n+1}_N(x_j)\right), \end{aligned} \tag{32}$$

$$u_N^0(x_j) = u_0(x_j), \delta_{\hat{t}} u_N^0(x_j) = u_1(x_j). \tag{33}$$

Applying the Fourier transformation to (24), we get the following form:

$$\begin{aligned}(\tau\kappa - \tau^2 M^\alpha i|\omega|^\alpha - 2i)\widehat{u}_{N\omega}^{n+1} &= (\tau\kappa + \tau^2 M^\alpha i|\omega|^\alpha + 2i)\widehat{u}_{N\omega}^{n-1} \\ &\quad - 4i\widehat{u}_{N\omega}^n + \frac{\beta\tau^2 i}{2}\widehat{F}_{N\omega}^n, \\ \left(\widehat{u_N^0}\right)_\omega = \left(\widehat{u_0}\right)_\omega, &\left(\widehat{\delta_t u_N^0}\right)_\omega = \left(\widehat{u_1}\right)_\omega,\end{aligned} \tag{34}$$

where $F_N^n = (|u_N^{n+1}|^2 u_N^{n+1} + |u_N^{n-1}|^2 u_N^{n-1} + |u_N^{n+1}|^2 u_N^{n-1} + |u_N^{n-1}|^2 u_N^{n+1})$, $\omega = -N/2, \cdots N/2 - 1$.

2.3. Theory Analysis of Conservation

Theorem 4. *The Crank-Nicolson Fourier Galerkin method (24) of solving equations (6)–(8) preserves the discrete mass and discrete energy:*

$$\begin{aligned}Q^n &= Q^0, \quad 0 \le n \le N_t, \\ E^n &= E^0, \quad 0 \le n \le N_t,\end{aligned} \tag{35}$$

where

$$\begin{aligned}Q^n &= \mathrm{Im}\,(\delta_t u_N^n, u_N^n) + \frac{\kappa}{4}\left(\|u_N^{n+1}\|^2 + \|u_N^n\|^2\right), \\ E^n &= \|\delta_t u_N^n\|^2 + \frac{M^\alpha}{2}\left(\|(-\Delta)^{\alpha/4} u_N^{n+1}\|^2 + \|(-\Delta)^{\alpha/4} u_N^n\|^2\right) \\ &\quad + \frac{\beta}{4}\left(\|u_N^{n+1}\|_{L^4(\Omega)}^4 + \|u_N^n\|_{L^4(\Omega)}^4\right).\end{aligned} \tag{36}$$

Proof. We derive the full discrete Fourier Galerkin method:

$$\begin{aligned}(i\kappa\delta_{\hat{t}} u_N^n, v) &= -\left(\delta_t^{\,2} u_N^n, v\right) - M^\alpha(-\Delta)^{\alpha/2}(\tilde{u}_N^n, v) \\ &\quad - \frac{\beta}{2}\left(\left(|u_N^{n-1}|^2 + |u_N^{n+1}|^2\right)\tilde{u}_N^n, v\right).\end{aligned} \tag{37}$$

Let $v = \tilde{u}_N^n$ in equation (37); it holds that

$$\begin{aligned}(i\kappa\delta_{\hat{t}} u_N^n, \tilde{u}_N^n) &= -\left(\delta_t^{\,2} u_N^n, \tilde{u}_N^n\right) - M^\alpha\left((-\Delta)^{\alpha/2}\tilde{u}_N^n, \tilde{u}_N^n\right) \\ &\quad - \frac{\beta}{2}\left(\left(|u_N^{n-1}|^2 + |u_N^{n+1}|^2\right)\tilde{u}_N^n, \tilde{u}_N^n\right).\end{aligned} \tag{38}$$

Taking the imaginary part of equation (38), due to

$$\begin{aligned}\mathrm{Im}\,(i\kappa\delta_{\hat{t}} u_N^n, \tilde{u}_N^n) &= \kappa\,\mathrm{Re}\,(\delta_{\hat{t}} u_N^n, \tilde{u}_N^n) \\ &= \kappa\,\mathrm{Re}\left(\frac{u_N^{n+1} - u_N^{n-1}}{2\tau}, \frac{u_N^{n-1} + u_N^{n+1}}{2}\right) \\ &= \frac{\kappa}{4\tau}\left(\|u_N^{n+1}\|^2 - \|u_N^{n-1}\|^2\right), \\ \mathrm{Im}\,(\delta_t^{\,2} u_N^n, \tilde{u}_N^n) &= \frac{1}{2\tau}\left[\mathrm{Im}\,(\delta_t u_N^n, u_N^{n-1} + u_N^{n+1})\right. \\ &\quad \left. - \mathrm{Im}\,(\delta_t u_N^{n-1}, u_N^{n-1} + u_N^{n+1})\right] \\ &= \frac{1}{2\tau}\left[2\,\mathrm{Im}\,(\delta_t u_N^n, u_N^n) + \mathrm{Im}\,(\delta_t u_N^n, u_N^{n+1} - u_N^n)\right. \\ &\quad - \mathrm{Im}\,(\delta_t u_N^n, u_N^n - u_N^{n-1}) - \mathrm{Im}\,(\delta_t u_N^{n-1}, u_N^{n+1} - u_N^n) \\ &\quad \left. - \mathrm{Im}\,(\delta_t u_N^{n-1}, u_N^n - u_N^{n-1}) - 2\,\mathrm{Im}\,(\delta_t u_N^{n-1}, u_N^{n-1})\right] \\ &= \frac{1}{\tau}\left[\mathrm{Im}\,(\delta_t u_N^n, u_N^n) - \mathrm{Im}\,(\delta_t u_N^{n-1}, u_N^{n-1})\right], \\ \mathrm{Im}\,&\left((-\Delta)^{\alpha/2}\tilde{u}_N^n, \tilde{u}_N^n\right) = 0, \\ \mathrm{Im}\,&\left(\left(|u_N^{n-1}|^2 + |u_N^{n+1}|^2\right)\tilde{u}_N^n, \tilde{u}_N^n\right) = 0.\end{aligned} \tag{39}$$

Therefore,

$$\begin{aligned}&\frac{\mathrm{Im}\,(\delta_t u_N^n, u_N^n) - \mathrm{Im}\,(\delta_t u_N^{n-1}, u_N^{n-1})}{\tau} \\ &\quad + \frac{\kappa}{4\tau}\left(\|u_N^{n+1}\|^2 - \|u_N^{n-1}\|^2\right) = 0,\end{aligned} \tag{40}$$

thus,

$$\begin{aligned}&\mathrm{Im}\,(\delta_t u_N^n, u_N^n) + \frac{\kappa}{4}\left(\|u_N^{n+1}\|^2 + \|u_N^n\|^2\right) \\ &\quad = \mathrm{Im}\,(\delta_t u_N^{n-1}, u_N^{n-1}) + \frac{\kappa}{4}\left(\|u_N^n\|^2 + \|u_N^{n-1}\|^2\right).\end{aligned} \tag{41}$$

The above equality indicates that the method (24) maintains the conservation of the discrete mass. The following items consider the conservation of the discrete energy.

Let $v = \delta_{\hat{t}} u_N^n$; according to equation (37), we also get

$$\begin{aligned}(i\kappa\delta_{\hat{t}} u_N^n, \delta_{\hat{t}} u_N^n) &= -\left(\delta_t^{\,2} u_N^n, \delta_{\hat{t}} u_N^n\right) - M^\alpha\left((-\Delta)^{\alpha/2}\tilde{u}_N^n, \delta_{\hat{t}} u_N^n\right) \\ &\quad - \frac{\beta}{2}\left(\left(|u_N^{n-1}|^2 + |u_N^{n+1}|^2\right)\tilde{u}_N^n, \delta_{\hat{t}} u_N^n\right).\end{aligned} \tag{42}$$

Taking the real part of (42), due to

$$\mathrm{Re}\,(i\kappa\delta_{\hat{t}} u_N^n, \delta_{\hat{t}} u_N^n) = 0, \tag{43}$$

$$\begin{aligned}\mathrm{Re}\,\left(\delta_t^{\,2} u_N^n, \delta_{\hat{t}} u_N^n\right) &= \frac{1}{2\tau}\,\mathrm{Re}\,\left(\delta_t u_N^n - \delta_t u_N^{n-1}, \delta_t u_N^n + \delta_t u_N^{n-1}\right) \\ &= \frac{\|\delta_t u_N^n\|^2 - \|\delta_t u_N^{n-1}\|^2}{2\tau},\end{aligned} \tag{44}$$

$$\begin{aligned}
&\operatorname{Re}\left((-\Delta)^{\alpha/2}\tilde{u}_N^n, \delta_{\hat{t}} u_N^n\right) \\
&= \operatorname{Re}\left((-\Delta)^{\alpha/2}\frac{u_N^{n-1}+u_N^{n+1}}{2}, \frac{u_N^{n+1}-u_N^{n-1}}{2\tau}\right) \\
&= \frac{1}{4\tau}\left(\left((-\Delta)^{\alpha/4}u_N^{n+1}, (-\Delta)^{\alpha/4}u_N^{n+1}\right)\right. \\
&\quad \left.-\left((-\Delta)^{\alpha/4}u_N^{n-1}, (-\Delta)^{\alpha/4}u_N^{n-1}\right)\right) \\
&= \frac{\left\|(-\Delta)^{\alpha/4}u_N^{n+1}\right\|^2 - \left\|(-\Delta)^{\alpha/4}u_N^{n-1}\right\|^2}{4\tau},
\end{aligned} \tag{45}$$

$$\begin{aligned}
&\operatorname{Re}\left(\left(\left|u_N^{n-1}\right|^2+\left|u_N^{n+1}\right|^2\right)\tilde{u}_N^n, \delta_{\hat{t}} u_N^n\right) \\
&= \operatorname{Re}\left(\left(\left|u_N^{n-1}\right|^2+\left|u_N^{n+1}\right|^2\right)\frac{u_N^{n-1}+u_N^{n+1}}{2}, \frac{u_N^{n+1}-u_N^{n-1}}{2\tau}\right) \\
&= \frac{1}{4\tau}\left(\left\|u_N^{n+1}\right\|_{L^4(\Omega)}^4 - \left\|u_N^{n-1}\right\|_{L^4(\Omega)}^4\right),
\end{aligned} \tag{46}$$

therefore, using (43)–(46), we obtain

$$\begin{aligned}
&\frac{\left\|\delta_t u_N^n\right\|^2 - \left\|\delta_t u_N^{n-1}\right\|^2}{2\tau} + \frac{M^\alpha}{4\tau}\left(\left\|(-\Delta)^{\alpha/4}u_N^{n+1}\right\|^2\right. \\
&\quad \left.-\left\|(-\Delta)^{\alpha/4}u_N^{n-1}\right\|^2\right) + \frac{\beta}{8\tau}\left(\left\|u_N^{n+1}\right\|_{L^4(\Omega)}^4 - \left\|u_N^{n-1}\right\|_{L^4(\Omega)}^4\right) = 0,
\end{aligned} \tag{47}$$

thus,

$$\begin{aligned}
&\left\|\delta_t u_N^n\right\|^2 + \frac{M^\alpha}{2}\left(\left\|(-\Delta)^{\alpha/4}u_N^{n+1}\right\|^2 + \left\|(-\Delta)^{\alpha/4}u_N^n\right\|^2\right) \\
&\quad + \frac{\beta}{4}\left(\left\|u_N^{n+1}\right\|_{L^4(\Omega)}^4 + \left\|u_N^n\right\|_{L^4(\Omega)}^4\right) \\
&= \left\|\delta_t u_N^{n-1}\right\|^2 + \frac{M^\alpha}{2}\left(\left\|(-\Delta)^{\alpha/4}u_N^n\right\|^2 + \left\|(-\Delta)^{\alpha/4}u_N^{n-1}\right\|^2\right) \\
&\quad + \frac{\beta}{4}\left(\left\|u_N^n\right\|_{L^4(\Omega)}^4 + \left\|u_N^{n-1}\right\|_{L^4(\Omega)}^4\right).
\end{aligned} \tag{48}$$

Based on the above analysis, the method (24) also maintains the conservation of the discrete energy. □

3. Theory Analysis of Convergence

In order to simplify the notation, we always assume that C is a positive constant in this article, which might be different in every formula.

Lemma 5 [20]. *For any discrete function u_N^n, it holds that*

$$\left\|u_N^{n+1}\right\|^2 - \left\|u_N^n\right\|^2 \le \tau\left(\left\|\delta_t u_N^n\right\|^2 + \frac{1}{2}\left(\left\|u_N^{n+1}\right\|^2 + \left\|u_N^n\right\|^2\right)\right). \tag{49}$$

Lemma 6. *For $u_N \in H_{per}^\mu(\Omega)$, there exists a positive constant C, such that*

$$\left\|u_N^n\right\| \le C, \left\|(-\Delta)^{\alpha/4}u_N^n\right\| \le C, \left\|\delta_t u_N^n\right\| \le C, \quad n = 0, 1, \cdots, N_t. \tag{50}$$

Proof. Using Theorem 4, it yields

$$\begin{aligned}
&\left\|\delta_t u_N^n\right\|^2 + \frac{M^\alpha}{2}\left(\left\|(-\Delta)^{\alpha/4}u_N^{n+1}\right\|^2 + \left\|(-\Delta)^{\alpha/4}u_N^n\right\|^2\right) \\
&\quad + \frac{\beta}{4}\left(\left\|u_N^{n+1}\right\|_{L^4(\Omega)}^4 + \left\|u_N^n\right\|_{L^4(\Omega)}^4\right) = E^n = E^0,
\end{aligned} \tag{51}$$

thus,

$$\begin{aligned}
&\left\|\delta_t u_N^n\right\|^2 + \frac{M^\alpha}{2}\left(\left\|(-\Delta)^{\alpha/4}u_N^{n+1}\right\|^2 + \left\|(-\Delta)^{\alpha/4}u_N^n\right\|^2\right) \\
&\quad = E^0 - \frac{\beta}{4}\left(\left\|u_N^{n+1}\right\|_{L^4(\Omega)}^4 + \left\|u_N^n\right\|_{L^4(\Omega)}^4\right).
\end{aligned} \tag{52}$$

Because of $\beta > 0$, it satisfies

$$\left\|\delta_t u_N^n\right\|^2 + \frac{M^\alpha}{2}\left(\left\|(-\Delta)^{\alpha/4}u_N^{n+1}\right\|^2 + \left\|(-\Delta)^{\alpha/4}u_N^n\right\|^2\right) \le C. \tag{53}$$

Sum the inequalities of Lemma 5 from 0 to n yields

$$\begin{aligned}
\left(1-\frac{\tau}{2}\right)\left\|u_N^{n+1}\right\|^2 &\le \left(1+\frac{\tau}{2}\right)\left\|u_N^0\right\|^2 + \tau\left\|\delta_t u_N^0\right\|^2 \\
&\quad + \tau\sum_{k=1}^n\left(\left\|\delta_t u_N^k\right\|^2 + \left\|u_N^k\right\|^2\right).
\end{aligned} \tag{54}$$

Adding (53) and (54), we can obtain the following items:

$$\begin{aligned}
&\left(1-\frac{\tau}{2}\right)\left\|u_N^{n+1}\right\|^2 + \left\|\delta_t u_N^n\right\|^2 \\
&\quad + \frac{M^\alpha}{2}\left(\left\|(-\Delta)^{\alpha/4}u_N^{n+1}\right\|^2 + \left\|(-\Delta)^{\alpha/4}u_N^n\right\|^2\right) \\
&\le C + \tau\sum_{k=1}^n\left(\left\|\delta_t u_N^k\right\|^2 + \left\|u_N^k\right\|^2\right).
\end{aligned} \tag{55}$$

For τ is sufficiently small ($\tau < 1$), this implies

$$\begin{aligned}
&\frac{1}{2}\left\|u_N^{n+1}\right\|^2 + \left\|\delta_t u_N^n\right\|^2 \\
&\quad + \frac{M^\alpha}{2}\left(\left\|(-\Delta)^{\alpha/4}u_N^{n+1}\right\|^2 + \left\|(-\Delta)^{\alpha/4}u_N^n\right\|^2\right) \\
&\le C + 2\tau\sum_{k=1}^n\left(\frac{1}{2}\left\|u_N^k\right\|^2 + \left\|\delta_t u_N^{k-1}\right\|^2\right. \\
&\quad \left.+ \frac{M^\alpha}{2}\left(\left\|(-\Delta)^{\alpha/4}u_N^k\right\|^2 + \left\|(-\Delta)^{\alpha/4}u_N^{k-1}\right\|^2\right)\right).
\end{aligned} \tag{56}$$

According to the discrete Gronwall's inequality, there is

$$\frac{1}{2}\left\|u_N^{n+1}\right\|^2+\left\|\delta_t u_N^n\right\|^2+\frac{M^\alpha}{2}\left(\left\|(-\Delta)^{\alpha/4}u_N^{n+1}\right\|^2+\left\|(-\Delta)^{\alpha/4}u_N^n\right\|^2\right)\le C. \tag{57}$$

Therefore,

$$\left\|u_N^n\right\|\le C, \left\|(-\Delta)^{\alpha/4}u_N^n\right\|^2\le C, \left\|\delta_t u_N^n\right\|\le C. \tag{58}$$

□

Theorem 7. *If $s\ge 1$, assume that $u\in C^2(I;H_{per}^\alpha(\Omega)\cap H^s(\Omega))$ is the exact solution of (6)–(8), and u_N^n is the numerical solution of (24). It possesses the following conclusion:*

$$\left\|u_N^n-u(x,t_n)\right\|\le C\left(\tau^2+N^{-s}\|u\|_s\right). \tag{59}$$

Proof. Let $u^*=P_N u$, $e=u-u_N$, $\xi=u-u^*$, and $\eta=u^*-u_N$; then, $e^n=\xi^n+\eta^n$. From triangle inequality and Lemma 1, it yields

$$\|e^n\|\le\|\xi^n\|+\|\eta^n\|\le CN^{-s}\|u\|_s+\|\eta^n\|. \tag{60}$$

According to the orthogonality of the projection operator P_N, we get

$$i(\kappa\delta_{\hat t}u^{*n},v)=-\left(\delta_t{}^2u^n,v\right)-M^\alpha\left((-\Delta)^{\alpha/2}\tilde u^n,v\right)-\frac{\beta}{2}\left(\left(\left|u^{n-1}\right|^2+\left|u^{n+1}\right|^2\right)\tilde u^n,v\right)+(R^n,v). \tag{61}$$

The authors derive the full discrete Fourier Galerkin method:

$$(i\kappa\delta_{\hat t}u_N^n,v)=-\left(\delta_t{}^2u_N^n,v\right)-M^\alpha\left((-\Delta)^{\alpha/2}\tilde u_N^n,v\right)-\frac{\beta}{2}\left(\left(\left|u_N^{n-1}\right|^2+\left|u_N^{n+1}\right|^2\right)\tilde u_N^n,v\right). \tag{62}$$

Subtracting equation (62) from equation (61), due to

$$\begin{aligned}
\delta_{\hat t}u^{*n}-\delta_{\hat t}u_N^n&=\frac{u^{*n+1}-u^{*n-1}}{2\tau}-\frac{u_N^{n+1}-u_N^{n-1}}{2\tau}=\frac{\eta^{n+1}-\eta^{n-1}}{2\tau},\\
-\delta_t{}^2u^n+\delta_t{}^2u_N^n&=-\frac{u^{n+1}-2u^n+u^{n-1}}{\tau^2}+\frac{u_N^{n+1}-2u_N^n+u_N^{n-1}}{\tau^2}\\
&=-\frac{e^{n+1}-2e^n+e^{n-1}}{\tau^2},\\
(-\Delta)^{\alpha/2}\tilde u_N^n-(-\Delta)^{\alpha/2}\tilde u^n&=(-\Delta)^{\alpha/2}\frac{u_N^{n-1}+u_N^{n+1}}{2}-(-\Delta)^{\alpha/2}\frac{u^{n-1}+u^{n+1}}{2}\\
&=-(-\Delta)^{\alpha/2}\frac{\left(u^{n-1}-u_N^{n-1}\right)+\left(u^{n+1}-u_N^{n+1}\right)}{2}\\
&=-(-\Delta)^{\alpha/2}\frac{e^{n-1}+e^{n+1}}{2},
\end{aligned} \tag{63}$$

thus,

$$i\kappa(\delta_{\hat t}\eta^n,v)=-\left(\delta_t^2e^n,v\right)-M^\alpha\left((-\Delta)^{\alpha/2}\tilde e^n,v\right)-\frac{\beta}{2}\left(\left(\left|u^{n-1}\right|^2+\left|u^{n+1}\right|^2\right)\tilde u^n-\left(\left|u_N^{n-1}\right|^2+\left|u_N^{n+1}\right|^2\right)\tilde u_N^n,v\right)+(R^n,v). \tag{64}$$

According to the orthogonality of operator P_N, i.e., $(P_N u-u,v)=0,\forall v\in S_N$. Therefore,

$$\left(e^j,\eta^k\right)=\left(\xi^j+\eta^j,\eta^k\right)=\left(\xi^j,\eta^k\right)+\left(\eta^j,\eta^k\right)=\left(\eta^j,\eta^k\right), j,k=0,1,\cdots,N_t. \tag{65}$$

Let $v=\delta_{\hat t}\eta^n$ in (64), and taking the real part, due to

$$\mathrm{Re}\,(i\kappa(\delta_{\hat t}\eta^n,\delta_{\hat t}\eta^n))=0, \tag{66}$$

$$\begin{aligned}
\mathrm{Re}\left(\delta_t{}^2e^n,\delta_{\hat t}\eta^n\right)&=\mathrm{Re}\left(\delta_t{}^2\eta^n,\delta_{\hat t}\eta^n\right)\\
&=\frac{1}{2\tau}\mathrm{Re}\left(\delta_t\eta^n-\delta_t\eta^{n-1},\delta_t\eta^n+\delta_t\eta^{n-1}\right)\\
&=\frac{\|\delta_t\eta^n\|^2-\left\|\delta_t\eta^{n-1}\right\|^2}{2\tau},
\end{aligned} \tag{67}$$

$$\begin{aligned}
&\mathrm{Re}\left((-\Delta)^{\alpha/2}\tilde e^n,\delta_{\hat t}\eta^n\right)\\
&=\mathrm{Re}\left((-\Delta)^{\alpha/2}\tilde\eta^n,\delta_{\hat t}\eta^n\right)\\
&=\mathrm{Re}\left((-\Delta)^{\alpha/2}\frac{\eta^{n-1}+\eta^{n+1}}{2},\frac{\eta^{n+1}-\eta^{n-1}}{2\tau}\right)\\
&=\frac{1}{4\tau}\left(\left((-\Delta)^{\alpha/4}\eta^{n+1},(-\Delta)^{\alpha/4}\eta^{n+1}\right)-\left((-\Delta)^{\alpha/4}\eta^{n-1},(-\Delta)^{\alpha/4}\eta^{n-1}\right)\right)\\
&=\frac{\left\|(-\Delta)^{\alpha/4}\eta^{n+1}\right\|^2-\left\|(-\Delta)^{\alpha/4}\eta^{n-1}\right\|^2}{4\tau},
\end{aligned} \tag{68}$$

therefore, using (66)-(68), this implies

$$\mathrm{Re}\,(R^n,\delta_{\hat t}\eta^n)=\frac{\|\delta_t\eta^n\|^2-\left\|\delta_t\eta^{n-1}\right\|^2}{2\tau}+\frac{M^\alpha}{4\tau}\left(\left\|(-\Delta)^{\alpha/4}\eta^{n+1}\right\|^2-\left\|(-\Delta)^{\alpha/4}\eta^{n-1}\right\|^2\right)+\mathrm{Re}\,(G^n,\delta_{\hat t}\eta^n), \tag{69}$$

where

$$\begin{aligned}
|G^n|&=\frac{\beta}{2}\left(\left(\left|u^{n-1}\right|^2+\left|u^{n+1}\right|^2\right)\tilde u^n-\left(\left|u_N^{n-1}\right|^2+\left|u_N^{n+1}\right|^2\right)\tilde u_N^n\right)\\
&=\left|\frac{\beta}{2}\left(\left|u^{n-1}\right|^2+\left|u^{n+1}\right|^2\right)\frac{u^{n-1}+u^{n+1}}{2}-\frac{\beta}{2}\left(\left|u^{n-1}-e^{n-1}\right|^2+\left|u^{n+1}-e^{n+1}\right|^2\right)\frac{u_N^{n-1}+u_N^{n+1}}{2}\right|
\end{aligned}$$

$$
\begin{aligned}
&= \left| \frac{\beta}{2} \left(\left|u^{n-1}\right|^2 \tilde{e}^n + \left|u^{n+1}\right|^2 \tilde{e}^n + e^{n-1} \bar{u}_N^{n-1} \tilde{u}_N^n \right.\right. \\
&\quad \left.\left. + u^{n-1} \bar{e}^{n-1} \tilde{u}_N^n + e^{n+1} \bar{u}_N^{n+1} \tilde{u}_N^n + u^{n+1} \bar{e}^{n+1} \tilde{u}_N^n \right) \right| \\
&\le \frac{\beta}{2} \left(\max \left\{ \left|u^{n-1}\right|, \left|u^{n+1}\right|, \left|u_N^{n-1}\right|, \left|u_N^{n+1}\right| \right\}^2 \right. \\
&\quad \left. \cdot \left(2\left|e^{n-1}\right| + 2\left|e^{n+1}\right| + 2\left|\tilde{e}^n\right| \right) \right).
\end{aligned} \tag{70}
$$

Thus, according to Lemma 6, we can get

$$
\left|G^n\right|^2 \le C\left(\left|e^{n-1}\right|^2 + \left|e^{n+1}\right|^2 \right). \tag{71}
$$

Note Lemma 1; it gives that

$$
\begin{aligned}
\|G^n\|^2 &= \int_\Omega \left|G^n\right|^2 dx \le C\left(\left\|e^{n-1}\right\|^2 + \left\|e^{n+1}\right\|^2 \right) \\
&\le C\left(\left\|\eta^{n-1}\right\|^2 + \left\|\eta^{n+1}\right\|^2 \right) + CN^{-2s} \|u\|_s^2.
\end{aligned} \tag{72}
$$

Then,

$$
\begin{aligned}
\operatorname{Re}\left(G^n, \delta_{\hat{t}} \eta^n\right) &\le \frac{\|G^n\|^2 + \|\delta_{\hat{t}} \eta^n\|^2}{2} \\
&\le \frac{1}{2} \|\delta_{\hat{t}} \eta^n\|^2 + C\left(\left\|\eta^{n-1}\right\|^2 + \left\|\eta^{n+1}\right\|^2 \right) \\
&\quad + CN^{-2s} \|u\|_s^2, \\
\operatorname{Re}\left(R^n, \delta_{\hat{t}} \eta^n\right) &\le \frac{\|R^n\|^2 + \|\delta_{\hat{t}} \eta^n\|^2}{2} \le \frac{1}{2} \|\delta_{\hat{t}} \eta^n\|^2 + C\left(\tau^4\right).
\end{aligned} \tag{73}
$$

Thus, (69) becomes

$$
\begin{aligned}
&\frac{\|\delta_t \eta^n\|^2 - \left\|\delta_t \eta^{n-1}\right\|^2}{2\tau} \\
&\quad + \frac{M^\alpha}{4\tau} \left(\left\|(-\Delta)^{\alpha/4} \eta^{n+1}\right\|^2 - \left\|(-\Delta)^{\alpha/4} \eta^{n-1}\right\|^2 \right) \\
&\le C\left(\left\|\eta^{n-1}\right\|^2 + \left\|\eta^{n+1}\right\|^2 \right) + \|\delta_{\hat{t}} \eta^n\|^2 + C\left(\tau^4 + N^{-2s} \|u\|_s^2\right).
\end{aligned} \tag{74}
$$

Because of

$$
\delta_{\hat{t}} \eta^n = \frac{\delta_t \eta^n + \delta_t \eta^{n-1}}{2}, \tag{75}
$$

and from Lemma 5, it gives that

$$
\frac{\|\eta^n\|^2 - \left\|\eta^{n-1}\right\|^2}{\tau} \le \|\delta_t \eta^n\|^2 + \frac{1}{2} \left(\|\eta^n\|^2 + \left\|\eta^{n-1}\right\|^2 \right). \tag{76}
$$

Then, combining (74) and (76) leads to

$$
\begin{aligned}
&\frac{\|\delta_t \eta^n\|^2 - \left\|\delta_t \eta^{n-1}\right\|^2}{2\tau} + \frac{M^\alpha}{4\tau} \left(\left\|(-\Delta)^{\alpha/4} \eta^{n+1}\right\|^2 \right. \\
&\quad \left. - \left\|(-\Delta)^{\alpha/4} \eta^{n-1}\right\|^2 \right) + \frac{\|\eta^n\|^2 - \left\|\eta^{n-1}\right\|^2}{\tau} \\
&\le C\left(\|\delta_t \eta^n\|^2 + \left\|\delta_t \eta^{n-1}\right\|^2 + \|\eta^n\|^2 + \left\|\eta^{n+1}\right\|^2 + \left\|\eta^{n-1}\right\|^2 \right) \\
&\quad + C\left(\tau^4 + N^{-2s} \|u\|_s^2\right).
\end{aligned} \tag{77}
$$

Summing above inequalities (77) from 1 to n yields

$$
\begin{aligned}
&\frac{1}{2} \|\delta_t \eta^n\|^2 + \frac{M^\alpha}{4} \left(\left\|(-\Delta)^{\alpha/4} \eta^{n+1}\right\|^2 + \left\|(-\Delta)^{\alpha/4} \eta^n\right\|^2 \right) + \|\eta^n\|^2 \\
&\le \tau C \sum_{i=1}^n \left(\frac{1}{2} \left\|\delta_t \eta^i\right\|^2 + \frac{M^\alpha}{4} \left(\left\|(-\Delta)^{\alpha/4} \eta^{i+1}\right\|^2 \right.\right. \\
&\quad \left.\left. + \left\|(-\Delta)^{\alpha/4} \eta^i\right\|^2 \right) + \left\|\eta^i\right\|^2 \right) + C\left(\tau^4 + N^{-2s} \|u\|_s^2\right).
\end{aligned} \tag{78}
$$

Hence, using the discrete Gronwall's inequality gives

$$
\begin{aligned}
&\frac{1}{2} \|\delta_t \eta^n\|^2 + \frac{M^\alpha}{4} \left(\left\|(-\Delta)^{\alpha/4} \eta^{n+1}\right\|^2 + \left\|(-\Delta)^{\alpha/4} \eta^n\right\|^2 \right) + \|\eta^n\|^2 \\
&\le C\left(\tau^4 + N^{-2s} \|u\|_s^2\right),
\end{aligned} \tag{79}
$$

thus,

$$
\|\eta^n\|^2 \le C\left(\tau^4 + N^{-2s} \|u\|_s^2\right). \tag{80}
$$

Substituting (80) into (60) can yield

$$
\|e^n\| \le C\left(N^{-s} \|u\|_s + \tau^2\right), \tag{81}
$$

which immediately gives conclusion. □

Similar to the proof of Theorem 7, we can obtain the following theorem.

Theorem 8. *Let $s \ge 1$; assume that $u \in C^2(I; H_{per}^\alpha(\Omega) \cap H^s(\Omega))$ is the exact solution of (6)–(8), and u_N^n is the numerical solution of (32). It possesses the following conclusion:*

$$
\left\|u_N^n - u(x, t_n)\right\| \le C\left(\tau^2 + N^{-s} \|u\|_s\right). \tag{82}
$$

4. Numerical Example

Numerical examples will be proposed in this section to verify the correctness of the theoretical analysis, that is, the convergence of the numerical method and its ability to maintain discrete mass and discrete energy.

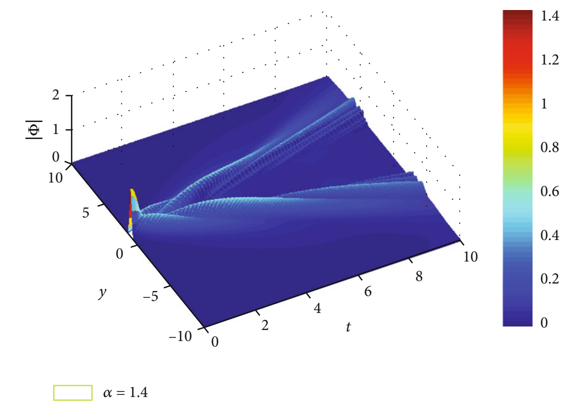

Figure 1: Numerical solutions for equation (83) with $\alpha = 1.4$ when $\tau = 0.01$ and $N = 128$.

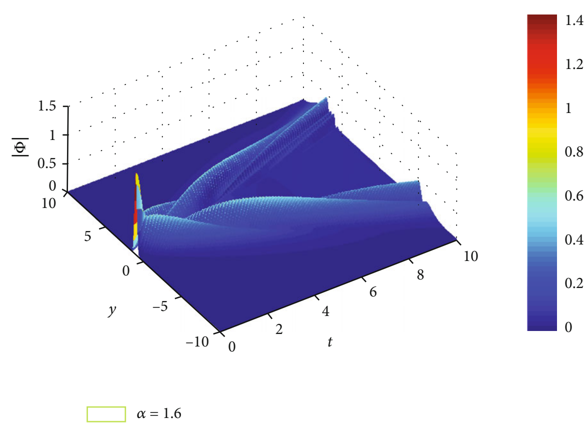

Figure 2: Numerical solutions for equation (83) with $\alpha = 1.6$ when $\tau = 0.01$ and $N = 128$.

Example 1. Consider the nonlinear fractional Schrödinger equation with the wave operator:

$$\begin{cases} \phi_{tt} + (-\Delta)^{\alpha/2}\phi + i\phi_t + |\phi(y,t)|^2\phi(y,t) = 0, & y \in (-10,10), t \in (0,T], \\ \phi(-10,t) = \phi(10,t) = 0, & t \in [0,T], \\ \phi(y,0) = (1+i)y \exp\left(-10(1-y)^2\right), \phi_t(y,0) = 0, & y \in (-10,10). \end{cases} \tag{83}$$

Let $\tau = 0.01$, $N = 128$, and $T = 10$. Figures 1 and 2 present the numerical solutions for $\alpha = 1.4$ and $\alpha = 1.6$. We can find that the order of α will affect the shape of the solution.

There is no exact solution of (83) known for $1 < \alpha < 2$. Therefore, numerical solution calculated by the method (24) with $N = 1024$ and $\tau = 2^{-10}$ is taken as the reference solution. Let Φ be the numerical solution, and calculate the error at $t = t_n$ in the sense of the discrete L^2 norm:

$$\text{error} = \|\phi^n - \Phi^n\|. \tag{84}$$

The convergence rates in the direction of time and space are calculated as

Table 1: Errors and convergence rates in time for $N = 512$ and $T = 1$.

α	τ	Error	Order
	2^{-5}	$1.9071e-2$	—
1.4	2^{-6}	$4.7789e-3$	1.9966
	2^{-7}	$1.1822e-3$	2.0152
	2^{-5}	$3.2228e-2$	—
1.6	2^{-6}	$7.8070e-3$	1.9952
	2^{-7}	$1.9320e-3$	2.0148

Table 2: Errors and convergence rates in space for $\tau = 2^{-10}$ and $T = 1$.

α	N	Error	Order
	128	$1.3922e-2$	—
1.4	256	$8.4405e-3$	0.7220
	512	$3.9789e-3$	1.0850
	128	$1.9930e-2$	—
1.6	256	$1.2081e-2$	0.7225
	512	$5.6949e-3$	1.0850

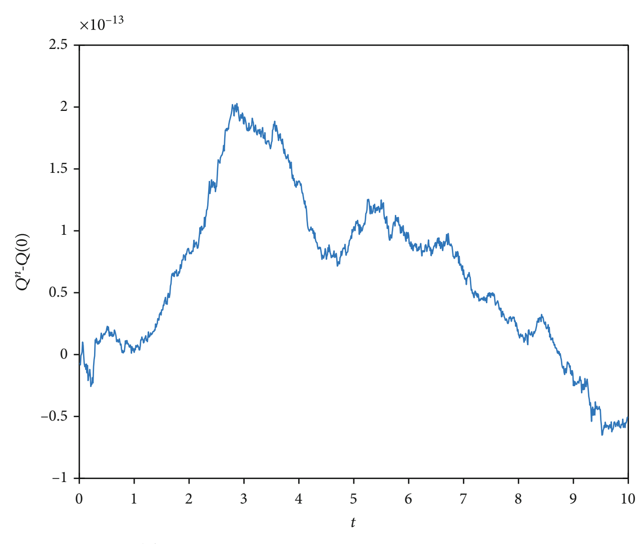

Figure 3: Discrete mass error when $\tau = 0.01$, $N = 128$, and $\alpha = 1.4$.

$$\text{order} = \begin{cases} \dfrac{\log\left(\|\text{error}(\tau_1)\|/\|\text{error}(\tau_2)\|\right)}{\log\left(\tau_1/\tau_2\right)}, \\ \dfrac{\log\left(\|\text{error}(N_1)\|/\|\text{error}(N_2)\|\right)}{\log\left(N_1/N_2\right)}. \end{cases} \tag{85}$$

Let $T = 1$. Tables 1 and 2 show that the numerical method is proven to have spectral accuracy in space and second-order accuracy in time for solving equation (83) with $\alpha = 1.4$ and $\alpha = 1.6$.

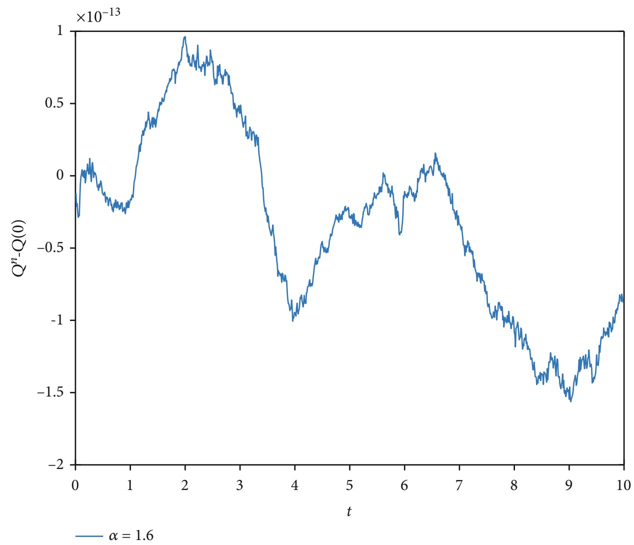

FIGURE 4: Discrete mass error when $\tau = 0.01$, $N = 128$, and $\alpha = 1.6$.

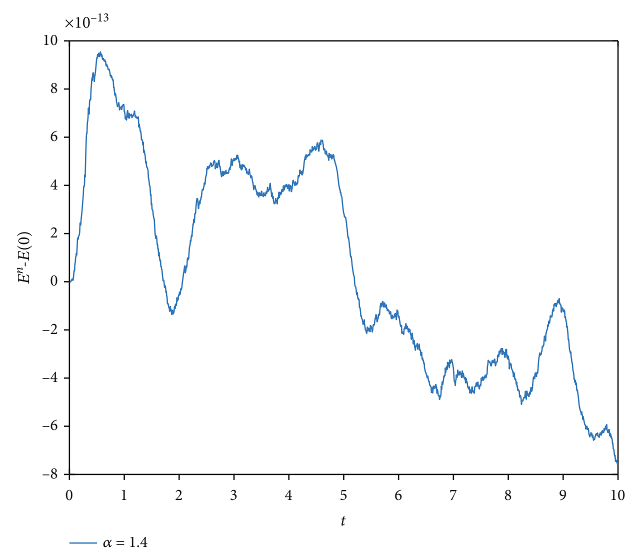

FIGURE 5: Discrete energy error when $\tau = 0.01$, $N = 128$, and $\alpha = 1.4$.

Figures 3–6 show the ability of the numerical method (24) to maintain the discrete mass and discrete energy when $\alpha = 1.4$ and $\alpha = 1.6$. It can be seen from the figure that the numerical method (24) maintains the discrete mass and discrete energy well.

5. Conclusion

For the fractional Schrödinger equation with wave operators, we successfully constructed the effective conservative Crank-Nicolson Fourier spectral method for solving this equation, based on the relative theory of a fractional-order derivative and its property. We give the strict theoretical derivation for the convergence rate of the numerical method, i.e., $O(\tau^2 + N^{-s}\|u\|_s)$. Finally, numerical examples are introduced to verify the correctness of the theoretical results and the validity of our numerical methods. Both theoretical derivation

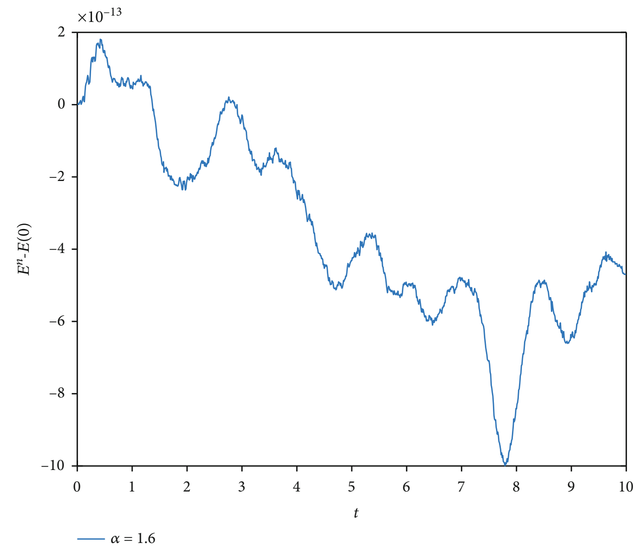

FIGURE 6: Discrete energy error when $\tau = 0.01$, $N = 128$, and $\alpha = 1.6$.

and numerical experiment verify that the numerical method can keep the energy conservation and mass conservation of the original fractional Schrödinger equation. Both environmental noise and regime switching are important factors [21–29]; we will introduce them in the model ((6)–(8)) in the future.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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Research Article

Inequalities for Unified Integral Operators via Strongly $(\alpha, h\text{-}m)$-Convexity

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In this paper, we give a generalized definition namely strongly $(\alpha, h\text{-}m)$-convex function that unifies many known definitions. By applying this new definition, we present inequalities for unified integral operators which have connection with many of the well-known results for different kinds of convex functions. Moreover, this paper at once provides refinements and generalizations of a lot of fractional integral inequalities which are identified in remarks.

1. Introduction

There are many applications of convexity in diverse fields of mathematics including operation research, mathematical statistics, optimization theory, and graph theory. In mathematical inequalities' point of view, convex functions are very important. They are extended and generalized in different ways to obtain corresponding generalizations and extensions of well-known inequalities. For the detail study of different kinds of convex functions, we refer the readers to [1–7].

In recent years, the researchers are working on fractional versions of mathematical inequalities by utilizing classical and new kinds fractional integral/derivative operators, see [8–11]. Also, several kinds of convex functions are applied to obtain these fractional versions, for example, see [1, 12–17] and references therein. The inequalities for fractional integrals and derivatives are very useful in the study of fractional differential equations. Using fractional differential and integral inequalities, qualitative properties of fractional differential equations involving the Riemann-Liouville and the Caputo derivatives can be found frequently in literature.

Our objective is to investigate integral inequalities for newly defined function called strongly $(\alpha, h-m)$-convex function. Integral operators (5) and (6) are used to establish these inequalities, and they have interesting consequences for distinctive fractional inequalities for various types of functions. Next, we give some definitions of known generalized fractional integral operators which can be directly obtained from operators (5) and (6).

Definition 1 (see [18]). Let $\psi : [a, b] \longrightarrow \mathbb{R}$ be an integrable function and ξ be an increasing positive function defined on $[a, b]$ has a continuous derivative ξ' on (a, b). The fractional integrals of a function ψ with respect to another function ξ on $[a, b]$ of order μ $(R(\mu) > 0)$ are defined by

$$
\begin{aligned}
{}^{\mu}_{\xi}I_{a^+}\psi(x) &= \frac{1}{\Gamma(\mu)}\int_a^x (\xi(x)-\xi(t))^{\mu-1}\xi'(t)\psi(t)dt, x > a,\\
{}^{\mu}_{\xi}I_{b^-}\psi(x) &= \frac{1}{\Gamma(\mu)}\int_x^b (\xi(t)-\xi(x))^{\mu-1}\xi'(t)\psi(t)dt, x < b,
\end{aligned}
\tag{1}
$$

here, $\Gamma(.)$ represents the gamma function.

One can see a k-analogue of Definition 1 in [16]. The following generalized integral operator is given in [19].

Definition 2. Let $\psi, \xi : [a, b] \longrightarrow \mathbb{R}$, $0 < a < b$ be the functions such that ψ be positive and $\psi \in L_1[a, b]$, and ξ be differentiable and increasing. Also let ϕ/x be an increasing function on $[a, \infty)$. Then, for $x \in [a, b]$, the left and right integral operators are defined by

$$\left(F_{a^+}^{\phi,\xi}\psi\right)(x) = \int_a^x \frac{\phi(\xi(x) - \xi(t))}{\xi(x) - \xi(t)}\xi'(t)\psi(t)dt, x > a,$$
$$\left(F_{b_-}^{\phi,\xi}\psi\right)(x) = \int_x^b \frac{\phi(\xi(t) - \xi(x))}{\xi(t) - \xi(x)}\xi'(t)\psi(t)dt, x < b. \quad (2)$$

The Mittag-Leffler function was introduced in 1903, which is generalization of the exponential function just by a single parameter with a convergence condition. The further generalization by another parameter was given by Wiman; further, it was extended by Prabhakar and then by other authors; to see the importance of these extensions, we suggest the reader to [20, 21]. By utilizing an extended generalized Mittag-Leffler function, we have defined a fractional integral operator.

Definition 3 (see [12]). Let $\alpha, \gamma, \mu, c, l, w \in \mathbb{C}$, $\mathfrak{R}(\alpha)$, $\mathfrak{R}(\mu)$, $\mathfrak{R}(l) > 0$, $\mathfrak{R}(c) > \mathfrak{R}(\gamma) > 0$ and $p \geq 0$, $\delta > 0$ with $0 < k \leq \delta + \mathfrak{R}(\mu)$. Let $\psi \in L_1[a, b]$ and $x \in [a, b]$. The generalized fractional integral operators $\varepsilon_{\mu,\alpha,l,w,a^+}^{\gamma,\delta,k,c}\psi$ and $\varepsilon_{\mu,\alpha,l,w,b^-}^{\gamma,\delta,k,c}\psi$ are defined by:

$$\left(\varepsilon_{\mu,\alpha,l,w,a^+}^{\gamma,\delta,k,c}\psi\right)(x;p) = \int_a^x (x-t)^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(w(x-t)^\mu;p)\psi(t)dt,$$
$$\left(\varepsilon_{\mu,\alpha,l,w,b^-}^{\gamma,\delta,k,c}\psi\right)(x;p) = \int_x^b (t-x)^{\alpha-1}E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(w(t-x)^\mu;p)\psi(t)dt, \quad (3)$$

where

$$E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(t;p) = \sum_{n=0}^{\infty}\frac{\eta_p(\gamma+nk, c-\gamma)}{\eta(\gamma, c-\gamma)}\frac{(c)_{nk}}{\Gamma(\mu n+\alpha)}\frac{t^n}{(l)_{n\delta}}, \quad (4)$$

is the extended generalized Mittag-Leffler function.

Unified integral operator is based on a kernel which also involves a real valued strictly increasing function along with two variables. This integral operator also unifies the above definitions.

Definition 4 (see [22]). Let $\psi, \xi : [a, b] \longrightarrow \mathbb{R}$, $0 < a < b$ be the functions such that ψ be positive and $\psi \in L_1[a, b]$, and ξ be differentiable and strictly increasing. Also, let ϕ/x be differentiable and strictly increasing. Also, let $[a, \infty)$ and $\tau, l, \gamma, c \in \mathbb{C}$, $\mathfrak{R}(\tau)$, $\mathfrak{R}(l) > 0$, $\mathfrak{R}(c) > \mathfrak{R}(\gamma) > 0$, $p, \mu, \delta \geq 0$ and $0 < k \leq \delta + \mu$. Then, for $x \in [a, b]$, the left and right integral operators are defined by

$$\left({}_\xi F_{\mu,\tau,l,a^+}^{\phi,\gamma,\delta,k,c}\psi\right)(x, w; p) = \int_a^x K_x^y\left(E_{\mu,\tau,l}^{\gamma,\delta,k,c}, \xi; \phi\right)\psi(y)d(\xi(y)), \quad (5)$$

$$\left({}_\xi F_{\mu,\tau,l,b^-}^{\phi,\gamma,\delta,k,c}\psi\right)(x, w; p) = \int_x^b K_y^x\left(E_{\mu,\tau,l}^{\gamma,\delta,k,c}, \xi; \phi\right)\psi(y)d(\xi(y)), \quad (6)$$

where we have

$$K_x^y\left(E_{\mu,\tau,l}^{\gamma,\delta,k,c}, \xi; \phi\right) = \frac{\phi(\xi(x) - \xi(y))}{\xi(x) - \xi(y)}E_{\mu,\tau,l}^{\gamma,\delta,k,c}(w(\xi(x) - \xi(y))^\mu; p). \quad (7)$$

Several recently defined fractional integrals studied in [12, 14, 18, 21, 23–30] can be reproduced from the above definition, see ([31], Remarks 6 and 7). The following results are obtained for strongly convex functions in [32].

Theorem 5. *Let $\psi : [a, b] \longrightarrow \mathbb{R}$ be a positive, integrable, and strongly convex function with modulus $\lambda \geq 0$. Let $\xi : [a, b] \longrightarrow \mathbb{R}$ be a strictly increasing and differentiable function, also let ϕ/x be an increasing function on $[a, b]$. If $\alpha, \gamma, \eta, c, l \in \mathbb{R}_+$, $c > \gamma$, $p, \mu, \delta \geq 0$ and $0 < k \leq \delta + \mu$, then for $x \in (a, b)$, the following inequality holds:*

$$\begin{aligned}&\left({}_\xi F_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c}\psi\right)(x, w; p) + \left({}_\xi F_{\mu,\eta,l,b^-}^{\phi,\gamma,\delta,k,c}\psi\right)(x, w; p)\\ &\leq K_x^a\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \xi; \phi\right)((\xi(x) - \xi(a))(\psi(x) + \psi(a))\\ &\quad -\lambda(x-a)(2I(a, x, I_d\xi) - (a+x)I(a, x, \xi)))\\ &\quad + K_b^x\left(E_{\mu,\eta,l}^{\gamma,\delta,k,c}, \xi; \phi\right)((\xi(b) - \xi(x))(\psi(b) + \psi(x))\\ &\quad -\lambda(b-x)(2I(x, b, I_d\xi) - (x+b)I(x, b, \xi))),\end{aligned} \quad (8)$$

where I_d is the identity function and $I(a, b, \psi) \coloneqq \int_a^b \psi(t)dt$.

Theorem 6. *Under the assumptions of Theorem 5, in addition, if $\psi(x) = \psi(a + b - x)$, then, we have*

$$\begin{aligned}&f\left(\frac{a+b}{2}\right)\left({}_\xi F_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c}1\right)(a, w; p) + \frac{\lambda}{4}\left({}_\xi F_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c}(a+b-2x)^2\right)\\ &\quad\times (a, w; p) + f\left(\frac{a+b}{2}\right)\left({}_\xi F_{\mu,\eta,l,a^+}^{\phi,\gamma,\delta,k,c}1\right)(b, w; p) + \frac{\lambda}{4}\\ &\quad\times\left({}_\xi F_{\mu,\eta,l,a^+}^{\phi,\gamma,\delta,k,c}(a+b-2x)^2\right)(b, w; p)\\ &\leq \left({}_\xi F_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c}\psi\right)(a, w; p) + \left({}_\xi F_{\mu,\eta,l,a^+}^{\phi,\gamma,\delta,k,c}\psi\right)(b, w; p)\\ &\leq \left(K_b^a\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, \xi; \phi\right) + K_b^a\left(E_{\mu,\eta,l}^{\gamma,\delta,k,c}, \xi; \phi\right)\right)\\ &\quad\times((\xi(b) - \xi(a))(\psi(b) + \psi(a)) - (b-a)\lambda(2I(a, b, I_d\xi)\\ &\quad - (a+b)I(a, b, \xi))).\end{aligned} \quad (9)$$

Theorem 7. *Let $\psi : [a, b] \longrightarrow \mathbb{R}$ be a differentiable function. If $|\psi'|$ is strongly convex with modulus $\lambda \geq 0$ and $\xi : [a, b] \longrightarrow \mathbb{R}$ be strictly increasing and differentiable, also let ϕ/x be an increasing function on $[a, b]$. If $\alpha, \gamma, \eta, c, l \in \mathbb{R}_+$, $c > \gamma$, $p, \mu, \delta \geq 0$ and $0 < k \leq \delta + \mu$, then for $x \in (a, b)$, the following* *inequality holds:*

$$\left|\left({}_{\xi}F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^{+}}\psi * \xi\right)(x,w;p) + \left({}_{\xi}F^{\phi,\gamma,\delta,k,c}_{\mu,\eta,l,b^{-}}\psi * \xi\right)(x,w;p)\right| \le K_{x}^{a}\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},\xi;\phi\right) \times \left((\xi(x)-\xi(a))\left(\left|\psi'(x)\right| + \left|\psi'(a)\right|\right) - \lambda(x-a)(2I(a,x,I_d\xi) - (a+x)I(a,x,\xi))\right) + K_{b}^{x}\left(E^{\gamma,\delta,k,c}_{\mu,\eta,l},\xi;\phi\right)\left((\xi(b)-\xi(x))\left(\left|\psi'(b)\right| + \left|\psi'(x)\right|\right) - \lambda(b-x)(2I(x,b,I_d\xi) - (x+b)I(x,b,\xi))\right). \quad (10)$$

Next, we give some definitions of convex functions. The definition of $(h\text{-}m)$-convex function is given as follows:

Definition 8 (see [5]). Let $J \subseteq \mathbb{R}$ be an interval containing $(0, 1)$ and let $h : J \longrightarrow \mathbb{R}$ be a nonnegative function. A nonnegative function $\psi : [0, \mathrm{b}] \longrightarrow \mathbb{R}$ is called $(h\text{-}m)$-convex function if

$$\psi(\zeta x + m(1-\zeta)y) \le h(\zeta)\psi(x) + mh(1-\zeta)\psi(y), \quad (11)$$

holds for all $x, y \in [0, b]$, $m \in [0, 1]$, and $\zeta \in (0, 1)$.

Remark 9.

(i) For $m = 1$, (11) gives the definition of h-convex function

(ii) For $h(\zeta) = \zeta$, (11) gives the definition of m-convex function

(iii) For $h(\zeta) = \zeta$ and $m = 1$, (11) gives the definition of convex function

(iv) For $h(\zeta) = 1$ and $m = 1$, (11) gives the definition of p-function

(v) For $h(\zeta) = \zeta^{s}$ and $m = 1$, (11) gives the definition of s-convex function

(vi) For $h(\zeta) = 1/\zeta$ and $m = 1$, (11) gives the definition of Godunova-Levin function

(vii) For $h(\zeta) = 1/\zeta^{s}$ and $m = 1$, (11) gives the definition of s-Godunova-Levin function of second

Definition 10 (see [4]). A function $\psi : [0, b] \longrightarrow \mathbb{R}, b > 0$ is said to be (α, m)-convex if

$$\psi(\zeta x + m(1-\zeta)y) \le \zeta^{\alpha}\psi(x) + m\left(1-\zeta^{\alpha}\right)\psi(y), \quad (12)$$

holds for all $x, y \in [0, b]$ $\zeta \in [0, 1]$ and $(\alpha, m) \in [0, 1]^2$.

Remark 11.

(i) For $(\alpha, m)=(1, m)$, (12) provides m-convex function

(ii) For $(\alpha, m)=(1, 1)$, (12) provides convex function

(iii) For $(\alpha, m)=(1, 0)$, (12) provides star-shaped function

Definition 12 see ([33]). A function $\psi : [0, b] \longrightarrow \mathbb{R}, b > 0$ is said to be (s, m)-convex, where $(s, m) \in [0, 1]^2$ if

$$\psi(\zeta x + m(1-\zeta)y) \le \zeta^{s}\psi(x) + m(1-\zeta)^{s}\psi(y), \quad (13)$$

holds for all $x, y \in [0, b]$ and $\zeta \in [0, 1]$.

The following definition unifies $(h\text{-}m)$-convex, (s, m)-convex, and (α, m)-convex functions in a single inequality.

Definition 13. Let $J \subseteq \mathbb{R}$ be an interval containing $(0, 1)$ and let $h : J \longrightarrow \mathbb{R}$ be a nonnegative function. A nonnegative function $\psi : [0, b] \longrightarrow \mathbb{R}$ is said to be $(\alpha, h\text{-}m)$-convex function if

$$\psi(\zeta x + m(1-\zeta)y) \le h\left(\zeta^{\alpha}\right)\psi(x) + mh\left(1-\zeta^{\alpha}\right)\psi(y), \quad (14)$$

holds for all $x, y \in [0, b], \zeta \in (0, 1), (\alpha, m) \in [0, 1]^2$.

Next, we give definitions of strongly convex, strongly (s, m)-convex, and strongly (α, m)-convex functions.

Definition 14 (see [34]). Let I be a nonempty convex subset of normed space $(X, \|.\|)$. A real valued function ψ is said to be strongly convex, with modulus $\lambda \ge 0$, on I if for each $a, b \in I$ and $\zeta \in [0, 1]$,

$$\psi(\zeta x + (1-\zeta)y) \le \zeta\psi(x) + (1-\zeta)\psi(y) - \lambda\zeta(1-\zeta)\|b-a\|^2. \quad (15)$$

Definition 15 (see [35]). A function $\psi : [0,+\infty) \longrightarrow \mathbb{R}$ is said to be strongly (s, m)-convex function, with modulus $\lambda \ge 0$, for $(s, m) \in [0, 1]^2$, if

$$\psi(\zeta x + m(1-\zeta)y) \le \zeta^{s}\psi(x) + m(1-\zeta)^{s}\psi(y) - \lambda m\zeta(1-\zeta)(y-x)^2, \quad (16)$$

holds for all $x, y \in [0,+\infty)$ and $\zeta \in [0, 1]$.

Definition 16 (see [36]). A function $\psi : [0, b] \longrightarrow \mathbb{R}, b > 0$ is said to be strongly (α, m)-convex, where $(\alpha, m) \in [0, 1]^2$ if

$$\psi(\zeta x + m(1-\zeta)y) \le \zeta^{\alpha}\psi(x) + m\left(1-\zeta^{\alpha}\right)\psi(y) - \lambda m\zeta^{\alpha}\left(1-\zeta^{\alpha}\right)|y-x|^2, \quad (17)$$

holds for all $x, y \in [0, b]$ and $\zeta \in [0, 1]$.

Next, we give a property of the kernel given in (7), which will be useful for finding the results of this paper.

P: Let ξ and ϕ/I be increasing functions. Then, for $u < t < v$, $u, v \in [a, b]$, the kernel $K_{u}^{v}(E^{\gamma,\delta,k,c}_{\mu,\tau,l}, \xi; \phi)$ satisfies the following inequality:

$$K_t^u\left(E_{\mu,\tau,l}^{\gamma,\delta,k,c},\xi;\phi\right)\xi'(t)\le K_v^u\left(E_{\mu,\tau,l}^{\gamma,\delta,k,c},\xi;\phi\right)\xi'(t). \quad (18)$$

It is easy to prove by using the following inequalities:

$$\frac{\phi(\xi(t)-\xi(u))}{\xi(t)-\xi(u)}\xi'(t)\le\frac{\phi(\xi(v)-\xi(u))}{\xi(v)-\xi(u)}\xi'(t),$$
$$E_{\mu,\tau,l}^{\gamma,\delta,k,c}\left(w(\xi(t)-\xi(u))^{\mu};p\right)\le E_{\mu,\tau,l}^{\gamma,\delta,k,c}\left(w(\xi(v)-\xi(u))^{\mu};p\right). \quad (19)$$

If ξ and ϕ/I are of opposite monotonicities, then (18) holds in reverse direction. For further properties, see [37].

In Section 2, we will define a new notion of strongly $(\alpha, h\text{-}m)$-convex function which unifies several kinds of convex functions in a single inequality. By applying this new definition, we give generalizations of results for strongly convex functions. The results established here will produce generalizations and refinements of fractional integral inequalities for different kinds of convex and strongly convex functions which have been published in various papers.

2. Main Results

We give the definition of a function will be called strongly $(\alpha, h\text{-}m)$-convex function.

Definition 17. Let $J\subseteq\mathbb{R}$ be an interval containing $(0,1)$ and let $h:J\longrightarrow\mathbb{R}$ be a nonnegative function. A nonnegative function $\psi:[0,b]\longrightarrow\mathbb{R}$ is said to be strongly $(\alpha, h\text{-}m)$-convex function with modulus $\lambda\ge0$ if

$$\psi(\zeta x+m(1-\zeta)y)\le h(\zeta^{\alpha})\psi(x)+mh(1-\zeta^{\alpha})\psi(y)-m\lambda h(\zeta^{\alpha})h(1-\zeta^{\alpha})|y-x|^2, \quad (20)$$

holds for all $x,y\in[0,b]$, $\zeta\in(0,1)$, $(\alpha,m)\in[0,1]^2$.

The definition of strongly $(h\text{-}m)$-convexity can be achieved by taking $\alpha=1$ in (20).

Definition 18. Let $J\subseteq\mathbb{R}$ be an interval containing $(0,1)$ and let $h:J\longrightarrow\mathbb{R}$ be a nonnegative function. A nonnegative function $\psi:[0,b]\longrightarrow\mathbb{R}$ is said to be strongly $(h\text{-}m)$-convex function with modulus $\lambda\ge0$ if

$$\psi(\zeta x+m(1-\zeta)y)\le h(\zeta)\psi(x)+mh(1-\zeta)\psi(y)-m\lambda h(\zeta)h(1-\zeta)|y-x|^2, \quad (21)$$

holds for all $x,y\in[0,b]$, $\zeta\in(0,1)$, $m\in[0,1]$.

One can obtain from (20) definitions of strongly convex, strongly s-convex, strongly m-convex, strongly h-convex, strongly (α,m)-convex, and strongly (s,m)-convex functions.

Theorem 19. *Let $\psi\in L_1[a,b]$ be a positive strongly $(\alpha,h\text{-}m)$-convex function with modulus $\lambda\ge0$, $m\in(0,1]$, $0<a<mb$. Let ξ be strictly increasing and differentiable function, also let ϕ/x be an increasing function on $[a,b]$ and $h(x)h(y)\le h(x+y)$. If $\tau,\eta,l,\gamma,c\in\mathbb{R}_+$, $c>\gamma$, $p,\mu,\delta\ge0$ and $0<k\le\delta+\mu$, then for $x\in(a,b)$, the following inequality holds:*

$$\begin{aligned}&\left({}_{\xi}F_{\mu,\tau,l,a^+}^{\phi,\gamma,\delta,k,c}\psi\right)(x,w;p)+\left({}_{\xi}F_{\mu,\eta,l,b^-}^{\phi,\gamma,\delta,k,c}\psi\right)(x,w;p)\\&\le K_x^a\left(E_{\mu,\tau,l}^{\gamma,\delta,k,c},\xi;\phi\right)(x-a)\left(\psi(a)X_x^a\left(r^{\alpha},h;\xi'\right)+m\psi\left(\frac{x}{m}\right)X_x^a\left(1-r^{\alpha},h;\xi'\right)-\frac{\lambda(x-ma)^2h(1)(\xi(x)-\xi(a))}{m(x-a)}\right)\\&\quad+K_b^x\left(E_{\mu,\eta,l}^{\gamma,\delta,k,c},\xi;\phi\right)(b-x)\times\left(\psi(b)X_x^b\left(r^{\alpha},h;\xi'\right)+m\psi\left(\frac{x}{m}\right)X_x^b\left(1-r^{\alpha},h;\xi'\right)-\frac{\lambda(bm-x)^2h(1)(\xi(b)-\xi(x))}{m(b-x)}\right),\end{aligned} \quad (22)$$

while $X_x^a(r^{\alpha},h;\xi')=\int_0^1h(r^{\alpha})\xi'(x-r(x-a))dr$, $X_x^a(1-r^{\alpha},h;\xi')=\int_0^1h(1-r^{\alpha})\xi'(x-r(x-a))dr$.

