Mathematical Modeling, Analysis, and Advanced Conirol of Complex Dynamical Systems

Guest Editors: Peng Shi, Hamid Reza Karimi, Xiaojie Su, Ronqni Yang, and Yuxin Zhao


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## Mathematical Problems in Engineering

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## Editorial

# Mathematical Modeling, Analysis, and Advanced Control of Complex Dynamical Systems 

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As most of technological environments or practical systems have a high complexity, complex dynamical systems have become the subject of intensive research in systems and control theory. The complexity of the system leads to severe difficulties that are encountered in the tasks of analyzing the system and designing and implementing control strategies algorithms. Here, the mathematical modeling and advanced control will provide a basis for the design and operation of complex dynamical systems, and these advanced techniques would result in potential and sustainable benefits.

This special issue contains four parts, that is, modeling and model approximation, stability analysis and robust control, filtering and state estimation, and engineering applications of complex dynamical systems. The contents of these parts are summarized as follows.
(1) Modeling and Model Approximation. "Frequency weighted model order reduction technique and error bounds for discrete time systems" by M. Imran et al. proposes a new frequency weighted technique for balanced model reduction of discrete time systems. The proposed technique guarantees stable reduced order models even for the case when two sided weightings are presented. "Modeling and optimal control of a class of warfare hybrid dynamic systems based on Lanchester $(n, 1)$ attrition model" by X. Chen and A. Zhang establishes a class of warfare hybrid dynamic systems by Lanchester equation in a $(n, 1)$ battle. This model can be characterized
by the interaction of continuous-time models (governed by Lanchester equation) and discrete event systems (described by variable tactics).
(2) Stability Analysis and Robust Control. "Stability and $l_{1}$ gain control of positive switched systems with time-varying delays via delta operator approach" by H. R. Karimi et al. investigates the problems of stability and $l_{1}$-gain controller design for positive switched systems with time-varying delays via delta operator approach. The purpose is to design a switching signal and a state feedback controller such that the resulting closed-loop system is exponentially stable with $l_{1}$-gain performance. "Localized and energy-efficient topology control in wireless sensor networks using fuzzy-logic control approaches" by J.-F. Martinez et al. aims at improving the network connectivity and fault-tolerant capability in response to node failures, while taking into account the fact that the control approach has to be localized and energy-efficient. Two fuzzy controllers are proposed in this paper: one is learning-based fuzzy-logic topology control, and the other one is rules-based fuzzy-logic topology control.
(3) Filtering and State Estimation. "Disturbance attraction domain estimation for saturated Markov jump systems with truncated Gaussian process" by K. L. Teo et al. studies the disturbance attraction domain estimation of saturated Markov jump systems with truncated Gaussian process.

The problem of the optimal disturbance attraction domain is solved through searching for the most appropriate auxiliary parameters in the defined domain. "Recursive estimation for dynamical systems with different delay rates sensor network and autocorrelated process noises" by J. Feng solves the recursive estimation problem for a class of uncertain dynamical systems with different delay rates sensor network and autocorrelated process noises. The desired recursive robust estimators including recursive robust filter, predictor, and smoother are proposed by using the orthogonal projection theorem and an innovation analysis approach.
(4) Engineering Applications. "Command filtered adaptive fuzzy neural network backstepping control for marine power system" by X. Zhang and L. Mu designs a novel commandfiltered adaptive fuzzy neural network backstepping control method to retrain chaotic oscillation of marine power system. The main result, command-filtered adaptive fuzzy neural network backstepping control law, is presented for marine power system, and the Lyapunov stability theory is applied to prove that the system can remain closed-loop and asymptotically stable with this proposed controller. "Robust parametric control of spacecraft rendezvous" by D. Gu and Y. Liu proposes a method to design the robust parametric control for autonomous rendezvous of spacecrafts. A novel concise control law for spacecraft rendezvous is given based on eigenstructure assignment and model reference theory. "Attitude analysis and robust adaptive backstepping sliding mode control of spacecrafts orbiting irregular asteroids" by C. Liang and Y. Li investigates attitude stability analysis and robust control algorithms for spacecrafts orbiting irregular asteroids with model uncertainties and external disturbances. "Rotor-flying manipulator: modeling, analysis, and control" by B. Yang et al. conducts the modeling, analysis, and control of the combined system, called rotor-flying multijoint manipulator (RF-MJM). The detailed dynamics model is constructed, and the full-state feedback linear quadratic regulator control problem is solved through obtaining linearized model near steady state.

Of course, the selected issues and papers are not a comprehensive representation of the area of this special issue. Nonetheless, they represent the rich and many-faceted knowledge that we have the pleasure of sharing with the readers.

## Acknowledgments

We would like to express our appreciation to all the authors for their contributions. We also thank all the reviewers for their time and help in assessing all the submissions.

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# Command Filtered Adaptive Fuzzy Neural Network Backstepping Control for Marine Power System 

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#### Abstract

In order to retrain chaotic oscillation of marine power system which is excited by periodic electromagnetism perturbation, a novel command-filtered adaptive fuzzy neural network backstepping control method is designed. First, the mathematical model of marine power system is established based on the two parallel nonlinear model. Then, main results of command-filtered adaptive fuzzy neural network backstepping control law are given. And the Lyapunov stability theory is applied to prove that the system can remain closed-loop asymptotically stable with this controller. Finally, simulation results indicate that the designed controller can suppress chaotic oscillation with fast convergence speed that makes the system return to the equilibrium point quickly; meanwhile, the parameter which induces chaotic oscillation can also be discriminated.


## 1. Introduction

Structure of modern marine power systems has become more complicated, especially the emergence of high-performance ship electric propulsion applications. The capacity of the ship power systems has been significantly improved, in which the reliability and stability of the ship power system made more high demand. In recent years, researchers found that chaotic oscillations appeared in marine power system during the voyage or paroxysmal bursts. Chaotic oscillations could lead to system instability, which poses a potential threat for the safe operation of the marine power grid $[1,2]$. At present, the power system chaos control method is mainly for landbased power system, the idea is generally the mature control methods such as adaptive compensation control, feedback control, and inverse system control, transplanted to control the chaotic system [3-6].

In control theory, backstepping is a technique developed in the 1990s for designing close-loop stabilizing control systems a special class of nonlinear dynamical systems [7]. These systems are built from subsystems that radiate out from an irreducible subsystem that can be stabilized using some other methods. Because of this recursive structure, the designer can start the design process at the known-stable system
and "back-out" new controllers that progressively stabilize each outer subsystem. The process terminates when the final external control is reached. Hence, this process is known as backstepping. So far, backstepping control has made many achievements, like adaptive backstepping control, adaptive sliding mode backstepping control, dynamic surface control, and so forth [8-14].

Recently, fuzzy logic [15-18] and neural networks [14, 19, 20] are increasingly receiving attention in solving complex and practical problems. Although both fuzzy logic and neural networks are universal approximators, there are some differences between them. The former has characteristics of linguistic information and logic control. The latter possesses characteristics of fault tolerance, parallelism, and learning if network training is carefully designed. However, fuzzy logic and neural networks have complementary characteristics. Thus, the development of integrated fuzzy neural networks (FNNs) that possess the merits of both fuzzy logic and neural networks has grown rapidly [21-24].

Based on the aforementioned works, this paper develops an adaptive backstepping control with command-filtered compensation and fuzzy neural network technology for marine power systems. In order to suppress the chaotic marine power system oscillations, based on a mathematical


Figure 1: Block diagram of the two parallel models.
model of two-machine parallel marine power system, command-filtered adaptive fuzzy neural network backstepping chaos controller is designed. Lyapunov stability theory proves that the controlled system can be maintained closedloop asymptotically stable. The rest of this paper is organized as follows. In Section 2, a brief description for two parallel nonlinear mathematical model and fuzzy neural network is given. In Section 3, main results of command-filtered adaptive fuzzy neural network backstepping control technique are developed. In Section 4, simulation results are presented to show the effectiveness of the proposed control technique. Finally, some conclusions are made in Section 5.

## 2. Background

2.1. Two Parallel Nonlinear Mathematical Model. The basic structure of the power supply network of marine power system can be expressed as in Figure 1, where $E_{1} \angle \delta_{1}$ and $E_{2} \angle \delta_{2}$ are emf of two generators in the system, respectively, $x_{d 1}^{\prime}$ and $x_{d 2}^{\prime}$ are synchronous reactance of two generators, respectively, $x_{l}$ and $r_{l}$ are the line resistance and reactance, respectively, and $P$ and $Q$ describe the system load. Because of the short-circuit in the marine power system, the line resistance is very small, which often can be neglected.

Consider same case of generator parameters. Let $\delta=\delta_{1}-$ $\delta_{2}$ and $\omega=\omega_{1}-\omega_{2}$ be relative power angle and relative power angle velocity of the two equivalent generators. Then, twomachine interconnected system can be described as follows:

$$
\begin{gather*}
\frac{d \delta}{d t}=\omega  \tag{1}\\
H \frac{d \omega}{d t}=P_{m}-D \omega-P_{e}(1+\Delta p \cos \beta t) \sin \delta
\end{gather*}
$$

where $H$ and $D$ are equivalent inertia and damping, respectively, $P_{m}$ is the input mechanical power of generator, and $P_{e}$ is the electromagnetic power of system output. $P_{e} \cdot \Delta p \cos \beta t$ is electromagnetic perturbation which is introduced to study chaotic motion for the marine power system under disturbance, where $P_{e} \cdot \Delta p$ describes the amplitude of disturbanc, and $\beta$ describes the frequency of disturbance.

Through the transformation $\tau=t \sqrt{P_{e} / H}, x(\tau)=\delta(t)$, and $y(\tau)=\sqrt{H / P_{e}} \omega(t)$. Equation (1) can be written as

$$
\begin{gather*}
\frac{d x}{d \tau}=y \\
\frac{d y}{d \tau}=-\sin x-\lambda y+\rho+\mu \cos \gamma \tau \sin x \tag{2}
\end{gather*}
$$

where $\lambda=D \sqrt{P_{e} / H}, \rho=P_{m} / P_{e}, \mu=\Delta p$, and $\gamma=\beta \sqrt{P_{e} / H}$. According to transformation, we know that the system state variables $x$ and $y$ were obtained by the transformation of $\delta$ and $\omega$, which have the physical meaning of power angle error and the power angle error relative velocity between the two generators.
2.2. Fuzzy-Neural Network for Approximation. Figure 2 depicts a functional link FNN which consists of fuzzy logic and neural network. The FLS can be divided into two parts: some fuzzy IF-THEN rules and a fuzzy inference engine. The fuzzy inference engine uses the fuzzy IF-THEN rules to perform a mapping form an input linguistic vector $z=$ $\left[z_{1}, \ldots, z_{m}\right]^{T} \in \mathfrak{R}^{m}$ to a scalar output variable $y_{f} \in \mathfrak{R}$. The $i$ th fuzzy IF-THEN rule can be characterized by the following form [20, 21]:

$$
\begin{align*}
& \text { IF } z_{1} \text { is } A_{1}^{i}, \ldots, z_{m} \text { is } A_{m}^{i}  \tag{3}\\
& \text { THEN } y_{f} \text { is } B^{i} \quad(i=1, \ldots, N),
\end{align*}
$$

where $A_{j}^{i}$ and $B^{i}$ are fuzzy sets. By using product inference, center-average, and singleton fuzzifier, $N$ is the total number of rules. Then, the output of the FNN can be expressed as

$$
\begin{equation*}
y_{f}=\frac{\sum_{i=1}^{N} \omega^{i}\left[\prod_{j}^{m} \mu_{A_{j}^{i}}\left(z_{j}\right)\right]}{\sum_{i=1}^{N}\left[\prod_{j}^{m} \mu_{A_{j}^{i}}\left(z_{j}\right)\right]}=W^{T} P(z), \tag{4}
\end{equation*}
$$

where $\mu_{A_{j}^{i}}\left(z_{j}\right)$ is the membership function value of the fuzzy variable, $\omega^{i}$ is the point at which $\mu_{B^{i}}\left(\omega^{i}\right)=1$, and $W=\left[\omega^{1}, \omega^{2}, \ldots, \omega^{N}\right]$ is an adjustable parameter vector. We assume that an upper limit $\|\varepsilon(z)\| \leq \varepsilon_{M}$ of the functional


Figure 2: Functional link a fuzzy-neural network structure.
reconstruction error is known. $P=\left[p^{1}, p^{2}, \ldots, p^{N}\right]$ is a fuzzy basis vector, where $p^{i}$ is defined as

$$
\begin{equation*}
p^{i}(z)=\frac{\prod_{j}^{m} \mu_{A_{j}^{i}}\left(z_{j}\right)}{\sum_{i=1}^{N}\left[\prod_{j}^{m} \mu_{A_{j}^{i}}\left(z_{j}\right)\right]} \tag{5}
\end{equation*}
$$

The truth value $p^{i}$ (layer III) of the antecedent part of the $i$ th implication is calculated by (5). Among the commonly used defuzzification strategies, the output (layer IV) of the FNN is expressed as (4). The fuzzy logic approximator based on the neural network can be established. The approximator has four layers. At layer I, nodes, which are input ones, stand for the input linguistic variables. At layer II, nodes represent the values of the membership function value. At layer III, nodes are the values of the fuzzy basis vector. Each node of layer III performs a fuzzy rule. The links between layer III and layer IV are fully connected by the weighting vector $\omega$, that is, the adjusted parameters. At layer IV, the output stands for the value of $y_{f}$.

## 3. Main Results

When the system is already in chaotic motion, the controller can control the chaotic system for any arbitrary unstable equilibrium points. In this section, command-filtered adaptive fuzzy neural network backstepping controller is designed for the chaotic motion of the marine power system. A control input $u$ is added to the equation of state (2), and the formula (6) can be described as follows:

Also, the system (6) can be described as an affine system with unknown parameter and disturbance by

$$
\begin{gather*}
\frac{d X_{1}}{d \tau}=X_{2}  \tag{7}\\
\frac{d X_{2}}{d \tau}=F(X)+H(x) \mu+u+d(t)
\end{gather*}
$$

where $F(X)=-\sin x-\lambda y+\rho, H(x)=\cos \gamma \tau \sin x$, and $d(t)$ is disturbance. In the following works, the $d(t)$ can be estimated by the fuzzy neural network (4) as follows:

$$
\begin{equation*}
d(t)=W^{T} P(x, y) \tag{8}
\end{equation*}
$$

Then, (7) can be also described as

$$
\begin{gather*}
\frac{d X_{1}}{d \tau}=X_{2} \\
\frac{d X_{2}}{d \tau}=F(X)+\bar{W}^{T} \bar{P}(x, y)+u+d(t) \tag{9}
\end{gather*}
$$

where $\bar{W}=[W, \mu]^{T}$ and $\bar{P}(x, y)=\left[P^{T}(x, y), H(x)\right]^{T}$. Define the state tracking error variables $E_{1}$ and $E_{2}$ that are introduced as follows:

$$
\begin{align*}
& E_{1}=X_{1}-X_{1}^{c} \\
& E_{2}=X_{2}-X_{2}^{c} \tag{10}
\end{align*}
$$

where $X_{1}^{c}$ and $X_{2}^{c}$ are the filtered-command of $X_{1}$ and $X_{2}$, respectively. From (7) and (10), we have

$$
\begin{gather*}
\frac{d E_{1}}{d \tau}=X_{2}-\frac{d X_{1}^{c}}{d \tau}  \tag{11}\\
\frac{d E_{2}}{d \tau}=F(X)+\bar{W}^{T} \bar{P}(x, y)+u-\frac{d X_{2}^{c}}{d \tau} \tag{12}
\end{gather*}
$$

The task is to stabilize (11) with respect to the Lyapunov function:

$$
\begin{equation*}
V_{1}=\frac{1}{2} E_{1}^{T} E_{1}, \tag{13}
\end{equation*}
$$

and the time derivative of $V_{1}$ with respect to time is given by

$$
\begin{equation*}
\frac{d V_{1}^{c}}{d \tau}=E_{1}^{T} \frac{d E_{1}}{d \tau}=E_{1}^{T}\left(X_{2}-\frac{d X_{1}^{c}}{d \tau}\right) \tag{14}
\end{equation*}
$$

The virtual controller can be designed as

$$
\begin{equation*}
X_{2}^{d}=\frac{d X_{1}^{c}}{d \tau}-K_{1} E_{1} \tag{15}
\end{equation*}
$$

where $K_{1}$ is a positive definite matrix to be designed. Substituting (15) into (14), we have $d V_{1}^{c} / d \tau \leq 0$.

To solve the derivative of the virtual control problems, the command filter is used to eliminate the impact of derivative of the virtual item and control saturation. Pass $X_{2}^{d}$ through a second-order filter for obtaining the $d X_{2}^{d} / d \tau$, the secondorder filter can be described as

$$
\begin{gather*}
\frac{d q_{1}}{d \tau}=q_{2} \\
\frac{d q_{2}}{d \tau}=-2 \xi q_{1}-\xi^{2}\left(q_{1}-X_{2}^{d}\right) \tag{16}
\end{gather*}
$$

where $q_{1}=X_{2}^{c}$ and $q_{2}=d X_{2}^{c} / d \tau$. Redefine tracking error $\bar{E}_{1}=E_{1}-\varepsilon$, and design

$$
\begin{equation*}
\frac{d \varepsilon}{d \tau}=-K_{1} \varepsilon+\left(X_{2}^{c}-X_{2}^{d}\right) \tag{17}
\end{equation*}
$$

We choose the Lyapunov function

$$
\begin{equation*}
V_{2}=\frac{1}{2}\left[\bar{E}_{1}^{T} \bar{E}_{1}+E_{2}^{T} E_{2}+\widetilde{\bar{W}}^{T} \Xi_{1} \widetilde{\bar{W}}\right] \tag{18}
\end{equation*}
$$

where $\widetilde{\bar{W}}=\widehat{\bar{W}}-\bar{W}$. Then the time derivative of $V_{2}$ is given by

$$
\begin{equation*}
\frac{d V_{2}}{d \tau}=\bar{E}_{1}^{T} \frac{d \bar{E}_{1}}{d \tau}+E_{2}^{T} \frac{d E_{2}}{d \tau}+\widetilde{\bar{W}}^{T} \Xi_{1} \frac{d \widetilde{\bar{W}}}{d \tau} \tag{19}
\end{equation*}
$$

We design the global control law and the parameter update law for $\mu$ as

$$
\begin{gather*}
u=\frac{d X_{2}^{c}}{d \tau}-F(X)-\bar{E}_{1}-K_{2} E_{2}-\widehat{\bar{W}}^{T} \bar{P}(x, y) \\
\frac{d \widehat{\bar{W}}}{d \tau}=\Xi_{1}^{-1} \bar{P}(x, y)^{T} E_{2} \tag{20}
\end{gather*}
$$



Figure 3: Chaotic attractor under $\mu=1.3$.
where $K_{2}$ is a positive constant to be designed. Substituting (17) and (20) into (19) yields

$$
\begin{align*}
\frac{d V_{2}}{d \tau}= & \bar{E}_{1}^{T}\left(-K_{1} \bar{E}_{1}+E_{2}\right) \\
& +E_{2}^{T}\left(F(X)+u+\bar{W}^{T} \bar{P}(x, y)-\frac{d X_{2}^{c}}{d \tau}\right)-\widetilde{\bar{W}}^{T} \Xi_{1} \frac{d \widehat{\bar{W}}}{d \tau} \\
= & -K_{1} \bar{E}_{1}^{T} \bar{E}_{1} \\
& +\bar{E}_{2}^{T}\left(F(X)+u+\bar{W}^{T} \bar{P}(x, y)+\bar{E}_{1}-\frac{d X_{2}^{c}}{d \tau}\right) \\
& -\widetilde{\bar{W}}^{T} \Xi_{1} \frac{d \widehat{\bar{W}}}{d \tau} \\
= & -K_{1} \bar{E}_{1}^{T} \bar{E}_{1}-K_{2} E_{2}^{T} E_{2}-\widetilde{\bar{W}}^{T}\left(H(X)^{T} E_{2}-\Xi_{1} \frac{d \widehat{\bar{W}}}{d \tau}\right) \\
= & -K_{1} \bar{E}_{1}^{T} \bar{E}_{1}-K_{2} E_{2}^{T} E_{2} \leq 0 . \tag{21}
\end{align*}
$$

## 4. Simulation Results

Simulations were performed in the MATLAB/SIMULINK environment. From numerical analysis of the marine power system's chaotic motion, we can obtain the results that when the amplitude $\mu=0.3$, chaotic behavior will occur in marine power system with $\lambda=0.4, \rho=0.2, \gamma=0.8$. We assume the disturbance $d(t)=0.2 \sin (0.02 t)$. We can obtain the motion state of the marine power system in Figures 3 and 4. Form Figure 3, it can be seen that the system power angle and the angular velocity of the phase diagram of movement are ergodicity, which shows that the system appeared in chaos. The system experiences a similar random but does not attenuation. It can determine that $\mu=1.3$ when the system enters the chaotic motion state.

If the perturbation amplitude $\mu$ is unknown, then chaotic oscillation appears in system at this time. In order to suppress


Figure 4: Timing diagram of power angle $\delta$ and relative power angle velocity $\omega$.


Figure 5: The curve of power angle $\delta$.
chaos, the parameter $\mu$ must be identified. Here, commandfiltered adaptive backstepping controller is used to control the marine power system after 100 s in chaotic state running. The controller parameters are designed as $K_{1}=2, K_{2}=2$, $\Xi_{1}=$ $I_{3 \times 3}$, and $W(0)=[0,0,0]^{T}$. The parameter of filter is designed as $\xi=100$. Figures 5 and 6 show the curve of power angle and the angular velocity of marine power system. And the phase diagram is shown in Figure 7.

It can be seen the results from Figures 5 and 6, before 100 seconds, power angle $\delta$, and relative power angle velocity $\omega$, are in a chaotic state. While the designed controller is added after 100 seconds, system is quickly stabilized. This indicates the proposed control algorithm has a very reliable stabilization ability for the marine power system's chaotic motion.

## 5. Conclusions

We have carried out a systematic study on commandfiltered adaptive fuzzy neural network backstepping control scheme for marine power system. Due to the adaptive fuzzy neural network, backstepping method can adaptively make the convergence of error to origin with external bounded


Figure 6: The curve of power angle velocity $\omega$.


Figure 7: Phase diagram of power angle $\delta$ and relative power angle velocity $\omega$.
disturbance. Therefore, the state error of the ship power system to the original can be asymptotically stable equilibrium point. Simulation results show that the proposed method not only guarantees closed-loop stability of the controlled marine power system, but also identifies well the caused chaotic system parameter.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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# Design of Logic Controllers Thanks to Symbolic Computation of Simultaneously Asserted Boolean Equations 

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#### Abstract

Formal methods can strongly contribute to improve dependability of controllers during design, by providing means to avoid flaws due to designers' omissions or specifications misinterpretations. This paper presents a synthesis method dedicated to logic controllers. Its goal is to obtain the control laws from specifications given in natural language by symbolic computation. The formal framework that underlies this method is the Boolean algebra of $n$-variable switching functions. In this algebra, thanks to relations and theorems presented in this paper, it is possible to formally express logical controllers specifications, to automatically detect inconsistencies in specifications, and to obtain automatically the set of solutions or to choose an optimal solution according to given optimization criteria. The application of this synthesis method to an example allows illustrating its main advantages.


## 1. Introduction

Programmable logic controllers (PLCs) are industrial automation components that receive input signals coming from sensors and send output signals to actuators, in accordance with control laws implemented into a user program (Figure 1). The control algorithms that allow the real time calculation of new output values, according to the current state of the PLC and the observation of new values of inputs, are written in standardized languages, such as ladder diagram (LD), structured text (ST) or instruction list (IL) [1]. A PLC cyclically performs three tasks: inputs reading, program execution, and outputs updating. The period of this task may be constant (periodic scan) or may vary (cyclic scan).

Because of their reliability, even in very severe conditions in terms of temperature, vibrations, electromagnetic perturbations, and so forth, PLCs are frequently used for the control of safety-critical systems (energy production, transport, chemical industry, etc.). In this context, improving the reliability of the user program has been one of the main challenges of the past two decades in the field of automation. Among the different techniques that can be used in this aim [2], formal verification and validation and formal synthesis are the most efficient. Verification is the proof that
the internal semantics of a model is correct, independently from the modeled system. The searched properties of the models are stability, deadlock existence, and so on. The validation determines if the model agrees with the designer's purpose [3]. Efficient validation/verification techniques of PLC programs [4], most often based on model-checking technique, have been proposed by researchers and are now widely used in industry [5], despite problems of state-space explosion that arise when treating large scale systems.

Contrary to verification techniques that aim at proving, after a PLC program has been more or less correctly designed by an expert, that control laws are safe, automatic synthesis methods aim at systematically generating control laws which guarantee by construction the respect of expected safety properties. The avoidance of human errors during the design of controllers is one of the main reasons for which synthesis is a very important subject of research in the field of discrete event systems (DES) since the end of 80 's.

Most part of recent works in this area are still based onto the Supervisory Control Theory (SCT) [6] and are aiming for the synthesis of a supervisor, and not directly to the controller of an automated system. Furthermore, the use of state models (Finite Automata, Petri Nets, etc.) and their composition for the construction of the models of the plant and of


Figure 1: PLC basic principle.
the specifications generates a complexity which remains problematic for the synthesis of a supervisor for complex systems [7]. It is therefore interesting to explore other ways for performing synthesis, such as algebraic approaches. In previous works, we proposed a method specifically developed to get the control laws that can be directly implemented into the controller [8]. We have chosen to synthesize these control laws under the form of recurrent Boolean equations because of the wide possibilities they offer for the formalization of safety requirements and for implementation.

Nevertheless, whatever is the used synthesis method, one of the weak links of the automatic generation of the control laws is the step of formal transcription by the designer (within state models or algebraic expressions) of the informal requirements and safety properties the controller has to satisfy. In the case of SCT, some authors have proposed more or less generic approaches for the construction of the models of the plant [9] or of the specifications [10]. But in any case, the hypothesis that requirements can be inconsistent has never been taken into account. Unfortunately in the framework of industrial collaborations we have been able to verify that it is always the case. In this paper we show how, in consideration of specific hypotheses, it is possible to install a correction loop for helping the designer to formalize these requirements and so improving the synthesis method robustness to the lack of precision of the specifications.

This paper is organized as follows. Some basics of algebraic synthesis given in Sections 2 and 3 recall the main steps of our method. Section 4 presents the mathematical framework of our approach and new results that allow us to accept inconsistencies in specifications. The strategy we developed for making the synthesis more robust to the lack of consistency of the specifications is described in Section 5, thanks to a case study.

## 2. Problem Statement

Figure 2 proposes a generic representation of a DES whose controller has $p$ Boolean inputs $\left(u_{i}\right), q$ Boolean outputs $\left(y_{j}\right)$, and $r$ Boolean state variables $\left(x_{l}\right)$. Plant and controller are connected through a closed loop exchanging inputs and outputs signals. The state variables, needed for expressing sequential behaviors of the controller, are represented by internal variables.


Figure 2: A sequential DES.

The algebraic modeling of the control laws of the controller necessitates the definition of $(q+r)$ switching functions of $(p+r)$ variables. Even if this representation is very compact (the $r$ Boolean state variables allow the representation of $2^{r}$ different states), the construction by hands of these switching functions is a very tedious and error-prone task [11]; the controller of Figure 2 admits $2^{p}$ inputs combinations can send $2^{q}$ outputs combinations and can express $\left(2^{2^{(p+r)}}\right)^{(q+r)}$ sequential behaviors. That is the reason why algebraic modeling approaches have been replaced by methods based on state models since the middle of 50's [12, 13]. Nevertheless, thanks to recent mathematical results obtained onto Boolean algebras [14, 15], the automatic algebraic synthesis of switching functions is now possible.

In [16] an interesting approach for the systematic construction of a reactive program from its formal specification is proposed. In this work, the program synthesis is considered as a theorem proving activity. A program with input $x$ and output $y$, specified by the formula $\varphi(x, y)$, is constructed as a byproduct of proving the theorem $(\forall x)(\exists y) \varphi(x, y)$. The specification $\varphi(x, y)$ characterizes the expected relation between the input $x$ and the output $y$ computed by the program. This approach is based on the observation that
the formula $(\forall x)(\exists y) \varphi(x, y)$ is equivalent to the secondorder formula $(\exists f)(\forall x) \varphi(x, f(x))$, stating the existence of a function $f$, such that $\varphi(x, f(x))$ holds for every $x$.

This approach provides a conceptual framework for the rigorous derivation of a program from its formal specification. It has also been used to synthesize specifications under the form of finite automata from their linear temporal logic (LTL) description [17].

The core of our approach is based on this strategy: we aim at deducing the $(q+r)$ switching functions of $(p+r)$ variables which define the behavior of the controller from a formula $\varphi\left(u_{i}[k], x_{l}[k-1], y_{j}[k], x_{l}[k]\right)$ that holds for every $k$, every $u_{i}[k]$, and every $x_{l}[k-1]$.

To cope with combinatorial explosion, switching functions will be handled through a symbolic representation (and not their truth-tables which contain $2^{(p+r)}$ Boolean values). Each input $u_{i}$ (resp., output $y_{j}$ ) of the controller will be represented by a switching function $U_{i}$ (resp., $Y_{j}$ ). To take into account the recursive aspect of state variables, each state variable $x_{l}$ will be represented by two switching functions: $X_{l}$ (for time $[k]$ ) and ${ }_{p} X_{l}($ for time $[k-1]$ ).

According to this representation, the synthesis of control laws of a logical system from its specification can now be transformed into the search of the solutions to the mathematical problem as follows:

$$
\begin{equation*}
\left(\forall U_{i}\right)\left(\forall_{p} X_{l}\right)\left(\exists Y_{j}\right)\left(\exists X_{l}\right) \varphi\left(U_{i},{ }_{p} X_{l}, Y_{j}, X_{l}\right), \tag{1}
\end{equation*}
$$

where $\left(U_{i},{ }_{p} X_{l}, Y_{j}, X_{l}\right)$ are $(p+q+2 r)$ switching functions of ( $p+r$ ) variables.

## 3. Overview of Our Method

The input data of the proposed method (Figure 3) are unformal functional and safety requirements given by the designer. In practice, these requirements are most often given in a textual form and/or by using technical Taylormade languages (Gantt diagrams, function blocks diagrams, Grafcet, etc.) or imposed standards.

All the steps of our synthesis method are implemented into a prototype software tool developed in Python (Case studies are available online: http://www.lurpa.ens-cachan.fr/226050.kjsp). The first step is the formalization of requirements within an algebraic description; examples are given in Section 5.2. Requirements expressed with a state model can directly be translated into recurrent Boolean equations, thanks to the algorithm proposed by Machado et al. [18]. In case where the knowhow of the designer enables him to build a priori the global form of the solution (or of a part of the whole solution) it is also possible to give fragments of solution as requirements [19].

The second step consists in checking the consistency of the set of requirements by symbolic calculation. The sufficient condition for checking this consistency has been given in [20] but no strategy has been proposed for coping with potential inconsistencies. In this paper we show that thanks to new theorems the causes of these inconsistencies can be pointed out. It is then possible for the designer to fix priority rules


Figure 3: The algebraic synthesis method step by step.
between the concerned requirements that will allow finding, if exist, solutions despite inconsistencies.

The core of the method is the third step, which consists in the synthesis of the control laws. This step is performed by solving the system of equations which represents the set of consistent requirements. The mathematical results we have obtained (Theorem 12 given in Section 4.3), allow finding a parametric expression of the set of solutions.

In the fourth step of the method, a particular solution has to be chosen among the set of solutions. For that, a specific value of each parameter of the general solution has to be fixed. In a previous work [19], we showed how well chosen heuristics can be used for fixing these parameters. In this paper, we show that the choice of a particular solution among the set of solutions can be expressed as an optimization problem. We propose new theorems that allow calculating the maximum and the minimum of a Boolean formula, and we show how optimal solutions can be automatically found. For ergonomic reasons, the synthesized control laws can finally be displayed under the form of a finite automaton [21].

After the mathematical background of the method has been recalled, we are going to show how, in consideration of specific hypotheses, the second step of the method can be improved by a correction loop helping the designer to formalize the requirements and so improving the robustness of our synthesis method to the lack of precision of the specifications. The strategy to find an optimal solution according to given criteria will be also presented.

## 4. Mathematical Foundations

This section is composed of five subsections. Sections 4.1 and 4.2 recall some classical results about Boolean algebras
and the Boolean algebra of $n$-variable switching functions. Section 4.3 presents how to solve Boolean equations. Sections 4.4 and 4.5 present specific results obtained for the algebraic synthesis of control laws.

### 4.1. Boolean Algebra: Typical Feature

Definition 1 (Boolean algebra). (Definition 15.5 of [22]) Let $\mathscr{B}$ be a nonempty set that contains two special elements 0 (the zero element) and 1 (the unity, or one, element) and on which we define two closed binary operations,$+ \cdot$, and an unary operation ${ }^{-}$. Then $(\mathscr{B},+, \cdot,-, 0,1)$ is called a Boolean algebra if the following conditions are satisfied for all $x, y, z \in \mathscr{B}$ :

$$
\begin{align*}
& \text { Commutative Laws: } \\
& \begin{array}{l}
x+y=y+x \\
x \cdot y=y \cdot x \\
\text { Distributive Laws: } \\
x \cdot(y+z)=(x \cdot y)+(x \cdot z) \\
x+(y \cdot z)=(x+y) \cdot(x+z) \\
\text { Identity Laws: } \\
x+0=x \\
x \cdot 1=x \\
\text { Inverse Laws: } \\
x+\bar{x}=1 \\
x \cdot \bar{x}=0 \\
0 \neq 1
\end{array}
\end{align*}
$$

Many Boolean algebras could be defined. The most known are the two-element Boolean algebra: $(\{0,1\}, \vee, \wedge, \neg$, $0,1)$ and the algebra of classes (set of subsets of a set $S$ ): $\left(2^{S}, \cup, \cap,-, \emptyset, S\right)$.

Definition 2 (Boolean formula). (From Section 3.6 of [15]) A Boolean formula (or a Boolean expression) on $\mathscr{B}$ is any formula which represents a combination of members of $\mathscr{B}$ by the operations + , $\cdot$, or ${ }^{-}$.

By construction, any Boolean formula on $\mathscr{B}$ represents one and only one member of $\mathscr{B}$. Two Boolean formulae are equivalent if and only if they represent the same member of $\mathscr{B}$. Later on, a Boolean formula $\mathscr{F}$ built with the members $\left(\alpha_{1}, \ldots, \alpha_{n}\right)$ of $\mathscr{B}$ is denoted $\mathscr{F}\left(\alpha_{1}, \ldots, \alpha_{n}\right)$.

Theorem 3 (Boole's expansion of a Boolean formula). Let $\left(\alpha_{1}, \ldots, \alpha_{n}\right)$ be $n$ members of $\mathscr{B} \backslash\{0,1\}$. Any Boolean Formula $\mathscr{F}\left(\alpha_{1}, \ldots, \alpha_{n}\right)$ can be expanded as

$$
\begin{equation*}
\mathscr{F}\left(\alpha_{1}, \ldots, \alpha_{n}\right)=\mathscr{F}_{0}\left(\alpha_{2}, \ldots, \alpha_{n}\right) \cdot \bar{\alpha}_{1}+\mathscr{F}_{1}\left(\alpha_{2}, \ldots, \alpha_{n}\right) \cdot \alpha_{1}, \tag{3}
\end{equation*}
$$

where $\mathscr{F}_{0}\left(\alpha_{2}, \ldots, \alpha_{n}\right)$ and $\mathscr{F}_{1}\left(\alpha_{2}, \ldots, \alpha_{n}\right)$ are Boolean formulae of only $\alpha_{2}, \ldots, \alpha_{n}$. These two formulae can be directly obtained from $\mathscr{F}\left(\alpha_{1}, \ldots, \alpha_{n}\right)$ as follows:

$$
\begin{align*}
& \mathscr{F}_{0}\left(\alpha_{2}, \ldots, \alpha_{n}\right)=\left.\mathscr{F}\left(\alpha_{1}, \ldots, \alpha_{n}\right)\right|_{\alpha_{1} \leftarrow 0}=\mathscr{F}\left(0, \alpha_{2}, \ldots, \alpha_{n}\right) \\
& \mathscr{F}_{1}\left(\alpha_{2}, \ldots, \alpha_{n}\right)=\left.\mathscr{F}\left(\alpha_{1}, \ldots, \alpha_{n}\right)\right|_{\alpha_{1} \leftarrow 1}=\mathscr{F}\left(1, \alpha_{2}, \ldots, \alpha_{n}\right) . \tag{4}
\end{align*}
$$

The relation equality is not the only defined relation on a Boolean algebra. It is also possible to define a partial order relation between members of $\mathscr{B}$. This relation is called Inclusion-Relation in [15].

Definition 4 (Inclusion-Relation). (Definition 15.6 of [22].) If $x, y \in \mathscr{B}$, define $x \leq y$ if and only if $x \cdot y=x$.

As Relation Inclusion is reflexive ( $x \leq x$ ), antisymmetric (if $x \leq y$ and $y \leq x$, then $x=y$ ), and transitive (if $x \leq y$ and $y \leq z$, then $x \leq z$ ), this relation defines a partial order between members of $\mathscr{B}$ (Theorem 15.4 of [22]).

Since in any Boolean algebra, $x \cdot y=x \Leftrightarrow x \cdot \bar{y}=0$, we also have $x \leq y \Leftrightarrow x \cdot \bar{y}=0$.

Remark 5. For the algebra of classes $\left(2^{S}, \cup, \cap,-, \emptyset, S\right)$, the Inclusion-Relation is the well-known relation $\subseteq$ and we have: $x \subseteq y \Leftrightarrow x \cap y=x$.

Theorem 6 (reduction of a set of relations). (Theorem 5.3.1 of [15].) Any set of simultaneously asserted relations built with the members $\left(\alpha_{1}, \ldots, \alpha_{n}\right)$ of $\mathscr{B}$ can be reduced to a single equivalent relation such as: $\mathscr{F}\left(\alpha_{1}, \ldots, \alpha_{n}\right)=0$.

To obtain this equivalent relation, it is necessary
(i) to rewrite each equality according to

$$
\begin{align*}
& \mathscr{F}_{1}\left(\alpha_{1}, \ldots, \alpha_{n}\right)=\mathscr{F}_{2}\left(\alpha_{1}, \ldots, \alpha_{n}\right) \\
& \Longleftrightarrow \mathscr{F}_{1}\left(\alpha_{1}, \ldots, \alpha_{n}\right) \cdot \overline{\mathscr{F}_{2}\left(\alpha_{1}, \ldots, \alpha_{n}\right)}  \tag{5}\\
& \quad+\overline{\mathscr{F}_{1}\left(\alpha_{1}, \ldots, \alpha_{n}\right)} \cdot \mathscr{F}_{2}\left(\alpha_{1}, \ldots, \alpha_{n}\right)=0
\end{align*}
$$

(ii) to rewrite each inclusion according to

$$
\begin{align*}
& \mathscr{F}_{1}\left(\alpha_{1}, \ldots, \alpha_{n}\right) \leq \mathscr{F}_{2}\left(\alpha_{1}, \ldots, \alpha_{n}\right) \\
& \Longleftrightarrow \mathscr{F}_{1}\left(\alpha_{1}, \ldots, \alpha_{n}\right) \cdot \overline{\mathscr{F}_{2}\left(\alpha_{1}, \ldots, \alpha_{n}\right)}=0, \tag{6}
\end{align*}
$$

(iii) to group together rewritten equalities as follows:

$$
\begin{align*}
& \left\{\begin{array}{l}
\mathscr{F}_{1}\left(\alpha_{1}, \ldots, \alpha_{n}\right)=0 \\
\mathscr{F}_{2}\left(\alpha_{1}, \ldots, \alpha_{n}\right)=0
\end{array}\right.  \tag{7}\\
& \quad \Longleftrightarrow \mathscr{F}_{1}\left(\alpha_{1}, \ldots, \alpha_{n}\right)+\mathscr{F}_{2}\left(\alpha_{1}, \ldots, \alpha_{n}\right)=0 .
\end{align*}
$$

4.2. The Boolean Algebra of n-Variable Switching Functions. To avoid confusion between Boolean variables and Boolean functions of Boolean variables, each Boolean variable $b_{i}$ is denoted by ${ }_{b} b_{i}$. The set of the two Boolean values ${ }_{b} 0$ and ${ }_{b} 1$ is denoted by $B=\left\{{ }_{b} 0,{ }_{b} 1\right\}$.

Definition 7 ( $N$-variable switching functions). (From Section 3.11 of [15].) An $n$-variable switching function is a mapping of the form

$$
\begin{align*}
& f: B^{n} \longrightarrow B \\
& \quad\left({ }_{b} b_{1}, \ldots, b b_{n}\right) \longmapsto f\left({ }_{b} b_{1}, \ldots,{ }_{b} b_{n}\right) \tag{8}
\end{align*}
$$

where $B=\left\{{ }_{b} 0_{b} 1\right\}$.

The domain of a $n$-variable switching function has $2^{n}$ elements and the codomain has 2 elements; hence, there are $2^{2^{n}} n$-variable switching functions. Let $F_{n}(B)$ be the set of the $2^{2^{n}} n$-variable switching functions.
$F_{n}(B)$ contains $(n+2)$ specific $n$-variable switching functions: the 2 constant functions $(0,1)$ and the $n$ projectionfunctions ( $f_{\text {Proj }}^{i}$ ). These functions are defined as follows:

$$
\begin{align*}
& 0: B^{n} \longrightarrow B \\
& \quad\left({ }_{b} b_{1}, \ldots,{ }_{b} b_{n}\right) \longmapsto{ }_{b} 0 \\
& 1: B^{n} \longrightarrow B \\
& \quad\left({ }_{b} b_{1}, \ldots,{ }_{b} b_{n}\right) \longmapsto{ }_{b} 1  \tag{9}\\
& f_{\text {Proj }}^{i}: B^{n} \longrightarrow B \\
& \quad\left({ }_{b} b_{1}, \ldots,{ }_{b} b_{n}\right) \longmapsto{ }_{b} b_{i},
\end{align*}
$$

$F_{n}(B)$ can be equipped with three closed operations (two binary and one unary operations)

$$
\begin{align*}
\text { Op. }+: F_{n}(B)^{2} & \longrightarrow F_{n}(B) \\
(f, g) & \longmapsto f+g \\
\text { Op. } \cdot: F_{n}(B)^{2} & \longrightarrow F_{n}(B)  \tag{10}\\
(f, g) & \longmapsto f \cdot g \\
\text { Op. }{ }^{-}: F_{n}(B) & \longrightarrow F_{n}(B) \\
f & \longmapsto \bar{f}
\end{align*}
$$

where $\forall\left({ }_{b} b_{1}, \ldots,{ }_{b} b_{n}\right) \in B^{n}$,

$$
\begin{align*}
& (f+g)\left({ }_{b} b_{1}, \ldots,{ }_{b} b_{n}\right) \\
& \quad=f\left({ }_{b} b_{1}, \ldots,{ }_{b} b_{n}\right) \vee g\left({ }_{b} b_{1}, \ldots,{ }_{b} b_{n}\right), \\
& (f \cdot g)\left({ }_{b} b_{1}, \ldots,{ }_{b} b_{n}\right)  \tag{11}\\
& \quad=f\left({ }_{b} b_{1}, \ldots,{ }_{b} b_{n}\right) \wedge g\left({ }_{b} b_{1}, \ldots,{ }_{b} b_{n}\right), \\
& \quad \bar{f}\left({ }_{b} b_{1}, \ldots,{ }_{b} b_{n}\right)=\neg f\left({ }_{b} b_{1}, \ldots,{ }_{b} b_{n}\right) .
\end{align*}
$$

$\left(F_{n}(B),+, \cdot,-, 0,1\right)$ is a Boolean algebra [22]. Then, it is possible to write a Boolean formula of $n$-variable switching functions and relations between Boolean formula of $n$-variable switching functions. In the case of $n$-variable switching functions, relations Equality and Inclusion can also be presented as follows:
(i) $f$ and $g$ are equal $(f=g)$ if and only if the columns of the truth-tables of $f, g$ are exactly the same, that is, $\forall\left({ }_{b} b_{1}, \ldots,{ }_{b} b_{n}\right) \in B^{n}, f\left({ }_{b} b_{1}, \ldots,{ }_{b} b_{n}\right)=g\left({ }_{b} b_{1}\right.$, $\left.\ldots,{ }_{b} b_{n}\right)$.
(ii) $f$ is included into $g(f \leq g)$ if and only if the value of $g$ is always ${ }_{b} 1$ when the value of $f$ is ${ }_{b} 1$, that is, $\forall\left({ }_{b} b_{1}, \ldots,{ }_{b} b_{n}\right) \in B^{n},\left[f\left({ }_{b} b_{1}, \ldots,{ }_{b} b_{n}\right)={ }_{b} 0\right]$, or $\left[g\left({ }_{b} b_{1}, \ldots,{ }_{b} b_{n}\right)={ }_{b} 1\right]$.

Remark 8. Each $n$-variable switching function can be expressed as a composition of $\left(f_{\text {Proj }}^{1}, \ldots, f_{\text {Proj }}^{n}, 0,1\right)$ by operations,$+ \cdot$ and ${ }^{-}$.

Therefore, the Boolean algebra $\left(F_{n}(B),+, \cdot,{ }^{-}, 0,1\right)$ is a mathematical framework which allows composing and to comparing switching functions. Thanks to the results presented in the next subsection, this framework allows also solving Boolean equations systems of switching functions.
4.3. Solutions of Boolean Equations over Boolean Algebra $F_{n}(B)$. In [15], Brown explains that many problems in the application of Boolean algebra may be reduced to solving an equation of the form

$$
\begin{equation*}
f(X)=0, \tag{12}
\end{equation*}
$$

over a Boolean algebra $\mathscr{B}$. Formal procedures for producing solution of this equation were developed by Boole himself as a way to treat problems of logical inference. Boolean equations have been studied extensively since Boole's initial work (a bibliography of nearly 400 sources is presented in [14]). These works concern essentially the two-element Boolean algebra $\left(\left\{{ }_{b} 0,{ }_{b} 1\right\}, \vee, \wedge, \neg,{ }_{b} 0,{ }_{b} 1\right)$.

In our case, we focus on the Boolean algebra of $n$ variable switching functions $F_{n}(B)$. We consider a Boolean system composed of $m$ relations among members of $F_{n}(B)$ for which $k$ of them are considered as unknowns. Theorems presented in this section permit to solve any system of Boolean equations as it exists in a canonic form of a Boolean system of $k$ unknowns and we are able to calculate solutions for this form.
4.3.1. Canonic Form of a Boolean System of $k$ Unknowns over Boolean Algebra $F_{n}(B)$. Consider the Boolean algebra of $n$ variable switching functions $\left(F_{n}(B),+, \cdot,-, 0,1\right)$.
(i) Let $\left(f_{\text {Proj }}^{1}, \ldots, f_{\text {Proj }}^{n}\right)$ be the $n$ projection-functions of $F_{n}(B)$.
(ii) Let $\left(x_{1}, \ldots, x_{k}\right)$ be $k$ elements of $F_{n}(B)$ considered as unknowns.

For notational convenience, we note " $X_{k}$ " as the vector $\left(x_{1}, \ldots, x_{k}\right)$ of the $k$ unknowns and "Proj" as the vector $\left(f_{\text {Proj }}^{1}, \ldots, f_{\text {Proj }}^{n}\right)$ of the $n$ projection-functions of $F_{n}(B)$.
Theorem 9 (reduction of a set of relations between $n$-variable switching functions). Any set of simultaneously asserted relations of switching functions can be reduced to a single equivalent relation such as

$$
\begin{equation*}
\mathscr{F}\left(X_{k}, \operatorname{Proj}\right)=0 . \tag{13}
\end{equation*}
$$

This theorem comes from Theorem 6.
In order to be able to write a canonic form for a Boolean system of $k$ unknowns over Boolean algebra $F_{n}(B)$, we introduce the following notation: for $x \in F_{n}(B)$ and $a \in\{0,1\}$, $x^{a}$ is defined by

$$
\begin{equation*}
x^{0}=\bar{x}, \quad x^{1}=x . \tag{14}
\end{equation*}
$$

This notation is extended to vectors as follows: for $X_{k}=$ $\left(x_{1}, \ldots, x_{k}\right) \in F_{n}(B)^{k}$ and $A_{k}=\left(a_{1}, \ldots, a_{k}\right) \in\{0,1\}^{k}, X_{k}^{A_{k}}$ is defined by

$$
\begin{equation*}
X_{k}^{A_{k}}=\prod_{i=1}^{i=k} x_{i}^{a_{i}}=x_{i}^{a_{1}} \cdots \cdots x_{k}^{a_{k}} . \tag{15}
\end{equation*}
$$

Theorem 10 (canonic form of a Boolean equation). Any Boolean equation $E q\left(X_{k}, \operatorname{Proj}\right)=0$ can be expressed within the canonic form

$$
\begin{equation*}
\sum_{A_{k} \in\{0,1\}^{k}} E q\left(A_{k}, \operatorname{Proj}\right) \cdot X_{k}^{A_{k}}=0, \tag{16}
\end{equation*}
$$

where $\operatorname{Eq}\left(A_{k}, \operatorname{Proj}\right)$ (with $A_{k} \in\{0,1\}^{k}$ ) are the $2^{k}$ discriminants of $E q\left(X_{k}, \operatorname{Proj}\right)=0$ according to $X_{k}$ (the term of "discriminant" comes from [15]).

This canonic form is obtained by expanding $\mathrm{Eq}\left(X_{k}, \operatorname{Proj}\right)$ according to the $k$ unknowns $\left(x_{1}, \ldots, x_{k}\right)$. For example, we have

$$
\begin{align*}
& \mathrm{Eq}(x, \operatorname{Proj})=\mathrm{Eq}(0, \operatorname{Proj}) \cdot \bar{x}+\mathrm{Eq}(1, \operatorname{Proj}) \cdot x \\
& \mathrm{Eq}\left(x_{1}, x_{2}, \operatorname{Proj}\right)= \mathrm{Eq}(0,0, \operatorname{Proj}) \cdot \overline{x_{1}} \cdot \overline{x_{2}} \\
&+\mathrm{Eq}(0,1, \operatorname{Proj}) \cdot \overline{x_{1}} \cdot x_{2}  \tag{17}\\
&+\mathrm{Eq}(1,0, \operatorname{Proj}) \cdot x_{1} \cdot \overline{x_{2}} \\
&+\mathrm{Eq}(1,1, \operatorname{Proj}) \cdot x_{1} \cdot x_{2}
\end{align*}
$$

4.3.2. Solution of a Single-Unknown Equation over $F_{n}(B)$. The following theorem has initially been demonstrated for the two-element Boolean algebra [14]. A generalization for all Boolean algebras can be found in [15], but no detailed demonstration is given. A new formalization of this theorem and its full demonstration are given below.

Theorem 11 (solution of a single-unknown equation). The Boolean equation over $F_{n}(B)$

$$
\begin{equation*}
E q(x, \operatorname{Proj})=0, \tag{18}
\end{equation*}
$$

for which the canonic form is

$$
\begin{equation*}
E q(0, \operatorname{Proj}) \cdot \bar{x}+E q(1, \operatorname{Proj}) \cdot x=0 \tag{19}
\end{equation*}
$$

is consistent (i.e., has at least one solution) if and only if the following condition is satisfied:

$$
\begin{equation*}
E q(0, \operatorname{Proj}) \cdot E q(1, \operatorname{Proj})=0 \tag{20}
\end{equation*}
$$

In this case, a general form of the solutions is

$$
\begin{equation*}
x=E q(0, \operatorname{Proj})+p \cdot \overline{E q(1, P r o j)}, \tag{21}
\end{equation*}
$$

where $p$ is an arbitrary parameter, that is, a freely-chosen member of $F_{n}(B)$.

This solution can also be expressed as

$$
\begin{align*}
x & =\overline{E q(1, \operatorname{Proj})} \cdot(E q(0, \operatorname{Proj})+p) \\
& =\bar{p} \cdot E q(0, \operatorname{Proj})+p \cdot \overline{E q(1, \operatorname{Proj})} \tag{22}
\end{align*}
$$

Proof. This theorem can be proved in four steps as follows:
(a) Equation (18) is consistent if and only if (20) is satisfied;
(b) Equation (21) is a solution of (18) if (20) is satisfied;
(c) each solution of (18) can be expressed as (21);
(d) if (20) is satisfied, the three parametric forms proposed are equivalent.

Step (a) can be proved as follows: Equation (20) is a sufficient condition for (18) to admit solutions since $x=$ $\mathrm{Eq}(0, \operatorname{Proj})$ is an obvious solution of (18). Equation (20) is also a necessary condition as if (18) admits a solution, then (18) can be also expressed thanks to the consensus theorem as $\mathrm{Eq}(0, \operatorname{Proj}) \cdot \bar{x}+\mathrm{Eq}(1, \operatorname{Proj}) \cdot x+\mathrm{Eq}(0, \operatorname{Proj}) \cdot \mathrm{Eq}(1, \operatorname{Proj})=0$ and we have necessarily $\mathrm{Eq}(0, \operatorname{Proj}) \cdot \mathrm{Eq}(1, \operatorname{Proj})=0$.

To prove Step (b), it is sufficient to substitute the expression for $x$ from (21) into (18) and to use (20) as follows:

$$
\begin{align*}
& \mathrm{Eq}(0, \operatorname{Proj}) \cdot \bar{x}+\mathrm{Eq}(1, \operatorname{Proj}) \cdot x \\
& =\mathrm{Eq}(0, \text { Proj }) \cdot \overline{(\mathrm{Eq}(0, \operatorname{Proj})+p \cdot \overline{\mathrm{Eq}(1, \text { Proj) })})} \\
& +E q(1, \operatorname{Proj}) \cdot(E q(0, \operatorname{Proj})+p \cdot \overline{\mathrm{Eq}(1, \operatorname{Proj})}) \\
& =\mathrm{Eq}(0, \operatorname{Proj}) \cdot \overline{\mathrm{Eq}(0, \text { Proj })} \cdot \overline{(p \cdot \overline{\mathrm{Eq}(1, \text { Proj })})}  \tag{23}\\
& +E q(0, \operatorname{Proj}) \cdot E q(1, \operatorname{Proj}) \\
& +p \cdot \mathrm{Eq}(1, \operatorname{Proj}) \cdot \overline{\mathrm{Eq}(1, \operatorname{Proj})} \\
& =0+0+0=0 \text {. }
\end{align*}
$$

To prove Step (c), it is sufficient to find one element $p$ of $F_{n}(B)$ for each solution for $x$ of (18). Let us consider $p$ defined by " $p=\overline{\mathrm{Eq}(0, \operatorname{Proj})} \cdot \overline{\mathrm{Eq}(1, \operatorname{Proj})} \cdot x$ " where $x$ is a solution to (18). Then we have

$$
\begin{align*}
& \left\{\begin{array}{c}
\mathrm{Eq}(0, \operatorname{Proj}) \cdot \mathrm{Eq}(1, \text { Proj })=0 \\
\mathrm{Eq}(0, \operatorname{Proj}) \cdot \bar{x}+\mathrm{Eq}(1, \text { Proj }) \cdot x=0 \\
p=\overline{\mathrm{Eq}(0, \operatorname{Proj})} \cdot \overline{\mathrm{Eq}(1, \text { Proj })} \cdot x
\end{array}\right.  \tag{24}\\
& \Longrightarrow x=\mathrm{Eq}(0, \text { Proj })+p \cdot \overline{\mathrm{Eq}(1, \text { Proj })}
\end{align*}
$$

as

$$
\begin{aligned}
x= & 1 \cdot x=(\mathrm{Eq}(0, \operatorname{Proj})+\mathrm{Eq}(1, \text { Proj }) \\
& +\overline{\mathrm{Eq}(0, \operatorname{Proj})} \cdot \overline{\mathrm{Eq}(1, \text { Proj })}) \cdot x \\
= & \mathrm{Eq}(0, \operatorname{Proj}) \cdot x+\mathrm{Eq}(1, \operatorname{Proj}) \cdot x \\
& +\overline{\mathrm{Eq}(0, \operatorname{Proj})} \cdot \overline{\mathrm{Eq}(1, \operatorname{Proj})} \cdot x \\
= & \mathrm{Eq}(0, \operatorname{Proj}) \cdot x+0+\overline{\mathrm{Eq}(1, \operatorname{Proj})} \\
& \cdot(\overline{\mathrm{Eq}(0, \operatorname{Proj})} \cdot \overline{\mathrm{Eq}(1, \operatorname{Proj})} \cdot x) \\
& \quad \text { as } \mathrm{Eq}(1, \operatorname{Proj}) \cdot x=0
\end{aligned}
$$

$$
\begin{align*}
= & \mathrm{Eq}(0, \operatorname{Proj}) \cdot x+\mathrm{Eq}(0, \operatorname{Proj}) \cdot \bar{x} \\
& +\overline{\mathrm{Eq}(1, \operatorname{Proj})} \cdot p \text { as } \mathrm{Eq}(0, \text { Proj }) \cdot \bar{x}=0 \\
= & \mathrm{Eq}(0, \operatorname{Proj}) \cdot(x+\bar{x})+p \cdot \overline{\mathrm{Eq}(1, \text { Proj })} \\
= & \mathrm{Eq}(0, \operatorname{Proj})+p \cdot \overline{\mathrm{Eq}(1, \text { Proj })} . \tag{25}
\end{align*}
$$

To prove Step (d), it is sufficient to rewrite (21) in the two other forms by using (20) as follows:

$$
\begin{align*}
x= & 1 \cdot \mathrm{Eq}(0, \operatorname{Proj})+p \cdot \overline{\mathrm{Eq}(1, \operatorname{Proj})} \\
= & (\mathrm{Eq}(1, \operatorname{Proj})+\overline{\mathrm{Eq}(1, \operatorname{Proj})}) \cdot \mathrm{Eq}(0, \operatorname{Proj}) \\
& +p \cdot \overline{\mathrm{Eq}(1, \operatorname{Proj})} \\
= & \mathrm{Eq}(0, \operatorname{Proj}) \cdot \mathrm{Eq}(1, \operatorname{Proj})+(\mathrm{Eq}(0, \operatorname{Proj})+p) \\
& \cdot \overline{\mathrm{Eq}(1, \operatorname{Proj})} \\
= & 0+\overline{\mathrm{Eq}(1, \operatorname{Proj})} \cdot(\mathrm{Eq}(0, \operatorname{Proj})+p) \\
= & \overline{\mathrm{Eq}(1, \operatorname{Proj})} \cdot(\mathrm{Eq}(0, \operatorname{Proj})+p), \\
x= & 1 \cdot \mathrm{Eq}(0, \operatorname{Proj})+p \cdot \overline{\mathrm{Eq}(1, \operatorname{Proj})} \\
= & (p+\bar{p}) \cdot \mathrm{Eq}(0, \operatorname{Proj})+p \cdot \overline{\mathrm{Eq}(1, \operatorname{Proj})} \\
= & \bar{p} \cdot \mathrm{Eq}(0, \operatorname{Proj})+p \cdot(\mathrm{Eq}(0, \operatorname{Proj})+\overline{\mathrm{Eq}(1, \operatorname{Proj})}) \\
= & \bar{p} \cdot \mathrm{Eq}(0, \operatorname{Proj}) \\
& +p \cdot(\mathrm{Eq}(0, \operatorname{Proj}) \cdot \mathrm{Eq}(1, \operatorname{Proj})+\overline{\mathrm{Eq}(1, \operatorname{Proj})}) \\
= & \bar{p} \cdot \mathrm{Eq}(0, \operatorname{Proj})+p \cdot(0+\overline{\mathrm{Eq}(1, \operatorname{Proj})}) \\
= & \bar{p} \cdot \mathrm{Eq}(0, \operatorname{Proj})+p \cdot \overline{\mathrm{Eq}(1, \operatorname{Proj})} . \tag{26}
\end{align*}
$$

4.3.3. Solution of $k$-Unknown Equations over $F_{n}(B)$. The global result presented in the following theorem can be found in [14] or [15]. However, in these works, the solution is not expressed with a parametric form, but with intervals only. The formulation presented in this paper is more adapted to symbolic computation and is mandatory for practice optimization.

A $k$-unknown equation can be solved by solving successively $k$ single-unknown equations. If we consider the $k$ unknown equation as a single-unknown equation of $x_{k}$, its consistence condition corresponds to a ( $k-1$ )-unknown equation. The process can be iterated until $x_{1}$. After substituting $S\left(x_{1}\right)$ for $x_{1}$ in the last equation, it is possible to find the solution for $x_{2}$. Then, it is sufficient to apply this procedure again $(k-2)$ times to obtain successively the solutions $S\left(x_{3}\right)$ to $S\left(x_{k}\right)$.

Theorem 12 (solution of a $k$-unknown equation). The Boolean equation over $F_{n}(B)$

$$
\begin{equation*}
E q_{0}\left(X_{k}, \operatorname{Proj}\right)=0 \tag{27}
\end{equation*}
$$

is consistent (i.e., has at least one solution) if and only if the following condition is satisfied:

$$
\begin{equation*}
\prod_{A_{k} \in\{0,1\}^{k}} E q_{0}\left(A_{k}, \operatorname{Proj}\right)=0 \tag{28}
\end{equation*}
$$

If (28) is satisfied, (27) admits one or more $k$-tuple solutions $\left(S\left(x_{1}\right), \ldots, S\left(x_{k}\right)\right)$ such each component $S\left(x_{i}\right)$ is defined by

$$
\begin{align*}
S\left(x_{i}\right)= & \prod_{A_{k-i} \in\{0,1\}^{k-i}} E q_{i-1}\left(0, A_{k-i}, \operatorname{Proj}\right) \\
& +p_{i} \cdot \prod_{A_{k-i} \in\{0,1\}^{k-i}} E q_{i-1}\left(1, A_{k-i}, \operatorname{Proj}\right), \tag{29}
\end{align*}
$$

with
(i) $E q_{i}\left(x_{i+1}, \ldots, x_{k}, \operatorname{Proj}\right)=E q_{i-1}\left(x_{i}, x_{i+1}, \ldots, x_{k}\right.$, Proj) $\left.\right|_{x_{i} \leftarrow S\left(x_{i}\right)}$
(ii) $p_{i}$ is an arbitrary parameter, that is, a freely-chosen member of $F_{n}(B)$.

The full demonstration of this theorem cannot be given in this paper because of lack of space (a full demonstration by mathematical induction can be found in [8]). A description of the different steps of the proof and the detail of the principal steps are given below.

Proof (elements of Proof). Equation (27) can be solved by applying Theorems 3 and $11 k$ times according to the unknowns $x_{k}$ to $x_{1}$ as follows.

According to Theorem 3, (27) is equivalent to

$$
\begin{equation*}
\mathrm{Eq}_{0}\left(X_{k-1}, 0, \operatorname{Proj}\right) \cdot \overline{x_{k}}+\mathrm{Eq}_{0}\left(X_{k-1}, 1, \operatorname{Proj}\right) \cdot x_{k}=0 \tag{30}
\end{equation*}
$$

According to Theorem 11, (30) admits solutions in $x_{k}$ if and only if

$$
\begin{equation*}
\mathrm{Eq}_{0}\left(X_{k-1}, 0, \operatorname{Proj}\right) \cdot \mathrm{Eq}_{0}\left(X_{k-1}, 1, \operatorname{Proj}\right)=0 \tag{31}
\end{equation*}
$$

Equation (31) is an equation with ( $k-1$ ) unknowns. Each term of (31) can be expanded according to $x_{k-1}$ and (31) can be written in the form

$$
\begin{align*}
& \left(\mathrm{Eq}_{0}\left(X_{k-2}, 0,0, \operatorname{Proj}\right) \cdot \mathrm{Eq}_{0}\left(X_{k-2}, 0,1, \operatorname{Proj}\right)\right) \cdot \overline{x_{k-1}} \\
& +\left(\mathrm{Eq}_{0}\left(X_{k-2}, 1,0, \operatorname{Proj}\right) \cdot \mathrm{Eq}_{0}\left(X_{k-2}, 1,1, \operatorname{Proj}\right)\right) \cdot x_{k-1}=0 . \tag{32}
\end{align*}
$$

According to Theorem 11, (32) admits solutions in $x_{k-1}$ if and only if

$$
\begin{equation*}
\prod_{A_{2} \in\{0,1\}^{2}} \mathrm{Eq}_{0}\left(X_{k-2}, A_{2}, \operatorname{Proj}\right)=0 \tag{33}
\end{equation*}
$$

Equation (33) is an equation with $(k-2)$ unknowns. Each term of (33) can be expanded according to $x_{k-2}$ and (33) can be written in the form

$$
\begin{align*}
& \left(\prod_{A_{2} \in\{0,1\}^{2}} \mathrm{Eq}_{0}\left(X_{k-3}, 0, A_{2}, \operatorname{Proj}\right)\right) \cdot \overline{x_{k-2}} \\
&  \tag{34}\\
& \quad+\left(\prod_{A_{2} \in\{0,1\}^{2}} \mathrm{Eq}_{0}\left(X_{k-3}, 1, A_{2}, \operatorname{Proj}\right)\right) \cdot x_{k-2}=0 .
\end{align*}
$$

In the end, we obtain an equation of only one unknown $x_{1}$ defined by

$$
\begin{align*}
& \left(\prod_{A_{k-1} \in\{0,1\}^{k-1}} \mathrm{Eq}_{0}\left(0, A_{k-1}, \operatorname{Proj}\right)\right) \cdot \overline{x_{1}} \\
& \quad+\left(\prod_{A_{k-1} \in\{0,1\}^{k-1}} \mathrm{Eq}_{0}\left(1, A_{k-1}, \operatorname{Proj}\right)\right) \cdot x_{1}=0 . \tag{35}
\end{align*}
$$

According to Theorem 11, (35) admits solutions if and only if

$$
\begin{equation*}
\prod_{A_{k} \in\{0,1\}^{k}} \mathrm{Eq}_{0}\left(A_{k}, \operatorname{Proj}\right)=0 \tag{36}
\end{equation*}
$$

When (36) is satisfied, the $k$ equations for $x_{1}$ to $x_{k}$ admit solutions. Equation (27) is then coherent and admits solutions.

When (36) is satisfied, solutions of (35) for $x_{1}$ are

$$
\begin{align*}
S\left(x_{1}\right)= & \prod_{A_{k-1} \in\{0,1\}^{k-1}} \mathrm{Eq}_{0}\left(0, A_{k-1}, \text { Proj }\right) \\
& +p_{1} \cdot \frac{\prod_{A_{k-1} \in\{0,1\}^{k-1}} \mathrm{Eq}_{0}\left(1, A_{k-1}, \operatorname{Proj}\right)}{} \tag{37}
\end{align*}
$$

After substituting $S\left(x_{1}\right)$ for $x_{1}$ into (27), we obtain a new equation $\mathrm{Eq}_{1}\left(x_{2}, \ldots, x_{k}, \operatorname{Proj}\right)=0$ involving the $(k-1)$ unknowns ( $x_{2}, \ldots, x_{k}$ ), where

$$
\begin{equation*}
\mathrm{Eq}_{1}\left(x_{2}, \ldots, x_{k}, \operatorname{Proj}\right)=\left.\mathrm{Eq}_{0}\left(x_{1}, x_{2}, \ldots, x_{k}, \operatorname{Proj}\right)\right|_{x_{1} \leftarrow S\left(x_{1}\right)} . \tag{38}
\end{equation*}
$$

By applying the previous procedure, we can obtain $S\left(x_{2}\right)$ and $\mathrm{Eq}_{2}\left(x_{3}, \ldots, x_{k}, \operatorname{Proj}\right)$. Then, it suffices to apply this procedure again $(k-2)$ times to obtain successively solutions $S\left(x_{3}\right)$ to $S\left(x_{k}\right)$.

It is important to note that the order in which unknowns are treated affects only the parametric form of the $k$-tuple solution. This is due to the fact that the same $k$-tuple solution can be represented with several distinct parametric forms.
4.3.4. Partial Conclusions. Thanks to theorems presented above, it is possible to obtain a parametric representation of all the solutions of any set of simultaneously asserted relations with $k$ unknowns, if a solution exists. In practice, due to the complexity of systems to be designed, proposed set of simultaneously asserted relations is generally inconsistent [23]. To simplify the work of the designer, we have proved complementary theorems to improve the robustness of our method to the lack of precision of the specifications (Section 4.4).

When several solutions exist, the comparison of solutions according to a given criterion can be envisaged since the Boolean algebra $F_{n}(B)$ is equipped with a partial order. To simplify the work of the designer too, we have developed a method to calculate the best solutions according to one or several criteria (Section 4.5).
4.4. Theorems to Cope with Inconsistencies of Specifications. In practice, it is very difficult for a designer to specify the whole requirements of a complex system without inconsistencies. It is the reason why requirements given by the designer are often declared as inconsistent according to Theorem 12. Since the inconsistency condition is a Boolean formula, it is possible to use it for the detection of the origin of inconsistencies. Two cases have to be considered as follows:
(i) Several requirements cannot be simultaneously respected. In this case, a hierarchy between requirements can be proposed in order to find a solution. The requirements which have the lower priority have to be corrected for becoming consistent with the requirements which have the higher priority. This strategy is based on Theorem 14.
(ii) The detected inconsistency refers to specific combinations of projection-functions for which the designer knows that they are impossible blocking the synthesis process, it is necessary to introduce new assumptions and to use Theorem 13.

Theorem 13 (solution of a Boolean equation according to an assumption among the projection-functions). The following problem

Equationtosolve:

$$
\begin{equation*}
E q_{0}\left(X_{k}, \operatorname{Proj}\right)=0 \tag{39}
\end{equation*}
$$

Assumptions:

$$
\mathscr{A}(\text { Proj })=0
$$

admits the same solutions as the following equation:

$$
\begin{equation*}
E q_{0}\left(X_{k}, \operatorname{Proj}\right) \leq \mathscr{A}(\text { Proj }) \tag{40}
\end{equation*}
$$

Proof. According to $\mathscr{A}(\operatorname{Proj})=0, \mathrm{Eq}_{0}\left(X_{k}\right.$, Proj $)=0$ can be rewritten as

$$
\begin{align*}
& \left\{\begin{array}{l}
\mathrm{Eq}_{0}\left(X_{k}, \operatorname{Proj}\right)=0 \\
\mathscr{A}(\text { Proj })=0
\end{array}\right. \\
& \mathscr{A}(\text { Proj })=0 \\
& \Longleftrightarrow \mathscr{A}(\operatorname{Proj})+\mathrm{Eq}_{0}\left(X_{k}, \operatorname{Proj}\right)=0 \\
& \Longleftrightarrow \mathscr{A}(\operatorname{Proj})+\overline{\mathscr{A}(\operatorname{Proj})} \cdot \mathrm{Eq}_{0}\left(X_{k}, \operatorname{Proj}\right)=0  \tag{41}\\
& \Longleftrightarrow\left\{\begin{array}{l}
\overline{\mathscr{A}(\operatorname{Proj})} \cdot \mathrm{Eq}_{0}\left(X_{k}, \text { Proj }\right)=0 \\
\mathscr{A}(\operatorname{Proj})=0
\end{array}\right. \\
& \Longleftrightarrow\left\{\begin{array}{l}
\mathrm{Eq}_{0}\left(X_{k}, \text { Proj }\right) \leq \mathscr{A}(\text { Proj }) \\
\mathscr{A}(\text { Proj })=0 .
\end{array}\right.
\end{align*}
$$

Equation $\overline{\mathscr{A}(\text { Proj })} \cdot \mathrm{Eq}_{0}\left(X_{k}\right.$, Proj $)=0$ is consistent if and only if the following condition is true (Theorem 12):

$$
\begin{equation*}
\overline{\mathscr{A}(\text { Proj })} \cdot \prod_{A_{k} \in\{0,1\}^{k}} \mathrm{Eq}_{0}\left(A_{k}, \text { Proj }\right)=0 . \tag{42}
\end{equation*}
$$

By construction, this new condition is the subset of the initial condition $\left(\prod_{A_{k} \in\{0,1\}^{k}} \mathrm{Eq}_{0}\left(A_{k}, \operatorname{Proj}\right)=0\right)$ for which the proposed assumption is satisfied. All the other terms have been removed.

If (42) is satisfied, (40) admits one or more $k$-tuple solutions where each component $S\left(x_{i}\right)$ is defined by

$$
\begin{align*}
S\left(x_{i}\right)= & \overline{\mathscr{A}(\text { Proj })} \\
& \cdot\left(\prod_{A_{k-i} \in\{0,1\}^{k-i}} \mathrm{Eq}_{i-1}\left(0, A_{k-i}, \text { Proj }\right)\right. \\
& \left.+p_{i} \cdot \frac{\prod_{A_{k-i} \in\{0,1\}^{k-i}} \mathrm{Eq}_{i-1}\left(1, A_{k-i}, \operatorname{Proj}\right)}{}\right)  \tag{43}\\
& +\mathscr{A}(\operatorname{Proj}) \cdot p_{i} .
\end{align*}
$$

As $\mathscr{A}(\operatorname{Proj})=0, S\left(x_{i}\right)$ can also be expressed as

$$
\begin{align*}
S\left(x_{i}\right)= & \prod_{A_{k-i} \in\{0,1\}^{k-i}} \mathrm{Eq}_{i-1}\left(0, A_{k-i}, \operatorname{Proj}\right)+p_{i} \\
& \cdot \prod_{A_{k-i} \in\{0,1\}^{k-i}} \mathrm{Eq}_{i-1}\left(1, A_{k-i}, \operatorname{Proj}\right) . \tag{44}
\end{align*}
$$

When $\mathscr{A}($ Proj $)=0$ is satisfied, the solutions of (40) are also solution to $E q_{0}\left(X_{k}, \operatorname{Proj}\right)=0$.

Theorem 14 (Solution of a Boolean equation system according to a priority rule between requirements). The following problem

## Equationssystemtosolve:

$$
\begin{array}{cc}
H R & \mathscr{F}_{\mathscr{H}}\left(X_{k}, \operatorname{Proj}\right)=0 \\
L R & \mathscr{F}_{\mathscr{L}}\left(X_{k}, \operatorname{Proj}\right)=0 \\
\text { OR } & \mathscr{F}_{\mathscr{O}}\left(X_{k}, \operatorname{Proj}\right)=0 \tag{45}
\end{array}
$$

Priorityrulebetweenrequirements:

$$
H R \gg L R,
$$

where
(i) $\mathscr{F}_{\mathscr{H}}\left(X_{k}\right.$, Proj $)=0$ is the formal expression of the requirements which have the higher priority ( $H R$ );
(ii) $\mathscr{F}_{\mathscr{L}}\left(X_{k}\right.$, Proj $)=0$ is the formal expression of the requirements which have the lower priority ( $L R$ );
(iii) $\mathscr{F}_{O}\left(X_{k}, P r o j\right)=0$ is the formal expression of the others requirements (OR);
(iv) $H R \gg L R$ is the priority rule between inconsistent requirements,
admits the same solutions as the system of equations as follows:

$$
\begin{align*}
& \mathscr{F}_{\mathscr{H}}\left(X_{k}, \operatorname{Proj}\right)=0 \\
& \mathscr{F}_{\mathscr{L}}\left(X_{k}, \operatorname{Proj}\right) \leq \mathscr{F}(\text { Proj })  \tag{46}\\
& \mathscr{F}_{\mathscr{O}}\left(X_{k}, \operatorname{Proj}\right)=0,
\end{align*}
$$

where $\mathscr{J}$ (Proj) is the inconsistency condition between requirements "HR" and "LR":

$$
\begin{equation*}
\mathscr{I}(\operatorname{Proj})=\prod_{A_{k} \in\{0,1\}^{k}}\left(\mathscr{F}_{\mathscr{L}}\left(A_{k}, \operatorname{Proj}\right)+\mathscr{F}_{\mathscr{L}}\left(A_{k}, \operatorname{Proj}\right)\right) \tag{47}
\end{equation*}
$$

Proof. Thanks to Theorem 12, the inconsistency condition $\mathcal{F}$ (Proj) between requirements "HR" and "LR" can be found by solving equation $\mathscr{F}_{\mathscr{H}}\left(X_{k}, \operatorname{Proj}\right)+\mathscr{F}_{\mathscr{L}}\left(X_{k}, \operatorname{Proj}\right)=0$. We have

$$
\begin{equation*}
\mathscr{I}(\operatorname{Proj})=\prod_{A_{k} \in\{0,1\}^{k}}\left(\mathscr{F}_{\mathscr{H}}\left(A_{k}, \operatorname{Proj}\right)+\mathscr{F}_{\mathscr{L}}\left(A_{k}, \operatorname{Proj}\right)\right) . \tag{48}
\end{equation*}
$$

To remove the inconsistency between requirements "HR" and "LR" according to the priority rule "HR $\gg$ LR", it is necessary to restrict the range of requirement "LR" to the part for which there is no inconsistency, that is, $\mathscr{J}(\operatorname{Proj})=0$. That is the case, when $\mathscr{F}_{\mathscr{L}}\left(X_{k}, \operatorname{Proj}\right)=0$ is replaced by $\mathscr{F}_{\mathscr{L}}\left(X_{k}, \operatorname{Proj}\right) \leq$ $\mathcal{I}$ (Proj).

Thanks to Theorem 12, (49) admits always one or more $k$-tuple solutions and it is impossible to find a less restrictive condition over requirement "LR".

$$
\begin{align*}
& \mathscr{F}_{\mathscr{H}}\left(X_{k}, \text { Proj }\right)=0  \tag{49}\\
& \mathscr{F}_{\mathscr{L}}\left(X_{k}, \operatorname{Proj}\right) \leq \mathscr{F}(\operatorname{Proj}) .
\end{align*}
$$

4.5. Optimal Solutions of Boolean Equations over $F_{n}(B)$. The goal of this step is to be able to obtain automatically the parametric form of the $k$-tuples solutions of $F_{n}(B)$ which satisfy not only a given equation $\left(\mathrm{Eq}\left(X_{k}, \operatorname{Proj}\right)=0\right)$ of Boolean functions but also which maximize (or minimize) a Boolean formula of these Boolean functions ( $\mathscr{F}_{C}\left(X_{k}, \operatorname{Proj}\right)$ ) corresponding to the desired optimization criterion.

Generally speaking, the search of the best solution tuples according to a given criterion when the space of solutions is composed of discrete values is a complex mathematical issue. It is sometimes necessary to make a side-by-side comparison of each solution in order to identify the best one. In our case, this exhaustive method which cannot be used as $F_{n}(B)$ is only provided by a partial order; two particular solutions cannot always be ordered between themselves.

Nevertheless, it is possible to obtain the researched parametric form of the $k$-tuples thanks to the following results.
(i) When an equation between Boolean functions has one or more solution tuples in $F_{n}(B)$, every Boolean formula onto these Boolean functions can be rewritten thanks to only projection-functions of $F_{n}(B)$ and free parameters of $F_{n}(B)$ which are describing these solution tuples.
(ii) Every Boolean formula expressed as a composition of projection-functions of $F_{n}(B)$ and free parameters of $F_{n}(B)$ has a unique maximum and a unique minimum. These extrema can be expressed thanks to only projection-functions of $F_{n}(B)$.

Hence it is then possible to rewrite the initial problem
Problem to solve:

$$
\begin{equation*}
\mathrm{Eq}\left(X_{k}, \operatorname{Proj}\right)=0 \tag{50}
\end{equation*}
$$

Optimization Criterion:

$$
\text { Maximization of } \mathscr{F}_{\mathrm{C}}\left(X_{k}, \operatorname{Proj}\right),
$$

into a 2-equation system to solve

$$
\begin{align*}
& \mathrm{Eq}\left(X_{k}, \text { Proj }\right)=0 \\
& \mathscr{F}_{\mathrm{C}}\left(X_{k}, \operatorname{Proj}\right)=\operatorname{Max}_{\left\{X_{k} \mid \operatorname{Eq}\left(X_{k}, \operatorname{Proj}\right)=0\right\}}\left(\mathscr{F}_{\mathrm{C}}\left(X_{k}, \operatorname{Proj}\right)\right) . \tag{51}
\end{align*}
$$

4.5.1. Extrema of a Boolean Formula according to Freely Chosen Members of $F_{n}(B)$. Considering the Boolean algebra of $n$-variable switching functions $\left(F_{n}(B),+, \cdot,-, 0,1\right)$,
(i) let $\left(f_{\text {Proj }}^{1}, \ldots, f_{\text {Proj }}^{n}\right)$ be the $n$ projection-functions of $F_{n}(B)$;
(ii) let $\left(p_{1}, \ldots, p_{k}\right)$ be $k$ elements of $F_{n}(B)$ considered as freely chosen members. Let " $P_{k}$ " be the corresponding vector.

Any formula $\mathscr{F}\left(P_{k}, \operatorname{Proj}\right)$ for which $P_{k}$ are freely chosen members of $F_{n}(B)$ defines a subset of $F_{n}(B)$. According to the relation $\leq$, elements of this subset can be compared.

In this specific case, the subset defined by $\mathscr{F}\left(P_{k}\right.$, Proj $)$ admits a minimal element and a maximal element.

Theorem 15 (minimum and maximum of a Boolean formula). Any formula $\mathscr{F}\left(P_{k}\right.$, Proj) for which $P_{k}$ are freely chosen members of $F_{n}(B)$ admits a minimum and a maximum defined as follows:

$$
\begin{align*}
& \operatorname{Min}_{P_{k} \in F_{n}(B)^{k}}\left(\mathscr{F}\left(P_{k}, \operatorname{Proj}\right)\right)=\prod_{A_{k} \in\{0,1\}^{k}} \mathscr{F}\left(A_{k}, \operatorname{Proj}\right) \\
& \operatorname{Max}_{P_{k} \in F_{n}(B)^{k}}\left(\mathscr{F}\left(P_{k}, P r o j\right)\right)=\sum_{A_{k} \in\{0,1\}^{k}} \mathscr{F}\left(A_{k}, P r o j\right), \tag{52}
\end{align*}
$$

Proof. To prove this theorem, it is necessary to establish that
(1) $\prod_{A_{k} \in\{0,1\}^{k}} \mathscr{F}\left(A_{k}, \operatorname{Proj}\right)$ is a lower bound of $\mathscr{F}\left(P_{k}\right.$, Proj);
(2) It exists at least one specific combination of $P_{k}$ for which $\mathscr{F}\left(P_{k}, \operatorname{Proj}\right)=\prod_{A_{k} \in\{0,1\}^{k}} \mathscr{F}\left(A_{k}\right.$, Proj $)$;
(3) $\sum_{A_{k} \in\{0,1\}^{k}} \mathscr{F}\left(A_{k}\right.$, Proj $)$ is an upper bound of $\mathscr{F}\left(P_{k}\right.$, Proj);
(4) It exists at least one specific combination of $P_{k}$ for which $\mathscr{F}\left(P_{k}, \operatorname{Proj}\right)=\sum_{A_{k} \in\{0,1\}^{k}} \mathscr{F}\left(A_{k}\right.$, Proj $)$.

Details of this proof can be found in [24].
4.5.2. Optimization Problem. Considering the Boolean algebra of $n$-variable switching functions $\left(F_{n}(B),+, \cdot,-, 0,1\right)$,
(i) let $\left(f_{\text {Proj }}^{1}, \ldots, f_{\text {Proj }}^{n}\right)$ be the $n$ projection-functions of $F_{n}(B)$. Let "Proj" be the corresponding vector;
(ii) Let $\left(x_{1}, \ldots, x_{k}\right)$ be $k$ elements of $F_{n}(B)$ considered as unknowns. Let " $X_{k}$ " be the corresponding vector;
(iii) Let $\left(p_{1}, \ldots, p_{k}\right)$ be $k$ elements of $F_{n}(B)$ considered as freely chosen members. Let " $P_{k}$ " be the corresponding vector.;
(iv) Let $\operatorname{Eq}\left(X_{k}, \operatorname{Proj}\right)=0$ be the Boolean equation to solve;
(v) Let $\mathscr{F}_{\mathrm{C}}\left(X_{k}\right.$, Proj) be the Boolean formula of the given criterion to optimize (maximization or minimization).

The method we propose, to obtain the parametric form of the $k$-tuple of switching functions solution of $\mathrm{Eq}\left(X_{k}, \operatorname{Proj}\right)=$ 0 according to a given optimization criterion $\mathscr{F}_{\mathrm{C}}\left(X_{k}, \operatorname{Proj}\right)$ is composed of five steps as follows.
(i) The first step is to establish the parametric form of the $k$-tuple solution to $\mathrm{Eq}\left(X_{k}, \operatorname{Proj}\right)=0$ only, thanks to Theorem 12.
(ii) The second step is to establish the parametric form of the given optimization criterion $\mathscr{F}_{\mathrm{C}}\left(X_{k}\right.$, Proj) by substituting $S\left(x_{i}\right)$ for $x_{i}$. Let $\mathscr{F}_{\mathrm{SC}}\left(P_{k}, \operatorname{Proj}\right)$ be the result of this substitution.
(iii) The third step is to calculate the extremum of $\mathscr{F}_{\mathrm{SC}}\left(P_{k}, \operatorname{Proj}\right)$ according to Theorem 15. Let $\mathscr{F}_{\mathrm{EC}}(\operatorname{Proj})$ be the Boolean formula of this extremum.


Figure 4: Structure of the water supply system.
(iv) The fourth step is to replace the given criterion by the equivalent relation

$$
\begin{equation*}
\mathscr{F}_{C}\left(X_{k}, \operatorname{Proj}\right)=\mathscr{F}_{E C}(\text { Proj }) . \tag{53}
\end{equation*}
$$

(v) The fifth step is to establish the parametric form of the $k$-tuple solution of the equivalent problem

$$
\begin{align*}
\mathrm{Eq}\left(X_{k}, \operatorname{Proj}\right) & =0 \\
\mathscr{F}_{\text {Crit }}\left(X_{k}, \operatorname{Proj}\right) & =\mathscr{F}_{\text {ExtCrit }}(\operatorname{Proj}) . \tag{54}
\end{align*}
$$

4.5.3. Partial Conclusions. Thanks to theorems presented in this section, it is now possible to obtain a parametric representation of the optimal solutions according to a given criterion, of any set of simultaneously asserted relations with $k$ unknowns if a solution exists.

The proposed method also permits to associate simultaneously or sequentially several criteria.
(i) When several criteria are treated simultaneously, the optimization problem can admit no solution. That is the case when the given criteria are antagonist.
(ii) When several criteria are treated sequentially, the obtained solutions satisfy the criteria with a given priority order. An example of optimization with several criteria treated sequentially is presented in the next section.

## 5. Algebraic Synthesis of Logical Controllers with Optimization Criteria and Incoherent Requirements

5.1. Control System Specifications. The studied system is the controller of a water supply system composed of two pumps which are working in redundancy (Figure 4). The water distribution is made when it is necessary according to the possible failures of elements (the pumps and the distributing system).

The expected behavior of the control system regarding the application requirements can be expressed by the set of assertions given hereafter:
(i) The two pumps never operate simultaneously.
(ii) A pump cannot operate if it is out of order.
(iii) When a global failure is detected, no pump can operate.
(iv) Pumps can operate if and only if a water distribution request is present.
(v) Priority is given according to "pr" (pump1 has priority when "pr" is true).
(vi) In order to reduce the wear of the pumps, it is necessary to restrict the number of starting of the pumps.
5.1.1. Inputs and Outputs of the Controller. The Boolean inputs and outputs of this controller are given in Figure 5(a). Each pump is controlled thanks to a Boolean output ("pl" and " p 2 "). The controller is informed of water distribution requests thanks to the input "req." Inputs "f1" and "f2" inform the controller of a failure of the corresponding pump and "gf" indicates a global failure of the installation. The values o or 1 of input "Pr" decide which pump has priority.
5.1.2. Control Laws to Synthetize. Our approach does not allow identifying automatically which state variables must be used. They are given by the designer according to its interpretation of the specification.

For the water distribution system, we propose to use 2 state variables, one for each output. According to this choice, 27 -variable switching functions ( $P 1$ and $P 2$ ) have to be synthesized (Figure 5(b)). They represent the unknowns of our problem. For this case study, the 7 projection-functions of $F_{7}(B)$ are therefore as follows.
(i) The 5 switching functions (Rq, F1, F2, GF, and Pr) which characterize the behavior of the inputs of the controller and are defined as follows:

$$
\begin{align*}
& \mathrm{Rq}: B^{7} \longrightarrow B \\
& \quad(\mathrm{rq}[k], \ldots, \mathrm{p} 2[k-1]) \longmapsto \mathrm{rq}[k] \tag{55}
\end{align*}
$$

(ii) The 2 switching functions ( ${ }_{p} P 1$ and ${ }_{p} P 2$ ) which characterize the previous behavior of the state variables of the controller and are defined as follows:

$$
\begin{align*}
& { }_{p} P 1: B^{7} \longrightarrow B  \tag{56}\\
& \quad(\mathrm{rq}[k], \ldots, \mathrm{p} 2[k-1]) \longmapsto \mathrm{p} 1[k-1] .
\end{align*}
$$

### 5.2. Algebraic Formalization of Requirements. The complete

 formalization of the behavior of the water distribution system is given in Figure 5(c). In order to illustrate the power of expression of relations Equality and Inclusion, several examples (generic assertions and equivalent formal relations illustrated in the case study) are given hereafter. It is important to note that the relation Inclusion permits to express distinctly necessary conditions and sufficient conditions. This relation is the cornerstone of our approach.[^0](a) Inputs and Outputs of the Controller

(b) General form of the Expected Control Laws
\[

$$
\begin{aligned}
& \mathrm{p} 1[k]=\mathrm{P} 1(\mathrm{rq}[k], \mathrm{f} 1[k], \mathrm{f} 2[k] \operatorname{gf}[k], \mathrm{pr}[k], \mathrm{p} 1[k-1], \mathrm{p} 2[k-1]) \\
& \mathrm{p} 2[k]=\mathrm{P} 2(\mathrm{rq}[k], \mathrm{f} 1[k], \mathrm{f} 2[k] \mathrm{gf}[k], \mathrm{pr}[k], \mathrm{p} 1[k-1], \mathrm{p} 2[k-1]) \\
& \mathrm{p} 1[0]={ }_{b} 0 \quad \mathrm{p} 2[0]={ }_{b} 0
\end{aligned}
$$
\]

(c) Formal Specification

Requirements:
R1 $\mathrm{P} 1 \cdot \mathrm{P} 2=0\left({ }^{*}\right.$ The two pumps never operate simultaneously. $\left.{ }^{*}\right)$
R2 $\mathrm{F} 1 \leq \overline{\mathrm{P} 1}$ ( ${ }^{*}$ Pump 1 cannot operate if it is out of order (F1). ${ }^{*}$ )
R3 F2 $\leq \overline{\mathrm{P} 2}$ ( ${ }^{*}$ Pump2 cannot operate if it is out of order (F2)..*)
$\mathrm{R} 4 \mathrm{GF} \leq(\overline{\mathrm{P} 1} \cdot \overline{\mathrm{P} 2})\left({ }^{*}\right.$ When a global failure is detected (GF), no pump can operate.*)
$\mathrm{R} 5 \quad(\mathrm{P} 1+\mathrm{P} 2) \leq \mathrm{Rq}\left({ }^{*}\right.$ It is necessary to have are quest for pumps operate. $\left.{ }^{*}\right)$
$\mathrm{R} 6 \quad \mathrm{Rq} \leq(\mathrm{P} 1+\mathrm{P} 2)\left({ }^{*}\right.$ It is sufficient to have a request for pumps operate. $\left.{ }^{*}\right)$
Priority rules:
R4 $\gg$ R6 ( ${ }^{*}$ Failure requirements has priority on a functional requirement. ${ }^{*}$ )
$\{\mathrm{R} 2, \mathrm{R} 3\} \gg \mathrm{R} 6$ ( ${ }^{*}$ Failure requirements has priority on a functional requirement. ${ }^{*}$ )
Optimization criteria:
(1) Minimization of: $\left(\left(\mathrm{P} 1 \cdot \overline{{ }_{p} P 1}\right)+\left(\mathrm{P} 2 \cdot \overline{{ }_{p} P 2}\right)\right)\left({ }^{*}\right.$ Minimization of the possibility to start a pump. $\left.{ }^{*}\right)$
(2) Maximization of: $((\operatorname{Pr} \cdot \mathrm{P} 1)+(\overline{\mathrm{Pr}} \cdot \mathrm{P} 2))\left({ }^{*}\right.$ Maximization of the priority order between the two pumps.*)
(d) Solution obtained by symbolic calculation

$$
\begin{aligned}
& \mathrm{P} 1=\mathrm{Rq} \cdot \overline{\mathrm{GF}} \cdot \overline{\mathrm{~F} 1} \cdot\left(\mathrm{~F} 2+\mathrm{Pr} \cdot\left({ }_{p} \mathrm{P} 1+\overline{{ }_{p} \mathrm{P} 2}\right)+{ }_{p} \mathrm{P} 1 \cdot \overline{{ }_{p} \mathrm{P} 2}\right) \\
& \mathrm{P} 2=\mathrm{Rq} \cdot \overline{\mathrm{GF}} \cdot \overline{\mathrm{~F} 2} \cdot\left(\mathrm{~F} 1+\overline{\mathrm{Pr}} \cdot\left({ }_{p} \mathrm{P} 2+{ }_{{ }_{p} P 1}^{P}\right)+{ }_{p} \mathrm{P} 2 \cdot \overline{{ }_{p} P 1}\right)
\end{aligned}
$$

(e) Control laws of the water distribution system

$$
\begin{aligned}
& \mathrm{p} 1[k]=\mathrm{rq}[k] \wedge \neg \mathrm{gf}[k] \wedge \neg \mathrm{f} 1[k] \wedge(\mathrm{f} 2[k] \vee \operatorname{pr}[k] \wedge(\mathrm{p} 1[k-1] \vee \neg \mathrm{p} 2[k-1]) \vee \mathrm{p} 1[k-1] \wedge \neg \mathrm{p} 2[k-1]) \\
& \mathrm{p} 2[k]=\mathrm{rq}[k] \wedge \neg \mathrm{gf}[k] \wedge \neg \mathrm{f} 1[k] \wedge(\mathrm{f} 1[k] \vee \operatorname{pr}[k] \wedge(\mathrm{p} 2[k-1] \vee \neg \mathrm{p} 1[k-1]) \vee \mathrm{p} 2[k-1] \wedge \neg \mathrm{p} 1[k-1]) \\
& \mathrm{p} 1[0]={ }_{b} 0 \quad \mathrm{p} 2[0]={ }_{b} 0
\end{aligned}
$$

Figure 5: Details of the case study.
(iii) It is necessary to have a request for pumps operate: $(P 1+P 2) \leq \mathrm{Rq}$;
(iv) It is sufficient to have a request for pumps operate: $\mathrm{Rq} \leq(P 1+P 2) ;$
(v) When Pumpl is failed, it is sufficient to have a request for Pump2 operate: $\mathrm{F} 1 \cdot \mathrm{Rq} \leq P 2$;
(vi) When Pumpl is failed, it is necessary to have a request for Pump2 operate: F1 • P2 $\leq$ Rq.
It is possible to prove that some of these formal expressions are equivalent (e.g., the first two). When a designer hesitates between two forms, he has the possibility, by using symbolic calculation, to check if the proposed relations are equivalent or not.

As $P 1$ and ${ }_{p} P 1$ represent the behavior of pumpl at, respectively, times $[k]$ and $[k-1]$, it is also possible to express relations about starts and stops of this pump as follows.
(i) It is necessary to have a request to start pumpl: ( $P 1$. $\left.\overline{{ }_{p} P 1}\right) \leq \mathrm{Rq}$.
(ii) When pumpl operates, it is sufficient to have a global failure to stop pumpl: $\left({ }_{p} P 1 \cdot \mathrm{GF}\right) \leq\left(\overline{P 1} \cdot{ }_{p} P 1\right)$.
5.3. Synthesis Process. In traditional design methods, the design procedure of a logic controller is not a linear process, but an iterative one converging to an acceptable solution. At the beginning of the design, requirements are neither complete nor without errors. Most often, new requirements are added during the search of solutions, and others are corrected. This complementary information is given by the designer after analysis of the partial solutions he found or when inconsistencies have been detected. If we do not make the hypothesis that the specifications are complete and consistent, designing a controller with a synthesis technique is also an iterative process in which the designer plays an important role.
5.3.1. Analysis of Requirements. For this case study, we choose to start with requirements R1 to R6. For this subset of requirements, the result given by your software tool was the following inconsistency condition: $\mathscr{J}=\mathrm{Rq} \cdot \mathrm{GF}+\mathrm{Rq} \cdot \mathrm{F} 1 \cdot \mathrm{~F} 2$.

Since requirements are declared inconsistent, we have to give complementary information to precise our specification. By analyzing each term of this formula, it is possible to detect the origin of the inconsistency:
(i) $\mathrm{Rq} \cdot \mathrm{GF}$ : what happens if we have simultaneously a request and a global failure? We consider that requirement R 4 is more important than requirement R6 (R4 $>\mathrm{R} 6$ ) as no pump can operate for this configuration.
(ii) $\mathrm{Rq} \cdot \mathrm{F} 1 \cdot \mathrm{~F}$ : what happens if we have simultaneously a request and a failure of each pump? We consider that requirements R2 and R3 are more important than requirement R 6 ( $\{\mathrm{R} 2, \mathrm{R} 3\} \gg \mathrm{R} 6$ ).

With these priority rules, all the requirements are now coherent and the set of all the solutions can be computed.
5.3.2. Optimal Solutions. For choosing a control law of the water supply system among this set of possible solutions, we will now take into account the given optimization criteria. The first criterion aims at minimizing the number of starting of each pump in order to reduce its wear. The second criterion aims at maximizing the use of the pump indicated by the value of parameter Pr. The method we propose allows proving that proposed criteria cannot be treated simultaneously since they are antagonist (to strictly the priority use of the pump fixed by parameter $\operatorname{Pr}$, it is necessary to permute pumps when $\operatorname{Pr}$ changes of value, implying a supplementary start of a pump). Details can be found in [25].

All the priorities rules and optimization criteria used for this case study are given in Figure 5(c). The solution we obtain is proposed in Figure 5(d).
5.3.3. Implementing Control Laws. The synthesized control laws presented in Figure 5(e) have been obtained by translating the expression of the two unknowns according to the projection-functions into relations between recurrent Boolean equations. These control laws can be automatically translated in the syntax of the ladder diagram language [1] before being implemented into a PLC. The code is composed of only 4 rungs (Figure 6).

The synthesized control laws can be given under the form of an automatically built input/output automaton with guarded transitions [21] (Figure 7).

## 6. Discussion

In our approach, the synthesis of control laws is based on the symbolic calculation, a prototype software tool has been developed to avoid tedious calculus and to aid the designer during the different steps of the synthesis. This tool (that can be obtained on request by the authors) performs all the computations required for inconsistencies detection between requirements and for control laws generation. In this tool, all the Boolean formulas are stored in the form of reduced ordered binary decision diagrams, which allows efficient calculations. For example, the computations for synthesizing a controller for the water supply system that we developed above have been made in less than 10 ms onto a classical laptop.

Our approach has been tested on several studies cases (some of them are available online: http://www.lurpa.ens-cachan.fr/-226050.kjsp). The feedbacks of these experiences allowed us to identify some of its limits and its possibilities; the most important are given below.

We have first to recall that our method can only be used for binary systems (systems whose inputs and outputs of their controller are Boolean values). Nevertheless, in practice many systems, like manufacturing systems, transport systems, and so on, are fully or partially binary.

In our opinion the main advantage of our approach is that, contrary to traditional engineering approaches, the synthesized control laws are not depending on designer's skill or of his correct interpretation of the system requirements. On the other hand, the quality of the synthesis results highly

Rung 3: update previous value of pump 1

Rung 4: update previous value of pump 2

| p2 | pp2 |
| :---: | :---: |
|  |  |
|  |  |

Rung 2: command of pump 2


Figure 6: Ladder diagram of the code to implement into the PLC.

Table 1: Futures concerning a same case study.
$\left.\begin{array}{lccc}\hline & \text { Formal requirements } & \text { Synthesized controller } & \text { PLC program (structured text) } \\ \hline \text { Supervisory Control Theory } & \begin{array}{c}\text { Plant behavior: 11 finite automata } \\ \text { (481 states and 1330 transitions) }\end{array} & \begin{array}{c}\text { Finite automaton of 45 } \\ \text { states and 70 transitions }\end{array} & 130 \text { lines } \\ \text { Specifications: 11 finite automata } & 26 \text {-variable switching } \\ \text { functions }\end{array}\right]$


Figure 7: State model of the obtained control law.
depends on the relevance of the requirements proposed by the designer. This step of formalization, by the designer, of the informal requirements of the system to be controlled is the Achilles heel of all synthesis methods, including the Supervisory Control Theory (SCT), and cannot be automated.

The objective comparison of our approach with other synthesis methods, and more especially with SCT, is very difficult because the models used and the theoretical basics are very different. Nevertheless, we tested both approaches on same study cases. One of them, the control of an automatic parking gate, has been published in [26]. The results obtained in this case are summarized in Table 1.

Furthermore, one may note that the supervisor that is synthesized by SCT is optimal in the sense where it
is the most permissive; that is, the one that reduces the less the plant behavior in order to force it to respect the specifications. As shown in this paper our method allows to cope with inconsistencies in specifications, what is not possible with SCT, and also allows to find optimal controllers by choosing different optimization criteria (most permissive, most restrictive, most safe controller, etc.).

## 7. Conclusion

Many research works in the field of DES aim at formalizing steps of the systems life cycle. Since 20 years, significant progresses have been obtained for the synthesis, verification, performance evaluation, and diagnosis of DESs. Nevertheless, one of the common difficulties of these works is the translation of informal expression of the knowledge of a system into formal requirements. Few works have paid attention to this important task which is very error prone. In this paper, we proposed an iterative process that allows coping with inconsistencies of the requirements during the synthesis of the controller. The framework in which we proposed this approach is an algebraic synthesis method. Since the problem is located in the frontier between formal and informal, intervention of the designer is necessary. Nevertheless, we have shown that this intervention can be guided by the results of the formal method provides.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Robust Parametric Control of Spacecraft Rendezvous 

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#### Abstract

This paper proposes a method to design the robust parametric control for autonomous rendezvous of spacecrafts with the inertial information with uncertainty. We consider model uncertainty of traditional C-W equation to formulate the dynamic model of the relative motion. Based on eigenstructure assignment and model reference theory, a concise control law for spacecraft rendezvous is proposed which could be fixed through solving an optimization problem. The cost function considers the stabilization of the system and other performances. Simulation results illustrate the robustness and effectiveness of the proposed control.


## 1. Introduction

With further exploration into the space, a set of complex missions is in the space development agenda such as large-scale structure assembling, sending and picking up astronauts, and repairing, saving, and docking, orbital propellant resupply based on the autonomous rendezvous technology [1]. Due to the essential position, many scholars have been focusing on the control problem during rendezvous and some results enlightened deeper research. In the approximately circular orbit, C-W equations [2], derived by Clohessy and Wiltshire, have been widely applied for the depiction of the relative motion between neighboring spacecrafts. The early stage of control design based on C-W equation revealed a number of open-loop methods such as V-bar, R-bar, dual impulsive, and multiple impulsive [3]. With the benefits of control theory flourishing, plenty of advanced control methods are used to solve the rendezvous problems such as using artificial potential function in [4], sliding mode control in [5], adaptive control in [6], and H-infinity theory in [7].

Though the C-W equation supplies an explicit description of the relative motion for spacecrafts, there is an obstacle when applied in reality that the real-time angle velocity of the target spacecraft could not be obtained accurately as result of detection errors and perturbation from environment. This parameter uncertainty affects the control force and system stability directly. It is necessary to investigate the uncertain model for spacecraft rendezvous not depending on accurate
value of real-time angle velocity. The traditional robust control method could deal with parametric uncertainty to recognize rendezvous but some expecting system characters are hard to be included during the control design.

In this paper, the spacecraft rendezvous problem with uncertain parameter would be solved by robust parametric method which allows freedom to improve system performance. The robust control integrates eigenstructure assignment and model reference theory to propose a concise control law for spacecraft rendezvous which takes into consideration the system performance such as the control constraints and fuel saving. In the rest of this paper, a relative motion model with uncertainty for the spacecraft rendezvous is to be established; the design of the robust parametric control law follows; besides, we apply the robust parametric control for an example to illustrate the effectiveness of this design approach.

## 2. Problem Formation

2.1. Equations of Motion. The coordinate frame for the two spacecrafts rendezvous is based on the target spacecraft orbit, described in Figure 1. We set the original point at the target's mass center; $x, y$, and $z$ indicate along-track, the radial, and out of plane components of the position vector of the chaser satellite in the target satellite's local-vertical-local-horizontal (LVLH) frame, respectively.


Figure 1: Orbital coordinate.

The spacecraft rendezvous in the circle orbit would obey the C-W equations

$$
\begin{gather*}
\ddot{x}-2 \omega \dot{y}=-f_{x} \\
\ddot{y}-2 \omega \dot{x}-3 \omega^{2}=-f_{y}  \tag{1}\\
\ddot{z}+\omega^{2} z=-f_{z},
\end{gather*}
$$

where $x, y$, and $z$ stand for the relative position between the chase spacecraft and the target spacecraft; $\omega$ represents the average angle velocity of the target spacecraft; $f_{x}, f_{y}$, and $f_{z}$ stand for the control acceleration on each axis.

According to the equation, the state and control vector can be described as

$$
\begin{align*}
X & =\left[\begin{array}{lllll}
x & y & z & \dot{x} & \dot{y} \\
\dot{z}
\end{array}\right]^{T} \\
u & =\left[\begin{array}{llll}
-f_{x} & -f_{y} & -f_{z}
\end{array}\right]^{T} \tag{2}
\end{align*}
$$

and output vector $Y$ can be

$$
Y=\left[\begin{array}{lll}
x & y & z \tag{3}
\end{array}\right]^{T}
$$

Then, we get

$$
\begin{gather*}
\dot{X}=A X+B u \\
Y=C X, \tag{4}
\end{gather*}
$$

where

$$
\begin{gather*}
A=\left[\right]  \tag{5}\\
B=\left[\begin{array}{l}
\mathbf{0}_{3} \\
\mathbf{I}_{3}
\end{array}\right], \quad C=\left[\begin{array}{ll}
\mathbf{I}_{3} & \mathbf{0}_{3}
\end{array}\right],
\end{gather*}
$$

and $\mathbf{0}_{3}$ represents the matrix with the values of all elements equal to zero; $\mathbf{I}_{3}$ represents the unit matrix.
2.2. Problem Description. The classical C-W equations need accurate angle velocity simultaneously which is difficult to obtain due to the detection error. Therefore, we consider the uncertain item $\theta$ to the angle velocity to make the system model closer to reality.

When the angle velocity changes are

$$
\begin{equation*}
\omega=\omega_{0}(1+\theta) \tag{6}
\end{equation*}
$$

the system model can be described as

$$
\begin{gather*}
\dot{X}=A_{c} X+B u  \tag{7}\\
Y=C X,
\end{gather*}
$$

where

$$
\begin{gather*}
A_{c}=A_{0}+\Delta A \\
A_{0}=\left[\begin{array}{cccccc} 
& \mathbf{0}_{3} & & \mathbf{I}_{3} & \\
0 & 0 & 0 & 0 & -2 \omega_{0} & 0 \\
0 & 3 \omega_{0}^{2} & 0 & 2 \omega_{0} & 0 & 0 \\
0 & 0 & -\omega_{0}^{2} & 0 & 0 & 0
\end{array}\right] .  \tag{8}\\
\Delta A=\left[\begin{array}{cccccc}
\mathbf{0}_{3} & 0 & 0 & -2 \theta \omega_{0} & 0 \\
0 & 0 & \mathbf{0}_{3} & 2 \theta \omega_{0} & 0 & 0 \\
0 & 3 \omega_{0}^{2}\left(2 \theta+\theta^{2}\right) & 0 & -\omega_{0}^{2}\left(2 \theta+\theta^{2}\right) & 0 & 0
\end{array}\right] .
\end{gather*}
$$

The object of the designing control law is to recognize

$$
\begin{equation*}
\lim _{t \rightarrow \infty}\left[Y(t)-Y_{r}(t)\right]=0, \tag{9}
\end{equation*}
$$

where $Y(t)$ is the output of the system and $Y_{r}(t)$ represent the reference relative position between chase spacecraft and target spacecraft. Meanwhile, the uncertainty brings trouble to the stability of the system which would be taken into consideration during designing the control law.

## 3. Design of Robust Parametric Control

The design of the control law aims at reaching the reference point of the chase spacecraft and keeping the closed loop system stable. It could be separated into two parts as stabilization controller and trajectory tracking controller.
3.1. Trajectory Tracking Controller. To begin with, we would design the tracking controller based on the model reference theory. Lemma 1 supplies theoretical evidence for the linear tracking problems referred to [8].

Lemma 1. For the system, if the stabilization feedback control law $K$ exists, the control law following the form as

$$
\begin{equation*}
u=K X+G Y_{r} \tag{10}
\end{equation*}
$$

would obtain the result of tracking reference signal, which means that

$$
\begin{equation*}
\lim _{t \rightarrow \infty}\left[Y(t)-Y_{r}(t)\right]=0 \tag{11}
\end{equation*}
$$

where feedforward control law $G$ could be calculated from the following equation:

$$
\begin{equation*}
G=U-K Z \tag{12}
\end{equation*}
$$

and $U, Z$ could be calculated as

$$
\left[\begin{array}{l}
Z  \tag{13}\\
U
\end{array}\right]=\left[\begin{array}{ll}
A & B \\
C & D
\end{array}\right]^{-1}\left[\begin{array}{l}
0 \\
I
\end{array}\right]
$$

According to Lemma 1, the rendezvous system could track the reference position when the feedback control law $K$ stabilizes the system. Then, the critical task of designed controller is to find a robust stabilization control law $K$. Regarding the eigenstructure assignment of linear system, some useful results would be utilized in the later part which are from [8].

Lemma 2. Suppose $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times r}$, and $(A, B)$ is controllable. $s_{i}, i=1,2, \ldots, n$, are a set of complex numbers, which are symmetric about the real axis. Then, the matrices $K \in \mathbb{R}^{r \times n}$ and $V \in \mathbb{C}^{n \times n}$ satisfying

$$
\begin{equation*}
\bar{A}=A+B K=V \operatorname{diag}\left(s_{1}, s_{2}, \ldots, s_{n}\right) V^{-1} \tag{14}
\end{equation*}
$$

are given by

$$
\begin{gather*}
K=W V^{-1} \\
V=\left[\begin{array}{llll}
v_{1} & v_{2} & \cdots & v_{n}
\end{array}\right], \quad v_{i}=N\left(s_{i}\right) f_{i}  \tag{15}\\
w=\left[\begin{array}{llll}
w_{1} & w_{2} & \cdots & w_{n}
\end{array}\right], \quad w_{i}=D\left(s_{i}\right) f_{i}
\end{gather*}
$$

where $f_{i} \in \mathbb{C}^{r}, i=1,2, \ldots, n$, are arbitrary vectors which satisfy

$$
\begin{gather*}
f_{i}=\bar{f}_{j} \quad \text { if } s_{i}=\bar{s}_{j}  \tag{16}\\
\operatorname{det}(V) \neq 0
\end{gather*}
$$

and $N(s)$ and $D(s)$ are right comprime polynomial matrices satisfying

$$
\begin{equation*}
(s I-A)^{-1} B=N(s) D^{-1}(s) \tag{17}
\end{equation*}
$$

For the rendezvous system in this paper, we could calculate according to Lemma 2 as

$$
\begin{align*}
& N(s)=\left[\begin{array}{cccccc}
-1 & 0 & 0 & -s & 0 & 0 \\
0 & -1 & 0 & 0 & -s & 0 \\
0 & 0 & -1 & 0 & 0 & -s
\end{array}\right]^{T} \\
& D(s)=\left[\begin{array}{ccc}
-s^{2} & -2 \omega_{0} s & 0 \\
-2 \omega_{0} s & 3 \omega_{0}^{2}-s^{2} & 0 \\
0 & 0 & \omega_{0}^{2}-s^{2}
\end{array}\right] . \tag{18}
\end{align*}
$$

Lemma 2 supplies a concise parametric formula for state feedback law $K$ in which the poles of the closed-loop system are included. Proper poles would not only guarantee the system stabilization but also enhance system characters through optimization in some specific fields. Besides, the parametric method offers all kinds of freedom to design the control system with the free parametric vectors $f_{i}, i=$ $1,2, \ldots, n$, which enable us to adjust these parameters for system stabilization.
3.2. Stabilization Controller. Using the control law

$$
\begin{equation*}
u=K X+G Y_{r} \tag{19}
\end{equation*}
$$

the closed-loop system can be described as

$$
\begin{equation*}
\dot{X}=\left(A_{k}+\Delta A\right) X+B G Y_{r} \tag{20}
\end{equation*}
$$

where

$$
\begin{equation*}
A_{k}=A_{0}+B K \tag{21}
\end{equation*}
$$

When $A_{k}$ is a nondefective matrix and the closed-loop system owns the required poles $s_{i}(i=1,2, \ldots, n)$, the sufficient condition for the system stabilization with the uncertainty item $\Delta A$ is [9]

$$
\begin{equation*}
\|\Delta A\|_{2}<\frac{1}{\|P\|_{2}} \tag{22}
\end{equation*}
$$

where $P$ is a symmetric positive definite solution of the following:

$$
\begin{equation*}
A_{k}^{T} P+P A_{k}=-2 I \tag{23}
\end{equation*}
$$

Lemma 3 provides the parametric expression for $P$ based on the eigenstucture of the system.

Lemma 3. The solution to (23) has the following parametric representation:

$$
\begin{equation*}
P=2 V^{-T} Q V^{-1} \tag{24}
\end{equation*}
$$

where

$$
\begin{equation*}
Q=\left[-\frac{v_{i}^{T} v_{j}}{s_{i}+s_{j}}\right]_{n \times n} \tag{25}
\end{equation*}
$$

and $s_{i}, v_{i}, i=1,2, \ldots, n$, are respectively the eigenvalues and corresponding eigenvectors of $A_{k}$.

For a better stabilization system, we hope to minimize $\|P\|_{2}$ which is the result of some adjustments for the parameters $s_{i}(i=1,2, \ldots, n)$ and $f_{i}(i=1,2, \ldots, n)$.
3.3. Optimization of Control Law. We have established the connection between the system characters and the parameters $s_{i}$ and $f_{i}$ through the design of the control law. Therefore, the design problem for the rendezvous system can be converted into the following nonlinear optimization problem:

$$
\begin{array}{ll}
\min & J\left(s_{i}, f_{i}\right) \\
\text { s.t. } & a_{i} \leq \operatorname{Re}\left(s_{i}\right) \leq b_{i}<0 \\
& c_{i} \leq \operatorname{Im}\left(s_{i}\right) \leq d_{i}, \quad i=1,2, \ldots, n \tag{26c}
\end{array}
$$

where $a_{i}, b_{i}, c_{i}$, and $d_{i}$ specify the desired areas of the closedloop eigenvalues.


Figure 2: Autonomous rendezvous trajectory of chaser.


Figure 3: Control inputs during rendezvous mission.
The performance index is chosen as follows:

$$
\begin{equation*}
J=\alpha L^{T} K^{T} K L+\beta\|K\|_{F}+\gamma\|P\|_{2} \tag{27}
\end{equation*}
$$

where $L$ denotes the initial state of system; $\alpha, \beta$, and $\gamma$ are the weighting factors. The first part of (27) is chosen due to the consideration of the input constraint. The second item of (27) takes into consideration fuel consumption. The last part of (27) is used for global stability of the rendezvous system.

The optimization discussed above could be solved resorting to the optimization tool in MATLAB for its convenience. Then, the poles $s_{i}(i=1,2, \ldots, n)$ of the system and free parametric vectors $f_{i}(i=1,2, \ldots, n)$ would be fixed to calculate the feedback matrix for the robust control.

## 4. Numerical Simulations

In this section, our control law designed through the method proposed above would be tested by an example of spacecrafts in the final approaching in rendezvous mission. With the assumption that the target is in the geosynchronous orbit,


Figure 4: Relative position of two spacecraft.
we set the standard angle velocity $\omega_{0}=7.2921 \times 10^{-5} \mathrm{rad} / \mathrm{s}$. Suppose the initial state vector is

$$
X(0)=\left[\begin{array}{llllll}
500 & -1500 & -800 & -1.0 & 0.1 & 0.1 \tag{28}
\end{array}\right]^{T}
$$

and the desired final state is

$$
X\left(t_{f}\right)=\left[\begin{array}{llllll}
0 & 0 & 0 & 0 & 0 & 0 \tag{29}
\end{array}\right]^{T}
$$

where $t_{f}$ is final time of the rendezvous mission. The weighting factors in the performance index $J$ are

$$
\begin{equation*}
\alpha=0.1, \quad \beta=15, \quad \gamma=0.0001 \tag{30}
\end{equation*}
$$

Specify the desired closed-loop eigenvalue regions as

$$
\begin{equation*}
-0.1 \leq \operatorname{Re}\left(s_{i}\right) \leq-0.001, \quad-0.1 \leq \operatorname{Im}\left(s_{i}\right) \leq 0.1 \tag{31}
\end{equation*}
$$

By solving the optimization problem (26a), (26b), and (26c), we gain the poles $s_{i}$ of the system and free parametric vectors $f_{i}, i=1,2, \ldots, 6$,

$$
\begin{gather*}
s_{1,2}=-0.0064 \mp 0.0008 \mathrm{i}, \quad s_{3,4}=-0.0456 \mp 0.0519 \mathrm{i}, \\
s_{5,6}=-0.0606 \mp 0.0094 \mathrm{i} \\
f_{1,2}=\left[\begin{array}{c}
-11.0091 \mp 50.3604 \mathrm{i} \\
-0.8446 \pm 20.1762 \mathrm{i} \\
-2.1281 \pm 0.5444 \mathrm{i}
\end{array}\right], \\
f_{3,4}=\left[\begin{array}{c}
14.0820 \pm 2.7142 \mathrm{i} \\
5.5407 \mp 20.7854 \mathrm{i} \\
-12.6640 \pm 26.1508 \mathrm{i}
\end{array}\right], \\
f_{5,6}=\left[\begin{array}{c}
31.5687 \pm 7.1917 \mathrm{i} \\
-1.5892 \pm 4.2163 \mathrm{i} \\
-0.3478 \pm 12.9690 \mathrm{i}
\end{array}\right] . \tag{32}
\end{gather*}
$$

According to Lemma 2 and (15), we get

$$
\begin{gather*}
V=\left[\begin{array}{cccccc}
11.0091+50.3604 \mathrm{i} & 11.0091-50.3604 \mathrm{i} & -14.0820-2.7142 \mathrm{i} & -14.0820+2.7142 \mathrm{i} & -31.5687-7.1917 \mathrm{i} & -31.5687+7.1917 \mathrm{i} \\
0.8446-20.1762 \mathrm{i} & 0.8446+20.1762 \mathrm{i} & -5.5407+20.7854 \mathrm{i} & -5.5407-20.7854 \mathrm{i} & 1.5892-4.2163 \mathrm{i} & 1.5892+4.2163 \mathrm{i} \\
2.1281-0.5444 \mathrm{i} & 2.1281+0.5444 \mathrm{i} & 12.6640-26.1508 \mathrm{i} & 12.6640+26.1500 \mathrm{i} & 0.3478-12.9690 \mathrm{i} & 0.3478+12.9690 \mathrm{i} \\
-0.1107-0.3135 \mathrm{i} & -0.1107+0.3135 \mathrm{i} & 0.7830-0.6071 \mathrm{i} & 0.7830+0.6071 \mathrm{i} & 1.9807+0.1391 \mathrm{i} & 1.9807-0.1391 \mathrm{i} \\
0.0107+0.1298 \mathrm{i} & 0.0107-0.1298 \mathrm{i} & -0.8261-1.2354 \mathrm{i} & -0.8261+1.2354 \mathrm{i} & -0.056+0.2704 \mathrm{i} & -0.056-0.2704 \mathrm{i} \\
-0.0132+0.0052 \mathrm{i} & -0.0132-0.0052 \mathrm{i} & 0.7797+1.8497 \mathrm{i} & 0.7797-1.8497 \mathrm{i} & 0.1008+0.7892 \mathrm{i} & 0.1008-0.7892 \mathrm{i}
\end{array}\right]  \tag{33}\\
\\
W=\left[\begin{array}{cccccc}
-0.0001+0.0022 \mathrm{i} & -0.0001-0.0022 \mathrm{i} & 0.0217-0.0651 \mathrm{i} & 0.0217+0.0651 \mathrm{i} & -0.1050-0.0617 \mathrm{i} & -0.1050+0.0617 \mathrm{i} \\
0.0002-0.0009 \mathrm{i} & 0.0002+0.0009 \mathrm{i} & -0.0949-0.0389 \mathrm{i} & -0.0949+0.0389 \mathrm{i} & 0.0108-0.0132 \mathrm{i} & 0.0108+0.0132 \mathrm{i} \\
0.0001 & 0.0001 & 0.1160+0.0760 \mathrm{i} & 0.1160-0.0760 \mathrm{i} & 0.0160+0.0461 \mathrm{i} & 0.0160-0.0461 \mathrm{i}
\end{array}\right] .
\end{gather*}
$$

Then, the stabilization control law $K$ to (15) is

$$
K=\left[\begin{array}{cccccc}
-0.0011 & -0.0020 & 0.0027 & -0.0685 & -0.0435 & -0.0271  \tag{34}\\
0.0005 & 0.0013 & -0.0028 & 0.0170 & 0.0149 & -0.0587 \\
0.0000 & -0.0009 & 0.0013 & 0.0053 & -0.1128 & -0.0040
\end{array}\right]
$$

Assume the uncertainty $\theta=0.01$ which leads to

$$
\begin{equation*}
\|\Delta A\|_{2}=1.4584 \times 10^{-6} \tag{35}
\end{equation*}
$$

Meanwhile, the closed-loop poles $s_{i}$ and parametric vectors $f_{i}$ could be used to calculate as

$$
\begin{equation*}
\frac{1}{\|P\|_{2}}=1.5037 \times 10^{-5} . \tag{36}
\end{equation*}
$$

It is obvious that the rendezvous process could reach the desired state with the control law $K$ when the inequality (22) has satisfied. The rendezvous trajectory and the relative position of the two spacecraft are showed in Figures 2 and 4 and the effectiveness could be proved simultaneously. Due to the proper optimization function, the control inputs have been constrained to $[-1,1]$ which can be seen in Figure 3. The motion in every axis direction changes smoothly so that the simulation system gets closer to the real engineering requirement showed in Figure 3.

## 5. Conclusion

This paper has proposed a method to design the robust control law for spacecraft rendezvous in the final approach subject to parameter uncertainty in near circle orbit. Based on the eigenstructure assignment and model reference theory, the control law is constructed with the closed-loop poles and design freedom. Through solving an optimization problem, we obtain the poles and parametric vectors to calculate the control law which has been proved useful by simulation.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Iterative Selection of Unknown Weights in Direct Weight Optimization Identification 

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#### Abstract

To the direct weight optimization identification of the nonlinear system, we add some linear terms about input sequences in the former linear affine function so as to approximate the nonlinear property. To choose the two classes of unknown weights in the more linear terms, this paper derives the detailed process on how to choose these unknown weights from theoretical analysis and engineering practice, respectively, and makes sure of their key roles between the unknown weights. From the theoretical analysis, the added unknown weights' auxiliary role can be known in the whole process of approximating the nonlinear system. From the practical analysis, we learn how to transform one complex optimization problem to its corresponding common quadratic program problem. Then, the common quadratic program problem can be solved by the basic interior point method. Finally, the efficiency and possibility of the proposed strategies can be confirmed by the simulation results.


## 1. Introduction

The theory of system identification can be divided into linear system and nonlinear system identification. In the classical reference [1], the identification of linear system is discussed in the time domain. Then, the whole system identification field can be divided into four procedures and the accuracy analyses corresponding to various identification algorithms are explained in the probability framework. The time domain identification can be extended to the frequency domain in [2]. Now, the research on the nonlinear system identification point out that the nonlinear system can be approximately regarded as a linear term adding a distortion term in [3]. All the nonlinear characteristic factors of the nonlinear system can be contained in this distortion term. In [4], many special nonlinear systems are studied, for example, Wiener system, Hammerstein system, and so forth. So, various identification methods are proposed to solve these nonlinear system identification problems, such as minimum probability method, covariance instrumental variable method, and blind maximum likelihood method. The most practical method that is used to identify the nonlinear system is the basis
function method. After prior selecting a group of basis functions, the nonlinear system is approximatively expanded under the prior basis functions. In order to attain the required accuracy, let the approximate error between the expansion and nonlinear system converge to zero by adjusting the unknown weights of each basis function. In [5], the process about how to construct the orthonormal basis functions using some prior poles of the denominator is given.

Based on the idea of adjusting the unknown weights to improve the approximate accuracy with basis function, a new nonlinear system identification method-direct weight optimization was proposed in [6]. The main core is that firstly we select an estimator that is linear in the observed output data of the nonlinear system and the adjusted weights are contained in this linear affine function expression. When disturbance noise exists, we get an optimization problem under the condition of the optimum approximate error. The optimum adjusted weights are derived in theory through the classical optimality KKT condition. In [7], the basic idea of the new direct weight optimization is applied to identify each weight that exists in the piecewise affine system. In [8], the effect of the perturb from the direct weight optimization is
analyzed. It points out that when one parameter's perturb range tends to infinity, the solution can be expressed as a piecewise linear solution path.

Based on the foundation idea of the references, we directly collect not only the observed output sequences but also the input sequences. Because the input sequences can be designed freely. So, the two sequences are all known as the prior information. From all above descriptions, we add the observed output and input sequences in the linear affine function simultaneously. Then, there exist two kinds of unknown weights about each observed input-output sequences. When compared with [3], many unknown weights corresponding to the all input sequences are added. These unknown weights can not only alleviate the dependence coming from the unknown weights of the only observed output sequences but also avoid negative effect from the perturbance. After adding some linear terms about the input sequences, the expected minimal mean square error is adopted as a criterion function to select those unknown weights. In the optimization problem of solving those unknown weight, the contribution of this paper is to deduce the selection strategy from the theory and engineering practice, respectively. We gain the unknown adjusted weights using optimality KKT sufficient and necessary condition and find that the second unknown weights that correspond to the observed output sequences are easy to get. Their concrete expressions of the second unknown weights do not depend on the first unknown weights corresponding to the input sequences. The whole selection process tells us that the second unknown weights undertake the key roles and the first unknown weights undertake the auxiliary roles. But this auxiliary effect coming from the first unknown weights may not be neglected.

This paper is organized as follows. In Section 2, we describe the problem discussed in this paper. In Section 3, we propose to add the input sequences to the linear affine function and derive an upper bound value of the objective function. In Section 4, we derive two kinds of unknown weights by using optimality KKT condition from [9]. In Section 5, the interior point algorithm is applied to solve a quadratic programming problem to get the unknown weights. The convergences of the two methods are analyzed, respectively, in Section 6. In Section 7, the numerical simulation results are given to validate the efficiency. Finally, the conclusions are drawn in Section 8.

## 2. Problem Description

Given the observed data $\{\varphi(t), y(t)\}_{t=1}^{N}$ from the nonlinear system,

$$
\begin{equation*}
y(t)=f_{0}(\varphi(t))+e(t) \tag{1}
\end{equation*}
$$

where $f_{0}(\varphi(t))$ is an unknown nonlinear system which need to be identified, $\varphi(t)$ is called the regression vector and $e(t)$ is an independent zero mean stochastic white noise with variance $\sigma_{e}^{2}$. When the regression vector $\varphi(t)$ is chosen as the
following form, the nonlinear system is called an exogenous input model:

$$
\varphi(t)=\left[\begin{array}{lllll}
u(t-1) & \cdots & u\left(t-n_{u}\right) & e(t-1) & \cdots  \tag{2}\\
e & e\left(t-n_{e}\right)
\end{array}\right]^{T} .
$$

Suppose a linear affine function is used to approximate the nonlinear system $f_{0}(\varphi(t))$ as follows:

$$
\begin{equation*}
\widehat{f}\left(\varphi^{*}(t)\right)=a_{0}+\sum_{t=1}^{N} a_{t} u(t)+\sum_{t=1}^{N} b_{t} y(t) \tag{3}
\end{equation*}
$$

In (3), a linear term comprised of $N$ terms of input sequences $\{u(t)\}_{t=1}^{N}$ is added. Then, we identify more $N$ unknown weights $\left\{a_{i}\right\}_{i=1}^{N}$ additionally. As the approximation performance depends tightly on the $2 N+1$ unknown weights. The main goal of this paper is to determine a parameter vector $\theta$ which is consisted of $2 N+1$ unknown weights:

$$
\begin{equation*}
\theta=\left[a_{0}, a_{1}, \ldots, a_{N}, b_{1}, \ldots, b_{N}\right]^{T} . \tag{4}
\end{equation*}
$$

## 3. Direct Weight Optimization Identification

As the nonlinear system $f_{0}(\varphi(t))$ is approximated by the linear affine function $\widehat{f}\left(\varphi^{*}(t)\right)$, we want to find a linear affine function $\widehat{f}\left(\varphi^{*}(t)\right)$ at an arbitrarily given point $\varphi^{*}(t)$. The approximation accuracy depends on the weights $\left\{a_{t}\right\}_{t=0}^{N}$ and $\left\{b_{t}\right\}_{t=1}^{N}$. A most commonly used criterion function would be the mean square error:

$$
\begin{equation*}
W\left(\varphi^{*}, f_{0}, \theta\right)=E\left[\widehat{f}\left(\varphi^{*}(t)\right)-f_{0}\left(\varphi^{*}(t)\right)\right]^{2} \tag{5}
\end{equation*}
$$

Substituting (3) into (5), we obtain

$$
\begin{equation*}
W\left(\varphi^{*}, f_{0}, \theta\right)=E\left[a_{0}+\sum_{t=1}^{N} a_{t} u(t)+\sum_{t=1}^{N} b_{t} y(t)-f_{0}\left(\varphi^{*}(t)\right)\right]^{2} . \tag{6}
\end{equation*}
$$

Substituting (1) into (6), the objection function is expanded to the following expression:

$$
\begin{align*}
W\left(\varphi^{*}, f_{0}, \theta\right)= & {\left[a_{0}+\sum_{t=1}^{N} a_{t} u(t)+\sum_{t=1}^{N} b_{t} f_{0}(\varphi(t))-f_{0}\left(\varphi^{*}(t)\right)\right]^{2} } \\
& +\sigma_{e}^{2} \sum_{t=1}^{N} b_{t}^{2} . \tag{7}
\end{align*}
$$

To simplify the description, we introduce the notation $\widetilde{\varphi}(t)=$ $\varphi(t)-\varphi^{*}(t)$. Then, after adding and subtracting the same two terms, the equality is not changed. Consider

$$
\begin{align*}
& W\left(\varphi^{*}, f_{0}, \theta\right) \\
& \begin{aligned}
&=\left[a_{0}+\sum_{t=1}^{N} a_{t} u(t)+\sum_{t=1}^{N} b_{t}\left(f_{0}(\varphi(t))-f_{0}\left(\varphi^{*}(t)\right)\right.\right. \\
&\left.-\nabla f_{0}\left(\varphi^{*}(t)\right) \widetilde{\varphi}(t)\right) \\
&\left.\quad+f_{0}\left(\varphi^{*}(t)\right)\left(\sum_{t=1}^{N} b_{t}-1\right)+\nabla f_{0}\left(\varphi^{*}(t)\right) \sum_{t}^{N} b_{t} \widetilde{\varphi}(t)\right]^{2} \\
& \quad+\sigma_{e}^{2} \sum_{t=1}^{N} b_{t}^{2}
\end{aligned}
\end{align*}
$$

In (8), the square term is called the square bias term and the last term is the variance error term caused by the unmodeled factor. From (8), we see that the bias term will be arbitrarily large, unless we impose two constraint conditions of the unknown weights $\left\{b_{t}\right\}_{t=1}^{N}$ :

$$
\begin{equation*}
\sum_{t=1}^{N} b_{t}=1, \quad \sum_{t=1}^{N} b_{t} \tilde{\varphi}(t)=0 \tag{9}
\end{equation*}
$$

Under (9), the objective function can be simplified to the following expression:

$$
\begin{align*}
& W\left(\varphi^{*}, f_{0}, \theta\right) \\
& =\left[a_{0}+\sum_{t=1}^{N} a_{t} u(t)+\sum_{t=1}^{N} b_{t}\left(f_{0}(\varphi(t))-f_{0}\left(\varphi^{*}(t)\right)\right.\right. \\
& \left.\left.-\nabla f_{0}\left(\varphi^{*}(t)\right) \widetilde{\varphi}(t)\right)\right]^{2}  \tag{10}\\
& \quad+\sigma_{e}^{2} \sum_{t=1}^{N} b_{t}^{2}
\end{align*}
$$

Expanding the nonlinear system $f_{0}(\varphi(t))$ with Taylor series around $f_{0}\left(\varphi^{*}\right)$ gives

$$
\begin{align*}
f_{0}(\varphi(t))= & f_{0}\left(\varphi^{*}\right)+\frac{d f_{0}(\varphi(t))}{d \varphi(t)}\left(\varphi(t)-\varphi^{*}(t)\right) \\
& +\frac{1}{2} \frac{d^{2} f_{0}(\varphi(t))}{d \varphi(t)^{2}}\left(\varphi(t)-\varphi^{*}(t)\right)^{2} \tag{11}
\end{align*}
$$

Assume that the nonlinear function $f_{0}$ satisfies the following Lipschitz condition:

$$
\begin{equation*}
\left\|f_{0}(\varphi(t))-f_{0}\left(\varphi^{*}(t)\right)-\nabla f_{0}\left(\varphi^{*}(t)\right) \widetilde{\varphi}(t)\right\| \leq \frac{L}{2} \widetilde{\varphi}^{2}(t) \tag{12}
\end{equation*}
$$

where $L$ is a constant; letting us combine the above three formulas, we obtain an upper bound on the mean square error (10). Consider

$$
\begin{align*}
W\left(\varphi^{*}, f_{0}, w^{N}\right) \leq & \left(\left|a_{0}\right|+\sum_{t=1}^{N}\left|a_{t}\right||u(t)|+\frac{L}{2} \sum_{t=1}^{N}\left|b_{t}\right|\|\widetilde{\varphi}(t)\|^{2}\right)^{2} \\
& +\sigma_{e}^{2} \sum_{t=1}^{N} b_{t}^{2} \tag{13}
\end{align*}
$$

The minimum mean square error expectations $W\left(\varphi^{*}, f_{0}, w^{N}\right)$ can be converted to the minimum upper bound value of the right side in (13). Hence, an optimization problem is getting

$$
\begin{align*}
& \min _{\theta^{2 N+1}}\left(\left|a_{0}\right|+\sum_{t=1}^{N}\left|a_{t}\right||u(t)|+\frac{L}{2} \sum_{t=1}^{N}\left|b_{t}\right|\|\widetilde{\varphi}(t)\|^{2}\right)^{2}+\sigma_{e}^{2} \sum_{t=1}^{N} b_{t}^{2} \\
& \text { subject to } \sum_{t=1}^{N} b_{t}=1, \quad \sum_{t=1}^{N} b_{t} \widetilde{\varphi}(t)=0 . \tag{14}
\end{align*}
$$

Because an additional term $\sum_{t=1}^{N}\left|a_{t}\right||u(t)|$ exists in (14), so the complexity of this paper increases.

## 4. Optimality KKT Sufficient and Necessary Condition

Notice that there exist some absolute operations in (14). Some slack variables $s_{t}, w_{t}$ are introduced to eliminate the absolute operations as follows:

$$
\begin{align*}
& \left|b_{t}\right| \leq s_{t}, \quad t=1,2, \ldots, N \\
& \left|a_{t}\right| \leq w_{t}, \quad t=0,1, \ldots, N \tag{15}
\end{align*}
$$

Using these slack variables $s_{t}, w_{t}$ in (14), the optimization problem can be formulated as

$$
\begin{gather*}
\min _{\theta^{2 N+1},\left\{s_{t}\right\}_{t=1}^{N},\left\{w_{t}\right\}_{t=0}^{N}}\left(w_{0}+\sum_{t=1}^{N} w_{t}|u(t)|+\frac{L}{2} \sum_{t=1}^{N} s_{t}\|\widetilde{\varphi}(t)\|^{2}\right)^{2} \\
+\sigma_{e}^{2} \sum_{t=1}^{N} s_{t}^{2} \\
\text { subject to } \quad s_{t} \geq b_{t}, \quad s_{t} \geq-b_{t}, \quad t=1, \ldots, N  \tag{16}\\
w_{t} \geq a_{t}, \quad w_{t} \geq-a_{t}, \quad t=0, \ldots, N \\
\sum_{t=1}^{N} b_{t}=1, \quad \sum_{t=1}^{N} b_{t} \widetilde{\varphi}(t)=0
\end{gather*}
$$

Now, the next problem is to solve the solutions of the optimization problem (16)

$$
\begin{equation*}
\left(a_{0}, a_{1}, \ldots, a_{N}, b_{1}, \ldots, b_{N},\left.s_{t}\right|_{1} ^{N},\left.w_{t}\right|_{0} ^{N}\right) \tag{17}
\end{equation*}
$$

Applying the optimality KKT sufficient and necessary condition to (16), the Lagrangian function is written as

$$
\begin{align*}
L & \left(\theta^{2 N+1},\left.s_{t}\right|_{1} ^{N},\left.w_{t}\right|_{0} ^{N}, \lambda_{1}, \lambda_{2},\left.\mu_{t}^{ \pm}\right|_{t=0} ^{N},\left.\gamma_{t}^{ \pm}\right|_{t=1} ^{N}\right) \\
= & \left(w_{0}+\sum_{t=1}^{N} w_{t}|u(t)|+\frac{L}{2} \sum_{t=1}^{N} s_{t}\|\widetilde{\varphi}(t)\|^{2}\right)^{2}+\sigma_{e}^{2} \sum_{t=1}^{N} s_{t}^{2} \\
& -\lambda_{1}\left(\sum_{t=1}^{N} b_{t}-1\right)-\lambda_{2}\left(\sum_{t=1}^{N} b_{t} \widetilde{\varphi}(t)\right)-\sum_{t=0}^{N} \mu_{t}^{+}\left(w_{t}-a_{t}\right) \\
& -\sum_{t=0}^{N} \mu_{t}^{-}\left(w_{t}+a_{t}\right)-\sum_{t=1}^{N} \gamma_{t}^{+}\left(s_{t}-b_{t}\right)-\sum_{t=1}^{N} \gamma_{t}^{-}\left(s_{t}+b_{t}\right), \tag{18}
\end{align*}
$$

where $\lambda_{1}$ and $\lambda_{2}$ are the Lagrangian multipliers corresponding to the equality constraint and $\left.\mu_{t}^{ \pm}\right|_{t=0} ^{N}$ and $\left.\gamma_{t}^{ \pm}\right|_{t=1} ^{N}$ are the Lagrangian multiplier vectors corresponding to the $2 N+1$ inequality constraint

$$
\begin{equation*}
\mu_{t}^{ \pm}=\left(\mu_{0}^{ \pm}, \mu_{1}^{ \pm}, \ldots, \mu_{N}^{ \pm}\right)^{T}, \quad \gamma_{t}^{ \pm}=\left(\gamma_{1}^{ \pm}, \ldots, \gamma_{N}^{ \pm}\right)^{T} \tag{19}
\end{equation*}
$$

From the optimality KKT condition, we find the equality relations for the optimal solution as follows:

$$
\begin{align*}
& \frac{\partial L}{\partial a_{t}}= \mu_{t}^{+}-\mu_{t}^{-}=0, \quad t=0, \ldots, N, \\
& \frac{\partial L}{\partial b_{t}}=-\lambda_{1}-\lambda_{2} \widetilde{\varphi}(t)+\gamma_{t}^{+}-\gamma_{t}^{-}=0, \quad t=1, \ldots, N, \\
& \frac{\partial L}{\partial s_{t}}= 2\left(w_{0}+\sum_{t=1}^{N} w_{t}|u(t)|+\frac{L}{2} \sum_{t=1}^{N} s_{t}\|\widetilde{\varphi}(t)\|^{2}\right) \\
& \times\|\widetilde{\varphi}(t)\|^{2}+2 \sigma_{e}^{2} s_{t}-\gamma_{t}^{+}-\gamma_{t}^{-}=0, \\
& \frac{\partial L}{\partial w_{0}}= 2\left(w_{0}+\sum_{t=1}^{N} w_{t}|u(t)|+\frac{L}{2} \sum_{t=1}^{N} s_{t}\|\widetilde{\varphi}(t)\|^{2}\right) \\
&-\mu_{0}^{+}-\mu_{0}^{-}=0, \\
& \frac{\partial L}{\partial w_{t}}= 2\left(w_{0}+\sum_{t=1}^{N} w_{t}|u(t)|+\frac{L}{2} \sum_{t=1}^{N} s_{t}\|\widetilde{\varphi}(t)\|^{2}\right)|u(t)| \\
&-\mu_{t}^{+}-\mu_{t}^{-}=0, \\
& \sum_{t=1}^{N} b_{t}= 1, \quad \sum_{t=1}^{N} b_{t} \widetilde{\varphi}(t)=0, \\
& \mu_{t}^{+}\left(w_{t}-a_{t}\right)=0, \quad \mu_{t}^{-}\left(w_{t}+a_{t}\right)=0, \\
& \gamma_{t}^{+}\left(s_{t}-b_{t}\right)=0, \quad \gamma_{t}^{-}\left(s_{t}+b_{t}\right)=0, \\
& \mu_{t}^{ \pm} \geq 0, \quad t=0, \ldots, N, \quad \gamma_{t}^{ \pm} \geq 0, \quad t=1, \ldots, N . \tag{20}
\end{align*}
$$

Through analyzing many subformulas in (20), we find many implicit optimal equalities:

$$
\begin{align*}
& \left|b_{t}\right|=s_{t}, \quad t=1,2, \ldots, N \\
& \left|a_{t}\right|=w_{t}, \quad t=0,1, \ldots, N \tag{21}
\end{align*}
$$

From the first subformula in (20), we see that $\mu_{t}^{+}=\mu_{t}^{-}$. Further, if $a_{t}>0$ in the ninth subformula in (20), then we see that $w_{t}+a_{t}=\left|a_{t}\right|+a_{t}=2 a_{t}>0$. The ninth subformula holds even when $\mu_{t}^{-}=0$, so from the first subformula we derive that

$$
\begin{equation*}
\mu_{t}^{+}=\mu_{t}^{-}=0 \tag{22}
\end{equation*}
$$

In the second subformula in (20), if $a_{t}<0$, it implies that

$$
\begin{equation*}
w_{t}-a_{t}=\left|a_{t}\right|-a_{t}=-2 a_{t}>0 \tag{23}
\end{equation*}
$$

If the eighth subformula in (20) holds, we make $\mu_{t}^{+}=0$ and, from the first subformula, we see that $\mu_{t}^{+}=\mu_{t}^{-}=0$.

When all the equalities $a_{t}=0$ hold, it means all unknown weights of the input sequences are equal to zeros. Synthesizing two cases $a_{t}>0$ and $a_{t}<0$, we obtain that

$$
\begin{gather*}
\mu_{t}^{+}=\mu_{t}^{-}=0 \\
\frac{\partial L}{\partial a_{t}}=0, \quad t=0, \ldots, N \tag{24}
\end{gather*}
$$

Substituting (24) into the each subformula in (20), every subformula in (20) can be simplified

$$
\begin{gather*}
\lambda_{1}+\lambda_{2} \widetilde{\varphi}(t)=\gamma_{t}^{+}-\gamma_{t}^{-}, \quad t=1, \ldots, N \\
2\left(\left|a_{0}\right|+\sum_{t=1}^{N}\left|a_{t}\right||u(t)|+\frac{L}{2} \sum_{t=1}^{N}\left|b_{t}\right|\|\widetilde{\varphi}(t)\|^{2}\right)\|\widetilde{\varphi}(t)\|^{2} \\
+2 \sigma_{e}^{2}\left|b_{t}\right|=\gamma_{t}^{+}+\gamma_{t}^{-} \\
2\left(\left|a_{0}\right|+\sum_{t=1}^{N}\left|a_{t}\right||u(t)|+\frac{L}{2} \sum_{t=1}^{N}\left|b_{t}\right|\|\widetilde{\varphi}(t)\|^{2}\right)=0  \tag{25}\\
\sum_{t=1}^{N} b_{t}=1, \quad \sum_{t=1}^{N} b_{t} \widetilde{\varphi}(t)=0 \\
\gamma_{t}^{+}\left(\left|b_{t}\right|-b_{t}\right)=0, \quad \gamma_{t}^{-}\left(\left|b_{t}\right|+b_{t}\right)=0
\end{gather*}
$$

The equality relations represented by the fourth and fifth subformula in (25) are completely implied in the constructed Lagrangian function. Substituting the third subformula into the second subformula, we get

$$
\begin{equation*}
2 \sigma_{e}^{2}\left|b_{t}\right|=\gamma_{t}^{+}+\gamma_{t}^{-} \tag{26}
\end{equation*}
$$

When $b_{t}>0$, from the seventh subformula in (25), we get $\left|b_{t}\right|+b_{t}=2 b_{t}>0$.

If the seventh subformula holds, let $\gamma_{t}^{-}=0$. Substituting $\gamma_{t}^{-}=0$ in the first subformula, we get

$$
\begin{equation*}
\gamma_{t}^{+}=\lambda_{1}+\lambda_{2} \widetilde{\varphi}(t) \tag{27}
\end{equation*}
$$

Substituting the above equality into (26), the following equality holds:

$$
\begin{equation*}
b_{t}=\frac{\gamma_{t}^{+}}{2 \sigma_{e}^{2}}=\frac{\lambda_{1}+\lambda_{2} \widetilde{\varphi}(t)}{2 \sigma_{e}^{2}} . \tag{28}
\end{equation*}
$$

When considering $b_{t}<0$, we get

$$
\begin{gather*}
\gamma_{t}^{+}=0 \\
\lambda_{1}+\lambda_{2} \widetilde{\varphi}(t)=-\gamma_{t}^{-}  \tag{29}\\
-2 \sigma_{e}^{2} b_{t}=\gamma_{t}^{-}=-\left(\lambda_{1}+\lambda_{2} \widetilde{\varphi}(t)\right)
\end{gather*}
$$

Formulating the above the equality relations, we get

$$
\begin{equation*}
b_{t}=\frac{\lambda_{1}+\lambda_{2} \widetilde{\varphi}(t)}{2 \sigma_{e}^{2}} \tag{30}
\end{equation*}
$$

All the above give us how to solve the unknown weights $\left\{b_{t}\right\}_{t=1}^{N}$. Substituting (28) into the third subformula in (25), we see that

$$
\begin{equation*}
\left|a_{0}\right|+\sum_{t=1}^{N}\left|a_{t}\right||u(t)|+\frac{L}{2} \sum_{t=1}^{N} \frac{\left|\lambda_{1}+\lambda_{2} \widetilde{\varphi}(t)\right|}{2 \sigma_{e}^{2}}\|\widetilde{\varphi}(t)\|^{2}=0 \tag{31}
\end{equation*}
$$

The following three equations are established:

$$
\begin{align*}
& \left|a_{0}\right|=0, \quad \sum_{t=1}^{N}\left|a_{t}\right||u(t)|=0  \tag{32}\\
& \sum_{t=1}^{N}\left(\lambda_{1}+\lambda_{2} \widetilde{\varphi}(t)\right)\|\widetilde{\varphi}(t)\|^{2}=0
\end{align*}
$$

From (32), we can see that

$$
\begin{gather*}
a_{0}=a_{1}=\cdots=a_{N}=0 \\
\lambda_{1} \sum_{t=1}^{N}\|\widetilde{\varphi}(t)\|^{2}+\lambda_{2} \sum_{t=1}^{N} \widetilde{\varphi}(t)\|\widetilde{\varphi}(t)\|^{2}=0 . \tag{33}
\end{gather*}
$$

Then,

$$
\begin{equation*}
\frac{\lambda_{1}}{\lambda_{2}}=-\frac{\sum_{t=1}^{N} \widetilde{\varphi}(t)\|\widetilde{\varphi}(t)\|^{2}}{\sum_{t=1}^{N}\|\widetilde{\varphi}(t)\|^{2}} \tag{34}
\end{equation*}
$$

Generally when considered in the complex domain, it is easy to get that

$$
\begin{equation*}
\left|a_{0}\right|+\sum_{t=1}^{N}\left|a_{t}\right||u(t)|=-\frac{L}{2} \sum_{t=1}^{N} \frac{\lambda_{1}+\lambda_{2} \widetilde{\varphi}(t)}{2 \sigma_{e}^{2}}\|\widetilde{\varphi}(t)\|^{2} \tag{35}
\end{equation*}
$$

as $|u(t)|$ represents the amplitude value of the input excite signal. When this amplitude is chosen to be constant $|u(t)|=$ $k$ ( $k$ is a constant), then (35) implies

$$
\begin{equation*}
\left|a_{0}\right|+k \sum_{t=1}^{N}\left|a_{t}\right|=-\frac{L}{2} \sum_{t=1}^{N} \frac{\lambda_{1}+\lambda_{2} \widetilde{\varphi}(t)}{2 \sigma_{e}^{2}}\|\widetilde{\varphi}(t)\|^{2} \tag{36}
\end{equation*}
$$

In the linear algebra from [10], the commonly used selection method is to impose a constrained condition about the unknown weights $\left\{a_{t}\right\}_{t=0}^{N}$ in order to guarantee uniqueness

$$
\begin{equation*}
\sum_{t=0}^{N} a_{t}=1 \tag{37}
\end{equation*}
$$

To eliminate the absolute notation in (36), assume that the former $k_{1}+1$ weights $\left\{a_{t}\right\}_{t=0}^{N}$ are positive and the latter $N-k_{1}$ weights are negative. Thus, we get

$$
\begin{align*}
& {\left[\begin{array}{lllllll}
1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 & 1 & \cdots & 1
\end{array}\right]\left[\begin{array}{c}
a_{0} \\
\vdots \\
a_{k} \\
a_{k+1} \\
\vdots \\
a_{N}
\end{array}\right]}  \tag{38}\\
& =\left[\begin{array}{l}
\frac{1}{2}-\frac{L}{4} \sum_{t=1}^{N} \frac{\lambda_{1}+\lambda_{2} \widetilde{\varphi}(t)}{2 \sigma_{e}^{2}}\|\widetilde{\varphi}(t)\|^{2} \\
\frac{1}{2}+\frac{L}{4} \sum_{t=1}^{N} \frac{\lambda_{1}+\lambda_{2} \widetilde{\varphi}(t)}{2 \sigma_{e}^{2}}\|\widetilde{\varphi}(t)\|^{2}
\end{array}\right] .
\end{align*}
$$

In the singular degradation linear equation (38), we get a group of unknown weight sequences $\left\{a_{t}\right\}_{t=0}^{N}$ through selecting $N-2$ free variables.

## 5. Solve the Unknown Weights Iteratively

To solve the unknown weights iteratively from the practice point, suppose $a_{0}=w_{0}=0$ in (16), and there exists three kinds of variables as the decision variables: $\theta^{2 N},\left\{s_{t}\right\}_{t=1}^{N}$, $\left\{w_{t}\right\}_{t=1}^{N}$.

For convenience, introduce a column vector whose dimension is $4 N$. Consider

$$
\begin{equation*}
\eta=\left(\theta^{2 N}, s_{1}, \ldots, s_{N}, w_{1}, \ldots, w_{N}\right)^{T} \tag{39}
\end{equation*}
$$

Formulating $4 N$ inequalities constrained conditions in (16) to a matrix product form,

$$
\begin{align*}
& {\left[\begin{array}{cccccccccccc}
-1 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 0 \\
& & & & \vdots & & & & & & & \\
0 & \cdots & -1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \\
1 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 0 \\
& & & & \vdots & & & & & & & \\
0 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \\
0 & \cdots & 0 & -1 & \cdots & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\
& & & & \vdots & & & & & & & \\
0 & \cdots & 0 & 0 & \cdots & -1 & 0 & \cdots & 1 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 1 & \cdots & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\
& & & & \vdots & & & & & & \\
0 & \cdots & 0 & 0 & \cdots & 1 & 0 & \cdots & 1 & 0 & \cdots & 0
\end{array}\right]\left[\begin{array}{c}
a_{1} \\
\vdots \\
a_{N} \\
b_{1} \\
\vdots \\
b_{N} \\
s_{1} \\
\vdots \\
s_{N} \\
w_{1} \\
\vdots \\
w_{N}
\end{array}\right]}  \tag{40}\\
& \geq \overrightarrow{0}, \\
&
\end{align*}
$$

where $\overrightarrow{0}$ is an $4 N \times 1$ zero vector, and denoting the above equality's left hand as matrix $A, A$ is $4 N \times 4 N$. The inequality constrain conditions can be simplified

$$
\begin{equation*}
A \eta \geq 0 . \tag{41}
\end{equation*}
$$

Similarly, the two equalities constrained conditions can be simplified to the matrix product form as follows:

$$
\begin{align*}
& {\left[\begin{array}{cccccccccccc}
1 & \cdots & 1 & -1 & \cdots & -1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & \widetilde{\varphi}(1) & \cdots & \widetilde{\varphi}(N) & 0 & \cdots & 0 & 0 & \cdots & 0
\end{array}\right]} \\
& \quad \times\left[\begin{array}{llllllllllll}
a_{1} & \cdots & a_{N} & b_{1} & \cdots & b_{N} & s_{1} & \cdots & s_{N} & w_{1} & \cdots & w_{N}
\end{array}\right]^{T} \\
& =0, \tag{42}
\end{align*}
$$

where 0 is a $2 \times 1$ zero vector, and denoting the above equality's left hand as matrix $B, B$ is $2 \times 4 N$. The equality constraint conditions can be simplified

$$
\begin{equation*}
B \eta=0 . \tag{43}
\end{equation*}
$$

It is obvious that the second term of the objective function can be rewritten as

$$
\sigma_{e}^{2} \sum_{t=1}^{N} s_{t}^{2}=\sigma_{e}^{2} \eta^{T}\left[\begin{array}{cccc}
0 & 0 & 0 & 0  \tag{44}\\
0 & 0 & 0 & 0 \\
0 & 0 & E & 0 \\
0 & 0 & 0 & 0
\end{array}\right] \eta=\sigma_{e}^{2} \eta^{T} C_{2} \eta
$$

Furthermore, the computation in the bracket of the objective function can be rewritten as

$$
\begin{align*}
& w_{0}+\sum_{t=1}^{N} w_{t}|u(t)|+\frac{L}{2} \sum_{t=1}^{N} s_{t}\|\widetilde{\varphi}(t)\|^{2} \\
& =\left[0 \cdots 0 \frac{L}{2}\|\widetilde{\varphi}(1)\|^{2} \cdots \frac{L}{2}\|\widetilde{\varphi}(N)\|^{2}|u(1)| \cdots|u(N)|\right] \eta \\
& =C_{1}^{T} \eta . \tag{45}
\end{align*}
$$

Squaring (45), we get

$$
\begin{equation*}
\left(w_{0}+\sum_{t=1}^{N} w_{t}|u(t)|+\frac{L}{2} \sum_{t=1}^{N} s_{t}\|\widetilde{\varphi}(t)\|^{2}\right)^{2}=\eta^{T} C_{1} C_{1}^{T} \eta . \tag{46}
\end{equation*}
$$

Combining (41), (43), (44), and (46), a new optimization problem is to get

$$
\begin{align*}
& \min _{\eta^{4 N}} \eta^{T}\left(C_{1} C_{1}^{T}+\sigma_{e}^{2} C_{2}\right) \eta  \tag{47}\\
& \text { subject to } A \eta \geq 0, \quad B \eta=0
\end{align*}
$$

as the new objective function (47) is a quadratic function about decision variable $\eta$. Also, the inequality and equality constraints are linear functions about $\eta$. Generally, (47) is a quadratic programming problem. The interior point method is applied to solve it.

Defining the Lagrangian function according to (47),

$$
\begin{equation*}
L(\eta, m, n)=\eta^{T}\left(C_{1} C_{1}^{T}+\sigma_{e}^{2} C_{2}\right) \eta-m A \eta-n B \eta . \tag{48}
\end{equation*}
$$

Setting the partial derivative with respect to the fact that $\eta$ is zero, we get the equality

$$
\begin{align*}
& \left(C_{1} C_{1}^{T}+\sigma_{e}^{2} C_{2}\right) \eta-A^{T} m-B^{T} n=0 \\
& m, n \geq 0, \quad m(A \eta)=0, \quad A \eta \geq 0, \quad B \eta=0 \tag{49}
\end{align*}
$$

Introducing a slack variable $z \geq 0$ to eliminate the inequality constraint $A \eta \geq 0$, we rewrite (49)

$$
\begin{align*}
& \left(C_{1} C_{1}^{T}+\sigma_{e}^{2} C_{2}\right) \eta-A^{T} m-B^{T} n=0,  \tag{50}\\
& B \eta=0, \quad A \eta-z=0, \quad z m^{T}=0 .
\end{align*}
$$

Suppose that the matrix comprised by (50) is

$$
F=\left[\begin{array}{c}
\left(C_{1} C_{1}^{T}+\sigma_{e}^{2} C_{2}\right) \eta-A^{T} m-B^{T} n  \tag{51}\\
A \eta-z \\
B \eta \\
Z \Lambda e-\varepsilon e
\end{array}\right]
$$

where

$$
\begin{align*}
& Z=\operatorname{diag}\left(z_{1}, \ldots, z_{4 N}\right) ; \quad \Lambda=\operatorname{diag}\left(m_{1}, \ldots, m_{4 N}\right) ; \\
& \varepsilon \in[0,1], \quad e=[1,1 \cdots 1]^{T} . \tag{52}
\end{align*}
$$

The constrained minimum is solved by updating unknown vector $\eta$ iteratively. This minimum solution is the stationary point of the Lagrangian function. During the minimal process, a new iteration value $\eta$ is updated by adding a correct term $\Delta \eta$ to the current estimation. When applying the constrained Gauss-Newton method, the $\Delta \eta$ must satisfy the solution of the following equality:

$$
\begin{align*}
& {\left[\begin{array}{cccc}
C_{1} C_{1}^{T}+\sigma_{e}^{2} C_{2} & 0 & -A & -B \\
A & -I & 0 & 0 \\
0 & B & 0 & 0 \\
0 & \Lambda & Z & 0
\end{array}\right]\left[\begin{array}{c}
\Delta \eta \\
\Delta z \\
\Delta m \\
\Delta n
\end{array}\right]} \\
& =-\left[\begin{array}{c}
\left(C_{1} C_{1}^{T}+\sigma_{e}^{2} C_{2}\right) \eta-A^{T} m-B^{T} n \\
A \eta-z \\
B \eta \\
Z \Lambda e-\varepsilon e
\end{array}\right]=-F . \tag{53}
\end{align*}
$$

At time $k+1$, the new iterate is defined as the vector

$$
\begin{align*}
& \left(\eta^{k+1}, z^{k+1}, m^{k+1}, n^{k+1}\right) \\
& \quad=\left(\eta^{k}, z^{k}, m^{k}, n^{k}\right)+v(\Delta \eta, \Delta z, \Delta m, \Delta n) \tag{54}
\end{align*}
$$

where the step length of the search direction must satisfy the following inequality:

$$
\begin{equation*}
\left(z^{k+1}, m^{k+1}\right)>0 \tag{55}
\end{equation*}
$$

The search direction is determined by (53). We may add a Levenberg-Marquardt parameter $\delta^{2}$ based on $C_{1} C_{1}^{T}+\sigma_{e}^{2} C_{2}$ in order to avoid the singular phenomenon. It makes the left top corner matrix $(1,1)$ of the left matrix in (53) change to the matrix $C_{1} C_{1}^{T}+\sigma_{e}^{2} C_{2}+\delta^{2} I$. So, it can guarantee that an inverse matrix exists and its inverse matrix is definite and bounded.

## 6. Algorithms Analysis

Now, we analyze the convergences of the two algorithms (20) and (54), respectively. From Sections 4 and 5, we see that the solution of (20) is derived from the optimality KKT sufficient and necessary condition and the solution of (54) is an iterative solution.

According to the optimality KKT necessary and sufficient condition which is similar to [11], the convergence of the algorithm used to identify the unknown weights is given.

Theorem 1. Assume that $\eta_{*}$ is a solution of the quadratic programming problem (47) which satisfies the optimality KKT necessary and sufficient condition (20). If Matrix $\left(C_{1} C_{1}^{T}+\right.$ $\left.\sigma_{e}^{2} C_{2}\right)$ is positive semidefinite for some Lagrangian multipliers $m$ and $n$, then $\eta_{*}$ is a global solution of quadratic programming problem (47).

Proof. If $\eta$ is any other feasible point for (47), we have that $A \eta \geq 0, B \eta=0$ for all $\eta \in R^{4 N}$. Hence, using the optimality KKT necessary and sufficient condition, we have that

$$
\begin{equation*}
\left(\eta-\eta_{*}\right)^{T}\left(C_{1} C_{1}^{T}+\sigma_{e}^{2} C_{2}\right)=\sum m_{i}\left(\eta-\eta_{*}\right)+\sum n_{i} B \eta \geq 0 . \tag{56}
\end{equation*}
$$

By elementary manipulation, we find that
$q(\eta)=\eta^{T}\left(C_{1} C_{1}^{T}+\sigma_{e}^{2} C_{2}\right) \eta$,
$q(\eta) \geq q\left(\eta_{*}\right)+\left(\eta-\eta_{*}\right)^{T}\left(C_{1} C_{1}^{T}+\sigma_{e}^{2} C_{2}\right)\left(\eta-\eta_{*}\right) \geq q\left(\eta_{*}\right)$,
where the first inequality follows from (56) and the second inequality follows from positive semidefinite of $\left(C_{1} C_{1}^{T}+\right.$ $\sigma_{e}^{2} C_{2}$ ). We have shown that $q(\eta) \geq q\left(\eta_{*}\right)$ for any feasible $\eta$, so $\eta_{*}$ is a global solution.

Theorem 1 tells us that if a solution which satisfies all the equality (20) can be found, then it will be a global solution for the original quadratic problem.

When the interior point algorithm is applied to solve (47) iteratively, its convergence conclusion can be gotten.

Theorem 2. Suppose that quadratic function $\eta^{T}\left(C_{1} C_{1}^{T}+\right.$ $\left.\sigma_{e}^{2} C_{2}\right) \eta$ and linear function $A \eta, B \eta$ are all continuous second differentiable functions in a neighborhood of a regular stationary point $\eta_{*}$ with associated multipliers $m_{*}, n_{*}$.

Suppose also that the functions $m(), n()$ used to set the value of $\eta$ satisfy $m\left(\eta_{*}\right)=m_{*}, n\left(\eta_{*}\right)=n_{*}$ and are continuous at $\eta_{*}$. Then, there exists a neighborhood $V$ of $\eta_{*}$ such that if the first iterates $\eta_{1} \in V$, the above interior point algorithm is well defined and generates a sequence $\left\{\eta_{k}\right\}$ iteratively by (54) converging superlinearly into $\eta_{*}$.

Proof. Simplifying (53) to emphasize the iterative number, the linear system (53) can be written as

$$
F^{\prime}\left(\eta_{k}, m_{k}, n_{k}\right)\left[\begin{array}{c}
\Delta \eta_{k}  \tag{58}\\
\Delta m_{k} \\
\Delta n_{k}
\end{array}\right]=-F\left(\eta_{k}, m_{k}, n_{k}\right)
$$

If $\eta_{k}$ is in some neighborhood of the regular stationary point $\eta_{*}$, with associated multipliers $m_{*}, n_{*}$ satisfies $\left(m_{k}, n_{k}\right) \rightarrow$ $\left(m_{*}, n_{*}\right)$.

Furthermore, $F^{\prime}\left(\eta_{k}, m_{k}, n_{k}\right)=F^{\prime}\left(\eta_{k}, m\left(\eta_{k}\right), n\left(\eta_{k}\right)\right)$ is nonsingular and has a bounded inverse on that neighborhood. With the notation

$$
z_{k+1}=\left[\begin{array}{c}
\eta_{k+1}  \tag{59}\\
m_{k} \\
n_{k}
\end{array}\right], \quad z_{k, *}=\left[\begin{array}{c}
\eta_{k} \\
m_{*} \\
n_{*}
\end{array}\right], \quad z_{*}=\left[\begin{array}{c}
\eta_{*} \\
m_{*} \\
n_{*}
\end{array}\right],
$$

and with the objective and constraint functions that are all continuous second differentiable functions, we have

$$
\begin{align*}
& z_{k+1}-z_{*} \\
& =z_{k, *}-z_{*}-F^{\prime}\left(\eta_{k}, m_{k}, n_{k}\right)^{-1} F\left(\eta_{k}, m_{k}, n_{k}\right) \\
& =F^{\prime}\left(\eta_{k}, m_{k}, n_{k}\right)^{-1} F\left(\eta_{k}, m_{k}, n_{k}\right)\left(z_{k, *}-z_{*}\right) \\
& \quad-F\left(z_{*}\right)-\int_{0}^{1} F^{\prime}\left(\eta_{*}+t\left(\eta_{k}-\eta_{*}\right), m_{k}, n_{k}\right)\left(z_{k, *}-z_{*}\right) d t . \tag{60}
\end{align*}
$$

Using $F\left(z_{*}\right)=0$ and taking norms, we get

$$
\begin{align*}
& \left\|z_{k+1}-z_{*}\right\| \\
& \leq \\
& \leq C\left(\int_{0}^{1} F^{\prime}\left(\eta_{k}, m_{k}, n_{k}\right)-F^{\prime}\left(\eta_{*}+t\left(\eta_{k}-\eta_{*}\right), m_{k}, n_{k}\right) d t\right)  \tag{61}\\
& \quad \times\left(\eta_{k}-\eta_{*}\right)
\end{align*}
$$

where $C$ is a positive constant. Since $F^{\prime}(\cdot, m(\eta), n(\eta))$ is continuous at $\eta_{*}$ and the last estimate gives $z_{k+1}-z_{*}=$ $o\left(\left\|\eta_{k}-\eta_{*}\right\|\right)$, it implies the superlinear convergence of $\eta_{k}$ to $\eta_{*}$ and

$$
\left[\begin{array}{c}
m_{k}  \tag{62}\\
n_{k}
\end{array}\right]-\left[\begin{array}{c}
m_{*} \\
n_{*}
\end{array}\right]=o\left(\left\|\eta_{k}-\eta_{*}\right\|\right)
$$



Figure 1: The relations between the friction force and the speed under sine position input signal.

## 7. Simulation Example

As the nonlinear system can be approximated by a linear affine function using direct weight optimization method, we apply this idea to approximate the Stribeck nonlinear friction which appears in the flight simulation turntable system.

The Stribeck nonlinear friction model is described as

$$
\begin{equation*}
f(t)=\left(f_{c}+\left(f_{s}-f_{c}\right) e^{\left.-(\dot{\theta}(t)) \dot{\theta}_{s}\right)^{2}}\right) \operatorname{sgn}(\dot{\theta}(t))+K \dot{\theta}(t), \tag{63}
\end{equation*}
$$

where $f_{s}$ is the maximum static friction force, $f_{c}$ is coulomb friction force, $K$ is a viscous friction coefficient, and $\dot{\theta}_{s}$ is the critical Stribeck speed. Let us regard $\dot{\theta}(t)$ in (56) as $\varphi(t)$ in (1) and apply the new linear affine function to approximate the Stribeck nonlinear friction model as follows:

$$
\begin{equation*}
\widehat{f}(t)=a_{0}+\sum_{t=1}^{N} a_{t} \dot{\theta}(t)+\sum_{t=1}^{N} b_{t} f(t) \tag{64}
\end{equation*}
$$

where $\dot{\theta}(t)$ is treated as the input signal. We minimize the performance function (10) to obtain the unknown parameter vector $\left(a_{0}, a_{1}, \ldots, a_{N}, b_{1}, \ldots, b_{N}\right)$. The interior point algorithm is applied to solve it and the number of $N$ is selected by trying test method. When $N$ is increased to some fixed value, we survey whether the performance index function will not change much. If not, then this fixed value is the number of $N$. Next, we make some simulations on the Stribeck nonlinear friction.

In Figure 1, we plot the relation curve between the friction force and the speed under sine position input signal. We compare the three curves of the true nonlinear friction with the proposed method, classical method. In Figure 1, the black curve represents the true nonlinear friction force, the green represents the linear affine curve proposed by our method, and the red curve represents the curve designed by [3]. From


Figure 2: The relations between the friction force and the speed under slope position input signal.

Figure 1, when the speed is low, the difference is very much obvious. But if the speed is increased, the black and green curve will coincide and the red curve starts to flutter away the black curve. It means that the relationship between the true nonlinear friction force and the linear affine friction force derived from our method will be equal. Then, if the speed is chosen sufficiently high, this paper's linear affine friction force can be used to replace the true nonlinear friction force. To the classical method, it should spend more time to approximate the true nonlinear friction force.

In Figure 2, we plot the relation curve between the friction force and the speed under slope position input signal in the flight simulation turntable. From Figure 2, we see that from the beginning, the linear affine function derived by our method can tightly approximate the nonlinear friction force and it has little swing. But to the classical method, the error is high even from the beginning and in the approximation process the curve has much more swings.

We plot the crawl phenomenon under slope position input signal in Figure 3. From Figure 3, each output corresponding to the nonlinear friction model is full of many irregular curves. And each output corresponding to the linear affine function model is full of many piecewise lines. The embodiment of the approximation is to use these piecewise lines to approximate the irregular curve at different time periods. In every time period, the approximation error is defined as the derivation between the line and the corresponding curve. At the beginning, this deviation error is bigger. As the time goes, the lines are close to the curve and the approximate error is small.

## 8. Conclusion

This paper derives how to choose the unknown weights from the theory and engineering, respectively, in the improved direct weight optimization method. Because the input


Figure 3: The crawl phenomenon under slope position input signal.
sequences should be designed to sufficiently excite the nonlinear system, further research on the optimal input signal design must be dealt with in future.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Spontaneous Synchronization in Two Mutually Coupled Memristor-Based Chua's Circuits: Numerical Investigations 

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#### Abstract

Chaotic dynamics of numerous memristor-based circuits is widely reported in literature. Recently, some works have appeared which study the problem of synchronization control of these systems in a master-slave configuration. In the present paper, the spontaneous dynamic behavior of two chaotic memristor-based Chua's circuits, mutually interacting through a coupling resistance, was studied via computer simulations in order to study possible self-organized synchronization phenomena. The used memristor is a flux controlled memristor with a cubic nonlinearity, and it can be regarded as a time-varying memductance. The memristor, in effect, retains memory of its past dynamic and any difference in the initial conditions of the two circuits results in different values of the corresponding memductances. In this sense, due to the memory effect of the memristor, even if coupled circuits have the same parameters they do not constitute two completely identical chaotic oscillators. As is known, for nonidentical chaotic systems, in addition to complete synchronizations (CS) other weaker forms of synchronization which provide correlations between the signals of the two systems can also occur. Depending on initial conditions and coupling strength, both chaotic and nonchaotic synchronization are observed for the system considered in this work.


## 1. Introduction

One of the most important topics of contemporary science focuses on the study of continuous and discrete dynamical systems [1-3], analysing their organization as nonlinear evolving structures [4-6] or as artificial agents in synthetic environments $[7,8]$. Chaos is the most striking feature of their behaviour. Chaotic systems are nonlinear deterministic systems that display highly complex dynamic with several peculiar features such as fractal properties of the motion in the phase space (strange attractors) and, especially, extraordinary sensitivity to initial conditions and system parameters variations. This implies that, even for two identical chaotic systems, a slight difference in the initial conditions grows exponentially in time resulting in completely different trajectories. Consequently, chaotic systems intrinsically would seem to defy synchronization. Nonetheless, two coupled chaotic systems also can exhibit some form of synchronization, meaning by that a dynamical state wherein a correlation exists among a given property of their motion $[9,10]$.

The synchronization between chaotic systems, either identical or nonidentical, is a fundamental phenomenon in nonlinear dynamics, observed in diverse areas of science and technology. Studies on chaos synchronization are of great interest, both from a theoretical and applicative point of view, due to their possible applications, for example, in cryptography and secure communications [11-13]. The synchronization of chaotic oscillators is also an important process in many biological systems [14].

Since the pioneering works of Pecora et al. [9, 15], it has become known that it is possible to force two chaotic systems to synchronize, and various methods for chaos control and synchronization have been developed [16] such as those based, for example, on sliding-mode control or linear matrix inequality, just to name a few [17-21].

On the other hand, spontaneous synchronization is also possible for nonlinear systems. More precisely, depending on the modalities of interaction between the systems, it is possible to distinguish between two configurations leading
to synchronization: unidirectional coupling (drive-response or master-slave configuration) and bidirectional coupling [10, 22].

In the first case, one of the two systems evolves freely and forces the other system to follow a certain function of the master dynamic, producing external synchronization. This approach is affected, in large part, by the point of view of the dynamics systems control theory. For example, the synchronization control of memristor-based chaotic systems in a drive-response configuration has been recently studied using adaptive control and fuzzy modelling [23, 24].

On the contrary, in the bidirectional coupling configuration spontaneous synchronization is due to the mutual interactions between the chaotic oscillators which self-organize their dynamics and, in this case, the synchronization is configured as an emergent phenomenon. In effect, spontaneous synchronization is recognized in various areas, ranging from physics to biology and social sciences [25]. A typical bidirectional coupling producing synchronization for many dynamical systems is the so-called "diffusive coupling," where the mutually forcing term is proportional to the differences between the states of the systems [26, 27]. The present work considers this coupling configuration and, for the first time as far as the authors' knowledge is concerned, results of numerical investigations on the spontaneous dynamics of two resistively coupled memristor-based Chua's circuit are presented. The "memristor" is the so-called fourth elementary circuit element, theorized by Chua in 1971 [28] in order to complete the mathematical relations connecting pairs of the four fundamental circuit variables (current, voltage, charge, and magnetic flux). It is a two-terminal circuit element in which the magnetic flux $\varphi$ between the terminals is a nonlinear function of the electric charge $q$ that passes through the device. Formally, a memristor is characterized by a relation $f(\varphi, q)=0$, called "the memristor constitutive relation," linking charge and flux, and its memductance is defined as $W(\varphi)=d q(\varphi) / d \varphi$. In the case of nonlinear constitutive relation, the memductance value depends upon the history of the device (i.e., taking into account the Lenz's law $V=d \varphi / d t$, the memductance varies according to the integral over time of the applied voltage). Therefore, the behavior of the memristor depends on its past history and the memristor retains memory of its state even when no current passes through it.

Despite its theorization in 1971, a physical realization of a memristor only occurred in 2008 in the form of a nanometer-sized solid-state two-terminal device, realized by Stan William's group at the Hewlett-Packard (HP) Labs [29]. After its discovery, studies on the special properties of the memristor as electric device have received increasing interest [30, 31]. Many papers focus on the possible technological applications of the memristor, for example, in order to build ultra-dense nonvolatile memories [32], or new kinds of high performance computers [33, 34]. Moreover, the special properties of the memristor appear useful in the modelling cognitive process $[35,36]$ and to emulate the human brain [37, 38]. The memristor is also of great interest in the field of chaotic dynamical systems. Due to the nonlinearity of
its constitutive relation, the memristor-based circuits can generate chaotic dynamics [39-44]. In particular, depending on the parameters and initial conditions of the memristor, a chaotic circuit with memory can produce transient chaos and intermittence [45-47].

The memristor used in this work is characterized by a cubic nonlinearity that makes the behavior of the single circuit chaotic. Since the actual memductance value depends on the history of the applied voltage, starting from different initial conditions the memristors in the two circuits have different memories, which results in different values of the memductance. In this sense, despite having the same circuit parameters, the two circuits can be viewed as nonperfectly identical chaotic oscillators.

It is worth noting that for nonequivalent chaotic oscillators, and depending on the coupling strength, several kinds of synchronization exist [10, 15, 48]. In particular, for identical systems complete synchronization (CS) is possible, and the trajectories of the two systems overlap perfectly. For example, it is known that two bidirectional coupled Chua's circuit reach a state of complete synchronization [49]. A weaker form of synchronization, also possible for nonidentical systems, is phase synchronization (PS), where only the phases of the interacting oscillators are correlated [50]. Other forms of synchronization are lag synchronizations (LG) [51,52] and rhythm synchronization (RS) [53], characterized by a fixed time lag between the trajectories of two coupled nonidentical oscillators. A more general synchronization state, that seems to be the chaos synchrony most frequently found in natural systems [54], is the generalized synchronization (GS). It is characterized by a functional relationship between the trajectories of two coupled systems [55,56], either identical or nonidentical. Therefore, generally speaking, chaos synchronization refers to a dynamic process in which two coupled chaotic systems adjust a given property of their motion to a common behavior, ranging from complete agreement of trajectories to a generic relationship between them.

In order to evaluate the presence of synchronization, the two-dimensional phase portrait between corresponding signals can be used. When CS occurs, the phase portrait consists in a straight line at $45^{\circ}$. Conversely, if two signals are uncorrelated there will be an isotropic cloud of points in the diagram. Between these two extremes, any "structure" in the phase diagram indicates the existence of some kind of correlation between the signals.

In this work, synchronization states, in the sense discussed above, were identified by the appearance of patterns in the phase portraits. This paper is organized as follows. In Section 2, the single memristor based Chua's circuit is presented. The diffusive coupling schema and the equations for the coupled circuits are derived in Section 3. Results of numerical simulations are presented in Section 4. Finally our main conclusions are summarized in Section 5.

## 2. The Memristor-Based Chua's Circuit

The memristor-based chaotic circuit considered in this work was proposed and described by Muthuswamy [57]. It consists


Figure 1: The memristor-based Chua's circuit: the Chua's diode is replaced by a flux-controlled active memristor.


Figure 2: 3D projection of the double-scroll type attractor generated by (3a)-(3d) for initial conditions [ $-24.33,-12480,-7294,2.948]$ and corresponding state variables $x, y, z$, and $w$ as a function of the time.


Figure 3: Peaks of the signal $w(t)$ vs $w(0)$ for initial conditions [ $0,23000,1250, w(0)$ ]. The arrows indicate progression of the dynamics described in the text. The zoom-in at the top shows a particular of the map with a period doubling scenario.


Figure 4: 3D projection of the Chua's spiral-type attractor generated by (3a)-(3d) for initial conditions $\left[\begin{array}{lll}0.0 & 23000 & 1250 \\ 1\end{array}\right]$, and corresponding time series of state variables $x, y, z$, and $w$.


Figure 5: 3D projection of the one-period limit cycle attractor generated by (3a)-(3d) for initial conditions [112.4047, 27015, 9360, -4.0542], and corresponding nonchaotic pseudosinusoidal oscillations of higher amplitude with respect to the chaotic dynamic.
of a Chua's circuit with the diode replaced by a flux-controlled active memristor (Figure 1) characterized by a cubic continuous nonlinearity for the $q-\varphi$ constitutive relation:

$$
\begin{equation*}
q(\varphi)=\alpha \varphi+\beta \varphi^{3} \tag{1}
\end{equation*}
$$

where $\alpha=-0.667 \cdot 10^{-3}$ and $\beta=0.029 \cdot 10^{-3}$. The memductance is given by

$$
\begin{equation*}
W(\varphi)=\alpha+3 \beta \varphi^{2} . \tag{2}
\end{equation*}
$$

Note that the memductance is negative for $\varphi \in$ $(-\sqrt{-\alpha /(3 \beta)}, \sqrt{-\alpha /(3 \beta)})$, therefore the considered memristor is an active element on this interval of magnetic flux [39, 57].

By applying the Kirchhoff's laws to the memristor-based Chua's circuit of Figure 1, the following state equations are obtained:

$$
\begin{equation*}
\frac{d x}{d t}=-\frac{y}{L} \tag{3a}
\end{equation*}
$$



Figure 6: The system of two memristor-based Chua's circuits bidirectionally coupled via a resistor.


Figure 7: Plot of state variables $z$ for both the coupled circuits with $R_{12}=17000$ (chaotic-chaotic initial conditions). Zoom is shown for both the regions of chaotic (box at the top) and more regular pseudosinusoidal oscillations (box at the bottom).

$$
\begin{align*}
\frac{d y}{d t} & =\frac{1}{C_{2}}\left(\frac{z-y}{R}+x\right)  \tag{3b}\\
\frac{d z}{d t} & =\frac{1}{C_{1}}\left(\frac{y-z}{R}-i_{M}\right)  \tag{3c}\\
\frac{d w}{d t} & =z \tag{3d}
\end{align*}
$$

where $x$ is the current through the inductor $L, y$ and $z$ represent the voltages across the capacitor $C_{2}$ and $C_{1}$, respectively, $w$ is the magnetic flux and $i_{M}=W(\varphi) \cdot V_{1}$ is the current through the memristor.

Note that (3a)-(3c) are formally identical to ones reported in the literature for the Chua's circuit [58] with the only difference that the current of the diode is replaced by the current through the memristor. Moreover, due to the presence of a new equation for the magnetic flux (or for the charge in the case of charge-controlled memristor), the substitution of the Chua's diode with a memristor augments the dimension of the equation set describing the original circuit. To obtain chaotic dynamic, the circuit parameters are set to $L=18 \mathrm{mH}, C_{2}=68 \mathrm{nF}, C_{1}=6.8 \mathrm{nF}$, and $R=2000 \Omega$ in original paper describing this circuit [57], and a chaotic attractor is found by numerical simulation of (3a)-(3d) starting from the following initial conditions: $x(0)=0$,
$y(0)=0.11, z(0)=0.11$, and $w(0)=0$. In order to study the dynamics of the memristor-based Chua's circuits, the MATLAB function ode45 implementing an explicit 4th and 5th order Runge-Kutta formula based on the DormandPrince method [59] is used in this work.

A chaotic dynamics was also found for initial conditions [ $-24.33,-12480,-7294,2.948$ ]. Figure 2 shows a 3D projection of the attractor with corresponding time series of the signals.

As is well known, however, chaotic systems are very sensitive to the changes of the initial conditions and different initial values can generate totally different behavior. In order to describe the diverse dynamics of the system (3a)-(3d) produced to vary the initial conditions, the peaks of the flux $w(t)$ as a function of its initial values $w(0)$ was calculated for initial conditions [ $0,23000,1250, w(0)]$. The resulting bifurcation diagram is shown in Figure 3 (peaks of $w(t)$ were recorded after transient).

Coexistence of multiple attractors in the phase space is evident, and a very interesting progression of the dynamics with varying $w(0)$ appears. For example, a scenario with Hopf-like bifurcations and period doubling bifurcations is evident between points $a$ and $b$ of Figure 3, with limit cycles of increasing period (a zoom-in image of this area is shown in the box at the top). For lower $w(0)$ values up to the point $c$, there is an area of fully developed chaos with Chua's spiraltype attractor (Figure 4). Windows of $n$-order limit cycles and trivial fixed points ( $0,0,0, w=$ const) corresponding to the damping of the system also appear.

Between positions $d$ and $e$, the expansion of points indicates the birth of a double-scroll type attractor (such as that shown in Figure 2). Finally, the straight lines at lower values of $w(0)$ indicate the saturation of the systems to a period one orbit, with nonchaotic pseudosinusoidal oscillations of higher amplitude.

Therefore, in addition to the chaotic behaviour and depending on the initial conditions of the circuit, nonchaotic oscillations and damped oscillations are also identified for the system (3a)-(3d). In particular, for initial conditions [112.4047, 27015, 9360, -4.0542] and [-178.1619, -66031, $-16762,4.5430$ ] the system produces nonchaotic and pseudosinusoidal oscillations, with a limit cycle of period 1 in the phase space (Figure 5). This dynamics corresponds to the saturation described above. For initial conditions [22, $10000,0.15,0.2]$ and $[-20,-10000,50000,-2]$ the circuit is damped, and a sink appears in the phase space. These values are just some of the initial conditions that have been considered in order to obtain a coarse characterization of the basins of attraction for the system (3a)-(3d). Further investigations are needed to adequately describe these basins of attraction but it is beyond the scope of the present work. However, the initial conditions indicated above produce the whole dynamic behaviors observed for the single circuit. They result in distant areas of the single circuit phase space and were used to simulate the coupling of circuits starting from significantly different initial conditions, as described in the following section.


FIGURE 8: Phase portraits between corresponding signals of the two coupled circuits during the steady state of nonchaotic oscillation (C-C initial conditions) for different coupling strength. From top: $R_{12}=100,10000,40000$.

## 3. The Coupling Scheme

In this study, two memristor-based Chua's circuits described above are mutually coupled through a resistor $R_{12}$ as shown in Figure 6.

The equations describing the dynamic of the system are:

$$
\begin{align*}
& \frac{d x_{1}}{d t}=-\frac{y_{1}}{L}  \tag{4a}\\
& \frac{d y_{1}}{d t}=\frac{1}{C_{2}}\left(\frac{z_{1}-y_{1}}{R}+x_{1}\right)  \tag{4b}\\
& \frac{d z_{1}}{d t}=\frac{1}{C_{1}}\left(\frac{y_{1}-z_{1}}{R}-\left(\frac{z_{1}-z_{2}}{R_{12}}\right)-W\left(w_{1}\right) \cdot z_{1}\right)  \tag{4c}\\
& \frac{d x_{2}}{d t}=-\frac{y_{2}}{L}  \tag{4d}\\
& \frac{d y_{2}}{d t}=\frac{1}{C_{2}}\left(\frac{z_{2}-y_{2}}{R}+x_{2}\right)  \tag{4e}\\
& \frac{d z_{2}}{d t}=\frac{1}{C_{1}}\left(\frac{y_{2}-z_{2}}{R}+\left(\frac{z_{1}-z_{2}}{R_{12}}\right)-W\left(w_{2}\right) \cdot z_{2}\right) \tag{4f}
\end{align*}
$$

$$
\begin{align*}
& \frac{d w_{1}}{d t}=z_{1}  \tag{4~g}\\
& \frac{d w_{2}}{d t}=z_{2} \tag{4h}
\end{align*}
$$

where symbols have the same meaning as in (3a)-(3d), and subscripts refer to the two circuits.

Accurate numerical integration of (4a)-(4h) was performed for different values of the coupling resistor $R_{12}$ in order to investigate the occurrence of self-induced synchronization phenomena during the free evolution of the system. Since, depending on the initial conditions, the behavior of the single memristor-based circuit can be chaotic (C), oscillating with pseudosinusoidal (PS) dynamics, and damped (D); there are 6 qualitatively different choices for the initial conditions of the two coupled circuits: C-C, C-D, C-PS, D-D, D-PS, and PS-PS. All of these cases were examined using the initial conditions given above. The range of variation for the coupling resistor was initially set to $R_{12} \in$ $\{0.1,1.0,2.0,3.0,4.0,5.0,6.0,7.0,8.0,9.0,10.0,15.0,20.0$, $25.0,30.0,35.0,40.0,45.0,50.0,60.0,70.0,80.0,90.0,100.00\}$ • $10^{3}$. Note that the presence of the resistor $R=1000$ in the circuit fixes a natural length scale for the resistances, and the


Figure 9: Pair of time series of the two mutually coupled circuits, for C-C initial conditions and $R_{12}=20000$. Signals are out of phase with different amplitudes, and some kind of AS is observed.
investigated range for the coupling factor $R_{12}$ correspond to $\sim 10^{-1} R \leq R_{12} \leq \sim 10^{2} R$, that is, a variation of a few orders of magnitude with respect to $R$. Numerical simulations were carried out for $t \in[0,1] \mathrm{s}$. Additional values of $R_{12}$ and longer integration times were investigated when deemed necessary, as detailed below.

## 4. Simulation Results

The results obtained in the $\mathrm{C}-\mathrm{C}$ and $\mathrm{C}-\mathrm{D}$ cases qualitatively reproduce the entire phenomenology observed in all the simulations performed for this study, and only these two cases will be presented below in detail.
4.1. Chaotic-Chaotic (C-C) Initial Conditions. For high coupling (low $R_{12}$ ) nonchaotic synchronization occurs. In more
details, for $R_{12} \in[100,40000]$ and $R_{12} \neq 20000$, after an initial transient during which the two circuits oscillate in a chaotic and uncorrelated way, they reach a state of nonchaotic synchronization. In Figure 7 the time series of the variables $y_{1}$ and $y_{2}$ in the case of $R_{12}=17000$ are depicted. The qualitative trend of other signals is similar to the presented one.

It is evident that an initial state of chaotic behavior exists, with a double-scroll type attractor such as that shown in Figure 2, followed by a situation in which the two circuits oscillate with larger amplitude in a pseudosinusoidal manner. In this latter case the trajectories of the two coupled circuits are limit cycles of order 1 in their respective phase space, as shown in Figure 4.

The existence of some kind of synchronization in this state of nonchaotic oscillation is evidenced by the appearance of well-defined curves in the phase portraits between


Figure 10: 3D projection $(y, z, w)$ of the attractors (a) of the two coupled circuits for the case of C-C initial conditions and $R_{12}=20000$. (b) 2D projection ( $y$ versus $z$ ) of the Chua's spiral type attractor is depicted.


Figure 11: Phase diagrams during the steady state of the systems for $R_{12}=20000$ and chaotic-chaotic initial conditions.


Figure 12: Plot of state variables $x$ for both the coupled circuits with $R_{12}=40000$ (C-C initial conditions). The duration of the chaotic and uncorrelated transient is different for the two circuits. Zoom is shown for the region of the first chaotic transient (box at the top) and for the nonchaotic synchronization final state (box at the bottom).


Figure 13: Phase diagrams during the chaotic oscillations of the two coupled circuits starting from chaotic-damped initial conditions. From top: $R_{12}=100,4000,9000$.


Figure 14: Phase diagrams for chaotic-chaotic initial conditions and $R_{12}=9000$. The signals are considered on a time interval of 0.02 s .
corresponding signals of the two coupled circuits (Figure 8). In particular, for $R_{12}=100$ the signals are practically coincident except for a constant bias for the flux, and a straight line appears in the phase diagrams (Figure 8(a)). For the other investigated values of $R_{12}$, the curves in the phase diagrams assume the form of a hysteresis-like loop with a single pinch (Figures 8(b) and 8(c))). It is worthy to note that during this steady state of pseudosinusoidal oscillation, the corresponding signals present a periodic phase shift. The amplitude difference between the signals and the initial phase shift change as $R_{12}$ varies, and this determines the different aspect of the phase diagrams shown in Figure 8.

For $R_{12}=20000$ the behavior of the system is quite different. After a chaotic and uncorrelated transient, the two coupled circuits achieve a steady state in which the corresponding signals are completely out of phase (Figure 9).

Some kind of antisynchronization [60] (AS) with significantly different amplitudes is observed in this case. Moreover, as shown in Figure 10, phase trajectories of the two systems evolve on different attractors; in particular, the circuit 2 is on a Chua's spiral-type attractor.

The phase diagrams are now more complex (Figure 11), but they still indicate the presence of some kind of synchronization [53].

The steady state behavior for $R_{12}=20000$ with the signals out of phase seems to be peculiar. Indeed, simulations carried out for $R_{12} \in\{19.1,19.2,19.5,19.9,19.999$, c20.001, 20.01, 20.1, 20.2\}•10 ${ }^{3}$ have produced results in accordance with that previously reported for $R_{12} \leq 40000$ and $R_{12} \neq 20000$ (in-phase pseudosinusoidal oscillations).

For $R_{12}=40000$, the system presents a new feature: the duration of the chaotic and uncorrelated transient is different for the two coupled circuits (Figure 12). Only after both


Figure 15: Plot of state variables $z$ for both the coupled circuits at $R_{12}=9000$ for C-D initial conditions. The signals pass through a progression of deconstruction and recomposition of the state of phase-synchronization.
circuits enter into high-amplitude pseudosinusoidal oscillation synchronization occurs, with characteristics similar to cases of $R_{12} \neq 20000$. In particular, the attractors of the two circuits are limit cycles of order 1 .

Finally, for $R_{12} \geq 45000$ the two circuits are practically uncoupled and the respective signals still remain uncorrelated. In more details, after a transient in which both circuits oscillate chaotically, only one circuit begins to oscillate in nonchaotic way with pseudosinusoidal oscillations and greater amplitude for $R_{12}=45000$. This uncorrelated coexistence of order and chaos remained unchanged in simulations of the dynamics of the system up to 80 s. For $R_{12} \geq 50000$ the two circuits remain in chaotic and uncorrelated oscillation.
4.2. Chaotic-Damped (C-D) Initial Conditions. Similarly as discussed above, a synchronization state with pseudosinusoidal oscillations is also observed in this case for low coupling. Moreover, a situation of chaotic oscillations with strong correlation between signals is also found at high coupling for $R_{12} \leq 9000$.

In more detail, for $R_{12}=100$ (Figure 13(a)) the phase portraits indicate a condition of CS , but as $R_{12}$ increases the correlation between the signals rapidly decreases (Figures 13(b) and 13(c)). In effect, the phase diagrams contain points whose dispersion around the diagonal depends on the value of $R_{12}$. For the lowest investigated value of $R_{12}$, trajectories in the phase portraits remain most of the time on the diagonal, and the synchronization is easy to recognize. As $R_{12}$ increases, the duration of periods of desynchronization, that is, the amount of points far from the diagonal, increases and it masks possible "structures" indicating correlation.

In order to highlight this behavior, in Figure 14 the phase diagrams are plotted for a time interval of 0.02 s with $R_{12}=9000$. Despite the cloud-like shape of the corresponding phase diagram in Figure 13(c), from Figure 14 it
is clear that the system passes through a sequence of phasesynchronized states. In effect, observing the signals of the two circuits (Figure 15) it is possible to note a rapid alternation of situations in which the signals are uncorrelated and situations in which the signals oscillate in phase, and a deconstruction and recomposition of the state of phase synchronization happen.

For $10000 \leq R_{12} \leq 20000$ the two circuits reach a state of nonchaotic synchronization with pseudosinusoidal oscillations. More precisely, as reported in Figure 16(a) for $R_{12}=10000$, after a chaotic transient the circuits have a behavior similar to that shown in Figure 7, with highamplitude pseudosinusoidal oscillations that result strongly correlated. The attractors of the synchronized circuit are limit cycles of order 1 (such as that shown in Figure 5).

For $R_{12} \in[15000,20000]$ the chaotic transient disappears and systems immediately synchronize with nonsaturated oscillations (phase diagrams depicted in Figures 16(b) and 16(c)). The attractors are limit cycles of periods 1 and 2, respectively, which result similar to the attractors displayed by the single system (3a)-(3d) during the period doubling bifurcations described in section 2. The amplitude of the oscillations for the circuit starting from damping initial condition is an order of magnitude lower than the other circuit.

Finally, for $R_{12} \geq 25000$ both the two systems remain in chaotic oscillation at different amplitudes evolving on Chua's spiral like attractors. The phase portraits for this case, shown in Figure 17, indicate very weak or null correlation. Unlike what happens for $R_{12} \leq 9000$, also at smaller time scale, coherent substructures do not emerge in the phase diagrams.

## 5. Conclusion

In this paper the problem of spontaneous selfsynchronization of two mutually coupled memristor-based Chuas circuits is investigated via numerical simulations. The great sensitivity of the single circuit on initial conditions was here investigated by means of a bifurcations diagram for the maxima of the flux $w(t)$ as a function of its initial values $w(0)$. A complex progression of the dynamics with varying initial conditions is evident. Beside chaotic (C) dynamics, pseudosinusoidal (PS) oscillations, and damped (D) oscillations were also identified for the single memristor based circuit.

A diffusive coupling between two of these circuits was realized with a resistor $R_{12}$ and accurate numerical simulations were performed for various values of the coupling resistor and for different initial conditions. Synchronization states were identified by the appearance of patterns in the phase portraits of the system which indicate correlation between the signals of the two circuits.

It was found that, depending on the initial conditions and on the coupling strength, both nonchaotic and chaotic synchronization is possible for the coupled circuits. Nonchaotic synchronization with positive correlation between signals seems to be the most frequent situation for the investigated system, and it was observed for all the initial conditions


Figure 16: Phase diagram during the pseudosinusoidal oscillations of the two coupled circuits starting from chaotic-damped initial conditions. From top: $R_{12}=10000,15000,20000$.
examined and for a wide range of $R_{12}$ values. In this case the circuits exhibit pseudosinusoidal waveform oscillations with a small periodic phase shift between corresponding signals of the two coupled circuits. This results in curves forming a hysteresis-like loop in the phase portraits.

Chaotic synchronization was found only for C-D initial conditions at high coupling (small values of $R_{12}$ ) and it is produced by a rapid succession of uncorrelated and phasecorrelated oscillations. Phase portraits show points scattered around the diagonal line, with positive correlation. The duration of the period during which the signals are uncorrelated increases with $R_{12}$ and more smeared phase portraits occur.

With respect to the whole numerical results obtained in this study, a peculiar situation was detected for C-C initial conditions at $R_{12}=9000$. In this case the oscillations of the two circuits are completely out of phase and with significantly different amplitudes. Phase portraits show complex patterns with negative correlation that clearly indicate some kind of synchronization.

Moreover, numerical integrations showed that transient chaos, already reported in literature for single memristorbased systems, also is possible for the coupled circuits
examined in this work. In fact, chaotic and uncorrelated oscillations may precede for a significant time the onset of pseudosinusoidal synchronization.

Finally, computer simulations also indicate the possibility of uncorrelated coexistence of chaos and order. In particular, this situation can be a transient state which precedes nonchaotic synchronization or, for low coupling strength, a stationary state of the coupled circuits.

