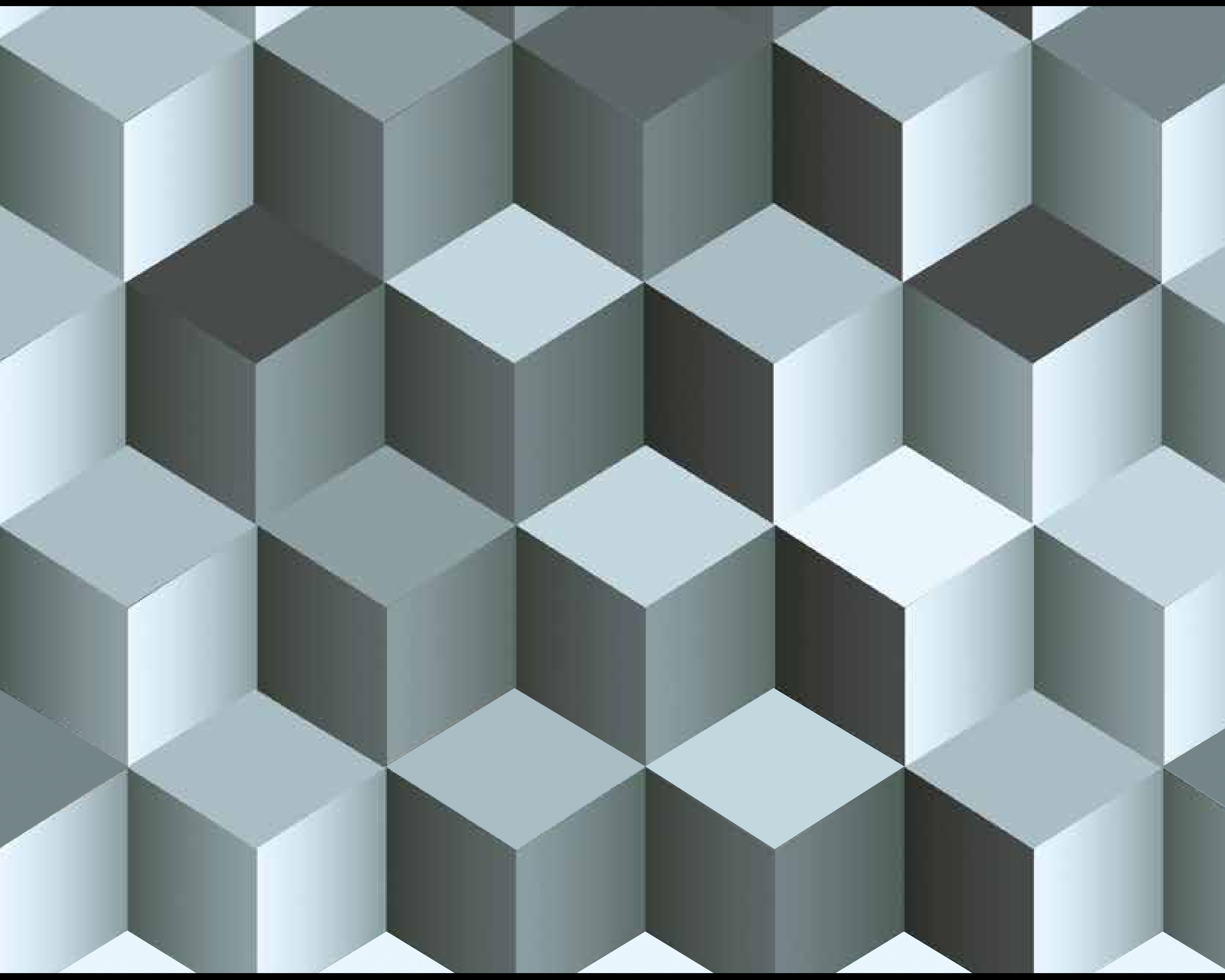
Journal of Function Spaces

Some Classes of Function Spaces, Their Properties, and Their Applications 2014

Guest Editors: Józef Banaś, Janusz Matkowski, Nelson Merentes, Jose Luis Sanchez, and Kishin Sadarangani



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Editorial Board

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Editorial

Some Classes of Function Spaces, Their Properties, and Their Applications 2014

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Miscellaneous types of function spaces appear very frequently in several mathematical investigations. For example, function spaces create the fundamentals of the study in functional analysis, theory of real functions, theories of differential and integral equations, operator theory, nonlinear analysis, and control theory. Let us also mention that such modern branches of mathematics as numerical analysis and probability theory exploit also methods and tools of the theory of function spaces.

This special issue presents a lot of ideas appearing in the above quoted branches of mathematics. It contains twenty-two papers devoted mainly to the study of function spaces and their various properties. Moreover, this special issue includes also a group of papers discussing some aspects of operator theory in connection with properties of function spaces in which those operators are investigated. Moreover, a part of papers included in this issue is dedicated to the solvability of some functional equations (differential, integral, etc.) and to properties of solutions of those equations.

The first part of the papers, which are devoted to various topics of operator theory in miscellaneous function spaces, contains eight papers. Below we describe briefly those papers. The paper of J. Huang and Y. Liu discusses a molecular characterization of the Hardy space associated with the so-called twisted convolutions. The results of the paper extend several ones obtained by the first author and other authors. An application to the boundedness of local Riesz transforms on the Hardy spaces is also presented. Another paper of the discussed part is authored by S. J. Chang et al. In that paper the analysis of a generalized analytic Feynman integral and a modified generalized analytic functions space associated with the Feynman integral is conducted. Some integration formulas for that integral are established and the applicability to physical circumstances is indicated. J. Dong et al. discuss in their paper the boundedness of singular integrals associated with Schrödinger operators on Hardy type function spaces. The main tool used in the investigations is a molecular characterization of Hardy spaces. The paper of T. Acar et al. describes a new type Stancu operators which create the generalization of Srivastava-Gupta operators. With help of those operators an approximation of functions being integrable on the interval $(0,\infty)$ can be realized. Moreover, the rate of convergence of the approximations in question for functions with derivatives of bounded variation is estimated. X. Feng et al. discuss in their paper a multiplication operator with a special symbol on the weighted Bergman space of the unit ball in $\mathbb{C}^n$. A few necessary and sufficient conditions for the compactness of the mentioned multiplication operator are given. In the paper of M. Nowak some general representation theorems for continuous linear operators acting from a suitable function space into a Banach space are obtained. Moreover, strongly bounded operators are also studied. The mentioned function space contains vector-valued continuous functions defined on a completely regular Hausdorff space with values in certain Banach space. The paper of J. Xu and X. Yang studies new type of Herz-Morrey-Hardy spaces with variable exponent. Those spaces are characterized in terms of atom. With the help of that characterization a few results on the boundedness of some singular integral operators defined on spaces in question are derived. The other paper included in the discussed group is authored by S. He et al. That paper contains some results concerning the boundedness of some fractional integrals on an infinitesimal generator of an analytic semigroup defined on the Hilbert space of Lebesgue type.

Now, we are going to present the group of six papers dedicated to investigations connected with the theory of function spaces. One paper included in this group is the paper of H. Wang and Z. Wu. The authors deal with the estimates of the L_p modulus of continuity of some classes of functions of bounded Waterman-Young variation. The obtained results are applied in obtaining some estimates of Fourier coefficients of functions of the mentioned classes, among others. A. M. Sarsenbi and P. A. Terekhin obtained in their paper general conditions ensuring that a complete biorthogonal conjugate system forms a Riesz basis. Moreover, affine Riesz bases are constructed with the help of the obtained results. The paper of J. Zhou discusses new spaces of Lebesque measurable functions on the unit circle. That space is closely related to a Sobolev space. A few results expressed in terms of Möbius boundedness in a Sobolev space are derived. Moreover, a dyadic characterization of functions of the introduced new space with the aim of dyadic arcs on the unit circle is also presented. In the paper of X. Guo the representation of g-frames as linear combination of simpler components (g-orthonormal bases, g-Riesz bases, and normalized tight g-frames) is considered. Moreover, the dual and pseudodual g-frames are investigated and the dual g-frames are characterized in a constructive way. Y. Niu and H. Wang study in their paper properties of functions in the class of functions with p-bounded Wiener variation for $0 < p < 1$. The main result asserts that each such function can be represented as the difference of two increasing functions from that class. The paper of Z. Pavić deals with convex functions which satisfy some global convexity properties. The classical ideas associated with Jensen approach to convexity are extended and studied in the paper in question.

Two papers published in this special issue are mainly devoted to operators acting in some function spaces. One paper of that kind authored by O. Mejía et al. deals with a necessary and sufficient condition on a real function $h = h(t)$ such that the composition operator H generated by the function h maps the space of functions with bounded Schramm-Korenblum variation into itself and is locally Lipschitzian. Another announced paper of L. Zhou and J. Lu contains a result which creates a generalization of the result of Krues and Zhu concerning the boundedness of an integral operator in the Lebesgue space L^p.

The fourth group of the papers included in this special issue is formed by six papers devoted thoroughly to some differential and integral equations in various function spaces. One paper written by Z. Dai et al. is dedicated to the Cauchy problem for the three-dimensional incompressible Boussinesq equation. A blow-up criterion for weak solutions of that equation in terms of the pressure is established in a homogeneous Besov space. Another paper by J. Wang et al. investigates a class of singular boundary value problems of a fractional q-difference equation. Using a fixed point theorem in partially ordered sets a few results on the existence and uniqueness of solutions of the mentioned equation are established. The paper of Y. Wu et al. shows how to obtain limit cycles for a family of generalized nilpotent systems of differential equations. The results of the paper are well motivated and appropriately illustrated. The paper of M. A. Darwish and B. Rzepka deals with the solvability of a generalized fractional quadratic functional-integral equation of Erdélyi-Kober type in the Banach space of functions being continuous and bounded on the real half-axis. The technique of measures of noncompactness is the main tool used in considerations. T. Zając studies in his paper the existence of nonnegative and monotonic solutions of a nonlinear quadratic Volterra-Stieltjes integral equation. That equation is considered in the classical space consisting of continuous real functions defined on a bounded, closed interval. The main tools used in considerations are the techniques of Stieltjes integrals and measures of noncompactness. The other paper included in the group in question is authored by N. K. Ashirbayev et al. In that paper it is shown that some classes of nonlinear integral equations (integral equations of fractional order, integral equations of Volterra-Wiener-Hopf type, integral equations of Erdélyi-Kober type, and integral equations of Volterra-Chandrasekhar type) can be treated as spacial cases of some nonlinear integral equation of Volterra-Stieltjes type. Some results concerning Volterra-Stieltjes integral equations in several variables are also discussed.

Acknowledgment

The guest editors of this special issue would like to express their immense gratitude to the authors who have submitted papers for considerations. We hope that results of the papers included in this special issue will inspire researchers for further study in a lot of branches of mathematical sciences and their applications in describing real world phenomena.

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Research Article

Multiplication Operator with BMO Symbols and Berezin Transform

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We discuss multiplication operator with a special symbol on the weighted Bergman space of the unit ball. We give the necessary and sufficient conditions for the compactness of multiplication operator on the weighted Bergman space of the unit ball.

1. Introduction

Let B_n denote the unit ball in $\mathbb{C}^n$, and let ν be the normalized Lebesgue volume measure on B_n. For $-1 < \alpha < \infty$, we denote by ν_α the measure on B_n defined by $d\nu_\alpha(z) = c_\alpha(1-|z|^2)^\alpha d\nu(z)$, where $c_\alpha = \Gamma(n+\alpha+1)/n!\Gamma(\alpha+1)$ is a normalizing constant such that $\nu_\alpha(B_n) = 1$. For $1 \le p < \infty$, we write $\|\cdot\|_{\alpha,p}$ for the norm on $L^p(B_n, d\nu_\alpha)$ and $\langle\cdot,\cdot\rangle_\alpha$ for the inner product on $L^2(B_n, d\nu_\alpha)$. The Bergman space $A^2_\alpha(B_n)$ is the space of holomorphic functions which are square-integrable with respect to measure $d\nu_\alpha$ on B_n. Reproducing kernels K^α_w and normalized reproducing kernels k^α_w in $A^2_\alpha(B_n)$ are given by

$$K^\alpha_w(z) = \frac{1}{(1-\langle z,w\rangle)^{n+\alpha+1}},$$
$$k^\alpha_w(z) = \frac{\left(1-|w|^2\right)^{(n+\alpha+1)/2}}{(1-\langle z,w\rangle)^{n+\alpha+1}}, \tag{1}$$

respectively, for $z, w \in B_n$. For every $h \in A^2_\alpha(B_n)$ we have $\langle h, K^\alpha_w\rangle_\alpha = h(w)$, for all $w \in B_n$. The orthogonal projection P_α of $L^2(B_n, d\nu_\alpha)$ onto $A^2_\alpha(B_n)$ is given by

$$\left(P_\alpha g\right)(w) = \left\langle g, K^\alpha_w\right\rangle_\alpha = \int_{B_n} g(z)\frac{1}{(1-\langle w,z\rangle)^{n+\alpha+1}} d\nu_\alpha(z), \tag{2}$$

for $g \in L^2(B_n, d\nu_\alpha)$ and $w \in B_n$.

Given $f \in L^1(B_n, d\nu_\alpha)$, the Toeplitz operator $T_f : L^2(B_n, d\nu_\alpha) \to A^2_\alpha(B_n)$, the Hankel operator $H_f : L^2(B_n, d\nu_\alpha) \to (A^2_\alpha(B_n))^\perp$, and the multiplication operator $M_f : A^2_\alpha(B_n) \to L^2(B_n, d\nu_\alpha)$ are given by

$$\left(T_f h\right)(z) = \int_{B_n} \frac{f(\omega)h(\omega)}{(1-\langle z,\omega\rangle)^{n+\alpha+1}} d\nu_\alpha(\omega),$$
$$\left(H_f h\right)(z) = f(z)g(z) - \left(T_f g\right)(z), \tag{3}$$
$$M_f(h) = fh,$$

respectively. For $f \in L^1(B_n, d\nu_\alpha)$, we define the Berezin transform of f to be the function $\widetilde{f}$; that is,

$$\widetilde{f}(z) = \int_{B_n} f(w)\left|k^\alpha_z(w)\right|^2 d\nu_\alpha(w). \tag{4}$$

If f is bounded, then $\widetilde{f}$ is a bounded function on B_n. Since the kernels k^α_z converge weakly to zero as z tends ∂B_n, we have that if f is compact, then $\widetilde{f} \to 0$ as $z \to \partial B_n$. The converse (in both cases) is not necessarily true. According to the definition of Berezin transform, the mean oscillation of f in the Bergman metric is the function $\mathrm{MO}(f)(z)$ defined on B_n by

$$\mathrm{MO}(f)(z) = \widetilde{|f|^2}(z) - \left|\widetilde{f}(z)\right|^2. \tag{5}$$

For $z \in B_n$, let ψ_z be the automorphism of B_n such that $\psi_z(0) = z$ and $\psi_z = (\psi_z)^{-1}$. Thus, we have the change-of-variable formula

$$\int_{B_n} h(\psi_z(w)) \left|k_z^\alpha(w)^2\right| dv_\alpha(w) = \int_{B_n} h(w)\, dv_\alpha(w), \quad (6)$$

for every $h \in L^1(B_n, dv_\alpha)$.

Multiplication operators are one of the most widely studied classes of concrete operators. The study of their behavior on the Hardy and Bergman spaces has generated an extensive list of results in the operator theory and in the theory of function spaces [1–6]. One of the useful approaches is the use of the Berezin transform [7–11]. This method is motivated by its connections with quantum physics and noncommutative geometry.

In general, Berezin transform $\widetilde{f}$ plays important role in giving necessary and sufficient conditions for the boundedness and compactness of the Toeplitz operator [12, 13]. However Berezin transform $\widetilde{|f|^2}$ or the mean oscillation $\mathrm{MO}(f)$ is used to obtain the necessary and sufficient conditions for the boundedness and compactness of the Hankel operator or multiplication operator [14, 15]. This work is partially motivated by using Berezin transform $\widetilde{f}$ to obtain necessary and sufficient conditions for the compactness of multiplication operator on the weighted Bergman space of the unit ball.

Throughout the paper, we will use the letter c to denote a generic positive constant that can change its value at each occurrence.

2. Main Results

In this section, we give the necessary and sufficient conditions for the compactness of multiplication operator on the weighted Bergman space of the unit ball. We furthermore obtain the necessary and sufficient conditions for the compactness of Toeplitz operator and Hankel operator.

Theorem 1. *Suppose $|f|/(1-|z|)^{4n+4\alpha+4}$ is bounded on B_n. Then M_f is compact operator on $A_\alpha^2(B_n)$ if and only if $\widetilde{|f|}(z) \to 0$ as $z \to \partial B_n$.*

Proof. Suppose $\widetilde{|f|}(z) \to 0$ as $z \to \partial B_n$.

Since

$$\left\langle M_f g, h\right\rangle_\alpha = \langle fg, h\rangle_\alpha = \left\langle g, \overline{f} h\right\rangle_\alpha, \quad (7)$$

it is clear that $(M_f)^* = M_{\overline{f}}$. It suffices to prove that the operator $(M_f)^*$ is compact by showing that $(M_f)^*$ can be approximated by compact operators in the operator norm.

Let $g \in L^2(B_n, dv_\alpha)$. Then $(M_f)^* g \in A_\alpha^2(B_n)$, so we have

$$\begin{aligned}\left(\left(M_f\right)^* g\right)(z) &= \left\langle (M_f)^* g, K_z^\alpha\right\rangle_\alpha \\ &= \int_{B_n} g(w)\, \overline{f(w) K_z^\alpha(w)}\, dv_\alpha(w),\end{aligned} \quad (8)$$

for $z \in B_n$.

We define for $0 < r < 1$ an operator S_r by

$$(S_r g)(z) = \int_{B_n} \chi_{rB_n}(z)\, g(w)\, \overline{f(w) K_z^\alpha(w)}\, dv_\alpha(w). \quad (9)$$

Since $|f|/(1-|z|)^{4n+4\alpha+4}$ is bounded on B_n, we prove that

$$\begin{aligned}&\int_{B_n}\int_{B_n} \left|\chi_{rB_n}(z)\, \overline{f(w) K_z^\alpha(w)}\right|^2 dv_\alpha(w)\, dv_\alpha(z) \\ &\quad = \int_{rB_n}\int_{B_n} \left|\overline{f(w) K_z^\alpha(w)}\right|^2 dv_\alpha(w)\, dv_\alpha(z) \\ &\quad < +\infty.\end{aligned} \quad (10)$$

Thus, the operator S_r is a Hilbert-Schmidt operator. Since

$$\begin{aligned}&\left(\left(M_f\right)^* - S_r\right) g(z) \\ &\quad = \int_{B_n} g(w)\, \chi_{B_n \setminus rB_n}(z)\, \overline{f(w) K_z^\alpha(w)}\, dv_\alpha(w),\end{aligned} \quad (11)$$

$(M_f)^* - S_r$ is an integral operator with kernel $K_r^f(z,w) = \chi_{B_n \setminus rB_n}(z)\overline{f(w) K_z^\alpha(w)}$.

By Schur's test, whenever there exists a positive measurable function h on B_n and constants c_1 and c_2 such that

$$\int_{B_n} \left|K_r^f(z,w)\right| h(z)\, dv_\alpha(z) \le c_1 h(w), \quad (12)$$

for all w in B_n, and

$$\int_{B_n} \left|K_r^f(z,w)\right| h(w)\, dv_\alpha(w) \le c_2 h(z), \quad (13)$$

for all z in B_n, we have

$$\left\|\left(M_f\right)^* - S_r\right\| \le c_1 c_2. \quad (14)$$

Let $h(z) = (1-|z|^2)^{(n+\alpha+1)/2}$. Since

$$\begin{aligned}K_z^\alpha(\psi_z(v))\, k_z^\alpha(v) &= \frac{1}{\left(1-|z|^2\right)^{(n+1+\alpha)/2}}, \\ 1 - \left|\psi_z(v)\right|^2 &= \frac{\left(1-|z|^2\right)\left(1-|v|^2\right)}{|1-\langle v,z\rangle|^2},\end{aligned} \quad (15)$$

and Hölder inequality, it is easy to prove that

$$\begin{aligned}
&\int_{B_n} \chi_{B_n \setminus rB_n}(z) \left| f(w) K_z^\alpha(w) \right| \frac{\left(1-|w|^2\right)^{(n+\alpha+1)/2}}{\left(1-|z|^2\right)^{(n+\alpha+1)/2}} dv_\alpha(w) \\
&= \int_{B_n} \chi_{B_n \setminus rB_n}(z) \left| f(w) k_z^\alpha(w) \right| \frac{\left(1-|w|^2\right)^{(n+\alpha+1)/2}}{\left(1-|z|^2\right)^{n+\alpha+1}} dv_\alpha(w) \\
&= \int_{B_n} \chi_{B_n \setminus rB_n}(z) \left| f(\psi_z(w)) k_z^\alpha(\psi_z(w)) \right| \\
&\quad \times \frac{\left(1-|\psi_z(w)|^2\right)^{(n+\alpha+1)/2}}{\left(1-|z|^2\right)^{n+\alpha+1}} \left| k_z^\alpha(w) \right|^2 dv_\alpha(w) \\
&= \int_{B_n} \chi_{B_n \setminus rB_n}(z) \left| f(\psi_z(w)) \right| \frac{\left(1-|w|^2\right)^{(n+\alpha+1)/2}}{|1-\langle z,w\rangle|^{2n+2\alpha+2}} dv_\alpha(w) \\
&= \int_{B_n} \chi_{B_n \setminus rB_n}(z) \sqrt{\left| f(\psi_z(w)) \right|} \sqrt{\left| f(\psi_z(w)) \right|} \\
&\quad \times \frac{\left(1-|w|^2\right)^{(n+\alpha+1)/2}}{|1-\langle z,w\rangle|^{2n+2\alpha+2}} dv_\alpha(w) \\
&\leq \chi_{B_n \setminus rB_n}(z) \left[\int_{B_n} \left| f(\psi_z(w)) \right| dv_\alpha(w) \right]^{1/2} \\
&\quad \times \left[\int_{B_n} \left| f(\psi_z(w)) \right| \frac{\left(1-|w|^2\right)^{n+\alpha+1}}{|1-\langle z,w\rangle|^{4n+4\alpha+4}} dv_\alpha(w) \right]^{1/2} \\
&\leq c\chi_{B_n \setminus rB_n}(z) \left[\int_{B_n} \left| f(\psi_z(w)) \right| dv_\alpha(w) \right]^{1/2} \\
&\quad \times \left[\int_{B_n} \left| f(\psi_z(w)) \right| \frac{1}{(1-|z|)^{3n+3\alpha+3}} dv_\alpha(w) \right]^{1/2} \\
&\leq c \left[\widetilde{|f|}(z) \right]^{1/2} \\
&= c_1,
\end{aligned} \tag{16}$$

where $c_1 = c[\widetilde{|f|}(z)]^{1/2}$, $r < |z| < 1$.

Since

$$\begin{aligned}
K_z^\alpha(\psi_z(v)) k_z^\alpha(v) &= \frac{1}{\left(1-|z|^2\right)^{(n+1+\alpha)/2}}, \\
1 - \langle \psi_z(w), z \rangle &= \frac{1-|z|^2}{1-\langle w,z\rangle},
\end{aligned} \tag{17}$$

then we obtain

$$\begin{aligned}
&\int_{B_n} \chi_{B_n \setminus rB_n}(z) \left| f(w) K_z^\alpha(w) \right| \frac{\left(1-|z|^2\right)^{(n+\alpha+1)/2}}{\left(1-|w|^2\right)^{(n+\alpha+1)/2}} dv_\alpha(z) \\
&= \int_{B_n} \chi_{B_n \setminus rB_n}(w) \left| f(z) K_z^\alpha(w) \right| \frac{\left(1-|w|^2\right)^{(n+\alpha+1)/2}}{\left(1-|z|^2\right)^{(n+\alpha+1)/2}} dv_\alpha(w) \\
&= \int_{B_n} \chi_{B_n \setminus rB_n}(\psi_z(w)) \left| f(z) K_z^\alpha(\psi_z(w)) \right| \\
&\quad \times \frac{\left(1-|\psi_z(w)|^2\right)^{(n+\alpha+1)/2}}{\left(1-|z|^2\right)^{(n+\alpha+1)/2}} \left| k_z^\alpha(w) \right|^2 dv_\alpha(w) \\
&= \int_{B_n} \chi_{B_n \setminus rB_n}(\psi_z(w)) \left| f(z) \right| \\
&\quad \times \frac{\left(1-|\psi_z(w)|^2\right)^{(n+\alpha+1)/2}}{|1-\langle z,w\rangle|^{4n+4\alpha+4}} dv_\alpha(w) \\
&\leq c_2,
\end{aligned} \tag{18}$$

where c_2 is positive number.

By the above analysis, we get (12) and (13). By Schur's test we get $\|(M_f)^* - S_r\| \leq c_1 c_2$, where $c_1 \to 0$ as $|z| \to \partial B_n$ and c_2 does not depend on r. So $\widetilde{|f|}(z) \to 0$ as $z \to \partial B_n$ implies that M_f is compact on $A_\alpha^2(B_n)$.

Suppose M_f is compact on $A_\alpha^2(B_n)$.

Since the kernels k_z^α converge weakly to zero as z tends ∂B_n, then we have $\|M_f k_z^\alpha\|_{\alpha,2}$ converges to zero as z tends ∂B_n. So we obtain

$$\widetilde{|f|}(z) \leq \left[\widetilde{|f|^2}(z) \right]^{1/2} = \left\| M_f k_z^\alpha \right\|_{\alpha,2} \longrightarrow 0, \tag{19}$$

as $z \to \partial B_n$. □

Let $f \in L^1(B_n, dv_\alpha)$ and let $p \geq 1$; we say that $f \in \mathrm{BMO}_\alpha^p(B_n)$ whenever

$$\|f\|_{\mathrm{BMO}_\alpha^p} = \sup_{z \in B_n} \left\| f \circ \psi_z - \tilde{f}(z) \right\|_{\alpha,p} < \infty. \tag{20}$$

Note that $\|\cdot\|_{\mathrm{BMO}_\alpha^p}$ does not distinguish constants, while $|||f|||_{\alpha,p} = \|f\|_{\mathrm{BMO}_\alpha^p} + |\tilde{f}(0)|$ is a norm in $\mathrm{BMO}_\alpha^p(B_n)$. By Theorem 5 in [16], we know that $\mathrm{BMO}_\alpha^p(B_n)$ is equivalent to BMO_∂^p (see the definition in [16]).

For any $p \geq 1$, let VMO_α^p denote the subspace of BMO_α^p consisting of functions f such that

$$\lim_{|z| \to 1^-} \left\| f \circ \psi_z - \tilde{f}(z) \right\|_{\alpha,p} = 0. \tag{21}$$

Theorem 2. *Suppose $f \in VMO_\alpha^1$ and $|f|/(1-|z|)^{4n+4\alpha+4}$ is bounded on B_n. Then the following are equivalent:*

(a) $\widetilde{f}(z) \to 0$ *as* $z \to \partial B_n$;

(b) M_f *is compact operator on* $A^2_\alpha(B_n)$;

(c) T_f *is compact operator on* $A^2_\alpha(B_n)$.

Proof. It suffices to prove that $(a) \Leftrightarrow (b)$ and $(a) \Leftrightarrow (c)$.

$(a) \Leftrightarrow (b)$. Since

$$\left|\widetilde{f}(z)\right| \le \widetilde{|f|}(z),$$

$$\begin{aligned}
&\widetilde{|f|}(z) - \left|\widetilde{f}(z)\right| \\
&= \int_{B_n} \left(|f(w)| - \left|\widetilde{f}(z)\right|\right) \frac{\left(1-|z|^2\right)^{n+\alpha+1}}{|1-\langle z,w\rangle|^{2n+2\alpha+2}} dv_\alpha(w) \\
&\le \int_{B_n} \left|f(w) - \widetilde{f}(z)\right| \frac{\left(1-|z|^2\right)^{n+\alpha+1}}{|1-\langle z,w\rangle|^{2n+2\alpha+2}} dv_\alpha(w) \\
&= \int_{B_n} \left|f \circ \psi_z(w) - \widetilde{f}(z)\right| dv_\alpha(w) \\
&= \left\|f \circ \psi_z - \widetilde{f}(z)\right\|_{\alpha,1},
\end{aligned} \tag{22}$$

then we obtain that $\widetilde{f}(z) \to 0$ as $z \to \partial B_n$ if and only if $\widetilde{|f|}(z) \to 0$ as $z \to \partial B_n$. By Theorem 1, we obtain that M_f is compact operator on $A^2_\alpha(B_n)$ if and only if $\widetilde{f}(z) \to 0$ as $z \to \partial B_n$.

$(a) \Leftrightarrow (c)$. It is clear that $\widetilde{f}(z) \to 0$ as $z \to \partial B_n$ if and only if T_f with BMO^1_α symbol is compact operator on $A^2_\alpha(B_n)$ in [12]. Since $\mathrm{VMO}^1_\alpha \subset \mathrm{BMO}^1_\alpha$, then it is clear that $\widetilde{f}(z) \to 0$ as $z \to \partial B_n$ if and only if T_f is compact operator on $A^2_\alpha(B_n)$. □

Corollary 3. *Suppose* $f \in VMO^1_\alpha$, $|f|/(1-|z|)^{4n+4\alpha+4}$ *is bounded on* B_n, *and* H_f *is compact operator on* $A^2_\alpha(B_n)$. *Then* $M_{f-\widetilde{f}}$ *is compact operator on* $A^2_\alpha(B_n)$.

Proof. Suppose H_f is compact operator on $A^2_\alpha(B_n)$. So we obtain $H_{f-\widetilde{f}}$ is compact operator on $A^2_\alpha(B_n)$. Since $\widetilde{f-\widetilde{f}}(z) \to 0$ as $z \to \partial B_n$, then $T_{f-\widetilde{f}}$ is compact operator on $A^2_\alpha(B_n)$. Since

$$M_{f-\widetilde{f}}g = T_{f-\widetilde{f}}g + H_{f-\widetilde{f}}g, \tag{23}$$

we obtain $M_{f-\widetilde{f}}$ is compact operator on $A^2_\alpha(B_n)$. □

By Lemma 17 and Theorem 19 in [16] and Theorem 2, we obtain the following theorem.

Theorem 4. *Suppose* $f \in VMO^1_\alpha$, $\widetilde{f}(0) = 0$, *and* $|f|/(1-|z|)^{4n+4\alpha+4}$ *is bounded on* B_n. *Then the following are equivalent:*

(a) M_f *is compact operator on* $A^2_\alpha(B_n)$;

(b) T_f *is compact operator on* $A^2_\alpha(B_n)$;

(c) H_f *is compact operator on* $A^2_\alpha(B_n)$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Functions Like Convex Functions

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The paper deals with convex sets, functions satisfying the global convexity property, and positive linear functionals. Jensen's type inequalities can be obtained by using convex combinations with the common center. Following the idea of the common center, the functional forms of Jensen's inequality are considered in this paper.

1. Introduction

Introduction is intended to be a brief overview of the concept of convexity and affinity. Let $\mathbb{X}$ be a real linear space. Let $a, b \in \mathbb{X}$ be points and let $\alpha, \beta \in \mathbb{R}$ be coefficients. Their binomial combination

$$\alpha a + \beta b \tag{1}$$

is convex if $\alpha, \beta \geq 0$ and if

$$\alpha + \beta = 1. \tag{2}$$

If $c = \alpha a + \beta b$, then the point c itself is called the combination center.

A set $\mathscr{S} \subseteq \mathbb{X}$ is convex if it contains all binomial convex combinations of its points. The convex hull $\operatorname{conv}\mathscr{S}$ of the set $\mathscr{S}$ is the smallest convex set containing $\mathscr{S}$, and it consists of all binomial convex combinations of points of $\mathscr{S}$.

Let $\mathscr{C} \subseteq \mathbb{X}$ be a convex set. A function $f : \mathscr{C} \to \mathbb{R}$ is convex if the inequality

$$f(\alpha a + \beta b) \leq \alpha f(a) + \beta f(b) \tag{3}$$

holds for all binomial convex combinations $\alpha a + \beta b$ of pairs of points $a, b \in \mathscr{C}$.

Requiring only the condition in (2) for coefficients and requiring the equality in (3), we get a characterization of the affinity.

Implementing mathematical induction, we can prove that all of the above applies to n-membered combinations for any positive integer n. In that case, the inequality in (3) is the famous Jensen's inequality obtained in [1]. Numerous papers have been written on Jensen's inequality; different types and variants can be found in [2, 3].

2. Positive Linear Functionals and Convex Sets of Functions

Let $\mathscr{X}$ be a nonempty set, and let $\mathbb{X}$ be a subspace of the linear space of all real functions on the domain $\mathscr{X}$. We assume that $\mathbb{X}$ contains the unit function $\mathbf{1}$ defined by $\mathbf{1}(x) = 1$ for every $x \in \mathscr{X}$.

Let $\mathscr{I} \subseteq \mathbb{R}$ be an interval, and let $\mathbb{X}_{\mathscr{I}}$ be the set containing all functions $g \in \mathbb{X}$ with the image in $\mathscr{I}$. Then, $\mathbb{X}_{\mathscr{I}}$ is convex set in the space $\mathbb{X}$. The same is true for convex sets of Euclidean spaces. Let $\mathscr{C} \subseteq \mathbb{R}^k$ be a convex set, and let $(\mathbb{X}^k)_{\mathscr{C}}$ be the set containing all function k-tuples $g = (g_1, \ldots, g_k) \in \mathbb{X}^k$ with the image in $\mathscr{C}$. Then, $(\mathbb{X}^k)_{\mathscr{C}}$ is convex set in the space $\mathbb{X}^k$.

A linear functional $L : \mathbb{X} \to \mathbb{R}$ is positive (nonnegative) if $L(g) \geq 0$ for every nonnegative function $g \in \mathbb{X}$, and L is unital (normalized) if $L(\mathbf{1}) = 1$. If $g \in \mathbb{X}$, then for every unital positive functional L the number $L(g)$ is in the closed interval of real numbers containing the image of g. Through the paper, the space of all linear functionals on the space $\mathbb{X}$ will be denoted with $\mathbb{L}(\mathbb{X})$.

Let $f : \mathbb{R} \to \mathbb{R}$ be an affine function, that is, the function of the form $f(x) := \kappa x + \lambda$ where κ and λ are real constants.

If $g_1, \ldots, g_n \in \mathbb{X}$ are functions and if $L_1, \ldots, L_n \in \mathbb{L}(\mathbb{X})$ are positive functionals providing the unit equality

$$\sum_{i=1}^{n} L_i(\mathbf{1}) = 1, \tag{4}$$

then

$$\begin{aligned} f\left(\sum_{i=1}^{n} L_i(g_i)\right) &= \kappa \sum_{i=1}^{n} L_i(g_i) + \lambda \sum_{i=1}^{n} L_i(\mathbf{1}) = \sum_{i=1}^{n} L_i(\kappa g_i + \lambda \mathbf{1}) \\ &= \sum_{i=1}^{n} L_i(f(g_i)). \end{aligned} \tag{5}$$

Respecting the requirement of unit equality in (4), the sum $\sum_{i=1}^{n} L_i(g_i)$ could be called the functional convex combination. In the case $n = 1$, the functional $L = L_1$ must be unital by the unit equality in (4).

In 1931, Jessen stated the functional form of Jensen's inequality for convex functions of one variable; see [4]. Adapted to our purposes, that statement is as follows.

Theorem A. *Let $\mathcal{I} \subseteq \mathbb{R}$ be a closed interval, and let $g \in \mathbb{X}_{\mathcal{I}}$ be a function.*

Then, a unital positive functional $L \in \mathbb{L}(\mathbb{X})$ ensures the inclusion

$$L(g) \in \mathcal{I} \tag{6}$$

and satisfies the inequality

$$f(L(g)) \le L(f(g)) \tag{7}$$

for every continuous convex function $f : \mathcal{I} \to \mathbb{R}$ providing that $f(g) \in \mathbb{X}$.

If f is concave, then the reverse inequality is valid in (7). *If f is affine, then the equality is valid in* (7).

The interval $\mathcal{I}$ must be closed, otherwise it could happen that $L(g) \notin \mathcal{I}$. The function f must be continuous, otherwise it could happen that the inequality in (7) does not apply. Such boundary cases are presented in [5].

In 1937, McShane extended the functional form of Jensen's inequality to convex functions of several variables. He has covered the generalization in two steps, calling them the geometric (the inclusion in (8)) and analytic (the inequality in (9)) formulation of Jensen's inequality; see [6, Theorems 1 and 2]. Summarized in a theorem, that generalization is as follows.

Theorem B. *Let $\mathcal{C} \subseteq \mathbb{R}^k$ be a closed convex set, and let $g = (g_1, \ldots, g_k) \in (\mathbb{X}^k)_{\mathcal{C}}$ be a function.*

Then, a unital positive functional $L \in \mathbb{L}(\mathbb{X})$ ensures the inclusion

$$(L(g_1), \ldots, L(g_k)) \in \mathcal{C} \tag{8}$$

and satisfies the inequality

$$f(L(g_1), \ldots, L(g_k)) \le L(f(g_1, \ldots, g_k)) \tag{9}$$

for every continuous convex function $f : \mathcal{C} \to \mathbb{R}$ providing that $f(g_1, \ldots, g_k) \in \mathbb{X}$.

If f is concave, then the reverse inequality is valid in (9). *If f is affine, then the equality is valid in* (9).

3. Main Results

3.1. Functions of One Variable. The main result of this subsection is Theorem 1 relying on the idea of a convex function graph and its secant line. Using functions that are more general than convex functions and positive linear functionals, we obtain the functional Jensen's type inequalities.

Through the paper, we will use an interval $\mathcal{I} \subseteq \mathbb{R}$ and a bounded closed subinterval $[a, b] \subseteq \mathcal{I}$ with endpoints $a < b$.

Every number $x \in \mathbb{R}$ can be uniquely presented as the binomial affine combination

$$x = \frac{b - x}{b - a} a + \frac{x - a}{b - a} b, \tag{10}$$

which is convex if and only if the number x belongs to the interval $[a, b]$. Let $f : \mathcal{I} \to \mathbb{R}$ be a function, and let $f_{\{a,b\}}^{\text{line}} : \mathbb{R} \to \mathbb{R}$ be the function of the line passing through the points $A(a, f(a))$ and $B(b, f(b))$ of the graph of f. Applying the affinity of the function $f_{\{a,b\}}^{\text{line}}$ to the combination in (10), we obtain its equation

$$f_{\{a,b\}}^{\text{line}}(x) = \frac{b - x}{b - a} f(a) + \frac{x - a}{b - a} f(b). \tag{11}$$

The consequence of the representations in (10) and (11) is the fact that every convex function $f : \mathcal{I} \to \mathbb{R}$ satisfies the inequality

$$f(x) \le f_{\{a,b\}}^{\text{line}}(x) \quad \text{for } x \in [a, b] \tag{12}$$

and the reverse inequality

$$f(x) \ge f_{\{a,b\}}^{\text{line}}(x) \quad \text{for } x \in \mathcal{I} \setminus (a, b). \tag{13}$$

In the following consideration, we use continuous functions satisfying the inequalities in (12)-(13).

Theorem 1. *Let $\mathcal{I} \subseteq \mathbb{R}$ be a closed interval, let $[a, b] \subseteq \mathcal{I}$ be a bounded closed subinterval, and let $g \in \mathbb{X}_{[a,b]}$ and $h \in \mathbb{X}_{\mathcal{I} \setminus (a,b)}$ be functions.*

Then, a pair of unital positive functionals $L, H \in \mathbb{L}(\mathbb{X})$ such that

$$L(g) = H(h), \tag{14}$$

satisfies the inequality

$$L(f(g)) \le H(f(h)) \tag{15}$$

for every continuous function $f : \mathcal{I} \to \mathbb{R}$ satisfying (12)-(13) *and providing that $f(g), f(h) \in \mathbb{X}$.*

Proof. The number $L(g)$ belongs to the interval $[a, b]$ by the inclusion in (6). Using the features of the function f and applying the affinity of the function $f_{\{a,b\}}^{\text{line}}$, we get

$$\begin{aligned} L(f(g)) &\le L\left(f_{\{a,b\}}^{\text{line}}(g)\right) = f_{\{a,b\}}^{\text{line}}(L(g)) \\ &= f_{\{a,b\}}^{\text{line}}(H(h)) = H\left(f_{\{a,b\}}^{\text{line}}(h)\right) \\ &\le H(f(h)) \end{aligned} \tag{16}$$

because $f_{\{a,b\}}^{\text{line}}(h(x)) \le f(h(x))$ for every $x \in \mathcal{X}$. □

It is obvious that a continuous convex function $f : \mathcal{I} \to \mathbb{R}$ satisfies Theorem 1 for every subinterval $[a,b] \subseteq \mathcal{I}$ with endpoints $a < b$. The function used in Theorem 1 is shown in Figure 1. Such a function satisfies only the global property of convexity on the sets $[a,b]$ and $\mathcal{I} \setminus (a,b)$.

Involving the binomial convex combination $\alpha a + \beta b$ with the equality in (14) by assuming that

$$L(g) = \alpha a + \beta b = H(h) \tag{17}$$

and inserting the term $\alpha f(a) + \beta f(b)$ in (16) via the double equality

$$f_{\{a,b\}}^{\text{line}}(L(g)) = \alpha f(a) + \beta f(b) = f_{\{a,b\}}^{\text{line}}(H(h)) \tag{18}$$

which is true because $f_{\{a,b\}}^{\text{line}}(\alpha a + \beta b) = \alpha f(a) + \beta f(b)$, we achieve the double inequality

$$L(f(g)) \le \alpha f(a) + \beta f(b) \le H(f(h)). \tag{19}$$

The functions used in Theorem 1 satisfy the functional form of Jensen's inequality in the following case.

Corollary 2. *Let $\mathcal{I} \subseteq \mathbb{R}$ be a closed interval, let $[a,b] \subseteq \mathcal{I}$ be a bounded closed subinterval, and let $h \in \mathbb{X}_{\mathcal{I}\setminus(a,b)}$ be a function.*

Then, a unital positive functional $H \in \mathbb{L}(\mathbb{X})$ such that

$$H(h) \in [a,b] \tag{20}$$

satisfies the inequality

$$f(H(h)) \le H(f(h)) \tag{21}$$

for every continuous function satisfying (12)-(13) *and providing that $f(h) \in \mathbb{X}$.*

Proof. Putting $\alpha a + \beta b = H(h)$, it follows that

$$\begin{aligned} f(H(h)) &= f(\alpha a + \beta b) \le f_{\{a,b\}}^{\text{line}}(\alpha a + \beta b) \\ &= \alpha f(a) + \beta f(b) \le H(f(h)) \end{aligned} \tag{22}$$

by the right inequality in (19). □

Now, we give a characterization of continuous convex functions by using unital positive functionals.

Proposition 3. *Let $\mathcal{I} \subseteq \mathbb{R}$ be a closed interval. A continuous function $f : \mathcal{I} \to \mathbb{R}$ is convex if and only if it satisfies the inequality*

$$L(f(g)) \le f_{\{a,b\}}^{line}(L(g)) \tag{23}$$

for every pair of interval endpoints $a, b \in \mathcal{I}$, every function $g \in \mathbb{X}_{[a,b]}$ such that $f(g) \in \mathbb{X}$, and every unital positive functional $L \in \mathbb{L}(\mathbb{X})$.

Proof. Let us prove the sufficiency. Let $c := \alpha a + \beta b$ be a convex combination of points $a, b \in \mathcal{I}$ where $a < b$. We take the constant function $g = c\mathbf{1}$ in $\mathbb{X}_{[a,b]}$ (actually $g(x) = c$ for every $x \in \mathcal{X}$) and a unital positive functional L. Then, connecting

$$\begin{aligned} L(f(g)) &= L(f(c)\mathbf{1}) = f(c) = f(\alpha a + \beta b), \\ f_{\{a,b\}}^{\text{line}}(L(g)) &= f_{\{a,b\}}^{\text{line}}(\alpha a + \beta b) = \alpha f(a) + \beta f(b) \end{aligned} \tag{24}$$

via (23), we get the convexity inequality in (3). □

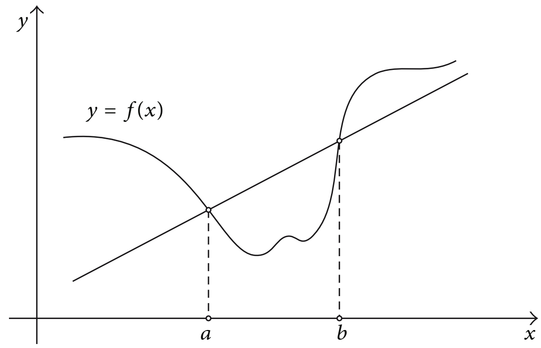

FIGURE 1: A continuous function satisfying (12)-(13).

3.2. Functions of Several Variables. We want to transfer the results of the previous subsection to higher dimensions. The main result in this subsection is Theorem 6 generalizing Theorem 1 to functions of several variables.

Let $\mathscr{C} \subseteq \mathbb{R}^2$ be a convex set, let $\triangle \subseteq \mathscr{C}$ be a triangle with vertices A, B, and C, and let $\triangle^o$ be its interior. In the following observation, we assume that $f : \mathscr{C} \to \mathbb{R}$ is a continuous function satisfying the inequality

$$f(P) \le f_{\{A,B,C\}}^{\text{plane}}(P) \quad \text{for } P \in \triangle \tag{25}$$

and the reverse inequality

$$f(P) \ge f_{\{A,B,C\}}^{\text{plane}}(P) \quad \text{for } P \in \mathscr{C} \setminus \triangle^o, \tag{26}$$

where $f_{\{A,B,C\}}^{\text{plane}}$ is the function of the plane passing through the corresponding points of the graph of f.

It should be noted that convex functions of two variables do not generally satisfy (26). The next example confirms this claim.

Example 4. We take the convex function $f(x,y) = x^2 + y^2$, the triangle with vertices $A(0,0)$, $B(1,0)$, and $C(0,2)$, and the outside point $P(1,1)$.

The valuation of functions f and $f_{\{A,B,C\}}^{\text{plane}}(x,y) = x + 2y$ at the point P is

$$2 = f(P) < f_{\{A,B,C\}}^{\text{plane}}(P) = 3 \tag{27}$$

as opposed to (26).

The generalization of Theorem 1 to two dimensions is as follows.

Lemma 5. *Let $\mathscr{C} \subseteq \mathbb{R}^2$ be a closed convex set, let $\triangle \subseteq \mathscr{C}$ be a triangle, and let $g = (g_1, g_2) \in (\mathbb{X}^2)_\triangle$ and $h = (h_1, h_2) \in (\mathbb{X}^2)_{\mathscr{C}\setminus\triangle^o}$ be functions.*

Then, a pair of unital positive functionals $L, H \in \mathbb{L}(\mathbb{X})$ such that

$$(L(g_1), L(g_2)) = (H(h_1), H(h_2)) \tag{28}$$

satisfies the inequality

$$L(f(g_1, g_2)) \leq H(f(h_1, h_2)) \tag{29}$$

for every continuous function satisfying (25)-(26) *and providing that $f(g_1, g_2), f(h_1, h_2) \in \mathbb{X}$.*

Proof. The proof is similar to that of Theorem 1. Using the triangle vertices A, B, and C, we apply the plane function $f^{\text{plane}}_{\{A,B,C\}}$ instead of the line function $f^{\text{line}}_{\{a,b\}}$. □

The previous lemma suggests how the results of the previous subsection can be transferred to higher dimensions.

Let $S_1, \ldots, S_{k+1} \in \mathbb{R}^k$ be points. Their convex hull

$$\mathscr{S} = \operatorname{conv}\{S_1, \ldots, S_{k+1}\} \tag{30}$$

is the k-simplex in $\mathbb{R}^k$ if the points $S_1 - S_{k+1}, \ldots, S_k - S_{k+1}$ are linearly independent.

Let $\mathscr{C} \subseteq \mathbb{R}^k$ be a convex set, and let $\mathscr{S} \subseteq \mathscr{C}$ be a k-simplex with vertices $S_1, \ldots, S_{k+1}$. In the consideration that follows, we use a function $f : \mathscr{C} \to \mathbb{R}$ satisfying the inequality

$$f(P) \leq f^{\text{hyperplane}}_{\{S_1,\ldots,S_{k+1}\}}(P) \quad \text{for } P \in \mathscr{S} \tag{31}$$

and the reverse inequality

$$f(P) \geq f^{\text{hyperplane}}_{\{S_1,\ldots,S_{k+1}\}}(P) \quad \text{for } P \in \mathscr{C} \setminus \mathscr{S}^o, \tag{32}$$

where $f^{\text{hyperplane}}_{\{S_1,\ldots,S_{k+1}\}}$ is the function of the hyperplane passing through the corresponding points of the graph of f.

Theorem 6. *Let $\mathscr{C} \subseteq \mathbb{R}^k$ be a closed convex set, let $\mathscr{S} \subseteq \mathscr{C}$ be a k-simplex, and let $g = (g_1, \ldots, g_k) \in (\mathbb{X}^k)_\mathscr{S}$ and $h = (h_1, \ldots, h_k) \in (\mathbb{X}^k)_{\mathscr{C}\setminus\mathscr{S}^o}$ be functions.*

Then, a pair of unital positive functionals $L, H \in \mathbb{L}(\mathbb{X})$ such that

$$(L(g_1), \ldots, L(g_k)) = (H(h_1), \ldots, H(h_k)) \tag{33}$$

satisfies the inequality

$$L(f(g_1, \ldots, g_k)) \leq H(f(h_1, \ldots, h_k)) \tag{34}$$

for every continuous function satisfying (31)-(32) *and providing that $f(g_1, \ldots, g_k), f(h_1, \ldots, h_k) \in \mathbb{X}$.*

Proof. Relying on the hyperplane function $f^{\text{hyperplane}}_{\{S_1,\ldots,S_{k+1}\}}$ where $S_1, \ldots, S_{k+1}$ are the simplex vertices, we can apply the proof similar to that of Theorem 1. □

Including the $(k+1)$-membered convex combination $\sum_{p=1}^{k+1} \gamma_p S_p$ with the equality in (33) in a way that

$$(L(g_1), \ldots, L(g_k)) = \sum_{p=1}^{k+1} \gamma_p S_p = (H(h_1), \ldots, H(h_k)) \tag{35}$$

and using the double equality

$$\begin{aligned} f^{\text{hyperplane}}_{\{S_1,\ldots,S_{k+1}\}}&(L(g_1), \ldots, L(g_k)) \\ &= \sum_{p=1}^{k+1} \gamma_p f(S_p) \\ &= f^{\text{hyperplane}}_{\{S_1,\ldots,S_{k+1}\}}(H(h_1), \ldots, H(h_k)), \end{aligned} \tag{36}$$

we can derive the double inequality

$$L(f(g_1, \ldots, g_k)) \leq \sum_{p=1}^{k+1} \gamma_p f(S_p) \leq H(f(h_1, \ldots, h_k)). \tag{37}$$

The following functional form of Jensen's inequality is true for functions of several variables.

Corollary 7. *Let $\mathscr{C} \subseteq \mathbb{R}^k$ be a closed convex set, let $\mathscr{S} \subseteq \mathscr{C}$ be a k-simplex, and let $h = (h_1, \ldots, h_k) \in (\mathbb{X}^k)_{\mathscr{C}\setminus\mathscr{S}^o}$ be a function.*

Then, a unital positive functional $H \in \mathbb{L}(\mathbb{X})$ such that

$$(H(h_1), \ldots, H(h_k)) \in \mathscr{S} \tag{38}$$

satisfies the inequality

$$f(H(h_1), \ldots, H(h_k)) \leq H(f(h_1, \ldots, h_k)) \tag{39}$$

for every continuous function satisfying (25)-(26) *and providing that $f(h_1, \ldots, h_k) \in \mathbb{X}$.*

Continuous convex functions of several variables can be characterized by unital positive functionals in the following way. The dimension of a convex set is defined as the dimension of its affine hull.

Proposition 8. *Let $\mathscr{C} \subseteq \mathbb{R}^k$ be a closed convex set of dimension k. A continuous function $f : \mathscr{C} \to \mathbb{R}$ is convex if and only if it satisfies the inequality*

$$L(f(g_1, \ldots, g_k)) \leq f^{\textit{hyperplane}}_{\{S_1,\ldots,S_{k+1}\}}(L(g_1), \ldots, L(g_k)) \tag{40}$$

for every $(k+1)$-tuple of k-simplex vertices $S_1, \ldots, S_{k+1} \in \mathscr{C}$, every function $g = (g_1, \ldots, g_k) \in (\mathbb{X}^k)_\mathscr{S}$ such that $f(g_1, \ldots, g_k) \in \mathbb{X}$, and every unital positive functional $L \in \mathbb{L}(\mathbb{X})$.

Proof. To prove the sufficiency, we take a convex combination $C = \sum_{p=1}^{k+1} \gamma_p S_p$ of k-simplex vertices $S_1, \ldots, S_{k+1} \in \mathscr{C}$. If $C = (c_1, \ldots, c_k)$, we take the constant mapping $g = (g_1, \ldots, g_k) \in (\mathbb{X}^k)_\mathscr{S}$ consisting of constant functions $g_i = c_i\mathbf{1}$ and continue the proof in the same way as in Proposition 3. Finally, we get Jensen's inequality

$$f\left(\sum_{p=1}^{k+1}\gamma_p S_p\right) \le \sum_{p=1}^{k+1}\gamma_p f\left(S_p\right) \tag{41}$$

confirming the convexity of the function f. □

4. Applications to Functional Quasiarithmetic Means

Functions investigated in Subsection 3.1 can be included to quasiarithmetic means by applying methods such as those for convex functions. The basic facts relating to quasiarithmetic and power means can be found in [7]. For more details on different forms of quasiarithmetic and power means, as well as their refinements, see [8].

The next generalization of Theorem 1 will be applied to the consideration of functional quasiarithmetic means.

Corollary 9. *Let $\mathcal{I} \subseteq \mathbb{R}$ be a closed interval, let $[a,b] \subseteq \mathcal{I}$ be a bounded closed subinterval, and let $g_1,\dots,g_n \in \mathbb{X}_{[a,b]}$ and $h_1,\dots,h_m \in \mathbb{X}_{\mathcal{I}\setminus(a,b)}$ be functions.*

Then, a pair of collections of positive functionals $L_i, H_j \in \mathbb{L}(\mathbb{X})$ providing the unit equalities $\sum_{i=1}^n L_i(\mathbf{1}) = \sum_{j=1}^m H_j(\mathbf{1}) = 1$ and the equality

$$\sum_{i=1}^{n} L_i\left(g_i\right) = \sum_{j=1}^{m} H_j\left(h_j\right) \tag{42}$$

satisfies the inequality

$$\sum_{i=1}^{n} L_i\left(f\left(g_i\right)\right) \le \sum_{j=1}^{m} H_j\left(f\left(h_j\right)\right) \tag{43}$$

for every continuous function satisfying (12)-(13) *and providing that all functions $f(g_i), f(h_j) \in \mathbb{X}$.*

Now, we present a way of introducing the functional quasiarithmetic means. Let $g_1,\dots,g_n \in \mathbb{X}_{\mathcal{I}}$ be functions, and let $\varphi : \mathcal{I} \to \mathbb{R}$ be a strictly monotone continuous function such that all $\varphi(g_i) \in \mathbb{X}$. Let $L_1,\dots,L_n : \mathbb{X} \to \mathbb{R}$ be positive linear functionals providing the unit equality $\sum_{i=1}^n L_i(\mathbf{1}) = 1$. The quasiarithmetic mean of functions g_i respecting the function φ and functionals L_i can be defined by

$$M_\varphi\left(L_1\dots,L_n; g_1,\dots,g_n\right) = \varphi^{-1}\left(\sum_{i=1}^{n} L_i\left(\varphi\left(g_i\right)\right)\right). \tag{44}$$

In what follows, we will use the abbreviation $M_\varphi(L_i, g_i)$ for the above mean. The term in parentheses belongs to the interval $\varphi(\mathcal{I})$, and therefore the quasiarithmetic mean $M_\varphi(L_i, g_i)$ belongs to the interval $\mathcal{I}$.

In applications of the function convexity, we use a pair of strictly monotone continuous functions $\varphi, \psi : \mathcal{I} \to \mathbb{R}$ such that ψ is convex with respect to φ (it also says that ψ is φ-convex), which means that the function $f = \psi(\varphi^{-1})$ is convex on the interval $\varphi(\mathcal{I})$. A similar notation is used for the concavity.

Instead of the convexity of f, we will apply the conditions in (12)-(13) via Corollary 9 as follows.

Theorem 10. *Let $\mathcal{I} \subseteq \mathbb{R}$ be a closed interval, let $[a,b] \subseteq \mathcal{I}$ be a bounded closed subinterval, and let $g_1,\dots,g_n \in \mathbb{X}_{[a,b]}$ and $h_1,\dots,h_m \in \mathbb{X}_{\mathcal{I}\setminus(a,b)}$ be functions. Let $L_i, H_j \in \mathbb{L}(\mathbb{X})$ be a pair of collections of positive functionals providing the unit equalities $\sum_{i=1}^n L_i(1) = \sum_{j=1}^m H_j(1) = 1$. Let $\varphi, \psi : \mathcal{I} \to \mathbb{R}$ be strictly monotone continuous functions such that all functions $\varphi(g_i), \varphi(h_j), \psi(g_i), \psi(h_j) \in \mathbb{X}$, and let $f = \psi(\varphi^{-1})$ be the composite function.*

If f satisfies (12)-(13) *and ψ is increasing and if the equality*

$$M_\varphi\left(L_i, g_i\right) = M_\varphi\left(H_j, h_j\right) \tag{45}$$

is valid, then we have the inequality

$$M_\psi\left(L_i, g_i\right) \le M_\psi\left(H_j, h_j\right). \tag{46}$$

Proof. We take $\mathcal{J} = \varphi(\mathcal{I})$ and $[c,d] = \varphi([a,b])$. We will apply Corollary 9 to the functions $u_i = \varphi(g_i) \in \mathbb{X}_{[c,d]}$ and $v_j = \varphi(h_j) \in \mathbb{X}_{\mathcal{J}\setminus(c,d)}$ and the function $f : \mathcal{J} \to \mathbb{R}$.

Using the equality $\varphi(M_\varphi(L_i, g_i)) = \varphi(M_\varphi(H_j, h_j))$ and including the functions u_i and v_j, we have

$$\sum_{i=1}^{n} L_i\left(u_i\right) = \sum_{j=1}^{m} H_j\left(v_j\right). \tag{47}$$

Then, the inequality

$$\sum_{i=1}^{n} L_i\left(f\left(u_i\right)\right) \le \sum_{j=1}^{m} H_j\left(f\left(v_j\right)\right) \tag{48}$$

follows from Corollary 9, and applying the increasing function ψ^{-1}, we get

$$\psi^{-1}\left(\sum_{i=1}^{n} L_i\left(f\left(u_i\right)\right)\right) \le \psi^{-1}\left(\sum_{j=1}^{m} H_j\left(f\left(v_j\right)\right)\right). \tag{49}$$

The above inequality is actually the inequality in (46) because $f(u_i) = \psi(g_i)$ and $f(v_j) = \psi(h_j)$. □

All the cases of the above theorem are as follows.

Corollary 11. *Let $f = \psi(\varphi^{-1})$ be the composite function satisfying the conditions of Theorem 10.*

If either f satisfies (12)-(13) *and ψ is increasing or $-f$ satisfies* (12)-(13) *and ψ is decreasing and if the equality in* (45) *is valid, then the inequality holds in* (46).

If either f satisfies (12)-(13) *and ψ is decreasing or $-f$ satisfies* (12)-(13) *and ψ is increasing and if the equality in* (45) *is valid, then the reverse inequality holds in* (46).

A special case of the quasiarithmetic means in (44) is power means depending on real exponents r. Thus, using the functions

$$\varphi_r\left(x\right) = \begin{cases} x^r, & r \ne 0 \\ \ln x, & r = 0, \end{cases} \tag{50}$$

where $x \in (0,\infty)$, we get the power means of order r in the form

$$M_r(L_i, g_i) = \begin{cases} \left(\sum_{i=1}^{n} L_i(g_i^r)\right)^{1/r}, & r \neq 0 \\ \exp\left(\sum_{i=1}^{n} L_i(\ln g_i)\right), & r = 0. \end{cases} \quad (51)$$

To apply Theorem 1 to the power means, we use a closed interval $\mathcal{I} = [\varepsilon,\infty)$ where ε is a positive number and the equality

$$M_1(L, g_i) = \sum_{i=1}^{n} L_i(g_i). \quad (52)$$

Corollary 12. *Let $\mathcal{I} = [\varepsilon,\infty)$ be an unbounded closed interval where $\varepsilon > 0$, let $[a,b] \subset \mathcal{I}$ be a bounded closed subinterval, and let $g_1,\dots,g_n \in \mathbb{X}_{[a,b]}$ and $h_1,\dots,h_m \in \mathbb{X}_{\mathcal{I}\setminus(a,b)}$ be functions. Let $L_i, H_j \in \mathbb{L}(\mathbb{X})$ be a pair of collections of positive functionals providing the unit equalities $\sum_{i=1}^{n} L_i(1) = \sum_{j=1}^{m} H_j(1) = 1$.*

If

$$M_1(L_i, g_i) = M_1(H_j, h_j), \quad (53)$$

then

$$\begin{aligned} M_r(L_i, g_i) &\leq M_r(H_j, h_j) \quad \text{for } r \geq 1, \\ M_r(L_i, g_i) &\geq M_r(H_j, h_j) \quad \text{for } r \leq 1. \end{aligned} \quad (54)$$

Proof. The proof follows from Theorem 10 and Corollary 11 by using convex and concave functions such as $\varphi(x) = x$ and $\psi(x) = x^r$ for $r \neq 0$, and $\psi(x) = \ln x$ for $r = 0$. □

5. Applications to Discrete and Integral Inequalities

Our aim is to use Theorem 6 to obtain certain discrete and integral inequalities concerning functions of several variables. The following is the application to discrete inequalities.

Proposition 13. *Let $\mathcal{C} \subseteq \mathbb{R}^k$ be a closed convex set, let $\mathcal{S} \subseteq \mathcal{C}$ be a k-simplex, let $\sum_{i=1}^{n} \alpha_i A_i$ be a convex combination of points $A_i \in \mathcal{S}$, and let $\sum_{j=1}^{m} \beta_j B_j$ be a convex combination of points $B_j \in \mathcal{C} \setminus \mathcal{S}^o$.*

If the above convex combinations have the common center

$$\sum_{i=1}^{n} \alpha_i A_i = \sum_{j=1}^{m} \beta_j B_j, \quad (55)$$

then the inequality

$$\sum_{i=1}^{n} \alpha_i f(A_i) \leq \sum_{j=1}^{m} \beta_j f(B_j) \quad (56)$$

holds for every continuous function $f : \mathcal{C} \to \mathbb{R}$ satisfying (31)-(32).

Proof. We take the set $\mathcal{X} = \mathcal{C}$ and the space $\mathbb{X}$ containing all real functions on $\mathcal{C}$. We also take any simplex vertex S and its coordinates $(s_1,\dots,s_k)$.

Let $g_p, h_p \in \mathbb{X}$ $(p = 1,\dots,k)$ be functions defined by

$$g_p(x_1,\dots,x_k) = \begin{cases} x_p, & (x_1,\dots,x_k) \in \mathcal{S} \\ s_p, & (x_1,\dots,x_k) \in \mathcal{C} \setminus \mathcal{S}, \end{cases} \quad (57)$$

$$h_p(x_1,\dots,x_k) = \begin{cases} s_p, & (x_1,\dots,x_k) \in \mathcal{S}^o \\ x_p, & (x_1,\dots,x_k) \in \mathcal{C} \setminus \mathcal{S}^o. \end{cases} \quad (58)$$

Then, $g = (g_1,\dots,g_k) \in (\mathbb{X}^k)_{\mathcal{S}}$ and $h = (h_1,\dots,h_k) \in (\mathbb{X}^k)_{\mathcal{C}\setminus\mathcal{S}^o}$.

Let $L, H \in \mathbb{L}(\mathbb{X})$ be summarizing unital positive functionals defined by

$$\begin{aligned} L(g) &= \sum_{i=1}^{n} \alpha_i g(A_i), \\ H(h) &= \sum_{j=1}^{m} \beta_j h(B_j). \end{aligned} \quad (59)$$

Applying the functional L to the functions g_p and the functional H to the functions h_p, we obtain

$$\begin{aligned} \sum_{i=1}^{n} \alpha_i A_i &= (L(g_1),\dots,L(g_k)) \\ &= (H(h_1),\dots,H(h_k)) = \sum_{j=1}^{m} \beta_j B_j. \end{aligned} \quad (60)$$

Now, we can apply Theorem 6 and get the inequality

$$\begin{aligned} \sum_{i=1}^{n} \alpha_i f(A_i) &= L(f(g_1,\dots,g_k)) \leq H(f(h_1,\dots,h_k)) \\ &= \sum_{j=1}^{m} \beta_j f(B_j) \end{aligned} \quad (61)$$

which concludes the proof. □

Proposition 13 does not generally hold for convex functions. The next example demonstrates a concrete planar case of $k = 2$.

Example 14. We take the convex function $f(x,y) = x^2 + y^2$, the triangle with vertices $A_1(-3,0)$, $A_2(3,0)$, and $A_3(0,3)$, and the outside points $B_1(-2,2)$, $B_2(0,-2)$, and $B_3(2,2)$.

Then, we have

$$\begin{aligned} \frac{1}{3}A_1 + \frac{1}{3}A_2 + \frac{1}{3}A_3 &= \frac{3}{8}B_1 + \frac{2}{8}B_2 + \frac{3}{8}B_3, \\ 9 = \frac{1}{3}f(A_1) &+ \frac{1}{3}f(A_2) + \frac{1}{3}f(A_3) \\ &> \frac{3}{8}f(B_1) + \frac{2}{8}f(B_2) + \frac{3}{8}f(B_3) = 7. \end{aligned} \quad (62)$$

More details on the behavior of a convex function of two variables on the triangle and outside the triangle can be found in [9, Theorem 3.2]. Triangle cones have a prominent part in these considerations.

The integral analogy of the concept of convex combination is the concept of barycenter. Let μ be a positive measure on $\mathbb{R}^k$, and let $\mathscr{A} \subseteq \mathbb{R}^k$ be a μ-measurable set with $\mu(\mathscr{A}) > 0$. Given the positive integer n, let $\mathscr{A} = \cup_{i=1}^{n} \mathscr{A}_{ni}$ be the partition of pairwise disjoint μ-measurable sets $\mathscr{A}_{ni}$. Taking points $A_{ni} \in \mathscr{A}_{ni}$, we determine the convex combination

$$A_n = \sum_{i=1}^{n} \frac{\mu(\mathscr{A}_{ni})}{\mu(\mathscr{A})} A_{ni} \tag{63}$$

whose center A_n belongs to conv$\mathscr{A}$. The μ-barycenter of the set $\mathscr{A}$ can be defined as the limit of the sequence $(A_n)_n$; that is,

$$\begin{aligned} M(\mathscr{A}, \mu) &= \lim_{n \to \infty} \left(\sum_{i=1}^{n} \frac{\mu(\mathscr{A}_{ni})}{\mu(\mathscr{A})} A_{ni} \right) \\ &= \frac{1}{\mu(\mathscr{A})} \left(\int_{\mathscr{A}} x_1 \, d\mu, \ldots, \int_{\mathscr{A}} x_k \, d\mu \right). \end{aligned} \tag{64}$$

As defined above, the point $M(\mathscr{A}, \mu)$ is in conv$\mathscr{A}$. So, the convex sets contain its barycenters.

The application of Theorem 6 to integral inequalities is as follows.

Proposition 15. *Let μ be a positive measure on $\mathbb{R}^k$. Let $\mathscr{C} \subseteq \mathbb{R}^k$ be a closed convex set, let $\mathscr{S} \subseteq \mathscr{C}$ be a k-simplex, and let $\mathscr{A} \subseteq \mathscr{S}$ and $\mathscr{B} \subseteq \mathscr{C} \setminus \mathscr{S}^o$ be sets of positive μ-measures.*

If the above sets have the common μ-barycenter

$$M(\mathscr{A}, \mu) = M(\mathscr{B}, \mu), \tag{65}$$

then the inequality

$$\frac{1}{\mu(\mathscr{A})} \int_{\mathscr{A}} f(x_1, \ldots, x_k) \, d\mu \le \frac{1}{\mu(\mathscr{B})} \int_{\mathscr{B}} f(x_1, \ldots, x_k) \, d\mu \tag{66}$$

holds for every continuous function $f : \mathscr{C} \to \mathbb{R}$ satisfying (31)-(32).

Proof. The proof is similar to that of Proposition 13 by using $\mathbb{X}$ as the space of all μ-integrable functions on $\mathscr{C}$. We apply the integrating unital positive functional L defined by

$$L(g) = \frac{1}{\mu(\mathscr{A})} \int_{\mathscr{A}} g(x_1, \ldots, x_k) \, d\mu \tag{67}$$

to the functions g_p of (57), as well as the integrating unital positive functional H defined by

$$H(h) = \frac{1}{\mu(\mathscr{B})} \int_{\mathscr{B}} h(x_1, \ldots, x_k) \, d\mu \tag{68}$$

to the functions h_p of (58). □

If $S_1, \ldots, S_{k+1}$ are the simplex vertices, then using the unique convex combination $\sum_{p=1}^{k+1} \gamma_p S_p$ satisfying

$$M(\mathscr{A}, \mu) = \sum_{p=1}^{k+1} \gamma_p S_p = M(\mathscr{B}, \mu) \tag{69}$$

and applying (37), we obtain the extension of (66) as the double inequality

$$\begin{aligned} \frac{1}{\mu(\mathscr{A})} \int_{\mathscr{A}} f(x_1, \ldots, x_k) \, d\mu &\le \sum_{p=1}^{k+1} \gamma_p f(S_p) \\ &\le \frac{1}{\mu(\mathscr{B})} \int_{\mathscr{B}} f(x_1, \ldots, x_k) \, d\mu. \end{aligned} \tag{70}$$

The above inequality is reminiscent of Hermite-Hadamard's inequality where discrete and integral terms are replaced, see the below inequality in (72).

Implementing convex combinations to the integral method, one may derive the following version of the Hermite-Hadamard inequality for convex functions on simplexes.

Proposition 16. *Let μ be a positive measure on $\mathbb{R}^k$. Let $\mathscr{S} \subset \mathbb{R}^k$ be a k-simplex of positive μ-measure, let $S_1, \ldots, S_{k+1}$ be simplex vertices, and let $\sum_{p=1}^{k+1} \gamma_p S_p$ be their convex combination.*

If the convex combination center and the μ-barycenter of $\mathscr{S}$ both fall at the same point

$$\sum_{p=1}^{k+1} \gamma_p S_p = M(\mathscr{S}, \mu), \tag{71}$$

then the double inequality

$$f\left(\sum_{p=1}^{k+1} \gamma_p S_p \right) \le \frac{1}{\mu(\mathscr{S})} \int_{\mathscr{S}} f(x_1, \ldots, x_k) \, d\mu \le \sum_{p=1}^{k+1} \gamma_p f(S_p) \tag{72}$$

holds for every μ-integrable convex function $f : \mathscr{S} \to \mathbb{R}$.

More on the important and interesting Hermite-Hadamard's inequality, including historical facts about its name, can be found in [10, 11].

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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Research Article

Operators on Spaces of Bounded Vector-Valued Continuous Functions with Strict Topologies

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Let X be a completely regular Hausdorff space, and let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be Banach spaces. Let $C_b(X, E)$ be the space of all E-valued bounded, continuous functions defined on X, equipped with the strict topologies β_z, where $z = \sigma, \infty, p, \tau, t$. General integral representation theorems of $(\beta_z, \|\cdot\|_F)$-continuous linear operators $T : C_b(X, E) \to F$ with respect to the corresponding operator-valued measures are established. Strongly bounded and $(\beta_z, \|\cdot\|_F)$-continuous operators $T : C_b(X, E) \to F$ are studied. We extend to "the completely regular setting" some classical results concerning operators on the spaces $C(X, E)$ and $C_o(X, E)$, where X is a compact or a locally compact space.

1. Introduction and Terminology

Throughout the paper let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be real Banach spaces, and let E' and F' denote the Banach duals of E and F, respectively. By $B_{F'}$ and B_E we denote the closed unit ball in F' and E, respectively. By $\mathscr{L}(E, F)$ we denote the space of all bounded linear operators $U : E \to F$. Given a locally convex space (L, ξ) by $(L, \xi)'$ or L'_ξ we will denote its topological dual. We denote by $\sigma(L, K)$ the weak topology on L with respect to a dual pair $\langle L, K\rangle$.

Assume that X is a completely regular Hausdorff space. Let $C_b(X, E)$ stand for the Banach space of all bounded continuous, E-valued functions on X provided with the uniform norm $\|\cdot\|$. We write $C_b(X)$ instead of $C_b(X, \mathbb{R})$. By $C_b(X, E)'$ we denote the Banach dual of $C_b(X, E)$. For $f \in C_b(X, E)$ let $\tilde{f}(t) = \|f(t)\|_E$ for $t \in X$.

Let $\mathscr{B}$ (resp., $\mathscr{B}a$) be the algebra (resp., σ-algebra) of Baire sets in X, which is the algebra (resp., σ-algebra) generated by the class $\mathscr{Z}$ of all zero sets of functions of $C_b(X)$. By $\mathscr{P}$ we denote the family of all cozero sets in X. Let $B(\mathscr{B}, E)$ stand for the Banach space of all totally $\mathscr{B}$-measurable functions $f : X \to E$ (the uniform limits of sequences of E-valued $\mathscr{B}$-simple functions) provided with the uniform norm $\|\cdot\|$ (see [1, 2]). We will write $B(\mathscr{B})$ instead of $B(\mathscr{B}, \mathbb{R})$.

Strict topologies β_z on $C_b(X)$ and $C_b(X, E)$ (for $z = \sigma, \infty, p, \tau, t$) play an important role in the topological measure theory (see [3–12] for definitions and more details). Recall that a subset H of $C_b(X, E)$ is said to be *solid* if $f_1 \in C_b(X, E)$ and $f_2 \in H$ with $\tilde{f}_1(t) \le \tilde{f}_2(t)$ for $t \in X$ imply that $f_1 \in H$. Then β_z are locally convex-solid topologies on $C_b(X, E)$; that is, they have a local base at 0 consisting of convex and solid sets (see [6, Theorem 8.1], [10, Theorem 5]). We have $\beta_t \subset \beta_\tau \subset \beta_\infty \subset \beta_\sigma \subset \mathscr{T}_{\|\cdot\|}$ and $\beta_t \subset \beta_p \subset \beta_\sigma$. For a net (f_α) in $C_b(X, E)$, $f_\alpha \to 0$ for β_z if and only if $\tilde{f}_\alpha \to 0$ for β_z in $C_b(X)$ (see [6, 10]).

Let $C_b(X) \otimes E$ stand for the algebraic tensor product of $C_b(X)$ and E; that is, $C_b(X) \otimes E$ is the space of all functions $\sum_{i=1}^n (u_i \otimes x_i)$, where $u_i \in C_b(X)$, $x_i \in E$ for $i = 1, \ldots, n$, and $(u_i \otimes x_i)(t) = u_i(t)x_i$ for $t \in X$. Then $C_b(X) \otimes E$ is dense in $(C_b(X, E), \beta_z)$ for $z = \infty, \tau, t$ (see [6, 8]). Moreover, $C_b(X) \otimes E$ is dense in $(C_b(X, E), \beta_\sigma)$ if X or E is a D-space (see [6, Theorem 5.2], [13]) and in $(C_b(X, E), \beta_p)$ if X is real-compact (see [10, Theorem 7]).

Let $C_{rc}(X, E)$ denote the Banach space of all continuous functions $h : X \to E$ such that $h(X)$ is a relatively compact set in E, provided with the uniform norm $\|\cdot\|$. Then $C_b(X) \otimes E \subset C_{rc}(X, E) \subset B(\mathscr{B}, E)$.

Linear operators from the spaces $C_{rc}(X,E)$ and $C_b(X,E)$, equipped with the strict topologies $\beta_z(z = \sigma,\infty,\tau)$ to a locally convex space (F,ξ), were studied by Katsaras and Liu [14], Aguayo-Garrido, Nova-Yanéz and Sanchez [15, 16], and Khurana [17]. In particular, Katsaras and Liu found an integral representation of weakly compact operators $S : C_{rc}(X,E) \to F$ and characterizations of (β_z,ξ)-continuous and weakly compact operators $S : C_{rc}(X,E) \to F$ for $z = \sigma,\tau$ (see [14, Theorems 3, 4, 5]). Aguayo-Arrido and Nova-Yanéz derived a Riesz representation theorem for (β_z,ξ)-continuous and weakly compact operators $T : C_b(X,E) \to F$ for $z = \infty,\tau$ in terms of their representing operator measures (see [15, Theorems 5 and 6]). If X is a locally compact space, continuous operators on $C_o(X,E)$ were studied by Dobrakov (see [18]) and Mitter and Young (see [19]).

In this paper we develop the theory of continuous linear operators from $C_b(X,E)$, equipped with the strict topologies β_z $(z = \sigma,\infty,p,\tau,t)$ to a Banach space $(F,\|\cdot\|_F)$. In particular, we extend to "the completely regular setting" some classical results of Brooks and Lewis (see [20, Theorem 5], [21, Theorem 5.2], [22, Theorem 2.1]) concerning operators on the spaces $C(X,E)$ and $C_o(X,E)$, where X is a compact or a locally compact space, respectively. In Section 2, using the device of embedding the space $B(\mathscr{B},E)$ into $C_{rc}(X,E)''$ (the Banach bidual of $C_{rc}(X,E)$), we state the integral representation of bounded linear operators from $C_{rc}(X,E)$ to F. In Section 3 we derive general Riesz representation theorems for $(\beta_z,\|\cdot\|_F)$-continuous linear operators $T : C_b(X,E) \to F$ $(z = \sigma,\infty,p,\tau,t)$ with respect to the corresponding measures $m : \mathscr{B} \to \mathscr{L}(E,F'')$ (see Theorems 9 and 14 below). Section 4 is devoted to the study of $(\beta_\sigma,\|\cdot\|_F)$-continuous and strongly bounded operators $T : C_b(X,E) \to F$.

2. Integral Representation of Bounded Linear Operators on $C_{rc}(X,E)$

Let $M(X)$ stand for the Banach lattice of all Baire measures on $\mathscr{B}$, provided with the norm $\|\nu\| = |\nu|(X)$ (= the total variation of ν). Due to the Alexandrov representation theorem $C_b(X)'$ can be identified with $M(X)$ through the lattice isomorphism $M(X) \ni \nu \mapsto \varphi_\nu \in C_b(X)'$, where $\varphi_\nu(u) = \int_X u\,d\nu$ for $u \in C_b(X)$ and $\|\varphi_\nu\| = \|\nu\|$ (see [4, Theorem 5.1]).

By $M(X,E')$ we denote the set of all finitely additive measures $\mu : \mathscr{B} \to E'$ with the following properties:

(i) for each $x \in E$, the function $\mu_x : \mathscr{B} \to \mathbb{R}$ defined by $\mu_x(A) = \mu(A)(x)$ belongs to $M(X)$,

(ii) $|\mu|(X) < \infty$, where $|\mu|(A)$ stands for the variation of μ on $A \in \mathscr{B}$.

In view of [23, Theorem 2.5] $C_{rc}(X,E)'$ can be identified with $M(X,E')$ through the linear mapping $M(X,E') \ni \mu \mapsto \Phi_\mu \in C_{rc}(X,E)'$, where $\Phi_\mu(h) = \int_X h\,d\mu$ for $h \in C_{rc}(X,E)$ and $\|\Phi_\mu\| = |\mu|(X)$. Then one can embed $B(\mathscr{B},E)$ into $C_{rc}(X,E)''$ by the mapping $\pi : B(\mathscr{B},E) \to C_{rc}(X,E)''$, where for $g \in B(\mathscr{B},E)$,

$$\pi(g)(\Phi_\mu) := \int_X g\,d\mu \quad \text{for } \mu \in M(X,E'). \tag{1}$$

Let $i_F : F \to F''$ denote the canonical embedding; that is, $i_F(y)(y') = y'(y)$ for $y \in F$, $y' \in F'$. Moreover, let $j_F : i_F(F) \to F$ stand for the left inverse of i_F; that is, $j_F \circ i_F = id_F$.

Assume that $S : C_{rc}(X,E) \to F$ is a bounded linear operator. Let

$$\widehat{S} := S'' \circ \pi : B(\mathscr{B},E) \longrightarrow F'', \tag{2}$$

where $S' : F' \to C_{rc}(X,E)'$ and $S'' : C_{rc}(X,E)'' \to F''$ denote the conjugate and biconjugate operators of S, respectively. Then we can define a measure $m : \mathscr{B} \to \mathscr{L}(E,F'')$ (called a *representing measure* of S) by

$$m(A)(x) := \widehat{S}(\mathbb{1}_A \otimes x) = (S'' \circ \otimes \pi)(\mathbb{1}_A \otimes x) \quad \text{for } A \in \mathscr{B},\ x \in E. \tag{3}$$

Then $\widetilde{m}(X) < \infty$, where the semivariation $\widetilde{m}(A)$ of m on $A \in \mathscr{B}$ is defined by $\widetilde{m}(A) := \sup \|\sum m(A_i)(x_i)\|_{F''}$, where the supremum is taken over all finite $\mathscr{B}$-partitions (A_i) of A and $x_i \in B_E$ for each i. For $y' \in F'$ let us put

$$m_{y'}(A)(x) := (m(A)(x))(y') \quad \text{for } A \in \mathscr{B}, x \in E. \tag{4}$$

Let $|m_{y'}|(A)$ stand for the variation of $m_{y'}$ on A. Then (see [1, Section 4, Proposition 5])

$$\widetilde{m}(A) = \sup\{|m_{y'}|(A) : y' \in B_{F'}\}. \tag{5}$$

The following general properties of the operator $\widehat{S} : B(\mathscr{B},E) \to F''$ are well known (see [1, Section 6], [2, Section 1], [13, 24]):

$$\widehat{S}(g) = \int_X g\,dm \quad \text{for } g \in B(\mathscr{B},E), \quad \|\widehat{S}\| = \widetilde{m}(X), \tag{6}$$

and for each $y' \in F'$,

$$\widehat{S}(g)(y') = \int_X g\,dm_{y'} \quad \text{for } g \in B(\mathscr{B},E). \tag{7}$$

For $A \in \mathscr{B}$ let

$$\int_A g\,dm := \int_X \mathbb{1}_A g\,dm \quad \text{for } g \in B(\mathscr{B},E). \tag{8}$$

From the general properties of $\widehat{S}$ it follows that

$$\widehat{S}(C_{rc}(X,E)) \subset i_F(F), \qquad S(h) = j_F\left(\int_X h\,dm\right) \quad \text{for } h \in C_{rc}(X,E). \tag{9}$$

Hence for each $y' \in F'$ we get

$$y'(S(h)) = \int_X h\,dm_{y'} \quad \text{for } h \in C_{rc}(X,E), \tag{10}$$

and hence $m_{y'} \in M(X, E')$. Moreover, we have

$$\begin{aligned} \|S\| &= \|S'\| \\ &= \sup\{\|S'(y')\| : y' \in B_{F'}\} \\ &= \sup\{\|y' \circ S\| : y' \in B_{F'}\} \\ &= \sup\{\|\Phi_{m_{y'}}\| : y' \in B_{F'}\} \\ &= \sup\{|m_{y'}|(X) : y' \in B_{F'}\}, \end{aligned} \tag{11}$$

and using (5) we get

$$\|S\| = \widetilde{m}(X). \tag{12}$$

By $M(X, \mathscr{L}(E, F''))$ we will denote the space of all measures $m : \mathscr{B} \to \mathscr{L}(E, F'')$ such that $\widetilde{m}(X) < \infty$ and $m_{y'} \in M(X, E')$ for each $y' \in F'$. Thus the representing measure m of S belongs to $M(X, \mathscr{L}(E, F''))$.

For any $x \in E$ define

$$\begin{aligned} S_x(u) &:= S(u \otimes x) \quad \text{for } u \in C_b(X), \\ m_x(A) &:= m(A)(x) \quad \text{for } A \in \mathscr{B}. \end{aligned} \tag{13}$$

Then $S_x : C_b(X) \to F$ is a bounded linear operator. Let $\chi : B(\mathscr{B}) \to C_b(X)''$ stand for the canonical embedding; that is, for $u \in B(\mathscr{B})$,

$$\chi(u)(\varphi_\nu) = \int_X u\, d\nu \quad \text{for } \nu \in M(X). \tag{14}$$

Let

$$\widehat{S}_x := (S_x)'' \circ \chi : B(\mathscr{B}) \longrightarrow F''. \tag{15}$$

Then

$$\begin{aligned} &\widehat{S}_x(C_b(X)) \subset i_F(F), \\ &S_x(u) = j_F(\widehat{S}_x(u)) \quad \text{for } u \in C_b(X). \end{aligned} \tag{16}$$

The following lemma will be useful.

Lemma 1. *Let $S : C_{rc}(X, E) \to F$ be a bounded linear operator. Then $S''(\pi(\mathbb{1}_A \otimes x)) = (S_x)''(\chi(\mathbb{1}_A))$ for any $x \in E$ and $A \in \mathscr{B}$.*

Proof. Let $y' \in F'$. Then for each $u \in C_b(X)$,

$$\begin{aligned} (y' \circ S_x)(u) &= y'(S(u \otimes x)) \\ &= \int_X (u \otimes x)\, dm_{y'} = \int_X u\, dm_{x,y'} \\ &= \varphi_{m_{x,y'}}(u). \end{aligned} \tag{17}$$

Hence we have

$$\begin{aligned} &(S_x)''(\chi(\mathbb{1}_A))(y') \\ &\quad = \chi(\mathbb{1}_A)(S_x'(y')) \\ &\quad = \chi(\mathbb{1}_A)(y' \circ S_x) = \chi(\mathbb{1}_A)(\varphi_{m_{x,y'}}) \\ &\quad = \int_X \mathbb{1}_A\, dm_{x,y'} = m_{x,y'}(\mathbb{1}_A) = m_x(\mathbb{1}_A)(y'). \end{aligned} \tag{18}$$

On the other hand, for each $h \in C_{rc}(X, E)$, $(y' \circ S)(h) = \int_X h\, dm_{y'} = \Phi_{m_{y'}}(h)$, and hence

$$\begin{aligned} &S''(\pi(\mathbb{1}_A \otimes x)) \\ &\quad = (\mathbb{1}_A \otimes x)(S'(y')) = \pi(\mathbb{1}_A \otimes x)(y' \circ S) \\ &\quad = \pi(\mathbb{1}_A \otimes x)(\Phi_{m_{y'}}) = \Phi_{m_{y'}}(\mathbb{1}_A \otimes x) \\ &\quad = \int_X (\mathbb{1}_A \otimes x)\, dm_{y'} = m_{y'}(A)(x) = m_x(\mathbb{1}_A)(y'). \end{aligned} \tag{19}$$

It follows that $S''(\pi(\mathbb{1}_A \otimes x)) = (S_x)''(\chi(\mathbb{1}_A))$, as desired. □

From Lemma 1 for $A \in \mathscr{B}$ and $x \in E$ we get

$$m_x(A) := \widehat{S}(\mathbb{1}_A \otimes x) = S''(\pi(\mathbb{1}_A \otimes x)) = (S_x)''(\chi(\mathbb{1}_A)); \tag{20}$$

that is,

$$m_x(A) = \widehat{S}_x(\mathbb{1}_A), \qquad \widehat{S}_x(u) = \int_X u\, dm_x \quad \text{for } u \in B(\mathscr{B}). \tag{21}$$

Now we are ready to prove the following Bartle-Dunford-Schwartz type theorem (see [25, Theorem 5, pages 153-154]).

Theorem 2. *Let $S : C_{rc}(X, E) \to F$ be a bounded linear operator and let $M(X, \mathscr{L}(E, F''))$ be its representing measure. Then for each $x \in E$ the following statements are equivalent.*

(i) *$S_x : C_b(X) \to F$ is weakly compact.*

(ii) *$m(A)(x) \in i_F(F)$ for each $A \in \mathscr{B}$ and $\{j_F(m(A)(x)) : A \in \mathscr{B}\}$ is a relatively weakly compact set in F.*

(iii) *$m_x : \mathscr{B} \to F''$ is strongly bounded.*

Proof. (i)⇒(ii) Assume that S_x is weakly compact. Then by the Gantmacher theorem $(S_x)''(C_b(X)'') \subset i_F(F)$ and $(S_x)'' : C_b(X)'' \to F''$ is weakly compact (see [26, Theorem 17.2]). Hence $\widehat{S}_x(B(\mathscr{B})) \subset i_F(F)$ and $\widehat{S}_x : B(\mathscr{B}) \to F''$ is weakly compact. In view of (21) for each $x \in E$, $m_x(A) \in i_F(F)$ for $A \in \mathscr{B}$ and $m_x : \mathscr{B} \to F''$ is strongly bounded (see [25, Theorem 1, page 148]). It follows that $\{j_F(m(A)(x)) : A \in \mathscr{B}\}$ is a relatively weakly compact subset of F (see [24, Theorem 7]).

(ii)⇒(iii) It follows from [24, Theorem 7].

(iii)⇒(i) Assume that $m_x : \mathscr{B} \to F''$ is strongly bounded. Then by (21) $\widehat{S}_x : B(\mathscr{B}) \to F''$ is weakly compact and in view of (16) we derive that S_x is weakly compact. □

3. Integral Representation of Continuous Linear Operators on $C_b(X,E)$

The spaces of all σ-additive, u-additive, perfect, τ-additive, and tight members of $M(X)$ will be denoted by $M_\sigma(X)$, $M_\infty(X)$, $M_p(X)$, $M_\tau(X)$, and $M_t(X)$, respectively (see [3, 4]). Then $(C_b(X), \beta_z)' = \{\varphi_\nu : \nu \in M_z(X)\}$ for $z = \sigma, \infty, p, \tau, t$.

For the integration theory of functions $f \in C_b(X, E)$ with respect to $\mu \in M_z(X, E')$ we refer the reader to [6, page 197], [5, Definition 3.10], [27, page 375]. For $z = \sigma, \infty, p, \tau, t$ let

$$M_z\left(X, E'\right) := \left\{\mu \in M\left(X, E'\right) : \mu_x \in M_z(X) \ \text{ for each } x \in E\right\}. \tag{22}$$

Then $|\mu| \in M_z(X)$ if $\mu \in M_z(X, E')$ (see [5, Proposition 3.9], [6, Theorem 3.1], [10, Theorem 1]). For $\Phi \in C_b(X, E)'$ let us put, for $u \in C_b(X)^+$,

$$|\Phi|(u) := \sup\left\{\left|\Phi(f)\right| : f \in C_b(X, E), \widetilde{f} \le u\right\}. \tag{23}$$

It is known that $|\Phi| : C_b(X)^+ \to \mathbb{R}^+$ is additive and positively homogeneous and can be extended to a linear functional on $C_b(X)$ (denoted by $|\Phi|$ again) by $|\Phi|(u) = |\Phi|(u^+) - |\Phi|(u^-)$ for $u \in C_b(X)$.

Theorem 3. *Assume that $z = \sigma$ and $C_b(X) \otimes E$ is dense in $(C_b(X, E), \beta_\sigma)$ (resp., $z = \infty$; $z = p$ and $C_b(X) \otimes E$ is dense in $(C_b(X, E), \beta_p)$; $z = \tau$; $z = t$). Then the following statements hold.*

(i) *For a linear functional Φ on $C_b(X, E)$ the following conditions are equivalent.*

(a) *Φ is β_z-continuous.*

(b) *There exists a unique $\mu \in M_z(X, E')$ such that*

$$\Phi(f) = \Phi_\mu(f) = \int_X f d\mu \quad \text{for } f \in C_b(X, E). \tag{24}$$

(ii) *For $\mu \in M_z(X, E')$, $|\Phi_\mu|(u) = \int_X u d|\mu| = \varphi_{|\mu|}(u)$ for $u \in C_b(X)$.*

Proof. (i) See [6, Theorems 5.3 and 4.2, Corollary 3.9], [5, Theorem 3.13], and [10, Theorem 8].

(ii) See [6, Theorem 2.1]. □

Assume that $\mathscr{M}$ is a subset of $M_z(X, E')$ and $\sup_{\mu\in\mathscr{M}} |\mu|(X) < \infty$, where $z = \sigma, \infty, p, \tau, t$. Then we say that $\mathscr{M}$ satisfies the condition (C_z) if we have the following:

(1) for $z = \sigma$: $\sup\{|\mu|(Z_n) : \mu \in \mathscr{M}\} \to 0$ whenever $Z_n \downarrow \emptyset$, $(Z_n) \subset \mathscr{Z}$;

(2) for $z = \infty$: for every partition of unity $(u_\alpha)_{\alpha\in\mathscr{A}}$ for X and every $\varepsilon > 0$ there exists a finite set $\mathscr{A}_\varepsilon$ in $\mathscr{A}$ such that $\sup_{\mu\in\mathscr{M}} \int_X (1 - \sum_{\alpha\in\mathscr{A}_\varepsilon} u_\alpha) d|\mu| < \varepsilon$;

(3) for $z = p$: for every continuous function f from X onto a separable metric space Y and every $\varepsilon > 0$, there is a compact subset K of Y such that $\sup_{\mu\in\mathscr{M}} |\mu|(X \setminus \overset{-1}{f}(K)) \le \varepsilon$;

(4) for $z = \tau$: $\sup\{|\mu|(Z_\alpha) : \mu \in \mathscr{M}\} \to 0$ whenever $Z_\alpha \downarrow \emptyset$, $(Z_\alpha) \subset \mathscr{Z}$;

(5) for $z = t$: for every $\varepsilon > 0$ there exists a compact subset K of X such that $\sup\{|\mu|(Z) : Z \in \mathscr{Z}, Z \subset X \setminus K\} \le \varepsilon$ for each $\mu \in \mathscr{M}$.

The following lemmas will be useful.

Lemma 4. *Assume that $\mathscr{M}$ is a subset of $M_z(X, E')$ and $\sup_{\mu\in\mathscr{M}} |\mu|(X) < \infty$, where $z = \sigma$ and $C_b(X) \otimes E$ is β_σ-dense in $C_b(X, E)$ (resp., $z = \infty$; $z = p$ and $C_b(X) \otimes E$ is β_p-dense in $C_b(X, E)$; $z = \tau$; $z = t$). Then the following statements are equivalent.*

(i) *$\{\Phi_\mu : \mu \in \mathscr{M}\}$ is β_z-equicontinuous.*

(ii) *$\{|\Phi_\mu| : \mu \in \mathscr{M}\}$ is β_z-equicontinuous.*

(iii) *$\{\varphi_{|\mu|} : \mu \in \mathscr{M}\}$ is β_z-equicontinuous.*

(iv) *The condition (C_z) holds.*

Proof. (i)⇔(ii) See [9, Lemma 2].

(ii)⇔(iii) It follows from Theorem 3.

(iii)⇔(iv) See [4, Theorem 11.14] for $z = \sigma$; [28, Proposition 3.6] for $z = \infty$; [28, Proposition 2.6] for $z = p$; [4, Theorem 11.24] for $z = \tau$; and [28, Proposition 1.1] for $z = t$. □

Lemma 5. *Assume that $z = \sigma$ and $C_b(X) \otimes E$ is β_σ-dense in $C_b(X, E)$ (resp., $z = \infty$; $z = p$, and $C_b(X) \otimes E$ is β_p-dense in $C_b(X, E)$; $z = \tau$; $z = t$). Let $\mu \in M_z(X, E')$. Then for $A \in \mathscr{B}$ the following statements hold.*

(i) *A functional $\Phi_A : C_{rc}(X, E) \to \mathbb{R}$ defined by $\Phi_A(h) = \int_A h d\mu$ is $\beta_z|_{C_{rc}(X,E)}$-continuous and can by uniquely extended to a β_z-continuous linear functional $\overline{\Phi_A} : C_b(X, E) \to \mathbb{R}$, and one will write the following:*

$$\int_A f d\mu := \overline{\Phi_A}(f) \quad \text{for } f \in C_b(X, E). \tag{25}$$

(ii) *$|\int_A f d\mu| \le \int_A \widetilde{f} d|\mu|$ for $f \in C_b(X, E)$.*

Proof. (i) Assume that (h_α) is a net in $C_{rc}(X, E)$ such that $h_\alpha \to 0$ for β_z. Then

$$\left|\Phi_A(h_\alpha)\right| = \left|\int_A h_\alpha d\mu\right| \le \int_A \widetilde{h}_\alpha d|\mu| \le \int_X \widetilde{h}_\alpha d|\mu|. \tag{26}$$

Since $\widetilde{h}_\alpha \to 0$ for β_z in $C_b(X)$ and $|\mu| \in M_z(X)$, we obtain that $\Phi_A(h_\alpha) \to 0$; that is, Φ_A is $\beta_z|_{C_{rc}(X,E)}$-continuous. Since $C_{rc}(X, E)$ is dense in $(C_b(X, E), \beta_z)$, Φ_A can be uniquely extended to a β_z-continuous linear functional $\overline{\Phi_A} : C_b(X, E) \to \mathbb{R}$ (see [29, Theorem 2.6]).

(ii) Assume that $f \in C_b(X,E)$. Choose a net (h_α) in $C_{rc}(X,E)$ such that $h_\alpha \to f$ for β_z. Then $\widetilde{h}_\alpha \to \widetilde{f}$ for β_z in $C_b(X)$. Then

$$\left|\int_A \widetilde{h}_\alpha d|\mu| - \int_A \widetilde{f} d|\mu|\right| \le \int_A \left|\widetilde{h}_\alpha - \widetilde{f}\right| d|\mu| \le \int_X \left|\widetilde{h}_\alpha - \widetilde{f}\right| d|\mu|, \tag{27}$$

and hence $\int_A \widetilde{f} d|\mu| = \lim_\alpha \int_A \widetilde{h}_\alpha d|\mu|$. Since $\int_A f d\mu = \overline{\Phi_A}(f) = \lim_\alpha \int_A h_\alpha d\mu$, we get

$$\left|\int_A f d\mu\right| = \lim_\alpha \left|\int_A h_\alpha d\mu\right| \le \lim_\alpha \int_A \widetilde{h}_\alpha d|\mu| = \int_A \widetilde{f} d|\mu|. \tag{28}$$

□

For $z = \sigma, \infty, p, \tau, t$ let us put

$$M_z\left(X, \mathscr{L}\left(E, F''\right)\right) := \left\{m \in M\left(X, \mathscr{L}\left(E, F''\right)\right) : m_{y'} \in M_z\left(X, E'\right) \text{ for each } y' \in F'\right\}. \tag{29}$$

Lemma 6. *Assume that $z = \sigma$ and $C_b(X) \otimes E$ is β_σ-dense in $C_b(X,E)$ (resp., $z = \infty$; $z = p$, and $C_b(X) \otimes E$ is β_p-dense in $C_b(X,E)$; $z = \tau$; $z = t$). Assume that $m \in M_z(X, \mathscr{L}(E,F''))$ and the set $\{m_{y'} : y' \in F'\}$ satisfies the condition (C_z). Then for $A \in \mathscr{B}$ the following statements hold.*

(i) *An operator $S_A : C_{rc}(X,E) \to F''$ defined by $S_A(h) = \int_A h dm$ is $(\beta_z|_{C_{rc}(X,E)}, \|\cdot\|_{F''})$-continuous and can be uniquely extended to a $(\beta_z, \|\cdot\|_{F''})$-continuous linear operator $\overline{S_A} : C_b(X,E) \to F''$, and one will write the following.*

$$\int_A f dm := \overline{S_A}(f) \quad \text{for } f \in C_b(X,E). \tag{30}$$

(ii) *For each $y' \in F'$, $(\int_A f dm)(y') = \int_A f dm_{y'}$ for $f \in C_b(X,E)$.*

Proof. (i) In view of Lemma 5 the set $\{\varphi_{|m_{y'}|} : y' \in B_{F'}\}$ is β_z-equicontinuous in $C_b(X)'_{\beta_z}$. Assume that (h_α) is a net in $C_{rc}(X,E)$ such that $h_\alpha \to 0$ for β_z. Let $\varepsilon > 0$ be given. Then there exists a neighborhood V_ε of 0 for β_z in $C_b(X)$ such that $\sup_{y' \in B_{F'}} |\int_X u d|m_{y'}|| \le \varepsilon$ for $u \in V_\varepsilon$. Since $\widetilde{h}_\alpha \to 0$ for β_z in $C_b(X)$, choose α_ε such that $h_\alpha \in V_\varepsilon$ for $\alpha \ge \alpha_\varepsilon$. Hence $\sup_{y' \in B_{F'}} \int_X \widetilde{h}_\alpha d|m_{y'}| \le \varepsilon$ for $\alpha \ge \alpha_\varepsilon$. It follows that, for $\alpha \ge \alpha_\varepsilon$ and each $y' \in B_{F'}$,

$$\left|\left(\int_A h_\alpha dm\right)(y')\right| = \left|\int_A h_\alpha dm_{y'}\right| \le \int_A \widetilde{h}_\alpha d\left|m_{y'}\right| \le \int_X \widetilde{h}_\alpha d\left|m_{y'}\right| \le \varepsilon, \tag{31}$$

and hence,

$$\|S_A(h_\alpha)\|_{F''} = \sup\left\{\left|S_A(h_\alpha)(y')\right| : y' \in B_{F'}\right\} \le \varepsilon. \tag{32}$$

This means that $S_A : C_{rc}(X,E) \to F''$ is $(\beta_z|_{C_{rc}(X,E)}, \|\cdot\|_{F''})$-continuous. Since $C_{rc}(X,E)$ is β_z-dense in $(C_b(X,E), \beta_z)$, S_A possesses a unique $(\beta_z, \|\cdot\|_{F''})$-continuous extension $\overline{S_A} : C_b(X,E) \to F''$ (see [29, Theorem 2.6]). Let

$$\int_A f dm := \overline{S_A}(f) \quad \text{for } f \in C_b(X,E). \tag{33}$$

(ii) Let $f \in C_b(X,E)$. Choose a net (h_α) in $C_{rc}(X,E)$ such that $h_\alpha \to f$ for β_z. By Lemma 5 and (7) for $y' \in F'$ we have

$$\left(\int_A f dm\right)(y') = \left(\lim_\alpha \left(\int_A h_\alpha dm\right)\right)(y') = \lim_\alpha \left(\int_A h_\alpha dm_{y'}\right)(y') = \lim_\alpha \int_A h_\alpha dm_{y'} = \int_A f dm_{y'}. \tag{34}$$

□

Corollary 7. *Assume that $z = \sigma$ and $C_b(X) \otimes E$ is β_σ-dense in $C_b(X,E)$ (resp., $z = \infty$; $z = p$ and $C_b(X) \otimes E$ is β_p-dense in $C_b(X,E)$; $z = \tau$; $z = t$). Assume that $m \in M_z(X, \mathscr{L}(E,F''))$ and the set $\{m_{y'} : y' \in B_{F'}\}$ satisfies the condition (C_z). Then for $A \in \mathscr{B}$ the following statements hold:*

$$\text{(a)}\ \left|m_{y'}\right|(A) = \sup\left\{\left|\int_A h dm_{y'}\right| : h \in C_b(X) \otimes E, \|h\| \le 1\right\} = \sup\left\{\left|\int_A f dm_{y'}\right| : f \in C_b(X,E), \|f\| \le 1\right\}.$$
$$\text{(b)}\ \widetilde{m}(A) = \sup\left\{\left\|\int_A h dm\right\|_{F''} : h \in C_b(X) \otimes E, \|h\| \le 1\right\} = \sup\left\{\left\|\int_A f dm\right\|_{F''} : f \in C_b(X,E), \|f\| \le 1\right\}. \tag{35}$$

In particular, if $U \in \mathscr{P}$, then

$$\text{(c)}\ \left|m_{y'}\right|(U) = \sup\left\{\left|\int_U h dm_{y'}\right| : h \in C_b(X) \otimes E, \|h\| \le 1,\ \operatorname{supp} h \subset U\right\} = \sup\left|\sum_{i=1}^n \int_X u_i dm_{x_i,y'}\right|, \tag{36}$$

where the supremum is taken over all finite disjoint supported collections $\{u_1,\dots,u_n\}\subset C_b(X)$ *with* $\|u_i\|\le 1$ *and* $\operatorname{supp} u_i\subset U$ *and* $\{x_1,\dots,x_n\}\subset B_E$. *One has*

$$\text{(d)}\ \widetilde{m}(U)=\sup\left\{\left\|\int_U h\,dm\right\|_{F''}: h\in C_b(X)\otimes E,\ \|h\|\le 1, \operatorname{supp} h\subset U\right\} = \sup\left\{\left\|\int_U f\,dm\right\|_{F''}: f\in C_b(X,E),\ \|f\|\le 1, \operatorname{supp} f\subset U\right\}. \tag{37}$$

Proof. Let $A\in\mathscr{B}$ and $y'\in F'$. Then by Lemma 5 for $f\in C_b(X,E)$ with $\|f\|\le 1$ we have

$$\left|\int_A f\,dm_{y'}\right|\le\int_A \widetilde{f}\,d\left|m_{y'}\right|\le\left|m_{y'}\right|(A). \tag{38}$$

On the other hand, let $\varepsilon>0$ be given. Then there exist a finite $\mathscr{B}$-partition $(A_i)_{i=1}^n$ of A and $x_i\in B_E$, $i=1,\dots,n$, such that

$$\left|m_{y'}\right|(A)-\frac{\varepsilon}{3}\le\left|\sum_{i=1}^n\left(m(A_i)(x_i)\right)(y')\right|=\left|\sum_{i=1}^n m_{x_i,y'}(A_i)\right|. \tag{39}$$

By the regularity of $m_{x_i,y'}\in M_z(X)$ for $i=1,\dots,n$, we can choose $Z_i\in\mathscr{Z}$, $Z_i\subset A_i$ such that $|m_{x_i,y'}|(A_i\setminus Z_i)\le\varepsilon/3n$ for $i=1,\dots,n$. Choose pairwise disjoint $V_i\in\mathscr{P}$ with $Z_i\subset V_i$ for $i=1,\dots,n$ such that $|m_{x_i,y'}|(V_i\setminus Z_i)\le\varepsilon/3n$. Then for $i=1,\dots,n$ we can choose $v_i\in C_b(X)$ with $0\le v_i\le \mathbb{1}_X$, $v_i|_{Z_i}\equiv 1$, and $v_i|_{X\setminus V_i}\equiv 0$ (see [4, page 115]). Define $h_o=\sum_{i=1}^n(v_i\otimes x_i)$. Then $\|h_o\|\le 1$ and $\int_A h_o\,dm_{y'}=\sum_{i=1}^n\int_A v_i\,dm_{x_i,y'}=\sum_{i=1}^n\int_{V_i\cap A}v_i\,dm_{x_i,y'}$. Hence we get

$$\begin{aligned}\left|m_{y'}\right|(A)-\frac{\varepsilon}{3}&\le\left|\sum_{i=1}^n m_{x_i,y'}(A_i)-\sum_{i=1}^n m_{x_i,y'}(Z_i)\right|\\&\quad+\left|\sum_{i=1}^n\int_{Z_i}v_i\,dm_{x_i,y'}-\sum_{i=1}^n\int_{V_i\cap A}v_i\,dm_{x_i,y'}\right|\\&\quad+\left|\int_A h_o\,dm_{y'}\right|\\&\le\sum_{i=1}^n\left|m_{x_i,y'}\right|(A_i\setminus Z_i)+\sum_{i=1}^n\left|m_{x_i,y'}\right|(V_i\setminus Z_i)\\&\quad+\left|\int_A h_o\,dm_{y'}\right|\\&\le\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\left|\int_A h_o\,dm_{y'}\right|\end{aligned} \tag{40}$$

and hence $|m_{y'}|(A)\le|\int_A h_o\,d\,m_{y'}|+\varepsilon$. Thus the proof of (a) is complete.

In view of (5), (a), and Lemma 6 we get

$$\begin{aligned}\widetilde{m}(A)&=\sup\left\{\left|m_{y'}\right|(A): y'\in B_{F'}\right\}\\&=\sup\left\{\left|\left(\int_A h\,dm\right)(y')\right|: h\in C_b(X)\otimes E,\ \|h\|\le 1, y'\in B_{F'}\right\}\\&=\sup\left\{\left|\left(\int_A f\,dm\right)(y')\right|: f\in C_b(X,E),\ \|f\|\le 1, y'\in B_{F'}\right\}\\&=\sup\left\{\left\|\left(\int_A h\,dm\right)\right\|_{F''}: h\in C_b(X)\otimes E, \|h\|\le 1\right\}\\&=\sup\left\{\left\|\left(\int_A f\,dm\right)\right\|_{F''}: f\in C_b(X,E), \|f\|\le 1\right\};\end{aligned} \tag{41}$$

that is, (b) holds.

Assume now that $U\in\mathscr{P}$. Let $U_i=V_i\cap U\in\mathscr{P}$ for $i=1,\dots,n$. Then $|m_{x_i,y'}|(U_i\setminus Z_i)\le|m_{x_i,y'}|(V_i\setminus Z_i)\le\varepsilon/3n$ for $i=1,\dots,n$. For $i=1,\dots,n$ choose $u_i\in C_b(X)$ with $0\le u_i\le\mathbb{1}_X$, $u_i|_{Z_i}\equiv 1$, and $u_i|_{X\setminus U_i}\equiv 0$. Let $h_o=\sum_{i=1}^n(u_i\otimes x_i)$. Then $\|h_o\|\le 1$ and $\operatorname{supp} h_o\subset U$; and hence by (a), $|m_{y'}|(U)\le|\int_U h_o\,dm_{y'}|+\varepsilon$. Note that $\int_U h_o\,dm_{y'}=\sum_{i=1}^n\int_X u_i\,dm_{x_i,y'}$, where $\operatorname{supp} u_i$ are pairwise disjoint and $\operatorname{supp} u_i\subset U$ for $i=1,\dots,n$. Thus (c) holds.

Using (c) we easily show that (d) holds. Thus the proof is complete. □

Definition 8. Let $T: C_b(X,E)\to F$ be a bounded linear operator. Then the measure $m\in M(X,\mathscr{L}(E,F''))$ defined by

$$m(A)(x):=\left(\left(T|_{C_{rc}(X,E)}\right)''\circ\pi\right)(\mathbb{1}_A\otimes x)\quad\text{for } A\in\mathscr{B}, x\in E \tag{42}$$

will be called a representing measure of T.

Now we state general Riesz representation theorems for continuous linear operators on $C_b(X,E)$, provided with the strict topologies β_z, where $z=\sigma,\infty,p,\tau,t$.

Theorem 9. *Assume that* $z=\sigma$ *and* $C_b(X)\otimes E$ *is* β_σ*-dense in* $C_b(X,E)$ *(resp.,* $z=\infty$*;* $z=p$*, and* $C_b(X)\otimes E$ *is* β_p*-dense in* $C_b(X,E)$*;* $z=\tau$*;* $z=t$*).*

(I) *Let* $T: C_b(X,E)\to F$ *be a* $(\beta_z,\|\cdot\|_F)$*-continuous linear operator and let* $m\in M(X,\mathscr{L}(E,F''))$ *be its representing measure. Then the following statements hold.*

(i) $m\in M_z(X,\mathscr{L}(E,F''))$ *and* $\{m_{y'}: y'\in B_{F'}\}$ *satisfies the condition* (C_z).

(ii) *For each* $y'\in F'$, $y'(T(f))=\int_X f\,dm_{y'}$ *for* $f\in C_b(X,E)$.

(iii) *For each* $f \in C_b(X,E)$ *and* $A \in \mathscr{B}$ *there exists a unique vector in* F'', *denoted by* $\int_A fdm$, *such that* $(\int_A fdm)(y') = \int_A fdm_{y'}$ *for each* $y' \in F'$.

(iv) *For each* $A \in \mathscr{B}$, *the mapping* $C_b(X,E) \ni f \mapsto \int_A fdm \in F''$ *is a* $(\beta_z, \|\cdot\|_{F''})$*-continuous linear operator.*

(v) *For* $f \in C_b(X,E)$, $\int_X fdm \in i_F(F)$ *and* $T(f) = j_F(\int_X fdm)$.

(vi) $\|T\| = \widetilde{m}(X)$.

(II) *Let* $m \in M_z(X,\mathscr{L}(E,F''))$ *and let the set* $\{m_{y'} : y' \in B_{F'}\}$ *satisfy the condition* (C_z). *Then the statements (iii) and (iv) hold and for* $f \in C_b(X,E)$, $\int_X fdm \in i_F(F)$ *and the mapping* $T : C_b(X,E) \to F$ *defined by* $T(f) := j_F(\int_X fdm)$ *is a* $(\beta_z, \|\cdot\|_F)$*-continuous linear operator. Moreover, m coincides with the representing measure of T and the statements (ii) and (vi) hold.*

Proof. (I) In view of (10) for each $y' \in F'$, $y'(T(h)) = \int_X hdm_{y'}$ for $h \in C_{rc}(X,E)$. By Theorem 3 for each $y' \in F'$ there exists a unique $\mu_{y'\circ T} \in M_z(X,E')$ such that $(y'\circ T)(f) = \int_X fd\mu_{y'\circ T}$ for $f \in C_b(X,E)$. It follows that, for each $y' \in F'$, $m_{y'} = \mu_{y'\circ T}$ (see [23, Theorem 2.5]) and this means that $m \in M_z(X,\mathscr{L}(E,F''))$. Hence

$$y'(T(f)) = \int_X fdm_{y'} \quad \text{for } f \in C_b(X,E). \tag{43}$$

Since $\{y' \circ T : y' \in B_{F'}\}$ is β_z-equicontinuous in $C_b(X,E)'_{\beta_z}$, by Lemma 4 the set $\{m_{y'} : y' \in B_{F'}\}$ satisfies the condition (C_z). Thus (i) and (ii) hold. In view of Lemma 6, (iii) and (iv) are satisfied.

According to (9) for each $h \in C_{rc}(X,E)$, $\int_X hdm \in i_F(F)$ and $T(h) = j_F(\int_X hdm)$. Hence by Lemma 6, $\int_X fdm \in i_F(F)$. Let $f \in C_b(X,E)$. Choose a net (h_α) in $C_{rc}(X,E)$ such that $h_\alpha \to f$ for β_z. Hence

$$\begin{aligned} T(f) &= \lim_\alpha T(h_\alpha) = \lim_\alpha j_F\left(\int_X h_\alpha dm\right) \\ &= j_F\left(\lim_\alpha \int_X h_\alpha dm\right) = j_F\left(\int_X fdm\right). \end{aligned} \tag{44}$$

Thus (v) holds. Using (v) and Corollary 7 we get $\|T\| = \widetilde{m}(X)$.

(II) By Lemma 6 the statements (iii) and (iv) are satisfied.

Now let $f \in C_b(X,E)$. Choose a net (h_α) in $C_{rc}(X,E)$ such that $h_\alpha \to f$ for β_z. Then by Lemma 6, $\int_X fdm = \overline{S_X}(f) = \lim_\alpha \int_X h_\alpha dm \in i_F(F)$ because $\int_X h_\alpha dm \in i_F(F)$, and it follows that $T(= j_F \circ \overline{S_X})$ is $(\beta_z, \|\cdot\|_F)$-continuous.

Let $m_o \in M(X,\mathscr{L}(E,F''))$ stand for the representing measure of T. Note that, for $A \in \mathscr{B}$, $x \in E$, and $y' \in F'$ we have

$$\begin{aligned} (m_o(A)(x))(y') &= \left(\left(\left(T|_{C_{rc}(X,E)}\right)'' \circ \pi\right)(\mathbb{1}_A \otimes x)\right)(y') \\ &= \pi(\mathbb{1}_A \otimes x)\left(\left(T|_{C_{rc}(X,E)}\right)'(y')\right) \\ &= \pi(\mathbb{1}_A \otimes x)\left(y' \circ \left(T|_{C_{rc}(X,E)}\right)\right) \\ &= \int_X (\mathbb{1}_A \otimes x)\, dm_{y'} = \int_X \mathbb{1}_A dm_{x,y'} \\ &= (m(A)(x))(y'); \end{aligned} \tag{45}$$

that is, $m_o = m$. By the first part of the proof (ii) and (vi) hold. Thus the proof is complete. □

Following [14, 27] by $M_\sigma(\mathscr{B}a)$ we denote the space of all bounded countably additive, real-valued, regular (with respect to zero sets) measures on $\mathscr{B}a$.

We define $M_\sigma(\mathscr{B}a,E')$ to be the set of all measures $\mu : \mathscr{B}a \to E'$ such that the following two conditions are satisfied.

(i) For each $x \in E$, the function $\mu_x : \mathscr{B}a \to \mathbb{R}$, defined by $\mu_x(A) = \mu(A)(x)$ for $A \in \mathscr{B}a$, belongs to $M_\sigma(\mathscr{B}a)$.

(ii) $|\mu|(X) < \infty$, where for each $A \in \mathscr{B}a$, we define $|\mu|(A) = \sup|\sum \mu(A_i)(x_i)|$, where the supremum is taken over all finite $\mathscr{B}a$-partitions (A_i) of A and all finite collections $x_i \in B_E$.

It is known that if $\mu \in M_\sigma(\mathscr{B}a,E')$, then $|\mu| \in M_\sigma(\mathscr{B}a)$ (see [27, Lemma 2.1]).

The following result will be of importance (see [27, Theorem 2.5]).

Theorem 10. *Let* $\mu \in M_\sigma(X,E')$. *Then* μ *possesses a unique extension* $\overline{\mu} \in M_\sigma(\mathscr{B}a,E')$ *and* $|\overline{\mu}|(X) = |\mu|(X)$.

Arguing as in the proof of Lemma 6 we can obtain the following lemma.

Lemma 11. *Assume that* $C_b(X) \otimes E$ *is* β_σ*-dense in* $C_b(X,E)$ *and* $\mu \in M_\sigma(X,E')$. *Then for* $A \in \mathscr{B}a$ *the following statements hold.*

(i) *A functional* $\Phi_A : C_{rc}(X,E) \to \mathbb{R}$ *defined by* $\Phi_A(h) = \int_A hd\overline{\mu}$ *is* $\beta_\sigma|_{C_{rc}(X,E)}$*-continuous and can be uniquely extended to a* β_σ*-continuous linear functional* $\overline{\Phi_A} : C_b(X,E) \to \mathbb{R}$, *and one will write the following:*

$$\int_A fd\overline{\mu} := \overline{\Phi_A}(f) \quad \textit{for } f \in C_b(X,E). \tag{46}$$

(ii) *For* $f \in C_b(X,E)$, $|\int_A fd\overline{\mu}| \le \int_A \tilde{f} d|\overline{\mu}|$.

By $M_\sigma(X,\mathscr{L}(E,F))$ we will denote the space of all operator measures $m : \mathscr{B} \to \mathscr{L}(E,F)$ such that $\widetilde{m}(X) < \infty$ and

$m_{y'} \in M_\sigma(X, E')$ for each $y' \in F'$. By $M_\sigma(\mathscr{B}a, \mathscr{L}(E,F))$ we will denote the space of all operator measures $m : \mathscr{B}a \to \mathscr{L}(E,F)$ with $\tilde{m}(X) < \infty$ such that $m_{y'} \in M_\sigma(\mathscr{B}a, E')$ for each $y' \in F'$.

Remark 12. Note that in view of the Orlicz-Pettis theorem every $m \in M_\sigma(\mathscr{B}a, \mathscr{L}(E,F))$ is countably additive in the strong operator topology; that is, for each $x \in E$, the measure $m_x : \mathscr{B}a \to F$ defined by $m_x(A) := m(A)(x)$ for $A \in \mathscr{B}a$ is countably additive. Moreover, in view of [30, Theorem 2] for each $x \in E$, m_x is inner regular by zero sets and outer regular by cozero sets; that is, for each $A \in \mathscr{B}a$ and $\varepsilon > 0$ there exist $Z \in \mathscr{Z}$ with $Z \subset A$ and $P \in \mathscr{P}$ with $A \subset \mathscr{P}$ such that $\|m_x\|(A \setminus Z) \le \varepsilon$ and $\|m_x\|(P \setminus A) \le \varepsilon$, ($\|m_x\|(A)$ denotes the semivariation of m_x on $A \in \mathscr{B}a$).

According to [14, Theorem 7] we have the following theorem.

Theorem 13. *Assume that $m \in M_\sigma(X, \mathscr{L}(E,F))$ and $\{m(A)(x) : A \in \mathscr{B}\}$ is a relatively weakly compact subset of F for each $x \in E$. Then m possesses a unique extension $\overline{m} \in M_\sigma(\mathscr{B}a, \mathscr{L}(E,F))$ such that $\tilde{\overline{m}}(X) = \tilde{m}(X)$.*

For a linear operator $T : C_b(X,E) \to F$ and $x \in E$ let $T_x(u) := T(u \otimes x)$ for $u \in C_b(X)$. For $m \in M_\sigma(\mathscr{B}, \mathscr{L}(E,F''))$ and $x \in E$ let $m_x(A) := m(A)(x)$ for $A \in \mathscr{B}$.

Theorem 14. *Assume that $C_b(X) \otimes E$ is β_σ-dense in $C_b(X,E)$.*

(I) *Let $T : C_b(X,E) \to F$ be a $(\beta_\sigma, \|\cdot\|_F)$-continuous linear operator such that $T_x : C_b(X) \to F$ is weakly compact for each $x \in E$, and let $m \in M(X, \mathscr{L}(E,F''))$ be the representing measure of T. Then the following statements hold.*

(i) *$m \in M_\sigma(X, \mathscr{L}(E,F''))$ and $\tilde{m}(Z_n) \to 0$ whenever $Z_n \downarrow \emptyset$, $(Z_n) \subset \mathscr{Z}$.*

(ii) *$m(A)(x) \in i_F(F)$, for each $A \in \mathscr{B}$, $x \in E$, and the measure $m_F : \mathscr{B} \to \mathscr{L}(E,F)$, defined by $m_F(A)(x) := j_F(m(A)(x))$ for $A \in \mathscr{B}$, $x \in E$, belongs to $M_\sigma(X, \mathscr{L}(E,F))$ and possesses a unique extension $\overline{m} \in M_\sigma(\mathscr{B}a, \mathscr{L}(E,F))$ with $\tilde{\overline{m}}(X) = \tilde{m}(X)$ which is countably additive both in the strong operator topology and in the weak star operator topology. Moreover, $\overline{m}_{y'} = \overline{m_{y'}}$ for $y' \in F'$.*

(iii) *For every $f \in C_b(X,E)$ and $A \in \mathscr{B}a$ there exists a unique vector in F, denoted by $\int_A f d\overline{m}$, such that, for each $y' \in F'$, $y'(\int_A f d\overline{m}) = \int_A f d\overline{m}_{y'}$.*

(iv) *For each $A \in \mathscr{B}a$, the mapping $T_A : C_b(X,E) \to F$ defined by $T_A(f) = \int_A f d\overline{m}$ is a $(\beta_\sigma, \|\cdot\|_F)$-continuous linear operator.*

(v) *$T(f) = T_X(f) = \int_X f d\overline{m}$ for $f \in C_b(X,E)$.*

(II) *Let $m \in M_\sigma(X, \mathscr{L}(E,F''))$ be such that $\tilde{m}(Z_n) \to 0$ whenever $Z_n \downarrow \emptyset$, $(Z_n) \subset \mathscr{Z}$ and for each $x \in E$, let $m_x : \mathscr{B} \to F''$ be strongly bounded. Then the operator $T : C_b(X,E) \to F$ defined by $T(f) = j_F(\int_X f dm)$ is $(\beta_\sigma, \|\cdot\|_F)$-continuous and $T_x : C_b(X) \to F$ is weakly compact for each $x \in E$, and the statements (ii)–(v) hold.*

Proof. (I) (i) It follows from Theorem 9.