Proof. Using (**P**), we can write the following inequalities

$$K_x^t\left(E_{\mu,\tau,l}^{\gamma,\delta,k,c},\xi;\phi\right)\xi'(t)\le K_x^a\left(E_{\mu,\tau,l}^{\gamma,\delta,k,c},\xi;\phi\right)\xi'(t),\ t\in(a,x), \quad (23)$$

$$K_t^x\left(E_{\mu,\tau,l}^{\gamma,\delta,k,c},\xi;\phi\right)\xi'(t)\le K_b^x\left(E_{\mu,\eta,l}^{\gamma,\delta,k,c},\xi;\phi\right)\xi'(t),\ t\in(x,b). \quad (24)$$

Using strongly $(\alpha,h\text{-}m)$-convexity of ψ, we have

$$\begin{aligned}\psi(t)&\le h\left(\frac{x-t}{x-a}\right)^{\alpha}\psi(a)+mh\left(1-\left(\frac{x-t}{x-a}\right)^{\alpha}\right)\psi\left(\frac{x}{m}\right)\\&\quad-\frac{\lambda(x-am)^2}{m}h\left(\frac{x-t}{x-a}\right)^{\alpha}h\left(1-\left(\frac{x-t}{x-a}\right)^{\alpha}\right),\end{aligned} \quad (25)$$

$$\begin{aligned}\psi(t)&\le h\left(\frac{t-x}{b-x}\right)^{\alpha}\psi(b)+mh\left(1-\left(\frac{t-x}{b-x}\right)^{\alpha}\right)\psi\left(\frac{x}{m}\right)\\&\quad-\frac{\lambda(bm-x)^2}{m}h\left(\frac{t-x}{b-x}\right)^{\alpha}h\left(1-\left(\frac{t-x}{b-x}\right)^{\alpha}\right).\end{aligned} \quad (26)$$

From (23) and (25), the following inequality is obtained:

$$\begin{aligned}&\int_a^xK_x^t\left(E_{\mu,\tau,l}^{\gamma,\delta,k,c},\xi;\phi\right)\psi(t)d(\xi(t))\\&\le\psi(a)K_x^a\left(E_{\mu,\tau,l}^{\gamma,\delta,k,c},\xi;\phi\right)\times\int_a^xh\left(\frac{x-t}{x-a}\right)^{\alpha}d(\xi(t))\\&\quad+m\psi\left(\frac{x}{m}\right)K_x^a\left(E_{\mu,\tau,l}^{\gamma,\delta,k,c},\xi;\phi\right)\times\int_a^xh\left(1-\left(\frac{x-t}{x-a}\right)^{\alpha}\right)d(\xi(t))\\&\quad-\frac{\lambda(bm-x)^2}{m}K_x^a\left(E_{\mu,\tau,l}^{\gamma,\delta,k,c},\xi;\phi\right)\times\int_a^xh\left(1-\left(\frac{x-t}{x-a}\right)^{\alpha}\right)h\left(\frac{x-t}{x-a}\right)^{\alpha}d(\xi(t)).\end{aligned} \quad (27)$$

By setting $r=(x-t)/(x-a)$ on the right side and using (5) on left side of above inequality, we get

$$\left({}_{\xi}F^{\phi,\gamma,\delta,k,c}_{\mu,\tau,l,a^+}\psi\right)(x,w;p) \le K^a_x\left(E^{\gamma,\delta,k,c}_{\mu,\tau,l},\xi;\phi\right)(x-a) \times\left(\psi(a)\int_0^1 h(r^\alpha)\xi'(x-r(x-a))dr + m\psi\left(\frac{x}{m}\right)\int_0^1 h(1-r^\alpha)\xi'(x-r(x-a))dr - \frac{\lambda(x-am)^2}{m}\int_0^1 h(1-r^\alpha)h(r^\alpha)\xi'(x-r(x-a))dr\right). \tag{28}$$

The inequality (28) can take the following form:

$$\left({}_{\xi}F^{\phi,\gamma,\delta,k,c}_{\mu,\tau,l,a^+}\psi\right)(x,w;p) \le K^a_x\left(E^{\gamma,\delta,k,c}_{\mu,\tau,l},\xi;\phi\right)(x-a) \times\left(\psi(a)X^a_x\left(r^\alpha,h;\xi'\right) + m\psi\left(\frac{x}{m}\right)X^a_x\left(1-r^\alpha,h;\xi'\right) - \frac{\lambda(x-ma)^2h(1)(\xi(x)-\xi(a))}{m(x-a)}\right). \tag{29}$$

On the other hand, multiplying (24) and (26), and adopting the same pattern as we did for (23) and (25), the following inequality holds true:

$$\left({}_{\xi}F^{\phi,\gamma,\delta,k,c}_{\mu,\eta,l,b^-}\psi\right)(x,w;p) \le K^x_b\left(E^{\gamma,\delta,k,c}_{\mu,\eta,l},\xi;\phi\right)(b-x) \times\left(\psi(b)\int_0^1 h(r^\alpha)\xi'(x-r(x-b))dr + m\psi\left(\frac{x}{m}\right)\int_0^1 h(1-r^\alpha)\xi'(x-r(x-b))dr - \frac{\lambda(bm-x)^2}{m}\int_0^1 h(1-r^\alpha)h(r^\alpha)\xi'(x-r(x-b))dr\right). \tag{30}$$

The inequality (30) can take the following form:

$$\left({}_{\xi}F^{\phi,\gamma,\delta,k,c}_{\mu,\eta,l,b^-}\psi\right)(x,w;p) \le K^x_b\left(E^{\gamma,\delta,k,c}_{\mu,\eta,l},\xi;\phi\right)(b-x) \times\left(\psi(b)X^b_x\left(r^\alpha,h;\xi'\right) + m\psi\left(\frac{x}{m}\right)X^b_x\left(1-r^\alpha,h;\xi'\right) - \frac{\lambda(bm-x)^2h(1)(\xi(b)-\xi(x))}{m(b-x)}\right). \tag{31}$$

By adding (29) and (31), (22) can be achieved.□

Corollary 20. *For $w=p=0$, (22) gives the following inequality obtained for fractional integral operator defined in [19]:*

$$\frac{\left(F^{\phi,\xi}_{a^+}\psi\right)(x)}{\Gamma(\tau)} + \frac{\left(F^{\phi,\xi}_{b^-}\psi\right)(x)}{\Gamma(\eta)} \le \frac{\phi(\xi(x)-\xi(a))(x-a)}{\Gamma(\tau)(\xi(x)-\xi(a))}\left(\psi(a)X^a_x\left(r^\alpha,h;\xi'\right) + m\psi\left(\frac{x}{m}\right)X^a_x\left(1-r^\alpha,h;\xi'\right) - \frac{\lambda(x-ma)^2h(1)(\xi(x)-\xi(a))}{m(x-a)}\right) + \frac{\phi(\xi(b)-\xi(x))(b-x)}{\Gamma(\eta)(\xi(b)-\xi(x))}\times\left(\psi(b)X^b_x\left(r^\alpha,h;\xi'\right) + m\psi\left(\frac{x}{m}\right)X^b_x\left(1-r^\alpha,h;\xi'\right) - \frac{\lambda(bm-x)^2h(1)(\xi(b)-\xi(x))}{m(b-x)}\right). \tag{32}$$

Remark 21.

(i) For $\lambda=0$, (22) gives ([38], Theorem 1)

(ii) For $\tau=\eta$ and $h(\zeta)=\zeta$, (22) gives ([39], Theorem 1)

(iii) For $\lambda=0$, $\tau=\eta$, $\alpha=m=1$, and $h(\zeta)=\zeta$, (22) gives ([31], Theorem 8)

(iv) For $\lambda=0$, $\phi(\zeta)=\Gamma(\tau)\zeta^{\tau/k}/k\Gamma_k(\tau)$, $h(\zeta)=\xi(t)=\zeta$, $\alpha=m=1$, and $w=p=0$, (22) gives ([40], Theorem 1)

(v) For $\lambda=0$, $\tau=\eta$, the result of (iv) gives ([40], Corollary 1)

(vi) For $\lambda=0$, $k=1$, and $x=a$ or $x=b$, the result of (v) gives ([40], Corollary 2)

(vii) For $\lambda=0$, $k=1$, and $x=(a+b)/2$, the result of (v) gives ([40], Corollary 3)

(viii) For $\lambda=0$, $\phi(\zeta)=\zeta^\tau$, $\alpha=1$, and $\xi(t)=\zeta$, (22) gives ([41], Theorem 1)

(ix) For $\lambda=0$, $\phi(\zeta)=\zeta^\tau$, $\alpha=m=1$, and $h(\zeta)=\xi(t)=\zeta$, (22) gives ([41], Corollary 1)

(x) For $\lambda=0$, $\phi(\zeta)=\Gamma(\tau)\zeta^\tau$, $\alpha=1$, $w=p=0$, and $\xi(\zeta)=\zeta$, (22) gives ([42], Theorem 2.1)

(xi) For $\lambda=0$, $\tau=\eta$, the result of (x) gives ([42], Corollary 2.2)

(xii) For $\lambda=0$, $\tau=\eta$, $\phi(\zeta)=\Gamma(\tau)\zeta^\tau$, $w=p=0$, $\xi(\zeta)=\zeta$, and $\alpha=1$ and using ([12], Remark 11), (22) gives ([42], Corollary 2.3)

(xiii) For $\lambda=0$, $\phi(\zeta)=\Gamma(\tau)\zeta^\tau$, $\alpha=1$, $w=p=0$, $\xi(\zeta)=\zeta$, and $h(\zeta)=1$, (22) gives inequality (26) of ([42], Corollary 2.4) similarly, under the same assumptions along with $h(\zeta)=\zeta^p$, (22) gives inequality (27) of ([42], Corollary 2.4)

(xiv) For $\lambda=0$, $\phi(\zeta)=\Gamma(\tau)\zeta^\tau$, $w=p=0$, $\alpha=m=1$, and $\xi(t)=h(\zeta)=\zeta$, (22) gives ([43], Theorem 1)

(xv) For $\lambda=0$, $\tau=\eta$, the result of (xiv) gives ([43], Corollary 1)

(xvi) For $\lambda=0$, $\phi(\zeta)=\Gamma(\tau)\zeta^\tau$, $w=p=0$, $\alpha=m=1$, and $h(\zeta)=\zeta$, (22) gives ([44], Theorem 1)

(xvii) For $\lambda=0$, $\tau=\eta$, $\phi(\zeta)=\Gamma(\tau)\zeta^\tau$, $w=p=0$, $\alpha=m=1$, and $h(\zeta)=\zeta$, (22) gives ([44], Corollary 1)

(xviii) For $\lambda=0$, $\phi(\zeta)=\zeta^\tau$, $\xi(\zeta)=\zeta$, and $h(\zeta)=\zeta^s$, $\alpha=m=1$, (22) gives ([45], Theorem 2.1)

(xix) For $\lambda=0$, $\tau=\eta$, $\phi(\zeta)=\zeta^\tau$, $\xi(\zeta)=\zeta$, $\alpha=m=1$, and $h(\zeta)=\zeta^s$, (22) gives ([45], Corollary 2.1)

(xx) For $\lambda=0$, $w=p=0$, $\alpha=1$, and $h(\zeta)=\zeta^s$, (22) gives ([46], Theorem 1)

(xxi) For $\lambda = 0$, $\phi(\zeta) = \zeta^{\tau}$, $h(\zeta) = \zeta^{s}$, $\alpha = 1$, and $\xi(\zeta) = \zeta$, (22) gives ([47], Theorem 1)

(xxii) For $\lambda = 0$, $\tau = \eta$, the result of (xxi) gives ([47], Corollary 1)

(xxiii) For $\lambda = 0$, $\phi(\zeta) = \Gamma(\tau)\zeta^{\tau/k}/k\Gamma_k(\tau)$, $h(\zeta) = z$, and $w = p = 0$, (22) gives ([15], Theorem 1)

(xxiv) For $\lambda = 0$, $\tau = \eta$, the result of (xxiii) gives ([15], Corollary 1)

(xxv) For $\phi(\zeta) = \zeta^{\tau}$, $\xi(\zeta) = \zeta$, and $h(\zeta) = \zeta$, (22) gives ([36], Theorem 4)

(xxvi) For $\tau = \eta$, the result of (xxiv) gives ([36], Corollary 1)

(xxvii) For $\psi \in L_{\infty}[a, b]$, the result of (xxiv) gives ([36], Corollary 2)

(xxviii) For $\tau = \eta$, the result of (xxvii) gives ([36], Corollary 3)

(xxix) For $\phi(\zeta) = \zeta^{\tau}$, $\xi(\zeta) = \zeta$, and $\lambda = 0$, (22) gives ([48], Theorem 1)

(xxx) For $\tau = \eta$, the result of (xxix) gives ([48], Corollary 1)

(xxxi) For $\psi \in L_{\infty}[a, b]$, the result of (xxix) gives ([48], Corollary 2)

(xxxii) For $\tau = \eta$, the result of (xxxi), gives ([48], Corollary 3)

For the proof of next theorem, we need the following lemma.

Lemma 22. *Let $\psi : [a, b] \longrightarrow \mathbb{R}$, be a strongly $(\alpha, h\text{-}m)$-convex function with modulus $\lambda \geq 0$, $m \in (0, 1]$, $0 \leq a < mb$. If $\psi(x) = \psi((a + mb - x)/m)$, then the following inequality holds:*

$$\psi\left(\frac{a+mb}{2}\right) \leq \left(h\left(\frac{1}{2^{\alpha}}\right) + mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\right)\psi(x) - \frac{\lambda}{m}h\left(\frac{1}{2^{\alpha}}\right)h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)(a - x + mb - mx)^2. \tag{33}$$

Proof. As ψ is strongly $(\alpha, h\text{-}m)$-convex function, we have

$$\psi\left(\frac{a+mb}{2}\right) \leq h\left(\frac{1}{2^{\alpha}}\right)\psi((1-t)a + mtb) + mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\psi\left(\frac{t + m(1-t)b}{m}\right) - \frac{\lambda}{m}h\left(\frac{1}{2^{\alpha}}\right)h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)(t(1+m)(a-mb) + mb - ma)^2. \tag{34}$$

Let $x = a(1-t) + mtb$. Then, we have $a + mb - x = ta + m(1-t)b$, and using $\psi((a + mb - x)/m) = \psi(x)$, the inequality (33) is obtained.□

The upcoming theorem provides the Hadamard inequality for strongly $(\alpha, h\text{-}m)$-convex function.

Theorem 23. *Under the assumptions of Theorem 19, in addition, if $\psi(x) = \psi((a + mb - x)/m)$, then, we have*

$$\begin{aligned}
&\frac{1}{h(1/2^{\alpha}) + mh((2^{\alpha}-1)/2^{\alpha})}\left(\psi\left(\frac{a+mb}{2}\right)\left(\left({}_{\xi}F^{\phi,\gamma,\delta,k,c}_{\mu,\tau,l,b^-}1\right)(a, w; p)\right.\right.\\
&\quad + \left({}_{\xi}F^{\phi,\gamma,\delta,k,c}_{\mu,\eta,l,a^+}1\right)(b, w; p) + \frac{\lambda}{m}h\left(\frac{1}{2^{\alpha}}\right)h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\\
&\quad \times \left(\left({}_{\xi}F^{\phi,\gamma,\delta,k,c}_{\mu,\tau,l,b^-}(a - x + mb - mx)^2\right)(a, w; p)\right.\\
&\quad \left.\left.+\left({}_{\xi}F^{\phi,\gamma,\delta,k,c}_{\mu,\eta,l,a^+}(a - x + mb - mx)^2\right)(b, w; p)\right)\right)\\
&\leq \left({}_{\xi}F^{\phi,\gamma,\delta,k,c}_{\mu,\eta,l,a^+}\psi\right)(b, w; p) + \left({}_{\xi}F^{\phi,\gamma,\delta,k,c}_{\mu,\tau,l,b^-}\psi\right)(a, w; p)\\
&\leq (b-a)\left(K^a_b\left(E^{\gamma,\delta,k,c}_{\mu,\tau,l}, \xi; \phi\right) + K^a_b\left(E^{\gamma,\delta,k,c}_{\mu,\eta,l}, \xi; \phi\right)\right)\\
&\quad \times \left(\psi(b)X^a_b\left(r^{\alpha}, h; \xi'\right) + m\psi\left(\frac{a}{m}\right)X^a_b\left(1 - r^{\alpha}, h; \xi'\right)\right.\\
&\quad \left. - \frac{\lambda(b-ma)^2h(1)(\xi(b) - \xi(a))}{m(b-a)}\right).
\end{aligned} \tag{35}$$

Proof. Using (**P**), we can write the following inequalities:

$$K^a_x\left(E^{\gamma,\delta,k,c}_{\mu,\tau,l}, \xi; \phi\right)\xi'(x) \leq K^a_b\left(E^{\gamma,\delta,k,c}_{\mu,\tau,l}, \xi; \phi\right)\xi'(x), x \in (a, b), \tag{36}$$

$$K^x_b\left(E^{\gamma,\delta,k,c}_{\mu,\eta,l}, \xi; \phi\right)\xi'(x) \leq K^a_b\left(E^{\gamma,\delta,k,c}_{\mu,\eta,l}, \xi; \phi\right)\xi'(x), x \in (a, b). \tag{37}$$

Using strongly $(\alpha, h\text{-}m)$-convexity of ψ, we have

$$\begin{aligned}
\psi(x) &\leq h\left(\frac{x-a}{b-a}\right)^{\alpha}\psi(b) + mh\left(1 - \left(\frac{x-a}{b-a}\right)^{\alpha}\right)\psi\left(\frac{a}{m}\right)\\
&\quad - \frac{\lambda(a-bm)^2}{m}h\left(\frac{x-a}{b-a}\right)^{\alpha}h\left(1 - \left(\frac{x-a}{b-a}\right)^{\alpha}\right).
\end{aligned} \tag{38}$$

Multiplying (36) and (38) and integrating the resulting inequality over $[a, b]$, we obtain

$$\begin{aligned}
&\int_a^b K^a_x\left(E^{\gamma,\delta,k,c}_{\mu,\tau,l}, \xi; \phi\right)\psi(x)d(\xi(x)) \leq \psi(b)K^a_b\left(E^{\gamma,\delta,k,c}_{\mu,\tau,l}, \xi; \phi\right)\\
&\quad \times \int_a^b h\left(\frac{x-a}{b-a}\right)^{\alpha}d(\xi(x)) + m\psi\left(\frac{a}{m}\right)K^a_b\left(E^{\gamma,\delta,k,c}_{\mu,\tau,l}, \xi; \phi\right)\\
&\quad \times \int_a^b h\left(1 - \left(\frac{x-a}{b-a}\right)^{\alpha}\right)d(\xi(x)) - \frac{\lambda(a-bm)^2}{m}\\
&\quad \times \int_a^b h\left(\frac{x-a}{b-a}\right)^{\alpha}h\left(1 - \left(\frac{x-a}{b-a}\right)^{\alpha}\right)d(\xi(x)).
\end{aligned} \tag{39}$$

By setting $r = ((x-a)/(b-a))$ on the right side and using

(5) on left side of above inequality, we get

$$\left({}_{\xi}F^{\phi,\gamma,\delta,k,c}_{\mu,\tau,l,b^-}\psi\right)(a,w;p)\leq K^a_b\left(E^{\gamma,\delta,k,c}_{\mu,\tau,l},\xi;\phi\right)(b-a)\times\left(\psi(b)\int_0^1 h(r^\alpha)\xi'(a+r(b-a))dr+m\psi\left(\frac{a}{m}\right)\cdot\int_0^1 h(1-r^\alpha)\xi'(a+r(b-a))dr-\frac{\lambda(a-bm)^2}{m}\cdot\int_0^1 h(r^\alpha)h(1-r^\alpha)\xi'(a+r(b-a))dr\right). \quad (40)$$

The inequality (40) can take the following form:

$$\left({}_{\xi}F^{\phi,\gamma,\delta,k,c}_{\mu,\tau,l,b^-}\psi\right)(a,w;p)\leq K^a_b\left(E^{\gamma,\delta,k,c}_{\mu,\tau,l},\xi;\phi\right)(b-a)\times\left(\psi(b)X^a_b\left(r^\alpha,h;\xi'\right)+m\psi\left(\frac{a}{m}\right)X^a_b\left(1-r^\alpha,h;\xi'\right)-\frac{\lambda(a-bm)^2h(1)(\xi(b)-\xi(a))}{m(b-a)}\right). \quad (41)$$

Adopting the same pattern of simplification as we did for (36) and (38), the following inequality can be observed for (38) and (37):

$$\left({}_{\xi}F^{\phi,\gamma,\delta,k,c}_{\mu,\eta,l,a^+}\psi\right)(b,w;p)\leq K^a_b\left(E^{\gamma,\delta,k,c}_{\mu,\eta,l},\xi;\phi\right)(b-a)\times\left(\psi(b)X^a_b\left(r^\alpha,h;\xi'\right)+m\psi\left(\frac{a}{m}\right)X^a_b\left(1-r^\alpha,h;\xi'\right)-\frac{\lambda(a-bm)^2h(1)(\xi(b)-\xi(a))}{m(b-a)}\right). \quad (42)$$

By adding (41) and (42), following inequality can be achieved:

$$\left({}_{\xi}F^{\phi,\gamma,\delta,k,c}_{\mu,\eta,l,a^+}\psi\right)(b,w;p)+\left({}_{\xi}F^{\phi,\gamma,\delta,k,c}_{\mu,\tau,l,b^-}\psi\right)(a,w;p)\leq(b-a)\left(K^a_b\left(E^{\gamma,\delta,k,c}_{\mu,\tau,l},\xi;\phi\right)+K^a_b\left(E^{\gamma,\delta,k,c}_{\mu,\eta,l},\xi;\phi\right)\right)\left(\psi(b)X^a_b\left(r^\alpha,h;\xi'\right)+m\psi\left(\frac{a}{m}\right)X^a_b\left(1-r^\alpha,h;\xi'\right)-\frac{\lambda(a-bm)^2h(1)(\xi(b)-\xi(a))}{m(b-a)}\right). \quad (43)$$

Multiplying both sides of (33) by $K^a_x(E^{\gamma,\delta,k,c}_{\mu,\tau,l},\xi;\phi)d(\xi(x))$ and integrating over $[a,b]$, one can get

$$\psi\left(\frac{a+mb}{2}\right)\int_a^b K^a_x\left(E^{\gamma,\delta,k,c}_{\mu,\tau,l},\xi;\phi\right)d(\xi(x))\leq\left(h\left(\frac{1}{2^\alpha}\right)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\right)\int_a^b K^a_x\left(E^{\gamma,\delta,k,c}_{\mu,\tau,l},\xi;\phi\right)\psi(x)d(\xi(x))-\frac{\lambda}{m}h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right)\int_a^b K^a_x\left(E^{\gamma,\delta,k,c}_{\mu,\tau,l},\xi;\phi\right)(a-x+mb-mx)^2d(\xi(x)). \quad (44)$$

By using the Definition 4, one can obtain the following inequality:

$$\frac{1}{h(1/2^\alpha)+mh((2^\alpha-1)/2^\alpha)}\left(\psi\left(\frac{a+mb}{2}\right)\left({}_{\xi}F^{\phi,\gamma,\delta,k,c}_{\mu,\tau,l,b^-}1\right)(a,w;p)+\frac{\lambda}{m}h\left(\frac{1}{2^\alpha}\right)\times h\left(\frac{2^\alpha-1}{2^\alpha}\right)\left({}_{\xi}F^{\phi,\gamma,\delta,k,c}_{\mu,\tau,l,b^-}(a-x+mb-mx)^2\right)(a,w;p)\right)\leq\left({}_{\xi}F^{\phi,\gamma,\delta,k,c}_{\mu,\tau,l,b^-}\psi\right)(a,w;p). \quad (45)$$

Now, multiplying by $K^x_b(E^{\gamma,\delta,k,c}_{\mu,\eta},\xi;\phi)d(\xi(x))$ on both sides of (33), then integrating over $[a,b]$, we get

$$\frac{1}{h(1/2^\alpha)+mh((2^\alpha-1)/2^\alpha)}\left(\psi\left(\frac{a+mb}{2}\right)\left({}_{\xi}F^{\phi,\gamma,\delta,k,c}_{\mu,\eta,l,a^+}1\right)(b,w;p)+\frac{\lambda}{m}h\left(\frac{1}{2^\alpha}\right)\times h\left(\frac{2^\alpha-1}{2^\alpha}\right)\left({}_{\xi}F^{\phi,\gamma,\delta,k,c}_{\mu,\eta,l,a^+}(a-x+mb-mx)^2\right)(b,w;p)\right)\leq\left({}_{\xi}F^{\phi,\gamma,\delta,k,c}_{\mu,\eta,l,a^+}\psi\right)(b,w;p). \quad (46)$$

From (43), (45), and (46), inequality (35) can be achieved.□

Corollary 24. *For $w=p=0$, (35) gives the following inequality obtained for the fractional integral operator that defined in [19]:*

$$\frac{1}{h(1/2^\alpha)+mh((2^\alpha-1)/2^\alpha)}\left(\psi\left(\frac{a+mb}{2}\right)\left(\frac{\left(F^{\phi,\xi}_{b^-}1\right)(a)}{\Gamma(\tau)}+\frac{\left(F^{\phi,\xi}_{a^+}1\right)(b)}{\Gamma(\eta)}\right)+\frac{\lambda}{m}h\left(\frac{1}{2^\alpha}\right)h\left(\frac{2^\alpha-1}{2^\alpha}\right)\left(\frac{\left(F^{\phi,\xi}_{b^-}(a-x+mb-mx)^2\right)(a)}{\Gamma(\tau)}+\frac{\left(F^{\phi,\xi}_{a^+}(a-x+mb-mx)^2\right)(b)}{\Gamma(\eta)}\right)\right)\leq\frac{\left(F^{\phi,\xi}_{a^+}\psi\right)(b)}{\Gamma(\eta)}+\frac{\left(F^{\phi,\xi}_{b^-}\psi\right)(a)}{\Gamma(\tau)}\leq\frac{(b-a)(\xi(b)-\xi(a))}{\xi(b)-\xi(a)}\left(\frac{1}{\Gamma(\tau)}+\frac{1}{\Gamma(\eta)}\right)\times\left(\psi(b)X^a_b\left(r^\alpha,h;\xi'\right)+m\psi\left(\frac{a}{m}\right)X^a_b\left(1-r^\alpha,h;\xi'\right)-\frac{\lambda(b-ma)^2h(1)(\xi(b)-\xi(a))}{m(b-a)}\right). \quad (47)$$

Remark 25.

(i) For $\lambda=0$, $\tau=\eta$, $\alpha=m=1$, and $h(\zeta)=\zeta$, (35) gives ([31], Theorem 22)

(ii) For $\lambda=0$, $\phi(\zeta)=\Gamma(\tau)\zeta^{(\tau/k)+1}$, $h(\zeta)=\xi(\zeta)=\zeta$, $\alpha=m=1$, and $w=p=0$, (35) gives ([40], Theorem 3)

(iii) For $\lambda=0$, $\tau=\eta$, the result of (ii) gives ([40], Corollary 6)

(iv) For $\lambda=0$, $\phi(\zeta)=\Gamma(\tau)\zeta^{\tau+1}$, $w=p=0$, $\alpha=m=1$, and $\xi(\zeta)=h(\zeta)=\zeta$, (35) gives ([43], Theorem 3)

(v) For $\lambda=0$, $\tau=\eta$, and $\phi(\zeta)=\Gamma(\tau)\zeta^{\tau}$ in the result of (v) gives ([43], Corollary 6)

(vi) For $\lambda=0$, $\phi(\zeta)=\Gamma(\tau)\zeta^{\tau+1}$, $w=p=0$, $\alpha=m=1$, and $h(\zeta)=\zeta$, (35) gives ([44], Theorem 3)

(vii) For $\lambda = 0$, $\tau = \eta$, the result of (vii) gives ([44], Corollary 3)

(viii) For $\lambda = 0$, $\phi(\zeta) = \zeta^{\tau+1}$, $\xi(\zeta) = \zeta$, and $h(\zeta) = \zeta^s$, $\alpha = m = 1$, (35) gives ([45], Theorem 2.4)

(ix) For $\lambda = 0$, $\tau = \eta$, the result of (ix) gives ([45], Corollary 2.6)

(x) For $\phi(\zeta) = \zeta^{\tau+1}$, $\xi(\zeta) = \zeta$, and $h(\zeta) = \zeta$, (35) gives ([36], Theorem 6)

(xi) For $\tau = \eta$, the result of (xi) gives ([36], Corollary 5)

(xii) For $\phi(\zeta) = \zeta^{\tau+1}$, $\xi(\zeta) = \zeta$, and $\lambda = 0$, (35) gives ([48], Theorem 4)

(xiii) For $\tau = \eta$, the result of (xiii) gives ([48], Corollary 5)

Theorem 26. *Let $\psi : [a, b] \longrightarrow \mathbb{R}$ be a differentiable function such that $|\psi'|$ is strongly $(\alpha, h\text{-}m)$-convex with modulus $\lambda \geq 0$, $m \in (0, 1]$, $0 < a < mb$. Let $\xi : [a, b] \longrightarrow \mathbb{R}$ be strictly increasing and differentiable function, also let ϕ/x be a function which is increasing on the interval $[a, b]$ and $h(x)h(y) \leq h(x+y)$. If $c > \gamma$, $p, \mu, \delta \geq 0$, $\tau, \eta, l, \gamma, c \in \mathbb{R}_+$, and $0 < k \leq \delta + \mu$, then for $x \in (a, b)$ the following inequality holds:*

$$\begin{aligned}
&\left|\left({}_{\xi}F^{\phi,\gamma,\delta,k,c}_{\mu,\tau,l,a^+}(\psi * \xi)\right)(x, w; p) + \left({}_{\xi}F^{\phi,\gamma,\delta,k,c}_{\mu,\eta,l,b^-}(\psi * \xi)\right)(x, w; p)\right| \\
&\leq K^a_x\left(E^{\gamma,\delta,k,c}_{\mu,\tau,l}, \xi; \phi\right) \times (x-a)\left(|\psi'(a)| X^a_x\left(r^\alpha, h; \xi'\right)\right. \\
&\quad \left. + m\left|\psi'\left(\frac{x}{m}\right)\right| X^a_x\left(1 - r^\alpha, h; \xi'\right) - \frac{\lambda(x-ma)^2 h(1)(\xi(x) - \xi(a))}{m(x-a)}\right) \\
&\quad + K^x_b\left(E^{\gamma,\delta,k,c}_{\mu,\eta,l}, \xi; \phi\right)(b-x) \times \left(|\psi'(b)| X^b_x\left(r^\alpha, h; \xi'\right)\right. \\
&\quad \left. + m\left|\psi'\left(\frac{x}{m}\right)\right| X^b_x\left(1 - r^\alpha, h; \xi'\right) - \frac{\lambda(b-mb)^2 h(1)(\xi(b) - \xi(x))}{m(b-x)}\right),
\end{aligned} \tag{48}$$

where

$$\begin{aligned}
\left({}_{\xi}F^{\phi,\gamma,\delta,k,c}_{\mu,\tau,l,a^+}\psi * \xi\right)(x, w; p) &:= \int_a^x K^t_x\left(E^{\gamma,\delta,k,c}_{\mu,\tau,l}, \xi; \phi\right)\psi'(t)d(\xi(t)), \\
\left({}_{\xi}F^{\phi,\gamma,\delta,k,c}_{\mu,\eta,l,b^-}\psi * \xi\right)(x, w; p) &:= \int_x^b K^x_t\left(E^{\gamma,\delta,k,c}_{\mu,\tau,l}, \xi; \phi\right)\psi'(t)d(\xi(t)).
\end{aligned} \tag{49}$$

Proof. Since $|\psi'|$ is strongly $(\alpha, h\text{-}m)$-convex function, one can have

$$\begin{aligned}
|\psi'(t)| &\leq h\left(\frac{x-t}{x-a}\right)^\alpha |\psi'(a)| + mh\left(1 - \left(\frac{x-t}{x-a}\right)^\alpha\right)\left|\psi'\left(\frac{x}{m}\right)\right| \\
&\quad - \frac{\lambda(x-am)^2}{m} h\left(\frac{x-t}{x-a}\right)^\alpha h\left(1 - \left(\frac{x-t}{x-a}\right)^\alpha\right).
\end{aligned} \tag{50}$$

The inequality (50) can take the following form:

$$\begin{aligned}
&-\left(h\left(\frac{x-t}{x-a}\right)^\alpha |\psi'(a)| + mh\left(1 - \left(\frac{x-t}{x-a}\right)^\alpha\right)\left|\psi'\left(\frac{x}{m}\right)\right|\right. \\
&\quad \left. - \frac{\lambda(x-am)^2}{m} h\left(\frac{x-t}{x-a}\right)^\alpha h\left(1 - \left(\frac{x-t}{x-a}\right)^\alpha\right)\right) \\
&\leq \psi'(t) \leq \left(h\left(\frac{x-t}{x-a}\right)^\alpha |\psi'(a)| + mh\left(1 - \left(\frac{x-t}{x-a}\right)^\alpha\right)\right. \\
&\quad \left. \cdot \left|\psi'\left(\frac{x}{m}\right)\right| - \frac{\lambda(x-am)^2}{m} h\left(\frac{x-t}{x-a}\right)^\alpha h\left(1 - \left(\frac{x-t}{x-a}\right)^\alpha\right)\right).
\end{aligned} \tag{51}$$

From inequality (51), we have

$$\begin{aligned}
\psi'(t) &\leq h\left(\frac{x-t}{x-a}\right)^\alpha |\psi'(a)| + mh\left(1 - \left(\frac{x-t}{x-a}\right)^\alpha\right)\left|\psi'\left(\frac{x}{m}\right)\right| \\
&\quad - \frac{\lambda(x-am)^2}{m} h\left(\frac{x-t}{x-a}\right)^\alpha h\left(1 - \left(\frac{x-t}{x-a}\right)^\alpha\right).
\end{aligned} \tag{52}$$

Multiplying (23) and (52) and integrating over $[a, x]$, we obtain

$$\begin{aligned}
&\int_a^x K^t_x\left(E^{\gamma,\delta,k,c}_{\mu,\tau,l}, \xi; \phi\right)\psi'(t)d(\xi(t)) \\
&\leq |\psi'(a)| K^a_x\left(E^{\gamma,\delta,k,c}_{\mu,\tau,l}, \xi; \phi\right) \times \int_a^x h\left(\frac{x-t}{x-a}\right)^\alpha d(\xi(t)) \\
&\quad + m\left|\psi'\left(\frac{x}{m}\right)\right| K^a_x\left(E^{\gamma,\delta,k,c}_{\mu,\tau,l}, \xi; \phi\right) \\
&\quad \times \int_a^x h\left(1 - \left(\frac{x-t}{x-a}\right)^\alpha\right) d(\xi(t)) \\
&\quad - \frac{\lambda(bm-x)^2}{m} K^a_x\left(E^{\gamma,\delta,k,c}_{\mu,\tau,l}, \xi; \phi\right) \\
&\quad \times \int_a^x h\left(1 - \left(\frac{x-t}{x-a}\right)^\alpha\right) h\left(\frac{x-t}{x-a}\right)^\alpha d(\xi(t)),
\end{aligned} \tag{53}$$

which gives

$$\begin{aligned}
&\left({}_{\xi}F^{\phi,\gamma,\delta,k,c}_{\mu,\tau,l,a^+}\psi * \xi\right)(x, w; p) \\
&\leq K^a_x\left(E^{\gamma,\delta,k,c}_{\mu,\tau,l}, \xi; \phi\right)(x-a) \times \left(|\psi'(a)| X^a_x\left(r^\alpha, h; \xi'\right)\right. \\
&\quad + m\left|\psi'\left(\frac{x}{m}\right)\right| X^a_x\left(1 - r^\alpha, h; \xi'\right) \\
&\quad \left. - \frac{\lambda(x-ma)^2 h(1)(\xi(x) - \xi(a))}{m(x-a)}\right).
\end{aligned} \tag{54}$$

Using the other inequality of (51) and doing on the same way as adopted for the right hand inequality, one can get

$$\begin{aligned}&\left({}_{\xi}F^{\phi,\gamma,\delta,k,c}_{\mu,\tau,l,a^+}\psi*\xi\right)(x,w;p)\\&\geq -K_x^a\left(E^{\gamma,\delta,k,c}_{\mu,\tau,l},\xi;\phi\right)(x-a)\times\left(|\psi'(a)|X_x^a\left(r^\alpha,h;\xi'\right)\right.\\&\quad+m\left|\psi'\left(\frac{x}{m}\right)\right|X_x^a\left(1-r^\alpha,h;\xi'\right)\\&\quad\left.-\frac{\lambda(x-ma)^2h(1)(\xi(x)-\xi(a))}{m(x-a)}\right).\end{aligned}\tag{55}$$

From (54) and (55), the following inequality is observed:

$$\begin{aligned}&\left|\left({}_{\xi}F^{\phi,\gamma,\delta,k,c}_{\mu,\tau,l,a^+}\psi*\xi\right)(x,w;p)\right|\\&\leq K_x^a\left(E^{\gamma,\delta,k,c}_{\mu,\tau,l},\xi;\phi\right)(x-a)\times\left(|\psi'(a)|X_x^a\left(r^\alpha,h;\xi'\right)\right.\\&\quad+m\left|\psi'\left(\frac{x}{m}\right)\right|X_x^a\left(1-r^\alpha,h;\xi'\right)\\&\quad\left.-\frac{\lambda(x-ma)^2h(1)(\xi(x)-\xi(a))}{m(x-a)}\right).\end{aligned}\tag{56}$$

By applying strongly $(\alpha, h\text{-}m)$-convexity of $|\psi'|$, one can get

$$\begin{aligned}|\psi'(t)|&\leq h\left(\frac{t-x}{b-x}\right)^\alpha|\psi'(b)|+mh\left(1-\left(\frac{t-x}{b-x}\right)^\alpha\right)\left|\psi'\left(\frac{x}{m}\right)\right|\\&\quad-\frac{\lambda(bm-x)^2}{m}h\left(\frac{t-x}{b-x}\right)^\alpha h\left(1-\left(\frac{t-x}{b-x}\right)^\alpha\right).\end{aligned}\tag{57}$$

By following the same steps as we did for (23) and (50), from (24) and (57), one can get the following inequality:

$$\begin{aligned}&\left|\left({}_{\xi}F^{\phi,\gamma,\delta,k,c}_{\mu,\eta,l,b^-}\psi*\xi\right)(x,w;p)\right|\\&\leq K_b^x\left(E^{\gamma,\delta,k,c}_{\mu,\eta,l},\xi;\phi\right)(b-x)\times\left(|\psi'(b)|X_x^b\left(r^\alpha,h;\xi'\right)\right.\\&\quad+m\left|\psi'\left(\frac{x}{m}\right)\right|X_x^b\left(1-r^\alpha,h;\xi'\right)\\&\quad\left.-\frac{\lambda(bm-x)^2h(1)(\xi(b)-\xi(x))}{m(b-x)}\right).\end{aligned}\tag{58}$$

By adding (56) and (58), inequality (48) can be achieved.□

Corollary 27. *For $w=p=0$, (48) gives the following inequality obtained for the fractional integral operator that defined in [19]:*

$$\begin{aligned}&\left|\frac{\left(F^{\phi,\xi}_{a^+}\psi*\xi\right)(x)}{\Gamma(\tau)}+\frac{\left(F^{\phi,\xi}_{b^-}\psi*\xi\right)(x)}{\Gamma(\eta)}\right|\\&\leq\frac{\phi(\xi(x)-\xi(a))(x-a)}{\Gamma(\tau)(\xi(x)-\xi(a))}\times\left(|\psi'(a)|X_x^a\left(r^\alpha,h;\xi'\right)\right.\\&\quad+m|\psi'(x/m)|X_x^a\left(1-r^\alpha,h;\xi'\right)\\&\quad\left.-\frac{\lambda(x-ma)^2h(1)(\xi(x)-\xi(a))}{m(x-a)}\right)\\&\quad+\frac{\phi(\xi(b)-\xi(x))(b-x)}{\Gamma(\eta)(\xi(b)-\xi(x))}\times\left(|\psi'(b)|X_x^b\left(r^\alpha,h;\xi'\right)\right.\\&\quad+m|\psi'(x/m)|X_x^b\left(1-r^\alpha,h;\xi'\right)\\&\quad\left.-\frac{\lambda(b-mb)^2h(1)(\xi(b)-\xi(x))}{m(b-x)}\right).\end{aligned}\tag{59}$$

Remark 28.

(i) For $\lambda=0$, (48) gives ([38], Theorem 4)

(ii) For $\lambda=0$, $\tau=\eta$, and $h(\zeta)=\zeta$, ((48) gives ([39], Theorem 3)

(iii) For $\lambda=0$, $\tau=\eta$, $\alpha=m=1$, and $h(\zeta)=\zeta$, (48) gives ([31], Theorem 25)

(iv) For $\lambda=0$, $\phi(\zeta)=\Gamma(\tau)\zeta^{(\tau/k)+1}$, $h(\zeta)=\xi(\zeta)=\zeta$, $\alpha=m=1$, and $w=p=0$, (48) gives ([40], Theorem 2)

(v) For $\lambda=0$, $\tau=\eta$, the result of (iv) gives ([40], Corollary 4)

(vi) For $\lambda=0$, $\phi(\zeta)=\zeta^\tau$, $\alpha=1$, and $\xi(\zeta)=\zeta$, (48) gives ([41], Theorem 2)

(vii) For $\lambda=0$, $m=1$, and $h(\zeta)=\zeta$, the result of (vi) gives ([41], Corollary 2)

(viii) For $\lambda=0$, $\phi(\zeta)=\Gamma(\tau)\zeta^{\tau+1}$, $\alpha=1$, $w=p=0$, and $\xi(\zeta)=\zeta$, (48) gives ([42], Theorem 2.6)

(ix) For $\lambda=0$, $\tau=\eta$, the result of (viii) gives ([42], Corollary 2.7)

(x) For $\lambda=0$, $\phi(\zeta)=\Gamma(\tau)\zeta^{\tau+1}$, $w=p=0$, $\alpha=m=1$, and $\xi(\zeta)=h(\zeta)=\zeta$, (48) gives ([43], Theorem 2)

(xi) For $\lambda=0$, $\tau=\eta$, and $\phi(\zeta)=\Gamma(\tau)\zeta^\tau$, the result of (x) gives ([43], Corollary 4)

(xii) For $\lambda=0$, $w=p=0$, $\phi(\zeta)=\Gamma(\tau)\zeta^{\tau+1}$, $\alpha=m=1$, and $h(\zeta)=\zeta$, (48) gives ([44], Theorem 2)

(xiii) For $\lambda=0$, $\tau=\eta$, the result of (xii) gives ([44], Corollary 2).

(xiv) For $\lambda=0$, $\xi(\zeta)=\zeta$ and $h(\zeta)=\zeta^s$, $\phi(\zeta)=\zeta^{\tau+1}$, $\alpha=m=1$, (48) gives ([45], Theorem 2.3)

(xv) For $\lambda = 0, \tau = \eta$, the result of (xiv) gives ([45], Corollary 2.5)

(xvi) For $\lambda = 0, w = p = 0, \alpha = 1$ and $h(\zeta) = \zeta^s$, (48) gives ([46], Theorem 2)

(xvii) For $\lambda = 0$, $h(\zeta) = \zeta^s$, $\phi(\zeta) = \Gamma(\tau)\zeta^\tau$, $\alpha = 1$, and $\xi(\zeta) = \zeta$, (48) gives ([47], Theorem 3)

(xviii) For $\lambda = 0$, $\tau = \eta$, the result of (xvii) gives ([47], Corollary 5)

(xix) For $\lambda = 0$, $h(\zeta) = \zeta$, $\phi(\zeta) = \Gamma(\tau)\zeta^{(\tau/k)+1}$, and $w = p = 0$, (48) gives ([15], Theorem 2)

(xx) For $\lambda = 0, \tau = \eta$, the result of (xix) gives ([15], Corollary 2)

(xxi) For $\phi(\zeta) = \zeta^\tau$, $\xi(\zeta) = \zeta$, and $h(\zeta) = \zeta$, (48) gives ([36], Theorem 5)

(xxii) For $\tau = \eta$ in the result of (xxi) gives ([36], Corollary 4)

(xxiii) For $\phi(\zeta) = \zeta^{\tau+1}$, $\xi(\zeta) = \zeta$, and $\lambda = 0$, (48) gives ([48], Theorem 3)

(xxiv) For $\tau = \eta$, the result of (xxiii) gives ([48], Corollary 4)

3. Concluding Remarks

A new definition is given and utilized to obtain some integral inequalities via a unified integral operator. The established results provide generalizations of many well-known inequalities. They also give refinements of recently published results for convex, m-convex, h-convex, s-convex, (α, m)-convex, (s, m)-convex, $(h$-$m)$-convex and $(\alpha, h$-$m)$-convex functions. The reader also can obtain more fractional integral inequalities by setting appropriate functions and parameters involved in the kernel of unified integral operators.

Data Availability

There is no data required for preparation of this article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

The Use of Mathematical Analysis in the Nursing Bed Design Evaluation

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In view of the lack of objective data support for product evaluation methods in the industry, a triangular verification method was proposed; it considered nursing beds as the study object and combined subjective evaluation with eye movement and electroencephalogram. Because the triangular validation method is based on the numerical value between the indicators and the frequency of ranking, this method is worth investigating for analyzing experimental data more scientifically. This paper focuses on the further analysis of the experimental data, especially the use of interval estimation method. After analysis, we obtain that proposal 2 is the optimal solution. This method is more suitable for product evaluation which will collect large amount of experimental data to obtain more accurate results. For industrial product designers, the evaluation of products by users is very important. In the design stage, how to grasp the user's evaluation of the product more accurately is a difficult problem. This paper takes nursing bed as the research object and studies the user participation design in order to make the product more acceptable to most people after it is launched.

1. Introduction

Nursing beds are designed as original ordinary steel beds, mechanical transmission beds, electric beds, or multifunctional beds. With the development of computer technology, development of multifunctional nursing beds is increasing. The development of multifunctional nursing beds is a breakthrough in realizing comprehensive nursing and is also an innovation in patient healthcare function [1]. With the development of information network, sensor, intelligent control, and bionic technology as well as the intersection of electromechanical technology and biotechnology, the development direction of multifunctional nursing bed is networking, digitalization, and intellectualization [2, 3]. The higher the level of medical treatment, the greater the pursuit of living standards and quality. Users not only require basic functions, safety, and practicality of the product but also pay more attention to comfort, aesthetics, and emotion of the product [4, 5]. However, the design of medical beds is obviously lacking in terms of Kansei engineering [6, 7]. As a result, patients not only suffer from illnesses but also feel inconvenienced due to unreasonable designs [8]. It is extremely important to improve and promote the design of medical beds for patients. In recent years, most studies on nursing beds focus on function and user experience. The research on function focuses on solving problems of patients and nurses when using nursing beds. For example, Enoi et al. [9] designed a smart bed to help nurses move overweight patients slowly and smoothly from the bed to other places. In addition, Takanokura et al. developed a systematic approach to use sensors around the nursing bed to prevent falls and secondary injuries.

The evaluation of nursing beds also focuses on functionality. Boorman et al. [10] assessed the value of a "Clinitron" air-fluidized bed in the setting of a general

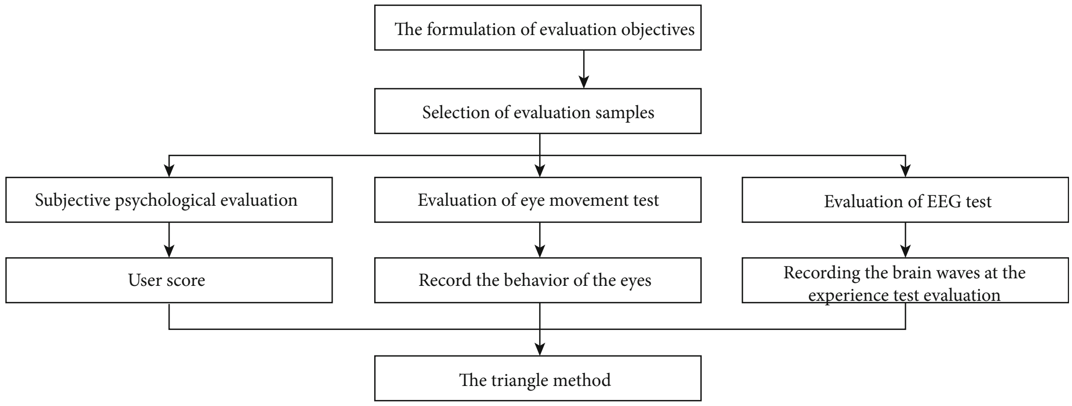

Figure 1: Experimental process of the triangular verification model.

plastic surgery unit by using pressure sensor data in 1981. Milward et al. [11] put forward the Walsall scoring system, which was designed with community patients in mind; it was later improved by Chaloner and Franks [12]. A scoring system is typically used to evaluate the medical system for both hospital and community staff. Some researchers pay more attention to decompressing equipment such as the mattress of the nursing bed, which can also be called a pressure-reducing foam mattress (PRFM); they evaluated the role of PRFMs [13–15]. They also focused on the evaluation of long-term clinical efficacy of PRFMs and found that PRFMs perform well after 3 and 4 years, respectively, in two different clinics [16]. In 2016, Gray et al. not just evaluated PRFMs but also electric bed frames [17]. All these studies focus on the functionality of equipment and the extent to which it can reduce the physical workload of nurses and improve the dignity and comfort of patients.

In terms of a product evaluation system, the current evaluation method is mainly based on expert opinions combined with random sampling. Subjective factors are major contributors in this evaluation method; it is impossible to determine whether the obtained evaluation is a true evaluation. Therefore, in a previous article, a triangular verification method, by combining subjective evaluation, electroencephalogram (EEG) data, and eye movement data, was proposed for a more convincing evaluation method [18].

Based on Kansei engineering [19], the psychological and physiological data collected from the experiment were combined, i.e., subjective evaluation, eye movement, and EEG data; a triangular validation system of nursing bed, which is based on the fact that there is correlation between subjective evaluation, eye movement, and EEG data, was established. The specific execution process is shown in Figure 1.

In Figure 1, we first identify the purpose of the evaluation, which is proposing four nursing bed designs. Participants were selected, and the Likert scale method [20] was used to obtain the subjective evaluation, eye movement, and EEG data synchronously using an instrument. Finally, the four proposals were ranked in terms of subjective evaluation, eye movement data, and EEG data; the final results were verified to improve the reliability of subjective evaluation. The four proposed nursing beds are marked as C1, C2, C3, and C4 as shown in Figure 2.

The experimental method is shown in Figure 3. Tobii X3-120 [21], a small eye movement tracking instrument developed by Sweden Tobii Company, was used in the experiment; its accuracy is 0.2 degrees and the sampling rate is 120 Hz. It can provide portability and large head movement range and ensure high-quality tracking accuracy and stable tracking. The EEG signal acquisition instrument used in the experiment was NeurOne innovative research system produced by Mega Electronics, USA. It has 24-bit analog-to-digital conversion with sensitivity of 51 nV/bit and input range of +430 mV, and the 40-channel amplifier includes 32 EEG+8 bipolar channels. This neuroscience measurement system provides a more accurate and cleaner signal, faster sampling, modular solutions, use of the latest processing in digital signal processing, more flexibility, and scalability.

Data were obtained from 20 participants with normal vision. All the data in this experiment were obtained according to relevant standards. The experimental process is as follows:

(1) Participants washed their hair with shampoo and dried it

(2) Participants watched and understood the experimental guidance and signed the statement

(3) Researchers prepared experimental instruments

(4) Researchers explained the experimental process to the participants

(5) At the beginning of the experiment, participants looked at the first randomly occurring proposal of the medical nursing bed and scored by pressing a button from 1 to 5 (1—worst; 2—worse; 3—normal; 4—better; 5—best). When the participants press the button, the first rendering experiment ends and the second rendering experiment begins until all the experimental materials are completed. To ensure the effectiveness of the experiment, after the first round of grading, four proposals will be

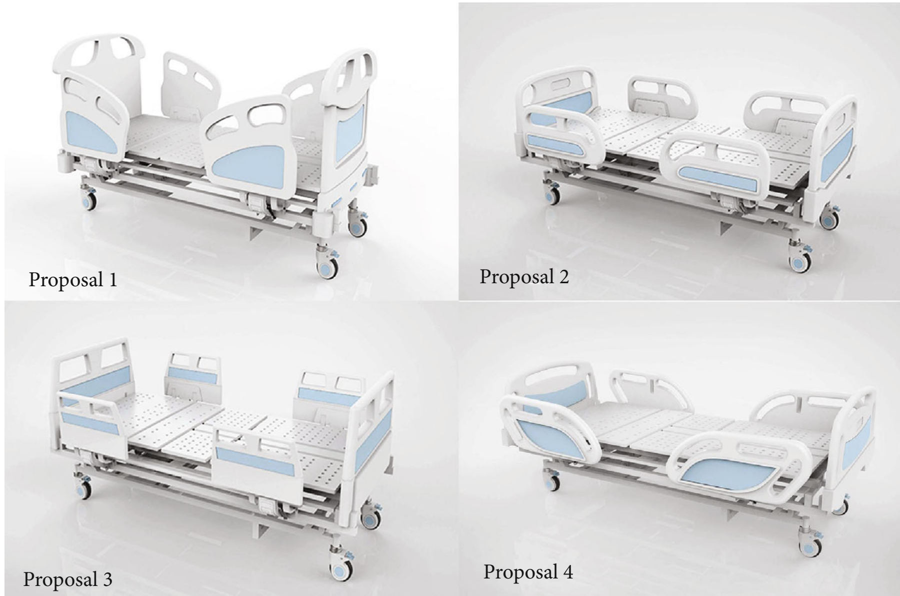

Figure 2: Four proposed nursing beds.

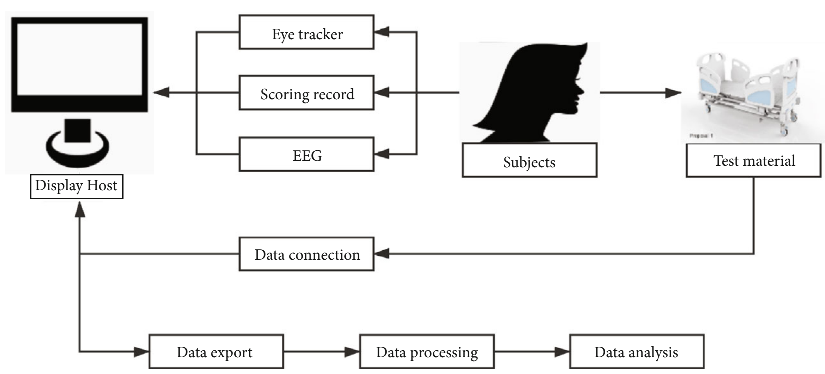

Figure 3: Experimental data acquisition model.

played randomly. The experiment was repeated 50 times

(6) After the experiment was completed, the eye tracker and brain instrument stopped recording

The experimental procedure is shown in Figure 4.

According to the experiment, we obtain the following data:

(1) The expected value of 20 people's subjective evaluation is

$$P = (P_1, P_2, P_3, P_4)^T. \quad (1)$$

P_i represents the expected value of the subjective evaluation of proposal i and P_{ij} represents the expected value of participant j's subjective evaluation of proposal i; the calculation process is as follows:

$$P_i = \frac{1}{20}\sum_{j=1}^{20} P_{ij} \, (1 \le i \le 4 ; 1 \le j \le 20),$$
$$P_{ij} = \frac{1}{50}\sum_{n=1}^{50} P_{ijn} \, (1 \le i \le 4 ; 1 \le j \le 20 ; 1 \le n \le 50). \quad (2)$$

(2) The expected value of 20 people's eye movement data is

$$E = \begin{pmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \\ E_{41} & E_{42} & E_{43} \end{pmatrix}. \quad (3)$$

Figure 4: Experimental setup.

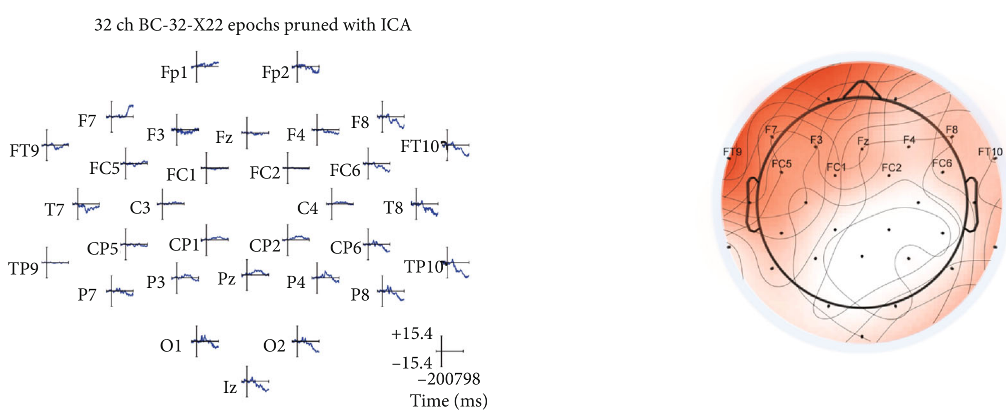

Figure 5: Electrode and topographic map.

E_{i1} represents the expected value of the fixation time for proposal i $(i = 1, 2, 3, 4)$ marked as e1. E_{i2} represents the expected value of the number of fixations for proposal i $(i = 1, 2, 3, 4)$ marked as e2. E_{i3} represents the expected value of the first fixation time for proposal i $(i = 1, 2, 3, 4)$ marked as e3; the calculation process is as follows:

$$E_{ij} = \frac{1}{20}\sum_{k=1}^{20} E_{ijk}\ (1 \le i \le 4\,;\, 1 \le j \le 3\,;\, 1 \le k \le 20),$$
$$E_{ijk} = \frac{1}{50}\sum_{n=1}^{50} E_{ijkn}\ (1 \le i \le 4\,;\, 1 \le j \le 3\,;\, 1 \le k \le 20\,;\, 1 \le n \le 50). \tag{4}$$

(3) The expected value of 20 people's EEG data is

$$D = \begin{pmatrix} D_{11} & D_{12} & \cdots & D_{1j} \\ D_{21} & D_{22} & \cdots & D_{2j} \\ D_{31} & D_{32} & \cdots & D_{3j} \\ D_{41} & D_{42} & \cdots & D_{4j} \end{pmatrix}. \tag{5}$$

D_{ij} represents the expected value of index j of EEG data for proposal i $(i = 1, 2, 3, 4)$. In this experiment, the observed electrodes and overlapping topographic maps are shown in Figure 5. According to the brain topographic map, 11 electrodes were selected in the most active area, named F4, F7, F8, FZ, FC1, FC2, FC5, FC6, ft9, ft10, and F3; therefore, in this experiment, $1 \le j \le 11$.

$$D_{ij} = \frac{1}{20}\sum_{k=1}^{20} D_{ijk}\,(1 \le i \le 4\,;1 \le j \le 3\,;1 \le k \le 20),$$
$$D_{ijk} = \frac{1}{50}\sum_{n=1}^{50} D_{ijkn}\,(1 \le i \le 4\,;1 \le j \le 3\,;1 \le k \le 20\,;1 \le n \le 50). \tag{6}$$

Based on the above analysis, we can obtain matrix Z as follows:

$$Z = (P, E, D). \tag{7}$$

The final data obtained are as follows:

$$Z = \begin{pmatrix} 3.65 & 1.86 & 6.69 & 1.28 & 2.46 & 2.9 & 2.67 & 3.56 & 2.45 & 4.11 & 1.74 & 3.15 & 1.67 & 6.04 & 3.97 \\ 3.76 & 2 & 7.88 & 1.21 & 3.08 & 2.97 & 3.1 & 4.09 & 3.43 & 4.76 & 1.99 & 3.31 & 2.11 & 6.32 & 4.94 \\ 3.07 & 1.75 & 6.33 & 1.54 & 1.69 & 2.64 & 2.37 & 2.63 & 2.27 & 3.76 & 1.39 & 2.48 & 1.39 & 5.4 & 3.31 \\ 3.45 & 1.85 & 6.11 & 1.37 & 2.01 & 2.74 & 2.51 & 3.01 & 2.55 & 3.73 & 1.36 & 2.43 & 1.61 & 5.47 & 3.49 \end{pmatrix}. \tag{8}$$

The correlation analysis of the data is carried out, and the results are shown in Tables 1–3.

It can be seen from the table that there is a certain correlation between subjective evaluation data, eye movement data, and EEG data. The first fixation time was negatively correlated with other indicators because the shorter the first fixation time, the more attention the participants paid, and vice versa.

Then, we need to analyze the results in terms of three different factors, namely, subjective evaluation, eye movement, and EEG data. First, proximity analysis is carried out.

Procedure 1. To calculate the maximum F_{ij}^{+} and minimum F_{ij}^{-} of each evaluation index.

$$F_{ij}^{+} = \max\left\{Z_{ij}\right\}(1 \le i \le 4\,;1 \le j \le 15),$$
$$F_{ij}^{-} = \min\left\{Z_{ij}\right\}(1 \le i \le 4\,;1 \le j \le 15). \tag{9}$$

Procedure 2. To calculate the distance from the maximum to minimum of each proposal.

$$d_{ij}^{+} = F_{ij}^{+} - Z_{ij}(1 \le i \le 4\,;1 \le j \le 15),$$
$$d_{ij}^{-} = Z_{ij} - F_{ij}^{+}(1 \le i \le 4\,;1 \le j \le 15). \tag{10}$$

Procedure 3. To calculate relative closeness of evaluative value and maximum value for each program.

We use the relative closeness of evaluative value and maximum value for each proposal as the foundation of the final evaluation for the design proposal.

$$Z'_{ij} = \frac{d_{ij}^{-}}{d_{ij}^{-} + d_{ij}^{+}}. \tag{11}$$

Through the above steps, we can obtain the results of the close degree analysis data as shown in Table 4. Because the first fixation time is negatively related to other indicators, the smaller the value, the closer will be the ranking.

Then, we use the frequency statistics method for the three factors and obtain the final ranking method according to the frequency of the four rankings. The specific calculation formula is as follows:

$$f_{ip} = \frac{R_{ip}}{Z_b}. \tag{12}$$

The frequency of proposal i appearing in the P ranking is the number of effective evaluation indexes in different dimensions. In this paper, in the subjective evaluation dimension $Z_b = 1$, in the eye movement evaluation dimension $Z_b = 3$, and in the EEG evaluation dimension $Z_b = 11$; $f_i p$ is the frequency of proposal i appearing in the P ranking. The specific frequency of the four proposals under different dimensions and ranking is shown in Tables 5–7.

From Tables 5–7, we can see that from the perspective of subjective evaluation, the ranking is C2>C1>C3>C4; from the perspective of eye movement data, the ranking is C2>C1>C4>C3; from the perspective of the EEG evaluation data, the ranking is C2>C1>C4>C3. We can see that from the three dimensions, which are subjective valuation, eye movement data, and EEG objective data to evaluate the four proposals, sorting results are the same. The triangular validation was passed, indicating that the experimental subject evaluation is highly reliable.

Because the original triangular validation method mainly relies on the size of the data value to arrange the data, which is not convincing to a certain extent, there is contingency; therefore, this study focuses on data processing, especially of the confidence interval validation method used in data

Table 1: Correlation analysis data 1.

	P	e1	e2	e3	F4	F7	F8	FZ
P	1	0.903	0.693	-0.998	0.919	0.966	0.862	0.944
e1	0.903	1	0.866	-0.926	0.963	0.903	0.973	0.949
e2	0.693	0.866	1	-0.731	0.921	0.822	0.954	0.890
e3	-0.998	-0.926	-0.731	1	-0.939	-0.973	-0.890	-0.959
F4	0.919	0.963	0.921	-0.939	1	0.97	0.988	0.996
F7	0.966	0.903	0.822	-0.973	0.97	1	0.921	0.988
F8	0.862	0.973	0.954	-0.89	0.988	0.921	1	0.970
FZ	0.944	0.949	0.890	-0.959	0.996	0.988	0.970	1

Table 2: Correlation analysis data 2.

	P	e1	e2	e3	F4	F7	F8	FZ
FC1	0.729	0.951	0.914	-0.767	0.895	0.767	0.951	0.855
FC2	0.773	0.905	0.993	-0.806	0.960	0.883	0.979	0.938
FC5	0.825	0.859	0.946	-0.847	0.962	0.940	0.944	0.959
FC6	0.835	0.784	0.865	-0.847	0.922	0.948	0.875	0.936
FT9	0.872	0.994	0.915	-0.899	0.972	0.898	0.990	0.952
FT10	0.887	0.872	0.902	-0.903	0.971	0.976	0.937	0.979
F3	0.838	0.951	0.973	-0.867	0.985	0.917	0.996	0.966

Table 3: Correlation analysis data 3.