Therefore, the two mutually coupled memristor-based chaotic circuits studied in this work display a complex dynamic with a great variety of both chaotic and nonchaotic synchronisms. A possible interpretation for this can be that, due to the presence of the memory effect of the memristor that results in different memductances values for the circuits, the two considered dynamical systems are not completely identical, and various kinds of synchronization are expected. In effect, if complete synchronization appears only at high coupling strength, some form of phase synchronization or generalized synchronization seems to be more suitable for interpreting most of the numerical results obtained in this work, which could provide new insights for further study


FIgURE 17: Phase diagram during the chaotic oscillations of the two circuits starting from chaotic-damped initial conditions and coupled with high values of $R_{12}$. From top: $R_{12}=25000,40000,100000$.
to better understand the spontaneous dynamic of coupled memristor-based chaotic systems. In particular, although the results presented here are not directly generalizable to the case of multiple mutually coupled oscillators, because emergent phenomena can occur in the case of collective dynamics, they may be a useful reference for studying multiple systems. For example, the spontaneous dynamics of multiple memristor-based Chua's circuits diffusively coupled in a ring geometry has been investigated in our recent paper [61]. In addition to chaotic and nonchaotic synchronization, also emerging chaotic steady waves and quasi-periodic traveling waves along the ring have been observed.

## Conflict of Interests

The authors declare that they have no conflict of interests regarding the publication of this paper.

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# Mathematical Foundations for Efficient Structural Controllability and Observability Analysis of Complex Systems 

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#### Abstract

The relationship between structural controllability and observability of complex systems is studied. Algebraic and graph theoretic tools are combined to prove the extent of some controller/observer duality results. Two types of control design problems are addressed and some fundamental theoretical results are provided. In addition new algorithms are presented to compute optimal solutions for monitoring large scale real networks.


## 1. Introduction

The controllability and observability analysis of dynamical systems has been an active area of research in control theory since the pioneer work of Kalman for the linear time invariant (LTI) case [1]. Since then, progress has been carried out in several directions such as the controllability/observability of a class of nonlinear systems [2-5], some types of fuzzy systems [ 6,7$]$, and the structural controllability/observability of LTI systems [8-10], aimed at robust system monitoring.

The structural controllability analysis of LTI systems was initially stated by [8]. Such analysis is intended to model those system properties which only rely on the existence or not of dependencies among inputs, outputs, and state variables; the existence of a dependency is reflected in the model by some nonzero system parameter (which multiplies the corresponding coupling term) but does not depend on the specific value of such parameter. In [8] both linear algebraic and graph characterizations of structural controllability are presented, the second one by means of analyzing the associated directed graph which precisely represents the dependencies among state variables and input signals.

This correspondence between some properties of system dynamics and the structure of the associated directed network has been analyzed in the context of large scale and distributed control systems [11, 12]. Conversely, the same correspondence has led to the study of complex networks from a control theoretic perspective [13]; there, the analysis of a graph has been identified with the structural controllability of an associated LTI system, where the controllability concept can be accordingly interpreted depending on the nature and meaning of the network under study. In this structural LTI system framework, some specific problems concerning the minimum number of required inputs (which corresponds to the number of required controllers or actuators) to guarantee controllability have attracted the attention of several researchers (see [10-16], where some computational solutions have been provided).

The present paper deepens on the relationship between network analysis and the controllability as well as the observability properties of associated dynamical systems. First, the analysis and design of systems regarding their structural properties are formalized. Then, the potential duality between controllability and observability is analyzed in the framework
of some design problems, providing new theoretical results which relate both concepts. Finally, properties of maximum matchings (MMs) and strongly connected components (SCCs) are demonstrated, which lead to new computational tools for analyzing complex networks [17-19].

The paper is organized as follows. Section 2 presents the main results on structural controllability of LTI systems. Two problems concerning the optimal design of the control matrix are addressed in Section 3; there, algebraic and graph theoretic tools are combined, and the corresponding computational algorithms are presented. Section 4 considers the observability problem and theoretically demonstrates several duality results which are confirmed via computational simulations. Some fundamental properties of maximum matchings and strongly connected components of the network are demonstrated in Section 5. The algorithms for computing several controllability and observability related properties in complex networks are presented in Section 6. Finally, concluding remarks are summarized in Section 8.

## 2. Structural Controllability of LTI Systems

This section presents several controllability results for LTI systems of the form

$$
\begin{equation*}
\dot{x}=A x+B u \tag{1}
\end{equation*}
$$

where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ are given a priori. This is the case for many engineering problems, where physical restrictions define both the relationship between state variables (matrix $A$ ) and the possible location of system actuators (matrix $B$ ).

First, the classical controllability problem is stated and the need to undertake a structural analysis perspective is motivated. Secondly, some useful results on the structural analysis of matrices are demonstrated; finally, the structural controllability problem is analyzed.
2.1. Classical Controllability. Roughly speaking, system (1) is controllable (in the classical sense) when it is possible to lead the system state variable $x(t)$ from any initial point $x_{0}$ to any arbitrary point $x_{1}$ in a finite time period.

Classical control theory states that system (1) is controllable if and only if the corresponding controllability matrix

$$
\begin{equation*}
\mathscr{C}(A, B)=\left(B|A B| A^{2} B \cdots \mid A^{n-1} B\right) \tag{2}
\end{equation*}
$$

satisfies $\operatorname{rank}(\mathscr{C})=n$ (see [1]). Hence, the classical controllability problem can be formulated as a linear algebra rank condition; this implies that, in some practical cases, the problem may be ill-conditioned and too sensitive to potential parameter variations. Hence, the need of performing robust analyses not affected by modelling errors and/or uncertainties motivates the study of structural properties.
2.2. Structural Properties. In practice, the elements of matrices $A$ and $B$ may not be precisely known. This leads to the definition of structural properties as those which do not change with variations in the nonzero values of the elements
of matrices $A$ and $B$. Structural analysis considers two types of entries in the matrices, zero and nonzero entries, and addresses those properties which are preserved no matter what the exact value of the nonzero entries is, except for a set of their values with zero Lebesgue measure in the parameter space; see [8]; such properties are called generic [9]. Hence, the nonzero entries may be represented by a 1 -value (defining then a binary matrix) or, alternatively, the $X$-symbol. This will allow for a straightforward graphical representation of the system as shown in Section 2.3.
2.2.1. Algebraic Properties: Generic Rank. We introduce here the concept of a matrix generic rank (denoted by $\mathrm{rank}_{g}$ ), which happens to play an important role in characterizing its structural properties. As mentioned earlier, the generic rank of a matrix $A$, say $\operatorname{rank}_{g} A$, is the rank of such matrix for all values of its nonzero entries except those that lie in a set of zero measures. We now define some basic concepts aimed to characterize the generic rank of an $n \times m$ matrix $A$ (with $n \leq m$ unless stated otherwise).

Definition 1 ([see 16]). An $n \times m$ matrix $A$ is of form $(t)$ for some $t, 1 \leq t \leq n$, if for some $k$ in the range $m-t<k \leq m, A$ contains a zero submatrix of order $(n+m-t-k+1) \times k$.

Remark 2 ([see 16]). If $A$ has form $(t)$, then clearly $A$ has form ( $j$ ) for $t<j \leq n$.

The following lemma will be employed in the proof of Theorem 5.

Lemma 3. Given a matrix $A$, let $A^{\prime}$ be a matrix structurally equivalent to $A$ except for a fixed zero of $A$ which has been replaced by an arbitrary nonzero entry in $A^{\prime}$. Then, if $A$ is not of form $(t)$, then $A^{\prime}$ is not of form $(t)$.

Proof. From Definition 1, we have that, given $t, \forall k$ in the range $m-t<k \leq m$, $A$ does not contain a zero submatrix of order $(n+m-t-k+1) \times k$. Hence, based on the way $A^{\prime}$ has been constructed from $A$, matrix $A^{\prime}$ does not contain a zero submatrix of order $(n+m-t-k+1) \times k$ either. This means that $A^{\prime}$ is not of form $(t)$.

We can now state the following theorem which provides an alternative way to define the generic rank of a matrix.

Theorem 4 (see [9], Theorem 2.2). For any $n \times m$ matrix $A$, it is $\mathrm{rank}_{g} A=t$,
(i) for $t=n$ if and only if $A$ is not form ( $n$ ),
(ii) for $1 \leq t<n$ if and only if $A$ is of form $(t+1)$ but not of form $(t)$.

We end up with the following generic result, which will be useful for structural controllability analysis.

Theorem 5. Given a matrix $A$, let $A^{\prime}$ be a matrix structurally equivalent to $A$ except for a fixed zero of $A$ which has been replaced by an arbitrary entry in $A^{\prime}$. Then $\operatorname{rank}_{g} A^{\prime} \geq$ $\operatorname{rank}_{g} A$.

$$
\begin{gathered}
A=\left(\begin{array}{lll}
0 & 0 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0
\end{array}\right) \\
B=\left(\begin{array}{ll}
1 & 0 \\
0 & 0 \\
0 & 1
\end{array}\right)
\end{gathered}
$$



Figure 1: Control configuration $(A, B)$ and its graph representation. Squared nodes represent control inputs, yellow nodes are directly controlled, and blue nodes are controlled by other nodes in the network.

Proof. Let us consider the case $\operatorname{rank}_{g} A=t=n$. Then $A$ is not of form ( $n$ ); considering Lemma 3, this implies that $A^{\prime}$ is not of form ( $n$ ) which is equivalent to $\operatorname{rank}_{g} A^{\prime}=t=n$. Let us now consider the case $\operatorname{rank}_{g} A=t<n$. Then $A$ is of form $(t+1)$, but not of form $(t)$. Hence $A^{\prime}$ is not of form $(t)$, which implies that $\mathrm{rank}_{g} A^{\prime} \geq t$.
(Note that, knowing that $A^{\prime}$ is not of form $(t)$, then if $A^{\prime}$ is of form $(t+1)$, then $\operatorname{rank}_{g} A^{\prime}=t$; and if $A^{\prime}$ is not of form $(t+1)$, then $\operatorname{rank}_{g} A^{\prime} \geq t+1>t=\operatorname{rank}_{g} A$.)
2.3. The Graph Perspective. The matrix binary form suggests a straightforward alternative representation of the system as a graph $G:=(V, E)$, where state variables appear as the nodes (or vertices belonging to set $V$ ) and the elements of $A$ are represented by the existence of a link or edge (nonzero entries correspond to an existing link belonging to set $E$ ). Concerning matrix $B$, nonzero entries are reflected as links from an external input to the corresponding node (see Figure 1).

Several system structural properties can be analyzed by referring to its associated graph; in the following, structural controllability is addressed and we emphasize its alternative analysis via a graph theoretic approach.
2.4. Structural Controllability Conditions. In [8] systems of the form $(A, b)$ are analyzed, where column $b$ represents the scalar input influence on the state variables. Structural controllability is analyzed via both matrix and graph theory perspectives. The system (network) is proved to be structurally controllable if and only if all nodes are accessible from the input and the network presents no dilation, which is equivalent to say that the graph is spanned by an input cactus $[8,10]$.

Structural controllability for multi-input systems defined by a given pair $(A, B)$ was first addressed in [9] by analyzing two properties of matrix $[A \mid B]$ : the first one is related to accessibility and the second one (which is $\operatorname{rank}_{g}\left[\begin{array}{lll}A & \mid\end{array}\right]$ ) relates to the absence of dilations. Fortunately, the problem can be reduced to solely computing the generic rank of the associated extended controllability matrix.

Again, from a graph theory perspective, the system (network) is structurally controllable if and only if there exists a vertex disjoint union of input cacti [10] that covers all the state vertices (see, for instance, [20]).


Figure 2: Dynamical system graph and its bipartite representation. Red links represent edges in the maximum matching. Adding control inputs to every right-unmatched node guarantees the controllability matrix to have full rank.
2.4.1. The Use of Maximum Matchings. In [21] the equivalence between computing the generic rank of a matrix and computing a maximum matching (MM) in $G:=(V, E)$ over the associated bipartite graph (see [15] for details) is indicated (see Figure 2). A matching is any subset of $E$ so that all nodes in $V$ have neither more than one incoming edge nor more than one outgoing edge belonging to the matching. A matching is maximum if there are no other larger matchings (i.e., a matching containing a larger number of edges); note that maximum matchings (MMs) need not be unique. A matching is perfect if all nodes of the network have an incoming edge belonging to the matching (i.e., the number of links belonging to the matching equals the number of nodes in the network). Maximum matchings (MMs) will be considered in detail in the following sections, where it will be shown that the equivalence between generic rank evaluation and the determination of a MM is in accordance with the fact that a MM provides a subgraph which guarantees the absence of dilations.

In the next section, some control design problems (on the matrix $B$ ) are presented, where both the algebraic and the graph theoretic perspectives can still be employed to address them. Again, the computation of MMs will prove to be an efficient step towards their solution.

## 3. Optimal Design of $B$

There are practical situations in which only matrix $A$ is known as a characterization of the system dynamics, and there is no a priori restriction about the structure of matrix $B$. This can be interpreted as if any state variable can be directly accessed by a control signal. Then, the selection of an appropriate matrix $B$ can be addressed as a design goal.

Different optimization criteria can be defined for the design of matrix $B$. In the following, we formulate two different problems aimed to minimize the control requirements. Both problems can be formulated either in the classical control context (with a specific $A$ matrix) or in the structural analysis framework considered in this paper.
3.1. Minimum Number of Required Inputs. The first problem is concerned with minimizing the number of inputs or actuators, independently of the fact that such actuators may need to be connected as an input to more than one state variable.

Problem 6. Find $B$ with a minimum number of columns (inputs or actuators) so that $(A, B)$ is controllable.

Note that, since a column of $B$ may have more than one nonzero entry, the number of inputs may be smaller than the number of states directly accessed by an input (i.e., the number of nonzero rows).

Obviously, the solutions to this problem are not unique; and it is straightforward to prove that, given two different solutions $B_{1}$ and $B_{2}$, the number of state variables directly accessed by each of them may be different.

The design of an optimal $B$ has not been an important issue in classical control theory since most of the time such matrix is given a priori (or it is restricted to access only a subset of state variables) in real engineering problems.

When structural controllability is considered, the main result concerning the minimum number of required inputs is stated in the following theorem.

Theorem 7. Let us consider the LTI system

$$
\begin{equation*}
\dot{x}=A x \tag{3}
\end{equation*}
$$

and let $n_{c}$ be the minimum number of inputs (c stands for controllers) to make it structurally controllable. Then

$$
\begin{equation*}
n_{c}=\max \left\{1, n-\operatorname{rank}_{g} A\right\} \tag{4}
\end{equation*}
$$

Proof. As stated in [8], the system will be structurally controllable if all its variables are accessible from the inputs and the system presents no dilation. The accessibility condition requires having at least one input to the system, which implies that $n_{c} \geq 1$. The condition of no dilation can be expressed as follows:

$$
\begin{equation*}
\operatorname{rank}_{g}(A \mid B)=n \tag{5}
\end{equation*}
$$

where $n$ is the number of state variables in the system. Since

$$
\begin{equation*}
\operatorname{rank}_{g} A \leq \operatorname{rank}_{g}(A \mid B) \tag{6}
\end{equation*}
$$

the structure of the system, described by $A$, determines the conditions imposed to $B$ to make the system controllable.

Given $A$, the problem of finding the minimum number of inputs of the system is thus reduced to finding the minimum number of column vectors forming a matrix $B$ that satisfies (5). To comply with the accessibility condition, we may face two different cases: if $\operatorname{rank}_{g} A=n$, we need $B$ to have at least one column with some nonzero entry; if $\operatorname{rank}_{g} A<$ $n$, the already nonzero matrix $B$ selected to satisfy the nodilation condition may need to add extra nonfixed values to its column vectors, but either of these operations will not affect the no-dilation condition since it will never reduce $\operatorname{rank}_{g}(A \mid B)$ as stated in Theorem 5. In other words, the range condition expressed in (5) will determine the minimum number of inputs of the system, regardless of the number of variables/vertices affected by them. This result reduces Problem 6 to the rank analysis of (5).

Therefore, $B$ can be chosen to comply with (5) just by constructing as many independent columns as $n-\operatorname{rank}_{g} A$,


Figure 3: Adding inputs to every right-unmatched node might leave inaccessible nodes (grey in the figure). To overcome this problem, one may either add wirings (dashed line) from any existing input or include new dedicated inputs $\left(u_{2}\right)$.
keeping in mind that if $n-\operatorname{rank}_{g} A=0$ we need $B$ to have one column. Hence

$$
\begin{align*}
n_{c} & =\max \left\{1, \min _{\operatorname{rank}_{g}(A \mid B)=n}\left\{\operatorname{rank}_{g} B\right\}\right\}  \tag{7}\\
& =\max \left\{1, n-\operatorname{rank}_{g} A\right\} .
\end{align*}
$$

3.1.1. Computation of $n_{c}$ : The Maximum Matching Alternative. A priori, the computation of $n_{c}$ would rely on calculating the generic rank of matrix $A$. Hence, only the no-dilation property must be taken into account to compute $n_{c}$, independently of accessibility issues. This implies that, once a matrix $B$ satisfying the rank condition has been selected, we may only further require changing some of its zero terms to one (without altering its generic rank and $n_{c}$ ) to cope with accessibility.

Alternatively, the network theory perspective provides a way of determining the value of $n_{c}$ by the calculation of MMs on the network associated bipartite graph (see [13]). Such $M M$, denoted by $\mathscr{M}$, need not be unique. Any MM provides a decomposition of the graph into paths and cycles; it can be proved that $n_{c}$ is the number of right-unmatched vertices of $\mathscr{M}$ (note also that $\left.n_{c}=|V|-|\mathscr{M}|\right)$ and such value does not depend on the specific $\mathscr{M}$ that we may have found. Note that any MM only takes into account the no-dilation property and it does not provide information about node accessibility; equivalently, once a set of control inputs has been connected to the right-unmatched nodes, in order to complete the control configuration, we may require adding some new wires from any input(s) to the nonaccessible nodes, without altering the number of required inputs, $n_{c}$ (see dashed line in Figure 3).

The computation of different $\mathscr{M}$ s has been analyzed in [14, 15].
3.2. Minimum Number of Directly Controlled States (or Dedicated Inputs). The second optimization problem associated
with matrix $B$ is concerned with the minimum number of states that have to be directly controlled with an input signal.

Problem 8. Find $B$ with a minimum number of columns so that each column of $B$ has only one nonzero entry (i.e., it represents a dedicated input) and $(A, B)$ is controllable.

In this case, the number of dedicated inputs $n_{d c}$ is exactly the same as the number of states directly accessed by an input. For example, in Figure 3, two states have to be directly accessed; hence, two dedicated inputs are required.
3.2.1. Computation of $n_{d c}$ : Again the Maximum Matching Alternative. In [10], Problem 8 has been formalized by considering a graph theoretic perspective. In fact, $n_{d c}$ is equal to the minimum number of disjoint state cacti that span the network. As stated there, $n_{d c}$ can be indirectly computed by resorting to the relationship between graph cacti decompositions and the more easily computable maximum matchings. One must remember that a MM provides an alternative decomposition of the graph into paths and cycles. Unfortunately, the accessibility information from right-unmatched nodes to cycles is lost in a MM. Hence further analysis is required, where the relationship between the information provided by the MM and the graph strongly connected components (SCCs) becomes crucial.

In [10] it is shown that the minimum number of dedicated inputs $n_{d c}$ is given by

$$
\begin{equation*}
n_{d c}=n_{c}+\beta_{c}-\alpha_{c} \tag{8}
\end{equation*}
$$

where $n_{c}$ again is the number of right-unmatched vertices with respect to the found maximum matching $\mathscr{M}, \beta_{c}$ is the number of nontop linked strongly connected components (SCC), and $\alpha_{c}$ is the so-called maximum assignability index of the network (to be explained below).

Each MM $\mathscr{M}$ found provides a set of right-unmatched nodes that are assigned an external control input. (As mentioned earlier, although the set of right-unmatched vertices may change from one MM to another, its size $n_{c}$ does not depend on the specific MM $\mathscr{M}$ found.) Concerning the cycles provided by the matching, some of them may be accessible from a control input and some others may not. Since this accessibility information is not provided by the matching, further analysis is required, knowing that the nonaccessible cycles can only show up within the nontop linked SCCs, in order to determine $\beta_{c}-\alpha_{c}$.

Let $S$ be the set of all SCCs and let $S_{\mathrm{nt}} \subset S$ be the set of all nontop linked SCCs $\left(\left|S_{\mathrm{nt}}\right|=\beta_{c}\right)$. Then each specific $\mathscr{M}$ defines a partition in $S_{\mathrm{nt}}=S_{\mathrm{nt}}^{\mathrm{ru}}(\mathscr{M}) \sqcup S_{\mathrm{nt}}^{\mathrm{rm}}(\mathscr{M})$ (where $\sqcup$ stands for the disjoint union of sets) so that elements of $S_{\mathrm{nt}}^{\mathrm{ru}}(\mathscr{M}) \subset S_{\mathrm{nt}}$ contain vertices which belong to the set of rightunmatched (ru) vertices provided by $\mathscr{M}$; one can interpret that the elements of $S_{\mathrm{nt}}^{\mathrm{ru}}(\mathscr{M})$ are directly assigned an external control by $\mathscr{M}$ so that their accessibility is guaranteed. In this context, the meaning of $\alpha_{c}$ as the maximum assignability index of the network is formally stated by

$$
\begin{equation*}
\alpha_{c}=\max _{\mathscr{M}}\left|S_{\mathrm{nt}}^{\mathrm{ru}}(\mathscr{M})\right| . \tag{9}
\end{equation*}
$$

On the other hand, the elements of $S_{\mathrm{nt}}^{\mathrm{rm}}(\mathscr{M})=S_{\mathrm{nt}}(\mathscr{M}) \backslash S_{\mathrm{nt}}^{\mathrm{ru}}(\mathscr{M})$ do not contain any of the right-unmatched vertices provided by $\mathscr{M}$; hence, additional dedicated input(s) (equivalent to a wiring from any input(s) in the minimum number of required inputs problem) to at least one node belonging to each one of such elements will be required to complete full node accessibility [13]. If dedicated inputs were to be employed to implement such specific matching and associated wiring, the total number of inputs would be

$$
\begin{equation*}
n_{d c}(\mathscr{M})=n_{c}+n_{w c}(\mathscr{M}) \tag{10}
\end{equation*}
$$

where $n_{w c}(\mathscr{M})$ stands for the number of additional wires required. Hence, Problem 8 can be formulated as finding a MM $\mathscr{M}^{*}$ which minimizes $n_{w c}(\mathscr{M})$. Since the number of required wires also satisfies $n_{w c}(\mathscr{M})=\left|S_{\mathrm{nt}}^{\mathrm{rm}}(\mathscr{M})\right|=\beta_{c}$ $\left|S_{\mathrm{nt}}^{\mathrm{ru}}(\mathscr{M})\right|$, we have that

$$
\begin{align*}
n_{w}^{*} & =n_{w c}\left(\mathscr{M}^{*}\right)=\min _{\mathscr{M}} n_{w c}(\mathscr{M})  \tag{11}\\
& =\beta_{c}-\max _{\mathscr{M}}\left|S_{\mathrm{nt}}^{\mathrm{ru}}(\mathscr{M})\right|=\beta_{c}-\alpha_{c} .
\end{align*}
$$

Since $\beta_{c}$ is solely determined by the network topology (being independent of the obtained $\mathscr{M}$ ), the solution of Problem 8 requires the computation of $\alpha_{c}$ (by solving a maximization problem over all possible $\mathscr{M} \mathrm{s}$ ).

## 4. Observability of LTI Systems and Duality Results

We now consider the LTI system defined by

$$
\begin{gather*}
\dot{x}=A x+B u, \\
y=C x, \tag{12}
\end{gather*}
$$

where again $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}$, and $C \in \mathbb{R}^{p \times n}$ are given a priori. This system is said to be observable (in the classical sense) if, for any known input $u$, the state space initial condition $x_{0}$ can be determined in finite time by measuring only the output vector $y(t)$.

It can be shown that for LTI systems matrix $B$ does not affect the observability property, which only depends on the relationship between matrices $A$ and $C$. Hence, the observability analysis can be addressed relying on a duality property (see [22] for details).

In the following, we address structural observability and associated design issues which will provide similar results to the controllability analysis performed earlier. In addition, duality issues are considered when referring to both structural controllability and observability properties.
4.1. Observability and Optimal Design of C. In the same way as for the controllability analysis, there are practical situations, where no restrictions on matrix $C$ exist, so that it can be freely selected. Therefore one can formulate diverse problems concerning the design of optimal $C$ matrices satisfying different minimality requirements.

Such matrix $C$ design problems can be related to the previously presented design problems for matrix $B$, invoking
duality. In the following we demonstrate some results concerning the design of both optimal $B$ and $C$ matrices.
4.2. Minimum Number of Required Inputs and Outputs. Given the LTI system (12), we state the following result concerning Problem 6 and its dual counterpart.

Theorem 9. Consider system (12), where only matrix $A$ is predefined (i.e., matrices $B$ and $C$ and the corresponding dimensions of $u$ and $y$ can be freely designed); let $n_{c}$ be the minimum number of inputs to make it structurally controllable and let $n_{o}$ be the minimum number of outputs (o will stand for observability) to make it structurally observable. Then

$$
\begin{equation*}
n_{c}=n_{o} . \tag{13}
\end{equation*}
$$

(This result was empirically noted in [14].)
Proof. By invoking the duality between the observability and controllability concepts [22], the observability analysis of the system defined by matrix $A$ can be performed by studying the controllability of the system defined by $A^{T}$. Since the structural properties are grounded on the classical ones, determining the minimum number of outputs to guarantee structural observability in a system defined by $A$ is equivalent to determining the minimum number of inputs to guarantee structural controllability of the system defined by $A^{T}$.

The dual system will be structurally controllable if

$$
\begin{equation*}
\operatorname{rank}_{g}\left(A^{T} C^{T}\right)=n \tag{14}
\end{equation*}
$$

And again, the minimum number of inputs for that new system would be

$$
\begin{align*}
n_{o} & =\max \left\{1, \min _{\operatorname{rank}_{g}\left(A^{T} \mid C^{T}\right)=n}\left\{\operatorname{rank}_{g} C^{T}\right\}\right\}  \tag{15}\\
& =\max \left\{1, n-\operatorname{rank}_{g} A^{T}\right\} .
\end{align*}
$$

Since

$$
\begin{equation*}
\operatorname{rank}_{g} A=\operatorname{rank}_{g} A^{T} \tag{16}
\end{equation*}
$$

we conclude that

$$
\begin{align*}
n_{o} & =\max \left\{1, n-\operatorname{rank}_{g} A^{T}\right\}  \tag{17}\\
& =\max \left\{1, n-\operatorname{rank}_{g} A\right\}=n_{c} .
\end{align*}
$$

This proof relies only on algebraic properties of $A$. An alternative proof can be constructed using graph theoretical results and the duality principle. Based on duality, the observability analysis in a given graph $G:=(V, E)$, with adjacency matrix $A$, is equivalent to the controllability analysis in a graph whose adjacency matrix is $A^{T}$; that is, a graph $G_{d}:=$ $\left(V, E^{\prime}\right)$ with the same set of nodes $V$ and whose links in $E^{\prime}$ have the directions of links in $E$ flipped. We call such a graph $G_{d}$ the dual graph of $G$.

Every MM $\mathscr{M}$ of $G$ (considered merely as a set of links, neglecting their directions) is also a MM of $G_{d}$. Also, $\mathscr{M}$ is composed by a disjoint union of paths and cycles, so that the number of required inputs $n_{c}$ is determined by the size of such paths and cycles. Since flipping the directions of links does not change the number and size of those paths and cycles, we have $n_{c}=n_{o}$. The $n_{o}$ sensors would be connected to the rightunmatched vertices determined by $\mathscr{M}$ in $G_{d}$ or equivalently to the left-unmatched vertices determined by $\mathscr{M}$ in $G$.

Note that this result does not imply that the number of required wirings should be the same, since it will depend on the accessibility of the cycles provided by $\mathscr{M}$, which can change from $G$ to $G_{d}$ (the directions of links do matter when determining accessibility), as illustrated in the following subsection.
4.3. Minimum Number of Dedicated Outputs. Given the LTI system (12), we now consider the dual counterpart of Problem 8, that is, the required dedicated outputs for guaranteeing observability.

Based on duality it can be shown that the minimum number of dedicated outputs (sensors) $n_{\text {do }}$ is given by

$$
\begin{equation*}
n_{\mathrm{do}}=n_{o}+\beta_{o}-\alpha_{o} \tag{18}
\end{equation*}
$$

where $n_{o}=n_{c}$ again corresponds to the left-unmatched vertices in $G$ provided by $\mathscr{M}, \beta_{o}$ is the size of the set $S_{\mathrm{nb}}$ composed by the nonbottom linked SCCs, and $\alpha_{o}$ is the maximum assignability index of the network (now also referred to as the nonbottom linked SCCs).

A parallel reasoning to the one carried out for controllability can be performed for the observability analysis, where the left-unmatched vertices play the role of the previous rightunmatched ones and the nonbottom linked SCCs play the role of the previous nontop linked ones.
4.4. Dedicated Inputs versus Dedicated Outputs. It is obvious that, in general, $\beta_{o}$ need not be equal to $\beta_{c}$; by the same way, $\alpha_{o}$ may not be equal to $\alpha_{c}$. Accordingly, the number of required wirings $n_{w o}(\mathscr{M})$ may be different from $n_{w c}(\mathscr{M})$. Therefore we get the following.

Remark 10. Consider system (12) and let $n_{d c}$ be the minimum number of dedicated inputs to make it structurally controllable, and let $n_{\mathrm{do}}$ be the minimum number of dedicated outputs to make it structurally observable. Then $n_{d c}$ may or may not to be equal to $n_{\text {do }}$.

For instance, if $A=\left[\begin{array}{ll}0 & 1 \\ 0 & 0\end{array}\right]$, then $n_{d c}=1=n_{\mathrm{do}}$, whereas if $A=\left[\begin{array}{lll}0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1\end{array}\right]$, then $n_{d c}=1 \neq 2=n_{\mathrm{do}}$ (see Figure 4).
(This result was also empirically discovered in [14].)
The difference of value between $n_{d c}$ and $n_{\text {do }}$ suggests that the relationship between these two quantities can shed some light on a further characterization of the network properties.


Figure 4: The system in the left can be controlled with only one dedicated input. However, its dual system for observability in the right needs an additional wiring to guarantee accessibility.

## 5. Properties of Maximum Matchings and Strongly Connected Components

In this section some fundamental results are presented for addressing the practical solution of the two problems presented in Section 3. In order to simplify the exposition, the controllability problem will be considered to illustrate the results. Note that the whole reasoning applies also to the observability analysis, which is performed by merely applying the same reasoning to the dual network.

As mentioned earlier, $n_{c}$ can be obtained via the computation of a MM. We will see that MMs are also crucial for the computation of $n_{d c}$ together with the properties of the SCCs. In the following some fundamental results concerning the properties of MMs and the network SCCs are presented.
5.1. Properties of Maximum Matchings. We begin by stating some properties which characterize the structure of the set of possible MMs; precisely, the construction of a MM from another one by only performing few changes is addressed, which will lead to characterize similarities between different MMs. In order to make the notation easy, the MM, as subgraph of $G:=(V, E)$, will be defined with $\mathscr{M}$ representing their set of links.

Given a MM $\mathscr{M}$ so that one of its right-unmatched nodes in $V^{\mathrm{ru}}(\mathscr{M})$ has an incoming link in $E$, the following results address the possibility of constructing a new MM $\mathscr{M}^{\prime}$ whose $V^{\text {ru }}\left(\mathscr{M}^{\prime}\right)$ is obtained by just swapping such node of $V^{\text {ru }}(\mathscr{M})$ by another node in $V^{\mathrm{rm}}(\mathscr{M})$.

Lemma 11. Let $\mathscr{M}$ be a $M M$ and let $V^{r u}(\mathscr{M})$ be the set of rightunmatched nodes of $\mathscr{M}$. Let $v_{1} \in V^{r u}(\mathscr{M})$ be such that there exists a link $\left(v_{2}, v_{1}\right) \in E$ (going from some node $v_{2}$ to $v_{1}$ ). Then, there exist a node $v_{3}$ and a $M M \mathscr{M}^{\prime}$ such that $\mathscr{M}^{\prime}=\mathscr{M} \sqcup$ $\left\{\left(v_{2}, v_{1}\right)\right\} \backslash\left\{\left(v_{2}, v_{3}\right)\right\}$ implying that $V^{r u}\left(\mathscr{M}^{\prime}\right)=V^{r u}(\mathscr{M}) \backslash\left\{v_{1}\right\} \sqcup$ $\left\{v_{3}\right\}$.

Proof. Let us consider the subgraph $G_{1}:=\left(V, E_{1}\right)$ with $E_{1}=$ $\mathscr{M} \sqcup\left\{\left(v_{2}, v_{1}\right)\right\}$. Obviously, $E_{1}$ must contain some ( $\left.v_{2}, v_{3}\right)$, a second outgoing link from $v_{2}$ (if not, $E_{1}$ would become a matching with more links than $\mathscr{M}$, leading to a contradiction
with the maximality of $\mathscr{M}$ ). By removing such link we obtain a new subgraph with the same number of links as $\mathscr{M}$ and satisfying again the no-dilation condition (i.e., a new MM), $\mathscr{M}^{\prime}=E_{1} \sqcup\left(v_{2}, v_{1}\right)=\mathscr{M} \sqcup\left(v_{2}, v_{1}\right) \backslash\left(v_{2}, v_{3}\right)$ which satisfies $V^{\text {ru }}\left(\mathscr{M}^{\prime}\right)=V^{\text {ru }}(\mathscr{M}) \backslash\left\{v_{1}\right\} \sqcup\left\{v_{3}\right\}$.

Remark 12. Note that if $v_{1} \in V^{\text {ru }}(\mathscr{M})$ and there exists $\mathscr{M}^{\prime \prime}$ such that $v_{1} \notin V^{\text {ru }}\left(\mathscr{M}^{\prime \prime}\right)$, then Lemma 11 does apply, implying the existence of $\mathscr{M}^{\prime}$, where $v_{1}$ has been swapped in $V^{\text {ru }}(\mathscr{M})$ by another single node to form $V^{\mathrm{ru}}\left(\mathscr{M}^{\prime}\right)$.

Lemma 13. Let $\mathscr{M}$ be a $M M$ and let $V^{r m}(\mathscr{M})$ be the set of right-matched nodes of $\mathscr{M}$. Let $v_{1} \in V^{r m}(\mathscr{M})$ be such that there exists $\mathscr{M}^{\prime \prime}$ with $v_{1} \in V^{r u}\left(\mathscr{M}^{\prime \prime}\right)$. Then, there exist a node $v_{j} \in V^{r u}(\mathscr{M}) \cap V^{r m}\left(\mathscr{M}^{\prime \prime}\right)$ and a $M M \mathscr{M}^{\prime}$ such that $V^{r u}\left(\mathscr{M}^{\prime}\right)=$ $V^{r u}(\mathscr{M}) \backslash\left\{v_{j}\right\} \sqcup\left\{v_{1}\right\}$.

Proof. Let us consider $\left(v_{2}, v_{1}\right) \in \mathscr{M}$, the link right-matching node $v_{1}$. Note that $\left(v_{2}, v_{1}\right) \notin \mathscr{M}^{\prime \prime}$ since $v_{1} \in V^{\text {ru }}\left(\mathscr{M}^{\prime \prime}\right)$. Let us now consider $v_{3} \neq v_{1}$ such that $\left(v_{2}, v_{3}\right) \in \mathscr{M}^{\prime \prime}$. Note that $\mathscr{M}^{\prime \prime}$ must contain such a link; otherwise, $\mathscr{M}^{\prime \prime} \sqcup\left\{\left(v_{2}, v_{1}\right)\right\}$ would be a valid matching, contradicting $\mathscr{M}^{\prime \prime}$ being maximum. If we construct $E_{1}=\mathscr{M} \backslash\left\{\left(v_{2}, v_{1}\right)\right\} \sqcup\left\{\left(v_{2}, v_{3}\right)\right\}$, then we face two possibilities.
(1) If $v_{3} \in V^{\mathrm{ru}}(\mathscr{M})$, then $E_{1}=\mathscr{M}^{\prime}$ would be the matching we are looking for such that $V^{\text {ru }}\left(\mathscr{M}^{\prime}\right)=V^{\text {ru }}(\mathscr{M}) \backslash$ $\left\{v_{3}\right\} \sqcup\left\{v_{1}\right\}$.
(2) If $v_{3} \notin V^{\mathrm{ru}}(\mathscr{M})$, then $E_{1}$ would have two incoming links to $v_{3}$. Let $v_{4}$ be such that $\left(v_{4}, v_{3}\right) \in \mathscr{M}$ (note that $v_{4} \neq v_{2}$ since $v_{3} \neq v_{1}$ and $\left.\left(v_{2}, v_{1}\right) \in \mathscr{M}\right)$. Let also $v_{5} \neq v_{1}$, $v_{3}$, such that $\left(v_{4}, v_{5}\right) \in \mathscr{M}^{\prime \prime}$ (note that such link must exist; otherwise, $\mathscr{M}^{\prime \prime} \sqcup\left\{\left(v_{4}, v_{3}\right),\left(v_{2}, v_{1}\right)\right\} \backslash\left\{\left(v_{2}, v_{3}\right)\right\}=$ $\mathscr{M}^{\prime \prime} \sqcup\left(\mathscr{M} \backslash E_{1}\right)$ would be a valid matching, leading to a contradiction). We construct $E_{2}=E_{1} \backslash\left\{\left(v_{4}, v_{3}\right)\right\}$ ப $\left\{\left(v_{4}, v_{5}\right)\right\}$, where again we can have two possibilities: if $v_{5} \in V^{\text {ru }}(\mathscr{M})$, then we are done with $E_{2}=\mathscr{M}^{\prime}$ and $V^{\text {ru }}\left(\mathscr{M}^{\prime}\right)=V^{\text {ru }}(\mathscr{M}) \backslash\left\{v_{5}\right\} \sqcup\left\{v_{1}\right\}$. Otherwise, we could apply the same reasoning recursively until some node $v_{j} \neq v_{1}, v_{3}, v_{5}, \ldots$ is encountered such that $v_{j} \in V^{\mathrm{ru}}(\mathscr{M}) \cap V^{\mathrm{rm}}\left(\mathscr{M}^{\prime \prime}\right)$ allowing its swapping with $v_{1}$.

Lemma 14. Let $\mathscr{M}$ be a $M M$ and let $V^{r m}(\mathscr{M})$ be the set of rightmatched nodes of $\mathscr{M}$. Let $\left\{v_{1}, \ldots, v_{k}\right\} \in V^{r m}(\mathscr{M})$ be such that there exists $\mathscr{M}^{\prime \prime}$ with $v_{1}, \ldots, v_{k} \in V^{r u}\left(\mathscr{M}^{\prime \prime}\right)$. Then, there exists a set of nodes $\left\{v_{1}^{\prime}, \ldots, v_{k}^{\prime}\right\} \subset V^{r u}\left(\mathscr{M} \cap V^{r m}\left(\mathscr{M}^{\prime \prime}\right)\right.$ and a $M M$ $\mathscr{M}^{\prime}$ such that $V^{r u}\left(\mathscr{M}^{\prime}\right)=V^{r u}(\mathscr{M}) \backslash\left\{v_{1}^{\prime}, \ldots, v_{k}^{\prime}\right\} \sqcup\left\{v_{1}, \ldots, v_{k}\right\}$.

Proof. From the previous Lemma 13, we can construct a MM $\mathscr{M}_{1}$ such that $V^{\mathrm{ru}}\left(\mathscr{M}_{1}\right)=V^{\mathrm{ru}}(\mathscr{M}) \backslash\left\{v_{1}^{\prime}\right\} \sqcup\left\{v_{1}\right\}$ for some $v_{1}^{\prime} \in$ $V^{\mathrm{rm}}\left(\mathscr{M}^{\prime \prime}\right)$. Applying again the same reasoning of Lemma 13 , we can construct a new MM $\mathscr{M}_{2}$ such that $V^{\text {ru }}\left(\mathscr{M}_{2}\right)=$ $V^{\mathrm{ru}}(\mathscr{M}) \backslash\left\{v_{1}^{\prime}, v_{2}^{\prime}\right\} \sqcup\left\{v_{1}, v_{2}\right\}$, where again $v_{2}^{\prime} \in V^{\mathrm{rm}}\left(\mathscr{M}^{\prime \prime}\right)$ which guarantees that $v_{2}^{\prime} \notin\left\{v_{1}, \ldots, v_{k}\right\}$. The procedure can be applied repeatedly for each $v_{j}$ to obtain a new $\mathscr{M}_{j}$ such that $V^{\mathrm{ru}}\left(\mathscr{M}_{j}\right)=V^{\mathrm{ru}}(\mathscr{M}) \backslash\left\{v_{1}^{\prime}, \ldots v_{j}^{\prime}\right\} \sqcup\left\{v_{1}, \ldots, v_{j}\right\}$ with
$v_{j}^{\prime} \in V^{\mathrm{rm}}\left(\mathscr{M}^{\prime \prime}\right)$ which guarantees that $v_{j}^{\prime} \notin\left\{v_{1}, \ldots v_{k}\right\}$. When $j=k$, the desired result holds.

These lemmas will allow for an efficient search of appropriate MMs.
5.2. Properties of the Elements of $S_{n t}$. We now address some properties of $S_{\mathrm{nt}}$, the set of nontop linked SCCs, from the point of view of their relationship with the different MMs which can be defined in the network.

For every $G_{i} \in S_{\mathrm{n}}$, let $V_{i}$ represent the set of its vertices or nodes. For any MM $\mathscr{M}$, let $V_{i}(\mathscr{M}) \subseteq V_{i}$ be the set of nodes of $G_{i}$ having an outgoing link in $\mathscr{M}$. Then we can define $V_{i}^{b}(\mathscr{M})$ and $V_{i}^{o}(\mathscr{M})$ to be a partition of $V_{i}$ into two subsets: nodes whose outgoing links in $\mathscr{M}$ are between nodes of $G_{i}$ and those whose outgoing links leave $G_{i}$, respectively, so that $\left|V_{i}(\mathscr{M})\right|=$ $\left|V_{i}^{b}(\mathscr{M})\right|+\left|V_{i}^{o}(\mathscr{M})\right| \leq\left|V_{i}\right|$. Note that $V_{i}^{\mathrm{ru}}(\mathscr{M})$, the set of right-unmatched nodes of $G_{i}$ for $\mathscr{M}$, satisfies $\left|V_{i}^{\text {ru }}(\mathscr{M})\right|=$ $\left|V_{i}\right|-\left|V_{i}^{b}(\mathscr{M})\right|$ (hence $\left.\left|V_{i}^{o}(\mathscr{M})\right| \leq\left|V_{i}^{\text {ru }}(\mathscr{M})\right|\right)$.

If $V_{i}^{\text {ru }}(\mathscr{M})=\emptyset$, all nodes of $G_{i}$ are right-matched for $\mathscr{M}$ and we define $G_{i} \in S_{\mathrm{nt}}^{\mathrm{rm}}(\mathscr{M})$. We can then define the following subset of $S_{\mathrm{nt}}$ :

$$
\begin{equation*}
S_{\mathrm{nt}}^{\mathrm{rm}}=\left\{G_{i} \in S_{\mathrm{nt}} \mid \exists \mathscr{M}, G_{i} \in S_{\mathrm{nt}}^{\mathrm{rm}}(\mathscr{M})\right\} \tag{19}
\end{equation*}
$$

We will see that the elements of $S_{\mathrm{nt}}^{\mathrm{rm}}$ accept a perfect matching; hence, they may end up being inaccessible from any input in a given MM , requiring an additional dedicated input. Therefore, further analysis of this type of subgraphs is required.

The following theorem analyzes the existence and similarity among different MMs when focused on the elements of $S_{\mathrm{nt}}^{\mathrm{rm}}$.

Theorem 15. If $G_{i} \in S_{n t}^{r m}$ (equivalently, $G_{i}$ accepts a perfect matching), then
(1) $\left|V_{i}(\mathscr{M})\right|=\left|V_{i}^{b}(\mathscr{M})\right|+\left|V_{i}^{o}(\mathscr{M})\right|=\left|V_{i}\right|$ (equivalently, $\left.\left|V_{i}^{o}(\mathscr{M})\right|=\left|V_{i}^{r u}(\mathscr{M})\right|\right)$ for all $\mathscr{M}$;
(2) given any $\mathscr{M}$, it is possible to construct an alternative $\mathscr{M}^{\prime}$ so that $V_{i}^{o}\left(\mathscr{M}^{\prime}\right)$ is any arbitrary subset of $V_{i}^{o}(\mathscr{M})$ $\left(\left|V_{i}^{o}\left(\mathscr{M}^{\prime}\right)\right|\right.$ taking the corresponding arbitrary value between 0 and $\left.\left|V_{i}^{o}(\mathscr{M})\right|\right)$ and $\mathscr{M}^{\prime}$ is the same as $\mathscr{M}$ for links not outgoing from nodes of $G_{i}$.
(In particular, one can construct such a $\mathscr{M}^{\prime}$ so that $V_{i}^{o}\left(\mathscr{M}^{\prime}\right)=$ $\emptyset$, meaning that $\left.G_{i} \in S_{n t}^{r m}\left(\mathscr{M}^{\prime}\right)\right)$.

Proof. (1) Let us first consider the existing $\mathscr{M}$ such that $G_{i} \in$ $S_{\mathrm{nt}}^{\mathrm{rm}}(\mathscr{M})$. Then every node of $G_{i}$ must have an input link belonging to $\mathscr{M}$, necessarily coming from another node of $G_{i}$. Therefore, $\left|V_{i}(\mathscr{M})\right|=\left|V_{i}\right|$ so that $\mathscr{M}$ defines a perfect matching in $G_{i}$; note also that $\left|V_{i}(\mathscr{M})\right|=\left|V_{i}^{b}(\mathscr{M})\right|$, so that all links outgoing from $G_{i}$ do not belong to $\mathscr{M}\left(\left|V_{i}^{o}(\mathscr{M})\right|=0\right)$. The same can be said for any other $\mathscr{M}$ satisfying $G_{i} \in S_{\mathrm{nt}}^{\mathrm{rm}}(\mathscr{M})$.

Let now $\mathscr{M}$ be an alternative MM so that $G_{i} \notin S_{\mathrm{nt}}^{\mathrm{rm}}(\mathscr{M})$ (i.e., $\left.1 \leq\left|V_{i}^{\text {ru }}(\mathscr{M})\right| \leq\left|V_{i}\right|\right)$. We will show now that $\left|V_{i}^{o}(\mathscr{M})\right|=$ $\left|V_{i}^{\text {ru }}(\mathscr{M})\right|$.

Note that $\left|V_{i}(\mathscr{M})\right|=\left|V_{i}^{b}(\mathscr{M})\right|+\left|V_{i}^{o}(\mathscr{M})\right|=\left|V_{i}\right|-$ $\left|V_{i}^{\mathrm{ru}}(\mathscr{M})\right|+\left|V_{i}^{o}(\mathscr{M})\right|$. On one hand, if $\left|V_{i}^{o}(\mathscr{M})\right|<\left|V_{i}^{\mathrm{ru}}(\mathscr{M})\right|$, we would have $\left|V_{i}(\mathscr{M})\right|<\left|V_{i}\right|$ and could define $\mathscr{M}^{\prime}$ with the (known existing) perfect matching in $G_{i}$ so that $\left|V_{i}^{b}\left(\mathscr{M}^{\prime}\right)\right|=$ $\left|V_{i}\right|$ and $\left|V_{i}^{o}\left(\mathscr{M}^{\prime}\right)\right|=0$ which would allow $\mathscr{M}^{\prime}$ to preserve the same links as $\mathscr{M}$ in the rest of the network; this would imply $\left|\mathscr{M}^{\prime}\right|>|\mathscr{M}|$ leading to a contradiction. On the other hand, if $\left|V_{i}^{o}(\mathscr{M})\right|>\left|V_{i}^{\text {ru }}(\mathscr{M})\right|$, we would have $\left|V_{i}(\mathscr{M})\right|>\left|V_{i}\right|$ leading also to a contradiction.

Therefore $\left|V_{i}^{o}(\mathscr{M})\right|=\left|V_{i}^{\text {ru }}(\mathscr{M})\right|$ and $\left|V_{i}(\mathscr{M})\right|=\left|V_{i}\right|$.
(2) Let $\mathscr{M}$ be any MM; note that, from $1,\left|V_{i}(\mathscr{M})\right|=\left|V_{i}\right|$. We begin by constructing $\mathscr{M}^{\prime}$ such that $\left|V_{i}^{o}\left(\mathscr{M}^{\prime}\right)\right|=0$, in two parts. On one hand, $\mathscr{M}^{\prime}$ would contain the (known existing) perfect matching in $G_{i}$ so that $\left|V_{i}\left(\mathscr{M}^{\prime}\right)\right|=\left|V_{i}^{b}\left(\mathscr{M}^{\prime}\right)\right|=\left|V_{i}\right|$. Since $\left|V_{i}^{o}\left(\mathscr{M}^{\prime}\right)\right|=0$, this would allow completing $\mathscr{M}^{\prime}$ by keeping the same links as $\mathscr{M}$ in the rest of the network (satisfying $\left|\mathscr{M}^{\prime}\right|=|\mathscr{M}|$ ).

We can now construct $\mathscr{M}^{\prime \prime}$ so that $V_{i}^{o}\left(\mathscr{M}^{\prime \prime}\right)$ is any arbitrary subset of $V_{i}^{o}(\mathscr{M})$. Since $V_{i}^{o}\left(\mathscr{M}^{\prime \prime}\right) \subset V_{i}^{b}\left(\mathscr{M}^{\prime}\right)$ for each of its nodes, we can remove the (known existing) outgoing link in $\mathscr{M}^{\prime}$ and restore the corresponding link in $\mathscr{M}$. This again would allow completing $\mathscr{M}^{\prime \prime}$ by keeping the same links as $\mathscr{M}$ and $\mathscr{M}^{\prime}$ in the rest of the network.

Finally, note the true equivalence in the theorem statement: for any $G_{i} \in S_{\mathrm{nt}}$, we have that $G_{i} \in S_{\mathrm{nt}}^{\mathrm{rm}}$ if and only if $G_{i}$ accepts a perfect match. If $G_{i} \in S_{\mathrm{nt}}^{\mathrm{rm}}(\mathscr{M})$, we have already seen that $\mathscr{M}$ defines a perfect matching in $G_{i}$. Alternatively, consider that $G_{i}$ accepts a perfect match. As shown above, given any $\mathscr{M}$, either $G_{i} \notin S_{\mathrm{nt}}^{\mathrm{rm}}(\mathscr{M})$ or we can construct $\mathscr{M}^{\prime}$ such that $G_{i} \in S_{\mathrm{nt}}^{\mathrm{rm}}\left(\mathscr{M}^{\prime}\right)$.

We now formulate a result illustrating the existence of MMs which can make or not, one by one, the elements of $S_{n t}$ to be right-unmatched.

Corollary 16. Let $G_{1}, G_{2} \in S_{n t}^{r m}(\mathscr{M})$ for some $\mathscr{M}$.
(1) If there exist $\mathscr{M}_{1}$ and $\mathscr{M}_{2}$ satisfying $S_{n t}^{r u}\left(\mathscr{M}_{1}\right) ~ \supseteq$ $S_{n t}^{r u}(\mathscr{M}) \cup\left\{G_{1}\right\}$ and $S_{n t}^{r u}\left(\mathscr{M}_{2}\right) \supseteq S_{n t}^{r u}(\mathscr{M}) \cup\left\{G_{2}\right\}$, then there may not exist $\mathscr{M}_{3}$ satisfying $S_{n t}^{r u}\left(\mathscr{M}_{3}\right) \supseteq S_{n t}^{r u}(\mathscr{M}) \cup$ $\left\{G_{1}, G_{2}\right\}$.
(2) The other way around, if there exists $\mathscr{M}_{3}$ satisfying $S_{n t}^{r u}\left(\mathscr{M}_{3}\right) \supseteq S_{n t}^{r u}(\mathscr{M}) \cup\left\{G_{1}, G_{2}\right\}$, then there must exist $\mathscr{M}_{1}$ and $\mathscr{M}_{2}$ satisfying $S_{n t}^{r u}\left(\mathscr{M}_{1}\right)=S_{n t}^{r u}\left(\mathscr{M}_{3}\right) \backslash\left\{G_{2}\right\}$ and $S_{n t}^{r u}\left(\mathscr{M}_{2}\right)=S_{n t}^{r u}\left(\mathscr{M}_{3}\right) \backslash\left\{G_{1}\right\}$.

Proof. (1) The first part of the corollary is obvious due to the interdependence of the outgoing links in the elements of $S_{\mathrm{nt}}^{\mathrm{rm}}(\mathscr{M})$. For instance, let us consider

$$
A=\left[\begin{array}{lllll}
0 & 1 & 0 & 0 & 0  \tag{20}\\
1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0
\end{array}\right]
$$

whose graphical representation can be found in Figure 5. There exists $\mathscr{M}$ such that $G_{1}$ is the subgraph gathering nodes


Figure 5: $G_{1}$ and $G_{2}$ are interdependent because they cannot both belong to $S_{\mathrm{nt}}^{\mathrm{ru}}(\mathscr{M})$ for any $\mathscr{M}$.
$\{1,2\}$, and $G_{2}$ gathers $\{4,5\}$. There exist $\mathscr{M}_{1}$ and $\mathscr{M}_{2}$ satisfying $S_{\mathrm{nt}}^{\mathrm{ru}}\left(\mathscr{M}_{1}\right)=S_{\mathrm{nt}}^{\mathrm{ru}}(\mathscr{M}) \cup\left\{G_{1}\right\}$ and $S_{\mathrm{nt}}^{\mathrm{ru}}\left(\mathscr{M}_{2}\right)=S_{\mathrm{nt}}^{\mathrm{ru}}(\mathscr{M}) \cup\left\{G_{2}\right\}$, but no $\mathscr{M}_{3}$ satisfying $S_{\mathrm{nt}}^{\mathrm{ru}}\left(\mathscr{M}_{3}\right)=S_{\mathrm{nt}}^{\mathrm{ru}}(\mathscr{M}) \cup\left\{G_{1}, G_{2}\right\}$.
(2) Given $\mathscr{M}_{3}$, such that $S_{\mathrm{nt}}^{\mathrm{ru}}\left(\mathscr{M}_{3}\right) \supseteq S_{\mathrm{nt}}^{\mathrm{ru}}(\mathscr{M}) \cup\left\{G_{1}, G_{2}\right\}$, then, by Theorem 15, we can construct a new MM (let us call it $\left.\mathscr{M}_{1}\right)$ such that $G_{2} \in S_{\mathrm{nt}}^{\mathrm{rm}}\left(\mathscr{M}_{1}\right), \mathscr{M}_{1}$ being the same as $\mathscr{M}_{3}$ for links not outgoing from nodes of $G_{2}$ (this includes all links involving nodes of $G_{1}$, since there cannot be links from nodes of $G_{2}$ to nodes of $\left.G_{1}\right)$. Therefore $S_{\mathrm{nt}}^{\mathrm{ru}}\left(\mathscr{M}_{1}\right)=S_{\mathrm{nt}}^{\mathrm{ru}}\left(\mathscr{M}_{3}\right) \backslash\left\{G_{2}\right\}$.

The same reasoning can be applied to justify the existence of $\mathscr{M}_{2}$.

We now consider the optimality with respect to $n_{d c}$ : a MM $\mathscr{M}^{*}$ is optimal if $n_{d c}\left(\mathscr{M}^{*}\right) \leq n_{d c}(\mathscr{M})$ for any other $\mathscr{M}$. The following result provides information about the existence of optimal solutions in a standard form.

Corollary 17. Let $G_{i} \in S_{n t}^{r m}$; then there exists an optimal $\mathscr{M}^{*}$ such that $\left|V_{i}^{r u}\left(\mathscr{M}^{*}\right)\right|=\left|V_{i}^{o}\left(\mathscr{M}^{*}\right)\right| \leq 1$.

Proof. Let us consider $\mathscr{M}^{*}$ being optimal and $\left|V_{i}^{\text {ru }}\left(\mathscr{M}^{*}\right)\right|=$ $\left|V_{i}^{o}\left(\mathscr{M}^{*}\right)\right|>1$. By Theorem 15, we can construct a new MM (let us call it $\left.\mathscr{M}^{* \prime}\right)$ such that $\left|V_{i}^{\text {ru }}\left(\mathscr{M}^{* \prime}\right)\right|=\left|V_{i}^{o}\left(\mathscr{M}^{* \prime}\right)\right|=$ $1, \mathscr{M}^{* \prime}$ being the same as $\mathscr{M}^{*}$ for links not outgoing from nodes of $G_{i}$. Note that neither $\mathscr{M}^{*}$ nor $\mathscr{M}^{* \prime}$ require a wiring on $G_{i}$, and the required wirings in the rest of the network remain unchanged. Hence, invoking (10), we have $n_{d c}\left(\mathscr{M}^{*}\right)=$ $n_{d c}\left(\mathscr{M}^{* \prime}\right)$.

Remark 18. Let $G_{i} \in S_{\mathrm{nt}}^{\mathrm{rm}}$ and let $\mathscr{M}^{*}$ be optimal with $\left|V_{i}^{\text {ru }}\left(\mathscr{M}^{*}\right)\right|=\left|V_{i}^{o}\left(\mathscr{M}^{*}\right)\right| \geq 1$. Let $\mathscr{M}^{\prime}$ be such that $\left|V_{i}^{\text {ru }}\left(\mathscr{M}^{\prime}\right)\right|=$ $\left|V_{i}^{o}\left(\mathscr{M}^{\prime}\right)\right|=0, \mathscr{M}^{\prime}$ being the same as $\mathscr{M}^{*}$ for links not outgoing from nodes of $G_{i}$. Then $\mathscr{M}^{\prime}$ is not optimal since a new wiring is required and $n_{d c}\left(\mathscr{M}^{\prime}\right)=n_{d c}\left(\mathscr{M}^{*}\right)+1$.

Nevertheless, there may exist another optimal $\mathscr{M}^{* \prime}$ such that $\left|V_{i}^{\text {ru }}\left(\mathscr{M}^{* \prime}\right)\right|=\left|V_{i}^{o}\left(\mathscr{M}^{* \prime}\right)\right|=0$, but $\mathscr{M}^{* \prime}$ should be necessarily different from $\mathscr{M}^{*}$ for links not outgoing from nodes of $G_{i}$.
5.3. Compatibility. We are now ready to state the final results which will determine the steps of the algorithms for searching optimal solutions $\mathscr{M}^{*}$.

Definition 19. Let $G_{i} \in S_{\mathrm{nt}}^{\mathrm{rm}}$. We say that $G_{i}$ is top-assignable if and only if there exists a MM $\mathscr{M}$ such that $\left|V_{i}^{o}(\mathscr{M})\right|=1$.

Note that we only need to consider top-assignable elements of $G_{i} \in S_{\mathrm{nt}}^{\mathrm{rm}}$ in the search for an optimum $\mathscr{M}^{*}$.

Definition 20. Let $G_{1}, \ldots, G_{k} \in S_{\mathrm{nt}}^{\mathrm{rm}}$ be top-assignable. We say that $\left\{G_{1}, \ldots, G_{k}\right\}$ are compatible if and only if there exists a MM $\mathscr{M}$ such that $G_{1}, \ldots, G_{k} \in S_{\mathrm{nt}}^{\mathrm{ru}}(\mathscr{M})$.

By Theorem 15 it is equivalent to guarantee that there exists a $\mathscr{M}^{\prime \prime}$ such that $\left|V_{i}^{o}\left(\mathscr{M}^{\prime \prime}\right)\right|=1, i=1, \ldots, k$. Note that all unitary sets of the form $\left\{G_{i}\right\}$ with $G_{i}$ assignable are also compatible; the definition provides new insights when being particularized for pairs $\left\{G_{1}, G_{2}\right\}$ (i.e., pairwise compatibility implicitly addressed in Corollary 16).

We say that $\left\{G_{1}, \ldots, G_{k}\right\}$ are incompatible if they are not compatible.

The following lemma proves a fundamental property of compatibility and incompatibility.

Lemma 21. Let $I=\left\{1, \ldots,\left|S_{n t}^{r m}\right|\right\}$ so that $S_{n t}^{r m}=\left\{G_{i}: i \in\right.$ $I\}$. Let $I_{1}, I_{2}$ be two different nonempty subsets of $I$, such that $\mathscr{G}_{1}=\left\{G_{i}: i \in I_{1}\right\}$ and $\mathscr{G}_{2}=\left\{G_{i}: i \in I_{2}\right\}$ are subsets of $S_{n t}^{r m}$ so that all elements of $\mathscr{G}_{1}$ are compatible among them and all elements of $\mathscr{G}_{2}$ are also compatible among them. Let us consider $\left|I_{1}\right|=k \leq l=\left|I_{2}\right|$ without loss of generality. Then, there exists $\mathscr{G}_{3}=\left\{G_{i}: i \in I_{3}\right\}$ compatible such that $\left|I_{2}\right| \leq\left|I_{3}\right|$ and $I_{1} \subseteq$ $I_{3} \subseteq I_{1} \cup I_{2}$ (equivalently, $\mathscr{G}_{1} \subseteq \mathscr{G}_{3} \subseteq \mathscr{G}_{1} \cup \mathscr{G}_{2}$ ).

Proof. From the hypotheses, there must exist the following MMs:
(i) $\mathscr{M}$ such that $\left|V_{i}^{o}(\mathscr{M})\right|=0, i \in I_{1} \cup I_{2}\left(\right.$ since $\mathscr{G}_{1}, \mathscr{G}_{1} \subset$ $S_{\mathrm{nt}}^{\mathrm{rm}}$;
(ii) $\mathscr{M}_{1}$ such that $\left|V_{i}^{o}\left(\mathscr{M}_{1}\right)\right|=1, i \in I_{1}$ (since $\mathscr{G}_{1}$ is compatible);
(iii) $\mathscr{M}_{2}$ such that $\left|V_{i}^{o}\left(\mathscr{M}_{2}\right)\right|=1, i \in I_{2}$ (since $\mathscr{G}_{2}$ is compatible).

Let us consider the subgraph $G_{1,2}=G-\cup_{i \in I_{1} \cup I_{2}} G_{i}$, where all nodes of $G_{i} \in \mathscr{M}_{1} \cup \mathscr{M}_{2}$ are removed from $G$ together with the links outgoing from them. Note that $\mathscr{M}$ restricted to each $G_{i}, i=I_{1} \cup I_{2}$ defines a perfect submatching $\mathscr{M}_{G_{i}}$ on it. Hence, $\mathscr{M}$ restricted to $G_{1,2}$ defines a maximum submatching $\mathscr{M}_{\mathrm{G}_{12}}$ on it; otherwise, a matching larger than $\mathscr{M}$ could be constructed on the whole graph by adding to the new larger submatching the subgraphs $G_{i}, i \in I_{1} \cup I_{2}$ with their corresponding perfect submatchings.

Let us consider now $\mathscr{M}_{1, G_{1,2}}$ and $\mathscr{M}_{2, G_{1,2}}$, the corresponding submatchings of $\mathscr{M}_{1}$ and $\mathscr{M}_{2}$ on $G_{1,2}$, respectively. By Theorem 15, all these submatchings are maximum in $G_{1,2}$ having all size $|\mathscr{M}|-\sum_{i \in I_{1} \cup I_{2}}\left|V_{i}(\mathscr{M})\right|$. By construction $\mathscr{M}_{1, \mathrm{G}_{1,2}}$ has $k$ right-unmatched nodes (let us call them $v_{i}, i \in I_{1}$ ) each one being the destination of the corresponding link outgoing
$G_{i} \in \mathscr{G}_{1}$; by the same way $\mathscr{M}_{2, G_{1,2}}$ has $l$ right-unmatched nodes (let us call them $v_{i}, i \in I_{2}$ ) destination of the links outgoing $G_{i} \in \mathscr{G}_{2}$ in $\mathscr{M}_{2}$. Let $I_{12}=I_{1}-I_{2}$; then for all $v_{i} \quad i \in I_{12}$, we have $v_{i} \in V^{\text {ru }}\left(\mathscr{M}_{1}\right) \cap V^{\mathrm{rm}}\left(\mathscr{M}_{2}\right), i \in I_{12}$, and we can apply Lemma 14 to $I_{12}$ obtaining $I_{12}^{\prime}$ (which satisfies $I_{12}^{\prime} \cap I_{1}=\emptyset$ ) and $\mathscr{M}_{3}$ such that $V^{\mathrm{ru}}\left(\mathscr{M}_{3}\right)=V^{\mathrm{ru}}\left(\mathscr{M}_{2}\right) \backslash\left\{v_{i}: \quad i \in\right.$ $\left.I_{12}^{\prime}\right\} \sqcup\left\{v_{i}: \quad i \in I_{12}\right\}$. Then, we have that, for $I_{3}=I_{2} \backslash I_{12}^{\prime} \cup I_{12}$, $I_{1} \subset I_{3}$ and $\left|I_{3}\right| \geq\left|I_{2}\right|$. Completing such submatching with the corresponding submatchings, $\mathscr{M}_{1, G_{i}}, i \in I_{1}, \mathscr{M}_{2, G_{i}}$, $i \in$ $I_{3}-I_{1}$, and $\mathscr{M}_{G_{i}}, i \in I_{12}^{\prime}$, we would complete the required MM to end the proof.

The following corollaries prove some relationships when the sets are modified element by element; they also show that pairwise incompatibility, besides being symmetric, is also a transitive property.

Corollary 22. Let $G_{1}, \ldots, G_{k}, G_{k+1} \in S_{n t}^{r m}$ be top-assignable, so that $\left\{G_{1}, \ldots, G_{k}\right\}$ are compatible and $\left\{G_{1}, \ldots, G_{k}, G_{k+1}\right\}$ are incompatible. Then, there exists $1 \leq j \leq k$ such that $\left\{G_{1}, \ldots, G_{j-1}, G_{j+1}, \ldots, G_{k+1}\right\}$ are compatible.

Proof. Calling $\mathscr{G}_{1}=\left\{G_{1}, \ldots, G_{k}\right\}$ and $\mathscr{G}_{2}=\left\{G_{k+1}\right\}$ we only need to apply Lemma 21.

Corollary 23. Let $G_{1}, G_{2}, G_{3} \in S_{n t}^{r m}$ be top-assignable, so that $G_{3}$ is incompatible with both $G_{1}$ and $G_{2}$, respectively. Then $G_{1}$ and $G_{2}$ are incompatible.

Proof. Keeping the assumptions, we will consider $G_{1}$ and $G_{2}$ to be compatible; then, applying Lemma 21, we know that either $\left\{G_{1}, G_{3}\right\}$ or $\left\{G_{1}, G_{3}\right\}$ must be compatible, which leads to a contradiction.

Corollary 24. Let $G_{1}, G_{2} \in S_{n t}^{r m}$ be top-assignable and incompatible. If $G_{1}$ is compatible with $G_{3}$, then $G_{2}$ is also compatible with $G_{3}$.

Proof. If we consider $G_{2}$ and $G_{3}$ incompatible, the first assumption and the transitivity property would imply $G_{1}$ and $G_{3}$ being incompatible, which contradicts the second assumption.

Finally, Lemma 21 allows for a useful characterization of the set of possible optimal matchings.

Theorem 25. Let $\mathscr{G}=\left\{G_{1}, \ldots, G_{k}\right\} \subset S_{n t}^{r m}$ be top-assignable and compatible. Then $\exists \mathscr{M}^{*}$ such that $\left|V_{i}^{r u}\left(\mathscr{M}^{*}\right)\right|=1$, for $i=$ $1, \ldots, k\left(\right.$ i.e., $\left.\mathscr{G} \subset S_{n t}^{r u}\left(\mathscr{M}^{*}\right)\right)$.

Proof. Let us consider $\exists \mathscr{M}^{* \prime}$ optimal, so that $S_{\mathrm{nt}}^{\mathrm{ru}}\left(\mathscr{M}^{* \prime}\right)=$ $\left\{G_{1}^{\prime}, \ldots, G_{l}^{\prime}\right\}=\mathscr{G}^{\prime}$, where obviously $k \leq l$ (otherwise $\mathscr{M}$ such that $\mathscr{G}=\left\{G_{1}, \ldots, G_{k}\right\} \subset S_{\mathrm{nt}}^{\mathrm{ru}}(\mathscr{M})$ would provide a larger set of right-unmatched components contradicting the optimality of $\mathscr{M}^{* \prime}$ ). Then applying Lemma 21 , we can construct a new compatible set $\mathscr{G}^{\prime \prime}=\left\{G_{1}^{\prime \prime}, G_{2}^{\prime \prime}, \ldots, G_{l}^{\prime \prime}\right\}$ satisfying $\mathscr{G} \subset \mathscr{G}^{\prime \prime}$ and $\left|\mathscr{G}^{\prime \prime}\right| \geq\left|\mathscr{G}^{\prime}\right|$. Since $\mathscr{M}^{\prime}$ is optimum, then $\left|\mathscr{G}^{\prime \prime}\right|=$ $\left|\mathscr{G}^{\prime}\right|$ and $\mathrm{MM} \mathscr{M}^{\prime \prime}$ is also optimum, satisfying the theorem statement.

## 6. A New Algorithm for Computing $n_{d c}$

The proposed algorithm for locating an optimal $\mathscr{M}^{*}$ is as follows.
(1) Determine $S_{n t}$.
(2) Determine all elements of $S_{n t}$ accepting a perfect matching; for each $G_{i} \in S_{\mathrm{nt}}$, we remove the links outgoing $G_{i}$ and compute a maximum submatching in $G_{i}$. If such matching is perfect, then $G_{i}$ accepts a perfect matching.
Let $S_{\mathrm{nt}}^{\mathrm{rm}}=\left\{G_{1}, \ldots, G_{k}\right\} \subset S_{\mathrm{nt}}$ be the elements of $S_{\mathrm{nt}}$ accepting a perfect matching. We call $G^{\prime}=G-G_{1}-$ $\cdots-G_{k}$ the subgraph, where $G_{1}, \ldots, G_{k}$ are removed from $G$ together with the links outgoing from them.
(3) For all elements of $S_{\mathrm{nt}}^{\mathrm{rm}}$, determine the set of topassignable elements $S_{\mathrm{nt}}^{\mathrm{rm} / \mathrm{ta}}=\left\{G_{1}, \ldots, G_{m}\right\}$; for doing so, we apply procedure P1.
If a given $G_{i}$ happens to be assignable, then a maximum submatching $\mathscr{M}_{i, G^{\prime}}$ is already available which might also give additional information about assignability of other $G_{j}$ 's as well as compatibility among them.
(4) Construct $S=\left\{G_{1}, \ldots, G_{l}\right\}$ as the maximum set of assignable and compatible elements provided by the previous step (the index being reordered without loss of generality). Note that if some assignable element has been found, $S$ will contain at least one element. If no $G_{i}$ is assignable, then $S=\emptyset$, implying that we are done since all elements of $S_{\mathrm{nt}}^{\mathrm{rm}}$ require a dedicated input.
(5) For $i=l+1, \ldots, m$, check compatibility of $S^{\prime}=S \cup\left\{G_{i}\right\}$ (applying procedure P 2 ). If $S^{\prime}$ is compatible, then $S=$ $S^{\prime}$ 。
(6) $\alpha_{c}=\left|S_{\mathrm{nt}}\right|-\left|S_{\mathrm{nt}}^{\mathrm{rm}}\right|+|S|$.
(7) By (8) and keeping in mind $\left|S_{\mathrm{nt}}\right|=\beta_{c}, n_{d c}$ is directly obtained as

$$
\begin{equation*}
n_{d c}=n_{c}+\left|S_{\mathrm{nt}}\right|-\alpha_{c} . \tag{21}
\end{equation*}
$$

The basic procedures are as follows.
(P1) Given $G_{i} \in S_{\mathrm{nt}}^{\mathrm{rm}}$, this procedure determines if it is top-assignable. We first compute a maximum submatching in $G^{\prime}$, called $\mathscr{M}_{G^{\prime}} ;\left|\mathscr{M}_{G^{\prime}}\right|$ is to be employed as a reference of the attainable MM size (note that $\left|V_{G^{\prime}}^{\mathrm{ru}}\left(\mathscr{M}_{\mathrm{G}^{\prime}}\right)\right|=\left|V_{G}^{\mathrm{ru}}\left(\mathscr{M}_{G}\right)\right|$, where $\mathscr{M}_{G}$ refers to the whole network, and it is obtained by adding to $\mathscr{M}_{G^{\prime}}$ the perfect submatchings corresponding to each $G_{i}$ ). Then, for all $G_{i}$, we check the existence of a maximum submatching in $G^{\prime}$ having one right-unmatched node belonging to the set of destination nodes of links outgoing $G_{i}$ (i.e., for each destination node, the links entering it are removed and the existence of a MM in such new graph is checked. Note that one may need to check for all the destination nodes associated with $G_{i}$ ). If such maximum submatching is found, we define $G_{i}$


Figure 6: The result of applying the proposed algorithm for minimizing the number of dedicated inputs.
to be assignable. If $G_{i}$ is top-assignable, this procedure will provide at least one maximum submatching $\mathscr{M}_{i, G^{\prime}}$ associated with one of such destination nodes.
(P2) Given $S$ and $G_{l+1} \in S_{\mathrm{nt}}$ top-assignable, determine if they are compatible. For doing so, we only need to consider the existence of a maximum submatching in $G^{\prime}$ which has one right-unmatched node belonging to each set of destination nodes of links outgoing each $G_{i}, i=1, \ldots, l$ and one right-unmatched node (different from the previous one) belonging to the set of destination nodes of links outgoing $G_{l+1}$. Note that one may need to check for all the possible pairings of different destination nodes associated with $S$ and $G_{l+1}$, respectively. If $S$ and $G_{l+1}$ are compatible, this procedure will provide at least one maximum submatching $\mathscr{M}_{1,2, \ldots, l+1}$ associated with a pair of such nodes.
The search of such maximum submatching $\mathscr{M}_{1,2, \ldots, l+1}$ can be (sometimes) simplified by using available maximum submatchings $\mathscr{M}_{1,2, \ldots, l}$ and $\mathscr{M}_{G_{l+1}}$ from P1 and following the procedure proposed in Lemma 13.
6.1. Suboptimal Solutions. Finding the control configuration with the minimum number of dedicated inputs might be computationally expensive for large networks; hence the consideration of a suboptimal solution can be useful. Considering the expression proposed in [10],

$$
\begin{equation*}
n_{d c}=n_{c}+\left|S_{\mathrm{nt}}\right|-\alpha_{c}, \tag{22}
\end{equation*}
$$

a suboptimal solution could be proposed, requiring $n_{c}+$ $\left|S_{\mathrm{nt}}\right|$ dedicated inputs. This upper bound can be computed by determining the nontop linked SCCs of $G(V, E)$ and performing a MM search on the network. Since the MM search dominates the complexity of the algorithm, computing such suboptimal solution takes $O(\sqrt{V} E)$ time.

Analogously, given the definition of $\alpha_{c}$ in step (10) of the algorithm proposed on Section 6, the minimum number of required inputs can also be expressed as

$$
\begin{equation*}
n_{d c}=n_{c}+\left|S_{\mathrm{nt}}^{\mathrm{rm}}\right|-|S| \tag{23}
\end{equation*}
$$

Note that a new smaller upper bound to $n_{d c}$ can also be derived from this expression by just computing $n_{c}$ and $\left|S_{\mathrm{nt}}^{\mathrm{rm}}\right|$. While the latter is already available at step (2) of the algorithm, $n_{c}$ can be obtained by just performing a MM search on $G^{\prime}$.

This suboptimal solution computes the MMs of subgraphs $G^{\prime}$ and $S_{\mathrm{nt}}^{\mathrm{rm}}=\left\{G_{1}, \ldots, G_{k}\right\} \subset S_{\mathrm{nt}}$ that define a partition on $G(V, E)$. Since the time complexity of a MM search is superlinear, finding a MM for each of the subgraphs is faster than computing a MM of the whole network. This means that this latter upper bound is not only closer to the optimal solution but also less computationally expensive.

## 7. A Comparative Example

The following example illustrates the behavior of the new proposed algorithm. If we consider the network in Figure 6, the application of a simple MM-based algorithm plus direct wiring keeping track of accessibility (see [15] for details) may provide (depending on the obtained MM) a different number of required dedicated input signals, ranging from four (corresponding to a solution with two unmatched nodes and only two wirings, in the most favourable case) to eight (two unmatched nodes plus six wirings, in the worst case). If we combine the MM-based algorithm with another one which also determines the SCCs for an ordered accessibility track (see [16]), we may obtain (again depending on the obtained MM) solutions ranging from four dedicated inputs (corresponding to an optimal solution) to six (two unmatched nodes plus four wirings, in the worst case).

The new proposed algorithm always leads to an optimal solution by first determining $S_{\mathrm{nt}}^{\mathrm{rm}}=\left\{G_{1}, G_{2}, G_{3}, G_{4}\right\}$; then, applying procedure P 1 , it finds that $G_{1}$ is not assignable whereas $G_{2}, G_{3}, G_{4}$ are assignable. Finally, applying procedure P2, it determines that $G_{2}, G_{3}, G_{4}$ are pairwise compatible but $\mathscr{G}=\left\{G_{2}, G_{3}, G_{4}\right\}$ is not compatible. These results lead to selecting (among others) the optimal MM $\mathscr{M}^{*}$ shown in the figure such that $\mathscr{G}_{1}=\left\{G_{2}, G_{3}\right\} \subset S_{\mathrm{nt}}^{\mathrm{ru}}\left(\mathscr{M}^{*}\right)$. (Note that nodes 3 and 4 are controlled since they constitute a cycle which is accessible from either $u_{1}$ or $u_{2}$.)

## 8. Concluding Remarks

Structural controllability and observability of complex directed networks have been analyzed by combining algebraic and graph theoretic tools. Two different design problems have been addressed and the extent of some controller/observer duality results has been demonstrated. In addition, some results concerning the structure of optimal solutions and their relationship with respect to MM have also been proved; these results have led to new algorithms to efficiently compute optimal and suboptimal solutions for monitoring large scale real networks.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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# Rotor-Flying Manipulator: Modeling, Analysis, and Control 

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#### Abstract

Equipping multijoint manipulators on a mobile robot is a typical redesign scheme to make the latter be able to actively influence the surroundings and has been extensively used for many ground robots, underwater robots, and space robotic systems. However, the rotor-flying robot (RFR) is difficult to be made such redesign. This is mainly because the motion of the manipulator will bring heavy coupling between itself and the RFR system, which makes the system model highly complicated and the controller design difficult. Thus, in this paper, the modeling, analysis, and control of the combined system, called rotor-flying multijoint manipulator (RF-MJM), are conducted. Firstly, the detailed dynamics model is constructed and analyzed. Subsequently, a full-state feedback linear quadratic regulator (LQR) controller is designed through obtaining linearized model near steady state. Finally, simulations are conducted and the results are analyzed to show the basic control performance.


## 1. Introduction

Rotor-flying robot (RFR) has been researched for several decades and achieved great development. To date, RFRs have shown their priority in many applications, such as search, rescue, and surveillance [1-6]. However, these applications are often passive. That means the RFR system can't manipulate the interested objects by a direct physical interaction.

Most recently, this problem has attained more and more attentions from many researches. For example, some researchers suggest to equip a gripper on the RFR system so that the RFR system can grasp as shown Figure 1 [7-9]. But it still has some disadvantages including that (1) the manipulation can only be implemented through controlling the attitude of the RFR system. However, the precise control of the RFR system is difficult due to its complicated dynamics, and the precise manipulation is impossible. (2) With this structure, in order to manipulate an object, the RFR must approach it. This, however, may both bring the so-called ground effect and "blow" the object, which makes the precise control much more difficult.

In this paper, a new system structure as shown in Figure 2 is proposed. The system is composed of an RFR system and
a multijoint manipulator and thus called rotor-flying multijoint manipulator (RF-MJM). Compared to the structure in Figure 1, the multiple-joint manipulator can be used to regulate position and attitude of the end-gripper. This is very useful to compensate the control imprecision of the RFR system and makes precise manipulation much easier.

However, it is obvious that the system shown in Figure 2 is very difficult to be controlled. This is mainly because the RFR system itself is very sensitive to the external disturbance (force/moment), especially for the time varying disturbance, for example, from a moving manipulator as the new proposed RF-MJM system. Thus, it is very important and necessary for us to study the detailed model that can describe the coupling between the RFR and the manipulator. What is more, to construct a full-state high fidelity dynamics model is also a benefit for optimizing the design parameters, for example, the mass, the joint number, and the configuration of the manipulator, and testing the designed control algorithms.

Thus in this paper, the dynamics model of the RFMJM system is constructed and analyzed to show its basic performance. Moreover, the linear LQR controller is designed to test the basic closed loop performance of it. The main contributions of this paper are as the following three aspects:


Figure 1: RFR system designed by Yale University (a) and DLR (b).


Figure 2: Sketch of the RF-MJM system.
(1) a detailed full-state high fidelity nonlinear dynamics model of the RF-MJM system is constructed through using Euler-Lagrange method; (2) the dynamics couplings between the RFR and the manipulator are analyzed in detail, which is good for the optimal design of the system configuration; (3) LQR controller is designed based on the linearized system model and simulations are conducted to show the basic control performance of the new proposed system.

## 2. Dynamics Model of RF-MJM

The dynamics model of the RF-MJM is composed of three parts as shown in Figure 3: the body dynamics model, the mid-dynamics model, and the actuator model. The body dynamics model describes the relation between the motion state and the external force/torque exerting on the body of the robot; the mid-dynamics model represents how the force/torque is produced, that is, the aerodynamics of the RFR, and the torque from the manipulator joint motor, while the actuator model depicts the dynamical characteristic of the actuator, for example, the motors of the manipulator and the steering engine of the RFR. For the RF-MJM system,


Figure 3: Model structure of the RF-MJM system.
the coupling between the RFR and the manipulator will mainly influence the body dynamics model, which, then, will be discussed in this paper.

A RF-MJM system is actually a multilink system shown as in Figure 4, where the cube denotes the RFR and the ellipsoids denote the link of the manipulator; $\Sigma_{0}, \Sigma_{I}, \Sigma_{E}$, and $\Sigma_{i}$ are RFR body-fixed reference frame, earth-fixed inertial frame, endgripper frame, and the frame of the $i$ th joint of manipulator; $J_{i}(i=1,2, \ldots, n)$ denote the joint of the manipulator; $p_{i}$ denotes its position vector in the frame of $\Sigma_{I} ; C_{0}$ and $C_{i}$ are the position vector of the center of mass (COM) of the link RFR and the link $i ; d_{0}$ and $d_{i}$ are the position vectors of $C_{0}$ and $C_{i} ; a_{i}$ is the vector from $J_{i}$ to $C_{i} ; b_{0}$ is the vector from COM of the RFR to the first joint; $b_{i}$ is vector from $C_{i}$ to $J_{i+1}$; $n$ is the number of the manipulator's link.
2.1. Kinematics Model. In this paper, we suppose that both the RFR system and the manipulator are rigid. Thus, the following geometric relations are satisfied:

$$
\begin{equation*}
d_{i}=d_{0}+b_{0}+\sum_{k=1}^{i-1}\left(a_{k}+b_{k}\right)+a_{i} \tag{1}
\end{equation*}
$$

Differentiate it with respect to time and we have

$$
\begin{equation*}
v_{i}=\dot{d}_{i}=v_{0}+\omega_{0} \times\left(d_{i}-d_{0}\right)+\sum_{k=1}^{i}\left\{k_{k} \times\left(\dot{d_{i}}-p_{k}\right) \dot{\theta}_{k}\right\} \tag{2}
\end{equation*}
$$



Figure 4: System structure of the RF-MJM system.
where $v_{0}$ and $v_{i}$ are the linear velocity of the COM of the RFR and the link $i$, respectively; $\omega_{0}$ is the angular velocity of the RFR in the frame of $\Sigma_{I} ; \theta_{k}$ is the angular position vector of the $k$ th joint; $k_{k}$ denotes the unit vector of the axis $Z_{i}$ of the $i$ th joint frame; $d_{i}, p_{k}$ represent the position vectors of the COM of the link $i$ and the $k$ th joint. The angular velocity of the $i$ th joint can be denoted as

$$
\begin{equation*}
\omega_{i}=\omega_{0}+\sum_{k=1}^{i} k_{k} \dot{\theta}_{k} . \tag{3}
\end{equation*}
$$

Combine (2) and (3), and the kinematic model of the RFMJM system is

$$
\left[\begin{array}{c}
v_{i}  \tag{4}\\
\omega_{i}
\end{array}\right]=\bar{J}_{b i}\left[\begin{array}{c}
v_{0} \\
\omega_{0}
\end{array}\right]+\bar{J}_{m i} \dot{\Theta}
$$

where $\bar{J}_{b i}$ is the Jacobian matrix of the RFR system and has the following form:

$$
\bar{J}_{b i}=\left[\begin{array}{cc}
E & -\widetilde{d}_{0 i}  \tag{5}\\
0 & E
\end{array}\right]+\left[\begin{array}{l}
\bar{J}_{b v_{i}} \\
\bar{J}_{b \omega_{i}}
\end{array}\right] .
$$

In (5), $E$ is the unity matrix with proper dimension:

$$
d_{0 i}=d_{i}-d_{0}=\left[\begin{array}{lll}
d_{0 i, x} & d_{0 i, y} & d_{0 i, z} \tag{6}
\end{array}\right]^{T}
$$

and $\tilde{d}_{0 i}$ is the skew-symmetric matrix of the vector $d_{0 i}$; that is,

$$
\widetilde{d}_{0 i}=\left[\begin{array}{ccc}
0 & -d_{0 i, z} & d_{0 i, y}  \tag{7}\\
d_{0 i, z} & 0 & -d_{0 i, x} \\
-d_{0 i, y} & d_{0 i, x} & 0
\end{array}\right] .
$$

$\bar{J}_{m i}$ in (4) is the Jacobian matrix of the manipulator system defined as

$$
\bar{J}_{m i}=\left[\begin{array}{ccc}
k_{1} \times\left(d_{i}-p_{1}\right) & \cdots & k_{i} \times\left(d_{i}-p_{i}\right)  \tag{8}\\
k_{1} & \cdots & k_{i}
\end{array}\right]=\left[\begin{array}{c}
\bar{J}_{m v_{i}} \\
\bar{J}_{m \omega_{i}}
\end{array}\right] .
$$

Also, we can transform the linear velocity of the RFR $v_{0}$ into the velocity in the RFR body frame; that is,

$$
v_{0}=\left[\begin{array}{ccc}
c \theta c \varphi & s \phi s \theta c \varphi-c \phi s \varphi & c \phi s \theta c \varphi+s \phi s \varphi  \tag{9}\\
c \theta s \varphi & c \phi c \varphi+s \theta s \phi s \varphi & c \phi s \theta s \varphi-s \phi c \varphi \\
-s \theta & s \phi c \theta & c \phi c \theta
\end{array}\right]\left[\begin{array}{c}
u \\
v \\
w
\end{array}\right],
$$

where $c$ and $s$ mean trigonometric function $\cos$ and $\sin$, respectively; $\Phi=[\phi, \theta, \psi]^{T}$ represents the attitude of the RFR; [ $u, v, w$ ] is the linear velocity of the RFR in the RFR body fixed reference frame.

Similarly, the relation between the angular velocity in the frame of $\Sigma_{I}$ and that in the body frame is as follows:

$$
\omega_{0}=\left[\begin{array}{ccc}
1 & \sin \phi \tan \theta & \cos \phi \tan \theta  \tag{10}\\
0 & \cos \phi & -\sin \phi \\
0 & \sin \phi \sec \theta & \cos \phi \sec \theta
\end{array}\right]\left[\begin{array}{l}
p \\
q \\
r
\end{array}\right]
$$

where $p, q$, and $r$ are the components of the angular velocity along the axes of the RFR body fixed reference frame.
2.2. Dynamics Model. In this section, the dynamics model of the RF-MJM system will be deduced using Euler-Lagrange method.
2.2.1. Kinetic Energy. Firstly, the kinetic energy of the system can be denoted as

$$
\begin{equation*}
E_{k}=\frac{1}{2} \sum_{i=1}^{n}\left(\omega_{i}^{T} I_{i} \omega_{i}+m_{i} v_{i}^{T} v_{i}\right), \tag{11}
\end{equation*}
$$

where $m_{i}$ and $I_{i}$ are the mass and the inertia tensor of the $i$ th part[10].

Substitute (2) and (3) into (11), we have

$$
E_{k}=\frac{1}{2}\left[\begin{array}{c}
v_{0}  \tag{12}\\
\omega_{0} \\
\dot{\Theta}
\end{array}\right]^{T}\left[\begin{array}{ccc}
\omega E & \omega \tilde{d}_{0 g}^{T} & J_{T \omega} \\
\omega \widetilde{d}_{0 g} & H_{\omega} & H_{\omega \phi} \\
J_{T \omega}^{T} & H_{\omega \phi}^{T} & H_{m}
\end{array}\right]\left[\begin{array}{c}
v_{0} \\
\omega_{0} \\
\dot{\Theta}
\end{array}\right]=\frac{1}{2}\left[\begin{array}{c}
v_{0} \\
\omega_{0} \\
\dot{\Theta}
\end{array}\right]^{T} H\left[\begin{array}{c}
v_{0} \\
\omega_{0} \\
\dot{\Theta}
\end{array}\right],
$$

where $H$ is called the inertia matrix of the RF-MJM system with

$$
J_{R i}=\left[\begin{array}{lllllll}
k_{1} & k_{2} & \cdots & k_{i} & 0 & \cdots & 0
\end{array}\right]
$$

$J_{T i}$

$$
=\left[\begin{array}{llllll}
k_{1} \times\left(d_{i}-p_{1}\right) & k_{2} \times\left(d_{i}-p_{2}\right) & \cdots & k_{i} \times\left(d_{i}-p_{i}\right) & 0 & \cdots
\end{array}\right],
$$

$$
\begin{equation*}
d_{g}=\frac{\sum_{1=0}^{n} m_{i} d_{i}}{\sum_{1=0}^{n} m_{i}} \tag{13}
\end{equation*}
$$

2.2.2. Potential Energy. In this part, the potential energy due to the gravity will be deduced. Firstly, based on Figure 2, (1) can be rewritten as

$$
\begin{equation*}
d_{i}=d_{0}+\sum_{k=1}^{i}\left({ }^{I} A_{k-1} C_{k-1, k}-{ }^{I} A_{k} C_{k, k}\right) \tag{14}
\end{equation*}
$$

where ${ }^{I} A_{0}$ and ${ }^{I} A_{k}$ denote the coordinate transformation matrix from the frame $\Sigma_{0}$ and the frame $\Sigma_{k}$ to the frame $\Sigma_{I}$, respectively; $C_{k, k}$ is the position vector from joint $i$ to the COM of the $i$ th partdenoted in the $i$ th joint coordinate system.

With (14), the potential energy due to gravity of the RFMJM system can be easily obtained as follows:

$$
\begin{align*}
E_{p} & =-\sum_{i=0}^{n} m_{i} G^{T} d_{i} \\
& =\sum_{i=0}^{n} m_{i} G^{T}\left(d_{0}+\sum_{k=1}^{i-1}\left({ }^{I} A_{k-1} C_{k-1, k}-{ }^{I} A_{k-1} C_{k, k}\right)\right), \tag{15}
\end{align*}
$$

where

$$
G=\left[\begin{array}{l}
0  \tag{16}\\
0 \\
g
\end{array}\right]
$$

and $g$ is the acceleration due to gravity.
2.2.3. Dynamics Model. The Euler-Lagrangian equation of the RF-MJM system is

$$
\begin{equation*}
L=E_{k}-E_{p} . \tag{17}
\end{equation*}
$$

From (12), the kinetic energy can be rewritten as

$$
\begin{align*}
E_{k}=\frac{1}{2}[ & \omega v_{0}^{T} E v_{0}+\omega \omega_{0}^{T} \widetilde{d}_{0 g} v_{0}+\dot{\Theta}^{T} J_{T \omega}^{T} v_{0} \\
& +\omega v_{0}^{T} \tilde{d}_{0 g}^{T} \omega_{0}+\omega_{0}^{T} H_{\omega} \omega_{0}+\dot{\Theta}^{T} H_{\omega \phi}^{T} \omega_{0}  \tag{18}\\
& \left.+v_{0}^{T} J_{T \omega} \dot{\Theta}+\omega_{0}^{T} H_{\omega \phi} \dot{\Theta}+\dot{\Theta}^{T} H_{m} \dot{\Theta}\right]
\end{align*}
$$

Thus we have

$$
\begin{gather*}
\frac{\partial E_{k}}{\partial \dot{q}}=\left[\begin{array}{l}
\frac{\partial E_{k}}{\partial v_{0}} \\
\frac{\partial E_{k}}{\partial \omega_{0}} \\
\frac{\partial E_{k}}{\partial \dot{\Theta}}
\end{array}\right]=\left[\begin{array}{c}
\omega E v_{0}+\omega \tilde{d}_{0 g}^{T} \omega_{0}+J_{T \omega} \dot{\Theta} \\
H_{\omega} \omega_{0}+\omega \tilde{d}_{0 g} v_{0}+H_{\omega \phi} \dot{\Theta} \\
H_{m} \dot{\Theta}+H_{\omega \phi}^{T} \omega_{0}+J_{T \omega}^{T} v_{0}
\end{array}\right],  \tag{19}\\
\frac{d}{d t}\left(\frac{\partial E_{k}}{\partial \dot{q}}\right)=\left[\begin{array}{c}
\frac{d}{d t}\left(\frac{\partial E_{k}}{\partial v_{0}}\right) \\
\frac{d}{d t}\left(\frac{\partial E_{k}}{\partial \omega_{0}}\right) \\
\frac{d}{d t}\left(\frac{\partial E_{k}}{\partial \dot{\Theta}}\right)
\end{array}\right]=\left[\begin{array}{c}
\omega E \dot{v}_{0}+\omega \dot{\tilde{d}}_{0 g}^{T} \omega_{0}+\omega \tilde{d}_{0 g}^{T} \dot{\omega}_{0}+J_{T \omega} \Theta{ }_{\Theta}+\dot{J}_{T \omega} \dot{\Theta} \\
\dot{H}_{\omega} \omega_{0}+H_{\omega} \dot{\omega}_{0}+\omega \dot{\tilde{d}}_{0 g} v_{0}+\omega \tilde{d}_{0 g} \dot{v}_{0}+\dot{H}_{\omega \phi} \dot{\Theta}+H_{\omega \phi} \Theta \\
\dot{H}_{m} \dot{\Theta}+H_{m} \ddot{\Theta}+\dot{H}_{\omega \phi}^{T} \omega_{0}+H_{\omega \phi}^{T} \dot{\omega}_{0}+\dot{J}_{T \omega}^{T} v_{0}+J_{T \omega}^{T} \dot{v}_{0}
\end{array}\right]  \tag{20}\\
\frac{\partial E_{k}}{\partial q}=\left[\begin{array}{l}
\frac{\partial E_{a}}{\partial p}+\frac{\partial E_{b}}{\partial p}+\frac{\partial E_{c}}{\partial p} \\
\frac{\partial E_{a}}{\partial \Phi}+\frac{\partial E_{b}}{\partial \Phi}+\frac{\partial E_{c}}{\partial \Phi} \\
\frac{\partial E_{a}}{\partial \Theta}+\frac{\partial E_{b}}{\partial \Theta}+\frac{\partial E_{c}}{\partial \Theta}
\end{array}\right] \tag{21}
\end{gather*}
$$

where

$$
q=\left[\begin{array}{l}
X  \tag{22}\\
\Phi \\
\Theta
\end{array}\right], \quad \dot{q}=\left[\begin{array}{c}
v_{0} \\
\omega_{0} \\
\dot{\Theta}
\end{array}\right]
$$

are the position vector and the velocity vector of the RF-MJM system, respectively; $X$ is the position vector of the RFR; and

$$
\begin{align*}
& E_{a}=\omega \omega_{0}^{T} \widetilde{d}_{0 g} v_{0} ; \quad E_{b}=\frac{1}{2} \omega_{0}^{T} H_{\omega} \omega_{0} \\
& E_{c}=\left(v_{0}^{T} J_{T \omega}+\omega_{0}^{T} H_{\omega \phi}+\frac{1}{2} \dot{\Theta}^{T} H_{m}\right) \dot{\Theta} \tag{23}
\end{align*}
$$

Similarly, with respect to the potential energy term, we have

$$
\frac{\partial E_{p}}{\partial X}=\sum_{i=0}^{n} m_{i}\left[\begin{array}{c}
\frac{\partial g^{T} d_{0}}{\partial x_{0}}  \tag{24}\\
\frac{\partial g^{T} d_{0}}{\partial y_{0}} \\
\frac{\partial g^{T} d_{0}}{\partial z_{0}}
\end{array}\right]
$$

$$
\begin{align*}
\frac{\partial E_{p}}{\partial \Theta}= & -\left(\sum_{i=0}^{n} m_{i} \partial g^{T}\right. \\
& \left.\times\left(d_{0}+\sum_{j=1}^{i}\left({ }^{I} A_{j-1} * C_{j-1, j}-{ }^{I} A_{j} * C_{j, j}\right)\right)\right) \\
& \times(\partial \Theta)^{-1} \tag{25}
\end{align*}
$$

$$
\begin{gather*}
\frac{\partial E_{p}}{\partial \Phi} \\
=-\left[\begin{array}{c}
\frac{\partial E_{p}}{\partial \phi} \\
\frac{\partial E_{p}}{\partial \theta} \\
\frac{\partial E_{p}}{\partial \varphi}
\end{array}\right] \\
=-\left[\frac{-\sum_{i=1}^{n} m_{i} \partial g^{T} \sum_{j=1}^{i}\left({ }^{I} A_{j-1} * C_{j-1, j}-{ }^{I} A_{j} * C_{j, j}\right)}{\partial \theta}\right]  \tag{26}\\
\left.\frac{-\sum_{i=1}^{n} m_{i} \partial g^{T} \sum_{j=1}^{i}\left({ }^{I} A_{j-1} * C_{j-1, j}-{ }^{I} A_{j} * C_{j, j}\right)}{\partial \varphi}\right]
\end{gather*}
$$

Thus, the dynamics model of the RF-MJM system can be obtained through the following Euler-Lagrange equation:

$$
\begin{equation*}
\frac{d}{d t} \frac{\partial E_{k}}{\partial \dot{q}}-\frac{\partial E_{k}}{\partial q}+\frac{\partial E_{P}}{\partial q}=\tau \tag{27}
\end{equation*}
$$

Substitute (20)-(21) and (24)-(26) into (27) and after simplifying we can obtain the following dynamics model of the RF-MJM system:

$$
\left[\begin{array}{cc}
H_{b} & H_{b m}  \tag{28}\\
H_{b m}^{T} & H_{m}
\end{array}\right]\left[\begin{array}{c}
\ddot{X}_{b} \\
\ddot{\Theta}
\end{array}\right]+\left[\begin{array}{c}
C_{b} \\
C_{m}
\end{array}\right]+\left[\begin{array}{c}
G_{b} \\
G_{m}
\end{array}\right]=\left[\begin{array}{c}
F_{p} \\
\tau_{m}
\end{array}\right]
$$

where

$$
\ddot{X}_{b}=\left[\begin{array}{c}
\dot{v}_{0}  \tag{29}\\
\dot{\omega}_{0}
\end{array}\right] .
$$

$C_{b}$ and $C_{m}$ are the Coriolis and centrifugal force of the system; $G_{b}$ and $G_{m}$ are the force due to gravity;

$$
F_{p}=\left[\begin{array}{c}
F_{B}  \tag{30}\\
M_{B}
\end{array}\right] ; \quad F_{B}=\left[\begin{array}{c}
F_{x} \\
F_{y} \\
F_{z}
\end{array}\right] ; \quad M_{B}=\left[\begin{array}{c}
L \\
M \\
N
\end{array}\right]
$$

are the force and moment produced by the RFR; the other terms are defined as follows:

$$
\begin{gather*}
H_{b}=\left[\begin{array}{ll}
H_{b_{11}} & H_{b_{12}} \\
H_{b_{21}} & H_{b_{22}}
\end{array}\right] \\
H_{b_{11}}=\omega E ; \quad H_{b_{12}}=\omega \tilde{d}_{o g}^{T} ; \\
H_{b_{21}}=\omega \widetilde{d}_{o g} ; \quad H_{b_{22}}=H_{\omega} \\
H_{b m}=\left[\begin{array}{l}
H_{b m v} \\
H_{b m \omega}
\end{array}\right]_{2 \times 1} \\
H_{b_{m v}}=J_{T \omega} ; \quad H_{b m \omega}=H_{\omega \phi} \\
C_{b v}=\left[\begin{array}{l}
C_{b v} \\
C_{b \omega}
\end{array}\right] \\
\dot{\tilde{d}}_{0 g}^{T} \omega_{0}+\dot{J}_{T \omega} \dot{\Theta}-\omega \omega_{0}^{T} \frac{\partial \widetilde{d}_{0 g}}{\partial X} v_{0}-v_{0}^{T} \frac{\partial J_{T \omega}}{\partial X} \dot{\Theta} \\
C_{b \omega}^{T} \frac{\partial H_{\omega \phi}}{\partial X} \dot{\Theta}-\frac{1}{2}\left(\dot{H}_{\omega} \omega_{0}^{T} \frac{\partial H_{\omega}}{\partial X} \omega_{0}+\dot{\Theta}^{T} \frac{\partial H_{m}}{\partial X} \dot{\Theta}\right) \\
\dot{\tilde{d}}_{0 g} v_{0}+\dot{H}_{\omega \phi} \dot{\Theta}-\omega \omega_{0}^{T} \frac{\partial \tilde{d}_{0 g}}{\partial \Phi} v_{0}-v_{0}^{T} \frac{\partial J_{T \omega}}{\partial \Phi} \dot{\Theta} \\
C_{m}^{T} \frac{\partial H_{\omega \phi}}{\partial \Phi} \dot{\Theta}-\frac{1}{2}\left(\dot{J}_{T \omega}^{T} v_{0}+\frac{1}{2} v_{0}^{T} \frac{\partial H_{\omega}}{\partial \Phi} \omega_{0}+\frac{1}{2} \dot{\Theta}^{T} \frac{\partial H_{m}^{T}}{\partial \Phi} \dot{\Theta}\right) . \\
\omega_{0}+\frac{1}{2} \omega_{0}^{T} \dot{H}_{\omega \phi}+\dot{H}{ }_{m} \dot{\Theta} . \tag{31}
\end{gather*}
$$

2.2.4. Extended Dynamics Model. When the manipulator contacts some external objects, the dynamics model (28) becomes

$$
\left[\begin{array}{cc}
H_{b} & H_{b m}  \tag{32}\\
H_{b m}^{T} & H_{m}
\end{array}\right]\left[\begin{array}{c}
\ddot{X}_{b} \\
\ddot{\Theta}
\end{array}\right]+\left[\begin{array}{c}
C_{b} \\
C_{m}
\end{array}\right]+\left[\begin{array}{c}
G_{b} \\
G_{b}
\end{array}\right]=\left[\begin{array}{c}
F_{p} \\
\tau_{m}
\end{array}\right]+\left[\begin{array}{c}
J_{b}^{T} \\
J_{m}^{T}
\end{array}\right] F_{e}
$$

where $F_{e}$ is the force and torque exerting on the end of the manipulator; $J_{b}$ and $J_{m}$ are the Jacobian matrix defined as

$$
\begin{gather*}
J_{b}=\left[\begin{array}{cc}
E & -\tilde{p}_{0 i} \\
0 & E
\end{array}\right] ; \quad p_{0 i}=p_{i}-d_{0} \\
J_{m}=\left[\begin{array}{cccc}
k_{1} \times\left(p_{e}-p_{1}\right) & k_{2} \times\left(p_{e}-p_{2}\right) & \cdots & k_{n} \times\left(p_{e}-p_{n}\right) \\
k_{1} & k_{2} & \cdots & k_{n}
\end{array}\right] . \tag{33}
\end{gather*}
$$

Furthermore, if we consider the aerodynamics of the RFR system, the force and moment produced by the RFR $F_{p}$ can be denoted as the following mathematical equations [11]:

$$
\begin{gather*}
F_{X}=-T_{M} \sin a_{1 s} ; \quad F_{Y}=T_{M} \sin b_{1 s}-T_{T} ; \\
F_{Z}=-T_{M} \cos a_{1 s} \cos b_{1 s} ; \\
L=-\left(\frac{\partial L_{M}}{\partial b_{1 s}}\right) b_{1 s}-Q_{M} \sin a_{1 s} ;  \tag{34}\\
M=\left(\frac{\partial M_{M}}{\partial a_{1 s}}\right) a_{1 s}-Q_{M} \sin b_{1 s}-Q_{T} ;
\end{gather*}
$$

$$
N=-Q_{M} \cos a_{1 s} \cos b_{1 s},
$$

where $T_{M}$ and $T_{T}$ are the forces derived from the main rotor and tail rotor of the RFR, and $a_{1 s}$ and $b_{1 s}$ stand for the longitudinal and lateral flapping angle of main rotor, respectively; the forces $T_{M}$ and $T_{T}$ and the moments $Q_{M}$ and $Q_{T}$ can be calculated as [12, 13].

Up to now, we have constructed the nonlinear dynamics model of the RF-MJM system.

## 3. Analysis of Dynamics Model and Linearization

3.1. Analysis of the Dynamics Model. In order to understand the coupling between the RFR and manipulator clearly, we rewrite the system model (28) as follows:

$$
\begin{align*}
& {\left[\begin{array}{cc}
H_{b}(\Phi, \Theta) & H_{b m}(\Phi, \Theta) \\
H_{b m}^{T}(\Phi, \Theta) & H_{m}(\Phi, \Theta)
\end{array}\right]\left[\begin{array}{c}
\ddot{X}_{b} \\
\ddot{\Theta}
\end{array}\right]+\left[\begin{array}{c}
C_{b}\left(v_{0}, \omega_{0}, \Theta, \Phi, \dot{\Phi}\right) \\
C_{m}\left(v_{0}, \omega_{0}, \Theta, \Phi, \Phi\right)
\end{array}\right]} \\
& \quad+\left[\begin{array}{c}
G_{b}(X, \Theta, \Phi) \\
G_{b}(\Theta, \Phi)
\end{array}\right]=\left[\begin{array}{c}
F_{p} \\
\tau_{m}
\end{array}\right]
\end{align*}
$$

From (35), it can be easily seen that the coupling between the RFR and the manipulator appears in all the terms except for the exerting force/moment. That means the RF-MJM system model is more complicated than the RFR system model as shown in [11]. These can be summarized as follows.
(1) Compared to the RFR model, there are some new terms in the system (28), such as $C_{b v}$. These new terms make the system model more complicated, and the result is that some control algorithm that has been shown to be fit for RFR system cannot be used directly in the RF-MJM system. For example, in reference [11], a RFR system is proved to be approximate feedback linearizable. This, unfortunately, cannot be implemented in the RF-MJM system.
(2) The system structure of the RF-MJM is of great complication compared to the RFR system. This can be easily seen through the preceding system equations. Again, the reason is because of the coupling between the RFR and the manipulator. The higher
complication results in heavier nonlinearity which makes the controller design of the RF-MJM systemextraordinarily challenging.
3.2. Linearization and $L Q R$ Controller Design. In the above section, we have obtained the detailed mathematical model of the RF-MJM system, which can be easily computed and simulated through using symbolic computation toolbox of MATLAB. However, this kind of controller is difficult to be used due to the high complexity and nonlinearities. Thus, in this section, we will try to find the linearized model of the RFMJM system and analyze the influence of the parameters on the system parameters.

The linearization can be implemented through the following steps: firstly, search the trim point of the RF-MJM system; secondly, compute the derivatives of the system model with respect to the state and input to obtain the system matrix of the desired linear model. With the linearized model, some linear controller design strategies, for example, the LQR controller, can be used to stabilize the original nonlinear system. In the following content of this section, taking one-joint RF-MJM system as an example, the linearization and the LQR control design will be conducted and system performance will be analyzed.

A trim point, also known as an equilibrium point, of a nonlinear system is a point in the state space of a dynamic system, and at this point, the derivatives of the states with respect to time are precisely zeros.

The state vector of the RF-MJM can be denoted as

$$
\left[\begin{array}{llllllllllllll}
x & y & z & \phi & \theta & \psi & u & v & w & p & q & r & \Theta & \dot{\Theta} \tag{36}
\end{array}\right]
$$

and the input vector is

$$
\left[\begin{array}{lllll}
a_{1 s} & b_{1 s} & \theta_{M} & \theta_{T} & \tau_{\Theta} \tag{37}
\end{array}\right]
$$

where $a_{1 s}$ and $b_{1 s}$ are the cyclic pitch angle of the main rotor, respectively; $\theta_{M}$ and $\theta_{T}$ are the collective pitch angle of the main rotor and the tilt rotor, respectively.

Based on the definition of trim point, all the velocity state should be set to zeros; that is,

$$
\begin{equation*}
u=v=w=p=q=r=\dot{\Theta}=0 \tag{38}
\end{equation*}
$$

Also, all the derivatives of the states with respect to time should be zeros. Under condition (38), this is equivalent to

$$
\left[\begin{array}{c}
\ddot{X}_{b}  \tag{39}\\
\ddot{\Theta}
\end{array}\right]=H^{-1}\left(\left[\begin{array}{c}
F_{p} \\
\tau_{m}
\end{array}\right]+\left[\begin{array}{c}
J_{b}^{T} \\
J_{m}^{T}
\end{array}\right] F_{e}-\left[\begin{array}{c}
C_{b} \\
C_{m}
\end{array}\right]-\left[\begin{array}{c}
G_{b} \\
G_{b}
\end{array}\right]\right)=0_{7 \times 1} .
$$

The right-hand side of (31) is only related to $\phi, \theta, \psi, \Theta$, and input vector, so we have 9 free variables and 7 equalities. Furthermore, if we define $\tau_{\Theta}=0$ and $\psi=0$, we will have only 7 free variables. Thus, the trim point can be obtained directly by solving the nonlinear equalities (39), which can be easily conducted using some searching function in MATLAB.

In order to evaluate the influence of the mass of the manipulator on the whole system, we list out the trim point

Table 1: Parameters of RF-MJM system.

| Parameter | Describe | Unit |
| :--- | :---: | :---: |
| $m_{0}=9.5$ | The mass of rotor-flying robot (RFR) | kg |
| $m_{1}=2.5$ | The mass of manipulator | kg |
| $I_{0 x x}=0.1634$ | Moment of inertia of RFR | $\mathrm{kgm}^{2}$ |
| $I_{0 y y}=0.5782$ | Moment of inertia of RFR | $\mathrm{kgm}^{2}$ |
| $I_{0 z z}=0.6306$ | Moment of inertia of RFR | $\mathrm{kgm}^{2}$ |
| $I_{1 x x}=0.1399$ | Moment of inertia of manipulator | $\mathrm{kgm}^{2}$ |
| $I_{1 y y}=0.1399$ | Moment of inertia of manipulator | $\mathrm{kgm}^{2}$ |
| $I_{1 z z}=0.00112$ | Moment of inertia of manipulator | $\mathrm{kgm}^{2}$ |
| $l_{0}=0.3$ | The length from the centroid of RFR | m |
| $l_{1}=0.4$ | The length of half of the first link | m |
| $\phi=0.0769$ | Roll angle | rad |
| $\theta=0.0211$ | Pitch angle | rad |
| $\psi=$ any value | Yaw angle | rad |
| $\theta_{1}=0.0211$ | Joint movement angle | rad |
| $F_{0}=\left[\begin{array}{lll}0 & 0 & 0\end{array}\right]^{T}$ | External force | N |
| $T_{0}=\left[\begin{array}{lll}0 & 0 & 0\end{array}\right]^{T}$ | External torque | Nm |

of the linearization system with different manipulator masses (the parameters of the system are listed out in Table 1, and the trim point is in Table 2 in the next section).

The linearization system model is

$$
\begin{equation*}
\Delta \dot{X}=A \Delta X+B \Delta u \tag{40}
\end{equation*}
$$

where

$$
\begin{gather*}
\Delta X=X-X_{\text {trim }} \\
A=\left[\begin{array}{cccccc}
0_{3 \times 3} & E_{3} & 0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 1} & 0_{3 \times 1} \\
A_{1} & A_{2} & A_{3} & A_{4} & A_{5} & A_{6} \\
0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} & E & 0_{3 \times 1} & 0_{3 \times 1} \\
A_{7} & A_{8} & A_{9} & A_{10} & A_{11} & A_{12} \\
0_{1 \times 3} & 0_{1 \times 3} & 0_{1 \times 3} & 0_{1 \times 3} & 0 & 1 \\
A_{13} & A_{14} & A_{15} & A_{16} & A_{17} & A_{18}
\end{array}\right]  \tag{41}\\
B=\left[\begin{array}{llllll}
0_{5 \times 3} & B_{1}^{T} & 0_{5 \times 3} & B_{2}^{T} & 0_{5 \times 1} & B_{3}^{T}
\end{array}\right]^{T} .
\end{gather*}
$$

Furthermore, in order to analyze the performance of system (40), the eigenvalues of $A$ matrix are given in Figure 5. From it, we can get the following results.
(1) The whole system is static-instable since it has positive eigenvalues.
(2) With increase of the manipulator mass, the distribution of eigenvalues will be more diverging.
3.3. $L Q R$ Controller Design. Next, we will design the full state-feedback linear quadratic regulation (LQR) controller for the RFM system. In the state-feedback version of the LQR problem [14], we assume that the whole state $x$ can be measured and therefore it is available to control. Solution to the optimal state-feedback LQR problem is to find $u(t)=$ $-K x(t)$ that minimizes

$$
\begin{equation*}
J_{\mathrm{LQR}}=\int_{0}^{\infty}\left(x^{T} \mathrm{Q} x+u^{T} R u\right) d t \tag{42}
\end{equation*}
$$



Figure 5: Eigenvalue distribution of $A$ matrix with different manipulator masses.

Table 2: Trim points corresponding to different masses of manipulator.

| Trim point | $m_{1}=2.5 \mathrm{~kg}$ | $m_{1}=2 \mathrm{~kg}$ | $m_{1}=1.5 \mathrm{~kg}$ | $m_{1}=1 \mathrm{~kg}$ | $m_{1}=0.5 \mathrm{~kg}$ |
| :--- | :---: | :---: | :---: | :---: | :---: |
| $\phi$ | 0.0769 | 0.0732 | 0.0697 | 0.0662 | 0.0626 |
| $\theta$ | 0.0211 | 0.0204 | 0.0197 | 0.0190 | 0.0183 |
| $\theta_{1}$ | 0.0211 | 0.0205 | 0.0198 | 0.0191 | 0.0183 |
| $\theta_{M}$ | 0.0436 | 0.0409 | 0.0381 | 0.0354 | 0.0326 |
| $\theta_{T}$ | -0.1282 | -0.1245 | -0.1207 | -0.1169 | -0.1130 |
| $a_{1 s}$ | 0.0211 | 0.0204 | 0.0197 | 0.0190 | 0.0183 |
| $b_{1 s}$ | 0.0195 | 0.0169 | 0.0145 | 0.0123 | 0.0103 |

where $K$ is given by $K=R^{-1} B^{T} P$ and $P$ is found by solving some continuous time algebraic Riccati equations. So we can easily get the eigenvalues of the open loop system and the closed loop system by the matrices $A$ and $A-B K$, respectively.

## 4. Simulations

In this section, simulations will be conducted using the preceding nonlinear system model and the LQR controller. In the simulation, the manipulator's mass is 2.5 kg , and the other parameters are given in Table 1.

And the trim point of the whole system is listed out in Table 2.

With these parameters, the system matrices $A$ and $B$ are as follows:

$$
\begin{align*}
& A=\left[\begin{array}{cccccc}
0_{3 \times 3} & E & 0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 1} & 0_{3 \times 1} \\
0_{3 \times 3} & 0_{3 \times 3} & \bar{A}_{3} & 0_{3 \times 3} & \bar{A}_{5} & 0_{3 \times 1} \\
0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} & E & 0_{3 \times 1} & 0_{3 \times 1} \\
0_{3 \times 3} & 0_{3 \times 3} & \bar{A}_{9} & 0_{3 \times 3} & \bar{A}_{11} & 0_{3 \times 1} \\
0_{1 \times 3} & 0_{1 \times 3} & 0_{1 \times 3} & 0_{1 \times 3} & 0 & 1 \\
0_{1 \times 3} & 0_{1 \times 3} & \bar{A}_{15} & 0_{1 \times 3} & \bar{A}_{17} & 0
\end{array}\right]  \tag{43}\\
& B=\left[\begin{array}{llllll}
0_{14 \times 3} & \bar{B}_{1}^{T} & 0_{14 \times 3} & \bar{B}_{2}^{T} & 0_{14 \times 1} & \bar{B}_{3}^{T}
\end{array}\right]^{T},
\end{align*}
$$

where

$$
\begin{aligned}
& \bar{A}_{3}=\left[\begin{array}{ccc}
0.0015 & 9.7636 & 0.9945 \\
-9.8000 & -0.0005 & 0.2102 \\
0.0001 & 0.0005 & 0.0014
\end{array}\right] \text {; } \\
& \bar{A}_{9}=\left[\begin{array}{ccc}
-0.0087 & -0.0083 & 0.1834 \\
-0.0130 & 0.0255 & -0.8587 \\
-0.2013 & -0.2456 & -0.0635
\end{array}\right] \text {; } \\
& \bar{A}_{5}=\left[\begin{array}{c}
3.5861 \\
-0.0077 \\
0.0310
\end{array}\right] ; \quad \bar{A}_{11}=\left[\begin{array}{c}
-0.0547 \\
-24.4857 \\
-0.8988
\end{array}\right] ; \\
& \bar{A}_{15}=\left[\begin{array}{lll}
0.0110 & -0.0924 & 2.6951
\end{array}\right] ; \\
& \bar{A}_{17}=40.1487 ; \\
& \bar{B}_{1}^{T}=\left[\begin{array}{ccc}
-0.5451 & -7.9183 & -147.7390 \\
-1.2434 & -9.3478 & 0.6158 \\
-21.7522 & 0.4534 & -0.2015 \\
-0.5829 & 7.2377 & -0.7650 \\
0.2990 & -0.0006 & 0.0009
\end{array}\right] ;
\end{aligned}
$$


(a)



(c)




(b)

(d)

$=b_{1}$
$--\theta_{M}$
$-\theta_{\mathrm{T}}$

(e)
(f)

Figure 6: States and inputs profile under LQR controller (46).

$$
\bar{B}_{2}^{T}=\left[\begin{array}{ccc}
16.7653 & -2.5282 & 168.3611  \tag{44}\\
-32.8659 & 14.0449 & 125.4196 \\
20.9082 & 178.3726 & 4.0958 \\
-121.2649 & 11.0285 & -0.2925 \\
0.0011 & 0.6932 & -0.0534
\end{array}\right]
$$

$$
\bar{B}_{3}^{T}=\left[\begin{array}{lllll}
-3.9371 & -9.0113 & -99.1817 & -4.2009 & 3.4137
\end{array}\right]
$$

From these equations, it can be seen that the coupling between the manipulator and RFR, denoted by $\bar{A}_{5}, \bar{A}_{11}, \bar{A}_{15}$,


Figure 7: Continued.


Figure 7: States response with disturbance (49) (blue and solid) and (50) (red and dashed) ${ }^{*}$; the subscript -1 means that the results are with the disturbance (49) and the subscript -2 means that the results are with the disturbance (50).
and $\bar{A}_{17}$ is heavy. Using the following QR parameters in the LQR controller,

$$
\begin{gather*}
Q=\operatorname{diag}(1,1,1,1,1,1,10,10,10,10,10,10,10,10) ; \\
R=\operatorname{diag}(1000,1000,1000,1000,1000) \tag{45}
\end{gather*}
$$

the LQR feedback control law is designed as

$$
\begin{gather*}
u=K \Delta x  \tag{46}\\
K=\left[\begin{array}{lllll}
K_{1} & K_{2} & K_{3} & K_{4} & K_{5}
\end{array}\right],
\end{gather*}
$$

where

$$
\begin{aligned}
& K_{1}=\left[\begin{array}{ccc}
0.0097 & -0.0037 & -0.0226 \\
0.0119 & 0.0058 & 0.0210 \\
-0.0232 & -0.0014 & 0.0013 \\
-0.0018 & 0.0308 & -0.0066 \\
0.0149 & -0.0006 & -0.0008
\end{array}\right] \\
& K_{2}=\left[\begin{array}{ccc}
0.0200 & -0.0071 & -0.0291 \\
0.0238 & 0.0116 & 0.0287 \\
-0.0544 & -0.0019 & 0.0031 \\
-0.0042 & 0.0575 & -0.0079 \\
0.0308 & -0.0013 & -0.0016
\end{array}\right] \\
& K_{3}=\left[\begin{array}{ccc}
0.0502 & 0.1489 & 0.1317 \\
-0.0816 & 0.1670 & 0.1279 \\
0.0053 & -0.4953 & -0.3462 \\
-0.3754 & -0.0384 & -0.0293 \\
0.0104 & 0.2401 & 0.0991
\end{array}\right]
\end{aligned}
$$

$$
\begin{gather*}
K_{4}=\left[\begin{array}{ccc}
0.0169 & 0.0637 & 0.0751 \\
-0.0226 & 0.0663 & 0.0827 \\
-0.0015 & -0.2717 & -0.0288 \\
-0.1208 & -0.0214 & -0.0058 \\
0.0040 & 0.1085 & 0.0073
\end{array}\right]  \tag{47}\\
K_{5}=\left[\begin{array}{cc}
0.6994 & 0.1310 \\
0.5973 & 0.1125 \\
-4.2835 & -0.8102 \\
-0.3735 & -0.0691 \\
1.1293 & 0.2157
\end{array}\right] .
\end{gather*}
$$

With the controller (46), the system can be stabilized with acceptable performance near the trim point, and the simulation results with initial state ( $0.1,0,-0.1,0,0,0,0.07$, $0.04,0,0,0,0.03,0)$ are shown in Figure 6.

From Figure 6, it can be seen that a linear LQR controller can stabilize the whole system near the trim point. However, the stabilizing region of LQR is very limited; we have tested that only when the attitude of the whole system satisfies the following conditions (all the initial velocities are set to zeros), the LQR control is effective (46):

$$
\begin{gather*}
-0.1389 \leq \theta \leq 0.3911 \\
-0.6731 \leq \phi \leq 0.5469  \tag{48}\\
-0.0619 \leq \theta_{1} \leq 0.0611
\end{gather*}
$$

Now in the next simulation, two periodic sinusoidal signals, as disturbances with different frequencies, are added


Figure 8: Decoupling between RFR and manipulator.
to the input of the manipulator to test motion influence of the manipulator on the whole system:

$$
\begin{align*}
& d_{1}=0.01 \sin (0.5 \pi t)  \tag{49}\\
& d_{2}=0.01 \sin (1.0 \pi t) \tag{50}
\end{align*}
$$

The results are as in Figure 7.
Simultaneously, in order to test the coupling between the RFR and the manipulator, the linear accelerations and the angular accelerations of the new RF-MJM system are compared to the helicopter system with the same parameters as in Table 1; that is,

$$
\begin{align*}
& {\left[\begin{array}{l}
\text { error 1 } \\
\text { error 2 } \\
\text { error 3 }
\end{array}\right]=\left[\begin{array}{c}
a_{\mathrm{RF}-\mathrm{MJM}, x} \\
a_{\mathrm{RF}-\mathrm{M} \mathrm{M}, y} \\
a_{\mathrm{RF}-\mathrm{MJM}, z}
\end{array}\right]-\left[\begin{array}{c}
a_{\mathrm{RFR}, x} \\
a_{\mathrm{RFR}, y} \\
a_{\mathrm{RFR}, z}
\end{array}\right] ;} \\
& {\left[\begin{array}{l}
\text { error 4 } \\
\text { error 5 } \\
\text { error 6 }
\end{array}\right]=\left[\begin{array}{c}
\dot{\omega}_{\mathrm{RF}-\mathrm{MJM}, x} \\
\dot{\omega}_{\mathrm{RF}-\mathrm{MJM}, y} \\
\dot{\omega}_{\mathrm{RF}-\mathrm{MJM}, z}
\end{array}\right]-\left[\begin{array}{c}
\dot{\omega}_{\mathrm{RFR}, x} \\
\dot{\omega}_{\mathrm{RFR}, y} \\
\dot{\omega}_{\mathrm{RFR}, z}
\end{array}\right] .} \tag{51}
\end{align*}
$$

The results are given in Figure 8, which presents the extra force and moment exerted on the RFR due to the manipulator and its motion.

## 5. Conclusions

In this paper, the detailed nonlinear dynamics model of a rotor-flying multijoint (RF-MJM) system is constructed through using Euler-Lagrange method. Compared to the rotor-flying vehicle system, the model nonlinearities and
complexity of the new RF-MJM are analyzed in detail. Moreover, linear analysis is conducted with respect to the constructed nonlinear model near its trim point, and the influence of the manipulator mass on the system's local performance is researched. Furthermore, LQR controller is designed based on the linearized system model. Finally, simulation results show that (1) a linear LQR controller is able to stabilize the system near steady state and presents acceptable performance; however, (2) the stabilization region of LQR controller is very limited, and the performance of LQR controller is sensitive to the external disturbance. Thus, in the future work, nonlinear and robust control scheme will be researched to overcome the disadvantages of the linear controller.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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# Analysis of a Delayed Internet Worm Propagation Model with Impulsive Quarantine Strategy 

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#### Abstract

Internet worms exploiting zero-day vulnerabilities have drawn significant attention owing to their enormous threats to Internet in the real world. To begin with, a worm propagation model with time delay in vaccination is formulated. Through theoretical analysis, it is proved that the worm propagation system is stable when the time delay is less than the threshold $\tau_{0}$ and Hopf bifurcation appears when time delay is equal to or greater than $\tau_{0}$. Then, a worm propagation model with constant quarantine strategy is proposed. Through quantitative analysis, it is found that constant quarantine strategy has some inhibition effect but does not eliminate bifurcation. Considering all the above, we put forward impulsive quarantine strategy to eliminate worms. Theoretical results imply that the novel proposed strategy can eliminate bifurcation and control the stability of worm propagation. Finally, simulation results match numerical experiments well, which fully supports our analysis.


## 1. Introduction

With the rapid growth of information technologies and network applications, severe challenges, in form of requirement of a suitable defense system, have been posed to make sure of the safety of the valuable information stored on system and in transit. For example, worms that exploit zeroday vulnerabilities have brought severe threats to Internet security in the real world. To date, none of the patches could effectively and reliably immunize the hosts thoroughly against being attacked by those worms. It may take a period of time for users to immunize their computers if they are in infected state. In addition, the failure of some vaccination measures or worm-variants may also lead to high risks that the hosts being immunized would be infected again. On the other hand, the propagation of worms in a system of interacting computers could be compared to contagious diseases in human population. In computer science field, computers are like individuals in an ecological system and thus the same mechanism of birth and death should be considered. Being infected by network worms or quarantined by IDS (intrusion detection systems), hosts will become
dangerous and their owners will have to reinstall the system. Another factor to consider is that when new computers are brought, most of them have preinstalled operating systems but without newest safety patches while old computers are discarded and recycled. Consequently, in order to imitate the real world, birth and death rates should be introduced to worm propagations model.

Considering all the above, we firstly construct a worm propagation model with time delay in vaccination based on the classical epidemic Kermack-Mckendrick model [1] to describe the current situation. Through theoretical analysis, it is proved that Hopf bifurcation appears when time delay is equal to or greater than the threshold $\tau_{0}$, which leads the number of infected hosts to be unpredictable and the propagation of worms to be out of control. In order to make up the deficiency of vaccination strategy and eliminate the negative impact of time delay, quarantine strategies are proposed to improve vaccination effect and eliminate bifurcation. The current quarantine strategy generally depends on the intrusion detection system, which can be classified into two categories: misuse and anomaly intrusion detection. Misuse intrusion detection system can accurately detect


Figure 1: State transition diagram of delayed model.
known worms. Based on misuse intrusion detection system, we propose constant quarantine strategy. Although it does improve vaccination effect, the system is still out of control and Hopf bifurcation is not eliminated either. Furthermore, the system fails to detect unknown worms and wormvariants. Anomaly intrusion detection system is of help in detecting these kinds of worm. However, it is always accompanied by high false-positive rate.

Consequently, this paper proposes a worm propagation model with impulsive quarantine strategy based on a hybrid intrusion detection system that combines both misuse and anomaly intrusion detection to make up for the gaps existing in the two systems. After adoption impulsive quarantine strategy, it is clearly proved that Hopf bifurcation is eliminated thoroughly so that the system is stable.

The rest of the paper is organized as follows. In the next section, related work on time delay and quarantine strategy is introduced. Section 3 provides a worm propagation model with time delay in vaccination. In Section 4, we construct a delayed worm propagation model with constant quarantine and analyze it in detail. Then, in Section 5, a delayed worm propagation model using impulsive quarantine strategy is proposed, and its analysis is performed. Section 6 presents numerical analyses and simulation experiments based on Slammer worm. Simulation results match well with numerical ones. Finally, Section 7 gives the conclusions.

## 2. Related Work

With the similarity between Internet worms and biological diseases, epidemiological models have been widely used in modeling the propagation of worms [2-6]. To make the worm transmission in computer network work as in the real word, the research within the data-driven framework has been done [7-9]. Although some human factors are included, these models cannot restrain worms effectively. Thus, a variety of containment strategies have been applied to worm propagation models. As far as we know, the use of quarantine strategies has produced a great effect on controlling disease. People use quarantine strategies widely in worm containment enlightened by this [10-16]. In addition, some scholars have done research on time delay [17-19].

However, previous studies have failed to consider the appropriate quarantine strategy to eliminate the negative effect of time delay. For instance, the pulse quarantine strategy that Yao has proposed [12] does lead to worm elimination with a relatively low value, but time delay is not considered, which leads to Hopf bifurcation so that the worm propagation system will be unstable and out of control. In this paper, constant quarantine and impulsive quarantine
strategies are proposed to constrain the worms spreading and even eliminate Hopf bifurcation.

## 3. Worm Propagation Model with Time Delay in Vaccination

With regard to worms exploiting zero-day vulnerabilities, none of the patches could effectively and reliably immunize the hosts. After the hosts are being infected, some measures, such as cutting off the network connection, running manual antivirus, or setting firewall, are taken to remove the worms. With these measures being carried out, the hosts cannot further infect other susceptible hosts, but they are in fact not vaccinated completely. Namely, detecting and cleaning worms take a period of time. Therefore, time delay should be considered in actual conditions. Since time delay exists, infected hosts go through a temporary state (delayed) after vaccination. Consequently, on the basis of KM model, we give a worm propagation model with time delay in vaccination. We assume all hosts are in one of four states: susceptible state $(S)$, infected state $(I)$, delayed state $(D)$, and vaccinated state $(V)$. The state transition diagram of the delayed model is given in Figure 1.

Let $S(t)$ denote the number of susceptible hosts at time $t$, $I(t)$ denote the number of infected hosts at time $t, D(t)$ denote the number of delayed hosts at time $t$, and $V(t)$ denote the number of vaccinated hosts at time $t . \beta$ is the infection rate at which susceptible hosts are infected by infected hosts and $\gamma$ is the rate of removal of infected from circulation. As worms and worm-variants exist, $\mu$ is the rate that vaccinated hosts back to susceptible hosts. The newborn hosts enter the system with the same rate $\nu$, of which a portion $1-p$ is recovered by installing patches at birth. Time delay is denoted by $\tau$.

In order to show it clearly, we list in Notations section some frequently used notations in this paper.
3.1. Description of Delayed Model. From the above definitions in the paper, we write down the complete differential equations of the delayed model:

$$
\begin{align*}
& \frac{d S(t)}{d t}=p \nu N-\beta S(t) I(t)-\nu S(t)+\mu V(t), \\
& \frac{d I(t)}{d t}=\beta S(t) I(t)-\gamma I(t)-\nu I(t),  \tag{1}\\
& \frac{d D(t)}{d t}=\gamma I(t)-\gamma I(t-\tau)-\nu D(t), \\
& \frac{d V(t)}{d t}=(1-p) \nu N+\gamma I(t-\tau)-\mu V(t)-\nu V(t) .
\end{align*}
$$

As mentioned above, the population size is set $N$, which is set to unity:

$$
\begin{equation*}
S(t)+I(t)+D(t)+V(t)=N \tag{2}
\end{equation*}
$$

### 3.2. Stability of the Positive Equilibrium and Bifurcation Analysis

Theorem 1. The system has a unique positive equilibrium $E^{*}\left(S^{*}, I^{*}, D^{*}, V^{*}\right)$ when it satisfies the following condition:

$$
\begin{gathered}
\left(H_{1}\right)(\beta N-\gamma-v)(\mu+\nu) / \beta(1-p) v N>1, \text { where } S^{*}= \\
(\gamma+v) / \beta, D^{*}=0, V^{*}=\left(\gamma I^{*}+(1-p) \nu N\right) /(\mu+v) .
\end{gathered}
$$

Proof. For system (1), if all the derivatives on the left of equal sign of the system are set to 0 , which implies that the system becomes stable, we can derive

$$
\begin{align*}
S & =\frac{\gamma+\nu}{\beta} \\
D & =0  \tag{3}\\
V & =\frac{\gamma I^{*}+(1-p) \nu N}{\mu+v}
\end{align*}
$$

Substituting the value of each variable in (3) for each of (2), then we can derive

$$
\begin{equation*}
S^{*}+I^{*}+\frac{\gamma I^{*}+(1-p) \nu N}{\mu+v}=N \tag{4}
\end{equation*}
$$

Obviously, if $\left(H_{1}\right)$ is satisfied, (4) has one unique positive root $I^{*}$ and there is one unique positive equilibrium $E^{*}\left(S^{*}, I^{*}, D^{*}, V^{*}\right)$ of system (1). The proof is completed.

According to (2), $V(t)=N-S(t)-I(t)-D(t)$; thus, system (1) can be simplified to

$$
\begin{align*}
\frac{d S(t)}{d t}= & p \nu N+\mu(N-S(t)-I(t)-D(t)) \\
& -\beta S(t) I(t)-v S(t) \\
\frac{d I(t)}{d t}= & \beta S(t) I(t)-\gamma I(t)-v I(t)  \tag{5}\\
\frac{d D(t)}{d t}= & \gamma I(t)-\gamma I(t-\tau)-v D(t)
\end{align*}
$$

The Jacobi matrix of (5) about $E^{*}\left(S^{*}, I^{*}, D^{*}\right)$ is given by

$$
J\left(E^{*}\right)=\left(\begin{array}{ccc}
-\mu-\beta I^{*}-\nu & -\mu-\beta S^{*} & -\mu  \tag{6}\\
\beta I^{*} & \beta S^{*}-\gamma-\nu & 0 \\
0 & \gamma-\gamma e^{-\lambda \tau} & -v
\end{array}\right)
$$

The characteristic equation of that matrix can be obtained by

$$
\begin{equation*}
P(\lambda)+Q(\lambda) e^{-\lambda \tau}=0 \tag{7}
\end{equation*}
$$

Let

$$
\begin{align*}
p_{2}= & \mu+\beta I^{*}+3 v-\beta S^{*}+\gamma, \\
p_{1}= & \beta(2 v-\mu+\gamma) I^{*}-\beta S^{*}(\mu+2 \nu)-2 \beta^{2} I^{*} S^{*} \\
& +\mu(2 \nu+\gamma)+\nu\left(3 \nu^{2}+2 \gamma\right), \\
p_{0}= & \beta I^{*}\left(\nu \gamma+\nu^{2}+\mu \gamma-\mu \nu\right)-\nu \beta S^{*}(\mu+\nu)  \tag{8}\\
& -2 \nu \beta^{2} S^{*} I^{*}+\nu \mu \gamma+\nu^{2}(\mu+\gamma+\nu),
\end{align*}
$$

$q_{0}=-\beta \mu \gamma I^{*}$.
Then $P(\lambda)=\lambda^{3}+p_{2} \lambda^{2}+p_{1} \lambda+p_{0}, Q(\lambda)=q_{0}$.
Theorem 2. The positive equilibrium $E^{*}$ is locally asymptotically stable without time delay, if the following holds:

$$
\left(H_{2}\right) p_{2}>0, p_{1} p_{2}-\left(p_{0}+q_{0}\right)>0, p_{0}+q_{0}>0 .
$$

Proof. If $\tau=0$, (7) reduces to

$$
\begin{equation*}
\lambda^{3}+p_{2} \lambda^{2}+p_{1} \lambda+\left(p_{0}+q_{0}\right)=0 \tag{9}
\end{equation*}
$$

According to Routh-Hurwitz criterion, all the roots of (9) have negative real parts. Therefore, it can be deduced that the positive equilibrium $E^{*}$ is locally asymptotically stable without time delay. The proof is completed.

Obviously, $\lambda=i \omega(\omega>0)$ is a root of (7). After separating the real and imaginary parts, it can be written as

$$
\begin{align*}
& -p_{2} \omega^{2}+p_{0}+q_{0} \cos (\omega \tau)=0  \tag{10}\\
& -\omega^{3}+p_{1} \omega-q_{0} \sin (\omega \tau)=0 \tag{11}
\end{align*}
$$

which implies

$$
\begin{equation*}
\omega^{6}+D_{3} \omega^{4}+D_{2} \omega^{2}+D_{1}=0 \tag{12}
\end{equation*}
$$

where

$$
\begin{align*}
& D_{3}=p_{2}^{2}-2 p_{1} \\
& D_{2}=p_{1}-2 p_{2} p_{0}  \tag{13}\\
& D_{1}=p_{0}^{2}-q_{0}^{2}
\end{align*}
$$

Let $z=\omega^{2}$; (12) can be written as

$$
\begin{equation*}
h(z)=z^{3}+D_{3} z^{2}+D_{2} z+D_{1} \tag{14}
\end{equation*}
$$

$\Delta$ is defined as $\Delta=D_{3}^{2}-3 D_{2}$. Hence, we can get a solution $z^{*}=\left(\sqrt{\Delta}-D_{3}\right) / 3$ of $h(z)$.

Lemma 3. Suppose that $\left(H_{2}\right) p_{2}>0, p_{1} p_{2}-\left(p_{0}+q_{0}\right)>0$; $p_{0}+q_{0}>0$ is satisfied.
(1) If one of the following holds: (a) $\Delta>0, z^{*}<0$; (b) $\Delta>0, z^{*}>0$; and $h\left(z^{*}\right)>0$, then all roots of (7) have negative real parts when $\tau \in\left[0, \tau_{0}\right)$ and $\tau_{0}$ is a certain positive constant.
(2) If the conditions (a) and (b) are not satisfied, then all roots of (7) have negative real parts for all $\tau \geq 0$.

Proof. When $\tau=0$, (7) can be reduced to

$$
\begin{equation*}
\lambda^{3}+p_{2} \lambda^{2}+p_{1} \lambda+\left(p_{0}+q_{0}\right)=0 \tag{15}
\end{equation*}
$$

By the Routh-Hurwitz criterion, all roots of (9) have negative real parts and only if

$$
\begin{equation*}
p_{2}>0, \quad p_{1} p_{2}-\left(p_{0}+q_{0}\right)>0, \quad p_{0}+q_{0}>0 \tag{16}
\end{equation*}
$$

Considering (14), it is easy to see from the characters of cubic algebraic equation that $h(z)$ is a strictly monotonically increasing function if $\Delta \leq 0$. If $\Delta>0, z^{*}<0$ or $\Delta>0$, $z^{*}>0$ and $h\left(z^{*}\right)>0$, then $h(z)$ has no positive root. Hence, (7) has no purely imaginary roots for any $\tau>0$, which implies that the positive equilibrium $E^{*}\left(S^{*}, I^{*}, D^{*}, V^{*}\right)$ of system (1) is absolutely stable. Therefore, the following theorem on the stability of positive equilibrium $E^{*}\left(S^{*}, I^{*}, D^{*}, V^{*}\right)$ can be easily obtained.

Theorem 4. Assume that $\left(H_{1}\right)$ and $\left(H_{2}\right)$ are satisfied, and $\Delta>0, z^{*}<0$ or $\Delta>0, z^{*}>0$. and $h\left(z^{*}\right)>0$, then the positive equilibrium $E^{*}\left(S^{*}, I^{*}, D^{*}, V^{*}\right)$ of system (1) is absolutely stable. Namely, $E^{*}\left(S^{*}, I^{*}, D^{*}, V^{*}\right)$ is asymptotically stable for any time delay $\tau>0$.

Assume that the coefficients in $h(z)$ satisfy the condition as follows:

$$
\left(H_{3}\right) \Delta>0, z^{*}>0, h\left(z^{*}\right)<0 .
$$

According to lemma, it is proved that (14) has at least a positive root $\omega_{0}$, namely, the characteristic equation (7) has a pair of purely imaginary roots $\pm i \omega_{0}$.

In view of the fact that (7) has a pair of purely imaginary roots $\pm i \omega_{0}$, the corresponding $\tau_{k}>0$ is given by eliminating $\sin (\omega \tau)$ in (10) and (11):

$$
\begin{equation*}
\tau_{k}=\frac{1}{\omega_{0}} \arccos \left[\frac{p_{2} \omega_{0}^{2}-p_{0}}{q_{0}}\right]+\frac{2 k \pi}{\omega_{0}} \quad(k=0,1,2, \ldots) . \tag{17}
\end{equation*}
$$

Let $\lambda(\tau)=v(\tau)+i \omega(\tau)$ be the root of (7), so that $v\left(\tau_{k}\right)=0$ and $\omega\left(\tau_{k}\right)=\omega_{0}$ are satisfied when $\tau=\tau_{k}$.

Lemma 5. Suppose $h^{\prime}\left(z_{0}\right) \neq 0$. If $\tau=\tau_{0}$, then $\pm i \omega_{0}$ is a pair of purely imaginary roots of (7). In addition, if the conditions in Lemma 3 are satisfied, then

$$
\begin{equation*}
\left.\frac{d(\operatorname{Re} \lambda)}{d \tau}\right|_{\tau=\tau_{k}}>0 \tag{18}
\end{equation*}
$$

This signifies that there exists at least one eigenvalue with positive real part for $\tau>\tau_{k}$. Differentiating both sides of (7) with respect to $\tau$, it can be written as

$$
\begin{equation*}
\left(\frac{d \lambda}{d \tau}\right)^{-1}=\frac{\left(3 \lambda^{2}+2 p_{2} \lambda+p_{1}\right)-q_{0} \tau e^{-\lambda \tau}}{q_{0} \lambda e^{-\lambda \tau}} \tag{19}
\end{equation*}
$$

Therefore,

$$
\begin{align*}
\operatorname{sgn}\left[\frac{d \operatorname{Re} \lambda}{d \tau}\right]_{\tau=\tau_{k}} & =\operatorname{sgn}\left[\operatorname{Re}\left(\frac{d \lambda}{d \tau}\right)^{-1}\right]_{\lambda=i \omega_{0}} \\
& =\operatorname{sgn} \frac{\omega_{0}^{2}}{\Lambda}\left(3 \omega_{0}^{4}+2 D_{2} \omega_{0}^{2}+D_{1}\right)  \tag{20}\\
& =\operatorname{sgn} \frac{\omega_{0}^{2}}{\Lambda}\left\{h^{\prime}\left(\omega_{0}^{2}\right)\right\} \\
& =\operatorname{sgn}\left\{h^{\prime}\left(\omega_{0}^{2}\right)\right\}
\end{align*}
$$

where $\Lambda=q_{0} \omega_{0}^{2}$; then it follows the hypothesis $\left(H_{3}\right)$ that $h^{\prime}\left(\omega_{0}^{2}\right) \neq 0$.

Hence,

$$
\begin{equation*}
\left.\frac{d(\operatorname{Re} \lambda)}{d \tau}\right|_{\tau=\tau_{k}}>0 \tag{21}
\end{equation*}
$$

The root of characteristic equation (7) crosses from left to right on the imaginary axis as $\tau$ continuously varies from a value less than $\tau_{k}$ to one greater than $\tau_{k}$ according to Routh's theorem. Therefore, according to the Hopf bifurcation theorem [20] for functional differential equations, the transverse condition holds and the conditions for Hopf bifurcation are satisfied at $\tau=\tau_{k}$. Then the following result can be obtained.

Theorem 6. Suppose that the conditions $\left(H_{1}\right)$ and $\left(H_{2}\right)$ are satisfied.
(1) The equilibrium $E^{*}\left(S^{*}, I^{*}, D^{*}, V^{*}\right)$ is locally asymptotically stable when $\tau \in\left[0, \tau_{0}\right)$, but unstable when $\tau>\tau_{0}$.
(2) If condition $\left(\mathrm{H}_{3}\right)$ is satisfied, the system will undergo Hopf bifurcation at the positive equilibrium $E^{*}\left(S^{*}, I^{*}, D^{*}, V^{*}\right)$ when $\tau=\tau_{k}(k=0,1,2, \ldots)$, where $\tau_{k}$ is defined by (17).

This implies that when time delay $\tau<\tau_{0}$, the system will stabilize at its infection equilibrium point, which is beneficial to implement a containment strategy; when $\tau \geq \tau_{0}$, the system will be unstable and worms cannot be effectively controlled.

## 4. A Delayed Worm Propagation Model with Constant Quarantine

Enlightened by the methods in disease control, quarantine is selected as an effective way to diminish the speed of worm propagation. The current quarantine strategy generally depends on the intrusion detection system, which can be classified into two categories: misuse and anomaly intrusion detection [12]. As the delayed model cannot make sure of the system stable and controlled, quarantine strategies should be taken into consideration to further control the worm propagation.
4.1. Using Constant Quarantine Strategy to Model a Delayed Worm Propagation. Misuse intrusion detection system builds a database with the feature of known attack behaviors.


Figure 2: State transition diagram of constant quarantine model.

The system can recognize the invaders once their behaviors agree with one of the databases and accurately detect known worms [12]. By applying misuse intrusion detection system for its relatively high accuracy, we add a new state called quarantine state (Q) [9], but only infected hosts will be quarantined. $Q(t)$ denote the number of quarantined hosts at time $t$. Unlike the quarantine strategy against epidemics, the implementation of constant quarantine strategy depends on the misuse intrusion detection system. Infected hosts will be quarantined at rate $\alpha$ which depends on the performance of intrusion detection system and network devices. When infected hosts are quarantined, they will get rid of worms and get patched at rate $\delta$. The state transition diagram of constant quarantine model is given in Figure 2.
4.2. Description of Constant Quarantine Model. According to the definitions above in the paper, the differential equations of constant quarantine model are given as follows:

$$
\begin{align*}
& \frac{d S(t)}{d t}=p \nu N-\beta S(t) I(t)+\mu V(t)-\nu S(t) \\
& \frac{d I(t)}{d t}=\beta S(t) I(t)-I(t)-\alpha I(t)-\nu I(t) \\
& \frac{d D(t)}{d t}=\gamma I(t)-\gamma I(t-\tau)-\nu D(t) \\
& \frac{d Q(t)}{d t}=\alpha I(t)-\delta Q(t)-\nu Q(t) \\
& \frac{d V(t)}{d t}=\gamma I(t-\tau)+\delta Q(t)-\mu V(t)+(1-p) \nu N-\nu V(t) \tag{22}
\end{align*}
$$

Similarly,

$$
\begin{equation*}
S(t)+I(t)+D(t)+Q(t)+V(t)=N \tag{23}
\end{equation*}
$$

### 4.3. Stability of the Positive Equilibrium and Bifurcation Analysis

Theorem 7. The system has a unique positive equilibrium $E^{*}\left(S^{*}, I^{*}, D^{*}, Q^{*}, V^{*}\right)$ when it satisfies the following condition:

$$
\begin{aligned}
& \left(H_{1}\right) \beta N(\mu+p \nu) / \mu(\mu+\nu)(\gamma+\alpha+\nu)>1, \text { where } S^{*}= \\
& (\gamma+\alpha+\nu) / \beta, D^{*}=0, Q^{*}=(\alpha /(\delta+\nu)) I^{*}, V^{*}= \\
& ((\gamma+\alpha+\nu) / \mu)\left(\nu / \beta+I^{*}\right)-p \nu N / \mu .
\end{aligned}
$$

Proof. For system (22), if all the derivatives on the left of equal sign of the system are set to 0 , which implies that the system becomes stable, we can get

$$
\begin{align*}
& S=\frac{\gamma+\alpha+v}{\beta}, \\
& D=0, \\
& Q=\frac{\alpha}{\delta+\nu} I^{*}  \tag{24}\\
& V=\frac{\gamma+\alpha+v}{\mu}\left(\frac{v}{\beta}+I^{*}\right)-\frac{p \nu N}{\mu} .
\end{align*}
$$

Substituting the value of each variable in (24) for each of (23), then we can get

$$
\begin{equation*}
S^{*}+\frac{\alpha}{\delta+\nu} I^{*}+I^{*}+\frac{\gamma+\alpha+\nu}{\mu}\left(\frac{v}{\beta}+I^{*}\right)-\frac{p v N}{\mu}=N . \tag{25}
\end{equation*}
$$

Obviously, if $\left(H_{1}\right)$ is satisfied, (25) has one unique positive root $I^{*}$, and there is one unique positive equilibrium $E^{*}\left(S^{*}, I^{*}, D^{*}, Q^{*}, V^{*}\right)$ of system (22). The proof is completed.

According to (23), $V(t)=N-S(t)-I(t)-D(t)-Q(t)$; thus, system (22) can be simplified to

$$
\begin{align*}
\frac{d S(t)}{d t}= & p \nu N+\mu(N-S(t)-I(t)-D(t)-Q(t)) \\
& -\beta S(t) I(t)-\nu S(t) \\
\frac{d I(t)}{d t}= & \beta S(t) I(t)-\gamma I(t)-\alpha I(t)-\nu I(t)  \tag{26}\\
\frac{d D(t)}{d t}= & \gamma I(t)-\gamma I(t-\tau)-\nu D(t) \\
\frac{d Q(t)}{d t}= & \alpha I(t)-\delta Q(t)-\nu \mathrm{Q}(t)
\end{align*}
$$

The Jacobi matrix of (26) about $E^{*}\left(S^{*}, I^{*}, D^{*}, Q^{*}\right)$ is given by

$$
J\left(E^{*}\right)=\left(\begin{array}{cccc}
-\beta I^{*}-\nu-\mu & -\beta S^{*}-\mu & -\mu & -\mu  \tag{27}\\
\beta I^{*} & \beta S^{*}-\gamma-\alpha-\nu & 0 & 0 \\
0 & \gamma-\gamma e^{-\lambda \tau} & -v & 0 \\
0 & \alpha & 0 & -\delta-\mu
\end{array}\right)
$$

The characteristic equation of that matrix can be obtained by

$$
\begin{equation*}
P(\lambda)+Q(\lambda) e^{-\lambda \tau}=0 \tag{28}
\end{equation*}
$$

Let

$$
\begin{align*}
& p_{3}=a+b+c+v \\
& p_{2}=a b+c v+(a+b)(v+c)+\beta I^{*} d \\
& p_{1}=a b(v+c)+v c(a+b)+\beta I^{*}(d(v+c)+\mu(\alpha+\gamma)) \\
& p_{0}=a b c v+\beta I^{*}(c d v+\alpha \mu v+c \mu \gamma) \\
& q_{1}=-\mu \gamma \beta I^{*} \\
& q_{0}=-\beta \mu c \gamma I^{*} \tag{29}
\end{align*}
$$

where

$$
\begin{align*}
& a=\beta I^{*}+\nu+\mu \\
& b=\gamma+\alpha+\mu-\beta S^{*},  \tag{30}\\
& c=\delta+v \\
& d=\beta S^{*}+\mu
\end{align*}
$$

then

$$
\begin{align*}
& P(\lambda)=\lambda^{4}+p_{3} \lambda^{3}+p_{2} \lambda^{2}+p_{1} \lambda+p_{0}  \tag{31}\\
& Q(\lambda)=q_{1} \lambda+q_{0}
\end{align*}
$$

Theorem 8. The positive equilibrium $E^{*}$ is locally asymptotically stable without time delay, if the following holds:

$$
\left(H_{2}\right) p_{3}>0, d_{1}>0, d_{2}>0,\left(p_{1}+q_{1}\right) d_{1}-p_{3}^{2} d_{2}>0
$$

where

$$
\begin{align*}
& d_{1}=p_{3} p_{2}-\left(p_{1}+q_{1}\right), \\
& d_{2}=p_{0}+q_{0} . \tag{32}
\end{align*}
$$

Proof. If $\tau=0$, (28) reduces to

$$
\begin{equation*}
\lambda^{4}+p_{3} \lambda^{3}+p_{2} \lambda^{2}+\left(p_{1}+q_{1}\right) \lambda+\left(p_{0}+q_{0}\right)=0 \tag{33}
\end{equation*}
$$

According to Routh-Hurwitz criterion, all the roots of (33) have negative real parts. Therefore, it can be deduced that the positive equilibrium $E^{*}$ is locally asymptotically stable without time delay. The proof is completed.

Obviously, $\lambda=i \omega(\omega>0)$ is a root of (28). After separating the real and imaginary parts, it can be written as

$$
\begin{gather*}
\omega^{4}-p_{2} \omega^{2}+p_{0}+q_{1} \omega \sin (\omega \tau)+q_{0} \cos (\omega \tau)=0 \\
-p_{3} \omega^{3}+p_{1} \omega+q_{1} \omega \cos (\omega \tau)-q_{0} \sin (\omega \tau)=0 \tag{34}
\end{gather*}
$$

which implies

$$
\begin{equation*}
\omega^{6}+D_{3} \omega^{4}+D_{2} \omega^{2}+D_{1}=0 \tag{35}
\end{equation*}
$$

where

$$
\begin{align*}
& D_{3}=p_{3}^{2}-2 p_{2}, \\
& D_{2}=p_{2}^{2}+2 p_{0}-2 p_{1} p_{3},  \tag{36}\\
& D_{1}=p_{1}^{2}-q_{1}^{2}-2 p_{2} p_{0} .
\end{align*}
$$

Let $z=\omega^{2}$, and (35) can be written as

$$
\begin{equation*}
h(z)=z^{3}+D_{3} z^{2}+D_{2} z+D_{1} \tag{37}
\end{equation*}
$$

$\Delta$ is defined as $\Delta=D_{3}^{2}-3 D_{2}$. Hence, we can get a solution $z^{*}=\left(\sqrt{\Delta}-D_{3}\right) / 3$ of $h(z)$.

Lemma 9. Suppose that $\left(H_{2}\right) p_{3}>0, d_{1}>0$, and $d_{2}>0$; $\left(p_{1}+q_{1}\right) d_{1}-p_{3}^{2} d_{2}>0$ is satisfied.
(1) If one of the following holds: (a) $\Delta>0, z^{*}<0$; (b) $\Delta>0, z^{*}>0$ and $h\left(z^{*}\right)>0$. Then all roots of (28) have negative real parts when $\tau \in\left[0, \tau_{0}\right), \tau_{0}$ is a certain positive constant.
(2) If the conditions (a) and (b) are not satisfied, then all roots of (28) have negative real parts for all $\tau \geq 0$.

Proof. when $\tau=0$, (28) can be reduced to

$$
\begin{equation*}
\lambda^{4}+p_{3} \lambda^{3}+p_{2} \lambda^{2}+\left(p_{1}+q_{1}\right) \lambda+\left(p_{0}+q_{0}\right)=0 \tag{38}
\end{equation*}
$$

By the Routh-Hurwitz criterion, all roots of (33) have negative real parts and only if

$$
\begin{equation*}
p_{3}>0, \quad d_{1}>0, \quad d_{2}>0, \quad\left(p_{1}+q_{1}\right) d_{1}-p_{3}^{2} d_{2}>0 \tag{39}
\end{equation*}
$$

Considering (37), it is easy to see from the characters of cubic algebraic equation that $h(z)$ is a strictly monotonically increasing function if $\Delta \leq 0$. If $\Delta>0, z^{*}<0$ or $\Delta>0, z^{*}>0$ and $h\left(z^{*}\right)>0$, then $h(z)$ has no positive root. Hence, (28) has no purely imaginary roots for any $\tau>0$, which implies that the positive equilibrium $E^{*}\left(S^{*}, I^{*}, D^{*}, Q^{*}, V^{*}\right)$ of system (22) is absolutely stable. Therefore, the following theorem on the stability of positive equilibrium $E^{*}\left(S^{*}, I^{*}, D^{*}, Q^{*}, V^{*}\right)$ can be easily obtained.

Theorem 10. Assume that $\left(H_{1}\right)$ and $\left(H_{2}\right)$ are satisfied, and $\Delta>0, z^{*}<0$ or $\Delta>0, z^{*}>0$, and $h\left(z^{*}\right)>0$, then the positive equilibrium $E^{*}\left(S^{*}, I^{*}, D^{*}, Q^{*}, V^{*}\right)$ of system (22) is absolutely stable. Namely, $E^{*}\left(S^{*}, I^{*}, D^{*}, Q^{*}, V^{*}\right)$ is asymptotically stable for any time delay $\tau>0$.

Assume that the coefficients in $h(z)$ satisfy the condition as follows:

$$
\left(H_{3}\right) \Delta>0, z^{*}>0, h\left(z^{*}\right)<0 .
$$

According to lemma, it is proved that (37) has at least a positive root $\omega_{0}$, namely, the characteristic equation (28) has a pair of purely imaginary roots $\pm i \omega_{0}$.

In view of the fact that (28) has a pair of purely imaginary roots $\pm i \omega_{0}$, the corresponding $\tau_{k}>0$ is given by eliminating $\sin (\omega \tau)$ in (34):

$$
\begin{align*}
& \tau_{k} \\
& =\frac{1}{\omega_{0}} \arccos \left[\frac{q_{0}\left(p_{2} \omega_{0}^{2}-\omega_{0}^{4}-p_{0}\right)+q_{1} \omega_{0}\left(p_{3} \omega_{0}^{3}-p_{1} \omega_{0}\right)}{q_{1}^{2} \omega_{0}^{2}+q_{0}^{2}}\right] \\
& \quad+\frac{2 k \pi}{\omega_{0}} \quad(k=0,1,2, \ldots) \tag{40}
\end{align*}
$$

Let $\lambda(\tau)=v(\tau)+i \omega(\tau)$ be the root of (28), so that $v\left(\tau_{k}\right)=0$ and $\omega\left(\tau_{k}\right)=\omega_{0}$ are satisfied when $\tau=\tau_{k}$.

Lemma 11. Suppose $h^{\prime}\left(z_{0}\right) \neq 0$. If $\tau=\tau_{0}$, then $\pm i \omega_{0}$ is a pair of purely imaginary roots of (28). In addition, if the conditions in Lemma 9 are satisfied, then

$$
\begin{equation*}
\left.\frac{d(\operatorname{Re} \lambda)}{d \tau}\right|_{\tau=\tau_{k}}>0 \tag{41}
\end{equation*}
$$

This signifies that there exists at least one eigenvalue with positive real part for $\tau>\tau_{k}$. Differentiating both sides of (28) with respect to $\tau$, it can be written as

$$
\begin{align*}
& \left(\frac{d \lambda}{d \tau}\right)^{-1} \\
& =\frac{\left(4 \lambda^{3}+3 p_{3} \lambda^{2}+2 p_{2} \lambda+p_{1}\right)+q_{1} e^{-\lambda \tau}-\left(q_{1} \lambda+q_{0}\right) \tau e^{-\lambda \tau}}{\left(q_{1} \lambda+q_{0}\right) \lambda e^{-\lambda \tau}} \tag{42}
\end{align*}
$$

Therefore

$$
\begin{align*}
\operatorname{sgn} & {\left[\frac{d \operatorname{Re} \lambda}{d \tau}\right]_{\tau=\tau_{k}} } \\
& =\operatorname{sgn}\left[\operatorname{Re}\left(\frac{d \lambda}{d \tau}\right)^{-1}\right]_{\lambda=i \omega_{0}} \\
& =\operatorname{sgn} \frac{\omega_{0}^{2}}{\Gamma}\left(4 \omega_{0}^{6}+3 D_{3} \omega_{0}^{4}+2 D_{2} \omega_{0}^{2}+D_{1}\right)  \tag{43}\\
& =\operatorname{sgn} \frac{\omega_{0}^{2}}{\Gamma}\left\{h^{\prime}\left(\omega_{0}^{2}\right)\right\} \\
& =\operatorname{sgn}\left\{h^{\prime}\left(\omega_{0}^{2}\right)\right\}
\end{align*}
$$

where $\Gamma=q_{1}{ }^{2} \omega_{0}^{4}+q_{0} \omega_{0}^{2}$; then it follows the hypothesis $\left(H_{3}\right)$ that $h^{\prime}\left(\omega_{0}{ }^{2}\right) \neq 0$.

Hence,

$$
\begin{equation*}
\left.\frac{d(\operatorname{Re} \lambda)}{d \tau}\right|_{\tau=\tau_{k}}>0 \tag{44}
\end{equation*}
$$

The root of characteristic equation (28) crosses from left to right on the imaginary axis as $\tau$ continuously varies from a value less than $\tau_{k}$ to one greater than $\tau_{k}$ according to Routh's theorem. Therefore, according to the Hopf bifurcation theorem for functional differential equations, the transverse condition holds and the conditions for Hopf bifurcation are satisfied at $\tau=\tau_{k}$. Then the following result can be obtained.

Theorem 12. Suppose that the conditions $\left(H_{1}\right)$ and $\left(H_{2}\right)$ are satisfied.
(1) Equilibrium $E^{*}\left(S^{*}, I^{*}, D^{*}, Q^{*}, V^{*}\right)$ is locally asymptotically stable when $\tau \in\left[0, \tau_{0}\right)$, but unstable when $\tau>\tau_{0}$.
(2) If condition $\left(H_{3}\right)$ is satisfied, the system will undergo Hopf bifurcation at the positive equilibrium $E^{*}\left(S^{*}, I^{*}, D^{*}, Q^{*}, V^{*}\right)$ when $\tau=\tau_{k}(k=0,1,2, \ldots)$, where $\tau_{k}$ is defined by (40).
This implies that when time delay $\tau<\tau_{0}$, the system will be stable at its infection equilibrium point so that it is easy to
control and eliminate worms; when $\tau \geq \tau_{0}$, the system will be unstable but the threshold $\tau_{0}$ is greater than delayed model's, which illustrates the model with constant quarantine strategy gets stable easier and the users have more time to remove worms.

## 5. A Delayed Worm Propagation Model with Impulsive Quarantine

5.1. Using Impulsive Quarantine Strategy to Model a Delayed Worm Propagation. Although constant quarantine strategy based on misuse intrusion detection does improve vaccination effect, the system is out of control and bifurcation is still not eliminated. In addition, the system fails to detect unknown worms and worm-variants. Anomaly intrusion detection system is of help in detecting these kinds of worm. However, the system is accompanied by high false-positive rate. To solve the problem of constant quarantine strategy and anomaly intrusion detection system, we proposed a novel quarantine strategy called impulsive quarantine based on a hybrid intrusion detection system, which can make up for the gaps existing in the two systems. Impulsive quarantine is implemented as follows: constant quarantine of infected hosts found by the misuse detection is performed, while susceptible and infected hosts detected by anomaly detection are quarantined in an impulsive way every $T$ units of time. The advantages of this strategy lie in both avoiding a high false-positive rate caused by anomaly detection and making up for the insufficiency of the misuse detection in detecting unknown worms [12]. Impulsive quarantine strategy adds two transitions as a result of the influence of the anomaly detection method. The susceptible and infected hosts detected by anomaly detection method are quarantined at rate $\theta_{1}$ and $\theta_{2}$, respectively. Other settings are identical to those of constant quarantine model.

The state transition diagram of impulsive quarantine model is given in Figure 3.
5.2. Description of Impulsive Quarantine Model. The complete differential equations of the impulsive quarantine model are showed as follows:

$$
\begin{aligned}
& \frac{d S(t)}{d t}=p \nu N-\beta S(t) I(t)-\nu S(t)+\mu V(t), \\
& \frac{d I(t)}{d t}=\beta S(t) I(t)-\nu I(t)-\gamma I(t)-\alpha I(t), \\
& \frac{d D(t)}{d t}=\gamma I(t)-\gamma I(t-\tau)-\nu D(t), \\
& \frac{d V(t)}{d t}=\gamma I(t-\tau)+\delta Q(t)-\nu V(t)-\omega V(t)+(1-p) \nu N, \\
& \frac{d Q(t)}{d t}=\alpha I(t)-\delta Q(t)-\nu Q(t), \\
& S\left(n T^{+}\right)=S\left(n T^{-}\right)-\theta_{1} S\left(n T^{-}\right), \\
& I\left(n T^{+}\right)=I\left(n T^{-}\right)-\theta_{2} I\left(n T^{-}\right),
\end{aligned}
$$



Figure 3: State transition diagram of impulsive quarantine model.

$$
\begin{aligned}
& D\left(n T^{+}\right)=D\left(n T^{-}\right) \\
& Q\left(n T^{+}\right)=Q\left(n T^{-}\right)+\theta_{1} S\left(n T^{-}\right)+\theta_{2} I\left(n T^{-}\right) \\
& V\left(n T^{+}\right)=V\left(n T^{-}\right)
\end{aligned}
$$

$$
\begin{equation*}
t=n T \tag{45}
\end{equation*}
$$

where $n=0,1,2, \ldots$, the impulsive strategy is applied at a discrete time $t=n T$, and $T$ is the interval time of impulsive quarantine. $n T^{+}$is the moment at which we apply the $n$th impulsive quarantine measure, whereas $n T^{-}$is the time just before the $n$th impulsive quarantine measure is applied.
5.3. Global Attractivity of Infection-Free Periodic Solution. We have

$$
\begin{equation*}
S(t)+I(t)+D(t)+Q(t)+V(t)=N \tag{46}
\end{equation*}
$$

Since $Q(t)=N-S(t)-I(t)-D(t)-V(t)$, then system (45) can be rewritten as

$$
\begin{aligned}
\frac{d S(t)}{d t}= & p \nu N-\beta S(t) I(t)-\nu S(t)+\mu V(t), \\
\frac{d I(t)}{d t}= & \beta S(t) I(t)-\nu I(t)-\gamma I(t)-\alpha I(t), \\
\frac{d D(t)}{d t}= & \gamma I(t)-\gamma I(t-\tau)-\nu D(t), \\
\frac{d V(t)}{d t}= & \gamma I(t-\tau)+\delta(N-S(t)-I(t)-D(t)-V(t)) \\
& -\mu V(t)-\nu V(t)+(1-p) \nu N, \\
\frac{d Q(t)}{d t}= & \alpha I(t)-\delta Q(t)-\nu Q(t), \\
S\left(n T^{+}\right)= & S\left(n T^{-}\right)-\theta_{1} S\left(n T^{-}\right), \\
I\left(n T^{+}\right)= & I\left(n T^{-}\right)-\theta_{2} I\left(n T^{-}\right),
\end{aligned}
$$

$$
\begin{aligned}
& D\left(n T^{+}\right)=D\left(n T^{-}\right) \\
& Q\left(n T^{+}\right)=Q\left(n T^{-}\right)+\theta_{1} S\left(n T^{-}\right)+\theta_{2} I\left(n T^{-}\right) \\
& V\left(n T^{+}\right)=V\left(n T^{-}\right)
\end{aligned}
$$

$$
\begin{equation*}
t=n T . \tag{47}
\end{equation*}
$$

We may see that the first four equations in (47) are independent of the fourth equation. Therefore, the fourth equation can be omitted without loss of generality [21]. Hence, model (47) can be rewritten as

$$
\begin{aligned}
\frac{d S(t)}{d t}= & p \nu N-\beta S(t) I(t)-\nu S(t)+\mu V(t), \\
\frac{d I(t)}{d t}= & \beta S(t) I(t)-\nu I(t)-\gamma I(t)-\alpha I(t), \\
\frac{d D(t)}{d t}= & \gamma I(t)-\gamma I(t-\tau)-v D(t), \\
\frac{d V(t)}{d t}= & \gamma I(t-\tau)+\delta(N-S(t)-I(t)-D(t)-V(t)) \\
& -\mu V(t)-\nu V(t)+(1-p) \nu N, \\
S\left(n T^{+}\right)= & S\left(n T^{-}\right)-\theta_{1} S\left(n T^{-}\right), \\
I\left(n T^{+}\right)= & I\left(n T^{-}\right)-\theta_{2} I\left(n T^{-}\right), \\
D\left(n T^{+}\right)= & D\left(n T^{-}\right), \\
V\left(n T^{+}\right)= & V\left(n T^{-}\right)
\end{aligned}
$$

$$
\begin{equation*}
t=n T . \tag{48}
\end{equation*}
$$

In the following, we introduce some notations and definitions in subsequent sections.

Let

$$
\begin{align*}
& R_{+}=[0, \infty) \\
& R_{+}^{4}=\left\{Z \in R^{4}: Z \geq 0\right\} \tag{49}
\end{align*}
$$

Denote $f=\left(f_{1}, f_{2}, f_{3}, f_{4}\right)^{T}$, the map defined by the right hand of the four equations of system (48).

Let $C$ be the space of continuous functions on $[-\omega, 0]$ with uniform norm. The initial conditions for (48) are

$$
\begin{array}{r}
\left(\phi_{1}(\zeta), \phi_{2}(\zeta), \phi_{3}(\zeta), \phi_{4}(\zeta)\right) \in C_{+}=C\left([-\omega, 0], R_{+}^{4}\right) \\
\phi_{i}(0)>0, i=1,2,3,4 \tag{50}
\end{array}
$$

Definition 13. System (48) is said to be permanent if there exists a compact region $\Omega_{0} \in \operatorname{int} \Omega$ such that every solution of system (48) with initial conditions (50) will eventually enter and remain in region $\Omega_{0}$.

The solution of system (48) is a piecewise continuous function $Z: R_{+} \rightarrow R_{+}^{4}, Z(t)$ is continuous on $[n T,(n+1) T]$, $k \in Z_{+}$, and $Z\left(n T^{+}\right)=\lim _{t \rightarrow n T^{+}} Z(t)$ exists. Obviously the smooth properties of $f$ guarantee the global existence and uniqueness of solutions of system (48) for detail on fundamental properties of impulsive systems [22, 23]. The following lemma is obtained.

Lemma 14. Suppose $Z(t)$ is a solution of system (48) with initial conditions (50), then $Z(t) \geq 0$ for all $t \geq 0$.

Denote

$$
\begin{equation*}
\Omega=\left\{(S, I, D, V) \in R^{4} \mid S \geq 0, I \geq 0, D \geq 0, V \geq 0\right\} \tag{51}
\end{equation*}
$$

It is easy to show that $\Omega$ is positively invariant with respect to (48) with initial conditions (48).

Lemma 15 (see [21, 22]). Consider the following equation:

$$
\begin{equation*}
\dot{x}(t)=a_{1} x(t-\omega)-a_{2} x(t), \tag{52}
\end{equation*}
$$

where $a_{1}, a_{2}, \omega>0 ; x(t)>0$ for $-\omega \leq t \leq 0$.
We have
(i) if $a_{1}<a_{2}$, then $\lim _{t \rightarrow \infty} x(t)=0$,
(ii) if $a_{1}>a_{2}$, then $\lim _{t \rightarrow \infty} x(t)=+\infty$.

The proofs of case (i) and case (ii) are given in Theorems 2.1 [24] and 2.2 [25], respectively.

We first demonstrate the existence of the infection-free periodic solution, in which infected individuals are entirely absent from the population permanently, that is, $I(t)=0$ for all $t \geq 0$. Under this condition, the $S, D$, and $V$ must satisfy

$$
\begin{align*}
& \frac{d S(t)}{d t}= p \nu N-\beta S(t) I(t)-\nu S(t)+\mu V(t), \\
& \frac{d D(t)}{d t}=\gamma I(t)-\gamma I(t-\tau)-\nu D(t), \\
& \frac{d V(t)}{d t}= \gamma I(t-\tau)+\delta(N-S(t)-I(t)-D(t)-V(t)) \\
&-\mu V(t)-\nu V(t)+(1-p) \nu N, \\
& S\left(n T^{+}\right)= S\left(n T^{-}\right)-\theta_{1} S\left(n T^{-}\right), \\
& D\left(n T^{+}\right)=D\left(n T^{-}\right), \\
& V\left(n T^{+}\right)=V\left(n T^{-}\right), \quad t \neq n T, \\
& \tag{53}
\end{align*}
$$

First we show below that the susceptible population $S$ oscillates with period $T$, in synchronization with the periodic
pulse vaccination. From the first and fourth equations of system (53), we have that

$$
\begin{equation*}
\widetilde{S}(t)=p N+\left(S^{*}-p N\right) e^{-\gamma(t-n T)}, \quad n T<t \leq(n+1) T \tag{54}
\end{equation*}
$$

is globally asymptotically stable, where

$$
\begin{equation*}
S^{*}=\frac{p N\left(1-\theta_{1}\right)\left(1-e^{-\nu T}\right)}{\left(1-\left(1-\theta_{1}\right) e^{-\nu T}\right)} \tag{55}
\end{equation*}
$$

From the second and fifth equations of system (53), we have $\lim _{t \rightarrow \infty} D(t)=0$. Further, it follows from the third and sixth equations of system (53) that $\lim _{t \rightarrow \infty} V(t)=([(1-p) v+\delta] N-$ $\delta \widetilde{S}(t)) /(\delta+\nu+\mu)$.

Therefore $(\widetilde{S}(t), 0,0,([(1-p) v+\delta] N-\delta \widetilde{S}(t)) /(\delta+v+\mu))$ is the infection-free periodic solution of system (48). In the rest of this section, we establish the global attractivity condition for the infection-free periodic solution.

Theorem 16. The infection-free periodic solution $(\widetilde{S}(t), 0$, $0,([(1-p) \nu+\delta] N-\delta \widetilde{S}(t)) /(\delta+\nu+\mu))$ of system (48) is globally attractive provided that $R^{*}<1$, where

$$
\begin{equation*}
R^{*}=\frac{\beta p N\left(1-\theta_{1}\right)\left(1-e^{-\nu T}\right)}{(\nu+\gamma+\alpha)\left(1-\left(1-\theta_{1}\right) e^{-\mu T}\right)} \tag{56}
\end{equation*}
$$

Proof. Since $R^{*}<1$, we can choose $\varepsilon_{0}>0$ sufficiently small such that

$$
\begin{equation*}
\beta\left(\frac{p N\left(1-\theta_{1}\right)\left(1-e^{-\nu T}\right)}{\left(1-\left(1-\theta_{1}\right) e^{-\nu T}\right)}+\varepsilon_{0}\right)<\nu+\gamma+\alpha . \tag{57}
\end{equation*}
$$

It follows from the first equation of (48) that

$$
\begin{equation*}
\dot{S}(t) \leq p \nu N-\nu S(t)+\mu V(t) . \tag{58}
\end{equation*}
$$

Thus we consider the comparison impulsive differential system

$$
\begin{align*}
\dot{x}(t) & =p v N-v x(t), \quad t \neq n T  \tag{59}\\
x\left(n T^{+}\right) & =\left(1-\theta_{1}\right) x\left(n T^{-}\right), \quad t=n T .
\end{align*}
$$

According to [26], we obtain that the periodic solution of system (59)

$$
\begin{array}{r}
\widetilde{x}(t)=\widetilde{S}(t)=p N+\left(S^{*}-p N\right) e^{-\nu(t-n T)},  \tag{60}\\
n T<t \leq(n+1) T
\end{array}
$$

is globally asymptotically stable, where

$$
\begin{equation*}
x^{*}=S^{*}=\frac{p N\left(1-\theta_{1}\right)\left(1-e^{-\nu T}\right)}{1-\left(1-\theta_{1}\right) e^{-\nu T}} . \tag{61}
\end{equation*}
$$

Let $(S(t), I(t), D(t), V(t))$ be the solution of system (48) with initial values (50) and let $S\left(0^{+}\right)=S_{0}>0, x(t)$ be the solution of system (59) with initial value $x\left(0^{+}\right)=S_{0}$.

In view of the comparison theorem in impulsive differential equations [18, 19], there exists an integer $n_{1}>0$ such that

$$
\begin{equation*}
S(t)<x(t)<\tilde{x}(t)+\varepsilon_{0}, \quad n T<t \leq(n+1) T, \tag{62}
\end{equation*}
$$

that is,

$$
\begin{gather*}
S(t)<\widetilde{S}(t)+\varepsilon_{0} \leq \frac{p N\left(1-\theta_{1}\right)\left(1-e^{-\nu T}\right)}{\left(1-\left(1-\theta_{1}\right) e^{-\nu T}\right)}+\varepsilon_{0} \triangleq S_{M}  \tag{63}\\
n T<t \leq(n+1) T, \quad n>n_{1}
\end{gather*}
$$

where $\widetilde{S}(t)$ is defined (55). Further, from the second equation of system (48), we know that (63) implies

$$
\begin{equation*}
\dot{I}(t) \leq \beta S_{M} I(t)-(\nu+\gamma+\alpha) I(t), \quad t>n T, n>n_{1} . \tag{64}
\end{equation*}
$$

Consider the following comparison differential system:

$$
\begin{equation*}
y(t)=\beta S_{M} y(t)-(v+\gamma+\alpha) y(t), \quad t>n T, n>n_{1} . \tag{65}
\end{equation*}
$$

From (57), we have $\beta S_{M}<\nu+\gamma+\alpha$. According to Lemma 15 we have $\lim _{t \rightarrow \infty} y(t)=0$.

Let $(S(t), I(t), D(t), V(t))$ be the solution of system (48) with initial values (50) and $I\left(0^{+}\right)=I_{0}>0$; let $y(t)$ be the solution of system (65) with initial value $y\left(0^{+}\right)=I_{0}$. Consider the second and the sixth equations of system (48); according to Lemma 15, we have lim $\sup _{t \rightarrow \infty} I(t) \leq \lim \sup _{t \rightarrow \infty} y(t)=$ 0 . Incorporating into the positivity of $I(t)$, we know that

$$
\begin{equation*}
\lim _{t \rightarrow \infty} I(t)=0 . \tag{66}
\end{equation*}
$$

Therefore, for any $\varepsilon_{1}>0$ (sufficiently small), there exists an integer $n_{2}>n_{1}$ such that $I(t)<\varepsilon_{1}$ for all $t>n_{2} T$.

For the third equation of system (48), we have

$$
\begin{equation*}
\dot{D}(t)<\gamma \varepsilon_{1}-\nu D(t) \quad \text { for } t>n_{2} T \text {. } \tag{67}
\end{equation*}
$$

Consider comparison differential equation, for $t>n_{2} T$,

$$
\begin{equation*}
\dot{z}(t)=\gamma \varepsilon_{1}-v z(t) . \tag{68}
\end{equation*}
$$

It is easy to see that $z(t)=\gamma \varepsilon_{1} / \nu$. According to the comparison theorem, there is a $n_{3}>n_{2}$ such that, for all $t>n_{3} T$,

$$
\begin{equation*}
D(t) \leq \frac{\gamma \varepsilon_{1}}{\nu}+\varepsilon_{1} . \tag{69}
\end{equation*}
$$

Therefore, in view of the positivity of $D(t)$ and sufficiently small $\varepsilon_{1}$, it follows from (69) that

$$
\begin{equation*}
\lim _{t \rightarrow \infty} D(t)=0 \tag{70}
\end{equation*}
$$

Moreover, for the first equation of system (48), we have

$$
\begin{equation*}
\dot{S}(t) \geq p \nu N-\left(\nu+\beta \varepsilon_{1}\right) S(t) \quad \text { for } n>n_{3} T \tag{71}
\end{equation*}
$$

Consider the following equations, for $t>n T$ and $n>n_{3}$ :

$$
\begin{align*}
& \dot{u}(t)=p \nu N-\left(\nu+\beta \varepsilon_{1}\right) u(t), \quad t \neq n T, \\
& u\left(n T^{+}\right)=\left(1-\theta_{1}\right) u\left(n T^{-}\right), \quad t=n T . \tag{72}
\end{align*}
$$

According to [27], we know that the periodic solution of system (72)

$$
\begin{array}{r}
\widetilde{u}(t)=\frac{p v N}{v+\beta \varepsilon_{1}}+\left(u^{*}-\frac{p v N}{v+\beta \varepsilon_{1}}\right) e^{-\left(v+\beta \varepsilon_{1}\right)(t-n T)},  \tag{73}\\
n T<t \leq(n+1) T
\end{array}
$$

is globally asymptotically stable, where

$$
\begin{equation*}
u^{*}=\frac{p v N}{v+\beta \varepsilon_{1}} \frac{\left(1-\theta_{1}\right)\left(1-e^{-\left(\nu+\beta \varepsilon_{1}\right) T}\right)}{\left(1-\left(1-\theta_{1}\right) e^{-\left(\nu+\beta \varepsilon_{1}\right) T}\right)} \tag{74}
\end{equation*}
$$

According to the comparison theorem in impulsive differential equations, there exists an integer $n_{4}>n_{3}$ such that

$$
\begin{equation*}
S(t)>\tilde{u}(t)-\varepsilon_{1}, \quad n T<t \leq(n+1), \quad n>n_{4} . \tag{75}
\end{equation*}
$$

Since that $\varepsilon_{1}$ is arbitrarily small, consider (63) and (75); we have that

$$
\begin{array}{r}
\widetilde{S}(t)=p N\left(1-\frac{\theta_{1}\left(1-e^{-v T}\right)}{1-\left(1-\theta_{1}\right) e^{-v T}} e^{-v(t-n T)}\right),  \tag{76}\\
n T<t \leq(n+1) T
\end{array}
$$

is globally attractive, that is,

$$
\begin{equation*}
\lim _{t \rightarrow \infty} S(t)=\widetilde{S}(t) \tag{77}
\end{equation*}
$$

For the fourth equation of system (48), we have

$$
\begin{equation*}
\dot{V}(t) \leq[\delta+(1-p) \nu] N-\delta S(t)-\delta V(t)-\mu V(t)-\nu V(t) \tag{78}
\end{equation*}
$$

for $t>n_{4} T$.
It is easy to obtain that there is a $n_{5}>n_{4}$ such that

$$
\begin{equation*}
V(t)<\frac{[\delta+(1-p) \nu] N-\delta S(t)}{\delta+\nu+\mu}+\varepsilon_{1} \quad \text { for } t>n_{5} T \tag{79}
\end{equation*}
$$

In a similar way, there is a $n_{6}>n_{5}$ :

$$
\begin{equation*}
V(t)>\frac{[\delta+(1-p) \nu] N-\delta S(t)}{\delta+v+\mu}-\varepsilon_{1} \quad \text { for } t>n_{6} T \tag{80}
\end{equation*}
$$

Since that $\varepsilon_{1}$ is arbitrarily small, consider (79) and (80); we have

$$
\begin{align*}
\lim _{t \rightarrow \infty} V & (t) \\
= & ([\delta+(1-p) \nu] N  \tag{81}\\
& \left.-\delta p N\left(1-\frac{\left(\theta_{1}\left(1-e^{-v T}\right)\right)}{\left(1-(1-\theta) e^{-v T}\right)} e^{-v(t-n T)}\right)\right) \\
& \times(\delta+\nu+\mu)^{-1}
\end{align*}
$$

It follows from (66), (70), (77), and (81) that the infection-free periodic solution $(\widetilde{S}(t), 0,0,([(1-p) \nu+\delta] N-\delta \widetilde{S}(t)) /(\delta+v+\mu))$ is globally attractive. The proof of Theorem 16 is complete.


Figure 4: Worm propagation trend of model with time delay when $\tau<\tau_{0}$.


Figure 5: Worm propagation trend of model with time delay when $\tau>\tau_{0}$.

## 6. Numerical and Simulation Experiments

In order to simulate the worm propagation in the real world, the parameters in the experiments are practical values. The Slammer worm is selected for experiments [10]. 750,000 hosts are picked as the population size, and the worm's average scan rate is 3300 per second. The worm infection rate can be calculated as $\alpha=\eta N / 2^{32}=0.5763$, which means that average 0.5763 hosts of all the hosts can be scanned by one host. The infection rate is $\beta=3300 / 2^{32}=0.00000077$, the recovery rate of infectious hosts is $\gamma=0.19$, the quarantine rate is $\alpha=0.15$, and the removal rate of quarantined hosts is $\delta=0.04$. The rest of the parameters are $p=0.9, \mu=0.031$, and $\nu=0.026$.


Figure 6: Number of infected hosts when $\tau$ is changed.

At the beginning, there are 50 infected hosts, while others are susceptible. The following numerical analyses are supplement for the above results.
6.1. Numerical Experiments of Worm Propagation Model with Time Delay in Vaccination. According to the above parameters, as shown in Figure 4, the curves of three kinds of host in system (1) are presented when $\tau=5<\tau_{0}$. All of the three kinds of host get stable quickly, which illustrates that $E^{*}$ is asymptotically stable. It implies that the number of infected hosts stays very low and can be predicted. Further strategies can be developed and utilized to eliminate worms.

However, when time delay $\tau$ gets increased and then reach the threshold $\tau_{0}, E^{*}$ will lose its stability and a bifurcation will occur. Figure 5 shows the susceptible, infected, and vaccinated hosts in system (1) when $\tau=100>\tau_{0}$. In this figure, we can clearly see that the number of infected hosts will outburst after a short period of peace and repeat again and again but not in the same period, which means that it is hard to predict the number of infected hosts and to develop further strategies to eliminate worms.

In order to see the influence of time delay, $\tau$ is set to a different value each time with other parameters remaining the same. Figure 6 shows the number of infected hosts in the same coordinate with time delays $\tau=5, \tau=15, \tau=45$, and $\tau=90$. Initially, the four curves are overlapped, which means that time delay has little effect in the initial stage of worm propagation. With time delay increasing, the curve begins to oscillate. When time delay passes through the threshold $\tau_{0}$, the infecting process gets unstable. Meanwhile, it can be discovered that the amplitude and period of the number of infected hosts get increased.

In Figure 7, the projection of the phase portrait of system (1) in (S, $I, V)$-space is presented when $\tau=35$ and $\tau=45$. In Figure 8, when $\tau=35$, it is clear that the curve converges to a fixed point which suggests that the system is stable.


Figure 7: The projection of the phase portrait of system (1) in $(S, I, V)$-space.


Figure 8: The phase portrait of susceptible hosts $s(t)$ and infected hosts $I(t)$.

When $\tau=45$, the curve converges to a limit circle which implies that the system is unstable. Figure 9 shows bifurcation diagram with $\tau$ from 1 to 100; Hopf bifurcation will occur when $\tau=\tau_{0}=38$.
6.2. Numerical Experiments of Worm Propagation Model with Constant Quarantine Strategy. In order to show the impact of
constant quarantine strategy, we analyze the numerical results after adopting the constant quarantine strategy. Further, we compare them with the worm propagation model with time delay.

Figure 10 shows the curves of three kinds of host in system (22) when $\tau=5<\tau_{0}$. All of the three kinds of host get stable quickly, which illustrates that $E^{*}$ is asymptotically stable.


Figure 9: Bifurcation diagram of system (1) with $\tau$ from 1 to 100.


Figure 10: Worm propagation trend of model with constant quarantine strategy when $\tau<\tau_{0}$.


Figure 11: Worm propagation trend of model with constant quarantine strategy when $\tau>\tau_{0}$.


Figure 12: Comparison of infected hosts before and after adopting constant quarantine strategy if $\tau>\tau_{0}$.

When time delay $\tau$ gets increased and then reach the threshold $\tau_{0}, E^{*}$ will lose its stability and a bifurcation will occur. Figure 11 shows the susceptible, infected, and vaccinated hosts in system (22) when $\tau=100>\tau_{0}$. In this figure, we can clearly see that the number of infected hosts will outburst after a short period of peace and repeat again and again but the range is much less than delayed model's. It implies that the constant quarantine strategy can't eliminate the Hopf bifurcation, but it can reduce the max number of infected hosts.

In Figure 12, when $\tau=100>\tau_{0}$, it is clear that the maximum of infected hosts is diminished sharply from 220,000 to 38,000 , which illustrates that constant quarantine strategy has much better inhibition impact than single vaccination. However, constant quarantine strategy cannot eliminate the Hopf bifurcation; the system is still unstable and out of control.

Figure 13 shows the projection of the phase portrait of system (22) in (S,I,V)-space when $\tau=40$ and $\tau=55$. In Figure 14, when $\tau=40$, it is clear that the curve converges to a fixed point which suggests that the system is stable. When $\tau=55$, the curve converges to a limit circle which implies that the system is unstable. Figure 15 shows bifurcation diagram with $\tau$ from 1 to 90 ; we find that Hopf bifurcation will occur when $\tau=\tau_{0}=46$. The threshold is greater than delayed model's, which illustrates the model gets stable easier and the users have more time to remove worms.
6.3. Numerical Experiments of Worm Propagation Model with Impulsive Quarantine Strategy. The paper performs the numerical experiments and compares the results with constant quarantine model after using impulsive quarantine strategy. The interval time of impulsive quarantine is set $T=10$. The susceptible and infected hosts detected by the anomaly intrusion detection method are quarantined at rate


Figure 13: The projection of the phase portrait of system (22) in ( $S, I, V$ )-space.


Figure 14: The phase portrait of susceptible hosts $S(t)$ and infected hosts $I(t)$.
$\theta_{1}=0.00002315$ and $\theta_{2}=0.6$, respectively. Other parameters are the same as constant quarantine model.

Figure 16 shows the curves of four kinds of host when $\tau=$ $5<\tau_{0}$. All of the four kinds of host get stable more quickly, which illustrates that $E^{*}$ is asymptotically stable. After using impulsive quarantine strategy, Figure 17 shows the curves of three kinds of hosts when $\tau=100>\tau_{0}$. All kinds of hosts get stable within 4 hours, which implies that Hopf bifurcation has been eliminated thoroughly. In Figure 18, the number of infected hosts has been shown without quarantine, adopting quarantine strategy, and impulsive quarantine strategy, respectively. It is clear that the number of infected hosts is
almost 0 after using the impulsive quarantine strategy, which is even much less than model using constant quarantine strategy. The result means that the impulsive quarantine strategy works well. Thus, the system will be stable and controlled so that the worm will not break out again.
6.4. Simulation Experiments. The discrete-time simulation is an expanded version of Zou's program [8] simulating Code Red worm propagation. The system in our simulation experiment consists of 750,000 hosts that can reach each other directly, which is consistent with the numerical experiments,


FIgURE 15: Bifurcation diagram of system (22) with $\tau$ from 1 to 90 .


Figure 16: Worm propagation trend of model with impulsive quarantine strategy when $\tau<\tau_{0}$.
and there is no topology issue in our simulation. At the beginning of simulation, 50 hosts are randomly chosen to be infected and the others are all susceptible. In the simulation experiments, the implement of transition rates of the model is based on probability. Under the propagation parameters of the Slammer worm, some simulation experiments are performed. Figure 19 shows that numerical and simulation curve of infected hosts match well when using the constant quarantine strategy and Figure 20 shows that numerical and simulation curve of infected hosts match well after using the impulsive quarantine strategy, whatever the value of $\tau$ is.

## 7. Conclusions

By considering that time delay leads to Hopf bifurcation so that the worm propagation system will be out of control, this paper proposes two quarantine strategies: constant


Figure 17: Worm propagation trend of model with impulsive quarantine strategy when $\tau>\tau_{0}$.


Figure 18: Comparison of infected hosts without quarantine, adopting constant quarantine strategy and impulsive quarantine strategy, respectively, when $\tau>\tau_{0}$.
quarantine and impulsive quarantine strategy to control the stability of worm propagation. Through theoretical analysis and simulation experiments, the following conclusions can be derived.
(1) In order to accord with actual facts in the real world, a worm propagation model with time delay in vaccination is constructed. The critical time delay $\tau_{0}$ where Hopf bifurcation appears is obtained. When time delay $\tau<\tau_{0}$, the worm propagation system will stabilize at its infection equilibrium point, which is beneficial to implement a containment strategy to eliminate the worm completely. When time delay $\tau \geq$


Figure 19: Comparison of numerical and simulation curve of the infected hosts of constant quarantine model.


Figure 20: Comparison of numerical and simulation curve of the infected hosts of impulsive quarantine model.
$\tau_{0}$, Hopf bifurcation appears, implying that the system will be unstable and the worm cannot be effectively controlled.
(2) Constant quarantine strategy based on misuse IDS has only some inhibition impact. Through theoretical analysis, the threshold $\tau_{0}$ is greater than delayed model's so that the users have more time to clean worms. Nevertheless, constant quarantine strategy cannot eliminate bifurcation.
(3) Impulsive quarantine strategy is proposed, which can both make up for the gaps existing in the misuse
and anomaly IDS and eliminate bifurcation. Through theoretical analysis and numerical experiments, the numerical results match theoretical ones well, which fully support our analysis.

Furthermore, various factors can affect worm propagation. The paper focuses on analyzing the influence of time delay. Other impact factors to worm propagation will be a major emphasis of our future research.

## Notations

$N$ : Total number of hosts in the network
$S(t)$ : Number of susceptible hosts at time $t$
$I(t)$ : Number of infected hosts at time $t$
$D(t)$ : Number of delayed hosts at time $t$
$Q(t)$ : Number of quarantined hosts at time $t$
$V(t)$ : Number of vaccinated hosts at time $t$
$\beta$ : Infection rate
$\gamma$ : Removal rate of infected hosts
$\mu$ : Rate from vaccinated to susceptible hosts
$\nu$ : Birth and death rates
$p$ : Birth ratio of susceptible hosts
$\alpha$ : $\quad$ Quarantine rate
$\delta$ : Removal rate of quarantined hosts
$T$ : The interval time of impulsive quarantine
$\theta_{1}$ : Quarantine rate of susceptible hosts using impulsive quarantine
$\theta_{2}$ : Quarantine rate of infected hosts using impulsive quarantine
$\tau$ : $\quad$ Time delay of detecting and removing worms.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Disturbance Attraction Domain Estimation for Saturated Markov Jump Systems with Truncated Gaussian Process 

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#### Abstract

This paper investigates the disturbance attraction domain estimation of saturated Markov jump systems with truncated Gaussian process. The aim is to estimate the disturbance domain of attraction so that the state is maintained in a neighbour around the origin by a state feedback controller regardless of bounded disturbance. The problem is formulated as parameter-dependent linear matrix inequalities (LMIs). The optimal disturbance attraction domain is obtained through searching for most appropriate auxiliary parameters in the defined domain. A numerical example is presented to show the potential application of the results.


## 1. Introduction

For a system subject to abrupt structural changes, such as component failures and sudden environmental changes, it is more appropriate to model it as a Markov jump linear system (MJS), where the switching behaviour amongst the different modes of the system is determined by its transition probability (TP) governed by a finite Markov chain. Many results related to controller design under the time-invariant transition probability are now available in the literature (see, e.g., $[1-11]$ and the references therein). However, the exact value of the transition probability cannot be easily obtainable. It is often that only partial information of the transition probability can be obtained. In this situation, questions on the stability analysis and controller design (see [12-14]) have also been addressed. In practice, the environment can be so complex that the transition probability of the MJS concerned can only be nonhomogeneous. For example, the delay and packet loss of a networked control system are distinct among different working time [15]. Similar phenomena are also observed in electronic circuits [16] and manpower systems [17]. For Markov systems with nonhomogeneous transition probability, some interesting results are now available (see [18, 19]). In [20], a new method for describing the time-varying transition probability in the statistic sense is proposed. This
approach covers the cases where the transition probabilities are known either exactly or partially as special cases.

On the other hand, saturation failure is widely encountered in engineering applications. In the presence of saturation nonlinearity, a linear system will become a highly complex nonlinear system [21]. It is well known that nonlinear systems do not have, in general, global stability property [22]. Thus, the problem of attraction domain estimation has become a fundamentally challenging problem in nonlinear control theory [23]. For a linear system with saturation, some results related to attraction domain estimation have been obtained (see, e.g., [24, 25]). However, it appears that the estimation of the attraction domain for a saturated Markov system with nonhomogeneous transition probability has not been fully investigated. The situation will become much worse when there is disturbance to the system, as the behavior of the system will be significantly degraded by disturbance. The difficulties mentioned above are the motivation behind this paper to study the disturbance attraction domain estimation for discrete-time Markov jump systems with saturation and subject to truncated Gaussian transition probability. Based on [20], the aim of this paper is to propose a novel approach to estimate the optimal domain of attraction which can restrain the states of system to be within the smallest neighborhood around the origin under the bounded disturbance.

The rest of the paper is organized as follows: in Section 2, the system is defined, Section 3 introduces the concept of stochastic stability, in Section 4, sufficient conditions for disturbance attraction domain estimation are derived, in Section 5, a numerical example is provided to illustrate the applicability of the results obtained, and Section 6 concludes the paper.

In the sequel, the notation $R^{n}$ stands for an $n$-dimensional Euclidean space; the transpose of the matrix $A$ is denoted by $A^{\mathrm{T}} ; E\{\cdot\}$ denotes the mathematical statistical expectation of the stochastic process or vector; $\partial$ is the boundary of a set; a positive-definite matrix is denoted by $P>0 ; I$ is the unit matrix with appropriate dimension; and $*$ means the symmetric term in a symmetric matrix.

## 2. Problem Statement and Preliminaries

Let $(M, F, P)$ be a probability space, where $M, F$, and $P$ represent, respectively, the sample space, the $\sigma$-algebra of events, and the probability measure defined on $F$. Consider the following discrete-time Markov jump system:

$$
\begin{equation*}
x_{k+1}=A\left(r_{k}\right) x_{k}+B\left(r_{k}\right) \sigma\left(u_{k}\right)+E\left(r_{k}\right) w_{k} \tag{1}
\end{equation*}
$$

where $x_{k} \in R^{n}$ is the state, $u_{k} \in R^{m}$ is the input, $w_{k} \in$ $\left\{w_{k}^{\mathrm{T}} w_{k} \leq 1\right\}$ is the bounded disturbance of the system, and $\sigma\left(u_{k}\right)=\left[\begin{array}{llll}\sigma\left(u_{1 k}\right) & \sigma\left(u_{2 k}\right) & \cdots & \sigma\left(u_{m k}\right)\end{array}\right]^{\mathrm{T}}$.