(ii) In view of Theorem 2 $m(A)(x) \in i_F(F)$ for $A \in \mathscr{B}$, $x \in E$, and $\{m_F(A)(x) : A \in \mathscr{B}\}$ is a relatively weakly compact in F for each $x \in E$. Since $m_F \in M_\sigma(X, \mathscr{L}(E,F))$, by Theorem 13 m_F possesses a unique extension $\overline{m} \in M_\sigma(\mathscr{B}a, \mathscr{L}(E,F))$ with $\tilde{\overline{m}}(X) = \tilde{m}(X)$. By the Orlicz-Pettis theorem $\overline{m}$ is countably additive in the strong operator topology. Moreover, since, for each $y' \in F'$, $|\overline{m}_{y'}| \in M_\sigma(\mathscr{B}a) = ca(\mathscr{B}a)$, we obtain that $\overline{m}_{y'} \in ca(\mathscr{B}a, E')$. This means that $\overline{m} : \mathscr{B}a \to \mathscr{L}(E,F)$ is countably additive in the weak star operator topology.

Let $y' \in F'$. Then for $A \in \mathscr{B}$ and $x \in E$ we have $\overline{m}_{y'}(A)(x) = m_{y'}(A)(x)$, and by Theorem 10, $\overline{m}_{y'} = \overline{m_{y'}}$.

(iii) For $A \in \mathscr{B}a$ let $S_A(h) := \int_A f d\overline{m}$ for $h \in C_{rc}(X,E)$. Proceeding as in the proof of Lemma 6 we can show that $S_A : C_{rc}(X,E) \to F$ is a $(\beta_\sigma|_{C_{rc}(X,E)}, \|\cdot\|_F)$-continuous linear operator, and hence S_A possesses a unique $(\beta_\sigma, \|\cdot\|_F)$-continuous linear extension $T_A : C_b(X,E) \to F$ (see [29, Theorem 2.6]). Let us write the following:

$$\int_A f d\overline{m} := T_A(f) \quad \text{for } f \in C_b(X,E). \tag{47}$$

Let $f \in C_b(X,E)$. Choose a net (h_α) in $C_{rc}(X,E)$ such that $h_\alpha \to f$ for β_σ. For each $y' \in F'$, $\overline{m}_{y'} = \overline{m_{y'}}$ (see (i)) and by Lemma 11 we have

$$\begin{aligned} y'\left(\int_A f d\overline{m}\right) &= y'\left(\lim_\alpha \int_A h_\alpha d\overline{m}\right) = \lim_\alpha \left(y'\left(\int_A h_\alpha d\overline{m}\right)\right) \\ &= \lim_\alpha \int_A h_\alpha d\overline{m}_{y'} = \lim_\alpha \int_A h_\alpha d\overline{m_{y'}} \\ &= \int_A f d\overline{m_{y'}} = \int_A f d\overline{m}_{y'}. \end{aligned} \tag{48}$$

(iv) It follows from the proof of (iii).

(v) Let $f \in C_b(X,E)$. In view of Theorem 9, for each $y' \in F'$, $y'(T(f)) = \int_X f dm_{y'}$. On the other hand by (ii) for $y' \in F'$ we have $y'(\int_X f d\overline{m}) = \int_X f d\overline{m}_{y'} = \int_X f dm_{y'}$. It follows that $T(f) = \int_X f d\overline{m}$.

(II) Since $\{m_{y'} : y' \in B_{F'}\}$ satisfies the condition (C_σ), by Theorem 9 for $f \in C_b(X,E)$, $\int_X f dm \in i_F(F)$ and the mapping $T : C_b(X,E) \to F$ defined by $T(f) := j_F(\int_X f dm)$ is a $(\beta_\sigma, \|\cdot\|_F)$-continuous linear operator, and m coincides with the representing measure of T. Hence in view of Theorem 2 $T_x : C_b(X) \to F$ is a weakly compact operator. Thus by the first part of the proof the statements (ii)–(v) are satisfied. □

4. Strongly Bounded Operators on $C_b(X,E)$

Definition 15. A bounded linear operator $T : C_b(X,E) \to F$ is said to be *strongly bounded* if its representing measure

$m \in M(X, \mathscr{L}(E, F''))$ is strongly bounded; that is, $\widetilde{m}(A_n) \to 0$ whenever (A_n) is a pairwise disjoint sequence in $\mathscr{B}$.

Note that $m \in M(X, \mathscr{L}(E, F''))$ is strongly bounded if and only if the family $\{|m_{y'}| : y' \in B_{F'}\}$ is uniformly strongly additive.

Now we are ready to state our main results that extend some classical results of Lewis (see [20, Theorem 5], [31, Lemma 1]) and Brooks and Lewis (see [22, Theorem 2.1], [21, Theorem 5.2]) concerning operators on the spaces $C(X, E)$ and $C_o(X, E)$, where X is a compact or a locally compact space, respectively.

Theorem 16. *Assume that $C_b(X) \otimes E$ is β_σ-dense in $C_b(X, E)$. Let $T : C_b(X, E) \to F$ be a $(\beta_\sigma, \|\cdot\|_F)$-continuous linear operator and let $m \in M(X, \mathscr{L}(E, F''))$ be its representing measure. Then $m \in M_\sigma(X, \mathscr{L}(E, F''))$ and the following statements are equivalent.*

(i) *T is strongly bounded.*

(ii) $\sup\{|\overline{m_{y'}}|(A_n) : y' \in B_{F'}\} \to 0$ *whenever $A_n \downarrow \emptyset$, $(A_n) \subset \mathscr{B}a$ (here $\overline{m_{y'}} \in M_\sigma(\mathscr{B}a, E')$ denotes the unique extension of $m_{y'} \in M_\sigma(X, E')$).*

(iii) *If (A_n) is a sequence in $\mathscr{B}a$ such that $A_n \downarrow \emptyset$, then there exists a nested sequence (U_n) in $\mathscr{P}$ such that $A_n \subset U_n$ for $n \in \mathbb{N}$ and* $\sup\{\|T(f)\|_F : f \in C_b(X, E),\ \|f\| \le 1$ *and* $\operatorname{supp} f \subset U_n\} \to 0$.

Proof. In view of Theorem 9 $m \in M_\sigma(X, \mathscr{L}(E, F''))$.

(i)⇒(ii) Assume that T is strongly bounded. Since the family $\{|m_{y'}| : y' \in B_{F'}\}$ is uniformly strongly additive, according to [25, Lemma 1, page 26] the family $\{|\overline{m_{y'}}| : y' \in B_{F'}\}$ is uniformly countably additive (see Theorem 16).

(ii)⇒(i) It follows from [25, Lemma 1, page 26].

(ii)⇒(iii) Assume that (ii) holds and (A_n) is a sequence in $\mathscr{B}a$ such that $A_n \downarrow \emptyset$. Then there exists $\lambda \in ca(\mathscr{B}a)^+$ such that $\{|\overline{m_{y'}}| : y' \in B_{F'}\}$ is uniformly λ-continuous (see [25, Theorem 4, pages 11-12]). Let $\varepsilon > 0$ be given. Hence there exists $\delta > 0$ such that $\sup\{|\overline{m_{y'}}|(A) : y' \in B_{F'}\} \le \varepsilon/2$ whenever $\lambda(A) \le \delta$ and $A \in \mathscr{B}a$. Since λ is zero-set regular, there exists a nested sequence (U_n) in $\mathscr{P}$ so that $A_n \subset U_n$ and $\lambda(U_n \setminus A_n) \le \delta$ for $n \in \mathbb{N}$. Hence $\sup\{|\overline{m_{y'}}|(U_n \setminus A_n) : y' \in B_{F'}\} \le \varepsilon/2$ for $n \in \mathbb{N}$. In view of (ii) there exists $n_\varepsilon \in \mathbb{N}$ such that $\sup\{|\overline{m_{y'}}|(A_n) : y' \in B_{F'}\} \le \varepsilon/2$ for $n \ge n_\varepsilon$. Hence $\sup\{|m_{y'}|(U_n) : y' \in B_{F'}\} \le \varepsilon$ for $n \ge n_\varepsilon$; that is, $\sup\{|m_{y'}|(U_n) : y' \in B_{F'}\} \to 0$.

Let $f \in C_b(X, E)$, $\|f\| \le 1$, and $\operatorname{supp} f \subset U_n$. Then by Theorem 9 we have

$$\begin{aligned} \|T(f)\|_F &= \sup\left\{\left|\int_X f\, dm_{y'}\right| : y' \in B_{F'}\right\} \\ &\le \sup\left\{\int_X \widetilde{f}\, d\left|m_{y'}\right| : y' \in B_{F'}\right\} \\ &\le \sup\left\{\left|m_{y'}\right|(U_n) : y' \in B_{F'}\right\}. \end{aligned} \tag{49}$$

It follows that $\sup\{\|T(f)\|_F : f \in C_b(X, E),\ \|f\| \le 1,\ \operatorname{supp} f \subset U_n\} \to 0$.

(iii)⇒(ii) Assume that (iii) holds and $A_n \downarrow \emptyset$, $(A_n) \subset \mathscr{B}a$. Then there exists a nested sequence (U_n) in $\mathscr{P}$ such that $A_n \subset U_n$ for $n \in \mathbb{N}$ and

$$\sup\left\{\|T(f)\|_F : f \in C_b(X, E),\ \|f\| \le 1,\ \operatorname{supp} f_n \subset U_n\right\} \longrightarrow 0. \tag{50}$$

Assume that (ii) does not hold. Then there exist $\varepsilon > 0$ and $n_\varepsilon \in \mathbb{N}$ such that $\sup\{|\overline{m_{y'}}|(A_{n_\varepsilon}) : y' \in B_{F'}\} \ge \varepsilon$ and $\|T(f)\|_F \le (1/8)\varepsilon$ whenever $f \in C_b(X, E)$, $\|f\| \le 1$, and $\operatorname{supp} f \subset U_{n_\varepsilon}$. It follows that there exists $y'_o \in B_{F'}$ such that $|\overline{m_{y'}}|(A_{n_\varepsilon}) \ge \varepsilon$. Hence there exist a finite $\mathscr{B}a$-partition $(B_i)_{i=1}^k$ of A_{n_ε} and $x_i \in B_E$, $i = 1, \ldots, k$, such that

$$\left|\overline{m_{y'_o}}\right|(A_{n_\varepsilon}) - \frac{\varepsilon}{4} \le \left|\sum_{i=1}^k \overline{m_{y'_o}}(B_i)(x_i)\right| = \left|\sum_{i=1}^k \left(\overline{m_{y'_o}}\right)_{x_i}(B_i)\right|. \tag{51}$$

Since $|(\overline{m_{y'_o}})_{x_i}| \in M_\sigma(\mathscr{B}a)$ is zero-set regular (see [4, page 118]), we can choose $Z_i \in \mathscr{Z}$, $Z_i \subset B_i$, such that $|(\overline{m_{y'_o}})_{x_i}|(B_i \setminus Z_i) \le \varepsilon/4k$ for $i = 1, \ldots, k$. Choose pairwise disjoint $V_i \in \mathscr{P}$ with $Z_i \subset V_i$ for $i = 1, \ldots, k$ such that $|m_{x_i, y'_o}|(V_i \setminus Z_i) \le \varepsilon/4k$. Let $U_i = V_i \cap U_{n_\varepsilon}$ for $i = 1, \ldots, k$. Then $U_i \in \mathscr{P}$ and $|m_{x_i, y'_o}|(U_i \setminus Z_i) \le \varepsilon/4k$ for $i = 1, \ldots, k$. For $i = 1, \ldots, k$ choose $u_i \in C_b(X)$ such that $0 \le u_i \le \mathbb{1}_X$, $u_i|_{Z_i} \equiv 0$, and $u_i|_{X \setminus U_i} \equiv 0$ (see [4, page 115]). Let $h_o = \sum_{i=1}^k (u_i \otimes x_i)$. Then $\|h_o\| \le 1$, $\operatorname{supp} h_o \subset U_{n_\varepsilon}$, and

$$\int_{U_{n_\varepsilon}} h_o\, dm_{y'_o} = \sum_{i=1}^k \int_{U_i} u_i\, dm_{x_i, y'_o}. \tag{52}$$

Hence we get

$$\begin{aligned} &\left|\overline{m_{y'_o}}\right|(A_{n_\varepsilon}) - \frac{\varepsilon}{4} \\ &\le \left|\sum_{i=1}^k \left(\overline{m_{y'_o}}\right)_{x_i}(B_i) - \sum_{i=1}^k \left(\overline{m_{y'_o}}\right)_{x_i}(Z_i)\right| \\ &\quad + \left|\sum_{i=1}^k \int_{Z_i} u_i\, dm_{x_i, y'_o} - \sum_{i=1}^k \int_{U_i} u_i\, dm_{x_i, y'_o}\right| \\ &\quad + \left|\int_{U_{n_\varepsilon}} h_o\, dm_{y'_o}\right| \\ &\le \sum_{i=1}^k \left|\left(\overline{m_{y'_o}}\right)_{x_i}\right|(B_i \setminus Z_i) + \sum_{k=1}^k \left|m_{x_i, y'_o}\right|(U_i \setminus Z_i) \\ &\quad + \left|\int_{U_{n_\varepsilon}} h_o\, dm_{y'_o}\right| \\ &\le \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \left|\int_{U_{n_\varepsilon}} h_o\, dm_{y'_o}\right|. \end{aligned} \tag{53}$$

Hence

$$\left|\int_{U_{n_\varepsilon}} h_o dm_{y'_o}\right| \geq \left|\overline{m_{y'_o}}\right|\left(A_{n_\varepsilon}\right) - \frac{3}{4}\varepsilon \geq \frac{1}{4}\varepsilon,$$
$$\|T(h_o)\|_F \geq \left|y'_o\left(T\left(h_o\right)\right)\right| = \left|\int_X h_o dm_{y'_o}\right| \quad (54)$$
$$= \left|\int_{U_{n_\varepsilon}} h_o dm_{y'_o}\right| \geq \frac{1}{4}\varepsilon.$$

Thus we get a contradiction to $\|T(h_o)\|_F \leq (1/8)\varepsilon$.

Thus the proof is complete. □

Theorem 17. *Assume that $C_b(X)\otimes E$ is β_σ-dense in $C_b(X,E)$. Let $T : C_b(X,E) \to F$ be a $(\beta_\sigma, \|\cdot\|_F)$-continuous and strongly bounded operator and let $m \in M(X, \mathscr{L}(E,F''))$ be its representing measure. Then the following statements hold.*

(i) *$m \in M_\sigma(X,\mathscr{L}(E,F''))$ and $m(A)(x) \in i_F(F)$ for $A \in \mathscr{B}$, $x \in E$, and the measure $m_F : \mathscr{B} \to \mathscr{L}(E,F)$, defined by $m_F(A)(x) := j_F(m(A)(x))$ for $A \in \mathscr{B}$, $x \in E$, belongs to $M_\sigma(X,\mathscr{L}(E,F))$ and possesses a unique extension $\overline{m} \in M_\sigma(\mathscr{B}a,\mathscr{L}(E,F))$ with $\widetilde{\overline{m}}(X) = \widetilde{m}_F(X) = \widetilde{m}(X)$ which is variationally semiregular; that is, $\widetilde{\overline{m}}(A_n) \to 0$ whenever $A_n \downarrow \emptyset$, $(A_n) \subset \mathscr{B}a$.*

(ii) *For every $f \in C_b(X,E)$ and $A \in \mathscr{B}a$ there exists a unique vector in F, denoted by $\int_A f d\overline{m}$, such that, for each $y' \in F'$, $y'(\int_A f d\overline{m}) = \int_A f d\overline{m}_{y'}$.*

(iii) *For each $A \in \mathscr{B}a$, $\int_A f_n d\overline{m} \to 0$ whenever (f_n) is a uniformly bounded sequence in $C_b(X,E)$ such that $f_n(t) \to 0$ for $t \in X$.*

(iv) *$T(f) = \int_X f d\overline{m}$ for $f \in C_b(X,E)$.*

(v) *$T(f_n) \to 0$ whenever (f_n) is a uniformly bounded sequence in $C_b(X,E)$ such that $f_n(t) \to 0$ for $t \in X$.*

Proof. (i) Note that, for $x \in E$, $\|m_x(A)\|_{F''} \leq \widetilde{m}(A)\|x\|_E$ for $A \in \mathscr{B}$. Hence $m_x : \mathscr{B} \to F''$ is strongly bounded, and by Theorems 2 and 14 $m(A)(x) \in i_F(F)$ and m_F possesses a unique extension $\overline{m} \in M_\sigma(\mathscr{B}a,\mathscr{L}(E,F))$ with $\widetilde{\overline{m}}(X) = \widetilde{m}_F(X) = \widetilde{m}(X)$. Since $\overline{m}_{y'} = \overline{m_{y'}}$ for $y' \in F'$, by Theorem 16 we have $\widetilde{\overline{m}}(A_n) = \sup\{|\overline{m}_{y'}|(A_n) : y' \in B_{F'}\} \to 0$ whenever $A_n \downarrow \emptyset, (A_n) \subset \mathscr{B}a$.

(ii) It follows from Theorem 14 because for each $x \in E$, $T_x : C_c(X) \to F$ is weakly compact (see Theorem 2).

(iii) In view of (i) there exists $\lambda \in ca(\mathscr{B}a)^+$ such that $\{|\overline{m}_{y'}| : y' \in B_{F'}\}$ is λ-continuous (see [25, Theorem 4, pages 11-12]). Let (f_n) be a sequence in $C_b(X,E)$ such that $\sup_n\|f_n\| = M < \infty$ and $f_n(t) \to 0$ for every $t \in X$. Let $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that $\sup\{|\overline{m}_{y'}|(A) : y' \in B_{F'}\} \leq \varepsilon/2M$ whenever $\lambda(A) \leq \delta$, $A \in \mathscr{B}a$. Since $\widetilde{f}_n \in B(\mathscr{B})$ for $n \in \mathbb{N}$, by the Egoroff theorem there exists $A_\delta \in \mathscr{B}a$ with $\lambda(X \setminus A_\delta) \leq \delta$ and $\sup_{t\in A_\delta}\widetilde{f}_n(t) \to 0$. Choose $n_\varepsilon \in \mathbb{N}$ such that $\sup_{t\in A_\delta}\widetilde{f}_n(t) \leq \varepsilon/2\widetilde{m}(X)$ for $n \geq n_\varepsilon$.

Let $A \in \mathscr{B}a$. Note that $\overline{m}_{y'} = \overline{m_{y'}}$ for $y' \in F'$. Then by Lemma 11 and (ii), for $n \geq n_\varepsilon$ and $y' \in B_{F'}$ we get

$$\begin{aligned}
&\left|y'\left(\int_A f_n d\overline{m}\right)\right| \\
&= \left|\int_A f_n d\overline{m}_{y'}\right| \\
&\leq \int_A \widetilde{f}_n d\left|\overline{m}_{y'}\right| \leq \int_X \widetilde{f}_n d\left|\overline{m}_{y'}\right| \\
&= \int_{A_\delta} \widetilde{f}_n d\left|\overline{m}_{y'}\right| + \int_{X\setminus A_\delta} \widetilde{f}_n d\left|\overline{m}_{y'}\right| \\
&\leq \frac{\varepsilon}{2\widetilde{m}(X)}\left|\overline{m}_{y'}\right|(A_\delta) + M\cdot\left|\overline{m}_{y'}\right|(X\setminus A_\delta) \\
&\leq \frac{\varepsilon}{2\widetilde{m}(X)}\left|m_{y'}\right|(X) + M\cdot\frac{\varepsilon}{2M} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned} \quad (55)$$

Hence $\|\int_A f_n d\overline{m}\|_F \leq \varepsilon$ for $n \geq n_\varepsilon$, as desired.

(iv) It follows from Theorem 14.

(v) It follows from (iii) and (iv). □

Let $\mathscr{L}^\infty(\mathscr{B}a, E)$ stand for the Banach space of all bounded strongly $\mathscr{B}a$-measurable functions $g : X \to E$, equipped with the uniform norm $\|\cdot\|$. Assume that $m : \mathscr{B} \to \mathscr{L}(E,F)$ with $\widetilde{m}(X) < \infty$ is variationally semiregular. Then every $g \in \mathscr{L}^\infty(\mathscr{B}a, E)$ is m-integrable (see [32, Definition 2, page 523 and Theorem 5, page 524]) and $\int_X g_n dm \to 0$ whenever (g_n) is a uniformly bounded sequence in $\mathscr{L}^\infty(\mathscr{B}a, E)$ converging pointwise to 0 (see [33, Proposition 2.2]).

Recall that a series $\sum_{i=1}^\infty z_i$ in a Banach space G is called *weakly unconditionally Cauchy* (wuc) if, for each $z' \in G'$, $\sum_{i=1}^\infty |z'(z_i)| < \infty$. We say that a linear operator $T : G \to F$ is *unconditionally converging* if for every weakly unconditionally Cauchy series $\sum_{i=1}^\infty z_i$ in G, the series $\sum_{i=1}^\infty T(z_i)$ converges unconditionally in a Banach space F.

As an application of Theorem 17 we have the following result.

Corollary 18. *Assume that $C_b(X)\otimes E$ is β_σ-dense in $C_b(X,E)$, where E is a separable Banach space which contains no isomorphic copy of c_o. Let $T : C_b(X,E) \to F$ be a $(\beta_\sigma, \|\cdot\|_F)$-continuous and strongly bounded operator. Then T is unconditionally converging.*

Proof. Assume that $\sum_{i=1}^\infty f_i$ is a wuc series in the Banach space $C_b(X,E)$. Hence $\sum_{i=1}^\infty |x'(f_i(t))| < \infty$ for each $t \in X$ and $x' \in E'$ because $\delta_{t,x'} \in C_b(X,E)'$, where $\delta_{t,x'}(f) = x'(f(t))$ for $f \in C_b(X,E)$. It follows that $\sum_{i=1}^\infty f_i(t)$ is an unconditionally convergent series in E for each $t \in X$ because E contains no isomorphic copy of c_o (see [34]). Let $g_o(t) = \lim_n S_n(t)$ for $t \in X$, where $S_n(t) = \sum_{i=1}^n f_i(t)$ for $t \in X, n \in \mathbb{N}$. Then $\sup_n\|S_n\| < \infty$ because $\sum_{i=1}^\infty f_i$ is wuc (see [34]) and $S_n \in \mathscr{L}^\infty(\mathscr{B}a, E)$ because E is assumed to be separable (see [2, Theorem 21, page 9]). Hence $g_o \in \mathscr{L}^\infty(\mathscr{B}a, E)$ (see [2, Theorem 10, page 6]).

Let $m \in M_\sigma(X,\mathscr{L}(E,F''))$ be the representing measure of T and let $\overline{m} \in M_\sigma(\mathscr{B}a,\mathscr{L}(E,F))$ be a unique extension of $m_F \in M_\sigma(\mathscr{B},\mathscr{L}(E,F))$ (see Theorem 17). Since $\overline{m}$ is variationally semiregular, in view of [33, Proposition 2.2] we have

$$\lim_n \sum_{i=1}^{n} T(f_i) = \lim_n \int_X S_n d\overline{m} = \int_X g_o d\overline{m} \in E. \quad (56)$$

Hence $\sum_{i=1}^{\infty} T(f_i) = \int_X g_o d\overline{m}$. Finally, if (n_j) is any permutation of $\mathbb{N}$, then $\lim_n \sum_{j=1}^{n} f_{n_j}(t) = g_o(t)$ for $t \in X$. Then $\sum_{j=1}^{\infty} T(f_{n_j}) = \int_X g_o d\overline{m}$, as desired. □

Remark 19. A related result to Corollary 18 for strongly bounded operators on the space $C_o(X, E)$ of E-valued continuous functions vanishing at infinity defined on a locally compact space X was obtained by Brooks and Lewis (see [21, Theorem 5.2]).

Recall that a Banach space E is said to be a Schur space if every weakly convergent sequence in E is norm convergent.

As a consequence of Theorem 17 we derive the following Dunford-Pettis type theorem for operators on $C_b(X, E)$.

Theorem 20. *Assume that $C_b(X) \otimes E$ is β_σ-dense in $C_b(X, E)$, where E is a Schur space. Let $T : C_b(X, E) \to F$ be a $(\beta_\sigma, \|\cdot\|_F)$-continuous and strongly bounded operator. Then $T(f_n) \to 0$ in F whenever (f_n) is a $\sigma(C_b(X, E), M_\sigma(X, E'))$ convergent to 0 sequence in $C_b(X, E)$.*

Proof. Assume that $f_n \to 0$ for $\sigma(C_b(X, E), M_\sigma(X, E'))$. Then according to [11, Corollary 5], we obtain that $\sup_n \|f_n\| < \infty$ and $f_n(t) \to 0$ in $\sigma(E, E')$ for each $t \in X$. It follows that $\|f_n(t)\|_E \to 0$ for $t \in X$ because E is supposed to be a Schur space. Using Theorem 17 we derive that $T(f_n) \to 0$ in F, as desired. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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Research Article

Positive Solutions for a Class of Singular Boundary Value Problems with Fractional q-Difference Equations

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We discuss a class of singular boundary value problems of fractional q-difference equations. Some existence and uniqueness results are obtained by a fixed point theorem in partially ordered sets. Finally, we give an example to illustrate the results.

1. Introduction

In recent years, many papers on fractional differential equations have appeared, because of their demonstrated applications in various fields of science and engineering; see [1–11] and the references therein. Based on the increasingly extensive application of discrete fractional calculus and the development of q-difference calculus or quantum calculus (see [12–19] and the references therein), fractional q-difference equations have attracted the attention of researchers for the numerous applications in a number of fields such as physics, chemistry, aerodynamics, biology, economics, control theory, mechanics, electricity, signal and image processing, biophysics, blood flow phenomena, aerodynamics, and fitting of experimental data; see [20–23]. Some recent work on the existence theory of fractional q-difference equations can be found in [24–29]. However, the study of singular boundary value problems (BVPs) with fractional q-difference equations is at its infancy and lots of work on the topic should be done.

Recently, in [25], Ferreira has investigated the existence of positive solution for the following fractional q-difference equations BVP

$$\begin{gathered}\left(D_q^\alpha y\right)(x)+f\left(x, y(t)\right)=0, \quad 0<x<1, \\ y(0)=\left(D_q\right) y(0)=0, \qquad \left(D_q\right) y(1)=\beta \geq 0,\end{gathered} \tag{1}$$

by applying a fixed point theorem in cones.

More recently, in [30], Caballero et al. have studied positive solutions for the following BVP:

$$\begin{gathered}\left(D_{0^+}^\alpha u\right)(t)+f(t, u(t))=0, \quad 0<t<1, \\ u(0)=u(1)=0,\end{gathered} \tag{2}$$

by fixed point theorem in partially ordered sets.

Motivated by the work above, in this paper, we discuss the existence and uniqueness of solutions for the singular BVPs of factional q-difference equations given by

$$\begin{gathered}\left(D_q^\alpha u\right)(t)+f(t, u(t))=0, \quad 0<t<1, \\ u(0)=u(1)=0, \qquad \left(D_q u\right)(0)=0,\end{gathered} \tag{3}$$

where $2<\alpha\leq 3$ and $f:[0,1]\times[0,\infty)\to[0,\infty)$ is continuous with $\lim_{t\to 0^+} f(t,\cdot)=\infty$ (i.e., f is singular at $t=0$).

2. Preliminary Results

For convenience, we present some definitions and lemmas which will be used in the proofs of our results.

Let $q\in(0,1)$ and define

$$[a]_q=\frac{1-q^a}{1-q}, \quad a\in R. \tag{4}$$

The q-analogue of the power function $(a-b)^n$ with $n \in N_0$ is

$$(a-b)^{(0)} = 1, \qquad (a-b)^{(n)} = \prod_{k=0}^{n-1}\left(a-bq^k\right), \quad n \in N, \quad a, b \in R. \tag{5}$$

More generally, if $\alpha \in R$, then

$$(a-b)^{(\alpha)} = a^\alpha \prod_{n=0}^{n-1} \frac{a-bq^n}{a-bq^{\alpha+n}}. \tag{6}$$

Note that if $b = 0$ then $a^{(\alpha)} = a^\alpha$. The q-gamma function is define by

$$\Gamma_q(x) = \frac{(1-q)^{(x-1)}}{(1-q)^{x-1}}, \quad x \in R \setminus \{0, -1, -2, \ldots\}, \tag{7}$$

and it satisfies $\Gamma_q(x+1) = [x]_q \Gamma_q(x)$.

Following, let us recall some basic concepts of q-calculus [12].

Definition 1. For $0 < q < 1$, we define the q-derivative of a real-value function f as

$$\left(D_q f\right)(x) = \frac{f(x) - f(qx)}{(1-q)x}, \qquad \left(D_q f\right)(0) = \lim_{x \to 0}\left(D_q f\right)(x). \tag{8}$$

Note that $\lim_{q\to 1^-} D_q f(x) = f'(x)$.

Definition 2. The higher order q-derivatives are defined inductively as

$$\left(D_q^0 f\right)(x) = f(x), \qquad \left(D_q^n f\right)(t) = D_q\left(D_q^{n-1} f\right)(t), \quad n \in N. \tag{9}$$

For example, $D_q(t^k) = [k]_q t^{k-1}$, where k is a positive integer and the bracket $[k]_q = (q^k - 1)/(q-1)$. In particular, $D_q(t^2) = (1+q)t$.

Definition 3. The q-integral of a function f in the interval $[0, b]$ is given by

$$\left(I_q f\right)(x) = \int_0^x f(t)\, d_q t = x(1-q)\sum_{n=0}^{\infty} f(xq^n) q^n, \quad x \in [0, b]. \tag{10}$$

If $a \in [0, b]$ and f is defined in the interval $[0, b]$, its integral from a to b is define by

$$\int_a^b f(t)\, d_q t = \int_0^b f(t)\, d_q t - \int_0^a f(t)\, d_q t. \tag{11}$$

Similarly as done for derivatives, an operator I_q^n can be define, namely,

$$\left(I_q^0 f\right)(x) = f(x), \qquad \left(I_q^n f\right)(x) = I_q\left(I_q^{n-1} f\right)(x), \quad n \in N. \tag{12}$$

Observe that

$$D_q I_q f(x) = f(x), \tag{13}$$

and if f is continuous at $x = 0$, then $I_q D_q f(x) = f(x) - f(0)$.

We now point out three formulas (${}_iD_q$ denotes the derivative with respect to variable i)

$$[a(t-s)]^{(\alpha)} = a^\alpha (t-s)^{(\alpha)}, \tag{14}$$

$${}_tD_q (t-s)^{(\alpha)} = [\alpha]_q (t-s)^{(\alpha-1)}, \qquad {}_xD_q \int_0^x f(x,t)\, d_q t = \int_0^x {}_xD_q f(x,t)\, d_q t + f(qx, x). \tag{15}$$

Remark 4. We note that if $\alpha \geq 0$ and $a \leq b \leq t$, then $(t-a)^{(\alpha)} \geq (t-b)^{(\alpha)}$ [24].

Definition 5 (see [21]). Let $\alpha \geq 0$ and f be a function defined on $[0, 1]$. The fractional q-integral of the Riemann-Liuville type is $(I_q^0 f)(x) = f(x)$ and

$$\left(I_q^\alpha f\right)(x) = \frac{1}{\Gamma_q(\alpha)} \int_0^x (x - qt)^{(\alpha-1)} f(t)\, d_q t, \quad \alpha > 0, \quad x \in [0, 1]. \tag{16}$$

Definition 6 (see [23]). The fractional q-derivative of the Riemann-Liuville type of $\alpha \geq 0$ is defined by $(D_q^0 f)(x) = f(x)$ and

$$\left(D_q^\alpha f\right)(x) = \left(D_q^m I_q^{m-\alpha} f\right)(x), \quad \alpha > 0, \tag{17}$$

where m is the smallest integer greater than or equal to α.

Lemma 7 (see [21, 23]). *Let $\alpha, \beta \geq 0$ and let f be a function define on $[0, 1]$. Then, the next formulas hold:*

(1) $(I_q^\beta I_q^\alpha f)(x) = (I_q^{\alpha+\beta} f)(x)$,

(2) $(D_q^\alpha I_q^\alpha f)(x) = f(x)$.

Lemma 8 (see [24]). *Let $\alpha > 0$ and let p be a positive integer. Then, the following equality holds:*

$$\left(I_q^\alpha D_q^p f\right)(x) = \left(D_q^p I_q^\alpha f\right)(x) - \sum_{k=0}^{p-1} \frac{x^{\alpha-p+k}}{\Gamma_q(\alpha+k-p+1)} \left(D_q^k f\right)(0). \tag{18}$$

Lemma 9. *Let $y(t) \in C[0,1] \bigcap L^1[0,1]$ and $2 < \alpha \leq 3$; then the BVP*

$$\left(D_q^\alpha u\right)(t) + y(t) = 0, \quad 0 < t < 1, \qquad u(0) = u(1) = 0, \qquad \left(D_q u\right)(0) = 0, \tag{19}$$

has a unique solution

$$u(t) = \int_0^1 G(t, qs)\, y(s)\, d_q s, \tag{20}$$

where

$$G(t,s) = \frac{1}{\Gamma_q(\alpha)} \begin{cases} (1-s)^{(\alpha-1)} t^{\alpha-1} - (t-s)^{(\alpha-1)}, & 0 \le s \le t \le 1, \\ (1-s)^{(\alpha-1)} t^{\alpha-1}, & 0 \le t \le s \le 1. \end{cases} \tag{21}$$

Proof. By Lemmas 7 and 8, we see that

$$\begin{aligned} \left(D_q^\alpha u\right)(t) &= -y(t) \\ &\Longleftrightarrow \left(I_q^\alpha D_q^3 I_q^{3-\alpha} u\right)(t) = -\left(I_q^\alpha y\right)(t) \\ &\Longleftrightarrow u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} \\ &\quad + c_3 t^{\alpha-3} - \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} \\ &\quad \times y(s)\, d_q s, \end{aligned} \tag{22}$$

where c_1, c_2, and c_3 are some constants to be determined. Since $u(0) = 0$, we must have $c_3 = 0$. Now, differentiating both sides of (22) and using (15), we obtain

$$\begin{aligned} \left(D_q u\right)(t) &= [\alpha-1]_q c_1 t^{\alpha-2} + [\alpha-2]_q c_2 t^{\alpha-3} \\ &\quad - \frac{1}{\Gamma_q(\alpha)} \int_0^t [\alpha-1]_q (t-qs)^{(\alpha-2)} y(s)\, d_q s. \end{aligned} \tag{23}$$

Using $(D_q u)(0) = 0$ and $u(1) = 0$, we must set $c_2 = 0$, and

$$c_1 = \frac{1}{\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} y(s)\, d_q s. \tag{24}$$

Finally, we obtain

$$\begin{aligned} u(t) &= \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} y(s)\, d_q s \\ &\quad - \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} y(s)\, d_q s \\ &= \int_0^1 G(t, qs)\, y(s)\, d_q s. \end{aligned} \tag{25}$$

The proof is complete. □

Lemma 10. *Function G defined above satisfies the following conditions:*

(i) *$G(t, qs)$ is a continuous function on $[0,1] \times [0,1]$;*

(ii) *$G(t, qs) \ge 0$ for $t, s \in [0,1]$.*

Proof. (i) Obviously, $G(t, qs)$ is continuous on $[0,1] \times [0,1]$.

(ii) Let

$$\begin{aligned} g_1(t,s) &= (1-s)^{(\alpha-1)} t^{\alpha-1} - (t-s)^{(\alpha-1)}, \\ &\qquad 0 \le s \le t \le 1, \\ g_2(t,s) &= (1-s)^{(\alpha-1)} t^{\alpha-1}, \quad 0 \le t \le s \le 1. \end{aligned} \tag{26}$$

It is clear that $g_2(t, qs) \ge 0$, for $t, s \in [0,1]$. Now, in view of Remark 4, for $t \ne 0$

$$\begin{aligned} g_1(t, qs) &= (1-qs)^{(\alpha-1)} t^{\alpha-1} - (t-qs)^{(\alpha-1)} \\ &= t^{\alpha-1} \left[(1-qs)^{(\alpha-1)} - \left(1 - \frac{qs}{t}\right)^{(\alpha-1)} \right] \\ &\ge t^{\alpha-1} \left[(1-qs)^{(\alpha-1)} - (1-qs)^{(\alpha-1)} \right] = 0. \end{aligned} \tag{27}$$

Therefore, $G(t, qs) \ge 0$. This proof is complete. □

By $\mathcal{J}$ we denote the class of those functions $\beta : [0, \infty) \to [0, 1)$ satisfying the following condition; $\beta(t_n) \to 1$ implies $t_n \to 0$.

Theorem 11 (see [31]). *Let $(X, \le)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $T : X \to X$ be a nondecreasing mapping such that there exists an element $x_0 \in X$ with $x_0 \le Tx_0$. Suppose that there exists $\beta \in \mathcal{J}$ such that*

$$\begin{aligned} &d(Tx, Ty) \le \beta(d(x,y)) \cdot d(x,y) \\ &\quad \text{for } x, y \in X \text{ with } x \ge y. \end{aligned} \tag{28}$$

Assume that either T is continuous or X is such that

$$\begin{aligned} &\text{if } \{x_n\} \text{ is a nondecreasing sequence in } X \\ &\quad \text{such that } x_n \longrightarrow x \text{ then } x_n \le x \quad \forall n \in N. \end{aligned} \tag{29}$$

Besides if

$$\begin{aligned} &\text{for each } x, y \in X \text{ there exists } z \in X \\ &\quad \text{which is comparable to } x \text{ and } y, \end{aligned} \tag{30}$$

then T has a unique fixed point.

Let $C[0,1] = \{x : [0,1] \to R, \text{continuous}\}$ be the Banach space with the classic metric given by $d(x,y) = \sup_{0 \le t \le 1} \{|x(t) - y(t)|\}$.

Notice that this space can be equipped with a partial order given by

$$x, y \in C[0,1], \quad x \le y \Longleftrightarrow x(t) \le y(t), \quad \text{for } t \in [0,1]. \tag{31}$$

In [32], it is proved that $(C[0,1], \le)$ satisfies condition (29) of Theorem 11. Moreover, for $x, y \in C[0,1]$, as the function $\max(x,y) \in C[0,1]$, $(C[0,1], \le)$ satisfies condition (30).

3. Main Result

In this section, we will consider the question of positive solutions for BVP (3). At first, we prove some lemmas required for the main result.

Lemma 12. *Let* $0 < \sigma < 1$, $2 < \alpha \leq 3$ *and* $F : (0,1] \to R$ *is a continuous function with* $\lim_{t\to 0^+} F(t) = \infty$. *Suppose that* $t^\sigma F(t)$ *is a continuous function on* $[0,1]$. *Then the function defined by*

$$H(t) = \int_0^1 G(t,qs) F(s)\, d_q s \tag{32}$$

is continuous on $[0,1]$, *where* $G(t,s)$ *is Green function be given in Lemma 9.*

Proof. We will divide the proof into three parts.

Case 1 ($t_0 = 0$). First, $H(0) = 0$. Since $t^\sigma F(t)$ is continuous on $[0,1]$, we can find a positive constant M such that $|t^\sigma F(t)| \leq M$ for any $t \in [0,1]$. Thus,

$$\begin{aligned}
&|H(t) - H(0)| \\
&\quad = |H(t)| \\
&\quad = \left|\int_0^1 G(t,qs) F(s)\, d_q s\right| \\
&\quad = \left|\int_0^1 G(t,qs) s^{-\sigma} s^{\sigma} F(s)\, d_q s\right| \\
&\quad = \left|\int_0^t \frac{(1-qs)^{(\alpha-1)} t^{\alpha-1} - (t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} s^{-\sigma} s^{\sigma} F(s)\, d_q s \right.\\
&\qquad \left. + \int_t^1 \frac{(1-qs)^{(\alpha-1)} t^{\alpha-1}}{\Gamma_q(\alpha)} s^{-\sigma} s^{\sigma} F(s)\, d_q s\right| \\
&\quad = \left|\int_0^1 \frac{(1-qs)^{(\alpha-1)} t^{\alpha-1}}{\Gamma_q(\alpha)} s^{-\sigma} s^{\sigma} F(s)\, d_q s \right.\\
&\qquad \left. - \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} s^{-\sigma} s^{\sigma} F(s)\, d_q s\right| \\
&\quad \leq M \int_0^1 \frac{(1-qs)^{(\alpha-1)} t^{\alpha-1}}{\Gamma_q(\alpha)} s^{-\sigma}\, d_q s \\
&\qquad + M \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} s^{-\sigma}\, d_q s \\
&\quad = \frac{M t^{\alpha-1}}{\Gamma_q(\alpha)} \left[\int_0^1 (1-qs)^{(\alpha-1)} s^{-\sigma}\, d_q s \right.\\
&\qquad \left. + \int_0^t \left(1 - \frac{qs}{t}\right)^{(\alpha-1)} s^{-\sigma}\, d_q s\right].
\end{aligned} \tag{33}$$

For $\int_0^t (1-(qs/t))^{(\alpha-1)} s^{-\sigma} d_q s$, let $u = s/t$; then we obtain

$$\int_0^t \left(1 - \frac{qs}{t}\right)^{(\alpha-1)} s^{-\sigma}\, d_q s = t^{1-\sigma} \int_0^1 (1-qu)^{(\alpha-1)} u^{-\sigma}\, d_q u. \tag{34}$$

Hence,

$$\begin{aligned}
|H(t)| &\leq \frac{M t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} s^{-\sigma}\, d_q s \\
&\quad + \frac{M t^{\alpha-\sigma}}{\Gamma_q(\alpha)} \int_0^1 (1-qu)^{(\alpha-1)} u^{-\sigma}\, d_q u \\
&= \left(\frac{M t^{\alpha-1}}{\Gamma_q(\alpha)} + \frac{M t^{\alpha-\sigma}}{\Gamma_q(\alpha)}\right) \beta_q(1-\sigma, \alpha),
\end{aligned} \tag{35}$$

where β_q denotes the q-beta function.

When $t \to 0$, we see that $H(t) \to H(0)$; that is $H(t)$ is continuous at $t_0 = 0$.

Case 2 ($t_0 \in (0,1)$). We should prove $H(t_n) \to H(t_0)$ when $t_n \to t_0$. Without loss of generality, we consider $t_n > t_0$ (it is the same argument for $t_n < t_0$). In fact,

$$\begin{aligned}
&|H(t_n) - H(t_0)| \\
&\quad = \left|\int_0^1 \frac{t_n^{\alpha-1}(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} s^{-\sigma} s^{\sigma} F(s)\, d_q s \right.\\
&\qquad - \int_0^{t_n} \frac{(t_n-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} s^{-\sigma} s^{\sigma} F(s)\, d_q s \\
&\qquad - \int_0^1 \frac{t_0^{\alpha-1}(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} s^{-\sigma} s^{\sigma} F(s)\, d_q s \\
&\qquad \left. + \int_0^{t_0} \frac{(t_0-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} s^{-\sigma} s^{\sigma} F(s)\, d_q s\right| \\
&\quad = \left|\int_0^1 \frac{\left(t_n^{\alpha-1} - t_0^{\alpha-1}\right)(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} s^{-\sigma} s^{\sigma} F(s)\, d_q s \right.\\
&\qquad - \int_0^{t_0} \frac{(t_n-qs)^{(\alpha-1)} - (t_0-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} s^{-\sigma} s^{\sigma} F(s)\, d_q s \\
&\qquad \left. - \int_{t_0}^{t_n} \frac{(t_n-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} s^{-\sigma} s^{\sigma} F(s)\, d_q s\right| \\
&\quad \leq \frac{M\left(t_n^{\alpha-1} - t_0^{\alpha-1}\right)}{\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} s^{-\sigma}\, d_q s \\
&\qquad + \frac{M}{\Gamma_q(\alpha)} \int_0^{t_0} \left((t_n-qs)^{(\alpha-1)} - (t_0-qs)^{(\alpha-1)}\right) s^{-\sigma}\, d_q s
\end{aligned}$$

$$+\frac{M}{\Gamma_q(\alpha)}\int_{t_0}^{t_n}(t_n-qs)^{(\alpha-1)}s^{-\sigma}d_qs$$
$$=\frac{M}{\Gamma_q(\alpha)}\beta_q(1-\sigma,\alpha)\left(t_n^{\alpha-1}-t_0^{\alpha-1}\right)+\frac{M}{\Gamma_q(\alpha)}(a_n+b_n), \tag{36}$$

where

$$a_n=\int_0^{t_0}\left((t_n-qs)^{(\alpha-1)}-(t_0-qs)^{(\alpha-1)}\right)s^{-\sigma}d_qs,$$
$$b_n=\int_{t_0}^{t_n}(t_n-qs)^{(\alpha-1)}s^{-\sigma}d_qs. \tag{37}$$

When $n\to\infty$, we verify $a_n\to 0$.

As $t_n\to t_0$, then $((t_n-qs)^{(\alpha-1)}-(t_0-qs)^{(\alpha-1)})s^{-\sigma}\to 0$, when $n\to\infty$. Moreover,

$$\left((t_n-qs)^{(\alpha-1)}-(t_0-qs)^{(\alpha-1)}\right)s^{-\sigma}\le 2s^{-\sigma},$$
$$\int_0^1 2s^{-\sigma}d_qs=\frac{2}{[1-\sigma]_q}s^{1-\sigma}\Big|_0^1=\frac{2}{[1-\sigma]_q}<\infty. \tag{38}$$

We have $((t_n-qs)^{(\alpha-1)}-(t_0-qs)^{(\alpha-1)})s^{-\sigma}$ converges pointwise to the zero function and $|(t_n-qs)^{(\alpha-1)}-(t_0-qs)^{(\alpha-1)}|s^{-\sigma}$ is bounded by a function belonging to $L^1[0,1]$, by Lebesgue's dominated convergence theorem $a_n\to 0$ when $n\to\infty$.

Now, we prove $b_n\to 0$ when $n\to\infty$.

In fact, as

$$b_n=\int_{t_0}^{t_n}(t_n-qs)^{(\alpha-1)}s^{-\sigma}d_qs$$
$$\le\int_{t_0}^{t_n}s^{-\sigma}d_qs=\frac{s^{1-\sigma}}{[1-\sigma]_q}\Big|_{t_0}^{t_n} \tag{39}$$
$$=\frac{1}{[1-\sigma]_q}\left(t_n^{1-\sigma}-t_0^{1-\sigma}\right),$$

and taking into account that $t_n\to t_0$, we get $b_n\to 0$ when $n\to\infty$.

Above all, we obtain $|H(t_n)-H(t_0)|\to 0$, when $n\to\infty$.

Case 3 ($t_0=1$). It is easy to check that $H(1)=0$ and $H(t)$ is continuous at $t_0=1$. The proof is the same as the proof of Case 1. □

Lemma 13. *Suppose that* $0<\sigma<1$. *Then,*

$$\max_{0\le t\le 1}\int_0^1 G(t,qs)s^{-\sigma}d_qs=\frac{A^{\alpha-1}-A^{\alpha-\sigma}}{\Gamma_q(\alpha)}\beta_q(1-\sigma,\alpha), \tag{40}$$

where $A=((\alpha-1)/(\alpha-\sigma))^{1/(1-\sigma)}$.

Proof.

$$\int_0^1 G(t,qs)s^{-\sigma}d_qs$$
$$=\int_0^t\frac{(1-qs)^{(\alpha-1)}t^{\alpha-1}-(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}s^{-\sigma}d_qs$$
$$+\int_t^1\frac{(1-qs)^{(\alpha-1)}t^{\alpha-1}}{\Gamma_q(\alpha)}s^{-\sigma}d_qs$$
$$=\int_0^1\frac{(1-qs)^{(\alpha-1)}t^{\alpha-1}}{\Gamma_q(\alpha)}s^{-\sigma}d_qs \tag{41}$$
$$-\int_0^t\frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}s^{-\sigma}d_qs$$
$$=\frac{t^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^1(1-qs)^{(\alpha-1)}s^{-\sigma}d_qs$$
$$-\frac{t^{\alpha-\sigma}}{\Gamma_q(\alpha)}\int_0^1(1-qs)^{(\alpha-1)}s^{-\sigma}d_qu$$
$$=\frac{t^{\alpha-1}-t^{\alpha-\sigma}}{\Gamma_q(\alpha)}\beta_q(1-\sigma,\alpha).$$

Let $g(t)=t^{\alpha-1}-t^{\alpha-\sigma}$, $t\in[0,1]$.

Since $g'(t)=(\alpha-1)t^{\alpha-2}-(\alpha-\sigma)t^{\alpha-\sigma-1}$, let $g'(t)=0$; we can get $g(t)$ has a maximum at the point $t_0=A=((\alpha-1)/(\alpha-\sigma))^{1/(1-\sigma)}$.

Hence,

$$\max_{0\le t\le 1}\int_0^1 G(t,qs)s^{-\sigma}d_qs=\frac{A^{\alpha-1}-A^{\alpha-\sigma}}{\Gamma_q(\alpha)}\beta_q(1-\sigma,\alpha). \tag{42}$$

□

For the convenience, we denote $\max_{0\le t\le 1}\int_0^1 G(t,qs)s^{-\sigma}d_qs$ by K.

Next, we denote the class of functions $\phi:[0,\infty)\to[0,\infty)$ by $\mathscr{A}$ satisfying

(i) ϕ is nondecreasing;

(ii) $\phi(x)<x$ for any $x>0$;

(iii) $\beta(x)=\phi(x)/x\in\mathscr{J}$, where $\mathscr{J}$ is the class of functions appearing in Theorem 11.

We give our main result as follows.

Theorem 14. *Let* $0<\sigma<1$, $2<\alpha\le 3$, $f:[0,1]\times[0,\infty)\to[0,\infty)$ *is continuous and* $\lim_{t\to 0^+}f(t,\cdot)=\infty$, *and* $t^\sigma f(t,y)$ *is a continuous function on* $[0,1]\times[0,\infty)$. *Assume that there exists* $0<\lambda\le 1/K$ *such that for* $x,y\in[0,\infty)$ *with* $y\ge x$ *and* $t\in[0,1]$,

$$0\le t^\sigma\left(f(t,y)-f(t,x)\right)\le\lambda\phi(y-x), \tag{43}$$

where $\phi\in\mathscr{A}$. *Then the BVP* (3) *has a unique positive solution (i.e.,* $x(t)>0$ *for* $t\in(0,1)$*).*

Proof. We define the cone P by

$$P = \{u \in C[0,1] : u(t) \geq 0\}. \tag{44}$$

It is clear that P is a complete metric space as P is a closed set of $C[0,1]$. It is also easy to check that P satisfies conditions (29) and (30) of Theorem 11.

We define the operator T by

$$\begin{aligned}(Tu)(t) &= \int_0^1 G(t,qs) f(s,u(s))\, d_q s \\ &= \int_0^1 G(t,qs) s^{-\sigma} s^{\sigma} f(s,u(s))\, d_q s.\end{aligned} \tag{45}$$

In view of Lemma 12, $Tu \in C[0,1]$. Moreover, it follows from the nonnegativeness of $G(t,qs)$ and $t^{\sigma} f(t,y)$ that $Tu \in P$ for $u \in P$. Thus, $T : P \to P$.

Next, we will prove that assumptions in Theorem 11 are satisfied.

First, for $u \geq v$, we have

$$\begin{aligned}(Tu)(t) &= \int_0^1 G(t,qs) f(s,u(s))\, d_q s \\ &= \int_0^1 G(t,qs) s^{-\sigma} s^{\sigma} f(s,u(s))\, d_q s \\ &\geq \int_0^1 G(t,qs) s^{-\sigma} s^{\sigma} f(s,v(s))\, d_q s \\ &= (Tv)(t).\end{aligned} \tag{46}$$

Hence, the operator T is nondecreasing. Besides, for $u \geq v$ and $u \neq v$,

$$\begin{aligned}&d(Tu,Tv) \\ &\quad= \max_{t\in[0,1]} |(Tu)(t) - (Tv)(t)| \\ &\quad= \max_{t\in[0,1]} ((Tu)(t) - (Tv)(t)) \\ &\quad= \max_{t\in[0,1]} \int_0^1 G(t,qs)\left(f(s,u(s)) - f(s,v(s))\right) d_q s \\ &\quad= \max_{t\in[0,1]} \int_0^1 G(t,qs) s^{-\sigma} s^{\sigma}\left(f(s,u(s)) - f(s,v(s))\right) d_q s \\ &\quad\leq \max_{t\in[0,1]} \int_0^1 G(t,qs) s^{-\sigma} \lambda\phi(u(s) - v(s))\, d_q s.\end{aligned} \tag{47}$$

Since ϕ is nondecreasing and $u(s) - v(s) \leq d(u,v)$,

$$\begin{aligned}d(Tu,Tv) &\leq \max_{t\in[0,1]} \int_0^1 G(t,qs) s^{-\sigma} \lambda\phi(d(u,v))\, d_q s \\ &= \lambda\phi(d(u,v)) \max_{t\in[0,1]} \int_0^1 G(t,qs) s^{-\sigma} d_q s \\ &= \lambda\phi(d(u,v)) K.\end{aligned} \tag{48}$$

Moreover, when $0 < \lambda \leq 1/K$, we get

$$\begin{aligned}d(Tu,Tv) &\leq \phi(d(u,v)) \\ &= \frac{\phi(d(u,v))}{d(u,v)} \cdot d(u,v) \\ &= \beta(d(u,v)) \cdot d(u,v).\end{aligned} \tag{49}$$

Obviously, the last inequality is satisfied for $u = v$.

Taking into account that the zero function satisfies $0 \leq T_0$, in view of Theorem 11, the operator T has a unique fixed point $x(t)$ in P.

At last, we will prove $x(t)$ is a positive solution. We assume that there exists $0 < t_1 < 1$ such that $x(t_1) = 0$. Since $x(t)$ of problem (3) is a fixed point of the operator T, we have

$$\begin{aligned}x(t) &= \int_0^1 G(t,qs) f(s,x(s))\, d_q s, \quad \text{for } 0 < t < 1, \\ x(t_1) &= \int_0^1 G(t_1,qs) f(s,x(s))\, d_q s = 0.\end{aligned} \tag{50}$$

For the nonnegative character of $G(t,qs)$ and $f(s,x)$, the last relation gives

$$G(t_1,qs) f(s,x(s)) = 0 \quad \text{a.e. } (s). \tag{51}$$

f is continuous and $\lim_{t\to 0^+} f(t,\cdot) = \infty$; then for $M > 0$, we can find $\delta > 0$, and, for $s \in [0,1] \cap (0,\delta)$, we have $f(s,0) > M$. It is clear that $[0,1] \cap (0,\delta) \subset \{s \in [0,1] : f(s,x(s)) > M\}$ and $\mu([0,1] \cap (0,\delta)) > 0$, where μ is the Lebesgue measure on $[0,1]$. That is to say, $G(t_1,qs) f(s,x(s)) = 0$ a.e. (s). This is a contradiction because $G(t_1,qs)$ is a rational function in s.

Therefore, $x(t) > 0$ for $t \in (0,1)$.

The proof is complete. □

4. Example

Consider the following singular BVP:

$$\begin{gathered}D_{1/2}^{5/2} u(t) + \frac{\lambda\left(t^2+1\right)\ln(1+u(t))}{t^{1/2}} = 0, \quad 0 < t < 1, \ \lambda > 0, \\ u(0) = u(1) = 0, \qquad \left(D_{1/2} u\right)(0) = 0.\end{gathered} \tag{52}$$

Here, $\alpha = 2.5$, $q = 1/2$, $\sigma = 1/2$, and $f(t,u) = \lambda(t^2 + 1)\ln(1 + u(t))/t^{1/2}$ for $(t,u) \in [0,1] \times [0,\infty)$. Notice that f is continuous in $[0,1] \times [0,\infty)$ and $\lim_{t\to 0^+} f(t,\cdot) = \infty$.

At first, we define ϕ by

$$\phi : [0,\infty) \longrightarrow [0,\infty), \qquad \phi(x) = \ln(1+x). \tag{53}$$

It is clear that $\phi(x) = \ln(1 + x)$ is a nondecreasing function; for $u \geq v$, we can get

$$\phi(u) - \phi(v) \geq 0. \tag{54}$$

Moreover, for $u \geq v$, ϕ also satisfies

$$\phi(u) - \phi(v) \leq \phi(u - v). \tag{55}$$

In fact, when $u \geq v$,

$$\begin{aligned} \phi(u-v)-(\phi(u)-\phi(v)) &= \ln(1+u-v) \\ &\quad -(\ln(1+u)-\ln(1+v)) \\ &= \ln\frac{(1+u-v)(1+v)}{(1+u)} \\ &= \ln\left(1+\frac{(u-v)v}{1+u}\right) \geq 0, \end{aligned} \tag{56}$$

equivalently

$$\phi(u)-\phi(v) \leq \phi(u-v). \tag{57}$$

Above all, $0 \leq \phi(u)-\phi(v) \leq \phi(u-v)$ for $u \geq v$.

Second, for $u \geq v$ and $t \in [0,1]$, we have

$$\begin{aligned} 0 &\leq t^{1/2}(f(t,u)-f(t,v)) \\ &= \lambda\left(t^2+1\right)[\ln(1+u)-\ln(1+v)] \\ &\leq \lambda\left(t^2+1\right)\ln(1+u-v) \\ &\leq 2\lambda\ln(1+u-v); \end{aligned} \tag{58}$$

that is, f satisfies assumptions of Theorem 14.

Third, we should prove $\phi(x)$ belongs to $\mathscr{A}$. By elemental calculus, it is easy to check that ϕ is nondecreasing and $\phi(x) < x$, for $x > 0$.

In order to prove $\beta(x) = \phi(x)/x \in \mathscr{J}$, we notice that if $\beta(t_n) \to 1$, then the sequence (t_n) is a bounded sequence because in contrary case, that is, $t_n \to \infty$, we get

$$\beta(t_n) = \frac{\ln(1+t_n)}{t_n} \longrightarrow 0. \tag{59}$$

Now, we assume that $t_n \nrightarrow 0$, and then we find $\varepsilon > 0$ such that for each $n \in N$ there exists $\rho_n \geq n$ with $t_{\rho_n} \geq \varepsilon$.

Since (t_n) is a bounded sequence, we can find a subsequence (t_{k_n}) of (t_{ρ_n}) with $t_{k_n} \to a$, for certain $a \in [0,1)$. When $\beta(t_n) \to 1$, it implies that

$$\beta(t_{k_n}) = \frac{\ln(1+t_{k_n})}{t_{k_n}} \longrightarrow 1. \tag{60}$$

and, as the unique solution of $\ln(1+x) = x$ is $x_0 = 0$, we deduce that $a = 0$. Therefore, $t_{k_n} \to 0$ and this contradicts the fact that $t_{k_n} \geq \varepsilon$ for every $n \in N$.

Thus, $t_n \to 0$ and this proves that $\beta \in \mathscr{J}$.

Finally, in view of Theorem 14,

$$\begin{aligned} 2\lambda &\leq \frac{1}{K} = \frac{1}{\left(\left((1/4)^{3/2}-(1/4)^{1/2}\right)/\Gamma_{1/2}(3/2)\right)\cdot\beta_{1/2}(1/2,3/2)} \\ &\approx 10.96511985; \end{aligned} \tag{61}$$

that is, when $\lambda \leq 5.48256$, boundary value problem (52) has a unique positive solution.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Properties of Functions in the Wiener Class $BV_p[a,b]$ for $0<p<1$

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We will investigate properties of functions in the Wiener class $BV_p[a,b]$ with $0 < p < 1$. We prove that any function in $BV_p[a,b]$ $(0 < p < 1)$ can be expressed as the difference of two increasing functions in $BV_p[a,b]$. We also obtain the explicit form of functions in $BV_p[a,b]$ and show that their derivatives are equal to zero a.e. on $[a,b]$.

1. Introduction

Let $0 < p < \infty$. We say that a real valued function f on $[a,b]$ is of bounded p-variation and is denoted by $f \in BV_p[a,b]$, if

$$V_p f = \sup_T \left(\sum_{k=1}^{n} \left| f(x_k) - f(x_{k-1}) \right|^p \right)^{1/p} < \infty, \qquad (1)$$

where the supremum is taken over all partitions $T : a = x_0 < x_1 < \cdots < x_n = b$. When $p = 1$, we get the well-known Jordan bounded variation $BV[a,b]$; and when $1 < p < \infty$, we get Wiener's definition of bounded p-variation. There are many other generalizations of BV, such as bounded Φ-variation in the sense of Young (see [1]) and Waterman's Λ-bounded variation (see [2]). The class BV_p and generalizations of BV have been studied mainly because of their applicability to the theory of Fourier series and some good approximative properties (see, e.g., [1–7]).

However, it should be mentioned that results of most papers deal mostly with the case $p \geq 1$. This is because that in this case $BV_p[a,b]$ is a Banach space with the norm $\|f\|_{BV_p} = |f(a)| + V_p f$ (see, e.g., [3]). In the case $0 < p < 1$, $BV_p[a,b]$ is no longer a Banach space and has not been studied as far as we know. Nevertheless, functions in $BV_p[a,b]$ $(0 < p < 1)$ have many interesting properties; for example, their derivatives are equal to zero a.e. on $[a,b]$.

In this paper, we will investigate properties of functions in the class $BV_p[a,b]$ with $0 < p < 1$. We will show that $BV_p[a,b]$ is a Frechet space with the quasinorm

$$q(f) = |f(a)|^p + (V_p f)^p. \qquad (2)$$

We will get the Jordan type decomposition theorem which says that any function in $BV_p[a,b]$ $(0 < p < 1)$ can be expressed as the difference of two increasing functions in $BV_p[a,b]$. We also get the representation theorem which gives the explicit form of functions in $BV_p[a,b]$ $(0 < p < 1)$.

2. Statement of Main Results

Clearly, for any fixed $p \in (0,1)$, the Wiener class $BV_p[a,b]$ is a linear space. We define the functional q on $BV_p[a,b]$ by

$$q(f) = |f(a)|^p + (V_p f)^p = |f(a)|^p + \sup_T \sum_{k=1}^{n} \left| f(x_k) - f(x_{k-1}) \right|^p, \quad f \in BV_p[a,b]. \qquad (3)$$

From the inequality $(a+b)^p \leq a^p + b^p$ $(a, b \geq 0,\ 0 < p < 1)$, we get that $q(f+g) \leq q(f) + q(g)$. It then follows that q is a quasinorm on $BV_p[a,b]$.

Our first result claims that $BV_p[a,b]$ $(0 < p < 1)$ equipped with the quasinorm q is a Frechet space.

Theorem 1. *The Wiener class $BV_p[a,b]$ $(0 < p < 1)$ equipped with the quasinorm q is a Frechet space.*

From the inequality

$$\left(\sum_{i=1}^{\infty} a_i^{p_2}\right)^{1/p_2} \le \left(\sum_{i=1}^{\infty} a_i^{p_1}\right)^{1/p_1}, \quad a_i \ge 0,\ 0 < p_1 \le p_2 < \infty, \tag{4}$$

we get that, for any $f \in BV_{p_1}[a,b]$,

$$V_{p_2} f \le V_{p_1} f, \tag{5}$$

which means that $BV_{p_1}[a,b] \subseteq BV_{p_2}[a,b]$. Specially, for $0 < p < 1$, $BV_p[a,b] \subseteq BV_1[a,b] \equiv BV[a,b]$. This implies that $BV_p[a,b]$ functions are bounded, and the discontinuities of a $BV_p[a,b]$ function are simple and, therefore, at most denumerable (see [8, Theorem 13.7 and Lemma 13.2]). By the Jordan decomposition theorem, we know that every function f in $BV[a,b]$ can be expressed as the difference of two increasing functions g and h defined on $[a,b]$ (see [8, Corollary 13.6]). If $f \in BV_p[a,b] \subseteq BV[a,b]$, we can require that the above increasing functions g and h are still in $BV_p[a,b]$. This is our next theorem.

Theorem 2 (Jordan type decomposition theorem). *Any function in $BV_p[a,b]$ $(0 < p < 1)$ can be expressed as the difference of two increasing functions in $BV_p[a,b]$.*

Let $t \in [a,b]$, $d > 0$, and $0 \le d' \le d$. We set

$$h_{t,d,d'}(x) = \begin{cases} 0, & x < t, \\ d', & x = t, \\ d, & x > t. \end{cases} \tag{6}$$

Then $h_{t,d,d'}(x)$ is increasing on $[a,b]$ with only one discontinuity point t. Also, $(h_{t,d,d'}(x))' = 0$ for $x \ne t$.

Let f be an increasing function in $BV_p[a,b]$ $(0 < p < 1)$. Denote by $A \equiv A(f)$ the set of points of discontinuity of f. Then A is at most countable (see [8, Theorem 2.17]). Since f is increasing, we get that, for any $t \in A$, the right and left limits $f(t+0)$ and $f(t-0)$ of the function f at t exist, $f(t+0) - f(t-0) > 0$, and $0 \le f(t) - f(t-0) \le f(t+0) - f(t-0)$. For $t \in A$, we define

$$\widetilde{h_t}(x) \equiv \widetilde{h_{t,f}}(x) = h_{t,f(t+0)-f(t-0),f(t)-f(t-0)}(x). \tag{7}$$

Our next theorem characterizes the form of an increasing function in $BV_p[a,b]$. Any increasing function f in $BV_p[a,b]$ must be as follows:

$$f(x) = \sum_{n=1}^{N} h_{t_n,d_n,d_n'}(x) + c, \tag{8}$$

where $N \le \infty$, $t_n \in [a,b]$, $d_n > 0$, $d_n' \in [0,d_n]$, and $\sum_{n=1}^{N} d_n^p < \infty$.

Theorem 3. *(1) If $f(x) = c + \sum_{n=1}^{N} h_{t_n,d_n,d_n'}(x)$, where $N \le \infty$, $t_n \in [a,b]$, $d_n > 0$, and $d_n' \in [0,d_n]$, then $f \in BV_p[a,b]$ $(0 < p < 1)$ if and only if $\sum_{n=1}^{N} d_n^p < \infty$. In this case,*

$$\left(\sum_{n=1}^{N} d_n^p\right)^{1/p} \le V_p(f) \le \left(2\sum_{n=1}^{N} d_n^p\right)^{1/p}. \tag{9}$$

(2) Let f be an increasing function in $BV_p[a,b]$ $(0 < p < 1)$. Then $f(x) = \sum_{t\in A} \widetilde{h_t}(x) + c$, where c is a constant, A is the set of points of discontinuity of f, and $\widetilde{h_t}(x)$ is defined by (7).

Finally, for an increasing function f in $BV_p[a,b]$ $(0 < p < 1)$, by Theorem 3 we have $f(x) = \sum_{t\in A} \widetilde{h_t}(x) + c$, where A is the set of points of discontinuity of f and at most countable. Since $(\widetilde{h_t}(x))' = 0$, a.e. $x \in [a,b]$, by the Fubini term by term differentiation theorem (see [9, Proposition 4.6]), we get $f'(x) = 0$, a.e. $x \in [a,b]$. By Theorem 2, any function f in $BV_p[a,b]$ can be expressed as the difference of two increasing functions $g(x)$ and $r(x)$ in $BV_p[a,b]$. Applying Theorem 3, we get the representation theorem of functions in $BV_p[a,b]$ $(0 < p < 1)$ as follows.

Corollary 4. *Let $f \in BV_p[a,b]$ $(0 < p < 1)$. Then f can be expressed in the following form:*

$$f(x) = g(x) - r(x) = \sum_{t\in A_1} \widetilde{h_{t,g}}(x) - \sum_{t\in A_2} \widetilde{h_{t,r}}(x) + c, \tag{10}$$

where c is a constant, $g(x)$, $r(x)$ are increasing functions in $BV_p[a,b]$, $\widetilde{h_{t,g}}(x)$ and $\widetilde{h_{t,r}}$ are defined by (7), $A_1, A_2 \subseteq A$, and A_1, A_2, A are the sets of points of discontinuity of g, r, and f, respectively. Furthermore, $f'(x) = 0$, a.e. $x \in [a,b]$.

3. Proofs of Theorems 1–3

Proof of Theorem 1. It suffices to prove that $BV_p[a,b]$ is complete. Let $\{f_n\}$ be a Cauchy sequence in $BV_p[a,b]$; that is, $q(f_n - f_m) = |f_n(a) - f_m(a)|^p + (V_p(f_n - f_m))^p \to 0$ as $n, m \to \infty$. For any $\xi \in [a,b]$, using the partition $T: a \le \xi \le b$ and the definition of $V_p f$, we get that $\{f_n(\xi)\}$ is a Cauchy sequence in $\mathbb{R}$ and converges to a number denoted by $f(\xi)$. For any $\varepsilon > 0$, there exists an integer N such that $q(f_n - f_m) \le \varepsilon$ for $m, n > N$. Let $T: a = x_0 < x_1 < \cdots < x_k = b$ be an arbitrary partition of $[a,b]$. Then

$$\begin{aligned} &|f_m(a) - f_n(a)|^p \\ &\quad + \sum_{i=1}^{k} |(f_m - f_n)(x_i) - (f_m - f_n)(x_{i-1})|^p \\ &\le q(f_n - f_m) \le \varepsilon. \end{aligned} \tag{11}$$

Letting $m \to \infty$, we get that

$$|f(a) - f_n(a)|^p + \sum_{i=1}^{k} |(f - f_n)(x_i) - (f - f_n)(x_{i-1})|^p \le \varepsilon. \tag{12}$$

Taking the supremum over all partitions T, we have $q(f - f_n) \le \varepsilon$ for $n > N$. This means that $f = (f - f_n) + f_n \in BV_p[a,b]$, and $q(f - f_n) \to 0$ as $n \to \infty$. Hence, $BV_p[a,b]$ $(0 < p < 1)$ is complete. Theorem 1 is proved. □

Proof of Theorem 2. Suppose that $f \in BV_p[a,b]$ $(0 < p < 1)$. Since $f \in BV_p[a,b] \subset BV[a,b]$, by the Jordan decomposition theorem (see [8, Corollary 13.6]), we have $f(x) = g(x) - r(x)$, where $g(x)$, $r(x)$ are increasing functions on $[a,b]$. Indeed, we can choose $g(x)$ to be $V_a^x(f)$, the total variation function of f defined by

$$V_a^x(f) = \sup_T \left\{ \sum_{i=1}^n |f(x_i) - f(x_{i-1})| \right\}, \tag{13}$$

where the supremum is taken over all partitions $T : a = x_0 < x_1 < \cdots < x_n = x$ of $[a,x]$, $r(x) = V_a^x(f) - f(x)$. It suffices to show that $g(x) = V_a^x(f) \in BV_p[a,b]$. For any fixed partition $T : a = x_0 < x_1 < \cdots < x_n = b$, we note that

$$\begin{aligned} |g(x_i) - g(x_{i-1})|^p &= \left|V_{x_{i-1}}^{x_i} f\right|^p \\ &= \sup_{T_i} \left(\sum_{j=1}^{m_i} \left|f(\xi_{i,j}) - f(\xi_{i,j-1})\right| \right)^p \\ &\le \sup_{T_i} \sum_{j=1}^{m_i} \left|f(\xi_{i,j}) - f(\xi_{i,j-1})\right|^p, \end{aligned} \tag{14}$$

where the supremum is taken over all partitions $T_i : x_{i-1} = \xi_{i,1} < \xi_{i,2} < \cdots < \xi_{i,m_i} = x_i$ of $[x_{i-1}, x_i]$. It follows that

$$\begin{aligned} \sum_{i=1}^n |g(x_i) - g(x_{i-1})|^p &\le \sum_{i=1}^n \sup_{T_i} \sum_{j=1}^{m_i} \left|f(\xi_{i,j}) - f(\xi_{i,j-1})\right|^p \\ &= \sup_{T_i,\ 1 \le i \le n} \sum_{i=1}^n \sum_{j=1}^{m_i} \left|f(\xi_{i,j}) - f(\xi_{i,j-1})\right|^p \\ &\le (V_p f)^p, \end{aligned} \tag{15}$$

which implies $g \in BV_p[a,b]$. This completes the proof of Theorem 2. □

To prove Theorem 3, we introduce the next lemma.

Lemma 5. *If $f \in BV_p[a,b] \bigcap C[a,b]$ $(0 < p < 1)$, then f is a constant function.*

Proof. It suffices to show that, for any $d \in [a,b]$, $f(d) = f(a)$. Assume that there exists $d \in (a,b]$ such that $f(d) \ne f(a)$. Without loss of generality, we assume that $f(a) < f(d)$. Since $f \in C[a,b]$, there exist $n - 1$ points $\xi_1, \xi_2, \ldots, \xi_{n-1}$ such that $a = \xi_0 < \xi_1 < \cdots < \xi_{n-1} < \xi_n = d$ and $f(\xi_i) = f(a) + ((f(d) - f(a))/n)i$. Hence,

$$\begin{aligned} (V_p f)^p &\ge \sum_{i=1}^n |f(\xi_i) - f(\xi_{i-1})|^p \\ &= n^{1-p} |f(d) - f(a)|^p \longrightarrow \infty, \end{aligned} \tag{16}$$

as $n \to \infty$, which implies that $f \notin BV_p[a,b]$. This leads to a contradiction. Lemma 5 is proved. □

Proof of Theorem 3. (1) Without loss of generality, we may assume that $N = \infty$. Let $T : a = y_0 < y_1 < \cdots < y_m = b$ be a partition of $[a,b]$. For j, $1 \le j \le m$, we note that

$$\begin{aligned} &\left|f(y_j) - f(y_{j-1})\right|^p \\ &\quad = \left|\sum_{n=1}^\infty \left(h_{t_n,d_n,d_n'}(y_j) - h_{t_n,d_n,d_n'}(y_{j-1})\right)\right|^p \\ &\quad = \left| \sum_{\substack{n \\ y_{j-1} < t_n < y_j}} d_n + \sum_{\substack{n \\ t_n = y_{j-1}}} (d_n - d_n') + \sum_{\substack{n \\ t_n = y_j}} d_n' \right|^p \\ &\quad \le \sum_{\substack{n \\ y_{j-1} \le t_n \le y_j}} d_n^p, \end{aligned} \tag{17}$$

where an empty sum denotes 0. It follows that

$$\sum_{j=1}^m \left|f(y_j) - f(y_{j-1})\right|^p \le \sum_{j=1}^m \left(\sum_{\substack{n \\ y_{j-1} \le t_n \le y_j}} d_n^p \right) \le 2 \sum_{n=1}^\infty d_n^p. \tag{18}$$

Taking the supremum over all partitions of $[a,b]$, we obtain that

$$(V_p f)^p \le 2 \sum_{n=1}^\infty d_n^p. \tag{19}$$

On the other hand, for any fixed m, by renumbering $\{t_n\}_{n=1}^m$ if necessary, we may assume that $a \le t_1 < t_2 < \cdots < t_m \le b$. We set $y_i = ((t_i + t_{i+1})/2)$ $(1 \le i \le m - 1)$. Then $T : a = y_0 < y_1 < y_2 < \cdots < y_{m-1} < y_m = b$ is a partition of $[a,b]$. It follows that

$$\begin{aligned} (V_p f)^p &\ge \sum_{j=1}^m \left|f(y_j) - f(y_{j-1})\right|^p \ge \sum_{j=1}^m \left(\sum_{\substack{n \\ y_{j-1} < t_n < y_j}} d_n \right)^p \\ &\ge \sum_{j=1}^m d_j^p. \end{aligned} \tag{20}$$

Letting $m \to \infty$, we get

$$V_p f \ge \left(\sum_{n=1}^\infty d_n^p \right)^{1/p}. \tag{21}$$

Combining (19) with (21), we get (9). Hence, $f \in BV_p[a,b]$ $(0 < p < 1)$ if and only if $\sum_{n=1}^\infty d_n^p < \infty$.

(2) Let f be an increasing function in $BV_p[a,b]$ $(0 < p < 1)$ and A the set of points of discontinuity of f on $[a,b]$. We set $h_f(x) = \sum_{t \in A} \widetilde{h_t}(x)$, where $\widetilde{h_t}(x)$ is defined by (7). Similar to the proof of (21), we have

$$\sum_{t \in A} (f(t+0) - f(t-0))^p \le (V_p f)^p < \infty. \tag{22}$$

Applying the above proved result, we obtain that $h_f(x) \in BV_p[a,b]$. We set $g(x) = f(x) - h_f(x)$; then $g \in BV_p[a,b]$. We will show that $g(x)$ is continuous on $[a,b]$.

Indeed, for $x \in [a,b]$, we have

$$\begin{aligned}\sum_{t\in A}\widetilde{h_t}(x) &\leq \sum_{t\in A}\left(f(t+0)-f(t-0)\right)\\ &\leq \left(\sum_{t\in A}\left(f(t+0)-f(t-0)\right)^p\right)^{1/p}\\ &\leq V_p f < \infty.\end{aligned} \tag{23}$$

By Weierstrass M-test (see [10, Theorem 7.10]), we get that the series $\sum_{t\in A}\widetilde{h_t}(x)$ converges uniformly on $[a,b]$. For $x_0 \in [a,b]\setminus A$, $\widetilde{h_t}(x)$ $(t \in A)$ is continuous at x_0, so $h_f(x) = \sum_{t\in A}\widetilde{h_t}(x)$ is also continuous at x_0. It follows that $g(x)$ is continuous at x_0 for $x_0 \in [a,b]\setminus A$.

For $x_0 \in A$, we set $u(x) = \sum_{t\in A\setminus\{x_0\}}\widetilde{h_t}(x)$. Then $u(x)$ is continuous at x_0 and $h_f(x) = u(x) + \widetilde{h_{x_0}}(x)$. Hence,

$$\begin{aligned}h_f(x_0+0) &= u(x_0) + \left(f(x_0+0) - f(x_0-0)\right),\\ h_f(x_0-0) &= u(x_0),\\ h_f(x_0) &= u(x_0) + \left(f(x_0) - f(x_0-0)\right).\end{aligned} \tag{24}$$

Thus,

$$g(x_0+0) = g(x_0) = g(x_0-0) = f(x_0-0) - u(x_0), \tag{25}$$

from which we can deduce that g is continuous at x_0. Hence, $g(x) \in C[a,b]$.

Since $g(x) \in C[a,b] \cap BV_p[a,b]$, it follows from Lemma 5 that $g(x)$ is a constant c. Thus $f(x) = h_f(x) + c = \sum_{t\in A}\widetilde{h_t}(x) + c$. The proof of Theorem 3 is complete. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Simultaneous Approximation for Generalized Srivastava-Gupta Operators

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We introduce a new Stancu type generalization of Srivastava-Gupta operators to approximate integrable functions on the interval $(0,\infty)$ and estimate the rate of convergence for functions having derivatives of bounded variation. Also we present simultenaous approximation by new operators in the end of the paper.

1. Introduction

To approximate integrable functions on the interval $(0,\infty)$, Srivastava and Gupta [1] introduced a general sequence of linear positive operators $G_{n,c}$ as follows:

$$G_{n,c}(f;x) = n\sum_{k=1}^{\infty} p_{n,k}(x;c)\int_0^{\infty} p_{n+c,k-1}(t;c)\, f(t)\,dt + p_{n,0}(x;c)\, f(0), \tag{1}$$

for a function $f \in H_\alpha(0,\infty)$, where $H_\alpha(0,\infty)$ $(\alpha \geq 0)$ is the class of locally integrable functions defined on $(0,\infty)$ and satisfying the growth condition

$$\left|f(t)\right| \leq Mt^{\alpha} \quad (M > 0;\ \alpha \geq 0;\ t \longrightarrow \infty), \tag{2}$$

$$p_{n,k}(x;c) = \frac{(-x)^k}{k!}\phi_{n,c}^{(k)}(x), \tag{3}$$

$$\phi_{n,c}(x) = \begin{cases} e^{-nx}, & c = 0 \\ (1+cx)^{-n/c}, & c \in \mathbb{N} := \{1,2,3,\ldots\}. \end{cases} \tag{4}$$

The general sequence of operators $G_{n,c}$ has many interesting properties in approximation theory, which is an interesting area of research in the present era, and several researchers have studied these operators; we can mention some important studies on these operators (see [1–3]). In [4], author introduced King and Stancu type generalization of Srivastava-Gupta operators and presented some direct results. Also, Verma and Agrawal [5] introduced a new generalization of Srivastava-Gupta operators and studied the rate of convergence for the functions having the derivatives of bounded variation (BV). The rate of convergence for the functions having the derivatives of (BV) is an active area of research and many researchers studied this direction. We refer the readers to [6–10] and references therein.

Stancu [11, 12] introduced generalizations of Bernstein polynomials with one and two parameters (resp.), satisfying the condition $0 \leq \alpha \leq \beta$, as

$$s_n^{\alpha}(f,x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}\frac{\prod_{s=0}^{k-1}(x+\alpha s)\prod_{s=0}^{n-k-1}(1-x+\alpha s)}{\prod_{s=0}^{n-1}(1+\alpha s)}, \quad 0 \leq x \leq 1,$$

$$s_n^{\alpha,\beta}(f,x)=\sum_{k=0}^{n} f\left(\frac{k+\alpha}{n+\beta}\right)\binom{n}{k}x^k(1-x)^{n-k}$$
$$0\le x\le 1, \tag{5}$$

for any $f\in C[0,1]$. Stancu type generalization of approximation operators present better approach depending on α,β. Therefore, this kind of generalizations and their approximation properties have been studied intensively. We refer the readers to [13–17] and references therein. Mishra et al. [18, 19], V. N. Mishra, and L. N. Mishra [20] have established very interesting results on approximation properties of various functional classes using different types of positive linear summability operators.

The purpose of this paper is to introduce a new Stancu type generalization of the operators defined in [5] as

$$G_{n,r,c}^{(\alpha,\beta)}(f;x)=\frac{n\Gamma((n/c)+r)\,\Gamma((n/c)-r+1)}{\Gamma((n/c)+1)\,\Gamma(n/c)}\sum_{k=0}^{\infty}p_{n+rc,k}(x;c)\times\int_0^{\infty}p_{n-(r-1)c,k+r-1}(t;c)\,f\left(\frac{nt+\alpha}{n+\beta}\right)dt. \tag{6}$$

By the definition of operators, it is clear that $G_{n,r,c}^{(\alpha,\beta)}(f;x)$ is positive and linear. For $\alpha=\beta=0$, $G_{n,r,c}^{(0,0)}(f;x)$ reduces to operators defined in [5]. In this study we obtain the rate of convergence for the functions having the derivatives of bounded variation. Also, in the end of the paper, we study the simultaneous approximation.

2. Auxiliary Results

In order to prove our main results, we need the following lemmas.

Lemma 1. *Let the mth order moment be defined as*

$$\begin{aligned}U_{n,r,m}^{\alpha,\beta}(x)&=G_{n,c}^{(\alpha,\beta)}\left((t-x)^m;x\right)\\&=(n-rc)\sum_{k=0}^{\infty}p_{n+rc,k}(x;c)\\&\quad\times\int_0^{\infty}p_{n-(r-1)c,k+r-1}(t;c)\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt,\end{aligned} \tag{7}$$

where $n,m\in\mathbb{N}\cup\{0\}$, and then, for $n>(m+r+1)c$, we have the following recurrence relation:

$$\begin{aligned}&(n-(r+m+1)c)(n+\beta)U_{n,r,m+1}(x)\\&=nx(1+cx)\left[\left(U_{n,r,m}^{\alpha,\beta}(x)\right)'+mU_{n,r,m-1}^{\alpha,\beta}(x)\right]\\&\quad+U_{n,r,m}^{\alpha,\beta}(x)\\&\quad\times\left[(m+r+(n+rc)x)n+(\alpha-(n+\beta)x)\right.\\&\qquad\left.\times(n-(r+2m+1)c)\right]\\&\quad+U_{n,r,m-1}^{\alpha,\beta}(x)\\&\quad\times\left[\frac{cm(\alpha-(n+\beta)x)^2-mn(\alpha-(n+\beta)x)}{n+\beta}\right],\end{aligned}$$

$$\begin{aligned}U_{n,r,0}^{\alpha,\beta}(x)&=1,\\U_{n,r,1}^{\alpha,\beta}(x)&=\frac{\alpha-(n+\beta)x}{n+\beta}+\frac{n(r+(n+rc)x)}{(n-(r+1)c)(n+\beta)},\\U_{n,r,2}^{\alpha,\beta}(x)&=\frac{nx(1+cx)}{(n-(r+1)c)(n-(r+2)c)(n+\beta)^2}\\&\quad+\left(\frac{\alpha}{n+\beta}-x+\frac{n(r+(n+rc)x)}{(n-(r+1)c)(n+\beta)}\right)\\&\quad\times\frac{n(1+r+(n+rc)x)}{(n-(r+2)c)(n+\beta)}\\&\quad+\left(\frac{\alpha}{n+\beta}-x+\frac{n(r+(n+rc)x)}{(n-(r+1)c)(n+\beta)^2}\right)\\&\quad\times(\alpha-(n+\beta)x)\\&\quad+\frac{(\alpha-(n+\beta)x)(c(\alpha-(n+\beta)x)-n)}{(n-(r+2)c)(n+\beta)^2}.\end{aligned} \tag{8}$$

Furthermore, $U_{n,r,m}^{\alpha,\beta}(x)$ is polynomial of degree m in x and

$$U_{n,r,m}^{\alpha,\beta}(x)=O\left((n+\beta)^{-[(m+1)/2]}\right). \tag{9}$$

Proof. By definition of $U_{n,r,m}^{\alpha,\beta}(x)$, taking the derivative of $U_{n,r,m}^{\alpha,\beta}(x)$, we get

$$\begin{aligned}&\left(U_{n,r,m}^{\alpha,\beta}(x)\right)'\\&=-(n-rc)m\sum_{k=0}^{\infty}p_{n+rc,k}(x;c)\\&\quad\times\int_0^{\infty}p_{n-(r-1)c,k+r-1}(t;c)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{m-1}dt\\&\quad+(n-rc)\sum_{k=0}^{\infty}p'_{n+rc,k}(x;c)\\&\quad\times\int_0^{\infty}p_{n-(r-1)c,k+r-1}(t;c)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{m}dt\\&=-mU_{n,r,m-1}^{\alpha,\beta}(x)+(n-rc)\sum_{k=0}^{\infty}p'_{n+rc,k}(x;c)\\&\quad\times\int_0^{\infty}p_{n-(r-1)c,k+r-1}(t;c)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{m}dt.\end{aligned} \tag{10}$$

Hence, using the identity

$$x(1+cx)\,p'_{n+rc,k}(x;c)=(k-(n+rc)x)\,p_{n+rc,k}(x;c) \tag{11}$$

we have

$$\begin{aligned}
&x(1+cx)\left[\left(U_{n,r,m}^{\alpha,\beta}(x)\right)' + mU_{n,r,m-1}^{\alpha,\beta}(x)\right]\\
&= (n-rc)\sum_{k=0}^{\infty}(k-(n+rc)x)\,p_{n+rc,k}(x;c)\\
&\quad\times\int_0^{\infty}p_{n-(r-1)c,k+r-1}(t;c)\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt\\
&= (n-rc)\sum_{k=0}^{\infty}k p_{n+rc,k}(x;c)\\
&\quad\times\int_0^{\infty}p_{n-(r-1)c,k+r-1}(t;c)\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt\\
&\quad-(n+rc)\,xU_{n,r,m}^{\alpha,\beta}(x)\\
&= I-(n+rc)\,xU_{n,r,m}^{\alpha,\beta}(x).
\end{aligned} \tag{12}$$

We can write I as

$$\begin{aligned}
I &= \Bigg[(n-rc)\sum_{k=0}^{\infty}p_{n+rc,k}(x;c)\\
&\quad\times\int_0^{\infty}[k+r-1-(n-(r-1)c)t]\,p_{n-(r-1)c,k+r-1}\\
&\qquad\times(t;c)\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt\\
&\quad+(n-rc)(n-(r-1)c)\sum_{k=0}^{\infty}p_{n+rc,k}(x;c)\\
&\quad\times\int_0^{\infty}p_{n-(r-1)c,k+r-1}(t;c)\,t\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt\\
&\quad-(r-1)(n-rc)\sum_{k=0}^{\infty}p_{n+rc,k}(x;c)\\
&\quad\times\int_0^{\infty}p_{n-(r-1)c,k+r-1}(t;c)\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt\Bigg]\\
&= I_1+I_2-(r-1)\,U_{n,r,m}^{\alpha,\beta}(x).
\end{aligned} \tag{13}$$

To estimate I_2 using $t=((n+\beta)/n)[(((nt+\alpha)/(n+\beta))-x)-((\alpha/(n+\beta))-x)]$, we have

$$\begin{aligned}
I_2 &= \frac{(n-(r-1)c)(n+\beta)}{n}\\
&\quad\times\Bigg[(n-rc)\sum_{k=0}^{\infty}p_{n+rc,k}(x;c)\\
&\qquad\times\int_0^{\infty}p_{n-(r-1)c,k+r-1}(t;c)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{m+1} dt\\
&\qquad-\left(\frac{\alpha}{n+\beta}-x\right)(n-rc)\sum_{k=0}^{\infty}p_{n+rc,k}(x;c)\\
&\qquad\times\int_0^{\infty}p_{n-(r-1)c,k+r-1}(t;c)\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt\Bigg],
\end{aligned}$$

$$\begin{aligned}
I_2 &= \frac{(n-(r-1)c)(n+\beta)}{n}\\
&\quad\times\Bigg[(n-rc)\sum_{k=0}^{\infty}p_{n+rc,k}(x;c)\\
&\qquad\times\int_0^{\infty}p_{n-(r-1)c,k+r-1}(t;c)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{m+1} dt\\
&\qquad-\left(\frac{\alpha}{n+\beta}-x\right)\\
&\qquad\times\Bigg((n-rc)\sum_{k=0}^{\infty}p_{n+rc,k}(x;c)\\
&\qquad\quad\times\int_0^{\infty}p_{n-(r-1)c,k+r-1}(t;c)\\
&\qquad\qquad\times\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt\Bigg)\Bigg]\\
&= \frac{(n-(r-1)c)(n+\beta)}{n}\\
&\quad\times\left[U_{n,r,m+1}^{\alpha,\beta}(x)-\left(\frac{\alpha}{n+\beta}-x\right)U_{n,r,m}^{\alpha,\beta}(x)\right].
\end{aligned} \tag{14}$$

Next to estimate I_1 using the equality

$$\begin{aligned}
&t(1+ct)\,p'_{n-(r-1)c,k+r-1}(t;c)\\
&\quad= [(k+r-1)-(n-(r-1)c)t]\,p_{n-(r-1)c,k+r-1}(t;c),
\end{aligned} \tag{15}$$

we have

$$\begin{aligned}
I_1 &= (n-rc)\sum_{k=0}^{\infty}p_{n+rc,k}(x;c)\\
&\qquad\times\int_0^{\infty}p'_{n-(r-1)c,k+r-1}(t;c)\,t\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt\\
&\quad+c(n-rc)\sum_{k=0}^{\infty}p_{n+rc,k}(x;c)\\
&\qquad\times\int_0^{\infty}p'_{n-(r-1)c,k+r-1}(t;c)\,t^2\left(\frac{nt+\alpha}{n+\beta}-x\right)^m dt\\
&= \mathcal{J}_1+\mathcal{J}_2.
\end{aligned} \tag{16}$$

Putting $t = ((n+\beta)/n)[(((nt+\alpha)/(n+\beta))-x)-((\alpha/(n+\beta))-x)]$, we get

$$\begin{aligned}\mathcal{J}_1 &= \frac{n+\beta}{n} \\ &\times \left[(n-rc)\sum_{k=0}^{\infty} p_{n+rc,k}(x;c) \right. \\ &\quad \times \int_0^{\infty} p'_{n-(r-1)c,k+r-1}(t;c)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{m+1} dt \\ &\quad - \left(\frac{\alpha}{n+\beta}-x\right)(n-rc)\sum_{k=0}^{\infty} p_{n+rc,k}(x;c) \\ &\quad \left. \times \int_0^{\infty} p'_{n-(r-1)c,k+r-1}(t;c)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{m} dt \right].\end{aligned} \tag{17}$$

Now integrating by parts, we get

$$\begin{aligned}\mathcal{J}_1 &= -(m+1)(n-rc)\sum_{k=0}^{\infty} p_{n+rc,k}(x;c) \\ &\quad \times \int_0^{\infty} p_{n-(r-1)c,k+r-1}(t;c)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{m} dt \\ &\quad + m\left(\frac{\alpha}{n+\beta}-x\right)(n-rc)\sum_{k=0}^{\infty} p_{n+rc,k}(x;c) \\ &\quad \times \int_0^{\infty} p_{n-(r-1)c,k+r-1}(t;c)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{m-1} dt \\ &= -(m+1) \\ &\quad \times \left[(n-rc)\sum_{k=0}^{\infty} p_{n+rc,k}(x;c) \right. \\ &\quad \left. \times \int_0^{\infty} p_{n-(r-1)c,k+r-1}(t;c)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{m} dt \right] \\ &\quad + m\left(\frac{\alpha}{n+\beta}-x\right) \\ &\quad \times \left[(n-rc)\sum_{k=0}^{\infty} p_{n+rc,k}(x;c) \right. \\ &\quad \left. \times \int_0^{\infty} p_{n-(r-1)c,k+r-1}(t;c)\left(\frac{nt+\alpha}{n+\beta}-x\right)^{m-1} dt \right] \\ &= -(m+1)U^{\alpha,\beta}_{n,r,m}(x) \\ &\quad + m\left(\frac{\alpha}{n+\beta}-x\right)U^{\alpha,\beta}_{n,r,m-1}(x).\end{aligned} \tag{18}$$

Proceeding in a similar manner, we obtain the estimate $\mathcal{J}_2$ as

$$\begin{aligned}\mathcal{J}_2 &= -\frac{c(m+2)(n+\beta)}{n}U_{n,r,m+1}(x) \\ &\quad + \frac{2c(m+1)(n+\beta)}{n}\left(\frac{\alpha}{n+\beta}-x\right)U_{n,r,m}(x) \\ &\quad - \frac{cm(n+\beta)}{n}\left(\frac{\alpha}{n+\beta}-x\right)^2 U^{\alpha,\beta}_{n,r,m-1}(x).\end{aligned} \tag{19}$$

Combining the equations, we have

$$\begin{aligned}&(n-(r+m+1)c)(n+\beta)U^{\alpha,\beta}_{n,r,m+1}(x) \\ &\quad = nx(1+cx)\left[\left(U^{\alpha,\beta}_{n,r,m}(x)\right)' + mU^{\alpha,\beta}_{n,r,m-1}(x)\right] \\ &\quad + U^{\alpha,\beta}_{n,r,m}(x) \\ &\quad \times \left[(m+r+(n+rc)x)n + (\alpha-(n+\beta)x)\right. \\ &\quad \left. \times (n-(r+2m+1)c)\right] + U^{\alpha,\beta}_{n,r,m-1}(x) \\ &\quad \times \left[\frac{cm(\alpha-(n+\beta)x)^2 - mn(\alpha-(n+\beta)x)}{n+\beta}\right]\end{aligned} \tag{20}$$

which is the desired result.

Moments for $m = 0, 1, 2$ can be easily obtained by using the above recurrence relation. □

Remark 2. For sufficiently large n, $C > 2$, and $x \in (0,\infty)$, it can be seen from Lemma 1 that

$$U^{\alpha,\beta}_{n,r,2}(x) \le \frac{C\sigma^{\alpha,\beta}_{r,c}(x)}{n+\beta}, \tag{21}$$

where $\sigma^{\alpha,\beta}_{r,c}(x) = [x(1+cx) + x(\alpha+\beta x + r(1+cx))]$ for the convenient notation.

Remark 3. By using Cauchy-Schwarz inequality, it follows from Remark 2 that, for sufficiently large n, $C > 2$, and $x \in (0,\infty)$,

$$\begin{aligned}&(n-rc)\sum_{k=0}^{\infty} p_{n+rc,k}(x;c) \\ &\quad \times \int_0^{\infty} p_{n-(r-1)c,k+r-1}(t;c)\left|\frac{nt+\alpha}{n+\beta}-x\right| dt \\ &\le \left[U^{\alpha,\beta}_{n,r,2}(x)\right]^{1/2} \le \sqrt{\frac{C\sigma^{\alpha,\beta}_{r,c}(x)}{n+\beta}}.\end{aligned} \tag{22}$$

Lemma 4. *Let $x \in (0,\infty)$ and $C > 2$; then, for sufficiently large n, we have*

$$\lambda_{n,r}(x,y) = (n-rc)\sum_{k=0}^{\infty} p_{n+rc,k}(x;c) \times \int_0^y p_{n-(r-1)c,k+r-1}(t;c)\,dt \leq \frac{Cx(1+cx)}{n(x-y)^2}, \quad 0 \leq y \leq x,$$
$$1-\lambda_{n,r}(x,z) = (n-rc)\sum_{k=0}^{\infty} p_{n+rc,k}(x;c) \times \int_z^{\infty} p_{n-(r-1)c,k+r-1}(t;c)\,dt \leq \frac{Cx(1+cx)}{n(z-x)^2}, \quad x \leq z \leq \infty. \tag{23}$$

Proof. We give the proof for only first inequality, and the other is similar. Using Remark 2 with $\alpha = \beta = 0$, for sufficiently large n and $0 \leq y \leq x$ and $((nt+\alpha)/(n+\beta)) \leq t$, we have

$$\begin{aligned}\lambda_{n,r}(x,y) &= (n-rc)\sum_{k=0}^{\infty} p_{n+rc,k}(x;c) \times \int_0^y p_{n-(r-1)c,k+r-1}(t;c)\,dt \\ &\leq (n-rc)\sum_{k=0}^{\infty} p_{n+rc,k}(x;c) \times \int_0^y p_{n-(r-1)c,k+r-1}(t;c)\frac{(t-x)^2}{(y-x)^2}dt \\ &\leq \frac{Cx(1+cx)}{n(x-y)^2}.\end{aligned} \tag{24}$$

□

Lemma 5. *Suppose f is s times differentiable on $[0,\infty)$ such that $f^{(s-1)}(t) = O(t^{\alpha})$, for some integer $\alpha > 0$ as $t \to \infty$. Then, for any $r, s \in \mathbb{N}_0$, and $n > \max\{\alpha, r+s\}$, we have*

$$D^s G_{n,r,c}^{(\alpha,\beta)}(f;x) = \left(\frac{n}{n+\beta}\right)^s G_{n,r+s,c}^{(\alpha,\beta)}(f;x)(D^s f,x). \tag{25}$$

Proof. Using the identity

$$p'_{n,k}(x) = n\left[p_{n+c,k-1}(x,c) - p_{n+c,k}(x,c)\right]. \tag{26}$$

One can observe that, even in case $k=0$, the above identity is true with the condition $p_{n+c,\text{negative}}(x,c) = 0$. Thus, applying (26), we have

$$\begin{aligned}D\left[G_{n,r,c}^{(\alpha,\beta)}\right](f;x) &= \frac{n\Gamma((n/c)+r)\Gamma((n/c)-r+1)}{\Gamma((n/c)+1)\Gamma(n/c)}\sum_{k=0}^{\infty} Dp_{n+rc,k}(x;c) \times \int_0^{\infty} p_{n-(r-1)c,k+r-1}(t;c) f\left(\frac{nt+\alpha}{n+\beta}\right)dt \\ &= \frac{n\Gamma((n/c)+r)\Gamma((n/c)-r+1)}{\Gamma((n/c)+1)\Gamma(n/c)} \times \sum_{k=0}^{\infty}(n+rc)\left[p_{n+(r+1)c,k-1}(x,c) - p_{n+(r+1)c,k}(x,c)\right] \times \int_0^{\infty} p_{n-(r-1)c,k+r-1}(t;c) f\left(\frac{nt+\alpha}{n+\beta}\right)dt \\ &= \frac{n(n+rc)\Gamma((n/c)+r)\Gamma((n/c)-r+1)}{\Gamma((n/c)+1)\Gamma(n/c)} \times \sum_{k=0}^{\infty} p_{n+(r+1)c,k}(x,c) \times \int_0^{\infty}\left[p_{n-(r-1)c,k+r}(t;c) - p_{n-(r-1)c,k+r-1}(t;c)\right] \times f\left(\frac{nt+\alpha}{n+\beta}\right)dt \\ &= \frac{-n(n+rc)\Gamma((n/c)+r)\Gamma((n/c)-r+1)}{(n-rc)\Gamma((n/c)+1)\Gamma(n/c)} \times \sum_{k=0}^{\infty} p_{n+(r+1)c,k}(x,c) \times \int_0^{\infty} Dp_{n-rc,k+r}(t;c) f\left(\frac{nt+\alpha}{n+\beta}\right)dt \\ &= \frac{n^2\Gamma((n/c)+r+1)\Gamma((n/c)-r)}{(n+\beta)\Gamma((n/c)+1)\Gamma(n/c)} \times \sum_{k=0}^{\infty} p_{n+(r+1)c,k}(x,c) \times \int_0^{\infty} p_{n-rc,k+r}(t;c) Df\left(\frac{nt+\alpha}{n+\beta}\right)dt \\ &= \frac{n}{(n+\beta)}\left[G_{n,r+1,c}^{(\alpha,\beta)}\right](Df;x),\end{aligned} \tag{27}$$

which means that the identity is satisfied for $s = 1$. Let us suppose that the result holds for $s = m$; that is,

$$\begin{aligned}
D^m G_{n,r,c}^{(\alpha,\beta)}(f;x) &= \left(\frac{n}{n+\beta}\right)^m G_{n,r+m,c}^{(\alpha,\beta)}(f;x)(D^m f, x) \\
&= \left(\frac{n}{n+\beta}\right)^m \\
&\quad \times \frac{n\Gamma((n/c)+r+m)\Gamma((n/c)-r-m+1)}{\Gamma((n/c)+1)\Gamma(n/c)} \\
&\quad \times \sum_{k=0}^{\infty} p_{n+(r+m)c,k}(x;c) \\
&\quad \times \int_0^\infty p_{n-(r+m-1)c,k+r+m-1}(t;c) D^m f\left(\frac{nt+\alpha}{n+\beta}\right) dt.
\end{aligned} \tag{28}$$

Also, from (26) we can write

$$\begin{aligned}
D^{m+1} G_{n,r,c}^{(\alpha,\beta)}(f;x) &= \left(\frac{n}{n+\beta}\right)^m \\
&\quad \times \frac{n\Gamma((n/c)+r+m)\Gamma((n/c)-r-m+1)}{\Gamma((n/c)+1)\Gamma(n/c)} \\
&\quad \times \sum_{k=0}^{\infty} D p_{n+(r+m)c,k}(x;c) \\
&\quad \times \int_0^\infty p_{n-(r+m-1)c,k+r+m-1}(t;c) D^m f\left(\frac{nt+\alpha}{n+\beta}\right) dt \\
&= \left(\frac{n}{n+\beta}\right)^m \\
&\quad \times \frac{n\Gamma((n/c)+r+m)\Gamma((n/c)-r-m+1)}{\Gamma((n/c)+1)\Gamma(n/c)} \\
&\quad \times \sum_{k=0}^{\infty} (n+(r+m)c) \\
&\quad \times [p_{n+(r+m+1)c,k-1}(x,c) - p_{n+(r+m+1)c,k}(x,c)] \\
&\quad \times \int_0^\infty p_{n-(r+m-1)c,k+r+m-1}(t;c) D^m f\left(\frac{nt+\alpha}{n+\beta}\right) dt \\
&= \left(\frac{n}{n+\beta}\right)^m \\
&\quad \times \frac{cn\Gamma((n/c)+r+m+1)\Gamma((n/c)-r-m+1)}{\Gamma((n/c)+1)\Gamma(n/c)} \\
&\quad \times \sum_{k=0}^{\infty} p_{n+(r+m+1)c,k}(x;c) \\
&\quad \times \int_0^\infty [p_{n-(r+m-1)c,k+r+m}(t;c) \\
&\qquad - p_{n-(r+m-1)c,k+r+m-1}(t;c)] D^m f\left(\frac{nt+\alpha}{n+\beta}\right) dt \\
&= -\left(\frac{n}{n+\beta}\right)^m \\
&\quad \times \frac{cn\Gamma((n/c)+r+m+1)\Gamma((n/c)-r-m+1)}{\Gamma((n/c)+1)\Gamma(n/c)} \\
&\quad \times \sum_{k=0}^{\infty} p_{n+(r+m+1)c,k}(x;c) \\
&\quad \times \int_0^\infty \frac{D p_{n-(r+m)c,k+r+m}(t;c)}{n-(r+m-1)c} D^m f\left(\frac{nt+\alpha}{n+\beta}\right) dt
\end{aligned} \tag{29}$$

and, integrating by parts the last integral, we have

$$\begin{aligned}
D^{m+1} G_{n,r,c}^{(\alpha,\beta)}(f;x) &= \left(\frac{n}{n+\beta}\right)^{m+1} \\
&\quad \times \frac{n\Gamma((n/c)+r+m+1)\Gamma((n/c)-r-m)}{\Gamma((n/c)+1)\Gamma(n/c)} \\
&\quad \times \sum_{k=0}^{\infty} p_{n+(r+m+1)c,k}(x;c) \\
&\quad \times \int_0^\infty p_{n-(r+m)c,k+r+m}(t;c) D^{m+1} f\left(\frac{nt+\alpha}{n+\beta}\right) dt.
\end{aligned} \tag{30}$$

Hence we have

$$\begin{aligned}
D^{m+1} G_{n,r,c}^{(\alpha,\beta)}(f;x) = \left(\frac{n}{n+\beta}\right)^{m+1} \\
\times G_{n,r+m+1,c}^{(\alpha,\beta)}(f;x)\left(D^{m+1} f, x\right),
\end{aligned} \tag{31}$$

in which the result is true for $s = m+1$, and hence by mathematical induction the proof of the lemma is completed. □

3. Main Results

Throughout the paper by $DB_q(0,\infty)$ we denote the class of absolutely continuous functions f on $(0,\infty)$ (where q is a some positive integer) satisfying the conditions:

(i) $|f(t)| \le C_1 t^q$ and $C_1 > 0$,

(ii) the function f has the first derivative on the interval $(0,\infty)$ which coincide almost everywhere with a function which is of bounded variation on every finite subinterval of $(0,\infty)$. It can be observed that for all functions $f \in DB_q(0,\infty)$ we can have the representation

$$f(x) = f(c) + \int_c^x \psi(t)\,dt, \quad x \ge c \ge 0. \tag{32}$$

Theorem 6. *Let $f \in DB_q(0,\infty)$, $q>0$, and $x \in (0,\infty)$. Then, for $C>2$ and sufficiently large n, we have*

$$\begin{aligned}
&\left|\frac{(\Gamma(n/c))^2}{\Gamma((n/c)+r)\Gamma((n/c)-r)} G_{n,r,c}^{(\alpha,\beta)}(f;x) - f(x)\right| \\
&\quad\leq \frac{C(1+cx)}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-(x/k)}^{x+(x/k)}\left(f'_x(x)\right) + \frac{x}{\sqrt{n}}\bigvee_{x-(x/\sqrt{n})}^{x+(x/\sqrt{n})}\left(f'_x(x)\right) \\
&\quad + \frac{C(1+cx)}{nx}\left|f(2x)-f(x)-xf'(x^+)+|f(x)|\right| \\
&\quad + O(n^{-q}) + \left|f'(x^+)\right|\sqrt{\frac{Cx(1+cx)}{n}} \\
&\quad + \sqrt{\frac{C\sigma_{r,c}^{\alpha,\beta}(x)}{n+\beta}}\frac{\left|f'(x^+)-f'(x^-)\right|}{2} + \frac{\left|f'(x^+)+f'(x^-)\right|}{2} \\
&\quad \times\left(\frac{(\alpha-\beta x)(n-c(r+1)) + 2nrcx + nxc + nr}{(n-(r+1)c)(n+\beta)}\right),
\end{aligned} \tag{33}$$

where C is a constant which may be different on each occurrence.

Proof. Using the mean value theorem, we have

$$\begin{aligned}
&\frac{(\Gamma(n/c))^2}{\Gamma((n/c)+r)\Gamma((n/c)-r)} G_{n,r,c}^{(\alpha,\beta)}(f;x) - f(x) \\
&\quad = (n-rc)\sum_{k=0}^{\infty} p_{n+rc,k}(x;c) \\
&\quad\quad \times \int_0^\infty p_{n-(r-1)c,k+r-1}(t;c)\left[f\left(\frac{nt+\alpha}{n+\beta}\right) - f(x)\right]dt \\
&\quad = \int_0^\infty\left(\int_x^{(nt+\alpha)/(n+\beta)}(n-rc)\sum_{k=0}^{\infty}p_{n+rc,k}(x;c)\right. \\
&\quad\quad \left.\times\, p_{n-(r-1)c,k+r-1}(t;c)\, f'(u)\,du\right)dt.
\end{aligned} \tag{34}$$

Also, using the identity

$$\begin{aligned}
f'(u) &= \frac{f'(x^+)+f'(x^-)}{2} + \left(f'\right)_x(u) \\
&\quad + \frac{f'(x^+)-f'(x^-)}{2}\operatorname{sgn}(u-x) \\
&\quad + \left[f'(x) - \frac{f'(x^+)+f'(x^-)}{2}\right]\chi_x(u),
\end{aligned} \tag{35}$$

where

$$\chi_x(u) = \begin{cases} 1, & u = x; \\ 0, & u \neq x, \end{cases} \tag{36}$$

we have

$$\begin{aligned}
&(n-rc)\sum_{k=0}^{\infty}p_{n+rc,k}(x;c) \\
&\quad\times\int_0^\infty\left(\int_x^t\left[f'(x) - \frac{f'(x^+)+f'(x^-)}{2}\right]\chi_x(u)\,du\right) \\
&\quad\times\, p_{n-(r-1)c,k+r-1}(t;c)\,dt = 0.
\end{aligned} \tag{37}$$

Thus, using the above identities, we can write

$$\begin{aligned}
&\left|\frac{(\Gamma(n/c))^2}{\Gamma((n/c)+r)\Gamma((n/c)-r)} G_{n,r,c}^{(\alpha,\beta)}(f;x) - f(x)\right| \\
&\quad\leq \left|\int_0^\infty\left(\int_x^t (n-rc)\sum_{k=0}^{\infty}p_{n+rc,k}(x;c)\right.\right. \\
&\quad\quad\times\, p_{n-(r-1)c,k+r-1}(t;c) \\
&\quad\quad\left.\left.\times\left[\frac{f'(x^+)+f'(x^-)}{2} + \left(f'\right)_x(u)\right]du\right)dt\right| \\
&\quad + \left|\int_0^\infty\left(\int_x^t (n-rc)\sum_{k=0}^{\infty}p_{n+rc,k}(x;c)\right.\right. \\
&\quad\quad\times\, p_{n-(r-1)c,k+r-1}(t;c) \\
&\quad\quad\left.\left.\times\left[\frac{f'(x^+)-f'(x^-)}{2}\operatorname{sgn}(u-x)\right]du\right)dt\right|.
\end{aligned} \tag{38}$$

Also, it can be verified that

$$\begin{aligned}
&\left|\int_0^\infty\left(\int_x^t (n-rc)\sum_{k=0}^{\infty}p_{n+rc,k}(x;c)\right.\right. \\
&\quad\times\, p_{n-(r-1)c,k+r-1}(t;c) \\
&\quad\left.\left.\times\left[\frac{f'(x^+)-f'(x^-)}{2}\operatorname{sgn}(u-x)\right]du\right)dt\right| \\
&\quad\leq \frac{\left|f'(x^+)-f'(x^-)\right|}{2}\left[U_{n,r,2}(x)\right]^{1/2},
\end{aligned} \tag{39}$$

$$\begin{aligned}
&\left|\int_0^\infty\left(\int_x^t (n-rc)\sum_{k=0}^{\infty}p_{n+rc,k}(x;c)\right.\right. \\
&\quad\times\, p_{n-(r-1)c,k+r-1}(t;c) \\
&\quad\left.\left.\times\left[\frac{f'(x^+)+f'(x^-)}{2}\right]du\right)dt\right| \\
&\quad\leq \frac{\left|f'(x^+)+f'(x^-)\right|}{2}U_{n,r,1}(x).
\end{aligned} \tag{40}$$

Combining (38)–(40), we get

$$\left|\frac{(\Gamma(n/c))^2}{\Gamma((n/c)+r)\Gamma((n/c)-r)}G_{n,r,c}^{(\alpha,\beta)}(f;x)-f(x)\right|$$
$$\leq\left|\int_x^\infty\left(\int_x^t f_x'(u)\,du\right)(n-rc)\times\sum_{k=0}^\infty p_{n+rc,k}(x;c)\,p_{n-(r-1)c,k+r-1}(t;c)\,dt\right.$$
$$\left.+\int_0^x\left(\int_x^t f_x'(u)\,du\right)(n-rc)\times\sum_{k=0}^\infty p_{n+rc,k}(x;c)\,p_{n-(r-1)c,k+r-1}(t;c)\,dt\right|\quad(41)$$
$$+\frac{\left|f'(x^+)-f'(x^-)\right|}{2}\left[U_{n,r,2}(x)\right]^{1/2}+\frac{\left|f'(x^+)+f'(x^-)\right|}{2}U_{n,r,1}(x)$$
$$=\left|A_{n,r}^{\alpha,\beta}(f,x)+B_{n,r}^{\alpha,\beta}(f,x)\right|+\frac{\left|f'(x^+)-f'(x^-)\right|}{2}\times\left[U_{n,r,2}(x)\right]^{1/2}+\frac{\left|f'(x^+)+f'(x^-)\right|}{2}U_{n,r,1}(x).$$

Applying Remark 2 and Lemma 1 in above equation, we have

$$\left|\frac{(\Gamma(n/c))^2}{\Gamma((n/c)+r)\Gamma((n/c)-r)}G_{n,r,c}^{(\alpha,\beta)}(f;x)-f(x)\right|$$
$$\leq\left|A_{n,r}^{\alpha,\beta}(f,x)\right|+\left|B_{n,r}^{\alpha,\beta}(f,x)\right|+\sqrt{\frac{C\sigma_{r,c}^{\alpha,\beta}(x)}{n+\beta}}\frac{\left|f'(x^+)-f'(x^-)\right|}{2}$$
$$+\frac{\left|f'(x^+)+f'(x^-)\right|}{2}\times\left(\frac{(\alpha-\beta x)(n-c(r+1))+2nrcx+nxc+nr}{(n-(r+1)c)(n+\beta)}\right).\quad(42)$$

In order to complete the proof of the theorem, it suffices to estimate the terms $A_{n,r}^{\alpha,\beta}(f,x)$ and $B_{n,r}^{\alpha,\beta}(f,x)$. Applying Remark 2 with $\alpha=\beta=0$, we get

$$\left|A_{n,r}^{\alpha,\beta}(f,x)\right|$$
$$=\left|\int_x^\infty\left(\int_x^t f_x'(u)\,du\right)(n-rc)\times\sum_{k=0}^\infty p_{n+rc,k}(x;c)\,p_{n-(r-1)c,k+r-1}(t;c)\,dt\right|$$
$$\leq\left|(n-rc)\sum_{k=0}^\infty p_{n+rc,k}(x;c)\times\int_{2x}^\infty\left(f(t)-f(x)\right)p_{n-(r-1)c,k+r-1}(t;c)\,dt\right|+\left|f'(x^+)\right|$$
$$\times\left|(n-rc)\sum_{k=0}^\infty p_{n+rc,k}(x;c)\times\int_x^{2x}p_{n-(r-1)c,k+r-1}(t;c)(t-x)\,dt\right|$$
$$+\left|\int_x^{2x}f_x'(u)\,du\right|\left|1-\lambda_{n,r}(x,2x)\right|+\left|\int_x^{2x}\left|f_x'(t)\right|\cdot\left|1-\lambda_{n,r}(x,t)\right|dt\right|$$
$$\leq(n-rc)\sum_{k=0}^\infty p_{n+rc,k}(x;c)\times\int_{2x}^\infty p_{n-(r-1)c,k+r-1}(t;c)\,C_1t^{2q}dt$$
$$+\frac{|f(x)|}{x^2}(n-rc)\sum_{k=0}^\infty p_{n+rc,k}(x;c)\times\int_0^\infty p_{n-(r-1)c,k+r-1}(t;c)(t-x)^2\,dt$$
$$+\left|f'(x^+)\right|(n-rc)\sum_{k=0}^\infty p_{n+rc,k}(x;c)\times\int_{2x}^\infty p_{n-(r-1)c,k+r-1}(t;c)\,|t-x|\,dt$$
$$+\frac{Cx(1+cx)}{nx^2}\left|f(2x)-(x)-xf'(x^+)\right|+\frac{C(1+cx)}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_x^{x+(x/k)}\left(f_x'(x)\right)+\frac{x}{\sqrt{n}}\bigvee_x^{x+(x/\sqrt{n})}\left(f_x'(x)\right).\quad(43)$$

For estimating the integral

$$(n-rc)\sum_{k=0}^\infty p_{n+rc,k}(x;c)\int_{2x}^\infty p_{n-(r-1)c,k+r-1}(t;c)\,C_1t^{2q}dt,\quad(44)$$

we proceed as follows: since $t \geq 2x$ implies that $t \leq 2(t-x)$ so by Schwarz inequality and Lemma 1,

$$\begin{aligned}
&(n-rc)\sum_{k=0}^{\infty}p_{n+rc,k}(x;c)\int_{2x}^{\infty}p_{n-(r-1)c,k+r-1}(t;c)\,C_1t^{2q}dt\\
&\leq C_12^q(n-rc)\sum_{k=0}^{\infty}p_{n+rc,k}(x;c)\\
&\quad\times\int_0^{\infty}p_{n-(r-1)c,k+r-1}(t;c)\,C_1(t-x)^{2q}dt\\
&\leq C_12^qU_{n,r,2q}(x)=O(n^{-q})\quad\text{as } n\longrightarrow\infty.
\end{aligned}\tag{45}$$

By using Hölder's inequality and Remark 2 ($\alpha=\beta=0$), we get the estimate as follows:

$$\begin{aligned}
&\left|f'(x^+)\right|(n-rc)\sum_{k=0}^{\infty}p_{n+rc,k}(x;c)\\
&\quad\times\int_{2x}^{\infty}p_{n-(r-1)c,k+r-1}(t;c)\,|t-x|\,dt\\
&\leq\left|f'(x^+)\right|\\
&\quad\times\left((n-rc)\sum_{k=0}^{\infty}p_{n+rc,k}(x;c)\right.\\
&\quad\quad\left.\times\int_0^{\infty}p_{n-(r-1)c,k+r-1}(t;c)(t-x)^2dt\right)^{1/2}\\
&\leq\left|f'(x^+)\right|\sqrt{\frac{Cx(1+cx)}{n}}.
\end{aligned}\tag{46}$$

Collecting the estimates from (43)–(46), we obtain

$$\begin{aligned}
\left|A_{n,r}^{\alpha,\beta}(f,x)\right|&\leq O(n^{-q})+\left|f'(x^+)\right|\\
&\quad\times\sqrt{\frac{Cx(1+cx)}{n}}+\frac{C(1+cx)}{nx}\\
&\quad\times\left|f(2x)-f(x)-xf'(x^+)+\left|f(x)\right|\right|\\
&\quad+\frac{C(1+cx)}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x}^{x+(x/k)}\left(f_x'(x)\right)\\
&\quad+\frac{x}{\sqrt{n}}\bigvee_{x}^{x+(x/\sqrt{n})}\left(f_x'(x)\right).
\end{aligned}\tag{47}$$

On the other hand, to estimate $B_{n,r}^{\alpha,\beta}(f,x)$ by applying Lemma 4 with $y=x-(x/\sqrt{n})$ and integration by parts, we have

$$\begin{aligned}
&\left|B_{n,r}^{\alpha,\beta}(f,x)\right|\\
&=\left|\int_0^x\int_x^t f_x'(u)\,d_t\lambda_{n,r}(x,t)\right|\\
&\leq\left(\int_0^y+\int_y^x\right)\left|f_x'(t)\right|\left|\lambda_{n,r}(x,t)\right|dt\\
&\leq\frac{Cx(1+cx)}{n}\int_0^y\bigvee_t^x\left(\left(f'\right)_x\right)\frac{1}{(x-t)^2}dt\\
&\quad+\int_y^x\bigvee_t^x\left(\left(f'\right)_x\right)dt\\
&=\frac{Cx(1+cx)}{n}\int_1^{\sqrt{n}}\bigvee_{(x-(x/u))}^{x}\left(\left(f'\right)_x\right)du\\
&\quad+\frac{x}{\sqrt{n}}\bigvee_{x-(x/\sqrt{n})}^{x}\left(\left(f'\right)_x\right)\\
&\leq\frac{Cx(1+cx)}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-(x/k)}^{x}\left(\left(f'\right)_x\right)\\
&\quad+\frac{x}{\sqrt{n}}\bigvee_{x-(x/\sqrt{n})}^{x}\left(\left(f'\right)_x\right),
\end{aligned}\tag{48}$$

where $u=(x/(x-t))$.

Combining (41), (47), and (48), we get the desired result. □

Corollary 7. *Let $f^{(s)}\in DB_q(0,\infty)$, $q>0$, and $x\in(0,\infty)$. Then, for $C>2$ and n sufficiently large, one has*

$$\begin{aligned}
&\left|\frac{(\Gamma(n/c))^2}{\Gamma((n/c)+r)\,\Gamma((n/c)-r)}\left(\frac{n+\beta}{n}\right)^s\right.\\
&\quad\left.\times D^sG_{n,r,c}^{(\alpha,\beta)}(f;x)-f^s(x)\right|\\
&\quad\leq\frac{C(1+cx)}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-(x/k)}^{x+(x/k)}\left(\left(D^{s+1}f\right)_x\right)\\
&\quad\quad+\frac{x}{\sqrt{n}}\bigvee_{x-(x/\sqrt{n})}^{x+(x/\sqrt{n})}\left(\left(D^{s+1}f\right)_x\right)+\frac{C(1+cx)}{nx}\\
&\quad\quad\times\left|f(2x)-(x)-xD^{s+1}f(x^+)+\left|f(x)\right|\right|\\
&\quad\quad+O(n^{-q})+\left|D^{s+1}f(x^+)\right|\sqrt{\frac{Cx(1+cx)}{n}}
\end{aligned}$$

$$+\sqrt{\frac{C\sigma_{r,c}^{\alpha,\beta}(x)}{n+\beta}}\frac{\left|D^{s+1}f(x^{+})-D^{s+1}f(x^{-})\right|}{2}$$
$$+\frac{\left|D^{s+1}f(x^{+})+D^{s+1}f(x^{-})\right|}{2}$$
$$\times\left(\frac{(\alpha-\beta x)(n-c(r+1))+2nrcx+nxc+nr}{(n-(r+1)c)(n+\beta)}\right), \tag{49}$$

where $\bigvee_a^b f_x$ *denotes the total variation of* f_x *on* $[a,b]$ *and the auxiliary function* $D^{s+1}f_x$ *is defined by*

$$D^{s+1}f_x(t)=\begin{cases}D^{s+1}f(t)-D^{s+1}f(x^{-}), & 0\le t\le x\\ 0, & t=x\\ D^{s+1}f(t)-D^{s+1}f(x^{+}), & x<t<\infty.\end{cases} \tag{50}$$

4. Conclusion

The results of our lemmas and theorems are more general rather than the results of any other previously proved lemmas and theorems, which will enrich the literature of applications of quantum calculus in operator theory and convergence estimates in the theory of approximations by positive linear operators. The researchers and professionals working or intend to work in areas of mathematical analysis and its applications will find this research paper to be quite useful. Consequently, the results so established may be found useful in several interesting situations appearing in the literature on mathematical analysis, pure and applied mathematics, and mathematical physics. Some interesting applications of the positive approximation linear operators can be seen in [21–24].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Herz-Morrey-Hardy Spaces with Variable Exponents and Their Applications

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The authors introduce Herz-Morrey-Hardy spaces with variable exponents and establish the characterization of these spaces in terms of atom. Applying the characterization, the authors obtain the boundedness of some singular integral operators on these spaces.