	FC1	FC2	FC5	FC6	FT9	FT10	F3
FC1	1	0.917	0.81	0.687	0.969	0.784	0.937
FC2	0.917	1	0.971	0.904	0.943	0.943	0.992
FC5	0.81	0.971	1	0.98	0.891	0.991	0.964
FC6	0.687	0.904	0.98	1	0.81	0.987	0.898
FT9	0.969	0.943	0.891	0.81	1	0.891	0.975
FT10	0.784	0.943	0.991	0.987	0.891	1	0.95
F3	0.937	0.992	0.964	0.898	0.975	0.95	1

Table 4: Data table for proximity analysis.

Index	C1	C2	C3	C4	Proposal sorting
P	0.84	1	0	0.55	C2>C1>C4>C3
e1	0.44	1	0	0.4	C2>C1>C4>C3
e2	0.33	1	0.12	0	C2>C1>C3>C4
e3	0.21	0	1	0.48	C2>C1>C4>C3*
F4	0.55	1	0	0.23	C2>C1>C4>C3
F7	0.79	1	0	0.3	C2>C1>C4>C3
F8	0.41	1	0	0.19	C2>C1>C4>C3
FZ	0.64	1	0	0.26	C2>C1>C4>C3
FC1	0.16	1	0	0.24	C2>C4>C1>C3
FC2	0.37	1	0.03	0	C2>C1>C3>C4
FC5	0.6	1	0.05	0	C2>C1>C3>C4
FC6	0.82	1	0.06	0	C2>C1>C3>C4
FT9	0.39	1	0	0.31	C2>C1>C4>C3
FT10	0.7	1	0	0.08	C2>C1>C4>C3
F3	0.4	1	0	0.11	C2>C1>C4>C3

Table 5: Frequency table of subjective evaluation.

Subjective evaluation proposal ranking	First place		Second place		Third place		Fourth place	
	R_{ip}	f_{ip}	R_{ip}	f_{ip}	R_{ip}	f_{ip}	R_{ip}	f_{ip}
C1	0	0.00	1	1.00	0	0.00	0	0.00
C2	1	1.00	0	0.00	0	0.00	0	0.00
C3	0	0.00	0	0.00	0	0.00	1	1.00
C4	0	0.00	0	0.00	1	1.00	0	0.00
Final ranking	C2		C1		C4		C3	

Table 6: Frequency table of eye movement test evaluation.

Subjective evaluation proposal ranking	First place		Second place		Third place		Fourth place	
	R_{ip}	f_{ip}	R_{ip}	f_{ip}	R_{ip}	f_{ip}	R_{ip}	f_{ip}
C1	0	0.00	3	1.00	0	0.00	0	0.00
C2	3	1.00	0	0.00	0	0.00	0	0.00
C3	0	0.00	0	0.00	1	0.33	2	0.67
C4	0	0.00	0	0.00	2	0.67	1	0.33
Final ranking	C2		C1		C4		C3	

Table 7: Frequency table of EEG test evaluation.

Subjective evaluation proposal ranking	First place		Second place		Third place		Fourth place	
	R_{ip}	f_{ip}	R_{ip}	f_{ip}	R_{ip}	f_{ip}	R_{ip}	f_{ip}
C1	0	0.00	10	0.91	1	0.09	0	0.00
C2	11	1.00	0	0.00	0	0.00	0	0.00
C3	0	0.00	0	0.00	3	0.27	8	0.73
C4	0	0.00	1	0.09	7	0.64	3	0.27
Final ranking	C2		C1		C4		C3	

analysis. We assume that the reader is familiar with the basic notions of statistical theory.

2. Methods

According to central limit theorem [22], for independent random variables, when the number is large, the data distribution follows a normal distribution. Then, according to the correlation and significance between the fifteen indicators of this experiment, F3 was selected as the specific analysis index.

The specific data of proposal 1 of the F3 indicator are as follows: F3-1 = (2.842 3.406 0.996 0.806 3.580 2.256 7.515 0.435 0.943 1.165 0.991 0.824 3.616 1.158 1.586 9.191 1.447 3.189 1.914 1.321).

The box diagram of the F3 index is shown in Figure 6. Because the experimental data of participant 7 and participant 16 deviated too much from other data, the data of these two individuals was rejected [23].

The final data and sample size n, sample mean $\bar{X}$, sample standard deviation S, and sample variance S^2 are shown in Table 8.

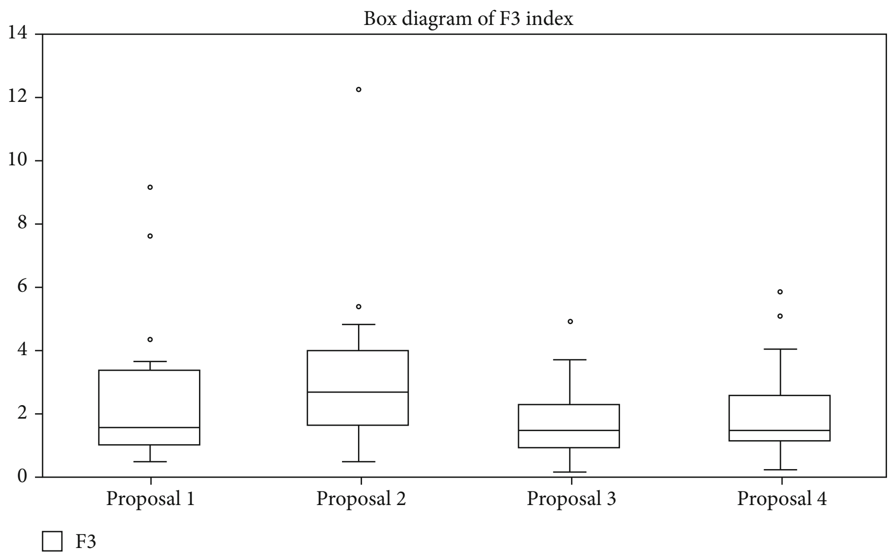

Figure 6: Box diagram of the F3 index.

Table 8: F3 index data sheet.

Participant	F3-1	F3-2	F3-3	F3-4
1	2.842	3.203	4.900	5.187
2	3.406	2.500	1.314	1.233
3	0.996	4.231	0.898	3.546
4	0.806	0.514	0.095	0.174
5	3.580	1.805	1.460	1.341
6	2.256	1.989	1.284	2.605
8	0.435	0.674	0.457	0.875
9	0.943	3.273	1.867	3.999
10	1.165	2.203	1.410	1.482
11	0.991	1.385	0.568	0.438
12	0.824	2.206	2.159	1.108
13	3.616	4.230	2.686	1.148
14	1.158	1.287	0.722	1.596
15	1.586	4.807	2.457	1.241
17	1.447	3.501	1.076	0.881
18	3.189	2.851	2.219	1.999
19	1.914	1.530	0.904	1.097
20	1.321	3.301	1.733	2.500
n	18	18	18	18
$\bar{X}$	1.804	2.527	1.567	1.803
S	1.067	1.237	1.097	1.309
S^2	1.138	1.530	1.204	1.713

According to Table 8, we know the specific data, sample average, and sample variance of F3-i ($i = 1, 2, 3, 4$). According to central limit theorem [22], F3-1 to F3-4 follows normal distribution $N(\mu_k, \sigma_k^2)(k = 1, 2, 3, 4)$ approximately. Population average μ_k and population variance σ_k^2 ($k = 1, 2, 3, 4$) are unknown. (Formulas (13)–(15) and (17)–(20) are quoted from references [22–27].) First, we analyze F3-1 and F3-2.

Table 9: Variance confidence interval data in the F3 index.

	$\frac{\sigma^2_{F3-1}}{\sigma^2_{F3-2}}$	$\frac{\sigma^2_{F3-1}}{\sigma^2_{F3-3}}$	$\frac{\sigma^2_{F3-1}}{\sigma^2_{F3-4}}$	$\frac{\sigma^2_{F3-2}}{\sigma^2_{F3-3}}$	$\frac{\sigma^2_{F3-2}}{\sigma^2_{F3-4}}$	$\frac{\sigma^2_{F3-3}}{\sigma^2_{F3-4}}$
Lower bound	0.393	0.500	0.351	0.673	0.473	0.372
Upper bound	1.405	1.786	1.255	2.402	1.688	1.328

We need to find a confidence interval with a confidence level of 0.90 for the variance ratio $\sigma^2_{F3-1}/\sigma^2_{F3-2}$ [23]. From Table 8, we know $n_1 = 18$, $S^2_{F3-1} = 1.138$, $n_2 = 18$, $S^2_{F3-1} = 1.530$, and $\alpha = 0.10$.

The F distribution has the following theorem:

$$F(n_1 - 1, n_2 - 1) \sim \frac{S_1^2/S_2^2}{\sigma_1^2/\sigma_2^2}. \tag{13}$$

Distribution $F(n_1 - 1, n_2 - 1)$ does not depend on any unknown parameters. Taking $(S_1^2/S_2^2)/(\sigma_1^2/\sigma_2^2)$ as the pivot amount, we can obtain the following formula:

$$P\left\{F_{1-\alpha/2}(n_1 - 1, n_2 - 1) < \frac{S_1^2/S_2^2}{\sigma_1^2/\sigma_2^2} < F_{\alpha/2}(n_1 - 1, n_2 - 1)\right\} = 1 - \alpha. \tag{14}$$

Thus, we get a confidence interval of σ_1^2/σ_2^2 with a confidence level of $1 - \alpha$:

$$\left(\frac{S_1^2}{S_2^2}\frac{1}{F_{\alpha/2}(n_1 - 1, n_2 - 1)}, \frac{S_1^2}{S_2^2}\frac{1}{F_{1-\alpha/2}(n_1 - 1, n_2 - 1)}\right). \tag{15}$$

According to the data table of F distribution, we can find that $F_{0.05}(17, 17) = 1.89$, $F_{0.95}(17, 17) = 1/1.89$. According to the above formula, it can be concluded that the confidence

Table 10: Confidence interval data of mean in the F3 index.

	$\mu_{F3-2}-\mu_{F3-1}$	$\mu_{F3-1}-\mu_{F3-3}$	$\mu_{F3-1}-\mu_{F3-4}$	$\mu_{F3-2}-\mu_{F3-3}$	$\mu_{F3-4}-\mu_{F3-3}$
Lower bound	-0.004	-0.467	-0.737	0.229	-0.507
Upper bound	1.450	0.942	0.740	1.692	0.979

Table 11: Confidence interval data of mean in indexes.

	$\mu_{F8-2}-\mu_{F8-1}$	$\mu_{F8-2}-\mu_{F8-3}$	$\mu_{FZ-2}-\mu_{FZ-1}$	$\mu_{FZ-2}-\mu_{FZ-3}$
Lower bound	0.071	0.278	0.584	0.240
Upper bound	1.671	1.828	2.033	1.878
	$\mu_{FT10-2}-\mu_{FT10-1}$	$\mu_{FT10-2}-\mu_{FT10-3}$	$\mu_{FT10-2}-\mu_{FT10-4}$	
Lower bound	0.031	0.276	0.197	
Upper bound	1.945	2.181	1.988	

interval of $\alpha = 0.10$ is as follows:

$$\left(\frac{1.138}{1.530}\times\frac{1}{1.89}, \frac{1.138}{1.530}\times 1.89\right), \tag{16}$$

that is, $(0.393, 1.405)$.

Because the confidence interval contains 1, we can assume that σ^2_{F3-1} and σ^2_{F3-2} are not significantly different. Similarly, we can conclude that there is no significant difference between the variance of the two proposals as shown in Table 9.

Based on the above analysis, we may consider the variance of each set of data to be equal and assume that the population variance of each data is equal. Next, we will verify the confidence interval of the population mean difference $\mu_{F3-i} - \mu_{F3-j}\,(1 \le i \ne j \le 4)$ [28]. We take $F3-2$ and $F3-3$ as examples. We already know that $\bar{X}_{F3-2} \sim N(\mu_{F3-2}, \sigma^2_{F3-2}/n_{F3-2})$ and $\bar{X}_{F3-3} \sim N(\mu_{F3-3}, \sigma^2_{F3-3}/n_{F3-3})$. Due to their independence, we know that

$$\bar{X}_{F3-2} - \bar{X}_{F3-3} \sim N\left(\mu_{F3-2} - \mu_{F3-3}, \frac{\sigma^2_{F3-2}}{n_{F3-2}} + \frac{\sigma^2_{F3-3}}{n_{F3-3}}\right), \tag{17}$$

or it can be written as

$$\frac{\left(\bar{X}_{F3-2} - \bar{X}_{F3-3}\right) - \left(\mu_{F3-2} - \mu_{F3-3}\right)}{\sqrt{\left(\sigma^2_{F3-2}/n_{F3-2}\right) + \left(\sigma^2_{F3-3}/n_{F3-3}\right)}} \sim N(0,1). \tag{18}$$

A confidence level of $\bar{X}_{F3-2} - \bar{X}_{F3-3}$ is obtained when $(\bar{X}_{F3-2} - \bar{X}_{F3-3}) - (\mu_{F3-2} - \mu_{F3-3})/\sqrt{(\sigma^2_{F3-2}/n_{F3-2}) + (\sigma^2_{F3-3}/n_{F3-3})}$ is chosen as a pivotal quantity, and the confidence interval of $1-\alpha$ is

$$\left(\bar{X}_{F3-2} - \bar{X}_{F3-3} \pm \sqrt{\frac{\sigma^2_{F3-2}}{n_{F3-2}} + \frac{\sigma^2_{F3-3}}{n_{F3-3}}}\right). \tag{19}$$

Although σ^2_{F3-2} and σ^2_{F3-3} are unknown but from formulas (12)–(15), we know that $\sigma^2_{F3-2} = \sigma^2_{F3-3}$; so we can prove that

$$\frac{\left(\bar{X}_{F3-2} - \bar{X}_{F3-3}\right) - \left(\mu_{F3-2} - \mu_{F3-3}\right)}{S_w\sqrt{\left(1/n_{F3-2}\right) + \left(1/n_{F3-3}\right)}} \sim t\left(n_{F3-2} + n_{F3-3} - 2\right). \tag{20}$$

In this formula, $S^2_w = ((n_{F3-2} - 1) \times S^2_{F3-2} + (n_{F3-3} - 1) \times S^2_{F3-3})/(n_{F3-2} + n_{F3-3} - 2)$, $S_w = \sqrt{S^2_w}$.

According to Table 8, we know that $\bar{X}_{F3-2} = 2.527$, $S_{F3-2} = 1.040$ and $\bar{X}_{F3-3} = 1.567$, $S_{F3-3} = 1.097$. We want to verify the confidence interval with a confidence level of 0.95 for the two population mean difference $\mu_{F3-2} - \mu_{F3-3}$. We know $1-\alpha = 0.95$, $\alpha/2 = 0.025$, $n_2 = 18$, $n_3 = 18$, $n_2 + n_3 - 2 = 34$, $t_{0.025}(34) = 2.0322$, $S^2_w = (17 \times 2.527 + 17 \times 1.567)/34$, and $S_w = \sqrt{S^2_w} = 1.0803$.

Based on the above formulas, we can obtain the confidence intervals as

$$\left(\bar{X}_{F3-2} - \bar{X}_{F3-3} \pm S_w \times t_{0.025}(34)\sqrt{\frac{1}{18} + \frac{1}{18}}\right) = (0.960 \pm 0.732), \tag{21}$$

that is, $(0.229, 1.189)$.

Because the lower bound of the confidence interval is greater than zero, μ_{F3-2} can be considered to be greater than μ_{F3-3}. Thus, we can infer that according to the F3 index, proposal 2 is better than proposal 3. Similarly, other results are shown in Table 10.

Similarly, in proposal 3, proposal 1, and proposal 4, the lower bound of confidence interval is lower than zero; therefore, we can conclude that according to the F3 index in this analysis method, there is no significant difference between proposal 3, proposal 1, and proposal 4.

3. Results

The same analysis method was adopted to verify the other 14 indexes; the nine indicators cannot draw a clear conclusion. Specific conclusions are shown in Table 11. According to the F8 index and FZ index, we can conclude that proposal 2 is better than proposal 1, and proposal 2 is better than proposal 3. According to the FT10 index, we can conclude that proposal 2 is better than proposal 1, proposal 3, and proposal 4.

According to the data obtained, we can see that proposal 2 is better than the others, that is, proposal 2 is the best, which is consistent with the previous conclusion.

4. Conclusions

In this study, we improved the way of data comparison in the triangular validation method. We apply the confidence interval in statistics to the analysis model and improve the problem of previous data comparison which is being too simple. Because less amount of data is collected in this experiment, a definite result cannot be acquired from a single indicator. However, according to statistical theory, when the number of data sample is large, the model constructed will draw a clear conclusion. This study shows that mathematical statistics can be well used in product evaluation and that triangular evaluation is deepened to make the evaluation model more convincing and applicable. With the development of intelligent wearable equipment, data acquisition will become more convenient in the future. Therefore, this evaluation model should have an extensive application and research value. We hope that more mathematical and statistical knowledge will be used for product evaluation to promote the development of industrial evaluation systems. Next, we will analyze the product evaluation in industrial design and guide the product design process through the analysis of a large number of data.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Exact Analytical Solutions of Generalized Fifth-Order KdV Equation by the Extended Complex Method

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The recently introduced technique, namely, the extended complex method, is used to explore exact solutions for the generalized fifth-order KdV equation. Appropriately, the rational, periodic, and elliptic function solutions are obtained by this technique. The 3D graphs explain the different physical phenomena to the exact solutions of this equation. This idea specifies that the extended complex method can acquire exact solutions of several differential equations in engineering. These results reveal that the extended complex method can be directly and easily used to solve further higher-order nonlinear partial differential equations (NLPDEs). All computer simulations are constructed by maple packages.

1. Introduction

In the 20 century, nonlinear science (NLS) plays a significant role in special inventions, for example, the invention of the radio, the discovery of DNA structure for biology, the development of quantum theory for theoretical physics and chemistry, and the invention of transister for computer engineering. It is well known that NLS belongs to the NLPDEs which are introduced in several areas such as fluid thermodynamics, plasma diffusion, biology, physics, geometry, and population dynamics.

Lots of studies are focused on the differential equations [1–10], and many effective techniques are used to acquire analytical and numerical solutions for NLPDEs such as sine-cosine method [11], extended sinh-Gorden equation expansion method [12], variation iteration algorithm [13], homotopy perturbation method [14], F-expansion method [15], Exp-function expansion method [16], first integral method [17], Ansatz method [18], generalized Kudryashov method [19], (G'/G)-expansion method [20], projective Riccati equation method [21], tanh method [22], nonpolynomial spline method [23], B-spline method [24], B-spline collocation [25], Weierstrass elliptic function method [26], Laplace decomposition method [27], extended direct algebraic method [28, 29], Sub-ODE method [30], Darboux transformation [31], and extended tanh-coth method [32, 33]. The generalized fifth-order KdV equation [34] is represented by

$$w_t + sww_x + fw^2w_x + ew_{xxx} + \mu w_{xxxxx} = 0, \tag{1}$$

where s, f, e, and μ are the arbitrary constants. This equation is a nonlinear model in many long wave physical phenomena. It is used in the shallow water wave with surface tension and magnetoacoustic wave in plasma. Several researchers have explored the analytical solutions of generalized fifth-order KdV equation such as Hedli and Kadem have attained a new analytical solution for the fifth-order KdV equation by the exponential expansion method [35]. Dinarvand et al. have found approximate analytical solutions of the sawada-kotera and Lax's fifth-order KdV equations by homotopy analysis technique [36]. Salas and Lugo have introduced extended tanh method to obtain the exact solutions of the general fifth-order KdV equation [37]. Alam and Xin et al. have attained new exact solutions by (G'/G)-expansion method of modified KdV-Zakharov-Kuznetsov equation [38]. Ganji and Abdollahzadeh have introduced the sech method and rational exp-function method to find the exact traveling wave solutions of the Lax's seventh-order KdV equation [39].

In the present work, our main purpose is to calculate the generalized fifth-order KdV equation by the extended complex method based on the concept of Yuan et al. [40–46]. It is a remarkable approach to attain exact analytical solutions. Our technique would be potentially applied to various processes of the engineering field. This article is organized as mentioned as follows. In Section 2, methods and materials are described. In Section 3, the application of the introduced method is determined. Section 4 deals with physical phenomena of important results. The comparison and conclusions are explained in Section 5.

2. Methods and Materials

Let us consider the general form of NLPDE

$$l(w, w_t, w_x, w_z, w_{tt}, w_{xx} \cdots\cdots\cdots), \tag{2}$$

where the unknown function is $w = w(x, t)$ and l is a polynomial in $w = w(x, t)$ and its derivatives.

Step 1. A transformation $T : w(x, t) \to W(z)$ is introduced, and (x, t) can be introduced in different standard; hence, we have used the transformation such as

$$w(x, t) = W(z), z = k(x - \omega t). \tag{3}$$

Step 2. The $w(x, t) = W(z), z = k(x - \omega t)$ transform Eq. (2) into nonlinear ODE:

$$T\left(W, W', W'', W''', \cdots\right) = 0, \tag{4}$$

in Eq. (4), where W primes are the derivatives w.r.t z. This equation is reduced by further integration.

Step 3. Let the meromorphic solutions W of Eq. (4) have at least one pole, and let us consider $p, q \in \mathbb{Z}$. For this condition, we substitute the Laurent series

$$W(z) = \sum_{k=-q}^{\infty} B_k z^k, q > 0, B_{-q} \neq 0, \tag{5}$$

into Eq. (4), if we can find p distinct Laurent singular parts:

$$\sum_{k=-q}^{-1} B_k z^k, \tag{6}$$

then the weak $\langle p, q\rangle$ condition of Eq. (4) holds. Weierstrass elliptic function $\wp(z) := \wp(z, g_2, g_3)$ with double periods of the equation is given as below:

$$\left(\wp'(z)\right)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3, \tag{7}$$

and the addition formula is mentioned as below:

$$\wp(z - z_0) = -\wp(z) + \frac{1}{4}\left[\frac{\wp'(z) + \wp'(z_0)}{\wp(z) - \wp(z_0)}\right]^2 - \wp(z_0). \tag{8}$$

Step 4. Putting the indeterminate forms

$$\begin{aligned} W(z) = {} & \sum_{i=1}^{y-1}\sum_{j=2}^{q} \frac{(-1)^j \delta_{-ij}}{(j-1)!}\frac{d^{j-2}}{dz^{j-2}}\left(\frac{1}{4}\left[\frac{\wp'(z) + G_i}{\wp(z) - H_i}\right]^2 - \wp(z)\right) \\ & + \sum_{i=1}^{y-1}\frac{\delta_{-i1}}{2}\frac{\wp'(z) + G_i}{\wp(z) - H_i} + \sum_{j=2}^{q}\frac{(-1)^j \delta_{-yj}}{(j-1)!}\frac{d^{j-2}}{dz^{j-2}}\wp(z) + \delta_0, \end{aligned} \tag{9}$$

$$W(z) = \sum_{i=1}^{y}\sum_{j=1}^{q}\frac{\delta_{ij}}{(z - z_i)^j} + \delta_0, \tag{10}$$

$$W(e^{\alpha z}) = \sum_{i=1}^{y}\sum_{j=1}^{q}\frac{\delta_{ij}}{(e^{\alpha z} - e^{\alpha z_i})^j} + \delta_0, \tag{11}$$

into Eq. (4); hence, the number of equations is computed by adjusting the coefficient to zero. These algebraic equations are calculated by the source of maple. Equation (9) is the elliptic solution W with pole at $z = 0$, where δ_{-ij} are attained by (4), $G_i^2 = 4H_i^3 - g_2H_i - g_3$, $\sum_{i=1}^{y}\delta_{-i1} = 0$. Equation (10) is the rational function, and Eq. (11) is the exponential function which are denoted as $W(z)$, $W(e^{\alpha z})(\alpha \in \mathbb{C})$, and they have y $(\leq p)$ distinct poles of multiplicity q.

Step 5. The meromorphic solutions are got with the arbitrary pole. Substitute inverse transformation T^{-1} into meromorphic solutions; then, we obtain the exact analytical solutions of NLPDEs.

3. Application of the Method

In this section, we would like to find the exact analytical solutions of a generalized fifth-order KdV equation by extended complex approach. Substitute

$$w(x, t) = W(z), z = k(x - \omega t), \tag{12}$$

into Eq. (1), then obtain

$$-\omega W' + sWW' + fW^2W' + ek^2W''' + \mu k^4W''''' = 0, \tag{13}$$

now, we integrate Eq. (13) w.r.t z; then, we attain new ODE

$$-\omega W + s\frac{W^2}{2} + f\frac{W^3}{3} + ek^2 W'' + \mu k^4 W'''' = 0. \tag{14}$$

Putting (5) into (14) then we have $p = 1$ and $q = 2$; hence, the weak $\langle 1, 2\rangle$ condition of (14) holds. By weak $\langle 1, 2\rangle$ and (10), then rational solutions with pole at $z = 0$ are

$$W_r(z) = \frac{\delta_{12}}{(z-1)^2} + \frac{\delta_{11}}{z-1} + \delta_{10}, \tag{15}$$

substituting the $W_r(z)$ into Eq. (14); then, we have

$$\frac{1}{6}\sum_{i=1}^{7} c_{1i} z^{(7-i)} (z-1)^{-6} = 0, \tag{16}$$

where

$$\begin{aligned}
c_{11} &= 2\delta_{10}^3 f + 3\delta_{10}^2 s - 6\delta_{10}\omega,\\
c_{12} &= 6\delta_{11}\delta_{10}^2 f - 12\delta_{10}^3 f + 6\delta_{11}\delta_{10}s - 18\delta_{10}^2 s - 6\delta_{11}\omega + 36\delta_{10}\omega,\\
c_{13} &= 6\delta_{12}\delta_{10}^2 f + 6\delta_{11}^2\delta_{10} f - 30\delta_{11}\delta_{10}^2 f + 30\delta_{10}^3 f + 6\delta_{12}\delta_{10}s\\
&\quad + 3\delta_{11}^2 s - 30\delta_{11}\delta_{10}s + 45\delta_{10}^2 s - 6\delta_{12}\omega + 30\delta_{11}\omega - 90\delta_{10}\omega,\\
c_{14} &= 12\delta_{12}\delta_{11}\delta_{10} f - 24\delta_{12}\delta_{10}^2 f + 2\delta_{11}^3 f - 24\delta_{11}^2\delta_{10} f + 60\delta_{11}\delta_{10}^2 f\\
&\quad + 12\delta_{11} ek^2 - 40\delta_{10}^3 f + 6\delta_{12}\delta_{11}s - 24\delta_{12}\delta_{10}s - 12\delta_{11}^2 s\\
&\quad + 60\delta_{11}\delta_{10}s - 60\delta_{12}^2 s + 24\delta_{12}\omega - 60\delta_{11}\omega + 120\delta_{10}\omega,\\
c_{15} &= 6\delta_{12}^2\delta_{10} f + 6\delta_{12}\delta_{11}^2 f - 36\delta_{12}\delta_{11}\delta_{10} f + 36\delta_{12}\delta_{10}^2 f + 36\delta_{12} ek^2\\
&\quad - 6\delta_{11}^3 f + 36\delta_{11}^2\delta_{10} f - 60\delta_{11}\delta_{10}^2 f - 36\delta_{11} ek^2 + 30\delta_{10}^3 f + 3\delta_{12}^2 s\\
&\quad - 18\delta_{12}\delta_{11}s + 36\delta_{12}\delta_{10}s + 18\delta_{11}^2 s - 60\delta_{11}\delta_{10}s + 45\delta_{10}^2 s\\
&\quad - 36\delta_{12}\omega + 60\delta_{11}\omega - 90\delta_{10}\omega,\\
c_{16} &= 144\delta_{11}k^4\mu + 6\delta_{12}^2\delta_{11} f - 12\delta_{12}^2\delta_{10} f - 12\delta_{12}\delta_{11}^2 f + 36\delta_{12}\delta_{11}\delta_{10} f\\
&\quad - 24\delta_{12}\delta_{10}^2 f - 72\delta_{12} ek^2 + 6\delta_{11}^3 f - 24\delta_{11}^2\delta_{10} f + 30\delta_{11}\delta_{10}^2 f\\
&\quad + 36\delta_{11} ek^2 - 12\delta_{10}^3 f - 6\delta_{12}^2 s + 18\delta_{12}\delta_{11}s - 24\delta_{12}\delta_{10}s\\
&\quad - 12\delta_{11}^2 s + 30\delta_{11}\delta_{10}s - 18\delta_{10}^2 s + 24\delta_{12}\omega - 30\delta_{11}\omega + 36\delta_{10}\omega,\\
c_{17} &= 720\delta_{12}k^4\mu - 144\delta_{11}k^4\mu + 2\delta_{12}^3 f - 6\delta_{12}^2\delta_{11} f + 6\delta_{12}^2\delta_{10} f\\
&\quad + 6\delta_{12}\delta_{11}^2 f - 12\delta_{12}\delta_{11}\delta_{10} f + 6\delta_{12}\delta_{10}^2 f + 36\delta_{12} ek^2\\
&\quad - 2\delta_{11}^3 f + 6\delta_{11}^2\delta_{10} f - 6\delta_{11}\delta_{10}^2 f - 12\delta_{11} ek^2 + 2\delta_{10}^3 f\\
&\quad + 3\delta_{12}^2 s - 6\delta_{12}\delta_{11}s + 6\delta_{12}\delta_{10}s + 3\delta_{11}^2 s - 6\delta_{11}\delta_{10}s\\
&\quad + 3\delta_{10}^2 s - 6\delta_{12}\omega + 6\delta_{11}\omega - 6\delta_{10}\omega.
\end{aligned} \tag{17}$$

By assuming that the coefficients of same powers concerning z in Eq. (16) are zero, then we have numbers of equations:

$$c_{1i} = 0, (i = 1, 2, \cdots, 7). \tag{18}$$

By solving number of these equations, we obtain

$$\delta_{12} = -\frac{6\sqrt{10}\sqrt{\mu}k^2}{\sqrt{f}}, \delta_{11} = 0, \delta_{10} = 0, \tag{19}$$

then

$$W_{r10}(z) = -\frac{6\sqrt{10}\sqrt{\mu}k^2/\sqrt{f}}{(z-1)^2}, \tag{20}$$

where $\omega = 0$ and $s = -(1/5)(\sqrt{f}e\sqrt{10}/\sqrt{\mu})$;

$$\delta_{12} = \frac{24k^2 e}{s}, \delta_{11} = 0, \delta_{10} = \frac{6}{5}\frac{e^2}{s\mu}, \tag{21}$$

then

$$W_{r20}(z) = \frac{24k^2 e/s}{(z-1)^2} + \frac{6}{5}\frac{e^2}{s\mu}, \tag{22}$$

where $f = -(5/8)(\mu s^2/e^2)$ and $\omega = (3/10)(e^2/\mu)$.

$W(z) = R(\eta)$ is a rational function of $\eta = e^{\alpha z} (\alpha \in \mathbb{C})$, applying it into Eq. (14) then

$$\begin{aligned}
&-\omega R + s\frac{R^2}{2} + f\frac{R^3}{3} + k^2 e\alpha^2\left(R''\eta^2 + R'\eta\right)\\
&\quad + k^4\alpha^4\mu\left(R^{(4)}\eta^4 + 6R'''\eta^3 + 7R''\eta^2 + R'\eta\right) = 0,
\end{aligned} \tag{23}$$

substituting

$$W_s(z) = \frac{\delta_{12}}{(\eta-1)^2} + \frac{\delta_{11}}{\eta-1} + \delta_{10}, \tag{24}$$

into the Eq. (23), we attain that

$$\frac{1}{6}\sum_{i=1}^{7} \frac{c_{2i}\alpha^2\eta^{7-i}}{(\eta-1)^6} = 0, \tag{25}$$

where

$$c_{21} = 2\delta_{10}^3 f + 3\delta_{10}^2 s - 6\delta_{10}\omega,$$

$$c_{22}=6\alpha^4\delta_{11}k^4\mu+6\alpha^2\delta_{11}ek^2+6\delta_{11}\delta_{10}^2f-12\delta_{10}^3f+6\delta_{11}\delta_{10}s-18\delta_{10}^2s-6\delta_{11}\omega+36\delta_{10}\omega,$$

$$c_{23}=96\delta_{12}\alpha^4k^4\mu+60\alpha^4\delta_{11}k^4\mu+24\delta_{12}\alpha^2ek^2-12\alpha^2\delta_{11}ek^2+6\delta_{12}\delta_{10}^2f+6\delta_{11}^2\delta_{10}f-30\delta_{11}\delta_{10}^2f+30\delta_{10}^3f+6\delta_{12}\delta_{10}s+3\delta_{11}^2s-30\delta_{11}\delta_{10}s+45\delta_{10}^2s-6\delta_{12}\omega+30\delta_{11}\omega-90\delta_{10}\omega,$$

$$c_{24}=396\delta_{12}\alpha^4k^4\mu-36\delta_{12}\alpha^2ek^2+12\delta_{12}\delta_{11}\delta_{10}f-24\delta_{12}\delta_{10}^2f+2\delta_{11}^3f-24\delta_{11}^2\delta_{10}f+60\delta_{11}\delta_{10}^2f-40\delta_{10}^3f+6\delta_{12}\delta_{11}s-24\delta_{12}\delta_{10}s-12\delta_{11}^2s+60\delta_{11}\delta_{10}s-60\delta_{10}^2s+24\delta_{12}\omega-60\delta_{11}\omega+120\delta_{10}\omega,$$

$$c_{25}=2\delta_{12}^3f-6\delta_{12}^2\delta_{11}f+6\delta_{12}^2\delta_{10}f+6\delta_{12}\delta_{11}^2f-12\delta_{12}\delta_{11}\delta_{10}f+6\delta_{12}\delta_{10}^2f-2\delta_{11}^3f+6\delta_{11}^2\delta_{10}f-6\delta_{11}\delta_{10}^2f+2\delta_{10}^3f+3\delta_{12}^2s-6\delta_{12}\delta_{11}s+6\delta_{12}\delta_{10}s+3\delta_{11}^2s-6\delta_{11}\delta_{10}s+3\delta_{10}^2s-6\delta_{12}\omega+6\delta_{11}\omega-6\delta_{10}\omega,$$

$$c_{26}=216\delta_{12}\alpha^4k^4\mu-60\alpha^4\delta_{11}k^4\mu+12\alpha^2\delta_{11}ek^2+6\delta_{12}^2\delta_{10}f+6\delta_{12}\delta_{11}^2f-36\delta_{12}\delta_{11}\delta_{10}f+36\delta_{12}\delta_{10}^2f-6\delta_{11}^3f+36\delta_{11}^2\delta_{10}f-60\delta_{11}\delta_{10}^2f+30\delta_{10}^3f+3\delta_{12}^2s-18\delta_{12}\delta_{11}s+36\delta_{12}\delta_{10}s+18\delta_{11}^2s-60\delta_{11}\delta_{10}s+45\delta_{10}^2s-36\delta_{12}\omega+60\delta_{11}\omega-90\delta_{10}\omega,$$

$$c_{27}=12\delta_{12}\alpha^4k^4\mu-6\alpha^4\delta_{11}k^4\mu+12\delta_{12}\alpha^2ek^2-6\alpha^2\delta_{11}ek^2+6\delta_{12}^2\delta_{11}f-12\delta_{12}^2\delta_{10}f-12\delta_{12}\delta_{11}^2f+36\delta_{12}\delta_{11}\delta_{10}f-24\delta_{12}\delta_{10}^2f+6\delta_{11}^3f-24\delta_{11}^2\delta_{10}f+30\delta_{11}\delta_{10}^2f-12\delta_{10}^3f-6\delta_{12}^2s+18\delta_{12}\delta_{11}s-24\delta_{12}\delta_{10}s-12\delta_{11}^2s+30\delta_{11}\delta_{10}s-18\delta_{10}^2s+24\delta_{12}\omega-30\delta_{11}\omega+36\delta_{10}\omega. \tag{26}$$

By assuming that the coefficients of the same powers concerning η in Eq. (25) are zero, then obtain the numbers of equations:

$$c_{2i}=0,\ (i=1,2,\cdots,7). \tag{27}$$

Solve the numbers of these equations, then attain

$$\delta_{12}=-\frac{6\sqrt{10}\sqrt{\mu}k^2\alpha^2}{\sqrt{f}},\ \delta_{11}=-\frac{6\sqrt{10}\sqrt{\mu}k^2\alpha^2}{\sqrt{f}},\ \delta_{10}=0, \tag{28}$$

where $\eta=e^{\alpha z}(\alpha\in\mathbb{C})$.

$$W_s(e^{\alpha z})=-\frac{6\sqrt{10}\sqrt{\mu}k^2\alpha^2/\sqrt{f}}{(e^{\alpha z}-1)^2}-\frac{6\sqrt{10}\sqrt{\mu}k^2\alpha^2/\sqrt{f}}{e^{\alpha z}-1},$$
$$W_s(e^{\alpha z})=-\frac{\left(6\sqrt{10}\sqrt{\mu}k^2\alpha^2/\sqrt{f}\right)e^{\alpha z}}{(e^{\alpha z}-1)^2}, \tag{29}$$

so, we obtain the simply periodic solutions of Eq. (14) with pole at $z=0$

$$W_{s10}(z)=-\frac{3\sqrt{10}\sqrt{\mu}k^2\alpha^2}{2\sqrt{f}}\left(\coth^2\frac{\alpha}{2}z-1\right), \tag{30}$$

where $\omega=\alpha^4k^4\mu+\alpha^2ek^2$ and $s=(1/5)((5\alpha^2k^2\mu+e)\sqrt{10}\sqrt{f}/\sqrt{\mu})$. Furthermore,

$$\delta_{12}=-\frac{12\alpha^2ek^2}{s},\ \delta_{11}=-\frac{12\alpha^2ek^2}{s},\ \delta_{10}=-\frac{2\alpha^2ek^2}{s}, \tag{31}$$

where $\eta=e^{\alpha z}(\alpha\in\mathbb{C})$.

$$W_s(e^{\alpha z})=-\frac{12\alpha^2ek^2/s}{(e^{\alpha z}-1)^2}-\frac{12\alpha^2ek^2/s}{e^{\alpha z}-1}-\frac{2\alpha^2ek^2}{s},$$
$$W_s(e^{\alpha z})=-\frac{\left(12\alpha^2ek^2/s\right)e^{\alpha z}}{(e^{\alpha z}-1)^2}-\frac{2\alpha^2ek^2}{s}, \tag{32}$$

so, we attain again the simply periodic solutions of Eq. (14) with pole at $z=0$

$$W_{s20}(z)=-\frac{3\alpha^2ek^2}{s}\left(\coth^2\frac{\alpha}{2}z\right)-\frac{5\alpha^2ek^2}{s}, \tag{33}$$

where $\omega=-\alpha^2ek^2$ and $f=0$.

By the weak $\langle 1,2\rangle$ condition, so, we introduce here the elliptic solutions by (9) with $z=0$ pole.

$$W_{d0}(z)=\delta_{12}\wp(z)+\delta_{10}, \tag{34}$$

substitute $W_{d0}(z)$ into Eq. (14); then, we have

$$\sum_{i=0}^{3}c_{3i}\wp^i(z)=0, \tag{35}$$

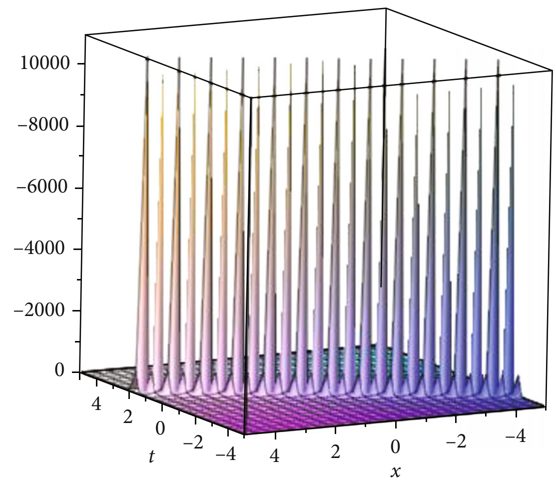

FIGURE 1: Perspective view of 3D graph of $W_{r,1}(z)$ for the fixed values $\omega = 2$, $z_0 = 0.5$, $\mu = 1$, $k = 1$, and $f = 1$ represents the exact solutions.

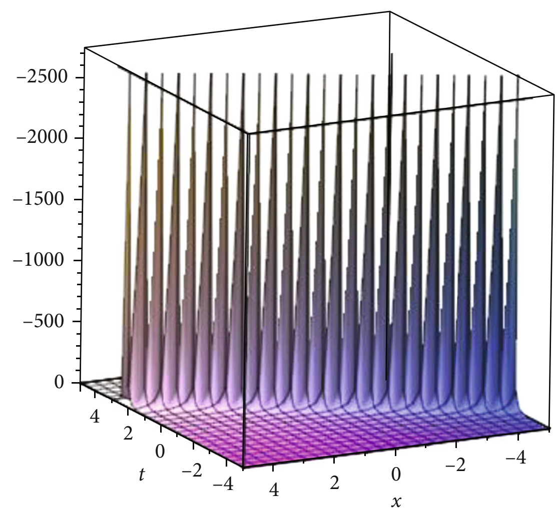

FIGURE 2: Perspective view of 3D graph of $W_{r,1}(z)$ for the fixed values $\omega = 2$, $z_0 = -0.5$, $\mu = 1$, $k = 1$, and $f = 1$ represents the exact solution.

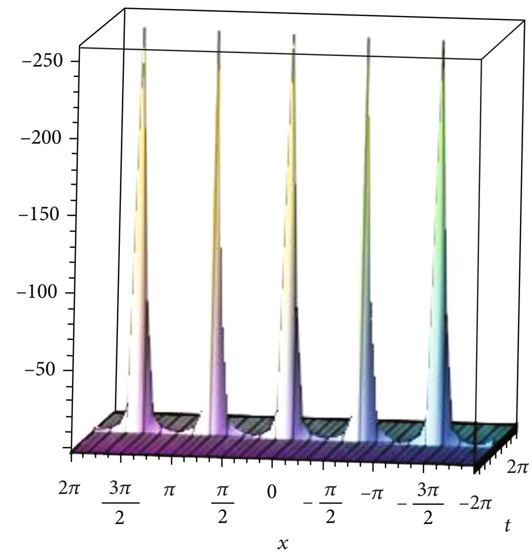

FIGURE 3: Perspective view of 3D graph of $W_{s,1}(z)$ for the fixed values $\omega = 9$, $z_0 = 1/6$, $\mu = 1$, $k = 1$, $f = 1$, and $\alpha = 1$ represents the exact solutions.

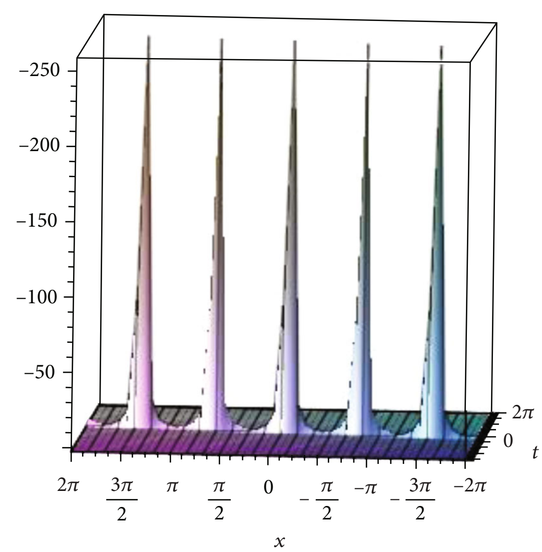

FIGURE 4: Perspective view of 3D graph of $W_{s,1}(z)$ for the fixed values $\omega = 9$, $z_0 = -1/6$, $\mu = 1$, $k = 1$, $f = 1$, and $\alpha = 1$ represents the exact solutions.

where

$$\begin{aligned} c_{30} &= -\delta_{10}\omega + \frac{1}{2}\delta_{10}^2 s + \frac{1}{3}\delta_{10}^3 f - \frac{1}{2}ek^2 a g_3 - 12\delta_{12} g_3 k^4 \mu, \\ c_{31} &= -18\delta_{12} g_2 k^4 \mu + \delta_{12}\delta_{10}^2 f + \delta_{12}\delta_{10} s - \delta_{10}\omega, \\ c_{32} &= \frac{1}{2}\delta_{12}^2 s + \delta_{12}^2 \delta_{10} f + 6ek^2 \delta_{12}, \\ c_{33} &= \frac{1}{3}\delta_{12}^3 f + 120\delta_{12}\mu k^4. \end{aligned} \tag{36}$$

By assuming that the coefficients of the same powers concerning $\wp(z)$ in Eq. (35) are zero, then obtain the numbers of equations:

$$c_{3i} = 0, (i = 0, 1, \cdots, 3). \tag{37}$$

Solve these equation; then, we have

$$\delta_{12} = -\frac{6\sqrt{10}\sqrt{\mu}k^2}{\sqrt{f}}, \delta_{10} = 0, \tag{38}$$

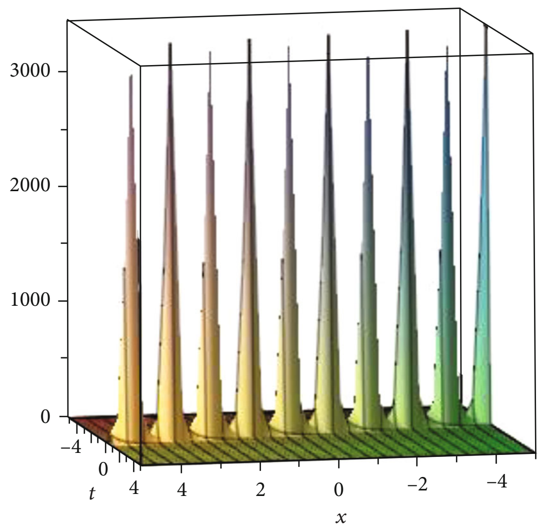

FIGURE 5: Perspective view of 3D graph of $W_{r,2}(z)$ for the fixed values $\omega = 5$, $z_0 = 1/3$, $\mu = 1$, $k = 1$, $s = 1$, and $e = 1$ represents exact solutions.

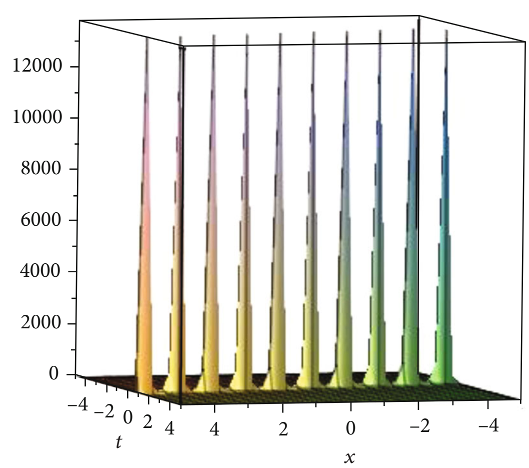

FIGURE 6: Perspective view of 3D graph of $W_{r,2}(z)$ for the fixed values $\omega = 5$, $z_0 = -1/3$, $\mu = 1$, $k = 1$, $s = 1$, and $e = 1$ represents exact solutions.

and then we have

$$W_{d0}(z) = -\frac{6\sqrt{10}\sqrt{\mu}k^2}{\sqrt{f}}\wp(z), \tag{39}$$

hence, the elliptic general solutions of Eq. (14) with arbitrary pole are expressed such as

$$W_{d,1}(z) = -\frac{6\sqrt{10}\sqrt{\mu}k^2}{\sqrt{f}}\wp(z - z_0), \tag{40}$$

where $z_0 \in \mathbb{C}$.

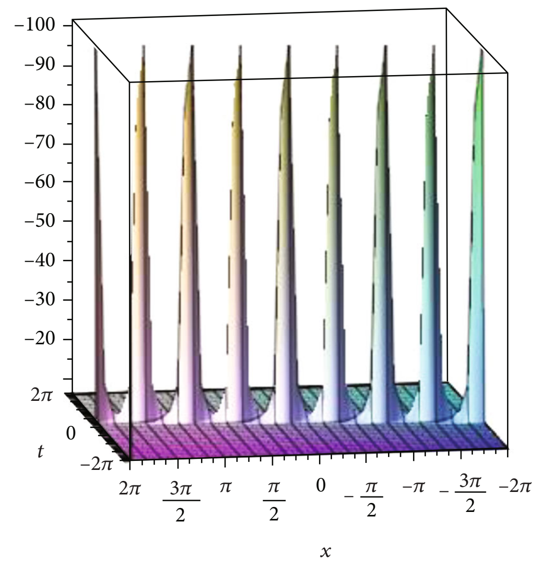

FIGURE 7: Perspective view of 3D graph of $W_{s,2}(z)$ for the fixed values $\omega = 6$, $z_0 = 1/8$, $\mu = 1$, $k = 1$, $s = 1$, $e = 1$, and $\alpha = 1$ represents exact solutions.

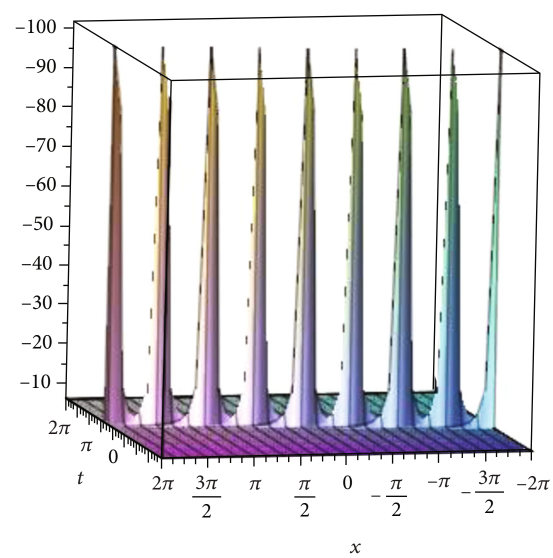

FIGURE 8: Perspective view of 3D graph of $W_{s,2}(z)$ for the fixed values $\omega = 6$, $z_0 = -1/8$, $\mu = 1$, $k = 1$, $s = 1$, $e = 1$, and $\alpha = 1$ represents exact solutions.

Applying the additional formula to the $W_{d,1}(z)$, and we attain

$$W_{d,1}(z) = -\frac{6\sqrt{10}\sqrt{\mu}k^2}{\sqrt{f}}\left(-\wp(z) + \frac{1}{4}\left(\frac{\wp'(z) + G_1}{\wp(z) - H_1}\right)^2\right) + \frac{6\sqrt{10}\sqrt{\mu}k^2}{\sqrt{f}}H_1, \tag{41}$$

where $e = -24k^2\mu$, $s = -(24/5)k^2\sqrt{\mu}\sqrt{10}\sqrt{f}$, $\omega = -18g_2k^4\mu$, $G_1 = 4H_1^3 - g_2H_1 - g_3$, and g_2 and g_3 are the arbitrary constants.

By the above approach, so, we obtain the meromorphic solutions of Eq. (14) with arbitrary pole as mention as follows:

$$W_{r,1}(z) = -\frac{6\sqrt{10}\sqrt{\mu}k^2/\sqrt{f}}{(z-z_0-1)^2}, \quad (42)$$

where $\omega = 0$, $s = -(1/5)(\sqrt{f}e\sqrt{10}/\sqrt{\mu})$, and $z_0 \in \mathbb{C}$.

$$W_{r,2}(z) = \frac{24k^2 e/s}{(z-z_0-1)^2} + \frac{6}{5}\frac{e^2}{s\mu}, \quad (43)$$

where $f = -(5/8)(\mu s^2/e^2)$, $\omega = (3/10)(e^2/\mu)$, and $z_0 \in \mathbb{C}$.

$$W_{s,1}(z) = -\frac{3\sqrt{10}\sqrt{\mu}k^2\alpha^2}{2\sqrt{f}}\left(\coth^2\frac{\alpha}{2}(z-z_0) - 1\right), \quad (44)$$

where $\omega = \alpha^4 k^4 \mu + \alpha^2 e k^2$, $s = (1/5)((5\alpha^2 k^2 \mu + e)\sqrt{10}\sqrt{f}/\sqrt{\mu}$), and $z_0 \in \mathbb{C}$.

$$W_{s,2}(z) = -\frac{3\alpha^2 e k^2}{s}\left(\coth^2\frac{\alpha}{2}(z-z_0)\right) - \frac{5\alpha^2 e k^2}{s}, \quad (45)$$

where $\omega = -\alpha^2 e k^2$, $f = 0$, and $z_0 \in \mathbb{C}$.

4. Description about Figures

Here, we display the exact solutions for $W_{r,1}(z)$, $W_{s,1}(z)$, $W_{r,2}(z)$, and $W_{s,2}(z)$ by graphical phenomena as in Figures 1–8. These graphs are represented by the source of maple to persuade important results. Figures 1–8 display different multisolitary wave solutions that are obtained by different values of z_0 and ω, whereas other parameters are constant.

Figures 1 and 2 indicate the exact solutions for $W_{r,1}(z)$, adjust the values $\omega = 2$, $z_0 = 0.5$, $\mu = 1$, $k = 1$, and $f = 1$ and $\omega = 2$, $z_0 = -0.5$, $\mu = 1$, $k = 1$, and $f = 1$.

Figures 3 and 4 indicate the exact solutions for $W_{s,1}(z)$, adjust the values $\omega = 9$, $z_0 = 1/6$, $\mu = 1$, $k = 1$, $f = 1$, and $\alpha = 1$ and $\omega = 9$, $z_0 = -1/6$, $\mu = 1$, $k = 1$, $f = 1$, and $\alpha = 1$.

Figures 5 and 6 indicate the exact solutions for $W_{r,2}(z)$, adjust the values $\omega = 5$, $z_0 = 1/3$, $\mu = 1$, $k = 1$, $s = 1$, and $e = 1$ and $\omega = 5$, $z_0 = -1/3$, $\mu = 1$, $k = 1$, $s = 1$, and $e = 1$.

Figures 7 and 8 indicate the exact solutions for $W_{s,2}(z)$, adjust the values $\omega = 6$, $z_0 = 1/8$, $\mu = 1$, $k = 1$, $s = 1$, $e = 1$, and $\alpha = 1$ and $\omega = 6$, $z_0 = -1/8$, $\mu = 1$, $k = 1$, $s = 1$, $e = 1$, and $\alpha = 1$.

5. Comparison and Conclusion

Khan et al. [25] represented the modified simple equation technique for the analytical treatment of generalized fifth-order KdV equation. This proposed technique provides fresh exact solutions in the area of engineering and mathematical physics. The results demonstrated the remarkable exact solutions for this technique. For this purpose, we create the comparison between the modified simple equation technique and the extended complex approach.

We employed the extended complex technique to explore the exact analytical solutions of the generalized fifth-order KdV equation. The graphical phenomena are showed by setting the values of arbitrary parameters, and the graphical representations are revealed the mechanism of wave behavior, for example, Figures 1–8 depict that different multisolitary wave solutions are attained by different values of z_0 and ω, whereas other parameters are constant. The extended complex approach is calculated by the source of maple software. This approach is a powerful analytical technique since it provides different new exact solutions which are indicated by the forms of rational, periodic, and elliptic function solutions. These results have been obtained by the extended complex technique to show a deeper understanding of physical structures and provide remarkable exact solutions of higher degree NPDEs.

Data Availability

The data used to support the finding of this study are mentioned in the article.

Conflicts of Interest

The authors mentioned here that they have no conflict of interests.

Acknowledgments

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Research Article

On Behavior Laplace Integral Operators with Generalized Bessel Matrix Polynomials and Related Functions

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Recently, the applications and importance of integral transforms (or operators) with special functions and polynomials have received more attention in various fields like fractional analysis, survival analysis, physics, statistics, and engendering. In this article, we aim to introduce a number of Laplace and inverse Laplace integral transforms of functions involving the generalized and reverse generalized Bessel matrix polynomials. In addition, the current outcomes are yielded to more outcomes in the modern theory of integral transforms.

1. Introduction

Recently, the integral transforms (or operators) have been extensively used tools in solving certain boundary value problems and certain integral equations. They are also useful in evaluating infinite integrals involving special functions or in solving differential equations of mathematical physics (see, e.g., [1–6] and the references cited therein). Laplace transform is a type of the integral transforms that is the most popular and widely used in several branches of astronomy, engineering, applied statistics, probability distributions, and applied mathematics (see, for instance, [7–13]).

A number of studies on the generalizations of Laplace transform associated with special polynomials have been contributed by Ortigueira and Machado [14], Jarad and Abdeljawad [15], Ganie and Jain [16], and Saifa et al. [17].

In 1949, Krall and Frink [18] introduced and discussed several properties of the generalized Bessel polynomials (GBPs), which are given by

$$\mathfrak{Y}_n(\alpha, \beta; \xi) = \sum_{s=0}^{n} \binom{n}{s} (n+\alpha-1)_s \left(\frac{\xi}{\beta}\right)^s. \tag{1}$$

These polynomials, which seem to have been considered first by Bochner [19], are also mentioned in Romanovsky [20] and Krall [21].

Recently, these polynomials have been investigated in diverse ways and turned out to be applicable in a number of research fields (see, to exemplify, [22–25]).

Additionally, various extensions of the classical orthogonal polynomials to matrix setting were investigated. The matrix generalization of the generalized Bessel polynomials $\mathscr{B}_n^{\theta,\phi}(z)$, $z \in \mathbb{C}$, for parameters (square) matrices θ and ϕ, was also introduced in diverse ways ([26]; see also [27]). Various studies of the generalized Bessel matrix polynomials have been presented and discussed (see [27, 28]).

Recently, many works established Laplace integral transforms of special functions like Gauss's and Kummer's functions [29], generalized hypergeometric functions [30, 31], Aleph-Functions [32], and Bessel functions [33]. Whereas, some formulas corresponding to integral transforms of orthogonal matrix polynomials are little known and traceless in the literature. This motivates us to discuss Laplace integral transforms for functions involving generalized Bessel matrix polynomials. In particular, we obtain a number of useful Laplace and inverse Laplace type integrals of the generalized Bessel matrix polynomials together with ceratin elementary matrix functions, exponential function, logarithmic function, generalized hypergeometric matrix functions, and Bessel functions and products of generalized Bessel matrix polynomials. We also discuss some interesting and special cases of our main results.

2. Preliminaries

Here, we state some basic definitions and preliminaries which will be used in the article (see, for details, [34–36]).

Here and in the following sections, C and N denote the sets of complex numbers and positive integers, respectively, and $N_0 = N \cup \{0\}$. We denote by $M_r(\mathbb{C})$ the space of $r \times r$ complex matrices endowed with classical norm defined by

$$\|\theta\| = \sup_{y\neq 0}\left\{\frac{\|\theta y\|}{\|y\|}\right\} = \sup\{\|\theta y\|, \|y\| = 1\}. \tag{2}$$

This norm satisfies the inequality $\|\theta\phi\| \le \|\theta\|\|\phi\|$, where θ and ϕ are in $M_r(\mathbb{C})$.

Definition 1. For any matrix θ in $M_r(\mathbb{C})$, the spectrum $\sigma(\theta)$ is the set of all eigenvalues of θ for which we denote

$$\alpha(\theta) = \max\{\Re(\eta)\colon \eta \in \sigma(\theta)\} \text{ and} \beta(\theta) = \min\{\Re(\eta)\colon \eta \in \sigma(\theta)\}, \tag{3}$$

where $\alpha(\theta)$ refers to the spectral abscissa of θ and for which $\beta(\theta) = -\alpha(-\theta)$. A matrix $\theta \in M_r(\mathbb{C})$ is said to be positive stable if and only if $\beta(\theta) > 0$.

Definition 2 (see [35, 36]). If $\theta \in M_r(\mathbb{C})$, and $w \in C$, then the matrix exponential $e^{\theta w}$ is given to be

$$e^{\theta w} = I + \theta w + \cdots + \frac{\theta^n}{n!} w^n + \cdots, \tag{4}$$

where I is the identity matrix in $M_r(\mathbb{C})$.

Definition 3 (see [37]). Let θ be a positive stable matrix in $M_r(\mathbb{C})$ with $\theta + nI$ is invertible for all integers $n \in N_0$, the Gamma matrix function $\Gamma(\theta)$ and the Digamma matrix function $\psi(\theta)$ are defined, respectively, as follows:

$$\Gamma(\theta) = \int_0^\infty e^{-u} u^{\theta - I} du\,; \quad u^{\theta-I} = \exp\left((\theta - I)\ln u\right). \tag{5}$$

$$\psi(\theta) = \Gamma^{-1}(\theta)\Gamma'(\theta), \tag{6}$$

where $\Gamma^{-1}(\theta)$ and $\Gamma'(\theta)$ are reciprocal and derivative of the Gamma matrix function.

Note that the scalar Gamma and Digamma functions are easily found when $r = 1$ in (5) and (6), respectively (see, e.g., [38, Section 1.1])).

Definition 4 (see [?]). For all θ in $M_r(\mathbb{C})$, we assume

$$\theta + kI \quad \text{is invertible for all} \quad k \in \mathbb{N}_0, \tag{7}$$

and the Pochhammer symbol (the shifted factorial) is defined by

$$(\theta)_r = \begin{cases} \theta(\theta + I)\cdots(\theta + (r-1)I) = \Gamma^{-1}(\theta)\Gamma(\theta + rI), & r \in \mathbb{N}, \\ I, & r = 0. \end{cases} \tag{8}$$

Lemma 5 (see [34]). *Let θ be a matrix in $M_r(\mathbb{C})$ such that $\|\theta\| < 1$ and $\|I\| = 1$. Then, $(I+\theta)^{-1}$ exists, and we have*

$$(I + \theta)^{-1} = I - \theta + \theta^2 - \theta^3 + \theta^4 - \theta^5 + \cdots. \tag{9}$$

Definition 6 (see [39]). Let m and n be finite positive integers, the generalized hypergeometric matrix function is given by

$${}_mF_n(\theta\,;\phi\,;z) = \sum_{k=0}^{\infty}\prod_{i=1}^{m}(\theta_i)_k \prod_{j=1}^{n}\left[\left(\varphi_j\right)_k\right]^{-1}\frac{z^k}{k!}, \tag{10}$$

where θ_i, $1 \le i \le m$ and ϕ_j, $1 \le j \le n$ are commutative matrices in $M_r(\mathbb{C})$ with $\phi_j + kI$ are invertible for all integers $k \in N_0$ and $1 \le i \le m$. In [39], Abdalla discussed regions of convergence of (2.6).

Note that for $m = 1$, $n = 0$ in (10), we have the Binomial type matrix function ${}_1F_0(\theta;-;z)$ [39] as follows:

$${}_1F_0(\theta;-;z) = (1-z)^{-\theta} = I + \theta z + \frac{\theta(\theta+I)z^2}{2!} + \cdots + \frac{(\theta)_n z^n}{n!} + \cdots, |z| < 1. \tag{11}$$

Also, for $m = 2$, $n = 1$ in (10), we get the hypergeometric matrix function ${}_2F_1$ (cf. [40]).

Further, the substitution $r = 1$ in (10) leads to the classical generalized hypergeometric functions [38, Section 1.5], see also, [41].

Definition 7 (see [26]). Let θ and ϕ be commuting matrices in $M_r(\mathbb{C})$ such that ϕ is an invertible matrix. For any natural number $n \ge 0$, the n^{th} generalized Bessel matrix polynomial $\mathscr{B}_n^{\theta,\theta}(z)$ is defined as

$$\begin{aligned}\mathscr{B}_n^{\theta,\phi}(z) &= \sum_{r=0}^{n}\binom{n}{r}(\theta+(n-1)I)_r\left(z\phi^{-1}\right)^r\\ &= \sum_{r=0}^{n}\frac{(-1)^r}{r!}(-nI)_r(\theta+(n-1)I)_r\left(z\phi^{-1}\right)^r\\ &= {}_2F_0\left(-nI,\theta+(n-1)I;-;-z\phi^{-1}\right).\end{aligned} \tag{12}$$

In addition, the n^{th} reverse generalized Bessel matrix polynomial $\Theta_n^{(\theta,\phi)}(z)$ is given by (see [27])

$$\begin{aligned}\Theta_n^{(\theta,\phi)}(z) &= z^n\mathscr{B}_n^{\theta,\phi}\left(z^{-1}\right) = (-1)^n\Gamma^{-1}(-\theta-(2n-2)I)\Gamma\\ &\cdot(-\theta+(n-2)I)\times{}_1F_1(-nI;-\theta-(2n-2)I;\phi z).\Theta_n^{(\theta,\phi)}(z)\\ &= z^n\mathscr{B}_n^{\theta,\phi}\left(z^{-1}\right) = (-1)^n\Gamma^{-1}(-\theta-(2n-2)I)\Gamma\\ &\cdot(-\theta+(n-2)I)\times{}_1F_1(-nI;-\theta-(2n-2)I;\phi z).\end{aligned} \tag{13}$$

Obviously, the n^{th} generalized Bessel matrix polynomial $\mathscr{B}_n^{(\theta,\phi)}(z)$ when $r=1$ is easily found to be the scalar generalized Bessel polynomials (1.1).