The system is driven by a random process $\left\{r_{k}, k \geq 0\right\}$ which takes values from a finite set $\Gamma=\{1,2,3, \ldots, s\}$, where $\pi_{r_{k} r_{k+1}}^{\left(\xi_{k}\right)}=\operatorname{Pr}\left(r_{k+1}=j \mid r_{k}=i, \xi_{k}\right)$ denotes the transition probability from mode $i$ at time $k$ to mode $j$ at time $k+1$. Here, it is assumed that the TP, which is nonhomogeneous, is approximated by a set of random variables driven by a truncated Gaussian stochastic process $\left\{\xi_{k}, k \geq 0\right\}$. The probability density function (PDF) of $\pi_{r_{k} r_{k+1}}^{\left(\xi_{k}\right)}$ is given as follows:

$$
\begin{equation*}
\pi_{r_{k} r_{k+1}}^{\left(\xi_{k}\right)}=\frac{\left(1 / \sigma_{r_{k} r_{k+1}}\right) f\left(\left(\pi_{r_{k} r_{k+1}}^{\left(\xi_{k}\right)}-\mu_{r_{k} r_{k+1}}\right) / \sigma_{r_{k} r_{k+1}}\right)}{F\left(\left(1-\mu_{r_{k} r_{k+1}}\right) / \sigma_{r_{k} r_{k+1}}\right)-F\left(\left(0-\mu_{r_{k} r_{k+1}}\right) / \sigma_{r_{k} r_{k+1}}\right)}, \tag{2}
\end{equation*}
$$

where $f(\cdot)$ is the PDF of the standard normal distribution, $F(\cdot)$ is the cumulative density function (CDF) of $f(\cdot)$, and $\mu_{r_{k} r_{k+1}}$ and $\sigma_{r_{k} r_{k+1}}^{2}$ are, respectively, the mean and variance of the Gaussian PDF. More specifically, the TP matrix is given by

$$
\pi=\left[\begin{array}{cccc}
n\left(\mu_{11}, \sigma_{11}^{2}\right) & n\left(\mu_{12}, \sigma_{12}^{2}\right) & \ldots & n\left(\mu_{1 s}, \sigma_{1 s}^{2}\right)  \tag{3}\\
n\left(\mu_{21}, \sigma_{21}^{2}\right) & n\left(\mu_{22}, \sigma_{22}^{2}\right) & \ldots & n\left(\mu_{2 s}, \sigma_{2 s}^{2}\right) \\
\vdots & \vdots & \ddots & \vdots \\
n\left(\mu_{s 1}, \sigma_{s 1}^{2}\right) & n\left(\mu_{s 2}, \sigma_{s 2}^{2}\right) & \ldots & n\left(\mu_{s s}, \sigma_{s s}^{2}\right)
\end{array}\right]
$$

where $n\left(\mu_{r_{k} r_{k+1}}, \sigma_{r_{k} r_{k+1}}^{2}\right)$ denotes the PDF of truncated Gaussian TP of $p\left(\pi_{r_{k} r_{k+1}}^{\left(\xi_{k}\right)}\right)$, which is assumed to be known a priori.

It is noted that a larger $\sigma^{2}$ implies a larger degree of uncertainty related to the TP. In this case, a larger $\sigma^{2}$ should
be chosen. Otherwise, a smaller $\sigma^{2}$ should be chosen. The random variables $\pi_{r_{k} r_{k+1}}^{\left(\xi_{k}\right)}$ which appeared in the TP matrix are continuous. Taking the expectation of the random variable yields

$$
\begin{align*}
& \hat{\pi}_{r_{k} r_{k+1}}^{\left(\xi_{k}\right)} \\
& \quad=E\left(\pi_{r_{k} r_{k+1}}^{\left(\xi_{k}\right)}\right) \\
& \quad=\int_{0}^{1} \pi_{r_{k} r_{k+1}}^{\left(\xi_{k}\right)} p\left(\pi_{r_{k^{r}} r_{k+1}}^{\left(\xi_{k}\right)}\right) d \pi_{r_{k} r_{k+1}}^{\left(\xi_{k}\right)} \\
& =\mu_{r_{k} r_{k+1}} \\
& \quad+\frac{f\left(\left(1-\mu_{r_{k} r_{k+1}}\right) / \sigma_{r_{k} r_{k+1}}\right)-f\left(\left(0-\mu_{r_{k} r_{k+1}}\right) / \sigma_{r_{k} r_{k+1}}\right)}{F\left(\left(1-\mu_{r_{k} r_{k+1}}\right) / \sigma_{r_{k} r_{k+1}}\right)-F\left(\left(0-\mu_{r_{k} r_{k+1}}\right) / \sigma_{r_{k} r_{k+1}}\right)} \sigma_{r_{k} r_{k+1}} . \tag{4}
\end{align*}
$$

Consequently, the desired TP matrix can be obtained as follows:

$$
\Pi=\left[\begin{array}{cccc}
\hat{\pi}_{11}^{\left(\xi_{k}\right)} & \hat{\pi}_{12}^{\left(\xi_{k}\right)} & \ldots & \hat{\pi}_{1 s}^{\left(\xi_{k}\right)}  \tag{5}\\
\hat{\pi}_{21}^{\left(\xi_{k}\right)} & \hat{\pi}_{22}^{\left(\xi_{k}\right)} & \ldots & \hat{\pi}_{2 s}^{\left(\xi_{k}\right)} \\
\vdots & \vdots & \ddots & \vdots \\
\hat{\pi}_{s 1}^{\left(\xi_{k}\right)} & \hat{\pi}_{s 2}^{\left(\xi_{k}\right)} & \ldots & \hat{\pi}_{s s}^{\left(\xi_{k}\right)}
\end{array}\right]
$$

where $\sum_{j}^{s} \hat{\pi}_{r_{k} r_{k+1}}^{\left(\xi_{k}\right)}=1, \hat{\pi}_{r_{k} r_{k+1}}^{\left(\xi_{k}\right)} \geq 0,1 \leq i$, and $j \leq s$.
To proceed further, we need some preliminaries.
Definition 1. Discrete-time Markov jump system (1) (with $w_{k}=0$ ) is said to be stochastically stable if

$$
\begin{equation*}
\lim _{\mathrm{T} \rightarrow \infty} E\left\{\sum_{k=0}^{\mathrm{T}} x_{k}^{\mathrm{T}} x_{k} \mid x_{0}, r_{0}\right\}<\infty \tag{6}
\end{equation*}
$$

Definition 2. Consider system (1); let $h_{q i}$ denote the $q$ th row of matrix $H_{i}$. Then

$$
\begin{equation*}
\Theta\left(H_{i}\right)=\left\{x_{k} \in R^{n}:\left|h_{q i} x_{k}\right| \leq 1, q=1,2, \ldots, m\right\} \tag{7}
\end{equation*}
$$

is a symmetric polyhedron set.
Lemma 3 (see [24]). Given matrices $u_{k} \in R^{m}$ and $v_{k} \in R^{m}$ for system (1), if $\left|v_{k}\right|<1$, then $\sigma\left(u_{k}\right)=\sum_{t=1}^{2^{m}} \theta_{t}\left(M_{t} u_{k}+M_{t}^{-} v_{k}\right)$, where $0 \leq \theta_{t} \leq 1, \sum_{t=1}^{2^{m}} \theta_{t}=1, M_{t}$, and $t=1, \ldots, 2^{m}$ are $m \times m$ diagonal matrices whose diagonal elements are either 1 or 0 , and $M_{t}^{-}=I-M_{t}$.

Lemma 4 (see [24]). Given matrices $v_{k}=H_{i} x_{k}$ for system (1), if $x_{k} \in \Theta\left(H_{i}\right)$, that is $\left|v_{k}\right|<1$, then $\sigma\left(F_{i} x_{k}\right)=\sum_{t=1}^{2^{m}} \theta_{t}\left(M_{t} F_{i}+\right.$ $\left.M_{t}^{-} H_{i}\right) x_{k}$.

Definition 5. For given symmetric matrices $P_{i}>0$, let us define a mode-dependent ellipsoid invariant set given below:

$$
\begin{equation*}
\varepsilon\left(P_{i}, 1\right)=\left\{x_{k} \in R^{n}: x_{k}^{\mathrm{T}} P_{i} x_{k} \leq 1\right\} . \tag{8}
\end{equation*}
$$

## 3. Estimation of the Attraction Domain

We first derive the sufficient condition for the estimation of the attraction domain for the case without disturbance. For simplicity, we assume that the mode at time instant $k$ is $r_{k}=i$ and the mode at time instant $k+1$ is $r_{k+1}=j$.

Theorem 6. Consider system (1) with nonhomogeneous TP matrix (5) under the condition $w_{k}=0$. Suppose that there exist a set of symmetric positive definite matrices $P_{i}>0$ and $F_{i}, H_{i}$, $\forall i \in \Gamma$, such that

$$
\begin{align*}
& \left(A_{i}+B_{i}\left(D_{t} F_{i}+D_{t}^{-} H_{i}\right)\right)^{T} \sum_{j \in \Gamma} \hat{\pi}_{i j} P_{j}\left(A_{i}+B_{i}\left(D_{t} F_{i}+D_{t}^{-} H_{i}\right)\right) \\
& \quad-P_{i}<0, \quad t \in\left[1,2^{m}\right] \tag{9}
\end{align*}
$$

$$
\begin{equation*}
\varepsilon\left(P_{i}, 1\right) \subset \Theta\left(H_{i}\right) \tag{10}
\end{equation*}
$$

Then the set $\cap_{i=1}^{s} \varepsilon\left(P_{i}, 1\right)$ is the domain of attraction of the closed-loop system (1).
Proof. Construct a potential Lyapunov function as

$$
\begin{equation*}
V\left(x_{k}, r_{k}=i\right)=x_{k}^{\mathrm{T}} P_{i} x_{k} \quad(i \in \Gamma) . \tag{11}
\end{equation*}
$$

For system (1), it follows from Lemmas 3 and 4 that

$$
\begin{align*}
& \Delta V\left(x_{k}, i\right) \\
& \quad=E\left\{V\left(x_{k+1}, j\right)\right\}-V\left(x_{k}, i\right) \\
& =x_{k+1}^{\mathrm{T}} \sum_{j \in \Gamma} \hat{\pi}_{i j} P_{j} x_{k+1}-x_{k}^{\mathrm{T}} P_{i} x_{k} \\
& =x_{k}^{\mathrm{T}}\left[\left(A_{i}+B_{i}\left(D_{t} F_{i}+D_{t}^{-} H_{i}\right)\right)^{\mathrm{T}}\right. \\
& \left.\quad \times \sum_{j \in \Gamma} \hat{\pi}_{i j} P_{j}\left(A_{i}+B_{i}\left(D_{t} F_{i}+D_{t}^{-} H_{i}\right)\right)-P_{i}\right] x_{k} \\
& = \tag{12}
\end{align*}
$$

Clearly, condition (9) implies

$$
\begin{equation*}
\Delta V\left(x_{k}, i\right)<0 \tag{13}
\end{equation*}
$$

Denote $\delta=\min _{t} \lambda_{\text {min }}\left(-\Phi_{i}(t)\right)$, for all $i \in \Gamma$, where $\lambda_{\text {min }}\left(-\Phi_{i}(t)\right)$ is the minimal eigenvalue of $\left(-\Phi_{i}(t)\right)$.

Hence,

$$
\begin{equation*}
\Delta V\left(x_{k}, i\right) \leq-\delta x_{k}^{\mathrm{T}} x_{k} \tag{14}
\end{equation*}
$$

Taking the sum on both sides from 0 to T gives

$$
\begin{align*}
E\left\{\sum_{k=0}^{\mathrm{T}} \Delta V\left(x_{k}, i\right)\right\}= & E\left\{V\left(x_{\mathrm{T}+1}, \mathrm{~T}+1\right)\right\} \\
& -V\left(x_{0}, r_{0}\right) \leq-\delta E\left\{\sum_{k=0}^{\mathrm{T}} x_{k}^{\mathrm{T}} x_{k}\right\} \tag{15}
\end{align*}
$$

which implies

$$
\begin{equation*}
\lim _{\mathrm{T} \rightarrow \infty} E\left\{\sum_{k=0}^{\mathrm{T}} x_{k}^{\mathrm{T}} x_{k}\right\} \leq \frac{1}{\delta} V\left(x_{0}, r_{0}\right)<\infty . \tag{16}
\end{equation*}
$$

This completes the proof. Clearly Theorem 6 implies stochastic stability (see Definition 1).

## 4. Estimation of Disturbance Attraction Domain

In this section, we will derive sufficient condition for the estimation of the attraction domain under bounded disturbance. This sufficient condition will ensure that the influence of disturbance is minimized. To move forward, we assume that the bounded disturbance satisfies $w_{k}^{\mathrm{T}} w_{k} \leq 1$.

Theorem 7. Consider system (1) with nonhomogeneous TP matrix (5); suppose that there exist symmetric positive definite matrices $P_{i}>0$, and $F_{i}, H_{i}$, for all $i \in \Gamma$, such that
$\min \alpha$,

$$
\begin{gather*}
\left(A_{i}+B_{i}\left(D_{t} F_{i}+D_{t}^{-} H_{i}\right)\right)^{T} \sum_{j \in \Gamma} \hat{\pi}_{i j} P_{j}\left(A_{i}+B_{i}\left(D_{t} F_{i}+D_{i}^{-} H_{i}\right)\right)  \tag{18}\\
+\frac{1}{1+\eta}\left(\frac{1+\eta}{\eta} \lambda_{\max }\left(E_{i}^{T} P_{j} E_{i}\right)-1\right) P_{i}<0 \\
t \in\left[1,2^{m}\right] \tag{19}
\end{gather*}
$$

$$
\begin{equation*}
\left|h_{i q} x\right| \leq 1, \quad \forall x \subset \cap \varepsilon\left(P_{i}, 1\right), i \in \Gamma, q \in[1, m] \tag{20}
\end{equation*}
$$

where $\chi_{0}$ is a reference set, $x_{0}$ is an initial state, and $\alpha>0$ is a scalar; then the subset $\cap_{i=1}^{\tau} \varepsilon\left(P_{i}, 1\right)$ is the disturbance attraction domain for system (1) which satisfies an optimal disturbance attenuation performance index $\alpha$.

Proof. Consider a candidate Lyapunov function $V(x)=$ $x_{k}^{\mathrm{T}} P_{i} x_{k}$. It is required to show that

$$
\begin{align*}
\Delta V_{k}= & x_{k}^{\mathrm{T}}\left[\left(A_{i}+B_{i}\left(\sigma\left(F_{i} x\right)\right)+E_{i} w_{k}\right)^{\mathrm{T}}\right. \\
& \left.\times \sum_{j \in \Gamma} \hat{\pi}_{i j} P_{j}\left(A_{i}+B_{i}\left(\sigma\left(F_{i} x\right)\right)+E_{i} w_{k}\right)\right] x_{k}  \tag{21}\\
& -x_{k}^{\mathrm{T}} P_{i} x_{k}<0 .
\end{align*}
$$

Noting that $(a+b)^{\mathrm{T}}(a+b) \leq(1+\eta) a^{\mathrm{T}} a+(1+(1 / \eta)) b^{\mathrm{T}} b$ and $w_{k}^{\mathrm{T}} w_{k} \leq 1$, it follows that

$$
\begin{aligned}
\left(A_{i}+\right. & \left.B_{i}\left(\sigma\left(F_{i} x\right)+E_{i} w_{k}\right)\right)^{\mathrm{T}} \\
& \times \sum_{j \in \Gamma} \hat{\pi}_{i j} P_{j}\left(A_{i}+B_{i}\left(\sigma\left(F_{i} x\right)+E_{i} w_{k}\right)\right) \\
\leq & \max _{t \in\left[1,2^{m}\right]} x_{k}^{\mathrm{T}}(1+\eta)\left(A_{i}+B_{i}\left(D_{t} F_{i}+D_{t}^{-} H_{i}\right)\right)^{\mathrm{T}} \\
& \times \sum_{j \in \Gamma} \hat{\pi}_{i j} P_{j}\left(A_{i}+B_{i}\left(D_{t} F_{i}+D_{t}^{-} H_{i}\right)\right) x_{k} \\
& +\left(1+\frac{1}{\eta}\right) w_{k}^{\mathrm{T}} E_{i}^{\mathrm{T}} \sum_{j \in \Gamma} \hat{\pi}_{i j} P_{j} E_{i} w_{k}-x_{k}^{\mathrm{T}} P_{i} x_{k} \\
\leq & \max _{t \in\left[1,2^{m}\right]} x_{k}^{\mathrm{T}}(1+\eta)\left(A_{i}+B_{i}\left(D_{t} F_{i}+D_{t}^{-} H_{i}\right)\right)^{\mathrm{T}} \\
& \times \sum_{j \in \Gamma}^{\hat{\pi}_{i j} P_{j}\left(A_{i}+B_{i}\left(D_{t} F_{i}+D_{t}^{-} H_{i}\right)\right) x_{k}} \\
& +\left(1+\frac{1}{\eta}\right) \lambda_{\max }\left(E_{i}^{\mathrm{T}} P_{j} E_{i}\right)-x_{k}^{\mathrm{T}} P_{i} x_{k} .
\end{aligned}
$$

To guarantee the attraction domain property for $x_{k} \in$ $\cap \varepsilon\left(P_{i}, 1\right)$, it suffices to show that there exists an $\eta$, for all $t \in\left[1,2^{m}\right]$ such that

$$
\begin{align*}
& x_{k}^{\mathrm{T}}(1+\eta)\left(A_{i}+B_{i}\left(D_{t} F_{i}+D_{t}^{-} H_{i}\right)\right)^{\mathrm{T}} \\
& \quad \times \sum_{j \in \Gamma} \widehat{\pi}_{i j} P_{j}\left(A_{i}+B_{i}\left(D_{t} F_{i}+D_{t}^{-} H_{i}\right)\right) x_{k}  \tag{23}\\
& \quad+\left(1+\frac{1}{\eta}\right) \lambda_{\max }\left(E_{i}^{\mathrm{T}} P_{j} E_{i}\right)-1<0 .
\end{align*}
$$

Noting that $1=x_{k}^{\mathrm{T}} P_{i} x_{k}$ on $\partial \varepsilon\left(P_{i}, 1\right),(23)$ is guaranteed by (19). By (18), the sufficient condition for the optimal disturbance attenuation performance index $\alpha$ is implied. This completes the proof.

Next, we show how to solve the problem by using LMIs.
Theorem 8. Consider system (1) with nonhomogeneous TP matrix (5) and let $\gamma=\alpha^{2}$ be a scalar; suppose that there exist symmetric positive definite matrices $Q_{i}=P_{i}^{-1}>0$ and $Y_{i}=$ $F_{i} Q_{i}, Z_{i}=H_{i} Q_{i}, \eta>0$, and $\lambda \in(0, \eta /(1+\eta))$, for all $i \in \Gamma$, such that

$$
\begin{gathered}
\min \gamma, \\
{\left[Q_{i}-\gamma * R^{-1}<0,\right.} \\
{\left[\begin{array}{cccc}
\left(\frac{\lambda}{\eta}-\frac{1}{1+\eta}\right) Q_{i} & * & * & * \\
\sqrt{\kappa_{i}^{1}}\left(A_{i}+B_{i}\left(D_{i} Y_{i}+D_{i}^{-} Z_{i}\right)\right) & -Q_{1} & * & * \\
\vdots & \vdots & \ddots & \vdots \\
\sqrt{\kappa_{i}^{l}}\left(A_{i}+B_{i}\left(D_{i} Y_{i}+D_{i}^{-} Z_{i}\right)\right) & * & * & -Q_{l}
\end{array}\right]} \\
<0, \quad \forall i \in \Gamma, j \in \pi_{j}^{k},
\end{gathered}
$$

$$
\begin{gather*}
{\left[\begin{array}{cc}
-\lambda & E_{i}^{T} \\
* & -Q_{k}
\end{array}\right]<0, \quad \forall i \in \Gamma, k \in \Gamma,}  \tag{27}\\
{\left[\begin{array}{cc}
-1 & Z_{i q} \\
* & -Q_{i}
\end{array}\right]<0, \quad \forall i \in \Gamma, q \in[1, m],} \tag{28}
\end{gather*}
$$

where $\chi_{0}$ is a reference set and $x_{0}$ is an initial state; then the subset $\cap_{i=1}^{\tau} \varepsilon\left(P_{i}, 1\right)$ is the disturbance attraction domain for system (1) which satisfies an optimal disturbance attenuation performance index $\alpha$.

Proof. Denote $\gamma=\alpha^{2}$. Choose an ellipsoid $\varepsilon(R, 1)$ as a reference set. Then condition (26) can be formulated as $R / \gamma \leq P_{i}$, which is implied by (25). By applying Schur complement, it is clear that (18) and (19) follow from (25) and (26), respectively. Equation (27) implies the existence of $\lambda_{\max }$. Equation (20) is equivalent to (28). This completes the proof.

Remark 9. If we choose a polyhedron $x_{0}=\left[x_{0}^{1}, \ldots, x_{0}^{n}\right]^{T}\left(x_{0}^{n}\right.$ is a point) as a reference set in Theorem 8, then condition (22) is converted into

$$
\left[\begin{array}{cc}
-\frac{1}{\alpha^{2}} & *  \tag{29}\\
x_{0}^{q} & -Q
\end{array}\right]<0, \quad \forall q \in[1, n]
$$

## 5. Illustrative Example

Consider a nonhomogeneous discrete-time jump system with four modes:

$$
\begin{array}{cc}
A_{1}=\left[\begin{array}{cc}
0.50 & -0.30 \\
0.10 & 0.60
\end{array}\right], \quad B_{1}=\left[\begin{array}{c}
-0.026 \\
0.247
\end{array}\right], \\
E_{1}=\left[\begin{array}{c}
0.0657 \\
0.0582
\end{array}\right], \quad A_{2}=\left[\begin{array}{cc}
0.36 & -0.30 \\
0.20 & 0.50
\end{array}\right], \\
B_{2}=\left[\begin{array}{c}
-0.030 \\
0.100
\end{array}\right], \quad E_{2}=\left[\begin{array}{c}
0.0308 \\
0.0453
\end{array}\right], \\
A_{3}=\left[\begin{array}{cc}
0.70 & -0.25 \\
0.10 & 0.70
\end{array}\right], \quad B_{3}=\left[\begin{array}{c}
-0.010 \\
0.320
\end{array}\right],  \tag{30}\\
E_{3}=\left[\begin{array}{c}
0.0236 \\
0.0292
\end{array}\right], \quad A_{4}=\left[\begin{array}{cc}
0.65 & -0.35 \\
0.25 & 0.65
\end{array}\right], \\
B_{4}=\left[\begin{array}{c}
-0.010 \\
0.220
\end{array}\right], \quad E_{4}=\left[\begin{array}{c}
0.0586 \\
0.0323
\end{array}\right] .
\end{array}
$$

Assume that the PDF matrix to describe the TP matrix in Table 1 is given by

$$
\pi_{N}=\left[\begin{array}{llll}
n\left(0.3, \sigma^{2}\right) & n\left(0.2, \sigma^{2}\right) & n\left(0.1, \sigma^{2}\right) & n\left(0.4, \sigma^{2}\right)  \tag{31}\\
n\left(0.3, \sigma^{2}\right) & n\left(0.2, \sigma^{2}\right) & n\left(0.3, \sigma^{2}\right) & n\left(0.2, \sigma^{2}\right) \\
n\left(0.1, \sigma^{2}\right) & n\left(0.1, \sigma^{2}\right) & n\left(0.5, \sigma^{2}\right) & n\left(0.3, \sigma^{2}\right) \\
n\left(0.2, \sigma^{2}\right) & n\left(0.2, \sigma^{2}\right) & n\left(0.1, \sigma^{2}\right) & n\left(0.5, \sigma^{2}\right)
\end{array}\right]
$$

Table 1 shows the obtained TP matrix with $\sigma^{2}=0.01$.

Table 1: Shows the obtained TP matrix with $\sigma^{2}=0.01$.

| $\sigma^{2}=0.01$ |  |  |  |
| :--- | :--- | :--- | :--- |
| 0.29917 | 0.19945 | 0.10248 | 0.39890 |
| 0.29994 | 0.20006 | 0.29994 | 0.20006 |
| 0.10495 | 0.10495 | 0.49381 | 0.29629 |
| 0.19881 | 0.19881 | 0.10559 | 0.49679 |



Figure 1: Disturbance attraction domain.

By Theorem 8, the feedback gains are calculated as

$$
\begin{array}{ll}
F_{1}=\left[\begin{array}{ll}
2.2177 & -3.6435
\end{array}\right], & F_{2}=\left[\begin{array}{ll}
2.7909 & -6.7110
\end{array}\right], \\
F_{3}=\left[\begin{array}{ll}
3.3680 & -3.3769
\end{array}\right], & F_{4}=\left[\begin{array}{ll}
2.9303 & -4.9250
\end{array}\right] . \tag{32}
\end{array}
$$

Figure 1 shows a state trajectory on the boundary of the disturbance attraction domain under the bounded disturbance $w_{k}=0.5 \sin (k)$. Though the bounded disturbance exists, the state trajectory is regulated to a small neighbourhood around the origin. When the disturbance disappears, the state is driven to the origin as expected (see Figure 2), implying the stochastic stability. Figure 3 shows a trajectory of mode evolution. Table 2 shows the optimal disturbance attenuation index.

## 6. Conclusions

This paper investigated the design of the disturbance attraction domain estimation for a class of nonhomogeneous discrete-time Markov jump systems with saturation and bounded disturbance. Furthermore, the optimal disturbance attenuation index is satisfied. The numerical example shows the applicability of the results obtained as expected. The results obtained may be extended to the systems with time delay.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Table 2

| Parameters | $\eta^{*}$ | $\lambda^{*}$ | $\alpha_{\min }^{*}$ |
| :---: | :---: | :---: | :---: |
|  | 0.998 | 0.2510 | 0.1782 |



Figure 2: Attraction domain without disturbance.


Figure 3: One sampled mode evolution.

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## Research Article

# Localized and Energy-Efficient Topology Control in Wireless Sensor Networks Using Fuzzy-Logic Control Approaches 

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#### Abstract

The sensor nodes in the Wireless Sensor Networks (WSNs) are prone to failures due to many reasons, for example, running out of battery or harsh environment deployment; therefore, the WSNs are expected to be able to maintain network connectivity and tolerate certain amount of node failures. By applying fuzzy-logic approach to control the network topology, this paper aims at improving the network connectivity and fault-tolerant capability in response to node failures, while taking into account that the control approach has to be localized and energy efficient. Two fuzzy controllers are proposed in this paper: one is Learning-based Fuzzy-logic Topology Control (LFTC), of which the fuzzy controller is learnt from a training data set; another one is Rules-based Fuzzy-logic Topology Control (RFTC), of which the fuzzy controller is obtained through designing if-then rules and membership functions. Both LFTC and RFTC do not rely on location information, and they are localized. Comparing them with other three representative algorithms (LTRT, List-based, and NONE) through extensive simulations, our two proposed fuzzy controllers have been proved to be very energy efficient to achieve desired node degree and improve the network connectivity when sensor nodes run out of battery or are subject to random attacks.


## 1. Introduction

The advent of Wireless Sensor Networks (WSNs) [1] stimulates a tremendous number of applications, such as forest monitoring, factory automation, secure installation, and battlefield surveillance. Unlike other conventional network devices, the nodes in WSNs are more likely to be disconnected from each other. On the one hand, the sensor nodes are usually battery powered, so they are prone to loss connectivity due to battery depletion. On the other hand, the sensor nodes are subject to unpredictable node failures, for example, deployment in a hostile environment. The WSNs operate properly only when all nodes are reachable to each other. Therefore, one of the major concerns when planning WSNs is to make sure that all nodes in a network are, directly or indirectly, connected together.

In addition, the WSNs are expected to be able to tolerate a certain amount of node failures. From the graph theory point of view, the fault-tolerant problem is a $k$-connected network
problem, where $k$ indicates that there are at least $k$ distinct paths from one node to any other node. A $k$-connected network is able to be constructed, maintained, or improved by means of topology control [2-4].

Furthermore, the energy issue is usually taken into account to make the sensor nodes functional as long as possible. Any algorithm or control system running in WSNs nodes is expected to be localized, because the cost of gathering global information is very time consuming and energy consuming.

In order to study the challenges mentioned above, this paper aims at developing an energy-efficient adaptive technique to improve the connectivity by means of adjusting the communication range under difficult events sensors may suffer from, such as battery depletion and malicious attacks. Some computational intelligence algorithms are applied to WSNs, such as fuzzy-logic, neural networks, and evolutionary algorithms [5]. The fuzzy-logic control is a very powerful
technique that helps designers to construct a control system, regardless of the lack of mathematical models to completely describe network dynamics. More specifically, by using the fuzzy-logic controller to adaptively adjust the communication range of each node, this paper aims at maintaining desired node degree (namely, the number of neighbors a node has), thus improving network connectivity and fault-tolerant capability in response to node failures in WSNs, while at the same time taking into account that the control approach has to be localized and energy efficient. In this paper, in the case that the node degree is characterized by the mathematical model, the Learning-based Fuzzy-logic Topology Control (LFTC) is proposed to learn the dynamics to construct a fuzzy controller; we also propose another fuzzy controller, named Rules-based Fuzzy-logic Topology Control (RFTC), which is dependent on a heuristic approach to design the membership functions and if-then rules. LFTC and RFTC are both localized, because all information the fuzzy-logic controllers needed can be obtained from the node itself and its one-hop neighbor(s).

By comparing LFTC and RFTC with other three algorithms (Local Tree-based Reliable Topology (LTRT) [6], Listbased topology control [7], and NONE) through extensive simulations, our two proposed fuzzy controllers are proved to be very energy efficient to improve network connectivity when node failure occurs (running out of battery and random attacks). First, due to the close-loop feedback of the control system, our two proposed approaches are able to trace the desired node degree as node density changes, while others are not able to do so; second, the average communication range, which is proportional to energy consumption, is lower than other algorithms, implying that our approaches are more energy efficient; third, our proposals are totally localized and the inputs that the fuzzy controller needs are very easy to obtain; fourth, the simulation results show that our approaches are able to respond to network dynamics, because the network is still able to maintain reasonable connectivity in the presence of random node failures. In short, our two proposed approaches are able to react to the network dynamics and outperform other three algorithms.

The main contributions of this paper are summarized as follows. (1) We have presented two control approaches based on the fuzzy logic to deal with network connectivity problem in WSNs. We have first presented the fuzzy controller based on the training dataset, called Learning-based Fuzzy-logic Topology Control (LFTC), and then proposed another fuzzy controller based on designing membership functions and ifthen rules, called Rules-based Fuzzy-logic Topology Control (RFTC). (2) We have performed extensive simulations to compare LFTC and RFTC with other algorithms, and also the comparison between LFTC and RFTC was made. (3) The simulation results show that our two proposed fuzzy controllers are proved to be very energy efficient to achieve desired node degree and improve network connectivity when the sensor nodes run out of battery and suffer random attacks.

The rest of this paper is organized as follows. An introduction of related works is provided in Section 2. Section 3 presents our two fuzzy control approaches in detail. In

Section 4, this paper evaluates the proposals by comparing them with other three representative algorithms. The potential applications of this work are discussed in Section 5. Section 6 concludes our work.

## 2. Related Works

The WSNs fault-tolerant and energy problems can be solved by means of appropriate topology control. For instance, deploy nodes in specific positions control the number of nodes deployed in the field or control communication range or transmission power of each node, and so forth. In a wireless environment, such as WSNs, adjusting communication range or transmission power is a very common approach. From the graph theory point of view, the faulttolerant problems are $k$-connected network problems. Unfortunately, many of them are proved to be NP-complete or NP-hard problems even when $k$ is very small, which means that the optimal solutions do not exist. For instance, the following optimization problems are NP hard: the minimum number of links required to obtain a 2 -connected network [8], minimizing the power while maintaining $k$-connected network [9], minimizing the number of node placement for $k$-connected network [10], minimizing the number of relay nodes for 2 -connected network [11, 12], and so forth. Therefore, heuristic algorithms are needed to obtain nearoptimal performance, which is the main goal of this paper. Some of existing solutions are nonlocalized, so they are very unlikely to be applied to WSNs due to limited processing capability of WSNs nodes. Some of other solutions, such as Local Tree-based Reliable Topology (LTRT) [6] and Listbased topology control [7], are two reprehensive heuristic localized approaches to achieve desired connectivity. Similar to LTRT and list-based topology control, our goals are to propose localized and energy-efficient control algorithms to manage the communication range, in order to maintain the network connectivity. Our proposals are evaluated through computer-based simulation and the simulation scenarios are relatively realistic, because we considered various network configurations that may influence the results, for example, energy dissipation model, undirected links, heterogeneous network, routing algorithm, random failures, and so forth.

From the graph theory point of view, the conditions of a network being $k$-connected have been studied. The upper bound of $k$ is given by Menger's theorem [13]: $k \leq 2|E| /|V|$, where $|E|$ is the number of links and $|V|$ is the number of vertices in a graph. The upper bound can be achieved by constructing the Harary graph. Assuming that all nodes in a network are randomly and uniformly deployed, the asymptotical condition that a network is 1-connected with high probability is the node degree of each node at least bigger than $0.5193 \log n$ as $n \rightarrow+\infty$, where $n$ is the total number of nodes in the network [14]. Once the network is 1 -connected with high probability, it is also $k$-connected with high probability $[14,15]$.

Fuzzy control has been proved to be very effective to deal with complex nonlinear and time-varying systems, such as [16] which applies Takagi Sugeno (TS) fuzzy logic for


Figure 1: Fuzzy-logic control system.
electromagnetic suspension systems. Moreover, the fuzzy control can integrate with other control techniques; for example, [17] applies the fuzzy logic system to the sliding mode control and $[18,19]$ integrate the fuzzy control with the proportional integral derivative (PID) technique. Lyapunov theory is a common approach being used to analyze the stability of fuzzy controller [20, 21]. Some literatures try to obtain the optimal controller parameters and analyze the fuzzy controller's performance. Reference [22] analyzes the output feedback controller such that the closed-loop discrete time TS fuzzy systems with time-varying delays are asymptotically stable. Similarly, [23] calculates the asymptotic stability conditions for the state feedback fuzzy controller. In this paper, the fuzzy control is applied to address the challenges in WSNs and the numerical simulations are carried out to evaluate the proposed approaches.

The fuzzy linguistical input variables can be residual energy, concentration, and centrality [24]; remaining battery power, number of neighbors, distance from cluster centroid, network traffic [25]; residual energy and local distance [26]; distance to base station and residual energy [27]; energy and traffic load [28]; remaining energy, distance to base station and node density [29]; consistency, completeness, QoI from the lower level [30]. The fuzzy outputs could be the communication range, the competition range to be a cluster head, or the probability to be a cluster head. In this paper, the first proposal takes the node degree and the probability to have that node degree as inputs and the communication range as fuzzy controller output; in the second approach, we take the residual energy and the node degree error as inputs and the incremental of communication range as output.

## 3. WSNs Topology Control Using Fuzzy Logic

This paper focuses on how to design the fuzzy controller. Figure 1 shows a typical fuzzy-logic control system. The "fuzzification" transforms a crisp input variable (e.g., node degree) into a linguistic variable, for example, high, medium, and low; the "inference engine" maps the linguistic inputs onto linguistic outputs based on "if-then" rules; "defuzzification" converts the linguistic outputs of inference engine to crisp variables. Both linguistic input and output are represented by the membership functions.

Sometimes, a dynamical and distributed network is able to be characterized by mathematical forms, but it may happen that there are no ways to describe a network mathematically. For the former case, the fuzzy-logic membership functions and rules can be learnt from those mathematic descriptions;


Figure 2: Learning-based Fuzzy-logic Topology Control System (LFTC).
for the latter case, the fuzzy-logic controller could be established by a heuristic approach. In this paper, we leverage both approaches to construct fuzzy-logic controller to control the communication range of each node, with the aim that the network maintains desired node degree and improves the network fault-tolerant capability. Besides, the location information of sensor nodes is not needed in our fuzzy controllers, since it is not always available for sensor nodes.

In Section 3.1, we propose a learning-based fuzzy-logic controller based on neuroadaptive learning technique; in Section 3.2, a fuzzy-logic controller based on heuristic rules is proposed. They are both localized controllers, because all information the fuzzy-logic controllers needed can be obtained from the node itself and its one-hop neighbor(s). Throughout this paper, we use following abbreviations.
(i) ND: node degree or number of neighbors a node has.
(ii) $\mathrm{ND}_{\text {ref }}$ : reference ND or desired ND. In this paper, sometimes we also use $k$ to represent $\mathrm{ND}_{\text {ref }}$.
(iii) $\mathrm{ND}_{\text {lost }}$ : number of lost neighbor(s).
(iv) $e_{\mathrm{ND}}$ : node degree error $e_{\mathrm{ND}}=\mathrm{ND}-\mathrm{ND}_{\text {ref }}$.
(v) CR: communication range.
(vi) $E$ : node residual energy.
(vii) BS: base station.
3.1. LFTC: Learning-Based Fuzzy-Logic Topology Control. Our proposed Learning-based Fuzzy-logic Topology Control (LFTC) is a localized controller, because all parameters can be obtained locally by the node itself or one-hop neighbor(s).

### 3.1.1. Learning-Based Fuzzy-Logic Control System Design.

 Figure 2 shows the control system design of LFTC. It is a PI fuzzy-logic controller. Provided a training dataset, the fuzzy controller is obtained through neuroadaptive learning technique. On the one hand, each node can detect the number of lost neighbor(s) in the network. The fuzzy inputs are $\mathrm{ND}_{\text {ref }}+\mathrm{ND}_{\text {lost }}$ and the probability Prob that a node has $\mathrm{ND}_{\text {ref }}+\mathrm{ND}_{\text {lost }}$. On the other hand, Prob is controlled by an integral controller. The parameter $K$ is set to 0.02 if ND $\geq$ $\mathrm{ND}_{\text {ref }}$ and 0.05 if $\mathrm{ND}<\mathrm{ND}_{\text {ref }} . \mathrm{Prob}_{0}$ is 0.8 .3.1.2. Training Dataset. If sensor nodes in WSNs are randomly and uniformly distributed, its node degree distribution is a Poisson distribution [31]. The probability of a node having $N$ neighbors is given by (1); therefore the probability that ND
is bigger than $k$ is (2), where $r$ is the communication range and $\rho$ is the node density which is defined as total nodes $n$ in WSNs divided by the area of the deployment field $A$; namely, $\rho=n / A$. However, in practice the node degree distribution in WSNs possibly is non-Poisson distribution. For instance, based on a realistic radio channel fading model, [32] shows that the degree distribution in WSNs is approximated by a binomial distribution if the average node degree is low (e.g., less than 18). So, we can use (3) to represent the probability that ND is bigger than $k$, where $p(r)$ represents the probability of two nodes having a link at distance $r$, which is given in [32]. In addition, a mobility model of sensor nodes can be included in $p(r)$ [33]. It must be recalled that $\mathrm{ND}_{\text {ref }}=k$ in this paper:

$$
\begin{align*}
P_{1}(\mathrm{ND}=N) & =\frac{\left(\rho \pi r^{2}\right)^{N}}{N!} e^{-\rho \pi r^{2}},  \tag{1}\\
P_{1}(\mathrm{ND} \geq k) & =f_{1}(r, k)=1-\sum_{N=0}^{k-1} \frac{\left(\rho \pi r^{2}\right)^{N}}{N!} e^{-\rho \pi r^{2}},  \tag{2}\\
P_{2}(\mathrm{ND} \geq k) & =f_{2}(r, k) \\
& =1-\sum_{N=0}^{k-1}\binom{n-1}{N} p(r)^{N}(1-p(r))^{n-1-N} . \tag{3}
\end{align*}
$$

Equations (2) and (3) can be used to generate training dataset. The learning process can be performed on this training dataset afterwards. Define set $k=\left\{k_{1}, k_{2}, \ldots, k_{s}\right\}, r=$ $\left\{r_{1}, r_{2}, \ldots, r_{t}\right\}$, and $p=\left\{p_{1}, p_{2}, \ldots, p_{w}\right\}$, where $w=s \cdot t$. Given node density $\rho$, according to (2), we have

$$
\begin{equation*}
p_{i \cdot j}=f_{1}\left(r_{j}, k_{i}\right) \tag{4}
\end{equation*}
$$

Define a $w \times 3$ matrix $\mathbf{T}_{w \times 3}$, three elements at row $m$ are formed by (5)

$$
\begin{equation*}
\mathbf{T}(m)=\left[k_{i}, p_{i \cdot j}, r_{j}\right] \tag{5}
\end{equation*}
$$

where $i, j$ are both integer, $0<i \leq t, 0<j \leq s$, and $m=i \cdot j$. Similarly, the training data for binomial distribution can be obtained from (3).

The matrix $\mathbf{T}$ is used as training dataset where the fuzzylogic controller can be learnt from. The learning technique employed in this paper is the adaptive neurofuzzy training provided by Matlab ANFIS tool. Depending on the network deployment, the node degree $k$ could range from 1 to tens in order to obtain a wide range training data. Regarding variable $r$, it could range from several meters to hundreds meters, depending on real sensor devices.

The benefit of this approach is that there is no need to design the membership functions and if-then rules; instead the membership functions and rules are learnt from the training dataset.

### 3.2. RFTC: Rules-Based Fuzzy-Logic Topology Control

3.2.1. Rules-Based Fuzzy-Logic Topology Control System Design. In this section, we propose another fuzzy logic
controller, called Rules-based Fuzzy-logic Topology Control (RFTC), because we need to design the if-then rules. Unlike LFTC, RFTC is shown in Figure 3(a). Here, the fuzzy controller of RFTC is not automatically generated from the training dataset, and the fuzzy rules and membership functions are generated by the heuristic approaches or experiences instead. The input parameters are different as well. The input variable probability Prob is replaced by residual energy $E$, and $e_{\mathrm{ND}}$ becomes input. The output is the CR incremental, $\triangle \mathrm{CR} . \mathrm{CR}_{0}$ is the initial value of a sensor node, which is random for each node. This paper leverages "Mamdani" type fuzzy inference system.
3.2.2. Membership Functions and If-Then Rules. In this paper, for each input and output, there are three fuzzy sets: high, medium, and low. Their membership functions are shown in Figures 3(b), 3(c), and 3(d). Intuitively, if $e_{\mathrm{ND}}$ is high and $E$ is high, $\triangle \mathrm{CR}$ should be low; if $e_{\mathrm{ND}}$ is low, no matter what $E$ is, $\Delta$ CR should be High, because maintaining the network connectivity is the top priority. The details of if-then rules are shown in Table 1.

The design of membership functions and if-then rules is heuristic. The change of membership functions and ifthen rules has significant impact on the performance. It is necessary to tune the membership shapes and positions and change the rules according to different network deployment strategies and network models.

## 4. Performance Evaluation

In this section, we evaluate our two fuzzy-logic approaches with other three localized algorithms by using Matlab. There are many topology control algorithms/protocols proposed in the literature. Some of the state-of-the-art topology control algorithms are Local Tree-based Reliable Topology (LTRT) [6], Local Minimum Spanning Tree (LMST) [34], and Faulttolerant Local Spanning Subgraph (FLSS) [35], which are similar to each other as they are based on the spanning tree algorithm. In this paper, we choose LTRT as the representative algorithm in this category. On the other hand, the list-based topology control [7] is selected to represent the algorithm that does not rely on constructing the spanning tree but utilizes the neighbors' information. In addition, we also compare them with the one without any control approaches or algorithms, which means that the CR is not changed during the simulation. It is called NONE algorithm in this paper.
4.1. LTRT: Local Tree-Based Reliable Topology. Local Treebased Reliable Topology (LTRT) [6] is a localized algorithm. Basically, it is a variant of spanning tree algorithm. When conducting spanning tree algorithm $k$ times, the resultant network is a $k$-edge-connected if the original network is at least $k$-edge-connected network. More specifically, it repeatedly processes the network $k$ times as follows: given a $s$ -edge-connected network $G(V, E)$, where $V$ is the set of nodes, $E$ is the set of links, and $s \geq k$. First one of its spanning tree $T\left(V, \widehat{E}_{1}\right)$ is calculated by a localized algorithm, then all


Figure 3: Rules-based Fuzzy-logic Topology Control (RFTC).
links in $\widehat{E}_{1}$ from $E$ are removed and the resulting network is denoted as $G\left(V, E-\widehat{E}_{1}\right)$. Next time, the same process is performed on $G\left(V, E-\widehat{E}_{1}\right)$. Repeating this process $k$ times, the resultant network will be $G\left(V, E-\widehat{E}_{1}-\widehat{E}_{2}-\cdots-\widehat{E}_{k}\right)$. The final $k$-edge-connectivity network is formed by combining all trees together; that is, $G\left(V, \widehat{E}_{1}+\widehat{E}_{2}+\cdots+\widehat{E}_{k}\right)$. The final CR of each node is selected from the maximum CR that connects to its neighbors in $G\left(V, \widehat{E}_{1}+\widehat{E}_{2}+\cdots+\widehat{E}_{k}\right)$.

The LTRT requires that the original network is at least a $k$-edge-connected network, and it requires the location information of its neighbors. LTRT needs that each node runs at its maximum CR before it starts running the algorithm. LTRT has been compared with Cone-Based distributed Topology Control CBTC $(\alpha)$ [36] and Fault-tolerant Local Spanning Subgraph $\left(\mathrm{FLSS}_{k}\right)$ [35]. $\mathrm{FLSS}_{k}$ is a near optimal algorithm with high complexity. The simulation results of LTRT show that LTRT achieves comparable performance as that of $\mathrm{FLSS}_{k}$, but at a much lower cost.
4.2. List-Based Topology Control. List-based topology control [7] is a cooperative algorithm. It is called list based because the change of CR relies on the list of its neighbors. Each node does not change its CR (increases or decreases) until its neighbors require its CR to be changed. In other words, each node is able to ask for its neighbors to change their CR when it needs them to do so. If a node has more neighbors than it needs, it will request the closer neighbors to change their $C R$, and other neighbors will remain their $C R$.

Table 1: Fuzzy-logic if-then rules.

|  |  | $E$ |  |
| :--- | :---: | :---: | :---: |
| $e_{\mathrm{ND}}$ | High | Medium | Low |
| High | Low | Medium | Medium |
| Medium | Medium | Medium | Low |
| Low | High | High | High |

For instance, if node $u$ wants its ND to be 4 , it broadcasts a request message. All nodes within its CR will receive the request, and they change their CR to reach node $u$. If there are more than 4 nodes that can reach $u$, only 4 nodes closer to $u$ finally increase their CR , and other nodes will not modify their CR.

The list-based topology control is a localized algorithm, but it needs the location information of its neighbors as well, because the length of CR needed is calculated according to their location information.
4.3. Network Model and Configurations. Before starting the simulation, we first introduce the network model and configurations.
(i) Training dataset can be obtained according to different network models. In this paper, we employ the disk model, which means that CR is modeled as a disk with radius $r$. A link exists between two distinct nodes
only when they are both in each other's CR; thus, all links are undirected. All nodes are randomly and uniformly deployed in a $100 \times 100 \mathrm{~m}^{2}$ field; therefore only (2), rather than (3), is used in the fuzzy-logic leaning process.
(ii) All nodes in the field are stationary after the deployment.
(iii) Each node is capable of adjusting its CR ranging from 10 m to 30 m . In addition, the initial CR of each node is a random value chosen from $[10,30] \mathrm{m}$ in order to simulate heterogeneous WSNs in terms of CR.
(iv) There is a special node in the network called base station (BS) located at the center of the field.
(v) Each node transmits sensor data to BS periodically. Each node updates its CR according to different control approaches or algorithms after transmitting 800000 bits packages. It is called one "round" simulation. Note that 800000 bits packages are not necessary to be transmitted at one time. They can be fragmented into many small packages.
(vi) The routing algorithm is the shortest distance to BS. The simulation is terminated when BS no longer receives packages.
(vii) The energy dissipation model is the same as [27]. Equations (6) and (7) represent power consumed when a node transmits/receives a $L$ bits package to/from another node at distance $d$. Constant $E_{\text {elec }}=$ $50 \mathrm{~nJ} / \mathrm{bit}$ and $\epsilon_{\text {amp }}=100 \mathrm{pJ} / \mathrm{bit} / \mathrm{m}^{2}$ are related to the circuit and antenna design of sensor nodes. Each node is charged with 1 J energy at the beginning of simulation. Nodes stop sending or receiving packages when there is no battery left:

$$
\begin{gather*}
E_{\mathrm{tx}}=L \times E_{\mathrm{elec}}+L \times \epsilon_{\mathrm{amp}} \times d^{2}  \tag{6}\\
E_{\mathrm{rcv}}=L \times E_{\text {elec }} \tag{7}
\end{gather*}
$$

(viii) Apart from running out of battery, in order to simulate random attacks or damage by malicious people or nodes, a configurable parameter called failure probability is introduced in the simulation. Each node experiences identical failure probability at each round.
4.4. Simulations and Discussions. Simulations are divided into two parts. In the first part, we only consider the effect of the topology control approaches or algorithms on the initial network topology. In other words, we only observe the topology changes after all nodes are deployed without any data transmission in the network. Therefore, some configurations in Section 4.3 are not applicable to this simulation part, such as the energy dissipation model, the routing algorithm, and the failure model. In the second part, we simulate the networks with packages being sent at each node. The main differences between two parts are in the second simulation part, the energy of each node will be decreasing, and some
of the nodes may run out of battery during the simulations. Particularly, the relay nodes deplete energy faster. As a result, the links between nodes are dynamic. Because $E$ is one of the inputs for RFTC, the energy status has an impact on the controller output. Besides, the failure probability also influences the connections among nodes. Therefore, the second part simulation is a more dynamic scenario.

For each part, the deployment area is fixed but the number of nodes deployed varies from 30 to 75 nodes to change the node density. Since the deployment is random, 50 different networks are generated for every algorithm with the same configurations (e.g., same number of nodes and failure probability). Obtained results are the average of 50 networks.
4.4.1. Topology Control on Initial Network. Figures 4(a), 4(b), and $4(c)$ show the average node degree, which is calculated by the sum of the node degrees of all nodes divided by the number of nodes in a network. As mentioned in Section 4.3, the link between two nodes is undirected. A node connected by a directed link is not counted as a neighbor. We observe that our proposed two approaches, LFTC and RFTC, are able to trace the reference $k=2,3,4$ as the number of nodes deployed in the field increases. But Figure 4(d) shows that the network is unable to trace $k=5$ when the node number is less than 60 . Because the maximum CR is limited, it is less likely that each node has node degree at least 5 if the network density is not high enough. On the contrary, other algorithms are unable to trace the desired $k$. LTRT has the highest ND, because it is the most aggressive algorithm, which needs each node to run at maximum CR before it starts running LTRT. Higher average ND is good for the network connectivity but also introduces higher signal inferences.

It is worth noting that LFTC and RFTC are very close to each other in Figures 4(a), 4(b), and 4(c), but in Figure 4(d) LFTC demonstrates better performance at tracing the desired $k$ than RFTC when the nodes deployed are less than 60. LFTC manifests better adaptivity than RFTC in response to the network dynamics. As we mentioned in Section 3.2, RFTC has to tune the control parameters to adapt the network dynamics, but the control parameters are the same for RFTC in the simulations.

As shown in (6), energy consumption is propositional to the squared distance between a transmitter and a receiver. Similar to the way of evaluating LTRT [6], we use average $\mathrm{CR}\left(\mathrm{CR}_{\text {avg }}\right)$, average maximum $\mathrm{CR}\left(\mathrm{CR}_{\max }\right)$, and Energy Expended Ratio (EER) (defined as EER $=100 \times$ $\left(\mathrm{CR}_{\text {avg }} / \mathrm{CR}_{\max }\right) \%$ ) to evaluate network energy consumption. Figures 5, 6, and 7 illustrate the average CR and average maximum CR, and its EER when $k=3$, respectively. The $\mathrm{CR}_{\text {avg }}$ of LFTC, RFTC, and LTRT decreases as the number of nodes increases, but the network running list-base algorithm and NONE, the $\mathrm{CR}_{\text {avg }}$ does not change to a great extent. They are expected results because
(1) for LFTC, RFTC, and LTRT, if there are more nodes deployed, it indicates that the node density becomes higher. Therefore, the lower CR can obtain desired $k$ neighbors. In other words, energy consumption


Figure 4: Average node degree with different $k$.
is higher when the network is sparse, while energy consumption is lower when the network is dense,
(2) for the algorithm NONE, because the CR does not change and the initial CR randomly chosen from [10, 30] m; therefore its $\mathrm{CR}_{\text {avg }}$ is always about 20 m ,
(3) for the list-base algorithm, a node can ask for other nodes, which are not its neighbors but within its CR , to increase their CR , but the maximum CR between any two nodes is limited by maximum CR between them. For instance, nodes $u$ and $v$ have communication ranges of 15 m and 20 m , respectively. The node $v$ may need $u$ to increase CR to reach $v$; however, the maximum between $u$ and $v$ is impossible bigger than maximum between them, that is, 20 m . In other words, the node CR can increase but the incremental is limited.

As far as the average energy is concerned, we conclude that the energy consumption of LFTC and RFTC is always lower than LTRT regardless of the node density, and LFTC and RFTC outperform list-based and NONE after the number of nodes higher than 40 . LFTC is slightly better than RFTC. As far as the most power consumption node is taken into account, as shown in Figure 6, LFTC has the lowest maximum CR. It indicates that the most power consuming node running LFTC in a network has the lowest power consumption than running other algorithms. EER in Figure 7 shows the same trend as Figure 5. We expect that EER is low. RFTC maintains lowest EER than other algorithms. LFTC is higher than RFTC due to the maximums being lower than RFTC, as shown in Figure 6.

In short, the simulations performed in this section only focus on the network connectivity and corresponding energy


Figure 5: Average communication range ( $k=3$ ).


Figure 6: Maximum communication range $(k=3)$.
consumption. The simulation results imply that our two proposals (LFTC and RFTC) are able to maintain the desired node degree, which perfectly shows the effectiveness of the feedback control loops, while the resulting node degree are higher than expected when using other conventional methods. On the other hand, from the energy consumption point of view, our proposals manifest lower energy consumption on the resulting networks than other algorithms.
4.4.2. Topology Control on Network with Random Failures. In this section, we evaluate the network performance when the nodes send and receive sensor data periodically. Moreover, we consider not only nodes running out of battery, but also nodes that are damaged on purpose. The node damage is modeled by introducing random failures at each round. In the simulation, there are 60 nodes deployed in the field. Since


Figure 7: Energy expended ratio (EER) $(k=3)$.
each node leaves the network randomly at each round, $\mathrm{ND}_{\text {avg }}$ is calculated in a different way. For any node $i$ in the network, let ND at round $j$ be denoted as $\mathrm{ND}(i, j)$, and let round $(i)$ be the number of rounds before node $i$ runs out of battery or is attacked. Average ND of node $i$, denoted as $\mathrm{ND}_{\text {avg }}(i)$, is calculated by the sum of ND at each round divided by the number of rounds for node $i$, as shown in (8). The $\mathrm{ND}_{\mathrm{avg}}$ of a specific network is the average of $\mathrm{ND}_{\text {avg }}(i)$ for all nodes, as shown in (9), where $n$ is the total number of nodes deployed:

$$
\begin{align*}
\mathrm{ND}_{\mathrm{avg}}(i) & =\frac{\sum_{j=1}^{\operatorname{round}(i)} \mathrm{ND}(i, j)}{\operatorname{round}(i)},  \tag{8}\\
\mathrm{ND}_{\mathrm{avg}} & =\frac{\sum_{i=1}^{n} \mathrm{ND}_{\mathrm{avg}}(i)}{n} \tag{9}
\end{align*}
$$

Figure 8 shows the $\mathrm{ND}_{\text {avg }}$ with failure probabilities $0 \%$, $4 \%, 8 \%$, and $12 \%$. First of all, $\mathrm{ND}_{\text {avg }}$ decreases for all algorithms, because the failure probability increases. Second, $\mathrm{ND}_{\text {avg }}$ of LFTC and RFTC decreases slower than others (especially when failure probabilities are $4 \%$ and $8 \%$ ), implying that our proposed approaches are able to effectively resist $\mathrm{ND}_{\text {avg }}$ (or, network connectivity) decreasing as the failure probability increases. RFTC outperforms LFTC, but both worse than LTRT. Third, ND $_{\text {avg }}$ of list-base and NONE is very close, while LTRT still has the highest $\mathrm{ND}_{\text {avg }}$.

In this section, the simulations are performed on the more realistic scenarios, where nodes transmitting, receiving, and routing packages, meanwhile considering that nodes are experiencing running out of battery and malicious attacks. The results indicate that the control approaches in our proposals are still valid. The performance degradation is slower than other algorithms in the case that random failures occur.


Figure 8: Average node degree for different failure probabilities ( $k=$ $3, n=60$ ).

## 5. Potential Applications

In this section, we discuss some potential applications, including fault-tolerant network topology design, sensor nodes power management, and routing protocol design.
5.1. Fault-Tolerant Network Topology Design. By applying the control approaches proposed in this paper, the WSNs are able to tolerate the desired amount of node failure(s). Literature [14] proved that the network will be asymptotical $k$-connected with high probability, provided that the node degree of each node is at least bigger than $0.5193 \log n$, where $n$ is the total number of nodes in the network. If a network is $k$-connected, then it means that the network can tolerate at most $k-1$ node failure(s), and the remaining network is still connected. For example, by configuring the desired node degree to be 6 in our control systems, our control approaches will automatically achieve this node degree through adaptively adjusting the communication range of each node, and the network will be able to keep connected as long as the number of node failures is no more than 5.
5.2. Sensor Nodes Power Management. In a real system, the communication range usually cannot be directly changed; instead the power transmission is the parameter that can be modified. The higher power transmission is, the longer communication range is, and also the higher energy consumption will be. The energy is one of the significant resources that WSNs nodes need to preserve. In the RFTC proposed in this paper, the energy is one of the controller inputs. Therefore, by appropriately defining the membership functions and if-then rules, the tradeoff between network connectivity and energyefficiency can be fulfilled.
5.3. Routing Protocol Design. As the energy consumption is one of the most critical resources for battery-powered
sensor nodes, an important goal in WSNs design is to maximize the lifetime of the network by choosing a routing path which consumes lower energy (such as EnergyBalanced Routing Protocol (EBRP) [37]), or by using energy efficient distributed algorithms (e.g., [38]). However, the routing algorithms/protocols usually do not change the communication range or power transmission itself, so our proposals possibly can be integrated with those energy-aware algorithms/protocols. For instance, EBRP forces packets to move toward the sink through the dense energy area, so our control approaches could be employed to balance the distribution of energy density in order to further extend the lifetime of the network.

## 6. Conclusions and Future Works

In order to improve the network connectivity of WSNs in the presence of node failures, this paper proposed two localized and energy-efficient approaches, called LFTC and RFTC, based on the fuzzy logic. As for LFTC, the fuzzy-logic controller is obtained through the training dataset, while the fuzzy controller is based on the heuristic if-then rules and membership functions for RFTC. However, both approaches can achieve almost the same goals. LFTC and RFTC both have strengths and weaknesses. The main benefit of LFTC is that it relies on a mathematical model, so there is no need to adjust fuzzy controller parameters, but the mathematical model may be not available or accurate enough for realistic sensor nodes deployment. In contrast, the control parameters of RFTC do not depend on the mathematical model, but the parameters have to be tuned according to specific node deployment to achieve best performance. It could be not an easy task, or even not feasible.

Compared LFTC and RFTC with other three algorithms NONE, LTRT, and list based by extensive simulations, our two proposals can achieve desired node degree and save more energy. Furthermore, in the case that random node failures exist, such as nodes running out of battery or suffering intentional attacks, LFTC and RFTC show their capability to resist node failures by adaptively adjusting the communication range. In our simulations, we employ the disk model which is an ideal wireless channel. Nonetheless, our approaches can be extended to more realistic models as long as the node degree is known, for example, binomial distribution [32]. The only difference is the way to obtain the training dataset.

In this paper, all nodes in the network are stationary once deployed, but it is possible that the nodes in WSNs can be relocated from one place to another. In addition, it is desirable to deploy our fuzzy controllers in real sensor nodes running in a harsh environment. The mobility and implementation in real sensor nodes will be considered in our future works.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Fault Isolation for Nonlinear Systems Using Flexible Support Vector Regression 

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#### Abstract

While support vector regression is widely used as both a function approximating tool and a residual generator for nonlinear system fault isolation, a drawback for this method is the freedom in selecting model parameters. Moreover, for samples with discordant distributing complexities, the selection of reasonable parameters is even impossible. To alleviate this problem we introduce the method of flexible support vector regression (F-SVR), which is especially suited for modelling complicated sample distributions, as it is free from parameters selection. Reasonable parameters for F-SVR are automatically generated given a sample distribution. Lastly, we apply this method in the analysis of the fault isolation of high frequency power supplies, where satisfactory results have been obtained.


## 1. Introduction

With the increasing use of complex systems, there has been great interest in the development of techniques to fault isolations. Generally, the major approaches for fault isolation can be divided into two categories, namely, modelbased and data-driven techniques. The fundamental aspect of a model-based fault isolation is a process model that runs parallel to the process [1]. With traditional methods like observers, approximating the function between state vectors and input/output vectors is successful due to precise mathematical modelling by the use of filters. While these methods have successfully modelled linear systems, when applied to nonlinear systems like chemical processing, precise devices, and aerodynamic systems, they often fail to construct a sufficient model because their mechanism models are hard to be formed. Model-based approaches have advantages in terms of on-board implementation considerations, but their reliability may decrease as the nonlinear system complexities increase [2].

Therefore, data-driven techniques have been introduced to more accurately construct process models as these
methods are free from the requirement to analytically derive equations for a given system, shown in Figure 1. One feasible method is to use the artificial neural network (ANN). ANN utilizes experience risk minimization (ERM) principle to construct the process model, where the target function is numerically approximated by minimizing residuals between function estimates and outputs of the process data. Applications of ANN-based fault isolation have been widely addressed in the literature. For example, Sadough Vanini et al. [2] used the dynamic neural networks to isolate the fault of a dual spool gas turbine engine. Filippetti et al. [3] applied fuzzy-NN to the fault isolation of induction motor drives. However, the learning ability of ANN is dependent on the number of training samples. It requires massive samples to ensure the modelling performance. But in most practical applications, not many of fault samples can be acquired.

More recently, the principle of structure risk minimization [4] has been introduced in fault isolation through the utilization of support vector regression (SVR) [5, 6] as it can provide more accurate results than using neural networks in condition of smaller training samples. It was constructed on the basis of statistics learning theory that provides
the theoretical proofs of learning from finite samples. Much has been addressed in the literature where SVR shows superiorities to ANN in process modeling [7].

However, the performance of SVR-based modelling is greatly affected by its parameters. Although SVR has been well studied and many remarkable achievements have been obtained, the theoretical estimation of regression parameter remains unsolved in the last decade. There is no general consensus on the selection of proper parameters, but only some practical recommendations on this issue. This greatly increases the difficulty for common operators to master the SVR-based approach. Moreover, in some complicated cases, there are even no reasonable parameter settings that could be found. A rigorous selection of regression parameters can lead to the overlearning of training samples, while slack selections can lead to underlearning. There exist no parameters that yield good trade-off between overlearning and underlearning.

In this paper we introduce a flexible SVR (F-SVR) approach [8] to more accurately implement models that construct different residual generators for fault isolation. By automatically dividing training samples into several regions, this method is not only free from parameter selection, but also able to learn well and to generalize well for complicated cases.

## 2. Problem Statement

Support vector regression (SVR) is a process modeling tool that approximates the function between inputs and outputs:

$$
\begin{equation*}
y=f(x)=w x+b \tag{1}
\end{equation*}
$$

Here $x$ and $y$ represent the input and output vectors, respectively, $f$ is the modeled function, $w$ is its weight vector, and $b$ is the bias decided by the vector $w$.

The SVR-based modelling can be viewed as a process of finding the optimal weight vector $w_{0}$ with a proper parameter vector $\alpha_{0}$ for a given data set $\left\{\left(x_{1}, y_{1}\right), \ldots,\left(x_{n}, y_{n}\right)\right\}$ :

$$
\begin{align*}
\left(\alpha_{0}, w_{0}\right) & =\underset{\alpha, w}{\arg \min }: R_{S R M}(\alpha, w) \\
& =\underset{\alpha, w}{\arg \min }: \text { Residual }+\phi(w) \\
& =\underset{\alpha, w}{\arg \min }: \int_{T_{o}}^{T_{k}} L(y, f(x, \alpha, w)) p(x, y) d x d y, \tag{2}
\end{align*}
$$

where $L\left(y_{i}, f\left(x_{i}, w, \alpha\right)\right)=C_{i} \cdot\left|y_{i}-\left(\sum_{i=1}^{l} \beta_{i} K\left(x, x_{i}\right)+b\right)\right|_{\varepsilon}$ is the loss function, $p(x, y)$ is the unknown joint distribution of $x$ and $y, C_{i}$ is the regularized parameter, $\varepsilon$ is the insensitive parameter, $K(\cdot)$ is the kernel function, $\phi(w)$ denotes the generalization ability for the regression, and $\alpha=\{C, \varepsilon, K(\cdot)\}$. The optimal weight vector $w_{0}$ could be obtained by Lagrangian approaches. Thus the core problem for the SVR modeling is the selection of parameters.

The framework for SVR-based fault isolation is shown in Figure 2. Different operating models are constructed by


Figure 1: Framework of data-driven fault isolation.


Figure 2: Scheme of SVR-based fault isolation.

SVRs with given parameters. The residuals between estimated outputs and real outputs are generated for fault isolation.

However, the selection of SVR parameters is not easy. With the fixed regression parameter $\alpha$, the drawback of SVR is the hardness of the trade-off between overlearning and underlearning. Moreover, in some complicated cases, even no reasonable parameters could be found. As shown in Figure 2, due to the different complexities of sample distributions, the requirements of parameters are discordant. If a rigorous parameter $\alpha_{1}$ is selected, the regression is overlearning in region $A$. In contrast, if a slack parameter $\alpha_{2}$ is selected, the regression fails to learn in region B (underlearning). No parameter that can adequately fit all of the cases exists. Consequently, we advocate using the F-SVR approach in order to overcome this drawback [8], as the regression parameter is automatically generated and is variable instead of fixed (Figure 3).


Figure 3: The drawback for SVR modeling.

## 3. Fault Isolation Using Flexible Support Vector Regression (F-SVR)

3.1. The Principle of Flexible Support Vector Regression. We proposed a parameter-free algorithm for process modeling, namely, flexible support vector regression. The F-SVR attempts to divide the training samples into $k$ regions according to the distribution complexity, and for the $i$ th region, parameter $\alpha_{i}$ is generated. By minimization (3), the function between $x$ and $y$ is approximated:

$$
\begin{align*}
R(w, \alpha)= & \iint_{T 1} L\left(y, f\left(x, w, \alpha_{1}\right)\right) p(x, y) d x d y \\
& +\iint_{T 2} L\left(y, f\left(x, w, \alpha_{2}\right)\right) p(x, y) d x d y \\
& +\cdots+\iint_{T k} L\left(y, f\left(x, w, \alpha_{k}\right)\right) p(x, y) d x d y . \tag{3}
\end{align*}
$$

With given parameters, (3) can be minimized by solving a quadratic programming (QP) problem. Supposing the training samples were divided into two areas, $\left(x_{i}, y_{i}\right)_{i=1}^{q}$ and $\left(x_{i}, y_{i}\right)_{i=q+1}^{l}$, and $\alpha=\{C, \sigma, \varepsilon\}$ are parameters for the two areas, the minimization of (3) could be termed as

$$
\begin{align*}
\min : & \frac{1}{2}(w, w)+C_{1}\left(\sum_{i=1}^{q} \xi_{i}^{*}+\sum_{i=1}^{q} \xi_{i}\right) \\
& +C_{2}\left(\sum_{i=q+1}^{l} \xi_{i}^{* 2}+\sum_{i=q+1}^{l} \xi_{i}^{2}\right) \tag{4}
\end{align*}
$$

S.t. $\quad y_{i}-\left(w, x_{i}\right)-b \leq \varepsilon+\xi_{i}^{*}$

$$
\begin{aligned}
& \left(w, x_{i}\right)+b-y_{i} \leq \varepsilon+\xi_{i} \\
& \xi_{i}^{*} \geq 0 ; \quad \xi_{i} \geq 0 .
\end{aligned}
$$

And the Lagrangian for (4) is

$$
\begin{align*}
& L\left(w, \xi^{*}, \xi, \alpha^{*}, \alpha, \gamma^{*}, \gamma\right) \\
& \quad=\frac{1}{2}(w, w)+C_{1}\left(\sum_{i=1}^{q} \xi_{i}^{*}+\sum_{i=1}^{q} \xi_{i}\right) \\
& \quad+C_{2}\left(\sum_{i=q+1}^{l} \xi_{i}^{* 2}+\sum_{i=q+1}^{l} \xi_{i}^{2}\right) \\
& \quad-\sum_{i=1}^{l} \alpha_{i}\left[y_{i}-w \cdot x_{i}-b+\varepsilon+\xi_{i}\right]  \tag{5}\\
& \quad-\sum_{i=1}^{l} \alpha_{i}^{*}\left[w \cdot x_{i}+b-y_{i}+\varepsilon+\xi_{i}^{*}\right] \\
& \quad-\sum_{i=1}^{l}\left(\gamma_{i}^{*} \xi_{i}^{*}+\gamma_{i} \xi_{i}\right) .
\end{align*}
$$

Taking the partial derivative for (5), we get

$$
\begin{gather*}
\frac{\partial L}{\partial w}=w+\sum_{i=1}^{l} \alpha_{i} x_{i}-\sum_{i=1}^{l} \alpha_{i}^{*} x_{i}=0, \\
\frac{\partial L}{\partial b}=\sum_{i=1}^{l}\left(\alpha_{i}-\alpha_{i}^{*}\right)=0, \\
\frac{\partial L}{\partial \xi_{i}}=\left\{\begin{array}{cc}
C_{1}-\alpha_{i}-\gamma_{i}, & 1 \leq i \leq q \\
2 C_{2} \xi_{i}-\alpha_{i}-\gamma_{i}, & q<i \leq l
\end{array}=0,\right.  \tag{6}\\
\frac{\partial L}{\partial \xi_{i}^{*}}=\left\{\begin{array}{cc}
C_{1}-\alpha_{i}^{*}-\gamma_{i}^{*}, & 1 \leq i \leq q \\
2 C_{2} \xi_{i}^{*}-\alpha_{i}^{*}-\gamma_{i}^{*}, & q<i \leq l
\end{array}=0 .\right.
\end{gather*}
$$

Make the dual problem for (4):
$\max : \quad-\varepsilon \sum_{i=1}^{l}\left(\alpha_{i}^{*}+\alpha_{i}\right)+\sum_{i=1}^{l} y_{i}\left(\alpha_{i}^{*}-\alpha_{i}\right)$
$-\frac{1}{2} \sum_{i, j=1}^{l}\left(\alpha_{i}^{*}-\alpha_{i}\right)\left(\alpha_{j}^{*}-\alpha_{j}\right)\left(x_{i} \cdot x_{j}\right)$

$$
\begin{equation*}
-\sum_{i=q+1}^{l} \frac{1}{4 C_{2}}\left(\alpha_{i}^{2}+\alpha_{i}^{* 2}\right) \tag{7}
\end{equation*}
$$

S.t. $\quad \sum_{i=1}^{l}\left(\alpha_{i}^{*}-\alpha_{i}\right)=0 ; \quad 0 \leq \alpha_{i}^{*} \leq C_{i} ; \quad 0 \leq \alpha_{i} \leq C_{i}$,
where

$$
C_{i}= \begin{cases}C_{1}, & 1 \leq i \leq q  \tag{8}\\ C_{2}, & q+1 \leq i \leq l\end{cases}
$$



Figure 4: The flowchart for F-SVR modeling.


Figure 5: Setting different parameters for different regions.

As $\alpha \cdot \alpha^{*} \equiv 0$, (7) could be written with the following form:

$$
\begin{align*}
\min : & \frac{1}{2}\left(\alpha-\alpha^{*}\right)^{T} Q\left(\alpha-\alpha^{*}\right)+y^{T}\left(\alpha-\alpha^{*}\right)  \tag{9}\\
& +\varepsilon e^{T}\left(\alpha+\alpha^{*}\right)-\left(\alpha-\alpha^{*}\right)^{T} v\left(\alpha-\alpha^{*}\right)
\end{align*}
$$

where $Q=K\left(x_{i}, x_{j}\right), v=\left[0, \ldots,(1 / 4) C_{q+1}^{-1}, \ldots,(1 / 4) C_{l}^{-1}\right]$, and $e$ is the unit vector. Further, the regression could be written as the QP problem

$$
\begin{align*}
\min : & \frac{1}{2}\left[\alpha^{T},\left(\alpha^{*}\right)^{T}\right]\left[\begin{array}{cc}
D & -D \\
-D & D
\end{array}\right]\left[\begin{array}{c}
\alpha \\
\alpha^{*}
\end{array}\right]+c^{T}\left[\begin{array}{c}
\alpha \\
\alpha^{*}
\end{array}\right] \\
\text { S.t. } & z^{T}\left[\begin{array}{c}
\alpha \\
\alpha^{*}
\end{array}\right]=0 ; \quad 0 \leq \alpha_{i} \leq C_{i} ; 0 \leq \alpha_{i}^{*} \leq C_{i} \tag{10}
\end{align*}
$$

where $D=Q-v, c=[\varepsilon e+y, \varepsilon e-y]$, and

$$
z_{i}= \begin{cases}1, & i=1,2 \ldots q  \tag{11}\\ -1, & i=q+1, \ldots, l\end{cases}
$$

This form of QP problem can be solved by an active set method [9]. The feasibility for complicated cases that more regions are divided can similarly be proved.
3.2. Detailed Process of F-SVR Modeling. F-SVR modeling contains three major steps, shown in Figure 4.

Step 1 (sample divisions). This section shows how $T_{i}$ in (3) is determined. Given training samples $\left(x_{i}, y_{i}\right)_{i=1}^{n}$, formula (12) is utilized to estimate the distribution complexity

$$
\begin{equation*}
C P=\frac{\sum_{i=1}^{n-1}\left|A_{i}\right| /(n-1)}{\sum_{i=1}^{n-2} \cos \theta_{i} /(n-2)}=\frac{n-2}{n-1} \cdot \frac{\sum_{i=1}^{n-1}\left|A_{i}\right|}{\sum_{i=1}^{n-2} \cos \theta_{i}} \tag{12}
\end{equation*}
$$

where $A_{i}=\left(x_{i+1}, y_{i+1}\right)-\left(x_{i}, y_{i}\right)$ and $\cos \theta_{i}=\left|A_{i} \cdot A_{i+1}\right| /\left(\left|A_{i}\right|\right.$. $\left.\left|A_{i+1}\right|\right)$. Supposing the training samples have been divided into $k$ areas, formula (13) is implemented to evaluate the performance of division

$$
\begin{equation*}
S=\sum_{i=1}^{k} \frac{-C_{i}}{\sum_{j=1}^{k} C_{j}} \log \frac{C_{i}}{\sum_{j=1}^{k} C_{j}} \tag{13}
\end{equation*}
$$

The samples are divided randomly for several times, and the division with the smallest $S$ value is treated as the best division:

$$
\begin{equation*}
j^{*}=\arg \min _{j} S_{j}, \quad j=1,2, \ldots m \tag{14}
\end{equation*}
$$

where $m$ is number of times that random division is made and $n$ is the number of training samples. And in this paper, we set $m=(k * n) / 10$. The $i$ th region denotes $x \in T_{i}$.

Step 2 (setting parameters for each region). Once the best division $j^{*}$ is obtained, the $C P$ values for all areas $C P_{j^{*}}=$ $\left\{C P_{j^{*} 1}, C P_{j^{*} 2}, \ldots, C P_{j^{*} k}\right\}$ can also be obtained. In flexible


Figure 6: F-SVR modeling for complicated distributions.


Figure 7: Scheme of F-SVR-based fault isolation.
support vector regression approach, the following empirical formulas are given to set the hyperparameter $\alpha_{i}=\left\{\varepsilon_{i}, C_{i}, \sigma_{i}\right\}$ :

$$
\alpha_{i}:\left\{\begin{align*}
\varepsilon_{i} & =\frac{0.5 * \operatorname{std}^{2}\left(X_{i}\right)}{C P_{j^{*} i}}  \tag{15}\\
C_{i} & =\frac{1000 C P_{j^{*} i}}{\sum_{i=1}^{k} C P_{j^{*} i}} \\
\sigma_{i} & =\frac{5}{C P_{j^{*} i}},
\end{align*}\right.
$$

where $X_{i}=\{x \backslash x \in$ the ith region $\}$.
Remark 1. The empirical setting of parameters for each region is referred to in Cherkassky's work [10] in 2004.

Step 3 (function approximation using selected support vectors). We use the conventional SVR with parameters $\alpha_{i}$ to extract informative samples for the $i$ th region. As shown in Figure 6(a), the red samples are selected as support vectors (SVs). If $m$ samples are selected as SVs for the training set $\left(x_{i}, y_{i}\right)_{i=1}^{n}$, the regression problem $f(x)=\sum_{i=1}^{n} \beta_{i} K\left(x, x_{i}\right)+b_{0}$
can be approximated by the regression problem of the SVs [8]:

$$
\begin{equation*}
f(x)=\sum_{i=1}^{m} \gamma_{i} K\left(x, \mathrm{SV}_{i}\right)+b_{0}^{\prime} \tag{16}
\end{equation*}
$$

Thus, the minimization of (3) can be simplified as

$$
\begin{equation*}
w^{*}=\underset{w}{\arg \min }: \int_{T_{o}}^{T_{k}} L(y, f(S V, \alpha, w)) p(S V, y) d_{S V} \cdot d y \tag{17}
\end{equation*}
$$

In (17), rigorous parameters $\alpha=\{0,1000,0.01\}$ are set to ensure the learning ability of regression. This problem can be solved by using the Lagrangian method in the same way that is used in conventional SVR (Figure 5).

As is shown in Figure 6(b), F-SVR (unlike conventional SVR) successfully approximates the distribution function without overlearning or underlearning. As we mentioned earlier, F-SVR is free from the issues that can arise in the manual selection of parameters as all of the parameters are generated automatically.


Figure 8: Parameter settings for LS-SVR.


Figure 9: Comparison of learning ability.
3.3. The Basic Scheme of F-SVR-Based Fault Isolation. Fault isolation algorithms attempt to reveal which fault is occurring in the operating system. In our method, we determine which model of the system is most likely to be accurate by initially constructing the modes for all faulty statuses and then calculating the deviations between the real outputs and all of the model outputs. As is shown in Figure 7, once the best model has been located, the fault type can then be isolated.

Compared with conventional SVR-based fault isolation, the most significant contribution of our work is that we have alleviated the problem of parameter setting. What is required for F-SVR-based fault isolation are only process samples for different operating cases.

The detailed process of F-SVR-based fault isolation is as follows. Given $\left(x_{i}, y_{i}\right)_{i=1}^{k}, x_{i}, y_{i} \in R^{n}$ as the training samples from $k$ different operating statuses concluding the normal
status and all faulty statuses, where $\left(x_{i}, y_{i}\right)$ represents the training samples for the $i$ th status, the function $G_{i}(\cdot)$ between input vector $x_{i}$ and output vector $y_{i}$ is initially approximated by the F-SVR method. After the training samples $(x, y)$ construct the input/output models for all of the statuses, they are sent to these models to generate the residuals between the real outputs and the model outputs, thereby forming the residual vector:

$$
\begin{align*}
\operatorname{Residual}_{i} & =y-G_{i}(x), \quad i=1, \ldots, k, \\
r & =\left[\begin{array}{c}
\text { Residual_1 } \\
\vdots \\
\text { Residual_ } k
\end{array}\right]_{k \times n} . \tag{18}
\end{align*}
$$

We define a function $R(i)$ to measure the deviation between the real output and the model output of the $i$ th status

$$
\begin{equation*}
R(i)=\sum\left|y-G_{i}(x)\right| . \tag{19}
\end{equation*}
$$

Faults can be isolated by analysis of the residual vector. In this paper, we simply regard the testing samples belonging to the $i^{*}$ th status:

$$
\begin{equation*}
i^{*}=\underset{i}{\arg \min R(i), \quad i=1, \ldots, k . . . . . . .} \tag{20}
\end{equation*}
$$

## 4. Experiments and Real Applications

4.1. Numerical Experiments. In our first attempt to validate our method by a numerical experiment, we used a data set with complicated distributions. White noise ( $\mathrm{SNR}=30 \mathrm{db}$ ) is added to the analytical equation shown in (21), where the training set consists of 600 samples extracted from $x \in$ $(-1,1]$. In order to approximate the distribution, both F-SVR and least-square SVR (LS-SVR) [11] are implemented:

$$
\begin{gather*}
y 1=(x-0.5)^{2}+4 \sin \left(3 x^{2}\right)+x \\
y 2=4 x(1-x)(2 \sin (30 x+24)+3) \\
t=\frac{(1-x)}{2},  \tag{21}\\
y=y 1(1-t)+y 2 * t ; \quad x \in(-1,1] .
\end{gather*}
$$

To provide a fair comparison, the parameters for LSSVR were optimized first using a grid search strategy. The evaluating index for LS-SVR is

$$
\begin{equation*}
F=\frac{d}{N} \cdot \sqrt{\frac{\sum_{i=1}^{N}\left(\hat{y}_{i}-y_{i}\right)^{2}}{N-1}}, \tag{22}
\end{equation*}
$$

where $d$ is the number of SVs and $\hat{y}$ is the function output of LS-SVR. A small $F$ value implies that the regression produces generalization and accurate learning. As is shown in Figure 8, the best parameter set for LS-SVR is $\left\{\gamma=10, \sigma^{2}=0.2\right\}$.

The performance of the two methods is shown in Figure 9. It is clear in this case that F-SVR is a more accurate method for modeling data compared to the well-known LS-SVR.


Figure 10: Process data for different operating cases of HFPS.


Figure 11: The testing samples.
4.2. Fault Isolation for High Frequency Power Supply. The high frequency power supply (HFPS) is a nonlinear device that has been widely used in power plants for dedusting purpose. As its structure is highly complicated, its precise model is usually hard to be constructed using classical analytical approaches [12-18]. Thus, data-driven approaches are utilized for process modeling and fault isolation. However, when applied to different power plants, the HFPS yields very different input/output functions due to the change of its loads and working environments. This means that there is no general process model for HFPS in all conditions.

For each HFPS that has been installed, a particular process model should be constructed. Therefore, for conventional SVR-based fault isolation, experienced operators are required to select the modeling parameters at site.

In this section, F-SVR was applied in order to isolate the fault of the high frequency power supply. Three operating cases were investigated: normal status, overcurrent fault, and learning excitation fault. Data for the 3 operating cases of HFPS was prepared in Figure 10 and an overcurrent fault sample was used as the input data for testing in Figure 11.

The basic scheme for HFPS fault isolation is designed in Figure 12. Firstly, process data for each operating case is acquired; then, F-SVR is implemented to approximate the unknown function between input (time) and output (the first primary current) for each operating case. As the models for all operating cases have been established, the residual vectors can be generated and then by finding the model with the smallest residual the fault can be isolated.

In this experiment, F-SVR is implemented to approximate the functions between the input and the output, namely, $G_{1}(x), G_{2}(x)$, and $G_{3}(x)$. As shown in Figure 13, the functions we obtained using F-SVR accurately describe the relationship between the input and the output of the unknown functions for the different operating cases without setting parameters.

Since $G_{1}(x), G_{2}(x)$, and $G_{3}(x)$ were already obtained by the F-SVR method, the residuals could be generated


Figure 12: Fault isolation scheme for HFPS.


Figure 13: Function approximation using F-SVR and LS-SVR.
using (12). The corresponding residuals are recorded in Table 1. According to Table 1, the testing sample belongs to the overcurrent fault ( $G_{2}(x)$ yields the smallest residual). Based on prior knowledge of the testing sample, the diagnostic result is consistent and shows the feasibility of our method.

The LS-SVR method [11, 19] is also implemented to give a comparison. As is shown in Table 1, the LS-SVR method
also makes a correct diagnosis and it has a better ability to generalize than our method (the number of SVs is smaller). However, our method yields a smaller value of residuals. This implies that our method produces a better modeling accuracy. Most importantly, all parameters are required to be selected manually in LS-SVR but are selected automatically in F-SVR.

TAble 1: Diagnostic performance of F-SVR and LS-SVR.

| Operating status | Residuals $\left(R_{i}\right)$ |  | Number of SVs $(d)$ |  | Parameters setting |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
|  | F-SVR | LS-SVR | F-SVR | LS-SVR | F-SVR | LS-SVR |
| Normal $\left(G_{1}\right)$ | 15.912 | 16.231 | 26 | 23 | Auto | Manual |
| Overcurrent $\left(G_{2}\right)$ | $\mathbf{1 1 . 2 9 6}$ | $\mathbf{1 4 . 1 3 1}$ | 38 | 28 | Auto | Manual |
| Learning excitation $\left(G_{3}\right)$ | 17.104 | 17.932 | 23 | 20 | Auto | Manual |

## 5. Conclusions

SVR is one of the most efficient tools for fault diagnosis because it is able to accurately model a function between the input and the output using process data. However, even though the SVR approach has been utilized for over a decade, there is still no consensus within the community on how to adequately select regression parameters. Given that the F-SVR method offers an automatic selection for regression parameters, we chose it to implement the fault isolation for nonlinear systems. We demonstrated on both a numerical experiment and the fault isolation for HFPS that F-SVR is especially suited for cases that yield complicated sample distributions. This is because this method generates reasonable parameters for each region by dividing the training samples into different regions according to the sample distribution complexity. Based on this work, we hope that F-SVR will become more widely recognized as a preferred fault isolation for nonlinear systems.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Hierarchical MPC Secondary Control for Electric Power System 

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#### Abstract

Although in electric power systems (EPS) the regulatory level guarantees a bounded error between the reference and the corresponding system variables, to keep its availability in time, optimizing the system operation is required for operational reasons such as, economic and/or environmental. In order to do this, there are the following alternative solutions: first, replacing the regulatory system with an optimized control system or simply adding an optimized supervisory level, without modifying the regulatory level. However, due to the high cost associated with the modification of regulatory controllers, the industrial sector accepts more easily the second alternative. In addition, a hierarchical supervisory control system improves the regulatory level through a new optimal signal support, without any direct intervention in the already installed regulatory control system. This work presents a secondary frequency control scheme in an electric power system, through a hierarchical model predictive control (MPC). The regulatory level, corresponding to traditional primary and secondary control, will be maintained. An optimal additive signal is included, which is generated from a MPC algorithm, in order to optimize the behavior of the traditional secondary control system.


## 1. Introduction

The growing complexity of the power systems, due to the increased interconnection, the use of new technologies, and the need for operating the system based on economic indicators, has motivated the creation of some tools that enable the system to operate with a high degree of security and very close to the limits of stability conditions. The use of advanced control techniques has been an effective way of extending the limits of stability and improving the operation of EPS [1,2]. Model predictive control is one of such advanced control techniques and is presented below.
1.1. Model Predictive Control. Model predictive control is a control technique defined as a collection of past and present information to predict the future behavior of a system, through the explicit use of a process model. The generation of the control signal is made through the minimization of an objective function. Essentially, this control technique is based on the use of a finite slide horizon control, which involves the calculation of the control sequence for the whole horizon, but only the first control signal in the sequence is applied to the
plant, and the process is repeated in the next sampling instant [3].

MPC is one of the few control techniques that allow the incorporation of variable constraints in the formulation. Furthermore, this control strategy is valid for a wide range of systems, linear and nonlinear, and has had a significant impact on the industry.

Guaranteeing the stability of the closed loop system is an essential aspect in the design of a controller. The stability analysis in the MPC is an aspect that has been evolving and now is considered as a mature field. The stability of the MPC controller is guaranteed due to the establishment of conditions (valid for most of the systems) [4]. These conditions are based on a formulation of the controller that includes cost terminal as well as terminal restrictions.

Determination of a more suitable hierarchical structure to control a power system is a very important task. The use of a single control system to operate the entire plant, or dividing the plant in a set of subsystems, and make the individual control of each one may represent a significant reduction in electricity costs if the best option is selected. In this paper we present the control techniques that gave better results in the problem.
1.2. Applications of MPC in EPS. One of the most important applications of MPC in EPS is the voltage control, which can be defined as operational activities to keep the voltage within a permissible range into a specific sector, providing appropriate reactive power flow through the transmission system, at a particular time, with the objective of maximizing the active power flow [2]. Some recent MPC algorithm applications to control voltage can be cited, such as [5], where a predictive controller is implemented with a mixed logical dynamical model to control the voltage in a 12 bar network. Discrete control actions are load shedding, control transformer taps, and capacitor connection. A similar work is made in [6, 7], where a MPC to control the voltage of a 10 bar system is presented. The implementation includes a terminal region, which assures the stability of the controller, and the solution is obtained solving a linear programming problem. In [8], a MPC approach is used to prevent voltage instability in the long term. The MPC algorithm is based on a linearized steady state system model, derived from power flow equations. Simulations have been made in an EPS Nordic32 system. In [6], a method for optimal coordination of load shedding, capacitor switching, and taps changer is presented, in order to preserve the voltage stability of long-term. A dynamic model is used and a search tree is used as a method of solution for the MPC algorithm.

Another type of application of MPC in EPS is the control of power oscillations, which are variations on the three-phase power, due to the advance and retreat of the relative angles of voltage between generators, due to changes in the magnitude of the loads, faults, and other disturbances in the system [2]. We can cite works as [9], where a new control of generator excitation is proposed, to assure the stability of an EPS with multiple generators. The MPC algorithm is implemented using a DSP to achieve stability in real time with adequate speed. The simulations show that the interarea oscillation arising after a large disturbance in a connection line of two areas can be damped quickly. Besides, simulation shows that the stability of the MPC for multiple generators has a similar quality compared with the optimum excitation control using an automatic voltage regulator (AVR) of high gain in addition to a finely tuned power system stabilizer (PSS).

The harmonic control is an important issue in EPS too, due to the fact that it is considered a main indicator of the service quality. In [10], author argues that the increasing nonlinear loads generate problems due to the effects of the harmonic components of currents and voltages in an EPS, for example, cables overheating transformers and motors, excessive currents in the neutral resonance phenomena between the circuit elements [11, 12], and considering the capacitor banks for power factor correction. So, in general, the quality of the power supply is deteriorated by this distortion in voltages and currents [13-15]. Regarding the applications of MPC in harmonic control, we can mention the work in [16], which presents a modulation method based on the MPC and the sliding Fourier transform, including a low switching frequency and less distortion in lower order harmonics. The results are similar to the algorithm of selective harmonic mitigation [17]. However, the proposed technique is computed online and presents an improvement in the
dynamic performances. The method can be applied to any converter topology with any number of levels in a simple manner. Results show that a large modulation index can be used to achieve excellent performance, even in the range of overmodulation.

Frequency control is defined as the ability of an EPS to keep the frequency constant after a disturbance, whose origin is a significant imbalance between generation and load. The literature related to the implementation of the MPC algorithm to control frequency of an EPS includes the area called automatic control of large scale systems. We can cite a decentralized frequency control presented in [18], using the MPC algorithm in a multiarea power. The MPC technique is designed such that the effect of the uncertainty due to variation of the parameters of the governor and turbine is reduced. Each local area controller is independently designed, so that the overall stability of the closed loop system is guaranteed. Model frequency response of EPS multizone is introduced, and the physical limitations of the governors and turbines have been considered. Note that in this decentralized controller the stability of the entire system is not guaranteed.

In [19] a MPC distributed control addressed the problem of cascading failures, which cause blackouts with high costs. A cascading failure can be thought of as an alternating sequence of equipment failures and violations of the dynamic constraints of the power system. The designed controller is described as a network of autonomous agents with quick response to reduce these sequences. Agents work in the elimination of violations before they can cause more interruptions. They make their decisions with a distributed MPC technique. Each agent has a set of models, specialized in its location in the network. The agent uses these models to predict what the other agents will do and how the network will respond. Then, each agent optimizes its decisions with respect to these predictions. A comparison study on the basic data-driven methods for process monitoring and fault diagnosis is presented in [20].

One remarkable work is the one in [21], which is the first establishing the distributed MPC systems stability. Based on noncooperative targets, this paper proposes a set of MPC subcontrollers, which transmits the information of the current state-entry trajectory to whole neighbors (MPC subcontrollers) with which it is interconnected and then competing subcontrollers have no knowledge of the other cost functions. From the game theory perspective, the balance of this strategy, if it exists, is called a noncooperative equilibrium or Nash equilibrium. The control objectives for each MPC subcontrol are often in conflict with the objectives the other MPC subcontrols have; that is, Nash equilibrium is usually suboptimal Pareto [22]. In addition, recent results on large scale system have been developed by [23].

Therefore, it is necessary to modify the objective function of each subcontroller, in order to provide a cooperation between them, which is achieved replacing in each subcontrol the noncooperative objective by one that measures the impact of control actions in the whole system. In this case, a convex combination of the objectives of the individual subsystems is used. With this modification, the best performance achievable by controllers is characterized by a Pareto optimal path,


Figure 1: Primary control block diagram with governor, prime motor, rotating mass, and load.
which represents the optimal trade-off set between their goals and the goals of other systems with which there is conflict. Then, it can be proved that the mediator iterations generated by these MPC algorithms, based on cooperation, are feasible and the control law state feedback MPC algorithm based on these mediating iterations is asymptotically stable. As an application the authors achieve a secondary control of four areas in an EPS.

In [24], a comparison is made between a centralized, decentralized, and distributed MPC for an EPS. An important work has been developed by [25], where a distributed stable nonlinear dynamic control system is proposed, based on a set of MPC controllers that share only updated information of its neighbors (without predictions of the behavior of neighbors). Asymptotic stability in an equilibrium point (origin) in the distributed MPC controller is achieved with the use of socalled structured Lyapunov function control, applied to the respective MPC subcontrollers. Authors present an application in secondary control to a CIGRE EPS of seven machines, using a distributed predictive control. However, all the machines were used for this purpose, which is not reasonable due to economic and technical reasons.

In relation to renewable energy, there are two interesting works in [26], where an economic dispatch with intermittent sources is presented and describes an objective function that penalizes performance indices related to generation costs and environmental costs. The flexibility of MPC algorithm allows the use of constraints that limit the speed ramp for entry into service. The prediction, through a solar power plant model and turbine units model, allows dispatching units with slower speed ramp, with the respective economical savings (the generating units with faster speed ramp generally has higher operating costs). The work in [27] also includes an analysis of frequency stability of an EPS with intermittent sources (generic name that includes renewable sources).

## 2. Classic Power System Control

In this section we present classic power system control, which includes primary and secondary frequency control.
2.1. Primary Frequency Control. Primary control corresponds to an integral control action of each unit, due to its speed
governors. This allows limiting frequency deviations from disturbance in the generation/load balance, in a few seconds of time response. However, the resulting frequency is not necessarily the nominal frequency, a result that is achieved with the secondary control, which is presented in Section 2.2. The block diagram of the governor, prime motor, rotating mass, and load is shown in Figure 1, where
$\Delta \omega=\omega_{r}-\omega_{0}$ with $\omega_{r}$ : real angular speed, and $\omega_{0}:$ nominal angular speed;
$D$ : damping constant;
K: constant PI control;
R: statism;
$M: 2 H$ with $H$ the inertia constant;
$\Delta Y, \Delta P_{m}, \Delta P_{L}, \Delta P_{\text {ref }}$, and $\Delta \omega$ are the variations in position of the valve, mechanical power, load power, power reference, and frequency, respectively;
$T_{C H}, T_{G}=(1 / K R)$ are the time constants of the turbine and governor, respectively.

Considering a set of $n$-machines interconnected by transmission lines [25], the equations that represent the primary frequency control for the $i$ th machine are

$$
\begin{gather*}
\Delta \dot{\delta}_{i}=\Delta \omega_{i}  \tag{1}\\
\Delta \dot{\omega}_{i}=\frac{1}{M_{i}}\left(\Delta P_{m_{i}}-D_{i} \Delta \omega_{i}-\Delta P_{L_{i}}-\sum_{\left\{j \mid\left(\varsigma_{i}, \varsigma_{j}\right) \in \overline{\}}\right\}} \Delta P_{\text {tie }_{i, j}}\right),  \tag{2}\\
\Delta \dot{P}_{m_{i}}=\frac{1}{T_{C H}}\left(\Delta Y_{i}-\Delta P_{m_{i}}\right)  \tag{3}\\
\Delta \dot{Y}_{i}=\frac{1}{T_{G_{i}}}\left(\Delta P_{\mathrm{ref}_{i}}-\Delta Y_{i}-\frac{1}{R_{i}} \Delta \omega_{i}\right) \text { con } T_{G_{i}}=\frac{1}{K_{i} R_{i}} . \tag{4}
\end{gather*}
$$

We assume that $P_{\text {tie }_{i, j}}$ is the transmitted power by the transmission line $i j$, where $\left(\varsigma_{i}, \varsigma_{j}\right)$ represents the transmission line between node $\varsigma_{i}$ and node $\varsigma_{j}$, belonging to the set of arcs $\bar{\varepsilon}$ and coupling (5) relating the angles of a load $\delta_{j}, j \in\{j \mid$
$\left.\varsigma_{j} \in S_{\text {Load }}\right\}$, with the angles and frequencies of the generators $\delta_{i}, \omega_{i}, i \in\left\{i \mid \varsigma_{i} \in S_{\text {Generator }}\right\}$. Consider

$$
\begin{align*}
& {\left[\begin{array}{c}
M_{1} \Delta \dot{\omega}_{1} \\
\vdots \\
M_{N g} \Delta \dot{\omega}_{N g} \\
\hline 0 \\
\vdots \\
0
\end{array}\right]} \\
& =\left[\begin{array}{c}
\Delta P_{m_{1}}-D_{1} \Delta \omega_{1}-\Delta P_{L_{1}} \\
\vdots \\
\frac{\Delta P_{m_{N g}}-D_{N g} \Delta \omega_{N g}-\Delta P_{L_{N g}}}{-\Delta P_{L_{N g+1}}} \\
\vdots \\
-\Delta P_{L_{\text {nod }}}
\end{array}\right]  \tag{5}\\
& -\left[\begin{array}{l|l}
B_{11} & B_{12} \\
\hline B_{21} & B_{22}
\end{array}\right]\left[\begin{array}{c}
\delta_{1} \\
\vdots \\
\delta_{\mathrm{Ng}} \\
\hline \delta_{\mathrm{Ng+1}} \\
\vdots \\
\delta_{\mathrm{nod}}
\end{array}\right],
\end{align*}
$$

where the admittance matrix

$$
B=\left[\begin{array}{ll}
B_{11} & B_{12}  \tag{6}\\
B_{21} & B_{22}
\end{array}\right]
$$

is suitably divided into submatrices $B_{11} \in M_{N g \times N g}$, $B_{12} \in M_{N g \times(\operatorname{nod}-N g)}, B_{21} \in M_{(\operatorname{nod}-N g) \times N g}$, and $B_{22} \in$ $M_{(\operatorname{nod}-\mathrm{Ng}) \times(\operatorname{nod}-\mathrm{Ng})}$, with Ng generators and nod nodes.

Eliminating the angles that represent the bar that does not have generators $\delta_{N g+1}, \ldots, \delta_{\text {nod }}$ [25],

$$
\begin{align*}
{\left[\begin{array}{c}
M_{1} \Delta \dot{\omega}_{1} \\
\vdots \\
M_{N g} \Delta \dot{\omega}_{N g}
\end{array}\right]=} & {\left[\begin{array}{c}
\Delta P_{m_{1}}-D_{1} \Delta \omega_{1} \\
\vdots \\
\Delta P_{m_{N g}}-D_{N g} \Delta \omega_{N g}
\end{array}\right] }  \tag{7}\\
& -\Gamma\left[\begin{array}{c}
\delta_{1} \\
\vdots \\
\delta_{N g}
\end{array}\right]+\Upsilon\left[\begin{array}{c}
\Delta P_{L_{1}} \\
\vdots \\
\Delta P_{L_{\text {nod }}}
\end{array}\right]
\end{align*}
$$

where

$$
\begin{align*}
\Gamma & :=\left(B_{11}-B_{11} B_{22}^{-1} B_{21}\right),  \tag{8}\\
\Upsilon & :=\left[\begin{array}{ll}
-I_{N g} & B_{12} B_{22}^{-1}
\end{array}\right] .
\end{align*}
$$

Then, to represent the continuous dynamic model of EPS with Ng generators and nod nodes we define the following matrices:

|  | ... 0 |  | ... | 0 |
| :---: | :---: | :---: | :---: | :---: |
|  | $\ddots \quad \vdots$ |  | $\cdot$ | . |
| 0 | ... 0 |  | . . | 0 |
| $1 / M_{1}$ | 0 | 0 | - ... | 0 |
| $\vdots$ | $\ddots$ |  |  | $\vdots$ |
| 0 | $\cdots 1 / M_{N g}$ |  | ... | 0 |
| $-1 / T_{\mathrm{CH}_{1}}$ | $\cdots 0$ | $1 / T_{C H_{1}}$ | $\ldots$ | 0 |
| : | $\ddots \quad \vdots$ | $\vdots$ |  | $\vdots$ |
| 0 | $\cdots-1 / T_{C H_{\text {Ng }}}$ | 0 |  | $1 / T_{C H_{N g}}$ |
| 0 | ... 0 | $-1 / T_{G_{1}}$ | $\cdots$ | 0 |
|  | $\ddots \quad \vdots$ | $\vdots$ | $\ddots$ | : |
|  | ) $\cdots 0$ | 0 |  | $-1 / T_{G_{\text {N } g}}$ |

where $\left[\Gamma / M_{i}\right] \equiv\left[\begin{array}{ccc}\Gamma_{11} / M_{1} & \cdots & \Gamma_{1 n} / M_{1} \\ \vdots & & \vdots \\ \Gamma_{n 1} M_{n} & \cdots & \Gamma_{n n} / M_{n}\end{array}\right]$, and
with $\left[\Upsilon / M_{i}\right] \equiv\left[\begin{array}{ccc}\Upsilon_{11} / M_{1} & \cdots & \Upsilon_{1 n} / M_{1} \\ \vdots & & \vdots \\ \Upsilon_{n 1} / M_{n} & \cdots & \Upsilon_{n n} / M_{n}\end{array}\right]$.
Then, the continuous dynamic model can be represented as

$$
\begin{equation*}
\dot{x}=A_{c} x+B_{\mathrm{aux}} u, \tag{11}
\end{equation*}
$$

with

$$
\begin{gather*}
x= \\
{\left[\begin{array}{llllllllllll}
\Delta \delta_{1} & \cdots & \Delta \delta_{N g} & \Delta \omega_{1} & \cdots & \Delta \omega_{N g} & \Delta P_{M_{1}} & \cdots & \Delta P_{M_{N g}} & \Delta Y_{1} & \cdots & \Delta Y_{N g}
\end{array}\right]^{\prime},} \\
u=\left[\begin{array}{lllllll}
\Delta P_{\mathrm{ref}_{1}} & \cdots & \Delta P_{\mathrm{ref}_{N g}} & \Delta P_{L_{1}} & \cdots & \Delta P_{L_{\mathrm{nod}}}
\end{array}\right]^{\prime} . \tag{12}
\end{gather*}
$$

2.2. Secondary Frequency Control. The role of secondary frequency control is to keep or restore the system frequency to its nominal value, the balance generation/load within a control area, and scheduled power exchanges with neighboring areas of control. This control is addressed modifying the set-point of active power units assigned to secondary control, which belong to the control area where the imbalance occurs.

The secondary frequency control can be manually accomplished by instructions of the plant operators or automatically by the automatic generation control. Unlike the primary control, its action is slow and coordinated, taking into account characteristics of the units, such as its speed of response (however for the current job this will not be considered). This control covers a time from the end of the action of the primary control to several minutes and should not interfere with the action of the secondary control.

Now, area error control (ACE) value necessary for the secondary frequency power control is developed. In Figure 2, the connection between neighbors areas $i$ and $i+1$ and also incoming and outgoing power flow are shown. Sets $\Omega_{i}$ and $\Psi_{i}$ are defined as follows:
$\Omega_{i}:$ set of neighbors areas to $i$ which inject power;
$\Psi_{i}:$ set of neighbors areas to $i$ which demand power.

For each area, the steady state dynamic equation depends on the variation of mechanical power, the incoming net flow,


Figure 2: Connection between areas.
and the outgoing net flow. Assuming that there is a load variation $\Delta P_{L_{i}}$ in the $i$ th area

$$
\begin{gather*}
\Delta P_{\text {mech }_{1}}+\sum_{j \in \Omega_{1}} \Delta P_{\text {tie }_{1, j}}-\sum_{k \in \Psi_{1}} \Delta P_{\text {tie }_{1, k}}=\Delta \omega D_{1}, \\
\vdots \\
\Delta P_{\text {mech }_{i}}+\sum_{j \in \Omega_{i}} \Delta P_{\text {tie }_{i, j}}-\sum_{k \in \Psi_{i}} \Delta P_{\text {tie }_{i, k}}-\Delta P_{L_{i}}=\Delta \omega D_{i}, \\
\vdots \\
\Delta P_{\text {mech }_{n}}+\sum_{j \in \Omega_{n}} \Delta P_{\text {tie }_{n, j}}-\sum_{k \in \Psi_{n}^{\prime}} \Delta P_{\text {tie }_{n, k}}=\Delta \omega D_{n} \tag{13}
\end{gather*}
$$

then for each area frequency power variation ratio is

$$
\begin{equation*}
\Delta P_{\operatorname{mech}_{i}}=-\frac{\Delta \omega_{i}}{R_{i}} \tag{14}
\end{equation*}
$$

where $R_{i}$ is equivalent statism of $i$ th area; also $\beta_{i}=\left(1 / R_{i}\right)+D_{i}$. Summing the equations in (13), then

$$
\begin{equation*}
\sum_{i=1}^{n} \Delta P_{\text {mech }_{i}}-\Delta P_{L_{i}}=\Delta \omega \sum_{i=1}^{n} D_{i} \tag{15}
\end{equation*}
$$

replacing $\Delta P_{\text {mech }_{i}}$ and $\beta_{i}=\left(1 / R_{i}\right)+D_{i}$, implies $-\Delta P_{L_{i}}=$ $\Delta \omega \sum_{i=1}^{n} \beta_{i}$. Then we come to the traditional result of the frequency variation for the complete system

$$
\begin{equation*}
\Delta \omega=-\frac{\Delta P_{L_{i}}}{\sum_{i=1}^{n} \beta_{i}} \tag{16}
\end{equation*}
$$

Now, using the power frequency variation ratio (14) in (13)

$$
\begin{gather*}
\sum_{j \in \Omega_{1}} \Delta P_{\mathrm{tie}_{1, j}}-\sum_{k \in \Psi_{1}} \Delta P_{\mathrm{tie}_{1, k}}=\Delta \omega \beta_{1} \\
\vdots  \tag{17}\\
\sum_{j \in \Omega_{i}} \Delta P_{\mathrm{ti}_{i, j}}-\sum_{k \in \Psi_{i}} \Delta P_{\mathrm{tie}_{i, k}}-\Delta P_{L_{i}}=\Delta \omega \beta_{i} \\
\vdots \\
\sum_{j \in \Omega_{n}} \Delta P_{\mathrm{ti}_{n, j}}-\sum_{k \in \Psi_{n}} \Delta P_{\mathrm{tie}_{n, k}}=\Delta \omega \beta_{n} .
\end{gather*}
$$

From equation of the $i$ th area, (16), we see that the power variation introduced in this area $-\Delta P_{L_{i}}=-\sum_{j \in \Omega_{i}} \Delta P_{\text {tie }_{i, j}}+$ $\sum_{k \in \Psi_{i}} \Delta P_{\text {tie }_{i, k}}+\Delta \omega \beta_{i}$, which provides a motivation to define the error area $\mathrm{AEC}_{i}$, should be fed back to the reference power of the secondary controller frequency of $i$ th area

$$
\begin{equation*}
\mathrm{ACE}_{i}=-\sum_{j \in \Omega_{i}} \Delta P_{\mathrm{ti}_{i, j}}+\sum_{k \in \Psi_{i}} \Delta P_{\mathrm{ti}_{i, k}}+\Delta \omega \beta_{i} . \tag{18}
\end{equation*}
$$

Now, as load variation $\Delta P_{L_{i}}$ occurs in the $i$ th area, summing all equations in (5) except the $i$ th, where load variation occurred $-\sum_{j \in \Omega_{i}} \Delta P_{\text {tie }_{i, j}}+\sum_{k \in \Psi_{i}} \Delta P_{\text {tie }_{i, k}}=\Delta \omega \sum_{k=1, k \neq i} \beta_{k}$, and using (16)

$$
\begin{equation*}
-\sum_{j \in \Omega_{i}} \Delta P_{\mathrm{tie}_{i, j}}+\sum_{k \in \Psi_{i}} \Delta P_{\mathrm{ti}_{i, k}}=-\Delta P_{L_{i}} \frac{\sum_{k=1, k \neq i}^{n} \beta_{k}}{\sum_{i=1}^{n} \beta_{i}} . \tag{19}
\end{equation*}
$$

Therefore in (6) using (7) and (4)

$$
\begin{align*}
\mathrm{ACE}_{i} & =-\Delta P_{L_{i}} \frac{\sum_{k=1, k \neq i}^{n} \beta_{k}}{\sum_{i=1}^{n} \beta_{i}}-\Delta P_{L_{i}} \frac{\beta_{i}}{\sum_{i=1}^{n} \beta_{i}} \\
& =-\Delta P_{L_{i}} \frac{\sum_{k=1}^{n} \beta_{k}}{\sum_{i=1}^{n} \beta_{i}}=-\Delta P_{L_{i}} . \tag{20}
\end{align*}
$$

Now, summing all equations in (17), except the equation of any area different from the $i$ th area where there was load variation $\Delta P_{L_{i}}$,

$$
\begin{equation*}
-\sum_{j \in \Omega_{i}} \Delta P_{\text {tie }_{i, j}}+\sum_{k \in \Psi_{i}} \Delta P_{\text {tie }_{i, k}}-\Delta P_{L_{i}}=\Delta \omega \sum_{\substack{k=1 \\ k \neq i}}^{n} \beta_{k} . \tag{21}
\end{equation*}
$$

Using (4)

$$
\begin{gather*}
-\sum_{j \in \Omega_{i}} \Delta P_{\text {tie }_{i, j}}+\sum_{k \in \Psi_{i}} \Delta P_{\text {tie }_{i, k}}+\Delta \omega \sum_{i=1}^{n} \beta_{i}=\Delta \omega \sum_{\substack{k=1 \\
k \neq i}}^{n} \beta_{k}  \tag{22}\\
-\sum_{j \in \Omega_{i}} \Delta P_{\text {tie }_{i, j}}+\sum_{k \in \Psi_{i}} \Delta P_{\mathrm{tie}_{i, k}}=-\Delta \omega \beta_{i} .
\end{gather*}
$$

Therefore

$$
\begin{equation*}
\mathrm{ACE}_{i}=-\Delta \omega \beta_{i}+\Delta \omega \beta_{i}=0 \tag{23}
\end{equation*}
$$

Summarizing,

$$
\begin{gather*}
\mathrm{ACE}_{1}=0 \\
\vdots \\
\mathrm{ACE}_{i}=-\Delta P_{i}  \tag{24}\\
\vdots \\
\mathrm{ACE}_{n}=0 .
\end{gather*}
$$

This shows that the selection of error area, as in (18), guarantees that the area where the power variation occurred provides the required power, and all this is in steady state.

Summarizing, given an interconnection of $N$ control areas, suppose a disturbance load $\Delta P_{i}$ appears in the $i$ th area. During the transient period, the dynamic phenomena involve generation of different frequencies in each area and deviations in the flow of power between them, calculating each area with its own area control error $\mathrm{ACE}_{i}$ (18). Then we defined the error in the net power exchanged $\Delta P_{\text {tie }_{i}}$ from all neighboring areas to the area $i$ :

$$
\begin{equation*}
\Delta P_{\mathrm{tie}_{i}}=-\sum_{j \in \Omega_{i}} \Delta P_{\text {tie }_{i, j}}+\sum_{k \in \Psi_{i}} \Delta P_{\text {tie }_{i, k}}, \tag{25}
\end{equation*}
$$

and then area control error can be written as

$$
\begin{equation*}
\mathrm{ACE}_{i}=\Delta P_{\mathrm{tie}_{i}}+\beta_{i} \Delta \omega \tag{26}
\end{equation*}
$$

Finally, the new reference $P_{\text {ref }_{i}}$ of generator $i$ is expressed by

$$
\begin{equation*}
P_{\mathrm{ref}_{i}}=-K_{i} \int_{0}^{\tau} \mathrm{ACE}_{i} d t \tag{27}
\end{equation*}
$$

## 3. Design of Hierarchical MPC for EPS

In this section we present the design of a frequency hierarchical MPC for EPS, including restrictions and optimization problem.
3.1. Problem Statement. The control strategy proposed in this paper is based on a hierarchical supervisor level, which determines the optimal set-point for a given regulatory system. The supervisor level dynamically optimizes a general objective function including equality and inequality constraints. Then, the described problem can be solved analytically with the predictive control theory and can be solved by numerical algorithms when working with restrictions [3].

Figure 3 shows how the hierarchical supervisor level delivery set-point $r$ is based on the optimization of the objective function $J$, the trajectory of an external reference $W$, controlled variable $y$, and manipulated variable $u$. The process is influenced by a nonmeasurable disturbance $p$.


Figure 3: Supervisory control diagram.
3.2. Application of Secondary Control of EPS. The implementation of MPC control for the hierarchical secondary control of an EPS is presented in Figure 4. The figure presents an area, bounded by a dotted line, which represents the linear EPS model including a primary frequency control. Besides, in the traditional secondary control, we have included an additive power signal $P_{\mathrm{MPC}}$, which provides an optimum correction
to signal of ACE. The correction is optimal because, in the optimization MPC problem, the objective function strongly penalizes the frequency variation and includes restrictions, which represent plant model and traditional secondary controller model.
3.3. Restrictions of EPS Model for the Optimization Problem. The corresponding restrictions for building optimization problem for the MPC algorithm are presented below. The involved variables must behave according to the dynamic model of the EPS. Then discretizing the model obtained in (11) we obtain

$$
\begin{equation*}
A\left(T_{s}\right)=e^{A_{c} T_{s}}, \quad B_{\mathrm{aux}}\left(T_{s}\right)=\left(e^{A_{c} T_{s}}-I\right)(A c)^{-1} B_{\mathrm{aux}_{c}}, \tag{28}
\end{equation*}
$$

with

$$
\begin{gather*}
T_{s}=1[\mathrm{~s}], \\
B_{\mathrm{aux}}=\left[\begin{array}{ll}
B & B_{L}
\end{array}\right] \\
u(k)=\left[\begin{array}{llll}
\Delta P_{\mathrm{ref}_{1}}(k) & \cdots & \Delta P_{\mathrm{ref}_{\mathrm{Ng}}}(k)
\end{array}\right]^{\prime}  \tag{29}\\
P_{L}(k)=\left[\begin{array}{llll}
\Delta P_{L_{1}}(k) & \cdots & \Delta P_{L_{\mathrm{nod}}}(k)
\end{array}\right]^{\prime} \\
x(k)=\left[\begin{array}{llllll}
\Delta \delta_{1}(k) & \cdots & \Delta \delta_{N g}(k) \Delta \omega_{1}(k) & \cdots & \Delta \omega_{N g}(k) & \Delta P_{M_{1}}(k) \\
\cdots & \cdots & \Delta P_{M_{N g}}(k) & \Delta Y_{1}(k) & \cdots & \Delta Y_{N g}(k)
\end{array}\right]^{\prime} .
\end{gather*}
$$

Hence, we have the following discrete model:

$$
\begin{equation*}
x(k+1)=A x(k)+B u(k)+B_{L} P_{L}(k) \tag{30}
\end{equation*}
$$

The group of restrictions to $N$ steps of EPS model used for the corresponding MPC optimization problem is

$$
\begin{align*}
& -B_{L} P_{L}(k)-A x(k)-B u(k)=-x(k+1) \\
& -B_{L} P_{L}(k+1)=-x(k+2)+A x(k+1)+B u(k+1) \\
& \vdots  \tag{31}\\
& -B_{L} P_{L}(k+N-1) \\
& =-x(k+N)+A x(k+N-1)+B u(k+N-1),
\end{align*}
$$

where $P_{L}$ model is obtained from a lineal model.
3.4. Constraints That Relate Angles. It is necessary to include the relationship between the angles of the bars without generators $\delta_{n g}$ with the angles of the bars containing the generators $\delta_{g}$.

Given the admittance matrix $\left[\begin{array}{ll}B_{11} & B_{12} \\ B_{21} & B_{22}\end{array}\right]$,

$$
\begin{align*}
B_{11} \in M_{N g \times N g}, & B_{12} \in M_{N g \times(\operatorname{nod}-N g)}, \\
B_{21} \in M_{(\text {nod }-N g) \times N g}, & B_{22} \in M_{(\operatorname{nod}-N g) \times(\operatorname{nod}-N g)} . \tag{32}
\end{align*}
$$

The group of restrictions to $N$ steps that shows that relationship between the angles can be deduced from (5) as follows:

$$
\begin{gather*}
-I_{n g} P_{L n g}(k)-B_{21} \delta_{g}(k)=-B_{22} \delta_{n g}(k) \\
0=-I_{n g} P_{L n g}(k+1)-B_{21} \delta_{g}(k+1)-B_{22} \delta_{n g}(k+1) \\
\vdots \\
0=-I_{n g} P_{L n g}(k+N)-B_{21} \delta_{g}(k+N)-B_{22} \delta_{n g}(k+N), \tag{33}
\end{gather*}
$$

with

$$
\begin{gather*}
I_{n g} \in M_{(\mathrm{nod}-\mathrm{Ng}) \times(\mathrm{nod}-\mathrm{Ng})}, \\
P_{L n g}(k)=\left[\begin{array}{lll}
\Delta P_{L_{N g+1}}(k) & \cdots & \Delta P_{L_{\mathrm{nod}}}(k)
\end{array}\right]^{\prime},  \tag{34}\\
\delta_{g}(k+N)=\left[\begin{array}{lll}
\Delta \delta_{1}(k) & \cdots & \Delta \delta_{N g}(k)
\end{array}\right]^{\prime}, \\
\delta_{n g}(k+N)=\left[\begin{array}{lll}
\Delta \delta_{N g+1}(k) & \cdots & \Delta \delta_{\mathrm{nod}}(k)
\end{array}\right]^{\prime} .
\end{gather*}
$$

3.5. Secondary Control Equations. The constraints that represent the secondary frequency control can be deduced applying the Laplace transform to the equation of secondary


Figure 4: Hierarchical MPC control diagram.
control in (26) including an additive signal power $P_{\mathrm{MPC}_{i}}$ to area control error $\mathrm{ACE}_{i}$ :

$$
\begin{equation*}
P_{\mathrm{ref}_{i}}=-K_{i} \frac{\mathrm{ACE}_{i}+P_{\mathrm{MPC}_{i}}}{s}, \quad i=1, \ldots, G 2 \tag{35}
\end{equation*}
$$

with $G 2$ number generators with secondary control.
Using the Tustin triangular approximation, $(1 / s)=$ $\left(T_{s}(z+1)\right) /(2(z-1))$, where $T_{s}$ is the sampling time, we obtain

$$
\begin{align*}
& \frac{P_{\mathrm{ref}_{i}}(k)}{\mathrm{ACE}_{i}(k)+P_{\mathrm{MPC}_{i}}(k)} \\
&=-K_{i} \frac{T_{s}\left(1+z^{-1}\right)}{2\left(1-z^{-1}\right)} P_{\mathrm{ref}_{i}}(k+1)-P_{\mathrm{ref}_{i}}(k) \\
&=-\frac{T_{s} K_{i}}{2} \mathrm{ACE}_{i}(k+1)-\frac{T_{s} K_{i}}{2} P_{\mathrm{MPC}_{i}}(k+1) \\
&-\frac{T_{s} K_{i}}{2} \mathrm{ACE}_{i}(k)-\frac{T_{s} K_{i}}{2} P_{\mathrm{MPC}_{i}}(k) \tag{36}
\end{align*}
$$

Then, we can generate the corresponding secondary control restrictions for $N$ steps. Finally the complete optimization problem is presented.
3.6. Formulation of MPC Optimization. The formulation of the optimization problem for the frequency MPC of the EPS is as follows.

Given in (30), the discrete model of the EPS is

$$
\begin{equation*}
x(k+1)=A x(k)+B u(k)+B_{L} P_{L}(k) . \tag{37}
\end{equation*}
$$

The frequency control of the EPS by the MPC algorithm involves solving optimization problem (38) to find the optimal set $\left\{u^{*}(k), \ldots, u^{*}\left(k+N_{p}-1\right)\right\}$ of control actions to $N_{p}$ steps and apply as control action the single signal $u^{*}(k)$ :

$$
\begin{equation*}
\min _{\left\{u(k), \ldots, u\left(k+N_{p}-1\right)\right\}} J=X(k)^{\prime} Q X(k)+U(k)^{\prime} R U(k), \tag{38}
\end{equation*}
$$

subject to the following:
constraints EPS model, (18),
constraints of relationship between $\delta_{n g}$ with $\delta_{g}$, (19),
constraints of secondary control, (22),
constraints of variables,
where
$Q \in M_{4 N_{p} N_{g} \times 4 N_{p} N_{g}}$, weights diagonal matrix;
$R \in M_{N_{p} N_{g} \times N_{p} N_{g}}$, weights diagonal matrix.

$$
\begin{align*}
& X(k)=\left[x(k)^{\prime} \cdots x\left(k+N_{p}\right)^{\prime}\right] \in \mathbb{R}^{4 N_{p} N_{g}}, \\
& U(k)=\left[\begin{array}{lll}
u(k)^{\prime} & \cdots & u\left(k+N_{p}\right)^{\prime}
\end{array}\right] \in \mathbb{R}^{N_{p} N_{g}}, \\
& x(k)=\left[\begin{array}{lllllll}
\Delta \delta_{1}(k) & \cdots & \Delta \delta_{N g}(k) & \Delta \omega_{1}(k) & \cdots & \Delta \omega_{N g}(k) \Delta P_{M_{1}}(k) & \cdots
\end{array} \Delta P_{M_{N g}}(k) \Delta Y_{1}(k) \cdots \Delta \Delta Y_{N g}(k)\right]^{\prime},  \tag{39}\\
& u(k)=\left[\begin{array}{llll}
P_{\mathrm{MPC}_{1}}(k) & \cdots & P_{\mathrm{MPC}_{\mathrm{G} 2}}(k)
\end{array}\right]^{\prime}, \\
& u^{*}(k)=\left[\begin{array}{llll}
P_{\mathrm{MPC}_{1}} & *(k) & \cdots & P_{\mathrm{MPC}_{\mathrm{G} 2}} \\
& \\
& (k)
\end{array}\right]^{\prime} .
\end{align*}
$$



Figure 5: EPS IEEE-39 diagram.

Table 1: Secondary control generator.

| Area | Generator |
| :--- | :---: |
| 1 | 10 |
| 2 | 3 |
| 3 | 4 |
| 4 | 8 |

## 4. Practical Application

As a practical application, a secondary MPC control to EPS IEEE-39 of 39 bars and 10 generators is developed. The system is divided into four interconnected areas. In each area, a generator is designated for the secondary control (see Table 1). The unifilar diagram of the EPS is shown in Figure 5. The characteristics of the test system, generators, lines, and transformers impedances are presented in [1].
4.1. Results. Because the objective of the MPC algorithm is a quadratic form function and constraints are linear equations, the solution of the optimization problem presented is obtained by quadratic programming. In particular, prediction was performed at 10 steps with computation time of 0.75 [s] per iteration (using the Matlab program Quadprog), time less than the sampling interval of $1[s]$, which makes feasible the real-time control.

Figure 6 shows the response of the secondary frequency MPC controller to an increased load of $10 \%$ on bar 18 in area 1 of EPS IEEE-39. It can be seen, in Figure 6(a), that the MPC controller, within a reasonable time of 50 [s], achieves the convergence to zero of the frequency variations of all the generators.

In Figure 6(b), it can be seen that the generators making the secondary control (generators $3,4,5$, and 11 ) are the ones that provide the required power (blue line) for satisfaction load variation, whereas the other generators only contribute in the initial stage. In Figures 6(c) and 6(d), the contribution


Figure 6: Response of the secondary frequency MPC controller.
of each generator separately can be seen, the ones that make secondary control (see Figure 6(c)) and the remaining generators (see Figure 6(d)). The greatest contribution of the power is given precisely by generator 3 , as can be seen in Figure 6(c), which made the secondary control in area 1 , where the load variation takes place.

Similar results (Figure 7) were achieved when simulating over 1500 [s], where the EPS was subjected to a series of power variations (Figure 7), not exceeding 15\%, in bar 18 in area 1.

Then $P_{L}$ model to (31) is obtained from a lineal model series of Figure 7.

Note that this work is in a framework of small signal analysis, implying linear models as an approximation of the system. However, by the necessity of using a predictive model of the system and for a more realistic treatment of the problem, we consider in future models nonlinear load variation, for example, Takagi and Sugeno or neural models; details of this type of model can be seen in [28-32].


Figure 7: Power variations series $\left(P_{L}\right)$.

In Figure 8(a) we can see how the frequency variation converges to zero for different power variations. Due to similarity of results between control techniques, classical and MPC secondary control, it is not possible to show differences from the graphic point of view. Then, these differences will be presented in data table (Tables 2 and 3).

Table 2 shows the values $\sum_{j=1}^{1500} \Delta \omega_{i}(j)^{2}$ for each generator $i$, using frequency variation $\Delta \omega_{i}(t)$ with $i=1, \ldots, 10$, in the case of applying the traditional secondary control and for the case of applying the secondary MPC control to load variation on bar 18 in area 1 of EPS IEEE-39.

In this case, the average value for the 10 generators using the traditional secondary control is $2.12 \%$ higher than the average for the corresponding MPC secondary controller, being a great improvement of the behavior of the system when using MPC control. Also we have load variation on bar 22 (area 3) and load variation on bar 29 (area 4); results are presented in Table 3.

Table 3 shows that there is an improvement in each area at least of $2 \%$ higher than the average for the corresponding MPC secondary controller. Demonstrating the effectiveness of the control method which is independent of the selected area (by definition EPS IEEE-39) does not consider loads in area 2.

Figure 9(a) shows the power reference for generators that perform secondary control. Figure 9(b) shows the additive power signal generated by the MPC, to support the secondary frequency control, which is small; however, its application achieves improved controller behavior. Figure 9(c) shows ACE signal, in response to varying load in area 1, also we see that the generator 10 (responsible for secondary control area 1) is the one that provides the most power between the four generators with secondary control. In addition, the expected convergence of (24) fails because the time intervals disposed for the load signal variation are small for this convergence.

Note that the works referenced in this paper [21, 25] perform applications in secondary MPC control using all the EPS machines, which is not reasonable due to economic and technical reasons, for example, higher cost of communication for all machines of system, absence of spinning reserve in
a specific generator, and energy cost generator incompatible with the economic dispatch as secondary frequency controller. In this paper, however, the MPC is applied only in the machines which have secondary control in the area.

## 5. Conclusions

A frequency hierarchical supervisory control system is presented for an EPS, which improves the regulatory level through a new optimal signal support, keeping fixed the entire regulatory process system. The frequency control is applied to an electric power system IEEE-39, where the regulatory levels corresponding to the primary and secondary control were maintained, and an additive signal generated from a MPC algorithm was added to the traditional secondary control system, in order to optimize their performance. The results show the feasibility of this solution. There is an improvement of the system performance when using MPC control over the use of traditional secondary control; that is, the average squared frequency variation for the traditional secondary control was at least $2 \%$ higher than in the case of using the MPC secondary controller.

Note that the works referenced in this paper perform applications in secondary MPC control using all the EPS machines, which is not reasonable due to economic and technical reasons. In this paper, however, the MPC is applied only in the machines which already have secondary control. In future works, we will consider other characteristics of the units, such as their response speed, which will add an economic component to the analysis.

Finally, considering the measurement of energy quality, European standard UNE-EN 50160 requires variations in voltage frequency less than or equal to $1 \%$ for 10 seconds in $95 \%$ of the week. Then designing a controller that achieves an improvement at least $2 \%$ over the control of traditional frequency variations is a promising result.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.


Figure 8: Secondary frequency MPC controller response.

Table 2: Accumulated squared frequency variation for secondary control.

| Secondary control | Generators with secondary control |  |  |  | Generators without secondary control |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $\begin{gathered} \text { Gen. } 10 \\ * 10^{-4} \end{gathered}$ | $\begin{aligned} & \text { Gen. } 3 \\ & * 10^{-4} \end{aligned}$ | $\begin{gathered} \text { Gen. } 4 \\ * 10^{-4} \end{gathered}$ | $\begin{gathered} \text { Gen. } 8 \\ * 10^{-4} \end{gathered}$ | $\begin{aligned} & \text { Gen. } 1 \\ & * 10^{-4} \end{aligned}$ | $\begin{gathered} \text { Gen. } 2 \\ * 10^{-4} \end{gathered}$ | $\begin{aligned} & \text { Gen. } 5 \\ & * 10^{-4} \end{aligned}$ | $\begin{aligned} & \text { Gen. } 6 \\ & * 10^{-4} \end{aligned}$ | $\begin{gathered} \text { Gen. } 7 \\ * 10^{-4} \end{gathered}$ | $\begin{gathered} \text { Gen. } 9 \\ * 10^{-4} \end{gathered}$ |
| Traditional | 0.905 | 0.597 | 0.648 | 0.605 | 0.705 | 0.417 | 0.315 | 0.553 | 0.788 | 0.970 |
| MPC | 0.886 | 0.583 | 0.604 | 0.591 | 0.700 | 0.414 | 0.314 | 0.551 | 0.786 | 0.939 |



Figure 9: Power reference, additive MPC power, and ACE for generators performing secondary control.

Table 3: Improvement secondary control.

| Place of load variation |  | Secondary control |  |  |
| :--- | :---: | :---: | :---: | :---: |
| Area | bar | Traditional mean | MPC mean | Improvement secondary MPC versus traditional \% |
|  |  | $* 10^{-4}$ | $* 10^{-4}$ |  |
| 1 | 18 | 6.503 | 6.368 | 2.12 |
| 3 | 22 | 7.053 | 6.908 | 2.06 |
| 4 | 29 | 7.872 | 7.707 | 2.10 |

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# Attitude Analysis and Robust Adaptive Backstepping Sliding Mode Control of Spacecrafts Orbiting Irregular Asteroids 

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#### Abstract

Attitude stability analysis and robust control algorithms for spacecrafts orbiting irregular asteroids are investigated in the presence of model uncertainties and external disturbances. Rigid spacecraft nonlinear attitude models are considered and detailed attitude stability analysis of spacecraft subjected to the gravity gradient torque in an irregular central gravity field is included in retrograde orbits and direct orbits using linearized system model. The robust adaptive backstepping sliding mode control laws are designed to make the attitude of the spacecrafts stabilized and responded accurately to the expectation in the presence of disturbances and parametric uncertainties. Numerical simulations are included to illustrate the spacecraft performance obtained using the proposed control laws.


## 1. Introduction

SMALL bodies including mainly asteroids and comets are studied by scientists because of the insight they can give into the history of the solar system. NASA missions are as follows: Galileo to Jupiter via asteroids Gaspra and Ida in 1989, Near Earth Asteroid Rendezvous (NEAR) Shoemaker to asteroid 433 Eros in 1996 [1, 2], and NASA Flyby Mission Deep Space 1 to asteroid Braille in 1998, Genesis-NASA Discovery Solar Wind Sample Return Mission in 2001. Hayabusa (Muses-C) is the Japan Aerospace Exploration Agency Sample Return Mission to Asteroid 25143 Itokawa [3, 4], and Rosetta is the ESA Comet Mission, flew by asteroids Steins and Lutetia [5].

While there is an increasing interest in such missions, the necessity and importance of orbital and attitude dynamics analyses of the small solar system bodies as the critical success factors of those missions are rising as well. The oblateness torque effects can be ignored for studying the attitude motion of spacecrafts around planetary bodies, while asteroids and comets usually have irregular shapes which lead to the complicated orbital and attitude dynamics in comparison with approximately spherical bodies such as the Earth. An asteroid's irregular shape, mass distribution, and the state of
its rotation (rapid or slow) have significant effects on the evolution of spacecraft orbit and attitude motion. Scheeres and his coworkers have made a large number of contributions to the study of orbital motion about asteroids [6-9]. These effects especially may deteriorate the attitude performance significantly, which lead to unstable attitude motion and thereby failure of the space mission. Wang and Xu find that the attitude stability domain is modified significantly due to the significantly nonspherical shape and rapid rotation of the asteroid, and attitude stability subjected to the disturbance of the gravity gradient torque is generalized to a rigid spacecraft on a stationary orbit around an asteroid [10, 11]. In order to solve this problem, it is important to understand the attitude motion of spacecrafts orbiting asteroids by deriving the stability conditions and thereupon develop effective control laws to neutralize the effects of asteroid shape and mass distributions. Riverin and Misra have proposed the attitude motion of the spacecraft depending heavily on the shape of the asteroid and the rotational state [12]. Then Misra and Panchenko have found the radius for which resonant pitch oscillations, considering the general three-dimensional attitude motion in 2006 [13]. Riverin and Misra have examined the spacecraft pitch motion assuming the spacecraft is in an equatorial orbit


Figure 1: Coordinate Frames.
but spacecraft attitude control algorithms have not been perfectly investigated. Kumar and Shah have set up the general formulation of the spacecraft equations of motion in an equatorial eccentric orbit using Lagrangian method and made some analysis about the stability. Then the control laws for three-axis attitude control of spacecrafts have been developed and a closed-form solution of the system has been derived in [14]. Mahmut et al. have designed Lyapunov-based nonlinear feedback laws to control the rotational and translational motion of the spacecraft for an asteroid orbiting spacecraft in [15]. However, in the above articles about orbiting attitude motion, external perturbations acting on the spacecraft are not taken into account and the control laws are not robust.

Backstepping is a systematic and recursive design methodology for nonlinear feedback control. The idea is to select recursively some appropriate functions of state variables as pseudocontrol inputs for lower dimension subsystems of the overall system. Each backstepping stage results in a new pseudocontrol design, expressed in terms of the pseudocontrol designs from preceding design stages. When the procedure terminates, a feedback design for the true control input is the result which achieves the original design objective by virtue of a final Lyapunov function, which is formed by summing up the Lyapunov functions associated with each individual design stage. Sliding mode control is a nonlinear robust control and applicable to solve the tracking of nonlinear system [16, 17]. The adaptive algorithm is adopted to estimate the external disturbances and uncertain parameters due to the highly complex environment in real time [18]. Moreover, owing to the robust control performance of adaptive backstepping control and sliding-mode control, many combined adaptive backstepping and sliding mode control schemes have appeared. Although good robust control strategies for uncertain nonlinear system and tracking problems have been
proposed in [19, 20], adaptive backstepping sliding mode control is also effective and easier for implementation in real time. In this paper, the performance of spacecrafts orbiting irregular asteroids with perturbations is overall analysed, and attitude motion is influenced seriously. Moreover, the robust adaptive backstepping sliding mode control laws are proposed to compensate the uncertainties and perturbations and make the attitude angles decay and reach the null state, which ensure orbiting motion and space mission.

This paper is organized as follows. In Section 2, gravity gradient torque of spacecraft orbiting irregular asteroids is derived and three-dimensional attitude motion equations of the rigid spacecraft are first examined considering the perturbations, which is followed by deriving the linearized system model. In Section 3, the stability analysis about the spacecraft is presented in retrograde orbits and direct orbits with the orbital radius. Then in Section 4, effective adaptive backstepping sliding mode control schemes for a spacecraft orbiting the asteroid Eros 433 are developed to stabilize the system. Computer simulations are carried out to illustrate the effectiveness of the control laws. Conclusions are presented in Section 5.

## 2. System Equations of Motion

2.1. Coordinate Frames. At first, the following Coordinate Frames are set up to make the problem clear, which are shown in Figure 1.
(1) Asteroid centered inertial frame $(\vec{I}, \vec{J}, \vec{K})$ : the origin of this frame is at the center of mass of the asteroid.
(2) Asteroid-fixed frame $(\vec{i}, \vec{j}, \vec{k})$ : the origin of this frame is at the center of mass of the asteroid, the vectors are aligned along the three centroidal principal axis of the smallest, the intermediate, and the largest moment of inertia, respectively. The asteroid rotational state relates the two frames, the unit vector $\vec{k}$ points in the same direction as $\vec{K}$. Asteroid-fixed frame $(\vec{i}, \vec{j}, \vec{k})$ is assumed to rotate with constant angular velocity $\vec{\Omega}=\Omega \cdot \vec{K}$.
(3) The spacecraft orbital frame $\left(\vec{o}_{1}, \vec{o}_{2}, \vec{o}_{3}\right)$ : the origin of this frame is at the center of mass of the spacecraft, $\vec{o}_{3}$ points towards the center of mass of the asteroid, $\vec{o}_{1}$ points towards the transverse direction in the orbital plane, and $\vec{o}_{2}$ satisfied the right hand rule. For equatorial orbits, the orbital frame is obtained from the inertial frame $(\vec{I}, \vec{J}, \vec{K})$ by a single rotation through an angle equal to the true anomaly $\eta$.
(4) The spacecraft-fixed frame $\left(\vec{b}_{1}, \vec{b}_{2}, \vec{b}_{3}\right)$ : aligned with the principal axes of the spacecraft, their orientation with respect to the orbital frame can be defined in terms of the attitude angles (roll, pitch, and yaw).

The sequence of rotations used here is yaw $(\lambda)$ around $\vec{b}_{1}$ axis, followed by pitch $(\theta)$ around $\vec{b}_{2}$ axis, and then followed by roll $(\gamma)$ around $\vec{b}_{3}$ axis.
2.2. Attitude Kinematics Model. The following assumptions are made in deriving the equations of motion firstly.
(1) The spacecraft is rigid.
(2) The gravitational attraction of the asteroid is the main disturbance force acting on the spacecraft, and the solar radiation and solar gravitation are considered perturbation force.
(3) The rotation rate of the asteroid is constant.
(4) The orbital motion of the spacecraft is not affected by attitude dynamics.
(5) Moment of inertias is affected by the irregular gravitational force of small bodies.
(6) Orbital motion of the spacecraft is fully described as a closed, planar, and periodic orbit.
(7) The asteroid is assumed to be a rotating triaxial ellipsoid.

In view of the first assumption, the attitude motion can be described by Euler's equations of motion for a rigid body:

$$
\begin{gather*}
I_{1} \dot{\omega}_{1}-\left(I_{2}-I_{3}\right) \omega_{2} \omega_{3}=M_{y}+M_{1}+M_{\Delta 1} \\
I_{2} \dot{\omega}_{2}-\left(I_{3}-I_{1}\right) \omega_{3} \omega_{1}=M_{p}+M_{2}+M_{\Delta 2}  \tag{1a}\\
I_{3} \dot{\omega}_{3}-\left(I_{1}-I_{2}\right) \omega_{1} \omega_{2}=M_{r}+M_{3}+M_{\Delta 3} \\
\dot{\lambda}=\omega_{b 1}+\tan \theta\left(\omega_{b 2} \sin \lambda+\omega_{b 3} \cos \lambda\right) \\
\dot{\theta}=\omega_{b 2} \cos \lambda-\omega_{b 3} \sin \lambda  \tag{lb}\\
\dot{\gamma}=\frac{1}{\cos \theta}\left(\omega_{b 2} \sin \lambda+\omega_{b 3} \cos \lambda\right)
\end{gather*}
$$

where $I_{1}, I_{2}, I_{3}$ are the principal moments of inertia of the spacecraft, $\omega_{1}, \omega_{2}, \omega_{3}$ are the components of the angular velocity along the principal axes in the spacecraft-fixed frame, $\omega_{b 1}, \omega_{b 2}, \omega_{b 3}$ are the relative angular velocity of the spacecraft with respect to the orbital frame $\left(\vec{o}_{1}, \vec{o}_{2}, \vec{o}_{3}\right)$ expressed in the spacecraft-fixed frame, and $\omega_{b 1}, \omega_{b 2}, \omega_{b 3}$ can be calculated by the coordinate transformation matrix $M_{B O}$ from the orbital frame to the spacecraft-fixed frame:

$$
\begin{align*}
& {\left[\begin{array}{lll}
\omega_{b 1} & \omega_{b 2} & \omega_{b 3}
\end{array}\right]^{T}=\left[\begin{array}{lll}
\omega_{1} & \omega_{2} & \omega_{3}
\end{array}\right]^{T}-M_{B O}\left[\begin{array}{ll}
0 & \dot{\eta}
\end{array}\right]^{T} . }  \tag{2a}\\
& M_{B O}= {\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos \lambda & \sin \lambda \\
0 & -\sin \lambda & \cos \lambda
\end{array}\right]\left[\begin{array}{ccc}
\cos \theta & 0 & -\sin \theta \\
0 & 1 & 0 \\
\sin \theta & 0 & \cos \theta
\end{array}\right] } \\
& \times\left[\begin{array}{ccc}
\cos \gamma & \sin \gamma & 0 \\
-\sin \gamma & \cos \gamma & 0 \\
0 & 0 & 1
\end{array}\right] \tag{2b}
\end{align*}
$$

And $M_{y}, M_{p}$, and $M_{r}$ are the components of the external control moment, $M_{1}, M_{2}, M_{3}$ are the components of the gravitational field of the asteroid, $M_{\Delta 1}, M_{\Delta 2}, M_{\Delta 3}$ are the components of the perturbation force, and $\dot{\eta}$ is the instantaneous orbital rate. Therefore, the full nonlinear equations of the attitude motion have been obtained by (1a), (1b), (2a), and (2b).

In view of the fifth assumption, the gravitational field of the asteroid is the primary and complex effect term which needs to be discussed in detail.
2.3. Gravity Gradient Torque. The gravitational potential of any arbitrary primary can be written in spherical harmonic series [21, 22]:

$$
\begin{align*}
U=\frac{G M}{R}\left\{1+\sum_{l=2}^{\infty}\left[C_{n 0}\left(\frac{R_{e}}{R}\right)^{l} P_{l}(\sin \varphi)\right.\right.
\end{align*} \quad \begin{array}{r}
+\sum_{m=1}^{l}\left(\left(\frac{R_{e}}{R}\right)^{l} P_{l m}(\sin \varphi)\right. \\
\\
\quad \times\left(C_{l m} \cos m \delta\right. \\
 \tag{3}\\
\left.\left.\left.\left.\quad+S_{l m} \sin m \delta\right)\right)\right]\right\}
\end{array}
$$

where $R$ is the distance of an orbiting particle from the center of mass of the primary and $R_{e}$ is the characteristic length of the primary, while $\varphi$ and $\delta$ are, respectively, the latitude and longitude of the orbiting particle measured in an asteroidfixed frame. The terms $P_{l}(\sin \varphi)$ are Legendre polynomials of degree $l$ and order 0 , and terms $P_{l m}(\sin \varphi)$ are associated Legendre polynomials of degree $l$ and order $m$. The two kinds of terms are given as

$$
\begin{align*}
& P_{l}(\sin \varphi)=\frac{1}{2^{l} l} \frac{d^{l}}{d(\sin \varphi)^{l}}\left\{\left[(\sin \varphi)^{2}-1\right]^{l}\right\}  \tag{4}\\
& P_{l m}(\sin \varphi)=\left[(\sin \varphi)^{2}-1\right]^{m / 2} \frac{d^{m} P_{l}(\sin \varphi)}{d(\sin \varphi)^{m}}
\end{align*}
$$

The corresponding $C_{l m}$ and $S_{l m}$ are known as harmonic coefficients. When $m=l \neq 0$, they are called sectorial harmonic coefficients, and $P_{l m}(\sin \varphi)=P_{l}(\sin \varphi)$; the corresponding $C_{l 0}$ are known as zonal harmonic coefficients of order 0 . The coefficients $C_{l 0}$ specify the oblateness of the asteroid while $C_{l m}$ characterize the ellipticity of the asteroid's equator. For the Earth, $C_{20}$ is $O\left(10^{-3}\right)$ and the other coefficients are $O\left(10^{-6}\right)$. However, for some familiar asteroids these coefficients can be as high as $O\left(10^{-2}\right)$. Thus, the irregular shape of an asteroid can have a much stronger effect on attitude dynamics. We approximate the small body is a homogeneous triaxial ellipsoid with axes $a, b$, and $c$ in order to simplify the problem. We can calculate the coefficients as follows.
$S_{l m}=0$ for all $l$ or $m, C_{l m}=0$ for $l$ or $m$ are odd and while other conditions

$$
\begin{aligned}
C_{l m}= & \frac{3}{R_{e}{ }^{l}} \frac{(l / 2)!(l-m)!}{2^{m}(l+3)(l+1)!}\left(2-\delta_{0 m}\right) \\
& \times \sum_{i=0}^{\operatorname{int}((l-m) / 4)}\left(\left(a^{2}-b^{2}\right)^{((m+4 i) / 2)}\right.
\end{aligned}
$$

$$
\begin{align*}
& \left.\times\left[c^{2}-\left(\frac{1}{2}\right)\left(a^{2}+b^{2}\right)\right]^{((l-m-4 i) / 2)}\right) \\
& \times\left(16^{i}\left(\frac{l-m-4 i}{2}\right)!\left(\frac{m+2 i}{2}\right)!i!\right)^{-1} \tag{5}
\end{align*}
$$

$\delta_{0 m}$ is Kronecker symbol, and the value is

$$
\delta_{0 m}= \begin{cases}0, & m=0  \tag{6}\\ 1, & m=1\end{cases}
$$

For our purposes we have stopped the expansion of (3) to the second order, so we get the following coefficient:

$$
\begin{equation*}
C_{20}=\frac{2 c^{2}-\left(a^{2}-b^{2}\right)}{10 R_{0}^{2}}, \quad C_{22}=\frac{a^{2}-b^{2}}{20 R_{0}^{2}} \tag{7}
\end{equation*}
$$

The gravitational force acting on a particle of mass $d m$ at a distance $R$ from the asteroid center of mass, having latitude $\varphi$ and longitude $\delta$, is given by

$$
\begin{equation*}
\overrightarrow{d F}=\left[\frac{\partial U}{\partial R} \vec{e}_{R}+\frac{1}{R \cos \varphi} \frac{\partial U}{\partial \delta} \vec{e}_{\delta}+\frac{1}{2} \frac{\partial U}{\partial \varphi} \vec{e}_{\varphi}\right] d m \tag{8}
\end{equation*}
$$

where $U$ is given in (3), while $\vec{e}_{R}, \vec{e}_{\varphi}, \vec{e}_{\delta}$ are unit vectors associated with the spherical coordinate system $R, \varphi, \delta$ as shown in Figure 1. The position vector $R$ of the element can be expressed as

$$
\begin{equation*}
\stackrel{\rightharpoonup}{R}=\vec{R}_{c}+\stackrel{\rightharpoonup}{r} \tag{9}
\end{equation*}
$$

where $\vec{R}_{c}$ is the position vector of the center of mass of the spacecraft relative to the asteroid center of mass, while $\vec{r}$ is the position vector of the element in the spacecraft frame. We assume that $R$ and $R_{c}$ are much greater than $r$. Clearly

$$
\begin{equation*}
R=\left|\stackrel{\rightharpoonup}{R}_{c}+\vec{r}\right|, \quad \vec{e}_{R}=\frac{\stackrel{\rightharpoonup}{R}_{c}+\vec{r}}{\left|\stackrel{\rightharpoonup}{R}_{c}+\vec{r}\right|}, \quad \vec{e}_{\delta}=\vec{e}_{R} \times \vec{e}_{\varphi} \tag{10}
\end{equation*}
$$

In conclusion, the gravity gradient torque on the spacecraft can then be determined from

$$
\begin{gather*}
\overrightarrow{d F}=\stackrel{\rightharpoonup}{F}_{R}+\stackrel{\rightharpoonup}{F}_{\delta}+d \stackrel{\rightharpoonup}{F}_{\varphi}  \tag{11}\\
M_{g}=\int r \times d f \tag{12}
\end{gather*}
$$

Evaluation of this torque involves expansion of the various powers of $\left|\vec{R}_{c}+\vec{r}\right|$ using the binomial theorem and neglecting terms involving third and higher powers of $|r| /\left|R_{c}\right|$.

Let $M_{g}=\left[\begin{array}{lll}M_{1} & M_{2} & M_{3}\end{array}\right]$ denote the gravity gradient torque in the spacecraft-fixed frame $\left(\vec{b}_{1}, \vec{b}_{2}, \vec{b}_{3}\right)$ and let $I$ denote the inertia matrix for the spacecraft, which is given as

$$
I=\left[\begin{array}{lll}
I_{1} & &  \tag{13}\\
& I_{2} & \\
& & I_{3}
\end{array}\right] .
$$

The unit vectors $\vec{e}_{R}, \vec{e}_{\varphi}, \vec{e}_{\delta}$ appearing in (8), (10), and (11) can now be expressed in terms of the yaw, pitch, and roll. The gravity-gradient torque components $M_{i}(i=1,2,3)$ in the spacecraft-fixed frame $\left(\vec{b}_{1}, \vec{b}_{2}, \vec{b}_{3}\right)$ can be written as follows after some algebra:

$$
\begin{align*}
& M_{1}=\frac{G M}{R_{c}^{3}}\left[(3+5 \alpha)\left(I_{3}-I_{2}\right) \cos \lambda \cos ^{2} \theta \sin \lambda\right. \\
& \\
& +5 \beta\left(-\frac{2}{5} I_{1} \cos \lambda \sin \gamma\right.  \tag{14}\\
& \\
& \left.\left.+\left(I_{1}-I_{2}+I_{3}\right) \sin \lambda \cos ^{2} \theta \cos \gamma\right)\right] \\
& \begin{aligned}
M_{2}=\frac{G M}{R_{c}^{3}}\left[(3+5 \alpha)\left(I_{3}-I_{1}\right) \cos \lambda \cos \theta \sin \theta\right.
\end{aligned} \\
& \\
& \\
& +5 \beta\left(-\frac{2}{5} I_{2}(\sin \lambda \sin \theta \sin \gamma-\cos \lambda \cos \gamma)\right.  \tag{15}\\
& \\
& \quad+\left(I_{2}-I_{1}+I_{3}\right) \\
& \\
& \times(\sin \lambda \sin \theta \sin \gamma \\
& \\
& \\
& \\
& \\
& \\
& +\left(I_{2}-I_{3}^{2} \theta \cos \lambda \cos \gamma\right) \\
&
\end{align*}
$$

$$
\begin{align*}
& M_{3}=\frac{G M}{R_{c}^{3}}\left[( 3 + 5 \alpha ) \left(I_{1}-\right.\right.\left.I_{2}\right) \sin \lambda \cos \theta \sin \theta \\
&+5 \beta\left(\frac{2}{5} I_{3}(\sin \lambda \cos \gamma-\cos \lambda \sin \theta \sin \gamma)\right. \\
&+\left(I_{2}-I_{1}+I_{3}\right) \\
& \times(\cos \lambda \sin \lambda \sin \gamma \\
&\left.-\sin ^{2} \theta \sin \lambda \cos \gamma\right) \\
&-\left(I_{1}-I_{2}+I_{3}\right) \\
&\left.\left.\times \cos ^{2} \theta \sin \lambda \cos \gamma\right)\right] \tag{16}
\end{align*}
$$

where $\alpha, \beta$ are defined as follows, respectively:

$$
\begin{gather*}
\alpha=\left[-\frac{3}{2} C_{20}+9 C_{22} \cos \left(2 \delta_{c}\right)\right]\left(\frac{R_{e}}{R_{c}}\right)^{2},  \tag{17}\\
\beta=\left[6 C_{22} \sin \left(2 \delta_{c}\right)\right]\left(\frac{R_{e}}{R_{c}}\right)^{2} . \tag{18}
\end{gather*}
$$

2.4. Three-Dimensional Motion for Equatorial Orbits. Threedimensional motion of a spacecraft in an equatorial orbit is considered, and the attitude motion is small. It is assumed that the asteroid is rotating with a constant angular velocity $\Omega \cdot \vec{K}$. Assuming the rotating orbit of the spacecraft is circular orbits, $\dot{\eta}=n$, where $n$ is constant and stands for the orbital angular velocity of the spacecraft. Therefore, the longitude of the center of mass of the spacecraft is then given by

$$
\begin{equation*}
\delta_{c}=(n \pm \Omega) t \tag{19}
\end{equation*}
$$

where the plus and minus signs apply for retrograde and direct orbits, respectively.

Furthermore, for small motion, the angular velocity components given in (1a), (1b), (2a), and (2b) become

$$
\left[\begin{array}{c}
\omega_{1}  \tag{20}\\
\omega_{2} \\
\omega_{3}
\end{array}\right]=\left[\begin{array}{c}
\dot{\lambda}-\dot{\eta} \gamma \\
\dot{\theta}-\dot{\eta} \\
\dot{\gamma}+\dot{\eta} \lambda
\end{array}\right] .
$$

Therefore, a set of linearized equations for small motion of spacecraft are obtained in (21)-(23) by introducing (14)(19) into (1a) and (1b):

$$
\begin{gather*}
\ddot{\lambda}+\dot{\eta}\left(k_{1}-1\right) \dot{\gamma}+\left[\frac{G M}{R_{c}^{3}}(3+5 \alpha)+\dot{\eta}^{2}\right] k_{1} \lambda  \tag{21}\\
- \\
\ddot{ }\left[\frac{1}{2} \frac{G M}{R_{c}^{3}} \beta\left(3+5 k_{1}\right)+\ddot{\eta}^{2}\right] \gamma=u_{\lambda}+\Delta_{\lambda}  \tag{22}\\
\ddot{\theta}-\ddot{\eta}+\frac{G M}{R_{c}^{3}}(3+5 \alpha) k_{2} \theta-\frac{1}{2} \frac{G M}{R_{c}^{3}} \beta\left(3+5 k_{2}\right)=u_{\theta}+\Delta_{\theta} \\
\ddot{\gamma}+\dot{\eta}\left(1-k_{3}\right) \dot{\lambda}+k_{3} \dot{\eta}^{2} \gamma  \tag{23}\\
\\
-\left[\ddot{\eta}+\frac{1}{2} \frac{G M}{R_{c}^{3}} \beta\left(3-5 k_{3}\right)\right] \lambda=u_{\gamma}+\Delta_{\gamma},
\end{gather*}
$$

where $k_{1}=\left(I_{2}-I_{3}\right) / I_{1}, k_{2}=\left(I_{1}-I_{3}\right) / I_{2}, k_{3}=\left(I_{2}-I_{1}\right) / I_{3}$, $u_{\lambda}, u_{\theta}, u_{\gamma}$ are control accelerations in three directions, and $\Delta_{\lambda}, \Delta_{\theta}, \Delta_{\gamma}$ are perturbation force accelerations consisting of gravitation higher order terms and solar radiation pressure, and so forth. Note that the pitch motion is decoupled from the roll and yaw motions, and this fact is similar to the case of a spacecraft orbiting symmetrically mass distributed planetary bodies.

## 3. Analysis of Motion for Orbiting Circular Orbits

3.1. Regular Resonance Analysis and Numerical Results of Pitch Motion. For circular orbits, $\dot{\eta}=n, \ddot{\eta}=0, G M / R_{c}^{3}=n^{2}$. For the understanding of the pitch behavior, let us consider small motion. Equation (22) then reduces to

$$
\begin{equation*}
\ddot{\theta}+n^{2}(3+5 \alpha) k_{2} \theta-\frac{1}{2} n^{2} \beta\left(3+5 k_{2}\right)=0 . \tag{24}
\end{equation*}
$$

One can cast (24) which represents a harmonically excited system with periodic stiffness. If $k_{2}$ is negative, the pitch

TABLE 1: Simulation parameters.

| Simulation parameters | Simulation value |
| :--- | :---: |
| Characteristic length of the asteroid $R_{e}$ <br> $(\mathrm{~km})$ | 9.933 |
| Harmonic coefficients $C_{20}$ | -0.0878 |
| Harmonic coefficients $C_{22}$ | 0.0439 |
| Spacecraft mass distribution | $1 / 3,1 / 3,1 / 3$ |
| parameters $k_{1}, k_{2}, k_{3}$ |  |
| Asteroid's gravitational constant | 876171 |
| parameter $\mu\left(\mathrm{km}^{3} / \mathrm{s}^{2}\right)$ | $(2 * 3.14) /(5.27 * 3600)$ |

motion is normally unstable; hence, the case of positive is considered in this paper.

Since $k_{2}<1,\left|C_{20}\right|<0.1$, choosing the minus sign in (19), parametric resonance occurs when the spacecraft is in a retrograde orbit when the asteroid and orbital angular velocities are related [14] approximately by

$$
\begin{equation*}
R_{c}=\left(\frac{G M}{\Omega^{2}}\right)^{1 / 3}\left[\frac{j \mp\left(3 k_{2}\right)^{1 / 2}}{j}\right]^{2 / 3}, \quad j=1,2,3, \ldots \tag{25}
\end{equation*}
$$

Regular resonance takes place when $j=2$; that is,

$$
\begin{equation*}
R_{c}=\left(\frac{G M}{\Omega^{2}}\right)^{1 / 3}\left[1 \mp \frac{\sqrt{3 k_{2}}}{2}\right]^{2 / 3} . \tag{26}
\end{equation*}
$$

3.2. Results for the Three-Dimensional Case. Equations (21)(23) are quite complex and must be solved numerically with the initial conditions of roll, pitch, and yaw $\lambda(0)=0.1 \mathrm{rad}$, $\dot{\lambda}(0)=0 ; \theta(0)=0.1 \mathrm{rad}, \dot{\theta}(0)=0 ; \gamma(0)=0.1 \mathrm{rad}, \dot{\gamma}(0)=0$.

Table 1 gives simulation parameters for three-dimensional motion about Eros 433. Figures 2, 3, 4, 5, 6, 7, 8, 9 , and 10 give the three-dimensional motions of a spacecraft orbiting Eros in equatorial circular retrograde orbits at $R_{c}=48 \mathrm{~km}, 31 \mathrm{~km}, 15 \mathrm{~km}$, respectively, without taking into account perturbation torque.

The pitch motion is quite regular with amplitude of 0.1 rad at $R_{c}=48 \mathrm{~km}$ with the above initial conditions, and amplitudes of the roll and yaw motions are all steady. When orbital radius is decrease the three-dimensional motions especially pitch are irregular, and the irregularity is becoming apparent when the spacecraft is nearer to the asteroid. Similarly Figures $11,12,13,14,15,16,17,18,19,20,21$, and 22 give the three-dimensional motions of a spacecraft orbiting Eros in equatorial circular direct orbits at $R_{c}=50 \mathrm{~km}, 35 \mathrm{~km}, 27 \mathrm{~km}$, 26 km , respectively, without taking into account perturbation torques. The three-dimensional motion has the same trend with retrograde orbits when the orbital radius are decrease, but the roll and raw motions become instable when $R_{c}=$ 26 km .

It is observed that irregularities of attitude angles are more obvious when the spacecraft is nearer to the small body, which make the vibration amplitude and frequency of the spacecrafts more strong. We can draw the conclusion from the simulation results that the irregular gravity-gradient


Figure 2: Roll $R_{c}=48 \mathrm{~km}, k_{2}=1 / 3$.


Figure 3: Pitch $R_{c}=48 \mathrm{~km}, k_{2}=1 / 3$.
torque of the asteroid has the primary and complex effect on the spacecraft orbiting motion. The spacecraft may get out of the orbit if external perturbations such as solar radiation pressure are taken into account. So it is essential to design the robust control algorithms to compensate the uncertainties and perturbations and stabilize the attitude angles.

## 4. Controller Design

In this section, we present adaptive sliding mode control laws based on backstepping which achieves three-axes stabilized nadir-pointing attitude. In other words, the control objective is to align the spacecraft-fixed axes with the orbital reference axes. The desired attitude angles yaw $(\lambda)$, pitch $(\theta)$, and roll $(\gamma)$ are zero.
4.1. Backstepping Control. The basic idea of backstepping method is decomposition of a complicated nonlinear system, then designing Lyapunov function and suppositional control


Figure 4: Yaw $R_{c}=48 \mathrm{~km}, k_{2}=1 / 3$.


Figure 5: Roll $R_{c}=31 \mathrm{~km}, k_{2}=1 / 3$.


Figure 6: Pitch $R_{c}=31 \mathrm{~km}, k_{2}=1 / 3$.


Figure 7: Yaw $R_{c}=31 \mathrm{~km}, k_{2}=1 / 3$.


Figure 8: Roll $R_{c}=15 \mathrm{~km}, k_{2}=1 / 3$.


Figure 9: Pitch $R_{c}=15 \mathrm{~km}, k_{2}=1 / 3$.


Figure 10: Yaw $R_{c}=15 \mathrm{~km}, k_{2}=1 / 3$.


Figure 11: Roll $R_{c}=50 \mathrm{~km}, k_{2}=1 / 3$.


Figure 12: Pitch $R_{c}=50 \mathrm{~km}, k_{2}=1 / 3$.


Figure 13: Yaw $R_{c}=50 \mathrm{~km}, k_{2}=1 / 3$.


Figure 14: Roll $R_{c}=35 \mathrm{~km}, k_{2}=1 / 3$.


Figure 15: Pitch $R_{c}=35 \mathrm{~km}, k_{2}=1 / 3$.


Figure 16: Yaw $R_{c}=35 \mathrm{~km}, k_{2}=1 / 3$.


Figure 17: Roll $R_{c}=27 \mathrm{~km}, k_{2}=1 / 3$.


FIGURE 18: Pitch $R_{c}=27 \mathrm{~km}, k_{2}=1 / 3$.


Figure 19: Raw $R_{c}=27 \mathrm{~km}, k_{2}=1 / 3$.
for the decomposed system. The final control laws are designed after backing to the overall system.

Regardless of perturbation, suppose (21), (22), and (23) are

$$
\begin{gather*}
\dot{x}_{1}=x_{2}, \quad \dot{x}_{2}=f\left(x_{1}, x_{2}\right)+b u \\
x_{1}=[\lambda, \theta, \gamma], \quad x_{2}=[\dot{\lambda}, \dot{\theta}, \dot{\gamma}] \\
f\left(x_{1}, x_{2}\right) \\
=\left[\dot{\eta}\left(k_{1}-1\right) \dot{\gamma}+\left[\frac{G M}{R_{c}^{3}}(3+5 \alpha)+\dot{\eta}^{2}\right] k_{1} \lambda\right. \\
-\left[\frac{1}{2} \frac{G M}{R_{c}^{3}} \beta\left(3+5 k_{1}\right)+\ddot{\eta}^{2}\right] \gamma-\ddot{\eta}+\frac{G M}{R_{c}^{3}}(3+5 \alpha) k_{2} \theta \\
-\frac{1}{2} \frac{G M}{R_{c}^{3}} \beta\left(3+5 k_{2}\right) \dot{\eta}\left(1-k_{3}\right) \dot{\lambda}+k_{3} \dot{\eta}^{2} \gamma \\
-\left[\frac{\left.\left.\ddot{\eta}+\frac{1}{2} \frac{G M}{R_{c}^{3}} \beta\left(3-5 k_{3}\right)\right] \lambda\right], \quad b=\left[\begin{array}{l}
1 \\
1 \\
1
\end{array}\right] .}{}\right. \tag{27}
\end{gather*}
$$

Define position error $z_{1}=x_{1}-z_{d} ; z_{d}$ is the expected trajectory $\dot{z}_{1}=\dot{x}_{1}-\dot{z}_{d}=x_{2}-\dot{z}_{d}$.

Assume the virtual control in

$$
\begin{equation*}
\alpha_{1}=-c_{1} z_{1}+\dot{z}_{d} \quad\left(c_{1}>0\right) \tag{28}
\end{equation*}
$$

Define $z_{2}=x_{2}-\alpha_{1}$ and Lyapunov function $V_{1}=(1 / 2) z_{1}^{2}$, so (29) is obtained:

$$
\begin{equation*}
\dot{V}_{1}=z_{1} \dot{z}_{1}=z_{1}\left(x_{2}-\dot{z}_{d}\right)=z_{1}\left(z_{2}+\alpha_{1}-\dot{z}_{d}\right) . \tag{29}
\end{equation*}
$$

Introducing (28) into (29), $\dot{V}_{1}=-c_{1} z_{1}^{2}+z_{1} z_{2}$ is obtained. If $z_{2}=0$, then $\dot{V}_{1} \leq 0$.


Figure 20: Roll $R_{c}=26 \mathrm{~km}, k_{2}=1 / 3$.


Figure 21: Pitch $R_{c}=26 \mathrm{~km}, k_{2}=1 / 3$.


Figure 22: Pitch $R_{c}=26 \mathrm{~km}, k_{2}=1 / 3$.


Figure 23: Roll response.

Define Lyapunov function $V_{2}=V_{1}+(1 / 2) z_{2}^{2}$; then

$$
\begin{align*}
\dot{V}_{2} & =\dot{V}_{1}+z_{2} \dot{z}_{2}  \tag{30}\\
& =-c_{1} z_{1}^{2}+z_{1} z_{2}+z_{2}\left[f\left(x_{1}, x_{2}\right)+b u+c_{1} \dot{z}_{1}-\ddot{z}_{d}\right] .
\end{align*}
$$

The following control laws are obtained in

$$
\begin{equation*}
u=\frac{1}{b}\left[-f\left(x_{1}, x_{2}\right)-c_{2} z_{2}-z_{1}-c_{1} \dot{z}_{1}+\ddot{z}_{d}\right]\left(c_{2}>0\right) . \tag{31}
\end{equation*}
$$

Thus, $\dot{V}_{2}=-c_{1} z_{1}^{2}-c_{2} z_{2}^{2} \leq 0$.
Spacecraft attitude angles pitch, roll, and yaw angles can reach regular resonance based on control law (31) with the initial conditions of roll, pitch, and yaw $\lambda(0)=0.1 \mathrm{rad}$, $\dot{\lambda}(0)=0 \mathrm{rad} ; \theta(0)=0.1 \mathrm{rad}, \dot{\theta}(0)=0 \mathrm{rad} ; \gamma(0)=0.1 \mathrm{rad}$, $\dot{\gamma}(0)=0 \mathrm{rad}$. In the case of the simulation parameters described as Table 1, the desired attitude motion of the spacecraft can be specified by periodic functions with the desired amplitudes. The corresponding attitude motions are determined numerically in Figures 23, 24, and 25. Figures 26, 27 , and 28 give the control accelerations of three-dimensional motions. Backstepping control laws can make roll, pitch, and yaw track the desired periodic trajectory without external disturbances when the spacecraft in circular retrograde or direct orbits. The desired attitude angles of spacecraft are adopted periodic function as $z d 1=0.1 * \cos (0.0002 * t)$ in the experiments.

From the experimental results in Figures 23, 24, and 25 , backstepping control law (31) can guarantee the output signals stable, tracking the desired attitude of spacecraft globally and asymptotically without external perturbances.
4.2. Adaptive Backstepping Sliding Mode Control. With external disturbances and uncertain parameters, adaptive backstepping sliding mode control schemes are developed, which have been applied in uncertain systems [23]. It introduces the sliding mode control in backstepping design to modify the last step of backstepping algorithm and simplify the design of controller.


Figure 24: Pitch response.

Without loss of generality, suppose (21), (22), and (23) are

$$
\begin{gather*}
\dot{x}_{1}=x_{2}, \quad \dot{x}_{2}=f\left(x_{1}, x_{2}\right)+b u+F, \\
y=x_{1}, \\
x_{1}=[\lambda, \theta, \gamma], \quad x_{2}=[\dot{\lambda}, \dot{\theta}, \dot{\gamma}], \\
=\left[\dot{\eta}\left(k_{1}-1\right) \dot{\gamma}+\left[\frac{G M}{R_{c}^{3}}(3+5 \alpha)+\dot{\eta}^{2}\right] k_{1} \lambda\right. \\
-\left[\frac{1}{2} \frac{G M}{R_{c}^{3}} \beta\left(3+5 k_{1}\right)+\ddot{\eta}^{2}\right] \gamma-\ddot{\eta}+\frac{G M}{R_{c}^{3}}(3+5 \alpha) k_{2} \theta \\
-\frac{1}{2} \frac{G M}{R_{c}^{3}} \beta\left(3+5 k_{2}\right) \dot{\eta}\left(1-k_{3}\right) \dot{\lambda}+k_{3} \dot{\eta}^{2} \gamma \\
\left.-\left[\ddot{\eta}+\frac{1}{2} \frac{G M}{R_{c}^{3}} \beta\left(3-5 k_{3}\right)\right] \lambda\right], \quad b=\left[\begin{array}{l}
1 \\
1 \\
1
\end{array}\right] .
\end{gather*}
$$

$|F| \leq \bar{F}$ is the whole external disturbances and uncertain parameters, and we suppose it changes slowly; that is $\dot{F}=0$.

To begin with, define the position error $z_{1}=y-y_{d} ; y_{d}$ is the expected position:

$$
\begin{equation*}
\dot{z}_{1}=\dot{y}-\dot{y}_{d}=x_{2}-\dot{y}_{d} . \tag{33}
\end{equation*}
$$

The stability term is $\alpha_{1}=c_{1} z_{1}$, and $c_{1}$ is positive constant.
Define Lyapunov function to be $V_{1}=(1 / 2) z_{1}^{2}$ and $z_{2}=$ $\dot{z}_{1}+\alpha_{1}=x_{2}-\dot{y}_{d}+\alpha_{1}$

$$
\begin{equation*}
\dot{V}_{1}=z_{1} \dot{z}_{1}=z_{1}\left(x_{2}-\dot{y}_{d}\right)=z_{1}\left(z_{2}-\alpha_{1}\right)=z_{1} z_{2}-c_{1} z_{1}^{2} . \tag{34}
\end{equation*}
$$

Then, $\dot{z}_{2}=\dot{x}_{2}-\ddot{y}_{d}+\dot{\alpha}_{1}=f\left(x_{1}, x_{2}\right)+b u+F-\ddot{y}_{d}+\dot{\alpha}_{1}$.
Define Lyapunov function $V_{2}=V_{1}+(1 / 2) \sigma^{2} ; \sigma=k_{1} z_{1}+$ $z_{2}\left(k_{1}>0\right)$ is the switching function.


Figure 25: Yaw response.


Figure 26: Roll control input.


Figure 27: Pitch control input.


Figure 28: Yaw control input.

Taking the derivative of $V_{2}$ (35) is obtained:

$$
\begin{align*}
\dot{V}_{2}= & \dot{V}_{1}+\sigma \dot{\sigma}=z_{1} z_{2}-c_{1} z_{1}^{2}+\sigma \dot{\sigma} \\
= & z_{1} z_{2}-c_{1} z_{1}^{2}+\sigma\left(k_{1} \dot{z}_{1}+\dot{z}_{2}\right) \\
= & z_{1} z_{2}-c_{1} z_{1}^{2} \\
& \quad+\sigma\left[k_{1}\left(z_{2}-c_{1} z_{1}\right)+f\left(x_{1}, x_{2}\right)+b u+F-\ddot{y}_{d}+\dot{\alpha}_{1}\right] \tag{35}
\end{align*}
$$

The control laws are deduced as follows supposing $\bar{F}$ is known:

$$
\begin{align*}
u=b^{-1}[ & -k_{1}\left(z_{2}-c_{1} z_{1}\right)-f\left(x_{1}, x_{2}\right)-\bar{F} \operatorname{sgn}(\sigma) \\
& \left.+\ddot{y}_{d}-\dot{\alpha}_{1}-h(\sigma+\beta \operatorname{sgn}(\sigma))\right] \tag{36}
\end{align*}
$$

Here $h$ and $\beta$ are all positive constants.
It is not easy to obtain the boundary of external disturbances and uncertain parameters due to the highly complex space environment. The adaptive algorithm is adopted to estimate the external disturbances and uncertainties $F$ in order to retain from the boundary.

Define Lyapunov function as $V_{3}=V_{2}+(1 / 2 \gamma) \widetilde{F}^{2}$; the error is $\widetilde{F}=F^{*}-\widehat{F}$, and $\widehat{F}$ is the estimated value of $F ; \gamma$ is a positive constant. Substituting $V_{2}$ into Equation $V_{3}$ as

$$
\begin{aligned}
\dot{V}_{3}= & \dot{V}_{2}-\frac{1}{\gamma} \widetilde{F} \dot{\widehat{F}} \\
= & z_{1} z_{2}-c_{1} z_{1}^{2}+\sigma[
\end{aligned} \quad k_{1}\left(z_{2}-c_{1} z_{1}\right)+f\left(x_{1}, x_{2}\right) .
$$

$$
\begin{align*}
=z_{1} z_{2}-c_{1} z_{1}^{2}+\sigma[ & k_{1}\left(z_{2}-c_{1} z_{1}\right)+f\left(x_{1}, x_{2}\right) \\
& \left.+b u+F-\ddot{y}_{d}+\dot{\alpha}_{1}\right]-\frac{1}{\gamma} \widetilde{F}(\dot{\widehat{F}}-\gamma \sigma) . \tag{37}
\end{align*}
$$

The adaptive controller laws are obtained as

$$
\begin{gather*}
u=b^{-1}\left[-k_{1}\left(z_{2}-c_{1} z_{1}\right)-f\left(x_{1}, x_{2}\right)\right. \\
\left.-\widehat{F}+\ddot{y}_{d}-\dot{\alpha}_{1}-h(\sigma+\beta \operatorname{sgn}(\sigma))\right]  \tag{38}\\
\dot{\widehat{F}}=\gamma \sigma .
\end{gather*}
$$

The stability of the controller is proved as follows.
Substituting (38) into (37),

$$
\begin{equation*}
\dot{V}_{3}=z_{1} z_{2}-c_{1} z_{1}^{2} h \sigma^{2}-h \beta|\sigma| \tag{39}
\end{equation*}
$$

Take

$$
\begin{gather*}
Q=\left[\begin{array}{cc}
c_{1}+h k_{1}^{2} & h k_{1}-\frac{1}{2} \\
h k_{1}-\frac{1}{2} & h
\end{array}\right] \\
z^{T} Q z=\left[\begin{array}{ll}
z_{1} & z_{2}
\end{array}\right]\left[\begin{array}{cc}
c_{1}+h k_{1}^{2} & h k_{1}-\frac{1}{2} \\
h k_{1}-\frac{1}{2} & h
\end{array}\right]\left[\begin{array}{ll}
z_{1} & z_{2}
\end{array}\right]^{T}  \tag{40}\\
=c_{1} z_{1}^{2}+h k_{1}^{2} z_{1}^{2}+2 h k_{1} z_{1} z_{2}-z_{1} z_{2}+h z_{2}^{2} \\
=c_{1} z_{1}^{2}-z_{1} z_{2}+h \sigma^{2}
\end{gather*}
$$

Rewriting (39) to

$$
\begin{equation*}
\dot{V}_{3}=z_{1} z_{2}-c_{1} z_{1}^{2}-h \sigma^{2}-h \beta|\sigma|=-z^{T} Q z-h \beta|\sigma| \leq 0 \tag{41}
\end{equation*}
$$

$Q$ is ensured to be a positive definite matrix while $h, c_{1}, k_{1}$ are appropriate values.

The desired attitude of spacecraft is adopted exponential function as

$$
\begin{equation*}
z d 1=(0.1+0.15 * n * t) * \exp (-1.5 * n * t) \tag{42}
\end{equation*}
$$

The higher order terms of gravitational potential are regarded as uncertainties and solar radiation pressures are regarded as disturbances which are assumed as the following equation:

$$
F\left(t, x_{1}, x_{2}\right)=\left[\begin{array}{l}
0.02 \sin (\omega t) \cdot V_{l x}  \tag{43}\\
0.02 \sin (\omega t) \cdot V_{l y} \\
0.02 \cos (\omega t) \cdot V_{l z}
\end{array}\right]
$$

To verify and visualize the efficacy of the developed control scheme, numerical simulations under external disturbances and uncertainties are conducted using (21)-(23) and control law (38). Parameters related to operating conditions are also given about Eros 433 in Table 1.


Figure 29: Roll response.


Figure 30: Pitch response.

Some experimental results are provided to demonstrate the effectiveness of the proposed adaptive backstepping sliding mode control laws. Figures 29, 30, and 31 give the spacecraft pitch, roll, and yaw attitude angles response motion around Eros 433 . We also can obtain the pitch controller response as Figure 32. Compared with the attitude stability analysis in references [10-13], closed-loop controllers are proposed to make the spacecraft attitude angles tracking the desired attitude as (42) and reach the null state as time increases. Moreover, from simulation results one can obtain the control law neutralizing the effects of asteroid shape and mass distributions and orbital eccentricity as well as external disturbances and uncertainties described as (43). The robust control performance of the proposed adaptive backstepping sliding-mode control system is obvious than references [14, 15], which ensure stable orbiting motion and space mission.


Figure 31: Yaw response.


Figure 32: Pitch control response.

## 5. Conclusions

This paper has focused on the attitude dynamics and effect control algorithms for spacecraft orbiting rotating asteroids. Firstly, three-dimensional attitude motion of the spacecraft is examined considering the perturbation force. Then stability analysis is presented in retrograde orbits and direct orbits using linearized system model. It appears that the nonspherical shape and the rotational state of asteroids can have important effects on the attitude motion. The adaptive backstepping sliding mode control laws are designed to make the attitude angles decay and reach the null state. Computer simulations are carried out for the asteroid Eros 433 to illustrate the effectiveness of the control laws.

## Nomenclature

| $a, b, c$ : | Major semiaxes of the asteroid considered as triaxial ellipsoid |
| :---: | :---: |
| $(\vec{I}, \vec{J}, \vec{K}):$ | Asteroid centered inertial frame |
| $(\vec{i}, \vec{j}, \vec{k})$ : | Asteroid-fixed frame |
| $\left(\vec{o}_{1}, \vec{o}_{2}, \vec{o}_{3}\right):$ | Spacecraft orbital frame |
| $\left(\vec{b}_{1}, \vec{b}_{2}, \vec{b}_{3}\right)$ : | Spacecraft-fixed frame |
| $I_{1}, I_{2}, I_{3}$ : | Principal moments of inertia of the spacecraft |
| $\omega_{1}, \omega_{2}, \omega_{3}:$ | Components of the angular velocity along the principal axes in the spacecraft-fixed frame |
| $\omega_{b 1}, \omega_{b 2}, \omega_{b 3}:$ | Relative angular velocity of the spacecraft with respect to the orbital frame |
| $M_{B O}$ : | Coordinate transformation matrix from the orbital frame to the spacecraft-fixed frame |
| $M_{y}, M_{p}, M_{r}:$ | Components of the external control moment |
| $M_{1}, M_{2}, M_{3}$ : | Components of the gravitational field of the asteroid |
| $M_{\Delta 1}, M_{\Delta 2}, M_{\Delta 3}$ : | Components of the perturbation force |
| $\dot{\eta}$ : | Instantaneous orbital rate |
| $R_{e}$ : | Characteristic length of the primary |
| $\varphi, \delta$ : | Latitude and longitude of the orbiting particle measured in an asteroid-fixed frame |
| $P_{l m}(\sin \varphi):$ | Legendre polynomials of degree $l$ and order $m$ |
| $C_{l m}, S_{l m}:$ | Harmonic coefficients |
| $m$ : | Spacecraft mass |
| M: | Asteroid mass |
| $R_{c}$ : | Orbital radius |
| R: | The distance of an orbiting particle from the center of mass of the primary |
| $\lambda, \theta, \gamma$ : | Spacecraft yaw, pitch, and roll angles |
| $\Omega$ : | Asteroid's spin rate |
| $\rho$ : | Density of the asteroid |
| $z d 1$ : | Desired attitude angle of spacecraft |

$\mu$ : Asteroid's gravitational constant parameter.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Robust Stabilization of Nonlinear Systems with Uncertain Varying Control Coefficient 

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#### Abstract

This paper investigates the stabilization problem for a class of nonlinear systems, whose control coefficient is uncertain and varies continuously in value and sign. The study emphasizes the development of a robust control that consists of a modified Nussbaum function to tackle the uncertain varying control coefficient. By such a method, the finite-time escape phenomenon has been prevented when the control coefficient is crossing zero and varying its sign. The proposed control guarantees the asymptotic stabilization of the system and boundedness of all closed-loop signals. The control performance is illustrated by a numerical simulation.


## 1. Introduction

The control design for nonlinear uncertain systems has been the research focus in the community for decades [1-3]. In recent years, the problem of uncertain control coefficient has attracted increasing research interests [4-17]. This special type of uncertainty is vital to control performance, because the control coefficient represents the system motion direction under any control, and unsuccessful controllers may lead to positive feedback and instability. Several methods have been proposed in literature to handle this problem [5, 7, 9, 10]. Among these studies, an adaptive function was proposed by Nussbaum [9] for linear time-invariant systems to deal with the uncertain but constant control coefficient. Now, the Nussbaum function has already become a standard technique targeting the uncertain control coefficient for both linear and nonlinear systems [18, 19]. In order to complete more complicated control tasks, the Nussbaum function has also been combined with other control techniques, such as robust control [8, 12], adaptive control [15-17], learning control [6, $11]$, and backstepping design $[8,15,16]$.

However, most previous results only investigate a relatively simple case that the sign of the uncertain control
coefficient is fixed, that is, either positive or negative. It is because of the fact that the conventional Nussbaum function requires the control coefficient to be sign-fixed or "bounded away from zero." Unfortunately, the general case of signvarying uncertain control coefficient has received much less attention and has not been fully solved yet. Undoubtedly, this problem is technically more challenging and cannot be directly handled by the conventional Nussbaum function. In particular, to design a successful controller, two critical issues have to be taken into full consideration. First, the sign may vary very rapidly and be difficult to track. Second, any control will lose its power when the control coefficient is crossing zero; that is, singular points of control exist and improper controllers probably result in finite-time escape phenomenon [20].

The first attempt in addressing the problem of signvarying uncertain control coefficient for a scalar nonlinear system was reported in [7]. Instead of using the Nussbaum function, an online estimator of the control coefficient was used in the robust control design. However, in order to launch the online estimation mechanism successfully, several restrictive assumptions are made in their work. For example,
besides assuming the known bounds of the control coefficient and its varying speed, it also requires the control coefficient to be a common term between the unknown dynamics and the control. These assumptions make their method specific instead of general, and thus they need to be relaxed.

In this paper, we propose a new Nussbaum function, which does not require such assumptions of [7] and is able to track the rapidly varying sign of control coefficient. Then, a robust controller is designed and then combined with the proposed Nussbaum function, such that the potential finitetime escape phenomenon is avoided. By such means, all closed-loop signals are bounded and asymptotic stabilization is achieved. The paper is organized as follows. The control problem is presented in Section 2. Section 3 designs the new Nussbaum function and the robust controller, followed by the convergence analysis in Section 4. The illustrative example is given in Section 5.

## 2. System Description

In order to highlight the development of the proposed control approach, we will only consider the following scalar uncertain nonlinear system in this paper:

$$
\begin{equation*}
\dot{x}(t)=f(x)+a(x, t) u, \tag{1}
\end{equation*}
$$

where $t \in[0,+\infty)$ is the time, $x \in R$ is the state, and $u \in R$ is the control. $f(x)$ is an uncertain continuous function, which denotes the dynamic part of the system and is assumed to be bounded by a known function $\rho(x)$ as follows:

$$
\begin{equation*}
|f(x)| \leq \rho(x) \tag{2}
\end{equation*}
$$

where $\rho(x)>0$ is well-defined in the sense that $\rho(x)$ is finite for any finite $x$.
$a(x, t)$ is an uncertain continuous function that denotes the sign-varying uncertain control coefficient. In this paper, let the control coefficient $a$ consist of two terms as follows:

$$
\begin{equation*}
a(x, t)=b(x, t) s(x, t) \tag{3}
\end{equation*}
$$

where $b(x, t)$ and $s(x, t)$ are both unknown continuous functions and govern, separately, the "amplitude" and "sign" of $a$. Let $b(x, t)$ be a bounded positive function satisfying

$$
\begin{equation*}
b(x, t) \in\left[b_{\text {low }}, b_{\mathrm{up}}\right]>0 \tag{4}
\end{equation*}
$$

where $b_{\text {low }}$ and $b_{\text {up }}$ are unknown positive constants. Without loss of any generality, let the "sign" function $s(k)$ be bounded by 1 as follows:

$$
\begin{equation*}
|s(x, t)| \leq 1 . \tag{5}
\end{equation*}
$$

Thus, we immediately have

$$
\begin{equation*}
|a(x, t)| \leq b_{\mathrm{up}} \tag{6}
\end{equation*}
$$

in previous literature $[8,9,15,16]$.
However, in this paper $s(x, t)$ is allowed to vary its value and change its sign. Compared with most previous
results, this study is more comprehensive and technically challenging. The focus of research will be put on handling the varying $s(x, t)$, which has been barely addressed. Since $s(x, t)$ continuously varies between positive and negative and will certainly cross zero, let the control singular point be denoted by the pair $\left(x_{i}, t_{i}\right) \in R \times[0,+\infty), i=1,2, \ldots$, such that,

$$
\begin{equation*}
s\left(x_{i}, t_{i}\right)=0 \tag{7}
\end{equation*}
$$

Note that at these points, the system is essentially out of control for any $u$; thus, the dynamics are solely determined by $f(x)$. Therefore, in general, the control singular points are considered to be separately located, so that the control $u$ only loses its power at these particular points; otherwise, finite-escape phenomenon may be produced by $f(x)$ when $u$ remains useless for a period of $s(x, t)=0$. According to this consideration, the following assumption is made, which in fact is quite weak and can be easily satisfied for most common continuous functions.

Assumption 1. Assume that there always exists an arbitrarily small positive constant $\delta$, such that one and only one control singular point locates in the neighborhood $|s(x, t)| \leq \delta$, while $|s(x, t)| \geq \delta$ outside these neighborhoods.

In addition, since $f(x)$ may not stabilize the system at the origin point $x=0$, it has to require that the control $u$ be effective when $x=0$, or equivalently, $s(0, t)$ is a nonzero constant instead of a control singular point for any $t$. In other words, $s(x, t)$ keeps varying between positive and negative when $x \neq 0$ and stops until $x=0$. To characterize this point, another assumption is imposed.

Assumption 2. Assume that the derivative of $s(x, t)$ can be written into

$$
\begin{equation*}
\frac{d s}{d t}=c(x, t) h(x) \tag{8}
\end{equation*}
$$

where $h(x)$ is a known function satisfying

$$
h(x) \begin{cases}\neq 0 & \text { when } x \neq 0  \tag{9}\\ =0 & \text { when } x=0\end{cases}
$$

Thus, $d s / d t=0$ when $x=0$ for any $t . c(x, t)$ stands for the "speed" of sign variation and is an unknown bounded positive function as follows:

$$
\begin{equation*}
c(x, t) \in\left[c_{\mathrm{low}}, c_{\mathrm{up}}\right]>0 \tag{10}
\end{equation*}
$$

where $c_{\text {low }}$ and $c_{\text {up }}$ are unknown positive constants.
Remark 3. Though $s(0, t)$ is a nonzero constant when $x=0$, $b(0, t)$ as well as $a(0, t)$ is not necessarily a constant and may still vary.

Remark 4. Though $h(x)$ needs to be known for control design in the next section, the other functions, that is, $a(x, t), b(x, t)$, $s(x, t)$, and $c(x, t)$, as well as their bounds, that is, $b_{\text {low }}, b_{\text {up }}, c_{\text {low }}$, $c_{\mathrm{up}}$, and $\delta$, are completely unknown. Therefore, it still provides us great flexibilities to model the control coefficient and to admit various kinds of uncertainties.

Remark 5. As stated before, the focus of this research is to tackle the continuously varying "sign" function $s(x, t)$. Note that since the unknown function $c(x, t)$ could be very large, $s(x, t)$ may vary very rapidly between positive and negative, which undoubtedly increases the difficulty to track the varying sign. In addition, because there exist some control singular points where the system is essentially out of control, the controller must be carefully designed to avoid the potential finite-time escape phenomenon.

## 3. Controller Design

Let us define the Lyapunov function $V(x)$ such that

$$
\begin{gather*}
V(x) \begin{cases}>0 & \text { when } x \neq 0 \\
=0 & \text { when } x=0,\end{cases}  \tag{11}\\
\frac{d V}{d x} \triangleq D(x)=\operatorname{sgn}(x) \frac{|h(x)|}{\rho(x)} \tag{12}
\end{gather*}
$$

Since $|h(x)| \geq 0$ and $\rho(x)>0$ are known functions, one can always solve $V(x)$ by using the differential equation (12) and $V(0)=0$ as initial condition. By using (12) to define a Lyapunov function, the information of $h(x)$ can be introduced into the controller design process; thus, it is able to avoid the finite-time escape phenomenon at control singular points as shown later. Consequently, the dynamics of $V(x)$ can be expressed as follows:

$$
\begin{equation*}
\dot{V}=\frac{d V}{d x} \dot{x}=D(x)(f(x)+a u) \tag{13}
\end{equation*}
$$

Now, design the control $u$ as follows:

$$
\begin{equation*}
u=g(\theta) z \tag{14}
\end{equation*}
$$

Conceptually, $z$ is a common robust controller [21] that can stabilize the system $\dot{x}(t)=f(x)+z$, that is, the simplified form of (1) when $a=1$, and $g(\theta)$ is the Nussbaum function that is used to deal with the uncertain and varying $a$.

Rewrite (13) as follows:

$$
\begin{equation*}
\dot{V}=D(x)(f(x)+z)+D(x)(a u-z) \tag{15}
\end{equation*}
$$

and let $z$ be given as follows:

$$
\begin{equation*}
z=-\frac{D(x) \rho^{2}(x)}{|D(x)| \rho(x)+e^{-t}}=-\frac{\operatorname{sgn}(x)|h(x)| \rho(x)}{|h(x)|+e^{-t}} \tag{16}
\end{equation*}
$$

Then, one can readily have

$$
\begin{align*}
D(x)(f(x)+z) & \leq|D(x) f(x)|+D(x) z \\
& \leq|D(x)| \rho(x)-D(x) \frac{\operatorname{sgn}(x)|h(x)| \rho(x)}{|h(x)|+e^{-t}} \\
& =|h(x)|-\frac{h^{2}(x)}{|h(x)|+e^{-t}} \\
& =\frac{|h(x)| e^{-t}}{|h(x)|+e^{-t}} \leq e^{-t} . \tag{17}
\end{align*}
$$

Taking (17) into (15) yields

$$
\begin{equation*}
\dot{V} \leq e^{-t}+D(x)(a u-z)=e^{-t}+(a g(\theta)-1) D(x) z \tag{18}
\end{equation*}
$$

Now, we will design the Nussbaum function $g(\theta)$, which can adjust its value between positive and negative according to the control performance index $\theta$. In this paper, $\theta$ is defined as follows:

$$
\begin{equation*}
\dot{\theta}=-D(x) z=\frac{h^{2}(x)}{|h(x)|+e^{-t}}, \quad \theta_{0}=0 \tag{19}
\end{equation*}
$$

It readily computes the lower and upper bounds of $\dot{\theta}$ as follows:

$$
\begin{equation*}
\dot{\theta} \in\left[\max \left(0,|h(x)|-e^{-t}\right),|h(x)|\right] \geq 0 \tag{20}
\end{equation*}
$$

Then (18) can be rewritten into

$$
\begin{equation*}
\dot{V} \leq(1-a g(\theta)) \dot{\theta}+e^{-t} \tag{21}
\end{equation*}
$$

and taking integration yields

$$
\begin{align*}
V_{t} & \leq \int_{0}^{t} \dot{\theta} d \tau-\int_{0}^{t} a g(\theta) \dot{\theta} d \tau+V_{0}+\int_{0}^{t} e^{-\tau} d \tau \\
& \leq \theta_{t}-\int_{0}^{\theta_{t}} a g(\theta) d \theta+V_{0}+1 \tag{22}
\end{align*}
$$

where $V_{t}=V(t), V_{0}=V(0)$, and $\theta_{t}=\theta(t)$.
Conventionally, the Nussbaum gain often adopts the form $g(\theta)=\exp \left(\theta^{2}\right) \cos (\theta), \exp \left(\theta^{2}\right) \sin (\theta), \theta^{2} \cos (\theta)$, or $\theta^{2} \sin (\theta)$, so that $g(\theta)$ can swing between positive infinite and negative infinite according to the control performance index $\theta$. Thus, it provides a possibility of correcting inappropriate deviation caused by erroneous previous control. However, the signvarying speed of the conventional Nussbaum gain is limited. For example, each variation of the $\operatorname{sign}$ of $\exp \left(\theta^{2}\right) \sin (\pi \theta)$ needs $\theta$ to increase 1 ; therefore, it may not be able to track the rapidly varying $s(k)$ in this paper. To deal with this, a new Nussbaum function is proposed as follows:

$$
\begin{equation*}
g(\theta)=2 \theta e^{\theta^{2 \eta}} \sin \left(\pi \theta^{2}\right) \tag{23}
\end{equation*}
$$

where the constant $\eta>1$ is a design parameter. Note that $\theta^{2}$ grows nonlinearly, and thus the sign-varying speed of $g(\theta)$ keeps increasing with the increase of $\theta$. Consequently, the proposed $g(\theta)$ varies much faster than those conventional ones when $\theta$ is sufficiently large. The property of the proposed $g(\theta)$ is presented below.

Lemma 6. Suppose that $|s(x, t)| \geq \varepsilon$, where $\varepsilon$ is an arbitrary positive constant that is smaller than 1. If $\operatorname{ag}(\theta)>0$ for $\theta^{2} \epsilon$ $\left[\left\lfloor\Theta^{2}\right\rfloor-1,\left\lfloor\Theta^{2}\right\rfloor\right]$ when $\Theta \rightarrow \infty$, then we have

$$
\begin{equation*}
\lim _{\Theta \rightarrow \infty} \sup \frac{1}{\Theta^{2}} \int_{0}^{\Theta} a g(\theta) d \theta=+\infty \tag{24}
\end{equation*}
$$

where $\lfloor\cdot\rfloor$ is the truncation operator; otherwise, if ag $(\theta)<0$ for $\theta^{2} \in\left[\left\lfloor\Theta^{2}\right\rfloor-1,\left\lfloor\Theta^{2}\right\rfloor\right]$ when $\Theta \rightarrow \infty$, then

$$
\begin{equation*}
\lim _{\Theta \rightarrow \infty} \inf \frac{1}{\Theta^{2}} \int_{0}^{\Theta} \operatorname{ag}(\theta) d \theta=-\infty \tag{25}
\end{equation*}
$$

The detailed proof of Lemma 6 is given in the Appendix. The property of $g(\theta)$ will be used to derive the asymptotic convergence in the next section.

## 4. Convergence Analysis

Before presenting the main theorem of asymptotic convergence, the following lemma is introduced first.

Lemma 7. If $\theta_{t}$ is bounded for any $t \in[0,+\infty)$, all closedloop signals are bounded; that is, $\theta_{r}$ and $V_{r}$ are bounded for $r \in[0, t)$, and the system is asymptotically stabilized; that is, $x \rightarrow 0$ when $t \rightarrow+\infty$.

Proof. The proof of Lemma 7 is straightforward. Since $|h(x)|$ is semipositive, from (19), we have

$$
\dot{\theta}=\frac{h^{2}(x)}{\|h(x)\|+e^{-t}} \begin{cases}>0 & \text { when } x \neq 0  \tag{26}\\ =0 & \text { when } x=0\end{cases}
$$

Clearly, $\theta$ is a semipositive and nondecreasing variable. Thus the boundedness of $\theta_{t}$ implies the boundedness of $\theta_{r}$ for $r \in[0, t)$. Then according to (22), since $a, g(\theta)$, and $V_{0}$ are all bounded, $V_{r}$ is also bounded when $\theta_{r}$ is bounded. On the other hand, the boundedness of $\theta_{t}$ also implies that $\dot{\theta} \rightarrow 0$ when $t \rightarrow+\infty$, since $\dot{\theta}$ is continuous and semipositive; or equivalently, $|h(x)| \rightarrow 0$ when $t \rightarrow+\infty$. Then from (9), we can readily conclude that $x \rightarrow 0$ when $t \rightarrow+\infty$. This completes the proof.

Now, the main result is stated below.
Theorem 8. The proposed controller in (14), (16), (20), and (23) guarantees the boundedness of all closed-loop signals and is able to drive $x \rightarrow 0$ when $t \rightarrow+\infty$.

Proof. As shown in Lemma 7, the key of the proof is to show the boundedness of $\theta_{t}$. Because $s(x, t)$ may cross zero and change its sign, we will consider two situations: (1) $s(x, t)$ is crossing zero, and (2) $s(x, t)$ is bounded away from zero. Next, we will prove that $\theta_{t}$ is bounded under both situations.

Situation 1. $s(x, t)$ is crossing zero. Let $s\left(x_{i}, t_{i}\right)=0$ and consider a small neighborhood of it; that is, $s(x, t) \mid \leq \varepsilon$, where $\varepsilon=$ $\delta / 2$ is a small positive constant. Since the positive constant $\delta$ in Assumption 1 can be arbitrarily small, it still has only one control singular point in the neighborhood $|s(x, t)| \leq \varepsilon=$ $\delta / 2$. Without loss of any generality, consider the situation that $s$ varies from negative to positive. Let $s=-\varepsilon$ at $t_{-\varepsilon}, s=\varepsilon$ at $t_{\varepsilon}$, and $t_{-\varepsilon}<t_{i}<t_{\varepsilon}$. From (8), one can readily yield

$$
\begin{gather*}
s\left(x, t_{-\varepsilon}\right)=\int_{0}^{t_{-\varepsilon}} c(x, \tau) h(x) d \tau+s_{0}=-\varepsilon  \tag{27}\\
s\left(x, t_{\varepsilon}\right)=\int_{0}^{t_{\varepsilon}} c(x, \tau) h(x) d \tau+s_{0}=\varepsilon \tag{28}
\end{gather*}
$$

Consequently, we have

$$
\begin{align*}
\int_{t_{-\varepsilon}}^{t_{\varepsilon}} h(x) d \tau & \leq \frac{\int_{t_{-\varepsilon}}^{t_{\varepsilon}} c(x, \tau) h(x) d \tau}{\min (c(x, t))} \leq \frac{\varepsilon-s_{0}-\left(-\varepsilon-s_{0}\right)}{c_{\text {low }}} \\
& =\frac{2 \varepsilon}{c_{\text {low }}} \tag{29}
\end{align*}
$$

Note that since $s$ varies from negative to positive, $d s / d t>0$ for $\tau \in\left[t_{-\varepsilon}, t_{\varepsilon}\right]$, or equivalently, $h(x)>0$ for $\tau \in\left[t_{-\varepsilon}, t_{\varepsilon}\right]$. Then, according to (20), (29) implies that

$$
\begin{equation*}
\theta_{t_{\varepsilon}}-\theta_{t_{-\varepsilon}} \leq \int_{t_{-\varepsilon}}^{t_{\varepsilon}}|h(x)| d \tau=\int_{t_{-\varepsilon}}^{t_{\varepsilon}} h(x) d \tau \leq \frac{2 \varepsilon}{c_{\mathrm{low}}} \tag{30}
\end{equation*}
$$

Since $\mathcal{c}_{\text {low }}$ is positive constant, $2 \varepsilon / \mathcal{c}_{\text {low }}$ is bounded. Equation (30) indicates that the increment of $\theta$ for $\tau \in\left[t_{-\varepsilon}, t_{\varepsilon}\right]$, or equivalently, in the neighborhoods of control singular points, is bounded. Note that (30) also implies that the finite-time escape phenomenon will not occur in these neighborhoods, since finite increment of $\theta$ insures finite increment of $V$ according to (22).