1. Introduction

The Herz spaces go back to Beurling and Herz; see [1, 2]. Firstly, they attracted a lot of authors' attention because they could be used to characterize Fourier multipliers for Hardy spaces; see [3]. Then, in 1989 Chen and Lau in [4] and García-Cuerva in [5] introduced now called nonhomogeneous Herz type Hardy spaces. They found that these Herz type Hardy spaces have a decomposition via central atoms. After that, Lu et al. considered homogeuous Herz type Hardy spaces and also obtained a central atomic decomposition for them. Since then Herz type spaces have been studied extensively; see monograph [6] for details. Meanwhile, in the last three decades, the interest of the study for variable exponent spaces has been increasing year by year. Variable exponent spaces have many applications: in electrorheological fluid [7], in differential equations [8] and references therein, and in image restoration [9–11], for instance. Indeed, many spaces with variable exponents appeared, such as: Lebesgue spaces, Sobolev spaces and Bessel potential spaces with variable exponent, Besov and Triebel-Lizorkin spaces with variable exponents, Morrey spaces with variable exponents, Campanato spaces with variable exponent, and Hardy spaces with variable exponent; see [12–23] and references therein. Moreover, the atomic, molecular, and wavelet decompositions of variable exponent Besov and Triebel-Lizorkin spaces were given in [13, 14, 20, 21, 24]. The duality and reflexivity of spaces $B^s_{p(\cdot),q}$ and $F^s_{p(\cdot),q}$ were discussed in [25]. The atomic and molecular decompositions of Hardy spaces with variable exponent and their applications for the boundedness of singular integral operators were obtained in [22, 26].

Recently, as a generalization of Lebesgue spaces with variable exponent, Herz spaces with variable exponents are introduced. In fact, in 2010 Izuki proved the boundedness of sublinear operators on Herz space with variable exponents $\dot{K}^{\alpha,q}_{p(\cdot)}$ and $K^{\alpha,q}_{p(\cdot)}$ in [27]. In 2012, Almeida and Drihem obtained boundedness results for a wide class of classical operators on Herz spaces $\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}$ and $K^{\alpha(\cdot),q}_{p(\cdot)}$ in [28]. Shi and the first author in [29] considered Herz type Besov and Triebel-Lizorkin spaces with one variable exponent. Then Dong and first author in [30] established the boundedness of vector-valued Hardy-Littlewood maximal operator in spaces $\dot{K}^{\alpha(\cdot),q}_{p(\cdot)}$ and $K^{\alpha(\cdot),q}_{p(\cdot)}$ and gave characterizations of Herz type Besov and Triebel-Lizorkin spaces with variable exponents by maximal functions. In [31], Wang and Liu introduced a certain Herz type Hardy spaces with variable exponent. In 2013, Samko introduced Herz spaces with three variable exponents and obtained the boundedness of Hardy-Littlewood maximal operator on them. In [32–34], the boundedness of singular integrals and their commutators of BMO functions are discussed in Herz Morrey spaces with variable exponents. The Herz-Morrey spaces with constants were considered in [35, 36]; however, there is no theory of Herz-Morrey type Hardy spaces. In this paper we fill the gap and introduce Herz-Morrey-Hardy spaces with variable exponents.

The outline of the paper is as follows. In the rest of the section we will recall some definitions and notions. In Section 2, we will define the Herz-Morrey-Hardy spaces with variable exponents $HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}$ and $HMK^{\alpha(\cdot),q}_{p(\cdot),\lambda}$ and give their atomic characterization. In Section 3, we obtain that certain singular integral operators are bounded from Herz-Morrey-Hardy spaces with variable exponents into Herz-Morrey spaces with variable exponents as an application of the atomic characterization.

Throughout this paper $|E|$ denotes the Lebesgue measure and χ_E the characteristic function for a measurable set $E \subset \mathbb{R}^n$. For a multi-index $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$, we denote $|\beta| = \beta_1 + \beta_2 + \cdots + \beta_n$. We also use the notation $a \lesssim b$ if there exists a constant $c > 0$ such that $a \leqslant cb$. If $a \lesssim b$ and $b \lesssim a$ we will write $a \approx b$. Finally we claim that C is always a positive constant but it may change from line to line.

Definition 1. Let E be a measurable set in $\mathbb{R}^n$ with $|E| > 0$. Let $p(\cdot) : E \to [1, \infty)$ be a measurable function. Denote

$$L^{p(\cdot)}(E) := \left\{ f \text{ is measurable on } E : \rho_{p(\cdot)}\left(\frac{f}{\lambda}\right) < \infty \right. \\ \left. \text{for some constant } \lambda > 0 \right\}, \tag{1}$$

where $\rho_{p(\cdot)}(f) := \int_E |f(x)|^{p(x)} dx$, and

$$\|f\|_{L^{p(\cdot)}(E)} := \inf\left\{\lambda > 0 : \rho_{p(\cdot)}\left(\frac{f}{\lambda}\right) \leqslant 1\right\}. \tag{2}$$

Then $L^{p(\cdot)}(E)$ is a Banach space with the norm $\|\cdot\|_{L^{p(\cdot)}(E)}$.

Let $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ be the collection of all locally integrable functions on $\mathbb{R}^n$. Given a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the Hardy-Littlewood maximal operator $\mathscr{M}$ is defined by

$$\mathscr{M}f(x) := \sup_{r>0} r^{-n} \int_{B(x,r)} |f(y)| \, dy, \quad \forall x \in \mathbb{R}^n, \tag{3}$$

where and what follows $B(x, r) := \{y \in \mathbb{R}^n : |x - y| < r\}$. We also use the following notation: $p_- := \operatorname{ess\,inf}\{p(x) : x \in \mathbb{R}^n\}$ and $p_+ := \operatorname{ess\,sup}\{p(x) : x \in \mathbb{R}^n\}$. The set $\mathscr{P}(\mathbb{R}^n)$ consists of all $p(\cdot)$ satisfying $p_- > 1$ and $p_+ < \infty$. $\mathscr{B}(\mathbb{R}^n)$ is the set of $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ satisfying the condition that $\mathscr{M}$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. It is well known that if $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ satisfies the following global log-Hölder continuous then $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$; see [37–42].

Definition 2. Let $\alpha(\cdot)$ be a real-valued function on $\mathbb{R}^n$. If there exists $C > 0$ such that, for all $x, y \in \mathbb{R}^n$, $|x - y| < 1/2$,

$$|\alpha(x) - \alpha(y)| \leqslant \frac{C}{-\log(|x - y|)}, \tag{4}$$

then $\alpha(\cdot)$ is said local log-Hölder continuous on $\mathbb{R}^n$.

If there exists $C > 0$, such that for all $x \in \mathbb{R}^n$,

$$|\alpha(x) - \alpha(0)| \leqslant \frac{C}{\log(e + 1/|x|)}, \tag{5}$$

then $\alpha(\cdot)$ is said log-Hölder continuous at origin.

If there exist $\alpha_\infty \in \mathbb{R}$ and a constant $C > 0$ such that for all $x \in \mathbb{R}^n$

$$|\alpha(x) - \alpha_\infty| \leqslant \frac{C}{\log(e + |x|)}, \tag{6}$$

then $\alpha(\cdot)$ is said log-Hölder continuous at infinity.

If $\alpha(\cdot)$ is both local log-Hölder continuous and log-Hölder continuous at infinity, then $\alpha(\cdot)$ is said global log-Hölder continuous.

The sets of log-Hölder continuous functions, log-Hölder continuous functions at origin, log-Hölder continuous functions at infinity, global log-Hölder continuous are denoted by $\mathscr{P}^{\log}_{\mathrm{loc}}(\mathbb{R}^n)$, $\mathscr{P}^{\log}_0(\mathbb{R}^n)$, $\mathscr{P}^{\log}_\infty(\mathbb{R}^n)$, and $\mathscr{P}^{\log}(\mathbb{R}^n)$, respectively.

We denote $p'(\cdot)$ by the conjugate exponent to $p(\cdot)$, which means $p'(\cdot) = p(\cdot)/(p(\cdot) - 1)$. It is also well known that $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$ is equivalent to $p'(\cdot) \in \mathscr{B}(\mathbb{R}^n)$; see [39].

For simplicity, we denote $L^{p(\cdot)}(\mathbb{R}^n)$ by $L^{p(\cdot)}$. We will use the following results.

Lemma 3 (see [43]). *Let $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$. If $f \in L^{p(\cdot)}$ and $g \in L^{p'(\cdot)}$, then fg is integrable on $\mathbb{R}^n$ and*

$$\int_{\mathbb{R}^n} |f(x) g(x)| \, dx \leqslant r_p \|f\|_{L^{p(\cdot)}} \|g\|_{L^{p'(\cdot)}}, \tag{7}$$

where $r_p = 1 + 1/p^- - 1/p^+$.

Lemma 4 (see [27]). *Let $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$. Then there exist $0 < \delta_1, \delta_2 < 1$, and a positive constant C depending only on $p(\cdot)$ and n such that for all balls B in $\mathbb{R}^n$ and all measurable subsets $S \subset B$,*

$$\frac{\|\chi_B\|_{L^{p(\cdot)}}}{\|\chi_S\|_{L^{p(\cdot)}}} \leqslant C\frac{|B|}{|S|}, \qquad \frac{\|\chi_S\|_{L^{p(\cdot)}}}{\|\chi_B\|_{L^{p(\cdot)}}} \leqslant C\left(\frac{|S|}{|B|}\right)^{\delta_1}, \\ \frac{\|\chi_S\|_{L^{p'(\cdot)}}}{\|\chi_B\|_{L^{p'(\cdot)}}} \leqslant C\left(\frac{|S|}{|B|}\right)^{\delta_2}. \tag{8}$$

Lemma 5 (see [27]). *Let $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$. Then there exists a positive constant C such that, for any ball B in $\mathbb{R}^n$,*

$$\|\chi_B\|_{L^{p(\cdot)}} \|\chi_B\|_{L^{p'(\cdot)}} \leqslant C|B|. \tag{9}$$

To give the definition of Herz-Morrey spaces with variable exponents, let us introduce the following notations. Let $k \in \mathbb{Z}$, $B_k := \{x \in \mathbb{R}^n : |x| \leqslant 2^k\}$, $L \in \mathbb{Z}$, $D_k := B_k \setminus B_{k-1}$, and $\chi_k := \chi_{D_k}$. The symbol $\mathbb{N}_0$ denotes the set of all nonnegative integers. For $m \in \mathbb{N}_0$, we denote $\widetilde{\chi}_m := \chi_{D_m}$ if $m \geq 1$ and $\widetilde{\chi}_0 := \chi_{B_0}$.

Definition 6. Let $0 < q \leqslant \infty$, $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, and $0 \leq \lambda < \infty$. Let $\alpha(\cdot)$ be a bounded real-valued measurable function on $\mathbb{R}^n$. The homogeneous Herz-Morrey space $M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}$ and nonhomogeneous Herz-Morrey space $MK^{\alpha(\cdot),q}_{p(\cdot),\lambda}$ are defined, respectively, by

$$M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda} := \left\{ f \in L^{p(\cdot)}_{\mathrm{loc}}\left(\mathbb{R}^n \setminus 0\right) : \|f\|_{M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}} < \infty \right\},$$
$$MK^{\alpha(\cdot),q}_{p(\cdot),\lambda} := \left\{ f \in L^{p(\cdot)}_{\mathrm{loc}}\left(\mathbb{R}^n\right) : \|f\|_{MK^{\alpha(\cdot),q}_{p(\cdot),\lambda}} < \infty \right\}, \tag{10}$$

where

$$\|f\|_{M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}} := \sup_{L\in\mathbb{Z}} 2^{-L\lambda} \left(\sum_{k=-\infty}^{L} \left\| 2^{\alpha(\cdot)k} f\chi_k \right\|^q_{L^{p(\cdot)}} \right)^{1/q},$$
$$\|f\|_{MK^{\alpha(\cdot),q}_{p(\cdot),\lambda}} := \sup_{L\in\mathbb{N}_0} 2^{-L\lambda} \left(\sum_{k=0}^{L} \left\| 2^{\alpha(\cdot)k} f\widetilde{\chi}_k \right\|^q_{L^{p(\cdot)}} \right)^{1/q}. \tag{11}$$

Here there is the usual modification when $q = \infty$.

Proposition 7. *Let $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, $q \in (0,\infty]$, and $\lambda \in [0,\infty)$. If $\alpha(\cdot) \in L^{\infty}(\mathbb{R}^n) \cap \mathscr{P}^{\log}_0(\mathbb{R}^n) \cap \mathscr{P}^{\log}_{\infty}(\mathbb{R}^n)$, then*

$$\begin{aligned}\|f\|_{M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}} \approx \max \Bigg\{ &\sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda} \left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q} \|f\chi_k\|^q_{L^{p(\cdot)}} \right)^{1/q}, \\ &\sup_{L>0, L\in\mathbb{Z}} \Bigg[2^{-L\lambda} \left(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q} \|f\chi_k\|^q_{L^{p(\cdot)}} \right)^{1/q} \\ &\quad + 2^{-L\lambda} \left(\sum_{k=0}^{L} 2^{k\alpha(\infty)q} \|f\chi_k\|^q_{L^{p(\cdot)}} \right)^{1/q} \Bigg] \Bigg\}.\end{aligned} \tag{12}$$

Proposition 7 is the generalization of Herz spaces with variable exponents in [28], and it was used in [33, 34].

Lemma 8. *Let $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$, $0 < q < \infty$, and $\lambda \in [0,\infty)$. Let $\alpha(\cdot)$ be bounded and log-Hölder continuous both at the origin and at infinity such that $-n\delta_1 < \alpha(0) \leqslant \alpha_{\infty} < n\delta_2$, where $0 < \delta_1, \delta_2 < 1$ are constants in Lemma 4. Suppose that T is a sublinear and bounded operator on $L^{p(\cdot)}$ satisfying size condition*

$$|Tf(x)| \leqslant C \int_{\mathbb{R}^n} |x-y|^{-n} |f(y)|\, dy \tag{13}$$

for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ with compact support and a.e. $x \notin \operatorname{supp} f$. Then there exists a positive constant C such that

$$\|Tf\|_{M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}} \leqslant C\|f\|_{M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}}, \qquad \|Tf\|_{MK^{\alpha(\cdot),q}_{p(\cdot),\lambda}} \leqslant C\|f\|_{MK^{\alpha(\cdot),q}_{p(\cdot),\lambda}} \tag{14}$$

for any function f belongs to $M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}$ and $MK^{\alpha(\cdot),q}_{p(\cdot),\lambda}$, respectively.

Lemma 8 is the generalization of Herz spaces with variable exponents in [28]. For a proof, see [33].

2. The Atomic Characterization

In this section, we will introduce Herz-Morrey-Hardy spaces with variable exponents $HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}$ and $HMK^{\alpha(\cdot),q}_{p(\cdot),\lambda}$. To do this, we need to recall some notations. $\mathscr{S}(\mathbb{R}^n)$ denotes the Schwartz space of all rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$, and $\mathscr{S}'(\mathbb{R}^n)$ denotes the dual space of $\mathscr{S}(\mathbb{R}^n)$. Let $G_N f$ be the grand maximal function of f defined by

$$G_N f(x) := \sup_{\phi\in\mathscr{A}_N} \left|\phi^*_{\triangledown}(f)(x)\right|, \quad x \in \mathbb{R}^n, \tag{15}$$

where $\mathscr{A}_N := \{\phi \in \mathscr{S}(\mathbb{R}^n) : \sup_{|\alpha|,|\beta|\leqslant N, \forall x\in\mathbb{R}^n} |x^{\alpha} D^{\beta}\phi(x)| \leqslant 1\}$ and $N > n+1$ and $\phi^*_{\triangledown}$ is the nontangential maximal operator defined by

$$\phi^*_{\triangledown}(f)(x) := \sup_{|y-x|<t} \left|\phi_t * f(y)\right|,$$
$$\forall x \in \mathbb{R}^n \quad \text{with } \phi_t(\cdot) = t^{-n}\phi\left(\frac{\cdot}{t}\right). \tag{16}$$

The grand maximal operator G_N was firstly introduced by Fefferman and Stein in [44] to study classical Hardy spaces. For classical Hardy spaces, one can also see [45–47]. Nakai and Sawano generalized them to variable exponent case in [22].

Definition 9. Let $\alpha(\cdot) \in L^{\infty}(\mathbb{R}^n)$, $0 < q \le \infty$, $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, $0 \le \lambda < \infty$, and $N > n+1$. The homogeneous Herz-Morrey-Hardy space with variable exponents $HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}$ and nonhomogeneous Herz-Morrey-Hardy space with variable exponents $HMK^{\alpha(\cdot),q}_{p(\cdot),\lambda}$ are defined, respectively, by

$$HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda} := \left\{ f \in \mathscr{S}'(\mathbb{R}^n) : \|f\|_{HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}} := \|G_N f\|_{M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}} < \infty \right\},$$
$$HMK^{\alpha(\cdot),q}_{p(\cdot),\lambda} := \left\{ f \in \mathscr{S}'(\mathbb{R}^n) : \|f\|_{HMK^{\alpha(\cdot),q}_{p(\cdot),\lambda}} := \|G_N f\|_{MK^{\alpha(\cdot),q}_{p(\cdot),\lambda}} < \infty \right\}. \tag{17}$$

Remark 10. If $\alpha(\cdot) \equiv \alpha$ and $\lambda = 0$, these spaces were considered by Wang and Liu in [31]. If $p(\cdot)$ and $\alpha(\cdot)$ are constant and $\lambda = 0$, these are the classical Herz type Hardy spaces; see [6].

Let $\psi(r) = 1$ for $r \in [0,1]$ and $\psi(r) = r^{-N}$ for $r \in (1,+\infty)$. Then there exists $C > 0$ such that $\phi(x) \le C\psi(|x|)$ for all $\phi \in \mathscr{A}_N$. Therefore, by [46, Proposition in Page 57], there exists $C > 0$ such that $G_N f(x) \leqslant CMf(x)$ for all $x \in \mathbb{R}^n$. This means that $G_N f$ satisfies the size condition in Lemma 8. By Lemma 8, if $-n\delta_1 < \alpha(0) \leqslant \alpha_{\infty} < n\delta_2$ and $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$, then

$$HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda} \cap L^{p(\cdot)}_{\mathrm{loc}}\left(\mathbb{R}^n \setminus \{0\}\right) = M\dot{K}^{\alpha(\cdot),q}_{p(\cdot,\lambda)},$$
$$HMK^{\alpha(\cdot),q}_{p(\cdot),\lambda} \cap L^{p(\cdot)}_{\mathrm{loc}}\left(\mathbb{R}^n\right) = MK^{\alpha(\cdot),q}_{p(\cdot),\lambda}. \tag{18}$$

Thus we are interested in the case $n\delta_2 \leqslant \alpha(0)$, $\alpha_\infty < \infty$. In this case, we will establish a characterization of the spaces $H M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}$ and $HMK^{\alpha(\cdot),q}_{p(\cdot),\lambda}$ in terms of central atom. For $u \in \mathbb{R}$ we denote by $[u]$ the largest integer less than or equal to u.

Definition 11. Let $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ and $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ be log-Hölder continuous both at the origin and infinity, and nonnegative integer $s \geqslant [\alpha_r - n\delta_2]$; here $\alpha_r = \alpha(0)$, if $r < 1$, and $\alpha_r = \alpha_\infty$, if $r \geqslant 1$, $n\delta_2 \leqslant \alpha_r < \infty$ and δ_2 as in Lemma 4.

(i) A function a on $\mathbb{R}^n$ is called a central $(\alpha(\cdot), p(\cdot))$-atom, if it satisfies (1) supp $a \subset B(0,r)$; (2) $\|a\|_{L^{p(\cdot)}} \leqslant |B(0,r)|^{-\alpha_r/n}$; (3) $\int_{\mathbb{R}^n} a(x)x^\beta \mathrm{d}x = 0$, $|\beta| \leqslant s$.

(ii) A function a on $\mathbb{R}^n$ is called a central $(\alpha(\cdot), p(\cdot))$-atom of restricted type, if it satisfies (1) supp $a \subset B(0,r)$, $r \geqslant 1$; (2) $\|a\|_{L^{p(\cdot)}} \leqslant |B(0,r)|^{-\alpha_r/n}$; (3) $\int_{\mathbb{R}^n} a(x)x^\beta \mathrm{d}x = 0$, $|\beta| \leqslant s$.

Remark 12. If $p(\cdot) \equiv p$ and $\alpha(\cdot) \equiv \alpha$ are constant, then taking $\delta_2 = 1 - 1/p$ we recover the classical case in [6].

Theorem 13. *Let $0 < q < \infty$, $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$, $0 \leq \lambda < \infty$, and $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ be log-Hölder continuous both at the origin and infinity, $2\lambda \leqslant \alpha(\cdot)$, $n\delta_2 \leqslant \alpha(0)$, $\alpha_\infty < \infty$, and δ_2 as in Lemma 4.*

(i) *$f \in H M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}$ if and only if $f = \sum_{k=-\infty}^{\infty} \lambda_k a_k$ in the sense of $\mathcal{S}'(\mathbb{R}^n)$, where each a_k is a central $(\alpha(\cdot), p(\cdot))$-atom with support contained in B_k and $\sup_{L\in\mathbb{Z}} 2^{-L\lambda} \sum_{k=-\infty}^{L} |\lambda_k|^q < \infty$. Moreover,*

$$\|f\|_{H M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}} \approx \inf \sup_{L\in\mathbb{Z}} 2^{-L\lambda} \left(\sum_{k=-\infty}^{L} |\lambda_k|^q\right)^{1/q}, \tag{19}$$

where the infimum is taken over all above decompositions of f.

(ii) *$f \in HMK^{\alpha(\cdot),q}_{p(\cdot),\lambda}$ if and only if $f = \sum_{k=0}^{\infty} \lambda_k a_k$ in the sense of $\mathcal{S}'(\mathbb{R}^n)$, where each a_k is a central $(\alpha(\cdot), p(\cdot))$-atom of restricted type with support contained in B_k and $\sup_{L\in\mathbb{Z}} 2^{-L\lambda} \sum_{k=0}^{L} |\lambda_k|^q < \infty$. Moreover*

$$\|f\|_{HMK^{\alpha(\cdot),q}_{p(\cdot),\lambda}} \approx \inf \sup_{L\in\mathbb{Z}} 2^{-L\lambda} \left(\sum_{k=0}^{L} |\lambda_k|^q\right)^{1/q}, \tag{20}$$

where the infimum is taken over all above decompositions of f.

Proof. We only prove (i). The proof of (ii) is similar. We use the ideas in [6]. To prove the necessity, we choose $\phi \in C_0^\infty(\mathbb{R}^n)$ such that $\phi \geqslant 0$, $\int_{\mathbb{R}^n} \phi(x)\mathrm{d}x = 1$, and supp $\phi \subset \{x : |x| \leqslant 1\}$. For $j \in \mathbb{N}_0$, let $\phi_{(j)}(x) := 2^{jn}\phi(2^j x)$, $\forall x \in \mathbb{R}^n$. For each $f \in \mathcal{S}'(\mathbb{R}^n)$, set $f^{(j)}(x) = f * \phi_{(j)}(x)$, $\forall x \in \mathbb{R}^n$. It is obvious that $f^{(j)} \in C^\infty(\mathbb{R}^n)$ and $\lim_{j\to\infty} f^{(j)} = f$ in $\mathcal{S}'(\mathbb{R}^n)$. Let ψ be a radial smooth function such that supp $\psi \subset \{x : 1/2 - \varepsilon \leqslant |x| \leqslant 1 + \varepsilon\}$ with $0 < \varepsilon < 1/4$, $\psi(x) = 1$ for $1/2 \leqslant |x| \leqslant 1$. Let $\psi_k(x) := \psi(2^{-k}x)$ for $k \in \mathbb{Z}$ and

$$\widetilde{A}_{k,\varepsilon} := \left\{x : 2^{k-1} - 2^k\varepsilon \leqslant |x| \leqslant 2^k + 2^k\varepsilon\right\}. \tag{21}$$

Observe that supp $\psi_k \subset \widetilde{A}_{k,\varepsilon}$ and $\psi_k(x) = 1$ for $x \in A_k := \{x : 2^{k-1} \leqslant |x| \leqslant 2^k\}$. Obviously, $1 \leqslant \sum_{k=-\infty}^{\infty} \psi_k(x) \leqslant 2$, $x \neq 0$. Let

$$\Phi_k(x) := \begin{cases} \dfrac{\psi_k(x)}{\sum_{l=-\infty}^{\infty} \psi_l(x)}, & x \neq 0 \\ 0, & x = 0. \end{cases} \tag{22}$$

Then $\sum_{k=-\infty}^{\infty} \Phi_k(x) = 1$ for $x \neq 0$. For each $m \in \mathbb{N}$, we denote by $\mathscr{P}_m$ the class of all the real polynomials with the degree less than m. Let $P_k^{(j)}(x) := P_{\widetilde{A}_{k,\varepsilon}}(f^{(j)}\Phi_k)(x)\chi_{\widetilde{A}_{k,\varepsilon}} \in \mathscr{P}_m(\mathbb{R}^n)$ be the unique polynomial satisfying

$$\int_{\widetilde{A}_{k,\varepsilon}} \left(f^{(j)}(x)\Phi_k(x) - P_k^{(j)}(x)\right) x^\beta \mathrm{d}x = 0,$$
$$|\beta| \leqslant m = \max\left\{[\alpha(0) - n\delta_2], [\alpha_\infty - n\delta_2]\right\}. \tag{23}$$

Write

$$f^{(j)}(x) = \sum_{k=-\infty}^{\infty} \left(f^{(j)}(x)\Phi_k(x) - P_k^{(j)}(x)\right) + \sum_{k=-\infty}^{\infty} P_k^{(j)}(x)$$
$$:= I_{(j)} + II_{(j)}. \tag{24}$$

For the term $I_{(j)}$, let $g_k^{(j)}(x) := f^{(j)}(x)\Phi_k(x) - P_k^{(j)}(x)$ and $a_k^{(j)}(x) := g_k^{(j)}(x)/\lambda_k$, where $\lambda_k := b|B_{k+1}|^{\alpha_{k+1}/n} \sum_{l=k-1}^{k+1} \|(G_N f)\chi_l\|_{L^{p(\cdot)}}$ and b is a constant which will be chosen later. Note that supp $a_k^{(j)} \subset B_{k+1}$, $I_{(j)} = \sum_{k=-\infty}^{\infty} \lambda_k a_k^{(j)}(x)$.

Now we estimate $\|g_k^{(j)}\|_{L^{p(\cdot)}}$. To do this, let $\{\phi_\gamma^k : |\gamma| \leqslant m\}$ be the orthogonal polynomials restricted to $\widetilde{A}_{k,\varepsilon}$ with respect to the weight $1/|\widetilde{A}_{k,\varepsilon}|$, which are obtained from $\{x^\beta : |\beta| \leqslant m\}$ by the Gram-Schmidt method, which means

$$\left\langle\phi_\nu^k, \phi_\mu^k\right\rangle = \frac{1}{\left|\widetilde{A}_{k,\varepsilon}\right|} \int_{\widetilde{A}_{k,\varepsilon}} \phi_\nu^k(x)\phi_\mu^k(x)\,\mathrm{d}x = \delta_{\nu\mu}, \tag{25}$$

where $\delta_{\nu\mu} = 1$ for $\nu = \mu$, otherwise 0.

It is easy to see that $P_k^{(j)}(x) = \sum_{|\gamma|\leqslant m} \langle f^{(j)}\Phi_k, \phi_\gamma^k\rangle \phi_\gamma^k(x)$ for $x \in \widetilde{A}_{k,\varepsilon}$. On the other hand, from $(1/|\widetilde{A}_{k,\varepsilon}|)\int_{\widetilde{A}_{k,\varepsilon}} \phi_\nu^k(x)\phi_\mu^k(x)\mathrm{d}x = \delta_{\nu\mu}$ we infer that

$$\frac{1}{\left|\widetilde{A}_{1,\varepsilon}\right|} \int_{\widetilde{A}_{1,\varepsilon}} \phi_\nu^k\left(2^{k-1}y\right)\phi_\mu^k\left(2^{k-1}y\right)\mathrm{d}y = \delta_{\nu\mu}. \tag{26}$$

Thus, we deduce $\phi_\nu^k(2^{k-1}y) = \phi_\nu^1(y)$ a.e. That is, $\phi_\nu^k(x) = \phi_\nu^1(2^{1-k}x)$ almost everywhere for $x \in \widetilde{A}_{k,\varepsilon}$. Therefore

$|\phi_\nu^k(x)| \leqslant C$ for $x \in \widetilde{A}_{k,\varepsilon}$. By the generalized Hölder inequality we have

$$\begin{aligned}\left|P_k^{(j)}(x)\right| &\lesssim \frac{1}{\left|\widetilde{A}_{k,\varepsilon}\right|}\int_{\widetilde{A}_{k,\varepsilon}}\left|f^{(j)}(x)\,\Phi_k(x)\right|\mathrm{d}x \\ &\lesssim \frac{1}{\left|\widetilde{A}_{k,\varepsilon}\right|}\left\|f^{(j)}\Phi_k\right\|_{L^{p(\cdot)}}\left\|\chi_{\widetilde{A}_{k,\varepsilon}}\right\|_{L^{p'(\cdot)}}.\end{aligned} \tag{27}$$

By Lemma 5 we have

$$\begin{aligned}\left\|g_k^{(j)}\right\|_{L^{p(\cdot)}} &\leqslant \left\|f^{(j)}\Phi_k\right\|_{L^{p(\cdot)}} + \left\|P_k^{(j)}\right\|_{L^{p(\cdot)}} \\ &\lesssim \left\|f^{(j)}\Phi_k\right\|_{L^{p(\cdot)}} + \frac{1}{\left|\widetilde{A}_{k,\varepsilon}\right|}\left\|f^{(j)}\Phi_k\right\|_{L^{p(\cdot)}} \\ &\quad\times\left\|\chi_{\widetilde{A}_{k,\varepsilon}}\right\|_{L^{p'(\cdot)}}\left\|\chi_{\widetilde{A}_{k,\varepsilon}}\right\|_{L^{p(\cdot)}} \\ &\lesssim \left\|f^{(j)}\Phi_k\right\|_{L^{p(\cdot)}} + \left\|f^{(j)}\Phi_k\right\|_{L^{p(\cdot)}} \\ &\lesssim \left\|(f*\phi_{(j)})\Phi_k\right\|_{L^{p(\cdot)}} \\ &\leqslant C\sum_{l=k-1}^{k+1}\left\|(G_N f)\chi_l\right\|_{L^{p(\cdot)}}.\end{aligned} \tag{28}$$

Choose $b = C$; then $\|a_k^{(j)}\|_{L^{p(\cdot)}} \leqslant |B_{k+1}|^{-\alpha_{k+1}/n}$ and each $a_k^{(j)}$ is a central $(\alpha(\cdot), p(\cdot))$-atom with support contained in B_{k+1}. Here and below we abuse $\alpha_k := \alpha_{2^k}$ and it is well defined in Definition 11. Thus,

$$\begin{aligned}&\sup_{L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L}\left|\lambda_k\right|^q \\ &\lesssim \sup_{L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L}\left|B_{k+1}\right|^{q\alpha_{k+1}/n}\left(\sum_{l=k-1}^{k+1}\left\|(G_N f)\chi_l\right\|_{L^{p(\cdot)}}\right)^q \\ &\lesssim \sup_{L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L}\left|B_{k+1}\right|^{q\alpha_{k+1}/n}\left\|(G_N f)\chi_k\right\|_{L^{p(\cdot)}}^q \\ &:= A.\end{aligned} \tag{29}$$

Now we estimate A. By the condition of $\alpha(\cdot)$ and Proposition 7 we consider it in two cases.

Case 1 ($L \leqslant 0$). Consider

$$\begin{aligned}&2^{-L\lambda q}\sum_{k=-\infty}^{L}\left|B_{k+1}\right|^{q\alpha_{k+1}/n}\left\|(G_N f)\chi_k\right\|_{L^{p(\cdot)}}^q \\ &\quad\lesssim \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{kq\alpha(0)}\left\|(G_N f)\chi_k\right\|_{L^{p(\cdot)}}^q \\ &\quad\lesssim \left\|G_N f\right\|_{MK_{p(\cdot),\lambda}^{\alpha(\cdot),q}}^q.\end{aligned} \tag{30}$$

Case 2 ($L > 0$). Consider

$$\begin{aligned}&2^{-L\lambda q}\sum_{k=-\infty}^{L}\left|B_{k+1}\right|^{q\alpha_{k+1}/n}\left\|(G_N f)\chi_k\right\|_{L^{p(\cdot)}}^q \\ &\quad= 2^{-L\lambda q}\sum_{k=-\infty}^{-2}2^{(k+1)q\alpha(0)}\left\|(G_N f)\chi_k\right\|_{L^{p(\cdot)}}^q \\ &\quad\quad+ 2^{-L\lambda q}\sum_{k=-1}^{L}2^{(k+1)q\alpha(\infty)}\left\|(G_N f)\chi_k\right\|_{L^{p(\cdot)}}^q \\ &\quad\lesssim 2^{-L\lambda q}\sum_{k=-\infty}^{-1}2^{kq\alpha(0)}\left\|(G_N f)\chi_k\right\|_{L^{p(\cdot)}}^q \\ &\quad\quad+ 2^{-L\lambda q}\sum_{k=0}^{L}2^{kq\alpha(\infty)}\left\|(G_N f)\chi_k\right\|_{L^{p(\cdot)}}^q \\ &\quad\lesssim \left\|G_N f\right\|_{MK_{p(\cdot),\lambda}^{\alpha(\cdot),q}}^q.\end{aligned} \tag{31}$$

Hence,

$$\sup_{L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L}\left|\lambda_k\right|^q \lesssim \left\|G_N f\right\|_{MK_{p(\cdot),\lambda}^{\alpha(\cdot),q}}^p. \tag{32}$$

It remains to estimate $II_{(j)}$. Let $\{\psi_d^k : |\gamma| \leqslant m\}$ be the dual basis of $\{x^\beta : |\beta| \leqslant m\}$ with respect to the weight $1/|\widetilde{A}_{k,\varepsilon}|$ on $\widetilde{A}_{k,\varepsilon}$, that is,

$$\left\langle \psi_\gamma^k, x^\beta\right\rangle = \frac{1}{\left|\widetilde{A}_{k,\varepsilon}\right|}\int_{\widetilde{A}_{k,\varepsilon}} x^\beta\psi_\gamma^k(x)\,\mathrm{d}x = \delta_{\beta\gamma}. \tag{33}$$

Similar to the method of [48], let

$$\begin{aligned}h_{k,\gamma}^{(j)}(x) &:= \sum_{l=-\infty}^{k}\left(\frac{\psi_\gamma^k(x)\,\chi_{\widetilde{A}_{k,\varepsilon}}(x)}{\left|\widetilde{A}_{k,\varepsilon}\right|} - \frac{\psi_\gamma^{k+1}(x)\,\chi_{\widetilde{A}_{k+1,\varepsilon}}(x)}{\left|\widetilde{A}_{k+1,\varepsilon}\right|}\right) \\ &\quad\times\int_{\mathbb{R}^n} f^{(j)}(y)\,\Phi_l(y)\,y^\gamma\mathrm{d}y.\end{aligned} \tag{34}$$

We write

$$\begin{aligned}II_{(j)} &= \sum_{k=-\infty}^{\infty}\sum_{|\gamma|\leqslant m}\left\langle f^{(j)}\Phi_k, x^\gamma\right\rangle\psi_\gamma^k(x)\,\chi_{\widetilde{A}_{k,\varepsilon}}(x) \\ &= \sum_{|\gamma|\leqslant m}\sum_{k=-\infty}^{\infty}\left(\int_{\mathbb{R}^n} f^{(j)}\Phi_k x^\gamma\mathrm{d}x\right)\frac{\psi_\gamma^k(x)\,\chi_{\widetilde{A}_{k,\varepsilon}}(x)}{\left|\widetilde{A}_{k,\varepsilon}\right|} \\ &= \sum_{|\gamma|\leqslant m}\sum_{k=-\infty}^{\infty}\left(\sum_{l=-\infty}^{k}\int_{\mathbb{R}^n} f^{(j)}(x)\,\Phi_l(x)\,x^\gamma\mathrm{d}x\right) \\ &\quad\times\left(\frac{\psi_\gamma^k(x)\,\chi_{\widetilde{A}_{k,\varepsilon}}(x)}{\left|\widetilde{A}_{k,\varepsilon}\right|} - \frac{\psi_\gamma^{k+1}(x)\,\chi_{\widetilde{A}_{k+1,\varepsilon}}(x)}{\left|\widetilde{A}_{k+1,\varepsilon}\right|}\right) \\ &=: \sum_{|\gamma|\leqslant m}\sum_{k=-\infty}^{\infty}\frac{\alpha_{k,\gamma}h_{k,\gamma}^{(j)}(x)}{\alpha_{k,\gamma}} \\ &=: \sum_{|\gamma|\leqslant m}\sum_{k=-\infty}^{\infty}\alpha_{k,\gamma}a_{k,\gamma}^{(j)}(x),\end{aligned} \tag{35}$$

where

$$\alpha_{k,\gamma} := \widetilde{b}\sum_{l=k-1}^{k+1}\left\|(G_N f)\chi_l\right\|_{L^{p(\cdot)}}\left|B_{k+2}\right|^{\alpha_{k+2}/n} \tag{36}$$

and $\widetilde{b}$ is a constant which will be chosen later. Note that

$$\int_{\mathbb{R}^n}\sum_{l=-\infty}^{k}\left|\Phi_l(x)x^\gamma\right|\mathrm{d}x=\sum_{l=-\infty}^{k}\int_{\widetilde{A}_{k,\varepsilon}}\left|\Phi_l(x)x^\gamma\right|\mathrm{d}x \lesssim 2^{k(n+|\gamma|)}. \tag{37}$$

By a computation we have

$$\left|\int_{\mathbb{R}^n}f^{(j)}(y)\sum_{l=-\infty}^{k}\Phi_l(y)y^\gamma \mathrm{d}y\right|\lesssim 2^{k(n+|\gamma|)}G_Nf(x), \quad x\in B_{k+2}. \tag{38}$$

Since

$$\left|\frac{\psi_\gamma^k(x)\chi_{\widetilde{A}_{k,\varepsilon}}(x)}{\left|\widetilde{A}_{k,\varepsilon}\right|}-\frac{\psi_\gamma^{k+1}(x)\chi_{\widetilde{A}_{k+1,\varepsilon}}(x)}{\left|\widetilde{A}_{k+1,\varepsilon}\right|}\right|\lesssim 2^{-k(n+|\gamma|)}\sum_{l=k-1}^{k+1}\chi_l(x), \tag{39}$$

it follows that

$$\left\|h_{k,\gamma}^{(j)}\right\|_{L^{p(\cdot)}}\leqslant C\sum_{l=k-1}^{k+1}\left\|(G_Nf)\chi_l\right\|_{L^{p(\cdot)}}. \tag{40}$$

Take $\widetilde{b}=C$. It is easy to show that each $a_{k,\gamma}^{(j)}$ is a central $(\alpha(\cdot), p(\cdot))$-atom with support contained in $\widetilde{A}_{k,\varepsilon}\cup\widetilde{A}_{k+1,\varepsilon}\subset B_{k+2}$, and

$$\alpha_{k,\gamma}=C\sum_{l=k-1}^{k+1}\left\|(G_Nf)\chi_l\right\|_{L^{p(\cdot)}}\left|B_{k+2}\right|^{\alpha_{k+2}/n}, \tag{41}$$

where C is a constant independent of j, f, k, and γ. Moreover, we have

$$\begin{aligned}\sup_{L\in\mathbb{Z}}2^{-L\lambda q}&\sum_{k=-\infty}^{L}\sum_{|\gamma|\leqslant m}\left|\alpha_{k,\gamma}\right|^q\\ &\lesssim\sup_{L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}\left|B_{k+2}\right|^{q\alpha_{k+2}/n}\left(\sum_{l=k-1}^{k+1}\left\|(G_Nf)\chi_l\right\|_{L^{p(\cdot)}}\right)^q\\ &\lesssim\sup_{L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}\left|B_{k+2}\right|^{q\alpha_{k+2}/n}\left\|(G_Nf)\chi_k\right\|_{L^{p(\cdot)}}^q\\ &:=B.\end{aligned} \tag{42}$$

Using the same argument as before for A, we obtain

$$B\lesssim\left\|G_Nf\right\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}}^q. \tag{43}$$

Therefore,

$$\sup_{L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}\left|\sum_{|\gamma|\leqslant m}\alpha_{k,\gamma}\right|^q\lesssim\left\|G_Nf\right\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}}^p. \tag{44}$$

Thus, we obtain that

$$f^{(j)}(x)=\sum_{d=-\infty}^{\infty}\lambda_d a_d^{(j)}(x), \tag{45}$$

where each $a_d^{(j)}$ is a central $(\alpha(\cdot), p(\cdot))$-atom with support contained in $\widetilde{A}_{d,\varepsilon}\cup\widetilde{A}_{d+1,\varepsilon}\subset B_{d+2}$, λ_d is independent of j and

$$\sup_{L\in\mathbb{Z}}2^{-L\lambda}\left(\sum_{d=-\infty}^{L}\left|\lambda_d\right|^q\right)^{1/q}\leqslant C\left\|G_Nf\right\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}}<\infty, \tag{46}$$

where C is independent of j and f.

Since

$$\sup_{j\in\mathbb{N}_0}\left\|a_0^{(j)}\right\|_{L^{p(\cdot)}}\leqslant\left|B_2\right|^{-\alpha_2/n}, \tag{47}$$

by the Banach-Alaoglu theorem we obtain a subsequence $\{a_0^{(j_{n_0})}\}$ of $\{a_0^{(j)}\}$ converging in the weak $*$ topology of $L^{p(\cdot)}$ to some $a_0\in L^{p(\cdot)}$. It is easy to verify that a_0 is a central $(\alpha(\cdot), p(\cdot))$-atom supported on B_2. Next, since

$$\sup_{j_{n_0}\in\mathbb{N}_0}\left\|a_1^{(j_{n_0})}\right\|_{L^{p(\cdot)}}\leqslant\left|B_3\right|^{-\alpha_3/n}, \tag{48}$$

another application of the Banach-Alaoglu theorem yields a subsequence $\{a_1^{(j_{n_1})}\}$ of $\{a_1^{(j_{n_0})}\}$ which converges weak $*$ in $L^{p(\cdot)}$ to a central $(\alpha(\cdot), p(\cdot))$-atom a_1 with support in B_3. Furthermore,

$$\sup_{j_{n_1}\in\mathbb{N}_0}\left\|a_{-1}^{(j_{n_1})}\right\|_{L^{p(\cdot)}}\leqslant\left|B_1\right|^{-\alpha_1/n}. \tag{49}$$

Similarly, there exists a subsequence $\{a_{-1}^{(j_{n_{-1}})}\}$ of $\{a_{-1}^{(j_{n_1})}\}$ which converges weak $*$ in $L^{p(\cdot)}$ to some $a_{-1}\in L^{p(\cdot)}$, and a_{-1} is a central $(\alpha(\cdot), p(\cdot))$-atom supported on B_1. Repeating the above procedure for each $d\in\mathbb{Z}$, we can find a subsequence $\{a_d^{(j_{n_d})}\}$ of $\{a_d^{(j)}\}$ converging weak $*$ in $L^{p(\cdot)}$ to some $a_d\in L^{p(\cdot)}$ which is a central $(\alpha(\cdot), p(\cdot))$-atom supported on B_{d+2}. By using the diagonal method we obtain a subsequence $\{j_\nu\}$ of $\mathbb{N}_0$ such that, for each $d\in\mathbb{Z}$, $\lim_{\nu\to\infty}a_d^{(j_\nu)}=a_d$ in the weak $*$ topology of $L^{p(\cdot)}$ and therefore in $\mathcal{S}'(\mathbb{R}^n)$.

Now we only need to prove that $f=\sum_{d=-\infty}^{\infty}\lambda_d a_d$ in the sense of $\mathcal{S}'(\mathbb{R}^n)$. For each $\varphi\in\mathcal{S}(\mathbb{R}^n)$, note that supp $a_d^{(j_\nu)}\subset(\widetilde{A}_{d,\varepsilon}\cup\widetilde{A}_{d+1,\varepsilon})\subset(A_{d-1}\cup A_d\cup A_{d+1}\cup A_{d+2})$. Using the same argument in [48], we have

$$\langle f,\varphi\rangle=\lim_{\nu\to\infty}\sum_{d=-\infty}^{\infty}\lambda_d\int_{\mathbb{R}^n}a_d^{(j_\nu)}(x)\varphi(x)\,\mathrm{d}x. \tag{50}$$

Recall that $m = \max\{[\alpha(0) - n\delta_2], [\alpha_\infty - n\delta_2]\}$. If $d \leqslant 0$, then by Lemmas 3 and 4 we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} a_d^{(j_\nu)}(x)\varphi(x)\,dx\right| \\
&= \left|\int_{\mathbb{R}^n} a_d^{(j_\nu)}(x)\left(\varphi(x) - \sum_{|\beta|\leqslant m}\frac{D^\beta\varphi(0)}{\beta!}x^\beta\right)dx\right| \\
&\lesssim \int_{\mathbb{R}^n}\left|a_d^{(j_\nu)}(x)\right|\cdot|x|^{m+1}\,dx \\
&\lesssim 2^{d(m+1)}\int_{\mathbb{R}^n}\left|a_d^{(j_\nu)}(x)\right|dx \\
&\lesssim 2^{d(m+1-\alpha_{d+2})}\left\|\chi_{B_{d+2}}\right\|_{L^{p'(\cdot)}} \\
&\lesssim 2^{d(m+1-\alpha_{d+2})}\left(\frac{|B_{d+2}|}{|B_2|}\right)^{\delta_2}\left\|\chi_{B_2}\right\|_{L^{p'(\cdot)}} \\
&\lesssim 2^{d(m+1-\alpha_{d+2}+n\delta_2)}\frac{|B_2|}{|B_0|}\left\|\chi_{B_0}\right\|_{L^{p'(\cdot)}} \\
&\lesssim 2^{d(m+1-\alpha_{d+2}+n\delta_2)}.
\end{aligned} \tag{51}$$

If $d > 0$, let $k_0 \in \mathbb{N}_0$ such that $\min\{k_0 + \alpha(0) - n, k_0 + \alpha_\infty - n\} > 0$; then by Lemmas 4 and 3 again we have

$$\begin{aligned}
\left|\int_{\mathbb{R}^n} a_d^{(j_\nu)}(x)\varphi(x)\,dx\right| &\lesssim \int_{\mathbb{R}^n}\left|a_d^{(j_\nu)}(x)\right||x|^{-k_0}\,dx \\
&\lesssim 2^{-d(k_0+\alpha_{d+2})}\left\|\chi_{B_{d+2}}\right\|_{L^{p'(\cdot)}} \\
&\lesssim 2^{-d(k_0+\alpha_{d+2})}\frac{|B_{d+2}|}{|B_0|}\left\|\chi_{B_0}\right\|_{L^{p'(\cdot)}} \\
&\lesssim 2^{-d(k_0+\alpha_{d+2}-n)}.
\end{aligned} \tag{52}$$

Let

$$\mu_d = \begin{cases} |\lambda_d|\,2^{d(m+1-\alpha_{d+2}+n\delta_2)}, & d \leqslant 0 \\ |\lambda_d|\,2^{-d(k_0+\alpha_{d+2}-n)}, & d > 0. \end{cases} \tag{53}$$

Then

$$\begin{aligned}
\sup_{L\in\mathbb{Z}} 2^{-L\lambda}\sum_{d=-\infty}^{L}|\mu_d| &\lesssim \left(\sup_{L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{d=-\infty}^{L}|\lambda_d|^q\right)^{1/q} \\
&\lesssim \|G_N f\|_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} < \infty, \\
|\lambda_d|\left|\int_{\mathbb{R}^n} a_d^{(j_\nu)}(x)\varphi(x)\,dx\right| &\lesssim |\mu_d|,
\end{aligned} \tag{54}$$

which implies that

$$\begin{aligned}
\langle f,\varphi\rangle &= \sum_{d=-\infty}^{\infty}\lim_{\nu\to\infty}\lambda_d\int_{\mathbb{R}^n} a_d^{(j_\nu)}(x)\varphi(x)\,dx \\
&= \sum_{d=-\infty}^{\infty}\lambda_d\int_{\mathbb{R}^n} a_d(x)\varphi(x)\,dx.
\end{aligned} \tag{55}$$

This establishes the identity we wanted.

To prove the sufficiency, for convenience, we denote $\sup_{L\in\mathbb{Z}} 2^{-L\lambda}\sum_{k=-\infty}^{L}|\lambda_k|^q = \Lambda$. Firstly we have

$$\begin{aligned}
&\|G_N f\|^q_{M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q}} \\
&\approx \max\left\{\sup_{L\leqslant 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{kq\alpha(0)}\left\|(G_N f)\chi_k\right\|^q_{L^{p(\cdot)}},\right. \\
&\qquad \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\left(\sum_{k=-\infty}^{-1}2^{kq\alpha(0)}\left\|(G_N f)\chi_k\right\|^q_{L^{p(\cdot)}}\right. \\
&\qquad\qquad \left.\left. + \sum_{k=0}^{L}2^{kq\alpha(\infty)}\left\|(G_N f)\chi_k\right\|^q_{L^{p(\cdot)}}\right)\right\} \\
&\lesssim \max\{I, II + III\},
\end{aligned} \tag{56}$$

where

$$\begin{aligned}
I &:= \sup_{L\leqslant 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{kq\alpha(0)}\left\|(G_N f)\chi_k\right\|^q_{L^{p(\cdot)}} \\
II &:= \sum_{k=-\infty}^{-1}2^{kq\alpha(0)}\left\|(G_N f)\chi_k\right\|^q_{L^{p(\cdot)}} \\
III &:= \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}2^{kq\alpha(\infty)}\left\|(G_N f)\chi_k\right\|^q_{L^{p(\cdot)}}.
\end{aligned} \tag{57}$$

Now we have

$$\begin{aligned}
I &= \sup_{L\leqslant 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{kq\alpha(0)}\left\|(G_N f)\chi_k\right\|^q_{L^{p(\cdot)}} \\
&\lesssim \sup_{L\leqslant 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{kq\alpha(0)}\left(\sum_{l=k}^{\infty}|\lambda_l|\left\|a_l\right\|_{L^{p(\cdot)}}\right)^q \\
&\quad + \sup_{L\leqslant 0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{kq\alpha(0)}\left(\sum_{l=-\infty}^{k-1}|\lambda_l|\left\|(G_N a_l)\chi_k\right\|_{L^{p(\cdot)}}\right)^q \\
&:= I_1 + I_2. \\
II &= \sum_{k=-\infty}^{-1}2^{kq\alpha(0)}\left\|(G_N f)\chi_k\right\|^q_{L^{p(\cdot)}} \\
&\lesssim \sum_{k=-\infty}^{-1}2^{kq\alpha(0)}\left(\sum_{l=k}^{\infty}|\lambda_l|\left\|a_l\right\|_{L^{p(\cdot)}}\right)^q \\
&\quad + \sum_{k=-\infty}^{-1}2^{kq\alpha(0)}\left(\sum_{l=-\infty}^{k-1}|\lambda_l|\left\|(G_N a_l)\chi_k\right\|_{L^{p(\cdot)}}\right)^q \\
&:= II_1 + II_2 \\
III &= \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}2^{kq\alpha_\infty}\left\|(G_N f)\chi_k\right\|^q_{L^{p(\cdot)}} \\
&\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}2^{kq\alpha_\infty}\left(\sum_{l=k}^{L}|\lambda_l|\left\|a_l\right\|_{L^{p(\cdot)}}\right)^q
\end{aligned}$$

$$+\sup_{L>0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=0}^{L}2^{kq\alpha_\infty}\left(\sum_{l=-\infty}^{k-1}|\lambda_l|\,\|(G_Na_l)\chi_k\|_{L^{p(\cdot)}}\right)^q$$
$$:= III_1+III_2. \tag{58}$$

To estimate I, II, and III we need a pointwise estimate for $G_Na_l(x)$ on D_k, where $k\geqslant l+2$. Let $\phi\in\mathscr{A}_N$, $m\in\mathbb{N}$ such that $\alpha_k-n\delta_2<m+1$. Denote by P_m the mth order Taylor series expansion of ϕ at y/t. If $|x-y|<t$, then from the vanishing moment condition of a_l we have

$$\begin{aligned}&|a_l*\phi_t(y)|\\&=t^{-n}\left|\int_{\mathbb{R}^n}a_l(z)\left(\phi\left(\frac{y-z}{t}\right)-P_m\left(-\frac{z}{t}\right)\right)dz\right|\\&\lesssim t^{-n}\int_{\mathbb{R}^n}|a_l(z)|\left|\frac{z}{t}\right|^{m+1}\left(1+\left|\frac{(y-\theta z)}{t}\right|\right)^{-(n+m+1)}dz\\&\lesssim\int_{\mathbb{R}^n}|a_l(z)|\,|z|^{m+1}(t+|y-\theta z|)^{-(n+m+1)}dz,\end{aligned}\tag{59}$$

where $0<\theta<1$. Since $x\in D_k$ for $k\in\mathbb{Z}$, we have $|x|\geqslant 2^{k-1}$. From $|x-y|<t$ and $|z|<2^l$, we have

$$t+|y-\theta z|\geqslant|x-y|+|y-\theta z|\geqslant|x|-|z|\geqslant\frac{|x|}{2}.\tag{60}$$

Thus,

$$\begin{aligned}&|a_l*\phi_t(y)|\\&\lesssim\int_{\mathbb{R}^n}|a_l(z)|\,|z|^{m+1}(|x-y|+|y-\theta z|)^{-(n+m+1)}dz\\&\lesssim 2^{l(m+1)}|x|^{-(n+m+1)}\int_{\mathbb{R}^n}|a(z)|\,dz\\&\lesssim 2^{l(m+1)}2^{-k(n+m+1)}|B_l|^{-\alpha_l/n}\|\chi_{B_l}\|_{L^{p'(\cdot)}}.\end{aligned}\tag{61}$$

Therefore, we have

$$G_Na_l(x)\lesssim 2^{l(m+1)}2^{-k(n+m+1)}|B_l|^{-\alpha_l/n}\|\chi_{B_l}\|_{L^{p'(\cdot)}},\quad x\in D_k,\quad k\geqslant l+2.\tag{62}$$

To proceed, we consider them into two cases $0<q\leqslant 1$ and $1<q<\infty$.

If $0<q\leqslant 1$,

$$\begin{aligned}I_1&=\sup_{L\leqslant0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{\alpha(0)kq}\left(\sum_{l=k}^{\infty}|\lambda_l|\,\|(Tb_l)\chi_k\|_{L^{p(\cdot)}}\right)^q\\&\lesssim\sup_{L\leqslant0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{\alpha(0)kq}\left(\sum_{l=k}^{\infty}|\lambda_l|\,2^{-\alpha_l l}\right)^q\\&\lesssim\sup_{L\leqslant0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{\alpha(0)kq}\\&\quad\times\left(\sum_{l=k}^{-1}|\lambda_l|^q2^{-\alpha(0)lq}+\sum_{l=0}^{\infty}|\lambda_l|^q2^{-\alpha_\infty lq}\right)\\&\lesssim\sup_{L\leqslant0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}\sum_{l=k}^{-1}|\lambda_l|^q2^{\alpha(0)(k-l)q}\\&\quad+\sup_{L\leqslant0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{\alpha(0)kq}\sum_{l=0}^{\infty}|\lambda_l|^q2^{-\alpha_\infty lq}\\&\lesssim\sup_{L\leqslant0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{l=-\infty}^{-1}|\lambda_l|^q\sum_{k=-\infty}^{l}2^{\alpha(0)(k-l)q}\\&\quad+\sup_{L\leqslant0,L\in\mathbb{Z}}\sum_{l=0}^{\infty}2^{-l\lambda q}|\lambda_l|^q2^{(\lambda-\alpha_\infty)lq}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{\alpha(0)kq}\\&\lesssim\Lambda+\sup_{L\leqslant0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{l=L}^{-1}|\lambda_l|^q\sum_{k=-\infty}^{l}2^{\alpha(0)(k-l)q}\\&\quad+\Lambda\sup_{L\leqslant0,L\in\mathbb{Z}}\sum_{l=0}^{\infty}2^{(\lambda-\alpha_\infty)lq}\sum_{k=-\infty}^{L}2^{\alpha(0)kq-L\lambda q}\\&\lesssim\Lambda+\sup_{L\leqslant0,L\in\mathbb{Z}}\sum_{l=L}^{-1}2^{-l\lambda q}|\lambda_l|^q2^{(l-L)\lambda q}\sum_{k=-\infty}^{l}2^{\alpha(0)(k-l)q}+\Lambda\\&\lesssim\Lambda+\Lambda\sup_{L\leqslant0,L\in\mathbb{Z}}\sum_{l=L}^{-1}2^{(l-L)\lambda q}\sum_{k=-\infty}^{l}2^{\alpha(0)(k-l)q}\\&\lesssim\Lambda.\end{aligned}$$

$$\begin{aligned}I_2&=\sup_{L\leqslant0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{kq\alpha(0)}\left(\sum_{l=-\infty}^{k-1}|\lambda_l|\,\|(G_Na_l)\chi_k\|_{L^{p(\cdot)}}\right)^q\\&\lesssim\sup_{L\leqslant0,L\in\mathbb{Z}}2^{-L\lambda q}\\&\quad\times\sum_{k=-\infty}^{L}2^{kq\alpha(0)}\left(\sum_{l=-\infty}^{k-1}|\lambda_l|\,2^{l(m+1)-k(n+m+1)}\right.\\&\qquad\left.\times|B_l|^{-\alpha_l/n}\|\chi_{B_l}\|_{L^{p'(\cdot)}}\|\chi_{B_k}\|_{L^{p(\cdot)}}\right)^q\\&\lesssim\sup_{L\leqslant0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}\left(\sum_{l=-\infty}^{k-1}|\lambda_l|\,2^{(l-k)(m+1+n\delta_2)-(l-k)\alpha(0)}\right)^q\\&\lesssim\sup_{L\leqslant0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}\left(\sum_{l=-\infty}^{k-1}|\lambda_l|\,2^{(l-k)(m+1+n\delta_2-\alpha(0))}\right)^q\\&=\sup_{L\leqslant0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{l=-\infty}^{L-1}\sum_{k=l+1}^{L}|\lambda_l|^q2^{(l-k)(m+1+n\delta_2-\alpha(0))q}\\&\lesssim\sup_{L\leqslant0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{l=-\infty}^{L}|\lambda_l|^q\\&\lesssim\Lambda.\end{aligned}\tag{63}$$

Then we turn to estimate II:

$$
\begin{aligned}
II_1 &= \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{l=k}^{\infty} |\lambda_l| \, \|a_l\|_{L^{p(\cdot)}} \right)^q \\
&\lesssim \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{l=k}^{\infty} |\lambda_l| \, |B_l|^{-\alpha_l/n} \right)^q \\
&\lesssim \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \left(\sum_{l=k}^{-1} |\lambda_l|^q \, 2^{-\alpha(0)lq} + \sum_{l=0}^{\infty} |\lambda_l|^q \, 2^{-\alpha_\infty lq} \right) \\
&\lesssim \sum_{k=-\infty}^{-1} \sum_{l=k}^{-1} |\lambda_l|^q \, 2^{\alpha(0)(k-l)q} \\
&\quad + \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \sum_{l=0}^{\infty} |\lambda_l|^q \, 2^{-\alpha_\infty lq} \\
&\lesssim \sum_{l=-\infty}^{-1} |\lambda_l|^q \sum_{k=-\infty}^{l} 2^{\alpha(0)(k-l)q} \\
&\quad + \sum_{l=0}^{\infty} |\lambda_l|^q \, 2^{-\alpha_\infty lq} \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \\
&\lesssim \sum_{l=-\infty}^{-1} |\lambda_l|^q + \sum_{l=0}^{\infty} 2^{-l\lambda q} |\lambda_l|^q \, 2^{-\alpha_\infty lq} \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \\
&\lesssim \Lambda + \Lambda \sum_{i=-\infty}^{l} |\lambda_i|^q \sum_{l=0}^{\infty} 2^{(\lambda-\alpha_\infty)lq} \sum_{k=-\infty}^{l} 2^{\alpha(0)kq} \\
&\lesssim \Lambda.
\end{aligned}
\tag{64}
$$

$$
\begin{aligned}
II_2 &= \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{l=-\infty}^{k-1} |\lambda_l| \, \|(G_N a_l)\chi_k\|_{L^{p(\cdot)}} \right)^q \\
&\lesssim \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{l=-\infty}^{k-1} |\lambda_l| \, 2^{l(m+1)-k(n+m+1)} \right. \\
&\qquad \left. \times |B_l|^{-\alpha_l/n} \, \|\chi_{B_l}\|_{L^{p'(\cdot)}} \, \|\chi_{B_k}\|_{L^{p(\cdot)}} \right)^q \\
&\lesssim \sum_{k=-\infty}^{-1} \left(\sum_{l=-\infty}^{k-1} |\lambda_l| \, 2^{(l-k)(m+1+n\delta_2)-(l-k)\alpha(0)} \right)^q \\
&\lesssim \sum_{k=-\infty}^{-1} \left(\sum_{l=-\infty}^{k-1} |\lambda_l| \, 2^{(l-k)(m+1+n\delta_2-\alpha(0))} \right)^q \\
&\lesssim \sum_{k=-\infty}^{-1} \left(\sum_{l=-\infty}^{k-1} |\lambda_l|^q \, 2^{(l-k)(m+1+n\delta_2-\alpha(0))q} \right) \\
&= \sum_{l=-\infty}^{-2} \sum_{k=l+1}^{-1} |\lambda_l|^q \, 2^{(l-k)(m+1+n\delta_2-\alpha(0))q} \\
&\lesssim \sum_{l=-\infty}^{-1} |\lambda_l|^q \leqslant \Lambda.
\end{aligned}
\tag{65}
$$

Third, we estimate III:

$$
\begin{aligned}
III_1 &= \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{kq\alpha_\infty} |B_k|^{\alpha_\infty q/n} \left(\sum_{l=k}^{\infty} |\lambda_l| \, \|a_l\|_{L^{p(\cdot)}} \right)^q \\
&\lesssim \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{kq\alpha_\infty} |B_k|^{\alpha_\infty q/n} \left(\sum_{l=k}^{\infty} |\lambda_l| \, |B_l|^{-\alpha_l/n} \right)^q \\
&\lesssim \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} \sum_{l=k}^{\infty} |B_k|^{\alpha_\infty q/n} |\lambda_l|^q \, |B_l|^{-\alpha_\infty q/n} \\
&\lesssim \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} \sum_{l=k}^{\infty} |\lambda_l|^q \, 2^{(k-l)\alpha_\infty q} \\
&= \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \\
&\quad \times \left[\sum_{l=0}^{L} |\lambda_l|^q \sum_{k=0}^{l} 2^{(k-l)\alpha_\infty q} + \sum_{l=L}^{\infty} |\lambda_l|^q \sum_{k=0}^{L} 2^{(k-l)\alpha_\infty q} \right] \\
&\lesssim \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{l=0}^{L} |\lambda_l|^q \\
&\quad + \sup_{L>0, L\in\mathbb{Z}} \sum_{l=L}^{\infty} 2^{(l\lambda q - L\lambda q)} 2^{-l\lambda q} \sum_{i=-\infty}^{l} |\lambda_i|^q \sum_{k=0}^{L} 2^{(k-l)\alpha_\infty q/2} \\
&\lesssim \Lambda + \Lambda \sup_{L>0, L\in\mathbb{Z}} \sum_{l=L}^{\infty} 2^{(l-L)\lambda q} 2^{(L-l)\alpha_\infty q/2} \\
&\lesssim \Lambda + \Lambda \sup_{L>0, L\in\mathbb{Z}} \sum_{l=L}^{\infty} 2^{(l-L)q(\lambda-\alpha_\infty/2)} \\
&\lesssim \Lambda.
\end{aligned}
$$

$$
\begin{aligned}
III_2 &= \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{kq\alpha_\infty} \left(\sum_{l=-\infty}^{k-1} |\lambda_l| \, \|(G_N a_l)\chi_k\|_{L^{p(\cdot)}} \right)^q \\
&\lesssim \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \\
&\quad \times \sum_{k=0}^{L} 2^{kq\alpha_\infty} \left(\sum_{l=-\infty}^{k-1} |\lambda_l| \, 2^{l(m+1)-k(n+m+1)} \right. \\
&\qquad \left. \times |B_l|^{-\alpha_l/n} \, \|\chi_{B_l}\|_{L^{p'(\cdot)}} \, \|\chi_{B_k}\|_{L^{p(\cdot)}} \right)^q \\
&\lesssim \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} \left(\sum_{l=-\infty}^{k-1} |\lambda_l| \, 2^{(l-k)(m+1+n\delta_2)-l\alpha_l+k\alpha_\infty} \right)^q \\
&\lesssim \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} \left(\sum_{l=-\infty}^{-1} |\lambda_l| \, 2^{(l-k)(m+1+n\delta_2)-l\alpha(0)+k\alpha_\infty} \right)^q \\
&\quad + \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} \left(\sum_{l=0}^{k-1} |\lambda_l| \, 2^{(l-k)(m+1+n\delta_2-\alpha_\infty)} \right)^q
\end{aligned}
$$

$$\begin{aligned}
&\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}2^{qk[\alpha_\infty-(m+1+n\delta_2)]}\\
&\quad\times\left(\sum_{l=-\infty}^{-1}|\lambda_l|\,2^{l(m+1+n\delta_2-\alpha(0))}\right)^q\\
&\quad+\sup_{L>0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=0}^{L}\sum_{l=0}^{k-1}|\lambda_l|^q\,2^{(l-k)(m+1+n\delta_2-\alpha_\infty)q}\\
&\lesssim \sup_{L>0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{l=-\infty}^{-1}|\lambda_l|^q\,2^{l(m+1+n\delta_2-\alpha(0))q}\\
&\quad+\sup_{L>0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{l=0}^{L-1}\sum_{k=l+1}^{L}|\lambda_l|^q\,2^{(l-k)(m+1+n\delta_2-\alpha_\infty)q}\\
&\lesssim \sup_{L>0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{l=-\infty}^{-1}|\lambda_l|^q+\sup_{L>0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{l=0}^{L}|\lambda_l|^q\\
&=\sup_{L>0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{l=-\infty}^{L}|\lambda_l|^q.
\end{aligned}\tag{66}$$

If $1<q<\infty$, we have

$$\begin{aligned}
I_1&=\sup_{L\leqslant0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{kq\alpha(0)}\left(\sum_{l=k}^{\infty}|\lambda_l|\,\|a_l\|_{L^{p(\cdot)}}\right)^q\\
&\lesssim\sup_{L\leqslant0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{kq\alpha(0)}\left(\sum_{l=k}^{\infty}|\lambda_l|\,|B_l|^{-\alpha_l/n}\right)^q\\
&\lesssim\sup_{L\leqslant0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}\left(\sum_{l=k}^{-1}|\lambda_l|\,2^{\alpha(0)(k-l)}\right)^q\\
&\quad+\sup_{L\leqslant0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{\alpha(0)kq}\left(\sum_{l=0}^{\infty}|\lambda_l|\,2^{-\alpha_\infty l}\right)^q\\
&\lesssim\sup_{L\leqslant0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}\left(\sum_{l=k}^{-1}|\lambda_l|^q\,2^{\alpha(0)(k-l)q/2}\right)\\
&\quad\times\left(\sum_{l=k}^{-1}2^{\alpha(0)(k-l)q'/2}\right)^{q/q'}\\
&\quad+\sup_{L\leqslant0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{\alpha(0)kq}\left(\sum_{l=0}^{\infty}|\lambda_l|^q\,2^{-\alpha_\infty jq/2}\right)\\
&\quad\times\left(\sum_{l=0}^{\infty}2^{-\alpha_\infty jq'/2}\right)^{q/q'}\\
&\lesssim\sup_{L\leqslant0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}\sum_{l=k}^{-1}|\lambda_l|^q\,2^{\alpha(0)(k-l)q/2}\\
&\quad+\sup_{L\leqslant0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{\alpha(0)kq}\sum_{l=0}^{\infty}|\lambda_l|^q\,2^{-\alpha_\infty lq/2}\\
&\lesssim\sup_{L\leqslant0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{l=-\infty}^{-1}|\lambda_l|^q\sum_{k=-\infty}^{l}2^{\alpha(0)(k-l)q/2}\\
&\quad+\sup_{L\leqslant0,L\in\mathbb{Z}}\sum_{l=0}^{\infty}2^{-l\lambda q}|\lambda_l|^q\,2^{(\lambda-\alpha_\infty/2)lq}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{\alpha(0)kq}\\
&\lesssim\sup_{L\leqslant0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{l=-\infty}^{L}|\lambda_l|^q\\
&\quad+\sup_{L\leqslant0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{l=L}^{-1}|\lambda_l|^q\sum_{k=-\infty}^{l}2^{\alpha(0)(k-l)q/2}\\
&\quad+\Lambda\sup_{L\leqslant0,L\in\mathbb{Z}}\sum_{l=0}^{\infty}2^{(\lambda-\alpha_\infty/2)lq}\sum_{k=-\infty}^{L}2^{\alpha(0)kq-L\lambda q}\\
&\lesssim\Lambda+\sup_{L\leqslant0,L\in\mathbb{Z}}\sum_{l=L}^{-1}2^{-l\lambda q}|\lambda_l|^q\,2^{(l-L)\lambda q}\\
&\quad\times\sum_{k=-\infty}^{l}2^{\alpha(0)(k-l)q/2}+\Lambda\\
&\lesssim\Lambda+\Lambda\sup_{L\leqslant0,L\in\mathbb{Z}}\sum_{l=L}^{-1}2^{(l-L)\lambda q}\sum_{k=-\infty}^{j}2^{\alpha(0)(k-l)q/2}\\
&\lesssim\Lambda.
\end{aligned}$$

$$\begin{aligned}
I_2&=\sup_{L\leqslant0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{kq\alpha(0)}\left(\sum_{l=-\infty}^{k-1}|\lambda_l|\,\|(G_Na_l)\chi_k\|_{L^{p(\cdot)}}\right)^q\\
&\lesssim\sup_{L\leqslant0,L\in\mathbb{Z}}2^{-L\lambda q}\\
&\quad\times\sum_{k=-\infty}^{L}2^{kq\alpha(0)}\Bigg(\sum_{l=-\infty}^{k-1}|\lambda_l|\,2^{l(m+1)-k(n+m+1)}\\
&\qquad\times|B_l|^{-\alpha_l/n}\,\|\chi_{B_l}\|_{L^{p'(\cdot)}}\,\|\chi_{B_k}\|_{L^{p(\cdot)}}\Bigg)^q\\
&\lesssim\sup_{L\leqslant0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}\left(\sum_{l=-\infty}^{k-1}|\lambda_l|\,2^{(l-k)(m+1+n\delta_2)-(l-k)\alpha(0)}\right)^q\\
&\lesssim\sup_{L\leqslant0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}\left(\sum_{l=-\infty}^{k-1}|\lambda_l|\,2^{(l-k)(m+1+n\delta_2-\alpha(0))}\right)^q\\
&\lesssim\sup_{L>0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}\left(\sum_{l=-\infty}^{k-1}|\lambda_l|^q\,2^{(l-k)(m+1+n\delta_2-\alpha(0))q/2}\right)\\
&\quad\times\left(\sum_{l=-\infty}^{k-1}2^{(l-k)(m+1+n\delta_2-\alpha(0))q'/2}\right)^{q/q'}\\
&\lesssim\sup_{L\leqslant0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}\left(\sum_{l=-\infty}^{k-1}|\lambda_l|^q\,2^{(l-k)(m+1+n\delta_2-\alpha(0))q/2}\right)\\
&=\sup_{L\leqslant0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{l=-\infty}^{L-1}\sum_{k=l+1}^{L}|\lambda_l|^q\,2^{(l-k)(m+1+n\delta_2-\alpha(0))q/2}\\
&\lesssim\sup_{L\leqslant0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{l=-\infty}^{L}|\lambda_l|^q\\
&\lesssim\Lambda.
\end{aligned}\tag{67}$$

Second, we estimate II. As the same argument before, we obtain that

$$II_1 = \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{l=k}^{\infty} |\lambda_l| \|a_l\|_{L^{p(\cdot)}} \right)^q$$

$$\lesssim \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{l=k}^{\infty} |\lambda_l| |B_l|^{-\alpha_l/n} \right)^q$$

$$\lesssim \sum_{k=-\infty}^{-1} \left(\sum_{j=k}^{-1} |\lambda_j| 2^{\alpha(0)(k-j)} \right)^q + \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \left(\sum_{j=0}^{\infty} |\lambda_j| 2^{-\alpha_\infty j} \right)^q$$

$$\lesssim \sum_{k=-\infty}^{-1} \left(\sum_{l=k}^{-1} |\lambda_l|^q 2^{\alpha(0)(k-l)q/2} \right) \left(\sum_{l=k}^{-1} 2^{\alpha(0)(k-l)q'/2} \right)^{q/q'} + \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq} \left(\sum_{l=0}^{\infty} |\lambda_j|^q 2^{-\alpha_\infty lq/2} \right) \left(\sum_{l=0}^{\infty} 2^{-\alpha_\infty lq'/2} \right)^{q/q'}$$

$$\lesssim \sum_{k=-\infty}^{-1} |\lambda_l|^q \sum_{k=-\infty}^{l} 2^{\alpha(0)(k-l)q/2} + \sum_{l=0}^{\infty} |\lambda_l|^q 2^{-\alpha_\infty lq/2} \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq}$$

$$\lesssim \sum_{l=-\infty}^{-1} |\lambda_l|^q + \sum_{l=0}^{\infty} 2^{(\lambda-\alpha_\infty/2)lq} 2^{-l\lambda q} \sum_{i=-\infty}^{l} |\lambda_i|^q \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq}$$

$$\lesssim \Lambda + \Lambda \sum_{l=0}^{\infty} 2^{(\lambda-\alpha_\infty/2)lq} \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq}$$

$$\lesssim \Lambda.$$

$$II_2 = \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{l=-\infty}^{k-1} |\lambda_l| \|(G_N a_l)\chi_k\|_{L^{p(\cdot)}} \right)^q$$

$$\lesssim \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left(\sum_{l=-\infty}^{k-1} |\lambda_l| 2^{l(m+1)-k(n+m+1)} \times |B_l|^{-\alpha_l/n} \|\chi_{B_l}\|_{L^{p'(\cdot)}} \|\chi_{B_k}\|_{L^{p(\cdot)}} \right)^q$$

$$\lesssim \sum_{k=-\infty}^{-1} \left(\sum_{l=-\infty}^{k-1} |\lambda_l| 2^{(l-k)(m+1+n\delta_2)-(l-k)\alpha(0)} \right)^q$$

$$\lesssim \sum_{k=-\infty}^{-1} \left(\sum_{l=-\infty}^{k-1} |\lambda_l| 2^{(l-k)(m+1+n\delta_2-\alpha(0))} \right)^q$$

$$\lesssim \sum_{k=-\infty}^{-1} \left(\sum_{l=-\infty}^{k-1} |\lambda_l|^q 2^{(l-k)(m+1+n\delta_2-\alpha(0))q/2} \right) \times \left(\sum_{l=-\infty}^{k-1} 2^{(l-k)(m+1+n\delta_2-\alpha(0))q'/2} \right)^{q/q'}$$

$$\lesssim \sum_{k=-\infty}^{-1} \left(\sum_{l=-\infty}^{k-1} |\lambda_l|^q 2^{(l-k)(m+1+n\delta_2-\alpha(0))q/2} \right)$$

$$= \sum_{l=-\infty}^{-2} \sum_{k=l+1}^{-1} |\lambda_l|^q 2^{(l-k)(m+1+n\delta_2-\alpha(0))q/2}$$

$$\lesssim \sum_{l=-\infty}^{-1} |\lambda_l|^q$$

$$\lesssim \Lambda. \tag{68}$$

Third, we estimate III. We have

$$III_1 = \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{kq\alpha_\infty} |B_k|^{\alpha_\infty q/n} \left(\sum_{l=k}^{\infty} |\lambda_l| \|a_l\|_{L^{p(\cdot)}} \right)^q$$

$$\lesssim \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{kq\alpha_\infty} |B_k|^{\alpha_\infty q/n} \left(\sum_{l=k}^{\infty} |\lambda_l| |B_l|^{-\alpha_l/n} \right)^q$$

$$\lesssim \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{kq\alpha_\infty} |B_k|^{\alpha_\infty q/n} \left(\sum_{l=k}^{\infty} |\lambda_l|^q |B_l|^{-\alpha_l q/(2n)} \right) \times \left(\sum_{l=k}^{\infty} |B_l|^{-\alpha_l q'/(2n)} \right)^{q/q'}$$

$$\lesssim \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} \sum_{l=k}^{\infty} |B_k|^{\alpha_\infty q/(2n)} |\lambda_l|^q |B_l|^{-\alpha_\infty q/(2n)}$$

$$\lesssim \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} \sum_{l=k}^{\infty} |\lambda_l|^q 2^{(k-l)\alpha_\infty q/2}$$

$$= \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \left[\sum_{l=0}^{L} |\lambda_l|^q \sum_{k=0}^{l} 2^{(k-l)\alpha_\infty q/2} + \sum_{l=L}^{\infty} |\lambda_l|^q \sum_{k=0}^{L} 2^{(k-l)\alpha_\infty q/2} \right]$$

$$\lesssim \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{l=0}^{L} |\lambda_l|^q + \sup_{L>0, L\in\mathbb{Z}} \sum_{l=L}^{\infty} 2^{(j\lambda q - L\lambda q)} 2^{-l\lambda q} \sum_{i=-\infty}^{l} |\lambda_i|^q \sum_{k=0}^{L} 2^{(k-l)\alpha_\infty q/2}$$

$$\lesssim \Lambda + \Lambda \sup_{L>0, L\in\mathbb{Z}} \sum_{l=L}^{\infty} 2^{(l-L)\lambda q} 2^{(L-l)\alpha_\infty q/2}$$

$$\lesssim \Lambda + \Lambda \sup_{L>0, L\in\mathbb{Z}} \sum_{l=L}^{\infty} 2^{(l-L)q(\lambda-\alpha_\infty/2)}$$

$$\lesssim \Lambda.$$

$$
\begin{aligned}
III_2 &= \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}2^{kq\alpha_\infty}\left(\sum_{l=-\infty}^{k-1}|\lambda_l|\,\|(G_N a_l)\chi_k\|_{L^{p(\cdot)}}\right)^q \\
&\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q} \\
&\quad\times\sum_{k=0}^{L}2^{kq\alpha_\infty}\left(\sum_{l=-\infty}^{k-1}|\lambda_l|\,2^{l(m+1)-k(n+m+1)}\right. \\
&\qquad\left.\times|B_l|^{-\alpha_l/n}\|\chi_{B_l}\|_{L^{p'(\cdot)}}\|\chi_{B_k}\|_{L^{p(\cdot)}}\right)^q \\
&\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}\left(\sum_{l=-\infty}^{k-1}|\lambda_l|\,2^{(l-k)(m+1+n\delta_2)-l\alpha_l+k\alpha_\infty}\right)^q \\
&\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}\left(\sum_{l=-\infty}^{-1}|\lambda_l|\,2^{(l-k)(m+1+n\delta_2)-l\alpha(0)+k\alpha_\infty}\right)^q \\
&\quad+\sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}\left(\sum_{l=0}^{k-1}|\lambda_l|\,2^{(l-k)(m+1+n\delta_2-\alpha_\infty)}\right)^q \\
&\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}2^{qk[\alpha_\infty-(m+1+n\delta_2)]} \\
&\quad\times\left(\sum_{l=-\infty}^{-1}|\lambda_l|\,2^{l(m+1+n\delta_2-\alpha(0))}\right)^q \\
&\quad+\sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}\left(\sum_{l=0}^{k-1}|\lambda_l|^q\,2^{(l-k)(m+1+n\delta_2-\alpha_\infty)q/2}\right) \\
&\quad\times\left(\sum_{l=0}^{k-1}2^{(l-k)(m+1+n\delta_2-\alpha_\infty)q'/2}\right)^{q/q'} \\
&\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\left(\sum_{l=-\infty}^{-1}|\lambda_l|^q\,2^{l(m+1+n\delta_2-\alpha(0))q/2}\right) \\
&\quad\times\left(\sum_{l=-\infty}^{-1}2^{l(m+1+n\delta_2-\alpha(0))q'/2}\right)^{q/q'} \\
&\quad+\sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}\sum_{l=0}^{k-1}|\lambda_l|^q\,2^{(l-k)(m+1+n\delta_2-\alpha_\infty)q/2} \\
&\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{l=-\infty}^{-1}|\lambda_l|^q\,2^{l(m+1+n\delta_2-\alpha(0))q/2} \\
&\quad+\sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{l=0}^{L-1}\sum_{k=l+1}^{L}|\lambda_l|^q\,2^{(l-k)(m+1+n\delta_2-\alpha_\infty)q/2} \\
&\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{l=-\infty}^{-1}|\lambda_l|^q+\sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{l=0}^{L}|\lambda_l|^q \\
&= \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{l=-\infty}^{L}|\lambda_l|^q \\
&\leqslant \Lambda.
\end{aligned}
\tag{69}
$$

Thus, we finish the proof of Theorem 13. □

3. Applications

As an application of the atomic decompositions, we will prove the following result.

Theorem 14. *Let $0<q<\infty$, $0\le\lambda<\infty$, $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, and $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)$ be log-Hölder continuous both at the origin and infinity, $2\lambda\le\alpha(\cdot)$, $n\delta_2\leqslant\alpha(0)$, $\alpha_\infty<\infty$, nonnegative integer and $s=\max\{[\alpha(0)-n\delta_2],[\alpha_\infty-n\delta_2]\}$, and δ_2 as in Lemma 4. If a sublinear operator T satisfies that*

(i) *T is bounded on $L^{p(\cdot)}$;*

(ii) *there exists a constant $\delta>0$ such that $s+\delta>\max\{\alpha(0)-n\delta_2,\alpha_\infty-n\delta_2\}$, and for any compact support function f with*

$$\int_{\mathbb{R}^n} f(x)\,x^\beta\,\mathrm{d}x=0,\qquad |\beta|\leqslant s, \tag{70}$$

Tf satisfies the size condition

$$
\begin{gathered}
|Tf(x)|\leqslant C(\operatorname{diam}(\operatorname{supp} f))^{s+\delta}|x|^{-(n+s+\delta)}\|f\|_1, \\
\text{when } \operatorname{dist}(x,\operatorname{supp} f)\geqslant\frac{|x|}{2}.
\end{gathered}
\tag{71}
$$

Then there exists a constant C such that

$$
\begin{gathered}
\|Tf\|_{M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}}\leqslant C\|f\|_{HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}}, \\
\|Tf\|_{MK^{\alpha(\cdot),q}_{p(\cdot),\lambda}}\leqslant C\|f\|_{HMK^{\alpha(\cdot),q}_{p(\cdot),\lambda}}
\end{gathered}
\tag{72}
$$

for $f\in HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}$ and $f\in HMK^{\alpha(\cdot),q}_{p(\cdot),\lambda}$, respectively.

Proof. It suffices to prove the homogeneous case. Suppose $f\in HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}$. By Theorem 13, $f=\sum_{j=-\infty}^{\infty}\lambda_j b_j$ converges in $\mathcal{S}'(\mathbb{R}^n)$, where each b_j is a central $(\alpha(\cdot),q(\cdot))$-atom with support contained in B_j and

$$\|f\|_{HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}}\approx\inf\sup_{L\in\mathbb{Z}}2^{-L\lambda}\left(\sum_{j=-\infty}^{L}|\lambda_j|^q\right)^{1/q}. \tag{73}$$

For simplicity, we denote $\Lambda=\sup_{L\in\mathbb{Z}}2^{-L\lambda}\sum_{j=-\infty}^{L}|\lambda_j|^q$. By Proposition 7, we have

$$
\begin{aligned}
&\|Tf\|^q_{M\dot{K}^{\alpha(\cdot),q}_{p(\cdot)\lambda}} \\
&\approx\max\left\{\sup_{L\leqslant0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{kq\alpha(0)}\|(Tf)\chi_k\|^q_{L^{p(\cdot)}},\right. \\
&\qquad\sup_{L>0,L\in\mathbb{Z}}2^{-L\lambda q}\left(\sum_{k=-\infty}^{-1}2^{kq\alpha(0)}\|(Tf)\chi_k\|^q_{L^{p(\cdot)}}\right. \\
&\qquad\qquad\left.\left.+\sum_{k=0}^{L}2^{kq\alpha(\infty)}\|(Tf)\chi_k\|^q_{L^{p(\cdot)}}\right)\right\} \\
&\lesssim\max\{I,II+III\},
\end{aligned}
\tag{74}
$$

where

$$I := \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{kq\alpha(0)} \left\|(Tf)\chi_k\right\|_{L^{p(\cdot)}}^{q},$$
$$II := \sum_{k=-\infty}^{-1} 2^{kq\alpha(0)} \left\|(Tf)\chi_k\right\|_{L^{p(\cdot)}}^{q}, \tag{75}$$
$$III := \sup_{L>0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=0}^{L} 2^{kq\alpha(\infty)} \left\|(Tf)\chi_k\right\|_{L^{p(\cdot)}}^{q}.$$

To complete our proof, we only need show that there exists a positive constant C such that $I, II, III \leqslant C\Lambda$.

First, we estimate I:

$$\begin{aligned}
I &= \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{kq\alpha(0)} \left\|(Tf)\chi_k\right\|_{L^{p(\cdot)}}^{q} \\
&\lesssim \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{kq\alpha(0)} \left(\sum_{j=k}^{\infty} \left|\lambda_j\right| \left\|(Tb_j)\chi_k\right\|_{L^{p(\cdot)}}\right)^{q} \\
&\quad + \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{kq\alpha(0)} \left(\sum_{j=-\infty}^{k-1} \left|\lambda_j\right| \left\|(Tb_j)\chi_k\right\|_{L^{p(\cdot)}}\right)^{q} \\
&:= I_1 + I_2.
\end{aligned} \tag{76}$$

By the boundedness of T in $L^{p(\cdot)}$, we have

$$\left\|(Tb_j)\chi_k\right\|_{L^{p(\cdot)}} \leqslant \left\|b_j\right\|_{L^{p(\cdot)}} \leqslant \left|B_j\right|^{-\alpha_j/n} = 2^{-\alpha_j j}. \tag{77}$$

Therefore, when $0 < q \leqslant 1$, we get

$$\begin{aligned}
I_1 &= \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{\alpha(0)kq} \left(\sum_{j=k}^{\infty} \left|\lambda_j\right| \left\|(Tb_j)\chi_k\right\|_{L^{p(\cdot)}}\right)^{q} \\
&\lesssim \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{\alpha(0)kq} \left(\sum_{j=k}^{\infty} \left|\lambda_j\right| 2^{-\alpha_j j}\right)^{q} \\
&\lesssim \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda q} \\
&\quad \times \sum_{k=-\infty}^{L} 2^{\alpha(0)kq} \left(\sum_{j=k}^{-1} \left|\lambda_j\right|^{q} 2^{-\alpha(0)jq} + \sum_{j=0}^{\infty} \left|\lambda_j\right|^{q} 2^{-\alpha_\infty jq}\right) \\
&\lesssim \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} \sum_{j=k}^{-1} \left|\lambda_j\right|^{q} 2^{\alpha(0)(k-j)q} \\
&\quad + \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{\alpha(0)kq} \sum_{j=0}^{\infty} \left|\lambda_j\right|^{q} 2^{-\alpha_\infty jq} \\
&\lesssim \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^{-1} \left|\lambda_j\right|^{q} \sum_{k=-\infty}^{j} 2^{\alpha(0)(k-j)q} \\
&\quad + \sup_{L\leqslant 0, L\in\mathbb{Z}} \sum_{j=0}^{\infty} 2^{-j\lambda q} \left|\lambda_j\right|^{q} 2^{(\lambda-\alpha_\infty)jq} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{\alpha(0)kq} \\
&\lesssim \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^{L} \left|\lambda_j\right|^{q} \\
&\quad + \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{j=L}^{-1} \left|\lambda_j\right|^{q} \sum_{k=-\infty}^{j} 2^{\alpha(0)(k-j)q} \\
&\quad + \Lambda \sup_{L\leqslant 0, L\in\mathbb{Z}} \sum_{j=0}^{\infty} 2^{(\lambda-\alpha_\infty)jq} \sum_{k=-\infty}^{L} 2^{\alpha(0)kq-L\lambda q} \\
&\lesssim \Lambda + \sup_{L\leqslant 0, L\in\mathbb{Z}} \sum_{j=L}^{-1} 2^{-j\lambda q} \left|\lambda_j\right|^{q} 2^{(j-L)\lambda q} \sum_{k=-\infty}^{j} 2^{\alpha(0)(k-j)q} + \Lambda \\
&\lesssim \Lambda + \Lambda \sup_{L\leqslant 0, L\in\mathbb{Z}} \sum_{j=L}^{-1} 2^{(j-L)\lambda q} \sum_{k=-\infty}^{j} 2^{\alpha(0)(k-j)q} \\
&\lesssim \Lambda.
\end{aligned} \tag{78}$$

When $1 < q < \infty$, let $1/q + 1/q' = 1$ and we obtain

$$\begin{aligned}
I_1 &= \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{\alpha(0)kq} \left(\sum_{j=k}^{\infty} \left|\lambda_j\right| \left\|(Tb_j)\chi_k\right\|_{L^{p(\cdot)}}\right)^{q} \\
&\lesssim \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{\alpha(0)kq} \left(\sum_{j=k}^{\infty} |\lambda_j| 2^{-\alpha_j j}\right)^{q} \\
&\lesssim \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} \left(\sum_{j=k}^{-1} \left|\lambda_j\right| 2^{\alpha(0)(k-j)}\right)^{q} \\
&\quad + \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{\alpha(0)kq} \left(\sum_{j=0}^{\infty} \left|\lambda_j\right| 2^{-\alpha_\infty j}\right)^{q} \\
&\lesssim \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} \left(\sum_{j=k}^{-1} \left|\lambda_j\right|^{q} 2^{\alpha(0)(k-j)q/2}\right) \\
&\quad \times \left(\sum_{j=k}^{-1} 2^{\alpha(0)(k-j)q'/2}\right)^{q/q'} \\
&\quad + \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{\alpha(0)kq} \left(\sum_{j=0}^{\infty} \left|\lambda_j\right|^{q} 2^{-\alpha_\infty jq/2}\right) \\
&\quad \times \left(\sum_{j=0}^{\infty} 2^{-\alpha_\infty jq'/2}\right)^{q/q'} \\
&\lesssim \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} \sum_{j=k}^{-1} \left|\lambda_j\right|^{q} 2^{\alpha(0)(k-j)q/2} \\
&\quad + \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{k=-\infty}^{L} 2^{\alpha(0)kq} \sum_{j=0}^{\infty} \left|\lambda_j\right|^{q} 2^{-\alpha_\infty jq/2} \\
&\lesssim \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda q} \sum_{j=-\infty}^{-1} \left|\lambda_j\right|^{q} \sum_{k=-\infty}^{j} 2^{\alpha(0)(k-j)q/2}
\end{aligned}$$

$$\begin{aligned}
&+\sup_{L\leqslant 0,L\in\mathbb{Z}}\sum_{j=0}^{\infty}2^{-j\lambda q}\left|\lambda_j\right|^q 2^{(\lambda-\alpha_\infty/2)jq}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{\alpha(0)kq}\\
&\leqslant \sup_{L\leqslant 0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{j=-\infty}^{L}\left|\lambda_j\right|^q\\
&\quad+\sup_{L\leqslant 0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{j=L}^{-1}\left|\lambda_j\right|^q\sum_{k=-\infty}^{j}2^{\alpha(0)(k-j)q/2}\\
&\quad+\Lambda\sup_{L\leqslant 0,L\in\mathbb{Z}}\sum_{j=0}^{\infty}2^{(\lambda-\alpha_\infty/2)jq}\sum_{k=-\infty}^{L}2^{\alpha(0)kq-L\lambda q}\\
&\lesssim \Lambda+\sup_{L\leqslant 0,L\in\mathbb{Z}}\sum_{j=L}^{-1}2^{-j\lambda q}\left|\lambda_j\right|^q 2^{(j-L)\lambda q}\sum_{k=-\infty}^{j}2^{\alpha(0)(k-j)q/2}+\Lambda\\
&\lesssim \Lambda+\Lambda\sup_{L\leqslant 0,L\in\mathbb{Z}}\sum_{j=L}^{-1}2^{(j-L)\lambda q}\sum_{k=-\infty}^{j}2^{\alpha(0)(k-j)q/2}\\
&\lesssim \Lambda.
\end{aligned}\tag{79}$$

So, we have $I_1 \lesssim \Lambda$.

Second, we estimate I_2. By (71) and Lemma 3, we get

$$\begin{aligned}
\left|Tb_j(x)\right| &\lesssim |x|^{-(n+s+\delta)}2^{j(s+\delta)}\int_{B_j}\left|b_j(y)\right|\mathrm{d}y\\
&\lesssim 2^{-k(n+s+\delta)}2^{j(s+\delta)}\left\|b_j\right\|_{L^{p(\cdot)}}\left\|\chi_{B_j}\right\|_{L^{p'(\cdot)}}\\
&\lesssim 2^{j(s+\delta-\alpha_j)-k(s+\delta+n)}\left\|\chi_{B_j}\right\|_{L^{p'(\cdot)}}.
\end{aligned}\tag{80}$$

So by Lemmas 3 and 4, we have

$$\begin{aligned}
&\left\|(Tb_j)\chi_k\right\|_{L^{p(\cdot)}}\\
&\quad\lesssim 2^{j(s+\delta-\alpha_j)-k(s+\delta+n)}\left\|\chi_{B_j}\right\|_{L^{p'(\cdot)}}\left\|\chi_{B_k}\right\|_{L^{p(\cdot)}}\\
&\quad\lesssim 2^{j(s+\delta-\alpha_j)-k(s+\delta)}2^{-kn}\left(\left|B_k\right|\left\|\chi_{B_k}\right\|_{L^{p'(\cdot)}}^{-1}\right)\left\|\chi_{B_j}\right\|_{L^{p'(\cdot)}}\\
&\quad\lesssim 2^{j(s+\delta-\alpha_j)-k(s+\delta)}\frac{\left\|\chi_{B_j}\right\|_{L^{p'(\cdot)}}}{\left\|\chi_{B_k}\right\|_{L^{p'(\cdot)}}}\\
&\quad\lesssim 2^{(s+\delta+n\delta_2)(j-k)-j\alpha_j}.
\end{aligned}\tag{81}$$

Therefore, when $0<q\leqslant 1$, by $n\delta_2\leqslant\alpha(0)<s+\delta+n\delta_2$ we get

$$\begin{aligned}
I_2&=\sup_{L\leqslant 0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{\alpha(0)kq}\left(\sum_{j=-\infty}^{k-1}\left|\lambda_j\right|\left\|(Tb_j)\chi_k\right\|_{L^{p(\cdot)}}\right)^q\\
&\lesssim\sup_{L\leqslant 0,L\in\mathbb{Z}}2^{-L\lambda q}\\
&\quad\times\sum_{k=-\infty}^{L}2^{\alpha(0)kq}\left(\sum_{j=-\infty}^{k-1}\left|\lambda_j\right|^q 2^{[(s+\delta+n\delta_2)(j-k)-j\alpha(0)]q}\right)\\
&=\sup_{L\leqslant 0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{j=-\infty}^{L}\left|\lambda_j\right|^q\sum_{k=j+1}^{-1}2^{(j-k)(s+\delta+n\delta_2-\alpha(0))q}\\
&\lesssim\Lambda.
\end{aligned}\tag{82}$$

When $1<q<\infty$, let $1/q+1/q'=1$. Since $n\delta_2\leqslant\alpha(0)<s+\delta+n\delta_2$, by Hölder's inequality, we have

$$\begin{aligned}
I_2&=\sup_{L\leqslant 0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{\alpha(0)kq}\left(\sum_{j=-\infty}^{k-1}\left|\lambda_j\right|\left\|(Tb_j)\chi_k\right\|_{L^{p(\cdot)}}\right)^q\\
&\lesssim\sup_{L\leqslant 0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=-\infty}^{L}2^{\alpha(0)kq}\left(\sum_{j=-\infty}^{k-1}|\lambda_j|2^{(s+\delta+n\delta_2)(j-k)-j\alpha(0)}\right)^q\\
&\lesssim\sup_{L\leqslant 0,L\in\mathbb{Z}}2^{-L\lambda q}\\
&\quad\times\sum_{k=-\infty}^{L}2^{\alpha(0)kq}\left(\sum_{j=-\infty}^{k-1}\left|\lambda_j\right|^q 2^{(j-k)(s+\delta+n\delta_2-\alpha(0))q/2}\right)\\
&\quad\times\left(\sum_{j=-\infty}^{k-1}2^{(j-k)(s+\delta+n\delta_2-\alpha(0))q'/2}\right)^{q/q'}\\
&\lesssim\sup_{L\leqslant 0,L\in\mathbb{Z}}2^{-L\lambda q}\\
&\quad\times\sum_{j=-\infty}^{L}2^{\alpha(0)kq}\left(\sum_{j=-\infty}^{k-1}\left|\lambda_j\right|^q 2^{(j-k)(s+\delta+n\delta_2-\alpha(0))q/2}\right)\\
&=\sup_{L\leqslant 0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{j=-\infty}^{L}\left|\lambda_j\right|^q\sum_{k=j+1}^{-1}2^{(j-k)(s+\delta+n\delta_2-\alpha(0))q/2}\\
&\lesssim\Lambda.
\end{aligned}\tag{83}$$

Hence, we have $I\lesssim\Lambda$.

Third, we estimate II. Consider

$$\begin{aligned}
II&=\sum_{k=-\infty}^{-1}2^{kq\alpha(0)}\left\|(Tf)\chi_k\right\|_{L^{p(\cdot)}}^q\\
&\lesssim\sum_{k=-\infty}^{-1}2^{kq\alpha(0)}\left(\sum_{j=k}^{\infty}\left|\lambda_j\right|\left\|(Tb_j)\chi_k\right\|_{L^{p(\cdot)}}\right)^q\\
&\quad+\sum_{k=-\infty}^{-1}2^{kq\alpha(0)}\left(\sum_{j=-\infty}^{k-1}\left|\lambda_j\right|\left\|(Tb_j)\chi_k\right\|_{L^{p(\cdot)}}\right)^q\\
&:=II_1+II_2.
\end{aligned}\tag{84}$$

When $0<q\leqslant 1$, we get

$$\begin{aligned}
II_1&=\sum_{k=-\infty}^{-1}2^{\alpha(0)kq}\left(\sum_{j=k}^{\infty}\left|\lambda_j\right|\left\|(Tb_j)\chi_k\right\|_{L^{p(\cdot)}}\right)^q\\
&\lesssim\sum_{k=-\infty}^{-1}2^{\alpha(0)kq}\left(\sum_{j=k}^{\infty}\left|\lambda_j\right|2^{-\alpha_j j}\right)^q
\end{aligned}$$

$$\begin{aligned}
&\lesssim \sum_{k=-\infty}^{-1} 2^{\alpha(0)kq}\left(\sum_{j=k}^{-1}\left|\lambda_j\right|^q 2^{-\alpha(0)jq}+\sum_{j=0}^{\infty}\left|\lambda_j\right|^q 2^{-\alpha_\infty jq}\right)\\
&\lesssim \sum_{k=-\infty}^{-1}\sum_{j=k}^{-1}\left|\lambda_j\right|^q 2^{\alpha(0)(k-j)q}+\sum_{k=-\infty}^{-1}2^{\alpha(0)kq}\sum_{j=0}^{\infty}\left|\lambda_j\right|^q 2^{-\alpha_\infty jq}\\
&\lesssim \sum_{j=-\infty}^{-1}\left|\lambda_j\right|^q\sum_{k=-\infty}^{j}2^{\alpha(0)(k-j)q}\\
&\quad+\sum_{j=0}^{\infty}\left|\lambda_j\right|^q 2^{-\alpha_\infty jq}\sum_{k=-\infty}^{-1}2^{\alpha(0)kq}\\
&\lesssim \sum_{j=-\infty}^{-1}\left|\lambda_j\right|^q+\sum_{j=0}^{\infty}2^{-j\lambda q}\left|\lambda_j\right|^q 2^{-\alpha_\infty jq}\sum_{k=-\infty}^{-1}2^{\alpha(0)kq}\\
&\lesssim \Lambda+\Lambda\sum_{i=-\infty}^{j}\left|\lambda_i\right|^q\sum_{j=0}^{\infty}2^{(\lambda-\alpha_\infty)jq}\sum_{k=-\infty}^{j}2^{\alpha(0)kq}\\
&\lesssim \Lambda.
\end{aligned}\tag{85}$$

When $1<q<\infty$, let $1/q+1/q'=1$ and we obtain

$$\begin{aligned}
II_1 &= \sum_{k=-\infty}^{-1}2^{\alpha(0)kq}\left(\sum_{j=k}^{\infty}\left|\lambda_j\right|\left\|(Tb_j)\chi_k\right\|_{L^{p(\cdot)}}\right)^q\\
&\lesssim \sum_{k=-\infty}^{-1}2^{\alpha(0)kq}\left(\sum_{j=k}^{\infty}\left|\lambda_j\right|2^{-\alpha_j j}\right)^q\\
&\lesssim \sum_{k=-\infty}^{-1}\left(\sum_{j=k}^{-1}\left|\lambda_j\right|2^{\alpha(0)(k-j)}\right)^q\\
&\quad+\sum_{k=-\infty}^{-1}2^{\alpha(0)kq}\left(\sum_{j=0}^{\infty}\left|\lambda_j\right|2^{-\alpha_\infty j}\right)^q\\
&\lesssim \sum_{k=-\infty}^{-1}\left(\sum_{j=k}^{-1}\left|\lambda_j\right|^q 2^{\alpha(0)(k-j)q/2}\right)\left(\sum_{j=k}^{-1}2^{\alpha(0)(k-j)q'/2}\right)^{q/q'}\\
&\quad+\sum_{k=-\infty}^{-1}2^{\alpha(0)kq}\left(\sum_{j=0}^{\infty}\left|\lambda_j\right|^q 2^{-\alpha_\infty jq/2}\right)\left(\sum_{j=0}^{\infty}2^{-\alpha_\infty jq'/2}\right)^{q/q'}\\
&\lesssim \sum_{k=-\infty}^{-1}\left|\lambda_j\right|^q\sum_{k=-\infty}^{j}2^{\alpha(0)(k-j)q/2}\\
&\quad+\sum_{j=0}^{\infty}\left|\lambda_j\right|^q 2^{-\alpha_\infty jq/2}\sum_{k=-\infty}^{-1}2^{\alpha(0)kq}\\
&\lesssim \sum_{j=-\infty}^{-1}\left|\lambda_j\right|^q\\
&\quad+\sum_{j=0}^{\infty}2^{(\lambda-\alpha_\infty/2)jq}2^{-j\lambda q}\sum_{i=-\infty}^{j}\left|\lambda_i\right|^q\sum_{k=-\infty}^{-1}2^{\alpha(0)kq}\\
&\lesssim \Lambda+\Lambda\sum_{j=0}^{\infty}2^{(\lambda-\alpha_\infty/2)jq}\sum_{k=-\infty}^{-1}2^{\alpha(0)kq}\\
&\lesssim \Lambda.
\end{aligned}\tag{86}$$

For II_2, when $0<q\leqslant 1$, by $n\delta_2\leqslant\alpha(0)<s+\delta+n\delta_2$ we get

$$\begin{aligned}
II_2 &= \sum_{k=-\infty}^{-1}2^{\alpha(0)kq}\left(\sum_{j=-\infty}^{k-1}\left|\lambda_j\right|\left\|\left(Tb_j\right)\chi_k\right\|_{L^{p(\cdot)}}\right)^q\\
&\lesssim \sum_{k=-\infty}^{-1}2^{\alpha(0)kq}\left(\sum_{j=-\infty}^{k-1}\left|\lambda_j\right|^q 2^{[(s+\delta+n\delta_2)(j-k)-j\alpha(0)]q}\right)\\
&= \sum_{k=-\infty}^{-1}\left|\lambda_j\right|^q\sum_{k=j+1}^{-1}2^{(j-k)(s+\delta+n\delta_2-\alpha(0))q}\\
&\lesssim \sum_{k=-\infty}^{-1}\left|\lambda_j\right|^q\\
&\leqslant \Lambda.
\end{aligned}\tag{87}$$

When $1<q<\infty$, let $1/q+1/q'=1$. Since $n\delta_2\leqslant\alpha(0)<s+\delta+n\delta_2$, by Hölder's inequality, we have

$$\begin{aligned}
II_2 &= \sum_{k=-\infty}^{-1}2^{\alpha(0)kq}\left(\sum_{j=-\infty}^{k-1}\left|\lambda_j\right|\left\|(Tb_j)\chi_k\right\|_{L^{p(\cdot)}}\right)^q\\
&\lesssim \sum_{k=-\infty}^{-1}2^{\alpha(0)kq}\left(\sum_{j=-\infty}^{k-1}\left|\lambda_j\right|2^{(s+\delta+n\delta_2)(j-k)-j\alpha(0)}\right)^q\\
&\lesssim \sum_{k=-\infty}^{-1}2^{\alpha(0)kq}\left(\sum_{j=-\infty}^{k-1}\left|\lambda_j\right|^q 2^{(j-k)(s+\delta+n\delta_2-\alpha(0))q/2}\right)\\
&\quad\times\left(\sum_{j=-\infty}^{k-1}2^{(j-k)(s+\delta+n\delta_2-\alpha(0))q'/2}\right)^{q/q'}\\
&\lesssim \sum_{j=-\infty}^{-1}2^{\alpha(0)kq}\left(\sum_{j=-\infty}^{k-1}\left|\lambda_j\right|^q 2^{(j-k)(s+\delta+n\delta_2-\alpha(0))q/2}\right)\\
&= \sum_{j=-\infty}^{-1}\left|\lambda_j\right|^q\sum_{k=j+1}^{-1}2^{(j-k)(s+\delta+n\delta_2-\alpha(0))q/2}\\
&\lesssim \Lambda.
\end{aligned}\tag{88}$$

So, we have $II\lesssim\Lambda$.

Finally, we estimate III:

$$\begin{aligned}
III &= \sup_{L>0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=0}^{L}2^{kq\alpha_\infty}\left\|(Tf)\chi_k\right\|_{L^{p(\cdot)}}^q\\
&\lesssim \sup_{L>0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=0}^{L}2^{kq\alpha_\infty}\left(\sum_{j=k}^{\infty}\left|\lambda_j\right|\left\|(Tb_j)\chi_k\right\|_{L^{p(\cdot)}}\right)^q\\
&\quad+\sup_{L>0,L\in\mathbb{Z}}2^{-L\lambda q}\sum_{k=0}^{L}2^{kq\alpha_\infty}\left(\sum_{j=-\infty}^{k-1}\left|\lambda_j\right|\left\|(Tb_j)\chi_k\right\|_{L^{p(\cdot)}}\right)^q\\
&:= III_1+III_2.
\end{aligned}\tag{89}$$

When $0 < q \leqslant 1$, by the boundedness of T in $L^{p(\cdot)}$, we have

$$\begin{aligned}
III_1 &= \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}2^{\alpha_\infty kq}\left(\sum_{j=k}^{\infty}\left|\lambda_j\right|\left\|(Tb_j)\chi_k\right\|_{L^{p(\cdot)}}\right)^q \\
&\leqslant \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}2^{\alpha_\infty kq}\sum_{j=k}^{\infty}\left|\lambda_j\right|^q\left\|(Tb_j)\chi_k\right\|_{L^{p(\cdot)}}^q \\
&\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}2^{\alpha_\infty kq}\sum_{j=k}^{\infty}\left|\lambda_j\right|^q 2^{-\alpha_j jq} \\
&\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}2^{\alpha_\infty kq}\sum_{j=k}^{\infty}\left|\lambda_j\right|^q 2^{-\alpha_\infty jq} \\
&= \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{j=0}^{L}\left|\lambda_j\right|^q\sum_{k=0}^{j}2^{(k-j)\alpha_\infty q} \\
&\quad + \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{j=L}^{\infty}\left|\lambda_j\right|^q\sum_{k=0}^{L}2^{(k-j)\alpha_\infty q} \\
&\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{j=0}^{L}\left|\lambda_j\right|^q \\
&\quad + \sup_{L>0,L\in\mathbb{Z}} \sum_{j=L}^{\infty}2^{(j\lambda q-L\lambda q)}2^{-j\lambda q}\sum_{i=-\infty}^{j}\left|\lambda_i\right|^q\sum_{k=0}^{L}2^{(k-j)\alpha_\infty q} \\
&\lesssim \Lambda + \Lambda \sup_{L>0,L\in\mathbb{Z}}\sum_{j=L}^{\infty}2^{(j-L)\lambda q}2^{(L-j)\alpha_\infty q} \\
&\lesssim \Lambda + \Lambda \sup_{L>0,L\in\mathbb{Z}}\sum_{j=L}^{\infty}2^{(j-L)q(\lambda-\alpha_\infty)} \\
&\lesssim \Lambda.
\end{aligned} \tag{90}$$

When $1 < q < \infty$, by the boundedness of T in $L^{p(\cdot)}$ and Hölder's inequality, we have

$$\begin{aligned}
III_1 &= \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}2^{\alpha_\infty kq}\left(\sum_{j=k}^{\infty}\left|\lambda_j\right|\left\|(Tb_j)\chi_k\right\|_{L^{p(\cdot)}}\right)^q \\
&\leqslant \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}2^{\alpha_\infty kq}\left(\sum_{j=k}^{\infty}\left|\lambda_j\right|^q\left\|(Tb_j)\chi_k\right\|_{L^{p(\cdot)}}^{q/2}\right) \\
&\quad \times\left(\sum_{j=k}^{\infty}\left\|(Tb_j)\chi_k\right\|_{L^{p(\cdot)}}^{q'/2}\right)^{q/q'} \\
&\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}2^{\alpha_\infty kq}\left(\sum_{j=k}^{\infty}\left|\lambda_j\right|^q\left\|b_j\right\|_{L^{p(\cdot)}}^{q/2}\right) \\
&\quad \times\left(\sum_{j=k}^{\infty}\left\|b_j\right\|_{L^{p(\cdot)}}^{q'/2}\right)^{q/q'} \\
&\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}2^{\alpha_\infty kq}\left(\sum_{j=k}^{\infty}\left|\lambda_j\right|^q\left|B_j\right|^{-\alpha_j q/(2n)}\right) \\
&\quad \times\left(\sum_{j=k}^{\infty}\left|B_j\right|^{-\alpha_j q'/(2n)}\right)^{q/q'} \\
&\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}2^{\alpha_\infty kq/2}\left(\sum_{j=k}^{\infty}\left|\lambda_j\right|^q\left|B_j\right|^{-\alpha_j q/(2n)}\right) \\
&= \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{j=0}^{L}\left|\lambda_j\right|^q\sum_{k=0}^{j}2^{(k-j)\alpha_\infty q/2} \\
&\quad + \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{j=L}^{\infty}\left|\lambda_j\right|^q\sum_{k=0}^{L}2^{(k-j)\alpha_\infty q/2} \\
&\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{j=0}^{L}\left|\lambda_j\right|^q \\
&\quad + \sup_{L>0,L\in\mathbb{Z}} \sum_{j=L}^{\infty}2^{(j\lambda q-L\lambda q)}2^{-j\lambda q}\sum_{i=-\infty}^{j}\left|\lambda_i\right|^q\sum_{k=0}^{L}2^{(k-j)\alpha_\infty q/2} \\
&\lesssim \Lambda + \Lambda \sup_{L>0,L\in\mathbb{Z}}\sum_{j=L}^{\infty}2^{(j-L)\lambda q}2^{(L-j)\alpha_\infty q/2} \\
&\lesssim \Lambda + \Lambda \sup_{L>0,L\in\mathbb{Z}}\sum_{j=L}^{\infty}2^{(j-L)q(\lambda-\alpha_\infty/2)} \\
&\lesssim \Lambda.
\end{aligned} \tag{91}$$

When $0 < q \leqslant 1$, by $n\delta_2 \leqslant \alpha(0)$, $\alpha_\infty < s + \delta + n\delta_2$ we get

$$\begin{aligned}
III_2 &= \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}2^{\alpha_\infty kq}\left(\sum_{j=-\infty}^{k-1}\left|\lambda_j\right|\left\|(Tb_j)\chi_k\right\|_{L^{p(\cdot)}}\right)^q \\
&\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q} \\
&\quad \times\sum_{k=0}^{L}2^{\alpha_\infty kq}\left(\sum_{j=-\infty}^{k-1}\left|\lambda_j\right|^q 2^{[(s+\delta+n\delta_2)(j-k)-j\alpha_j]q}\right) \\
&= \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q} \\
&\quad \times\sum_{k=0}^{L}2^{\alpha_\infty kq}\left(\sum_{j=-\infty}^{-1}\left|\lambda_j\right|^q 2^{[(s+\delta+n\delta_2)(j-k)-j\alpha(0)]q}\right) \\
&\quad + \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q} \\
&\quad \times\sum_{k=0}^{L}2^{\alpha_\infty kq}\left(\sum_{j=0}^{k-1}\left|\lambda_j\right|^q 2^{[(s+\delta+n\delta_2)(j-k)-j\alpha_\infty]q}\right)
\end{aligned}$$

$$\begin{aligned}
&\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}2^{[\alpha_\infty-(s+\delta+n\delta_2)]kq}\\
&\quad\times\sum_{j=-\infty}^{-1}\left|\lambda_j\right|^q 2^{(s+\delta+n\delta_2-\alpha(0))jq}\\
&\quad+\sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}\left|\lambda_j\right|^q\sum_{k=j+1}^{\infty}2^{(j-k)(s+\delta+n\delta_2-\alpha_\infty)q}\\
&\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{j=-\infty}^{-1}\left|\lambda_j\right|^q+\sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{j=0}^{L-1}\left|\lambda_j\right|^q\\
&\lesssim \Lambda.
\end{aligned} \tag{92}$$

When $1 < q < \infty$, let $1/q + 1/q' = 1$. Since $n\delta_2 \leqslant \alpha(0)$, $\alpha_\infty < s+\delta+n\delta_2$, by Hölder's inequality, we have

$$\begin{aligned}
III_2 &= \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}2^{\alpha_\infty kq}\left(\sum_{j=-\infty}^{k-1}\left|\lambda_j\right|\left\|(Tb_j)\chi_k\right\|_{L^{p(\cdot)}}\right)^q\\
&\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}2^{\alpha_\infty kq}\left(\sum_{j=-\infty}^{k-1}\left|\lambda_j\right|2^{[(s+\delta+n\delta_2)(j-k)-j\alpha_j]}\right)^q\\
&\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}2^{\alpha_\infty kq}\left(\sum_{j=-\infty}^{-1}\left|\lambda_j\right|2^{[(s+\delta+n\delta_2)(j-k)-j\alpha(0)]}\right)^q\\
&\quad+\sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}2^{\alpha_\infty kq}\left(\sum_{j=0}^{k-1}\left|\lambda_j\right|2^{[(s+\delta+n\delta_2)(j-k)-j\alpha_\infty]}\right)^q\\
&\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}2^{[\alpha_\infty-(s+\delta+n\delta_2)]kq}\\
&\quad\times\left(\sum_{j=-\infty}^{-1}\left|\lambda_j\right|2^{(s+\delta+n\delta_2-\alpha(0))j}\right)^q\\
&\quad+\sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}\left(\sum_{j=0}^{k-1}\left|\lambda_j\right|2^{(j-k)(s+\delta+n\delta_2-\alpha_\infty)}\right)^q\\
&\lesssim \left(\sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{j=-\infty}^{-1}\left|\lambda_j\right|^q 2^{(s+\delta+n\delta_2-\alpha(0))jq/2}\right)\\
&\quad\times\left(\sum_{j=-\infty}^{-1}2^{(s+\delta+n\delta_2-\alpha(0))jq'/2}\right)^{q/q'}\\
&\quad+\sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}\left(\sum_{j=0}^{k-1}\left|\lambda_j\right|^q 2^{(j-k)(s+\delta+n\delta_2-\alpha_\infty)q/2}\right)\\
&\quad\times\left(\sum_{j=0}^{k-1}2^{(j-k)(s+\delta+n\delta_2-\alpha_\infty)q'/2}\right)^{q/q'}\\
&\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{j=-\infty}^{-1}\left|\lambda_j\right|^q 2^{(s+\delta+n\delta_2-\alpha(0))jq/2}\\
&\quad+\sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{k=0}^{L}\sum_{j=0}^{k-1}\left|\lambda_j\right|^q 2^{(j-k)(s+\delta+n\delta_2-\alpha_\infty)q/2}\\
&\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{j=-\infty}^{-1}\left|\lambda_j\right|^q\\
&\quad+\sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{j=0}^{L-1}\left|\lambda_j\right|^q\sum_{k=j+1}^{L}2^{(j-k)(s+\delta+n\delta_2-\alpha_\infty)q/2}\\
&\lesssim \sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{j=-\infty}^{-1}\left|\lambda_j\right|^q+\sup_{L>0,L\in\mathbb{Z}} 2^{-L\lambda q}\sum_{j=0}^{L-1}\left|\lambda_j\right|^q\\
&\lesssim \Lambda.
\end{aligned} \tag{93}$$

Thus we have $III \lesssim \Lambda$. This finishes the proof of Theorem 14. □

Definition 15. Let K be a locally integrable function on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{x = y\}$. Then K is called a standard kernel if there exist $\delta \in (0, 1]$ and $C > 0$, such that

$$\begin{gathered}
\left|K(x,y)\right| \leqslant \frac{C}{|x-y|^n}, \quad x \neq y;\\
\left|K(x,y)-K(x,y')\right| \leqslant C\frac{|y-y'|^\delta}{|x-y|^{n+\delta}},\\
|y-y'| \leqslant \frac{1}{2}|x-y|;\\
\left|K(x,y)-K(x',y)\right| \leqslant C\frac{|x-x'|^\delta}{|x-y|^{n+\delta}},\\
|x-x'| \leqslant \frac{1}{2}|x-y|.
\end{gathered} \tag{94}$$

A linear operator $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is called a Calderón-Zygmund operator associated to a standard kernel K if

(i) T can be extended to a bounded operator on $L^2(\mathbb{R}^n)$;

(ii) for any $f \in L^2$ with compact support and almost everywhere $x \notin \operatorname{supp} f$,

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y)\,dy. \tag{95}$$

It is well known that a Calderón-Zygmund operator is also bounded in $L^{p(\cdot)}$ for any $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$; for example, see [38].

Corollary 16. *Let $\alpha(\cdot)$ be a bounded and log-Hölder continuous both at the origin and infinity such that $n\delta_2 \leqslant \alpha(0)$,*

$\alpha_\infty < n\delta_2 + \delta$ with δ_2 as in Lemma 4. Suppose T is a Calderón-Zygmund operator associated to a standard kernel K. If $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$, $0 < q < \infty$, and $0 \leq \lambda < \infty$, then there exists a constant C such that

$$\begin{aligned} \|Tf\|_{M\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}} &\leqslant C\|f\|_{HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}}, \\ \|Tf\|_{MK^{\alpha(\cdot),q}_{p(\cdot),\lambda}} &\leqslant C\|f\|_{HMK^{\alpha(\cdot),q}_{p(\cdot),\lambda}} \end{aligned} \tag{96}$$

for $f \in HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}$ and $f \in HMK^{\alpha(\cdot),q}_{p(\cdot),\lambda}$, respectively.

Proof. It is easy to know that $s = \max\{[\alpha(0) - n\delta_2], [\alpha_\infty - n\delta_2]\} = 0$ when $n\delta_2 \leqslant \alpha(0)$ and $\alpha_\infty < n\delta_2 + \delta$. Then the result follows from Theorem 14. □

Remark 17. For Hardy type spaces, there are some characterizations: maximal function, square function, atomic decomposition, and molecular decomposition. To discuss the boundedness of singular integrals in Hardy type spaces, we use the atomic decomposition for the domain Hardy space, while it is convenient to use another characterization of Hardy space for the target Hardy space; for example, see the proof of Theorem 6.7.4 in [49] and [50, 51]. In a future paper, we will give the molecular decomposition of spaces $HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}$ and $HMK^{\alpha(\cdot),q}_{p(\cdot),\lambda}$. Then, by the atomic and molecular decompositions, we will obtain the boundedness of T in Corollary 16 from $HM\dot{K}^{\alpha(\cdot),q}_{p(\cdot),\lambda}$ and $HMK^{\alpha(\cdot),q}_{p(\cdot),\lambda}$ into themselves, respectively.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Decompositions of g-Frames and Duals and Pseudoduals of g-Frames in Hilbert Spaces

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Firstly, we study the representation of g-frames in terms of linear combinations of simpler ones such as g-orthonormal bases, g-Riesz bases, and normalized tight g-frames. Then, we study the dual and pseudodual of g-frames, which are critical components in reconstructions. In particular, we characterize the dual g-frames in a constructive way; that is, the formulae for dual g-frames are given. We also give some g-frame like representations for pseudodual g-frame pairs. The operator parameterizations of g-frames and decompositions of bounded operators are the key tools to prove our main results.

1. Introduction

A sequence $(f_i)_{i\in I}$ of elements of a Hilbert space H is called a *frame* for H if there are constants $A, B > 0$ so that

$$A\|f\|^2 \leq \sum_{i\in I}\left|\langle f, f_i\rangle\right|^2 \leq B\|f\|^2, \quad \forall f \in H. \tag{1}$$

The numbers A and B are called the *lower* (resp., *upper*) frame bounds. The frame is a *tight frame* if $A = B$ and a *normalized tight frame* if $A = B = 1$.

The concept of frame first appeared in the late 40s and early 50s (see [1–3]). The development and study of wavelet theory during the last decades also brought new ideas and attention to frames because of their close connections. There are many related references on this topic, see [4–8].

In [9], Sun raised the concept of g-frame as follows, which generalized the concept of frame extensively. A sequence $\{\Lambda_i \in B(H, H_i) : i \in \mathcal{N}\}$ is called a *g-frame* for H with respect to $\{H_i : i \in \mathcal{N}\}$, which is a sequence of closed subspaces of a Hilbert space K, if there exist two positive constants A and B such that, for any $f \in H$,

$$A\|f\|^2 \leq \sum_{i\in\mathcal{N}}\|\Lambda_i f\|^2 \leq B\|f\|^2, \tag{2}$$

where A is called the *lower g-frame bound* and B is called the *upper g-frame bound*. The largest lower frame bound and the smallest upper frame bound are called the *optimal lower g-frame bound* and the *optimal upper g-frame bound*, respectively. We simply call $\{\Lambda_i : i \in \mathcal{N}\}$ a g-frame for H whenever the space sequence $\{H_i : i \in \mathcal{N}\}$ is clear. The tight g-frame and normalized tight g-frame are defined similarly. We call $\{\Lambda_i : i \in \mathcal{N}\}$ a *g-frame sequence*, if it is a g-frame for $\overline{\text{span}}\{\Lambda_i^*(H_i)\}_{i\in\mathcal{N}}$. We call $\{\Lambda_i : i \in \mathcal{N}\}$ a *g-Bessel sequence*, if only the right inequality is satisfied. A g-frame $\{\Gamma_j : j \in \mathcal{N}\}$ for H is called an *alternate dual g-frame* of $\{\Lambda_j : j \in \mathcal{N}\}$, if for every $f \in H$, we have

$$f = \sum_{j\in N}\Lambda_j * \Gamma_j f = \sum_{j\in\mathcal{N}}\Gamma_j^*\Lambda_j f. \tag{3}$$

If $\{\Lambda_j : j \in \mathcal{N}\}$ is a g-frame for H, then the operator $S \in B(H)$ such that

$$Sf = \sum_{j\in\mathcal{N}}\Lambda_j^*\Lambda_j f, \quad \forall f \in H \tag{4}$$

is called the *g-fame operator* associated with $\{\Lambda_j : j \in \mathcal{N}\}$. It is well-known that $\{\Lambda_j S^{-1} : j \in \mathcal{N}\}$ is a dual g-frame of $\{\Lambda_j : j \in \mathcal{N}\}$, which is called the *canonical dual g-frame* associated with $\{\Lambda_j : j \in \mathcal{N}\}$. In this paper, we use dual of g-frames to denote any of the duals. Recently, g-frames in Hilbert spaces have been studied intensively; for more details, see [10–16] and the references therein.