Definition 8. Let $g(\tau)$ be a function of τ specified for $\tau>0$. Then, the Laplace transform of $g(\tau)$ is defined by

$$\mathscr{G}(\lambda)=\mathscr{L}\{g(\tau):\lambda\}=\int_0^{\infty}e^{-\lambda\tau}g(\tau)d\tau,\quad \Re(\lambda)>0, \tag{14}$$

provided that the improper integral exists, $e^{-\lambda u}$ is the kernel of the transformation and the function $g(\tau)$ is called the inverse Laplace transform of $\mathscr{G}(\lambda)$ (see [1, Chapter 3]; see also [7]).

The following Lemma, which may be easily derivable from (14), will be desired in the sequel.

Lemma 9. *Let θ be a positive stable and invertible matrix in $M_r(\mathbb{C})$ and $\Re(\lambda)>0$. Then, we have*

$$\mathscr{L}\left\{\tau^{\theta}:\lambda\right\}=\int_0^{\infty}e^{-\lambda\tau}\tau^{\theta}d\tau=\lambda^{-(\theta+I)}\Gamma(\theta+I), \tag{15}$$

$$\mathscr{L}\left\{\tau^{\theta}(\tau+1)^{-1}:\lambda\right\}=\Gamma(\theta+I)e^{\lambda}\Gamma(-\theta,\lambda), \tag{16}$$

where $\Gamma(\theta,\lambda)$ is the incomplete Gamma matrix function [42].

$$\begin{aligned}\mathscr{L}\left\{g(\tau)e^{\theta\tau}:\lambda\right\}&=\mathscr{G}(\lambda I-\theta),\\ \mathscr{L}^{-1}\left\{\lambda^{-\theta}:\tau\right\}&=\tau^{(\theta-I)}\Gamma^{-1}(\theta).\end{aligned} \tag{17}$$

3. Laplace Type Integrals of Functions Involving $\mathscr{B}_n^{\theta,\phi}(z)$ and $\Theta_n^{\theta,\phi}(z)$

In this section, we investigate several Laplace-type transforms of functions involving generalized and reverse generalized Bessel matrix polynomials asserted in the following theorems:

Theorem 10. *Let $z,\lambda\in\mathbb{C}$, $\Re(\lambda)>0$, $n\in\mathbb{N}_0$, and $r\in\mathbb{N}$. Also, let θ,ϕ and A be matrices in $M_r(\mathbb{C})$ such that $\beta(A)>0$ and $\phi+kI$ are invertible for all $k\in\mathbb{N}_0$. For the function*

$$g_1(z)=z^{A-I}\mathscr{B}_n^{\theta,\phi}(z), \tag{18}$$

we have

$$\mathscr{G}_1(\lambda)=\mathscr{L}\{g_1(z):\lambda\}=\lambda^{-A}\Gamma(A)_3F_0\left[\begin{matrix}-nI,\theta+(n-1)I,A\\ \\ -\end{matrix};-(\lambda\phi)^{-1}\right]. \tag{19}$$

Proof. From the expansion series of the $\mathscr{B}_n^{\theta,\phi}(z)$ in (12) and upon using (15) in Lemma 9, we obtain

$$\begin{aligned}\mathscr{G}_1(\lambda)&=\sum_{s=0}^{n}\frac{(-nI)_s(\theta+(n-1)I)_s\left(-\phi^{-1}\right)^s}{s!}\mathscr{L}\left\{z^{A+(s-1)I}\right\}\\ &=\sum_{s=0}^{n}\frac{(-nI)_s(\theta+(n-1)I)_s\left(-\phi^{-1}\right)^s}{s!}\lambda^{-(A+sI)}\Gamma(A+sI)\\ &=\lambda^{-A}\Gamma(A)\sum_{s=0}^{n}\frac{(-nI)_s(\theta+(n-1)I)_s(A)_s\left(-(\lambda\phi)^{-1}\right)^s}{s!}.\end{aligned} \tag{20}$$

Thus, we get the required result (19).

Theorem 11. *Let $z,\lambda\in\mathbb{C}$, $\Re(\lambda)>0$, $n\in\mathbb{N}_0$, and $r\in\mathbb{N}$. Also, let θ,ϕ and A be matrices in $M_r(\mathbb{C})$ such that $\beta(A)>0$,$\phi+kI$ are invertible for all $k\in\mathbb{N}_0$ and $I-A$ satisfies the spectral condition (7). Further, let*

$$g_2(z)=z^{A-(n+1)I}\Theta_n(\theta,\phi;z). \tag{21}$$

Then,

$$\mathscr{G}_2(\lambda)=\mathscr{L}\{g_2(z):\lambda\}=\lambda^{-A}\Gamma(A)_2F_1\left[\begin{matrix}-nI,\theta+(n-1)I\\ \\ I-A\end{matrix};\lambda\theta^{-1}\right]. \tag{22}$$

Proof. Starting from Definition 7, and applying the relation (15), it follows that

$$\begin{aligned}\mathcal{G}_2(\lambda) &= \sum_{s=0}^{n} \frac{(-nI)_s(\theta+(n-1)I)_s\left(-\phi^{-1}\right)^s}{s!} \mathcal{L}\left\{z^{A-(s+1)I}\right\} \\ &= \sum_{s=0}^{n} \frac{(-nI)_s(\theta+(n-1)I)_s\left(-\phi^{-1}\right)^s}{s!} \lambda^{-(A-sI)}\,\Gamma(A-sI) \\ &= \lambda^{-A}\,\Gamma(A) \sum_{s=0}^{n} \frac{(-nI)_s(\theta+(n-1)I)_s\left[(I-A)_s\right]^{-1}\left(\lambda\phi^{-1}\right)^s}{s!}.\end{aligned} \tag{23}$$

Thus, the result (22) is established.

Theorem 12. *Let $z,\mu,\lambda \in \mathbb{C}$, $\mathfrak{R}(\lambda-\mu)>0$, $n\in\mathbb{N}_0$, and $r\in\mathbb{N}$. Also, let θ,ϕ and A be matrices in $M_r(\mathbb{C})$ such that $\beta(A)>0$, $\phi+kI$ are invertible for all $k\in\mathbb{N}_0$ and $I-A$ satisfies the spectral condition (7). If*

$$g_3(z) = z^{A-I}\, e^{\mu z}\, \mathcal{B}_n^{\theta,\phi}\left(z^{-1}\right). \tag{24}$$

Then,

$$\mathcal{G}_3(\lambda) = \mathcal{L}\{g_3(z)\colon \lambda\} = (\lambda-\mu)^{-A}\,\Gamma(A)\,{}_2F_1\left[\begin{matrix}-nI, \theta+(n-1)I \\ \\ I-A\end{matrix};(\lambda-\mu)\phi^{-1}\right]. \tag{25}$$

Proof. For convenience, let the left-hand side of (25) be denoted by S and by invoking the series expression of (12) to S, we obtain

$$\begin{aligned}S &= \sum_{k=0}^{n} \frac{(-nI)_s\,(\theta+(n-1)I)_k\left(-\phi^{-1}\right)^s}{s!} \int_0^\infty z^{A-(s-1)I}\, e^{-(-\mu+\lambda)z}\, dz \\ &= \sum_{s=0}^{n} \frac{(-nI)_s\,(\theta+(n-1)I)_s\left(-\phi^{-1}\right)^s}{s!} (-\mu+\lambda)^{-(A-sI)}\,\Gamma(A-sI) \\ &= \Gamma(A)\,(\lambda-\mu)^{-A} \sum_{s=0}^{n} (-nI)_s\,(\theta+(n-1)I)_s \\ &\quad \cdot \left[(I-A)_s\right]^{-1} \frac{\left((\lambda-\mu)\,\phi^{-1}\right)^s}{s!},\end{aligned} \tag{26}$$

therefore, (25) as desired.

Theorem 13. *Let $z,w,\lambda\in\mathbb{C}$, $\mathfrak{R}(\lambda)>0$, $n\in\mathbb{N}_0$, and $r\in\mathbb{N}$. Also, let θ,ϕ and A be matrices in $M_r(\mathbb{C})$ such that $\beta(A)>0$ and $\phi+kI$ are invertible for all $k\in\mathbb{N}_0$. For the function*

$$g_4(z) = z^{A-I}\,(z+w)^{-1}\,\mathcal{B}_n^{\theta,\phi}(z), \tag{27}$$

we have

$$\begin{aligned}\mathcal{G}_4(\lambda) &= \mathcal{L}\{g_4(z)\colon \lambda\} = w^{A-I}\,\Gamma(A)\,e^{\lambda w} \\ &\quad \times \sum_{k=0}^{n} \frac{(-nI)_k\,(\theta+(n-1)I)_k\,(A)_k}{k!}\,\Gamma \\ &\quad \cdot (I-A-kI\,;\lambda w)\left(-w\,\phi^{-1}\right)^k,\end{aligned} \tag{28}$$

where $\Gamma(A,z)$ is the incomplete Gamma matrix function defined in [42].

Proof. To prove (28), we consider

$$\begin{aligned}\mathcal{G}_4(\lambda) &= \int_0^\infty z^{A-I}\,(z+w)^{-1}\,\mathcal{B}_n^{\theta,\phi}(z)e^{-\lambda z}dz \\ &= \sum_{k=0}^{n} \frac{(-nI)_k(\theta+(n-1)I)_k}{k!}\left(-\phi^{-1}\right)^k \\ &\quad \times \int_0^\infty z^{A+(k-1)I}\,(z+w)^{-1}\,e^{-\lambda z}dz.\end{aligned} \tag{29}$$

According to (16) in Lemma 9, we get

$$\begin{aligned}\mathcal{G}_4(\lambda) &= \sum_{k=0}^{n} \frac{(-nI)_k(\theta+(n-1)I)_k}{k!}\,\Gamma(A+kI) \\ &\quad \times w^{A+(k-1)I}\,e^{\lambda w}\Gamma((1-k)I-A, w\lambda)\left(-\phi^{-1}\right)^k \\ &= \Gamma(A)\,w^{A-I}e^{w\lambda} \sum_{k=0}^{n} \frac{(-nI)_k(\theta+(n-1)I)_k\,(A)_k}{k!} \\ &\quad \times \Gamma((1-k)I-A, w\lambda)\left(-w\phi^{-1}\right)^k.\end{aligned} \tag{30}$$

This completes the proof of Theorem 13.

Theorem 14. *Let $z,\lambda,\nu\in\mathbb{C}$, $\mathfrak{R}(\lambda)>0$, $\mathfrak{R}(\nu)>0$,$n,m\in\mathbb{N}_0$, and $r\in\mathbb{N}$. Also let θ,ϕ and A be matrices in $M_r(\mathbb{C})$ such that $\beta(A)>0$,$\phi+kI$ are invertible for all $k\in\mathbb{N}_0$,$(1+n)I-A$ and $(2-n)I-A-\theta$ satisfies the spectral condition (7). Further, let*

$$g_5(z) = z^{A-I}\,\mathcal{B}_n^{\theta,\lambda I}\left(z^{-1}\right)\mathcal{B}_m^{\nu I,\phi}\left(z^{-1}\right). \tag{31}$$

Then,

$$\begin{aligned}\mathcal{G}_5(\lambda) &= \{g_5(z)\colon \lambda\} = \lambda^{-A}\,\Gamma(A)\Gamma(I-A)\Gamma(2I-A-\theta)\times\Gamma^{-1} \\ &\quad \cdot ((1+n)I-A)\Gamma^{-1}((2-n)I-A-\theta)\times{}_3F_2 \\ &\quad \cdot \left[\begin{matrix}-mI, (\nu+n-1)I, 2I-A-\theta \\ \\ (1+n)I-A, (2-n)I-A-\theta\end{matrix};\lambda\,\phi^{-1}\right].\end{aligned} \tag{32}$$

Proof. To prove (32), we require the relation (15) and Definition 7, thus we arrive at

$$\begin{aligned}
\mathscr{G}_5(\lambda) &= \sum_{s=0}^{n}\sum_{j=0}^{m} \frac{(-nI)_s(\theta+(n-1)I)_s\left(-\lambda^{-1}\right)^s}{s!} \\
&\quad\times \frac{(-mI)_j(\nu I+(m-1)I)_j\left(-\phi^{-1}\right)^j}{j!}\,\mathscr{L}\left\{z^{A-(s+j+1)I}\right\} \\
&= \sum_{s=0}^{n}\sum_{j=0}^{m} \frac{(-nI)_s(\phi+(n-1)I)_s\left(-\lambda^{-1}\right)^s}{s!} \\
&\quad\times \frac{(-mI)_j(\nu I+(m-1)I)_j\left(-\phi^{-1}\right)^r}{r!}\,\Gamma(A-(s+j)I)\,\lambda^{-(A-(s+j)I)} \\
&= \lambda^{-(A)}\Gamma(A)\sum_{j=0}^{m}\frac{(-mI)_j(\nu I+(m-1)I)_j\left(-\phi^{-1}\right)^j}{j!}\left[(I-A)_j\right]^{-1} \\
&\quad\times \sum_{s=0}^{n}\frac{(-nI)_s(\phi+(n-1)I)_s}{s!}\left[((1-j)I-A)_s\right]^{-1} \\
&= \lambda^{-A}\,\Gamma(A)\Gamma(I-A)\Gamma(2I-A-\theta)\,\Gamma^{-1}(I-\theta+nI)\,\Gamma^{-1} \\
&\quad\cdot(2I-A-\theta-nI)\times\sum_{j=0}^{m}\frac{(-mI)_j(\nu I+(m-1)I)_j\left(-\lambda\phi^{-1}\right)^j}{j!} \\
&\quad\cdot(2I-A-\theta)_j\times\left[((1+n)I-A)_r\right]^{-1}\left[((2-n)I-A-\theta)_j\right]^{-1}.
\end{aligned} \tag{33}$$

This completes the proof of Theorem 14.

Theorem 15. *Let $z,\lambda\in\mathbb{C}$, $\Re(\lambda)>0$, $n,m\in\mathbb{N}_0$, and $r\in\mathbb{N}$. Also, let θ,ϑ,ϕ and A be matrices in $M_r(\mathbb{C})$ such that $\beta(A)>0$, $\phi+kI$, are invertible for all $k\in\mathbb{N}_0$, ϑ, $(\theta+A)$ and $\theta+A-I$ satisfies the spectral condition (7). Further, let*

$$g_6(z)=z^{A-I}\,\mathscr{B}_n^{\theta,\lambda zI}(1)\,\mathscr{B}_m^{\vartheta,\phi}(z)\,\mathscr{B}_m^{\vartheta,\phi}(-z). \tag{34}$$

Then,

$$\begin{aligned}
\mathscr{G}_6(\lambda)&=\mathscr{L}\{g_6(z):\lambda\}=\frac{2^{A-I}}{\sqrt{\pi}}(\theta+A-I)_n\,\Gamma(A)\,\lambda^{-A}\left[(I-A)_n\right]^{-1} \\
&\quad\times{}_8F_3\left[\begin{matrix}-mI,\vartheta+(m-1)I,\frac{1}{2}(\theta-I),\frac{1}{2}\vartheta,\frac{1}{2}(A+(1-n)I),\\ \frac{1}{2}(A-nI),\frac{1}{2}(\theta+A+nI),\frac{1}{2}(\theta+A+(n-1)I)\\ \vartheta I,\frac{1}{2}(\theta+A),\frac{1}{2}(\theta+A-I)\end{matrix};16(\lambda\phi)^{-2}\right].
\end{aligned} \tag{35}$$

Proof. Applying the following formula (see [39])

$$\mathscr{B}_m^{\vartheta,\phi}(z)\,\mathscr{B}_m^{\vartheta,\phi}(-z)={}_4F_1\left[\begin{matrix}-mI,\vartheta+(m-1)I,\frac{1}{2}(\vartheta-I),\frac{1}{2}\vartheta\\ \vartheta-I\end{matrix};4z^2\phi^{-2}\right]. \tag{36}$$

We thus find that

$$\begin{aligned}
\mathscr{G}_6(\lambda)&=\mathscr{L} \\
&\quad\cdot\left\{z^{A-I}\,\mathscr{B}_n^{\theta,\lambda zI}(1)\,{}_4F_1\left[\begin{matrix}-mI,\vartheta+(m-1)I,\frac{1}{2}(\vartheta-I),\frac{1}{2}\vartheta\\ \vartheta-I\end{matrix};4z^2\phi^{-2}\right]\right\} \\
&=\sum_{s=0}^{n}\frac{(-nI)_s(\theta+(n-1)I)_s}{s!}\left(-\lambda^{-1}\right)^s\times\sum_{j=0}^{m}\frac{(-mI)_j(\vartheta+(m-1)I)_j}{j!} \\
&\quad\cdot\left(\frac{1}{2}(\vartheta-I)\right)_j\left(\frac{1}{2}\vartheta\right)_j\left[(\vartheta-I)_j\right]^{-1}\left(4\phi^{-2}\right)^j\times\mathscr{L}\left\{z^{A-(s+1+2j)I}\right\}.
\end{aligned} \tag{37}$$

Making use of (15), we observe that

$$\begin{aligned}
\mathscr{G}_6(\lambda)&=\sum_{s=0}^{n}\frac{(-nI)_s(\theta+(n-1)I)_s}{s!}\left(-\lambda^{-1}\right)^s \\
&\quad\times\sum_{j=0}^{m}\frac{(-mI)_j(\vartheta+(m-1)I)_j}{j!}\left(\frac{1}{2}(\vartheta-I)\right)_j\left(\frac{1}{2}\vartheta\right)_j \\
&\quad\cdot\left[(\vartheta-I)_j\right]^{-1}\times\left(4\phi^{-2}\right)^j\lambda^{-(A+(s-2j)I)}\,\Gamma(A-(s-2j)I) \\
&=\lambda^{-A}\Gamma(A)\sum_{j=0}^{m}\frac{(-mI)_j(\vartheta+(m-1)I)_j}{j!}\left(\frac{1}{2}(\vartheta-I)\right)_j \\
&\quad\cdot\left(\frac{1}{2}\vartheta\right)_j\times\left[(\vartheta-I)_j\right]^{-1}(A)_{2j}\left(4(\lambda\phi)^{-2}\right)^j \\
&=\sum_{s=0}^{n}\frac{(-nI)_s(\theta+(n-1)I)_s}{s!}\left[(I-A-2jI)_s\right]^{-1} \\
&=\lambda^{-A}\frac{2^{A-I}}{\sqrt{\pi}}\,\Gamma(A)\Gamma(I-A)\Gamma(2I-A-\theta)\,\Gamma^{-1} \\
&\quad\cdot(I-A+nI)\Gamma^{-1}(2I-A-\theta-nI) \\
&\quad\times\sum_{j=0}^{m}\frac{(-mI)_j(\vartheta+(m-1)I)_j}{j!}\left(\frac{1}{2}(\vartheta-I)\right)_r\left(\frac{1}{2}\vartheta\right)_j \\
&\quad\cdot\left(\frac{1}{2}A\right)_j\left[(\vartheta-I)_j\right]^{-1}\times\left(\frac{1}{2}(A+I)\right)_j \\
&\quad\cdot\left(\frac{1}{2}(A+(1-n)I)\right)_j\left(\frac{1}{2}(A-nI)\right)_j \\
&\quad\times\left(\frac{1}{2}(A+\theta+nI)\right)_j\left(\frac{1}{2}(A+\theta+(n-1)I)\right)_j \\
&\quad\cdot\left[\left(\frac{1}{2}(A+I)\right)_j\right]^{-1}\times\left[\left(\frac{1}{2}A\right)_j\right]^{-1}\left[\left(\frac{1}{2}(A+\theta)\right)_j\right]^{-1} \\
&\quad\cdot\left[\left(\frac{1}{2}(A+\theta-I)\right)_j\right]^{-1}.\left(16(\lambda\phi)^{-2}\right)^j.
\end{aligned} \tag{38}$$

Thus, after a simplification, we get the required result (35).

Theorem 16. *Let $z,\lambda\in\mathbb{C}$, $\Re(\lambda)>0$, $n\in\mathbb{N}_0$, and $r\in\mathbb{N}$. Also, let θ,ϕ and A be matrices in $M_r(\mathbb{C})$ such that $\beta(A)>0$ and $\phi+kI$ are invertible for all $k\in\mathbb{N}_0$. For the function*

$$g_7(z) = z^{A-I} \log z\, \mathscr{B}_n^{\theta,\phi}(z), \tag{39}$$

then, we have

$$\mathscr{G}_7(\lambda) = \mathscr{L}\{g_7(z): \lambda\} = \lambda^{-A}\, \Gamma(A) \sum_{s=0}^{n} (-nI)_s\, (\theta + (n-1)I)_s\, (A)_s \times \frac{\left(-(\lambda\phi)^{-1}\right)^s}{s!}\, (\psi(A+sI) - \log \lambda), \tag{40}$$

where $\psi(A)$ is the Digamma matrix function defined in (6).

Proof. The proof of this Theorem is quite straight forward as

$$\begin{aligned}\mathscr{G}_7(\lambda) &= \int_0^\infty z^{A-I} \log z\, \mathscr{B}_n^{\theta,\phi}(z) e^{-\lambda z} dz \\ &= \sum_{s=0}^{n} \frac{(-nI)_s(\theta + (n-1)I)_k}{s!} \left(-\phi^{-1}\right)^s \\ &\quad \times \int_0^\infty z^{A+(s-1)I} \log z\, e^{-\lambda z} dz.\end{aligned} \tag{41}$$

Upon using (2,2), we have

$$\Gamma(A+sI) = \int_0^\infty z^{A+(s-1)I} e^{-z}\, dz. \tag{42}$$

Hence,

$$\Gamma'(A+sI) = \int_0^\infty z^{A+(s-1)I} e^{-z} \log z dz. \tag{43}$$

We thus arrive at

$$\begin{aligned}\Psi(A+sI) &= \Gamma'(A+sI)\Gamma^{-1}(A+sI) \\ &= \Gamma^{-1}(A+sI) \int_0^\infty z^{A+(s-1)I} e^{-z} \log z dz.\end{aligned} \tag{44}$$

Therefore, we get

$$\begin{aligned}\Psi(A+sI) &= \lambda^{A+sI}\, \Gamma^{-1}(A+sI) \int_0^\infty z^{A+(s-1)I} e^{-\lambda z} \log(\lambda z) dz \\ &= \lambda^{A+sI}\, \Gamma^{-1}(A+sI) \int_0^\infty z^{A+(s-1)I} e^{-\lambda z} [\log(\lambda) + \log(z)] dz \\ &= \lambda^{A+sI}\, \Gamma^{-1}(A+sI) \int_0^\infty z^{A+(s-1)I} e^{-\lambda z} \log(\lambda) dz \\ &\quad + \lambda^{A+sI}\, \Gamma^{-1}(A+sI) \int_0^\infty z^{A+(s-1)I} e^{-\lambda z} \log(z) dz \\ &= \log(\lambda) + \lambda^{A+sI}\, \Gamma^{-1}(A+sI) \int_0^\infty z^{A+(s-1)I} e^{-\lambda z} \log(z) dz.\end{aligned} \tag{45}$$

We thus have

$$\int_0^\infty z^{A+(s-1)I} e^{-\lambda z} \log(z) dz = \lambda^{-(A+sI)}\, \Gamma(A+sI)\, [\Psi(A+sI) - \log \lambda]. \tag{46}$$

From the above equations, we get the required result as follows:

$$\begin{aligned}\mathscr{G}_7(\lambda) &= \sum_{s=0}^{n} \frac{(-nI)_s(\theta + (n-1)I)_s\, (A)_s}{s!} \\ &\quad \cdot \left(-(\lambda\phi)^{-1}\right)^s \times \lambda^{-A}\, \Gamma(A)\, [\Psi(A+sI) - \log \lambda] \\ &= \lambda^{-A}\, \Gamma(A) \sum_{s=0}^{n} \frac{(-nI)_s(\theta + (n-1)I)_s\, (A)_s}{s!} \\ &\quad \times \left(-(\lambda\phi)^{-1}\right)^s [\Psi(A+sI) - \log \lambda].\end{aligned} \tag{47}$$

Theorem 17. *Let $z, \lambda \in \mathbb{C}$, $\Re(\lambda) > 0$, $n, m, q \in \mathbb{N}_0$, and $r \in \mathbb{N}$. Also, let θ, ϕ, E, D and A be matrices in $M_r(\mathbb{C})$ such that $\beta(A) > o$, and $\phi + kI$ are invertible for all $k \in \mathbb{N}_0$. Further, let*

$$g_8(z) = z^{2A-I}\, {}_mF_q\left(E; D; z^2\right) \mathscr{B}_n^{\theta,\phi}\left(z^2\right). \tag{48$$

Then,

$$\begin{aligned}\mathscr{G}_8(\lambda) = \mathscr{L}\{g_8 : \lambda\} &= \frac{2^{2A-I}}{\sqrt{\pi}}\, \Gamma(A)\, \Gamma\left(A + \frac{1}{2}\right) \lambda^{-2A} \\ &\quad \times \sum_{k=0}^{n} \frac{1}{k!} (-nI)_k(\theta + (n-1)I)_k\, (A)_k \left(A + \frac{1}{2}\right)_k \left(-4\left(\lambda^2\phi\right)^{-1}\right)^k \\ &\quad \times {}_{m+2}F_q\left(E, A + kI, A + \left(k + \frac{1}{2}\right) I; D; 4(\lambda)^{-2}\right),\end{aligned} \tag{49}$$

where ${}_mF_q(E; D; z)$ is the generalized hypergeometric type matrix functions defined in (10) such that Re $(\lambda) > 0$ *if $m < q - 1$ and* Re $(\lambda) > |\beta(A)|$ *if $m = q - 1$.*

Proof. Using Definitions (10) and (12) and upon using (15), we obtain

$$\begin{aligned}\mathscr{G}_8(\lambda) &= \sum_{k=0}^{n} \frac{1}{k!} (-nI)_k(\theta + (n-1)I)_k \left(4(\phi)^{-1}\right)^k \\ &\quad \times \sum_{r=0}^{\infty} \prod_{i=1}^{m} (E_i)_r \prod_{j=1}^{q} \left[(D_j)_r\right]^{-1} \frac{1}{k!} \mathscr{L}\left\{z^{2A-(1-2k-2r)I}\right\} \\ &= \sum_{k=0}^{n} \frac{1}{k!} (-nI)_k(\theta + (n-1)I)_k \left(4(\phi)^{-1}\right)^k \\ &\quad \times \sum_{r=0}^{\infty} \prod_{i=1}^{m} (E_i)_r \prod_{j=1}^{q} \left[(D_j)_r\right]^{-1} \frac{1}{k!} \times \lambda^{-2A-(2k+2r)I}\, \Gamma \\ &\quad \cdot (2A + (2k+2r)I).\end{aligned} \tag{50}$$

Thus, after a simplification, we obtain the result (49) in Theorem 3.11.

Theorem 18. *Let $z, \upsilon, \sigma, \lambda \in \mathbb{C}$, $\Re(\lambda) > 0, \Re(\upsilon) > -1, \Re(\sigma) > 0$, $n, m \in \mathbb{N}_0$, and $r \in \mathbb{N}$. Also, let θ be matrix in $M_r(\mathbb{C})$ such that $\beta(A) > 0$ and $\phi + kI$ are invertible for all $k \in \mathbb{N}_0$. For the function*

$$g_9(z) = z^{\upsilon/2} J_\upsilon\left(2(\sigma z)^{1/2}\right) \mathscr{B}_n^{\theta,\lambda z}(1). \tag{51}$$

Then, we have

$$\mathscr{G}_9(\lambda) = \mathscr{L}\{g_9(z)\colon \lambda\} = \sigma^{\upsilon/2} (\theta + \upsilon I)_n \left(\frac{1}{(-\upsilon)_n}\right) \lambda^{-(\upsilon+1)} \times {}_2F_2\left[\begin{matrix} 1+\upsilon-m, \theta+(n+\upsilon)I \\ \\ 1+\upsilon, \theta+\upsilon I \end{matrix} ; -\frac{\sigma}{\lambda}\right], \tag{52}$$

where $J_\upsilon(z)$ is the Bessel function of the first kind of order υ defined by (see, e.g., [38, 41, 43])

$$J_\upsilon(z) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma(1+\upsilon+s)} \left(\frac{z}{2}\right)^{\upsilon+2s}. \tag{53}$$

Proof. According to (12) and (53) and upon sing (15), it follows that

$$\begin{aligned}
\mathscr{G}_9(\lambda) &= \mathscr{L}\left\{z^{\upsilon/2} J_\upsilon\left(2(\sigma z)^{1/2}\right) \mathscr{B}_n^{\theta,\lambda z}(1)\right\} = \sum_{m=0}^{\infty} \frac{(-1)^m (\sigma)^{m+(\upsilon/2)}}{m!\,\Gamma(1+\upsilon+m)} \\
&\quad \times \sum_{k=0}^{n} \frac{(-nI)_k(\theta+(n-1)I)_k}{k!} \left(-\lambda^{-1}\right)^k \mathscr{L}\left\{z^{\frac{\upsilon}{2}+\frac{\upsilon}{2}-k+m}\right\} \\
&= (\sigma)^{\upsilon/2} \sum_{m=0}^{\infty} \frac{(-\sigma)^m}{m!\,\Gamma(1+\upsilon+m)} \times \sum_{k=0}^{n} \frac{(-nI)_k(\theta+(n-1)I)_k}{k!} \\
&\quad \cdot \left(-\lambda^{-1}\right)^k \Gamma(1+\upsilon+m-k).\lambda^{\upsilon-m+k-1} \\
&= (\sigma)^{\upsilon/2} \lambda^{\upsilon-1} \sum_{m=0}^{\infty} \frac{(-\sigma)^m \lambda^{-m}\Gamma(1+\upsilon+m)}{m!\,\Gamma(1+\upsilon+m)} \\
&\quad \times \sum_{k=0}^{n} \frac{(-nI)_k(\theta+(n-1)I)_k}{(-(\upsilon+m))_k\,k!} = (\sigma)^{\upsilon/2} \lambda^{-\upsilon-1} \frac{(\theta+\upsilon I)_n}{(-\upsilon)_n} \\
&\quad \cdot \sum_{m=0}^{\infty} \frac{(1+\upsilon-n)_m((\upsilon+n)I+\theta)_m\left[(\theta+\upsilon I)_m\right]^{-1}}{m!\,(1+\upsilon)_m} \left(\frac{-\sigma}{\lambda}\right)^m.
\end{aligned} \tag{54}$$

This completes the proof of Theorem 18.

4. Inverse Laplace Type Integrals of Functions Involving $\mathscr{B}_n^{P,Q}(z)$

Here, we obtain the following inverse Laplace type transforms of generalized Bessel matrix polynomials with products of some functions in the following theorem:

Theorem 19. *Let $z, \lambda, \sigma \in \mathbb{C}$, $\Re(\lambda) > 1/2\,|\,\Re(\sigma)\,|$, $n \in \mathbb{N}_0$, and $r \in \mathbb{N}$. Also, let A be matrix in $M_r(\mathbb{C})$ such that $\beta(A) > 0$. If*

$$\mathscr{G}_{10}(\lambda) = \Gamma(A) \left(\lambda + \frac{1}{2}\sigma\right)^{-A} \mathscr{B}_n^{A-(n+1)I,\frac{1}{\lambda+1/2\sigma}}(-\sigma). \tag{55}$$

Then,

$$g_{10}(z) = z^{A-I} \exp\left(\frac{-1}{2}\sigma z\right)(1-\sigma z)^n. \tag{56}$$

Proof. It is sufficient to find Laplace transform of $g_{10}(z)$

$$\begin{aligned}
\mathscr{G}_{10}(\lambda) &= \mathscr{L}\left\{z^{A-I} \exp\left(\frac{-1}{2}\sigma z\right)(1-\sigma z)^n\right\} \\
&= \mathscr{L}\left\{z^{A-I} \exp\left(\frac{-1}{2}\sigma z\right) {}_1F_0\left(\begin{matrix}-n \\ -\end{matrix}; \sigma z\right)\right\} \\
&= \sum_{k=0}^{n} \frac{(-nI)_k \sigma^k}{k!} \mathscr{L}\left\{z^{A-(1-k)I} \exp\left(\frac{-1}{2}\sigma z\right)\right\} \\
&= \sum_{k=0}^{n} \frac{(-nI)_k \sigma^k}{k!} \Gamma(A+kI) \left(\lambda + \frac{1}{2}\sigma\right)^{-(A+kI)} \\
&= \Gamma(A) \left(\lambda + \frac{1}{2}\sigma\right)^{-A} \sum_{k=0}^{n} \frac{(-nI)_k (A)_k}{k!} \left(\frac{\sigma}{(\lambda+1/2\sigma)}\right)^k,
\end{aligned} \tag{57}$$

This finalizes the proof of Theorem 19.

Theorem 20. *Let $z, \lambda, \sigma \in \mathbb{C}$, $\Re(\lambda) > 0, \Re(\sigma) > 0$, $n \in \mathbb{N}_0$, and $r \in \mathbb{N}$. Also, let A be matrix in $M_r(\mathbb{C})$ such that $\beta(A+nI) > 0$. Further, let*

$$\mathscr{G}_{11}(\lambda) = (-1)^n \sigma^{\frac{1}{2}A+nI} \lambda^{-(A+(2n+1)I)} \exp\left(\frac{-\sigma}{\lambda}z\right) \mathscr{B}_n^{I-A-2nI,\sigma}(\lambda). \tag{58}$$

Then,

$$g_{11}(z) = z^{\frac{A}{2}+nI} J_\upsilon\left(2(\sigma z)^{1/2}\right). \tag{59}$$

Proof. By invoking to (15) and (53), we consider

$$
\begin{aligned}
\mathscr{G}_{11}(\lambda) &= \mathscr{L}\left\{z^{\frac{A}{2}+nI} J_v\left(2(\sigma z)^{1/2}\right)\right\} \\
&= \sum_{r=0}^{\infty} \frac{\Gamma^{-1}(A+(1+r)I)\,(-\sigma)^r\,\sigma^{A/2}}{r!}\,\mathscr{L}\left\{z^{A+(n+r)I}\right\} \\
&= \sigma^{A/2}\,\Gamma^{-1}(A+I)\sum_{r=0}^{\infty}\frac{(-)^r\left[(A+I)_r\right]^{-1}}{r!}\,\Gamma \\
&\quad\cdot (A+(r+n+1)I)\,\lambda^{-(A+(r+n+1)I)} \\
&= \sigma^{A/2}\,(A+I)_n\,\lambda^{-(A+(n+1)I)}\,\exp\left(\frac{-\sigma}{\lambda}\right) \\
&\quad\cdot \sum_{r=0}^{n}\frac{(-nI)_r\left[(A+I)_r\right]^{-1}}{r!}\left(\frac{\sigma}{\lambda}\right)^r \\
&= \sigma^{\frac{A}{2}+nI}\,(A+I)_n\,\lambda^{-(A+(2n+1)I)}\,\exp\left(\frac{-\sigma}{\lambda}\right) \\
&\quad\cdot \sum_{r=0}^{n}\frac{(-nI)_r\left[(A+I)_r\right]^{-1}}{r!}\left(\frac{\sigma}{\lambda}\right)^{r-n}.
\end{aligned}
\tag{60}
$$

Putting $n-r=k$, we obtain

$$
\begin{aligned}
\mathscr{G}_{11}(\lambda) &= (-1)^n\,\sigma^{\frac{A}{2}+nI}\,\lambda^{-(A+(2n+1)I)}\,\exp\left(\frac{-\sigma}{\lambda}\right) \\
&\quad\times \sum_{k=0}^{n}\frac{(-nI)_k\,(-(A+nI))_k}{k!}\left(\frac{-\lambda}{\sigma}\right)^k \\
&= (-1)^n\,\sigma^{\frac{1}{2}A+nI}\,\lambda^{-(A+(2n+1)I)}\,\exp\left(\frac{-\sigma}{\lambda}z\right)\mathscr{B}_n^{I-A-2nI,\sigma}(\lambda).
\end{aligned}
\tag{61}
$$

This finalizes the proof of Theorem 20.

The remaining results, which are given in the following theorems, can also be proven in a similar way. So we prefer to omit the details.

Theorem 21. *Let $z, \lambda \in \mathbb{C}$, $\mathfrak{R}(\lambda) > 0$, $n \in \mathbb{N}_0$, and $r \in \mathbb{N}$. Also, let θ and ϕ be matrices in $M_r(\mathbb{C})$ such that $\phi + kI$ are invertible for all $k \in \mathbb{N}_0$. Further, let*

$$
\mathscr{G}_{12}(\lambda) = (-\phi)^n\,\lambda^{\theta+(2n-2)I}\,\Gamma(2I-\theta)\quad\mathscr{B}_n^{2I-\theta-2nI,\frac{(\phi-\lambda I)}{\lambda}}(-n).
\tag{62}
$$

Then,

$$
g_{12}(z) = z^{-(\theta+(n-1)I)}\,\mathscr{B}_n^{\theta,\phi}\left(z^{-1}\right).
\tag{63}
$$

Theorem 22. *Let $z, \lambda \in \mathbb{C}$, $\mathfrak{R}(\lambda) > 0$, $n \in \mathbb{N}_0$, and $r \in \mathbb{N}$. Also let θ and ϕ be matrices in $M_r(\mathbb{C})$ such that $\phi + kI$ are invertible for all $k \in \mathbb{N}_0$. Further, let*

$$
\mathscr{G}_{13}(\lambda) = \frac{1}{\lambda_2}F_0\left[\begin{matrix}-n, \theta-(n+1)I \\ -\end{matrix}\;;\lambda\phi^{-1}\right].
\tag{64}
$$

Then,

$$
g_{13}(z) = \mathscr{B}_n^{\theta,\phi}\left(z^{-1}\right).
\tag{65}
$$

Theorem 23. *Let $z, \lambda, \mu \in \mathbb{C}$, $\mathfrak{R}(\lambda) > \mathfrak{R}(\mu) > 0$, $n \in \mathbb{N}_0$, and $r \in \mathbb{N}$. Also let θ and ϕ be matrices in $M_r(\mathbb{C})$ such that $\phi + k$ I are invertible for all $k \in \mathbb{N}_0$. Further, let*

$$
\mathscr{G}_{14}(\lambda) = (\lambda-\mu)^{-1}{}_2F_0\left[\begin{matrix}-n, \theta-(1-n)I \\ -\end{matrix}\;;(\lambda-\mu)\phi^{-1}\right].
\tag{66}
$$

Then,

$$
g_{14}(z) = \exp(\mu z)\mathscr{B}_n^{\theta,\phi}\left(z^{-1}\right).
\tag{67}
$$

5. Conclusion

In fact, this work is a continuation of the recent paper by Abdalla [44]. In the current manuscript, the authors introduced various Laplace integral formulas of generalized Bessel matrix polynomials with certain elementary matrix functions, Binomial matrix functions exponential function, logarithmic function, generalized hypergeometric matrix functions, and Bessel function of the first kind. We also presented inverse Laplace transforms of generalized Bessel matrix polynomials with some functions. It is obvious that the results presented here which are involved in certain matrices in $M_r(\mathbb{C})$ may reduce to yield the corresponding scalar ones when $r = 1$. Furthermore, the results derived in this article yields to many special cases; the interested reader may be referred to (see, e.g., [1, 7, 45]).

A remarkably large number of Laplace transforms and inverse Laplace transforms involving a variety of functions and polynomials have been presented (see, e.g., [45, pp. 129–299]). In this connection, we tried to give matrix versions of those outcomes for Laplace transforms and inverse Laplace formulas involving a variety of functions and polynomials (see, [45, pp. 129–299]).

Data Availability

No data were used to support the study.

Conflicts of Interest

This work does not have any conflicts of interest.

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Research Article

Blow-Up for a Stochastic Viscoelastic Lamé Equation with Logarithmic Nonlinearity

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In this paper, we consider an initial boundary value problem of stochastic viscoelastic wave equation with nonlinear damping and logarithmic nonlinear source terms. We proved a blow-up result for the solution with decreasing kernel.

1. Introduction

In recent years, stochastic partial differential equations in a separable Hilbert space have been studied by many authors, and various results on the existence, uniqueness, stability, blow-up, and other quantitative and qualitative properties of solutions have been established.

In this work, we consider the following problem of stochastic wave equation:

$$\begin{cases} u_{tt} - \mu\Delta u - (\lambda+\mu)\nabla(divu) + \int_0^t h(t-s)\Delta u(s)\,ds + |u_t|^{q-2}u_t = u|u|^{p-2}\ln|u|^k + \varepsilon\sigma(x,t)W_t(x,t) \text{ in } \mathscr{D}\times\,]0,+\infty[, \\ u(x,t) = 0 \text{ on } \partial\mathscr{D}\times[0,+\infty[, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) \text{ in } \bar{\mathscr{D}}, \end{cases} \tag{1}$$

where $\mathscr{D}$ is a bounded domain in IR^n, $n \in IN^*$, with a smooth boundary $\partial\mathscr{D}$; μ, λ are the Lamé constants which satisfy $\mu > 0$, $\lambda + \mu \geq 0$; h is a positive function, $p > q \geq 2$; the constant k is a small nonnegative real number; and $L^2(\mathscr{D})$ is the set of square integrable function on $\mathscr{D}$ equipped with the inner product $\langle .,. \rangle$ and its norm $\|.\|_2$.

$W(x,t)$ is an infinite dimensional Wiener process, $\sigma(x,t)$ is $L^2(\mathscr{D})$ valued progressively measurable, and ε is a positive constant which measures the strength of noise.

It is common to observe a wave motion as a physical phenomenon which is mathematically modeled by a partial differential equation of hyperbolic type. Much has been written about such equations regarding their widespread applications to engineering and sciences. However, for more realistic models, the random fluctuation had been taken into consideration which led to introduced stochastic wave equation in 1960's. Several examples of linear stochastic wave propagation and applications can be found in [1]. Mueller [2] was the first who investigate the existence of explosive solutions for some stochastic wave equation. Motivated by Mueller [2], Chow [3] was interested by knowing how does a random perturbation affect the solution behavior for a wave equation with a polynomial nonlinearity. He was concerned with the existence of local and global solutions of the stochastic equation:

$$\begin{cases} u_{tt} = \Delta u + f(u) + \sigma(u) W_t(x,t) \text{ in } x \in \mathbb{R}^d, & t > 0, \\ u(x,0) = g(x), u_t(x,0) = h(x), \end{cases} \tag{2}$$

where the initial data g and h are given functions and the nonlinear terms $f(u)$ and $\sigma(u)$ are assumed to be polynomials in u. Four years later, he [4] established an energy inequality and the exponential bound for a linear stochastic equation and gave the existence theorem for a unique global solution for the randomly perturbed wave equation:

$$\begin{cases} u_{tt} + 2\alpha u_t - A(x,\partial x)u(x,t) = f(x,t) + \sigma(x,t) W_t(x,t) \text{ in } x \in \mathbb{R}^d, & t > 0, \\ u(x,0) = u_0(x), u_t(x,0) = v_0(x). \end{cases} \tag{3}$$

In 2009, Chow [5] studied the problem of explosive solutions for a class of nonlinear stochastic wave equation in a domain $\mathscr{D} \subset \mathbb{R}^d$ for $d \geq 3$,

$$\begin{cases} u_{tt} = \left(c^2\Delta - \alpha\right) u + f(u) + \sigma(u,x,t) W_t(x,t) \text{ in } x \in \mathscr{D}, & t > 0, \\ u(x,0) = g(x), u_t(x,0) = h(x). \end{cases} \tag{4}$$

We can mention some other works such as Cheng et al. [6] who studied the existence of a global solution and blow-up solutions for the nonlinear stochastic viscoelastic wave equation with nonlinear damping and source terms:

$$\begin{cases} u_{tt} - \Delta u + \int_0^t h(t-s)\Delta u(s) ds + |u_t|^{q-2} u_t = u|u|^{p-2} + \varepsilon\sigma(x,t) W_t(x,t) \text{ in } \mathscr{D} \times]0,+\infty[, \\ u(x,t) = 0 \text{ on } \partial\mathscr{D} \times [0,+\infty[, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) \text{ in } \bar{\mathscr{D}}. \end{cases} \tag{5}$$

The authors proved that finite time blow-up with nonnegative probability is explosive or it is explosive in energy sense for $p > q$.

Moreover, Kim et al. [7] considered the stochastic quasilinear viscoelastic wave equation with nonlinear damping and source terms:

$$\begin{cases} |u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t h(t-s)\Delta u(s) ds + |u_t|^{q-2} u_t = u|u|^{p-2} + \varepsilon\sigma(x,t) W_t(x,t) \text{ in } \mathscr{D} \times]0,+\infty[, \\ u(x,t) = 0 \text{ on } \partial\mathscr{D} \times [0,+\infty[, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) \text{ in } \bar{\mathscr{D}}. \end{cases} \tag{6}$$

They showed the existence of a global solution and blow-up in finite time.

Recently, Yang et al. [8] treated the following stochastic nonlinear viscoelastic wave equation:

$$\begin{cases} |u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t h(t-s)\Delta u(s)\, ds = \sigma(x,t) W_t(x,t) \text{ in } \mathscr{D} \times]0,+\infty[, \\ u(x,t) = 0 \text{ on } \partial\mathscr{D} \times [0,+\infty[, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) \text{ in } \bar{\mathscr{D}}. \end{cases} \tag{7}$$

They established the existence of global solution and asymptotic stability of the solution by using some properties of the convex function.

However, it was noticed that the logarithmic nonlinearity appears naturally in many branches of physics such as nuclear physics, optics, and geophysics (see [9, 10]). These specific applications in physics and other fields attract a lot of mathematical scientists to work with such problems. In the deterministic case, Al-Gharabli [11] investigated the stability of the solution of a viscoelastic plate equation with a logarithmic nonlinearity source term for the following problem:

$$\begin{cases} u_{tt} + \Delta^2 u + u + \int_0^t h(t-s)\Delta u^2(s)ds = u \ln |u|^k \text{ in } \mathscr{D} \times]0, +\infty[, \\ u = \dfrac{\partial u}{\partial \nu} = 0 \text{ in } \partial\mathscr{D} \times]0, +\infty[, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) \text{ in } \bar{\mathscr{D}}, \end{cases} \tag{8}$$

where $\mathscr{D} \subseteq \mathbb{R}^2$ is a bounded domain with a smooth boundary $\partial\mathscr{D}$. The vector ν is the unit outer normal to $\partial\mathscr{D}$, and h is the nondecreasing nonnegative function.

Mezouar et al. [12] treated a more general problem where they considered the following nonlinear viscoelastic Kirchhoff equation with a time-varying delay term:

$$\begin{cases} |u_t|^l u_{tt} - M\left(\|\nabla u\|^2\right)\Delta u - \Delta u_{tt} + \int_0^t h(t-s)\Delta u(s)\, ds + \mu_1 g_1(u_t(x,t)) + \mu_2 g_2(u_t(x, t-\tau(t))) = ku \ln |u| \text{ in } \mathscr{D} \times]0, +\infty[, \\ u(x,t) = 0 \text{ on } \partial\mathscr{D} \times [0, +\infty[, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) \text{ in } \mathscr{D}, \\ u_t(x, t-\tau(0)) = f_0(x, t-\tau(0)) \text{ in } \mathscr{D} \times]0, \tau(0)[. \end{cases} \tag{9}$$

The paper is organized as follows: in Section 2, we introduce some basic definitions, necessary assumptions, and lemmas that are helpful in proving our main result. Section 3 is devoted to show the blow-up of the solution of our problem.

2. Preliminaries

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space for which a filtration $\{\mathscr{F}_t, t \geq 0\}$ of increasing sub $\sigma -$ fields $\mathscr{F}_t$ is given and $W(x,t)$ be a continuous Wiener random field in this space with a mean zero and the covariance operator Q satisfying

$$Tr(Q) = \sum_{i\geq 1} \lambda_i < \infty. \tag{10}$$

$W(x,t)$ is defined by

$$W(x,t) = \sum_{j=1}^{\infty} \sqrt{\lambda_j} \beta_j(t) e_j(t), \quad j \in \mathrm{IN}^*, t \geq 0, \tag{11}$$

where $\beta_j(t)$ is a sequence of real-valued standard Brownian motions mutually independent on the probability space $(\Omega, \mathscr{F}, P)$, λ_j are the eigenvalues of Q, and e_j are the corresponding eigenvectors. That is,

$$Qe_j = \lambda_j e_j. \tag{12}$$

Note $E(.)$ stands for expectation with respect to probability measure P. Let $\mathscr{H}$ be the set of $L_2^0 = L^2(Q^{1/2}V, V)$-valued processes with the norm

$$\|\phi(t)\|^2_{\mathscr{H}} = E\int_0^t \|\phi(s)\|^2_{L_2^0} ds = E\int_0^t Tr(\phi(s)Q\phi^*(s))ds < \infty, \tag{13}$$

where $\phi^*(s)$ denotes the adjoint operator of $\phi(s)$ and $V = H_0^1(\mathscr{D})$ which is equivalent to $H^1(\mathscr{D})$. For any process $\phi(s) \in \mathscr{H}$, we can define the stochastic integral with respect to the Q-Wiener process as $\int_0^t \phi(s)dW(s)$ which is a martingale. For more details about the infinite dimension Wiener process and stochastic integral, we refer to Da Prato and Zabczyk (pp. 90-96, [13]).

To state and prove our result, we need some assumptions.

A1. Assume that $h : IR^+ \longrightarrow IR^+$ is a C^1 nonincreasing function satisfying

$$h(0) > 0, \mu - \int_0^\infty h(s)ds = l > 0, \tag{14}$$

and there exist tow nonnegative constants ς_1 and ς_2 such that

$$-\varsigma_1 h(t) \leq h'(t) \leq -\varsigma_2 h(t), \quad t \geq 0. \tag{15}$$

A2.

$$\int_0^\infty h(s)ds < \mu \frac{(p-2)p}{(p-1)^2}. \quad (16)$$

A3. $p > q \geq 2$ and

$$\begin{cases} 2 < p \leq \dfrac{2(n-1)}{n-2}, & \text{if } n \geq 3, \\ 2 < k \leq +\infty, & \text{if } n = 1, 2. \end{cases} \quad (17)$$

The following theorem states the existence and uniqueness of a local solution of our problem; the proof can be established by combining the proof given in [6, 12].

Theorem 1. *Assume that (A1) and (A3) hold. If* $(u_0, u_1) \in H_0^1(\mathscr{D}) \times L^2(\mathscr{D})$ *and* $E\int_0^t \|\sigma(t)\|_2^2 dt < \infty$, *then there exists a solution in which* u *holds (1) on the interval* $[0, T]$ *in the sense of distributions over* $(0, T) \times \mathscr{D}$ *for almost all* w *a test function such that*

$$(u, u_t) \in L^2\left(\Omega; L^\infty\left([0, T]; \left(H^2(\mathscr{D}) \cap H_0^1(\mathscr{D})\right) \times H_0^1(\mathscr{D})\right)\right) \cap L^2\left(\Omega; C\left([0, T]; H_0^1(\mathscr{D}) \times L^2(\mathscr{D})\right)\right). \quad (18)$$

We define the energy associated to the solution of system (1) by

$$\begin{aligned} e(t) = {} & \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\left(\mu - \int_0^t h(s)ds\right)\|\nabla u\|_2^2 \\ & + \frac{\lambda+\mu}{2}\|\text{div } u\|_2^2 + \frac{1}{2}(ho\nabla u)(t)^{k} dx, \\ & + \frac{k}{p^2}\|u\|_p^p - \frac{1}{p}\int_{\mathscr{D}} |u|^p \ln|u| \end{aligned} \quad (19)$$

where

$$(hov)(t) = \int_0^t h(t-s)\|v(.,t) - v(.,s)\|^2 ds. \quad (20)$$

We rewrite (1) as an equivalent Itô's system

$$\begin{cases} du = v dt, \\ dv = \left[\mu\Delta u + (\lambda+\mu)\nabla(\text{div } u) - \displaystyle\int_0^t h(t-s)\Delta u(s)\, ds - |v|^{q-2}v + u|u|^{p-2} \ln |u|^k\right] dt + \varepsilon\sigma(x,t) dW_t(x,t) \text{ in } \mathscr{D} \times]0, +\infty[, \\ u(x,t) = 0 \text{ on } \partial\mathscr{D} \times [0, +\infty[, \\ u(x,0) = u_0(x), v(x,0) = u_1(x) \text{ in } \bar{\mathscr{D}}, \end{cases} \quad (21)$$

which can be written as the integral equation

$$\begin{cases} u(t) = u_0 + \displaystyle\int_0^t v(s)ds, \\ v(t) = v(0) + \displaystyle\int_0^t \left[\mu\Delta u + (\lambda+\mu)\nabla(\text{div } u) - \int_0^t h(s-r)\Delta u(r)\, dr - |v|^{q-2}v + u|u|^{p-2} \ln |u|^k\right] ds + \int_0^t \varepsilon\sigma(x,s) dW_s(x,t) \text{ in } \mathscr{D} \times]0, +\infty[, \\ u(x,t) = 0 \text{ on } \partial\mathscr{D} \times [0, +\infty[, \\ u(x,0) = u_0(x), v(x,0) = u_1(x) \text{ in } \bar{\mathscr{D}}. \end{cases} \quad (22)$$

Lemma 2 [14] (Sobolev-Poincaré's inequality). *Let* m *be a number with*

$$2 \leq m \leq +\infty (n = 1, 2) \quad (23)$$

or

$$\frac{2 \leq m \leq 2n}{(n-2)(n \geq 3)}. \quad (24)$$

Then there exists a constant $C_s = C_s(\mathscr{D}, m)$ *such that*

$$\|u\|_m \le C_s \|\nabla u\|_2, \quad \text{for } u \in H_0^1(\mathscr{D}). \tag{25}$$

Lemma 3 [15]. *For* $h, \varphi \in C^1([0,+\infty[, IR)$*, we have*

$$\int_{\mathscr{D}} h * \varphi\varphi_t dx = -\frac{1}{2}h(t)\|\varphi(t)\|_2^2 + \frac{1}{2}\left(h' o\varphi\right)(t) - \frac{1}{2}\frac{d}{dt}\left[(ho\varphi)(t) - \left(\int_0^t h(s)ds\right)\|\varphi\|^2\right]. \tag{26}$$

Lemma 4. *Let* (u, v) *be a solution of the problem (21) with the initial data* $(u_0, v_0) \in H_0^1(\mathscr{D}) \times L^2(\mathscr{D})$, $E\int_0^t \|\sigma(s)\|_2^2 ds < \infty$. *Then, the energy functional defined by (19) satisfies*

$$\begin{aligned} e(t) = e(0) &- \int_0^t \|v\|_q^q ds - \frac{1}{2}\int_0^t h(s)\|\nabla u(s)\|_2^2 ds \\ &+ \frac{1}{2}\int_0^t \left(h' o\nabla u\right)(s)ds + \int_0^t \langle v(s), \varepsilon\sigma(x,s)dW_s\rangle \\ &+ \frac{\varepsilon^2}{2}\sum_{j=1}^{\infty}\int_0^t\int_{\mathscr{D}} \lambda_j e_j^2(x)\sigma^2(x,s)dxds. \end{aligned} \tag{27}$$

Proof. We can apply the Itô's formula to (21) for each $x \in \mathscr{D}$ after integrating the above equation over $\mathscr{D}$ to get

$$\begin{aligned} \|v(t)\|_2^2 = \|v(0)\|_2^2 &+ 2\int_{\mathscr{D}}\int_0^t v(s)[\mu\Delta u + (\lambda+\mu)\nabla(\text{div } u) \\ &- \int_0^s h(s-\tau)\Delta u(\tau)d\tau - |v|^{q-2}v + u|u|^{p-2}\ln|u|^k]dsdx \\ &+ 2\int_0^t \langle v(s), \varepsilon\sigma(x,s)dW_s\rangle \\ &+ \varepsilon^2\sum_{j=1}^{\infty}\int_0^t\int_{\mathscr{D}} \lambda_j e_j^2(x)\sigma^2(x,s)dxds. \end{aligned} \tag{28}$$

By using integration by parts, we get

$$\begin{aligned} \mu\int_{\mathscr{D}}\int_0^t \Delta u v(s)dxds &= -\mu\int_{\mathscr{D}}\int_0^t \nabla u\nabla v dsdx \\ &= -\frac{\mu}{2}\left(\|\nabla u(t)\|_2^2 - \|\nabla u(0)\|_2^2\right), \end{aligned} \tag{29}$$

$$\begin{aligned} &\int_{\mathscr{D}}\int_0^t (\lambda+\mu)\nabla(\text{div } u(s))v(s)dsdx \\ &= -(\lambda+\mu)\int_{\mathscr{D}}\int_0^t \text{div } u(s) \text{ div } v(s)dsdx \\ &= -\frac{\lambda+\mu}{2}\left[\|\text{div } u(t)\|_2^2 - \|\text{ div } u(0)\|_2^2\right]. \end{aligned} \tag{30}$$

By applying Lemma 3, we have

$$\begin{aligned} &\int_0^t\int_{\mathscr{D}}\int_0^s h(s-\tau)\Delta u(\tau)v(s)d\tau dxds \\ &= -\int_0^t\int_{\mathscr{D}}\int_0^s h(s-\tau)\nabla u(\tau)\nabla v(s)d\tau dxds \\ &= \int_0^t\left(\frac{1}{2}h(s)\|\nabla u(s)\|_2^2 - \frac{1}{2}\left(h' o\nabla u\right)(s)\right. \\ &\left. + \frac{1}{2}\frac{d}{ds}\left[(h\, o\nabla u)(s) - \int_0^s h(\tau)d\tau\|\nabla u(s)\|_2^2\right]\right)ds. \end{aligned} \tag{31}$$

We have

$$\begin{aligned} &\int_{\mathscr{D}}\int_0^t u|u|^{p-2}\ln|u|^k u_s dsdx \\ &= \int_0^t\int_{\mathscr{D}} \frac{1}{p}\frac{d}{ds}\left(|u(s)|^p\right)\ln|u|^k dxds \\ &= \int_0^t\left\{\int_{\mathscr{D}}\left\{\frac{1}{p}\frac{d}{ds}\left(|u(s)|^p\ln|u|^k\right)\right.\right. \\ &\left.\left. - \frac{1}{p}|u(s)|^p\frac{d}{ds}\left(\ln|u|^k\right)\right\}dx\right\}ds \\ &= \int_{\mathscr{D}}\left(\frac{1}{p}\left(|u|^p\ln|u|^k\right)\right)dx - \frac{k}{p^2}\|u\|_p^p. \end{aligned} \tag{32}$$

By replacing (29)–(32) in (28) and multiplying equation (28) by 1/2, we arrive at (27).

3. Blow-Up

We prove our main result for $p > q$; we purpose

$$E\int_0^\infty\int_{\mathscr{D}} \sigma^2(x,t)dxdt < \infty, \tag{33}$$

$$G(t) = \frac{\varepsilon^2}{2}\sum_{j=1}^{\infty} E\int_0^t\int_{\mathscr{D}} \lambda_j e_j^2(x)\sigma^2(x,s)dxds, \tag{34}$$

$$\begin{aligned} G(\infty) &= \frac{\varepsilon^2}{2}\sum_{j=1}^{\infty} E\int_0^\infty\int_{\mathscr{D}} \lambda_j e_j^2(x)\sigma^2(x,s)dxds \\ &\le \frac{\varepsilon^2}{2}Tr(Q)c_0^2 E\int_0^\infty\int_{\mathscr{D}} \sigma^2(x,s)dxds := E_1 < \infty, \end{aligned} \tag{35}$$

where

$$Tr(Q) = \sum_{j=1}^{\infty} \lambda_j < \infty \text{and } c_0 = \sup_{j\ge 1}\|e_j\|_\infty < \infty. \tag{36}$$

Lemma 5. *Let (u, v) be a solution of system (21) with initial data $(u_0, v_0) \in H_0^1(\mathscr{D}) \times L^2(\mathscr{D})$. Then, we have*

$$\frac{d}{dt}Ee(t) = -E\|v(t)\|_q^q - \frac{1}{2}h(t)E\|\nabla u(t)\|_2^2 + \frac{1}{2}E\left(h' o\nabla u\right)(t) + \frac{\varepsilon^2}{2}\sum_{j=1}^{\infty} E\int_{\mathscr{D}} \lambda_j e_j^2(x)\sigma^2(x,t)dx, \tag{37}$$

$$\begin{aligned} E\langle u(t), v(t)\rangle = {} & E\langle u_0, u_1\rangle - \mu\int_0^t E\|\nabla u(s)\|_2^2 ds \\ & - (\lambda+\mu)\int_0^t E\|\text{div } u(s)\|_2^2 ds \\ & + E\int_0^t\int_0^s h(s-r)\langle\nabla u(r), \nabla u(s)\rangle dr ds \\ & - E\int_0^t \left\langle u(s), |v(s)|^{q-2}v(s)\right\rangle ds \\ & + E\int_0^t \left\langle u(s), u(s)|u(s)|^{p-2}\ln|u(s)|^k\right\rangle ds \\ & + E\int_0^t \|v(s)\|_2^2 ds. \end{aligned} \tag{38}$$

Proof. Using the Itô's formula and by following the same way as our discussions in Lemma 4 with taking the expectations, we obtain (37).

We multiply the second equation in (22) by u and integrate the result over $\mathscr{D}$, and we take expectation; we obtain (38).

We set $H(t) = G(t) - Ee(t)$. As h is a positive decreasing function so

$$H'(t) = G'(t) - \frac{d}{dt}Ee(t) = E\|v\|_q^q + \frac{1}{2}h(t)E\|\nabla u(t)\|_2^2 - \frac{1}{2}E\left(h' o\nabla u\right)(t) \geq E\|v\|_q^q. \tag{39}$$

Consequently,

$$H'(t) \geq 0. \tag{40}$$

Lemma 6. *Let (u, v) be a solution of system (21). Assume that (A1) holds. Then, there exists a positive constant C such that*

$$E\|u(t)\|_{p+1}^s \leq C\left(G(t) - H(t) - \frac{1}{2}E\|v\|_2^2 + \frac{1}{p}E\int_{\mathscr{D}} |u|^p \ln|u|^k dx - \frac{1}{2}E(h\, o\nabla u)(t) - \frac{\lambda+\mu}{2}E\|\text{div } u\|_2^2 + E\|u\|_{p+1}^{p+1}\right), \tag{41}$$

where $2 \leq s \leq p+1$.

Proof.

$$\begin{aligned} & G(t) - H(t) - \frac{1}{2}E\|v\|_2^2 + \frac{1}{p}E\int_{\mathscr{D}} |u|^p \ln|u|^k dx \\ & \quad - \frac{1}{2}E(ho\nabla u)(t) - \frac{\lambda+\mu}{2}E\|\text{div } u\|_2^2 + E\|u\|_{p+1}^{p+1} \\ & = Ee(t) - \frac{1}{2}E\|v\|_2^2 + \frac{1}{\text{p}}E\int_{\mathscr{D}} |u|^p \ln|u|^k dx \\ & \quad - \frac{1}{2}E(ho\nabla u)(t) + E\|u\|_{p+1}^{p+1} - \frac{\lambda+\mu}{2}\|\text{div } u\|_2^2 \\ & = \frac{1}{2}E\|u_t\|_2^2 + \frac{1}{2}E\left(\mu - \int_0^t h(s)ds\right)\|\nabla u\|_2^2 \\ & \quad + \frac{\lambda+\mu}{2}E\|\text{div } u\|_2^2 + \frac{1}{2}E(h\, o\nabla u)(t) + \frac{k}{p^2}E\|u\|_p^p \\ & \quad - \frac{1}{p}E\int_{\mathscr{D}} |u|^p \ln|u|^k dx - \frac{1}{2}E\|v\|_2^2 \\ & \quad + \frac{1}{p}E\int_{\mathscr{D}} |u|^p \ln|u|^k dx - \frac{1}{2}E(h\, o\nabla u)(t) \\ & \quad - \frac{\lambda+\mu}{2}E\|\text{div } u\|_2^2 + E\|u\|_{p+1}^{p+1} \\ & = \frac{1}{2}\left(\mu - \int_0^t h(s)ds\right)E\|\nabla u\|_2^2 + \frac{k}{p^2}E\|u\|_p^p + E\|u\|_{p+1}^{p+1} \\ & \geq \frac{1}{2}lE\|\nabla u\|_2^2 + E\|u\|_{p+1}^{p+1}. \end{aligned} \tag{42}$$

The last inequality is getting from (A1).