Situation 2. $s(x, t)$ is bounded away from zero. Consider a continuous interval $|s| \geq \varepsilon$ between two successive control singular points. Note that from Assumption 1, the sign of $s$ will not change and $\max (|s|) \geq \delta=2 \varepsilon$ in this interval. Without loss of any generality, let $s=\varepsilon$ at $t_{\varepsilon}$ and $s=\delta=2 \varepsilon$ at $t$. The boundedness of $\theta_{t}$ within this interval will be proven by the contradiction method; that is, we will first assume that $\theta_{t}$ is unbounded and then derive some contradictions.

Since $\theta \geq 0$, if $\theta_{t}$ is unbounded, we have $\theta_{t} \rightarrow+\infty$. As $V_{t} \geq 0$, (22) gives

$$
\begin{equation*}
0 \leq V_{t} \leq \theta_{t}-\int_{0}^{\theta_{t}} a g(\theta) d \theta+V_{0}+1 \tag{31}
\end{equation*}
$$

and we further have

$$
\begin{equation*}
\frac{1}{\theta_{t}^{2}} \int_{0}^{\theta_{t}} a g(\theta) d \theta \leq \frac{1}{\theta_{t}^{2}}+\frac{1}{\theta_{t}}+\frac{V_{0}}{\theta_{t}^{2}} \tag{32}
\end{equation*}
$$

which implies that

$$
\begin{equation*}
\lim _{\theta_{t} \rightarrow+\infty} \frac{1}{\theta_{t}^{2}} \int_{0}^{\theta_{t}} a g(\theta) d \theta \leq 0 \tag{33}
\end{equation*}
$$

Next, we will deduce a result for $\theta_{t} \rightarrow+\infty$ that contradicts (33), separately, for two cases: (a) $\theta_{\varepsilon}$ is bounded; (b) $\theta_{\varepsilon}$ is unbounded, where $\theta_{\varepsilon}=\theta\left(t_{\varepsilon}\right)$.
(a) $\theta_{\varepsilon}$ Is Bounded. Since $\theta_{t} \rightarrow+\infty$, we conclude that the increment between $\theta_{t}$ and $\theta_{\varepsilon}$ is infinite; that is, $\theta_{t}-\theta_{\varepsilon} \rightarrow+\infty$. It implies that $g(\theta)$ swings infinite times between positive and negative for $\theta \in\left[\theta_{\varepsilon}, \theta_{t}\right]$, or equivalently for $\tau \in\left[t_{\varepsilon}, t\right]$. Since the sign of $s$ is fixed for $\tau \in\left[t_{\varepsilon}, t\right]$, the sign of $\operatorname{ag}(\theta)$ is solely determined by $g(\theta)$ for $\tau \in\left[t_{\varepsilon}, t\right]$. Without loss of any generality, let $\operatorname{ag}(\theta)>0$ for $\theta^{2} \in\left[\left\lfloor\theta_{t}^{2}\right\rfloor-1,\left\lfloor\theta_{t}^{2}\right\rfloor\right]$, which satisfies the condition of Lemma 6. Consequently, according to (24), we have

$$
\begin{equation*}
\lim _{\theta_{t} \rightarrow+\infty} \sup \frac{1}{\theta_{t}^{2}} \int_{0}^{\theta_{t}} a g(\theta) d \theta=+\infty \tag{34}
\end{equation*}
$$

Clearly (34) contradicts (33), which implies that $\theta_{t}$ must be bounded if $\theta_{\varepsilon}$ is bounded.
(b) $\theta_{\varepsilon}$ Is Unbounded. Since $\theta \geq 0$, we have $\theta_{\varepsilon} \rightarrow+\infty$. Note that from Assumption 1, the sign of $h(x)$ is fixed for $\tau \in\left[t_{\varepsilon}, t\right]$.

Without loss of any generality, let $h(x)>0$. Similar to (28), we have

$$
\begin{align*}
& s\left(x, t_{\varepsilon}\right)=\int_{0}^{t_{\varepsilon}} c(x, \tau) h(x) d \tau+s_{0}=\varepsilon \\
& s\left(x, t_{\sigma}\right)=\int_{0}^{t_{\sigma}} c(x, \tau) h(x) d \tau+s_{0}=2 \varepsilon \tag{35}
\end{align*}
$$

Then, similar to (30), we further have

$$
\begin{align*}
\int_{t_{\varepsilon}}^{t} h(x) d \tau & \geq \frac{\int_{t_{\varepsilon}}^{t} c(x, \tau) h(x) d \tau}{\max (c(x, t))} \geq \frac{\int_{t_{\varepsilon}}^{t} c(x, \tau) h(x) d \tau}{c_{\mathrm{up}}}  \tag{36}\\
& \geq \frac{\delta-\varepsilon}{c_{\mathrm{up}}}=\frac{\varepsilon}{c_{\mathrm{up}}}
\end{align*}
$$

which can be simplified as follows:

$$
\begin{equation*}
\int_{t_{\varepsilon}}^{t} h(x) d \tau \geq \gamma \tag{37}
\end{equation*}
$$

where $\gamma \triangleq \varepsilon / c_{\text {up }}$ is a positive constant since $\varepsilon$ could be an arbitrarily small positive constant.

Then, we will estimate the increment of $\theta_{t}^{2}-\theta_{\varepsilon}^{2}$. Note that $h(x)>0$ for $\tau \in\left[t_{\varepsilon}, t\right]$ and then take integration for (20) as follows:

$$
\begin{align*}
\theta_{t}-\theta_{\varepsilon} & =\int_{t_{\varepsilon}}^{t} \dot{\theta} d \tau \geq \int_{t_{\varepsilon}}^{t} h(x)-e^{-\tau} d \tau  \tag{38}\\
& \geq \int_{t_{\varepsilon}}^{t} h(x) d \tau-\left(e^{-t_{\varepsilon}}-e^{-t}\right)>\gamma-e^{-t_{\varepsilon}}
\end{align*}
$$

We can always choose a sufficiently large constant $\bar{t}$, such that for any $t_{\varepsilon} \geq \bar{t}$, we have

$$
\begin{equation*}
e^{-t_{\varepsilon}} \leq \frac{\gamma}{2} \tag{39}
\end{equation*}
$$

Thus the increment of $\theta_{t}^{2}-\theta_{\varepsilon}^{2}$ for $t_{\varepsilon} \geq \bar{t}$ can be readily estimated by taking (39) into account as follows:

$$
\begin{equation*}
\theta_{t}^{2}-\theta_{\varepsilon}^{2}=\left(\theta_{t}+\theta_{\varepsilon}\right)\left(\theta_{t}-\theta_{\varepsilon}\right)>2 \theta_{\varepsilon}\left(\gamma-\frac{\gamma}{2}\right)=\theta_{\varepsilon} \gamma \tag{40}
\end{equation*}
$$

Again, note that the sign of $s$ is fixed for $\tau \in\left[t_{\varepsilon}, t\right]$ and the sign of $\operatorname{ag}(\theta)$ is determined by $g(\theta)$. Since $\gamma$ is a positive constant and $\theta_{\varepsilon}$ is unbounded, with the increase of $\theta_{\varepsilon}$, we will certainly have

$$
\begin{equation*}
\theta_{t}^{2}-\theta_{\varepsilon}^{2}>\theta_{\varepsilon} \gamma \geq 3 \tag{41}
\end{equation*}
$$

Clearly, (41) implies that $g(\theta)$ has completed at least a positive round and a negative round when $\theta$ increases from $\theta_{\varepsilon}$ to $\theta_{t}$. Consequently, the conditions of Lemma 6 have been fulfilled. Similar to case (a), we could assume that $\operatorname{ag}(\theta)>0$ for $\theta^{2} \epsilon$ $\left[\left\lfloor\theta_{t}^{2}\right\rfloor-1,\left\lfloor\theta_{t}^{2}\right\rfloor\right]$ without loss of any generality. Then according to (24), we have the same result as (34) as follows:

$$
\begin{equation*}
\lim _{\theta_{t} \rightarrow+\infty} \sup \frac{1}{\theta_{t}^{2}} \int_{0}^{\theta_{t}} \operatorname{ag}(\theta) d \theta=+\infty \tag{42}
\end{equation*}
$$

which still contradicts (33). Therefore, $\theta_{t}$ must be bounded to avoid any contradiction.

In summary, $\theta_{t}$ must be bounded for the intervals of $|s| \geq$ $\varepsilon$ from Situation 2 and its increment is also bounded for the intervals of $|s| \leq \varepsilon$ from Situation 1. As a result, $\theta_{t}$ is always bounded, and thus all closed-loop signals are bounded and $V_{t} \rightarrow 0$ when $t \rightarrow+\infty$. This completes the proof.

Remark 9. Since $\theta_{t}$ is bounded, $\int_{0}^{t} h(x) d \tau$ is also bounded because

$$
\begin{equation*}
\int_{0}^{t} h(x) d \tau \leq \int_{0}^{t} \dot{\theta}+e^{-\tau} d \tau \leq \theta_{t}+1 \tag{43}
\end{equation*}
$$

On the other hand, (37) indicates that the integral of $h(x)$ needs to increase at least $\gamma$ to trigger a sign variation of control coefficient. Since $\gamma$ is a positive constant, $\int_{0}^{t} h(x) d \tau / \gamma$ is also bounded, which essentially implies that the control coefficient will only undergo finite number of sign variations in the entire control process.

Remark 10. The proof of the Theorem is organized as follows. First, the finite increment of $\theta$ has been proven in Situation 1 and case (a) of Situation 2, separately, at the control singular points and in the intervals between these points. That is, these two parts show that a complete sign variation process corresponds to finite increment of $\theta$. Second, case (b) of Situation 2 further shows that the control coefficient only has finite number of sign variations. In such a way, the proof of finite $\theta_{t}$ has been completed.

Remark 11. Note that according to Lemma 7, the boundedness of $\theta_{t}$ for $t \in[0,+\infty)$ also implies that no finite-time escape phenomenon will occur, even at the control singular points.

## 5. Simulation

An example is used to illustrate the performance of the proposed control. Consider system (1) with the following setup:

$$
\begin{align*}
& f(x)=\cos (x) x^{2} \\
& b(x, t)=\exp (\cos (x)+\sin (t)) \\
& s(k)=\cos (k)  \tag{44}\\
& c(x, t)=\exp (\sin (x t)) \\
& h(x)=|x|
\end{align*}
$$

Let the initial state be $x_{0}=2$ and the known bounding function be $\rho(x)=x^{2}+1$. Note that only $\rho(x)$ and $h(x)$ are known for controller design. Let the controller in (14), (16), (20), and (23) be used with the parameter settings $\eta=1.01$ and $\dot{\theta}=-D(x) z / 10$. It is shown by Figure 1 that the proposed controller can drive $x$ to 0 asymptotically and Figure 2 shows the profile of other bounded variables.


Figure 1: The profiles of $a$ and $x$.


Figure 2: The profiles of $\theta, g$, and $u$.

## 6. Conclusion

In this paper, the control problem is studied for a class of nonlinear uncertain systems with the uncertain control coefficient, which is allowed to vary continuously between positive and negative. A new Nussbaum gain is designed and integrated with robust controller to tackle this problem. By following the Lyapunov-fashion controller design procedure, the potential finite-time escape phenomenon is avoided. It is proven that the proposed control approach yields asymptotic stability and guarantees the boundedness of the closed-loop signals.

## Appendix

Proof of Lemma 6. For the concise of the proof, we will only consider the case of $\Theta \rightarrow+\infty$, while the same results can be obtained similarly for $\Theta \rightarrow-\infty$. Now define

$$
\begin{equation*}
G_{i}=\int_{\sqrt{i}}^{\sqrt{i+1}} g(\theta) d \theta \tag{A.1}
\end{equation*}
$$

where $i$ is a positive integer. Further calculation yields

$$
\begin{align*}
G_{i} & =\int_{\sqrt{i}}^{\sqrt{i+1}} 2 \theta e^{\theta^{2 \eta}} \sin \left(\pi \theta^{2}\right) d \theta=\int_{\sqrt{i}}^{\sqrt{i+1}} e^{\theta^{2 \eta}} \sin \left(\pi \theta^{2}\right) d \theta^{2} \\
& =\int_{i}^{i+1} e^{w^{\eta}} \sin (\pi w) d w \tag{A.2}
\end{align*}
$$

where $w=\theta^{2}$. Clearly, $G_{i}$ is positive if $i$ is even and negative if $i$ is odd.

Because $w$ is positive and $\eta>1$, it is readily to show that

$$
\begin{equation*}
(w+1)^{\eta}>w^{\eta}+\eta w^{\eta-1} . \tag{A.3}
\end{equation*}
$$

Then, according to (A.3), the ratio between $G_{i+1}$ and $G_{i}$ is

$$
\begin{align*}
\frac{G_{i+1}}{G_{i}} & =\frac{\int_{i+1}^{i+2} e^{w^{\eta}} \sin (\pi w) d w}{\int_{i}^{i+1} e^{w^{\eta}} \sin (\pi w) d w} \\
& =\frac{\int_{i}^{i+1} e^{(w+1)^{\eta}} \sin (\pi(w+1)) d w}{\int_{i}^{i+1} e^{w^{\eta}} \sin (\pi w) d w} \\
& =\frac{-\int_{i}^{i+1} e^{(w+1)^{\eta}} \sin (\pi w) d w}{\int_{i}^{i+1} e^{w^{\eta}} \sin (\pi w) d w}  \tag{A.4}\\
& <\frac{-\int_{i}^{i+1} e^{w^{\eta}+\eta w^{\eta-1}} \sin (\pi w) d w}{\int_{i}^{i+1} e^{w^{\eta}} \sin (\pi w) d w} \\
& <\frac{-e^{\eta i^{\eta-1}} \int_{i}^{i+1} e^{w^{\eta}} \sin (\pi w) d w}{\int_{i}^{i+1} e^{w^{\eta}} \sin (\pi w) d w} \\
& =-e^{\eta^{i \eta-1}} \cdot
\end{align*}
$$

Because $\eta i^{\eta-1}>1$ and $e^{\eta i^{\eta-1}}>e$, it is clear that $G_{i} \rightarrow \infty$ and $G_{i+1} / G_{i} \rightarrow-\infty$, when $i \rightarrow+\infty$. Let us investigate the situation of $\Theta^{2}$ being an integer and $\Theta^{2}=\left\lfloor\Theta^{2}\right\rfloor=i+1$; that is, $\operatorname{ag}(\theta)>0$ for $\theta^{2} \in[i, i+1]$. Since $\varepsilon \leq|s(x, t)| \leq 1$, we have $\varepsilon b_{\text {low }} \leq|a| \leq b_{\text {up }}$. Then, we will firstly prove (24). Consider the following:

$$
\begin{align*}
\int_{0}^{\sqrt{i+1}} a g(\theta) d \theta & \geq-\int_{0}^{\sqrt{i}}|a g(\theta)| d \theta+\int_{\sqrt{i}}^{\sqrt{i+1}} a g(\theta) d \theta \\
& \geq-b_{\text {up }} \int_{0}^{\sqrt{i}}|g(\theta)| d \theta+\varepsilon b_{\text {low }} \int_{\sqrt{i}}^{\sqrt{i+1}} a g(\theta) d \theta \\
& \geq-b_{\text {up }} \sum_{l=0}^{i-1}\left|G_{l}\right|+\varepsilon b_{\text {low }}\left|G_{i}\right| \\
& \geq-b_{\text {up }}(i-1)\left|G_{i-1}\right|+\varepsilon b_{\text {low }} e^{\eta i^{\eta-1}}\left|G_{i-1}\right| \\
& =\left(-b_{\text {up }}(i-1)+\varepsilon b_{\text {low }} e^{\eta i^{\eta-1}}\right)\left|G_{i-1}\right| \tag{A.5}
\end{align*}
$$

Clearly,

$$
\begin{equation*}
\frac{1}{i+1} \int_{0}^{\sqrt{i+1}} a g(\theta) d \theta \geq\left(-b_{\mathrm{up}} \frac{i-1}{i+1}+\varepsilon b_{\mathrm{low}} \frac{1}{i+1} e^{\eta i^{\eta-1}}\right)\left|G_{i-1}\right| \tag{A.6}
\end{equation*}
$$

Because $b_{\text {low }}, b_{\text {up }}$, and $\varepsilon$ are positive and finite and $\eta>1$, when $i \rightarrow+\infty$, we have

$$
\begin{align*}
& \lim _{i \rightarrow+\infty}\left(-b_{\mathrm{up}} \frac{i-1}{i+1}+\varepsilon b_{\text {low }} \frac{1}{i+1} e^{\eta i^{\eta-1}}\right)\left|G_{i-1}\right| \\
& \quad=\left(-b_{\mathrm{up}}+\varepsilon b_{\mathrm{low}_{i \rightarrow+\infty}} \lim _{i \rightarrow+} \frac{1}{i+1} e^{\eta i^{\eta-1}}\right)\left|G_{i-1}\right| \longrightarrow+\infty \tag{A.7}
\end{align*}
$$

Consequently, (A.7) implies that

$$
\begin{equation*}
\lim _{i \rightarrow+\infty} \frac{1}{i+1} \int_{0}^{\sqrt{i+1}} \operatorname{ag}(\theta) d \theta \longrightarrow+\infty \tag{A.8}
\end{equation*}
$$

which is the result of (24) since $\Theta^{2}=i+1$.
Similarly, when $|a| \geq \varepsilon b_{\text {low }}$ if $\operatorname{ag}(\theta)<0$ for $\theta^{2} \in\left[\left\lfloor\Theta^{2}\right\rfloor-\right.$ $\left.1,\left\lfloor\Theta^{2}\right\rfloor\right]$ with $\Theta^{2}=\left\lfloor\Theta^{2}\right\rfloor=i+1$, we can also show that

$$
\begin{equation*}
\lim _{i \rightarrow+\infty} \frac{1}{i+1} \int_{0}^{\sqrt{i+1}} a g(\theta) d \theta \longrightarrow-\infty \tag{A.9}
\end{equation*}
$$

which is the result of (25). For the case of $\Theta \rightarrow-\infty$, the same results can also be obtained by similar procedures. The proof is complete.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Encryption in Chaotic Systems with Sinusoidal Excitations 

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#### Abstract

In this contribution an encryption method using a chaotic oscillator, excited by " $n$ " sinusoidal signals, is presented. The chaotic oscillator is excited by a sum of " $n$ " sinusoidal signals and a message. The objective is to encrypt such a message using the chaotic behavior and transmit it, and, as the chaotic system is perturbed by the sinusoidal signal, the transmission security could be increased due to the effect of such a perturbation. The procedure is based on the regulation theory and consider that the receiver knows the frequencies of the perturbing signal, with this considerations the algorithm estimates the excitation in such a way that the receiver can cancel out the perturbation and all the undesirable dynamics in order to produce only the message. In this way we consider that the security level is increased.


## 1. Introduction

Since Pecora and Carroll presented their work about chaos synchronization [1], the investigation on chaos has received significant attention and the phenomenon has spread to the application in communications security ([2-4]). The use of chaotic systems for encoding and decoding is a new method different from the conventional techniques; see, for example, the work presented by Cuomo and Oppenheim in which a voice signal is encrypted by chaos and then is sent by the communication channel and is retrieved successfully [5].

In recent years, encryption schemes are being studied and increasing demand exists to develop a safer encryption system for transmitting data in real time via the Internet, wireless networks, and other devices ( $[6,7]$ ).

The traditional standard encryption algorithm for images and data (DES) has a disadvantage when handling large amounts of data [8] and is performed digitally (first on a PC), and then sends the encrypted signal.

The online encryption in a dynamic system has the advantage of processing the signal in real time, so the analog signal (message) is encrypted while being sent.

In recent years some works were presented for the synchronization of master-slave structure ([9-11]), in the cites references a perturbation signal was introduced in the slave
system and no perturbation was introduced in the master system. In different away, in the present work, a perturbation defined by the combination of the message and a sinusoidal signal is introduced in the master system.

In this work and algorithm is proposed in order to encrypt the message using a nonlinear chaotic system; this message is combined with " $n$ " sinusoidal signals whose amplitudes, phases, and bias are all unknown.

These signals excite the chaotic oscillator (the master) in order to encrypt the message more safely in the sense that the message signal is combined with sinusoidal signals; this combination increases the harmonics produced by the chaotic system and then the spectrum of the sending signal has more frequencies in the bandwidth of the signal.

In the proposed scheme the receiver (the slave) knows the frequencies to be used and estimates the sinusoidal signals which perturb the message and retrieve the message in exacta away.

In many other works (see [2-5]) only the message is used to excite the chaotic dynamical system and then is retrieved by the receptor. This is the principal difference in this work.

The paper is organized as follows: in Section 2 the problem is presented, the nonlinear estimator is designed in Section 3, in Section 4 several examples are shown, and finally in Section 5 some conclusions are presented.

## 2. Statement of the Problem

In this paper a nonlinear system is proposed, which is excited by a signal given by the following equation:

$$
\begin{equation*}
p(t)=B+\sum_{i=1}^{n} A_{i} \sin \left(\alpha_{i} t+\varphi_{i}\right) \tag{1}
\end{equation*}
$$

where the amplitudes $A_{i} \neq 0$, the phases $\varphi_{i}$, and the constant bias $B$ are unknown. In addition to the sinusoidal perturbation (1), an information signal " $m(t)$ " will be added in order to encrypt this message by the nonlinear dynamical chaotic system.

It is observed that the signal perturbation $p(t)$ can be generated by a linear system

$$
\begin{equation*}
\dot{w}=S w \tag{2}
\end{equation*}
$$

which is called exosystem [12].
Various types of chaotic systems can be treated under the impulse of a sinusoidal input signal. In general the chaotic system to the encryption of the message can be seen in (3), where $f\left(x_{1}, x_{2}, \eta\right)$ and $\alpha\left(x_{1}, x_{2}\right)$ are nonlinear functions of the states, $p(t)$ is the sinusoidal perturbation, and $m(t)$ is the message to be encrypted:

$$
\begin{align*}
& \dot{x}_{1}=x_{2} \\
& \dot{x}_{2}=-a_{1} x_{1}-a_{2} x_{2}+f\left(x_{1}, x_{2}, \eta\right)+m(t)+p(t),  \tag{3}\\
& \dot{\eta}=-a_{3} \eta+\alpha\left(x_{1}, x_{2}\right) .
\end{align*}
$$

Having described the chaotic system model in general, we proceed to select the outputs of the system, which are given by

$$
\begin{align*}
& y_{0}=x_{1}+x_{2}+f\left(x_{1}, x_{2}, \eta\right)+m(t)  \tag{4}\\
& y_{1}=x_{1} .
\end{align*}
$$

Some chaotic oscillators are studied in the following paragraphs and will be adapted to realize the desired encryption and take the form (3).

It is proposed that the outputs given by (4) can recover the message $m(t)$ and the disturbance signal $p(t)$ as shown in Figure 1.

If an intruder is capable of obtaining the transmission signal and can eliminate the chaos, find that the message will be disturbed by a sinusoidal signal and its harmonics as this was injected into the chaotic system, so the message is not decoded and the information will be preserved.

Note. Note that even though two outputs are taken, it is possible to transmit a single signal, if we perform a frequency multiplexing to send the two signals into one channel and they can be recovered later in the receiver.

In what follows some chaotic systems and how they are processed to take a proper structure are presented.


Figure 1: Estimation of disturbance $p(t)$ and message $m(t)$ in the chaotic system.
2.1. The Van der Pol Chaotic System. The Van der Pol equation provides an example of a nonlinear oscillator system. The system can be written as

$$
\begin{equation*}
\ddot{x}-\mu\left(1-x^{2}\right) \dot{x}+x=p+m ; \tag{5}
\end{equation*}
$$

taking $x_{1}=x$ and $x_{2}=\dot{x}$, the system takes the form (2):

$$
\begin{align*}
& \dot{x}_{1}=x_{2}  \tag{6}\\
& \dot{x}_{2}=-x_{1}+\mu x_{2}-\mu x_{1}^{2} x_{2}+p+m
\end{align*}
$$

with the following coefficients:

$$
\begin{equation*}
a_{1}=1, \quad a_{2}=-\mu, \quad f\left(x_{1}, x_{2}, \eta\right)=-\mu x_{1}^{2} x_{2} \tag{7}
\end{equation*}
$$

2.2. The Duffing Chaotic System. Duffing equation is introduced in 1918 as a nonlinear oscillator model. The equation is defined as

$$
\begin{equation*}
\ddot{x}+\delta \dot{x}-\beta x+x^{3}=p+m \tag{8}
\end{equation*}
$$

taking $x_{1}=x$ and $x_{2}=\dot{x}$, the system takes form (2):

$$
\begin{align*}
& \dot{x}_{1}=x_{2} \\
& \dot{x}_{2}=\beta x_{1}-\delta x_{2}-x_{1}^{3}+p+m \tag{9}
\end{align*}
$$

with the following coefficients:

$$
\begin{equation*}
a_{1}=-\beta, \quad a_{2}=\delta, \quad f\left(x_{1}, x_{2}, \eta\right)=-x_{1}^{3} \tag{10}
\end{equation*}
$$

2.3. The Lorenz Chaotic System. This system is known as a simplified model of multiple physical systems ([13, 14]). This system is described by

$$
\begin{align*}
& \dot{x}=a(y-x) \\
& \dot{y}=c x-y-x z+\frac{(m+p)}{a}  \tag{11}\\
& \dot{z}=-b z+x y
\end{align*}
$$

If we take $x_{1}=x, x_{2}=a(y-x)$, and $\eta=z$, the system (11) takes the form

$$
\begin{gather*}
\dot{x}_{1}=x_{2} \\
\dot{x}_{2}=-a(1-c) x_{1}-(a+1) x_{2}-a x_{1} \eta+m+p  \tag{12}\\
\dot{\eta}=-b \eta+x_{1}\left(\frac{x_{2}}{a}+x_{1}\right)
\end{gather*}
$$

and then takes the same structure as (2) where the respective coefficients are

$$
\begin{gather*}
a_{1}=a(1-c), \quad a_{2}=(a+1), \quad a_{3}=b, \\
f\left(x_{1}, x_{2}, \eta\right)=-a x_{1} \eta, \quad \alpha\left(x_{1}, x_{2}\right)=x_{1}\left(\frac{x_{2}}{a}+x_{1}\right) . \tag{13}
\end{gather*}
$$

Note. It is not difficult to see that chaotic systems of Lu, Chen, and Raylegh can be also put in form (2). Also in some cases the state $\eta$ does not exist ([11]).

For this class of systems some assumptions are considered.

Assumption 1. The constants $\left(a_{1}, a_{2}, a_{3}\right) \in \mathbb{R}$, the function $f\left(x_{1}, x_{2}, \eta\right)$, and $\eta$ dynamics are known.

Assumption 2. The $\alpha_{i}$ frequencies are known and $\alpha_{i} \neq \alpha_{j}$ for $i \neq j$.

Assumption 3. The message and the perturbation do not destroy the chaos.

With this assumption outputs (4) are chaotic and then this signal has an infinite number of frequencies.

## 3. Receiver Design

Now in order to recover the transmitted message, a state estimator for system (3) is proposed. In the estimator the information is decrypted and separated from the signal perturbation $p(t)$.

Consider the next theorem.
Theorem 4. Take the outputs signals given by (4) and consider Assumptions 1 to 3; then the estimator given by

$$
\begin{aligned}
& \dot{\hat{x}}_{1}=\widehat{x}_{2}+g_{1}\left(y_{1}-\widehat{y}_{1}\right) \\
& \dot{\hat{x}}_{2}=-a_{1} \widehat{x}_{1}-a_{2} \widehat{x}_{2}+\left(y_{0}-\widehat{y}_{0}\right)+g_{2}\left(y_{1}-\widehat{y}_{1}\right)+\widehat{p} \\
& \dot{\hat{\eta}}=-a_{3} \widehat{\eta}+\alpha\left(\widehat{x}_{1}, \widehat{x}_{2}\right) \\
& \dot{\hat{w}}=S \widehat{w}+G_{0}\left(y_{1}-\widehat{y}_{1}\right),
\end{aligned}
$$

with

$$
\begin{gather*}
\widehat{p}=\sum_{k=0}^{n} \widehat{w}_{2 k+1}, \quad S=\left[\begin{array}{ccccc}
S_{1} & 0 & \cdots & 0 & 0 \\
0 & S_{2} & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & S_{n} & 0 \\
0 & 0 & \cdots & 0 & 0
\end{array}\right],  \tag{15}\\
S_{j}=\left[\begin{array}{cc}
0 & \alpha_{j} \\
-\alpha_{j} & 0
\end{array}\right], \quad \widehat{m}=y_{0}-\widehat{y}_{0}-f\left(\widehat{x}_{1}, \widehat{x}_{2}, \widehat{\eta}\right)
\end{gather*}
$$

and outputs

$$
\begin{align*}
& \widehat{y}_{0}=\widehat{x}_{1}+\widehat{x}_{2}, \\
& \widehat{y}_{1}=\widehat{x}_{1}, \tag{16}
\end{align*}
$$

where the constants $g_{1}, g_{2}$, and $G_{0}$ are properly selected, is such that the signals are estimated; it is

$$
\begin{gather*}
\widehat{x}_{1} \longrightarrow x_{1}, \quad \hat{x}_{2} \longrightarrow x_{2}, \quad \hat{\eta} \longrightarrow \eta \\
\hat{p} \longrightarrow p, \quad \widehat{m} \longrightarrow m \tag{17}
\end{gather*}
$$

Proof. Consider the error system between (3) and (14) and the exosystem (2):

$$
\begin{align*}
& \dot{e}_{1}=e_{2}-g_{1} e_{1} \\
& \dot{e}_{2}=-\left(a_{1}+1\right) e_{1}-\left(a_{2}+1\right) e_{2}+\sum_{k=0}^{n} e_{w_{2 k+1}}-g_{2} e_{1},  \tag{18}\\
& \dot{e}_{3}=-a_{3} e_{3}+\alpha\left(x_{1}, x_{2}\right)-\alpha\left(\widehat{x}_{1}, \widehat{x}_{2}\right), \\
& \dot{e}_{w}=S e_{w}-G_{0} e_{1}
\end{align*}
$$

where the errors are defined as

$$
\begin{align*}
e_{1}=x_{1}-\widehat{x}_{1}, \quad e_{2} & =x_{2}-\widehat{x}_{2}, \quad e_{3}=\eta-\hat{\eta} \\
e_{w} & =w-\widehat{w} \tag{19}
\end{align*}
$$

From (16) consider the subsystem

$$
\begin{gather*}
{\left[\begin{array}{c}
\dot{e}_{1} \\
\dot{e}_{2} \\
\dot{e}_{w}
\end{array}\right]=\left[\begin{array}{ccc}
-g_{1} & 1 & 0 \\
-\left(a_{1}+1+g_{2}\right) & -\left(a_{2}+1\right) & V_{0} \\
-G_{0} & 0 & S
\end{array}\right]\left[\begin{array}{l}
e_{1} \\
e_{2} \\
e_{w}
\end{array}\right],}  \tag{20}\\
V_{0}=\left[\begin{array}{llllll}
1 & 0 & 1 & \cdots & 0 & 1
\end{array}\right]
\end{gather*}
$$

Then using control theory [15], it is possible to find constants $g_{1}, g_{2}$, and $G_{0}$ such that $e_{1} \rightarrow 0, e_{2} \rightarrow 0$, and $e_{w} \rightarrow 0$.

For the dynamics of $e_{3}$, it is observed that the difference between the nonlinear functions is an input to a stable linear system and it also tends to zero, so $e_{3}$ also tend to zero.

Then $\widehat{x}_{1} \rightarrow x_{1}, \widehat{x}_{2} \rightarrow x_{2}, \hat{\eta} \rightarrow \eta$, and $\hat{p} \rightarrow p$; for the message we can write

$$
\begin{align*}
& \widehat{m}=y_{0}-\widehat{y}_{0}-f\left(\widehat{x}_{1}, \widehat{x}_{2}, \widehat{\eta}\right) \\
& \widehat{m}=x_{1}+x_{2}+f\left(x_{1}, x_{2}, \eta\right)+m-\left(\widehat{x}_{1}+\widehat{x}_{2}\right)-f\left(\widehat{x}_{1}, \widehat{x}_{2}, \widehat{\eta}\right) \\
& \widehat{m}=e_{1}+e_{2}+m+f\left(x_{1}, x_{2}, \eta\right)-f\left(\widehat{x}_{1}, \widehat{x}_{2}, \widehat{\eta}\right) \tag{21}
\end{align*}
$$

and then $\widehat{m} \rightarrow m$, and the proof is finished.


Figure 2: Estimation of disturbance on chaotic system. $(p(t)=$ $\sin (t)+\sin (4 t))$.


Figure 3: Lorenz attractor with sinusoidal signal and audio signal.

## 4. Simulation

4.1. Audio Signals. Consider the Lorenz oscillator given by equations

$$
\begin{align*}
& \dot{x}=a(y-x) \\
& \dot{y}=c x-y-x z+\frac{(m+p)}{a}  \tag{22}\\
& \dot{z}=-b z+x y
\end{align*}
$$

where $m$ represents an audio signal from the song Un bel di vedremo (taken from the opera Madama Butterfly, Act II) and the disturbance signal with two frequencies is $p(t)=\sin (t)+$ $\sin (4 t)$.

Applying (14) with the following parameters $a=10, b=$ $28, c=8 / 3$ (these values generate chaotic behavior), and $g_{1}=$ 18; $g_{2}=392 ; G_{0}=[1085,441,904,707]$ we can see how the disturbance $p(t)$ is estimated in Figure 2.

Figure 3 presents the chaotic behavior of the Lorenz attractor with the sinusoidal signal and audio signal.


Figure 4: (a) the original audio signal to be encrypted. ((b), (c)) the outputs $y_{0}$ and $y_{1}$, of the encryption system, (d) the original audio signal recovered by the algorithm.

Figure 4 shows the original audio signal, chaotic signals ( $y_{0}$ and $y_{1}$ ), and audio recovery. The time axis is scaled. It is clear how the message is recovered in a very good manner.

In Figure 5, you can see how the recovered audio signal is the same as the original audio signal.

Figure 6 presents the error signal between the original audio signal and retrieved message signal. It is possible to see that the error tends to zero.

It is observed how the message and the perturbation are retrieved in exact manner, it is, the synchronization is exact and does not exist an error as other algorithms.
4.2. Digital Images. The method can also be applied to encrypt and to transmit digital images; for encryption, the image is modified to be sent in vector form. In this case, the receiver knows the number of rows and columns.

In this example we use the Duffing chaotic system with the perturbation given by $p(t)=\sin (0.5 t)$ :

$$
\begin{align*}
& \dot{x}_{1}=x_{2} \\
& \dot{x}_{2}=\beta x_{1}-\delta x_{2}-x_{1}^{3}+p+m \tag{23}
\end{align*}
$$

with the gain parameters being $g_{1}=20 ; g_{2}=100$; and $G_{0}=$ [350.7, 89.3516].

Figure 7 shows the sinusoidal signal estimated in comparison with the original signal.

Figure 8 represents the chaotic behavior of the Duffing attractor with sinusoidal signal and digital image (the reference image is Frida Kahlo).


Figure 5: Original audio signal and recovered signal from the decoder implementation.


Figure 6: Error signal between the original message and the message retrieved.

Figure 9 represents the chaotic behavior of the Duffing attractor with sinusoidal signal and digital image (the reference image is Chichen-Itzá).

The differences between Figures 8 and 9 are due to the information of the images being different and therefore the perturbation changes the chaotic attractor in different manner; however, both figures show chaotic behavior.

Two examples are shown in Figures 10 and 11. First, image (a) is the original information and the recovered image. Second, image (b) is the chaotic signals $y_{0}$ and $y_{1}$.


Figure 7: Sinusoidal signal retrieved and original signal.

## 5. Conclusions

In this contribution we presented a methodology to encrypt and transmit audio and image signals using chaos and regulation theory. The algorithm consisted in perturbing a chaotic system with a signal composed of $n$ sinusoidal signals with


Figure 8: Duffing attractor with sinusoidal signal and digital image (Frida Kahlo).


Figure 9: Duffing attractor with sinusoidal signal and digital image (Chichen-Itzá).
known frequencies and with a signal message. The main result is that using a perturbing sinusoidal signal the encryption level could be increased since the message signal is hidden not only in the spread spectrum of the chaotic system but also by the frequencies of the disturbance. This fact is what we consider an increment on the encryption level and security on the transmission. We corroborate the encryption using two examples: transmitting and audio signal and an image; in both cases the recovery information is the same as the original message, except in the transition time. In practical terms this algorithm could be implemented in real time electronic circuit and also in a DSP or a fast processor since the number of equations to be solved is small, depending on the chaotic system and the number of sinusoidal signals used. Finally, an extension of this work is the encryption with sinusoidal signals with unknown frequencies, in addition to the problem of encrypting and transmitting multiple messages.


Figure 10: (a) Comparison between the original information and the recovered image. (b) Chaotic signals $y_{0}$ and $y_{1}$.


Figure 11: (a) Comparison between the original information and the recovered image. (b) Chaotic signals $y_{0}$ and $y_{1}$.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Reconfigurability Analysis Method for Spacecraft Autonomous Control 

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#### Abstract

As a critical requirement for spacecraft autonomous control, reconfigurability should be considered in design stage of spacecrafts by involving effective reconfigurability analysis method in guiding system designs. In this paper, a novel reconfigurability analysis method is proposed for spacecraft design. First, some basic definitions regarding spacecraft reconfigurability are given. Then, based on function tree theory, a reconfigurability modeling approach is established to properly describe system's reconfigurability characteristics, and corresponding analysis procedure based on minimal cut set and minimal path set is further presented. In addition, indexes of fault reconfigurable degree and system reconfigurable rate for evaluating reconfigurability are defined, and the methodology for analyzing system's week links is also constructed. Finally, the method is verified by a spacecraft attitude measuring system, and the results show that the presented method cannot only implement the quantitative reconfigurability evaluations but also find the weak links, and therefore provides significant improvements for spacecraft reconfigurability design.


## 1. Introduction

Nowadays, autonomous control has become a key technology for increasing spacecraft survival capability. The reason is that autonomous control, regarding fault detection, identification, and reconfiguration, will be automatically activated to reduce the fault effect when faults emerge in a spacecraft. Therefore, how to increase the ability of fault processing has become a key issue for autonomous control of spacecraft. However, it can be concluded by many recent serious spacecraft incidents that certain deficiencies exist in their fault diagnosis and processing procedure. Further analysis reveals that these deficiencies are caused by reconfigurability lack of spacecraft. From this viewpoint, excellent reconfigurability has been becoming more and more critical for autonomous control to ensure the increasing requirements of spacecraft safety and reliability. In order to improve spacecraft autonomous control ability of tolerating faults, reconfigurability should be considered in design stage of spacecrafts and effective reconfigurability analysis method must be presented to guide the system design.

As far as the authors know, regarding reconfigurability design, mass research, aiming at enhancing flexibility about
environment changes and function variations, has been conducted in computing and manufacturing fields [1, 2]. For spacecraft, although extensive attention to reconfigurability design has been devoted to controller designs after faults [3-9], or to system function changes [10] to satisfy other mission requirements, little improvement has been achieved regarding function recovery of faulty spacecraft by reconfigurability design. Meanwhile, some scholars have studied control reconfigurability from the intrinsic and performance-based perspectives. The intrinsic reconfigurability of LTI systems can be evaluated by the controllability and observability Gramians [11], or by the smallest secondorder mode which is the smallest eigenvalue of the combination of controllability and observability Gramians [12]. The performance-based control reconfigurability is regarded as the ability of the considered system to keep/recover some admissible system performance when certain fault occurs. Staroswiecki discussed the reconfigurability under energy limitation constraints in [13]. However, all the studies mentioned above did not consider system's components and configuration, and thus they cannot settle reconfigurability analysis and design problems for complex systems such as spacecrafts. Consequently, the critical objective of this study
is to construct an effective reconfigurability analysis method based on the function tree theory, which can synthesize components and reconfiguration strategies of spacecraft and estimate quantitative evaluation indexes.

The rest of this paper is organized as follows. Section 2 presents some basic definitions, and Section 3 constructs a reconfigurability modeling and analyzing method. In Sections 4 and 5, reconfigurability evaluation indexes and weak link analysis procedure for reconfiguration design are discussed, respectively. In Section 6, the proposed approach is illustrated by a practical application regarding spacecraft attitude measuring system. Some conclusions and relevant remarks are given in Section 7.

## 2. Basic Definitions

Siddiqi indicated that different definitions exist in different fields in [14]. By summing up a series of definitions, he defined reconfigurable system and reconfigurability as follows. Reconfigurable system is a system that can reversibly achieve distinct configurations (or states), through alteration of system form or function, in order to achieve a desired outcome within acceptable reconfiguration; while, reconfigurability is a system architectural property that defines the ease and extent to which a system is reconfigurable. Considering spacecraft, reconfiguration is the problem of replacing the faulty part of the system by a nonfaulty one, so as to still achieve control objectives, and reconfigurability is the ability of recovering all the functions or achieving degraded objectives by reconfiguration when faults appear.

System configuration is one of the basic factors that affect reconfigurability. Two relevant definitions, reconfiguration unit (RU) and minimal reconfiguration unit (MRU), should be explained here. RU is a combination of spacecraft components to achieve the anticipant function by reconfiguration itself or by switching to other RUs when the current RU fails. MRU is a combination of spacecraft components to achieve the anticipant function only by switching to other RUs when the current RU fails. It is the minimal unit in the reconfiguration analysis.

A novel reconfigurability model is established based on the function tree theory in this study. Function tree is a tree diagram whose vertex corresponds to the system function and whose branches are subfunctions decomposed from the system function, and its roots are the MRUs. Higher level functions and lower level functions in a function tree are connected by AND gates or OR gates. The relationship between function and MRUs can be clearly explained by the corresponding function tree. A typical function tree is illustrated in Figure 1.

In order to evaluate the reconfigurability quantitatively, definitions including cut set (CS), minimal cut set (MCS), path set (PS), and minimal path set (MPS) of a function tree are involved. A CS is a set of MRUs. When all MRUs in a CS are healthy, the system functions can be achieved. MCS is a special CS, and, if and only if all MRUs in MCS are in good condition, the system functions can be achieved. A PS is also a set of MRUs. When all MRUs in a PS fail, the system will lose


Figure 1: Function tree schematic diagram.
its function. MPS is a special PS, and, if and only if failure appears in every MRU in MPS, the system function should have been lost. Furthermore, the MCS set or MPS set is called MCS family or MPS family.

## 3. Reconfigurability Modeling

For reconfigurability evaluating and designing, one first needs to build an effective reconfigurability model and establish relationships between reconfigurability and MRUs. Then, evaluation indexes and weak links of the spacecraft reconfigurability can be analyzed.

We define a reconfigurability model from viewpoint of function tree, which is similar to theory of fault tree. The modeling processes are discussed as below.

Step 1. According to the system function, define the reconfiguration strategy based on the system observability and controllability.

For example, consider the LTI deterministic system

$$
\begin{gather*}
\dot{x}(t)=A x(t)+B u(t), \\
y(t)=C x(t) . \tag{1}
\end{gather*}
$$

We adopt the observability criterion and controllability criterion

$$
\begin{align*}
& \operatorname{rank}\left[\begin{array}{llll}
C & C A & \cdots & C A^{n-1}
\end{array}\right]^{\prime}=n  \tag{2}\\
& \operatorname{rank}\left[\begin{array}{llll}
B & B A & \cdots & B A^{n-1}
\end{array}\right]=n
\end{align*}
$$

to confirm the reconfiguration strategy by changing $B$ or $C$ in the system model and then obtain the component set $C_{\text {com }}$, each one of which can perform the system function.

Step 2. If any redundancy is involved in a system component, decompose it to the functional module. According to the redundancy relationship between the modules, determine the MRUs. Furthermore, according to the MRUs functions, the MRUs function set $F_{\text {MRU }}$ can be obtained. And the elements in $F_{\mathrm{MRU}}$ are the lowest level function in the function tree.


Figure 2: Structure decomposition of gyro.


Figure 3: Function decomposition of gyro.

To get a better understanding, a gyro system is utilized as an example to illustrate this procedure. A gyro can be decomposed to several modules, such as power supply module, data processing module, I/O module, and gyro sensor module. If the power supply module is redundant, while others are not, any single power supply module can be considered as MRU, and the rest can be treated as MRU. Consequently, $F_{\text {MRU }}$ of a gyro is \{power supply, measure and data process\}. Figure 2 shows the decomposition structure.

Step 3. From the system function, decompose higher level functions into lower level functions (or subfunctions) until the functions are contained in $F_{\mathrm{MRU}}$.

Return to the example of gyro. "Angle velocity measure" is the function of a gyro. It can be decomposed into two subfunctions, "power supply" and "measure and data process". Then the decomposition process can be terminated, because "power supply" and "measure and data process" belong to $F_{\mathrm{MRU}}$. The decomposition process is illustrated in Figure 3.

Step 4. Build a function tree by AND gate and OR gate. The vertex of this function tree is the system function, the branches are the subfunctions, and the roots are the MRUs. AND gate and OR gate connect the higher layers and the lower layers according to the relationship between the subfunctions.

AND gate and OR gate in function trees are depicted in Figure 4. The AND gate in Figure 4(a) shows that the upper level function $Y$ can only be achieved when all the subfunctions $x_{i}$ have been realized, $i=1,2, \ldots, n$, while for OR gate in Figure 4(b), it can be concluded that the upper level function $Y$ can be realized when any single or multiple or all subfunctions $x_{i}$ are achieved, $i=1,2, \ldots, n$.


Figure 4: AND gate and OR gate.


Figure 5: Function tree of gyro.

According to the steps mentioned above, the function tree of a gyro can be formed, which is shown in Figure 5.

In order to analyze the reconfigurability quantitatively, the MCS and MPS of function tree should be obtained firstly.

Let $\mathrm{C}_{i}\left(x_{j}\right)$ denote the $i$ th MCS for the $j$ th level function $x_{j}$, and let $\mathbb{C}(Y)$ denote the CS family for the upper level function $Y$. For AND gate,

$$
\begin{align*}
\mathbb{C}(Y)=\left\{\mathbf{C}_{i}\left(x_{1}\right) \cup\right. & \left.\mathbf{C}_{j}\left(x_{2}\right) \cup \cdots \cup \mathbf{C}_{k}\left(x_{n}\right)\right\}, \\
& i \in\left(1,2, \ldots,\left|\mathbb{C}\left(x_{1}\right)\right|\right),  \tag{3}\\
& j \in\left(1,2, \cdots,\left|\mathbb{C}\left(x_{2}\right)\right|\right) \\
& k \in\left(1,2, \cdots,\left|\mathbb{C}\left(x_{n}\right)\right|\right) .
\end{align*}
$$

For OR gate,

$$
\begin{equation*}
\mathbb{C}(Y)=\mathbb{C}\left(x_{1}\right) \cup \mathbb{C}\left(x_{2}\right) \cup \cdots \cup \mathbb{C}\left(x_{n}\right), \tag{4}
\end{equation*}
$$

where $\left|\mathbb{C}\left(x_{i}\right)\right|, i=1,2, \ldots, n$, is the cardinal number of $\mathbb{C}\left(x_{i}\right)$, which indicates MCS number in the MCS family for the subfunction $x_{i}$.

Let $\mathbf{R}_{i}\left(x_{j}\right)$ be the $i$ th MPS for the $j$ th level function $x_{j}$, and let $\mathbb{R}(Y)$ be the PS family of the upper level function $Y$. For AND gate,

$$
\begin{equation*}
\mathbb{R}(Y)=\mathbb{R}\left(x_{1}\right) \cup \mathbb{R}\left(x_{2}\right) \cup \cdots \cup \mathbb{R}\left(x_{n}\right) \tag{5}
\end{equation*}
$$

For OR gate,

$$
\begin{align*}
\mathbb{R}(Y)=\left\{\mathbf{R}_{i}\left(x_{1}\right) \cup\right. & \left.\mathbf{R}_{j}\left(x_{2}\right) \cup \cdots \cup \mathbf{R}_{k}\left(x_{n}\right)\right\}, \\
& i \in\left(1,2, \ldots,\left|\mathbb{R}\left(x_{1}\right)\right|\right),  \tag{6}\\
& j \in\left(1,2, \ldots,\left|\mathbb{R}\left(x_{2}\right)\right|\right), \\
& k \in\left(1,2, \ldots,\left|\mathbb{R}\left(x_{n}\right)\right|\right),
\end{align*}
$$

where $\left|\mathbb{R}\left(x_{i}\right)\right|, i=1,2, \ldots, n$, is the cardinal number of $\mathbb{R}\left(x_{i}\right)$, which corresponds to the MPS number of the MPS family for the subfunction $x_{i}$.

Although $\mathbb{C}(Y)$ or $\mathbb{R}(Y)$ derived by (3) to (6) may not be MCS family or MPS family, the MCS and MPS are needed in the upper level function analysis according to (3) to (6). Consequently, the MCS and MPS of function $Y$ can be calculated by the following steps.

Step 1. Initialize $\mathbb{C}_{\min }(Y)$ or $\mathbb{R}_{\min }(Y)$ to be a null set.
Step 2. Choose $\mathbf{C}_{\text {min }}(Y)$ or $\mathbf{R}_{\min }(Y)$ with a minimum cardinal number in all sets in $\mathbb{C}(Y)$ or $\mathbb{R}(Y)$ and transform it into $\mathbb{C}_{\text {min }}(Y)$ or $\mathbb{R}_{\text {min }}(Y)$.

Step 3. Check all remaining sets in $\mathbb{C}(Y)$ or $\mathbb{R}(Y)$. If there is a set containing all the MRUs in $\mathbf{C}_{\text {min }}(Y)$ or $\mathbf{R}_{\text {min }}(Y)$, delete it from $\mathbb{C}(Y)$ or $\mathbb{R}(Y)$ and go back to Step 2 otherwise.

Step 4. Execute Steps 2 and 3 repeatedly until $\mathbb{C}(Y)$ or $\mathbb{R}(Y)$ turns to a null set. Then elements $\mathbf{C}_{i}(Y)$ or $\mathbf{R}_{i}(Y)$ in $\mathbb{C}_{\text {min }}(Y)$ or $\mathbb{R}_{\text {min }}(Y)$ are the expected MCS or MPS.

## 4. Reconfigurability Evaluation Indexes

Based on the reconfigurability model constructed in the preceding section, reconfigurability evaluation indexes for spacecrafts are given as follows.
4.1. Fault Reconfigurable Degree (FRD). FRD describes whether the system has available resources and methods for reconfigurations after certain faults as

$$
\gamma= \begin{cases}1 & \text { fault is reconfigurable }  \tag{7}\\ 0 & \text { fault is unreconfigurable }\end{cases}
$$

When certain faults emerge, the MCS family should be activated by deleting all the MCSs including the fault reconfigurable units. Consider $\gamma=0$ if the MCS family is empty; consider $\gamma=1$ otherwise.
4.2. System Reconfigurable Rate (SRR). SRR indicates the rate of reconfigurable faults with respect to all faults in the system

$$
\begin{equation*}
r=\frac{\sum_{i=1}^{m} w_{i} \gamma_{i}}{\sum_{i=1}^{m} w_{i}} \tag{8}
\end{equation*}
$$

where $\gamma_{i}$ is the FRD of the $i$ th fault $f_{i}, m$ is the number of all the system fault modes, and $w_{i}$ is the weight of fault $f_{i}$ according to its severity and occurrence probability. The major fault has a bigger weight than a minor one; and the fault with high occurrence probability has a bigger weight than the one with low occurrence probability. If the fault severity can be defined as four levels, as listed in Table 1, and the occurrence probability can be divided into five levels, as listed in Table 2, then $w_{i}$ can be determined from Table 3. $S$ denotes the fault severity level and $P$ indicates the fault occurrence probability in Table 3.

Table 1: Fault severity level definition.

| Level | Definition |
| :--- | :--- |
| I | System function is lost or service life is shortened seriously. |
| II | System function is degraded seriously or service life is <br> reduced by $1 / 4$ to $1 / 2$. |
| III | System function is degraded partially or service life is <br> reduced below $1 / 4$. |
| IV | There is little affection in system function and service life. |

TAble 2: Fault occurrence probability definition.

| Level | Definition |
| :---: | :---: |
| A | MRU fault probability $\geq 20 \% \times$ total fault probability |
| B | $20 \% \times$ total fault probability $>$ MRU fault probability $\geq$ $10 \% \times$ total fault probability |
| C | $10 \% \times$ total fault probability $>$ MRU fault probability $\geq 1 \%$ $\times$ total fault probability |
| D | $1 \% \times$ total fault probability $>$ MRU fault probability $\geq 0.1 \%$ $\times$ total fault probability |
| E | MRU fault probability $<0.1 \% \times$ total fault probability |

Table 3: $w_{i}$ matrix.

| $P$ |  | $S$ |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | I | II | III | IV |  |
| A | 1 | $1 / 3$ | $1 / 7$ | $1 / 13$ |  |
| B | $1 / 2$ | $1 / 5$ | $1 / 9$ | $1 / 16$ |  |
| C | $1 / 4$ | $1 / 6$ | $1 / 11$ | $1 / 18$ |  |
| D | $1 / 8$ | $1 / 10$ | $1 / 14$ | $1 / 19$ |  |
| E | $1 / 12$ | $1 / 15$ | $1 / 17$ | $1 / 20$ |  |

## 5. Weak Link Analysis in Reconfigurability Design

For better reconfigurability, the reconfiguration weak links should be improved in the design phase of a spacecraft. Based on the established configurability model, the following two indexes are proposed to determine weak links in reconfiguration.
5.1. Importance Degree of MRU (IDMRU). IDMRU denotes the rate of the number of MCSs that includes the MRU with respect to the number of all MCSs as

$$
\begin{equation*}
I_{M}=\frac{N_{M}}{N_{T}} \tag{9}
\end{equation*}
$$

where $I_{M}$ is the IDMRU of MRU $M, N_{M}$ is the number of MCSs that comprise the MRU, and $N_{T}$ is the number of all MCSs.

For any system, the MRU with maximal IDMRU contributes most in system function realization. Consequently, necessary redundancy or special reliability design should be considered for this MRU.
5.2. System Fault Tolerance Degree (SFTD). SFTD represents the maximal number of failure MRUs that the system can
tolerate without loss of system functions. SFTD reflects the system reconfigurability as

$$
\begin{equation*}
T=\min \left(\left|\mathbf{R}_{i}\right|\right)-1 \quad\left|\mathbf{R}_{i}\right| \in \mathbb{R}, i=1,2, \ldots,|\mathbb{R}| \tag{10}
\end{equation*}
$$

where $T$ denotes SFTD, $\mathbf{R}_{i}$ is the $i$ th minimal path set of the function tree, $\left|\mathbf{R}_{i}\right|$ is the cardinal number of $\mathbf{R}_{i}$.

In a system, the path set with the minimum number of MPSs is the weakest link. And for this part, necessary redundancy or special reliability design should be considered according to the subfunctions of MRUs in the MPS.

The four indexes proposed above are closely connected to each other. Let $f_{i}$ be a fault whose corresponding reconfigurable degree is equal to zero, $\gamma_{i}=0$; namely, the corresponding MRU cannot be reconfigured; then the importance degree $I_{M}$ of the MRU will be equal to one and the system fault tolerance degree $T$ will become zero. Otherwise, if all fault reconfigurable degrees are one, namely, all the MRU can be reconfigured, then we can conclude that all the importance degrees will be less than one, the system fault tolerance degree will be not less than one, and the system reconfigurable rate will be equal to $100 \%$.

## 6. Empirical Results

In this section, we focus on the practical performance of the proposed method. Our experiment is presented for the reconfigurability analysis of an attitude measuring system in a spacecraft. The dynamic functions regarding momentum devices are shown in (11). The spacecraft is considered as rigid body systems, and the body coordinate system coincides with the principle axes of inertia as

$$
\begin{align*}
& I_{x} \dot{\omega}_{x}-\left(I_{y}-I_{z}\right) \omega_{y} \omega_{z}-h_{y} \omega_{z}+h_{z} \omega_{y}=-\dot{h}_{x}+T_{x} \\
& I_{y} \dot{\omega}_{y}-\left(I_{z}-I_{x}\right) \omega_{z} \omega_{x}-h_{z} \omega_{x}+h_{x} \omega_{z}=-\dot{h}_{y}+T_{y}  \tag{11}\\
& I_{z} \dot{\omega}_{z}-\left(I_{x}-I_{y}\right) \omega_{x} \omega_{y}-h_{x} \omega_{y}+h_{y} \omega_{x}=-\dot{h}_{z}+T_{z}
\end{align*}
$$

where $I_{x}, I_{y}$ and $I_{z}$ are moments of inertia along axes $O x$, $O y$ and $O z$, respectively; $\boldsymbol{\omega}=\left[\omega_{x}, \omega_{y}, \omega_{z}\right]^{T}$ is the angular velocity vector; $\mathbf{h}=\left[h_{x}, h_{y}, h_{z}\right]^{T}$ is the synthesizing angular momentum vector of all the momentum devices; $\mathbf{T}=$ $\left[T_{x}, T_{y}, T_{z}\right]^{T}$ is the control torque vector applied on the spacecraft except for the torque from the momentum devices. Therefore, the control torque vector $\mathbf{T}=\left[T_{x}, T_{y}, T_{z}\right]^{T}$ in (11) includes torques from thrusters, other space torques, and disturbing torques.

If all attitudes vary in a small scale, the dynamic functions can be simplified as

$$
\begin{align*}
& \omega_{x}=\dot{\varphi}-\omega_{0} \psi \\
& \omega_{y}=\dot{\theta}-\omega_{0}  \tag{12}\\
& \omega_{z}=\dot{\psi}+\omega_{0} \varphi
\end{align*}
$$

where $\varphi, \theta$ and $\psi$ are Euler angles; $\omega_{0}$ denotes the orbit angular velocity with which the spacecraft circles around the center body.

Then, the linearization form of the attitude dynamic function can be derived based on (11) and (12) as

$$
\begin{gather*}
I_{x} \ddot{\varphi}+\left[\left(I_{y}-I_{z}\right) \omega_{0}^{2}-\omega_{0} h_{y}\right] \varphi \\
\\
+\left[\left(I_{y}-I_{z}-I_{x}\right) \omega_{0}-h_{y}\right] \dot{\psi} \\
=  \tag{13}\\
-\dot{h}_{x}+\omega_{0} h_{z}+T_{x}, \\
I_{y} \ddot{\theta}+h_{x}\left(\dot{\psi}+\omega_{0} \varphi\right)-h_{z}\left(\dot{\varphi}-\omega_{0} \psi\right)=-\dot{h}_{y}+T_{y}, \\
I_{x} \ddot{\psi}+\left[\left(I_{y}-I_{x}\right) \omega_{0}^{2}-\omega_{0} h_{y}\right] \psi \\
\\
-\left[\left(I_{y}-I_{z}-I_{x}\right) \omega_{0}-h_{y}\right] \dot{\varphi} \\
= \\
-\dot{h}_{z}-\omega_{0} h_{x}+T_{z} .
\end{gather*}
$$

Accordingly, the dynamic function of the spacecraft can be expressed by a state space form, as shown in (1), with the following notations:

$$
\begin{gather*}
x=\left[\begin{array}{lllll}
\varphi & \dot{\varphi} & \theta & \dot{\theta} & \psi \\
\dot{\psi}
\end{array}\right]^{T}, \\
A=\left[\begin{array}{cccccc}
0 & 1 & 0 & 0 & 0 & 0 \\
M_{21} & 0 & 0 & 0 & 0 & M_{26} \\
0 & 0 & 0 & 1 & 0 & 0 \\
M_{41} & M_{42} & 0 & 0 & M_{45} & M_{46} \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & M_{62} & 0 & 0 & M_{65} & 0
\end{array}\right], \\
M_{21}=I_{x}^{-1}\left[\left(I_{y}-I_{z}\right) \omega_{0}^{2}-\omega_{0} h_{y}\right], \\
M_{26}=I_{x}^{-1}\left[\left(I_{y}-I_{z}-I_{x}\right) \omega_{0}-h_{y}\right],  \tag{14}\\
M_{41}=I_{y}^{-1} h_{x} \omega_{0}, \\
M_{42}=-I_{y}^{-1} h_{z}, \\
M_{45}=I_{y}^{-1} h_{z} \omega_{0}, \\
M_{46}=I_{y}^{-1} h_{x}, \\
M_{62}=-I_{z}^{-1}\left[\left(I_{y}-I_{z}-I_{x}\right) \omega_{0}-h_{y}\right], \\
M_{65}=I_{z}^{-1}\left[\left(I_{y}-I_{x}\right) \omega_{0}^{2}-\omega_{0} h_{y}\right] .
\end{gather*}
$$

Matrixes $B$ and $C$ in (1) can be determined according to the detailed configuration of the system. For example, a system, with two infrared earth sensors, three orthogonal gyros, and one main backup thruster, can be described as

$$
\begin{array}{r}
u(t)=\left[\begin{array}{llllll}
T_{x 1} & T_{x 2} & T_{y 1} & T_{y 2} & T_{z 1} & T_{z 2}
\end{array}\right]^{T} \\
y(t)=\left[\begin{array}{lllllll}
\varphi_{h 1} & \theta_{h 1} & \varphi_{h 2} & \theta_{h 2} & g_{x} & g_{y} & g_{z}
\end{array}\right]^{T}
\end{array}
$$

$$
\begin{align*}
B & =\left[\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
I_{x}^{-1} & I_{x}^{-1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I_{y}^{-1} & I_{y}^{-1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I_{z}^{-1} & I_{z}^{-1}
\end{array}\right], \\
C & =\left[\begin{array}{cccccc}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -\omega_{0} & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
\omega_{0} & 0 & 0 & 0 & 0 & 1
\end{array}\right] . \tag{15}
\end{align*}
$$

Considering a spacecraft system described by (1), when faults appear, the premise of achieving system reconfigurability is that the remaining of the system is observable and controllable. The corresponding criterion is given by (2). According to engineering experience, one can assume that $I_{x} \neq I_{y} \neq I_{z}$ and $\omega_{0} \neq 0$. Consider the following.
(1) Only one infrared earth sensor is employed for attitude determination as

$$
C_{1}=\left[\begin{array}{llllll}
1 & 0 & 0 & 0 & 0 & 0  \tag{16}\\
0 & 0 & 1 & 0 & 0 & 0
\end{array}\right], \quad \operatorname{rank}\left[\begin{array}{c}
C_{1} \\
C_{1} A \\
\vdots \\
C_{1} A^{5}
\end{array}\right]=6
$$

(2) Three gyros are employed for attitude determination as
$C_{2}=\left[\begin{array}{cccccc}0 & 1 & 0 & 0 & -\omega_{0} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \omega_{0} & 0 & 0 & 0 & 0 & 1\end{array}\right], \quad \operatorname{rank}\left[\begin{array}{c}C_{2} \\ C_{2} A \\ \vdots \\ C_{2} A^{5}\end{array}\right]=5$.
(3) One infrared earth sensor and three gyros are employed for attitude determination as

$$
C_{3}=\left[\begin{array}{cccccc}
1 & 0 & 0 & 0 & 0 & 0  \tag{18}\\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -\omega_{0} & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
\omega_{0} & 0 & 0 & 0 & 0 & 1
\end{array}\right], \quad \operatorname{rank}\left[\begin{array}{c}
C_{3} \\
C_{3} A \\
\vdots \\
C_{3} A^{5}
\end{array}\right]=6
$$

From (16) to (18), the attitude can be measured in the following two ways:

M1: by infrared earth sensors;
M2: by infrared earth sensors and gyros.
In addition, it is assumed that two infrared earth sensors share one power supply and three gyros share another power supply; then Table 4 lists the MRUs and their corresponding subfunctions.

Table 4: MRUs and their corresponding functions.

| MRU | Functions |
| :--- | :---: |
| Infrared earth sensor power | Power supply for infrared earth |
| (ESP) | sensor (PS for ES) |
| Infrared earth sensor 1 (ES1) | $\varphi$ and $\theta$ measure |
| Infrared earth sensor 2 (ES2) | $\varphi$ and $\theta$ measure |
| Gyro power (GPower) | Power supply for gyros |
| Gyro $x\left(G_{x}\right)$ | (PS for gyro) |
| Gyro $y\left(G_{y}\right)$ | measure $\omega_{x}$ |
| Gyro $z\left(G_{z}\right)$ | measure $\omega_{y}$ |

Table 5: Results of reconfigurability analysis.

| MRU | $\gamma$ | $I$ |
| :---: | :---: | :---: |
| ESPower | 0 | 1 |
| ES1 | 1 | 0.5 |
| ES2 | 1 | 0.5 |
| GPower | 1 | 0 |
| $G_{x}$ | 1 | 0 |
| $G_{y}$ | 1 | 0 |
| $G_{z}$ | 1 | 0 |

Figure 6 illustrates the function tree constructed by the reconfigurability modeling process. The MCS family and the MPS family could be derived by analyzing the function tree in Figure 6 as

$$
\begin{gather*}
\mathbb{C}=\{\{\mathrm{ESP}, \mathrm{ES} 1\},\{\mathrm{ESP}, \mathrm{ES} 2\}\}, \\
\mathbb{R}=\{\{\mathrm{ESP}\},\{\mathrm{ES} 1, \mathrm{ES} 2\}\} . \tag{19}
\end{gather*}
$$

Thus, reconfigurability indexes can be calculated by (7) to (10). Table 5 lists the FRD and IDMEU of all the MRUs. Furthermore, suppose that the severity and occurrence possibility for all MRUs are the same; then $w_{i}=1, r=6 / 7$, and $T=0$.

According to the analysis results of IDMRU and SFTD of all MRUs, the weakest link of this system is the power of infrared earth sensors. Consequently, it is better to store a backup in this link.

## 7. Conclusion

To involve reconfigurability in spacecraft design phase for potential faults, a novel reconfigurability analysis method is investigated in this paper. First, on the basis of observability and controllability, the reconfigurability criterion is given for spacecraft that is considered as a rigid body system. Then, the function tree is built for modeling reconfigurability, and evaluation indexes are proposed. After that, according to minimal cut set and minimal path set of the function tree, a quantitative evaluation method for reconfigurability indexes and an approach for determining system weak links


Figure 6: Function tree for attitude determinations.
are summarized. Theoretical research and empirical study both illustrate the benefit of the constructed methodology for spacecraft reconfigurability design on reliability criterions.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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# On the Identification of Coupled Pitch and Heave Motions Using Opposition-Based Particle Swarm Optimization 

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#### Abstract

A mathematical model must be established to study the motions of ships in order to control them effectively. An assessment of the model depends on the accuracy of hydrodynamic parameters. An algorithm for the parameter identification of the coupled pitch and heave motions in ships is, thus, put forward in this paper. The algorithm proposed is based on particle swarm optimization (PSO) and the opposition-based learning theory known as opposition-based particle swarm optimization (OPSO). A definition of the opposition-based learning algorithm is given first of all, with ideas on how to improve this algorithm and its process being presented next. Secondly, the design of the parameter identification algorithm is put forward, modeling the disturbing force and disturbing moment of the identification system and the output parameters of the identification system. Then, the problem involving the hydrodynamic parameters of motions is identified and the coupled pitch and heave motions of a ship described as an optimization problem with constraints. Finally, the numerical simulations of different sea conditions with unknown parameters are carried out using the PSO and OPSO algorithms. The simulation results show that the OPSO algorithm is relatively stable in terms of the hydrodynamic parameters identification of the coupled pitch and heave motions.


## 1. Introduction

The model concerned with the motions of ships describes the process of movement of the response characteristics of the control input, and this is the precondition to understanding the motions of ships. Widely used system identification techniques for hydrodynamic parameters and the hydrodynamic parameters themselves can be identified by observing the data relating to the motions of ships. This directly establishes the mathematical model of the ship's hydrodynamic parameters and motions between the state models. Classic identification algorithms, such as the maximum likelihood identification method and the prediction error used for parameter estimation are too low in terms of sensitivity and are thus inadequate as methods used in order to solve true values. Abkowitz [1] extended the Kalman Filter method to estimate the hydrodynamic parameters of a ship's motions. Clarke et al. [2] identified the nonlinear parameters of the motions of ships by using an artificial neural network. Haddara used free attenuation from the free response-signal method to identify
the roll $[3,4]$ and extended it to the parameter identification of sway and yaw coupled equations [5] and pitch and heave coupled equations [6]. Haddara and Xu [7] additionally put forward a ship's longitudinal motions state as a Markov chain process and, in order to simplify the longitudinal motions equation, they used a neural network for the identification of a ship's longitudinal motions parameters. Mahfouz and Haddara [8] mixed the classic recognition algorithm and neural network methods, putting forward a hydrodynamic parameter identification method using RDLRNNT technology, a method that appears to be fairly sound.

Bhattacharyya and Haddara [9] used artificial neural networks (ANN) and spectral analysis methods to identify the hydrodynamic derivatives in the mathematical model involving ship and marine vehicle motions. ANN has also some defects, however, such as bad generalization performance, easily falling into a local minimum. Luo and Zou [10] applied support vector machines (SVM) to identify the hydrodynamic derivatives of Abkowitz's model from the simulated free-running model test results and then used the regressive

Abkowitz model to predict zigzag tests. Zhang and Zou [11] proposed a novel method of artificial intelligence technology in the shape of support vector machines in order to estimate the hydrodynamic coefficients in the mathematical models of ship maneuvering motions. A comparison between the predicted hydrodynamic forces and the test results shows that the identified hydrodynamic mathematical model has a good generalization performance.

In recent years, with the rapid development of intelligent algorithms, a number of scholars have used the swarm intelligent algorithm, applying it to hydrodynamic parameters in order to identify problems related to the motions of ships. Through the state equation of underwater bodies and observation equation, Chen et al. [12] has used intelligent recognition technology for simulation recognition of an underwater navigation body and has obtained ten hydrodynamic parameters. In 2008, Chen et al. [13] proposed a new recognition algorithm based on intelligence technology. The leastsquare criterion together with the Differential Swarm Intelligent (DS) algorithm is employed to identify hydrodynamic parameters. In 2011, Dai et al. [14, 15] used an improved PSO algorithm and continuous domain ant colony optimization algorithm to identify the hydrodynamic parameters of heave and pitch with satisfactory results. In order to determine a water diving device parameter test and for the theoretical calculation of difficult problems, Gao and Li [16] put forward a method based on the basic artificial bee colony algorithm and the improved artificial bee colony method used to identify a potential coefficient method. Experiments show that the use of artificial colony algorithm identification device parameters is indeed feasible.

The PSO algorithm was proposed by Kennedy and Eberhart in 1995 [17, 18]. In order to overcome the existing problems in the practical use of the PSO algorithm and improve the performance of the algorithm, an improved algorithm has been put forward by number of scholars [19-26]. Generally speaking, the improved PSO algorithm strategy currently includes two aspects, namely, the improvement of the strategy of the PSO algorithm and its fusion with other algorithms. The improved PSO algorithm mainly concerns the variation of the particles, the multipopulation cooperation, and the design of parameters. These methods can potentially prevent particle aggregation and conflict, and avoid premature convergence to local optima. PSO algorithms are integrated with other algorithms, improving the strategy of the PSO. Some scholars are currently advocating the opposition-based learning particle swarm optimization algorithm. Wang et al. [21] have introduced opposition-based learning into the PSO algorithm, then they proposed opposition-based learning and the Cauchy mutation PSO algorithm (OPSO), using opposition-based learning to initialize the group. Omran and Al-Sharhan [24] have used dynamic shrinkage factors to generate opposition-based solutions. Shahzad [25] have presented three kinds of opposition-based PSO algorithm: the first version of the OPSO algorithm using only oppositionbased learning in order to initialize groups and the second version (IOPSO) in addition to opposition-based learning produced the opposition-based particles as a replacement for the worst particle in the group during each iteration. In the
third version of the algorithm the initialization of oppositionbased learning is removed from IOPSO. Shi and Eberhart [26] have controlled the velocity of PSO by using oppositionbased learning and proposed the speed clamping PSO algorithm (OVCPSO) based on opposition-based learning, which achieved good results.

Owing to the limitations of conventional identification methods in coupled pitch and heave motions parameter identification, in this paper, we are proposing to identify ship coupled heave and pitch motions using opposition-based PSO. In order to achieve this, we have designed a model involving wave disturbance force and torque disturbance, using the design methods for an output parameter identification system, with an opposition-based PSO algorithm for parameter identification.

## 2. Pitch and Heave Motions Model

The coupled pitch and heave motions of a ship in a realistic sea can be described by two linear second order ordinary coupled differential equations as follows [7]:

$$
\begin{align*}
& \left(a_{33}+m_{0}\right) \ddot{z}+b_{33} \dot{z}+c_{33} z+a_{35} \ddot{\theta}+b_{35} \dot{\theta}+c_{35} \theta \\
& \quad=F_{R}+F_{3} \\
& a_{53} \ddot{z}+b_{53} \dot{z}+c_{53} z+\left(I_{5}+a_{55}\right) \ddot{\theta}+b_{55} \dot{\theta}+c_{55} \theta  \tag{1}\\
& \quad=F_{R} X_{R}+M_{5},
\end{align*}
$$

where $z$ is heave; $\theta$ is pitch; $I_{5}$ is the pitching moment of inertia; $m_{0}$ is the ship's mass; $F_{R}$ is the force of the rudder; $X_{R}$ is advanced from the rudder lifting center to the ship's center of gravity; $F_{3}$ is the disturbance force of heave; $M_{5}$ is the disturbance moment of pitch, $a_{33}$ is the added mass, $a_{35}$ and $a_{53}$ are mass moment; $a_{55}$ is the moment of inertia; $b_{33}, b_{35}$ are the damping coefficients; $c_{33}, c_{35}$ are the resilience coefficient; $b_{53}, b_{55}$ are the damping moment coefficient; $c_{53}, c_{55}$ are the righting moment coefficients; and $a_{33}, a_{35}, a_{53}, a_{55}, b_{33}, b_{35}$, $b_{53}, b_{55}, c_{33}, c_{35}, c_{53}$, and $c_{55}$ are the hydrodynamic parameters.

Set $x_{1}=z, x_{3}=\theta$ as the state variables

$$
X=\left[\begin{array}{llll}
x_{1} & x_{2} & x_{3} & x_{4}
\end{array}\right]^{T}=\left[\begin{array}{llll}
z & \dot{z} & \theta & \dot{\theta} \tag{2}
\end{array}\right]^{T}
$$

The system state equation is then obtained as follows:

$$
\begin{equation*}
\dot{X}=A X+B u+C W \tag{3}
\end{equation*}
$$

Among $A=E^{-1} A^{*}, B=E^{-1} B^{*}, C=E^{-1} C^{*}$, where

$$
E=\left(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & a_{33}+\Delta & 0 & a_{35} \\
0 & 0 & 1 & 0 \\
0 & a_{53} & 0 & I_{5}+a_{55}
\end{array}\right)
$$

$$
\begin{gather*}
A^{*}=\left(\begin{array}{cccc}
0 & 1 & 0 & 0 \\
-c_{33} & -b_{33} & -c_{35} & -b_{35} \\
0 & 0 & 0 & 1 \\
-c_{53} & -b_{53} & -c_{55} & -b_{55}
\end{array}\right) \\
B=\left[\begin{array}{c}
0 \\
1 \\
0 \\
X_{R}
\end{array}\right], \quad u=F_{R}, \quad C=\left[\begin{array}{ll}
0 & 0 \\
1 & 0 \\
0 & 0 \\
0 & 1
\end{array}\right], \quad W=\left[\begin{array}{c}
F_{3} \\
M_{5}
\end{array}\right] \tag{4}
\end{gather*}
$$

para $=\left[a_{33}, a_{35}, a_{53}, a_{55}, b_{33}, b_{35}, b_{53}, b_{55}, c_{33}, c_{35}, c_{53}, c_{55}\right]$ as the parameters to be identified, $W$ is the wave disturbing force and moment.

## 3. Opposition-Based Particle Swarm Optimization

3.1. Particle Swarm Optimization Algorithm. PSO learns from the natural phenomenon of birds looking for food made by a class of population-based stochastic global optimization techniques. With respect to birds in flight, in its initial state every bird is in a random position and flies randomly in all directions, but as time goes on, these initial random state birds form a small community, through mutual learning, mutual tracking, and self-organization, fly at the same speed in the same direction, and ultimately the entire group gathers in one place, namely, the food source.

In the PSO algorithm, each individual is called a "particle," and each particle represents a potential solution. In continuous spatial coordinates, the PSO algorithm is described as follows.

Suppose that the size of the swarm is $N$ and the search space is $D$-dimensional, then the position of the $i$ th particle is presented as $X_{i}=\left(X_{i 1}, X_{i 2}, \ldots, X_{i D}\right)$, the velocity of this particle is presented as $V_{i}=\left(V_{i 1}, V_{i 2}, \ldots, V_{i D}\right)$, the $T$ fitness value of each particle in its current position is fitness $_{i}=$ fitness $\left(X_{i}\right)$, its corresponding optimal value is $P_{i}=\left(P_{i 1}, P_{i 2}, \ldots, P_{i D}\right)$ and the population current optimal experience value is recorded as $P_{g}=\left(P_{g 1}, P_{g 2}, \ldots, P_{g D}\right)$. Each particle adjusts its speed dynamically according to the comprehensive analysis of both individual and population flying experience and flies to the best position that it and other particles have experienced. Each particle updates its speed and position according to the formula equations:

$$
\begin{gather*}
V_{i}(t+1)=w V_{i}(t)+c_{1} r_{1}\left(P_{i}(t)-X_{i}(t)\right) \\
+c_{2} r_{2}\left(P_{g}(t)-X_{i}(t)\right)  \tag{5}\\
X_{i}(t+1)=X_{i}(t)+V_{i}(t+1) \tag{6}
\end{gather*}
$$

where $t$ is iterative times, $d=1,2 \cdots D . r_{1}, r_{2}$ are random numbers between 0 and 1 and $C_{1}, C_{2}$ are nonnegative constants. This is called the learning factor and each iteration step is justified accordingly.
3.2. Opposition-Based Learning. Opposition-based learning was put forward by Professor Tizhoosh [19] in 2005.

He argues that intelligent algorithms are based on a random guess value concerning the initial population, with each generation coming close to the solution with the optimal solution or a close approximation of the optimal solution eventually being found. Thus, the initial guess value greatly influences the algorithm and, if the random guess value is very close to the optimal solution, the algorithm may converge quickly, but if the value is far away from the solution or is even the reverse, the algorithm will take much more time. If the current solution and its opposite are sought simultaneously, a better solution will be chosen and the efficiency of the algorithm will be greatly enhanced. According to the theory of probability, there is a $50 \%$ probability that the current solution is more remote from the optimal solution than its opposite [20].

Definition 1 (definition of the opposite number). Let $x \in R$ be a real number within a defined interval, where $x \in[a, b]$. The opposite number $x_{0}$ can be defined as

$$
\begin{equation*}
x_{0}=a+b-x \tag{7}
\end{equation*}
$$

Definition 2 (definition of the opposite point). In the high dimensional space, if $p=\left(x_{1}, x_{2}, \ldots, x_{D}\right)$ is a set of points in the $D$-dimensional search space where $x_{1}, x_{2}, \ldots, x_{D} \in R$, $x_{j} \in\left[a_{j}, b_{j}\right]$, then the points in the opposition set $p_{o}=$ $\left(x_{1 o}, x_{20}, \ldots, x_{D o}\right)$ can be defined as

$$
\begin{equation*}
x_{j o}=a_{j}+b_{j}-x_{j} . \tag{8}
\end{equation*}
$$

Definition 3 (opposition-based optimization). For a point in the $D$-dimensional space $X=\left(x_{1}, x_{2}, \ldots, x_{D}\right)$, suppose that $f(X)$ is the function used to measure the performance of a candidate solution, according to the opposition theorem, $X_{o}=\left(x_{1 o}, x_{2 o}, \ldots, x_{D o}\right)$ will be the opposition set for $X=$ $\left(x_{1}, x_{2}, \ldots, x_{D}\right)$, If $f\left(X_{0}\right)<f(X)$, then the set of points $X$ can be replaced by $X_{0}$, or else $X$ is maintained.
3.3. Opposition-Based Particle Swarm Optimization. In PSO, each particle adjusts its search direction on the basis of the optimum location of all particles. In the initial stage, the algorithm converges quickly but slows down later on or even stops. These particles lose the ability to evolve when the speed of all the particles approaches zero and the algorithm is thought to represent convergence. Sometimes the algorithm does not converge to global extreme values, however, not even local extreme values. This is because the high aggregate and deficiency diversity of the particles takes a long time or an infinite time to skip from the focusing point.

In order to solve this problem, the opposition-based learning mechanism is introduced into the basic PSO, and a new random optimization algorithm is constructed, the opposition-based particle swarm optimization (OPSO) algorithm. In the OPSO algorithm, a variable is set and referred to as conNum. If the global best fitness is not updated during a single iteration, then the conNum $=$ conNum +1 , and when conNum reaches a constant set number setNum, it shows that there is a high concentration of particles, and the algorithm cannot find a better solution in the current position and speed. At this point, opposition-based learning is brought

```
%% Initialization
(1) Initialize swarm size N and constant number C}\mp@subsup{C}{1}{}\mathrm{ and }\mp@subsup{C}{2}{};\mathrm{ space dimension D; maximum
(2) Initialize the iteration number for opposition calculation conNum = 0, setnum = 100;
(3) for i=1: particleNum
(4) initialize Xi randomly with the search range ( }\mp@subsup{X}{\operatorname{min}}{},\mp@subsup{X}{\operatorname{max}}{}
(5) initialize V}\mp@subsup{V}{i}{}\mathrm{ randomly with the velocity range ( }\mp@subsup{V}{\operatorname{min}}{,},\mp@subsup{V}{\operatorname{max}}{}\mathrm{ )
(6) End for
(7) Evaluate each particle's fitness }\mp@subsup{f}{i}{}=\mathrm{ fitness( (Xi), and the best fitness PbestValue}\mp@subsup{i}{i}{}=\mp@subsup{f}{i}{
(8) Identify the best particle's position Pg and its fitness GbestValue = min(PbestValue}\mp@subsup{}{i}{});%%Loo
(9) While (t< n max )
(10) If conNum > setNum then
(11) for i=1 to N
(12) Calculate the opposite particle }\mp@subsup{X}{0}{}\mathrm{ using (8);
(13) Evaluate fitness in opposite vector f}\mp@subsup{f}{io}{}=\mathrm{ fitness ( }\mp@subsup{X}{io}{})\mathrm{ ;
(14) end for
(15) Select N fittest particles Newfitness from F (f, f, f1,\ldots, f
(16) create a population of size N;
(17) Else
(18) for i=1 to N
(19) Calculate particle velocity }\mp@subsup{V}{i}{}(t+1)\mathrm{ using (5)
(20) update particle position }\mp@subsup{X}{i}{}(t+1)\mathrm{ using (6)
(21) Evaluate newfitness }=\mathrm{ fitness (Xi}(t+1)
(22) End for
(23) End if
(24) update Pbest 
(25) for i=1 to N
(26) if (newFitness < GbestValue)
(27) update Gbest
(28) conNum = 0;
(29) end if
(30) end for
(31) conNum = conNum + 1;
(32) end while
```

Algorithm 1: The process of OPSO.
into play. According to (8) the position-based position $X_{o}$ of the particle's current position is produced, the smaller values $N$ are selected from $X$ and $X_{o}$ which have $2 N$ locations and a new swarm is then formed. If the conNum is less than setNum, then according to (5) and (6) the particle velocity and position are updated. The process of OPSO is shown in Algorithm 1.

## 4. Design of Pitch and Heave Parameter Identification

The value of pitch and heave can be measured, so select state $x_{1}$ and $x_{3}$ in (2) as the measurement state, with the observation equation as

$$
Y=H X+V, \quad H=\left(\begin{array}{llll}
1 & 0 & 0 & 0  \tag{9}\\
0 & 0 & 1 & 0
\end{array}\right)
$$

where $Y$ is an observation vector and $V$ is the twodimensional measurement of noise, and this is generally considered to be white noise.

Before computer simulation, it was necessary to disperse the state equation and observation equation. This paper does
not consider the steering input but only the wave disturbance on the ship, with the state equation being formulated as follows:

$$
\begin{gather*}
X_{k+1}=\boldsymbol{\Phi} X_{k}+\Gamma W_{k} \\
Y_{k+1}=H X_{k+1}+V_{k+1} \tag{10}
\end{gather*}
$$

where, $\Phi=e^{A T_{s}}, \Gamma=\int_{0}^{T_{s}} e^{A t} d t C$.
In this paper, the single-parameter ITTC spectrum is chosen for wave disturbance simulation, with the expression being formulated as follows:

$$
\begin{equation*}
S_{\zeta}(\omega)=\frac{8.1 \times 10^{-3} g^{2}}{\omega^{5}} \exp \left(\frac{-3.11}{h_{1 / 3}^{2} \omega^{4}}\right) \tag{11}
\end{equation*}
$$

where $g$ is the acceleration of gravity, $w$ is the natural angular frequency, and $h_{1 / 3}$ is a third significant wave height, for a level 4 and 5 sea condition, and the ITTC recommended value is 2.5 m and 3.75 m , respectively.

By (10), the observed quantity is the function of $X$, para, namely:

$$
\begin{equation*}
y_{i}=g\left(x_{i}, \operatorname{para}_{i}, W_{i}\right)+V_{i}, \quad i=1,2, \ldots, \text { dataNum } \tag{12}
\end{equation*}
$$

where dataNum is the times of observation, $g(\cdot)$ is the model output vector, and $y_{i}$ is the observation vector. The minimum mean square errors between the observed data and identical value are setting as follows:

$$
\begin{array}{r}
\left.F_{j}=\operatorname{sqrt}\left(\frac{1}{\operatorname{dataNum}} \sum_{i=1}^{\text {dataNum }}\left(y_{i}-g\left(x_{i}, \text { para }_{i}, W_{i}\right)\right)\right)^{2}\right), \\
j=1,2 . \tag{13}
\end{array}
$$

The hydrodynamic parameters of our requirements should mean that the two components in (13) obtain their minimum value simultaneously under constrained conditions (10). We must set different weight coefficients to ensure that each parameter has the same effect on the objective function, taking into consideration the different order of magnitude of pitch and heave. The fitness function is shown as follows:

$$
\begin{equation*}
\text { Fit }=F_{1}+\alpha F_{2}, \tag{14}
\end{equation*}
$$

where $\alpha$ is the weight coefficient.

## 5. Experiment Testing and Comparisons

Here the experimental ship's parameters are as follows: the hull quality $m$ is 442000 kg , the beam $B$ is 7.2 m , and the draft is 2.25 m . The waterline $L_{p p}$ is $60 \mathrm{~m} . I_{5}=(0.25 *$ $\left.L_{P P}\right), m_{0}=99450000\left(\mathrm{~kg} \cdot \mathrm{~m}^{2}\right), X_{R}=25.8 \mathrm{~m}$, and $Z_{R}=$ 1.32 m . The number of rudder $n$ is 2 and the hydrostatic resilience factor $C_{44}$ is 3370000 . For the ship, $C_{33}, C_{35}, C_{55}$, and $C_{53}$ are constants and can be obtained by calculating the structural parameters of the ship. $V$ is Gaussian white noise $V$ is Gaussian white noise and its covariance matrix can be taken as so that the accuracy of the sensor can be taken as $Q_{V V}=$ $\operatorname{diag}\left[\begin{array}{lll}20.3 e-4 & 2.25 e-6\end{array}\right]$.

A comparative algorithm is the OPSO algorithm proposed in this paper and the PSO-w (PSO with inertia weight [26]), with the parameters being set as shown in Table 1.

The identification results by PSO and OPSO are shown in Tables 2, 3, and 4, for level 4 sea condition, speed 18 kn and course angle $90^{\circ}, 135^{\circ}$, and $180^{\circ}$, respectively.

As we can see from Table 2 to Table 4, the parameters of pitch and heave are correctly identified by using the PSO algorithm, with the OPSO algorithm clearly obtaining better results than the PSO, especially for level 4 sea condition, speed 18 kn , and course angle $135^{\circ}$, and the relative error of the OPSO is smaller than for the PSO algorithm.

From the identification of the parameters of pitch and heave motions carried out by using the OPSO algorithm, we can obtain the mathematical model for pitch and heave which is constructed according to the identification of the hydrodynamic parameters. The model in which the pitch and

Table 1: Parameters and their range of values used in our proposed algorithm.

| Parameters | Value/range |
| :--- | :---: |
| Population size of swarm | 100 |
| Acceleration constants $c_{1}, c_{2}$ | $c_{1}=c_{2}=1.4962$ |
| Inertia weight $w$ | $[0.4,0.9]$ |
| Random number $r_{1}, r_{2}$ | $[0,1]$ |
| Maximum iteration times $n_{\max }$ | 1000 |
| Range of velocity $V_{\max }$ | $\left(x_{\text {Max }}-x_{\text {Min }}\right) / 2$ |
| setNum for OPSO | 100 |
| Sampling dot number dataNum | 200 |

heave for sea condition 4 , speed 18 kn , and course angle $90^{\circ}$, $135^{\circ}$, and $180^{\circ}$ is shown in Figures 1, 2, and 3, respectively. There are three curves in each diagram; the first curve "-ロ-" represents the observed values of the coupled pitch and heave motions, the second curve " $-\boldsymbol{\nabla}-$ " represents the model constructed by PSO-w, and the last curve "--" represents the model output values constructed by OPSO.

It is obvious in these figures that there was an agreement between the pitch and heave model estimated by the PSOw model and observed values for a level 4 sea condition, ship speed 18 kn , and course angle $90^{\circ}$ and $180^{\circ}$, but when the course angle was $90^{\circ}$, the results were not accurate. However, the model estimated by the OPSO algorithm tallies completely with the observed values.

Tables 5, 6, and 7 and Figures 4, 5, and 6 show similar results for a level 5 sea condition. It is clear in Tables 5-7 that the parameters identified by the OPSO algorithm are close to the real ones while those identified by PSO-w are not, especially the results for level 5 sea condition, ship speed 18 kn , and course angle 90 degrees, as the parameters have greater relative errors. Because of this, the corresponding pitch and heave models do not tally with the observed values. However, for other course angles, both the PSO-w and OPSO algorithms agree with observed values.

## 6. Conclusions

The identification of the hydrodynamic parameters of ships is an important way of obtaining these parameters. In this paper, we have used OPSO to design the method of identifying the hydrodynamic parameters of the coupled pitch and heave motions of ships. This paper introduces in detail the opposition-basedlearning algorithm and puts forward an improved idea and process for the oppositionbased algorithm. In addition, this paper introduces the process involving the hydrodynamic parameter identification algorithm. Here, we have established wave disturbance as the model input, with the algorithm's fitness function being the output model. The hydrodynamic parameter identification problem was then converted into a constrained optimization problem and the OPSO algorithm was used to find the optimal solution. Finally, we made use of computer simulation, with the simulation results showing that the OPSO algorithm is relatively stable in terms of identifying the hydrodynamic

TAbLe 2: Parameter identification of pitch and heave motions for level 4 sea condition, ship speed 18 kn and course angle $90^{\circ}$.

| Parameters | Theoretical values | PSO-w |  |  | OPSO |
| :--- | :---: | :---: | :---: | :---: | :---: |
|  |  | Identification value | Relative error | Identification value | Relative error |
| $a_{33}$ | $8.045 E+05$ | $9.446 E+05$ | $17.42 \%$ | $8.141 E+05$ | $1.19 \%$ |
| $b_{33}$ | $9.369 E+05$ | $7.901 E+05$ | $15.66 \%$ | $9.447 E+05$ | $0.83 \%$ |
| $a_{35}$ | $7.753 E+06$ | $8.902 E+06$ | $14.82 \%$ | $8.037 E+06$ | $3.66 \%$ |
| $b_{35}$ | $1.159 E+07$ | $1.480 E+07$ | $27.72 \%$ | $1.222 E+07$ | $5.39 \%$ |
| $a_{55}$ | $2.595 E+08$ | $2.830 E+08$ | $9.05 \%$ | $2.706 E+08$ | $4.27 \%$ |
| $b_{55}$ | $2.885 E+08$ | $4.077 E+08$ | $41.31 \%$ | $2.892 E+08$ | $0.26 \%$ |
| $a_{53}$ | $1.479 E+07$ | $1.777 E+07$ | $20.17 \%$ | $1.500 E+07$ | $1.39 \%$ |
| $b_{53}$ | $3.324 E+06$ | $2.296 E+06$ | $30.91 \%$ | $3.068 E+06$ | $7.70 \%$ |

Table 3: Parameter identification of pitch and heave motions for level 4 sea condition, ship speed 18 kn and course angle $135^{\circ}$.

| Parameters | Theoretical values | Identification value | Relative error | Identification value | Relative error |
| :--- | :---: | :---: | :---: | :---: | :---: |
|  |  | $5.569 E+05$ | $11.77 \%$ | $6.177 E+05$ | $2.12 \%$ |
| $b_{33}$ | $8.033 E+05$ | $7.635 E+05$ | $4.96 \%$ | $8.245 E+05$ | $2.64 \%$ |
| $a_{35}$ | $3.281 E+06$ | $5.248 E+06$ | $59.95 \%$ | $3.057 E+06$ | $6.84 \%$ |
| $b_{35}$ | $9.429 E+06$ | $7.879 E+06$ | $16.44 \%$ | $9.598 E+06$ | $1.80 \%$ |
| $a_{55}$ | $1.525 E+08$ | $1.541 E+08$ | $1.01 \%$ | $1.533 E+08$ | $0.52 \%$ |
| $b_{55}$ | $2.091 E+08$ | $2.716 E+08$ | $29.88 \%$ | $2.115 E+08$ | $1.14 \%$ |
| $a_{53}$ | $7.865 E+06$ | $9.509 E+06$ | $20.90 \%$ | $7.772 E+06$ | $1.18 \%$ |
| $b_{53}$ | $2.267 E+06$ | $1.616 E+06$ | $28.72 \%$ | $2.423 E+06$ | $6.87 \%$ |

Table 4: Parameter identification of pitch and heave motions for level 4 sea condition, ship speed 18 kn and course angle $180^{\circ}$.

| Parameters | Theoretical values | PSO-w |  | OPSO |  |
| :--- | :---: | :---: | :---: | :---: | :---: |
|  |  | Identification value | Relative error | Identification value | Relative error |
| $a_{33}$ | $6.101 E+05$ | $5.815 E+05$ | $4.69 \%$ | $6.148 E+05$ | $0.77 \%$ |
| $b_{33}$ | $7.400 E+05$ | $7.782 E+05$ | $-5.16 \%$ | $7.366 E+05$ | $0.46 \%$ |
| $a_{35}$ | $2.663 E+06$ | $2.396 E+06$ | $10.03 \%$ | $2.758 E+06$ | $3.56 \%$ |
| $b_{35}$ | $8.951 E+06$ | $8.835 E+06$ | $1.30 \%$ | $8.873 E+06$ | $0.87 \%$ |
| $a_{55}$ | $1.377 E+08$ | $1.348 E+08$ | $2.14 \%$ | $1.367 E+08$ | $0.74 \%$ |
| $b_{55}$ | $1.873 E+08$ | $1.944 E+08$ | $-3.79 \%$ | $1.838 E+08$ | $1.87 \%$ |
| $a_{53}$ | $6.611 E+06$ | $6.594 E+06$ | $0.25 \%$ | $6.568 E+06$ | $0.65 \%$ |
| $b_{53}$ | $2.356 E+06$ | $3.123 E+06$ | $-32.57 \%$ | $2.113 E+06$ | $10.32 \%$ |

Table 5: Parameter identification for pitch and heave motions for level 5 sea condition, ship speed 18 kn and course angle $90^{\circ}$.

| Parameters | Theoretical values | PSO-w |  | OPSO |  |
| :--- | :---: | :---: | :---: | :---: | :---: |
|  |  | Identification value | Relative error | Identification value | Relative error |
| $a_{33}$ | $9.375 E+05$ | $1.178 E+06$ | $25.64 \%$ | $9.111 E+05$ | $2.82 \%$ |
| $b_{33}$ | $9.089 E+05$ | $8.644 E+05$ | $4.89 \%$ | $9.635 E+05$ | $6.01 \%$ |
| $a_{35}$ | $1.137 E+07$ | $2.102 E+07$ | $-84.86 \%$ | $1.030 E+07$ | $9.39 \%$ |
| $b_{35}$ | $1.274 E+07$ | $1.831 E+07$ | $43.71 \%$ | $1.295 E+07$ | $1.65 \%$ |
| $a_{55}$ | $3.613 E+08$ | $4.017 E+08$ | $11.19 \%$ | $3.532 E+08$ | $2.25 \%$ |
| $b_{55}$ | $3.202 E+08$ | $4.593 E+08$ | $43.43 \%$ | $3.428 E+08$ | $7.07 \%$ |
| $a_{53}$ | $1.942 E+07$ | $2.430 E+07$ | $25.14 \%$ | $1.964 E+07$ | $1.11 \%$ |
| $b_{53}$ | $4.641 E+06$ | $3.829 E+06$ | $17.50 \%$ | $5.348 E+06$ | $-15.24 \%$ |



FIGURE 1: Pitch and heave observed value. Model identified by PSO-w and model identified by OPSO for level 4 sea condition, ship speed 18 kn and course angle $90^{\circ}$.


FIgure 2: Pitch and heave observed value. Model identified by PSO-w and model identified by OPSO for level 4 sea condition, ship speed 18 kn and course angle $135^{\circ}$.


FIgure 3: Pitch and heave observed value. Model identified by PSO-w and model identified by OPSO for level 4 sea condition, ship speed 18 kn and course angle $180^{\circ}$.


Figure 4: Pitch and heave observed value. Model identified by PSO-w and model identified by OPSO for level 5 sea condition, ship speed 18 kn and course angle $90^{\circ}$.


Figure 5: Pitch and heave observed value. Model identified by PSO-w and model identified by OPSO for level 5 sea condition, ship speed 18 kn and course angle $135^{\circ}$.


Figure 6: Pitch and heave observed value. Model identified by PSO-w and model identified by OPSO for level 5 sea condition, ship speed 18 kn and course angle $180^{\circ}$.

Table 6: Parameter identification of pitch and heave motions for level 5 sea condition, ship speed 18 kn and course angle $135^{\circ}$.

| Parameters | Theoretical values | PSO-w <br> Identification value | Relative error | Identification value | Relative error |
| :--- | :---: | :---: | :---: | :---: | :---: |
|  |  | $7.515 E+05$ | $-4.34 \%$ | $7.106 E+05$ | $1.34 \%$ |
| $b_{33}$ | $8.553 E+05$ | $8.283 E+05$ | $3.15 \%$ | $8.921 E+05$ | $4.31 \%$ |
| $a_{35}$ | $5.539 E+06$ | $5.923 E+06$ | $-6.93 \%$ | $5.426 E+06$ | $2.05 \%$ |
| $b_{35}$ | $1.046 E+07$ | $9.616 E+06$ | $8.10 \%$ | $1.043 E+07$ | $0.35 \%$ |
| $a_{55}$ | $2.051 E+08$ | $2.086 E+08$ | $-1.70 \%$ | $2.079 E+08$ | $1.34 \%$ |
| $b_{55}$ | $2.451 E+08$ | $2.473 E+08$ | $-0.90 \%$ | $2.467 E+08$ | $0.64 \%$ |
| $a_{53}$ | $1.124 E+07$ | $1.115 E+07$ | $0.82 \%$ | $1.167 E+07$ | $3.85 \%$ |
| $b_{53}$ | $2.876 E+06$ | $2.297 E+06$ | $20.13 \%$ | $2.895 E+06$ | $0.64 \%$ |

Table 7: Parameter identification of pitch and heave motions for level 5 sea condition, ship speed 18 kn and course angle $180^{\circ}$.

| Parameters | Theoretical values | PSO-w |  |  | OPSO |
| :--- | :---: | :---: | :---: | :---: | :---: |
|  |  | Identification value | Relative error | Identification value | Relative error |
| $a_{33}$ | $6.787 E+05$ | $6.812 E+05$ | $-0.38 \%$ | $6.762 E+05$ | $0.37 \%$ |
| $b_{33}$ | $8.182 E+05$ | $7.952 E+05$ | $2.82 \%$ | $7.939 E+05$ | $2.97 \%$ |
| $a_{35}$ | $4.428 E+06$ | $3.406 E+06$ | $23.09 \%$ | $4.638 E+06$ | $4.74 \%$ |
| $b_{35}$ | $9.914 E+06$ | $9.902 E+06$ | $0.12 \%$ | $9.849 E+06$ | $0.66 \%$ |
| $a_{55}$ | $1.783 E+08$ | $1.798 E+08$ | $-0.89 \%$ | $1.742 E+08$ | $2.30 \%$ |
| $b_{55}$ | $2.246 E+08$ | $2.317 E+08$ | $-3.17 \%$ | $2.284 E+08$ | $1.70 \%$ |
| $a_{53}$ | $9.511 E+06$ | $9.738 E+06$ | $-2.38 \%$ | $9.351 E+06$ | $1.69 \%$ |
| $b_{53}$ | $2.664 E+06$ | $2.228 E+06$ | $16.37 \%$ | $2.974 E+06$ | $11.62 \%$ |

parameters connected with the problem of the coupled pitch and heave motions of ships. In addition, the identified coupled pitch and heave model values and the observed values are consistent. This method may provide a new solution for the identification of coupled pitch and heave motions. This paper has not taken the disturbance of the rudder angle into consideration, something which the author intends to research in due course.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Stability and Stabilization of Continuous-Time Markovian Jump Singular Systems with Partly Known Transition Probabilities 

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#### Abstract

This paper investigates the problem of the stability and stabilization of continuous-time Markovian jump singular systems with partial information on transition probabilities. A new stability criterion which is necessary and sufficient is obtained for these systems. Furthermore, sufficient conditions for the state feedback controller design are derived in terms of linear matrix inequalities. Finally, numerical examples are given to illustrate the effectiveness of the proposed methods.


## 1. Introduction

In practice, many dynamical systems cannot be represented by the class of linear time-invariant model since the dynamics of these systems are random with some features, for example, abrupt changes, breakdowns of components, changes in the interconnections of subsystems, and so forth. Such class of dynamical systems can be adequately described by the class of stochastic hybrid systems. A special class of hybrid systems referred to as Markovian jump systems (MJS), a class of multimodel systems in which the transitions among different modes are governed by a Markov chain, have attracted a lot of researchers and many problems have been solved, such as stability, stabilization, and $H_{\infty}$ control problems; see [1-7].

However, in most of the studies, complete knowledge of the mode transitions is required as a prerequisite for analysis and synthesis of MJS. This means that the transition probabilities of the underlying Markov chain are assumed to be completely known. However, in practice, incomplete transition probabilities are often encountered especially if adequate samples of the transitions are costly or time consuming to obtain. So, it is necessary to further consider more general jump systems with partial information on transition probabilities. The concept for MJS with partially unknown transition probabilities is first proposed in [8] and a series of studies have been carried out [ $9-12$ ] recently. A new approach for the analysis and synthesis for Markov jump linear systems
with incomplete transition descriptions has been proposed in [12], which can be further used for other analysis and synthesis issues, such as the stability of Markovian jump singular systems (MJSS).

A lot of attention has already been focused on robust stability, robust stabilization, and $H_{\infty}$ control problems for MJSS in recent years, such as the works in [13-17]. However, to the best of the authors' knowledge, the necessary and sufficient conditions for the stochastic stability and stabilization problems of MJSS have not been fully investigated, especially when the transition probabilities are partially known. The authors in $[15,16]$ have, respectively, studied the problems of stability and stabilization for a class of continuous-time (discrete-time) singular hybrid systems. New sufficient and necessary conditions for these singular hybrid systems to be regular, impulse-free (causal), and stochastically stable have been proposed in terms of a set of coupled strict linear matrix inequalities (LMIs). But the case of systems with partly known transition probabilities still needs to be considered. In addition to this, it is important to mention that the derivation of strict LMIs for MJSS with incomplete transition probabilities renders the synthesis of the state feedback controllers easier. These problems are important and challenging in both theory and practice, which motivates us for this study.

In this paper, the problem of the stability and stabilization of MJSS with partly known transition probabilities is
addressed. Inspired by the ideas in [12], which fully unitized the properties of the transition rate matrix (TRM) and the convexity of the uncertain domains, we explore a new sufficient and necessary condition in terms of strict linear matrix inequalities (LMIs) for the MJSS to be regular, impulsive, and stochastically stable. Then, based on the proposed stability criterion, the conditions for state feedback controller are derived. Finally, numerical examples are given to illustrate the effectiveness of the proposed method.

Compared with the existing works about the stability and stabilization of Markovian jump systems, the current paper has the following novel features. First, the current paper deals with the stability and stabilization problems for MJSS with partly known transition probabilities, while most literatures (e.g., [8-12]) focused on those of normal ones that are special cases of MJSS. Second, the conservatism in the conventional studies [15] is eliminated by considering the fact that the unknown elements of each row in TRM exist. Moreover, the difficulty that the unknown elements contain diagonal elements is also overcome by introducing a lower bound of the diagonal element without additional conservatism.

Notation. The notation used in this technical note is standard. The superscript " $T$ " stands for matrix transposition; $\mathbb{R}^{n}$ denotes the $n$ dimensional Euclidean space; $\mathbb{Z}^{+}$represents the sets of positive integers, respectively. For the notation $(\Omega, \mathscr{F}, \mathscr{P}), \Omega$ represents the sample space, $\mathscr{F}$ is the $\sigma$-algebra of subsets of the sample space, and $\mathscr{P}$ is the probability measure on $\mathscr{F}$. $\mathrm{E}[\cdot]$ stands for the mathematical expectation. In addition, in symmetric block matrices or long matrix expressions, we use $*$ as an ellipsis for the terms that are introduced by symmetry and $\operatorname{diag}\left\{X_{1}, X_{2}, \ldots, X_{N}\right\}$ stands for a block-diagonal matrix constituted by $X_{1}, X_{2}, \ldots, X_{N}$. The notation $X>0$ means $X$ is real symmetric positive definite, and $X_{i}$ is adopted to denote $X(i)$ for brevity. $I$ and 0 represent, respectively, identity matrix and zero matrix. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

## 2. Preliminaries and Problem Formulation

Consider the following continuous-time MJSS with Markovian jump parameters:

$$
\begin{equation*}
E \dot{x}(t)=A\left(r_{t}\right) x(t)+B\left(r_{t}\right) u(t) \tag{1}
\end{equation*}
$$

where $x(t) \in \mathbb{R}^{n}$ is the state vector and $u(t) \in \mathbb{R}^{m}$ is the control input. The matrix $E \in \mathbb{R}^{n \times n}$ is supposed to be singular with $\operatorname{rank}(E)=r<n$. The stochastic process $\left\{r_{t}, t \geq 0\right\}$ taking values in a finite set $S=\{1,2, \ldots, N\}$ is described by a continuous-time, discrete-state homogeneous Markov process and has the following mode transition probabilities:

$$
\operatorname{Pr}\left\{r_{t+h}=j \mid r_{t}=i\right\}= \begin{cases}\lambda_{i j} h+o(h), & \text { if } j \neq i,  \tag{2}\\ 1+\lambda_{i i} h+o(h), & \text { if } j=i\end{cases}
$$

where $h>0, \lim _{h \rightarrow 0}(o(h) / h)=0$, and $\lambda_{i j} \geq 0(i, j \in S, j \neq i)$ denotes the switching rate from mode $i$ at time $t$ to mode $j$ at
time $t+h$, and $\lambda_{i i}=-\sum_{j \in S, j \neq i} \lambda_{i j}$ for all $i \in S$. The TRM is given by

$$
\Lambda=\left[\begin{array}{cccc}
\lambda_{11} & \lambda_{12} & \cdots & \lambda_{1 N}  \tag{3}\\
\lambda_{21} & \lambda_{22} & \cdots & \lambda_{2 N} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_{N 1} & \lambda_{N 2} & \cdots & \lambda_{N N}
\end{array}\right]
$$

The set $S$ contains $N$ modes of system (1) and for $r_{t}=$ $i \in S$, the system matrices of the $i$ th mode are denoted by $A_{i}, B_{i}$, which are known real-valued constant matrices of appropriate dimensions that describe the nominal system.

The transition rates described above are considered to be partially available; that is, some elements in matrix $\Lambda$ are unknown. Take system (1) with 4 operation modes for example; the TRM $\Lambda$ may be written as

$$
\Lambda=\left[\begin{array}{llll}
\lambda_{11} & \lambda_{12} & \hat{\lambda}_{13} & \hat{\lambda}_{14}  \tag{4}\\
\hat{\lambda}_{21} & \widehat{\lambda}_{22} & \lambda_{23} & \lambda_{24} \\
\hat{\lambda}_{31} & \hat{\lambda}_{32} & \lambda_{33} & \lambda_{34} \\
\lambda_{41} & \lambda_{42} & \hat{\lambda}_{43} & \lambda_{44}
\end{array}\right]
$$

where " $\mu$ " denotes the unknown element.
For $\forall i \in S$, we denote

$$
S=S_{\mathscr{K}}^{i}+S_{\mathscr{K}}^{i}
$$

$S_{\mathscr{K}}^{i} \triangleq\left\{j: \lambda_{i j}\right.$ is known $\}, \quad S_{\mathscr{K}}^{i} \triangleq\left\{j: \lambda_{i j}\right.$ is unknown $\}$.

If $S_{\mathscr{K}}^{i} \neq \emptyset, S_{\mathscr{K}}^{i}$ is further described as

$$
\begin{equation*}
S_{\mathscr{K}}^{i}=\left\{\mathscr{K}_{1}^{i}, \mathscr{K}_{2}^{i}, \ldots, \mathscr{K}_{m}^{i}\right\}, \quad 1 \leq m \leq N \tag{6}
\end{equation*}
$$

where $\mathscr{K}_{m}^{i} \in \mathbb{Z}^{+}$represents the index of the $m$ th known element in the $i$ th row of matrix $\Lambda$. Also, throughout the technical note, we denote

$$
\begin{equation*}
\lambda_{\mathscr{K}}^{i}=\sum_{j \in S_{\mathscr{C}}^{i}} \lambda_{i j} \tag{7}
\end{equation*}
$$

When $\hat{\lambda}_{i i}$ is unknown, it is necessary to provide a lower bound $\lambda_{d}^{i}$ for it and $\lambda_{d}^{i} \leq-\lambda_{\mathscr{K}}^{i}$.

Now, we introduce the following definition for the continuous-time MJSS (1) (with $u(t) \equiv 0)$.

Definition 1 (see [17]).
(i) The continuous-time MJSS in (1) is said to be regular if, for each $i \in S, \operatorname{det}\left(s E-A_{i}\right)$ is not identically zero.
(ii) The continuous-time MJSS in (1) is said to be impulsive if, for each $i \in S, \operatorname{deg}\left(\operatorname{det}\left(s E-A_{i}\right)\right)=\operatorname{rank}(E)$.
(iii) The continuous-time MJSS in (1) is said to be stochastically stable if, for any $x_{0} \in \mathbb{R}^{n}$ and $r_{0} \in S$, there exists a scalar $M\left(x_{0}, r_{0}\right)>0$ such that

$$
\begin{equation*}
\mathbf{E}\left\{\int_{0}^{\infty}\|x(t)\|^{2} \mid x_{0}, r_{0}\right\} \leq M\left(x_{0}, r_{0}\right) \tag{8}
\end{equation*}
$$

where $\mathbf{E}$ is the mathematical expectation, and $x\left(t, x_{0}, r_{0}\right)$ denotes the solution to system (1) at time $t$ under the initial conditions $x_{0}$ and $r_{0}$.
(iv) The continuous-time MJSS in (1) is said to be stochastically admissible if it is regular, impulsive, and stochastically stable.

The following lemma is recalled, which will be used in what follows.

Lemma 2 (see [18]). Let $P \in R^{n \times n}$ be symmetric such that $E_{R}^{T} P E_{R}>0, \Phi \in R^{n \times n}$, and $S$ are nonsingular. Then, $P E+$ $S^{T} \Phi R^{T}$ is nonsingular and its inverse is expressed as

$$
\begin{equation*}
\left(P E+S^{T} \Phi R^{T}\right)^{-1}=\bar{P} E^{T}+R \bar{\Phi} S \tag{9}
\end{equation*}
$$

where $E_{L}$ and $E_{R}$ are full column rank with $E=E_{L} E_{R}^{T}, R \in$ $R^{(n-r) \times n}$, and $S \in R^{n \times(n-r)}$ satisfies $R E=0$ and $E S=0$, respectively. $\bar{P}$ is symmetric and $S$ is nonsingular such that

$$
\begin{gather*}
E_{L}^{T} \bar{P} E_{L}=\left(E_{R}^{T} P E_{R}\right)^{-1}, \\
\bar{\Phi}=\left(R R^{T}\right)^{-1} \Phi^{-1}\left(S S^{T}\right)^{-1} \tag{10}
\end{gather*}
$$

## 3. Main Results

In this section, we will derive the stochastic stability criteria for system (1) when the transition probabilities are partially unknown and design a state-feedback controller and a static output feedback controller such that the closed-loop system is stochastically stabilizable. The mode-dependent controller considered here has the form

$$
\begin{equation*}
u(t)=K\left(r_{t}\right) x(t) \tag{11}
\end{equation*}
$$

where $K_{i}=K\left(r_{t}\right) \in R^{m \times n}\left(\forall r_{t}=i \in S\right)$ are the controller gains to be determined. The closed-loop systems obtained by applying controllers (11) to system (1) are

$$
\begin{equation*}
E \dot{x}(t)=\left(A_{i}+B_{i} K_{i}\right) x(t) \tag{12}
\end{equation*}
$$

First, we provide the following lemma which presents a necessary and sufficient condition for the continuous-time MJSS with completely known transition probabilities matrix to be stochastically admissible.

Lemma 3 (see [15]). System (1) with $u(t)=0$ is stochastically admissible if and only if there exist matrices $P_{i} \in R^{n \times n}>0$, $i \in S$, and $\Phi_{i} \in R^{(n-r) \times(n-r)}$, such that following coupled LMIs hold for each $i \in S$ :

$$
\begin{align*}
& A_{i}^{T}\left(P_{i} E+R^{T} \Phi_{i} S^{T}\right)+\left(P_{i} E+R^{T} \Phi_{i} S^{T}\right)^{T} A_{i} \\
& \quad+\sum_{j \in S} \lambda_{i j} E^{T} P_{j} E<0 \tag{13}
\end{align*}
$$

Let us first give the stability result for the unforced system (1) (with $u(t) \equiv 0$ ). The following theorem presents a necessary and sufficient condition on the stochastic admissibility of the considered system with partially unknown transition probabilities.

Theorem 4. Consider the unforced system (1) with partially unknown transition probabilities. The corresponding system is stochastically admissible if and only if there exist matrices $P_{i} \in \mathbb{R}^{n \times n}>0$ and nonsingular symmetric matrices $\Phi_{i} \in$ $\mathbb{R}^{(n-r)(n-r)}$, such that for each $i \in S$

$$
\begin{align*}
& A_{i}^{T}\left(P_{i} E+R^{T} \Phi_{i} S^{T}\right)+\left(P_{i} E+R^{T} \Phi_{i} S^{T}\right)^{T} A_{i} \\
& +E^{T} \mathscr{P}_{\mathscr{K}}^{i} E-\lambda_{\mathscr{K}}^{i} E^{T} P_{j} E<0,  \tag{14}\\
& \forall j \in S_{\mathscr{K}}^{i}, \text { if } i \in S_{\mathscr{K}}^{i}, \\
& A_{i}^{T}\left(P_{i} E+R^{T} \Phi_{i} S^{T}\right)+\left(P_{i} E+R^{T} \Phi_{i} S^{T}\right)^{T} A_{i} \\
& +E^{T} \mathscr{P}_{\mathscr{K}}^{i} E+E^{T}\left(\lambda_{d}^{i} P_{i}-\lambda_{d}^{i} P_{j}-\lambda_{\mathscr{K}}^{i} P_{j}\right) E<0,  \tag{15}\\
& \forall j \in S_{\mathscr{K}}^{i}, \text { if } i \in S_{\mathscr{K}}^{i},
\end{align*}
$$

where $\mathscr{P}_{\mathscr{K}}^{i}=\sum_{j \in S_{\mathscr{K}}^{i}} \lambda_{i j} P_{j}$ and $\lambda_{d}^{i}$ is a given lower bound for the unknown diagonal element.

Proof. Consider two cases, $i \in S_{\mathscr{K}}^{i}$ and $i \in S_{\mathscr{K}}^{i}$, and note that system (1) is stochastically stable if and only if (13) holds.

Case $1\left(i \in S_{\mathscr{K}}^{i}\right)$. It should be noted that in this case one has $\lambda_{\mathscr{K}}^{i} \leq 0$. We only need to consider $\lambda_{\mathscr{K}}^{i}<0$ since $\lambda_{\mathscr{K}}^{i}=0$ means the elements in the $i$ th row of the TRM are known, so it is not considered here. Now the left-hand side of (13) in Lemma 3 can be rewritten as

$$
\begin{align*}
\Theta_{i} \triangleq & A_{i}^{T}\left(P_{i} E+R^{T} \Phi_{i} S^{T}\right)+\left(P_{i} E+R^{T} \Phi_{i} S^{T}\right)^{T} A_{i} \\
& +\sum_{j \in S_{\mathscr{K}}^{i}} \lambda_{i j} E^{T} P_{j} E+\sum_{j \in S_{\mathscr{K}}^{i}} \hat{\lambda}_{i j} E^{T} P_{j} E \\
= & A_{i}^{T}\left(P_{i} E+R^{T} \Phi_{i} S^{T}\right)+\left(P_{i} E+R^{T} \Phi_{i} S^{T}\right)^{T} A_{i}  \tag{16}\\
& +E^{T} \mathscr{P}_{\mathscr{K}}^{i} E-\lambda_{\mathscr{K}}^{i} \sum_{j \in S_{\mathscr{U K}}^{i}} \frac{\hat{\lambda}_{i j}}{-\lambda_{\mathscr{K}}^{i}} E^{T} P_{j} E,
\end{align*}
$$

where the elements $\hat{\lambda}_{i j}, j \in S_{\mathscr{K}}^{i}$ are unknown. Since $0 \leq$ $\hat{\lambda}_{i j} /\left(-\lambda_{\mathscr{K}}^{i}\right) \leq 1$ and $\sum_{j \in S_{\mathscr{K}}^{i}} \hat{\lambda}_{i j} /\left(-\lambda_{\mathscr{K}}^{i}\right)=1$, we know that

$$
\begin{align*}
\Theta_{i}= & \sum_{j \in S_{\mathscr{H}}^{i}} \frac{\hat{\lambda}_{i j}}{-\lambda_{\mathscr{K}}^{i}} \\
& \times\left[A_{i}^{T}\left(P_{i} E+R^{T} \Phi_{i} S^{T}\right)+\left(P_{i} E+R^{T} \Phi_{i} S^{T}\right)^{T} A_{i}\right.  \tag{17}\\
& \left.\quad+E^{T} \mathscr{P}_{\mathscr{K}}^{i} E-\lambda_{\mathscr{K}}^{i} E^{T} P_{j} E\right] .
\end{align*}
$$

Therefore, for $0 \leq \hat{\lambda}_{i j} \leq-\lambda_{\mathscr{K}}^{i}, \Theta_{i}<0$ is equivalent to $A_{i}^{T}\left(P_{i} E+\right.$ $\left.R^{T} \Phi_{i} S^{T}\right)+\left(P_{i} E+R^{T} \Phi_{i} S^{T}\right)^{T} A_{i}+E^{T} \mathscr{P}_{\mathscr{K}}^{i} E-\lambda_{\mathscr{K}}^{i} E^{T} P_{j} E<$ $0, \forall j \in S_{\mathscr{K}}^{i}$, which implies that, in the presence of unknown
elements $\hat{\lambda}_{i j}$, the system stochastic admissibility is ensured if and only if (14) holds.

Case $2\left(i \in S_{\mathscr{K}}^{i}\right)$. In this case, $\hat{\lambda}_{i i}$ is unknown, $\lambda_{\mathscr{K}}^{i} \geq 0$, and $\hat{\lambda}_{i i} \leq-\lambda_{\mathscr{K}}^{i}$. We also only consider $\widehat{\lambda}_{i i}<-\lambda_{\mathscr{K}}^{i}$ since $\widehat{\lambda}_{i i}=-\lambda_{\mathscr{K}}^{i}$; then the $i$ th row of the TRM is completely known. Now the left-hand side of (15) can be rewritten as

$$
\begin{align*}
\Theta_{i} \triangleq & A_{i}^{T}\left(P_{i} E+R^{T} \Phi_{i} S^{T}\right)+\left(P_{i} E+R^{T} \Phi_{i} S^{T}\right)^{T} A_{i} \\
& +E^{T} \mathscr{P}_{\mathscr{K}}^{i} E+\hat{\lambda}_{i i} E^{T} P_{i} E+\sum_{j \in S_{\mathscr{U}}} \hat{\lambda}_{i j} E^{T} P_{j} E \\
= & A_{i}^{T}\left(P_{i} E+R^{T} \Phi_{i} S^{T}\right)+\left(P_{i} E+R^{T} \Phi_{i} S^{T}\right)^{T} A_{i}+E^{T} \mathscr{P}_{\mathscr{K}}^{i} E \\
& +E^{T}\left[\hat{\lambda}_{i i} P_{i}+\left(-\hat{\lambda}_{i i}-\lambda_{\mathscr{K}}^{i}\right) \sum_{j \in S_{\mathscr{U K}} ; j \neq i} \frac{\hat{\lambda}_{i j}}{-\hat{\lambda}_{i i}-\lambda_{\mathscr{K}}^{i}} P_{j}\right] E . \tag{18}
\end{align*}
$$

Likewise, since we have $0 \leq \widehat{\lambda}_{i j} /\left(-\widehat{\lambda}_{i i}-\lambda_{\mathscr{K}}^{i}\right) \leq 1$ and $\sum_{j \in S_{U S \mathscr{S}^{i}} j \neq i} \hat{\lambda}_{i j} /\left(-\hat{\lambda}_{i i}-\lambda_{\mathscr{K}}^{i}\right)=1$, we know that

$$
\begin{align*}
\Theta_{i}=\sum_{j \in S_{\mathscr{H}}^{i}, j \neq i} \frac{\hat{\lambda}_{i j}}{-\widehat{\lambda}_{i i}-\lambda_{\mathscr{K}}^{i}} & {\left[A_{i}^{T}\left(P_{i} E+R^{T} \Phi_{i} S^{T}\right)\right.} \\
& +\left(P_{i} E+R^{T} \Phi_{i} S^{T}\right)^{T} A_{i}+E^{T} \mathscr{P}_{\mathscr{K}}^{i} E \\
& \left.+E^{T}\left(\hat{\lambda}_{i i} P_{i}-\hat{\lambda}_{i i} P_{j}-\lambda_{\mathscr{K}}^{i} P_{j}\right) E\right] \tag{19}
\end{align*}
$$

which means that $\Theta_{i}<0$ is equivalent to $\forall j \in S_{\mathscr{K}}, j \neq i$,

$$
\begin{align*}
& A_{i}^{T}\left(P_{i} E+R^{T} \Phi_{i} S^{T}\right)+\left(P_{i} E+R^{T} \Phi_{i} S^{T}\right)^{T} A_{i}  \tag{20}\\
& \quad+E^{T} \mathscr{P}_{\mathscr{K}}^{i} E+E^{T}\left(\hat{\lambda}_{i i} P_{i}-\hat{\lambda}_{i i} P_{j}-\lambda_{\mathscr{K}}^{i} P_{j}\right) E<0 .
\end{align*}
$$

As $\hat{\lambda}_{i i}$ is lower bounded by $\lambda_{d}^{i}$, we have

$$
\begin{equation*}
\lambda_{d}^{i} \leq \hat{\lambda}_{i i}<-\lambda_{\mathscr{K}}^{i} \tag{21}
\end{equation*}
$$

which implies that

$$
\begin{equation*}
\lambda_{d}^{i} \leq \hat{\lambda}_{i i}<-\lambda_{\mathscr{K}}^{i}+\epsilon \tag{22}
\end{equation*}
$$

for some $\epsilon<0$ arbitrarily small. Then $\hat{\lambda}_{i i}$ can be further written as a convex combination

$$
\begin{equation*}
\hat{\lambda}_{i i}=-\alpha \lambda_{\mathscr{K}}^{i}+\alpha \epsilon+(1-\alpha) \lambda_{d}^{i}, \tag{23}
\end{equation*}
$$

where $\alpha$ takes value arbitrarily in $[0,1]$. Thus, (14) holds if and only if $\forall j \in S_{\mathscr{K}}^{i}, i \neq j$,

$$
\begin{align*}
& A_{i}^{T}\left(P_{i} E+R^{T} \Phi_{i} S^{T}\right)+\left(P_{i} E+R^{T} \Phi_{i} S^{T}\right)^{T} A_{i}  \tag{24}\\
& \quad+E^{T} \mathscr{P}_{\mathscr{K}}^{i} E+E^{T}\left(-\lambda_{\mathscr{K}}^{i} P_{i}+\epsilon\left(P_{i}-P_{j}\right)\right) E<0, \\
& A_{i}^{T}\left(P_{i} E+R^{T} \Phi_{i} S^{T}\right)+\left(P_{i} E+R^{T} \Phi_{i} S^{T}\right)^{T} A_{i}  \tag{25}\\
& \quad+E^{T} \mathscr{P}_{\mathscr{K}}^{i} E+E^{T}\left(\lambda_{d}^{i} P_{i}-\lambda_{d}^{i} P_{j}-\lambda_{\mathscr{K}}^{i} P_{j}\right) E<0
\end{align*}
$$

simultaneously hold. Since $\epsilon$ is arbitrarily small, (24) holds if and only if

$$
\begin{align*}
& A_{i}^{T}\left(P_{i} E+R^{T} \Phi_{i} S^{T}\right)+\left(P_{i} E+R^{T} \Phi_{i} S^{T}\right)^{T} A_{i}  \tag{26}\\
& \quad+E^{T} \mathscr{P}_{\mathscr{K}}^{i} E-\lambda_{\mathscr{K}}^{i} E^{T} P_{i} E<0,
\end{align*}
$$

which is the case in (25) when $j=i, \forall j \in S_{\mathscr{K}}^{i}$. Hence (20) is equivalent to (15).

Therefore, we can conclude that the unforced system (1) with unknown elements in the TRM is stochastically admissible if and only if (14) and (15) hold for $i \in S_{\mathscr{K}}^{i}$ and $i \in S_{\chi \mathscr{K}}^{i}$, respectively.

Remark 5. Theorem 4 presents a new necessary and sufficient condition of stochastic admissibility criterion for the MJSS (1). The approach adopted in Theorem 4, which uses the TRM property (the sum of each row is zero), has extended the result of Theorem 1 in [12] to the MJSS. Note that the lower bound, $\lambda_{d}^{i}$, of $\lambda_{i i}$ is allowed to be arbitrarily negative.

Now let us consider the stabilization problem of system (1) in the presence of unknown elements in the TRM. The following theorem presents a condition for the existence of a mode-dependent stabilizing controller of the form in (11).

Theorem 6. Let $\varepsilon_{i}$ be given scalars. Consider the closed-loop system (12) with partially unknown transition probabilities. If there exist matrices $\bar{P}_{i} \in \mathbb{R}^{n \times n}>0$ and nonsingular matrices $\bar{\Phi}_{i} \in \mathbb{R}^{(n-r) \times(n-r)}$, matrices $L_{i} \in \mathbb{R}^{n \times m}$ and $H_{i} \in \mathbb{R}^{m \times(n-r)}$ such that, for each $i \in S$, the following LMIs hold:

$$
\left[\begin{array}{ccc}
A_{i} Y_{i}+Y_{i}^{T} A_{i}^{T}+W_{i}+\lambda_{i i}\left(\varepsilon_{i} E Y_{i}+\varepsilon_{i} Y_{i}^{T} E^{T}-\varepsilon_{i}^{2} E \bar{P}_{i} E^{T}\right) & Y_{i}^{T} F_{i}^{T}(E) & \sqrt{-\lambda_{\mathscr{K}}^{i}} Y_{i}^{T} E_{R}  \tag{27}\\
* & -X_{i}(\bar{P}) & 0 \\
* & * & -E_{R}^{T} \bar{P}_{j} E_{R}
\end{array}\right]<0
$$

$$
\left[\begin{array}{ccc}
A_{i} Y_{i}+Y_{i}^{T} A_{i}^{T}+W_{i}+\lambda_{d}^{i}\left(\varepsilon_{i} E Y_{i}+\varepsilon_{i} Y_{i}^{T} E^{T}-\varepsilon_{i}^{2} E \overline{P_{i}} E^{T}\right) & Y_{i}^{T} F_{i}^{T}(E) & \sqrt{-\lambda_{d}^{i}-\lambda_{\mathscr{H}}^{i}} Y_{i}^{T} E_{R}  \tag{28}\\
* & -X_{i}(\bar{P}) & 0 \\
* & * & -E_{R}^{T} \bar{P}_{j} E_{R}
\end{array}\right]<0,
$$

where

$$
\begin{align*}
Y_{i} & =\bar{P}_{i} E^{T}+R \bar{\Phi}_{i} S \\
W_{i} & =B_{i}\left(L_{i} E^{T}+H_{i} R\right)+\left(L_{i} E^{T}+H_{i} R\right)^{T} B_{i}^{T}  \tag{29}\\
F_{i}(E) & =\left[\sqrt{\lambda_{i \mathscr{K}_{1}}} E_{R}, \ldots, \sqrt{\lambda_{i \mathscr{K}_{m}^{i}}} E_{R}\right]^{T}, \quad \mathscr{K}_{m}^{i} \neq i \\
X_{i}(\bar{P}) & =\operatorname{diag}\left\{E_{R}^{T} \bar{P}_{\mathscr{K}_{1}} E_{R}, \ldots, E_{R}^{T} \bar{P}_{\mathscr{K}_{m}^{i}} E_{R}\right\}, \quad \mathscr{K}_{m}^{i} \neq i .
\end{align*}
$$

Then there exists a mode-dependent stabilizing controller of the form in (11) such that the closed-loop system is
stochastically admissible. The gain of the stabilizing state feedback controller is given by

$$
\begin{equation*}
K_{i}=\left(L_{i} E^{T}+H_{i} R\right)\left(\bar{P}_{i} E^{T}+R \bar{\Phi}_{i} S\right)^{-1} \tag{30}
\end{equation*}
$$

Proof. Consider the closed-loop system (12) and replace $A_{i}$ by $A_{i}+B_{i} K_{i}$ in (14) and (15), respectively. Then, if $i \in S_{\mathscr{K}}^{i}$, by Schur complement and performing a congruence transformation to (14) by $\left[\begin{array}{cc}Y_{i}^{T} & 0 \\ 0 & I\end{array}\right]$, with $Y_{i}=\left(P_{i} E+S^{T} \Phi_{i} R^{T}\right)^{-1}=$ $\bar{P}_{i} E^{T}+R \bar{\Phi}_{i} S$, we can obtain

$$
\left[\begin{array}{ccc}
A_{i} Y_{i}+Y_{i}^{T} A_{i}^{T}+B_{i} K_{i} Y_{i}+Y_{i}^{T} K_{i}^{T} B_{i}^{T}+\lambda_{i i} Y_{i}^{T} E_{R}\left(E_{R}^{T} \bar{P}_{i} E_{R}\right)^{-1} E_{R}^{T} Y_{i} & Y_{i}^{T} F_{i}^{T}(E) & \sqrt{-\lambda_{\mathscr{}}^{i}} Y_{i}^{T} E_{R}  \tag{31}\\
* & -X_{i}(\bar{P}) & 0 \\
* & * & -E_{R}^{T} \bar{P}_{j} E_{R}
\end{array}\right]<0
$$

Let $L_{i}=K_{i} \bar{P}_{i}$ and $H_{i}=K_{i} S \bar{\Phi}_{i}$; we have

$$
K_{i}=\left(L_{i} E^{T}+H_{i} R\right) Y_{i}^{-1}=\left(L_{i} E^{T}+H_{i} R\right)\left(\bar{P}_{i} E^{T}+R \bar{\Phi}_{i} S\right)^{-1}
$$

$$
\begin{align*}
B_{i} K_{i} Y_{i}+Y_{i}^{T} K_{i}^{T} B_{i}^{T}= & B_{i}\left(L_{i} E^{T}+H_{i} R\right)  \tag{32}\\
& +\left(L_{i} E^{T}+H_{i} R\right)^{T} B_{i}^{T}=W_{i}
\end{align*}
$$

So (31) becomes

$$
\left[\begin{array}{ccc}
A_{i} Y_{i}+Y_{i}^{T} A_{i}^{T}+W_{i}+\lambda_{i i} Y_{i}^{T} E_{R}\left(E_{R}^{T} \bar{P}_{i} E_{R}\right)^{-1} E_{R}^{T} Y_{i} & Y_{i}^{T} F_{i}^{T}(E) & \sqrt{-\lambda_{\mathscr{K}}^{i}} Y_{i}^{T} E_{R}  \tag{33}\\
* & -X_{i}(\bar{P}) & 0 \\
* & * & -E_{R}^{T} \bar{P}_{j} E_{R}
\end{array}\right]<0
$$

Considering the nonlinear term in the above inequalities, the following inequalities are introduced. For any scalars $\varepsilon_{i}$, $i \in S$, by Lemma 2, the following inequalities hold:

$$
\begin{align*}
0 \leq & {\left[Y_{i}^{T} E_{R}-\varepsilon_{i} E_{L}\left(E_{R}^{T} \bar{P}_{i} E_{R}\right)\right]\left(E_{R}^{T} \bar{P}_{i} E_{R}\right)^{-1} } \\
& \times\left[Y_{i}^{T} E_{R}-\varepsilon_{i} E_{L}\left(E_{R}^{T} \bar{P}_{i} E_{R}\right)\right]^{T} \\
= & Y_{i}^{T} E_{R}\left(E_{R}^{T} \bar{P}_{i} E_{R}\right)^{-1} E_{R}^{T} Y_{i}-\varepsilon_{i} E Y_{i}-\varepsilon_{i} Y_{i}^{T} E^{T}+\varepsilon_{i}^{2} E \bar{P}_{i} E^{T} \tag{34}
\end{align*}
$$

Note that $\lambda_{i i} \leq 0$; we have

$$
\begin{align*}
& \lambda_{i i} Y_{i}^{T} E_{R}\left(E_{R}^{T} \bar{P}_{i} E_{R}\right)^{-1} E_{R}^{T} Y_{i} \\
& \quad \leq \lambda_{i i}\left(\varepsilon_{i} E Y_{i}+\varepsilon_{i} Y_{i}^{T} E^{T}-\varepsilon_{i}^{2} E \bar{P}_{i} E^{T}\right) \tag{35}
\end{align*}
$$

So (33) holds if (27) is fulfilled. In a similar way, if $i \in S_{\mathscr{K}}^{i}$, (28) can be worked out from (15). Therefore, the closed-loop system is stochastically admissible, and the desired controller gain is given by (30).

Remark 7. It should be pointed out that if the diagonal elements in the TRM contain unknown ones, the system

Table 1

| Mode | 1 | 2 | 3 | 4 |
| :--- | :---: | :---: | :---: | :---: |
| 1 | -1.2 | $\hat{\lambda}_{12}$ | $\hat{\lambda}_{13}$ | 0.6 |
| 2 | 0.3 | -0.8 | 0.1 | 0.4 |
| 3 | $\hat{\lambda}_{31}$ | $\hat{\lambda}_{32}$ | -0.6 | 0.3 |
| 4 | $\widehat{\lambda}_{41}$ | $\widehat{\lambda}_{42}$ | $\hat{\lambda}_{43}$ | -0.9 |

admissibility, the existence of the admissible controller, and the controller gains solution will be dependent on $\lambda_{d}^{i}$. The conditions of Theorem 6 are strict LMIs; hence they can be easily tractable by Matlab LMI toolbox.

## 4. Examples

Example 1. Consider system (1) with four operation modes and the following system matrices:

$$
\begin{gather*}
E=\left[\begin{array}{ccc}
4 & 0 & 0 \\
0 & 0.8 & 0 \\
0 & 0 & 0
\end{array}\right], \quad E_{L}=\left[\begin{array}{cc}
2 & 0 \\
0 & 0.4 \\
0 & 0
\end{array}\right], \\
E_{R}=\left[\begin{array}{ll}
2 & 0 \\
0 & 2 \\
0 & 0
\end{array}\right], \quad R=\left[\begin{array}{lll}
0 & 0 & 2
\end{array}\right], \quad S=\left[\begin{array}{l}
0 \\
0 \\
1
\end{array}\right], \\
A_{1}=\left[\begin{array}{ccc}
2 & -7 & 1 \\
-5 & -2 & -1 \\
2 & 4 & -5
\end{array}\right], \quad A_{2}=\left[\begin{array}{lll}
5 & 3 & 7 \\
7 & 9 & 3 \\
2 & 4 & 5
\end{array}\right],  \tag{36}\\
A_{3}=\left[\begin{array}{ccc}
2 & -5 & 4 \\
-1 & -3 & 3 \\
4 & -6 & 8
\end{array}\right], \quad A_{4}=\left[\begin{array}{lll}
1 & 4 & 3 \\
2 & 4 & 1 \\
6 & 1 & 4
\end{array}\right], \\
B_{1}=\left[\begin{array}{cc}
0 & 6 \\
-7 & 9 \\
1 & 0
\end{array}\right], \quad B_{2}=\left[\begin{array}{ll}
5 & 2 \\
0 & 5 \\
6 & 0
\end{array}\right], \\
B_{3}=\left[\begin{array}{ll}
3 & 5 \\
0 & 4 \\
2 & 0
\end{array}\right], \quad B_{4}=\left[\begin{array}{ll}
0 & 4 \\
7 & 6 \\
3 & 0
\end{array}\right] .
\end{gather*}
$$

The transition rate matrix is given as shown in Table 1.
Let $\varepsilon_{1}=1.2, \varepsilon_{2}=-1, \varepsilon_{3}=-0.2, \varepsilon_{4}=2$, and $\widehat{\lambda}_{i j}$ denote the unknown elements. Using Theorem 6 and the LMI control toolbox of Matlab, we obtain the controller gains for the system as follows:

$$
\begin{aligned}
& K_{1}=\left[\begin{array}{lll}
3.7123 & 3.7708 & 0.0005 \\
2.1986 & 2.2325 & 0.0006
\end{array}\right] \times 10^{4}, \\
& K_{2}=\left[\begin{array}{ccc}
-0.7952 & -3.3671 & -0.0001 \\
1.1211 & 4.7407 & 0.0002
\end{array}\right] \times 10^{4}, \\
& K_{3}=\left[\begin{array}{ccc}
2.5210 & 1.2413 & -0.0000 \\
0.5945 & 0.2927 & -0.0000
\end{array}\right] \times 10^{5}, \\
& K_{4}=\left[\begin{array}{ccc}
5.1907 & -7.2130 & -0.0013 \\
1.7600 & -2.4473 & 0.0008
\end{array}\right] \times 10^{3} .
\end{aligned}
$$

Table 2

| Mode | 1 | 2 | 3 |
| :--- | :---: | :---: | :---: |
| 1 | -1.2 | $\hat{\lambda}_{12}$ | $\hat{\lambda}_{13}$ |
| 2 | $\hat{\lambda}_{21}$ | $\hat{\lambda}_{22}$ | 0.4 |
| 3 | 0.3 | 0.5 | -0.8 |

The closed-loop dynamic responses and the Markovian chain are shown in Figure 1 with the initial condition $x(0)=$ $[0.7,0.5,-2.3]^{T}$.

Example 2. Consider system (1) with three operation modes and the following system matrices:

$$
\begin{gather*}
E=\left[\begin{array}{ll}
2 & 0 \\
0 & 0
\end{array}\right], \quad E_{L}=\left[\begin{array}{l}
2 \\
0
\end{array}\right], \quad E_{R}=\left[\begin{array}{l}
1 \\
0
\end{array}\right], \\
R=\left[\begin{array}{ll}
0 & 1
\end{array}\right], \quad S=\left[\begin{array}{l}
0 \\
2
\end{array}\right], \quad A_{1}=\left[\begin{array}{cc}
1.5 & -1.4 \\
0.1 & 0.2
\end{array}\right], \\
A_{2}=\left[\begin{array}{cc}
-0.5 & -0.3 \\
1 & -1.2
\end{array}\right], \quad A_{3}=\left[\begin{array}{cc}
-0.1 & 0.2 \\
1 & 1
\end{array}\right],  \tag{38}\\
B_{1}=\left[\begin{array}{l}
2 \\
0
\end{array}\right], \quad B_{2}=\left[\begin{array}{l}
-1 \\
-3
\end{array}\right], \quad B_{3}=\left[\begin{array}{c}
3 \\
-2
\end{array}\right] .
\end{gather*}
$$

The transition rate matrix is given as shown in Table 2.
Let $\varepsilon_{1}=1.2, \varepsilon_{2}=-1, \varepsilon_{3}=-0.2, \lambda_{d}^{2}=-1$. In the 2nd row of TRM, the diagonal element $\hat{\lambda}_{22}$ is unknown; we assign its lower bound $\lambda_{d}^{2}$ a priori with different values ( $\lambda_{d}^{2} \epsilon$ $(-\infty,-0.4])$. Using Theorem 6 and LMI control toolbox in Matlab, the controller gains for the system are given by

$$
\begin{align*}
& K_{1}=\left[\begin{array}{ll}
-7.6834 & 0.0014
\end{array}\right] \times 10^{5}, \\
& K_{2}=\left[\begin{array}{ll}
-114.1162 & -0.4001
\end{array}\right],  \tag{39}\\
& K_{3}=\left[\begin{array}{ll}
529.6195 & 0.5013
\end{array}\right] .
\end{align*}
$$

When $\lambda_{d}^{2}=-2$, we obtain the controller gains differently for the system as follows:

$$
\begin{align*}
& K_{1}=\left[\begin{array}{ll}
-2.9825 & 0.0003
\end{array}\right] \times 10^{6}, \\
& K_{2}=\left[\begin{array}{ll}
504.0862 & -0.4000
\end{array}\right],  \tag{40}\\
& K_{3}=\left[\begin{array}{ll}
3.0048 & 0.0005
\end{array}\right] \times 10^{3} .
\end{align*}
$$

It is seen from above that the obtained controller gains are dependent on $\lambda_{d}^{2}$. The closed-loop dynamic responses and the Markovian chain are shown in Figure 2 with the initial condition $x(0)=[0.7,2.89]^{T}$ and $\lambda_{d}^{2}=-1$.

Remark 8. Notice that, in Example 1, all the diagonal elements of TRM are known and, in Example 2, there are unknown diagonal elements in the TRM which illustrate that the controller design is dependent on the lower bound $\lambda_{d}^{i}$ of the corresponding unknown diagonal element. So they cannot be solved by the stabilization criterions developed in [15] which lack considering the case of systems with partly


Figure 1: System states and Markovian chain.


Figure 2: System states and Markovian chain.
known transition probabilities. Moreover, here examples are for MJSS, while the stabilization criterions developed in [12] which focused on those of normal ones that are special cases of MJSS.

## 5. Conclusion

The problems of stability and state feedback control for continuous-time MJSS with partly known transition probabilities have been studied. A new sufficient and necessary condition for this class of system to be stochastically admissible has been proposed in terms of strict LMIs. Furthermore, sufficient conditions for the state feedback controller are derived, and numerical examples have also been given to illustrate the main results. However, the study of stability and stabilization of continuous-time MJSS with partly known transition probabilities is a basic problem which only serves as a stepping stone to investigate more complicated systems. However, time-delay appears commonly in various practical systems, and researchers have been paying remarkable attention to the problems of analysis and synthesis for timedelay systems [18-24]. The approaches proposed in this paper could be further extended to time-delay systems in our future work. It is expected that the approach can be further used for other analysis and synthesis issues such as $H_{\infty}$ analysis, $H_{\infty}$
synthesis, and other applications such as Markov jumping neural networks with incomplete transition descriptions.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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# Modeling and Optimal Control of a Class of Warfare Hybrid Dynamic Systems Based on Lanchester $(n, 1)$ Attrition Model 

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#### Abstract

For the particularity of warfare hybrid dynamic process, a class of warfare hybrid dynamic systems is established based on Lanchester equation in a $(n, 1)$ battle, where a heterogeneous force of $n$ different troop types faces a homogeneous force. This model can be characterized by the interaction of continuous-time models (governed by Lanchester equation), and discrete event systems (described by variable tactics). Furthermore, an expository discussion is presented on an optimal variable tactics control problem for warfare hybrid dynamic system. The optimal control strategies are designed based on dynamic programming and differential game theory. As an example of the consequences of this optimal control problem, we take the $(2,1)$ case and solve the optimal strategies in a $(2,1)$ case. Simulation results show the feasibility of warfare hybrid system model and the effectiveness of the optimal control strategies designed.


## 1. Introduction

In 1914, Lanchester [1] first proposed a mathematical model to describe and forecast quantitatively the development trend of battle. Since then, Lanchester equation has been used to analyze real wars [2-5] and determine tactics for deploying forces in war game simulations, as they produce reasonably good predictions. The popularity and wide acceptance of the Lanchester models are due to their amenability to simple analysis and the fact that they, by and large, reflect the actual conflict situation. Undoubtedly, even in the modern hightech war, Lanchester equation can still make a comprehensive assessment and decision-making for a variety of battlefield factors.

To establish the mathematical model describing warfare process is the basis for researching quantitatively decisionmaking problems in conflicts. So far, warfare system models based on Lanchester equation have penetrated into many fields of military problems; the research core mainly focuses on extending and modifying Lanchester equation. For example, in [6], Sha introduced morale parameter based on
the conventional Lanchester equation of casualty rate to set up mathematic models. In [7, 8], by introducing battlefield sensing coefficient and information superiority coefficient to modify the casualty rates of Lanchester equation, the generalized model for information warfare was proposed. In [9], considering the relationship between electronic jamming and operational efficiency, an expanded Lanchester square law model with variable efficiency factors was established. In [10, 11], a spatial modeling of Lanchester equations was conceptualized on the basis of explicit movement dynamics and balance of forces, ensuring stability and theoretical consistency with the original model. In [12, 13], some extensions of the Lanchester square law to inhomogeneous forces with an application to force allocation methodology were studied. However, all the models above have some limitations in war game simulations and tactical decision-making application, especially without regard to the interaction between the discrete event-driven tactics and the continuous force changes and without revealing better the complex operation mechanism, which is more close to the actual warfare dynamic process.

In fact, warfare dynamic process is a hybrid dynamic process, which is characterized by the interaction of continuous time dynamic process (described, e.g., fighting strength changes) and discrete event dynamic process (described, e.g., fighting strength scheduling, variable tactics). Thus, how to establish warfare hybrid dynamic system model, which integrates the discrete events with continuous time, has become a problem to be solved urgently. In [14, 15], Xu first proposed the modeling idea of warfare hybrid dynamic system; the basic frame and method to solve this kind of problems are investigated and a series of key technologies are emerged. Thus, in this paper, we first study the modeling of warfare hybrid dynamic system based on Lanchester equation. Introducing the discrete event variable, which can trigger the occurrence of variable tactics, the terms concerned with operational losses within Lanchester equation are modified, and a new warfare hybrid dynamic system is established.

Optimal control problem of warfare dynamic system has been an area of considerable research interest and has been an absolutely necessary tache on using Lanchester equation to research the tactic decision-making problem. So far, a wide variety of research achievements on this problem have been obtained, such as. In [16], Taylor considered a class of optimal resource allocation problems as a time sequential resource allocation problem and presented a solution in the optimal control framework based on Lanchester equation. In [17, 18], Taylor and Issacc used differential game theory to study the tactic decision-making problem, respectively. In [19, 20], Sha and Zeng and Li et al. solved the firepower assignment problem based on Lanchester equation and differential game theory, and they also validated, from another aspect, the principle of concentrating superior firepower in attack. In [21-25], optimal strategies of force resource complementary were obtained based on optimal control and differential game. However, the recent study results are confined to determine the optimal control problem of warfare continuous dynamic process. And there are not the effective theories and methods to solve the optimal tactic control problem of discrete event dynamic process. Thus, in this paper, we study the problem of building the optimal variable tactics control of warfare hybrid dynamic system based on Lanchester $(n, 1)$ model $[26,27]$. By using dynamic programming and differential game theory, we present the approach to obtain the optimal control strategies.

The paper is organized as follows. In Section 2, warfare hybrid dynamic process description and modeling are studied. In Section 3, the optimal control strategies of warfare hybrid dynamic system based on Lanchester $(n, 1)$ model are given, and the simulation results demonstrate the effectiveness of proposed optimal control schemes. And finally some concluding remarks are given in Section 4.

## 2. Warfare Hybrid Dynamic Process Description and Modeling

In order to directly understand the basic frame of warfare hybrid dynamic system, the evolution analysis of warfare process is given in Figure 1.

Firstly, let $X$ denote the attacking force and let $Y$ denote the defending force. It is assumed that the attacking force consists of one type of forces and the defending force consists of $n$ type of forces. Then we can make out that, from Figure 1, the composition of warfare hybrid dynamic process can be considered as the following key elements of two; they are the event-driven tactics and the continuous force strength changes. The variable tactics evolution process can be described as follows:
(1) firstly, $Y$ will concentrate all superior firepower to attack $X$ throughout the period of battle. However, at the time $t_{0}, X$ decides the initial encounter according to each original situation;
(2) at the time $t_{1}$, after surveys, $X$ detects the opponent's targets and motives and responds with variable tactics, and a new belligerent encounter accordingly is established;
(3) at the time $t_{2}$, after fresh surveys, $X$ responds with variable tactics for good and consequently establishes a new belligerent encounter;
(4) the above-mentioned process continues.

There is no difficulty in deducing the conclusion that variable tactics of the decision-maker $X$ happen at the discrete moment, which can lead to the changes of the belligerent encounter and structural changes of the system, shown in features of discrete event dynamic system.

The warfare process evolution also involves the continuous control of force strengths on both combat units. Based on decision-maker's instructions and detected situation, each combat unit adjusts the control variables with the purpose of changing the force strengths; however, the warfare task is certain herein. It falls into a category of continuous control process with the systematic structure unchanged. Therefore, warfare dynamic process can be considered as warfare hybrid dynamic system, which is that force strengths change on both sides from one continuous system to another via certain variable tactics (those that change combat encounter), and every variable tactic happens; the whole system operates following the later's rules.

Inspired by [21], we can give some reasonable assumptions as follows.

Assumption 1. $X$ and $Y$ have, respectively, one and $n$ combat units, $x_{1}(t)$ and $y_{j}(t)(j=1, \ldots, n)$ are the strengths of one and $j$ th combat unit on both sides surviving at time $t$, and the original states are $x_{1}(0)=x_{10}$ and $y_{j}(0)=y_{j 0}$.

Assumption 2. Suppose that variable tactics happen at time $t_{k}(1 \leq k \leq \infty)$, where $t_{k} \in\left[t_{0}, t_{f}\right], t_{0} \leq \cdots \leq \cdots \leq t_{f}$, and $t_{0}$ is the initial time. $e_{s}(s=1, \ldots, k, \ldots)$ is the discrete event variable happening at time $t_{s}$; then the route of variable tactics can be described as

$$
\begin{equation*}
r=\left(\left(t_{1}, e_{1}\right), \ldots,\left(t_{s}, e_{s}\right)\right) \tag{1}
\end{equation*}
$$

where $\left(t_{s}, e_{s}\right)$ means the occurrence of variable tactics at time $t_{s}$.


FIGURE 1: Evolution analysis of warfare hybrid dynamic process in a $(n, 1)$ battle.

Motivated by the above discussions, a class of warfare hybrid dynamic systems based on Lanchester equation can be established as follows:

$$
\begin{gather*}
\dot{x}_{1}(t)=-\sum_{j=1}^{n} \alpha_{j 1} \psi_{j 1_{e_{s}}} y_{j}(t)+u_{1}(t)  \tag{2}\\
\dot{y}_{j}(t)=-\beta_{1 j} \phi_{1 j_{e_{s}}} x_{1}(t)+v_{j}(t)
\end{gather*}
$$

where $\beta_{1 j}$ is the nonnegative attrition coefficient of $x_{1}$ to $y_{j}$ and $\alpha_{j 1}$ is the nonnegative attrition coefficient of $y_{j}$ to $x_{1}$; $u_{1}(t)$ and $v_{j}(t)$ are corresponding control input; $\psi_{j 1_{e_{s}}} \in\{0,1\}$ and $\phi_{1 j_{e_{s}}} \in\{0,1\}$ are corresponding encounter tactics; then $\Psi_{e_{s}}$ and $\Phi_{e_{s}}$ are variable tactic matrices driven by discrete events:

$$
\Psi_{e_{s}}=\left[\begin{array}{llll}
\psi_{11_{e_{s}}} & \psi_{21_{e_{s}}} & \cdots & \psi_{n 1_{e_{s}}}
\end{array}\right] ; \quad \Phi_{e_{s}}=\left[\begin{array}{c}
\phi_{11_{e_{s}}}  \tag{3}\\
\phi_{12_{e_{s}}} \\
\vdots \\
\phi_{1 n_{e_{s}}}
\end{array}\right]
$$

From Assumption 2, it is known that the variable tactics driven by discrete events will satisfy

$$
\begin{gather*}
\Psi_{e_{s-1}} \times e_{s} \longrightarrow \Psi_{e_{s}}  \tag{4}\\
\Phi_{e_{s-1}} \times e_{s} \longrightarrow \Phi_{e_{s}}
\end{gather*}
$$

Remark 3. The values of switch variable $\psi_{j 1_{e_{s}}} \in\{0,1\}$ and $\phi_{1 j_{e_{s}}} \in\{0,1\}$ are given according to the encounter relation between two combat units on both sides:

$$
\begin{align*}
\psi_{j 1_{e_{s}}} & = \begin{cases}0, & \text { No encounter between } y_{j} \text { and } x_{1} \\
1, & y_{j} \text { encounters } x_{1} \text { with all forces }\end{cases}  \tag{5}\\
\phi_{i j_{e_{s}}} & = \begin{cases}0, & \text { No encounter between } y_{j} \text { and } x_{1} \\
1, & x_{1} \text { encounters } y_{j} \text { with all forces. }\end{cases}
\end{align*}
$$

Remark 4. $\Psi_{e_{s}}$ and $\Phi_{e_{s}}$ are driven to change by discrete event $e_{s}$, which therefore affects and changes system (2). If (4) is tenable, then continuous subsystems of $\Psi_{e_{s-1}}$ and $\Phi_{e_{s-1}}$ are changed to that of $\Psi_{e_{s}}$ and $\Phi_{e_{s}}$, which tell that variable tactics decide the number of continuous subsystems.

From Remarks 3 and 4, this model has a better description of the interaction of continuous-time models (governed by Lanchester equations) and of logic rules and discrete event systems (described, e.g., by variable tactics). And it is known that the discrete part makes the decision for the whole system to switch to another set of control rules if conditions are favorable, and the continuous part as a result works according to the new rules.

## 3. Optimal Control of Warfare Hybrid System via Lanchester ( $n, 1$ ) Model

3.1. Problem Statement. In this section, the optimal variable tactics control problem of warfare hybrid dynamic system in a $(n, 1)$ battle, in which a heterogeneous force of $n$ different troop types faces a homogeneous force, is investigated. With what is mentioned above, we present some assumptions which will be used.

Assumption 5. The values of switch variables $\psi_{j 1_{e_{s}}}$ and $\phi_{1 j_{e_{s}}}$ satisfy the following conditions:

$$
\begin{equation*}
\psi_{j 1_{e_{s}}}=1 ; \quad \sum_{j=1}^{n} \phi_{i j_{e_{s}}}=1 \tag{6}
\end{equation*}
$$

Meanwhile, we suppose that the most effective battle stage is $[0, T]$, where $T$ is the end time of battle, $x_{1}(T)$ and $y_{j}(T)$ are the residual of strengths on both sides in the terminal time $T$, $x_{1}(T) \neq 0$, and $y_{j}(T) \neq 0$.

From Assumption 5, the system model is rewritten to be

$$
\begin{gather*}
\dot{x}_{1}(t)=-\sum_{j=1}^{n} \alpha_{j 1} y_{j}(t)+u_{1}(t)  \tag{7}\\
\dot{y}_{j}(t)=-\beta_{1 j} \phi_{1 j_{e_{s}}} x_{1}(t)+v_{j}(t) .
\end{gather*}
$$

The objective function associated with system (6) is of the following form:

$$
\begin{equation*}
J=\eta_{1} x_{1}(T)-\sum_{j=1}^{n} \theta_{j} y_{j}(T), \tag{8}
\end{equation*}
$$

where $\eta_{1}$ and $\theta_{j}$ are the relative operation indices, which is the weight of the importance of the corresponding units on both sides. Then, $\eta_{1} x_{1}(T)$ and $\sum_{j=1}^{n} \theta_{j} y_{j}(T)$ are the residual of actual strengths on both sides in the terminal time $T$.

Now, the optimal variable tactics control problem can be described as follows. The attacking side $X$ selects the number of tactics changes, the time of every variable tactics, and the sequences of corresponding variable tactics $\Phi_{e_{s}}^{*}(s=1, \ldots, k)$ to maximize the objective function $J$.
3.2. Solving Method for the Optimal Control Strategies. In this subsection, we analyze the conditions of the optimal variable tactics and give a quantitative analysis of the variable tactics process. Finally, a solving method for the optimal control strategies is designed.

For the above optimal control problem, we introduce the adjoint function as follows:

$$
\left[\begin{array}{l}
\lambda  \tag{9}\\
\mu
\end{array}\right]=\left(\lambda_{1}, \mu_{1}, \ldots, \mu_{n}\right)^{T}
$$

and construct the Hamilton function to be

$$
\begin{align*}
& H(x, y, \lambda, \mu, \Phi, t) \\
& \quad=-\sum_{j=1}^{n}\left(\lambda_{1} \alpha_{j 1}\right) y_{j}-\sum_{j=1}^{n} \mu_{j} \beta_{1 j} \phi_{1 j e_{s}} x_{1}  \tag{10}\\
& \quad+\lambda_{1} u_{1}(t)+\sum_{j=1}^{n} \mu_{j} v_{j}(t) .
\end{align*}
$$

Then, by using the Minimax principle of differential game, the necessary conditions about the optimal tactics are that there exist the corresponding adjoint functions $\lambda^{*}(t)$ and $u^{*}(t)$, which satisfy

$$
\begin{gather*}
\dot{\lambda}_{1}(t)=-\frac{\partial H}{\partial x_{1}}=\sum_{j=1}^{n} \mu_{j} \beta_{i j} \phi_{i j_{e_{s}}} \\
\lambda_{1}(T)=\frac{\partial J}{\partial x_{1}}=\eta_{1} \\
\dot{\mu}_{j}(t)=-\frac{\partial H}{\partial y_{j}}=\lambda_{1} \alpha_{j 1}  \tag{11}\\
\mu_{j}(T)=\frac{\partial J}{\partial y_{j}}=-\theta_{j} .
\end{gather*}
$$

From (11), we have

$$
\begin{array}{ll}
\dot{\lambda}_{1}(t)<0, & \dot{\mu}_{j}(t)>0  \tag{12}\\
\lambda_{1}(T)>0, & \mu_{j}(T)<0 .
\end{array}
$$

Therefore, for any time $t \in[0, T]$, it is easy to get

$$
\begin{equation*}
\lambda_{1}(t)>0, \quad \mu_{j}(t)<0 . \tag{13}
\end{equation*}
$$

Then, we can obtain that

$$
\begin{align*}
& H\left(x^{*}, y^{*}, \lambda^{*}, \mu^{*}, \Phi_{e_{s}}^{*}, t\right) \\
& \quad=\max _{\Phi_{e_{s}}} H\left(x^{*}, y^{*}, \lambda^{*}, \mu^{*}, \Phi_{e_{s}}, t\right)  \tag{14}\\
& \quad=\max _{\Phi_{e_{s}}}\left(\sum_{j=1}^{n}-\mu_{j}^{*} \beta_{1 j} \phi_{1 j_{e_{s}}} x_{1}^{*}\right) .
\end{align*}
$$

From (14) and $x_{1}^{*}>0$, we know that $\sum_{j=1}^{n}\left(-\mu_{j}^{*}\right) \beta_{i j} \phi_{i j_{e_{s}}}$ is the weighted average of $-\mu_{j}^{*} \beta_{i j}$; then the optimal tactics satisfy

$$
\phi_{i j_{e_{s}}}^{*}=\left\{\begin{array}{lc}
1, & \max _{j}\left(-\mu_{j}^{*} \beta_{1 j}\right)=-\mu_{j^{*}}^{*} \beta_{1 j^{*}}  \tag{15}\\
0, & \max _{j}\left(-\mu_{j}^{*} \beta_{1 j}\right) \neq-\mu_{j^{*}}^{*} \beta_{1 j^{*}}
\end{array}\right.
$$

Remark 6. Since $\mu_{j}(T)$ is a continuous function, $\mu_{j}^{*}(T) \beta_{1 j}$ is also continuous; thus, the optimal tactic strategy $\phi_{i j_{e_{s}}}^{*}$ remains stable on a period of time; that is, $\phi_{i j_{s}}^{*}$ remains stable at the time interval $[T-\delta, T]$.

Based on the above analysis, we discuss the variable tactics process about the attacking side $X$. And the existence conditions of variable tactics are investigated in the following theorem.

Theorem 7. If there exist at least two functions $\mu_{j_{l}}(t) \beta_{1 j_{l}}$ and $\mu_{j_{g}}(t) \beta_{1 j_{g}}\left(l, g \in\{1, \ldots, n\}\right.$ and $\left.j_{l} \neq j_{g}\right)$ at the time interval $\left[\Delta_{k}, T\right]$, such that

$$
\begin{align*}
&-\mu_{j_{l}}(t) \beta_{1 j_{l}}<-\mu_{j_{g}}(t) \beta_{1 j_{g}}  \tag{16}\\
&-\mu_{j_{l}}\left(\Delta_{k}\right) \beta_{1 j_{l}}=-\mu_{j_{g}}\left(\Delta_{k}\right) \beta_{1 j_{g}},  \tag{17}\\
&-\dot{\mu}_{j_{l}}\left(\Delta_{k}\right) \beta_{1 j_{l}}<-\dot{\mu}_{j_{g}}\left(\Delta_{k}\right) \beta_{1 j_{g}}, \tag{18}
\end{align*}
$$

hold, then one obtains that

$$
\begin{equation*}
-\mu_{j_{l}}(t) \beta_{1 j_{l}}>-\mu_{j_{g}}(t) \beta_{1 j_{g}} \tag{19}
\end{equation*}
$$

at the left neighborhood of $t=\Delta_{k}$. That is, there exists a variable tactic for $X$ at $[t, T]\left(t<\Delta_{k}\right)$, and $\Delta_{k}$ is the minimum time of tactic change.

Proof. From (15), it is not difficult to show that

$$
\phi_{i j_{s}^{*}}^{*}=\left\{\begin{array}{lc}
1, & \max _{j}\left(-\mu_{j}^{*}(T) \beta_{1 j}\right)=-\mu_{j^{*}}^{*}(T) \beta_{1 j^{*}}  \tag{20}\\
0, & \max _{j}\left(-\mu_{j}^{*}(T) \beta_{1 j}\right) \neq-\mu_{j^{*}}^{*}(T) \beta_{1 j^{*}}
\end{array}\right.
$$

So, there exist at least two functions $\mu_{j_{l}}(t) \beta_{1 j_{l}}$ and $\mu_{j_{g}}(t) \beta_{1 j_{g}}\left(j_{l} \neq j_{g}\right)$ at the time interval $\left[\Delta_{k}, T\right]$, which satisfy

$$
\begin{equation*}
-\mu_{j_{l}}(t) \beta_{1 j_{l}}<-\mu_{j_{g}}(t) \beta_{1 j_{g}} \tag{21}
\end{equation*}
$$

From (18), we have that

$$
\begin{align*}
& \lim _{\varepsilon^{-} \rightarrow 0} \frac{-\mu_{j_{l}}\left(\Delta_{k}\right) \beta_{1 j_{l}}-\left(-\mu_{j_{l}}\left(\Delta_{k}-\varepsilon\right) \beta_{1 j_{l}}\right)}{\varepsilon} \\
& \quad<\lim _{\varepsilon^{-} \rightarrow 0} \frac{-\mu_{j_{g}}\left(\Delta_{k}\right) \beta_{1 j_{g}}-\left(-\mu_{j_{g}}\left(\Delta_{k}-\varepsilon\right) \beta_{1 j_{g}}\right)}{\varepsilon} \tag{22}
\end{align*}
$$

Combining the aforementioned inequality with (17) yields

$$
\begin{equation*}
-\mu_{j_{l}}\left(\Delta_{k}-\varepsilon\right) \beta_{1 j_{l}}>-\mu_{j_{g}}\left(\Delta_{k}-\varepsilon\right) \beta_{1 j_{g}} \tag{23}
\end{equation*}
$$

Since the following inequality

$$
\begin{equation*}
-\mu_{j_{l}}\left(\Delta_{k}+\varepsilon\right) \beta_{i j_{l}}<-\mu_{j_{g}}\left(\Delta_{k}+\varepsilon\right) \beta_{i j_{g}} \tag{24}
\end{equation*}
$$

holds at the right neighborhood of $t=\Delta_{k}$, therefore, there exists a variable tactic for $X$ at the time $\Delta_{k}\left(0<\Delta_{k}<T\right)$.

Now, we will investigate that $\Delta_{k}$ is the minimum time of tactic change. Firstly, we suppose that $\Delta_{k 1}\left(\Delta_{k 1}<\Delta_{k}\right)$ is the minimum time of tactic change; then we have

$$
\begin{align*}
& -\mu_{j_{l}}\left(\Delta_{k 1}+\varepsilon\right) \beta_{1 j_{l}}<-\mu_{j_{g}}\left(\Delta_{k 1}+\varepsilon\right) \beta_{1 j_{g}} \\
& -\mu_{j_{l}}\left(\Delta_{k 1}-\varepsilon\right) \beta_{1 j_{l}}>-\mu_{j_{g}}\left(\Delta_{k 1}-\varepsilon\right) \beta_{1 j_{g}} . \tag{25}
\end{align*}
$$

From (17), (18), and (19), we easily get

$$
\begin{equation*}
-\mu_{j_{l}}(t) \beta_{1 j_{l}}<-\mu_{j_{g}}(t) \beta_{1 j_{g}}, \quad t \in\left(\Delta_{k 1}, T\right] ; \tag{26}
\end{equation*}
$$

then there exists the contradiction between (23) and (26). Thus, $\Delta_{k}$ is the minimum time of tactic change when it draws near the termination time $T$. The proof is completed.

According to Theorem 7, we give a solving method of the optimal control strategies.

Step 1. Using $\mu_{j}(T) \beta_{1 j}$, we solve the optimal tactics $\Phi_{e_{k}}$ at the time interval near the termination time $T$. Furthermore, we seek the minimum time of tactic change $\Delta_{k}$, which satisfies Theorem 7.

Step 2. Using $\mu_{j}\left(\Delta_{k}\right) \beta_{1 j}$, the optimal tactics $\Phi_{e_{k-1}}$ are obtained at the time interval near $\Delta_{k}$, and we seek the minimum time of tactic change $\Delta_{k-1}$.

Step 3. The aforementioned process continues. When the conditions of tactic change cannot hold in the time interval $\left[0, \Delta_{1}\right]$ near the initial time, the solving process stops.

Step 4. Sorting $\Delta_{1}, \ldots, \Delta_{k}$, we obtain that the discrete event variable $e=\left\{e_{1}, \ldots, e_{f}\right\}(0 \leq f<\infty)$ and the route of variable tactic $r=\left(\left(t_{1}, e_{1}\right), \ldots,\left(t_{f}, e_{f}\right)\right)$, where, $t_{f} \in$ $\left\{\Delta_{1}, \ldots, \Delta_{k}\right)$ is the time of tactic change and $e_{f}$ is a discrete event.
3.3. Application Example Analysis. As an example of the consequences of the optimal control problem, we take the $(2,1)$ case and solve the optimal strategies in a $(2,1)$ case. In this subsection, we consider the warfare dynamic system model that is described by

$$
\begin{align*}
& \dot{x}_{1}=-\sum_{j=1}^{2} \alpha_{j 1} y_{j}+u_{1} \\
& \dot{y}_{1}=-\beta_{11} \phi_{11_{e_{k}}} x_{1}+v_{1}  \tag{27}\\
& \dot{y}_{2}=-\beta_{12} \phi_{12_{e_{k}}} x_{1}+v_{2},
\end{align*}
$$

where $x_{1}(t), y_{1}(t)$, and $y_{2}(t)$ are the strengths of two opposing forces surviving at time $t$.

The objective function associated with the system (27) is of the following form:

$$
\begin{equation*}
J=\eta_{1} x_{1}-\theta_{1} y_{1}-\theta_{2} y_{2} \tag{28}
\end{equation*}
$$

The relevant parameters of system are as follows. The initial force strengths are $x_{10}=100, y_{10}=30, y_{20}=30 ; \alpha_{11}=$ 9, $\alpha_{21}=1, \beta_{11}=\beta_{12}=1$ are the nonnegative attrition coefficients; the battle terminal time is $T=0.489$; the relative operation indices are $\eta_{1}=9, \theta_{1}=1, \theta_{2}=9$, and we choose that $u_{1}=0, v_{1}=v_{2}=0$. In the proposed solving algorithm, we set the initial values $t_{0}=T, \lambda_{1}(T)=9, \mu_{1}(T)=-1$, and $\mu_{2}(t)=-9$ and the step length $\kappa=0.001$; then we can know that $t=t_{0}-\chi \kappa$, where $\chi$ is the cycle number.


Figure 2: Strength change curves of each combat unit on both sides.

Solving the optimal control problem by Matlab Toolbox yields that when $\chi=105$, the variable tactics occur, and the corresponding time is $\Delta_{1}=0.384$; then it is easy to get that the optimal tactics of $X$ are

$$
\begin{align*}
& \phi_{11_{e_{k}}}(t)=1, \quad \phi_{12_{e_{k}}}(t)=0, \quad 0<t \leq \Delta_{1}, \\
& \phi_{11_{e_{k}}}(t)=0, \quad \phi_{12_{e_{k}}}(t)=1, \quad \Delta_{1}<t \leq T . \tag{29}
\end{align*}
$$

So the discrete event variable is $e_{1}=\left\{\Phi_{e_{0}}, \Phi_{e_{1}}\right\}$, where $\Phi_{e_{0}}=$ $\left[\begin{array}{l}1 \\ 0\end{array}\right] ; \Phi_{e_{1}}=\left[\begin{array}{l}0 \\ 1\end{array}\right]$. Therefore, the solution of optimal control problem in a $(2,1)$ battle can be gotten as follows. $X$ has a variable tactic in $\Delta_{1}=0.384$, and the optimal tactics $\Phi_{e_{s}}^{*}$ for $X$ satisfy

$$
\begin{array}{lll}
\phi_{11_{e_{0}}}^{*}(t)=1 ; & \phi_{12_{e_{0}}}^{*}(t)=0, & 0<t \leq \Delta_{1}, \\
\phi_{11_{e_{0}}}^{*}(t)=0 ; & \phi_{12_{e_{1}}}^{*}(t)=1, & \Delta_{1} \leq t \leq T . \tag{30}
\end{array}
$$

And the route of variable tactic $r=\left(\left(0, e_{0}\right), \ldots,\left(\Delta_{1}, e_{1}\right)\right)$, and the optimal value of $J$ is $J^{*}=44.152$.

Figure 2 shows the change of state trajectories of the units on both warring sides. It is easy to see that the state values change in $t=\Delta_{1}$; meanwhile, the state changes are nonnegative, and the change of variable tactics always holds in the time interval $[0, T]$.

## 4. Conclusions

In this paper, we established a class of warfare hybrid dynamic systems based on Lanchester equation in a battle between an attacker with one type of force and a defender with $n$ types of forces. For the attacking side, an optimal control problem of warfare hybrid dynamic system in a $(n, 1)$ battle was investigated. Then the optimum condition and the solving
method about the game problem are given. Simulation results illustrate the effectiveness of proposed optimal strategies. This is of great significance in analyzing quantitatively military actions. However, the proposed warfare hybrid dynamic model in this paper fails to consider the warfare dynamic process in a $(n, m)$ battle, in which an attacker has $m$ type of forces and a defender has $n$ types of forces. Thus, constructing a more reasonable model and employing advanced control techniques to investigate the warfare dynamic game problem in a $(n, m)$ battle are our future research directions.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Direct Self-Repairing Control for Quadrotor Helicopter Attitude Systems 

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#### Abstract

A quadrotor helicopter with uncertain actuator faults, such as loss of effectiveness and lock-in-place, is studied in this paper. An adaptive fuzzy sliding mode controller based on direct self-repairing control is designed for such nonlinear system to track the desired output signal, when any actuator of this quadrotor helicopter is loss of effectiveness or stuck at some place. Moreover, using the Lyapunov stability theory, the stability of the whole system and the convergence of the tracking error can be guaranteed. Finally, the availability of the proposed method is verified by simulation on 3-DOF hover to ensure that the system performance under faulty conditions can be quickly recovered to its normal level. And this proposed method is also proved to be better than that of LQR through simulation.


## 1. Introduction

Quadrotor helicopter is one kind of electric VTOL. Compared with the conventional rotor helicopter, quadrotor can generate more lift force and its structure is more compact. Especially, its four rotors can counteract the reaction torque mutually, so the propellers against reaction torque are not needed [1]. Due to these properties, it makes quadrotor monitor and detect the targets close to the ground so that it has broader military and civilian prospect.

On the other hand, quadrotor, which is the underactuated system with $6-$ DOF and 4 outputs, has the properties of multivariety, nonlinearity, strong coupling, and sensitivity to disturbance. Once it has some faults, it may lead to the loss of performance of flight, even loss of control. Thus, selfrepairing control is born.

Self-repairing control, which utilizes the redundancy of the control system under normal working condition to improve the adaptability to the fault of the flight control system, can avoid catastrophes and make the faulty aircraft operate safely. Then, self-repairing control consists of the direct one and the indirect one. Direct self-repairing control does not need accurate system parameters, while system parameters and several control strategies are the necessity in indirect self-repairing control.

As is known to all, attitude control is the key point of the whole flight control. In addition, the attitude and position of quadrotor helicopter have the direct coupling. Therefore, the research on attitude controller with the capability of selfrepairing from fault is imperative.

Recently, research on the flight control of mini quadrotor helicopter has got some achievements. For instance, Bouabdallah from EPFL has developed several control methods, such as PID, LQR, and Backstepping, based on OS4 [2], one kind of mini quadrotor helicopter, and realized the control on attitude during flight. But Altug from University of Pennsylvania has designed the controller of the quadrotor helicopter HMX4 [3] based on Backstepping and, moreover, actualized the autonomous hover control with the help of vision orientation. Afterwards, vertical takeoff and landing of quadrotor helicopter based on neural network control were achieved by J. Dunfied. Then, Wang has focused on the robust control method based on $H_{\infty}$ [4], which can guarantee the tracking performance and noise immunity of quadrotor helicopter.

However, the existing works on fault diagnosis and faulttolerant control of quadrotor helicopter are quite few at present. A Backstepping fault-tolerant controller for quadrotor helicopter system based on the estimation of compound interference and partial FDI was proposed in [5]. And one
kind of robust fault detection module with observer based on model was designed to reconfigure control law in [6]. In addition, a state estimator for fault detection is proposed in [7] to reconfigure the structure of controller for quadrotor so that it can recover some of control performances when a fault occurs. Yet there is still some weakness in the articles above; firstly, the models of quadrotor are obviously inaccurate. Then, the control strategies are somewhat complicated, which should contain fault identification; thus, it is difficult for application.

The need for effective and realizable fault-tolerant control for quadrotor helicopter with uncertain actuator faults motivates this research. In this paper, we develop an attitude system for quadrotor based on an adaptive fuzzy sliding mode tracking control to compensate the actuator fault such as loss of effectiveness and lock-in-place. The main contributions of this paper are as follows.
(1) The nonlinear model of quadrotor helicopter is put forward in detail, while the linear model ignoring the nonlinear factors such as gyroscopic effect is inaccurate when designing the fault-tolerant control system.
(2) An adaptive fuzzy sliding mode controller without fault identification is designed to track the desired output signal so that quadrotor can finish its mission safely even when any actuator of this quadrotor is loss of effectiveness or stuck at some place.

The rest of this paper is organized as follows. In Section 3.2, the stability of the whole system is guaranteed by Lyapunov stable principle. In Section 4, the proposed method is verified by simulation on 3-DOF hover and also proved to be better than the LQR method which is often used in the attitude system of quadrotor. Finally, conclusions follow in Section 5.

## 2. Modeling Process

2.1. System Model. The attitude and position of quadrotor helicopter are operated by the rotor's rotation rate, without the auto bank unit. The structure schematic is shown in Figure 1.

Three attitude angles are controlled in these principles shown in Figure 2: the roll moment is generated by the difference between the speed of right and left rotor so that the roll angle changes. In the same way, the pitch angle is controlled by the front and back rotor. While the yaw angle changes, the rotors in the diagonal rotate in the same speed and the speed of rotors in different diagonal differs.

To limit the complexity of the dynamics modeling, the following assumptions are adopted [8].
(1) The whole structure is rigid and symmetrical.
(2) Thrust and drag forces are proportional to the square of propellers speed rotation.
(3) The variable range of attitude angles is small (generally less than $5^{\circ}$ ).


Figure 1: The structure schematic of quadrotor helicopter.

Under these assumptions, using the Newton-Euler Equation, the dynamics equations are written in the following way:

$$
\begin{gather*}
m \ddot{\xi}=F_{L}+F_{D}+G,  \tag{1}\\
J \dot{\Omega}=-\Omega \times J \Omega+\Gamma_{f}-\Gamma_{a}-\Gamma_{g}, \tag{2}
\end{gather*}
$$

where $\xi=(x, y, z) \in R^{3}$ is the position of the centre of mass with respect to the inertial frame, $m$ is the total mass of this structure, and $J \in R^{3 \times 3}$ is a constant inertia matrix of quadrotor with respect to the body fixed frame. That is,

$$
J=\left(\begin{array}{ccc}
J_{\phi} & 0 & 0  \tag{3}\\
0 & J_{\theta} & 0 \\
0 & 0 & J_{\psi}
\end{array}\right)
$$

where $J_{\phi}, J_{\theta}$, and $J_{\psi}$ represent rotational inertia of the roll axis, pitch axis, and yaw axis, respectively. $\Omega$ represents the angular velocity of quadrotor expressed in the body fixed frame such as

$$
\Omega=\left(\begin{array}{ccc}
1 & 0 & -\sin \theta  \tag{4}\\
0 & \cos \phi & \cos \theta \sin \phi \\
0 & -\sin \phi & \cos \phi \cos \theta
\end{array}\right)\left(\begin{array}{c}
\dot{\phi} \\
\dot{\theta} \\
\dot{\varphi}
\end{array}\right),
$$

where $\phi, \theta$, and $\varphi$ are roll, pitch, and yaw angles, respectively. When the attitude angles are small, $\Omega=(p, q, r) \approx$ $(\dot{\phi}, \dot{\theta}, \dot{\varphi})$.

There are some details about all terms in (1) and (2) below.
In (1), firstly, $F_{L}$ represents the total lift generated by the four rotors and the expression is

$$
F_{L}=\left(\begin{array}{c}
\cos \phi \sin \theta \cos \psi+\sin \phi \sin \psi  \tag{5}\\
\cos \phi \sin \theta \sin \psi-\sin \phi \cos \psi \\
\cos \phi \cos \theta
\end{array}\right)\left(F_{r}+F_{l}+F_{f}+F_{b}\right)
$$

where $F_{r}=K_{p} \omega_{r}^{2}, F_{l}=K_{p} \omega_{l}^{2}, F_{f}=K_{p} \omega_{f}^{2}$, and $F_{b}=K_{p} \omega_{b}^{2}$, which are the lift generated by right rotor, left one, front one,


Figure 2: The attitude control principle of quadrotor helicopter.
and back one, respectively. Then $K_{p}$ is the lift coefficient, and $\omega_{r}, \omega_{l}, \omega_{f}$, and $\omega_{b}$ represent the rotation speed of right rotor, left one, front one, and back one, respectively.

Secondly, $F_{D}$, the total drag, can be described by

$$
F_{D}=\left(\begin{array}{ccc}
-K_{f d x} & 0 & 0  \tag{6}\\
0 & -K_{f d y} & 0 \\
0 & 0 & -K_{f d z}
\end{array}\right) \dot{\xi}
$$

where $-K_{f d x},-K_{f d y}$, and $-K_{f d z}$ are the component of drag coefficient in $x, y$, and $z$ axis, respectively.

Then, $G$ is the gravity, which is

$$
G=\left(\begin{array}{c}
0  \tag{7}\\
0 \\
-m g
\end{array}\right)
$$

In (2), firstly, $\Gamma_{f}$, the moment of lift, when decomposed into each axis of the body fixed frame, can be described by

$$
\Gamma_{f}=\left(\begin{array}{c}
l\left(F_{r}-F_{l}\right)  \tag{8}\\
l\left(F_{f}-F_{b}\right) \\
C_{D}\left(\omega_{l}^{2}+\omega_{r}^{2}-\omega_{b}^{2}-\omega_{f}^{2}\right)
\end{array}\right)
$$

where $l$ is the distance between the axis of any rotor and the centre of mass and $C_{D}$ is the coefficient of drag moment.

Secondly, $\Gamma_{a}$ represents the total pneumatic friction torque, whose expression is

$$
\Gamma_{a}=\left(\begin{array}{ccc}
K_{\mathrm{fax}} & 0 & 0  \tag{9}\\
0 & K_{\mathrm{fay}} & 0 \\
0 & 0 & K_{\mathrm{faz}}
\end{array}\right)\|\Omega\|^{2}
$$

where $K_{\text {fax }}, K_{\text {fay }}$, and $K_{\text {faz }}$ are the component of pneumatic friction coefficient in $x, y$, and $z$ axis, respectively.

Next, $\Gamma_{g}$ is the resultant moment under gyroscopic effect and can be defined as

$$
\Gamma_{g}=\Omega \times J_{r}\left(\begin{array}{c}
0  \tag{10}\\
0 \\
\hline \Omega
\end{array}\right)
$$

where $J_{r}$ is the moment of inertia of the rotor; moreover, $\bar{\Omega}=$ $\omega_{l}+\omega_{r}-\omega_{b}-\omega_{f}$.

In conclusion, the dynamic model of quadrotor helicopter can be written as

$$
\begin{align*}
\ddot{\phi} & =\frac{1}{J_{\phi}}\left[\left(J_{\theta}-J_{\psi}\right) \dot{\psi} \dot{\theta}-K_{\mathrm{fax}} \dot{\phi}^{2}-J_{r} \bar{\Omega} \dot{\theta}+l\left(F_{r}-F_{l}\right)\right] \\
\ddot{\theta} & =\frac{1}{J_{\theta}}\left[\left(J_{\psi}-J_{\phi}\right) \dot{\psi} \dot{\phi}-K_{\mathrm{fay}} \dot{\theta}^{2}+J_{r} \bar{\Omega} \dot{\phi}+l\left(F_{f}-F_{b}\right)\right] \\
\ddot{\psi} & =\frac{1}{J_{\psi}}\left[\left(J_{\phi}-J_{\theta}\right) \dot{\phi} \dot{\theta}-K_{\mathrm{faz}} \dot{\psi}^{2}+\frac{C_{D}}{K_{p}}\left(F_{r}+F_{l}-F_{f}-F_{b}\right)\right], \tag{11}
\end{align*}
$$

where $F_{r}, F_{f}, F_{l}$, and $F_{b}$ defined previously are the inputs of this system.

To simplify the representations, we define

$$
\begin{align*}
\left(\begin{array}{c}
U_{\phi} \\
U_{\theta} \\
U_{\psi}
\end{array}\right) & =\left(\begin{array}{cccc}
1 & 0 & -1 & 0 \\
0 & 1 & 0 & -1 \\
\frac{C_{D}}{K_{P}} & -\frac{C_{D}}{K_{P}} & \frac{C_{D}}{K_{P}} & -\frac{C_{D}}{K_{P}}
\end{array}\right)\left(\begin{array}{c}
F_{r} \\
F_{f} \\
F_{l} \\
F_{b}
\end{array}\right)  \tag{12}\\
& =L\left(\begin{array}{c}
F_{r} \\
F_{f} \\
F_{l} \\
F_{b}
\end{array}\right)
\end{align*}
$$

Due to the limit of the power of each electromotor, there exists a maximum rotation speed $\omega_{\max }$ for each rotor. It is assumed that each $\omega_{\max }$ is equal because of the property of symmetry in the quadrotor helicopter. Therefore, the inputs meet the following conditions:

$$
\begin{align*}
&-K_{p} \omega_{\max }^{2} \leq U_{\phi} \leq K_{p} \omega_{\max }^{2} \\
&-K_{p} \omega_{\max }^{2} \leq U_{\theta} \leq K_{p} \omega_{\max }^{2}  \tag{13}\\
&-2 C_{D} \omega_{\max }^{2} \leq U_{\psi} \leq 2 C_{D} \omega_{\max }^{2}
\end{align*}
$$

Table 1: Fault mode.

| Fault parameter | State of system |
| :--- | :---: |
| $\lambda_{i}=1, \sigma_{i}=0$ | Normal |
| $0<\lambda_{i}<1, \sigma_{i}=0$ | Loss of effectiveness |
| $\lambda_{i}=0, \sigma_{i}=1$ | Lock-in-place |

In addition, the dynamic of the DC-electromotor which drives rotors is shown below:

$$
\begin{align*}
V & =R I+L \frac{d I}{d t}+K_{e} \omega,  \tag{14}\\
K_{m} I & =J_{r} \frac{d \omega}{d t}+K_{r} \omega^{2}+C_{s},
\end{align*}
$$

where $R$ is the internal resistance of electromotor and $K_{e}, K_{m}$, and $K_{r}$ are the electric torque constant, mechanical torque constant, and load constant torque, respectively. Then, $C_{s}$ denotes the solid friction.

Based on this analysis, let $x=[\phi, \dot{\phi}, \theta, \dot{\theta}, \psi, \dot{\psi}]^{T} \in R^{6}$, $u=\left[F_{r}, F_{f}, F_{l}, F_{b}\right]^{T}=\left[u_{1}, u_{2}, u_{3}, u_{4}\right]^{T} \in R^{4}$ be the state and the control input vectors, respectively. The state equation of this system can be written in the following affine nonlinear representation:

$$
\begin{equation*}
\dot{x}=f(x)+g(x) u, \tag{15}
\end{equation*}
$$

where

$$
\begin{align*}
& f(x)=\left(\begin{array}{c}
x_{2} \\
a_{1} x_{4} x_{6}+a_{2} x_{2}^{2}+a_{3} \bar{\Omega} x_{4} \\
x_{4} \\
a_{4} x_{2} x_{6}+a_{5} x_{4}^{2}+a_{6} \bar{\Omega} x_{2} \\
x_{6} \\
a_{7} x_{2} x_{4}+a_{8} x_{6}^{2}
\end{array}\right) \\
& g(x)=\left(\begin{array}{cccc}
0 & 0 & 0 & 0 \\
b_{1} & 0 & -b_{1} & 0 \\
0 & 0 & 0 & 0 \\
0 & b_{2} & 0 & -b_{2} \\
0 & 0 & 0 & 0 \\
b_{3} & -b_{3} & b_{3} & -b_{3}
\end{array}\right) \tag{16}
\end{align*}
$$

with

$$
\begin{gather*}
a_{1}=\frac{J_{\theta}-J_{\psi}}{J_{\phi}}, \quad a_{2}=-\frac{K_{\mathrm{fax}}}{J_{\phi}}, \quad a_{3}=-\frac{J_{r}}{J_{\phi}} \\
a_{4}=\frac{J_{\psi}-J_{\phi}}{J_{\theta}}, \quad a_{5}=-\frac{K_{\mathrm{fay}}}{J_{\theta}}, \quad a_{6}=\frac{J_{r}}{J_{\theta}}, \\
a_{7}=\frac{J_{\phi}-J_{\theta}}{J_{\psi}}, \quad a_{8}=-\frac{K_{\mathrm{faz}}}{J_{\psi}}, \\
b_{1}=\frac{l}{J_{\phi}}, \quad b_{2}=\frac{l}{J_{\theta}}, \quad b_{3}=\frac{C_{D}}{J_{\psi} K_{p}} . \tag{17}
\end{gather*}
$$

2.2. Actuator Fault Model. According to report of research, the actuator of the helicopter can be easily damaged. In view of quadrotor helicopter, when the actuator fault occurs, the rotation speed of rotors will be abruptly changed so that the attitude system of quadrotor will vary rapidly or even lose control.

In this paper, we consider actuator faults including loss of effectiveness and lock-in-place. When any actuator has failed, we can denote a general actuator fault model as [9]

$$
\begin{equation*}
u=\lambda \underline{u}+\sigma(\bar{u}-\lambda \underline{u}) \tag{18}
\end{equation*}
$$

where $\lambda=\operatorname{diag}\left[\lambda_{1}, \lambda_{2}, \lambda_{3}, \lambda_{4}\right], \nu=\left[\nu_{1}, \nu_{2}, \nu_{3}, \nu_{4}\right]^{T}, \sigma=$ $\operatorname{diag}\left[\sigma_{1}, \sigma_{2}, \sigma_{3}, \sigma_{4}\right]$, and $\bar{u}=\left[\bar{u}_{1}, \bar{u}_{2}, \bar{u}_{3}, \bar{u}_{4}\right]^{T}$. $\lambda_{i}$ denotes the percentage of the remaining effective part of the corresponding actuator. $\underline{u}_{i}$ denotes the applied control vector, while $\sigma_{i}\left(\sigma_{i}=0\right.$ or 1$)$ is used to describe the lock-in-place fault. If $\lambda_{i}=1$ and $\sigma_{i}=0$, no fault occurs in this actuator. When $0<\lambda_{i}<1$ and $\sigma_{i}=0$, the corresponding actuator loses partial effectiveness. The case of $\lambda_{i}=0, \sigma_{i}=1$ means that the actuator is stuck at some unknown place where $\bar{u}_{i}$ is the constant value. The considered faults can be synthesized by Table 1.

Inspired from [10], we can define another input vector $v=$ $\left[U_{\phi}, U_{\theta}, U_{\psi}\right]^{T} \in R^{3}$ so that the number of the output which will be given in the next section is equal to that of the input.

Thus, we design the control vector $v$ instead of $\underline{u}$. However, the applied input $\underline{u}$ can be achieved by

$$
\begin{equation*}
\underline{u}=L^{-} v, \tag{19}
\end{equation*}
$$

where $L^{-}=\left(L^{T} L\right)^{-1} L^{T}$ is the generalized inverse matrix of $L$.
To attain the control objective, we propose to use a proportional actuator structure as follows [11]:

$$
\begin{equation*}
\nu=c v_{0} \tag{20}
\end{equation*}
$$

where $c=\operatorname{diag}\left[c_{1}, c_{2}, c_{3}\right]$ represents the proportional actuation gain matrix and $\nu_{0}=\left[v_{01}, v_{02}, v_{03}\right]^{T}$ is adaptive fuzzy controller that we proposed.

Using (18), (19), and (20), the system (15) can be described by

$$
\begin{equation*}
\dot{x}=f^{\prime}(x)+g^{\prime}(x) v_{0} \tag{21}
\end{equation*}
$$

where $f^{\prime}(x)=f(x)+g(x) \sigma \bar{u}, g^{\prime}(x)=g(x)(I-\sigma) \lambda L^{-} c$.

## 3. Direct Self-Repairing Control Strategy

The direct adaptive fuzzy control based on sliding mode is proposed to actualize self-repairing control in this paper. Adaptive fuzzy sliding mode control combines the advantages between adaptive fuzzy control and sliding mode control, which can not only adjust the adaptation law on line when uncertain function exists but also ensure the robustness of the considered nonlinear system.

The control block diagram is shown in Figure 3.
We consider the output of this quadrotor helicopter attitude system as $y=[\phi, \theta, \psi]^{T} \in R^{3}$.


Figure 3: The control block diagram for adaptive fuzzy sliding mode control.

Then, the dynamic equation of the output can be rewritten in the following form:

$$
\begin{align*}
y^{(2)} & =F(x)+G(x) v_{0} \\
& =\left[\begin{array}{lllllll}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]\left[f^{\prime}(x)+g^{\prime}(x) v_{0}\right] . \tag{22}
\end{align*}
$$

Our task is to design a robust adaptive fuzzy controller based on sliding mode. There is no couple between each output vector, so we can consider the sliding surface in the state error space as

$$
\begin{equation*}
s=e+k \dot{e}, \tag{23}
\end{equation*}
$$

where $s=\left[s_{1}, s_{2}, s_{3}\right]^{T}$ and $k=\operatorname{diag}\left[k_{1}, k_{2}, k_{3}\right]\left(k_{i}>0 i=\right.$ $1,2,3) e=\left[e_{1}, e_{2}, e_{3}\right]^{T}$ with

$$
\begin{align*}
& e_{1}(t)=\phi_{d}-\phi \\
& e_{2}(t)=\theta_{d}-\theta,  \tag{24}\\
& e_{3}(t)=\psi_{d}-\psi
\end{align*}
$$

with the desired output signal denoted by $y_{d}=\left[\phi_{d}, \theta_{d}, \psi_{d}\right]^{T}$.
The time derivative of (23) can be written as

$$
\begin{equation*}
\dot{s}=\alpha-k\left[F(x)+G(x) v_{0}\right] \tag{25}
\end{equation*}
$$

where $\alpha$ is given as follows:

$$
\begin{equation*}
\alpha=\left[k_{1} \ddot{\phi}_{d}+\dot{e}_{1}, k_{2} \ddot{\theta}_{d}+\dot{e}_{2}, k_{3} \ddot{\theta}_{d}+\dot{e}_{3}\right]^{T} \tag{26}
\end{equation*}
$$

If the functions $F(x)$ and $G(x)$ are known, to achieve the control objective, one can use the following ideal nonlinear control law:

$$
\begin{equation*}
v_{0}^{*}=G^{-1}(x)\left(-F(x)+k^{-1} \alpha+\frac{s}{\gamma^{2}}\right) . \tag{27}
\end{equation*}
$$

Effectively, when we select the control input as $\nu_{0}=\nu_{0}^{*}$, (25) simplifies to

$$
\begin{equation*}
\dot{s}=-k \frac{s}{\gamma^{2}} . \tag{28}
\end{equation*}
$$

Here we design a Lyapunov function as $V=(1 / 2) s^{T} s \geq 0$. Then we have $\dot{V}=\dot{s}^{T} s=-\left(1 / \gamma^{2}\right) s^{T} k s \leq 0$, which indicates that the sliding mode defined can be achieved.

Thus, we can conclude that $s(t) \rightarrow 0$ as $t \rightarrow \infty$; therefore, $e(t)$ converges to zero, and the whole system is stable.
3.1. Adaptive Fuzzy Control Law. However, when the actuators of quadrotor helicopter have faults, the functions $F(x)$ and $G(x)$ are unknown so that the ideal controller designed previously cannot be used.

To overcome this problem, we propose to use an adaptive fuzzy system to approximate this ideal control law. Moreover, the parameters of this fuzzy controller are updated by the error between the fuzzy controller and the desired one.

According to the approximation theorem [12], there exists an optimal input based on fuzzy control approximating uniformly the ideal control law (27) such that

$$
\begin{equation*}
\nu_{0}=\xi(x) \theta \tag{29}
\end{equation*}
$$

Then the fuzzy approximation error is

$$
\begin{equation*}
\varepsilon=v_{0}^{*}-\nu_{0}=\nu_{0}^{*}-\xi(x) \theta \tag{30}
\end{equation*}
$$

where $\theta=\left[\theta_{1}, \theta_{2}, \theta_{3}\right]^{T}$ is an unknown parameter vector which minimizes the approximation error on a compact set $\Omega$. And $\xi(x)=\operatorname{diag}\left[\xi_{1}(x), \xi_{2}(x), \xi_{3}(x)\right]$ is the matrix of fuzzy basis function suitably selected.

In this paper, we assume that the fuzzy controller proposed satisfies the universal approximation property over the compact set $\Omega$, which is assumed to be large enough so that the state variables remain inside it under closed-loop control. Therefore, it is reasonable to assume that the fuzzy approximation error is bounded for all $x \in \Omega$.
3.2. Parameter Adaptation Law. In the preview, we recall that the parameter vector $\theta$ is unknown. So the parameter estimate $\widehat{\theta}$ based on a gradient descent adaptation algorithm will be developed in this subsection.

To design a suitable adaptation law, our goal is to minimize the approximation error between $\nu_{0}^{*}$ and $\nu_{0}$.

Hence, the parameter estimate $\widehat{\theta}$ is obtained according to the following theorem.