Frames and g-frames have advantages of allowing decomposing and reconstructing elements in Hilbert spaces, in which the dual and pseudodual of frames (g-frames) play important roles. Characterizing dual frames and general frame decompositions is an important problem in pure and applied fields. In [17–22], the authors study the dual frames and frame-like decompositions in Hilbert spaces. In particular, Li derived a general parametric and algebraic formula for all duals of a frame in [17] and introduced the pseudoframe decompositions in [18]. In this paper, motivated by these works on frames, we consider similar problems on g-frames in Hilbert spaces and generalize the corresponding results on frames to g-frames. Another interesting problem in frame theory is representing general g-frames in terms of special and more simpler g-frames such as g-orthonormal bases, g-Riesz bases, and normalized tight g-frames. In [23], the authors study similar questions on frames in Hilbert spaces by applying the techniques of decomposing linear bounded operators. In this paper, we will study the decompositions of g-frames in Hilbert spaces by using similar techniques combing with what we have obtained on the operator parameterizations for g-frames in [24].

Throughout this paper, we use $\mathscr{N}$ to denote the set of natural numbers and $\mathscr{C}$ to denote the complex plane. All Hilbert spaces in this paper are assumed to be separable complex Hilbert spaces. This paper is organized as follows. In Section 2, we give some definitions and lemmas which are needed to understand the following sections. In Section 3, we consider the decomposition of g-frames. In Section 4, the dual and pseudodual of g-frames are considered.

2. Preliminary Definitions and Lemmas

In this section, we introduce some basic definitions and lemmas which are necessary for the following sections.

Definition 1. Let $\Lambda_i \in B(H, H_i)$, $i \in \mathscr{N}$.

(i) If $\{f : \Lambda_i f = 0, i \in \mathscr{N}\} = \{0\}$, then we say that $\{\Lambda_i : i \in \mathscr{N}\}$ is *g-complete*.

(ii) If $\{\Lambda_i : i \in \mathscr{N}\}$ is g-complete and there are positive constants A and B such that, for any finite subset $J \subset \mathscr{N}$ and $g_j \in H_j$, $j \in J$,

$$A\sum_{j\in J}\left\|g_j\right\|^2 \le \left\|\sum_{j\in J}\Lambda_j^* g_j\right\|^2 \le B\sum_{j\in J}\left\|g_j\right\|^2, \tag{5}$$

then we say that $\{\Lambda_i : i \in \mathscr{N}\}$ is a *g-Riesz basis* for H with respect to $\{H_i : i \in \mathscr{N}\}$.

(iii) We say $\{\Lambda_i : i \in \mathscr{N}\}$ is a *g-orthonormal basis* for H with respect to $\{H_i : i \in \mathscr{N}\}$ if it satisfies the following:

$$\begin{gathered}\left\langle \Lambda_i^* g_i, \Lambda_j^* g_j\right\rangle = \delta_{ij}\left\langle g_i, g_j\right\rangle, \\ \forall i, j \in \mathscr{N}, \quad g_i \in H_i, \quad g_j \in H_j, \\ \sum_{j\in\mathscr{N}}\left\|\Lambda_j f\right\|^2 = \|f\|^2, \quad \forall f \in H.\end{gathered} \tag{6}$$

Remark 2. It is obvious that any g-frame $\{\Lambda_i : i \in \mathscr{N}\}$ is g-complete and any g-orthonormal basis is a normalized tight g-frame.

Definition 3. Suppose that $\Lambda_j \in B(H, H_j)$, $\Gamma_j \in B(H, H_j)$ for any $j \in \mathscr{N}$. If, for any $x, y \in H$, we have $\langle x, y\rangle = \sum_{j\in\mathscr{N}}\langle \Lambda_j x, \Gamma_j y\rangle$, then we call $\{\Lambda_j : j \in \mathscr{N}\}$ and $\{\Gamma_j : j \in \mathscr{N}\}$ *a pair of pseudodual g-frames* for H. In particular, if $\{\Lambda_j : j \in \mathscr{N}\}$ is a g-frame for H, we call $\{\Gamma_j : j \in \mathscr{N}\}$ a *pseudodual g-frame* of $\{\Lambda_j : j \in \mathscr{N}\}$.

Lemma 4 (see [25]). *Let H be a Hilbert space. Then,*

(1) *for every invertible operator $U \in B(H)$, there exists a unique decomposition $U = WP$, where W is a unitary operator and P is a positive operator.*

(2) *for every positive operator $P \in B(H)$ with $\|P\| \le 1$, $P = (1/2)(W + W^*)$, where $W = P + i\sqrt{I - P^2}$ is a unitary operator.*

Lemma 5 (see [26]). *Given Hilbert space H and a sequence of closed subspaces $\{H_j : j \in \mathscr{N}\}$ of a Hilbert space K, then there exists a g-orthonormal basis $\{\Lambda_j \in B(H, H_j) : j \in \mathscr{N}\}$ for H with respect to $\{H_j : j \in \mathscr{N}\}$ if and only if $\sum_{j\in\mathscr{N}} \dim H_j = \dim H$.*

Lemma 6 (see [24]). *Let $\{\theta_j \in B(H, H_j) : j \in \mathscr{N}\}$ be a g-orthonormal basis for H with respect to $\{H_j : j \in \mathscr{N}\}$. Then, the sequence $\{\Lambda_j : j \in \mathscr{N}\}$ is a g-Bessel sequence for H if and only if there is a unique bounded operator $V \in B(H)$ such that $\Lambda_j = \theta_j V^*$ for all $j \in \mathscr{N}$.*

Remark 7. Given the g-orthonormal basis $\{\theta_i \in B(H, H_i) : i \in N\}$, the operator V in Lemma 6 is called the *g-preframe operator* associated with $\{\Lambda_i \in B(H, H_i) : i \in N\}$.

Lemma 8 (see [24]). *Suppose that $\{\theta_i \in B(H, H_i) : i \in N\}$ is a g-orthonormal basis for H, $\{\Lambda_i \in B(H, H_i) : i \in N\}$ is a g-Bessel sequence for H, and V and S are the g-preframe operator and g-frame operator associated with $\{\Lambda_i \in B(H, H_i) : i \in N\}$, respectively. Then*

(1) *$\{\Lambda_i \in B(H, H_i) : i \in N\}$ is a g-frame if and only if V is onto;*

(2) *$\{\Lambda_i \in B(H, H_i) : i \in N\}$ is a normalized tight g-frame if and only if V is a coisometry;*

(3) *$\{\Lambda_i \in B(H, H_i) : i \in N\}$ is a g-Riesz basis if and only if V is invertible;*

(4) *$\{\Lambda_i \in B(H, H_i) : i \in N\}$ is a g-orthonormal basis if and only if V is unitary.*

Lemma 9 (see [23]). *Let $T \in B(H)$ be onto; then T can be written as a linear combination of two unitary operators if and only if T is invertible.*

3. Decompositions of g-Frames

In this section, we do some research on the decompositions of g-frames in Hilbert spaces by using similar techniques in [23] combing with what we have established on the operator parameterizations for g-frames in [24].

Theorem 10. *Suppose that $\{\Lambda_j \in B(H, H_j) : j \in \mathcal{N}\}$ is a g-Bessel sequence for H. Let T be the g-preframe operator associated with $\{\Lambda_j : j \in \mathcal{N}\}$. Then, for any $\varepsilon \in (0, 1)$, there exist three g-orthonormal bases $\{\theta_j^l : j \in \mathcal{N}\}$ $(l = 1, 2, 3)$ such that $\Lambda_j = (1-\varepsilon)^{-1}\|T\|(\theta_j^1 + \theta_j^2 + \theta_j^3)$ for any $j \in \mathcal{N}$.*

Proof. Since we have assumed that all Hilbert spaces are separable, the g-orthonormal bases for H with respect to $\{H_j : j \in \mathcal{N}\}$ exist by Lemma 5. Let $\{\theta_j \in B(H, H_j) : j \in \mathcal{N}\}$ be a g-orthonormal basis for H with respect to $\{H_j : j \in \mathcal{N}\}$. Since $\{\Lambda_j \in B(H, H_j) : j \in \mathcal{N}\}$ is a g-Bessel sequence for H, there exists a bounded operator $T \in B(H)$ such that $\Lambda_j = \theta_j T^*$ for any $j \in \mathcal{N}$ by Lemma 6. Define an operator $U \in B(H)$ by $U = (1/2)I + ((1-\varepsilon)/2) \cdot (T^*/\|T\|)$, where I is the identity operator on H. Since

$$\begin{aligned} \|I - U\| &= \left\| \frac{1}{2} I - \frac{1-\varepsilon}{2} \cdot \frac{T^*}{\|T\|} \right\| \\ &\le \left\| \frac{1}{2} I \right\| + \left\| \frac{1-\varepsilon}{2} \cdot \frac{T^*}{\|T\|} \right\| \\ &= \frac{1}{2} + \frac{1-\varepsilon}{2} = 1 - \frac{\varepsilon}{2} < 1, \end{aligned} \tag{7}$$

U is invertible. By Lemma 4, there exist a unitary V and a positive operator P such that $U = VP$. Since

$$\begin{aligned} \|P\| &= \left\| V^{-1} U \right\| \le \|U\| \\ &\le \left\| \frac{1}{2} I \right\| + \left\| \frac{1-\varepsilon}{2} \cdot \frac{T^*}{\|T\|} \right\| \\ &= \frac{1}{2} + \frac{1-\varepsilon}{2} \le 1, \end{aligned} \tag{8}$$

$P = (1/2)(W^* + W)$, where $W = P + i\sqrt{I - P^2}$ is a unitary operator by Lemma 4. So

$$\begin{aligned} T^* &= \frac{2\|T\|}{1-\varepsilon} \left(U - \frac{1}{2} I \right) \\ &= \frac{2\|T\|}{1-\varepsilon} \left(\frac{V}{2} (W + W^*) - \frac{I}{2} \right) \\ &= \frac{\|T\|}{1-\varepsilon} (VW + VW^* - I). \end{aligned} \tag{9}$$

Hence,

$$\begin{aligned} \Lambda_j &= \theta_j T^* = \theta_j \cdot \frac{\|T\|}{1-\varepsilon} (VW + VW^* - I) \\ &= \frac{\|T\|}{1-\varepsilon} \left(\theta_j VW + \theta_j VW^* - \theta_j \right). \end{aligned} \tag{10}$$

Denote $\theta_j^1 = \theta_j VW$, $\theta_j^2 = \theta_j VW^*$, and $\theta_j^3 = -\theta_j$ for any $j \in \mathcal{N}$. Then, it is easy to see that $\{\theta_j^l : j \in \mathcal{N}\}$ $(l = 1, 2, 3)$ are g-orthonormal bases for H, since V and W are unitary operators. So $\Lambda_j = (1-\varepsilon)^{-1}\|T\|(\theta_j^1 + \theta_j^2 + \theta_j^3)$ for any $j \in \mathcal{N}$. □

Since a g-frame is of course a g-Bessel sequence, the following corollary is obvious.

Corollary 11. *Every g-frame can be represented as a multiple of sum of three g-orthonormal bases.*

Theorem 12. *A g-frame $\{\Lambda_j \in B(H, H_j) : j \in \mathcal{N}\}$ for H can be written as a linear combination of two g-orthonormal bases for H if and only if $\{\Lambda_j : j \in \mathcal{N}\}$ is a g-Riesz basis for H.*

Proof. $\Rightarrow$: Suppose that $\{\Gamma_j : j \in \mathcal{N}\}$ and $\{L_j : j \in \mathcal{N}\}$ are g-orthonormal bases for H such that $\Lambda_j = a \cdot \Gamma_j + b \cdot L_j$ for any $j \in \mathcal{N}$. By Lemma 8, there exist surjective operator T and unitary operator U such that $\Lambda_j = \Gamma_j T^*$ and $L_j = \Gamma_j U$ for any $j \in \mathcal{N}$. So, $\Gamma_j T^* = a \cdot \Gamma_j + b \cdot \Gamma_j U, \forall j \in \mathcal{N}$. Hence, $T\Gamma_j^* = \overline{a} \cdot \Gamma_j^* + \overline{b} \cdot U^* \Gamma_j^*, \forall j \in \mathcal{N}$. It implies that $T = \overline{a} \cdot I + \overline{b} \cdot U^*$, since $\overline{\operatorname{Span}}\{\Lambda_j^*(H_j) : j \in \mathcal{N}\} = H$. So T is invertible by Lemma 9. Hence, $\{\Lambda_j : j \in \mathcal{N}\}$ is a g-Riesz basis for H.

$\Leftarrow$: Since $\{\Lambda_j : j \in \mathcal{N}\}$ is a g-Riesz basis for H, there exist a g-orthonormal basis $\{\theta_j \in B(H, H_j) : j \in \mathcal{N}\}$ and an invertible operator $T \in B(H)$ such that $\Lambda_j = \theta_j T^*$ for any $j \in \mathcal{N}$ by Lemma 8. There exist two unitary operators U_1 and U_2 in $B(H)$ and constants a, b such that $T^* = a \cdot U_1 + b \cdot U_2$ by Lemma 9. So $\Lambda_j = \theta_j T^* = \theta_j(a \cdot U_1 + b \cdot U_2) = a \cdot \theta_j U_1 + b \cdot \theta_j U_2$ for any $j \in \mathcal{N}$. Since $\{\theta_j U_1 : j \in \mathcal{N}\}$ and $\{\theta_j U_2 : j \in \mathcal{N}\}$ are g-orthonormal bases for H by Lemmas 8 and 9, the result follows. □

Theorem 13. *Every g-frame for H is a multiple of two normalized tight g-frames for H.*

Proof. Suppose that $\{\Lambda_j \in B(H, H_j) : j \in \mathcal{N}\}$ is a g-frame for H and $\{\theta_j \in B(H, H_j) : j \in \mathcal{N}\}$ is a g-orthonormal basis for H. Then, there exists a surjective operator $T \in B(H)$ such that $\Lambda_j = \theta_j T^*$ for any $j \in \mathcal{N}$ by Lemma 8. Let $U = T/2\|T\|$. Then, $\|U\| = 1/2 < 1$ and U is also surjective. Suppose that $U = VP$ is the polar decomposition of U, where V is a coisometry and P is a positive operator in $B(H)$. Since $\|P\| = \|V^* U\| \le \|U\| < 1$, then $P = (1/2)(W + W^*)$ with $W = P + i\sqrt{I - P^2}$ being a unitary operator. So $U = VP = (1/2)V(W + W^*)$. It follows that $T = 2\|T\|U = \|T\|(VW + VW^*)$. So

$$\begin{aligned} \Lambda_j &= \theta_j T^* = \theta_j \|T\| \left(W^* V^* + W V^* \right) \\ &= \|T\| \left(\theta_j W^* V^* + \theta_j W V^* \right). \end{aligned} \tag{11}$$

Since VW and VW^* are coisometries, $\{\theta_j W^* V^* : j \in \mathcal{N}\}$ and $\{\theta_j W V^* : j \in \mathcal{N}\}$ are normalized tight g-frames for H by Lemma 8. This finishes the proof. □

Theorem 14. *Every g-frame for H is a multiple of the sum of a g-orthonormal basis for H and a g-Riesz basis for H.*

Proof. Suppose that $\{\Lambda_j \in B(H, H_j) : j \in \mathcal{N}\}$ is a g-frame for H and $\{\theta_j \in B(H, H_j) : j \in \mathcal{N}\}$ is a g-orthonormal basis for H. Let T be the g-preframe operator associated with $\{\Lambda_j : j \in \mathcal{N}\}$; then $\Lambda_j = \theta_j T^*$ for any $j \in \mathcal{N}$. Define operator S by $S = (3/4)I + (1/4)(1-\varepsilon) \cdot (T^*/\|T\|)$; then

$$\begin{aligned} \|I - S\| &= \left\| \frac{I}{4} - \frac{1}{4} \cdot (1-\varepsilon) \cdot \frac{T^*}{\|T\|} \right\| \\ &\le \left\| \frac{1}{4} \cdot I \right\| + \frac{1-\varepsilon}{4} < 1, \\ \|S\| &\le \frac{3}{4} + \frac{1-\varepsilon}{4} < 1. \end{aligned} \tag{12}$$

So S is invertible. Let $S = VP$ be the polar decomposition of S. Then, V is a unitary operator and P is a positive operator by Lemma 4. Since $P = V^* S$, $\|P\| = \|V^* S\| \le \|V^*\| \cdot \|S\| < 1$. So $P = (1/2)(W + W^*)$ by Lemma 4, where $W = P + i\sqrt{I - P^2}$ is a unitary operator. So $S = VP = (1/2)(VW + VW^*)$. It implies that

$$\begin{aligned} T^* &= \frac{4\|T\|}{1-\varepsilon} \left(S - \frac{3}{4} \cdot I \right) \\ &= \frac{4\|T\|}{1-\varepsilon} \left(\frac{1}{2} \cdot (VW + VW^*) - \frac{3}{4} \cdot I \right) \\ &= \frac{2\|T\|}{1-\varepsilon} \left(VW + VW^* - \frac{3}{2} \cdot I \right) \\ &= \frac{2\|T\|}{1-\varepsilon} (VW + R), \end{aligned} \tag{13}$$

where $R = VW^* - (3/2) \cdot I$. Since

$$\begin{aligned} \left\| I - \frac{-1}{2} \cdot R \right\| &= \left\| \frac{I}{4} + \frac{1}{2} VW^* \right\| \\ &\le \left\| \frac{I}{4} \right\| + \left\| \frac{1}{2} \cdot VW^* \right\| \\ &= \frac{1}{4} + \frac{1}{2} = \frac{3}{4} < 1, \end{aligned} \tag{14}$$

$-(1/2) \cdot R$ is invertible. Hence, R is invertible. So

$$\begin{aligned} \Lambda_j &= \theta_j T^* = \theta_j \cdot \frac{2\|T\|}{1-\varepsilon} \cdot (VW + R) \\ &= \frac{2\|T\|}{1-\varepsilon} \left(\theta_j VW + \theta_j R \right). \end{aligned} \tag{15}$$

Since $\{\theta_j VW : j \in \mathcal{N}\}$ is a g-orthonormal basis for H and $\{\theta_j R : j \in \mathcal{N}\}$ is a g-Riesz basis for H by Lemma 8, $\{\Lambda_j : j \in \mathcal{N}\}$ is a multiple of a sum of a g-orthonormal basis and a g-Riesz basis for H. □

4. Dual and Pseudodual g-Frames

In this section, we consider the characterizations of dual and pseudodual g-frames. The algebraic formula about the dual of g-frames for a given g-frame will be given and some properties on dual and pseudodual g-frames will be established.

Theorem 15. *Suppose that $\{\Lambda_j \in B(H, H_j) : j \in \mathcal{N}\}$ is a g-frame for H and $\{\theta_j \in B(H, H_j) : j \in \mathcal{N}\}$ is a g-orthonormal basis for H. Suppose that the g-preframe operator associated with $\{\Lambda_j : j \in \mathcal{N}\}$ is T; that is, $\Lambda_j = \theta_j T^*$ for any $j \in \mathcal{N}$. Then, $\{\Gamma_j \in B(H, H_j) : j \in \mathcal{N}\}$ is a dual g-frame of $\{\Lambda_j : j \in \mathcal{N}\}$ if and only if $\Gamma_j = \theta_j V^*$ for any $j \in \mathcal{N}$, where V is a bounded left inverse of T^*.*

Proof. ⇒: Suppose that $\{\Gamma_j : j \in \mathcal{N}\}$ is a dual g-frame of $\{\Lambda_j : j \in \mathcal{N}\}$. Let V be the g-preframe operator of $\{\Gamma_j : j \in \mathcal{N}\}$. Then, $\Gamma_j = \theta_j V^*$ for any $j \in \mathcal{N}$ and V is bounded. Since, for any $f \in H$, we have $f = \sum_{j\in\mathcal{N}} \Lambda_j^* \Gamma_j f$,

$$f = \sum_{j\in\mathcal{N}} T\theta_j^* \theta_j V^* f = T \sum_{j\in\mathcal{N}} \theta_j^* \theta_j V^* f = TV^* f. \tag{16}$$

It implies that $TV^* = I$. Hence, $VT^* = I$. It follows that V is a bounded left inverse of T^*.

⇐: Since $VT^* = I$, V is bounded surjective operator in $B(H)$. Hence, $\{\Gamma_j : j \in \mathcal{N}\}$ is a g-frame for H by Lemma 8. Since

$$\begin{aligned} \sum_{j\in\mathcal{N}} \Lambda_j^* \Gamma_j f &= \sum_{j\in\mathcal{N}} T\theta_j^* \theta_j V^* f = T \sum_{j\in\mathcal{N}} \theta_j^* \theta_j V^* f \\ &= TV^* f = f, \quad \forall f \in H, \end{aligned} \tag{17}$$

$\{\Gamma_j : j \in \mathcal{N}\}$ is a dual g-frame for H. □

Lemma 16. *Suppose that $\{\Lambda_j \in B(H, H_j) : j \in \mathcal{N}\}$ is a g-frame for H, whose g-preframe operator is T. Then, V is a linear bounded left inverse of T^* if and only if*

$$V = S^{-1}T + W\left(I - T^* S^{-1} T\right), \tag{18}$$

where S is the g-frame operator associated with $\{\Lambda_j : j \in \mathcal{N}\}$, $W \in B(H)$, and I is the identity operator in $B(H)$.

Proof. ⇒: Suppose V is a linear bounded left inverse of T^*. Let $W = V$. Then

$$\begin{aligned} S^{-1}T + V\left(I - T^* S^{-1} T\right) &= S^{-1}T + V - VT^* S^{-1} T \\ &= S^{-1}T + V - S^{-1}T = V. \end{aligned} \tag{19}$$

⇐: Suppose $V = S^{-1}T + W(I - T^* S^{-1} T)$. Then

$$\begin{aligned} VT^* &= S^{-1}TT^* + W\left(I - T^* S^{-1} T\right) T^* \\ &= S^{-1}S + W\left(T^* - T^* S^{-1} TT^*\right) = I. \end{aligned} \tag{20}$$

Hence, V is a linear bounded left inverse of T^*. □

Theorem 17. *Suppose $\{\Lambda_j \in B(H, H_j) : j \in \mathcal{N}\}$ is a g-frame for H, T is its g-preframe operator, and S is its g-frame operator. Let $\{\theta_j \in B(H, H_j) : j \in \mathcal{N}\}$ be a g-orthonormal basis for H. Then, $\{\Gamma_j \in B(H, H_j) : j \in \mathcal{N}\}$ is a dual g-frame of $\{\Lambda_j : j \in \mathcal{N}\}$ if and only if there exists a bounded operator $W \in B(H)$ such that*

$$\Gamma_j = \Lambda_j S^{-1} + \theta_j W^* - \Lambda_j S^{-1} T W^*, \quad \forall j \in \mathcal{N}. \tag{21}$$

Proof. $\Rightarrow$: Suppose that $\{\Gamma_j \in B(H, H_j) : j \in \mathcal{N}\}$ is a dual g-frame of $\{\Lambda_j \in B(H, H_j) : j \in \mathcal{N}\}$. Then, by Theorem 15, we know that $\Gamma_j = \theta_j V^*$ for any $j \in \mathcal{N}$, where V is a linear bounded left inverse of T^*. By Lemma 16, $V = S^{-1}T + W(I - T^*S^{-1}T)$ for some linear bounded operator $W \in B(H)$. Hence, for any $j \in \mathcal{N}$, we have

$$\begin{aligned}\Gamma_j &= \theta_j V^* = \theta_j\left(S^{-1}T + W\left(I - T^*S^{-1}T\right)\right)^* \\ &= \theta_j\left(T^*S^{-1} + W^* - T^*S^{-1}TW^*\right) \\ &= \theta_j T^*S^{-1} + \theta_j W^* - \theta_j T^*S^{-1}TW^* \\ &= \Lambda_j S^{-1} + \theta_j W^* - \Lambda_j S^{-1}TW^*.\end{aligned} \tag{22}$$

$\Leftarrow$: Suppose that there exists a linear bounded operator $W \in B(H)$ such that $\Gamma_j = \Lambda_j S^{-1} + \theta_j W^* - \Lambda_j S^{-1}TW^*$. Then,

$$\begin{aligned}\Gamma_j &= \theta_j T^*S^{-1} + \theta_j W^* - \theta_j T^*S^{-1}TW^* \\ &= \theta_j\left(T^*S^{-1} + W^* - T^*S^{-1}TW^*\right) \\ &= \theta_j\left(S^{-1}T + W - WT^*S^{-1}T\right)^*.\end{aligned} \tag{23}$$

So $\{\Gamma_j : j \in \mathcal{N}\}$ is a g-Bessel sequence for H and the g-preframe operator associated with $\{\Gamma_j : j \in \mathcal{N}\}$ is

$$\begin{aligned}V &= S^{-1}T + W - WT^*S^{-1}T \\ &= S^{-1}T + W\left(I - T^*S^{-1}T\right).\end{aligned} \tag{24}$$

Since V is a linear bounded left inverse of T^* by Lemma 16, $VT^* = I$. Therefore, $\{\Gamma_j : j \in \mathcal{N}\}$ is a dual g-frame of $\{\Lambda_j : j \in \mathcal{N}\}$ by Theorem 15. □

Theorem 18. *Suppose that $\{\Lambda_j \in B(H, H_j) : j \in \mathcal{N}\}$ is a g-frame for H. If $\{\Gamma_j \in B(H, H_j) : j \in \mathcal{N}\}$ is a pseudodual g-frame of $\{\Lambda_j : j \in \mathcal{N}\}$, then $\{\Gamma_j : j \in \mathcal{N}\}$ has lower g-frame bound.*

Proof. Since $\{\Gamma_j : j \in \mathcal{N}\}$ is a pseudodual g-frame of $\{\Lambda_j : j \in \mathcal{N}\}$, $\langle x, y\rangle = \sum_{j\in\mathcal{N}}\langle \Lambda_j x, \Gamma_j y\rangle$ for any $x, y \in H$. In particular, $\langle x, x\rangle = \sum_{j\in\mathcal{N}}\langle \Lambda_j x, \Gamma_j x\rangle$; that is, $\|x\|^2 = \sum_{j\in\mathcal{N}}\langle \Lambda_j x, \Gamma_j x\rangle$. Since

$$\begin{aligned}\sum_{j\in\mathcal{N}}\left\langle \Lambda_j x, \Gamma_j x\right\rangle &\le \sum_{j\in\mathcal{N}}\left\|\Lambda_j x\right\| \cdot \left\|\Gamma_j x\right\| \\ &\le \left(\sum_{j\in\mathcal{N}}\left\|\Lambda_j x\right\|^2\right)^{1/2} \cdot \left(\sum_{j\in\mathcal{N}}\left\|\Gamma_j x\right\|^2\right)^{1/2} \\ &\le \left(B\|x\|^2\right)^{1/2} \cdot \left(\sum_{j\in\mathcal{N}}\left\|\Gamma_j x\right\|^2\right)^{1/2},\end{aligned} \tag{25}$$

where B is the upper g-frame bound of $\{\Lambda_j : j \in \mathcal{N}\}$, hence,

$$\sum_{j\in\mathcal{N}}\left\|\Gamma_j x\right\|^2 \ge \frac{1}{B} \cdot \|x\|^2. \tag{26}$$

So $\{\Gamma_j : j \in \mathcal{N}\}$ has lower g-frame bound. □

Corollary 19. *Suppose that $\{\Lambda_j \in B(H, H_j) : j \in \mathcal{N}\}$ is a g-frame for H and $\{\Gamma_j : j \in \mathcal{N}\}$ is a pseudodual g-frame of $\{\Lambda_j : j \in \mathcal{N}\}$; then $\{\Gamma_j : j \in \mathcal{N}\}$ is g-complete.*

Proof. Since $\{\Gamma_j : j \in \mathcal{N}\}$ has lower g-frame bound by Theorem 18, there exists a constant $B > 0$ such that

$$\sum_{j\in\mathcal{N}}\left\|\Gamma_j x\right\|^2 \ge B\|x\|^2, \quad \forall x \in H. \tag{27}$$

If $\Gamma_j x = 0$, for all $j \in \mathcal{N}$, then $B\|x\|^2 = 0$; it follows that $x = 0$. So $\{\Gamma_j : j \in \mathcal{N}\}$ is g-complete. □

Theorem 20. *Suppose that $\{\Lambda_j \in B(H, H_j) : j \in \mathcal{N}\}$ and $\{\Gamma_j \in B(H, H_j) : j \in \mathcal{N}\}$ are a pair of pseudodual g-frames for H. Then, for any $x \in H$, $x = \sum_{j\in\mathcal{N}}\Lambda_j^*\Gamma_j x$ if and only if $x = \sum_{j\in\mathcal{N}}\Gamma_j^*\Lambda_j x$, where the series converge in norm of H.*

Proof. It is obvious that we only need to prove one direction; the other direction is identical. Now, suppose that $x = \sum_{j\in\mathcal{N}}\Lambda_j^*\Gamma_j x$. Since $\{\Lambda_j \in B(H, H_j) : j \in \mathcal{N}\}$ and $\{\Gamma_j \in B(H, H_j) : j \in \mathcal{N}\}$ are a pair of pseudodual g-frames for H, we have $\langle x, y\rangle = \sum_{j\in\mathcal{N}}\langle \Lambda_j x, \Gamma_j y\rangle \forall x, y \in H$. It is obvious that $f_N(y) = |\sum_{j=1}^N\langle \Lambda_j x, \Gamma_j y\rangle - \langle x, y\rangle|$ is a weakly continuous function on H and $\lim_{N\to\infty}f_N(y) = 0$ for each $x \in H$. Since the closed unit ball of H is weakly compact, for any $\varepsilon > 0$, there exists $N_0 > 0$ such that, for any $\|y\| \le 1$ and any $N > N_0$, we have $f_N(y) < \varepsilon$. So whenever $N > N_0$, we have

$$\begin{aligned}\left\|\sum_{j=1}^N\Gamma_j^*\Lambda_j x - x\right\| &= \sup_{\|y\|=1}\left|\left\langle \sum_{j=1}^N\Gamma_j^*\Lambda_j x - x, y\right\rangle\right| \\ &= \sup_{\|y\|=1}\left|\left\langle \sum_{j=1}^N\Gamma_j^*\Lambda_j x, y\right\rangle - \langle x, y\rangle\right| \\ &= \sup_{\|y\|=1} f_N(y) \le \varepsilon.\end{aligned} \tag{28}$$

Hence, $x = \sum_{j\in\mathcal{N}}\Gamma_j^*\Lambda_j x$. □

Corollary 21. *Suppose that $\{\Lambda_j \in B(H, H_j) : j \in \mathcal{N}\}$ and $\{\Gamma_j \in B(H, H_j) : j \in \mathcal{N}\}$are a pair of pseudo g-frames, $x_0 \in H$. If $\sum_{j\in\mathcal{N}}\Lambda_j^*\Gamma_j x_0$ is convergent, then*

$$x_0 = \sum_{j\in\mathcal{N}}\Lambda_j^*\Gamma_j x_0 = \sum_{j\in\mathcal{N}}\Gamma_j^*\Lambda_j x_0. \tag{29}$$

Proof. Since $\{\Lambda_j \in B(H, H_j) : j \in \mathcal{N}\}$ and $\{\Gamma_j \in B(H, H_j) : j \in \mathcal{N}\}$are a pair of pseudo g-frames, for any $y \in H$, we have

$$\left\langle y, x_0\right\rangle = \sum_{j\in\mathcal{N}}\left\langle \Lambda_j y, \Gamma_j x_0\right\rangle = \left\langle y, \sum_{j\in\mathcal{N}}\Lambda_j^*\Gamma_j x_0\right\rangle. \tag{30}$$

So $x_0 = \sum_{j\in\mathcal{N}}\Lambda_j^*\Gamma_j x_0$. It follows that $x_0 = \sum_{j\in\mathcal{N}}\Lambda_j^*\Gamma_j x_0 = \sum_{j\in\mathcal{N}}\Gamma_j^*\Lambda_j x_0$ by Theorem 20. □

Theorem 22. *Suppose that $\{\theta_j \in B(H, H_j) : j \in \mathcal{N}\}$ is a g-orthonormal basis for H and T_1 and T_2 are g-preframe operators associated with g-frames $\{\Lambda_j \in B(H, H_j) : j \in \mathcal{N}\}$ and $\{\Gamma_j \in B(H, H_j) : j \in \mathcal{N}\}$, respectively. Then, $\{\Lambda_j : j \in \mathcal{N}\}$ and $\{\Gamma_j : j \in \mathcal{N}\}$ are g-biorthogonal if and only if $T_1^* T_2 = I$, where I is the identity operator in $B(H)$.*

Proof. Since T_1 and T_2 are g-preframe operators associated with g-frames $\{\Lambda_j : j \in \mathcal{N}\}$ and $\{\Gamma_j : j \in \mathcal{N}\}$, respectively, $\Lambda_j = \theta_j T_1^*$ and $\Gamma_j = \theta_j T_2^*$ for any $j \in \mathcal{N}$. So, for any $i, j \in \mathcal{N}$ and any $g_i \in H_i$, $g_j \in H_j$, we have

$$\left\langle \Lambda_i^* g_i, \Gamma_j^* g_j \right\rangle = \left\langle T_1 \theta_i^* g_i, T_2 \theta_j^* g_j \right\rangle = \left\langle \theta_i^* g_i, T_1 T_2^* \theta_j^* g_j \right\rangle. \tag{31}$$

If $\{\Lambda_j : j \in \mathcal{N}\}$ and $\{\Gamma_j : j \in \mathcal{N}\}$ are g-biorthogonal, then

$$\begin{gathered} \left\langle \Lambda_i^* g_i, \Gamma_j^* g_j \right\rangle = \delta_{i,j} \left\langle g_i, g_j \right\rangle = \left\langle \theta_i^* g_i, \theta_j^* g_j \right\rangle, \\ \forall i, j \in \mathcal{N}, \quad \forall g_i \in H_i, \quad g_j \in H_j. \end{gathered} \tag{32}$$

So

$$\begin{gathered} \left\langle \theta_i^* g_i, T_1 T_2^* \theta_j^* g_j \right\rangle = \left\langle \theta_i^* g_i, \theta_j^* g_j \right\rangle, \\ \forall i, j \in \mathcal{N}, \quad \forall g_i \in H_i, \quad g_j \in H_j. \end{gathered} \tag{33}$$

It implies that $T_1 T_2^* = I$.

Conversely, if $T_1 T_2^* = I$, then

$$\begin{aligned} \langle \Lambda_i^* g_i, \Gamma_j^* g_j \rangle &= \langle \theta_i^* g_i, T_1 T_2^* \theta_j^* g_j \rangle \\ &= \langle \theta_i^* g_i, \theta_j^* g_j \rangle = \delta_{i,j} \langle g_i, g_j \rangle, \\ &\forall i, j \in \mathcal{N}, \quad \forall g_i \in H_i, \quad g_j \in H_j. \end{aligned} \tag{34}$$

So $\{\Lambda_j : j \in \mathcal{N}\}$ and $\{\Gamma_j : j \in \mathcal{N}\}$ are g-biorthogonal. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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Research Article

Boundedness of Singular Integrals on Hardy Type Spaces Associated with Schrödinger Operators

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Let $L = -\Delta + V$ be a Schrödinger operator on $\mathbb{R}^n, n \geq 3$, where $V \not\equiv 0$ is a nonnegative potential belonging to the reverse Hölder class $B_{n/2}$. The Hardy type spaces $H_L^p, n/(n+\delta) < p \leq 1$, for some $\delta > 0$, are defined in terms of the maximal function with respect to the semigroup $\{e^{-tL}\}_{t>0}$. In this paper, we investigate the bounded properties of some singular integral operators related to L, such as $L^{i\gamma}$ and $\nabla L^{-1/2}$, on spaces H_L^p. We give the molecular characterization of H_L^p, which is used to establish the H_L^p-boundedness of singular integrals.

1. Introduction

Let $L = -\Delta + V$ be a Schrödinger operator on $\mathbb{R}^n$, $n \geq 3$, where $V \not\equiv 0$ is a nonnegative potential belonging to the reverse Hölder class B_q for some $q \geq n/2$; that is, there exists a constant $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B V^q(x)\,dx\right)^{1/q} \leq C\left(\frac{1}{|B|}\int_B V(x)\,dx\right) \qquad (1)$$

holds for every ball B in $\mathbb{R}^n$. It is well known that if $V \in B_q$ then $V \in B_{q+\varepsilon}$ for some $\varepsilon > 0$. Also obviously, $B_{q_1} \subset B_{q_2}$ when $q_1 > q_2$.

Some singular integral operators related to L, such as the imaginary power $L^{i\gamma}$, and the Riesz transform $\nabla L^{-1/2}$ have been studied by Shen [1]. Some of his results are following. The operator $L^{i\gamma}$ is a Calderón-Zygmund operator for any $\gamma \in \mathbb{R}$. $\nabla L^{-1/2}$ is a Calderón-Zygmund operator if $q \geq n$. When $n/2 \leq q < n$, $\nabla L^{-1/2}$ is bounded on L^p for $1 < p \leq p_0$, where $1/p_0 = 1/q - 1/n$. The above range of p is optimal. Earlier results were given by Fefferman [2] and Zhong [3].

The Hardy type spaces H_L^p, $n/(n+\delta) < p \leq 1$ for some $\delta > 0$, associated with L, are studied by Dziubański and Zienkiewicz [4, 5]. They establish the $H_L^{p,\infty}$ atomic decomposition theorem and the Riesz transform characterization of H_L^1. Specifically, $\nabla L^{-1/2}$ is bounded from H_L^1 to L^1. We will investigate the bounded properties of the operators $L^{i\gamma}$ and $\nabla L^{-1/2}$ on spaces H_L^p. To do this, we give the molecular characterization of H_L^p.

Without loss of generalization, we assume that $V \in B_{q_0}$ for some $q_0 > n/2$ and set $\delta = \min(2 - n/q_0, 1)$. When $q_0 > n$, we set $\eta = 1 - n/q_0$. Throughout the paper, we will use A and C to denote the positive constants, which are independent of main parameters and may be different at each occurrence. By $B_1 \sim B_2$, we mean that there exists a constant $C > 1$ such that $1/C \leq B_1/B_2 \leq C$.

Let $\{T_t^L\}_{t>0} = \{e^{-tL}\}_{t>0}$ be the semigroup of linear operators generated by $-L$ and $K_t^L(x, y)$ their kernels. Since V is nonnegative, the Feynman-Kac formula implies that

$$0 \leq K_t^L(x, y) \leq K_t(x - y) = (4\pi t)^{-n/2} e^{-(4t)^{-1}|x-y|^2}, \qquad (2)$$

where $K_t(x)$ is the convolution kernels of the heat semigroup $\{T_t\}_{t>0} = \{e^{t\Delta}\}_{t>0}$. The estimate (2) can be improved as follows. We introduce the auxiliary function $\rho(x,V)=\rho(x)$ defined by

$$\rho(x)=\sup\left\{r>0:\frac{1}{r^{n-2}}\int_{B(x,r)}V(y)\,dy\le 1\right\}. \quad (3)$$

It is known that $0<\rho(x)<\infty$. For every $N>0$,

$$K_t^L(x,y)\le C_N t^{-n/2}e^{-(5t)^{-1}|x-y|^2}\left(1+\frac{\sqrt{t}}{\rho(x)}+\frac{\sqrt{t}}{\rho(y)}\right)^{-N}, \quad (4)$$

(cf. [6, Theorem 4.10]). Let $0<\delta'<\delta$; for every $N>0$ and all $|h|\le\sqrt{t}$,

$$\begin{aligned}&\left|K_t^L(x+h,y)-K_t^L(x,y)\right|\\&\quad\le C_N\left(\frac{|h|}{\sqrt{t}}\right)^{\delta'}t^{-n/2}e^{-At^{-1}|x-y|^2}\left(1+\frac{\sqrt{t}}{\rho(x)}+\frac{\sqrt{t}}{\rho(y)}\right)^{-N}\end{aligned} \quad (5)$$

(cf. [6, Proposition 4.11]).

We define the Hardy type spaces H_L^p, $n/(n+\delta)<p\le 1$, in terms of the maximal function with respect to the semigroup $\{T_t^L\}_{t>0}$.

For $p=1$, the Hardy space H_L^1 is defined, according to Dziubański and Zienkiewicz [4], by

$$H_L^1=\left\{f\in L^1: M^Lf\in L^1\right\}, \quad (6)$$

where

$$M^Lf(x)=\sup_{t>0}\left|T_t^Lf(x)\right|. \quad (7)$$

The norm of a function $f\in H_L^1$ is defined to be $\|f\|_{H_L^1}=\|M^Lf\|_{L^1}$.

The Hardy spaces, H_L^p, $n/(n+\delta)<p<1$, consist of some kind of distributions. But $M^Lf(x)$ may have no meaning for a tempered distribution f because $K_t^L(x,y)$ are not smooth. Let f be a locally integrable function. $B=B(x,r)$ is the ball of radius r centered at x. Set

$$\begin{aligned}f_B&=\frac{1}{|B|}\int_B f(y)\,dy,\\ f(B,V)&=\begin{cases}f_B, & \text{if } r<\rho(x),\\ 0, & \text{if } r\ge\rho(x).\end{cases}\end{aligned} \quad (8)$$

Let $n/(n+\delta)<p<1$, $1\le q'\le\infty$. A locally integrable function f is said to be in the Campanato type space $\Lambda^L_{1/p-1,q'}$ if

$$\|f\|_{\Lambda^L_{1/p-1,q'}}=\sup_{B\subset\mathbb{R}^d}\left\{|B|^{1-1/p}\left(\int_B|f-f(B,V)|^{q'}\frac{dx}{|B|}\right)^{1/q'}\right\}<\infty. \quad (9)$$

All spaces $\Lambda^L_{1/p-1,q'}$ are mutually coincident with equivalent norms and will be simply denoted by $\Lambda^L_{1/p-1}$ (cf. [7]). Due to (4) and (5), for every $t>0$,

$$\sup_{x\in\mathbb{R}^d}\left\|K_t^L(x,\cdot)\right\|_{\Lambda^L_{1/p-1}}<Ct^{-d/2p} \quad (10)$$

(cf. [7, Lemma 1]). Thus the semigroup maximal function Mf is well defined for distributions in $(\Lambda^L_{1/p-1})^*$. We define the Hardy space, H_L^p, $n/(n+\delta)<p<1$, by

$$H_L^p=\left\{f\in\left(\Lambda^L_{1/p-1}\right)^*: M^Lf\in L^p\right\}, \quad (11)$$

and set $\|f\|_{H_L^p}=\|M^Lf\|_{L^p}$.

Similar to the classical case, the Hardy space H_L^p admits an atomic decomposition. Let $n/(n+\delta)<p\le 1\le q\le\infty$, $p\ne q$. A function a is called an $H_L^{p,q}$-atom associated with a ball $B(x_0,r)$ if

(1) supp $a\subset B(x_0,r)$,

(2) $\|a\|_{L^q}\le|B(x_0,r)|^{1/q-1/p}$,

(3) *if* $r<\rho(x_0)$, then $\int a(x)dx=0$.

Proposition 1 (see [7, Theorem 1]). *Given p,q as above, then $f\in H_L^p$ if and only if f can be written as $f=\sum_j\lambda_ja_j$, where a_j are $H_L^{p,q}$-atoms and $\sum_j|\lambda_j|^p<\infty$. The sum converges in H_L^p norm and also in $(\Lambda^L_{1/p-1})^*$ when $p<1$. Moreover,*

$$\|f\|_{H_L^p}\sim\|f\|_{H_L^{p,q,a}}=\inf\left\{\left(\sum_j\left|\lambda_j\right|^p\right)^{1/p}\right\}, \quad (12)$$

where the infimum is taken over all decompositions of f into $H_L^{p,q}$-atoms.

Now we state the main results in this paper.

Theorem 2. *For any $\gamma\in\mathbb{R}$, the imaginary power $L^{i\gamma}$ is bounded on H_L^p for $n/(n+\delta)<p\le 1$. When $q_0>n$, the Riesz transform $\nabla L^{-1/2}$ is bounded on H_L^p for $n/(n+\eta)<p\le 1$. Moreover, $\nabla L^{-1/2}$ is bounded on H_L^1 whenever $q_0>n/2$.*

Remark 3. When $n/2<q_0<n$, the kernel of Riesz transform $\nabla L^{-1/2}$ only satisfies the Hörmander condition with respect to the second variable, which is weaker than the smoothness condition of standard kernels. Thus we cannot expect, in general consideration, to deal with the boundedness of $\nabla L^{-1/2}$ for the case of $p<1$.

In order to prove Theorem 2, we give the molecular characterization of H_L^p.

Let $n/(n+\delta)<p\le 1\le q\le\infty$, $p\ne q$ and $\epsilon>1/p-1$. Set $a=1-1/p+\epsilon$, $b=1-1/q+\epsilon$. A function $M\in L^q$ is called an $H_L^{p,q,\epsilon}$-molecule with the center x_0 if

(1) $|x|^{nb}M(x)\in L^q$,

(2) $\mathcal{N}(M) = \mu_1^{b-a}\|M\|_{L^q}^{a/b} \quad \||\cdot - x_0|^{nb} M\|_{L^q}^{1-a/b} \leq 1$,

(3) *if* $\|M\|_{L^q}^{1/(a-b)} < \mu_1 \rho(x_0)^n$, then $\int M(x)dx = 0$,

where μ_1 is the volume of the unit ball.

Theorem 4. *Given p, q, ϵ as above, then $f \in H_L^p$ if and only if f can be written as $f = \sum_j \lambda_j M_j$, where M_j are $H_L^{p,q,\epsilon}$-molecules and $\sum_j |\lambda_j|^p < \infty$. The sum converges in H_L^p norm and also in $(\Lambda_{1/p-1}^L)^*$ when $p < 1$, where M_j are H_L^p-molecules. Moreover,*

$$\|f\|_{H_L^p} \sim \|f\|_{H_L^{p,q,\epsilon,M}} = \inf\left\{\left(\sum_j |\lambda_j|^p\right)^{1/p}\right\}, \qquad (13)$$

where the infimum is taken over all decompositions of f into $H_L^{p,q,\epsilon}$-molecules.

Remark 5. It is easy to verify that any $H_L^{p,q}$-atom is an $H_L^{p,q,\epsilon}$-molecule with a constant factor less than or equal to 1. We will see that the image of an $H_L^{p,q}$-atom under the action of a singular integral operator may not be an $H_L^{p,q,\epsilon}$-molecule but is a sum of two $H_L^{p,q,\epsilon}$-molecules up to constant factors. This is different from the classical case.

This paper is organized as follows. In Section 2, we collect some useful facts and results about the potential V, the auxiliary function $\rho(x)$ and the kernels of operators $L^{i\gamma}$, and $\nabla L^{-1/2}$, which will be used in the sequel. Most of these results are already known. In Section 3, we prove Theorem 4. The proof of Theorem 2 is given in the last two sections. The H_L^p-boundedness for $p < 1$ is proved in Section 4 while H_L^1-boundedness is proved in Section 5.

2. Preliminaries

First we list some known facts and results about the potential V, the auxiliary function $\rho(x)$, and the kernels of operators $L^{i\gamma}$ and $\nabla L^{-1/2}$.

Lemma 6. *$V(x)dx$ is a doubling measure; that is, there exists a constant $C_0 > 0$ such that*

$$\int_{B(x,2r)} V(y)\, dy \leq C_0 \int_{B(x,r)} V(y)\, dy. \qquad (14)$$

Lemma 7. *Consider*

$$\frac{1}{r^{n-2}} \int_{B(x,r)} V(y)\, dy \leq C\left(\frac{R}{r}\right)^{n/q_0-2} \frac{1}{R^{n-2}} \int_{B(x,R)} V(y)\, dy, \quad 0 < r < R < \infty. \qquad (15)$$

Lemma 8. *There exists $m_0 > 0$ such that*

$$\frac{1}{R^{n-2}} \int_{B(x,R)} V(y)\, dy \leq C\left(1 + \frac{R}{\rho(x)}\right)^{m_0}. \qquad (16)$$

Lemma 9. *There exists $k_0 > 0$ such that*

$$\frac{1}{C}\left(1 + \frac{|x-y|}{\rho(x)}\right)^{-k_0} \leq \frac{\rho(y)}{\rho(x)} \leq C\left(1 + \frac{|x-y|}{\rho(x)}\right)^{k_0/(k_0+1)}. \qquad (17)$$

In particular, $\rho(y) \sim \rho(x)$ if $|x-y| < C\rho(x)$.

Let $F_\gamma^L(x,y)$ and $F_\gamma(x,y)$ be the kernels of $L^{i\gamma}$ and $(-\Delta)^{i\gamma}$, respectively, and $R^L(x,y)$ and $R(x,y)$ the kernels of $\nabla L^{-1/2}$ and $\nabla(-\Delta)^{-1/2}$, respectively. Set $\widetilde{F}_\gamma(x,y) = F_\gamma^L(x,y) - F_\gamma(x,y)$, $\widetilde{R}(x,y) = R^L(x,y) - R(x,y)$.

Lemma 10. *$L^{i\gamma}$ is a Calderón-Zygmund operator. It does not matter to assume that $n/2 < q_0 < n$. The kernel $F_\gamma^L(x,y)$ satisfies*

$$\left|F_\gamma^L(x, y+h) - F_\gamma^L(x,y)\right| \leq \frac{Ce^{\pi|\gamma|/2}|h|^\delta}{|x-y|^{n+\delta}}, \quad |h| \leq \frac{|x-y|}{2}, \qquad (18)$$

and, for any $N > 0$,

$$\left|F_\gamma^L(x,y)\right| \leq \frac{C_N e^{\pi|\gamma|/2}}{|x-y|^n}\left(1 + \frac{|x-y|}{\rho(y)}\right)^{-N}. \qquad (19)$$

In addition,

$$\left|\widetilde{F}_\gamma(x,y)\right| \leq \frac{Ce^{\pi|\gamma|/2}}{|x-y|^n}\left(\frac{|x-y|}{\rho(y)}\right)^\delta. \qquad (20)$$

Lemma 11. *When $n/2 < q_0 < n$, $\nabla L^{-1/2}$ is bounded on L^p for $1 < p \leq p_0$, where $1/p_0 = 1/q_0 - 1/n$. The kernel $R^L(x,y)$ satisfies, for any $N > 0$,*

$$\left|R^L(x,y)\right| \leq \frac{C_N}{|x-y|^{n-1}}\left(\int_{B(x,|x-y|/4)} \frac{V(z)\, dz}{|z-x|^{n-1}} + \frac{1}{|x-y|}\right) \times \left(1 + \frac{|x-y|}{\rho(y)}\right)^{-N}. \qquad (21)$$

In addition,

$$\left|\widetilde{R}(x,y)\right| \leq \frac{C}{|x-y|^{n-1}} \times \left(\int_{B(x,|x-y|/4)} \frac{V(z)\, dz}{|z-x|^{n-1}} + \frac{1}{|x-y|}\left(\frac{|x-y|}{\rho(y)}\right)^\delta\right). \qquad (22)$$

Lemma 12. *When $q_0 > n$, $\nabla L^{-1/2}$ is a Calderón-Zygmund operator. The kernel $R^L(x,y)$ satisfies*

$$\left|R^L(x,y+h)-R^L(x,y)\right| \le \frac{C|h|^\eta}{|x-y|^{n+\eta}}, \quad |h| \le \frac{|x-y|}{2}, \tag{23}$$

and, for any $N>0$,

$$\left|R^L(x,y)\right| \le \frac{C_N}{|x-y|^n}\left(1+\frac{|x-y|}{\rho(y)}\right)^{-N}. \tag{24}$$

In addition, for any $\delta' < 1$,

$$\left|\widetilde{R}(x,y)\right| \le \frac{C}{|x-y|^n}\left(\frac{|x-y|}{\rho(y)}\right)^{\delta'}. \tag{25}$$

For Lemmas 6–12, we refer readers to [1]. We also need the following estimates about $\widetilde{F}_\gamma(x,y)$ and $\widetilde{R}(x,y)$.

Lemma 13. *When $n/2 < q_0 < n$,*

$$\left|\widetilde{F}_\gamma(x,y+h)-\widetilde{F}_\gamma(x,y)\right| \le \frac{Ce^{\pi|\gamma|/2}}{|x-y|^n}\left(\frac{|h|}{\rho(y)}\right)^{\delta}, \quad |h| \le \frac{|x-y|}{2}. \tag{26}$$

When $q_0 > n$, for any $\delta' < 1$,

$$\left|\widetilde{R}(x,y+h)-\widetilde{R}(x,y)\right| \le \frac{C}{|x-y|^n}\left(\frac{|h|}{\rho(y)}\right)^{\delta'}, \quad |h| \le \frac{|x-y|}{2}. \tag{27}$$

Proof. It is well known that

$$\begin{gathered}\left|F_\gamma(x,y+h)-F_\gamma(x,y)\right| \le \frac{C|h|}{|x-y|^{n+1}}, \quad |h| \le \frac{|x-y|}{2},\\ \left|R(x,y+h)-R(x,y)\right| \le \frac{C|h|}{|x-y|^{n+1}}, \quad |h| \le \frac{|x-y|}{2}.\end{gathered} \tag{28}$$

Therefore, we also have the estimates

$$\left|\widetilde{F}_\gamma(x,y+h)-\widetilde{F}_\gamma(x,y)\right| \le \frac{Ce^{\pi|\gamma|/2}|h|^\delta}{|x-y|^{n+\delta}}, \quad |h| \le \frac{|x-y|}{2}, \tag{29}$$

$$\left|\widetilde{R}(x,y+h)-\widetilde{R}(x,y)\right| \le \frac{C|h|^\eta}{|x-y|^{n+\eta}}, \quad |h| \le \frac{|x-y|}{2}. \tag{30}$$

We may assume that $|x-y|<\rho(y)$. Otherwise, Lemma 13 is obvious.

We will use the following known facts (cf. [1]). Let $\Gamma^L(x,y,\tau)$ and $\Gamma(x,y,\tau)$ denote, respectively, the fundamental solutions for the operators $L+i\tau$ and $-\Delta+i\tau$ in $\mathbb{R}^n$, where $\tau\in\mathbb{R}$. They satisfy the following estimates. For any $k>0$ and $|h|\le|x-y|/2$,

$$\begin{aligned}\left|\Gamma(x,y,\tau)\right| &\le \frac{C_k}{\left(1+|\tau|^{1/2}|x-y|\right)^k}\frac{1}{|x-y|^{n-2}},\\ &\left|\Gamma^L(x,y+h,\tau)-\Gamma^L(x,y,\tau)\right|\\ &\le \frac{C_k}{\left(1+|\tau|^{1/2}|x-y|\right)^k}\frac{|h|^\delta}{|x-y|^{n-2+\delta}}\left(1+\frac{|x-y|}{\rho(y)}\right)^{-k}\end{aligned} \tag{31}$$

when $n/2<q_0<n$. Set $\widetilde{\Gamma}(x,y,\tau)=\Gamma^L(x,y,\tau)-\Gamma(x,y,\tau)$. Then $\widetilde{\Gamma}(x,y,\tau)$ is expressed as

$$\widetilde{\Gamma}(x,y,\tau) = -\int_{\mathbb{R}^n}\Gamma(x,z,\tau)V(z)\Gamma^L(z,y,\tau)\,dz. \tag{32}$$

Thus,

$$\begin{aligned}&\left|\widetilde{\Gamma}(x,y+h,\tau)-\widetilde{\Gamma}(x,y,\tau)\right|\\ &\le \int_{\mathbb{R}^n}\left|\Gamma(x,z,\tau)\right|V(z)\left|\Gamma^L(z,y+h,\tau)-\Gamma^L(z,y,\tau)\right|dz\\ &\le \int_{\mathbb{R}^n}\Big(C_k|h|^\delta V(z)\left(1+\rho(y)^{-1}|z-y|\right)^{-k}dz\Big)\\ &\quad\times\left(\left(1+|\tau|^{1/2}|x-z|\right)^k\left(1+|\tau|^{1/2}|z-y|\right)^k\right.\\ &\qquad\left.\times|x-z|^{n-2}|z-y|^{n-2+\delta}\right)^{-1}\\ &= \int_{|z-x|<|x-y|/2}(\cdots)+\int_{|z-y|<|x-y|/2}(\cdots)\\ &\quad+\int_{|z-x|\ge|x-y|/2,|z-y|\ge|x-y|/2}(\cdots)\\ &= I_1+I_2+I_3.\end{aligned} \tag{33}$$

Note that $V\in B_{q_0+\varepsilon}$ for some $\varepsilon>0$. Using Hölder inequality and $B_{q_0+\varepsilon}$ condition, it is easy to see that, for $0\le\sigma\le\delta$,

$$\int_{B(x,R)}\frac{V(y)}{|x-y|^{n-2+\sigma}}\,dy \le \frac{C}{R^{n-2+\sigma}}\int_{B(x,R)}V(y)\,dy. \tag{34}$$

Note that $\rho(x) \sim \rho(y)$ when $|x-y| < \rho(y)$. Making use of (34), we get

$$\begin{aligned} I_1 &\le \frac{C_k|h|^\delta}{\left(1+|\tau|^{1/2}|x-y|\right)^k |x-y|^{n-2+\delta}} \int_{|z-x|<|x-y|/2} \frac{V(z)\,dz}{|x-z|^{n-2}} \\ &\le \frac{C_k|h|^\delta}{\left(1+|\tau|^{1/2}|x-y|\right)^k |x-y|^{n-2+\delta}} \frac{1}{|x-y|^{n-2}} \\ &\quad \times \int_{|z-x|<|x-y|/2} V(z)\,dz \\ &\le \frac{C_k}{\left(1+|\tau|^{1/2}|x-y|\right)^k |x-y|^{n-2}} \left(\frac{|h|}{\rho(y)}\right)^\delta, \end{aligned} \tag{35}$$

where we have used Lemma 7 in the last inequality. Similarly,

$$\begin{aligned} I_2 &\le \frac{C_k|h|^\delta}{\left(1+|\tau|^{1/2}|x-y|\right)^k |x-y|^{n-2}} \int_{|z-y|<|x-y|/2} \frac{V(z)\,dz}{|z-y|^{n-2+\delta}} \\ &\le \frac{C_k|h|^\delta}{\left(1+|\tau|^{1/2}|x-y|\right)^k |x-y|^{n-2+\delta}} \frac{1}{|x-y|^{n-2}} \\ &\quad \times \int_{|z-y|<|x-y|/2} V(z)\,dz \\ &\le \frac{C_k}{\left(1+|\tau|^{1/2}|x-y|\right)^k |x-y|^{n-2}} \left(\frac{|h|}{\rho(y)}\right)^\delta. \end{aligned} \tag{36}$$

To estimate I_3, we write

$$\begin{aligned} I_3 &\le \frac{C_k|h|^\delta}{\left(1+|\tau|^{1/2}|x-y|\right)^k} \\ &\quad \times \int_{|z-y|\ge|x-y|/2} \left(1+\frac{|x-y|}{\rho(y)}\right)^{-k} \frac{V(z)\,dz}{|z-y|^{2n-4+\delta}} \\ &\le \frac{C_k|h|^\delta}{\left(1+|\tau|^{1/2}|x-y|\right)^k} \\ &\quad \times \left(\int_{|x-y|/2\le|z-y|<\rho(y)} \frac{V(z)\,dz}{|z-y|^{2n-4+\delta}} \right. \\ &\qquad \left. + \rho(y)^k \int_{|z-y|\ge\rho(y)} \frac{V(z)\,dz}{|z-y|^{2n-4+\delta+k}} \right). \end{aligned} \tag{37}$$

Using Hölder inequality and B_{q_0} condition, we obtain

$$\begin{aligned} &\int_{|x-y|/2\le|z-y|<\rho(y)} \frac{V(z)\,dz}{|z-y|^{2n-4+\delta}} \\ &\quad \le \left(\int_{|z-y|<\rho(y)} V(z)^{q_0}dz\right)^{1/q_0} \\ &\quad \times \left(\int_{|z-y|\ge|x-y|/2} \frac{dz}{|z-y|^{(2n-4+\delta)q_0'}}\right)^{1/q_0'} \\ &\quad \le \frac{C}{|x-y|^{n-2}\rho(y)^\delta}. \end{aligned} \tag{38}$$

Using Lemma 6 and taking k sufficiently large, we get

$$\begin{aligned} &\rho(y)^k \int_{|z-y|\ge\rho(y)} \frac{V(z)\,dz}{|z-y|^{2n-4+\delta+k}} \\ &\quad \le C\rho(y)^{4-2n-\delta} \sum_{j=1}^{\infty} 2^{-j(2n-4+\delta+k)} \int_{|z-y|\le 2^j\rho(y)} V(z)\,dz \\ &\quad \le C\rho(y)^{4-2n-\delta} \sum_{j=1}^{\infty} 2^{-j(2n-4+\delta+k)} C_0^j \int_{|z-y|\le\rho(y)} V(z)\,dz \\ &\quad \le C\rho(y)^{2-n-\delta} \\ &\quad \le \frac{C}{|x-y|^{n-2}\rho(y)^\delta}. \end{aligned} \tag{39}$$

Therefore,

$$\begin{aligned} &\left|\widetilde{\Gamma}(x,y+h,\tau) - \widetilde{\Gamma}(x,y,\tau)\right| \\ &\quad \le \frac{C_k}{\left(1+|\tau|^{1/2}|x-y|\right)^k |x-y|^{n-2}} \left(\frac{|h|}{\rho(y)}\right)^\delta. \end{aligned} \tag{40}$$

We also have

$$\nabla_x \widetilde{\Gamma}(x,y,\tau) = -\int_{\mathbb{R}^n} \nabla_x \Gamma(x,z,\tau)\,V(z)\,\Gamma^L(z,y,\tau)\,dz, \tag{41}$$

where $\nabla_x \Gamma(x,z,\tau)$ satisfies the estimate

$$\left|\nabla_x \Gamma(x,y,\tau)\right| \le \frac{C_k}{\left(1+|\tau|^{1/2}|x-y|\right)^k} \frac{1}{|x-y|^{n-1}}. \tag{42}$$

If $q_0 > n$, by the same argument as (40), for any $\delta' < 1$,

$$\begin{aligned} &\left|\nabla_x \widetilde{\Gamma}(x,y+h,\tau) - \nabla_x \widetilde{\Gamma}(x,y,\tau)\right| \\ &\quad \le \frac{C_k}{\left(1+|\tau|^{1/2}|x-y|\right)^k |x-y|^{n-1}} \left(\frac{|h|}{\rho(y)}\right)^{\delta'}. \end{aligned} \tag{43}$$

By the functional calculus and making use of (40), we obtain

$$\begin{aligned}&\left|\widetilde{F}_{\gamma}(x,y+h)-\widetilde{F}_{\gamma}(x,y)\right|\\&=\frac{1}{2\pi}\left|\int_{\mathbb{R}}(-i\tau)^{i\gamma}\left(\widetilde{\Gamma}(x,y+h,\tau)-\widetilde{\Gamma}(x,y,\tau)\right)\,d\tau\right|\\&\leq\frac{Ce^{\pi|\gamma|/2}}{|x-y|^{n}}\left(\frac{|h|}{\rho(y)}\right)^{\delta}.\end{aligned}\quad(44)$$

This proves (26).

Similarly, it follows from (43) that

$$\begin{aligned}&\left|\widetilde{R}(x,y+h)-\widetilde{R}(x,y)\right|\\&=\frac{1}{2\pi}\left|\int_{\mathbb{R}}(-i\tau)^{-1/2}\left(\nabla_{x}\widetilde{\Gamma}(x,y+h,\tau)-\nabla_{x}\widetilde{\Gamma}(x,y,\tau)\right)d\tau\right|\\&\leq\frac{C}{|x-y|^{n}}\left(\frac{|h|}{\rho(y)}\right)^{\delta'}.\end{aligned}\quad(45)$$

This proves (27). □

3. Molecular Characterization

Essentially, the proof of Theorem 4 is the same as the usual molecular theory.

Proof of Theorem 4. By Proposition 1, it is sufficient to prove that for any $H_L^{p,q,\epsilon}$-molecule $M(x)$ admits an atomic decomposition $M = \sum_j \lambda_j a_j$, where a_j are $H_L^{p,q}$-atoms and $\sum_j |\lambda_j|^p < C$.

We will give the proof in case $q = 2$. The proof is similar in the case of $q \neq 2$. Suppose $M(x)$ is an $H_L^{p,2,\epsilon}$-molecule centered at x_0. Let $\sigma = \|M\|_{L^2}^{1/(a-b)}$, where $\epsilon > 1/p - 1$, $a = 1 - 1/p + \epsilon$, $b = 1/2 + \epsilon$. If $\sigma < \mu_1 \rho(x_0)^n$, we return the usual molecular theory (cf. [8]). Thus nothing needs to be proved. Suppose $\sigma \geq \mu_1 \rho(x_0)^n$. Set

$$\begin{aligned}&B_k=\left\{x:\left|x-x_0\right|\leq 2^k\mu_1^{-1/n}\sigma^{1/n}\right\},\quad k=0,1,2,\ldots,\\&E_0=B_0,\qquad E_k=B_k\setminus B_{k-1},\quad k=1,2,\ldots.\end{aligned}\quad(46)$$

Then

$$M(x)=\sum_{k=0}^{\infty}M(x)\,\chi_{E_k}(x)=\sum_{k=0}^{\infty}M_k(x).\quad(47)$$

Note that supp $M_k \subset B_k$ and $2^k\mu_1^{-1/n}\sigma^{1/n} \geq \rho(x_0)$, $k = 0, 1, 2, \ldots$. Also we have

$$\left\|M_0\right\|_{L^2}\leq\|M\|_{L^2}=\sigma^{a-b}=\left|B_0\right|^{1/2-1/p},\quad(48)$$

$$\begin{aligned}\left\|M_k\right\|_{L^2}&\leq 2^{-(k-1)nb}\,\mu_1^{b}\sigma^{-b}\left\|\left|\cdot-x_0\right|^{nb}M\right\|_{L^2}\\&\leq 2^{-(k-1)nb}\,\sigma^{-b}\,\|M\|_{L^2}^{a/(a-b)}\\&=2^{-(k-1)nb}\,\sigma^{a-b}\\&=2^{nb-kna}\,\left|B_k\right|^{1/2-1/p}.\end{aligned}\quad(49)$$

Thus $M_k(x) = \lambda_k a_k(x)$, $k = 0, 1, 2, \ldots$, where a_k are $H_L^{p,q}$-atoms and $\sum_{k=0}^{\infty} |\lambda_k|^p < C$.

Originally, the sum in (47) converges pointwise. When $p = 1$, it is easy to see that the sum in (47) converges in L^1. If $n/(n+\delta) < p < 1$, for any $g \in \Lambda^L_{1/p-1}$,

$$\begin{aligned}&\left\|\left(1+|x|^{nb}\right)^{-1}g\right\|_{L^2}\\&\leq\left(\int_{|x|<\rho(0)}\left|g(x)\right|^2dx\right)^{1/2}+\sum_{k=1}^{\infty}2^{-(k-1)nb}\rho(0)^{-nb}\\&\quad\times\left(\int_{2^{k-1}\rho(0)\leq|x|<2^k\rho(0)}\left|g(x)\right|^2dx\right)^{1/2}\\&\leq C\sum_{k=0}^{\infty}2^{-kna}\,\|g\|_{\Lambda^L_{1/p-1}}\\&\leq C\,\|g\|_{\Lambda^L_{1/p-1}}.\end{aligned}\quad(50)$$

Therefore,

$$\|Mg\|_{L^1}\leq\left\|\left(1+|x|^{nb}\right)M\right\|_{L^2}\left\|\left(1+|x|^{nb}\right)^{-1}g\right\|_{L^2}<\infty.\quad(51)$$

It follows that the sum in (47) converges in $(\Lambda^L_{1/p-1})^*$. The proof of Theorem 4 is completed. □

4. H_L^p-Boundedness

In this section, we prove the boundedness of $L^{i\gamma}$ on H_L^p, $n/(n+\delta) < p \leq 1$. When $q_0 > n$, the boundedness of $\nabla L^{-1/2}$ on H_L^p, $n/(n+\eta) < p \leq 1$, can be proved by the same method. In fact, their kernels satisfy similar estimates.

Let $a(x)$ be an $H_L^{p,q}$-atom associated with a ball $B(x_0, r)$ for some suitable q. If $r \geq \rho(x_0)$, we will prove that $L^{i\gamma}a(x)$ is an $H_L^{p,q,\epsilon}$-molecule up to a constant factor. If $r < \rho(x_0)$, $L^{i\gamma}a(x)$ may be not an $H_L^{p,q,\epsilon}$-molecule up to a constant factor but $(-\Delta)^{i\gamma}a(x)$is (cf. [9]). We will prove that $(L^{i\gamma}-(-\Delta)^{i\gamma})a(x)$ is an $H_L^{p,q,\epsilon}$-molecule up to a constant factor for some suitable ϵ. This means that $\|L^{i\gamma}a(x)\|_{H_L^p} \leq C$ uniformly. Because the semigroup maximal function $M^L f$ is subadditive, by Proposition 1, $L^{i\gamma}$ is bounded on H_L^p, $n/(n+\delta) < p \leq 1$.

First, let $r \geq \rho(x_0)$. Because

$$\left\|L^{i\gamma}a(x)\right\|_{L^q}\leq C\|a(x)\|_{L^q}\leq C\left|B(x_0,r)\right|^{a-b},\quad(52)$$

where $\epsilon > 1/p - 1$, $a = 1 - 1/p + \epsilon$, $b = 1 - 1/q + \epsilon$, we have

$$\left\|L^{i\gamma}a(x)\right\|_{L^q}^{1/(a-b)}\geq\frac{1}{C}\,\rho(x_0)^n.\quad(53)$$

Thus there needs no the cancelation condition. We only need to estimate $\mathcal{N}(L^{i\gamma}a)$. Write

$$\begin{aligned}
&\left(\int_{\mathbb{R}^n}|x-x_0|^{nbq}\left|L^{i\gamma}a(x)\right|^q dx\right)^{1/q}\\
&\leq\left(\int_{B(x_0,2r)}|x-x_0|^{nbq}\left|L^{i\gamma}a(x)\right|^q dx\right)^{1/q}\\
&\quad+\left(\int_{|x-x_0|\geq 2r}|x-x_0|^{nbq}\left|L^{i\gamma}a(x)\right|^q dx\right)^{1/q}\\
&=I_1+I_2.
\end{aligned}\tag{54}$$

It is obvious that

$$\begin{aligned}
I_1&\leq r^{nb}\left\|L^{i\gamma}a(x)\right\|_{L^q}\leq C\left|B(x_0,r)\right|^b\\
&\quad\times\|a(x)\|_{L^q}\leq C\left|B(x_0,r)\right|^a.
\end{aligned}\tag{55}$$

For $y\in B(x_0,r)$, if $\rho(y)>r$, by Lemma 9, $\rho(y)\leq C\,\rho(x_0)\leq Cr$. Note that $|x-y|\sim|x-x_0|$ when $x\notin B(x_0,2r)$, $y\in B(x_0,r)$. Using Lemma 10, we get

$$\begin{aligned}
I_2&=\left(\int_{|x-x_0|\geq 2r}\left(\int_{B(x_0,r)}|x-x_0|^{nb}\right.\right.\\
&\qquad\left.\left.\times\left|F^L_\gamma(x,y)a(y)\right|dy\right)^q dx\right)^{1/q}\\
&\leq\int_{B(x_0,r)}|a(y)|\,dy\left(\int_{|x-x_0|\geq 2r}|x-x_0|^{nbq}\right.\\
&\qquad\left.\times\left|F^L_\gamma(x,y)\right|^q dx\right)^{1/q}\\
&\leq C\int_{B(x_0,r)}|a(y)|\;r^N dy\\
&\quad\times\left(\int_{|x-x_0|\geq 2r}|x-x_0|^{(nb-n-N)q}\;dx\right)^{1/q}\\
&\leq C\,r^{n-n/p+N}r^{nb-n/q'-N}\\
&\leq C\left|B(x_0,r)\right|^a
\end{aligned}\tag{56}$$

and provide $N>n\epsilon$. Therefore,

$$\left\||x-x_0|^{nb}L^{i\gamma}a(x)\right\|_{L^q}\leq C\left|B(x_0,r)\right|^a.\tag{57}$$

It follows that

$$\mathcal{N}\left(L^{i\gamma}a\right)=\mu_1^{b-a}\left\|L^{i\gamma}a(x)\right\|_{L^q}^{a/b}\left\||x-x_0|^{nb}L^{i\gamma}a(x)\right\|_{L^q}^{1-a/b}\leq C.\tag{58}$$

Next, suppose $r<\rho(x_0)$. Let us estimate $\|(L^{i\gamma}-(-\Delta)^{i\gamma})a(x)\|_{L^q}$. Consider

$$\begin{aligned}
&\left(\int_{\mathbb{R}^n}\left|\left(L^{i\gamma}-(-\Delta)^{i\gamma}\right)a(x)\right|^q dx\right)^{1/q}\\
&\leq\left(\int_{B(x_0,2r)}\left|\left(L^{i\gamma}-(-\Delta)^{i\gamma}\right)a(x)\right|^q dx\right)^{1/q}\\
&\quad+\left(\int_{2r\leq|x-x_0|<2\rho(x_0)}\left|\left(L^{i\gamma}-(-\Delta)^{i\gamma}\right)a(x)\right|^q dx\right)^{1/q}\\
&\quad+\left(\int_{|x-x_0|\geq 2\rho(x_0)}\left|\left(L^{i\gamma}-(-\Delta)^{i\gamma}\right)a(x)\right|^q dx\right)^{1/q}\\
&=J_1+J_2+J_3.
\end{aligned}\tag{59}$$

Note that $\rho(y)\sim\rho(x_0)$, when $y\in B(x_0,r)$ and by Lemma 10, we have

$$\begin{aligned}
&J_1\\
&=\left(\int_{B(x_0,2r}\left(\int_{B(x_0,r)}\left|\widetilde{F}_\gamma(x,y)a(y)\right|\;dy\right)^q dx\right)^{1/q}\\
&\leq\int_{B(x_0,r)}|a(y)|\left(\int_{B(x_0,2r)}\left|\widetilde{F}_\gamma(x,y)\right|^q dx\right)^{1/q}dy\\
&\leq C\int_{B(x_0,r)}|a(y)|\,\rho(x_0)^{-\delta}\left(\int_{B(x_0,2r)}\frac{dx}{|x-y|^{(n-\delta)q}}\right)^{1/q}dy\\
&\leq C\rho(x_0)^{-\delta}r^{n-n/p}\left(\int_{B(0,3r)}\frac{dx}{|x|^{(n-\delta)q}}\right)^{1/q}\\
&\leq C\rho(x_0)^{-\delta}r^{n/q-n/p+\delta}\\
&\leq C\rho(x_0)^{n(a-b)}.
\end{aligned}\tag{60}$$

Here we choose q such that $1<q<n/(n-\delta)$ and $n/q-n/p+\delta>0$ or, equivalently, $1<q<np/(n-p\delta)$. When $2r\leq|x-x_0|<2\rho(x_0)$, using the cancelation condition of a and Lemma 13, we obtain

$$\begin{aligned}
&\left|\left(L^{i\gamma}-(-\Delta)^{i\gamma}\right)a(x)\right|\\
&\quad=\left|\int_{B(x_0,r)}\left(\widetilde{F}_\gamma(x,y)-\widetilde{F}_\gamma(x,x_0)\right)a(y)\,dy\right|\\
&\quad\leq\frac{C\rho(x_0)^{-\delta}}{|x-x_0|^n}\int_{B(x_0,r)}|a(y)|\,|y-x_0|^\delta dy\\
&\quad\leq\frac{C\rho(x_0)^{-\delta}r^{n-n/p+\delta}}{|x-x_0|^n}.
\end{aligned}\tag{61}$$

It follows that

$$\begin{aligned} J_2 &\le C\rho(x_0)^{-\delta} r^{n-n/p+\delta}\left(\int_{2r\le|x-x_0|<2\rho(x_0)} \frac{dx}{|x-x_0|^{nq}}\right)^{1/q} \\ &\le C\rho(x_0)^{-\delta} r^{n/q-n/p+\delta} \\ &\le C\rho(x_0)^{n(a-b)}. \end{aligned} \tag{62}$$

When $|x-x_0|\ge 2\rho(x_0)$, by (29), we have

$$\begin{aligned} &\left|\left(L^{i\gamma}-(-\Delta)^{i\gamma}\right)a(x)\right| \\ &\quad = \left|\int_{B(x_0,r)}\left(\widetilde{F}_\gamma(x,y)-\widetilde{F}_\gamma(x,x_0)\right)a(y)\,dy\right| \\ &\quad \le \frac{C}{|x-x_0|^{n+\delta}}\int_{B(x_0,r)}|a(y)|\,|y-x_0|^\delta dy \\ &\quad \le \frac{C\, r^{n-n/p+\delta}}{|x-x_0|^{n+\delta}}. \end{aligned} \tag{63}$$

Then

$$\begin{aligned} J_3 &\le Cr^{n-n/p+\delta}\left(\int_{|x-x_0|\ge 2\rho(x_0)} \frac{dx}{|x-x_0|^{(n+\delta)q}}\right)^{1/q} \\ &\le Cr^{n-n/p+\delta}\rho(x_0)^{-n/q'-\delta} \\ &\le C\rho(x_0)^{n(a-b)}. \end{aligned} \tag{64}$$

We have seen that

$$\left\|\left(L^{i\gamma}-(-\Delta)^{i\gamma}\right)a(x)\right\|_{L^q}^{1/(a-b)} \ge \frac{1}{C}\,\rho(x_0)^n. \tag{65}$$

As above, there needs no the cancelation condition . To finish the proof, we only need to prove $\mathcal{N}((L^{i\gamma}-(-\Delta)^{i\gamma})a)\le C$ or, equivalently,

$$\left\||x-x_0|^{nb}\left(L^{i\gamma}-(-\Delta)^{i\gamma}\right)a(x)\right\|_{L^q} \le C\,\rho(x_0)^{na}. \tag{66}$$

Write

$$\begin{aligned} &\left(\int_{\mathbb{R}^n}|x-x_0|^{nbq}\left|L^{i\gamma}-(-\Delta)^{i\gamma}\,a(x)\right|^q dx\right)^{1/q} \\ &\quad\le \left(\int_{B(x_0,2\rho(x_0))}|x-x_0|^{nbq}\left|L^{i\gamma}-(-\Delta)^{i\gamma}a(x)\right|^q dx\right)^{1/q} \\ &\qquad + \left(\int_{|x-x_0|\ge 2\rho(x_0)}|x-x_0|^{nbq}\left|L^{i\gamma}-(-\Delta)^{i\gamma}a(x)\right|^q dx\right)^{1/q} \\ &\quad = H_1+H_2. \end{aligned} \tag{67}$$

It is clear that

$$H_1 \le C\rho(x_0)^{nb}\left\|L^{i\gamma}-(-\Delta)^{i\gamma}a(x)\right\|_{L^q} \le C\rho(x_0)^{na}. \tag{68}$$

By (63),

$$\begin{aligned} H_2 &\le Cr^{n-n/p+\delta}\left(\int_{|x-x_0|\ge 2\rho(x_0)}|x-x_0|^{(nb-n-\delta)q}dx\right)^{1/q} \\ &\le Cr^{n-n/p+\delta}\rho(x_0)^{nb-\delta-n/q'} \\ &\le C\rho(x_0)^{na}, \end{aligned} \tag{69}$$

where we have taken ϵ such that $1/p-1<\epsilon<\delta/n$, which implies that $(nb-n-\delta)q+n<0$. The proof is complete.

5. H_L^1-Boundedness

In this section we prove the boundedness of $\nabla L^{-1/2}$ on H_L^1 when $n/2<q_0<n$.

Let $a(x)$ be an $H_L^{1,q}$-atom associated with a ball $B(x_0,r)$ for some suitable q. As the above section, if $r\ge\rho(x_0)$, we will prove that $\nabla L^{-1/2}a(x)$ is an $H_L^{1,q,\epsilon}$-molecule up to a constant factor. If $r<\rho(x_0)$, we will prove that $(\nabla L^{-1/2}-\nabla(-\Delta)^{-1/2})a(x)$ is an $H_L^{1,q,\epsilon}$-molecule up to a constant factor for some suitable ϵ. In any case we have $\|\nabla L^{-1/2}a(x)\|_{H_L^p}\le C$ uniformly.

Suppose $r\ge\rho(x_0)$. It follows from Lemma 11 that

$$\left\|\nabla L^{-1/2}a(x)\right\|_{L^q} \le C\|a(x)\|_q \le C\,|B(x_0,r)|^{a-b}, \tag{70}$$

provide $1<q\le p_0$, where $1/p_0=1/q_0-1/n$, $a=\epsilon>0$, $b=1-1/q+\epsilon$. Thus there needs no the cancelation condition. Write

$$\begin{aligned} &\left(\int_{\mathbb{R}^n}|x-x_0|^{nbq}\left|\nabla L^{-1/2}a(x)\right|^q dx\right)^{1/q} \\ &\quad\le \left(\int_{B(x_0,2r)}|x-x_0|^{nbq}\left|\nabla L^{-1/2}a(x)\right|^q dx\right)^{1/q} \\ &\qquad + \left(\int_{|x-x_0|\ge 2r}|x-x_0|^{nbq}\left|\nabla L^{-1/2}a(x)\right|^q dx\right)^{1/q} \\ &\quad = I_1+I_2. \end{aligned} \tag{71}$$

It is obvious that

$$\begin{aligned} I_1 &\le r^{nb}\left\|\nabla L^{-1/2}a(x)\right\|_{L^q} \\ &\le C\,|B(x_0,r)|^b\,\|a(x)\|_{L^q} \le C\,|B(x_0,r)|^a. \end{aligned} \tag{72}$$

On the other hand, we have

$$\begin{aligned} I_2 &\leq \left(\int_{|x-x_0|\geq 2r}\left(\int_{B(x_0,r)}|x-x_0|^{nb}\right.\right. \\ &\qquad \left.\left.\times\left|R^L(x,y)\right|\left|a(y)\right|\right)^q dx\right)^{1/q} dy \\ &\leq \int_{B(x_0,r)}|a(y)|\left(\int_{|x-x_0|\geq 2r}|x-x_0|^{nbq}\right. \\ &\qquad \left.\times\left|R^L(x,y)\right|^q dx\right)^{1/q} dy \\ &= \int_{B(x_0,r)} G(y)\,|a(y)|\,dy. \end{aligned} \tag{73}$$

Note that $|x-y| \sim |x-x_0|$ when $|x-x_0| \geq 2r$ and $y \in B(x_0,r)$ and by Lemma 11,

$$\begin{aligned} &G(y) \\ &\leq C\rho(y)^N \\ &\quad\times\left\{\left(\int_{|x-x_0|\geq 2r}\left(\int_{B(x,|x-y|/4)}\frac{V(z)\,dz}{|z-x|^{n-1}}\right)^q\right.\right. \\ &\qquad\left.\times\frac{dx}{|x-x_0|^{(N-nb+n-1)q}}\right)^{1/q} \\ &\qquad\left.+\left(\int_{|x-x_0|\geq 2r}\frac{dx}{|x-x_0|^{(N-nb+n)q}}\right)^{1/q}\right\}. \end{aligned} \tag{74}$$

Since $\rho(y) \leq Cr$ for $y \in B(x_0,r)$, it is clear that

$$\begin{aligned} &\rho(y)^N\left(\int_{|x-x_0|\geq 2r}\frac{dx}{|x-x_0|^{(N-nb+n)q}}\right)^{1/q} \\ &\quad\leq Cr^{nb-n/q'} \leq C\left|B(x_0,r)\right|^a \end{aligned} \tag{75}$$

provide $N > na$. We have taken q such that $1 < q \leq p_0$, where $1/p_0 = 1/q_0 - 1/n$. Let $1/q = 1/s - 1/n$. Then $s \leq q_0$. Using the theorem on fractional integrals, B_s condition, and Lemma 8, we obtain

$$\begin{aligned} &\rho(y)^N\left(\int_{|x-x_0|\geq 2r}\left(\int_{B(x,|x-y|/4)}\frac{V(z)\,dz}{|z-x|^{n-1}}\right)^q\right. \\ &\qquad\left.\times\frac{dx}{|x-x_0|^{(N-nb+n-1)q}}\right)^{1/q} \\ &\leq Cr^N\sum_{j=1}^{\infty}\left(\int_{2^jr\leq|x-x_0|<2^{j+1}r}\frac{1}{|x-x_0|^{(n+N-nb-1)q}}\right. \\ &\qquad\left.\cdot\left(\int_{B(x,|x-y|/4)}\frac{V(z)\,dz}{|z-x|^{n-1}}\right)^q dx\right)^{1/q} \\ &\leq C\sum_{j=1}^{\infty}2^{-jN}\left(2^jr\right)^{-n+nb+1} \\ &\quad\times\left(\int_{B(x_0,2^{j+1}r)}\left(\int_{B(x,|x-y|/4)}\frac{V(z)\,dz}{|z-x|^{n-1}}\right)^q dx\right)^{1/q} \\ &\leq C\sum_{j=1}^{\infty}2^{-jN}\left(2^jr\right)^{-n+nb+1}\left(\int_{B(x_0,2^{j+1}r)}V(x)^s dx\right)^{1/s} \\ &\leq C\sum_{j=1}^{\infty}2^{-jN}\left(2^jr\right)^{-n+nb+1}\left|B(x_0,2^{j+1}r)\right|^{1/s-1} \\ &\quad\times\int_{B(x_0,2^{j+1}r)}V(x)\,dx \\ &\leq C\sum_{j=1}^{\infty}2^{-j(N-m_0)}\left(2^jr\right)^{-n+nb+n/s-1} \\ &= C\sum_{j=1}^{\infty}2^{-j(N-m_0-na)}r^{na} \\ &\leq C\left|B(x_0,r)\right|^a \end{aligned} \tag{76}$$

provide N sufficiently large. Thus $G(y) \leq C\,|B(x_0,r)|^a$. It follows that

$$\begin{aligned} I_2 &\leq \int_{B(x_0,r)} G(y)\,|a(y)|\,dy \\ &\leq C\left|B(x_0,r)\right|^a\|a\|_{L^1} \leq C\left|B(x_0,r)\right|^a. \end{aligned} \tag{77}$$

Therefore, $\||\cdot-x_0|^{nb}\nabla L^{-1/2}a\|_{L^q} \leq C\,|B(x_0,r)|^a$ and $\mathcal{N}(\nabla L^{-1/2}a) \leq C$.

In case $r < \rho(x_0)$, we need to prove that $(\nabla L^{-1/2} - \nabla(-\Delta)^{-1/2})a(x)$ is an $H_L^{1,q,\epsilon}$-molecule up to a constant factor for some suitable ϵ. First we give the estimate of $\|(\nabla L^{-1/2} - \nabla(-\Delta)^{-1/2})a(x)\|_{L^q}$. Write

$$\begin{aligned} &\left(\int_{\mathbb{R}^n}\left|\left(\nabla L^{-1/2}-\nabla(-\Delta)^{-1/2}\right)a(x)\right|^q dx\right)^{1/q} \\ &\leq\left(\int_{B(x_0,2\rho(x_0))}\left|\left(\nabla L^{-1/2}-\nabla(-\Delta)^{-1/2}\right)a(x)\right|^q dx\right)^{1/q} \\ &\quad+\left(\int_{|x-x_0|\geq 2\rho(x_0)}\left|\left(\nabla L^{-1/2}-\nabla(-\Delta)^{-1/2}\right)a(x)\right|^q dx\right)^{1/q} \\ &= J_1 + J_2. \end{aligned} \tag{78}$$

We have

$$J_1 \le \int_{B(x_0,r)} |a(y)| \left(\int_{B(x_0,2\rho(x_0))} \left|\widetilde{R}(x,y)\right|^q dx \right)^{1/q} dy = \int_{B(x_0,r)} \widetilde{G}(y) |a(y)| dy. \tag{79}$$

By Lemma 11,

$$\begin{aligned}\widetilde{G}(y) \le & \left(\int_{B(x_0,2\rho(x_0))} \frac{C}{|x-y|^{(n-1)q}} \times \left(\int_{B(x,|x-y|/4)} \frac{V(z)\,dz}{|z-x|^{n-1}} \right)^q dx \right)^{1/q} \\ &+ C\rho(y)^{-\delta} \left(\int_{B(x_0,2\rho(x_0))} \frac{dx}{|x-y|^{(n-\delta)q}} \right)^{1/q} \\ = & \widetilde{G}_1(y) + \widetilde{G}_2(y).\end{aligned} \tag{80}$$

Note that $\rho(y) \sim \rho(x_0)$ when $y \in B(x_0,r)$. It is obvious that

$$\widetilde{G}_2(y) \le C\rho(x_0)^{-n/q'} = C\rho(x_0)^{n(a-b)}, \tag{81}$$

provide $1 < q < n/(n-\delta)$. On the other hand, using the theorem on fractional integrals and B_s condition with $s \le q_0$, $1/q = 1/s - 1/n$, and $1 < q < n/(n-\delta)$, we get

$$\begin{aligned}\widetilde{G}_1(y) \le & \sum_{j=0}^{\infty} \left(\int_{2^{-j+1}\rho(x_0)\le|x-y|<2^{-j+2}\rho(x_0)} \frac{C}{|x-y|^{(n-1)q}} \cdot \left(\int_{B(x,|x-y|/4)} \frac{V(z)\,dz}{|z-x|^{n-1}} \right)^q dx \right)^{1/q} \\ \le & C\sum_{j=0}^{\infty} \left(2^{-j}\rho(x_0)\right)^{-n+1} \times \left(\int_{|x-y|<2^{-j+2}\rho(x_0)} \times \left(\int_{B(x,|x-y|/4)} \frac{V(z)\,dz}{|z-x|^{n-1}} \right)^q dx \right)^{1/q} \\ \le & C\sum_{j=0}^{\infty} \left(2^{-j}\rho(x_0)\right)^{-n+1} \left(\int_{|x-y|<2^{-j+3}\rho(x_0)} V(x)^s \, dx \right)^{1/s} \\ \le & C\sum_{j=0}^{\infty} \left(2^{-j}\rho(x_0)\right)^{-2n+n/s+1} \int_{|x-y|<2^{-j+3}\rho(x_0)} V(x)\,dx \\ \le & C\sum_{j=0}^{\infty} \left(2^{-j}\rho(x_0)\right)^{-n+n/s-1} 2^{-j\delta} \\ \le & C\rho(x_0)^{n(a-b)},\end{aligned} \tag{82}$$

where we have used Lemma 7 in the last second inequality. Thus,

$$J_1 \le \int_{B(x_0,r)} \widetilde{G}(y) |a(y)| dy \le C\rho(x_0)^{n(a-b)}. \tag{83}$$

Since $|x-y| \sim |x-x_0|$ when $|x-x_0| \ge 2\rho(x_0)$ and $y \in B(x_0,r)$, $|\widetilde{R}(x,y)| \le C/|x-y|^n$, it is easy to see that

$$\begin{aligned}J_2 \le & \int_{B(x_0,r)} |a(y)| \left(\int_{|x-x_0|\ge 2\rho(x_0)} \left|\widetilde{R}(x,y)\right|^q dx \right)^{1/q} dy \\ \le & C \int_{B(x_0,r)} |a(y)| \left(\int_{|x-x_0|\ge 2\rho(x_0)} \frac{dx}{|x-y|^{nq}} \right)^{1/q} dy \\ \le & C \int_{B(x_0,r)} |a(y)| \rho(x_0)^{n(a-b)} dy \\ \le & C\rho(x_0)^{n(a-b)}.\end{aligned} \tag{84}$$

Therefore we have

$$\left\| \left(\nabla L^{-1/2} - \nabla(-\Delta)^{-1/2} \right) a(x) \right\|_{L^q} \le C\rho(x_0)^{n(a-b)}. \tag{85}$$

As above, there needs no the cancelation condition. Write

$$\begin{aligned}&\left(\int_{\mathbb{R}^n} |x-x_0|^{nbq} \left| \left(\nabla L^{-1/2} - \nabla(-\Delta)^{-1/2} \right) a(x) \right|^q dx \right)^{1/q} \\ &\le \left(\int_{B(x_0,2\rho(x_0))} |x-x_0|^{nbq} \times \left| \left(\nabla L^{-1/2} - \nabla(-\Delta)^{-1/2} \right) a(x) \right|^q dx \right)^{1/q} \\ &\quad + \left(\int_{|x-x_0|\ge 2\rho(x_0)} |x-x_0|^{nbq} \left| \nabla L^{-1/2} a(x) \right|^q dx \right)^{1/q} \\ &\quad + \left(\int_{|x-x_0|\ge 2\rho(x_0)} |x-x_0|^{nbq} \left| \nabla(-\Delta)^{-1/2} a(x) \right|^q dx \right)^{1/q} \\ &= H_1 + H_2 + H_3.\end{aligned} \tag{86}$$

It is obvious that

$$H_1 \le C\rho(x_0)^{nb} \left\| \left(\nabla L^{-1/2} - \nabla(-\Delta)^{-1/2} \right) a(x) \right\|_{L^q} \le C\rho(x_0)^{na}. \tag{87}$$

We have

$$\begin{aligned}H_2 \le & \int_{B(x_0,r)} |a(y)| \times \left(\int_{|x-x_0|\ge 2\rho(x_0)} |x-x_0|^{nbq} \times \left| R^L(x,y) \right|^q dx \right)^{1/q} dy \\ = & \int_{B(x_0,r)} G_0(y) |a(y)| \, dy.\end{aligned} \tag{88}$$

Since $|x-y| \sim |x-x_0|$ and $\rho(y) \sim \rho(x_0)$ when $|x-x_0| \geq 2\rho(x_0)$ and $y \in B(x_0, r)$, by Lemma 11,

$$\begin{aligned} & G_0(y) \\ & \quad \leq C\rho(x_0)^N \\ & \quad\quad \times \left\{ \left(\int_{|x-x_0|\geq 2\rho(x_0)} \left(\int_{B(x,|x-y|/4)} \frac{V(z)\,dz}{|z-x|^{n-1}} \right)^q \right.\right. \\ & \quad\quad\quad \left. \times \frac{dx}{|x-x_0|^{(N-nb+n-1)q}} \right)^{1/q} \\ & \quad\quad\quad \left. + \left(\int_{|x-x_0|\geq 2\rho(x_0)} \frac{dx}{|x-x_0|^{(N-nb+n)q}} \right)^{1/q} \right\}. \end{aligned} \tag{89}$$

Similar to $G(y)$ in the proof of (77), we obtain $G_0(y) \leq C\rho(x_0)^{na}$ by the same argument. It follows that

$$H_2 \leq \int_{B(x_0,r)} G_0(y)\,|a(y)|\,dy \leq C\,\rho(x_0)^{na}. \tag{90}$$

Using the cancelation condition of a,

$$\begin{aligned} H_3 \leq & \int_{B(x_0,r)} |a(y)| \left(\int_{|x-x_0|\geq 2\rho(x_0)} |x-x_0|^{nbq} \right. \\ & \quad \left. \times |R(x,y) - R(x,x_0)|^q dx \right)^{1/q} dy \\ \leq & \; C \int_{B(x_0,r)} |a(y)| \\ & \quad \times \left(\int_{|x-x_0|\geq 2\rho(x_0)} \frac{|y-x_0|^q dx}{|x-x_0|^{(n+1-nb)q}} \right)^{1/q} dy \\ \leq & \; C \int_{B(x_0,r)} |a(y)|\,\rho(x_0)^{na} dy \\ \leq & \; C\rho(x_0)^{na}, \end{aligned} \tag{91}$$

where we have taken ϵ such that $0 < \epsilon < 1/n$. This proves that

$$\left\| |x-x_0|^{nb} \left(\nabla L^{-1/2} - \nabla(-\Delta)^{-1/2} \right) a(x) \right\|_{L^q} \leq C\rho(x_0)^{na}. \tag{92}$$

It follows that $\mathcal{N}((\nabla L^{-1/2} - \nabla(-\Delta)^{-1/2})a) \leq C$. The proof is completed.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Estimates for Multilinear Commutators of Generalized Fractional Integral Operators on Weighted Morrey Spaces

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Let L be the infinitesimal generator of an analytic semigroup on $L^2(\mathbb{R}^n)$ with Gaussian kernel bounds, and let $L^{-\alpha/2}$ be the fractional integrals of L for $0 < \alpha < n$. Assume that $\vec{b} = (b_1, b_2, \ldots, b_m)$ is a finite family of locally integrable functions; then the multilinear commutators generated by $\vec{b}$ and $L^{-\alpha/2}$ are defined by $L_{\vec{b}}^{-\alpha/2} f = [b_m, \ldots, [b_2, [b_1, L^{-\alpha/2}]], \ldots] f$. Assume that b_j belongs to weighted BMO space, $j = 1, 2, \ldots, m$; the authors obtain the boundedness of $L_{\vec{b}}^{-\alpha/2}$ on weighted Morrey spaces. As a special case, when $L = -\Delta$ is the Laplacian operator, the authors also obtain the boundedness of the multilinear fractional commutator $I_\alpha^{\vec{b}}$ on weighted Morrey spaces. The main results in this paper are substantial improvements and extensions of some known results.

1. Introduction and Main Results

Assume that L is a linear operator on $L^2(\mathbb{R}^n)$, which generates an analytic semigroup e^{-tL} with a kernel $p_t(x, y)$ satisfying a Gaussian upper bound; that is,

$$\left|p_t(x, y)\right| \leq \frac{C}{t^{n/2}} e^{-c(|x-y|^2/t)}, \tag{1}$$

for $x, y \in \mathbb{R}^n$ and all $t > 0$.

The property (1) is satisfied by a large amount of differential operators. One can see [1] for details and examples.

For $0 < \alpha < n$, the fractional integral $L^{-\alpha/2}$ generated by the operator L is defined by

$$L^{-\alpha/2} f(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty e^{-tL}(f) \frac{dt}{t^{-\alpha/2+1}}(x). \tag{2}$$

Let $\vec{b} = (b_1, b_2, \ldots, b_m)$ be a finite family of locally integrable functions; then the multilinear commutators generated by $L^{-\alpha/2}$ and $\vec{b}$ are defined by

$$L_{\vec{b}}^{-\alpha/2} f = \left[b_m, \ldots, \left[b_2, \left[b_1, L^{-\alpha/2}\right]\right], \ldots\right] f, \tag{3}$$

where $m \in \mathbb{Z}^+$.

Note that if $L = -\Delta$, which is Laplacian on $\mathbb{R}^n$, then $L^{-\alpha/2}$ is the classical fractional integral I_α:

$$I_\alpha f(x) = \frac{\Gamma((n-\alpha)/2)}{\pi^{n/2} 2^\alpha \Gamma(\alpha/2)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \tag{4}$$

while $L_{\vec{b}}^{-\alpha/2}$ is the iterated commutator generated by $\vec{b}$ and I_α:

$$I_\alpha^{\vec{b}} f = [b_m, \ldots, [b_2, [b_1, I_\alpha]], \ldots] f, \tag{5}$$

where $m \in \mathbb{Z}^+$.

When $m = 1$, it is easy to see that $L_{\vec{b}}^{-\alpha/2} f = [b, L^{-\alpha/2}] f$ is the commutator generated by $L^{-\alpha/2}$ and b, and when $b_1 = b_2 = \cdots = b_m$, $L_{\vec{b}}^{-\alpha/2}$ is the higher commutator.

As we all know, if $b \in$ BMO, the commutator of fractional integral operator $[b, I_\alpha]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, where $1 < p < n/\alpha$ and $1/q = 1/p -$

α/n (see [2]). In 2004, Duong and Yan [1] generalized the above classical result and obtained the (L^p, L^q) boundedness of the commutator $[b, L^{-\alpha/2}]$ under the same conditions. Simultaneously, the theory on multilinear integral operators and multilinear commutators has attracted much attention as a rapid developing field in harmonic analysis. Mo and Lu [3] studied the (L^p, L^q) boundedness of the multilinear commutators $L_{\vec{b}}^{-\alpha/2}$, where $\vec{b} = (b_1, \ldots, b_m)$, $b_j \in \mathrm{BMO}$, and $j = 1, 2, \ldots, m$.

On the other hand, Muckenhoupt and Wheeden [4] gave some definitions of weighted bounded mean oscillation and obtained some equivalent conditions for them.

Definition 1 (see [4]). Let $1 \leq p < \infty$ and w be locally integral in $\mathbb{R}^n$ and let $w \geq 0$. A locally integrable function b is said to be in $\mathrm{BMO}_p(w)$ if

$$\|b\|_{\mathrm{BMO}_p(w)} = \sup_Q \left(\frac{1}{w(Q)} \int_Q \left| b(x) - b_Q \right|^p w(x)^{1-p} \, dx \right)^{1/p} \leq C, \tag{6}$$

where $b_Q = (1/|Q|) \int_Q b(y) dy$ and the supremum is taken over all balls $Q \in \mathbb{R}^n$.

We may note that other weighted definitions for the bounded mean oscillation also have been given by Muckenhoupt and Wheeden in [4].

Definition 2 (see [4]). Let w be locally integral in $\mathbb{R}^n$ and $w \geq 0$. A locally integrable function b is said to be in $\mathrm{BMO}(w)$ if the norm of $\mathrm{BMO}(w)$: $\|\cdot\|_{*,w}$ satisfies

$$\|b\|_{*,w} = \sup_Q \frac{1}{w(Q)} \int_Q \left| b(x) - b_{Q,w} \right| w(x) \, dx \leq C, \tag{7}$$

where $b_{Q,w} = (1/w(Q)) \int_Q b(z) w(z) dz$ and the supremum is taken over all balls $Q \in \mathbb{R}^n$.

The above two definitions cannot contain each other; throughout this paper, we will make some investigations on the basis of Definition 2.

Recently, Wang [5] obtained some estimates for the commutator $[b, I_\alpha]$ on weighted Morrey space (see Definitions 3 and 4), where $b \in \mathrm{BMO}_1(w)$. Furthermore, Wang and Si [6] obtained the necessary and sufficient conditions for the boundedness of $[b, L^{-\alpha/2}]$ on weighted Morrey spaces when $b \in \mathrm{BMO}_1(w)$.

Motivated by [1, 3, 5, 6], it is natural to raise the following question: how to establish corresponding boundedness of the multilinear commutator $L_{\vec{b}}^{-\alpha/2} f$ on the weighted Morrey space, where $\vec{b} = (b_1, \ldots, b_m)$, $b_j \in \mathrm{BMO}(w)$?

The question is not motivated only by a mere quest to extend the multilinear commutator $L_{\vec{b}}^{-\alpha/2} f$ from the classical commutator $[b, I_\alpha]$ but rather by their natural appearance in analysis (see [3]).

To state the main results, we now give some definitions and notations.

A weight is a locally integrable function on $\mathbb{R}^n$ which takes values in $(0, \infty)$ almost everywhere. For a weight w and a measurable set E, we define $w(E) = \int_E w(x) dx$, the Lebesgue measure of E, by $|E|$ and the characteristic function of E by χ_E. For a real number p, $1 < p < \infty$; p' is the conjugate of p; that is, $1/p + 1/p' = 1$. The letter C denotes a positive constant that may vary at each occurrence but is independent of the essential variable.

Definition 3 (see [7]). Let $1 \leq p < \infty$, $0 < \kappa < 1$; let w be a weight; then weighted Morrey space is defined by

$$L^{p,\kappa}(w) := \left\{ f \in L^p_{\mathrm{loc}}(w) : \|f\|_{L^{p,\kappa}(w)} < \infty \right\}, \tag{8}$$

where

$$\|f\|_{L^{p,\kappa}(w)} = \sup_B \left(\frac{1}{w(B)^\kappa} \int_B |f(x)|^p w(x) \, dx \right)^{1/p}, \tag{9}$$

and the supremum is taken over all balls B in $\mathbb{R}^n$.

Definition 4 (see [7]). Let $1 \leq p < \infty$, $0 < \kappa < 1$; let u, v be weight; then two weights weighted Morrey space are defined by

$$L^{p,\kappa}(u, v) := \left\{ f : \|f\|_{L^{p,\kappa}(u,v)} < \infty \right\}, \tag{10}$$

where

$$\|f\|_{L^{p,\kappa}(u,v)} = \sup_B \left(\frac{1}{v(B)^\kappa} \int_B |f(x)|^p u(x) \, dx \right)^{1/p}, \tag{11}$$

and the supremum is taken over all balls B in $\mathbb{R}^n$. If $u = v$, then we denote $L^{p,\kappa}(u)$ for short.

Remark 5. (1) If $w = 1$, $\kappa = \lambda/n$, and $0 < \lambda < n$, then $L^{p,\kappa}(w) = L^{p,\lambda}(\mathbb{R}^n)$, the classical Morrey space.

(2) If $\kappa = 0$, then $L^{p,0}(w) = L^p(w)$, the weighted Lebesgue space; if $w = 1$, $\kappa = 0$, then $L^{p,\kappa}(w) = L^p(\mathbb{R}^n)$, the Lebesgue space.

Definition 6 (see [8]). A weight function w is in the Muckenhoupt class A_p with $1 < p < \infty$ if for every ball B in $\mathbb{R}^n$, there exists a positive constant C which is independent of B such that

$$\left(\frac{1}{|B|} \int_B w(x) \, dx \right) \left(\frac{1}{|B|} \int_B w(x)^{-1/(p-1)} \, dx \right)^{p-1} \leq C. \tag{12}$$

When $p = 1$, $w \in A_1$, if

$$\frac{1}{|B|} \int_B w(x) \, dx \leq C \operatorname*{ess\,inf}_{x \in B} w(x). \tag{13}$$

When $p = \infty$, $w \in A_\infty$, if there exist positive constants δ and C such that given a ball B and E is a measurable subset of B, then

$$\frac{w(E)}{w(B)} \leq C \left(\frac{|E|}{|B|} \right)^\delta. \tag{14}$$

Definition 7 (see [9]). A weight function w belongs to the reverse Hölder class RH_r, if there exist two constants $r > 1$ and $C > 0$ such that the following reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B w(x)^r\,dx\right)^{1/r} \le C\frac{1}{|B|}\int_B w(x)\,dx, \tag{15}$$

holds for every ball B in $\mathbb{R}^n$.

It is well known that if $w \in A_p$ with $1 \le p < \infty$, then there exists $r > 1$ such that $w \in \mathrm{RH}_r$. It follows from Hölder's inequality that $w \in \mathrm{RH}_r$ implies $w \in \mathrm{RH}_s$ for all $1 < s < r$. Moreover, if $w \in \mathrm{RH}_r$, $r > 1$, then we have $w \in \mathrm{RH}_{r+\varepsilon}$ for some $\varepsilon > 0$. We thus write $r_w \equiv \sup\{r > 1 : w \in \mathrm{RH}_r\}$ to denote the critical index of w for the reverse Hölder condition.

Definition 8. The Hardy-Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{x\in B}\frac{1}{|B|}\int_B |f(y)|\,dy. \tag{16}$$

Let w be a weight. The weighted maximal operator M_w is defined by

$$M_w f(x) = \sup_{x\in B}\frac{1}{w(B)}\int_B |f(y)|\,w(y)\,dy. \tag{17}$$

For $0 < \alpha < n$, $r \ge 1$, the fractional maximal operator $M_{\alpha,r}$ is defined by

$$M_{\alpha,r} f(x) = \sup_{x\in B}\left(\frac{1}{|B|^{1-\alpha r/n}}\int_B |f(y)|^r\,dy\right)^{1/r}. \tag{18}$$

And the fractional weighted maximal operator $M_{\alpha,r,w}$ is defined by

$$M_{\alpha,r,w} f(x) = \sup_{x\in B}\left(\frac{1}{w(B)^{1-\alpha r/n}}\int_B |f(y)|^r\,w(y)\,dy\right)^{1/r}. \tag{19}$$

If $\alpha = 0$, we denote $M_{r,w}$ for short.

Definition 9. A family of operators $\{A_t : t > 0\}$ is said to be an "approximation to identity" if, for every $t > 0$, A_t is represented by the kernel $a_t(x, y)$, which is a measurable function defined on $\mathbb{R}^n\times\mathbb{R}^n$, in the following sense: for every $f \in L^p(\mathbb{R}^n)$, $p \ge 1$,

$$\begin{gathered} A_t f(x) = \int_{\mathbb{R}^n} a_t(x,y)\,f(y)\,dy, \\ |a_t(x,y)| \le h_t(x,y) = t^{-n/2} g\left(\frac{|x-y|^2}{t}\right), \end{gathered} \tag{20}$$

for $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, $t > 0$. Here, g is a positive, bounded, decreasing function satisfying

$$\lim_{r\to\infty} r^{n+\varepsilon} g(r^2) = 0, \tag{21}$$

for some $\varepsilon > 0$.

Associated with an "approximation to identity" $\{A_t : t > 0\}$, Martell [10] introduced the sharp maximal function as follows:

$$M_A^\sharp f(x) = \sup_{x\in B}\frac{1}{|B|}\int_B \left|f(y) - A_{t_B} f(y)\right| dy, \tag{22}$$

where $t_B = r_B^2$, r_B is the radius of the ball B, and $f \in L^p(\mathbb{R}^n)$ for some $p \ge 1$.

Notice that our analytic semigroup $\{e^{-tL} : t > 0\}$ is an "approximation to identity." In particular, denote

$$M_L^\sharp f(x) = \sup_{x\in B}\frac{1}{|B|}\int_B \left|f(y) - e^{-t_B L} f(y)\right| dy. \tag{23}$$

Next, we make some conventions on notation. Given any positive integer m, for all $0 \le j \le m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma_1, \sigma_2, \dots, \sigma_j\}$ of $\{1, 2, \dots, m\}$ of different elements, and, for any $\sigma \in C_j^m$, let $\sigma' = \{1, 2, \dots, m\} \setminus \sigma$. Let $\vec{b} = (b_1, b_2, \dots, b_m)$; then, for any $\sigma = \{\sigma_1, \sigma_2, \dots, \sigma_j\} \in C_j^m$, we denote $\vec{b}_\sigma = (b_{\sigma_1}, b_{\sigma_2}, \dots, b_{\sigma_j})$, $b_\sigma(x) = \prod_{\sigma_j\in\sigma} b_{\sigma_j}(x)$ and $\|\vec{b}_\sigma\|_{\mathrm{BMO}(w)} = \prod_{\sigma_j\in\sigma}\|b_{\sigma_j}\|_{\mathrm{BMO}(w)}$, and $\|\vec{b}\|_{\mathrm{BMO}(w)} = \prod_{\sigma_j\in\{1,2,\dots,m\}}\|b_{\sigma_j}\|_{\mathrm{BMO}(w)} = \prod_{j=1}^m \|b_j\|_{\mathrm{BMO}(w)}$.

In this paper, our main results are stated as follows.

Theorem 10. *Assume the condition (1) holds. Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $0 \le \kappa < p/q$, $w^{q/p} \in A_1$, and $r_w > (1-\kappa)/(p/q-\kappa)$, where r_w denotes the critical index of w for the reverse Hölder condition. If $b_j \in BMO(w)$, $j = 1, 2, \dots, m$, then*

$$\left\|L_{\vec{b}}^{-\alpha/2} f\right\|_{L^{q,\kappa q/p}(w^{q/p},w)} \le C\left\|\vec{b}\right\|_{BMO(w)}\|f\|_{L^{p,\kappa}(w)}. \tag{24}$$

Theorem 11. *Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $w^{q/p} \in A_1$, and $r_w > q/p$, where r_w denotes the critical index of w for the reverse Hölder condition. If $b_j \in BMO(w)$, $j = 1, 2, \dots, m$, then*

$$\left\|L_{\vec{b}}^{-\alpha/2} f\right\|_{L^q(w^{q/p})} \le C\left\|\vec{b}\right\|_{BMO(w)}\|f\|_{L^p(w)}. \tag{25}$$

Moreover, if $L = -\Delta$ is the Laplacian, then

$$\left\|I_\alpha^{\vec{b}} f\right\|_{L^q(w^{q/p})} \le C\left\|\vec{b}\right\|_{BMO(w)}\|f\|_{L^p(w)}. \tag{26}$$

Remark 12. We note that our results extend some results in [1, 3]. To be specific, if we take $m = 1$, $w = 1$, and $\kappa = 0$ in Theorem 10, it is easy to see that our conclusion is the main result of Duong and Yan [1]. If we only take $w = 1$ and $\kappa = 0$ in Theorem 10, our result contains the corresponding conclusion of [3].

The remaining part of this paper will be organized as follows. In Section 2, we will give some known results and prove some requisite lemmas. Section 3 is devoted to proving the theorems of this paper.

2. Requisite Lemmas

In this section, we will prove some lemmas and state some known results about weights and weighted Morrey space.

Lemma 13 (see [11]). *Let $s > 1$, $1 \le p < \infty$, and $A_p^s = \{w : w^s \in A_p\}$. Then*

$$A_p^s = A_{1+(p-1)/s} \cap RH_s. \tag{27}$$

In particular, $A_1^s = A_1 \cap RH_s$.

Lemma 14 (see [10]). *Assume that the semigroup e^{-tL} has a kernel $p_t(x,y)$ which satisfies the upper bound (1). Take $\lambda > 0$, $f \in L_0^1(\mathbb{R}^n)$ (the set of functions in $L^1(\mathbb{R}^n)$ with bounded support) and a ball B_0 such that there exists $x_0 \in B_0$ with $Mf(x_0) \le \lambda$. Then, for every $w \in A_\infty$, $0 < \eta < 1$, we can find $\gamma > 0$ (independent of λ, B_0, f, x_0) and constants $C, r > 0$ (which only depend on w) such that*

$$w\left(\left\{x \in B_0 : Mf(x) > A\lambda, M_L^\sharp f(x) \le \gamma\lambda\right\}\right) \le C\eta^r w(B_0), \tag{28}$$

where $A > 1$ is a fixed constant which depends only on n.

As a result, using the above good-λ inequality together with the standard arguments, we have the following estimates:

For every $f \in L^{p,\kappa}(u,v)$, $1 < p < \infty$, $0 \le \kappa < 1$; if $u, v \in A_\infty$, then

$$\|f\|_{L^{p,\kappa}(u,v)} \le \|Mf\|_{L^{p,\kappa}(u,v)} \le \left\|M_L^\sharp f\right\|_{L^{p,\kappa}(u,v)}. \tag{29}$$

In particular, when $u = v = w$, $w \in A_\infty$, we have

$$\|f\|_{L^{p,\kappa}(w)} \le \|Mf\|_{L^{p,\kappa}(w)} \le \left\|M_L^\sharp f\right\|_{L^{p,\kappa}(w)}. \tag{30}$$

Lemma 15 (Kolmogorov's inequality; see [9, page 455]). *Let $0 < r < l < \infty$, for $f \ge 0$; define $\|f\|_{L^{l,\infty}} = \sup_{t>0} t|\{x \in \mathbb{R}^n : |f(x)| > t\}|^{1/l}$, $N_{l,r}(f) = \sup_E(\|f\chi_E\|_r / \|\chi_E\|_h)$, and $1/h = 1/r - 1/l$; then*

$$\|f\|_{L^{l,\infty}} \le N_{l,r}(f) \le \left(\frac{l}{l-r}\right)^{1/r} \|f\|_{L^{l,\infty}}. \tag{31}$$

Lemma 16 (see [4]). *Let $w \in A_\infty$. Then the norm of $BMO(w)$: $\|\cdot\|_{*,w}$, is equivalent to the norm of $BMO(\mathbb{R}^n)$: $\|\cdot\|_*$; that is, if b is a locally integrable function, then*

$$\|b\|_{*,w} = \sup_Q \frac{1}{w(Q)} \int_Q \left|b(x) - b_{Q,w}\right| w(x)\,dx \le C \tag{32}$$

is equivalent to

$$\|b\|_* = \sup_Q \frac{1}{|Q|} \int_Q \left|b(x) - b_Q\right| dx \le C. \tag{33}$$

Lemma 17 (see [5]). *Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, and $w^{q/p} \in A_1$; if $0 < \kappa < p/q$, $r_w > (1-\kappa)/(p/q-\kappa)$, then*

$$\left\|M_{\alpha,1} f\right\|_{L^{q,\kappa q/p}(w^{q/p},w)} \le C\|f\|_{L^{p,\kappa}(w)}. \tag{34}$$

It also holds for I_α.

Lemma 18 (see [5]). *Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, and $w^{q/p} \in A_1$; if $0 < \kappa < p/q$, $1 < r < p$, $r_w > (1-\kappa)/(p/q-\kappa)$, then*

$$\left\|M_{r,w} f\right\|_{L^{q,\kappa q/p}(w^{q/p},w)} \le C\|f\|_{L^{q,\kappa q/p}(w^{q/p},w)}. \tag{35}$$

Lemma 19 (see [5]). *Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $0 < \kappa < p/q$, and $w \in A_\infty$. Then, for $1 < r < p$,*

$$\left\|M_{\alpha,r,w} f\right\|_{L^{q,\kappa q/p}(w)} \le C\|f\|_{L^{p,\kappa}(w)}. \tag{36}$$

Remark 20. It is easy to see that Lemmas 17, 18, and 19 still hold for $\kappa = 0$.

Lemma 21. *Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, and $w^{q/p} \in A_1$; if $0 \le \kappa < p/q$, $r_w > (1-\kappa)/(p/q-\kappa)$, then*

$$\left\|L^{-\alpha/2} f\right\|_{L^{q,\kappa q/p}(w^{q/p},w)} \le C\|f\|_{L^{p,\kappa}(w)}. \tag{37}$$

Proof. Since semigroup e^{-tL} has a kernel $p_t(x,y)$ that satisfies the upper bound (1), it is easy to see that, for $x \in \mathbb{R}^n$, $L^{-\alpha/2} f(x) \le I_\alpha(|f|)(x)$. From the boundedness of I_α on weighted Morrey space (see Lemma 17), we get

$$\left\|L^{-\alpha/2} f\right\|_{L^{q,\kappa q/p}(w^{q/p},w)} \le \left\|I_\alpha f\right\|_{L^{q,\kappa q/p}(w^{q/p},w)} \le C\|f\|_{L^{p,\kappa}(w)}. \tag{38}$$

□

Remark 22. Since I_α is of weak $(1, n/(n-\alpha))$ type, from the above proof, we can obtain that $L^{-\alpha/2}$ is of weak $(1, n/(n-\alpha))$ type.

Lemma 23 (see [1]). *Assume the semigroup e^{-tL} has a kernel $p_t(x,y)$ which satisfies the upper bound (1). Then, for $0 < \alpha < n$, the differential operator $L^{-\alpha/2} - e^{-tL}L^{-\alpha/2}$ has an associated kernel $\widetilde{K}_{\alpha,t}(x,y)$ which satisfies*

$$\widetilde{K}_{\alpha,t}(x,y) \le \frac{C}{|x-y|^{n-\alpha}} \frac{t}{|x-y|^2}. \tag{39}$$

Lemma 24. *Assume the semigroup e^{-tL} has a kernel $p_t(x,y)$ which satisfies the upper bound (1), $b \in BMO(w)$, and $w \in A_1$. Then, for $f \in L^p(\mathbb{R}^n)$, $p > 1$, $\sigma \in C_j^m$ $(j = 1, 2, \ldots, m)$, $1 < \tau < \infty$, and*

$$\begin{aligned}&\sup_{x\in B} \frac{1}{|B|} \int_B \left|e^{-t_B L}\left((b-b_B)_\sigma f\right)(y)\right| dy \\ &\quad \le C\left\|\vec{b}_\sigma\right\|_{BMO(w)} M_{\tau,w} f(x),\end{aligned} \tag{40}$$

where $t_B = r_B^2$ and r_B is the radius of B.

Proof. For any $f \in L^p(\mathbb{R}^n)$, $x \in B$, we have

$$\begin{aligned}
&\frac{1}{|B|}\int_B \left|e^{-t_B L}\left((b-b_B)_\sigma f\right)(y)\right| dy \\
&\quad\leq \frac{1}{|B|}\int_B\int_{\mathbb{R}^n}\left|p_{t_B}(y,z)\right|\left|(b(z)-b_B)_\sigma f(z)\right| dz\, dy \\
&\quad\leq \frac{1}{|B|}\int_B\int_{2B}\left|p_{t_B}(y,z)\right|\left|(b(z)-b_B)_\sigma f(z)\right| dz\, dy \\
&\qquad+\frac{1}{|B|}\int_B\sum_{k=1}^{\infty}\int_{2^{k+1}B\setminus 2^k B}\left|p_{t_B}(y,z)\right| \\
&\qquad\qquad\times\left|(b(z)-b_B)_\sigma f(z)\right| dz\, dy \\
&\quad:= I + II.
\end{aligned} \tag{41}$$

Noticing that $y \in B$, $z \in 2B$, from (1), we get

$$\left|p_{t_B}(y,z)\right| \leq C t_B^{-n/2} \leq C|2B|^{-1}. \tag{42}$$

Thus,

$$\begin{aligned}
I &\leq \frac{C}{|2B|}\int_{2B}\left|(b(z)-b_B)_\sigma\right|\left|f(z)\right| dz \\
&= \frac{C}{|2B|}\int_{2B}\prod_{\sigma_j\in\sigma}\left|b_{\sigma_j}(z)-\left(b_{\sigma_j}\right)_B\right|\left|f(z)\right| dz.
\end{aligned} \tag{43}$$

For simplicity, we only consider the case of $j = 2$. We also want to point out that although we state our results on the case of $j = 2$, all results are valid on the multilinear case ($j > 2$) without any essential difference and difficulty in the proof. So it follows that

$$\begin{aligned}
I &\leq \frac{C}{|2B|}\int_{2B}\left(\left|b_{\sigma_1}(z)-\left(b_{\sigma_1}\right)_{2B}\right|+\left|\left(b_{\sigma_1}\right)_{2B}-\left(b_{\sigma_1}\right)_B\right|\right) \\
&\quad\times\left(\left|b_{\sigma_2}(z)-\left(b_{\sigma_2}\right)_{2B}\right|+\left|\left(b_{\sigma_2}\right)_{2B}-\left(b_{\sigma_2}\right)_B\right|\right)\left|f(z)\right| dz \\
&\leq \frac{C}{|2B|}\int_{2B}\left|b_{\sigma_1}(z)-\left(b_{\sigma_1}\right)_{2B}\right|\left|b_{\sigma_2}(z)-\left(b_{\sigma_2}\right)_{2B}\right|\left|f(z)\right| dz \\
&\quad+\frac{C}{|2B|}\int_{2B}\left|b_{\sigma_1}(z)-\left(b_{\sigma_1}\right)_{2B}\right|\left|\left(b_{\sigma_2}\right)_{2B}-\left(b_{\sigma_2}\right)_B\right|\left|f(z)\right| dz \\
&\quad+\frac{C}{|2B|}\int_{2B}\left|\left(b_{\sigma_1}\right)_{2B}-\left(b_{\sigma_1}\right)_B\right|\left|b_{\sigma_2}(z)-\left(b_{\sigma_2}\right)_{2B}\right|\left|f(z)\right| dz \\
&\quad+\frac{C}{|2B|}\int_{2B}\left|\left(b_{\sigma_1}\right)_{2B}-\left(b_{\sigma_1}\right)_B\right|\left|\left(b_{\sigma_2}\right)_{2B}-\left(b_{\sigma_2}\right)_B\right|\left|f(z)\right| dz \\
&:= I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{44}$$

We split I_1 as follows:

$$\begin{aligned}
I_1 &\leq \frac{C}{|2B|}\int_{2B}\left\{\left|b_{\sigma_1}(z)-\left(b_{\sigma_1}\right)_{2B,w}\right|+\left|\left(b_{\sigma_1}\right)_{2B,w}-\left(b_{\sigma_1}\right)_{2B}\right|\right\} \\
&\quad\times\left\{\left|b_{\sigma_2}(z)-\left(b_{\sigma_2}\right)_{2B,w}\right|\right. \\
&\qquad\left.+\left|\left(b_{\sigma_2}\right)_{2B,w}-\left(b_{\sigma_2}\right)_{2B}\right|\right\}\left|f(z)\right| dz \\
&= \frac{C}{|2B|}\int_{2B}\left|b_{\sigma_1}(z)-\left(b_{\sigma_1}\right)_{2B,w}\right| \\
&\qquad\times\left|b_{\sigma_2}(z)-\left(b_{\sigma_2}\right)_{2B,w}\right|\left|f(z)\right| dz \\
&\quad+\frac{C}{|2B|}\int_{2B}\left|b_{\sigma_1}(z)-\left(b_{\sigma_1}\right)_{2B,w}\right| \\
&\qquad\times\left|\left(b_{\sigma_2}\right)_{2B,w}-\left(b_{\sigma_2}\right)_{2B}\right|\left|f(z)\right| dz \\
&\quad+\frac{C}{|2B|}\int_{2B}\left|\left(b_{\sigma_1}\right)_{2B,w}-\left(b_{\sigma_1}\right)_{2B}\right| \\
&\qquad\times\left|b_{\sigma_2}(z)-\left(b_{\sigma_2}\right)_{2B,w}\right|\left|f(z)\right| dz \\
&\quad+\frac{C}{|2B|}\int_{2B}\left|\left(b_{\sigma_1}\right)_{2B,w}-\left(b_{\sigma_1}\right)_{2B}\right| \\
&\qquad\times\left|\left(b_{\sigma_2}\right)_{2B,w}-\left(b_{\sigma_2}\right)_{2B}\right|\left|f(z)\right| dz \\
&:= I_{11} + I_{12} + I_{13} + I_{14}.
\end{aligned} \tag{45}$$

We now consider the four terms, respectively. Choose $\tau_1, \tau_2, \tau, s > 1$ satisfing $1/\tau_1 + 1/\tau_2 + 1/\tau + 1/s = 1$. According to Hölder's inequality and $w \in A_1$, we have

$$\begin{aligned}
I_{11} &= \frac{C}{|2B|}\int_{2B}\left|b_{\sigma_1}(z)-\left(b_{\sigma_1}\right)_{2B,w}\right| w(z)^{1/\tau_1} \\
&\qquad\times\left|b_{\sigma_2}(z)-\left(b_{\sigma_2}\right)_{2B,w}\right| w(z)^{1/\tau_2}\left|f(z)\right| \\
&\qquad\times w(z)^{1/\tau} w(z)^{-1+1/s} dz \\
&\leq \frac{C}{|2B|}\left(\int_{2B}\left|b_{\sigma_1}(z)-\left(b_{\sigma_1}\right)_{2B,w}\right|^{\tau_1} w(z)\, dz\right)^{1/\tau_1} \\
&\quad\times\left(\int_{2B}\left|b_{\sigma_2}(z)-\left(b_{\sigma_2}\right)_{2B,w}\right|^{\tau_2} w(z)\, dz\right)^{1/\tau_2} \\
&\quad\times\left(\int_{2B}\left|f(z)\right|^{\tau} w(z)\, dz\right)^{1/\tau}\left(\int_{2B} w(z)^{-s+1} dz\right)^{1/s} \\
&\leq \frac{C}{|2B|}\left\|\vec{b}_\sigma\right\|_{*,w} M_{\tau,w} f(x)\, w(2B)^{1-1/s} w(x)^{-1+1/s}|2B|^{1/s} \\
&= C\left\|\vec{b}_\sigma\right\|_{*} M_{\tau,w} f(x)\left(\frac{w(2B)}{|2B|}\right)^{1-1/s} w(x)^{-1+1/s} \\
&\leq C\left\|\vec{b}_\sigma\right\|_{*} M_{\tau,w} f(x).
\end{aligned} \tag{46}$$

In the above inequalities, we use the fact that if $w \in A_1$, then $w \in A_\infty$. Thus, the norm of $\mathrm{BMO}(w)$ is equivalent to the norm of $\mathrm{BMO}(\mathbb{R}^n)$ (see Lemma 16).