Case 7. If $\|u\|_{p+1} \leq 1$, then $\|u\|_{p+1}^s \leq \|u\|_{p+1}^2$.

By applying Lemma 2, we obtain $\|u\|_{p+1}^s \leq c\|\nabla u\|_2^2$, then

$$\frac{1}{2}lE\|\nabla u\|_2^2 + \frac{k}{p^2}E\|u\|_{p+1}^{p+1} \geq \frac{1}{2}lE\|u\|_{p+1}^s + \frac{k}{p^2}E\|u\|_{p+1}^{p+1} \geq E\|u\|_{p+1}^s. \tag{43}$$

Case 8. If $\|u\|_{p+1} \geq 1$, then $\|u\|_{p+1}^{p+1} \geq \|u\|_{p+1}^s$.

Hence,

$$\frac{1}{2}lE\|\nabla u\|_2^2 + E\|u\|_{p+1}^{p+1} \geq \frac{1}{2}lE\|\nabla u\|_2^2 + E\|u\|_{p+1}^s \geq E\|u\|_{p+1}^s. \tag{44}$$

Consequently, we obtain (41).

We are ready to state and prove our main result for $p > q$. For this purpose, we define

$$L(t) := H^{1-\alpha}(t) + \delta E\langle u, v\rangle, \tag{45}$$

where

$$0 < \alpha < \min\left\{\frac{p-1}{2(p+1)}, \frac{p+1-q}{(p+1)q}\right\} \tag{46}$$

and δ is a very small constant determined later.

Theorem 9. *Assume (A1) and (A2) hold. Let (u, v) be a solution of system (21) with initial data $(u_0, v_0) \in H_0^1(\mathscr{D}) \times L^2(\mathscr{D})$ satisfying*

$$Ee(0) \leq -(1+\beta)E_1, \tag{47}$$

where β is a nonnegative constant and E_1 is given in (35). If $p > q$, then there exists a positive time $T_0 \in [0, T]$ such that

$$\lim_{t \longrightarrow T_0^-} E(e(t)) = +\infty, \tag{48}$$

where

$$T_0 = \frac{1-\alpha}{\alpha K L^{\alpha/1-\alpha}(0)}, \quad L(0) = H^{1-\alpha}(0) + \delta E\langle u_0, u_1\rangle > 0, \tag{49}$$

and K is given later.

Proof. Let

$$L(t) = H^{1-\alpha}(t) + \delta E\langle u, v\rangle. \tag{50}$$

A direct differentiation of $L(t)$ gives

$$\begin{aligned} L'(t) &= (1-\alpha)H^{-\alpha}(t)H'(t) + \delta\big[-\mu E\|\nabla u(t)\|_2^2 \\ &\quad - (\lambda+\mu)E\|divu\|_2^2 + E\int_0^t h(t-r)\langle \nabla u(r), \nabla u(t)\rangle dr \\ &\quad - E\langle u(t), |v(t)|^{q-2}v(t)\rangle \\ &\quad + E\left\langle u(t), u(t)|u(t)|^{p-2}\ln|u(t)|^k\right\rangle + E\|v(t)\|_2^2\big] \\ &= (1-\alpha)H^{-\alpha}(t)H'(t) + \delta\big[-\mu E\|\nabla u(t)\|_2^2 \\ &\quad - (\lambda+\mu)E\|\text{div } u\|_2^2 + E\int_0^t h(t-r)\langle \nabla u(r), \nabla u(t)\rangle dr \\ &\quad - E\langle u(t), |v(t)|^{q-2}v(t)\rangle + E\left\langle u(t), u(t)|u(t)|^{p-2}\ln|u(t)|^k\right\rangle \\ &\quad + E\|v(t)\|_2^2\big] + \delta p[H(t) - G(t) + Ee(t)]. \end{aligned} \tag{51}$$

Recalling (39) and (19), (51) leads to

$$\begin{aligned} L'(t) &\geq (1-\alpha)H^{-\alpha}(t)E\|v\|_q^q + \delta p(H(t) - G(t)) \\ &\quad + \delta\left(\frac{\mu p}{2} - \mu\right)E\|\nabla u(t)\|_2^2 + \delta\left(\frac{p}{2}+1\right)E\|v\|_2^2 \\ &\quad - \delta E\langle u(t), |v(t)|^{q-2}v(t)\rangle - \frac{\delta p}{2}E\int_0^t h(s)ds\|\nabla u\|_2^2 \\ &\quad + \delta E\int_0^t h(t-r)\langle \nabla u(r), \nabla u(t)\rangle dr \\ &\quad + (\lambda+\mu)\delta\left(\frac{\text{p}}{2} - 1\right)E\|\text{div } u\|_2^2 + \frac{\delta k}{p}E\|u\|_p^p \\ &\quad + \frac{\delta p}{2}E(h\, o\nabla u)(t). \end{aligned} \tag{52}$$

By using Young's and Hölder's inequalities, we get

$$\begin{aligned} &E\int_0^t h(t-r)\langle \nabla u(r), \nabla u(t)\rangle dr \\ &\quad = E\int_0^t h(t-r)\langle \nabla u(r) - \nabla u(t), \nabla u(t)\rangle dr + E\int_0^t h(s)ds\|\nabla u(t)\|_2^2 \\ &\quad \geq -\frac{p}{2}E(h\, o\nabla u)(t) - \frac{1}{2p}E\int_0^t h(s)ds\|\nabla u(t)\|_2^2 \\ &\qquad + E\int_0^t h(s)ds\|\nabla u(t)\|_2^2. \end{aligned} \tag{53}$$

Hence,

$$\begin{aligned} L'(t) &\geq (1-\alpha)H^{-\alpha}(t)E\|v\|_q^q + \delta p(H(t) - G(t)) \\ &\quad + \delta\left(\frac{\mu p}{2} - \mu\right)E\|\nabla u(t)\|_2^2 + \delta\left(\frac{p}{2}+1\right)E\|v\|_2^2 \\ &\quad - \delta E\langle u(t), |v(t)|^{q-2}v(t)\rangle \\ &\quad + \delta\left(1 - \frac{1}{2\text{p}} - \frac{p}{2}\right)E\int_0^t h(s)ds\|\nabla u(t)\|_2^2 \\ &\quad - \frac{\delta p}{2}E(h\, o\nabla u)(t) + \frac{\delta p}{2}E(h\, o\nabla u)(t) \\ &\quad + (\lambda+\mu)\delta\left(\frac{p}{2} - 1\right)E\|\text{div } u\|_2^2 + \frac{\delta k}{p}E\|u\|_p^p \\ &\geq (1-\alpha)H^{-\alpha}(t)E\|v\|_q^q + \delta p(H(t) - G(t)) \\ &\quad + \delta\left(\frac{\mu p}{2} - \mu\right)E\|\nabla u(t)\|_2^2 + \delta\left(\frac{p}{2}+1\right)E\|v\|_2^2 \\ &\quad - \delta E\langle u(t), |v(t)|^{q-2}v(t)\rangle \\ &\quad + \delta\left(1 - \frac{1}{2p} - \frac{p}{2}\right)E\int_0^t h(s)ds\|\nabla u(t)\|_2^2 \\ &\quad + (\lambda+\mu)\delta\left(\frac{p}{2} - 1\right)E\|\text{div } u\|_2^2 + \frac{\delta k}{p}E\|u\|_p^p. \end{aligned} \tag{54}$$

As $q < p+1$, then $E\|u(t)\|_q^q \leq cE\|u(t)\|_{p+1}^{p+1}$ so by using Young's and Hölder's inequality; we obtain

$$\begin{aligned} E\langle u(t), |v(t)|^{q-2}v(t)\rangle &\leq \left(E\|v(t)\|_q^q\right)^{q-1/q}\left(E\|u(t)\|_q^q\right)^{1/q} \\ &\leq c\left(E\|v(t)\|_q^q\right)^{q-1/q}\left(E\|u(t)\|_{p+1}^{p+1}\right)^{1/p+1} \\ &\leq c\left(E\|v(t)\|_q^q\right)^{q-1/q}\left(E\|u(t)\|_{p+1}^{p+1}\right)^{(1/p+1)-(1/q)}\left(E\|u(t)\|_{p+1}^{p+1}\right)^{(1/q)} \\ &\leq c\left(\frac{q-1}{q}\xi\left(E\|v(t)\|_q^q\right) + \frac{\xi^{1-q}}{q}\left(E\|u(t)\|_{p+1}^{p+1}\right)\right) \\ &\quad \times \left(E\|u(t)\|_{p+1}^{p+1}\right)^{(1/p+1)-(1/q)}, \end{aligned} \tag{55}$$

where ξ and c are constants.

We consider the following partition of $\mathscr{D}$:

$$\mathscr{D}_1 = \{x \in \mathscr{D} : |u| > 1\}, \mathscr{D}_2 = \{x \in \mathscr{D} : |u| \leq 1\}. \tag{56}$$

We have

$$E\int_{\mathscr{D}}|u|^p \ln |u|^k dx = E\int_{\mathscr{D}_1}|u|^p \ln |u|^k dx + E\int_{\mathscr{D}_2}|u|^p \ln |u|^k dx \le E\int_{\mathscr{D}_1}|u|^p \ln |u|^k dx \le E\int_{\mathscr{D}_1} k|u|^{p+1} dx \le kE\|u\|_{p+1}^{p+1}. \quad (57)$$

By (40), (47), and $-Ee(0) = H(0)$, we have

$$(1+\beta)G(t) < (1+\beta)E_1 \le H(0) \le H(t) \le G(t) + \frac{1}{p}E\int_{\mathscr{D}}|u|^p \ln |u|^k dx. \quad (58)$$

Therefore,

$$G(\mathrm{t}) \le \frac{1}{1+\beta}H(t). \quad (59)$$

From (57), (58), and (59), we get

$$kE\|u(t)\|_{p+1}^{p+1} \ge E\int_{\mathscr{D}_1} k|u|^{p+1} dx \ge p(H(t) - G(t)) \ge p\frac{\beta}{1+\beta}H(t). \quad (60)$$

As H is increasing positive nonnegative function and by recalling (46), we get

$$\left(E\|u(t)\|_{p+1}^{p+1}\right)^{(1/p+1)-(1/q)} \le \left(p\frac{\beta}{k(1+\beta)}\right)^{(1/p+1)-(1/q)} H^{(1/p+1)-(1/q)}(t) \le \left(p\frac{\beta}{k(1+\beta)}\right)^{(1/p+1)-(1/q)} H^{-\alpha}(t) \le \left(p\frac{\beta}{k(1+\beta)}\right)^{(1/p+1)-(1/q)} H^{-\alpha}(0). \quad (61)$$

Taking into account (61) in (55), we find

$$E\left\langle u(t), |v(t)|^{q-2}v(t)\right\rangle \le \left(c\left(\frac{p\beta}{k(1+\beta)}\right)^{(1/p+1)-(1/q)}\right)\frac{\mathrm{q}-1}{q}\xi\left(E\|v(t)\|_q^q\right)H^{-\alpha}(t) + \left(c\left(\frac{p\beta}{k(1+\beta)}\right)^{(1/p+1)-(1/q)}\right)\frac{\xi^{1-q}}{q}\left(E\|u(t)\|_{p+1}^{p+1}\right)H^{-\alpha}(0). \quad (62)$$

Substituting (62) into (54), we get

$$L'(t) \ge (1-\alpha)H^{-\alpha}(t)E\|v\|_q^q + \delta p(H(t) - G(t)) + \delta\mu\left(\frac{p}{2}-1\right)E\|\nabla u(t)\|_2^2 + \delta\left(\frac{p}{2}+1\right)E\|v\|_2^2 + \delta\left(1 - \frac{1}{2p} - \frac{p}{2}\right)E\int_0^t h(s)ds\|\nabla u(t)\|_2^2 + (\lambda+\mu)\delta\left(\frac{p}{2}-1\right)E\|\mathrm{div}\ u\|_2^2 - \delta\frac{a_1(q-1)}{q}\xi\left(E\|v\|_q^q\right)H^{-\alpha}(t) - \delta\frac{a_1}{q}\xi^{1-q}\left(E\|u\|_{p+1}^{p+1}\right)H^{-\alpha}(0) + \frac{\delta k}{p}E\|u\|_p^p, \quad (63)$$

where $a_1 = c(p\beta/(k(1+\beta)))^{(1/p+1)-(1/q)}$.

Using Lemma 6, we arrive at

$$L'(t) \ge \left(1 - \alpha - \delta\frac{a_1(q-1)}{q}\xi\right)H^{-\alpha}(t)E\|v\|_q^q + \delta p(H(t) - G(t)) + \delta\mu\left(\frac{p}{2}-1\right)E\|\nabla u(t)\|_2^2 + \delta\left(\frac{p}{2}+1\right)E\|v\|_2^2 + \delta\left(1 - \frac{1}{2p} - \frac{p}{2}\right)E\int_0^t h(s)ds\|\nabla u(t)\|_2^2 + (\lambda+\mu)\delta\left(\frac{p}{2}-1\right)\|\mathrm{div}\ u\|_2^2 - \delta\frac{a_1}{q}\xi^{1-q}H^{-\alpha}(0)C \cdot \left(G(t) - H(t) - \frac{1}{2}E\|v\|_2^2 + \frac{1}{p}E\int_{\mathscr{D}}|u|^p \ln |u|^k dx + E\|u\|_{p+1}^{p+1} - \frac{1}{2}E(h\, o\nabla u)(t) - \frac{\lambda+\mu}{2}E\|\mathrm{div}\ u\|_2^2\right) + \frac{\delta k}{p}E\|u\|_p^p. \quad (64)$$

Once ξ is fixed, we pick δ small enough so that

$$1 - \alpha - \delta\frac{a_1(q-1)}{q}\xi \ge 0, \quad (65)$$

It implies that

$$L'(t) \ge \delta\left(p + a_2\xi^{1-q}\right)(H(t) - G(t)) + \delta\left(\frac{p}{2} + 1 + a_2\frac{1}{2}\xi^{1-q}\right)E\|v\|_2^2 - \delta a_2\xi^{1-q}\frac{1}{p}E\int_{\mathscr{D}}|u|^p \ln |u|^k dx + \delta(\lambda+\mu) \cdot \left(\xi^{1-q}\frac{a_2}{2} + \left(\frac{p}{2}-1\right)\right)E\|\mathrm{div}\ u\|_2^2 + \delta a_2\xi^{1-q}\frac{1}{2}E(h\, o\nabla u)(t) + \delta a_3 E\|\nabla u(t)\|_2^2 + \frac{\delta k}{p}E\|u\|_p^p - \delta a_2\xi^{1-q}E\|u\|_{p+1}^{p+1}. \quad (66)$$

where $a_2 = C(a_1/q)H^{-\alpha}(0)$ and $a_3 = \mu((p/2)-1) + (1-(1/2p)-(p/2))\int_0^\infty h(s)ds$ which is positive from (A2).

From (A1), (19), and Lemma 2, we have

$$\begin{aligned} H(t) - G(t) \geq & -\frac{1}{2}E\|v\|_2^2 - \left(\frac{\mu}{2} + 2C_s\right)E\|\nabla u\|_2^2 \\ & - \frac{\lambda+\mu}{2}E\|\text{div } u\|_2^2 - \frac{1}{2}E(h\, o\nabla u)(t) \\ & - \frac{k}{p^2}E\|u\|_p^p + \frac{1}{p}E\int_{\mathscr{D}}|u|^p \ln |u|^k dx + E\|u\|_{p+1}^{p+1}. \end{aligned} \tag{67}$$

Now we add and subtract $\delta a_4(H(t)-G(t))$ in (66), and using (67), we find

$$\begin{aligned} L'(t) \geq & \delta\left(p - a_4 + a_2\xi^{1-q}\right)(H(t)-G(t)) \\ & + \delta\left(\frac{p}{2} + 1 - \frac{a_4}{2} + a_2\frac{1}{2}\xi^{1-q}\right)E\|v\|_2^2 \\ & + \delta(\lambda+\mu)\left(\xi^{1-q}\frac{a_2}{2} + \left(\frac{p}{2}-1\right) - \frac{a_4}{2}\right)E\|\text{div } u\|_2^2 \\ & + \delta\left(a_2\xi^{1-q}\frac{1}{2} - a_4\right)E(h\, o\nabla u)(t) \\ & + \delta\left(a_3 - a_4\left(\frac{\mu+4C_s}{2}\right)\right)E\|\nabla u(t)\|_2^2 \\ & + \frac{\delta k}{p}\left(1 - \frac{a_4}{p}\right)E\|u\|_p^p + \frac{\delta}{p}\left(a_4 - a_2\xi^{1-q}\right)E\int_{\mathscr{D}}|u|^p \ln |u|^k dx \\ & + \delta\left(a_4 - a_2\xi^{1-q}\right)E\|u\|_{p+1}^{p+1}, \end{aligned} \tag{68}$$

where $a_4 = \min\{a_2\xi^{1-q}, (2a_3/(\mu+4C_s))\} > 0$.

Using (60), we obtain

$$\begin{aligned} L'(\mathrm{t}) \geq & \delta p\frac{\beta}{1+\beta}H(t) + \delta\left(\frac{p}{2} + 1 - \frac{a_4}{2} + a_2\frac{1}{2}\xi^{1-q}\right)E\|v\|_2^2 \\ & + \delta(\lambda+\mu)\left(\xi^{1-q}\frac{a_2}{2} + \left(\frac{p}{2}-1\right) - \frac{a_4}{2}\right)E\|\text{div } u\|_2^2 \\ & + \frac{\delta}{2}\left(a_2\xi^{1-q} - a_4\right)E(h\, o\nabla u)(t) \\ & + \delta\left(a_3 - a_4\left(\frac{\mu+4C_s}{2}\right)\right)E\|\nabla u(t)\|_2^2 \\ & + \frac{\delta k}{p}\left(1 - \frac{a_4}{p}\right)E\|u\|_p^p + \delta\left(a_4 - a_2\xi^{1-q}\right)E\|u\|_{p+1}^{p+1} \\ \geq & \gamma\big(H(t) + E\|v\|_2^2 + E\|\text{div } u\|_2^2 + E(h\, o\nabla u)(t) \\ & + E\|\nabla u(t)\|_2^2 + E\|u\|_p^p + E\|u\|_{p+1}^{p+1}\big) \geq 0, \end{aligned} \tag{69}$$

where $\gamma > 0$ is the minimum of the coefficients of $H(t)$, $E\|v\|_2^2$, $E\|\text{div } u\|_2^2$, $E(h\, o\nabla u)(t)$, $E\|\nabla u(t)\|_2^2$, and $E\|u\|_p^p$ in (69).

Consequently,

$$L(t) \geq L(0) > 0, \quad \forall t > 0. \tag{70}$$

Next, we have

$$\begin{aligned} (L(t))^{1/1-\alpha} &= \left(H^{1-\alpha}(t) + \delta E\langle u, v\rangle\right)^{1/1-\alpha} \\ &\leq 2^{1/1-\alpha}\left(H(t) + \delta^{1/1-\alpha}\left|E\int_{\mathscr{D}} uv dx\right|^{1/1-\alpha}\right). \end{aligned} \tag{71}$$

Therefore, by using Hölder's and Young's inequalities, we obtain

$$\begin{aligned} \left|E\int_{\mathscr{D}} uv dx\right|^{1/1-\alpha} &\leq \left(c\left(E\|u\|_{p+1}^2\right)^{1/2}\left(E\|v\|_2^2\right)^{1/2}\right)^{1/1-\alpha} \\ &\leq c\left(E\|u\|_{p+1}^2\right)^{1/(2(1-\alpha))}\left(E\|v\|_2^2\right)^{1/(2(1-\alpha))} \\ &\leq c\left[\frac{\left(E\|u\|_{p+1}^2\right)^{\eta/(2(1-\alpha))}}{\eta} + \frac{\left(E\|v\|_2^2\right)^{\zeta/(2(1-\alpha))}}{\zeta}\right], \end{aligned} \tag{72}$$

with $(1/\eta) + (1/\zeta) = 1$.

We choose $= 2(1-\alpha), \eta = (2(1-\alpha))/1-2\alpha$, and we use (46), so (72) becomes

$$\begin{aligned} \left|E\int_{\mathscr{D}} uv dx\right|^{1/1-\alpha} &\leq c\left[(1-2\alpha)E\|u\|_{p+1}^{2/1-2\alpha} + E\|v\|_2^2\right] \\ &\leq c\left[E\|u\|_{p+1}^{2/1-2\alpha} + E\|v\|_2^2\right]. \end{aligned} \tag{73}$$

By applying Lemma 6 with $s = 2/1-2\alpha$ and recalling (19), we obtain

$$\begin{aligned} \left|E\int_{\mathscr{D}} uv dx\right|^{1/1-\alpha} \leq & c\Big[G(t) - H(t) - \frac{1}{2}E\|v\|_2^2 \\ & + \frac{1}{p}E\int_{\mathscr{D}}|u|^p \ln |u|^k dx - \frac{1}{2}E(ho\nabla u)(t) \\ & - \frac{\lambda+\mu}{2}E\|\text{div } u\|_2^2 + E\|u\|_{p+1}^{p+1} + E\|v\|_2^2\Big] \\ \leq & c\Big[\frac{1}{2}E\|v\|_2^2 + \frac{1}{2}\left(\mu - \int_0^t h(s)ds\right)E\|\nabla u\|_2^2 \\ & + \frac{\lambda+\mu}{2}E\|\text{div } u\|_2^2 + \frac{k}{p^2}E\|u\|_p^p + \frac{1}{2}E(ho\nabla u)(t) \\ & - \frac{1}{p}E\int_{\mathscr{D}}|u|^p \ln |u|^k dx - \frac{1}{2}E\|v\|_2^2 + \frac{1}{p}E\int_{\mathscr{D}}|u|^p \ln |u|^k dx \\ & - \frac{1}{2}E(ho\nabla u)(t) - \frac{\lambda+\mu}{2}E\|\text{div } u\|_2^2 + E\|v\|_2^2 + E\|u\|_{p+1}^{p+1}\Big] \\ \leq & c\Big[E\|v\|_2^2 + \frac{1}{2}\mu E\|\nabla u\|_2^2 + \frac{k}{p^2}E\|u\|_p^p + \frac{\lambda+\mu}{2}E\|\text{div } u\|_2^2 \\ & + \frac{1}{2}E(ho\nabla u)(t) + E\|u\|_{p+1}^{p+1}\Big]. \end{aligned} \tag{74}$$

Hence,

$$(L(t))^{1/1-\alpha} \leq 2^{1/1-\alpha}\Bigg(H(t) + \delta^{1/1-\alpha}c\Bigg[E\|v\|_2^2 + \frac{1}{2}\mu E\|\nabla u\|_2^2 + \frac{\lambda+\mu}{2}E\|\text{div } u\|_2^2 + \frac{k}{p^2}E\|u\|_p^p + \frac{1}{2}E(ho\nabla u)(t) + E\|u\|_{p+1}^{p+1}\Bigg]\Bigg) \leq \tilde{C}\Big[H(t) + E\|v\|_2^2 + E\|\nabla u\|_2^2 + E\|\text{div } u\|_2^2 + E\|u\|_p^p + E(ho\nabla u)(t) + E\|u\|_{p+1}^{p+1}\Big], \tag{75}$$

where $\tilde{C} = 2^{1/1-\alpha} \max\{1, c\delta^{1/1-\alpha}, c\delta^{1/1-\alpha}((\lambda+\mu)/2), c\delta^{1/1-\alpha}(k/p^2)\}$.

According to (69) and (75), we have

$$(L(t))^{1/1-\alpha} \leq \frac{\tilde{C}}{\gamma}L'(t) \leq \tilde{K}L'(t). \tag{76}$$

In a direct integration of (76), we get

$$(L(t))^{1/1-\alpha} \geq \frac{1}{(L(0))^{1/1-\alpha} - \left(\tilde{K}\alpha t/(1-\alpha)\right)}. \tag{77}$$

Therefore, $L(t)$ blows up in time $T \leq T_0 = (1-\alpha)/(\alpha K L^{\alpha/1-\alpha}(0))$, and the proof is completed.

Data Availability

No data were used to support the study.

Conflicts of Interest

This work does not have any conflicts of interest.

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Research Article

Solving Fractional Differential Equations by Using Triangle Neural Network

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In this paper, numerical methods for solving fractional differential equations by using a triangle neural network are proposed. The fractional derivative is considered Caputo type. The fractional derivative of the triangle neural network is analyzed first. Then, based on the technique of minimizing the loss function of the neural network, the proposed numerical methods reduce the fractional differential equation into a gradient descent problem or the quadratic optimization problem. By using the gradient descent process or the quadratic optimization process, the numerical solution to the FDEs can be obtained. The efficiency and accuracy of the presented methods are shown by some numerical examples. Numerical tests show that this approach is easy to implement and accurate when applied to many types of FDEs.

1. Introduction

Fractional differential equations (FDEs) have been a hot topic in many scientific fields, such as dynamical system control theory, fluid flow, modelling in rheology, dynamic process of self-similar porous structure, diffusion transport similar to diffusion, electric network, and probability statistics [1–9]. These problems in science and engineering sometimes require us to get the solutions of various fractional differential equations. But as we know, it is difficult to find the exact solutions in most cases. So, we have to use numerical methods to solve fractional differential equations.

In the literature, some numerical methods for solving FDEs have been proposed, such as nonlinear functional analysis methods, including monotone iterative technique [10], topological degree theory [11], and fixed point theorem [12]. In addition, someone proposed the following numerical methods: random walk [13], Adomian decomposition method and variational iteration method [14], homotopy perturbation method [15–17], etc.

In recent years, some scholars try to use the neural network to solve differential equations [18–20]. Lagaris et al. [21] proposed an artificial neural network method for solving initial and boundary value problems. In their work, a trial solution is adopted and written as the sum of two parts. The first part satisfies the initial or boundary conditions and does not contain adjustable parameters while the construction of the second part does not affect the initial and boundary conditions. Then, the neural network is trained to satisfy the differential equation at many selected points. The question for this method is that it is difficult to construct the first part of the trial solution and this method cannot be applied to fractional partial differential equations.

Piscopo et al. [22] also introduced a method to find the numerical solutions of many types of differential equations. The proposed method does not depend on the trial solution and therefore has more flexibility in many cases. It can be used for solving many types of ODE and PDE. The two mentioned neural network techniques motivate us to develop more neural network methods to solve FDEs, but how to get the fractional order of the neural network is a difficult problem.

To overcome this difficulty, in this work, we use a triangle base neural network as basis function to propose an alternative method called triangle neural network methods. This paper is organized as follows. In Section 2, we study the fractional derivative of the triangle base neural network and present the numerical method for solving many types of FDEs. In Section 3, we show the efficiency of the proposed method by some numerical examples. Section 4 is the conclusion.

2. Fractional Derivative of Triangle Neural Network and Numerical Algorithm

2.1. Ordinary Fractional Differential Equation. To solve the following fractional initial value problem (1) and boundary value problem (2),

$$\begin{cases} D^{(\alpha)}y = f(x,y), \\ y^{(k)}(0) = y_k, \quad k = 0, 1, \cdots, m-1, \end{cases} \tag{1}$$

where $m = [\alpha]$:

$$\begin{cases} a(x)D^{(\alpha)}y + b(x)D^{(\beta)}y = f(x), \\ y(0) = y_0, y(l) = y_1, \end{cases} \tag{2}$$

where $a(x)$ and $b(x)$ are real functions, $1 < \alpha \le 2$, $0 < \beta \le 1$. $D^{(\alpha)}$ and $D^{(\beta)}$ are Caputo fractional derivative operators.

We consider the following triangle base neural network 1 (see Figure 1) to approximate the solution of problems (1) and (2), where w_j are weights for the neural networks and $C_j(x)$ are triangle base functions as the following:

$$C_j(x) = \begin{cases} \cos(jx), & j = 0, 1, 2, \cdots, N, \\ \sin[(j-N)x], & j = N+1, N+2, \cdots, 2N, \end{cases} \tag{3}$$

where $C_j(x)$ are activation function of neurons in the hidden layer of the above neural network and N is an integer and $x \in [0, \pi]$.

Let the weight matrix be $w = [w_0, w_1, \cdots, w_{2N}]^T$ and the activation matrix be $C(x) = [c_0(x), c_1(x), \cdots, c_{2N}(x)]^T$. The triangle base neural network can be written as

$$y = \sum_{j=0}^{2N} w_j C_j(x). \tag{4}$$

When this neural network is used to be the numerical solution of problem (1), the loss function is

$$J = \frac{1}{2}\sum_{k=1}^{m} e^2(t) = \frac{1}{2}\sum_{k=1}^{m}\left[D^{(\alpha)}y(x_t) - f(x_t, y(x_t))\right]^2. \tag{5}$$

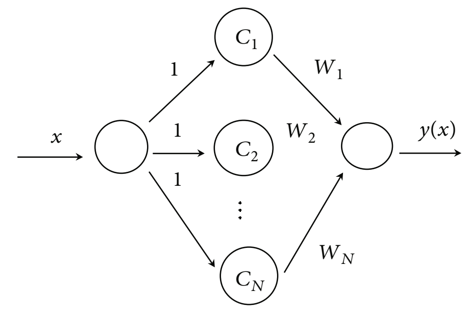

Figure 1: Triangle neural network 1.

For problem (2), The loss function is

$$J = \frac{1}{2}\sum_{k=1}^{m} e^2(t) = \frac{1}{2}\sum_{k=1}^{m}\left[a(x_t)D^{(\alpha)}y(x_t) + b(x_t)D^{(\beta)}y(x_t) - f(x_t)\right]^2, \tag{6}$$

where $x_t, t = 1, 2, \cdots, m$ are training points. We have two methods to minimize the loss function to get the corresponding numerical solution. One is the gradient descent algorithm, and another one is the optimization process. For both methods, we need to compute the α derivative of the triangle neural network. For this purpose, we have the following theorems.

Theorem 1. *For given $\alpha \in R^+$, $f(x) \in C^1(R)$, then*

$$D^{(\alpha)}f(\lambda x) = \lambda^\alpha f(u)|_{u=\lambda x}. \tag{7}$$

Proof. Since $D^{(\alpha)}f(x) = 1/\Gamma(1-\alpha)\int_0^x f'(\tau)/(x-\tau)^\alpha d\tau$.

We have

$$D^{(\alpha)}f(\lambda x) = \frac{1}{\Gamma(1-\alpha)}\int_0^x \frac{[f(\lambda\tau)]'}{(x-\tau)^\alpha}d\tau. \tag{8}$$

Let $\lambda\tau = u$, we have

$$\begin{aligned} D^{(\alpha)}f(\lambda x) &= \frac{1}{\Gamma(1-\alpha)}\int_0^{\lambda x}\frac{\lambda f'(u)}{(x-u/\lambda)^\alpha}du \\ &= \frac{1}{\Gamma(1-\alpha)}\int_0^{\lambda x}\frac{f'(u)}{(\lambda x-u)^\alpha\lambda^{-\alpha}}du \\ &= \frac{1}{\Gamma(1-\alpha)}\int_0^{\lambda x}\frac{\lambda^\alpha f'(u)}{(\lambda x-u)^\alpha}du \\ &= \frac{\lambda^\alpha}{\Gamma(1-\alpha)}\int_0^{\lambda x}\frac{f'(u)}{(\lambda x-u)^\alpha}du = \lambda^\alpha D^{(\alpha)}f(\lambda x). \end{aligned} \tag{9}$$

We also have the following.

Theorem 2. *Given $\alpha \in R^+$, $\lambda, b \in R$, then*

$$D^{(\alpha)}[\sin(\lambda x + b)] = \lambda^\alpha \sin\left(\lambda x + b + \frac{\alpha\pi}{2}\right),$$
$$D^{(\alpha)}[\cos(\lambda x + b)] = \lambda^\alpha \cos\left(\lambda x + b + \frac{\alpha\pi}{2}\right). \quad (10)$$

Proof.

$$D^{(\alpha)}[\sin(\lambda x + b)] = D^{(\alpha)}[\sin(\lambda x)\cos b + \cos(\lambda x)\sin b]. \quad (11)$$

From Theorem 1, we have

$$\begin{aligned}&\cos b D^{(\alpha)}\sin(\lambda x) + \sin b D^{(\alpha)}\cos(\lambda x)\\ &= \cos b \lambda^\alpha \sin\left(\lambda x + \frac{\alpha\pi}{2}\right) + \sin b \lambda^\alpha \sin\left(\lambda x + \frac{\alpha\pi}{2}\right)\\ &= \lambda^\alpha \sin\left(\lambda x + b + \frac{\alpha\pi}{2}\right).\end{aligned} \quad (12)$$

The second part of this theorem can be verified in the same way. Based on Theorems 1 and 2, we can get the α derivative of the triangle base neural network.

In fact, let the solution to problems (1) and (2) be

$$\begin{aligned}y &= \sum_{j=0}^{N} w_j \cos jx + \sum_{j=0}^{N} w_{N+j}\sin jx\\ &= w_0 + \sum_{j=1}^{N} w_j \cos jx + \sum_{j=1}^{N} w_{N+j}\sin jx.\end{aligned} \quad (13)$$

We can get

$$D^{(\alpha)}y = \sum_{j=1}^{N} j^\alpha w_j \cos\left(jx + \frac{\alpha}{2}\pi\right) + \sum_{j=1}^{N} j^\alpha w_{N+j}\sin\left(jx + \frac{\alpha}{2}\pi\right). \quad (14)$$

Thus, we get the loss function for problem (1):

$$\begin{aligned}J &= \frac{1}{2}\sum_{k=1}^{m} e^2(t) = \frac{1}{2}\sum_{k=1}^{m}\left[D^{(\alpha)}y(x_t) - f(x_t, y(x_t))\right]^2\\ &= \frac{1}{2}\sum_{t=1}^{m}\sum_{j=1}^{N}\left[\left(j^\alpha w_j \cos\left(jx_t + \frac{\alpha}{2}\pi\right)\right.\right.\\ &\left.\left.\quad + j^\alpha w_{N+j}\sin\left(jx_t + \frac{\alpha}{2}\pi\right)\right) - f(x_t, y(x_t))\right]^2.\end{aligned} \quad (15)$$

To carry out the gradient descent process, we have

$$\begin{aligned}\frac{\partial J}{\partial w_k} &= \left[\sum_{t=1}^{m}\sum_{j=0}^{N} j^\alpha\left(w_j \cos\left(jx_t + \frac{\alpha}{2}\pi\right)\right.\right.\\ &\left.\left.\quad + w_{N+j}\sin\left(jx + \frac{\alpha}{2}\pi\right)\right) - f(x, y)\right]\\ &\quad \cdot\left(k^\alpha \cos\left(kx_t + \frac{\alpha}{2}\pi\right) - f_y(x_t, y)\cos(kx_t)\right)\end{aligned} \quad (16)$$

for $j = 0, 1, 2, \cdots, N$, and we also have

$$\begin{aligned}\frac{\partial J}{\partial w_k} &= \left[\sum_{t=1}^{m}\sum_{j=0}^{N} j^\alpha w_j \cos\left(jx_t + \frac{\alpha}{2}\pi\right)\right.\\ &\left.\quad + w_{N+j}\sin\left(jx_t + \frac{\alpha}{2}\pi\right)\right) - f(x_t, y)\Bigg]\\ &\quad \cdot\left(k^\alpha \sin\left(kx_t + \frac{\alpha}{2}\pi\right) - f_y(x_t, y)\sin(k - N)x_t\right)\end{aligned} \quad (17)$$

for $j = N + 1, N + 2, \cdots, 2N$.

So, we can see that getting the numerical solution of (1) is equivalent to finding w_js by minimizing the loss function J. Usually, we have two methods to do this work. One is the gradient descent method, and another one is adopting the optimization process.

The gradient descent method is as below:

$$w_k^{(n+1)} = w_k^{(n)} - \eta\frac{\partial J}{\partial w_k}, \quad k = 1, 2, \cdots, 2N, \quad (18)$$

where η is the step size for the gradient descent. If the function $f(x, y)$ is a linear function of y, the initial value problem can also be reduced to

$$\begin{aligned}&\min \quad J = \frac{1}{2}\sum_{k=1}^{m}\left[D^{(\alpha)}y_k - f(x_k, y_k)\right]^2\\ &\text{s.t.} \qquad y(0) = y_0.\end{aligned} \quad (19)$$

That is,

$$\begin{aligned}&\min \quad J = \frac{1}{2}\sum_{k=1}^{m}\left[D^{(\alpha)}y_k - f(x_k, y_k)\right]^2\\ &\text{s.t.} \qquad \sum_{i=0}^{N} w_i = y_0,\end{aligned} \quad (20)$$

which is a quadratic optimization problem. For fractional

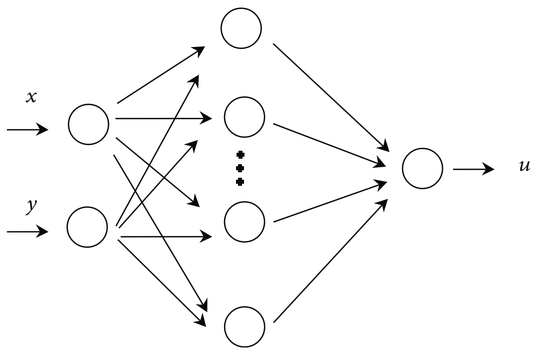

Figure 2: Neural network 2.

boundary value problem (2), the numerical solution can be reduced to the following optimization process:

$$\min \quad J = \frac{1}{2}\sum_{k=1}^{m}\left[a(x_k)D^{(\alpha)}y_k + b(x_k)D^{(\beta)}y_k - f(x_k)\right]^2$$
$$\text{s.t.} \quad \begin{cases} \sum_{i=1}^{N} w_i = y_0, \\ \sum_{i=0}^{N} \cos(il)w_i + \sin\,(N+i)lw_{N+i} = y_1. \end{cases} \tag{21}$$

So, there are two methods to solve this problem. One method is to get the solution through the gradient descent method. Another method is using the optimization technique.

2.2. Fractional Partial Differential Equation. For fractional partial differential equation problem

$$\begin{cases} \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = a\frac{\partial^2 u(x,t)}{\partial x^2} + f(t,x), \\ u(0,t) = u_0(t), u(1,t) = u_1(t), u(x,0) = v(x), \end{cases} \tag{22}$$

where $0 < \alpha \le 1$, and problem

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = a\frac{\partial^\beta u(x,t)}{\partial x^\beta} + f(t,x), \\ u(0,t) = u_0(t), u(1,t) = u_1(t), u(x,0) = v(x), \end{cases} \tag{23}$$

where $1 < \beta \le 2$. We use the triangle base neural network (see Figure 2) to approximate the solution of problems (22) and (23). The triangle base neural network can be written as

$$u(x,y) = \sum_{i=1}^{N} w_i \sin\,(a_{i1}x + a_{i2}y - b_i), \tag{24}$$

where a_{i1}, a_{i2} are weights for the import layer in the neural network, b_i are bias parameters for the hidden layer in the neural network, w_i are weights for the export layer in the neural network, and sin (x) is the activation function of neurons in the hidden layer.

Based on Theorems 1 and 2, we can get the fractional derivative of the neural network as below:

$$\begin{aligned} \frac{\partial^\alpha u(x,y)}{\partial x^\alpha} &= \sum_{i=1}^{N} w_i a_{i1}^\alpha \sin\left(a_{i1}x + a_{i2}y + b_i + \frac{\alpha}{2}\pi\right), \\ \frac{\partial^\beta u(x,y)}{\partial y^\beta} &= \sum_{i=1}^{N} w_i a_{i2}^\beta \sin\left(a_{i1}x + a_{i2}y + b_i + \frac{\alpha}{2}\pi\right). \end{aligned} \tag{25}$$

The loss function for problem (22) is

$$\min J = \frac{1}{2}\sum_{k=1}^{m}\left[\frac{\partial^\alpha u}{\partial t_k^\alpha} - \frac{\partial^2 u}{\partial x_k^2} - f(t_k, x_k)\right]^2 + \frac{1}{2}\sum_{(t_k,x_k)\in B}^{m}[u(t_k,x_k) - B(t_k,x_k)]^2, \tag{26}$$

where (t_k, x_k) are training points and B is the boundary of problem (22) and

$$\begin{aligned} B(0,t) &= u_0(t), \\ B(l,t) &= u_1(t), \\ B(x,0) &= v(x). \end{aligned} \tag{27}$$

The loss function for problem (23) can be given in the same way.

We use the gradient descent algorithm to train the neural network. In fact, we can train the neural network by layers. First, we train the export layer to get w_is, then the bias parameters to get b_is in the hidden layer, and finally, we train the import layer to get a_{i1}s and a_{i2}s.

3. Numerical Experiment

3.1. Numerical Test 1. Consider the following example 1:

$$\begin{cases} D^{(\alpha)}y = \frac{1}{x^2+1}y + \frac{\Gamma(3)}{\Gamma(3-\alpha)}x^{2-\alpha}, \\ y(0) = 1, \end{cases} \tag{28}$$

where $0 < \alpha \le 1$. The exact solution to this problem is $y = x^2 + 1$. We let α be $0.5, 0.9$ and use the optimization method when $N = 5, N = 15$ and gradient descent method when $N = 5, N = 15$, respectively. The computational results are listed in Tables 1 and 2.

3.2. Numerical Test 2. Consider the following example 2 for boundary value problem:

$$\begin{cases} D^{(1+\alpha)}y + D^{(\alpha)}y = \sqrt{2}\sin\left(x+\frac{\pi}{4}\right)\cos\frac{\alpha}{2} + \sqrt{2}\cos\left(x+\frac{\pi}{4}\right)\sin\frac{\alpha}{2}, \\ y(0) = 0, y\left(\frac{\pi}{2}\right) = 1, \end{cases} \tag{29}$$

Table 1: Computational errors for example 1 where $\alpha = 0.5, h = 0.1$.

x	1 h	2 h	3 h	4 h	5 h	6 h	7 h	8 h	9 h	10 h
optimi ($N = 5$)	0.010	0.013	0.015	0.016	0.018	0.019	0.021	0.022	0.024	0.026
optimi ($N = 15$)	0.003	0.003	0.008	0.009	0.012	0.014	0.013	0.014	0.015	0.016
grad, $N = 5$	0.011	0.013	0.017	0.019	0.025	0.021	0.020	0.019	0.027	0.030
grad, $N = 15$	0.008	0.008	0.007	0.004	0.005	0.009	0.012	0.018	0.015	0.013

Table 2: Computational errors for example 1 where $\alpha = 0.9, h = 0.1$.

x	1 h	2 h	3 h	4 h	5 h	6 h	7 h	8 h	9 h	10 h
optimi ($N = 5$)	0.000	0.010	0.010	0.011	0.012	0.014	0.018	0.017	0.018	0.020
optimi ($N = 15$)	0.000	0.002	0.006	0.008	0.010	0.010	0.011	0.010	0.012	0.012
grad, $N = 5$	0.009	0.013	0.012	0.014	0.021	0.020	0.020	0.016	0.021	0.021
grad, $N = 15$	0.005	0.006	0.003	0.002	0.007	0.012	0.013	0.010	0.010	0.010

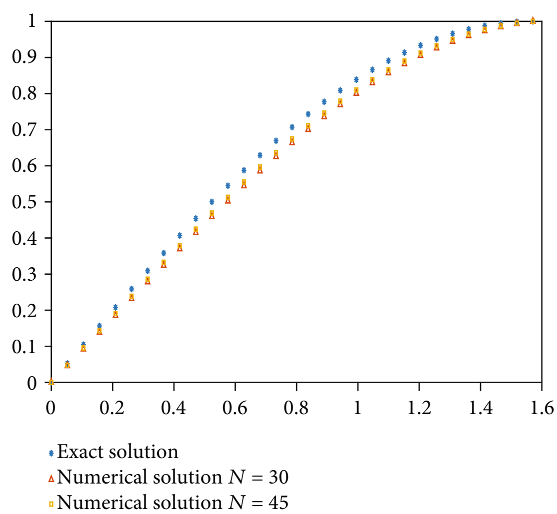

Figure 3: Solution of Test 2 when $\alpha = 0.5$.

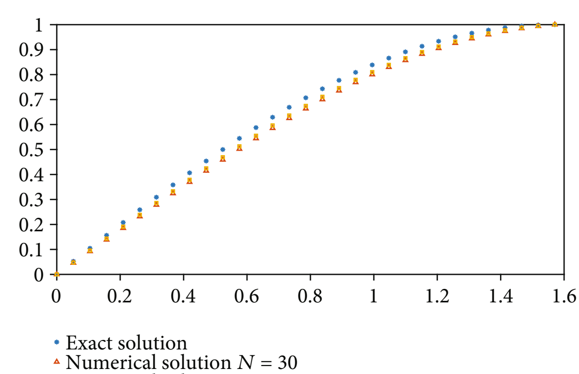

Figure 4: Solution of Test 2 when $\alpha = 0.8$.

where $0 < \alpha \leq 1$. The exact solution to this problem is $y = \sin x$.

We let α be 0.5 and $N = 30, N = 45$, respectively; the computational error is listed in Figure 3.

We let α be 0.8 and $N = 20, N = 30$. The computational error is listed in Figure 4.

As we see in example 1, the solution becomes more accurate when N is increased. And for the boundary value problem, we use two constraints when we use the optimization process.

3.3. Numerical Test 3. Consider the following example 3:

$$\begin{cases} \dfrac{\partial^\alpha u(x,t)}{\partial t^\alpha} = -\dfrac{\Gamma(4)}{\Gamma(4-\alpha)} t^{-\alpha} \dfrac{\partial^2 u(x,t)}{\partial x^2}, \\ u(0,t) = u(\pi,t) = 0, u(x,0) = 0, \end{cases} \tag{30}$$

where $0 < \alpha \leq 1$. The exact solution to this problem is $u = t^3 \sin x$.

We use the gradient descent method to solve this problem, and the computational error is listed in Figure 5. In the training process of the neural network, we set a stopping criteria $J < 10^{-5}$ for the computing process to stop. If this stopping criteria cannot be achieved, the computing will be stopped when 10^5 times of training is completed.

3.4. Numerical Test 4. Consider the following example 4:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \dfrac{\partial^\beta u(x,t)}{\partial x^\beta} - t^3 \sin\left(x + \dfrac{\beta}{2}\pi\right) + 3t^2 \sin x, \\ u(0,t) = u(\pi,t) = 0, u(x,0) = 0, \end{cases} \tag{31}$$

where $1 < \beta \leq 2$. The exact solution to this problem is $u = t^3 \sin x$.

We use the gradient descent method to solve this problem, and the computational error is listed in Figure 6.

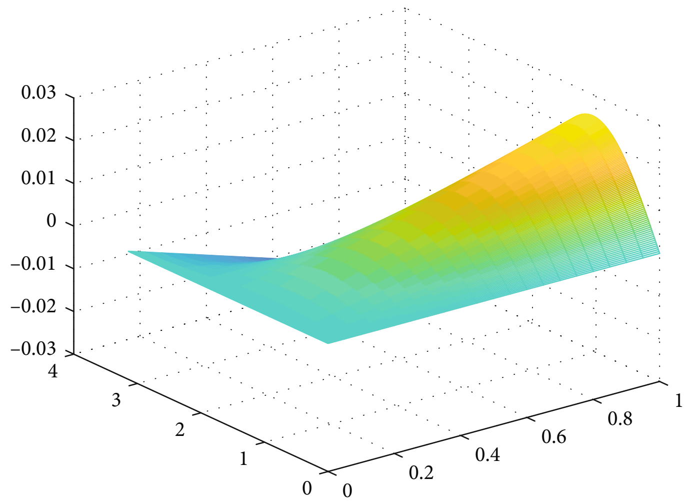

FIGURE 5: Solution of Test 3 when $\alpha = 0.8$.

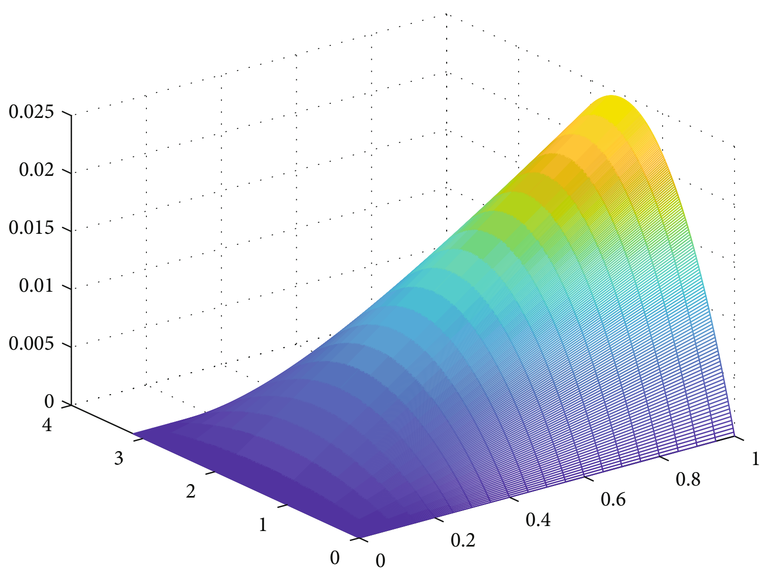

FIGURE 6: Solution of Test 4 when $\alpha = 0.8$.

4. Conclusion

The neural network method is a promising approach for solving fractional differential equations. The difficulty for this method is how to calculate the fractional derivatives of the involved neural network. In this paper, we propose numerical methods for solving fractional differential equations including the initial problem, boundary value problem, and partial FDEs by using the triangle base neural network and gradient descending method. All the involved fractional derivatives in this work are considered as Caputo type. We first analyze the fractional derivative of the triangle base neural network. Then, based on the loss function, the proposed numerical methods reduce the fractional differential equation into the gradient descent process or the quadratic optimization problem. By carrying out the gradient descent process or the quadratic optimization process, we can get the numerical solutions. Numerical tests show that this approach is easy to implement and the solution is accurate when applied to many types of FDEs.

Data Availability

All the data supporting this study are available within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly to this paper. All authors read and approved the final manuscript.

Acknowledgments

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Research Article

Toeplitz Operators whose Symbols Are Borel Measures

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In this paper, we are concerned with Toeplitz operators whose symbols are complex Borel measures. When a complex Borel measure μ on the unit circle is given, we give a formal definition of a Toeplitz operator T_μ with symbol μ, as an unbounded linear operator on the Hardy space. We then study various properties of T_μ. Among them, there is a theorem that the domain of T_μ is represented by a trichotomy. Also, it was shown that if the domain of T_μ contains at least one polynomial, then T_μ is densely defined. In addition, we give evidence for the conjecture that T_μ with a singular measure μ reduces to a trivial linear operator.

1. Introduction

A classical Toeplitz operator is the compression of a multiplication operator on the Lebesgue space $L^2(\mathbb{T})$ of the unit circle $\mathbb{T}$ to the Hardy space $H^2(\mathbb{T})$. The study of Toeplitz operators seems to have originated from the paper of Toeplitz [1]. In the paper [2], he used Toeplitz matrices to characterize nonnegative continuous functions on the unit circle in terms of their Fourier coefficients. The remarkable paper of Brown and Halmos [3] started the systematic study of spectral properties of Toeplitz operators. Since then, the theory of Toeplitz operators has been studied in various ways. Recently, the theory of Toeplitz operators has been studied in a variety of settings and connections with other fields. One direction is to deal with Toeplitz operators on reproducing kernel spaces like Bergman spaces, Dirichlet spaces, or Fock spaces (cf. [4–8]). Another direction is to study Toeplitz operators with operator-valued symbols (cf. [9–11]). Also, truncated Toeplitz operators have attracted attention. A systematic approach on truncated Toeplitz operators can be found in the paper of Sarason in 2007 [12]. In that paper, he has used "compatible" measures to describe bounded truncated Toeplitz operators. The boundedness of infinite Hankel matrices is also related to the compatibility of measures: the infinite Hankel matrix of the moment of a nonnegative Carleson measure is bounded and vice versa [13]. (For related recent studies, see [14].) These works inspired us to consider Toeplitz operators whose symbols are measures. The Toeplitz operators whose symbols are measures have been studied in the setting of Bergman spaces and other spaces (cf. [15], chapter 7).

In this paper, we consider Toeplitz operators on the Hardy space, whose symbols are measures. In this study, unbounded Toeplitz operators arise naturally. When studying unbounded Toeplitz operators, it was usually considered that the symbols come from $L^2(\mathbb{T})$. In 2008, Sarason [16] treated not only the case of $L^2(\mathbb{T})$-symbols but the case of analytic functions on the open unit disk $\mathbb{D}$. It is natural to attempt to extend the symbols of Toeplitz operators to measures, because the initial reasearch for them was related to the moment problem. As mentioned before, Toeplitz and Hankel operators associated with measures can be seen in the papers [13] and [12]. In this paper, we provide an explicit definition of Toeplitz operators whose symbols are complex Borel measures and then consider their unbounded operator theory. As the study on Toeplitz operators whose symbols are functions shows the interplay between function theory and operator theory, the study on Toeplitz operators whose symbols are measures is also expected to show the interplay between measure theory and operator theory.

Our consideration for the symbol of a Toeplitz operator, denoted by T_μ, is a complex Borel measure μ on the unit circle. When we study an unbounded linear operator, we usually assume that its domain is dense, i.e., the operator is densely defined. Hence, one may ask if T_μ is densely defined, i.e., the domain is dense in H^2. Toeplitz operators with L^2-symbols are always densely defined. Unlike when the symbol is a function, it does not seem easy to answer the question. Nonetheless, we will show that the domain of T_μ is represented by a trichotomy (Theorem 8). In particular, we can show that if the domain of T_μ contains at least one polynomial, then T_μ is densely defined (Proposition 10). We also give evidence for the conjecture that the cases of singular measures induce trivial linear operators (Theorem 15).

The organization of this paper is as follows. In Section 2, we give notations, definitions, and preliminary facts, which will be used in the sequel. In Section 3, we give a formal definition of Toeplitz operators whose symbols are complex Borel measures on $\mathbb{T}$ and then investigate their properties in the viewpoint of unbounded linear operator theory.

2. Preliminaries

Let $\mathbb{T}$ be the unit circle in the complex plane. Let m be the normalized Lebesgue measure on $\mathbb{T}$, so that $m(\mathbb{T})=1$. For $1\le p\le\infty$, we write $L^p(\mathbb{T})=L^p(\mathbb{T},m)$ for the Lebesgue space on $\mathbb{T}$ and $H^p(\mathbb{T})$ for the Hardy space on $\mathbb{T}$. Note that $H^p(\mathbb{T})$ is a closed subspace of $L^p(\mathbb{T})$.

Let $\mathbb{D}$ be the open unit disk and let $\bar{\mathbb{D}}$ be the closed unit disk in the complex plane. Let $C_A(\mathbb{D})$ denote the disk algebra, i.e., the set of all continuous functions on $\bar{\mathbb{D}}$ which is analytic on $\mathbb{D}$.

For $1\le p\le\infty$, we write $H^p(\mathbb{D})$ for the Hardy space on $\mathbb{D}$. Two spaces $H^p(\mathbb{D})$ and $H^p(\mathbb{T})$ are identified via nontangential limits and Poisson integral. Thus, we often write H^p to denote the both of them. The norm in $L^p(\mathbb{T})$ (or $H^p(\mathbb{D})$) will be denoted by $\|\cdot\|_p$ and the inner product in $L^2(\mathbb{T})$ (or $H^2(\mathbb{D})$) will be denoted by $\langle\cdot,\cdot\rangle$. We refer the reader to the texts [17–19] and [20] for details of Hardy spaces.

The shift operator and its adjoint are one of the most interesting operators on the Hardy space. For convenience, we define them on $H(\mathbb{D})$, the class of all analytic functions on $\mathbb{D}$. For $f\in H(\mathbb{D})$, define

$$\begin{aligned} Sf(z) &= zf(z)(z\in\mathbb{D}),\\ S^*f(z) &= \frac{f(z)-f(0)}{z}(z\in\mathbb{D}). \end{aligned} \tag{1}$$

The operators S and S^* are often called the unilateral shift and the backward shift, respectively. We refer the reader to the text [21] which treats the shift operator in great detail.

One of the most remarkable theorems in analysis is Beurling's theorem (cf. [18, 20, 22]), which characterizes all S-invariant subspaces of H^2. (We use the term "subspace" for a closed linear subspace.) For a nonzero subspace M of H^2, M is S-invariant if and only if

$$M=\theta H^2=\{\theta f: f\in H^2\}, \tag{2}$$

for some inner function $\theta\in H^\infty$. A bounded analytic function θ on $\mathbb{D}$ is called an inner function if its radial limit $\theta^*(e^{it})=\lim_{r\to1-}\theta(re^{it})$ has a unit modulus for almost all $e^{it}\in\mathbb{T}$. If an inner function has no zero in $\mathbb{D}$, we call it a singular inner function.

Let $M(\mathbb{T})$ be the set of all complex (finite) Borel measures on $\mathbb{T}$. Note that $M(\mathbb{T})$ is a Banach space with the total variation norm $\|\mu\|=|\mu|(\mathbb{T})$, where $|\mu|$ is the total variation measure of μ. We may regard the normalized Lebesgue measure m as a finite positive Borel measure. Hence, $m\in M(\mathbb{T})$. We write $\mathscr{B}_\mathbb{T}$ for the σ-algebra of all Borel sets in $\mathbb{T}$. We say μ is singular if $\mu\perp m$.

Suppose that $\mu\in M(\mathbb{T})$. For any function $f\in L^1(\mathbb{T},|\mu|)$, let $f\cdot\mu$ denote the complex Borel measure on $\mathbb{T}$ defined by

$$(f\cdot\mu)(E)=\int_E f\,d\mu(E\in\mathscr{B}_\mathbb{T}). \tag{3}$$

Then, $|f\cdot\mu|=|f|\cdot|\mu|$. Hence, $\|f\cdot\mu\|=\|f\|_{L^1(\mathbb{T},|\mu|)}$. In particular, for every $f\in C(\mathbb{T})$, the measure $f\cdot\mu$ is defined and $\|f\cdot\mu\|\le\|f\|_\infty\|\mu\|$.

For $\mu\in M(\mathbb{T})$, the nth Fourier–Stieltjes coefficient of μ is given by

$$\widehat{\mu}(n)=\int_\mathbb{T}\bar{\zeta}^n\,d\mu(\zeta)(n\in\mathbb{Z}). \tag{4}$$

For any $\mu\in M(\mathbb{T})$, the bilateral sequence $\widehat{\mu}=\{\widehat{\mu}(n)\}_{n\in\mathbb{Z}}$ is bounded and the mapping $\mu\mapsto\widehat{\mu}$ is a bounded linear transformation from $M(\mathbb{T})$ into $\ell^\infty(\mathbb{Z})$. Note that the mapping $\mu\mapsto\widehat{\mu}$ is one-to-one, and hence, a measure $\mu\in M(\mathbb{T})$ is completely determined by its Fourier–Stieltjes coefficients. By the theorem of F. and M. Riesz, if $\mu\in M(\mathbb{T})$ is analytic, i.e., $\widehat{\mu}(n)=0$ for all $n\le0$, then $\mu\ll m$ and $d\mu/dm\in H^1(\mathbb{T})$; in other words, $\mu=f\cdot m$ for some $f\in H^1(\mathbb{T})$.

For the definition of Toeplitz operators whose symbols are measures, we use the Cauchy transform as the "projection" of measures. For this reason, we use the notation $P\mu$ instead of $K\mu$ for the Cauchy transform of μ. We refer the reader to the text [23] for thorough treatments of the Cauchy transform. For $\mu\in M(\mathbb{T})$, the analytic function $P\mu$ on $\mathbb{D}$, given by

$$(P\mu)(z)=\int_\mathbb{T}\frac{1}{1-\bar{\zeta}z}\,d\mu(\zeta)=\sum_{n=0}^\infty\widehat{\mu}(n)z^n(z\in\mathbb{D}), \tag{5}$$

is called the Cauchy transform of μ. Clearly, the mapping P is a linear transformation from $M(\mathbb{T})$ into $H(\mathbb{D})$. We may regard $f\in L^1(\mathbb{T})$ as the absolutely continuous measure $f\cdot m\in M(\mathbb{T})$. Hence, we denote $P(f\cdot m)$ by Pf, i.e.,

$$(Pf)(z)=\int_\mathbb{T}\frac{f(\zeta)}{1-\bar{\zeta}z}\,dm(\zeta)=\sum_{n=0}^\infty\widehat{f}(n)z^n(z\in\mathbb{D}). \tag{6}$$

(Clearly, $\widehat{f\cdot m}(n)=\widehat{f}(n)$.) As we have identified $H^2(\mathbb{D})$ with $H^2(\mathbb{T})$, the mapping P may be regarded as the orthogonal projection of $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$ (the so-called Riesz projection).

Let $\varphi \in L^2(\mathbb{T})$. The *Toeplitz operator T_φ with symbol φ* is the linear operator on H^2 with domain

$$\mathscr{D}(T_\varphi) = \{f \in H^2(\mathbb{D}): P(\varphi f) \in H^2(\mathbb{D})\}, \quad (7)$$

given by

$$T_\varphi f = P(\varphi f)(f \in \mathscr{D}(T_\varphi)). \quad (8)$$

(Recall that every function in $H^2(\mathbb{D})$ may be identified with its nontangential limit function which belongs to $H^2(\mathbb{T})$.) Clearly, $C_A(\mathbb{D}) \subseteq \mathscr{D}(T_\varphi)$. Hence, T_φ is densely defined. Also, T_φ is closed. Observe that

$$_\varphi z^j, z^i = \varphi, z^{i-j} = \widehat{\varphi}(i-j), \quad (9)$$

for every $i, j \in \mathbb{N} \cup \{0\}$. Hence, the matrix representation of T_φ with respect to the orthonormal basis $\{1, z, z^2, \cdots\}$ is

$$\begin{bmatrix} \widehat{\varphi}(0) & \widehat{\varphi}(-1) & \widehat{\varphi}(-2) & \cdots \\ \widehat{\varphi}(1) & \widehat{\varphi}(0) & \widehat{\varphi}(-1) & \cdots \\ \widehat{\varphi}(2) & \widehat{\varphi}(1) & \widehat{\varphi}(0) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \quad (10)$$

A matrix of this form is called a *Toeplitz matrix*; in other words, an infinite matrix $\{\alpha_{i,j}\}_{i,j\geq 0}$ is called a Toeplitz matrix if

$$\alpha_{i,j} = \alpha_{i+1,j+1}, \quad (11)$$

for every $i, j \in \mathbb{N} \cup \{0\}$.

For a bilateral sequence $s = \{s_n\}_{n\in\mathbb{Z}}$ of complex numbers, we denote by $T(s)$ the infinite Toeplitz matrix corresponding to s, i.e., $T(s)$ is the infinite matrix whose (i, j)-entry is s_{i-j}. Note that if $\varphi \in L^2(\mathbb{T})$, then the matrix representations of T_φ is $T(\widehat{\varphi})$. For $n \in \mathbb{N} \cup \{0\}$, we denote by $T_n(s)$ the $(n+1) \times (n+1)$ Toeplitz matrix corresponding to s, i.e.,

$$T_n(s) = \begin{bmatrix} s_0 & s_{-1} & \cdots & s_{-n} \\ s_1 & s_0 & \cdots & s_{-n+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_n & s_{n-1} & \cdots & s_0 \end{bmatrix}. \quad (12)$$

3. The Main Results

Let μ be a complex Borel measure on $\mathbb{T}$. For any function $f \in C_A(\mathbb{D})$, $f \cdot \mu$ is a complex Borel measure on $\mathbb{T}$, and hence, the Cauchy transform $P(f \cdot \mu)$ is an analytic function on $\mathbb{D}$. Define

$$\mathscr{D}(T_\mu) = \{f \in C_A(\mathbb{D}): P(f \cdot \mu) \in H^2(\mathbb{D})\}. \quad (13)$$

It is easy to show that $\mathscr{D}(T_\mu)$ is a linear manifold of $H^2(\mathbb{D})$. Now define

$$T_\mu f = P(f \cdot \mu)(f \in \mathscr{D}(T_\mu)). \quad (14)$$

Then, T_μ is a linear operator on $H^2(\mathbb{D})$ with domain $\mathscr{D}(T_\mu)$.

Definition 1. The operator T_μ is called the Toeplitz operator with symbol μ.

We begin with the following:

Proposition 2. *Suppose that $\mu \ll m$ and the Radon–Nikodym derivative $\varphi = d\mu/dm$ belongs to $L^2(\mathbb{T})$. Then, $\mathscr{D}(T_\mu) = C_A(\mathbb{D})$ and*

$$T_\mu f = T_\varphi f, \quad (15)$$

for every $f \in C_A(\mathbb{D})$.