Theorem 1. The adaptive fuzzy control law

$$
\begin{equation*}
\hat{v}_{0}=\xi(x) \hat{\theta} \tag{31}
\end{equation*}
$$

equipped with the following adaption law:

$$
\begin{equation*}
\dot{\hat{\theta}}=\eta \xi(x)\left(\dot{s}+k \frac{s}{\gamma^{2}}\right)-\eta \tau \widehat{\theta} \tag{32}
\end{equation*}
$$

where $\eta, \tau$ are positive constant parameters, guarantees the stability and the robustness of the system with actuator failures (18).

Proof. Firstly, the estimate error $\tilde{\theta}$ should be defined by

$$
\begin{equation*}
\widetilde{\theta}=\theta-\widehat{\theta} \tag{33}
\end{equation*}
$$

Then, we substitute the proposed control law (31) to (25), and the derivative of sliding surface changes to be

$$
\begin{equation*}
\dot{s}=-k \frac{s}{\gamma^{2}}+G(x)\left(v_{0}^{*}-\widehat{\nu}_{0}\right)=-k \frac{s}{\gamma^{2}}+G(x) \varepsilon^{\prime}, \tag{34}
\end{equation*}
$$

where $\varepsilon^{\prime}=\varepsilon+\xi(x) \widetilde{\theta}$, denoted to be the error between desired input and actual input.

Then, another Lyapunov function is designed by the following form:

$$
\begin{equation*}
V^{\prime}=\frac{1}{2} s^{T} s+\frac{1}{2 \eta} \widetilde{\theta}^{T} \widetilde{\theta} \tag{35}
\end{equation*}
$$

Obviously,

$$
\begin{equation*}
\dot{V}^{\prime}=s^{T} \dot{s}+\frac{1}{\eta} \widetilde{\theta}^{T} \dot{\tilde{\theta}}=s^{T} \dot{s}-\frac{1}{\eta} \widetilde{\theta}^{T} \dot{\hat{\theta}} . \tag{36}
\end{equation*}
$$

Substituting (32) and (34), we can get

$$
\begin{align*}
\dot{V}^{\prime} & =-\frac{1}{\gamma^{2}} s^{T} k s+s^{T} G(x) \varepsilon^{\prime}-\widetilde{\theta}^{T} \xi G(x) \varepsilon^{\prime}+\tau \widetilde{\theta}^{T} \hat{\theta} \\
& =-\frac{1}{\gamma^{2}} s^{T} k s+s^{T} G(x) \varepsilon^{\prime}-(\xi \widetilde{\theta})^{T} G(x) \varepsilon^{\prime}+\tau \widetilde{\theta}^{T} \widehat{\theta} \\
& =-\frac{1}{\gamma^{2}} s^{T} k s+s^{T} G(x) \varepsilon^{\prime}-\varepsilon^{\prime T} G(x) \varepsilon^{\prime}+\varepsilon^{T} G(x) \varepsilon^{\prime}+\tau \widetilde{\theta}^{T} \widehat{\theta} \tag{37}
\end{align*}
$$

using the inequalities

$$
\begin{align*}
\widetilde{\theta}^{T} \widehat{\theta}= & -\frac{1}{2}\|\widetilde{\theta}\|^{2}-\frac{1}{2}\|\widehat{\theta}\|^{2}+\frac{1}{2}\|\widetilde{\theta}+\widehat{\theta}\|^{2} \\
\leq & -\frac{1}{2}\|\widetilde{\theta}\|^{2}+\frac{1}{2}\|\theta\|^{2}, \\
s^{T} G(x) \varepsilon^{\prime}= & -\left(\frac{1}{2} \varepsilon^{\prime}+s\right)^{T} G(x)\left(\frac{1}{2} \varepsilon^{\prime}+s\right)  \tag{38}\\
& +\frac{1}{4} \varepsilon^{\prime T} G(x) \varepsilon^{\prime}+s^{T} G(x) s \\
\leq & \frac{1}{4} \varepsilon^{\prime T} G(x) \varepsilon^{\prime}+s^{T} G(x) s, \\
\varepsilon^{T} G(x) \varepsilon^{\prime} \leq & \frac{1}{4} \varepsilon^{\prime T} G(x) \varepsilon^{\prime}+\varepsilon^{T} G(x) \varepsilon .
\end{align*}
$$

Equation (37) can be bounded as

$$
\begin{align*}
\dot{V}^{\prime} \leq & -\frac{1}{\gamma^{2}} s^{T} k s-\frac{\tau}{2}\|\widetilde{\theta}\|^{2}-\frac{1}{2} \varepsilon^{\prime T} G(x) \varepsilon^{\prime}+\varepsilon^{T} G(x) \varepsilon  \tag{39}\\
& +s^{T} G(x) s+\frac{\tau}{2}\|\theta\|^{2} .
\end{align*}
$$

Using the fact that the control gain matrix is positive definite, so there exists a positive constant $\beta$ such that $G(x) \leq \beta I$; then, let another positive constant denote $k_{0}=$ $\min \left\{k_{1}, k_{2}, k_{3}\right\}$ so that $k \geq k_{0} I$; moreover, we select $\tau=$ $2\left(\left(k_{0} / \gamma^{2}\right)-\beta\right)$ to be positive, and this leads to

$$
\begin{aligned}
\dot{V}^{\prime} \leq & -\frac{\tau}{2}\left(s^{T} s+\widetilde{\theta}^{T} \widetilde{\theta}\right)-\frac{1}{2} \varepsilon^{\prime T} G(x) \varepsilon^{\prime} \\
& +\beta \varepsilon^{T} \varepsilon+\frac{\tau}{2}\|\theta\|^{2} \\
\leq & -\frac{\tau}{2} V^{\prime}+\beta \varepsilon^{T} \varepsilon+\frac{\tau}{2}\|\theta\|^{2} .
\end{aligned}
$$



Figure 4: 3-DOF hover experimental platform.

Since the desired parameter vector $\theta$ is a constant vector, $\varepsilon$ is assumed to be bounded. Also we can define a positive constant bound $\beta_{0}$ as

$$
\begin{equation*}
\beta_{0}=\sup \left(\frac{\tau}{2}\|\theta\|^{2}+\beta \varepsilon^{T} \varepsilon\right) . \tag{41}
\end{equation*}
$$

Then the inequality of $\dot{V}^{\prime}$ is simplified to

$$
\begin{equation*}
\dot{V}^{\prime} \leq-\frac{\tau}{2} V^{\prime}+\beta_{0} \tag{42}
\end{equation*}
$$

By quoting a theorem in [10], we can prove that the parameter error vector $\widetilde{\theta}$ and the tracking error vector $e(t)$ are bounded. Furthermore, due to the bound of $\dot{e}(t)$, we can also conclude that the state vector $x$ is bounded as well. Therefore, this adaptive fuzzy control system based on sliding mode for quadrotor helicopter is stable and has good performance of tracking.

## 4. Simulation Results

In this paper, we take the 3-DOF hover helicopter shown in Figure 4 which is produced by Quanser Company as the research object to simulate the operation of quadrotor helicopter attitude system. The 3-DOF hover helicopter consists of electric motors, rotors, helicopter body, power-supply module, encoders (sensors), and so forth. In the existing software platform, we can design the direct adaptive fuzzy sliding mode control system under the circumstance of MATLAB REAL-TIME. With the help of Quanser's supporting software, the block diagrams of MATLAB Simulink can be directly encoded into $C$ language which downloads to realtime simulation system from supporting PIC card through the parallel port [13]. After that, we can do the simulation experiment to verify the practicability of the control method proposed.


Figure 5: The output response curves with no fault.

According to the user's guide of Quanser Hover, the system parameters are given by

$$
\begin{gathered}
K_{\mathrm{fax}}=0.0080, \quad K_{\mathrm{fay}}=0.0080, \quad K_{\mathrm{faz}}=0.0091 \\
J_{\phi}=0.0552,
\end{gathered} J_{\theta}=0.0552, \quad J_{\varphi}=0.11,
$$

$$
\begin{equation*}
l=0.197, \quad J_{r}=0.1188, \quad K_{p}=0.0036, \quad C_{D}=0.0036 \tag{43}
\end{equation*}
$$

Furthermore, to illustrate the superiority of the method proposed, system dynamic performance under the adaptive fuzzy sliding mode control will be compared with that under the LQR method when these three cases occur: normal, the loss of effectiveness of actuators, and lock-in-place fault.

The design procedure of LQR is shown below.
Firstly, the affine nonlinear model of quadrotor helicopter should be linearized. Next, we can select suitable weight
matrices $Q, R$, where $Q=\operatorname{diag}([125,250,250,0,10,10])$, $R=0.01 \cdot \operatorname{diag}([1,1,1])$. The control matrix $K$ is achieved by using LQR commands in MATLAB, which is

$$
K=\left[\begin{array}{cccccc}
0 & 0 & 158.1139 & 0 & 0 & 41.7939  \tag{44}\\
0 & 158.1139 & 0 & 0 & 41.7939 & 0 \\
122.4745 & 0 & 0 & 86.6881 & 0 & 0
\end{array}\right]
$$

Hence, the LQR controller is described by $u=K\left(x-x_{d}\right)$, where $x_{d}=\left[\phi_{d}, 0, \theta_{d}, 0, \varphi_{d}, 0\right]^{T}$ is the desired state vector.

In all simulation cases, the desired rolling angle, pitching angle, and yawing angle are selected to be square wave with the amplitude $2^{\circ}$ and square wave with the amplitudes $2^{\circ}$ and $0^{\circ}$, respectively.

Three fuzzy systems in the form of (29) are used to generate the control signals $u_{1}, u_{2}$, and $u_{3}$. Each system has the input as $z_{1}=\left[e_{1}(t), \dot{e}_{1}(t)\right]^{T}, z_{2}=\left[e_{2}(t), \dot{e}_{2}(t)\right]^{T}$,


Figure 6: The output response curves with the loss of effectiveness in $u_{2}$.
and $z_{3}=\left[e_{3}(t), \dot{e}_{3}(t)\right]^{T}$, respectively. For each input variable in $z_{1}, z_{2}$, and $z_{3}$, seven Gaussian membership functions which give 49 fuzzy rules are defined as

$$
\begin{aligned}
& \mu_{F_{1}}\left(z_{i j}\right)=\exp \left(-\frac{1}{2}\left(\frac{z_{i j}+1}{0.14}\right)^{2}\right) \\
& \mu_{F_{2}}\left(z_{i j}\right)=\exp \left(-\frac{1}{2}\left(\frac{z_{i j}+0.67}{0.14}\right)^{2}\right) \\
& \mu_{F_{3}}\left(z_{i j}\right)=\exp \left(-\frac{1}{2}\left(\frac{z_{i j}+0.33}{0.14}\right)^{2}\right) \\
& \mu_{F_{4}}\left(z_{i j}\right)=\exp \left(-\frac{1}{2}\left(\frac{z_{i j}}{0.14}\right)^{2}\right)
\end{aligned}
$$

$$
\begin{align*}
& \mu_{F_{5}}\left(z_{i j}\right)=\exp \left(-\frac{1}{2}\left(\frac{z_{i j}-0.33}{0.14}\right)^{2}\right), \\
& \mu_{F_{6}}\left(z_{i j}\right)=\exp \left(-\frac{1}{2}\left(\frac{z_{i j}-0.67}{0.14}\right)^{2}\right) \\
& \mu_{F_{7}}\left(z_{i j}\right)=\exp \left(-\frac{1}{2}\left(\frac{z_{i j}-1}{0.14}\right)^{2}\right) \tag{45}
\end{align*}
$$

where $i=1,2, j=1,2$.
In addition, the time of simulation and step size are set to be 30 s and 0.001 s , respectively. The other design parameters used in this simulation are chosen as follows:

$$
\begin{equation*}
k=\operatorname{diag}[10,10,10], \quad \gamma=1, \quad \tau=0.001, \quad \eta=20 . \tag{46}
\end{equation*}
$$



Figure 7: The output response curves with the lock-in-place in $u_{2}$.
(1) When no fault happened in this system, the simulation curves on adaptive fuzzy sliding mode controller and LQR controller are shown in Figure 5.

From Figure 5, it can be seen that the system can track the desired output well with the help of both controllers and has good dynamic performance. However, as for yawing angle, under the adaptive fuzzy controller based on sliding mode, the system can track signal almost without error while LQR controller cannot achieve that, which demonstrates that the proposed adaptive fuzzy controller has a better tracking performance than LQR controller when the system is healthy.
(2) We suppose that the system is subject to loss of effectiveness at 10 s in input $u_{2}$, and this may lead to the changes of pitching angle and yawing angle. The rest of the simulation settings are the same. Thus, we
can get the simulation curves under adaptive fuzzy sliding mode controller and LQR controller shown in Figure 6.
Refer to Figure 6; the conclusion is that, when the loss of effectiveness occurred in $u_{2}$, the system nearly has no influence and still tracks the signal very well with the help of adaptive fuzzy sliding mode controller. But the performance of LQR controller falls with the obvious tracking errors. So the superiority of adaptive fuzzy controller is proven again.
(3) Suppose that a lock-in-place fault occurs in $u_{2}$ at 10 s . The rest of the simulation settings are unchanged. The simulation curves under adaptive fuzzy sliding mode controller and LQR controller are given in Figure 7.
We can obtain the same conclusion from Figure 7.
Without the loss of generality, another case is set to indicate that adaptive fuzzy sliding mode controller still


Figure 8: The output response curves with the loss of effectiveness in $u_{1}$.
works and is better than LQR controller when the actuator fault occurs in other inputs.
(4) We suppose that the system loses effectiveness at 10 s in input $u_{1}$. The rest of the simulation settings are unchanged. Then the simulation curves under both controllers are given in Figure 8.

Remark. Compared with the related work in [6, 13], the good features of this paper are in 3 aspects: (1) fault diagnosis is not needed in the proposed method so that it can be easier to be applied to the engineering practice; (2) even without the process of fault diagnosis, the proposed control system can still deal with a variety of actuator faults such as loss of effectiveness and lock-in-place; (3) the dynamic performance of the attitude system based on adaptive fuzzy sliding mode control when fault occurs is more smooth.

To sum up, when the actuator faults such as loss of effectiveness and lock-in-place occur in the attitude system of quadrotor helicopter, under the adaptive fuzzy sliding mode controller, this system can still track the desired output signal very well and return to the normal performance very rapidly, which implies that the whole system has the certain capability of self-repairing.

## 5. Conclusion

In this paper, firstly, we built the affine nonlinear model for the quadrotor helicopter attitude system, which is MIMO. With the consideration of unknown actuator faults such as loss of effectiveness and lock-in-place, an adaptive fuzzy controller based on sliding mode has been proposed to realize the direct self-repairing control for this attitude system. Through a series of simulations, it has verified the availability
of the proposed method which can make the system recover from the actuator faults and has good tracking performance.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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# A Robust Recursive Filter for Nonlinear Systems with Correlated Noises, Packet Losses, and Multiplicative Noises 

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#### Abstract

A robust filtering problem is formulated and investigated for a class of nonlinear systems with correlated noises, packet losses, and multiplicative noises. The packet losses are assumed to be independent Bernoulli random variables. The multiplicative noises are described as random variables with bounded variance. Different from the traditional robust filter based on the assumption that the process noises are uncorrelated with the measurement noises, the objective of the addressed robust filtering problem is to design a recursive filter such that, for packet losses and multiplicative noises, the state prediction and filtering covariance matrices have the optimized upper bounds in the case that there are correlated process and measurement noises. Two examples are used to illustrate the effectiveness of the proposed filter.


## 1. Introduction

In recent years, the state estimation theory has received extensive attention in many fields of application, such as attitude estimation [1], target tracking [2], signal processing [3], and integrated navigation [4]. State estimation refers to a methodology that is used for estimating the state of a time-varying system through noisy measurements, which are different from other methods [5-9]. So far, various kinds of filtering algorithms for state estimation have been presented, for example, Kalman filter [10], extended Kalman filter (EKF) [11], unscented Kalman filter (UKF) [12], and so forth. As is well known, among those filters, Kalman filter is an optimal solution based on the minimum mean square error rule for linear systems and EKF is an effective way for softly nonlinear system to estimate the state by using linearization techniques. Although EKF is a popular estimating algorithm in engineering practice, its use must satisfy the following two assumptions: (1) the system model should be accurate and (2) the additive noises should be Gaussian and uncorrelated. Otherwise, the performance of EKF can be degraded severely, even unstable. Unfortunately, in real world, the model uncertainty is an unavoidable and crucial problem for
nonlinear systems. Therefore it is required to develop a more general filtering algorithm. To this end, the robust filtering technique has been developed to reduce the unfavorable effect of model uncertainties by establishing an appropriate uncertain model in consideration of uncertainties. Up to now, a lot of literatures on the robust filtering problem with model uncertainties have been published, such as the $H_{\infty}$ filter [13-16], set-valued nonlinear filter [17, 18], mixed $H_{2} / H_{\infty}$ filter [19, 20], and robust extended Kalman filter design [21, 22]. In these reports, the robust recursive filter design has been investigated to be available for handling the nonlinear filtering problem with model uncertainties. For instance, a discrete-time robust extended Kalman filter has been presented for uncertain systems with sum quadratic constraints in [21]. Due to the influence of the misalignments of star sensors, by considering the model uncertainties, a nonlinear robust filter for satellite attitude determination is developed and verified in [22].

In literature mentioned above, however, only additive noises are considered for nonlinear systems. Actually, another important noise called multiplicative noise is often encountered in many engineering systems, such as attitude estimation systems and airborne synthetic aperture radar systems. It
is coupled with the state and has an unknown noise variance, which results in a negative impact on the state estimation. Hence, the multiplicative noise is usually viewed as a model uncertainty. Currently, the nonlinear robust filtering problem with multiplicative noises has been much less researched. In $[23,24]$, by utilizing linear matrix inequality approach, a robust Kalman filter is derived for linear systems. Different from them, another robust Kalman filter is proposed for linear systems by finding two Riccati differential equations and determining the filter parameters in [25]. Then, [26] extends the work to nonlinear systems. Apart from multiplicative noises, signal transmissions in the sensor networks are often unreliable. For example, sudden sensor failure, random communication delays, and packet losses appear in the practical system frequently [27-31]. All these lead to the measurement mode uncertainty. Accordingly, the filtering problem with packet losses has stirred considerable research attention and many research results have been published recently; see, for example [32-35]. In most literatures, the packet loss is described as a random variable in the distribution of Bernoulli, which may not be available because of the existence of the different transmission process in multiple sensors. In [36, 37], a diagonal matrix composed of Bernoulli random variables is introduced to the measurement equation, which means that individual sensor might have different missing rates. Meanwhile, it is not difficult to find that the most existing filtering researches concerning packet losses are subject to linear systems. However, as we all know, nonlinearity is inevitable in almost all engineering applications, which will directly degrade the quality of the filtering performance. For this purpose, a quantized recursive filtering is presented for a class of nonlinear systems with missing measurements, multiplicative noises, and quantization effects in [37]. Though missing measurements and multiplicative noises are taken into consideration at the same time, this work endures the limitation that the measurement equation must be linear, which makes that the algorithm in [37] cannot be extended to solve the general nonlinear filtering problems in the case that the process and measurement model are all nonlinear. But note that the multiplicative noise case is just a special case of the stochastic nonlinearities considered in [36]. Therefore, an explicit and systematic solution to this problem can be extended.

In addition, the correlation of additive noises is one of the key factors to the filtering algorithm. Disturbed by the complicated environment, the additive noises often show the characteristic of correlation in the practical application. Unluckily, the design procedures of all the above filters for multiplicative noises or packet losses are based on the assumption that there are uncorrelated additive noises in the system. In fact, this assumption does not always come into existence, and the process noise might be correlated with the measurement noise in real applications. In [38], a modified UKF for nonlinear systems with correlated additive noises is proposed. Wang et al. [39] extend the work to develop a Gaussian approximation recursive filter framework to deal with correlated noises. But model uncertainties are not considered in these works. To the best of the authors' knowledge, up to the present, based on the assumption that the process noise is
correlated with the measurement noise, the nonlinear robust filtering problem with multiplicative noises and packet losses has not been reported. Therefore, in order to better reflect the actual situation and consider the complex dynamical systems, there is a strong desire to develop a robust recursive filter to handle the robust filtering problem with correlated additive noises, multiplicative noises, and packet losses.

Motivated by the above discussion, we present a robust recursive filter for a class of nonlinear systems with correlated additive noises, multiplicative noises, and packet losses. In this paper, multiplicative noises are assumed as zero mean Gaussian white noises and the packet losses are modeled as independent Bernoulli random variables. Based on the structure of the extended Kalman filter with correlated noises, the proposed filter designs an optimal upper bound of the prediction error and the filtering error covariance matrices, respectively. The main contributions of the paper are as follows. (1) In the case that the process noise is correlated with the measurement noise, a recursive filter framework is established to deal with the robust filtering problem for nonlinear systems in the presence of multiplicative noises and packet losses. (2) The addressed robust recursive filter problem is new especially when the correlated additive noises appear in the system. (3) The developed robust filter is recursive, which is suitable for online applications. The remainder of the paper is organized as follows. In Section 2, the problem is formulated. In Section 3, the robust recursive filter with correlated additive noises, multiplicative noises, and packet losses is developed. In Section 4, two simulation examples are employed, and the simulation analysis is given. In Section 5, some conclusions are drawn.

## 2. Problem Formulation and Preliminaries

Consider a general class of discrete time-varying systems with multiplicative noises, correlated additive noises, and packet losses:

$$
\begin{align*}
& \mathbf{x}_{k+1}=f\left(\mathbf{x}_{k}\right)+\sum_{i=1}^{q} \mathbf{A}_{i k}^{s} \boldsymbol{\eta}_{i k} \mathbf{x}_{k}+\mathbf{w}_{k},  \tag{1}\\
& \mathbf{y}_{k}=\boldsymbol{\Sigma}_{k} h\left(\mathbf{x}_{k}\right)+\sum_{i=1}^{r} \mathbf{C}_{i k}^{s} \xi_{i k} \mathbf{x}_{k}+\mathbf{v}_{k}, \tag{2}
\end{align*}
$$

where $\mathbf{x}_{k} \in \mathbf{R}^{n}$ is the state vector, $\mathbf{y}_{k} \in \mathbf{R}^{m}$ is the measurement vector, $\boldsymbol{\eta}_{i k}$ and $\boldsymbol{\xi}_{i k}$ are the uncorrelated zero mean Gaussian multiplicative noises, and $\mathbf{A}_{i k}^{s}$ and $\mathbf{C}_{i k}^{s}$ are known matrices with appropriate dimension. The diagonal matrix $\boldsymbol{\Sigma}_{k}$ is denoted as $\boldsymbol{\Sigma}_{k}=\operatorname{diag}\left\{\lambda_{k}^{1}, \lambda_{k}^{2}, \ldots, \lambda_{k}^{m}\right\}$, where $\lambda_{k}^{i}(i=1,2, \ldots, m)$ are independent Bernoulli random variables. It is assumed that $\lambda_{k}^{i}$ has the probability density function $p\left(\lambda_{k}^{i}\right)$ on the interval $[0,1]$ with mean $\mu_{k}^{i}$ and covariance $\left(\sigma_{k}^{i}\right)^{2}$. The process noise $\mathbf{w}_{k}$ and the measurement
noise $\mathbf{v}_{k}$ are correlated zero mean Gaussian white noises, which satisfies

$$
\begin{gather*}
E\left(\mathbf{w}_{k}\right)=0, \quad \operatorname{cov}\left(\mathbf{w}_{k}, \mathbf{w}_{j}^{T}\right)=\mathbf{Q}_{k} \delta_{k j} \\
E\left(\mathbf{v}_{k}\right)=0, \quad \operatorname{cov}\left(\mathbf{v}_{k}, \mathbf{v}_{j}^{T}\right)=\mathbf{R}_{k} \delta_{k j}  \tag{3}\\
\operatorname{cov}\left(\mathbf{w}_{k}, \mathbf{v}_{j}^{T}\right)=\mathbf{S}_{k} \delta_{k j} .
\end{gather*}
$$

The deterministic nonlinear functions $f\left(\mathbf{x}_{k}\right): \mathbf{R}^{n} \rightarrow \mathbf{R}^{n}$ and $h\left(\mathbf{x}_{k}\right): \mathbf{R}^{n} \rightarrow \mathbf{R}^{m}$ are known. According to the known measurement equation, we employ the assumption in [36] as the following form:

$$
\begin{equation*}
\left\|h\left(\mathbf{x}_{k}\right)\right\| \leq a_{1}\left\|\mathbf{x}_{k}\right\|+a_{2} \tag{4}
\end{equation*}
$$

where $a_{1}$ and $a_{2}$ are the nonnegative scalars.
Because of existing correlated additive noises, for system (1)-(2), a recursive filter with correlated noises to be designed is constitutive of the following two steps including the state prediction and correction:

State prediction:

$$
\begin{gather*}
\widehat{\mathbf{x}}_{k+1 \mid k-1}=f\left(\widehat{\mathbf{x}}_{k \mid k-1}\right)  \tag{5}\\
\widehat{\mathbf{x}}_{k+1 \mid k}=\widehat{\mathbf{x}}_{k+1 \mid k-1}+\mathbf{L}_{k}\left[\mathbf{y}_{k}-\overline{\mathbf{\Sigma}}_{k} h\left(\widehat{\mathbf{x}}_{k \mid k-1}\right)\right] \tag{6}
\end{gather*}
$$

State correction:

$$
\begin{equation*}
\widehat{\mathbf{x}}_{k+1 \mid k+1}=\widehat{\mathbf{x}}_{k+1 \mid k}+\mathbf{K}_{k+1}\left[\mathbf{y}_{k+1}-\overline{\boldsymbol{\Sigma}}_{k+1} h\left(\widehat{\mathbf{x}}_{k+1 \mid k}\right)\right] \tag{7}
\end{equation*}
$$

where $\overline{\boldsymbol{\Sigma}}_{k}=E\left(\boldsymbol{\Sigma}_{k}\right)=\operatorname{diag}\left(\mu_{k}^{1}, \mu_{k}^{2}, \ldots, \mu_{k}^{m}\right) ; \widehat{\mathbf{x}}_{k \mid k-1}$ is the onestep state prediction at time $k-1$ with $\widehat{\mathbf{x}}_{0 \mid-1}=\widehat{\mathbf{x}}_{0 \mid 0} ; \widehat{\mathbf{x}}_{k+1 \mid k-1}$ is the two-step state prediction at time $k-1 ; \mathbf{L}_{k}$ and $\mathbf{K}_{k+1}$ are the gain parameters to be determined; $\widehat{\mathbf{x}}_{k+1 \mid k+1}$ is the state estimation at time $k+1$.

The aim of the paper is to design a recursive filter for the structures (5)-(7), which make the filtering prediction and estimation covariance have upper bounds in the presence of multiplicative noises and packet losses. Suppose that two positive definite matrices $\boldsymbol{\Xi}_{k+1 \mid k}$ and $\boldsymbol{\Xi}_{k+1 \mid k+1}$ satisfy

$$
\begin{gather*}
E\left[\left(\mathbf{x}_{k+1}-\widehat{\mathbf{x}}_{k+1 \mid k}\right)\left(\mathbf{x}_{k+1}-\widehat{\mathbf{x}}_{k+1 \mid k}\right)^{T}\right] \leq \boldsymbol{\Xi}_{k+1 \mid k}  \tag{8}\\
E\left[\left(\mathbf{x}_{k+1}-\widehat{\mathbf{x}}_{k+1 \mid k+1}\right)\left(\mathbf{x}_{k+1}-\widehat{\mathbf{x}}_{k+1 \mid k+1}\right)^{T}\right] \leq \boldsymbol{\Xi}_{k+1 \mid k+1}
\end{gather*}
$$

The addressed filtering problem is that the designed filter parameters $\mathbf{L}_{k}$ and $\mathbf{K}_{k+1}$ in (5)-(7) should minimize the upper bounds $\boldsymbol{\Xi}_{k+1 \mid k}$ and $\boldsymbol{\Xi}_{k+1 \mid k+1}$.

Remark 1. In engineering applications, multiplicative noises constantly existing in the systems depend on the real state value, which results in the unknown noise variance. As discussed in [26], it should be seen as a model uncertainty. Moreover, the definition (3) shows that the process noise is correlated with the measurement noise. This requires employing a new state prediction step in (5)-(6), which is different from the state prediction of the recursive filter form
in $[36,37]$. Meanwhile, the unknown prediction gain $\mathbf{L}_{k}$ does not exist in the literature [36,37], which will lead to the different estimation results. Subsequently, the packet losses are described by utilizing the diagonal matrix $\boldsymbol{\Sigma}_{k}$ in (2), which indicates that the different sensors have different failure rates. Since multiplicative noises, correlated additive noises, and packet losses are taken into account, the system (1)-(2) is more generalized to describe the realistic situations in engineering.

## 3. Design of Robust Recursive Filter

3.1. The Error Covariance Matrix. Denote the two-step prediction error as $\widetilde{\mathbf{x}}_{k+1 \mid k-1}=\mathbf{x}_{k+1}-\widehat{\mathbf{x}}_{k+1 \mid k-1}$ and the one-step prediction error as $\widetilde{\mathbf{x}}_{k+1 \mid k}=\mathbf{x}_{k+1}-\widehat{\mathbf{x}}_{k+1 \mid k}$. From (1), (2), (5), and (6), they can be calculated as

$$
\begin{align*}
& \widetilde{\mathbf{x}}_{k+1 \mid k-1}=f\left(\mathbf{x}_{k}\right)-f\left(\widehat{\mathbf{x}}_{k \mid k-1}\right) \sum_{i=1}^{q} \mathbf{A}_{i k}^{s} \boldsymbol{\eta}_{i k} \mathbf{x}_{k}+\mathbf{w}_{k},  \tag{9}\\
& \widetilde{\mathbf{x}}_{k+1 \mid k}= \\
& \quad \widetilde{\mathbf{x}}_{k+1 \mid k-1}  \tag{10}\\
& \quad-\mathbf{L}_{k}\left[\boldsymbol{\Sigma}_{k} h\left(\mathbf{x}_{k}\right)+\sum_{i=1}^{r} \mathbf{C}_{i k}^{s} \boldsymbol{\xi}_{i k} \mathbf{x}_{k}+\mathbf{v}_{k}-\overline{\mathbf{\Sigma}}_{k} h\left(\widehat{\mathbf{x}}_{k \mid k-1}\right)\right] .
\end{align*}
$$

The nonlinear functions $f\left(\mathbf{x}_{k}\right)$ and $h\left(\mathbf{x}_{k}\right)$ can be linearized by utilizing the Taylor series expansion around $\widehat{\mathbf{x}}_{k \mid k-1}$ :

$$
\begin{align*}
& f\left(\mathbf{x}_{k}\right)=f\left(\widehat{\mathbf{x}}_{k \mid k-1}\right)+\mathbf{A}_{k} \widetilde{\mathbf{x}}_{k \mid k-1}+o\left(\left|\widetilde{\mathbf{x}}_{k \mid k-1}\right|\right),  \tag{11}\\
& h\left(\mathbf{x}_{k}\right)=h\left(\widehat{\mathbf{x}}_{k \mid k-1}\right)+\mathbf{C}_{k} \widetilde{\mathbf{x}}_{k \mid k-1}+\widehat{o}\left(\left|\widetilde{\mathbf{x}}_{k \mid k-1}\right|\right), \tag{12}
\end{align*}
$$

where $\mathbf{A}_{k}=\partial f\left(\mathbf{x}_{k}\right) /\left.\partial \mathbf{x}_{k}\right|_{\mathbf{x}_{k}=\widehat{\mathbf{x}}_{k \mid k-1}} ; \mathbf{C}_{k}=\partial h\left(\mathbf{x}_{k}\right) /\left.\partial \mathbf{x}_{k}\right|_{\mathbf{x}_{k}=\widehat{\mathbf{x}}_{k \mid k-1}} ;$ $o\left(\left|\widetilde{\mathbf{x}}_{k \mid k-1}\right|\right)$ and $\widehat{o}\left(\left|\widetilde{\mathbf{x}}_{k \mid k-1}\right|\right)$ represent the high-order terms of the Taylor series expansion. According to the literature [26], $o\left(\left|\widetilde{x}_{k \mid k-1}\right|\right)$ and $\widehat{o}\left(\left|\widetilde{\mathbf{x}}_{k \mid k-1}\right|\right)$ can be expressed as

$$
\begin{align*}
& o\left(\left|\widetilde{\mathbf{x}}_{k \mid k-1}\right|\right)=\mathbf{B}_{k} \boldsymbol{\beta}_{k} \mathbf{E}_{k} \widetilde{\mathbf{x}}_{k \mid k-1} \\
& \widehat{o}\left(\left|\widetilde{\mathbf{x}}_{k \mid k-1}\right|\right)=\mathbf{D}_{k} \boldsymbol{\alpha}_{k} \mathbf{E}_{k} \widetilde{\mathbf{x}}_{k \mid k-1} \tag{13}
\end{align*}
$$

where $\mathbf{B}_{k} \in \mathbf{R}^{n \times n}$ and $\mathbf{D}_{k} \in \mathbf{R}^{m \times n}$ are known scaling matrices, $\mathbf{E}_{k} \in \mathbf{R}^{n \times n}$ is a known tuning matrix, and $\boldsymbol{\beta}_{k} \in \mathbf{R}^{n \times n}$ and $\boldsymbol{\alpha}_{k} \in \mathbf{R}^{n \times n}$ are unknown time-varying matrices accounting for the linearization errors of the system model that satisfies

$$
\begin{equation*}
\boldsymbol{\beta}_{k} \boldsymbol{\beta}_{k}^{T} \leq \mathbf{I}, \quad \boldsymbol{\alpha}_{k} \boldsymbol{\alpha}_{k}^{T} \leq \mathbf{I} \tag{14}
\end{equation*}
$$

According to (9), (11), and (13), the two-step prediction error can be written as

$$
\begin{equation*}
\widetilde{\mathbf{x}}_{k+1 \mid k-1}=\left(\mathbf{A}_{k}+\mathbf{B}_{k} \boldsymbol{\beta}_{k} \mathbf{E}_{k}\right) \widetilde{\mathbf{x}}_{k \mid k-1}+\sum_{i=1}^{q} \mathbf{A}_{i k}^{s} \boldsymbol{\eta}_{i k} \mathbf{x}_{k}+\mathbf{w}_{k} . \tag{15}
\end{equation*}
$$

Substituting (12), (13), and (15) into (9), the one-step prediction error can be determined as

$$
\begin{align*}
\widetilde{\mathbf{x}}_{k+1 \mid k}= & \left(\mathbf{A}_{k}+\mathbf{B}_{k} \boldsymbol{\beta}_{k} \mathbf{E}_{k}\right) \widetilde{\mathbf{x}}_{k \mid k-1}+\sum_{i=1}^{q} \mathbf{A}_{i k}^{s} \boldsymbol{\eta}_{i k} \mathbf{x}_{k}+\mathbf{w}_{k} \\
- & \mathbf{L}_{k}\left[\boldsymbol{\Sigma}_{k} h\left(\mathbf{x}_{k}\right)+\sum_{i=1}^{r} \mathbf{C}_{i k}^{s} \boldsymbol{\xi}_{i k} \mathbf{x}_{k}+\mathbf{v}_{k}\right. \\
& \left.-\overline{\boldsymbol{\Sigma}}_{k}\left(h\left(\mathbf{x}_{k}\right)-\left(\mathbf{C}_{k}+\mathbf{D}_{k} \boldsymbol{\alpha}_{k} \mathbf{E}_{k}\right) \widetilde{\mathbf{x}}_{k \mid k-1}\right)\right] \\
= & {\left[\left(\mathbf{A}_{k}+\mathbf{L}_{k} \overline{\boldsymbol{\Sigma}}_{k} \mathbf{C}_{k}\right)+\left(\mathbf{B}_{k} \boldsymbol{\beta}_{k}+\mathbf{L}_{k} \overline{\boldsymbol{\Sigma}}_{k} \mathbf{D}_{k} \boldsymbol{\alpha}_{k}\right) \mathbf{E}_{k}\right] \widetilde{\mathbf{x}}_{k \mid k-1} } \\
& -\mathbf{L}_{k}\left(\boldsymbol{\Sigma}_{k}-\overline{\boldsymbol{\Sigma}}_{k}\right) h\left(\mathbf{x}_{k}\right)-\mathbf{L}_{k}\left(\sum_{i=1}^{r} \mathbf{C}_{i k}^{s} \boldsymbol{\xi}_{i k} \mathbf{x}_{k}+\mathbf{v}_{k}\right) \\
& +\sum_{i=1}^{q} \mathbf{A}_{i k}^{s} \boldsymbol{\eta}_{i k} \mathbf{x}_{k}+\mathbf{w}_{k} \\
= & {\left[\left(\mathbf{A}_{k}+\mathbf{L}_{k} \overline{\boldsymbol{\Sigma}}_{k} \mathbf{C}_{k}\right)+\mathbf{H}_{k} \mathbf{F}_{k} \mathbf{J}_{k} \mathbf{E}_{k}\right] \widetilde{\mathbf{x}}_{k \mid k-1} } \\
& -\mathbf{L}_{k}\left(\mathbf{\Sigma}_{k}-\overline{\boldsymbol{\Sigma}}_{k}\right) h\left(\mathbf{x}_{k}\right)-\mathbf{L}_{k}\left(\sum_{i=1}^{r} \mathbf{C}_{i k}^{s} \boldsymbol{\xi}_{i k} \mathbf{x}_{k}+\mathbf{v}_{k}\right) \\
& +\sum_{i=1}^{q} \mathbf{A}_{i k}^{s} \boldsymbol{\eta}_{i k} \mathbf{x}_{k}+\mathbf{w}_{k}, \tag{16}
\end{align*}
$$

where it is assumed that $\mathbf{H}_{k}=\left[\begin{array}{ll}\mathbf{B}_{k} & \mathbf{L}_{k} \overline{\boldsymbol{\Sigma}}_{k} \mathbf{D}_{k}\end{array}\right], \mathbf{F}_{k}=\left[\begin{array}{cc}\boldsymbol{\beta}_{k} & \mathbf{0}_{n \times n} \\ \mathbf{0}_{n \times n} & \boldsymbol{\alpha}_{k}\end{array}\right]$, and $\mathbf{J}_{k}=\left[\begin{array}{l}\mathbf{I}_{n \times n} \\ \mathbf{I}_{n \times n}\end{array}\right]$ and from (14) we have $\mathbf{F}_{k} \mathbf{F}_{k}^{T} \leq \mathbf{I}$.

The one-step prediction error covariance can be obtained as

$$
\begin{align*}
\mathbf{P}_{k+1 \mid k}= & E\left(\widetilde{\mathbf{x}}_{k+1 \mid k} \widetilde{\mathbf{x}}_{k+1 \mid k}^{T}\right) \\
= & {\left[\left(\mathbf{A}_{k}+\mathbf{L}_{k} \overline{\mathbf{\Sigma}}_{k} \mathbf{C}_{k}\right)+\mathbf{H}_{k} \mathbf{F}_{k} \mathbf{J}_{k} \mathbf{E}_{k}\right] \mathbf{P}_{k \mid k-1} } \\
& \times\left[\left(\mathbf{A}_{k}+\mathbf{L}_{k} \overline{\mathbf{\Sigma}}_{k} \mathbf{C}_{k}\right)+\mathbf{H}_{k} \mathbf{F}_{k} \mathbf{J}_{k} \mathbf{E}_{k}\right]^{T}-\mathbf{S}_{k} \mathbf{L}_{k}^{T}-\mathbf{L}_{k} \mathbf{S}_{k}^{T} \\
& +\mathbf{L}_{k}\left[\breve{\mathbf{\Sigma}}_{k} \circ E\left(h\left(\mathbf{x}_{k}\right) h^{T}\left(\mathbf{x}_{k}\right)\right)\right. \\
& \left.+\sum_{i=1}^{r} \mathbf{C}_{i k}^{s} E\left(\mathbf{x}_{k} \mathbf{x}_{k}^{T}\right)\left(\mathbf{C}_{i k}^{s}\right)^{T}+\mathbf{R}_{k}\right] \mathbf{L}_{k}^{T} \\
& +\sum_{i=1}^{q} \mathbf{A}_{i k}^{s} E\left(\mathbf{x}_{k} \mathbf{x}_{k}^{T}\right)\left(\mathbf{A}_{i k}^{s}\right)^{T}+\mathbf{Q}_{k}, \tag{17}
\end{align*}
$$

where $\breve{\Sigma}_{k}=\operatorname{diag}\left\{\left(\sigma_{k}^{1}\right)^{2},\left(\sigma_{k}^{2}\right)^{2}, \ldots,\left(\sigma_{k}^{m}\right)^{2}\right\}$.
Denote the estimation error as

$$
\begin{equation*}
\widetilde{\mathbf{x}}_{k+1 \mid k+1}=\mathbf{x}_{k+1}-\widehat{\mathbf{x}}_{k+1 \mid k+1} . \tag{18}
\end{equation*}
$$

From (1), (2), and (7), it can be rewritten as

$$
\begin{align*}
& \widetilde{\mathbf{x}}_{k+1 \mid k+1} \\
&= \widetilde{\mathbf{x}}_{k+1 \mid k}-\mathbf{K}_{k+1}\left[\mathbf{y}_{k+1}-\overline{\boldsymbol{\Sigma}}_{k+1} h\left(\widehat{\mathbf{x}}_{k+1 \mid k}\right)\right] \\
&= \widetilde{\mathbf{x}}_{k+1 \mid k} \\
&-\mathbf{K}_{k+1}\left[\boldsymbol{\Sigma}_{k+1} h\left(\mathbf{x}_{k+1}\right)+\sum_{i=1}^{r} \mathbf{C}_{i k+1}^{s} \boldsymbol{\xi}_{i k+1} \mathbf{x}_{k+1}+\mathbf{v}_{k+1}\right. \\
&\left.\quad-\overline{\mathbf{\Sigma}}_{k+1} h\left(\widehat{\mathbf{x}}_{k+1 \mid k}\right)\right]  \tag{19}\\
&=\left(\mathbf{I}-\mathbf{K}_{k+1} \overline{\boldsymbol{\Sigma}}_{k+1} \overline{\mathbf{C}}_{k+1}-\mathbf{K}_{k+1} \overline{\boldsymbol{\Sigma}}_{k+1} \overline{\mathbf{D}}_{k+1} \overline{\boldsymbol{\alpha}}_{k+1} \overline{\mathbf{E}}_{k+1}\right) \widetilde{\mathbf{x}}_{k+1 \mid k} \\
& \quad-\mathbf{K}_{k+1}\left(\sum_{i=1}^{r} \mathbf{C}_{i k+1}^{s} \boldsymbol{\xi}_{i k+1} \mathbf{x}_{k+1}+\mathbf{v}_{k+1}\right) \\
&-\mathbf{K}_{k+1}\left(\mathbf{\Sigma}_{k+1}-\overline{\boldsymbol{\Sigma}}_{k+1}\right) h\left(\mathbf{x}_{k+1}\right),
\end{align*}
$$

where $\overline{\mathbf{C}}_{k+1}=\partial h\left(\mathbf{x}_{k}\right) /\left.\partial \mathbf{x}_{k}\right|_{\mathbf{x}_{k}=\widehat{\mathbf{x}}_{k+1 \mid k}} ; \overline{\mathbf{D}}_{k+1}$ is a known problemdependent scaling matrix; $\overline{\mathbf{E}}_{k+1}$ is a known tuning matrix; $\overline{\boldsymbol{\alpha}}_{k+1}$ is an unknown time-varying matrix that satisfies

$$
\begin{equation*}
\overline{\boldsymbol{\alpha}}_{k+1} \overline{\boldsymbol{\alpha}}_{k+1}^{T} \leq \mathbf{I} . \tag{20}
\end{equation*}
$$

Subsequently, in the light of (19), the filtering error covariance can be expressed as

$$
\begin{align*}
& \mathbf{P}_{k+1 \mid k+1} \\
& =E\left(\widetilde{\mathbf{x}}_{k+1 \mid k+1} \widetilde{\mathbf{x}}_{k+1 \mid k+1}^{T}\right) \\
& =\left(\mathbf{I}-\mathbf{K}_{k+1} \overline{\boldsymbol{\Sigma}}_{k+1} \overline{\mathbf{C}}_{k+1}-\mathbf{K}_{k+1} \overline{\boldsymbol{\Sigma}}_{k+1} \overline{\mathbf{D}}_{k+1} \overline{\boldsymbol{\alpha}}_{k+1} \overline{\mathbf{E}}_{k+1}\right) \\
& \quad \times \mathbf{P}_{k+1 \mid k}\left(\mathbf{I}-\mathbf{K}_{k+1} \overline{\mathbf{\Sigma}}_{k+1} \overline{\mathbf{C}}_{k+1}-\mathbf{K}_{k+1} \overline{\mathbf{\Sigma}}_{k+1} \overline{\mathbf{D}}_{k+1} \overline{\boldsymbol{\alpha}}_{k+1} \overline{\mathbf{E}}_{k+1}\right)^{T} \\
& \quad+\mathbf{K}_{k+1}\left[\breve{\mathbf{\Sigma}}_{k+1} \circ E\left(h\left(\mathbf{x}_{k+1}\right) h^{T}\left(\mathbf{x}_{k+1}\right)\right)\right. \\
& \quad+\sum_{i=1}^{r} \mathbf{C}_{i k+1}^{s} E\left(\mathbf{x}_{k+1} \mathbf{x}_{k+1}^{T}\right)\left(\mathbf{C}_{i k+1}^{s}\right)^{T} \\
& \left.\quad+\mathbf{R}_{k+1}\right] \mathbf{K}_{k+1}^{T} . \tag{21}
\end{align*}
$$

Remark 2. Since there are the high-order errors, the matrices $\boldsymbol{\beta}_{k}, \boldsymbol{\alpha}_{k}$, and $\overline{\boldsymbol{\alpha}}_{k+1}$ are unknown, which makes the fact that the prediction covariance $\mathbf{P}_{k+1 \mid k}$ and the filtering error covariance $\mathbf{P}_{k+1 \mid k+1}$ from (17) and (21) cannot be computed directly. In order to complete the design of the filter, an effective way is to calculate the upper bounds for the $\mathbf{P}_{k+1 \mid k}$ and $\mathbf{P}_{k+1 \mid k+1}$ and then design the prediction gain $\mathbf{L}_{k}$ and the filtering gain $\mathbf{K}_{k+1}$ to minimize the upper bounds. Due to the influence correlated noises and unknown prediction gain $\mathbf{L}_{k}$, there is a striking contrast between the prediction covariance $\mathbf{P}_{k+1 \mid k}$ in this paper and the counterpart in the literature [36, 37].
3.2. The Robust Recursive Filter Design. To develop the robust recursive filter, the following lemmas are given.

Lemma 3 (see [40]). Let $\mathbf{A}=\left[a_{i j}\right]_{n \times n}$ be a real matrix and let $\mathbf{B}=\operatorname{diag}\left(b_{1}, b_{2}, \ldots, b_{n}\right)$ be a diagonal random matrix. Then

$$
E\left\{\mathbf{B A B}^{T}\right\}=\left[\begin{array}{cccc}
E\left\{b_{1}^{2}\right\} & E\left\{b_{1} b_{2}\right\} & \cdots & E\left\{b_{1} b_{n}\right\}  \tag{22}\\
E\left\{b_{2} b_{1}\right\} & E\left\{b_{2}^{2}\right\} & \cdots & E\left\{b_{2} b_{n}\right\} \\
\vdots & \vdots & \ddots & \vdots \\
E\left\{b_{n} b_{1}\right\} & E\left\{b_{n} b_{2}\right\} & \cdots & E\left\{b_{n}^{2}\right\}
\end{array}\right] \circ \mathbf{A},
$$

where $\circ$ is the Hadamard product.
Lemma 4 (see [41]). Given matrices A, H, E, and F with compatible dimensions such that $\mathbf{F F}^{T} \leq \mathbf{I}$, let $\mathbf{X}$ be a symmetric positive definite matrix and let $\gamma$ be an arbitrary positive constant such that

$$
\begin{equation*}
\gamma^{-1} \mathbf{I}-\mathbf{E X E}^{T}>0 \tag{23}
\end{equation*}
$$

Then the following matrix inequality holds:

$$
\begin{align*}
(\mathbf{A}+\mathbf{H F E}) \mathbf{X}(\mathbf{A}+\mathbf{H F E})^{T} \leq & \mathbf{A}\left(\mathbf{X}^{-1}-\gamma \mathbf{E}^{T} \mathbf{E}\right)^{-1} \mathbf{A}^{T} \\
& +\gamma^{-1} \mathbf{H} \mathbf{H}^{T} \tag{24}
\end{align*}
$$

Lemma 5 (see [42]). For $0 \leq k \leq n$, suppose that $\mathbf{X}=\mathbf{X}^{T}>0$, $\mathbf{e}_{k}(\mathbf{X})=\mathbf{e}_{k}^{T}(\mathbf{X}) \in \mathbf{R}^{n \times n}$, and $\mathbf{g}_{k}(\mathbf{X})=\mathbf{g}_{k}^{T}(\mathbf{X}) \in \mathbf{R}^{n \times n}$. If there exists $\mathbf{Y} \geq \mathbf{X}$ such that

$$
\begin{align*}
& \mathbf{e}_{k}(\mathbf{Y}) \geq \mathbf{e}_{k}(\mathbf{X}),  \tag{25}\\
& \mathbf{g}_{k}(\mathbf{Y})>\mathbf{e}_{k}(\mathbf{Y}),
\end{align*}
$$

then the solutions $\mathbf{M}_{k}$ and $\mathbf{N}_{k}$ to the following difference equations,

$$
\begin{gather*}
\mathbf{M}_{k+1}=\mathbf{e}_{k}\left(\mathbf{M}_{k}\right), \quad \mathbf{N}_{k+1}=\mathbf{g}_{k}\left(\mathbf{N}_{k}\right),  \tag{26}\\
\mathbf{M}_{0}=\mathbf{N}_{0}>0
\end{gather*}
$$

satisfy $\mathbf{M}_{k} \leq \mathbf{N}_{k}$.
According to these lemmas, the following theorem is given to obtain the main results of the robust recursive filter.

Theorem 6. Consider the covariance matrices of the one-step prediction errorand the filtering error in (17) and (21). Assume that the conditions shown in (14) and (20) come into existence.

Let $\lambda_{1}, \lambda_{2}, \varepsilon_{1}$, and $\varepsilon_{2}$ be positive scalars. If the following two discrete-time Riccati difference equations,

$$
\begin{align*}
& \mathbf{\Xi}_{k+1 \mid k} \\
& \begin{aligned}
&=\left(\mathbf{A}_{k}+\mathbf{L}_{k} \overline{\boldsymbol{\Sigma}}_{k} \mathbf{C}_{k}\right)\left(\boldsymbol{\Xi}_{k \mid k-1}^{-1}-2 \lambda_{1} \mathbf{E}_{k}^{T} \mathbf{E}_{k}\right)^{-1}\left(\mathbf{A}_{k}+\mathbf{L}_{k} \overline{\boldsymbol{\Sigma}}_{k} \mathbf{C}_{k}\right)^{T} \\
&+\lambda_{1}^{-1}\left(\mathbf{B}_{k} \mathbf{B}_{k}^{T}+\mathbf{L}_{k} \overline{\boldsymbol{\Sigma}}_{k} \mathbf{D}_{k} \mathbf{D}_{k}^{T} \overline{\boldsymbol{\Sigma}}_{k}^{T} \mathbf{L}_{k}^{T}\right)+\mathbf{Q}_{k}-\mathbf{S}_{k} \mathbf{L}_{k}^{T}-\mathbf{L}_{k} \mathbf{S}_{k}^{T} \\
&+\mathbf{L}_{k}\left\{\breve{\mathbf{\Sigma}}_{k} \circ\left[2\left(a_{1}^{2} \operatorname{tr}\left(\boldsymbol{\Omega}_{k \mid k-1}\right)+a_{2}^{2}\right)\right] \mathbf{I}\right. \\
&\left.+\sum_{i=1}^{r} \mathbf{C}_{i k}^{s} \mathbf{\Omega}_{k \mid k-1}\left(\mathbf{C}_{i k}^{s}\right)^{T}+\mathbf{R}_{k}\right\} \mathbf{L}_{k}^{T}+\sum_{i=1}^{q} \mathbf{A}_{i k}^{s} \mathbf{\Omega}_{k \mid k-1}\left(\mathbf{A}_{i k}^{s}\right)^{T}, \\
& \mathbf{\Xi}_{k+1 \mid k+1} \\
&=\left(\mathbf{I}-\mathbf{K}_{k+1} \overline{\boldsymbol{\Sigma}}_{k+1} \overline{\mathbf{C}}_{k+1}\right)\left(\mathbf{\Xi}_{k+1 \mid k}^{-1}-\lambda_{2} \overline{\mathbf{E}}_{k+1}^{T} \overline{\mathbf{E}}_{k+1}\right)^{-1} \\
& \quad \times\left(\mathbf{I}-\mathbf{K}_{k+1} \overline{\boldsymbol{\Sigma}}_{k+1} \overline{\mathbf{C}}_{k+1}\right)^{T}+\lambda_{2}^{-1} \mathbf{K}_{k+1} \overline{\boldsymbol{\Sigma}}_{k+1} \overline{\mathbf{D}}_{k+1} \overline{\mathbf{D}}_{k+1}^{T} \overline{\boldsymbol{\Sigma}}_{k+1}^{T} \mathbf{K}_{k+1}^{T} \\
&+\mathbf{K}_{k+1}\left\{\breve{\boldsymbol{\Sigma}}_{k+1} \circ\left[2\left(a_{1}^{2} \operatorname{tr}\left(\boldsymbol{\Delta}_{k+1 \mid k}\right)+a_{2}^{2}\right)\right] \mathbf{I}\right. \\
&\left.\quad+\sum_{i=1}^{r} \mathbf{C}_{i k+1}^{s} \boldsymbol{\Delta}_{k+1 \mid k}\left(\mathbf{C}_{i k+1}^{s}\right)^{T}+\mathbf{R}_{k+1}\right\} \mathbf{K}_{k+1}^{T}
\end{aligned}
\end{align*}
$$

with initial covariance $\boldsymbol{\Xi}_{0 \mid-1}=\boldsymbol{\Xi}_{0 \mid 0}>0$ have positive definite solution, such that for $0 \leq k \leq N$

$$
\begin{gather*}
\lambda_{1}^{-1} \mathbf{I}-\mathbf{J}_{k} \mathbf{E}_{k} \boldsymbol{\Xi}_{k \mid k-1}\left(\mathbf{J}_{k} \mathbf{E}_{k}\right)^{T}>0 \\
\lambda_{2}^{-1} \mathbf{I}-\overline{\mathbf{E}}_{k+1} \boldsymbol{\Xi}_{k+1 \mid k} \overline{\mathbf{E}}_{k+1}^{T}>0 \tag{29}
\end{gather*}
$$

where

$$
\begin{align*}
& \boldsymbol{\Omega}_{k \mid k-1}=\left(1+\varepsilon_{1}\right) \boldsymbol{\Xi}_{k \mid k-1}+\left(1+\varepsilon_{1}^{-1}\right) \widehat{\mathbf{x}}_{k \mid k-1} \widehat{\mathbf{x}}_{k \mid k-1}^{T}  \tag{30}\\
& \boldsymbol{\Delta}_{k+1 \mid k}=\left(1+\varepsilon_{2}\right) \boldsymbol{\Xi}_{k+1 \mid k}+\left(1+\varepsilon_{2}^{-1}\right) \widehat{\mathbf{x}}_{k+1 \mid k} \widehat{\mathbf{x}}_{k+1 \mid k}^{T}
\end{align*}
$$

then the prediction gain $\mathbf{L}_{k}$ and the filtering gain $\mathbf{K}_{k+1}$ are given by

$$
\begin{aligned}
\mathbf{L}_{k}= & {\left[\mathbf{S}_{k}-\mathbf{A}_{k}\left(\Xi_{k \mid k-1}^{-1}-2 \lambda_{1} \mathbf{E}_{k}^{T} \mathbf{E}_{k}\right)^{-1} \mathbf{C}_{k}^{T} \overline{\boldsymbol{\Sigma}}_{k}\right] } \\
& \times\left\{\overline{\boldsymbol{\Sigma}}_{k} \mathbf{C}_{k}\left(\boldsymbol{\Xi}_{k \mid k-1}^{-1}-2 \lambda_{1} \mathbf{E}_{k}^{T} \mathbf{E}_{k}\right)^{-1} \mathbf{C}_{k}^{T} \overline{\boldsymbol{\Sigma}}_{k}\right. \\
& +\lambda_{1}^{-1} \overline{\boldsymbol{\Sigma}}_{k} \mathbf{D}_{k} \mathbf{D}_{k}^{T} \overline{\boldsymbol{\Sigma}}_{k}^{T}
\end{aligned}
$$

$$
\begin{gather*}
+\breve{\mathbf{\Sigma}}_{k} \circ\left[2\left(a_{1}^{2} \operatorname{tr}\left(\boldsymbol{\Omega}_{k \mid k-1}\right)+a_{2}^{2}\right)\right] \mathbf{I} \\
 \tag{31}\\
\left.+\sum_{i=1}^{r} \mathbf{C}_{i k}^{s} \boldsymbol{\Omega}_{k \mid k-1}\left(\mathbf{C}_{i k}^{s}\right)^{T}+\mathbf{R}_{k}\right\}^{-1}, \\
\mathbf{K}_{k+1}=\left(\boldsymbol{\Xi}_{k+1 \mid k}^{-1}-\lambda_{2} \overline{\mathbf{E}}_{k+1}^{T} \overline{\mathbf{E}}_{k+1}\right)^{-1} \overline{\mathbf{C}}_{k+1}^{T} \overline{\boldsymbol{\Sigma}}_{k+1} \\
\times\left\{\begin{array}{c}
\boldsymbol{\Sigma} \quad \overline{\mathbf{C}}_{k+1}\left(\mathbf{\Xi}_{k+1 \mid k}^{-1}-\lambda_{2} \overline{\mathbf{E}}_{k+1}^{T} \overline{\mathbf{E}}_{k+1}\right)^{-1} \overline{\mathbf{C}}_{k+1}^{T} \overline{\boldsymbol{\Sigma}}_{k+1} \\
\\
\\
+\lambda_{2+1}^{-1} \overline{\boldsymbol{\Sigma}}_{k+1} \overline{\mathbf{D}}_{k+1} \overline{\mathbf{D}}_{k+1}^{T} \overline{\mathbf{\Sigma}}_{k+1}^{T}+\sum_{i=1}^{r} \mathbf{C}_{i k+1}^{s} \mathbf{\Delta}_{k+1 \mid k}\left(\mathbf{C}_{i k+1}^{s}\right)^{T} \\
\\
\\
\left.+\breve{\mathbf{\Sigma}}_{k+1} \circ\left[2\left(a_{1}^{2} \operatorname{tr}\left(\Delta_{k+1 \mid k}\right)+a_{2}^{2}\right)\right] \mathbf{I}+\mathbf{R}_{k+1}\right\}^{-1}
\end{array}\right.
\end{gather*}
$$

and the matrices $\Xi_{k+1 \mid k}$ and $\Xi_{k+1 \mid k+1}$ are upper bounds for $\mathbf{P}_{k+1 \mid k}$ and $\mathbf{P}_{k+1 \mid k+1}$; that is,

$$
\begin{equation*}
\mathbf{P}_{k+1 \mid k} \leq \boldsymbol{\Xi}_{k+1 \mid k}, \quad \mathbf{P}_{k+1 \mid k+1} \leq \boldsymbol{\Xi}_{k+1 \mid k+1} \tag{33}
\end{equation*}
$$

Moreover, the prediction gain $\mathbf{L}_{k}$ given by (31) minimizes the upper bound $\boldsymbol{\Xi}_{k+1 \mid k}$ and the filtering gain $\mathbf{K}_{k+1}$ given by (32) minimizes the upper bound $\boldsymbol{\Xi}_{k+1 \mid k+1}$.

Proof. According to (17) and (21), the prediction covariance $\mathbf{P}_{k+1 \mid k}$ and the filtering error covariance $\mathbf{P}_{k+1 \mid k+1}$ can be expressed as the functions of $\mathbf{P}_{k \mid k-1}$ and $\mathbf{P}_{k+1 \mid k}$ :

$$
\begin{align*}
\mathbf{P}_{k+1 \mid k} & =\mathbf{P}_{k+1 \mid k}\left(\mathbf{P}_{k \mid k-1}\right) \\
\mathbf{P}_{k+1 \mid k+1} & =\mathbf{P}_{k+1 \mid k+1}\left(\mathbf{P}_{k+1 \mid k}\right) . \tag{34}
\end{align*}
$$

Assume that $\varepsilon_{1}$ is a positive scalar. The matrix inequality can be obtained as

$$
\begin{equation*}
\widetilde{\mathbf{x}}_{k \mid k-1} \widehat{\mathbf{x}}_{k \mid k-1}^{T}+\widehat{\mathbf{x}}_{k \mid k-1} \widetilde{\mathbf{x}}_{k \mid k-1}^{T} \leq \varepsilon_{1} \widetilde{\mathbf{x}}_{k \mid k-1} \widetilde{\mathbf{x}}_{k \mid k-1}^{T}+\varepsilon_{1}^{-1} \widehat{\mathbf{x}}_{k \mid k-1} \widehat{\mathbf{x}}_{k \mid k-1}^{T} . \tag{35}
\end{equation*}
$$

From (35), we have

$$
\begin{align*}
E\left(\mathbf{x}_{k} \mathbf{x}_{k}^{T}\right) & \leq E\left[\left(\widetilde{\mathbf{x}}_{k \mid k-1}+\widehat{\mathbf{x}}_{k \mid k-1}\right)\left(\widetilde{\mathbf{x}}_{k \mid k-1}+\widehat{\mathbf{x}}_{k \mid k-1}\right)^{T}\right] \\
& \leq\left(1+\varepsilon_{1}\right) \mathbf{P}_{k \mid k-1}+\left(1+\varepsilon_{1}^{-1}\right) \widehat{\mathbf{x}}_{k \mid k-1} \widehat{\mathbf{x}}_{k \mid k-1}^{T}  \tag{36}\\
& =\Psi_{k \mid k-1} .
\end{align*}
$$

According to the literature [36], based on (4), we obtain

$$
\begin{align*}
E\left\{h\left(\mathbf{x}_{k}\right) h^{T}\left(\mathbf{x}_{k}\right)\right\} & \leq E\left\{\left\|h\left(\mathbf{x}_{k}\right)\right\|^{2}\right\} \mathbf{I} \leq E\left\{\left(a_{1}\left\|\mathbf{x}_{k}\right\|+a_{2}\right)^{2}\right\} \mathbf{I} \\
& \leq\left(2 a_{1}^{2} E\left\{\left\|\mathbf{x}_{k}\right\|^{2}\right\}+2 a_{2}^{2}\right) \mathbf{I} \\
& =2\left[a_{1}^{2} \operatorname{tr}\left(E\left(\mathbf{x}_{k} \mathbf{x}_{k}^{T}\right)\right)+a_{2}^{2}\right] \mathbf{I} . \tag{37}
\end{align*}
$$

Substituting (36) into (37), it can be rewritten as

$$
\begin{equation*}
E\left\{h\left(\mathbf{x}_{k}\right) h^{T}\left(\mathbf{x}_{k}\right)\right\} \leq 2\left[a_{1}^{2} \operatorname{tr}\left(\boldsymbol{\Psi}_{k \mid k-1}\right)+a_{2}^{2}\right] \mathbf{I} \tag{38}
\end{equation*}
$$

Furthermore, inserting (36) and (38) into (17), the onestep prediction error covariance can be rearranged as

$$
\begin{align*}
\mathbf{P}_{k+1 \mid k} \leq & {\left[\left(\mathbf{A}_{k}+\mathbf{L}_{k} \overline{\boldsymbol{\Sigma}}_{k} \mathbf{C}_{k}\right)+\mathbf{H}_{k} \mathbf{F}_{k} \mathbf{J}_{k} \mathbf{E}_{k}\right] \mathbf{P}_{k \mid k-1} } \\
& \times\left[\left(\mathbf{A}_{k}+\mathbf{L}_{k} \overline{\boldsymbol{\Sigma}}_{k} \mathbf{C}_{k}\right)+\mathbf{H}_{k} \mathbf{F}_{k} \mathbf{J}_{k} \mathbf{E}_{k}\right]^{T}-\mathbf{S}_{k} \mathbf{L}_{k}^{T}-\mathbf{L}_{k} \mathbf{S}_{k}^{T} \\
& +\mathbf{L}_{k}\left[\breve{\mathbf{\Sigma}}_{k} \circ 2\left[a_{1}^{2} \operatorname{tr}\left(\mathbf{\Psi}_{k \mid k-1}\right)+a_{2}^{2}\right] \mathbf{I}\right. \\
& \left.+\sum_{i=1}^{r} \mathbf{C}_{i k}^{s} \boldsymbol{\Psi}_{k \mid k-1}\left(\mathbf{C}_{i k}^{s}\right)^{T}+\mathbf{R}_{k}\right\} \mathbf{L}_{k}^{T} \\
& +\sum_{i=1}^{q} \mathbf{A}_{i k}^{s} \boldsymbol{\Psi}_{k \mid k-1}\left(\mathbf{A}_{i k}^{s}\right)^{T}+\mathbf{Q}_{k} \tag{39}
\end{align*}
$$

Similar to (36) and (38), let $\varepsilon_{2}$ be a positive scalar; we have

$$
\begin{align*}
E\left(\mathbf{x}_{k+1} \mathbf{x}_{k+1}^{T}\right) & \leq\left(1+\varepsilon_{2}\right) \mathbf{P}_{k+1 \mid k}+\left(1+\varepsilon_{2}^{-1}\right) \widehat{\mathbf{x}}_{k+1 \mid k} \widehat{\mathbf{x}}_{k+1 \mid k}^{T} \\
& =\boldsymbol{\Lambda}_{k+1 \mid k} \\
E\left\{h\left(\mathbf{x}_{k+1}\right)\right. & \left.h^{T}\left(\mathbf{x}_{k+1}\right)\right\} \leq 2\left[a_{1}^{2} \operatorname{tr}\left(\boldsymbol{\Lambda}_{k+1 \mid k}\right)+a_{2}^{2}\right] \mathbf{I} \tag{40}
\end{align*}
$$

Substituting (40) into (21), we have

$$
\begin{align*}
& \mathbf{P}_{k+1 \mid k+1} \\
& =\left(\mathbf{I}-\mathbf{K}_{k+1} \overline{\boldsymbol{\Sigma}}_{k+1} \overline{\mathbf{C}}_{k+1}-\mathbf{K}_{k+1} \overline{\boldsymbol{\Sigma}}_{k+1} \overline{\mathbf{D}}_{k+1} \overline{\boldsymbol{\alpha}}_{k+1} \overline{\mathbf{E}}_{k+1}\right) \mathbf{P}_{k+1 \mid k} \\
& \quad \times\left(\mathbf{I}-\mathbf{K}_{k+1} \overline{\boldsymbol{\Sigma}}_{k+1} \overline{\mathbf{C}}_{k+1}-\mathbf{K}_{k+1} \overline{\boldsymbol{\Sigma}}_{k+1} \overline{\mathbf{D}}_{k+1} \overline{\boldsymbol{\alpha}}_{k+1} \overline{\mathbf{E}}_{k+1}\right)^{T} \\
& \quad+\mathbf{K}_{k+1}\left\{\breve{\boldsymbol{\Sigma}}_{k+1} \circ\left[2\left(a_{1}^{2} \operatorname{tr}\left(\boldsymbol{\Lambda}_{k+1 \mid k}\right)+a_{2}^{2}\right)\right] \mathbf{I}\right. \\
& \left.\quad+\sum_{i=1}^{r} \mathbf{C}_{i k+1}^{s} \boldsymbol{\Lambda}_{k+1 \mid k}\left(\mathbf{C}_{i k+1}^{s}\right)^{T}+\mathbf{R}_{k+1}\right\} \mathbf{K}_{k+1}^{T} . \tag{41}
\end{align*}
$$

From (27) and (28), $\boldsymbol{\Xi}_{k+1 \mid k}$ and $\boldsymbol{\Xi}_{k+1 \mid k+1}$ can be rewritten as the functions of $\boldsymbol{\Xi}_{k \mid k-1}$ and $\Xi_{k+1 \mid k}$ :

$$
\begin{align*}
\Xi_{k+1 \mid k} & =\Xi_{k+1 \mid k}\left(\Xi_{k \mid k-1}\right) \\
\Xi_{k+1 \mid k+1} & =\Xi_{k+1 \mid k+1}\left(\Xi_{k+1 \mid k}\right) . \tag{42}
\end{align*}
$$

Assume that there exist $\lambda_{1}>0$ and $\lambda_{2}>0$. Let the matrices $\mathbf{E}_{k}$ and $\overline{\mathbf{E}}_{k+1}$ satisfy the inequalities

$$
\begin{align*}
& \lambda_{1}^{-1} \mathbf{I}-\mathbf{J}_{k} \mathbf{E}_{k} \mathbf{\Xi}_{k \mid k-1}\left(\mathbf{J}_{k} \mathbf{E}_{k}\right)^{T}>0 \\
& \quad \lambda_{2}^{-1} \mathbf{I}-\overline{\mathbf{E}}_{k+1} \boldsymbol{\Xi}_{k+1 \mid k} \overline{\mathbf{E}}_{k+1}^{T}>0 \tag{43}
\end{align*}
$$

According to Lemma 4, we have

$$
\begin{align*}
& {\left[\left(\mathbf{A}_{k}+\mathbf{L}_{k} \overline{\boldsymbol{\Sigma}}_{k} \mathbf{C}_{k}\right)+\mathbf{H}_{k} \mathbf{F}_{k} \mathbf{J}_{k} \mathbf{E}_{k}\right] \mathbf{\Xi}_{k \mid k-1}} \\
& \times\left[\left(\mathbf{A}_{k}+\mathbf{L}_{k} \overline{\boldsymbol{\Sigma}}_{k} \mathbf{C}_{k}\right)+\mathbf{H}_{k} \mathbf{F}_{k} \mathbf{J}_{k} \mathbf{E}_{k}\right]^{T} \\
& \quad \leq\left(\mathbf{A}_{k}+\mathbf{L}_{k} \overline{\boldsymbol{\Sigma}}_{k} \mathbf{C}_{k}\right)\left(\mathbf{\Xi}_{k \mid k-1}^{-1}-2 \lambda_{1} \mathbf{E}_{k}^{T} \mathbf{E}_{k}\right)^{-1} \\
& \quad \times\left(\mathbf{A}_{k}+\mathbf{L}_{k} \overline{\boldsymbol{\Sigma}}_{k} \mathbf{C}_{k}\right)^{T}+\lambda_{1}^{-1}\left(\mathbf{B}_{k} \mathbf{B}_{k}^{T}+\mathbf{L}_{k} \overline{\mathbf{\Sigma}}_{k} \mathbf{D}_{k} \mathbf{D}_{k}^{T} \overline{\boldsymbol{\Sigma}}_{k}^{T} \mathbf{L}_{k}^{T}\right), \\
& \left(\mathbf{I}-\mathbf{K}_{k+1} \overline{\boldsymbol{\Sigma}}_{k+1} \overline{\mathbf{C}}_{k+1}-\mathbf{K}_{k+1} \overline{\boldsymbol{\Sigma}}_{k+1} \overline{\mathbf{D}}_{k+1} \overline{\boldsymbol{\alpha}}_{k+1} \overline{\mathbf{E}}_{k+1}\right) \boldsymbol{\Xi}_{k+1 \mid k} \\
& \times\left(\mathbf{I}-\mathbf{K}_{k+1} \overline{\mathbf{\Sigma}}_{k+1} \overline{\mathbf{C}}_{k+1}-\mathbf{K}_{k+1} \overline{\mathbf{\Sigma}}_{k+1} \overline{\mathbf{D}}_{k+1} \overline{\boldsymbol{\alpha}}_{k+1} \overline{\mathbf{E}}_{k+1}\right)^{T} \\
& \quad \leq\left(\mathbf{I}-\mathbf{K}_{k+1} \overline{\boldsymbol{\Sigma}}_{k+1} \overline{\mathbf{C}}_{k+1}\right)\left(\mathbf{\Xi}_{k+1 \mid k}^{-1}-\lambda_{2} \overline{\mathbf{E}}_{k+1}^{T} \overline{\mathbf{E}}_{k+1}\right)^{-1} \\
& \quad \times\left(\mathbf{I}-\mathbf{K}_{k+1} \overline{\mathbf{\Sigma}}_{k+1} \overline{\mathbf{C}}_{k+1}\right)^{T} \\
& \quad+\lambda_{2}^{-1} \mathbf{K}_{k+1} \overline{\mathbf{\Sigma}}_{k+1} \overline{\mathbf{D}}_{k+1} \overline{\mathbf{D}}_{k+1}^{T} \overline{\boldsymbol{\Sigma}}_{k+1}^{T} \mathbf{K}_{k+1}^{T} . \tag{44}
\end{align*}
$$

Combining (39) and (41)-(44), the condition (25) can be satisfied in Lemma 5. Thus, according to Lemmas 4 and 5, we have

$$
\begin{equation*}
\mathbf{P}_{k+1 \mid k} \leq \boldsymbol{\Xi}_{k+1 \mid k}, \quad \mathbf{P}_{k+1 \mid k+1} \leq \boldsymbol{\Xi}_{k+1 \mid k+1} \tag{45}
\end{equation*}
$$

To minimize the upper bounds, constructingthe optimized prediction gain $\mathbf{L}_{k}$ and the filtering gain $\mathbf{K}_{k+1}$ is to minimize the upper bounds $\boldsymbol{\Xi}_{k+1 \mid k}$ and $\boldsymbol{\Xi}_{k+1 \mid k+1}$; according to (27) and (28), we have

$$
\begin{aligned}
& \frac{\partial \operatorname{tr}\left(\boldsymbol{\Xi}_{k+1 \mid k}\right)}{\partial \mathbf{L}_{k}}= 2\left(\mathbf{A}_{k}+\mathbf{L}_{k} \overline{\boldsymbol{\Sigma}}_{k} \mathbf{C}_{k}\right)\left(\mathbf{\Xi}_{k \mid k-1}^{-1}-2 \lambda_{1} \mathbf{E}_{k}^{T} \mathbf{E}_{k}\right)^{-1} \\
& \times \mathbf{C}_{k}^{T} \overline{\boldsymbol{\Sigma}}_{k}-2 \mathbf{S}_{k} \\
&+2 \mathbf{L}_{k}\left\{\lambda_{1}^{-1} \overline{\boldsymbol{\Sigma}}_{k} \mathbf{D}_{k} \mathbf{D}_{k}^{T} \overline{\boldsymbol{\Sigma}}_{k}^{T}\right. \\
&+\breve{\mathbf{\Sigma}}_{k} \circ\left[2\left(a_{1}^{2} \operatorname{tr}\left(\mathbf{\Omega}_{k \mid k-1}\right)+a_{2}^{2}\right)\right] \mathbf{I} \\
&\left.+\sum_{i=1}^{r} \mathbf{C}_{i k}^{s} \mathbf{\Omega}_{k \mid k-1}\left(\mathbf{C}_{i k}^{s}\right)^{T}+\mathbf{R}_{k}\right\}=0, \\
& \begin{aligned}
\frac{\partial \operatorname{tr}\left(\boldsymbol{\Xi}_{k+1 \mid k+1}\right)}{\partial \mathbf{K}_{k+1}}
\end{aligned} \\
&=-2\left(\mathbf{I}-\mathbf{K}_{k+1} \overline{\boldsymbol{\Sigma}}_{k+1} \overline{\mathbf{C}}_{k+1}\right)\left(\mathbf{\Xi}_{k+1 \mid k}^{-1}-\lambda_{2} \overline{\mathbf{E}}_{k+1}^{T} \overline{\mathbf{E}}_{k+1}\right)^{-1} \\
& \times \overline{\mathbf{C}}_{k+1}^{T} \overline{\boldsymbol{\Sigma}}_{k+1}
\end{aligned}
$$

$$
\begin{align*}
+2 \mathbf{K}_{k+1}\{ & \lambda_{2}^{-1} \overline{\boldsymbol{\Sigma}}_{k+1} \overline{\mathbf{D}}_{k+1} \overline{\mathbf{D}}_{k+1}^{T} \overline{\boldsymbol{\Sigma}}_{k+1}^{T} \\
& +\breve{\boldsymbol{\Sigma}}_{k+1} \circ\left[2\left(a_{1}^{2} \operatorname{tr}\left(\Delta_{k+1 \mid k}\right)+a_{2}^{2}\right)\right] \mathbf{I} \\
& \left.+\sum_{i=1}^{r} \mathbf{C}_{i k+1}^{s} \Delta_{k+1 \mid k}\left(\mathbf{C}_{i k+1}^{s}\right)^{T}+\mathbf{R}_{k+1}\right\}=0 \tag{46}
\end{align*}
$$

where $\boldsymbol{\Omega}_{k \mid k-1}$ and $\boldsymbol{\Delta}_{k+1 \mid k}$ are defined in (30).
Considering (46), the optimized prediction gain $\mathbf{L}_{k}$ and the filtering gain $\mathbf{K}_{k+1}$ can be obtained in (31) and (32). The proof is completed.

For the sake of clarity, the robust recursive filter is summarized as follows.

Step 1. Given $\widehat{\mathbf{x}}_{k \mid k-1}$ and $\Xi_{k \mid k-1}$ and from (31), the prediction gain $\mathbf{L}_{k}$ is computed. Using (5) and (31), the one-step state prediction $\widehat{\mathbf{x}}_{k \mid k-1}$ and the upper bound $\boldsymbol{\Xi}_{k+1 \mid k}$ can be obtained by (6) and (27).

Step 2. The filtering gain $\mathbf{K}_{k+1}$ can be given by (32). The state estimation $\widehat{\mathbf{x}}_{k+1 \mid k+1}$ and the upper bound $\boldsymbol{\Xi}_{k+1 \mid k+1}$ can be given by (7) and (28).

Step 3. Repeat Step 1 to update the one-step state prediction and its upper bound and use Step 2 to obtain the state estimation.

Remark 7. The robust recursive filter problem is removed by using Theorem 6 for nonlinear systems with multiplicative noises, correlated additive noises, and packet losses. Different from the most existing robust filter literature, the robust recursive filter design proposed in this paper is based on the structure including state prediction and state correction in the presence of the correlated additive noises. Note that the phenomenon of correlated additive noises, multiplicative noises, and packet losses arises in the engineering applications. In order to solve this problem, a robust recursive filter is derived to find the upper bound of the prediction error covariance and the filtering error covariance and design the filter parameters to minimize the upper bounds. It is worth mentioning that, though the correction terms in (28) and (32) are similar to the corresponding component in [36], there is a clear difference between the prediction terms in (27) and (31) caused by correlated additive noises and the counterpart in [36, 37], which will directly affect the estimation results. This distinguishes our work from the work in [36, 37].

## 4. Simulation

To show the effectiveness of the proposed robust recursive filter (RRF), it is compared with the finite-horizon extended Kalman filter (FEKF) in the literature [36] by employing the following examples.

Example 8. The discretized maneuvering target tracking example in [36,37] is presented including correlated additive noises, multiplicative noises, and packet losses:

$$
\begin{align*}
& \mathbf{x}_{k+1}=\left[\begin{array}{l}
\mathbf{x}_{1, k+1} \\
\mathbf{x}_{2, k+1}
\end{array}\right]= {\left[\begin{array}{l}
0.8 \mathbf{x}_{1, k}+\mathbf{x}_{1, k} \mathbf{x}_{2, k} \\
1.5 \mathbf{x}_{2, k}-\mathbf{x}_{1, k} \mathbf{x}_{2, k}
\end{array}\right] } \\
&+\left[\begin{array}{ll}
0.06 & 0.08 \\
0.09 & 0.12
\end{array}\right] \boldsymbol{\eta}_{k}\left[\begin{array}{l}
\mathbf{x}_{1, k} \\
\mathbf{x}_{2, k}
\end{array}\right]+\left[\begin{array}{l}
0.01 \\
0.03
\end{array}\right] \mathbf{w}_{k}, \\
& \mathbf{y}_{k}=\mathbf{\Sigma}_{k} \times 7.5 \sin \left(\mathbf{x}_{2, k}\right)+\left[\begin{array}{ll}
0.15 & 0.2
\end{array}\right] \boldsymbol{\xi}_{k}\left[\begin{array}{l}
\mathbf{x}_{1, k} \\
\mathbf{x}_{2, k}
\end{array}\right]+\mathbf{v}_{k}, \tag{47}
\end{align*}
$$

where the state $\mathbf{x}_{k}=\left[\begin{array}{ll}\mathbf{x}_{1, k}^{T} & \mathbf{x}_{2, k}^{T}\end{array}\right]^{T}$ represents the position and velocity of target; $\boldsymbol{\eta}_{k}$ and $\boldsymbol{\xi}_{k}$ are independent zero mean Gaussian white noises with covariance 1; $\mathbf{w}_{k}$ and $\mathbf{v}_{k}$ are correlated zero mean Gaussian noises with $\mathbf{Q}_{k}=0.05$ and $\mathbf{R}_{k}=0.05$. Let $\mathbf{S}_{k}=0.02$. The mean and covariance of $\boldsymbol{\Sigma}_{k}$ are determined as $\mu_{k}=0.9$ and $\left(\sigma_{k}\right)^{2}=0.065$.

The initial state and covariance are set as $\widehat{\mathbf{x}}_{0 \mid 0}=\left[\begin{array}{ll}1.8 & 0.2\end{array}\right]^{T}$ and $\boldsymbol{\Xi}_{0 \mid 0}=20 \mathbf{I}_{2}$. Let $\varepsilon_{1}=0.4, \varepsilon_{2}=0.35, \lambda_{1}=\lambda_{2}=0.002$, $\mathbf{D}_{k}=\overline{\mathbf{D}}_{k+1}=\left[\begin{array}{ll}0.1 & 0.15\end{array}\right]^{T}, \mathbf{E}_{k}=\overline{\mathbf{E}}_{k+1}=0.01 \mathbf{I}_{2}$, and $\mathbf{B}_{k}=$ $\operatorname{diag}\{0.1,0.2\}$.

To evaluate the performance of the proposed robust recursive filter, the mean square error (MSE) is employed. And it can be expressed as

$$
\begin{equation*}
\mathrm{MSE}=\frac{1}{N} \sum_{k-1}^{N}\left(\mathbf{x}_{k}-\widehat{\mathbf{x}}_{k \mid k}\right)^{2}, \tag{48}
\end{equation*}
$$

where $N$ is the sample number.
Simulation results are shown in Figures 1-4. From Figures 1 and 2, it can be seen that, compared with the FEKF in [36], the proposed robust recursive filter performs better when the model is correlated with additive noises, multiplicative noises, and packet losses. Both true position and velocity are tracked well. This is because the effect of the proposed algorithm can compensate for the correlated noise, while the FEKF cannot. Shown in Figures 3 and 4 are the comparisons of MSE of the estimated states with the corresponding diagonal elements of the estimation error covariance. Obviously, for the proposed algorithm, the MSE of the estimated state is always lower than the upper bound. This confirms the results of Theorem 6. Meanwhile, the MSE of the RRF stays below the MSE of the FEKF, which further illustrates that the proposed algorithm has higher precision than the FEKF in presence of correlated additive noises, multiplicative noises, and packet losses.

Example 9. According to the literature [26], the robust recursive filter is considered to handle the attitude estimation problem with correlated additive noises, multiplicative


Figure 1: The trajectory of the actual state $\mathbf{x}_{1}$ and its estimate.


Figure 2: The trajectory of the actual state $\mathbf{x}_{2}$ and its estimate.
noises, and packet losses. The system process models are expressed as follows:

$$
\begin{align*}
\mathbf{x}_{k+1}=\left[\begin{array}{l}
\mathbf{q}_{k+1} \\
\boldsymbol{\beta}_{k+1}
\end{array}\right]= & {\left[\begin{array}{cc}
\mathbf{I}_{4 \times 4}+\frac{\Delta t}{2} \boldsymbol{\Omega}\left(\widetilde{\boldsymbol{w}}_{k}-\boldsymbol{\beta}_{k}\right) & \mathbf{0}_{4 \times 3} \\
\mathbf{0}_{3 \times 4} & \mathbf{I}_{3 \times 3}
\end{array}\right]\left[\begin{array}{l}
\mathbf{q}_{k} \\
\boldsymbol{\beta}_{k}
\end{array}\right] } \\
& +\left[\begin{array}{c}
-\frac{\Delta t}{2} \boldsymbol{\Xi}\left(\mathbf{q}_{k}\right) \\
\mathbf{0}_{3 \times 3}
\end{array}\right] \boldsymbol{\eta}_{v}+\left[\begin{array}{c}
\mathbf{0}_{4 \times 1} \\
\boldsymbol{\eta}_{u}
\end{array}\right] \\
= & f\left(\mathbf{x}_{k}, \widetilde{\boldsymbol{w}}_{k}\right)+\sum_{i=1}^{s} \mathbf{A}_{i k} \boldsymbol{\eta}_{i k} \mathbf{x}_{k}+\left[\begin{array}{c}
\mathbf{0}_{4 \times 1} \\
\sigma_{u} \sqrt{\Delta t} \mathbf{I}_{3 \times 1}
\end{array}\right] \mathbf{w}_{k}, \tag{49}
\end{align*}
$$



Figure 3: MSE of the estimated state $\mathbf{x}_{1}$ and upper bound.


Figure 4: MSE of the estimated state $\mathbf{x}_{2}$ and upper bound.
where the state $\mathbf{x}_{k}$ consisted of the quaternion vector $\mathbf{q}_{k}$ and the gyro bias vector $\boldsymbol{\beta}_{k} ; \widetilde{\boldsymbol{w}}_{k}$ is the gyro measured angular rate at time $k ; \Delta t$ is the gyros sampling interval; $\boldsymbol{\eta}_{v}$ is the Gaussian white-noise process with zero mean and covariance $\sigma_{v}^{2} ; \boldsymbol{\eta}_{u}$ is the zero mean Gaussian white-noise process with covariance $\sigma_{u}^{2} \Delta t ; s=3 ; \boldsymbol{\eta}_{i k}$ is the zero mean multiplicative noise with covariance $1 ; \mathbf{w}_{k}$ is the zero mean Gaussian noise with covariance $1 ;[\omega \times]$ is a cross-product matrix defined by

$$
[\boldsymbol{\omega} \times]=\left[\begin{array}{ccc}
0 & -\omega_{3} & \omega_{2}  \tag{50}\\
\omega_{3} & 0 & -\omega_{1} \\
-\omega_{2} & \omega_{1} & 0
\end{array}\right], \quad \Omega(\boldsymbol{\omega})=\left[\begin{array}{cc}
-[\boldsymbol{\omega} \times] & \boldsymbol{\omega} \\
-\boldsymbol{\omega}^{T} & 0
\end{array}\right]
$$

$\mathbf{A}_{i k}$ are known scaling matrices with appropriate dimension, which can be expressed as

$$
\mathbf{A}_{i k}=-\frac{\Delta t \sigma_{v}}{2}\left[\begin{array}{cc}
\mathbf{A}_{i k}^{1} & \mathbf{0}_{4 \times 3}  \tag{51}\\
\mathbf{0}_{3 \times 4} & \mathbf{0}_{3 \times 3}
\end{array}\right]
$$

where

$$
\begin{gather*}
\mathbf{A}_{1 k}^{1}=\left[\begin{array}{cccc}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{array}\right], \quad \mathbf{A}_{2 k}^{1}=\left[\begin{array}{cccc}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{array}\right],  \tag{52}\\
\mathbf{A}_{3 k}^{1}=\left[\begin{array}{cccc}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{array}\right] .
\end{gather*}
$$

The measurement model can be described as

$$
\mathbf{z}_{k}=\left[\begin{array}{c}
\mathbf{z}_{k}^{1}  \tag{53}\\
\mathbf{z}_{k}^{2} \\
\mathbf{z}_{k}^{3}
\end{array}\right]=\boldsymbol{\Sigma}_{k}\left[\begin{array}{l}
\mathbf{A}\left(\mathbf{q}_{k}\right) \overrightarrow{\mathbf{r}}^{1} \\
\mathbf{A}\left(\mathbf{q}_{k}\right) \overrightarrow{\mathbf{r}}^{2} \\
\mathbf{A}\left(\mathbf{q}_{k}\right) \overrightarrow{\mathbf{r}}^{3}
\end{array}\right]+\left[\begin{array}{c}
\mathbf{v}_{k}^{1} \\
\mathbf{v}_{k}^{2} \\
\mathbf{v}_{k}^{3}
\end{array}\right]=\boldsymbol{\Sigma}_{k} h\left(\mathbf{x}_{k}\right)+\sigma_{s} \mathbf{I}_{9 \times 1} \mathbf{v}_{k},
$$

where $\mathbf{z}_{k}^{i}(i=1,2,3)$ is the measurement vector; $\mathbf{A}\left(\mathbf{q}_{k}\right)$ is the real attitude matrix at time $k ; \overrightarrow{\mathbf{r}}^{i}(i=1,2,3)$ is the reference vector of the star sensors; $i$ is the number of star sensors. $\mathbf{v}_{k}^{i}$ is a zero mean Gaussian white noise with covariance matrix $\sigma_{s}^{2} \mathbf{I}_{3 \times 3} ; \mathbf{v}_{k}$ is the zero mean Gaussian noise with covariance 1 ; if $\mathbf{q}=\left[q_{1}, q_{2}, q_{3}, q_{4}\right]^{T}=\left[\boldsymbol{\rho}^{T}, q_{4}\right]^{T}$, the attitude matrix can be written as

$$
\begin{equation*}
\mathbf{A}(\mathbf{q})=\left(q_{4}^{2}-\boldsymbol{\rho}^{T} \boldsymbol{\rho}\right) \mathbf{I}_{3 \times 3}+2 \boldsymbol{\rho} \boldsymbol{\rho}^{T}-2 q_{4}[\boldsymbol{\rho} \times] \tag{54}
\end{equation*}
$$

The simulation conditions are set as follows: the gyro sampling interval is $\Delta t=0.25 \mathrm{~s}$; the standard deviation of gyros' measurement noise is $\sigma_{v}=1.45444 \times 10^{-6} \mathrm{rad} / \mathrm{s}^{1 / 2}$; the standard deviation of gyros' drift noise is $\sigma_{u}=1.3036 \times$ $10^{-9} \mathrm{rad} / \mathrm{s}^{3 / 2} ; \mathbf{w}_{k}$ and $\mathbf{v}_{k}$ are correlated zero mean Gaussian noises with $\mathbf{Q}_{k}=1$ and $\mathbf{R}_{k}=1$; let $\mathbf{S}_{k}=0.5$; the standard deviation of star sensors' measurement noise is all $\sigma_{s}=18^{\prime \prime}$; due to using three star sensors, the reference vectors of the star sensors are $\overrightarrow{\mathbf{r}}^{1}=\left[\begin{array}{lll}1 & 0 & 0\end{array}\right]^{T}, \overrightarrow{\mathbf{r}}^{2}=\left[\begin{array}{lll}0 & 1 & 0\end{array}\right]^{T}$, and $\overrightarrow{\mathbf{r}}^{3}=$ $\left[\begin{array}{lll}0 & 0 & 1\end{array}\right]^{T}$; because of the high precision of the star sensors, the estimation error is rather small in the attitude estimation system. Therefore, set $\mathbf{B}_{k}=\mathbf{0}, \mathbf{D}_{k}=\overline{\mathbf{D}}_{k+1}=\mathbf{0}$. The random variables $\mu_{k}^{i}(\mathrm{i}=1,2, \ldots, 9)$ satisfy the Bernoulli distribution with $\bar{\Gamma}_{k}=\operatorname{diag}\{0.8,0.8,0.8,0.9,0.9,0.9,0.95,0.95,0.95\}$; let $\varepsilon_{1}=\varepsilon_{2}=0.1$ and $\lambda_{1}=\lambda_{2}=0.0001$; let $\mathbf{C}_{i k+1}=\mathbf{0}, a_{1}=1$, and $a_{2}=0.05$.

The simulated results are shown in Figures 5 and 6. In Figure 5, blue lines represent the quaternion estimation errors of the RRF, green dashed lines represent the quaternion estimation errors of the FEKF, and red dashed lines show the corresponding diagonal elements of the error covariance of the RRF. It can be seen clearly that the estimation errors of the quaternion vector part of the RRF are generally within


Figure 5: The quaternion estimation errors of the proposed filter.


Figure 6: RMSE of attitude angles in the proposed filter.
the boundaries of the computed covariance, which indicates that the proposed filter can control correlated additive noises, multiplicative noises, and packet losses. Besides, the quaternion estimation errors of the RRF are smaller than the quaternion estimation errors of the FEKF. These show that the RRF performs better than the FEKF. Furthermore, since it is very important for attitude estimation to get the angle information, the estimated quaternion needs to be converted as the form of Euler angles. In Figure 6, root mean square error (RMSE) is employed to express the quality of the Euler angle estimation. For the RRF, the means of RMSE of the attitude angles are $3.199^{\prime \prime}, 3.125^{\prime \prime}$, and $3.195^{\prime \prime}$, respectively, which are lower than the REKF obviously. The reason why our method has such advantages is that the effects of arising multiplicative noises, packet losses, and correlated additive noises are all compensated for without loss of generality.

## 5. Conclusion

Due to the fact that existing robust filtering algorithms are difficult in dealing with correlated additive noises, a robust recursive filter is developed in this paper for nonlinear systems with consideration of correlated additive noises, multiplicative noises, and packet losses. The proposed algorithm is designed to minimize the upper bound on the prediction covariance and the filtering covariance. Simulated results demonstrate that the proposed filter provides effective performance for controlling correlated additive noises, multiplicative noises, and packet losses.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Channel Estimation and Information Symbol Detection for DS-UWB Communication Systems 

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#### Abstract

The UWB channel estimation and multiuser detection problem are investigated. The information symbol and channel parameter are considered as unknown variables. The multiuser detector and UWB channel estimator are designed jointly. For symbol detection, the one-step predictor of channel parameter is used and the estimation error is treated as a multiplicative noise; then a Riccati equation and a Lyapunov equation will be needed. If the transmitted symbols are uncorrelated and identically distributed random variables with zero mean and unit variance, only a Riccati equation needs to be solved. For UWB channel estimation, the one-step predictor of information symbol is used and the estimation error is also considered as a multiplicative noise. The solutions to the above two problems are obtained by solving a couple of Riccati equations together with two Lyapunov equations.


## 1. Introduction

In the communications literature, a number of different algorithms have been proposed for channel estimation problems with accurate models [1-19]. In [1], a subspace-based estimation algorithm is developed. The algorithms in $[2,3]$ are based on the maximum likelihood estimation method. Due to the performance benefits of the Kalman algorithms, many works have focused on the Kalman-filter-based channel estimation algorithms [13-15]. These algorithms require a state-space model for the random process to be estimated.

As for the UWB channel, many different types of channel models have been proposed. In general, the UWB propagation channel models are characterized by a dense multipath propagation and the clustering phenomenon and can be classified as deterministic and statistical [17, 18]. Deterministic models apply an electromagnetic simulation tool to obtain exact propagation characteristics for a specified geometry. Statistical models are normally less complex than the deterministic models and can provide sufficiently accurate channel information. In [20], three channel models were considered, namely, the Rayleigh tap delay line model, the $\Delta$-K model, and the Saleh-Valenzuela (S-V) model. The comparisons showed that the S-V model gives the best fit to the measured
channel characteristics. This double exponential channel model is commonly used for UWB realistic indoor channel, and the channel impulse response is given by

$$
\begin{equation*}
\mathscr{H}(t)=\sum_{l_{c}=0}^{L_{c}-1} \sum_{l_{r}=0}^{L_{r}-1} \alpha_{l_{c} l_{r}} \delta\left(t-T_{l_{c}}-\tau_{l_{c} l_{r}}\right) \tag{1}
\end{equation*}
$$

where
(i) $\left\{\alpha_{l_{c} l_{r}}\right\}$ are the multipath gain coefficients, $l_{c}$ refers to the cluster, and $l_{r}$ refers to the rays in one cluster;
(ii) $T_{l_{c}}$ is the delay of the $l_{c}$ th cluster which is defined as the TÖA of the first arriving multipath component within the $l_{c}$ th cluster;
(iii) $\tau_{l_{c} l_{r}}$ is the delay of the $l_{r}$ th multipath component relative to the $l_{c}$ th cluster arrival time $T_{l_{c}}$.

The clustering channel model relies on two classes of the parameters, namely, intercluster and intracluster parameters, which characterize the cluster and multipath component, respectively. In the above model, $\left\{L_{c}, T_{l_{c}}\right\}$ and $\left\{T_{r}, \tau_{l_{c}, l_{r}}, \alpha_{l_{c}, l_{r}}\right\}$ are classified as the intercluster and intracluster parameters, respectively. The distributions of the cluster arrive time $T_{l_{c}}$
and the ray arrive time $\tau_{l_{c} l_{r}}$ can be given by two Poisson processes:

$$
\begin{gather*}
p\left(T_{l_{c}} \mid T_{l_{c}-1}\right)=\Lambda \exp \left[-\Lambda\left(T_{l_{c}}-T_{l_{c}-1}\right)\right], \quad l_{c}>0,  \tag{2}\\
p\left(\tau_{l_{c} l_{r}} \mid \tau_{l_{c} l_{r}-1}\right)=\lambda \exp \left[-\lambda\left(\tau_{l_{c} l_{r}}-\tau_{l_{c} l_{r}-1}\right)\right], \\
l_{r}>0,
\end{gather*}
$$

where $\Lambda$ and $\lambda$ are mean cluster arrival rate and mean ray arrival rate, respectively. The channel coefficients are defined as follows:

$$
\begin{equation*}
\alpha_{l_{c} l_{r}}=p_{l_{c} l_{r} l_{r}} \beta_{l_{c}, l_{r}}, \tag{3}
\end{equation*}
$$

where $p_{l_{c} l_{r}}$ is equiprobable to $\pm 1$ to account for signal inversion due to reflection; $\beta_{l_{c}, l_{r}}$ correspond to the fading associated with the $l_{c}$ th ray of the $l_{r}$ th cluster, which can be modeled as a log-normal distribution:

$$
\begin{equation*}
20 \log 10\left(\beta_{l_{c} l_{r}}\right) \propto \operatorname{Normal}\left(\mu_{l_{c}}, l_{r}, \sigma^{2}\right) \tag{4}
\end{equation*}
$$

where $\mu_{l_{c}, l_{r}}$ is given by

$$
\begin{equation*}
\mu_{l_{c} l_{r}}=\frac{10 \ln \left(\Omega_{0}\right)-10 T_{l_{c}} / \Gamma-10 \tau_{l_{c}, l_{r}} / \gamma}{\ln (10)}-\frac{\sigma^{2} \ln (10)}{20} \tag{5}
\end{equation*}
$$

where $\Omega_{0}$ is the mean power of the first path of the first cluster. The behavior of the averaged power delay profile is

$$
\begin{equation*}
E\left[\left|\beta_{l_{c} l_{r}}\right|^{2}\right]=\Omega_{0} e^{-T_{l_{c}} / \Gamma} e^{-\tau_{l_{c} l_{r}} / \gamma} \tag{6}
\end{equation*}
$$

which reflect the exponential decay of each cluster.
With mapping, the above two-dimensional channel model can be reduced to a one-dimensional channel model:

$$
\begin{equation*}
\mathscr{H}(t)=\sum_{l=0}^{L} \alpha^{l} \delta\left(t-\tau_{l}\right) \tag{7}
\end{equation*}
$$

where $l=l_{c} L_{c}+l_{r}$ and $L=L_{c} L_{r}-1$ are the number of the resolvable multipath components; $\tau_{l}=T_{l_{c}}+\tau_{l_{c} l_{r}}$ is the delay of the $l$ th path relative to the first path; $\alpha^{l}=\alpha_{l_{c} l_{r}}$ is the fading coefficient of path $l$.

After mapping, the one-dimensional model can be dealt with by using some conventional channel estimation algorithm that is used for narrowband systems, such as maximum likelihood approach and least mean square approach. In this paper, we will pursue a Kalman-filter-based approach with information symbols unknown.

## 2. UWB System Model

In this paper, we consider a binary DS-CDMA UWB communication system with $K$ multiple access users. The transmitted baseband signal of the $k$ th user is given by $[15,16]$

$$
\begin{equation*}
x_{k}(t)=\sqrt{A_{k}} \sum_{n=-\infty}^{\infty} b_{k}(n) s_{k}\left(t-n T_{s}\right), \tag{8}
\end{equation*}
$$

where $A_{k}$ is the transmitted bit energy, $T_{s}$ is symbol duration, $b_{k}(n)$ is the modulated information symbol of the $k$ th user and
is chosen randomly from the set $\{-1,+1\}$, and $s_{k}(t)$ represents the transmitted waveform and has the form

$$
\begin{equation*}
s_{k}(t)=\sum_{i=0}^{N} \widetilde{c}_{k}(i) \psi\left(t-i T_{c}\right) \tag{9}
\end{equation*}
$$

where $N$ is the spreading gain, $\widetilde{c}_{k}(i)$ is the spreading code of the $k$ th user with period $N$, and $\psi(t)$ is the real transmitted monocycle waveform shape in the time interval $0 \leq t \leq T_{c}$, that is, $\psi(t)=0$ if $t \notin\left[0, T_{c}\right]$, and has energy $(1 / N)$.

Note that the channel coefficient $\alpha^{l}$ in (7) is fading with respect to time $t$; the channel impulse response for the $k$ th user can be described by

$$
\begin{equation*}
\mathscr{H}_{k}(t)=\sum_{l=0}^{L} \alpha_{k}^{l}(t) \delta\left(t-\tau_{k, l}\right), \tag{10}
\end{equation*}
$$

where $\tau_{k, l}$ is the time delay for the $l$ th path of user $k$. Then the received signal component from the $k$ th user can be represented as

$$
\begin{align*}
& y_{k}(t)=x_{k}(t) * \mathscr{H}_{k}(t) \\
& =\sqrt{A_{k}} \sum_{n=-\infty}^{\infty} b_{k}(n) \sum_{i=0}^{N-1} \widetilde{c}_{k}(i) \\
&  \tag{11}\\
& \quad \times \sum_{j=0}^{L} \alpha_{k}^{l}(t) \psi\left(t-n T_{s}-i T_{c}-\tau_{k, j}\right) \\
& =\sqrt{A_{k}} \sum_{n=-\infty}^{\infty} b_{k}(n) \\
& \quad \times \sum_{i=0}^{N-1} \widetilde{c}_{k}(i) g_{k}\left(t, t-n T_{s}-i T_{c}\right),
\end{align*}
$$

where $*$ denotes the convolution, and

$$
\begin{equation*}
g_{k}(t, \tau) \triangleq \sum_{j=0}^{L} \alpha_{k}^{l}(t) \psi\left(\tau-\tau_{k, j}\right) \tag{12}
\end{equation*}
$$

The total received signal at the receiver is the superposition of the signals of the $K$ users, given by

$$
\begin{equation*}
r(t)=\sum_{k=0}^{K} y_{k}(t)+v(t) \tag{13}
\end{equation*}
$$

where $v(t)$ is a white Gaussian noise with zero mean. The discrete-time signal is generated by sampling the output of a pulse-matched filter (PMF) at the monocycle rate (as shown in Figure 1) and given by [13]

$$
\begin{aligned}
& y_{k}(n N+j) \\
& \qquad=\int_{n T_{s}+j T_{p}}^{n T_{s}+(j+1) T_{p}} y_{k}(t) \psi\left(t-n T_{s}-j T_{p}\right) d t
\end{aligned}
$$



Figure 1: Received discrete-time signal.

$$
\begin{align*}
& =\int_{n T_{s}+j T_{p}}^{n T_{s}+(j+1) T_{p}} \sum_{m=-\infty}^{\infty} \sqrt{A_{k}} b_{k}(m) \\
& \times \sum_{i=0}^{N_{c}-1} \widetilde{c}_{k}(i) g_{k}\left(t, t-m T_{s}-i T_{c}\right) \\
& \times \psi\left(t-n T_{s}-j T_{p}\right) d t \\
& =\sqrt{A_{k}} \sum_{m=-\infty}^{\infty} b_{k}(m) \\
& \times \int_{0}^{T_{p}} \sum_{i=0}^{N_{c}-1} \widetilde{c}_{k}(i) g_{k} \\
& \times\left(t+n T_{s}+j T_{p}, t+(n-m) T_{s}\right. \\
& \left.+\left(j-i N_{p}\right) T_{p}\right) \psi(t) d t . \tag{14}
\end{align*}
$$

In this paper the multipath delay $\left\{\tau_{k, j}\right\}$ of the UWB channel is assumed to be an integral multiple of the monocycle length $T_{p}$; then the above equation can be rewritten as

$$
\begin{aligned}
& y_{k}(n N+j) \\
& \quad \sqrt{A_{k}} \sum_{m=-\infty}^{\infty} b_{k}(m) \\
& \quad \times \int_{0}^{T_{p}} \sum_{i=0}^{N_{c}-1} \widetilde{c}_{k}(i) \\
& \quad \times \sum_{l=0}^{L} \alpha_{k}^{l}\left(t+n T_{s}+j T_{p}\right) \\
& \\
& \quad \times \psi\left(t+(n-m) T_{s}\right. \\
& \\
& \left.\quad+\left(j-i N_{p}-l\right) T_{c}\right)
\end{aligned}
$$

$\times \psi(t) d t$

$$
\begin{align*}
&=b_{k}(n) \sum_{l=0}^{L} c_{k}\left(\left\lfloor\frac{j-l}{N_{p}}\right\rfloor\right) \\
& \times \int_{0}^{T_{p}} \alpha_{k}^{l}\left(t+n T_{s}+j T_{p}\right) d t \\
&+b_{k}(n-1) \sum_{l=0}^{L} c_{k}\left(\left\lfloor\frac{N+j-l}{N_{p}}\right\rfloor\right) \\
& \times \int_{0}^{T_{p}} \alpha_{k}^{l}\left(t+n T_{s}+j T_{p}\right) d t \\
&= b_{k}(n) \sum_{l=0}^{L} c_{k}\left(\left\lfloor\frac{j-l}{N_{p}}\right\rfloor\right) h_{k}^{l}(n, j) \\
&+b_{k}(n-1) \sum_{l=0}^{L} c_{k}\left(\left\lfloor\frac{N+j-l}{N_{p}}\right\rfloor\right) h_{k}^{l}(n, j) \tag{15}
\end{align*}
$$

where

$$
\begin{align*}
h_{k}^{l}(n, j) & =\int_{0}^{T_{p}} \alpha_{k}^{l}\left(t+n T_{s}+j T_{c}\right) d t \\
c_{k}(i) & = \begin{cases}\frac{\sqrt{A_{k}}}{N} \widetilde{c}_{k}(i), & 0 \leq i \leq N-1 \\
0, & \text { otherwise }\end{cases} \tag{16}
\end{align*}
$$

Further assuming that $h_{k}^{l}(n, j)$ is invariant during a symbol interval and using $h_{k}^{l}(n)$ to denote the channel parameter in the $n$th symbol, then we have

$$
\begin{align*}
y_{k}(n N+j)= & b_{k}(n) \sum_{l=0}^{L} c_{k}\left(\left\lfloor\frac{j-l}{N_{p}}\right\rfloor\right) h_{k}^{l}(n) \\
& +b_{k}(n-1) \sum_{l=0}^{L} c_{k}\left(\left\lfloor\frac{N+j-l}{N_{p}}\right\rfloor\right) h_{k}^{l}(n) . \tag{17}
\end{align*}
$$

By collecting $N$ successive samples, the channel output from the $k$ th user at the $n$th symbol can be expressed as

$$
\begin{align*}
\mathbf{y}_{k}(n) & =\left[\begin{array}{llll}
y_{k}(n N) & y_{k}(n N+1) & \cdots & y_{k}(n N+N-1)
\end{array}\right]^{T} \\
& =b_{k}(n) C_{k}^{0} \mathbf{h}_{k}(n)+b_{k}(n-1) C_{k}^{1} \mathbf{h}_{k}(n), \tag{18}
\end{align*}
$$

where $C_{k}^{0}$ and $C_{k}^{1}$ are the signature sequence matrices with dimension $N \times(L+1)$ and have the forms

$$
\left.C_{k}^{0}=\left[\begin{array}{cccc}
c_{k}(0) & 0 & \cdots & 0  \tag{19}\\
c_{k}\left(\left\lfloor\frac{l}{N_{p}}\right\rfloor\right) & c_{k}(0) & \cdots & 0 \\
c_{k}\left(\left\lfloor\frac{2}{N_{p}}\right\rfloor\right) & c_{k}\left(\left\lfloor\frac{1}{N_{p}}\right\rfloor\right) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
c_{k}\left(\left\lfloor\frac{N-1}{N_{p}}\right\rfloor\right) & c_{k}\left(\left\lfloor\frac{N-2}{N_{p}}\right\rfloor\right) & \cdots & c_{k}\left(\left\lfloor\frac{N-L-1}{N_{p}}\right\rfloor\right.
\end{array}\right)\right],
$$

and $\mathbf{h}_{k}(n)$ is the parameter collection of all multipath components

$$
\mathbf{h}_{k}(n)=\left[\begin{array}{llll}
h_{k}^{0}(n) & h_{k}^{1}(n) & \cdots & h_{k}^{L}(n) \tag{20}
\end{array}\right]^{T}
$$

The total received discrete-time signal of all users can be given by

$$
\begin{align*}
\mathbf{r}(n) & =\left[\begin{array}{llll}
r(n N) & r(n N+1) & \cdots & r(n N+N-1)
\end{array}\right]^{T} \\
& =\sum_{k=1}^{K} \mathbf{y}_{k}(n)+\mathbf{v}(n) \tag{21}
\end{align*}
$$

where

$$
\begin{align*}
\mathbf{v}(n) & =\left[\begin{array}{llll}
v(n N) & v(n N+1) & \cdots & v(n N+N-1)
\end{array}\right]^{T}, \\
v(n N+j) & =\int_{n T_{s}+j T_{c}}^{n T_{s}+(j+1) T_{c}} v(t) \psi\left(t-n T_{s}-j T_{c}\right) d t . \tag{22}
\end{align*}
$$

Considering that the channel parameters $\left\{h_{k}^{l}(n)\right\}$ and the user information symbols $\left\{b_{k}(n)\right\}$ are unknown, in this paper, we pursue a joint design method for user detection and channel parameter estimation.

The problems investigated in this paper can be stated as follows.

Problem I. Given the received signal sequence $\{\mathbf{r}(s)\}_{s=0}^{n}$, with $\left\{\mathbf{h}_{k}(s), k=1, \ldots, K\right\}_{s=0}^{n}$ not exactly known, find an optimal symbol detector $\left\{\widehat{b}_{k}(n \mid n), k=1, \ldots, K\right\}$ by using a priori estimate $\left\{\widehat{\mathbf{h}}_{k}(s \mid s-1), k=1, \ldots, K\right\}_{s=0}^{n}$ of channel parameter which is recursively calculated in Problem-II.

Problem II. Given the received signal sequence $\{\mathbf{r}(s)\}_{s=0}^{n}$, with the information symbol $\left\{b_{k}(s), k=1, \ldots, K\right\}_{s=0}^{n}$ not exactly known, find a channel estimator $\left\{\widehat{\mathbf{h}}_{k}(n \mid n), k=1, \ldots, K\right\}$ by using a priori estimate $\left\{\widehat{b}_{k}(s \mid s-1), k=1, \ldots, K\right\}_{s=0}^{n}$ which is recursively calculated in Problem-I.

Remark 1. In this section, we have adopted a signal model for DS-CDMA communication systems similar to that in [15, 16]. Different from [15], in (7) the information symbol matrix is unknown which will be detected together with the channel parameter. In most relevant works, the information symbol is considered known for channel estimation [14], or channel parameter is known for user detection [12] and only few works investigate a joint estimation scheme considering both of the aforementioned unknown variables [13]. This paper will propose a Kalman-filter-based joint design method for multiuser detection and channel estimation.

Remark 2. The above two problems cannot be solved separately, because the channel parameters and information
symbols are not exactly known. Different from [13], in this paper the symbol detector and channel estimator are designed simultaneously. The solutions of the detector and channel estimator will be obtained via solving coupled Riccati equations together with two Lyapunov equations.

## 3. Multiuser Detector

In this section, a first-order state-space model is applied to symbol detection for the proposed UWB system, where the channel parameter is not exactly known. Then we can
employ the Kalman filter to estimate all users' symbols simultaneously. In view of (7), for multiuser detection the total received discrete-time signal $\mathbf{r}(n)$ can be expressed as

$$
\begin{align*}
\mathbf{r}(n) & =\left[\begin{array}{lll}
r(n N) & r(n N+1) \cdots r(n N+N-1)
\end{array}\right]^{T} \\
& =\sum_{k=1}^{K} \mathbf{y}_{k}(n)+\mathbf{v}(n)  \tag{23}\\
& =C H(n) \mathbf{b}(n)+\mathbf{v}(n),
\end{align*}
$$

where

$$
\begin{align*}
& C=\left[\begin{array}{llllllll}
C_{1}^{0} & C_{2}^{0} & \cdots & C_{K}^{0} & C_{1}^{1} & C_{2}^{1} & \cdots & C_{K}^{1}
\end{array}\right], \\
& H(n)=\operatorname{diag}\left\{\mathbf{h}_{1}(n), \mathbf{h}_{2}(n), \ldots, \mathbf{h}_{K}(n), \mathbf{h}_{1}(n), \mathbf{h}_{2}(n), \cdots, \mathbf{h}_{K}(n)\right\},  \tag{24}\\
& \mathbf{b}(n)=\left[\begin{array}{lllllll}
b_{1}(n) & b_{2}(n) & \cdots & b_{K}(n) & b_{1}(n-1) & b_{2}(n-1) & \cdots
\end{array} b_{K}(n-1)\right]^{T} .
\end{align*}
$$

Note the symbol vector $\mathbf{b}(n)$ defined in (23); the first-order non-Gaussian Markov transition model is defined as

$$
\begin{equation*}
\mathbf{b}(n+1)=\Phi \mathbf{b}(n)+\mathbf{w}(n), \tag{25}
\end{equation*}
$$

where

$$
\begin{gather*}
\Phi=\left[\begin{array}{cc}
0_{K, K} & 0_{K, K} \\
I_{K} & 0_{K, K}
\end{array}\right]  \tag{26}\\
\mathbf{w}(n)=\left[\begin{array}{llll}
b_{1}(n+1) & \cdots & b_{K}(n+1) & 0_{1, K}
\end{array}\right]^{T},
\end{gather*}
$$

where $0_{m, n}$ denotes the $m \times n$ all-zero matrix, and $I_{m}$ is the $m \times m$ identity matrix; the noise vector $\mathbf{w}(n)$ is white with zero mean and covariance matrix

$$
\begin{equation*}
Q_{w}(n)=E\left\{\mathbf{w}(n) \mathbf{w}^{T}(n)\right\} \tag{27}
\end{equation*}
$$

For the convenience of discussion, we first give the following definitions.

Definition 3. For a given symbol $n$, let $\widehat{\xi}(n \mid n-1)$ denote the optimal estimation of $\xi(n)$, which is the projection of $\xi(n)$ onto the linear space

$$
\begin{equation*}
\mathscr{L}\{\mathbf{r}(0) \cdots \mathbf{r}(n-1)\} \tag{28}
\end{equation*}
$$

Definition 4. For multiuser detection with unknown UWB channel parameters, define

$$
\begin{equation*}
\mathbf{e}_{b}(n) \triangleq \mathbf{r}(n)-\widehat{\mathbf{r}}(n \mid n-1) \tag{29}
\end{equation*}
$$

For UWB channel estimation with unknown information symbols, define

$$
\begin{equation*}
\mathbf{e}_{h}(n) \triangleq \mathbf{r}(n)-\widehat{\mathbf{r}}(n \mid n-1) \tag{30}
\end{equation*}
$$

where $\widehat{\mathbf{r}}(n \mid n-1)$ is defined as in Definition 3.

As in the standard Kalman filtering, we define the onestep prediction error covariance matrix of the information symbol and channel parameter as

$$
\begin{align*}
& P_{b}(n) \triangleq E\left\{\widetilde{\mathbf{b}}(n \mid n-1) \widetilde{\mathbf{b}}^{T}(n \mid n-1)\right\} \\
& P_{h}(n) \triangleq E\left\{\widetilde{\mathbf{h}}(n \mid n-1) \widetilde{\mathbf{h}}^{T}(n \mid n-1)\right\} \tag{31}
\end{align*}
$$

where

$$
\begin{align*}
& \widetilde{\mathbf{b}}(n \mid n-1) \triangleq \mathbf{b}(n)-\widehat{\mathbf{b}}(n \mid n-1), \\
& \widetilde{\mathbf{h}}(n \mid n-1) \triangleq \mathbf{h}(n)-\widehat{\mathbf{h}}(n \mid n-1) \tag{32}
\end{align*}
$$

where $\mathbf{h}(n)$ is the stack of channel parameters of all users

$$
\mathbf{h}(n)=\left[\begin{array}{llll}
\mathbf{h}_{1}^{T}(n) & \mathbf{h}_{2}^{T}(n) & \cdots & \mathbf{h}_{K}^{T}(n) \tag{33}
\end{array}\right]^{T}
$$

and $\widehat{\mathbf{b}}(n \mid n-1)$ and $\widehat{\mathbf{h}}(n \mid n-1)$ are defined as in Definition 3.
Note that the elements of UWB channel parameter matrix are unknown. In this section, we will use the one-step prediction $\widehat{H}(n \mid n-1)$ instead of $H(n)$ and consider the estimation error $\widetilde{H}(n \mid n-1)$ as a multiplicative noise for symbol detection. The optimal detector is given according to the following theorem.

Theorem 5. Consider the discrete-time state-space signal model (23) and (25); when the channel parameter matrix $H(n)$ is unknown, the information symbol detector is given by

$$
\begin{align*}
\widehat{\mathbf{b}}(n \mid n)= & {\left[I_{2 K}-K_{b}(n) C \widehat{H}(n \mid n-1)\right] \widehat{\mathbf{b}}(n \mid n-1) }  \tag{34}\\
& +K_{b}(n) \mathbf{r}(n),
\end{align*}
$$

where $\widehat{H}(n \mid n-1)$ is the one-step prediction of UWB channel parameter obtained from the next section, and $K_{b}(n)$ is the detector gain matrix

$$
\begin{equation*}
K_{b}(n)=P_{b}(n) \widehat{H}^{T}(n \mid n-1) C^{T}\left[Q_{e}^{b}(n)\right]^{-1} \tag{35}
\end{equation*}
$$

where $Q_{e}^{b}(n)$ is the covariance matrix of innovation $\mathbf{e}_{b}(n)$

$$
\begin{align*}
Q_{e}^{b}(n)= & C \widehat{H}(n \mid n-1) P_{b}(n) \widehat{H}^{T}(n \mid n-1) C^{T} \\
& +C\left(\left[\frac{P_{h}(n) \mid P_{h}(n)}{P_{h}(n) \mid P_{h}(n)}\right] \circ\left[\prod_{b}(n) \otimes I_{L+1}\right]\right) C^{T} \\
& +Q_{v}(n), \tag{36}
\end{align*}
$$

where $\circ$ denotes the Hadamard product and $\otimes$ is the Kronecker product. $\Pi_{b}(n)$ satisfies the following Lyapunov equation:

$$
\begin{equation*}
\prod_{b}(n+1)=\Phi \prod_{b}(n) \Phi^{T}+Q_{w}(n), \tag{37}
\end{equation*}
$$

where $P_{b}(n)$ is the symbol estimation error covariance matrix and satisfies the following Riccati equation:

$$
\begin{equation*}
P_{b}(n+1)=\Phi P_{b}(n) \Phi-\Phi K_{b}(n) Q_{e}^{b}(n) K_{b}^{T}(n) \Phi+Q_{w}(n), \tag{38}
\end{equation*}
$$

where $P_{h}(n)$ is the UWB channel parameter estimation error covariance matrix which will be calculated recursively in the next section. The one-step prediction of the information symbol is given by

$$
\begin{equation*}
\widehat{\mathbf{b}}(n+1 \mid n)=\Phi \widehat{\mathbf{b}}(n \mid n), \tag{39}
\end{equation*}
$$

which will be used for channel estimator design.
Proof. From Definition 3 we know that the a priori estimate $\widehat{\mathbf{r}}(n \mid n-1)$ is the projection of $\mathbf{r}(n)$ onto the linear space $\mathscr{L}\{\mathbf{r}(0), \ldots, \mathbf{r}(n-1)\}$ and consider the channel parameter matrix $H(n)$ as an unknown variable; then we have

$$
\begin{align*}
\widehat{\mathbf{r}}(n \mid n-1) & =\operatorname{Proj}\{\mathbf{r}(n) \mid \mathbf{r}(0), \ldots, \mathbf{r}(n-1)\} \\
& =C \widehat{H}(n \mid n-1) \widehat{\mathbf{b}}(n \mid n-1) . \tag{40}
\end{align*}
$$

In view of (7) and Definition 4, we obtain

$$
\begin{align*}
\mathbf{e}_{b}(n)= & \mathbf{r}(n)-\widehat{\mathbf{r}}(n \mid n-1) \\
= & C H(n) \mathbf{b}(n)-C \widehat{H}(n \mid n-1) \widehat{\mathbf{b}}(n \mid n-1)+\mathbf{v}(n) \\
= & C \widehat{H}(n \mid n-1) \widetilde{\mathbf{b}}(n \mid n-1) \\
& +C \widetilde{H}(n \mid n-1) \mathbf{b}(n)+\mathbf{v}(n) . \tag{41}
\end{align*}
$$

It is apparent that $\mathbf{e}_{b}(n)$ is with zero mean and $E\left\{\mathbf{e}_{b}(s) \mathbf{e}_{b}(j)\right\}=$ 0 if $s \neq j$. The stochastic process $\left\{\mathbf{e}_{b}(s)\right\}_{s=0}^{n}$ is termed as the innovation sequence associated with the received signal sequence. The covariance matrix of $\mathbf{e}_{b}(n)$, denoted as $Q_{e}^{b}(n)$, is calculated as follows:

$$
\begin{aligned}
Q_{e}^{b}(n) & \triangleq\langle\mathbf{e}(n), \mathbf{e}(n)\rangle \\
& =\langle C \widehat{H}(n \mid n-1) \widetilde{\mathbf{b}}(n \mid n-1)
\end{aligned}
$$

$$
\begin{align*}
& \quad+C \widetilde{H}(n \mid n-1) \mathbf{b}(n)+\mathbf{v}(n), \\
& \quad C \widehat{H}(n \mid n-1) \widetilde{\mathbf{b}}(n \mid n-1) \\
& +C \widetilde{H}(n \mid n-1) \mathbf{b}(n)+\mathbf{v}(n)\rangle \\
& =C \widehat{H}(n \mid n-1) P_{b}(n) \widehat{H}^{T}(n \mid n-1) C^{T} \\
& +C\langle\widetilde{H}(n \mid n-1) \mathbf{b}(n), \\
& \quad \widetilde{H}(n \mid n-1) \mathbf{b}(n)\rangle C^{T}+Q_{v}(n), \tag{42}
\end{align*}
$$

where $\langle$,$\rangle denotes the inner product, and$

$$
\begin{align*}
& \langle\widetilde{H}(n \mid n-1) \mathbf{b}(n), \widetilde{H}(n \mid n-1) \mathbf{b}(n)\rangle \\
& =\underset{\{\widetilde{H}, \mathbf{b}\}}{E}\left\{\widetilde{H}(n \mid n-1) \mathbf{b}(n) \mathbf{b}^{T}(n) \widetilde{H}^{T}(n \mid n-1)\right\} \\
& =\underset{\{\tilde{\mathbf{h}}, b\}}{E}\left\{\left[\begin{array}{c}
\widetilde{\mathbf{h}}_{1}(n \mid n-1) b_{1}(n) \\
\vdots \\
\widetilde{\mathbf{h}}_{K}(n \mid n-1) b_{K}(n) \\
\widetilde{\mathbf{h}}_{1}(n \mid n-1) b_{1}(n-1) \\
\vdots \\
\widetilde{\mathbf{h}}_{K}(n \mid n-1) b_{K}(n-1)
\end{array}\right]\right. \\
& \left.\times\left[\begin{array}{c}
\widetilde{\mathbf{h}}_{1}(n n-1) b_{1}(n) \\
\vdots \\
\widetilde{\mathbf{h}}_{K}(n n-1) b_{K}(n) \\
\widetilde{\mathbf{h}}_{1}(n n-1) b_{1}(n-1) \\
\vdots \\
\widetilde{\mathbf{h}}_{K}(n n-1) b_{K}(n-1)
\end{array}\right]^{T}\right\}  \tag{43}\\
& =\underset{\{\widetilde{\mathbf{h}}, b\}}{E}\left[\begin{array}{l|l}
M_{11}(n) & M_{12}(n) \\
\hline M_{12}^{T}(n) & M_{22}(n)
\end{array}\right] \\
& =\left[\begin{array}{c|c}
P_{h}(n) & P_{h}(n) \\
\hline P_{h}(n) & P_{h}(n)
\end{array}\right] \circ\left[\prod_{b}(n) \otimes I_{L+1}\right],
\end{align*}
$$

where $\circ$ denotes the Hadamard product and $\otimes$ is the Kronecker product; $\Pi_{b}(n)$ denotes the inner product of symbol vector $\mathbf{b}(n)$, given by

$$
\begin{equation*}
\prod_{b}(n) \triangleq\langle\mathbf{b}(n), \mathbf{b}(n)\rangle, \tag{44}
\end{equation*}
$$

and satisfies the following Lyapunov equation:

$$
\begin{equation*}
\prod_{b}(n+1)=\Phi \prod_{b}(n+1) \Phi^{T}+Q_{w}(n), \tag{45}
\end{equation*}
$$

where $P_{h}(n)$ is the parameter estimation error covariance matrix which will be calculated recursively in the next section. In the third step $\left\{M_{i j}(n), i, j=1,2\right\}$ are as follows:

$$
\begin{gather*}
M_{11}(n)=\left[\begin{array}{cccc}
\widetilde{\mathbf{h}}_{1} b_{1}^{2}(n) \widetilde{\mathbf{h}}_{1}^{T} & \widetilde{\mathbf{h}}_{1} b_{1}(n) b_{2}(n) \widetilde{\mathbf{h}}_{2}^{T} & \cdots & \widetilde{\mathbf{h}}_{1} b_{1}(n) b_{K}(n) \widetilde{\mathbf{h}}_{K}^{T} \\
\widetilde{\mathbf{h}}_{2} b_{2}(n) b_{1}(n) \widetilde{\mathbf{h}}_{1}^{T} & \widetilde{\mathbf{h}}_{2} b_{2}^{2}(n) \widetilde{\mathbf{h}}_{2}^{T} & \cdots & \widetilde{\mathbf{h}}_{2} b_{2}(n) b_{K}(n) \widetilde{\mathbf{h}}_{K}^{T} \\
\vdots & \vdots & \ddots & \vdots \\
\widetilde{\mathbf{h}}_{K} b_{K}(n) b_{1}(n) \widetilde{\mathbf{h}}_{1}^{T} & \widetilde{\mathbf{h}}_{K} b_{K}(n) b_{2}(n) \widetilde{\mathbf{h}}_{2}^{T} & \cdots & \widetilde{\mathbf{h}}_{K} b_{K}^{2}(n) \widetilde{\mathbf{h}}_{K}^{T}
\end{array}\right], \\
M_{12}(n)=\left[\begin{array}{cccc}
\widetilde{\mathbf{h}}_{1} b_{1}(n) b_{1}(n-1) \widetilde{\mathbf{h}}_{1}^{T} & \widetilde{\mathbf{h}}_{1} b_{1}(n) b_{2}(n-1) \widetilde{\mathbf{h}}_{2}^{T} & \cdots & \widetilde{\mathbf{h}}_{1} b_{1}(n) b_{K}(n-1) \widetilde{\mathbf{h}}_{K}^{T} \\
\widetilde{\mathbf{h}}_{2} b_{2}(n) b_{1}(n-1) \widetilde{\mathbf{h}}_{1}^{T} & \widetilde{\mathbf{h}}_{2} b_{2}(n) b_{2}(n-1) \widetilde{\mathbf{h}}_{2}^{T} & \cdots & \widetilde{\mathbf{h}}_{2} b_{2}(n) b_{K}(n-1) \widetilde{\mathbf{h}}_{K}^{T} \\
\vdots & \vdots & \ddots & \vdots \\
\widetilde{\mathbf{h}}_{K} b_{K}(n) b_{1}(n-1) \widetilde{\mathbf{h}}_{1}^{T} & \widetilde{\mathbf{h}}_{K} b_{K}(n) b_{2}(n-1) \widetilde{\mathbf{h}}_{2}^{T} & \cdots & \widetilde{\mathbf{h}}_{K} b_{K}(n) b_{K}(n-1) \widetilde{\mathbf{h}}_{K}^{T}
\end{array}\right], \\
M_{22}(n)=\left[\begin{array}{ccccc}
\widetilde{\mathbf{h}}_{1} b_{1}^{2}(n-1) \widetilde{\mathbf{h}}_{1}^{T} & \widetilde{\mathbf{h}}_{1} b_{1}(n-1) b_{2}(n-1) \widetilde{\mathbf{h}}_{2}^{T} & \cdots & \widetilde{\mathbf{h}}_{1} b_{1}(n-1) b_{K}(n-1) \widetilde{\mathbf{h}}_{K}^{T} \\
\widetilde{\mathbf{h}}_{2} b_{2}(n-1) b_{1}(n-1) \widetilde{\mathbf{h}}_{1}^{T} & \widetilde{\mathbf{h}}_{2} b_{2}(n-1) \widetilde{\mathbf{h}}_{2}^{T} & \cdots & \widetilde{\mathbf{h}}_{2} b_{2}(n-1) b_{K}(n-1) \widetilde{\mathbf{h}}_{K}^{T} \\
\vdots & \vdots & \ddots & \vdots \\
\widetilde{\mathbf{h}}_{K} b_{K}(n-1) b_{1}(n-1) \widetilde{\mathbf{h}}_{1}^{T} & \widetilde{\mathbf{h}}_{K} b_{K}(n-1) b_{2}(n-1) \widetilde{\mathbf{h}}_{2}^{T} & \cdots & \widetilde{\mathbf{h}}_{K} b_{K}^{2}(n-1) \widetilde{\mathbf{h}}_{K}^{T}
\end{array}\right], \tag{46}
\end{gather*}
$$

where, for the convenience, $\left\{\widetilde{\mathbf{h}}_{i}(n \mid n-1), i=1, \ldots, K\right\}$ have been replaced by $\left\{\widetilde{\mathbf{h}}_{i}\right\}$ without confusion. Then substituting (43) into (42), we can get (36).

In terms of the definition of projection, we know that

$$
\begin{equation*}
\widehat{\mathbf{b}}(n+1 \mid n)=\operatorname{Proj}\{\mathbf{b}(n+1) \mid \mathbf{r}(0), \ldots, \mathbf{r}(n)\} \tag{47}
\end{equation*}
$$

From the linear estimation theory, the linear space spanned by the innovation sequence $\left\{\mathbf{e}_{b}(s)\right\}_{s=0}^{n}$ contains the same information as the one spanned by the received signal sequence $\{\mathbf{r}(s)\}_{s=0}^{n}$; that is,

$$
\begin{equation*}
\mathscr{L}\{\mathbf{r}(0), \ldots, \mathbf{r}(n)\}=\mathscr{L}\left\{\mathbf{e}_{b}(0), \ldots, \mathbf{e}_{b}(n)\right\} \tag{48}
\end{equation*}
$$

Then the projection in (48) can be rewritten as

$$
\begin{align*}
\widehat{\mathbf{b}}(n+1 \mid n)= & \operatorname{Proj}\left\{\mathbf{b}(n+1) \mid \mathbf{e}_{b}(0), \ldots, \mathbf{e}_{b}(n)\right\} \\
= & \operatorname{Proj}\left\{\Phi \mathbf{b}(n)+\mathbf{w}(n) \mid \mathbf{e}_{b}(0), \ldots, \mathbf{e}_{b}(n-1)\right\} \\
& +\operatorname{Proj}\left\{\Phi \mathbf{b}(n)+\mathbf{w}(n) \mid \mathbf{e}_{b}(n)\right\} \\
= & \Phi \widehat{\mathbf{b}}(n \mid n-1)+\Phi K_{b}(n) \mathbf{e}_{b}(n), \tag{49}
\end{align*}
$$

where $K_{b}(n)$ is the parameter of the projection of $\mathbf{b}(n)$ onto $\mathbf{e}_{b}(n)$, which yields the stationary point of the following error Gramian matrix:

$$
\begin{equation*}
\left\langle\mathbf{b}(n)-K_{b}(n) \mathbf{e}_{b}(n), \mathbf{b}(n)-K_{b}(n) \mathbf{e}_{b}(n)\right\rangle, \tag{50}
\end{equation*}
$$

and satisfies

$$
\begin{align*}
K_{b}(n) Q_{e}^{b}(n)= & \left\langle\mathbf{b}(n), e_{b}(n)\right\rangle \\
= & \langle\mathbf{b}(n), C \widehat{H}(n \mid n-1) \widetilde{\mathbf{b}}(n \mid n-1)  \tag{51}\\
& +C \widetilde{H}(n \mid n-1) \mathbf{b}(n)+\mathbf{v}(n)\rangle \\
= & P_{b}(n) \widehat{H}^{T}(n \mid n-1) C^{T} .
\end{align*}
$$

Thus we have

$$
\begin{equation*}
K_{b}(n)=P_{b}(n) \widehat{H}^{T}(n \mid n-1) C^{T}\left[Q_{e}^{b}(n)\right]^{-1} \tag{52}
\end{equation*}
$$

In view of (52) and (53), it is apparent that we have

$$
\begin{equation*}
P_{b}(n+1)+\Phi K_{b}(n) Q_{e}^{b}(n) K_{b}^{T}(n) \Phi=\Phi P_{b}(n) \Phi+Q_{w}(n) \tag{53}
\end{equation*}
$$

which is (38).
In view of (49), the projection in (48) can be further given by

$$
\begin{align*}
\widehat{\mathbf{b}}(n+1 \mid n)= & \Phi \widehat{\mathbf{b}}(n \mid n-1)+\Phi K_{b}(n) \mathbf{e}_{b}(n) \\
= & \Phi\left[I_{2 K}-K_{b}(n) C \widehat{H}(n \mid n-1)\right] \\
& \times \widehat{\mathbf{b}}(n \mid n-1)+\Phi K_{b}(n) \mathbf{r}(n)  \tag{54}\\
= & \Phi \widehat{\mathbf{b}}(n \mid n),
\end{align*}
$$

where we have defined detector as

$$
\begin{align*}
\widehat{\mathbf{b}}(n \mid n) \triangleq & {\left[I_{2 K}-K_{b}(n) C \widehat{H}(n \mid n-1)\right] }  \tag{55}\\
& \times \widehat{\mathbf{b}}(n \mid n-1)+K_{b}(n) \mathbf{r}(n),
\end{align*}
$$

which is (34).

Remark 6. In Theorem 5, the UWB channel estimation error $\widetilde{H}(n \mid n-1)$ is considered as a multiplicative noise which is in matrix form, and the transmitted symbols may be colored and cross-correlated for different users. Their statistic characteristics are represented in the Lyapunov equation in (37). If the transmitted symbols are uncorrelated and identically distributed random variables with zero mean and unit variance, then the above result is equivalent to that proposed in [13] where the channel estimation error is considered as an additive noise.

Corollary 7. If one assumes the transmitted symbols are uncorrelated and identically distributed random variables with zero mean and unit variance, that is, $E\left\{\mathbf{b}(n) \mathbf{b}^{T}(n)\right\}=I_{2 K}$, then the information symbol detector is given by

$$
\begin{align*}
\widehat{\mathbf{b}}(n \mid n)= & {\left[I_{2 K}-K_{b}(n) C \widehat{H}(n \mid n-1)\right] } \\
& \times \widehat{\mathbf{b}}(n \mid n-1)+K_{b}(n) \mathbf{r}(n), \tag{56}
\end{align*}
$$

where

$$
\begin{align*}
K_{b}(n)= & P_{b}(n) \widehat{H}^{T}(n \mid n-1) C^{T}\left[Q_{e}^{b}(n)\right]^{-1}, \\
Q_{e}^{b}(n)= & C \widehat{H}(n \mid n-1) P_{b}(n) \widehat{H}^{T}(n \mid n-1) C^{T} \\
& +C\left(\left[\begin{array}{c|c}
P_{h}(n) & 0 \\
\hline 0 & P_{h}(n)
\end{array}\right] \circ\left[I_{2 K} \otimes J_{L+1}\right]\right) C^{T} \\
& +Q_{v}(n), \\
P_{b}(n+1)= & \Phi P_{b}(n) \Phi-\Phi K_{b}(n) Q_{e}^{b}(n) K_{b}^{T}(n) \Phi+Q_{w}(n), \tag{57}
\end{align*}
$$

with $J_{L+1}$ being the all-one matrix with dimension $(L+1) \times(L+$ 1) and $Q_{w}(n)=\left[\begin{array}{ccc}I_{K} & 0_{K, K} \\ 0_{K, K} & 0_{K, K}\end{array}\right]$.

Proof. The proof is straightforward. Note that the transmitted symbols are uncorrelated and identically distributed random variables with zero mean and unit variance; then we can easily obtain that $\Pi_{b}(n)=I_{2 K}$ and $E_{\{\tilde{\mathbf{h}}, b\}}\left\{M_{12}\right\}=0$. It is apparent that $Q_{w}(n)=\left[\begin{array}{cc}I_{K} & 0_{K, K} \\ 0_{K, K} & 0_{K, K}\end{array}\right]$.

## 4. Channel Estimator

For UWB channel estimation, the received discrete-time signal in (23) can be reexpressed as

$$
\begin{align*}
\mathbf{r}(n) & =\left[\begin{array}{llll}
r(n N) & r(n N+1) & \cdots & r(n N+N-1)
\end{array}\right]^{T} \\
& =\sum_{k=1}^{K} \mathbf{y}_{k}(n)+\mathbf{v}(n)  \tag{58}\\
& =C B(n) \mathbf{h}(n)+\mathbf{v}(n),
\end{align*}
$$

where $B(n)$ is the information symbol matrix and is defined as

$$
\begin{gather*}
B(n)=[\bar{B}(n) \bar{B}(n-1)]^{T}  \tag{59}\\
\bar{B}(n)=\operatorname{diag}\left\{b_{1}(n) I_{L+1}, b_{2}(n) I_{L+1}, \ldots, b_{K}(n) I_{L+1}\right\},
\end{gather*}
$$

and $\mathbf{h}(n)$ is as shown in (33), which can be modeled by using a first-order autoregressive (AR) model as

$$
\begin{equation*}
\mathbf{h}(n+1)=\Gamma \mathbf{h}(n)+\mathbf{u}(n), \tag{60}
\end{equation*}
$$

where $\mathbf{u}(n)$ is a white random variable with zero mean and covariance matrix $Q_{u}(n)$, and $\Gamma$ is the channel correlation matrix, given by

$$
\begin{equation*}
\Gamma=\operatorname{diag}\left\{a_{1}^{0}, \ldots, a_{1}^{L}, \ldots, a_{K}^{0}, \ldots, a_{K}^{L}\right\} \tag{61}
\end{equation*}
$$

where the scalar factor $\left\{a_{k}^{l}, k=1, \ldots, K, l=0, \ldots, L\right\}$ denotes the state transition coefficient of the $k$ th user in the $l$ th path. The above AR model for the channel parameter is only an approximation to the actual statistics of these random processes.

Similar to multiuser detection, for the UWB channel estimation the symbol matrix is treated as an unknown variable and uses the one-step prediction $\widehat{B}(n \mid n-1)$ instead of $B(n)$ and considers the estimation error $\widetilde{B}(n \mid n-1)$ as a multiplicative noise for channel estimation. The optimal estimation is given according to the following theorem.

Theorem 8. Consider the discrete-time state-space signal model (60) and (62); when the information symbol matrix $B(n)$ is unknown, the channel estimator is given by

$$
\begin{align*}
\widehat{\mathbf{h}}(n \mid n)= & {\left[I_{K(L+1)}-K_{h}(n) C \widehat{B}(n \mid n-1)\right] \widehat{\mathbf{h}}(n \mid n-1) } \\
& +K_{h}(n) \mathbf{r}(n), \tag{62}
\end{align*}
$$

where $\widehat{B}(n \mid n-1)$ is the one-step prediction of information symbol obtained from the previous section, and $K_{h}(n)$ is the estimator gain matrix:

$$
\begin{equation*}
K_{h}(n)=P_{h}(n) \widehat{B}^{T}(n \mid n-1) C^{T}\left[Q_{e}^{h}(n)\right]^{-1} \tag{63}
\end{equation*}
$$

where $Q_{e}^{h}(n)$ is the covariance matrix of innovation $\mathbf{e}_{h}(n)$ :

$$
\begin{align*}
Q_{e}^{h}(n)= & C \widehat{B}(n \mid n-1) P_{h}(n) \widehat{B}^{T}(n \mid n-1) C^{T} \\
& +C\left(\left[\frac{\Pi_{h}(n) \mid \Pi_{h}(n)}{\Pi_{h}(n) \mid \Pi_{h}(n)}\right] \circ\left[P_{b}(n) \otimes I_{L+1}\right]\right) C^{T} \\
& +Q_{v}(n) \tag{64}
\end{align*}
$$

where $P_{b}(n)$ is the information symbol estimation error covariance matrix which is obtained in the previous section, and $\Pi_{h}(n)$ satisfies the following Lyapunov equation:

$$
\begin{equation*}
\prod_{h}(n+1)=\Gamma \prod_{h}(n) \Gamma^{T}+Q_{u}(n) \tag{65}
\end{equation*}
$$

where $P_{b}(n)$ is the channel estimation error covariance matrix and satisfies the following Riccati equation:

$$
\begin{equation*}
P_{h}(n+1)=\Phi P_{h}(n) \Phi-\Gamma K_{h}(n) Q_{e}^{h}(n) K_{h}^{T}(n) \Gamma+Q_{u}(n) \tag{66}
\end{equation*}
$$

The one-step prediction of the information symbol is given by

$$
\begin{equation*}
\widehat{\mathbf{h}}(n+1 \mid n)=\Gamma \widehat{\mathbf{h}}(n \mid n) \tag{67}
\end{equation*}
$$

which will be used for the design of the user detector.


Figure 2: The proposed algorithm structure.

Proof. Consider the information symbol matrix $B(n)$ as an unknown variable; then we have

$$
\begin{equation*}
\widehat{\mathbf{r}}(n \mid n-1)=C \widehat{B}(n \mid n-1) \widehat{\mathbf{h}}(n \mid n-1) \tag{68}
\end{equation*}
$$

In view of (7) and Definition 4, we obtain

$$
\begin{align*}
\mathbf{e}_{h}(n)= & C \widehat{B}(n \mid n-1) \widetilde{\mathbf{h}}(n \mid n-1)  \tag{69}\\
& +C \widetilde{B}(n \mid n-1) \mathbf{h}(n)+\mathbf{v}(n)
\end{align*}
$$

It is apparent that $\mathbf{e}_{h}(n)$ is with zero mean and $E\left\{\mathbf{e}_{h}(s) \mathbf{e}_{h}(j)\right\}=$ 0 if $s \neq j$. The covariance matrix of $\mathbf{e}_{h}(n)$ is denoted as $Q_{e}^{h}(n)$. The following proof of this theorem is very similar to that of Theorem 5, so we omit it here.

Remark 9. Different from [7, 10], for UWB channel estimation, the users' symbols are also considered as unknown variables in this paper. The one-step prediction of symbol matrix is used, and the estimation error is treated as a multiplicative noise in matrix form. The detector and channel estimator are designed jointly and cannot be solved separately. The algorithm structure is as shown in Figure 2.

## 5. Conclusions

The information symbol and channel parameter are considered as unknown variables in this paper. The multiuser detector and UWB channel estimator are designed jointly. For symbol detection, the one-step predictor of channel parameter is used and the estimation error is treated as a multiplicative noise; then a Riccati equation and a Lyapunov equation will be needed. If the transmitted symbols are uncorrelated and identically distributed random variables with zero mean and unit variance, only a Riccati equation needs to be solved. For UWB channel estimation, the one-step predictor of information symbol is used and the estimation error is also considered as a multiplicative noise. The solutions to the above two problems are obtained by solving a couple of Riccati equations together with two Lyapunov equations.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding to the publication of this paper.

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# Research Article 

# Simulation Analysis and Model of Current Retrieval Based on Marine Radar Sea Clutter Images 

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#### Abstract

Using the sea clutter image from X-Band radar for current retrieval is an effective way of obtaining information on ocean currents. Traditional methods used for current retrieval have been based on the least squares algorithm, which is not only simple and efficient but also generally speaking accurate. In order to improve the precision of current retrieval, this paper has, as its goal, the study of the used radar connected with sea clutter imaging for current retrieval, with the particle swarm optimization (PSO) algorithm being proposed. This method is achieved by obtaining a three-dimensional image spectrum, taking the high-order dispersion relation model as the theoretical distribution model of the wave energy points of three-dimensional image spectra, using a forward model within the PSO framework, and considering the requirements of the order of the model, weights and optimal distribution of the energy points, and so on in fitness function. Simulation results show that, compared with the traditional ILSM methods, the method provided in this paper is more flexible, with a capacity for a high dispersion relationship order, higher precision, and an increased stability in terms of current inversion.


## 1. Introduction

Ocean currents are the result of a variety of physical effects arising from a relatively stable large-scale flow. Ocean currents have a very close relationship for marine exploration, marine development, and marine navigation safety. Existing methods for obtaining the sea surface current include moored wave buoys, an analysis of stereo images, and analyses of satellite altimetry data and marine radar images. Although buoys provide reliable measurements, they are easily subject to damage and loss. The method for collecting stereo images from synthetic aperture radar is costly and time-consuming. Furthermore, orbiting satellites cannot obtain continuous data around a specified zone [1-3]. The marine radar of current telemetry is an effective means of obtaining currents. Compared with other methods, it has the advantages of wide range of detection, high safety, all-day observation, convenience, and the low price.

Current retrieval based on marine radar related to sea clutter images is essentially an optimization problem, with the least squares method (LSM) being used to implement
the current strategy for solving the problem. In 1985, Young et al. were the first to suggest the use of marine radar sea clutter imaging for current retrieval [4]. A three-dimensional fast Fourier transformation (3-D FFT) was used to transform the time domain of the sea clutter image sequence into the frequency domain spectra of the three-dimensional image, and then the LSM was used for the current retrieval. As far as the high currents in the dispersion relation of the inversion effect were concerned, in 2001 Senet and others proposed a method for current retrieval based on the iterative least squares method (ILSM) [5], increasing the accuracy of sea surface stream retrieval, based on methods put forward in the literature on the subject [4]. In 2002, in their work concerning the effect of the three-dimensional image spectrum of the energy value of each point, Gangeskar proposed a method of current retrieval based on the three-dimensional image spectrum weighted least squares and applied the equation based on the regularization method, thus improving the precision of current retrieval [6]. In 2010, using the dispersion relation retrieval current target function, Tang proposed an iterative method for current estimation based on the

| Sea clutter image sequence | 3D FFT | Three-dimensional <br> image spectrum | Dispersion relation | Sea surface <br> current |
| :---: | :---: | :---: | :---: | :---: |

Figure 1: Schematic diagram illustrating current retrieval.


Figure 2: Sea clutter image data for current retrieval.
minimum variance error sequence [7] and emphasized the effect of overall energy collection on the estimated curve on the basis of the relevant literature [5]. In 2011, using the threedimensional image spectrum to extract information related to the wave spectrum as accurately as possible in order to improve the retrieval accuracy of currents, Yuan and others proposed an iterative method for estimating currents based on adaptive threshold selection [8] on the basis of the relevant literature [5].

The least squares strategy, when adopted as the method for current retrieval, has the advantage of a simple, fast solution and certainty, but, in the terms of constraint handling, processing complex optimization problems, nonlinear optimization, and so on, it has obvious shortcomings. In addition, its objective function is inflexible. In order to rectify this problem, the PSO algorithm is proposed here as a model for current retrieval. Using this algorithm, with the flexible design of this objective function, constraint condition positive expression, self-organization solving, and other characteristics for current retrieval, this model for current retrieval is discussed with a view to assess the observational data selection, the initial selection, fitness function design, process of algorithm design, and other related issues in detail. The numerical simulation results verify the correctness of the proposed method. It improves the precision and stability of the current retrieval effectively and provides a new solution for the modeling method of the current retrieval based on marine radar.

## 2. Current Retrieval Fundamentals

The method used for current retrieval based on the marine X-Band radar related to sea clutter images was proposed by Young et al. [4] in 1985, using the Fourier transforms spectrum analysis framework, as illustrated in Figure 1.
2.1. Three-Dimensional Image Spectrum Acquisition. In order to obtain a radar three-dimensional image spectrum, threedimensional fast Fourier transformation (3-D FFT) for sea clutter images sequence continuously measured by marine radar is required, to enable us to transform the time and space domain of radar sea clutter image sequencing into the frequency domain of a radar image spectrum.

As shown in Figure 2, before the transformation operation can be performed, a rectangular analysis area from the sea clutter image sequence must be selected, and, through filtering and interpolation, the polar coordinates of the image sequence are transformed into $\eta(x, y, t)$, namely, the Cartesian coordinate system connected with grid image sequencing.

With the grid image sequence $\eta(x, y, t)$ being used for the 3-D FFT, the three-dimensional radar image spectrum $F\left(k_{x}, k_{y}, \omega\right)$ is obtained:

$$
\begin{align*}
& F\left(k_{x}, k_{y}, \omega\right) \\
& \quad=\int_{0}^{L_{x}} \int_{0}^{L_{y}} \int_{0}^{T} \eta(x, y, t) \cdot e^{-i\left(k_{x} \cdot x+k_{y} \cdot y-\omega \cdot t\right)} d x d y d t \tag{1}
\end{align*}
$$

where $L_{x}$ and $L_{y}$ are the rectangular analysis area, $x$ and $y$ are the direction of the spatial scale, respectively, and $T$ is the time scale of the image sequence, with $k_{x}, k_{y}$ being the wave number of $x$ and $y$, respectively, and $\omega$ the frequency. The resolutions of wave number and frequency are

$$
\begin{equation*}
d k_{x}=\frac{2 \pi}{L_{x}}, \quad d k_{y}=\frac{2 \pi}{L_{y}}, \quad d \omega=\frac{2 \pi}{T} \tag{2}
\end{equation*}
$$

Considering the symmetry of the Fourier transforms and in order to eliminate the $180^{\circ}$ ambiguity problems, $\omega>0$ are the only parts retained. The power spectrum density of the image is thus obtained:

$$
\begin{equation*}
I\left(k_{x}, k_{y}, \omega\right)=\frac{1}{L_{x} L_{y} T}\left|F\left(k_{x}, k_{y}, \omega\right)\right|^{2} \tag{3}
\end{equation*}
$$

2.2. Relation of Dispersion. Assuming that the waves satisfy the linear wave theory, homogeneous space, and stable time of the sea surface current filed in the analysis area, then when the water depth is greater than or equal to half the wavelength, a first-order approximation gravity wave will satisfy the following dispersion relation equation [9-12]:

$$
\begin{equation*}
\omega_{0}(\vec{k})=\sqrt{g|\vec{k}| \tanh (d|\vec{k}|)} \tag{4}
\end{equation*}
$$

where $\omega_{0}$ represents the wave frequency of gravity, $\vec{k}$ represents the wave number of the wave, $d$ represents depth, and $g$ represents the acceleration of gravity.

Considering that the presence of the currents $\vec{u}$ will result in the generation of the Doppler frequency-shift, then the current $\vec{u}$ will be the same as wave number $\vec{k}$ and the angular frequency $\omega$ generating a frequency-shift affecting the image spectrum $F\left(k_{x}, k_{y}, \omega\right)$, with the dispersion relation equation as follows:

$$
\begin{equation*}
S(\stackrel{\rightharpoonup}{k})=\sqrt{g|\vec{k}| \tanh (d|\vec{k}|)}+\stackrel{\rightharpoonup}{k} \cdot \stackrel{\rightharpoonup}{u} \tag{5}
\end{equation*}
$$

The surface schematics of dispersion relation in various cases are shown in Figure 3.

The detection process carried out by the marine radar found that nonlinear effects are caused by the influence of sea surface imaging and the relative weakness of the sea surface waves themselves. The wave energy of the threedimensional image spectrum $F\left(k_{x}, k_{y}, \omega\right)$ of the radar exist not only in the base (level 0 ) dispersion relation equation but also in the higher-order dispersion relation equation, which is referred to as the high-order wave phenomena. The high-order dispersion relation equation can be expressed as follows:

$$
\begin{equation*}
\omega_{p}=(p+1) \sqrt{\frac{g|\vec{k}|}{p+1} \cdot \tanh \left(\frac{d|\vec{k}|}{p+1}\right)}+\vec{k} \cdot \vec{u} \tag{6}
\end{equation*}
$$

where $p$ is the order of the dispersion relation and $\omega_{p}$ is the frequency of the wave of $p$ order.
2.3. The Basic Idea behind Current Retrieval. After $n$ sea clutter images have been measured consecutively by marine radar using the 3-D FFT, $\omega>0$ is the reserved part and the resulting spectrum of the radar image is the energy point set $I\left(k_{x}, k_{y}, \omega\right)$ distributed on the spatial grid $k_{x} \times k_{y} \times(n / 2)$. The current retrieval process involves the extraction of the energy points of the wave spectrum from the energy point collection, with (6) being used to determine the current $\vec{u}$, so that all the energy points of the wave spectrum are distributed in the parameters $\vec{u}$ of the dispersion relation surface.

## 3. Current Retrieval Based on the Least Squares Algorithm

Current retrieval was first proposed by Young et al. in 1985 in the literature [4] on the subject and was based on the average weighted least squares. All the energy points in $I\left(k_{x}, k_{y}, \omega\right)$ of this method are in line with the 0 -order dispersion relation. For any point $(\vec{k}, \omega)$, the theory of the frequency of the point $\vec{k}$ is calculated by (5). The square of the SSE of the difference between $\Delta \omega$ is

$$
\begin{equation*}
\mathrm{SSE}=\sum\left(\omega-\sqrt{g|\vec{k}| \tanh (d|\vec{k}|)}-k_{x} u_{x}-k_{y} u_{y}\right)^{2} \tag{7}
\end{equation*}
$$

The principle of LSM is used. The minimum of SSE is needed to find out the optimal value of $u_{x}$ and $u_{y}$ and can be obtained as follows:

$$
\begin{align*}
{\left[\begin{array}{l}
u_{x} \\
u_{y}
\end{array}\right]=} & {\left[\begin{array}{cc}
\sum k_{x}^{2} & \sum k_{x} k_{y} \\
\sum k_{x} k_{y} & \sum k_{y}^{2}
\end{array}\right]^{-1} } \\
& \times\left[\begin{array}{l}
\sum(\omega-\sqrt{g|\vec{k}| \tanh (d|\vec{k}|)}) k_{x} \\
\sum(\omega-\sqrt{g|\vec{k}| \tanh (d|\vec{k}|)}) k_{y}
\end{array}\right] \tag{8}
\end{align*}
$$

In 2001, in their consideration of the impact of the higherorder dispersion relation based on the LSM method, Senet et al. proposed a current estimation method based on the Iterative Least Squares Method (ILSM) [5]. First of all, a larger threshold $C_{\mathrm{FG}}$ is selected, with the observed data ( $\vec{k}, \omega$ ) of spectral energy greater than $C_{\mathrm{FG}}$ being retained in $I\left(k_{x}, k_{y}, \omega\right)$ in accordance with the 0 -order dispersion relation of energy points, and the initial estimate currents $\vec{u}_{0}$ are calculated by using (8). Then the dispersion relation order $p$ included in the algorithm is determined, $\vec{u}_{0}$ is substituted, and the corresponding values of $\vec{k}$ of the order of theoretical frequency $\omega_{i}(i=0,1, \ldots, p)$ are obtained. A threshold value $C_{\mathrm{IT}}$, which is much smaller than $C_{\mathrm{FG}}$, is then selected; the spectral energy of the energy point that is greater than $C_{\text {IT }}$ is reserved in $I\left(k_{x}, k_{y}, \omega\right)$, with the observed data $(\vec{k}$ , $\omega^{\prime}$ ) being obtained. By comparing the distance of $\omega^{\prime}$ and


Figure 3: Surface schematics of dispersion relation.
$\omega_{i}(i=0,1, \ldots, p)$, the order of the observed data is decided. In line with the new 0 -order dispersion relation of energy points, current $\overrightarrow{u_{0}^{\prime}}$ is calculated by using (8); the above process is then repeated, with the current being calculated by iteration.

In 2002, Gangeskar put forward the weighted least squares method [6] for currents estimation based on the radar three-dimensional image spectrum considering the influence of the power spectrum on the basis of the LSM method. The objective function pulls in the spectral energy $I\left(k_{x}, k_{y}, \omega\right)$ as shown in (7), so that

$$
\begin{equation*}
\mathrm{SSE}=\sum I \cdot\left(\omega-\sqrt{g|\vec{k}| \tanh (d|\vec{k}|)}-k_{x} u_{x}-k_{y} u_{y}\right)^{2} \tag{9}
\end{equation*}
$$

The LSM used to obtain the current estimation value is

$$
\begin{align*}
{\left[\begin{array}{l}
u_{x} \\
u_{y}
\end{array}\right]=} & {\left[\begin{array}{cc}
\sum I k_{x}^{2} & \sum I k_{x} k_{y} \\
\sum I k_{x} k_{y} & \sum I k_{y}^{2}
\end{array}\right]^{-1} } \\
& \times\left[\begin{array}{l}
\sum I k_{x}(\omega-\sqrt{g|\vec{k}| \tanh (d|\vec{k}|)}) \\
\sum I k_{y}(\omega-\sqrt{g|\vec{k}| \tanh (d|\vec{k}|)})
\end{array}\right] \tag{10}
\end{align*}
$$

In 2010, Tang considered the use of the overall attributes belonging to the dispersion relation set and,by improving the objective function in the framework of the ILSM algorithm, they proposed a current retrieval method of minimum
variance based on error sequence [7]. In this method, the objective function is written; thus

$$
\begin{equation*}
\mathrm{SSE}=\sum(\Delta \omega-\Delta \bar{\omega})^{2} \tag{11}
\end{equation*}
$$

where $\Delta \omega=\omega-\sqrt{g|\vec{k}| \tanh (d|\vec{k}|)}-k_{x} u_{x}-k_{y} u_{y}, \Delta \bar{\omega}=$ $(1 /(M \cdot N)) \sum \Delta \omega$.

## 4. Current Retrieval Model Based on PSO

4.1. Observational Data Selection. Current retrieval using observational data is the point set of energy in the power spectral density $I\left(k_{x}, k_{y}, \omega\right)$. These energy points can be divided into three kinds: the wave energy points of the 0 order dispersion relation, the wave energy points of the high-order dispersion relation, and the energy points of the background noise. The energy points of background noise must be filtered out before the current retrieval is carried out, with only the energy points of the wave being retained. As far as the filtering out of background noise is concerned, three different methods are discussed below.
(1) Overall Selection. The total number of energy points in $I\left(k_{x}, k_{y}, \omega\right)$ is used in current retrieval. When background noise points represent lesser and their energy value are low, the overall energy of the noise is much less than the overall wave energy, making the selection of the total number of energy points simple and effective. When there is a lot of background noise or the energy value is high, the overall energy of the noise cannot be ignored, so that this selection method is effective in terms of current retrieval, and the ability to adapt to the retrieval method is greater. In addition,
the selection of all the energy points extends the computation time of current retrieval.
(2) Fixed Number Selection. Assuming $I\left(k_{x}, k_{y}, \omega\right)$, the energy value of the high points corresponds to the wave energy points, with all the energy points in descending order. This is followed by the selection of a certain number of higher energy points to represent current retrieval of observational data. With this method, the observed data points are more complex. When the wave is small, the wave energy point is also less, so that it is only necessary to select a small number of high energy points. When the waves are large, the number selected must be increased or the wave energy will not be calculated.
(3) Selection of Energy Percentage. The maximum energy value of $I\left(k_{x}, k_{y}, \omega\right)$ up to a certain percentage is representative of the threshold, and all the energy points above the threshold are retained as the observational data of current retrieval. This approach assumes that, regardless of the circumstances, the energy of the noise points is lower. The highest relative energy value selected as the threshold is selected.
4.2. Initial Selection. As far as the PSO algorithm is concerned, the location of the information of particles is the optimization object of the algorithm. In this paper, for the current retrieval, the particle's position is the current component of $u_{x}$ and $u_{y}$. The particle dimensions should thus be two-dimensional.

The selection of the initial value of the position and speed has a bearing on the PSO. As far as the initial value of the position is concerned, if it is relatively close between the initial position and the distance of the optimal point, the initial position of the fitness value of the particle is high, which is easy to find the optimal solution for particles quickly, and if it is far between the initial position and the distance of the optimal point, the algorithm will increase the optimization time. For the initial value of the speed, if the initial value of the speed is much big, the particle can jump over a wide range in the search space and it is easy for these particles to exceed the permissible range. As far as the initial value of the speed is small, a particle that only moves within a small area is not conducive to global optimization. Generally, the initial value of position and speed of the actual solution are randomly selected within the permissible range.

In order to speed up the current retrieval and shorten the time of optimization as much as possible, the calculation of the value of the algorithm using the formula specified in (10) as the benchmark positions $\bar{u}_{x}$ and $\bar{u}_{y}$, as the initial position of the particle, is chosen as follows:

$$
\begin{align*}
& u_{x i}=\bar{u}_{x}+\frac{\left|\bar{u}_{x}\right|}{2} N(0,1) \\
& u_{y i}=\bar{u}_{y}+\frac{\left|\bar{u}_{y}\right|}{2} N(0,1) \tag{12}
\end{align*}
$$

Due to the fact that the selection of the initial position has a particular directional meaning, the initial speed value can also be appropriately small selected.
4.3. Fitness Function. In the process of current retrieval, a PSO algorithm with an adaptive value function evaluates the advantages and disadvantages of the position (current component) of each particle. The design of the fitness function is, therefore, particularly important. In this paper, the algorithm fitness function design is defined as follows:

$$
\begin{align*}
& \operatorname{Fit}_{k}\left(u_{x}, u_{y}\right) \\
& =\sum_{p} A \cdot \min _{p} \left\lvert\, \omega-(p+1) \sqrt{\frac{g|\vec{k}|}{p+1} \cdot \tanh \left(\frac{d|\vec{k}|}{p+1}\right)}\right. \\
&  \tag{13}\\
& \quad-k_{x} u_{x}-k_{y} u_{y} \mid
\end{align*}
$$

where $A$ is weight, $p$ is the order of the dispersion relation when the fitness value is taken into account, and $n$ is the order of the deviation. The weight of $A$ can be selected as any number greater than 0 , for example, 1 or $I\left(k_{x}, k_{y}, \omega\right)$. The dispersion relation order $p$ is associated with actual highorder effects, with generally 2 -order being taken as the highest order. The deviation order $n$ is able to adjust the deviation and weight $A$ for the influence of the fitness function.

Let us assume that the weight $A$ of each observational data point $\left(k_{x}, k_{y}, \omega\right)$ is given and remains unchanged. When $n=1$, the deviation order is able truly to reflect the deviation value effect on the fitness function; when $n>1$, it is equivalent to amplifying the effects of the deviation value on fitness function as well as being equivalent to weakening the role of weight $A$. When $n \rightarrow \infty$, it is equivalent to the weight $A=1$, so that all the retrieval results relative to the observed data points are evenly distributed in the vicinity of the surface dispersion relation. When $0<n<1$, it is equivalent to weakening the image of the deviation on the fitness function and to strengthening the role of the weight $A$. When $n \rightarrow 0$, it is equivalent to the role of the point with the greatest weight being magnified infinitely, so that the results of the current retrieval make the observational data points of the larger weights distribute evenly close to the surface of dispersion relation.

As far as the method of current retrieval is concerned, in the framework based on the LSM, the deviation coefficient of the objective function SSE is 2 , which is equivalent to weakening the role of the weights.
4.4. Algorithmic Process. The process based on the current retrieval of PSO used in this paper is shown in Figure 4.

Step 1 (initialization). Set the learning factors $c_{1}, c_{2}$, the inertia weight $w$, the initial position of the particle (currents), and the speed.


Figure 4: The algorithmic process.

Step 2. The fitness function is used to assess the current position of each particle to obtain the fitness value of the current position of each particle.

Step 3. The current position fitness value of each particle and the individual extreme pbest are compared and, if it is better, the fitness value is then used with the current position fitness value to update the individual extreme pbest and record the current position.

Step 4. The individual extreme value pbest of each particle is compared with the group's global extreme value gbest. If it is better, gbest is updated, and the best position recorded.

Step 5. If the termination condition is deemed to have been satisfied, then the global extremum gbest and its corresponding global positions are output; otherwise, the speed and position of the particle are updated, and "Step 2" is implemented.

## 5. Simulation Analyses

5.1. Simulation Based on Three-Dimensional Image Spectrum. Current retrieval essentially uses the energy points of a threedimensional image spectrum as observed data, according to the dispersion relation for optimal estimation. The method
used for the dispersion relation used to simulate a threedimensional image spectrum is outlined as follows.
(1) Three-Dimensional Image Spectrum Simulation. It is very difficult to acquire a large number of sea clutter image sequences from marine radar and their corresponding true value. A point on a three-dimensional radar image spectrum imitatively generated based on a dispersion relation equation is proposed in this paper, and simulation experiments of current retrieval precision are carried out using the image spectrum.

Based on the previous analysis, we already know that the energy points of the three-dimensional image spectrum can be divided into two categories in line with the wave energy points of a dispersion relation and do not meet the noise energy points of the dispersion relation. These two types of energy points are therefore sufficient when the threedimensional image spectrum is generated in the simulation. This simulation takes place for a three-dimensional image spectrum according to the following principles.

Principle 1. The dispersion relation only considers three times of the 0 -order, 1 -order, and 2 -order.

Principle 2. The dispersion relation of energy points of the 0 -order is 1,1 -order is 0.2 , and 2 -order is 0.04 .

TABLE 1: Operating parameters of PSO algorithm.

| $m$ | $c_{1}$ | $c_{2}$ | $w_{\text {ini }}$ | $w_{\text {end }}$ | $T_{\max }$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 10 | 2.05 | 2.05 | 0.9 | 0.4 | 100 |

Principle 3. The energy point standard for background noise is 0.01 , the number of wave energy points is $20 \%$, and the location is set at random.

The simulation process of the three-dimensional image spectrum is shown in Figure 5.

The formula of each energy value during the simulation of energy points of three-dimensional image spectrum is as follows:

$$
\begin{equation*}
E\left(k_{x}, k_{y}, \omega\right)=\bar{E}+\frac{\bar{E}}{2} N(0,1) \tag{14}
\end{equation*}
$$

where $E\left(k_{x}, k_{y}, \omega\right)$ is the energy value of the point and $\bar{E}$ is energy reference value.
(2) Adaptive Ability of Dispersion Relation Order Simulation. The three-dimensional images spectrum is generated considering 0 -order and 1 -order of dispersion relation by using the above method. Within the three-dimensional image spectrum, the current speed changes from $0 \mathrm{~m} / \mathrm{s}$ to $6 \mathrm{~m} / \mathrm{s}$, while the current direction changes from 0 to $2 \pi$; then the current speed changes from $6 \mathrm{~m} / \mathrm{s}$ to $0 \mathrm{~m} / \mathrm{s}$, while the current direction changes from 0 back to $2 \pi$ again. 32 groups of corresponding data points are generated.

Both the ILSM method and the method proposed here (using the PSO representation) are used for current retrieval. Among the three cases of the highest order 0,1 , and 2 of dispersion relations are, respectively, taken into consideration using the ILSM method, and, in this paper, 2-order is only considered as being the highest order. The solutions of wave power point distribution on the different orders of dispersion in three-dimensional image spectrum are counted in different simulation experiments.

150 maximum energy points from the three-dimensional image spectrum are selected as the observed data on which to carry out the simulation experiments. If the difference of current speed is continuous less than 0.1 two times, the ILSM method will be stopped. When the current speed of optimal particles remains unchanged for consecutive 10 times, the PSO method will be stopped. In the PSO algorithm, $X_{d}^{\max }$ is taken as $5, V_{d}^{\max }$ is taken as 2 , and the deviation order $n$ is 1 in the fitness function, with the empirical parameters given in [13] being used for the other parameters. The specific values are shown in Table 1.

Different simulation results are given in Table 2. The wave energy points of the three-dimensional image spectrum at the different orders of dispersion relation are distributed over the surface.

Table 2 shows that, although the three-dimensional image imitatively generated just contains 0 -order and 1 -order, some points are still attributed to the second-order dispersion relation when ILSM method considers the highest order two. In other words, in terms of the relation of dispersion order,


Figure 5: Simulation chart representing three-dimensional image spectrum.
the ILSM does not have the adaptive capacity. From the simulation results of the PSO, although the highest order of dispersion relation takes into account the second-order, the method has the adaptive ability for the order and it can identify the three-dimensional image spectrum that does not contain two order data points for the dispersion relation.

It is apparent that, when ILSM is used for current retrieval, the requested dispersion relation order should correspond to the actual dispersion relation order contained in three-dimensional image spectrum, while the order is not necessary correspondent in the PSO for the current retrieval, so that only the highest dispersion relation order needs to be set.
(3) Simulation of Precision for Current Retrieval. In this simulation, ILSM and PSO are used to simulate a threedimensional image spectrum for current retrieval by calculating the variance of current speed and current direction and evaluating the precision of the current retrieval in the two methods. The simulation methods and parameters are selected as above; the main difference is only in the following two aspects.
(a) The simulation of the three-dimensional image spectrum is taken into 2 -order, with the ILSM and PSO methods also being taken into a 2 -order situation.
(b) The PSO method is random. In order to evaluate the method as accurately as possible, it is necessary to count it 10 times. The mean of the 10 results and the results of 10 times for the optimum value of the standard of current speed are used to calculate the variance.

The simulation curve representing the simulation and variance of current retrieval are given in Figure 6 and Table 3, respectively.


Figure 6: Simulation experiment results of current retrieval precision based on simulated three-dimensional image spectrum.

The results of the simulation show that, compared with the current retrieval results obtained by the ILSM method, the PSO method obtains better results, especially at current speed parameters. When the current speed is greater than 4 m per second, the current retrieval precision using PSO is obviously superior to the ILSM.
5.2. Simulation Based on Imitation Sea Clutter Images. The imitative radar sea clutter image sequence derived from literature [14] is provided in this section as the simulation radar data. In the process of generating imitative radar sea clutter image sequences, the method of adding current information is the same as in Section 5.1, generating 32 sets of files of sea clutter image sequences.

Due to the fact that the dispersion relationship in the three-dimensional image spectra is unknown, corresponding
to simulation-generated radar sea clutter images, 0 -order, 1 -order, and 2 -order are taken into account in the ILSM method, with only 2 -order being taken into account in the PSO method. Simulation parameter selection and methods are consistent with Section 5.1. The simulation curve and current retrieval variance are given in Figure 7 and Table 4. The ILSM results are given in Figure 7 in which the case of 0 -order is considered.

The simulation results show that PSO method obtains a higher degree of precision than the ILSM method for curve retrieval of simulative radar sea clutter images.

### 5.3. Simulation Experiment Based on Real Sea Clutter Images.

 In the simulation experiment, real data from a radar sea clutter image sequence measured in Pingtan, Fujian, Haitan Island, China, on October 23, 2010, were used.

Figure 7: Results of simulation experimental of current retrieval precision based on imitation sea clutter images.

Table 2: Distributive point statistics on curve of different orders of dispersion relation.

| Method for estimated current | Order of dispersion relation | Total number of distribution points |
| :--- | :---: | :---: |
| ILSM |  |  |
| 0 -order only considered | 0 -order | 5473 |
| 1-order the highest consideration | 0 -order | 5201 |
|  | 1-order | 272 |
| 2 -order the highest consideration | 0 -order | 5201 |
|  | 1 -order | 233 |
|  | 2-order | 39 |
| PSO (2-order the highest consideration) | 0-order | 5139 |
|  | 1-order | 334 |




Figure 8: Simulation results of different data selection method.
Table 3: The variance of current retrieval based on a simulated three-dimensional image.

| Method | Current speed variance | Current direction variance |
| :--- | :---: | :---: |
| ILSM | 2.3651 | 2.2066 |
| PSO (mean) | 0.0470 | 1.3481 |
| PSO (optimal value) | 0.0025 | 1.4861 |

(1) Design of Evaluation Indicators. Generally speaking, the current retrieval method is assessed by comparing the error between current retrieval results and the true value. However, due to the fact that the true value of the corresponding region and currents could not be obtained from the real sea clutter image sequence from radar in the simulation experiment, the algorithms could not be properly assessed.

Consider that, for certain area current filed, its speed and directional values within the space range are uniform with small changes occurring over time. That is to say, the current changes occurring at adjacent times were small. In view of this, the mean of the differences in the continuous current retrieval results was taken as the performance evaluation indicator of the current retrieval. The calculations were made according to

$$
\begin{equation*}
e=\frac{1}{n-1} \sum_{i=1}^{n-1}\left|u_{i+1}-u_{i}\right| \tag{15}
\end{equation*}
$$

(2) Simulated Comparison of Dispersion Relationship Order Based on the Method of LSM. The ILSM method only considering three solutions of 0 -order, 1 -order of the highest order, and 2 -order of the highest order is used for this simulation to carry out the current retrieval. The different situations related to current retrieval performance indicators are compared in order to determine the image sequence for the real sea clutter data from radar. The ILSM method should be considered the highest dispersion relationship order.

The simulated parameter selection is consistent with Section 5.1. Table 5 provides the evaluation indicator for the simulation experiment.

The ILSM method can obtain the best current estimation results which are shown in current speed indicators in Table 5, only considering the 0 -order dispersion relationship. The ILSM method is sometimes the same as the LSM method when the simulation using real radar sea clutter images sequence data contains only 0 -order dispersion relationships.

## (3) Observable Data Selection Method Simulation Compari-

 son. Simulation experiments performed on the three observational data selection methods discussed in Section 4.1 are carried out in order to evaluate the performance of the LSM and PSO methods using a different selected strategy for the observational data. The simulation curve and evaluation indicators are given in Figure 8 and Table 6.Simulation results show that, as the selected data points increase, the current retrieval results using the PSO method are improved in terms of the polymerization and stability of the data in question. When all the data points are selected, the current retrieval is not affected by noise points, but better results are obtained. As far as the LSM method is concerned, the stability of the current retrieval results is best when 1000 points are selected. When 150 data points and higher $1 \%$ of maximum energy value are selected, current retrieval results deteriorate slightly and when all energy points are selected, current retrieval results deviate markedly from the

TABLE 4: Inversion variance based on simulation sea clutter images.

| Method | Current speed variance | Current direction variance |
| :--- | :---: | :---: |
| ILSM (0-order) | 1.2745 | 2.0465 |
| ILSM (1-order) | 1.0989 | 2.0656 |
| ILSM (2-order) | 1.0648 | 2.0656 |
| PSO (average) | 0.9972 | 1.8284 |
| PSO (optimal value) | 0.8553 | 1.9255 |

TAble 5: ILSM method evaluation indicators of different orders.

| Method | Current speed indicators | Current direction indicators |
| :--- | :---: | :---: |
| ILSM (0-order) | 0.0937 | 0.8708 |
| ILSM (1-order) | 0.1139 | 0.6960 |
| ILSM (2-order) | 0.1165 | 0.6918 |

true value, although an increased stability can be obtained. If we compare the PSO and LSM methods, the PSO method is slightly less stable than the LSM method when 150 points are selected, while the performance of the PSO method is better in the other cases.
(4) Initial Value Method Simulation Comparison. Simulation is carried out using the different initial values selected for the PSO method and discussed in Section 4.2. The initial value is selected using the random initialization method and the algorithm initial current speed being randomly selected with the range $[-3,3]$. When the initial current speed is selected for the LSM initialization method, the initial current speed is shown in (12). Simulation data points are selected according to the percentage of energy. Selection parameters are consistent with Section 5.1. Evaluation indicators of the simulation experiment are given in Table 7.

Evaluation indicators show that the stability of current retrieval is consistent in the two strategies with speed optimization showing that the strategy for the initial value of the LSM results is faster and the number of algorithm iterations is less.

## (5) Deviation Order Selection Method Simulation Comparison.

 The simulation experiments are carried out using different fitness function deviation orders to evaluate the stability of the PSO method in different deviation orders. The current retrieval is simulated when $n$ is 2,1 , and 0.5 , respectively. The selected parameters are consistent with Section 5.1. The simulation curve and evaluation indicators are given in Figure 9 and Table 8.The simulation results show that, when deviation order value is small, the current order is more stable. That is to say, it can be beneficial to the stability of the current retrieval when there is a focus on the three-dimensional image spectrum related to large energy points.

## 6. Conclusion

This paper had, as its goal, an improvement in the accuracy of current retrieval methods, with a study concerning radar related to sea clutter images used for current retrieval. The principle of current retrieval and methods used for current retrieval based on the least squares algorithm were introduced in this paper with the PSO algorithm being proposed as a viable method for current retrieval. Observational data and the selection strategy of the position of initial particles constituted its main focus, with the fitness function of the design, taking into account the impact of a higher dispersion relationship order and providing the framework for execution of the algorithms. Simulation experiments were based on three cases related to the three-dimensional image spectrum, sea clutter images analog, and real sea clutter in order to verify several aspects of the algorithms under investigation, namely, the adaptive capacity of the order of higherorder dispersion relations, the observational data selection method, the particle initialization selection method, the order bias selection method, and the current retrieval accuracy performance. Simulation results show that, compared with the traditional ILSM methods, the method provided in this paper is more flexible, with a capacity for high dispersion relationship order, higher precision, and an increased stability in terms of current inversion.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Figure 9: Simulation results of different deviation order value.

TABLE 6: Evaluation indicators of different observational data selection methods.

|  | Method | Current speed indicators | Current director indicators |
| :--- | :---: | :---: | :---: |
| LSM | 150 points of maximum energy are selected | 0.0937 | 0.8708 |
| PSO |  | 0.2002 | 0.5269 |
| LSM | 1000 points of maximum energy are selected | 0.0887 | 0.6025 |
| PSO |  | 0.0875 | 0.3556 |
| LSM | 0.0947 | 0.2957 |  |
|  |  | 0.0840 | 0.1908 |
| LSM |  | 0.0284 | 0.2289 |
| PSO | All points | 0.0662 | 0.2466 |

Table 7: Evaluation indicators of different initial data selection methods.

| Method | Current speed indicators | Current direction indicators | $T_{\min }$ | $T_{\max }$ | $T_{\text {avg }}$ |
| :--- | :---: | :---: | :---: | :---: | :---: |
| PSO (initialization at random) | 0.0842 | 0.1564 | 11 | 100 | 33.4 |
| PSO (initial results reference LSM) | 0.0840 | 0.1908 | 11 | 100 | 27.5 |

Table 8: Evaluation indicators of different deviation order values.

| Method | Current speed indicators | Current director indicators |
| :--- | :---: | :---: |
| PSO $(n=2)$ | 0.1864 | 0.2521 |
| PSO $(n=1)$ | 0.0840 | 0.1908 |
| PSO $(n=0.5)$ | 0.0750 | 0.1614 |

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# Control System Design of Shunt Active Power Filter Based on Active Disturbance Rejection and Repetitive Control Techniques 

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#### Abstract

To rely on joint active disturbance rejection control (ADRC) and repetitive control (RC), in this paper, a compound control law for active power filter (APF) current control system is proposed. According to the theory of ADRC, the uncertainties in the model and from the circumstance outside are considered as the unknown disturbance to the system. The extended state observer can evaluate the unknown disturbance. Next, RC is introduced into current loop to improve the steady characteristics. The ADRC is used to get a good dynamic performance, and RC is used to get a good static performance. A good simulation result is got through choosing and changing the parameters, and the feasibility, adaptability, and robustness of the control are testified by this result.


## 1. Introduction

The proliferation of nonlinear loads caused by more and more modem electronic equipments results in deterioration of power quality in power transmission or distribution systems. Harmonic, reactive, negative sequence and flickers are the reasons of various undesirable phenomena in the operation of power system. In order to solve these problems, the concept of active power filter (APF) was presented. Active power filters, which compensate harmonic and reactive current component for the power supplies, can improve the power qualities and enhance the reliabilities and stabilities on power utility [1-3]. In recent 30 years from APF presented, the continual innovation of control strategies mainly impels the APF techniques to be developed rapidly [4-7].

Active disturbance rejection control (ADRC) is a robust control method that is based on extension of the system model with an additional and fictitious state variable, representing everything that the user does not include in the mathematical description of the plant [8-11]. Different from other disturbances and states estimation [12-15], this virtual state (sum of internal and external disturbances, usually denoted as a "total disturbance") is estimated online with
a state observer and used in the control signal in order to decouple the system from the actual perturbation acting on the plant. This disturbance rejection feature allows user to treat the considered system with a simpler model, since the negative effects of modeling uncertainty are compensated in real time. As a result, the operator does not need a precise analytical description of the system, as one can assume the unknown parts of dynamics as the internal disturbance in the plant. Robustness and the adaptive ability of this method make it an interesting solution in scenarios where the full knowledge of the system is not available.

Repetitive control is a control method developed by a group of Japanese scholars in 1980s. It is based on the Internal Model Principle and used specifically in dealing with periodic signals, for example, tracking periodic reference or rejecting periodic disturbances. The repetitive control system has been proven to be a very effective and practical method dealing with periodic signals [15-18]. Repetitive control has some similarities with iterative learning control.

This paper addresses the electric current tracking control problem for shunt APF. The control law is joint ADRC and RC which can deal with the static and dynamic performance. The rest of this paper is organized as follows. In Section 2,


Figure 1: Block diagram of the ADRC.
a brief description of the ADRC is presented. In Section 3, main results of $\mathrm{ADRC}+\mathrm{RC}$ control technique are developed. In Section 4, simulation results are presented to show the effectiveness of the proposed control technique. Finally, some conclusions are made in Section 5.

## 2. Active Disturbance Rejection Control

In ADRC, the tracking differentiator (TD) is used to deal with the reference input and the extended state observer (ESO) is used to deal with the output of controlled system. Then the ADRC control law can be selected through the appropriate nonlinear combination of state errors. The general structure of $A D R C$ is shown in Figure 1. In Figure 1 of ADRC, the transient profile generator is used to obtain each order derivative $\dot{y}_{*}(t), \ddot{y}_{*}(t), \ldots, y_{*}^{n}(t)$ of reference trajectory $y_{*}(t)$. Next, brief description of ADRC is given as follows.

Consider a class SISO nonlinear system as

$$
\begin{equation*}
y^{n}=f\left(y, \dot{y}, \ldots, y^{(n-1)}, t\right)+b u(t)+d(t) . \tag{1}
\end{equation*}
$$

Equation (1) also can be described as

$$
\begin{align*}
\dot{x}_{1} & =x_{2} \\
& \vdots \\
x_{n-1} & =x_{n}  \tag{2}\\
\dot{x}_{n} & =f\left(x_{1}, x_{2}, \ldots, x_{n-1}, t\right)+b u(t)+d(t) \\
y & =x_{1},
\end{align*}
$$

where $f\left(x_{1}, x_{2}, \ldots, x_{n-1}, t\right)$ is unknown function, $d(t)$ is unknown disturbance, and $u(t)$ is control input.

Construct the following ESO for nonlinear systems (2):

$$
\begin{align*}
\dot{z}_{1}= & z_{2}-g_{1}\left(z_{1}-y\right) \\
& \vdots \\
\dot{z}_{n}= & z_{n+1}-g_{n}\left(z_{1}-y\right)+b u(t)  \tag{3}\\
\dot{z}_{n+1}= & -g_{n+1}\left(z_{1}-y\right) .
\end{align*}
$$

Let $a(t)=f\left(x_{1}, x_{2}, \ldots, x_{n-1}, t\right)+d(t)$, so we can obtain the following conclusion:

$$
\begin{equation*}
z_{1} \longrightarrow x_{1}, z_{2} \longrightarrow x_{2}, \ldots, z_{n} \longrightarrow x_{n}, z_{n+1} \longrightarrow a \tag{4}
\end{equation*}
$$

through selecting appropriate nonlinear function $g_{1}, g_{2}, \ldots$, $g_{n+1}$. Defining that $\widehat{a}(t)$ is the estimation value of $a(t)$, we can obtain $z_{n+1}=\widehat{a}(t)$.

From the above brief description of ESO, it can be seen that ESO can be used to estimate the states and the sum of model uncertainty $f\left(x_{1}, x_{2}, \ldots, x_{n-1}, t\right)$ and disturbance $d(t)$. So, ESO is such a link, which uses the output $y(t)$ of plant to get each order derivative signal $z_{1}, z_{2}, \ldots, z_{n}$ and estimation value of disturbance.

Using $z_{1}, z_{2}, \ldots, z_{n}$ from ESO and $\dot{y}_{*}(t), \ddot{y}_{*}(t), \ldots, y_{*}^{n}(t)$ from TD, we get the state errors as

$$
\begin{equation*}
\varepsilon_{i}=y_{*}^{i}(t)-z_{i}, \quad i=1,2, \ldots, n \tag{5}
\end{equation*}
$$

So the following nonlinear combination can be gotten by state errors (5):

$$
\begin{equation*}
u_{0}(t)=k_{1} \mathrm{fal}\left(\varepsilon_{1}, \alpha, \delta\right)+\cdots+k_{n} \mathrm{fal}\left(\varepsilon_{n}, \alpha, \delta\right) \tag{6}
\end{equation*}
$$

where $k_{i}, \alpha$, and $\delta$ are adjustable parameters. And nonlinear function fal is defined as follows:

$$
\operatorname{fal}\left(\varepsilon_{i}, \alpha, \delta\right)= \begin{cases}\left|\varepsilon_{i}\right|^{\alpha} \operatorname{sgn}\left(\varepsilon_{i}\right) & \left|\varepsilon_{i}\right|>\delta  \tag{7}\\ \frac{\varepsilon_{i}}{\delta^{1-\alpha}} & \left|\varepsilon_{i}\right| \leq \delta\end{cases}
$$

Using the nonlinear state errors feedback (6) and estimation value $\widehat{a}(t)$, the ADRC law can be given by

$$
\begin{equation*}
u(t)=\frac{u_{0}(t)-\widehat{a}(t)}{b}, \quad i=1,2, \ldots, n . \tag{8}
\end{equation*}
$$

## 3. Main Results

Shunt APF circuit schematic is shown in Figure 2; the upper and lower arm of the shunt APF can be considered as ideal switch from the APF working principle. The equivalent circuit of APF is shown in Figure 3. Since the switching operation can control voltage size of the AC side. So shunt APF can be considered as a controllable voltage source and a parallel impedance in the circuit, and to compensate harmonic current and reactive current can be achieved.

So we can obtain the model of shunt APF as follows:

$$
\begin{equation*}
L \frac{d i_{c}}{d t}=u_{i}-R i_{c}-u_{c} \tag{9}
\end{equation*}
$$



Figure 2: Block diagram of shunt APF.


Figure 3: Equivalent circuit of shunt APF.

Define PWM as a proportional part, namely, $u_{c}=u V_{c}$, where $u$ is modulation amount. Let $u$ be the control input of system. $V_{c}$ is voltage of DC side. For the supply current, we know

$$
\begin{equation*}
i_{s}=i_{c}+i_{L} \tag{10}
\end{equation*}
$$

Substituting (10) into (9), we have

$$
\begin{equation*}
\left(L+L_{s}\right) \frac{d i_{s}}{d t}=-\left(R+R_{s}\right) i_{s}-u V_{c}+u_{s}+R i_{L}+L \frac{d i_{L}}{d t} \tag{11}
\end{equation*}
$$

Designed system controller can be considered by a DC voltage outer-loop control and an inner-loop current control. Since the response speed of inner-loop is much faster than the DC voltage outer-loop, it can be considered that DC voltage is constant when the inner current controls. Ignore the impedance of the power line; we let $\bar{d}(t)=u_{s}+R i_{L}+$ $L\left(d i_{L} / d t\right)$; system (11) can be written as

$$
\begin{equation*}
\left(L+L_{s}\right) \frac{d i_{s}}{d t}=-\left(R+R_{s}\right) i_{s}-u V_{c}+\bar{d}(t) \tag{12}
\end{equation*}
$$

The APF is a first-order system. ADRC does not need to detect the load current and supply voltage and only uses them as unknown disturbances. A PI controller is used to control the outer-loop DC voltage, which is order to obtain a given current value $i_{s}^{*}(t) . i_{s}^{*}(t)$ can be seen as the reference input $y_{*}(t)$ of ADRC. The control objective is to make the supply current $i_{s}$ able to track the given current value $i_{s}^{*}(t)$ through controlling the modulation amount $u$ of PWM. Set an order TD output as

$$
\begin{equation*}
\dot{z}_{1,1}=-k_{0} \mathrm{fal}\left(\left(z_{1,1}-i_{s}^{*}(t)\right), \alpha_{0}, \delta_{0}\right), \tag{13}
\end{equation*}
$$

where $k_{0}, \alpha_{0}$, and $\delta_{0}$ are selected parameters. Construct of the following formula ESO:

$$
\begin{align*}
& \dot{z}_{1}=z_{2}-k_{11} \mathrm{fal}\left(\left(z_{1}-i_{s}^{*}(t)\right), \alpha_{1}, \delta_{1}\right)-V_{c} u(t) \\
& \dot{z}_{2}=-k_{12} \mathrm{fal}\left(\left(z_{1}-i_{s}^{*}(t)\right), \alpha_{1}, \delta_{1}\right) \tag{14}
\end{align*}
$$

where $k_{11}, k_{12}, \alpha_{1}$, and $\delta_{1}$ are selected parameters. So we can obtain the ADRC law as

$$
\begin{gather*}
u_{0}(t)=k_{2} \operatorname{fal}\left(\left(i_{s}^{*}(t)-z_{1}\right), \alpha_{2}, \delta_{2}\right), \\
u(t)=\frac{u_{0}(t)-\dot{z}_{2}}{V_{c}}, \tag{15}
\end{gather*}
$$

where $k_{2}, \alpha_{2}$, and $\delta_{2}$ are also selected parameters. All selected parameters of ADRC controller must try to get in simulation.

RC is mainly used in continuous processes for tracking or rejecting periodic exogenous signals. In most cases, the period of the exogenous signal is known. The internal model principle is the theoretical foundation of RC. According to internal model principle, to track or reject a certain signal without steady-state error, the signal can be regarded as the output of an autonomous generator that is inside the control system.

Although RC system can still get a good static performance, it cannot get a good dynamic performance of the system. RC is usually used to meet up with other control strategies. Actually, RC is only used to restrain the tracking error. But ADRC can improve the rapid response of the system. After being coupled with the repetitive controller, controller can detect the tracking error and accumulate a correction on the basis of the original command to reduce the error. Repetitive controller can be seen as an embedded component, so this system is called embedded repetitive control system (ERCS). Figure 4 is a block diagram of a parallel ADRC with RC. Next, how to select the controller parameters of RC is shown as follows.
(1) Cycle delay factor $N$ : $N$ is sampled beat number of sinusoidal cycle and can be described as fundamental frequency $f_{s}$ and the switching frequency $f_{c}$.
(2) Compensation link $Q(z): Q(z)$ characterizes the steady precision of repetitive controller. In general, $Q(z)$ is a constant. When $Q(z)=1$, the open-loop gain of system is infinite, and steady-state error is zero. But this may likely cause system instability. So we usually select a constant that is less than but close to $1 . Q(z)$ is also preferably chosen zero phase low pass filter.
(3) Compensation link $S(z)$ of plant: $S(z)$ is used to reform the controlled plant. After reformation, the amplitude-frequency characteristics of the plant has zero gain in the low frequency band. Generally, the series correction part $S_{1}(z)$ is first selected to correct the low-frequency gain of controlled plant. Then, in order to improve system stability, the second-order low-pass filter is selected to attenuate high frequency gain.
(4) Phase compensation factor $k$ : the aim of phase compensation factor $k$ is to compensate phase lag for reformed controlled plant in the low frequency.
(5) Repetitive controller gain $K_{r}: K_{r}$ is used to ensure the stability of the system in the high frequency band. The smaller $K_{r}$ can cause the better stability, but the speed of convergence will become slow and the steady-state


Figure 4: Block diagram of a parallel ADRC with RC for shunt APF.


Figure 5: Load electric current.
error will increase. In general, $K_{r}$ is chosen to be close to 1 as possible under maintaining the well stability of the RC.

## 4. Simulation Results

In this section, we use Matlab/Simulink for testing and verifying the proposed APF control method. The parameters of chosen APF are $L=1 \mathrm{H}, R=8 \Omega, T=100 \mu \mathrm{~s}$, and $f_{c}=10 \mathrm{kHz}$. The ADRC controller parameters are designed as $\alpha_{0}=2, \alpha_{1}=0.5, \alpha_{0}=1, \delta_{0}=0.00001, \delta_{1}=0.001$, $\delta_{0}=0.00001, k_{0}=8000, k_{11}=10000, k_{12}=50000$, and $k_{2}=500$. The RC controller parameters are designed as $N=$ 200, $S_{1}(z)=(z-0995) /[0.1248(z-06)], S_{2}(z)=\left(0.0675 z^{2}+\right.$ $01349 z+00675) /\left(z^{2}-1143 z+0.4128\right), Q(z)=0.97, k=3$, and $K_{r}=O .8$. First, we consider the 150 Hz sine wave for load. Figures 5 and 6 show the load electric current which is third harmonic and the output current $i(t)$ of controlled APF.


Figure 6: Reference current $i^{*}(t)$ and output current $i(t)$ of controlled APF.

Figure 7 shows the total harmonic distortion (THD) analysis for grid current. It can be seen that the proposed control method of the APF can better restrain harmonic of grid. The value of TDH can achieve $0.26 \%$.

First, we consider the 200 Hz square wave signal for load. Figures 8 and 9 show the load electric current which is fourth harmonic and the output current $i(t)$ of controlled APF. Figure 10 shows the total harmonic distortion (THD) analysis for grid current. It can be seen that the proposed control method of the APF can better restrain harmonic of grid. The value of TDH can achieve $3.88 \%$.

In simulation process, we do not correct the controller parameters, just change the distortion form of load; the simulation results are provided to show that the proposed


Figure 7: THD analysis for grid current.


Figure 8: Load electric current.


Figure 9: Reference current $i^{*}(t)$ and output current $i(t)$ of controlled APF.


Figure 10: THD analysis for grid current.
control algorithm of APF has a very reliable robustness and adaptability for the different distortion forms.

## 5. Conclusions

Since the switch voltage drops, the drive circuit delay differs and dead zones are affected, there are a large number of low-order harmonics in the output current of a single-phase grid-connected APF. The traditional PI controller exists the capacity deficiencies in the harmonic suppression, and unable realizes the static error tracking for the sine command current. It can effectively improve the grid current waveform through $A D R C+R C$ controller. In this paper, we give the composite control law design method for single-phase grid connected APF current loop. Theory and simulation are provided to show that the proposed control algorithm has a very reliable tracking ability and satisfactory robustness to different harmonics of the load.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# The Quadrotor Dynamic Modeling and Indoor Target Tracking Control Method 

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#### Abstract

A reliable nonlinear dynamic model of the quadrotor is presented. The nonlinear dynamic model includes actuator dynamic and aerodynamic effect. Since the rotors run near a constant hovering speed, the dynamic model is simplified at hovering operating point. Based on the simplified nonlinear dynamic model, the PID controllers with feedback linearization and feedforward control are proposed using the backstepping method. These controllers are used to control both the attitude and position of the quadrotor. A fully custom quadrotor is developed to verify the correctness of the dynamic model and control algorithms. The attitude of the quadrotor is measured by inertia measurement unit (IMU). The position of the quadrotor in a GPS-denied environment, especially indoor environment, is estimated from the downward camera and ultrasonic sensor measurements. The validity and effectiveness of the proposed dynamic model and control algorithms are demonstrated by experimental results. It is shown that the vehicle achieves robust vision-based hovering and moving target tracking control.


## 1. Introduction

As an emerging platform for unmanned aerial vehicle (UAV) research, the quadrotor has recently gained most attention from the community. With some specific capabilities, such as vertical take-off and landing (VTOL), hovering, fly alone or in team, autonomously fly, it has been envisaged for a wide of applications including military reconnaissance, search and rescue, meteorological survey, environmental monitoring, and wireless mobile senor networks [1]. The quadrotor has several advantages compared to other rotorcrafts. First, the quadrotor does not require swash plate and mechanical linkages as it equips fixed pitch propellers and uses speed variation for vehicle control. This makes it more convenient to design, manufacture, maintain, and recover from incidents. Second, its propellers are smaller in diameter relative to the airframe size and can be enclosed within a frame. This makes it safer and brings more benefit for indoor flight and in obstacle-dense environments. In addition, the quadrotor has greater thrust-weight ratio and thus better maneuver performance. All these advantages promote the development of a number of commercial and research quadrotor platforms [2-5].

Although the quadrotor has a series of advantages, it is an absolutely unstable and underactuated dynamic system with sophisticated nonlinearity and strong coupling. Moreover, it is easily affected by near-surface airstream. Because of these difficulties, the intensive study on dynamical modeling, analysis, and advanced control of quadrotor needs to be done to improve the flight quality. In particular, the actuator dynamic and aerodynamic effects must be investigated to establish a reliable dynamic model of quadrotor. Control method dealing with the nonlinearity and coupling property of quadrotor has to be proposed for precise flight control. Bouabdallah and Siegwart mentioned the importance of actuator dynamic and analyzed forces and moments caused by aerodynamic effects. But they simplified the dynamic model and omitted those effects [6]. Huang et al. researched two important aerodynamic effects and presented control techniques compensating for them accordingly. However, only the altitude controller was designed based on nonlinear method [7]. Minh and Ha linearized the nonlinear dynamic model about a trim in hover and applied the LQG method to stabilize the quadrotor with vision-based pose estimation. Nevertheless, only simulation results were displayed [8]. A variety of control
algorithms have been attempted to handle the nonlinearity and coupling property, such as neural networks control [9], integral predictive/nonlinear $H_{\infty}$ control [10], sliding mode control [11, 12], and fuzzy tracking control [13]. Although simulation results demonstrated effectiveness of those control algorithms, most of them were developed based on dynamic model without the actuator dynamic and aerodynamic effect. Besides, the stochastic method and fault detection algorithms are investigated [14-18], whereas those methods are difficult to be implemented in the real system. As the quadrotor is a cascade system, it is proven that the backstepping control method has an excellent performance [19]. Recently, interests in the quadrotor research have transferred to autonomous flight. While the success of laser-based autonomous indoor flight [20-22] has made a huge impact on the development of quadrotor, the vision-based autonomous indoor flight [23] is an immediate area of research focus. Moreover, heterogeneous multiagent problems such as UAV and unmanned ground vehicle (UGV) indoor/outdoor coordination control are expected to be the next technical breakthrough. Based on the research actuality and trend on quadrotor, this paper studies the quadrotor dynamic modeling and indoor target tracking control method. These research results will build foundations for precise flight control and heterogeneous multiagent study.