For I_{12}, we first estimate the term that contains $(b_{\sigma_2})_{2B}$. In fact, it follows from the John-Nirenberg lemma that there exist $C_1 > 0$ and $C_2 > 0$ such that, for any ball B and $\alpha > 0$,

$$\left|\left\{z \in 2B : \left|b_{\sigma_2}(z)-\left(b_{\sigma_2}\right)_{2B}\right| > \alpha\right\}\right| \leq C_1|2B|\, e^{-C_2\alpha/\|b_{\sigma_2}\|_*}, \tag{47}$$

since $b \in \mathrm{BMO}(\mathbb{R}^n)$. Using the definition of A_∞, we get

$$w\left(\left\{z \in 2B : \left|b_{\sigma_2}(z) - \left(b_{\sigma_2}\right)_{2B}\right| > \alpha\right\}\right) \leq Cw(2B)\, e^{-C_2\alpha\delta/\|b_{\sigma_2}\|_*}, \tag{48}$$

for some $\delta > 0$. We can see that (48) yields

$$\begin{aligned}\int_{2B} &\left|b_{\sigma_2}(z) - \left(b_{\sigma_2}\right)_{2B}\right| w(z)\, dz \\ &= \int_0^\infty w\left(\left\{z \in 2B : \left|b_{\sigma_2}(z) - \left(b_{\sigma_2}\right)_{2B}\right| > \alpha\right\}\right) d\alpha \\ &\leq Cw(2B) \int_0^\infty e^{-C_2\alpha\delta/\|b_{\sigma_2}\|_*}\, d\alpha \\ &= Cw(2B)\left\|b_{\sigma_2}\right\|_*.\end{aligned} \tag{49}$$

Thus,

$$\begin{aligned}&\left|\left(b_{\sigma_2}\right)_{2B,w} - \left(b_{\sigma_2}\right)_{2B}\right| \\ &\quad\leq \frac{1}{w(2B)} \int_{2B} \left|b_{\sigma_2}(z) - \left(b_{\sigma_2}\right)_{2B}\right| w(z)\, dz \\ &\quad\leq \frac{C}{w(2B)} w(2B)\left\|b_{\sigma_2}\right\|_* = C\left\|b_{\sigma_2}\right\|_*.\end{aligned} \tag{50}$$

On the basis of (50), we now estimate I_{12}. For the above τ, select u, v such that $1/u + 1/v + 1/\tau = 1$; by virtue of Hölder's inequality and $w \in A_1$, we have

$$\begin{aligned}I_{12} &\leq C\left\|b_{\sigma_2}\right\|_* \frac{1}{|2B|} \int_{2B} \left|b_{\sigma_1}(z) - \left(b_{\sigma_1}\right)_{2B,w}\right| |f(z)|\, dz \\ &= C\left\|b_{\sigma_2}\right\|_* \frac{1}{|2B|} \int_{2B} \left|b_{\sigma_1}(z) - \left(b_{\sigma_1}\right)_{2B,w}\right| \\ &\qquad\qquad \times w(z)^{1/u} |f(z)|\, w(z)^{1/\tau} w(z)^{-1+1/v}\, dz \\ &\leq C\left\|b_{\sigma_2}\right\|_* \frac{1}{|2B|} \left(\int_{2B} \left|b_{\sigma_1}(z) - \left(b_{\sigma_1}\right)_{2B,w}\right|^u w(z)\, dz\right)^{1/u} \\ &\quad \times \left(\int_{2B} |f(z)|^\tau w(z)\, dz\right)^{1/\tau} \left(\int_{2B} w(z)^{-v+1}\, dz\right)^{1/v} \\ &\leq C\left\|\vec{b}_\sigma\right\|_* M_{\tau,w} f(x) \frac{1}{|2B|} w(2B)^{1/u+1/\tau} w(x)^{-1+1/v} |2B|^{1/v} \\ &= C\left\|\vec{b}_\sigma\right\|_* M_{\tau,w} f(x) \left(\frac{w(2B)}{|2B|}\right)^{1-1/v} w(x)^{-1+1/v} \\ &= C\left\|\vec{b}_\sigma\right\|_* M_{\tau,w} f(x).\end{aligned} \tag{51}$$

Analogous to the estimate of I_{12}, we also have $I_{13} \leq C\|\vec{b}_\sigma\|_* M_{\tau,w} f(x)$.

As for I_{14}, taking advantage of Hölder's inequality and (50), we get

$$\begin{aligned}I_{14} &\leq C\left\|\vec{b}_\sigma\right\|_* \frac{1}{|2B|} \int_{2B} |f(z)|\, dz \\ &= C\left\|\vec{b}_\sigma\right\|_* \frac{1}{|2B|} \int_{2B} |f(z)|\, w(z)^{1/\tau} w(z)^{1/\tau'-1}\, dz \\ &\leq C\left\|\vec{b}_\sigma\right\|_* \frac{1}{|2B|} \left(\int_{2B} |f(z)|^\tau w(z)\, dz\right)^{1/\tau} \\ &\quad \times \left(\int_{2B} w(z)^{1-\tau'}\, dz\right)^{1/\tau'} \\ &= C\left\|\vec{b}_\sigma\right\|_* \frac{1}{|2B|} M_{\tau,w} f(x)\, w(2B)^{1/\tau} w(x)^{1/\tau'-1} |2B|^{1/\tau'} \\ &= C\left\|\vec{b}_\sigma\right\|_* M_{\tau,w} f(x) \left(\frac{w(2B)}{|2B|}\right)^{1/\tau} w(x)^{1/\tau'-1} \\ &\leq C\left\|\vec{b}_\sigma\right\|_* M_{\tau,w} f(x).\end{aligned} \tag{52}$$

Collecting the estimates of I_{11}, I_{12}, I_{13}, and I_{14}, it is easy to see that

$$I_1 \leq C\left\|\vec{b}_\sigma\right\|_* M_{\tau,w} f(x). \tag{53}$$

For term I_2, using the fact that $|b_{2B} - b_B| \leq C\|b\|_*$ (see [12]), we now get

$$\begin{aligned}I_2 &\leq \left\|b_{\sigma_2}\right\|_* \frac{C}{|2B|} \int_{2B} \left(\left|b_{\sigma_1}(z) - \left(b_{\sigma_1}\right)_{2B,w}\right|\right. \\ &\qquad\qquad \left. + \left|\left(b_{\sigma_1}\right)_{2B,w} - \left(b_{\sigma_1}\right)_{2B}\right|\right) \\ &\qquad\qquad \times |f(z)|\, dz \\ &= \frac{C\left\|b_{\sigma_2}\right\|_*}{|2B|} \int_{2B} \left|b_{\sigma_1}(z) - \left(b_{\sigma_1}\right)_{2B,w}\right| |f(z)|\, dz \\ &\quad + \frac{C\left\|b_{\sigma_2}\right\|_*}{|2B|} \int_{2B} \left|\left(b_{\sigma_1}\right)_{2B,w} - \left(b_{\sigma_1}\right)_{2B}\right| |f(z)|\, dz \\ &:= I_{21} + I_{22}.\end{aligned} \tag{54}$$

By some estimates similar to those used in the estimate for I_{12}, we conclude that

$$I_{21} \leq C\left\|b_\sigma\right\|_* M_{\tau,w} f(x). \tag{55}$$

For I_{22}, using the same method as in dealing with I_{14}, we get

$$I_{22} \leq C\left\|b_\sigma\right\|_* M_{\tau,w} f(x). \tag{56}$$

Therefore,

$$I_2 \leq C\left\|b_\sigma\right\|_* M_{\tau,w} f(x). \tag{57}$$

Analogously,

$$I_3 \leq C\left\|b_\sigma\right\|_* M_{\tau,w} f(x). \tag{58}$$

An argument similar to that used in the estimate for I_{14} leads to

$$I_4 \leq C \left\|b_\sigma\right\|_* M_{\tau,w} f(x). \tag{59}$$

Hence,

$$\begin{aligned} I &\leq I_1 + I_2 + I_3 + I_4 \\ &\leq C \left\|b_\sigma\right\|_* M_{\tau,w} f(x). \end{aligned} \tag{60}$$

Next, we will consider the second term II in (41). For any $y \in B$, $z \in 2^{k+1}B \setminus 2^k B$, it is easy to get that $|y-z| \geq 2^{k-1} r_B$ and

$$\left|p_{t_B}(y,z)\right| \leq C \frac{e^{-C2^{2(k-1)}} 2^{(k+1)n}}{\left|2^{k+1}B\right|}. \tag{61}$$

Thus,

$$\begin{aligned} II &\leq C \sum_{k=1}^{\infty} \frac{e^{-C2^{2(k-1)}} 2^{(k+1)n}}{\left|2^{k+1}B\right|} \int_{2^{k+1}B} \left|\left(b(z) - b_B\right)_\sigma\right| \left|f(z)\right| dz \\ &= C \sum_{k=1}^{\infty} \frac{e^{-C2^{2(k-1)}} 2^{(k+1)n}}{\left|2^{k+1}B\right|} \\ &\quad \times \int_{2^{k+1}B} \left|b_{\sigma_1}(z) - \left(b_{\sigma_1}\right)_B\right| \cdots \left|b_{\sigma_j}(z) - \left(b_{\sigma_j}\right)_B\right| \left|f(z)\right| dz. \end{aligned} \tag{62}$$

For simplicity, we also consider the case of $j = 2$.

$$\begin{aligned} II &\leq C \sum_{k=1}^{\infty} \frac{e^{-C2^{2(k-1)}} 2^{(k+1)n}}{\left|2^{k+1}B\right|} \\ &\quad \times \int_{2^{k+1}B} \left(\left|b_{\sigma_1}(z) - \left(b_{\sigma_1}\right)_{2^{k+1}B}\right| + \left|\left(b_{\sigma_1}\right)_{2^{k+1}B} - \left(b_{\sigma_1}\right)_B\right|\right) \\ &\quad \times \left(\left|b_{\sigma_2}(z) - \left(b_{\sigma_2}\right)_{2^{k+1}B}\right| + \left|\left(b_{\sigma_2}\right)_{2^{k+1}B} - \left(b_{\sigma_2}\right)_B\right|\right) \left|f(z)\right| dz \\ &= C \sum_{k=1}^{\infty} \frac{e^{-C2^{2(k-1)}} 2^{(k+1)n}}{\left|2^{k+1}B\right|} \\ &\quad \times \int_{2^{k+1}B} \left|b_{\sigma_1}(z) - \left(b_{\sigma_1}\right)_{2^{k+1}B}\right| \\ &\quad \times \left|b_{\sigma_2}(z) - \left(b_{\sigma_2}\right)_{2^{k+1}B}\right| \left|f(z)\right| dz \\ &\quad + C \sum_{k=1}^{\infty} \frac{e^{-C2^{2(k-1)}} 2^{(k+1)n}}{\left|2^{k+1}B\right|} \\ &\quad \times \int_{2^{k+1}B} \left|\left(b_{\sigma_1}\right)_{2^{k+1}B} - \left(b_{\sigma_1}\right)_B\right| \\ &\quad \times \left|b_{\sigma_2}(z) - \left(b_{\sigma_2}\right)_{2^{k+1}B}\right| \left|f(z)\right| dz \\ &\quad + C \sum_{k=1}^{\infty} \frac{e^{-C2^{2(k-1)}} 2^{(k+1)n}}{\left|2^{k+1}B\right|} \\ &\quad \times \int_{2^{k+1}B} \left|b_{\sigma_1}(z) - \left(b_{\sigma_1}\right)_{2^{k+1}B}\right| \\ &\quad \times \left|\left(b_{\sigma_2}\right)_{2^{k+1}B} - \left(b_{\sigma_2}\right)_B\right| \left|f(z)\right| dz \\ &\quad + C \sum_{k=1}^{\infty} \frac{e^{-C2^{2(k-1)}} 2^{(k+1)n}}{\left|2^{k+1}B\right|} \\ &\quad \times \int_{2^{k+1}B} \left|\left(b_{\sigma_1}\right)_{2^{k+1}B} - \left(b_{\sigma_1}\right)_B\right| \\ &\quad \times \left|\left(b_{\sigma_2}\right)_{2^{k+1}B} - \left(b_{\sigma_2}\right)_B\right| \left|f(z)\right| dz \\ &:= II_1 + II_2 + II_3 + II_4. \end{aligned} \tag{63}$$

For II_1, similar to the estimate of I_1, we have

$$\begin{aligned} II_1 &\leq C \left\|\vec{b}_\sigma\right\|_* M_{\tau,w} f(x) \sum_{k=1}^{\infty} e^{-C2^{2(k-1)}} 2^{(k+1)n} \\ &\leq C \left\|\vec{b}_\sigma\right\|_* M_{\tau,w} f(x). \end{aligned} \tag{64}$$

Noticing that we could use similar methods of the estimates of I_2, I_3, and I_4 in the estimates of II_2, II_3, and II_4, respectively. From this together with the fact that $|b_{2^{j+1}B} - b_B| \leq 2^n(j+1)\|b\|_*$ (see [12]), it is easy to get

$$II_2 + II_3 + II_4 \leq C \left\|\vec{b}_\sigma\right\|_* M_{\tau,w} f(x). \tag{65}$$

Therefore,

$$II \leq C \|\vec{b}_\sigma\|_* M_{\tau,w} f(x). \tag{66}$$

According to the estimates of I and II, the lemma has been proved. □

Now, we will establish a lemma which plays an important role in the proof of Theorem 10.

Lemma 25. *Let $0 < \alpha < n$, $w \in A_1$, and $b \in BMO(w)$; then, for all $r > 1$, $\tau > 1$, and $x \in \mathbb{R}^n$, we have*

$$\begin{aligned} & M_L^\sharp \left(L_{\vec{b}}^{-\alpha/2} f\right)(x) \\ & \leq C \left\{ \left\|\vec{b}\right\|_* M_{r,w} \left(L^{-\alpha/2} f\right)(x) \right. \\ & \quad + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} C_{j,m} \left\|\vec{b}_\sigma\right\|_* M_{\tau,w} \left(L_{\vec{b}_{\sigma'}}^{-\alpha/2} f\right)(x) \\ & \quad + \left\|\vec{b}\right\|_* w(x)^{-\alpha/n} M_{\alpha,r,w} f(x) \\ & \quad \left. + \left\|\vec{b}\right\|_* M_{\alpha,1} f(x) \right\}. \end{aligned} \tag{67}$$

Proof. For any given $x \in \mathbb{R}^n$, take a ball $B = B(x_0, r_B)$ which contains x. For $f \in L^p(\mathbb{R}^n)$, let $f_1 = f\chi_{2B}$, $f_2 = f - f_1$.

Denote the kernel of $L^{-\alpha/2}$ by $K_\alpha(x,y)$, $\vec{\lambda}=(\lambda_1,\lambda_2,\dots,\lambda_m)$, where $\lambda_j\in\mathbb{R}^n$, $j=1,2,\dots,m$. Then $L_{\vec{b}}^{-\alpha/2}f$ can be written in the following form:

$$\begin{aligned} L_{\vec{b}}^{-\alpha/2}f(y) &= \int_{\mathbb{R}^n}\prod_{j=1}^{m}\left(b_j(y)-b_j(z)\right)K_\alpha(y,z)f(z)\,dz \\ &= \int_{\mathbb{R}^n}\prod_{j=1}^{m}\left(\left(b_j(y)-\lambda_j\right)-\left(b_j(z)-\lambda_j\right)\right)K_\alpha(y,z)f(z)\,dz \\ &= \sum_{i=0}^{m}\sum_{\sigma\in C_i^m}(-1)^{m-i}(b(y)-\lambda)_\sigma \\ &\qquad\times\int_{\mathbb{R}^n}(b(z)-\lambda)_{\sigma'}K_\alpha(y,z)f(z)\,dz. \end{aligned} \tag{68}$$

Now expanding $(b(z)-\lambda)_{\sigma'}$ as

$$(b(z)-\lambda)_{\sigma'}=\left((b(z)-b(y))+(b(y)-\lambda)\right)_{\sigma'}, \tag{69}$$

and combining (68) with (69), it is easy to see that

$$\begin{aligned} e^{-t_BL}\left(L_{\vec{b}}^{-\alpha/2}f\right)(y) &= e^{-t_BL}\left(\prod_{j=1}^{m}\left(b_j-\lambda_j\right)L^{-\alpha/2}f\right)(y) \\ &\quad+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}C_{j,m}e^{-t_BL}\left((b-\lambda)_\sigma L_{\vec{b}_{\sigma'}}^{-\alpha/2}f\right)(y) \\ &\quad+(-1)^m e^{-t_BL}\left(L^{-\alpha/2}\left(\prod_{j=1}^{m}\left(b_j-\lambda_j\right)f_1\right)\right)(y) \\ &\quad+(-1)^m e^{-t_BL}\left(L^{-\alpha/2}\left(\prod_{j=1}^{m}\left(b_j-\lambda_j\right)f_2\right)\right)(y), \end{aligned} \tag{70}$$

where $t_B=r_B^2$ and r_B is the radius of ball B.

Take $\lambda_j=(b_j)_B$, $j=1,2,\dots,m$, and denote $\vec{b}_B=((b_1)_B,(b_2)_B,\dots,(b_m)_B)$; then

$$\begin{aligned} &\frac{1}{|B|}\int_B\left|L_{\vec{b}}^{-\alpha/2}f(y)-e^{-t_BL}\left(L_{\vec{b}}^{-\alpha/2}f\right)(y)\right|dy \\ &\le\frac{1}{|B|}\int_B\left|\prod_{j=1}^{m}\left(b_j(y)-\left(b_j\right)_B\right)L^{-\alpha/2}f(y)\right|dy \\ &\quad+\frac{1}{|B|}\int_B\left|\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}C_{j,m}\left(b(y)-b_B\right)_\sigma L_{\vec{b}_{\sigma'}}^{-\alpha/2}f(y)\right|dy \\ &\quad+\frac{1}{|B|}\int_B\left|L^{-\alpha/2}\left(\prod_{j=1}^{m}\left(b_j-\left(b_j\right)_B\right)f_1\right)(y)\right|dy \\ &\quad+\frac{1}{|B|}\int_B\left|e^{-t_BL}\left(\prod_{j=1}^{m}\left(b_j-\left(b_j\right)_B\right)L^{-\alpha/2}f\right)(y)\right|dy \\ &\quad+\frac{1}{|B|}\int_B\left|\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}C_{j,m}e^{-t_BL}\left(\left(b-b_B\right)_\sigma L_{\vec{b}_{\sigma'}}^{-\alpha/2}f\right)(y)\right|dy \\ &\quad+\frac{1}{|B|}\int_B\left|e^{-t_BL}\left(L^{-\alpha/2}\left(\prod_{j=1}^{m}\left(b_j-\left(b_j\right)_B\right)f_1\right)\right)(y)\right|dy \\ &\quad+\frac{1}{|B|}\int_B\left|L^{-\alpha/2}\left(\prod_{j=1}^{m}\left(b_j-\left(b_j\right)_B\right)f_2\right)(y)\right. \\ &\qquad\left.-e^{-t_BL}\left(L^{-\alpha/2}\left(\prod_{j=1}^{m}\left(b_j-\left(b_j\right)_B\right)f_2\right)\right)(y)\right|dy \\ &:=\sum_{i=1}^{7}G_i. \end{aligned} \tag{71}$$

We now estimate the above seven terms, respectively. We take $m=2$ as an example; the estimate for the case $m>2$ is the same. For the first term G_1, we split it as follows:

$$\begin{aligned} G_1 &= \frac{1}{|B|}\int_B\left|b_1(y)-(b_1)_{B,w}\right|\left|b_2(y)-(b_2)_{B,w}\right|\left|L^{-\alpha/2}f(y)\right|dy \\ &\quad+\frac{1}{|B|}\int_B\left|b_1(y)-(b_1)_{B,w}\right|\left|(b_2)_{B,w}-(b_2)_B\right|\left|L^{-\alpha/2}f(y)\right|dy \\ &\quad+\frac{1}{|B|}\int_B\left|(b_1)_{B,w}-(b_1)_B\right|\left|b_2(y)-(b_2)_{B,w}\right|\left|L^{-\alpha/2}f(y)\right|dy \\ &\quad+\frac{1}{|B|}\int_B\left|(b_1)_{B,w}-(b_1)_B\right|\left|(b_2)_{B,w}-(b_2)_B\right|\left|L^{-\alpha/2}f(y)\right|dy \\ &:= G_{11}+G_{12}+G_{13}+G_{14}. \end{aligned} \tag{72}$$

Choose $r_1,r_2,r,q>1$ such that $1/r_1+1/r_2+1/r+1/q=1$. Noticing that $w\in A_1$, by Hölder's inequality and the similar estimate of I_1, we have

$$G_1\le C\left\|\vec{b}\right\|_* M_{r,w}\left(L^{-\alpha/2}f\right)(x). \tag{73}$$

For G_2, take $\tau_1,\dots,\tau_j,\tau,\nu>1$ that satisfy $1/\tau_1+\cdots+1/\tau_j+1/\tau+1/\nu=1$; then following Hölder's inequality and the same idea as that of G_1 yields

$$G_2\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}C_{j,m}\left\|\vec{b}_\sigma\right\|_* M_{\tau,w}\left(L_{\vec{b}_{\sigma'}}^{-\alpha/2}f\right)(x). \tag{74}$$

To estimate G_3, applying Lemma 15 (Kolmogorov's inequality), weak $(1, n/(n-\alpha))$ boundedness of $L^{-\alpha/2}$ (see Remark 22), and Hölder's inequality, we have

$$\begin{aligned} G_3 &= \frac{1}{|B|}\int_B \left| L^{-\alpha/2}\left(\prod_{j=1}^{m}\left(b_j - (b_j)_B\right) f_1\right)(y)\right| dy \\ &\le \frac{C}{|B|^{1-\alpha/n}} \left\| L^{-\alpha/2}\left(\prod_{j=1}^{m}\left(b_j - (b_j)_B\right) f_1\right)\right\|_{L^{n/(n-\alpha),\infty}} \\ &\le \frac{C}{|B|^{1-\alpha/n}} \int_{2B} \left|\prod_{j=1}^{m}\left(b_j(y) - (b_j)_B\right) f(y)\right| dy. \end{aligned} \tag{75}$$

We consider the case of $m = 2$, for example, and we split G_3 as follows:

$$\begin{aligned} G_3 &\le \frac{C}{|B|^{1-\alpha/n}} \int_{2B} \left(|b_1(y) - (b_1)_{2B}| + |(b_1)_{2B} - (b_1)_B|\right) \\ &\qquad \times \left(|b_2(y) - (b_2)_{2B}| + |(b_2)_{2B} - (b_2)_B|\right) |f(y)|\, dy \\ &\le \frac{C}{|B|^{1-\alpha/n}} \int_{2B} |b_1(y) - (b_1)_{2B}|\, |b_2(y) - (b_2)_{2B}|\, |f(y)|\, dy \\ &\quad + \frac{C}{|B|^{1-\alpha/n}} \int_{2B} |b_1(y) - (b_1)_{2B}|\, |(b_2)_{2B} - (b_2)_B|\, |f(y)|\, dy \\ &\quad + \frac{C}{|B|^{1-\alpha/n}} \int_{2B} |(b_1)_{2B} - (b_1)_B|\, |b_2(y) - (b_2)_{2B}|\, |f(y)|\, dy \\ &\quad + \frac{C}{|B|^{1-\alpha/n}} \int_{2B} |(b_1)_{2B} - (b_1)_B|\, |(b_2)_{2B} - (b_2)_B|\, |f(y)|\, dy \\ &:= G_{31} + G_{32} + G_{33} + G_{34}. \end{aligned} \tag{76}$$

Take $r_1, r_2, r, q > 1$ such that $1/r_1 + 1/r_2 + 1/r + 1/q = 1$; by virtue of Hölder's inequality and the same manner as that used in dealing with I_1, I_2, I_3 in Lemma 24, we get

$$G_{31} + G_{32} + G_{33} \le C \|\vec{b}\|_* M_{\alpha,r,w} f(x). \tag{77}$$

For G_{34},

$$\begin{aligned} G_{34} &\le C \|\vec{b}\|_* \frac{1}{|B|^{1-\alpha/n}} \int_{2B} |f(y)|\, dy \\ &\le C \|\vec{b}\|_* M_{\alpha,1} f(x). \end{aligned} \tag{78}$$

Therefore,

$$G_3 \le C \|\vec{b}\|_* M_{\alpha,r,w} f(x) + C \|\vec{b}\|_* M_{\alpha,1} f(x). \tag{79}$$

By Lemma 24, we have

$$\begin{aligned} G_4 &\le C \|\vec{b}\|_* M_{r,w}\left(L^{-\alpha/2} f\right)(x), \\ G_5 &\le C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} C_{j,m} \|\vec{b}_\sigma\|_* M_{\tau,w}\left(L_{\vec{b}_{\sigma'}}^{-\alpha/2} f\right)(x). \end{aligned} \tag{80}$$

Next, we consider the term G_6:

$$\begin{aligned} G_6 &= \frac{1}{|B|}\int_B \left| \int_{\mathbb{R}^n} p_{t_B}(y,z) \times \left(L^{-\alpha/2}\left(\prod_{j=1}^{m}\left(b_j - (b_j)_B\right) f_1\right)\right)(z)\, dz\right| dy \\ &\le \frac{1}{|B|}\int_B \int_{2B} |p_{t_B}(y,z)| \times \left| L^{-\alpha/2}\left(\prod_{j=1}^{m}\left(b_j - (b_j)_B\right) f_1\right)(z)\right| dz\, dy \\ &\le \frac{1}{|B|}\int_B \int_{\mathbb{R}^n\setminus 2B} |p_{t_B}(y,z)| \times \left| L^{-\alpha/2}\left(\prod_{j=1}^{m}\left(b_j - (b_j)_B\right) f_1\right)(z)\right| dz\, dy \\ &:= G_{61} + G_{62}. \end{aligned} \tag{81}$$

For G_{61}, since $y \in B$, $z \in 2B$, and $|p_{t_B}(y,z)| \le C|2B|^{-1}$, it follows that

$$G_{61} \le \frac{C}{|2B|} \int_{2B} \left| L^{-\alpha/2}\left(\prod_{j=1}^{m}\left(b_j - (b_j)_B\right) f_1\right)(z)\right| dz. \tag{82}$$

Analogous to the estimate of G_3, we have

$$G_{61} \le C \|\vec{b}\|_* M_{\alpha,r,w} f(x) + C \|\vec{b}\|_* M_{\alpha,1} f(x). \tag{83}$$

For G_{62}, note that $y \in B$, $z \in 2^{k+1}B \setminus 2^k B$, and $|p_{t_B}(y,z)| \le C(e^{-C2^{2(k-1)}} 2^{(k+1)n}/|2^{k+1}B|)$. Hence, the estimate of G_{62} runs as that of G_{61} yields that

$$G_{62} \le C \|\vec{b}\|_* M_{\alpha,r,w} f(x) + C \|\vec{b}\|_* M_{\alpha,1} f(x). \tag{84}$$

For the last term G_7, applying Lemma 23, we have

$$\begin{aligned} G_7 &\le \frac{1}{|B|}\int_B \left| \left(L^{-\alpha/2} - e^{-t_B L} L^{-\alpha/2}\right) \times \left(\prod_{j=1}^{m}\left(b_j - (b_j)_B\right) f_2\right)(y)\right| dy \\ &\le \frac{1}{|B|}\int_B \int_{\mathbb{R}^n\setminus 2B} |\widetilde{K}_{\alpha,t_B}(y,z)| \times \left|\prod_{j=1}^{m}\left(b_j(z) - (b_j)_B\right) f(z)\right| dz\, dy \end{aligned}$$

$$\leq C\sum_{k=1}^{\infty}\int_{2^{k+1}B\setminus 2^kB}\frac{t_B}{|x_0-z|^{n-\alpha+2}} \times\left|\prod_{j=1}^{m}\left(b_j(z)-(b_j)_B\right)f(z)\right|dz$$
$$\leq C\sum_{k=1}^{\infty}\frac{2^{-2k}}{|2^kB|^{1-\alpha/n}}\int_{2^{k+1}B}\left|\prod_{j=1}^{m}\left(b_j(z)-(b_j)_B\right)\right||f(z)|\,dz. \tag{85}$$

The same manner as that of G_3 gives us that

$$G_7\leq C\sum_{k=1}^{\infty}2^{-2k}\left(\|\vec{b}\|_* M_{\alpha,r,w}f(x)+\|\vec{b}\|_* M_{\alpha,1}f(x)\right) \leq C\|\vec{b}\|_*\left(M_{\alpha,r,w}f(x)+M_{\alpha,1}f(x)\right). \tag{86}$$

According to the above estimates, we have completed the proof of Lemma 25. □

3. Proof of Theorems

At first, we give the proof of Theorem 10.

Proof. It follows from Lemmas 13, 14, 25, and 17–21 that

$$\begin{aligned}
&\left\|L_{\vec{b}}^{-\alpha/2}f\right\|_{L^{q,\kappa q/p}(w^{q/p},w)}\\
&\leq\left\|ML_{\vec{b}}^{-\alpha/2}f\right\|_{L^{q,\kappa q/p}(w^{q/p},w)}\\
&\leq\left\|M_L^{\sharp}(L_{\vec{b}}^{-\alpha/2}f)\right\|_{L^{q,\kappa q/p}(w^{q/p},w)}\\
&\leq C\Big\{\|\vec{b}\|_*\left\|M_{r,w}(L^{-\alpha/2}f)\right\|_{L^{q,\kappa q/p}(w^{q/p},w)}\\
&\quad+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}C_{j,m}\|\vec{b}_\sigma\|_*\\
&\quad\times\left\|M_{\tau,w}(L_{\vec{b}_{\sigma'}}^{-\alpha/2}f)\right\|_{L^{q,\kappa q/p}(w^{q/p},w)}\\
&\quad+\|\vec{b}\|_*\left\|w^{-\alpha/n}M_{\alpha,r,w}(f)\right\|_{L^{q,\kappa q/p}(w^{q/p},w)}\\
&\quad+\|\vec{b}\|_*\left\|M_{\alpha,1}(f)\right\|_{L^{q,\kappa q/p}(w^{q/p},w)}\Big\}\\
&\leq C\|\vec{b}\|_*\left\|L^{-\alpha/2}f\right\|_{L^{q,\kappa q/p}(w^{q/p},w)}\\
&\quad+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}C_{j,m}\|\vec{b}_\sigma\|_*\left\|L_{\vec{b}_{\sigma'}}^{-\alpha/2}f\right\|_{L^{q,\kappa q/p}(w^{q/p},w)}\\
&\quad+C\|\vec{b}\|_*\left\|M_{\alpha,r,w}f\right\|_{L^{q,\kappa q/p}(w)}+C\|\vec{b}\|_*\|f\|_{L^{p,\kappa}(w)}\\
&\leq C\|\vec{b}\|_*\|f\|_{L^{p,\kappa}(w)}\\
&\quad+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}C_{j,m}\|\vec{b}_\sigma\|_*\left\|(L_{\vec{b}_{\sigma'}}^{-\alpha/2}f)\right\|_{L^{q,\kappa q/p}(w^{q/p},w)}.
\end{aligned}\tag{87}$$

Then, we can make use of an induction on $\sigma\subseteq\{1,2,\ldots,m\}$ to get that

$$\left\|L_{\vec{b}}^{-\alpha/2}f\right\|_{L^{q,\kappa q/p}(w^{q/p},w)}\leq C\|\vec{b}\|_*\|f\|_{L^{p,\kappa}(w)}. \tag{88}$$

This completes the proof of Theorem 10. □

Now, we are in the position of proving Theorem 11. If we take $\kappa=0$ in Theorem 10, we will immediately get our desired results.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Limit Cycles and Analytic Centers for a Family of $4n - 1$ Degree Systems with Generalized Nilpotent Singularities

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With the aid of computer algebra system *Mathematica* 8.0 and by the integral factor method, for a family of generalized nilpotent systems, we first compute the first several quasi-Lyapunov constants, by vanishing them and rigorous proof, and then we get sufficient and necessary conditions under which the systems admit analytic centers at the origin. In addition, we present that seven amplitude limit cycles can be created from the origin. As an example, we give a concrete system with seven limit cycles via parameter perturbations to illustrate our conclusion. An interesting phenomenon is that the exponent parameter n controls the singular point type of the studied system. The main results generalize and improve the previously known results in Pan.

1. Introduction

A famous problem for the plane analytic systems of differential equations is under what conditions the local phase portrait at a critical point p is topologically equivalent to the local phase portrait of the linear part of the system at p. This problem has been solved by Poincaré and Bendixson for hyperbolic critical points and for elementary critical points, that is, for points having zero determinant and nonzero trace linear part. Another problem is to characterize the local phase portrait at an isolated critical point p.

When the matrix of the linear part at the origin is not identically null but has its eigenvalues which are equal to zero, at this moment, the origin is a nilpotent critical point. An analytic system having an isolated nilpotent singularity at the origin, in some suitable coordinates, can be written as follows:

$$\begin{aligned} \frac{dx}{dt} &= y + \sum_{i+j=2}^{\infty} a_{ij} x^i y^j = y + X(x, y), \\ \frac{dy}{dt} &= \sum_{i+j=2}^{\infty} b_{ij} x^i y^j = Y(x, y). \end{aligned} \tag{1}$$

Suppose that the function $y = y(x)$ satisfies $X(x, y) = 0$, $y(0) = 0$. Amelkin et al. proved (see, for instance, [1]) that the origin of system (1) is a monodromic critical point (i.e., a center or a focus) if and only if

$$\begin{gathered} Y(x, y(x)) = \alpha x^{2n-1} + o\left(x^{2n-1}\right), \quad \alpha < 0, \\ \left[\frac{\partial X(x, y)}{\partial x} + \frac{\partial Y(x, y)}{\partial x}\right]_{y=y(x)} = \beta x^{n-1} + o\left(x^{n-1}\right), \\ \beta^2 + 4n\alpha < 0, \end{gathered} \tag{2}$$

where n is a positive integer. Andreev [2] shows what the behavior of the solutions in a neighborhood of the nilpotent critical point is, except if it is a center or a focus (nilpotent center problem). This last result can not distinguish between a focus and a center. Takens [3] and Bogdanov [4] find easy formal normal forms for the nilpotent critical points. Moussu [5] has found the C^∞ normal form for a nilpotent center and asks if there exists an analytic normal form for the nilpotent centers. Takens [3] proves that for any analytic nilpotent center there exists an analytic change of variables such that the new system can be written as a system of the form (1) and it is a *time-reversible* nilpotent center. However, the aforementioned result is difficult to implement because, in general, in order to decide if an analytic system has a nilpotent center, we must know explicitly the analytic change of variables which writes system (1) in the Berthier-Moussu normal form. Writing systems in a convenient normal form, Teixeira and Yang [6] study the relationship between time-reversibility and the center-focus problem for elementary singular points and nilpotent singular points. Giné [7] develops a method which provides necessary conditions for obtaining a local analytic integral in a neighborhood of a generalized nilpotent singular point. García and Giné [8] give a necessary condition to have local analytic integrability in an analytic nilpotent center. Giacomini et al. [9] studied the centers of planar analytic vector fields which are a limit of linear-type centers.

In this paper, we consider the system of differential equations in the plane whose origin is a generalized nilpotent singular point

$$\begin{aligned}\frac{dx}{dt} &= y^{2n-1} + y^{n-1}\left(a_{30}x^3 + a_{21}x^2y^n + a_{12}xy^{2n} + a_{03}y^{3n}\right),\\ \frac{dy}{dt} &= -2x^3 + b_{02}y^{2n} + b_{12}xy^{2n} + b_{03}y^{3n}.\end{aligned} \tag{3}$$

The origin of system (3) is a third-order nilpotent singular point when $n = 1$, while it is a total degenerate singular point when $n > 1$.

In Section 2, we give some preliminary knowledge concerning the nilpotent critical point. In Section 3, we transform the origin into a third-order nilpotent singular point by a homeomorphism. Then, we compute the first several quasi-Lyapunov constants and derive the sufficient and necessary conditions for the origin to be an analytic center. In the last section, we prove that there exist seven small amplitude limit cycles in the neighborhood of the origin.

2. Computation of Quasi-Lyapunov Constants and Determination of Analytic Centers

In this section, we first introduce some definitions, notations, and symbols in order to make the paper compact and clear, followed by an algorithm to obtain the necessary conditions for the third-order nilpotent critical point of system (1) to be an analytic center, and then we present several methods to prove the sufficiency. More details are due to [10, 11].

If the origin of system (1) is a high-order critical point, it is called a $(2n-1)$th-order critical point when (2) is satisfied, and it could be broken into $2n-1$ elementary critical points in the neighborhood of the origin in the complex plane. It is easy to show that the origin of system (1) is a third-order monodromic critical point if and only if $b_{20} = 0$ and $(2a_{20} - b_{11})^2 + 8b_{30} < 0$.

Without loss of generality, we assume that

$$a_{20} = \mu, \qquad b_{20} = 0, \qquad b_{11} = 2\mu, \qquad b_{30} = -2. \tag{4}$$

Otherwise, by letting $(2a_{20} - b_{11})^2 + 8b_{30} = -16\lambda^2$, $2a_{20} + b_{11} = 4\lambda\mu$, and taking the transformation $\xi = \lambda x$, $\eta = \lambda y + (1/4)(2a_{20} - b_{11})\lambda x^2$, we obtain the desired form. Under (4), system (1) becomes the following real autonomous planar system:

$$\begin{aligned}\frac{dx}{dt} &= y + \mu x^2 + \sum_{i+2j=3}^{\infty} a_{ij}x^iy^j = X(x, y),\\ \frac{dy}{dt} &= -2x^3 + 2\mu xy + \sum_{i+2j=4}^{\infty} b_{ij}x^iy^j = Y(x, y).\end{aligned} \tag{5}$$

In the following, we are going to classify the third-order nilpotent critical point.

Theorem 1. *For system* (5)*, there exists a series with nonzero convergence radius:*

$$\begin{aligned}u(x, y) &= x + \sum_{\alpha+\beta=2}^{\infty} a'_{\alpha\beta}x^\alpha y^\beta,\\ v(x, y) &= y + \sum_{\alpha+\beta=2}^{\infty} b'_{\alpha\beta}x^\alpha y^\beta, \quad b'_{20} = -\mu,\\ \zeta(x, y) &= 1 + \sum_{\alpha+\beta=1}^{\infty} c'_{\alpha\beta}x^\alpha y^\beta,\end{aligned} \tag{6}$$

such that, by the transformation

$$u = u(x, y), \qquad v = v(x, y), \qquad dt = \zeta(x, y)\,d\tau, \tag{7}$$

system (5) *is reduced to the following Liénard equation:*

$$\begin{aligned}\frac{du}{d\tau} &= v + \sum_{k=1}^{\infty} A_k u^{4k} + \sum_{k=0}^{\infty} B_k u^{4k+2} + \sum_{k=1}^{\infty} C_k u^{2k+1} = U(u, v),\\ \frac{dv}{d\tau} &= -2\left(1 + \mu^2\right)u^3 = V(u, v),\end{aligned} \tag{8}$$

where $B_0 = 2\mu$. In addition, the origin of system (5) *is a center if and only if $C_k = 0$ for all k.*

The following two definitions are taken from [10].

Definition 2. (1) If $\mu \neq 0$, then the origin of system (5) is called a third-order nilpotent critical point of zero-class.

(2) If $\mu = 0$ and there exists a positive integer s, such that $B_0 = B_1 = \cdots = B_{s-1} = 0$, but $B_s \neq 0$, then the origin of system (5) is called a third-order nilpotent critical point of s-class.

(3) If $\mu = 0$ and $B_s = 0$ for all positive integer s, then the origin of system (5) is called a third-order nilpotent critical point of ∞-class.

Definition 3. Let f_k, g_k be continuous and bounded functions with respect to μ and a_{ij}, b_{ij}, $k = 1, 2, \ldots$. If, for any integer m, there exist continuous and bounded functions of μ and all a_{ij}, b_{ij}: $\xi_1^{(m)}, \xi_2^{(m)}, \ldots, \xi_{m-1}^{(m)}$, such that

$$f_m = g_m + \left(\xi_1^{(m)}f_1 + \xi_2^{(m)}f_2 + \cdots + \xi_{m-1}^{(m)}f_{m-1}\right), \tag{9}$$

then we say that f_m and g_m are equivalent, denoted by $f_m \sim g_m$. If, for any integer m, we have $f_m \sim g_m$, we say that the sequences of functions $\{f_m\}$ and $\{g_m\}$ are equivalent, denoted by $\{f_m\} \sim \{g_m\}$.

Remark 4. It is easy to see from Definition 3 that the following conclusions hold.

(1) The equivalence relationship of two sequences of functions is self-reciprocal, symmetric, and transmissible.

(2) If, for some integer m, $f_m \sim g_m$, then when $f_1 = f_2 = \cdots = f_{m-1} = 0$, we have $f_m = g_m$.

(3) The relationship $f_1 \sim g_1$ implies that $f_1 = g_1$.

The following three theorems were proved in [10].

Theorem 5. *If the origin of system* (5) *is s-class, then the origin of system* (5) *is a center if and only if there is an inverse integrating factor* M^{s+1}.

If the origin of system (5) *is* ∞*-class, then the origin of system* (5) *is a center if and only if for any natural number l there exists an inverse integrating factor* M^{l+1}*, where M is a power series given by*

$$M = x^4 + y^2 + \sum_{k+2j=5}^{\infty} c_{kj} x^k y^j. \tag{10}$$

Theorem 6. *For any natural number s and a given number sequence*

$$\{c_{0\beta}\}, \quad \beta \geq 3, \tag{11}$$

terms with the coefficients $c_{\alpha\beta}$ *satisfying* $\alpha \neq 0$ *of the formal series* (10) *can be constructed successively such that*

$$\frac{\partial}{\partial x}\left(\frac{X}{M^{s+1}}\right) + \frac{\partial}{\partial y}\left(\frac{Y}{M^{s+1}}\right) = \frac{1}{M^{s+2}} \sum_{m=6}^{\infty} \omega_m x^m; \tag{12}$$

that is,

$$\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) M - (s+1)\left(\frac{\partial M}{\partial x} X + \frac{\partial M}{\partial y} Y\right) = \sum_{m=6}^{\infty} \omega_m x^m, \tag{13}$$

where $s\mu = 0$.

Theorem 7. *For* $\alpha \geq 1$, $\alpha + \beta \geq 3$ *in* (11) *and* (12), $c_{\alpha\beta}$ *is uniquely determined by the recursive formula*

$$c_{\alpha\beta} = \frac{1}{(s+1)\alpha}\left(A_{\alpha-1,\beta+1} + B_{\alpha-1,\beta+1}\right). \tag{14}$$

For $m \geq 1$, $\omega_m(s, \mu)$ *is uniquely determined by the recursive formula*

$$\omega_m = A_{m,0} + B_{m,0}, \tag{15}$$

where

$$\begin{aligned} A_{\alpha\beta} &= \sum_{k+j=2}^{\alpha+\beta-1} [k - (s+1)(\alpha - k + 1)] a_{kj} c_{\alpha-k+1,\beta-j}, \\ B_{\alpha\beta} &= \sum_{k+j=2}^{\alpha+\beta-1} [j - (s+1)(\beta - j + 1)] b_{kj} c_{\alpha-k,\beta-j+1}. \end{aligned} \tag{16}$$

In (15), *one sets*

$$\begin{gathered} c_{00} = c_{10} = c_{01} = 0, \\ c_{20} = c_{11} = 0, \qquad c_{02} = 1, \\ c_{\alpha\beta} = 0, \quad \text{if } \alpha < 0 \text{ or } \beta < 0. \end{gathered} \tag{17}$$

It follows from Theorems 5–7 that if the origin of system (5) is a center with s-class or ∞-class, then, by choosing $\{c_{0\beta}\}$, such that

$$\omega_k = 0, \quad k = 6, 7, \ldots. \tag{18}$$

The following conclusion holds (see Theorem 3.4 of [10]).

Theorem 8. *If the origin of system* (5) *is* ∞*-class, then when the origin of system* (5) *is a center, in a neighborhood of the origin, system* (5) *has an analytic inverse integrating factor*

$$M_\infty = 1 + h.o.t. \tag{19}$$

and an analytic first integral given by

$$F(x, y) = x^4 + y^2 + \sum_{k+2j=5}^{\infty} C_{kj} x^k y^j. \tag{20}$$

According to [11], we have the following lemma.

Lemma 9. *For system* (5), *if there exists a first integral which is the power series* (20) *in the neighborhood of the origin, then the origin of system* (5) *is a center.*

By Theorem 1, if the origin of system (5) is a center, there exist analytic transformations in the neighborhood of the origin such that system (5) can be transformed into the Liénard equations:

$$\begin{aligned} \frac{du}{d\tau} &= v + \sum_{k=1}^{\infty} A_k u^{4k} + \sum_{k=0}^{\infty} B_k u^{4k+2}, \\ \frac{dv}{d\tau} &= -2\left(1 + \mu^2\right) u^3, \quad B_0 = 2\mu. \end{aligned} \tag{21}$$

The vector field defined in (21) is symmetrical with respect to the v-axis. Further, by the transformation

$$w = u^2, \qquad v = v, \qquad d\tau = -\frac{d\tau'}{2u}, \tag{22}$$

system (21) is reduced to

$$\begin{aligned} \frac{dw}{d\tau'} &= -v - \sum_{k=1}^{\infty} A_k w^{2k} - \sum_{k=0}^{\infty} B_k w^{2k+1}, \\ \frac{dv}{d\tau'} &= \left(1 + \mu^2\right) w, \quad B_0 = 2\mu. \end{aligned} \tag{23}$$

In the (w,v)-phase plane, the origin of system (23) is an elementary critical point (focus or center). Obviously, we have the following.

Lemma 10. *If there exists a power series $\mathcal{F}=\mathcal{F}(u,v)$ in u, v satisfying*

$$\left.\frac{d\mathcal{F}}{d\tau}\right|_{(21)}=0, \tag{24}$$

then $\mathcal{F}$ can be written as a power series in u^2, v; namely, $\mathcal{F}(u,v)=G(u^2,v)$.

Then, we have the following theorem.

Theorem 11. *The origin of system (5) is an analytic center if and only if the origin of system (5) is a center and ∞-class; namely, the origin of system (5) is a center and for any natural number k, $B_k=0$.*

Corollary 12. *If $\mu\neq 0$, the origin of system (5) is not an analytic center.*

Theorem 13. *The origin of system (5) is an analytic center if and only if there exists an analytic first integral $F(x,y)$ in the neighborhood of the origin, which is the power series (23).*

From Theorems 8 and 13, we further have the following.

Theorem 14. *The origin of system (5) is an analytic center if and only if in the neighborhood of the origin of system (5) there exists an analytic inverse integrating factor*

$$M_\infty=1+\sum_{k+j=1}^{\infty}C'_{kj}x^k y^j. \tag{25}$$

Similarly, by Theorems 5 and 11, we have the following.

Theorem 15. *The origin of system (5) is an analytic center, if and only if, for any natural number s, there exists an inverse integrating factor M^{s+1}, where M is the power series (10).*

Moreover, Theorems 7 and 15 imply the following.

Theorem 16. *The origin of system (5) is an analytic center, if and only if, for any natural number s, there exists a power series M satisfying $\omega_m=0$ in (12) for all m.*

Therefore, a new method of determining analytic nilpotent center for a given system has been obtained in Theorem 16.

Theorem 17. *If the origin of system (5) is a nilpotent center and system (5) is symmetric with respect to the origin, namely,*

$$\begin{aligned}X(-x,-y)&=-X(x,y),\\ Y(-x,-y)&=-Y(x,y),\end{aligned} \tag{26}$$

then the origin of system (5) is an analytic center.

Theorem 18. *If system (5) is symmetric with respect to the x-axis, then the origin of system (5) is an analytic center.*

Remark 19. If system (5) is symmetric with respect to the y-axis, then the origin of system (5) may not be an analytic center. For example, system (21) is symmetric with respect to the v-axis, but the origin is an analytic center if and only if $B_k=0$ for all k.

Eventually, by Theorem 11, we have the following.

Theorem 20. *The origin of system (5) is an analytic center if and only if system (5) can be changed into*

$$\frac{du}{d\tau}=v+\sum_{k=1}^{\infty}A_k u^{4k},\qquad \frac{dv}{d\tau}=-2u^3 \tag{27}$$

by the analytic transformation (7).

3. Analytic Center Conditions

In this section we will derive conditions for the origin of system (3) to be an analytic center.

After the change

$$x_1=x,\qquad y_1=\frac{1}{\sqrt{n}}y^n,\qquad dt_1=\sqrt{n}y^{n-1}dt, \tag{28}$$

and renaming (x_1,y_1,t_1) with (x,y,t), system (3) takes the form

$$\begin{aligned}\frac{dx}{dt}&=y+\frac{1}{\sqrt{n}}a_{30}x^3+a_{21}x^2y+\sqrt{n}a_{12}xy^2+na_{03}y^3,\\ \frac{dy}{dt}&=-2x^3+nb_{02}y^2+nb_{12}xy^2+n^{3/2}b_{03}y^3.\end{aligned} \tag{29}$$

According to Theorem 7, we have the following.

Lemma 21. *Assume that s is a natural number. One can derive a power series (10) for system (29) under which (12) is satisfied, where*

$$\begin{gathered}c_{00}=0,\qquad c_{10}=0,\qquad c_{01}=0,\qquad c_{20}=0,\\ c_{11}=0,\qquad c_{02}=1.\end{gathered} \tag{30}$$

In addition, for any natural numbers α, β, $c_{\alpha\beta}$ is given by the following recursive formula:

$$\begin{aligned}c_{\alpha\beta}=\Big(&2(1+s)(2+\beta)c_{-4+\alpha,2+\beta}\\ &+\frac{1}{\sqrt{n}}a_{30}(3-(1+s)(-3+\alpha))c_{-3+\alpha,1+\beta}\\ &+a_{21}(2-(1+s)(-2+\alpha))c_{-2+\alpha,\beta}\end{aligned}$$

$$\begin{aligned}&+b_{12}n\left(2-(1+s)\beta\right)c_{-2+\alpha,\beta}\\&+a_{12}\sqrt{n}\left(1-(1+s)(-1+\alpha)\right)c_{-1+\alpha,-1+\beta}\\&+b_{03}n^{3/2}\left(3-(1+s)(-1+\beta)\right)c_{-1+\alpha,-1+\beta}\\&+b_{02}n\left(2-(1+s)\beta\right)c_{-1+\alpha,\beta}\\&-a_{03}n(1+s)\alpha c_{\alpha,-2+\beta}\Big)\\&\times((s+1)\alpha)^{-1},\end{aligned} \tag{31}$$

and, for any natural number m, ω_m is given by the following recursive formula:

$$\begin{aligned}\omega_m={}&2(1+s)c_{-3+m,1}\\&+\frac{1}{\sqrt{n}}a_{30}\left(3-(-2+m)(1+s)\right)c_{-2+m,0}\\&+b_{12}n(3+s)c_{-1+m,-1}\\&+a_{21}\left(2-(-1+m)(1+s)\right)c_{-1+m,-1}\\&+b_{03}n^{3/2}\left(3+2(1+s)\right)c_{m,-2}\\&+a_{12}\sqrt{n}\left(1-m(1+s)\right)c_{m,-2}+b_{02}n(3+s)c_{m,-1}\\&-a_{03}(1+m)n(1+s)c_{1+m,-3}.\end{aligned} \tag{32}$$

Applying Lemma 21 and computing with Mathematica, we have

$$\begin{aligned}&\omega_6=-\frac{4s-1}{\sqrt{n}}a_{30},\qquad \omega_7\sim 3(s+1)c_{03},\\&\omega_8\sim-\frac{2}{5}\sqrt{n}(4s-3)\left(a_{12}+3nb_{03}\right),\\&\omega_9\sim-4n^{5/2}(s-1)b_{02}b_{03}.\end{aligned} \tag{33}$$

From $\omega_9=0$, the analytic center problem of system (29) can be broken down into three cases: (1) $b_{02}=0$, (2) $b_{03}=0$, and (3) $s=1$.

Case 1. Consider $b_{02}=0$.

In this case, further calculation gives the following.

Theorem 22. *For system* (29), *the first three quasi-Lyapunov constants at the origin are given by*

$$\begin{aligned}&\lambda_1=\frac{1}{\sqrt{n}}a_{30},\qquad \lambda_2\sim\frac{2}{5}\sqrt{n}\left(a_{12}+3nb_{03}\right),\\&\lambda_3\sim\frac{4}{7}n^{3/2}b_{03}\left(a_{21}+nb_{12}\right).\end{aligned} \tag{34}$$

In the above expression of λ_k, one has already let $\lambda_1=\lambda_2=\cdots=\lambda_{k-1}=0$, $k=2,3$.

From Theorem 22, we obtain the following assertion.

Theorem 23. *For system* (29), *all the quasi-Lyapunov constants at the origin are zero if and only if the first three quasi-Lyapunov constants at the origin are zero; that is, one of the following two conditions holds:*

$$a_{30}=a_{12}=b_{02}=b_{03}=0; \tag{35}$$

$$a_{30}=b_{02}=0,\qquad a_{12}+3nb_{03}=0,\qquad a_{21}+nb_{12}=0. \tag{36}$$

Relevantly, both conditions are the analytic center conditions of the origin.

Proof. When condition (35) is satisfied, system (29) can be brought to

$$\begin{aligned}\frac{dx}{dt}&=y\left(1+a_{21}x^2+na_{03}y^2\right),\\\frac{dy}{dt}&=-x\left(2x^2-nb_{12}y^2\right),\end{aligned} \tag{37}$$

whose vector field is symmetric with respect to the origin.

When condition (36) is satisfied, system (29) can be brought to

$$\begin{aligned}\frac{dx}{dt}&=y\left(1-nb_{12}x^2-3n^{3/2}b_{03}xy+na_{03}y^2\right),\\\frac{dy}{dt}&=-2x^3+nb_{12}xy^2+n^{3/2}b_{03}y^3,\end{aligned} \tag{38}$$

which is Hamiltonian and possesses the analytic first integral

$$F_1(x,y)=x^4+y^2-nb_{12}x^2y^2-2n^{3/2}b_{03}xy^3+\frac{1}{2}na_{03}y^4. \tag{39}$$

□

Case 2. Consider $b_{03}=0$.

In this case, $\lambda_1=\lambda_2=0$ yields that

$$a_{30}=a_{12}=b_{03}=0. \tag{40}$$

Under this condition, system (29) becomes

$$\begin{aligned}\frac{dx}{dt}&=y\left(1+a_{21}x^2+na_{03}y^2\right),\\\frac{dy}{dt}&=-2x^3+nb_{02}y^2+nb_{12}xy^2,\end{aligned} \tag{41}$$

whose vector field is symmetric with respect to axis x. Thus, system (29) has an analytic center at the origin when condition (40) is satisfied.

Obviously, condition (35) is a special case of condition (40).

Case 3. Consder $s=1$.

In this case, we have the following.

Proposition 24. *For system* (29)*, one can determine successively the terms of the formal series* $M(x,y) = x^4 + y^2 + o(r^4)$*, such that*

$$\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) M - 2\left(\frac{\partial M}{\partial x} X + \frac{\partial M}{\partial y} Y\right) = \sum_{m=1}^{7} \lambda_m \left[(2m-5)\, x^{2m+4} + o\left(r^{18}\right)\right], \tag{42}$$

where λ_m *is the mth quasi-Lyapunov constant at the origin of system* (29)*,* $m = 1, 2, \ldots, 7$*.*

Further calculation gives the following.

Theorem 25. *For system* (29)*, the first seven quasi-Lyapunov constants at the origin are given by*

$$\begin{aligned}
\lambda_1 &= \frac{1}{\sqrt{n}} a_{30}, \\
\lambda_2 &\sim \frac{2}{5}\sqrt{n}\left(a_{12} + 3nb_{03}\right), \\
\lambda_3 &\sim \frac{4}{35} n^{3/2} b_{03}\left(5a_{21} + 5nb_{12} + 11n^2 b_{02}^2\right), \\
\lambda_4 &\sim -\frac{4}{7875} n^{9/2} b_{02}^2 b_{03}\left(-1475 b_{12} + 744 n b_{02}^2\right), \\
\lambda_5 &\sim -\frac{8}{2512846875} n^{9/2} b_{02}^2 b_{03} \\
&\quad \times \left(589594375 a_{03} + 395195814 n^3 b_{02}^4\right), \\
\lambda_6 &\sim -\frac{24}{4352606774140625} n^{13/2} b_{02}^2 b_{03} \\
&\quad \times \left(-1131416865765625 b_{03}^2 + 529893701720802 n^3 b_{02}^6\right), \\
\lambda_7 &\sim -\frac{1057311533261767 6917216}{38365073074138357421875} n^{23/2} b_{02}^{10} b_{03}.
\end{aligned} \tag{43}$$

In the above expression of λ_k*, we have already let* $\lambda_1 = \lambda_2 = \cdots = \lambda_{k-1} = 0$*,* $k = 2, 3, 4, 5, 6, 7$*.*

λ_i, $i = 1, 2, 3, 4, 5, 6, 7$, in expression (43) vanish if and only if one of conditions (36) and (40) holds.

Therefore, we see from Cases 1–3 the following.

Corollary 26. *The origin of system* (29) *((3)) is an analytic center if and only if condition* (36) *or* (40) *holds.*

Actually, when condition (36) holds, system (3) goes over to

$$\begin{aligned}
\frac{dx}{dt} &= y^{2n-1}\left(1 - nb_{12}x^2 - 3nb_{03}xy^n + a_{03}y^{2n}\right), \\
\frac{dy}{dt} &= -2x^3 + b_{12}xy^{2n} + b_{03}y^{3n},
\end{aligned} \tag{44}$$

which is Hamiltonian and possesses the analytic first integral

$$F_2(x,y) = x^4 + \frac{1}{n}y^{2n} - b_{12}x^2y^{2n} - 2b_{03}xy^{3n} + \frac{1}{2n}a_{03}y^{4n}. \tag{45}$$

When condition (40) holds, system (3) goes over to

$$\begin{aligned}
\frac{dx}{dt} &= y^{2n-1}\left(1 + a_{21}x^2 + a_{03}y^{2n}\right), \\
\frac{dy}{dt} &= -2x^3 + b_{02}y^{2n} + b_{12}xy^{2n},
\end{aligned} \tag{46}$$

whose vector field is symmetric with respect to axis x.

4. Multiple Bifurcation of Limit Cycles

In this section, we are going to establish the conditions for $O(0,0)$ to be at most a weak focus of order seven, and, more, we will prove that the perturbed system of (29) can generate seven limit cycles enclosing an elementary node at the origin.

From the fact $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = 0$, $\lambda_7 \neq 0$, we obtain the following.

Theorem 27. *For system* (29)*, the origin is a weak focus of order seven if and only if*

$$\begin{gathered}
a_{30} = 0, \qquad a_{12} = -3nb_{03}, \\
a_{21} = -\frac{1}{5}n\left(5b_{12} + 11nb_{02}^2\right), \qquad b_{12} = \frac{744}{1475}nb_{02}^2, \\
a_{03} = -\frac{395195814}{589594375}n^3 b_{02}^4, \\
b_{03}^2 = \frac{529893701720802}{1131416865765625}n^3 b_{02}^6, \qquad b_{02}b_{03} \neq 0.
\end{gathered} \tag{47}$$

Consider the perturbed system of (29):

$$\begin{aligned}
\frac{dx}{dt} &= \delta(\varepsilon)\,x + y + \frac{1}{\sqrt{n}}a_{30}(\varepsilon)\,x^3 + a_{21}(\varepsilon)\,x^2y \\
&\quad + \sqrt{n}a_{12}(\varepsilon)\,xy^2 + na_{03}(\varepsilon)\,y^3, \\
\frac{dy}{dt} &= 2\delta(\varepsilon)\,y - 2x^3 + nb_{02}y^2 + nb_{12}(\varepsilon)\,xy^2 \\
&\quad + n^{3/2}b_{03}(\varepsilon)\,y^3.
\end{aligned} \tag{48}$$

In order to get seven limit cycles, we only need to show that, when condition (47) holds, the Jacobian of the function group $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)$ with respect to the parameter group $(a_{30}, a_{12}, a_{21}, b_{12}, a_{03}, b_{03})$ does not equal zero. A direct calculation gives that

$$\left.\frac{\partial(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)}{\partial(a_{30}, a_{12}, a_{21}, b_{12}, a_{03}, b_{03})}\right|_{(47)} = -\frac{12778943607392678363862335633 8176}{1452605210698421132685852050781 25}n^{23}b_{02}^{18}b_{03} \neq 0. \tag{49}$$

The above discussions indicate the following.

Theorem 28. *If the origin of system* (29) *((3)) is a weak focus of order seven, for* $0 < \delta \ll 1$*, making a small perturbation to the coefficient group* $(a_{30}, a_{12}, a_{21}, b_{12}, a_{03}, b_{03})$*, then, for system* (48)*, in a small neighborhood of the origin, there exist exactly seven small amplitude limit cycles enclosing the origin* $O(0,0)$*, which is an elementary node.*

Example 29. Take

$$\begin{aligned}
&\delta(\varepsilon) = \varepsilon^{56}, \qquad a_{30}(\varepsilon) = -\varepsilon^{42},\\
&a_{12}(\varepsilon) = -\frac{9\sqrt{58877077968978/20801689}}{7375} n^{5/2} c^3 \operatorname{sign}(c) - 3n\varepsilon^2 + \varepsilon^{30},\\
&a_{21}(\varepsilon) = -\frac{3989}{1475} n^2 c^2 - n\varepsilon^{12} - \varepsilon^{20},\\
&b_{12}(\varepsilon) = \frac{744}{1475} nc^2 + \varepsilon^{12},\\
&a_{03}(\varepsilon) = -\frac{395195814}{589594375} n^3 c^4 + \varepsilon^6,\\
&b_{03}(\varepsilon) = \frac{3\sqrt{58877077968978/20801689}}{7375} n^{3/2} c^3 \operatorname{sign}(c) + \varepsilon^2,\\
&b_{02} = c \operatorname{sign}(c),
\end{aligned} \tag{50}$$

where c is an arbitrary nonzero real constant.

Straightforward computations by using Theorem 25 give the first seven quasi-Lyapunov constants at the origin of system (48):

$$\begin{aligned}
\lambda_1 &= -\frac{1}{\sqrt{n}} \varepsilon^{42} + o\left(\varepsilon^{42}\right),\\
\lambda_2 &\sim \frac{2}{5} \sqrt{n} \varepsilon^{30} + o\left(\varepsilon^{30}\right) = 0.4\sqrt{n}\varepsilon^{30} + o\left(\varepsilon^{30}\right),\\
\lambda_3 &\sim -\frac{12\sqrt{58877077968978/20801689}}{51625} n^3 c^3 \operatorname{sign}(c)\,\varepsilon^{20} + o\left(\varepsilon^{20}\right) \approx -0.391061 n^3 c^3 \operatorname{sign}(c)\,\varepsilon^{20} + o\left(\varepsilon^{20}\right),\\
\lambda_4 &\sim \frac{4\sqrt{19625692656326/62405067}}{4375} n^6 c^5 \operatorname{sign}(c)\,\varepsilon^{12} + o\left(\varepsilon^{12}\right) \approx 0.512725 n^6 c^5 \operatorname{sign}(c)\,\varepsilon^{12} + o\left(\varepsilon^{12}\right),\\
\lambda_5 &\sim -\frac{8\sqrt{15955688129593038/76759}}{2839375} n^6 c^5 \operatorname{sign}(c)\,\varepsilon^{6} + o\left(\varepsilon^{6}\right) \approx -1.28458 n^6 c^5 \operatorname{sign}(c)\,\varepsilon^{6} + o\left(\varepsilon^{6}\right),\\
\lambda_6 &\sim \frac{25434897682598496}{4352606774140625} n^{19/2} c^8 \varepsilon^2 + o\left(\varepsilon^2\right) \approx 5.8436 n^{19/2} c^8 \varepsilon^2 + o\left(\varepsilon^2\right),\\
\lambda_7 &\sim -\frac{3171934599785303075164 8\sqrt{58877077968978/20801689}}{28294241392177038598632 8125} n^{13} c^{13} \operatorname{sign}(c) + o(1) \approx -0.188604 n^{13} c^{13} \operatorname{sign}(c) + o(1).
\end{aligned} \tag{51}$$

Then, for $0 < \varepsilon \ll 1$, system (48) has seven limit cycles Γ_k : $r = \widetilde{r}(\theta, h_k(\varepsilon))$ in a small neighborhood of the origin, where $h_k(\varepsilon) = O(\varepsilon^k)$, $k = 1, 2, 3, 4, 5, 6, 7$.

Conflict of Interests

The authors declare that they have no conflict of interests.

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Research Article

The Improvement on the Boundedness and Norm of a Class of Integral Operators on L^p Space

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We prove the condition "c is neither 0 nor a negative integer" can be dropped on the boundedness of a class of integral operators $S_{a,b,c}$ on L^p space, which improves the result by Krues and Zhu. Besides, the exact norm of $S_{a,b,c}$ on L^p space is also obtained under the assumption $c = n + 1 + a + b$.

1. Introduction

Let $\mathbb{B}_n$ be the open unit ball in the complex space $\mathbb{C}^n$. The measure,

$$d\nu_t = \left(1 - |z|^2\right)^t d\nu(z), \tag{1}$$

denotes the weighted Lebesgue measure on $\mathbb{B}_n$, where t is real parameter and ν is the normalized Lebesgue measure on $\mathbb{B}_n$ such that $\nu(\mathbb{B}_n) = 1$. It is easy to know $d\nu_t$ is finite if and only if $t > -1$. Suppose $1 \leq p < \infty$; to simplify the notation, we write $L_t^p := L^p(\mathbb{B}_n, \nu_t)$ for the weighted L^p-space under the measure ν_t on $\mathbb{B}_n$ and $L^p := L_0^p$ for the usual L^p-space under the measure ν.

Suppose a, b, c are real numbers, and a class of integral operators is defined by

$$S_{a,b,c} f(z) = \left(1 - |z|^2\right)^a \int_{\mathbb{B}_n} \frac{\left(1 - |w|^2\right)^b}{|1 - \langle z, w\rangle|^c} f(w)\, d\nu(w). \tag{2}$$

The class of integral operators is introduced by Kures and Zhu [1]. And it is closely related to "maximal Bergman projection" and Berezin transform. In fact, the boundedness of Bergman projection on L_α^p comes from the boundedness of the operator

$$P_\alpha^\sharp f(z) = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)} \int_{\mathbb{B}_n} \frac{f(w)}{|1 - \langle z, w\rangle|^{n+1+\alpha}} d\nu_\alpha(w), \quad \alpha > -1, \tag{3}$$

on L_α^p; see [2]. Therefore, we can call $P_\alpha^\sharp$ by "maximal Bergman projection," which is the particular case of $S_{a,b,c}$. Berezin transforms, whatever the case of the unit disk [3, page 141] or the case of unit ball ([4, page 76], [5, page 383]), are all concluded in the form of $S_{a,b,c}$ with special a, b, c.

In [1], Krues and Zhu gave the sufficient and necessary conditions of the boundedness of operator $S_{a,b,c}$.

Theorem A (see [1]). *Suppose c is neither 0 nor a negative integer.*

(1) *The operator $S_{a,b,c}$ is bounded on L_t^p $(1 < p < \infty)$ if and only if $-pa < t + 1 < p(b + 1)$, $c \leq n + 1 + a + b$.*

(2) *The operator $S_{a,b,c}$ is bounded on L_t^1 if and only if $-a < t + 1 < b + 1, c = n + 1 + a + b$ or $-a < t + 1 \leq b + 1, c < n + 1 + a + b$.*

The main purposes of this note contain two parts. One part is to prove the condition "c is neither 0 nor a negative integer" in Theorem A can be removed; see Section 3.

The other part is to give the accurate norm of the operator $S_{a,b,c}$ on L_t^p under the assumption $c = n+1+a+b$, which can be seen from the following two theorems.

Theorem 1. *Suppose $c = n+1+a+b$. If $1 \le p < \infty$ and $-pa < t+1 < p(b+1)$, then*

$$\|S_{a,b,c}\|_{L_t^p \to L_t^p} = \frac{n!\Gamma(a+(t+1)/p)\,\Gamma(b+1-(t+1)/p)}{\Gamma^2((n+1+a+b)/2)}. \tag{4}$$

Else, we also give the sufficient and necessary conditions of the operator $S_{a,b,c}$ on L^∞ and the accurate norm under $c = n+1+a+b$ of this case, where L^∞ denotes the set of all essentially bounded and measurable functions under the measure ν_t on $\mathbb{B}_n$.

Theorem 2. *The operator $S_{a,b,c}$ is bounded on L^∞ if and only if $a > 0$, $b > -1$, and $c = n+1+a+b$ or $a \ge 0$, $b > -1$, and $c < n+1+a+b$. Moreover, when $c = n+1+a+b$, we have*

$$\|S_{a,b,c}\|_{L^\infty \to L^\infty} = \frac{n!\Gamma(a)\,\Gamma(1+b)}{\Gamma^2((n+1+a+b)/2)}. \tag{5}$$

Notice $S_{a,b,c}$ is the generalization of "maximal Bergman projection" and Berezin transform which was first introduced by Berezin [6]. The boundedness of Berezin transform of $f \in L^1(\mathbb{D})$ is a well-known fact; see [7, Proposition 2.2]. But the norm of it was not calculated out until 2008 by Dostanić; see [8, Corollary 2]. Recently, the result by Dostanić has been extended to several complex variables in [9, Theorem 1.1]. Thus, Theorems 1 and 2 promote the main results in [8, 9]. And they also imply the following corollary.

Corollary 3. *Suppose $1 \le p < \infty$, $\alpha > -1$, and the norm of $P_\alpha^\sharp$ on L_α^p can be*

$$\left\|P_\alpha^\sharp\right\|_{L_\alpha^p \to L_\alpha^p} = \frac{\Gamma((\alpha+1)/p)\,\Gamma((\alpha+1)-(\alpha+1)/p)\,\Gamma(n+\alpha+1)}{\Gamma^2((n+1+\alpha)/2)\,\Gamma(\alpha+1)}, \tag{6}$$

which implies $\|P_\alpha^\sharp\|_{L_\alpha^p \to L_\alpha^p}$ grows at most like $(\alpha+1)^{-1}$ as $\alpha \to -1$.

Next, we will see that the boundedness of an operator called Berezin-type transform on L_t^p can also be obtained from our main results. The Berezin-type transform is defined by

$$\mathscr{B}_{k,\alpha,\beta} f(z) = C_{k,\alpha,\beta} \times \int_{\mathbb{B}_n} \frac{\left(1-|z|^2\right)^{n+\alpha+\beta+k+1}\left(1-|w|^2\right)^k}{(1-\langle z,w\rangle)^{n+\alpha+k+1}(1-\langle w,z\rangle)^{n+\beta+k+1}} f(w)\,d\nu(w), \tag{7}$$

where

$$C_{k,\alpha,\beta} = \frac{\Gamma(n+\alpha+k+1)\,\Gamma(n+\beta+k+1)}{\Gamma(n+1)\,\Gamma(k+1)\,\Gamma(n+\alpha+\beta+k+1)}, \tag{8}$$

and $n+\alpha+\beta > 0$, $n+\alpha > 0$, $n+\beta > 0$, and $k > -1$. The transform was introduced by Li and Liu [10] when they discuss whether the mean-value property implies (α,β)-harmonicity for integrable functions on the unit ball in $\mathbb{C}^n$. Notice that

$$\left|\mathscr{B}_{k,\alpha,\beta} f(z)\right| \le C_{k,\alpha,\beta} S_{a,b,c} |f|(z) \tag{9}$$

with $a = n+\alpha+\beta+k+1$, $b = k$, and $c = n+1+a+b$. And $\mathscr{B}_{k,\alpha,\alpha} f(z) = C_{k,\alpha,\alpha} S_{a,b,c} f(z)$ as $\alpha = \beta$. Therefore, the boundedness of Berezin-type transform $\mathscr{B}_{k,\alpha,\beta}$ on L_t^p comes from the boundedness of the operator $S_{a,b,c}$ on L_t^p. Thus, we have the following result, which extends Propositions 3.3 and 3.4 in [10] combining the fact of Lemma 2.4 in [10] therein.

Corollary 4. *If $1 \le p < \infty$ such that $-p(n+\alpha+\beta+k+1) < t+1 < p(k+1)$, then the Berezin-type $\mathscr{B}_{k,\alpha,\beta}$ is bounded on L_t^p and*

$$\left\|\mathscr{B}_{k,\alpha,\beta}\right\|_{L_t^p \to L_t^p} \le \lambda_{k,\alpha,\beta,p} \frac{\Gamma(n+\alpha+k+1)\,\Gamma(n+\beta+k+1)}{\Gamma^2(n+1+(\alpha+\beta)/2+k)}, \tag{10}$$

where

$$\lambda_{k,\alpha,\beta,p} = \frac{\Gamma(n+\alpha+\beta+k+1+(t+1)/p)\,\Gamma(k+1-(t+1)/p)}{\Gamma(n+\alpha+\beta+k+1)\,\Gamma(k+1)}. \tag{11}$$

Moreover, the Berezin-type transform is bounded on L^∞, and

$$\left\|\mathscr{B}_{k,\alpha,\beta}\right\|_{L^\infty \to L^\infty} \le \frac{\Gamma(n+\alpha+k+1)\,\Gamma(n+\beta+k+1)}{\Gamma^2(n+1+(\alpha+\beta)/2+k)}. \tag{12}$$

2. Preliminaries

A number of hypergeometric functions will appear throughout. We use the classical notation ${}_2F_1(\alpha,\beta;\gamma;z)$ to denote

$${}_2F_1(\alpha,\beta;\gamma;z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{z^k}{k!}, \tag{13}$$

with $\gamma \ne 0, -1, -2, \ldots$, where

$$(\alpha)_0 = 1, \qquad (\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1) \quad \text{for } k \ge 1. \tag{14}$$

And the hypergeometric series in (13) converges absolutely for all the value of $|z| < 1$. Moreover, as $|z| \to 1^-$, it is easy to know that

$${}_2F_1(\alpha,\beta;\gamma;z) \approx \begin{cases} 1, & \text{if } \gamma-\alpha-\beta > 0; \\ \log\dfrac{1}{1-|z|}, & \text{if } \gamma-\alpha-\beta = 0; \\ (1-|z|)^{\gamma-\alpha-\beta}, & \text{if } \gamma-\alpha-\beta < 0, \end{cases} \tag{15}$$

where $a(z) \approx b(z)$ represents the ratio and $a(z)/b(z)$ has a positive finite limit as $|z| \to 1^-$. Now we list a few formulas for easy reference (see [11, Chapter II]):

$$ {}_2F_1(\alpha,\beta;\gamma;1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}, \quad \operatorname{Re}(\gamma-\alpha-\beta) > 0, \tag{16} $$

$$ {}_2F_1(\alpha,\beta;\gamma;z) = (1-z)^{\gamma-\alpha-\beta}\, {}_2F_1(\gamma-\alpha,\gamma-\beta;\gamma;z), \tag{17} $$

$$ \begin{aligned} {}_2F_1(\alpha,\beta;\gamma;z) &= \frac{\Gamma(\gamma)}{\Gamma(\lambda)\Gamma(\gamma-\lambda)} \\ &\quad\times \int_0^1 t^{\lambda-1}(1-t)^{\gamma-\lambda-1}\, {}_2F_1(\alpha,\beta;\lambda;tz)\,dt, \\ \operatorname{Re}\gamma > \operatorname{Re}\lambda > 0; &\quad |\arg(1-z)| < \pi; \quad z \neq 1. \end{aligned} \tag{18} $$

Lemma 5. *Suppose* $\operatorname{Re}\delta > 0$ *and* $\operatorname{Re}(\lambda+\delta-\alpha-\beta) > 0$. *Then*

$$ \int_0^1 t^{\lambda-1}(1-t)^{\delta-1}\, {}_2F_1(\alpha,\beta;\lambda;t)\,dt = \frac{\Gamma(\lambda)\Gamma(\delta)\Gamma(\lambda+\delta-\alpha-\beta)}{\Gamma(\lambda+\delta-\alpha)\Gamma(\lambda+\delta-\beta)}. \tag{19} $$

Proof. Note that, under the assumption of the lemma, both sides of (18) are continuous at $z = 1$. The lemma then follows by letting $z \to 1$ in (18) and applying (16). □

The following integral formulae concerning the hypergeometric function are significant for our main results. And all these formulae are contained in [12]. Now we list them.

Lemma 6 (see [12, Corollary 2.4]). *For* $\alpha \in \mathbb{R}$ *and* $\gamma > -1$, *we have*

$$ \int_{\mathbb{B}_n} \frac{(1-|w|^2)^{\gamma}}{|1-\langle z,w\rangle|^{2\alpha}}\,d\nu(w) = \frac{n!\Gamma(1+\gamma)}{\Gamma(n+1+\gamma)}\, {}_2F_1(\alpha,\alpha;n+1+\gamma;|z|^2). \tag{20} $$

Lemma 6 is also contained implicitly in the proof of Theorem 1.4.10 in [13] (see the formula in page 19, line 5 of [13]).

Lemma 7 (see [12, Corollary 2.5]). *Suppose that* $\alpha, \beta > 0$, $\gamma \in \mathbb{R}$, *and* $n+\alpha+\beta-2\gamma > 0$. *Then*

$$ \int_{\mathbb{B}_n} |z|^{2\beta}(1-|z|^2)^{\alpha-1} \left\{ \int_{\mathbb{B}_n} \frac{(1-|w|^2)^{\beta-1}}{|1-\langle z,w\rangle|^{2\gamma}}\,d\nu(w) \right\} d\nu(z) = \frac{n(n!)\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\alpha+\beta-2\gamma)}{\Gamma^2(n+\alpha+\beta-\gamma)}. \tag{21} $$

Proof. Using Lemma 6 in the inner integral, we have

$$ \begin{aligned} &\frac{n!\Gamma(\beta)}{\Gamma(n+\beta)} \int_{\mathbb{B}_n} |z|^{2\beta}(1-|z|^2)^{\alpha-1} \times {}_2F_1(\gamma,\gamma;n+\beta;|z|^2)\,d\nu(z) \\ &= \frac{n(n!)\Gamma(\beta)}{\Gamma(n+\beta)} \int_0^1 r^{n+\beta-1}(1-r)^{\alpha-1} \times {}_2F_1(\gamma,\gamma;n+\beta;|z|^2)\,dr. \end{aligned} \tag{22} $$

Then (19) gives the result. □

The following result, usually called Schur's test, is a very effective tool in proving the L^p-boundedness of integral operators. See, for example, [3].

Lemma 8. *Suppose that* (X,μ) *is a* σ*-finite measure space,* $K(x,y)$ *is a nonnegative measurable function on* $X \times X$, *and* T *is the associated integral operator:*

$$ Tf(x) = \int_X K(x,y) f(y)\,d\mu(y). \tag{23} $$

Let $1 < p < \infty$ *and* $1/p + 1/q = 1$. *If there exist a positive constant* C *and a positive measurable function* u *on* X *such that*

$$ \int_X K(x,y) u(y)^q\,d\mu(y) \le Cu(x)^q, \tag{24} $$

for almost every x *in* X, *and*

$$ \int_X K(x,y) u(x)^p\,d\mu(x) \le Cu(y)^p, \tag{25} $$

for almost every y *in* X, *then* T *is bounded on* $L^p(X,\mu)$ *with* $\|T\| \le C$.

3. The Improvement

The section mainly proposes the condition "c is neither 0 nor a negative integer" can be omitted in Theorem A. Notice the condition is only used to give $c \le n+1+a+b$ while proving the necessity for the boundedness of the operator $S_{a,b,c}$ on L_t^p $(1 \le p < \infty)$; see [1, lemma 12]. Now we will give a new proof of the necessity for the boundedness of $S_{a,b,c}$ on L_t^p in Propositions 9 and 11 to introduce the condition can be put off.

Proposition 9. *Suppose the operator* $S_{a,b,c}$ *is bounded on* L_t^p $(1 < p < \infty)$, *and then* $-pa < t+1 < p(b+1)$, $c \le n+1+a+b$.

Proof. Let q be the number such that $1/p + 1/q = 1$. For any fixed $\epsilon > 0$, define

$$ \begin{aligned} g_\epsilon(w) &= C_1(\epsilon)(1-|w|^2)^{(\epsilon-(t+1))/p}, \\ h_\epsilon(z) &= C_2(\epsilon)(1-|z|^2)^{(\epsilon-(t+1))/q}|z|^{2(b+1+(\epsilon-t-1)/p)}, \end{aligned} \tag{26} $$

where

$$C_1(\epsilon) = \left\{ \frac{\Gamma(\epsilon)\Gamma(n+1)}{\Gamma(n+\epsilon)} \right\}^{-1/p}, \tag{27}$$

$$C_2(\epsilon) = \left\{ \frac{n\Gamma(\epsilon)\Gamma(n+q(b+1)+q(\epsilon-(t+1))/p)}{\Gamma(n+q(b+1)+q(\epsilon-(t+1))/p+\epsilon)} \right\}^{-1/q}. \tag{28}$$

Easy calculation shows $\|g_\epsilon\|_{p,t} = \|h_\epsilon\|_{q,t} = 1$. Notice the fact

$$\begin{aligned} &\|S_{a,b,c}\|_{L_t^p \to L_t^p} \\ &= \sup_{\substack{\|f\|_{p,t}=1 \\ \|g\|_{q,t}=1}} \\ &\times \left\{ \left| \int_{\mathbb{B}_n} \left(\int_{\mathbb{B}_n} (1-|z|^2)^a \right.\right.\right. \\ &\qquad \left.\left.\left. \times \frac{(1-|w|^2)^{b-t}}{|1-\langle z,w\rangle|^c} f(w)\, dv_t(w) \right) \overline{g(z)} dv_t(z) \right| \right\}. \end{aligned} \tag{29}$$

Then the boundedness of the operator $S_{a,b,c}$ on L_t^p leads to the integral

$$\begin{aligned} &\left| \int_{\mathbb{B}_n} \left\{ \int_{\mathbb{B}_n} \frac{(1-|z|^2)^a (1-|w|^2)^{b-t}}{|1-\langle z,w\rangle|^c} g_\epsilon(w)\, dv_t(w) \right\} \right. \\ &\qquad \left. \times \overline{h_\epsilon(z)} dv_t(z) \right| \\ &\le \|S_{a,b,c}\|_{L_t^p \to L_t^p} < +\infty. \end{aligned} \tag{30}$$

Hence, using Lemma 7 with $\alpha = a + \epsilon/q + (t+1)/p$, $\beta = b + 1 + (\epsilon - (t+1))/p$, and $\gamma = c/2$, we can conclude that

$$\begin{gathered} a + \frac{\epsilon}{q} + \frac{t+1}{p} > 0, \qquad b + 1 + \frac{\epsilon - (t+1)}{p} > 0, \\ n + 1 + a + b + \epsilon - c > 0. \end{gathered} \tag{31}$$

Then the arbitrariness of ϵ gives

$$-pa \le t+1 \le p(b+1), \qquad c \le n+1+a+b. \tag{32}$$

Now, we will give the proof by dividing into the following two cases.

When $c = n+1+a+b$, by Lemma 7, the integral in (30) equals

$$\begin{aligned} &\frac{n!\Gamma(a+(\epsilon/q)+((t+1)/p))\Gamma(b+1+((\epsilon-(t+1))/p))}{\Gamma^2((n+1+a+b)/2+\epsilon)} \\ &\times \left\{ \frac{\Gamma(n+\epsilon)}{\Gamma(n)} \right\}^{1/p} \\ &\times \left\{ \frac{\Gamma(n+q(b+1)+q(\epsilon-(t+1))/p+\epsilon)}{\Gamma(n+q(b+1)+q(\epsilon-(t+1))/p)} \right\}^{1/q}. \end{aligned} \tag{33}$$

Then letting $\epsilon \to 0^+$, by (30), we can know the limits

$$\begin{aligned} 0 &\le \lim_{\epsilon \to 0^+} \frac{n!\Gamma(a+(\epsilon/q)+((t+1)/p))\Gamma(b+1+((\epsilon-(t+1))/p))}{\Gamma^2((n+1+a+b)/2+\epsilon)} \\ &\le \|S_{a,b,c}\|_{L_t^p \to L_t^p}. \end{aligned} \tag{34}$$

Then the boundedness of the operator $S_{a,b,c}$ gives $-pa < t+1 < p(b+1)$.

When $c < n+1+a+b$, take the function

$$f_\lambda(z) = (1-|z|^2)^\lambda, \tag{35}$$

with $\lambda > a$. The condition (32) implies the function $f_\lambda \in L_t^p$. And using Lemma 6, we have

$$\begin{aligned} S_{a,b,c} f_\lambda(z) &= (1-|z|^2)^a \int_{\mathbb{B}_n} \frac{(1-|w|^2)^b}{|1-\langle z,w\rangle|^c} f_\lambda(w)\, dv(w) \\ &= (1-|z|^2)^a \frac{n!\Gamma(1+b+\lambda)}{\Gamma(n+1+b+\lambda)} \\ &\quad \times {}_2F_1\left(\frac{c}{2}, \frac{c}{2}; n+1+b+\lambda; |z|^2\right). \end{aligned} \tag{36}$$

According to (15), we can obtain that $S_{a,b,c} f_\lambda(z) \approx (1-|z|^2)^a$. Thus the boundedness of the operator $S_{a,b,c}$ on $L_t^p(\mathbb{B}_n)$ gives that $pa + t > -1$; that is, $-pa < t+1$. Now we consider the adjoint operator $S_{a,b,c}^*$ of the operator $S_{a,b,c}$; that is,

$$S_{a,b,c}^* f(z) = (1-|z|^2)^{b-t} \int_{\mathbb{B}_n} \frac{(1-|w|^2)^{a+t}}{|1-\langle z,w\rangle|^c} f(w)\, dv(w). \tag{37}$$

The boundedness of $S_{a,b,c}$ on L_t^p implies the boundedness of $S_{a,b,c}^*$ on L_t^q. With the similar discussion above, we can obtain that $q(b-t) + t > -1$; that is, $t+1 < p(b+1)$. □

When $c = n+1+a+b$, (34) implies the following result.

Corollary 10. *Suppose $c = n+1+a+b$ and $1 < p < \infty$, $-pa < t+1 < p(b+1)$, and then*

$$\|S_{a,b,c}\|_{L_t^p \to L_t^p} \ge \frac{n!\Gamma(a+(t+1)/p)\Gamma(b+1-(t+1)/p)}{\Gamma^2((n+1+a+b)/2)}. \tag{38}$$

Proposition 11. *The operator $S_{a,b,c}$ is bounded on L_t^1 if and only if $-a < t+1 < b+1, c = n+1+a+b$ or $-a < t+1 \le b+1, c < n+1+a+b$. And when $c = n+1+a+b$, we have*

$$\|S_{a,b,c}\|_{L_t^1 \to L_t^1} = \frac{n!\Gamma(1+a+t)\Gamma(b-t)}{\Gamma^2((n+1+a+b)/2)}. \tag{39}$$

When $c < n+1+a+b$, $-a < t+1 = b+1$, we have

$$\|S_{a,b,c}\|_{L_t^1 \to L_t^1} = \frac{n!\Gamma(1+a+b)\Gamma(\sigma)}{\Gamma^2((n+1+a+b+\sigma)/2)}, \tag{40}$$

where $\sigma = (n+1+a+b) - c$.

Proof. By Lemma 6, we have

$$\begin{aligned}\|S_{a,b,c}\|_{L_t^1\to L_t^1} &= \left\|S_{a,b,c}^*\right\|_{L^\infty\to L^\infty} \\ &= \sup_{z\in\mathbb{B}_n}\left(1-|z|^2\right)^{b-t}\int_{\mathbb{B}_n}\frac{\left(1-|w|^2\right)^{a+t}}{|1-\langle z,w\rangle|^c}d\nu(w) \\ &= \frac{n!\Gamma(1+a+t)}{\Gamma(n+1+a+t)}\sup_{z\in\mathbb{B}_n}\left(1-|z|^2\right)^{b-t} \\ &\quad\times {}_2F_1\left(\frac{c}{2},\frac{c}{2};n+1+a+t;|z|^2\right),\end{aligned} \tag{41}$$

where $S_{a,b,c}^*$ denotes the adjoint operator of $S_{a,b,c}$. Then, using (15), we can obtain that the operator $S_{a,b,c}$ is bounded on L_t^1 if and only if

$$\begin{gathered}1+a+t>0,\\ b-t>0,\\ n+1+a+t-c\geq t-b,\end{gathered} \tag{42}$$

or

$$\begin{gathered}1+a+t>0,\\ b-t=0,\\ n+1+a+t-c>0,\end{gathered} \tag{43}$$

which gives the first part of the proposition.

Now we will give the second part. When $c<n+1+a+b$ and $-a<t+1=b+1$, the hypergeometric function in (41) is increasing since its Taylor coefficients are all positive. Applying (16), we have (40). When $c=n+1+a+b$, (17) gives

$$\begin{aligned}&{}_2F_1\left(\frac{n+1+a+b}{2},\frac{n+1+a+b}{2};n+1+a+t;|z|^2\right)\\ &\quad=\left(1-|z|^2\right)^{t-b}\\ &\quad\times{}_2F_1\left(\frac{n+1+a-b}{2}+t,\frac{n+1+a-b}{2}+t;\right.\\ &\qquad\left.n+1+a+t;|z|^2\right).\end{aligned} \tag{44}$$

Thus (41), the increase of the last hypergeometric function, and (16) lead to

$$\begin{aligned}&\|S_{a,b,c}\|_{L_t^1\to L_t^1}\\ &\quad=\left\|S_{a,b,c}^*\right\|_{L^\infty\to L^\infty}\\ &\quad=\frac{n!\Gamma(1+a+t)}{\Gamma(n+1+a+t)}\\ &\quad\times{}_2F_1\left(\frac{n+1+a-b}{2}+t,\frac{n+1+a-b}{2}+t;\right.\\ &\qquad\left.n+1+a+t;1\right)\\ &\quad=\frac{n!\Gamma(1+a+t)\Gamma(b-t)}{\Gamma^2((n+1+a+b)/2)}.\end{aligned} \tag{45}$$

□

4. The Proof of Theorems 1 and 2

Proof of Theorems 1 and 2. Since

$$\|S_{a,b,c}\|_{L^\infty\to L^\infty}=\sup_{z\in\mathbb{B}_n}\left(1-|z|^2\right)^a\int_{\mathbb{B}_n}\frac{\left(1-|w|^2\right)^b}{|1-\langle z,w\rangle|^c}d\nu(w), \tag{46}$$

therefore Theorem 2 comes out as the same discussion as Proposition 11.

Next, we will concentrate on the proof of Theorem 1. Remember the hypothesis $c=n+1+a+b$ throughout the following proof. Since (39) gives the case of $p=1$, for the case $1<p<\infty$, Corollary 10 gives the lower bound of $\|S_{a,b,c}\|_{L_t^p\to L_t^p}$. Thus we only show the fact

$$\|S_{a,b,c}\|_{L_t^p\to L_t^p}\leq\frac{n!\Gamma(a+(t+1)/p)\Gamma(b+1-(t+1)/p)}{\Gamma^2((n+1+a+b)/2)}. \tag{47}$$

To this end, we will use Schur's test (Lemma 8) with

$$K(z,w)=\frac{\left(1-|z|^2\right)^a\left(1-|w|^2\right)^{b-t}}{|1-\langle z,w\rangle|^{n+1+a+b}}. \tag{48}$$

Set

$$u_t(z)=\left(1-|z|^2\right)^{-(t+1)/(pq)}, \tag{49}$$

where q is the conjugate exponent of p such that $1/p+1/q=1$. It then suffices to show

$$\begin{aligned}&\left(1-|z|^2\right)^a\int_{\mathbb{B}_n}\frac{\left(1-|w|^2\right)^{b-t}}{|1-\langle z,w\rangle|^{n+1+a+b}}u_t(w)^q d\nu_t(w)\\ &\quad\leq\frac{n!\Gamma(a+(t+1)/p)\Gamma(b+1-(t+1)/p)}{\Gamma^2((n+1+a+b)/2)}u_t(z)^q,\end{aligned} \tag{50}$$

for all $z\in\mathbb{B}_n$, and

$$\begin{aligned}&\left(1-|w|^2\right)^{b-t}\int_{\mathbb{B}_n}\frac{\left(1-|z|^2\right)^a}{|1-\langle z,w\rangle|^{n+1+a+b}}u_t(z)^p d\nu_t(z)\\ &\quad\leq\frac{n!\Gamma(a+(t+1)/p)\Gamma(b+1-(t+1)/p)}{\Gamma^2((n+1+a+b)/2)}u_t(w)^p\end{aligned} \tag{51}$$

for all $w \in \mathbb{B}_n$. We only prove (50), since (51) comes from the same way as (50). Applying Lemma 6 and (17), we have

$$\begin{aligned}
&\left(1-|z|^2\right)^a \int_{\mathbb{B}_n} \frac{\left(1-|w|^2\right)^{b-t}}{|1-\langle z,w\rangle|^{n+1+a+b}} u_t(w)^q d\nu_t(w) \\
&= \left(1-|z|^2\right)^a \frac{n!\Gamma\left(b+1-(t+1)/p\right)}{\Gamma\left(n+b+1-(t+1)/p\right)} \\
&\quad \times {}_2F_1\left(\frac{n+1+a+b}{2}, \frac{n+1+a+b}{2}; \right. \\
&\quad \left. n+b+1-\frac{t+1}{p}; |z|^2\right) \\
&= \frac{n!\Gamma\left(b+1-(t+1)/p\right)}{\Gamma\left(n+b+1-(t+1)/p\right)} \left(1-|z|^2\right)^{-(t+1)/p} \\
&\quad \times {}_2F_1\left(\frac{n+1+b-a}{2} - \frac{t+1}{p}, \frac{n+1+b-a}{2} \right. \\
&\quad \left. -\frac{t+1}{p}; n+1+b-\frac{t+1}{p}; |z|^2\right).
\end{aligned} \tag{52}$$

By (16), the last hypergeometric function is bounded from the above by

$$\begin{aligned}
&{}_2F_1\left(\frac{n+1+b-a}{2} - \frac{t+1}{p}, \frac{n+1+b-a}{2} \right. \\
&\quad \left. -\frac{t+1}{p}; n+1+b-\frac{t+1}{p}; 1\right) \\
&= \frac{\Gamma\left(n+1+b-(t+1)/p\right)\Gamma\left(a+(t+1)/p\right)}{\Gamma^2\left((n+1+a+b)/2\right)},
\end{aligned} \tag{53}$$

since it is increasing on the interval $[0,1)$. This proves (50), which in turn implies (47). The proof is completed. □

5. Remark

The topic on the exact norm of an operator is an interesting but difficult problem. In this note, we only give the accurate norm of the generalized operator $S_{a,b,c}$ on L_t^p under $c = n + 1 + a + b$. But for other cases, except the particular case (40), we can give an upper bound of $\|S_{a,b,c}\|_{L_t^p \to L_t^p}$ by Theorem 1 according to the fact

$$\frac{\left(1-|z|^2\right)^a\left(1-|w|^2\right)^{b-t}}{|1-\langle z,w\rangle|^c} \le \frac{2^\sigma\left(1-|z|^2\right)^a\left(1-|w|^2\right)^{b-t}}{|1-\langle z,w\rangle|^{n+1+a+b}} \tag{54}$$

and a lower bound for one fixed $\epsilon > 0$ by (30) and Lemma 7; thus the problem of the norm of other cases may be left as an open problem to consider.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Blow-Up Criterion of Weak Solutions for the 3D Boussinesq Equations

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The Boussinesq equations describe the three-dimensional incompressible fluid moving under the gravity and the earth rotation which come from atmospheric or oceanographic turbulence where rotation and stratification play an important role. In this paper, we investigate the Cauchy problem of the three-dimensional incompressible Boussinesq equations. By commutator estimate, some interpolation inequality, and embedding theorem, we establish a blow-up criterion of weak solutions in terms of the pressure p in the homogeneous Besov space $\dot{B}^{0}_{\infty,\infty}$.

1. Introduction

This paper is devoted to establish a blow-up criterion of weak solutions to the Cauchy problem for 3-dimensional Boussinesq equations:

$$u_t + u\cdot\nabla u - \eta\Delta u + \nabla p = \theta e_3, \tag{1}$$

$$\theta_t + u\cdot\nabla\theta - \nu\Delta\theta = 0, \tag{2}$$

$$\nabla\cdot u = 0, \tag{3}$$

$$t = 0 : u = u_0(x), \qquad \theta = \theta_0(x), \tag{4}$$

where u is the velocity, p is the pressure, and θ is the small temperature deviations which depends on the density. $\eta \geq 0$ is the viscosity, $\nu \geq 0$ is called the molecular diffusivity, and $e_3 = (0,0,1)^T$. The above systems describe the evolution of the velocity field u for a three-dimensional incompressible fluid moving under the gravity and the earth rotation which come from atmospheric or oceanographic turbulence where rotation and stratification play an important role.

When the initial density θ_0 is identically zero (or constant) and $\eta = 0$, then (1)–(4) reduces to the classical incompressible Euler equation:

$$\begin{gathered} u_t + u\cdot\nabla u + \nabla p = 0, \\ \nabla\cdot u = 0, \\ u(x,t)|_{t=0} = u_0(x). \end{gathered} \tag{5}$$

From the investigation of (5), we cannot expect to have a better theory for the Boussinesq system than that of the Euler equation. For the Euler equation, a well-known criterion for the existence of global smooth solutions is the Beale-Kato-Majda criterion [1]. It states that the control of the vorticity of the fluid $\omega = \operatorname{curl} u$ in $L^1(0,T;L^\infty)$ is sufficient to get the global well posedness.

The Boussinesq equations (1)–(4) are of relevance to study a number of models coming from atmospheric or oceanographic turbulence where rotation and stratification play an important role. The scalar function θ may for instance represent temperature variation in a gravity field and θe_3

the buoyancy force. For the regularity criteria of the Navier-Stokes equations, we can refer to Zhou et al. [2–9], Fan and Ozawa [10], He [11], Zhang and Chen [12], and Escauriaza et al. [13].

From the mathematical point of view, the global well posedness for two-dimensional Boussinesq equations which has recently drawn much attention seems to be in a satisfactory state. More precisely, global well posedness has been shown in various function spaces and for different viscosities; we refer, for example, to [14–19]. In contrast, in the case when $\eta = \nu = 0$, the Boussinesq system exhibits vorticity intensification and the global well-posedness issue remains an unsolved challenging open problem (except if θ_0 is a constant, of course) which may be formally compared to the similar problem for the three-dimensional axisymmetric Euler equations with swirl.

In the three-dimensional case, there are only few results (see [20–24]). Hmidi and Rousset [23] proved the global well-posedness for the three-dimensional Euler-Boussinesq equations with axisymmetric initial data without swirl. Danchin and Paicu [20] obtained a global existence and uniqueness result for small data in Lorentz space.

Our purpose of this paper is to obtain a blow-up criterion of weak solutions in terms of Besov space.

Now, we state our result as follows.

Theorem 1. *Assume that* $(u_0, \theta_0) \in H^3(R^3)$ *with* $\operatorname{div} u_0 = 0$ *in* R^3. *Assume that the pressure p satisfies the condition*

$$\int_0^T \frac{\|\nabla p(t)\|_{\dot{B}^0_{\infty,\infty}}^{2/3}}{\left(1 + \ln\left(1 + \|\nabla p(t)\|_{\dot{B}^0_{\infty,\infty}}\right)\right)^{2/3}} dt < +\infty; \tag{6}$$

then the solution (u, θ) *can be extended smoothly only up to* T.

The paper is organized as follows. We first state some important inequalities in Section 2. We will prove Theorem 1 in Section 3.

2. Preliminaries

Throughout this paper, we use the following usual notations. $L^p(R^3)$ denotes the Lebesgue space and $H^m(R^3)$ denotes the standard Sobolev space. BMO denotes the space of bounded mean oscillations. $\dot{B}^0_{m,n}$ is the homogeneous Besov space, where $0 \le m, n \le +\infty$.

Lemma 2. *There exists a uniform positive constant C, such that*

$$\|f\|_{L^4}^2 \le C \|f\|_{L^2} \|f\|_{BMO}, \tag{7}$$

$$\|f\|_{\dot{B}^0_{\infty,2}} \le C\left(1 + \|f\|_{\dot{B}^0_{\infty,\infty}} \ln^{1/2}\left(e + \|f\|_{H^{s-1}}\right)\right) \tag{8}$$

hold for all vectors $f \in H^{s-1}(R^3)$ *with* $s > 5/2$.

Proof. See, for example, [19] or [25]. □

Lemma 3. *From (1), one has*

$$\begin{aligned} \|\nabla p\|_{L^2} &\le C\left(\|u \cdot \nabla u\|_{L^2} + \|\theta\|_{L^2}\right), \\ \|\nabla p\|_{L^2}^{1/2} &\le C\left(\|u \cdot \nabla u\|_{L^2}^{1/2} + \|\theta\|_{L^2}^{1/2}\right). \end{aligned} \tag{9}$$

Lemma 4. *Assume that* $\Lambda = (-\Delta)^{1/2}$; *one has the commutator estimate due to Kato and Ponce [24]:*

$$\begin{aligned} &\|\Lambda^s(fg) - f\Lambda^s g\|_{L^p} \\ &\quad \le C\left(\|\nabla f\|_{L^{p_1}} \left\|\lambda^{s-1} g\right\|_{L^{q_1}} + \|\Lambda^s f\|_{L^{p_2}} \|g\|_{L^{q_2}}\right), \end{aligned} \tag{10}$$

with $s > 0$, $1/p = 1/p_1 + 1/q_1 = 1/p_2 + 1/q_2$.

Lemma 5 (the Gagliardo-Nirenberg inequality). *Consider*

$$\|\nabla f\|_{L^4} \le C \|f\|_{L^4}^{1/5} \|\Delta f\|_{L^2}^{4/5}, \tag{11}$$

$$\|\nabla f\|_{L^3} \le C \|\nabla f\|_{L^2}^{3/4} \left\|\Lambda^3 f\right\|_{L^2}^{1/4}, \tag{12}$$

$$\left\|\Lambda^3 f\right\|_{L^3} \le C \|\nabla f\|_{L^2}^{1/6} \left\|\Lambda^4 f\right\|_{L^2}^{5/6}. \tag{13}$$

3. Proof of Theorem 1

Proof of Theorem 1. Multiplying (1) by u, using (3), and integrating in R^3, we derive

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \eta\|\nabla u\|_{L^2}^2 \\ &\quad = \int_{R^3} \theta e_3 \cdot u \, dx \le \|\theta\|_{L^2} \|u\|_{L^2} \\ &\quad \le \frac{1}{2}\|\theta\|_{L^2}^2 + \frac{1}{2}\|u\|_{L^2}^2. \end{aligned} \tag{14}$$

Multiplying (2) by θ, using (3), and integrating in R^3, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\theta\|_{L^2}^2 + \nu\|\nabla\theta\|_{L^2} = 0. \tag{15}$$

Combining (14) and (15), using the Gronwall inequality, we deduce that

$$\begin{aligned} \|u\|_{L^\infty(0,T;L^2)} + \|u\|_{L^2(0,T;H^1)} &\le C, \\ \|\theta\|_{L^\infty(0,T;L^2)} + \|\theta\|_{L^2(0,T;H^1)} &\le C. \end{aligned} \tag{16}$$

Multiplying (1) by $|u|^2 u$, using (3) and (7), and integrating in R^3, we derive

$$\begin{aligned}
&\int \left[|u|^2 \cdot u \left(u_t + u \cdot \nabla u - \eta \Delta u + \nabla p \right) \right] \\
&= \frac{1}{4} \frac{d}{dt} \int |u|^4 \, dx + \int |u|^2 \cdot u^2 \cdot \nabla u \, dx \\
&\quad + \frac{\eta}{2} \int \left(\nabla |u|^2 \right)^2 dx \\
&\quad + \eta \int |u|^2 |\nabla u|^2 \, dx + \int (u \cdot \nabla p) |u|^2 \, dx \\
&= \int |u|^2 \cdot u \cdot \theta e_3 dx \\
&\leq C \int \left(|u|^4 + |\theta|^4 \right) dx;
\end{aligned} \tag{17}$$

that is,

$$\begin{aligned}
&\frac{1}{4} \frac{d}{dt} \int |u|^4 \, dx + \frac{\eta}{2} \int \left(\nabla |u|^2 \right)^2 dx + \eta \int |u|^2 |\nabla u|^2 \, dx \\
&\quad \leq - \int (u \cdot \nabla p) |u|^2 \, dx + C \int |u|^4 + |\theta|^4 \, dx \\
&\quad \leq \|u\|_{L^4}^3 \|\nabla p\|_{L^4} + C \|u\|_{L^4}^4 + C \|\theta\|_{L^4}^4 \\
&\quad \leq C \|u\|_{L^4}^3 \|\nabla p\|_{L^2}^{1/2} \|\nabla p\|_{\mathrm{BMO}}^{1/2} + C \|u\|_{L^4}^4 + C \|\theta\|_{L^4}^4 .
\end{aligned} \tag{18}$$

Multiplying (2) by $|\theta|^2 \theta$, using (3), and integrating in R^3, we arrive at

$$\begin{aligned}
&\int \left(|\theta|^2 \cdot \theta \cdot \theta_t + |\theta|^2 \theta \cdot u \cdot \nabla \theta - \nu |\theta|^2 \cdot \theta \cdot \Delta \theta \right) dx \\
&\quad = \frac{1}{4} \frac{d}{dt} \int |\theta|^4 \, dx + \nu \int |\theta|^2 (\nabla \theta)^2 \, dx \\
&\qquad + \frac{\nu}{2} \int |\theta|^2 (\operatorname{div} \theta)^2 \, dx.
\end{aligned} \tag{19}$$

Combining (18) and (19), using (9) and (16), we derive that

$$\begin{aligned}
&\frac{1}{4} \frac{d}{dt} \int \left(|\theta|^4 + |u|^4 \right) dx + \frac{\eta}{2} \int \left(\nabla |u|^2 \right)^2 dx \\
&\qquad + \eta \int |u|^2 |\nabla u|^2 \, dx + \nu \int |\theta|^2 (\nabla \theta)^2 \, dx \\
&\qquad + \frac{\nu}{2} \int |\theta|^2 (\operatorname{div} \theta)^2 \, dx, \\
&\quad \leq C \|u\|_{L^4}^3 \|\nabla p\|_{L^2}^{1/2} \|\nabla p\|_{\mathrm{BMO}}^{1/2} + C \|u\|_{L^4}^4 + C \|\theta\|_{L^4}^4 \\
&\quad \leq C \|u\|_{L^4}^3 \left(\|u \cdot \nabla u\|_{L^2}^{1/2} + \|\theta\|_{L^2}^{1/2} \right) \|\nabla p\|_{\mathrm{BMO}}^{1/2} \\
&\qquad + C \|u\|_{L^4}^4 + C \|\theta\|_{L^4}^4 \\
&\quad \leq 2C \|u\|_{L^4}^4 \|\nabla p\|_{\mathrm{BMO}}^{2/3} + \frac{\eta}{2} \||u| \nabla u\|_{L^2}^2 \\
&\qquad + C \|\theta\|_{L^2}^2 + C \|u\|_{L^4}^4 + C \|\theta\|_{L^4}^4 ,
\end{aligned} \tag{20}$$

which implies

$$\begin{aligned}
&\frac{d}{dt} \int \left(|\theta|^4 + |u|^4 \right) dx + \eta \int \left(\nabla |u|^2 \right)^2 dx \\
&\qquad + \eta \int |u|^2 |\nabla u|^2 \, dx + \nu \int |\theta|^2 (\nabla \theta)^2 \, dx \\
&\qquad + \nu \int |\theta|^2 (\operatorname{div} \theta)^2 \, dx \\
&\quad \leq 8C \|u\|_{L^4}^4 \|\nabla p\|_{\mathrm{BMO}}^{2/3} + 4C \|\theta\|_{L^2}^2 \\
&\qquad + 4C \|u\|_{L^4}^4 + 4C \|\theta\|_{L^4}^4 \\
&\quad \leq 8C \|u\|_{L^4}^4 \|\nabla p\|_{\dot{B}^0_{\infty,\infty}}^{2/3} \ln^{1/3} \left(1 + \|\nabla p\|_{H^2} \right) \\
&\qquad + 4C \|\theta\|_{L^2}^2 + 4C \|u\|_{L^4}^4 + 4C \|\theta\|_{L^4}^4 \\
&\quad \leq 8C \|u\|_{L^4}^4 \|\nabla p\|_{\dot{B}^0_{\infty,\infty}}^{2/3} \\
&\qquad \times \ln^{1/3} \left(1 + \|\nabla \Delta u\|_{L^2} + \|\Delta \theta\|_{L^2} \right) \\
&\qquad + 4C \|\theta\|_{L^2}^2 + 4C \|u\|_{L^4}^4 + 4C \|\theta\|_{L^4}^4 \\
&\quad \leq 8C \|u\|_{L^4}^4 \frac{\|\nabla p\|_{\dot{B}^0_{\infty,\infty}}^{2/3}}{\left(1 + \ln \left(1 + \|\nabla p\|_{\dot{B}^0_{\infty,\infty}} \right) \right)^{2/3}} \\
&\qquad \times \ln \left(1 + \|\nabla \Delta u\|_{L^2} + \|\Delta \theta\|_{L^2} \right) \\
&\qquad + 4C \|\theta\|_{L^2}^2 + 4C \|u\|_{L^4}^4 + 4C \|\theta\|_{L^4}^4 .
\end{aligned} \tag{21}$$

Choosing $t \in [T_*, T]$ and setting

$$y(t) = \sup_{t \in [T_*, T]} \left(\|\nabla \cdot \Delta u(t)\|_{L^2} + \|\Delta \theta\|_{L^2} \right), \tag{22}$$

we have

$$\sup_{t \in [T_*, T]} \left(\|u\|_{L^4} + \|\theta\|_{L^4} \right) \leq C_* \left(1 + y(t) \right)^{C\varepsilon}, \tag{23}$$

where ε is a small enough constant, such that

$$\int_{T_*}^{T} \frac{\|\nabla p\|_{\dot{B}^0_{\infty,\infty}}^{2/3}}{\left(1 + \ln \left(1 + \|\nabla p\|_{\dot{B}^0_{\infty,\infty}} \right) \right)^{2/3}} dt < \varepsilon. \tag{24}$$

Next, we want to estimate the L^2-norm of ∇u and $\nabla \theta$.

Multiplying (1) by $-\Delta u$, integrating in R^3, and using (3) and (11), we derive that

$$\begin{aligned}
&\int u_t \cdot (-\Delta u) \, dx + \int (u \cdot \nabla u) (-\Delta u) \, dx \\
&\qquad + \eta \|\Delta u\|_{L^2}^2 + \int \nabla p \cdot (-\Delta u) \, dx \\
&\quad = - \int \theta e_3 \cdot \Delta u \, dx;
\end{aligned} \tag{25}$$

that is,

$$\frac{1}{2}\frac{d}{dt}\int|\nabla u|^2\,dx+\eta\int|\Delta u|^2\,dx$$
$$=\int(u\cdot\nabla u)\,\Delta u\,dx-\int\theta e_3\Delta u\,dx$$
$$\leq\|u\|_{L^4}\|\nabla u\|_{L^4}\|\Delta u\|_{L^2}+\|\Delta u\|_{L^2}\|\theta\|_{L^2}$$
$$\leq C\|u\|_{L^4}\|u\|_{L^4}^{1/5}\|\Delta u\|_{L^2}^{4/5}\|\Delta u\|_{L^2}+\frac{\eta}{4}\|\Delta u\|_{L^2}^2+C\|\theta\|_{L^2}^2$$
$$\leq\frac{\eta}{4}\|\Delta u\|_{L^2}^2+C\|u\|_{L^4}^{12}+\frac{\eta}{4}\|\Delta u\|_{L^2}^2+C\|\theta\|_{L^2}\|\Delta\theta\|_{L^2}$$
$$\leq\frac{\eta}{2}\|\Delta u\|_{L^2}^2+C\|u\|_{L^4}^{12}+\frac{\nu}{4}\|\Delta\theta\|_{L^2}^2+C\|\theta\|_{L^2}^2. \tag{26}$$

Multiplying (2) by $-\Delta\theta$, integrating in R^3, and using (3) and (11), we derive that

$$\int\theta_t\cdot(-\Delta\theta)\,dx+\int(u\cdot\nabla\theta)(-\Delta\theta)\,dx+\nu\|\Delta\theta\|_{L^2}^2=0; \tag{27}$$

that is,

$$\frac{1}{2}\frac{d}{dt}\int|\nabla\theta|^2\,dx+\nu\int|\Delta\theta|^2\,dx$$
$$=\int(u\cdot\nabla\theta)\,\Delta\theta+\frac{\eta}{4}\|\Delta u\|_{L^2}^2+C\|\theta\|_{L^2}^2\,dx$$
$$\leq\|u\|_{L^4}\|\nabla\theta\|_{L^4}\|\Delta\theta\|_{L^2} \tag{28}$$
$$\leq C\|u\|_{L^4}\|\theta\|_{L^4}^{1/5}\|\Delta\theta\|_{L^2}^{4/5}\|\Delta\theta\|_{L^2}$$
$$\leq\frac{\nu}{4}\|\Delta\theta\|_{L^2}^2+C\|u\|_{L^4}^{10}\|\theta\|_{L^4}^2.$$

Combining (26) and (28), using (16), we deduce

$$\frac{1}{2}\frac{d}{dt}\int\left(|\nabla u|^2+|\nabla\theta|^2\right)dx+\eta\int|\Delta u|^2\,dx+\nu\int|\Delta\theta|^2\,dx$$
$$\leq\frac{\eta}{2}\|\Delta u\|_{L^2}^2+C\|u\|_{L^4}^{12}+\frac{\nu}{4}\|\Delta\theta\|_{L^2}^2+C\|\theta\|_{L^2}^2$$
$$+\frac{\nu}{4}\|\Delta\theta\|_{L^2}^2+C\|u\|_{L^4}^{10}\|\theta\|_{L^4}^2$$
$$=\frac{\eta}{2}\|\Delta u\|_{L^2}^2+\frac{\nu}{2}\|\Delta\theta\|_{L^2}^2+C\|u\|_{L^4}^{12}+C\|\theta\|_{L^2}^2$$
$$+C\|u\|_{L^4}^{10}\|\theta\|_{L^4}^2; \tag{29}$$

that is,

$$\frac{d}{dt}\int\left(|\nabla u|^2+|\nabla\theta|^2\right)dx+\eta\int|\Delta u|^2\,dx+\nu\int|\Delta\theta|^2\,dx \tag{30}$$
$$\leq 2C\|u\|_{L^4}^{12}+2C\|\theta\|_{L^2}^2+2C\|u\|_{L^4}^{10}\|\theta\|_{L^4}^2,$$

which implies that

$$\|\nabla u(t,\cdot)\|_{L^2}^2+\|\nabla\theta(t,\cdot)\|_{L^2}^2\leq C\left(1+y(t)\right)^{C\varepsilon}. \tag{31}$$

Last, we will estimate the H^3-norm and H^4-norm of u and θ and use the operator Λ to derive our goal.

Applying $\Lambda^3=(-\Delta)^{3/2}$ to (1) and then multiplying (1) with $\Lambda^3 u$, we deduce

$$\int\Lambda^3u_t\cdot\Lambda^3u\,dx+\int\Lambda^3(u\cdot\nabla u)\cdot\Lambda^3u\,dx$$
$$-\eta\int\Lambda^3\Delta u\cdot\Lambda^3u\,dx+\int\Lambda^3\nabla p\left(\Lambda^3u\right)dx \tag{32}$$
$$=\int\Lambda^3\theta e_3\cdot\Lambda^3u\,dx;$$

that is,

$$\frac{1}{2}\frac{d}{dt}\int\left|\Lambda^3u\right|^2dx+\eta\int\left|\Lambda^4u\right|^2dx$$
$$=-\int\left[\Lambda^3(u\cdot\nabla u)-u\cdot\nabla\Lambda^3u\right]\cdot\Lambda^3u\,dx+\int\Lambda^3\theta e_3\Lambda^3u\,dx$$
$$\leq C\left(\|\nabla u\|_{L^3}\left\|\Lambda^3u\right\|_{L^3}^2+\left\|\Lambda^3u\right\|_{L^2}\left\|\Lambda^4u\right\|_{L^2}\right)+\left\|\Lambda^3u\right\|_{L^2}\|\theta\|_{L^2}$$
$$\leq C\|\nabla u\|_{L^2}^{3/4}\left\|\Lambda^3u\right\|_{L^2}^{1/4}\|\nabla u\|_{L^2}^{1/3}\left\|\Lambda^4u\right\|_{L^2}^{5/3}$$
$$+\left\|\Lambda^3u\right\|_{L^2}\|\theta\|_{L^2}+C\left\|\Lambda^3u\right\|_{L^2}^2+\frac{\eta}{16}\left\|\Lambda^4u\right\|_{L^2}$$
$$\leq C\|\nabla u\|_{L^2}^{13/12}\left\|\Lambda^3u\right\|_{L^2}^{1/4}\left\|\Lambda^4u\right\|_{L^2}^{5/3}$$
$$+C\left\|\Lambda^3u\right\|_{L^2}^2+C\|\theta\|_{L^2}+C\left\|\Lambda^3u\right\|_{L^2}^2+\frac{\eta}{16}\left\|\Lambda^4u\right\|_{L^2}$$
$$\leq\frac{\eta}{16}\left\|\Lambda^4u\right\|_{L^2}+C\|\nabla u\|_{L^2}^{13/10}\left\|\Lambda^3u\right\|_{L^2}^{1/2}$$
$$+2C\left\|\Lambda^3u\right\|_{L^2}^2+C\|\theta\|_{L^2}+\frac{\eta}{16}\left\|\Lambda^4u\right\|_{L^2}$$
$$\leq C\|\nabla u\|_{L^2}^{13/12}\left\|\Lambda^3u\right\|_{L^2}^{1/4}\left\|\Lambda^4u\right\|_{L^2}^{5/3}+C\left\|\Lambda^3u\right\|_{L^2}^2$$
$$+C\|\theta\|_{L^2}+C\left\|\Lambda^3u\right\|_{L^2}^2+\frac{\eta}{8}\left\|\Lambda^4u\right\|_{L^2}$$
$$=\frac{\eta}{4}\left\|\Lambda^4u\right\|_{L^2}^2+C\|\nabla u\|_{L^2}^{13/10}\left\|\Lambda^3u\right\|_{L^2}^{1/2}$$
$$+2C\left\|\Lambda^3u\right\|_{L^2}^2+C\|\theta\|_{L^2}. \tag{33}$$

Similarly, applying Λ^3 to (2) and multiplying (2) by $\Lambda^3\theta$, we derive

$$\int\Lambda^3\theta_t\left(\Lambda^3\theta\right)dx+\int\Lambda^3(u\cdot\nabla\theta)\,\Lambda^3\theta\,dx \tag{34}$$
$$-\nu\int\Lambda^3\Delta\theta\cdot\Lambda^3\theta\,dx=0;$$

that is,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\left|\Lambda^3\theta\right|^2 dx+\nu\int\left|\Lambda^4\theta\right|^2 dx\\
&\quad=-\int\Lambda^3\left(u\cdot\nabla\theta\right)\Lambda^3\theta\,dx\\
&\quad=C\left\|\nabla u\right\|_{L^3}\left\|\Lambda^3\theta\right\|_{L^3}^2+\left\|\Lambda^3 u\right\|_{L^3}\left\|\nabla\theta\right\|_{L^3}\left\|\Lambda^3\theta\right\|_{L^3}\\
&\quad\le C\left\|\nabla u\right\|_{L^2}^{3/4}\left\|\Lambda^3 u\right\|_{L^2}^{1/4}\left\|\nabla\theta\right\|_{L^2}^{1/3}\left\|\Lambda^4\theta\right\|_{L^2}^{5/3}\\
&\qquad+C\left\|\nabla u\right\|_{L^2}^{1/6}\left\|\Lambda^4 u\right\|_{L^2}^{5/6}\left\|\nabla\theta\right\|_{L^2}^{3/4}\left\|\Lambda^3\theta\right\|_{L^2}^{1/4}\\
&\qquad\times\left\|\nabla\theta\right\|_{L^2}^{1/6}\left\|\Lambda^4\theta\right\|_{L^2}^{5/6}\\
&\quad\le C\left\|\nabla u\right\|_{L^2}^{9/2}\left\|\Lambda^3 u\right\|_{L^2}^{3/2}\left\|\nabla\theta\right\|_{L^2}^{2}+\frac{\nu}{4}\left\|\Lambda^4\theta\right\|_{L^2}^2\\
&\qquad+C\left\|\nabla u\right\|_{L^2}^{1/3}\left\|\Lambda^4 u\right\|_{L^2}^{5/3}\\
&\qquad+C\left\|\nabla\theta\right\|_{L^2}^{3/2}\left\|\Lambda^3\theta\right\|_{L^2}^{1/2}\left\|\nabla\theta\right\|_{L^2}^{1/3}\left\|\Lambda^4\theta\right\|_{L^2}^{5/3}\\
&\quad\le C\left\|\nabla u\right\|_{L^2}^{9/2}\left\|\Lambda^3 u\right\|_{L^2}^{3/2}\left\|\nabla\theta\right\|_{L^2}^{2}+\frac{\nu}{4}\left\|\Lambda^4\theta\right\|_{L^2}^2\\
&\qquad+C\left\|\nabla u\right\|_{L^2}^{2}+\frac{\eta}{4}\left\|\Lambda^4 u\right\|_{L^2}^2\\
&\qquad+C\left\|\nabla\theta\right\|_{L^2}^{9}\left\|\Lambda^3\theta\right\|_{L^2}^{3}\left\|\nabla\theta\right\|_{L^2}^{2}+\frac{\nu}{4}\left\|\Lambda^4\theta\right\|_{L^2}^2.
\end{aligned}\tag{35}$$

Combining (33) and (35), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\int\left|\Lambda^3 u\right|^2\left|\Lambda^3\theta\right|^2 dx\right)\\
&\qquad+\eta\int\left|\Lambda^4 u\right|^2 dx+\nu\int\left|\Lambda^4\theta\right|^2 dx\\
&\quad\le\frac{\eta}{2}\left\|\Lambda^4 u\right\|_{L^2}^2+C\left\|\nabla u\right\|_{L^2}^{13/10}\left\|\Lambda^3 u\right\|_{L^2}^{1/2}\\
&\qquad+2C\left\|\Lambda^3 u\right\|_{L^2}^2+C\left\|\theta\right\|_{L^2}\\
&\qquad+C\left\|\nabla u\right\|_{L^2}^{9/2}\left\|\Lambda^3 u\right\|_{L^2}^{3/2}\left\|\nabla\theta\right\|_{L^2}^{2}+\frac{\nu}{2}\left\|\Lambda^4\theta\right\|_{L^2}^2\\
&\qquad+C\left\|\nabla u\right\|_{L^2}^{2}+C\left\|\nabla\theta\right\|_{L^2}^{9}\left\|\Lambda^3\theta\right\|_{L^2}^{3}\left\|\nabla\theta\right\|_{L^2}^{2};
\end{aligned}\tag{36}$$

that is,

$$\begin{aligned}
&\frac{d}{dt}\left(\int\left|\Lambda^3 u\right|^2\left|\Lambda^3\theta\right|^2 dx\right)\\
&\qquad+\eta\int\left|\Lambda^4 u\right|^2 dx+\nu\int\left|\Lambda^4\theta\right|^2 dx\\
&\quad\le 2C\left\|\nabla u\right\|_{L^2}^{13/10}\left\|\Lambda^3 u\right\|_{L^2}^{1/2}\\
&\qquad+4C\left\|\Lambda^3 u\right\|_{L^2}^2+2C\left\|\theta\right\|_{L^2}\\
&\qquad+2C\left\|\nabla u\right\|_{L^2}^{9/2}\left\|\Lambda^3 u\right\|_{L^2}^{3/2}\left\|\nabla\theta\right\|_{L^2}^{2}+2C\left\|\nabla u\right\|_{L^2}^{2}\\
&\qquad+2C\left\|\nabla\theta\right\|_{L^2}^{9}\left\|\Lambda^3\theta\right\|_{L^2}^{3}\left\|\nabla\theta\right\|_{L^2}^{2}.
\end{aligned}\tag{37}$$

Choosing ε small enough, using (16), (23), and (24), we conclude that

$$\begin{aligned}
&\|u\|_{L^\infty(0,T;H^3)}+\|u\|_{L^2(0,T;H^4)}\le C,\\
&\|\theta\|_{L^\infty(0,T;H^3)}+\|\theta\|_{L^2(0,T;H^4)}\le C.
\end{aligned}\tag{38}$$

We complete the proof. □

Conflict of Interests

The authors declare that they have no competing interests.