Proof. Suppose that $\mu = \varphi \cdot m$, where $\varphi \in L^2(\mathbb{T})$. Let f be an arbitrary function in $C_A(\mathbb{D})$. Then,

$$P(f \cdot \mu)(z) = \int_{\mathbb{T}} \frac{f(\zeta)}{1 - \bar{\zeta} z} d\mu(\zeta) = \int_{\mathbb{T}} \frac{f(\zeta)\varphi(\zeta)}{1 - \bar{\zeta} z} dm(\zeta) = P(\varphi f)(z), \quad (16)$$

for every $z \in \mathbb{D}$, and so, $P(f \cdot \mu) = P(\varphi f)$. Since $\varphi f \in L^2(\mathbb{T})$, it follows that $P(f \cdot \mu) \in H^2(\mathbb{D})$. Hence, $f \in \mathscr{D}(T_\mu)$ and

$$T_\mu f = P(f \cdot \mu) = P(\varphi f) = T_\varphi f. \quad (17)$$

This completes the proof.

Proposition 2 shows that the notion of T_μ is a kind of generalization of the Toeplitz operators whose symbols are L^2-functions.

Remark 3.

(a) *Toeplitz operators with L^1 -symbols*: every function $\varphi \in L^1(\mathbb{T})$ would be regarded as the absolutely continuous measure $\varphi \cdot m \in M(\mathbb{T})$. Hence, we may use Definition 1 to define Toeplitz operators with L^1-symbols: if $\varphi \in L^1(\mathbb{T})$ and $\mu = \varphi \cdot m$, then

$$\mathscr{D}(T_\mu) = \{f \in C_A(\mathbb{D}): P(\varphi f) \in H^2(\mathbb{D})\}, \quad (18)$$

$$T_\mu f = P(\varphi f), \quad (19)$$

for $f \in \mathscr{D}(T_\mu)$.

(b) *Toeplitz operators with H^1 -symbols*: let $\varphi \in H^1(\mathbb{T})$ and put $\mu = \varphi \cdot m \in M(\mathbb{T})$. For every $f \in C_A(\mathbb{D})$, $\varphi f \in H^1(\mathbb{T})$. Hence, $P(\varphi f) = \varphi f$ (if we view φ in the right-hand side as a function in $H^1(\mathbb{D})$). It follows that

$$\mathcal{D}(T_\mu) = \{f \in C_A(\mathbb{D}) : \varphi f \in H^2(\mathbb{D})\}, \tag{20}$$

$$T_\mu f = \varphi f, \tag{21}$$

for $f \in \mathcal{D}(T_\mu)$. This shows that a Toeplitz operator with H^1-symbol behaves as a multiplication. Notice that the action of T_μ is the same as that of T_φ defined in ([16], Section 5). (In that paper, the domain of T_φ is given by $\mathcal{D}(T_\varphi) = \{f \in H^2(\mathbb{D}) : \varphi f \in H^2(\mathbb{D})\}$.) Moreover, since φ is of Smirnov class, $\varphi = b/a$ for some $a, b \in H^\infty(\mathbb{D})$ such that a is an outer function, $a(0) > 0$, and $|a|^2 + |b|^2 = 1$ on $\mathbb{T}$. In this case, $\mathcal{D}(T_\varphi) = aH^2(\mathbb{D})$ (cf. [16]). It follows that

$$\mathcal{D}(T_\mu) = \mathcal{D}(T_\varphi) \cap C_A(\mathbb{D}) = aH^2(\mathbb{D}) \cap C_A(\mathbb{D}). \tag{22}$$

Since a is an outer function, it follows that $aH^2(\mathbb{D})$ is dense in $H^2(\mathbb{D})$.

Question: is $aH^2 \cap C_A(\mathbb{D})$ dense in H^2?

We give some concrete examples.

Example 4.

(a) Let φ be the analytic function on $\mathbb{D}$ such that $(\varphi(z))^2 = (1-z)^{-1}$ and $\varphi(0) = 1$. Then, $\varphi \in H^1(\mathbb{D})$ but $\varphi \notin H^2(\mathbb{D})$. Put $\mu = \varphi \cdot m$. By Remark 3, (b), we have

$$\mathcal{D}(T_\mu) = \{f \in C_A(\mathbb{D}) : \varphi f \in H^2(\mathbb{D})\}. \tag{23}$$

How large is the domain $\mathcal{D}(T_\mu)$? Suppose that $g \in C_A(\mathbb{D})$ and $g(1) \neq 0$. Then, there exists a constant $c > 0$ such that $|g| \geq c$ on a neighborhood of $\zeta = 1$. It follows that $\varphi g \notin H^2(\mathbb{D})$. Hence, $g \notin \mathcal{D}(T_\mu)$. This shows that

$$\mathcal{D}(T_\mu) \subseteq \{f \in C_A(\mathbb{D}) : f(1) = 0\}. \tag{24}$$

On the other hand, if $r > 0$ and if ψ_r is the function in $C_A(\mathbb{D})$ which satisfies $(\psi_r(z))^{1/r} = 1 - z$ and $\psi_r(0) = 1$, then, for every $g \in C_A(\mathbb{D})$,

$$\begin{aligned}
\|\varphi\psi_r g\|_2^2 &= \int_{\mathbb{T}} |\varphi(\zeta)|^2 |\psi_r(\zeta)|^2 |g(\zeta)|^2 \, dm(\zeta) \\
&= \int_{\mathbb{T}} \frac{|1-\zeta|^{2r}}{|1-\zeta|} |g(\zeta)|^2 \, dm(\zeta) \\
&\leq \|g\|_\infty^2 \cdot \int_{\mathbb{T}} |1-\zeta|^{2r-1} \, dm(\zeta) \\
&= \frac{\|g\|_\infty^2}{2\pi} \int_{-\pi}^{\pi} \left|1 - e^{it}\right|^{2r-1} dt \\
&\leq \frac{\|g\|_\infty^2}{2\pi} \int_{-\pi}^{\pi} |t|^{2r-1} \, dt \\
&= \frac{\|g\|_\infty^2}{\pi} \frac{\pi^{2r}}{2r},
\end{aligned} \tag{25}$$

and hence, $\varphi\psi_r g \in H^2(\mathbb{D})$, i.e., $\psi_r g \in \mathcal{D}(T_\mu)$. It follows that

$$\bigcup_{r>0} \psi_r C_A(\mathbb{D}) \subseteq \mathcal{D}(T_\mu). \tag{26}$$

Since $\psi_1 = 1 - z$, we have

$$(1-z) \cdot C_A(\mathbb{D}) \subseteq \mathcal{D}(T_\mu). \tag{27}$$

In particular, $\mathcal{D}(T_\mu)$ contains all polynomials vanishing at $\zeta = 1$.

(b) Let $\mu = \delta_1$ be the unit point mass concentrated at $\zeta = 1$. Note that the measure μ is discrete. Observe that, for $f \in C_A(\mathbb{D})$,

$$P(f \cdot \mu)(z) = \int_{\mathbb{T}} \frac{f(\zeta)}{1 - \bar{\zeta} z} \, d\mu(\zeta) = \frac{f(1)}{1-z} (z \in \mathbb{D}). \tag{28}$$

Since $1/(1-z) = \sum_{n=0}^{\infty} z^n$, the function $1/(1-z)$ does not belong to $H^2(\mathbb{D})$. It follows that $P(f \cdot \mu) \in H^2(\mathbb{D})$ if and only if $f(1) = 0$. Therefore,

$$\mathcal{D}(T_\mu) = \{f \in C_A(\mathbb{D}) : f(1) = 0\}. \tag{29}$$

Also, we have

$$T_\mu f = 0, \tag{30}$$

for all $f \in \mathcal{D}(T_\mu)$. Hence, T_μ is trivial, i.e., $T_\mu f = 0$ for all $f \in \mathcal{D}(T_\mu)$. Consequently, T_μ is bounded (on $\mathcal{D}(T_\mu)$). Notice that $\mathcal{D}(T_\mu)$ does not contain the constant function 1. We show later (see Remark 11) that $\mathcal{D}(T_\mu)$ is dense in $H^2(\mathbb{D})$.

(c) *The Cantor middle-third measure*: let C denote the Cantor ternary set and let φ be the Cantor function, i.e., for $x = \sum_{j=1}^{\infty} (a_j/3^j) \in C$,

$$\varphi(x)=\sum_{j=1}^{\infty}\frac{a_j/2}{2^j}, \tag{31}$$

and $\varphi(x)=\sup\{\varphi(y): y<x, y\in C\}$ for $x\notin C$. Then, φ is continuous and monotonically increasing. Hence, there exists a positive Borel measure μ on $\mathbb{T}$ such that

$$\mu\left(\left\{e^{2\pi i\theta}: 0\le\theta<t\right\}\right)=\varphi(t)(0\le t\le 1). \tag{32}$$

The measure μ (the so-called Cantor middle-third measure) is a typical example of a singular continuous measure. We refer the reader to the papers [24] and [25] which treat measures of the Cantor type. It is known that

$$\widehat{\mu}(n)=(-1)^n\prod_{j=1}^{\infty}\cos\frac{2\pi n}{3^j}\ (n\in\mathbb{Z}). \tag{33}$$

Hence,

$$|\mu\wedge(n)|^2=\prod_{j=1}^{\infty}\left(1-\sin^2\frac{2\pi n}{3^j}\right)(n\in\mathbb{Z}). \tag{34}$$

Since $0\le\sin^2(2\pi n/3^j)<1$ for each j and $\sum_{j=1}^{\infty}\sin^2(2\pi n/3^j)<\infty$, it follows that $\widehat{\mu}(n)\ne 0$. Note also that $\widehat{\mu}(-n)=\widehat{\mu}(n)$ and $\widehat{\mu}(3n)=\widehat{\mu}(n)$ for every $n\in\mathbb{Z}$. We may here ask the following questions:

(a) What is $\mathscr{D}(T_\mu)$? Is $\mathscr{D}(T_\mu)$ dense in $H^2(\mathbb{D})$?

(b) What is T_μ? Is T_μ trivial?

We next ask: *when is the domain $\mathscr{D}(T_\mu)$ dense in $H^2(\mathbb{D})$?* It does not seem easy to answer this question in general. The following lemma is used to derive some properties of $\mathscr{D}(T_\mu)$ which are helpful to determine the density of $\mathscr{D}(T_\mu)$ in $H^2(\mathbb{D})$. Recall that S is the shift operator on $H(\mathbb{D})$, i.e., if $f\in H(\mathbb{D})$, then $Sf(z)=zf(z)$ for $z\in\mathbb{D}$.

We then have the following:

Lemma 5. *For every $\mu\in M(\mathbb{T})$ and $f\in C_A(\mathbb{D})$,*

$$P(Sf\cdot\mu)=SP(f\cdot\mu)+P(Sf\cdot\mu)(0). \tag{35}$$

Proof. For each $z\in\mathbb{D}$,

$$\begin{aligned}P(Sf\cdot\mu)(z)-P(Sf\cdot\mu)(0)&=\int_{\mathbb{T}}\frac{\zeta f(\zeta)}{1-\bar\zeta z}\,d\mu(\zeta)-\int_{\mathbb{T}}\zeta f(\zeta)\,d\mu(\zeta)\\&=\int_{\mathbb{T}}\frac{\bar\zeta z}{1-\bar\zeta z}\zeta f(\zeta)\,d\mu(\zeta)\\&=z\int_{\mathbb{T}}\frac{f(\zeta)}{1-\bar\zeta z}\,d\mu(\zeta)\\&=zP(f\cdot\mu)(z)\\&=SP(f\cdot\mu)(z).\end{aligned} \tag{36}$$

The following proposition gives an important information for the domain of T_μ.

Proposition 6. *Let $\mu\in M(\mathbb{T})$ and let α be a complex number such that $|\alpha|\ne 1$. Then, the following statements hold:*

(a) For $f\in C_A(\mathbb{D})$, $f\in\mathscr{D}(T_\mu)$ if and only if $(S-\alpha)f\in\mathscr{D}(T_\mu)$

(b) For $f\in H^2(\mathbb{D})$, $f\in cl_{H^2}(\mathscr{D}(T_\mu))$ if and only if $(S-\alpha)f\in cl_{H^2}(\mathscr{D}(T_\mu))$

Proof. (a) Suppose that $f\in C_A(\mathbb{D})$. Then, by Lemma 5,

$$\begin{aligned}P((S-\alpha)f\cdot\mu)&=P(Sf\cdot\mu)-P(\alpha f\cdot\mu)\\&=SP(f\cdot\mu)+P(Sf\cdot\mu)(0)-\alpha P(f\cdot\mu)\\&=(S-\alpha)P(f\cdot\mu)+P(Sf\cdot\mu)(0).\end{aligned} \tag{37}$$

Hence, $P((S-\alpha)f\cdot\mu)\in H^2(\mathbb{D})$ if and only if $(S-\alpha)P(f\cdot\mu)\in H^2(\mathbb{D})$. Since $P(f\cdot\mu)\in H(\mathbb{D})$ and $|\alpha|\ne 1$, it follows that $P(f\cdot\mu)\in H^2(\mathbb{D})$ if and only if $(S-\alpha)P(f\cdot\mu)\in H^2(\mathbb{D})$. Therefore, $f\in\mathscr{D}(T_\mu)$ if and only if $(S-\alpha)f\in\mathscr{D}(T_\mu)$. This proves (a).

(b) Suppose that $f\in H^2(\mathbb{D})$ and $f\in cl_{H^2}(\mathscr{D}(T_\mu))$. Then, there exists a sequence $\{f_j\}$ in $\mathscr{D}(T_\mu)$ such that $\|f-f_j\|_2\longrightarrow 0$. Since $S-\alpha$ is a bounded operator on $H^2(\mathbb{D})$, we have

$$\|(S-\alpha)f-(S-\alpha)f_j\|_2=\|(S-\alpha)\left(f-f_j\right)\|_2\longrightarrow 0. \tag{38}$$

By (a), each $(S-\alpha)f_j$ belongs to $\mathscr{D}(T_\mu)$. It follows that $(S-\alpha)f\in cl_{H^2}(\mathscr{D}(T_\mu))$.

Conversely, suppose that $f\in H^2(\mathbb{D})$ and $(S-\alpha)f\in cl_{H^2}(\mathscr{D}(T_\mu))$. Then, there exists a sequence $\{g_j\}$ in $\mathscr{D}(T_\mu)$ such that

$$\|(S-\alpha)f-g_j\|_2\longrightarrow 0. \tag{39}$$

We want to show that $f\in cl_{H^2}(\mathscr{D}(T_\mu))$. To see this we consider two cases.

Case 1. ($|\alpha|<1$). Assume first that $g_j(\alpha)=0$ for all j. Then,

$$g_j=(S-\alpha)f_j, \tag{40}$$

where $f_j\in C_A(\mathbb{D})$. Since $g_j\in\mathscr{D}(T_\mu)$, it follows from (a) that $f_j\in\mathscr{D}(T_\mu)$. Note that the approximate point spectrum of the operator S on $H^2(\mathbb{D})$ is $\sigma_{\text{ap}}(S)=\mathbb{T}$ (cf. [26]). Since α does not belong to $\mathbb{T}$, the operator $S-\alpha$ is bounded below on $H^2(\mathbb{D})$. It follows that there exists a constant $c>0$ such that

$$\|(S-\alpha)f-g_j\|_2=\|(S-\alpha)\left(f-f_j\right)\|_2\ge c\cdot\|f-f_j\|_2 \tag{41}$$

for all j. This implies that $\|f-f_j\|_2\to 0$. Therefore, $f\in cl_{H^2}(\mathscr{D}(T_\mu))$.

In the case that $g_j(\alpha) \neq 0$ for some j, we may assume that $g_1(\alpha) \neq 0$. Note that $g_j \longrightarrow (S-\alpha)f$ weakly. Hence, $g_j(z) \longrightarrow ((S-\alpha)f)(z)$ for each $z \in \mathbb{D}$. In particular, we have

$$g_j(\alpha) \longrightarrow 0. \quad (42)$$

Now put

$$h_j = g_j - \frac{g_j(\alpha)}{g_1(\alpha)} g_1 \quad (j = 1, 2, 3, \cdots). \quad (43)$$

Then, $h_j \in \mathscr{D}(T_\mu)$ and $h_j(\alpha) = 0$ for all j. Observe that

$$\|(S-\alpha)f - h_j\|_2 \leq \|(S-\alpha)f - g_j\|_2 + |\frac{g_j(\alpha)}{g_1(\alpha)}|\|g_1\|_2. \quad (44)$$

It follows that

$$\|(S-\alpha)f - h_j\|_2 \rightarrow 0. \quad (45)$$

Hence, by the preceding paragraph, we conclude that $f \in cl_{H^2}(\mathscr{D}(T_\mu))$.

Case 2. ($|\alpha| > 1$). The operator $S-\alpha$ on $H^2(\mathbb{D})$ is invertible. Hence,

$$\|f - (S-\alpha)^{-1} g_j\|_2 \longrightarrow 0. \quad (46)$$

Since $(S-\alpha)^{-1} = -\sum_{n=0}^{\infty} S^n/\alpha^{n+1}$ and $\mathscr{D}(T_\mu)$ is S-invariant by (a), each $(S-\alpha)^{-1} g_j$ belongs to $cl_{H^2}(\mathscr{D}(T_\mu))$. It follows that $f \in cl_{H^2}(\mathscr{D}(T_\mu))$, and the proof is complete.

Remark 7. If we take $\alpha = 0$ in Proposition 6, then the linear subspaces $\mathscr{D}(T_\mu)$ and its closure $cl_{H^2}(\mathscr{D}(T_\mu))$ are S-invariant. Also, the equality in Lemma 5 can be rewritten as $S^* P(Sf \cdot \mu) = P(f \cdot \mu)$. Consequently, we have $S^* T_\mu S f = T_\mu f$ for every $f \in \mathscr{D}(T_\mu)$.

As a consequence of Proposition 6, we derive the following theorem which describes the domain $\mathscr{D}(T_\mu)$. Recall that an inner function is said to be singular if it has no zero in the unit disk.

Theorem 8. *Let $\mu \in M(\mathbb{T})$. Then, one of the following holds:*

(i) $\mathscr{D}(T_\mu) = \{0\}$

(ii) $\mathscr{D}(T_\mu)$ is dense in $H^2(\mathbb{D})$

(iii) $cl_{H^2}(\mathscr{D}(T_\mu)) = \theta H^2(\mathbb{D})$, where θ is a singular inner function

Proof. By Proposition 6, $cl_{H^2}(\mathscr{D}(T_\mu))$ is an S-invariant subspace of $H^2(\mathbb{D})$. It follows from Beurling's theorem that

$$cl_{H^2}(\mathscr{D}(T_\mu)) = \theta H^2(\mathbb{D}), \quad (47)$$

where θ is an inner function or $\theta = 0$. If $\theta = 0$, then the case (i) occures. If θ is a nonzero constant function, case (ii) occurs. Now, suppose that θ is nonconstant. We show that θ has no zero in $\mathbb{D}$. To see this, choose any nonzero function f in $\mathscr{D}(T_\mu)$. Fix an arbitrary point α of $\mathbb{D}$ and let n be the multiplicity of the zero of f at α. Then,

$$f(z) = (z-\alpha)^n g(z) (z \in \mathbb{D}), \quad (48)$$

where $g \in C_A(\mathbb{D})$ and $g(\alpha) \neq 0$. Hence, by a repeated application of Proposition 6(a), we have

$$g \in \mathscr{D}(T_\mu) \subseteq \theta H^2(\mathbb{D}). \quad (49)$$

It follows that $g = \theta h$ for some $h \in H^2(\mathbb{D})$. Thus, $\theta(\alpha)$ cannot be 0. Since α was arbitrary, we conclude that θ has no zero in $\mathbb{D}$. Therefore θ is a singular inner function.

Remark 9. Unfortunately, we cannot find a concrete example for the third case. It would be possible that the third case never occurs.

The following proposition is another consequence of Proposition 6 which gives a sufficient condition for the domain $\mathscr{D}(T_\mu)$ to be dense in $H^2(\mathbb{D})$.

Proposition 10. *If $cl_{H^2}(\mathscr{D}(T_\mu))$ contains a polynomial, then $\mathscr{D}(T_\mu)$ is dense in $H^2(\mathbb{D})$.*

Proof. Suppose that $cl_{H^2}(\mathscr{D}(T_\mu))$ contains a polynomial. Then, by Proposition 6, (b), there exists a polynomial $p \in cl_{H^2}(\mathscr{D}(T_\mu))$, all of whose zeros are in $\mathbb{T}$, such that $p(0) = 1$. Let $\zeta_1, \cdots, \zeta_N \in \mathbb{T}$ be the zeros of p, listed according to their multiplicities. Then,

$$p(z) = \left(1 - \bar{\zeta_1} z\right) \cdots \left(1 - \bar{\zeta_N} z\right). \quad (50)$$

Choose a sequence $\{k_n\}$ in $\mathbb{N}$ such that $k_{n+1} > N k_n$ (e.g., $k_n = (N+1)^n$). For each $n \in \mathbb{N}$, define

$$p_n(z) = \frac{1}{n} \sum_{j=1}^{n} \left(1 - \left(\bar{\zeta_1} z\right)^{k_j}\right) \cdots \left(1 - \left(\bar{\zeta_N} z\right)^{k_j}\right). \quad (51)$$

All of them are polynomials, divisible by p. Since $cl_{H^2}(\mathscr{D}(T_\mu))$ is S-invariant, the polynomials p_n belong to $\mathscr{D}(T_\mu)$. It follows by a direct computation that

$$\|1 - p_n\|_2^2 \leq \frac{n}{n^2} \left[\binom{N}{1} \binom{N}{1} + \cdots + \binom{N}{N} \binom{N}{N} \right], \quad (52)$$

for every $n \in \mathbb{N}$. This implies that $p_n \longrightarrow 1$ in $H^2(\mathbb{D})$.

Therefore, the constant function 1 belongs to $cl_{H^2}(\mathscr{D}(T_\mu))$. Since $cl_{H^2}(\mathscr{D}(T_\mu))$ is S-invariant, we conclude that $cl_{H^2}(\mathscr{D}(T_\mu)) = H^2(\mathbb{D})$; in other words, $\mathscr{D}(T_\mu)$ is dense in $H^2(\mathbb{D})$.

Remark 11. Proposition 10 shows that the domains $\mathscr{D}(T_\mu)$, presented in (a) and (b) of Example 4, are dense in $H^2(\mathbb{D})$, because they contain the polynomial $p(z) = 1 - z$. The proof of Proposition 10 shows that every polynomial, all of whose zeros are in $\mathbb{T}$, is an outer function.

In order to consider the matrix representation of a linear operator on $H^2(\mathbb{D})$, it is necessary that its domain contains all polynomials. Let us interpret the condition that $\mathscr{D}(T_\mu)$ contains all polynomials. Note that this is equivalent to the condition that $\mathscr{D}(T_\mu)$ contains any polynomial which does not vanish on $\mathbb{T}$, by Proposition 6, (a).

Lemma 12. *Let $\mu \in M(\mathbb{T})$. Then, the following are equivalent:*

(i) $\mathscr{D}(T_\mu)$ contains all polynomials, or equivalently, $\mathscr{D}(T_\mu)$ contains the constant function 1

(ii) $\mu \ll m$ and $d\mu/dm \in H^2(\mathbb{T}) + \overline{H_0^1}(\mathbb{T})$

Proof. (i) ⇒ (ii): suppose that the constant function 1 belongs to $\mathscr{D}(T_\mu)$. Then, $P\mu = P(1\cdot\mu) \in H^2(\mathbb{D})$. Let ψ denote the non-tangential limit function of $P\mu$. Since $P\mu = \sum_{n=0}^{\infty} \widehat{\mu}(n) z^n$, it follows that $\widehat{\psi}(n) = \widehat{\mu}(n)$ for all $n \in \mathbb{N}\cup\{0\}$. Put $\nu = \mu - \psi\cdot m$. Then, $\nu \in M(\mathbb{T})$ and

$$\widehat{\nu}(n) = \widehat{\mu}(n) - \widehat{\psi}(n) = 0, \tag{53}$$

for all $n \in \mathbb{N}\cup\{0\}$. It follows from the F. and M. Riesz theorem that $\nu \ll m$ and $\nu = \chi\cdot m$ for some $\chi \in \overline{H_0^1}(\mathbb{T})$. Thus, we have $\mu = \nu + \psi\cdot m = (\chi+\psi)\cdot m$. This proves (ii).

(ii) ⇒ (i): suppose that (ii) holds so that $\mu = (\psi+\chi)\cdot m$ for some $\psi \in H^2(\mathbb{T})$ and $\chi \in \overline{H_0^1}(\mathbb{T})$. Then, $\widehat{\mu}(n) = \widehat{\psi}(n)$ for all $n \in \mathbb{N}\cup\{0\}$. Hence, we have

$$\sum_{n=0}^{\infty} |\mu\wedge(n)|^2 < \infty. \tag{54}$$

Since $P\mu = \sum_{n=0}^{\infty} \widehat{\mu}(n) z^n$, it follows that $P(1\cdot\mu) = P\mu \in H^2(\mathbb{D})$. Clearly, the constant function 1 belongs to $C_A(\mathbb{D})$. Therefore, $1 \in \mathscr{D}(T_\mu)$. Now, Proposition 6, (a), implies that $\mathscr{D}(T_\mu)$ contains all polynomials.

Corollary 13. *Let $\mu \in M(\mathbb{T})$ be a real measure. Then, $\mathscr{D}(T_\mu) = C_A(\mathbb{D})$ if and only if $\mu \ll m$ and $d\mu/dm \in L^2(\mathbb{T})$.*

Proof. Suppose that $\mathscr{D}(T_\mu) = C_A(\mathbb{D})$. Then, $\mu \ll m$ and $\mu = (\psi+\chi)\cdot m$ for some $\psi \in H^2(\mathbb{T})$ and $\chi \in \overline{H_0^1}(\mathbb{T})$ by Lemma 12. Since μ is a real measure, we have

$$\widehat{\mu}(-n) = \int_{\mathbb{T}} \bar{z}^{-n}\, d\mu = \overline{\int_{\mathbb{T}} \bar{z}^{n}\, d\mu} = \overline{\widehat{\mu}(n)}, \tag{55}$$

for every $n \in \mathbb{Z}$. Thus, $\widehat{\chi}(-n) = \overline{\widehat{\psi}(n)}$ for every $n \in \mathbb{N}$. Since $\psi \in H^2(\mathbb{T})$, we have

$$\sum_{n=-\infty}^{-1} |\chi\wedge(n)|^2 = \sum_{n=1}^{\infty} |\psi\wedge(n)|^2 < \infty. \tag{56}$$

It follows that $\chi \in \overline{H_0^2}(\mathbb{T})$. Therefore, $d\mu/dm = \psi + \chi \in L^2(\mathbb{T})$.

The converse is a part of Proposition 2.

On the other hand, we would like to conjecture the following:

Conjecture 14. *Every Toeplitz operator with a singular symbol is trivial.*

We give evidence for Conjecture 14 by using the known fact about the Cauchy transform. Let E be a closed subset of $\mathbb{T}$ and let

$$F(E) = \{g \in H^2(\mathbb{D})\colon g = P\mu \text{ for some } \mu \in M(E)\}. \tag{57}$$

Then, it is known that $F(E) = \{0\}$ if and only if $m(E) = 0$ (cf. [23], Theorem 5.5.2).

We then have the following:

Theorem 15. *If $\mu \in M(\mathbb{T})$ is singular and $m(\sup p\mu) = 0$, then T_μ is trivial.*

Proof. Let $E = \sup p\mu$. By assumption, $m(E) = 0$. Thus, $F(E) = \{0\}$. Suppose that $f \in \mathscr{D}(T_\mu)$, i.e., $f \in C_A(\mathbb{D})$ and $P(f\cdot\mu) \in H^2(\mathbb{D})$. Note that $\sup p(f\cdot\mu) \subseteq \sup p\mu = E$. Hence, $f\cdot\mu \in M(E)$. So the function $P(f\cdot\mu) \in H^2(\mathbb{D})$ belongs to $F(E) = \{0\}$. It follows that $P(f\cdot\mu) = 0$. We have shown that $P(f\cdot\mu) \in H^2(\mathbb{D})$ implies $P(f\cdot\mu) = 0$. In other words,

$$f \in \mathscr{D}(T_\mu)\, T_\mu f = 0. \tag{58}$$

Therefore T_μ is trivial (on its domain).

Remark 16. Conjecture 14 seems to be known when μ is a positive singular measure. Indeed, if μ is a positive singular measure, then its Poisson integral is the real part of $(1+\theta)/(1-\theta)$ for some inner function θ (cf. [23], Remark 9.1.4). Now, if $f \in C_A(\mathbb{D})$ and $P(f\cdot\mu) \in H^2(\mathbb{D})$, then the function

$g = (1-\theta)P(f \cdot \mu)$ belongs to $H^2(\mathbb{D})!\theta H^2(\mathbb{D})$ (cf. [27], Chapter III), and hence, $\theta\bar{g} \in zH^2(\mathbb{D})$. Since $1-\theta$ is the outer H^2-function, it follows that

$$P(\bar{f} \cdot \mu) = \frac{\bar{g}}{1\theta} = \frac{\bar{g}}{1-\bar{\theta}} = -\frac{\theta\bar{g}}{1-\theta}, \tag{59}$$

which implies that $P(\bar{f} \cdot \mu) \in zH^2(\mathbb{D})$. Therefore, $P(f \cdot \mu) = 0$.

The Cantor-middle-third measure μ in Example 4, (c), is a singular continuous measure, and its support is the Cantor set (in $\mathbb{T}$) whose Lebesgue measure is 0. Hence, Theorem 15 implies that T_μ is trivial.

We have seen that the Toeplitz operator T_μ in Example 4, (b), is a densely defined trivial linear operator. This result can be extended to the case that μ has a finite support. In this case, the fact that T_μ is trivial may follow from Theorem 15. However, we give a direct proof and also show that T_μ is densely defined.

Proposition 17. *Let $\mu \in M(\mathbb{T})$ be a discrete measure whose support is a finite set. Then, the Toeplitz operator T_μ is a densely defined trivial linear operator with domain*

$$\mathcal{D}(T_\mu) = \{f \in C_A(\mathbb{D}) : f(\zeta) = 0 \textit{ for every } \zeta \in \sup p\mu\}. \tag{60}$$

Proof. Suppose that $\sup p\mu$ consists of N distinct points $\zeta_1, \cdots, \zeta_N$ of $\mathbb{T}$. Then,

$$\mu = c_1\delta_{\zeta_1} + \cdots + c_N\delta_{\zeta_N}, \tag{61}$$

where $c_1, \cdots, c_N$ are nonzero complex numbers and δ_ζ is the unit point mass concentrated at ζ.

We first show that

$$\mathcal{D}(T_\mu) = \{f \in C_A(\mathbb{D}) : f(\zeta_1) = \cdots = f(\zeta_N) = 0\}. \tag{62}$$

For any $f \in C_A(\mathbb{D})$,

$$P(f \cdot \mu)(z) = \sum_{j=1}^{N} c_j P\left(f \cdot \delta_{\zeta_j}\right)(z) = \sum_{j=1}^{N} \frac{c_j f(\zeta_j)}{1-\bar{\zeta}_j z} (z \in \mathbb{D}). \tag{63}$$

It follows that

$$\{f \in C_A(\mathbb{D}) : f(\zeta_1) = \cdots = f(\zeta_N) = 0\} \subseteq \mathcal{D}(T_\mu). \tag{64}$$

Conversely, let $f \in \mathcal{D}(T_\mu)$. Then, $P(f \cdot \mu) \in H^2(\mathbb{D})$. For each j, put

$$F_j(\zeta) = \frac{c_j f(\zeta_j)}{1-\bar{\zeta}_j \zeta} (\zeta \in \mathbb{T}). \tag{65}$$

Then, $F = \sum_{j=1}^{N} F_j$ is the nontangential limit function of $P(f \cdot \mu)$. Thus, $F \in H^2(\mathbb{T})$. Choose disjoint open arcs $I_j \subseteq \mathbb{T}$ with $\zeta_j \in I_j$. Fix an index j_0 and let χ denote the characteristic function of I_{j_0}. Then, $\chi \cdot F \in L^2(\mathbb{T})$. Also, $\chi \cdot F_j \in L^\infty(\mathbb{T})$ for each $j \neq j_0$. Hence,

$$\chi \cdot F_{j_0} = \chi \cdot F - \sum_{j \neq j_0} \left(\chi \cdot F_j\right) \in L^2(\mathbb{T}). \tag{66}$$

Since $(1-\chi) \cdot F_{j_0} \in L^\infty(\mathbb{T})$, it follows that

$$F_{j_0} = \chi \cdot F_{j_0} + (1-\chi) \cdot F_{j_0} \in L^2(\mathbb{T}). \tag{67}$$

This implies that $f(\zeta_{j_0}) = 0$, because otherwise, $F_{j_0} \notin L^2(\mathbb{T})$. Since j_0 was arbitrary, we have $f(\zeta_j) = 0$ for each j. It follows that

$$\mathcal{D}(T_\mu) \subseteq \{f \in C_A(\mathbb{D}) : f(\zeta_1) = \cdots = f(\zeta_N) = 0\}. \tag{68}$$

This proves (62). In particular, $\mathcal{D}(T_\mu)$ contains the polynomial $p(z) = (z-\zeta_1) \cdots (z-\zeta_N)$. Hence, by Proposition 10, $\mathcal{D}(T_\mu)$ is dense in $H^2(\mathbb{D})$.

Equations (62) and (63) imply that $T_\mu f = 0$ for all $f \in \mathcal{D}(T_\mu)$, i.e., T_μ is trivial. This completes the proof.

Example 18. Let $\mu \in M(\mathbb{T})$ be a discrete measure whose support has only finitely many limit points, for example,

$$\mu = \sum_{n=1}^{\infty} \frac{1}{2^n} \delta_{\zeta_n}, \tag{69}$$

where $\zeta_n = e^{\pi i/2^n}$. By an argument similar to the proof of Proposition 17, we may show that

$$\mathcal{D}(T_\mu) = \{f \in C_A(\mathbb{D}) : f(\zeta) = 0 \text{ for every } \zeta \in \sup p\mu\}, \tag{70}$$

and $T_\mu f = 0$ for all $f \in \mathcal{D}(T_\mu)$. Hence, T_μ is trivial. Note that every polynomial has only finitely many zeros. It follows that $\mathcal{D}(T_\mu)$ cannot contain any polynomial. Nevertheless, $\mathcal{D}(T_\mu)$ contains a nonzero function by Fatou's theorem for $C_A(\mathbb{D})$, which says that, for any given closed set $K \subseteq \mathbb{T}$ with $m(K) = 0$, there exists a function in $C_A(\mathbb{D})$ which vanishes precisely on K (cf. [19]). Hence by Theorem 8, $\mathcal{D}(T_\mu)$ is dense in $H^2(\mathbb{D})$ or $cl_{H^2}(\mathcal{D}(T_\mu)) = \theta H^2(\mathbb{D})$ for some singular inner function θ. But it does not seem easy to determine whether $\mathcal{D}(T_\mu)$ is dense in $H^2(\mathbb{D})$ or not.

To each Toeplitz operator T_μ, there corresponds an infinite Toeplitz matrix $T(\widehat{\mu})$. In general, however, it is a bit awkward to call $T(\widehat{\mu})$ as the matrix representation of T_μ, because the domain $\mathcal{D}(T_\mu)$ may not contain the monomials z^n. Nevertheless, often, information about T_μ gives information about $T(\widehat{\mu})$. The following is one of such example.

Corollary 19. *Let $\mu \in M(\mathbb{T})$ be a discrete measure whose support consists of N points of $\mathbb{T}$. Then,*

$$\det T_n(\widehat{\mu}) = 0, \tag{71}$$

for all $n \geq N$.

Proof. Suppose that μ is the discrete measure given by (61). Then, the domain $\mathscr{D}(T_\mu)$ is given by (62). Choose any polynomial p in $\mathscr{D}(T_\mu)$ whose degree is N (e.g., $p(z) = (z - \zeta_1) \cdots (z - \zeta_N)$). Write $p = \sum_{k=0}^N a_k z^k$. Since $T_\mu z^k = \sum_{n=0}^\infty \widehat{\mu}(n - k) z^n$, it follows that

$$\begin{aligned} 0 = T_\mu p &= \sum_{k=0}^N a_k T_\mu z^k = \sum_{k=0}^N a_k \sum_{n=0}^\infty \widehat{\mu}(n-k) z^n \\ &= \sum_{n=0}^\infty \left(\sum_{k=0}^N a_k \widehat{\mu}(n-k) \right) z^n. \end{aligned} \tag{72}$$

Hence, we have

$$\sum_{k=0}^N a_k \widehat{\mu}(n-k) = 0, \tag{73}$$

for all $n \geq 0$. Now, let $n \geq N$ and put

$$x = [a_0 \ \cdots \ a_N \ 0 \ \cdots \ 0]^T \in \mathbb{C}^{n+1}. \tag{74}$$

Then, by (73), $T_n(\widehat{\mu})x = 0$, i.e., $x \in \ker T_n(\widehat{\mu})$. Since $x \neq 0$, the square matrix $T_n(\widehat{\mu})$ is not invertible, or equivalently, $\det T_n(\widehat{\mu}) = 0$.

Lastly, we may ask: what is the adjoint of T_μ ? To answer this question, we need the following:

Lemma 20. *Let $\mu \in M(\mathbb{T})$. Then,*

$$\langle T_\mu f, g \rangle = \int_\mathbb{T} f \bar{g} \, d\mu, \tag{75}$$

for every $f \in \mathscr{D}(T_\mu)$ and $g \in C_A(\mathbb{D})$.

Proof. Suppose that $f \in \mathscr{D}(T_\mu)$ and $g \in C_A(\mathbb{D})$. Then, $T_\mu f \in H^2(\mathbb{D})$. Write $T_\mu f = \sum_{n=0}^\infty a_n z^n$ and $g = \sum_{n=0}^\infty b_n z^n$. Then,

$$\langle T_\mu f, g \rangle = \sum_{n=0}^\infty a_n \bar{b}_n. \tag{76}$$

Observe that, for each $z \in \mathbb{D}$,

$$\begin{aligned} (T_\mu f)(z) &= \int_\mathbb{T} \frac{f(\zeta)}{1 - \bar{\zeta} z} \, d\mu(\zeta) = \int_\mathbb{T} f(\zeta) \sum_{n=0}^\infty \bar{\zeta}^n z^n \, d\mu(\zeta) \\ &= \sum_{n=0}^\infty \left[\int_\mathbb{T} f(\zeta) \bar{\zeta}^n \, d\mu(\zeta) \right] z^n. \end{aligned} \tag{77}$$

Hence, we have

$$a_n = \int_\mathbb{T} f(\zeta) \bar{\zeta}^n \, d\mu(\zeta). \tag{78}$$

Observe that, for each $0 < r < 1$,

$$g_r = \sum_{n=0}^\infty b_n r^n z^n \in C_A(\mathbb{D}). \tag{79}$$

It follows that

$$\begin{aligned} \langle T_\mu f, g_r \rangle &= \sum_{n=0}^\infty a_n \bar{b}_n r^n = \sum_{n=0}^\infty \int_\mathbb{T} f(\zeta) \bar{\zeta}^n \bar{b}_n r^n \, d\mu(\zeta) \\ &= \int_\mathbb{T} f(\zeta) \sum_{n=0}^\infty \overline{b_n r^n \zeta^n} \, d\mu(\zeta) = \int_\mathbb{T} f \bar{g}_r \, d\mu. \end{aligned} \tag{80}$$

If we let $r \longrightarrow 1$, then $\|g - g_r\|_\infty \longrightarrow 0$, and hence, $\langle T_\mu f, g_r \rangle \longrightarrow \langle T_\mu f, g \rangle$ and $\int_\mathbb{T} f \bar{g}_r \, d\mu \to \int_\mathbb{T} f \bar{g} \, d\mu$. This proves (75).

Assume that $\mu \in M(\mathbb{T})$ and $\mathscr{D}(T_\mu)$ is dense in $H^2(\mathbb{D})$. Then, the adjoint T_μ^* of T_μ can be defined; the domain of T_μ^* is

$$\mathscr{D}\left(T_\mu^*\right) = \left\{ g \in H^2(\mathbb{D}) \colon \exists h \in H^2(\mathbb{D}) \text{s.t.} \langle T_\mu f, g \rangle = \langle f, h \rangle \forall f \in \mathscr{D}(T_\mu) \right\}, \tag{81}$$

and, for each $g \in \mathscr{D}(T_\mu^*)$, $T_\mu^* g$ is the (unique) element of $H^2(\mathbb{D})$ such that

$$\langle T_\mu f, g \rangle = \left\langle f, T_\mu^* g \right\rangle, \tag{82}$$

for every $f \in \mathscr{D}(T_\mu)$.

If $\varphi \in L^\infty(\mathbb{T})$, then $T_\varphi^* = T_{\bar{\varphi}}$. Hence, it is reasonable to expect that the adjoint of T_μ is the Toeplitz operator induced by the "complex conjugation" of μ. For $\mu \in M(\mathbb{T})$, define

$$\bar{\mu}(E) = \overline{\mu(E)} (E \in \mathscr{B}_\mathbb{T}). \tag{83}$$

Then, $\bar{\mu} \in M(\mathbb{T})$. Of course, $\mu \in M(\mathbb{T})$ is a real measure if and only if $\bar{\mu} = \mu$. Note that

$$\widehat{\bar{\mu}}(n) = \overline{\widehat{\mu}(-n)}, \tag{84}$$

for every $n \in \mathbb{Z}$.

We now have the following:

Proposition 21. *Let $\mu \in M(\mathbb{T})$. Assume that $\mathscr{D}(T_\mu)$ is dense in $H^2(\mathbb{D})$. Then,*

$$T_{\bar{\mu}} \subseteq T_\mu^*, \tag{85}$$

that is $\mathscr{D}(T_{\bar{\mu}}) \subseteq \mathscr{D}(T_\mu^*)$ and $T_{\bar{\mu}} = T_\mu^*$ on $\mathscr{D}(T_{\bar{\mu}})$.

Proof. Let $g \in \mathscr{D}(T_{\bar{\mu}})$. By Lemma 20, it follows that

$$\langle T_\mu f, g\rangle = \int_{\mathbb{T}} f\bar{g}\, d\mu = \int_{\mathbb{T}} \overline{g\bar{f}\, d\bar{\mu}} = \left\langle f, T_{\bar{\mu}} g\right\rangle, \tag{86}$$

for every $f \in \mathscr{D}(T_\mu)$. It follows that $g \in \mathscr{D}(T_\mu^*)$ and $T_\mu^* g = T_{\bar{\mu}} g$. Therefore, we conclude that

$$\mathscr{D}\left(T_{\bar{\mu}}\right) \subseteq \mathscr{D}\left(T_\mu^*\right), \tag{87}$$

and $T_\mu^* g = T_{\bar{\mu}} g$ for every $g \in \mathscr{D}(T_{\bar{\mu}})$. This completes the proof.

If $\mu \in M(\mathbb{T})$, and T is the restriction of the Toeplitz operator T_μ to $cl_{H^2}(\mathscr{D}(T_\mu))$, then T is a densely defined linear operator. In this case, T^* is a linear operator from $H^2(\mathbb{D})$ onto $cl_{H^2}(\mathscr{D}(T_\mu))$. By the same argument as the proof of Proposition 21, we have $\mathscr{D}(T_{\bar{\mu}}) \subseteq \mathscr{D}(T^*)$ and $T^* g = T_{\bar{\mu}} g$ for $g \in \mathscr{D}(T_{\bar{\mu}})$.

We also have the following:

Proposition 22. *Let $\mu \in M(\mathbb{T})$ be positive. Then, the following hold:*

(a) T_μ is positive, i.e., $_\mu f, f \geq 0$ for all $f \in \mathscr{D}(T_\mu)$

(b) $\ker T_\mu = \{f \in C_A(\mathbb{D}) : f(\zeta) = 0$ for every $\zeta \in \operatorname{supp} \mu\}$

Proof. (a) Suppose that $\mu \geq 0$. Then, by Lemma 20, we have

$$\left\langle T_\mu f, f\right\rangle = \int_{\mathbb{T}} |f|^2\, d\mu \geq 0, \tag{88}$$

for every $f \in \mathscr{D}(T_\mu)$.

(b) Suppose that $\mu \in M(\mathbb{T})$ is positive. If $f \in \ker T_\mu$, then $\int_{\mathbb{T}} |f|^2\, d\mu = \langle T_\mu f, f\rangle = 0$. Hence, $f = 0\mu$-a.e. on $\mathbb{T}$. We show that $f = 0$ on supp μ. Assume to the contrary that $f(\zeta_0) \neq 0$ for some $\zeta_0 \in \sup p\mu$. Since $f \in C_A(\mathbb{D})$, there exist a constant $\varepsilon > 0$ and an open arc $I \subseteq \mathbb{T}$ with center ζ_0 such that $|f(\zeta)\,| \geq \varepsilon$ for all $\zeta \in I$. Since $\zeta_0 \in \operatorname{supp} \mu$, we have $\mu(I) > 0$. It follows that

$$\int_{\mathbb{T}} |f|^2\, d\mu \geq \int_I |f|^2\, d\mu \geq \varepsilon \cdot \mu(I) > 0, \tag{89}$$

which is a contradiction. Hence, $f(\zeta) = 0$ for all $\zeta \in \operatorname{supp} \mu$. Therefore,

$$\ker T_\mu \subseteq \{f \in C_A(\mathbb{D}) : f = 0 \text{ on } \operatorname{supp} \mu\}. \tag{90}$$

The reverse inclusion is trivial.

The operator T_μ may be positive even though μ is complex. For example, for any complex number α, the measure $\alpha \cdot \delta_1$ is trivial, and hence, it is positive.

We conclude with a remark on the boundedness of T_μ. It is well known (cf. [3]) that for $\varphi \in L^2(\mathbb{T})$, T_φ is bounded if and only if $\varphi \in L^\infty(\mathbb{T})$, in which case, $\|T_\varphi\| = \|\varphi\|_\infty$. If $\mu \geq 0$ and T_μ is bounded, then

$$\int_{\mathbb{T}} |f|^2\, d\mu \leq c \cdot \|f\|_2^2 \quad \left(f \in \mathscr{D}\left(T_\mu\right)\right). \tag{91}$$

Let us call a positive measure $\mu \in M(\mathbb{T})$ a compatible measure if μ satisfies (91) for all $f \in C_A(\mathbb{D})$. The word "compatible" comes from the paper [12]. One can show that the following statements are equivalent:

(i) μ is a compatible measure

(ii) $\mu \ll m$ and $d\mu/dm \in L^\infty(\mathbb{T})$

(iii) $\mathscr{D}(T_\mu)$ contains all polynomials and T_μ is bounded

If these conditions are satisfied and if $\varphi = d\mu/dm$, then $\mathscr{D}(T_\mu) = C_A(\mathbb{D})$ and

$$T_\mu f = T_\varphi f, \tag{92}$$

for every $f \in C_A(\mathbb{D})$. In (iii), we cannot reduce the condition that $\mathscr{D}(T_\mu)$ contains all polynomials to the condition that $\mathscr{D}(T_\mu)$ is dense in $H^2(\mathbb{D})$: there is a measure $\mu \in M(\mathbb{T})$ which is not compatible such that T_μ is densely defined and bounded (see Example 4, (b)).

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares there are no conflicts of interest.

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Research Article

Global Existence and Decay Estimates of Energy of Solutions for a New Class of p-Laplacian Heat Equations with Logarithmic Nonlinearity

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The present research paper is related to the analytical studies of p-Laplacian heat equations with respect to logarithmic nonlinearity in the source terms, where by using an efficient technique and according to some sufficient conditions, we get the global existence and decay estimates of solutions.

1. A Brief History and Contribution

Consider the following nonlinear p-Laplacian problem: equation with logarithmic nonlinearity:

$$\begin{cases} u_t - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + |u|^{p-2}u = |u|^{p-2}u \ln |u|, \ x \in \Omega, \ t > 0, \\ u(x,0) = u_0(x), x \in \Omega, \\ u(x,t) = 0, \ x \in \partial\Omega, \ t \geq 0, \end{cases} \tag{1}$$

where $\Omega \subset R^n$ is a bounded domain with smooth boundary $\partial\Omega$, the function u_0 is given initial data and exponent p verify

$$\begin{cases} 2 < p < \infty, & \text{if } n \leq p, \\ 2 < p < \dfrac{np}{n-p}, & \text{if } n > p. \end{cases} \tag{2}$$

In the last few decades, the researchers have shown significant interest in polynomial nonlinear terms in different areas, such as edge detection, viscoelasticity, engineering, electromagnetic, electrochemistry, cosmology, signal processing material science, turbulence, diffusion, physics, and acoustics. Many other problems in applied sciences are also modeled by linear and nonlinear evolutionary partial differential equations [1–13]. Various dynamical systems in physics and engineering are also modeled by using evolutionary differential equations. Many researchers have contributed a lot to provide an outstanding history of the evolutionary differential partial equations related to $p(x)$-Laplacian such as [13–17].

The majority of problems in science are nonlinear, and it is not easy to find its analytical solutions. The physical problems are mostly designed by using higher nonlinear partial differential equations (PDEs). It is found to be very difficult to find the exact or analytical solutions for such problems.

However, in the last several centuries, many scientists have made significant progress and adopted different techniques to study the analytical side of the nonlinear PDEs. Through recent years and in the literature on nonlinear PDEs, logarithmic nonlinearity has received much interest from mathematicians and physicists. If we read in recent research, we notice that logarithmic nonlinearity has been entered into nonrelativistic wave equations that describe spinning particles that move in an external electromagnetic field and in the relativistic wave equation for spinless particles (see, for example, [2, 4, 18, 19]). In addition to what we mentioned above, this type of nonlinearity is used in various branches of physics such as optics, nuclear physics, geophysics, and inflationary cosmology (to read about this in detail, see [18–31]). Given all the basic previous meanings in physics, the study of universal solutions of this type of nonlinear logarithms is of great interest on the part of mathematicians.

Recently, Wu and Xue in [32] gave the uniformly proof of energy decay of the solution using the multiplier method of the following problem:

$$u_{tt} - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) - \Delta u_t + |u_t|^{q-1}u_t = |u|^{p-1}u. \quad (3)$$

Moreover, the author in [33] studied the exponential and polynomial decay rate of solutions for seminar problem (3) by applying the inequality of Nakao.

On the another handle, for a Laplacian parabolic equation related to the logarithmic in the right-hand side, the authors in [24] gave the analytical side of the following problem:

$$u_t - \Delta u - \Delta u_t = u \ln u. \quad (4)$$

Then, in [27], Nhan and Truong studied the global existence, decay together with the blow up the solutions of the following problem:

$$u_t - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) - \Delta u_t = |u|^{p-2}u \ln |u|, \quad (5)$$

where $p > 2$. In addition, in [25], Cao and Liu gave for $1 < p < 2$, the blow up and global boundedness results of problem (5).

Most recently, in [14], Piskin et al. studied the p-Laplacian hyperbolic case

$$u_{tt} - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + |u|^{p-2}u + u_t = |u|^{p-2}u \ln |u|, \ x \in \Omega, \ t > 0. \quad (6)$$

Motivated by the last mentioned papers, especially [14], in this current research, we consider problem (1) with the presence of nonlinear diffusion $\Delta_p = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, logarithmic nonlinearity $|u|^{p-2}u \ln |u|$ together with a damping term which is an extension of the previous recent analytical study in [14], where the authors considered the hyperbolic case without damping terms. Our goal is to exploit a potential well method for problem (1) in order to obtain global existence and decay estimate of solutions. More precisely, we give the global existence and decay estimates of solutions under some sufficient conditions.

2. Preliminaries

In this section, we put the definitions and lemmas that we need in the rest of the paper:

$$\|u\|_p = \|u\|_{L^p(\Omega)}, \|u\|_{1,s} = \|u\|_{W_0^{1,p}(\Omega)} = \left(\|u\|_p + \|\nabla u\|_p\right)^{1/p}, \quad (7)$$

for $1 < p < \infty$. We denote the positive constants by C and C_i $(i = 1, 2, \cdots)$.

We give the function of energy by

$$E(t) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{p}\|u\|_p^p - \frac{1}{p}\int_\Omega \ln |u| u^p dx + \frac{1}{p^2}\|u\|_p^p. \quad (8)$$

Lemma 1. *$E(t)$ is a nonincreasing function, for $t \geq 0$*

$$E'(t) = -\|u_t\|^2 \leq 0. \quad (9)$$

Proof. Multiplying equation (1) by u_t and using the integration on Ω, we have

$$-\int_\Omega \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)u_t dx + \int_\Omega |u|^{p-2}uu_t dx + \int_\Omega u_t u_t dx = \int_\Omega u^{p-2}u \ln |u| u_t dx, \quad (10)$$

$$\frac{d}{dt}\left(\frac{1}{p}\|\nabla u\|_p^p + \frac{1}{p}\|u\|_p^p - \frac{1}{p}\int_\Omega \ln |u| u^p dx + \frac{1}{p^2}\|u\|_p^p\right) = -\|u_t\|^2. \quad (11)$$

Thus,

$$E'(t) = -\|u_t\|^2. \quad (12)$$

Lemma 2 (see [5, 14]). *Let u be any function $u \in W_0^{1,p}(R^n) \setminus \{0\}$. Then, for $p > 1$, $\mu > 0$*

$$p\int_{R^n} u^p \ln\left(\frac{|u|}{\|u\|_{L^p(R^n)}}\right)dx \leq \mu\int_{R^n} |\nabla u|^p dx - \frac{n}{p}\ln\left(\frac{p\mu e}{n\mathscr{L}_p}\right)\int_{R^n} |u|^p dx, \quad (13)$$

where

$$\mathscr{L}_p = \frac{p}{n}\left(\frac{p-1}{e}\right)^{p-1}\pi^{-p/2}\left[\frac{\Gamma(n/2+1)}{\Gamma(n(p-1)/p+1)}\right]^{p/n}. \quad (14)$$

Remark 3. Let $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ and by defining $u(x) = 0$ *for* $x \in R^n \setminus \Omega$, we can write

$$p\int_\Omega u^p \ln\left(\frac{|u|}{\|u\|_{L^p(\Omega)}}\right)dx \le \mu\int_\Omega |\nabla u|^p dx - \frac{n}{p}\ln\left(\frac{p\mu e}{n\mathscr{L}_p}\right)\int_\Omega |u|^p dx. \quad (15)$$

Lemma 4 (see [27]). *Let $\vartheta > 0$. Therefore, we can easy give the following result:*

$$\log s \le Cs^{\vartheta}, \quad (16)$$

$\forall s \in [1,\infty)$, *such as* $C = e^{-1}/\vartheta$.

Remark 5. According to Lemma 4, we have

$$s^p \log s \le Cs^{p+\vartheta}, s \in [1,\infty). \quad (17)$$

Lemma 6 (see [34]).

(i) For all function $u \in W_0^{1,p}(\Omega)$, we have

$$\|u\|_q \le B_{q,p}\|\nabla u\|_p, \quad (18)$$

for every $q \in [1,\infty]$ if $n \le p$, and $1 \le q \le np/(n-p)$ if $n > p$. We choose constant $B_{q,p}$ related only on Ω, p and q. Denote $B_{p,p}$ by B_p.

(ii) For every $u \in W_0^{1,p}(\Omega)$, $p \ge 1$ with $r \ge 1$, we get

$$\|u\|_q \le C\|\nabla u\|_p^{\alpha}\|u\|_r^{1-\alpha}, \quad (19)$$

where $C > 0$,

$$\alpha = \left(\frac{1}{r} - \frac{1}{q}\right)\left(\frac{1}{n} - \frac{1}{p} + \frac{1}{r}\right)^{-1}, \quad (20)$$

and we have the following:

(i) For $p \ge n = 1, r \le q \le \infty$

(ii) For $n > 1$ and $p < n, q \in [r, (np/n-p)]$ if $r \le np/n-p$ and $q \in [r, (np/n-p)]$ if $r \le np/n-p$

(iii) For $p = n > 1, r \le q < \infty$

(iv) For $p > n > 1, r \le q \le \infty$

3. Result of the Global Existence

We give in this section the proof of the global existence for (1). First, putting the following functionals:

$$J(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{p+1}{p^2}\|u\|_p^p - \frac{1}{p}\int_\Omega \ln|u|u^p dx, \quad (21)$$

$$I(u) = \|\nabla u\|_p^p + \|u\|_p^p - \int_\Omega \ln|u|u^p dx. \quad (22)$$

Hence, (21) and (22) give

$$J(u) = \frac{1}{p}I(u) + \frac{1}{p^2}\|u\|_p^p, \quad (23)$$

and we have

$$E(u) = J(u). \quad (24)$$

As in [35], the potential depth of the well is given as

$$0 < d = \inf_u\left\{\sup_{\lambda\ge 0} J(\lambda u) : u \in W_0^{1,p}(\Omega), \|u\|_p^p \ne 0\right\}, \quad (25)$$

$$0 < d = \inf_{u\in\aleph} J(t). \quad (26)$$

Hence, two sets can be assigned, the first stable W and the second V unstable by

$$W = \left\{u \in W_0^{1,p}(\Omega) : J(u) < d, I(u) > 0\right\} \cup \{0\}, \quad V = \left\{u \in W_0^{1,p}(\Omega) : J(u) < d, I(u) < 0\right\}. \quad (27)$$

Lemma 7. *Let u be all function $u \in W_0^{1,p}(\Omega) \setminus \{0\}$, $\|u\|_p^p \ne 0$ and let $(\lambda) = J(\lambda u)$. Hence, we have*

(i) $\lim_{\lambda\to\infty} g(\lambda) = -\infty, \lim_{\lambda\to 0^+} g(\lambda) = 0$

(ii) $I(\lambda u) = \lambda g'(\lambda)\begin{cases} >0, & 0 \le \lambda < \lambda^*, \\ =0, & \lambda = \lambda^*, \\ <0, & \lambda < \lambda^* < \infty, \end{cases}$

where

$$\lambda^* = \exp\left(\frac{\|\nabla u\|_p^p + \|u\|_p^p - \int_\Omega \ln|u||u|^p dx}{\|u\|_p^p}\right). \quad (28)$$

Proof.

(i) From $g(\lambda)$ which we get

$$g(\lambda) = J(\lambda u) = \frac{1}{p}\|\lambda\nabla u\|_p^p + \frac{p+1}{p^2}\|\lambda u\|_p^p - \frac{1}{p}\int_\Omega \ln|\lambda u|(\lambda u)^p dx = \frac{\lambda^p}{p}\|\nabla u\|_p^p + \frac{\lambda^p}{p}\left(\frac{p+1}{p} - \ln|\lambda|\right)\|u\|_p^p - \frac{\lambda^p}{p}\int_\Omega \ln|u||u|^p dx. \tag{29}$$

According to $\|u\|_p^p \neq 0$, we find $\lim_{\lambda\to\infty} g(\lambda) = -\infty$, and $\lim_{\lambda\to 0} g(\lambda) = 0$.

(ii) From the derivative of $g(\lambda)$, we get

$$g'(\lambda) = \frac{d}{d\lambda}J(\lambda u) = \lambda^{p-1}\left(\|\nabla u\|_p^p + (1-\ln|\lambda|)\|u\|_p^p - \int_\Omega \ln|u||u|^p dx\right). \tag{30}$$

There exists a unique λ^* verify $(d/d\lambda)J(\lambda u)|_{\lambda=\lambda^*}$, by taking

$$\lambda^* = \exp\left(\frac{\|\nabla u\|_p^p + \|u\|_p^p - \int_\Omega \ln|u||u|^p dx}{\|u\|_p^p}\right). \tag{31}$$

Of course, we note that the recent property is the result of the following:

$$\lambda\frac{dJ(\lambda u)}{d\lambda} = \lambda g'(\lambda) = I(\lambda u). \tag{32}$$

Thus, we have the desired results such that

$$I(\lambda u) = \lambda g'(\lambda)\begin{cases} >0, & 0 \le \lambda < \lambda^*, \\ =0, & \lambda = \lambda^*, \\ <0, & \lambda < \lambda^* < \infty. \end{cases} \tag{33}$$

Lemma 8. *For every $u \in W_0^{1,p}(\Omega)\setminus\{0\}$ and $l = e^{(n+p^2)/p^2}(p^2/n\mathscr{L}_p)^{n/p^2}$, we get*

(i) If $0 < \|u\|_p < l$, then $I(u) > 0$

(ii) If $I(u) < 0$, then $\|u\|_p > l$

(iii) If $I(u) = 0$, then $\|u\|_p \ge l$

Proof. According to inequality of logarithmic Sobolev, it can be found

$$\begin{aligned} I(u) &= \|\nabla u\|_p^p + \|u\|_p^p - \int_\Omega \left(\ln\frac{|u|}{\|u\|_p} + \ln\|u\|_p\right)|u|^p dx \\ &\ge \|\nabla u\|_p^p + \left(1 - \ln\|u\|_p\right)\|u\|_p^p \\ &\quad - \left[\frac{\mu}{p}\int_\Omega |\nabla u|^p dx - \frac{n}{p^2}\ln\left(\frac{p\mu e}{n\mathscr{L}_p}\right)\int_\Omega |u|^p dx\right] \\ &\ge \left(1-\frac{\mu}{p}\right)\|\nabla u\|_p^p + \left(1 - \ln\|u\|_p + \frac{n}{p^2}\ln\left(\frac{p\mu e}{n\mathscr{L}_p}\right)\right)\|u\|_p^p. \end{aligned} \tag{34}$$

Selecting $\mu = p$ in (34) gives

$$I(u) \ge \left(1 - \ln\|u\|_p + \frac{n}{p^2}\ln\left(\frac{p^2 e}{n\mathscr{L}_p}\right)\right)\|u\|_p^p. \tag{35}$$

Thus, we have

(i) If $0 < \|u\|_p < l$, then $I(u) > 0$ using the last inequality

(ii) Suppose that $I(u) < 0$. This is due to (35), and it

$$\|u\|_p \ge e^{(n+p^2)/p^2}\left(\frac{p^2}{n\mathscr{L}_p}\right)^{n/p^2} = l \tag{36}$$

(iii) Similar to the proof of (ii), we prove (iii)

As for functional J, it represents the Nehari manifold

$$\aleph = \left\{u \in W_0^{1,p}(\Omega)\setminus\{0\}: I(u) = 0\right\}. \tag{37}$$

Using Lemma 7 in order to prove that $\aleph$ is an unempty set, consider that if $u \in \aleph$, we obtain

$$J(u) = \frac{1}{p^2}\|u\|_p^p. \tag{38}$$

We use (23). Further, it proves that J is coercive with respect to $\aleph$. In addition, if we give Ω_1 and Ω_2 such that

$$\begin{aligned} \Omega_1 &= \{x \in \Omega : |u(x) < 1|\}, \\ \Omega_2 &= \{x \in \Omega : |u(x) \ge 1|\}. \end{aligned} \tag{39}$$

From Remark 5, we can get that

$$\int_\Omega |u|^p \ln |u| dx \le \int_{\Omega_1} |u|^p \ln |u| dx + \int_{\Omega_2} |u|^p \ln |u| dx \le C\int_{\Omega_2} |u|^{p+\zeta} dx \le C\|u\|_{p+\zeta}^{p+\zeta}, \quad (40)$$

where $\zeta > 0$. Under Lemma 6, we get

$$\int_\Omega \ln |u||u|^p dx \le C\|u\|_{p+\zeta}^{p+\zeta} \le C\|\nabla u\|_p^{\alpha(p+\zeta)} \|u\|_p^{(1-\alpha)(p+\zeta)}, \quad (41)$$

where

$$\alpha = \left(\frac{1}{p} - \frac{1}{p+\zeta}\right)\left(\frac{1}{n} - \frac{1}{p} + \frac{1}{p}\right)^{-1} = \frac{n\zeta}{p(p+\zeta)}. \quad (42)$$

Choosing $\zeta < p^2/n$, we obtain

$$\alpha(p+\zeta) < p. \quad (43)$$

By using Young's inequality together with (41), we get

$$\int_\Omega |u|^p \ln |u| dx \le \varepsilon\|\nabla u\|_p^p + C_\varepsilon\left(\|u\|_p^p\right)^\beta, \quad (44)$$

where $\varepsilon > 0$ and $\beta = (1-\alpha)(p+\zeta)/p - \alpha(p+\zeta) > 1$. As $u \in \aleph$, by (22) and (44), we get

$$\|u\|_p^p + \|\nabla u\|_p^p = \int_\Omega |u|^p \ln |u| dx,$$

$$\|u\|_p^p + \|\nabla u\|_p^p \le \varepsilon\|\nabla u\|_p^p + C_\varepsilon\left(\|u\|_p^p\right)^\beta,$$

$$\|\nabla u\|_p^p \le \varepsilon\|\nabla u\|_p^p + C_\varepsilon\left(\|u\|_p^p\right)^\beta,$$

$$(1-\varepsilon)\|\nabla u\|_p^p \le C_\varepsilon\left(\|u\|_p^p\right)^\beta. \quad (45)$$

Select $\varepsilon < 1$. Then, combining (38) and (44), we find

$$J(t) = \frac{1}{p^2}\|u\|_p^p \ge C_\varepsilon\left(\|\nabla u\|_p^p\right)^{1/\beta}. \quad (46)$$

Hence, the coercivity of J on $\aleph$.