The main contributions of this paper are the following. First, a reliable nonlinear dynamic model is presented based on the analysis of actuator dynamic, aerodynamic effect, and rigid body dynamic. The gyroscope effect of the rotors is considered by dividing the quadrotor into body part and rotor part. It makes the dynamic model more reliable to take actuator dynamic and aerodynamic effect into account. Second, the PID controllers with feedback linearization and feedforward control are proposed to control both the attitude and position of the quadrotor. The dynamic model is explicitly expressed as a cascade system of three subsystems to be suitable for the backstepping method. The control algorithms are realized on a fully custom quadrotor and vision-based autonomous indoor moving target tracking flight is achieved.

This paper is structured as follows. In Section 2, we first analyze the actuator dynamic and aerodynamic effect. The actuator dynamic is the derivation of the Kirchhoff laws and the law of rotation. The aerodynamic effect is mainly about blade flapping which has a significant effect on attitude tracking control. Then, a reliable nonlinear dynamic model is addressed using Newton-Euler method. The dynamic model is a combination of actuator dynamic, aerodynamic effect, and rigid body dynamic. In Section 3, a general PID controller with feedforward control is proposed. Then, based on the simplified nonlinear dynamic model, decoupling nonlinear control laws are presented using feedback linearization and the backstepping control strategy is applied to the position control. Section 4 describes the system design of our fully custom quadrotor and discusses the experimental results. The fully custom quadrotor is equipped with an IMU, a downward camera, and a downward ultrasonic sensor. Full control experiments are executed in the order of attitude control, altitude control, hovering control, and tracking control. At last, we outline the conclusion in Section 5.


Figure 1: Body-fixed frame and Earth-fixed frame.

## 2. Mathematical Modeling

Most researchers used to regard the whole quadrotor as a rigid model [24], neglecting the propeller gyroscope effect and aerodynamic effect. Moreover, models containing actuator dynamic are rarely investigated. However, studies show that aerodynamic effect is obvious even with moderate speed [7] and that actuator dynamic has a strong influence on the attitude stabilization [6]. Thus, a detailed analysis of those effects is necessary.

The coordinate system is defined in Figure 1. $E$ is an earthfixed frame and $B$ is a body-fixed frame. The body fixed coordinates origin locates at the center of gravity (CoG) of the quadrotor. $h$ is the distance between propeller plane and CoG.
2.1. Actuator Dynamic. Actuator dynamic describes the relationship between rotor speed and actuator voltage. The latter is our real control input. Basically, the actuator response speed is most interested by designers. Based on Kirchhoff laws and the law of rotation, the simplified actuator dynamic model is [25]

$$
\begin{equation*}
\dot{\Omega}_{i}=-\frac{1}{\tau} \Omega_{i}+\frac{k_{\Omega}}{\tau} u_{i}, \tag{1}
\end{equation*}
$$

where $\Omega_{i}, \tau$, and $K_{\Omega}$ represent the speed of actuator $i$, delay coefficient, and gain coefficient, respectively.

Actuator delay is curial especially when the attitude control loop runs at a low frequency.
2.2. Aerodynamic Effect. Aerodynamic effect dramatically increases with the variation from equilibrium state. Some literatures show excellent performance on a test bench [11]. Whereas, the attitude control result without the test bench is worse than the attitude control result on the test bench. This is because the aerodynamic effect occurs when flight without the test bench. Aerodynamic effect is mainly caused by blade flapping, as shown in Figure 2.

During forward flight, the advancing blade has a higher velocity relative to the free stream and the retreating blade sees a lower effective airspeed. This brings imbalance of lift and results in the propeller plane deflecting from position 1


Figure 2: Motor 1 is the front of the quadrotor; speed-up motor 3 and speed-down motor 1 will result in forward flight.
to position 2. The deflection angle $\alpha_{x}$ is proportional to the velocity in body $x$-axis $v_{x}^{b}$ :

$$
\begin{equation*}
\alpha_{x}=k_{\alpha} v_{x}^{b} . \tag{2}
\end{equation*}
$$

The deflection of the propeller plane causes an extra moment on the $y$-axis [7]:

$$
\begin{equation*}
M_{b f, y}=4\left(k_{\beta} \alpha_{x}+T h \sin \alpha_{x}\right) \approx 4\left(k_{\beta} \alpha_{x}+T h \alpha_{x}\right) \tag{3}
\end{equation*}
$$

where $T$ is the total thrust and $k_{\alpha}$ and $k_{\beta}$ are experimentally measured constants. The blade flapping effect on the $x$-axis is the same.

Air friction is relative to the velocity of quadrotor and can be expressed as

$$
\begin{equation*}
f=-\frac{1}{2} C A_{c} \rho v|v|, \tag{4}
\end{equation*}
$$

where $C, A_{c}, \rho$, and $v$ represent damping coefficient, active area, air density, and relative speed, respectively.

Theoretical and experimental results demonstrate that aerodynamic effect is not trivial even with moderate speed and will be crucially important in aerobatic flight.
2.3. Nonlinear Dynamic Model. We separate the quadrotor into two portions, the body part and the rotor part. The body part includes the frame structure and equipments. The rotor part includes motors and propellers. Manifestly, the relative position between the rotor part and the body part varies as the rotor spins. Hence we cannot assume the whole quadrotor as a rigid body. The mathematical model is based on following assumptions.
(1) The body part and rotor part are rigid, respectively.
(2) The quadrotor is symmetric.
(3) Thrust and drag are proportional to the square of propeller's speed.
(4) Actuator dynamic is identical.
(5) The center of gravity (CoG) coincides with the body fixed coordinates origin.

Apply Newton-Euler equation to body part:

$$
\begin{gather*}
F^{E}=m \ddot{X}^{E} \\
M^{B}=I_{b} \dot{\omega}^{B}+\omega^{B} \times I_{b} \omega^{B} \tag{5}
\end{gather*}
$$

where $F^{E}$ is the total force and $X^{E}$ is the position of the quadrotor expressed in earth-fixed frame $E . M^{B}$ is the total moment expressed in body-fixed frame $B . I_{b}$ is the rotational inertia of the body part. $\omega^{B}=\left[\begin{array}{lll}p & q & r\end{array}\right]^{T}$ is the angular speed expressed in body-fixed frame.

Here we employ the Euler angle representation of orientation. We consequently rotate about $Z-X-Y$ axes by yaw angle $\psi$, roll angle $\phi$. and pitch angle $\theta$.

Let $\eta=\left[\begin{array}{lll}\phi & \theta & \psi\end{array}\right]^{T}$; the relationship between $\dot{\eta}$ and $\omega^{B}$ is

$$
\dot{\eta}=\left[\begin{array}{ccc}
c \theta & 0 & s \theta  \tag{6}\\
\frac{s \phi s \theta}{c \phi} & 1 & \frac{-s \phi c \theta}{c \phi} \\
\frac{-s \theta}{c \phi} & 0 & \frac{c \theta}{c \phi}
\end{array}\right] \omega^{B}
$$

where $s(\cdot)$ and $c(\cdot)$ represent $\sin (\cdot)$ and $\cos (\cdot)$, respectively.
According to the third assumption, rotor thrust $T$ and drag $Q$ are

$$
\begin{align*}
& T=\sum_{i=1}^{4} T_{i}=k_{T} \Omega_{i}^{2} \\
& Q=\sum_{i=1}^{4} Q_{i}=k_{Q} \Omega_{i}^{2} \tag{7}
\end{align*}
$$

Utilizing the analysis of aerodynamic effect, the total force and moment on the body part are

$$
\begin{aligned}
F^{E} & =\left[\begin{array}{l}
F_{z} \\
F_{x} \\
F_{y}
\end{array}\right] \\
& =\left[\begin{array}{c}
m g-(c \psi c \phi) T c \alpha_{x} c \alpha_{y} \\
(s \psi s \phi+c \psi s \theta c \phi) T c \alpha_{x} c \alpha_{y}-\frac{1}{2} C_{x} A_{c} \rho \dot{x}|x| \\
(-c \psi s \phi+s \psi s \theta c \phi) T c \alpha_{x} c \alpha_{y}-\frac{1}{2} C_{x} A_{c} \rho \dot{y}|y|
\end{array}\right],
\end{aligned}
$$

$$
\begin{align*}
M^{B} & =\left[\begin{array}{l}
M_{x} \\
M_{y} \\
M_{z}
\end{array}\right] \\
& =\left[\begin{array}{c}
J_{r} q \Omega_{r}+l\left(T_{4}-T_{2}\right)+M_{b f, x} \\
J_{r} p \Omega_{r}+l\left(T_{1}-T_{3}\right)+M_{b f, y} \\
Q+l\left(T_{2}-T_{4}\right) s \alpha_{y}+l\left(T_{3}-T_{1}\right) s \alpha_{x}
\end{array}\right] \\
& \approx\left[\begin{array}{c}
J_{r} q \Omega_{r}+l\left(T_{4}-T_{2}\right)+M_{b f, x} \\
J_{r} p \Omega_{r}+l\left(T_{1}-T_{3}\right)+M_{b f, y} \\
Q+l\left(T_{2}-T_{4}\right) \alpha_{y}+l\left(T_{3}-T_{1}\right) \alpha_{x}
\end{array}\right] \tag{8}
\end{align*}
$$

where $J_{r}$ is rotational inertia of the rotor part and $\Omega_{r}=$ $\Omega_{1}-\Omega_{2}+\Omega_{3}-\Omega_{4}$ is the sum of rotor speed. Note that the moment for the rotor angular acceleration $J_{r} \dot{\Omega}_{r}$ is produced by electromagnetic force and unrelated to the body part. Based on the second assumption, $I_{i j}=0, i \neq j ; i, j \in\left\{\begin{array}{lll}x & y & z\end{array}\right\}$ and $I_{x x}=I_{y y}$. We regard the body part as controlled object and by integrating the body part dynamic and actuator dynamic we can reach the global nonlinear dynamic model. Here we simplify the nonlinear dynamic model about the hovering operating point. $\Omega_{h}$ is the speed of rotor at the hovering operating point

$$
\begin{equation*}
\Omega_{h}=\sqrt{\frac{m g}{4 k_{T}}} \tag{9}
\end{equation*}
$$

Let

$$
\begin{align*}
& \Delta \Omega_{\phi}=\Omega_{4}-\Omega_{2}, \quad \Delta \Omega_{\theta}=\Omega_{1}-\Omega_{3}, \\
& \Delta \Omega_{\psi}=\Omega_{1}-\Omega_{2}+\Omega_{3}-\Omega_{4}, \quad \lambda=\frac{\Delta \Omega_{T}}{4 \Omega_{h}}, \\
& \Delta \Omega_{T}=\Omega_{1}-\Omega_{h}+\Omega_{2}-\Omega_{h}+\Omega_{3}-\Omega_{h}+\Omega_{4}-\Omega_{h}, \\
& U_{2}=\left[\begin{array}{llll}
u_{1} & u_{2} & u_{3} & u_{4}
\end{array}\right]^{T} \text {, } \\
& U_{1}=X_{2}=\left[\begin{array}{llll}
\Delta \Omega_{\phi} & \Delta \Omega_{\theta} & \Delta \Omega_{\psi} & \Delta \Omega_{T}
\end{array}\right]^{T}, \\
& X_{1}=\left[\begin{array}{lllllllllll}
\phi & p & \theta & q & \psi & r & z & \dot{z} & x & \dot{x} & y \\
y
\end{array}\right]^{T} \text {. } \tag{10}
\end{align*}
$$

Hence

$$
\begin{align*}
\dot{X}_{2}= & f\left(X_{2}, U_{2}\right) \\
= & {\left[\begin{array}{cccc}
\frac{-1}{\tau} & 0 & 0 & 0 \\
0 & \frac{-1}{\tau} & 0 & 0 \\
0 & 0 & \frac{-1}{\tau} & 0 \\
0 & 0 & 0 & -\frac{\lambda+1}{\lambda \tau}
\end{array}\right]\left[\begin{array}{l}
\Delta \Omega_{\phi} \\
\Delta \Omega_{\theta} \\
\Delta \Omega_{\psi} \\
\Delta \Omega_{T}
\end{array}\right] }  \tag{11}\\
& +\left[\begin{array}{cccc}
0 & -\frac{k_{\Omega}}{\tau} & 0 & \frac{k_{\Omega}}{\tau} \\
\frac{k_{\Omega}}{\tau} & 0 & -\frac{k_{\Omega}}{\tau} & 0 \\
\frac{k_{\Omega}}{\tau} & -\frac{k_{\Omega}}{\tau} & \frac{k_{\Omega}}{\tau} & -\frac{k_{\Omega}}{\tau} \\
\frac{k_{\Omega}}{\tau} & \frac{k_{\Omega}}{\tau} & \frac{k_{\Omega}}{\tau} & \frac{k_{\Omega}}{\tau}
\end{array}\right]\left[\begin{array}{l}
u_{1} \\
u_{2} \\
u_{3} \\
u_{4}
\end{array}\right],
\end{align*}
$$

$$
\dot{X}_{1}=f\left(X_{1}, U_{1}\right)=\left[\begin{array}{c}
p c \theta+r s \theta  \tag{12}\\
\frac{\left[\left(I_{y y}-I_{z z}\right) q r+J_{r} q \Delta \Omega_{\psi}+2 l k_{T} \Omega_{h} \Delta \Omega_{\phi}+k_{\alpha} k_{\beta} \dot{x}+T h k_{\alpha} \dot{x}\right]}{I_{x x}} \\
\frac{(p s \phi s \theta+q c \phi-r s \phi c \theta)}{c \phi} \\
\frac{\left[\left(I_{z z}-I_{x x}\right) p r+J_{r} p \Delta \Omega_{\psi}+2 l k_{T} \Omega_{h} \Delta \Omega_{\theta}+k_{\alpha} k_{\beta} \dot{y}+T h k_{\alpha} \dot{y}\right]}{I_{y y}} \\
\frac{(r c \theta-p s \theta)}{c \phi} \\
2\left[k_{Q} \Omega_{h} \Delta \Omega_{\psi}+l k_{\alpha} k_{T}\left(\Delta \Omega_{\phi} \dot{x}-\Delta \Omega_{\theta} \dot{y}\right)\right] \\
I_{z z} \\
\dot{z} \\
m g-2 k_{T} \Omega_{h} c \phi c \psi \Delta \Omega_{T} c\left(k_{\alpha} \dot{x}\right) c\left(k_{\alpha} \dot{y}\right) \\
\dot{x} \\
2 k_{T} \Omega_{h}(s \phi s \psi+s \theta c \phi c \psi) \Delta \Omega_{T} c\left(k_{\alpha} \dot{x}\right) c\left(k_{\alpha} \dot{y}\right)-\frac{1}{2} C_{x} A_{c} \rho \dot{x}|\dot{x}| \\
\dot{y} \\
2 k_{T} \Omega_{h}(-s \phi c \psi+s \theta c \phi s \psi) \Delta \Omega_{T} c\left(k_{\alpha} \dot{x}\right) c\left(k_{\alpha} \dot{y}\right)-\frac{1}{2} C_{y} A_{c} \rho \dot{y}|\dot{y}|
\end{array}\right] .
$$



Figure 3: Backstepping control scheme.


Figure 4: Off-board process of the image.

## 3. Control Implementation

First, we define a general PID controller with feedforward control

$$
\begin{equation*}
\operatorname{PID}(\zeta)=K_{p}\left(\zeta^{d}-\zeta\right)+k_{d}\left(\dot{\zeta}^{d}-\dot{\zeta}\right)+k_{I} \int\left(\zeta^{d}-\zeta\right) d t+\ddot{\zeta}^{d} \tag{13}
\end{equation*}
$$

in which $\ddot{\zeta}^{d}$ is the feedforward part.
Then we apply feedback linearization to design the attitude and position controllers. Finally, we describe the backstepping control scheme.
3.1. Position Controllers. The outputs of the position controllers are $\theta, \phi, \Delta \Omega_{T}$. During the hovering and tracking flight, $\psi=0$; thus, we choose the control law as

$$
\begin{gather*}
s \theta=\frac{\operatorname{PID}(x)+C_{x} A_{c} \rho \dot{x}|\dot{x}| / 2}{2 k_{T} \Omega_{h} \Delta \Omega_{T} c\left(k_{\alpha} \dot{x}\right) c\left(k_{\beta} \dot{y}\right)}, \\
s \phi=\frac{\operatorname{PID}(y)+C_{y} A_{c} \rho \dot{y}|\dot{y}| / 2}{-2 k_{T} \Omega_{h} c \theta \Delta \Omega_{T} c\left(k_{\alpha} \dot{x}\right) c\left(k_{\beta} \dot{y}\right)},  \tag{14}\\
\Delta \Omega_{T}=\frac{m g-\operatorname{PID}(z)}{2 k_{T} \Omega_{h} c \phi c \theta c\left(k_{\alpha} \dot{x}\right) c\left(k_{\beta} \dot{y}\right)} .
\end{gather*}
$$

Substituting (14) into (12), we get

$$
\begin{align*}
& \ddot{x}=\operatorname{PID}(x), \\
& \ddot{y}=\operatorname{PID}(y),  \tag{15}\\
& \ddot{z}=\operatorname{PID}(z) .
\end{align*}
$$

This guarantees asymptotic stability and has robustness to some uncertainties. Solving (14), we get the decoupling form:

$$
\begin{align*}
\theta= & -\arctan \\
\times( & \left(\operatorname{PID}(x)+\frac{C_{x} A_{c} \rho \dot{x}|\dot{x}|}{2}\right) \\
& \times\left(\left[(\operatorname{PID}(z)-m g)^{2}\right.\right. \\
& \left.\left.\left.+\left(\operatorname{PID}(y)+\frac{C_{y} A_{c} \rho \dot{y}|\dot{y}|}{2}\right)^{2}\right]^{1 / 2}\right)^{-1}\right), \\
\phi= & \arctan \left(\frac{\operatorname{PID}(y)+C_{y} A_{c} \rho \dot{y}|\dot{y}| / 2}{\operatorname{PID}(z)-m g}\right), \\
= & (m g-\operatorname{PID}(z))^{3} \\
& \times\left(2 k_{T} \Omega_{h} c\left(k_{\alpha} \dot{x}\right) c\left(k_{\alpha} \dot{y}\right)\right. \\
& \times\left[(\operatorname{PID}(z)-m g)^{2}\right. \\
& \left.\left.\quad+\left(\operatorname{PID}(y)+\frac{C_{y} A_{c} \rho \dot{y}|\dot{y}|}{2}\right)^{2}\right]\right) . \tag{16}
\end{align*}
$$



FIgURe 5: A fully custom quadrotor and autonomous moving target tracking.


Figure 6: The HIL simulation: manually change the quadrotor, the 3D model changes, and the curves show the outputs of the controllers.
3.2. Attitude Controllers. The outputs of the position controllers are the inputs of the attitude controllers. The outputs of the attitude controllers are $\Delta \Omega_{\phi}, \Delta \Omega_{\theta}$, and $\Delta \Omega_{\psi}$. Utilizing the similar design method, we choose the control law as

$$
\begin{aligned}
& \Delta \Omega_{\phi}= \frac{1}{2 l k_{T} \Omega_{h}} \\
& \times\left\{[\cdot]_{\phi}-\frac{J_{r} q \Omega_{h}}{2 k_{Q} \Omega_{h}^{2}+k_{\alpha} J_{r}(p \dot{y}-q \dot{x})}\right. \\
&\left.\times\left(\operatorname{PID}(\psi) I_{z z}-\frac{k_{\alpha} \dot{x}[\cdot]_{\phi}}{\Omega_{h}}+\frac{k_{\alpha} \dot{y}[\cdot]_{\theta}}{\Omega_{h}}\right)\right\}, \\
& \Delta \Omega_{\theta}= \frac{1}{2 l k_{T} \Omega_{h}} \\
& \times\left\{[\cdot]_{\theta}-\frac{J_{r} q \Omega_{h}}{2 k_{Q} \Omega_{h}^{2}+k_{\alpha} J_{r}(p \dot{y}-q \dot{x})}\right. \\
&\left.\times\left(\operatorname{PID}(\psi) I_{z z}-\frac{\left.k_{\alpha} \dot{x} \cdot \cdot\right]_{\phi}}{\Omega_{h}}+\frac{k_{\alpha} \dot{y}[\cdot]_{\theta}}{\Omega_{h}}\right)\right\}, \\
& \Delta \Omega_{\psi}= \frac{\Omega_{h}}{2 k_{Q} \Omega_{h}^{2}+k_{\alpha} J_{r}(p \dot{y}-q \dot{x})} \\
& \times\left\{\operatorname{PID}(\psi) I_{z z}-\frac{k_{\alpha} \dot{x}}{\Omega_{h}}[\cdot]_{\phi}+\frac{k_{\alpha} \dot{y}}{\Omega_{h}}[\cdot]_{\theta}\right\}, \\
& {[\cdot]_{\theta}=}\left.\operatorname{PID}(\phi) I_{x x}-k_{\alpha} k_{\beta} \dot{x}-T h k_{\alpha} \dot{x}-\left(I_{y y}-I_{z z}\right) q r\right) I_{y y}-k_{\alpha} k_{\beta} \dot{y}-T h k_{\alpha} \dot{y}-\left(I_{y y}-I_{z z}\right) p r .
\end{aligned}
$$

3.3. Backstepping Control Algorithms. The outputs of attitude controllers are the inputs of the actuator controllers. The outputs of the actuator controllers are $U_{2}=\left[\begin{array}{llll}u_{1} & u_{2} & u_{3} & u_{4}\end{array}\right]^{T}$. Considering the relationship between position controllers, attitude controllers, and actuator controllers, we proposed the backstepping control scheme shown in Figure 3.

## 4. Verification and Results

The nonlinear dynamic model and control algorithms are verified on a fully custom quadrotor. First, the design and manufacture of our fully custom quadrotor is briefly described. Then, results of different experiments are discussed consequently.
4.1. Verification. We developed a fully custom quadrotor using the optimal design algorithm, shown in Figure 4. The frame is made from carbon fiber composite (CFC), and the structure is designed with CATIA v5. The inherent frequency of the frame is more than 100 Hz while the vibration frequency caused by actuators is about 60 Hz . This avoids resonance and reduces the accelerometer measurement noise. The downward camera and ultrasonic sensor are utilized to obtain the physical position, running at 25 Hz and 10 Hz , respectively. The image is off-board processed as shown in Figure 4. Furthermore, there is a remote control UGV with a colored mark, playing the role as a moving target, Figure 5.


Figure 7: Attitude control results.


Figure 8: Altitude control results.

The parameters of the rigid body dynamic are calculated during the design process. The parameters of the aerodynamic effect are estimated by empirical model and data fitting. The actuator dynamic is obtained by system identification. The parameters used in the verification experiments are listed in Table 1.
4.2. Experiment Results. The hardware in loop (HIL) simulation is executed before flight experiment. Manually change the pitch angle, roll angle, and yaw angle of the quadrotor;


Figure 9: Hovering control results.
the speed command of each motor changes, respectively. Manually change the altitude of the quadrotor; the throttle command for four motors changes. Manually change the position of the quadrotor; the attitude commands change. The simulation results verify the correctness of the dynamic model and control algorithms. The HIL simulation is shown in Figure 6.

Attitude control, altitude control, hovering control, and moving target tracking control experiments are consequently performed. As shown in Figure 7, the pitch and roll control error is less than 2 degrees while yaw control error is less than 5 degrees. Altitude control experiment is executed with a switch. The quadrotor takes off manually and then switches to autoaltitude control. From Figure 8 we can know that the altitude control error is less than 5 cm . Hovering and moving target tracking control experiments are conducted with a switch too. As shown in Figures 9 and 10, both the hovering control and tracking control errors are less than 10 cm .

## 5. Conclusions and Future Works

We aim at precise modeling, analysis, and control of a sophisticated nonlinear system. This paper presented the newest research on quadrotor of our project. First, we analyzed the actuator dynamic and aerodynamic effect of the quadrotor. Then, we established a reliable nonlinear dynamic model of the quadrotor. As the backstepping control algorithm is well fit for the cascaded structured systems such as the quadrotor, we designed a series of PID controllers with feedforward control and feedback linearization using the backstepping method. Real experiments were executed and the effectiveness of the proposed dynamic model and control method is demonstrated by the experimental result. The future works include two directions. Firstly, the quaternion representation of orientation needs to be employed since the Euler angle representation is subject to problematic singularities. The global stable controllers are expected to be proposed based on


Figure 10: Moving target tracking control results.

Table 1: Parameters used in the verification experiments.

| Parameter | Description | Value | Units |
| :---: | :---: | :---: | :---: |
| $g$ | Gravity | 9.81 | $\mathrm{m} / \mathrm{s}^{2}$ |
| m | Mass | 1.17 | kg |
| $l$ | Distance between CoG and motor | 0.25 | m |
| $h$ | Distance between CoG and propeller plane | 0.05 | m |
| $I_{x x}$ | Roll inertia | $1.27 \times 10^{-2}$ | $\mathrm{kg} \cdot \mathrm{m}^{2}$ |
| $I_{y y}$ | Pitch inertia | $1.27 \times 10^{-2}$ | $\mathrm{kg} \cdot \mathrm{m}^{2}$ |
| $I_{z z}$ | Yaw inertia | $2.29 \times 10^{-2}$ | $\mathrm{kg} \cdot \mathrm{m}^{2}$ |
| $J_{r}$ | Rotor inertia | $3.8 \times 10^{-5}$ | $\mathrm{kg} \cdot \mathrm{m}^{2}$ |
| $k_{T}$ | Thrust coefficient | $2.1 \times 10^{-5}$ | $\mathrm{N} \cdot \mathrm{s}^{2}$ |
| $k_{\text {Q }}$ | Drag coefficient | $1.2 \times 10^{-6}$ | $\mathrm{N} \cdot \mathrm{s}^{2}$ |
| $\tau$ | Motor time constant | 0.09 | $s$ |
| $k_{\alpha}$ | Velocity to angle constant | $6.1 \times 10^{-3}$ | $\mathrm{rad} \cdot \mathrm{s} / \mathrm{m}$ |
| $k_{\beta}$ | Angle to moment constant | $6.1 \times 10^{-3}$ | $\mathrm{N} \cdot \mathrm{m} / \mathrm{rad}$ |
| $A_{c}$ | Active area | 0.25 | $\mathrm{m}^{2}$ |
| $\rho$ | Air density | 1.205 | $\mathrm{kg} / \mathrm{m}^{3}$ |
| C | Damping coefficient | 0.09 |  |

the quaternion representation. Secondly, more efforts need to be done to promote the moving target tracking system more like a heterogeneous multiagent system. Problems within heterogeneous multiagent system are expected to be the next technical breakthrough.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Recursive Estimation for Dynamical Systems with Different Delay Rates Sensor Network and Autocorrelated Process Noises 

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#### Abstract

The recursive estimation problem is studied for a class of uncertain dynamical systems with different delay rates sensor network and autocorrelated process noises. The process noises are assumed to be autocorrelated across time and the autocorrelation property is described by the covariances between different time instants. The system model under consideration is subject to multiplicative noises or stochastic uncertainties. The sensor delay phenomenon occurs in a random way and each sensor in the sensor network has an individual delay rate which is characterized by a binary switching sequence obeying a conditional probability distribution. By using the orthogonal projection theorem and an innovation analysis approach, the desired recursive robust estimators including recursive robust filter, predictor, and smoother are obtained. Simulation results are provided to demonstrate the effectiveness of the proposed approaches.


## 1. Introduction

The Kalman filter is very popular for estimating the system states of a class of linear systems which are characterized by state-space models. Since its inception in the early 1960s, it has played an important role in the research fields of target tracking, communication, control engineering, and signal processing. An implied assumption of traditional Kalman filter is that the system model and measurement model are exactly known. Unfortunately, this assumption does not always hold due to the constrained knowledge and the variation of the system and environment. When the system model and measurement model under consideration are not exactly known, the performance of traditional Kalman filter can deteriorate appreciably $[1-3]$. Therefore, in the past decades, the recursive robust state-space estimation problem has become a hot topic of the estimation theory. There are many different ways to describe the model uncertainty. Multiplicative noise is an important stochastic uncertainty which is commonly encountered in aerospace systems [4], communication systems [5], and image processing systems $[6,7]$. Different from the additive noise, the second-order statistics of the multiplicative noise are usually unknown and
this property leads to more difficulties in the research. Up to now, there are several solutions to treat with the estimation and control problems for systems with multiplicative noises, including linear matrix inequality approach [8], Riccati equation approach $[9,10]$, and game-theoretic method [11], to name just a few.

In traditional state estimation theory, the process noises are usually assumed to be Gaussian and uncorrelated with each other. However, this assumption is not always realistic, correlated noises are commonly encountered in practical applications. For example, in a target tracking system, the system state is usually consecutive (i.e., the system state at time $k$ is correlated with its neighbors); thus, when the process noises are dependent on the system state, the process noises are usually autocorrelated across time. So far, there have been several approaches to deal with the estimation problem for systems with correlated noises [12-16]. The optimal Kalman filtering fusion problem for dynamic systems with crosscorrelated measurement noises has been dealt with in [1315]. In [16], the state estimation for discrete-time systems with cross-correlated noises has been treated based on an optimal weighted matrix sequence, where the process noises and measurement noises are cross correlated. It should be
pointed out that the estimators mentioned previously are only suited for the correlated noises at the same time instant. In [17, 18], a Kalman-type recursive filter has been proposed for dynamic systems with finite-step autocorrelated process noises, where the autocorrelation property is described by the covariances between different time instants. The filtering problem with finite-step cross-correlated process noises and measurement noises has been investigated in [19]. In [20], the optimal robust nonfragile Kalman-type recursive filter has been designed for a class of uncertain systems with finite-step autocorrelated noises.

On another research frontier, with the development of network technologies, the sensor network has attracted increasing attention from many researchers in different fields due to their wide scope applications in surveillance, environment monitoring, information collection, wireless networks, robotics, and so on. In the sensor network, the network-induced time-delay or/and packet dropouts cannot be avoided due to limited single-sensor energy and communication capability and these have brought us new challenges in the design of the desired state estimators. The binary switching sequence is a popular way to describe the networkinduced time-delay or/and packet dropouts since the timedelay or/and packet dropouts in the sensor network are inherently random [21-24]. The least-mean-square filtering problem for one-step random sampling delay has been studied in [25, 26]. Unfortunately, the filters designed in [25, 26] are suboptimal since a colored noise due to augmentation has been treated as a white noise. The filtering problem for systems with random measurement delays and multiple packet dropouts has also been discussed in [24]. In [27], the problem of robust filtering for uncertain systems with missing measurements and finite-step correlated process noises has been investigated. It should be noted that, in all the aforementioned literature, sensors involved in the sensor network have the same delay characteristics. Recently, Hounkpevi and Yaz [28, 29] present minimum variance state estimators for multiple sensors with different delay or failure rates. The leastsquare filtering problem for systems with one- or two-step random delay has been studied in [30], where the algorithms are derived without requiring the knowledge of the state space model but only the means and covariance functions of the processes involved in the observation equations. The optimal unbiased filtering problem for uncertain systems with different delay rates sensor network and autocorrelated process noises has also been discussed in [31]. However, the estimator obtained in [31] is nonrecursive and a colored noise due to augmentation has been treated as white noise. Up to now, to the best of the authors' knowledge, the recursive robust estimation problem has not yet been addressed for uncertain systems with different delay rates sensor network and autocorrelated noises, and this situation motivates our current study.

Motivated by the above analysis, in this paper, we aim to investigate the recursive robust estimation problem for uncertain systems with different delay rates sensor network and autocorrelated noises. The system model and measurement model under consideration are both subject to stochastic uncertainties or multiplicative noises. Different
sensors in the sensor network have different delay rates and different delay rates are described by different binary switching sequences. The process noises are assumed to be one-step autocorrelated across time and the autocorrelation property is described by the covariances between different time instants. Based on an innovation analysis approach (IAA) and the orthogonal projection theorem (OPT), recursive robust estimators including filter, predictor, and smoother are obtained. This paper extends the results in [31], in two directions: (1) the autocorrelated measurement noise due to augmentation leads to more difficulties in the design of the recursive robust estimators; however, in [31], the measurement noise is treated as a white noise; and (2) the filter obtained in [31] is actually a nonrecursive filter; however, in our current work, we do not only derive a recursive robust filter, but also derive a recursive robust predictor and a recursive robust smoother. Also, the current paper differs from [28,30] for the model uncertainties considered and for the autocorrelated process noises considered to derive the desired recursive robust estimators.

The remainder of the paper is organized as follows. In Section 2, the recursive robust estimation problem is formulated for a class of uncertain systems with autocorrelated noises and different delay rates sensor network. The recursive robust estimators including filter, predictor, and smoother are derived in Section 3. In Section 4, a simulation example is provided to illustrate the usefulness of the theory developed in this paper. We end the paper with some concluding remarks in Section 5.

Notation 1. The notation used in the paper is fairly standard. The superscript " $T$ " stands for matrix transposition, the notation $\mathbb{R}^{n}$ denotes the $n$-dimensional Euclidean space, the notation $\mathbb{R}^{m \times n}$ is the set of all real matrices of dimension $m \times n$, and $I$ and 0 represent the identity matrix and zero matrix, respectively. The notation $P>0$ means that $P$ is real symmetric and positive definite, and $\operatorname{diag}(\cdots)$ stands for block-diagonal matrix. The notation $\delta_{k-j}$ is the Kronecker delta function, which is equal to unity for $k=j$ and zero for $k \neq j$. In addition, $\mathscr{E}\{x\}$ means mathematical expectation of $x$ and $\operatorname{Prob}\{\cdot\}$ represents the occurrence probability of the event ".". Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

## 2. Problem Formulation

Consider the following system model and measurement model:

$$
\begin{gather*}
\breve{x}_{k+1}=\left(\breve{A}_{k}+\breve{A}_{s, k} \mu_{k}\right) \breve{x}_{k}+\breve{B}_{k} \omega_{k} \\
\breve{y}_{k}^{i}=\left(\breve{C}_{k}^{i}+\breve{C}_{s, k}^{i} \eta_{k}^{i}\right) \breve{x}_{k}+\breve{v}_{k}^{i}  \tag{1}\\
y_{k}^{i}=\left(1-\lambda_{k}^{i}\right) \breve{y}_{k}^{i}+\lambda_{k}^{i} \breve{y}_{k-1}^{i}, \quad i=1,2, \ldots, N
\end{gather*}
$$

where $\breve{x}_{k} \in \mathbb{R}^{n}$ is the state to be estimated, the vector $\breve{y}_{k}^{i} \in$ $\mathbb{R}$ is the actual output vector of the $i$ th sensor, the vector $y_{k}^{i} \in \mathbb{R}$ is the measured output vector of the $i$ th sensor, the vector $\omega_{k} \in \mathbb{R}^{m}$ is the process noise, the vectors $\mu_{k} \in \mathbb{R}$
and $\eta_{k}^{i} \in \mathbb{R}$ are multiplicative noises, the vector $\breve{v}_{k}^{i} \in \mathbb{R}$ is the measurement noise of the $i$ th sensor, the matrices $\breve{A}_{k}$, $\breve{A}_{s, k}, \breve{B}_{k}, \breve{C}_{k}^{i}$, and $\breve{C}_{s, k}^{i}$ are known real time-varying matrices of appropriate dimensions, and the variable $\lambda_{k}^{i} \in \mathbb{R}$ is a mutually uncorrelated binary switching sequence (and uncorrelated with other random variables) taking values on 1 and 0 with

$$
\begin{gather*}
\operatorname{Prob}\left\{\lambda_{k}^{i}=1\right\}=\mathscr{E}\left\{\lambda_{k}^{i}\right\}=\beta_{k}^{i}, \\
\operatorname{Prob}\left\{\lambda_{k}^{i}=0\right\}=1-\mathscr{E}\left\{\lambda_{k}^{i}\right\}=1-\beta_{k}^{i} . \tag{2}
\end{gather*}
$$

Remark 1. The measurement model (1) is a popular way to model the random sensor delay. It can be seen that if $\lambda_{k}^{i}=1$ then $y_{k}^{i}=\breve{y}_{k-1}^{i}$ which means that the measurement of the $i$ th sensor is delayed; if $\lambda_{k}^{i}=0$, then $y_{k}^{i}=\breve{y}_{k}^{i}$; that is to say, the measurement of the $i$ th sensor is up to date.

The noise signals $\mu_{k}, \breve{v}_{k}^{i}$, and $\eta_{k}^{i}$ are all zero-mean Gaussian white noises. They, together with the initial state $\breve{x}_{0}$ and the process noise $\omega_{k}$, have the following statistical properties:

$$
\begin{gather*}
\mathscr{E}\left\{\breve{x}_{0}\right\}=\overline{\breve{x}}_{0}, \quad \mathscr{E}\left\{\left(\breve{x}_{0}-\overline{\breve{x}}_{0}\right)\left(\breve{x}_{0}-\overline{\breve{x}}_{0}\right)^{T}\right\}=\breve{P}_{0} \\
\left.\mathscr{E}\left\{\left[\begin{array}{c}
\omega_{k} \\
\mu_{k} \\
\breve{v}_{k}^{i} \\
\eta_{k}^{i} \\
\breve{x}_{0}
\end{array}\right]\left[\begin{array}{c}
\omega_{l} \\
\mu_{l} \\
\breve{v}_{l}^{j} \\
\eta_{l}^{j} \\
\breve{x}_{0}
\end{array}\right]\right\}\right\}  \tag{3}\\
\\
=\left[\begin{array}{ccccc}
Y_{k, l} & 0 & 0 & 0 & 0 \\
0 & \delta_{k-l} & 0 & 0 & 0 \\
0 & 0 & \breve{R}_{k}^{i} \delta_{k-l} \delta_{i-j} & 0 & 0 \\
0 & 0 & 0 & \delta_{k-l} \delta_{i-j} & 0 \\
0 & 0 & 0 & 0 & \breve{X}_{0}
\end{array}\right]
\end{gather*}
$$

where $Y_{k, l}=Q_{k} \delta_{k-l}+Q_{k, l} \delta_{k-l-1}+Q_{k, l} \delta_{k-l+1}, \breve{X}_{0}=\breve{P}_{0}+\overline{\breve{x}}_{0} \overline{\breve{x}}_{0}^{T}$. By defining

$$
\begin{gathered}
x_{k}=\left[\begin{array}{c}
\breve{x}_{k} \\
\breve{x}_{k-1}
\end{array}\right], \quad A_{k}=\left[\begin{array}{cc}
\breve{A}_{k} & 0 \\
I & 0
\end{array}\right], \\
A_{s, k}=\left[\begin{array}{cc}
\breve{A}_{s, k} & 0 \\
0 & 0
\end{array}\right], \quad B_{k}=\left[\begin{array}{c}
\breve{B}_{k} \\
0
\end{array}\right], \\
V_{k}=\left[\begin{array}{c}
\breve{v}_{k} \\
\breve{v}_{k-1}
\end{array}\right], \quad R_{k}=\left[\begin{array}{cc}
\breve{R}_{k} & 0 \\
0 & \breve{R}_{k-1}
\end{array}\right],
\end{gathered}
$$

$$
\begin{gather*}
R_{k, k-1}=\left[\begin{array}{cc}
0 & 0 \\
\breve{R}_{k-1} & 0
\end{array}\right], \quad R_{k, k+1}=\left[\begin{array}{cc}
0 & \breve{R}_{k} \\
0 & 0
\end{array}\right] \\
y_{k}=\left[\begin{array}{lll}
\left(y_{k}^{1}\right)^{T} & \cdots & \left(y_{k}^{N}\right)^{T}
\end{array}\right]^{T} \\
D_{k}=\left[\begin{array}{ll}
\left(I-J_{k}\right) & J_{k}
\end{array}\right] \\
C_{k}=\left[\begin{array}{lll}
\left(I-J_{k}\right) \breve{C}_{k} & J_{k} \breve{C}_{k-1}
\end{array}\right] \\
C_{s, k}=\left[\begin{array}{lll}
\left(I-J_{k}\right) \breve{C}_{s, k} \eta_{k} & J_{k} \breve{C}_{s, k-1} \eta_{k-1}
\end{array}\right] \tag{4}
\end{gather*}
$$

where

$$
\begin{gather*}
\breve{C}_{k}=\left[\left(\breve{C}_{k}^{1}\right)^{T} \ldots\left(\breve{C}_{k}^{N}\right)^{T}\right]^{T}, \\
\breve{v}_{k}=\left[\left(\breve{v}_{k}^{1}\right)^{T} \cdots\left(\breve{v}_{k}^{N}\right)^{T}\right]^{T}, \\
J_{k}=\operatorname{diag}\left(\lambda_{k}^{1}, \ldots, \lambda_{k}^{N}\right), \\
\eta_{k}=\operatorname{diag}\left(\eta_{k}^{1}, \ldots, \eta_{k}^{N}\right), \quad \breve{C}_{s, k}=\left[\left(\breve{C}_{s, k}^{1}\right)^{T} \ldots\left(\breve{C}_{s, k}^{N}\right)^{T}\right]^{T}, \\
\breve{R}_{k}=\operatorname{diag}\left(\breve{R}_{k}^{1}, \ldots, \breve{R}_{k}^{N}\right), \tag{5}
\end{gather*}
$$

a compact representation of (1) can be expressed as follows:

$$
\begin{align*}
x_{k+1} & =\left(A_{k}+A_{s, k} \mu_{k}\right) x_{k}+B_{k} \omega_{k}  \tag{6}\\
y_{k} & =C_{k} x_{k}+C_{s, k} x_{k}+D_{k} V_{k} \tag{7}
\end{align*}
$$

where $V_{k}$ is the measurement noise of the newly obtained auxiliary system (6) and (7). It follows readily from (4) that $V_{k}$ has the statistic properties as follows:

$$
\begin{gather*}
\mathscr{E}\left\{V_{k}\right\}=0 \\
\mathscr{E}\left\{V_{k} V_{t}^{T}\right\}=R_{k} \delta_{k-t}+R_{k, k-1} \delta_{k-t-1}+R_{k, k+1} \delta_{k-t+1} \tag{8}
\end{gather*}
$$

Remark 2. It can be seen from (3) and (8) that the process noise $\omega_{k}$ and the measurement noise $V_{k}$ are both one-step autocorrelated across time. For example, the process noise at time $k$ is correlated with the process noises at times $k-1$ and $k+1$ with covariances $Q_{k, k-1}$ and $Q_{k, k+1}$, respectively. The measurement noise at time $k$ is correlated with the measurement noises at times $k-1$ and $k+1$ with covariances $R_{k, k-1}$ and $R_{k, k+1}$, respectively.

Remark 3. Observe that the system model and measurement model of system (6) and (7) are both subject to stochastic uncertainties and $C_{k}, C_{s, k}$, and $D_{k}$ involve the stochastic variable $\lambda_{k}^{i}$. Thus, system (6) and (7) is actually a stochastic uncertain system. On the other hand, the process noise $\omega_{k}$ and the measurement noise $V_{k}$ are both one-step autocorrelated across time. Therefore, the traditional recursive robust estimation approaches may not satisfy the performance requirements here.

Remark 4. A seemingly natural way of handling the autocorrelated noises is the augmentation of the system states. However, such a state augmentation approach gives rise to significant increase in the system dimension, which would inevitably lead to computational burden. In addition, in the state augmentation method, the noises are treated as components of the auxiliary system state, generally, it is difficult for an estimator to track noise signals, and this will affect the estimation of other components of the auxiliary system state. Without resorting to state augmentation, in our current work, we treat system (6) and (7) directly by using an IAA and the OPT.

## 3. The Main Results

For convenience of later development, let us introduce the following lemmas, which are very useful in establishing our main results.

Lemma 5. For stochastic matrices $J_{k}, C_{k}, D_{k}$, and $C_{s, k}$, one has the following results:

$$
\begin{gather*}
\bar{J}_{k}=\mathscr{E}\left\{J_{k}\right\}=\operatorname{diag}\left(\beta_{k}^{1}, \ldots, \beta_{k}^{N}\right), \quad \widetilde{J}_{k}=J_{k}-\bar{J}_{k}, \\
\Sigma_{k}=\mathscr{E}\left\{\widetilde{J}_{k} \widetilde{J}_{k}^{T}\right\}=\operatorname{diag}\left(\left(1-\beta_{k}^{1}\right) \beta_{k}^{1}, \ldots,\left(1-\beta_{k}^{N}\right) \beta_{k}^{N}\right), \\
\bar{C}_{k}=\mathscr{E}\left\{C_{k}\right\}=\left[\left(I-\bar{J}_{k}\right) \check{C}_{k} \bar{J}_{k} \check{C}_{k-1}\right], \\
\widetilde{C}_{k}=C_{k}-\bar{C}_{k}=\left(J_{k}-\bar{J}_{k}\right)\left[-\breve{C}_{k} \check{C}_{k-1}\right]=\widetilde{J}_{k} C_{e, k},  \tag{9}\\
\bar{D}_{k}=\mathscr{E}\left\{D_{k}\right\}=\left[\left(I-\bar{J}_{k}\right) \bar{J}_{k}\right], \\
\widetilde{D}_{k}=D_{k}-\bar{D}_{k}=\left(J_{k}-\bar{J}_{k}\right)[-I I]=\widetilde{J}_{k} D_{e, k}, \\
C_{e, k}=\left[-\breve{C}_{k} \breve{C}_{k-1}\right], \quad D_{e, k}=[-I \quad I], \\
\mathscr{E}\left\{\widetilde{J}_{k}\right\}=\mathscr{E}\left\{\widetilde{C}_{k}\right\}=\mathscr{E}\left\{\widetilde{D}_{k}\right\}=\mathscr{E}\left\{C_{s, k}\right\}=0 .
\end{gather*}
$$

Proof. Lemma 5 follows directly from (2), (4), and (5) and the fact that $\eta_{k}$ is zero mean.

Lemma 6. For system state $x_{k}$ and the process noise $\omega_{k}$, one has the following result:

$$
\begin{equation*}
\mathscr{E}\left\{x_{k} \omega_{k}^{T}\right\}=B_{k-1} Q_{k-1, k} \tag{10}
\end{equation*}
$$

Proof. Lemma 6 follows directly from (3) and (6).
Lemma 7. The state covariance matrix $X_{k}=\mathscr{E}\left\{x_{k} x_{k}^{T}\right\}$ has the following recursion:

$$
\begin{align*}
X_{k+1}= & A_{k} X_{k} A_{k}^{T}+A_{k} B_{k-1} Q_{k-1, k} B_{k}^{T}+A_{s, k} X_{k} A_{s, k}^{T}  \tag{11}\\
& +B_{k} Q_{k, k-1} B_{k-1}^{T} A_{k}^{T}+B_{k} Q_{k} B_{k}^{T} .
\end{align*}
$$

Proof. Lemma 7 follows directly from (3), (6), and Lemma 6.

Furthermore, defining $\breve{X}_{k+1}=\mathscr{E}\left\{\breve{x}_{k+1} \breve{x}_{k+1}^{T}\right\}$ and $\breve{X}_{k+1, k}=$ $\mathscr{E}\left\{\breve{x}_{k+1} \breve{x}_{k}^{T}\right\}$, one has from (4) and Lemma 7 the following:

$$
\begin{align*}
\breve{X}_{k+1}= & \breve{A}_{k} \breve{X}_{k} \breve{A}_{k}^{T}+\breve{A}_{k} \breve{B}_{k-1} Q_{k-1, k} \breve{B}_{k}^{T}+\breve{A}_{s, k} \breve{X}_{k} \breve{A}_{s, k}^{T} \\
& +\breve{B}_{k} Q_{k, k-1} \breve{B}_{k-1}^{T} \breve{A}_{k}^{T}+\breve{B}_{k} Q_{k} \breve{B}_{k}^{T},  \tag{12}\\
\breve{X}_{k+1, k}= & A_{k} \breve{X}_{k}+\breve{B}_{k} Q_{k, k-1} \breve{B}_{k-1}^{T} .
\end{align*}
$$

Lemma 8 (see [32]). If $A \in \mathscr{R}^{p \times p}$ is a real matrix and $B=$ $\operatorname{diag}\left(b_{1}, \ldots, b_{p}\right)$ is a diagonal stochastic matrix, then

$$
\mathscr{E}\left\{\mathrm{BAB}^{T}\right\}=\left[\begin{array}{ccc}
\mathscr{E}\left\{b_{1}^{2}\right\} & \cdots & \mathscr{E}\left\{b_{1} b_{p}\right\}  \tag{13}\\
\vdots & \cdots & \vdots \\
\mathscr{E}\left\{b_{p} b_{1}\right\} & \cdots & \mathscr{E}\left\{b_{p}^{2}\right\}
\end{array}\right] \otimes A
$$

where $\otimes$ is the Hadamard product (this product is defined as $\left.[A \otimes B]_{i, j}=A_{i, j} \cdot B_{i, j}\right)$.

### 3.1. Recursive Robust Filter

Theorem 9. For the addressed system (6) and (7), one has the following recursive robust filter:

$$
\begin{align*}
& \widehat{x}_{k \mid k-1}= A_{k-1} \widehat{x}_{k-1 \mid k-1}+B_{k-1} Q_{k-1, k-2} B_{k-2}^{T} \bar{C}_{k-1}^{T} \Pi_{k-1}^{-1} \varepsilon_{k-1},  \tag{14}\\
& P_{k \mid k-1}= A_{k-1} P_{k-1 \mid k-1} A_{k-1}^{T}+B_{k-1} Q_{k-1} B_{k-1}^{T} \\
&+A_{s, k-1} X_{k-1} A_{s, k-1}^{T} \\
&+A_{k-1}\left(B_{k-2} Q_{k-2, k-1}-F_{k-1, k-1}\right. \\
&\left.\times \Pi_{k-1}^{-1} \bar{C}_{k-1} B_{k-2} Q_{k-2, k-1}\right) B_{k-1}^{T} \\
&+B_{k-1}\left(B_{k-2} Q_{k-2, k-1}-F_{k-1, k-1}\right.  \tag{15}\\
&\left.\quad \times \Pi_{k-1}^{-1} \bar{C}_{k-1} B_{k-2} Q_{k-2, k-1}\right)^{T} A_{k-1}^{T} \\
& \quad-B_{k-1} Q_{k-1, k-2} B_{k-2}^{T} \bar{C}_{k-1}^{T} \Pi_{k-1}^{-1} \bar{C}_{k-1} B_{k-2} \\
& \times Q_{k-1, k-2}^{T} B_{k-1}^{T}, \\
& \varepsilon_{k}= y_{k}-\bar{C}_{k} \widehat{x}_{k \mid k-1}-\bar{D}_{k} R_{k, k-1} \bar{D}_{k-1}^{T} \Pi_{k-1}^{-1} \varepsilon_{k-1},  \tag{16}\\
& F_{k, k}= P_{k \mid k-1} \bar{C}_{k}^{T} \\
&-\left(A_{k-1} F_{k-1, k-1}+B_{k-1} Q_{k-1, k-2} B_{k-2}^{T} \bar{C}_{k-1}^{T}\right)  \tag{17}\\
& \times \Pi_{k-1}^{-1} \bar{D}_{k-1} R_{k, k-1} \bar{D}_{k}^{T}
\end{align*}
$$

$$
\begin{align*}
& \Pi_{k}= \bar{C}_{k} P_{k \mid k-1} \bar{C}_{k}^{T}-\bar{C}_{k} \\
& \times\left[\left(A_{k-1} F_{k-1, k-1}+B_{k-1} Q_{k-1, k-2} B_{k-2}^{T} \bar{C}_{k-1}^{T}\right)\right. \\
&\left.\times \Pi_{k-1}^{-1} \bar{D}_{k-1} R_{k-1, k}\right] \bar{D}_{k}^{T}+\Sigma_{k} \otimes\left(C_{e, k} X_{k} C_{e, k}^{T}\right) \\
&+\left(I-\bar{J}_{k}\right) \breve{C}_{s, k} \breve{X}_{k} C_{s, k}^{T}\left(I-\bar{J}_{k}\right)^{T}+\Sigma_{k} \otimes\left(\breve{C}_{s, k} \breve{X}_{k} C_{s, k}^{T}\right) \\
&+ \bar{J}_{k} \breve{C}_{s, k-1} \breve{X}_{k-1} \breve{C}_{s, k-1}^{T} \bar{J}_{k}^{T}+\Sigma_{k} \otimes\left(\breve{C}_{s, k-1} \breve{X}_{k-1} C_{s, k-1}^{T}\right) \\
&-\bar{D}_{k}\left[\left(A_{k-1} F_{k-1, k-1}+B_{k-1} Q_{k-1, k-2} B_{k-2}^{T} \bar{C}_{k-1}^{T}\right)\right. \\
&\left.\times \Pi_{k-1}^{-1} \bar{D}_{k-1} R_{k-1, k}\right]^{T} \bar{C}_{k}^{T}+\bar{D}_{k} R_{k} \bar{D}_{k}^{T} \\
&+\Sigma_{k} \otimes\left(D_{e, k} R_{k} D_{e, k}^{T}\right)-\bar{D}_{k} R_{k, k-1} \bar{D}_{k-1}^{T} \\
& \times \Pi_{k-1}^{-1} \bar{D}_{k-1} R_{k, k-1}^{T} \bar{D}_{k}^{T},  \tag{18}\\
& \hat{x}_{k \mid k}=\widehat{x}_{k \mid k-1}+F_{k, k} \Pi_{k}^{-1} \varepsilon_{k},  \tag{19}\\
& P_{k \mid k}=P_{k \mid k-1}-F_{k, k} \Pi_{k}^{-1} F_{k, k}^{T},
\end{align*}
$$

where $\varepsilon_{k}$ is the innovation with covariance $\Pi_{k}$, the matrix $F_{k, k}$ is the covariance between $x_{k}$ and $\varepsilon_{k}$, the vectors $\widehat{x}_{k \mid k}$ and $\widehat{x}_{k \mid k-1}$ are the filter and one-step predictor, and the matrices $P_{k \mid k}$ and $P_{k \mid k-1}$ are the filter error covariance and one-step prediction error covariance. The initial values are $\widehat{x}_{0 \mid 0}=\left[\begin{array}{ll}\bar{x}_{0}^{T} & 0\end{array}\right]^{T}, P_{0 \mid 0}=$ $\operatorname{diag}\left(\breve{P}_{0}, 0\right)$, and $\varepsilon_{1}=y_{1}-\bar{C}_{1} \widehat{x}_{1 \mid 0}$.

Proof. Please see Appendix A.
Remark 10. In the traditional recursive estimation problem, the innovation is calculated as $\varepsilon_{k}=y_{k}-C_{k} \widehat{x}_{k \mid k-1}$. However, due to possible sensor delay which occurs in a random way, this is not true for the problem at hand; thus, we have to recalculate the innovation as in (16). Furthermore, it can be seen that the second term on the right-hand side of (14), the last four terms on the right-hand side of (15), the second term of the right-hand side of (17), and the last ten terms on the right-hand side of (18) are caused by the random delays, the stochastic uncertainties, and autocorrelated noises. These terms constitute the main differences between our work and the traditional Kalman filter.

Next, we will derive the recursive robust predictor and recursive robust smoother based on Theorem 9.

### 3.2. Recursive Robust Predictor

Theorem 11. For the addressed system (6) and (7), one has the following L-step $(L \geq 2)$ recursive robust predictor:

$$
\widehat{x}_{k+L \mid k}=A_{k+L-1} \widehat{x}_{k+L-1 \mid k}
$$

$$
\begin{align*}
P_{k+L \mid k}= & A_{k+L-1} P_{k+L-1 \mid k} A_{k+L-1}^{T} \\
& +A_{k+L-1} B_{k+L-2} Q_{k+L-2, k+L-1} B_{k+L-1}^{T}+A_{s, k+L-1} \\
& \times X_{k+L-1} A_{s, k+L-1}^{T}+B_{k+L-1} Q_{k+L-1, k+L-2} \\
& \times B_{k+L-2}^{T} A_{k+L-1}^{T}+B_{k+L-1} Q_{k+L-1} B_{k+L-1}^{T} \tag{20}
\end{align*}
$$

where the initial values $\widehat{x}_{k+1 \mid k}$ and $P_{k+1 \mid k}$ can be calculated as in Theorem 9.

Proof. Please see Appendix B.

### 3.3. Recursive Smoother

Theorem 12. For the addressed system (6) and (7), one has the following robust recursive $L$-step $(L>0)$ fixed-lag smoother:

$$
\begin{gather*}
\widehat{x}_{k \mid k+L}=\widehat{x}_{k \mid k+L-1}+F_{k, k+L} \Pi_{k+L}^{-1} \varepsilon_{k+L}, \\
F_{k, k+L}=\Psi_{k+L} \bar{C}_{k+L}^{T}-F_{k, k+L-1} \Pi_{k+L-1}^{-1} \\
\times \bar{D}_{k+L-1} R_{k+L, k+L-1}^{T} \bar{D}_{k+L}^{T}, \\
\Psi_{k+1}= \\
P_{k \mid k-1} A_{k}^{T}-F_{k, k} \Pi_{k}^{-1} F_{k, k}^{T} A_{k}^{T}+B_{k-1} Q_{k-1, k} B_{k}^{T}  \tag{21}\\
-F_{k, k} \Pi_{k}^{-1} \bar{C}_{k} B_{k-1} Q_{k, k-1}^{T} B_{k}^{T}, \\
\Psi_{k+L}= \\
\Psi_{k+L-1} A_{k+L-1}^{T}-F_{k, k+L-1} \Pi_{k+L-1}^{-1} F_{k+L-1, k+L-1} \\
-F_{k, k+L-1} \Pi_{k+L-1}^{-1} \bar{C}_{k+L-1} B_{k+L-2} \\
\times Q_{k+L-1, k+L-2}^{T} B_{k+L-1}^{T}, \quad(L>1), \\
P_{k \mid k+L}=P_{k \mid k+L-1}-F_{k, k+L} \Pi_{k+L}^{-1} F_{k, k+L}^{T},
\end{gather*}
$$

where the initial values $\hat{x}_{k \mid k}, P_{k \mid k}$, and $F_{k, k}$ are supplied by Theorem 9.

Proof. Please see Appendix C.

## 4. An Illustrative Example

Consider the following uncertain system with different delay rates sensor network and autocorrelated process noises:

$$
\begin{gather*}
\breve{x}_{k+1}=\left(\left[\begin{array}{cc}
0.95 & 0.1 \\
0 & 0.95
\end{array}\right]+\left[\begin{array}{cc}
0.1 & 0 \\
0 & 0.1
\end{array}\right] \mu_{k}\right) \breve{x}_{k}+\left[\begin{array}{l}
0.3 \\
0.1
\end{array}\right] \omega_{k}  \tag{22}\\
\omega_{k}=\zeta_{k}+\zeta_{k-1}  \tag{23}\\
\breve{y}_{k}^{i}=\left(\breve{C}_{k}^{i}+\breve{C}_{k, s}^{i} \eta_{k}^{i}\right) \breve{x}_{k}+\breve{v}_{k}^{i}, \quad i=1,2  \tag{24}\\
y_{k}^{i}=\left(1-\lambda_{k}^{i}\right) \breve{y}_{k}^{i}+\lambda_{k}^{i} \breve{y}_{k-1}^{i}, \quad i=1,2 \tag{25}
\end{gather*}
$$

where $\breve{x}_{k} \in \mathbb{R}^{2}$ is the state to be estimated. The vectors $\zeta_{k} \in \mathbb{R}$, $\mu_{k} \in \mathbb{R}, \eta_{k}^{i} \in \mathbb{R}$, and $v_{k}^{i} \in \mathbb{R}, i=1,2$ are zero-mean Gaussian white noises with covariances $0.5,1,1$, and 1 , respectively.


Figure 1: MSE1 filter, predictor, and smoother.

Without loss of generality, the process noise $\omega_{k}$ is chosen to be as defined in (23).

In the simulation, the initial value $\breve{x}_{0}$ has mean $\mathscr{E}\left\{\breve{x}_{0}\right\}=$ $\overline{\breve{x}}_{0}^{T}=\left[\begin{array}{ll}100 & 10\end{array}\right]^{T}$ and covariance $\breve{P}_{0}=\operatorname{diag}(20,1)$. The variables $\lambda_{k}^{i} \in \mathbb{R}, i=1,2$ are binary switching sequences taking values on 1 with $\operatorname{Prob}\left\{\lambda_{k}^{1}=1\right\}=\mathscr{E}\left\{\lambda_{k}^{1}\right\}=\beta_{k}^{1}=0.15$ and $\operatorname{Prob}\left\{\lambda_{k}^{2}=1\right\}=\mathscr{E}\left\{\lambda_{k}^{2}\right\}=\beta_{k}^{2}=0.25$, respectively, and the matrices are set as $\breve{C}_{k}^{1}=\left[\begin{array}{ll}0 & 1\end{array}\right], \breve{C}_{k}^{2}=\left[\begin{array}{ll}1 & 0\end{array}\right]$, $\breve{C}_{k, s}^{1}=\left[\begin{array}{ll}0 & 0.1\end{array}\right]$, and $\breve{C}_{k, s}^{2}=\left[\begin{array}{ll}0.1 & 0\end{array}\right]$. The newly obtained recursive robust estimators and the filter of Zeng et al. [31] are compared in the simulation. Let MSE1 denotes the meansquare error for estimation of the first component of $\breve{x}_{k}$; that is, $\left.(1 / K) \sum_{k=1}^{K}\left\{\begin{array}{ll}1 & 0\end{array}\right]\left(\breve{x}_{k}-\widehat{\tilde{x}}_{k \mid k}\right)\right\}$, where $K$ is the number of the samples. Similarly, MSE2 denotes the mean-square error for estimation of the second component of $\breve{x}_{k}$; that is, $(1 / K) \sum_{k=1}^{K}\left\{\left[\begin{array}{ll}0 & 1\end{array}\right]\left(\breve{x}_{k}-\widehat{\tilde{x}}_{k \mid k}\right)\right\}$.

From Figures 1 and 2, we can see that the smoother has the best performance and the predictor has the worst performance. This is due to the fact that smoother uses the most measurement information and the predictor uses the least measurement information.

From Figures 3 and 4, we can see that the filter developed in this work has better performance than the filter of Zeng et al. [31]. This is due to the fact that the autocorrelated measurement noise $V_{k}$ was treated as zero-mean Gaussian white noise in the filter of Zeng et al. [31].

## 5. Conclusions

In this paper, we have studied the recursive robust estimation problem for a class of uncertain systems with autocorrelated process noises and different delay rates sensor network. The system model and measurement model are both subject


Figure 2: MSE2 filter, predictor, and smoother.


Figure 3: MSE1 filter of this work and Zeng et al. [31].
to stochastic uncertainties. The process noises are one-step autocorrelated across time. Each sensor in the sensor network has a different delay rate and the delay rate has been described by an individual binary switching sequence obeying a conditional probability distributed. Based on an IAA and the OPT, recursive robust estimators including filter, predictor, and smoother have been obtained. Simulation results have indicated that the smoother has the best performance and the predictor has the worst performance, and the filter obtained in this work has better performance than the filter of Zeng et al. [31].


Figure 4: MSE2 filter of this work and Zeng et al. [31].

## Appendices

## A. The Proof of Theorem 9

Proof. Using the OPT, the one-step measurement prediction $\widehat{y}_{k \mid k-1}$ can be calculated as follows:

$$
\begin{align*}
\hat{y}_{k \mid k-1} & =\sum_{i=1}^{k-1} \mathscr{E}\left\{y_{k} \varepsilon_{i}^{T}\right\} \Pi_{i}^{-1} \varepsilon_{i} \\
& =\sum_{i=1}^{k-1} \mathscr{E}\left\{C_{k} x_{k} \varepsilon_{i}^{T}\right\} \Pi_{i}^{-1} \varepsilon_{i}+\sum_{i=1}^{k-1} \mathscr{E}\left\{D_{k} V_{k} \varepsilon_{i}^{T}\right\} \Pi_{i}^{-1} \varepsilon_{i}  \tag{A.1}\\
& =\bar{C}_{k} \hat{x}_{k \mid k-1}+\sum_{i=1}^{k-1} \mathscr{E}\left\{D_{k} V_{k} \varepsilon_{i}^{T}\right\} \Pi_{i}^{-1} \varepsilon_{i} .
\end{align*}
$$

Taking into account the fact that $V_{k}$ is one-step autocorrelated, we have from (4), (8), and (9) the following:

$$
\begin{gather*}
\mathscr{E}\left\{D_{k} V_{k} \varepsilon_{i}^{T}\right\}=0, \quad i \leq k-2, \\
\mathscr{E}\left\{D_{k} V_{k} \varepsilon_{k-1}^{T}\right\}=\mathscr{E}\left\{D_{k} V_{k}\left(y_{k-1}-\widehat{y}_{k-1 \mid k-2}\right)^{T}\right\}  \tag{A.2}\\
\\
=\mathscr{E}\left\{D_{k} V_{k} V_{k-1}^{T} D_{k-1}^{T}\right\} \\
\\
=\bar{D}_{k} R_{k, k-1} \bar{D}_{k-1}^{T}
\end{gather*}
$$

Substituting (A.2) into (A.1), we have

$$
\begin{equation*}
\widehat{y}_{k \mid k-1}=\bar{C}_{k} \widehat{x}_{k \mid k-1}+\bar{D}_{k} R_{k, k-1} \bar{D}_{k-1}^{T} \Pi_{k-1}^{-1} \varepsilon_{k-1} \tag{A.3}
\end{equation*}
$$

Therefore, the innovation $\varepsilon_{k}$ can be calculated as follows:

$$
\begin{align*}
\varepsilon_{k}= & y_{k}-\widehat{y}_{k \mid k-1} \\
= & y_{k}-\bar{C}_{k} \widehat{x}_{k \mid k-1}-\bar{D}_{k} R_{k, k-1} \bar{D}_{k-1}^{T} \Pi_{k-1}^{-1} \varepsilon_{k-1} \\
= & \left(\bar{C}_{k}+\widetilde{C}_{k}\right) x_{k}+C_{s, k} x_{k}+D_{k} V_{k}-\bar{C}_{k} \widehat{x}_{k \mid k-1} \\
& -\bar{D}_{k} R_{k, k-1} \bar{D}_{k-1}^{T} \Pi_{k-1}^{-1} \varepsilon_{k-1}  \tag{A.4}\\
= & \bar{C}_{k} \widetilde{x}_{k \mid k-1}+\widetilde{C}_{k} x_{k}+C_{s, k} x_{k}+D_{k} V_{k} \\
& -\bar{D}_{k} R_{k, k-1} \bar{D}_{k-1}^{T} \Pi_{k-1}^{-1} \varepsilon_{k-1}
\end{align*}
$$

where $\tilde{x}_{k \mid k-1}=x_{k}-\widehat{x}_{k \mid k-1}$.
Again, according to the OPT, the state prediction $\widehat{x}_{k \mid k-1}$ can be obtained as follows:

$$
\begin{align*}
\widehat{x}_{k \mid k-1}= & \sum_{i=1}^{k-1} \mathscr{E}\left\{x_{k} \varepsilon_{i}^{T}\right\} \Pi_{i}^{-1} \varepsilon_{i} \\
= & \sum_{i=1}^{k-1} \mathscr{E}\left\{\left(A_{k-1} x_{k-1}+A_{s, k-1} \mu_{k-1} x_{k-1}\right.\right.  \tag{A.5}\\
& \left.\left.+B_{k-1} \omega_{k-1}\right) \varepsilon_{i}^{T}\right\} \Pi_{i}^{-1} \varepsilon_{i} \\
= & A_{k-1} \widehat{x}_{k-1 \mid k-1}+B_{k-1} \sum_{i=1}^{k-1} \mathscr{C}\left\{\omega_{k-1} \varepsilon_{i}^{T}\right\} \Pi_{i}^{-1} \varepsilon_{i}
\end{align*}
$$

Taking (3) into consideration, the expectation $\mathscr{E}\left\{\omega_{k-1} \varepsilon_{i}^{T}\right\}$ can be calculated as follows:

$$
\begin{align*}
\mathscr{E}\left\{\omega_{k-1} \varepsilon_{i}^{T}\right\}= & 0, \quad i \leq k-2, \\
\mathscr{E}\left\{\omega_{k-1} \varepsilon_{k-1}^{T}\right\}= & \mathscr{E}\left\{\omega_{k-1}\left(y_{k-1}-\widehat{y}_{k-1 \mid k-2}\right)^{T}\right\} \\
= & \mathscr{E}\left\{\omega_{k-1} x_{k-1}^{T} C_{k-1}^{T}\right\}, \\
= & \mathscr{E}\left\{\omega _ { k - 1 } \left(A_{k-2} x_{k-2}+A_{s, k-2} \mu_{k-2} x_{k-2}\right.\right.  \tag{A.6}\\
& \left.\left.+B_{k-2} \omega_{k-2}\right)^{T}\right\} \bar{C}_{k-1}^{T} \\
= & Q_{k-1, k-2} B_{k-2}^{T} \bar{C}_{k-1}^{T} .
\end{align*}
$$

Substituting (A.6) into (A.5), we have

$$
\begin{align*}
\widehat{x}_{k \mid k-1}= & A_{k-1} \widehat{x}_{k-1 \mid k-1}+B_{k-1} Q_{k-1, k-2} \\
& \times B_{k-2}^{T} \bar{C}_{k-1}^{T} \Pi_{k-1}^{-1} \varepsilon_{k-1} . \tag{A.7}
\end{align*}
$$

Therefore, the one-step prediction error $\tilde{x}_{k \mid k-1}$ can be calculated as follows:

$$
\begin{align*}
\tilde{x}_{k \mid k-1}= & x_{k}-\widehat{x}_{k \mid k-1} \\
= & \left(A_{k-1}+A_{s, k-1} \mu_{k-1}\right) x_{k-1} \\
& +B_{k-1} \omega_{k-1}-A_{k-1} \widehat{x}_{k-1 \mid k-1} \\
& -B_{k-1} Q_{k-1, k-2} B_{k-2}^{T} \bar{C}_{k-1}^{T} \Pi_{k-1}^{-1} \varepsilon_{k-1}, \\
= & A_{k-1} \widetilde{x}_{k-1 \mid k-1}+A_{s, k-1} \mu_{k-1} x_{k-1} \\
& +B_{k-1} \omega_{k-1}-B_{k-1} Q_{k-1, k-2} B_{k-2}^{T} \bar{C}_{k-1}^{T} \Pi_{k-1}^{-1} \varepsilon_{k-1}, \tag{A.8}
\end{align*}
$$

where $\widetilde{x}_{k-1 \mid k-1}$ is the filter error at time instant $k-1$. Taking into account the fact that $\omega_{k}$ is one-step autocorrelated across time and $\mu_{k}$ is uncorrelated with other signals, the onestep prediction error covariance $P_{k \mid k-1}$ can be calculated as follows:

$$
\begin{align*}
P_{k \mid k-1}= & \mathscr{E}\left\{\tilde{x}_{k \mid k-1} \tilde{x}_{k \mid k-1}^{T}\right\} \\
= & A_{k-1} P_{k-1 \mid k-1} A_{k-1}^{T}+A_{k-1} \mathscr{E}\left\{\widetilde{x}_{k-1 \mid k-1} \omega_{k-1}^{T}\right\} B_{k-1}^{T} \\
& +A_{s, k-1} X_{k-1} A_{s, k-1}^{T}+B_{k-1} \mathscr{E}\left\{\omega_{k-1} \widetilde{x}_{k-1 \mid k-1}^{T}\right\} A_{k-1}^{T} \\
& +B_{k-1} Q_{k-1} B_{k-1}^{T}-B_{k-1} \mathscr{E}\left\{\omega_{k-1} \varepsilon_{k-1}^{T}\right\} \Pi_{k-1}^{-1} \bar{C}_{k-1} \\
& \times B_{k-2} Q_{k-1 \mid k-2}^{T} B_{k-1}^{T} \\
& -B_{k-1} Q_{k-1, k-2} B_{k-2}^{T} \bar{C}_{k-1}^{T} \Pi_{k-1}^{-1} \mathscr{E}\left\{\varepsilon_{k-1} \omega_{k-1}^{T}\right\} B_{k-1}^{T} \\
& +B_{k-1} Q_{k-1, k-2} B_{k-2}^{T} \bar{C}_{k-1}^{T} \Pi_{k-1}^{-1} \bar{C}_{k-1} \\
& \times B_{k-2} Q_{k-1 \mid k-2}^{T} B_{k-1}^{T}, \tag{A.9}
\end{align*}
$$

where the expectation $\mathscr{E}\left\{\omega_{k-1} \varepsilon_{k-1}^{T}\right\}$ can be calculated as in (A.6) and expectation $\mathscr{E}\left\{\widetilde{x}_{k-1 \mid k-1} \omega_{k-1}^{T}\right\}$ can be obtained as follows:

$$
\begin{align*}
\mathscr{E}\left\{\tilde{x}_{k-1 \mid k-1} \omega_{k-1}^{T}\right\}= & \mathscr{E}\left\{x_{k-1} \omega_{k-1}^{T}\right\}-\mathscr{E}\left\{\widehat{x}_{k-1 \mid k-1} \omega_{k-1}^{T}\right\} \\
= & B_{k-2} \mathscr{E}\left\{\omega_{k-2} \omega_{k-1}^{T}\right\} \\
& -\mathscr{E}\left\{\left(\sum_{i=1}^{k-1} \mathscr{E}\left\{x_{k-1} \varepsilon_{i}^{T}\right\} \Pi_{i}^{-1} \varepsilon_{i}\right) \omega_{k-1}^{T}\right\} \\
= & B_{k-2} Q_{k-2, k-1}-\mathscr{E}\left\{x_{k-1} \varepsilon_{k-1}^{T}\right\} \Pi_{k-1}^{-1} \\
& \times \mathscr{E}\left\{\varepsilon_{k-1} \omega_{k-1}^{T}\right\} \\
= & B_{k-2} Q_{k-2, k-1}-F_{k-1, k-1} \Pi_{k-1}^{-1} \bar{C}_{k-1} \\
& \times B_{k-2} Q_{k-2, k-1}, \tag{A.10}
\end{align*}
$$

where the third equality in (A.10) holds since $\omega_{k}$ is onestep autocorrelated across time. Substituting (A.10) into (A.9) yields (15).

Noting the fact that $\widetilde{x}_{k \mid k-1}$ is orthogonal to $\widehat{x}_{k \mid k-1}$, we have from (9) and (A.4) the following:

$$
\begin{align*}
F_{k, k}= & \mathscr{E}\left\{x_{k} \varepsilon_{k}^{T}\right\} \\
= & \mathscr{E}\left\{x _ { k } \left(\bar{C}_{k} \tilde{x}_{k \mid k-1}+\widetilde{C}_{k} x_{k}+C_{s, k} x_{k}+D_{k} V_{k}\right.\right. \\
& \left.\left.\quad-\bar{D}_{k} R_{k, k-1} \bar{D}_{k-1}^{T} \Pi_{k-1}^{-1} \varepsilon_{k-1}\right)^{T}\right\} \\
= & P_{k \mid k-1} \bar{C}_{k}^{T}-\mathscr{E}\left\{x_{k} \varepsilon_{k-1}^{T}\right\} \Pi_{k-1}^{-1} \bar{D}_{k-1} R_{k, k-1} \bar{D}_{k}^{T} \\
= & P_{k \mid k-1} \bar{C}_{k}^{T}-\left(A_{k-1} \mathscr{E}\left\{x_{k-1} \varepsilon_{k-1}^{T}\right\}+B_{k-1} \mathscr{E}\left\{\omega_{k-1} \varepsilon_{k-1}^{T}\right\}\right) \\
& \times \Pi_{k-1}^{-1} \bar{D}_{k-1} R_{k, k-1} \bar{D}_{k}^{T} \\
= & P_{k \mid k-1} \bar{C}_{k}^{T}-\left(A_{k-1} F_{k-1, k-1}+B_{k-1} Q_{k-1, k-2}\right. \\
& \left.\quad \times B_{k-2}^{T} \bar{C}_{k-1}^{T}\right) \Pi_{k-1}^{-1} \bar{D}_{k-1} R_{k, k-1} \bar{D}_{k}^{T} . \tag{A.11}
\end{align*}
$$

It implies from (9), (A.4), and Lemmas 5 and 8 that the expectation $\Pi_{k}$ can be obtained as follows:

$$
\begin{align*}
\Pi_{k}= & \mathscr{E}\left\{\varepsilon_{k} \varepsilon_{k}^{T}\right\} \\
= & \mathscr{E}\left\{\left(\bar{C}_{k} \widetilde{x}_{k \mid k-1}+\widetilde{C}_{k} x_{k}+C_{s, k} x_{k}+D_{k} V_{k}\right.\right. \\
& \left.-\bar{D}_{k} R_{k, k-1} \bar{D}_{k-1}^{T} \Pi_{k-1}^{-1} \varepsilon_{k-1}\right) \\
& \times\left(\bar{C}_{k} \widetilde{x}_{k \mid k-1}+\widetilde{C}_{k} x_{k}+C_{s, k} x_{k}+D_{k} V_{k}\right. \\
& \left.\left.\quad-\bar{D}_{k} R_{k, k-1} \bar{D}_{k-1}^{T} \Pi_{k-1}^{-1} \varepsilon_{k-1}\right)^{T}\right\} \\
= & \bar{C}_{k} P_{k \mid k-1} \bar{C}_{k}^{T}+\bar{C}_{k} \mathscr{E}\left\{\widetilde{x}_{k \mid k-1} V_{k}^{T}\right\} \bar{D}_{k}^{T}  \tag{A.12}\\
& +\mathscr{E}\left\{\widetilde{C}_{k} x_{k} x_{k}^{T} \widetilde{C}_{k}^{T}\right\}+\mathscr{E}\left\{C_{s, k} x_{k} x_{k}^{T} C_{s, k}^{T}\right\} \\
+ & \bar{D}_{k} \mathscr{E}\left\{V_{k} \widetilde{x}_{k \mid k-1}^{T}\right\} \bar{C}_{k}^{T}+\mathscr{E}\left\{D_{k} V_{k} V_{k}^{T} D_{k}^{T}\right\} \\
& -\mathscr{E}\left\{D_{k} V_{k} \varepsilon_{k-1}^{T}\right\} \Pi_{k-1}^{-1} \bar{D}_{k-1} R_{k, k-1}^{T} \bar{D}_{k}^{T} \\
& -\bar{D}_{k} R_{k, k-1} \bar{D}_{k-1}^{T} \Pi_{k-1}^{-1} \mathscr{E}\left\{\varepsilon_{k-1} V_{k}^{T} D_{k}^{T}\right\} \\
& +\bar{D}_{k} R_{k, k-1} \bar{D}_{k-1}^{T} \Pi_{k-1}^{-1} \bar{D}_{k-1} R_{k, k-1}^{T} \bar{D}_{k}^{T},
\end{align*}
$$

where the remaining expectations can be obtained as follows:

$$
\begin{aligned}
& \mathscr{E}\left\{\tilde{x}_{k \mid k-1} V_{k}^{T}\right\} \\
& \quad=\mathscr{E}\left\{x_{k} V_{k}^{T}\right\}-\mathscr{E}\left\{\hat{x}_{k \mid k-1} V_{k}^{T}\right\} \\
& \quad=0-\mathscr{E}\left\{\left(\sum_{i=1}^{k-1} \mathscr{E}\left\{x_{k} \varepsilon_{i}^{T}\right\} \Pi_{i}^{-1} \varepsilon_{i}\right) V_{k}^{T}\right\}
\end{aligned}
$$

$$
\begin{align*}
& =-\mathscr{E}\left\{x_{k} \varepsilon_{k-1}^{T}\right\} \Pi_{i}^{-1} \mathscr{E}\left\{\varepsilon_{k-1} V_{k}^{T}\right\} \\
& =-\left(A_{k-1} \mathscr{E}\left\{x_{k-1} \varepsilon_{k-1}^{T}\right\}\right. \\
& \left.+B_{k-1} \mathscr{E}\left\{\omega_{k-1} \varepsilon_{k-1}^{T}\right\}\right) \\
& \times \Pi_{k-1}^{-1} \bar{D}_{k-1} R_{k-1, k} \\
& =-\left(A_{k-1} F_{k-1, k-1}\right. \\
& \left.+B_{k-1} Q_{k-1, k-2} B_{k-2}^{T} \bar{C}_{k-1}^{T}\right) \\
& \times \Pi_{k-1}^{-1} \bar{D}_{k-1} R_{k-1, k}, \\
& \mathscr{E}\left\{\widetilde{C}_{k} x_{k} x_{k}^{T} \widetilde{C}_{k}^{T}\right\} \\
& =\mathscr{E}\left\{\tilde{J}_{k} C_{e, k} x_{k} x_{k}^{T} C_{e, k}^{T} \tilde{J}_{k}^{T}\right\} \\
& =\Sigma_{k} \otimes\left(C_{e, k} X_{k} C_{e, k}^{T}\right), \\
& \mathscr{E}\left\{C_{s, k} x_{k} x_{k}^{T} C_{s, k}^{T}\right\}  \tag{A.13}\\
& =\mathscr{E}\left\{\left[\left(I-J_{k}\right) \breve{C}_{s, k} \eta_{k} J_{k} \breve{C}_{s, k-1} \eta_{k-1}\right]\right. \\
& \times\left[\begin{array}{cc}
\breve{x}_{k} \breve{x}_{k}^{T} & \breve{x}_{k} \check{x}_{k-1}^{T} \\
\breve{x}_{k-1} \breve{x}_{k}^{T} & \breve{x}_{k-1} \check{x}_{k-1}^{T}
\end{array}\right] \\
& \left.\times\left[\begin{array}{ll}
\left(I-J_{k}\right) \breve{C}_{s, k} \eta_{k} & J_{k} \check{C}_{s, k-1} \eta_{k-1}
\end{array}\right]^{T}\right\} \\
& =\mathscr{E}\left\{\left(I-J_{k}\right) \check{C}_{s, k} \mathscr{E}\left\{\eta_{k} \breve{x}_{k} \check{x}_{k}^{T} \eta_{k}^{T}\right\}\right.  \tag{A.14}\\
& \left.\times C_{s, k}^{T}\left(I-J_{k}\right)^{T}\right\} \\
& +\mathscr{E}\left\{\left(I-J_{k}\right) \breve{C}_{s, k} \mathscr{E}\left\{\eta_{k}\right\} \breve{X}_{k, k-1}\right. \\
& \left.\times \mathscr{E}\left\{\eta_{k-1}^{T}\right\} \breve{S}_{s, k-1}^{T} J_{k}^{T}\right\} \\
& +\mathscr{E}\left\{J_{k} \breve{C}_{s, k-1} \mathscr{E}\left\{\eta_{k-1}\right\} \breve{X}_{k-1, k} \mathscr{E}\left\{\eta_{k}\right\}^{T}\right. \\
& \left.\times \breve{C}_{s, k}^{T}\left(1-J_{k}\right)^{T}\right\} \\
& +\mathscr{E}\left\{J_{k} \breve{C}_{s, k-1} \mathscr{E}\left\{\eta_{k-1} \breve{x}_{k-1} \breve{x}_{k-1}^{T} \eta_{k-1}^{T}\right\}\right.  \tag{A.15}\\
& \left.\times \breve{C}_{s, k-1}^{T} I_{k}^{T}\right\} \\
& =\mathscr{E}\left\{\left(I-J_{k}\right) \breve{C}_{s, k} \breve{X}_{k} C_{s, k}^{T}\left(I-J_{k}\right)^{T}\right\} \\
& +\mathscr{E}\left\{J_{k} \breve{C}_{s, k-1} \breve{X}_{k-1} \breve{C}_{s, k-1}^{T} T_{k}^{T}\right\} \\
& =\mathscr{E}\left\{\left[\left(I-\bar{J}_{k}\right)-\widetilde{J}_{k}\right] \check{C}_{s, k} \breve{X}_{k} C_{s, k}^{T}\right. \\
& \left.\times\left[\left(I-\bar{J}_{k}\right)-\tilde{J}_{k}\right]^{T}\right\}  \tag{A.16}\\
& +\mathscr{E}\left\{\left(\bar{J}_{k}+\widetilde{J}_{k}\right) \breve{C}_{s, k-1} \breve{X}_{k-1}\right.
\end{align*}
$$

$$
\begin{gathered}
\left.\times \breve{C}_{s, k-1}^{T}\left(\widetilde{J}_{k}+\bar{J}_{k}\right)^{T}\right\} \\
=\left(I-\bar{J}_{k}\right) \check{C}_{s, k} \breve{X}_{k} C_{s, k}^{T}\left(I-\bar{J}_{k}\right)^{T} \\
+\Sigma_{k} \otimes\left(\breve{C}_{s, k} \breve{X}_{k} C_{s, k}^{T}\right) \\
+\bar{J}_{k} \breve{C}_{s, k-1} \breve{X}_{k-1} \breve{C}_{s, k-1}^{T} \bar{J}_{k}^{T} \\
+\Sigma_{k} \otimes\left(\breve{C}_{s, k-1} \breve{X}_{k-1} C_{s, k-1}^{T}\right), \\
\mathscr{E}\left\{D_{k} V_{k} V_{k}^{T} D_{k}^{T}\right\} \\
=\mathscr{E}\left\{\left(\bar{D}_{k}+\widetilde{D}_{k}\right) V_{k} V_{k}^{T}\left(\bar{D}_{k}+\widetilde{D}_{k}\right)^{T}\right\} \\
=\mathscr{E}\left\{\bar{D}_{k} V_{k} V_{k}^{T} \bar{D}_{k}^{T}\right\} \\
+ \\
=\mathscr{E}\left\{\widetilde{J}_{k} D_{e, k} V_{k} V_{k}^{T} D_{e, k}^{T} \widetilde{J}_{k}^{T}\right\} \\
R_{k} \bar{D}_{k}^{T}+\Sigma_{k} \otimes\left(D_{e, k} R_{k} D_{e, k}^{T}\right),
\end{gathered}
$$

where Lemmas 5-8 have been used. Substituting (A.13) into (A.12) yields to (18).

Again, by using the OPT, the state estimation $\widehat{x}_{k \mid k}$ can be calculated as follows:

$$
\begin{aligned}
\hat{x}_{k \mid k} & =\sum_{i=1}^{k} \mathscr{E}\left\{x_{k} \varepsilon_{i}^{T}\right\} \Pi_{i}^{-1} \varepsilon_{i} \\
& =\sum_{i=1}^{k-1} \mathscr{E}\left\{x_{k} \varepsilon_{i}^{T}\right\} \Pi_{i}^{-1} \varepsilon_{i}+\mathscr{E}\left\{x_{k} \varepsilon_{k}^{T}\right\} \Pi_{k}^{-1} \varepsilon_{k} \\
& =\widehat{x}_{k \mid k-1}+F_{k, k} \Pi_{k}^{-1} \varepsilon_{k} .
\end{aligned}
$$

Therefore, the estimation error $\tilde{x}_{k \mid k}$ can be obtained as follows:

$$
\begin{aligned}
\tilde{x}_{k \mid k} & =x_{k}-\widehat{x}_{k \mid k} \\
& =x_{k}-\widehat{x}_{k \mid k-1}-F_{k, k} \Pi_{k}^{-1} \varepsilon_{k} \\
& =\tilde{x}_{k \mid k-1}-F_{k, k} \Pi_{k}^{-1} \varepsilon_{k} .
\end{aligned}
$$

From (A.15), the estimation error covariance $P_{k \mid k}$ can be calculated as follows:

$$
\begin{aligned}
P_{k \mid k}= & \mathscr{E}\left\{\tilde{x}_{k \mid k} \tilde{x}_{k \mid k}^{T}\right\} \\
= & P_{k \mid k-1}-\mathscr{E}\left\{\tilde{x}_{k \mid k-1} \varepsilon_{k}^{T}\right\} \Pi_{k}^{-1} F_{k, k}^{T} \\
& -F_{k, k} \Pi_{k}^{-1} \mathscr{E}\left\{\varepsilon_{k} \tilde{x}_{k \mid k-1}^{T}\right\}+F_{k, k} \Pi_{k}^{-1} F_{k, k}^{T},
\end{aligned}
$$

where the remaining expectation $\mathscr{E}\left\{\widetilde{x}_{k \mid k-1} \varepsilon_{k}^{T}\right\}$ can be calculated as follows:

$$
\begin{align*}
\mathscr{E}\left\{\tilde{x}_{k \mid k-1} \varepsilon_{k}^{T}\right\} & =\mathscr{E}\left\{x_{k} \varepsilon_{k}^{T}\right\}-\mathscr{E}\left\{\widehat{x}_{k \mid k-1} \varepsilon_{k}^{T}\right\} \\
& =F_{k, k}-\mathscr{E}\left\{\sum_{i=1}^{k-1} \mathscr{E}\left\{x_{k} \varepsilon_{i}^{T}\right\} \Pi_{i}^{-1} \varepsilon_{i} \varepsilon_{k}^{T}\right\}  \tag{A.17}\\
& =F_{k, k} .
\end{align*}
$$

Substituting (A.17) into (A.16), we have

$$
\begin{equation*}
P_{k \mid k}=P_{k \mid k-1}-F_{k, k} \Pi_{k}^{-1} F_{k, k}^{T}, \tag{A.18}
\end{equation*}
$$

which completes the proof of Theorem 9.

## B. The Proof of Theorem 11

Proof. Taking into account the fact that the process noise $\omega_{k}$ is one-step autocorrelated across time, the $L$-step prediction $\widehat{x}_{k+L \mid k}$ can be calculated as follows:

$$
\begin{align*}
\widehat{x}_{k+L \mid k}= & \sum_{i=1}^{k} \mathscr{E}\left\{x_{k+L} \varepsilon_{i}^{T}\right\} \Pi_{i}^{-1} \varepsilon_{i} \\
= & \sum_{i=1}^{k} \mathscr{E}\left\{\left[\left(A_{k+L-1}+A_{s, k+L-1} \mu_{k+L-1}\right) x_{k+L-1}\right.\right. \\
& \left.\left.\quad+B_{k+L-1} \omega_{k+L-1}\right] \varepsilon_{i}^{T}\right\} \Pi_{i}^{-1} \varepsilon_{i} \\
= & A_{k+L-1} \widehat{x}_{k+L-1 \mid k} . \tag{B.1}
\end{align*}
$$

Therefore, the $L$-step prediction error $\tilde{x}_{k+L \mid k}$ can be obtained as follows:

$$
\begin{align*}
\widetilde{x}_{k+L \mid k}= & x_{k+L}-\widehat{x}_{k+L \mid k} \\
= & \left(A_{k+L-1}+A_{s, k+L-1} \mu_{k+L-1}\right) x_{k+L-1} \\
& +B_{k+L-1} \omega_{k+L-1}-A_{k+L-1} \widehat{x}_{k+L-1 \mid k}  \tag{B.2}\\
= & A_{k+L-1} \widetilde{x}_{k+L-1 \mid k}+A_{s, k+L-1} \mu_{k+L-1} x_{k+L-1} \\
& +B_{k+L-1} \omega_{k+L-1} .
\end{align*}
$$

Thus, the $L$-step prediction error covariance $P_{k+L-1 \mid k}$ can be calculated as follows:

$$
\begin{aligned}
P_{k+L \mid k}= & \mathscr{E}\left\{\tilde{x}_{k+L \mid k} \tilde{x}_{k+L \mid k}^{T}\right\} \\
= & A_{k+L-1} P_{k+L-1 \mid k} A_{k+L-1}^{T}+A_{k+L-1} \\
& \times \mathscr{E}\left\{\widetilde{x}_{k+L-1 \mid k} \omega_{k+L-1}^{T}\right\} B_{k+L-1}^{T}+A_{s, k+L-1} \\
& \times X_{k+L-1} A_{s, k+L-1}^{T}+B_{k+L-1} \mathscr{E}\left\{\omega_{k+L-1} \widetilde{x}_{k+L-1 \mid k}^{T}\right\} \\
& \times A_{k+L-1}^{T}+B_{k+L-1} Q_{k+L-1} B_{k+L-1}^{T} \\
= & A_{k+L-1} P_{k+L-1 \mid k} A_{k+L-1}^{T}
\end{aligned}
$$

$$
\begin{align*}
& +A_{k+L-1} B_{k+L-2} Q_{k+L-2, k+L-1} B_{k+L-1}^{T}+A_{s, k+L-1} \\
& \times X_{k+L-1} A_{s, k+L-1}^{T}+B_{k+L-1} Q_{k+L-1, k+L-2} \\
& \times B_{k+L-2}^{T} A_{k+L-1}^{T}+B_{k+L-1} Q_{k+L-1} B_{k+L-1}^{T}, \tag{B.3}
\end{align*}
$$

which completes the proof of Theorem 11.

## C. The Proof of Theorem 12

Proof. According to the OPT, the $L$-step fixed-lag smoother $\widehat{x}_{k \mid k+L}$ can be calculated as follows:

$$
\begin{align*}
\widehat{x}_{k \mid k+L} & =\sum_{i=1}^{k+L} \mathscr{E}\left\{x_{k} \varepsilon_{i}^{T}\right\} \Pi_{i}^{-1} \varepsilon_{i} \\
& =\sum_{i=1}^{k+L-1} \mathscr{E}\left\{x_{k} \varepsilon_{i}^{T}\right\} \Pi_{i}^{-1} \varepsilon_{i}+\mathscr{E}\left\{x_{k} \varepsilon_{k+L}^{T}\right\} \Pi_{k+L}^{-1} \varepsilon_{k+L}  \tag{C.1}\\
& =\widehat{x}_{k \mid k+L-1}+F_{k, k+L} \Pi_{k+L}^{-1} \varepsilon_{k+L}
\end{align*}
$$

where $F_{k, k+L}$ can be calculated as follows:

$$
\begin{align*}
F_{k, k+L}= & \mathscr{E}\left\{x_{k} \varepsilon_{k+L}^{T}\right\} \\
= & \mathscr{E}\left\{x _ { k } \left(\bar{C}_{k+L} \widetilde{x}_{k+L \mid k+L-1}+\widetilde{C}_{k+L} x_{k+L}+C_{s, k+L} x_{k+L}\right.\right. \\
& +D_{k+L} V_{k+L}-\bar{D}_{k+L} R_{k+L, k+L-1} \bar{D}_{k+L-1}^{T} \\
& \left.\left.\times \Pi_{k+L-1}^{-1} \varepsilon_{k+L-1}\right)^{T}\right\} \\
= & \mathscr{E}\left\{x _ { k } \left(\bar{C}_{k+L} \widetilde{x}_{k+L \mid k+L-1}-\bar{D}_{k+L} R_{k+L, k+L-1}\right.\right. \\
& \left.\left.\times \bar{D}_{k+L-1}^{T} \Pi_{k+L-1}^{-1} \varepsilon_{k+L-1}\right)^{T}\right\} \\
= & \Psi_{k+L} \bar{C}_{k+L}^{T}-F_{k, k+L-1} \Pi_{k+L-1}^{-1} \\
& \times \bar{D}_{k+L-1} R_{k+L, k+L-1}^{T} \bar{D}_{k+L}^{T} \tag{C.2}
\end{align*}
$$

where the third equality holds since $\widetilde{C}_{k+L}, C_{s, k+L}$, and $V_{k+L}$ are zero-mean stochastic matrices and they are all uncorrelated with $x_{k}$. From (A.8) the expectation $\Psi_{k+L}=\mathscr{E}\left\{x_{k} \tilde{x}_{k+L \mid k+L-1}^{T}\right\}$ can be obtained as follows:

$$
\begin{aligned}
\Psi_{k+1}= & \mathscr{E}\left\{x_{k} \tilde{x}_{k+1 \mid k}^{T}\right\} \\
= & \mathscr{E}\left\{x _ { k } \left(A_{k} \tilde{x}_{k \mid k}+A_{s, k} \mu_{k} x_{k}+B_{k} \omega_{k}\right.\right. \\
& \left.\left.-B_{k} Q_{k, k-1} B_{k-1}^{T} \bar{C}_{k}^{T} \Pi_{k}^{-1} \varepsilon_{k}\right)^{T}\right\} \\
= & \mathscr{E}\left\{x_{k} \tilde{x}_{k \mid k}^{T}\right\} A_{k}^{T}+\mathscr{E}\left\{x_{k} \omega_{k}^{T}\right\} B_{k}^{T} \\
& -\mathscr{E}\left\{x_{k} \varepsilon_{k}^{T}\right\} \Pi_{k}^{-1} \bar{C}_{k} B_{k-1} Q_{k, k-1}^{T} B_{k}^{T}
\end{aligned}
$$

$$
\begin{align*}
= & \mathscr{E}\left\{x_{k}\left(\tilde{x}_{k \mid k-1}-\digamma_{k, k} \Pi_{k}^{-1} \varepsilon_{k}\right)^{T}\right\} A_{k}^{T} \\
& +B_{k-1} Q_{k-1, k} B_{k}^{T}-\digamma_{k, k} \Pi_{k}^{-1} \bar{C}_{k} B_{k-1} Q_{k, k-1}^{T} B_{k}^{T} \\
= & P_{k \mid k-1} A_{k}^{T}-\digamma_{k, k} \Pi_{k}^{-1} F_{k, k}^{T} A_{k}^{T}+B_{k-1} Q_{k-1, k} B_{k}^{T} \\
& -\digamma_{k, k} \Pi_{k}^{-1} \bar{C}_{k} B_{k-1} Q_{k, k-1}^{T} B_{k}^{T} . \tag{C.3}
\end{align*}
$$

Similarly, when $L \geq 2$, the expectation $\Psi_{k+L}$ can be calculated as follows:

$$
\begin{align*}
\Psi_{k+L}= & \Psi_{k+L-1} A_{k+L-1}^{T}-\digamma_{k, k+L-1} \Pi_{k+L-1}^{-1} F_{k+L-1, k+L-1} \\
& -\digamma_{k, k+L-1} \Pi_{k+L-1}^{-1} \bar{C}_{k+L-1} B_{k+L-2}  \tag{C.4}\\
& \times Q_{k+L-1, k+L-2}^{T} B_{k+L-1}^{T} .
\end{align*}
$$

From (6) and (C.1), the smoother error can be obtained as follows:

$$
\begin{align*}
\tilde{x}_{k \mid k+L} & =x_{k}-\widehat{x}_{k \mid k+L-1}-F_{k, k+L} \Pi_{k+L}^{-1} \varepsilon_{k+L}  \tag{C.5}\\
& =\tilde{x}_{k \mid k+L-1}-F_{k, k+L} \Pi_{k+L}^{-1} \varepsilon_{k+L} .
\end{align*}
$$

Therefore, the smoother error covariance can be obtained as follows:

$$
\begin{align*}
P_{k \mid k+L}= & \mathscr{E}\left\{\tilde{x}_{k \mid k+L} \tilde{x}_{k \mid k+L}^{T}\right\} \\
= & \mathscr{E}\left\{\left(\tilde{x}_{k \mid k+L-1}-F_{k, k+L} \Pi_{k+L}^{-1} \varepsilon_{k+L}\right)\right. \\
& \left.\times\left(\tilde{x}_{k \mid k+L-1}-F_{k, k+L} \Pi_{k+L}^{-1} \varepsilon_{k+L}\right)^{T}\right\} \\
= & P_{k \mid k+L-1}-\mathscr{E}\left\{\tilde{x}_{k \mid k+L-1} \varepsilon_{k+L}^{T}\right\} \Pi_{k+L}^{-1} F_{k, k+L}^{T}  \tag{C.6}\\
& -F_{k, k+L} \Pi_{k+L}^{-1} \mathscr{E}\left\{\varepsilon_{k+L} \widetilde{x}_{k \mid k+L-1}^{T}\right\} \\
& +F_{k, k+L} \Pi_{k+L}^{-1} F_{k, k+L}^{T} \\
= & P_{k \mid k+L-1}-F_{k, k+L} \Pi_{k+L}^{-1} F_{k, k+L}^{T},
\end{align*}
$$

which completes the proof of Theorem 12.

## Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Frequency Weighted Model Order Reduction Technique and Error Bounds for Discrete Time Systems 

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#### Abstract

Model reduction is a process of approximating higher order original models by comparatively lower order models with reasonable accuracy in order to provide ease in design, modeling and simulation for large complex systems. Generally, model reduction techniques approximate the higher order systems for whole frequency range. However, certain applications (like controller reduction) require frequency weighted approximation, which introduce the concept of using frequency weights in model reduction techniques. Limitations of some existing frequency weighted model reduction techniques include lack of stability of reduced order models (for two sided weighting case) and frequency response error bounds. A new frequency weighted technique for balanced model reduction for discrete time systems is proposed. The proposed technique guarantees stable reduced order models even for the case when two sided weightings are present. Efficient technique for frequency weighted Gramians is also proposed. Results are compared with other existing frequency weighted model reduction techniques for discrete time systems. Moreover, the proposed technique yields frequency response error bounds.


## 1. Introduction

Model reduction has played a significant role in modern control system design and caught a lot of attention in the last few decades [1-5]. It is desirable that reduced order model preserves the fundamental properties of original system like stability, passivity, and so forth. Moreover, the approximation error between original and reduced order system is required to be small. Balanced truncation [6] is prominent model reduction scheme, which not only ensures stability of reduced order systems but also provides frequency response a priori error bounds. Other schemes like Hankel norm approximation [7], Pade approximation [8], Krylov technique $[9,10]$, linear matrix inequality (LMI) technique [11, 12], and so forth, are also useful for model reduction. Hankel norm approximation has complex implementation and does not preserve steady state [13]. Pade and Krylov approximation sometimes provide unstable reduced order models and there exist no global error bounds [14, 15]. LMI technique is based on mathematical iterative methods (i.e., convex optimization and bisection algorithm); therefore,
much computational power is consumed [16]. However, LMI technique has been applied on various model reduction problems for different types of systems (including time delay [17], discrete state delay [18, 19], switched hybrid [20], and nonlinear stochastic [21]). Applications of model reduction are not only restricted to control engineering but also find utility in medical [22] and text summarization [23] areas, and so forth. Most model reduction algorithms tend to minimize the reduction error over the whole frequency range; however there are certain situations (like controller reduction [24]), wherein the approximation error is more critical over a certain band of frequencies.

Enns [25] has enhanced the balanced truncation [6] to incorporate frequency weights for model reduction of higher order systems. This technique can work with input, output, and two sided weightings. However, in two sided weighting case, this technique may give unstable reduced order models [26]. To circumvent instability issue in the presence of two sided weightings, many techniques appear in the literature (including Lin and Chiu [27], Wang et al. [28], Varga and Anderson [29], Ghafoor and Sreeram [30], etc.).

In Lin and Chiu [27] technique weightings are strictly proper. It was improved to include more general proper weightings in [26]. Varga and Anderson [29] technique yields proper model for strictly proper systems also. Ghafoor and Sreeram [30] technique is a parameterized technique. Wang et al.'s technique is relatively useful, since other techniques [27, 29] are not applicable for controller order reduction. Moreover, Wang et al.s technique yields easily computable expression for a priori error bounds.

Various partial fraction based techniques appear in the literature [31-35] that works for continuous as well as discrete time systems. Unfortunately, most of these techniques yield large frequency response error [30] as compared to Enns technique. However, [35] incorporates free parameters to reduce the approximation error.

Most frequently, frequency weighted model reduction problem is treated in continuous time; however, there are few papers (like [36-38]) which deal with this problem explicitly in discrete time. Sahlan et al.'s [36] technique (discrete time version of [31]) is a modified version of Lin and Chiu [27] technique. Campbell et al.'s [37] technique is discrete time counterpart of Wang et al.s [28] technique, which not only provides stability for two sided case but also gives easily computable error bounds. Campbell et al.'s [37] technique involves taking absolute values of eigenvalues. This may introduce a large change to the system and hence a larger error if it contains some negative eigenvalues. Varga and Anderson's [38] tend to minimize the distance between Enns and Campbell et al.s techniques Gramians by eliminating negative eigenvalues.

In this work, a technique is developed for frequency weighted balanced model order reduction for discrete time systems. The large change in eigenvalues is avoided by yielding similar effect on all eigenvalues. The stability is guaranteed even for double sided weighting. The proposed technique provides comparable frequency response error and yields easily computable a priori error bounds. Numerical examples are given to show the usefulness and comparison of proposed technique with the existing frequency weighted balanced reduction techniques.

## 2. Preliminaries

In this section we review some of the existing frequency weighted model reduction techniques for discrete time systems which include Enns [25], Campbell et al.s [37], and Varga and Anderson's [38].

Consider a stable full order original system with transfer function $H(z)=C(z I-A)^{-1} B+D$, a stable input weighting system with input transfer function $V_{i}(z)=C_{i}\left(z I-A_{i}\right)^{-1} B_{i}+$ $D_{i}$, and a stable output weighting system with output transfer function $W_{o}(z)=C_{o}\left(z I-A_{o}\right)^{-1} B_{o}+D_{o}$; the augmented systems are given by

$$
\begin{gather*}
H(z) V_{i}(z)=C_{d i}\left(z I-A_{d i}\right)^{-1} B_{d i}+D_{d i}  \tag{1}\\
W_{o}(z) H(z)=C_{d o}\left(z I-A_{d o}\right)^{-1} B_{d o}+D_{d o}
\end{gather*}
$$

where

$$
\begin{align*}
& {\left[\begin{array}{c|c}
A_{d i} & B_{d i} \\
\hline C_{d i} & D_{d i}
\end{array}\right]=\left[\begin{array}{cc|c}
A & B C_{i} & B D_{i} \\
0 & A_{i} & B_{i} \\
\hline C & D C_{i} & D D_{i}
\end{array}\right]}  \tag{2}\\
& {\left[\begin{array}{c|c}
A_{d o} & B_{d o} \\
\hline C_{d o} & D_{d o}
\end{array}\right]=\left[\begin{array}{cc|c}
A_{o} & B_{o} C & B_{o} D \\
0 & A & B \\
\hline C_{o} & D_{o} C & D_{o} D
\end{array}\right] .}
\end{align*}
$$

Let the Gramians

$$
P_{d i}=\left[\begin{array}{cc}
P_{e n} & P_{12}  \tag{3}\\
P_{12}^{T} & P_{V}
\end{array}\right], \quad Q_{d o}=\left[\begin{array}{ll}
Q_{W} & Q_{12}^{T} \\
Q_{12} & Q_{e n}
\end{array}\right]
$$

satisfy the following Lyapunov equations:

$$
\begin{gather*}
A_{d i} P_{d i} A_{d i}^{T}-P_{d i}+B_{d i} B_{d i}^{T}=0,  \tag{4}\\
A_{d o}^{T} Q_{d o} A_{d o}-Q_{d o}+C_{d o}^{T} C_{d o}=0 . \tag{5}
\end{gather*}
$$

Expanding the $(1,1)$ and $(2,2)$ blocks of above equations, we get

$$
\begin{align*}
& A P_{e n} A^{T}-P_{e n}+X_{e n}=0 \\
& A^{T} Q_{e n} A-Q_{e n}+Y_{e n}=0 \tag{6}
\end{align*}
$$

where

$$
\begin{align*}
X_{e n}=A P_{12} C_{i}^{T} B^{T} & +B C_{i} P_{12}^{T} A^{T}+B C_{i} P_{V} C_{i}^{T} B^{T}+B D_{i} D_{i}^{T} B^{T} \\
Y_{e n}= & C^{T} B_{o}^{T} Q_{12}^{T} A+A^{T} Q_{12} B_{o} C \\
& +C^{T} B_{o}^{T} Q_{W} B_{o} C+C^{T} D_{o}^{T} D_{o} C \tag{7}
\end{align*}
$$

2.1. Enns Technique [25]. Let $T$ be contragredient matrix obtained as

$$
\begin{equation*}
T^{T} Q_{e n} T=T^{-1} P_{e n} T^{-T}=\operatorname{diag}\left\{\sigma_{1}, \sigma_{2}, \sigma_{3}, \ldots, \sigma_{n}\right\} \tag{8}
\end{equation*}
$$

where $\sigma_{j} \geq \sigma_{j+1}, j=1,2,3, \ldots, n-1$ and $\sigma_{l}>\sigma_{l+1}$. By transforming and then partitioning the original system, we have

$$
\begin{gather*}
\widehat{A}=T^{-1} A T=\left[\begin{array}{ll}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{array}\right], \quad \widehat{B}=T^{-1} B=\left[\begin{array}{l}
B_{1} \\
B_{2}
\end{array}\right],  \tag{9}\\
\widehat{C}=C T=\left[\begin{array}{ll}
C_{1} & C_{2}
\end{array}\right], \quad \widehat{D}=D,
\end{gather*}
$$

where $A_{11} \in R^{l \times l}$. The reduced order system is obtained as follows: $H_{l}(z)=C_{1}\left(z I-A_{11}\right)^{-1} B_{1}+D$.

Remark 1. It is not guaranteed to ensure that $X_{e n} \geq 0$ and $Y_{e n} \geq 0$; the reduced order models obtained using Enns technique may not remain stable for both sided weightings.
2.2. Campbell et al.'s Technique [37]. Campbell et al.'s [37] (a discrete time version of [28]) guarantees the positive semidefiniteness of symmetric matrices $X_{e n}$ and $Y_{e n}$ to ensure stability. Let the new controllability $P_{C S}$ and observability $Q_{C S}$ Gramians, respectively, be calculated by solving the following Lyapunov equations:

$$
\begin{gather*}
A P_{C S} A^{T}-P_{C S}+B_{C S} B_{C S}^{T}=0 \\
A^{T} Q_{C S} A+Q_{C S}+C_{C S}^{T} C_{C S}=0 \tag{10}
\end{gather*}
$$

which are used to obtain contragredient matrix $T$ as

$$
\begin{equation*}
T^{T} Q_{C S} T=T^{-1} P_{C S} T^{-T}=\Sigma \tag{11}
\end{equation*}
$$

where $\Sigma=\operatorname{diag}\left\{\sigma_{1}, \sigma_{2}, \sigma_{3}, \ldots, \sigma_{n}\right\}$ and $\sigma_{j} \geq \sigma_{j+1}, j=$ $1,2,3, \ldots, n-1$ and $\sigma_{l}>\sigma_{l+1}$. The fictitious input $B_{C S}$ and output $C_{C S}$ matrices shown in the above Lyapunov equations are defined as $B_{C S}=U_{C S}\left|S_{C S}\right|^{1 / 2}$ and $C_{C S}=$ $\left|R_{C S}\right|^{1 / 2} V_{C S}^{T}$, respectively. Since the expressions $U_{C S}, S_{C S}, V_{C S}$, and $R_{C S}$ are calculated by orthogonal eigen decomposition $X_{e n}=U_{C S} S_{C S} U_{C S}^{T}$ and $Y_{e n}=V_{C S} R_{C S} V_{C S}^{T}$, where $S_{C S}=$ $\operatorname{diag}\left(s_{1}, s_{2}, \ldots, s_{n}\right), R_{C S}=\operatorname{diag}\left(r_{1}, r_{2}, \ldots, r_{n}\right),\left|s_{1}\right| \geq\left|s_{2}\right| \geq$ $\cdots \geq\left|s_{n}\right| \geq 0$ and $\left|r_{1}\right| \geq\left|r_{2}\right| \geq \cdots \geq\left|r_{n}\right| \geq 0$. The reduced order systems are calculated by transforming and then partitioning the original system.

Remark 2. The stability of reduced order models in the presence of both input and output weightings is guaranteed and the following error bound holds [37]

$$
\begin{align*}
& \left\|W_{o}(z)\left(H(z)-H_{l}(z)\right) V_{i}(z)\right\|_{\infty} \\
& \quad \leq 2\left\|W_{o}(z) L_{C S}\right\|_{\infty}\left\|K_{C S} V_{i}(z)\right\|_{\infty} \sum_{j=l+1}^{n} \sigma_{j} \tag{12}
\end{align*}
$$

where

$$
\begin{align*}
& L_{C S}=C V_{C S} \operatorname{diag}\left(\left|r_{1}\right|^{-1 / 2},\left|r_{2}\right|^{-1 / 2}, \ldots,\left|r_{l i}\right|^{-1 / 2}, 0, \ldots, 0\right), \\
& K_{C S}=\operatorname{diag}\left(\left|s_{1}\right|^{-1 / 2},\left|s_{2}\right|^{-1 / 2}, \ldots,\left|s_{k o}\right|^{-1 / 2}, 0, \ldots, 0\right) U_{C S}^{T} B . \tag{13}
\end{align*}
$$

$l i=\operatorname{rank}\left[X_{e n}\right]$ and $k o=\operatorname{rank}\left[Y_{e n}\right]$.
2.3. Varga and Anderson's Technique [38]. Note that the Gramians satisfy $P_{e n} \leq P_{C S}$ and $Q_{e n} \leq Q_{C S}$. For minimizing the distances between the Gramians, $P_{e n}-P_{C S}$ and $Q_{e n}-Q_{C S}$, Varga and Anderson proposed the following technique.

Let new controllability and observability Gramians $P_{V d}$ and $Q_{V d}$, respectively, be calculated as the solutions to Lyapunov equations

$$
\begin{align*}
& A P_{V d} A^{T}-P_{V d}+B_{V d} B_{V d}^{T}=0 \\
& A^{T} Q_{V d} A-Q_{V d}+C_{V d}^{T} C_{V d}=0 \tag{14}
\end{align*}
$$

which are used obtain contragredient matrix $T$ as

$$
\begin{equation*}
T^{T} Q_{V d} T=T^{-1} P_{V d} T^{-T}=\Sigma \tag{15}
\end{equation*}
$$

where $\Sigma=\operatorname{diag}\left\{\sigma_{1}, \sigma_{2}, \sigma_{3}, \ldots, \sigma_{n}\right\}$, and $\sigma_{j} \geq \sigma_{j+1}, j=$ $1,2, \ldots, n-1, \sigma_{l}>\sigma_{l+1}$. The new fictitious input $B_{V d}$ and output $C_{V d}$ matrices in the above Lyapunov equations are defined as $B_{V d}=U_{V d_{1}} S_{V d_{1}}^{1 / 2}$ and $C_{V d}=R_{V d_{1}}^{1 / 2} V_{V d_{1}}^{T}$, respectively. The terms $U_{V d_{1}}, S_{V d_{1}}, V_{V d_{1}}$, and $R_{V d_{1}}$ are calculated from the orthogonal eigen decomposition of symmetric matrices

$$
\begin{align*}
& X_{e n}=\left[\begin{array}{ll}
U_{V d_{1}} & U_{V d_{2}}
\end{array}\right]\left[\begin{array}{cc}
S_{V d_{1}} & 0 \\
0 & S_{V d_{2}}
\end{array}\right]\left[\begin{array}{c}
U_{V d_{1}}^{T} \\
U_{V d_{2}}^{T}
\end{array}\right], \\
& Y_{e n}=\left[\begin{array}{ll}
V_{V d_{1}} & V_{V d_{2}}
\end{array}\right]\left[\begin{array}{cc}
R_{V d_{1}} & 0 \\
0 & R_{V d_{2}}
\end{array}\right]\left[\begin{array}{c}
V_{V d_{1}}^{T} \\
V_{V d_{2}}^{T}
\end{array}\right], \tag{16}
\end{align*}
$$

where $\left[\begin{array}{cc}S_{V d_{1}} & 0 \\ 0 & s_{V d_{2}}\end{array}\right]=\operatorname{diag}\left\{s_{1}, s_{2}, \ldots, s_{n}\right\},\left[\begin{array}{cc}R_{V d_{1}} & 0 \\ 0 & R_{V d_{2}}\end{array}\right]=$ $\operatorname{diag}\left\{r_{1}, r_{2}, r_{3}, \ldots, r_{n}\right\}, s_{1} \geq s_{2} \geq s_{3} \geq \cdots \geq s_{n}, r_{1} \geq$ $r_{2} \geq r_{3} \geq \cdots \geq r_{n}, S_{V d_{1}}=\operatorname{diag}\left\{s_{1}, s_{2}, s_{3}, \ldots, s_{k}\right\}, R_{V d_{1}}=$ $\operatorname{diag}\left\{r_{1}, r_{2}, r_{3}, \ldots, r_{k}\right\}, s_{1} \geq s_{2} \geq s_{3} \geq \cdots \geq s_{k} \geq 0, r_{1} \geq$ $r_{2} \geq r_{3} \geq \cdots \geq r_{k} \geq 0$.

The reduced order systems are calculated by transforming and then partitioning the original system.

Remark 3. The stability of the reduced system is guaranteed and the following error bound holds [38]

$$
\begin{align*}
& \left\|W_{o}(z)\left(H(z)-H_{l}(z)\right) V_{i}(z)\right\|_{\infty} \\
& \quad \leq 2\left\|W_{o}(z) L_{V d}\right\|_{\infty}\left\|K_{V d} V_{i}(z)\right\|_{\infty} \sum_{j=l+1}^{n} \sigma_{j} \tag{17}
\end{align*}
$$

where $L_{V d}=C V_{V d_{1}} R_{V d_{1}}^{-1 / 2}$ and $K_{V d}=S_{V d_{1}}^{-1 / 2} U_{V d_{1}}^{T} B$.

## 3. Main Results

In Campbell et al.s [37] technique, the symmetric matrices $X_{e n}$ and $Y_{e n}$ are guaranteed positive/semipositive definite by taking the square root of absolute values of the eigenvalues obtained by eigen decomposition of symmetric matrices $X_{e n}$ and $Y_{e n}$. This may lead to a large change in some eigenvalues and may not affect other eigenvalues. Although in Varga and Anderson's [38] technique, this large change was slightly improved by eliminating negative eigenvalues, but the problem persists with the other eigenvalues. In the following, a new technique is proposed in which a similar effect on all eigenvalues of indefinite matrices $X_{e n}$ and $Y_{e n}$ guarantees stability, error bound, and improved frequency response error.
3.1. Proposed Technique. Let a new controllability $P_{I G}$ and observability $Q_{I G}$ Gramians, respectively, be calculated by solving the following Lyapunov equations:

$$
\begin{align*}
& A P_{I G} A^{T}-P_{I G}+B_{I G} B_{I G}^{T}=0  \tag{18}\\
& A^{T} Q_{I G} A-Q_{I G}+C_{I G}^{T} C_{I G}=0 \tag{19}
\end{align*}
$$

The matrices $B_{I G}$ and $C_{I G}$ are the new fictitious input and output matrices, respectively, and are defined as

$$
\begin{align*}
& B_{I G}= \begin{cases}U_{I G}\left(S_{I G}-s_{n} I\right)^{1 / 2} & \text { for } s_{n}<0 \\
U_{I G} S_{I G}^{1 / 2} & \text { for } s_{n} \geq 0,\end{cases}  \tag{20}\\
& C_{I G}= \begin{cases}\left(R_{I G}-r_{n} I\right)^{1 / 2} V_{I G}^{T} & \text { for } r_{n}<0 \\
R_{I G}^{1 / 2} V_{I G}^{T} & \text { for } r_{n} \geq 0 .\end{cases}
\end{align*}
$$

The terms $U_{I G}, S_{I G}, V_{I G}$, and $R_{I G}$ are calculated by the orthogonal eigen decomposition of symmetric matrices $X_{e n}=U_{I G} S_{I G} U_{I G}^{T}$ and $Y_{e n}=V_{I G} R_{I G} V_{I G}^{T}$, where $S_{I G}=$ $\operatorname{diag}\left(s_{1}, s_{2}, s_{3}, \ldots, s_{n}\right), R_{I G}=\operatorname{diag}\left(r_{1}, r_{2}, r_{3}, \ldots, r_{n}\right), s_{1} \geq s_{2} \geq$ $\cdots \geq s_{n}$, and $r_{1} \geq r_{2} \geq \cdots \geq r_{n}$.

Let a contragradient transformation matrix $T$ be obtained as

$$
\begin{equation*}
T^{T} Q_{I G} T=T^{-1} P_{I G} T^{-T}=\Sigma \tag{21}
\end{equation*}
$$

where $\Sigma=\operatorname{diag}\left\{\sigma_{1}, \sigma_{2}, \sigma_{3}, \ldots, \sigma_{n}\right\}$, and $\sigma_{j} \geq \sigma_{j+1}, j=$ $1,2, \ldots, n-1, \sigma_{l}>\sigma_{l+1}$. The reduced order system is calculated by transforming and partitioning the original system.

Remark 4. Since $X_{e n} \leq B_{I G} B_{I G}^{T} \geq 0, Y_{e n} \leq C_{I G}^{T} C_{I G} \geq 0$, $P_{I G}>0$ and $Q_{I G}>0$, therefore, the realization $\left(A, B_{I G S}, C_{I G S}\right)$ is minimal. Moreover, the reduced order models are stable.

Theorem 5. The following error bound for the proposed technique holds if the rank conditions $\operatorname{rank}\left[\begin{array}{ll}B_{I G} & B\end{array}\right]=\operatorname{rank}\left[B_{I G}\right]$ and $\operatorname{rank}\left[\begin{array}{c}C_{I G} \\ C\end{array}\right]=\operatorname{rank}\left[C_{I G}\right]$ (which follows from $[2,28,29$, 38]) are satisfied
(i) $\left\|W_{o}(z)\left(H(z)-H_{l}(z)\right) V_{i}(z)\right\|_{\infty}$

$$
\leq 2\left\|W_{o}(z) L_{I G}\right\|_{\infty}\left\|K_{I G} V_{i}(z)\right\|_{\infty} \sum_{j=l+1}^{n} \sigma_{j}
$$

(ii) $\left\|\left(H(z)-H_{l}(z)\right) V_{i}(z)\right\|_{\infty} \leq 2\left\|K_{I G} V_{i}(z)\right\|_{\infty} \sum_{j=l+1}^{n} \sigma_{j}$
(iii) $\left\|W_{o}(z)\left(H(z)-H_{l}(z)\right)\right\|_{\infty}$ $\leq 2\left\|W_{o}(z) L_{I G}\right\|_{\infty} \sum_{j=l+1}^{n} \sigma_{j}$,
where

$$
\begin{align*}
& L_{I G}= \begin{cases}C V\left(R_{I G}-r_{n} I\right)^{-1 / 2} & \text { for } r_{n}<0 \\
C V R_{I G}^{-1 / 2} & \text { for } r_{n} \geq 0,\end{cases}  \tag{23}\\
& K_{I G}= \begin{cases}\left(S_{I G}-s_{n} I\right)^{-1 / 2} U^{T} B & \text { for } s_{n}<0 \\
S_{I G}^{-1 / 2} U^{T} B & \text { for } s_{n} \geq 0 .\end{cases}
\end{align*}
$$

Proof. We show proof of (i) (whereas (ii) and (iii) are special cases of (i)). Since the rank conditions $\operatorname{rank}\left[\begin{array}{ll}B_{I G} & B\end{array}\right]=$ $\operatorname{rank}\left[B_{I G}\right]$ and $\operatorname{rank}\left[\begin{array}{c}C_{I G} \\ C\end{array}\right]=\operatorname{rank}\left[C_{I G}\right]$ are satisfied,
the relationships $B=B_{I G} K_{I G}$ and $C=L_{I G} C_{I G}$ hold. By partitioning $B_{I G}=\left[\begin{array}{l}B_{I G_{1}} \\ B_{I G_{2}}\end{array}\right], C_{I G}=\left[\begin{array}{ll}C_{I G_{1}} & C_{I G_{2}}\end{array}\right]$ and then substituting $B_{1}=B_{I G_{1}} K_{I G}, C_{1}=L_{I G} C_{I G_{1}}$, respectively, yields

$$
\begin{align*}
& \left\|W_{o}(z)\left(H(z)-H_{l}(z)\right) V_{i}(z)\right\|_{\infty} \\
& =\left\|W_{o}(z)\left(C(z I-A)^{-1} B-C_{1}\left(z I-A_{11}\right)^{-1} B_{1}\right) V_{i}(z)\right\|_{\infty} \\
& =\| W_{o}(z)\left(L_{I G} C_{I G}(z I-A)^{-1} B_{I G} K_{I G}\right. \\
& \left.\quad-L_{I G} C_{I G_{1}}\left(z I-A_{11}\right)^{-1} B_{I G_{1}} K_{I G}\right) V_{i}(z) \|_{\infty} \\
& =\| W_{o}(z) L_{I G}\left(C_{I G}(z I-A)^{-1} B_{I G}\right. \\
& \left.\quad-C_{I G_{1}}\left(z I-A_{11}\right)^{-1} B_{I G_{1}}\right) K_{I G} V_{i}(z) \|_{\infty} \\
& = \\
& \quad\left\|W_{o}(z) L_{I G}\right\|_{\infty} \\
& \quad \times\left\|\left(C_{I G}(z I-A)^{-1} B_{I G}-C_{I G_{1}}\left(z I-A_{11}\right)^{-1} B_{I G_{1}}\right)\right\|_{\infty}  \tag{24}\\
& \quad \times\left\|K_{I G} V_{i}(z)\right\|_{\infty} .
\end{align*}
$$

If $\left\{A_{11}, B_{I G_{1}}, C_{I G_{1}}\right\}$ is reduced order model obtained by partitioning a balanced realization $\left\{A, B_{I G}, C_{I G}\right\}$, we have [7,39]

$$
\begin{equation*}
\left\|\left(C_{I G}(z I-A)^{-1} B_{I G}-C_{I G_{1}}\left(z I-A_{11}\right)^{-1} B_{I G_{1}}\right)\right\|_{\infty} \leq 2 \sum_{j=l+1}^{n} \sigma_{j} . \tag{25}
\end{equation*}
$$

Therefore,

$$
\begin{align*}
& \left\|W_{o}(z)\left(H(z)-H_{l}(z)\right) V_{i}(z)\right\|_{\infty} \\
& \quad \leq 2\left\|W_{o}(z) L_{I G}\right\|_{\infty}\left\|K_{I G} V_{i}(z)\right\|_{\infty} \sum_{j=l+1}^{n} \sigma_{j} . \tag{26}
\end{align*}
$$

Remark 6. For the case when symmetric matrices $X_{e n} \geq$ 0 and $Y_{e n} \geq 0$, then $P_{e n}=P_{C S}=P_{V d}=P_{I G}$ and $Q_{e n}=Q_{C S}=Q_{V d}=Q_{I G}$. However, when matrices $X_{e n}$ and $Y_{e n}$ are indefinite, then $P_{e n}<P_{I G}$ and $Q_{e n}<Q_{I G}$. Moreover, frequency weighted Hankel singular values satisfy $\left(\lambda_{j}\left[P_{e n} Q_{e n}\right]\right)^{1 / 2} \leq\left(\lambda_{j}\left[P_{I G} Q_{I G}\right]\right)^{1 / 2}$.

Remark 7. For the case when input $V_{i}(z)$ weights are co-inner and output $W_{o}(z)$ weights are inner [40], then $P=P_{e n}=$ $P_{C S}=P_{V d}=P_{I G}$ and $Q=Q_{e n}=Q_{C S}=Q_{V d}=Q_{I G}$, where $P$ and $Q$ are unweighted Gramians defined as

$$
\begin{align*}
& A P A^{T}-P+B B^{T}=0 \\
& A^{T} Q A-Q+C^{T} C=0 \tag{27}
\end{align*}
$$

Remark 8. For the case when the symmetric matrices $X_{e n} \geq 0$ and $Y_{e n} \geq 0$, the reduced order models obtained using Enns [25], Campbell et al.'s [37], Varga and Anderson's [38], and proposed technique are same.
3.2. Computational Aspects. The frequency weighted balanced truncation model reduction techniques balance the original system and then truncate the balanced realization to get the desired reduced order system. The balancing procedure involves computation of transformation matrix from controllability and observability Gramians. Sometimes these matrices become numerically low rank especially in large scale systems (possibly) due to rapid decay of their eigenvalues [3]. Due to this reason, balancing procedure becomes inefficient. Accuracy enhancing techniques for different frequency weighted model reduction techniques appear in [4, 29].

For unweighted case, Hammarling's technique [41] is used to obtained Cholesky factors of Gramian matrices from original system realization without actually computing controllability and observability Gramian matrices, respectively.

In frequency weighted techniques, Cholesky factors of the Gramian matrices are obtained from the augmented system realizations. Let $\bar{S}$ and $\bar{R}$ be the Cholesky factors of the augmented system Gramians matrices $P_{b i}$ and $Q_{b o}$ of (4) and (5), respectively,

$$
\begin{align*}
P_{b i} & =\bar{S} \bar{S}^{T}=\left[\begin{array}{cc}
S_{11} & S_{12} \\
0 & S_{22}
\end{array}\right]\left[\begin{array}{cc}
S_{11}^{T} & 0 \\
S_{12}^{T} & S_{22}^{T}
\end{array}\right] \\
& =\left[\begin{array}{cc}
S_{11} S_{11}^{T}+S_{12} S_{12}^{T} & S_{12} S_{22}^{T} \\
S_{22} S_{12}^{T} & S_{22} S_{22}^{T}
\end{array}\right]=\left[\begin{array}{ll}
P_{e n} & P_{12} \\
P_{12}^{T} & P_{V}
\end{array}\right], \\
Q_{b o} & =\bar{R}^{T} \bar{R}=\left[\begin{array}{cc}
R_{11}^{T} & 0 \\
R_{12}^{T} & R_{22}^{T}
\end{array}\right]\left[\begin{array}{cc}
R_{11} & R_{12} \\
0 & R_{22}
\end{array}\right]  \tag{28}\\
& =\left[\begin{array}{cc}
R_{11}^{T} R_{11} & R_{11}^{T} R_{12} \\
R_{12}^{T} R_{11} & R_{22}^{T} R_{22}+R_{12}^{T} R_{12}
\end{array}\right]=\left[\begin{array}{ll}
Q_{W} & Q_{12}^{T} \\
Q_{12} & Q_{e n}
\end{array}\right] .
\end{align*}
$$

By making use of Cholesky factors $\bar{S}$ and $\bar{R}$ calculated above, the Cholesky factors corresponding to Gramians in frequency weighted model reduction techniques like Enns [25], Campbell et al.'s [37], Varga and Anderson's [38], and proposed technique can be obtained as follows:
(1) Enns Technique. The Cholesky factors $S_{e n}=\left[\begin{array}{ll}S_{11} & S_{12}\end{array}\right]$ and $R_{e n}=\left[\begin{array}{l}R_{12} \\ R_{22}\end{array}\right]$ satisfy [29]

$$
\begin{gather*}
P_{e n}=S_{e n} S_{e n}^{T}=S_{11} S_{11}^{T}+S_{12} S_{12}^{T}=\left[\begin{array}{ll}
S_{11} & S_{12}
\end{array}\right]\left[\begin{array}{l}
S_{11}^{T} \\
S_{22}^{T}
\end{array}\right], \\
Q_{e n}=R_{e n}^{T} R_{e n}=R_{22}^{T} R_{22}+R_{12}^{T} R_{12}=\left[\begin{array}{ll}
R_{22}^{T} & R_{12}^{T}
\end{array}\right]\left[\begin{array}{l}
R_{22} \\
R_{12}
\end{array}\right] . \tag{29}
\end{gather*}
$$

(2) Campbell et al.'s Technique. The Cholesky factors $\bar{S}_{C S}$ and $\bar{R}_{C S}$ satisfy $P_{C S}=\bar{S}_{C S} \bar{S}_{C S}^{T}$ and $Q_{C S}=\bar{R}_{C S}^{T} \bar{R}_{C S}[4]$.
(3) Varga and Anderson's Technique. The Cholesky factors $\bar{S}_{V d}$ and $\bar{R}_{V d}$ satisfy $P_{V d}=\bar{S}_{V d} \bar{S}_{V d}^{T}$ and $Q_{V d}=\bar{R}_{V d}^{T} \bar{R}_{V d}[4]$.
(4) Proposed Technique. The Cholesky factors $\bar{S}_{I G}$ and $\bar{R}_{I G}$ satisfy $P_{I G}=\bar{S}_{I G} \bar{S}_{I G}^{T}$ and $Q_{I G}=\bar{R}_{I G}^{T} \bar{R}_{I G}$.

In the following we establish a relationship between Cholesky factors of Gramian matrices used in Enns and proposed techniques. Equations (18) and (19) can be expressed as

$$
\begin{align*}
& A\left(P_{e n}+P_{a d}\right) A^{T}-\left(P_{e n}+P_{a d}\right) \\
& \quad+\left(X_{e n}-s_{n} I\right)=0, \quad \text { for } s_{n}<0, \\
& A P_{e n} A^{T}-P_{e n}+X_{e n}=0, \quad \text { for } s_{n} \geq 0 \\
& A^{T}\left(Q_{e n}+Q_{a d}\right) A-\left(Q_{e n}+Q_{a d}\right) \\
& \quad+\left(Y_{e n}-r_{n} I\right)=0, \quad \text { for } r_{n}<0 \\
& A^{T} Q_{e n} A-Q_{e n}+Y_{e n}=0, \quad \text { for } r_{n} \geq 0 \\
& A P_{a d} A^{T}-P_{a d}-s_{n} I=0, \quad \text { for } s_{n}<0 \\
& A^{T} Q_{a d} A-Q_{a d}-r_{n} I=0, \quad \text { for } r_{n}<0 \tag{30}
\end{align*}
$$

Since

$$
\begin{gather*}
X_{I G}=U\left(S-s_{n} I\right)^{1 / 2}\left(S-s_{n} I\right)^{1 / 2} U^{T} \\
=X_{e n}-s_{n} I, \quad \text { for } s_{n}<0, \\
X_{I G}=U(S)^{1 / 2}(S)^{1 / 2} U^{T}=X_{e n}, \quad \text { for } s_{n} \geq 0, \\
Y_{I G}=V^{T}\left(R-r_{n} I\right)^{1 / 2}\left(R-r_{n} I\right)^{1 / 2} V=Y_{e n}-r_{n} I, \quad \text { for } r_{n}<0 \\
Y_{I G}=V^{T}(R)^{1 / 2}(R)^{1 / 2} V=Y_{e n}, \quad \text { for } r_{n} \geq 0 . \tag{31}
\end{gather*}
$$

By using Hammarling technique to calculate Cholesky factors of Gramians $P_{a d}$ and $Q_{a d}$ from realization $\left\{A, \sqrt{-s_{n}} I, \sqrt{-r_{n}} I, D\right\}$, we can write $P_{a d}=\bar{S}_{a d} \bar{S}_{a d}^{T}$ and $Q_{a d}=\bar{R}_{a d}^{T} \bar{R}_{a d}$. Therefore, frequency weighted controllability $P_{I G}$ (18) and observability $Q_{I G}$ (19) Gramians can be expressed as

$$
\begin{align*}
P_{I G} & =\bar{S}_{I G} \bar{S}_{I G}^{T}=P_{e n}+P_{a d}=S_{11} S_{11}^{T}+S_{12} S_{12}^{T}+S_{a d} S_{a d}^{T} \\
& =\left[\begin{array}{lll}
S_{11} & S_{12} & S_{a d}
\end{array}\right]\left[\begin{array}{c}
S_{11}^{T} \\
S_{12}^{T} \\
S_{a d}^{T}
\end{array}\right] \tag{32}
\end{align*}
$$

Table 1: Frequency weighted errors and error bounds comparison for reduced order models.

| Weighting Order |  | Enns technique [25] | Campbell et al.s technique [37] |  | Varga and Anderson's technique [38] |  | Proposed technique |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | Error | Error bound | Error | Error bound | Error | Error bound |
| Two sided | 1 |  | 21.1254 | 20.5953 | 1634.2 | 21.6491 | 725.7718 | 15.6065 | 2549.0 |
|  | 2 | 31.9647 | 32.8319 | 978.34 | 32.8863 | 433.8123 | 18.4571 | 1623.9 |
|  | 3 | 35.0441 | 32.3860 | 590.48 | 33.9063 | 255.7017 | 26.1274 | 998.92 |
|  | 4 | 28.7611 | 31.4710 | 236.41 | 30.4331 | 102.7518 | 30.7929 | 427.13 |
|  | 5 | 12.7538 | 29.5760 | 117.81 | 12.7660 | 50.4647 | 25.6547 | 203.36 |
| Input | 1 | 7.0257 | 7.1275 | 145.811 | 7.2356 | 92.6748 | 7.0140 | 242.5983 |
|  | 2 | 10.4643 | 10.7354 | 87.9789 | 10.7694 | 55.8603 | 10.6714 | 149.7017 |
|  | 3 | 11.2055 | 10.3852 | 53.0816 | 10.8477 | 32.9346 | 9.9857 | 90.8656 |
|  | 4 | 8.9654 | 10.0342 | 21.8067 | 9.6079 | 13.5877 | 8.4277 | 41.1182 |
|  | 5 | 2.4435 | 3.1720 | 10.4718 | 2.8446 | 6.3761 | 3.2061 | 16.5445 |

$$
\begin{align*}
Q_{I G} & =\bar{R}_{I G}^{T} \bar{R}_{I G}=Q_{e n}+Q_{a d}=R_{22}^{T} R_{22}+R_{12}^{T} R_{12}+R_{a d}^{T} R_{a d} \\
& =\left[\begin{array}{lll}
R_{22}^{T} & R_{12}^{T} & R_{a d}^{T}
\end{array}\right]\left[\begin{array}{c}
R_{22} \\
R_{12} \\
R_{a d}
\end{array}\right] . \tag{33}
\end{align*}
$$

Remark 9. Note that, Cholesky factors for Enns and proposed technique are computed directly from augmented system realization using Hammarling technique without calculating augmented system realization Gramian matrices $P_{b i}$ and $Q_{b o}$.

## 4. Illustrative Examples

In this section, using numerical illustrative examples we show the usefulness of the proposed technique in comparison with existing frequency weighted balanced model reduction techniques for discrete time systems. Note that, proposed work deals with frequency weighted model reduction problem for discrete time systems, therefore, comparison is done with existing frequency weighted balanced model reduction techniques only.

Example 1. Consider (example C appeared in [26]) a 4th order stable discrete time system

$$
\begin{equation*}
H(z)=\frac{z^{3}}{z^{4}+1.1 z^{3}-0.01 z^{2}-0.275 z-0.06} \tag{34}
\end{equation*}
$$

with the following weightings

$$
\begin{equation*}
V_{i}(z)=W_{o}(z)=\frac{z+0.9}{z+0.1} \tag{35}
\end{equation*}
$$

The first order reduced model obtained by Enns [25] technique is unstable while reduced order model obtained by Campbell et al.'s, Varga and Anderson's, and proposed techniques is stable yielding frequency response errors 112.9338, 100.8739, and 94.116, respectively. Note that, proposed technique provides stability with relatively lower error when compared to other techniques.

Example 2. Consider a 6 th order stable low pass digital elliptic filter with 0.2 dB of peak-to-peak ripple and a minimum stopband attenuation of 20 dB represented by

$$
\begin{align*}
H(z)= & \left(0.1054 z^{6}-0.1944 z^{5}+0.1187 z^{4}\right. \\
& \left.-0.1187 z^{2}+0.1944 z-0.1054\right) \\
& \times\left(z^{6}-2.9621 z^{5}+4.8325 z^{4}-4.9819 z^{3}\right.  \tag{36}\\
& \left.\quad+3.5245 z^{2}-1.5262 z+0.3657\right)^{-1}
\end{align*}
$$

with the following input and output weightings, respectively,

$$
\begin{align*}
V_{i}(z) & =\frac{z^{3}+3.0081 z^{2}+1.9944 z+1.0325}{z^{3}+0.2 z^{2}+0.75 z+0.2}  \tag{37}\\
W_{o}(z) & =\frac{z^{3}+2.97 z^{2}+2.9403 z+0.9703}{z^{3}+1.1619 z^{2}+0.6959 z+0.1378}
\end{align*}
$$

Table 1 gives the comparison of error and error bounds for reduced order systems obtained by Enns, Campbell et al.'s, Varga and Anderson's, and proposed techniques for the input and two sided weighting cases. Note that, the proposed technique mostly yields lower error as compared to other techniques.

Example 3. Consider a 4th order stable discrete time system [42]

$$
\begin{equation*}
H(z)=\frac{10^{-3}\left(3.315 z^{3}-4.9695 z^{2}+2.1668 z-0.24002\right)}{z^{4}-3.7035 z^{3}+5.1957 z^{2}-3.2718 z+0.77986} \tag{38}
\end{equation*}
$$

with the following input weighting:

$$
\begin{equation*}
V_{i}(z)=\frac{z^{2}-0.1 z-0.05}{z^{2}-0.9 z+0.75} \tag{39}
\end{equation*}
$$

Table 2 gives the comparison of error and error bounds for reduced order systems obtained by Enns, Campbell et al.'s Varga and Anderson's and proposed techniques for the input weighting case. Note that, the proposed technique compares well and yields relatively lower error as compared to other techniques.

Table 2: Frequency weighted errors and error bounds comparison for reduced order models.

| Weighting Order | Enns technique [25] | Campbell et al.s technique [37] <br> Error |  | Varga and Anderson's technique [38] <br> Error bound | Proposed technique <br> Error | Error bound | Error | Error bound |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |

Table 3: Frequency weighted errors and error bounds comparison for reduced order models.

| Weighting Order | Enns technique [25] | Campbell et al.s technique [37] <br> Error |  | Varga and Anderson's technique [38] |  | Proposed technique <br> Error bound | Error | Error bound |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |

Example 4. Consider a 4th order stable low pass digital Chebychev type 1 filter with 0.1 dB of peak-to-peak ripples in the passband represented by

$$
\begin{equation*}
H(z)=\frac{0.49 z^{4}-0.9799 z^{2}+0.49}{z^{4}-0.2893 z^{3}-0.6629 z^{2}+0.0246 z+0.2904} \tag{40}
\end{equation*}
$$

with the following output weighting:

$$
\begin{equation*}
W_{o}(z)=\frac{z-0.2}{z^{2}-0.4 z+0.5} \tag{41}
\end{equation*}
$$

Table 3 gives the comparison of error and error bounds for reduced order systems obtained by Enns, Campbell et al.'s Varga and Anderson's and proposed techniques for output weighting. Note that, the proposed technique compares well and yields relatively lower error as compared to other techniques.

## 5. Conclusion

A new frequency weighted technique for model reduction of discrete time systems is explored. The reduced order models obtained in the presence of input, output, and two sided weightings are stable. A comparison with existing schemes shows that proposed technique provides comparable results (mostly produces lower error) for reduced order models.

## Conflict of Interests

The authors declare that there is no conflict of interests.

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## Research Article

# Gain-Scheduled $\mathscr{H}_{2}$ Controller Synthesis for Continuous-Time Polytopic LPV Systems 

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#### Abstract

This paper is concerned with the problem of gain-scheduled $\mathscr{H}_{2}$ controller synthesis for continuous-time linear parameter-varying systems. In this problem, the system matrices in the state-space form are polytopic and patameterized and the admissible values of the parameters are assumed to be measurable on-line in a polytope space. By employing a basis-parameter-dependent Lyapunov function and introducing some slack variables to the well-established performance conditions, sufficient conditions for the existence of the desired gain-scheduled $\mathscr{H}_{2}$ state feedback and dynamic output feedback controllers are established in terms of parameterized linear matrix inequalities. Based on the polytopic characteristic of the dependent parameters and a convexification method, the corresponding controller synthesis problem is then cast into finite-dimensional convex optimization problem which can be efficiently solved by using standard numerical softwares. Numerical examples are given to illustrate the effectiveness and advantage of the proposed methods.


## 1. Introduction

It is well known that linear parameter-varying (LPV) systems are a class of linear systems whose state-space matrices depend on a set of time-varying parameters which are not known in advance but can be measured or estimated upon operation of the systems. Gain-scheduled control strategies for LPV systems have been studied intensively in the last two decades and significant progress has been made in this area (see, e.g., [1-27]). There are many examples of parameter-dependent systems in practice, such as in aeronautics, aerospace, mechanics, and industrial processes (see, e.g., $[2,4,12-14,26]$ ). This has motivated extensive studying of the gain-scheduled analysis and controller synthesis methods for various LPV systems from many aspects, including continuous-time and discrete-time systems, statespace formula and linear fractional transformation representation, affine-type and polytopic systems, state feedback and output feedback controllers, quadratic Lyapunov functionbased and parameter-dependent Lyapunov function-based methods, and robust $\mathscr{H}_{2}$ and $\mathscr{H}_{\infty}$ control performances.

In most of the existing work, the gain-scheduled LPV control synthesis problems are performed through semidefinite programming and especially linear matrix inequality (LMI) techniques [2,5-8, 10, 15-17, 19, 26]. This is due mainly to the fact that a number of methods of gain-scheduled control design for LPV systems proposed in the literature are based on small-gain approach (see, e.g., $[1-4,17]$ ) or on the notions of quadratic Lyapunov function (see, e.g., $[1,7,28]$ ). The advantage of small-gain approach and quadratic Lyapunov function-based methods is that the associated computation is relatively straightforward (e.g., standard numerical software, LMI Control Toolbox [29]). However, the drawback of these methods is in that a single parameter-independent Lyapunov function must be used to guarantee both stability and control performance for all parameter values, and it can produce conservative results (see, e.g., $[5,10,15,26]$ ). For the sake of reducing the abovementioned conservativeness, several control methods have been developed in the past decade, such as gain-scheduled $\mathscr{H}_{2}$ and $\mathscr{H}_{\infty}$ control based on parameter-dependent Lyapunov function (PDLF) method (see, e.g., $[4,5,9,10,15,18-21,26-28,30])$. Until now,
some researches have been carried out on $\mathscr{H}_{2}$ performance synthesis problems for continuous-time LPV systems (see, e.g., $[9,10,26]$ ). In $[9,10]$, de Souza et al. presented two novel state feedback $\mathscr{H}_{2}$ controllers for affine LPV systems based on quadratic PDLF frameworks. Also, Xie recently developed a gain-scheduled $\mathscr{H}_{2}$ state feedback for polytopic LPV systems with new LMI formulation by introducing additional slack variables [24, 26, 27]. Despite the recent development of gain-scheduled $\mathscr{H}_{2}$ analysis and controller synthesis for LPV systems, the issue associated with gain-scheduled $\mathscr{H}_{2}$ control via PDLF is not well documented so far, even in the case of state feedback.

From the above analysis, it is clear that the gain-scheduled $\mathscr{H}_{2}$ control problem for LPV systems has not been studied thoroughly, and this leads to the first objective of this paper for reducing the conservativeness of the existing state feedback controller. The second objective of this paper is to realize the dynamic output feedback control design for continuoustime polytopic LPV systems. It has to be stressed that the determination of a dynamic output feedback controller for polytopic LPV systems is indeed a difficult problem. The dynamic output feedback control is more flexible than static output feedback since additional dynamics of the controller are introduced $[18-24]$. Both are new contributions to the existing literature. To begin with, the result of gain-scheduled $\mathscr{H}_{2}$ analysis in [26] is introduced with some explanations. To reduce the conservativeness of the state feedback controller synthesis, an improved sufficient condition for the existence of desired gain-scheduled $\mathscr{H}_{2}$ state feedback controller is obtained in terms of parameterized linear matrix inequalities (PLMIs). Furthermore, based on polytopic characteristic of the parameter-dependent system, the corresponding controller synthesis problem is cast into a finite-dimensional convex optimization problem by a convexification method.

Considering the commonly encountered case in practice that the full state variables are unavailable for state feedback control design, we make further research on the gainscheduled $\mathscr{H}_{2}$ dynamic output feedback controller synthesis. Inspired by the work of the basis-PDLF approach proposed in $[20,21,31,32]$, some auxiliary slack matrix variables are introduced in the process of expressing the relationships among the terms of the system equation. A sufficient condition for the existence of desired gain-scheduled $\mathscr{H}_{2}$ dynamic output feedback controller is initially obtained by PLMIs. Such a sufficient condition can guarantee that the closedloop system is exponentially stable and has a prescribed $\mathscr{H}_{2}$ disturbance attenuation performance. Similar to the case of state feedback control design, the new convexification method is introduced to derive a finite-dimensional PLMI condition for the dynamic output feedback controller synthesis, which can be efficiently solved by using standard numerical software. Finally, simulation results of an example in a gain-scheduled $\mathscr{H}_{2}$ state feedback control indicate that our approach can generate less conservativeness than the existing results. The effectiveness of the proposed dynamic output feedback controller is verified by the other numerical example. It should be noticed that $\mathscr{H}_{2}$ performance in this paper can be used to capture both the response to stationary noise and the transient response of the closed-loop system.

Different from the Lyapunov-based robust $\mathscr{H}_{2}$ performance, other methods lean too heavily on robustness and sacrifice an adequate view of performance. For example, robust $\mathscr{H}_{\infty}$ method treats disturbances or commands as being the worst in a very broad class which is often unrealistic.

The rest of this paper is organized as follows. The problem formulation and some preliminary results are presented in the next section. Section 3 gives our main results of gainscheduled $\mathscr{H}_{2}$ state feedback and dynamic output feedback control design, respectively. Two numerical examples are given in Section 4 and we conclude this paper in Section 5.
Notations. We use the following notations throughout this paper. The superscript " $T$ " stands for matrix transposition, $\mathbb{R}^{n}$ denotes the $n$-dimensional Euclidean space, and the notation $P>0(\geq 0)$ means that $P$ is real symmetrical and positive definite (semidefinite). $\operatorname{Tr}(\cdot)$ denotes the matrix trace, and $\operatorname{Her}\{A\}$ stands for $A+A^{T}$. In symmetric block matrices or long matrix expressions, we use an asterisk ( $*$ ) to represent a term that is induced by symmetry, and $\operatorname{diag}\{\cdots\}$ stands for a blockdiagonal matrix. In addition, $\mathbf{I}$ and $\mathbf{0}$ denote identity matrix and zero matrix, respectively. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

## 2. Problem Description and Preliminaries

2.1. Problem Description. Consider the following class of continuous-time LPV systems:

$$
\begin{align*}
\mathcal{S}: \dot{x}(t)= & A(\theta(t)) x(t) \\
& +B_{1}(\theta(t)) w(t)+B_{2}(\theta(t)) u(t), \\
z(t) & =C_{1}(\theta(t)) x(t)+D_{1}(\theta(t)) u(t),  \tag{1}\\
y(t) & =C_{2}(\theta(t)) x(t),
\end{align*}
$$

where $x(t) \in \mathbb{R}^{n}$ is the state vector, $y(t) \in \mathbb{R}^{m}$ is the measured output, $z(t) \in \mathbb{R}^{p}$ is the controlled output, $u(t) \in \mathbb{R}^{q}$ is the control input, and $w(t) \in \mathbb{R}^{l}$ is the disturbance input. All the system matrices have compatible dimensions. The system matrices $A(\theta), B_{1}(\theta), B_{2}(\theta), C_{1}(\theta)$, $D_{1}(\theta)$, and $C_{2}(\theta)$ are parameter-dependent matrices with respect to the time-varying scheduling parameter $\theta(t)=$ $\left[\theta_{1}(t), \theta_{2}(t), \ldots, \theta_{r}(t)\right]^{T} \in \mathbb{R}^{r}$. Similar to the previous gainscheduled $\mathscr{H}_{2}$ control problems for LPV systems [10, 26], the following assumptions are given.

Assumption 1. The state-space matrices $A(\theta), B_{1}(\theta), B_{2}(\theta)$, $C_{1}(\theta), D_{1}(\theta)$, and $C_{2}(\theta)$ are continuous and bounded functions and depend affinely on $\theta(t)$.

Assumption 2. The parameter values of vector $\theta(t)$ are not known in advance but are measurable in real time. In addition, the parameter $\theta(t)$ is limited to a given convex bounded polyhedral domain $\mathscr{P}$ described by $N$ vertices as

$$
\theta(t) \in \mathscr{P} \triangleq \operatorname{Co}\left\{\omega_{1}, \omega_{2}, \ldots, \omega_{N}\right\}
$$

$$
\begin{equation*}
=\left\{\sum_{i=1}^{N} \alpha_{i}(t) \omega_{i}: \alpha_{i}(t) \geqslant 0, \sum_{i=1}^{N} \alpha_{i}(t)=1, N=2^{r}\right\} \tag{2}
\end{equation*}
$$

and the rate of variation $\dot{\theta}(t)$ is well defined over the time horizon and varies in a polytope $\mathscr{V}$ as

$$
\begin{align*}
\dot{\theta}(t) \in \mathscr{V} & \triangleq \operatorname{Co}\left\{v_{1}, v_{2}, \ldots, v_{N}\right\} \\
& =\left\{\sum_{k=1}^{N} \beta_{k}(t) v_{k}: \beta_{k}(t) \geqslant 0, \sum_{k=1}^{N} \beta_{k}(t)=1, N=2^{r}\right\} . \tag{3}
\end{align*}
$$

Given the above sets $\mathscr{P}$ and $\mathscr{V}$, we define the parameter set $\mathscr{F}_{\mathscr{P}}^{\mathscr{V}}$ described by $N$ vertices as

$$
\begin{equation*}
\mathscr{F}_{\mathscr{P}}^{\mathscr{V}} \triangleq\left\{\theta(t) \in \mathscr{C}^{1}\left(\mathbb{R}_{+}, \mathbb{R}^{N}\right): \theta(t) \in \mathscr{P}, \dot{\theta}(t) \in \mathscr{V}, \forall t \geqslant 0\right\} . \tag{4}
\end{equation*}
$$

Moreover, the LPV system $\mathcal{S}$ in (1) is called polytopic, when it ranges in a matrix polytope, that is, the LPV system (1) can be expressed as

$$
\begin{equation*}
\Omega(\theta) \triangleq\left(A(\theta), B_{1}(\theta), B_{2}(\theta), C_{1}(\theta), D_{1}(\theta), C_{2}(\theta)\right) \in \mathscr{R} \tag{5}
\end{equation*}
$$

where $\theta(t) \in \mathscr{F}_{\mathscr{P}}^{\mathscr{V}}$ and $\mathscr{R}$ is also a given convex bounded polyhedral domain described by $N$ vertices:

$$
\begin{equation*}
\mathscr{R} \triangleq\left\{\sum_{k=1}^{N} \alpha_{i}(t) \Omega_{i}: \alpha_{i}(t) \geqslant 0, \sum_{i=1}^{N} \alpha_{i}(t)=1, N=2^{r}\right\} . \tag{6}
\end{equation*}
$$

Here, we are interested in designing both gain-scheduled $\mathscr{H}_{2}$ state feedback controller and dynamic output feedback controller for the system $\mathcal{S}$ described by (1). Therefore, two gain-scheduled $\mathscr{H}_{2}$ control laws are described by $\mathscr{K}_{\mathcal{F}}$ and $\mathscr{K}_{\mathscr{D} G F}$, respectively, as follows:

$$
\begin{gather*}
\mathscr{K}_{S \mathscr{F}}: u(t)=K(\theta) x(t),  \tag{7}\\
\mathscr{K}_{\mathscr{D O F}}: \dot{x}_{K}(t)=A_{K}(\theta) x_{K}(t)+B_{K}(\theta) y(t),  \tag{8}\\
u(t)=C_{K}(\theta) x_{K}(t)+D_{K}(\theta) y(t),
\end{gather*}
$$

where $x_{K}(t) \in \mathbb{R}^{n}$ is controller state vector and $K(\theta)$ and $\left(A_{K}(\theta), B_{K}(\theta), C_{K}(\theta), D_{K}(\theta)\right)$ are appropriately dimensioned LPV controller matrices to be determined.

Substituting the state-feedback control law $\mathscr{K}_{\mathcal{F} F}$ into (1), the closed-loop system can be obtained as

$$
\begin{gather*}
\mathscr{C}_{\mathcal{F} F}: \dot{x}(t)=A_{\mathrm{cl}}(\theta) x(t)+B_{\mathrm{cl}}(\theta) w(t) \\
z(t)=C_{\mathrm{cl}}(\theta) x(t) \tag{9}
\end{gather*}
$$

where

$$
\begin{gather*}
A_{\mathrm{cl}}(\theta)=A(\theta)+B_{2}(\theta) K(\theta) \\
B_{\mathrm{cl}}(\theta)=B_{1}(\theta)  \tag{10}\\
C_{\mathrm{cl}}(\theta)=C_{1}(\theta)+D_{1}(\theta) K(\theta)
\end{gather*}
$$

Augmenting the model of $\mathcal{S}$ to include the state of the gain-scheduled dynamic output feedback control $\mathscr{K}_{\mathscr{D} \mathscr{F}}$, we obtain the closed-loop LPV system $\mathscr{C}_{\mathscr{D} O F}$ :

$$
\begin{gather*}
\mathscr{C}_{\mathscr{D O F} F}: \dot{x}_{\mathrm{cl}}(t)=A_{\mathrm{cl}}(\theta) x_{\mathrm{cl}}(t)+B_{\mathrm{cl}}(\theta) w(t),  \tag{11}\\
z(t)=C_{\mathrm{cl}}(\theta) x_{\mathrm{cl}}(t)
\end{gather*}
$$

where $x_{\mathrm{cl}}(t)=\left[x^{T}(t), x_{K}^{T}(t)\right]^{T}$ and

$$
\begin{gather*}
A_{\mathrm{cl}}(\theta)=\left[\begin{array}{cc}
A(\theta)+B_{2}(\theta) D_{K}(\theta) C_{2}(\theta) & B_{2}(\theta) C_{K}(\theta) \\
B_{K}(\theta) C_{2}(\theta) & A_{K}(\theta)
\end{array}\right], \\
B_{\mathrm{cl}}(\theta)=\left[\begin{array}{cc}
B_{1} & (\theta) \\
0
\end{array}\right], \\
C_{\mathrm{cl}}(\theta)=\left[C_{1}(\theta)+D_{1}(\theta) D_{K}(\theta) C_{2}(\theta) D_{1}(\theta) C_{K}(\theta)\right] \tag{12}
\end{gather*}
$$

Then, the problems of gain-scheduled $\mathscr{H}_{2}$ control design for LPV systems can be expressed as follows.
Gain-Scheduled $\mathscr{H}_{2}$ Controller Synthesis Problem. Given the polytopic LPV system $\mathcal{S}$ in (1), our concerned problem is to determine the gain-scheduled state feedback controller $\mathscr{K}_{\delta \mathscr{F}}$ in (7) and dynamic output feedback controller $\mathscr{K}_{\mathscr{D O F}}$ in (8), such that both the closed-loop LPV systems $\mathscr{C}_{\mathcal{S F}}$ and $\mathscr{C}_{\mathscr{D} G \mathscr{F}}$ in (9) and (11) are exponentially stable for all $(\theta, \dot{\theta}) \in \mathscr{F}_{\mathscr{P}}^{\mathscr{V}}$, and $z(t)$ reaches the desired controlled output in the sense of the $\mathscr{H}_{2}$ performance with respect to the disturbance input $w(t)$.
2.2. Preliminaries. In order to solve the gain-scheduled $\mathscr{H}_{2}$ control design problem, we present some preliminary results for later use.

First, we introduce the notion of $\mathscr{H}_{2}$ norm for LPV systems borrowed from linear time-varying (LTV) systems (see $[9,33]$ for details). In this paper we use the stationary white noise approach and the average output variance to define the $\mathscr{H}_{2}$ norm. This kind of $\mathscr{H}_{2}$ norm can be used to capture both the transient response of the system and the response to stationary noise. Furthermore, it has also been proven to be the most appropriate for the Lyapunov-based $\mathscr{H}_{2}$ performance analysis and control design methods that will be developed in this paper $[9,10]$.

Definition 3 (see [10]). Let the closed-loop LPV system $\mathscr{C}$ in (9) or (11) be exponentially stable. The $\mathscr{H}_{2}$ norm of system $\mathscr{C}$ can be defined as

$$
\begin{equation*}
\|\mathscr{C}\|_{2}^{2}=\lim _{h \rightarrow \infty} E\left\{\frac{1}{h} \int_{0}^{h} z^{T}(t) z(t) d t\right\} \tag{13}
\end{equation*}
$$

when $x_{\mathrm{cl}}(0)=0$ and $w(t)$ is a stationary zero-mean white process with an identity power spectrum density matrix, where $E\{\cdot\}$ denotes the mathematical expectation.

From the above definition, $\mathscr{H}_{2}$ norm performance can be regarded as an index or criterion assessing the elimination of the disturbance or noise. It is clear that the optimal $\mathscr{H}_{2}$ attenuation levels by the latest approaches are less conservative than that by the approach in the existing literature,
and the improvement on conservativeness of the optimal $\mathscr{H}_{2}$ attenuation level is more apparent [ $9,10,33$ ]. Based on Definition 3, we introduce an important result based on the conclusion in [26] using a PDLF, which is a preliminary result for solving the gain-scheduled $\mathscr{H}_{2}$ controller synthesis problem in this paper.

Lemma 4 (see [26]). Given a scalar $\gamma>0$, the closedloop system $\mathscr{C}$ in (9) or (11) is exponentially stable with the prescribed performance index $\|\mathscr{C}\|_{2}<\gamma$ if there exist matrices $P(\theta)>0, \Pi(\theta)>0$, and $W(\theta)$ and a sufficiently small positive scalar $\varepsilon>0$ satisfying

$$
\begin{gather*}
\operatorname{Tr}(\Pi(\theta))<\gamma  \tag{14}\\
{\left[\begin{array}{ccc}
W(\theta)+W^{T}(\theta)-P(\theta) & W^{T}(\theta) C_{\mathrm{cl}}^{T}(\theta) \\
* & \Pi(\theta)
\end{array}\right]>0,}  \tag{15}\\
{\left[\begin{array}{ccc}
\Theta_{1} & \Theta_{2} & B_{\mathrm{cl}}(\theta) \\
* & -\varepsilon\left(W(\theta)+W^{T}(\theta)\right) & \mathbf{0} \\
* & * & -\mathbf{I}
\end{array}\right]<0,} \tag{16}
\end{gather*}
$$

where

$$
\begin{gather*}
\Theta_{1} \triangleq A_{\mathrm{cl}}(\theta) W(\theta)+W^{T}(\theta) A_{\mathrm{cl}}^{T}(\theta)+\frac{d P(\theta)}{d t}  \tag{17}\\
\Theta_{2} \triangleq P(\theta)-W^{T}(\theta)+\varepsilon A_{\mathrm{cl}}(\theta) W(\theta)
\end{gather*}
$$

Remark 5. Note that there exists a term $d P(\theta) / d t$ in condition (16) which cannot be implemented since it is not convex in the parameter $\theta(t)$. However, for the polytopic LPV system (1), we can solve this difficulty by using the following method from [11, 26]. Choose the parameter-dependent matrix $P(\theta)$ as

$$
\begin{equation*}
P(\theta)=\sum_{i=1}^{N} \theta_{i}(t) P_{i}, \quad(\theta, \dot{\theta}) \in \mathscr{F}_{\mathscr{P}}^{\mathscr{V}} . \tag{18}
\end{equation*}
$$

Based on this expression, its time derivation can be derived as

$$
\begin{equation*}
\frac{d P(\theta)}{d t}=\dot{\theta}_{1}(t) P_{1}+\dot{\theta}_{2}(t) P_{2}+\cdots+\dot{\theta}_{N}(t) P_{N}=P(\dot{\theta}) \tag{19}
\end{equation*}
$$

From Assumption 2, we have

$$
\begin{equation*}
\frac{d P(\theta)}{d t}=P(\dot{\theta})=\sum_{k=1}^{N} \beta_{k}(t) P_{t}\left(v_{k}\right) \tag{20}
\end{equation*}
$$

and then $d P(\theta) / d t$ in condition (16) can be substituted by convex parameter-dependent matrix $P_{t}(\theta)$.

It can be seen from Lemma 4 that, by introducing an extra matrix variable $W(\theta)$, the LMIs (14)-(16) provide a decoupling between Lyapunov function matrix and system matrices and will be useful for a gain-scheduled $\mathscr{H}_{2}$ controller synthesis for LPV systems. In addition, it has been shown, both theoretically and numerically, that the parameterdependent approach is less conservative than the results in
the quadratic framework, where a common Lyapunov matrix is used for the entire uncertainty domain [26]. However, we have to mention that the result developed in [26] is also conservative due to the imposition of $W(\theta) \equiv W$ when the result is used to synthesise a gain-scheduled $\mathscr{H}_{2}$ state feedback controller. The reason is that the introduced slack matrix $W(\theta)$ has been involved in the products with system matrices.

Then, there is a natural question: whether the conservativeness could be further reduced if we adopt different approaches other than the imposition of parameter independence as described above in the process of controller synthesis? The answer is affirmative. For state feedback controller synthesis, a possible alternative is the introduction of a new approach in [31] that helps to cast the infinite-dimensional LMI condition into finite-dimensional one. In the case of dynamic output feedback controller synthesis, a possible alternative is the introductions of structural block matrices and basis-parameter-dependent Lyapunov function [20, 21]. To the best of the authors' knowledge, these approaches have not been investigated for gain-scheduled $\mathscr{H}_{2}$ control problem of LPV systems so far. In the following, we will present some methods to solve both the state feedback and dynamic output feedback controller syntheses for the gain-scheduled $\mathscr{H}_{2}$ control of LPV systems.

## 3. Gain-Scheduled $\mathscr{H}_{2}$ Control Design

To reduce the conservativeness of state feedback controller synthesis mentioned above, we present a new sufficient condition for the existence of desired $\mathscr{H}_{2}$ state feedback controller in terms of finite-dimensional PLMIs. In order to solve the gain-scheduled $\mathscr{H}_{2}$ dynamic output feedback controller syntheses, a decoupling technique and a convexification method are introduced to obtain some sufficient conditions for the existence of the desired controller. The following two parts present the main results of gain-scheduled $\mathscr{H}_{2}$ state feedback and dynamic output feedback controller syntheses, respectively.

### 3.1. State Feedback Control Design

Theorem 6. Consider the system $\mathcal{S}$ in (1). Given a scalar $\gamma>0$ and a sufficiently small positive scalar $\varepsilon>0$, there exists a gain-scheduled $\mathscr{H}_{2}$ state feedback controller $\mathscr{K}_{\delta \mathscr{F}}$ in the form of (7) such that the resulting closed-loop system $\mathscr{C}_{\delta \mathscr{F}}$ in (9) is exponentially stable with a prescribed $\mathscr{H}_{2}$ disturbance attenuation level $\gamma$ if there exist matrices $\Pi(\theta)>0, P(\theta)>0$, $P_{t}(\theta)>0, W(\theta)$, and $M(\theta)$ satisfying

$$
\begin{gather*}
\operatorname{Tr}(\Pi(\theta))<\gamma,  \tag{21}\\
\Phi(\theta) \triangleq\left[\begin{array}{cc}
\Delta_{1} & \Delta_{2} \\
* & \Pi(\theta)
\end{array}\right]>0,  \tag{22}\\
\Psi(\theta) \triangleq\left[\begin{array}{ccc}
\Delta_{3} & \Delta_{4} & B_{1}(\theta) \\
* & -\varepsilon W(\theta)-\varepsilon W^{T}(\theta) & 0 \\
* & * & -\mathbf{I}
\end{array}\right]<0, \tag{23}
\end{gather*}
$$

where

$$
\begin{gathered}
\Delta_{1} \triangleq W(\theta)+W^{T}(\theta)-P(\theta) \\
\Delta_{2} \triangleq W^{T}(\theta) C_{1}^{T}(\theta)+M^{T}(\theta) D_{1}^{T}(\theta) \\
\Delta_{3} \triangleq \operatorname{Her}\left\{A(\theta) W(\theta)+B_{2}(\theta) M(\theta)\right\}+P_{t}(\theta) \\
\Delta_{4} \triangleq P(\theta)-W^{T}(\theta)+\varepsilon A(\theta) W(\theta)+\varepsilon B_{2}(\theta) M(\theta)
\end{gathered}
$$

In this case, a desired gain-scheduled $\mathscr{H}_{2}$ state feedback gain $K(\theta)$ is given by

$$
\begin{equation*}
K(\theta)=M(\theta) W^{-1}(\theta) \tag{25}
\end{equation*}
$$

Proof. Define $M(\theta) \triangleq K(\theta) W(\theta)$. With (9) and (10), the results can be derived easily from Lemma 4.

Note that the matrix variables $W(\theta)$ and $M(\theta)$ in Theorem 6 are dependent on time-varying parameter $\theta(t)$ and are not assumed to be constant matrices, which makes the new state feedback controller design conditions (21)-(23) less conservative than the results in [26]. However, the LMI conditions (21)-(23) in Theorem 6 cannot be implemented since they are not convex in the parameter $\theta(t)$. To solve this problem, we will introduce a new technique that helps convexify the matrix inequalities in Theorem 6 based on the polytopic characteristic of the dependent parameters. Then, we have the main result in the following theorem.

Theorem 7. Consider the system $\mathcal{\delta}$ in (1). Given a scalar $\gamma>0$ and a sufficiently small positive scalar $\varepsilon>0$, an admissible gain-scheduled $\mathscr{H}_{2}$ state feedback controller in the form of $\mathscr{K}_{\text {S耳्F }}$ in (7) exists if there exist matrices $\Pi_{i}>0, P_{i}>0$, $P_{t k}>0, W_{i}$, and $M_{i}$ satisfying

$$
\begin{gather*}
\operatorname{Tr}\left(\Pi_{i}\right)<\gamma, \quad i=1, \ldots, N  \tag{26}\\
\Phi_{i j}+\Phi_{j i}-\Lambda_{i j}-\Lambda_{i j}^{T} \geqslant 0, \quad 1 \leqslant i<j \leqslant N  \tag{27}\\
\Psi_{i j k}+\Psi_{j i k}-\Upsilon_{i j k}-\Upsilon_{i j k}^{T} \leqslant 0  \tag{28}\\
1 \leqslant i<j \leqslant N, \quad k=1, \ldots, N \\
\Lambda \triangleq\left[\begin{array}{cccc}
\Phi_{11} & \Lambda_{12} & \cdots & \Lambda_{1 N} \\
* & \Phi_{22} & \cdots & \Lambda_{2 N} \\
* & * & \ddots & \vdots \\
* & * & * & \Phi_{N N}
\end{array}\right]>0  \tag{29}\\
\Upsilon \triangleq\left[\begin{array}{cccc}
\Psi_{11 k} & \Upsilon_{12 k} & \cdots & \Upsilon_{1 N k} \\
* & \Psi_{22 k} & \cdots & \Upsilon_{2 N k} \\
* & * & \ddots & \vdots \\
* & * & * & \Psi_{N N k}
\end{array}\right]<0 \tag{30}
\end{gather*}
$$

where

$$
\begin{gather*}
\Phi_{i j} \triangleq\left[\begin{array}{cc}
W_{i}+W_{i}^{T}-P_{i} & W_{i}^{T} C_{1 j}^{T}+M_{i}^{T} D_{1 j}^{T} \\
* & \Pi_{i}
\end{array}\right], \\
\Psi_{i j k} \triangleq\left[\begin{array}{ccc}
\mathscr{L}_{1} & \mathscr{L}_{2} & B_{1 j} \\
* & -\varepsilon W_{i}-\varepsilon W_{i}^{T} & \mathbf{0} \\
* & * & -\mathbf{I}
\end{array}\right]  \tag{31}\\
\mathscr{L}_{1} \triangleq \operatorname{Her}\left\{A_{j} W_{i}+B_{2 j} M_{i}\right\}+P_{t k} \\
\mathscr{L}_{2} \triangleq P_{i}-W_{i}^{T}+\varepsilon A_{j} W_{i}+\varepsilon B_{2 j} M_{i}
\end{gather*}
$$

Moreover, under the above conditions, the admissible gainscheduled $\mathscr{H}_{2}$ state feedback gain $K(\theta)$ is given by

$$
\begin{equation*}
K(\theta)=\left(\sum_{i=1}^{N} \alpha_{i} M_{i}\right)\left(\sum_{i=1}^{N} \alpha_{i} W_{i}\right)^{-1} \tag{32}
\end{equation*}
$$

Proof. From Theorem 6, an admissible gain-scheduled $\mathscr{H}_{2}$ state feedback controller in the form of $\mathscr{K}_{\mathcal{F}}$ in (7) exists if there exist matrices $\Pi(\theta)>0, P(\theta)>0, P_{t}(\theta)>0, W(\theta)$, and $M(\theta)$ satisfying (21)-(23). Now, assume that the above matrix functions are of the following forms:

$$
\begin{gather*}
\Pi(\theta)=\sum_{i=1}^{N} \alpha_{i} \Pi_{i}, \quad W(\theta)=\sum_{i=1}^{N} \alpha_{i} W_{i}, \\
P(\theta)=\sum_{i=1}^{N} \alpha_{i} P_{i}, \quad P_{t}(\theta)=\sum_{k=1}^{N} \beta_{k} P_{t k},  \tag{33}\\
M(\theta)=\sum_{i=1}^{N} \alpha_{i} M_{i}
\end{gather*}
$$

Then, with (33), we rewrite $\Phi(\theta)$ and $\Psi(\theta)$ in (22)-(23) as

$$
\begin{align*}
\Phi(\theta) & =\sum_{j=1}^{N} \sum_{i=1}^{N} \alpha_{i} \alpha_{j} \Phi_{i j} \\
& =\sum_{i=1}^{N} \alpha_{i}^{2} \Phi_{i i}+\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \alpha_{i} \alpha_{j}\left(\Phi_{i j}+\Phi_{j i}\right) \\
\Psi(\theta) & =\sum_{k=1}^{N} \sum_{j=1}^{N} \sum_{i=1}^{N} \alpha_{i} \alpha_{j} \beta_{k} \Phi_{i j k} \\
& =\sum_{k=1}^{N} \beta_{k}\left(\sum_{i=1}^{N} \alpha_{i}^{2} \Psi_{i i k}+\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \alpha_{i} \alpha_{j}\left(\Psi_{i j k}+\Psi_{j i k}\right)\right) . \tag{34}
\end{align*}
$$

On the other hand, (27)-(28) are equivalent to

$$
\begin{gather*}
\Phi_{i j}+\Phi_{j i} \geqslant \Lambda_{i j}+\Lambda_{i j}^{T}, \quad 1 \leqslant i<j \leqslant N \\
\Psi_{i j k}+\Psi_{j i k} \leqslant \Upsilon_{i j k}+\Upsilon_{i j k}^{T},  \tag{35}\\
1 \leqslant i<j \leqslant N, \quad k=1, \ldots, N .
\end{gather*}
$$

Then, from (34)-(35), we have

$$
\begin{align*}
\Phi(\theta) \geqslant & \sum_{i=1}^{N} \alpha_{i}^{2} \Phi_{i i} \\
& +\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \alpha_{i} \alpha_{j}\left(\Lambda_{i j}+\Lambda_{i j}^{T}\right)=\eta^{T} \Lambda \eta  \tag{36}\\
\Psi(\theta) \leqslant & \sum_{k=1}^{N} \beta_{k}\left(\sum_{i=1}^{N} \alpha_{i}^{2} \Psi_{i i k}+\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \alpha_{i} \alpha_{j}\left(\Upsilon_{i j k}+\Upsilon_{i j k}^{T}\right)\right) \\
= & \eta^{T} \Upsilon \eta
\end{align*}
$$

where $\eta \triangleq\left[\alpha_{1} I, \alpha_{2} I, \ldots, \alpha_{N} I\right]^{T}$. Inequalities (29)-(30) guarantee $\Phi(\theta)>0$ and $\Psi(\theta)<0$, respectively. As to (26), since $\Pi(\theta)=\sum_{i=1}^{N} \alpha_{i} \Pi_{i}$ and $\operatorname{Tr}(\Pi(\theta))=\sum_{i=1}^{N} \alpha_{i} \operatorname{Tr}\left(\Pi_{i}\right)$, if (26) is satisfied, we can get (21). By substituting the matrices defined in (33) into (25), we readily obtain (32), and the proof is completed.

Remark 8. From the proof of Theorems 6 and 7, it can be seen that in the process of solving the gain-scheduled $\mathscr{H}_{2}$ state feedback control problem, we actually define parameterdependent Lyapunov function-based matrix for $\mathscr{H}_{2}$ performance objective. In other words, $P(\theta)$ takes the form of $P(\theta)=\sum_{i=1}^{N} \alpha_{i} P_{i}$. The gain-scheduled $\mathscr{H}_{2}$ state feedback control design for continuous-time LPV systems has been investigated in [26], where parameter-dependent idea is realized at the expense of setting an additional slack variable to be constant for each vertex of the polytope. Notably, in Theorem 7, we do not set any matrix variable to be constant for the whole polytope domain. Therefore, Theorem 7 has the potential to yield less conservative results in the applications of gain-scheduled $\mathscr{H}_{2}$ state feedback controller synthesis.

Remark 9. The idea behind Theorem 7 is to use convex combinations of vertex matrices in the form of (33) to substitute the matrix functions in Theorem 6. By introducing these matrices and by means of the convexification method used in the proof of Theorem 7, the infinite-dimensional nonlinear matrix inequality conditions in Theorem 6 are cast into finite-dimensional PLMIs conditions, which depend only on the vertex matrices of the polytope $\mathscr{R}$. Therefore, these PLMIs conditions can be readily checked by using standard numerical software [29]. Note that the conditions in Theorem 7 are PLMIs for prescribed scalar $\varepsilon$ not only over the matrix variables but also over the scalar $\gamma$. This implies that the $\mathscr{H}_{2}$ performance index $\gamma$ can be included as optimization variable to obtain a reduction of the attenuation level bound. As in [26,31], it is usually desired to design $\mathscr{H}_{2}$ controller with minimized performance $\gamma^{*}$, which can be readily found by solving the following convex optimization problem:

Minimize $\gamma$
subject to (26)-(30) for given scalar $\varepsilon$.

### 3.2. Dynamic Output Feedback Control Design

Theorem 10. Consider the system $\mathcal{S}$ in (1). Given a scalar $\gamma>0$ and a sufficiently small positive scalar $\varepsilon>0$, there exists a gain-scheduled $\mathscr{H}_{2}$ dynamic output feedback controller $\mathscr{K}_{\text {DOF }}$ in the form of (8) such that the resulting closed-loop system $\mathscr{C}_{\text {DOF }}$ in (11) is exponentially stable with a prescribed $\mathscr{H}_{2}$ disturbance attenuation level $\gamma$ if there exist matrices $\Pi(\theta)>0$, $\widetilde{P}(\theta) \triangleq\left[\begin{array}{cc}\widetilde{P}_{1}(\theta) & \widetilde{P}_{2}(\theta) \\ * & \widetilde{P}_{3}(\theta)\end{array}\right]>0, \widetilde{P}_{t}(\theta) \triangleq\left[\begin{array}{cc}\widetilde{P}_{11}(\theta) & \widetilde{T}_{t 2}(\theta) \\ * & \widetilde{P}_{t 3}(\theta)\end{array}\right]>0, R(\theta), S(\theta)$, $T(\theta), \widetilde{A}_{K}(\theta), \widetilde{B}_{K}(\theta), \widetilde{C}_{K}(\theta)$, and $\widetilde{D}_{K}(\theta)$ satisfying
$\operatorname{Tr}(\Pi(\theta))<\gamma$,

$$
\Phi(\theta) \triangleq\left[\begin{array}{ccc}
\Gamma_{1} & \Gamma_{2} & \Gamma_{3}  \tag{38}\\
* & \Gamma_{4} & \Gamma_{5} \\
* & * & \Pi(\theta)
\end{array}\right]>0
$$

$$
\Psi(\theta) \triangleq\left[\begin{array}{ccccc}
\Xi_{1} & \Xi_{2} & \Xi_{3} & \Xi_{4} & B_{1}(\theta)  \tag{40}\\
* & \Xi_{5} & \Xi_{6} & \Xi_{7} & S^{T}(\theta) B_{1}(\theta) \\
* & * & \Xi_{8} & \Xi_{9} & \mathbf{0} \\
* & * & * & \Xi_{10} & \mathbf{0} \\
* & * & * & * & -\mathbf{I}
\end{array}\right]<0
$$

where

$$
\begin{gather*}
\Gamma_{1} \triangleq R(\theta)+R^{T}(\theta)-\widetilde{P}_{1}(\theta), \\
\Gamma_{2} \triangleq \mathbf{I}+T(\theta)-\widetilde{P}_{2}(\theta), \\
\Gamma_{3} \triangleq R^{T}(\theta) C_{1}^{T}(\theta)+\widetilde{C}_{K}^{T}(\theta) D_{1}^{T}(\theta), \\
\Gamma_{4} \triangleq S(\theta)+S^{T}(\theta)-\widetilde{P}_{3}(\theta), \\
\Gamma_{5} \triangleq C_{1}^{T}(\theta)+C_{2}^{T}(\theta) \widetilde{D}_{K}^{T}(\theta) D_{1}^{T}(\theta), \\
\Xi_{1} \triangleq \operatorname{Her}\left\{A(\theta) R(\theta)+B_{2}(\theta) \widetilde{C}_{K}(\theta)\right\}+\widetilde{P}_{t 1}(\theta), \\
\Xi_{2} \triangleq A(\theta)+B_{2}(\theta) \widetilde{D}_{K}(\theta) C_{2}(\theta)+\widetilde{A}_{K}^{T}(\theta)+\widetilde{P}_{t 2}(\theta), \\
\Xi_{3} \triangleq \widetilde{P}_{1}(\theta)-R^{T}(\theta)+\varepsilon A(\theta) R(\theta)+\varepsilon B_{2}(\theta) \widetilde{C}_{K}(\theta),  \tag{41}\\
\Xi_{4} \triangleq \widetilde{P}_{2}(\theta)-T(\theta)+\varepsilon A(\theta)+\varepsilon B_{2}(\theta) \widetilde{D}_{K}(\theta) C_{2}(\theta), \\
\Xi_{5} \triangleq \operatorname{Her}\left\{S^{T}(\theta) A(\theta)+\widetilde{B}_{K}(\theta) C_{2}(\theta)\right\}+\widetilde{P}_{t 3}(\theta), \\
\Xi_{6} \triangleq \widetilde{P}_{2}^{T}(\theta)-\mathbf{I}+\varepsilon \widetilde{A}_{K}(\theta), \\
\Xi_{7} \triangleq \widetilde{P}_{3}(\theta)-S+\varepsilon S^{T}(\theta) A(\theta)+\varepsilon \widetilde{B}_{K}(\theta) C_{2}(\theta), \\
\Xi_{8} \triangleq-\varepsilon R(\theta)-\varepsilon R^{T}(\theta), \\
\Xi_{9} \triangleq-\varepsilon \mathbf{I}-\varepsilon T(\theta), \\
\Xi_{10} \triangleq-\varepsilon S(\theta)-\varepsilon S^{T}(\theta)
\end{gather*}
$$

In this case, a desired gain-scheduled $\mathscr{H}_{2}$ dynamic output feedback controller $\mathscr{K}_{\mathscr{D} O F}$ in the form of (8) can be obtained by solving the following equations:

$$
\begin{aligned}
\widetilde{A}_{K}(\theta)= & S^{T}(\theta) A(\theta) R(\theta)+G^{T}(\theta) A_{K}(\theta) F(\theta) \\
& +G^{T}(\theta) B_{K}(\theta) C_{2}(\theta) R(\theta)
\end{aligned}
$$

$$
\begin{gather*}
+S^{T}(\theta) B_{2}(\theta) C_{K}(\theta) F(\theta) \\
+S^{T}(\theta) B_{2}(\theta) D_{K}(\theta) C_{2}(\theta) R(\theta), \\
\widetilde{B}_{K}(\theta)=G^{T}(\theta) B_{K}(\theta)+S^{T}(\theta) B_{2}(\theta) D_{K}(\theta) \\
\widetilde{C}_{K}(\theta)=C_{K}(\theta) F(\theta)+D_{K}(\theta) C_{2}(\theta) R(\theta), \\
\widetilde{D}_{K}(\theta)=D_{K}(\theta) \tag{42}
\end{gather*}
$$

where $F(\theta)$ and $G(\theta)$ can be obtained by taking any full-rank factorization of $F^{T}(\theta) G(\theta)=T(\theta)-R^{T}(\theta) S(\theta)$.

Proof. It can be seen from Lemma 4 that the matrix $W(\theta)$ is nonsingular if (15) holds since $W(\theta)+W^{T}(\theta)-P(\theta)>0$ and $P(\theta)>0$. For notational simplicity, we denote $V(\theta) \triangleq W^{-1}(\theta)$ in the following. Then $W(\theta)$ and $V(\theta)$ can be partitioned as follows:

$$
\begin{align*}
& W(\theta) \triangleq\left[\begin{array}{ll}
W_{1}(\theta) & W_{2}(\theta) \\
W_{4}(\theta) & W_{3}(\theta)
\end{array}\right],  \tag{43}\\
& V(\theta) \triangleq W^{-1}(\theta)=\left[\begin{array}{ll}
V_{1}(\theta) & V_{2}(\theta) \\
V_{4}(\theta) & V_{3}(\theta)
\end{array}\right] .
\end{align*}
$$

Without loss of generality, we assume that $W_{4}(\theta)$ and $V_{4}(\theta)$ are nonsingular (if not, $W_{4}(\theta)$ and $V_{4}(\theta)$ may be perturbed, respectively, by matrices $\Delta W_{4}(\theta)$ and $\Delta V_{4}(\theta)$ with sufficiently small norms such that $W_{4}(\theta)+\Delta W_{4}(\theta)$ and $V_{4}(\theta)+\Delta V_{4}(\theta)$ are nonsingular and satisfy (15)). Then we can define the following nonsingular matrices:

$$
\mathscr{J}_{W}(\theta) \triangleq\left[\begin{array}{ll}
W_{1}(\theta) & \mathbf{I}  \tag{44}\\
W_{4}(\theta) & \mathbf{0}
\end{array}\right], \quad \mathscr{J}_{V}(\theta) \triangleq\left[\begin{array}{cc}
\mathbf{I} & V_{1}(\theta) \\
\mathbf{0} & V_{4}(\theta)
\end{array}\right]
$$

Note that

$$
\begin{gather*}
W(\theta) \mathscr{J}_{V}(\theta)=\mathscr{J}_{W}(\theta), \quad V(\theta) \mathscr{J}_{W}(\theta)=\mathscr{J}_{V}(\theta), \\
W_{1}(\theta) V_{1}(\theta)+W_{2}(\theta) V_{4}(\theta)=\mathbf{I} . \tag{45}
\end{gather*}
$$

Performing congruence transformations to (15) and (16) by matrices $\operatorname{diag}\left\{\mathscr{J}_{V}(\theta), \mathbf{I}\right\}$ and $\operatorname{diag}\left\{\mathscr{J}_{V}(\theta), \mathscr{J}_{V}(\theta), \mathbf{I}\right\}$, respectively, we have

$$
\begin{gather*}
{\left[\begin{array}{c}
\operatorname{Her}\left\{\mathscr{J}_{V}^{T}(\theta) \mathscr{J}_{W}(\theta)\right\}-\widetilde{P}(\theta) \\
* \\
\mathscr{J}_{W}^{T}(\theta) C_{\mathrm{cl}}^{T}(\theta) \\
\Pi(\theta)
\end{array}\right]>0}  \tag{46}\\
{\left[\begin{array}{ccc}
\Delta_{1} & \Delta_{2} & \mathscr{J}_{V}^{T}(\theta) B_{\mathrm{cl}}(\theta) \\
* & \Delta_{3} & \mathbf{0} \\
* & * & -\mathbf{I}
\end{array}\right]<0} \tag{47}
\end{gather*}
$$

where

$$
\begin{gather*}
\Delta_{1} \triangleq \operatorname{Her}\{\Phi(\theta)\}+\widetilde{P}_{t}(\theta), \\
\Delta_{2} \triangleq \widetilde{P}(\theta)-\mathscr{J}_{W}^{T}(\theta) \mathscr{J}_{V}(\theta)+\varepsilon \Phi_{1}(\theta), \\
\Delta_{3} \triangleq-\varepsilon \operatorname{Her}\left\{\mathscr{J}_{V}^{T}(\theta) \mathscr{J}_{W}(\theta)\right\}, \\
\Phi(\theta) \triangleq \mathscr{J}_{V}^{T}(\theta) A_{\mathrm{cl}}(\theta) \mathscr{J}_{W}(\theta),  \tag{48}\\
\widetilde{P}(\theta) \triangleq\left[\begin{array}{cc}
\widetilde{P}_{1}(\theta) & \widetilde{P}_{2}(\theta) \\
* & \widetilde{P}_{3}(\theta)
\end{array}\right]=\mathscr{J}_{V}^{T}(\theta) P(\theta) \mathscr{J}_{V}(\theta)>0, \\
\widetilde{P}_{t}(\theta) \triangleq\left[\begin{array}{cc}
\widetilde{P}_{t 1}(\theta) & \widetilde{P}_{t 2}(\theta) \\
* & \widetilde{P}_{t 3}(\theta)
\end{array}\right]=\mathscr{J}_{V}^{T}(\theta) P_{t}(\theta) \mathscr{J}_{V}(\theta)>0 .
\end{gather*}
$$

Define the following matrices:

$$
\begin{gather*}
R(\theta) \triangleq W_{1}(\theta), \quad S(\theta) \triangleq V_{1}(\theta)  \tag{49}\\
T(\theta) \triangleq W_{1}^{T}(\theta) V_{1}(\theta)+W_{4}^{T}(\theta) V_{4}(\theta) \\
\widetilde{A}_{K}(\theta) \triangleq V_{1}^{T}(\theta) A(\theta) W_{1}(\theta)+V_{4}^{T}(\theta) A_{K}(\theta) W_{4}(\theta) \\
+V_{4}^{T}(\theta) B_{K}(\theta) C_{2}(\theta) W_{1}(\theta) \\
+V_{1}^{T}(\theta) B_{2}(\theta) C_{K}(\theta) W_{4}(\theta) \\
+V_{1}^{T}(\theta) B_{2}(\theta) D_{K}(\theta) C_{2}(\theta) W_{1}(\theta)  \tag{50}\\
\widetilde{B}_{K}(\theta) \triangleq V_{4}^{T}(\theta) B_{K}(\theta)+V_{1}^{T}(\theta) B_{2}(\theta) D_{K}(\theta) \\
\widetilde{C}_{K}(\theta) \triangleq C_{K}(\theta) W_{4}(\theta)+D_{K}(\theta) C_{2}(\theta) W_{1}(\theta) \\
\widetilde{D}_{K}(\theta) \triangleq D_{K}(\theta)
\end{gather*}
$$

Then, substituting (12) into (46) and (47) and considering (44) and

$$
\begin{gather*}
\mathscr{J}_{V}^{T}(\theta) \mathscr{J}_{W}(\theta)=\left[\begin{array}{cc}
R(\theta) & \mathbf{I} \\
T^{T}(\theta) & S^{T}(\theta)
\end{array}\right] \\
\mathscr{J}_{V}^{T}(\theta) A_{\mathrm{cl}}(\theta) \mathscr{J}_{W}(\theta)=\left[\begin{array}{cc}
\mathscr{A}_{1} & \mathscr{A}_{2} \\
\widetilde{A}_{K}(\theta) & \mathscr{A}_{3}
\end{array}\right], \\
\mathscr{J}_{V}^{T}(\theta) B_{\mathrm{cl}}(\theta)=\left[\begin{array}{c}
B_{1}(\theta) \\
S^{T}(\theta) B_{1}(\theta)
\end{array}\right],  \tag{51}\\
\mathscr{J}_{W}^{T}(\theta) C_{\mathrm{cl}}^{T}(\theta)=\left[\begin{array}{c}
R^{T}(\theta) C_{1}^{T}(\theta)+\widetilde{C}_{K}^{T}(\theta) D_{1}^{T}(\theta) \\
C_{1}^{T}(\theta)+C_{2}^{T}(\theta) \widetilde{D}_{K}^{T}(\theta) D_{1}^{T}(\theta)
\end{array}\right],
\end{gather*}
$$

where

$$
\begin{align*}
& \mathscr{A}_{1} \triangleq A(\theta) R(\theta)+B_{2}(\theta) \widetilde{C}_{K}(\theta) \\
& \mathscr{A}_{2} \triangleq A(\theta)+B_{2}(\theta) \widetilde{D}_{K}(\theta) C_{2}(\theta)  \tag{52}\\
& \mathscr{A}_{3} \triangleq S^{T}(\theta) A(\theta)+\widetilde{B}_{K}(\theta) C_{2}(\theta)
\end{align*}
$$

we obtain that (39) and (40) hold.
Define the following matrices:

$$
\begin{equation*}
F(\theta) \triangleq W_{4}(\theta), \quad G(\theta) \triangleq V_{4}(\theta) . \tag{53}
\end{equation*}
$$

It is noted that

$$
\begin{equation*}
F^{T}(\theta) G(\theta)=T(\theta)-R^{T}(\theta) S(\theta) \tag{54}
\end{equation*}
$$

Finally, from (50), (47), and (54), (10) holds. This completes the proof.

Remark 11. Theorem 10 casts the nonlinear matrix inequality condition of dynamic output feedback control design based on Lemma 4 into an LMI condition by using linearization procedures. Based on these procedures, the desired LPV controllers can be constructed by using the obtained matrix functions $\Pi(\theta), \widetilde{P}(\theta), \widetilde{P}_{t}(\theta), R(\theta), S(\theta), T(\theta), \widetilde{A}_{K}(\theta), \widetilde{B}_{K}(\theta)$, $\widetilde{C}_{K}(\theta)$, and $\widetilde{D}_{K}(\theta)$. However, these LMI conditions of testing the feasibility in Theorem 10 are infinite-dimensional constraints in terms of the parameter $\theta(t)$. Therefore, these conditions still cannot be implemented since they are not convex in the parameter $\theta(t)$. It is noted that if we set $\Pi(\theta) \equiv$ $\Pi, \widetilde{P}(\theta) \equiv \widetilde{P}, \widetilde{P}_{t}(\theta) \equiv \widetilde{P}_{t}, R(\theta) \equiv R, S(\theta) \equiv S, T(\theta) \equiv T$, $\widetilde{A}_{K}(\theta) \equiv \widetilde{A}_{K}, \widetilde{B}_{K}(\theta) \equiv \widetilde{B}_{K}, \widetilde{C}_{K}(\theta) \equiv \widetilde{C}_{K}$, and $\widetilde{D}_{K}(\theta) \equiv \widetilde{D}_{K}$, we will readily obtain a new gain-scheduled $\mathscr{H}_{2}$ output feedback control result in a quadratic framework. Then, we can obtain the following corollary based on Theorem 10 immediately.

Corollary 12. Consider the system $\mathcal{S}$ in (1). Given a scalar $\gamma>0$ and a sufficiently small positive scalar $\varepsilon>0$, then there exists a gain-scheduled $\mathscr{H}_{2}$ dynamic output feedback controller $\mathscr{K}_{\text {DOFF }}$ in the form of (8) such that the resulting closed-loop system $\mathscr{C}_{\mathscr{D O F}}$ in (11) is exponentially stable with a prescribed $\mathscr{H}_{2}$ disturbance attenuation level $\gamma$ if there exist matrices $\Pi>$ $0, \widetilde{P} \triangleq\left[\begin{array}{cc}\widetilde{P}_{1} & \widetilde{P}_{2} \\ * & \widetilde{P}_{3}\end{array}\right]>0, \widetilde{P}_{t} \triangleq\left[\begin{array}{ccc}\widetilde{P}_{t 1} & \widetilde{P}_{t 2} \\ * & \widetilde{P}_{t 3}\end{array}\right]>0, R, S, T, \widetilde{A}_{K}, \widetilde{B}_{K}, \widetilde{C}_{K}$, and $\widetilde{D}_{K}$ satisfying

$$
\begin{gather*}
\operatorname{Tr}(\Pi)<\gamma, \\
\Phi_{i} \triangleq\left[\begin{array}{ccc}
\Gamma_{1} & \Gamma_{2} & \Gamma_{3} \\
* & \Gamma_{4} & \Gamma_{5} \\
* & * & \Pi
\end{array}\right]>0, \quad i=1, \ldots, N, \\
\Psi_{i} \triangleq\left[\begin{array}{ccccc}
\Xi_{1} & \Xi_{2} & \Xi_{3} & \Xi_{4} & B_{1 i} \\
* & \Xi_{5} & \Xi_{6} & \Xi_{7} & S^{T} B_{1 i} \\
* & * & \Xi_{8} & \Xi_{9} & \mathbf{0} \\
* & * & * & \Xi_{10} & \mathbf{0} \\
* & * & * & * & -\mathbf{I}
\end{array}\right]<0, \quad i=1, \ldots, N, \tag{55}
\end{gather*}
$$

where

$$
\begin{gathered}
\Gamma_{1} \triangleq R+R^{T}-\widetilde{P}_{1}, \\
\Gamma_{2} \triangleq \mathbf{I}+T-\widetilde{P}_{2}, \\
\Gamma_{3} \triangleq R^{T} C_{1 i}^{T}+\widetilde{C}_{K}^{T} D_{1 i}^{T}, \\
\Gamma_{4} \triangleq S+S^{T}-\widetilde{P}_{3}, \\
\Gamma_{5} \triangleq C_{1 i}^{T}+C_{2 i}^{T} \widetilde{D}_{K}^{T} D_{1 i}^{T}, \\
\Xi_{1} \triangleq \operatorname{Her}\left\{A_{i} R+B_{2 i} \widetilde{C}_{K}\right\}+\widetilde{P}_{t 1}, \\
\Xi_{2} \triangleq A_{i}+B_{2 i} \widetilde{D}_{K} C_{2 i}+\widetilde{A}_{K}^{T}+\widetilde{P}_{t 2},
\end{gathered}
$$

$$
\begin{gather*}
\Xi_{3} \triangleq \widetilde{P}_{1}-R^{T}+\varepsilon A_{i} R+\varepsilon B_{2 i} \widetilde{C}_{K}, \\
\Xi_{4} \triangleq \widetilde{P}_{2}-T+\varepsilon A_{i}+\varepsilon B_{2 i} \widetilde{D}_{K} C_{2 i}, \\
\Xi_{5} \triangleq \operatorname{Her}\left\{S^{T} A_{i}+\widetilde{B}_{K} C_{2 i}\right\}+\widetilde{P}_{t 3}, \\
\Xi_{6} \triangleq \widetilde{P}_{2}^{T}-\mathbf{I}+\varepsilon \widetilde{A}_{K} \\
\Xi_{7} \triangleq \widetilde{P}_{3}-S+\varepsilon S^{T} A_{i}+\varepsilon \widetilde{B}_{K} C_{2 i} \\
\Xi_{8} \triangleq-\varepsilon R-\varepsilon R^{T} \\
\Xi_{9} \triangleq-\varepsilon \mathbf{I}-\varepsilon T \\
\Xi_{10} \triangleq-\varepsilon S-\varepsilon S^{T} \tag{56}
\end{gather*}
$$

In this case, a desired gain-scheduled $\mathscr{H}_{2}$ dynamic output feedback controller $\mathscr{K}_{\mathscr{D} G F}$ in the form of (8) can be obtained by solving the following equations:

$$
\begin{align*}
\widetilde{A}_{K}= & S^{T} A(\theta) R+G^{T} A_{K}(\theta) F \\
& +G^{T} B_{K}(\theta) C_{2}(\theta) R+S^{T} B_{2}(\theta) C_{K}(\theta) F \\
& +S^{T} B_{2}(\theta) D_{K}(\theta) C_{2}(\theta) R \\
\widetilde{B}_{K}= & G^{T} B_{K}(\theta)+S^{T} B_{2}(\theta) D_{K}(\theta),  \tag{57}\\
\widetilde{C}_{K}= & C_{K}(\theta) F+D_{K}(\theta) C_{2}(\theta) R, \\
\widetilde{D}_{K}= & D_{K}(\theta)
\end{align*}
$$

where $F$ and $G$ can be obtained by taking any full-rank factorization of $F^{T} G=T-R^{T} S$.