Authors' Contribution

The authors declare that the work was realized in collaboration with the same responsibility. All authors read and approved the final paper.

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Research Article

A Unified Approach to Some Classes of Nonlinear Integral Equations

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We are going to discuss some important classes of nonlinear integral equations such as integral equations of Volterra-Chandrasekhar type, quadratic integral equations of fractional orders, nonlinear integral equations of Volterra-Wiener-Hopf type, and nonlinear integral equations of Erdélyi-Kober type. Those integral equations play very significant role in applications to the description of numerous real world events. Our aim is to show that the mentioned integral equations can be treated from the view point of nonlinear Volterra-Stieltjes integral equations. The Riemann-Stieltjes integral appearing in those integral equations is generated by a function of two variables. The choice of a suitable generating function enables us to obtain various kinds of integral equations. Some results concerning nonlinear Volterra-Stieltjes integral equations in several variables will be also discussed.

1. Introduction

In the theory of integral equations and their numerous applications, one can encounter some classes of integral equations having an important significance. This fact is mainly connected with applications of the mentioned classes of integral equations to the description of several real world events which appear in engineering, mechanics, physics, mathematical physics, electrochemistry, bioengineering, porous media, viscoelasticity, control theory, transport theory, kinetic theory of gases, radiative transfer, and other important branches of exact science and applied mathematics (cf. [1–12]).

Let us distinguish and describe some important classes of nonlinear integral equations mentioned tacitly above.

The first class we are going to present is the class of the so-called quadratic integral equations of Volterra-Chandrasekhar type (see [1, 7, 13, 14], e.g.). The interest in the study of those integral equations was initiated around 1950 by the famous astrophysicist Chandrasekhar [1], who investigated the following integral equation:

$$x(t) = l + x(t)\int_0^1 \frac{t}{t+s}\varphi(s)\,x(s)\,ds, \tag{1}$$

being the so-called quadratic (nonlinear) integral equation and called the *Chandrasekhar integral equation*.

Nowadays, integral equation (1) has been generalized in a few directions but in general two principal types of generalizations of (1) are investigated, namely, the *quadratic integral equation of Fredholm-Chandrasekhar type*

$$x(t) = a(t) + f(t, x(s))\int_0^a \frac{v(t,s,x(s))}{t+s}ds \tag{2}$$

and *the quadratic integral equation of Volterra-Chandrasekhar type*

$$x(t) = a(t) + f(t, x(t))\int_0^t \frac{v(t,s,x(s))}{t+s}ds. \tag{3}$$

We will focus on integral equations having form (3), that is, on nonlinear integral equations of Volterra-Chandrasekhar type.

The second class of nonlinear integral equations which will be discussed is the class of the so-called *nonlinear integral equations of fractional order*. Such equations have the form

$$x(t) = a(t) + \frac{f(t, x(t))}{\Gamma(\alpha)}\int_a^t \frac{v(t,s,x(s))}{(t-s)^{1-\alpha}}ds, \tag{4}$$

where $\alpha \in (0,1)$ is a fixed number and $\Gamma(\alpha)$ denotes the gamma function.

Observe that (4) is the so-called singular integral equation (of Abel type). These equations were very intensively studied during the last three decades and found a vast number of applications. Mathematicians working in the theory of integral equations of fractional orders wrote several papers and monographs devoted to those equations [4, 5, 8–11, 15–20].

The next, third class of nonlinear integral equations which we would like to present, is associated with the so-called *nonlinear integral equations of Volterra-Wiener-Hopf type*. Such equations are a special case of integral equations with kernels depending on the difference of arguments and they also play very important role in applications (cf. [3, 12, 21–23]).

The Volterra-Wiener-Hopf integral equation has the form

$$x(t) = a(t) + \int_0^t k(t-s)\, v(s, x(s))\, ds, \tag{5}$$

where $t \in \mathbb{R}_+ = [0, \infty)$ or $t \in [0, T]$ with $T > 0$.

Now, let us describe the fourth class of nonlinear integral equations being the object of our study as well as being recently very intensively investigated with regard to its numerous applications [24–30]. That class comprises integral equations called the *nonlinear Erdélyi-Kober integral equations* and having the form

$$x(t) = a(t) + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{m s^{m-1} s^p v(t, s, x(s))}{(t^m - s^m)^{1-\alpha}} ds, \tag{6}$$

where α, m, and p are positive constant and $\alpha \in (0,1)$. Moreover, $t \in I = [0,1]$ (or $I = [a,b]$).

Obviously, the integral equation of Erdélyi-Kober type creates the generalization of the integral equation of fractional order (4). Indeed, putting in (6) $m = 1$ and including the factor s^p into the function $v(t, s, x)$, we obtain (4) with $f(t, x) \equiv 1$.

Our aim in this paper is to show that all four classes of nonlinear integral equations (3)–(6) can be treated from one point of view. More precisely, we show that with help of nonlinear Volterra-Stieltjes integral equations we are able to unify all those classes in such a way that they are particular cases of the mentioned Volterra-Stieltjes integral equations.

The paper has a review character and is based on the results from [14, 21, 25, 31].

2. Notation, Definitions, and Auxiliary Results

In this section, we provide notation, definitions, and auxiliary results which will be needed in our further considerations. Firstly, we recall a few facts concerning functions of bounded variation [32]. Thus, assume that x is a real function defined on the fixed interval $[a,b]$. Then, the symbol $\bigvee_a^b x$ denotes the variation of the function x on the interval $[a,b]$. If $\bigvee_a^b x < \infty$, we say that x is of bounded variation on $[a,b]$. Similarly, if we have a function $u(t,s) = u : [a,b] \times [c,d] \to \mathbb{R}$, then we denote by $\bigvee_{t=p}^q u(t,s)$ the variation of the function $t \to u(t,s)$ on the interval $[p,q] \subset [a,b]$, where s is a fixed number in $[c,d]$. In a similar way, we define the quantity $\bigvee_{s=p}^q u(t,s)$.

Now, assume that x and φ are two real functions defined on the interval $[a,b]$. Then, we can define the Stieltjes integral (in the Riemann-Stieltjes sense)

$$\int_a^b x(t)\, d\varphi(t), \tag{7}$$

under appropriate assumptions on the functions x and φ (cf. [32]). For example, if we require that x is continuous and φ is of bounded variation on $[a,b]$, then the Stieltjes integral (7) does exist [32].

Let us mention that in our considerations we will often use the following two important lemmas [32].

Lemma 1. *If x is Stieltjes integrable on the interval $[a,b]$ with respect to a function φ of bounded variation, then*

$$\left| \int_a^b x(t)\, d\varphi(t) \right| \le \int_a^b |x(t)|\, d\left(\bigvee_a^t \varphi \right). \tag{8}$$

Lemma 2. *Let x_1, x_2 be Stieltjes integrable functions on the interval $[a,b]$ with respect to a nondecreasing function φ, such that $x_1(t) \le x_2(t)$ for $t \in [a,b]$. Then,*

$$\int_a^b x_1(t)\, d\varphi(t) \le \int_a^b x_2(t)\, d\varphi(t). \tag{9}$$

Obviously, in a similar way we can also consider Stieltjes integrals of the form

$$\int_a^b x(s)\, d_s g(t,s), \tag{10}$$

where $g : [a,b] \times [a,b] \to \mathbb{R}$ and the symbol d_s indicates the integration with respect to s. The details concerning the integral of this type will be given later.

Now, assume that x is a real function defined on the interval $[a,b]$. Denote by $\omega(x, \varepsilon)$ *the modulus of continuity* of the function x defined by the formula

$$\begin{aligned} \omega(x, \varepsilon) = \sup \{ &|x(t) - x(s)| : \\ &t, s \in [a,b], |t-s| \le \varepsilon \}. \end{aligned} \tag{11}$$

Similarly, if $p(t,s) = p : [a,b] \times [c,d] \to \mathbb{R}$, then we can define the modulus of continuity of the function $p(t,s)$ with respect to each variable separately. For example,

$$\begin{aligned} \omega(p(t,\cdot), \varepsilon) = \sup \{ &|p(t,u) - p(t,v)| : \\ &u, v \in [c,d], |u-v| \le \varepsilon \}, \end{aligned} \tag{12}$$

where t is a fixed number in the interval $[a,b]$.

In what follows, we recall some facts concerning measures of noncompactness which will be used later on [33].

To this end, assume that E is an infinite dimensional Banach space with the norm $\|\cdot\|$ and zero element θ. Denote by $B(x,r)$ the closed ball centered at x and of radius r. The symbol B_r will denote the ball $B(\theta, r)$.

For a given nonempty bounded subset X of E, we denote by $\chi(X)$ the so-called *Hausdorff measure of noncompactness* of the set X [33]. This quantity is defined by the formula

$$\chi(X) = \inf \{\varepsilon > 0 : X \text{ has a finite } \varepsilon\text{-net in } E\}. \quad (13)$$

Let us mention that the function χ has several useful properties and is often applied in nonlinear analysis [33]. Obviously, the concept of a measure of noncompactness may be defined in a more general way [33, 34], but for our purposes the Hausdorff measure of noncompactness defined by (13) will be completely sufficient.

Indeed, in our further considerations, we will work in the Banach space $C(I)$ consisting of real functions defined and continuous on the interval $I = [a, b]$, with the standard maximum norm. If X is a nonempty and bounded subset of $C(I)$, then the Hausdorff measure of noncompactness of X can be expressed by the formula [33]

$$\chi(X) = \frac{1}{2}\omega_0(X), \quad (14)$$

where

$$\omega_0(X) = \lim_{\varepsilon \to 0} \omega(X, \varepsilon), \quad (15)$$

and the symbol $\omega(X, \varepsilon)$ stands for the modulus of continuity of the set X defined in the following way:

$$\omega(X, \varepsilon) = \sup \{\omega(x, \varepsilon) : x \in X\}. \quad (16)$$

In our further considerations, we will utilize the fixed point theorem of Darbo type [33], which is formulated below.

Theorem 3. *Let Ω be a nonempty, bounded, closed, and convex subset of the space E and let $Q : \Omega \to \Omega$ be a continuous mapping such that there exists a constant $k \in [0, 1)$ for which $\chi(QX) \le k\chi(X)$ for an arbitrary nonempty subset X of Ω. Then, Q has at least one fixed point in the set Ω.*

Further on, we recall some facts concerning the so-called superposition operator [35]. To this end, assume that $I = [a, b]$ and $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is a given function. Then, to every function $x : I \to \mathbb{R}$, we may assign the function Fx defined by the formula

$$(Fx)(t) = f(t, x(t)), \quad (17)$$

for $t \in I$. The operator F defined in such a way is called *the superposition operator* generated by the function $f = f(t, x)$. For our further purposes, we will need the following result concerning the behaviour of the superposition operator F in the space $C(I)$ [35].

Lemma 4. *The superposition operator F defined by* (17) *transforms the space $C(I)$ into itself and is continuous if and only if the function f generating the operator F is continuous on the set $I \times \mathbb{R}$.*

3. A Nonlinear Volterra-Stieltjes Integral Equation and Its Special Cases

The considerations of this section are focused on the following nonlinear Volterra-Stieltjes integral equation:

$$x(t) = a(t) + \frac{f(t, x(t))}{\Gamma(\alpha)} \int_0^t v(t, s, x(s))\, d_s g(t, s), \quad (18)$$

where $t \in I = [0, 1]$ and $\Gamma(\alpha)$ (similarly as earlier) denotes the gamma function. Moreover, α is a fixed number in the interval $(0, 1)$. Let us notice that the interval $[0, 1]$ can be replaced by any interval $[a, b]$.

The details concerning assumptions imposed on the components of (18) will be given later. Now, we show that integral equation (18) unifies all previously considered integral equations (3)–(6).

At the beginning, denote by Δ the triangle

$$\Delta = \{(t, s) : 0 \le s \le t \le 1\}, \quad (19)$$

and consider the function $g(t, s) = g : \Delta \to \mathbb{R}$ defined in the following way:

$$g(t, s) = \begin{cases} t \ln \dfrac{t + s}{t} & \text{for } 0 < s \le t \\ 0 & \text{for } t = 0. \end{cases} \quad (20)$$

It is easy to see that the above function $g(t, s)$ is continuous on the triangle Δ. On the other hand, we get

$$d_s g(t, s) = \left(\frac{\partial}{\partial s} g(t, s)\right) ds = \frac{t}{t + s} ds. \quad (21)$$

Hence, we see that the integral equation of Volterra-Chandrasekhar type (3) (or (1), in the simplest case) can be treated as a special case of (18).

Further, consider the function $g(t, s)$ defined by the formula

$$g(t, s) = \frac{1}{\alpha} \left[t^\alpha - (t - s)^\alpha\right], \quad (22)$$

where $(t, s) \in \Delta$. Obviously, we have

$$d_s g(t, s) = \frac{1}{(t - s)^{1-\alpha}} ds, \quad (23)$$

which shows that the integral equation of fractional order (4) is also a particular case of (18).

To show that the Volterra-Wiener-Hopf integral equation (5) is a special case of (18), let us consider the function $g(t, s)$ given by the formula

$$g(t, s) = \int_0^s k(t - z)\, dz, \quad (24)$$

under appropriate assumptions imposed on the function $k = k(u)$ (cf. [21]). Obviously, we have

$$d_s g(t, s) = \frac{\partial}{\partial s} \left(\int_0^s k(t - z)\, dz\right) ds = k(t - s)\, ds, \quad (25)$$

and we see that (5) is in fact a special case of (18).

Finally, let us take into account the nonlinear Erdélyi-Kober integral equation (6). Then, putting

$$g(t,s) = t^{\alpha m} - (t^m - s^m)^{\alpha}, \tag{26}$$

for $(t,s) \in \Delta$, we have that

$$d_s g(t,s) = \frac{\alpha m s^{m-1}}{(t^m - s^m)^{1-\alpha}} ds. \tag{27}$$

Thus, we see that the integral equation (6) is also a special case of (18).

Now, we formulate theorem on the existence of solutions of Volterra-Stieltjes integral equation (18) imposing assumptions of such a type that the obtained theorem will ensure also the existence of solutions of all particular cases of (18) mentioned above.

We will consider the existence of solutions of (18) under the following hypotheses.

(i) The function $a = a(t)$ is continuous on the interval I.

(ii) The function $f(t,x) = f : I \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the Lipschitz condition with respect to the second variable; that is, there exists a constant $k > 0$ such that

$$|f(t,x) - f(t,y)| \le k|x-y|, \tag{28}$$

for all $t \in I$ and $x, y \in \mathbb{R}$.

(iii) The function $g(t,s) = g : \Delta \to \mathbb{R}$ is continuous.

(iv) The function $s \to g(t,s)$ is of bounded variation on the interval $[0,t]$ for each fixed $t \in I$.

(v) For any $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $t_1, t_2 \in I$, $t_1 < t_2$, and $t_2 - t_1 \le \delta$, the following inequality holds:

$$\bigvee_{s=0}^{t_1} [g(t_2,s) - g(t_1,s)] \le \varepsilon. \tag{29}$$

(vi) $g(t,0) = 0$ for any $t \in I$.

(vii) $v : \Delta \times \mathbb{R} \to \mathbb{R}$ is continuous such that $|v(t,s,x)| \le \phi(|x|)$ for all $(t,s) \in \Delta$ and for each $x \in \mathbb{R}$, where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function.

Now, we provide a few properties of the function $g = g(t,s)$ which will be needed in our further considerations. Obviously, we will assume that g satisfies assumptions (iii)–(vi).

Let us notice that these properties were proved in [14].

Lemma 5. *Let assumptions (iii)–(v) be satisfied. Then, for an arbitrarily fixed number $t_2 \in I$ ($t_2 > 0$) and for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $t_1, t_2 \in I$, $t_1 < t_2$, and $t_2 - t_1 \le \delta$ then*

$$\bigvee_{s=t_1}^{t_2} g(t_2,s) \le \varepsilon. \tag{30}$$

Lemma 6. *Under assumptions (iii)–(v), the function*

$$t \longrightarrow \bigvee_{s=0}^{t} g(t,s) \tag{31}$$

is continuous on the interval I.

Corollary 7. *There exists a finite positive constant K such that*

$$K = \sup\left\{\bigvee_{s=0}^{t} g(t,s) : t \in I\right\}. \tag{32}$$

In fact, the above statement is an immediate consequence of the continuity of the function

$$t \longrightarrow \bigvee_{s=0}^{t} g(t,s). \tag{33}$$

Further, let us denote by F_1 the finite constant (cf. assumption (iii)) defined by the formula

$$F_1 = \max\{|f(t,0)| : t \in I\}. \tag{34}$$

Now, we are prepared to formulate the last assumption utilized in our considerations.

(viii) There exists a positive solution r_0 of the inequality

$$\|a\| + K(kr + F_1)\phi(r) \le r, \tag{35}$$

such that $kK\phi(r_0) < 1$.

Our main result is formulated in the form of the following theorem.

Theorem 8. *Under assumptions (i)–(viii), there exists at least one solution $x = x(t)$ of (18) belonging to the space $C(I)$.*

Proof. At the beginning, let us introduce two functions $M(\varepsilon)$, $N(\varepsilon)$ defined in the following way:

$$M(\varepsilon) = \sup\left\{\bigvee_{s=0}^{t_1}[g(t_2,s) - g(t_1,s)] : t_1, t_2 \in I, t_1 < t_2, t_2 - t_1 \le \varepsilon\right\},$$
$$N(\varepsilon) = \sup\left\{\bigvee_{s=t_1}^{t_2} g(t_2,s) : t_1, t_2 \in I, t_1 < t_2, t_2 - t_1 \le \varepsilon\right\}. \tag{36}$$

Notice that in view of assumption (v) we have that $M(\varepsilon) \to 0$ as $\varepsilon \to 0$. Similarly, $N(\varepsilon) \to 0$ as $\varepsilon \to 0$ which is an easy consequence of Lemma 5.

Next, for a fixed $x \in C(I)$ and $t \in I$, let us denote

$$(Fx)(t) = f(t, x(t)),$$
$$(Vx)(t) = \int_0^t v(t,s,x(s))\, d_s g(t,s), \tag{37}$$
$$(Qx)(t) = a(t) + (Fx)(t)(Vx)(t).$$

Further, fix arbitrarily $\varepsilon > 0$ and take $t_1, t_2 \in I$ such that $t_1 < t_2$ and $t_2 - t_1 \leq \varepsilon$. Then, in view of our assumptions and Lemmas 1 and 2, for a fixed $x \in C(I)$, we obtain

$$\begin{aligned}
&\left|(Vx)(t_2) - (Vx)(t_1)\right| \\
&\leq \left|\int_0^{t_2} v(t_2, s, x(s))\, d_s g(t_2, s) - \int_0^{t_1} v(t_2, s, x(s))\, d_s g(t_2, s)\right| \\
&\quad + \left|\int_0^{t_1} v(t_2, s, x(s))\, d_s g(t_2, s) \right.\\
&\qquad \left. - \int_0^{t_1} v(t_1, s, x(s))\, d_s g(t_2, s)\right| \\
&\quad + \left|\int_0^{t_1} v(t_1, s, x(s))\, d_s g(t_2, s) \right.\\
&\qquad \left. - \int_0^{t_1} v(t_1, s, x(s))\, d_s g(t_1, s)\right| \\
&\leq \int_{t_1}^{t_2} \left|v(t_2, s, x(s))\right| d_s \left(\bigvee_{p=0}^{s} g(t_2, p)\right) \\
&\quad + \int_0^{t_1} \left|v(t_2, s, x(s)) - v(t_1, s, x(s))\right| d_s \left(\bigvee_{p=0}^{s} g(t_2, p)\right) \\
&\quad + \int_0^{t_1} \left|v(t_1, s, x(s))\right| d_s \left(\bigvee_{p=0}^{s} \left[g(t_2, p) - g(t_1, p)\right]\right) \\
&\leq \phi(\|x\|) \int_{t_1}^{t_2} d_s \left(\bigvee_{p=0}^{s} g(t_2, p)\right) \\
&\quad + \int_0^{t_1} \left|v(t_2, s, x(s)) - v(t_1, s, x(s))\right| d_s \left(\bigvee_{p=0}^{s} g(t_2, p)\right) \\
&\quad + \phi(\|x\|) \int_0^{t_1} d_s \left(\bigvee_{p=0}^{s} \left[g(t_2, p) - g(t_1, p)\right]\right) \\
&\leq \phi(\|x\|) \left[\bigvee_{s=0}^{t_2} g(t_2, s) - \bigvee_{s=0}^{t_1} g(t_2, s)\right] \\
&\quad + \omega(\varepsilon) \bigvee_{s=0}^{t_1} g(t_2, s) + \phi(\|x\|) \bigvee_{s=0}^{t_1} \left[g(t_2, s) - g(t_1, s)\right] \\
&\leq \phi(\|x\|) \bigvee_{s=t_1}^{t_2} g(t_2, s) \\
&\quad + \omega(\varepsilon) \bigvee_{s=0}^{t_2} g(t_2, s) + \phi(\|x\|) M(\varepsilon) \\
&\leq \phi(\|x\|) N(\varepsilon) + K\omega(\varepsilon) + \phi(\|x\|) M(\varepsilon),
\end{aligned} \tag{38}$$

where we denoted

$$\begin{aligned}
&\omega(\varepsilon) \\
&\quad = \sup \left\{\left|v(t_2, s, y) - v(t_1, s, y)\right| : \right.\\
&\qquad\qquad \left.(t_1, s), (t_2, s) \in \Delta, \left|t_2 - t_1\right| \leq \varepsilon, y \in [-\|x\|, \|x\|]\right\}.
\end{aligned} \tag{39}$$

Moreover, the functions $M(\varepsilon)$, $N(\varepsilon)$ are defined by (36) and the constant K is defined by (32).

Observe that in view of the uniform continuity of the function v on the set $\Delta\times[-\|x\|, \|x\|]$ we infer that $\omega(\varepsilon) \to 0$ as $\varepsilon \to 0$. Linking this fact with Lemma 5 and the properties of the functions $M(\varepsilon)$ and $N(\varepsilon)$ indicated previously, we deduce from (38) that the function Vx is continuous on the interval I.

On the other hand, the function Fx is continuous on I which is an easy consequence of assumption (ii) and Lemma 4. Thus, keeping in mind the above established facts, assumption (i), and (37), we conclude that the function Qx is continuous on the interval I. This means that the operator Q transforms the space $C(I)$ into itself.

In what follows, we show that the operator Q is continuous on the space $C(I)$. To this end, let us first observe that in view of the properties of the superposition operator F (cf. Lemma 4) it is sufficient to show that the operator V defined by (37) is continuous on $C(I)$.

To do this, fix $\varepsilon > 0$ and $x \in C(I)$. Next, take an arbitrary function $y \in C(I)$ with $\|x - y\| \leq \varepsilon$. Then, in view of Lemma 1, for an arbitrary fixed $t \in I$, we obtain

$$\begin{aligned}
&\left|(Vx)(t) - (Vy)(t)\right| \\
&\quad \leq \int_0^{t} \left|v(t, s, x(s)) - v(t, s, y(s))\right| d_s \left(\bigvee_{p=0}^{s} g(t, p)\right).
\end{aligned} \tag{40}$$

Now, let us denote

$$P = \|x\| + \varepsilon,$$

$$\begin{aligned}
\omega_P(v, \varepsilon) = \sup &\left\{\left|v(t, s, w) - v(t, s, u)\right| : \right.\\
&\qquad \left.(t, s) \in \Delta, w, u \in [-P, P], |w - u| \leq \varepsilon\right\}.
\end{aligned} \tag{41}$$

Then, from (40), we derive the following inequalities:

$$\begin{aligned}
&\left|(Vx)(t) - (Vy)(t)\right| \\
&\quad \leq \int_0^{t} \omega_P(v, \varepsilon)\, d_s \left(\bigvee_{z=0}^{s} g(t, z)\right) \\
&\quad \leq \omega_P(v, \varepsilon) \bigvee_{s=0}^{t} g(t, s) \leq K\omega_P(v, \varepsilon).
\end{aligned} \tag{42}$$

Hence, in virtue of the uniform continuity of the function v on the set $\Delta \times [-P, P]$, we deduce that V is continuous on the space $C(I)$.

In what follows, let us fix arbitrarily $x \in C(I)$. Then, taking into account the imposed assumptions and applying Lemmas 1 and 2, for a fixed $t \in I$, we get

$$\begin{aligned}
|(Qx)(t)| &\le |a(t)| \\
&\quad + |f(t, x(t))| \int_0^t |v(t, s, x(s))| \, d_s \left(\bigvee_{p=0}^{s} g(t, p) \right) \\
&\le \|a\| + [|f(t, x(t)) - f(t, 0)| + |f(t, 0)|] \\
&\quad \times \int_0^t \phi(\|x\|) \, d_s \left(\bigvee_{p=0}^{s} g(t, p) \right) \\
&\le \|a\| + (k\|x\| + F_1) \phi(\|x\|) \bigvee_{s=0}^{t} g(t, s).
\end{aligned} \tag{43}$$

Hence, in view of Corollary 7, we derive the following estimate:

$$\|Qx\| \le \|a\| + (k\|x\| + F_1) K \phi(\|x\|). \tag{44}$$

Then, keeping in mind assumption (viii), we deduce that there exists a number r_0 such that Q transforms the ball B_{r_0} into itself and $kK\phi(r_0) < 1$.

In what follows, let us take a nonempty subset X of the ball B_{r_0} and $x \in X$. Next, fix $\varepsilon > 0$ and choose $t_1, t_2 \in I$ such that $t_1 < t_2$ and $t_2 - t_1 \le \varepsilon$. Then, applying (38), we obtain

$$\begin{aligned}
&|(Qx)(t_2) - (Qx)(t_1)| \\
&\quad \le |a(t_2) - a(t_1)| \\
&\qquad + |(Fx)(t_2)(Vx)(t_2) - (Fx)(t_2)(Vx)(t_1)| \\
&\qquad + |(Fx)(t_2)(Vx)(t_1) - (Fx)(t_1)(Vx)(t_1)| \\
&\quad \le \omega(a, \varepsilon) \\
&\qquad + |(Fx)(t_2)| \, |(Vx)(t_2) - (Vx)(t_1)| \\
&\qquad + |(Vx)(t_1)| \, |(Fx)(t_2) - (Fx)(t_1)| \\
&\quad \le \omega(a, \varepsilon) + [|f(t_2, x(t_2)) - f(t_2, 0)| + |f(t_2, 0)|] \\
&\qquad \times \{\phi(\|x\|) N(\varepsilon) + K\omega(\varepsilon) + \phi(\|x\|) M(\varepsilon)\} \\
&\qquad + \left| \int_0^{t_1} v(t_1, s, x(s)) \, d_s g(t_1, s) \right| \\
&\qquad \times \{|f(t_2, x(t_2)) - f(t_2, x(t_1))| \\
&\qquad\quad + |f(t_2, x(t_1)) - f(t_1, x(t_1))|\} \\
&\quad \le \omega(a, \varepsilon) + (k\|x\| + F_1) \\
&\qquad \times \{\phi(\|x\|) N(\varepsilon) + K\omega(\varepsilon) + \phi(\|x\|) M(\varepsilon)\} \\
&\qquad + \int_0^{t_1} |v(t_1, s, x(s))| \, d_s \left(\bigvee_{p=0}^{s} g(t_1, p) \right) \\
&\qquad \times \{k|x(t_2) - x(t_1)| + \omega_{r_0}^1(f, \varepsilon)\},
\end{aligned} \tag{45}$$

where we denoted

$$\begin{aligned}
\omega_{r_0}^1(f, \varepsilon) = \sup \{&|f(t_2, x) - f(t_1, x)| : \\
&t_1, t_2 \in I, |t_2 - t_1| \le \varepsilon, x \in [-r_0, r_0]\}.
\end{aligned} \tag{46}$$

Further, from (45), we get

$$\begin{aligned}
&|(Qx)(t_2) - (Qx)(t_1)| \\
&\quad \le \omega(a, \varepsilon) + (kr_0 + F_1) \\
&\qquad \times \{\phi(r_0) N(\varepsilon) + K\omega(\varepsilon) + \phi(r_0) M(\varepsilon)\} \\
&\qquad + \phi(r_0) \int_0^{t_1} d_s \left(\bigvee_{p=0}^{s} g(t_1, p) \right) \{k\omega(x, \varepsilon) + \omega_{r_0}^1(f, \varepsilon)\} \\
&\quad \le \omega(a, \varepsilon) + (kr_0 + F_1) \\
&\qquad \times \{\phi(r_0) N(\varepsilon) + K\omega(\varepsilon) + \phi(r_0) M(\varepsilon)\} \\
&\qquad + K\phi(r_0) \{k\omega(x, \varepsilon) + \omega_{r_0}^1(f, \varepsilon)\}.
\end{aligned} \tag{47}$$

Hence, we have

$$\begin{aligned}
&\omega(Qx, \varepsilon) \\
&\quad \le \omega(a, \varepsilon) + (kr_0 + F_1) \\
&\qquad \times \{\phi(r_0) N(\varepsilon) + K\omega(\varepsilon) + \phi(r_0) M(\varepsilon)\} \\
&\qquad + K\phi(r_0) \{k\omega(x, \varepsilon) + \omega_{r_0}^1(f, \varepsilon)\}.
\end{aligned} \tag{48}$$

Consequently, we derive the following inequality:

$$\begin{aligned}
&\omega(QX, \varepsilon) \\
&\quad \le \omega(a, \varepsilon) + (kr_0 + F_1) \\
&\qquad \times \{\phi(r_0) N(\varepsilon) + K\omega(\varepsilon) + \phi(r_0) M(\varepsilon)\} \\
&\qquad + K\phi(r_0) \{k\omega(X, \varepsilon) + \omega_{r_0}^1(f, \varepsilon)\}.
\end{aligned} \tag{49}$$

Now, taking into account the fact that $\omega(\varepsilon) \to 0$, $M(\varepsilon) \to 0$, and $N(\varepsilon) \to 0$ as $\varepsilon \to 0$ and keeping in mind that the function f is uniformly continuous on the set $I \times [-r_0, r_0]$, we derive from (49) the following estimate:

$$\omega_0(QX) \le kK\phi(r_0) \omega_0(X). \tag{50}$$

From the above estimate, assumption (viii), and Theorem 3, we infer that there exists at least one fixed point x of the operator Q in the ball B_{r_0}. Obviously, the function $x = x(t)$ is a solution of (18). This completes the proof. □

In order to illustrate the result contained in Theorem 8, we provide an example.

Example 9. Let us consider the following nonlinear integral equation of Erdélyi-Kober type:

$$x(t) = t\exp t + \frac{1}{\Gamma(1/2)}\int_0^t \frac{(4/3)\,s^{7/3}\left(t + \sin s^2 + \sqrt[3]{x^2(s)}\right)}{\left(t^{4/3} - s^{4/3}\right)^{1/2}}ds, \tag{51}$$

for $t \in I = [0, 1]$. At first, let us observe that this equation can be written in the form (6). Indeed, we have

$$x(t) = t\exp t + \frac{1}{\Gamma(1/2)}\int_0^t \frac{(4/3)\,s^{1/3}s^2\left(t + \sin s^2 + \sqrt[3]{x^2(s)}\right)}{\left(t^{4/3} - s^{4/3}\right)^{1/2}}ds. \tag{52}$$

Thus, (52) is a particular case of (6) if we put $a(t) = t\exp t$, $\alpha = 1/2$, $m = 4/3$, $p = 2$, and

$$v(t, s, x) = t + \sin s^2 + x^{2/3}. \tag{53}$$

Further, let us notice that (52) can be treated as a particular case of Volterra-Stieltjes integral equation (18) if we take into account the fact that the function $g = g(t, s)$ appearing in (18) has the form (26); that is,

$$g(t, s) = t^{2/3} - \left(t^{4/3} - s^{4/3}\right)^{1/2}. \tag{54}$$

It is easily seen that such a function $g(t, s)$ satisfies assumptions (iii)–(vi) of Theorem 8. Moreover, we see that $f(t, x) \equiv 1$ and $|v(t, s, x)| \le 2 + x^{2/3}$.

Thus, applying Theorem 8, we can accept that $\phi(r) = 2 + r^{2/3}$. We omit further, technical details (cf. [25]) but the final conclusion asserts that (52) has a solution in the space $C(I)$ belonging to the ball B_4.

4. Further Results and Remarks

The result contained in Theorem 8 does not cover some cases being important with regard to applications. Obviously, we can also formulate a more general theorem than that presented above and concerning the existence of solutions of (18) which are defined, continuous, and bounded on $\mathbb{R}_+$ and are satisfying some other conditions (e.g., having a limit at infinity).

On the other hand, we can always adapt a suitable version of Theorem 8 in combination with the considered particular class of integral equations discussed above.

For example, if we consider the Volterra-Wiener-Hopf integral equation (5), then its generalized Volterra-Stieltjes counterpart has the form

$$x(t) = a(t) + \int_0^t v(s, x(s))\, d_s g(t, s), \tag{55}$$

with the function $g(t, s)$ of the form (24). Then, we can formulate the following existence result concerning (55) [21] (cf. also [36]).

Theorem 10. *Assume that the following hypotheses are satisfied.*

(i) *The function $a = a(t)$ is continuous and bounded on $\mathbb{R}_+$. Moreover, there exists the limit $\lim_{t\to\infty} a(t)$ (of course, this limit is finite).*

(ii) *$v : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ being nondecreasing on $\mathbb{R}_+$, $\psi(0) = 0$, and $\lim_{t\to 0}\psi(t) = 0$ such that*

$$|v(s, x) - v(s, y)| \le \psi(|x - y|), \tag{56}$$

for all $s \in \mathbb{R}_+$ and $x, y \in \mathbb{R}$.

(iii) *The function $s \to v(s, 0)$ is bounded on $\mathbb{R}_+$.*

(iv) *$g(t, s) = g : \Delta \to \mathbb{R}$ is uniformly continuous on the triangle $\Delta = \{(t, s) : 0 \le s \le t\}$.*

(v) *The function $s \to g(t, s)$ is of bounded variation on the interval $[0, t]$ for each fixed $t \in \mathbb{R}_+$.*

(vi) *For any $\varepsilon > 0$, there is $\delta > 0$ such that, for all $t_1, t_2 \in \mathbb{R}_+$, $t_1 < t_2$, and $t_2 - t_1 \le \delta$, the inequality*

$$\bigvee_{s=0}^{t_1}[g(t_2, s) - g(t_1, s)] \le \varepsilon \tag{57}$$

holds.

(vii) *$g(t, 0) = 0$ for all $t \ge 0$.*

(viii) *The function $t \to \bigvee_{s=0}^{t} g(t, s)$ is bounded on $\mathbb{R}_+$.*

(ix) *There exists a positive solution r_0 of the inequality*

$$\|a\| + (\psi(r) + V_1)K \le r, \tag{58}$$

where $\|a\| = \sup\{|a(t)| : t \ge 0\}$, $V_1 = \sup\{|v(s, 0)| : s \ge 0\}$, and $K = \sup\{\bigvee_{s=0}^{t} g(t, s) : t \ge 0\}$.

Then, (55) *has at least one solution $x = x(t)$ which is defined, continuous, and bounded on $\mathbb{R}_+$ and has a finite limit at infinity.*

Further, let us mention that the crucial role in Theorem 8 is played by assumption (v) (the same assumption appears as assumption (vi) in Theorem 10). That assumption seems to be rather difficult to be verified in practice. But it turns out that, in considerations which cover all our particular classes of the above indicated integral equations, we can replace the mentioned assumption by less restrictive ones which are very convenient in verification.

For example, we formulate below the assumption of such a type which is connected with Theorem 10 (see [21]).

(x) For arbitrary $t_1, t_2 \in \mathbb{R}_+$ such that $t_1 < t_2$, the function $s \to g(t_2, s) - g(t_1, s)$ is nonincreasing on the interval $[0, t_1]$.

Then, we have the following lemma [21].

Lemma 11. *Let assumptions (iv) and (vii) of Theorem 10 be satisfied. Moreover, we assume that the function $g = g(t,s)$ satisfies condition (x). Then, g satisfies assumption (vi) of Theorem 10.*

It can be shown that Lemma 11 enables us to formulate convenient requirements concerning, for example, the function $k = k(u)$ appearing in (6), which guarantee that the Volterra-Wiener-Hopf counterpart of (55) satisfies assumptions imposed in Theorem 10. We omit details which can be found in [21].

5. Remarks concerning Nonlinear Volterra-Stieltjes Integral Equations in Two Variables

In this final section, we indicate the possibility of investigations concerning the nonlinear Volterra-Stieltjes integral equations with an unknown function of two or more variables (cf. [31]). For example, the Volterra-Stieltjes integral equation in two variables has the form

$$u(t,x) = a(t,x) + f(t,x,u(t,x)) \times \int_0^t \int_0^x v(t,s,x,y,u(s,y))\, d_y g_2(x,y)\, d_s g_1(t,s), \tag{59}$$

for $(t,x) \in I^2$, where $I = [0,1]$. Obviously, the interval $[0,1]$ can be replaced by any closed and bounded interval $[a,b]$.

We will not formulate in detail assumptions concerning the functions involved in (59). Those assumptions are combinations and a refinement of assumptions imposed in Theorem 8 (cf. [31]).

It is worthwhile mentioning that the Volterra-Stieltjes integral equation in two variables (59) covers a lot of particular cases being a combination of nonlinear integral equations of the type (3)–(6). For example, we can consider the functional integral equation with functions involved depending on two variables which has the form

$$u(t,x) = a(t,x) + \frac{f(t,x,u(t,x))}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \int_0^x \frac{v(t,s,x,y,u(s,y))}{(t-s)^{1-\alpha}(x-y)^{1-\beta}} ds\, dy, \tag{60}$$

for $t, x \in I$ and for α, β being fixed numbers in the interval $(0,1)$. Obviously, (60) is a particular case of (59) if we put

$$g_1(t,s) = \frac{1}{\alpha}\left[t^\alpha - (t-s)^\alpha\right], \qquad g_2(x,y) = \frac{1}{\beta}\left[x^\beta - (x-y)^\beta\right], \tag{61}$$

for $(t,s) \in \Delta_1$ and $(x,y) \in \Delta_2$, where $\Delta_1 = \{(t,s) : 0 \le s \le t \le 1\}$ and $\Delta_2 = \{(x,y) : 0 \le y \le x \le 1\}$.

On the other hand, we can also consider the functional integral equation with functions depending on two variables, which has other mixed forms composed of functions $g = g(t,s)$ appearing in previously investigated integral equations (3)–(6).

Thus, we can consider the nonlinear Volterra-Stieltjes integral equation with an unknown function depending on two variables and having the form

$$u(t,x) = a(t,x) + \frac{f(t,x,u(t,x))}{\Gamma(\alpha)} \int_0^t \int_0^x \frac{t v(t,s,x,y,u(s,y))}{(t+s)(x-y)^{1-\alpha}} ds\, dy, \tag{62}$$

for $t, x \in I$ and for α being a fixed number in the interval $(0,1)$.

Observe that (62) is a particular case of (59) if we put

$$g(t,s) = \begin{cases} t \ln \dfrac{t+s}{t} & \text{for } 0 < s \le t \\ 0 & \text{for } t = 0, \end{cases} \qquad g_2(x,y) = \frac{1}{\alpha}\left[x^\alpha - (x-y)^\alpha\right], \tag{63}$$

for $(x,y) \in \Delta_2$.

Hence, we see that (62) represents the mixed type of Chandrasekhar and fractional order integral equations.

Obviously, it is not difficult to construct other mixed types of nonlinear integral equations with unknown functions in two variables which are particular cases of (59). For example, we can construct nonlinear integral equation in two variables of mixed type of Erdélyi-Kober and fractional order, of Erdélyi-Kober and Wiener-Hopf type, and so on.

The details are rather involved and we will not present details (cf. [31]).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Locally Lipschitz Composition Operators in Space of the Functions of Bounded $\kappa\Phi$-Variation

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We give a necessary and sufficient condition on a function $h : \mathbb{R} \to \mathbb{R}$ under which the nonlinear composition operator H, associated with the function h, $Hu(t) = h(u(t))$, acts in the space $\kappa\Phi BV[a,b]$ and satisfies a local Lipschitz condition.

1. Introduction

Given a function $h : \mathbb{R} \to \mathbb{R}$, the composition operator H associated with the function h maps each function $u : [a,b] \to \mathbb{R}$ into the composition function $Hu : [a,b] \to \mathbb{R}$ defined by

$$Hu(t) := h(u(t)), \quad (t \in [a,b]). \tag{1}$$

More generally, given $h : [a,b]\times\mathbb{R} \to \mathbb{R}$, we consider the operator H, defined by

$$Hu(t) := h(t, u(t)), \quad (t \in [a,b]). \tag{2}$$

This operator is also called *superposition operator* or *substitution operator* or *Nemytskij operator* associated with h. In what follows, we will refer to (1) as the *autonomous case* and to (2) as the *nonautonomous case*. For an extensive treatment of composition operator and function spaces we refer to the monographs Appell et al. [1], Appell and Zabrejko [2], and Runst and Sickel [3].

In 1984, Sobolevskij [4] proved the following statement: "the autonomous composition operator associated with $h : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz in the space $\mathrm{Lip}[a,b]$ if and only if the derivative h' exists and is locally Lipschitz." In recent articles Appell et al. [5] and Merentes et al. [6] obtained several results of the Sobolevskij type. As the authors explain in the introduction, the significance of these results lies in the fact that in most applications to many nonlinear problems it is sufficient to impose a local Lipschitz condition, instead of a global Lipschitz condition. In fact they proved that Sobolevskij's result is valid in the spaces $BV_\varphi[a,b]$, $HBV[a,b]$, $RV_\varphi[a,b]$, and $\Phi BV[a,b]$.

Motivated by the work done in the papers [5, 6], we establish a similar result to the one given by Sobolevskij, in the space of functions $\kappa\Phi BV[a,b]$.

Although the composition operator (or Nemytskij operator) is very simple, it turns out to be one of the most interesting and important operators studied in nonlinear functional analysis; the behavior of this operator exhibits many surprising and even pathological features in various function spaces. For example, about 35 years ago Dahlberg [7] proved the following: for $1 \le p \le \infty$ and $1+(1/p) < m < n/p$ integer, if H maps the Sobolev space $W_p^m(\mathbb{R}^n)$ into itself, then h is a linear function. Among these pathologies there is one called *degeneracy phenomenon*, which states that the global Lipschitz condition necessarily leads to affine functions in various functions spaces. This property was first proved in [8] for the space $\mathrm{Lip}[a,b]$. Additional information about the degeneracy phenomena can be found in [9, 10].

This paper is organized as follows: Section 2 contains definitions, notations, and necessary background about the class of functions of bounded $\kappa\Phi$-variation in the sense of Schramm-Korenblum; Section 3 contains the main theorem. Also in this section we state and prove a Helly-type theorem, which plays a crucial role in the demonstration of our Sobolevskij-type result.

2. Some Function Spaces

The concept of functions of bounded variation has been well known since C. Jordan gave the complete characterization of functions of bounded variation as a difference of two increasing functions in 1881. This class of functions exhibits so many interesting properties that it makes a suitable class of functions in a variety of contexts with wide applications in pure and applied mathematics [1, 11].

Definition 1. Let $f : [a, b] \to \mathbb{R}$ be a function. For a given partition $\pi : a = t_0 < t_1 < \cdots < t_n = b$ of the interval $[a, b]$,

$$\sigma(f, \pi) = \sigma(f, \pi; [a, b]) := \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| \quad (3)$$

is called the variation of f on $[a, b]$ with respect to π.

The (possibly infinite) number,

$$V(f; [a, b]) := \sup_{\pi} \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|, \quad (4)$$

where the supremum is taken over all partitions π of the interval $[a, b]$ is called the total variation of f on $[a, b]$. If $V(f; [a, b]) < \infty$, we say that f has bounded variation. The collection of all functions of bounded variation on $[a, b]$ is denoted by $BV[a, b]$.

This notion of a function of bounded variation has been generalized by several authors. One of these generalized versions was given by Korenblum in 1975 [12]. He considered a new kind of variation, called κ-variation, and introduced a function κ for distorting the expression $|t_j - t_{j-1}|$ in the partition itself, rather than the expression $|f(t_j) - f(t_{j-1})|$ in the range. One advantage of this alternative approach is that a function of bounded κ-variation may be decomposed into the difference of two simpler functions called κ-decreasing functions.

Definition 2. A function $\kappa : [0, 1] \to [0, 1]$ is called a distortion function (κ-function) if κ satisfies the following properties:

(1) κ is continuous with $\kappa(0) = 0$ and $\kappa(1) = 1$;

(2) κ is concave and increasing;

(3) $\lim_{t \to 0^+}(\kappa(t)/t) = \infty$.

Korenblum (see [12]) introduced the definition of bounded κ-variation as follows.

Definition 3. Let κ be a distortion function, f a real function $f : [a, b] \to \mathbb{R}$, and $\pi : a = t_0 < t_1 < \cdots < t_n = b$ a partition of the interval $[a, b]$. Let one consider

$$\kappa(f, \pi) := \frac{\sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|}{\sum_{i=1}^{n} \kappa((t_i - t_{i-1})/(b - a))},$$
$$\kappa V(f) = \kappa V(f; [a, b]) := \sup_{\pi} \kappa(f, \pi), \quad (5)$$

where the supremum is taken over all partitions π of the interval $[a, b]$. In the case $\kappa V(f; [a, b]) < \infty$ one says that f has bounded κ-variation on $[a, b]$ and one will denote by $\kappa BV[a, b]$ the space of functions of bounded κ-variation on $[a, b]$.

Schramm in 1985 [13] considered a Φ-sequence as follows.

Definition 4 (Φ-sequence). Let $\Phi = \{\phi_n\}_{n \geq 1}$ be a sequence of increasing convex functions, defined on $\mathbb{R}_+ = [0, \infty)$ such that

(1) $\phi_n(0) = 0$, $n \geq 1$;

(2) $\phi_n(t) > 0$ for $t > 0$.

We will say that Φ is a Φ^*-sequence if $\phi_{n+1}(t) \leq \phi_n(t)$ for all n and t and a Φ-sequence if in addition $\sum_n \phi_n(t)$ diverges for $t > 0$.

From now on, all sequences considered in this work will be Φ-sequences. We will consider a nonoverlapping family of subintervals $I_n = [t_{n-1}, t_n]$ of the interval $I = [a, b]$, ($n = 1, 2, \ldots$); it means that $I_i \cap I_j$ either is empty or contains a single point for $i, j = 1, 2, \ldots, i \neq j$.

Definition 5. If Φ is a Φ-sequence, one says that a function $f : [a, b] \to \mathbb{R}$ is of bounded Φ-variation if the Φ-sums $\sum_n \phi_n(|f(t_n) - f(t_{n-1})|) < \infty$ for any nonoverlapping collection $\{I_n\}$ of the interval I.

Definition 6 (condition δ_2 generalized for small values G_{δ_2}). The Φ-sequence $\Phi = \{\phi_n\}_{n \geq 1}$ satisfies condition G_{δ_2} if and only if there exist $t_0 > 0$ and $M(t_0) > 0$ such that

$$\sum_{n=1}^{m} \phi_n(2t) \leq M(t_0) \sum_{n=1}^{m} \phi_n(t) \quad 0 \leq t \leq t_0, \ m \geq 1. \quad (6)$$

We may define, for f of bounded Φ-variation, the total Φ-variation of f by

$$V_\Phi(f) = V_\Phi(f; [a, b]) := \sup \sum_n \phi_n(|f(t_n) - f(t_{n-1})|), \quad (7)$$

where the supremum is taken over all $\{I_n\}, I_n \subseteq [a, b]$. Hernández and Rivas (see [14]) showed that if $\Phi = \{\phi_n\}_{n \geq 1}$ is a Φ-sequence and Φ satisfies condition G_{δ_2}, then $V_\Phi[a, b]$ is a linear space. We denote by $\Phi BV[a, b]$ the collection of all functions f such that cf is of bounded Φ-variation for some $c > 0$.

S. K. Kim and J. Kim in 1986 [15] considered a bounded $\kappa\Phi$-variation as follows.

Definition 7. Let $\kappa : [0, 1] \to [0, 1]$ be a distortion function and $\Phi = \{\phi_n\}_{n \geq 1}$ a Φ-sequence and let $f : [a, b] \to \mathbb{R}$. One defines

$$\kappa\sigma_\phi(f, I_n) := \frac{\sum_{n=1}^{m} \phi_n(|f(t_n) - f(t_{n-1})|)}{\sum_{n=1}^{m} \kappa((t_n - t_{n-1})/(b - a))},$$
$$\kappa V_\Phi(f) = \kappa V_\Phi(f; [a, b]) := \sup \kappa\sigma_\phi(f, I_n). \quad (8)$$

If $\kappa V_\Phi(f;[a,b]) < \infty$, we say that f has bounded $\kappa\Phi$-variation in the interval $[a,b]$ and this number denotes the $\kappa\Phi$-variation of f in Schramm-Korenblum's sense in $[a,b]$. The class of functions that has bounded $\kappa\Phi$-variation in the interval $[a,b]$ is denoted by $\kappa V_\Phi[a,b]$. The vector space generated by this class is denoted by $\kappa\Phi BV[a,b]$.

Let us consider $\kappa V_\Phi(cf)$ as a function of variable c. If $\Phi = \{\phi_n\}_{n\geq 1}$ is a sequence of increasing convex functions, $\phi_n(0) = 0$, $t \geq 0$, we have $\phi_n(ct) \leq c\phi_n(t)$, $0 \leq c \leq 1$. Let $\kappa V_\Phi(f) < \infty$ and let $0 < c \leq 1$. Then $\kappa V_\Phi(cf) \leq c\kappa V_\Phi(f) \to 0$ as $c \to 0$. With this in mind, we define a norm in the space $\kappa\Phi BV_0 = \{f \in \kappa\Phi BV \mid f(a) = 0\}$ as follows:

$$\|f\| = \inf\left\{c > 0 \mid \kappa V_\Phi\left(\frac{f}{c}\right) \leq 1\right\}. \qquad (9)$$

We will consider the following norm in the space $\kappa\Phi BV[a,b]$:

$$\|f\|_{\kappa\Phi} = \|f\|_\infty + \mu_\phi(f) \quad (f \in \kappa\Phi BV[a,b]), \qquad (10)$$

where $\mu_\phi(f) = \inf\{c > 0 \mid \kappa V_\Phi(f/c) \leq 1\}$ and $\|\cdot\|_\infty$ denotes the supremum norm.

By the above definition, we have the following.

Theorem 8 (see [16]). *Let $\{f_n\} \subset \kappa\Phi BV_0$ be a sequence such that f_n converges to f almost everywhere with $f \in \kappa\Phi BV_0$. Then*

$$\|f\| \leq \lim_{n\to\infty} \inf \|f_n\|; \qquad (11)$$

that is, the Luxemburg norm is lower semicontinuous on $\kappa\Phi BV_0$.

Theorem 9 (see [15]). *$(\kappa\Phi BV_0, \|\cdot\|)$ is a Banach space.*

Definition 10 (see [17]). Let $\Phi = \{\phi_n\}_{n\geq 1}$ be a Φ-sequence. A real function $f : [a,b] \to \mathbb{R}$ is said to be $\kappa\Phi$-decreasing on $[a,b]$ if there exists a positive constant c such that for each subinterval I of $[a,b]$

$$\phi_n(|f(I)|) \leq c\kappa\left(\frac{|I|}{b-a}\right). \qquad (12)$$

Lemma 11 (see [16]). *For any κ-function and any Φ-sequence $\Phi = \{\phi_n\}_{n\geq 1}$, one has the following:*

(1) $\kappa V_\Phi(f/\|f\|_{\kappa\Phi}) \leq 1$, $f \in \kappa\Phi BV$,

(2) *if $\|f\|_{\kappa\Phi} \leq 1$, then $\kappa V_\Phi(f) \leq \|f\|_{\kappa\Phi}$, $f \in \kappa\Phi BV$.*

Lemma 12 (see [18]). *Let $\kappa : [0,1] \to [0,1]$ be a distortion function and $\Phi = \{\phi_n\}_{n\geq 1}$ a Φ-sequence and let $f \in \kappa\Phi BV[a,b]$ and $c > 0$. Then $\mu_\phi(f) < c$ if and only if $\kappa V_\Phi(f/c) < 1$.*

Theorem 13 (see [15] or [17]). *If a function f is $\kappa\Phi$-decreasing on $[a,b]$, then one has the following properties.*

(1) *f is of bounded $\kappa\Phi$-variation.*

(2) *$f(x_0^+)$ and $f(y_0^-)$ exist for any $a \leq x_0 < b$ and $a < y_0 \leq b$.*

(3) *f is continuous on $[a,b]$.*

Theorem 14 (see [18]). *Let $\kappa : [0,1] \to [0,1]$ be a distortion function, let $\Phi = \{\phi_n\}_{n\geq 1}$ be a Φ-sequence, let $h : \mathbb{R} \to \mathbb{R}$, and let H be the composition operator associated with h. H maps the space* Lip $[0,1]$ *into the space $\kappa\Phi BV[0,1]$ or $\kappa BV[0,1]$ if and only if h is locally Lipschitz. Furthermore, the operator H is bounded.*

The following lemma is basic for our main result.

Lemma 15 (invariance principle). *Let $h : \mathbb{R} \to \mathbb{R}$ be a function. Then the composition operator* (1) *maps the space $\kappa\Phi BV[a,b]$ into itself if and only if it maps, for any other choice of $c < d$, the space $\kappa\Phi BV[c,d]$ into itself.*

Proof. Suppose that the composition operator defined by $Hu = h\circ u$ maps the space $\kappa\Phi BV[a,b]$ into itself. The function $\alpha : [c,d] \to [a,b]$ defined by

$$\alpha(t) := \frac{b-a}{d-c}(t-c) + a \quad (c \leq t \leq d) \qquad (13)$$

is a strictly increasing homeomorphism between $[c,d]$ and $[a,b]$ with inverse

$$\alpha^{-1}(s) = \frac{d-c}{b-a}(s-a) + c \quad (a \leq s \leq b) \qquad (14)$$

which satisfies $\alpha(c) = a$ and $\alpha(d) = b$. Let $\mathscr{P}([a,b])$ denote the family of all partitions of $[a,b]$. Thus, $\alpha : \mathscr{P}([c,d]) \to \mathscr{P}([a,b])$ with

$$\begin{aligned}\alpha(\{t_0, t_1, \ldots, t_{m-1}, t_m\}) \\ = \{\alpha(t_0), \alpha(t_1), \ldots, \alpha(t_{m-1}), \alpha(t_m)\}\end{aligned} \qquad (15)$$

defines a one-to-one correspondence between all partitions of $[c,d]$ and all partitions of $[a,b]$.

Given $v \in \kappa\Phi BV[c,d]$, the function $u := v\circ\alpha^{-1}$ belongs to $\kappa\Phi BV[a,b]$, by the definition of functions of bounded $\kappa\Phi$-variation, and so $Hu = h\circ v\circ\alpha^{-1}$ belongs to $\kappa\Phi BV[a,b]$, by assumption. But for $P \in \mathscr{P}([c,d])$ and $\alpha(P) \in \mathscr{P}([a,b])$ as above we have

$$\begin{aligned}&\kappa\sigma_\phi(h\circ u, \alpha(P)) \\ &= \kappa\sigma_\phi\left(h\circ v\circ\alpha^{-1}, P\right) \\ &= \frac{\sum_{j=1}^m \phi_j\left(\left|h\left(u\left(\alpha\left(t_j\right)\right)\right) - h\left(u\left(\alpha\left(t_{j-1}\right)\right)\right)\right|\right)}{\sum_{j=1}^m c\kappa\left(\left|u\left(\alpha\left(t_j\right)\right) - u\left(\alpha\left(t_{j-1}\right)\right)\right| / (b-a)\right)} \\ &= \frac{\sum_{j=1}^m \phi_j\left(\left|h\left(v\left(t_j\right)\right) - h\left(v\left(t_{j-1}\right)\right)\right|\right)}{\sum_{j=1}^m c\kappa\left(\left|v\left(t_j\right) - v\left(t_{j-1}\right)\right| / (d-c)\right)} \\ &= \kappa\sigma_\phi(h\circ v, P).\end{aligned} \qquad (16)$$

Passing to the supremum with respect to $P \in \mathscr{P}([c,d])$ and $\alpha(P) \in \mathscr{P}([a,b])$ we conclude that $\kappa V_\Phi(h\circ v;[c,d]) = \kappa V_\Phi(h\circ u;[a,b])$. □

3. Main Results

In the proof of the main result of this paper, we will employ a compactness result, for instance, Helly's selection principle or second Helly's theorem. Helly's theorem for functions of generalized variation has been of some importance for a long time. Helly's selection principle has been the subject of intensive research, and many applications, generalizations, and improvements of them can be found in the literature (see, e.g., [19–21] and the references therein).

In this part we will state and prove our main results. In the proof of our main result we make use of a Helly-type selection theorem for a $\kappa\Phi$-decreasing function.

In the paper [22] Cyphert and Kelingos proved the same result for an arbitrary infinite family of functions defined on $[0,1]$ which is both uniformly bounded and uniformly κ-decreasing.

Theorem 16 (Helly-type selection theorem). *An arbitrary infinite family of functions defined on $[0,1]$ which is both uniformly bounded and uniformly $\kappa\Phi$-decreasing contains a subsequence which converges at every point of $[0,1]$ to a $\kappa\Phi$-decreasing function.*

Proof. Let us denote by $\mathcal{F}$ an arbitrary infinite family of functions defined on $[0,1]$, which is both uniformly bounded and uniformly $\kappa\Phi$-decreasing. Then, there exists a constant $c > 0$ such that for every $f \in \mathcal{F}$ and every pair $0 \le x < y \le 1$

$$|f(x)| \le c, \tag{17}$$

$$\phi_n(f(y) - f(x)) \le c\kappa(y - x). \tag{18}$$

Using (17) we can, by means of the standard Cantor diagonalization technique, find a sequence of functions f_k in $\mathcal{F}$ which converges pointwise at each rational point of $[0,1]$, to a function g. Since each f_k satisfies (18), so does g, for all rational numbers $x, y \in [0,1]$.

Define g at irrational points x by

$$g(x) = \lim_{y \to x^-} g(y), \quad y \text{ rational}. \tag{19}$$

The existence of this limit can be seen as follows:

$$A = \liminf_{\mathbb{Q} \ni y \to x^-} g(y) \le \limsup_{\mathbb{Q} \ni y \to x^-} g(y) = B \quad \text{as } y \to x^-, \quad y \text{ rational}. \tag{20}$$

Let $\{y_i\}$ and $\{y_i'\}$ be two sequences of rational points converging to x, arranged so that $y_1 < y_1' < y_2 < y_2' < \cdots < x$ and such that $g(y_i) \to A$ and $g(y_i') \to B$ as $i \to \infty$. Then

$$\begin{gathered} \phi_n(g(y_i') - g(y_i)) \le c\kappa(y_i' - y_i), \\ \phi_n(g(y_i') - g(y_i)) = \phi_n\left(\lim_{y \to y_i'^-} g(y) - \lim_{y \to y_i^-} g(y)\right) \\ = \phi_n(B - A) \le 0. \end{gathered} \tag{21}$$

Then $\phi_n(B - A) = 0$, and hence $A = B$.

From (19) we obtain, by taking limits of rational points in inequality (18), that g satisfies (18) for all pairs of positive real numbers; that is, g is $\kappa\Phi$-decreasing with constant c on $[0,1]$. By Theorem 13 g is of bounded $\kappa\Phi$-variation and g is continuous. Hence, by another Cantor diagonalization process, a convergent subsequence of the functions f_k can be found.

Now, let us consider $0 < t < 1$ and $\varepsilon > 0$. Then, we fix two rational numbers y_1 and y_2 with $y_1 < t < y_2$ such that

$$\begin{gathered} |g(y_i) - g(t)| < \frac{\varepsilon}{3}, \quad i = 1, 2, \\ c\kappa(|y_i - t|) < \frac{\varepsilon}{3}, \quad i = 1, 2. \end{gathered} \tag{22}$$

Since the sequence $\{f_k\}$, $k = 1, 2, \ldots$, converges to g in the rational numbers, there exists $N > 0$ such that

$$|f_k(y_i) - g(y_i)| < \frac{\varepsilon}{3}, \quad i = 1, 2, \ k \ge N. \tag{23}$$

Now, from (22) and (23) we obtain

$$\begin{aligned} g(t) - f_k(t) = {} & (f_k(y_2) - f_k(t)) + (g(t) - g(y_2)) \\ & + (g(y_2) - f_k(y_2)) \\ \le {} & c\kappa(|y_2 - t|) + (g(t) - g(y_2)) \\ & + (g(y_2) - f_k(y_2)) \\ < {} & \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned} \tag{24}$$

Similarly,

$$\begin{aligned} f_k(t) - g(t) = {} & (f_k(t) - f_k(y_1)) + (g(y_1) - g(t)) \\ & + (f_k(y_1) - g(y_1)) \\ \le {} & c\kappa(|t - y_1|) \\ & + (g(y_1) - g(t)) + (f_k(y_1) - g(y_1)) \\ < {} & \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned} \tag{25}$$

Then, $|f_k(t) - g(t)| < \varepsilon$. □

We are now in a position to formulate and prove our main result.

Theorem 17. *Let us suppose that the composition operator H associated with h maps the space $\kappa\Phi BV[a,b]$ into itself. Then H is locally Lipschitz if and only if h' exists and is locally Lipschitz in $\mathbb{R}$.*

Proof. First let us assume that h' is locally Lipschitz in $\mathbb{R}$. Given $u \in \kappa\Phi BV[a,b]$, for $r > 0$, we denote by $K_1(r)$ the minimal Lipschitz constant of h' and by $K_2(r)$ the supremum of $|h'|$ on the bounded set

$$B_r := \bigcup_{a \le t \le b} \{u(t) : \|u\|_{\kappa\Phi} \le r\} \subset \mathbb{R}. \tag{26}$$

The finiteness of $K_2(r)$ implies that H satisfies a local Lipschitz condition with respect to the norm $\|\cdot\|_\infty$, so we only have to prove a local Lipschitz condition for H with respect to the $\kappa\Phi$-variation norm. We will prove this by applying twice the mean value theorem.

In fact, let us fix $u, v \in \kappa\Phi BV[a,b]$ with $u \neq v$ and $\|u\|_{\kappa\Phi} \leq r$, $\|v\|_{\kappa\Phi} \leq r$. Given a partition $P = \{t_0, t_1, \ldots, t_m\}$ of $[a,b]$, we split the index set $\{1,\ldots,m\}$ into a union $I \cup J$ of disjoint sets I and J by defining the following:

$j \in I$ if

$$\begin{aligned}&\left|u(t_j) - v(t_j)\right| + \left|u(t_{j-1}) - v(t_{j-1})\right| \\ &\quad \leq \left|u(t_j) - u(t_{j-1})\right| + \left|v(t_j) - v(t_{j-1})\right|\end{aligned} \tag{27}$$

and $j \in J$ if

$$\begin{aligned}&\left|u(t_j) - v(t_j)\right| + \left|u(t_{j-1}) - v(t_{j-1})\right| \\ &\quad > \left|u(t_j) - u(t_{j-1})\right| + \left|v(t_j) - v(t_{j-1})\right|.\end{aligned} \tag{28}$$

By the classical mean value theorem we find α_j between $v(t_j)$ and $u(t_j)$ such that

$$\begin{aligned}&Hu(t_j) - Hv(t_j) = h'(\alpha_j)\left[u(t_j) - v(t_j)\right] \\ &\qquad (j = 1, 2, \ldots, m).\end{aligned} \tag{29}$$

Now, by definition of I we have

$$\begin{aligned}&\left|\alpha_j - \alpha_{j-1}\right| \leq 2\left|u(t_j) - u(t_{j-1})\right| + 2\left|v(t_j) - v(t_{j-1})\right| \\ &\qquad (j \in I).\end{aligned} \tag{30}$$

A straightforward calculation shows then that

$$\begin{aligned}&\left|Hu(t_j) - Hv(t_j) - Hu(t_{j-1}) + Hv(t_{j-1})\right| \\ &= \left|h'(\alpha_j)\left[u(t_j) - v(t_j)\right] - h'(\alpha_{j-1})\left[u(t_{j-1}) - v(t_{j-1})\right]\right| \\ &= \left|\left(h'(\alpha_j) - h'(\alpha_{j-1})\right)\left[u(t_j) - v(t_j)\right]\right. \\ &\quad \left. + h'(\alpha_{j-1})\left[u(t_j) - v(t_j) - u(t_{j-1}) + v(t_{j-1})\right]\right| \\ &\leq K_1(r)\left|\alpha_j - \alpha_{j-1}\right| \|u - v\|_\infty \\ &\quad + K_2(r)\left|u(t_j) - v(t_j) - u(t_{j-1}) + v(t_{j-1})\right| \\ &\leq \left[2K_1(r)\|u - v\|_\infty + K_2(r)\right] \\ &\quad \times \left[\left|u(t_j) - u(t_{j-1})\right| + \left|v(t_j) - v(t_{j-1})\right|\right] \\ &= K_3(r)\left[\left|u(t_j) - u(t_{j-1})\right| + \left|v(t_j) - v(t_{j-1})\right|\right].\end{aligned} \tag{31}$$

Since $\phi_n(t_1) \leq \phi_n(t_2)$ for $t_1 \leq t_2$, we obtain that

$$\begin{aligned}&\phi_n\left(\left|Hu(t_j) - Hv(t_j) - Hu(t_{j-1}) + Hv(t_{j-1})\right|\right) \\ &\quad \leq \phi_n\left(K_3(r)\left[\left|u(t_j) - u(t_{j-1})\right| + \left|v(t_j) - v(t_{j-1})\right|\right]\right),\end{aligned} \tag{32}$$

and dividing by $\sum_{j=1}^m \kappa(|t_j - t_{j-1}|/(b-a))$ and adding on $j \in I$ we get that

$$\begin{aligned}&\sum_{j\in I}\left(\phi_j\left(\left|Hu(t_j) - Hv(t_j) - Hu(t_{j-1}) + Hv(t_{j-1})\right|\right)\right. \\ &\qquad \left.\times\left(\sum_{j=1}^m \kappa\left(\frac{\left|t_j - t_{j-1}\right|}{b-a}\right)\right)^{-1}\right) \\ &\leq \sum_{j\in I}\left(\phi_j\left(K_3(r)\left[\left|u(t_j) - u(t_{j-1})\right| + \left|v(t_j) - v(t_{j-1})\right|\right]\right)\right. \\ &\qquad \left.\times\left(\sum_{j=1}^m \kappa\left(\frac{\left|t_j - t_{j-1}\right|}{b-a}\right)\right)^{-1}\right) \\ &\leq \sum_{j\in I}\left(\frac{(1/2)\,\phi_j\left(2K_3(r)\left|u(t_j) - u(t_{j-1})\right|\right)}{\sum_{j=1}^m \kappa\left(\left|t_j - t_{j-1}\right|/(b-a)\right)}\right. \\ &\qquad \left. + \frac{(1/2)\,\phi_j\left(2K_3(r)\left|v(t_j) - v(t_{j-1})\right|\right)}{\sum_{j=1}^m \kappa\left(\left|t_j - t_{j-1}\right|/(b-a)\right)}\right) \\ &= \frac{1}{2}\kappa\sigma_\phi\left(2K_3(r)\,u, P\right) + \frac{1}{2}\kappa\sigma_\phi\left(2K_3(r)\,v, P\right) \\ &\leq K_3(r)\left(\|u\|_{\kappa\Phi} + \|v\|_{\kappa\Phi}\right) \leq K_4(r)\|u - v\|_{\kappa\Phi}.\end{aligned} \tag{33}$$

Again, by the mean value theorem, we find β_j between $u(t_j)$ and $u(t_{j-1})$ and γ_j between $v(t_j)$ and $v(t_{j-1})$ such that

$$\begin{aligned}&Hu(t_j) - Hu(t_{j-1}) = h'(\beta_j)\left[u(t_j) - u(t_{j-1})\right] \\ &\qquad (j = 1, 2, \ldots, m), \\ &Hv(t_j) - Hv(t_{j-1}) = h'(\gamma_j)\left[v(t_j) - v(t_{j-1})\right] \\ &\qquad (j = 1, 2, \ldots, m).\end{aligned} \tag{34}$$

By definition of J we have

$$\left|\beta_j - \gamma_j\right| < 2\left|u(t_j) - v(t_j)\right| + 2\left|u(t_{j-1}) - v(t_{j-1})\right|. \tag{35}$$

A straightforward calculation shows that

$$\begin{aligned}&\left|Hu(t_j) - Hv(t_j) - Hu(t_{j-1}) + Hv(t_{j-1})\right| \\ &= \left|h'(\beta_j)\left[u(t_j) - u(t_{j-1})\right] - h'(\gamma_j)\left[v(t_j) - v(t_{j-1})\right]\right|\end{aligned}$$

$$\begin{aligned}
&= \left|\left(h'\left(\beta_j\right) - h'\left(\gamma_j\right)\right)\left[u\left(t_j\right) - u\left(t_{j-1}\right)\right]\right.\\
&\quad \left. + h'\left(\gamma_j\right)\left[u\left(t_j\right) - u\left(t_{j-1}\right) - v\left(t_j\right) + v\left(t_{j-1}\right)\right]\right|\\
&\le K_1(r)\left|\beta_j - \gamma_j\right|\left|u\left(t_j\right) - u\left(t_{j-1}\right)\right|\\
&\quad + K_2(r)\left|u\left(t_j\right) - v\left(t_j\right) - u\left(t_{j-1}\right) + v\left(t_{j-1}\right)\right|\\
&< 2K_1(r)\left[\left|u\left(t_j\right) - v\left(t_j\right)\right| + \left|u\left(t_{j-1}\right) - v\left(t_{j-1}\right)\right|\right]\\
&\quad \times \left|u\left(t_j\right) - u\left(t_{j-1}\right)\right|\\
&\quad + K_2(r)\left[\left|u\left(t_j\right) - u\left(t_{j-1}\right)\right| + \left|v\left(t_j\right) - v\left(t_{j-1}\right)\right|\right]\\
&\le 4K_1(r)\|u - v\|_\infty \left|u\left(t_j\right) - u\left(t_{j-1}\right)\right|\\
&\quad + K_2(r)\left|u\left(t_j\right) - u\left(t_{j-1}\right)\right|\\
&\quad + K_2(r)\left|v\left(t_j\right) - v\left(t_{j-1}\right)\right|\\
&\le \left(4K_1(r)\|u - v\|_\infty + K_2(r)\right)\left|u\left(t_j\right) - u\left(t_{j-1}\right)\right|\\
&\quad + K_2(r)\left|v\left(t_j\right) - v\left(t_{j-1}\right)\right|\\
&\le K_5(r)\left|u\left(t_j\right) - u\left(t_{j-1}\right)\right| + K_2(r)\left|v\left(t_j\right) - v\left(t_{j-1}\right)\right|.
\end{aligned} \tag{36}$$

Since $\phi_n(t_1) \le \phi_n(t_2)$ for $t_1 \le t_2$, we obtain that

$$\begin{aligned}
&\phi_n\left(\left|Hu\left(t_j\right) - Hv\left(t_j\right) - Hu\left(t_{j-1}\right) + Hv\left(t_{j-1}\right)\right|\right)\\
&\quad \le \phi_n\left(K_5(r)\left|u\left(t_j\right) - u\left(t_{j-1}\right)\right| + K_2(r)\left|v\left(t_j\right) - v\left(t_{j-1}\right)\right|\right),
\end{aligned} \tag{37}$$

and dividing by $\sum_{j=1}^m \kappa(|t_j - t_{j-1}|/(b-a))$ and adding on $j \in J$ we get that

$$\begin{aligned}
&\sum_{j\in J}\left(\frac{\phi_j\left(\left|Hu\left(t_j\right) - Hv\left(t_j\right) - Hu\left(t_{j-1}\right) + Hv\left(t_{j-1}\right)\right|\right)}{\sum_{j=1}^m \kappa\left(\left|t_j - t_{j-1}\right|/(b-a)\right)}\right)\\
&\le \sum_{j\in J}\left(\phi_j\left(K_5(r)\left|u\left(t_j\right) - u\left(t_{j-1}\right)\right|\right.\right.\\
&\qquad \left. + K_2(r)\left|v\left(t_j\right) - v\left(t_{j-1}\right)\right|\right)\\
&\qquad \left. \times \left(\sum_{j=1}^m \kappa\left(\frac{\left|t_j - t_{j-1}\right|}{b-a}\right)\right)^{-1}\right)\\
&\le \sum_{j\in J}\left(\frac{(1/2)\,\phi_j\left(2K_5(r)\left|u\left(t_j\right) - u\left(t_{j-1}\right)\right|\right)}{\sum_{j=1}^m \kappa\left(\left|t_j - t_{j-1}\right|/(b-a)\right)}\right.\\
&\qquad \left. + \frac{(1/2)\,\phi_j\left(2K_2(r)\left|v\left(t_j\right) - v\left(t_{j-1}\right)\right|\right)}{\sum_{j=1}^m \kappa\left(\left|t_j - t_{j-1}\right|/(b-a)\right)}\right)\\
&= \frac{1}{2}\kappa\sigma_\phi\left(2K_5(r)u, P\right) + \frac{1}{2}\kappa\sigma_\phi\left(2K_2(r)v, P\right)\\
&\le K_6(r)\left(\|u\|_{\kappa\Phi} + \|v\|_{\kappa\Phi}\right)\\
&\le K_7(r)\|u - v\|_{\kappa\Phi}.
\end{aligned} \tag{38}$$

Summing up both partial sums and observing that $K_4(r)$ and $K_7(r)$ do not depend on the partition P we conclude that

$$\kappa V_\Phi\left(\frac{Hu - Hv}{\left(K_4(r) + K_7(r)\right)\|u - v\|_{\kappa\Phi}}\right) \le 1 \tag{39}$$

which proves the assertion.

Conversely, suppose that H satisfies a Lipschitz condition. By assumption, the constant

$$\begin{aligned}
K(r) := \sup\left\{\frac{\|Hu - Hv\|_{\kappa\Phi}}{\|u - v\|_{\kappa\Phi}} : u, v \in \kappa\Phi BV[a,b],\right.\\
\left. \|u\|_{\kappa\Phi} \le r, \|v\|_{\kappa\Phi} \le r, u \ne v\right\}
\end{aligned} \tag{40}$$

is finite for each $r > 0$. Considering, in particular, both functions u and v in (40) constant, we see that

$$|h(u) - h(v)| \le K(r)|u - v| \quad (u, v \in \mathbb{R}, |u| \le r, |v| \le r). \tag{41}$$

This shows that h is locally Lipschitz, and so the derivative h' exists almost everywhere in $\mathbb{R}$. It remains to prove that h' exists everywhere in $\mathbb{R}$ and is locally Lipschitz. For the proof of the first claim we show that h' exists in any closed interval $I = [a,b]$.

Given $r > 0$, we consider $z \in \kappa\Phi BV[a,b]$ with $\|z\|_{\kappa\Phi} \le r/2$. Let $\{\alpha_n\}_{n=1}^\infty$ be a decreasing sequence of positive real numbers converging to 0; without loss of generality, we may assume that $\alpha_n \le r/2$ for all $n \in \mathbb{N}$. We define a sequence of functions $h_{\alpha_n,z} : [a,b] \to \mathbb{R}$ by

$$h_{\alpha_n,z}(t) = \frac{h\left(z(t) + \alpha_n\right) - h(z(t))}{\alpha_n} \quad (t \in [a,b]). \tag{42}$$

Since the composition operator H associated with h acts in the space $\kappa\Phi BV[a,b]$, by assumption, the functions $h_{\alpha_n,z}$ given by (42) belong to $\kappa\Phi BV[a,b]$.

Now, we show that the sequence $\{h_{\alpha_n,z}\}_{n=1}^\infty$ has uniformly bounded $\kappa\Phi$-variation for all $z \in \kappa\Phi BV[a,b]$ with $\|z\|_{\kappa\Phi} \le r/2$. In fact, let $\pi = \{t_0, t_1, \ldots, t_m\}$ be a partition of the interval $[a,b]$. For each $n \in \mathbb{N}$ we define functions u_n and v by

$$u_n(t) = z(t) + \alpha_n, \quad v(t) = z(t) \quad (t \in [a,b]). \tag{43}$$

Then $\|u_n\|_{\kappa\Phi} \le r$ and $\|v\|_{\kappa\Phi} \le r$. Furthermore, from Lemma 11, (42), and (43), we obtain the estimates

$$\begin{aligned}
&\frac{\sum_{j=1}^{m}\phi_j\left(\left|\alpha_n\left[h_{\alpha_n,z}\left(t_j\right)-h_{\alpha_n,z}\left(t_{j-1}\right)\right]\right|/\|Hu_n-Hv\|_{\kappa\Phi}\right)}{\sum_{j=1}^{m}\kappa\left(b-a\right)}\\
&=\frac{\sum_{j=1}^{m}\phi_j\left(\left|h\left(z\left(t_j\right)+\alpha_n\right)-h\left(z\left(t_j\right)\right)-h\left(z\left(t_{j-1}\right)+\alpha_n\right)+h\left(z\left(t_{j-1}\right)\right)\right|/\|Hu_n-Hv\|_{\kappa\Phi}\right)}{\sum_{j=1}^{m}\kappa\left(b-a\right)}\\
&=\frac{\sum_{j=1}^{m}\phi_j\left(\left|h\left(u_n\left(t_j\right)\right)-h\left(v\left(t_j\right)\right)-h\left(u_n\left(t_{j-1}\right)\right)+h\left(v\left(t_{j-1}\right)\right)\right|/\|Hu_n-Hv\|_{\kappa\Phi}\right)}{\sum_{j=1}^{m}\kappa\left(b-a\right)}\\
&=\frac{\sum_{j=1}^{m}\phi_j\left(\left(Hu_n-Hv\right)/\|Hu_n-Hv\|_{\kappa\Phi}\right)}{\sum_{j=1}^{m}\kappa\left(b-a\right)}\\
&\le\kappa V_\Phi\left(\frac{Hu_n-Hv}{\|Hu_n-Hv\|_{\kappa\Phi}};[a,b]\right)\le 1.
\end{aligned}\tag{44}$$

Since the partition $\pi = \{t_0, t_1, \ldots, t_m\}$ was arbitrary, the inequality

$$\kappa V_\Phi\left(\frac{\alpha_n h_{\alpha_n,z}}{\|Hu_n-Hv\|_{\kappa\Phi}};[a,b]\right)\le 1 \tag{45}$$

holds for every $n \in \mathbb{N}$ and each $z \in \kappa\Phi BV[a,b]$ with $\|z\|_{\kappa\Phi} \le r/2$. From Lemma 11, the definition of the function $h_{\alpha_n,z}$ in (42), and the definition of the functions u_n and v in (43), we further get

$$\begin{aligned}
\left\|\alpha_n h_{\alpha_n,z}\right\|_{\kappa\Phi} &= \left\|h\left(z+\alpha_n\right)-h\left(z\right)\right\|_{\kappa\Phi}\\
&= \left\|h\left(u_n\right)-h\left(v\right)\right\|_{\kappa\Phi}\\
&\le K\left(r\right)\left\|u_n-v\right\|_{\kappa\Phi} = K\left(r\right)\alpha_n
\end{aligned}\tag{46}$$

and hence $\|h_{\alpha_n,z}\|_{\kappa\Phi} \le K(r)$. By Lemma 11, we conclude that

$$\kappa V_\Phi\left(h_{\alpha_n,z}\right)\le K\left(r\right), \tag{47}$$

which shows that the sequence $\{h_{\alpha_{n,z}}\}_{n=1}^{\infty}$ satisfies the hypotheses of Theorem 16.

Theorem 16 ensures the existence of a pointwise convergent subsequence of $\{h_{\alpha_n,z}\}_{n=1}^{\infty}$; without loss of generality we assume that the whole sequence $\{h_{\alpha_n,z}\}_{n=1}^{\infty}$ converges pointwise on $[a,b]$ to some function $f \in \kappa\Phi BV[a,b]$.

Now we define $z(t) := \lambda t$, where $\lambda > 0$ is so small that $\|z\|_{\kappa\Phi} \le r/2$. By (43) we see that

$$\begin{aligned}
f\left(t\right) &= \lim_{n\to\infty}\frac{h\left(z\left(t\right)+\alpha_n\right)-h\left(z\left(t\right)\right)}{\alpha_n}\\
&= \lim_{n\to\infty}\frac{h\left(\lambda t+\alpha_n\right)-h\left(\lambda t\right)}{\alpha_n} = \lambda h'\left(\lambda t\right)
\end{aligned}\tag{48}$$

for almost all $t \in [a,b]$. Since the primitive of f and the function $t \mapsto h(\lambda t)$ are both absolutely continuous and have the same derivative on $[a,b]$, we conclude that they differ only by some constant on $[a,b]$, and so h' exists everywhere on $[a,b]$. From the invariance principle (Lemma 15) we deduce that the derivative h' of h exists on any interval and so everywhere in $\mathbb{R}$.

It remains to prove that h' satisfies a local Lipschitz condition. Denoting by F the composition operator associated with the function f from (48), we claim that, for $z \in \kappa\Phi BV[a,b]$ with $\|z\|_{\kappa\Phi} \le r/2$, we have

$$\|Fz\|_{\kappa\Phi} \le K\left(r\right), \tag{49}$$

where $K(r)$ is the Lipschitz constant from (40). In fact, we conclude that

$$\|f\|_{\kappa\Phi} \le \lim_{n\to\infty}\inf\|h_n\|_{\kappa\Phi}, \tag{50}$$

whenever the sequence $\{h_n\}_{n=1}^{\infty}$ of functions $h_n \in \kappa\Phi BV[a,b]$ converges pointwise on $[a,b]$ to a function f. Combining this with (47) and the observation that the sequence $\{h_{\alpha_n,z}(a)\}$ converges as $n \to \infty$, we obtain (49). We conclude that the composition operator F maps the space $\kappa\Phi BV[a,b]$ into itself, and so the corresponding function f is locally Lipschitz on $\mathbb{R}$. By (48), the same is true for the function h'. □

Conflict of Interests

The authors declare that there is no conflict of interests.

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Research Article

Q_K Spaces on the Unit Circle

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We introduce a new space $Q_K(\partial\mathbb{D})$ of Lebesgue measurable functions on the unit circle connecting closely with the Sobolev space. We obtain a necessary and sufficient condition on K such that $Q_K(\partial\mathbb{D}) = \mathrm{BMO}(\partial\mathbb{D})$, as well as a general criterion on weight functions K_1 and K_2, $K_1 \le K_2$, such that $Q_{K_1}(\partial\mathbb{D}) \subsetneq Q_{K_2}(\partial\mathbb{D})$. We also prove that a measurable function belongs to $Q_K(\partial\mathbb{D})$ if and only if it is Möbius bounded in the Sobolev space $L^2_K(\partial\mathbb{D})$. Finally, we obtain a dyadic characterization of functions in $Q_K(\partial\mathbb{D})$ spaces in terms of dyadic arcs on the unit circle.

1. Introduction

In recent years a new class of Möbius invariant function spaces, called Q spaces, has attracted a lot of attention. These spaces were originally defined in [1] as spaces of analytic functions in the unit disc $\mathbb{D}$ in the complex plane $\mathbb{C}$. Later on, some further generalizations such as $F(p,q,s)$ and Q_K appeared; see [2, 3], for example. Let $\partial\mathbb{D}$ be the boundary of $\mathbb{D}$. For $p \in (-\infty,\infty)$, Xiao studied the space $Q_p(\partial\mathbb{D})$ in paper [4], consisting of all Lebesgue measurable functions $f : \partial\mathbb{D} \to \mathbb{C}$ with

$$\|f\|_{Q_p(\partial\mathbb{D})} = \sup_{I\subset\partial\mathbb{D}} \left(\frac{1}{|I|^p} \iint_I \frac{|f(u) - f(v)|^2}{|u-v|^{2-p}} |du|\,|dv| \right)^{1/2} < \infty, \tag{1}$$

where the supremum is taken over all subarcs $I \subset \partial\mathbb{D}$ and $|I|$ is the arc length of I. A series of results of $Q_p(\partial\mathbb{D})$ can be found in [4–6]. Note that if $p = 2$, then $Q_p(\partial\mathbb{D})$ coincides with $\mathrm{BMO}(\partial\mathbb{D})$, the space of measurable functions of bounded mean oscillation on $\partial\mathbb{D}$ introduced by John and Nirenberg in [7]. For any given arc $I \subset \partial\mathbb{D}$ and $L^2(\partial\mathbb{D})$ function f, the square mean oscillation of f on I is defined by

$$\Phi_f(I) = \frac{1}{|I|} \int_I |f(u) - f_I|^2 |du|, \tag{2}$$

where

$$f_I = \frac{1}{|I|} \int_I f(u) |du|. \tag{3}$$

Then a function $f \in L^2(\partial\mathbb{D})$ is said to belong to the space $\mathrm{BMO}(\partial\mathbb{D})$ if and only if $\|f\|^2_{\mathrm{BMO}(\partial\mathbb{D})} = \sup_{I\subset\partial\mathbb{D}} \Phi_f(I) < \infty$.

In paper [2], Essén and Wulan studied Q_K spaces of holomorphic functions on the unit disc $\mathbb{D}$ and developed their general theory. Later on, Wulan and Zhou gave a decomposition theorem on Q_K spaces and built a relationship between Q_K spaces of analytic functions and the Morrey type space; see [8, 9], for example. Our aim in this paper will be to extend these ideas to the real Q_K spaces so that we may obtain related results on the "real Q_K spaces" by using known results on real Hardy spaces. Historically, the "real variable" theory of Hardy spaces has proved to be important in the development of harmonic analysis. We feel that these spaces are intrinsically interesting and that understanding them better will help inform our study of spaces of holomorphic functions.

As a continuation of [2], Essén et al. described the boundary values behavior of analytic functions in Q_K spaces [10] as follows.

Theorem EWX. *Let $K : [0,\infty) \to [0,\infty)$ be nondecreasing and satisfy the conditions*

$$\int_0^1 \varphi_K(t)\frac{dt}{t} < \infty, \tag{4}$$

$$\int_1^\infty \varphi_K(t)\frac{dt}{t^2} < \infty, \tag{5}$$

where

$$\varphi_K(s) = \sup_{0<t\le 1}\frac{K(st)}{K(t)}, \quad 0 < s < \infty. \tag{6}$$

Then $f \in H^2$ belongs to the space Q_K if and only if

$$\sup_{I\subset\partial\mathbb{D}}\left(\iint_I \frac{|f(u)-f(v)|^2}{|u-v|^2}K\left(\frac{|u-v|}{|I|}\right)|du||dv|\right)^{1/2} < \infty. \tag{7}$$

The above theorem suggests the following definition of $Q_K(\partial\mathbb{D})$ spaces on the unit circle. Let $K : [0,\infty) \to [0,\infty)$ be a nondecreasing function. The space $Q_K(\partial\mathbb{D})$ consists of all Lebesgue measurable functions f on $\partial\mathbb{D}$ for which (7) holds. If $K(t) = t^p$, $0 \le p < \infty$, $Q_K(\partial\mathbb{D})$ coincides with $Q_p(\partial\mathbb{D})$. The space $Q_K(\partial\mathbb{D})$ first appeared in [11], where Pau gave that the Szegö projection from $Q_K(\partial\mathbb{D})$ to Q_K is bounded and surjective. By [10] and [11] we know that $Q_K = H^2 \cap Q_K(\partial\mathbb{D})$ if the weight function K satisfies conditions (4) and (5).

In addition, $f \lesssim g$ (for two functions f and g) means that there is a constant $C > 0$ (independent of f and g) such that $f \le Cg$. We say that $f \approx g$ (i.e., f is comparable with g) whenever $f \lesssim g \lesssim f$. In the whole paper we assume that K is doubling; that is, $K(2t) \approx K(t)$.

2. BMO and $Q_K(\partial\mathbb{D})$ Spaces

In this section, we investigate the relationship between spaces $Q_K(\partial\mathbb{D})$ and $\mathrm{BMO}(\partial\mathbb{D})$ and study how $Q_K(\partial\mathbb{D})$ depends on the weight function K.

The following identity is easily verified:

$$\frac{1}{|I|^2}\iint_I |f(u)-f(v)|^2|du||dv| = 2\Phi_f(I). \tag{8}$$

Proposition 1. *$Q_K(\partial\mathbb{D})$ is a subset of BMO$(\partial\mathbb{D})$ for all K.*

Proof. For $I \subset \partial\mathbb{D}$, it is easy to see that

$$\begin{aligned} &I\times I \\ &\quad = \{(z,w) : 0 < |z-w| < |I|, z,w \in I\}\bigcup\{(z,z), z \in I\}. \end{aligned} \tag{9}$$

Note that the area measure of $\{(z,z), z \in I\}$ is zero. For $z,w \in I$, we have

$$\begin{aligned} &\{(z,w) : 0 < |z-w| < |I|, z,w \in I\} \\ &\quad\subset \bigcup_{k=1}^\infty\left\{(z,w) : \frac{|I|}{2^k} < |z-w| \le \frac{|I|}{2^{k-1}}\right\}. \end{aligned} \tag{10}$$

Suppose that $f \in Q_K(\partial\mathbb{D})$. For $I \subset \partial\mathbb{D}$ and integer k, denote by $2^{-k}I$ the subarcs of I with arc length $2^{-k}|I|$. Then

$$\begin{aligned} &\iint_I |f(u)-f(v)|^2|du||dv| \\ &\quad\le \sum_{k=1}^\infty\iint_{|I|/2^k<|u-v|\le|I|/2^{k-1}}|f(u)-f(v)|^2|du||dv| \\ &\quad\lesssim \frac{1}{K(1)}\sum_{k=1}^\infty\left(\frac{|I|}{2^k}\right)^2 \\ &\qquad\times\iint_{|u-v|\le|I|/2^{k-1}}\frac{|f(u)-f(v)|^2}{|u-v|^2}K\left(\frac{|u-v|}{2^{-k}|I|}\right) \\ &\qquad\times|du||dv| \\ &\quad\lesssim \frac{1}{K(1)}\sum_{k=1}^\infty\left(\frac{|I|}{2^k}\right)^2 \\ &\qquad\times 2^k\iint_{I/2^{k-1}}\frac{|f(u)-f(v)|^2}{|u-v|^2} \\ &\qquad\times K\left(\frac{|u-v|}{2^{1-k}|I|}\right)|du||dv| \\ &\quad\lesssim |I|^2\|f\|^2_{Q_K(\partial\mathbb{D})}. \end{aligned} \tag{11}$$

We have $f \in \mathrm{BMO}(\partial\mathbb{D})$ by (8). □

Corollary 2. *The space $Q_K(\partial\mathbb{D})$ is Banach with the norm of $\|f\| = |f(0)| + \|f\|_{Q_K(\partial\mathbb{D})}$, where $\|f\|_{Q_K(\partial\mathbb{D})}$ is the supremum of (7).*

Proof. Let $\{f_n\}$ be a Cauchy sequence in $Q_K(\partial\mathbb{D})$. By Proposition 1 we know that $Q_K(\partial\mathbb{D})$ is subset of $\mathrm{BMO}(\partial\mathbb{D})$. Hence $\{f_n\}$ is a Cauchy sequence in $\mathrm{BMO}(\partial\mathbb{D})$ as well and $f_n \to f$ in $\mathrm{BMO}(\partial\mathbb{D})$ for some f. It follows from Fatou's lemma that, for every integer $n \ge 1$,

$$\|f - f_n\|_{Q_K(\partial\mathbb{D})} \le \limsup_{j\to\infty}\|f_j - f_n\|_{Q_K(\partial\mathbb{D})}. \tag{12}$$

This gives $f_n \to f$ in $Q_K(\partial\mathbb{D})$. □

Theorem 3. *$Q_K(\partial\mathbb{D}) = \mathrm{BMO}(\partial\mathbb{D})$ if and only if*

$$\int_0^1 \frac{K(s)}{s^2}ds < \infty. \tag{13}$$

Proof. Assume that $f \in \mathrm{BMO}(\partial\mathbb{D})$ and (13) holds. We use nI for the arc in $\partial\mathbb{D}$ which has the same center as I and length

$n|I|$ for a nonnegative integer n. For any given $I \subset \partial\mathbb{D}$ and $|t| \le |I|$, then

$$\begin{aligned} &\int_I \left|f\left(e^{i(\theta+t)}\right) - f\left(e^{i\theta}\right)\right|^2 d\theta \\ &\quad \lesssim \int_I \left\{\left|f\left(e^{i(\theta+t)}\right) - f_I\right|^2 + \left|f\left(e^{i\theta}\right) - f_I\right|^2\right\} d\theta \\ &\quad \lesssim |I| \|f\|^2_{\mathrm{BMO}(\partial\mathbb{D})} + \int_I \left|f\left(e^{i(\theta+t)}\right) - f_{3I}\right|^2 d\theta \\ &\qquad + |I| \left|f_{3I} - f_I\right|^2 \\ &\quad \lesssim |I| \|f\|^2_{\mathrm{BMO}(\partial\mathbb{D})}. \end{aligned} \tag{14}$$

By the inequality $2x/\pi < \sin x < x$ for $0 < x < \pi/2$ and the above estimate, we have

$$\begin{aligned} &\iint_I \frac{\left|f\left(e^{i\theta}\right) - f\left(e^{i\varphi}\right)\right|^2}{\left|e^{i\theta} - e^{i\varphi}\right|^2} K\left(\frac{\left|e^{i\theta} - e^{i\varphi}\right|}{|I|}\right) d\theta\, d\varphi \\ &\quad \lesssim \int_{|t|\le|I|} \int_I \frac{\left|f\left(e^{i(\theta+t)}\right) - f\left(e^{i\theta}\right)\right|^2}{\left|e^{i(\theta+t)} - e^{i\theta}\right|^2} \\ &\qquad\qquad \times K\left(\frac{\left|e^{i(\theta+t)} - e^{i\theta}\right|}{|I|}\right) d\theta\, dt \\ &\quad \lesssim \|f\|^2_{\mathrm{BMO}(\partial\mathbb{D})} |I| \int_{|t|\le|I|} \frac{K\left(|\sin(t/2)| / |I|\right)}{(\sin(t/2))^2} dt \\ &\quad \lesssim \|f\|^2_{\mathrm{BMO}(\partial\mathbb{D})} |I| \int_0^{|I|} \frac{K(t/|I|)}{(t/2)^2} dt \\ &\quad \lesssim \|f\|^2_{\mathrm{BMO}(\partial\mathbb{D})} \int_0^1 \frac{K(s)}{s^2} ds. \end{aligned} \tag{15}$$

The above estimate shows that $\mathrm{BMO}(\partial\mathbb{D}) \subset Q_K(\partial\mathbb{D})$. This and Proposition 1 imply $\mathrm{BMO}(\partial\mathbb{D}) = Q_K(\partial\mathbb{D})$.

Conversely, suppose that $Q_K(\partial\mathbb{D}) = \mathrm{BMO}(\partial\mathbb{D})$. If

$$\int_0^1 \frac{K(s)}{s^2} ds = \infty, \tag{16}$$

we can choose an integer sequence $\{\lambda_j\}_{j=1}^{\infty}$ such that

$$\int_{2\pi 2^{-\lambda_j}}^1 \frac{K(s)}{s^2} ds \ge j, \quad j = 1, 2, 3, \ldots. \tag{17}$$

Define a function f as follows:

$$f\left(e^{i\theta}\right) = \sum_{j=1}^{\infty} \frac{1}{j} e^{i2^{\lambda_j}\theta}. \tag{18}$$

Then $f \in \mathrm{BMO}(\partial\mathbb{D})$ ([12], page 178). By assumption we have $f \in Q_K(\partial\mathbb{D})$. It is easy to see that

$$\int_0^{2\pi} \left|f\left(e^{i(\theta+t)}\right) - f\left(e^{i\theta}\right)\right|^2 d\theta = \sum_{j=1}^{\infty} \frac{1}{j^2} \left|e^{i2^{\lambda_j}t} - 1\right|^2. \tag{19}$$

We give the following estimate which will be proved later:

$$\int_0^{\pi} \sin^2 \frac{nt}{2} \frac{K(t)}{t^2} dt \gtrsim \int_{2\pi/n}^{\pi} \frac{K(t)}{t^2} dt, \quad n = 1, 2, \ldots. \tag{20}$$

By (17), (19), and (20), we have

$$\begin{aligned} &\iint_0^{2\pi} \frac{\left|f\left(e^{i\theta}\right) - f\left(e^{i\varphi}\right)\right|^2}{\left|e^{i\theta} - e^{i\varphi}\right|^2} K\left(\frac{\left|e^{i\theta} - e^{i\varphi}\right|}{2\pi}\right) d\theta\, d\varphi \\ &\quad = \sum_{j=1}^{\infty} \frac{1}{j^2} \int_0^{2\pi} \frac{\left|e^{i2^{\lambda_j}t} - 1\right|^2}{\left|e^{it} - 1\right|^2} K\left(\frac{\left|e^{it} - 1\right|}{2\pi}\right) dt \\ &\quad \gtrsim \sum_{j=1}^{\infty} \frac{1}{j^2} \int_0^{\pi} \frac{\sin^2\left(2^{\lambda_j}t/2\right)}{\sin^2(t/2)} K(t)\, dt \\ &\quad \gtrsim \sum_{j=1}^{\infty} \frac{1}{j^2} \int_0^{\pi} \sin^2 \frac{2^{\lambda_j}t}{2} \frac{K(t)}{t^2} dt \\ &\quad \gtrsim \sum_{j=1}^{\infty} \frac{1}{j^2} \int_{2\pi 2^{-\lambda_j}}^{\pi} \frac{K(t)}{t^2} dt \\ &\quad \gtrsim \sum_{j=1}^{\infty} \frac{1}{j} = \infty. \end{aligned} \tag{21}$$

We now prove (20). Note that $j/n \ge (j+1)/2n$ is valid for all $j, n = 1, 2, \ldots$ and $K(2t) \approx K(t)$. Then

$$\begin{aligned} &\int_0^{\pi} \sin^2 \frac{nt}{2} \frac{K(t)}{t^2} dt \\ &\quad = \sum_{j=0}^{n-1} \int_{j\pi/n}^{((j+1)/n)\pi} \frac{1 - \cos nt}{2} \frac{K(t)}{t^2} dt \\ &\quad \ge \sum_{j=0}^{n-1} \frac{K(j\pi/n)}{((j+1)\pi/n)^2} \int_{j\pi/n}^{((j+1)/n)\pi} \frac{1 - \cos nt}{2} dt \\ &\quad \gtrsim \sum_{j=1}^{n-1} \frac{K((j+1)\pi/n)}{((j+1)\pi/n)^2} \frac{\pi}{2n} \\ &\quad \ge \frac{1}{2} \sum_{j=1}^{n-1} \left(\frac{j}{j+1}\right)^2 \int_{j\pi/n}^{((j+1)/n)\pi} \frac{K(t)}{t^2} dt \\ &\quad \gtrsim \int_{2\pi/n}^{\pi} \frac{K(t)}{t^2} dt. \end{aligned} \tag{22}$$

The proof is complete. □

It is reasonable to assume that $\lim_{r\to 0^+} K(r) = 0$ for otherwise weight function K basically dose not play any role. Moreover, the function f must be at least locally L^2 on the boundary when f belongs to the Q_K spaces. Therefore the weight function K plays a role only if t is small. Then the following result is obvious.

Theorem 4. *Let $r_0 \in (0,1)$ such that $K(r_0) > 0$, and set $K_1(r) = \inf(K(r), K(r_0))$. Then $Q_{K_1}(\partial\mathbb{D}) = Q_K(\partial\mathbb{D})$.*

Proof. Since $K_1 \le K$ and K_1 is nondecreasing, it is easy to see that $Q_K(\partial\mathbb{D}) \subset Q_{K_1}(\partial\mathbb{D})$. We now prove $Q_{K_1}(\partial\mathbb{D}) \subset Q_K(\partial\mathbb{D})$. Note that there exists an integer $m \in \mathbb{N}$ such that $m^{-1} \le r_0/2$. If $f \in Q_{K_1}(\partial\mathbb{D})$, then $f \in \mathrm{BMO}(\partial\mathbb{D})$ by Proposition 1. For any $I \subset \partial\mathbb{D}$, divide I into the m subarcs of length $|I|/m$. For $1 \le j \le m$, denote I_j the jth subarcs, arranged in the natural order. Let $I_{j,k}$ be the smallest subarcs containing I_j and I_k. Then we have

$$\begin{aligned}
A &= \sum_{\substack{j,k=1\\ k-1\le j\le k+1}}^{m} \int_{I_j}\int_{I_k} \frac{|f(u)-f(v)|^2}{|u-v|^2} K\left(\frac{|u-v|}{|I|}\right)|du|\,|dv| \\
&= \sum_{\substack{j,k=1\\ k-1\le j\le k+1}}^{m} \int_{I_j}\int_{I_k} \frac{|f(u)-f(v)|^2}{|u-v|^2} K_1\left(\frac{|u-v|}{|I|}\right)|du|\,|dv| \\
&\le \iint_I \frac{|f(u)-f(v)|^2}{|u-v|^2} K_1\left(\frac{|u-v|}{|I|}\right)|du|\,|dv| \\
&\le \|f\|_{Q_{K_1}(\partial\mathbb{D})}, \\
B &= \sum_{\substack{j,k=1\\ j>k+1,\, j<k-1}}^{m} \int_{I_j}\int_{I_k} \frac{|f(u)-f(v)|^2}{|u-v|^2} K\left(\frac{|u-v|}{|I|}\right)|du|\,|dv| \\
&\le K(1) \sum_{\substack{j,k=1\\ j>k+1,\, j<k-1}}^{m} \left(\frac{m}{I}\right)^2 \int_{I_j}\int_{I_k} |f(u)-f(v)|^2\,|du|\,|dv| \\
&\lesssim \sum_{\substack{j,k=1\\ j>k+1,\, j<k-1}}^{m} \frac{m}{|I|}\left(\int_{I_j}\left|f(u)-f_{I_{j,k}}\right|^2|du| \right. \\
&\qquad \left. + \int_{I_k}\left|f(v)-f_{I_{j,k}}\right|^2|dv|\right) \\
&\lesssim \|f\|^2_{\mathrm{BMO}(\partial\mathbb{D})} \lesssim \|f\|^2_{Q_{K_1}(\partial\mathbb{D})}.
\end{aligned} \tag{23}$$

The above estimate gives

$$\begin{aligned}
&\iint_I \frac{|f(u)-f(v)|^2}{|u-v|^2} K\left(\frac{|u-v|}{|I|}\right)|du|\,|dv| \\
&\quad = A + B \lesssim \|f\|^2_{Q_{K_1}(\partial\mathbb{D})}.
\end{aligned} \tag{24}$$

Hence $f \in Q_K(\partial\mathbb{D})$. So we have $Q_{K_1}(\partial\mathbb{D}) \subset Q_K(\partial\mathbb{D})$. The proof is complete. □

The following result is natural in view of Proposition 1 and Theorem 3.

Theorem 5. *Let $K_1 \le K_2$ and assume that $K_1(r)/K_2(r) \to 0$ as $r \to 0$. If*

$$\int_0^1 \frac{K_2(s)}{s^2}\,ds = \infty, \tag{25}$$

then $Q_{K_2}(\partial\mathbb{D}) \subsetneq Q_{K_1}(\partial\mathbb{D})$.

Proof. Obviously, we have $Q_{K_2}(\partial\mathbb{D}) \subset Q_{K_1}(\partial\mathbb{D})$. We assume that $Q_{K_1}(\partial\mathbb{D}) = Q_{K_2}(\partial\mathbb{D})$. The open mapping theorem tells us that the identity map from one of those spaces into the other one is continuous. Therefore there exists a constant C such that $\|\cdot\|_{Q_{K_2}(\partial\mathbb{D})} \le C\|\cdot\|_{Q_{K_1}(\partial\mathbb{D})}$. By the assumption, there exists an integer m such that $K_1(t) \le (2C)^{-1}K_2(t)$ for $t \le m^{-1}$. For any $I \subset \partial\mathbb{D}$, divide I into the $2m$ subarcs of length $|I|/(2m)$. For $1 \le j \le 2m$, denote by I_j the jth subarcs, arranged in the natural order. Applying the same manner in handing A and B in the proof of Theorem 4, we can deduce that if $f \in Q_{K_2}(\partial\mathbb{D})$, then

$$\begin{aligned}
&\|f\|^2_{Q_{K_2}(\partial\mathbb{D})} \\
&\quad \le C \sup_{I\subset\partial\mathbb{D}} \iint_I \frac{\left|f(e^{i\theta})-f(e^{i\varphi})\right|^2}{\left|e^{i\theta}-e^{i\varphi}\right|^2} \\
&\qquad \times K_1\left(\frac{\left|e^{i\theta}-e^{i\varphi}\right|}{|I|}\right) d\theta\, d\varphi \\
&\quad = C \sup_{I\subset\partial\mathbb{D}} \left(\sum_{\substack{j,k=1\\ k-1\le j\le k+1}}^{2m} + \sum_{\substack{j,k=1\\ j<k-1,\, j>k+1}}^{2m} \right) \\
&\qquad \times \int_{I_j}\int_{I_k} \frac{\left|f(e^{i\theta})-f(e^{i\varphi})\right|^2}{\left|e^{i\theta}-e^{i\varphi}\right|^2} K_1\left(\frac{\left|e^{i\theta}-e^{i\varphi}\right|}{|I|}\right) d\theta\, d\varphi \\
&\quad \le C \sup_{I\subset\partial\mathbb{D}} \left(\frac{1}{2C} \sum_{\substack{j,k=1\\ k-1\le j\le k+1}}^{2m} \int_{I_j}\int_{I_k} \frac{|f(u)-f(v)|^2}{|u-v|^2} K_2 \right. \\
&\qquad \left. \times \left(\frac{|u-v|}{|I|}\right)|du|\,|dv| \right) \\
&\qquad + C \sup_{I\subset\partial\mathbb{D}} \left(\sum_{\substack{j,k=1\\ j<k-1,\, j>k+1}}^{2m} \int_{I_j}\int_{I_k} \frac{|f(u)-f(v)|^2}{|u-v|^2} K_1 \right. \\
&\qquad \left. \times \left(\frac{|u-v|}{|I|}\right)|du|\,|dv| \right) \\
&\quad = \frac{1}{2}\|f\|^2_{Q_{K_2}(\partial\mathbb{D})} + M\|f\|^2_{\mathrm{BMO}(\partial\mathbb{D})},
\end{aligned} \tag{26}$$

where M is a constant which is dependent on C. Consequently, for any $f \in Q_{K_2}(\partial\mathbb{D})$ and $I \subset \partial\mathbb{D}$, we have

$$\iint_I \frac{|f(u)-f(v)|^2}{|u-v|^2} K_2\left(\frac{|u-v|}{|I|}\right)|du|\,|dv| \lesssim \|f\|^2_{\mathrm{BMO}(\partial\mathbb{D})}. \tag{27}$$

A simple computation shows that $z^n \in Q_{K_2}(\partial\mathbb{D})$ for $n = \pm1, \pm2, \ldots$. So all polynomials belong to $Q_{K_2}(\partial\mathbb{D})$ spaces. For any given $g(u) = \sum_{j=-\infty}^{\infty} a_j u^j \in \mathrm{BMO}(\partial\mathbb{D})$, denote by $g_n(u) = \sum_{j=-n}^{n} a_j u^j$ the truncation of the function g. Then $g_n \in Q_{K_2}(\partial\mathbb{D})$ and $\|g_n\|_{\mathrm{BMO}(\partial\mathbb{D})} \leq \|g\|_{\mathrm{BMO}(\partial\mathbb{D})}$. Applying Fatou's lemma, we deduce that

$$\begin{aligned}\sup_{I\subset\partial\mathbb{D}} \iint_I \frac{|g(u)-g(v)|^2}{|u-v|^2} K_2\left(\frac{|u-v|}{|I|}\right)|du|\,|dv| \\ \lesssim \|g\|^2_{\mathrm{BMO}(\partial\mathbb{D})}.\end{aligned} \tag{28}$$

Equation (28) and Proposition 1 show $\mathrm{BMO}(\partial\mathbb{D}) = Q_{K_2}(\partial\mathbb{D})$. It follows from Theorem 3 that the integral (13) with $K = K_2$ must be convergent, which contradicts our assumption. We conclude that we must have $Q_{K_2}(\partial\mathbb{D}) \subsetneq Q_{K_1}(\partial\mathbb{D})$. □

3. Möbius Invariant $Q_K(\partial\mathbb{D})$ Spaces

Let $K : [0,\infty) \to [0,\infty)$ be a nondecreasing function. The Sobolev type space $L^2_K(\partial\mathbb{D})$ consists of those Lebesgue measurable functions $f : \partial\mathbb{D} \to \mathbb{C}$ satisfying

$$\begin{aligned}&\|f\|_{L^2_K(\partial\mathbb{D})} \\ &= \left(\iint_{\partial\mathbb{D}} \frac{|f(u)-f(v)|^2}{|u-v|^2} K(|u-v|)\,|du|\,|dv|\right)^{1/2} \\ &< \infty.\end{aligned} \tag{29}$$

If $K(t) = t^p, 0 \leq p < \infty$, then $L^2_K(\partial\mathbb{D}) = L^2_p(\partial\mathbb{D})$ are sobolev spaces and are introduced in [4]. See [13] about the theory of Sobolev spaces. If $K(t) = t^p, p > 1$, then $L^2(\partial\mathbb{D})$ is a subspace of $L^2_K(\partial\mathbb{D})$. From Section 2 it turns out that $Q_K(\partial\mathbb{D})$ is closely related to the Sobolev type space $L^2_K(\partial\mathbb{D})$ on the unit circle. By (7) and (29) it follows that $Q_K(\partial\mathbb{D})$ is a subspace of $L^2_K(\partial\mathbb{D})$. As a matter of fact, we have the following result.

Theorem 6. *Let K satisfy condition* (4). *Then $f \in Q_K(\partial\mathbb{D})$, if and only if*

$$|||f|||_{Q_K(\partial\mathbb{D})} = \sup_{a\in\mathbb{D}} \|f\circ\phi_a\|_{L^2_K(\partial\mathbb{D})} < \infty, \tag{30}$$

where $\phi_a(z) = (a-z)/(1-\bar{a}z)$ is a Möbius transformation of the unit disk for $a \in \mathbb{D}$.

Proof. We acknowledge that this proof is suggested by the technique of [4]. Firstly, we give the following equality for $u = \phi_a(z)$ and $v = \phi_a(w)$:

$$\begin{aligned}&\iint_{\partial\mathbb{D}} \frac{|f\circ\phi_a(z)-f\circ\phi_a(w)|^2}{|z-w|^2} K(|z-w|)\,|dz|\,|dw| \\ &= \iint_{\partial\mathbb{D}} \frac{|f(u)-f(v)|^2}{|u-v|^2} \\ &\quad\times K\left(\frac{|u-v|\left(1-|a|^2\right)}{|1-\bar{a}u|\,|1-\bar{a}v|}\right)|du|\,|dv|.\end{aligned} \tag{31}$$

Sufficiency. Suppose that (30) holds. Choose an arc $I \subset \partial\mathbb{D}$. Without loss of generality, we assume $|I| < 1/4$. We choose a point of $a \in \mathbb{D}$ $(a \neq 0)$ such that $a/|a|$ and $2\pi(1-|a|)$ are the center and arc length of I, respectively. We have the following estimate:

$$\frac{1}{|1-\bar{a}u|} \approx \frac{1}{|I|}, \quad u \in I. \tag{32}$$

Then

$$\begin{aligned}&\iint_I \frac{|f(u)-f(v)|^2}{|u-v|^2} K\left(\frac{|u-v|}{|I|}\right)|du|\,|dv| \\ &\lesssim \iint_{\partial\mathbb{D}} \frac{|f(u)-f(v)|^2}{|u-v|^2} \\ &\quad\times K\left(\frac{|u-v|\left(1-|a|^2\right)}{|1-\bar{a}u|\,|1-\bar{a}v|}\right)|du|\,|dv|.\end{aligned} \tag{33}$$

By (31) we complete the proof of sufficiency.

Necessity. We assume that $f \subset Q_K(\partial\mathbb{D})$. For any $a \in \mathbb{D}$, let I_a be the arc in $\partial\mathbb{D}$ with the midpoint of $a/|a|$ and the arc length of $2\pi(1-|a|)$. If $a = 0$, we set $I_a = \partial\mathbb{D}$. Also, define

$$I_n = 2^n I_a, \quad n = 0, 1, \ldots, N-1, \tag{34}$$

where N is the smallest integer such that $2^N|I_a| \geq 2\pi$; that is, $I_N = \partial\mathbb{D}$. Then

$$\begin{aligned}&\iint_{\partial\mathbb{D}} \frac{|f(u)-f(v)|^2}{|u-v|^2} \\ &\quad\times K\left(\frac{|u-v|\left(1-|a|^2\right)}{|1-\bar{a}u|\,|1-\bar{a}v|}\right)|du|\,|dv| \\ &= \iint_{I_0} + \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} \int_{I_{n+1}\setminus I_n}\int_{I_{m+1}\setminus I_m} \{\cdots\} \\ &\leq \|f\|^2_{Q_K(\partial\mathbb{D})} + A + B,\end{aligned} \tag{35}$$

where

$$A=\sum_{n=0}^{N-1}\sum_{m\le n}^{N-1}\int_{I_{n+1}\setminus I_n}\int_{I_{m+1}\setminus I_m}\frac{\left|f(u)-f(v)\right|^2}{|u-v|^2}\times K\left(\frac{|u-v|\left(1-|a|^2\right)}{|1-\bar{a}u|\,|1-\bar{a}v|}\right)|du|\,|dv|,$$

$$B=\sum_{n=0}^{N-1}\sum_{m>n}^{N-1}\int_{I_{n+1}\setminus I_n}\int_{I_{m+1}\setminus I_m}\frac{\left|f(u)-f(v)\right|^2}{|u-v|^2}\times K\left(\frac{|u-v|\left(1-|a|^2\right)}{|1-\bar{a}u|\,|1-\bar{a}v|}\right)|du|\,|dv|. \tag{36}$$

For any given $u\in I_{n+1}\setminus I_n, n=1,2,\ldots$, we have

$$\frac{1}{|1-\bar{a}u|}\lesssim\frac{1}{2^n(1-|a|)}. \tag{37}$$

By (32) and (37), we obtain that

$$\begin{aligned}
&\sum_{n=0}^{N-1}\iint_{I_{n+1}\setminus I_n}\frac{\left|f(u)-f(v)\right|^2}{|u-v|^2}\times K\left(\frac{|u-v|\left(1-|a|^2\right)}{|1-\bar{a}u|\,|1-\bar{a}v|}\right)|du|\,|dv|\\
&\lesssim\sum_{n=0}^{N-1}\iint_{I_{n+1}\setminus I_n}\frac{\left|f(u)-f(v)\right|^2}{|u-v|^2}\times K\left(\frac{|u-v|}{2^{2n}\left|I_0\right|}\right)|du|\,|dv|\\
&\lesssim\sum_{n=0}^{N-1}\varphi_K\left(\frac{1}{2^n}\right)\iint_{I_{n+1}}\frac{\left|f(u)-f(v)\right|^2}{|u-v|^2}\times K\left(\frac{|u-v|}{2^{n+1}\left|I_0\right|}\right)|du|\,|dv|\\
&\lesssim\|f\|^2_{Q_K(\partial\mathbb{D})}\int_0^1\frac{\varphi_K(s)}{s}ds.
\end{aligned} \tag{38}$$

On the other hand, by Lemma 2.1 of [10], condition (4) implies that $K(t)\lesssim t^c$ for small enough $c>0$. Then

$$\begin{aligned}
&\sum_{n=1}^{N-1}\sum_{m<n-1}\int_{I_{n+1}\setminus I_n}\int_{I_{m+1}\setminus I_m}\frac{\left|f(u)-f(v)\right|^2}{|u-v|^2}\times K\left(\frac{|u-v|\left(1-|a|^2\right)}{|1-\bar{a}u|\,|1-\bar{a}v|}\right)|du|\,|dv|\\
&\lesssim\sum_{n=1}^{N-1}\sum_{m<n-1}\int_{I_{n+1}\setminus I_n}\int_{I_{m+1}\setminus I_m}\frac{\left|f(u)-f(v)\right|^2}{\left(2^n-2^m\right)^2\left|I_0\right|^2}\times K\left(\frac{\left(2^n-2^m\right)\left|I_0\right|^2}{2^{m+n}\left|I_0\right|^2}\right)|du|\,|dv|\\
&\lesssim\sum_{n=1}^{N-1}\sum_{m<n-1}K\left(\frac{1}{2^m}\right)\times\int_{I_{n+1}\setminus I_n}\int_{I_{m+1}\setminus I_m}\left(\left|f(u)-f_{I_m}\right|^2+\left|f(v)-f_{I_m}\right|^2\right)\times\left(2^{2n}\left|I_0\right|^2\right)^{-1}|du|\,|dv|\\
&\lesssim\sum_{n=1}^{N-1}\sum_{m<n-1}\frac{2^{m(1-c)}}{2^n}\times\left(\|f\|^2_{\mathrm{BMO}(\partial\mathbb{D})}+\frac{1}{2^n\left|I_0\right|}\times\int_{I_{n+1}\setminus I_n}\left|f(u)-f_{I_m}\right|^2|du|\right)\\
&\lesssim\sum_{n=1}^{N-1}\sum_{m<n-1}\frac{2^{m(1-c)}}{2^n}\|f\|^2_{\mathrm{BMO}(\partial\mathbb{D})}\left(1+(n-m)^2\right)\\
&\lesssim\|f\|^2_{\mathrm{BMO}(\partial\mathbb{D})}\sum_{n=1}^{\infty}\frac{n^2}{2^{nc}}.
\end{aligned} \tag{39}$$

Here we apply the following estimate:

$$\begin{aligned}
&\frac{1}{\left|I_{n+1}\right|}\int_{I_{n+1}}\left|f(u)-f_{I_m}\right|^2|du|\\
&\quad\le\frac{1}{\left|I_{n+1}\right|}\int_{I_{n+1}}\left|f(u)-f_{I_{n+1}}\right|^2|du|+\left|f_{I_m}-f_{I_{n+1}}\right|^2\\
&\quad\le\frac{1}{\left|I_{n+1}\right|}\int_{I_{n+1}}\left|f(u)-f_{I_{n+1}}\right|^2|du|+\left(\sum_{j=m+1}^{n+1}\left|f_{I_{j-1}}-f_{I_j}\right|\right)^2\\
&\quad\lesssim(n-m)^2\|f\|^2_{\mathrm{BMO}(\partial\mathbb{D})}.
\end{aligned} \tag{40}$$

The above estimate gives that

$$A = \left(\sum_{n=0}^{N-1} \iint_{I_{n+1}\setminus I_n} + \sum_{n=1}^{N-1} \int_{I_{n+1}\setminus I_n} \int_{I_n \setminus I_{n-1}} \right) \{\cdots\} + \sum_{n=1}^{N-1} \sum_{m<n-1} \int_{I_{n+1}\setminus I_n} \int_{I_{m+1}\setminus I_m} \{\cdots\} \lesssim \|f\|^2_{Q_K(\partial\mathbb{D})}. \quad (41)$$

Applying the same manner in handing A, we have $B \lesssim \|f\|^2_{Q_K(\partial\mathbb{D})}$. Therefore, we obtain

$$\iint_{\partial\mathbb{D}} \frac{|f(u) - f(v)|^2}{|u-v|^2} \times K\left(\frac{|u-v|\left(1-|a|^2\right)}{|1-\bar{a}u|\,|1-\bar{a}v|} \right) |du|\,|dv| \lesssim \|f\|^2_{Q_K}. \quad (42)$$

The proof is complete. □

Corollary 7. *$Q_K(\partial\mathbb{D})$ is a Möbius invariant space in the sense that $\||f\||_{Q_K(\partial\mathbb{D})} = \||f \circ \phi_a\||_{Q_K(\partial\mathbb{D})}$ for any $f \in Q_K(\partial\mathbb{D})$ and $a \in \mathbb{D}$.*

Proof. Corollary 7 is obvious by Theorem 6. □

4. Dyadic Characterization

For given arc $I \subset \partial\mathbb{D}$, denote by I_n the set of the 2^n arcs of length $2^{-n}|I|$ obtained by n successive bipartition of I. The discrete characterization of $Q_p(\partial\mathbb{D})$ space is given in [5]. We will prove a discrete characterization of $Q_K(\partial\mathbb{D})$ spaces. The following is the principle result of this section.