Lemma 9.

(i) The depth of the potential well is given by

$$d = \inf_{u\in\aleph} J(u) = \inf\left\{\sup_{\lambda\ge 0} J(\lambda u): u \in W_0^{1,p}(\Omega)/\{0\}, \|u\|_p^p \neq 0\right\} \quad (47)$$

(ii) d admis a positive lower bound, given by

$$d \ge \frac{1}{p^2} e^{(n+p^2)/p}\left(\frac{p^2}{n\mathscr{L}_p}\right)^{n/p} = \frac{l^p}{p^2} = K, \quad (48)$$

where $\mathscr{L}_p$ is given as in Lemma 2

(iii) There exists a positive function $u \in \aleph$, verify $J(u) = d$

Proof.

(i) According to Lemma 7, it implies that for every $u \in W_0^{1,p}(\Omega) \setminus \{0\}$, there exists a λ^*, verify $I(\lambda^* u) = 0$, that is $\lambda^* u \in \aleph$. Using (47) gives

$$J(\lambda^* u) \ge \inf_{u\in\aleph} J(u) = d. \quad (49)$$

From Lemma 7, the maximizer of $J(\lambda u)$ is exact λ^*, such that

$$\sup_{\lambda\ge 0} J(\lambda u) = J(\lambda^* u) = \frac{1}{p} I(\lambda^* u) + \frac{1}{p^2}\|\lambda^* u\|_p^p = \frac{1}{p^2}\|\lambda^* u\|_p^p. \quad (50)$$

By the combination of (50) and (49), we find

$$\inf_{u\in W_0^{1,p}(\Omega)/\{0\}} \sup_{\lambda\ge 0} J(\lambda u) = \inf_{u\in W_0^{1,p}(\Omega)/\{0\}} J(\lambda^* u) \ge d. \quad (51)$$

So that, as $u \in W_0^{1,p}(\Omega) \setminus \{0\}$, we have $d \neq 0$. And if $u \in \aleph$ by (30), we obtain that λ^* is the only critical point in $(0, \infty)$ of the mapping $g(\lambda)$. Therefore,

$$\sup_{\lambda>0} J(\lambda u) = J(u), \quad (52)$$

for any $u \in \aleph$. Then,

$$\inf_{u\in W_0^{1,p}(\Omega)/\{0\}} \sup_{\lambda>0} J(\lambda u) \le \inf_{u\in\aleph} \sup_{\lambda>0} J(\lambda u) = \inf_{u\in\aleph} J(u) = d. \quad (53)$$

By (51) and (53), (i) is obtained.

(ii) From Lemma 7, $\forall u \in W_0^{1,p}(\Omega) \setminus \{0\}$, we get $I(\lambda^* t) = 0$. Lemma 8 gives

$$\|\lambda^* u\|_p \geq e^{\left(n+p^2\right)/p^2}\left(\frac{p^2}{n\mathscr{L}_p}\right)^{n/p^2} = l. \tag{54}$$

By using (50) and (54), we get

$$\sup_{\lambda>0} J(\lambda u) \geq \frac{l^p}{p^2} = K. \tag{55}$$

According to (i), we find that $d \geq K$.

(iii) Consider the minimize sequence $\{u_k\}_k^\infty \subset u \in \aleph$ for J, verify

$$\lim_{k\to\infty} J(u_k) = d. \tag{56}$$

Hence, we have $\{|u_k|\}_k^\infty \subset u \in \aleph$ is also a minimizing sequence for J due to $|u_k| \subset u \in \aleph$ and $J(|u_k|) = J(u_k)$. For this, we can suppose that $u_k > 0$ a.e. Ω for any $k \in u \in \aleph$.

From it, we note that J is coercive on $u \in \aleph$; in other words, $\{u_k\}_k^\infty$ is bounded in $W_0^{1,p}(\Omega)$. Since $W_0^{1,p}(\Omega)^\circ L^p(\Omega)$ is compact embedding, $\exists u$ is a function and a subsequence of $\{u_k\}_k^\infty$, still given by $\{u_k\}_k^\infty$, such that

$$\begin{aligned} &u_k \to u \text{ weakly in } W_0^{1,p}(\Omega),\\ &u_k \to u \text{ strongly in } L^p(\Omega),\\ &u_k(x) \to u(x) \text{ a.e.in } \Omega. \end{aligned} \tag{57}$$

Hence, $u \geq 0$ on Ω and

$$\begin{aligned} J(t) &= \frac{1}{p}\|\nabla u\|_p^p + \frac{p+1}{p^2}\|u\|_p^p - \frac{1}{p}\int_\Omega \ln|u|u^p dx\\ &\leq \lim_{k\to\infty}\inf\left(\frac{p+1}{p^2}\|u\|_p^p + \frac{1}{p}\|\nabla u\|_p^p - \frac{1}{p}\int_\Omega \ln|u|u^p dx\right)\\ &= \lim_{k\to\infty}\inf J(u_k) = d. \end{aligned} \tag{58}$$

We apply Lebesgue dominated convergence theorem and weak lower semicontinuity.

As $u_k \in u \in \aleph$, we have $u_k \in W_0^{1,p}(\Omega)\setminus\{0\}$ and $I(u_k)$ which implies

$$\|u_k\|_p \geq e^{\left(n+p^2\right)/p^2}\left(\frac{p^2}{n\mathscr{L}_p}\right)^{n/p^2} = l. \tag{59}$$

According to Lemma 8, we have $\|u\|_p \neq 0$ converge strongly in $L^p(\Omega)$; that is to say, that $u \in W_0^{1,p}(\Omega)\setminus\{0\}$. Moreover, using weak lower continuity, we find

$$\begin{aligned} I(u) &= \|u\|_p^p + \|\nabla u\|_p^p - \int_\Omega \ln|u|u^p dx\\ &\leq \lim_{k\to\infty}\inf\left(\|u\|_p^p + \|\nabla u\|_p^p - \int_\Omega \ln|u|u^p dx\right)\\ &= \lim_{k\to\infty}\inf I(u_k) = 0. \end{aligned} \tag{60}$$

As a final stage of proof (iii), we prove that $I(u) = 0$. If this is false, we get $I(u) < 0$; hence, by Lemma 7, $\exists\lambda^* < 1$ which verifying $I(\lambda^* u) = 0$. Further, we find

$$\begin{aligned} 0 < d \leq J(\lambda^* u) &= \frac{1}{p^2}\|\lambda^* u\|_p^p \leq \frac{(\lambda^*)^p}{p^2}\lim_{k\to\infty}\inf\|u_k\|_p^p\\ &= (\lambda^*)^p \lim_{k\to\infty}\inf J(u_k) = (\lambda^*)^p d < d. \end{aligned} \tag{61}$$

And it produces a stark contrast. Meaning that the proof of Lemma 9 has ended.

Definition 10. We say that function $u(t)$ represents a weak solution to problem (1) on $\Omega\times[0,T)$, if

$$\begin{aligned} u \in C\left((0,T);W_0^{1,p}(\Omega)\right) \cap C^1\left((0,T);L^2(\Omega)\right),\\ u_t \in L^\infty\left((0,T);L^2(\Omega)\right) \end{aligned} \tag{62}$$

satisfies

$$\begin{cases} \int_\Omega |\nabla u|^{p-2}\nabla u\nabla w dx + \int_\Omega |u|^{p-2}u w(x)dx + \int_\Omega u_t w(x)dx = k\int_\Omega \ln|u(x,t)|u^{p-2}(x,t)w(x)dx,\ \forall w \in H_0^1(\Omega),\\ u(x,0) = u_0(x). \end{cases} \tag{63}$$

Lemma 11. *Let $u_0 \in W_0^{1,p}(\Omega)\setminus\{0\}$ and $l = e^{(n+p^2)/p^2}(p^2/n\mathscr{L}_p)^{n/p^2}$. Suppose that $0 < E(0) < l^p/p^2 < d$.*

(i) If $u_0 \in W$, then $u \in W$ for $0 \leq t \leq T$

(ii) If $u_0 \in V$, then $u \in V$ for $0 \leq t \leq T$,

such that T is the maximum time of existence of $u(t)$.

Proof.

(i) We put T is the maximum time of existence of solution u. From (24) combined with (47), we find

$$J(u) \leq J(u_0) < d, \forall t \in [0, T). \quad (64)$$

Then, we have $u(t) \in W$ for every $t \in 0, T)$. If it is false, hence $\exists t_0 \in [0, T)$ verify $u(t_0) \in \partial W$, we get either $I(u_0) = 0$ and $\|\Delta(u_0)\| \neq 0$ or (b) $J(u_0) = d$.

According to (64), (b) is impossible, that is, $I(u_0) = 0$ and $\|\Delta(u_0)\| \neq 0$. But it is $\exists J(u_0) \geq d$ if $0 < d = \inf_{u \in \aleph} J(u)$. From this, we have a stark contrast, $u(t) \in W$ is obtained for $\forall t \in 0, T)$.

(ii) In the same way, we prove case (ii)

Theorem 12. *Consider $u_0(x) \in W_0^{1,p}(\Omega) \setminus \{0\}$. If $I(u_0) > 0$ and $E(0) < d$ or $\|u_0\|_p^p = 0$. Therefore, problem (1) admits a weak global solution $u(t) \in L^\infty(0,\infty; W_0^{1,p}(\Omega) \setminus \{0\})$, $u_t(t) \in L^\infty(0,\infty; L^2(\Omega))$.*

Proof. Consider the orthogonal basis $\{w_j\}_{j=1}^\infty$ of the "separable" space $W_0^{1,p}(\Omega)$ which is orthonormal in $L^2(\Omega)$. Let the following subspace V_m on the finite dimensional

$$V_m = \text{span}\{w_1, w_2, \cdots, w_m\}, \quad (65)$$

where the projections of the initial data be defined by

$$u_0^m(x) = \sum_{j=1}^m a_j w_j(x) \longrightarrow u_0 \text{ in } H_0^2(\Omega), \quad (66)$$

for all $j = 1, 2, \cdots, m$.

Now, we can see the approximated solutions of (1) as in the following form

$$u^m(x,t) = \sum_{j=1}^m h_j^m(t) w_j(x), \quad (67)$$

of the approximate problem in V_m

$$\begin{cases} \int_\Omega |\nabla u^m|^{p-2} \nabla u^m \nabla w dx + \int_\Omega |u^m|^{p-2} u^m w(x) dx + \int_\Omega u_t^m w(x) dx = k \int_\Omega |u^m|^{p-2}(x,t) \ln |u^m(x,t)| w(x) dx, w \in V_m, \\ u^m(0) = u_0^m = \sum_{j=1}^m (u_0, w_j) w_j. \end{cases} \quad (68)$$

It produces an ordinary differential equation system (ODE) made up of unknown functions $h_j^m(t)$. Starting from the standard theory of existence, there are functions

$$h_j : [0, t_m) \longrightarrow R, j = 1, 2, \cdots, m, \quad (69)$$

which verify (68) in a maximal interval $[0, t_m), 0 < t_m \leq T$. Next, we prove that $t_m = T$ and that the local solution is uniformly bounded independent of m and t. For this purpose, let us replace w by u_t^m in (68) and integrate by parts, we get

$$\frac{d}{dt} E^m(t) = -\|u_t^m\|^2 \leq 0, \quad (70)$$

such as

$$E^m(t) = \frac{1}{p}\|\nabla u^m\|_p^p + \frac{p+1}{p^2}\|u^m\|_p^p - \frac{1}{p}\int_\Omega |u^m|^p \ln |u^m| dx. \quad (71)$$

Integrating (70) from 0 to t, and using (24), we obtain

$$J(u^m) + \int_0^t \|u_s^m\|^2 ds = E^m(0). \quad (72)$$

According to (68), with $m \longrightarrow \infty$, we find $E^m(0) \longrightarrow E(0)$. We select m large enough; we find

$$J(u^m) + \int_0^t \|u_s^m\|^2 ds < d. \quad (73)$$

Hence, by (23), we have

$$J(u) = \frac{1}{p} I(u) + \frac{1}{p^2}\|u\|_p^p. \quad (74)$$

By $u_0 \in W$,

$$J(u^m(0)) = E(0) ; \quad (75)$$

we select m large enough and $0 \leq t < \infty$; we find $u^m(0) \in W$. By (24) and Lemma 11, by picking m large enough and $0 \leq t < \infty$, we get $^m(t) \in W$. Further, according to (24) and (21), we obtain

$$\frac{1}{p}\|\nabla u^m\|_p^p + \frac{p+1}{p^2}\|u^m\|_p^p - \frac{1}{p}\int_\Omega |u^m|^p \ln |u^m| dx + \int_0^t \|u_s^m\|^2 ds < d, \quad (76)$$

where $0 \leq t < \infty$. By choosing m large enough and $0 \leq t < \infty$ (76), we get

$$\|\nabla u^m\|_p^p < pd,$$

$$\|u^m\|_p^p < \frac{p^2}{p+1} d,$$

$$\int_0^t \|u_s^m\|^2 ds < d. \tag{77}$$

According to Remark 5, we find

$$\begin{aligned} \int_\Omega |u^m|^p \ln |u^m| dx &\leq \int_{\Omega_1} |u^m|^p \ln |u^m| dx \\ &+ \int_{\Omega_2} |u^m|^p \ln |u^m| dx \\ &\leq C \int_{\Omega_2} |u^m|^{p+\zeta} dx \leq C \|u^m\|_{p+\zeta}^{p+\zeta}, \end{aligned} \tag{78}$$

where ζ is pick satisfying $p+\zeta < np/(n-p)$ as $p < n$ and $\zeta > 0$ as $p \geq n$ and $\Omega_1 = \{x \in \Omega : |u^m(x) < 1|\}$ and $\Omega_2 = \{x \in \Omega : | u^m(x) \geq 1|\}$.

Applying the embedding theorem, Lemma 6 and Young's inequality, gives from (78):

$$\begin{aligned} &\int_\Omega \ln |u^m||u^m|^p dx \\ &\quad \leq C\|u^m\|_{p+\zeta}^{p+\zeta} \\ &\quad \leq C\|\nabla u^m u\|_p^{\alpha(p+\zeta)} \|u^m\|_p^{(1-\alpha)(p+\zeta)} \\ &\quad \leq \varepsilon \|\nabla u^m u\|_p^p + C_\varepsilon \left(\|u^m\|_p^p\right)^{(1-\alpha)p(p+\zeta)/p[p-\alpha(p+\zeta)]} \\ &\quad \leq C_\varepsilon \|\nabla u^m u\|_p^p. \end{aligned} \tag{79}$$

Therefore, we choose $0 < \zeta$ for $p > \alpha(p+\zeta)$, where $\varepsilon \in (0, 1)$ with

$$\alpha = \left(\frac{1}{p} - \frac{1}{p+\zeta}\right)\left(\frac{1}{n} - \frac{1}{p} + \frac{1}{p}\right)^{-1}, \frac{(1-\alpha)p(p+\zeta)}{p-\alpha(p+\zeta)} > 1. \tag{80}$$

Using (79) and (76), for $0 \leq t < \infty$, we find

$$\int_\Omega \ln |u^m||u^m|^p dx < C_\varepsilon pd. \tag{81}$$

Hence, we get

$$\begin{cases} u^m \text{ is uniformly bounded in } L^\infty\left(0,\infty; W_0^{1,p}(\Omega)\right), \\ u_t^m \text{ is uniformly bounded in } L^\infty\left(0,\infty; L^2(\Omega)\right). \end{cases} \tag{82}$$

Using the integration on (68), we get for $0 \leq t < \infty$

$$\begin{aligned} \int_\Omega u^m w_s dx = &\int_\Omega u_0 w_s dx + \int_0^t \int_\Omega \ln |u^m|^k |u^m|^{p-1} w_s dx ds \\ &- \int_0^t \int_\Omega |\nabla u^m|^{p-2}(s) \nabla u^m(s) \nabla w_s dx ds \\ &- \int_0^t \int_\Omega |u^m|^{p-2}(s) u^m(s) w_s dx. \end{aligned} \tag{83}$$

Further, after passing through the limit in (ref 4030), we arrive at the weak solution left$(u$ right) to the problem (ref 300). According to the initial data in (ref 300), we conclude that $(u(x,0)) = (u_0)$ in $W_0^{1,p}$.

4. Decay of Solution

In this section, by using the Lyapunov functional, we show the decay of solution to (1).

First, we define the Lyapunov functional by

$$L(t) = E(t) + \frac{\varepsilon}{2}\int_\Omega u^2 dx, \tag{84}$$

where $\varepsilon > 0$. We will prove the equivalence between $L(t)$ and $E(t)$.

Lemma 13. *For $\varepsilon > 0$ small enough, we have*

$$\beta_1 L(t) \leq E(t) \leq \beta_2 L(t), \tag{85}$$

where $\beta_1, \beta_2 > 0$.

We find $L \sim E$ by choosing ε small enough.

Theorem 14. *Let $u_0 \in V$. Assume further $0 < E(0) < (p+1)/p^2 \mu l^p < d$, where*

$$\begin{aligned} &l = e^{n+p^2/p^2}\left(\frac{p^2}{n\mathscr{L}_p}\right)^{n/p^2}, \\ &\mu^{n-p^2/np} e^{(1-p)(n+p^2)}\left(\frac{p}{n\mathscr{L}_p}\right)^{1-p/p} < \mu < \frac{p(\beta-p)+\beta C^*}{(\beta-p)}; \end{aligned} \tag{86}$$

hence, $\exists c_1, c_2 > 0$ satisfies

$$0 < E(t) \leq c_1 e^{-c_2 t}, \ t \geq 0. \tag{87}$$

Proof. A differentiation of $L(t)$ and equation (1) gives

$$\begin{aligned} L'(t) = E'(t) + \varepsilon\int_\Omega u u_t dx = &-\|u_t\|^2 \\ &- \varepsilon\left(\|\nabla u\|_p^p + \|u\|_p^p\right) + \varepsilon\int_\Omega \ln |u| u^p dx. \end{aligned} \tag{88}$$

Adding and subtracting $\varepsilon\beta E(t)$ into (88) ($\beta > 0$), we obtain

$$\begin{aligned} L'(t) &= -\|u_t\|^2 + \varepsilon\left(\frac{\beta-p}{p}\right)\|\nabla u\|_p^p + \varepsilon\left(\frac{\beta-p}{p}\right)\|u\|_p^p \\ &\quad + \varepsilon\left(1-\frac{\beta}{p}\right)\int_\Omega \ln|u|u^p dx + \frac{1}{p^2}\varepsilon\beta\|u\|_p^p - \varepsilon\beta E(t) \\ &\le -\|u_t\|^2 + \varepsilon\left(\frac{\beta-p}{p}\right)\left(1+\frac{C^*\beta}{p(\beta-p)}\right)\|\nabla u\|_p^p \\ &\quad + \varepsilon\left(\frac{\beta-p}{p}\right)\|u\|_p^p + \varepsilon\left(1-\frac{\beta}{p}\right)\int_\Omega \ln|u|u^p dx. \end{aligned} \tag{89}$$

Using the inequality of logarithmic Sobolev together with $\|u\|_p^p \le C^*\|\nabla u\|_p^p (C^* > 0)$ gives

$$\begin{aligned} L'(t) &\le -\|u_t\|^2 + \varepsilon\left(\frac{\beta-p}{p}\right)\left(1+\frac{C^*\beta}{p(\beta-p)}\right)\|\nabla u\|_p^p \\ &\quad + \varepsilon\left(\frac{\beta-p}{p}\right)\|u\|_p^p + \varepsilon\left(1-\frac{\beta}{p}\right)\int_\Omega \ln|u|u^p dx - \varepsilon\beta E(t) \\ &\le -\varepsilon\beta E(t) - \|u_t\|^2 + \varepsilon\left(\frac{\beta-p}{p}\right)\left(1+\frac{C^*\beta}{p(\beta-p)}-\frac{\mu}{p}\right)\|\nabla u\|_p^p \\ &\quad - \varepsilon\left(\frac{\beta-p}{p}\right)\left[\ln\|u\|_p - \left(\frac{n}{p^2}\ln\left(\frac{p\mu e}{n\mathscr{L}_p}\right)+1\right)\right]\|u\|_p^p. \end{aligned} \tag{90}$$

Noting that $0 < \beta < p$ and using (21) and Theorem 12, we find

$$\begin{aligned} \ln\|u\|_p &\le \ln\left(\frac{p^2}{p+1}J(u)\right) \le \ln\left(\frac{p^2}{p+1}E(t)\right) \\ &\le \ln\left(\frac{p^2}{p+1}E(0)\right) \le \ln\left(\mu l^p\right) \\ &= \ln\left(\mu e^{n+p^2/p}\left(\frac{p\mu}{n\mathscr{L}_p}\right)^{n/p}\right). \end{aligned} \tag{91}$$

By μ satisfying

$$\mu^{n-p^2/np} e^{(1-p)\left(n+p^2\right)}\left(\frac{p}{n\mathscr{L}_p}\right)^{1-p/p} < \mu < \frac{p(\beta-p)+\beta C^*}{(\beta-p)}, \tag{92}$$

we guarantee

$$\left(1+\frac{C^*\beta}{p(\beta-p)}-\frac{\mu}{p}\right) > 0, \tag{93}$$

$$\ln\|u\|_p - \left(\frac{n}{p^2}\ln\left(\frac{p\mu e}{n\mathscr{L}_p}\right)+1\right) > 0\,; \tag{94}$$

then, we obtain

$$L'(t) \le -\varepsilon\beta E(t) - \|u_t\|^2. \tag{95}$$

Hence, inequality (95) becomes

$$L'(t) \le -\varepsilon\beta E(t). \tag{96}$$

According to (85), we get

$$L'(t) \le -\varepsilon\beta\beta_2 L(t). \tag{97}$$

Setting $c_2 = \varepsilon\beta\beta_2 > 0$ and integrating (97) yield

$$L(t) \le c_1 e^{-c_2 t}. \tag{98}$$

Finally, by (85), we obtain (87). This is the end of the proof.

5. Conclusion

As mentioned earlier in the introduction, the majority of problems in science are nonlinear and their analytical solutions are not easy to find, and most physical problems mostly use higher nonlinear partial differential equations (PDEs). It has been found to be extremely difficult to find accurate or analytical solutions to such problems. However, in the past several centuries, many scientists have made great progress and adopted various techniques to study the analytical side of these famous problems, and nonlinear logarithmic has also received much attention from physicists and mathematicians. Log nonlinearity was introduced into the relativistic wave equation describing spinning particles moving in an external electromagnetic field and in the relativistic wave equation (see, for example, [1–3, 6, 14, 18, 19, 29, 36, 37]); in this contribution, under some sufficient initial and boundary conditions, we have studied the analytical side of p -Laplacian heat equations with respect to logarithmic nonlinearity in the right-hand side, where the global existence and decay estimates of weak solutions are proved. In the next work, we extend our recent work to the coupled system for this important problem. Also, some numerical examples will be given in order to ensure the theory study by using some famous algorithms which are presented in [38, 39].

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Existence Results for Fractional Semilinear Integrodifferential Equations of Mixed Type with Delay

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In this paper, we discuss a class of fractional semilinear integrodifferential equations of mixed type with delay. Based on the theories of resolvent operators, the measure of noncompactness, and the fixed point theorems, we establish the existence and uniqueness of global mild solutions for the equations. An example is provided to illustrate the application of our main results.

1. Introduction

Fractional calculus can be used to describe some nonclassical phenomena in natural science and engineering applications. Fractional differential equations have been applied in different fields ranging from engineering, finance, and physics in the past few decades. Researchers have conducted extensive explorations on this subject and have achieved fruitful results for the fractional differential equations [1–13]. Zhu and Han [10] and Chadha and Pandey [11] studied the fractional integrodifferential equations and discussed the existence of mild solutions. Based on the theory of the resolvent family and fixed point theorems, Chen et al. [14–17] analyzed nonautonomous evolution equations in a Banach space. Moreover, some researchers considered sufficient conditions on the existence of mild solutions for fractional differential equations by the measure of noncompactness [4, 18, 19]. The initial boundary value problem for the fractional integrodifferential equations with delay has been investigated by using fixed point theorems [4, 5, 18, 20]. In [3, 21–24], differential equations of mixed type have been studied and some results have been concluded.

Chen [22] studied the fractional nonautonomous evolution equations of mixed type:

$$\begin{cases} {}^cD_t^\beta u(t) + A(t)u(t) = f(t, u(t), Tu(t), Su(t)), & t \in (0, a], \\ u(0) = A^{-1}(0)u_0, \end{cases} \tag{1}$$

where

$$Tu(t) = \int_0^t K(t,s)u(s)ds, \qquad Su(t) = \int_0^a H(t,s)u(s)ds, \tag{2}$$

where the kernels K and H are linear functions. The operator T is an integral with a variable upper limit, and the operator S is an ordinary definite integral; accordingly, problem (1) is called fractional semilinear integrodifferential equations of mixed type.

Li and Jia [25] investigated the existence of mild solutions for abstract delay fractional differential equations:

$$\begin{cases} {}^cD_t^\beta u(t) = Au(t) + J_t^{1-\beta} f(t, u_t), & t \in [0, T], \\ u(t) = \varphi(t), & t \in [-r, 0], \end{cases} \tag{3}$$

where $\beta \in (0,1)$, $J_t^{1-\beta}$ is the Riemann-Liouville fractional integral, the linear operator A is independent on t, and the Lipschitz coefficient of f is constant.

To the best of our knowledge, there are no results on the fractional integrodifferential equations of mixed type with delay. Motivated by this idea, we consider the following problem:

$$\begin{cases} {}^cD_t^\beta x(t) = A(t)x(t) + J_t^{1-\beta} f(t, x_t, \mathscr{K}x_t, \mathscr{H}x_t), & t \in [0, T_0], \\ x(t) = \phi(t), & t \in [-r, 0], \end{cases} \tag{4}$$

where $\beta \in (0,1]$, ${}^cD_t^\beta$ is the Caputo fractional derivative of order β, $A(t)$ is a closed and linear operator with domain $D(A)$ defined on a Banach space E, $J_t^{1-\beta}$ is the Riemann-Liouville fractional integral of order $1-\beta$, $\mathscr{K}$ and $\mathscr{H}$ are defined by

$$\mathscr{K}x_t = \int_0^t K(t, s, x_s) ds, \qquad \mathscr{H}x_t = \int_0^{T_0} H(t, s, x_s) ds, \tag{5}$$

where $K : D \times C([-r, 0]; E) \longrightarrow E$ and $H : D_0 \times C([-r, 0]; E) \longrightarrow E$ are continuous and nonlinear functions, $D = \{(t, s) \in R^2 : 0 \le s \le t \le T_0\}$, $J = [0, T_0]$, $D_0 = \{(t, s) \in R^2 : 0 \le t, s \le T_0\}$, $\phi \in C[-r, 0]$, f is to be specified later, and x_t means the element of $C([-r, 0]; E)$ defined by $x_t(\theta) = x(t+\theta)$, $-r \le \theta \le 0$, for $x \in C([-r, T_0]; E)$, $t \in J$.

We demonstrate the existence and uniqueness of global mild solutions for problem (4) under the conditions of the compact resolvent operator and noncompact resolvent operator, respectively. The kernels K and H of the operators $\mathscr{K}$ and $\mathscr{H}$ are nonlinear functions. In addition, the operator $A(t)$ is dependent on t. The rest of this paper is organized as follows. Basic definitions and auxiliary results are presented in Section 2. In Section 3, we prove the existence and uniqueness of mild solutions via various fixed point theorems, the measure of noncompactness, and the Banach contraction mapping principle. An example is provided to illustrate the main theorems in Section 4. Finally, Section 5 is the summary of our results.

2. Preliminaries

Definition 1 [6, 26]. The Riemann-Liouville fractional integral J_t^β and derivative D_t^β of a function $f : (0,\infty) \longrightarrow R$ of order $\beta > 0$ are defined by

$$J_t^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s) ds,$$
$$D_t^\beta f(t) = \frac{1}{\Gamma(n-\beta)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\beta-1} f(s) ds, \quad n-1 < \beta \le n, \tag{6}$$

where $f(t) \in L^1((0, T_0); E)$, $\Gamma(\cdot)$ denotes the gamma function, and $n \in N$.

Remark 2 [25]. $D_t^\beta f(t) = D_t^m J_t^{m-\beta} f(t)$, where $D_t^m = d^m/dt^m$ and $J_t^{m-\beta} f(t) \in W^{m,1}((0, T_0); E)$.

Definition 3 [26, 27]. The Caputo fractional derivative of order $\beta > 0$ of a function $f : (0,\infty) \longrightarrow R$ is given by

$${}^cD_t^\beta f(t) = \frac{1}{\Gamma(n-\beta)} \int_0^t (t-s)^{n-\beta-1} f^{(n)}(s) ds, \quad n-1 < \beta < n. \tag{7}$$

Remark 4 [25]. For the Riemann-Liouville fractional integral operator and the Caputo fractional derivative operator, the following conclusions are obtained:

$${}^cD_t^\beta f(t) = D_t^\beta \left(f(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} f^{(k)}(0) \right),$$
$${}^cD_t^\beta \left(J_t^\beta f(t) \right) = f(t),$$
$$J_t^\beta \left({}^cD_t^\beta (f(t)) \right) = f(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} f^{(k)}(0). \tag{8}$$

Definition 5 [28, 29]. Let $A(t)$ be a closed and linear operator with domain $D(A)$ defined on a Banach space E and $\beta > 0$. Let $\rho[A(t)]$ be the resolvent set of $A(t)$. $A(t)$ is called the generator of a β-resolvent family if there exist $\omega \ge 0$ and a strongly continuous function $U_\beta : R_+^2 \longrightarrow B(E)$ such that $\{\lambda^\beta : \operatorname{Re} \lambda > \omega\} \subset \rho(A)$ and

$$\left(\lambda^\beta I - A(s)\right)^{-1} x = \int_0^\infty e^{-\lambda(t-s)} U_\beta(t, s) x dt, \quad \operatorname{Re}(\lambda) > \omega, x \in E. \tag{9}$$

In this case, $U_\beta(t, s)$ is called the β-resolvent family generated by $A(t)$.

Remark 6 [29, 30]. $U_\beta(t, s)$ satisfies the following properties:

(1) $U_\beta(s, s) = I$ and $U_\beta(t, s) = U_\beta(t, r) U_\beta(r, s)$, for $0 \le s \le r \le t \le a$

(2) $(t, s) \longrightarrow U_\beta(t, s)$ is strongly continuous for $0 \le s \le t \le a$

(3) If $U_\beta(t, s)$ is compact for $t, s > 0$, then $U_\beta(t, s)$ is continuous in the uniform operator topology

Lemma 7 [21]. *Let $B \subset C[J, E]$ be equicontinuous and bounded; then, $\bar{C}oB \subset C[J, E]$ is also equicontinuous and bounded.*

Lemma 8 [24]. *Let $B \subset C[J, E]$ be equicontinuous and bounded; then, $\alpha(B(t))$ is continuous on J and*

$$\alpha\left(\int_J B(s)ds\right) \leq \int_J \alpha(B(s)ds), \quad \alpha(B) = \max_{t\in J}\alpha(B(t)), \tag{10}$$

where α denotes the measure of noncompactness.

Lemma 9 [21]. *Let E be a Banach space and $D \subset E$ be bounded; then, there exists a countable set $D_0 \subset D$ such that $\alpha(D) \leq 2\alpha(D_0)$.*

Lemma 10 [31]. *Let E be a Banach space and $D \subset E$ be a bounded closed and convex set. Assume that $Q : D \longrightarrow D$ is a strict set contraction mapping; then, Q has at least one fixed point in D.*

Definition 11. A function $x \in C([-r, T_0]; E)$ is a mild solution of problem (4), if x satisfies the following equations:

$$x(t) = \begin{cases} U_\beta(t,0)\phi(0) + \int_0^t U_\beta(t,s) f(s, x_s, \mathscr{K}x_s, \mathscr{H}x_s)ds, & t \in [0, T_0], \\ \phi(t), & t \in [-r, 0]. \end{cases} \tag{11}$$

3. Main Results

Let us introduce the operator $\Psi : C([-r, T_0]; E) \longrightarrow C([-r, T_0]; E)$ by

$$\Psi x(t) = \begin{cases} U_\beta(t,0)\phi(0) + \int_0^t U_\beta(t,s) f(s, x_s, \mathscr{K}x_s, \mathscr{H}x_s)ds, & t \in [0, T_0], \\ \phi(t), & t \in [-r, 0]. \end{cases} \tag{12}$$

Theorem 12. *Assume that the following conditions hold:*

(H_1). The resolvent operator $U_\beta(t, s)$ is compact for all $t, s > 0$, $M^\star = \max \|U_\beta(t, s)\| < +\infty, 0 \leq s \leq t \leq T_0$.

(H_2). $K : D \times C([-r, 0]; E) \longrightarrow E$ and $H : D_0 \times C([-r, 0]; E) \longrightarrow E$ are continuous; there exist nonnegative Lebesgue integrable functions $p_i \in L(J, R_+)(i = 1, 2)$ such that $\|K(t, s, x)\| \leq p_1(t)\|x\|_{C([-r,0];E)}$ and $\|H(t, s, x)\| \leq p_2(t)\|x\|_{C([-r,0];E)}$, for all $(t, s) \in D$, $(t, s) \in D_0$, $x \in C([-r, 0]; E)$.

(H_3). $f : J \times C([-r, 0]; E) \times C([-r, 0]; E) \times C([-r, 0]; E) \longrightarrow E$ is continuous; there exist nonnegative Lebesgue integrable functions $a, L_i \in L(J, R_+)(i = 1, 2, 3)$ such that $\|f(t, x_1, x_2, x_3)\| \leq a(t) + \sum_{i=1}^3 L_i(t)\|x_i\|_{C([-r,0];E)}$, for all $t \in J$, $x_i \in C([-r, 0]; E)$.

Then, problem (4) has at least one mild solution $x \in C([-r, T_0]; E)$.

Proof. Let us set the notation $R_1 > 0$ such that

$$R_1 \geq \frac{M^\star\phi_0 + M^\star\int_0^{T_0}a(s)ds}{1 - M^\star\left(\int_0^{T_0}L_1(s)ds + \int_0^{T_0}L_2(s)\int_0^{T_0}p_1(\nu)d\nu ds + \int_0^{T_0}L_3(s)\int_0^{T_0}p_2(\nu)d\nu ds\right)}, \tag{13}$$

where $\phi_0 = \|\phi(0)\|$ and $(\int_0^{T_0}L_1(s)ds + \int_0^{T_0}L_2(s)\int_0^{T_0}p_1(\nu)d\nu ds + \int_0^{T_0}L_3(s)\int_0^{T_0}p_2(\nu)d\nu ds)^{-1} > M^\star$.

First of all, we consider the set $B_{R_1} = \{x \in C([-r, T_0]; E) : \|x\|_{C([-r,T_0];E)} \leq R_1\}$ and show that $\Psi B_{R_1} \subset B_{R_1}$. By using conditions ($H_2$) and ($H_3$), for all $x \in B_{R_1}$, we have

$$\begin{aligned} \|(\Psi x)(t)\| &\leq \|U_\beta(t,0)\phi(0)\| + \int_0^t \|U_\beta(t,s) f(s, x_s, \mathscr{K}x_s, \mathscr{H}x_s)\|ds \\ &\leq M^\star\phi_0 + M^\star\int_0^t \|f(s, x_s, \mathscr{K}x_s, \mathscr{H}x_s)\|ds \leq M^\star\phi_0 \\ &\quad + M^\star\int_0^t (a(s) + L_1(s)\|x_s\| + L_2(s)\|\mathscr{K}x_s\| + L_3(s)\|\mathscr{H}x_s\|)ds \\ &\leq M^\star\phi_0 + M^\star\int_0^t a(s)ds + M^\star\left(\int_0^t L_1(s)ds + \int_0^t L_2(s)\right. \\ &\quad \left.\cdot \int_0^s p_1(\nu)d\nu ds + \int_0^t L_3(s)\int_0^{T_0} p_2(\nu)d\nu ds\right)\|x\|_{C([-r,0];E)} \\ &\leq M^\star\phi_0 + M^\star\int_0^{T_0} a(s)ds + M^\star\left(\int_0^{T_0} L_1(s)ds + \int_0^{T_0} L_2(s)\right. \\ &\quad \left.\cdot \int_0^{T_0} p_1(\nu)d\nu ds + \int_0^{T_0} L_3(s)\int_0^{T_0} p_2(\nu)d\nu ds\right)\|x\|_{C([-r,T_0];E)} \leq R_1. \end{aligned} \tag{14}$$

So, we conclude that Ψ maps B_{R_1} into itself.

Second, we prove that $\Psi : B_{R_1} \longrightarrow B_{R_1}$ is continuous.

Let $\{x_n\}_0^\infty \subset C([-r, T_0]; E)$, with $x_n \longrightarrow x(n \longrightarrow \infty)$, $x \in C([-r, T_0]; E)$. Using the fact that $K : D \times C([-r, 0]; E) \longrightarrow E$, $H : D_0 \times C([-r, 0]; E) \longrightarrow E$, and $f : J \times C([-r, 0]; E) \times C([-r, 0]; E) \times C([-r, 0]; E) \longrightarrow E$ are continuous, we obtain

$$f\left(t, (x_n)_t, \mathscr{K}(x_n)_t, \mathscr{H}(x_n)_t\right) \longrightarrow f(t, x_t, \mathscr{K}x_t, \mathscr{H}x_t)(n \longrightarrow \infty), \tag{15}$$

for any $t \in J$ uniformly. That is, for any $\varepsilon > 0$, there exists a natural number N_0, for $n > N_0$, $t \in J$, such that

$$\left\|f\left(t, (x_n)_t, \mathscr{K}(x_n)_t, \mathscr{H}(x_n)_t\right) - f(t, x_t, \mathscr{K}x_t, \mathscr{H}x_t)\right\| \leq \frac{\varepsilon}{M^\star T_0}, \tag{16}$$

which implies that

$$\begin{aligned} \|(\Psi x_n)(t) - (\Psi x)(t)\| &= \left\|\int_0^t U_\beta(t,s) f\left(s, (x_n)_s, \mathscr{K}(x_n)_s, \mathscr{H}(x_n)_s\right)ds \right. \\ &\quad \left. - \int_0^t U_\beta(t,s) f(s, x_s, \mathscr{K}x_s, \mathscr{H}x_s)ds\right\| \\ &\leq M^\star\int_0^t \left\|f\left(s, (x_n)_s, \mathscr{K}(x_n)_s, \mathscr{H}(x_n)_s\right)\right. \\ &\quad \left. - f(s, x_s, \mathscr{K}x_s, \mathscr{H}x_s)\right\|ds \leq M^\star T_0 \frac{\varepsilon}{M^\star T_0} = \varepsilon. \end{aligned} \tag{17}$$

In consequence, $\Psi : B_{R_1} \longrightarrow B_{R_1}$ is continuous.

Furthermore, we prove that $\Psi(B_{R_1})$ is equicontinuous.

To do this, let $L(s) = L_1(s) + L_2(s)\int_0^{T_0} p_1(\nu)d\nu + L_3(s)\int_0^{T_0} p_2(\nu)d\nu$. Obviously, it is a nonnegative Lebesgue integrable function. For all $x \in B_{R_1}$, $t_1, t_2 \in J(t_1 < t_2)$, we have

$$\begin{aligned}
&\|(\Psi x)(t_2) - (\Psi x)(t_1)\| \le \|(U_\beta(t_2,0) - U_\beta(t_1,0))\phi(0)\| \\
&\quad + \left\|\int_{t_1}^{t_2} U_\beta(t_2,s) f(s,x_s,\mathscr{K}x_s,\mathscr{H}x_s)ds\right\| \\
&\quad + \left\|\int_0^{t_1} (U_\beta(t_2,s) - U_\beta(t_1,s)) f(s,x_s,\mathscr{K}x_s,\mathscr{H}x_s)ds\right\| \\
&\le \phi_0 \|U_\beta(t_2,0) - U_\beta(t_1,0)\| + M^\star \int_{t_1}^{t_2} \|f(s,x_s,\mathscr{K}x_s,\mathscr{H}x_s)\|ds \\
&\quad + \sup_{s\in J}\|U_\beta(t_2,s) - U_\beta(t_1,s)\| \int_0^{t_1} \|f(s,x_s,\mathscr{K}x_s,\mathscr{H}x_s)\|ds \\
&\le \phi_0 \|U_\beta(t_2,0) - U_\beta(t_1,0)\| + M^\star \int_{t_1}^{t_2} (a(s) + L(s)R_1)ds \\
&\quad + \sup_{s\in J}\|U_\beta(t_2,s) - U_\beta(t_1,s)\| \int_0^{t_1} (a(s) + L(s)R_1)ds \\
&=: I_1 + I_2 + I_3.
\end{aligned} \tag{18}$$

In view of condition (H_1), compactness of the resolvent operator $U_\beta(t,s)(t,s) > 0$ implies the continuity in the uniform operator topology. That is, for any $\varepsilon > 0$, there exists $\delta_1 > 0$, for any $|t_2 - t_1| < \delta_1$, $t_1, t_2 \in J$, such that $I_3 < \varepsilon/3$. Hence, for the above $\varepsilon > 0$, by using properties of $U_\beta(t,s)$ and the above inequalities, there exists $\delta > 0$ $(\delta < \delta_1)$ such that $\|(\Psi x)(t_2) - (\Psi x)(t_1)\| < \varepsilon$, for any $|t_2 - t_1| < \delta$, $t_1, t_2 \in J$. Consequently, $\Psi(B_{R_1})$ is equicontinuous.

In the end, we prove that $\Psi(B_{R_1})$ is precompact.

For any fixed $t (t \in [-r, T_0])$ and $0 < \varepsilon < t$, the operator $(\Psi_\varepsilon x)(t)$ is defined by

$$(\Psi_\varepsilon x)(t) = \begin{cases} U_\beta(t,0)\phi(0) + \int_0^{t-\varepsilon} U_\beta(t,s) f(s,x_s,\mathscr{K}x_s,\mathscr{H}x_s)ds, & t \in [0,T_0], \\ \phi(t), & t \in [-r,0]. \end{cases} \tag{19}$$

Since $U_\beta(t,s)(t,s) > 0$ is a compact resolvent operator, then the set $Y_\varepsilon(t) = \{(\Psi_\varepsilon x)(t): x \in B_{R_1}\}$ is relatively compact in E for any ε $(0 < \varepsilon < t)$.

Moreover, for any $x \in B_{R_1}$, one can find that

$$\begin{aligned}
\|(\Psi x)(t) - (\Psi_\varepsilon x)(t)\| &= \left\|\int_{t-\varepsilon}^t U_\beta(t,s) f(s,x_s,\mathscr{K}x_s,\mathscr{H}x_s)ds\right\| \\
&\le M^\star \left\|\int_{t-\varepsilon}^t f(s,x_s,\mathscr{K}x_s,\mathscr{H}x_s)ds\right\| \\
&\le M^\star \int_{t-\varepsilon}^t (a(s) + L(s)R_1)ds \\
&\le M^\star (\|a(s)\| + \|L(s)\|R_1)\varepsilon.
\end{aligned} \tag{20}$$

Thus, $Y(t) = \{(\Psi x)(t): x \in B_{R_1}\}$ is totally bounded. Hence, $Y(t)$ is relatively compact in E, and so, based on the Arzelà-Ascoli theorem, $\Psi : B_{R_1} \longrightarrow B_{R_1}$ is completely continuous. As all the assumptions of the Schauder fixed point theorem are satisfied, the conclusion implies that the operator Ψ has a fixed point x in $C([-r; T_0], E)$, which is a global mild solution of problem (4). This completes the proof.

Next, we develop the existence of global mild solutions for problem (4) via the measure of noncompactness and fixed point theorem. Furthermore, we employ the notations: $T_R = \{x \in C([-r, T_0]; E): \|x\|_{C([-r,T_0];E)} \le R\}$, $k_0 = \sup\{\|K(t,s,x_s)\|: (t,s,x_s) \in D \times C([-r,0]; E)\}$, $h_0 = \sup\{\|H(t,s,x_s)\|: (t,s,x_s) \in D_0 \times C([-r,0]; E)\}$, and $R \ge \max\{T_0 k_0, T_0 h_0\}$.

Theorem 13. *Assume that (H_1) and the following conditions hold:*

(H_4). The function $f : J \times T_R \times T_R \times T_R \longrightarrow E$ is bounded and continuous, which satisfies

$$\lim_{R\to\infty} \sup \frac{M(R)}{R} < \frac{1}{T_0 M^\star}, \tag{21}$$

where $M(R) = \sup\{\|f(t,x_1,x_2,x_3)\|: (t,x_1,x_2,x_3) \in J \times T_R \times T_R \times T_R\}$.

(H_5). For any R, there exist nonnegative Lebesgue integrable functions $q_i \in L(J, R_+)$, $(i = 1,2,3,4,5)$ such that for any equicontinuous and countable set $D_i \subset T_R (i = 1,2,3)$, $\alpha(f(t,D_1,D_2,D_3)) \le \sum_{i=1}^3 q_i(t)\alpha(D_i)$, $\alpha(K(t,s,D_2)) \le q_4(t)\alpha(D_2)$, and $\alpha(H(t,s,D_3)) \le q_5(t)\alpha(D_3)$.

(H_6). $2M^\star \int_0^{T_0}(q_1(s) + q_2(s)\int_0^{T_0} q_4(\nu)d\nu + q_3(s)\int_0^{T_0} q_5(\nu)d\nu)ds < 1$.

Then, problem (4) has at least one mild solution.

Proof. By (H_4), there exists $0 < \mu < 1/T_0 M^\star$ and $R_0 > 0$, for any $R \ge R_0$, such that

$$M(R) < \mu R. \tag{22}$$

Let $R^\star = \max\{R_0, M^\star\phi_0(1 - M^\star T_0\mu)^{-1}\}$; we first consider the set $B_{R^\star} = \{x \in C([-r,T_0]; E): \|x\|_{C([-r,T_0];E)} \le R^\star\}$ and show that $\Psi B_{R^\star} \subset B_{R^\star}$. From the above inequality, for all $x \in B_{R^\star}$, we have

$$\begin{aligned}
\|\Psi x\|_{C([-r,T_0];E)} &\le \|U_\beta(t,0)\phi(0)\| + \int_0^t \|U_\beta(t,s)\|\|f(s,x_s,\mathscr{K}x_s,\mathscr{H}x_s)\|ds \\
&\le M^\star\phi_0 + M^\star T_0 M(R^\star) \le M^\star\phi_0 + M^\star T_0 \mu R^\star \le R^\star.
\end{aligned} \tag{23}$$

Meanwhile, applying the arguments employed in the proof of Theorem 12, we conclude that Ψ is a continuous and bounded operator on $B_{R^\star}$.

Then, we prove that $\Psi(B_{R^\star})$ is equicontinuous. For any $x \in B_{R^\star}, t_1, t_2 \in J (t_1 < t_2)$, we have

$$\begin{aligned}
\|(\Psi x)(t_2) - (\Psi x)(t_1)\| &\le \left\| \left(U_\beta(t_2,0) - U_\beta(t_1,0)\right)\phi(0) \right\| \\
&\quad + \int_{t_1}^{t_2} \left\| U_\beta(t_2,s) f(s, x_s, \mathcal{K}x_s, \mathcal{H}x_s) \right\| ds \\
&\quad + \int_0^{t_1} \left\| \left(U_\beta(t_2,s) - U_\beta(t_1,s)\right) f(s, x_s, \mathcal{K}x_s, \mathcal{H}x_s) \right\| ds \\
&\le \phi_0 \left\| U_\beta(t_2,0) - U_\beta(t_1,0) \right\| + M^\star (t_2 - t_1) M(R^\star) \\
&\quad + \sup_{s\in J} \left\| U_\beta(t_2,s) - U_\beta(t_1,s) \right\| M(R^\star) t_1.
\end{aligned} \tag{24}$$

By (H_1), the compactness of $U_\beta(t,s)$, for $(t,s) > 0$, implies the continuity in the uniform operator topology. Namely, for any $\varepsilon > 0$, there exists $\delta_1 > 0$, for any $|t_2 - t_1| < \delta_1, t_1, t_2 \in J$, such that

$$\sup_{s\in J} \left\| U_\beta(t_2,s) - U_\beta(t_1,s) \right\| M(R^\star) t_1 < \frac{\varepsilon}{3}. \tag{25}$$

Therefore, for the above $\varepsilon > 0$, there exists $\delta > 0$ $(\delta < \delta_1)$ such that $\|(\Psi x)(t_2) - (\Psi x)(t_1)\| < \varepsilon$, for all $x \in B_{R^\star}, |t_2 - t_1| < \delta, t_1, t_2 \in J$, which shows that $\Psi(B_{R^\star})$ is equicontinuous. In view of Lemma 7, $\bar{C}o\Psi(B_{R^\star}) \subset B_{R^\star}$ is bounded and equicontinuous.

Finally, we prove that $\Psi : \bar{C}o\Psi(B_{R^\star}) \longrightarrow \bar{C}o\Psi(B_{R^\star})$ is a condensing operator. By Lemma 9, for any $D \subset \bar{C}o\Psi(B_{R^\star})$, there exists a countable set $D_0 = \{x_n\} \subset D$ such that

$$\alpha(\Psi(D)) \le 2\alpha(\Psi(D_0)). \tag{26}$$

By using condition (H_5) and Lemma 8, we obtain

$$\begin{aligned}
\alpha(\Psi(D_0)(t)) &= \alpha\left(\int_0^t U_\beta(t,s) f\left(s, (D_0)_s, \mathcal{K}(D_0)_s, \mathcal{H}(D_0)_s\right) ds \right) \\
&\le M^\star \int_0^t \alpha\left(f\left(s, (D_0)_s, \mathcal{K}(D_0)_s, \mathcal{H}(D_0)_s\right)\right) ds \\
&\le M^\star \int_0^t \left(q_1(s)\alpha\left((D_0)_s\right) + q_2(s)\alpha\left(\mathcal{K}(D_0)_s\right) \right. \\
&\quad \left. + q_3(s)\alpha\left(\mathcal{H}(D_0)_s\right)\right) ds \\
&\le M^\star \int_0^t \left(q_1(s) + q_2(s)\int_0^s q_4(\nu) d(\nu) \right. \\
&\quad \left. + q_3(s) \int_0^{T_0} q_5(\nu) d(\nu) \right) ds\, \alpha(D).
\end{aligned} \tag{27}$$

In addition, using Lemma 8, we have

$$\alpha(\Psi(D_0)) = \max_{t\in J} \alpha(\Psi(D_0)(t)). \tag{28}$$

Consequently,

$$\begin{aligned}
\alpha(\Psi(D)) &\le 2M^\star \int_0^{T_0} \left(q_1(s) + q_2(s) \int_0^{T_0} q_4(\nu) d(\nu) \right. \\
&\quad \left. + q_3(s) \int_0^{T_0} q_5(\nu) d(\nu) \right) ds\, \alpha(D).
\end{aligned} \tag{29}$$

By (H_6), we obtain that Ψ is a condensing operator on $\bar{C}o\Psi(B_{R^\star})$. By Lemma 10, there exists at least one fixed point $x \in \bar{C}o\Psi(B_{R^\star}) \subset C([-r, T_0]; E)$ for Ψ. In conclusion, problem (4) has at least one global mild solution. This completes the proof.

Remark 14. Theorems 12 and 13 above are concluded under the conditions that $U_\beta(t,s)$ is compact for $t, s > 0$ and the functions f, K, and H satisfy corresponding conditions; in contrast, when the resolvent operator $\mathrm{U}_\beta(t,s)$ is noncompact, we could obtain Theorem 15 if f, K, and H meet the Lipschitz conditions.

Theorem 15. *Assume that the following conditions hold:*

(H_7). $f : J \times C([-r, 0]; E) \times C([-r, 0]; E) \times C([-r, 0]; E) \longrightarrow E$ is continuous; there exist nonnegative Lebesgue integrable functions $g_i \in L(J, R_+) (i = 1, 2, 3)$, for all $t \in J$, $u_i, v_i \in E$, such that

$$\|f(t, u_1, u_2, u_3) - f(t, v_1, v_2, v_3)\| \le \sum_{i=1}^{3} g_i(t) \|u_i - v_i\|_{C([-r,0];E)}. \tag{30}$$

(H_8). $K : D \times C([-r, 0]; E) \longrightarrow E$ and $H : D_0 \times C([-r, 0]; E) \longrightarrow E$; there exist nonnegative Lebesgue integrable functions $g_4, g_5 \in L(J, R_+)$, for all $u, v \in E$, $(t,s) \in D$, $(t,s) \in D_0$ such that

$$\begin{aligned}
\|K(t,s,u) - K(t,s,v)\| &\le g_4(t) \|u - v\|_{C([-r,0];E)}, \\
\|H(t,s,u) - H(t,s,v)\| &\le g_5(t) \|u - v\|_{C([-r,0];E)}.
\end{aligned} \tag{31}$$

(H_9). $M^\star \int_0^{T_0} (g_1(s) + g_2(s) \int_0^{T_0} g_4(\nu) d\nu + g_3(s) \int_0^{T_0} g_5(\nu) d\nu) ds < 1$.

Then, problem (4) has a unique mild solution.

Proof. For any $u, v \in C([-r, T_0]; E)$,

$$\begin{aligned}
&\|(\Psi u)(t) - (\Psi v)(t)\| \le M^\star \int_0^t \|f(s, u_s, \mathcal{K}u_s, \mathcal{H}u_s) \\
&\quad - f(s, v_s, \mathcal{K}v_s, \mathcal{H}v_s)\| ds \le M^\star \int_0^t (g_1(s)\|u_s - v_s\| \\
&\quad + g_2(s)\|\mathcal{K}u_s - \mathcal{K}v_s\| + g_3(s)\|\mathcal{H}u_s - \mathcal{H}v_s\|) ds \le M^\star \int_0^{T_0} \\
&\quad \cdot \left(g_1(s) + g_2(s) \int_0^{T_0} g_4(\nu) d\nu + g_3(s) \int_0^{T_0} g_5(\nu) d\nu \right) ds \|u - v\|_{C([-r,T_0];E)}.
\end{aligned} \tag{32}$$

By (H_9), we have $\|\Psi u - \Psi v\|_{C([-r,T_0];E)} < \|u - v\|_{C([-r,T_0];E)}$.

These arguments enable us to conclude that the operator Ψ is a contraction mapping. Hence, the operator Ψ has a unique fixed point $x^* \in C([-r, T_0]; E)$, which implies that problem (4) has a unique global mild solution. This completes the proof.

Remark 16. In Theorem 15, we develop the uniqueness of the mild solution for problem (4) via the Banach contraction mapping principle. In conditions (H_7) and (H_8), $g_i \in L(J, R_+)(i = 1, 2, 3, 4, 5)$ turn out to be nonnegative Lebesgue integrable functions instead of constants.

4. An Application

In order to show the application of the main results, we consider the following problem:

$$\begin{cases} {}^cD_t^\beta x(z,t) = t^2 \dfrac{\partial^2}{\partial z^2} x(z,t) + J_t^{1-\beta}\left(\dfrac{t}{1+t^2} x(z,t+\theta) + \dfrac{1}{1+t^2}\displaystyle\int_0^t a(s)x(z,s+\theta)ds + \dfrac{1}{1+e^t}\displaystyle\int_0^1 b(s)x(z,s+\theta)ds\right), \\ 0 < t \le 1, \quad z \in \Omega, \theta \in [-r,0], \\ x(z,\theta) = \varphi(z,\theta), \quad z \in \Omega, \theta \in [-r,0], \end{cases} \tag{33}$$

where $0 < \beta < 1$, ${}^cD_t^\beta$ is the Caputo fractional derivative of order β, $J_t^{1-\beta}$ is the Riemann-Liouville fractional integral of order $1-\beta$, $\Omega \subset \mathbb{R}^n$ is a bounded domain with regular boundary $\partial\Omega$, and $\varphi \in C([-r,0]; E)$, $E = C(\bar{\Omega}; \mathbb{R})$, $\bar{\Omega} = \Omega \bigcup \partial\Omega$.

By setting $x(t) = x(\cdot, t)$, problem (33) can be rewritten as the following abstract form:

$$\begin{cases} {}^cD_t^\beta x(t) = A(t)x(t) + J_t^{1-\beta} f(t, x_t, \mathscr{K}x_t, \mathscr{H}x_t), \quad t \in [0,1], \\ x(t) = \varphi(t), \quad t \in [-r, 0], \end{cases} \tag{34}$$

where $x_t = x(t+\theta)$, $f(t, x_t, \mathscr{K}x_t, \mathscr{H}x_t) = ((t/(1+t^2))x_t + (1/(1+t^2))\int_0^t a(s)x_s ds + (1/(1+e^t))\int_0^1 b(s)x_s ds)$, and

$$\begin{cases} D(A) = \left\{x \in C(\bar{\Omega}, R) : x'' \in C(\bar{\Omega}, R)\right\}, \\ A(t)x = x'', \quad t \in [-r, 0]. \end{cases} \tag{35}$$

It is well known that the operator $A(t)$ generates a β-resolvent family $U_\beta(t,s)$ [23, 25]. Let equation (34) satisfy the conditions of Theorems 12–15; then, problem (34) has a global mild solution, which means that problem (33) has a mild solution.

5. Conclusion

In this paper, we study the existence and uniqueness of the global mild solutions for the fractional integrodifferential equations of mixed type with delay. Under the condition of the compact resolvent operator, we obtain Theorems 12 and 13, respectively, via various fixed point theorems and the measure of noncompactness. Theorem 15 is established by using the Banach contraction mapping principle under the condition of the noncompact resolvent operator. Furthermore, an example is provided to illustrate the main theorems. The kernels K and H of the operators $\mathscr{K}$ and $\mathscr{H}$ are nonlinear functions; meanwhile, the operator $A(t)$ is dependent on t. As a consequence, our main theorems improve and generalize many corresponding results by using different methods.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Research Article

On the System of Coupled Nondegenerate Kirchhoff Equations with Distributed Delay: Global Existence and Exponential Decay

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This paper studies the system of coupled nondegenerate viscoelastic Kirchhoff equations with a distributed delay. By using the energy method and Faedo-Galerkin method, we prove the global existence of solutions. Furthermore, we prove the exponential stability result.

1. Introduction

Let $\mathscr{H} = \Omega \times (\tau_1, \tau_2) \times (0, \infty)$, in this work, we consider

$$\begin{cases} |u|^l u_{tt} - M(\|\nabla u\|^2)\Delta u - \Delta u_{tt} + \int_0^t g_1(t-s)\Delta u(s)ds - \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| \Delta u_t(x, t-\varrho) d\varrho + f_1(u, v) = 0, \\ |v|^l v_{tt} - M(\|\nabla v\|^2)\Delta v - \Delta v_{tt} + \int_0^t g_2(t-s)\Delta v(s)ds - \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| \Delta v_t(x, t-\varrho) d\varrho + f_2(u, v) = 0, \end{cases} \tag{1}$$

where

$$(x, \varrho, t) \in \mathscr{H}, \tag{2}$$

under the initial and boundary conditions

$$\begin{cases} (u(x,0), v(x,0)) = (u_0(x), v_0(x)), \text{ in } \Omega, \\ (u_t(x,0), v_t(x,0)) = (u_1(x), v_1(x)), \text{ in } \Omega, \\ (u_t(x,-t), v_t(x,-t)) = (f_0(x,t), g_0(x,t)), \text{ in } \Omega \times (0, \tau_2), \\ u(x,t) = v(x,t) = 0, \text{ in } \partial\Omega \times (0, \infty), \end{cases} \tag{3}$$

where Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $l > 0$ and Δ is the Laplacian operator, and the functions $\mu_1, \mu_2 : [\tau_1, \tau_2] \longrightarrow \mathbb{R}$ are bounded, with $0 \le \tau_1 < \tau_2$, and the relaxation functions are denoted by g_1, g_2. The function M is given by

$$\begin{aligned} M &: \mathbb{R}_+ \longrightarrow \mathbb{R}_+, \\ r &\mapsto M(r) = a + br^\gamma, \end{aligned} \tag{4}$$

with $a, b > 0$, and $\gamma \ge 1$, and the functions f_1, f_2 will be defined later.

In 1976, Kirchhoff developed an equation describing the vibrations produced by a fixed series at its end, since it is considered a generalization of the d'Alembert equation, and it belongs to the wave equation models. Over time, many researchers and authors addressed these issues and problems with their continuous and rapid development, for example, see [1–4].

As for viscoelasticity, it is possible to delve into the following works for further clarification [3–10].

Also, the time or delay recorded in many natural and physical phenomena, especially problems resulting from vibrations, is an important factor for stability in general. And it has been studied extensively by many authors, including [5–7, 11–21]. Recently, in the presence of the varying delay, Mezouar and Boularrass studied system (1); for more information, see [22]. Based on these works, we in this work expand the results in [22] by adding the term of distributed delay.

We, under appropriate conditions, obtained the global existence of solutions, and we proved the exponential stability result of the system.

And we divided the paper into the following: in the second part, we set out the necessary hypotheses and the main result; in the third part, we prove the global existence of solutions, while in the fourth part, we present our result for exponential stability.

2. Preliminaries

In this section, we set the necessary hypotheses for proving the main result.

We need the following assumptions:

(A1) $g_i : \mathbb{R}_+ \longrightarrow \mathbb{R}_+, i = 1, 2$ are C^1 functions satisfying

$$g(0) > 0, a - \int_0^\infty g_i(s)ds \geq k > 0, i = 1, 2. \tag{5}$$

(A2) $\exists \xi_i > 0$ satisfying

$$g_i'(t) \leq -\xi_i g_i(t), i = 1, 2, t \geq 0. \tag{6}$$

(A3) The number l satisfying $0 < l \leq \gamma$ and

$$\begin{cases} \leq \dfrac{2}{n-2} & \text{if } n > 2, \\ \gamma < \infty & \text{if } \quad n \leq 2. \end{cases} \tag{7}$$

(A4)

$$\begin{cases} f_1(u, v) = a_1 v + b_1 |v|^{q+1} |v|^{p-1} u, \\ f_2(u, v) = a_1 u + b_2 |u|^{q+1} |v|^{p-1} v, \end{cases} \tag{8}$$

where $a_1 > 0, b_1 = (p+1)(p+q), b_2 = (q+1)(p+q)$ such that $p$and q are onjugate $((1/p) + (1/q) = 1), p, q < \gamma - (1/2)$ and satisfy

$$2 \leq p, q \leq \begin{cases} \sqrt{\dfrac{n}{2(n-2)}} & \text{if} \quad n > 2, \\ \infty & \text{if} \quad n \leq 2. \end{cases} \tag{9}$$

We set the notations

$$(g \circ \Psi)(t) := \int_0^t g(t-s) \|\Psi(t) - \Psi(s)\|^2 ds. \tag{10}$$

As in [17], we introduce the new variables

$$\begin{cases} u_t(x, t - \varrho\rho) = \mathscr{Z}(x, \rho, \varrho, t), \\ v_t(x, t - \varrho\rho) = \mathscr{Y}(x, \rho, \varrho, t). \end{cases} \tag{11}$$

We have

$$\begin{cases} \varrho \mathscr{Z}_t(x, \rho, \varrho, t) + \mathscr{Z}_\rho(x, \rho, \varrho, t) = 0, \\ u_t(x, t) = \mathscr{Z}(x, 0, \varrho, t), \end{cases} \\ \begin{cases} \varrho \mathscr{Y}_t(x, \rho, \varrho, t) + \mathscr{Y}_\rho(x, \rho, \varrho, t) = 0, \\ v_t(x, t) = \mathscr{Y}(x, 0, \varrho, t). \end{cases} \tag{12}$$

Consequently, problem (1) is equivalent to

$$\begin{cases} |u|^l u_{tt} - M(\|\nabla u\|^2)\Delta u - \Delta u_{tt} + \displaystyle\int_0^t g_1(t-s)\Delta u(s)ds - \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| \Delta \mathscr{Z}(x, 1, \varrho, t) d\varrho + f_1(u, v) = 0, \\ |v|^l v_{tt} - M(\|\nabla v\|^2)\Delta v - \Delta v_{tt} + \displaystyle\int_0^t g_2(t-s)\Delta v(s)ds - \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| \Delta \mathscr{Y}(x, 1, \varrho, t) d\varrho + f_2(u, v) = 0, \\ \rho \mathscr{Z}_t(x, \rho, \varrho, t) + \mathscr{Z}_\rho(x, \rho, \varrho, t) = 0, \\ \rho \mathscr{Y}_t(x, \rho, \varrho, t) + \mathscr{Y}_\rho(x, \rho, \varrho, t) = 0, \end{cases} \tag{13}$$

where

$$(x,\rho,s,t)\in\Omega\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty), \tag{14}$$

with the initial and boundary conditions

$$\begin{cases}(u(x,0),v(x,0))=(u_0(x),v_0(x)), \text{ in } \Omega,\\ (u_t(x,0),v_t(x,0))=(u_1(x),v_1(x)), \text{ in } \Omega,\\ (u_t(x,-t),v_t(x,-t))=(f_0(x,t),g_0(x,t)), \text{ in } \Omega\times(0,\tau_2),\\ u(x,t)=v(x,t)=0, \text{ in } \partial\Omega\times(0,\infty),\\ \mathscr{Z}(x,\rho,\varrho,0)=f_0(x,\rho\varrho), \text{ in } \Omega\times(0,1)\times(0,\tau_2),\\ \mathscr{Y}(x,\rho,\varrho,0)=g_0(x,\rho\varrho).\end{cases} \tag{15}$$

We need the following lemma.