Although Corollary 12 gives a finite-dimensional LMI approach to design a gain-scheduled $\mathscr{H}_{2}$ dynamic output feedback controller $\mathscr{K}_{\mathscr{D} \mathcal{F}}$ in the form of (8), all vertices of the polytope share a common Lyapunov function, which may lead to the conservative result. In order to reduce the conservativeness and derive a solvable condition, we develop a new PLMI condition in the following. Similar to the introduction of the convexification method used in Theorem 7, a finitedimensional PLMI condition that depends on the vertices of the polytope $\mathscr{R}$ is presented in the following theorem, which can be efficiently solved by using standard numerical software.

Theorem 13. Consider the system $\mathcal{S}$ in (1). Given a scalar $\gamma>0$ and a sufficiently small positive scalar $\varepsilon>0$, then an admissible gain-scheduled $\mathscr{H}_{2}$ dynamic output feedback controller in the form of $\mathscr{K}_{\mathscr{D} \mathscr{F}}$ in (8) exists if there exist matrices $\Pi_{i}>0, \widetilde{P}_{i} \triangleq$ $\left[\begin{array}{cc}\widetilde{P}_{1 i} & \widetilde{P}_{2 i} \\ * & \widetilde{P}_{3 i}\end{array}\right]>0, \widetilde{P}_{t k} \triangleq\left[\begin{array}{cc}\widetilde{P}_{t 1 k} & \widetilde{P}_{t 2 k} \\ * & \widetilde{P}_{t 3 k}\end{array}\right]>0, R_{i}, S_{i}, T_{i}, \widetilde{A}_{K i}, \widetilde{B}_{K i}, \widetilde{C}_{K i}$, $\widetilde{D}_{K i}, \Lambda_{i j}$, and $\Upsilon_{i j k}$ satisfying

$$
\begin{equation*}
\operatorname{Tr}\left(\Pi_{i}\right)<\gamma, \quad i=1, \ldots, N \tag{58}
\end{equation*}
$$

$$
\begin{gather*}
\Phi_{i j}+\Phi_{j i}-\Lambda_{i j}-\Lambda_{i j}^{T} \geqslant 0, \quad 1 \leqslant i<j \leqslant N  \tag{59}\\
\Psi_{i j k}+\Psi_{j i k}-\Upsilon_{i j k}-\Upsilon_{i j k}^{T} \leqslant 0  \tag{60}\\
1 \leqslant i<j \leqslant N, \quad k=1, \ldots, N \\
\Lambda \triangleq\left[\begin{array}{cccc}
\Phi_{11} & \Lambda_{12} & \cdots & \Lambda_{1 N} \\
* & \Phi_{22} & \cdots & \Lambda_{2 N} \\
* & * & \ddots & \vdots \\
* & * & * & \Phi_{N N}
\end{array}\right]>0  \tag{61}\\
\Upsilon \triangleq\left[\begin{array}{cccc}
\Psi_{11 k} & \Upsilon_{12 k} & \cdots & \Upsilon_{1 N k} \\
* & \Psi_{22 k} & \cdots & \Upsilon_{2 N k} \\
* & * & \ddots & \vdots \\
* & * & * & \Psi_{N N k}
\end{array}\right]<0 \tag{62}
\end{gather*}
$$

where

$$
\begin{gathered}
\Phi_{i j} \triangleq\left[\begin{array}{cccc}
R_{i}+R_{i}^{T}-\widetilde{P}_{1 i} & \mathbf{I}+T_{i}-\widetilde{P}_{2 i} & \Theta_{1} \\
* & & S_{i}+S_{i}^{T}-\widetilde{P}_{3 i} & \Theta_{2} \\
* & & * & \Pi_{i}
\end{array}\right], \\
\Psi_{i j k} \triangleq\left[\begin{array}{ccccc}
\Sigma_{1} & \Sigma_{2} & \Sigma_{3} & \Sigma_{4} & B_{1 j} \\
* & \Sigma_{5} & \Sigma_{6} & \Sigma_{7} & S_{i}^{T} B_{1 j} \\
* & * & \Sigma_{8} & \Sigma_{9} & \mathbf{0} \\
* & * & * & \Sigma_{10} & \mathbf{0} \\
* & * & * & * & -\mathbf{I}
\end{array}\right], \\
\Theta_{1} \triangleq R_{i}^{T} C_{1 j}^{T}+\widetilde{C}_{K i}^{T} D_{1 j}^{T}, \\
\Theta_{2} \triangleq C_{1 j}^{T}+C_{2 j}^{T} \widetilde{D}_{K i}^{T} D_{1 j}^{T}, \\
\Sigma_{2} \triangleq \operatorname{Her}_{2}\left\{A_{j} R_{i}+B_{2 j} \widetilde{C}_{K i}\right\}+\widetilde{P}_{t 1 k}+B_{2 j} \widetilde{D}_{K i} C_{2 j}+\widetilde{A}_{K i}^{T}+\widetilde{P}_{t 2 k}, \\
\Sigma_{3} \triangleq \widetilde{P}_{1 i}-R_{i}^{T}+\varepsilon A_{j} R_{i}+\varepsilon B_{2 j} \widetilde{C}_{K i}, \\
\Sigma_{4} \triangleq \widetilde{P}_{2 i}-T_{i}+\varepsilon A_{j}+\varepsilon B_{2 j} \widetilde{D}_{K i} C_{2 j}, \\
\Sigma_{5} \triangleq \operatorname{Her}\left\{S_{i}^{T} A_{j}+\widetilde{B}_{K i} C_{2 j}\right\}+\widetilde{P}_{t 3 k}, \\
\Sigma_{6} \triangleq \widetilde{P}_{2 i}^{T}-\mathbf{I}+\varepsilon \widetilde{A}_{K i}, \\
\Sigma_{7} \triangleq \widetilde{P}_{3 i}-S_{i}^{T}+\varepsilon S_{i}^{T} A_{j}+\varepsilon \widetilde{B}_{K i} C_{2 j}, \\
\Sigma_{8} \triangleq-\varepsilon R_{i}-\varepsilon R_{i}^{T}, \\
\Sigma_{9} \triangleq-\varepsilon \mathbf{I}-\varepsilon T_{i}, \\
\Sigma_{10} \triangleq-\varepsilon S_{i}-\varepsilon S_{i}^{T} .
\end{gathered}
$$

Moreover, under the above conditions, the matrix functions for an admissible gain-scheduled $\mathscr{H}_{2}$ dynamic output feedback controller $\mathscr{K}_{\mathscr{D O F}}$ in (8) are given by solving the following equations:

$$
\begin{aligned}
\widetilde{A}_{K}(\theta)= & S^{T}(\theta) A(\theta) R(\theta)+G^{T}(\theta) A_{K}(\theta) F(\theta) \\
& +G^{T}(\theta) B_{K}(\theta) C_{2}(\theta) R(\theta)
\end{aligned}
$$

$$
\begin{gather*}
+S^{T}(\theta) B_{2}(\theta) C_{K}(\theta) F(\theta) \\
+S^{T}(\theta) B_{2}(\theta) D_{K}(\theta) C_{2}(\theta) R(\theta), \\
\widetilde{B}_{K}(\theta)=G^{T}(\theta) B_{K}(\theta)+S^{T}(\theta) B_{2}(\theta) D_{K}(\theta), \\
\widetilde{C}_{K}(\theta)=C_{K}(\theta) F(\theta)+D_{K}(\theta) C_{2}(\theta) R(\theta), \\
\widetilde{D}_{K}(\theta)=D_{K}(\theta), \tag{64}
\end{gather*}
$$

where $R(\theta)=\sum_{i=1}^{N} \alpha_{i} R_{i}, S(\theta)=\sum_{i=1}^{N} \alpha_{i} S_{i}, T(\theta)=\sum_{i=1}^{N} \alpha_{i} T_{i}$, $\widetilde{A}_{K}(\theta)=\sum_{i=1}^{N} \alpha_{i} \widetilde{A}_{K i}, \widetilde{B}_{K}(\theta)=\sum_{i=1}^{N} \alpha_{i} \widetilde{B}_{K i}, \widetilde{C}_{K}(\theta)=$ $\sum_{i=1}^{N} \alpha_{i} \widetilde{\mathrm{C}}_{K i}, \widetilde{D}_{K}(\theta)=\sum_{i=1}^{N} \alpha_{i} \widetilde{D}_{K i}$, and $F(\theta)$ and $G(\theta)$ can be obtained by taking any full-rank factorization $F^{T}(\theta) G(\theta)=$ $T(\theta)-R^{T}(\theta) S(\theta)$.

Proof. From Theorem 10, an admissible gain-scheduled $\mathscr{H}_{2}$ output feedback controller in the form of $\mathscr{K}_{\mathscr{D} \mathscr{F}}$ in (8) exists if there exist matrices $\Pi(\theta)>0, \widetilde{P}(\theta)>0, \widetilde{P}_{t}(\theta)>0$, $R(\theta), S(\theta), T(\theta), \widetilde{A}_{K}(\theta), \widetilde{B}_{K}(\theta), \widetilde{C}_{K}(\theta)$, and $\widetilde{D}_{K}(\theta)$ and a scalar $\varepsilon>0$ satisfying (38)-(40). Now, assume that the above matrix functions are of the following form:

$$
\begin{align*}
& \widetilde{P}(\theta)=\sum_{i=1}^{N} \alpha_{i} \widetilde{P}_{i}=\sum_{i=1}^{N} \alpha_{i}\left[\begin{array}{cc}
\widetilde{P}_{1 i} & \widetilde{P}_{2 i} \\
* & \widetilde{P}_{3 i}
\end{array}\right], \\
& \widetilde{P}_{t}(\theta)=\sum_{k=1}^{N} \beta_{k} \widetilde{P}_{t k}=\sum_{k=1}^{N} \beta_{k}\left[\begin{array}{cc}
\widetilde{P}_{t 1 k} & \widetilde{P}_{t} \\
* & \widetilde{P}_{t 3 k}
\end{array}\right], \\
& \Pi(\theta)=\sum_{i=1}^{N} \alpha_{i} \Pi_{i}, \quad R(\theta)=\sum_{i=1}^{N} \alpha_{i} R_{i}, \\
& S(\theta)=\sum_{i=1}^{N} \alpha_{i} S_{i}, \quad T(\theta)=\sum_{i=1}^{N} \alpha_{i} T_{i},  \tag{65}\\
& \widetilde{A}_{K}(\theta)=\sum_{i=1}^{N} \alpha_{i} \widetilde{A}_{K i}, \quad \widetilde{B}_{K}(\theta)=\sum_{i=1}^{N} \alpha_{i} \widetilde{B}_{K i}, \\
& \widetilde{C}_{K}(\theta)=\sum_{i=1}^{N} \alpha_{i} \widetilde{C}_{K i}, \quad \widetilde{D}_{K}(\theta)=\sum_{i=1}^{N} \alpha_{i} \widetilde{D}_{K i} .
\end{align*}
$$

Then, with (65), we rewrite $\Phi(\theta)$ and $\Psi(\theta)$ in (39)-(40) as

$$
\begin{aligned}
\Phi(\theta) & =\sum_{j=1}^{N} \sum_{i=1}^{N} \alpha_{i} \alpha_{j} \Phi_{i j} \\
& =\sum_{i=1}^{N} \alpha_{i}^{2} \Phi_{i i}+\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \alpha_{i} \alpha_{j}\left(\Phi_{i j}+\Phi_{j i}\right),
\end{aligned}
$$

$\Psi(\theta)$

$$
=\sum_{k=1}^{N} \sum_{j=1}^{N} \sum_{i=1}^{N} \alpha_{i} \alpha_{j} \beta_{k} \Phi_{i j k}
$$

$$
\begin{equation*}
=\sum_{k=1}^{N} \beta_{k}\left(\sum_{i=1}^{N} \alpha_{i}^{2} \Psi_{i i k}+\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \alpha_{i} \alpha_{j}\left(\Psi_{i j k}+\Psi_{j i k}\right)\right) \tag{66}
\end{equation*}
$$

On the other hand, (59)-(60) are equivalent to

$$
\begin{gather*}
\Phi_{i j}+\Phi_{j i} \geqslant \Lambda_{i j}+\Lambda_{i j}^{T}  \tag{67}\\
\Psi_{i j k}+\Psi_{j i k} \leqslant \Upsilon_{i j k}+\Upsilon_{i j k}^{T}
\end{gather*}
$$

where $1 \leqslant i<j \leqslant N$ and $k=1, \ldots, N$. Then, from (66)-(67), we have

$$
\begin{align*}
\Phi(\theta) & \geqslant \sum_{i=1}^{N} \alpha_{i}^{2} \Phi_{i i}+\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \alpha_{i} \alpha_{j}\left(\Lambda_{i j}+\Lambda_{i j}^{T}\right)=\eta^{T} \Lambda \eta \\
\Psi(\theta) & \leqslant \sum_{k=1}^{N} \beta_{k}\left(\sum_{i=1}^{N} \alpha_{i}^{2} \Psi_{i i k}+\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \alpha_{i} \alpha_{j}\left(\Upsilon_{i j k}+\Upsilon_{i j k}^{T}\right)\right) \\
& =\eta^{T} \Upsilon \eta \tag{68}
\end{align*}
$$

where $\eta \triangleq\left[\alpha_{1} I, \alpha_{2} I, \ldots, \alpha_{N} I\right]^{T}$. Inequalities (61)-(62) guarantee $\Phi(\theta)>0$ and $\Psi(\theta)<0$, respectively. As to (58), since $\Pi(\theta)=\sum_{i=1}^{N} \alpha_{i} \Pi_{i}$ and $\operatorname{Tr}(\Pi(\theta))=\sum_{i=1}^{N} \alpha_{i} \operatorname{Tr}\left(\Pi_{i}\right)$, if (58) satisfies, we can get (38). By substituting the matrices defined in (65) into (10), we readily obtain (13), and the proof is completed.

Remark 14. From the proof of Theorem 13, it can be seen that the Lyapunov function-based matrix for $\mathscr{H}_{2}$ performance objective is also parameter dependent; that is, $P(\theta)$ takes the form of $P(\theta)=\sum_{i=1}^{N} \alpha_{i} P_{i}$. Notably, here in Theorem 13, we do not set any matrix variable to be constant for the whole polytope domain, and therefore, Theorem 13 has the potential to yield less conservative results in the applications of gainscheduled $\mathscr{H}_{2}$ LPV control design than the ones presented in Corollary 12. Similar to the case of state feedback control design, the gain-scheduled $\mathscr{H}_{2}$ dynamic output feedback controller synthesis problem can be also cast into a finitedimensional convex optimization problem as follows:

Minimize $\quad \gamma$
subject to (58)-(62) for given scalar $\varepsilon$.
Remark 15. From the results in the existing literature, it is known that the definition of quadratic Lyapunov function is simple, and the computation cost in the design procedure is low. However, the common quadratic Lyapunov functions tend to be conservative and might not exist for some highly nonlinear systems. To reduce the conservatism and to establish well-performance condition, parameter-dependent Lyapunov functions and new slack variables have been adopted in this paper. It can be seen that the quadratic Lyapunov function is a special case of PDLF. Thus, the proposed method in this paper is more general and less conservative.

Meanwhile, the PDLF-based approach also has some disadvantages: the design procedures become more complex, and the computational requirement is usually demanding. The number of PLMIs conditions in Theorems 7 and 13 increases rapidly with the number of system dimensions. Thus, a computational problem might arise for high-order systems. One effective way to solve this problem is to try to reduce the number of variables with the tradeoff between computational burden and conservativeness. For example, Theorem 13 can be replaced by Corollary 12 with increasing the conservativeness and decreasing computational burden.

## 4. Illustrative Example

In this section, we use Example 1 to show the less conservativeness of the result developed in Theorem 7. Example 2 is provided to show the effectiveness of the gain-scheduled $\mathscr{H}_{2}$ dynamic output feedback controller proposed in Theorem 13.

Example 1. Consider the following numerical example borrowed from [26, 29]. The problem is to control the yaw angles of a satellite system. The satellite system consisting of two rigid bodies joined by a flexible link has the state-space representation as follows:

$$
\begin{gather*}
\dot{x}=\left[\begin{array}{cccc}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-k & k & -f & f \\
k & -k & f & -f
\end{array}\right] x+\left[\begin{array}{l}
0 \\
0 \\
0 \\
1
\end{array}\right] w+\left[\begin{array}{l}
0 \\
0 \\
1 \\
0
\end{array}\right] u,  \tag{70}\\
z=\left[\begin{array}{llll}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right] x+\left[\begin{array}{c}
0 \\
0.01
\end{array}\right] u,
\end{gather*}
$$

where $x=\left[\begin{array}{llll}\theta_{1} & \theta_{2} & \dot{\theta}_{1} & \dot{\theta}_{2}\end{array}\right]^{T}$ and $k$ and $f$ are torque constant and viscous damping, and they vary in the following uncertainty ranges: $k \in\left[\begin{array}{cc}0.09 & 0.4\end{array}\right] ; f \in\left[\begin{array}{cc}0.0038 & 0.04\end{array}\right]$.

For this system, our purpose is to design a gain-scheduled $\mathscr{H}_{2}$ state feedback control $u(t)$ in the form of (7), such that the closed-loop system is exponentially stable with a minimized $\mathscr{H}_{2}$ disturbance attenuation level $\gamma$.

Define

$$
\begin{align*}
x & \triangleq \frac{0.4-k(t)}{0.4-0.09}, \quad y \triangleq \frac{f(t)-0.0038}{0.04-0.0038}, \\
\alpha_{1}(t) & =x y, \quad \alpha_{2}(t)=(1-x) y,  \tag{71}\\
\alpha_{3}(t) & =x(1-y), \quad \alpha_{4}(t)=(1-x)(1-y) .
\end{align*}
$$

It is easy to check that $\alpha_{i}(t), i=1, \ldots, 4$, are convex coordinates, since they satisfy $0 \leqslant \alpha_{i}(t) \leqslant 1$ and $\sum_{i=1}^{4} \alpha_{i}(t)=$ 1. It should be noted that the choice of scalar $\varepsilon$ is important to converge to minimum $\mathscr{H}_{2}$ performance [26]. In this example, the value of minimum guaranteed $\mathscr{H}_{2}$ performance $\gamma^{*}$ is 1.0883 with fixed $\varepsilon=0.11$ and 1.4156 with fixed $\varepsilon=0.5$ by the method in [26], and 0.8892 with fixed $\varepsilon=0.11$ and 1.0531 with fixed $\varepsilon=0.5$ by using Theorem 7. Table 1 shows the minimum $\mathscr{H}_{2}$ performance and the numbers of decision variables when different methods are used. It is clearly shown in Table 1 that the guaranteed performance obtained by our

Table 1: Minimum $\mathscr{H}_{2}$ performance for different cases.

| $\varepsilon$ | 0.11 |  | 0.5 |  |
| :--- | :---: | :---: | :---: | :---: |
| Method | $[26]$ | Theorem 7 | $[26]$ | Theorem 7 |
| $\gamma^{*}$ | 1.0883 | 0.8892 | 1.4156 | 1.0531 |
| Complexity | 14 | 257 | 14 | 257 |

approach is much better than that obtained by the method in [26], which indicates the less conservativeness of the controller design result developed in this paper, even though this procedure increases some numerical complexity. From Table 1, we can also see that the smaller the value of $\varepsilon$, the better the value of $\gamma^{*}$. Although the computational complexity of Theorem 7 for this example is relatively more than the approach in [26], the reduced conservatism is significant. With the rapid development of computer technology and computational method, the computational burden problem may be solved.

Example 2. Consider a continuous-time LPV system $\mathcal{S}$ in (1) with the following matrix functions:

$$
\begin{aligned}
& A(\theta)=\left[\begin{array}{ccc}
1.5+0.5 \theta & 3 & \theta \\
-2.2+\theta & -1.8+0.5 \theta & 0.2 \theta \\
0.1 & 0.5 & -\theta
\end{array}\right] \\
& B_{1}(\theta)=\left[\begin{array}{c}
0.2 \\
0.02 \\
0.1
\end{array}\right],
\end{aligned} B_{2}(\theta)=\left[\begin{array}{c}
2 \theta \\
0.1+\theta \\
0.2
\end{array}\right],
$$

where $\theta(t)=\sin (0.2 t)$ is a time-varying parameter, $|\theta(t)| \leqslant$ 1 , and $|\dot{\theta}(t)| \leqslant 0.2$. Let the exogenous disturbance input $w(t)$ be

$$
\begin{equation*}
w(t)=\exp (-t) \sin (0.5 t), \quad t \geqslant 0 \tag{73}
\end{equation*}
$$

It can be checked that the above system with $u(t)=0$ is unstable, and the states of open-loop system are shown in Figure 1 with the initial condition given by $x(0)=$ $\left[\begin{array}{lll}-0.1 & -0.1 & 0.1\end{array}\right]^{T}$. Therefore, our purpose is to design a gainscheduled $\mathscr{H}_{2}$ dynamic output feedback control $u(t)$ in the form of (8), such that the closed-loop system is exponentially stable with a minimized $\mathscr{H}_{2}$ disturbance attenuation level $\gamma$.

To solve the synthesis problem, we transform system (72) into the polytopic form. The system matrices $A(\theta)$ and $B_{2}(\theta)$ of LPV system (72) can be expressed as

$$
\begin{equation*}
A(\theta)=\sum_{i=1}^{2} \alpha_{i} A_{i}, \quad B(\theta)=\sum_{i=1}^{2} \alpha_{i} B_{i}, \tag{74}
\end{equation*}
$$



Figure 1: States of the open-loop system.
where

$$
\begin{array}{cc}
A_{1}=\left[\begin{array}{ccc}
1 & 3 & -1 \\
-3.2 & -2.3 & -0.2 \\
0.1 & 0.5 & 1
\end{array}\right], & B_{21}=\left[\begin{array}{c}
-2 \\
-0.9 \\
0.2
\end{array}\right], \\
A_{2}=\left[\begin{array}{ccc}
2 & 3 & 1 \\
-1.2 & -1.3 & 0.2 \\
0.1 & 0.5 & -1
\end{array}\right], & B_{22}=\left[\begin{array}{c}
2 \\
1.1 \\
0.2
\end{array}\right] \tag{75}
\end{array}
$$

with

$$
\begin{equation*}
\alpha_{1}(t)=\frac{\theta_{\max }-\theta}{\theta_{\max }-\theta_{\min }}, \quad \alpha_{2}(t)=\frac{\theta-\theta_{\min }}{\theta_{\max }-\theta_{\min }} \tag{76}
\end{equation*}
$$

It is easy to check that $\alpha_{i}(t), i=1,2$, are convex coordinates, since they satisfy $0 \leq \alpha_{1}(t) \leq 1,0 \leq \alpha_{2}(t) \leq 1$, and $\alpha_{1}(t)+$ $\alpha_{2}(t)=1$. In this example, we set $\varepsilon=0.5$. Using Corollary 12, it is found that the LMIs (55) are infeasible. However, using Theorem 13 and solving the LMIs (58)-(62) by using the same standard LMI-Toolbox in the Matlab environment [29], we obtain that the LMIs constraints (58)-(62) are feasible. Furthermore, by solving the convex optimization problem of (69) in Remark 14, we obtain that the minimum achievable noise attenuation level for the gain-scheduled $\mathscr{H}_{2}$ dynamic output feedback control problem is $\gamma^{*}=0.5806$ and the corresponding matrices are as follows:

$$
\begin{aligned}
& R_{1}=\left[\begin{array}{ccc}
0.1096 & 0.0090 & 0.0292 \\
-0.1255 & 0.1156 & -0.0502 \\
0.0289 & 0.0046 & 0.0101
\end{array}\right], \\
& R_{2}=\left[\begin{array}{ccc}
0.1469 & -0.0229 & 0.0384 \\
-0.1468 & 0.1029 & 0.0219 \\
0.0045 & 0.0141 & 0.0324
\end{array}\right],
\end{aligned}
$$

$$
\left.\begin{array}{l}
S_{1}=10^{5} \times\left[\begin{array}{ccc}
1.9878 & -0.7346 & -3.8287 \\
3.8991 & 8.3985 & -9.4741 \\
-4.7517 & -0.2094 & 9.5476
\end{array}\right], \\
S_{2}=10^{6} \times\left[\begin{array}{ccc}
0.2857 & -0.1633 & -0.5386 \\
0.2507 & 0.9838 & -0.6979 \\
-0.6214 & 0.1299 & 1.2169
\end{array}\right], \\
T_{1}=\left[\begin{array}{ccc}
-1.8487 & -1.8817 & 3.3794 \\
4.9104 & 3.5711 & -7.6522 \\
-0.4836 & -0.8968 & 1.0365
\end{array}\right], \\
T_{2}=\left[\begin{array}{ccc}
1.8140 & 0.0634 & -1.0335 \\
-2.0647 & -0.4457 & 3.1531 \\
0.3720 & 0.8677 & 0.3924
\end{array}\right], \\
\widetilde{A}_{K 1}=\left[\begin{array}{ccc}
-6.4403 & -7.0131 & -2.0779 \\
-3.5879 & -4.9302 & -0.1886 \\
11.6185 & 11.0381 & 2.1956
\end{array}\right], \\
\widetilde{A}_{K 2}=\left[\begin{array}{ccc}
-0.7134 & 2.3105 & -3.0523 \\
-0.8594 & 0.6710 & -3.0487 \\
-3.4059 & -6.9564 & 1.4208
\end{array}\right], \\
\widetilde{B}_{K 1}=10^{6} \times\left[\begin{array}{ccc}
0.8267 & 0.9623 \\
0.3636 & 2.0002 \\
-1.7249 & -2.3245
\end{array}\right], \\
\widetilde{B}_{K 2}=10^{6} \times\left[\begin{array}{ccc}
-0.0142 & -0.4723 \\
0.1378 & 1.1325 \\
-0.0000 & 0.7169
\end{array}\right], \\
\widetilde{C}_{K 1}=[-0.0118
\end{array}-0.0185-0.0224\right],
$$

Setting $F(\theta)=I_{3}$, we obtain $G(\theta)=T(\theta)-R^{T}(\theta) S(\theta)$ by the process of proof in Theorem 10. Therefore, from (13) and Theorem 13, the matrices $\left(A_{K}(\theta), B_{K}(\theta), C_{K}(\theta), D_{K}(\theta)\right)$ for the desired $\mathscr{H}_{2}$ dynamic output feedback controller $\mathscr{K}_{\mathscr{O G F}}$ in (8) can be obtained by the Matlab symbolic computation.

Figures 2 and 3 give the state responses of the closedloop system $\mathscr{C}_{\mathscr{D} \mathscr{O F}}$ and the dynamic output feedback control system $\mathscr{K}_{\mathscr{D O F}}$, respectively. The control input $u$ in (8) is shown in Figure 4. From the above results, we can conclude that the desired stability of the closed-loop system is verified.

Based on the results of disturbance attenuation in Example 1 and the characteristic curves of Figures 2-4 in Example 2, it is shown that $\mathscr{H}_{2}$ performance can be used to capture both the response to stationary noise and the transient response of the closed-loop system.

## 5. Conclusions

In this paper, the problems of gain-scheduled $\mathscr{H}_{2}$ controller designs for continuous-time polytopic LPV systems have


Figure 2: States of the closed-loop system.


Figure 3: Controller states of $\mathscr{D} \mathscr{O}$ system.
been addressed. Based on a basis-dependent Lyapunov function and the introduction of some auxiliary slack variables, sufficient conditions for both state feedback and dynamic output feedback controller synthesis problems have been established in terms of PLMIs, which guarantee the exponential stability and a prescribed $\mathscr{H}_{2}$ performance level of the closed-loop system over the given polytope. Moreover, the controller design problems have been cast into a convex optimization problem on the basis of the polytopic characteristic of the dependent parameters and the convexification method, which can be readily solved via standard LMI Toolbox.


Figure 4: Control input.

Numerical examples have been provided to illustrate the effectiveness and advantage of the proposed design methods.

Several works may be needed in the future to improve the current results. First of all, in this paper, sufficient conditions for both state feedback and dynamic output feedback controller synthesis problems have been established in terms of PLMIs. With the increasing number of system dimension, how to solve these complicated PLMIs conditions quickly and efficiently is a good problem which should be further studied. Second, the developed results are expected to extend to the domain of practical application, such as designing the stabilized controller for the flight control system with good performances.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Review Article

# An Overview on Study of Identification of Driver Behavior Characteristics for Automotive Control 

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#### Abstract

Driver characteristics have been the research focus for automotive control. Study on identification of driver characteristics is provided in this paper in terms of its relevant research directions and key technologies involved. This paper discusses the driver characteristics based on driver's operation behavior, or the driver behavior characteristics. Following the presentation of the fundamental of the driver behavior characteristics, the key technologies of the driver behavior characteristics are reviewed in detail, including classification and identification methods of the driver behavior characteristics, experimental design and data acquisition, and model adaptation. Moreover, this paper discusses applications of the identification of the driver behavior characteristics which has been applied to the intelligent driver advisory system, the driver safety warning system, and the vehicle dynamics control system. At last, some ideas about the future work are concluded.


## 1. Introduction

In the driver-vehicle-road closed-loop system, the driver plays the role of not only the controller, but also the major evaluator of the quality of the vehicle path-following. Due to variant of driving experiences, emotions, driving preferences, and so on between drivers, the driver becomes the weakest part in the driver-vehicle-road closed-loop system. And different drivers display distinct behaviors; that is to say, every driver has his/her unique driving characteristic (also referred to as driving style). To improve the performance of the driver-vehicle-road closed-loop system, research on the driver characteristics includes (1) driver characteristics identification based on head movement and facial features, such as the eye movement recognition, which identifies the driver's driving status (fatigue/drunk/drowsy/distracted driving) and warns the driver in order to improve the active safety performance [1-3]; (2) physiological-based and psychologicalbased driver characteristics identification, which enhances the human-machine interactive performance and/or improve the driver's operation comfort [4-7]; (3) driver characteristics identification based on driver's operation behavior, which detects abnormal driving behavior and then alarm the driver,
design control method for driving comfort, and/or design the human-centered driving assistance systems [8-11]; and (4) the research on the driver characteristics based on dynamics simulations of the driver-vehicle-road closed-loop system, which is aimed at the optimal design on dynamic performance of the closed-loop system [12-16].

This paper focuses on the identification of driver characteristics based on driver's operation behavior, namely, the identification of driver behavior characteristics. In the current research, the identification is generally realized based on the measured real-time driver behavior and vehicle states, or by monitoring driver's head movements and/or facial expressions. It is believed that on the completion of the identification of driver behavior characteristics, the following task can be performed: (1) the vehicle would follow the current driver's operation, trigger the appropriate driving assistant device, in order to achieve smooth transition of the semiautonomous human-machine control modes, and to realize coordination of driver's operation and automatic control [17]; (2) the parameters of electronic control system would be automatically modified [18], or appropriate characteristics of the ideal reference models established are chosen to achieve
the ideal dynamic response of the vehicle, driver adaptive control, and personalized driving [19, 20]; (3) the systems that real-time monitor and assess the driver's driving behaviors and driving status (fatigue/drunk/drowsy/distracted driving) can early detect possible operating errors and warn the driver to avoid traffic accidents in visual, auditory, or tactile approaches $[8,21]$. Since identification of the driver behavior characteristics is of great importance to improve automotive active safety and to achieve intelligent driving, more and more researchers have been committed to studying the related fields, mainly Ford [12, 13, 18, 22, 23], Nissan Institute [17, 24, 25], Columbia University [8], Vienna University of Technology [26], Nagoya University [27-32], Tsinghua University [10, 33-36], Jilin University [20, 37-40], and Chinese University of Hong Kong [41].

This paper is organized as follows: Section 2 introduces the fundamental of the driver behavior characteristics. Section 3 reviews key technologies about driver characteristics identification. In Section 4, applications of the identification of the driver behavior characteristics are introduced. Conclusions are drawn in Section 5 and future research work is suggested.

## 2. The Driver Behavior Characteristics

In driving a vehicle, the driver makes his/her driving intentions and selects a series of operation behaviors that are most suitable for the current driving conditions. Even very simple driving intentions (long-term driving intention) can be subdivided into a series of simpler driving operation behaviors (short-term driving behavior); that is, the driving intention is achieved by a series of driving behaviors [37]. It is widely accepted that the driving behaviors vary between drivers according to their ages, genders, ethnicities, driving experiences, emotions, and so forth [17, 42-44]. Even for the same driver, driving behavior may alter from situation to situation [27], which can be attributed to the driver behavior characteristics. Wahab et al. [45] believe that the differences of each driver in the driver characteristics are due to the way drivers' subconscious mind works and responds, and the conversion from subconscious to conscious minds would also generate unique responds on how the brains work.

There are a lot of literature investigating the uniqueness of driving behaviors in vehicles and the possibility to use it for identifying the driver characteristics, with the objectives to achieve safer and personalized driving, to detect driver's abnormal operation and then alarm, to realize integration between the driver and the electronic control systems, or to build identification models of the driver behavior characteristics [22, 28, 46-53]. For example, data collected by a set of vehicle sensors can be processed by certain recognition methods to recognize a series of driving maneuvers, and the parameters of these driving maneuvers could be extracted and used to classify driver characteristics or evaluate driver's abilities [47]. Figure 1 gives the structure of neural net pattern recognition for classifying driving behavior during carfollowing condition. Closing-in is referred to the following vehicle close to the leading vehicle, and falling-behind is


Figure 1: Model by MacAdam et al., adopted from [48].


Figure 2: Overall framework of the IVDR system, adopted from [50].
referred to the leading vehicle away from the following vehicle.

Some monitoring and analysis systems are also developed and available to research driving behaviors and finally driver behavior characteristics are obtained. Tomer Toledo develops the in-vehicle data recorder (IVDR) system [49, 50] to monitor and analyze driving behaviors, as shown in Figure 2. It can be seen from the figure that this system can identify various maneuvers that occur in the measurements, and the results can be used to evaluate risk indices that show the safety on the overall trip and to classify the drivers' characteristics. Some other relevant reviews can be found in [51,52].

## 3. Key Technologies for Identification of Driver Behavior Characteristics

According to previous studies, identification of the driver behavior characteristics is modeled based on certain pattern
recognition methods by use of simulation or field test data. Therefore, the selection of pattern recognition methods, experimental design, and data acquisition are of great importance to build identification models of the driver behavior characteristics. In general, the driver behavior characteristics need to be classified before identified.
3.1. Classification of Driver Behavior Characteristics. Fuzzy control theory and K-means algorithm are generally used to cluster the feature parameters that reflect the driver behavior characteristics, in order to achieve classification of the driver behavior characteristics. To reach a feasible classification of the driver behavior characteristics, it is necessary to take the following aspects into consideration [33]. First, it is essential to select feasible metrics which can describe the driver characteristics. It should be noted that the metrics that represent the driver characteristics are chosen on purpose so that they can be expressed by use of the defined and measurable parameters. Second, the classification results are directly affected by the clustering method selected. For example, being a kind of learning method without surveillance, the K-means clustering algorithm has rapid convergence speed and concise structure; however, measurement errors and uncertainties are ignored. Third, the sample size of the parameters affects the clustering results. In general, an increase in the amount of data (or number of drivers) would improve the accuracy of the classification.

Lu et al. propose that the driving behaviors can be divided into four categories with respect to the handling limit conditions (conditions beyond the limits of tire adhesion): cautious, average, expert, and reckless [12]. A cautious driver is interpreted as someone who usually drives without frequent aggressive maneuvers, for example, rapid steering, high speed, and quickly stepping on the pedal. An average driver features driving a car with a higher level of handling risk factor (HRF, the parameter that evaluates how a driving condition is close to the handling limit) than a cautious driver does. An expert driver is defined as who can control the vehicle under a rather high level of HRF for a long duration and will not have the vehicle exceed the handling limit. A driver is considered as reckless if he/she behaves careless and unpredictable during his/her driving tour. Since these driving behaviors cannot be well defined, fuzzy control method is used to identify the four categories of drivers above, as shown in Figure 3. Besides, the author also uses zero-speed-gap velocity $v_{\text {zsg }}$ for steady-state car-following to classify driving behaviors into three types-normal, cautious, and aggressive [23]. Similarly, the drivers are characterized by use of response time and the damping ratio for transient carfollowing, which is treated as a 2 nd order system.

Raz et al. present a system for analyzing and evaluating the performance and attitude of a motor vehicle driver [47]. In this work, factor $g$, on the interval $[0,1]$, represents the weights of available maneuvers (safely executed maneuver and dangerously executed maneuver), and the combination of the two maneuvers is compared against current maneuver to find the closest value of $g$ for reproducing the original maneuver, as shown in Figure 4. Thus, $g$ represents the


Figure 3: Membership functions characterizing the four driver categories based on the handling risk factor, adopted from [12].


Figure 4: A conceptual block diagram of an arrangement for assessing driver attitude, adopted from [47].
driver's attitude for the current maneuver, and fuzzy logic method combined with statistical method is used to classify the driving maneuvers, in terms of the value of $g$. In [54, 55], a consolidated fuzzy clustering algorithm is developed and implemented to classify different car-following conditions including stable following, acceleration, approaching, braking, and opening using the pretreated data.

A common disadvantage of fuzzy-algorithm-based is that the thresholds are solely defined by the a priori knowledge of modelers, possibly with bias. A consolidated method, which can calibrate some psychological thresholds based on properties in real data, has not been developed ever [56]. One possible approach to solving the above-mentioned problem is to use supervised classification methods [57], for example, Bayesian classification, but it requires detailed a priori knowledge (e.g., probability distributions of certain variables) in different maneuvers.


Figure 5: The structure of driver classifier based on Back Propagation neural network (BPNN), adopted from [35].

K-means clustering algorithm, also known as ISODATA (Iterative Self-Organizing Data Analysis Techniques Algorithm) [58], is a widely used unsupervised clustering algorithm, which can classify multidimensional data into different groups on the basis of certain dissimilarity measures. Wang et al. use the efficient K-means clustering algorithm to classify the determinants of longitudinal driving behavior, which is acquired from 11 systems and control-related parameters, with the indicated opposite extreme values: aggressive versus prudent, unstable versus stable, risk prone versus risk infrequent, nonskillful versus skillful [33]. Specifically, according to the data sequence with car-following condition, the time to collision (TTC) data of the driver releasing the accelerator pedal and starting braking are extracted and utilized to classify drivers into three categories by clustering analysis method, namely, cautious, normal and aggressive [34].

Besides, driver classifier is designed by Zhang based on neural network [35]. In this work, the author presents that driver behavior inputs are obtained through the humanmachine interface, and the system automatically classifies the driver to achieve self-learning and the parameters automatically match for driver's abnormal behavior characteristic, as shown in Figure 5. In addition, Quintero et al. propose the driving behaviors classifier based on the existing intelligent driving diagnosis system, classifying drivers into two types, aggressive and moderate [8]. Furthermore, Ishibashi et al. develop "Driving Style Questionnaire" (DSQ) to characterize drivers [59].
3.2. Methods of Building Identification Models of Driver Behavior Characteristics. Identification of driver behavior characteristics is a pattern recognition process. Since driver behavior characteristics differ in different road surfaces, driving maneuvers, driver profiles, and vehicle dynamics [60], some requirements to the selected modeling methods are necessary: (1) offer a robust processing, that is, with the abilities to detect, approximate, and classify, and with a high reject ratio for the noise and (2) work based on learned cases. The existing attempts to model the driver behavior characteristics are dominated by models that are inspired by neural network (NN) [8, 36, 45, 61-64], Hidden Markov Model (HMM) [17, 24, 25, 38-41, 65-67], fuzzy control theory
[12, 18, 23, 68, 69], Gaussian Mixture Model (GMM) [28$32,70]$, and other models [17,21,71-74]. In the following paragraphs, the work exemplified will be reviewed in some detail.
3.2.1. Neural Network (NN) Model. The motivation of using an NN approach to behavior identification stems from the desire to conduct efficient searches of various types of driving behaviors located within relatively large amounts of stored time history data. The precision of feature parameters is crucial to the accuracy of NN . If feature parameters of different types are similar or overlapped, the model may not be capable of obtaining demanded accuracy. The pattern recognition ability of certain NN architectures is well known and lends itself well to this type of task $[8,36,45,61-64]$.

Through implementing and testing two artificial neural network (ANN) topologies: Back Propagation (BP) and Learning Vector Quantization (LVQ) [8], Quintero et al. take advantage of BP to build driver identification model. Besides, the author finds that topology of feed-forward neural network (FFNN) algorithms trained with BP is expert in designing intelligent diagnostic systems and is able to offer a strong learning ability, even with considerably less training samples [63]. The cerebellar model articulation controller (CMAC), developed by Albus, is one of NN architecture and has the advantages of fast learning and a high convergence rate [64]. Thus Wahab et al. propose the use of CMAC to model each driver's behaviors [45].
3.2.2. Hidden Markov Model (HMM). HMM is fundamentally a statistical model. Since the construction of such a model involves assuming a Markov process, it has the ability to determine the hidden states from the observable states of certain systems [75]. An HMM is capable of capturing the dynamic movement of a time series (series arrayed in chronological order), and the states of HMM can be hierarchically organized to describe both short-term and long-term driving behaviors. For example, in the case of driving a vehicle, the long-term driving behaviors represent driving intentions (e.g., accelerating/turning/following/changing lane), while the short-term driving behaviors represent driver's operation behaviors, for example, hitting the steering wheel and pressing the gas pedal [24].

Table 1: Rules for driving behavior characterization.

| If gap time <br> is | If accelerator <br> pedal rate STD <br> is | If brake <br> pedal rate STD <br> is | Then driver <br> index is |
| :--- | :---: | :---: | :---: |
| Low | Low | Low | Less <br> aggressive |
| High | Low | Low | Cautious |
| Low | High | Low | Aggressive |
| Low | Low | High | Aggressive <br> Low |
| High | High | Aggressive <br> Less |  |
| High | High | High | aggressive <br> Cautious |
| High | Low | High | Less <br> Highressive |

Takano et al. propose a hierarchical model with one HMM characterizing the short-term driving behaviors in the lower layer, and the other HMM characterizing the longterm driving behaviors which are represented in the HMM space [25]. This structure makes the vehicles intelligent by storing the knowledge of driving behaviors as the symbols of driving intention through observing the driving behavior given by expert drivers. Baum-Welch algorithm (a maximum likelihood estimation method) which trains parameters of HMMs is applied to optimize three HMMs-driving straight, normal steering, and emergency steering [38, 39]. In addition, the model [40] based on the combination of HMM and NN model is presented, which can achieve the driving intention recognition and the driving behavior prediction. In [67], driver behaviors are modeled by using HMM in two alternative ways. Using the measured data of driving behaviors, an HMM consisting of three recognition categories-emergency lane change (LCE), ordinary lane change (LCN), and lane keeping (LKN)—is developed [17].
3.2.3. Fuzzy Control Theory. For the pattern recognition systems whose parameters' range is difficult to determine but can be divided according to the a priori knowledge or commonsense, fuzzy control theory is available to model it.

Lu et al. [12] use fuzzy subsets to category drivers by introducing the HRF (see Figure 3). First degree of membership is calculated to each of the four categories (i.e., cautious, average, expert, and reckless) for each event of a specific driver; then a probabilistic method is used to calculate the possibilities that are generated by multiple events and to aggregate the overall possibilities in order to characterize the driver.

Lu also proposes a driver-in-the-loop system (see Figure 6) and uses three methods to characterize driver's driving behaviors or control structure in real time. Table 1 illustrates fuzzy rules of a Takagi-Sugeno model to realize semistructured driving behavior characterization. In addition, an evolving Takagi-Sugeno fuzzy model is presented for
capturing the evolving characteristics of driving behaviors [23].
3.2.4. Gaussian Mixture Model (GMM). GMM is a parametric approach to density estimation [76]. GMM is known for its ability to generate arbitrarily shaped densities, and it has experienced extensive use in pattern recognition, such as speech recognition and speaker recognition. Miyajima et al. has been working on modeling driving behavior based on GMM and written several related papers [27-32].

In [29], GMM is applied to identify drivers in the case of car-following condition, with the accuracy of $76.8 \%$ by field test data. In [30], driving patterns of each driver are modeled based on GMM. In this work, the GMM is trained as a joint probability distribution of following distance, velocity, pedal position signals, and their dynamics. Experiments conducted using a driving simulator show that car-following conditions reproduced by the GMMs for three different drivers maintain these drivers' individual driving styles.

In addition, by comparing the performance of present driver-behavior models for car-following condition based on GMM and based on piecewise auto regressive exogenous (PWARX) algorithms, Miyajima et al. find that the PWARXbased model takes slightly advantage over the GMM-based model for all cases [29]. Furthermore, the literature [31] certifies that the GMM-based model performs better when the measured parameters of diving behaviors are used, but under such circumstance the model becomes more sensitive to the approximation errors of the input parameters as in the recursive prediction. The literature also confirms that the PWARX-based model performs better than the GMMbased model at the long-term prediction but not at the short-term prediction. This is because of the feature of the PWARX; namely, it captures the relationship among the driving behaviors at long-time duration, finally makes the PWARX-based model more generalized, and sets abound to the prediction errors.
3.2.5. Other Models. Apart from the aforementioned methods, there are other ways to represent a driver's behavior. Bifulco et al. analyze and compare the performance of linear, polynomial, and FFNN approach to model driving behaviors, proposes that a linear model is not actually overperformed by more complicated approaches and that it is worth adopting in light of its great simplicity [72, 73]. Adaptive Network based Fuzzy Inference System (ANFIS) [74] draws the predominance of NN and fuzzy control efficiently and has high learning efficiency, high training efficiency, and good network generalization capability. The hybrid driver model [71] is able to reproduce the driving behaviors of different driver characteristics, so it can adapt to various types of drivers (e.g., good, novice, and fast).
3.2.6. Summary. It is increasingly recognized that intelligent and personalized vehicle systems are developed based on certain intelligent algorithms to which have a knowledge base. Such systems could process the received sensory data not only quantitatively but also qualitatively, for example,


Figure 6: Cognitive (solid line) and subjective (broken line) flow of information in a driver-vehicle system, adopted from [18].
interpret the driving behavior data, then compare it with the stored data in the behavior base, in some case add this new driving behavior data to the behavior base, and finally identify the driver behavior characteristics. Several methods of building identification models of the driver behavior characteristics have been introduced above. Each method has its own pros and cons, as shown in Table 2.
3.3. Experimental Design and Data Acquisition. In general, during the design of experiment, it is necessary to consider experimental participants, experimental vehicles, test maneuvers, and data acquisition. In particular, for building identification models of the driver behavior characteristics, the above aspects should be carefully determined.
3.3.1. Experimental Participants. First, experimental participants must have their own driving licenses. In addition, the number of the selected drivers should be large enough to cover all possible drivers' characteristics researched. For example, four driver characteristics-cautious, average, expert, and reckless-are described in [12], so the drivers with these driving styles should be chosen as experimental participants. It should be noted that even for the same driver, driver characteristic may alter from situation to situation, for example, fatigue, drunk, and drowsy. Therefore, experiment participants should participate in experiments under not overly tension and fatigue status.
3.3.2. Experimental Test. In order to obtain a diverse range of driver characteristics, a large number of experiments on vehicles with different dynamics and on different roads should be conducted. This is believed to be one of the reasons that the driving simulator experiment is used instead of field test. In [8], the Racer Simulator (created by Ruud van Gaal) is used to conduct experiments in Figure 7. Other relevant reviews include [79, 80].

In addition to the driving simulator, the instrumented vehicles have been developed to conduct experiments [67, 70, 81]. The UTDrive Vehicle converted from a Toyota RAV4 (see Figure 8) is used to collect data from real-road experiments. Additionally, a data collection vehicle, which is called TOYOTA REGIUS [81], has been specially designed


Figure 7: Driving platform for simulated experiments, adopted from [8].
for data collection in the Center for Integrated Acoustic Information Research (CIAIR) project.
3.3.3. Test Maneuvers. Driver behavior characteristics can be subdivided into steering characteristic, acceleration characteristic, and braking characteristic. In order to have full access to the driver behavior characteristics, it is necessary to design experiments with all test maneuvers in connection with certain characteristic. For the steering characteristic, the test track should be designed to one with a variety of bend radiuses, as shown in Figure 9. For the acceleration characteristic and the braking characteristic, most existing references $[18,29,30,82,83]$ are available with optional carfollowing maneuvers.

Especially, when conducting driving simulator experiments, using computer-generated time history curve of certain variables about test maneuvers to design experiments, is easier to achieve all test maneuvers. Figure 10 gives two velocity patterns of the lead vehicle which are used. The upper velocity pattern is recorded in an express way driving scenario in the driving simulator, and the lower pattern is an artificial velocity pattern by use of software aimed at obtaining all velocity ranges. Obviously, the test maneuver generated by computer is more comprehensive than that by driving simulator.

### 3.3.4. Data Acquisition

(1) Feature Parameters Chosen. Since identification of driver behavior characteristics is a pattern recognition process, it can be modeled based on pattern recognition methods by

Table 2: Features of the four methods of building identification models.

| Identification methods | NN | HMM | Fuzzy control theory | GMM |
| :---: | :---: | :---: | :---: | :---: |
| Algorithm features | To identify various types of driving behaviors located within relatively large amounts of stored time history data. The quality of feature parameters is crucial to the accuracy of NN. | To describe the statistical properties of stochastic processes and to identify inherent invisible states through external observation sequence. | To formulate fuzzy rules based on previous experience and then design model performance in accordance with the expectations of the designer. | A parametric approach to density estimation is able to generate arbitrarily shaped densities. |
| Model accuracy | Very high | Very high | High | High |
| Real-time performance | Fair [37] | Very good [37] | Fair [77] | The traditional GMM is poor, and the advanced GMM is good [78]. |
| Model adaptive | - | - | - | Using the maximum a posteriori (MAP) or Bayesian adaptive algorithm to adjust parameters of GMM, personalized driver behavior model will be obtained. |
| Disadvantages | There is not a unified feasible method to adjust parameters (e.g., the number of NN layers) but generally subjective adjustments based on the simulation results of the models; training time is long. | HMM is not suitable for long-term forecasting system and requires artificial hypothesis for the sequence distribution of the current states. | Since its fuzzy rules are formulated based on a priori knowledge, the simulation results may deviate from the actual values. | GMM cannot obtain more efficient modeling of the time series of feature vectors than other methods do. |
| Applications | NN is suitable for pattern recognition that is easy to access to acquire the feature parameters, such as music recognition and speech recognition. | HMM is suitable for pattern recognition with strong time series data, such as driver's intention recognition and speech recognition. | Fuzzy control theory is suitable for pattern recognition whose parameter range is difficult to determine. | GMM is expert in identifying short-term driving behaviors but not in long-term driving behaviors. If combined with PWARX, the model can have a good performance both in the short- and long-term driving behaviors. |

using feature parameters of the drivers' driving behaviors. Therefore, the parameters must be selected in a way that it is relevant to a human driver's characteristics, and the number of feature parameters is crucial to the accuracy of the identification models of the driver behavior characteristics. Particularly in practice when the amount of individual driving data used to establish the identification models is relatively smaller than that of the development data, the models obtained from such sparse data may not be able to represent the driver behavior characteristics in a typical manner [27]. In general, the data includes driver's maneuvers and vehicle states information, such as vehicle speed, yaw rate, lateral and longitudinal accelerations, steering angle, steering angle velocity, brake pedal position and its derivative, acceleration pedal position and its derivative, and others relative to a certain test maneuver.

Through a large number of tests and analysis, the rules selecting feature parameters are summarized. Wakita et al. [84] find that the nonparametric models take advantage over the parametric models and explains that the driver's operation behaviors are better than both adhesion conditions and vehicle states information. According to previous work [85], the gas and brake pedal signals chosen as the modeling data are adopted to build identification model, with best computational efficiency and high identification accuracy. However, the experimental results also show higher accuracy of driver identification when the modeling data is combined with the derivative of the gas and brake pedal signals (e.g., rate of the gas pedal), instead of the original gas and brake pedal signals collected [45]. Compared with raw driving signals, frequency responses calculated by spectral analysis of driving behaviors could better capture the individualities in driving


Figure 8: UTDrive data collection vehicle, adopted from [67].


Figure 9: The test track for the steering characteristic, derived from the test maneuvers of CarSim.
behaviors and could receive better performance in identifying the drivers [81].
(2) Data Processing. Data would generally be filtered, converted, grouped, and so forth. In addition, different pattern recognition methods have different requirements to data. With regard to NN, in order to build the intelligent driver behavior models and to learn different drivers' characteristics, input data to NN should be representative parameters of driver behaviors under different maneuvers. The following statistical measures of central tendency may be used: range, media, standard deviation, variance, and mean absolute deviation [8]. While for PWARX model, the input and output parameters are first clustered and categorized into different driving modes [31].
3.4. Model Adaptation. The identification models of driver behavior characteristics mentioned above are static models. Overall, the prediction performance of these models decreases as the driving maneuvers change from the simulation scenarios with controlled environment to the real-world driving environments under diverse uncontrolled factors [27]. In the real-world driving environments, the profiles and driving intentions are unknown; the driver may not behave exactly as the identification models represent, which results in bias of the models. In addition, the inaccuracy of signals collected from the sensory systems should be taken into account. Therefore, to solve these problems and to develop a more reliable model, model adaptation has been presented and regarded as one of the foremost solutions.

In general, the main objectives of model adaptation are to cope with (1) relatively small amount of observed driving data of individual driver, (2) individual driving style or characteristic that differs from the designed average style, and (3) mismatch of driving maneuvers between development and usage stages.

Angkititrakul et al. propose a method to execute drivermodel adaptation [31]. The universal driver-behavior models are first built by use of driving data of several drivers based on certain identification method (e.g., HMM). The universal driver-behavior models represent average/common driver characteristics shared by these drivers. The author takes further measures to adjust the parameters of the universal driver-behavior models in the following two scenarios: (1) driver adaptation: the driving data of each driver are used to adjust the universal driver-behavior model to build the adapted driver models, namely, driver-dependent or personalized driver models. (2) On-line adaptation: the driving data at the beginning of each driving event (e.g., car-following condition) are used to adjust the universal driver-behavior model and subsequently the on-line adapted driver model


Figure 10: Velocity patterns of lead vehicle, adopted from [30].
is utilized to represent driving behaviors for the rest of that driving event.

## 4. Applications in Identification of Driver Behavior Characteristics for Automotive Control

4.1. Intelligent Driver Advisory System. The main objective of the state-of-the-art vehicle electronic control systems is to improve vehicle performance by identifying driver intentions/characteristics and controlling the vehicle to realize the driving intentions safely, robustly, and smoothly. The performance of electronic control systems can be significantly improved when the driver and the electronic control systems could cooperate with each other to pursue the same hazard avoidance goal and to maximize the accident avoidance capability of the driver-in-the-loop vehicle as a system.

Lu develops a driver advisory system based on the driver characteristics that can warn the drivers of driving conditions when the vehicle approaches to its limit, which is a part of a cluster of warning functions defined as an Intelligent Personal Minder (IPM) system. Figure 11 depicts the interaction of the IPM system with the other subsystems and functions. The electronic control system follows the driver intentions and the driver responds to the advisory information from the electronic control system to modify his/her operation inputs (e.g., rising braking pedal, increasing steering wheel angle, etc.). In this way a seamless coordination between the driver and the electronic control system could be realized and it is likely to minimize the effect of the potential safety hazards resulted from driver's operation errors.
4.2. Driver Safety Warning System. Introducing the driver behavior characteristics to design the driver safety warning system, the redundancy alarm rate can be reduced and the negative interference on the driver will be improved. Wang et al. present vehicle collision warning/avoidance algorithms based on the driver behavior characteristics [34, 86]. By changing parameters, the algorithms can better adapt to the behavior of different drivers. Take the longitudinal driving maneuver for example, the different level safe thresholds, warning rules, and warning logic are determined using time headway (THW) and time to collision (TTC).

Kentaro Ogchi et al. invent a system for predicting driver behavior and generating control and/or warning signals. As shown in Figure 12, the driver predictor is comprised of
an initialization module, an updating module, a transfer module, a hierarchical temporal memory (HTM), and a prediction retrieval module. The initialization module processes the data from the database and uses it to create initial states or definitions for different driver behaviors in the HTM. After the updating module identifies the driver, the HTM is adjusted to match the driver of the vehicle. The transfer module standardizes the format of real-time data and inputs it to the HTM. In accordance with the information of the initialization module, the updating module and the transfer module, the HTM uses a hierarchical temporal memory construct to predict vehicle states. Then the prediction retrieval module queries the HTM to generate warning control signals to alert the driver of potentially dangerous conditions, to generate collision control signals to prevent or avoid collisions, and to generate acceleration control signals for the adaptive cruise control.

The functional view of the HTM is shown in Figure 13. The HTM consists of long-term storage, intermediate-term storage, short-term storage, an intermediate-to-long-term (ILT) converter and a short-to-intermediate-term (SIT) converter. The HTM has a memory mechanism with a hierarchical structure of temporal memory based on the memory mechanism of the human brain. The information stored in the three storages of the HTM, respectively, represents a current state of the vehicle and other vehicles surrounding it, that is, the short-term storage stores short paths of vehicle action, the intermediate-term storage stores trajectory types, and the long-term storage stores behavior types.

The prediction retrieval module is made up of a storage for driver preferences, a training module, a warning control module, a collision control module, and acceleration control module, as shown in Figure 14. Each of these modules accesses and retrieves data from the HTM. The storage for driver preferences contains sample sets of preferences for different kinds of drivers. Typical preferences stored in the storage for driver preferences include the types and levels of settings for warning control, collision control, and acceleration control. The training module is used to train the warning control module, the collision control module, and the acceleration control module.
4.3. Vehicle Dynamics Control System. X-by-wire control technology comprises steer-by-wire, brake-by-wire, drive-by-wire, and suspension-by-wire (active suspensions) subsystems and has higher accurate and more complex control algorithms than traditional automotive control systems do.


Figure 11: The block diagram of a vehicle control system including an IPM, adopted from [12].


Figure 12: A functional view of a driver predictor system, adopted from [21].


Figure 13: A functional view of a hierarchical temporal memory, adopted from [21].

Introducing the intelligent control algorithm to identify driver's intention and driver characteristics, the performance of the driver-vehicle-road closed-loop system will be enhanced, and finally different driving characteristics and personalized ideal vehicle dynamics characteristics will be realized, that is, converting from "driver adapts to car" to "car adapts to driver" $[19,20]$.

The x-by-wire vehicle's controller controls the actuators according to the driver's operation signals, and the vehicle
reference models in the controller provide control objectives for the x -by-wire vehicle according to the driver operation signals and vehicle states information. The vehicle reference models play a very important role for controlling the x-bywire vehicle. In order to realize different driving characteristics of the x -by-wire vehicle, we present the control principle of "car adapts to driver" $x$-by-wire vehicle's ideal dynamics characteristics (see Figure 15). The dotted area is an integrated controller of the x-by-wire vehicle, which exports steering angles, driving, and braking torques of four wheels. The dash dot area shows the identification system of driver (behavior) characteristics. Initially the vehicle uses the original vehicle reference model before the identification system of driver characteristics identifies the driver's characteristics based on the driver's operation signals and vehicle states; thereafter the vehicle reference model is switched to matching characteristic of the ideal reference models, which can control actuators of the x-by-wire vehicle through the integrated controller and the optimal distribution of tire forces, and finally realizes the personalized ideal dynamics outputs for the x -by-wire vehicle.

The identification system of driver characteristics includes the driver characteristics identification models and the ideal reference models. According to driver's operation behaviors, the driver characteristics can be decomposed into steering characteristic, acceleration characteristic, and braking characteristic; correspondingly the identification


Figure 14: A block diagram illustrating a functional view of a prediction retrieval module, adopted from [21].


FIGURE 15: Control principle of "car adapts to driver" x-by-wire vehicle's ideal dynamics characteristics.
system is divided into three subsystems, which are the steering module, the acceleration module, and the braking module, as shown in Figure 16. Each module, respectively, identifies the driver's each characteristic; for example, steering characteristic is cautious, acceleration characteristic is average, and braking characteristic is reckless; then the corresponding ideal dynamics reference modelscautious steering, average acceleration, and reckless braking reference model-are automatically matched. Driver's final ideal dynamics reference models consist of the steering, acceleration, and braking models matched and, respectively, provide reference outputs (e.g., yaw rate, sideslip angle, acceleration, and deceleration) to integrated controller for the x-by-wire vehicle. The identification models are built based on NN or HMM considering these two intelligent algorithms' advantages mentioned above.

## 5. Conclusion

Identification and applications of the driver characteristics for automotive control are widely ranging and informative. In the present work, the fundamental of driver behavior characteristics is introduced; the intrinsic link among the driving behavior, the driving intention, and the driver behavior characteristics is explained; the whole process during establishing the identification models of the driver behavior characteristics is summarized and analyzed in detail, including driver characteristics classification, identification methods, experimental design and data acquisition, and model adaptation. On this basis, applications of the driver characteristics for automotive control have been introduced on three aspects, namely, the intelligent driver advisory system, the driver safety warning system, and the vehicle dynamics control system.


Figure 16: Principle of the driver characteristics identification system.

The driver is a complex and uncertain individual, which might exhibit different driving characteristics in different driving situations (fatigue/drunk/drowsy/distracted driving). In addition, different road adhesion, traffic conditions, and weather conditions will also impact driver characteristics. Therefore, extensive experiments are required to be conducted on potential user groups (the groups with the driver characteristics being studied) for the purpose of establishing a comprehensive human driving behavior library for greater precision and wider applications. It will be necessary to ensure robustness of the models in actual driving situations, in addition to improving recognition performance by resolving the aforementioned issues. On-line adaptation of the driver behavior characteristics is considered as one of the foremost solutions. In fact the driver behavior characteristics can be influenced and reaccustomed through learning, so it is of great importance to build online adaptive models of the driver behavior characteristics by using online field test data to revise the parameters of the established models, instead of the static models. In addition, there may exist some cross correlations among the classifications of driver behavior characteristics, which is a topic that deserves further study. Since the existing classifications are very rough, the classification should be further refined, and the ultimate goal is to acquire the driver's personal preference feature.

Driver characteristics have been used to identify driver, to detect driver's abnormal behaviors, to design driver assistance systems which adapt to individual driver, to establish different types of driver models to intelligently assist individual driver, and so forth. Designing ideal dynamics reference models adaptation to driver characteristics for the x-by-wire vehicles, then making it realize that "driver adapts to car" changes
to "car adapts to driver", and finally achieving personalized driving are a very interesting and promising application of the driver characteristics for automotive control. With further research, applications of the driver characteristics will be broadened in the future.

## Conflict of Interests

The authors declare that they have no conflict of interests regarding the publication of this paper.

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## Research Article

# Inverse Optimal Control with Speed Gradient for a Power Electric System Using a Neural Reduced Model 

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#### Abstract

This paper presented an inverse optimal neural controller with speed gradient (SG) for discrete-time unknown nonlinear systems in the presence of external disturbances and parameter uncertainties, for a power electric system with different types of faults in the transmission lines including load variations. It is based on a discrete-time recurrent high order neural network (RHONN) trained with an extended Kalman filter (EKF) based algorithm. It is well known that electric power grids are considered as complex systems due to their interconections and number of state variables; then, in this paper, a reduced neural model for synchronous machine is proposed for the stabilization of nine bus system in the presence of a fault in three different cases in the lines of transmission.


## 1. Introduction

Many physical systems, such as electric power grids, computer and communication networks, networked dynamical systems, transportation systems, and many others, are complex large-scale interconnected systems [1]. To control such large scale systems, centralized control schemes are proposed in the literature assuming available global information for the overall system. Another problem in complex large-scale interconnected systems is the effect of delays that typically are unknown and time-variable [2,3]. While using control centralization has theoretical advantages, it is very difficult for a complex large-scale system with interconnections due to technical and economic reasons [4]. Furthermore, centralized control designs are dependent upon the system structure and cannot handle structural changes. If subsystems are added or removed, the controller for the overall system should be redesigned. Therefore decentralized control for interconnected power systems has also attracted considerable attention of researchers in the field of complex and largescale systems like multiarea interconnected power systems. Besides, due to physical configuration and high dimensionality of interconnected systems, centralized control is neither economically feasible nor even necessary. These facts
motivate the design of decentralized controllers, using only local information while guaranteeing stability for the whole system [1].

The main issue in this paper is the analysis of a fault in the electric power system in different lines of transmission, the recurrent high order neural networks (RHONN) allow the identification of nonlinear systems, and then the RHONN model can be used for the controller design. Recently, some works have been published about synchronous generators in which reduced models have been proposed, such models are able to reproduce full order dynamics for synchronous generators $[1,5]$. The system under study consists of three synchronous generators interconnected (nine bus system) and there are cases of study of power electric system, where a three-phase fault is introduced at the end of the line 7 [6]; in this paper, the analysis for the system is focused in other lines, at the end of buses 8 and 9 , the fault is proposed and tested via simulation and the purpose is the production and distribution of a reliable and robust electric energy.

On the other hand, a model in discrete time has been proposed [7], in which a recurrent high order neural network has been incorporated to implement a control law as this reduced model allows the stabilization through the inverse optimal control law SG. In this work, a neural model of
the multimachine system is proposed, which results useful, because it is focused in the variable states that are more relevant for this paper: position, velocity, and voltage rotor [7]; further, the control law is implemented for the power electric system that consists of three interconnected synchronous generators. A solution is proposed for the destabilization problem of multimachine power electric system in the presence of a fault in one of its lines of transmission that occurs at 10 seconds of simulation. A system identification of the complete multimachine power electric system model (nine bus system) is presented through a neural reduced model and this allows the design of a neural inverse optimal SG control law. Finally the results obtained are shown, in which it can be seen that the control law stabilizes the system in presence of the fault in the three cases of fault that are presented.

In literature, there are works that report the parameter identification for synchronous machines for full order models [5] as well as for reduced order ones [8]; however, these models are for nominal condition; that is, they do not consider fault scenarios; in [1], a reduced order neural model is considered; however, it is developed for continuous time; nevertheless, the need to real-time implementations makes necessary the use of digital models, besides, in [9], has been developed a discrete-time neural controller, which is proposed for a single machine system. Then, the paper main contributions can be stated as follows: first a RHONN is used to establish a discrete-time reduced order mathematical model for a multimachine power electric system model. Then this neural model is used to synthesize an inverse optimal SG control law to stabilize the system and, finally, three fault scenarios are considered in order to illustrate the applicability of the proposed scheme.

## 2. Mathematical Preliminaries

2.1. Discrete-Time High Order Neural Networks. The use of multilayer neural networks is well known for pattern recognition and for static systems modelling. The NN is trained to learn an input-output map. Theoretical works have proven that, even with just one hidden layer, a NN can uniformly approximate any continuous function over a compact domain, provided that the NN has a sufficient number of synaptic connections [10]. To implement the neural network (NN) design, a RHONN is used [7] and this model turns out to be very flexible because it allows incorporating priory information to the model:

$$
\begin{equation*}
\widehat{x}_{i}(k+1)=\omega_{i}^{T} z_{i}\left(x_{i}(k), u(k)\right), \quad i=1, \ldots, n \tag{1}
\end{equation*}
$$

where $\widehat{x}_{i}(i=1,2, \ldots, n)$ is the state of the $i$ th neuron and $\omega_{i}(i=1,2 \ldots, n)$ is the respective online adapted weight vector. Now we define the vector:

$$
z_{i}\left(x_{i}(k), u(k)\right)=\left[\begin{array}{c}
z_{i_{1}} \\
z_{i_{2}} \\
\vdots \\
z_{i_{L_{i}}}
\end{array}\right]=\left[\begin{array}{c}
\prod_{j \in I_{1}} \xi_{i_{j}}^{d i_{j}(1)} \\
\prod_{j \in I_{2}} \xi_{i_{j}}^{d i_{j}(2)} \\
\vdots \\
\prod_{j \in I_{L_{i}}} \xi_{i_{j}}^{d i_{j}\left(L_{i}\right)}
\end{array}\right] .
$$

$L_{i}$ is the respective number of high-order connections, $\left\{I_{1}, I_{2}, \ldots, I_{L_{i}}\right\}$ is a collection of nonordered subsets of $\{1,2, \ldots, n+m\}, n$ is the state dimension, and $m$ is the number of external inputs, with $d_{i}(k)$ being nonnegative integers and $\xi_{i}$ defined as follows:

$$
\xi_{i}=\left[\begin{array}{c}
\xi_{i_{1}}  \tag{3}\\
\vdots \\
\xi_{i_{n}} \\
\xi_{i_{n+1}} \\
\vdots \\
\xi_{i_{n+m}}
\end{array}\right]=\left[\begin{array}{c}
S\left(x_{1}\right) \\
\vdots \\
S\left(x_{n}\right) \\
u_{1} \\
\vdots \\
S\left(u_{m}\right)
\end{array}\right]
$$

$u=\left[u_{1}, u_{2}, \ldots, u_{m}\right]^{T}$ is the input vector to the neural network and $S(\bullet)$ is defined by

$$
\begin{equation*}
S(\varsigma)=\frac{1}{1+\exp (-\beta \varsigma)}, \quad \beta>0 \tag{4}
\end{equation*}
$$

where $\varsigma$ is any real value variable.
2.2. The EKF Training Algorithm. The best well-known training approach for recurrent neural networks (RNN) is the backpropagation through time learning [11]. However, it is a first order gradient descent method and hence its learning speed could be very slow [12]. Recently, Extended Kalman Filter (EKF) based algorithms have been introduced to train neural networks [7, 9, 13, 14]. With the EKF based algorithm, the learning convergence is improved [14]. The EKF training of neural networks, both feedforward and recurrent ones, has proven to be reliable and practical for many applications over the past years [14]. It is known that Kalman filtering (KF) estimates the state of a linear system with additive state and output white noises $[15,16]$. For KF-based neural network training, the network weights become the states to be estimated. In this case, the error between the neural network output and the measured plant output can be considered as additive white noise. Due to the fact that the neural network mapping is nonlinear, an EKF-type is required (see [17] and references therein). The training goal is to find the optimal weight values which minimize the prediction error. The EKFbased training algorithm is described by [15]:

$$
\begin{gather*}
K_{i}(k)=\rho_{i}(k) H_{i}(k) M_{i}^{-1}(k) \\
\omega_{i}(k+1)=\omega_{i}(k)+\eta_{i} K_{i}(k) e_{i}(k)  \tag{5}\\
\rho_{i}(k+1)=\rho_{i}(k)-K_{i}(k) H_{i}^{T}(k) \rho_{i}(k)+\phi_{i}(k)
\end{gather*}
$$

with

$$
\begin{gather*}
M_{i}(k)=\left[\tau_{i}(k)+H_{i}^{T}(k) \rho_{i}(k) H_{i}(k)\right]^{-1}  \tag{6}\\
e_{i}(k)=x_{i}(k)-\widehat{x}_{i}(k),
\end{gather*}
$$

where $\rho_{i} \in \Re^{L_{i} \times L_{i}}$ is the prediction error associated covariance matrix, $\omega_{i} \in \mathfrak{R}^{L_{i}}$ is the weight (state) vector, $x_{i} \in \mathfrak{R}$ is the $i$ th plant state component, $\widehat{x}_{i} \in \mathfrak{R}$ is the $i$ th neural
state component, $\eta_{i}$ is a design parameter, $K_{i} \in \mathfrak{R}^{L_{i} \times m}$ is the Kalman gain matrix, $\phi_{i} \in \mathfrak{R}^{L_{i} \times L_{i}}$ is the state noise associated covariance matrix, $\tau_{i} \in \Re^{m \times m}$ is the measurement noise associated covariance matrix, and $H_{i j} \in \mathfrak{R}^{L_{i} \times m}$ is a matrix, for which each entry $\left(H_{i j}\right)$ is the derivative of one of the neural network output, ( $\widehat{x}_{i j}$ ), with respect to one neural network weight, $\left(\omega_{i j}\right)$, as follows:

$$
\begin{equation*}
H_{i j}(k)=\left[\frac{\partial \widehat{x}_{i}(k)}{\partial \omega_{i j}(k)}\right]_{\omega_{i}(k)=\widehat{\omega}_{i(k+1)}} i=1, \ldots, n, j=1, \ldots, L_{i} . \tag{7}
\end{equation*}
$$

Usually $\rho_{i}, \phi_{i}$, and $\tau_{i}$, are initialized as diagonal matrices, with entries $\rho_{i}(0), \phi_{i}(0)$, and $\tau_{i}(0)$, respectively.

## 3. Controller Design

Optimal control is related to finding a control law for a given system such that a performance criterion is minimized. This criterion is usually formulated as a cost functional, which is a function of the state and control variables. The optimal control problem can be solved using Pontryagin's maximum principle (a necessary condition) [18] and the method of dynamic programming developed by Bellman [19, 20], which can lead to a nonlinear partial differential equation called the Hamilton-Jacobi-Bellman (HJB) equation (a sufficient condition); nevertheless, solving the HJB equation is not a feasible task [21, 22].
3.1. Inverse Optimal Control via CLF. In this paper, the inverse optimal control and its solution by proposing a quadratic control Lyapunov function (CLF) are used [23] and the CLF depends on a fixed parameter in order to satisfy stability and optimality condition. A posteriori, the speed gradient algorithm is established to compute this CLF parameter and it is used to solve the inverse optimal control problem. Motivated by the favorable stability margins of optimal control systems, a stabilizing feedback control law is proposed, which will be optimal with respect to a meaningful cost functional. At the same time, it is desirable to avoid the difficult task of solving the HJB partial differential equation. In the inverse optimal control problem, a candidate CLF is used to construct an optimal control law directly without solving the associated HJB equation [24]. Inverse optimality is selected, because it avoids solving the HJB partial differential equations and still allows obtaining Kalman-type stability margins [21].

In contrast to the inverse optimal control via passivity approach, in which a storage function is used as a candidate CLF and the inverse optimal control law is selected as the output feedback, for the inverse optimal control via CLF, the control law is obtained as a result of solving the Bellman equation. Then, a candidate CLF for the obtained control law is proposed such that it stabilizes the system and a posteriori a meaningful cost functional is minimized.

In this paper, a quadratic candidate CLF is used to synthesize the inverse optimal control law. The following
assumptions and definitions allow the inverse optimal control solution via the CLF approach.

The full state of system

$$
\begin{equation*}
x(k+1)=f(x(k))+g(x(k)) u(k) \tag{8}
\end{equation*}
$$

is measurable.
Definition 1 (inverse optimal control law). Let us define the control law [23]

$$
\begin{equation*}
u(k)=-\frac{1}{2} R^{-1}(x(k)) g^{T}(x(k)) \frac{\partial V(x(k+1))}{\partial x(k+1)} \tag{9}
\end{equation*}
$$

to be inverse optimal (globally) stabilizing if
(1) it achieves (global) asymptotic stability of $x=0$ for system (8);
(2) $V(x(k))$ is (radially unbounded) positive definite function such that inequality

$$
\begin{equation*}
\bar{V}:=V(x(k+1))-V(x(k))+u(k)^{T} R(x(k)) u(k) \leq 0 \tag{10}
\end{equation*}
$$

is satisfied. When $l(x(k):=\bar{V} \geq 0$, is selected; then $V(x(k))$ is a solution for the HJB equation

$$
\begin{align*}
& l(x(k)+V(x(k+1))-V(k)) \\
& \quad+\frac{1}{4} V^{T *} R^{-1}(x(k)) g^{T}(x(k)) V^{*}=0 \tag{11}
\end{align*}
$$

where

$$
\begin{equation*}
V^{T *}=\frac{\partial V^{T}(x(k+1))}{\partial x(k+1)}, \quad V^{*}=\frac{\partial V(x(k+1))}{\partial x(k+1)} \tag{12}
\end{equation*}
$$

It is possible to establish the main conceptual differences between optimal control and inverse optimal control as follows.
(i) For optimal control, the meaningful cost indexes $l(x(k)) \geq 0$ and $R(x(k))>0$ are given a priory; then, they are used to calculate $u(x(k))$ and $V(x(k))$ by means of HJB equation solution.
(ii) For inverse optimal control, a candidate CLF $V(x(k))$ and the meaningful cost index $R(x(k))$ are given a priory, and then these functions are used to calculate the inverse control law $u(k)$ and the meaningful cost index $(x(k))$, defined as $l(x(k)):=-\bar{V}(x(k))$.

As established in Definition 1, the inverse optimal control problem is based on the knowledge of $V(x(k))$. Thus, it is proposed as a CLF $V(x(k))$, such that (1) and (2) are guaranteed. That is, instead of solving (11) for $V(x(k))$, it is proposed a control Lyapunov function $V(x(k))$ as

$$
\begin{equation*}
V(x(k))=\frac{1}{2} x^{T}(k) P x(k) \tag{13}
\end{equation*}
$$

for control law (9), in order to ensure stability of the equilibrium point $x(k)=0$ of system (8), which will be
achieved by defining an appropriate matrix $P$. Moreover, it will be established that control law (9) with (13), which is referred to as the inverse optimal control law, optimizes a meaningful cost functional of the form:

$$
\begin{equation*}
J(x(k))=\sum_{0}^{\infty}\left(l(x(k))+u^{T}(k) R(x(k)) u(k)\right) . \tag{14}
\end{equation*}
$$

Consequently, by considering $V(x(k))$ as in (13), the control law takes the following form:

$$
\begin{equation*}
\alpha(x(k)):=u(k)=-\frac{1}{2}\left(R(x(k))+P_{2}(x(k))\right)^{-1} P_{1}(x(k)), \tag{15}
\end{equation*}
$$

where $P_{1}(x(k))=g^{T}(x(k)) P f(x(k))$ and $P_{2}(x(k))=(1 / 2) g^{T}$ $(x(k)) P g(x(k))$. It is worth pointing out that $P$ and $R(x(k))$ are positive definite and symmetric matrices; thus, the existence of the inverse in (15) is ensured.
3.2. Speed-Gradient SG Algorithm. Given that (15) $P$ is redefined as $P(k)$ where $P_{1}(x(k))=g^{T}(x(k)) P(k) f(x(k))$ and $P_{2}(x(k))=(1 / 2) g^{T}(x(k)) P(k) g(x(k))$, this will allow us to compute a time variant value in time for $P(k)$, which ensures stability to the system (8) by means of the algorithm SG.

In [25] a discrete-time application of the SG algorithm is formulated to find a control law $u(k)$ which ensures the control goal:

$$
\begin{equation*}
Q(x(k+1)) \leq \Delta, \quad \text { for } k \geq k^{*} \tag{16}
\end{equation*}
$$

where $Q$ is a control goal function, a constant $\Delta>0$, and $k^{*} \in \mathbb{Z}^{+}$is the time at which the control goal is achieved. $Q$ ensures stability if it is a positive definite function.

Based on the SG application proposed in [25], the control law given by (15) is considered, with $\Delta$ in (16) a state dependent function $\Delta(x(k))$.

Consider the control law redefined for the speed gradient algorithm which at every time depends on the matrix $P(k)$. Let us define the matrix $P(k)$ at every time $k$ as

$$
\begin{equation*}
P(k)=p(k) P^{\prime}, \tag{17}
\end{equation*}
$$

where $P^{\prime}=P^{\prime T}>0$ is a given constant matrix and $p(k)$ is a scalar parameter to be adjusted by the SG algorithm. Then the control law is transformed as follows:

$$
\begin{equation*}
u(k)=-\frac{p(k)}{2}\left(R(x(k))+\frac{p(k)}{2} P_{1}^{*}\right)^{-1} P_{2}^{*} \tag{18}
\end{equation*}
$$

where

$$
\begin{equation*}
P_{1}^{*}=g^{T}(x(k)) P^{\prime} g(x(k)), \quad P_{2}^{*}=g^{T}(x(k)) P^{\prime} f(x(k)) . \tag{19}
\end{equation*}
$$

The SG algorithm is now reformulated for the inverse optimal control problem.

Definition 2 (SG goal function). Consider a time-varying parameter $p(k) \in \mathscr{P} \subset \mathfrak{R}^{+}$with $p(k)>0$ for all $k$, and $\mathscr{P}$
is the set of admissible values for $p(k)$ [23]. A nonnegative function $Q: \Re^{n} \times \Re \rightarrow \Re$ of the form

$$
\begin{equation*}
Q(x(k), p(k))=V_{\mathrm{SG}}(x(k+1)) \tag{20}
\end{equation*}
$$

where $V_{\mathrm{SG}}(x(k+1))=-(1 / 2) x^{T}(k+1) P^{\prime} x(k+1)$ with $x(k+1)$ as defined in (8), is referred to as SG goal function for system (8), with $Q(k(p)):=Q(x(k), p(k))$.

Definition 3 (SG control goal). Consider a constant $p^{*} \in \mathscr{P}$. The SG control goal for system (8) with (18) is defined as finding $p(k)$, so that the SG goal function $Q(k(p))$ [23], as in (20), fulfills

$$
\begin{equation*}
Q(k(p)) \leq \Delta(x(k)), \quad \text { for } k \geq k^{*}, \tag{21}
\end{equation*}
$$

where

$$
\begin{equation*}
\Delta(x(k))=V_{\mathrm{SG}}(x(k))-\frac{1}{p(k)} u^{T}(k) R(x(k)) u(k) \tag{22}
\end{equation*}
$$

with $V_{\mathrm{SG}}(x(k))=-(1 / 2) x^{T}(k) P^{\prime} x(k)$ and $u(k)$ as defined in (18); $k^{*} \in Z^{+}$is the time at which the SG control goal is achieved.

Solution $p(k)$ must guarantee that $V_{\mathrm{SG}}(x(k))>$ $(1 / p(k)) u^{T}(k) R(x(k)) u(k)$ in order to obtain a positive definite function $\Delta(x(k))$.

To conclude, the SG algorithm is used to calculate $p(k)$ in order to achieve the SG control goal defined above.

Proposition 4. Consider a discrete-time nonlinear system of the form (8) with (18) as input [23]. Let Q be a SG goal function as defined in (2) and denoted by $Q(k(p))$. Let $\bar{p}, p^{*} \in \mathscr{P}$ be positive constant values and let $\Delta(x(k))$ be a positive definite function with $\Delta(0)=0$ and let $\epsilon^{*}$ be a sufficiently small positive constant. Assume the following.
(i) There exist $p^{*}$ and $\epsilon^{*}$ such that

$$
\begin{gather*}
Q\left(k\left(p^{*}\right)\right) \leq \epsilon^{*} \ll \Delta(x(k)), \\
\frac{1-\epsilon^{*}}{\Delta(x(k))} \approx 1 . \tag{23}
\end{gather*}
$$

(ii) For all $p(k) \in \mathscr{P}$,

$$
\begin{equation*}
\left(p^{*}-P(k)\right)^{T} \nabla(p) Q(k(p)) \leq \epsilon^{*}-\Delta(x(k))<0 \tag{24}
\end{equation*}
$$

where $\nabla(p) Q(k(p))$ denotes the gradient of $Q(k(p)$ with respect to $p(k)$. Then, for any initial condition $p(0)>0$, there exists a $k^{*} \in \mathfrak{R}^{+}$such that the $S G$ control goal (16) is achieved by means of the following dynamic variation of parameter $p(k)$ :

$$
\begin{equation*}
p(k+1)=p(k)-\gamma_{d(k)} \nabla(p) Q(k(p)) \tag{25}
\end{equation*}
$$

with

$$
\begin{gather*}
\gamma_{d(k)}=\gamma_{c} \delta(k)|\nabla(p) Q(k(p))|^{-2} \\
0<\gamma_{c} \leq 2 \Delta(x(k)),  \tag{26}\\
\delta(k)= \begin{cases}1 & \text { for } Q(p(k))>\Delta(x(k)) \\
0 & \text { otherwise } .\end{cases}
\end{gather*}
$$

Finally, for $k \geq k^{*}, p(k)$ becomes a constant value denoted by $\bar{p}$ and the $S G$ algorithm is completed.

With $Q(p(k))$ as defined in (20), the dynamic variation of parameter $p(k)$ in (25) results in

$$
\begin{equation*}
p(k+1)=p(k)+\Theta^{*} \tag{27}
\end{equation*}
$$

where

$$
\begin{align*}
\Theta^{*}= & 8 \gamma_{d(k)} \\
& \times \frac{f^{T}(x(k)) P^{\prime} g(x(k)) R(x(k))^{2} g^{T}(x(k)) f(x(k))}{\left(2 R(x(k))+p(k) g^{T}(x(k)) P^{\prime} g(x(k))\right)^{3}} \tag{28}
\end{align*}
$$

which is positive for all time $k$ if $p(0)>0$. Therefore positiveness for $p(k)$ is ensured and requirement $P(k)=$ $P^{T}(k)>0$ para $V(x(k))=(1 / 2) x^{T}(k) P(k) x(k)$ with $P(k)=$ $P^{T}(k)>0$ is guaranteed. When SG control goal (21) is achieved, then $p(k)=p$ for $k \geq k^{*}$. Thus, matrix $P(k)$ in (18) is considered constant and $P(k)=P$ where $P$ is computed as $P=\bar{p} P^{\prime}$, with $P^{\prime}$ a design positive definite matrix. Under these constraints, we obtain

$$
\begin{equation*}
\alpha(x(k)):=u(k)=-\frac{1}{2}\left(R(x(k))+P_{2}(x(k))\right)^{-1} P_{1}(x(k)) \tag{29}
\end{equation*}
$$

where $P_{1}(x(k))=g^{T}(x(k)) P f(x(k))$ and $P_{2}(x(k))=$ $(1 / 2) g^{T}(x(k)) P g(x(k))$.
3.3. Tracking Reference. In the case of tracking reference, the control law is defined as follows [23]:

$$
\begin{equation*}
u(k)=-\frac{1}{2}\left(R(x(k))+P_{2}(x(k))\right)^{-1} P_{1}(x(k)) \tag{30}
\end{equation*}
$$

where $P_{1}(x(k))=g^{T}(x(k)) P f\left(x(k)-x_{\mathrm{ref}}(k+1)\right)$ and $P_{2}(x(k))=(1 / 2) g^{T}(x(k)) P g(x(k))$.

## 4. Multimachine Power System Control

4.1. Multimachine Power System Complete Model. In this work, the proposed decentralized identification and control scheme is tested with the Western System Coordinating Council (WSCC) 3-machine, 9-bus system [6, 26]. The differential and algebraic equations which represent the $i$ th generator dynamics and power flow constraints respectively $[1,6]$ are given by

$$
\begin{gathered}
\dot{x}_{1 i}=x_{2 i}-\omega_{s} \\
\dot{x}_{2 i}=\left(\frac{\omega_{s}}{2 H_{i}}\right)\left(T_{m i}-\left(\psi_{d i} I_{q i}-\psi_{q i} I_{d i}\right)\right) \\
\dot{x}_{3 i}=\left(\frac{1}{T_{d 0 i}^{\prime}}\right)\left(-x_{3 i}-X_{d d}\right) \\
\left.\left.\times\left[I_{d i}-X_{d i}^{*}\left(x_{5 i}+X_{d l s}\right)\right]+E_{f d i}\right)\right]
\end{gathered}
$$

$$
\begin{gather*}
\dot{x}_{4 i}=\left(\frac{1}{T_{q 0 i}^{\prime}}\right)\left(-x_{4 i}-X_{q q}\right) \\
\left.\times\left[I_{q i}-X_{q i}^{*}\left(x_{6 i}+X_{q l s}\right)\right]+E_{d i}^{\prime}\right) \\
\dot{x}_{5 i}=\left(\frac{1}{T_{d 0 i}^{\prime \prime}}\right)\left(-x_{5 i}+x_{3 i}-\left(X_{d i}^{\prime}-X_{l s i}\right) I_{d i}\right) \\
\dot{x}_{6 i}=\left(\frac{1}{T_{q 0 i}^{\prime \prime}}\right)\left(-x_{6 i}-x_{4 i}-\left(X_{q i}^{\prime}-X_{l s i}\right) I_{q i}\right), \tag{31}
\end{gather*}
$$

where $x_{1}$ is the power angle of the $i$ th generator in rad, $x_{2}$ is the rotating speed of the $i$ th generator in $\mathrm{rad} / \mathrm{s}, x_{3}$ is the $q$-axis internal voltage of the $i$ th generator in p.u., $x_{4}$ is the $d$ axis internal voltage of the $i$ th generator in p.u., $x_{5}$ is the $1 d$ axis flux linkage of the $i$ th generator in p.u., $x_{6}$ is the $2 q$-axis flux linkage of the $i$ th generator in p.u., $E_{f d i}$ is the excitation control input, and $\psi_{d i}$ and $\psi_{q i}$ are the $d$-axis flux linkage and $q$-axis flux linkage of the $i$ th generator in p.u., respectively; $\omega_{s}$ is the synchronous rotor speed in rad $/ \mathrm{s}, I_{d i}$ and $I_{q i}$ are the $d$ axis and $q$-axis currents of the $i$ th generator in p.u., and $E_{d i}^{\prime}$ is the transient voltage in $d$-axis of the $i$ th generator. Besides, (4.1) is complemented with

$$
\begin{array}{cc}
X_{d i}^{*}=\frac{X_{d i}^{\prime}-X_{d i}^{\prime \prime}}{\left(X_{d i}^{\prime}-X_{l s i}\right)^{2}}, & X_{q i}^{*}=\frac{X_{q i}^{\prime}-X_{q i}^{\prime \prime}}{\left(X_{q i}^{\prime}-X_{l s i}\right)^{2}} \\
X_{d d}=X_{d i}-X_{d i}^{\prime}, & X_{q q}=X_{q i}-X_{q i}^{\prime}  \tag{32}\\
X_{d l s}=\left(X_{d i}^{\prime}-X_{l s i}\right) I_{d i}, & X_{q l s}=\left(X_{q i}^{\prime}-X_{l s i}\right) I_{q i}
\end{array}
$$

being parameters for each synchronous generator. It is important to consider that each machine model considered is a flux decay model (one axis model) given in [1, 6]; exciters and governors are not included in this model $[1,8]$.

### 4.2. Reduced Neural Model of Multimachine Power System.

 The model mentioned above [1] is in continuous time and due to this fact, we proceed to discretize the states using Euler methodology; with the state variables discretized, the reduced neural model is proposed [7] as follows:$$
\begin{gather*}
\widehat{x}_{1}(k+1)=f_{1}(k) \\
\widehat{x}_{2}(k+1)=f_{2}(k)  \tag{33}\\
\widehat{x}_{3}(k+1)=f_{3}(k)+w_{34} u(k) \\
f_{1}(k)=w_{11}(k) S\left(\widehat{x}_{1}(k)\right)+w_{12}(k) S\left(\widehat{x}_{2}(k)\right) \\
f_{2}(k)=w_{21}(k) S\left(\widehat{x}_{1}(k)\right)^{6}+w_{22}(k) S\left(\widehat{x}_{2}(k)\right) \\
+w_{23}(k) S\left(\widehat{x}_{3}(k)\right)  \tag{34}\\
f_{3}(k)=w_{31}(k) S\left(\widehat{x}_{1}(k)\right)+w_{32}(k) S\left(\widehat{x}_{2}(k)\right) \\
+w_{33}(k) S\left(\widehat{x}_{3}(k)\right),
\end{gather*}
$$

where $\widehat{x}_{i}$ estimates $x_{i}(i=1,2,3)$. Given the neural reduced model, the inverse optimal SG control law is applied to the reduced neural model to each synchronous generator, that is, in a decentralized way. Thus, the control law is established from (30) where the matrix $P$ is given for different values for each fault as follows: in the case of the fault at the end of bus $7(100 \times I, 5 \times I, 20 \times I)$, in the case of the fault at the end of bus $8(80 \times I, 5 \times I, 700 \times I)$, and in the case of the fault at the end of bus $9(100 \times I, 5 \times I, 10 \times I)$ for generators 1,2 , and 3 respectively, where $I$ is an identity matrix of $3 \times 3$.

From (33) $(x(k)), f(x(k))$, the control law for the neural network is defined as

$$
\begin{gather*}
g(x(k))=\left[\begin{array}{c}
0 \\
0 \\
\omega_{34}
\end{array}\right]  \tag{35}\\
f(x(k))=\left[\begin{array}{l}
f_{1}(k) \\
f_{2}(k) \\
f_{3}(k)
\end{array}\right] .
\end{gather*}
$$

It is important to note, that [5] proves that low-order models are well-suited for stability analysis and feedback control design for industrial power generators. Moreover, the use of neural networks allows modelling system interconnections using only local information, as well as not modeled dynamics for the reduced model [1].

## 5. Preliminary Calculations for Faults

For the design of the fault, a system data preparation is required and the following preliminary calculations are taken from [6],considering the parameters of the generators given in Tables 7 and 8.
(1) All system data are converted to a common base; a system base of 100 MVA is frequently used.
(2) The loads are converted to equivalent impedances or admittances. The needed data for this step are obtained from the load-flow study. Thus if a certain load bus has a voltage $\bar{V}_{L}$, power $P_{L}$, reactive power $Q_{L}$, and current flowing into a load admittance $\bar{Y}_{L}=$ $G_{L}+j B_{L}$, then
$P_{L}+j Q_{L}=\bar{V}_{L} \bar{I}_{L}^{*}=\bar{V}_{L}\left[\bar{V}_{L}^{*}\left(G_{L}-j B_{L}\right)\right]=V_{L}^{2}\left(G_{L}-j B_{L}\right)$.

The equivalent shunt admittance at that bus is given by

$$
\begin{equation*}
\bar{Y}_{L}=\frac{P_{L}}{V_{L}^{2}}-j\left(\frac{Q_{L}}{V_{L}^{2}}\right) \tag{37}
\end{equation*}
$$

(3) The internal voltages of the generators $E_{i} \angle \delta_{i 0}$ are calculated from the load-flow data. These internal angles may be computed from the pretransient terminal voltages $V \angle \alpha$ as follows. Let the terminal voltage be used temporarily as a reference, as shown in Figure 1. If $\bar{I}=I_{1}+j I_{2}$, then, from the relation $P+j Q=\bar{V} \bar{I}^{*}$


Figure 1: Generator representation for computing $\delta_{0}$.
it is possible to obtain $I_{1}+j I_{2}=(P-j Q) / V$. Since $E \angle \delta^{\prime}=\bar{V}+j x_{d}^{\prime} \bar{I}$, then

$$
\begin{equation*}
E \angle \delta^{\prime}=\left(V+\frac{Q x_{d}^{\prime}}{V}\right)+j\left(\frac{P x_{d}^{\prime}}{V}\right) \tag{38}
\end{equation*}
$$

The initial generator angle $\delta_{0}$ is then obtained by adding the pretransient voltage angle $\alpha$ to $\delta^{\prime}$, or

$$
\begin{equation*}
\delta_{o}=\delta^{\prime}+\alpha \tag{39}
\end{equation*}
$$

(4) The $\bar{Y}$ matrix for each network condition is calculated. The following steps are usually needed.
(a) The equivalent load impedances (or admittances) are connected between the load buses and the reference node; additional nodes are provided for the internal generator voltages (nodes $1,2, \ldots, n$ in Figure 2) and the appropriate values of $x_{d}^{\prime}$ are connected between these nodes and the generator terminal nodes. Also, simulation of the fault impedance is added as required, and the admittance matrix is determined for each switching condition.
(b) All impedance elements are converted to admittances.
(c) Elements of the $\bar{Y}$ matrix are identified as follows: $\bar{Y}_{i j}$ is the sum of all the admittances connected to node $i$, and $\bar{Y}_{i j}$ is the negative of the admittance between node $i$ and node $j$.
(5) Finally, all the nodes except for the internal generator nodes are eliminated and obtain the $\bar{Y}$ matrix for the reduced network. The reduction can be achieved by matrix operation recalling all the nodes that have zero injection currents except for the internal generator nodes. This property is used to obtain the network reduction as shown below. Let

$$
\begin{equation*}
I=Y V \tag{40}
\end{equation*}
$$

where

$$
I=\left[\begin{array}{c}
I_{n}  \tag{41}\\
0
\end{array}\right] .
$$



FIGURE 2: Representation of a multimachine system (classical model).


Figure 3: Nine bus system.

Now the matrices $Y$ and $V$ are partitioned accordingly to get

$$
\left[\begin{array}{c}
I_{n}  \tag{42}\\
0
\end{array}\right]=\left[\begin{array}{cc}
Y_{n n} & Y_{n r} \\
Y_{r n} & Y_{r r}
\end{array}\right]\left[\begin{array}{l}
V_{n} \\
V_{r}
\end{array}\right],
$$

where the subscript $n$ is used to denote generator nodes and the subscript $r$ is used for the remaining nodes. Thus for the
network in Figure 2, $V_{n} \in \Re^{n \times 1}$ and $V \in \mathfrak{R}^{r \times 1}$. Expanding (42),

$$
\begin{equation*}
I_{n}=Y_{n n} V_{n}+Y_{n r} V_{r}, \quad 0=Y_{r n} V_{n}+Y_{r} r V_{r} \tag{43}
\end{equation*}
$$

from which we eliminate $V_{r}$ to find

$$
\begin{equation*}
I_{n}=\left(Y_{n n} V_{n}-Y_{n r} Y_{r r}^{-1} Y_{r n}\right) V_{n} \tag{44}
\end{equation*}
$$



Figure 4: Generator 1 response with a fault at bus 7.


Figure 5: Generator 2 response with a fault at bus 7.

The matrix $\left(Y_{n n} V_{n}-Y_{n r} Y_{r r}^{-1} Y_{r n}\right)$ is the desired reduced matrix $Y \in \Re^{n \times n}$, where $n$ is the number of the generators. The network reduction illustrated by (43)-(44) is a convenient analytical technique that can be used only when the loads are treated as constant impedances. If the loads are not considered to be constant impedances, the identity of the load buses must be retained. Network reduction can be applied only to those nodes that have zero injection current.

Once the preliminaries calculations are made to obtain the $Y$ matrix for each fault in the correspondent bus, the network reduction for each fault is applied. For the first case of the analysis, the fault occurs at bus 7 and then the correspondent $Y$ matrix is obtained as shown in Tables 9, 10,


Figure 6: Generator 3 response with a fault at bus 7.


Figure 7: Generator 1 response with a fault at bus 8.
and 11 included at the Appendix. Then the network reduction of $Y$ matrix is applied and is defined as in Table 1.

For the second case of the analysis, the fault occurs at bus 8 and then the correspondent $Y$ matrix is obtained as shown in Tables 12, 13, and 14 included at the Appendix after the network reduction of $Y$ matrix is realized to obtain the reduced networks defined as in Table 2.

For the third case of the analysis, the fault occurs at bus 9 and then the correspondent $Y$ matrix is obtained as shown in Tables 15, 16 and 17 included at the Appendix after the network reduction of $Y$ matrix is realized to obtain the reduced networks defined as in Table 3.

Table 1: Reduced Y Matrices at bus 7.

| Type of network | Node | 1 | 2 | 3 |
| :--- | :---: | :---: | :---: | :---: |
|  | 1 | $0.846-j 2.988$ | $0.287+j 1.513$ | $0.210+j 1.226$ |
| Pre-fault | 2 | $0.287+j 1.513$ | $0.420-j 2.724$ | $0.213+j 1.088$ |
|  | 3 | $0.210+j 1.226$ | $0.213+j 1.088$ | $0.277-j 2.368$ |
| Faulted | 1 | $0.657-j 3.816$ | $0.000+j 0.000$ | $0.070+j 0.631$ |
|  | 2 | $0.000+j 0.000$ | $0.000-j 5.486$ | $0.000+j 0.000$ |
|  | 3 | $0.070+j 0.631$ | $0.000-j 0.000$ | $0.174-j 2.796$ |
|  | 1 | $1.181-j 2.229$ | $0.138+j 0.726$ | $0.191+j 1.079$ |
| Fault cleared | 2 | $0.138+j 0.726$ | $0.389-j 1.953$ | $0.199+j 1.229$ |
|  | 3 | $0.191+j 1.079$ | $0.199+j 1.229$ | $0.273-j 2.342$ |

Table 2: Reduced $Y$ Matrices at bus 8 .

| Type of network | Node | 1 | 2 | 3 |
| :--- | :---: | :---: | :---: | :---: |
|  | 1 | $0.938-j 2.798$ | $0.325+j 1.588$ | $0.251+j 1.315$ |
| Pre-fault | 2 | $0.325+j 1.588$ | $0.436-j 2.694$ | $0.230+j 1.123$ |
|  | 3 | $0.251+j 1.315$ | $0.230+j 1.123$ | $0.296-j 2.325$ |
|  | 1 | $0.736-j 3.569$ | $0.082+j 0.535$ | $0.063+j 0.534$ |
| Faulted | 2 | $0.082+j 0.535$ | $0.146-j 4.128$ | $0.006+j 0.058$ |
|  | 3 | $0.063+j 0.534$ | $0.006+j 0.058$ | $0.122-j 3.115$ |
|  | 1 | $0.850-j 3.252$ | $0.334+j 1.346$ | $0.075+j 0.569$ |
| Fault cleared | 2 | $0.334+j 1.346$ | $0.687-j 2.061$ | $0.032+j 0.148$ |
|  | 3 | $0.075+j 0.569$ | $0.032+j 0.148$ | $0.124-j 3.111$ |



Figure 8: Generator 2 response with a fault at bus 8.

## 6. Fault Simulation

The power electric system used in this paper is presented in Figure 3. It corresponds to the nine bus system. Figure 3 also includes the bus interconnection and the related parameters in the transmission lines. Data for simulation is given in Tables 7 and 8, respectively [6], where the modeling of the


Figure 9: Generator 3 response with a fault at bus 8.
system is explained and the related parameters for each synchronous generator are described.

In this paper, the 18 state variables related to 3 synchronous generators are stabilized, using the neural reduced model [7], reaching stabilization for the system with the fault in three different lines of transmission, for simulation the sample time is fitted to 0.005 ms .

Table 3: Reduced Y Matrices at bus 9.

| Type of network | Node | 1 | 2 | 3 |
| :--- | :---: | :---: | :---: | :---: |
| Pre-fault | 1 | $0.938-j 2.798$ | $0.325+j 1.588$ | $0.251+j 1.315$ |
|  | 2 | $0.325+j 1.588$ | $0.436-j 2.694$ | $0.230+j 1.123$ |
|  | 3 | $0.251+j 1.315$ | $0.230+j 1.123$ | $0.296-j 2.325$ |
| Faulted | 1 | $0.727-j 3.735$ | $0.135+j 0.787$ | $-0.004+j 0.001$ |
|  | 2 | $0.135+j 0.787$ | $0.263-j 3.377$ | $-0.002+j 0.000$ |
|  | 3 | $-0.004+j 0.001$ | $-0.002+j 0.000$ | $-0.010-j 4.168$ |
|  | 1 | $1.271-j 1.980$ | $0.290+j 1.247$ | $0.102+j 0.344$ |
|  | Fault cleared | 2 | $0.290+j 1.247$ | $0.380-j 2.957$ |
|  | 3 | $0.102+j 0.344$ | $0.149+j 0.702$ | $0.149+j 0.702$ |
|  |  |  | $0.209-j 2.853$ |  |



Figure 10: Generator 1 response with a fault at bus 9.


Figure 11: Generator 2 response with a fault at bus 9 .


Figure 12: Generator 3 response with a fault at bus 9.

TAble 4: Admittance loads.

| Load | Admittance |
| :--- | :---: |
| $A$ | $\bar{y}_{L 5}=1.2610-j 0.5044$ |
| $B$ | $\bar{y}_{L 6}=0.8777-j 0.2926$ |
| $C$ | $\bar{y}_{L 8}=0.9690-j 0.3391$ |

Table 5: Initial conditions of the generators.

| Initial conditions | Generator 1 | Generator 2 | Generator 3 |
| :--- | :---: | :---: | :---: |
| $x_{01}$ | 0.0396 | 0.3444 | 0.23 |
| $x_{02}$ | 377 | 377 | 377 |
| $x_{03}$ | 1.056 | 1.0502 | 1.0170 |
| $x_{04}$ | 0 | 0.622 | 0.624 |
| $x_{05}$ | 1.0478 | 0.7007 | 0.7078 |
| $x_{06}$ | -0.0425 | -0.7568 | -0.7328 |

There are three cases contemplated in the system simulation.
(1) The fault occurs near bus 7 at the end of the lines 5-7. Results are depicted in Figure 4 for generator 1, Figure 5 for generator 2, and Figure 6 for generator 3 .

Table 6: References for the system.

| References | Generator 1 | Generator 2 | Generator 3 |
| :--- | :---: | :---: | :---: |
| $x_{1 \text { ref }}$ | 0.0396 | 0.3444 | 0.23 |
| $x_{2 \text { ref }}$ | 377 | 377 | 377 |
| $x_{\text {3ref }}$ | 0.5 | 1.0502 | 1.0170 |

(2) The fault occurs near bus 8 at the end of the lines 8-9. Results are depicted in Figure 7 for generator 1, Figure 8 for generator 2, and Figure 9 for generator 3 .
(3) The fault occurs near bus 9 at the end of the lines 69. Results are depicted in Figure 10 for generator 1, Figure 11 for generator 2, and Figure 12 for generator 3.

For the cases above mentioned, the fault is incepted at 10 seconds of simulation and then it is possible to see that the system has a prefault state (before 10 seconds), a fault state (at 10 seconds), and a postfault state (after 10 seconds). The admittances for the loads are given in p.u. in Table 4.

The initial conditions for the system are given in Table 5.
It is important to note that initial conditions of the generators are defined by their respective parameters [1]; however, in order to test the NN approximation capabilities, it is common to use signals that can represent a wide range of frequencies; then, it is possible that plant signals can exhibit a high frequency behavior [10].

The control goal is to stabilize the power electric system and this is why the references given for each state variable of the neural reduced model for the multimachine system are proposed as in Table 6.

## 7. Conclusions

In this paper a SG discrete-time inverse optimal controller is synthesized for a reduced order neural model to stabilize a multimachine power electric system in the presence of a fault at line 7 , at line 8 , and at line 9 ; from simulation results, it can be seen that the proposed controller allows stabilizing the state in an efficient way in the three different cases, allowing the system stabilization after the fault occurs. As future work authors are considering the stability analysis including the neural decentralized controller, besides the analysis of control delay for closed loop system.

## Appendix

In this appendix, parameters used for simulations are presented. Tables 7 and 8 show the parameters for generators and transmission lines, respectively. Tables 9,10 and 11 display the $Y$ matrix of network with fault near to bus 7 for prefault, fault, and fault cleared conditions. Tables 12,13 and 14 show the $Y$ matrix of network with fault near to bus 8 for prefault, fault, and fault cleared conditions. Tables 15,16 and 17 present the $Y$

TABLE 7: Parameters of the generators.

| Parameter | Generator 1 | Generator 2 | Generator 3 |
| :--- | :---: | :---: | :---: |
| $H(\mathrm{sec})$ | 23.6400 | 6.4000 | 3.0100 |
| $T_{m}(\mathrm{pu})$ | 0.7160 | 1.6300 | 0.8500 |
| $T_{d 0}^{\prime}(\mathrm{sec})$ | 8.9600 | 6.0000 | 5.8900 |
| $T_{d 0}^{\prime \prime}(\mathrm{sec})$ | 0.2000 | 0.3000 | 0.4000 |
| $T_{q 0}^{\prime}(\mathrm{sec})$ | 0.3100 | 0.5350 | 0.6000 |
| $T_{q 0}^{\prime \prime}(\mathrm{sec})$ | 0.2000 | 0.3000 | 0.4000 |
| $X_{d}(\mathrm{pu})$ | 0.1460 | 0.8958 | 1.3125 |
| $X_{d}^{\prime}(\mathrm{pu})$ | 0.0608 | 0.1198 | 0.1813 |
| $X_{d}^{\prime \prime}(\mathrm{pu})$ | 0.0200 | 0.0500 | 0.0800 |
| $X_{q}(\mathrm{pu})$ | 0.0969 | 0.8645 | 1.2578 |
| $X_{q}^{\prime}(\mathrm{pu})$ | 0.0969 | 0.1969 | 0.2500 |
| $X_{q}^{\prime \prime}(\mathrm{pu})$ | 0.0200 | 0.500 | 0.0800 |
| $X_{l_{s}}(\mathrm{pu})$ | 0.0336 | 0.0521 | 0.0742 |

Table 8: Parameters of the transmission lines.

| Bus $i$ | Bus $j$ | $R_{i j}$ | $X_{i j}$ | $G_{i j}$ | $B_{i j}$ |
| :--- | :---: | :---: | :---: | :---: | :---: |
| 1 | 4 | 0.000 | 0.1184 | 0.000 | -8.4459 |
| 2 | 7 | 0.000 | 0.1823 | 0.000 | -5.4855 |
| 3 | 9 | 0.000 | 0.2399 | 0.000 | -4.1684 |
| 4 | 5 | 0.0100 | 0.0850 | 1.3652 | -11.6041 |
| 4 | 6 | 0.0170 | 0.0920 | 1.9522 | -10.5107 |
| 5 | 7 | 0.0320 | 0.1610 | 1.1876 | -5.9751 |
| 6 | 9 | 0.0390 | 0.1700 | 1.2820 | -5.5882 |
| 7 | 8 | 0.0085 | 0.0720 | 1.6171 | -9.7843 |
| 8 | 9 | 0.0119 | 0.1008 | 1.1551 | -9.7843 |
| 5 | 0 | 0.000 | 0.000 | 1.2610 | -0.2634 |
| 6 | 0 | 0.000 | 0.000 | 0.8777 | -0.0346 |
| 8 | 0 | 0.000 | 0.000 | 0.9690 | -1.1601 |
| 4 | 0 | 0.000 | 0.000 | 0.000 | 0.1670 |
| 7 | 0 | 0.000 | 0.000 | 0.000 | 0.2275 |
| 9 | 0 | 0.000 | 0.000 | 0.000 | 0.2835 |

matrix of network with fault near to bus 9 for prefault, fault, and fault cleared conditions.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Table 9: Y matrix of pre-faulted Network near to bus 7.

| Node | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | -j8.4459 | 0 | 0 | $j 8.4459$ | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | -j5.4855 | 0 | 0 | 0 | 0 | j5.4855 | 0 | 0 |
| 3 | 0 | -j4.1684 | 0 | 0 | 0 | 0 | 0 | 0 | j4.1684 |
| 4 | j8.4459 | 0 | 0 | $3.3074-30.3937$ | $-1.3652+j 11.6041$ | $-1.9422+j 10.5107$ | 0 | 0 | 0 |
| 5 | 0 | 0 | 0 | $-1.3652+j 11.6041$ | 3.8138-j17.8426 | 0 | $-1.1876+j 5.9751$ | 0 | 0 |
| 6 | 0 | 0 | 0 | $-1.9422+j 10.5107$ | 0 | 4.1019 - j15.4225 | 0 | 0 | $-1.2820+j 5.5882$ |
| 7 | 0 | j5.4855 | 0 | 0 | $-1.1876+j 5.9751$ | 0 | 2.8047 - j24.9311 | $-1.6171+j 13.6980$ | 0 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0 | $-1.6171+j 13.6980$ | 3.7412 - j23.6423 | $-1.1551+j 9.7843$ |
| 9 | 0 | 0 | j4.1684 | 0 | 0 | $-1.2820+j 5.5882$ | 0 | $-1.1551+j 9.7843$ | 2.4371-j19.2574 |

Table 10: $Y$ matrix of faulted Network near to bus 7.

| Node | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 0 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | $-j 8.4459$ | 0 | 0 | $j 8.4459$ | 0 | 0 | 0 | 0 |
| 2 | 0 | $-j 5.4855$ | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | 0 | $-j 4.1684$ | 0 | 0 | $3.3074-j 30.3937$ | $-1.3652+j 11.6041$ | $-1.9422+j 10.5107$ | 0 |
| 4 | $j 8.4459$ | 0 | 0 | 0 | $-1.3652+j 11.6041$ | $3.8138-j 17.8426$ | 0 | 0 |
| 5 | 0 | 0 | 0 | $-1.9422+j 10.5107$ | 0 | 0.0100 | 0 | 0 |
| 6 | 0 | 0 | 0.0100 | 0 | 0 | 0.0100 | 0 | 0 |
| 7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 8 | 0 | 0 | $j 4.1684$ | 0 | 0 | 0 | 0 | 0 |
| 9 | 0 |  | $-1.1019-j 15.4225$ | 0 | 0 | 0 |  |  |

Table 11: $Y$ matrix of fault cleared Network near to bus 7.

| Node | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | - j 8.4459 | 0 | 0 | $j 8.4459$ | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | -j5.4855 | 0 | 0 | 0 | 0 | j5.4855 | 0 | 0 |
| 3 | 0 | -j4.1684 | 0 | 0 | 0 | 0 | 0 | 0 | j4.1684 |
| 4 | j8.4459 | 0 | 0 | $3.3074-j 30.3937$ | $-1.3652+j 11.6041$ | $-1.9422+j 10.5107$ | 0 | 0 | 0 |
| 5 | 0 | 0 | 0 | $-1.3652+j 11.6041$ | $2.6262-j 11.8675$ | 0 | 0.01 | 0 | 0 |
| 6 | 0 | 0 | 0 | $-1.9422+j 10.5107$ | 0 | 4.1019 - j15.4225 | 0 | 0 | $-1.2820+j 5.5882$ |
| 7 | 0 | j5.4855 | 0 | 0 | 0.0100 | 0 | 1.6171 - j18.9559 | $-1.6171+j 13.6980$ | 0 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0 | $-1.6171+j 13.6980$ | 3.7412 - j23.6423 | $-1.1551+j 9.7843$ |
| 9 | 0 | 0 | j4.1684 | 0 | 0 | $-1.2820+j 5.5882$ | 0 | $-1.1551+j 9.7843$ | 2.4371-j19.2574 |

Table 12: $Y$ matrix of pre-faulted Network near to bus 8 .

| Node | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | - j 8.4459 | 0 | 0 | j8.4459 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | -j5.4855 | 0 | 0 | 0 | 0 | j5.4855 | 0 | 0 |
| 3 | 0 | -j4.1684 | 0 | 0 | 0 | 0 | 0 | 0 | j4.1684 |
| 4 | j8.4459 | 0 | 0 | 3.3074 - j30.3937 | $-1.3652+j 11.6041$ | $-1.9422+j 10.5107$ | 0 | 0 | 0 |
| 5 | 0 | 0 | 0 | $-1.3652+j 11.6041$ | 3.8138-j17.8426 | 0 | $-1.1876+j 5.9751$ | 0 | 0 |
| 6 | 0 | 0 | 0 | $-1.9422+j 10.5107$ | 0 | 4.1019 - j15.4225 | 0 | 0 | $-1.2820+j 5.5882$ |
| 7 | 0 | j5.4855 | 0 | 0 | $-1.1876+j 5.9751$ | 0 | 2.8047 - j24.9311 | $-1.6171+j 13.6980$ | 0 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0 | $-1.6171+j 13.6980$ | $3.7412-j 23.6423$ | $-1.1551+j 9.7843$ |
| 9 | 0 | 0 | j4.1684 | 0 | 0 | $-1.2820+j 5.5882$ | 0 | $-1.1551+j 9.7843$ | 2.4371-j19.2574 |

Table 13: $Y$ matrix of faulted Network near to bus 8.

| Node | 1 | 2 | 3 | 4 | 5 | 6 | 9 |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | $-j 8.4459$ | 0 | 0 | $j 8.4459$ | 0 | 0 | 0 | 0 |
| 2 | 0 | $-j 5.4855$ | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | 0 | $-j 4.1684$ | 0 | 0 | $3.3074-j 30.3937$ | $-1.3652+j 11.6041$ | $-1.9422+j 10.5107$ | 0 |
| 4 | $j 8.4459$ | 0 | 0 | 0 | $-1.3652+j 11.6041$ | $3.8138-j 17.8426$ | 0 | 0 |
| 5 | 0 | 0 | 0 | $-1.9422+j 10.5107$ | 0 | $-1.1876+j 5.9751$ |  |  |
| 6 | 0 | 0 | $j 5.4855$ | 0 | 0 | $-1.1876+j 5.9751$ | 0 | 0 |
| 7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 8 | 0 | 0 | $j 4.1684$ | 0 | 0 | 0 | 0 | 0 |
| 9 | 0 |  | $-1.1019-j 15.4225$ | 0 | 0 | 0 |  |  |

Table 14: $Y$ matrix of fault cleared Network near to bus 8.

| Node | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | - $ز 8.4459$ | 0 | 0 | $j 8.4459$ | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | -j5.4855 | 0 | 0 | 0 | 0 | j5.4855 | 0 | 0 |
| 3 | 0 | -j4.1684 | 0 | 0 | 0 | 0 | 0 | 0 | j4.1684 |
| 4 | j8.4459 | 0 | 0 | $3.3074-j 30.3937$ | $-1.3652+j 11.6041$ | $-1.9422+j 10.5107$ | 0 | 0 | 0 |
| 5 | 0 | 0 | 0 | $-1.3652+j 11.6041$ | $3.8138-j 17.8426$ | 0 | $-1.1876+j 5.9751$ | 0 | 0 |
| 6 | 0 | 0 | 0 | $-1.9422+j 10.5107$ | 0 | 4.1019-j15.4225 | 0 | 0 | $-1.2820+j 5.5882$ |
| 7 | 0 | j5.4855 | 0 | 0 | $-1.1876+j 5.9751$ | 0 | 2.8047 - j24.9311 | $-1.6171+j 13.6980$ | 0 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0 | $-1.6171+j 13.6980$ | $2.5861-j 13.8580$ | 0 |
| 9 | 0 | 0 | j4.1684 | 0 | 0 | $-1.2820+j 5.5882$ | 0 | 0 | $2.4371-j 19.2574$ |

Table 15: $Y$ matrix of pre-faulted Network near to bus 9.

| Node | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | -j8.4459 | 0 | 0 | j8.4459 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | -j5.4855 | 0 | 0 | 0 | 0 | j5.4855 | 0 | 0 |
| 3 | 0 | -j4.1684 | 0 | 0 | 0 | 0 | 0 | 0 | j4.1684 |
| 4 | $j 8.4459$ | 0 | 0 | $3.3074-j 30.3937$ | $-1.3652+j 11.6041$ | $-1.9422+j 10.5107$ | 0 | 0 | 0 |
| 5 | 0 | 0 | 0 | $-1.3652+j 11.6041$ | $3.8138-j 17.8426$ | 0 | $-1.1876+j 5.9751$ | 0 | 0 |
| 6 | 0 | 0 | 0 | $-1.9422+j 10.5107$ | 0 | 4.1019 - j15.4225 | 0 | 0 | $-1.2820+j 5.5882$ |
| 7 | 0 | j5.4855 | 0 | 0 | $-1.1876+j 5.9751$ | 0 | 2.8047 - j24.9311 | $-1.6171+j 13.6980$ | 0 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0 | $-1.6171+j 13.6980$ | $3.7412-j 23.6423$ | $-1.1551+j 9.7843$ |
| 9 | 0 | 0 | j4.1684 | 0 | 0 | $-1.2820+j 5.5882$ | 0 | $-1.1551+j 9.7843$ | 2.4371-j19.2574 |

Table 16: $Y$ matrix of faulted Network near to bus 9.

| Node | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | -j8.4459 | 0 | 0 | j8.4459 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | - j5.4855 | 0 | 0 | 0 | 0 | j5.4855 | 0 | 0 |
| 3 | 0 | -j4.1684 | 0 | 0 | 0 | 0 | 0 | 0 | 0.0100 |
| 4 | j8.4459 | 0 | 0 | $3.3074-j 30.3937$ | $-1.3652+j 11.6041$ | $-1.9422+j 10.5107$ | 0 | 0 | 0.0100 |
| 5 | 0 | 0 | 0 | $-1.3652+j 11.6041$ | 3.8138-j17.8426 | 0 | $-1.1876+j 5.9751$ | 0 | 0.0100 |
| 6 | 0 | 0 | 0 | $-1.9422+j 10.5107$ | 0 | 4.1019 - j15.4225 | 0 | 0 | 0.0100 |
| 7 | 0 | j5.4855 | 0 | 0 | $-1.1876+j 5.9751$ | 0 | 2.8047 - j24.9311 | $-1.6171+j 13.6980$ | 0.0100 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0 | $-1.6171+j 13.6980$ | $3.7412-j 23.6423$ | 0.0100 |
| 9 | 0 | 0 | 0.0100 | 0 | 0 | 0.0100 | 0 | 0.0100 | 0.0100 |

Table 17: $Y$ matrix of fault cleared Network near to bus 9 .

| Node | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | - j 8.4459 | 0 | 0 | $j 8.4459$ | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | -j5.4855 | 0 | 0 | 0 | 0 | j5.4855 | 0 | 0 |
| 3 | 0 | -j4.1684 | 0 | 0 | 0 | 0 | 0 | 0 | j4.1684 |
| 4 | j8.4459 | 0 | 0 | $3.3074-j 30.3937$ | $-1.3652+j 11.6041$ | $-1.9422+j 10.5107$ | 0 | 0 | 0 |
| 5 | 0 | 0 | 0 | $-1.3652+j 11.6041$ | $2.6262-j 11.8675$ | 0 | $-1.1876+j 5.9751$ | 0 | 0 |
| 6 | 0 | 0 | 0 | $-1.9422+j 10.5107$ | 0 | $2.8199-j 9.8343$ | 0 | 0 | 0 |
| 7 | 0 | j5.4855 | 0 | 0 | 0.0100 | 0 | 1.6171 - j18.9559 | $-1.6171+j 13.6980$ | 0 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0 | $-1.6171+j 13.6980$ | $3.7412-j 23.6423$ | $-1.1551+j 9.7843$ |
| 9 | 0 | 0 | j4.1684 | 0 | 0 | 0 | 0 | $-1.1551+j 9.7843$ | $2.4371-j 19.2574$ |

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## Research Article

# Passengers' Evacuation in Ships Based on Neighborhood Particle Swarm Optimization 

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#### Abstract

A new intelligent model to simulate evacuation behavior in ships called neighborhood particle swarm optimization is proposed. This model determines the rules of behavior and velocity updating formulas to solve staff conflicts. The individuals in evacuation are taken as particles in PSO and update their behaviors by individual attributes, neighborhood attributes, and social attributes. Putting the degree of freedom movement of ships into environment factor and using the real Ro-Ro ship information and IMO test scenarios to simulate the evacuation process, the model in this paper can truly simulate the behavior of persons in emergency and provide a new idea to design excellent evacuation model.


## 1. Introduction

In recent ten years, with the rapid development of shipbuilding industry, the number of high speed and high load of ship has increased sharply. It brings us convenience and wealth and meanwhile increases the frequency of marine perils. Therefore, a security system which can ensure all shipboard personnel evacuation needs to be established urgently. The security system should be matched with set routes, procedures, effective decision support and management, rescue apparatus, and so forth. An optimized design scheme should be considered in ship design, which can enhance passengers' evacuation performance. Even in the situation of safety sailing measure failures, available people assembly and evacuation can also be considered as the final safety shelter to avoid the disaster and the economic loss. Thus, the design of efficient evacuation model has become one of the hottest research areas focusing on ship industry.

In the personnel evacuation progress, dynamic programming for behavior of personnel is needed. With the wide use of evacuation model, more and more research institutions have been dedicated to exploit it. The majority of evacuation models adopt implicit actions, based on function programming, rule-based behavior, or basic behavior of intelligent agent to express evacuation behavior. During the evacuation
progress, Helbing's social force model [1] is a typical representation of function programming. Using differential equation to describe the behavior of personnel is too complex to fit large-scale personnel evacuation. Cellular automaton model [2] is a regular basic model, which possesses a good simulation effect but lacks theoretical foundation. Briefly speaking, describing the behavior of personnel in reality by means of several rules is obviously lack of authority. Agent's [3] model is the representation of intelligent agent technology. Izquierdo et al. put forward the idea of applying particle swarm optimization algorithm to simulate human behavior [4]; this algorithm is adopted to large-scale personnel evacuation yet described roughly. Based on above ideas Zheng et al. improve the rules of behavior using the particle swarm optimization algorithm to simulate the process of evacuation [5], while the simulation results lack persuasion because the freedom movement of ship is not taken into consideration. There are many domestic scholars doing research on the passengers' evacuation under the marine environment [6, 7]. In this paper, on the basis of using the particle swarm optimization algorithm to simulate the process of passengers' evacuation, we improve the simulation system of algorithm and present the simulation method based on the neighborhood particle swarm optimization algorithm. In the meantime we present the update rules of passengers' velocity, the solutions of staff
conflicts, and take the freedom movement of ship as one of the environment factors. The model in this paper can simulate the process of passengers' evacuation under the condition of ship's freedom movement, and the simulation results show that the algorithm has a high practicability, which provides the ideas for the design of evacuation models with high efficiency and extreme stability.

## 2. Particle Swarm Optimization

The particle in PSO personnel evacuation model can be described as follows: for the $i$ th particle, we consider the candidate solutions of particle swarm optimization algorithm for personnel microscopic evacuation problems as points of space. Even though the solution space can be N -dimensional in the application of particle swarm optimization algorithm, while simulating the process of personnel evacuation, we only need two-dimensional space or three-dimensional space. In the whole particle movement process of particle, the attributes of the $i$ th particle can be described by the following three variables:

The current position:

$$
\begin{equation*}
X_{i}=\left(x_{i 1}, x_{i 2}, \ldots, x_{i N}\right) \tag{1}
\end{equation*}
$$

The best evacuation individual location reached in the past:

$$
\begin{equation*}
P_{i}=\left(P_{i 1}, P_{i 2}, \ldots, P_{i N}\right) \tag{2}
\end{equation*}
$$

The current velocity of the individual:

$$
\begin{equation*}
V_{i}=\left(v_{i 1}, v_{i 2}, \ldots, v_{i N}\right) \tag{3}
\end{equation*}
$$

In the application of standard particle swarm optimization algorithm to simulate the personnel evacuation, the best positions of particles are updated in each iteration. The best position of particles in the whole group is crucial in the searching process; $P_{i}$ represents the best position of each particle. In each iteration, the particles not only learn from the global best position but also learn from the best position of their own. The combination of individual attributes and social attributes for a particle has significant effect on the problems of simulating the personnel evacuation. In the process of simulating the personnel evacuation, the update of particle's velocity and position can be represented by the following formula:

$$
\begin{gather*}
X_{i+1}=X_{i}+V_{i+1},  \tag{4}\\
V_{i+1}=w \cdot V_{i}+c_{1} \cdot \operatorname{rand}()\left(P_{i}-X_{i}\right)+c_{2} \cdot \operatorname{rand}()\left(P_{g}-X_{i}\right) \tag{5}
\end{gather*}
$$

From the updating formula (5), we can see that the particles obey the inertia of their own firstly to retain part of their own attributes $w \cdot V_{i}$ and then update their behavior according to the best cognitive ability of the environment of their own $P_{i}$; the social cognitive ability of particle movement $P_{g}$ is the global best position $[8,9]$. This mechanism optimization reflects people's actual behavior reasonably.

In the optimization process, particles update their position in each iteration until they reach a special region [10]. In the personnel microscopic evacuation problems described in this paper, the best position of the individual is the safety exit of evacuation. The safety exit can be a single outlet and can also be the selection of multiple exports. When the individual reaches an evacuation exit, we consider that the evacuation individual has found the optimal position, at the same time the current individual withdraws the evacuation sequence, and we record its trajectory and evacuation time. When all the evacuation individuals evacuate safely through the evacuation exit and one process of evacuation has finished, then we record the time of evacuation and the scheme of evacuation.
2.1. Neighborhood Particle Swarm Optimization. Basic particle swarm optimization algorithm has a significant effect on solving the problems about single door evacuation because the targets of all the particles are consistent, moving towards the same evacuation exit [4]. However, when it comes to the complex problems about network, the environment is abstract and there are more than one evacuation exit. In the basic particle swarm optimization algorithm, the particles determine the next position merely according to the individual cognition and social behavior that will cause the problem of all the particles moving toward the best global position. Those problems will lead to large-scale congestion and staff conflicts, which is contrary to the actual evacuation. The behaviors of individuals in actual evacuation mainly depend on the individual cognitive ability for the environment, social behavior, and the behavior of their neighborhood. Therefore, in this paper we will use the improved particle swarm optimization algorithm that contains the neighborhood learning factor to simulate the evacuation, which is out of the limitation of the simple network environment and suitable for the complex evacuation.

In neighborhood particle swarm optimization algorithm, the particles update their velocity and position according to individual behavior, the behavior of the whole group, and the best individual experience in the neighborhood. In the process of evacuation, the neighborhood of a particle is the channel node area which is connected with the position of current particle. Putting the neighborhood learning mechanism into the PSO algorithm not only can simulate the herd behavior of the crowd in the process of evacuation successfully, but also can effectively avoid the problem of all the particles moving toward the local optimal point which could lead to large-scale staff conflicts and congestion. According to the above idea, we modify the velocity updating formula of particle swarm as follows:

$$
\begin{gather*}
V_{i+1}=w V_{i}+c_{1} \operatorname{rand}()\left(P_{\text {best }}-X_{i}\right)+c_{2} \operatorname{rand}()\left(G_{\text {best }}-X_{i}\right) \\
+c_{3} \operatorname{rand}()\left(N_{\text {best }}-X_{i}\right)  \tag{6}\\
X_{i+1}=X_{i}+V_{i+1} . \tag{7}
\end{gather*}
$$

$c_{1}, c_{2}$ are learning factors in standard particle swarm optimization algorithm, while $c_{3}$ is neighborhood learning
factor. $X_{i}$ is the current position of particles. $P_{\text {best }}$ is the best location of particles reached in the past, $G_{\text {best }}$ is the global best position of the particle swarm, and $N_{\text {best }}$ is the best position of the neighborhood of the particle. The application of neighborhood particle swarm optimization algorithm can simulate the behavior of staff conflicts effectively and can solve the problems on the behavior of staff congestion in some extent. In this paper, we consider the circle with the diameter of 0.6 meters as the individual in evacuation, and the neighborhood is the circular area with the evacuation individual as its center and diameter of 2 meters.
2.2. Behavior Process. Firstly, formula (5) presents how the current particle can determine the movement at next moment through the calculation, and then we will judge the feasibility of the movement through the social attributes of populations. If the movement exceeds the environment, it will be restricted and even canceled. The parameter $w$ in formula (5) is

$$
\begin{equation*}
w=0.5+\frac{1}{2 *(\ln (k)+1)} \tag{8}
\end{equation*}
$$

This setting for $w$ is according to [11], and $k$ is the iterative number. As the iteration proceeds, $w$ decreases from 1 to 0.5 . This can ensure the global convergence of particles. At the beginning of iteration, the great change of the speed variation can ensure that particles moved to the best position quickly. The change of particles' velocity decreases, with the iteration number increasing, which ensures stronger local convergence when particles arrive at the optimal point. Then $c_{1}=3, c_{2}=2$, and $c_{3}=3$.

Apart from inertia factors, calculation of personnel's velocity is contained by the following factor: environmental layout, individual, and society-related factor. We mainly consider two aspects in evacuation. The first is the choice of exits. In the space of multiple exits, people choose exit generally depending on individuals' ability to perceive information from environment. The second is the queuing phenomenon caused by human-following behavior. In the passenger evacuation progress, congestion occurs frequently.

## 3. Velocity Updating Rules

In the process of actual evacuation, the velocity of people's movement will not increase unlimitedly, whereas the velocity will distribute in a fixed interval. Therefore, the velocity definitely has the upper limit in the PSO model; namely, $V_{i} \leq V_{\max }$. From another perspective, the crowd density of passengers' current space has effect on the moving speed of passengers. According to IMO determination of personnel's speed, this paper takes $1.2 \mathrm{~m} / \mathrm{s}$ as the moving speed of personnel in normal circumstance, and the moving speed will decrease when the crowd density increases. The maximum moving velocity of passengers can be described by the following formula [5]:

$$
V_{\max }= \begin{cases}1.2 \mathrm{~m} / \mathrm{s} & \rho \leq 20  \tag{9}\\ 0.9 \mathrm{~m} / \mathrm{s} & \rho>20\end{cases}
$$



Figure 1: Schematic diagram of the staff conflicts.
$\rho$ is defined as the particle's density for the current position; it is the number of people in the circular neighborhood which takes the current particle as center and radius of 2 meters. The individuals in evacuation are taken as circles with the diameter of 0.6 meters. Every particle forms a circular area in which simulates the staff conflicts and updates the velocity. The density of particles changes greatly in different time and different locations. Within the process of evacuation, the positions of passengers and the number of particles around change constantly.

In the evacuation process of particles, each particle has its own unique occupy space that other particles are unable to invade. The occupy space of particles will decrease when the density of particles increases. We use $D$ to represent the diameter of particle's occupy space. The relationship between $D$ and the density of particles can be assumed as follows:

$$
D= \begin{cases}0.6 \mathrm{~m} & \rho<12  \tag{10}\\ 0.4 \mathrm{~m} & 12 \leq \rho \leq 20 \\ 0.3 \mathrm{~m} & \rho>20\end{cases}
$$

Conflicts among the passengers will still exist even though in the application of ideal evacuation process. If a particle's updating position has been occupied by others, the particle will produce a new velocity to avoid staff conflicts (see Figure 1).

Particle A updates its position during the process of evacuation, and its updating position is $D$, while $D$ has a confliction of physical location with $B$. Therefore, the particle A should create a new velocity to make itself move to position C so as to avoid the conflicts with $B$.

## 4. Passengers' Evacuation Based on PSO Model

Simulating the process of evacuation with particle swarm optimization algorithm can obtain the behavior characteristics of many individuals successfully. It considers not only the perception ability of individuals on the environment, but also the social attributes of groups in the process of evacuation to make the simulation results more suitable to the real process.

In the process of evacuation, various behaviors of personnel will have an effect on the evacuation velocity of
individuals, for instance, aggregation behavior, individuals in evacuation move towards to the center of population or move towards to the position with fewer people; people will maintain a certain distance between each other, which means a person will have the same velocity with other persons who are nearby, people will obey the evacuation signal, and so forth. A person's individual property such as age, gender, and health status will have effects on their walking speed. An excellent simulation model can reflect the influence of personnel's psychological factors such as pain, fear, and negative emotions. People's cognition degree of sound, environment, and facilities will have influences on the process of evacuation. We can establish the comprehensive consideration of the above situation with the application of the fuzzy comprehensive evaluation. Simultaneously, it will have great influence on the evacuation behavior of population if passengers' psychological effects are considered too much. We suppose that the passengers have slight psychological effects in the research process so that they will not lose the behavior of evacuation because of panic caused by emergencies.

For the selection of evaluation function, it is expressed as the distance between the person and the nearest evacuation exit to him or her. $X$ is the position of current particle and $E$ is the collection of evacuation exit:

$$
\begin{equation*}
F(X)=d(X, E)=\min \{d(X, e), e \in E\} \tag{11}
\end{equation*}
$$

Function $F$ is obviously a nonlinear function in most conditions. The procedures of applying the PSO model to solve the problems of passengers' evacuation can be described as follows.

Step 1. Set the number of particles (the summation of individuals in evacuation). Distribute the initial position randomly. Set the number and the positions of evacuation exits.

Step 2. Set the value of inertia factor $w$, learning factors $c_{1}$ and $c_{2}$, and neighborhood learning factor $c_{3}$. Record all the particles' in current position and velocity.

Step 3. Calculate the value of fitness function for all the particles. If the distance between a particle and a safety exit is 0 , then the current particle evacuates successfully and this particle is no longer retained in the following process of evacuation. Record the particles whose distance to the safety exits is more than 0 .

Step 4. Update particles' velocity according to formula (6); judge whether the condition of the velocity's upper limit is satisfied. If it is satisfied, update the velocity as usual; if it is not satisfied, the velocity's upper limit is assigned to the current particle.

Step 5. Judge whether the particles' conflicts exist; namely, whether the movement direction of current particle has been occupied. If it is not occupied, update the position as usual; if it is occupied, generate the velocity increment randomly and update the position.

Step 6. Judge if all the particles have been reached to the safety exits: if it is satisfied, go to Step 7; else go to Step 3.

Step 7. When the process of evacuation is finished, output the evacuation time and the dynamic graph of evacuation.

The algorithm will finish if the individuals in evacuation have evacuated successfully, and then output the evacuation time.

## 5. Simulations

In the following context, we will discuss the effectiveness of the PSO evacuation model proposed in this paper through the numerical experiment. Firstly, we set the parameters of neighborhood particle swarm optimization algorithm to simulate the problem of evacuation, $w=0.5+0.5 /[\ln (k)+1]$, $w$ varies from 1.0 to 0.5 , and $c_{1}=3, c_{2}=2$, and $c_{3}=3$. The iteration step $T=0.5$ [12].

This research selects a cabin with the 2 meters wide door as the research object to simulate the process of microscopic passengers' evacuation by the software of Visual Studio 2010.

Figure 2 shows that individuals in evacuation distribute randomly in the evacuation space at the beginning of evacuation. The individuals that are close to the safety exits of evacuation can find the evacuation plan quickly, and the individuals that are far from the safety exits of evacuation can also find the direction of evacuation fleetly. With the learning strategy of particle swarm, the individuals in evacuation can make quick and effective evacuation judgments in order to find the best evacuation routes which are suitable to their current locations.

### 5.1. The Influence of Doors' Dimension for the Process of

 Evacuation. Different sizes of the door have quite different evacuation capacities. Obviously, the larger size can get the well effect of evacuation. But in the period of ship design, the size of doors cannot be randomly designed due to the limitation of space and size of ship. Hence, the most appropriate size for doors needs to be designed under limitation of surroundings and conditions to improve the evacuation capacity to the ultimate extreme. The influence of doors' dimension for the process of evacuation is shown in Figures 3 and 4.By fixing the number of escaping people at 200 as initial value, Figure 3 shows the changing curve of egress time of 200 people with the size of the door increasing. From the figure we can see obviously that, with the size of the door increasing, the evacuation time is effectively reduced. When the size is larger than 4 meters, it has little effect on the evacuation time. It shows that, in this condition, the size of the door has lower influence on the evacuation time, and there exists the best threshold for the size.

For different sizes of the door, with the number of people increasing, the changing curve of the evacuation time is shown in Figure 4. When the door width is set to 2, 3, 4 , and 6 meters, respectively, the evacuation time significantly increases with the number of peopleincreasing. When


Figure 2: The distribution of the evacuation of people in different time.


Figure 3: Effect of the door size on evacuation time.
the door width is set to 4 or 6 meters, the difference of the evacuation time is small. But when setting the dimension for 2 or 3 meters, the difference is huge. The evacuation time for 2 meters is nearly 1.5 times longer than 6 meters of the doors' dimension. However when the doors' dimension is over


Figure 4: Effect of door size and number of people on evacuation time.

4 meters, the difference was not significant. This is verified by Figure 3. In the premise of the suitable designing dimensions of the ship, there exists the best dimension for evacuation.


Figure 5: The time influences of the ship's heel and trim.

If the doors' dimension is fixed, the evacuation time obviously increases with the increasing number of people.
5.2. The Influences of the Ship's Heel and Trim. An experiment is conducted to test the time influences of the ship's degree of freedom. It is conducted under the condition that the angle of ship's heel changes from 0 to 35 degrees and the angle of ship's trim changes from -20 to 20 degrees. The number of passengers is fixed as 200 as well. Figure 5 reflects the influences of the ship's DOF (degree of freedom) movement on passengers' evacuation time. Figure 5(a) is in the condition of ship's heel. When the angle of ship's heel changes from 0 to 15 degrees, the effect on passengers' behavior of evacuation is very little. However, the influence is enhanced gradually when the angle of ship's heel is more than 20 degrees, and when it turns to more than 30 degrees, the evacuation time of the whole 200 persons is $3-6$ times of that in steady state. The above fact proves that it is not of any significance without the consideration of ship's movement in the simulation of passengers' evacuation process.

Figure 5(b) exhibits the influences of the ship's trim on passengers' evacuation time. When the angle of ship's trim changes from -15 to 15 degrees, it has fewer effect on passengers' evacuation time, while when the angle of ship's trim increases to 20 degrees, it has a tremendous impact on individuals in evacuation. The angle of ship's trim can also be interpreted as the angle of lateral inclination. With the increasing of trim's angle, the left-right force among the persons increases, which will greatly hinder the movement of passengers; the conflicts among the individuals and the congestion caused from the conflicts are increased obviously. However, in the condition of ship's heel, it is the pre-post force that increases, the influence of pre-post force caused by ship's heel will be significant when the crowd density is high in evacuation. If the distribution of individuals is not
intensive, the force caused by ship's heel will have a slight effect on the population movement, but it will seriously affect the behavior of individuals in evacuation. The increasing of the heel's angle and the inclination of walking plane will increase the difficulty for the individuals in evacuation greatly when they are walking and at the same time it depends on the individual differences in ability. It will make the whole process of evacuation more difficult when the angle of ship's heel and trim increases. Thus, ship's heel and trim are the main factors that influence people's behavior in evacuation.
5.3. The Process of Simulating the Evacuation System with Multiple Doors. Also in the evacuation space of $25 \times 25 \mathrm{~m}$, four evacuation exits are settled in the evacuation space. In the designation of ships, the size of the door is restricted, so in order to match the actual situation, the width of the door is set to 2 meters. The initial distribution of persons is random and the crowd is divided into three parts, the youth, the elderly, and the children. Meanwhile, all of them are given different walking speed and initial speed. The red circles represent the elderly, purple circles represent the children, and yellow circles represent the youth. The distributions of the crowd at $0 \mathrm{~s}, 40 \mathrm{~s}, 80 \mathrm{~s}$, and 120 s can be seen from Figure 6; the results of simulation clearly show the aggregation behavior of adults and children, as well as the walking condition of elderly people.

The results can be seen from Figure 6 that the algorithm of PSO model is still remarkable on the evacuation system with multiples doors. The initial distribution of persons in the space is random. But with the evacuation process continuing, the passengers find their own evacuation directions immediately. The behavior of group is significant and the staff conflicts are lower in the process of evacuation. Figure 7 shows the curve of passengers' evacuation time under the condition of four doors with the increasing number


Figure 6: The distributions of the persons in the case of four doors.


Figure 7: The relationship between evacuation time and the number of passengers.
of passengers, and the results reflect that the evacuation time increases significantly when the number of passengers increases.

We explore the influence of the ship's DOF movement for the conditions of four doors with the fixed 200 passengers. The angle of ship's heel is set to change from 0 degrees to 35 degrees and the angle of ship's trim changes from -20 degrees to 20 degrees. Figure 8 shows the effects of ship's DOF movement on the evacuation time. The curve shape of evacuation time in the condition of four doors is roughly equal to that of single door. But the whole evacuation time decreases because the evacuation space is larger, and the staff conflicts are lower in the condition of four doors. Meanwhile, the more the evacuation exits are, the stronger the evacuation capacity of the cabin is. If we only take the evacuation ability of a cabin into consideration, the evacuation time should be a quarter of that in the condition of single door. However, the actual situation is not the same as what we have mentioned above. It is because the personnel microscopic behavior simulation algorithm based on the PSO model takes the individuals' movement into consideration. The congestion, conflicts, and queuing phenomenon of persons will increase the whole evacuation time, so that it will make the simulation results more realistic.
5.4. IMO Test 8: The Simulation Test of Personnel Convection. In the process of passengers' evacuation in marine environment, the phenomenon of personnel convection is always


Figure 8: The time influences of the ship's heel and trim in the case of four doors.


Figure 9: Simulation environment of test 8.
accompanied by the evacuation process, and it is one of the main factors affecting personnel behavior. Then we will simulate the scene of IMO convection test with the PSO evacuation model. The scenes of convection test regulated by IMO are as follows: two evacuation groups are arranged in the adjacent compartments, respectively, and there is a passage which connects the two compartments. People in room 1 need to move to room 2, and people in room 2 need to move to room 1 . The initial number of individuals in room 1 is 100 [13], and the characteristics of the individuals are $30-50$-year-old evacuation persons. The passage is 10 meters long and 2 meters wide. The phenomenon of convection occurs in a large range when passengers in room 2 are trying to move to room 1 , and the initial number of individuals in room 2 is set, respectively, as $0,10,50$, and 100 (see Figure 9). The results of simulation are shown in Table 1; the evacuation time significantly increases when the number of the convective population increases. Figure 10 shows the process of evacuation as the time goes by under the condition when the number of convection population is 100 .

The experiment results reflect the fact that the evacuation time will increase greatly when the number of individuals

Table 1: The contrast of PSO and EVI.

| Parameters | Evacuation time (s) |  |
| :--- | :---: | :---: |
| Convection passengers | PSO | EVI |
| 0 | 83.7 | 88.9 |
| 10 | 92.6 | 125.6 |
| 50 | 136.9 | 229.1 |
| 100 | 215.1 | 327.9 |

in evacuation in room 2 increases. In the meanwhile, in the process of evacuation, the personnel convection is largely reduced because of the attributes of social learning of PSO model. We can see from Table 1 that the research results in this paper are significantly enhanced [14] compared with other similar researches, and the whole evacuation time has greatly improved.

## 6. Conclusions

This paper mainly studies how to simulate the process of passengers' evacuation under the circumstance of emergency evacuation and establishes the simulation model of the passengers based on the neighborhood particle swarm optimization algorithm. In this paper, we consider the particles in the swarm as the individuals in evacuation, and the behaviors of the particles are directed by their own learning abilities, the perception abilities of environment, and the social attributes, which can simulate the real process of evacuation. We use the scene of IMO convection test as well as the real data of a ship to test the properties of the model presented in this paper. The simulation results show that the model of passengers' behavior based on PSO algorithm is very effective and has a strong promotional value, and it provides ideas for designing a stable and efficient evacuation behavior model.


Figure 10: Simulation effects of test 8 with all 100 persons in room 1 and room 2.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Energy Flow Chart-Based Energy Efficiency Analysis of a Range-Extended Electric Bus 

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#### Abstract

This paper puts forward an energy flow chart analysis method for a range-extended electric bus. This method uses dissipation and cycle energy, recycle efficiency, and fuel-traction efficiency as evaluation indexes. In powertrain energy efficiency analysis, the range-extended electric bus is developed by Tsinghua University, the driving cycle based on that of Harbin, a northern Chinese city. The CD-CS and blended methods are applied in energy management strategies. Analysis results show with average daily range of 200 km , auxiliary power of 10 kW , CD-CS strategy, recycle ability and fuel-traction efficiency are higher. The input-recycled efficiency using the blended strategy is $24.73 \%$ higher than CD-CS strategy, while the output-recycled efficiency when using the blended strategy is $7.83 \%$ lower than CD-CS strategy.


## 1. Introduction

Compared with conventional fuel vehicles, application of electric vehicle decreases the dependency on petroleum and has the advantages of high energy efficiency and low environmental impact [1-3]. For a pure electric bus, the cost of a battery pack that can meet the driving range is too high; meanwhile, vehicle weight is too large for adding a large battery pack. A range-extended electric vehicle is regarded as one of the most suitable solutions for powertrain schemes, because of the maximum utility of the electric drive and the minimum capacity requirement of battery packs.

The main powertrain configurations of range-extended electric vehicles are series plug-in hybrid electric vehicles [4] and the Chevrolet Volt, produced by General Motors Corporation (GM) [5]. This paper will analyze a series plugin hybrid electric bus.

In the studies of energy efficiency and fuel economy of range-extended electric vehicles, vehicle performance is analyzed on the basis of energy consumption and greenhouse gas emissions on the well-to-wheel and tank-to-wheel paths [ 6,7$]$. Well-to-wheel fuel economy and greenhouse gas emissions data were obtained using the greenhouse gases, regulated emissions, and energy use in transportation (GREET)
software model. The tank-to-wheel process is characterized by the recuperation and fuel-traction efficiencies, which are quantified and compared for two optimization-based energy management strategies.

The improvement of fuel economy for a range-extended electric vehicle can be realized by matching powertrain parts and a model selection method [8-12]. An optimal gen-set operating line method can minimize fuel consumption at a set level of electric output power. Series hybrid vehicles with direct injected stratified charge (DISC) rotary engines are proven to be more efficient in pure electric mode in terms of energy consumption and greenhouse gases (GHG) emissions than in vehicles with reciprocating engines.

Energy efficiency and fuel economy of range-extend electric vehicles can be improved by studying energy management strategy [13-16]. Researchers use dynamic programming strategies and equivalent consumption minimization strategies as well as Pontryagin's minimum principle strategy to analyze energy efficiency and fuel economy of rangeextended electric vehicles, and results show that optimized energy management strategy can improve energy efficiency and fuel economy to a certain extent.

In conclusion, current studies on configuration analysis and energy management strategy on range-extended electric


Figure 1: Powertrain configuration of the range-extended electric bus.
vehicles mainly focus on passenger vehicles, but work is rarely conducted into range-extended electric buses. Reference [17] asserts that a driving cycle significantly influences the energy efficiency and fuel economy of the vehicle. It proposes that construction and optimization of energy management strategy should consider different driving cycles. A system of energy efficiency analysis based on a certain driving cycle is the foundation of an optimal control strategy.

This paper focuses on the application requirements of the range-extended electric bus developed by Tsinghua University in Harbin and establishes the powertrain model of the bus based on the construction of the Harbin driving cycle. It examines the energy efficiency of the range-extended electric bus with two different energy management strategies (CD-CS and blended) and proposes improvement methods for energy efficiency.

## 2. Configuration and Principle of the Range-Extended Electric Bus

The range-extended electric vehicle lies between the plugin hybrid electric vehicle and pure electric vehicle. Compared with a pure electric vehicle, a range-extended electric vehicle is supplemented with an onboard power generation system (range-extender) $[18,19]$. The range-extender consists of engine, generator, and rectifier. The engine continually charges the power battery, so the driving range can be greatly increased to the level of a conventional internal combustion engine vehicle. The engine and power battery of a rangeextended electric bus can be optimized at the same time. The working area of the engine can be optimized according to the driving cycle and the engine efficiency can be improved. The engine can operate with low pollution and fuel consumption. As for the battery, working condition of the power battery can also be optimized. If the power battery can continually work in good condition without overcharging or overdischarging, battery life can be extended. Braking energy can be recycled and energy consumption is decreased. The range-extender
solves the problems of the energy consumption of airconditioning, lighting, heating, and other electric auxiliaries, making the range-extended electric bus the most suitable solution for city buses.

A typical range-extended electric powertrain is shown in Figure 1. In a range-extended electric vehicle, wheels are driven directly by an electric motor. The motor draws energy from a battery pack and drives in pure electric mode when battery energy is available. Once the battery has been mostly depleted, the motor draws power from the range-extender, composed of an internal combustion engine and generator, in conjunction with a battery. Range-extended electric vehicles are designed with a predetermined all-electric range (AER). The AER represents the distance that the vehicle can travel using the energy stored in its battery only, without the engine and generator. Vehicles with a higher AER must have larger, heavier, and more expensive battery systems. The rangeextended electric powertrain configuration is one of the most attractive applications for the diesel engine.

We can see the powertrain configuration of the rangeextended electric bus discussed in this paper; the energy flow conditions of different driving modes are analyzed, as is shown in Figure 2. There are three driving modes: pure electric drive mode, range-extended mode, and regenerative brake mode.

Pure electric drive mode is shown in Figure 2(a). If SOC is high, the powertrain begins pure electric mode, whereby the engine stops and the motor will be driven by a power battery. Cheaper electric energy from the power grid is fully utilized. Fuel consumption and pollution are reduced in this mode. If SOC decreases to the set starting value, the range-extender begins and the powertrain works in range-extended mode.

Powertrain working in range-extended is shown in Figure 2(b). To increase driving range, if SOC decreases to the set starting value, the range-extender starts to generate power, reducing the rate of electricity loss and ensuring the motor can work to drive the bus. This mode can be divided into two kinds. One, if the output power of range-extender is lower than the motor required power, the lacking electric energy is provided by battery; the battery discharges. Two, if the output


Figure 2: Driving modes of the range-extended electric bus.


Figure 3: The range-extended electric bus developed by Tsinghua University.
power of the range-extender is higher than the required motor power, the redundant electric energy is reserved in battery, charging the battery. The output power of the rangeextender is not directly influenced by the driving conditions and can be optimized in the high efficiency working areas of the engine and motor.

If the bus is braking, the motor can work in regenerative brake mode, as is shown in Figure 2(c). The motor provides braking torque for the vehicle wheels, and braking energy is transferred into electric energy reserved in the battery. Braking energy is not transferred into heat and lost; it is recycled.

For an individual axle drive bus, only wheels driven by the motor can recycle braking energy. Other wheels are stopped by mechanical braking. Braking energy is partly recycled and mechanical braking is also used on driven wheels for safety.

The range-extended electric bus analyzed in this paper is developed by Tsinghua University, shown in Figure 3. The powertrain is designed based on matching powertrain parts and model selection found in [20]. The generator is a permanent magnet generator and the traction motor is an asynchronous motor. Key parameters of the powertrain are listed in Table 1.

## 3. System Modeling of the Range-Extended Electric Bus

To analyze energy efficiency and fuel economy of the rangeextended electric bus, system models based on benchmarks and modeling lines of [21-24] are built. The basic model of the range-extended electric system can be divided into four modules: range-extender module, traction motor module, power battery module, and the vehicle longitudinal dynamics module. Considering the high complexity of a diesel engine, permanent magnet synchronous generator, and the rectifier and driving motor, relevant components are tested by benchmarks and the characteristics MAP are determined according to the test results. Then, the simulation models are built based on the MAP, which replaces the complex mathematical description, reduces modeling complexity, and therefore improves model credibility.

Table 1: Powertrain parameters of range-extended electric bus.

| Vehicle | Size (length $\times$ width $\times$ height)/mm | $11980 \times 2550 \times 3200$ |
| :---: | :---: | :---: |
|  | Vehicle mass/kg | 13000 |
|  | Rated passengers | 78 |
|  | Windward area/m ${ }^{2}$ | 7.5 |
|  | Air resistance coefficient $C_{D}$ | 0.75 |
|  | Rolling resistance coefficient $f$ | $0.0076+0.00056 u_{a}$ |
|  | Rolling radius $r / \mathrm{m}$ | 0.512 |
|  | Speed ratio of main reducer $i_{0}$ | 6.2 |
|  | Speed ratio of transmission <br> $i_{g}$ | 2.34 |
| Motor | Continuous power/kW | 100 |
|  | Peak power/kW | 180 |
|  | Maximum torque/ $\mathrm{N} \cdot \mathrm{m}$ | 860 |
|  | Maximum speed/r/min | 4500 |
|  | Operating voltage/V | 300~450 |
| Engine | Displacement/L | 1.9 |
|  | Power/kW | 82/4000 r/min |
| Generator | Rated power/kW | 50 |
|  | Rated torque/N•m | 220 |
| Power Battery | Capacity | 180 Ah |
|  | Operating voltage/V | 350~460 |

3.1. Range-Extender. The range-extender includes a diesel engine, a permanent magnet synchronous generator, and rectifier. System features can be described by the following equations:

$$
\begin{gather*}
n_{\mathrm{eng}}=n_{r} \frac{1}{\tau_{e} s+1}, \\
T_{\mathrm{eng}}=f_{1}\left(\alpha, n_{\mathrm{eng}}\right),  \tag{1}\\
C_{\mathrm{eng}}=f_{2}\left(n_{\mathrm{eng}}, T_{\mathrm{eng}}\right), \\
\eta_{\mathrm{gr}}=f_{3}\left(n_{\mathrm{eng}}, \lambda\right) \cdot \eta_{r},
\end{gather*}
$$

where $f_{1}$ is the accelerator characteristic MAP of the engine, $f_{2}$ is fuel consumption characteristic MAP of the engine, $f_{3}$ is the generator efficiency MAP, $n_{r}$ is the engine's target speed, $\zeta_{e}$ is a time constant, $\alpha$ is the accelerator signal, $n_{\text {eng }}$ is the engine's actual speed, $T_{\text {eng }}$ is the engine torque, $\lambda$ is the generator loading rate, $\eta_{r}$ is the rectifier efficiency, $C_{\text {eng }}$ is the engine's instantaneous fuel consumption, and $\eta_{\mathrm{gr}}$ is the total rate of the generator and rectifier.
3.2. Traction Motor. The traction motor module includes the motor and motor controller. The motor model consists of the


Figure 4: Schema of recycle energy flow.
steady state efficiency characteristic MAP and a first-order process:

$$
\begin{gather*}
\eta_{m}=f_{m 1}\left(n_{m}, T_{m}\right) \\
T_{m}=\min \left(T_{r}, T_{\max }\right) \cdot \frac{1}{\tau_{m} s+1},  \tag{2}\\
T_{\max }=f_{m 2}\left(n_{m}\right)
\end{gather*}
$$

where $\eta_{m}$ is the motor's electric efficiency, $n_{m}$ is the motor's rotational speed, $\zeta_{m}$ is a time constant, and $T_{m}, T_{r}$, and $T_{\max }$ are the motor's actual torque, target torque, and torque capacity, respectively. The function $f_{m 1}$ denotes the motor's efficiency MAP, and $f_{m 2}$ denotes the motor's maximum output torque characteristic MAP.
3.3. Power Battery. The power battery model is built based on the $R_{\text {int }}$ model, which is equivalent to a variable voltage source and a variable resistance in series. According to the battery internal resistance equivalent circuit, the following equation can be established:

$$
\begin{equation*}
U_{\mathrm{oc}}=E(\mathrm{SOC}, T)-I \cdot R(\mathrm{SOC}, T), \tag{3}
\end{equation*}
$$

where SOC is the state of charge of the battery, $T$ is the temperature, and $I$ is the battery current. $E$ stands for the open circuit voltage of the battery, which is a function of SOC, $T$ is determined by the test, and $R$ is the internal resistance of the battery.

In this model, the battery's SOC state uses ampere-hour integral method to estimate [25]. That is, when the vehicle is
in operation, it will use SOC as $\mathrm{SOC}_{\text {int }}$ and at $t$ moment it will use SOC formula (4) to decide the following:

$$
\begin{equation*}
\mathrm{SOC}=\mathrm{SOC}_{\mathrm{int}}-\frac{1}{\mathrm{Q}_{b}} \int_{t_{0}}^{t} \eta_{\mathrm{bat}} I_{\mathrm{bat}} d t \tag{4}
\end{equation*}
$$

where $Q_{b}$ is rated capacity, $\eta_{\text {bat }}$ is the battery's columbic capacity, and $I_{\text {bat }}$ is its charging and discharging current.
3.4. Vehicle Longitudinal Dynamics. The road load characteristic is assumed to be ideal in simulation, that is, zero air speed and good adhesion. As the vehicle is traveling on the road, traction motor needs to overcome driving resistance $\left(F_{t}\right)$, rolling resistance $\left(F_{f}\right)$, air resistance $\left(F_{w}\right)$, slope resistance $\left(F_{i}\right)$, and acceleration resistance $\left(F_{j}\right)$. Consider

$$
\begin{gather*}
F_{t}=F_{f}+F_{w}+F_{i}+F_{j}, \\
F_{f}=f m g \cos (a \tan i), \\
F_{w}=\frac{1}{2} C_{d} A \rho u_{a}^{2}, \\
F_{i}=m g \sin (a \tan i),  \tag{5}\\
F_{j}=0.28 \delta m \frac{d u_{a}}{d t}, \\
F_{t}=\frac{3.6 \eta_{T} P_{\text {motor }}}{u_{a}},
\end{gather*}
$$

where $f$ is the rolling resistance coefficient, $m$ is the vehicle mass, $g$ is the acceleration of gravity, $i$ is the road slope, $C_{d}$ is the coefficient of air resistance, $A$ is the windward area, $\rho$ is the air density, $u_{a}$ is the motor speed, $\delta$ is the correction coefficient of rotating mass, $\eta_{T}$ is the overall efficiency of drive system, and $P_{\text {motor }}$ is the output power of the traction motor.


Figure 5: Energy flow chart in energy efficiency analysis.

## 4. Energy Efficiency Analysis Using Energy Flow Chart

According to the energy efficiency analysis method of plugin hybrid electric powertrain in [7], energy efficiency analysis can be divided into the following three parts.
4.1. Dissipation and Cycle Energy. Traction power is used to drive the wheels and vehicle auxiliaries; the calculation equation is as follows:

$$
\begin{equation*}
P(t)=P_{\mathrm{ae}}(t)+P_{\mathrm{rol}}(t)+P_{\mathrm{au}}(t)+P_{\mathrm{ac}}(t)+P_{\mathrm{gr}}(t) \tag{6}
\end{equation*}
$$

where $P_{\mathrm{ae}}(t)=\rho_{\mathrm{air}} A_{f} \mathcal{c}_{d} v^{3}(t) / 2$ is the power to overcome air resistance, $\rho_{\text {air }}$ is the air density, $A_{f}$ is the frontal area, $c_{d}$ is the air resistance coefficient, $v$ is the vehicle speed, $P_{\text {rol }}(t)=\left(m_{v}+m_{p}\right) g c_{r} \cos (\alpha(t)) v(t)$ is the power to overcome rolling resistance, $P_{\mathrm{ac}}(t)=\left(m_{v}+m_{p}\right) v(t) v(t)(d v(t) / d t)$ is the acceleration/deceleration power, $P_{\mathrm{gr}}(t)=\left(m_{v}+\right.$ $\left.m_{p}\right) g \sin (\alpha(t)) v(t)$ is the up/down hill power, $v$ is the vehicle speed, $\alpha$ is the road gradient, $m_{p}$ is the battery mass, and $P_{\mathrm{au}}(t)$ is the auxiliaries power, including air condition, battery heat management system, heating (seat heating and windshield heating), lighting, control system, and braking steer consumption. The average power of the auxiliaries is 10 kW [26], assuming air-conditioning is working.


Figure 6: Construction process of Harbin city driving cycle.


Figure 7: Harbin city driving cycle.
$P(t)$ can be divided into two parts: one is the dissipated power $P_{\mathrm{dis}}(t)=P_{\mathrm{ae}}(t)+P_{\mathrm{rol}}(t)+P_{\mathrm{au}}(t)$ and the other is conserved power $P_{\text {cons }}(t)=P_{\mathrm{ac}}(t)+P_{\mathrm{gr}}(t)$. As the initial and final altitude and speed are the same in a whole driving cycle, the reserved power is zero. If braking energy can be fully recycled, required traction energy $E_{\text {trac }}$ should be the same as the dissipated energy $E_{\text {dis }}$

$$
\begin{equation*}
E_{\mathrm{trac}}=\int_{t_{0}}^{t_{f}} P(t) d t=\int_{t_{0}}^{t_{f}} P_{\mathrm{dis}}(t) d t=E_{\mathrm{dis}}, \tag{7}
\end{equation*}
$$

where $t_{0}$ and $t_{f}$ are initial and final time. If there is no barking energy recycled,

$$
\begin{align*}
E_{\text {trac }} & =\int_{P(t)>0} P(t) d t \\
& =\int_{t_{0}}^{t_{f}} P_{\text {dis }}(t) d t+\int_{P(t)<0}(-P(t)) d t  \tag{8}\\
& =E_{\text {dis }}+\int_{P(t)<0}(-P(t)) d t,
\end{align*}
$$

where $\int_{P(t)<0}(-P(t)) d t$ represents the cycle energy $E_{\text {cir }}$, which is the temporal vehicle cycle energy in the form of kinetic or potential energy and ultimately dissipated during friction
braking. Therefore, (8) for vehicle without energy recycle can be calculated as

$$
\begin{equation*}
E_{\mathrm{trac}}=E_{\mathrm{dis}}+E_{\mathrm{cir}} . \tag{9}
\end{equation*}
$$

In actual driving, energy balance equation can be calculated as

$$
\begin{equation*}
E_{\mathrm{trac}}=E_{\mathrm{dis}}+E_{\mathrm{cir}}-E_{\mathrm{rec}} \tag{10}
\end{equation*}
$$

where $E_{\text {rec }}$ is the recycled net energy that is usable for traction. According to (7) and (10), it can be found that $E_{\text {rec }}=E_{\text {cir }}$.
4.2. Recycle Efficiency of Barking Energy. Recycle efficiency is defined as

$$
\begin{equation*}
\eta_{\mathrm{rec}}=\frac{E_{\mathrm{rec}}}{E_{\mathrm{cir}}} \tag{11}
\end{equation*}
$$

As $E_{\text {cir }}$ can be easily obtained in driving cycle, the key mission is to calculate $E_{\text {rec }}$. Recycle energy consists of two flow methods: input and output. As is shown in Figure 4, recycle ability is determined by energy dissipation, motor torque, current of power battery, and threshold charge value.

Time set $S$ is defined as

$$
\begin{equation*}
S=\left\{t \mid t \in\left[t_{0}, t_{f}\right], P(t)-P_{\mathrm{au}}(t)<0, P_{b}(t)<0\right\} \tag{12}
\end{equation*}
$$

where $P_{\text {au }}$ is physical load in converter. $E_{w, \text { in }}$ is absolute input energy during $S$ and is calculated by

$$
\begin{equation*}
E_{w, \text { in }}=\int_{S}\left|P(t)-P_{\mathrm{au}}(t)\right| d t \tag{13}
\end{equation*}
$$

The energy reserved in battery is

$$
\begin{equation*}
E_{b, \text { in }}=\int_{S}\left|P_{b}(t)+I(t)^{2} R n_{c}\right| d t . \tag{14}
\end{equation*}
$$

The energy loss is

$$
\begin{equation*}
E_{\text {loss, in }}=E_{w, \text { in }}-E_{b, \text { in }}-\int_{S} P_{\mathrm{au}}(t) d t . \tag{15}
\end{equation*}
$$



Figure 8: The SOC curve with the two different energy management strategies.

The cycle-average wheel to battery energy efficiency is

$$
\begin{equation*}
\eta_{\mathrm{wb}, \text { in }}=1-\frac{E_{\mathrm{loss}, \text { in }}}{E_{w, \text { in }}} \tag{16}
\end{equation*}
$$

The recycle energy reserved in battery is

$$
\begin{equation*}
E_{\mathrm{rec}, \mathrm{in}}=E_{\mathrm{cir}} \eta_{\mathrm{wb}, \mathrm{in}} \eta_{\mathrm{cir}, w}, \tag{17}
\end{equation*}
$$

where $\eta_{\mathrm{cir}, w}$ is the ratio between $E_{w}$ and $E_{\mathrm{cir}}$.
Time set $D$ is defined as
$D=\left\{t \mid t \in\left[t_{0}, t_{f}\right], P(t)-P_{\mathrm{au}}(t) \geq 0, P_{b}(t) \geq 0\right\}$.
The associated motor energy loss in propelling is

$$
\begin{equation*}
E_{\text {loss }, m}=\int_{D} P_{\text {loss }, m}(t) d t \tag{19}
\end{equation*}
$$

The average motor efficiency is
$\eta_{m, \text { out }}=1-\frac{E_{\text {loss }, m} d}{\int_{D} P_{b}(t) \eta_{\text {con }}+P_{\text {apu }}(t) \eta_{\text {con }}-P_{\text {au }}(t) d t}$.
The battery energy loss during time $D$ is

$$
\begin{equation*}
E_{\text {loss }, b}=\int_{D} I(t)^{2} R n_{c} d t \tag{21}
\end{equation*}
$$

The battery efficiency in output way is

$$
\begin{equation*}
\eta_{b, \text { out }}=1-\frac{E_{\text {loss }, b} d}{\int_{D} P_{b}(t)+I(t)^{2} R n_{b} d t} . \tag{22}
\end{equation*}
$$

The cycle-average battery to wheel energy efficiency is

$$
\begin{equation*}
\eta_{\text {bw, out }}=\eta_{b, \text { out }} \eta_{\text {con }} \eta_{m, \text { out }} \eta_{f d} \eta_{b, m}, \tag{23}
\end{equation*}
$$

where $\eta_{b, m}$ is the ratio between the power that battery provides to the motor and the power emitted by battery. According to (16) and (23), the recycle energy for traction is

$$
\begin{equation*}
E_{\text {rec }}=E_{\text {rec, in }} \eta_{\text {bw, out }} . \tag{24}
\end{equation*}
$$

The cycle-average recycle efficiency is

$$
\begin{equation*}
\eta_{\mathrm{rec}}=\frac{E_{\mathrm{rec}}}{E_{\mathrm{cir}}}=\eta_{\mathrm{cir}, w} \eta_{\mathrm{wb}, \mathrm{in}} \eta_{\mathrm{bw}, \mathrm{out}} \tag{25}
\end{equation*}
$$

4.3. Fuel-Traction Efficiency. Fuel-traction efficiency is defined as

$$
\begin{equation*}
\eta_{\mathrm{ft}}=\frac{E_{\mathrm{trac}}}{E_{\mathrm{ef}}}=\frac{E_{\mathrm{dis}}+\left(1-\eta_{\mathrm{rec}}\right) E_{\mathrm{cir}}}{E_{\mathrm{ef}}} \tag{26}
\end{equation*}
$$

where $E_{\text {ef }}$ is the equal fuel energy, the average consumption of the sum of diesel, and electric energy. $\eta_{\mathrm{ft}}$ is the cycle-average conversion efficiency from the total consumed energy to the mechanical energy at the wheels and the electrical energy for the auxiliaries. Based on the initial and final SOC, $E_{\text {ef }}$ can be divided into diesel energy and electric energy.

An energy flow chart can clearly show the direction of energy flow, showing the sizes of losses from each individual part. It is one of the most effective methods for analyzing flow in the context of system performance. Referring to the analysis method in [27], this paper puts forward a powertrain energy analysis method with energy flow chart for rangeextended electric bus. The powertrain energy flow chart is shown in Figure 5.

## 5. Analysis Example with the Driving Cycle of Harbin City

To provide a credible reference for the match and control of the range-extended electric bus, energy efficiency analysis


Figure 9: Energy flow chart with CD-CS strategy.
should be carried out within a driving cycle. Based on the authors' location, the Harbin city driving cycle has been chosen for this paper, and the construction process is shown in Figure 6.

The constructed Harbin city driving cycle is shown in Figure 7; acceleration and deceleration are frequent. The maximum acceleration is $1.94 \mathrm{~m} / \mathrm{s}^{2}$, the idle proportion is $22.3 \%$, the maximum speed is $50 \mathrm{~km} / \mathrm{h}$, and the average speed is $14.5 \mathrm{~km} / \mathrm{h}$.

The analysis process mainly compares the CD-CS strategy with the switching control method on the range-extender and blended strategy with power following control method on the range-extender. According to the research, the daily average range for Harbin city bus line 101 is $150-180 \mathrm{~km}$, the initial SOC is $100 \%$, and the auxiliaries' power is 10 kW . SOC curves with the two different driving cycles are shown in Figure 8.

Figure 9 shows the energy flow chart with CD-CS strategy, and Figure 10 shows the energy flow chart with blended strategy.

Based on Figure 5, the input and output recycle efficiencies can be obtained, as is shown in Figure 11. The inputrecycled efficiency of the blended strategy is $24.73 \%$ higher than that of CD-CS strategy. The output-recycled efficiency of the blended strategy is $7.83 \%$ lower than that of the CD-CS strategy.

Figure 12 consists of recycle efficiency, fuel-traction efficiency, and energy consumption with the two different energy management strategies. As is shown in Figure 12(a), the recycle efficiency is $36.80 \%$ with CD-CS strategy and $51.32 \%$ with blended strategy. With CD-CS strategy, the fuel-traction efficiency is $38.91 \%$, but, with blended strategy, it is $33.26 \%$. The fuel-traction efficiency is limited by engine efficiency and


Figure 10: Energy flow chart with blended strategy.


Figure 11: Comparison of powertrain input- and output-recycled efficiency with different energy management strategies.


FIGURE 12: Comparison of recycle efficiency, fuel-traction efficiency, and energy consumption with different energy management strategies.
is also influenced by other electric and mechanical losses, such as the battery, power converter, and motor and drive system. Energy consumption is 664.43 kWh with the CDCS strategy and 47.87 kWh lower than that with the blended strategy. The CD-CS strategy has a better recycle ability and fuel-traction efficiency. It is worth noting that the yearly range of one bus in Harbin is nearly 70000 km , and the energy saved with the CD-CS strategy would be considerable.

## 6. Conclusions

For a range-extended electric bus developed by Tsinghua University, this paper analyzes the energy efficiency with two different energy management strategies (CD-CS and blended) using an energy flow chart method. Harbin city driving cycle is taken for analysis. The recycle efficiency and fuel-traction efficiency are evaluation indexes.

Analysis results from the energy flow chart show that the energy loss mainly occurs at the engine. Engine energy loss reaches $187.59 \%$ of the whole driving energy using the CD-CS strategy and $209.47 \%$ with the blended strategy. As the CD-CS strategy uses a thermostat control method, its charging loss is $3.85 \%$ of total driving energy, while the blended strategy only has a $2 \%$ charging loss. Of these two energy management strategies, the CD-CS strategy can effectively reduce the engine loss but has a higher charging loss compared with the blended strategy.

Energy efficiency results show that over the Harbin city driving cycle, the input-recycled efficiency of the blended strategy is $24.73 \%$ higher than that of the CD-CS strategy and that the output-recycled efficiency of the blended strategy is $7.83 \%$ lower than that of the CD-CS strategy. With the CD-CS strategy, the recycle efficiency is $36.80 \%$, but, with blended strategy, it is $51.32 \%$. Using the CD-CS strategy, the fuel-traction efficiency is $38.91 \%$, but, with blended strategy, it is $33.26 \%$. In comparison of the two nonoptimized energy management strategies, powertrain energy efficiency is better
with CD-CS than with blended strategy. We suggest that a CD-CS energy management strategy is more appropriate for the driving conditions on urban Harbin roads.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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# A Cooperative Harmony Search Algorithm for Function Optimization 

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#### Abstract

Harmony search algorithm (HS) is a new metaheuristic algorithm which is inspired by a process involving musical improvisation. HS is a stochastic optimization technique that is similar to genetic algorithms (GAs) and particle swarm optimizers (PSOs). It has been widely applied in order to solve many complex optimization problems, including continuous and discrete problems, such as structure design, and function optimization. A cooperative harmony search algorithm (CHS) is developed in this paper, with cooperative behavior being employed as a significant improvement to the performance of the original algorithm. Standard HS just uses one harmony memory and all the variables of the object function are improvised within the harmony memory, while the proposed algorithm CHS uses multiple harmony memories, so that each harmony memory can optimize different components of the solution vector. The CHS was then applied to function optimization problems. The results of the experiment show that CHS is capable of finding better solutions when compared to HS and a number of other algorithms, especially in high-dimensional problems.


## 1. Introduction

Optimization is a very ancient problem which has been researched by numerous mathematicians since time immemorial, including Newton, Gauss, and John von Neumann. They developed numerous mathematical theories and methods that made a considerable contribution to optimization. However, almost all mathematical methods require firstorder derivatives or second-order derivatives, or even higher derivatives. When the object function is not too complex, we can compute its derivatives easily, but unfortunately most object functions in the real world are so complex that we cannot compute the derivatives, and even worse, they may have no derivatives at all. In this case, it is very difficult to implement mathematical methods.

What should we do, then, to solve complex, nonlinear, nondifferentiable, and multimodal problems? We can perhaps learn from the nature. Many natural phenomena as well as the activities of animals provide us with inspiration and we are able to imitate these phenomena or activities in order to
solve complex problems. Over the last four decades, a large number of methods inspired by nature have been developed in order to solve very complex optimization problems [1], and these were called metaheuristic algorithms. All these metaheuristic algorithms imitate natural phenomena, such as evolutionary algorithms (EAs), which include genetic algorithms (GAs) [2], evolutionary programming (EP) [3], and evolution strategy (ES) [4], all of which simulate biological evolution. PSO and ant colony optimization (ACO), for instance, mimic the foraging behavior of animals [5, 6], and simulated annealing (SA) simulates physical annealing process [7]. As a metaheuristic algorithm, HS is no exception, it is inspired by the improvisation process of musical performers [8]. Although the algorithms mentioned above imitate different phenomena, they have some common factors: (1) they all have a random process; (2) the solutions that they show us are just approximate results; (3) they all suffer from the "curse of dimensionality" [9], meaning that the search space will increase exponentially when the number of the dimensions increases. The probabilities of finding the
optimality region will thus decrease. One way of overcoming this drawback is to partition higher dimensional search space into lower dimensional subspaces.

Potter and De Jong [ 10,11 ] have proposed a general framework for cooperative coevolutionary algorithms and then they applied the method to the GA (CCGA). They suggested that the higher dimensional solution vectors should be split into smaller vectors, with each vector just being a part of a complete solution, so that they must cooperate with each other to constitute a potential solution. In this paper, we apply Potter and De Jong's cooperative method to the HS to enhance the performance of standard HS.

The rest of this paper is organized as follows. Section 2 presents an overview of the standard HS and some other improved HS algorithms and Section 3 demonstrates the convergence of HS and explains why the improved HS performs better than the standard HS. In Section 4, the cooperative method is briefly introduced, and then the proposed CHS algorithm is elaborated. Section 5 describes the benchmark functions used to test the new algorithm and the experimental results can be found in Section 6. Finally, the conclusions are given in Section 7.

## 2. Harmony Search Algorithms

The HS algorithm, originally conceived by Geem et al. in 2001 [8], was inspired by musical improvisation. There are always three ways open to a musician [12], when he or she is improvising. The first is when he or she plays a piece of music that he or she remembers exactly; the second is when a musician plays something that is similar to what he or she remembers exactly, the musician possibly being engaged in improvisation based on the original harmony by adjusting the pitch slightly. The last one involves a composition that is new. The process employed by the musicians in order to find the best harmony is likely to be the process of optimization. In fact, the HS algorithm mimics the process used to solve optimization problems, with this algorithm being widely applied in order to solve optimization problems, including water network design [13, 14], PID controller design [15], Cluster analysis, and function optimization [16, 17]. Several approaches have been taken in order to improve the performance of the standard HS, some of which are discussed in the following subsections.
2.1. The Standard HS (SHS). When we use an algorithm to solve problems, firstly we must know what the problems are. Assuming that there is an unconstrained optimization problem which can be described as follows:

$$
\begin{array}{ll}
\min & f(X)  \tag{1}\\
\text { s.t. } & L x_{i} \leq x_{i} \leq U x_{i}, \quad i=1,2, \ldots, n
\end{array}
$$

where $f(X)$ is the object function, $X$ is the set of each decision variable, $x_{i}$ is the $i$ th decision variable, $L x_{i}$ and $U x_{i}$ are the lower and upper bounds of the $i$ th decision variable, and $n$ is the number of decision variables. To apply SHS to solve the
optimization problem mentioned above, five steps should be taken [1] as follows.

Step 1. Initialize the problem algorithm parameters. The problem parameters like $n, L x_{i}, U x_{i}$ should be initialized in this step. In addition, four algorithm parameters should be initialized, including harmony memory size (HMS), harmony memory considering rate (HMCR), pitch adjusting rate (PAR), and the maximum number of improvisation ( $T_{\text {max }}$ ), or stopping criterion.

Step 2. Initialize the harmony memory. The SHS is similar to GAs. GAs are population based optimization algorithms, but the "population" in SHS is referred to as harmony memory (HM), which is composed of solution vectors. The structure of HM is as follows:

$$
\mathrm{HM}=\left[\begin{array}{cccc}
x_{1}^{1} & x_{2}^{1} & \cdots & x_{n}^{1}  \tag{2}\\
x_{1}^{2} & x_{1}^{2} & \cdots & x_{n}^{2} \\
\vdots & \vdots & \ddots & \vdots \\
x_{1}^{\mathrm{HMS}} & x_{2}^{\mathrm{HMS}} & \cdots & x_{n}^{\mathrm{HMS}}
\end{array}\right]
$$

where $x_{j}^{i}$ is the $j$ th decision variable of the $i$ th solution vector. Generally, we initialize the HM randomly. For $x_{j}^{i}$, it can be generated by using the following equation:

$$
\begin{equation*}
x_{j}^{i}=L x_{j}+\left(U x_{j}-L x_{j}\right) \times \operatorname{rand}(), \tag{3}
\end{equation*}
$$

where $\operatorname{rand}()$ is a random number between 0 and 1 .
Step 3. Improvise a new harmony. This step is the core step of SHS. A new harmony vector $X^{\text {new }}=\left(x_{1}^{\text {new }}, x_{2}^{\text {new }}, \ldots, x_{n}^{\text {new }}\right)$ is improvised according to these three rules: (1) harmony memory consideration; (2) pitch adjustment; (3) randomization. The probabilities of harmony consideration and pitch adjustment are dependent on HMCR and PAR. For instance, the $i$ th variable $x_{i}^{\text {new }}$ of the new harmony vector can be improvised as follows:

$$
\begin{gather*}
x_{i}^{\text {new }}=x_{i}^{\text {new }} \in\left\{x_{i}^{1}, x_{i}^{2}, \ldots, x_{i}^{\mathrm{HMS}}\right\}  \tag{4}\\
x_{i}^{\text {new }}=L x_{j}+\left(U x_{i}-L x_{i}\right) \times \operatorname{rand}() \tag{5}
\end{gather*}
$$

In fact, before the variable is generated, we should generate a random number $r_{1}$ between 0 and 1, then we compare $r_{1}$ with HMCR, if $r_{1} \leq$ HMCR; (4) is used to improvise the variable; otherwise, the variable will be improvised by using (5). For example, if HMCR $=0.95$, then the HS algorithm will choose a decision variable from the HM with a $95 \%$ probability. And if the variable is chosen from the HM, then it will be adjusted with probability PAR as follows:

$$
\begin{equation*}
x_{i}^{\text {new }}=x_{i}^{\text {new }} \pm \mathrm{bw} \times \operatorname{rand}() \quad \text { with probability PAR, } \tag{6}
\end{equation*}
$$

where bw is an arbitrary distance bandwidth and rand() is a random number between 0 and 1 .

Step 4. Update the HM. If the new improvised harmony vector is better than the worst harmony vector in the HM in

```
Initialize parameters and HM
while \(t<T_{\text {max }}\) do
    For \(i=1\) to \(n\) do
        if \(\operatorname{rand}() \leq\) HMCR then
            \(x_{i}^{\text {new }}=x_{i}^{j}\), where \(j \sim U(1,2, \ldots\), HMS \()\)
            if \(\operatorname{rand}() \leq\) PAR then
                    \(x_{i}^{\text {new }}=x_{i}^{\text {new }} \pm \mathrm{bw} \times \operatorname{rand}()\)
            end if
        else
            \(x_{i}^{\mathrm{new}}=L x_{j}+\left(U x_{i}-L x_{i}\right) \times \operatorname{rand}()\)
        end if
    end for
    Update HM
end while
```

Pseudocode 1: Pseudocode for the SHS algorithm.
terms of the object function value, the worst harmony in the HM is superseded by the new harmony vector.

Step 5. Check stopping criterion. If the stopping criterion is satisfied, the iteration is terminated. If not, Steps 3 and 4 are repeated.

The pseudocode of SHS is shown in Pseudocode 1.
2.2. Improved HSs. The SHS algorithm was introduced in the last subsection, and in this subsection several improved HSs are introduced. PAR and bw are two important parameters of HS, deciding the accuracy of the solution. As the number of iteration increases, the HM becomes better and the solution is closer to the global optimal position. We should use a smaller bw to adjust the pitches, and this adjustment should be with a higher probability, but all parameters are fixed within the SHS algorithm, meaning that they cannot change. If we choose a low PAR and a narrow bw, the SHS will converge slowly, but on the other hand, if we choose a very high PAR and a wide bw, the SHS will converge fast, although the solution may scatter around some potential optimals as a random search. A dynamic adjustment strategy for parameters, especially for PAR and bw, is therefore very necessary.

A dynamic adjustment strategy for PAR and bw was proposed by Mahdavi et al. in 2007 [18]. They suggested that PAR should increase linearly and bw should index decrease. They can change with generation number as shown in Figure 1 by using the following equations:

$$
\begin{gather*}
\operatorname{PAR}(t)=\mathrm{PAR}_{\min }+\frac{\mathrm{PAR}_{\max }-\mathrm{PAR}_{\min }}{T_{\max }} \times t, \\
\mathrm{bw}(t)=\mathrm{bw}_{\max } \times \exp \left(\frac{\ln \left(\mathrm{bw}_{\min } / \mathrm{bw}_{\max }\right)}{T_{\max }} \times t\right), \tag{7}
\end{gather*}
$$

where $\operatorname{PAR}(t)$ and $\mathrm{bw}(t)$ are the pitch adjusting rate and bandwidth for each generation, $\mathrm{PAR}_{\min }$ and $\mathrm{bw}_{\min }$ are the minimum pitch adjusting rate and bandwidth, $\mathrm{PAR}_{\text {max }}$ and $\mathrm{bw}_{\max }$ are the maximum pitch adjusting rate and bandwidth, and $t$ is the generation number. The drawback of the IHS
is that we have to specify the $\mathrm{PAR}_{\text {min }}, \mathrm{PAR}_{\text {max }}, \mathrm{bw}_{\text {min }}$, and $\mathrm{bw}_{\text {max }}$, which is essentially very difficult, and we are unable to guess without experience.

The IHS algorithm merely changes the parameters dynamically, while some other improved HSs are capable of changing the search strategy of SHS, such as the global best HS (GHS) proposed by Omran and Mahdavi [19]. The GHS is inspired by the concept of swarm intelligence as proposed in PSO. In PSO, the position of each particle presents a candidate solution, so that, when a particle flies through the search space, the position of the particle is influenced by the best position itself has visited so far and the position of the best particle among the swarm, in other words, there are some similarities between the new position and the best position. In GHS, the new harmony vector can mimic the best harmony in the HM. The difference between GHS and SHS is the pitch adjustment strategy, with the variable being adjusted in SHS using (6), while in GHS it is adjusted using the following equation:

$$
\begin{equation*}
x_{i}^{\text {new }}=x_{k}^{\text {best }} \quad \text { with probability PAR, } \tag{8}
\end{equation*}
$$

where best is the index of the best harmony in the HM and $k \sim U(1, n)$.

The results of the experiment show that these two improved harmony search algorithms can find better solutions when compared with SHS.

## 3. Convergence of HS Algorithm

As a metaheuristic algorithm, HS is capable of finding local optima and the global optimum, but why can it converge on local or global optima, and which operator or parameter may have effects on the speed of convergence? All of these problems are unsolved. In this section, we endeavor to solve the problems.

As we noted in the last section, HS has an operator referred to as "HM updating," the fourth step of HS. This is an indispensable step for HS, because this operator guarantees the convergence of HS, without it HS may not find even a local minimum. In order to explain this, some definitions and theorems are necessary.

Definition 1 (monotone sequences). A sequence $\left\{x_{n}\right\}$ is said to be monotonically increasing provided that

$$
\begin{equation*}
x_{n+1} \geq x_{n} \tag{9}
\end{equation*}
$$

for every natural number $n$.
A sequence $\left\{x_{n}\right\}$ is said to be monotonically decreasing provided that

$$
\begin{equation*}
x_{n+1} \leq x_{n} \tag{10}
\end{equation*}
$$

for every natural number $n$.
A sequence $\left\{x_{n}\right\}$ is called monotone if it is either monotonically increasing or monotonically decreasing.

Theorem 2 (the monotone convergence theorem). A monotone sequence converges if and only if it is bounded.


Figure 1: Variation of PAR and bw versus generation number.

Proof. Firstly, let us suppose that the sequence $\left\{x_{n}\right\}$ is a monotonically increasing sequence. Then we can define $S=$ $\left\{x_{n} \mid n \in \mathbb{N}\right\}$, assuming that $S$ is bounded above. According to the Completeness Axiom, $S$ has a least upper bound, we can call it $x$. We claim that the sequence $\left\{x_{n}\right\}$ converges to $x$. To claim this, we must prove that for all integers, $n \geq N$ and $\epsilon>0$ provided that

$$
\begin{equation*}
\left|x_{n}-x\right|<\epsilon, \tag{11}
\end{equation*}
$$

that is,

$$
\begin{equation*}
x-\epsilon<x_{n}<x+\epsilon \quad \forall \text { integers } n \geq N . \tag{12}
\end{equation*}
$$

Since $x$ is the least upper bound for the set $S$, so

$$
\begin{equation*}
x_{n} \leq x<x+\epsilon \quad \forall \text { integers } n \geq N \tag{13}
\end{equation*}
$$

Furthermore, $x-\epsilon$ is not an upper bound for set $S$, so there must be a natural number $N$ such that $x-\epsilon<x_{N}$, since the sequence is monotonically increasing, so

$$
\begin{equation*}
x-\epsilon<x_{N} \leq x_{n} \quad \forall \text { integers } n \geq N . \tag{14}
\end{equation*}
$$

We have now proved that a monotonically increasing sequence must converge to its least upper bound and, in the same way, we can also claim that a monotonically decreasing sequence must converge to its greatest lower bound.

To return to the HS algorithm mentioned in the previous section: when HS finds a solution which is better than the worst harmony in the current HM, then the operator "HM updating" is implemented, which means that for the current iteration, the algorithm finds a solution that is not worse than the one that was found by the algorithm in the last iteration. Suppose that the problem to be solved is the minimum problem described as (1), the current solution offered is $x_{n+1}$ and the last found solution is $x_{n}$, then we have $x_{n+1} \leq x_{n}$, and if we let the best solution in HM for each iteration constitute a sequence $S=\left\{x_{n} \mid n \in N\right\}$, the sequence must be a
monotonically decreasing sequence as in Definition 1, and as we have proved, if the sequence has a greatest lower bound, the sequence must converge to the greatest lower bound.

In fact, for a continuous function, suppose that the function is named $f(X)$, and if $X \in[a, b]^{n}$, where $n$ is the dimension of $X$, then there must be a number $f\left(X_{1}\right)$ and a number $f\left(X_{2}\right)$ (assuming that $\left.f\left(X_{1}\right) \leq f\left(X_{2}\right)\right)$ satisfying the following inequality:

$$
\begin{equation*}
f\left(X_{1}\right) \leq f(X) \leq f\left(X_{2}\right) . \tag{15}
\end{equation*}
$$

So in the case that the object function $f(X)$ is continuous and $x_{i} \in\left[a_{i}, b_{i}\right]$, where $i=1,2, \ldots, n$ and $X=\left(x_{1}, x_{2}, \ldots, x_{n}\right)$, then the sequence $\left\{f_{k}\right\}$ which is composed of the best solution $f_{k}$ in the HM for the $k$ th iteration is bounded and $f\left(X_{1}\right)<$ $f_{k}<f\left(X_{2}\right)$, as we have mentioned above for a minimum problem, the sequence $\left\{f_{k}\right\}$ is monotonically decreasing, so $\left\{f_{k}\right\}$ must converge.

We have explained that HS must converge, but can it converge to $f\left(X_{1}\right)$ ? In fact, $f\left(X_{1}\right)$ is the global optimum of the object function. From Theorem 2, we know that if $f\left(X_{1}\right)$ is the greatest lower bound, $\left\{f_{k}\right\}$ must converge to $f\left(X_{1}\right)$. The problem is whether $f\left(X_{1}\right)$ is the greatest lower bound for $\left\{f_{k}\right\}$. This is, in fact, difficult to solve. HS is a stochastic algorithm and for each iteration, the best solution in HM is random so that it is not a fixed predictable number, but we are able to calculate the probability that the HM is updated with. An example would be as follows: suppose that the object function is a minimum problem, the expression is $f(x)=x^{2}$ which has only one dimension, and $x \in\left[x_{L}, x_{U}\right]$, and HMS $=1$, for the $k$ th iteration, the best solution is $f\left(x_{k}\right)$ (suppose that $x_{k}>0$ ), so the HM is updated with probability HMCR $\times$ PAR $\times 0.5+(1-\operatorname{HMCR}) \times\left(x_{k} /\left(x_{U}-x_{L}\right)\right)$ when $2 x_{k}>$ bw or HMCR $\times \operatorname{PAR} \times 0.5 \times\left(2 x_{k} / \mathrm{bw}\right)+(1-\mathrm{HMCR}) \times\left(x_{k} /\left(x_{U}-x_{L}\right)\right)$ when $2 x_{k} \leq$ bw. The higher the probability is, the more quickly the HM is updated. From the expression, it is clear that the probability decreases by $x_{k}$, so that as the iteration number increases, the probability decreases sharply, and the convergence curve becomes flatter.

```
Initialize parameters of algorithm
for each subpopulation do
    Initialize all the subpoputions randomly
    Evaluate the fitness of each individual
end for
while termination condition is false do
    for each subpopulation do
        Using GA operators to generate offsprings
        Evaluate the fitness of each off spring
    end for
end while
```

Pseudocode 2: Pseudocode for the CCGA.

This might also explain why IHS performs better than SHS. The two parameters PAR and bw are very important for HS. In SHS, the two parameters are fixed while in IHS, PAR increases linearly and the bw index decreases, so that, from the expression of the HM updated probability, we can see that this is very reasonable, owing to the fact that this strategy is capable of making the probability decrease slowly, with IHS generally performing better. The results of this experiment are demonstrated in Section 6.

## 4. CHS Algorithm

4.1. A Brief Overview of Cooperative Method. The natural world is a very complex system, and evolution is one element within it. EAs imitate the evolution of species, but they are just a simple simulation and almost all EAs just mimic the evolution of one species. There are a large number of species on the earth, with some of them being closely associated [20]. In order to survive, different individuals from different species or the same species may compete or cooperate with each other. Cooperative coevolutionary algorithms just act as models of symbiotic coevolution. The individuals are from different species or subpopulations, and they have to cooperate with some others. The difficulty involved in cooperative coevolutationary algorithms is how to split the fitness that is achieved by collective effort definitively [21, 22].

Potter and De Jong applied the method to GA, coming up with CCGA. They evolved the solutions in different subpopulations, and each subpopulation has just one dimension of the solution vector. The pseudocode for the CCGA is given in Pseudocode 2. The initial fitness of each subpopulation member is computed by combining it with a random individual from each of the other subpopulations and then using the object function to evaluate the complete solution. After initializing, one individual of a subpopulation cooperates only with the best individual from each of the other subpopulations.

Potter and De Jong also found that the cooperative approach does not perform well when problem parameters are strongly interdependent, due to the greediness of the credit assignment approach [11]. To reduce greediness, Potter and De Jong suggested that random collaboration should be used to generate a complete solution. An individual from

```
Initialize the parameters
Divide the }n\mathrm{ decision variables into }m\mathrm{ groups
for each HM do
    Initialize HM randomly
end for
while termination condition is false do
    calculate PAR and bw by using
    Equation (7)
    for each HM do
        generate a new harmony
        evaluate the new harmony
    end for
end while
```

Pseudocode 3: Pseudocode for the CHS algorithm.
one subpopulation can collaborate with the best individual of each of the other subpopulations and then constitute one complete solution, or the individual can cooperate with one individual of each of the other subpopulations randomly and then generate another solution. The fitness of the individual is equal to the better fitness between the two solutions.
4.2. The CHS. In the CHS algorithm, to reduce the complexity, we use $m(m \leq n)$ HMs instead of $n$ HMs, where $n$ is the number of decision variables. The pseudocode for CHS algorithm is shown in Pseudocode 3. Each HM may thus contain more than one decision variables, so that, for each HM, we can use the HS operators including harmony consideration and pitch adjustment to generate a new harmony vector, but the vector is just a part of a complete solution vector, and it must collaborate with harmony vectors from every other HM, and as we have mentioned above, when initializing HMs, the cooperation is random. When calculating a new harmony vector which is improvised by using HS operators, it cooperates only with the best ones of each HM. This is similar to several musicians working together to find the best harmony, with each musician being in charge of a part of the whole symphony, assuming that all of them are selfless. When they work together, each one shares the best harmony that he or she has found by himself or herself. In our daily life, teamwork is very universal, and generally it is more effective than work being carried out alone.

We should ask why it brings better solutions when using cooperative method. There are few mathematical foundations for the method, but this can be explained by using an example. HM updating is very important for the HS algorithm, when the HM is updated, it indicates that a better solution may be found. If the HM is updated with a higher probability or in other words, the algorithm has a higher probability of finding better solutions, it will converge faster. For instance, suppose that the object function is $\min f(x, y)=x^{2}+y^{2}$, it has two decision variables $x$ and $y$, and $x \in\left[x_{\min }, x_{\text {max }}\right]$, $y \in\left[y_{\min }, y_{\max }\right]$. It is quite obvious that the origin is the global optimum. Assume that the worst solution vector in the initial HM is $\left(x_{0}, y_{0}\right)$, which is called point $A$ and is shown in Figure 2. If we use the CHS algorithm, two HMs will be


FIGURE 2: Diagram illustrating the advantage of CHS algorithm.
used, one $\mathrm{HM}\left(\mathrm{HM}_{1}\right)$ representing $x$ and another $\left(\mathrm{HM}_{2}\right)$ representing $y$. In one cycle, the HMs are used one by one and when $\mathrm{HM}_{1}$ is used to generate a new harmony vector, this new vector must cooperate with the best one in $\mathrm{HM}_{2}$ in order to constitute a complete solution vector, meaning that $y$ is fixed when the algorithm searches in the subspace $x$, and it is the same for $y$.