Theorem 8. *Let K satisfy condition* (4). *Then $f \in L^2(\partial\mathbb{D})$ belongs to the space $Q_K(\partial\mathbb{D})$, if and only if*

$$\sup_{I\subset\partial\mathbb{D}} \sum_{n=0}^{\infty} \sum_{J\in I_n} K\left(\frac{1}{2^n}\right) \Phi_f(J) < \infty. \quad (43)$$

We first acknowledge that this proof is suggested by the technique of [5]. To prove Theorem 8, we need the following lemmas.

Lemma 9. *Let $I \subset \partial\mathbb{D}$ be an arc. If $f \in L^2(\partial\mathbb{D})$, then*

$$\Psi_{f,K}(I) \approx \sum_{J\in I_1} \Psi_{f,K}(J) + \sum_{J\in I_1} |f_J - f_I|^2, \quad (44)$$

where

$$\Psi_{f,K}(I) = \sum_{n=0}^{\infty} \sum_{J\in I_n} K\left(\frac{1}{2^n}\right) \Phi_f(J). \quad (45)$$

Proof. The following result can be found in [5]:

$$\Phi_f(I) = \frac{1}{2} \sum_{J\in I_1} \Phi_f(J) + \frac{1}{2^2} \sum_{J\in I_1} |f_J - f_I|^2. \quad (46)$$

Note that $I_k = \cup_{J\in I_1} J_{k-1}$ and $K(2^{-k}) \approx K(2^{-k-1}), k = 0, 1, 2 \ldots$. By (46), we have

$$\begin{aligned} \Psi_{f,K}(I) &= \Phi_f(I) + \sum_{k=1}^{\infty} \sum_{J\in I_1} \sum_{U\in I_{k-1}} K\left(\frac{1}{2^k}\right) \Phi_f(U) \\ &\approx \sum_{J\in I_1} \left(\Psi_{f,K}(J) + \Phi_f(J) + |f_J - f_I|^2 \right) \\ &\approx \sum_{J\in I_1} \left(\Psi_{f,K}(J) + |f_J - f_I|^2 \right). \end{aligned} \quad (47)$$

The proof is complete. □

Lemma 10. *Let K satisfy condition* (4). *Let I, I', I'' be three arcs of equal length: $|I| = |I'| = |I''|$, such that I' and I'' are adjacent and $I \subset I' \cup I''$. Then for any $f \in L^2(\partial\mathbb{D})$, we have*

$$\Psi_{f,K}(I) \lesssim \Psi_{f,K}\left(I'\right) + \Psi_{f,K}\left(I''\right) + |f_{I'} - f_{I''}|. \quad (48)$$

Proof. See [5] about the proof of the following inequality:

$$\Phi_f(I) \lesssim \Phi_f\left(I'\right) + \Phi_f\left(I''\right) + |f_{I'} - f_{I''}|^2. \quad (49)$$

Without loss of generality, we assume that $I' = [0, 1)$ and $I'' = [1, 2)$. For each integer $j \geq 0$, let $\{I_{j,k}\}_{k=1}^{2^j+1}$ be the set of the 2^{j+1} dyadic arcs of length 2^{-j} contained in $I' \cup I''$, arranged in the natural order. If $J \in I_j$, then $J \in I_{j,k} \cup I_{j,k+1}$ for some k; by (48) we have

$$\Phi_f\left(I_j\right) \lesssim \Phi_f\left(I_{j,k}\right) + \Phi_f\left(I_{j,k+1}\right) + \left|f_{I_{j,k}} - f_{I_{j,k+1}}\right|^2. \quad (50)$$

The different choices of $J \in I_j$ yield different k. Summing over all j and J, we have

$$\begin{aligned} \Psi_{f,K}(I) &= \sum_{j=0}^{\infty} \sum_{J\in I_j} K\left(\frac{1}{2^j}\right) \Phi_f(J) \\ &\leq 2 \sum_{j=0}^{\infty} \sum_{k=1}^{2^{j+1}} K\left(\frac{1}{2^j}\right) \Phi_f\left(I_{j,k}\right) \\ &\quad + \sum_{j=0}^{\infty} \sum_{k=1}^{2^{j+1}-1} K\left(\frac{1}{2^j}\right) \left|f_{I_{j,k}} - f_{I_{j,k+1}}\right|^2. \end{aligned} \quad (51)$$

It is easy to see that

$$\sum_{j=0}^{\infty} \sum_{k=1}^{2^{j+1}} K\left(\frac{1}{2^j}\right) \Phi_f\left(I_{j,k}\right) \lesssim \Psi_{f,K}\left(I'\right) + \Psi_{f,K}\left(I''\right). \quad (52)$$

The following estimate about the final double sum first appeared in Lemma 1 of [5]. Consider

$$\sum_{k=1}^{2^{j+1}-1} \left|f_{I_{j,k}} - f_{I_{j,k+1}}\right|^2 \lesssim \sum_{l=1}^{j} \sum_{J\in I'_{j-l}\cup I''_{j-l}} l^2 \Phi_f(J) + |f_{I'} - f_{I''}|^2. \quad (53)$$

If K satisfies condition (4), Lemma 2.1 in [10] implies that there exists some small enough $c > 0$ such that $t^{-c}K(t)$ is nondecreasing. Substituting $j = m + l$ and summing over j, we finally obtain

$$\begin{aligned}
&\sum_{j=0}^{\infty}\sum_{k=1}^{2^{j+1}-1} K\left(\frac{1}{2^j}\right)\left|f_{I_{j,k}} - f_{I_{j,k+1}}\right|^2 \\
&\quad \lesssim \sum_{l=1}^{\infty}\sum_{m=0}^{\infty}\sum_{J\in I'_m\cup I''_m} K\left(\frac{1}{2^{m+l}}\right) l^2 \Phi_f(J) \\
&\qquad + \sum_{j=0}^{\infty} K\left(\frac{1}{2^j}\right)\left|f_{I'} - f_{I''}\right|^2 \\
&\quad \lesssim \sum_{l=1}^{\infty}\frac{l^2}{2^{lc}}\sum_{m=0}^{\infty}\sum_{J\in I'_m\cup I''_m} K\left(\frac{1}{2^m}\right)\Phi_f(J) \\
&\qquad + \left|f_{I'} - f_{I''}\right|^2\int_0^1 \varphi_K(t)\,\frac{dt}{t} \\
&\quad \lesssim \Psi_{f,K}\left(I'\right) + \Psi_{f,K}\left(I''\right) + \left|f_{I'} - f_{I''}\right|^2.
\end{aligned} \tag{54}$$

Thus we have proved (49) and hence the proof is complete. □

Lemma 11. *If K satisfies condition* (4), *then there exists a $p \in (0,\infty)$ such that $K(t)/t^p$ is nonincreasing. Furthermore, $K(t) \approx K(2t)$ for any $0 < t < \infty$.*

Proof. Lemma 11 can be found in [14]. □

Proof of Theorem 8. We now prove the necessity. It is easy to see that

$$\frac{1}{|I|^2}\iint_I \left|f(u) - f(v)\right|^2 |du|\,|dv| = 2\Phi_f(I). \tag{55}$$

By (55), we have

$$\begin{aligned}
\Psi_{f,K}(I) &= \frac{1}{2}\sum_{k=0}^{\infty}\sum_{J\in I_k} K\left(\frac{1}{2^k}\right)\frac{2^{2k}}{|I|^2}\iint_J \left|f(u) - f(v)\right|^2 |du|\,|dv| \\
&= \iint_{\partial\mathbb{D}} \alpha_I(u,v)\left|f(u) - f(v)\right|^2 |du|\,|dv|,
\end{aligned} \tag{56}$$

where

$$\alpha_I(u,v) = \frac{1}{2}\sum_{k=0}^{\infty}\sum_{J\in I_k} K\left(\frac{1}{2^k}\right)\frac{2^{2k}}{|I|^2}\chi_J(u)\,\chi_J(v) \tag{57}$$

and $\chi_J(u) = 1$, for $u \in J$, and $\chi_J(u) = 0$, for $u \in \partial\mathbb{D}\setminus J$.

Note that $|u - v| \le 2^{-k}|I|$ because of $u, v \in J \in I_k$. Since K satisfies condition (4), by Lemma 11 we may assume that $t^{-2}K(t)$ is nonincreasing. In fact, if $p \ge 2$, we can replace $K(t)$ with t^2 by Theorem 3. Then

$$\begin{aligned}
\alpha_I(u,v) &= \frac{1}{2}\sum_{k=0}^{\infty}\sum_{J\in I_k} K\left(\frac{|I|}{2^k|I|}\right)\frac{2^{2k}}{|I|^2}\chi_J(u)\,\chi_J(v) \\
&\lesssim \sum_{2^k|u-v|\le|I|} K\left(\frac{|I|}{2^k|I|}\right)\frac{2^{2k}}{|I|^2} \\
&\lesssim \frac{K(|u-v|/|I|)}{|u-v|^2}.
\end{aligned} \tag{58}$$

This gives

$$\Psi_{f,K}(I) \lesssim \iint_I \frac{\left|f(u) - f(v)\right|^2}{|u-v|^2}K\left(\frac{|u-v|}{|I|}\right)|du|\,|dv|. \tag{59}$$

For sufficiency, we claim that

$$\begin{aligned}
&\iint_I \frac{\left|f(u) - f(v)\right|^2}{|u-v|^2}K\left(\frac{|u-v|}{|I|}\right)|du|\,|dv| \\
&\quad \lesssim \frac{1}{|I|}\int_{-|I|}^{|I|}\Psi_{f,K}(\{I+t\})\,dt + \Psi_{f,K}(I),
\end{aligned} \tag{60}$$

where $\{I+t\} = \{z + e^{it}, z \in I\}$ for $I \subset \partial\mathbb{D}$.

In fact, by (56) and Fubini's theorem, we have

$$\begin{aligned}
&\frac{1}{|I|}\int_{-|I|}^{|I|}\Psi_{f,K}(\{I+t\})\,dt \\
&\quad = \iint_{\partial\mathbb{D}}\frac{1}{|I|}\int_{-|I|}^{|I|}\alpha_{\{I+t\}}(u,v)\,dt \\
&\qquad \times \left|f(u) - f(v)\right|^2 |du|\,|dv|.
\end{aligned} \tag{61}$$

This and (56) show that it suffices to verify

$$\frac{K(|u-v|/|I|)}{|u-v|^2} \lesssim \frac{1}{|I|}\int_{-|I|}^{|I|}\alpha_{\{I+t\}}(u,v)\,dt + \alpha_I(u,v). \tag{62}$$

First, suppose that $u, v \in I$ with $|u-v| \le |I|/2$ and let $l \in \mathbb{N}\cup 0$ be such that $2^{-l-2}|I| < |u-v| \le 2^{-l-1}|I|$. Noting that $u \notin \{I+t\}$ and thus $\alpha_{\{I+t\}}(u,v) = 0$ when $|t| > |I|$,

$$\begin{aligned}
&\frac{1}{|I|}\int_{-|I|}^{|I|}\alpha_{\{I+t\}}(u,v)\,dt \\
&\quad \ge \frac{1}{2|I|}\int_{\partial\mathbb{D}}\sum_{J\in\{I+t\}_l} K\left(\frac{1}{2^l}\right)\frac{1}{|2^{-l}I|^2}\chi_J(u)\,\chi_J(v)\,dt \\
&\quad = \frac{2^{2l}}{|I|^3}K\left(\frac{1}{2^l}\right)\sum_{J\in I_l}\int_{\partial\mathbb{D}}\chi_{\{J+t\}}(u)\,\chi_{\{J+t\}}(y)\,dt \\
&\quad \gtrsim \frac{1}{|I|}\frac{K(|u-v|/|I|)}{|u-v|^2}\sum_{J\in I_l}\int_{\partial\mathbb{D}}\chi_{\{J+t\}}(u)\,\chi_{\{J+t\}}(v)\,dt.
\end{aligned} \tag{63}$$

For each J, the final integral equals $|J| - |u-v| \ge |J|/2$. Hence the sum over J is at least $|I|/2$ and (62) holds for $|u-v| \le |I|/2$.

If $u, v \in I$ with $|u-v| > |I|/2$, by (57) we have

$$\alpha_I(u,v) \geq K\left(\frac{1}{2}\right)\frac{1}{|I|^2} \gtrsim \frac{K(|u-v|/|I|)}{|u-v|^2}. \qquad (64)$$

Hence (62) holds in this case.

We now assume that f is defined on $\mathbb{R}$ with constant f_I outside I. Let I_+ and I_- be the two arcs of the same length as I that are adjacent to I on the left and right, respectively. Note that $\Psi_{f,K}(I_-) = \Psi_{f,K}(I_+) = 0$ and $f_{I_+} = f_{I_-} = f_I$. Note that $\{I+t\} \subset I \cup I_+$ for $0 < t < |I|$ and $\{I+t\} \subset I \cup I_-$ for $-|I| < t < 0$. Lemma 10 and (60) give

$$\begin{aligned} &\iint_I \frac{|f(u)-f(v)|}{|u-v|^2} K\left(\frac{|u-v|}{|I|}\right)|du|\,|dv| \\ &\lesssim \frac{1}{|I|}\int_{-|I|}^{|I|} \Psi_{f,K}(\{I+t\})\,dt + \Psi_{f,K}(I) \\ &\lesssim \frac{1}{|I|}\int_{-|I|}^{|I|} \Psi_{f,K}(I)\,dt + \Psi_{f,K}(I) \\ &\lesssim \Psi_{f,K}(I). \end{aligned} \qquad (65)$$

The proof is complete. □

Corollary 12. *Let $0 < p < \infty$. Then $f \in L^2(\partial\mathbb{D})$ belongs to $Q_p(\partial\mathbb{D})$ if and only if*

$$\sup_{I \subset \partial\mathbb{D}} \sum_{n=0}^{\infty} \sum_{J \in I_n} \left(\frac{1}{2^n}\right)^p \Phi_f(J) < \infty. \qquad (66)$$

Proof. The nondecreasing function K satisfies condition (4) if $K(t) = t^p, 0 < p < \infty$. The desired result follows from Theorem 8. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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Research Article

Riesz Basicity for General Systems of Functions

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In this paper we find the general conditions for a complete biorthogonal conjugate system to form a Riesz basis. We show that if a complete biorthogonal conjugate system is uniformly bounded and its coefficient space is solid, then the system forms a Riesz basis. We also construct affine Riesz bases as an application to the main result.

1. Main Result

The aim of this paper is to find the general conditions for a complete biorthogonal conjugate system to form a Riesz basis, following the results obtained by Bari [1], Christensen [2], Sarsenbi with coauthors [3–5], San Antolin and Zalik [6], and Guo [7].

Let $\{u_n(x)\}_{n=1}^{\infty}$ and $\{v_n(x)\}_{n=1}^{\infty}$ be a complete biorthogonal conjugate system of functions from $L^2(0,1)$ space.

By system coefficient space $\{u_n(x)\}_{n=1}^{\infty}$ we denote the space $X(u)$ of all the numeric sequences $a=\{a_n\}_{n=1}^{\infty}$ such that the series $\sum_{n=1}^{\infty} a_n u_n(x)$ converges in $L^2(0,1)$. It is evident that coefficient space $X(u)$ is complete under the norm $\|a\|_{X(u)} = \sup_{n\in\mathbb{N}} \|\sum_{k=1}^{n} a_k u_k(x)\|$, and a natural basis $\varepsilon_i = \{\delta_{ij}\}_{j=1}^{\infty}$, $i\in\mathbb{N}$, where δ_{ij} is a Kronecker delta, forms a $X(u)$ space basis.

A Banach coordinate space X of numeric sequences $a=\{a_n\}_{n=1}^{\infty}$ is said to be solid if $b\in X$ follows from $a\in X$ and $|b_n|\le|a_n|$, $n\in\mathbb{N}$ (the inequality $\|b\|_X\le\|a\|_X$, as it is put by the precise definition, is not required here).

It is clear that $X(u)$ space is solid if natural basis is an unconditional basis for $X(u)$. The latter follows from unconditional basicity for a system $\{u_n(x)\}_{n=1}^{\infty}$.

Theorem 1. *Let $\{u_n(x)\}_{n=1}^{\infty}$ and $\{v_n(x)\}_{n=1}^{\infty}$ be a complete biorthogonal conjugate system of functions that is uniformly bounded:*

$$\int_0^1 |u_n(x)|^2 dx \le C, \qquad \int_0^1 |v_n(x)|^2 dx \le C, \quad n\in\mathbb{N}. \quad (1)$$

Let there be given coefficient spaces $X(u)$ and $X(v)$ which are both solid. Then $\{u_n(x)\}_{n=1}^{\infty}$ and $\{v_n(x)\}_{n=1}^{\infty}$ system form a Riesz basis.

Proof. We consider the series $\sum_{n=1}^{\infty} a_n u_n(x)$ for a numeric sequence $\{a_n\}_{n=1}^{\infty} \in \ell^2$ and show that series converges for almost all choices of signs, that is, series

$$\sum_{n=1}^{\infty} a_n r_n(t)\, u_n(x), \quad (2)$$

where $\{r_n(t)\}_{n=1}^{\infty}$ is the Rademacher system and converges for almost all $t\in[0,1]$ in L^2 metrics by variable x (e.g., [8, Chapter 2]).

We use the results from [8] claiming that convergence of series $\sum_{n=1}^{\infty} f_n(x)$ for almost all choices of signs is equivalent to

$$\left(\sum_{n=1}^{\infty} |f_n(x)|^2\right)^{1/2} \in L^2(0,1). \quad (3)$$

For the series considered $\sum_{n=1}^{\infty} a_n u_n(x)$, by Levi's theorem we have

$$\int_0^1 \sum_{n=1}^{\infty} |a_n u_n(x)|^2 dx = \sum_{n=1}^{\infty} |a_n|^2 \int_0^1 |u_n(x)|^2 dx \le C \sum_{n=1}^{\infty} |a_n|^2 < \infty, \quad (4)$$

meaning

$$\left(\sum_{n=1}^{\infty}\left|a_n u_n(x)\right|^2\right)^{1/2} \in L^2(0,1). \quad (5)$$

Convergence of the series $\sum_{n=1}^{\infty} a_n u_n(x)$ for almost all choices of signs is shown.

Now take a fixed $t_0 \in [0,1]$ such that the series $\sum_{n=1}^{\infty} a_n r_n(t_0) u_n(x)$ converges in $L^2(0,1)$ space. By the solidity condition for coefficient space $X(u)$ in $L^2(0,1)$, the series $\sum_{n=1}^{\infty} a_n u_n(x)$ converges, too.

Thus for any numeric sequence $a = \{a_n\}_{n=1}^{\infty} \in \ell^2$ the series $\sum_{n=1}^{\infty} a_n u_n(x)$ converges in $L^2(0,1)$. Then the following equivalent inequalities are satisfied:

$$\begin{gathered}\left\|\sum_{n=1}^{\infty} a_n u_n\right\|^2 \leq B\sum_{n=1}^{\infty}\left|a_n\right|^2, \quad \{a_n\}_{n=1}^{\infty} \in \ell^2, \\ \sum_{n=1}^{\infty}\left|f, u_n\right|^2 \leq B\|f\|^2, \quad f \in L^2(0,1).\end{gathered} \quad (6)$$

This means that $\{u_n(x)\}_{n=1}^{\infty}$ is Bessel system.

Besselian property for a system $\{v_n(x)\}_{n=1}^{\infty}$ is proved in the same way. It is clear that Besselian property for both biorthogonal conjugate systems $\{u_n(x)\}_{n=1}^{\infty}$ and $\{v_n(x)\}_{n=1}^{\infty}$ implies the Riesz basicity for these systems. □

Remark 2. Note that in Theorem 1 we can replace the coefficient space $X(u)$ with $X(u) \cap \ell^2$ and $X(v)$ with $X(v) \cap \ell^2$.

2. Affine Riesz bases

Let function $u : \mathbb{R} \to \mathbb{R}$ have a support $\operatorname{supp} u \subset [0,1]$. Using the representation $n = 2^k + j$, $k = 0, 1, \ldots$, $j = 0, \ldots, 2^k - 1$ for $n \in \mathbb{N}$, we assume

$$u_n(x) = u_{k,j}(x) = 2^{k/2} u\left(2^k x - j\right). \quad (7)$$

Besides, we suppose $u_0(x) = 1$, $x \in [0,1]$. System of functions $\{u_n(x)\}_{n=0}^{\infty}$ is called an affine system generated by a function u. Here and elsewhere we assume

$$u \in L^2(0,1), \qquad \int_0^1 u(x)\,dx = 0. \quad (8)$$

Note that the classic example of an affine system of functions is the Haar wavelet $\{h_n(x)\}_{n=0}^{\infty}$ generated by the function

$$h(x) = \begin{cases} 1, & x \in \left[0, \frac{1}{2}\right), \\ -1, & x \in \left[\frac{1}{2}, 1\right), \\ 0, & x \notin [0,1). \end{cases} \quad (9)$$

We enumerate the functions of Rademacher system $\{r_k\}_{k=0}^{\infty}$

$$r_k = 2^{-k/2}\sum_{j=0}^{2^k-1} h_{k,j}, \quad k = 0, 1, \ldots. \quad (10)$$

We suppose that an affine system $\{u_n(x)\}_{n=0}^{\infty}$ generator u can be represented by Rademacher system

$$u = \sum_{k=0}^{\infty} a_k r_k, \quad \sum_{k=0}^{\infty}\left|a_k\right|^2 < \infty. \quad (11)$$

In this case we have the following completeness criterion for a system $\{u_n(x)\}_{n=0}^{\infty}$. Let the function

$$U(z) = \sum_{k=0}^{\infty} a_k z^k, \quad |z| < 1, \quad (12)$$

be analytic in the unit disk with coefficients a_k from (11).

Theorem 3 (see [9]). *A necessary and sufficient condition for an affine system $\{u_n(x)\}_{n=0}^{\infty}$ to be complete in $L^2(0,1)$ space is that analytic function $U(z)$ is outer function.*

The following results are true for function u in the form (11).

Theorem 4. *System $\{v_n(x)\}_{n=0}^{\infty}$ that is biorthogonal conjugate to the affine system $\{u_n(x)\}_{n=0}^{\infty}$ exists and is complete in $L^2(0,1)$ space if $a_0 \neq 0$.*

Proof. Suppose

$$V(z) = \frac{1}{U(z)} = \sum_{k=0}^{\infty} b_k z^k, \quad (13)$$

that is,

$$a_0 b_0 = 1, \qquad \sum_{\nu=0}^{k} a_\nu b_{k-\nu} = 0, \quad k \geq 1. \quad (14)$$

Then it follows from the results of [10] that

$$v_n = v(\alpha_1, \ldots, \alpha_k) = \sum_{\nu=0}^{k} 2^{-(k-\nu)/2} b_{k-\nu} h(\alpha_1, \ldots, \alpha_\nu), \quad (15)$$

where $n \in \mathbb{N}$ and $n = 2^k + \sum_{\nu=1}^{k} \alpha_\nu 2^{k-\nu}$ is binary expansion, $h(\alpha_1, \ldots, \alpha_\nu) = h_m$ is the Haar function for $m = 2^\nu + \sum_{\mu=1}^{\nu} \alpha_\mu 2^{\nu-\mu}$, and $v_0(x) = 1$, $x \in [0,1]$. The explicit representation (15) shows that v_n is a Haar polynomial of degree n. Hence it follows that the system $\{v_n(x)\}_{n=0}^{\infty}$ is complete. □

Now we can formulate the Riesz basicity test for affine system $\{u_n(x)\}_{n=0}^{\infty}$ with form (11) generator, based on Theorem 1.

Theorem 5. *Let analytic function $U(z)$ have an absolutely convergent Taylor-series expansion*

$$\sum_{k=0}^{\infty}\left|a_k\right| < \infty \quad (16)$$

and $U(z)$ does not vanish in the closed unit disk ($|z| \leq 1$). Then an affine system of functions $\{u_n(x)\}_{n=0}^{\infty}$ forms a Riesz basis.

Proof. By the conditions of the theorem, $U(z)$ is outer function. By Theorem 3, an affine system $\{u_n(x)\}_{n=0}^{\infty}$ is complete in $L^2(0,1)$ space. By Theorem 4, biorthogonal conjugate system $\{v_n(x)\}_{n=0}^{\infty}$ is complete, too.

Obviously, $\|u_n\| \le \max\{1, \|u\|\}$. From representation (15) we get

$$\|v_n\|^2 \le \sum_{k=0}^{\infty} 2^{-k} |b_k|^2 < \infty, \quad n \in \mathbb{N}. \tag{17}$$

We need to take into account that by Wiener theorem on absolutely convergent Taylor series we have

$$\sum_{k=0}^{\infty} |b_k| < \infty. \tag{18}$$

Finally, from results of [11] it follows that $X(u) \cap \ell^2 = \ell^2$ and $X(v) \cap \ell^2 = \ell^2$, so all the conditions from Theorem 1 including the Remark are satisfied. □

Conflict of Interests

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Research Article

Molecular Characterization of Hardy Spaces Associated with Twisted Convolution

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We give a molecular characterization of the Hardy space associated with twisted convolution. As an application, we prove the boundedness of the local Riesz transform on the Hardy space.

1. Introduction

In this paper, we consider the $2n$ linear differential operators

$$Z_j = \frac{\partial}{\partial z_j} + \frac{1}{4}\overline{z}_j, \qquad \overline{Z}_j = \frac{\partial}{\partial \overline{z}_j} - \frac{1}{4}z_j, \qquad \text{on } \mathbb{C}^n,\ j = 1, 2, \dots, n. \tag{1}$$

Together with the identity they generate a Lie algebra h^n which is isomorphic to the $2n+1$ dimensional Heisenberg algebra. The only nontrivial commutation relations are

$$\left[Z_j, \overline{Z}_j\right] = -\frac{1}{2}I, \quad j = 1, 2, \dots, n. \tag{2}$$

The operator L defined by

$$L = -\frac{1}{2}\sum_{j=1}^{n}\left(Z_j\overline{Z}_j + \overline{Z}_j Z_j\right) \tag{3}$$

is nonnegative, self-adjoint, and elliptic. Therefore, it generates a diffusion semigroup $\{T_t^L\}_{t>0} = \{e^{-tL}\}_{t>0}$. The operators in (1) generate a family of "twisted translations" τ_w on $\mathbb{C}^n$ defined on measurable functions by

$$\begin{aligned}(\tau_w f)(z) &= \exp\left(\frac{1}{2}\sum_{j=1}^{n}\left(w_j z_j + \overline{w}_j \overline{z}_j\right)\right) f(z) \\ &= f(z+w)\exp\left(\frac{i}{2}\operatorname{Im}(z\cdot s\overline{w})\right).\end{aligned} \tag{4}$$

The "twisted convolution" of two functions f and g on $\mathbb{C}^n$ can now be defined as

$$\begin{aligned}(f \times g)(z) &= \int_{\mathbb{C}^n} f(w)\,\tau_{-w}g(z)\,dw \\ &= \int_{\mathbb{C}^n} f(z-w)\,g(w)\,\overline{\omega}(z,w)\,dw,\end{aligned} \tag{5}$$

where $\omega(z,w) = \exp((i/2)\operatorname{Im}(z\cdot\overline{w}))$. More about twisted convolution can be found in [1–3].

In [4], the authors defined the Hardy space $H_L^1(\mathbb{C}^n)$ associated with twisted convolution. They gave several characterizations of $H_L^1(\mathbb{C}^n)$ via maximal functions, the atomic decomposition, and the behavior of the local Riesz transform. As applications, the boundedness of Hömander multipliers on Hardy spaces is considered in [5]. The "twisted cancellation" and Weyl multipliers were introduced for the first time in [6]. Recently, Huang and Wang [7] defined the Hardy space $H_L^p(\mathbb{C}^n)$ associated with twisted convolution for $2n/(2n+1) < p < 1$. Huang gave the characterizations of the Hardy space associated with twisted convolution by the Lusin area integral function and Littlewood-Paley function in [8] and established the boundedness of the Weyl multiplier on the Hardy space associated with twisted convolution by these characterizations in [9]. The purpose of this paper is to give a molecular characterization for $H_L^p(\mathbb{C}^n)$. As an application, we prove the boundedness of the local Riesz transform on the Hardy space $H_L^p(\mathbb{C}^n)$.

We first give some basic notations about $H_L^p(\mathbb{C}^n)$. Let $\mathscr{B}$ denote the class of C^∞-functions φ on $\mathbb{C}^n$, supported on the ball $B(0,1)$ such that $\|\varphi\|_\infty \le 1$ and $\|\nabla\varphi\|_\infty \le 2$. For $t > 0$, let $\varphi_t(z) = t^{-2n}\varphi(z/t)$. Given $\sigma > 0$, $0 < \sigma \le +\infty$, and a tempered distribution f, define the grand maximal function

$$M_\sigma f(z) = \sup_{\varphi\in\mathscr{B}} \sup_{0<t<\sigma} \left|\varphi_t \times f(z)\right|. \tag{6}$$

Then, the Hardy space $H_L^p(\mathbb{C}^n)$ can be defined by

$$H_L^p(\mathbb{C}^n) = \left\{f \in \mathscr{S}'(\mathbb{C}^n) : M_\infty f \in L^p(\mathbb{C}^n)\right\}. \tag{7}$$

For any $f \in H_L^p(\mathbb{C}^n)$, define $\|f\|_{H_L^p(\mathbb{C}^n)} = \|M_\infty f\|_{L^p}$.

Definition 1. Let $0 < p \le 1 \le q \le \infty$ and $p \ne q$. A function $a(z)$ is a $H_L^{p,q}$-atom for the Hardy space $H_L^p(\mathbb{C}^n)$ associated to a ball $B(z_0, r)$ if

(1) $\operatorname{supp} a \subset B(z_0, r)$;

(2) $\|a\|_q \le |B(z_0, r)|^{1/q-1/p}$;

(3) $\int_{\mathbb{C}^n} a(w)\overline{\omega}(z_0, w)dw = 0$.

We define the atomic Hardy space $H_L^{p,q}(\mathbb{C}^n)$ to be the set of all tempered distributions of the form $\sum_j \lambda_j a_j$ (the sum converges in the topology of $\mathscr{S}'(\mathbb{C}^n)$), where a_j are $H_L^{p,q}$-atoms and $\sum_j |\lambda_j|^p < +\infty$.

The atomic quasinorm in $H_L^{p,q}(\mathbb{C}^n)$ is defined by

$$\|f\|_{L\text{-atom}} = \inf\left\{\left(\sum_j \left|\lambda_j\right|^p\right)^{1/p}\right\}, \tag{8}$$

where the infimum is taken over all decompositions $f = \sum_j \lambda_j a_j$ and a_j are $H_L^{p,q}$-atoms.

The following result has been proved in [4, 7].

Proposition 2. *Let $2n/(2n+1) < p \le 1$. Then, for a tempered distribution f on $\mathbb{C}^n$, the following are equivalent:*

(i) $M_\infty f \in L^p(\mathbb{C}^n)$;

(ii) *for some σ, $0 < \sigma < +\infty$, $M_\sigma f \in L^p(\mathbb{C}^n)$;*

(iii) *for some radial function $\varphi \in \mathscr{S}$, such that $\int_{\mathbb{C}^n} \varphi(z)dz \ne 0$, we have*

$$\sup_{0<t<1} \left|\varphi_t \times f(z)\right| \in L^p(\mathbb{C}^n); \tag{9}$$

(iv) *f can be decomposed as $f = \sum_j \lambda_j a_j$, where a_j are $H_L^{p,q}$-atoms and $\sum_j |\lambda_j|^p < +\infty$.*

Corollary 3. *Let $2n/(2n+1) < p \le 1$ and $1 < q \le \infty$. Then, $H_L^{p,q}(\mathbb{C}^n) = H_L^p(\mathbb{C}^n)$ with equivalent norms.*

In order to give the main result of this paper, we need the dual space of Hardy space $H_L^p(\mathbb{C}^n)$.

Definition 4. Let $0 \le \alpha < 1/2n$; a locally integrable function f is said to be in the Campanato type space Λ_α^L if there exists a constant $K > 0$ such that, for every ball $B = B(z, r)$,

$$|B|^{-\alpha}\left(\int_B \left|f(v) - \left(\frac{1}{|B|}\int_B f(u)\overline{\omega}(z,u)\,du\right)\omega(z,v)\right|^2 \times \frac{dv}{|B|}\right)^{1/2} \le K. \tag{10}$$

The norm $\|f\|_{\Lambda_\alpha^L}$ of f is the least value of K for which the above inequality holds.

The dual space of $H_L^1(\mathbb{C}^n)$ is the BMO type space $\mathrm{BMO}_L(\mathbb{C}^n)$ (cf. [4]). Note that Λ_0^L is identified with BMO_L. Let $\mathscr{H}_L^{p,2,a}$ denote the space of finite linear combinations of $H_L^{p,2}$-atoms, which coincides with $L_c^2(\mathbb{C}^n)$, the space of square integrable functions with compact support. By Proposition 2, $\mathscr{H}_L^{p,2,a}$ is a dense subspace of $H_L^p(\mathbb{C}^n)$. Set

$$\mathscr{L}_g(f) = \int_{\mathbb{C}^n} f(z)\overline{g}(z)\,dz, \quad f \in \mathscr{H}_L^{p,2,a},\ g \in L^2_{\mathrm{loc}}(\mathbb{C}^n). \tag{11}$$

Similar to the classical case in [10], we immediately obtain the following theorem which proves that $\Lambda_{1/p-1}^L$ is the dual space of $H_L^p(\mathbb{C}^n)$ for $2n/(2n+1) < p < 1$.

Theorem 5. *Let $2n/(2n+1) < p < 1$. Then*

(a) *suppose $g \in \Lambda_{(1/p)-1}^L$; then $\mathscr{L}_g$ given by* (11) *extends to a bounded linear functional on $H_L^p(\mathbb{C}^n)$ and satisfies*

$$\left\|\mathscr{L}_g\right\| \le C\,\|g\|_{\Lambda_{(1/p)-1}^L}; \tag{12}$$

(b) *conversely, every bounded linear functional $\mathscr{L}$ on $H_L^p(\mathbb{C}^n)$ can be realized as $\mathscr{L} = \mathscr{L}_g$ with $g \in \Lambda_{(1/p)-1}^L$ and*

$$\|g\|_{\Lambda_{(1/p)-1}^L} \le C\,\|\mathscr{L}\|. \tag{13}$$

Remark 6. We may define the space $\Lambda_{(1/p)-1,q'}^L$, $2n/(2n+1) < p < 1$, $1 \le q' \le \infty$, by

$$|B|^{1-(1/p)}\left(\int_B \left|f(v) - \left(\frac{1}{|B|}\int_B f(u)\overline{\omega}(z,u)\,du\right)\omega(z,v)\right|^{q'} \times \frac{dv}{|B|}\right)^{1/q'} \le K, \tag{14}$$

where $B = B(z, r)$. The norm $\|f\|_{\Lambda_{(1/p)-1,q'}^L}$ of f is the least value of K for which the above inequality holds. Due to Theorem 5, $\Lambda_{(1/p)-1,q'}^L$ is also identified with the dual space of $H_L^p(\mathbb{C}^n)$. The proof is almost the same as that of Theorem 5. Thus, the space $\Lambda_{(1/p)-1,q'}^L$ coincides with $\Lambda_{(1/p)-1}^L$ and $\|f\|_{\Lambda_{(1/p)-1,q'}^L} \sim \|f\|_{\Lambda_{(1/p)-1}^L}$.

Definition 7. Let $2n/(2n+1) < p \le 1 \le q \le \infty, p \ne q$, and $\epsilon > (1/p) - 1$. Set $a = 1 - (1/p) + \epsilon$, $b = 1 - (1/p) + \epsilon$. A function $M \in L^q$ is called a $H_L^{p,q,\epsilon}$-molecule with the center z_0 if

(1) $|z|^{2nb} M(z) \in L^q$,

(2) $\mathcal{N}(M) = \|M\|_{L^q}^{a/b} \, \||\cdot - z_0|^{2nb} M\|_{L^q}^{1-(a/b)} < \infty$,

(3) $\int_{\mathbb{C}^n} M(z)\overline{\omega}(z_0, z)\, dz = 0$.

Then, we can obtain a molecular characterization of $H_L^p(\mathbb{C}^n)$ as follows.

Theorem 8. *Given p, q, ϵ as in Definition 7, then $f \in H_L^p$ if and only if f can be written as $f = \sum_j \lambda_j M_j$, where M_j are $H_L^{p,q,\epsilon}$-molecules and $\sum_j |\lambda_j|^p < \infty$. The sum converges in H_L^p norm and also in $(\Lambda^L_{(1/p)-1})^*$ when $2n/(2n+1) < p < 1$. Moreover,*

$$\|f\|_{H_L^p} \sim \|f\|_{H_L^{p,q,\epsilon,M}} = \inf \left\{ \left(\sum_j |\lambda_j|^p \right)^{1/p} \right\}, \tag{15}$$

where the infimum is taken over all decompositions of f into $H_L^{p,q,\epsilon}$-molecules.

Let ψ be a C^∞-function on $\mathbb{C}^n$ with compact support and such that $\psi \equiv 1$ on a neighborhood of zero. Define

$$R_j(z) = \frac{z_j}{|z|^{2n+1}} \psi(z), \qquad \overline{R}_j(z) = \frac{\overline{z}_j}{|z|^{2n+1}} \psi(z), \tag{16}$$

for $j = 1, 2, \ldots, n$.

We refer to the singular integral operators R_j, $\overline{R}_j$ defined by left twisted convolution with these kernels as the local Riesz transforms. The terminology is motivated by the fact that they are essentially the operators which are formally defined as $Z_j L^{-1/2}$, $\overline{Z}_j L^{-1/2}$, $j = 1, 2, \ldots, n$.

As an application of Theorem 8, we can prove the following.

Theorem 9. *The local Riesz transforms R_j, $\overline{R}_j$, $j = 1, 2, \ldots, n$ are bounded on $H_L^p(\mathbb{C}^n)$, where $2n/(2n+1) < p \le 1$.*

Remark 10. When $p = 1$, Theorem 9 is proved by the connection between $H_L^1(\mathbb{C}^n)$ and Hardy space on the Heisenberg group $H^1(\mathbb{H}_n)$ (cf. Lemma 4.9 in [4]).

Throughout the paper, we will use C to denote a positive constant, which is independent of main parameters and may be different at each occurrence. By $B_1 \sim B_2$, we mean that there exists a constant $C > 1$ such that $1/C \le B_1/B_2 \le C$.

2. Molecule Characterization of $H_L^p(\mathbb{C}^n)$

In this section, we prove the main result of this paper. Firstly, we have the following lemma.

Lemma 11. *If a is a $H_L^{p,q}$-atom for $2n/(2n+1) < p \le 1 \le q \le +\infty$ supported in $B(z_0, r)$, then a is a $H_L^{p,q,\epsilon}$-molecule centered at z_0 and*

$$\mathcal{N}(a) \le C, \tag{17}$$

where $\epsilon > 0$ and C is a positive constant that is independent of a.

Proof. Since

$$\|a\|_q \le |B|^{1/q - 1/p} = |B|^{a-b}, \tag{18}$$

we get

$$\left\| |\cdot - z_0|^{2nb} a(\cdot) \right\|_q \le r^{2nb} \|a\|_q \le C|B|^b |B|^{a-b} = C|B|^a. \tag{19}$$

Therefore,

$$\mathcal{N}(a) \le C|B|^{(a/b)(a-b)} |B|^{a(1-(a/b))} = C. \tag{20}$$

This proves that a is a molecule with center at z_0. □

The following lemma is the key step for the proof of Theorem 8.

Lemma 12. *If M is a $H_L^{p,q,\epsilon}$-molecule with center at z_0, then $M \in H_L^{p,q}(\mathbb{C}^n)$ and*

$$\|M\|_{H_L^p} \le C\mathcal{N}(M), \tag{21}$$

where C is independent of M.

Proof. If $q = 2$, let $\sigma = \|M\|_2^{1/(2n(a-b))}$, $E_0 = \{z \in \mathbb{C}^n : |z - z_0| \le \sigma\}$, and $E_k = \{z \in \mathbb{C}^n : 2^{k-1}\sigma < |z - z_0| \le 2^k \sigma\}$. Denote $M_k = M\chi_k$, where χ_k is the characteristic function of E_k.

Let

$$P_k(z) = \frac{1}{|E_k|} \int_{E_k} M(u)\, \overline{\omega}(z_0, u)\, du\, \omega(z_0, z)\, \chi_k(z). \tag{22}$$

Then

$$\begin{aligned} &\int_{\mathbb{C}^n} (M_k(z) - P_k(z))\, \overline{\omega}(z_0, z)\, dz \\ &\quad = \int_{E_k} M(z)\, \overline{\omega}(z_0, z)\, dz - \int_{E_k} M(z)\, \overline{\omega}(z_0, z)\, dz = 0. \end{aligned} \tag{23}$$

Without loss of generality, we can assume that $\mathcal{N}(M) = 1$. Then

$$\left\| |\cdot - z_0|^{2nb} M(\cdot) \right\|_2^{1-(a/b)} = \|M\|_2^{-a/b}. \tag{24}$$

Therefore,

$$\left\| |\cdot - z_0|^{2n((1/2)+\epsilon)} M(\cdot) \right\|_2 = \|M\|_2^{-a/(b-a)} = \sigma^{2na}. \tag{25}$$

Let $B_k = \{z \in \mathbb{C}^n : |z - z_0| \le 2^k \sigma\}$. Then

$$\operatorname{supp}(M_k - P_k) \subseteq E_k \subseteq B_k. \tag{26}$$

In the following, we will prove

$$\frac{1}{|B_k|}\int_{B_k}\left|M_k-P_k\right|^2dz \le \frac{C}{|E_k|}\int_{E_k}\left|M_k(z)\right|^2dz. \tag{27}$$

In fact, by

$$\int_{E_k}\left(M_k(z)-P_k(z)\right)\overline{P}_k(z)\,dz=0, \tag{28}$$

we have

$$\begin{aligned}\frac{1}{|B_k|}\int_{B_k}&\left|M_k(z)-P_k(z)\right|^2dz\\ &=\frac{1}{|B_k|}\int_{B_k}\left(M_k(z)-P_k(z)\right)\left(\overline{M}_k(z)-\overline{P}_k(z)\right)dz\\ &=\frac{1}{|B_k|}\int_{B_k}\left(M_k(z)-P_k(z)\right)\overline{M}_k(z)\,dz.\end{aligned} \tag{29}$$

Since

$$\begin{aligned}\int_{E_k}&\overline{M}_k(z)\,P_k(z)\,dz\\ &=\int_{E_k}\overline{M}_k(z)\frac{1}{|E_k|}\int_{E_k}M(u)\,\overline{\omega}(z_0,u)\,du\,\omega(z_0,z)\,dz\\ &=\frac{1}{|E_k|}\int_{E_k}M(u)\,\overline{\omega}(z_0,u)\,du\int_{E_k}\overline{M}_k(z)\,\omega(z_0,z)\,dz\\ &=|E_k|\left|P_k(z)\right|^2,\end{aligned} \tag{30}$$

we get

$$\begin{aligned}\frac{1}{|B_k|}\int_{B_k}&\left|M_k(z)-P_k(z)\right|^2dz\\ &=\frac{1}{|B_k|}\int_{E_k}\left|M_k(z)\right|^2dz-\int_{E_k}\frac{|E_k|}{|B_k|}\left|P_k(z)\right|^2dz\\ &\le\frac{1}{|B_k|}\int_{E_k}\left|M_k(z)\right|^2dz\le\frac{C}{|E_k|}\int_{E_k}\left|M_k(z)\right|^2dz.\end{aligned} \tag{31}$$

Therefore, (27) holds true. In particular, we have

$$\begin{aligned}\frac{1}{|B_0|}\int_{B_0}&\left|M_0(z)-P_0(z)\right|^2dz\\ &\le\frac{C}{|E_0|}\int_{E_0}\left|M_0(z)\right|^2dz\\ &\le C\sigma^{-2n}\sigma^{4n((1/2)-(1/p))}=C\left|B_0\right|^{-2/p}.\end{aligned} \tag{32}$$

For $k\ge1$,

$$\begin{aligned}\frac{1}{|B_k|}\int_{B_k}&\left|M_k(z)-P_k(z)\right|^2dz\\ &\le\frac{C}{|E_k|}\int_{E_k}\left|M_k(z)\right|^2dz\\ &=\frac{C}{|E_k|}\int_{E_k}\left|M_k(z)\right|^2\left|z-z_0\right|^{2n(1+2\epsilon)}\\ &\qquad\times\left|z-z_0\right|^{-2n(1+2\epsilon)}dz\\ &\le C\left(2^k\sigma\right)^{-2n}\left(2^{k-1}\sigma\right)^{-2n(1+2\epsilon)}\\ &\quad\times\int_{E_k}\left|M_k(z)\right|^2\left|z-z_0\right|^{2n(1+2\epsilon)}dz\\ &\le C\sigma^{-4n(1+\epsilon)}2^{-4kn-4kn\epsilon}\sigma^{4na}\\ &\le C2^{-4kn-4kn\epsilon-4kn/p}\left(2^k\sigma\right)^{-2n/p}\\ &=C2^{-4kna}\left|B_k\right|^{-2/p},\end{aligned} \tag{33}$$

where C depends on n,ϵ. This proves that $M_k-P_k=\lambda_k a_k$, where a_k is a $H_L^{p,2}$-atom supported on B_k and $|\lambda_k|\le C2^{-2kna}$.

Now, we prove that $\sum_{k=1}^{\infty}P_k(z)$ has atomic decomposition. For $k\ge1$,

$$\begin{aligned}\left|P_k(z)\right|&\le\frac{1}{|E_k|}\int_{E_k}|M(u)|\,du\\ &=\frac{1}{|E_k|}\int_{E_k}\left|u-z_0\right|^{2nb}|M(u)|\left|u-z_0\right|^{-2nb}du\\ &\le C(2^k\sigma)^{-2nb}\frac{1}{|E_k|}\left\|\left|\cdot-z_0\right|^{2nb}M(\cdot)\right\|_2\left|E_k\right|^{1/2}\\ &\le C\left(2^k\sigma\right)^{-2nb-n}\sigma^{2na}\\ &=C2^{-kn(1+2b)}\sigma^{-2n/p}.\end{aligned} \tag{34}$$

Therefore,

$$P_k(z)=\sum_{l=k+1}^{\infty}\left(P_{l-1}(z)-P_l(z)\right). \tag{35}$$

Let

$$N^k=\sum_{l=k}^{\infty}\int_{E_l}M(u)\,\overline{\omega}(z_0,u)\,du. \tag{36}$$

Then,

$$\begin{aligned}N^0&=\sum_{l=0}^{\infty}\int_{E_l}M(u)\,\overline{\omega}(z_0,u)\,du\\ &=\int_{\mathbb{C}^n}M(u)\,\overline{\omega}(z_0,u)\,du=0.\end{aligned} \tag{37}$$

Thus, by Abel transform,

$$\begin{aligned}\sum_{k=0}^{\infty} P_k(z) &= \sum_{k=0}^{\infty}\sum_{l=k+1}^{\infty}\left(P_{l-1}(z)-P_l(z)\right)\\ &= \sum_{k=0}^{\infty} N^{k+1}\left\{\left|E_k\right|^{-1}\omega(z_0,z)\chi_k(z)\right.\\ &\qquad\left.-\left|E_{k+1}\right|^{-1}\omega(z_0,z)\chi_{k+1}(z)\right\}.\end{aligned} \tag{38}$$

Following from (34), we obtain

$$\begin{aligned}&\left|N^{k+1}\left\{\left|E_k\right|^{-1}\omega(z_0,z)\chi_k(z)\right.\right.\\ &\quad\left.\left.-\left|E_{k+1}\right|^{-1}\omega(z_0,z)\chi_{k+1}(z)\right\}\right|\\ &\qquad\le C2^{-2n(k+1)\epsilon}\sigma^{2n-(2n/p)}\left|B_{k+1}\right|^{-1}\\ &\qquad= C2^{-2n(k+1)\epsilon}\sigma^{2n-(2n/p)}\left(2^{k+1}\sigma\right)^{-1}\\ &\qquad= C2^{-2na(k+1)}\left|B_{k+1}\right|^{-1/p}.\end{aligned} \tag{39}$$

Let $\mu_k = C2^{-2na(k+1)}$ and

$$\begin{aligned}b_k(z) = C^{-1}2^{2na(k+1)}N^{k+1}&\left\{\left|E_k\right|^{-1}\omega(z_0,z)\chi_k(z)\right.\\ &\left.-\left|E_{k+1}\right|^{-1}\omega(z_0,z)\chi_{k+1}(z)\right\}.\end{aligned} \tag{40}$$

Then, b_k are $H_L^{p,\infty}$-atoms, $\sum_{k=0}^{\infty}|\mu_k|^p<\infty$, and

$$\sum_{k=0}^{\infty} P_k(z) = \sum_{k=0}^{\infty}\mu_k b_k(z). \tag{41}$$

Therefore,

$$M(z) = \sum_{k=0}^{\infty}\lambda_k a_k(z) + \sum_{k=0}^{\infty}\mu_k b_k(z) \tag{42}$$

holds pointwise, where a_k are $H_L^{p,2}$-atoms and b_k are $H_L^{p,\infty}$-atoms, and

$$\sum_{k=0}^{\infty}\left\{\left|\lambda_k\right|^p + \left|\mu_k\right|^p\right\} < \infty. \tag{43}$$

When $p=1$, it is easy to see that the sum in (42) converges in L^1.

To prove $M \in H_L^{p,q}$ for $2n/(2n+1) < p < 1$, we need to show that, for every $g \in \Lambda^L_{(1/p)-1}$,

$$\begin{aligned}&\int_{\mathbb{C}^n} M(z)g(z)\,dz\\ &\quad= \lim_{m\to\infty}\sum_{k=0}^{m}\int_{\mathbb{C}^n}\left\{\lambda_k a_k(z)+\mu_k b_k(z)\right\}g(z)\,dz.\end{aligned} \tag{44}$$

In fact, (44) implies that (42) holds in $\mathcal{S}'(\mathbb{C}^n)$.

For any $z\in\mathbb{C}^n$, there exists $k\ge 0$ such that $z\in E_k$. If $k=0$, then

$$M(z) = \lambda_0 a_0(z) + \mu_0 b_0(z). \tag{45}$$

If $k \ge 1$, then

$$M(z) = \lambda_k(z)a_k(z) + \sum_{j=k-1}^{k}\mu_j b_j(z). \tag{46}$$

Therefore, when $|z-z_0| \le 2^m\sigma$,

$$\begin{aligned}&\sum_{k=0}^{\infty}\left(\lambda_k a_k(z)+\mu_k b_k(z)\right)\\ &\quad= \sum_{k=0}^{m}\left(\lambda_k a_k(z)+\mu_k b_k(z)\right) = M(z).\end{aligned} \tag{47}$$

Thus,

$$\begin{aligned}&\int_{\{z:|z-z_0|\le 2^m\sigma\}}\sum_{k=0}^{m}\left(\lambda_k a_k(z)+\mu_k b_k(z)\right)g(z)\,dz\\ &\quad= \int_{\{z:|z-z_0|\le 2^m\sigma\}} M(z)g(z)\,dz.\end{aligned} \tag{48}$$

Let $m\to\infty$; the right side is $\int_{\mathbb{C}^n} M(z)g(z)dz$. The left side is

$$\begin{aligned}&\lim_{m\to\infty}\sum_{k=0}^{m}\int_{\{z:|z-z_0|\le 2^m\sigma\}}\left(\lambda_k a_k(z)+\mu_k b_k(z)\right)g(z)\,dz\\ &\quad= \lim_{m\to\infty}\sum_{k=0}^{m}\int_{\mathbb{C}^n}\left(\lambda_k a_k(z)+\mu_k b_k(z)\right)g(z)\,dz.\end{aligned} \tag{49}$$

This proves (42) and the case of $q=2$ for Lemma 12 is proved. Similarly, the case of $q\ne 2$ can be proved as the case of $q=2$. Lemma 12 is proved. □

Proof of Theorem 8. Theorem 8 follows from Lemmas 11 and 12. □

3. The Boundedness of Local Riesz Transform on $H_L^p(\mathbb{C}^n)$

In this section, we prove the boundedness of local Riesz transform on $H_L^p(\mathbb{C}^n)$ by using Theorem 8.

Proof of Theorem 9. By Theorem 8, it is sufficient to prove that, for any $H_L^{p,2}$-atom a, $R_j(a)$ is a $H_L^{p,2,\varepsilon}$-molecule and the norm $\mathcal{N}(R_j(a)) \le C$, where C is independent of a.

Assume that $\operatorname{supp} a \subset B(z_0, r)$; then

$$\begin{aligned}&\int_{\mathbb{C}^n} R_j(a)(z)\,\overline{\omega}(z_0,z)\,dz\\ &\quad= \int_{\mathbb{C}^n}\left(\int_{\mathbb{C}^n} a(z-u)\frac{u_i}{|u|^{2n+1}}\psi(u)\,\overline{\omega}(z,u)\,du\right)\\ &\qquad\times\overline{\omega}(z_0,z)\,dz\end{aligned}$$

$$= \int_{\mathbb{C}^n} \frac{u_i}{|u|^{2n+1}} \psi(u) \left(\int_{\mathbb{C}^n} a(z-u) \overline{\omega}(u+z_0, z)\, dz \right) \times \overline{\omega}(z_0, z)\, du = 0, \tag{50}$$

where the last equality is valid because $a(\cdot - u)$ is an atom supported on $B(u + z_0, r)$. This proves that $R_j(a)$ satisfies moment condition.

Denote $M(z) = R_j(a)(z)$. Then, we have

$$\begin{aligned} \|M\|_2 &= \left\|R_j(a)\right\|_2 \le C\|a\|_2 \\ &\le |B|^{1/2-1/p} = C|B|^{a-b}. \end{aligned} \tag{51}$$

Let $B^* = \{z \in \mathbb{C}^n : |z - z_0| \le 2r\}$. Then,

$$\begin{aligned} &\int_{\mathbb{C}^n} \left|z-z_0\right|^{2n(1+2\epsilon)} |M(z)|^2 dz \\ &\quad = \int_{B^*} \left|z-z_0\right|^{2n(1+2\epsilon)} M(z)^2 dz \\ &\qquad + \int_{(B^*)^c} \left|z-z_0\right|^{2n(1+2\epsilon)} |M(z)|^2 dz = I + II. \end{aligned} \tag{52}$$

For I,

$$I \le C|B|^{1+2\epsilon} \int_{\mathbb{C}^n} |M(z)|^2 dz \le C|B|^{2+2\epsilon-(2/p)} = C|B|^{2a}. \tag{53}$$

For II, since

$$\begin{aligned} &\left|R_j(a)(z)\right| \\ &\quad = \left| \int_{\mathbb{C}^n} a(u) \frac{z_j - u_i}{|z-u|^{2n+1}} \psi(z-u) \overline{\omega}(z,u)\, du \right| \\ &\quad = \left| \int_{\mathbb{C}^n} a(u) \overline{\omega}(z_0, u) \left(\frac{z_j - u_i}{|z-u|^{2n+1}} \psi(z-u) \right.\right. \\ &\qquad\qquad \times \omega(z_0 - z, u) \\ &\qquad\qquad - \frac{z_j - z_{0,j}}{\left|z-z_0\right|^{2n+1}} \psi(z - z_0) \\ &\qquad\qquad \left.\left. \times \omega(z_0 - z, z_0) \right) du \right| \\ &\quad \le C \int_{\mathbb{C}^n} \frac{\left|u-z_0\right|}{\left|z-z_0\right|^{2n+1}} |a(u)|\, du \\ &\quad \le Cr|B|^{1-1/p} \frac{1}{\left|z-z_0\right|^{2n+1}}, \end{aligned} \tag{54}$$

we get

$$II \le Cr^2 |B|^{2-2/p} \int_{(B^*)^c} \frac{\left|z-z_0\right|^{2n(1+2\epsilon)}}{\left|z-z_0\right|^{4n+2}} dz. \tag{55}$$

Let $0 < \epsilon < 1/2n$. Then

$$II \le C|B|^{2+2\epsilon-2/p} = C|B|^{2a}. \tag{56}$$

Therefore,

$$\begin{aligned} \mathcal{N}(M) &= \|M\|_2^{a/b} \left\| \left|\cdot - z_0\right|^{n(1+2\epsilon)} M(\cdot) \right\|_2^{1-(a/b)} \\ &\le C|B|^{(a/b)(a-b)} |B|^{a(1-(a/b))} = C. \end{aligned} \tag{57}$$

This completes the proof of Theorem 9. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

A Modified Analytic Function Space Feynman Integral and Its Applications

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We analyze the generalized analytic function space Feynman integral and then defined a modified generalized analytic function space Feynman integral to explain the physical circumstances. Integration formulas involving the modified generalized analytic function space Feynman integral are established which can be applied to several classes of functionals.

1. Introduction

Let $C_0[0,T]$ denote the one-parameter Wiener space, that is, the space of continuous real-valued functions x on $[0,T]$ with $x(0)=0$, and let m denote Wiener measure. Since the concept of the Feynman integral was introduced by Feynman and Kac, many mathematicians studied the "analytic" Feynman integral of functionals in several classes of functionals [1–7]. Recently the authors have introduced an approach to the solutions of the diffusion equation and the Schrödinger equation via the Fourier-type functionals on Wiener space [6].

The function space $C_{a,b}[0,T]$, induced by a generalized Brownian motion, was introduced by Yeh in [8] and studied extensively in [9–11]. In [11] the authors have studied the generalized analytic Feynman integral for functionals in a very general function space $C_{a,b}[0,T]$.

In this paper, we present an analysis of the generalized analytic Feynman integral on function space. We define a modified generalized analytic function space Feynman integral (AFSFI) and then explain the physical circumstances with respect to an anharmonic oscillator using the concept of the modified generalized analytic Feynman integral on function space.

The Wiener process used in [1–7] is stationary in time and is free of drift while the stochastic process used in this paper, as well as in [9–12], is nonstationary in time, is subject to a drift $a(t)$, and can be used to explain the position of the Ornstein-Uhlenbeck process in an external force field [13].

2. Preliminaries

Let $a(t)$ be an absolutely continuous real-valued function on $[0,T]$ with $a(0)=0$, $a'(t)\in L^2[0,T]$, and let $b(t)$ be a strictly increasing, continuously differentiable real-valued function with $b(0)=0$ and $b'(t)>0$ for each $t\in[0,T]$. The generalized Brownian motion process Y determined by $a(t)$ and $b(t)$ is a Gaussian process with mean function $a(t)$ and covariance function $r(s,t) = \min\{b(s),b(t)\}$. By Theorem 14.2 in [14], the probability measure μ induced by Y, taking a separable version, is supported by $C_{a,b}[0,T]$ (which is equivalent to the Banach space of continuous functions x on $[0,T]$ with $x(0)=0$ under the sup norm). Hence, $(C_{a,b}[0,T],\mathscr{B}(C_{a,b}[0,T]),\mu)$ is the function space induced by Y where $\mathscr{B}(C_{a,b}[0,T])$ is the Borel σ-algebra of $C_{a,b}[0,T]$. We then complete this function space to obtain $(C_{a,b}[0,T],\mathscr{W}(C_{a,b}[0,T]),\mu)$ where $\mathscr{W}(C_{a,b}[0,T])$ is the set of all Wiener measurable subsets of $C_{a,b}[0,T]$.

A subset A of $C_{a,b}[0,T]$ is said to be scale-invariant measurable provided $\rho A\in\mathscr{W}(C_{a,b}[0,T])$ for all $\rho>0$, and a scale-invariant measurable set N is said to be a scale-invariant null set provided $\mu(\rho N) = 0$ for all $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere(s-a.e.) [15].

Let $L^2_{a,b}[0,T]$ be the Hilbert space of functions on $[0,T]$ which are Lebesgue measurable and square integrable with respect to the Lebesgue Stieltjes measures on $[0,T]$ induced by $a(\cdot)$ and $b(\cdot)$; that is,

$$L^2_{a,b}[0,T] = \left\{ v : \int_0^T v^2(s)\,db(s) < \infty, \int_0^T v^2(s)\,d|a|(s) < \infty \right\}, \tag{1}$$

where $|a|(t)$ denotes the total variation of the function a on the interval $[0,t]$.

For $u, v \in L^2_{a,b}[0,T]$, let

$$(u,v)_{a,b} = \int_0^T u(t)\,v(t)\,d[b(t) + |a|(t)]. \tag{2}$$

Then $(\cdot,\cdot)_{a,b}$ is an inner product on $L^2_{a,b}[0,T]$ and $\|u\|_{a,b} = \sqrt{(u,u)_{a,b}}$ is a norm on $L^2_{a,b}[0,T]$. In particular note that $\|u\|_{a,b} = 0$ if and only if $u(t) = 0$ a.e. on $[0,T]$. Furthermore $(L^2_{a,b}[0,T], \|\cdot\|_{a,b})$ is a separable Hilbert space. Note that all functions of bounded variation on $[0,T]$ are elements of $L^2_{a,b}[0,T]$. Also note that if $a(t) \equiv 0$ and $b(t) = t$, then $L^2_{a,b}[0,T] = L^2[0,T]$. In fact,

$$\begin{aligned}\left(L^2_{a,b}[0,T], \|\cdot\|_{a,b}\right) &\subset \left(L^2_{0,b}[0,T], \|\cdot\|_{0,b}\right) \\ &= \left(L^2[0,T], \|\cdot\|_2\right)\end{aligned} \tag{3}$$

since the two norms $\|\cdot\|_{0,b}$ and $\|\cdot\|_2$ are equivalent.

For $v \in L^2_{a,b}[0,T]$ and $x \in C_{a,b}[0,T]$ we let

$$\langle v, x\rangle = \int_0^T v(t)\,dx(t) \tag{4}$$

denote the Paley-Wiener-Zygmund (PWZ) stochastic integral. Following are some facts about the PWZ stochastic integral [10–12].

(1) The PWZ stochastic integral $\langle v, x\rangle$ is essentially independent of the complete orthonormal set $\{\phi_j\}_{j=1}^{\infty}$.

(2) If v is of bounded variation on $[0,T]$, then the PWZ stochastic integral $\langle v, x\rangle$ equals the Riemann-Stieltjes integral $\int_0^T v(t)\,dx(t)$ for s-a.e. $x \in C_{a,b}[0,T]$.

(3) The PWZ integral has the expected linearity properties.

(4) For all $v \in L^2_{a,b}[0,T]$, $\langle v, x\rangle$ is a Gaussian random variable with mean $\int_0^T v(s)\,da(s)$ and variance $\int_0^T v^2(s)\,db(s)$.

Throughout this paper we will assume that each functional $F : C_{a,b}[0,T] \to \mathbb{C}$ we consider is scale-invariant measurable and that

$$\int_{C_{a,b}[0,T]} |F(\rho x)|\,d\mu(x) < \infty \tag{5}$$

for each $\rho > 0$.

We finish this section by stating the notion of generalized analytic function space Feynman integral, cf. [10, 11].

Definition 1. Let $\mathbb{C}$ denote the complex numbers, let $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}$, and let $\widetilde{\mathbb{C}}_+ = \{\lambda \in \mathbb{C} : \lambda \neq 0 \text{ and } \mathrm{Re}(\lambda) \geq 0\}$. Let $F : C_{a,b}[0,T] \to \mathbb{C}$ be a measurable functional such that, for each $\lambda > 0$, the function space integral

$$J(\lambda) = \int_{C_{a,b}[0,T]} F\left(\lambda^{-1/2}x\right) d\mu(x) \tag{6}$$

exists. If there exists a function $J^*(\lambda)$ analytic in $\mathbb{C}_+$ such that $J^*(\lambda) = J(\lambda)$ for all $\lambda > 0$, then $J^*(\lambda)$ is defined to be the analytic function space integral of F over $C_{a,b}[0,T]$ with parameter λ, and for $\lambda \in \mathbb{C}_+$ we write

$$J^*(\lambda) = \int_{C_{a,b}[0,T]}^{an_\lambda} F(x)\,d\mu(x). \tag{7}$$

Let $q \neq 0$ be a real number and let F be a functional such that $J^*(\lambda)$ exists for all $\lambda \in \mathbb{C}_+$. If the following limit exists, we call it the generalized AFSFI of F with parameter q and we write

$$\int_{C_{a,b}[0,T]}^{anf_q} F(x)\,d\mu(x) = \lim_{\lambda \to -iq} \int_{C_{a,b}[0,T]}^{an_\lambda} F(x)\,d\mu(x), \tag{8}$$

where $\lambda \to -iq$ through values in $\mathbb{C}_+$.

3. Analogue of the Generalized AFSFI

The differential equation

$$\frac{\partial}{\partial t}\psi(u,t) = \frac{1}{2\lambda}\Delta\psi(u,t) - V(u)\,\psi(u,t) \tag{9}$$

is called the diffusion equation with initial condition $\psi(u,0) = \varphi(u)$, where Δ is the Laplacian and V is an appropriate potential function. Many mathematicians have considered the Wiener integral of functionals of the form

$$F\left(\lambda^{-1/2}x + u\right), \tag{10}$$

where u is a real number. It is a well-known fact that the Wiener integral of the functional having the form

$$\exp\left\{-\int_0^T V\left(\lambda^{-1/2}x(t) + u\right)dt\right\}\varphi\left(\lambda^{-1/2}x(T) + u\right) \tag{11}$$

forms the solution of the diffusion equation (9) by the Feynman-Kac formula. If time is replaced by an imaginary time, this diffusion equation becomes the Schrödinger equation

$$i\frac{\partial}{\partial t}\psi(u,t) = -\frac{1}{2}\Delta\psi(u,t) + V(u)\,\psi(u,t) \tag{12}$$

with the initial condition $\psi(u,0)=\varphi(u)$. Hence the solution to the Schrödinger equation (12) can be obtained via the analytic Feynman integral. An approach to finding the solution to the diffusion equation (9) and the Schrödinger equation (12) involves the harmonic oscillator $V(u)=(k/2)u^2$; for a more detailed study, see [6]. However, it can be difficult to obtain the solution for the diffusion equation (9) and the Schrödinger equation (12) with respect to anharmonic oscillators.

In this paper, we consider the following functional:

$$\begin{aligned}&\exp\left\{-\int_0^T V\left(\lambda^{-1/2}x(t)+c(\lambda)h(t)\right)dt\right\}\\&\times\varphi\left(\lambda^{-1/2}x(T)+c(\lambda)h(t)\right),\end{aligned}\tag{13}$$

where $c(\lambda)$ is a real number with respect to λ and $h(t)$ is a real-valued function on $[0,T]$. When $h(t)=u$ for all $t\in[0,T]$ and $c(\lambda)$ is independent of the value λ, the functional in (13) reduces the functional in (11). That is to say, our functional (13) is more generalized compared with the functional in (11). Hence, all results and formulas for the functional in (11) are special cases of our results and formulas.

We will now explain the importance of the functionals given by (13). For a positive real number k, when the potential function is $V(u)=(k/2)u^2$, the diffusion equation (9) is called the diffusion equation for a harmonic oscillator with V. For $\xi\in\mathbb{R}$,

$$V_1(u)\equiv V(u+\xi)=\frac{k}{2}(u+\xi)^2\tag{14}$$

is just the translation of V; thus, it is called the diffusion equation for a harmonic oscillator with V_1. However, for an appropriate function $h(t)$ on $[0,T]$,

$$V_2(u)\equiv V(u+h(u))=\frac{k}{2}(u+h(u))^2\tag{15}$$

may be an anharmonic oscillator. For example, consider the following.

(1) If $h(t)=u^2$ on $[0,T]$, then

$$V_3(u)=\frac{k}{2}\left(u^2+2u^3+u^4\right).\tag{16}$$

In this case, the diffusion equation (9) is called the diffusion equation for anharmonic oscillator with V_3 because it contains the "u^3-term." This means that the status of the harmonic oscillator can be exchanged for the status of the anharmonic oscillator under certain physical circumstances. We can explain this phenomenon by considering the Wiener integral of the functional in (13).

(2) For a real number γ, if $h(t)=-u+\sqrt{u^2(u^2-\gamma^2)}$ on $[0,T]$, then

$$V_4(u)=\frac{k}{2}u^2\left(u^2-\gamma^2\right).\tag{17}$$

In this case, the diffusion equation (9) is called the diffusion equation for double-well potential with V_4. As such, it is a harmonic oscillator.

(3) Furthermore, we see that, for $v\in L^2_{a,b}[0,T]$, $h\in C_{a,b}[0,T]$, and $u\in\mathbb{R}$,

$$\begin{aligned}\langle v,x+u\rangle&=\langle v,x\rangle,\\\langle v,x+h\rangle&=\langle v,x\rangle+\langle v,h\rangle\end{aligned}\tag{18}$$

provided $\langle v,h\rangle\neq 0$. Thus, the functionals presented in this paper are more meaningful than the functionals given in previous papers [6, 11]. This also has implications regarding the generalizations of our research observations.

We are now ready to state the definition of the modified generalized AFSFI.

Definition 2. Let $h\in C_{a,b}[0,T]$ be given. Let $F:C_{a,b}[0,T]\to\mathbb{C}$ be such that, for each $\lambda>0$, the function space integral

$$J(\lambda)=\int_{C_{a,b}[0,T]}F\left(\lambda^{-1/2}x+c(\lambda)h\right)d\mu(x)\tag{19}$$

exists for all $\lambda>0$ where $c(\lambda)$ is a nonnegative real number which depends on λ. If there exists a function $J^*(\lambda)$ analytic in $\mathbb{C}_+$ such that $J^*(\lambda)=J(\lambda)$ for all $\lambda>0$, then $J^*(\lambda)$ is defined to be the modified analytic function space integral of F over $C_{a,b}[0,T]$ with parameter λ, and for $\lambda\in\mathbb{C}_+$ we write

$$J^*(\lambda)=\int_{C_{a,b}[0,T]}^{an_\lambda^{c(\lambda)},h}F(x)\,d\mu(x).\tag{20}$$

Let $q\neq 0$ be a real number and let F be a functional such that $\int_{C_{a,b}[0,T]}^{an_\lambda^{c(\lambda)},h}F(x)\,d\mu(x)$ exists for all $\lambda\in\mathbb{C}_+$. If the following limit exists, we call it the modified generalized AFSFI of F with parameter q and we write

$$\int_{C_{a,b}[0,T]}^{anf_q^{c(q)},h}F(x)\,d\mu(x)=\lim_{\lambda\to -iq}\int_{C_{a,b}[0,T]}^{an_\lambda^{c(\lambda)},h}F(x)\,d\mu(x),\tag{21}$$

where λ approaches $-iq$ through values in $\mathbb{C}_+$.

Remark 3. We have the following assertions with respect to the modified generalized AFSFI.

(1) If $h(t)\equiv 0$ on $[0,T]$ or $c(\lambda)=0$, then we can write

$$\begin{aligned}\int_{C_{a,b}[0,T]}^{an_\lambda^{c(\lambda)},h}F(x)\,d\mu(x)&=\int_{C_{a,b}[0,T]}^{an_\lambda}F(x)\,d\mu(x),\\\int_{C_{a,b}[0,T]}^{anf_q^{c(q)},h}F(x)\,d\mu(x)&=\int_{C_{a,b}[0,T]}^{anf_q}F(x)\,d\mu(x).\end{aligned}\tag{22}$$

(2) In the setting of classical Wiener space (in our research, when $a(t)\equiv 0$ and $b(t)=t$ on $[0,T]$), our modified generalized AFSFI, the generalized AFSFI, and the analytic Feynman integral coincide. Hence all results and formulas in [2, 3, 5, 6, 16] are corollaries of our results and formulas in this paper.

We conclude this section by listing several integration formulas for simple functionals to compare with the generalized AFSFI and the modified generalized AFSFI. For all nonzero real number q, we have Tables 1 and 2.

Table 1: Modified generalized AFSFI ($j=1,2$).

	$\int_{C_{a,b}[0,T]}^{anf_q^{c(q)},h} F_j(x)\,d\mu(x)$
$F_1(x)=x(T)$	$\left(\frac{i}{q}\right)^{1/2} a(T)+c(q)h(T)$
$F_2(x)=e^{x(T)}$	$\exp\left\{\left(\frac{i}{q}\right)^{1/2} a(T)+\frac{i}{2q}b(T)+c(q)h(T)\right\}$

Table 2: Generalized AFSFI ($j=1,2$).

	$\int_{C_{a,b}[0,T]}^{anf_q} F_j(x)\,d\mu(x)$
$F_1(x)=x(T)$	$\left(\frac{i}{q}\right)^{1/2} a(T)$
$F_2(x)=e^{x(T)}$	$\exp\left\{\left(\frac{i}{q}\right)^{1/2} a(T)+\frac{i}{2q}b(T)\right\}$

4. Some Properties for the Modified Generalized AFSFI

In this section we establish a Fubini theorem for the modified analytic function space integrals and the modified generalized AFSFIs for functionals on $C_{a,b}[0,T]$. We also use these Fubini theorems to establish various modified generalized analytic Feynman integration formulas.

First, we define a function to simply express many results and formulas in this paper. For $n\ge 2$, define a function $H_n:\widetilde{\mathbb{C}}_+^n\to\widetilde{\mathbb{C}}_+$ by

$$H_n(z_1,\ldots,z_n)=\sum_{j=1}^n z_j^{-1/2}-\left(\sum_{j=1}^n z_j^{-1}\right)^{1/2}, \tag{23}$$

where $\sum_{j=1}^n z_j^{-1/2}\neq 0$ and $\sum_{j=1}^n z_j^{-1}\neq 0$. Note that H_n is a symmetric function for all $n=2,3,\ldots$. In this paper we will assume that, for all $(z_1,\ldots,z_n)\in\widetilde{\mathbb{C}}_+^n$ and $(\sum_{j=1}^n z_j^{-1})^{1/2}$, $n=1,2,\ldots$, and $z_j^{-1/2}$, $j=1,2,\ldots,n$, are always chosen to have positive real parts.

In our first theorem, we show that the modified generalized AFSFIs are commutative.

Theorem 4. *Let h_1 and h_2 be elements of $C_{a,b}[0,T]$ and let F be a functional defined on $C_{a,b}[0,T]$ such that*

$$\begin{aligned}\int_{C_{a,b}^2[0,T]} &|F(\gamma x+\beta y+c(\gamma)h_1+c(\beta)h_2)|\\ &\times d(\mu\times\mu)(x,y)<\infty,\end{aligned} \tag{24}$$

for all nonzero real numbers γ and β. Then for all $q_1,q_2\in\mathbb{R}-\{0\}$,

$$\begin{aligned}&\int_{C_{a,b}[0,T]}^{anf_{q_2}^{c(q_2)},h_2}\left(\int_{C_{a,b}[0,T]}^{anf_{q_1}^{c(q_1)},h_1} F(x+y)\,d\mu(x)\right)d\mu(y)\\ &\quad\doteq\int_{C_{a,b}[0,T]}^{anf_{q_1}^{c(q_1)},h_1}\left(\int_{C_{a,b}[0,T]}^{anf_{q_2}^{c(q_2)},h_2} F(x+y)\,d\mu(y)\right)d\mu(x),\end{aligned} \tag{25}$$

where $\doteq$ means that if either side exists, both sides exist and equality holds.

Proof. First, using the symmetric property, for all $\lambda,\beta>0$,

$$\begin{aligned}&\int_{C_{a,b}^2[0,T]} F\left(\lambda^{-1/2}x+\beta^{-1/2}y+c(\lambda)h_1\right.\\ &\qquad\left.+c(\beta)h_2\right)d(\mu\times\mu)(x,y)\\ &\quad=\int_{C_{a,b}^2[0,T]} F\left(\beta^{-1/2}y+\lambda^{-1/2}x+c(\beta)h_2\right.\\ &\qquad\left.+c(\lambda)h_1\right)d(\mu\times\mu)(y,x).\end{aligned} \tag{26}$$

This can be analytically continued in λ and β for (λ,β) and so we have, for all $(\lambda,\beta)\in\mathbb{C}_+\times\mathbb{C}_+$,

$$\begin{aligned}&\int_{C_{a,b}[0,T]}^{an_\beta^{c(\beta)},h_2}\left(\int_{C_{a,b}[0,T]}^{an_\lambda^{c(\lambda)},h_1} F(x+y)\,d\mu(x)\right)d\mu(y)\\ &\quad\doteq\int_{C_{a,b}[0,T]}^{an_\lambda^{c(\lambda)},h_1}\left(\int_{C_{a,b}[0,T]}^{an_\beta^{c(\beta)},h_2} F(x+y)\,d\mu(y)\right)d\mu(x).\end{aligned} \tag{27}$$

Next, let E be a subset of $\widetilde{\mathbb{C}}_+\times\widetilde{\mathbb{C}}_+$ containing the point $(-iq_1,-iq_2)$ and it is such that $(\lambda,\beta)\in E$ implies that $\lambda+\beta\neq 0$. Note that the function

$$\mathscr{H}(\lambda,\beta)\equiv\int_{C_{a,b}[0,T]}^{an_\beta^{c(\beta)},h_2}\left(\int_{C_{a,b}[0,T]}^{an_\lambda^{c(\lambda)},h_1} F(y+z)\,d\mu(y)\right)d\mu(z) \tag{28}$$

is continuous on E and is uniformly continuous on E provided E is compact. Then by the continuity of $\mathscr{H}$ and (27), we can establish (25) as desired. □

The following theorem was established in [12, 17]. Formula (29) is called the Fubini theorem with respect to the function space integrals.

Theorem 5. *Let F be as in Theorem 4 above. Then*

$$\begin{aligned}&\int_{C_{a,b}^2[0,T]} F(\gamma x+\beta y)\,d(\mu\times\mu)(x,y)\\ &\quad=\int_{C_{a,b}[0,T]} F\left(\sqrt{\gamma^2+\beta^2}z\right.\\ &\qquad\left.+\left(\gamma+\beta-\sqrt{\gamma^2+\beta^2}\right)a\right)d\mu(z)\\ &\quad=\int_{C_{a,b}[0,T]} F\left(\sqrt{\gamma^2+\beta^2}z+H_2\left(\gamma^{-2},\beta^{-2}\right)a\right)d\mu(z).\end{aligned} \tag{29}$$

To establish Theorem 7, we need the following lemma.

Lemma 6. *Let F be as in Theorem 4 above. Then for all* $(\lambda, \beta) \in \mathbb{C}_+ \times \mathbb{C}_+$ *with* $\lambda + \beta \neq 0$,

$$\int_{C_{a,b}[0,T]}^{an_\beta} \left(\int_{C_{a,b}[0,T]}^{an_\lambda} F(x+y)\, d\mu(x) \right) d\mu(y) \doteq \int_{C_{a,b}[0,T]}^{an_\gamma^{c(\gamma)},a} F(z)\, d\mu(z), \tag{30}$$

where $\gamma = \lambda\beta/(\lambda+\beta)$ *and* $c(\gamma) = H_2(\lambda, \beta)$.

Proof. Using (29), it follows that for $\lambda > 0$ and $\beta > 0$

$$\int_{C_{a,b}^2[0,T]} F\left(\lambda^{-1/2}x + \beta^{-1/2}y\right) d(\mu \times \mu)(x, y) = \int_{C_{a,b}[0,T]} F\left(\sqrt{\lambda^{-1} + \beta^{-1}}z + H_2(\lambda, \beta)\, a\right) d\mu(z). \tag{31}$$

This last expression is defined for $\lambda > 0$ and $\beta > 0$. For $\beta > 0$, it can be analytically continued in $\lambda \in \mathbb{C}_+$. Also for $\lambda > 0$, it can be analytically continued in $\beta \in \mathbb{C}_+$. Therefore since $\lambda \in \mathbb{C}_+$ and $\beta \in \mathbb{C}_+$ implies that $\lambda\beta/(\lambda+\beta) \in \mathbb{C}_+$, we conclude that the last expression in proof of Lemma 6 can be analytically continued into $\mathbb{C}_+$ to equal the analytic function space integral

$$\int_{C_{a,b}[0,T]}^{an_\gamma^{c(\gamma)},a} F(z)\, d\mu(z), \tag{32}$$

which completes the proof of Lemma 6 as desired. □

The following theorem is the main result with respect to the modified generalized AFSFI.

Theorem 7. *Let F be as in Lemma 6 above. Then for all* $q_1, q_2 \in \mathbb{R} - \{0\}$ *with* $q_1 + q_2 \neq 0$,

$$\int_{C_{a,b}[0,T]}^{anf_{q_2}} \left(\int_{C_{a,b}[0,T]}^{anf_{q_1}} F(x+y)\, d\mu(x) \right) d\mu(y) \doteq \int_{C_{a,b}[0,T]}^{anf_{q_3}^{c(q_3)},a} F(z)\, d\mu(z), \tag{33}$$

where $q_3 = q_1q_2/(q_1+q_2)$ *and* $c(q_3) = H_2(-iq_1, -iq_2)$.

Proof. First note that, for all $q_1, q_2 \in \mathbb{R} - \{0\}$ with $q_1 + q_2 \neq 0$, if $\lambda \to -iq_1$ and $\beta \to -iq_2$, then $\lambda\beta/(\lambda+\beta) \to -i(q_1q_2/(q_1+q_2))$. Now using this fact and (30) it follows that

$$\begin{aligned}
&\int_{C_{a,b}[0,T]}^{anf_{q_2}} \left(\int_{C_{a,b}[0,T]}^{anf_{q_1}} F(x+y)\, d\mu(x) \right) d\mu(y) \\
&\doteq \lim_{\beta \to -iq_2} \lim_{\lambda \to -iq_1} \int_{C_{a,b}[0,T]}^{an_\gamma^{c(\gamma)},a} F(z)\, d\mu(z) \\
&\doteq \lim_{\lambda\beta/(\lambda+\beta) \to -i(q_1q_2/(q_1+q_2))} \int_{C_{a,b}[0,T]}^{an_\gamma^{c(\gamma)},a} F(z)\, d\mu(z) \\
&\doteq \int_{C_{a,b}[0,T]}^{anf_{q_3}^{c(q_3)},a} F(z)\, d\mu(z),
\end{aligned} \tag{34}$$

where γ and $c(\gamma)$ are as in Lemma 6. Hence we complete the proof as desired. □

Equations (35)–(37) below follow by mathematical induction and Theorem 7 above.

Corollary 8. *Let F be as in Theorem 7 above. Then one has the following assertions.*

(1) *For all* $q \in \mathbb{R} - \{0\}$,

$$\int_{C_{a,b}[0,T]}^{anf_q} \left(\int_{C_{a,b}[0,T]}^{anf_q} F(x+y)\, d\mu(x) \right) d\mu(y) \doteq \int_{C_{a,b}[0,T]}^{anf_{q/2}^{c(q)},a} F(z)\, d\mu(z), \tag{35}$$

where $c(q) = H_2(-iq, -iq)$.

(2) *For all* $q_1, \ldots, q_n \in \mathbb{R} - \{0\}$ *with* $\sum_{j=1}^k (q_1 \cdots q_k/q_j) \neq 0$ *for* $k = 2, \ldots, n$,

$$\int_{C_{a,b}[0,T]}^{anf_{q_n}} \cdots \int_{C_{a,b}[0,T]}^{anf_{q_1}} F(x_1 + \cdots + x_n)\, d(\mu \times \cdots \times \mu)(\vec{x}) \doteq \int_{C_{a,b}[0,T]}^{anf_{\beta_n}^{c(\beta_n)},a} F(z)\, d\mu(z), \tag{36}$$

where $\vec{x} = (x_1, \ldots, x_n)$, $\beta_n = q_1 \cdots q_n / \sum_{j=1}^n (q_1 \cdots q_n / q_j)$, *and* $c(\beta_n) = H_n(-iq_1, \ldots, -iq_n)$. *Furthermore,*

$$\int_{C_{a,b}[0,T]}^{anf_q} \cdots \int_{C_{a,b}[0,T]}^{anf_q} F(x_1 + \cdots + x_n)\, d(\mu \times \cdots \times \mu)(\vec{x}) \doteq \int_{C_{a,b}[0,T]}^{anf_{q/n}^{c(q/n)},a} F(z)\, d\mu(z), \tag{37}$$

where $c(q/n) = H_2(-iq, \ldots, -iq)$.

Next we establish some integration formulas with respect to the modified generalized AFSFIs.

(1) A formula showing that the double modified generalized AFSFIs can be expressed by just one modified

generalized AFSFI. For all $q_1, q_2 \in \mathbb{R} - \{0\}$ with $q_1 + q_2 \neq 0$,

$$\begin{aligned}&\int_{C_{a,b}[0,T]}^{anf_{q_2}^{c(q_2)},a}\left(\int_{C_{a,b}[0,T]}^{anf_{q_1}^{c(q_1)},a} F(x+y)\,d\mu(x)\right)d\mu(y)\\&\quad\doteq\int_{C_{a,b}[0,T]}^{anf_{q_1q_2/(q_1+q_2)}^{c(q_1,q_2)},a} F(z)\,d\mu(z),\end{aligned}\tag{38}$$

where $c(q_1, q_2) = H_2(-iq_1, -iq_2) + c(q_1) + c(q_2)$. Furthermore, if $c(q_1) + c(q_2) = -H_2(-iq_1, -iq_2)$, then

$$\begin{aligned}&\int_{C_{a,b}[0,T]}^{anf_{q_2}^{c(q_2)},a}\left(\int_{C_{a,b}[0,T]}^{anf_{q_1}^{c(q_1)},a} F(x+y)\,d\mu(x)\right)d\mu(y)\\&\quad\doteq\int_{C_{a,b}[0,T]}^{anf_{q_1q_2/(q_1+q_2)}} F(z)\,d\mu(z).\end{aligned}\tag{39}$$

(2) A relationship between the modified generalized AFSFI and the generalized AFSFI. For all $q_1, q_2 \in \mathbb{R} - \{0\}$ with $q_1 + q_2 \neq 0$,

$$\begin{aligned}&\int_{C_{a,b}[0,T]}^{anf_{q_1}^{c(q_1)},a}\left(\int_{C_{a,b}[0,T]}^{anf_{q_2}} F(x+y)\,d\mu(x)\right)d\mu(y)\\&\quad\doteq\int_{C_{a,b}[0,T]}^{anf_{q_3}^{c(q_3)},a} F(z)\,d\mu(z),\end{aligned}\tag{40}$$

where $q_3 = q_1q_2/(q_1+q_2)$ and $c(q_3) = H_2(-iq_1, -iq_2) + c(q_1)$.

(3) A formula relating the modified generalized AFSFI and the generalized AFSFI. For all $q_1, q_2 \in \mathbb{R} - \{0\}$,

$$\begin{aligned}&\int_{C_{a,b}[0,T]}^{anf_{q_1}^{c(q_1)},a}\left(\int_{C_{a,b}[0,T]}^{anf_{q_2}} F(x+y)\,d\mu(x)\right)d\mu(y)\\&\quad\doteq\int_{C_{a,b}[0,T]}^{anf_{q_2}}\left(\int_{C_{a,b}[0,T]}^{anf_{q_1}^{c(q_1)},a} F(x+y)\,d\mu(x)\right)d\mu(y).\end{aligned}\tag{41}$$

5. Examples

In this section, we provide several brief examples in which we apply our formulas and results.

5.1. Banach Algebra $\mathcal{S}(L^2_{a,b}[0,T])$. Let $M(L^2_{a,b}[0,T])$ be the space of complex-valued, countably additive Borel measures on $L^2_{a,b}[0,T]$. The Banach algebra $\mathcal{S}(L^2_{a,b}[0,T])$ consists of those functionals F on $C_{a,b}[0,T]$ expressible in the form

$$F(x) = \int_{L^2_{a,b}[0,T]} \exp\{i\langle v, x\rangle\}\,df(v) \tag{42}$$

for s-a.e. $x \in C_{a,b}[0,T]$ where the associated measure f is an element of $M(L^2_{a,b}[0,T])$.

Example 1. Let q_0 be a fixed nonzero real number. Let $F \in \mathcal{S}(L^2_{a,b}[0,T])$ be given by (42) above. Suppose that corresponding measure f of F satisfies the condition

$$\int_{L^2_{a,b}[0,T]} \exp\left\{\frac{4}{\sqrt{2|q_0|}}\int_0^T |v(s)|\,d|a|(s)\right\}|df(v)| < \infty. \tag{43}$$

Then for all nonzero real number q with $|q| \geq |q_0|$,

$$\begin{aligned}&\int_{C_{a,b}[0,T]}^{anf_q^{c(q)},a} F(x)\,d\mu(x)\\&\quad=\int_{L^2_{a,b}[0,T]}\exp\left\{-\frac{i}{2q}(v^2,b')\right.\\&\qquad\left.+i\left(c(q)+\left(\frac{i}{q}\right)^{1/2}\right)(v,a')\right\}df(v),\end{aligned}\tag{44}$$

where

$$(v,a') = \int_0^T v(t)\,da(t), \qquad (v^2,b') = \int_0^T v^2(t)\,db(t). \tag{45}$$

Next, using Theorem 7, we can compute the double generalized AFSFIs of $F \in \mathcal{S}(L^2_{a,b}[0,T])$ by just one modified generalized AFSFI. That is to say, for all $q_1, q_2 \in \mathbb{R}$ with $|q_1| \geq |q_0|, |q_2| \geq |q_0|$ and $q_1 + q_2 \neq 0$,

$$\begin{aligned}&\int_{C_{a,b}[0,T]}^{anf_{q_2}}\left(\int_{C_{a,b}[0,T]}^{anf_{q_1}} F(x+y)\,d\mu(x)\right)d\mu(y)\\&\quad=\int_{C_{a,b}[0,T]}^{anf_\gamma^{c(\gamma)},a} F(z)\,d\mu(z)\\&\quad=\int_{L^2_{a,b}[0,T]}\exp\left\{-\frac{i}{2}\left(\frac{1}{q_1}+\frac{1}{q_2}\right)(v^2,b')\right.\\&\qquad+i\left(\left(\frac{i}{q_1}\right)^{1/2}+\left(\frac{i}{q_2}\right)^{1/2}\right)\\&\qquad\left.\times(v,a')\right\}df(v),\end{aligned}\tag{46}$$

where $\gamma = q_1q_2/(q_1 + q_2)$ and $c(\gamma) = H_2(-iq_1, -iq_2)$. Furthermore the last expression in (46) equals the expression

$$\int_{C_{a,b}[0,T]}^{anf_{q_1}}\left(\int_{C_{a,b}[0,T]}^{anf_{q_2}} F(x+y)\,d\mu(y)\right)d\mu(x). \tag{47}$$

5.2. The Fourier Transform of a Complex-Valued Measure. For given $\vec{m} = (m_1, \ldots, m_n) \in \mathbb{R}^n$ and $\vec{\sigma^2} = (\sigma_1^2, \ldots, \sigma_n^2) \in \mathbb{R}^n$

with $\sigma_j^2 > 0$, $j = 1, \ldots, n$, let $\nu_{\vec{m},\vec{\sigma^2}}$ be the Gaussian measure given by

$$\nu_{\vec{m},\vec{\sigma^2}}(B) = \left(\prod_{j=1}^{n} 2\pi\sigma_j^2\right)^{-1/2} \int_B \exp\left\{-\sum_{j=1}^{n}\frac{(u_j - m_j)^2}{2\sigma_j^2}\right\} d\vec{u}, \tag{48}$$

where $B \in \mathscr{B}(\mathbb{R}^n)$. Then $\nu_{\vec{m},\vec{\sigma^2}}$ is a complex-valued Borel measure on $\mathbb{R}^n$ and

$$\widehat{\nu_{\vec{m},\vec{\sigma^2}}}(\vec{u}) = \exp\left\{-\frac{1}{2}\sum_{j=1}^{n}\sigma_j^2 u_j^2 + i\sum_{j=1}^{n} m_j u_j\right\}, \tag{49}$$

where $\widehat{\nu_{\vec{m},\vec{\sigma^2}}}$ is the Fourier transform of the Gaussian measure $\nu_{\vec{m},\vec{\sigma^2}}$.

Example 2. Let $\{\alpha_1, \ldots, \alpha_n\}$ be any orthonormal set in $L^2_{a,b}[0,T]$ and let $F : C_{a,b}[0,T] \to \mathbb{R}^n$ be the functional defined by

$$F(x) = \widehat{\nu_{\vec{m},\vec{\sigma^2}}}\left(\langle\alpha_1, x\rangle, \ldots, \langle\alpha_n, x\rangle\right), \tag{50}$$

where $\mathrm{Var}[\langle\alpha_j, x\rangle^2] = 1$ for all $j = 1, 2, \ldots, n$. Then for all nonzero real number q,

$$\begin{aligned} &\int_{C_{a,b}[0,T]}^{anf_q^{c(q)},a} F(x)\, d\mu(x) \\ &= \left(\prod_{j=1}^{n}\frac{1}{2(1-(-iq)^{-1/2}\sigma_j^2)}\right)^{1/2} \\ &\quad\times \exp\Bigg\{\sum_{j=1}^{n}\Big((-iq)^{-1/2} \\ &\qquad\times\left[i4A_j m_j - (-iq)^{-1/2}m_j^2 + 2A_j^2\sigma_j^2\right]\Big) \\ &\qquad\times\left(2\left(2-(-iq)^{-1/2}\sigma_j^2\right)\right)^{-1}\Bigg\} \\ &\quad\times \exp\left\{-\frac{c(q)}{2}\sum_{j=1}^{n}\sigma_j^2\langle\alpha_j, h\rangle + ic(q)\sum_{j=1}^{n} m_j\langle\alpha_j, h\rangle\right\}, \end{aligned} \tag{51}$$

where $A_j = \int_0^T \alpha_j(t)\,da(t)$. Using Theorem 7, we can compute the double generalized AFSFIs of F given by (50) by just one modified generalized AFSFI. That is to say, for all $q_1, q_2 \in \mathbb{R}$ with $q_1 + q_2 \neq 0$,

$$\begin{aligned} &\int_{C_{a,b}[0,T]}^{anf_{q_2}}\left(\int_{C_{a,b}[0,T]}^{anf_{q_1}} F(x+y)\, d\mu(x)\right) d\mu(y) \\ &= \left(\prod_{j=1}^{n}\frac{1}{2\left(1-(-iQ)^{-1/2}\sigma_j^2\right)}\right)^{1/2} \\ &\quad\times \exp\Bigg\{\sum_{j=1}^{n}\Big((-iQ)^{-1/2} \\ &\qquad\times\left[i4A_j m_j - (-iQ)^{-1/2}m_j^2 + 2A_j^2\sigma_j^2\right]\Big) \\ &\qquad\times\left(2\left(2-(-iQ)^{-1/2}\sigma_j^2\right)\right)^{-1}\Bigg\} \\ &\quad\times \exp\left\{-\frac{c(Q)}{2}\sum_{j=1}^{n}\sigma_j^2\langle\alpha_j, h\rangle + ic(Q)\sum_{j=1}^{n} m_j\langle\alpha_j, h\rangle\right\}, \end{aligned} \tag{52}$$

where $Q = q_1 q_2/(q_1 + q_2)$ and $c(Q) = H_2(-iq_1, -iq_2)$. Furthermore, the last expression in (52) equals the expression

$$\int_{C_{a,b}[0,T]}^{anf_{q_1}}\left(\int_{C_{a,b}[0,T]}^{anf_{q_2}} F(x+y)\, d\mu(y)\right) d\mu(x). \tag{53}$$

5.3. The Generalized Fourier-Hermite Functional on Function Space. For each $m = 0, 1, 2, \ldots$, and for each $j = 1, 2, \ldots$, let $H_m^j(u)$ denote the generalized Hermite polynomial

$$\begin{aligned} H_m^j(u) &\equiv (-1)^m (m!)^{-1/2}(B_j)^{m/2}\exp\left\{\frac{(u-A_j)^2}{2B_j}\right\} \\ &\quad\times\frac{d^m}{du^m}\left(\exp\left\{-\frac{(u-A_j)^2}{2B_j}\right\}\right). \end{aligned} \tag{54}$$

Then for each $j = 1, 2, \ldots$, the set

$$\left\{(2\pi B_j)^{-1/4}H_m^j(u)\exp\left\{-\frac{(u-A_j)^2}{4B_j}\right\} : m = 0, 1, \ldots\right\} \tag{55}$$

is a complete orthonormal set in $L_2(\mathbb{R})$. Now we define

$$\Phi_{(m_1,\ldots,m_k)}(x) = \prod_{j=1}^{k} H_{m_j}^j\left(\langle\alpha_j, x\rangle\right). \tag{56}$$

The functionals in (56) are called the generalized Fourier-Hermite functionals. It is known that these functionals form a complete orthonormal set in $L^2(C_{a,b}[0,T])$; that is to say, let $F \in L^2(C_{a,b}[0,T])$ and, for $N = 1, 2, \ldots$, let

$$F_N(x) = \sum_{m_1,\ldots,m_N=0}^{N} A^F_{(m_1,\ldots,m_N)}\Phi_{(m_1,\ldots,m_N)}(x), \tag{57}$$

where $A^F_{(m_1,\dots,m_N)}$ is the generalized Fourier-Hermite coefficient,

$$A^F_{(m_1,\dots,m_N)} \equiv \int_{C_{a,b}[0,T]} F(x)\,\Phi_{(m_1,\dots,m_N)}(x)\,d\mu(x). \quad (58)$$

Then

$$\begin{aligned} F(x) &= \lim_{N\to\infty} F_N(x) \\ &= \lim_{N\to\infty} \sum_{m_1,\dots,m_N=0}^{N} A^F_{(m_1,\dots,m_N)} \Phi_{(m_1,\dots,m_N)}(x) \end{aligned} \quad (59)$$

is called the generalized Fourier-Hermite series expansion of F. In (59), the limit is taken in the $L^2(C_{a,b}[0,T])$-sense.

Example 3. Let q_0 be a nonzero real number and let $\Phi_{(m_1,\dots,m_N)}$ be the generalized Fourier-Hermite functional given by (56) above. Then for all nonzero real number q with $|q| \geq |q_0|$, the modified generalized AFSFI of $\Phi_{(m_1,\dots,m_N)}$ exists and it is given by the formula

$$\int_{C_{a,b}[0,T]}^{anf_q^{c(q)},h} \Phi_{(m_1,\dots,m_N)} d\mu(x) = \prod_{j=1}^{N} \int_{C_{a,b}[0,T]}^{anf_q^{c(q)},h} \phi_{(m_j,j)} d\mu(x), \quad (60)$$

where

$$\begin{aligned} \phi_{(m_j,j)}(x) &= H^j_{m_j}\left(\left\langle \alpha_j, x\right\rangle\right), \\ \int_{C_{a,b}[0,T]}^{anf_q^{c(q)},h} \phi_{(m_j,j)} d\mu(x) &= (2\pi B_j)^{-1/2} \int_R H^j_m\left((-iq)^{-1/2}u + c(q)\langle\vec{\alpha}, h\rangle\right) \\ &\quad \times \exp\left\{-\frac{\left(u - A_j\right)^2}{2B_j}\right\} du. \end{aligned} \quad (61)$$

The last expression is valid because the generalized Hermite functional is a polynomial with degree m_j and hence it has an analytic extension.

Remark 9. Since the set of generalized Fourier-Hermite functionals

$$\mathscr{M} \equiv \{\Phi_{(m_1,\dots,m_k)}\}_{k=1}^{\infty} \quad (62)$$

is a complete orthonormal set in $L^2(C_{a,b}[0,T])$, we could extend the results for functionals in $L^2(C_{a,b}[0,T])$ under the appropriate conditions.

6. Conclusions

In Section 3, we presented our analysis of the generalized AFSFI and defined the modified generalized AFSFI. Furthermore we explained the physical circumstances with respect to an anharmonic oscillator using the concept of the modified generalized AFSFI. That is to say, we introduced some new concepts in order to explain various physical circumstances. In Section 4, we established some relationships with respect to the modified generalized AFSFI involving the generalized AFSFI; see Theorem 7. Finally, we applied our results to various classes of functionals studied in [2, 4, 10, 11].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

On Monotonic and Nonnegative Solutions of a Nonlinear Volterra-Stieltjes Integral Equation

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We study the existence of monotonic and nonnegative solutions of a nonlinear quadratic Volterra-Stieltjes integral equation in the space of real functions being continuous on a bounded interval. The main tools used in our considerations are the technique of measures of noncompactness in connection with the theory of functions of bounded variation and the theory of Riemann-Stieltjes integral. The obtained results can be easily applied to the class of fractional integral equations and Volterra-Chandrasekhar integral equations, among others.

1. Introduction

The aim of this paper is to study of monotonic and nonnegative solutions of the nonlinear quadratic Volterra-Stieltjes integral equation having the form

$$x(t) = (F_1 x)(t) + (F_2 x)(t) \int_0^t u(t, \tau, (Tx)(\tau))\, d_\tau g(t, \tau), \tag{1}$$

where $t \in [a, b]$ and F_1, F_2 are superposition operators defined on the function space $C[a, b]$. The precise definitions will be given later. We show the existence of such solutions of the previous equation under some reasonable and handy assumptions. In our considerations, we use the technique associated with measures of noncompactness and the Riemann-Stieltjes integral with a kernel depending on two variables. Moreover, the theory of functions of bounded variation is also employed.

The main result of the paper is contained in Theorem 8. That theorem covers, as particular cases, the classical Volterra integral equation, the integral equation of fractional order, and the Volterra counterpart of the famous integral equation of Chandrasekhar type. It is worth pointing out that differential and integral equations of fractional order create an important branch of nonlinear analysis and the theory of integral equations. Moreover, these equations have found a lot of applications connected with real world problems. Integral equations of Chandrasekhar type can be often encountered in several applications as well.

This paper can be considered as a continuation of [1, 2] (cf. also [3–5]).

2. Preliminaries

At the beginning, we provide some basic facts concerning functions of bounded variation and the Riemann-Stieltjes integral. We refer to [6] or [7] for more information about this subject. Assume that x is a real function defined on the interval $[a, b]$. The symbol $\bigvee_a^b x$ stands for the variation of the function x on the interval $[a, b]$. In case of a function $u(t, \tau) = u : A \to \mathbb{R}$, where $A \subset \mathbb{R}^2$, the symbol $\bigvee_{\tau=p}^q u(t, \tau)$ denotes the variation of the function $\tau \to u(t, \tau)$ on the interval $[p, q]$ which is contained in the domain of this function, where the variable t is fixed. Further, assume that x, φ are given real functions defined on the interval $[a, b]$. Then, under some additional conditions imposed on x and φ, we can define the Riemann-Stieltjes integral

$$\int_a^b x(t)\, d\varphi(t) \tag{2}$$

of the function x with respect to the function φ. In such a case, we say that x is integrable in the Riemann-Stieltjes sense on the interval $[a, b]$ with respect to φ.

Now, we recall two useful properties of the Riemann-Stieltjes integral, which will be employed in the sequel.

Theorem 1. *(a) If x is a continuous function and φ is a function of bounded variation on the interval $[a, b]$, then x is Riemann-Stieltjes integrable on $[a, b]$ with respect to φ.*

(b) Suppose that x_1 and x_2 are functions being Riemann-Stieltjes integrable on the interval $[a, b]$ with respect to a nondecreasing function φ and $x_1(t) \le x_2(t)$, for $t \in [a, b]$. Then,

$$\int_a^b x_1(t)\, d\varphi(t) \le \int_a^b x_2(t)\, d\varphi(t). \tag{3}$$

In what follows we will use the Riemann-Stieltjes integral of the form

$$\int_a^b x(\tau)\, d_\tau g(t, \tau), \tag{4}$$

where the symbol d_τ indicates the integration with respect to the variable τ and t is fixed. Let us mention that, in some situations, lower and upper limit of the integration can also depend upon the variable t.

Now, we deal with the discussion of basic facts connected with measures of noncompactness. We refer to [8] (see also [9]) for a more detailed discussion. Assume that E is a real Banach space. Denote by $B(x, r)$ the closed ball centered at x and with radius r. Instead of $B(0, r)$, we will write B_r. If X is a subset of E, then the symbols $\overline{X}$ and Conv$X$ denote the closure and convex closed hull of the set X, respectively. Further, denote by $\mathfrak{M}_E$ the family of all nonempty and bounded subsets of E. The symbol $\mathfrak{N}_E$ stands for the subfamily of $\mathfrak{M}_E$ consisting of all relatively compact sets. We will accept the following definition of a measure of noncompactness.

Definition 2. A mapping $\mu : \mathfrak{M}_E \to \mathbb{R}_+ = [0, +\infty)$ will be called a measure of noncompactness in the space E if it satisfies the following conditions:

(1) the family $\ker \mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathfrak{N}_E$;

(2) $X \subset Y \Rightarrow \mu(X) \le \mu(Y)$;

(3) $\mu(X) = \mu(\overline{X}) = \mu(\text{Conv}\, X)$;

(4) $\mu(\lambda X + (1-\lambda)Y) \le \lambda\mu(X) + (1-\lambda)\mu(Y)$, for $\lambda \in [0, 1]$;

(5) if (X_n) is a sequence of closed sets belonging to $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$, for $n = 1, 2, \ldots$, and if $\lim_{n\to\infty} \mu(X_n) = 0$, then the intersection $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

An important example of a measure of noncompactness is the Hausdorff measure of noncompactness defined by the formula

$$\chi(X) = \inf\{\varepsilon > 0 : X \text{ has a finite } \varepsilon - \text{net in } E\}, \quad X \in \mathfrak{M}_E. \tag{5}$$

The key role in our studies will be played by Darbo's fixed point theorem.

Theorem 3. *Let Ω be a nonempty, bounded, closed, and convex subset of the space E and let $Q : \Omega \to \Omega$ be a continuous transformation. Assume that there exists a constant $k \in [0, 1)$ such that $\mu(QX) \le k\mu(X)$ for any nonempty subset X of Ω. Then, Q has at least one fixed point in the set Ω. Moreover, the set* Fix Q *of all fixed points of Q belonging to Ω is a member of the family* $\ker \mu$.

The considerations in this paper will be placed in the Banach space $C[a, b]$ consisting of all real functions defined and continuous on the bounded interval $[a, b]$ with the standard maximum norm.

Finally, we turn our attention to the *superposition* (or *Nemytskii*) operator which appears very frequently in nonlinear analysis. We refer to monographs [6, 10] for detailed information covering the properties of this operator. To define the operator in question, suppose that $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is a given function. For any function $x(t) = x : [a, b] \to \mathbb{R}$, we can define the function Fx by putting

$$(Fx)(t) = f(t, x(t)), \quad t \in [a, b]. \tag{6}$$

The operator F defined in such a way is called the superposition operator generated by the function f.

3. Main Result

In this section, we will investigate the nonlinear quadratic Volterra-Stieltjes integral equation which has the form

$$x(t) = f_1(t, x(t)) + f_2(t, x(t)) \times \int_0^t u(t, \tau, (Tx)(\tau))\, d_\tau g(t, \tau), \quad t \in I = [0, M], \tag{7}$$

where $M > 0$ is fixed number. Obviously, in our further considerations the interval $I = [0, M]$ can be replaced by any interval $[a, b]$. We look for monotonic and nonnegative solutions of this equation in the space $C[0, M]$. In our study, we will need some results obtained in [1, 2].

At the beginning, let us consider the following conditions.

(i) The functions $f_i : I \times \mathbb{R} \to \mathbb{R}$ $(i = 1, 2)$ are continuous and there exist nondecreasing functions $k_i : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|f_i(t, x) - f_i(t, y)| \le k_i(r)|x - y| \quad (i = 1, 2), \tag{8}$$

for any $t \in I$ and for all $x, y \in [-r, r]$, where $r \ge 0$ is an arbitrary fixed number.

Observe that, on the basis of the above condition, we may define the finite constants $\overline{F_1}, \overline{F_2}$ by putting

$$\overline{F_i} = \max\{|f_i(t, 0)| : t \in I\} \quad (i = 1, 2). \tag{9}$$

Let Δ_M denote the following triangle:

$$\Delta_M = \{(t, \tau) \in \mathbb{R}^2 : 0 \le \tau \le t \le M\}. \tag{10}$$

(ii) The function $u : \Delta_M \times \mathbb{R} \to \mathbb{R}$ is continuous. Moreover, there exists a continuous function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|u(t,\tau,x)| \le \Phi(|x|), \tag{11}$$

for all $(t,\tau) \in \Delta_M$ and $x \in \mathbb{R}$.

(iii) The function $g : \Delta_M \to \mathbb{R}$ is continuous with respect to the variable τ on the interval $[0,t]$, where $t \in I$ is fixed.

(iv) For any $t \in I$, the function $\tau \to g(t,\tau)$ is of bounded variation on the interval $[0,t]$.

(v) For each $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $t, s \in I$ and $|s-t| \le \delta$, the following inequality holds

$$\bigvee_{\tau=0}^{\min\{t,s\}} [g(s,\tau) - g(t,\tau)] \le \varepsilon. \tag{12}$$

Remark 4. It can be shown (see [1, 2]) that the constant

$$K = \max\left\{\bigvee_{\tau=0}^{t} g(t,\tau) : t \in I\right\} \tag{13}$$

is well defined and finite.

(vi) The operator $T : C(I) \to C(I)$ is continuous and there exists a nondecreasing function $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\|Tx\| \le \Psi(\|x\|)$, for any $x \in C(I)$.

(vii) There exists a positive real number r_0 which satisfies the inequalities

$$\begin{gathered} rk_1(r) + \overline{F_1} + K\left(rk_2(r) + \overline{F_2}\right)\Phi(\Psi(r)) \le r, \\ k_1(r) + Kk_2(r)\Phi(\Psi(r)) < 1. \end{gathered} \tag{14}$$

Remark 5. Observe that if r_0 is a positive solution of the first inequality from condition (vii) and if one of the terms $\overline{F_1}$ and $K\overline{F_2}\Phi(\Psi(r_0))$ does not vanish, then the second inequality from (vii) is automatically satisfied.

Now, let us consider the operators F_i $(i = 1, 2)$, U, and V defined on the space $C(I)$ by the following formulas:

$$\begin{aligned} (F_i x)(t) &= f_i(t, x(t)) \quad (i = 1,2), \\ (Ux)(t) &= \int_0^t u(t,\tau,(Tx)(\tau))\, d_\tau g(t,\tau), \\ (Vx)(t) &= (F_1 x)(t) + (F_2 x)(t)(Ux)(t). \end{aligned} \tag{15}$$

Theorem 6. *Let conditions (i)–(vii) hold. Then, the operator $V_{|B_{r_0}} : B_{r_0} \to B_{r_0}$ is well defined and continuous and has at least one fixed point, which gives that* (7) *has at least one solution in the ball B_{r_0}, where r_0 is a number appearing in condition (vii).*

The basic idea of the proof of Theorem 6 is to study behaviour of the operator V with respect to the Hausdorff measure of noncompactness in connection with Theorem 3.

Remark 7. Additionally, all solutions of (7) from the ball B_{r_0} are equicontinuous. This observation results directly from the Arzela-Ascoli theorem and Theorem 3.

We can now formulate our main result about monotonicity and nonnegativity of the solutions of (7). In our study, we will consider the following conditions.

(i$'$) The functions $f_i : I \times \mathbb{R} \to \mathbb{R}$ $(i = 1, 2)$ are such that

(1) $f_i(I \times \mathbb{R}_+) \subset \mathbb{R}_+$;
(2) the function $t \to f_i(t,x)$ is nondecreasing on I, for any fixed $x \in \mathbb{R}_+$;
(3) the function $x \to f_i(t,x)$ is nondecreasing on $\mathbb{R}_+$, for any fixed $t \in I$.

(ii$'$) (a) The function $u : \Delta_M \times \mathbb{R} \to \mathbb{R}$ is such that

(1) $u(\Delta_M \times \mathbb{R}_+) \subset \mathbb{R}_+$;
(2) the function $t \to u(t,\tau,x)$ is nondecreasing on $[\tau, M]$, for any fixed $\tau \in I$ and $x \in \mathbb{R}_+$;
(3) for each $t, s \in I$ such that $t < s$, the function $\tau \to g(s,\tau) - g(t,\tau)$ is nondecreasing on $[0,t]$;
(4) for any function $x \in B_{r_0}$ which is nonnegative and nondecreasing on I, the function Tx is nonnegative on I, where r_0 is a number appearing in condition (vii).

Or

(b) The function $u : \Delta_M \times \mathbb{R} \to \mathbb{R}$ is such that

(1) $u(\Delta_M \times \mathbb{R}_+) \subset \mathbb{R}_+$;
(2) the function $t \to u(t,\tau,x)$ is nondecreasing on $[\tau, M]$, for any fixed $\tau \in I$ and $x \in \mathbb{R}_+$;
(3) the function $\tau \to u(t,\tau,x)$ is nondecreasing on $[0,t]$, for any fixed $t \in I$ and $x \in \mathbb{R}_+$;
(4) the function $x \to u(t,\tau,x)$ is nondecreasing on $\mathbb{R}_+$ for any fixed $(t,\tau) \in \Delta_M$;
(5) for each $t, s \in I$ such that $t < s$, the function $\tau \to g(s,\tau) - g(t,\tau)$ is nondecreasing on $[0,t]$;
(6) $g(s,s) - g(t,t) + g(t,0) - g(s,0) \ge 0$;
(7) for any function $x \in B_{r_0}$ which is nonnegative and nondecreasing on I, the function Tx is nonnegative and nondecreasing on I, where r_0 is a number appearing in condition (vii).

(iii$'$) For each $t \in I$ the function $\tau \to g(t,\tau)$ is nondecreasing on $[0,t]$.

The following theorem is a completion of Theorem 6.

Theorem 8. *Suppose that conditions (i)–(vii) and (i$'$)–(iii$'$) are fulfilled. Then,* (7) *has at least one solution in B_{r_0} which is nonnegative and nondecreasing, where r_0 is a number appearing in condition (vii).*

Proof. Let $B^+_{r_0}$ denote set of all nonnegative and nondecreasing functions from the ball B_{r_0}. It is clear that $B^+_{r_0}$ is nonempty, bounded, closed, and convex. From Theorem 6, we conclude that the operator $V_{|B^+_{r_0}}$ is continuous. We show that $V(B^+_{r_0}) \subset B^+_{r_0}$. To this end, fix $x \in B^+_{r_0}$ and take $t, s \in I$ such that $s > t$. Since $(F_i x)(t) = f_i(t, x(t)) \geq 0$ and

$$\begin{aligned}(F_i x)(s) - (F_i x)(t) &= f_i(s, x(s)) - f_i(t, x(t)) \\ &\geq f_i(t, x(s)) - f_i(t, x(t)) \geq 0,\end{aligned} \tag{16}$$

we obtain $F_i(B^+_{r_0}) \subset B^+_{r_0}$, for $i = 1, 2$.

It is easily seen that $(Ux)(t) \geq 0$ so it suffices to check monotonicity of the operator U. We get

$$\begin{aligned}
&(Ux)(s) - (Ux)(t) \\
&\quad = \int_0^s u(s,\tau,(Tx)(\tau))\, d_\tau g(s,\tau) \\
&\qquad - \int_0^t u(t,\tau,(Tx)(\tau))\, d_\tau g(t,\tau) \\
&\quad = \int_0^t u(s,\tau,(Tx)(\tau))\, d_\tau g(s,\tau) \\
&\qquad + \int_t^s u(s,\tau,(Tx)(\tau))\, d_\tau g(s,\tau) \\
&\qquad - \int_0^t u(t,\tau,(Tx)(\tau))\, d_\tau g(t,\tau) \\
&\quad \geq \int_0^t u(s,\tau,(Tx)(\tau))\, d_\tau g(s,\tau) \\
&\qquad - \int_0^t u(s,\tau,(Tx)(\tau))\, d_\tau g(t,\tau) \\
&\qquad + \int_t^s u(s,\tau,(Tx)(\tau))\, d_\tau g(s,\tau) \\
&\quad = \int_0^t u(s,\tau,(Tx)(\tau))\, d_\tau \left[g(s,\tau) - g(t,\tau)\right] \\
&\qquad + \int_t^s u(s,\tau,(Tx)(\tau))\, d_\tau g(s,\tau).
\end{aligned} \tag{17}$$

Further proving process depends on which of conditions (ii$'$(a)) or (ii$'$(b)) is satisfied.

Assume that condition (ii$'$(a)) holds. Then based on Theorem 1, the two last integrals in estimation (17) are nonnegative and indeed $(Ux)(s) - (Ux)(t) \geq 0$.

Now, assume that condition (ii$'$(b)) is satisfied. Coming back to estimation (17), we obtain

$$\begin{aligned}
&(Ux)(s) - (Ux)(t) \\
&\quad \geq \int_0^t u(s,t,(Tx)(t))\, d_\tau \left[g(s,\tau) - g(t,\tau)\right] \\
&\qquad + \int_t^s u(s,t,(Tx)(t))\, d_\tau g(s,\tau) \\
&\quad = u(s,t,(Tx)(t)) \left[g(s,t) - g(t,t) - \left(g(s,0) - g(t,0)\right)\right. \\
&\qquad \left. + g(s,s) - g(s,t)\right] \\
&\quad = u(s,t,(Tx)(t)) \left[g(s,s) - g(t,t) + g(t,0)\right. \\
&\qquad \left. - g(s,0)\right] \geq 0
\end{aligned} \tag{18}$$

and, consequently, $U(B^+_{r_0}) \subset B^+_{r_0}$. Finally, we have $V(B^+_{r_0}) \subset B^+_{r_0}$. Using Theorems 3 and 6, we obtain the existence of a fixed point of the operator V in $B^+_{r_0}$. This means that (7) has at least one nonnegative and nondecreasing solution in B_{r_0}, and the proof is complete. □

Remark 9. It can be shown (see for instance [1]) that if the function $g : \Delta_M \to \mathbb{R}$ is continuous on the triangle Δ_M and for arbitrarily fixed $t, s \in I$ such that $t < s$, the function $\tau \to g(s,\tau) - g(t,\tau)$ is monotonic (nondecreasing or nonincreasing) on the interval $[0, t]$; then g satisfies condition (v).

4. Applications and an Example

The topic of this section is to present some applications of Theorem 8 in the situation of the classical integral equations.

Let us consider the equation

$$\begin{aligned}x(t) = f_1(t, x(t)) &+ \frac{\widetilde{f}_2(t, x(t))}{\Gamma(\alpha)} \\ &\times \int_0^t \frac{u(t,\tau,(Tx)(\tau))}{(t-\tau)^{1-\alpha}} d\tau, \quad t \in I,\end{aligned} \tag{19}$$

where Γ denotes the Euler gamma function and $\alpha > 0$. It is the well-known *integral equation of fractional order*. If we take on the set Δ_M the function g defined by

$$g(t,\tau) = \frac{1}{\alpha}\left[t^\alpha - (t-\tau)^\alpha\right], \tag{20}$$

then it is easy to check that (19) is a special case of (7). Using Remark 9 and the standard methods of differential calculus, we can show that the function g satisfies conditions (iii)–(v), (ii$'$), and (iii$'$). Additionally, we have $K = (1/\alpha)M^\alpha$, where K is the constant appearing in Remark 4. Making use of the fact that $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$ for $\alpha > 0$, condition (vii) in this situation takes the following form:

(vii*) there exists a positive real number r_0 which satisfies the inequalities

$$\begin{aligned}rk_1(r) + \overline{F_1} + \frac{M^\alpha}{\Gamma(\alpha+1)}\left(r\widetilde{k}_2(r) + \widetilde{F}_2\right)\Phi(\Psi(r)) \leq r, \\ k_1(r) + \frac{M^\alpha}{\Gamma(\alpha+1)}\widetilde{k}_2(r)\Phi(\Psi(r)) < 1,\end{aligned} \tag{21}$$

where $\widetilde{F}_2 = \max\{|\widetilde{f}_2(t,0)| : t \in I\}$ and $\widetilde{k}_2$ is a function chosen for $\widetilde{f}_2$ based on condition (i).

Obviously, when $\alpha = 1$, (19) reduces to the classical *nonlinear quadratic Volterra integral equation*.

Now, let us consider the equation

$$x(t) = f_1(t, x(t)) + f_2(t, x(t)) \times \int_0^t \frac{t}{t+\tau} u(t, \tau, (Tx)(\tau))\, d\tau, \quad t \in I. \tag{22}$$

It is *the Volterra counterpart of the quadratic integral equation of Chandrasekhar type*. This equation is also a special case of (7), in which

$$g(t, \tau) = \begin{cases} t \ln\left(1 + \dfrac{\tau}{t}\right), & (t, \tau) \in \Delta_M \setminus \{(0,0)\} \\ 0, & t = \tau = 0. \end{cases} \tag{23}$$

Using, as before, Remark 9 and the standard methods of differential calculus, we can show that this function satisfies conditions (iii)–(v), (ii′(a)), and (iii′). Additionally, we have $K = M \ln 2$, where K is the constant appearing in Remark 4.

Let us observe that if we put $f_2(t, x) \equiv 0$ in (7), we obtain the classical functional equation of the first order on the interval I.

We finish by providing an example illustrating Theorem 8.

Example 1. Let us consider the following integral equation:

$$x(t) = te^{-t} + \frac{t^2 + x(t)}{\Gamma(2/3)} \times \int_0^t \frac{1}{2\pi} \frac{\sqrt{|x(\tau)|} \operatorname{arctg}\left(4 + t^2 + \tau^2\right)}{\sqrt[3]{t - \tau}}\, d\tau, \quad t \in [0, 1]. \tag{24}$$

Obviously, this equation is a special case of (19) if we put $\alpha = 2/3$ and

$$\begin{gathered} f_1(t, x) = te^{-t}, \\ \widetilde{f}_2(f, x) = t^2 + x, \\ u(t, \tau, x) = \frac{1}{2\pi} \sqrt{|x|} \operatorname{arctg}\left(4 + t^2 + \tau^2\right), \\ Tx = x. \end{gathered} \tag{25}$$

In is easy to check that conditions (i)–(vi), (i′), (ii′(b)), and (iii′) of Theorem 8 are satisfied and $k_1(r) = 0$, $\overline{F_1} = 1/e$, $\widetilde{k}_2(r) = 1$, $\widetilde{F}_2 = 1$, $\Phi(r) = (1/4)\sqrt{r}$, and $\Psi(r) = r$. Using standard estimation $\Gamma(\alpha) > 0.8856$ for $\alpha > 0$ and taking $r_0 = 1$, we verify that condition (vii*) is also satisfied. Therefore, in case of (24), we can apply Theorem 8. This means that (24) has at least one nonnegative and nondecreasing solution belonging to the ball B_1 of the space $C[0, 1]$.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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Research Article

Asymptotically Stable Solutions of a Generalized Fractional Quadratic Functional-Integral Equation of Erdélyi-Kober Type

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We study a generalized fractional quadratic functional-integral equation of Erdélyi-Kober type in the Banach space $BC(\mathbb{R}_+)$. We show that this equation has at least one asymptotically stable solution.

1. Introduction

Quadratic integral equations with nonsingular kernels have received a lot of attention because of their useful applications in describing numerous events and problems of the real world. For example, quadratic integral equations are often applicable in kinetic theory of gases, in the theory of neutron transport, and in the traffic theory; see [1–8]. The existence of solutions for several classes of nonlinear quadratic integral equations with nonsingular kernels has been studied by several authors, for example, Argyros [9], Banaś et al. [10–12], Benchohra and Darwish [13, 14], Caballero et al. [15–17], Darwish et al. [18, 19], Leggett [20], and Stuart [21]. There is a great interest in studying singular quadratic integral equations by many authors, after the appearance of Darwish's paper [22], for example, Banaś and O'Regan [23], Banaś and Rzepka [24, 25], Darwish [26, 27], Darwish and Sadarangani [28], Darwish and Ntouyas [29], Darwish et al. [30], and Wang et al. [31, 32].