Lemma 1. *The energy functional E, given by*

$$\begin{aligned}E(t)&=\frac{1}{l+2}\left(\|u_t\|_{l+2}^{l+2}+\|v_t\|_{l+2}^{l+2}\right)+\frac{b}{2(\gamma+2)}\\&\cdot\left(\|\nabla u\|^{2(\gamma+2)}+\|\nabla v\|^{2(\gamma+2)}\right)+\frac{1}{2}\left(a-\int_0^t g_1(s)ds\right)\|\nabla u\|^2\\&+\frac{1}{2}\left(a-\int_0^t g_2(s)ds\right)\|\nabla v\|^2+\frac{1}{2}\left(\|\nabla u_t\|^2+\|\nabla v_t\|^2\right)\\&+\frac{1}{2}(g_1\circ\nabla u)(t)+\frac{1}{2}(g_2\circ\nabla v)(t)+\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}\rho(|\mu_1(\rho)|\\&\cdot\|\nabla\mathscr{Z}\|^2+|\mu_2(\rho)|\|\nabla\mathscr{Y}\|^2)d\rho d\rho+\alpha\int_\Omega uvdx\\&+(p+q)\int_\Omega|u|^{p+1}|v|^{q+1}dx,\end{aligned} \tag{16}$$

satisfies

$$\begin{aligned}E'(t)&\le-\beta\int_{\tau_1}^{\tau_2}\left(|\mu_1(\varrho)|\|\nabla\mathscr{Z}(x,1,\varrho,t)\|^2+|\mu_2(\varrho)|\right.\\&\cdot\left.\|\nabla\mathscr{Y}(x,1,\varrho,t)\|^2\right)d\varrho+\lambda\left(\|\nabla u_t\|^2+\|\nabla v_t\|^2\right)\\&+\frac{1}{2}\left(g_1'\circ\nabla u\right)(t)+\frac{1}{2}\left(g_2'\circ\nabla v\right)(t)-\frac{1}{2}g_1(t)\\&\cdot\|\nabla u(t)\|^2-\frac{1}{2}g_2(t)\|\nabla v(t)\|^2,\end{aligned} \tag{17}$$

where $\beta=((1-\delta_1)/2)>0$,

and $\lambda=\max\{\lambda_1=((\delta_1+1)/2)\int_{\tau_1}^{\tau_2}|\mu_1(\rho)|d\rho,\lambda_2=((\delta_1+1)/2)\int_{\tau_1}^{\tau_2}|\mu_2(\rho)|d\rho\}$, $\delta_1<1$.

Proof. Multiplying equation $(13)_{1,2}$ by u_t, v_t, and we use (15), one gets

$$\begin{aligned}&\frac{d}{dt}\left\{\frac{1}{l+2}\|u_t\|_{l+2}^{l+2}+\frac{b}{2(\gamma+1)}\|\nabla u_t\|^{2(\gamma+1)}+\frac{1}{2}\left(a-\int_0^t g_1(s)ds\right)\right.\\&\left.\cdot\|\nabla u\|^2+\frac{1}{2}\|\nabla u_t\|^2+\frac{1}{2}(g_1\circ\nabla u)(t)\right\}-\frac{1}{2}\left(g_1'\circ\nabla u\right)(t)\\&+\frac{1}{2}g_1(t)\|\nabla u(t)\|^2+\int_\Omega u_t\int_{\tau_1}^{\tau_2}|\mu_1(\varrho)|\Delta\mathscr{Z}(x,1,\varrho,t)d\varrho dx\\&+\int_\Omega u_tvdx+b_1\int_\Omega u_tu|u|^{p-1}|v|^{q+1}dx+\frac{d}{dt}\left\{\frac{1}{l+2}\|v_t\|_{l+2}^{l+2}\right.\\&+\frac{b}{2(\gamma+1)}\|\nabla v_t\|^{2(\gamma+1)}+\frac{1}{2}\left(a-\int_0^t g_2(s)ds\right)\|\nabla v\|^2\\&\left.+\frac{1}{2}\|\nabla v_t\|^2+\frac{1}{2}(g_2\circ\nabla v)(t)\right\}-\frac{1}{2}\left(g_2'\circ\nabla v\right)(t)\\&+\frac{1}{2}g_2(t)\|\nabla v(t)\|^2+\int_\Omega v_t\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|\Delta\mathscr{Y}(x,1,\varrho,t)d\varrho dx\\&+\int_\Omega v_tudx+b_2\int_\Omega v_tv|v|^{p-1}|u|^{q+1}dx.\end{aligned} \tag{18}$$

And multiplying equation $(13)_3$ by $-\Delta\mathscr{Z}|\mu_1(\varrho)|$, and integrating the result over $\Omega\times(0,1)\times(\tau_1,\tau_2)$, one gets

$$\begin{aligned}&\frac{d}{dt}\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}\varrho|\mu_1(\varrho)|(\nabla\mathscr{Z})^2d\varrho d\rho dx\\&=-\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}|\mu_1(\varrho)|\nabla\mathscr{Z}\nabla\mathscr{Z}_\rho d\varrho d\rho dx\\&=-\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}|\mu_1(\varrho)|\frac{d}{d\rho}(\nabla\mathscr{Z})^2d\varrho d\rho dx\\&=\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_1(\varrho)|\left((\nabla\mathscr{Z}(x,0,\varrho,t))^2-(\nabla\mathscr{Z}(x,1,\varrho,t))^2\right)d\varrho dx\\&=\frac{1}{2}\int_{\tau_1}^{\tau_2}|\mu_1(\varrho)|d\rho\int_\Omega|\nabla u_t|^2dx-\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_1(\varrho)|\\&\cdot(\nabla\mathscr{Z}(x,1,\varrho,t))^2d\varrho dx=\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}|\mu_1(\varrho)|d\varrho\right)\\&\cdot\|\nabla u_t\|^2-\frac{1}{2}\int_{\tau_1}^{\tau_2}|\mu_1(\varrho)|\|\nabla\mathscr{Z}(x,1,\varrho,t)\|^2d\varrho.\end{aligned} \tag{19}$$

Similarly, multiplying equation $(13)_4$ by $-\Delta\mathscr{Y}\,|\,\mu_2(\rho)\,|$, we find

$$\begin{aligned}&\frac{d}{dt}\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}\varrho|\mu_2(\varrho)|(\nabla\mathscr{Y})^2d\varrho d\rho dx\\&=\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|d\varrho\right)\|\nabla v_t\|^2\\&-\frac{1}{2}\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|\|\nabla\mathscr{Y}(x,1,\varrho,t)\,\|^2d\varrho,\end{aligned} \tag{20}$$

by using the inequalities of Young and Cauchy-Schwartz for $\delta_1 > 0$, we have

$$\begin{aligned}\int_\Omega \nabla u_t \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| \nabla \mathscr{Z}(x,1,\varrho,t) d\varrho dx \\ \le \frac{\delta_1}{2}\left(\int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| d\varrho\right) \|\nabla u_t\|^2 \\ + \frac{\delta_1}{2}\int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| \|\nabla \mathscr{Z}(x,1,\varrho,t)\|^2 d\varrho.\end{aligned} \tag{21}$$

Similarly, we get

$$\begin{aligned}\int_\Omega \nabla v_t \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| \nabla \mathscr{Y}(x,1,\varrho,t) d\varrho dx \\ \le \frac{\delta_1}{2}\left(\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho\right) \|\nabla v_t\|^2 + \frac{\delta_1}{2}\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| \\ \cdot \|\nabla \mathscr{Y}(x,1,\varrho,t)\|^2 d\varrho.\end{aligned} \tag{22}$$

By summing (18)–(20) and using (21) and (22), and choosing δ_1 such that $\delta_1 < 1$, we find (16) and (17). This completes the proof.

3. Global Existence

Theorem 2. *Suppose that (5)–(8) hold. Then, given* $(u_0, v_0) \in (H^2(\Omega) \cap H_0^1(\Omega))^2$, $(u_1, v_1) \in (H_0^1(\Omega))^2$, *and* $(f_0, g_0) \in (H^1(\Omega, (0,1), (\tau_1, \tau_2)))^2$, *there exists a weak solution* $(u, v, \mathscr{Z}, \mathscr{Y})$ *of problem (13)–(15) such that*

$$\begin{aligned}(u, v, \mathscr{Z}, \mathscr{Y}) \in L^\infty(\mathbb{R}_+, \mathscr{H}_1), u_t, v_t \\ \in L^\infty(\mathbb{R}_+, H_0^1(\Omega)), u_{tt}, v_{tt} \\ \in L^2(\mathbb{R}_+, H_0^1(\Omega)),\end{aligned} \tag{23}$$

where

$$\mathscr{H}_1 = (H^2(\Omega) \cap H_0^1(\Omega))^2 \times (H^1(\Omega, (0,1), (\tau_1, \tau_2)))^2. \tag{24}$$

Proof. Let the Galerkin basis $u_j, v_j, \mathscr{Z}_j, \mathscr{Y}_j$, for $n \ge 1$, we set

$$\begin{aligned}W_n = \operatorname{span}\{u_1, u_2, \cdots, u_n\}, \\ K_n = \operatorname{span}\{v_1, v_2, \cdots, v_n\}.\end{aligned} \tag{25}$$

The sequences $\mathscr{Z}_j(x,\tau,p), \mathscr{Y}_j(x,\tau,p)$ are defined for $1 \le j \le n$ by

$$\mathscr{Z}_j(x,0,p) = u_j(x), \mathscr{Y}_j(x,0,p) = v_j(x). \tag{26}$$

Then, taking $\mathscr{Z}_j(x,0,p), \mathscr{Y}_j(x,0,p)$ by over $L^2((0,1) \times (0,1) \times (\tau_1, \tau_2))$ and denoting

$$\begin{aligned}Z_n = \operatorname{span}\{\mathscr{Z}_1, \mathscr{Z}_2, \cdots, \mathscr{Z}_n\}, \\ Y_n = \operatorname{span}\{\mathscr{Y}_1, \mathscr{Y}_2, \cdots, \mathscr{Y}_n\}.\end{aligned} \tag{27}$$

Given initial data $u_0, v_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $u_1, v_1 \in H_0^1(\Omega)$, and $f_0, g_0 \in L^2((\Omega) \times (0,1) \times (\tau_1, \tau_2))$, we define the approximations

$$\begin{aligned}u_m &= \sum_{j=1}^n g_{jm}(t) u_j(x), \\ v_m &= \sum_{j=1}^n h_{jm}(t) v_j(x), \\ \mathscr{Z}_m &= \sum_{j=1}^n f_{jm}(t) \mathscr{Z}_j(x,\tau,p), \\ \mathscr{Y}_m &= \sum_{j=1}^n k_{jm}(t) \mathscr{Y}_j(x,\tau,p).\end{aligned} \tag{28}$$

It investigates the following problem:

$$\begin{aligned}&\left(|u_{mt}|^l u_{mtt}, u_j\right) + M(\|\nabla u_m(t)\|)\left(\nabla u_m, \nabla u_j\right) \\ &\quad + \left(\nabla u_{mtt}, \nabla u_j\right) + \left(f_1(u_m, v_m), u_j\right) \\ &\quad - \int_0^t g_1(t-s)\left(\nabla u_m(s), \nabla u_j\right) ds \\ &\quad + \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| \left(\nabla \mathscr{Z}_m(x,1,\varrho,t), \nabla u_j\right) d\varrho = 0, \\ &\left(|v_{mt}|^l v_{mtt}, v_j\right) + M(\|\nabla v_m(t)\|)\left(\nabla v_m, \nabla v_j\right) \\ &\quad + \left(\nabla v_{mtt}, \nabla v_j\right) + \left(f_2(u_m, v_m), v_j\right) \\ &\quad - \int_0^t g_2(t-s)\left(\nabla v_m(s), \nabla v_j\right) ds \\ &\quad + \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| \left(\nabla \mathscr{Y}_m(x,1,\varrho,t), \nabla v_j\right) d\varrho = 0, \\ &\left(\varrho \mathscr{Z}_{mt}(x,\rho,\varrho,t), \mathscr{Z}_j\right) + \left(\mathscr{Z}_{m\rho}(x,\rho,\varrho,t), \mathscr{Z}_j\right) = 0, \\ &\left(\varrho \mathscr{Y}_{mt}(x,\rho,\varrho,t), \mathscr{Y}_j\right) + \left(\mathscr{Y}_{m\rho}(x,\rho,\varrho,t), \mathscr{Y}_j\right) = 0,\end{aligned} \tag{29}$$

with initial conditions

$$\begin{aligned}u_m(0) = u_0^m, u_{mt}(0) = u_1^m, \\ v_m(0) = v_1^m, v_{mt}(0) = v_1^m, \\ \mathscr{Z}_m(0) = \mathscr{Z}_0^m, \mathscr{Y}_m(0) = \mathscr{Y}_0^m,\end{aligned} \tag{30}$$

which satisfies

$$
\begin{aligned}
u_0^m &\longrightarrow u_0, \text{ in } H^2(\Omega)\cap H_0^1(\Omega),\\
u_1^m &\longrightarrow u_1, \text{ in } H_0^1(\Omega),\\
v_0^m &\longrightarrow v_0, \text{ in } H^2(\Omega)\cap H_0^1(\Omega),\\
v_1^m &\longrightarrow v_1, \text{ in } H_0^1(\Omega),\\
\mathscr{Z}_0^m &\longrightarrow \mathscr{Z}_0, \text{ in } L^2(\Omega\times(0,1)\times(\tau_1,\tau_2)),\\
\mathscr{Y}_0^m &\longrightarrow \mathscr{Y}_0, \text{ in } L^2(\Omega\times(0,1)\times(\tau_1,\tau_2)).
\end{aligned}
\tag{31}
$$

Noting that $(l/(2(l+1))) + (1/(2(l+1))) + (1/2) = 1$, by using Hölder's inequality, we get

$$
\begin{aligned}
\left(|u_{mt}|^l u_{mtt}, u_j\right) &= \int_\Omega |u_{mt}|^l u_{mtt} u_j dx\\
&\le \left(\int_\Omega |u_{mt}|^{2(l+1)} dx\right)^{1/(2(l+1))}\\
&\quad\cdot \|u_{mtt}\|_{2(l+1)} \|u_j\|_2.
\end{aligned}
\tag{32}
$$

As (8) holds, using the embedding of Sobolev, the terms $(|u_{mt}|^l u_{mtt}, u_j)$ and $(|v_{mt}|^l v_{mtt}, v_j)$ in (29) make sense (see [22]).

First estimate.

As the sequences $u_0^m, v_0^m, u_1^m, v_1^m, \mathscr{Z}_0^m(.,.,0)$ and $\mathscr{Y}_0^m(.,.,0)$ converge and from (17) and Gronwall's lemma, we get $C_1 > 0$ independent of m such that

$$
\begin{aligned}
E_m(t) &+ \beta \int_{\tau_1}^{\tau_2} \varrho\big(|\mu_1(\varrho)| \|\nabla \mathscr{Z}_m(x,1,\varrho,t)\|^2\\
&+ |\mu_2(\varrho)| \|\nabla \mathscr{Y}_m(x,1,\varrho,t)\|^2\big) d\varrho \le C_1,
\end{aligned}
\tag{33}
$$

where

$$
\begin{aligned}
E_m(t) &= \frac{1}{l+2}\left(\|u_{mt}\|_{l+2}^{l+2} + \|v_{mt}\|_{l+2}^{l+2}\right) + \frac{b}{2(\gamma+2)}\left(\|\nabla u_m\|^{2(\gamma+2)}\right.\\
&\left.+ \|\nabla v_m\|^{2(\gamma+2)}\right) + \frac{1}{2}\left(a - \int_0^t g_1(s)ds\right)\|\nabla u_m\|^2\\
&+ \frac{1}{2}\left(a - \int_0^t g_2(s)ds\right)\|\nabla v_m\|^2 + \frac{1}{2}\left(\|\nabla u_{mt}\|^2 + \|\nabla v_{mt}\|^2\right)\\
&+ \frac{1}{2}(g_1 \circ \nabla u_m)(t) + \frac{1}{2}(g_2 \circ \nabla v_m)(t)\\
&+ \frac{1}{2}\int_0^1 \int_{\tau_1}^{\tau_2} \varrho\left(|\mu_1(\varrho)| \|\nabla \mathscr{Z}_m\|^2 + |\mu_2(\varrho)| \|\nabla \mathscr{Y}_m\|^2\right) d\varrho d\rho\\
&+ \alpha \int_\Omega u_m v_m dx + (p+q)\int_\Omega |u_m|^{p+1} |v_m|^{q+1} dx,
\end{aligned}
\tag{34}
$$

using (33) and (8), one gets

$$
\begin{aligned}
&u_m, v_m \text{ are bounded in } L^\infty_{\text{loc}}\left(\mathbb{R}_+, H_0^1(\Omega)\right),\\
&u_{mt}, v_{mt} \text{ are bounded in } L^\infty_{\text{loc}}\left(\mathbb{R}_+, H_0^1(\Omega)\right),\\
&\mathscr{Z}_m(x,\rho,\varrho,t), \mathscr{Y}_m(x,\rho,\varrho,t) \text{ are bounded in}\\
&L^\infty_{\text{loc}}\left(\mathbb{R}_+, H_0^1(\Omega\times(0,1)\times(\tau_1,\tau_2))\right).
\end{aligned}
\tag{35}
$$

The second estimate.

We multiply equation $(29)_{1,2}$ by g_{jmtt}, h_{jmtt}; by summing j from 1 to n, one gets

$$
\begin{aligned}
&\int_\Omega |u_{mt}|^l |u_{mtt}|^2 dx + \int_\Omega M(\|\nabla u_m(t)\|) \nabla u_m \nabla u_{mtt} dx\\
&\quad + \int_\Omega |\nabla u_{mtt}|^2 dx + \int_\Omega f_1(u_m, v_m) u_{mtt} dx\\
&\quad - \int_\Omega \int_0^t g_1(t-s) \nabla u_m(s) \nabla u_{mtt} ds dx\\
&\quad + \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_1(\rho)| \nabla \mathscr{Z}_m(x,1,\rho,t) \nabla u_{mtt} d\rho dx = 0,\\
&\int_\Omega |v_{mt}|^l |v_{mtt}|^2 dx + \int_\Omega M(\|\nabla v_m(t)\|) \nabla v_m \nabla v_{mtt} dx\\
&\quad + \int_\Omega |\nabla v_{mtt}|^2 dx + \int_\Omega f_2(u_m, v_m) v_{mtt} dx\\
&\quad - \int_\Omega \int_0^t g_2(t-s) \nabla v_m(s) \nabla v_{mtt} ds dx\\
&\quad + \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| \nabla \mathscr{Y}_m(x,1,\varrho,t) \nabla v_{mtt} d\varrho dx = 0.
\end{aligned}
\tag{36}
$$

By differentiating $(29)_{3,4}$, we get

$$
\begin{aligned}
&\left(\varrho \mathscr{Z}_{mtt}(x,\rho,\varrho,t) + \mathscr{Z}_{mt\rho}(x,\rho,\varrho,t), \mathscr{Z}_j\right) = 0,\\
&\left(\varrho \mathscr{Y}_{mtt}(x,\rho,\varrho,t) + \mathscr{Y}_{mt\rho}(x,\rho,\varrho,t), \mathscr{Y}_j\right) = 0.
\end{aligned}
\tag{37}
$$

And we multiply $(37)_1$ by $\mathscr{Z}_{jmt}$ and $(37)_2$ by $\mathscr{Y}_{jmt}$; by summing j from 1 to n, we have

$$
\begin{aligned}
&\frac{1}{2}\varrho \frac{d}{dt}\|\mathscr{Z}_{mt}\|^2 + \frac{1}{2}\frac{d}{d\rho}\|\mathscr{Z}_{mt}\|^2 = 0,\\
&\frac{1}{2}\varrho \frac{d}{dt}\|\mathscr{Y}_{mt}\|^2 + \frac{1}{2}\frac{d}{d\rho}\|\mathscr{Y}_{mt}\|^2 = 0.
\end{aligned}
\tag{38}
$$

Integrating the result (38) over $(0,1)$ with respect to ρ, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 \varrho \|\mathscr{Z}_{mt}\|^2 d\rho + \frac{1}{2}\|\mathscr{Z}_{mt}(x,1,\varrho,t)\|^2 - \frac{1}{2}\|u_{mtt}(x,t)\|^2 = 0,\\
&\frac{1}{2}\frac{d}{dt}\int_0^1 \varrho \|\mathscr{Y}_{mt}\|^2 d\rho + \frac{1}{2}\|\mathscr{Y}_{mt}(x,1,\varrho,t)\|^2 - \frac{1}{2}\|v_{mtt}(x,t)\|^2 = 0.
\end{aligned}
\tag{39}
$$

Summing (36) and (39) and using $M(r) \geq a$, we get

$$\begin{aligned}
&\int_\Omega |u_{mt}|^l |u_{mtt}|^2 dx + \|\nabla u_{mtt}\|^2 \\
&\quad + \frac{1}{2}\frac{d}{dt}\int_0^1 \varrho \|\mathscr{Z}_{mt}\|^2 d\rho + \frac{1}{2}\|\mathscr{Z}_{mt}(x,1,\rho,t)\|^2 \\
&\leq \frac{1}{2}\|u_{mtt}\|^2 - \int_\Omega a\nabla u_m \nabla u_{mtt} dx - \int_\Omega f_1(u_m, v_m) u_{mtt} dx \\
&\quad + \int_\Omega \int_0^t g_1(t-s)\nabla u_m(s)\nabla u_{mtt} ds dx \\
&\quad - \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| \nabla \mathscr{Z}_m(x,1,\varrho,t)\nabla u_{mtt} d\varrho dx, \\
&\int_\Omega |v_{mt}|^l |v_{mtt}|^2 dx + \|\nabla v_{mtt}\|^2 \\
&\quad + \frac{1}{2}\frac{d}{dt}\int_0^1 \varrho \|\mathscr{Y}_{mt}\|^2 d\rho + \frac{1}{2}\|\mathscr{Y}_{mt}(x,1,\varrho,t)\|^2 \\
&\leq \frac{1}{2}\|v_{mtt}(x,t)\|^2 - \int_\Omega a\nabla v_m \nabla v_{mtt} dx \\
&\quad - \int_\Omega f_2(u_m, v_m) v_{mtt} dx + \int_\Omega \int_0^t g_2(t-s)\nabla v_m(s)\nabla v_{mtt} ds dx \\
&\quad - \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| \nabla \mathscr{Y}_m(x,1,\varrho,t)\nabla v_{mtt} d\rho dx.
\end{aligned} \tag{40}$$

At this point, we estimate the RHS of (40).

Integrating by parts, and using Young's and Poincare' inequalities, one gets

$$\begin{aligned}
\left| -\int_\Omega f_1(u_m, v_m) u_{mtt} dx \right| &\leq \frac{C_*^2 \alpha}{2}\left(\|\nabla u_{mt}\|^2 + \|\nabla v_{mt}\|^2\right) \\
&\quad + \frac{b_1 \eta C_*^{4q(q+1)}}{2} |\Omega|^{\frac{q-1}{2q}} \|\nabla v_m\|^{4q(q+1)} \\
&\quad + \frac{b_1 C_*^{2p^2}}{8\eta}\|\nabla u_m\|^{2p^2} + \frac{b_1 C_*^2}{2}\|\nabla u_{mtt}\|^2.
\end{aligned} \tag{41}$$

Similarly, we get

$$\begin{aligned}
\left| -\int_\Omega f_2(u_m, v_m) v_{mtt} dx \right| &\leq \frac{C_*^2 \alpha}{2}\left(\|\nabla u_{mt}\|^2 + \|\nabla v_{mt}\|^2\right) \\
&\quad + \frac{b_1 \eta C_*^{4p(p+1)}}{2} |\Omega|^{(p-1)/2p} \|\nabla u_m\|^{4p(p+1)} \\
&\quad + \frac{b_1 C_*^{2q^2}}{8\eta}\|\nabla v_m\|^{2q^2} + \frac{b_1 C_*^2}{2}\|\nabla v_{mtt}\|^2.
\end{aligned} \tag{42}$$

And, by using the inequality of Young, we get

$$\begin{aligned}
\left|\int_\Omega a\nabla u_m \nabla u_{mtt} dx\right| &\leq \eta\|\nabla u_{mtt}\|^2 + \frac{a^2}{4\eta}\|\nabla u_m\|^2, \\
\left|\int_\Omega a\nabla v_m \nabla v_{mtt} dx\right| &\leq \eta\|\nabla v_{mtt}\|^2 + \frac{a^2}{4\eta}\|\nabla v_m\|^2,
\end{aligned} \tag{43}$$

we have

$$\begin{aligned}
&\left|\int_\Omega \int_0^t g_1(t-s)\nabla u_m(s)\nabla u_{mtt} ds dx\right| \\
&\quad \leq \eta\|\nabla u_{mtt}\|^2 + \frac{(a-k)g_1(0)}{4\eta}\int_0^t \|\nabla u_m(s)\|^2 ds, \\
&\left|\int_\Omega \int_0^t g_2(t-s)\nabla v_m(s)\nabla v_{mtt} ds dx\right| \\
&\quad \leq \eta\|\nabla v_{mtt}\|^2 + \frac{(a-k)g_2(0)}{4\eta}\int_0^t \|\nabla v_m(s)\|^2 ds.
\end{aligned} \tag{44}$$

Similarly, we get

$$\begin{aligned}
&\left|\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| \nabla \mathscr{Z}_m(x,1,\varrho,t)\nabla u_{mtt} d\varrho dx\right| \\
&\quad \leq \eta\lambda_1\|\nabla u_{mtt}\|^2 + \frac{1}{4\eta}\int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| \|\nabla \mathscr{Z}_m(x,1,\varrho,t)\|^2 d\varrho \\
&\quad \cdot \left|\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| \nabla \mathscr{Y}_m(x,1,\varrho,t)\nabla v_{mtt} d\varrho dx\right| \\
&\quad \leq \eta\lambda_2\|\nabla v_{mtt}\|^2 + \frac{1}{4\eta}\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| \|\nabla \mathscr{Y}_m(x,1,\varrho,t)\|^2 d\varrho,
\end{aligned} \tag{45}$$

substituting (41)–(45) into (40), and using (17), one gets

$$\begin{aligned}
&\int_\Omega |u_{mt}|^l |u_{mtt}|^2 dx + \left(1 - \left\{\eta(\lambda_1 + 2) + \frac{(1+b_1)C_*^2}{2}\right\}\right) \\
&\quad \cdot \|\nabla u_{mtt}\|^2 + \frac{1}{2}\frac{d}{dt}\int_0^1 \varrho \|\mathscr{Z}_{mt}\|^2 d\varrho + \frac{1}{2}\|\mathscr{Z}_{mt}(x,1,\varrho,t)\|^2 \\
&\leq C_2 + \frac{1}{4\eta}(a-k)g_1(0)C_1 T, \\
&\int_\Omega |v_{mt}|^l |v_{mtt}|^2 dx + \left(1 - \left\{\eta(\lambda_2 + 2) + \frac{(1+b_2)C_*^2}{2}\right\}\right) \\
&\quad \cdot \|\nabla v_{mtt}\|^2 + \frac{1}{2}\frac{d}{dt}\int_0^1 \varrho \|\mathscr{Y}_{mt}\|^2 d\varrho + \frac{1}{2}\|\mathscr{Y}_{mt}(x,1,\varrho,t)\|^2 \\
&\leq C_2 + \frac{1}{4\eta}(a-k)g_2(0)C_1 T,
\end{aligned} \tag{46}$$

where $C_2 > 0$ depends on $\eta, \alpha, a, C_*, b_1, b_2, p, q, C_1$.

Integrating (41) over $(0, t)$, we get

$$\begin{aligned}
&\int_0^t\int_\Omega |u_{mt}(\sigma)|^l |u_{mtt}(\sigma)|^2 dx d\sigma \\
&\quad + \left(1 - \left\{\eta(\lambda_1 + 2) + \frac{(1+b_1)C_*^2}{2}\right\}\right) \\
&\quad \cdot \int_0^t \|\nabla u_{mtt}(\sigma)\|^2 d\sigma + \frac{1}{2}\int_0^1 \varrho \|\mathscr{Z}_{mt}\|^2 d\varrho \\
&\quad + \frac{1}{2}\int_0^t \|\mathscr{Z}_{mt}(x,1,\varrho,\sigma)\|^2 d\sigma \\
&\le \left(C_2 + \frac{1}{4\eta}(a-k)g_1(0)C_1 T\right)T, \\
&\int_0^t\int_\Omega \left|v_{mt}(\sigma)^l\right| v_{mtt}(\sigma)\Big|^2 dx d\sigma \\
&\quad + \left(1 - \left\{\eta(\lambda_2 + 2) + \frac{(1+b_2)C_*^2}{2}\right\}\right) \\
&\quad \cdot \int_0^t \|\nabla v_{mtt}(\sigma)\|^2 d\sigma + \frac{1}{2}\int_0^1 \varrho \|\mathscr{Y}_{mt}\|^2 d\varrho \\
&\quad + \frac{1}{2}\int_0^t \|\mathscr{Y}_{mt}(x,1,\varrho,\sigma)\|^2 d\sigma \\
&\le \left(C_2 + \frac{1}{4\eta}(a-k)g_2(0)C_1 T\right)T.
\end{aligned} \tag{47}$$

At this stage, choosing $\eta > 0$ such that

$$\left(1 - \left\{\eta(\lambda_i + 2) + \frac{(1+b_i)C_*^2}{2}\right\}\right) > 0, \text{ for } i = 1, 2, \tag{48}$$

we find

$$\begin{aligned}
&\int_0^t \left(\|\nabla u_{mtt}(\sigma)\|^2 + \|\nabla v_{mtt}(\sigma)\|^2\right) d\sigma \\
&\quad + \frac{1}{2}\int_0^1 \rho\left(\|\mathscr{Z}_{mt}\|^2 + \|\mathscr{Y}_{mt}\|^2\right) d\rho \le C_3.
\end{aligned} \tag{49}$$

We have from (17) and (49) that there exist subsequences (u_k) of (u_m) and (v_k) of (v_m) such that

$$\begin{aligned}
&(u_k, v_k) \rightharpoonup (u, v) \text{ weakly star in } L^\infty\left(0, T, H_0^1(\Omega)\right), \\
&(u_{kt}, v_{kt}) \rightharpoonup (u_t, v_t) \text{ weakly star in } L^\infty\left(0, T, H_0^1(\Omega)\right), \\
&(u_{ktt}, v_{ktt}) \rightharpoonup (u_{tt}, v_{tt}) \text{ weakly star in } L^2\left(0, T, H_0^1(\Omega)\right), \\
&(\mathscr{Z}_k, \mathscr{Y}_k) \rightharpoonup (\mathscr{Z}, \mathscr{Y}) \text{ weakly star in } \\
&\qquad L^\infty\left(0, T, L^2(\Omega \times (0,1) \times (\tau_1, \tau_2)\right), \\
&(\mathscr{Z}_{kt}, \mathscr{Y}_{kt}) \rightharpoonup (\mathscr{Z}_t, \mathscr{Y}_t) \textit{ weakly star } \text{in} \\
&\qquad L^\infty\left(0, T, L^2(\Omega \times (0,1) \times (\tau_1, \tau_2)\right).
\end{aligned} \tag{50}$$

We work now with the nonlinear term. From (17), we find

$$\begin{aligned}
\left\| |u_{kt}|^l u_{kt} \right\|_{L^2(0,T,L^2(\Omega))} &= \int_0^T \|u_{kt}\|_{2(l+1)}^{2(l+1)} dt \\
&\le C_*^{2(l+1)} \int_0^T \|u_{kt}\|_2^{2(l+1)} dt \le C_4,
\end{aligned} \tag{51}$$

where C_4 depends only on C_*, C_1, T, l.

And from the theorem of Aubin-Lions (see Lions [23]), we deduce that there exists a subsequence of (u_k), given by (u_k), such that

$$u_{kt} \longrightarrow u_t \text{ stongly in } L^2\left(0, T, L^2(\Omega)\right), \tag{52}$$

we get

$$u_{kt} \longrightarrow u_t \text{ almost everywhere in } \Omega \times \mathbb{R}_+. \tag{53}$$

Hence,

$$|u_{kt}|^l u_{kt} \longrightarrow |u_t|^l u_t \text{ almost everywhere in } \Omega \times \mathbb{R}_+. \tag{54}$$

Thus, using (46) and (48) and the Lions lemma, we derive

$$|u_{kt}|^l u_{kt} \rightharpoonup |u_t|^l u_t \text{ weakly in } L^2\left(0, T, L^2(\Omega)\right). \tag{55}$$

Similarly,

$$|v_{kt}|^l v_{kt} \rightharpoonup |v_t|^l v_t \text{ weakly in } L^2\left(0, T, L^2(\Omega)\right), \tag{56}$$

$$\begin{aligned}
&(\mathscr{Z}_k, \mathscr{Y}_k) \longrightarrow (\mathscr{Z}, \mathscr{Y}) \text{ stongly in} \\
&\qquad L^2\left(0, T, L^2(\Omega \times (0,1) \times (\tau_1, \tau_2)\right),
\end{aligned} \tag{57}$$

which implies

$$\begin{aligned}
&(\mathscr{Z}_k, \mathscr{Y}_k) \longrightarrow (z, y) \text{ almost everywhere in } \Omega \times (0,1) \\
&\qquad \times (\tau_1, \tau_2) \times \mathbb{R}_+.
\end{aligned} \tag{58}$$

The sequences (u_k) and (v_k) satisfy

$$\begin{aligned}
&f_1(u_k, v_k) \longrightarrow f_1(u, v) \text{ stongly in } L^2\left(0, T, L^2(\Omega)\right), \\
&f_2(u_k, v_k) \longrightarrow f_2(u, v) \text{ stongly in } L^2\left(0, T, L^2(\Omega)\right).
\end{aligned} \tag{59}$$

We have

$$\|f_1(u_k, v_k) - f_1(u, v)\|^2 = \int_\Omega \left| |v_m|^{q+1} |u_m|^p u_m - |v|^{q+1} |u|^p u \right|^2 dx. \tag{60}$$

Noting that $(l/2p) + (1/2q) + (1/2) = 1$, by applying the generalized Hölder's and Young's inequalities, and (8), we get

$$\|f_1(u_k, v_k) - f_1(u, v)\|^2 \le C\left[\|\nabla(u_m - u)\|^2 + \|\nabla(v_m - v)\|^2\right]. \tag{61}$$

As (u_k) and (v_k) are Cauchy sequences in $L^{\infty}(0, T, H_0^1(\Omega))$ (prove it as in [1]), then we get $(59)_1$. Similarly, we get the convergence $(59)_2$.

Multiplying (29) by $\Psi(t) \in \mathcal{D}(0, T)$ and integrating the result over $(0, T)$, we get

$$
\begin{aligned}
&-\frac{1}{l+1}\int_0^T \left(|u_{mt}|^l u_{mtt}, u_j\right)\Psi'(t)dt \\
&\quad + \int_0^T M(\|\nabla u_m(t)\|)(\nabla u_m, \nabla u_j)\Psi(t)dt \\
&\quad + \int_0^T (\nabla u_{mtt}, \nabla u_j)\Psi(t)dt + \int_0^T \left(f_1(u_m, v_m), u_j\right)\Psi(t)dt \\
&\quad - \int_0^T\int_0^t g_1(t-s)(\nabla u_m(s), \nabla u_j)\Psi(t)dsdt \\
&\quad + \int_0^T\int_{\tau_1}^{\tau_2} |\mu_1(\varrho)|(\nabla \mathscr{Z}_m(x, 1, \varrho, t), \nabla u_j)\Psi(t)d\varrho dt = 0, \\
&-\frac{1}{l+1}\int_0^T \left(|v_{mt}|^l v_{mtt}, v_j\right)\Psi'(t)dt \\
&\quad + \int_0^T M(\|\nabla v_m(t)\|)(\nabla v_m, \nabla v_j)\Psi(t)dt \\
&\quad + \int_0^T (\nabla v_{mtt}, \nabla v_j)\Psi(t)dt + \int_0^T \left(f_2(u_m, v_m), v_j\right)\Psi(t)dt \\
&\quad - \int_0^T\int_0^t g_2(t-s)(\nabla v_m(s), \nabla v_j)\Psi(t)dsdt \\
&\quad + \int_0^T\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|(\nabla \mathscr{Y}_m(x, 1, \varrho, t), \nabla v_j)\Psi(t)d\varrho dt = 0, \\
&\int_0^T (\varrho \mathscr{Z}_{mt}(x, \rho, \varrho, t) + \mathscr{Z}_{m\rho}(x, \rho, \varrho, t), \mathscr{Z}_j)\Psi(t)dt = 0, \\
&\int_0^T (\rho \mathscr{Y}_{mt}(x, \rho, \varrho, t) + \mathscr{Y}_{m\rho}(x, \rho, \varrho, t), \mathscr{Y}_j)\Psi(t)dt = 0, \\
&\qquad \forall j = 1, \cdots, m.
\end{aligned} \tag{62}
$$

We obtain (62) by the convergence of (50), (54), (56), and (59). This completes the proof.

4. Exponential Decay

In this section, the stability result of the system (13)–(15) is proved.

We need the following lemmas.

Lemma 3. *The functional*

$$
\begin{aligned}
F_1(t) &:= \frac{1}{l+1}\int_\Omega \left(|u_t|^l u_t u + |v_t|^l v_t v\right)dx \\
&\quad + \int_\Omega (\nabla u_t \nabla u + \nabla v_t \nabla v)dx,
\end{aligned} \tag{63}
$$

satisfies

$$
\begin{aligned}
F_1(t) &\le \frac{1}{l+2}\left(\|u_t\|_{l+2}^{l+2} + \|v_t\|_{l+2}^{l+2}\right) + \frac{1}{2}\left(\|\nabla u_t\|^2 + \|\nabla v_t\|^2\right) \\
&\quad + \left(\frac{(l+1)^{-1}}{l+2}C_*^{l+2} + \frac{c}{2}\right)\left(\|\nabla u\|^{l+2} + \|\nabla v\|^{l+2}\right),
\end{aligned} \tag{64}
$$

$$
\begin{aligned}
F_1'(t) &\le \frac{1}{l+1}\left(\|u_t\|_{l+2}^{l+2} + \|v_t\|_{l+2}^{l+2}\right) + \left(\|\nabla u_t\|^2 + \|\nabla v_t\|^2\right) \\
&\quad + \left\{\varepsilon_1(a - k + \lambda) - k + \left(\frac{b_1 + b_2}{2} + \alpha\right)C_*^2\right\} \\
&\quad \cdot \left(\|\nabla u\|^2 + \|\nabla v\|^2\right) + \frac{1}{4\varepsilon_1}\int_{\tau_1}^{\tau_2}\left(|\mu_1(\varrho)|\|\nabla \mathscr{Z}(x, 1, \varrho, t)\|^2\right. \\
&\quad \left. + |\mu_2(\varrho)|\|\nabla \mathscr{Y}(x, 1, \varrho, t)\|^2\right)d\varrho + \frac{1}{4\varepsilon_1}(g_1 \circ \nabla u + g_2 \circ \nabla v).
\end{aligned} \tag{65}
$$

Proof.

(1) By applying the inequalities of Young and Poincare', we find

$$
\begin{aligned}
|F_1(t)| &\le \frac{1}{l+2}\|u_t\|_{l+2}^{l+2} + \frac{(l+1)^{-1}}{l+2}\|u\|^{l+2} + \frac{1}{l+2}\|v_t\|_{l+2}^{l+2} \\
&\quad + \frac{(l+1)^{-1}}{l+2}\|v\|^{l+2} + \frac{1}{2}\left(\|\nabla u_t\|^2 + \|\nabla u\|^2\right) \\
&\quad + \frac{1}{2}\left(\|\nabla v_t\|^2 + \|\nabla v\|^2\right) \le \frac{1}{l+2}\left(\|u_t\|_{l+2}^{l+2} + \|v_t\|_{l+2}^{l+2}\right) \\
&\quad + \frac{1}{2}\left(\|\nabla u_t\|^2 + \|\nabla v_t\|^2\right) + \left(\frac{(l+1)^{-1}}{l+2}C_*^{l+2} + \frac{c}{2}\right) \\
&\quad \cdot \left(\|\nabla u\|^{l+2} + \|\nabla v\|^{l+2}\right)
\end{aligned} \tag{66}
$$

(2) Direct computation using integration by parts, we get

$$
\begin{aligned}
F_1'(t) &= \int_\Omega \left(|u_t|^l u_{tt}\right)udx + \frac{1}{l+1}\|u_t\|_{l+2}^{l+2} + \int_\Omega \left(|v_t|^l v_{tt}\right)vdx \\
&\quad + \frac{1}{l+1}\|v_t\|_{l+2}^{l+2} - \int_\Omega \Delta u_{tt} u dx + \|\nabla u_t\|^2 \\
&\quad - \int_\Omega \Delta v_{tt} v dx + \|\nabla v_t\|^2 = \frac{1}{l+1}\left(\|u_t\|_{l+2}^{l+2} + \|v_t\|_{l+2}^{l+2}\right) \\
&\quad - M\left(\|\nabla u\|^2\right)\|\nabla u\|^2 - M\left(\|\nabla v\|^2\right)\|\nabla v\|^2 \\
&\quad + \int_\Omega \nabla u \int_0^t g_1(t-s)\nabla u(s)dsdx \\
&\quad + \int_\Omega \nabla v \int_0^t g_2(t-s)\nabla v(s)dsdx
\end{aligned}
$$

$$-\int_\Omega \nabla u \int_{\tau_1}^{\tau_2} |\mu_1(\varrho)| \nabla \mathscr{Z}(x,1,\varrho,t)\, d\varrho\, dx + \|\nabla u_t\|^2 - \int_\Omega \nabla v \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| \nabla \mathscr{Y}(x,1,\varrho,t)\, d\varrho\, dx + \|\nabla v_t\|^2 - (b_1+b_2)\int_\Omega |v|^{q+1}|u|^{p+1}dx - 2\alpha \int_\Omega uv\,dx, \tag{67}$$

estimate (65) easily follows by using $M(r) \geq a$, Young's inequality for $\varepsilon_1 > 0$, and (8).

Lemma 4. *The functional*

$$F_2(t) := \int_\Omega \left(\Delta u_t - \frac{1}{l+1}|u_t|^l u_t\right) \int_0^t g_1(t-s)(u(t)-u(s))\,ds\,dx + \int_\Omega \left(\Delta v_t - \frac{1}{l+1}|v_t|^l v_t\right) \int_0^t g_2(t-s)(v(t)-v(s))\,ds\,dx, \tag{68}$$

satisfies

$$F_2(t) \leq \frac{1}{l+2}\left(\|u_t\|_{l+2}^{l+2} + \|v_t\|_{l+2}^{l+2}\right) + \frac{1}{2}\left(\|\nabla u_t\|^2 + \|\nabla v_t\|^2\right) + \left(\frac{(l+1)^{-1}}{l+2}(a-k)^{l+2}C_*^{l+2}2^{2l+1}\right) \cdot \left(\|\nabla u\|^{2(l+1)} + \|\nabla v\|^{2(l+1)}\right) + \frac{1}{2}(a-k) \cdot \left\{1 + \frac{(l+1)^{-1}}{l+2}(a-k)^l C_*^{l+2}\right\}(g_1 \circ \nabla u + g_2 \circ \nabla v), \tag{69}$$

and for any $\varepsilon_2 > 0$,

$$\begin{aligned}
F_2'(t) \leq{}& \frac{1}{l+1}\left[\left(1 - \int_0^t g_1(s)ds\right)\|u_t\|_{l+2}^{l+2} + \left(1 - \int_0^t g_2(s)ds\right) \cdot \|v_t\|_{l+2}^{l+2}\right] + \left(2\varepsilon_2(a-k)^2 + \frac{\alpha C_*^2}{2}\right)\left(\|\nabla u\|^2 + \|\nabla v\|^2\right) \\
&+ \varepsilon_2\left\{(a-k) + \frac{(l+1)^{-1}}{l+2}(g_1(0))^{l+2}C_*^{l+2}2^{2(l+1)} + b_2\frac{C_*^{4(p+1)}}{2} + b_1\frac{C_*^{2p}}{2}\right\}M\left(\|\nabla u\|^2\right)\|\nabla u\|^2 \\
&+ \varepsilon_2\left\{(a-k) + \frac{(l+1)^{-1}}{l+2}(g_2(0))^{l+2}C_*^{l+2}2^{2(l+1)} + b_1\frac{C_*^{4(q+1)}}{2} + b_2\frac{C_*^{2q}}{2}\right\}M\left(\|\nabla v\|^2\right)\|\nabla v\|^2 \\
&+ \left(\varepsilon_2 - \int_0^t g_1(s)ds\right)\|\nabla u_t\|^2 + \varepsilon_2\int_{\tau_1}^{\tau_2}|\mu_1(\rho)| \cdot \|\nabla \mathscr{Z}(x,1,\rho,t)\|^2 d\rho + \left(\varepsilon_2 - \int_0^t g_2(s)ds\right)\|\nabla v_t\|^2 \\
&+ \varepsilon_2\int_{\tau_1}^{\tau_2}|\mu_2(\rho)|\|\nabla \mathscr{Y}(x,1,\rho,t)\|^2 d\rho + \left\{\frac{M\left(\|\nabla u\|^2\right)}{4\varepsilon_2} + \left(2\varepsilon_2 + \frac{\lambda_1}{4\varepsilon_2} + \frac{\alpha C_*^2}{2}\right)(a-k)\right\} \\
&\cdot (g_1 \circ \nabla u) + \left\{\frac{M\left(\|\nabla v\|^2\right)}{4\varepsilon_2} + \left(2\varepsilon_2 + \frac{\lambda_2}{4\varepsilon_2} + \frac{\alpha C_*^2}{2}\right) \cdot (a-k)\right\}(g_2 \circ \nabla v) - \frac{g_1(0)}{4\varepsilon_2}\left(1 + \frac{(l+1)^{-1}}{l+2} \cdot (g_1(0))^l C_*^{l+2}\right)\left(g_1' \circ \nabla u\right) \\
&- \frac{g_2(0)}{4\varepsilon_2} \cdot \left(1 + \frac{(l+1)^{-1}}{l+2}(g_2(0))^l C_*^{l+2}\right)\left(g_2' \circ \nabla v\right).
\end{aligned} \tag{70}$$

Proof.

(1) By using Young's inequality and the conjugate exponents $p' = (l+2)/(l+1)$, $q' = l+2$, and Hölder's inequality, we obtain

$$\left|-\int_\Omega \frac{1}{l+1}|u_t|^l u_t \int_0^t g_1(t-s)(u(t)-u(s))\,ds\,dx\right| \leq \frac{1}{l+2}\|u_t\|_{l+2}^{l+2} + \frac{(l+1)^{-1}}{l+2}\left[(a-k)^{l+1}C_*^{l+2} \cdot \left(2^{2l+1}(a-k)\|\nabla u\|^{2(l+1)} + \frac{1}{2}(g_1 \circ \nabla u)\right)\right], \tag{71}$$

$$\left|-\int_\Omega \nabla u_t \int_0^t g_1(t-s)(\nabla u(t) - \nabla u(s))\,ds\,dx\right| \leq \frac{1}{2}\|\nabla u_t\|^2 + \frac{1}{2}(a-k)(g_1 \circ \nabla u_t). \tag{72}$$

Similarly, we get

$$\left|-\int_\Omega \frac{1}{l+1}|v_t|^l v_t \int_0^t g_2(t-s)(v(t)-v(s))\,ds\,dx\right| \leq \frac{1}{l+2}\|v_t\|_{l+2}^{l+2} + \frac{(l+1)^{-1}}{l+2}\left[(a-k)^{l+1}C_*^{l+2} \cdot \left(2^{2l+1}(a-k)\|\nabla v\|^{2(l+1)} + \frac{1}{2}(g_2 \circ \nabla v)\right)\right], \tag{73}$$

$$\left|-\int_\Omega \nabla v_t \int_0^t g_2(t-s)(\nabla v(t)-\nabla v(s))dsdx\right| \leq \frac{1}{2}\|\nabla v_t\|^2 + \frac{1}{2}(a-k)(g_2\circ\nabla v_t) \quad (74)$$

By combining (71)–(74), we find (69).

(2) By derivation of F_2, and integrating by parts and (15), we find

$$\begin{aligned}
F_2'(t) &= \int_\Omega M(\|\nabla u\|^2)\nabla u \int_0^t g_1(t-s)(\nabla u(t)-\nabla u(s))dsdx \\
&- \int_\Omega \int_0^t g_1(t-s)\nabla u(s)ds \int_0^t g_1(t-s) \\
&\cdot (\nabla u(t)-\nabla u(s))dsdx + \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_1(\rho)|\nabla \mathcal{Z}(x,1,\rho,t) \\
&\cdot \left(\int_0^t g_1(t-s)(\nabla u(t)-\nabla u(s))ds\right) d\rho dx \\
&+ \int_\Omega f_1(u,v)\int_0^t g_1(t-s)(u(t)-u(s))dsdx \\
&- \int_\Omega \nabla u_t \int_0^t g_1'(t-s)(\nabla u(t)-\nabla u(s))dsdx \\
&- \int_\Omega \frac{1}{l+1}|u_t|^l u_t \int_0^t g_1'(t-s)(u(t)-u(s))dsdx \\
&- \left(\|\nabla u_t\|^2 + \frac{1}{l+1}\|u_t\|_{l+2}^{l+2}\right)\left(\int_0^t g_1(s)ds\right) \\
&+ \int_\Omega M(\|\nabla v\|^2)\nabla v \int_0^t g_2(t-s)(\nabla v(t)-\nabla v(s))dsdx \\
&- \int_\Omega \int_0^t g_2(t-s)\nabla v(s)ds \int_0^t g_2(t-s) \\
&\cdot (\nabla v(t)-\nabla v(s))dsdx + \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(\rho)|\nabla \mathcal{Y}(x,1,\rho,t) \\
&\cdot \left(\int_0^t g_2(t-s)(\nabla v(t)-\nabla v(s))ds\right) d\rho dx \\
&+ \int_\Omega f_2(u,v)\int_0^t g_2(t-s)(v(t)-v(s))dsdx \\
&- \int_\Omega \nabla v_t \int_0^t g_2'(t-s)(\nabla v(t)-\nabla v(s))dsdx \\
&- \int_\Omega \frac{1}{l+1}|v_t|^l v_t \int_0^t g_2'(t-s)(v(t)-v(s))dsdx \\
&- \left(\|\nabla v_t\|^2 + \frac{1}{l+1}\|v_t\|_{l+2}^{l+2}\right)\left(\int_0^t g_2(s)ds\right)
\end{aligned} \quad (75)$$

Using Young's, Cauchy-Schwarz, Hölder's, and Poincaré's inequalities, and $l \leq \gamma$, we obtain (70).

At this point, let us introduce the functional given by

Lemma 5. *The functional*

$$F_3(t) := \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} \rho e^{-\rho p}\left(|\mu_1(\rho)|\mathcal{Z}^2 + |\mu_2(\rho))\mathcal{Y}^2|\right) d\rho d\rho dx, \quad (76)$$

satisfies

$$F_3(t) \leq \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} \rho\left(|\mu_1(\rho)|\mathcal{Z}^2 + |\mu_2(\rho))\mathcal{Y}^2|\right) d\rho d\rho dx, \quad (77)$$

$$\begin{aligned}
F_3'(t) &\leq -\eta_1 \int_0^1 \int_{\tau_1}^{\tau_2} \rho\left(|\mu_1(\rho)|\|\mathcal{Z}\|^2 + |\mu_2(\rho)|\|\mathcal{Y}\|^2\right) \\
&\cdot d\rho d\rho + \lambda\left(\|\nabla u_t\|^2 + \|\nabla v_t\|^2\right) - \eta_1 \int_{\tau_1}^{\tau_2} (|\mu_1(\rho)| \\
&\cdot \|\mathcal{Z}(x,1,\rho,t)\|^2 + |\mu_2(\rho)|\|\mathcal{Y}(x,1,\rho,t)\|^2) d\rho d\rho,
\end{aligned} \quad (78)$$

where $\eta_1 > 0$.

Proof. By derivation of F_3, and using equations $(13)_3$ and $(13)_4$, we get

$$\begin{aligned}
F_3'(t) &= -2\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} e^{-\rho p}|\mu_1(\rho)|\nabla \mathcal{Z}\nabla \mathcal{Z}_\rho(x,\rho,\rho,t) d\rho d\rho dx \\
&- 2\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} e^{-\rho p}|\mu_2(\rho)|\nabla \mathcal{Y}\nabla \mathcal{Y}_\rho(x,\rho,\rho,t) d\rho d\rho dx \\
&= -\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} \rho e^{-\rho p}|\mu_1(\rho)|\nabla \mathcal{Z}^2 d\rho d\rho dx \\
&- \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_1(\rho)|\left[e^{-\rho}\nabla \mathcal{Z}^2(x,1,\rho,t) - \nabla \mathcal{Z}^2(x,0,\rho,t)\right] \\
&\cdot d\rho dx - \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} \rho e^{-\rho p}|\mu_2(\rho)|\nabla \mathcal{Y}^2 d\rho d\rho dx \\
&- \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(\rho)|\left[e^{-\rho}\nabla \mathcal{Y}^2(x,1,\rho,t)\right. \\
&\left. - \nabla \mathcal{Y}^2(x,0,\rho,t)\right] d\rho dx.
\end{aligned} \quad (79)$$

Applying the equality $\mathcal{Z}(x,0,\rho,t) = u_t(x,t)$, $\mathcal{Y}(x,0,\rho,t) = v_t(x,t)$, and $e^{-\rho} \leq e^{-\rho p} \leq 1$, for any $0 < \rho < 1$, we get

$$\begin{aligned}
F_3'(t) &= -\int_0^1 \int_{\tau_1}^{\tau_2} \rho e^{-\rho p}\left(|\mu_1(\rho)|\|\nabla \mathcal{Z}\|^2 + |\mu_2(\rho)|\|\nabla \mathcal{Y}\|^2\right) \\
&\cdot d\rho d\rho - \int_{\tau_1}^{\tau_2} e^{-\rho}\left(|\mu_1(\rho)|\|\nabla \mathcal{Z}(x,1,\rho,t)\|^2\right. \\
&\left. + |\mu_2(\rho)|\|\nabla \mathcal{Y}(x,1,\rho,t)\|^2\right) d\rho \\
&+ \left(\int_{\tau_1}^{\tau_2} |\mu_1(\rho)| d\rho\right)\|\nabla u_t\|^2 + \left(\int_{\tau_1}^{\tau_2} |\mu_2(\rho)| d\rho\right) \\
&\cdot \|\nabla v_t\|^2.
\end{aligned} \quad (80)$$

As $-e^{-\rho}$ is an increasing function, we have $-e^{-\rho} \le -e^{-\tau_2}$, for any $\rho \in [\tau_1, \tau_2]$.

Then, setting $\eta_1 = e^{-\tau_2}$, we find (78).

Theorem 6. *Assume (5)–(8) hold, then* $\exists \zeta_1, \zeta_2 > 0$ *such that the energy functional (16) satisfies*

$$E(t) \le \zeta_2 e^{-\zeta_1 t}, \ \forall t \ge t_0. \tag{81}$$

Proof. We define the functional of Lyapunov

$$\mathscr{L}(t) := NE(t) + F_1(t) + N_2 F_2(t) + F_3(t), \tag{82}$$

where $N, N_2 > 0$.

First, if we let

$$\mathscr{K}(t) = F_1(t) + N_2 F_2(t) + F_3(t), \tag{83}$$

then, by (64), (69), and (77), we get

$$|\mathscr{K}(t)| \le cE(t). \tag{84}$$

Consequently,

$$|\mathscr{K}(t)| = |\mathscr{L}(t) - NE(t)| \le cE(t), \tag{85}$$

which yields

$$(N - c)E(t) \le \mathscr{L}(t) \le (N + c)E(t). \tag{86}$$

By derivation (82) and applying (17), (65), (70), (78), and (6), one gets

$$\begin{aligned}
\mathscr{L}'(t) \le{} & \frac{1}{l+1}\{(1-h_0)+N_1\}\left[\|u_t\|_{l+2}^{l+2} + \|v_t\|_{l+2}^{l+2}\right] \\
& + \{\lambda(1+N) + N_1 + \varepsilon_2 - h_0\}\left(\|\nabla u_t\|^2 + \|\nabla v_t\|^2\right) \\
& + \left\{\varepsilon_2 M_0\left((a-k) + \frac{(l+1)^{-1}}{l+2}(h_2 C_*)^{l+2} 2^{2(l+1)} + R_1\right)\right. \\
& + N_1\left(\varepsilon_1(a-k+\lambda) - k + \left(\frac{b_1+b_2}{2} + \alpha\right)C_*^2\right) \\
& \left. + \left(2\varepsilon_2(a-k)^2 + \frac{\alpha C_*^2}{2}\right)\right\}\left[\|\nabla u\|^2 + \|\nabla v\|^2\right] \\
& + \left\{-\frac{1}{\xi}\left(\frac{M_0}{4\varepsilon_2} + \left(2\varepsilon_2 + \frac{\lambda}{4\varepsilon_2} + \frac{\alpha C_*^2}{2}\right)(a-k) + \frac{N_1}{4\varepsilon_1}\right)\right. \\
& \left. + \frac{N}{2} - \frac{h_1}{4\varepsilon_2}\left(1 + \frac{(l+1)^{-1}}{l+2}(h_1)^l C_*^{l+2}\right)\right\} \\
& \cdot \left[\left(g_1' \circ \nabla u\right) + \left(g_2' \circ \nabla v\right)\right] - \eta_1 \int_0^1 \int_{\tau_1}^{\tau_2} \rho\left(|\mu_1(\rho)| \|\mathscr{Z}\|^2\right. \\
& \left. + |\mu_2(\rho)| \|\mathscr{Y}\|^2\right) d\rho d\rho - \left\{\eta_1 + N\beta - \varepsilon_2 - \frac{N_1}{4\varepsilon_1}\right\} \\
& \cdot \int_{\tau_1}^{\tau_2} \left(|\mu_1(\rho)| \|\mathscr{Z}(x,1,\rho,t)\|^2\right. \\
& \left. + |\mu_2(\rho)| \|\mathscr{Y}(x,1,\rho,t)\|^2\right) d\rho d\rho,
\end{aligned} \tag{87}$$

where $h_0 = \min\left(\int_0^{t_0} g_1(s)ds, \int_0^{t_0} g_2(s)ds\right)$, $M_0 = \max\left(M(\|\nabla u\|^2), M(\|\nabla v\|^{2\cdot})\right)$, $h_1 = \min(g_1(0), g_2(0))$, $h_2 = \max(g_1(0), g_2(0))$, $\xi = \max(\xi_1, \xi_2)$, and $R_1 = \min\left(b_1(C_*^{4(q+1)}/2) + b_2(C_*^{4q}/2), b_2(C_*^{4(p+1)}/2) + b_1(C_*^{4p}/2)\right)$.

At this stage, choosing two fixed numbers N, N_1, such that $N - c > 0$, and

$$\begin{aligned}
& h_1 - \lambda(1+N) - N_1 > 0, \\
& \alpha_1 = h_1 - 1 - N_1 > 0,
\end{aligned} \tag{88}$$

we choose ε_2 small enough such that

$$\alpha_2 = h_1 - \lambda(1+N) - N_1 - \varepsilon_2 > 0. \tag{89}$$

After that, we choose ε_1 small enough such that

$$\begin{aligned}
\alpha_3 ={} & \eta_1 + N\beta - \varepsilon_2 - \frac{N_1}{4\varepsilon_1} < 0, \\
\alpha_4 ={} & \left\{-\varepsilon_2 M_0\left((a-k) + \frac{(l+1)^{-1}}{l+2}(h_2 C_*)^{l+2} 2^{2(l+1)} + R_1\right)\right. \\
& + N_1\left(k - \varepsilon_1(a-k+\lambda) - \left(\frac{b_1+b_2}{2} + \alpha\right)C_*^2\right) \\
& \left. - \left(2\varepsilon_2(a-k)^2 + \frac{\alpha C_*^2}{2}\right)\right\} > 0, \\
\alpha_5 ={} & \left\{\frac{1}{\xi}\left(\frac{N_1}{4\varepsilon_1} + \frac{M_0}{4\varepsilon_2} + \left(2\varepsilon_2 + \frac{\lambda}{4\varepsilon_2} + \frac{\alpha C_*^2}{2}\right)(a-k)\right)\right. \\
& \left. - \frac{N}{2} + \frac{h_1}{4\varepsilon_2}\left(1 + \frac{(l+1)^{-1}}{l+2}(h_1)^l C_*^{l+2}\right)\right\} > 0.
\end{aligned} \tag{90}$$

Thus, we get

$$\begin{aligned}
\mathscr{L}'(t) \le{} & \frac{-1}{l+1}\alpha_1\left[\|u_t\|_{l+2}^{l+2} + \|v_t\|_{l+2}^{l+2}\right] - \alpha_2\left(\|\nabla u_t\|^2 + \|\nabla v_t\|^2\right) \\
& - \alpha_4\left[\|\nabla u\|^2 + \|\nabla v\|^2\right] - \alpha_5\left[\left(g_1' \circ \nabla u\right) + \left(g_2' \circ \nabla v\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&-\eta_1 \int_0^1 \int_{\tau_1}^{\tau_2} \rho\left(|\mu_1(\rho)|\|\mathscr{Z}\|^2 + |\mu_2(\rho)|\|\mathscr{Y}\|^2\right) d\rho d\rho \\
&+ \alpha_3 \int_{\tau_1}^{\tau_2} \left(|\mu_1(\rho)|\|\mathscr{Z}(x,1,\rho,t)\|^2 + |\mu_2(\rho)|\right. \\
&\left.\cdot \|\mathscr{Y}(x,1,\rho,t)\|^2\right) d\rho d\rho,
\end{aligned} \tag{91}
$$

$$
c_1 E(t) \le \mathscr{L}(t) \le c_2 E(t), \ \forall t \ge 0, \tag{92}
$$

using (16), estimates (91) and (86), respectively, we get

$$
\mathscr{L}'(t) \le -k_1 E(t) - k_2 E'(t), \forall t \ge t_0, \tag{93}
$$

for some $k_1, k_2, c_1, c_2 > 0$.

By the combination of (93) with (92), we obtain

$$
\mathscr{R}'(t) \le -\lambda_1 \mathscr{R}(t), \tag{94}
$$

where

$$
\mathscr{R}(t) = \mathscr{L}(t) + k_2 E(t) \sim E(t). \tag{95}
$$

Integrating the result (94) over (t_0, t), we find

$$
\mathscr{R}(t) \le \mathscr{R}(t_0) e^{-\zeta_1 (t-t_0)}, \ \forall t_0 \ge t. \tag{96}
$$

It follows from (95) that (81) holds. This completes the proof.

Data Availability

No data were used to support the study.

Conflicts of Interest

This work does not have any conflicts of interest.

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