In this paper, we will study the quadratic functional-integral equation of fractional order

$$x(t) = a(t) + f\left(t, \frac{\beta g(t, x(t))}{\Gamma(\alpha)} \int_0^t \frac{s^{\beta-1}}{\left(t^\beta - s^\beta\right)^{1-\alpha}} u(t, s, x(s))\, ds\right), \quad t \in \mathbb{R}_+, \tag{1}$$

where $\alpha \in (0,1)$ and $\beta > 0$.

If $\beta = 1$ and $f(t,u) = u$, we obtain a quadratic Urysohn-Volterra integral equation of fractional order studied by Banas' and O'Regan in [23] while in the case where $\beta = 1$, $f(t,u) = u$, and $u(t,s,x) = v(t,x)$, we get a fractional quadratic integral equation of Hammerstein-Volterra type studied by Darwish in [22]. Moreover, in the case where $\beta = 1$, we obtain the quadratic functional-integral equation of fractional order studied by Darwish and Sadarangani in [28].

The aim of this paper is to prove the existence of solutions of (1) in the space of real functions, defined, continuous, and bounded on an unbounded interval. Moreover, we will obtain some asymptotic characterization of solutions of (1). Our proof depends on suitable combination of the technique of measures of noncompactness and the Schauder fixed point principle.

2. Notation and Auxiliary Facts

This section is devoted to collecting some definitions and results which will be needed further on. First, we recall from [33–35] that the Erdélyi-Kober fractional integral of a continuous function f is defined as

$$I_\beta^\gamma f(t) = \frac{\beta}{\Gamma(\gamma)} \int_0^t \frac{s^{\beta-1} f(s)}{\left(t^\beta - s^\beta\right)^{1-\gamma}} ds, \quad \beta > 0,\ 0 < \gamma < 1. \tag{2}$$

When $\beta = 1$, we obtain Riemann-Liouville fractional integral; that is,

$$I^{\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{f(s)}{(t-s)^{1-\gamma}} ds, \quad 0 < \gamma < 1. \tag{3}$$

Now, let $(E, \|\cdot\|)$ be an infinite dimensional Banach space with zero element θ. Let $B(x, r)$ denote the closed ball centered at x with radius r. The symbol B_r stands for the ball $B(\theta, r)$.

If X is a subset of E, then $\overline{X}$ and $\mathrm{Conv}X$ denote the closure and convex closure of X, respectively. Moreover, we denote by $\mathscr{M}_E$ the family of all nonempty and bounded subsets of E and by $\mathscr{N}_E$ its subfamily consisting of all relatively compact subsets.

Next we give the definition of the concept of a measure of noncompactness [36].

Definition 1. A mapping $\mu : \mathscr{M}_E \to \mathbb{R}_+ = [0, \infty)$ is said to be a measure of noncompactness in E if it satisfies the following conditions.

(1) The family $\ker \mu = \{X \in \mathscr{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathscr{N}_E$.

(2) $X \subset Y \Rightarrow \mu(X) \le \mu(Y)$.

(3) $\mu(\overline{X}) = \mu(\mathrm{Conv}X) = \mu(X)$.

(4) $\mu(\lambda X + (1-\lambda)Y) \le \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $0 \le \lambda \le 1$.

(5) If $X_n \in \mathscr{M}_E$, $X_n = \overline{X}_n$, $X_{n+1} \subset X_n$ for $n = 1, 2, 3, \ldots$ and $\lim_{n\to\infty} \mu(X_n) = 0$, then $X_\infty = \bigcap_{n=1}^{\infty} X_n \ne \emptyset$.

The family $\ker \mu$ described above is called the kernel of the measure of noncompactness μ. Let us observe that the intersection set X_∞ from (5) belongs to $\ker \mu$. In fact, since $\mu(X_\infty) \le \mu(X_n)$ for every, then we have that $\mu(X_\infty) = 0$.

In what follows we will work in the Banach space $BC(\mathbb{R}_+)$ consisting of all real functions defined, bounded, and continuous on $\mathbb{R}_+$. This space is equipped with the standard norm

$$\|x\| = \sup \{|x(t)| : t \ge 0\}. \tag{4}$$

Next, we give the construction of the measure of noncompactness in $BC(\mathbb{R}_+)$ which will be used as main tool of the proof of our main result; see [37, 38] and references therein.

Let us fix a nonempty and bounded subset X of $BC(\mathbb{R}_+)$ and numbers $\varepsilon > 0$ and $T > 0$. For arbitrary function $x \in X$ let us denote by $\omega^T(x, \varepsilon)$ the modulus of continuity of the function x on the interval $[0, T]$; that is,

$$\omega^T(x, \varepsilon) = \sup \{|x(t) - x(s)| : t, s \in [0, T], \ |t - s| \le \varepsilon\}. \tag{5}$$

Further, let us put

$$\begin{aligned} \omega^T(X, \varepsilon) &= \sup \left\{\omega^T(x, \varepsilon) : x \in X\right\}, \\ \omega_0^T(X) &= \lim_{\varepsilon\to 0} \omega^T(X, \varepsilon), \\ \omega_0^{\infty}(X) &= \lim_{T\to\infty} \omega_0^T(X). \end{aligned} \tag{6}$$

Moreover, for a fixed number $t \in \mathbb{R}_+$ let us define

$$\begin{aligned} X(t) &= \{x(t) : x \in X\}, \\ \operatorname{diam} X(t) &= \sup \{|x(t) - y(t)| : x, y \in X\}, \\ c(X) &= \limsup_{t\to\infty} \operatorname{diam} X(t). \end{aligned} \tag{7}$$

Let us mention that the kernel $\ker \omega_0^{\infty}$ consists of all nonempty and bounded sets X such that functions belonging to X are locally equicontinuous on $\mathbb{R}_+$. On the other hand, the kernel $\ker c$ is the family containing all nonempty and bounded sets X in the space $BC(\mathbb{R}_+)$ such that the thickness of the bundle formed by the graphs of functions belonging to X tends to zero at infinity.

Finally, with the help of the above quantities we can define a measure of noncompactness as

$$\mu(X) = \omega_0^{\infty}(X) + c(X). \tag{8}$$

The function μ is a measure of noncompactness in the space $BC(\mathbb{R}_+)$ [36, 37].

In the end of this section, we recall the definition of the asymptotic stability solutions which will be used in the proof of our main result. To this end we assume that Ω is a nonempty subset of the space $BC(\mathbb{R}_+)$. Let $Q : \Omega \to BC(\mathbb{R}_+)$ be a given operator. We consider the following operator equation:

$$x(t) = (Qx)(t), \quad t \in \mathbb{R}_+. \tag{9}$$

Definition 2. One says that solutions of (9) are asymptotically stable if there exists a ball $B(x_0, r)$ such that $\Omega \cap B(x_0, r) \ne \emptyset$ and such that for each $\varepsilon > 0$ there exists $T > 0$ such that for arbitrary solutions $x = x(t)$, $y = y(t)$ of this equation belonging to $\Omega \cap B(x_0, r)$ the inequality $|x(t) - y(t)| \le \varepsilon$ is satisfied for any $t \ge T$.

3. The Existence and Asymptotic Stability of Solutions

In this section we will study (1) assuming that the following hypotheses are satisfied.

(h_1) $a : \mathbb{R}_+ \to \mathbb{R}$ is a continuous and bounded function on $\mathbb{R}_+$.

(h_2) $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and the function $t \to f(t, 0)$ is bounded on $\mathbb{R}_+$ with $f^* = \sup\{|f(t, 0)| : t \in \mathbb{R}_+\}$. Moreover, there exists a continuous function $m(t) = m : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|f(t, x) - f(t, y)| \le m(t)|x - y| \tag{10}$$

for all $x, y \in \mathbb{R}$ and for any $t \in \mathbb{R}_+$.

(h_3) $g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a continuous function $n(t) = n : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|g(t, x) - g(t, y)| \le n(t)|x - y| \tag{11}$$

for all $x, y \in \mathbb{R}$ and for any $t \in \mathbb{R}_+$.

(h_4) $u : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Moreover, there exist a function $l(t) = l : \mathbb{R}_+ \to \mathbb{R}_+$ being continuous on $\mathbb{R}_+$ and a function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ being continuous and nondecreasing on $\mathbb{R}_+$ with $\Phi(0) = 0$ such that

$$\left|u(t,s,x) - u(t,s,y)\right| \le l(t)\,\Phi\left(\left|x-y\right|\right) \tag{12}$$

for all $t, s \in \mathbb{R}_+$ such that $t \ge s$ and for all $x \in \mathbb{R}$.

For further purpose let us define the function $u^* : \mathbb{R}_+ \to \mathbb{R}_+$ by $u^*(t) = \max\{|u(t,s,0)| : 0 \le s \le t\}$.

(h_5) The functions $\phi, \psi, \xi, \eta : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $\phi(t) = m(t)n(t)l(t)t^{\alpha\beta}$, $\psi(t) = m(t)n(t)u^*(t)t^{\alpha\beta}$, $\xi(t) = m(t)l(t)|g(t,0)|t^{\alpha\beta}$, and $\eta(t) = m(t)u^*(t)|g(t,0)|t^{\alpha\beta}$ are bounded on $\mathbb{R}_+$ and the functions ϕ and ξ vanish at infinity; that is, $\lim_{t\to\infty}\phi(t) = \lim_{t\to\infty}\xi(t) = 0$.

(h_6) There exists a positive solution r_0 of the inequality

$$\begin{aligned}&\left(\|a\| + f^*\right)\Gamma(\alpha+1)\\ &\quad + \left[\phi^* r\Phi(r) + \psi^* r + \xi^*\Phi(r) + \eta^*\right]\\ &\le r\Gamma(\alpha+1)\end{aligned} \tag{13}$$

and $\phi^*\Phi(r_0) + \psi^* < \Gamma(\alpha+1)$, where $\phi^* = \sup\{\phi(t) : t \in \mathbb{R}_+\}$, $\psi^* = \sup\{\psi(t) : t \in \mathbb{R}_+\}$, $\xi^* = \sup\{\xi(t) : t \in \mathbb{R}_+\}$, and $\eta^* = \sup\{\eta(t) : t \in \mathbb{R}_+\}$.

Now, we are in a position to state and prove our main result.

Theorem 3. *Let the hypotheses (h_1)–(h_6) be satisfied. Then* (1) *has at least one solution $x \in BC(\mathbb{R}_+)$ and all solutions of this equation belonging to the ball B_{r_0} are asymptotically stable.*

Proof. Denote by $\mathcal{F}$ the operator associated with the right-hand side of (1). Then, (1) takes the form

$$x = \mathcal{F}x, \tag{14}$$

where

$$\begin{aligned}\mathcal{F}x &= a + F\mathcal{H}x,\\ (\mathcal{H}x)(t) &= (Gx)(t)\cdot(\mathcal{U}x)(t),\\ (\mathcal{U}x)(t) &= \frac{\beta}{\Gamma(\alpha)}\int_0^t \frac{s^{\beta-1}u(t,s,x(s))}{\left(t^\beta - s^\beta\right)^{1-\alpha}}ds, \quad t \in \mathbb{R}_+.\end{aligned} \tag{15}$$

Here, F and G are the superposition operators, generated by the functions $f = f(t,x)$ and $g = g(t,x)$ involved in (1), defined by

$$(Fx)(t) = f(t,x(t)), \tag{16}$$

$$(Gx)(t) = g(t,x(t)), \tag{17}$$

respectively, where $x = x(t)$ is an arbitrary function defined on $\mathbb{R}_+$ (see [39]).

Solving (1) is equivalent to finding a fixed point of the operator $\mathcal{F}$ defined on the space $BC(\mathbb{R}_+)$.

For convenience, we divide the proof into several steps.

Step 1 ($\mathcal{F}x$ is continuous on $\mathbb{R}_+$). To prove the continuity of the function $\mathcal{F}x$ on $\mathbb{R}_+$ it suffices to show that if $x \in BC(\mathbb{R}_+)$, then $\mathcal{U}x$ is continuous function on $\mathbb{R}_+$, thanks to (h_1), (h_2), and (h_3). For this purpose, take an arbitrary $x \in BC(\mathbb{R}_+)$ and fix $\varepsilon > 0$ and $T > 0$. Assume that $t_1, t_2 \in [0,T]$ are such that $|t_2 - t_1| \le \varepsilon$. Without loss of generality we can assume that $t_2 > t_1$. Then we get

$$\begin{aligned}&\left|(\mathcal{U}x)(t_2) - (\mathcal{U}x)(t_1)\right|\\
&= \left|\frac{\beta}{\Gamma(\alpha)}\int_0^{t_2}\frac{s^{\beta-1}u(t_2,s,x(s))}{\left(t_2^\beta - s^\beta\right)^{1-\alpha}}ds - \frac{\beta}{\Gamma(\alpha)}\int_0^{t_1}\frac{s^{\beta-1}u(t_1,s,x(s))}{\left(t_1^\beta - s^\beta\right)^{1-\alpha}}ds\right|\\
&\le \left|\frac{\beta}{\Gamma(\alpha)}\int_0^{t_2}\frac{s^{\beta-1}u(t_2,s,x(s))}{\left(t_2^\beta - s^\beta\right)^{1-\alpha}}ds - \frac{\beta}{\Gamma(\alpha)}\int_0^{t_1}\frac{s^{\beta-1}u(t_2,s,x(s))}{\left(t_2^\beta - s^\beta\right)^{1-\alpha}}ds\right|\\
&\quad + \left|\frac{\beta}{\Gamma(\alpha)}\int_0^{t_1}\frac{s^{\beta-1}u(t_2,s,x(s))}{\left(t_2^\beta - s^\beta\right)^{1-\alpha}}ds - \frac{\beta}{\Gamma(\alpha)}\int_0^{t_1}\frac{s^{\beta-1}u(t_1,s,x(s))}{\left(t_2^\beta - s^\beta\right)^{1-\alpha}}ds\right|\\
&\quad + \left|\frac{\beta}{\Gamma(\alpha)}\int_0^{t_1}\frac{s^{\beta-1}u(t_1,s,x(s))}{\left(t_2^\beta - s^\beta\right)^{1-\alpha}}ds - \frac{\beta}{\Gamma(\alpha)}\int_0^{t_1}\frac{s^{\beta-1}u(t_1,s,x(s))}{\left(t_1^\beta - s^\beta\right)^{1-\alpha}}ds\right|\\
&\le \frac{\beta}{\Gamma(\alpha)}\int_{t_1}^{t_2}\frac{s^{\beta-1}\left|u(t_2,s,x(s))\right|}{\left(t_2^\beta - s^\beta\right)^{1-\alpha}}ds\\
&\quad + \frac{\beta}{\Gamma(\alpha)}\int_0^{t_1}\frac{s^{\beta-1}\left[\left|u(t_2,s,x(s)) - u(t_1,s,x(s))\right|\right]}{\left(t_2^\beta - s^\beta\right)^{1-\alpha}}ds\\
&\quad + \frac{\beta}{\Gamma(\alpha)}\int_0^{t_1}s^{\beta-1}\left|u(t_1,s,x(s))\right|\\
&\quad \times \left[\left(t_1^\beta - s^\beta\right)^{\alpha-1} - \left(t_2^\beta - s^\beta\right)^{\alpha-1}\right]ds.\end{aligned} \tag{18}$$

Let us denote

$$\begin{aligned}
&\omega_d^T(u,\varepsilon)\\
&\quad = \sup\{|u(t_2,s,y)-u(t_1,s,y)| : s,t_1,t_2\in[0,T],\\
&\qquad t_1\ge s,\ t_2\ge s,\ |t_2-t_1|\le\varepsilon,\\
&\qquad y\in[-d,d];\ d\ge 0\};
\end{aligned}\tag{19}$$

then we obtain

$$\begin{aligned}
&|(\mathcal{U}x)(t_2)-(\mathcal{U}x)(t_1)|\\
&\le\frac{\beta}{\Gamma(\alpha)}\\
&\quad\times\int_{t_1}^{t_2}\frac{s^{\beta-1}\left[|u(t_2,s,x(s))-u(t_2,s,0)|+|u(t_2,s,0)|\right]}{\left(t_2^\beta-s^\beta\right)^{1-\alpha}}ds\\
&\quad+\frac{\beta}{\Gamma(\alpha)}\int_0^{t_1}\frac{s^{\beta-1}\omega_{\|x\|}^T(u,\varepsilon)}{\left(t_2^\beta-s^\beta\right)^{1-\alpha}}ds\\
&\quad+\frac{\beta}{\Gamma(\alpha)}\\
&\quad\times\int_0^{t_1}s^{\beta-1}\left[|u(t_1,s,x(s))-u(t_1,s,0)|+|u(t_1,s,0)|\right]\\
&\qquad\times\left[\left(t_1^\beta-s^\beta\right)^{\alpha-1}-\left(t_2^\beta-s^\beta\right)^{\alpha-1}\right]ds\\
&\le\frac{\beta}{\Gamma(\alpha)}\int_{t_1}^{t_2}\frac{s^{\beta-1}\left[l(t_2)\Phi(|x(s)|)+u^*(t_2)\right]}{\left(t_2^\beta-s^\beta\right)^{1-\alpha}}ds\\
&\quad+\frac{\omega_{\|x\|}^T(u,\varepsilon)}{\Gamma(\alpha+1)}\left[t_2^{\alpha\beta}-\left(t_2^\beta-t_1^\beta\right)^\alpha\right]\\
&\quad+\frac{\beta}{\Gamma(\alpha)}\int_0^{t_1}s^{\beta-1}\left[l(t_1)\Phi(|x(s)|)+u^*(t_1)\right]\\
&\qquad\times\left[\left(t_1^\beta-s^\beta\right)^{\alpha-1}-\left(t_2^\beta-s^\beta\right)^{\alpha-1}\right]ds\\
&\le\frac{\left[l(t_2)\Phi(\|x\|)+u^*(t_2)\right]}{\Gamma(\alpha+1)}\left(t_2^\beta-t_1^\beta\right)^\alpha\\
&\quad+\frac{\omega_{\|x\|}^T(u,\varepsilon)}{\Gamma(\alpha+1)}t_2^{\alpha\beta}\\
&\quad+\frac{l(t_1)\Phi(\|x\|)+u^*(t_1)}{\Gamma(\alpha+1)}\left[t_1^{\alpha\beta}-t_2^{\alpha\beta}+\left(t_2^\beta-t_1^\beta\right)^\alpha\right].
\end{aligned}\tag{20}$$

Thus

$$\begin{aligned}
&\omega^T(\mathcal{U}x,\varepsilon)\\
&\le\frac{1}{\Gamma(\alpha+1)}\left\{2\varepsilon^{\alpha\beta}\left[\hat{l}(T)\Phi(\|x\|)+\hat{u}(T)\right]+T^{\alpha\beta}\omega_{\|x\|}^T(u,\varepsilon)\right\},
\end{aligned}\tag{21}$$

where

$$\begin{aligned}
\hat{l}(T)&=\max\{l(t):t\in[0,T]\},\\
\hat{u}(T)&=\max\{u^*(t):t\in[0,T]\}.
\end{aligned}\tag{22}$$

In view of the uniform continuity of the function u on $[0,T]\times[0,T]\times[-\|x\|,\|x\|]$ we have that $\omega_{\|x\|}^T(u,\varepsilon)\to 0$ as $\varepsilon\to 0$. From the above inequality we infer that the function $\mathcal{U}x$ is continuous on the interval $[0,T]$ for any $T>0$. This yields the continuity of $\mathcal{U}x$ on $\mathbb{R}_+$ and, consequently, the function $\mathcal{F}x$ is continuous on $\mathbb{R}_+$.

Step 2 ($\mathcal{F}x$ is bounded on $\mathbb{R}_+$). In view of our hypotheses for arbitrary $x\in BC(\mathbb{R}_+)$ and for a fixed $t\in\mathbb{R}_+$ we have

$$\begin{aligned}
&|(\mathcal{F}x)(t)|\\
&\le\left|a(t)+f\left(t,\frac{\beta g(t,x(t))}{\Gamma(\alpha)}\int_0^t\frac{s^{\beta-1}u(t,s,x(s))}{\left(t^\beta-s^\beta\right)^{1-\alpha}}ds\right)\right|\\
&\le\|a\|+\frac{\beta}{\Gamma(\alpha)}m(t)\left[|g(t,x(t))-g(t,0)|+|g(t,0)|\right]\\
&\quad\times\int_0^t\frac{s^{\beta-1}\left[|u(t,s,x(s))-u(t,s,0)|+|u(t,s,0)|\right]}{\left(t^\beta-s^\beta\right)^{1-\alpha}}ds\\
&\quad+|f(t,0)|\\
&\le\|a\|+f^*+\frac{\beta m(t)\left[n(t)\|x\|+|g(t,0)|\right]}{\Gamma(\alpha)}\\
&\quad\times\int_0^t\frac{s^{\beta-1}\left[l(t)\Phi(|x(s)|)+u^*(t)\right]}{\left(t^\beta-s^\beta\right)^{1-\alpha}}ds\\
&\le\|a\|+f^*+\frac{m(t)\left[n(t)\|x\|+|g(t,0)|\right]}{\Gamma(\alpha+1)}\\
&\quad\times\left[l(t)\Phi(\|x\|)+u^*(t)\right]t^{\alpha\beta}\\
&=\|a\|+f^*\\
&\quad+\frac{1}{\Gamma(\alpha+1)}\left[\phi(t)\|x\|\Phi(\|x\|)\right.\\
&\qquad\left.+\psi(t)\|x\|+\xi(t)\Phi(\|x\|)+\eta(t)\right].
\end{aligned}\tag{23}$$

Hence, $\mathcal{F}x$ is bounded on $\mathbb{R}_+$, thanks to hypothesis (h_5).

Step 3 ($\mathscr{F}$ maps the ball B_{r_0} into itself). Steps 2 and 3 allow us to conclude that the operator $\mathscr{F}$ transforms $BC(\mathbb{R}_+)$ into itself. Moreover, from the last estimate we have

$$\begin{aligned}&\|\mathscr{F}x\|\\&\leq \|a\| + f^*\\&\quad+\frac{1}{\Gamma(\alpha+1)}\left[\phi^*\|x\|\Phi(\|x\|)+\psi^*\|x\|+\xi^*\Phi(\|x\|)+\eta^*\right].\end{aligned}\tag{24}$$

From the last estimate with hypothesis (h_6) we deduce that there exists $r_0 > 0$ such that the operator $\mathscr{F}$ maps B_{r_0} into itself.

Step 4 (an estimate of $\mathscr{F}$ with respect to the quantity c). Let us take a nonempty set $X \subset B_{r_0}$. Then, for arbitrary $x, y \in X$ and for a fixed $t \in \mathbb{R}_+$, we obtain

$$\begin{aligned}
&\left|(\mathscr{F}x)(t)-(\mathscr{F}y)(t)\right|\\
&\leq \frac{\beta m(t)}{\Gamma(\alpha)}\left|g(t,x(t))\int_0^t \frac{s^{\beta-1}u(t,s,x(s))}{(t^\beta-s^\beta)^{1-\alpha}}ds\right.\\
&\qquad\left.-g(t,y(t))\int_0^t \frac{s^{\beta-1}u(t,s,y(s))}{(t^\beta-s^\beta)^{1-\alpha}}ds\right|\\
&\leq \frac{\beta m(t)\left|g(t,x(t))-g(t,y(t))\right|}{\Gamma(\alpha)}\\
&\quad\times\int_0^t \frac{s^{\beta-1}\left|u(t,s,x(s))\right|}{(t^\beta-s^\beta)^{1-\alpha}}ds\\
&\quad+\frac{\beta m(t)\left|g(t,y(t))\right|}{\Gamma(\alpha)}\\
&\quad\times\int_0^t \frac{s^{\beta-1}\left|u(t,s,x(s))-u(t,s,y(s))\right|}{(t^\beta-s^\beta)^{1-\alpha}}ds\\
&\leq \frac{\beta m(t)n(t)\left|x(t)-y(t)\right|}{\Gamma(\alpha)}\\
&\quad\times\int_0^t \frac{s^{\beta-1}\left[\left|u(t,s,x(s))-u(t,s,0)\right|+\left|u(t,s,0)\right|\right]}{(t^\beta-s^\beta)^{1-\alpha}}ds\\
&\quad+\frac{\beta m(t)\left[n(t)\left|y(t)\right|+\left|g(t,0)\right|\right]}{\Gamma(\alpha)}\\
&\quad\times\int_0^t \frac{s^{\beta-1}l(t)\Phi\left(\left|x(s)-y(s)\right|\right)}{(t^\beta-s^\beta)^{1-\alpha}}ds\\
&\leq \frac{\beta m(t)n(t)\left|x(t)-y(t)\right|}{\Gamma(\alpha)}\\
&\quad\times\int_0^t \frac{s^{\beta-1}\left[l(t)\Phi(\left|x(s)\right|)+u^*(t)\right]}{(t^\beta-s^\beta)^{1-\alpha}}ds\\
&\quad+\frac{\beta m(t)\left[n(t)\left|y(t)\right|+\left|g(t,0)\right|\right]}{\Gamma(\alpha)}\\
&\quad\times\int_0^t \frac{s^{\beta-1}l(t)\Phi\left(\left|x(s)\right|+\left|y(s)\right|\right)}{(t^\beta-s^\beta)^{1-\alpha}}ds\\
&\leq \frac{\beta m(t)n(t)l(t)\left(\left|x(t)\right|+\left|y(t)\right|\right)}{\Gamma(\alpha)}\\
&\quad\times\int_0^t \frac{s^{\beta-1}\Phi\left(\left|x(s)\right|\right)}{(t^\beta-s^\beta)^{1-\alpha}}ds\\
&\quad+\frac{\beta m(t)n(t)u^*(t)\left|x(t)-y(t)\right|}{\Gamma(\alpha)}\\
&\quad\times\int_0^t \frac{s^{\beta-1}}{(t^\beta-s^\beta)^{1-\alpha}}ds\\
&\quad+\frac{\beta m(t)n(t)l(t)\left|y(t)\right|}{\Gamma(\alpha)}\\
&\quad\times\int_0^t \frac{s^{\beta-1}\Phi\left(\left|x(s)\right|+\left|y(s)\right|\right)}{(t^\beta-s^\beta)^{1-\alpha}}ds\\
&\quad+\frac{\beta m(t)l(t)\left|g(t,0)\right|}{\Gamma(\alpha)}\\
&\quad\times\int_0^t \frac{s^{\beta-1}\Phi\left(\left|x(s)\right|+\left|y(s)\right|\right)}{(t^\beta-s^\beta)^{1-\alpha}}ds\\
&\leq \frac{2\beta m(t)n(t)l(t)r_0\Phi(r_0)}{\Gamma(\alpha)}\int_0^t \frac{s^{\beta-1}}{(t^\beta-s^\beta)^{1-\alpha}}ds\\
&\quad+\frac{\beta m(t)n(t)u^*(t)\operatorname{diam}X(t)}{\Gamma(\alpha)}\int_0^t \frac{s^{\beta-1}}{(t^\beta-s^\beta)^{1-\alpha}}ds\\
&\quad+\frac{\beta m(t)n(t)l(t)r_0\Phi(2r_0)}{\Gamma(\alpha)}\int_0^t \frac{s^{\beta-1}}{(t^\beta-s^\beta)^{1-\alpha}}ds\\
&\quad+\frac{\beta m(t)l(t)\left|g(t,0)\right|\Phi(2r_0)}{\Gamma(\alpha)}\int_0^t \frac{s^{\beta-1}}{(t^\beta-s^\beta)^{1-\alpha}}ds\\
&\leq \frac{2\phi(t)r_0\Phi(r_0)}{\Gamma(\alpha+1)}+\frac{\psi(t)}{\Gamma(\alpha+1)}\operatorname{diam}X(t)\\
&\quad+\frac{\phi(t)r_0\Phi(2r_0)}{\Gamma(\alpha+1)}+\frac{\xi(t)\Phi(2r_0)}{\Gamma(\alpha+1)}.
\end{aligned}\tag{25}$$

Hence, we can easily deduce the following inequality:

$$\begin{aligned}\operatorname{diam}(\mathscr{F}X)(t) &\leq \frac{2\phi(t)r_0\Phi(r_0)}{\Gamma(\alpha+1)}+\frac{\psi(t)}{\Gamma(\alpha+1)}\operatorname{diam}X(t)\\&\quad+\frac{\phi(t)r_0\Phi(2r_0)}{\Gamma(\alpha+1)}+\frac{\xi(t)\Phi(2r_0)}{\Gamma(\alpha+1)}.\end{aligned}\tag{26}$$

Now, taking into account hypothesis (h_5) we obtain

$$c(\mathcal{F}X) \leq qc(X), \tag{27}$$

where $q = (\phi^*\Phi(r_0)+\psi^*)/\Gamma(\alpha+1) \geq \psi^*/\Gamma(\alpha+1)$. Obviously, in view of hypothesis (h_6) we have that $q < 1$.

Step 5 (an estimate of $\mathcal{F}$ with respect to the modulus of continuity ω_0^∞). Take arbitrary numbers $\varepsilon > 0$ and $T > 0$. Choose a function $x \in X$ and take $t_1, t_2 \in [0,T]$ such that $|t_2 - t_1| \leq \varepsilon$. Without loss of generality we can assume that $t_2 > t_1$. Then, taking into account our hypotheses and (21), we have

$$\begin{aligned}
&\left|(\mathcal{F}x)(t_2) - (\mathcal{F}x)(t_1)\right| \\
&\leq \left|a(t_2) - a(t_1)\right| \\
&\quad + m(t_2)\left|(Gx)(t_2)(\mathcal{U}x)(t_2) - (Gx)(t_1)(\mathcal{U}x)(t_2)\right| \\
&\quad + m(t_2)\left|(Gx)(t_1)(\mathcal{U}x)(t_2) - (Gx)(t_1)(\mathcal{U}x)(t_1)\right| \\
&\quad + \left|f(t_2,(Gx)(t_1)(\mathcal{U}x)(t_1))\right. \\
&\qquad \left.- f(t_1,(Gx)(t_1)(\mathcal{U}x)(t_1))\right| \\
&\leq \omega^T(a,\varepsilon) \\
&\quad + \frac{\beta m(t_2)\left|g(t_2,x(t_2)) - g(t_1,x(t_1))\right|}{\Gamma(\alpha)} \\
&\quad \times \int_0^{t_2} \frac{s^{\beta-1}\left[\left|u(t_2,s,x(s)) - u(t_2,s,0)\right| + \left|u(t_2,s,0)\right|\right]}{\left(t_2^\beta - s^\beta\right)^{1-\alpha}} ds \\
&\quad + \frac{m(t_2)\left[\left|g(t_1,x(t_1)) - g(t_1,0)\right| + \left|g(t_1,0)\right|\right]}{\Gamma(\alpha+1)} \\
&\quad \times \left\{2\varepsilon^{\alpha\beta}\left[\hat{l}(T)\Phi(\|x\|) + \hat{u}(T)\right] + T^{\alpha\beta}\omega^T_{\|x\|}(u,\varepsilon)\right\} \\
&\quad + \left|f(t_2,(Gx)(t_1)(\mathcal{U}x)(t_1))\right. \\
&\qquad \left.- f(t_1,(Gx)(t_1)(\mathcal{U}x)(t_1))\right| \\
&\leq \omega^T(a,\varepsilon) + \frac{\beta m(t_2)\left[n(t_2)\left|x(t_2) - x(t_1)\right| + \omega_g^T(\varepsilon)\right]}{\Gamma(\alpha)} \\
&\quad \times \int_0^{t_2} \frac{s^{\beta-1}\left[l(t_2)\Phi(|x(s)|) + u^*(t_2)\right]}{\left(t_2^\beta - s^\beta\right)^{1-\alpha}} ds \\
&\quad + \frac{m(t_2)\left[n(t_1)\left|x(t_1)\right| + \left|g(t_1,0)\right|\right]}{\Gamma(\alpha+1)} \\
&\quad \times \left\{2\varepsilon^{\alpha\beta}\left[\hat{l}(T)\Phi(\|x\|) + \hat{u}(T)\right] + T^{\alpha\beta}\omega^T_{\|x\|}(u,\varepsilon)\right\} \\
&\quad + \left|f(t_2,(Gx)(t_1)(\mathcal{U}x)(t_1))\right. \\
&\qquad \left.- f(t_1,(Gx)(t_1)(\mathcal{U}x)(t_1))\right| \\
&\leq \omega^T(a,\varepsilon) + \frac{t_2^{\alpha\beta}}{\Gamma(\alpha+1)} m(t_2) \\
&\quad \times \left[n(t_2)\omega^T(x,\varepsilon) + \omega_g^T(\varepsilon)\right]\left[l(t_2)\Phi(r_0) + u^*(t_2)\right] \\
&\quad + \frac{\widehat{m}(T)\left[n(t_1)r_0 + \widehat{g}(T)\right]}{\Gamma(\alpha+1)} \\
&\quad \times \left\{2\varepsilon^{\alpha\beta}\left[\hat{l}(T)\Phi(r_0) + \hat{u}(T)\right] + T^{\alpha\beta}\omega^T_{\|x\|}(u,\varepsilon)\right\} \\
&\quad + \left|f(t_2,(Gx)(t_1)(\mathcal{U}x)(t_1))\right. \\
&\qquad \left.- f(t_1,(Gx)(t_1)(\mathcal{U}x)(t_1))\right| \\
&\leq \omega^T(a,\varepsilon) + \frac{\left[\phi(t_2)\Phi(r_0) + \psi(t_2)\right]}{\Gamma(\alpha+1)}\omega^T(x,\varepsilon) \\
&\quad + \frac{T^{\alpha\beta}\omega_g^T(\varepsilon)}{\Gamma(\alpha+1)}\widehat{m}(T)\left[\hat{l}(T)\Phi(r_0) + \hat{u}(T)\right] \\
&\quad + \frac{\widehat{m}(T)\left[\hat{n}(T)r_0 + \widehat{g}(T)\right]}{\Gamma(\alpha+1)} \\
&\quad \times \left\{2\varepsilon^{\alpha\beta}\left[\hat{l}(T)\Phi(r_0) + \hat{u}(T)\right] + T^{\alpha\beta}\omega^T_{\|x\|}(u,\varepsilon)\right\} \\
&\quad + \left|f(t_2,(Gx)(t_1)(\mathcal{U}x)(t_1))\right. \\
&\qquad \left.- f(t_1,(Gx)(t_1)(\mathcal{U}x)(t_1))\right| \\
&\leq \omega^T(a,\varepsilon) + \frac{\left[\phi^*\Phi(r_0) + \psi^*\right]}{\Gamma(\alpha+1)}\omega^T(x,\varepsilon) \\
&\quad + \frac{T^{\alpha\beta}\omega_g^T(\varepsilon)}{\Gamma(\alpha+1)}\widehat{m}(T)\left[\hat{l}(T)\Phi(r_0) + \hat{u}(T)\right] \\
&\quad + \frac{\widehat{m}(T)\left[\hat{n}(T)r_0 + \widehat{g}(T)\right]}{\Gamma(\alpha+1)} \\
&\quad \times \left\{2\varepsilon^{\alpha\beta}\left[\hat{l}(T)\Phi(r_0) + \hat{u}(T)\right] + T^{\alpha\beta}\omega^T_{\|x\|}(u,\varepsilon)\right\} \\
&\quad + \left|f(t_2,(Gx)(t_1)(\mathcal{U}x)(t_1))\right. \\
&\qquad \left.- f(t_1,(Gx)(t_1)(\mathcal{U}x)(t_1))\right|.
\end{aligned} \tag{28}$$

In the last estimates, we have denoted by

$$\begin{aligned}
\omega_g^T(\varepsilon) &= \sup\left\{\left|g(t_2,x) - g(t_1,x)\right| : t_1,t_2 \in [0,T],\right. \\
&\qquad \left.\left|t_2 - t_1\right| \leq \varepsilon, x \in [-r_0, r_0]\right\}, \\
\hat{n}(T) &= \max\{n(t) : t \in [0,T]\}, \\
\widehat{m}(T) &= \max\{m(t) : t \in [0,T]\}, \\
\widehat{g}(T) &= \max\{\left|g(t,0)\right| : t \in [0,T]\}.
\end{aligned} \tag{29}$$

Hence,

$$
\begin{aligned}
&\omega^T(\mathscr{F}x,\varepsilon) \\
&\le \omega^T(a,\varepsilon) + \frac{[\phi^*\Phi(r_0)+\psi^*]}{\Gamma(\alpha+1)}\omega^T(x,\varepsilon) \\
&\quad + \frac{T^{\alpha\beta}\omega_g^T(\varepsilon)}{\Gamma(\alpha+1)}\widehat{m}(T)\left[\widehat{l}(T)\Phi(r_0)+\widehat{u}(T)\right] \\
&\quad + \frac{\widehat{m}(T)\left[\widehat{n}(T)r_0+\widehat{g}(T)\right]}{\Gamma(\alpha+1)} \\
&\quad \times\left\{2\varepsilon^{\alpha\beta}\left[\widehat{l}(T)\Phi(r_0)+\widehat{u}(T)\right]+T^{\alpha\beta}\omega^T_{\|x\|}(u,\varepsilon)\right\} \\
&\quad + \sup_{t_1,t_2\in[0,T],\ \|x\|\le r_0}\left|f(t_2,(Gx)(t_1)(\mathscr{U}x)(t_1))\right. \\
&\qquad\qquad \left. -f(t_1,(Gx)(t_1)(\mathscr{U}x)(t_1))\right|.
\end{aligned}
\tag{30}
$$

Since the function $f(t,y)$ is uniformly continuous on the set $[0,T]\times[-H,H]$, the function $g(t,x)$ is uniformly continuous on the set $[0,T]\times[-r_0,r_0]$ and the function $u(t,s,x)$ is uniformly continuous on the set $[0,T]\times[0,T]\times[-r_0,r_0]$, where

$$
\begin{aligned}
H = \sup\Bigg\{&\frac{\beta\left|g(t_1,x(t_1))\right|}{\Gamma(\alpha)} \\
&\times\int_0^{t_1}\frac{s^{\beta-1}\left|u(t_1,s,x(s))\right|}{\left(t_1^\beta-s^\beta\right)^{1-\alpha}}ds : t_1\in[0,T], \\
&\|x\|\le r_0\Bigg\};
\end{aligned}
\tag{31}
$$

we have

$$
\begin{aligned}
&\sup\left\{\left|f(t_2,y)-f(t_1,y)\right| : t_1,t_2\in[0,T],\ \left|t_2-t_1\right|\le\varepsilon,\right. \\
&\qquad \left.\left|y\right|\le H\right\}\longrightarrow 0 \quad \text{as}\ \varepsilon\longrightarrow 0.
\end{aligned}
\tag{32}
$$

It is easy to see that $H<\infty$ because $u(t,s,x)$ is bounded on $[0,T]\times[0,T]\times[-r_0,r_0]$, $g(t,x)$ is bounded on $[0,T]\times[-r_0,r_0]$, and $(\beta/\Gamma(\alpha))\int_0^{t_1}(s^{\beta-1}/(t_1^\beta-s^\beta)^{1-\alpha})ds\le T^{\alpha\beta}/\Gamma(\alpha+1)$.

Therefore, from the last estimate we derive the following one:

$$
\omega_0^T(\mathscr{F}X)\le q\omega_0^T(X). \tag{33}
$$

Hence we have

$$
\omega_0^\infty(\mathscr{F}X)\le q\omega_0^\infty(X). \tag{34}
$$

Step 6 ($\mathscr{F}$ is contraction with respect to the measure of noncompactness μ). From (27) and (34) and the definition of the measure of noncompactness μ given by formula (8), we obtain

$$
\mu(\mathscr{F}X)\le q\mu(X). \tag{35}
$$

Step 7. We construct a nonempty, bounded, closed, and convex set Y on which we will apply a fixed point theorem.

In the sequel let us put $B^1_{r_0} = \mathrm{Conv}\mathscr{F}(B_{r_0})$, $B^2_{r_0} = \mathrm{Conv}\mathscr{F}(B^1_{r_0})$, and so on. In this way we have constructed a decreasing sequence of nonempty, bounded, closed, and convex subsets $(B^n_{r_0})$ of B_{r_0} such that $\mathscr{F}(B^n_{r_0})\subset B^n_{r_0}$ for $n=1,2,\ldots$. Hence, in view of (35) we obtain

$$
\mu\left(B^n_{r_0}\right)\le q^n\mu\left(B_{r_0}\right), \quad \text{for any } n=1,2,3,\ldots. \tag{36}
$$

This implies that $\lim_{n\to\infty}\mu(B^n_{r_0}) = 0$. Hence, taking into account Definition 1 we infer that the set $Y = \bigcap_{n=1}^\infty B^n_{r_0}$ is nonempty, bounded, closed, and convex subset of B_{r_0}. Moreover, $Y\in\ker\mu$. Also, the operator $\mathscr{F}$ maps Y into itself.

Step 8 ($\mathscr{F}$ is continuous on the set Y). Let us fix a number $\varepsilon>0$ and take arbitrary functions $x,y\in Y$ such that $\|x-y\|\le\varepsilon$. Using the fact that $Y\in\ker\mu$ and keeping in mind the structure of sets belonging to $\ker\mu$ we can find a number $T>0$ such that for each $z\in Y$ and $t\ge T$ we have that $|z(t)|\le\varepsilon$. Since $\mathscr{F}$ maps Y into itself, we have that $\mathscr{F}x,\mathscr{F}y\in Y$. Thus, for $t\ge T$ we get

$$
\left|(\mathscr{F}x)(t)-(\mathscr{F}y)(t)\right|\le\left|(\mathscr{F}x)(t)\right|+\left|(\mathscr{F}y)(t)\right|\le 2\varepsilon. \tag{37}
$$

On the other hand, let us assume $t\in[0,T]$. Then we obtain

$$
\begin{aligned}
&\left|(\mathscr{F}x)(t)-(\mathscr{F}y)(t)\right| \\
&\quad\le\frac{\beta m(t)n(t)\left|x(t)-y(t)\right|}{\Gamma(\alpha)} \\
&\qquad\times\int_0^t\frac{s^{\beta-1}\left[l(t)\Phi(|x(s)|)+u^*(t)\right]}{\left(t^\beta-s^\beta\right)^{1-\alpha}}ds \\
&\qquad+\frac{\beta m(t)\left[n(t)\left|y(t)\right|+\left|g(t,0)\right|\right]}{\Gamma(\alpha)} \\
&\qquad\times\int_0^t\frac{s^{\beta-1}l(t)\Phi\left(\left|x(s)-y(s)\right|\right)}{\left(t^\beta-s^\beta\right)^{1-\alpha}}ds \\
&\quad\le\frac{\left[m(t)n(t)l(t)\Phi(r_0)+m(t)n(t)u^*(t)\right]\varepsilon\beta}{\Gamma(\alpha)} \\
&\qquad\times\int_0^t\frac{s^{\beta-1}}{\left(t^\beta-s^\beta\right)^{1-\alpha}}ds \\
&\qquad+\frac{\left[m(t)n(t)l(t)r_0+m(t)l(t)\left|g(t,0)\right|\right]\Phi(\varepsilon)\beta}{\Gamma(\alpha)} \\
&\qquad\times\int_0^t\frac{s^{\beta-1}}{\left(t^\beta-s^\beta\right)^{1-\alpha}}ds \\
&\quad\le\frac{\phi(t)\Phi(r_0)+\psi(t)}{\Gamma(\alpha+1)}\varepsilon+\frac{\phi(t)r_0+\xi(t)}{\Gamma(\alpha+1)}\Phi(\varepsilon) \\
&\quad\le\frac{\phi^*\Phi(r_0)+\psi^*}{\Gamma(\alpha+1)}\varepsilon+\frac{\phi^*r_0+\xi^*}{\Gamma(\alpha+1)}\Phi(\varepsilon).
\end{aligned}
\tag{38}
$$

Now, taking into account (37) and (38) and hypothesis (h_5) we conclude that the operator $\mathscr{F}$ is continuous on the set Y.

Step 9 (application of Schauder fixed point principle). Linking all above-obtained facts about the set Y and the operator $\mathscr{F} : Y \to Y$ and using the classical Schauder fixed point principle we deduce that the operator $\mathscr{F}$ has at least one fixed point x in the set Y. Obviously the function $x = x(t)$ is a solution of the quadratic integral equation (1). Moreover, since $Y \in \ker \mu$ we have that all solutions of (1) belonging to B_{r_0} are asymptotically stable in the sense of Definition 2. This completes the proof. □

4. Example

In this section, we present an example as an application of Theorem 3.

Consider the following integral equation of fractional order:

$$x(t) = te^{-t} + \frac{1}{1+t^3} + \arctan\left[\frac{1}{t^2+1}\cdot\frac{\sin(xt)}{2\Gamma(1/2)}\int_0^t \frac{\sqrt{1+\delta|x(s)|}}{\sqrt{s}\sqrt{\sqrt{t}-\sqrt{s}}}ds\right], \quad t \in \mathbb{R}_+. \tag{39}$$

Equation (39) is a special case of (1), where $\alpha = 1/2$, $\beta = 1/2$, δ is a positive constant, and

$$\begin{gathered} a(t) = te^{-t}, \\ f(t,x) = \frac{1}{1+t^3} + \arctan\left(\frac{1}{t^2+1}\cdot x\right), \\ g(t,x) = \sin(xt), \\ u(t,s,x) = \sqrt{1+\delta|x|}. \end{gathered} \tag{40}$$

It is easy to check that the assumptions of Theorem 3 are satisfied. In fact we have that the function $a(t) = te^{-t}$ is continuous and bounded on $\mathbb{R}_+$ and $\|a\| = 1/e$.

The function $f(t,x) = (1/(1+t^3)) + \arctan((1/(t^2+1))\cdot x)$ satisfies assumption (h_2) with $m(t) = 1/(t^2+1)$ and $|f(t,0)| = f(t,0) = 1/(1+t^3)$, being $f^* = 1$.

Moreover, the function $g(t,x) = \sin(xt)$ satisfies assumption (h_3) with $n(t) = t$.

The function $u(t,s,x) = \sqrt{1+\delta|x|}$ satisfies assumption (h_4) with $l(t) = 1$, $\Phi(r) = \sqrt{\delta r}$, $u(t,s,0) = 1$, and $u^* = 1$.

Next, we are going to check that assumption (h_5) is satisfied. The functions ϕ, ψ, ξ, and η appearing in assumption (h_5) take the form

$$\begin{gathered} \phi(t) = \frac{t^{5/4}}{t^2+1}; \qquad \psi(t) = \frac{t^{5/4}}{t^2+1}; \\ \xi(t) = 0; \qquad \eta(t) = 0. \end{gathered} \tag{41}$$

It is easy to see that $\lim_{t\to\infty}\phi(t) = \lim_{t\to\infty}\xi(t) = 0$.

Moreover we have $\phi^* = \psi^* = (3/8)\cdot(5/3)^{5/8}$, $\xi^* = \eta^* = 0$, and $\Gamma(3/2) = (1/2)\sqrt{\pi}$.

Therefore the inequality in assumption (h_6)

$$\|a\| + f^* + \frac{1}{\Gamma(\alpha+1)} \times \left[\phi^* r\Phi(r) + \psi^* r + \xi^*\Phi(r) + \eta^*\right] \le r \tag{42}$$

has the form

$$\frac{1}{e} + 1 + \frac{2}{\sqrt{\pi}}\left[\phi^*\sqrt{\delta}r^{3/2} + \psi^* r\right] \le r. \tag{43}$$

We can easily check that the number $r_0 = 7$ is a solution of the inequality (43) for $\delta \le 0,02$. Now, by Theorem 3, we infer that our equation has a solution in $B_{r_0} \subset BC(\mathbb{R}_+)$ and all solutions of (39) which belongs to B_{r_0} are asymptotically stable in the sense of the Definition 2.

Conflict of Interests

The authors declare that there is no conflict of interests in the submitted paper.

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Research Article

Estimates of L_p Modulus of Continuity of Generalized Bounded Variation Classes

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Some sharp estimates of the $L_p\,(1 \le p < \infty)$ modulus of continuity of classes of Λ_φ-bounded variation are obtained. As direct applications, we obtain estimates of order of Fourier coefficients of functions of Λ_φ-bounded variation, and we also characterize some sufficient and necessary conditions for the embedding relations $H_p^\omega \subset \Lambda_\varphi BV$. Our results include the corresponding known results of the class ΛBV as a special case.

1. Introduction and Main Results

To generalize the notion of functions of bounded variation, Wiener [1] introduced the class BV_β ($\beta > 1$) of functions of β-bounded variation. Young [2] introduced the notion of functions of φ-bounded variation, and Waterman [3] studied a class of Λ-bounded variations. Combining the notion of Λ-bounded variation with that of φ-bounded variation, Leindler [4] introduced the class $\Lambda_\varphi BV$ of functions of Λ_φ-bounded variation, and both classes of Λ-bounded variation and φ-bounded variation are its special cases. Actually the class $\Lambda_\varphi BV$ first appeared in Schramm and Waterman's paper [5], and some restrictions are imposed on φ in their definition. Here we adopt Leindler's definition.

Definition 1. Let $\varphi : [0,\infty) \to \mathbb{R}$ be a nondecreasing function with $\varphi(0) = 0$, and let $\Lambda =: \{\lambda_k\}$ be a nondecreasing sequence of positive numbers such that $\sum_{k=1}^{\infty}(1/\lambda_k) = +\infty$. Let Γ be the set of all sequences of nonoverlapping subintervals $[a_k, b_k]$ in $[a,b]$. If for any $\Delta = \{(a_k, b_k) \subset [a,b],\ k = 1,2,\ldots,n,\ n \in \mathbf{Z}^+\} \in \Gamma$, a real valued function $f : [a,b] \to \mathbb{R}$ satisfies the condition

$$\sum_{k=1}^{n} \frac{\varphi\left(\left|f(b_k) - f(a_k)\right|\right)}{\lambda_k} < \infty, \tag{1}$$

then f is said to be of Λ_φ-bounded variation, and this fact is denoted by $f \in \Lambda_\varphi BV$. And the quantity

$$V_{\Lambda_\varphi}(f;[a,b]) := \sup_{\Delta\in\Gamma}\left\{\sum_{k=1}^{n}\frac{\varphi\left(\left|f(b_k) - f(a_k)\right|\right)}{\lambda_k}\right\} \tag{2}$$

is said to be Λ_φ-total variation of f.

In the special case when $\varphi(x) = x^\beta$ ($\beta \ge 1$), f is said to be of Λ_β-bounded variation, and we write $f \in \Lambda_\beta BV$ and $V_{\Lambda_\beta}(f;[a,b]) = V_{\Lambda_\varphi}(f;[a,b])$, and if $\beta = 1$, f is said to be of Λ-bounded variation, and we denote $f \in \Lambda BV$ and $V_\Lambda(f;[a,b]) = V_{\Lambda_1}(f;[a,b])$.

In the case $\Lambda = \{1\}$, we get the class of φ-bounded variation, and f is said to be of φ-bounded variation, and we denote $V_\varphi(f;[a,b]) = V_{\Lambda_\varphi}(f;[a,b])$. More specifically, when $\varphi(x) = x^\beta$ ($\beta \ge 1$), we say that f is of β-bounded variation, and we denote $f \in BV_\beta$ and $V_\beta(f;[a,b]) = V_\varphi(f;[a,b])$. The class BV_β is also called the Wiener class and BV_1 is the well-known class of bounded variation BV.

It is easily seen from the definition that $\Lambda_\varphi BV$ functions are bounded; that is, $\Lambda_\varphi BV[a,b] \subseteq B[a,b]$, and the discontinuities of a $\Lambda_\varphi BV$ function are simple and therefore at most denumerable, where $B[a,b]$ denotes the class of bounded real valued functions on $[a,b]$.

Let $\Lambda =: \{\lambda_k\}$ be a nondecreasing sequence of positive numbers such that $\sum_{k=1}^{\infty}(1/\lambda_k) = +\infty$. If a continuous and nondecreasing function $\lambda(s)$ on $[0,\infty)$ such that $\lambda(s) \equiv \lambda_1$, $0 \le s \le 1$ and $\lambda(k) = \lambda_k$, $k = 1, 2, \ldots$, then we say that $\lambda(s)$ generates Λ. By the nondecreasing property of Λ, it is easily verified that if $\lambda(s)$ generates Λ, then

$$\frac{1}{2}\int_0^n \frac{ds}{\lambda(s)} \le \sum_{k=1}^{n} \frac{1}{\lambda_k} \le \int_0^n \frac{ds}{\lambda(s)}, \quad n \ge 1. \tag{3}$$

Let $\lambda(x)$ be a nonnegative real-valued function on $[0,\infty)$. If there exists $a > 0$ such that $\lambda(2x) \le a\lambda(x)$, $x \in (0, A)$ for some positive constant A, then we say that $\lambda(x)$ satisfies the condition Δ_2. If $\lambda(x)$ generates $\Lambda = \{\lambda_k\}$ and satisfies the condition Δ_2, we say that Λ satisfies the condition Δ_2. Obviously the condition Δ_2 here is a very weak restriction on $\lambda(x)$ and Λ.

Let $\omega(t)$ be a modulus of continuity, that is, a continuous and nondecreasing function on $[0, +\infty)$ satisfying $\omega(0) = 0$ and $\omega(t_1 + t_2) \le \omega(t_1) + \omega(t_2)$ for nonnegative t_1 and t_2. As usual, for $1 \le p \le \infty$, denote by $H_p^\omega \equiv H_p^{\omega(t)}$ the class of functions for which $\|f\|_{H_p^\omega} := \sup_{t>0}(\omega(f,t)_p/\omega(t)) < \infty$, where

$$\omega(f;t)_p := \begin{cases} \sup\limits_{0\le h\le t} \left\{ \displaystyle\int_a^b |f(x+h) - f(x)|^p dx \right\}^{1/p}, & 1 \le p < \infty, \\ \sup\limits_{0\le h\le t} \sup\limits_{x\in[a,b]} |f(x+h) - f(x)|, & p = \infty \end{cases} \tag{4}$$

is the L_p modulus of continuity of f. We write H^ω instead of H_∞^ω and H_p^α $(0 < \alpha \le 1)$ instead of $H_p^{t^\alpha} \equiv \mathrm{Lip}(\alpha, p)$, the Lipschitz class, for brevity.

Functions of classes BV_β, φBV, ΛBV, and $\Lambda_\varphi BV$ are considered in trigonometric Fourier series and some of them share good approximative properties (see [1–3, 6–11], etc.). What we mention here is the following theorem proved by Shiba [12], Schramm and Waterman [5], and Wang [13]:

Theorem A. *(a) If $f \in \Lambda_\beta BV$, $1 \le \beta < 2r$, $1 \le r < \infty$, and*

$$\frac{\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} (1/\lambda_k)\right)^{-1/2r} \left(\omega_{\beta+(2-\beta)s}(f;\pi/n)\right)^{1-\beta/2r}}{n^{1/2}} < \infty, \tag{5}$$

where $1/r + 1/s = 1$, then the Fourier series of f converges absolutely.

(b) If φ is Δ_2, $f \in \Lambda_\varphi BV$, $1 \le \beta < 2r$, $1 \le r < \infty$, and

$$\frac{\sum_{n=1}^{\infty} \left[\varphi^{-1}\left(\left(\sum_{k=1}^{n}\left(\frac{1}{\lambda_k}\right)\right)^{-1} \omega_{\beta+(2-\beta)s}^{2r-\beta}\left(f;\frac{\pi}{n}\right)\right)\right]^{1/2r}}{n^{1/2}} < \infty, \tag{6}$$

where $1/r + 1/s = 1$, then the Fourier series of f converges absolutely.

Embedding relations between various generalized bounded variation classes and the class H_p^ω (or the Lipschitz class H_p^α) are also investigated in recent years. It iswell known that $H_p^1 \subset H_1^1 = BV \subset \Lambda BV$. For $f \in \Lambda BV$, the estimates of L_p modulus of continuity of f had been given in [13] for $p = 1$ and in [9, 14] for $1 < p < \infty$. Furthermore, Goginava in [15] and Li and Wang in [9] proved that, for $1 \le p < \infty$, $0 < \alpha, \delta \le 1$,

$$\left\{n^\delta\right\} BV \subset H_p^\alpha \quad \text{iff } \alpha \le \min\left\{\frac{1}{p}, 1-\delta\right\}. \tag{7}$$

For more detailed results on this topic, we refer readers to [4, 9, 10, 14–25].

In this paper, we obtain some sharp estimates of L_p $(1 \le p < \infty)$ modulus of continuity of the classes $\Lambda_\varphi BV$ in the case of that φ is convex. More specifically, our results include estimates of L_1 modulus of continuity of the classes $\Lambda_\varphi BV$, estimates of L_p $(1 \le p < \infty)$ modulus of continuity of the classes $\Lambda_\beta BV$, and specially estimates of L_p $(1 \le p < \infty)$ modulus of the classes $\{(n+1)^\alpha \ln^\gamma (n+1)\}_\beta BV$ $(0 \le \alpha \le 1, \beta \ge 1, \gamma \in \mathbb{R})$. Our results extend and include the corresponding known results of the class ΛBV as a special case and are also sharp in most cases. As direct applications, we obtain estimates of order of Fourier coefficients of functions of Λ_φ-bounded variation, and we also characterize some sufficient and necessary conditions for the embedding relations $H_p^\omega \subset \Lambda_\varphi BV$ and $H_p^\omega \subset \varphi BV$.

Now we state our main results, and in what follows, without loss of generality, we always assume that $[a,b] = [0, 2\pi]$, and functions in various generalized bounded variation classes are 2π periodic.

Theorem 2. *Let $\Lambda_\varphi BV$ be the class of functions of Λ_φ-bounded variation, and let $\lambda(s)$ generate Λ, where φ is convex and φ^{-1} is the inverse function of φ. Then, for $f \in \Lambda_\varphi BV$,*

$$\omega(f;t)_1 \le 4\pi\varphi^{-1}\left(\frac{V_{\Lambda_\varphi}(f;[0,2\pi])}{\int_0^{1/t}(ds/\lambda(s))}\right), \tag{8}$$

and this estimate is sharp in the sense of order provided that Λ satisfies the condition Δ_2.

Corollary 3 is a direct result of Theorem 2 by choosing $\varphi(x) = x$ and $\Lambda = \{1\}$, respectively.

Corollary 3. *(a) Let ΛBV be the class of functions of Λ-bounded variation, and let $\lambda(s)$ generate Λ. Then, for $f \in \Lambda BV$,*

$$\omega(f;t)_1 \le \frac{4\pi V_\Lambda(f;[0,2\pi])}{\int_0^{1/t}(ds/\lambda(s))}, \tag{9}$$

and this estimate is sharp in the sense of order provided that Λ satisfies the condition Δ_2.

(b) Let φBV be the class of functions of φ-bounded variation, where φ is convex. Then, for $f \in \varphi BV$,

$$\omega(f;t)_1 \le 4\pi\varphi^{-1}\left(tV_\varphi(f;[0,2\pi])\right), \tag{10}$$

and this estimate is sharp in the sense of order.

The first part of Corollary 3(a) is due to Wang [13].

For special Λ's, one has

Corollary 4. *Let $f \in \Lambda_\varphi BV$ and φ be convex. Then*

(a) *for* $\Lambda = \{n^\alpha\}$ $(0 \le \alpha < 1)$,

$$\omega(f;t)_1 \le c\varphi^{-1}\left(t^{1-\alpha}V_{\Lambda_\varphi}(f;[0,2\pi])\right); \tag{11}$$

(b) *for* $\Lambda = \{n\}$,

$$\omega(f;t)_1 \le c\varphi^{-1}\left(\frac{V_{\Lambda_\varphi}(f;[0,2\pi])}{\ln(1/t)}\right); \tag{12}$$

(c) *for* $\Lambda = \{(n+1)\ln(n+1)\}$,

$$\omega(f;t)_1 \le c\varphi^{-1}\left(\frac{V_{\Lambda_\varphi}(f;[0,2\pi])}{\ln\ln\ (1/t)}\right); \tag{13}$$

(d) *for* $\Lambda = \{\ln^\gamma(n+1)\}$ $(\gamma > 0)$,

$$\omega(f;t)_1 \le c\varphi^{-1}\left(t\ln^{-\gamma}\left(\frac{1}{t}\right)V_{\Lambda_\varphi}(f;[0,2\pi])\right); \tag{14}$$

(e) *for* $\Lambda = \{(n+1)\ln^\gamma(n+1)\}$ $(-\infty < \gamma < 1)$,

$$\omega(f;t)_1 \le c\varphi^{-1}\left(\frac{V_{\Lambda_\varphi}(f;[0,2\pi])}{\ln^{1-\gamma}(1/t)}\right); \tag{15}$$

(f) *for* $\Lambda = \{(n+1)^\alpha\ln^\gamma(n+1)\}$ $(0 < \alpha < 1, \gamma \in \mathbb{R}\setminus\{0\})$,

$$\omega(f;t)_1 \le c\varphi^{-1}\left(t^{1-\alpha}\ln^{-\gamma}\left(\frac{1}{t}\right)V_{\Lambda_\varphi}(f;[0,2\pi])\right). \tag{16}$$

The above estimates are sharp in the sense of order.

Let $f(x)$ be an integrable function on $[0,2\pi]$ and let its Fourier coefficients be defined as follows:

$$\begin{aligned} a_n(f) &= \frac{1}{\pi}\int_0^{2\pi} f(t)\cos nt\,dt, \\ b_n(f) &= \frac{1}{\pi}\int_0^{2\pi} f(t)\sin nt\,dt, \\ c_n(f) &= a_n(f) + ib_n(f). \end{aligned} \tag{17}$$

We note that

$$\begin{aligned} |c_n(f)| &\le \frac{1}{2\pi}\int_0^{2\pi}\left|f(t) - f\left(t+\frac{\pi}{n}\right)\right|dt \\ &\le c\omega_1\left(f;\frac{1}{n}\right), \quad c > 0. \end{aligned} \tag{18}$$

Theorem 2 implies the following estimates of Fourier coefficients.

Corollary 5. *(a) Let $\Lambda_\varphi BV$ be the class of functions of Λ_φ-bounded variation, and let $\lambda(s)$ generate Λ, where φ is convex and φ^{-1} is the inverse function of φ. Then, for $f \in \Lambda_\varphi BV$,*

$$\left.\begin{matrix}|a_n(f)| \\ |b_n(f)|\end{matrix}\right\} = \mathbf{O}\left(\varphi^{-1}\left(\frac{1}{\int_0^n(ds/\lambda(s))}\right)\right), \quad n \ge 1. \tag{19}$$

(b) Let φBV be the class of functions of φ-bounded variation, where φ is convex and φ^{-1} is the inverse function of φ. Then for $f \in \varphi BV$

$$\left.\begin{matrix}|a_n(f)| \\ |b_n(f)|\end{matrix}\right\} = \mathbf{O}\left(\varphi^{-1}\left(\frac{1}{n}\right)\right), \quad n \ge 1. \tag{20}$$

Corollary 5(a) includes Wang's result [13] for the class ΛBV as a special case of $\varphi(x) = x$.

Theorem 6. *Let $\Lambda_\beta BV$ $(\beta \ge 1)$ be class of functions of Λ_β-bounded variation, and let $\lambda(s)$ generate $\Lambda = \{\lambda_n\}$. Set $\phi(z) = \int_0^z(ds/\lambda(s))$ and assume $f \in \Lambda_\beta BV$.*

(a) For $1 \le p \le \beta$,

$$\omega(f;t)_p \le c\left(\frac{V_{\Lambda_\beta}(f;[0,2\pi])}{\int_0^{1/t}(ds/\lambda(s))}\right)^{1/\beta}, \tag{21}$$

and this estimate is sharp in the sense of order provided that Λ satisfies the condition Δ_2.

(b) For $1 < p < \infty$,

$$\omega(f;t)_p \le c\left(V_{\Lambda_\beta}(f;[0,2\pi])\right)^{1/\beta}\left(t\int_0^{1/t}\frac{dz}{\phi(z)^{p/\beta}}\right)^{1/p}. \tag{22}$$

(c) If, for some $\beta < p < \infty$, $\phi(z)^{(\beta-p)/\beta}\lambda(z)$ is bounded on $[1,\infty)$, then

$$\omega(f;t)_p \le c\left(V_{\Lambda_\beta}(f;[0,2\pi])\right)^{1/\beta}t^{1/p}. \tag{23}$$

Unfortunately we cannot assert the sharpness of our estimates in (b) and (c) of Theorem 6 for the case $\beta < p < \infty$. Our next theorems concern some important special case of the class $\Lambda_\beta BV$ and elaborate on the estimates in Theorem 6. Among the classes considered here are $\{n^\alpha\}_\beta BV$ $(0 \le \alpha \le 1, \beta \ge 1)$ and $\{(n+1)^\alpha\ln^\gamma(n+1)\}_\beta BV$ $(0 \le \alpha \le 1, \beta \ge 1, \gamma \in \mathbb{R})$.

Theorem 7. *(a) Let $\{n^\alpha\}_\beta BV$ $(0 \le \alpha \le 1, \beta \ge 1)$ be the class of functions of $\{n^\alpha\}_\beta$-bounded variation. Then, for $f \in \{n^\alpha\}_\beta BV$ and $1 \le p < \infty$, one has*

(i) $\omega(f;t)_p \le c(V_{\Lambda_\beta}(f;[0,2\pi]))^{1/\beta}t^{\min\{(1-\alpha)/\beta,1/p\}}$, $0 \le \alpha < 1$;

(ii) $\omega(f;t)_p \le c(V_{\Lambda_\beta}(f;[0,2\pi]))^{1/\beta}(1/\ln t^{-1})^{1/\beta}$, $\alpha = 1$.

(b) Let BV_β $(\beta \ge 1)$ be the class of β-bounded variation, that is, the Wiener class. Then, for $f \in BV_\beta$ and $1 \le p < \infty$,

$$\omega(f;t)_p \le c\left(V_\beta(f;[0,2\pi])\right)^{1/\beta}t^{\min\{1/\beta,1/p\}}, \tag{24}$$

and both estimates in (a) and (b) are sharp in the sense of order.

Li and Wang's results in [9] are extended in the Theorems 6 and 7, which can be treated as the case $\beta = 1$ of our theorems.

Theorem 8. *Let $f \in \Lambda_\beta BV$ $(\beta \geq 1)$. The following assertions are true.*

(a) *For $\Lambda = \{\ln^\gamma(n+1)\}$ $(\gamma > 0)$,*

$$\omega(f;t)_p \leq \begin{cases} c\left(V_{\Lambda_\beta}(f;[0,2\pi])\right)^{1/\beta}\left(t\ln^\gamma\left(\frac{1}{t}\right)\right)^{1/\beta}, & 1 \leq p \leq \beta, \\ c\left(V_{\Lambda_\beta}(f;[0,2\pi])\right)^{1/\beta} t^{1/p}, & \beta < p < \infty. \end{cases} \quad (25)$$

(b) *For $\Lambda = \{(n+1)\ln(n+1)\}$,*

$$\omega(f;t)_p \leq c\left(V_{\Lambda_\beta}(f;[0,2\pi])\right)^{1/\beta}\left(\ln\ln\frac{1}{t}\right)^{-1/\beta}, \quad 1 \leq p < \infty. \quad (26)$$

(c) *For $\Lambda = \{(n+1)\ln^\gamma(n+1)\}$ $(-\infty < \gamma < 1)$,*

$$\omega(f;t)_p \leq c\left(V_{\Lambda_\beta}(f;[0,2\pi])\right)^{1/\beta}\left(\ln\frac{1}{t}\right)^{-(1-\gamma)/\beta}, \quad 1 \leq p < \infty. \quad (27)$$

(d) *For $\Lambda = \{(n+1)^\alpha/\ln^\gamma(n+1)\}$ $(0 < \alpha < 1, \gamma > 0)$,*

$$\omega(f;t)_p \leq \begin{cases} c\left(V_{\Lambda_\beta}(f;[0,2\pi])\right)^{1/\beta} t^{(1-\alpha)/\beta}\left(\ln\frac{1}{t}\right)^{-\gamma/\beta}, & 1 \leq p < \dfrac{\beta}{1-\alpha}, \\ c\left(V_{\Lambda_\beta}(f;[0,2\pi])\right)^{1/\beta} t^{1/p}, & p \geq \dfrac{\beta}{1-\alpha}. \end{cases} \quad (28)$$

(e) *For $\Lambda = \{(n+1)^\alpha\ln^\gamma(n+1)\}$ $(0 < \alpha < 1, \gamma > 0)$*

$$\omega(f;t)_p \leq \begin{cases} c\left(V_{\Lambda_\beta}(f;[0,2\pi])\right)^{1/\beta} t^{(1-\alpha)/\beta}\left(\ln\frac{1}{t}\right)^{\gamma/\beta}, & 1 \leq p < \dfrac{\beta}{1-\alpha}, \\ c\left(V_{\Lambda_\beta}(f;[0,2\pi])\right)^{1/\beta} t^{1/p}, & p > \dfrac{\beta}{1-\alpha}. \end{cases} \quad (29)$$

And all above estimates are sharp in the sense of order.

Remark 9. In (e), for $p = \beta/(1-\alpha)$, we only have

$$\omega(f;t)_p \leq c\left(V_{\Lambda_\beta}(f;[0,2\pi])\right)^{1/\beta} t^{1/p}\left(\ln\frac{1}{t}\right)^{1/p+\gamma/\beta}. \quad (30)$$

We do not know whether this estimate is sharp in the sense of order. However there exists $f_n \in \Lambda_\beta BV$ such that

$$V_{\Lambda_\beta}(f_n;[0,2\pi]) \asymp 1, \quad n \longrightarrow +\infty,$$
$$\omega\left(f_n;\frac{\pi}{n}\right)_p \geq c\left(\frac{\pi}{n}\right)^{1/p}, \quad n \geq 10. \quad (31)$$

This exception indicates that the estimates of the L_p modulus of continuity $\omega(f;t)_p$ $(\beta < p < +\infty)$ of classes of Λ_β-bounded variation are complicated, and Theorem 6 cannot cover all cases of the class $\Lambda_\beta BV$ $(\beta \geq 1)$.

As direct applications of the above theorems, we characterize some sufficient and necessary conditions for the embedding relations between the generalized bounded variation classes and the class H_p^ω (or H_p^δ).

Corollary 10. *On the embedding relations between the generalized bounded variation classes and the class H_p^ω or H_p^δ, the following assertions are true.*

(a) *Let $\Lambda_\varphi BV$ be the class of functions of Λ_φ-bounded variation and let φ be convex; let $\lambda(s)$ generate Λ and Λ satisfy the condition Δ_2. Then*

$$\Lambda_\varphi BV \subset H_1^\omega \quad \text{iff} \quad \omega(t) = \mathbf{O}\left(\varphi^{-1}\left(\left(\left(\int_0^{1/t}\frac{ds}{\lambda(s)}\right)^{-1}\right)\right)\right), \quad t \longrightarrow 0. \quad (32)$$

(b) *Let φBV be the class of functions of φ-bounded variation and let φ be convex. Then*

$$\varphi BV \subset H_1^\omega \quad \text{iff} \quad \omega(t) = \mathbf{O}\left(\varphi^{-1}(t)\right), \quad t \longrightarrow 0. \quad (33)$$

(c) *Let $\Lambda_\beta BV$ be the class of functions of Λ_β-bounded variation and let $\lambda(s)$ generate Λ.*

(i) *If, for $1 \leq p \leq \beta$, Λ satisfies the condition Δ_2, then*

$$\Lambda_\beta BV \subset H_p^\omega \quad \text{iff} \quad \omega(t) = \mathbf{O}\left(\left(\left(\int_0^{1/t}\frac{ds}{\lambda(s)}\right)^{-1/\beta}\right)\right), \quad t \longrightarrow 0. \quad (34)$$

(ii) *If, for $\beta < p < \infty$, $\phi(z)^{(\beta-p)/\beta}\lambda(z)$ is bounded on $[1,\infty)$ and $0 < \delta \leq 1/p$, then $\Lambda_\beta BV \subset H_p^\delta$.*

(d) *Let $\{n^\alpha\}_\beta BV$ $(0 \leq \alpha \leq 1, \beta \geq 1)$ be the class of functions of $\{n^\alpha\}_\beta$-bounded variation. Then, for $1 \leq p < \infty$,*

(i) *$\{n^\alpha\}_\beta BV \subset H_p^\delta$ $(0 \leq \alpha < 1)$ if and only if $\delta \leq \min\{(1-\alpha)/\beta, 1/p\}$.*

(ii) *$\{n\}_\beta BV \subset H_p^\omega$ if and only if $\omega(t) = \mathbf{O}((\ln(1/t))^{-1/\beta})$, $t \to 0$.*

(e) *Let BV_β $(\beta \geq 1)$ be the class of functions of β-bounded variation, that is, Wiener class; then, for $1 \leq p < \infty$, $BV_\beta \subset H_p^\delta$ if and only if $\delta \leq \min\{1/\beta, 1/p\}$.*

This paper is organized as follows. In Section 2, we first state three lemmas, and then by them we prove Theorem 2. Lemma 12 provides our proofs crucial upper estimates and Lemma 13 will be used repeatedly in proving the sharpness of our estimates. In Section 3, we prove Theorem 6, and the same estimate technique in [9] is partly used in our proof. Theorems 7 and 8 are proved in Section 4. For the case $\beta < p < \infty$, the difficulty in our proofs of Theorems 7 and 8 is to prove the sharpness of our estimates and the key is to construct extreme functions by Lemma 13.

2. Proof of Theorem 2

Before we start our proof of Theorem 2, we prove three lemmas. Lemmas 11 and 12 will also be used in the proof of Theorem 6. Lemma 13 will be employed repeatedly in the proof of the sharpness of our estimates. Lemmas 12 and 13 are of independent interest for functions of Λ_φ-bounded variation.

Lemma 11. *Let $f \in L_p[a,b]$ $(p \geq 1)$. Then*

$$F(h) = \left(\int_a^b |f(x+h) - f(x)|^p dx \right)^{1/p} \tag{35}$$

is continuous on $[0,\infty)$.

Proof. Using triangle inequality of L_p norm, for any $h_1, h_2 > 0$, we have

$$\begin{aligned} |F(h_1) - F(h_2)| &\leq \|f(\cdot + h_1) - f(\cdot + h_2)\|_p \\ &\leq \omega(f; |h_1 - h_2|)_p. \end{aligned} \tag{36}$$

The right continuity of $\omega(f;t)_p$ at $t=0$ implies the continuity of $F(h)$. □

Lemma 12. *Let $\Delta = \{[a_k, b_k] : 1 \leq k \leq n, n \in \mathbf{Z}^+\} \in \Gamma$ be an arbitrary sequence of nonoverlapping intervals in $[a,b]$, $\Lambda = \{\lambda_k\}$, φ convex, and φ^{-1} the function of φ. Then, for $f \in \Lambda_\varphi BV$,*

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| \leq n\varphi^{-1} \left(\frac{V_{\Lambda_\varphi}(f; [a,b])}{\sum_{k=1}^n \lambda_k^{-1}} \right). \tag{37}$$

Specifically, if $f \in \varphi BV$, then $\sum_{k=1}^n |f(b_k) - f(a_k)| \leq n\varphi^{-1}(V_\varphi(f;[a,b])/n)$.

Proof. By the definition of Λ_φ-total variation and letting $A_i = 1/\lambda_i$ and $B_k = \varphi(|f(b_k) - f(a_k)|)$ in summation transform

$$\begin{aligned} &\left(\sum_{i=1}^n A_i\right)\left(\sum_{k=1}^n B_k\right) \\ &\quad = \sum_{k=0}^{n-1} \left\{ \sum_{i=1}^{n-k} A_i B_{i+k} + \sum_{i=n-k+1}^{n} A_i B_{i+k-n} \right\}, \end{aligned} \tag{38}$$

we have

$$\sum_{k=1}^n \varphi(|f(b_k) - f(a_k)|) \leq \frac{n}{\sum_{k=1}^n \lambda_k^{-1}} V_{\Lambda_\varphi}(f;[a,b]). \tag{39}$$

Note that if φ is an increasing convex function on $[0,\infty)$, then φ^{-1} is increasing and concave on $[0,\infty)$. The concavity of φ^{-1} implies that $\varphi^{-1}(ax) \geq a\varphi^{-1}(x)$, $0 < a \leq 1$. Therefore, by Jensen's inequality and the above inequality, we finally get

$$\begin{aligned} &\sum_{k=1}^n |f(b_k) - f(a_k)| \\ &\quad = n \sum_{k=1}^n \frac{\varphi^{-1}(\varphi(|f(b_k) - f(a_k)|))}{n} \\ &\quad \leq n\varphi^{-1}\left(\frac{1}{n} \sum_{k=1}^n \varphi(|f(b_k) - f(a_k)|) \right) \\ &\quad \leq n\varphi^{-1}\left(\frac{V_{\Lambda_\varphi}(f;[a,b])}{\sum_{k=1}^n \lambda_k^{-1}} \right). \end{aligned} \tag{40}$$

This completes the proof of Lemma 12. □

Lemma 13. *Let $\mathbf{a}_n = \{a_k\}_{k=1}^n$ be a given set of nonnegative and nonincreasing numbers. Define a function $f(x)$ on $[0,2\pi]$ as*

$$f_{\mathbf{a}_n}(x) = \begin{cases} a_i, & \dfrac{(2i-1)\pi}{n} \leq x < \dfrac{2i\pi}{n}, \quad i = 1,2,\ldots,n, \\ 0, & \text{other points of } [0,2\pi], \end{cases} \tag{41}$$

and extend it to $\mathbb{R}$ with period 2π. Then

(a) $f \in \Lambda_\varphi BV[0,2\pi]$ *and* $V_{\Lambda_\varphi}(f_{\mathbf{a}_n};[0;2\pi]) = \sum_{i=1}^n (1/\lambda_{2i-1} + 1/\lambda_{2i})\varphi(a_i)$;

(b) $\omega(f_{\mathbf{a}_n}; \pi/n)_p = ((2\pi/n)\sum_{k=1}^n a_k^p)^{1/p}$, $p \geq 1$;

(c) $\sum_{i=1}^n (\varphi(a_i)/\lambda_i) \leq V_{\Lambda_\varphi}(f_{\mathbf{a}_n};[0;2\pi]) \leq 2\sum_{i=1}^n(\varphi(a_i)/\lambda_i)$.

Proof. By the definition of Λ_φ-total variation and the nonnegative and nonincreasing properties of $\mathbf{a}_n$, direct computation proves (a). Since $\Lambda = \{\lambda_k\}$ is nondecreasing, it is obvious that (a) implies (c). For (b), computation shows that

$$\begin{aligned} &\omega\left(f_{\mathbf{a}_n}; \frac{\pi}{n}\right)_p^p \\ &\quad = \sum_{k=1}^n \int_{(2(k-1)\pi)/n}^{2k\pi/n} \left| f_{\mathbf{a}_n}\left(x + \frac{\pi}{n}\right) - f_{\mathbf{a}_n}(x) \right|^p dx \\ &\quad = \frac{2\pi}{n} \sum_{k=1}^n a_k^p. \end{aligned} \tag{42}$$

□

Now we prove Theorem 2.

Proof of Theorem 2. We write $V_{\Lambda_\varphi}(f) = V_{\Lambda_\varphi}(f;[0,2\pi])$ for simplicity. By Lemma 11, there exists an $h_t \in (0,t]$ such that

$$\omega(f;t)_1 = \int_0^{2\pi} |f(x+h_t) - f(x)| dx. \tag{43}$$

If we set $N = [2\pi/h_t]$ and consider the periodicity of $f(x)$, then

$$\omega(f;t)_1 = \frac{1}{N}\int_0^{2\pi} F_t(x)\,\mathrm{d}x, \tag{44}$$

where

$$F_t(x) = \sum_{k=1}^{N} \left| f(x+kh_t) - f(x+(k-1)h_t) \right|. \tag{45}$$

From Jensen's inequality and Lemma 12, for all $x \in [0,2\pi]$, the concavity of φ^{-1} implies

$$F_t(x) \le N\varphi^{-1}\left(\frac{V_{\Lambda_\varphi}(f)}{\sum_{k=1}^{N}\lambda_k^{-1}}\right) \le 2N\varphi^{-1}\left(\frac{V_{\Lambda_\varphi}(f)}{\int_0^N (\mathrm{d}s/\lambda(s))}\right). \tag{46}$$

Substituting (46) into (44), we get

$$\omega(f;t)_1 \le 4\pi\varphi^{-1}\left(\frac{V_{\Lambda_\varphi}(f)}{\int_0^N (\mathrm{d}s/\lambda(s))}\right). \tag{47}$$

Since the right of (47) is decreasing with respect to N and $N = [2\pi/h_t] \ge [2\pi/t] \ge 1/t$, the estimate in Theorem 2 is obtained from (47) directly.

Now we show that our estimate is sharp in the sense of order under the assumption that Λ satisfies the condition Δ_2.

We choose

$$a_k = \varphi^{-1}\left(\frac{1}{2\pi\int_0^{n/\pi}(\mathrm{d}s/\lambda(s))}\right), \quad k = 1,\dots,n, \tag{48}$$

and consider function $f_{\mathbf{a}_n}(x)$ defined in Lemma 13.

We have

$$\begin{aligned}\frac{\int_0^n (\mathrm{d}s/\lambda(s))}{4\int_0^n (\mathrm{d}s/\lambda(s/\pi))} &= \frac{\int_0^n (\mathrm{d}s/\lambda(s))}{4\pi\int_0^{n/\pi}(\mathrm{d}s/\lambda(s))} \\ &\le V_{\Lambda_\varphi}(f_{\mathbf{a}_n}) \\ &\le \frac{\int_0^n (\mathrm{d}s/\lambda(s))}{\pi\int_0^{n/\pi}(\mathrm{d}s/\lambda(s))} \\ &= \frac{\int_0^n (\mathrm{d}s/\lambda(s))}{\int_0^n (\mathrm{d}s/\lambda(s/\pi))}, \quad n \ge 10,\end{aligned} \tag{49}$$

$$\omega\left(f_{\mathbf{a}_n};\frac{\pi}{n}\right)_1 = 2\pi\varphi^{-1}\left(\frac{1}{2\pi\int_0^{n/\pi}(\mathrm{d}s/\lambda(s))}\right), \quad n \ge 10. \tag{50}$$

If Λ satisfies the condition Δ_2, that is, there exists $a > 0$ such that $\lambda(2x) \le a\lambda(x)$, $x > 0$, then

$$\lambda\left(\frac{s}{\pi}\right) \le \lambda(s) = \lambda\left(\pi\cdot\frac{s}{\pi}\right) \le \lambda\left(4\cdot\frac{s}{\pi}\right) \le a^2\lambda\left(\frac{s}{\pi}\right), \tag{51}$$

and this yields

$$\int_0^n \frac{\mathrm{d}s}{\lambda(s)} \le \int_0^n \frac{\mathrm{d}s}{\lambda(s/\pi)} \le a^2\int_0^n \frac{\mathrm{d}s}{\lambda(s)}. \tag{52}$$

From (49) and (52), we have

$$0 < \frac{1}{4a^2} \le V_{\Lambda_\varphi}(f_{\mathbf{a}_n}) \le 1, \quad n \ge 10. \tag{53}$$

On the other hand, (50), the concavity and the monotonicity of φ^{-1} imply that

$$\begin{aligned}\omega\left(f_{\mathbf{a}_n};\frac{\pi}{n}\right)_1 &\ge \varphi^{-1}\left(\frac{1}{\int_0^{n/\pi}(\mathrm{d}s/\lambda(s))}\right) \\ &\ge \varphi^{-1}\left(\frac{V_{\Lambda_\varphi}(f_{\mathbf{a}_n})}{\int_0^{n/\pi}(\mathrm{d}s/\lambda(s))}\right), \quad n \ge 10.\end{aligned} \tag{54}$$

Obviously, (53) and (54) mean the sharpness of our estimate in Theorem 2. □

Proof of Corollary 4. Obviously the Λ's in Corollary 4 satisfy the condition Δ_2. The proofs of (a), (b), (c), and (e) in Corollary 4 are obvious. In (d), we have

$$\begin{aligned}&\lambda(s) = \ln^\gamma(s+1), \quad s \ge 1, \\ &\int_0^z \frac{\mathrm{d}s}{\lambda(s)} \asymp \frac{z}{\ln^\gamma z}, \quad z \longrightarrow +\infty,\end{aligned} \tag{55}$$

and in (f), we have

$$\begin{aligned}&\lambda(s) = (s+1)^\alpha\ln^\gamma(s+1), \quad s \ge 1, \\ &\int_0^z \frac{\mathrm{d}s}{\lambda(s)} \asymp \frac{z^{1-\alpha}}{\ln^\gamma z}, \quad z \longrightarrow +\infty,\end{aligned} \tag{56}$$

which complete the proofs of (d) and (f) of Corollary 4. □

3. Proof of Theorem 6

In this section we prove Theorem 6. The proof of Theorem 6(a) is based on Hölder's inequality and Lemma 12. We use techniques used by Li and Wang [9] in the proofs of (b) and (c) of Theorem 6. Lemma 12 plays a crucial role in the whole proof of Theorem 6.

Proof of Theorem 6. As in the proof of Theorem 2, it is easily seen from Lemma 11 that, for $1 \le p < \infty$, there exists an $h_t \in (0,t]$ such that

$$\omega(f;t)_p^p = \frac{1}{N}\int_0^{2\pi} F_t(x)\,\mathrm{d}x, \tag{57}$$

where

$$\begin{aligned}&F_t(x) = \sum_{k=1}^{N}\left| f(x+kh_t) - f(x+(k-1)h_t)\right|^p, \\ &N = \left[\frac{2\pi}{h_t}\right].\end{aligned} \tag{58}$$

We first prove (a). We note that $r = \beta/p > 1$ for $1 \le p < \beta$. Let $s > 0$ satisfy $1/r + 1/s = 1$. By Hölder's inequality and Lemma 12, we have

$$\begin{aligned} F_t(x) &\le \left(\sum_{k=1}^{N} \left|f(x+kh_t) - f(x+(k-1)h_t)\right|^{\beta}\right)^{1/r} \left(\sum_{k=1}^{N} 1^s\right)^{1/s} \\ &\le \left(\frac{NV_{\Lambda_\beta}(f)}{\sum_{k=1}^{N}\lambda_k^{-1}}\right)^{1/r} N^{1/s} = N\left(\frac{V_{\Lambda_\beta}(f)}{\sum_{k=1}^{N}\lambda_k^{-1}}\right)^{p/\beta}. \end{aligned} \tag{59}$$

For $p = \beta$, Lemma 12 directly yields

$$F_t(x) \le N\left(\frac{V_{\Lambda_\beta}(f)}{\sum_{k=1}^{N}\lambda_k^{-1}}\right). \tag{60}$$

Thus, for $1 \le p \le \beta$, we obtain

$$F_t(x) \le N\left(\frac{V_{\Lambda_\beta}(f)}{\sum_{k=1}^{N}\lambda_k^{-1}}\right)^{p/\beta}. \tag{61}$$

Substituting (61) into (57) and noting that $N = [2\pi/h_t] \ge [2\pi/t] \ge 1/t$, we prove the desired estimate in Theorem 6(a).

If Λ satisfies the condition Δ_2, then from the proof of Theorem 2, we know that

$$\int_0^n \frac{ds}{\lambda(s)} \asymp \int_0^{n/\pi} \frac{ds}{\lambda(s)}, \quad n \longrightarrow \infty. \tag{62}$$

If we choose

$$\begin{aligned} a_k &= \varepsilon_n^{1/\beta}\left(\frac{1}{\int_0^{n/\pi}(ds/\lambda(s))}\right)^{1/\beta}, \quad k = 1, 2, \ldots, n, \\ \text{where } \varepsilon_n &= \frac{\int_0^{n/\pi}(ds/\lambda(s))}{2\int_0^{n}(ds/\lambda(s))} \asymp 1, \quad n \longrightarrow +\infty, \end{aligned} \tag{63}$$

and consider the functions $f_{\mathbf{a}_n}(x)$ defined in Lemma 13, then we have

$$\frac{1}{4} \le V_{\Lambda_\beta}\left(f_{\mathbf{a}_n}\right) \le 1, \quad n \ge 10, \tag{64}$$

$$\begin{aligned} \omega\left(f_{\mathbf{a}_n}; \frac{\pi}{n}\right)_p^p &= 2\pi\varepsilon_n^{p/\beta}\left(\frac{1}{\int_0^{\pi/n}(ds/\lambda(s))}\right)^{p/\beta} \\ &\ge c\left(\frac{V_{\Lambda_\beta}(f_{\mathbf{a}_n})}{\int_0^{\pi/n}(ds/\lambda(s))}\right)^{p/\beta}, \quad n \ge 10. \end{aligned} \tag{65}$$

Equations (64) and (65) show that the estimate in Theorem 6(a) is sharp.

Now we prove (b). Without loss of generality, we assume that $\lambda_1 = 1$ and denote $\psi(x) = x^{\beta}$ $(\beta \ge 1)$ and $\psi^{-1}(x) = x^{1/\beta}$. From the definition of $V_{\Lambda_\beta}(f)$, we first have

$$\left|f(x+kh_t) - f(x+(k-1)h_t)\right| \le \psi^{-1}\left(V_{\Lambda_\beta}(f)\right). \tag{66}$$

Denote by σ_m $(m \ge 0)$ the set of integers k $(1 \le k \le N)$ for which

$$\begin{aligned} 2^{-m-1}\psi^{-1}\left(V_{\Lambda_\beta}(f)\right) &< \left|f(x+kh_t) - f(x+(k-1)h_t)\right| \\ &\le 2^{-m}\psi^{-1}\left(V_{\Lambda_\beta}(f)\right). \end{aligned} \tag{67}$$

Then $\sum_{m=0}^{\infty}|\sigma_m| = N$, and there are at most N nonempty σ_m, where $|\sigma_m|$ denotes the number of the elements in σ_m. Obviously $|\sigma_m| \le N$, $m \ge 0$.

Hence we have

$$\begin{aligned} F_t(x) &= \sum_{m=0}^{\infty}\sum_{k\in\sigma_m}\left|f(x+kh_t) - f(x+(k-1)h_t)\right|^p \\ &\le \left(\psi^{-1}\left(V_{\Lambda_\beta}(f)\right)\right)^p \sum_{m=0}^{\infty} 2^{-mp}|\sigma_m| \\ &= \left(V_{\Lambda_\beta}(f)\right)^{p/\beta}\left\{\sum_{m\le M} 2^{-mp}|\sigma_m| + \sum_{m>M} 2^{-mp}|\sigma_m|\right\} \\ &=: \left(V_{\Lambda_\beta}(f)\right)^{p/\beta}(A_1 + A_2), \end{aligned} \tag{68}$$

with $M > 0$ to be determined.

From (67) and Lemma 12, we have, for $|\sigma_m| \ne 0$,

$$\begin{aligned} 2^{-m-1}\psi^{-1}\left(V_{\Lambda_\beta}(f)\right)|\sigma_m| &\le \sum_{k\in\sigma_m}\left|f(x+kh_t) - f(x+(k-1)h_t)\right| \\ &\le |\sigma_m|\,\psi^{-1}\left(\frac{V_{\Lambda_\beta}(f)}{\sum_{i=1}^{|\sigma_m|}\lambda_i^{-1}}\right), \end{aligned} \tag{69}$$

and thus

$$\sum_{i=1}^{|\sigma_m|}\frac{1}{\lambda_i} \le \frac{V_{\Lambda_\beta}(f)}{\psi\left(2^{-m-1}\psi^{-1}\left(V_{\Lambda_\beta}(f)\right)\right)} = 2^{\beta(m+1)}. \tag{70}$$

Therefore,

$$\phi\left(|\sigma_m|\right) = \int_0^{|\sigma_m|}\frac{ds}{\lambda(s)} \le 2\sum_{i=1}^{|\sigma_m|}\frac{1}{\lambda_i} \le 2^{\beta(m+1)+1}. \tag{71}$$

If we set $z_m = \phi^{-1}(2^{\beta(m+1)+1})$, where ϕ^{-1} is the inverse function of ϕ, then

$$|\sigma_m| \le z_m, \qquad 2^{-m} = \frac{2^{1+1/\beta}}{\phi(z_m)^{1/\beta}}. \tag{72}$$

From the monotonicity of $\lambda(x)$, it is easily verified that $\phi(2z) \le 2\phi(z)$. And from this we also have

$$\phi\left(2z_{m-1}\right) \le 2\phi\left(z_{m-1}\right) = 2^{1-\beta}\phi\left(z_m\right) \le \phi\left(z_m\right), \quad m \ge 1. \tag{73}$$

This yields

$$2z_{m-1} \le z_m, \quad z_m \le 2\left(z_m - z_{m-1}\right), \quad m \ge 1. \tag{74}$$

Now we estimate A_1 and A_2 in (68).

By means of (72) and (74), we first have

$$\begin{aligned} A_1 &\le \sum_{m\le M} 2^{(2+1/\beta)p} \frac{z_m}{\phi(z_m)^{p/\beta}} \\ &\le 2^{(2+1/\beta)p+1} \sum_{m\le M} \frac{z_m - z_{m-1}}{\phi(z_m)^{p/\beta}} \\ &\le c_p \int_0^{z_M} \frac{\mathrm{d}z}{\phi(z)^{p/\beta}}. \end{aligned} \tag{75}$$

If we choose $M = [\log_2^{\phi(N)/2}/\beta] - 1$, then $z_M \le N \le z_{M+1}$.

From (72) and the monotonicity of $\phi(z)$, we have

$$\begin{aligned} A_2 &\le N \sum_{m>M} 2^{-pm} \\ &= N \cdot \frac{2^{-p(M+1)}}{1-2^{-p}} \\ &\le 2^{(1+1/\beta)p+1} \frac{N}{\phi(z_{M+1})^{p/\beta}} \\ &\le c_p \frac{N}{\phi(N)^{p/\beta}}. \end{aligned} \tag{76}$$

Inserting (75) and (76) into (68) and noting that $\int_0^N (\mathrm{d}z/\phi(z)^{p/\beta}) \ge (N/\phi(N)^{p/\beta})$, for $x \in [0, 2\pi]$, we get

$$\begin{aligned} F_t(x) &\le c_3 \Big(V_{\Lambda_\beta}(f)\Big)^{p/\beta} \left[\int_0^N \frac{\mathrm{d}z}{\phi(z)^{p/\beta}} + \frac{N}{\phi(N)^{p/\beta}}\right] \\ &\le 2c_p \Big(V_{\Lambda_\beta}(f)\Big)^{p/\beta} \int_0^N \frac{\mathrm{d}z}{\phi(z)^{p/\beta}}. \end{aligned} \tag{77}$$

Combining (77) and (57), we finally obtain

$$\omega(f;t)_p^p \le c_p \Big(V_{\Lambda_\beta}(f)\Big)^{p/\beta} \frac{1}{N} \int_0^N \frac{\mathrm{d}z}{\phi(z)^{p/\beta}}. \tag{78}$$

Since $N = [2\pi/h_t] \ge [2\pi/t] \ge 1/t$ and the right of (78) is decreasing with respect to N, the proof of Theorem 6(b) is complete.

Finally we prove Theorem 6(c). Notice that Λ is nondecreasing. It follows from (67) that

$$\begin{aligned} &F_t(x) \\ &= \sum_{m=0}^{\infty} \sum_{k\in\sigma_m} \left|f\left(x+kh_t\right) - f\left(x+(k-1)h_t\right)\right|^{p-\beta} \\ &\qquad \times \left|f\left(x+kh_t\right) - f\left(x+(k-1)h_t\right)\right|^{\beta} \\ &\le \Big(V_{\Lambda_\beta}(f)\Big)^{(p-\beta)/\beta} \\ &\qquad \times \sum_{m=0}^{\infty} 2^{-m(p-\beta)} \sum_{k\in\sigma_m} \Big|f\left(x+kh_t\right) \\ &\qquad\qquad - f\left(x+(k-1)h_t\right)\Big|^{\beta} \\ &\le \Big(V_{\Lambda_\beta}(f)\Big)^{(p-\beta)/\beta} \sum_{m=0}^{\infty} 2^{-m(p-\beta)} \lambda_{k_m} \Omega_m, \end{aligned} \tag{79}$$

where

$$\begin{aligned} k_m &= \left|\sigma_0\right| + \left|\sigma_1\right| + \cdots + \left|\sigma_m\right|, \\ \Omega_m &= \sum_{j=k_{m-1}+1}^{k_m} \frac{\left|f\left(x+\sigma_m(j)h_t\right) - f\left(x+\left(\sigma_m(j)-1\right)h_t\right)\right|^{\beta}}{\lambda_j}, \end{aligned} \tag{80}$$

and $\sigma_m(j) \in \sigma_m$, $k_{-1} = 0$. It is obvious that $\sum_{m=0}^{\infty} \Omega_m \le V_{\Lambda_\beta}(f)$.

By (72) and (74), we know that

$$k_m \le \sum_{j=0}^{m} z_j \le \sum_{j=1}^{m} 2\left(z_j - z_{j-1}\right) + z_0 \le 2z_m. \tag{81}$$

Assume that $\phi(z)^{(\beta-p)/\beta}\lambda(z)$ is bounded on $[1,\infty)$. From (72) and the fact $\phi(2z_m) \le 2\phi(z_m)$, $m \ge 0$, we obtain

$$\begin{aligned} 2^{-m(p-\beta)}\lambda_{k_m} &\le \frac{2^{(1+1/\beta)(p-\beta)}\lambda\left(2z_m\right)}{\phi(z_m)^{(p-\beta)/\beta}} \\ &\le c\phi(2z_m)^{(\beta-p)/\beta}\lambda\left(2z_m\right) \le c, \quad m \ge 0. \end{aligned} \tag{82}$$

Finally, (79) yields

$$\begin{aligned} F_t(x) &\le c\Big(V_{\Lambda_\beta}(f)\Big)^{(p-\beta)/\beta} \sum_{m=0}^{\infty} \Omega_m \\ &\le c\Big(V_{\Lambda_\beta}(f)\Big)^{p/\beta}, \quad x \in [0, 2\pi]. \end{aligned} \tag{83}$$

Since $N \ge 1/t$, we prove Theorem 6(c) from (57). □

4. Proofs of Theorems 7 and 8

In this section, we prove Theorems 7 and 8.

Proof of Theorem 7. Theorem 7(a) implies Theorem 7(b) by letting $\alpha = 0$. We only need to prove Theorem 7(a). Obviously $\Lambda = \{n^\alpha\}$ $(0 \le \alpha < 1)$ satisfies the condition Δ_2 and we have

$$\lambda(z) = z^\alpha, \quad z \ge 1,$$
$$\phi(z) = \int_0^z \frac{ds}{\lambda(s)} \asymp z^{1-\alpha}, \quad z \ge 1. \tag{84}$$

For $1 \le p \le \beta$, it follows from Theorem 6(a) that $\omega(f;t)_p \le c(V_{\Lambda_\beta}(f))^{1/\beta} t^{(1-\alpha)/\beta}$, and the order of this estimate is sharp.

For $\beta < p < \beta/(1-\alpha)$, $1-(p/\beta)(1-\alpha) > 0$, we have

$$\int_0^N \frac{dz}{\phi(z)^{p/\beta}} \asymp \int_1^N \frac{dz}{z^{p(1-\alpha)/\beta}} \asymp N^{1-(p/\beta)(1-\alpha)}, \quad N \longrightarrow \infty. \tag{85}$$

Substituting into Theorem 6(b), we get $\omega(f;t)_p \le c(V_{\Lambda_\beta}(f))^{1/\beta} t^{(1-\alpha)/\beta}$.

For $\beta/(1-\alpha) \le p < \infty$, $\alpha + (1-\alpha)(\beta-p)/\beta \le 0$, we have

$$\phi(z)^{(\beta-p)/\beta}\lambda(z) \le c_1 z^{\alpha+(1-\alpha)(\beta-p)/\beta} \le C < \infty, \quad z \ge 1. \tag{86}$$

By Theorem 6(c), we get the estimate

$$\omega(f;t)_p \le \left(V_{\Lambda_\beta}(f)\right)^{1/\beta} t^{1/p}. \tag{87}$$

If $\Lambda = \{n\}$, then Λ satisfies the condition Δ_2, and

$$\lambda(z) = z, \quad z \ge 1;$$
$$\phi(z) = \int_0^z \frac{ds}{\lambda(s)} \asymp \ln(z+1) \asymp \ln z, \quad z \ge 1. \tag{88}$$

For $1 \le p \le \beta$, (ii) follows from Theorem 6(a) and the order in the estimate (ii) is also sharp.

For $\beta < p < \infty$, we have

$$\frac{1}{N}\int_0^N \frac{dz}{\phi(z)^{p/\beta}} \asymp \frac{1}{N}\int_1^N \frac{dz}{\ln(z+1)^{p/\beta}} \asymp \frac{1}{(\ln z)^{p/\beta}}, \quad N \longrightarrow \infty. \tag{89}$$

Substituting into Theorem 6(b), we get (ii).

Now we show the sharpness of the estimates in (i) and (ii) for $\beta < p < \infty$.

For this purpose, we consider $f_{\mathbf{a}_n}(x)$ defined in Lemma 13.

In (i), for $\beta < p < \beta/(1-\alpha)$, we choose

$$a_k = \varepsilon_n^{1/\beta}\left(\frac{\pi}{n}\right)^{(1-\alpha)/\beta}, \quad k = 1,2,\ldots,n,$$
$$\text{where } \varepsilon_n = \frac{n^{1-\alpha}}{2\pi^{1-\alpha}\sum_{k=1}^n k^{-\alpha}} \asymp 1, \quad n \longrightarrow \infty. \tag{90}$$

From Lemma 13, we know that

$$\frac{1}{2} \le V_{\Lambda_\beta}\left(f_{\mathbf{a}_n}\right) \le 1, \quad n \ge 10,$$
$$\omega\left(f_{\mathbf{a}_n}; \frac{\pi}{n}\right)_p^p = 2\pi\varepsilon_n^{p/\beta}\left(\frac{\pi}{n}\right)^{p(1-\alpha)/\beta} \ge c\left(V_{\Lambda_\beta}\left(f_{\mathbf{a}_n}\right)\right)^{p/\beta}\left(\frac{\pi}{n}\right)^{p(1-\alpha)/\beta}, \quad n \ge 10. \tag{91}$$

For $\beta/(1-\alpha) \le p < \infty$, $p(1-\alpha)/\beta \ge 1$, we choose

$$a_k = \varepsilon_n^{1/\beta} k^{-(1-\alpha)/\beta}(\ln(k+1))^{-(1+\sigma)/\beta}\ (\sigma > 0), \quad k = 1,2,\ldots,n, \tag{92}$$

where

$$\varepsilon_n = \left(2\sum_{k=1}^n k^{-1}(\ln(k+1))^{-(1+\sigma)}\right)^{-1} \asymp 1, \quad n \longrightarrow +\infty. \tag{93}$$

From Lemma 13, we have

$$\frac{1}{2} \le V_{\Lambda_\beta}\left(f_{\mathbf{a}_n}\right) \le 1, \quad n \ge 10,$$
$$\omega\left(f_{\mathbf{a}_n}; \frac{\pi}{n}\right)_p^p = \frac{2\pi}{n}\varepsilon_n^{p/\beta}\sum_{k=1}^n k^{-(1-\alpha)p/\beta}(\ln(1+k))^{-(1+\sigma)p/\beta} \ge c\left(V_{\Lambda_\beta}\left(f_{\mathbf{a}_n}\right)\right)^{p/\beta}\frac{\pi}{n}, \quad n \ge 10. \tag{94}$$

In (ii), for $\beta < p < \infty$, we choose

$$a_k = \varepsilon_n^{1/\beta}\left(\frac{n}{\pi}\right)^{1/\beta}, \quad k = 1,2,\ldots,n, \tag{95}$$

where

$$\varepsilon_n = \frac{\ln(n/\pi)}{2\sum_{k=1}^n k^{-1}} \asymp 1, \quad n \longrightarrow +\infty. \tag{96}$$

From Lemma 13, we know that

$$\frac{1}{2} \le V_{\Lambda_\beta}\left(f_{\mathbf{a}_n}\right) \le 1, \quad n \ge 10,$$
$$\omega\left(f_{\mathbf{a}_n}; \frac{\pi}{n}\right)_p^p = 2\pi\varepsilon_n^{p/\beta}\left(\ln\left(\frac{n}{\pi}\right)\right)^{-p/\beta} \ge c\left(V_{\Lambda_\beta}\left(f_{\mathbf{a}_n}\right)\right)^{-p/\beta}\left(\ln\frac{n}{\pi}\right)^{-p/\beta}, \quad n \ge 10. \tag{97}$$

Obviously the above functions $f_{\mathbf{a}_n}$ chosen prove the sharpness of our estimates in Theorem 7 for $\beta < p < \infty$. □

The proof of Theorem 8 is similar to that of Theorem 7, but computations are more complicated.

Proof of Theorem 8. Let $\lambda(s)$ generate Λ and $\phi(z) = \int_0^z (ds/\lambda(s))$. It is readily seen that Λ's in (a), (b), (c), (d), and (e) satisfy the condition Δ_2.

In (a), we have

$$\lambda(s) = \ln^{\gamma}(s+1) \quad (\gamma > 0), \; s \geq 1,$$
$$\phi(z) \asymp \frac{z+1}{\ln^{\gamma}(z+1)} \asymp \frac{z}{\ln^{\gamma} z}, \quad z \geq 2. \tag{98}$$

In (b), we have

$$\lambda(s) = (s+1)\ln(s+1), \quad s \geq 1,$$
$$\phi(z) \asymp \ln\ln(z+1) \asymp \ln\ln z, \quad z \geq 10. \tag{99}$$

In (c), we have

$$\lambda(s) = (s+1)\ln^{\gamma}(s+1) \quad (-\infty < \gamma < 1), \; s \geq 1,$$
$$\phi(z) \asymp (\ln(z+1))^{1-\gamma} \asymp (\ln z)^{1-\gamma}, \quad z \geq 2. \tag{100}$$

In (d), we have

$$\lambda(s) = \frac{(s+1)^{\alpha}}{\ln^{\gamma}(s+1)} \quad (0 < \alpha < 1, \gamma > 0), \; s \geq 1,$$
$$\phi(z) \asymp (z+1)^{1-\alpha}(\ln^{\gamma}(z+1)) \asymp z^{1-\alpha}\ln^{\gamma} z, \quad z \geq 2. \tag{101}$$

In (e), we have

$$\lambda(s) = (s+1)^{\alpha}\ln^{\gamma}(s+1) \quad (0 < \alpha < 1, \gamma > 0), \; s \geq 2,$$
$$\phi(z) \asymp \frac{(z+1)^{1-\alpha}}{\ln^{\gamma}(z+1)} \asymp \frac{z^{1-\alpha}}{\ln^{\gamma} z}, \quad z \geq 2. \tag{102}$$

From Theorem 6(a) we obtain sharp estimates in (a), (b), (c), (d), and (e) for $1 \leq p \leq \beta$.

Now we prove the estimates in Theorem 8 for $\beta < p < \infty$. We apply (b) and (c) in Theorem 6 for upper estimates. The technique used here to show the sharpness is the same as that of the proof of Theorem 7 and the key is to choose $\mathbf{a}_n$ to construct extreme functions $f_{\mathbf{a}_n}(x)$ defined in Lemma 13.

In (a), for $\beta < p < \infty$, we have

$$\phi(z)^{(\beta-p)/\beta}\lambda(z) \asymp \frac{(\ln(z+1))^{p\gamma/\beta}}{(z+1)^{(p-\beta)/\beta}} < +\infty, \quad z \geq 1. \tag{103}$$

From Theorem 6(b), we obtain

$$\omega(f;t)_p \leq c\left(V_{\Lambda_\beta}(f)\right)^{1/\beta} t^{1/p}. \tag{104}$$

If we choose

$$a_k = \varepsilon_n^{1/\beta}(k+1)^{-1/\beta}(\ln(k+1))^{-(1+\gamma+\sigma)/\beta}$$
$$(\sigma > 0), \quad k = 1, 2, \ldots, n, \tag{105}$$

where

$$\varepsilon_n = \left(2\sum_{k=1}^{n}(k+1)^{-1}(\ln(k+1))^{-(1+\sigma)}\right)^{-1} \asymp 1, \quad n \longrightarrow +\infty, \tag{106}$$

then

$$\frac{1}{2} \leq V_{\Lambda_\beta}\left(f_{\mathbf{a}_n}\right) \leq 1, \quad n \geq 10, \tag{107}$$

$$\omega\left(f_{\mathbf{a}_n}; \frac{\pi}{n}\right)_p^p = 2\varepsilon_n^{p/\beta}\sum_{k=1}^{n}\left((k+1)^{-p/\beta}(\ln(k+1))^{-(p(1+\sigma)/\beta)}\right)$$
$$\cdot \frac{\pi}{n} \geq c\left(V_{\Lambda_\beta}\left(f_{\mathbf{a}_n}\right)\right)^{p/\beta} \cdot \frac{\pi}{n}, \quad n \geq 10. \tag{108}$$

Equations (107) and (108) imply the sharpness of (104).

In (b), we have

$$\frac{1}{N}\int_0^N \frac{dz}{\phi(z)^{p/\beta}} \asymp \frac{1}{N}\int_1^N \frac{dz}{(\ln\ln(z+1))^{p/\beta}}$$
$$\asymp \frac{1}{(\ln\ln N)^{p/\beta}}, \quad N \longrightarrow +\infty. \tag{109}$$

Again from Theorem 6(b), we have

$$\omega(f;t)_p \leq c\left(V_{\Lambda_\beta}(f)\right)^{1/\beta}\left(\ln\ln\frac{1}{t}\right)^{-1/\beta}. \tag{110}$$

If we choose

$$a_k = \varepsilon_n^{1/\beta}\left(\ln\ln\frac{n}{\pi}\right)^{-1/\beta}, \quad k = 1, 2, \ldots, n, \tag{111}$$

where

$$\varepsilon_n = \frac{\ln\ln(n/\pi)}{2\sum_{k=1}^{n}\left[(k+1)(\ln(k+1))\right]^{-1}} \asymp 1, \quad n \longrightarrow +\infty, \tag{112}$$

then

$$\frac{1}{2} \leq V_{\Lambda_\beta}\left(f_{\mathbf{a}_n}\right) \leq 1, \quad n \geq 10, \tag{113}$$

$$\omega\left(f_{\mathbf{a}_n}; \frac{\pi}{n}\right)_p^p = \frac{2\pi}{n}\varepsilon_n^{p/\beta}\sum_{k=1}^{n}\left(\ln\ln\frac{n}{\pi}\right)^{-p/\beta}$$
$$\geq c\left(V_{\Lambda_\beta}\left(f_{\mathbf{a}_n}\right)\right)^{p/\beta}\left(\ln\ln\frac{n}{\pi}\right)^{-p/\beta}, \quad n \geq 10. \tag{114}$$

Equations (113) and (114) imply the sharpness of (110).

In (c), for $\beta < p < \infty$, we have

$$\frac{1}{N}\int_0^N \frac{dz}{\phi(z)^{p/\beta}} \asymp \frac{1}{N}\int_1^N \frac{dz}{(\ln(z+1))^{p(1-\gamma)/\beta}}$$
$$\asymp \frac{1}{(\ln N)^{p(1-\gamma)/\beta}}, \quad N \longrightarrow \infty. \tag{115}$$

From Theorem 6(b), we get

$$\omega(f;t)_p \leq c\left(V_{\Lambda_\beta}(f)\right)^{1/\beta}\left(\ln\frac{1}{t}\right)^{-(1-\gamma)/\beta}. \tag{116}$$

If we choose

$$a_k = \varepsilon_n^{1/\beta}\left(\ln \frac{n}{\pi}\right)^{-(1-\gamma)/\beta}, \quad k = 1, 2, \ldots, n, \tag{117}$$

where

$$\varepsilon_n = \frac{(\ln (n/\pi))^{1-\gamma}}{2\sum_{k=1}^{n}(k+1)^{-1}(\ln (k+1))^{-\gamma}} \asymp 1, \quad n \longrightarrow +\infty, \tag{118}$$

then

$$\frac{1}{2} \le V_{\Lambda_\beta}\left(f_{\mathbf{a}_n}\right) \le 1, \quad n \ge 10, \tag{119}$$

$$\begin{aligned} \omega\left(f_{\mathbf{a}_n}; \frac{\pi}{n}\right)_p^p &= 2\pi\varepsilon_n^{p/\beta}\left(\ln \frac{n}{\pi}\right)^{-p(1-\gamma)/\beta} \\ &\ge c\left(V_{\Lambda_\beta}\left(f_{\mathbf{a}_n}\right)\right)^{p/\beta}\left(\ln \frac{n}{\pi}\right)^{-p(1-\gamma)/\beta}, \quad n \ge 10. \end{aligned} \tag{120}$$

Equations (119) and (120) imply the sharpness of (116).

In (d), for $\beta < p < \beta/(1-\alpha)$, we have

$$\begin{aligned} \frac{1}{N}\int_0^N \frac{\mathrm{d}z}{\phi(z)^{p/\beta}} &\asymp \frac{1}{N}\int_1^N \frac{\mathrm{d}z}{(z+1)^{p(1-\alpha)/\beta}(\ln (z+1))^{p\gamma/\beta}} \\ &\asymp \frac{1}{N^{p(1-\alpha)/\beta}(\ln N)^{p\gamma/\beta}}, \quad N \longrightarrow \infty. \end{aligned} \tag{121}$$

From Theorem 6(b), we get

$$\omega(f;t)_p \le c\left(V_{\Lambda_\beta}(f)\right)^{1/\beta} t^{(1-\alpha)/\beta}\left(\ln \frac{1}{t}\right)^{-\gamma/\beta}. \tag{122}$$

If we choose

$$a_k = \varepsilon_n^{1/\beta}\left(\frac{\pi}{n}\right)^{(1-\alpha)/\beta}\left(\ln \frac{n}{\pi}\right)^{-\gamma/\beta}, \quad k = 1, 2, \ldots, n, \tag{123}$$

where

$$\varepsilon_n = \frac{(n/\pi)^{1-\alpha}(\ln (n/\pi))^{\gamma}}{2\sum_{k=1}^{n}(k+1)^{-\alpha}\ln^{\gamma}(k+1)} \asymp 1, \quad n \longrightarrow +\infty, \tag{124}$$

then

$$\frac{1}{2} \le V_{\Lambda_\beta}\left(f_{\mathbf{a}_n}\right) \le 1, \quad n \ge 10, \tag{125}$$

$$\begin{aligned} \omega\left(f_{\mathbf{a}_n}; \frac{\pi}{n}\right)_p^p &= 2\pi\varepsilon_n^{p/\beta}\left(\frac{\pi}{n}\right)^{p(1-\gamma)/\beta}\left(\ln \frac{n}{\pi}\right)^{-p\gamma/\beta} \\ &\ge c\left(V_{\Lambda_\beta}\left(f_{\mathbf{a}_n}\right)\right)^{p/\beta}\left(\frac{\pi}{n}\right)^{p(1-\gamma)/\beta}\left(\ln \frac{n}{\pi}\right)^{-p\gamma/\beta}, \\ &\qquad n \ge 10. \end{aligned} \tag{126}$$

Equations (125) and (126) imply the sharpness of (122).

For $\beta/(1-\alpha) \le p < \infty$, $\beta - p(1-\alpha) \le 0$, we have

$$\phi(z)^{(\beta-p)/\beta}\lambda(z) \asymp \frac{(z+1)^{\beta-p(1-\alpha)/\beta}}{(\ln (z+1))^{p\gamma/\beta}} < +\infty, \quad z \ge 1. \tag{127}$$

From Theorem 6(c), we get

$$\omega(f;t)_p \le c\left(V_{\Lambda_\beta}(f)\right)^{1/\beta} t^{1/p}. \tag{128}$$

If we choose

$$\begin{aligned} a_k = \varepsilon_n^{1/\beta}&(k+1)^{-(1-\alpha+\sigma)/\beta}(\ln(k+1))^{-\gamma/\beta} \\ &(\sigma > 0), \quad k = 1, 2, \ldots, n, \end{aligned} \tag{129}$$

where

$$\varepsilon_n = \left(2\sum_{k=1}^{n}(k+1)^{-(1+\sigma)}\right)^{-1} \asymp 1, \quad n \longrightarrow +\infty, \tag{130}$$

then

$$\frac{1}{2} \le V_{\Lambda_\beta}\left(f_{\mathbf{a}_n}\right) \le 1, \quad n \ge 10, \tag{131}$$

$$\begin{aligned} \omega\left(f_{\mathbf{a}_n}; \frac{\pi}{n}\right)_p^p &= 2\varepsilon_n^{p/\beta}\sum_{k=1}^{n}\left((k+1)^{-(p(1-\alpha+\sigma)/\beta)}(\ln (k+1))^{-p\gamma/\beta}\right) \\ &\cdot \frac{\pi}{n} \ge c\left(V_{\Lambda_\beta}\left(f_{\mathbf{a}_n}\right)\right)^{p/\beta}\cdot\frac{\pi}{n}, \quad n \ge 10. \end{aligned} \tag{132}$$

Equations (131) and (132) imply the sharpness of (128).

In (e), for $\beta < p < \beta/(1-\alpha)$, we have

$$\begin{aligned} \frac{1}{N}\int_0^N \frac{\mathrm{d}z}{\phi(z)^{p/\beta}} &\asymp \frac{1}{N}\int_1^N \frac{(\ln (z+1))^{p\gamma/\beta}}{(z+1)^{p(1-\alpha)/\beta}}\mathrm{d}z \\ &\asymp \frac{(\ln N)^{p\gamma/\beta}}{N^{p(1-\alpha)/\beta}}, \quad N \longrightarrow \infty. \end{aligned} \tag{133}$$

From Theorem 6(b), we get

$$\omega(f;t)_p \le c\left(V_{\Lambda_\beta}(f)\right)^{1/\beta} t^{(1-\alpha)/\beta}\left(\ln \frac{1}{t}\right)^{\gamma/\beta}. \tag{134}$$

If we choose

$$a_k = \varepsilon_n^{1/\beta}\left(\frac{\pi}{n}\right)^{(1-\alpha)/\beta}\left(\ln \frac{n}{\pi}\right)^{\gamma/\beta}, \quad k = 1, 2, \ldots, n, \tag{135}$$

where

$$\varepsilon_n = \frac{(n/\pi)^{1-\alpha}(\ln (n/\pi))^{-\gamma}}{2\sum_{k=1}^{n}(k+1)^{-\alpha}(\ln (k+1))^{-\gamma}} \asymp 1 \quad n \longrightarrow +\infty, \tag{136}$$

then

$$\frac{1}{2} \le V_{\Lambda_\beta}\left(f_{\mathbf{a}_n}\right) \le 1, \quad n \ge 10, \tag{137}$$

$$\begin{aligned} \omega\left(f_{\mathbf{a}_n}; \frac{\pi}{n}\right)_p^p &= 2\pi\varepsilon_n^{p/\beta}\left(\frac{n}{\pi}\right)^{p(1-\gamma)/\beta}\left(\ln \frac{n}{\pi}\right)^{p\gamma/\beta} \\ &\ge c\left(V_{\Lambda_\beta}\left(f_{\mathbf{a}_n}\right)\right)^{p/\beta}\left(\frac{n}{\pi}\right)^{p(1-\gamma)/\beta}\left(\ln \frac{n}{\pi}\right)^{p\gamma/\beta}, \\ &\qquad n \ge 10. \end{aligned} \tag{138}$$

Equations (137) and (138) imply the sharpness of (134).

For $\beta/(1-\alpha) < p < \infty$, $\beta - p(1-\alpha) < 0$, we have

$$\phi(z)^{(\beta-p)/\beta}\lambda(z) \asymp \frac{(\ln(z+1))^{p\gamma/\beta}}{(z+1)^{(p(1-\alpha)-\beta)/\beta}} < +\infty, \quad z \geq 1. \tag{139}$$

From Theorem 6(c), we get

$$\omega(f;t)_p \leq c\left(V_{\Lambda_\beta}(f)\right)^{1/\beta} t^{1/p}. \tag{140}$$

If we choose

$$a_k = \varepsilon_n^{1/\beta}(k+1)^{-(1-\alpha+\sigma)/\beta}(\ln(k+1))^{\gamma/\beta} \quad (\sigma > 0), \quad k = 1, 2, \ldots, n, \tag{141}$$

where

$$\varepsilon_n = \left(2\sum_{k=1}^{n}(k+1)^{-(1+\sigma)}\right)^{-1} \asymp 1, \quad n \longrightarrow +\infty, \tag{142}$$

then

$$\frac{1}{2} \leq V_{\Lambda_\beta}\left(f_{\mathbf{a}_n}\right) \leq 1, \quad n \geq 10, \tag{143}$$

$$\omega\left(f_{\mathbf{a}_n}; \frac{\pi}{n}\right)_p^p = 2\varepsilon_n^{p/\beta}\sum_{k=1}^{n}(k+1)^{-p(1-\alpha+\sigma)/\beta}(\ln(k+1))^{p\gamma/\beta} \cdot \frac{\pi}{n} \geq c\left(V_{\Lambda_\beta}\left(f_{\mathbf{a}_n}\right)\right)^{p/\beta} \cdot \frac{\pi}{n}, \quad n \geq 10. \tag{144}$$

Equations (143) and (144) imply the sharpness of (140).

But, for $p = \beta/(1-\alpha)$, we have

$$\phi(z)^{(\beta-p)/\beta}\lambda(z) \asymp (\ln(z+1))^{p\gamma/\beta} \longrightarrow +\infty, \quad \text{as } z \longrightarrow +\infty,$$
$$\frac{1}{N}\int_0^N \frac{\mathrm{d}z}{\phi(z)^{p/\beta}} \asymp \frac{1}{N}\int_1^N \frac{(\ln(z+1))^{p\gamma/\beta}}{z+1}\mathrm{d}z \asymp \frac{(\ln N)^{1+p\gamma/\beta}}{N}, \quad N \longrightarrow +\infty. \tag{145}$$

Theorem 6(c) is not applicable for this case. From Theorem 6(b), we obtain

$$\omega(f;t)_p \leq c\left(V_{\Lambda_\beta}(f;[0,2\pi])\right)^{1/\beta} t^{1/p}\left(\ln\frac{1}{t}\right)^{1/p+\gamma/\beta}. \tag{146}$$

Unfortunately this estimate is not sharp in the sense of order. However, if we choose

$$a_k = \varepsilon_n^{1/\beta}(k+1)^{-(1-\alpha)/\beta}(\ln(k+1))^{-(1-\gamma+\sigma)/\beta} \quad (\sigma > \max\{0, \gamma-\alpha\}), \quad k = 1, 2, \ldots, n, \tag{147}$$

where

$$\varepsilon_n = \left[\sum_{k=1}^{n}(k+1)^{-1}(\ln(k+1))^{-(1+\sigma)}\right]^{-1} \asymp 1, \quad n \longrightarrow +\infty, \tag{148}$$

then we have

$$\frac{1}{2} \leq V_{\Lambda_\beta}\left(f_{\mathbf{a}_n}\right) \leq 1, \quad n \geq 10,$$
$$\omega\left(f_{\mathbf{a}_n}; \frac{\pi}{n}\right)_p^p = 2\varepsilon_n^{p/\beta}\sum_{k=1}^{n}(k+1)^{-1}(\ln(k+1))^{-(1-\gamma+\sigma)/(1-\alpha)} \cdot \frac{\pi}{n} \geq c \cdot \frac{\pi}{n}, \quad n \geq 10. \tag{149}$$

In other words, there exists $f_n \in \Lambda_\beta BV$ such that

$$V_{\Lambda_\beta}(f) \asymp 1, \quad n \longrightarrow +\infty,$$
$$\omega\left(f_n; \frac{\pi}{n}\right)_p \geq c\left(\frac{\pi}{n}\right)^{1/p}, \quad n \geq 10. \tag{150}$$

This exception indicates that our methods used in this paper cannot cover all cases of estimates of L_p $(1 \leq p < \infty)$ modulus of continuity of classes of functions of Λ_β-bounded variation. □

Proof of Corollary 10. Obviously Theorem 2 implies (a) and (b). (c) follows from Theorem 6. Finally, (d) and (e) are obtained from Theorem 7 directly. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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