When $y$ is fixed, the CHS algorithm is able to find a better solution with probability $d_{2} / L_{2}$, and when $x$ is fixed, the probability is $d_{1} / L_{1}$, so after one cycle, the CHS algorithm has the $P_{\text {CHS }}$ probability of updating the HM , where $P_{\mathrm{CHS}}=$ $d_{1} / L_{1}+d_{2} / L_{2}$, but for SHS algorithm, if the HM is updated, the algorithm must find a solution which is located in the circular region, so the HM is updated with probability $P_{\mathrm{SHS}}$, where $P_{\text {SHS }}=\pi R^{2} / L_{1} L_{2}$. Also, $P_{\text {CHS }}=d_{1} / L_{1}+d_{2} / L_{2}=$ $\left(d_{2} L_{1}+d_{1} L_{2}\right) / L_{1} L_{2}$, because of $d_{2} L_{1}+d_{1} L_{2}>d_{2}^{2}+d_{1}^{2}>\pi R^{2}$, so $P_{\text {CHS }}>P_{\text {SHS }}$, which means that the CHS algorithm has a higher probability of updating HM than the SHS algorithm, meaning that the CHS is able to find better solutions than the SHS algorithm.

The cooperative method also has disadvantages, however, with one being that the CHS algorithm is easier to trap in a local minimum. This phenomenon is shown in Figure 3, in which point $A$ is a local minimum point and point $O$ or the origin is the global minimum point; the area where $A$ is located in is local minimum area and the global minimum region is where $O$ is located. If the algorithm is trapped in point $A$, when we use the CHS algorithm in every iteration, only one subspace is searched. Assuming that $y$ is fixed and $x$ is searched, irrespective of the value of $x$, the CHS algorithm has no way of reaching the global minimum region, and it is the same for $y$. In this case, the CHS algorithm will never escape from the local minimum. For the SHS algorithm, however, it is possible to reach the global minimum area, although the probability is low.


Figure 3: Diagram illustrating the disadvantage of CHS algorithm.

## 5. Experimental Setup

5.1. Benchmark Functions. Ten benchmark functions (five unrotated and five rotated) have been chosen to test the performance of the CHS algorithm, and SHS, his, and GHS algorithms are also tested for the sake of comparison. The unrotated functions are as follows.

The Quadric function is

$$
\begin{equation*}
f_{1}(X)=\sum_{i=1}^{n}\left(\sum_{j=1}^{i} x_{j}\right)^{2} \tag{16}
\end{equation*}
$$

where $n=30, X \in[-100,100]^{30}$. This function is a nonseparable unimodal function, with the global minimum point being $X^{*}=(0,0, \ldots, 0)$.

Ackley's function is

$$
\begin{align*}
f_{2}(X)= & -20 \exp \left(-0.2 \sqrt{\frac{1}{n} \sum_{i=1}^{n} x_{i}^{2}}\right)+20  \tag{17}\\
& -\exp \left(\frac{1}{n} \sum_{i=1}^{n} \cos \left(2 \pi x_{i}\right)\right)+e
\end{align*}
$$

where $n=30, X \in[-30,30]^{30}$. Ackley's function is a separable multimodal function, with the global optimum being $X^{*}=(0,0, \ldots, 0)$.

The Generalized Rastrigin function is

$$
\begin{equation*}
f_{3}(X)=\sum_{i=1}^{n}\left(x_{i}^{2}-10 \cos \left(2 \pi x_{i}\right)+10\right) \tag{18}
\end{equation*}
$$

where $n=30, X \in[-5.12,5.12]^{30}$. The Rastrigin function is a separable multimodal function, with the global optimum being $X^{*}=(0,0, \ldots, 0)$.

The Generalized Griewank function is

$$
\begin{equation*}
f_{4}(X)=\frac{1}{4000} \sum_{i=1}^{n} x_{i}^{2}-\prod_{i=1}^{n} \cos \left(\frac{x_{i}}{\sqrt{i}}\right)+1 \tag{19}
\end{equation*}
$$

where $n=30, X \in[-600,600]^{30}$. This function is a very complex, nonlinear, separable, and multimodal function and is very difficult for optimization algorithms. The global optimum is $X^{*}=(0,0, \ldots, 0)$.

Rosenbrock's function (or Banana-valley function) is

$$
\begin{equation*}
f_{5}(X)=\sum_{i=1}^{n-1}\left(100\left(x_{i}^{2}-x_{i+1}\right)^{2}+\left(1-x_{i}\right)^{2}\right) \tag{20}
\end{equation*}
$$

where $n=30, X \in[-2.048,2.048]^{30}$. Rosenbrock's function is a naturally nonseparable function, the global optimum being $X^{*}=(1,1, \ldots, 1)$ which is located in a narrow valley; it is very hard for algorithms to find the correct direction and reach to the global minimum area.

The parameters of rotated functions are the same as unrotated functions and all the orthogonal matrices are fixed, meaning that all functions are rotated at a fixed angle.
5.2. Configuration of Algorithms. The iteration number of each experiment is $3 \times 10^{4}$, with all the experiments being ran independently 30 times. The results reported are averages, and the best solutions have been calculated from all 30 runs. The parameters of the algorithms are as follows.
(i) Parameters of the SHS algorithm: HMS $=30$ and HMCR $=0.95$. This was suggested by Yang [12]. $\operatorname{PAR}=0.3$ and $\mathrm{bw}=0.01$. The values of the last two parameters (PAR, bw) were suggested by Omran and Mahdavi [19].
(ii) Parameters of the IHS algorithm: HMS $=30$ and HMCR $=0.95$, with PAR increasing linearly over time (by using (7)), and $\mathrm{PAR}_{\text {min }}=0.01, \mathrm{PAR}_{\max }=$ 0.99 which were suggested by Omran and Mahdavi [19]. bw decreases by using (6), and $\mathrm{bw}_{\min }=1 e^{-005}$, $\mathrm{bw}_{\text {max }}=5$.
(iii) Parameters of the GHS algorithm: HMS $=30$, $\mathrm{HMCR}=0.95, \mathrm{PAR}_{\text {min }}=0.01$, and $\mathrm{PAR}_{\text {max }}=0.99$.
(iv) Parameters of the CHS algorithm: HMS $=30$, $\mathrm{HMCR}=0.95, \mathrm{PAR}_{\min }=0.01, \mathrm{PAR}_{\max }=0.99$, $\mathrm{bw}_{\text {min }}=1 e^{-005}, \mathrm{bw}_{\max }=5$, and the number of groups $m=6,10,15,30$.

## 6. Results

This section shows the experimental results gathered by allowing all of the methods tested to run for a fixed number of function evaluations, that is, $3 \times 10^{4}$. All the results are shown in Table 1 to Table 10, for each table, the first column represents the algorithms used, the second lists the mean error and $95 \%$ confidence interval after $3 \times 10^{4}$ times iteration, and the third lists the best solution found by the algorithm after 30 runs. We should not forget that all the functions have a minimum function value of 0 .

Table 1 shows the experimental results of the unrotated Quadric function; this is a nonseparable function, and it is hard to optimize for HS algorithms, but GHS is better able to solve the problem during the 30 runs; $\mathrm{CHS}_{30}$ also finds a

Table 1: Results of unrotated Quadric function.

| Algorithm | Mean | Best |
| :--- | :---: | :---: |
| SHS | $5.46 e+003 \pm 5.97 e+002$ | $2.38 e+003$ |
| IHS | $5.87 e+003 \pm 7.37 e+002$ | $1.67 e+003$ |
| GHS | $2.55 e+003 \pm 2.11 e+003$ | $3.64 e+000$ |
| $\mathrm{CHS}_{6}$ | $1.91 e+003 \pm 2.61 e+001$ | $5.53 e+002$ |
| $\mathrm{CHS}_{10}$ | $2.05 e+003 \pm 3.82 e+002$ | $9.13 e+002$ |
| $\mathrm{CHS}_{15}$ | $2.29 e+003 \pm 6.42 e+002$ | $3.97 e+002$ |
| $\mathrm{CHS}_{30}$ | $1.24 e+003 \pm 4.95 e+002$ | $6.36 e+001$ |

Table 2: Results of rotated Quadric function.

| Algorithm | Mean | Best |
| :--- | :---: | :---: |
| SHS | $8.09 e+002 \pm 3.04 e+002$ | $9.72 e+002$ |
| IHS | $1.09 e+003 \pm 4.35 e+002$ | $1.05 e+002$ |
| GHS | $8.99 e+002 \pm 6.22 e+002$ | $7.78 e-002$ |
| $\mathrm{CHS}_{6}$ | $1.78 e+002 \pm 1.16 e+002$ | $6.24 e-008$ |
| $\mathrm{CHS}_{10}$ | $1.61 e+002 \pm 1.84 e+002$ | $2.39 e-010$ |
| $\mathrm{CHS}_{15}$ | $7.24 e+001 \pm 5.43 e+001$ | $3.81 e-011$ |
| $\mathrm{CHS}_{30}$ | $4.33 e+000 \pm 8.31 e+000$ | $1.04 e-011$ |

Table 3: Results of unrotated Ackley's function.

| Algorithm | Mean | Best |
| :--- | :---: | :---: |
| SHS | $5.54 e-001 \pm 1.16 e-001$ | $2.96 e-002$ |
| IHS | $1.08 e-002 \pm 1.76 e-002$ | $3.90 e-005$ |
| GHS | $1.78 e-002 \pm 5.05 e-003$ | $5.78 e-006$ |
| $\mathrm{CHS}_{6}$ | $8.96 e-006 \pm 1.79 e-007$ | $7.29 e-006$ |
| $\mathrm{CHS}_{10}$ | $4.86 e-006 \pm 1.31 e-007$ | $4.01 e-006$ |
| $\mathrm{CHS}_{15}$ | $1.40 e-006 \pm 4.58 e-008$ | $1.18 e-006$ |
| $\mathrm{CHS}_{30}$ | $7.06 e-008 \pm 3.57 e-009$ | $5.18 e-008$ |

Table 4: Results of rotated Ackley's function.

| Algorithm | Mean | Best |
| :--- | :---: | :---: |
| SHS | $3.95 e+000 \pm 2.37 e-001$ | $2.62 e+000$ |
| IHS | $1.72 e-003 \pm 3.22 e-003$ | $4.02 e-005$ |
| GHS | $3.48 e-001 \pm 2.16 e-001$ | $6.64 e-005$ |
| $\mathrm{CHS}_{6}$ | $3.54 e+000 \pm 3.54 e-001$ | $1.50 e+000$ |
| $\mathrm{CHS}_{10}$ | $5.33 e+000 \pm 6.98 e-001$ | $3.03 e+000$ |
| $\mathrm{CHS}_{15}$ | $7.15 e+000 \pm 9.90 e-001$ | $2.41 e+000$ |
| $\mathrm{CHS}_{30}$ | $6.86 e+000 \pm 1.07 e+000$ | $3.57 e+000$ |

Table 5: Results of unrotated Rastrigin function.

| Algorithm | Mean | Best |
| :--- | :---: | :---: |
| SHS | $4.39 e-001 \pm 1.65 e-001$ | $2.96 e-002$ |
| IHS | $1.44 e+000 \pm 3.12 e-001$ | $2.31 e-002$ |
| GHS | $2.20 e-003 \pm 1.28 e-003$ | $4.41 e-007$ |
| $\mathrm{CHS}_{6}$ | $3.01 e-008 \pm 1.31 e-009$ | $2.14 e-008$ |
| $\mathrm{CHS}_{10}$ | $8.88 e-009 \pm 4.35 e-010$ | $6.14 e-009$ |
| $\mathrm{CHS}_{15}$ | $7.59 e-010 \pm 6.88 e-011$ | $4.63 e-010$ |
| $\mathrm{CHS}_{30}$ | $1.69 e-012 \pm 2.22 e-013$ | $7.99 e-013$ |

satisfying solution. Table 2 shows the experimental results of

Table 6: Results of rotated Rastrigin function.

| Algorithm | Mean | Best |
| :--- | :---: | :---: |
| SHS | $1.92 e+002 \pm 2.99 e+000$ | $1.77 e+002$ |
| IHS | $1.92 e+002 \pm 2.85 e+000$ | $1.73 e+002$ |
| GHS | $1.86 e+002 \pm 3.81 e+000$ | $1.65 e+002$ |
| CHS $_{6}$ | $8.22 e+001 \pm 8.38 e+000$ | $4.38 e+001$ |
| $\mathrm{CHS}_{10}$ | $1.09 e+002 \pm 1.38 e+001$ | $4.28 e+001$ |
| $\mathrm{CHS}_{15}$ | $1.37 e+002 \pm 1.14 e+001$ | $8.86 e+001$ |
| $\mathrm{CHS}_{30}$ | $1.71 e+002 \pm 1.38 e+001$ | $6.96 e+001$ |

Table 7: Results of unrotated Griewangk function.

| Algorithm | Mean | Best |
| :--- | :---: | :---: |
| SHS | $1.06 e+000 \pm 7.92 e-003$ | $1.02 e+000$ |
| IHS | $1.05 e+000 \pm 5.35 e-003$ | $1.02 e+000$ |
| GHS | $3.52 e-002 \pm 2.78 e-002$ | $3.31 e-006$ |
| $\mathrm{CHS}_{6}$ | $5.02 e-002 \pm 9.63 e-003$ | $6.61 e-012$ |
| $\mathrm{CHS}_{10}$ | $7.06 e-002 \pm 2.14 e-002$ | $1.82 e-012$ |
| $\mathrm{CHS}_{15}$ | $7.79 e-002 \pm 1.49 e-002$ | $9.86 e-003$ |
| $\mathrm{CHS}_{30}$ | $7.07 e-002 \pm 2.03 e-002$ | $1.72 e-002$ |

Table 8: Results of rotated Griewangk function.

| Algorithm | Mean | Best |
| :--- | :---: | :---: |
| SHS | $1.06 e+000 \pm 8.65 e-003$ | $1.03 e+000$ |
| IHS | $1.05 e+000 \pm 6.78 e-003$ | $1.00 e+000$ |
| GHS | $4.79 e-001 \pm 7.81 e-002$ | $1.36 e-004$ |
| $\mathrm{CHS}_{6}$ | $9.61 e-003 \pm 3.90 e-003$ | $4.96 e-012$ |
| $\mathrm{CHS}_{10}$ | $1.50 e-002 \pm 5.09 e-003$ | $2.04 e-012$ |
| $\mathrm{CHS}_{15}$ | $1.22 e-002 \pm 5.19 e-003$ | $2.53 e-013$ |
| $\mathrm{CHS}_{30}$ | $1.09 e-002 \pm 5.78 e-003$ | $8.00 e-016$ |

Table 9: Results of unrotated Rosenbrock function.

| Algorithm | Mean | Best |
| :--- | :---: | :---: |
| SHS | $5.54 e+001 \pm 7.88 e+000$ | $3.85 e+000$ |
| IHS | $4.92 e+001 \pm 9.34 e+000$ | $1.32 e+001$ |
| GHS | $6.89 e+001 \pm 1.63 e+001$ | $2.67 e+001$ |
| $\mathrm{CHS}_{6}$ | $3.16 e+001 \pm 6.67 e+000$ | $1.18 e-002$ |
| $\mathrm{CHS}_{10}$ | $3.57 e+001 \pm 8.76 e+000$ | $2.36 e+000$ |
| $\mathrm{CHS}_{15}$ | $2.77 e+001 \pm 7.78 e+000$ | $4.49 e+000$ |
| $\mathrm{CHS}_{30}$ | $1.51 e+001 \pm 2.20 e+000$ | $1.53 e-003$ |

Table 10: Results of rotated Rosenbrock function.

| Algorithm | Mean | Best |
| :--- | :---: | :---: |
| SHS | $2.81 e+001 \pm 3.80 e-001$ | $2.58 e+001$ |
| IHS | $2.80 e+001 \pm 3.40 e-001$ | $2.61 e+001$ |
| GHS | $2.91 e+001 \pm 9.03 e-001$ | $2.78 e+001$ |
| $\mathrm{CHS}_{6}$ | $2.99 e+001 \pm 5.48 e+000$ | $2.11 e+001$ |
| $\mathrm{CHS}_{10}$ | $2.55 e+001 \pm 4.57 e+000$ | $1.64 e+001$ |
| $\mathrm{CHS}_{15}$ | $2.24 e+001 \pm 1.45 e+000$ | $1.67 e+001$ |
| $\mathrm{CHS}_{30}$ | $1.89 e+001 \pm 1.18 e+000$ | $1.63 e+001$ |

the rotated Quadric function. The results are very different from those shown in Table 1.

The results of the experiment show that the rotated Quadric function is more easily optimized by HS algorithms and that the CHS can always find more satisfying solutions than other HS algorithms. As the number of groups $m$ increases, the solution becomes better and better, but at the same time, the algorithm becomes more and more complex. Figure 4 shows the best function value profile of each algorithm both for unrotated and rotated Quadric functions. This figure is a more visual representation. It shows the best results among 30 independent runs for each algorithm. From Figure 4, we can deduce that in unrotated cases, all the algorithms converge very sharply, while in rotated cases, the solutions found by CHS are better.

The results of Ackley's function are very interesting and very different from the results of the Quadric function. Table 3 shows the experimental results of unrotated Ackley's function; this is a multimodal function and is easily optimized by HS algorithms. Almost all the algorithms are capable of finding a satisfactory solution, but among the tested algorithms, CHS performs better with both the mean and the best results than the other algorithms, and the higher $m$ is, the better the solution is. However, in terms of the rotated Ackley's function, CHS does not perform as well as it is illustrated in Table 4. In terms of the rotated Ackley's function, IHS and GHS perform better, especially the IHS algorithm. This can be deduced from Figure 5. When the search space is rotated, all the algorithms perform worse, especially CHS, the performance of which is as bad as SHS. GHS converges very fast to a local optimum, although it is capable of finding a better solution. This is because GHS resembles the standard PSO, which is not good at dealing with multimodal problems.

The Rastrigin function is also a multimodal function with many local optima. In an unrotated case, CHS performs relatively well, especially $\mathrm{CHS}_{30}$. It is the performance leader, as shown in Table 5. The results of the experiment are very similar to those observed with unrotated Ackley's function: the higher $m$ is, the better the solution found by CHS will be. In the rotated case, however, none of the algorithms performs so well, as shown in Table 6. For CHS, with the increase of $m$, the mean of the solutions becomes worse, which is the reverse of the unrotated case. Figure 6 shows the best function value profile. From Figure 6(a), we can find that, generally, CHS converges, offering a better solution faster, while Figure 6(b) shows a very different result.

Tables 7 and 8 show the results of the Griewank function, which is a nonlinear multimodal and complex function. In unrotated cases, CHS performs better than both SHS and IHS, but not so well as GHS. Furthermore, when $m$ increases, the performance of CHS is very similar. This is very different from the results of unrotated Ackley's function and unrotated Rastrigin function, which are shown from Table 3 to Table 6. In rotated cases, the mean for each algorithm is very similar, but the best solutions for each algorithm are distinct from each other, as shown in Table 8. We might also observe that the higher $m$ is, the better the solution found by CHS. Figure 7


FIgURe 4: Quadric $\left(f_{1}\right)$ best function value profile. (a) Unrotated Quadric best function value profile. (b) Rotated Quadric best function value profile.


(a)


| $\rightarrow$ SHS | - CHS10 |
| :---: | :---: |
| $\bigcirc$ - IHS | $\triangle$ CHS15 |
| $\square$ GHS | $\checkmark$ CHS30 |
| * CHS6 |  |

(b)

Figure 5: Ackley $\left(f_{2}\right)$ best function value profile. (a) Unrotated Ackley best function value profile. (b) Rotated Ackley best function value profile.


Figure 6: Rastrigin $\left(f_{3}\right)$ best function value profile. (a) Unrotated Rastrigin best function value profile. (b) Rotated Rastrigin best function value profile.


| $\rightarrow$ SHS | $\mp$ CHS10 |
| :--- | :--- |
| $\rightarrow$ IHS | $\square$ CHS15 |
| $\square$ GHS | $\nabla$ CHS30 |
| $\rightarrow$ CHS6 |  |

(a)


(b)

Figure 7: Griewangk $\left(f_{4}\right)$ best function value profile. (a) Unrotated Griewangk best function value profile. (b) Rotated Griewangk best function value profile.


Figure 8: Rosenbrock $\left(f_{5}\right)$ best function value profile. (a) Unrotated Rosenbrock best function value profile. (b) Rotated Rosenbrock best function value profile.
shows the best function value profile for each algorithm. For both the unrotated Griewank function and the rotated Griewank function, SHS and IHS trap in a local optimum easily, but GHS is capable of finding a better solution.

The results of the Rosenbrock function are shown in Tables 9 and 10. The global optimum of the Rosenbrock function is located in a narrow valley so that it is very difficult for the algorithms to find the correct direction in order to reach the global region, and thus, none of the algorithms is satisfactory, especially in rotated cases. This is shown more visually in Figure 8.

In this section, we have examined the results of the experiments and from these results, it has been possible to deduce that CHS is capable of solving all the problems that can be solved by SHS. In most cases, the results found by CHS are better than those found by the SHS algorithm or the IHS algorithm. We should, however, bear in mind the "no free lunch" theory, which acts as a reminder that no algorithm is capable of performing satisfactorily for all problems. Each algorithm, thus, has its own advantages and, as a result, is able to deal with one kind of problem only.

## 7. Conclusions

A cooperative harmony search algorithm has been presented in this paper, and as the results have shown, this cooperative approach constitutes a very real improvement in terms of the performance of SHS. This is especially so regarding the quality of the solutions, and in most cases, the higher the number of groups $m$, the better the solution. CHS can furthermore be applied as a solution to all the same problems
as the SHS algorithm, so that, when the dimensionality of the problem increases, the performance of the CHS algorithm is significantly better.

Despite the fact that the CHS algorithm has numerous advantages, there are also some shortcomings. As we mentioned in Section 4, the CHS algorithm may be more easily trapped in a local optimum. As the CHS algorithm is not an answer to every problem, we must ask ourselves the question of when exactly it is that the CHS algorithm can perform better and which kinds of problems can be solved satisfactorily by the CHS algorithm. This is a problem that should be solved urgently. Another unsolved problem is why the cooperative method is actually better. We explained this in Section 4 by means of examples, but the experiment still requires mathematical analysis and proof.

Another aspect to which attention should be paid is the mathematical analysis of the HS algorithm. Although we have proved that the SHS algorithm must converge, can it in fact converge to the global optimum or to a local optimum? In addition, it is not clear that what the influences the parameters have on HS algorithm are, as all the parameters used here have been suggested by other scholars or selected by experience. If someone were to provide a mathematical analysis for the parameters, it would be very helpful for the future development of the HS algorithm.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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# Modeling and Chaotic Dynamics of the Laminated Composite Piezoelectric Rectangular Plate 

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#### Abstract

This paper investigates the multipulse heteroclinic bifurcations and chaotic dynamics of a laminated composite piezoelectric rectangular plate by using an extended Melnikov method in the resonant case. According to the von Karman type equations, Reddy's third-order shear deformation plate theory, and Hamilton's principle, the equations of motion are derived for the laminated composite piezoelectric rectangular plate with combined parametric excitations and transverse excitation. The method of multiple scales and Galerkin's approach are applied to the partial differential governing equation. Then, the four-dimensional averaged equation is obtained for the case of $1: 3$ internal resonance and primary parametric resonance. The extended Melnikov method is used to study the Shilnikov type multipulse heteroclinic bifurcations and chaotic dynamics of the laminated composite piezoelectric rectangular plate. The necessary conditions of the existence for the Shilnikov type multipulse chaotic dynamics are analytically obtained. From the investigation, the geometric structure of the multipulse orbits is described in the four-dimensional phase space. Numerical simulations show that the Shilnikov type multipulse chaotic motions can occur. To sum up, both theoretical and numerical studies suggest that chaos for the Smale horseshoe sense in motion exists for the laminated composite piezoelectric rectangular plate.


## 1. Introduction

The need for high-speed, light-weight, and energy-saving structures in the aerospace and aviation industry has led to the composite materials instead of traditional materials. Additional requirements for multifunctionality, active vibration, shape control, vibration suppression, and acoustic control have made the development of smart and intelligent structures. A piezoelectric composite laminate is composed of piezoelectric layers which are embedded in laminated composite structures or are boned on the surface of structures. The direct and converse piezoelectric effects are used to suppress the transient vibration and to control the deformation, shape, and buckling of the structures. Such lightweight flexible structures generate large deformations, geometrical nonlinearity, and structural instability when piezoelectric composite laminates are subjected to the coupling between the mechanical and electrical loads. Therefore, it is necessary to study geometrically nonlinear effects on dynamic characteristics of structures in order to accurately design
and effectively control vibrations of piezoelectric composite laminate structures. It is very important to investigate the large amplitude nonlinear vibrations of smart structures with piezoelectric materials in order to achieve and predict the desired performance of the systems.

Recently, the studies on dynamics of composite structures with piezoelectric materials have made some progress. Tzou et al. [1] used spatially distributed orthogonal piezoelectric actuators to perform the distributed structural control of elastic shell. They utilized a gain factor and a spatially distributed mode actuator function to describe modal feedback functions. Purekar et al. [2] presented phased array filters with piezoelectric sensors to detect damage in isotropic plates and adopted wave propagation to describe plate dynamics. Ishihara and Noda [3] took into account the effect of transverse shear to analyze the dynamic behavior of the laminate composed of fiber-reinforced laminae and piezoelectric layers constituting a symmetric cross-ply laminate rectangular plate with simply supported edges. Oh [4] considered snap-through thermopiezo-elastic behaviors to
examine the buckling bifurcation and sling-shot buckling of active piezo-laminated plates. Lee et al. [5] employed thirdorder shear deformation theory and nonlinear finite element to canvass deflection suppression characteristics of laminated composite shell structures with smart material laminae. Panda and Ray [6] exploited the first-order shear deformation theory and the three-dimensional finite element method to delve into the open-loop and closed-loop nonlinear dynamics of functionally graded plates with the piezoelectric fiberreinforced composite material under the thermal environment. Dumir et al. [7] used the extended Hamilton's principle to derive the coupled nonlinear equations of motion and the boundary conditions for buckling and vibration of symmetrically laminated hybrid angle-ply piezoelectric panels under in-plane electrothermomechanical loading. Yao and Zhang [8] employed the third-order shear deformation plate theory to explore the bifurcations and chaotic dynamics of the fouredge simply supported laminated composite piezoelectric rectangular plate in the case of the $1: 2$ internal resonances.

The global bifurcations and chaotic dynamics of highdimensional nonlinear systems have been at the forefront of nonlinear dynamics for the past two decades. There are two ways of solutions on Shilnikov type chaotic dynamics of high-dimensional nonlinear systems. One is Shilnikov type single-pulse chaotic dynamics and the other is Shilnikov type multipulse chaotic dynamics. Most researchers focused on Shilnikov type single-pulse chaotic dynamics of highdimensional nonlinear systems. Much research in this field has concentrated on Shilnikov type single-pulse chaotic dynamics of thin plate structures. Feng and Sethna [9] utilized the global perturbation method to study the global bifurcations and chaotic dynamics of the thin plate under parametric excitation and obtained the conditions in which the Shilnikov type homoclinic orbits and chaos can occur. Tien et al. [10] applied the Melnikov method to investigate the global bifurcation and chaos for the Smale horseshoe sense of a two-degree-of-freedom shallow arch subjected to simple harmonic excitation for the case of $1: 2$ internal resonance. Malhotra and Sri Namachchivaya [11] employed the averaging method and Melnikov technique to canvass the local, global bifurcations and chaotic motions of a two-degree-of-freedom shallow arch subjected to simple harmonic excitation for the case of $1: 1$ internal resonance. The global bifurcations and chaotic dynamics were investigated by Zhang [12] for the simply supported rectangular thin plates subjected to the parametrical-external excitation and the parametrical excitation. Yeo and Lee [13] made use of the global perturbation technique to examine the global dynamics of an imperfect circular plate for the case of 1:1 internal resonance and obtained the criteria for chaotic motions of homoclinic orbits and heteroclinic orbits. Yu and Chen [14] adopted the global perturbation method to explore the global bifurcations of a simply supported rectangular metallic plate subjected to a transverse harmonic excitation for the case of $1: 1$ internal resonance.

While most of studies are on the Shilnikov type single-pulse global bifurcations and chaotic dynamics of high-dimensional nonlinear systems, there are researchers investigating the Shilnikov type multipulse homoclinic and
heteroclinic bifurcations and chaotic dynamics. So far, there are two theories of the Shilnikov type multipulse chaotic dynamics. One is the extended Melnikov method and the other theory is the energy phase method. Much achievement is made in the former theory of high-dimensional nonlinear systems. In 1996, Kovačič and Wettergren [15] used a modified Melnikov method to investigate the existence of the multipulse jumping of homoclinic orbits and chaotic dynamics in resonantly forced coupled pendula. Furthermore, Kaper and Kovačič [16] studied the existence of several classes of the multibump orbits homoclinic to resonance bands for completely integral Hamiltonian systems subjected to small amplitude Hamiltonian and damped perturbations. Camassa et al. [17] presented a new Melnikov method which is called the extended Melnikov method to explore the multipulse jumping of homoclinic and heteroclinic orbits in a class of perturbed Hamiltonian systems. Until recently, Zhang and Yao [18] introduced the extended Melnikov method to the engineering field. They came up with a simplification of the extended Melnikov method in the resonant case and utilized it to analyze the Shilnikov type multipulse homoclinic bifurcations and chaotic dynamics for the nonlinear nonplanar oscillations of the cantilever beam.

The study on the second theory of the Shilnikov type multipulse chaotic dynamics was stated by Haller and Wiggins [19]. They presented the energy phase method to investigate the existence of the multipulse jumping homoclinic and heteroclinic orbits in perturbed Hamiltonian systems. Up to now, few researchers have made use of the energy phase method to study the Shilnikov type multipulse homoclinic and heteroclinic bifurcations and chaotic dynamics of highdimensional nonlinear systems in engineering applications. Malhotra et al. [20] used the energy-phase method to investigate multipulse homoclinic orbits and chaotic dynamics for the motion of flexible spinning discs. Yu and Chen [21] made use of the energy-phase method to examine the Shilnikov type multipulse homoclinic orbits of a harmonically excited circular plate.

This paper focuses on the Shilnikov type multipulse orbits and chaotic dynamics for a simply supported laminated composite piezoelectric rectangular plate under combined parametric excitations and transverse load. Based on the von Karman type equations and Reddy's third-order shear deformation plate theory, Hamilton's principle is employed to obtain the governing nonlinear equations of the laminated composite piezoelectric rectangular plate with combined parametric excitation and transverse load. We apply Galerkin's approach and the method of multiple scales to the partial differential governing equations to obtain the fourdimensional averaged equation for the case of $1: 3$ internal resonance and primary parametric resonance. From the averaged equation, the theory of normal form is used to find the explicit formulas of normal form. We study the heteroclinic bifurcations of the unperturbed system and the characteristic of the hyperbolic dynamics of the dissipative system, respectively. Finally, we employ the extended Melnikov method to analyze the Shilnikov type multipulse orbits and chaotic dynamics in the laminated composite piezoelectric plate. In this paper, the extended Melnikov function


Figure 1: The model of a laminated composite piezoelectric rectangular plate is given.
can be simplified in the resonant case and does not depend on the perturbation parameter. We have used the extended Melnikov method to investigate heteroclinic bifurcations and multipulse chaotic dynamics of the laminated composite piezoelectric plate under the case of $1: 3$ internal resonances. The analysis indicates that there exist the Shilnikov type multipulse jumping orbits in the perturbed phase space for the averaged equations. We present the geometric structure of the multipulse orbits in the four-dimensional phase space. The results from numerical simulation also show that the chaotic motion can occur in the motion of the laminated composite piezoelectric plate, which verifies the analytical prediction. The Shilnikov type multipulse orbits are discovered from the results of numerical simulation. In summary, both theoretical and numerical studies demonstrate that chaos for the Smale horseshoe sense in the motion exists. This paper demonstrates how to employ the extended Melnikov method to analyze the Shilnikov type multipulse heteroclinic bifurcations and chaotic dynamics of high-dimensional nonlinear systems in engineering applications.

The laminated composite piezoelectric rectangular plates are widely applied in space stations, satellite solar panels, sensors, and actuators for the active control of structures and so on. In this paper, we have investigated the multipulse global bifurcations and chaotic dynamics of a laminated composite piezoelectric rectangular plate by using an extended Melnikov method and numerical simulations in detail. We have understood nonlinear vibration characteristics of a laminated composite piezoelectric rectangular plate. Our theoretical results can be used to solve some engineering problems. Since these smart structures are generally light weight and relatively large structural flexibility, laminated composite piezoelectric rectangular plates can induce large vibration deformation during the rapid deployment. In order to eliminate or suppress large vibration and chaotic motion, theoretical results can help optimize the design of the structural parameters of laminated composite piezoelectric rectangular plates. Therefore, the theoretical studies on the multipulse global bifurcations and chaotic dynamics of laminated composite piezoelectric rectangular plates play a very
important role in applications in aerospace and mechanical engineering.

## 2. Equations of Motion and Perturbation Analysis

We consider a four-edge simply supported laminated composite piezoelectric rectangular plate, where the length, the width, and the thickness are denoted by $a, b$, and $h$, respectively. The laminated composite piezoelectric rectangular plate is subjected to in-plane excitation, transverse excitation, and piezoelectric excitation, as shown in Figure 1. We consider the laminated composite piezoelectric rectangular plate as regular symmetric cross-ply laminates with $n$ layers with respect to principal material coordinates alternatively oriented at $0^{\circ}$ and $90^{\circ}$ to the laminated coordinate axes. Some of layers are made of the piezoelectric materials as actuators, and the other layers are made of fiber-reinforced composite materials. It is assumed that different layers of the symmetric cross-ply composite laminated piezoelectric rectangular plate are perfectly clung to each other, and piezoelectric actuator layers are embedded in the plate. The fiber direction of odd-numbered layers is the $x$-direction of the laminate. The fiber direction of even-numbered layers is the $y$-direction of the laminate. Simply supported plate with immovable edges satisfies the symmetry requirement that eliminates the coupling between bending and extension. However, the displacement of $x$ is free to move at the edge of $y=0$, and the displacement of $y$ is free to move at the edge of $x=0$. Therefore, the membrane stress is smaller and there exists the coupling between bending and extension. A Cartesian coordinate system $O x y z$ is located in the middle surface of the composite laminated piezoelectric rectangular plate. Assume that ( $w, v, u$ ) and $\left(w_{0}, v_{0}, u_{0}\right)$ describe the displacements of an arbitrary point and a point in the middle surface of the composite laminated piezoelectric rectangular plate in the $x, y$, and $z$ directions, respectively. It is also assumed that in-plane excitations of the composite laminated piezoelectric rectangular plate are loaded along the $y$-direction at $x=0$
and the $x$-direction at $y=0$ with the form of $q_{0}+q_{x} \cos \Omega_{1} t$ and $q_{1}+q_{y} \cos \Omega_{2} t$, respectively. Transverse excitation loaded to the composite laminated piezoelectric rectangular plate is expressed as $q=q_{3} \cos \Omega_{3}$ t. The dynamic electrical loading is represented by $E_{z}=E_{z} \cos \Omega_{4} t$.

Considering Reddy's third-order shear deformation description of the displacement field, we have

$$
\begin{align*}
u(x, y, z, t)= & u_{0}(x, y, t)+z \phi_{x}(x, y, t) \\
& -z^{3} \frac{4}{3 h^{2}}\left(\phi_{x}+\frac{\partial w_{0}}{\partial x}\right)  \tag{1a}\\
v(x, y, z, t)= & v_{0}(x, y, t)+z \phi_{y}(x, y, t) \\
& -z^{3} \frac{4}{3 h^{2}}\left(\phi_{y}+\frac{\partial w_{0}}{\partial y}\right)  \tag{lb}\\
w(x, y, z, t)= & w_{0}(x, y, t) \tag{1c}
\end{align*}
$$

where $\left(u_{0}, v_{0}, w_{0}\right)$ are the deflection of a point on the middle surface, $(u, v, w)$ are the displacement components along the $(x, y, z)$ coordinate directions, and $\phi_{x}$ and $\phi_{y}$ represent the rotation components of normal to the middle surface about the $y$ and $x$ axes, respectively.

The nonlinear strain-displacement relations are assumed to have the following form:

$$
\begin{gather*}
\varepsilon_{x x}=\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^{2}, \quad \varepsilon_{x z}=\frac{1}{2}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) \\
\varepsilon_{x y}=\frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x} \frac{\partial w}{\partial y}\right) \\
\varepsilon_{y y}=\frac{\partial v}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^{2}, \quad \varepsilon_{y z}=\frac{1}{2}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right) \\
\varepsilon_{z z}=\frac{\partial w}{\partial z} \tag{2}
\end{gather*}
$$

Stress constitutive relations are presented as follows:

$$
\begin{equation*}
\sigma_{i j}=\sigma_{i j k l}^{s} \varepsilon_{k l}-e_{i j k} E_{k}, \quad(i, j, k, l=x, y, z), \tag{3}
\end{equation*}
$$

where $\sigma_{i j}$ and $\varepsilon_{k l}$ denote the mechanical stresses and strains in extended vector notation, $\sigma_{i j k l}^{s}$ represents the elastic stiffness tensor, $E_{k}$ stands for the electric field vector, and $e_{i j}$ is the piezoelectric tensor.

According to Hamilton's principle, the nonlinear governing equations of motion in terms of generalized displacements $\left(u_{0}, v_{0}, w_{0}, \phi_{x}, \phi_{y}\right)$ for the composite laminated
piezoelectric rectangular plate are given in the previous studies as follows [8]:

$$
\begin{align*}
& A_{11} \frac{\partial^{2} u_{0}}{\partial x^{2}}+A_{66} \frac{\partial^{2} u_{0}}{\partial y^{2}}+\left(A_{12}+A_{66}\right) \frac{\partial^{2} v_{0}}{\partial x \partial y} \\
& +A_{11} \frac{\partial w_{0}}{\partial x} \frac{\partial^{2} w_{0}}{\partial x^{2}}+A_{66} \frac{\partial w_{0}}{\partial x} \frac{\partial^{2} w_{0}}{\partial y^{2}}  \tag{4a}\\
& +\left(A_{12}+A_{66}\right) \frac{\partial w_{0}}{\partial y} \frac{\partial^{2} w_{0}}{\partial x \partial y} \\
& =I_{0} \ddot{u}_{0}+J_{1} \ddot{\phi}_{x}-c_{1} I_{3} \frac{\partial \ddot{w}_{0}}{\partial x} \text {, } \\
& A_{66} \frac{\partial^{2} v_{0}}{\partial x^{2}}+A_{22} \frac{\partial^{2} v_{0}}{\partial y^{2}}+\left(A_{21}+A_{66}\right) \frac{\partial^{2} u_{0}}{\partial x \partial y} \\
& +A_{66} \frac{\partial w_{0}}{\partial y} \frac{\partial^{2} w_{0}}{\partial x^{2}}+A_{22} \frac{\partial w_{0}}{\partial y} \frac{\partial^{2} w_{0}}{\partial y^{2}}  \tag{4b}\\
& +\left(A_{21}+A_{66}\right) \frac{\partial w_{0}}{\partial x} \frac{\partial^{2} w_{0}}{\partial x \partial y} \\
& =I_{0} \ddot{v}_{0}+J_{1} \ddot{\phi}_{y}-c_{1} I_{3} \frac{\partial \ddot{w}_{0}}{\partial y} \text {, } \\
& A_{66} \frac{\partial w_{0}}{\partial x} \frac{\partial^{2} u_{0}}{\partial y^{2}}-H_{22} 2_{1}^{2} \frac{\partial^{4} w_{0}}{\partial y^{4}} \\
& +c_{1}\left(2 F_{66}+F_{12}-2 H_{66} c_{1}-H_{12} c_{1}\right) \frac{\partial^{3} \phi_{y}}{\partial y \partial x^{2}} \\
& +c_{1}\left(F_{22}-H_{22} c_{1}\right) \frac{\partial^{3} \phi_{y}}{\partial y^{3}}-H_{11} c_{1} \frac{\partial^{4} w_{0}}{\partial x^{4}} \\
& +A_{11} \frac{\partial w_{0}}{\partial x} \frac{\partial^{2} u_{0}}{\partial x^{2}}+\left(F_{44} c_{2}^{2}-2 D_{44} \mathcal{c}_{2}+A_{44}\right) \frac{\partial \phi_{y}}{\partial y} \\
& +c_{1}\left(F_{21}+2 F_{66}-H_{21} c_{1}-2 H_{66} c_{1}\right) \frac{\partial^{3} \phi_{x}}{\partial y^{2} \partial x} \\
& -c_{1}^{2}\left(H_{21}+4 H_{66}+H_{12}\right) \frac{\partial^{4} w_{0}}{\partial y^{2} \partial x^{2}} \\
& +\left(A_{44}-N_{y}^{P} \cos \left(\Omega_{4} t\right)+F_{44} c_{2}^{2}-2 D_{44} \mathcal{c}_{2}\right) \frac{\partial^{2} w_{0}}{\partial y^{2}} \\
& -\frac{\partial N_{y}^{P}}{\partial y} \cos \left(\Omega_{2} t\right) \frac{\partial w_{0}}{\partial y}+\left(A_{21}+4 A_{66}+A_{12}\right) \\
& \times \frac{\partial w_{0}}{\partial x} \frac{\partial w_{0}}{\partial y} \frac{\partial^{2} w_{0}}{\partial y \partial x}+c_{1}\left(F_{11}-H_{11} c_{1}\right) \frac{\partial^{3} \phi_{x}}{\partial x^{3}} \\
& +\left(A_{21}+A_{66}\right) \frac{\partial w_{0}}{\partial y} \frac{\partial^{2} u_{0}}{\partial y \partial x}+A_{21} \frac{\partial u_{0}}{\partial x} \frac{\partial^{2} w_{0}}{\partial y^{2}} \\
& \times A_{66} \frac{\partial w_{0}}{\partial y} \frac{\partial^{2} v_{0}}{\partial x^{2}}+A_{22} \frac{\partial w_{0}}{\partial y} \frac{\partial^{2} v_{0}}{\partial y^{2}}
\end{align*}
$$

$$
\begin{align*}
& +\frac{1}{2}\left(A_{12}+2 A_{66}\right)\left(\frac{\partial w_{0}}{\partial y}\right)^{2} \frac{\partial^{2} w_{0}}{\partial x^{2}}+A_{22} \frac{\partial^{2} w_{0}}{\partial y^{2}} \frac{\partial v_{0}}{\partial y} \\
& +\left(A_{12}+A_{66}\right) \frac{\partial w_{0}}{\partial x} \frac{\partial^{2} v_{0}}{\partial y \partial x}+\frac{1}{2}\left(A_{21}+2 A_{66}\right) \frac{\partial^{2} w_{0}}{\partial y^{2}} \\
& \times\left(\frac{\partial w_{0}}{\partial x}\right)^{2}+\frac{3}{2} A_{11}\left(\frac{\partial w_{0}}{\partial x}\right)^{2} \frac{\partial^{2} w_{0}}{\partial x^{2}}+A_{11} \frac{\partial^{2} w_{0}}{\partial x^{2}} \frac{\partial u_{0}}{\partial x} \\
& +A_{12} \frac{\partial^{2} w_{0}}{\partial x^{2}} \frac{\partial v_{0}}{\partial y}+2 A_{66} \frac{\partial^{2} w_{0}}{\partial y \partial x} \frac{\partial v_{0}}{\partial x} \\
& +2 A_{66} \frac{\partial^{2} w_{0}}{\partial y \partial x} \frac{\partial u_{0}}{\partial y}+\frac{3}{2} A_{22}\left(\frac{\partial w_{0}}{\partial y}\right)^{2} \frac{\partial^{2} w_{0}}{\partial y^{2}}  \tag{4e}\\
& +\left(A_{55}+q_{x} \cos \left(\Omega_{1} t\right)-N_{x}^{P} \cos \left(\Omega_{4} t\right)\right. \\
& \left.+F_{55} c_{2}^{2}-2 D_{55} \mathcal{c}_{2}\right) \frac{\partial^{2} w_{0}}{\partial x^{2}}-\frac{\partial N_{x}^{P}}{\partial x} \cos \left(\Omega_{3} t\right) \frac{\partial w}{\partial x}  \tag{5a}\\
& +\left(F_{55} c_{2}^{2}-2 D_{55} \mathcal{c}_{2}+A_{55}\right) \frac{\partial \phi_{x}}{\partial x} \\
& -q \cos \left(\Omega_{3} t\right)+k f \frac{\partial w_{0}}{\partial t}  \tag{5b}\\
& =I_{0} \ddot{w}_{0}-c_{1}^{2} I_{6}\left(\frac{\partial^{2} \ddot{w}_{0}}{\partial x^{2}}+\frac{\partial^{2} \ddot{w}_{0}}{\partial y^{2}}\right)  \tag{5c}\\
& +c_{1} I_{3}\left(\frac{\partial \ddot{u}_{0}}{\partial x}+\frac{\partial \ddot{v}_{0}}{\partial y}\right)+c_{1} J_{4}\left(\frac{\partial \ddot{\phi}_{x}}{\partial x}+\frac{\partial \ddot{\phi}_{y}}{\partial y}\right),  \tag{5d}\\
& \left(D_{11}-2 F_{11} c_{1}+H_{11} c_{1}^{2}\right) \frac{\partial^{2} \phi_{x}}{\partial x^{2}}+\left(D_{66}-2 F_{66} c_{1}+H_{66} c_{1}^{2}\right) \frac{\partial^{2} \phi_{x}}{\partial y^{2}}  \tag{5f}\\
& -c_{1}\left(F_{11}-H_{11} c_{1}\right) \frac{\partial^{3} w_{0}}{\partial x^{3}} \\
& -\left(F_{55} c_{2}^{2}-2 D_{55} \mathcal{c}_{2}+A_{55}\right) \frac{\partial w_{0}}{\partial x} \\
& +\left(D_{12}+D_{66}+H_{66} c_{1}^{2}-2 F_{66} c_{1}+H_{12} c_{1}^{2}-2 F_{12} c_{1}\right) \\
& \times \frac{\partial^{2} \phi_{y}}{\partial y \partial x}-c_{1}\left(2 F_{66}+F_{12}-2 H_{66} c_{1}-H_{12} c_{1}\right)  \tag{6a}\\
& \times \frac{\partial^{3} w_{0}}{\partial y^{2} \partial x}+\left(2 D_{55} \mathcal{c}_{2}-A_{55}-F_{55} c_{2}^{2}\right) \phi_{x}  \tag{6b}\\
& =J_{1} \ddot{u}_{0}+K_{2} \ddot{\phi}_{x}-c_{1} J_{4} \frac{\partial \ddot{w}_{0}}{\partial x},  \tag{6c}\\
& \left(D_{66}-2 F_{66} c_{1}+H_{66} c_{1}^{2}\right) \frac{\partial^{2} \phi_{y}}{\partial x^{2}}  \tag{6d}\\
& -c_{1}\left(F_{21}+2 F_{66}-H_{21} c_{1}-2 H_{66} c_{1}\right) \frac{\partial^{3} w_{0}}{\partial y \partial x^{2}} \tag{6e}
\end{align*}
$$

The simply supported boundary conditions of the composite laminated piezoelectric rectangular plate can be represented as follows [8, 22]:

$$
\begin{aligned}
& x=0: \quad w=0, \quad \phi_{y}=0, \quad N_{x y}=0, \quad M_{x x}=0, \\
& x=a: \quad w=0, \quad \phi_{y}=0, \quad N_{x y}=0, \quad M_{x x}=0, \\
& y=0: \quad w=0, \quad \phi_{x}=0, \quad N_{x y}=0, \quad M_{y y}=0, \\
& y=b: \quad w=0, \quad \phi_{x}=0, \quad N_{x y}=0, \quad M_{y y}=0, \\
& \left.\int_{0}^{h} N_{x x}\right|_{x=0} d z=-\int_{0}^{h}\left(q_{0}+q_{x} \cos \Omega_{1} t\right) d z, \\
& \left.\int_{0}^{h} N_{y y}\right|_{y=0} d z=-\int_{0}^{h}\left(q_{1}+q_{y} \cos \Omega_{2} t\right) d z .
\end{aligned}
$$

The boundary condition (5f) includes the influence of the in-plane load. We consider complicated nonlinear dynamics of the composite laminated piezoelectric rectangular plate in the first two modes of $u_{0}, v_{0}, w_{0}, \phi_{x}$, and $\phi_{y}$. It is desirable that we select an appropriate mode function to satisfy the boundary condition. Thus, we can rewrite $u_{0}, v_{0}, w_{0}, \phi_{x}$, and $\phi_{y}$ in the following forms:

$$
\begin{aligned}
& u_{0}=u_{01}(t) \cos \frac{\pi x}{2 a} \cos \frac{\pi y}{2 b}+u_{02}(t) \cos \frac{3 \pi x}{2 a} \cos \frac{\pi y}{2 b} \\
& v_{0}=v_{1}(t) \cos \frac{\pi y}{2 b} \cos \frac{\pi x}{2 a}+v_{2}(t) \cos \frac{\pi y}{2 b} \cos \frac{3 \pi x}{2 a} \\
& w_{0}=w_{1}(t) \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}+w_{2}(t) \sin \frac{3 \pi x}{a} \sin \frac{\pi y}{b} \\
& \phi_{x}=\phi_{1}(t) \cos \frac{\pi x}{a} \sin \frac{\pi y}{b}+\phi_{2}(t) \cos \frac{3 \pi x}{a} \sin \frac{\pi y}{b} \\
& \phi_{y}=\phi_{3}(t) \cos \frac{\pi y}{b} \sin \frac{\pi x}{a}+\phi_{4}(t) \cos \frac{\pi y}{b} \sin \frac{3 \pi x}{a}
\end{aligned}
$$

By means of the Galerkin method, substituting (6a), (6b), $(6 \mathrm{c}),(6 \mathrm{~d}),(6 \mathrm{e})$ into (4a), (4b), (4c), (4d), (4e), integrating,
and neglecting all inertia terms in (4a), (4b), (4d), and (4e), we obtain the expressions of $u_{01}, u_{02}, v_{1}, v_{2}, \phi_{1}, \phi_{2}, \phi_{3}$, and $\phi_{4}$ via $w_{1}$ and $w_{2}$ as follows:

$$
\begin{align*}
& u_{01}=k_{1} w_{1}^{2}+k_{2} w_{2}^{2}+k_{3} w_{1} w_{2}  \tag{7a}\\
& u_{02}=k_{4} w_{1}^{2}+k_{5} w_{2}^{2}+k_{6} w_{1} w_{2},  \tag{7b}\\
& v_{1}=k_{7} w_{1}^{2}+k_{8} w_{2}^{2}+k_{9} w_{1} w_{2},  \tag{7c}\\
& v_{2}=k_{10} w_{1}^{2}+k_{11} w_{2}^{2}+k_{12} w_{1} w_{2}  \tag{7d}\\
& \phi_{1}=k_{19} w_{1}, \quad \phi_{2}=k_{20} w_{2},  \tag{7e}\\
& \phi_{3}=k_{21} w_{1}, \quad \phi_{4}=k_{22} w_{2}, \tag{7f}
\end{align*}
$$

where the coefficients presented in (7a), (7b), (7c), (7d), (7e), (7f) can be found in the previous studies [8].

In order to obtain the dimensionless governing equations of motion, we introduce the transformations of the variables and parameters

$$
\begin{gather*}
\bar{u}=\frac{u_{0}}{a}, \quad \bar{v}=\frac{v_{0}}{b}, \quad \bar{w}=\frac{w_{0}}{h}, \\
\bar{\phi}_{x}=\phi_{x}, \quad \bar{\phi}_{y}=\phi_{y}, \quad \bar{x}=\frac{x}{a}, \quad \bar{y}=\frac{y}{b}, \\
\bar{q}=\frac{b^{2}}{E h^{3}} q, \quad \bar{q}_{x}=\frac{b^{2}}{E h^{3}} q_{x}, \quad \bar{q}_{y}=\frac{b^{2}}{E h^{3}} q_{y}, \\
\bar{t}=\pi^{2}\left(\frac{E}{a b \rho}\right)^{1 / 2} t, \quad \bar{\Omega}_{i}=\frac{1}{\pi^{2}}\left(\frac{a b \rho}{E}\right)^{1 / 2} \Omega_{i} \quad(i=1,2), \\
\bar{A}_{i j}=\frac{(a b)^{1 / 2}}{E h^{2}} A_{i j}, \quad \bar{B}_{i j}=\frac{(a b)^{1 / 2}}{E h^{3}} B_{i j}, \\
\bar{D}_{i j}=\frac{(a b)^{1 / 2}}{E h^{4}} D_{i j}, \quad \bar{E}_{i j}=\frac{(a b)^{1 / 2}}{E h^{5}} E_{i j}, \\
\bar{F}_{i j}=\frac{(a b)^{1 / 2}}{E h^{6}} F_{i j}, \quad \bar{H}_{i j}=\frac{(a b)^{1 / 2}}{E h^{8}} H_{i j}, \\
\bar{I}_{i}=\frac{1}{(a b)^{(i+1) / 2} \rho} I_{i} . \tag{8}
\end{gather*}
$$

For simplicity, we drop the overbar in the following analysis. Substituting (5a), (5b), (5c), (5d), (5e), (5f)-(8) into (4c) and applying the Galerkin procedure, we obtain the governing equations of motion of the composite laminated piezoelectric rectangular plate for the dimensionless as follows:

$$
\begin{align*}
\ddot{w}_{1}+ & \mu_{1} \dot{w}_{1}+\omega_{1}^{2} w_{1} \\
& +\left(a_{2} \cos \Omega_{1} t+a_{3} \cos \Omega_{2} t-a_{4} \cos \Omega_{4} t\right) w_{1} \\
& +a_{5} w_{1}^{2} w_{2}+a_{6} w_{2}^{2} w_{1}+a_{7} w_{1}^{3}+a_{8} w_{2}^{3}=f_{1} \cos \Omega_{3} t \tag{9a}
\end{align*}
$$

$$
\begin{align*}
\ddot{w}_{2}+ & \mu_{2} \dot{w}_{2}+\omega_{2}^{2} w_{2} \\
& +\left(b_{2} \cos \Omega_{1} t+b_{3} \cos \Omega_{2} t+b_{4} \cos \Omega_{4} t\right) w_{2} \\
& +b_{5} w_{2}^{2} w_{1}+b_{6} w_{1}^{2} w_{2}+b_{7} w_{2}^{3}+b_{8} w_{1}^{3}=f_{2} \cos \Omega_{3} t \tag{9b}
\end{align*}
$$

where the coefficients presented in (9a), (9b) are given in the previous studies [8].

The above equations include the cubic terms, in-plane excitation, transverse excitation, and piezoelectric excitation. Equation (9a), (9b) can describe the nonlinear transverse oscillations of the composite laminated piezoelectric rectangular plate. We only study the case of primary parametric resonance and $1: 3$ internal resonances. In this resonant case, there are the following resonant relations:

$$
\begin{align*}
& \omega_{1}^{2}=\frac{\omega^{2}}{9}+\varepsilon \sigma_{1}, \quad \omega_{2}^{2}=\omega^{2}+\varepsilon \sigma_{2} \\
& \Omega_{3}=\omega, \quad \Omega_{1}=\Omega_{2}=\Omega_{4}=\frac{2 \omega}{3}  \tag{10}\\
& \omega_{2} \approx 3 \omega_{1},
\end{align*}
$$

where $\sigma_{1}$ and $\sigma_{2}$ are two detuning parameters.
The method of multiple scales [23] is employed to (9a), (9b) to find the uniform solutions in the following form:

$$
\begin{align*}
& w_{1}(t, \varepsilon)=x_{10}\left(T_{0}, T_{1}\right)+\varepsilon x_{11}\left(T_{0}, T_{1}\right)+\cdots,  \tag{11a}\\
& w_{2}(t, \varepsilon)=x_{20}\left(T_{0}, T_{1}\right)+\varepsilon x_{21}\left(T_{0}, T_{1}\right)+\cdots, \tag{11b}
\end{align*}
$$

where $T_{0}=t, T_{1}=\varepsilon t$.
Substituting (10) and (1la), (11b) into (9a), (9b) and balancing the coefficients of corresponding powers of $\varepsilon$ on the left-hand and right-hand sides of equations, the fourdimensional averaged equations in the Cartesian form are obtained as follows:

$$
\begin{align*}
\dot{x}_{1}= & -\frac{1}{2} \mu_{1} x_{1}-\frac{1}{2} \sigma_{1} x_{2}+\frac{1}{4}\left(a_{2}+a_{3}-a_{4}\right) x_{2} \\
& -\frac{3}{2} a_{7} x_{2}\left(x_{1}^{2}+x_{2}^{2}\right)-\frac{1}{2} a_{5} x_{4}\left(x_{1}^{2}-x_{2}^{2}\right)  \tag{12a}\\
& -a_{6} x_{2}\left(x_{3}^{2}+x_{4}^{2}\right)+a_{5} x_{1} x_{2} x_{3}, \\
\dot{x}_{2}= & -\frac{1}{2} \mu_{1} x_{2}+\frac{1}{2} \sigma_{1} x_{1}+\frac{1}{4}\left(a_{2}+a_{3}-a_{4}\right) x_{1} \\
& +\frac{3}{2} a_{7} x_{1}\left(x_{1}^{2}+x_{2}^{2}\right)+\frac{1}{2} a_{5} x_{3}\left(x_{1}^{2}-x_{2}^{2}\right)  \tag{12b}\\
& +a_{6} x_{1}\left(x_{3}^{2}+x_{4}^{2}\right)+a_{5} x_{1} x_{2} x_{4}, \\
\dot{x}_{3}= & -\frac{1}{2} \mu_{2} x_{3}-\frac{1}{6} \sigma_{2} x_{4}-\frac{1}{3} b_{6} x_{4}\left(x_{1}^{2}+x_{2}^{2}\right)  \tag{12c}\\
& -\frac{1}{2} b_{7} x_{4}\left(x_{3}^{2}+x_{4}^{2}\right)-\frac{1}{6} b_{8} x_{2}\left(3 x_{1}^{2}-x_{2}^{2}\right),
\end{align*}
$$

$$
\begin{align*}
\dot{x}_{4}= & -\frac{1}{2} \mu_{2} x_{4}+\frac{1}{6} \sigma_{2} x_{3}+\frac{1}{3} b_{6} x_{3}\left(x_{1}^{2}+x_{2}^{2}\right)  \tag{12d}\\
& +\frac{1}{2} b_{7} x_{3}\left(x_{3}^{2}+x_{4}^{2}\right)+\frac{1}{6} b_{8} x_{1}\left(x_{1}^{2}-3 x_{2}^{2}\right)-\frac{1}{12} f_{2} .
\end{align*}
$$

## 3. Computation of Normal Form

In order to assist the analysis of the Shilnikov type multipulse orbits and chaotic dynamics of the laminated composite piezoelectric rectangular plate, it is necessary to reduce the averaged equation (12a), (12b), (12c), (12d) to a simpler normal form. It is found that there are $Z_{2} \oplus Z_{2}$ and $D_{4}$ symmetries in the averaged equation (12a), (12b), (12c), (12d) without the parameters. Therefore, these symmetries are also held in normal form.

We take into account the excitation amplitude $f_{2}$ as a perturbation parameter. Amplitude $f_{2}$ can be considered as an unfolding parameter when the Shilnikov type multipulse orbits are investigated. Obviously, when we do not consider the perturbation parameter, (12a), (12b), (12c), (12d) become

$$
\begin{align*}
\dot{x}_{1}= & -\frac{1}{2} \mu_{1} x_{1}+\left(f_{0}-\frac{1}{2} \sigma_{1}\right) x_{2}-\frac{3}{2} a_{7} x_{2}\left(x_{1}^{2}+x_{2}^{2}\right) \\
& -\frac{1}{2} a_{5} x_{4}\left(x_{1}^{2}-x_{2}^{2}\right)  \tag{13a}\\
& -a_{6} x_{2}\left(x_{3}^{2}+x_{4}^{2}\right)+a_{5} x_{1} x_{2} x_{3}, \\
\dot{x}_{2}= & -\frac{1}{2} \mu_{1} x_{2}+\left(f_{0}+\frac{1}{2} \sigma_{1}\right) x_{1}+\frac{3}{2} a_{7} x_{1}\left(x_{1}^{2}+x_{2}^{2}\right) \\
& +\frac{1}{2} a_{5} x_{3}\left(x_{1}^{2}-x_{2}^{2}\right)  \tag{13b}\\
& +a_{6} x_{1}\left(x_{3}^{2}+x_{4}^{2}\right)+a_{5} x_{1} x_{2} x_{4}, \\
\dot{x}_{3}= & -\frac{1}{2} \mu_{2} x_{3}-\frac{1}{6} \sigma_{2} x_{4}-\frac{1}{3} b_{6} x_{4}\left(x_{1}^{2}+x_{2}^{2}\right)  \tag{13c}\\
& -\frac{1}{2} b_{7} x_{4}\left(x_{3}^{2}+x_{4}^{2}\right)-\frac{1}{6} b_{8} x_{2}\left(3 x_{1}^{2}-x_{2}^{2}\right), \\
\dot{x}_{4}= & -\frac{1}{2} \mu_{2} x_{4}+\frac{1}{6} \sigma_{2} x_{3}+\frac{1}{3} b_{6} x_{3}\left(x_{1}^{2}+x_{2}^{2}\right) \\
& +\frac{1}{2} b_{7} x_{3}\left(x_{3}^{2}+x_{4}^{2}\right)+\frac{1}{6} b_{8} x_{1}\left(x_{1}^{2}-3 x_{2}^{2}\right), \tag{13d}
\end{align*}
$$

where $f_{0}=(1 / 4)\left(a_{2}+a_{3}-a_{4}\right)$.
Executing the Maple program given by Zhang et al. [24], the nonlinear transformation used here is given as follows:

$$
\begin{align*}
x_{1}= & y_{1}-\frac{1}{4} a_{7} y_{1}^{3}+\frac{3 a_{5}\left(\sigma_{2}-6\right)}{\sigma_{2}^{2}} y_{1}^{2} y_{3}-a_{6} y_{1} y_{3}^{2} \\
& -a_{6} y_{1} y_{4}^{2}-\frac{3 a_{5}\left(\sigma_{2}^{3}-18 \sigma_{2}^{2}+216 \sigma_{2}-1296\right)}{\sigma_{2}^{4}} y_{2}^{2} y_{3} \\
& +\frac{a_{5}\left(6 \sigma_{2}^{2}-72 \sigma_{2}+432\right)}{\sigma_{2}^{3}} y_{1} y_{2} y_{4} \tag{14a}
\end{align*}
$$

$$
\begin{align*}
x_{2}= & y_{2}+\frac{3}{2} a_{7} y_{2}^{3}+\frac{3}{4} a_{7} y_{1}^{2} y_{2}+\frac{3 a_{5}}{\sigma_{2}} y_{1}^{2} y_{4} \\
& -\frac{3 a_{5}\left(\sigma_{2}^{2}-12 \sigma_{2}+72\right)}{\sigma_{2}^{3}} y_{2}^{2} y_{4}-\frac{6 a_{5}\left(\sigma_{2}-6\right)}{\sigma_{2}^{2}} y_{1} y_{2} y_{3}, \tag{14b}
\end{align*}
$$

$$
\begin{align*}
& x_{3}=y_{3}-\frac{b_{8}}{\sigma_{2}} y_{1}^{3}+\frac{3 b_{8}\left(\sigma_{2}^{2}-12 \sigma_{2}+72\right)}{\sigma_{2}^{3}} y_{1} y_{2}^{2}-\frac{1}{3} b_{6} y_{1} y_{2} y_{4},  \tag{14c}\\
& x_{4}=y_{4}+\frac{b_{8}\left(\sigma_{2}^{3}-18 \sigma_{2}^{2}+216 \sigma_{2}-1296\right)}{\sigma_{2}^{4}} y_{2}^{3}  \tag{14d}\\
& -\frac{3 b_{8}\left(\sigma_{2}-6\right)}{\sigma_{2}^{2}} y_{1}^{2} y_{2}+\frac{1}{3} b_{6} y_{1} y_{2} y_{3} \text {. }
\end{align*}
$$

Substituting (14a), (14b), (14c), (14d) into (13a), (13b), (13c), (13d) yields a simpler 3rd-order normal form with the parameters for averaged equation (12a), (12b), (12c), (12d) as follows:

$$
\begin{gather*}
\dot{y}_{1}=-\bar{\mu}_{1} y_{1}+\left(1-\bar{\sigma}_{1}\right) y_{2}  \tag{15a}\\
\dot{y}_{2}=\bar{\sigma}_{1} y_{1}-\bar{\mu}_{1} y_{2}+a_{6} y_{1}\left(y_{3}^{2}+y_{4}^{2}\right)+\frac{3}{2} a_{7} y_{1}^{3}  \tag{15b}\\
\dot{y}_{3}=-\bar{\mu}_{2} y_{3}-\bar{\sigma}_{2} y_{4}-\frac{1}{3} b_{6} y_{1}^{2} y_{4}-\frac{1}{2} b_{7} y_{4}\left(y_{3}^{2}+y_{4}^{2}\right),  \tag{15c}\\
\dot{y}_{4}=\bar{\sigma}_{2} y_{3}-\bar{\mu}_{2} y_{4}+\frac{1}{3} b_{6} y_{1}^{2} y_{3}+\frac{1}{2} b_{7} y_{3}\left(y_{3}^{2}+y_{4}^{2}\right)-\bar{f}_{2}, \tag{15d}
\end{gather*}
$$

where the coefficients are $\bar{\mu}_{1}=(1 / 2) \mu_{1}, \bar{\mu}_{2}=(1 / 2) \mu_{2}, \bar{\sigma}_{2}=$ $(1 / 6) \sigma_{2}$, and $\bar{f}_{2}=(1 / 12) f_{2}$, respectively.

Further, let

$$
\begin{equation*}
y_{3}=I \cos \gamma, \quad y_{4}=I \sin \gamma \tag{16}
\end{equation*}
$$

Substituting (16) into (15a), (15b), (15c), (15d) yields

$$
\begin{gather*}
\dot{y}_{1}=-\bar{\mu}_{1} y_{1}+\left(1-\bar{\sigma}_{1}\right) y_{2}  \tag{17a}\\
\dot{y}_{2}=\bar{\sigma}_{1} y_{1}-\bar{\mu}_{1} y_{2}+a_{6} y_{1} I^{2}+\frac{3}{2} a_{7} y_{1}^{3},  \tag{17b}\\
\dot{I}=-\bar{\mu}_{2} I-\bar{f}_{2} \sin \gamma,  \tag{17c}\\
I \dot{\gamma}=\bar{\sigma}_{2} I+\frac{1}{3} b_{6} y_{1}^{2} I+\frac{1}{2} b_{7} I^{3}-\bar{f}_{2} \cos \gamma . \tag{17d}
\end{gather*}
$$

In order to get the unfolding of (17a), (17b), (17c), (17d) a linear transformation is introduced:

$$
\left[\begin{array}{l}
y_{1}  \tag{18}\\
y_{2}
\end{array}\right]=\sqrt{3} \frac{\sqrt{\left|a_{6}\right|}}{\sqrt{\left|b_{6}\right|}}\left[\begin{array}{cc}
1-\bar{\sigma}_{1} & 0 \\
\bar{\mu}_{1} & 1
\end{array}\right]\left[\begin{array}{l}
u_{1} \\
u_{2}
\end{array}\right] .
$$

Substituting (18) into (17a), (17b), (17c), (17d) and canceling nonlinear terms including the parameter $\bar{\sigma}_{1}$ yield the unfolding as follows:

$$
\begin{gather*}
\dot{u}_{1}=u_{2}  \tag{19a}\\
\dot{u}_{2}=-\mu u_{1}-\mu_{3} u_{2}+\eta_{1} u_{1}^{3}+a_{6} u_{1} I^{2},  \tag{19b}\\
\dot{I}=-\bar{\mu}_{2} I-\bar{f}_{2} \sin \gamma,  \tag{19c}\\
I \dot{\gamma}=\bar{\sigma}_{2} I+a_{6} u_{1}^{2} I+\alpha_{2} I^{3}-\bar{f}_{2} \cos \gamma, \tag{19d}
\end{gather*}
$$

where $\mu=\bar{\mu}_{1}^{2}-\bar{\sigma}_{1}\left(1-\bar{\sigma}_{1}\right), \mu_{3}=2 \bar{\mu}_{1}, \eta_{1}=9 a_{6} a_{7} / 2 b_{6}$ and $\alpha_{2}=(1 / 2) b_{7}$.

The scale transformations to be introduced into (19a), (19b), (19c), (19d) are

$$
\begin{align*}
\bar{\mu}_{2} & \longrightarrow \varepsilon \bar{\mu}_{2}, & \mu_{3} \longrightarrow \varepsilon \mu_{3}, & \bar{f}_{2} \longrightarrow \varepsilon \bar{f}_{2}  \tag{20}\\
\eta_{1} & \longrightarrow \eta_{1}, & \alpha_{2} \longrightarrow \alpha_{2}, & a_{6} \longrightarrow a_{6}
\end{align*}
$$

Then, normal form (19a), (19b), (19c), (19d) can be rewritten in the form with the perturbations

$$
\begin{gather*}
\dot{u}_{1}=\frac{\partial H}{\partial u_{2}}+\varepsilon g^{u_{1}}=u_{2}  \tag{21a}\\
\dot{u}_{2}=-\frac{\partial H}{\partial u_{1}}+\varepsilon g^{u_{2}}=-\mu u_{1}+\eta_{1} u_{1}^{3}+a_{6} u_{1} I^{2}-\varepsilon \mu_{3} u_{2}  \tag{21b}\\
\dot{I}=\frac{\partial H}{\partial \gamma}+\varepsilon g^{I}=-\varepsilon \bar{\mu}_{2} I-\varepsilon \bar{f}_{2} \sin \gamma  \tag{21c}\\
I \dot{\gamma}=-\frac{\partial H}{\partial I}+\varepsilon g^{\gamma}=\bar{\sigma}_{2} I+\alpha_{2} I^{3}+a_{6} I u_{1}^{2}-\varepsilon \bar{f}_{2} \cos \gamma \tag{21d}
\end{gather*}
$$

where the Hamiltonian function $H$ is of the form

$$
\begin{align*}
H\left(u_{1}, u_{2}, I, \gamma\right)= & \frac{1}{2} u_{2}^{2}+\frac{1}{2} \mu u_{1}^{2}-\frac{1}{4} \eta_{1} u_{1}^{4} \\
& -\frac{1}{2} a_{6} I^{2} u_{1}^{2}-\frac{1}{2} \bar{\sigma}_{2} I^{2}-\frac{1}{4} \alpha_{2} I^{4}, \tag{22}
\end{align*}
$$

and $g^{u_{1}}, g^{u_{2}}, g^{I}$, and $g^{\gamma}$ are the perturbation terms induced by the dissipative effects

$$
\begin{gather*}
g^{u_{1}}=0, \quad g^{u_{2}}=-\mu_{3} u_{2}, \\
g^{I}=-\bar{\mu}_{2} I-\bar{f}_{2} \sin \gamma, \quad g^{\gamma}=-\bar{f}_{2} \cos \gamma . \tag{23}
\end{gather*}
$$

## 4. Heteroclinic Bifurcations of Unperturbed System

In this section, we focus on studying the nonlinear dynamics characteristic of the unperturbed system. When $\varepsilon=0$, it can be known that system from (21a), (21b), (21c), (21d) is an uncoupled two-degree-of-freedom nonlinear system. The variable $I$ appears in the subspace ( $u_{1}, u_{2}$ ) of (21a), (21b), (21c), (21d) as a parameter since $\dot{I}=0$. Consider the first two decoupled equations of (21a), (21b), (21c), (21d),

$$
\begin{gather*}
\dot{u}_{1}=u_{2}  \tag{24a}\\
\dot{u}_{2}=-\mu u_{1}+\eta_{1} u_{1}^{3}+a_{6} u_{1} I^{2} \tag{24b}
\end{gather*}
$$

Since $\eta_{1}>0$, (24a), (24b) can exhibit the heteroclinic bifurcations. It is obvious from (24a), (24b) that when $\mu$ $a_{6} I^{2}<0$, the only solution to (24a), (24b) is the trivial zero solution, $\left(u_{1}, u_{2}\right)=(0,0)$, which is the saddle point. On the curve defined by $\mu=a_{6} I^{2}$, that is,

$$
\begin{equation*}
I_{1,2}= \pm\left[\frac{\bar{\mu}_{1}^{2}-\bar{\sigma}_{1}\left(1-\bar{\sigma}_{1}\right)}{a_{6}}\right]^{1 / 2} \tag{25}
\end{equation*}
$$

the trivial zero solution bifurcates into three solutions through a pitchfork bifurcation, which are given by $q_{0}=(0,0)$ and $q_{ \pm}(I)=(B, 0)$, respectively, where

$$
\begin{equation*}
B= \pm\left\{\frac{1}{\eta_{1}}\left[\bar{\mu}_{1}^{2}-\bar{\sigma}_{1}\left(1-\bar{\sigma}_{1}\right)-a_{6} I^{2}\right]\right\}^{1 / 2} . \tag{26}
\end{equation*}
$$

From the Jacobian matrix evaluated at the nonzero solutions, it can be found that the singular point $q_{0}$ is the center point and the singular points $q_{ \pm}(I)$ are saddle points. It is observed that $I$ and $\gamma$ actually represent the amplitude and phase of vibrations. Therefore, we assume that $I \geq 0$ and (25) becomes

$$
\begin{equation*}
I_{1}=0, \quad I_{2}=\left[\frac{\bar{\mu}_{1}^{2}-\bar{\sigma}_{1}\left(1-\bar{\sigma}_{1}\right)}{a_{6}}\right]^{1 / 2} \tag{27}
\end{equation*}
$$

such that for all $I \in\left[I_{1}, I_{2}\right]$, (24a), (24b) have two hyperbolic saddle points, $q_{ \pm}(I)$, which are connected by a pair of heteroclinic orbits, $u_{ \pm}^{h}\left(T_{1}, I\right)$; that is, $\lim _{T_{1} \rightarrow \pm \infty} u_{ \pm}^{h}\left(T_{1}, I\right)=$ $q_{ \pm}(I)$. Thus, in the full four-dimensional phase space, the set defined by

$$
\begin{equation*}
M=\left\{(u, I, \gamma) \mid u=q_{ \pm}(I), I_{1}<I<I_{2}, 0 \leq \gamma<2 \pi\right\} \tag{28}
\end{equation*}
$$

is a two-dimensional invariant manifold.
From the results obtained by Feng et al. [9-11], it is known that two-dimensional invariant manifold $M$ is normally hyperbolic. The two-dimensional normally hyperbolic invariant manifold $M$ has the three-dimensional stable and unstable manifolds represented as $W^{s}(M)$ and $W^{u}(M)$, respectively. The existence of the heteroclinic orbit of (24a), $(24 \mathrm{~b})$ to $q_{ \pm}(I)=(B, 0)$ indicates that $W^{s}(M)$ and $W^{u}(M)$ intersect nontransversally along a three-dimensional manifold denoted by $\Gamma$, which can be written as

$$
\begin{gather*}
\Gamma=\left\{(u, I, \gamma) \mid u=u_{ \pm}^{h}\left(T_{1}, I\right), I_{1}<I<I_{2}\right. \\
\left.\quad \gamma=\int_{0}^{T_{1}} D_{I} H\left(u_{ \pm}^{h}\left(T_{1}, I\right), I\right) d s+\gamma_{0}\right\} . \tag{29}
\end{gather*}
$$

We analyze the dynamics of the unperturbed system of (21a), (21b), (21c), (21d) restricted to $M$. Considering the unperturbed system of (21a), (21b), (21c), (21d) restricted to $M$ yields

$$
\begin{gather*}
\dot{I}=0,  \tag{30a}\\
I \dot{\gamma}=D_{I} H\left(q_{ \pm}(I), I\right), \quad I_{1}<I<I_{2}, \tag{30b}
\end{gather*}
$$

where

$$
\begin{equation*}
D_{I} H\left(q_{ \pm}(I), I\right)=-\frac{\partial H\left(q_{ \pm}(I), I\right)}{\partial I}=\bar{\sigma}_{2} I+\alpha_{2} I^{3}+a_{6} I u_{1}^{2} \tag{31}
\end{equation*}
$$

From the results obtained by Feng et al. [9-11], it is known that if $D_{I} H\left(q_{ \pm}(I), I\right) \neq 0, I=$ constant is called a periodic orbit, and if $D_{I} H\left(q_{ \pm}, I\right)=0, I=$ constant is known as a circle of the singular points. Any value of $I \in\left[I_{1}, I_{2}\right]$ at which $D_{I} H\left(q_{ \pm}, I\right)=0$ is a resonant value $I$ and these singular points are resonant singular points. We denote a resonant value by $I_{r}$ such that

$$
\begin{equation*}
D_{I} H\left(q_{ \pm}, I\right)=\bar{\sigma}_{2} I_{r}+\alpha_{2} I_{r}^{3}+a_{6} I_{r} u_{1}^{2}=0 \tag{32}
\end{equation*}
$$

Then, we obtain

$$
\begin{equation*}
I_{r}= \pm\left\{\frac{\bar{\sigma}_{2} \eta_{1}+a_{6}\left[\bar{\mu}_{1}^{2}-\bar{\sigma}_{1}\left(1-\bar{\sigma}_{1}\right)\right]}{a_{6}^{2}-\alpha_{2} \eta_{1}}\right\}^{1 / 2} \tag{33}
\end{equation*}
$$

The geometric structure of the stable and unstable manifolds of $M$ in the full four-dimensional phase space for the unperturbed system of (21a), (21b), (21c), (21d) is given in Figure 2. Since $\gamma$ represents the phase of the oscillations, when $I=I_{r}$, the phase shift $\Delta \gamma$ of oscillations is defined by

$$
\begin{equation*}
\Delta \gamma=\gamma\left(+\infty, I_{r}\right)-\gamma\left(-\infty, I_{r}\right) \tag{34}
\end{equation*}
$$

The physical interpretation of the phase shift is the phase difference between the two end points of the orbit. In the subspace $\left(u_{1}, u_{2}\right)$, there exists a pair of heteroclinic orbits connecting to saddle points. Therefore, the homoclinic orbit in the subspace $(I, \gamma)$ is, in fact, a heteroclinic connecting in the full four-dimensional space $\left(u_{1}, u_{2}, I, \gamma\right)$. The phase shift denotes the difference of the value $\gamma$ when a trajectory leaves and returns to the basin of attraction of $M$. We will use the phase shift in subsequent analysis to obtain the condition for the existence of the Shilnikov type multipulse orbit. The phase shift will be calculated later in the heteroclinic orbit analysis.

We consider the heteroclinic orbits of (24a), (24b). Let $\varepsilon_{1}=\mu-a_{6} I^{2}$ and $\mu_{3}=\varepsilon_{2}$, (24a), (24b) can be rewritten as

$$
\begin{gather*}
\dot{u}_{1}=u_{2}  \tag{35a}\\
\dot{u}_{2}=-\varepsilon_{1} u_{1}+\eta_{1} u_{1}^{3}-\varepsilon \varepsilon_{2} u_{2} . \tag{35b}
\end{gather*}
$$

Set $\varepsilon=0 ;(35 \mathrm{a}),(35 \mathrm{~b})$ is a system with the Hamiltonian function

$$
\begin{equation*}
\bar{H}\left(u_{1}, u_{2}\right)=\frac{1}{2} u_{2}^{2}+\frac{1}{2} \varepsilon_{1} u_{1}^{2}-\frac{1}{4} \eta_{1} u_{1}^{4} \tag{36}
\end{equation*}
$$

When $\bar{H}=0$, there is a heteroclinic loop $\Gamma^{0}$ which consists of the two hyperbolic saddle points $q_{ \pm}(I)$ and a pair of heteroclinic orbits $u_{ \pm}\left(T_{1}\right)$. In order to calculate the phase shift and the extended Melnikov function, it is necessary to
obtain the equations of a pair of heteroclinic orbits, which are given as follows:

$$
\begin{align*}
& u_{1}\left(T_{1}\right)= \pm \sqrt{\frac{\varepsilon_{1}}{\eta_{1}}} \tanh \left(\frac{\sqrt{2 \varepsilon_{1}}}{2} T_{1}\right)  \tag{37a}\\
& u_{2}\left(T_{1}\right)= \pm \frac{\varepsilon_{1}}{\sqrt{2 \eta_{1}}} \operatorname{sech}^{2}\left(\frac{\sqrt{2 \varepsilon_{1}}}{2} T_{1}\right) \tag{37b}
\end{align*}
$$

We turn our attention to the computation of the phase shift. Substituting the first equation of (37a), (37b) into the fourth equation of the unperturbed system of (21a), (21b), (21c), (21d) and integrating yield

$$
\begin{equation*}
\gamma\left(T_{1}\right)=\omega_{r} T_{1}-\frac{a_{6} \sqrt{2 \varepsilon_{1}}}{\eta_{1}} \tanh \left(\frac{\sqrt{2 \varepsilon_{1}}}{2} T_{1}\right)+\gamma_{0} \tag{38}
\end{equation*}
$$

where $\omega_{r}=\bar{\sigma}_{2}+\alpha_{2} I^{2}+a_{6} \varepsilon_{1} / \eta_{1}$.
At $I=I_{r}$, there is $\omega_{r} \equiv 0$. Therefore, the phase shift may be expressed as

$$
\begin{align*}
\Delta \gamma & =\left[-\frac{2 a_{6} \sqrt{2 \varepsilon_{1}}}{\eta_{1}}\right]_{I=I_{r}}  \tag{39}\\
& =-\frac{2 a_{6}}{\eta_{1}} \sqrt{2\left[\bar{\mu}_{1}^{2}-\bar{\sigma}_{1}\left(1-\bar{\sigma}_{1}\right)-a_{6} I_{r}^{2}\right]}
\end{align*}
$$

## 5. Existence of Multipulse Orbits

After obtaining detailed information on the nonlinear dynamic characteristics of the subspace $\left(u_{1}, u_{2}\right)$ for the unperturbed system from (21a), (21b), (21c), (21d), the next step is to examine the effects of small perturbation terms $(0<$ $\varepsilon \ll 1$ ) on the unperturbed system from (21a), (21b), (21c), (21d). The extended Melnikov method developed by Kovačič et al. [15-17] is utilized to discover the existence of the multipulse orbits and chaotic dynamics of the nonlinear vibration for the laminated composite piezoelectric rectangular plate. We start by studying the influence of such small perturbations on the manifold $M$. The objective of the research is to identify the parameter regions where the existence of the multipulse orbits is possible in the perturbed phase space. The main aim is to verify whether these parameters satisfy the transversality condition of multipulse chaotic dynamics. It will be shown that these multipulse orbits can occur in the Hamilton system with dissipative perturbations if the parameters meet the transversality condition. The existence of such multipulse orbits provides a robust mechanism for the existence of the complicated dynamics in the perturbed system. In this section, the emphasis is put on the application aspects of the extended Melnikov method to (21a), (21b), (21c), (21d).
5.1. Dissipative Perturbations of the Homoclinic Loop. We analyze dynamics of the perturbed system and the influence of small perturbations on $M$. Based on the analysis by Kovačič et al. [15-17], we know that $M$ along with its stable and unstable manifolds is invariant under small, sufficiently differentiable perturbations. It is noticed that $q_{ \pm}(I)$ in (24a),


Figure 2: The geometric structure of manifolds $M, W^{s}(M)$, and $W^{u}(M)$ is given in the full four-dimensional phase space.
(24b) maintains the characteristic of the hyperbolic singular point under small perturbations, in particular, $M \rightarrow M_{\varepsilon}$. Therefore, we obtain

$$
\begin{equation*}
M=M_{\varepsilon}=\left\{(u, I, \gamma) \mid u=q_{ \pm}(I), I_{1}<I<I_{2}, 0 \leq \gamma<2 \pi\right\} . \tag{40}
\end{equation*}
$$

Considering the last two equations of (21a), (21b), (21c), (21d) yields

$$
\begin{gather*}
\dot{I}=-\bar{\mu}_{2} I-\bar{f}_{2} \sin \gamma,  \tag{41a}\\
\dot{\gamma}=\bar{\sigma}_{2}+\alpha_{2} I^{2}+a_{6} u_{1}^{2}-\frac{\bar{f}_{2} \cos \gamma}{I} . \tag{41b}
\end{gather*}
$$

It is known from the above analysis that the last two equations of (21a), (21b), (21c), (21d) are of a pair of pure imaginary eigenvalues. Therefore, the resonance can occur in (41a), (41b). Also introduce the scale transformations

$$
\begin{array}{ll}
\bar{\mu}_{2} \longrightarrow \varepsilon \bar{\mu}_{2}, & I=I_{r}+\sqrt{\varepsilon} h, \\
\bar{f}_{2} \longrightarrow \varepsilon \bar{f}_{2}, & T_{1} \longrightarrow \frac{T_{1}}{\sqrt{\varepsilon}} \tag{42}
\end{array}
$$

Substituting the above transformations into (41a), (41b) yields

$$
\begin{array}{r}
\dot{h}=-\bar{\mu}_{2} I_{r}-\bar{f}_{2} \sin \gamma-\sqrt{\varepsilon} \bar{\mu}_{2} h \\
\dot{\gamma}=-\frac{2 \delta}{\eta_{1}} I_{r} h-\sqrt{\varepsilon}\left(\frac{\delta}{\eta_{1}} h^{2}+\frac{\bar{f}_{2}}{I_{r}} \cos \gamma\right), \tag{43b}
\end{array}
$$

where $\delta=a_{6}^{2}-\alpha_{2} \eta_{1}$.
When $\varepsilon=0$, (43a), (43b) become

$$
\begin{gather*}
\dot{h}=-\bar{\mu}_{2} I_{r}-\bar{f}_{2} \sin \gamma,  \tag{44a}\\
\dot{\gamma}=-\frac{2 \delta}{\eta_{1}} I_{r} h . \tag{44b}
\end{gather*}
$$

The unperturbed system from (44a), (44b) is a Hamilton system with the function

$$
\begin{equation*}
\widehat{H}_{D}(h, \gamma)=-\bar{\mu}_{2} I_{r} \gamma+\bar{f}_{2} \cos \gamma+\frac{\delta}{\eta_{1}} I_{r} h^{2} \tag{45}
\end{equation*}
$$

The singular points of (44a), (44b) are given as

$$
\begin{gather*}
P_{0}=\left(0, \gamma_{c}\right)=\left(0,-\arcsin \left(\frac{\bar{\mu}_{2} I_{r}}{\bar{f}_{2}}\right)\right),  \tag{46}\\
Q_{0}=\left(0, \gamma_{s}\right)=\left(0, \pi+\arcsin \left(\frac{\bar{\mu}_{2} I_{r}}{\bar{f}_{2}}\right)\right) .
\end{gather*}
$$

Based on the characteristic equations evaluated at the two singular points $P_{0}$ and $Q_{0}$, we can know the stabilities of these singular points. Therefore, it is known that the singular point $P_{0}$ is a center point. The singular point $Q_{0}$ is a saddle which is connected to itself by a homoclinic orbit. The phase portrait of system for (44a), (44b) is shown in Figure 3(a).

It is found that for the sufficiently small parameter $\varepsilon$, the singular point $Q_{0}$ remains a hyperbolic singular point $Q_{\varepsilon}$ of the saddle stability type. For small perturbations, the singular point $P_{0}$ becomes a hyperbolic sink $P_{\varepsilon}$. The phase portrait of the perturbed system from (43a), (43b) is depicted in Figure 3(b).

Using the function (45), at $h=0$, and substituting $\gamma_{s}$ in (46) into (45), the estimate of the basin of the attractor for $\gamma_{\text {min }}$ is obtained as

$$
\begin{equation*}
\gamma_{\min }-\frac{\bar{f}_{2}}{\bar{\mu}_{2} I_{r}} \cos \gamma_{\min }=\pi+\arcsin \frac{\bar{\mu}_{2} I_{r}}{\bar{f}_{2}}+\frac{\sqrt{\bar{f}_{2}^{2}-\bar{\mu}_{2}^{2} I_{r}^{2}}}{\bar{\mu}_{2} I_{r}} \tag{47}
\end{equation*}
$$

Define an annulus $A_{\varepsilon}$ near $I=I_{r}$ as

$$
\begin{equation*}
A_{\varepsilon}=\left\{\left(u_{1}, u_{2}, I, \gamma\right)\left|u_{1}=B, u_{2}=0,\left|I-I_{r}\right|<\sqrt{\varepsilon} C, \gamma \in T^{L}\right\},\right. \tag{48}
\end{equation*}
$$

where $C$ is a constant and is sufficiently large so that the unperturbed orbit is enclosed within the annulus.

It is noticed that the three-dimensional stable and unstable manifolds of $A_{\varepsilon}$, denoted as $W^{s}\left(A_{\varepsilon}\right)$ and $W^{u}\left(A_{\varepsilon}\right)$, are the subsets of the manifolds $W^{s}\left(M_{\varepsilon}\right)$ and $W^{u}\left(M_{\varepsilon}\right)$, respectively. We will indicate that for the perturbed system, the saddle focus $P_{\varepsilon}$ on $A_{\varepsilon}$ has the multipulse orbits which come out of the annulus $A_{\varepsilon}$ and can return to the annulus in the full four-dimensional space. These orbits, which are asymptotic to some invariant manifolds in the slow manifold $M_{\varepsilon}$, leave and


Figure 3: Dynamics on the normally hyperbolic manifold is described; (a) the unperturbed case; (b) the perturbed case.
enter a small neighborhood of $M_{\varepsilon}$ multiple times and finally return and approach an invariant set in $M_{\varepsilon}$ asymptotically, as shown in Figure 4. In Figure 4, this is an example of the threepulse jumping orbit which depicts the formation mechanism of the multipulse orbits.
5.2. The k-Pulse Melnikov Function. Most researchers focused on Shilnikov type single-pulse chaotic dynamics of the high-dimensional nonlinear systems from the thin plate structures in the past. There exist multipulse chaotic dynamics in the practical engineering systems. The extended Melnikov method is a kind of theory which can be used to investigate the multipulse jumping orbits in the high-dimensional nonlinear systems. Since the theory on multipulse chaotic dynamics is very esoteric and abstract, it is difficult to be extended to solve the engineering problems. Up to now, few researchers have made use of the extended Melnikov method to study the Shilnikov type multipulse homoclinic and heteroclinic bifurcations and chaotic dynamics of highdimensional nonlinear systems in engineering applications.

The extended Melnikov method was first presented by Kovačič et al. [15-17], which is an extension of the global perturbation method developed by Feng et al. [9-11]. Camassa et al. [17] gave the detailed procedure of mathematical proof on the extended Melnikov method, which unifies several disjoint perturbation theoretical methods. This method can be also utilized to detect the Shilnikov type multipulse homoclinic or heteroclinic orbits to the slow manifolds of fourdimensional, near-integrable Hamilton systems or higherdimensional, nonlinear systems. The extended Melnikov function is different from the usual Melnikov function and describes slow dynamics of the multipulse orbits on the hyperbolic manifold.

The key of the extended Melnikov method is how to calculate the extended Melnikov method. The extended Melnikov function is computed by a recursion procedure from the usual 1-pulse Melnikov function and depends on the small perturbation parameter $\varepsilon$ through a logarithmic function which calculates the asymptotic in the particularly delicate small $\varepsilon$ limit. In this paper, the extended Melnikov function


Figure 4: The Shilnikov type three-pulse orbits are obtained.
can be simplified in the resonant case and does not depend on the perturbation parameter. We have used the extended Melnikov method to investigate heteroclinic bifurcations and multipulse chaotic dynamics of the laminated composite piezoelectric rectangular plate.

We use the extended Melnikov method described by Kovačič et al. [15-17] to find the Shilnikov type multipulse orbits for nonlinear vibration for the laminated composite piezoelectric rectangular plate. We search for the multipulse excursions to find the nondegenerate zeroes of the extended Melnikov function $M_{k}\left(\varepsilon, I, \gamma_{0}, \bar{\mu}_{2}\right)$ with the certain combination of parameters $\varepsilon, I, \gamma_{0}$, and $\bar{\mu}_{2}$, which we name the $k$-pulse Melnikov function.

It is important to obtain the detailed expression of the $k$ pulse Melnikov function. We compute the 1-pulse Melnikov function based on the formula obtained by Kovačič et al. [15-17] at the resonant case $I=I_{r}$. The 1-pulse Melnikov function $M_{1}\left(\varepsilon, I_{r}, \gamma_{0}, \bar{\mu}_{2}\right)$ coincides with the standard Melnikov function $M\left(I_{r}, \gamma_{0}, \bar{\mu}_{2}\right)$. The 1-pulse Melnikov function $M\left(I_{r}, \gamma_{0}, \bar{\mu}_{2}\right)$ on both heteroclinic manifolds $W^{s}(M)$ and $W^{u}(M)$ is given as follows:

$$
\begin{aligned}
M & \left(I_{r}, \gamma_{0}, \bar{\mu}_{2}, \eta_{1}, a_{6}, \varepsilon_{1}\right) \\
\quad & =\int_{-\infty}^{+\infty}\left\langle\mathbf{n}\left(p^{h}(t)\right), \mathbf{g}\left(p^{h}(t), \bar{\mu}_{2}, 0\right)\right\rangle d T_{1}
\end{aligned}
$$

$$
\begin{align*}
= & \int_{-\infty}^{+\infty}\left(\frac{\partial H}{\partial u_{1}} g^{u_{1}}+\frac{\partial H}{\partial u_{2}} g^{u_{2}}+\frac{\partial H}{\partial I} g^{I}+\frac{\partial H}{\partial \gamma} g^{\gamma}\right) d T_{1} \\
= & -\frac{2 \sqrt{2} \mu_{3}}{3 \eta_{1}} \varepsilon_{1}^{3 / 2}-2 \sqrt{2} a_{6} \bar{\mu}_{2} I_{r}^{2} \frac{\varepsilon_{1}^{1 / 2}}{\eta_{1}} \\
& -\bar{f}_{2} I_{r}\left[\cos \left(\gamma_{0}-a_{6} \frac{\sqrt{2 \varepsilon_{1}}}{\eta_{1}}\right)-\cos \left(\gamma_{0}+a_{6} \frac{\sqrt{2 \varepsilon_{1}}}{\eta_{1}}\right)\right] . \tag{49}
\end{align*}
$$

Based on the results given by Kovačič et al. [1517], it is known that the $k$-pulse Melnikov function $M_{k}\left(\varepsilon, I_{r}, \gamma_{0}, \bar{\mu}_{2}\right)(k=1,2, \ldots)$ is defined as

$$
\begin{align*}
& M_{k}\left(\varepsilon, I_{r}, \gamma_{0}, \bar{\mu}_{2}\right) \\
& \quad=\sum_{j=0}^{k-1} M\left(I_{r}, j \Delta \gamma\left(I_{r}\right)+\Gamma_{j}\left(\varepsilon, I_{r}, \gamma_{0}, \bar{\mu}_{2}\right)+\gamma_{0}, \bar{\mu}_{2}\right) \tag{50}
\end{align*}
$$

where

$$
\begin{equation*}
\Gamma_{j}\left(\varepsilon, I_{r}, \gamma_{0}, \bar{\mu}_{2}\right)=\frac{\Omega\left(\bar{x}_{0}\left(I_{r}\right), I_{r}\right)}{\lambda\left(I_{r}\right)} \sum_{r=1}^{j} \log \left|\frac{\varsigma\left(I_{r}\right)}{\varepsilon M_{r}\left(\varepsilon, I_{r}, \gamma_{0}, \bar{\mu}_{2}\right)}\right|, \tag{51}
\end{equation*}
$$

for $j=1, \ldots, k-1$ and $\Gamma_{0}\left(\varepsilon, I_{r}, \gamma_{0}, \bar{\mu}_{2}\right)=0$.
It is noticed that the angle function $\Gamma_{j}\left(\varepsilon, I_{r}, \gamma_{0}, \bar{\mu}_{2}\right)$ is the complex formula where $M_{k}\left(\varepsilon, I_{r}, \gamma_{0}, \bar{\mu}_{2}\right)$ appears as the argument of a logarithm. When resonance occurs, the periodic orbit corresponding to the value $I_{r}$ degenerates into a circle of equilibria. Under this case, there exists $\Gamma_{j}\left(\varepsilon, I_{r}, \gamma_{0}, \bar{\mu}_{2}\right)=0,(j=0,1, \ldots, k-1)$. Based on the expression obtained by Kovačič et al. [15-17], the $k$-pulse Melnikov function can be written as follows:

$$
\begin{aligned}
& M_{k}\left(I_{r}, \gamma_{0}, \bar{\mu}_{2}, \eta_{1}, a_{6}, \varepsilon_{1}\right) \\
& =\sum_{j=0}^{k-1} M\left(I_{r}, \gamma_{0}+j \Delta \gamma\left(I_{r}\right), \bar{\mu}_{2}, \eta_{1}, a_{6}, \varepsilon_{1}\right) \\
& =-\bar{f}_{2} I_{r}\left[\cos \left(\gamma_{0}-\frac{a_{6} \sqrt{2 \varepsilon_{1}}}{\eta_{1}}\right)\right. \\
& \left.\quad-\cos \left(\gamma_{0}+\frac{a_{6} \sqrt{2 \varepsilon_{1}}}{\eta_{1}}\right)\right] \\
& \quad-\frac{2 \sqrt{2} \mu_{3}}{3 \eta_{1}} \varepsilon_{1}^{3 / 2}-2 \sqrt{2} a_{6} \bar{\mu}_{2} I_{r}^{2} \frac{\varepsilon_{1}^{1 / 2}}{\eta_{1}} \\
& -\bar{f}_{2} I_{r}\left[\cos \left(\gamma_{0}-\frac{a_{6} \sqrt{2 \varepsilon_{1}}}{\eta_{1}}-\frac{2 a_{6} \sqrt{2 \varepsilon_{1}}}{\eta_{1}}\right)\right. \\
& \left.\quad-\cos \left(\gamma_{0}+\frac{a_{6} \sqrt{2 \varepsilon_{1}}}{\eta_{1}}-\frac{2 a_{6} \sqrt{2 \varepsilon_{1}}}{\eta_{1}}\right)\right]
\end{aligned}
$$

$$
\begin{align*}
& -\frac{2 \sqrt{2} \mu_{3}}{3 \eta_{1}} \varepsilon_{1}^{3 / 2}-2 \sqrt{2} a_{6} \bar{\mu}_{2} I_{r}^{2} \frac{\varepsilon_{1}^{1 / 2}}{\eta_{1}}+\cdots \\
& -\bar{f}_{2} I_{r}\left[\cos \left(\gamma_{0}-\frac{a_{6} \sqrt{2 \varepsilon_{1}}}{\eta_{1}}-2(k-1) a_{6} \frac{\sqrt{2 \varepsilon_{1}}}{\eta_{1}}\right)\right. \\
& \\
& \left.\quad-\cos \left(\gamma_{0}+\frac{a_{6} \sqrt{2 \varepsilon_{1}}}{\eta_{1}}-2(k-1) \frac{a_{6} \sqrt{2 \varepsilon_{1}}}{\eta_{1}}\right)\right] \\
& =-\frac{2 \sqrt{2} \mu_{3}}{3 \eta_{1}} \varepsilon_{1}^{3 / 2}-2 \sqrt{2} a_{6} \bar{\mu}_{2} I_{r}^{2} \frac{\varepsilon_{1}^{1 / 2}}{\eta_{1}} \\
& = \\
& -\bar{f}_{2} I_{r}\left[\cos \left(\gamma_{0}-\frac{a_{6} \sqrt{2 \varepsilon_{1}}}{\eta_{1}}-2(k-1) \frac{a_{6} \sqrt{2 \varepsilon_{1}}}{\eta_{1}}\right)\right.  \tag{52}\\
& \left.\quad-\cos \left(\gamma_{0}+\frac{a_{6} \sqrt{2 \varepsilon_{1}}}{\eta_{1}}\right)\right] \\
& -\frac{2 \sqrt{2} k \mu_{3}}{3 \eta_{1}} \varepsilon_{1}^{3 / 2}-2 \sqrt{2} k a_{6} \bar{\mu}_{2} I_{r}^{2} \frac{\varepsilon_{1}^{1 / 2}}{\eta_{1}} .
\end{align*}
$$

If we set $\Delta \gamma=-2 a_{6}\left(\sqrt{2 \varepsilon_{1}} / \eta_{1}\right)$ and $\gamma_{k-1}=\gamma_{0}+(k-$ 1) ( $\Delta \gamma / 2$ ), (52) can be rewritten as follows:

$$
\begin{align*}
& M_{k}\left(I_{r},\right. \gamma_{0}, \\
&\left.\mu_{2}, \eta_{1}, a_{6}, \varepsilon_{1}\right) \\
&= M_{k}\left(I_{r}, \gamma_{k-1}-(k-1) \frac{\Delta \gamma}{2}, \bar{\mu}_{2}, \eta_{1}, a_{6}, \varepsilon_{1}\right) \\
&=-\bar{f}_{2} I_{r}\left[\cos \left(\gamma_{k-1}+\frac{1}{2} k \Delta \gamma\right)\right.  \tag{53}\\
&\left.\quad-\cos \left(\gamma_{k-1}-\frac{1}{2} k \Delta \gamma\right)\right] \\
&+\frac{k \mu_{3} \varepsilon_{1}}{3 a_{6}} \Delta \gamma+\bar{\mu}_{2} I_{r}^{2}(k \Delta \gamma) \\
&= 2 \bar{f}_{2} I_{r} \sin \gamma_{k-1} \sin \left(\frac{1}{2} k \Delta \gamma\right) \\
&+\frac{\mu_{3} \varepsilon_{1}}{3 a_{6}}(k \Delta \gamma)+\bar{\mu}_{2} I_{r}^{2}(k \Delta \gamma) .
\end{align*}
$$

Based on Proposition 3.1 given by Kovačič et al. [15-17], the nonfolding condition is always satisfied in the resonant case. We obtain the following two conditions:

$$
\begin{array}{r}
\left|\frac{(1 / 2) k \Delta \gamma}{\sin ((1 / 2) k \Delta \gamma)} \frac{\left(\mu_{3} \varepsilon_{1}+3 a_{6} \bar{\mu}_{2} I_{r}^{2}\right)}{3 a_{6} \bar{f}_{2} I_{r}}\right|<1  \tag{54}\\
\frac{1}{2} k \Delta \gamma \neq n \pi, \quad n=0, \pm 1, \pm 2, \ldots
\end{array}
$$

The main aim of the following analysis focuses on identifying simple zeroes of the $k$-pulse Melnikov function. Define a set that contains all such simple zeroes to be

$$
\begin{equation*}
Z_{-}^{n}=\left\{\left(I_{r}, \gamma_{k-1}, \bar{\mu}_{2}, \eta_{1}, a_{6}, \varepsilon_{1}\right) \mid M_{k}=0, D_{\gamma_{0}} M_{k} \neq 0\right\} . \tag{55}
\end{equation*}
$$

The $k$-pulse Melnikov function has two simple zeroes in the interval $\gamma_{k-1} \in[0, \pi]$

$$
\begin{gather*}
\bar{\gamma}_{k-1,1}=-\arcsin \frac{(1 / 2) k \Delta \gamma}{\sin ((1 / 2) k \Delta \gamma)} \frac{\left(\mu_{3} \varepsilon_{1}+3 a_{6} \bar{\mu}_{2} I_{r}^{2}\right)}{\left(3 a_{6} \bar{f}_{2} I_{r}\right)},  \tag{56}\\
\bar{\gamma}_{k-1,2}=\pi-\bar{\gamma}_{k-1,1} .
\end{gather*}
$$

5.3. Geometric Structure of Multipulse Orbits. Based on the aforementioned analysis, we obtain the following conclusions. When the parameters of $k, \mu_{3}, \varepsilon_{1}, \bar{\mu}_{2}, a_{6}$, and $\bar{f}_{2}$ satisfy condition (54), the $k$-pulse Melnikov function (53) has simple zeroes at $\gamma_{k-1}=\bar{\gamma}_{k-1,1}$ and $\gamma_{k-1}=\bar{\gamma}_{k-1,2}=$ $\pi-\bar{\gamma}_{k-1,1}$. For $i=1$ or $i=2$, when the $j$-pulse Melnikov function $M_{j}\left(I_{r}, \bar{\gamma}_{0, i}, \bar{\mu}_{2}, \eta_{1}, a_{6}, \varepsilon_{1}\right)$ has no simple zeroes, the stable and unstable manifolds $W^{s}\left(M_{\varepsilon}\right)$ and $W^{u}\left(M_{\varepsilon}\right)$ intersect transversely along a symmetric pair of the two-dimensional, $k$-pulse surfaces $\sum_{ \pm, \varepsilon}^{\bar{\mu}_{2}, \eta_{1}, a_{6}, \varepsilon_{1}}\left(\bar{\gamma}_{k-1, i}\right)$. This signifies that the presence of the Shilnikov type $n$-pulse orbits leads to chaotic dynamics in the sense of the Smale horseshoes for the nonlinear motion for the laminated composite piezoelectric rectangular plate. In the phase space of the unperturbed system from (21a), (21b), (21c), (21d), this symmetric pair of the two-dimensional, $k$-pulse surfaces breaks down smoothly onto a pair of limiting $k$-pulse surfaces, $\sum_{ \pm, 0}^{\bar{\mu}_{2}, \eta_{1}, a_{6}, \varepsilon_{1}}\left(\bar{\gamma}_{k-1, i}\right)$, parametrized by (37a), (37b), and (38) with $I=I_{r}, \gamma_{0}=$ $\bar{\gamma}_{k-1, i}-(k-1)(\Delta \gamma / 2)+j \Delta \gamma$, and an arbitrary $h$. The sign in (49) is determined by the sign of the corresponding $j$-pulse Melnikov function $M_{j}\left(I_{r}, \bar{\gamma}_{0, i}, \bar{\mu}_{2}, \eta_{1}, a_{6}, \varepsilon_{1}\right)$.

From the discussion given by Kovačič et al. [15-17], it is easily found that for $\bar{\gamma}_{0, i}=\bar{\gamma}_{k-1, i}-(k-1)(\Delta \gamma / 2)+j \Delta \gamma$ ( $i=1$ or $i=2$ ), the values of the $j$-pulse Melnikov functions $M_{j}\left(I_{r}, \bar{\gamma}_{0, i}, \bar{\mu}_{2}, \eta_{1}, a_{6}, \varepsilon_{1}\right)$ are not zero for all $j=$ $1, \ldots, k-1$, and all $j$ have the same sign. It is known that this sign is negative for $\bar{\gamma}_{0,1}$ and positive for $\bar{\gamma}_{0,2}$. Therefore, the $k$-pulse heteroclinic surfaces $\sum_{ \pm, \varepsilon}^{\bar{\mu}_{2}, \eta_{1}, a_{6}, \varepsilon_{1}}\left(\bar{\gamma}_{k-1,1}\right)$ and $\sum_{ \pm, \varepsilon}^{\bar{\mu}_{2}, \eta_{1}, a_{6}, \varepsilon_{1}}\left(\bar{\gamma}_{k-1,2}\right)$ indeed exist, and the limiting $k$-pulse surfaces $\sum_{ \pm, 0}^{\bar{\mu}_{2}, \eta_{1}, a_{6}, \varepsilon_{1}}\left(\bar{\gamma}_{k-1,1}\right)$ and $\sum_{ \pm, 0}^{\bar{\mu}_{2}, \eta_{1}, a_{6}, \varepsilon_{1}}\left(\bar{\gamma}_{k-1,2}\right)$ also exist when $\varepsilon=0$. Since the regions enclosed by the stable and unstable heteroclinic manifolds $W^{s}(M)$ and $W^{u}(M)$ are both convex, and the normal vector

$$
\begin{equation*}
\mathbf{n}=\left(\left(-\mu u_{1}+\eta_{1} u_{1}^{3}+a_{6} I^{2} u_{1}\right),-u_{2}, 0,0\right) \tag{57}
\end{equation*}
$$

is known to point out of these manifolds, it demonstrates that the orbits forming each of the surfaces $\sum_{ \pm, 0}^{\bar{\mu}_{2}, \eta_{1}, a_{6}, \varepsilon_{1}}\left(\bar{\gamma}_{k-1,1}\right)$ are parametrized by (37a), (37b), and (38) with the alternating signs, and the orbits forming each of the surfaces $\sum_{ \pm, 0}^{\bar{\mu}_{2}, \eta_{1}, a_{6}, \varepsilon_{1}}\left(\bar{\gamma}_{k-1,2}\right)$ are parametrized by (37a), (37b), and (38) with the same signs.

For the parameter $\bar{\mu}_{2}=\bar{\mu}$, there exist $N-1$ orbit segments $O_{i}(\bar{\mu})(i=2, \ldots, N)$ on the annulus $M$, where the end points of the segments $O_{i}(\bar{\mu})$ are $d_{i}(\bar{\mu})$ and $c_{i}(\bar{\mu})$, respectively. The trajectories of (44a), (44b) on the segments $O_{i}(\bar{\mu})$ travel from the end points $d_{i}(\bar{\mu})$ to $c_{i}(\bar{\mu})$ in forward time. Therefore, the end points $d_{i}(\bar{\mu})$ and $c_{i}(\bar{\mu})$ are, respectively, referred to as
the departure and landing points of the heteroclinic jumping $\Gamma_{i}$. In addition, the line $\gamma=\bar{\gamma}_{0, i}\left(I_{r}, \bar{\mu}\right)-\Delta \gamma^{-}\left(I_{r}\right)$ transversely intersects the segments $O_{i}(\bar{\mu})$ at the end point $c_{i}(\bar{\mu})$ for $i=$ $2, \ldots, N$, while the line $\gamma=\bar{\gamma}_{0, i}\left(I_{r}, \bar{\mu}\right)+\Delta \gamma^{+}\left(I_{r}\right)$ transversely intersects the segments $O_{i+1}(\bar{\mu})$ at the end point $d_{i+1}(\bar{\mu})$ when $i=1, \ldots, N-1$. For all $i=2, \ldots, N-1$, the difference in the coordinates $h$ of two end points $c_{i}(\bar{\mu})$ and $d_{i+1}(\bar{\mu})$ is zero, namely,

$$
\begin{equation*}
h\left(c_{i}(\bar{\mu})\right)-h\left(d_{i+1}(\bar{\mu})\right)=0 \tag{58}
\end{equation*}
$$

For each $i=2, \ldots, N-1$, one of the heteroclinic orbits represented by $\Gamma_{i}$ and contained in the limiting surfaces $\sum_{0}^{\bar{\mu}_{2}, \eta_{1}, a_{6}, \varepsilon_{1}}\left(\bar{\gamma}_{0, i}\right)$ at the value $\mu=\bar{\mu}$, connects two intersection points $c_{i}(\bar{\mu})$ and $d_{i+1}(\bar{\mu})$. Therefore, a heteroclinic orbit $\Gamma_{1}$ on the limiting surfaces $\sum_{0}^{\bar{\mu}_{2}, \eta_{1}, a_{6}, \varepsilon_{1}}\left(\bar{\gamma}_{0,1}\right)$ connects the certain point $c_{1}(\bar{\mu})$ on the annulus $M$ to the end point $d_{2}(\bar{\mu})$ on the segment $O_{2}(\bar{\mu})$. It is also known that a heteroclinic orbit $\Gamma_{N}$ on the limiting surfaces $\sum_{0}^{\bar{\mu}_{2}, \eta_{1}, a_{6}, \varepsilon_{1}}\left(\bar{\gamma}_{0, N}\right)$ connects the end point $c_{N}(\bar{\mu})$ on the segments $O_{N}(\bar{\mu})$ to the certain point $d_{N+1}(\bar{\mu})$ on the annulus $M$. According to the study of Kovačič et al. [15-17], there exists an $n$-bump singular transition orbit or a modified $N$-bump singular transition orbit. The 3-bump jumping orbit depicted in Figure 5 consists of the heteroclinic orbits $\Gamma_{i}(i=1,2,3)$ on the limiting surfaces $\sum_{0}^{\bar{\mu}_{2}, \eta_{1}, a_{6}, \varepsilon_{1}}\left(\bar{\gamma}_{0, i}\right)(i=1,2,3)$ at the parameter $\mu=\bar{\mu}$ and the orbit segments $O_{1}(\bar{\mu})$ and $O_{2}(\bar{\mu})$ of (44a), (44b). It is known from the above analysis that the orbit segments $O_{i}(\bar{\mu})(i=$ $2, \ldots, N)$ intersect transversely with the lines $\gamma=\bar{\gamma}_{0, i}\left(I_{r}, \bar{\mu}\right)+$ $\Delta \gamma^{+}\left(I_{r}\right)$ and $\gamma=\bar{\gamma}_{0, i}\left(I_{r}, \bar{\mu}\right)-\Delta \gamma^{-}\left(I_{r}\right)$.

The 2-bump singular surface shown in Figure 6 is composed of two single-pulse singular intersection surfaces $\sum_{0}^{\bar{\mu}_{2}, \eta_{1}, a_{6}, \varepsilon_{1}}\left(\bar{\gamma}_{k-1,1}\right)$ and $\sum_{0}^{\bar{\mu}_{2}, \eta_{1}, a_{6}, \varepsilon_{1}}\left(\bar{\gamma}_{k-1,2}\right)$. This surface connects the singular points of (44a), (44b) that lie on the line $\gamma=\bar{\gamma}_{0,1}-$ $\Delta \gamma^{-}$to those of (44a), (44b) that lie on the line $\gamma=\bar{\gamma}_{0,1}-\Delta \gamma^{+}$ on the annulus $M$.

We obtain a countable infinity of the singular heteroclinic jumping orbits as follows. Each orbit starts along one branch of the manifold $W\left(Q_{0}\right)$ of the saddle $Q_{0}$ on the annulus $M$. Then, the singular heteroclinic jumping orbit departs from the annulus $M$, goes along one of the singular $k$-pulse orbits $\Gamma_{k}$, and lands back at a point on the separatrix that connects the saddle $Q_{0}$ to itself on the annulus $M$. After traveling along the separatrix for a while, the singular heteroclinic jumping orbit takes off again along the singular $l$-pulse orbit $\Gamma_{l}$ and continues such process. Eventually, the singular heteroclinic jumping orbit lands back on the separatrix.

Therefore, it is concluded that the multipulse orbits of (21a), (21b), (21c), (21d) consist of several portions of the slow time scale on the hyperbolic manifold $M_{\varepsilon}$ and many fast time scale heteroclinic pulses leaving from the manifold $M_{\varepsilon}$, and these multipulse heteroclinic orbits form a consecutive and recurrence process.

## 6. Numerical Results of Chaotic Motions

Based on the above qualitative analysis for the multipulse orbits and chaotic dynamics of the laminated composite


Figure 5: The 3-bump orbit with the single-pulse is depicted.
piezoelectric rectangular plate, the conditions of the chaotic motion under the sense of the Smale horses are obtained. The heteroclinic bifurcations of (12a), (12b), (12c), (12d) appear when $\eta_{1}>0$. Therefore, the above theoretical analysis is focused on the situation which there exist heteroclinic bifurcations in (12a), (12b), (12c), (12d). The parameter $\eta_{1}$ is the combination of the parameters $a_{6}, a_{7}$, and $b_{6}$, where $\eta_{1}=\left(9 a_{6} a_{7}\right) / 2 b_{6}$. In this section, we have only performed numerical simulations of the multipulse chaotic motions of the laminated composite piezoelectric rectangular plate under heteroclinic bifurcations in order to further verify the theoretical analysis. Consequently, the parameters $a_{6}, a_{7}$, and $b_{6}$ are chosen to satisfy $\eta_{1}>0$.

We choose the averaged equation (12a), (12b), (12c), (12d) to conduct numerical simulations. A numerical approach through the computer software Matlab is utilized to explore the existence of the Shilnikov type multipulse chaotic motions in the laminated composite piezoelectric rectangular plate. Based on the above qualitative analysis, it is found that the damping coefficients $\mu_{1}, \mu_{2}$ and transverse excitation $f_{2}$ play an important role in the multipulse chaotic motions of the laminated composite piezoelectric rectangular plate. In addition, the parameters $a_{2}$ and $a_{3}$ are related to the inplane excitation in the $x$-direction and the in-plane excitation in the $y$-direction, respectively. The parameter $a_{4}$ is the piezoelectric excitation which reflects the characteristics of the piezoelectric material. Hence, the parameters $\mu_{1}, a_{2}, a_{4}$, and $f_{2}$ are selected as the controlling parameters to discover the law for complicated nonlinear dynamics of the laminated composite piezoelectric rectangular plate.

We begin to draw bifurcation diagrams of the parameters $f_{2}, \mu_{1}, a_{2}$, and $a_{4}$. Bifurcation diagrams describe the vibration law of the modal displacements $x_{1}$ and $x_{3}$, respectively, when the parameters $f_{2}, \mu_{1}, a_{2}$, and $a_{4}$ change in a certain region. We draw bifurcation diagrams according to the rules of the Runge-Kutta algorithm and the Poincaré map theory. For the periodic motions, Poincaré map is of several separate points. For a chaotic motion, the Poincaré map consists of a number of points on the limited Poincaré section. Therefore, it can be observed that chaotic motion and periodic motion of nonlinear system appear from bifurcation diagrams. The chaotic and


Figure 6: The 2-pulse singular surfaces $\sum\left(\bar{\gamma}_{0,1}, \bar{\gamma}_{0,2}\right)$ are depicted.
periodic responses can be identified by several conventional criteria such as phase portraits and Poincaré map. Based on the response law of bifurcation diagrams, phase portraits and Poincaré map are utilized to further verify the existence of the chaotic motions and the periodic motions. In order to compare the influence of these parameters $f_{2}, \mu_{1}, a_{2}$, and $a_{4}$ on nonlinear vibration in the laminated composite piezoelectric rectangular plate, we choose the same initial conditions to carry out numerical simulation.

Figure 7 illustrates the bifurcation diagram of the laminated composite piezoelectric rectangular plate when the excitation $f_{2}$ varies in the interval $f_{2}=2 \sim 200$. Other parameters and initial conditions are chosen as $\sigma_{1}=$ 1.83, $\sigma_{2}=1.97, \mu_{1}=0.2, \mu_{2}=0.2, a_{2}=23.0, a_{3}=$ 12.0, $a_{4}=13.0, a_{5}=-1.01, a_{6}=-2.03, a_{7}=-2.05, b_{6}=$ 4.07, $b_{7}=-3.08, b_{8}=1.09, x_{10}=-0.01, x_{20}=-0.05, x_{30}=$ $-0.01, x_{40}=-0.01$. Figures 7(a) and 7(b) represent the bifurcation diagram on the plane $\left(x_{1}, f_{2}\right)$ and $\left(x_{3}, f_{2}\right)$, respectively. It is observed from Figure 7 that the excitation $f_{2}$ is an important parameter that influences on the nonlinear dynamic responses of the laminated composite piezoelectric rectangular plate. Figure 7 shows that the chaotic motion of the laminated composite piezoelectric rectangular plate appears first, followed by a periodic motion of that. With the increase of excitation $f_{2}$, Figure 7 presents the following law: chaotic motion $\rightarrow$ multi-period motion.

We study the impact of the damping parameter on the nonlinear dynamic responses of the laminated composite piezoelectric rectangular plate. Figure 8 is the bifurcation diagram of the laminated composite piezoelectric rectangular plate with the damping coefficient $\mu_{1}$. The figure demonstrates that system is beginning to enter into the region of the chaotic motion then appears the periodic motion window and finally comes into the region of the chaotic motion again as the damping coefficient $\mu_{1}$ varies in the interval $\mu_{1}=$ $0.01 \sim 0.7$. Other parameters and initial conditions are the same as those in Figure 7 when excitation is chosen as $f_{2}=$ 82.7. Figures 8(a) and 8(b) describe the nonlinear motion of the laminated composite piezoelectric rectangular plate on


FIgure 7: The bifurcation diagram is obtained for the excitation $f_{2}=2 \sim 200$, and initial conditions $x_{10}=-0.01, x_{20}=-0.05, x_{30}=$ $-0.01, x_{40}=-0.01$; (a) the bifurcation diagram on the plane $\left(x_{1}, f_{2}\right)$; (b) the bifurcation diagram on the plane $\left(x_{3}, f_{2}\right)$.


Figure 8: The bifurcation diagram is obtained for the damping coefficient $\mu_{1}=0.01 \sim 0.7$, the excitation $f_{2}=82.7$, and initial conditions $x_{10}=-0.01, x_{20}=-0.05, x_{30}=-0.01, x_{40}=-0.01$; (a) the bifurcation diagram on the plane $\left(x_{1}, \mu_{1}\right)$; (b) the bifurcation diagram on the plane $\left(x_{3}, \mu_{1}\right)$.
the planes $\left(x_{1}, \mu_{1}\right)$ and $\left(x_{3}, \mu_{1}\right)$, respectively, as well as the impact of the damping coefficient $\mu_{1}$ on the system.

Figure 9 portraysthe bifurcation diagram for the laminated composite piezoelectric rectangular plate when the inplane excitation $a_{2}$ in the $x$-direction varies in the interval $a_{2}=2 \sim 65$. Other parameters and initial conditions remain the same as those in Figure 8 when the damping coefficient $\mu_{1}$ is selected as $\mu_{1}=0.2$. Figures 9(a) and 9(b) display the bifurcation diagram on the plane $\left(x_{1}, a_{2}\right)$ and $\left(x_{3}, a_{2}\right)$, respectively. Figure 9 presents that the beginning movement of the system is the periodic motion; then the system appears the chaotic motion. With the increase of the excitation $a_{2}$, Figure 9 shows the following evolution law: periodic motion $\rightarrow$ chaotic motion.

Figure 10 indicates the bifurcation diagram for the laminated composite piezoelectric rectangular plate when the piezoelectric excitation $a_{4}$ varies from $a_{4}=2$ to $a_{4}=120$. Other parameters and initial conditions remain the same as those in Figure 9 when the in-plane excitation $a_{2}$ is chosen as $a_{2}=23$. Figures $10(\mathrm{a})$ and $10(\mathrm{~b})$ demonstrate the bifurcation diagram on the planes $\left(x_{1}, a_{4}\right)$ and $\left(x_{3}, a_{4}\right)$, respectively. It is observed from Figure 10 that the piezoelectric excitation $a_{4}$ has a significant influence on the complicated nonlinear dynamic behaviors of the laminated composite piezoelectric rectangular plate. As the piezoelectric excitation $a_{4}$ increases, Figure 10 reveals the following law: chaotic motion $\rightarrow$ multiperiod motion $\rightarrow$ one-period motion $\rightarrow$ multiperiod motion $\rightarrow$ chaotic motion.


Figure 9: The bifurcation diagram is obtained for the in-plane excitation $a_{2}=2 \sim 65$, the damping coefficients $\mu_{1}=0.2$ and $\mu_{2}=0.2$, the excitation $f_{2}=82.7$, and initial conditions $x_{10}=-0.01, x_{20}=-0.05, x_{30}=-0.01, x_{40}=-0.01$; (a) the bifurcation diagram on the plane $\left(x_{1}, a_{2}\right)$; (b) the bifurcation diagram on the plane $\left(x_{3}, a_{2}\right)$.


FIgure 10: The bifurcation diagram is obtained for the piezoelectric excitation $a_{4}=2 \sim 120$, the excitation $f_{2}=82.7$, the damping coefficients $\mu_{1}=0.2$ and $\mu_{2}=0.2$, the in-plane excitations $a_{2}=23$ and $a_{3}=12.0$, and initial conditions $x_{10}=-0.01, x_{20}=-0.05, x_{30}=-0.01, x_{40}=$ -0.01 ; (a) the bifurcation diagram on the plane ( $x_{1}, a_{4}$ ); (b) the bifurcation diagram on the plane ( $x_{3}, a_{4}$ ).

Based on the above bifurcation diagram, the excitations $f_{2}, a_{2}, a_{4}$, and the damping coefficient $\mu_{1}$ are selected as specific values in order to find the multipulse chaotic motions of the laminated composite piezoelectric rectangular plate. Figure 11 indicates existence of the multipulse chaotic motion of the laminated composite piezoelectric rectangular plate when the excitation $f_{2}$ is 82.7 . In this case, the chosen parameters and initial conditions are the same as those in Figure 7. Figures $11(\mathrm{a})$ and $11(\mathrm{~b})$ are the three-dimensional phase portrait in the space $\left(x_{1}, x_{2}, x_{3}\right)$ and the Poincare map on the plane $\left(x_{1}, x_{2}\right)$, respectively. Figure 11 shows that the excitation $f_{2}$ has a noticeable effect on the existence of the multipulse chaotic motions on the laminated composite piezoelectric rectangular plate.

Besides the excitations $f_{2}, a_{2}, a_{4}$ and the damping coefficient $\mu_{1}$, the multipulse chaotic motions of the laminated composite piezoelectric rectangular plate also depend on other parameters. Figure 12 is obtained when the parameters and initial conditions are chosen as $\sigma_{1}=14.37, \sigma_{2}=$ $11.42, \mu_{1}=0.2, \mu_{2}=0.2, a_{2}=30.0, a_{3}=75.0, a_{4}=45.0$, $a_{5}=-11.66, a_{6}=12.27, f_{2}=122.7, a_{7}=-2.68, b_{6}=$ $-2.2, b_{7}=-9.69, b_{8}=-22.32, x_{10}=-1.08, x_{20}=0.5$, $x_{30}=-0.01, x_{40}=9.16$. Comparing with Figures 11 and 12 , it is found that there are differences in the phase portrait and the Poincaré map, respectively. From the three-dimensional phase portrait in Figure 12, we can see that there exists obvious multipulse jumping phenomenon. The three-dimensional phase portrait is composed of the four


Figure 11: The multipulse chaotic motion is obtained when $f_{2}=82.7$ and $\mu_{1}=0.2$; (a) the phase portrait in the three-dimensional space $\left(x_{1}, x_{2}, x_{3}\right)$; (b) Poincaré map on the plane $\left(x_{1}, x_{2}\right)$.


Figure 12: The multipulse chaotic motion is obtained when $\sigma_{1}=14.37, \sigma_{2}=11.42, a_{2}=30.0, a_{3}=75.0, a_{4}=45.0, a_{5}=-11.66, a_{6}=$ 12.27, $f_{2}=122.7, a_{7}=-2.68, b_{6}=-2.2, b_{7}=-9.69, b_{8}=-22.32, x_{10}=-1.08, x_{20}=0.5, x_{30}=-0.01, x_{40}=9.16$; (a) the phase portrait in the three-dimensional space $\left(x_{1}, x_{2}, x_{3}\right)$; (b) Poincare map on the plane $\left(x_{1}, x_{2}\right)$.
regions. The different regions are connected by the multipulse orbits.

In the following numerical simulations, several different sets of parameters and initial conditions are given in order to investigate the different shapes of the multipulse chaotic motion. Figure 13 demonstrates the multipulse chaotic response in the laminated composite piezoelectric rectangular plate for $f_{2}=92.38$. Some parameters and initial conditions are chosen as $\sigma_{1}=3.61, \sigma_{2}=3.13, a_{5}=$ $-15.01, b_{8}=4.09, x_{10}=-0.01, x_{20}=-0.09, x_{30}=$ $-0.05, x_{40}=-0.05$. In this case, other parameters are the same as those in Figure 7. From Figure 13, we can see that there is another shape for the multipulse chaotic motion. It is found that the shapes of these two phenomena depicted in Figures 12 and 13 are completely different. From the threedimensional phase portrait in Figure 13, it is found that multipulse jumping phenomenon is more prominent.

## 7. Conclusions

In this paper, the nonlinear vibrations of the laminated composite piezoelectric rectangular plate are studied by applying the theories of the global bifurcations and chaotic dynamics for high-dimensional nonlinear systems. The multipulse heteroclinic orbits and chaotic dynamics are investigated using the extended Melnikov method for the case where the averaged equations have one nonsemisimple double zero and a pair of pure imaginary eigenvalues. The extended Melnikov method can be applied to study the Shilnikov type multipulse heteroclinic bifurcations and chaotic dynamics of highdimensional nonlinear systems in engineering applications. Analysis of the multipulse heteroclinic orbits in the laminated composite piezoelectric rectangular plate demonstrates that such an analysis is a typical singular perturbation problem in which there are two different time scales. Dynamics on the hyperbolic manifold $M_{\varepsilon}$ are of the slow time scale and


FIGURE 13: The multipulse chaotic motion is obtained when $\sigma_{1}=3.61, \sigma_{2}=3.13, a_{2}=23.0, a_{3}=12.0, a_{4}=13.0, a_{5}=-15.01, a_{6}=$ $-2.03, f_{2}=92.38, a_{7}=-2.05, b_{6}=4.07, b_{7}=-3.08, b_{8}=4.09, x_{10}=-0.01, x_{20}=-0.09, x_{30}=-0.05, x_{40}=-0.05$; (a) the phase portrait in the three-dimensional space $\left(x_{1}, x_{2}, x_{3}\right)$; (b) Poincaré map on the plane $\left(x_{1}, x_{2}\right)$.
the multipulse jumping orbits taking off from this manifold are of the fast time scale. It is shown that the transfer of energy between the two different modes occurs through the multipulse jumping orbits. The studies have led to the following conclusions.
(1) There exist the Shilnikov type multipulse chaotic motions in nonlinear vibration of the laminated composite piezoelectric rectangular plate. The geometric interpretation of the $k$-pulse Melnikov function is a signed distance measured along the normal to a heteroclinic manifold, which gives the more delicate local estimates near the hyperbolic manifold. In the resonant case, the $k$-pulse extended Melnikov function $M_{k}\left(\varepsilon, I, \gamma_{0}, \bar{\mu}_{2}\right)$ does not depend on the small perturbation parameter $0<\varepsilon \ll 1$, and the nonfolding condition is automatically satisfied, resulting in the angle function $\Gamma_{j}\left(\varepsilon, I_{r}, \gamma_{0}, \bar{\mu}_{2}\right)(j=0,1, \ldots, k-1)$ being zero. Therefore, the computing procedure of the extended Melnikov function can be simplified.
(2) In order to verify the theoretical predictions, numerical simulation is used to examine the bifurcations and chaotic motions of the laminated composite piezoelectric rectangular plate. Several types of the bifurcation diagrams are obtained when the transverse excitation $f_{2}$, the in-plane excitation $a_{2}$, the piezoelectric excitation $a_{4}$, and the damping coefficient $\mu_{1}$ are chosen as several different kinds of control parameters. Based on the bifurcation diagrams, the nonlinear complicated dynamic behavior of the laminated composite piezoelectric rectangular plate is controlled by varying the excitations $f_{2}, a_{2}, a_{4}$ and the damping coefficient $\mu_{1}$, respectively. Therefore, the excitations $f_{2}, a_{2}, a_{4}$ and the damping coefficient $\mu_{1}$ have important influence on the nonlinear dynamics responses of the laminated composite piezoelectric rectangular plate.
(3) There exist different shapes of the chaotic motions in the nonlinear oscillations of the laminated composite piezoelectric rectangular plate under different excitations, parameters, and initial conditions. It is found from numerical
simulations that the shapes of the chaotic motions are completely different. From the three-dimensional phase portraits in Figures 12 and 13, it is found that there exist obvious multipulse jumping phenomena. Therefore, parameters and initial conditions impact the shapes of the multipulse chaotic motions.
(4) There exist multipulse chaotic motions in the averaged equations. It is well known that the multipulse chaotic motions in the averaged equations can lead to the multipulse amplitude modulated chaotic vibrations in the original system under certain conditions. Therefore, the multipulse amplitude modulated chaotic motions occur in the laminated composite piezoelectric rectangular plate.

In summary, both theoretical and numerical studies suggest that chaos for the Smale horseshoe sense in nonlinear motion of the simply supported laminated composite piezoelectric rectangular plate exists.

## Conflict of Interests

The authors declare that there is no conflict of interests in this paper.

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# The Effects of Using Chaotic Map on Improving the Performance of Multiobjective Evolutionary Algorithms 

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#### Abstract

Chaotic maps play an important role in improving evolutionary algorithms (EAs) for avoiding the local optima and speeding up the convergence. However, different chaotic maps in different phases have different effects on EAs. This paper focuses on exploring the effects of chaotic maps and giving comprehensive guidance for improving multiobjective evolutionary algorithms (MOEAs) by series of experiments. NSGA-II algorithm, a representative of MOEAs using the nondominated sorting and elitist strategy, is taken as the framework to study the effect of chaotic maps. Ten chaotic maps are applied in MOEAs in three phases, that is, initial population, crossover, and mutation operator. Multiobjective problems (MOPs) adopted are ZDT series problems to show the generality. Since the scale of some sequences generated by chaotic maps is changed to fit for MOPs, the correctness of scaling transformation of chaotic sequences is proved by measuring the largest Lyapunov exponent. The convergence metric $\gamma$ and diversity metric $\Delta$ are chosen to evaluate the performance of new algorithms with chaos. The results of experiments demonstrate that chaotic maps can improve the performance of MOEAs, especially in solving problems with convex and piecewise Pareto front. In addition, cat map has the best performance in solving problems with local optima.


## 1. Introduction

Multiobjective evolutionary algorithms have attracted widespread attention and have been applied successfully in many areas, such as test task scheduling problem (TTSP) [1], reservoir operation [2], proportional integral and derivative (PID) controller [3], and distribution feeder reconfiguration (DFR) [4]. One key challenge in multiobjective evolutionary algorithms is the problem of resolving local optima and the speed of convergence. There are different solutions for improving evolutionary algorithms. Some approaches have been devoted to propose new algorithms, such as MOEA/D [5], SPEA-2 [6], and NSGA-II [7]. Other researchers have proposed a variety of hybrid algorithms, which combined the advantages of two different methods. For example, a new hybrid evolutionary algorithm (EA) based on the combination of the honey bee mating optimization (HBMO) and the discrete particle swarm optimization (DPSO), called DPSOHBMO, is applied to solve the multiobjective distribution
feeder reconfiguration (DFR) problem [4]. Another approach has focused on modifying original algorithms. For example, new particle swarm optimization (PSO) methods were proposed by using chaotic maps for parameter adaptation [8]. The results showed that chaos embedded PSO can improve the quality of results in some optimization problems. Chaos variables are loaded into the variable colony of the immune algorithm in the immune evolutionary algorithm, and the experimental results indicate that the new immune evolutionary algorithm improves the convergence performance and search efficiency [9]. Due to the characteristics such as randomness, regularity, ergodicity, and initial value sensitiveness, chaos has been widely applied in the original evolutionary algorithms to improve the performance.

Recently researches have been done to the chaos embed in evolutionary algorithms. For example, Alatas et al. [8] applied seven chaotic maps to generate seven new chaotic artificial bee colony algorithms. Three phases were adopted in generating these algorithms to solve three different benchmark
single objective problems. Results showed that these methods have somewhat improved the solution quality. Tavazoei and Haeri [10] introduced ten chaotic maps in weighted gradient direction to solve two test functions. Results showed that none of these maps transcends other maps for all of the problems and desired criteria. Those researches demonstrated that chaotic sequences replacing the random parameters in three phases, including initial population, crossover operator, and mutation operator, can improve the performance of evolutionary algorithms. However, questions remain that for a given MOP, which chaotic map should be chosen in order to achieve the best performance. It is also not clean what kinds of combination of chaotic maps used in a particular phase have the best property. Therefore, it is difficult to give comprehensive guidance to improve the performance of evolutionary algorithms.

In addition, from the problems solved by COA, it can be seen that single objective optimization problems are the focus. Comparisons of different chaotic maps in improving the effects of COAs for solving single objective problems are common, but it is rare in solving multiobjective optimization problems (MOPs). Yu et al. [11] revealed that COA is not effective for solving MOPs, whereas the experiments in Alatas and Akin [12] showed the opposite. The results on these foregoing researches demonstrate that COAs are successful and competitive for solving single objective optimization problem, but effects of COAs on solving MOPs are not consistent.

In summary, although there have been many researches about the chaos and its application in COAs, the effects of different chaotic maps used in different phases on the performance of evolutionary algorithms have not yet been fully evaluated, especially for the multiobjective evolutionary algorithms.

In this paper, we explore the relationships of chaotic maps and phases on improving multiobjective evolutionary algorithms by a series of experiments. We will answer the question whether chaotic maps are suitable to improve the evolutionary algorithms in solving MOPs. We also investigate which phase should be chosen when one chaotic map is used to improve a multiobjective evolutionary algorithm.

In this research, NSGA-II is chosen as the main optimization algorithm, because it captures the core ideas and characteristics of MOEAs with the properties of a fast nondominated sorting procedure, an elitist strategy, a parameterless approach, and a simple yet efficient constrainthandling method [7]. Despite these good aspects of NSGAII for solving MOPs, it may be entrapped into local optimal solutions. Thus, the properties of chaos can help to improve the performance of NSGA-II.

In order to reflect the diversity of chaotic maps, ten chaotic maps that have been widely used in pioneering researches are studied in this paper. They are circle map, cubic map, Gauss map, ICMIC map, logistic map, sinusoidal map, tent map, Baker's map, cat map, and Zaslavskii map. Each chaotic map has its own property and has its own effect on improving the performance of evolutionary algorithms. For example, logistic map has Chebyshev-type distribution but not uniform distribution. As a result, it is necessary for optimal solution to go through multiple iterations.

Similar to past researches, chaotic maps are used in three common phases in evolutionary algorithms in experiments, that is, chaotic sequences for initial population, chaotic sequences for crossover operator, and chaotic sequences for mutation operator.

Five benchmark MOPs including ZDT1, ZDT2, ZDT3, ZDT4, and ZDT6 [7] are chosen as test problems. These MOPs have different characteristics and can reflect the property of evolutionary algorithms from different aspects. For example, we can use problem ZDT4 to evaluate the performance of evolutionary algorithms for resolving local optimal, because ZDT4 has different local Pareto-optimal solutions in the search space.

In addition, ranges of chaotic maps are not always fit for test problems. Scaling transformation is needed to apply chaotic sequences. For example, Coelho and Mariani [13] adopted Zaslavskii's map by changing its range to $(0,1)$ and Alatas [12, 14, 15] took a similar approach. The problem is whether the chaotic sequences through scaling transformation still maintain the properties of chaos. In this paper, the correctness of scaling transformation of chaotic sequences is proved by measuring the largest Lyapunov exponent.

Finally, the criteria of convergence and distribution proposed by Deb et al. [7] are adopted in this paper to evaluate the effects of the combinations of phases and chaotic maps on improving the performance of multiobjective evolutionary algorithms. One is metric $\gamma$, which measures the extent of convergence to a known set of Pareto-optimal solutions. The other is metric $\Delta$, which measures the extent of spread achieved among the obtained solutions.

From the results of experiments, it can be seen that NSGA-II embedded with chaotic maps in most cases get better results with regard to the metrics $\gamma$ and $\Delta$. The effects of using chaotic maps depend on which chaotic map is selected and in which phase it is used. In particular, chaos can improve the ability of NSGA-II in solving ZDT3 and ZDT6, which are difficult for the original NSGA-II algorithm. Besides, cat map is good at solving problems with local optima, such as ZDT4.

The rest of paper is organized as follows. Section 2 gives a summary of related work on applying chaos to improve evolutionary algorithms. Section 3 shows the phases in which chaos can be embedded in evolutionary algorithms. Section 4 defines ten chaotic maps which are embedded in NSGA-II in the experiments. Section 5 proves that the chaotic sequences through scaling transformation still hold the properties of chaos. Section 6 describes the test problems and metrics used in the experiments. Section 7 presents the performance results of the experiments. Section 8 concludes the paper.

## 2. Related Work

Applying chaotic maps to improve evolutionary algorithms has been studied for a while. There are two different strategies to apply the chaotic maps in the evolutionary algorithms.

One is to use chaotic sequences generated by chaotic maps to replace the random parameters needed by evolutionary algorithms. Coelho [16] proposed a quantum-behaved particle swarm optimization (QPSO). Random sequences
of mutation operator in QPSO were replaced with chaotic sequences based on Zaslavskii map. The results demonstrated that it is a powerful strategy to diversify the population and improve the performance in preventing premature convergence to local minima. Dos Coelho and Alotto considered the chaotic crossover operator using the Zaslavskii map to solve multiobjective optimization problems [17]. Zhang et al. [18] proposed three chaotic sequences based multiobjective differential evolution (CS-MODE) to solve short-term hydrothermal optimal scheduling with economic emission (SHOSEE). In the modified mutation operator, chaotic theory is used to increase the population diversity, and some adaptive tuning parameters are produced by chaotic mappings to control the evolution.

The other strategy is to use the chaos optimization as an operator. For example, Alatas [14] applied chaotic search in case that a solution does not obtain improvements in artificial bee colony ( ABC ) algorithm. The results showed that the strategy has better performance than that of ABC algorithm. Wang and Zhang [19] employed chaos analogously. When the value of objective function had no improvement in continuous iterations, one chaotic system was applied to reinitialize half of the population. It replaced the worst half part of the population in order to jump out of the local optimum, whereas the best half part is kept unchanged.

Since evolutionary algorithms have sensitive dependence on their initial condition and parameters, the improvements on these parameters can have a good effect. That may be one of the reasons that the first strategy is widely adopted. In the first strategy, it is necessary to consider the phases of replacing random sequences with chaotic sequences and the different chaotic maps adopted.

For the phases of the evolutionary algorithms, Caponetto et al. [20] introduced chaotic sequences instead of random ones during all the phases of the evolution process. Results showed that the behaviors of all operators were influenced by chaotic sequences. Alatas [15], Ahmadi and Mojallali [21], and Ma [22] focused on random parameters in initial population. Coelho [16] and Zhang et al. [18] did their research on mutation operator. However, which phase is the best choice was not discussed.

To study the performance of different chaotic maps, some researchers give the comparisons of different chaotic maps solving both single objective optimization problems and MOPs. Talatahari et al. [23] proposed a novel chaotic improved imperialist competitive algorithm (CICA) for global optimization. Seven chaotic maps were utilized to improve the movement step of the algorithm, and the logistic and sinusoidal maps were found as the best choices. Caponetto et al. [20] proposed an experimental analysis on the convergence of evolutionary algorithms. Six chaotic maps, four phases, and single-objective statistical tests showed an improvement of evolutionary algorithms when chaotic sequences were used instead of random processes. Lu et al. [1] proposed a chaotic nondominated sorting genetic algorithm (CNSGA) to solve the automatic test task scheduling problem (TTSP). According to the different capabilities of the logistic and the cat chaotic operators, the CNSGA approach using the cat population initialization, the cat or
logistic crossover operator, and the logistic mutation operator performs well and is very suitable for solving the TTSP. The comparisons of the performance of chaotic maps in these researches are based on solving one specific problem, so the results cannot be generalized to offer guidance on how to choose a chaotic map for solving other problems. Furthermore, most researches focus on single objective problems.

In contrast, this paper performs extensive experiments on genetic multiobjective evolutionary algorithms embedded with chaotic sequences. It focuses on exploring the relationships of phases and chaotic maps on improving multiobjective evolutionary algorithms. As mentioned above ten chaotic maps and three phases of evolutionary algorithms are considered. Five general benchmark problems are used to demonstrate that the conclusions can be generalized. Finally the guidance is presented to help researchers choose the suitable chaotic map and phases in multiobjective evolutionary algorithms for different MOPs.

## 3. Phases in Chaos Embedded Evolutionary Algorithms

With the ergodic property, chaos is adopted to enrich the searching behavior and to avoid solutions being trapped into local optimum in optimization problems. In this section three key phases in evolutionary algorithms, initialization, crossover, and mutation, are chosen to be embedded with chaos. Those three phases are described as follows.
3.1. Initialization. Initial population is the starting point of iterations. Ergodicity and diversity of initial population are very important for making sure that the individuals in the population spread in the search spaces uniformly as far as possible. In this case, initial population is generated by chaotic maps which can form a feasible solution space with good distribution by the properties of randomicity and ergodicity of chaos. Chaotic sequences can guarantee the diversity of the initial population, speed up its convergence, and enhance global search capability.

More specifically, a chaotic map, such as logistic map or cat map, is adopted instead of random population initialization of evolutionary algorithms. In the experiments of multiobjective evolutionary algorithms with chaos, the initial population is generated by chaos maps. For example, one of the individuals can be denoted by $x_{s}=\left\{x_{s}^{1}, x_{s}^{2}, \ldots x_{s}^{i}, \ldots x_{s}^{n}\right\}$, $s=1,2, \ldots N, i=1,2, \ldots N$. For the logistic map initialization, $x_{s}^{i+1}=4 x_{s}^{i}\left(1-x_{s}^{i}\right)$.
3.2. Crossover Operator. Crossover operator is most important for evolutionary algorithms. Most of the offsprings are generated through the crossover operator. It has a great influence on the convergence speed. A good crossover operator may prevent premature convergence. Ergodicity of chaos helps search all the solutions, avoid solutions from falling into local optimum, and gain the global optimum.

There are many different crossover operators, such as simulated binary crossover operator [7] in NSGA-II algorithm and multiparent arithmetic crossover operator. Chaotic
sequences substitute random parameters in the crossover operators. Chaotic sequences do not change the randomness of the parameter but display better randomness and therefore enhance the global performance of evolutionary algorithms.

In this paper, simulated binary crossover (SBX) operator is adopted in the experiment. According to SBX, two child individuals $x_{c 1}=\left\{x_{c 1}^{1}, \ldots, x_{c 1}^{i}, \ldots, x_{c 1}^{n}\right\}$ and $x_{c 2}=$ $\left\{x_{c 2}^{1}, \ldots, x_{c 2}^{i}, \ldots, x_{c 2}^{n}\right\}$ are generated by a pair of parents $x_{p 1}=$ $\left\{x_{p 1}^{1}, \ldots, x_{p 1}^{i}, \ldots, x_{p 1}^{n}\right\}$ and $x_{p 2}=\left\{x_{p 2}^{1}, \ldots, x_{p 2}^{i}, \ldots, x_{p 2}^{n}\right\}$ as follows:

$$
\begin{align*}
& x_{c 1}^{i}=\frac{1}{2}\left[(1-\beta) x_{p 1}^{i}+(1+\beta) x_{p 2}^{i}\right], \\
& x_{c 2}^{i}=\frac{1}{2}\left[(1+\beta) x_{p 1}^{i}+(1-\beta) x_{p 2}^{i}\right], \tag{1}
\end{align*}
$$

and $\beta$ is generated in the following manner:

$$
\beta= \begin{cases}(2 u)^{1 /\left(\eta_{c}+1\right)}, & \text { if } u \leq 0.5  \tag{2}\\ \left(\frac{1}{2(1-u)}\right)^{1 /\left(\eta_{c}+1\right)}, & \text { others }\end{cases}
$$

where $u$ is a random number in the range $[0,1] . \eta_{c}$ is the distribution index for the crossover operator.

Since $u$ is a random number, $u$ can be generated by chaotic maps. For instance, if the chaotic map is a logistic map and in the $i$ th iteration $u=u_{i}$, then in the $(i+1)$ th iteration, $u_{s}=$ $u_{i+1}=4 \times u_{i}\left(1-u_{i}\right)$.
3.3. Mutation Operator. Mutation operator is indispensable in the process of evolutionary algorithms. This mechanism avoids solutions from falling into local optimum and guarantees more possibilities of obtaining global optimum. The properties of chaos, like randomness and sensitivity to initial conditions, contribute to preventing solutions from being trapped into local optimum.

Random parameters in mutation operators, for instance, polynomial variation, are replaced by chaotic sequences. For a solution $x_{s}$, the polynomial mutation is described as

$$
\begin{equation*}
x_{s}^{*}=x_{s}+\left(x_{s}^{u}-x_{s}^{l}\right) \times \delta_{s}, \tag{3}
\end{equation*}
$$

where $x_{s}^{u}$ and $x_{s}^{l}$ are the upper and lower bounds of $x_{s}$, and

$$
\delta_{s}= \begin{cases}\left(2 u_{s}\right)^{1 /\left(\eta_{m}+1\right)}-1, & \text { if } u_{s}<0.5  \tag{4}\\ 1-\left(2 \times\left(1-u_{s}\right)\right)^{1 /\left(\eta_{m}+1\right)}, & \text { others }\end{cases}
$$

where $u_{s}$ is a random number ranging from 0 to $1 . \eta_{m}$ is the distribution index for the mutation operator.

The phase for mutation is that $u_{s}$ is calculated by chaotic maps in iterations. For example, if the chaotic map is logistic map, and in the $i$ th iteration $u_{s}=u_{i}$, then in the $(i+1)$ th iteration, $u_{s}=u_{i+1}=4 \times u_{i}\left(1-u_{i}\right)$.

As a representative of MOEAs, the framework of NSGAII algorithm is adopted in the experiments. In order to eliminate the effect of NSGA-II algorithm, other two different mutation operators, that is, Gauss mutation and Cauchy mutation, are chosen to replace polynomial variation.
3.3.1. Gauss Mutation. If random variable $X$ has the probability density function:

$$
\begin{equation*}
p(x)=\frac{1}{\sqrt{2 \pi \sigma}} e^{-\left((x-\mu)^{2} / 2 \sigma^{2}\right)}, \quad-\infty<x<+\infty \tag{5}
\end{equation*}
$$

then $X$ obeys Gauss normal distribution with the parameters $\mu, \sigma$; that is, $X \sim N\left(\mu, \sigma^{2}\right)$.

Gauss mutation means that the random numbers obeying gauss distribution substitute $\delta_{s}$ in polynomial mutation; that is, $\delta_{s} \sim N\left(\mu, \sigma^{2}\right)$.

### 3.3.2. Cauchy Mutation. The probability density function of

 Cauchy distribution concentrated near the origin. It is defined as$$
\begin{equation*}
f(x)=\frac{1}{\pi} \frac{t}{t^{2}+x^{2}}, \quad-\infty<x<+\infty, t>0 \tag{6}
\end{equation*}
$$

It is similar to Gauss probability density function. The difference is that the value of Cauchy distribution is lower than the value of Gauss distribution in the vertical direction, and Cauchy distribution is closer to the horizontal axis in the horizontal direction. Cauchy mutation means that the random numbers obeying Cauchy distribution substitute $\delta_{s}$ in polynomial mutation.

## 4. Chaotic Maps

Chaotic maps generate chaotic sequences in the process of evolutionary algorithms. Ten chaotic maps including both one-dimensional maps and two-dimensional maps are introduced in this section. They will be used to improve the performance of MOP algorithms.

### 4.1. One-Dimensional Maps

(1) Circle Map. Circle map is a member of a family of dynamical systems on the circle first defined by Andrey Kolmogorov. He proposed this family as a simplified model for driven mechanical rotors specifically, a free-spinning wheel weakly coupled by a spring to a motor. The circle map equations also describe a simplified model of the phaselocked loop in electronics. The circle map [24] is given by iterating the map:

$$
\begin{equation*}
x_{k+1}=\left\{x_{k}+b-\left(\frac{a}{2 \pi}\right) \sin \left(2 \pi x_{k}\right)\right\} \bmod (1) \tag{7}
\end{equation*}
$$

with $a=0.5$ and $b=0.2$; it generates chaotic sequence in $(0,1)$.
(2) Cubic Map. Cubic map is one of the most commonly used maps in generating chaotic sequences in various applications. This map is formally defined by the following equation [25]:

$$
\begin{equation*}
x_{k+1}=\rho x_{k}\left(1-x_{k}^{2}\right), \quad x_{k} \in(0,1) . \tag{8}
\end{equation*}
$$

Cubic map generates chaotic sequences in $(0,1)$ with $\rho=$ 2.59 .
(3) Gauss Map. Gauss map is also one of the well-known and commonly employed maps in generating chaotic sequences [26] as follows:

$$
x_{k+1}= \begin{cases}0, & x_{k}=0  \tag{9}\\ \frac{1}{x_{k}} \bmod (1), & \text { otherwise }\end{cases}
$$

This map also generates chaotic sequences in $(0,1)$.
(4) ICMIC Map. The iterative chaotic map with infinite collapses (ICMIC) [27] is defined by the following equation:

$$
\begin{equation*}
x_{k+1}=\sin \left(\frac{a}{x_{k}}\right), \quad a \in(0, \infty), x_{k} \in(-1,1) \tag{10}
\end{equation*}
$$

The parameter " $a$ " is an adjustable parameter. This paper chooses $a=2$. Because the range of $x_{k}$ is not $(0,1)$, the chaotic sequences need to be transformed to change the range.
(5) Logistic Map. As a well-known chaotic map, logistic map is one of the simplest maps and was introduced by May in 2004 [28]. It is often cited as an example of how complex behavior can arise from a very simple nonlinear dynamical equation. Logistic map generates chaotic sequences in $(0,1)$. This map is formally defined by the following equation:

$$
\begin{equation*}
x_{k+1}=a x_{k}\left(1-x_{k}\right) \tag{11}
\end{equation*}
$$

Parameter $a$ is set to 4 in the simulation.
(6) Sinusoidal Iterator. The sinusoidal iterator [29] is formally defined by the following equation:

$$
\begin{equation*}
x_{k+1}=a x_{k}^{2} \sin \left(\pi x_{k}\right), \quad x_{k} \in(0,1) \tag{12}
\end{equation*}
$$

In this paper the simplified equation is used in the following iteration:

$$
\begin{equation*}
x_{k+1}=\sin \left(\pi x_{k}\right), \quad x_{k} \in(0,1) \tag{13}
\end{equation*}
$$

(7) Tent Map. Tent chaotic map is very similar to the logistic map, which displays specific chaotic effects [30]. This map is defined by the following equation:

$$
x_{k+1}= \begin{cases}2 x_{k}, & x_{k}<0.5  \tag{14}\\ 2\left(1-x_{k}\right), & x_{k} \geq 0.5\end{cases}
$$

where $x_{k}$ is ranging from 0 to 1 .
Tent map generates chaotic sequences in $(0,1)$.

### 4.2. Two-Dimensional Maps

(1) Baker's Map. The Baker map [31] is described by the following formulas:

$$
B(x, y)= \begin{cases}(2 x, 2 y), & \text { for } 0 \leq x<0.5  \tag{15}\\ \left(2-2 x, 1-\frac{y}{2}\right), & \text { for } 0.5 \leq x<1\end{cases}
$$

In the following simulations, one dimension of Baker's map, which is similar to tent map, is adopted. The equation is defined by

$$
x_{k+1}= \begin{cases}2 x_{k}, & \text { for } 0 \leq x_{k}<0.5  \tag{16}\\ 2-2 x_{k}, & \text { for } 0.5 \leq x_{k}<1\end{cases}
$$

This map generates chaotic sequences in $(0,1)$.
(2) Arnold's Cat Map. Arnold's cat map is named after Vladimir Arnold, who demonstrated its effects in the 1960s using an image of a cat. It is represented by [32]

$$
\begin{align*}
& x_{k+1}=x_{k}+y_{k} \bmod (1)  \tag{17}\\
& y_{k+1}=x_{k}+2 y_{k} \bmod (1)
\end{align*}
$$

It is obvious that the sequences $x_{k} \in(0,1)$ and $y_{k} \in(0,1)$.
(3) Zaslavskii Map. Zaslavskii map [33] is an interesting dynamic system with chaotic behavior. The discretized equation is given by

$$
\begin{gather*}
x_{k+1}=\left(x_{k}+v+a y_{k+1}\right) \bmod (1), \\
y_{k+1}=\cos \left(2 \pi x_{k}\right)+e^{-r} y_{k} \tag{18}
\end{gather*}
$$

The Zaslavskii map shows a strange attractor with the largest Lyapunov exponent for $v=400, r=3$, and $a=12.6695$. In this case, it can be calculated that $y_{k+1} \in$ [ $-1.0512,1.0512$ ]. Only one dimension is chosen in the following simulation. Since the scale of $y_{k+1}$ is not $[0,1]$, the chaotic sequences generated need scale transformation.

## 5. Chaotic Properties of Sequences Generated by Scale Transformation

As mentioned in the previous sections, the scale of sequences generated by chaotic maps is not always fit for the problems to be solved. Some sequences have to change their scale, and some sequences are generated by one dimension of a twodimension chaotic map. Hence, it is necessary to demonstrate the chaotic properties of sequences after these changes.

Detecting the presence of chaos in a dynamical system is usually solved by measuring the largest Lyapunov exponent which describes quantitatively the speed of index divergence or convergence between the adjacent phase space orbits. A positive largest Lyapunov exponent indicates chaos. Since the chaotic sequences adopted in this paper are discrete, the Lyapunov exponent of discrete series can be calculated by small data sets arithmetic [34]. This method makes full use of all the data, obtains higher accuracy, and has stronger robustness for the amount of data, the embedding dimension, and the time delay.
5.1. Small Data Sets Arithmetic. The reconstructed trajectory, $X$, can be expressed as a matrix where each row is a phasespace vector; that is,

$$
\begin{equation*}
X=\left(X_{1}, X_{2}, \ldots, X_{M}\right)^{T} \tag{19}
\end{equation*}
$$

where $X_{i}$ is the state of the system at discrete time $i$. For an $N$-point time series, $\left\{x_{1}, x_{2}, \ldots, x_{N}\right\}$, each $X_{i}$ is given by

$$
\begin{equation*}
X_{i}=\left(x_{i}, x_{i+J}, \ldots, x_{i+(m-1) J}\right), \tag{20}
\end{equation*}
$$

where $J$ is the lag or reconstruction delay, and $m$ is the embedding dimension. Thus, $X$ is an $M \times m$ matrix, and the constants $m, M, J$, and $N$ are related as

$$
\begin{equation*}
M=N-(m-1) J . \tag{21}
\end{equation*}
$$

After reconstructing the dynamics, the algorithm locates the nearest neighbor of each point on the trajectory. The nearest neighbor $X_{\hat{j}}$, where $\widehat{j} \in\{1,2, \ldots M\}$, is found by searching for the point that minimizes the distance to the particular reference point $X_{j}$. This is expressed as

$$
\begin{equation*}
d_{j}(0)=\min _{X_{\hat{j}}}\left\|X_{j}-X_{\hat{j}}\right\|, \tag{22}
\end{equation*}
$$

where $d_{j}(0)$ is the initial distance from the $j$ th point to its nearest neighbor, and |||| denotes the Euclidean norm. We impose an additional constraint that the nearest neighbors have a temporal separation greater than the mean period of the time series:

$$
\begin{equation*}
|j-j|>p, \tag{23}
\end{equation*}
$$

where $p$ is the mean period of time series. $p$ can be estimated by the reciprocal of the mean frequency of the power spectrum. This allows us to consider each pair of neighbors as nearby initial conditions for different trajectories. The largest Lyapunov exponent is estimated as the mean rate of separation of the nearest neighbors.

For each reference point $X_{j}, d_{j}(i)$ is the distance between the $j$ th pair of nearest neighbors after $i$ discrete time:

$$
\begin{equation*}
d_{j}(i)=\left\|X_{j+i}-X_{\hat{j}+i}\right\|, \quad i=1,2, \ldots, \min (M-j, M-\widehat{j}) . \tag{24}
\end{equation*}
$$

Assume that reference point $X_{j}$ and its nearest neighbor $X_{\hat{j}}$ have index divergence rate $\lambda_{1}$; then

$$
\begin{equation*}
d_{j}(i)=C_{j} e^{\lambda_{1}(i \cdot \Delta t)}, \quad C_{j}=d_{j}(0) \tag{25}
\end{equation*}
$$

where $C_{j}$ is the initial separation. By taking the logarithm of both sides of (25) we get

$$
\begin{equation*}
\ln d_{j}(i) \approx \ln C_{j}+\lambda_{1}(i \cdot \Delta t) . \tag{26}
\end{equation*}
$$

Equation (26) represents a set of approximately parallel lines (for $j=1,2, \ldots, M$ ), each with a slope $s$ roughly proportional to $\lambda_{1}$. The largest Lyapunov exponent is easily and accurately calculated using a least square fit to the "average" line defined by

$$
\begin{equation*}
y(i)=\frac{1}{\Delta t}\left\langle\ln d_{j}(i)\right\rangle, \tag{27}
\end{equation*}
$$

where $\rangle$ denotes the average over all values of $j$. So

$$
\begin{equation*}
y(i)=\frac{1}{q \Delta t} \sum_{j=1}^{q} \ln d_{j}(i) \tag{28}
\end{equation*}
$$

where $q$ is the number of $d_{j}(i)$ with $d_{j}(i) \neq 0$.
Choose a linear area of the curve $y(i) \sim i$, and apply the least square method to get the regression straight line. Then the slope of the regression straight line is the largest Lyapunov exponent $\lambda_{1}$.
5.2. The Lyapunov Exponent of Sequences. In the calculation process, the embedding dimension $m$ is calculated through the method of false nearest neighbors (FNN). For the time delay $J$, a good approximation of $J$ is equal to the number lagging where the autocorrelation function drops to $1-1 / e$ of its initial value.

Since different test problems have different ranges, chaotic sequences need to be changed to different scales. Two kinds of sequences used in experiments need to be investigated: sequences with scales changed and sequences generated by one dimension of a two-dimension chaotic map.
5.2.1. Sequences with Scales Changed. Since the sequence $x_{1}$ to $x_{100}$ generated by ICMIC is not in $(0,1)$, the new sequence $y_{1}$ to $y_{100}$ has to be generated by the following function:

$$
\begin{equation*}
y_{i}=\frac{1}{2}\left(x_{i}+1\right), \quad i \in[1,100] . \tag{29}
\end{equation*}
$$

The sequence $y_{1}$ to $y_{100}$ is in the range of $(0,1)$. The Lyapunov exponent of the new sequence is calculated through small data sets arithmetic. The average Lyapunov exponent of 10 runs is 0.0744 . Since it is a positive number, the new sequence $y_{1}$ to $y_{100}$ conforms to the chaotic nature.
5.2.2. Sequences Generated by One Dimension of a TwoDimension Chaotic Map. For the Zaslavskii map, one dimension $y_{k}$ is chosen in the following simulation. The sequence $y_{1}$ to $y_{100}$ is generated by 100 iterations through Zaslavskii map. The new sequence $z_{1}$ to $z_{100}$ is generated by the following function:

$$
\begin{equation*}
z_{i}=\frac{\left(y_{i}+1.0513\right)}{2.1026}, \quad i \in[1,100] \tag{30}
\end{equation*}
$$

Then the sequence $z_{1}$ to $z_{100}$ is in $(0,1)$. By a similar processing with ICMIC, the average Lyapunov exponent is 0.00194 . Then the new sequence $z_{1}$ to $z_{100}$ conforms to the chaotic nature.

## 6. Test Problem and Performance Measures

6.1. Test Problems. Two-objective optimization problems are chosen to test and measure the performance improvement of the evolutionary algorithms using chaotic maps in three phases. We use well-defined benchmark functions as objective functions. Their properties are shown in Table 1.
6.2. Performance Measures. Two criteria are used to evaluate the performance of multiobjective optimization: (1) convergence to the Pareto-optimal set and (2) maintenance of diversity in solutions of the Pareto-optimal set [7]. Two metrics are adopted to evaluate the effects of the combinations of phases and chaotic maps.

The first metric $\gamma$ measures the extent of convergence to a known set of Pareto-optimal solutions. It is defined as

$$
\begin{equation*}
\gamma=\frac{1}{N} \sum_{i=1}^{N} d_{i}, \tag{31}
\end{equation*}
$$

Table 1: Test problems.

| Problem | $n$ | Variable bounds | Objective functions | Optimal solutions |
| :---: | :---: | :---: | :---: | :---: |
| ZDT1 | 30 | $[0,1]$ | $\begin{gathered} f_{1}(x)=x_{1} \\ f_{2}(x)=g(x)\left[1-\sqrt{x_{1} / g(x)}\right] \\ g(x)=1+\left(9\left(\sum_{i=2}^{n} x_{i}\right) /(n-1)\right) \end{gathered}$ | $\begin{gathered} x_{1} \in[0,1] \\ x_{i}=0, \\ i=2, \ldots, n \\ \hline \end{gathered}$ |
| ZDT2 | 30 | $[0,1]$ | $\begin{gathered} f_{1}(x)=x_{1} \\ f_{2}(x)=g(x)\left[1-\left(x_{1} / g(x)\right)^{2}\right] \\ g(x)=1+\left(9\left(\sum_{i=2}^{n} x_{i}\right) /(n-1)\right) \end{gathered}$ | $\begin{gathered} x_{1} \in[0,1] \\ x_{i}=0, \\ i=2, \ldots, n \end{gathered}$ |
| ZDT3 | 30 | $[0,1]$ | $\begin{gathered} f_{1}(x)=x_{1} \\ f_{2}(x)=g(x)\left[1-\sqrt{x_{1} / g(x)}-\left(x_{1} / g(x)\right) \sin \left(10 \pi x_{1}\right)\right] \\ g(x)=1+\left(9\left(\sum_{i=2}^{n} x_{i}\right) /(n-1)\right) \end{gathered}$ | $\begin{gathered} x_{1} \in[0,1] \\ x_{i}=0, \\ i=2, \ldots, n \end{gathered}$ |
| ZDT4 | 10 | $\begin{gathered} x_{1} \in[0,1] \\ x_{i} \in[-5,5], \\ i=2, \ldots, n \end{gathered}$ | $\begin{gathered} f_{1}(x)=x_{1} \\ f_{2}(x)=g(x)\left[1-\sqrt{x_{1} / g(x)}\right] \\ g(x)=1+\left(10(n-1)+\sum_{i=2}^{n}\left[x_{i}^{2}-10 \cos \left(4 \pi x_{i}\right)\right]\right) \end{gathered}$ | $\begin{gathered} x_{1} \in[0,1] \\ x_{i}=0, \\ i=2, \ldots, n \end{gathered}$ |
| ZDT6 | 10 | $[0,1]$ | $\begin{gathered} f_{1}(x)=1-\exp \left(-4 x_{1}\right) \sin ^{6}\left(6 \pi x_{1}\right) \\ f_{2}(x)=g(x)\left[1-\left(f_{1}(x) / g(x)\right)^{2}\right] \\ g(x)=1+\left(9\left[\left(\sum_{i=2}^{n} x_{i}\right) /(n-1)\right]^{0.25}\right) \end{gathered}$ | $\begin{gathered} x_{1} \in[0,1] \\ x_{i}=0, \\ i=2, \ldots, n \end{gathered}$ |

where $d_{i}$ is the minimum Euclidean distance of every obtained solution to the Pareto-optimal front. The smaller the value of this metric is, the nearer the convergence toward Pareto-front is.

The other metric $\Delta$ measures the extent of spread achieved among the obtained solutions. The metric $\Delta$ is defined by

$$
\begin{equation*}
\Delta=\frac{d_{f}+d_{l}+\sum_{i=1}^{N-1}\left|d_{i}-\bar{d}\right|}{d_{f}+d_{l}+(N-1) \bar{d}} . \tag{32}
\end{equation*}
$$

The parameter $d_{i}$ is the Euclidean distance between consecutive solutions in the obtained nondominated set of solutions. The parameters $d_{l}$ and $d_{f}$ are the Euclidean distances between the extreme solutions and the boundary solutions of the obtained nondominated set. The parameter $\bar{d}$ is the average of all distances $d_{i}, i=1,2, \ldots, N-1$, assuming that there are $N$ solutions on the best nondominated front.

## 7. Experiments and Results

To explore the relationship of phases and chaotic maps to solve MOPs, NSGA-II algorithm is chosen as the main framework. The ten chaotic maps mentioned in Section 4 are embedded in three different phases in the original NSGAII algorithm. Each time only one parameter is modified. For example, if initial population is generated by chaotic map, the crossover and mutation operator are not changed. Similarly, if crossover operator is modified by a chaotic map, the initial population and mutation operator are not changed. The solutions, generated by the chaos embedded NSGA-II algorithm, are evaluated by two metrics: $\gamma$ and $\Delta$. For reader's convenience, the new algorithms with different combinations of chaotic maps and phases are named as "cns_[chaotic map]_[phase]," and the results of different algorithms on test problems are named as "cns_[chaotic map]_[phase]_[test problem]." In addition, "i" represents the
phase for initial population, " $c$ " represents the phase for crossover operator, and " $m$ " represents the phase for mutation operator. For example, the results through modified initial population by logistic map solving ZDT1 problem are named as "cns_logistic_i_zdtl."

Each combination of one chaotic map and one phase needs one experiment. In this research, 10 chaotic maps with 3 different phases based on 2 metrics solving 5 test problems need 150 basic experiments and obtain 300 results. Each experiment obtains a Pareto front. The values of convergence metric $\gamma$ and the diversity metric $\Delta$ are also calculated.

In order to compare with the results of original NSGA-II algorithm, we focused on the difference of the $\gamma$ and $\Delta$ values of the original NSGA-II algorithm and the new algorithm. For example, the $\gamma$ of results of "cns_sinusoidal_i_zdt1" is named as "cns_sinusoidal_i_zdt1_gama," and the $\gamma$ of results of NSGA-II solving ZDT1 problem is named as "ns_zdt1_gama." Then the difference is named as "ns_zdt1_gama-cns_sinusoidal_i_zdt1_gama." When the processes of algorithms get to convergence, the difference is very small. The properties of convergence and diversity in the process of iterations need to be taken into account, so the $\gamma$ values of each generation in the iterations are recorded and the differences of $\gamma$ of each generation are obtained. This process also applies to $\Delta$.

Some main parameters in the process of NSGA-II algorithm are introduced in the following paragraphs. Then the results of experiments are shown and analyzed.
7.1. The Main Parameters. The main parameters in the process of NSGA-II algorithm are presented in this section. Choosing an appropriate representation of a chromosome is very important for solving problems. Real numbers are chosen to represent the genes. One chromosome represents one individual. The initial population has 100 individuals, and each chromosome has a certain number of genes which are represented by a real number. Each individual of the initial population is generated randomly with the range

Table 2: Parameters in the process of algorithms.

|  | ZDT1 | ZDT2 | ZDT3 | ZDT4 | ZDT6 |
| :--- | :---: | :---: | :---: | :---: | :---: |
| $n_{\text {iter }}$ |  |  | 250 |  |  |
| $n_{\text {pop }}$ |  |  | 100 |  |  |
| $n_{\text {var }}$ | 30 | 30 | 30 | 10 | 10 |
| $p_{c}$ |  |  | 0.9 |  |  |
| $p_{m}$ | $1 / 30$ | $1 / 30$ | $1 / 30$ | $1 / 10$ | $1 / 10$ |

based on the test problems. The iteration will not terminate until the number of iterations gets to 250 . For the process of NSGA-II algorithm, a parent population is selected by tournament selection depending on the nondominated rank and the crowed-comparison operator. Then the new population is generated by crossover and mutation operators. The crossover operation is executed with the probability of $p_{c}=$ 0.9. The probability of mutation $p_{m}$ is equal to the reciprocal of $n_{\mathrm{var}}$, which is the dimension number of a chromosome; that is, $p_{m}=1 / n_{\text {var }}$.

Those parameters are summarized in Table 2. In the table, $n_{\text {iter }}$ is the number of iterations, $n_{\text {pop }}$ is the scale of the population, $n_{\mathrm{var}}$ is the number of dimensions of a chromosome, and $p_{c}$ and $p_{m}$ are the probabilities of crossover and mutation operations.
7.2. Convergence Performance. It is known that the $\gamma$ difference is used to evaluate the performance of the chaotic maps in different phases in multiobjective evolutionary algorithms. An example is chosen for further explanation in detail. As in Figure 1, the graph shows the results of solving ZDT1 problems with Baker's map in crossover operator in NSGA-II. The differences of $\gamma$ between the experiment "cns_baker_c_zdtl" and the experiment "ns_zdt1" in the 250 iterations are given. As seen from the figure, the black line is above the red line which represents 0 , so the new algorithm "cns_bakers_c" is better than NSGA-II algorithm in solving ZDT1 problem with regard to the convergence metric.

The $\gamma$ results of all the experiments are given similar to Figure 1. Since it is difficult to show so many graphs in this paper, the results of three typical problems are chosen, that is, ZDT1, which is a simple convex problem, ZDT3, whose Pareto front is piecewise, and ZDT4, which has local optima. The graphs in Figures 2, 3, and 4 provide a comparison of the performance of solving different MOPs with chaotic maps in initial population. ZDT4 is chosen to show the performance of chaotic maps in different phases on solving the same MOP, as shown in Figures 4, 5, and 6. Each subgraph is labeled with the name of the chaotic map used.

In order to quantify the effect of chaotic maps and phases with regard to the metric $\gamma$, the average of $\gamma$ difference in 250 generations is calculated to represent the effect of the new algorithms.

Since the order of magnitude of $\gamma$ is not the same, the comparison of these $\gamma$ values is not convenient. The normalized values are obtained by dividing the $\gamma$ values by the maximum of the absolute values of the $\gamma$ based on one test problem. The results of normalization are shown in Table 3.


Figure 1: Performance of Baker's maps in crossover operator in solving ZDT1.

Table 3 can be presented in a more intuitive way. If $\gamma \geq$ 0.3 , the numerical value of $\gamma$ is replaced by " ++ ." Similarly, " + " represents $0.1 \leq \gamma<0.3$, " 0 " represents $-0.1 \leq \gamma<$ 0.1 , "-" represents $-0.3 \leq \gamma<-0.1$, and " - -" represents $\gamma<-0.3$. Therefore " ++ " means that the effect of the new algorithm with chaotic maps is much better, whereas "--" is much worse. Table 4 shows the results.

As shown in Table 4, most of the combinations of chaotic maps and phases have a positive effect on improving the performance of NSGA-II algorithm. The effect of some chaotic maps is very good, especially in some particular phases. For example, Baker's map in crossover operator, Gauss map in crossover operator and initial population, ICMIC map in initial population, sinusoidal map in initial population, tent map in crossover operation, and Zaslavskii map in initial population have very good effect.

Since ZDT4 problem has $21^{9}$ or $7.94 \times 10^{11}$ different local Pareto-optimal fronts in the search space, the solutions easily get entrapped into local optimum. As seen from Table 4, chaotic maps used for crossover and mutation operator have significant improvement on evolutionary algorithms solving ZDT4 problem; especially cat map has the best performance in ten maps. Circle map and cubic map have less contribution in solving those MOPs. The distribution of cat map is relatively uniform. It is probably the reason for the good performance in solving problems with local optima.

The original NSGA-II algorithm is not good at solving ZDT3 and ZDT6 problems, because Pareto-optimal front of ZDT3 is disconnected and solutions of ZDT6 are nonuniformly spaced. However, it can be seen in Table 4 that chaotic maps can improve NSGA-II especially in crossover operation and initial population in solving ZDT3 and ZDT6 problem.

In order to eliminate the special effect of the NSGAII algorithm, the polynomial mutation operator in NSGAII is changed by the Gauss mutation and Cauchy mutation operators. Four typical chaotic maps, which include two chaotic maps with best performance and two chaotic maps


Figure 2: Performance of chaotic maps in initial population in solving ZDT1.


FIgURe 3: Performance of chaotic maps in initial population in solving ZDT3.


Figure 4: Performance of chaotic maps in initial population in solving ZDT4.


Figure 5: Performance of chaotic maps in crossover operator in solving ZDT4.

Table 3: The normalized results of $\gamma$.

|  | ZDT1 |  |  | ZDT2 |  |  | ZDT3 |  |  | ZDT4 |  |  | ZDT6 |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | c | i | m | c | i | m | c | i | m | c | i | m | c | i | m |
| Baker | 0.617 | 0.080 | 0.049 | 0.682 | 0.149 | 0.094 | 0.668 | 0.109 | 0.003 | 0.220 | -0.040 | 0.167 | 0.567 | 0.105 | -0.090 |
| Cat | -0.060 | 0.076 | 0.101 | 0.013 | 0.084 | 0.024 | -0.007 | 0.090 | 0.174 | 0.191 | 0.109 | 0.158 | -0.028 | 0.022 | -0.096 |
| Circle | -0.142 | -0.040 | -0.016 | 0.013 | -0.121 | -0.007 | -0.153 | 0.028 | -0.016 | 0.151 | -0.069 | 0.098 | -0.176 | -0.072 | $-0.002$ |
| Cubic | -0.544 | 0.064 | -0.025 | -0.626 | -0.041 | 0.006 | $-0.334$ | -0.049 | 0.077 | 0.032 | -0.781 | 0.071 | -0.288 | 0.040 | 0.008 |
| Gauss | 0.307 | 0.513 | 0.089 | 0.306 | 0.507 | 0.070 | 0.454 | 0.585 | 0.037 | 0.159 | 0.005 | 0.191 | 0.114 | -0.010 | 0.132 |
| ICMIC | -0.415 | 0.558 | 0.132 | -0.280 | 0.609 | 0.144 | -0.295 | 0.510 | 0.004 | 0.003 | -0.380 | 0.088 | -0.162 | 0.252 | 0.163 |
| Logistic | 0.070 | 0.242 | 0.189 | 0.072 | 0.204 | -0.031 | 0.012 | 0.158 | 0.127 | 0.017 | -0.819 | 0.183 | 0.152 | 0.129 | 0.132 |
| Sinusoidal | 0.077 | 1 | 0.616 | 0.121 | 1 | 0.742 | 0.169 | 1 | 0.734 | -0.091 | -1 | -0.138 | 0.148 | 0.688 | 1 |
| Tent | 0.655 | 0.177 | 0.062 | 0.704 | 0.103 | -0.005 | 0.731 | 0.043 | -0.088 | 0.190 | -0.008 | -0.058 | 0.569 | 0.124 | -0.003 |
| Zaslavskii | -0.051 | 0.462 | 0.032 | -0.150 | 0.518 | 0.064 | -0.086 | 0.499 | 0.110 | -0.103 | -0.339 | 0.108 | -0.060 | 0.174 | 0.183 |

Table 4: The visualized results of $\gamma$.

|  | ZDT1 |  |  | ZDT2 |  |  | ZDT3 |  |  | ZDT4 |  |  | ZDT6 |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | c | i | m | c | i | m | c | i | m | c | i | m | c | i | m |
| Baker | ++ | 0 | 0 | ++ | + | 0 | ++ | + | 0 | + | 0 | $+$ | ++ | + | 0 |
| Cat | 0 | 0 | + | 0 | 0 | 0 | 0 | 0 | $+$ | + | + | $+$ | 0 | 0 | 0 |
| Circle | - | 0 | 0 | 0 | - | 0 | - | 0 | 0 | + | 0 | 0 | - | 0 | 0 |
| Cubic | - | 0 | 0 | - | 0 | 0 | - | 0 | 0 | 0 | - | 0 | - | 0 | 0 |
| Gauss | ++ | ++ | 0 | ++ | ++ | 0 | ++ | ++ | 0 | + | 0 | + | + | 0 | + |
| ICMIC | - | ++ | $+$ | - | ++ | + | - | ++ | 0 | 0 | - | 0 | - | + | + |
| Logistic | 0 | $+$ | $+$ | 0 | $+$ | 0 | 0 | $+$ | $+$ | 0 | - | $+$ | $+$ | $+$ | $+$ |
| Sinusoidal | 0 | ++ | ++ | + | ++ | ++ | $+$ | ++ | ++ | 0 | - | - | $+$ | ++ | ++ |
| Tent | ++ | $+$ | 0 | ++ | $+$ | 0 | ++ | 0 | 0 | + | 0 | 0 | ++ | + | 0 |
| Zaslavskii | 0 | ++ | 0 | - | ++ | 0 | 0 | ++ | + | - | - | $+$ | 0 | + | + |

Table 5: Results of Gauss mutation.

|  | ZDT1 |  | ZDT2 |  | ZDT3 |  |  | ZDT4 |  | ZDT6 |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | c | i | c | i | c | i | c | i | c | i |  |
| Circle | -0.4886 | -0.0170 | 1.9394 | -0.3130 | 0.0971 | -0.4370 | 185.6683 | 75.0688 | 2.8693 | -0.5529 |  |
| Cubic | -2.4674 | -0.0589 | -6.2676 | -0.0630 | -2.5351 | -1.2051 | 49.1193 | -448.985 | -2.3071 | 5.7609 |  |
| Sinusoidal | 1.0277 | 6.0982 | 0.4343 | 9.4173 | -0.0396 | 4.5810 | -106.768 | -525.536 | 5.84566 | 28.5351 |  |
| Tent | 3.5035 | 0.6784 | 5.9502 | 1.1086 | 1.8616 | 0.0623 | -272.428 | 101.4576 | 15.0061 | 11.2005 |  |

Table 6: Results of Cauchy mutation.

|  | ZDT1 |  | ZDT2 |  | ZDT3 |  | ZDT4 |  | ZDT6 |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | c | i | c | i | c | i | c | i | c | i |
| Circle | 1.3978 | 0.5012 | 1.3997 | 0.7723 | 0.7187 | 0.4043 | 135.4511 | -136.938 | 0.1105 | -2.6454 |
| Cubic | -2.6201 | -0.4932 | -4.2148 | -0.5317 | -2.4831 | -0.7487 | -77.1621 | -456.939 | -5.7560 | 5.4533 |
| Sinusoidal | 0.1470 | 4.6977 | 0.8327 | 8.2820 | 0.3179 | 4.5376 | -202.995 | -468.243 | 6.5462 | 23.8545 |
| Tent | 2.7613 | 0.3380 | 5.2687 | 0.9699 | 2.6813 | -0.1916 | -283.172 | 57.9606 | 12.7523 | 5.2806 |

with worst performance, are chosen to be used in the experiments. These chaotic maps are circle map, cubic map, sinusoidal map, and tent map. The values of $\gamma$ differences are shown in Tables 5 and 6. As seen from Tables 5 and 6, the performance of sinusoidal map and tent map is better than the performance of circle map and cubic map. Sinusoidal map in initial population is better than that in crossover operation, and tent map in crossover operation is better than that in initial population. This means the rules of combinations of
chaotic maps and phases in solving MOPs are almost the same as in the previous observations. So the rules based on the framework of NSGA-II algorithm are applicable to other MOEAs.

In general, Baker's map with a phase for crossover operator, sinusoidal map with phases for initial population and mutation operator, and tent map with a phase for crossover operator could be the best choice for improving evolutionary algorithms for MOPs without local optimum. For problems


Figure 6: Performance of chaotic maps in mutation operator in solving ZDT4.


Figure 7: Performance of chaotic maps in crossover operator in solving ZDT1 with metric $\Delta$.
Table 7: The average results of $\Delta$.

|  | ZDT1 |  |  | ZDT2 |  |  | ZDT3 |  |  | ZDT4 |  |  | ZDT6 |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | c | i | m | c | i | m | c | i | m | c | i | m | c | i | m |
| Baker | -0.0337 | -0.0026 | 0.0015 | 0.0226 | 0.0310 | 0.0024 | -0.0096 | 0.0069 | 0.0006 | -0.0603 | 0.0343 | -0.0513 | 0.0125 | 0.0024 | -0.0015 |
| Cat | 0.0005 | 0.0017 | 0.0042 | 0.0234 | -0.1021 | -0.0145 | -0.0019 | -0.0052 | 0.0006 | 0.0277 | -0.0090 | -0.0148 | -0.0001 | -0.0002 | -0.0068 |
| Circle | -0.0325 | -0.0035 | -0.0064 | -0.2019 | -0.0632 | -0.1410 | -0.0052 | -0.0065 | 0.0010 | -0.0255 | -0.0259 | -0.0136 | 0.0031 | -0.0020 | 0.0064 |
| Cubic | -0.0047 | 0.0121 | -0.0022 | -0.0439 | -0.0148 | 0.0361 | 0.0000 | -0.0006 | 0.0031 | -0.0052 | -0.0279 | 0.0238 | -0.0170 | 0.0054 | -0.0017 |
| Gauss | 0.0121 | 0.0075 | -0.0180 | -0.1075 | 0.0329 | -0.0711 | -0.0024 | -0.0032 | 0.0019 | 0.0330 | -0.0214 | 0.0155 | 0.0180 | 0.0003 | 0.0064 |
| ICMIC | 0.0081 | 0.0092 | 0.0002 | -0.0734 | -0.0850 | -0.0917 | -0.0036 | 0.0008 | -0.0054 | -0.0356 | -0.0497 | 0.0238 | 0.0123 | 0.0001 | 0.0085 |
| Logistic | -0.0118 | 0.0076 | 0.0075 | -0.1935 | 0.0239 | 0.0509 | -0.0084 | -0.0014 | 0.0080 | 0.0275 | -0.0100 | -0.0232 | -0.0069 | 0.0020 | 0.0077 |
| Sinusoidal | 0.0149 | 0.0069 | -0.0564 | -0.1750 | -0.0067 | -0.2902 | -0.0014 | 0.0029 | -0.0286 | -0.0390 | -0.0164 | -0.1945 | 0.0129 | 0.0088 | -0.0488 |
| Tent | -0.0350 | 0.0093 | 0.0016 | 0.0215 | -0.0173 | -0.0261 | -0.0059 | -0.0168 | 0.0071 | -0.0761 | -0.0139 | 0.0075 | 0.0125 | 0.0028 | -0.0002 |
| Zaslavskii | 0.02307 | 0.0056 | -0.0014 | -0.0292 | -0.0189 | -0.0013 | 0.0099 | 0.0040 | -0.0037 | 0.0201 | -0.0056 | 0.0090 | 0.0038 | 0.0027 | 0.0084 |

TABLE 8: Statistical data for combinations of chaotic maps and phases on different problems.

| Threshold | ZDT1 | ZDT2 | ZDT3 | ZDT4 | ZDT6 |
| :--- | :---: | :---: | :---: | :---: | :---: |
| 0 | 18 | 9 | 15 | 10 | 20 |
| 0.001 | 16 | 9 | 9 | 10 | 18 |
| 0.002 | 13 | 9 | 7 | 10 | 18 |
| 0.003 | 13 | 8 | 6 | 10 | 14 |
| 0.004 | 13 | 8 | 5 | 10 | 12 |
| 0.005 | 12 | 8 | 4 | 10 | 12 |
| 0.006 | 11 | 8 | 4 | 10 | 11 |



Figure 8: Performance of Baker's maps in crossover operator in solving ZDT1 with metric $\Delta$.
with local optimum, cat map has good performance on improving evolutionary algorithms.
7.3. Diversity Performance. Similar to the convergence metric $\gamma$, the differences of the diversity metric $\Delta$ between the results of the new algorithm with chaotic maps and the original NSGA-II algorithm are used to measure the performance. Figure 7 shows the performance of chaotic maps in crossover operator in solving ZDT1 with metric $\Delta$. Each subgraph shows the effect of one chaotic map. To be seen more clearly, the first subgraph in Figure 7 is shown in Figure 8.

As seen from Figures 7 and 8 , the $\Delta$ difference is not stable in 250 generations. The average values of $\Delta$ difference in 250 generations are calculated to represent the effect of the new algorithms. For brevity, the rest of results are not shown in graphs but in Table 7.

As seen from Table 7, the $\Delta$ values have little difference. We count the number of combinations of chaotic maps and phases for solving one problem in different threshold values. For example, there are 18 combinations whose values of $\Delta$ are greater than zero. Based on the number of the combinations of chaotic maps and phases in different threshold values, the values of diversity metric $\Delta$ are summarized in Table 8.

In Table 8 the rank of the number of $\Delta$ in different threshold values is ZDT1 > ZDT6 > ZDT4 > ZDT2 >

ZDT3, especially for larger threshold. ZDT1 problem, which is a convex function and has no local optima, is a relatively easy problem. Chaotic maps bring the biggest improvements on solving ZDT1. Though the solutions of ZDT6 are nonuniformly spaced, chaotic maps can find better spread of solutions. While ZDT4 problem is a complex problem and the solutions are easily trapped into local optima, chaotic maps can improve the distribution of the solutions. ZDT2 problem is a convex function, and the solutions sometimes fall into the local optimum. The effects of chaotic maps can be generalized. The Pareto front of ZDT3 problem is segmented, so the $\Delta$ value of ZDT3 is larger and the ranking of ZDT3 is lower. It is our observation that $\Delta$ is not fit for evaluating the solutions to problems which are disconnected.

Based on the diversity metric $\Delta$, chaotic maps have the best improvement on solving convex problems without local optima and have better effect on solving problems which have nonuniform solutions. For problems with local minimum, chaotic maps embedded algorithms can improve the performance with regard to metric $\Delta$.

A short summary can be given according to the above experiments. First, chaotic maps can improve the performance of MOEAs, but the results showed that no one chaotic map outperforms other maps for all of the problems. The results in this paper give some guidance on how to choose a chaotic map and a phase in MOEAs. Second, an interesting discovery is that cat map has best performance on solving problems with local optima. Uniformity of cat map may be one of the reasons for the good performance of solving ZDT4.

## 8. Conclusion

The focus of this paper is to explore the relationships of chaotic maps and phases in MOEAs in solving MOPs. The main framework of algorithms in experiments is the NSGAII algorithm. The combinations of ten chaotic maps and three phases are chosen in the experiments. Two metrics, convergence metric $\gamma$ and diversity metric $\Delta$, are used to evaluate the convergence and diversity properties of the algorithms with chaotic maps. The test problems are ZDT series which were all MOPs. The ergodicity and initial value sensitivity of chaotic maps can help evolutionary algorithms avoid solutions from falling into local optimal and get better convergence. In the experimental results, almost all chaotic maps have good effects on improving the performance of evolutionary algorithms to solve MOPs without local optima.

Cat map has best performance on solving problems with local optimum. This work gives insight on choosing chaotic maps and phases for different problems. Our future work will perform further experiments with more chaotic maps on other MOEAs and formulate the theory analysis.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Closed-Loop Estimation for Randomly Sampled Measurements in Target Tracking System 

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#### Abstract

Many tracking applications need to deal with the randomly sampled measurements, for which the traditional recursive estimation method may fail. Moreover, getting the accurate dynamic model of the target becomes more difficult. Therefore, it is necessary to update the dynamic model with the real-time information of the tracking system. This paper provides a solution for the target tracking system with randomly sampling measurement. Here, the irregular sampling interval is transformed to a time-varying parameter by calculating the matrix exponential, and the dynamic parameter is estimated by the online estimated state with YuleWalker method, which is called the closed-loop estimation. The convergence condition of the closed-loop estimation is proved. Simulations and experiments show that the closed-loop estimation method can obtain good estimation performance, even with very high irregular rate of sampling interval, and the developed model has a strong advantage for the long trajectory tracking comparing the other models.


## 1. Introduction

Target tracking is the most important preliminary step for many higher-level analysis applications. Nowadays, some new sensors have been used in the tracking systems, such as the radio frequency identification (RFID) readers. The RFID stores and retrieves data through the electromagnetic transmission to an RF compatible integrated circuit. Once the tag gets close to the readers, the distance between the readers and tags can be got and sent to the data processing center. The measurements of RFID are randomly sampled [1] because of the data-driven measurement mechanisms. Datadriven approach has been used in many applications [2], and the irregular sampling is one of the important issues in this approach.

In general, the video tracking system has to extract the visual information at each frame [3], which costs much computing amount. In [4], the target is tracked by some selected frames to reduce the calculation cost and achieve the real-time tracking, which also results in the randomly sampled tracking problem. If the output measurements are obtained at a set of irregular sampling times, the traditional
recursive estimation from $K$ to $K+1$ may fail in general [5]. Both the model and the estimation method should be reconsidered.

Reference [6] transformed the randomly sampled measurement tracking to some time-varying parameters and used the current model to describe the processing model [7-11], which assumes a priori probability density of the acceleration as Rayleigh density. Due to the randomly sampled measurement, this assumption is no longer satisfied.

Except the current model, there were several other models used in the tracking, such as constant-velocity (CV) model, constant acceleration model (CA), and Singer model (zero mean first-order Markov model) [12, 13]. The CV models [1] emphasize that the accelerations are small. In maneuvering target tracking, the inclusion of acceleration in the state vector would degrade tracking performance. The main attractive feature of this model is its simplicity. It is sometimes used in the maneuvering target tracking techniques, such as the so-called noise-level adjustment, when the maneuver is quite small or random. It is also simply referred to as the CA model or more precisely the nearly CA model. The Singer model regards the target acceleration
as a first-order semi-Markov process with zero mean, which is in essence a priori model since it does not use online information on the target maneuver. Again because of the irregular sampling time, the priori model does not meet the actual dynamic model of the target.

The approach to update the system model online has attracted great interest of the researchers. For example, the interacting multiple model (IMM) $[14,15]$ method considers the change of the system dynamics as a Markovian parameter, whose transition probability is set based on the online estimation and then fusions several models for the tracking, while IMM suffers heavy computational burden on condition that the maneuvering target has complex motion. Moreover, the complex movement can also lead to frequent switch between different models, which can cause the tracking performance to decline. Another model $[16,17]$ estimated the state of a power system, where the bus voltages are transformed to a system parameter. But the works of this closed-loop estimation have not yet been involved in the randomly sampled tracking system.

This paper will develop a joint state-and-parameter estimation method for the target tracking system with randomly sampled measurements, where the estimation problem is reformulated as two loosely coupled linear subproblems. This paper is organized as follows. Section 2 derives the system dynamic model under the random sampling time and gives the estimation method based on Kalman filter. The convergence of the algorithm is proved in Section 3. The simulations and experiments are provided in Section 4. Finally, some concluding remarks are given in Section 5.

## 2. System Model and Closed-Loop Estimation Method

We begin with the continuous dynamic model of the moving target. Let $x, \dot{x}$, and $\ddot{x}$ be the target location, velocity, and acceleration along a generic direction, and the state is expressed as $x=[x, \dot{x}, \ddot{x}]^{T}$. Assume the nonzero mean acceleration satisfies $\ddot{x}(t)=\bar{a}(t)+a(t)$, where $\bar{a}(t)$ is the mean of acceleration in the interval $[0 t]$ and $a(t)$ is a zero mean first-order stationary Markov process with variance $\delta_{a}^{2}$. We have $\dot{a}(t)=-\alpha a(t)+w(t) ; \alpha$ is maneuver frequency and $w(t)$ is zero mean processing white noise with variance $\delta_{w}^{2}=2 \alpha \delta_{a}^{2}$. The parameter $\alpha=1 / \tau$ is the reciprocal of the maneuver time constant $\tau$ and thus depends on how long the maneuver lasts. For example, for an aircraft $\tau \approx 60 \mathrm{~s}$ for a lazy turn and $\tau \approx 10-20 \mathrm{~s}$ for an evasive maneuver. The parameter $\delta_{a}^{2}=E\left[a^{2}(t)\right]$ is the "instantaneous variance" of the acceleration.

Then, we can obtain the acceleration satisfying $\dot{\ddot{x}}(t)=$ $-\alpha \ddot{x}(t)+\alpha \bar{a}(t)+w(t)$ and the following state-space representation of the continuous time model can be obtained:

$$
\begin{equation*}
\dot{x}(t)=A x(t)+U \bar{a}(t)+B w(t), \tag{1}
\end{equation*}
$$

where

$$
A=\left[\begin{array}{ccc}
0 & 1 & 0  \tag{2}\\
0 & 0 & 1 \\
0 & 0 & -\alpha
\end{array}\right], \quad U=\left[\begin{array}{l}
0 \\
0 \\
\alpha
\end{array}\right], \quad B=\left[\begin{array}{l}
0 \\
0 \\
1
\end{array}\right]
$$

and $w(t)$ is process noise with the covariance matrix given by $w(t) \sim N\left(0,2 \alpha \delta_{a}^{2}\right)$. Assume the measurement data is obtained at the sampling time $t_{i}$ and the measurement equation is as follows:

$$
\begin{equation*}
z\left(t_{i}\right)=H\left(t_{i}\right) x\left(t_{i}\right)+v\left(t_{i}\right), \quad i=0,1,2, \ldots \tag{3}
\end{equation*}
$$

where $H\left(t_{i}\right)$ is measurement matrix and $v\left(t_{i}\right)$ is measurement noise with the known variance $R$; that is, $v\left(t_{i}\right) \sim N(0, R)$.
2.1. Model Discretization. We can get the following by the differential equation (1):

$$
\begin{align*}
x(t)= & e^{A\left(t-t_{0}\right)} x\left(t_{0}\right)+\int_{t_{0}}^{t} e^{A(t-\lambda)} U \bar{a}(\lambda) d \lambda  \tag{4}\\
& +\int_{t_{0}}^{t} e^{A(t-\lambda)} B w(\lambda) d \lambda
\end{align*}
$$

We can see that for any known integration interval $\left[t_{0}, t\right]$, $x(t)$ can be gotten at any time $t$ if the initial state $x\left(t_{0}\right)$, the parameters $A, U, B, \bar{a}(t)$, and $w(t)$ in $\left[t_{0}, t\right]$ are known.

Consider the time interval from $t_{i-1}$ to $t_{i}$ and assume

$$
\begin{align*}
\bar{a}(\lambda)=\bar{a}\left(t_{i-1}\right), \quad w(\lambda) & =w\left(t_{i-1}\right)  \tag{5}\\
\lambda & \in\left[\begin{array}{ll}
t_{i-1} & t_{i}
\end{array}\right]
\end{align*}
$$

we can have

$$
\begin{align*}
& \int_{t_{i-1}}^{t_{i}} e^{A\left(t_{i}-\lambda\right)} U \bar{a}(\lambda) d \lambda=\int_{t_{i-1}}^{t_{i}} e^{A\left(t_{i}-\lambda\right)} U d \lambda \bar{a}\left(t_{i-1}\right) \\
& \int_{t_{i-1}}^{t_{i}} e^{A\left(t_{i}-\lambda\right)} B w(\lambda) d \lambda=\int_{t_{i-1}}^{t_{i}} e^{A\left(t_{i}-\lambda\right)} B d \lambda w\left(t_{i-1}\right) \tag{6}
\end{align*}
$$

Set $\mathrm{th}_{i}=t_{i}-t_{i-1}$; we have the system matrix as $A_{d}\left(t_{i-1}\right)=e^{A t \mathrm{~h}_{i}}, U_{d}\left(t_{i-1}\right)=\int_{t_{i-1}}^{t_{i}} e^{A\left(t_{i}-\lambda\right)} U d \lambda$ and the noise $w_{d}\left(t_{i-1}\right)=\int_{t_{i-1}}^{t_{i}} e^{A\left(t_{i}-\lambda\right)} B d \lambda w\left(t_{i-1}\right)$ with the covariance $Q_{d}\left(t_{i-1}\right)=E\left[w_{d}\left(t_{i-1}\right) w_{d}^{T}\left(t_{i-1}\right)\right]$.

Because the process matrix $A$ in (2) is not a full-rank matrix, we cannot calculate the matrix exponential $e^{A \text { th }_{i}}$ by the Lagrange-Hermite interpolation. Here, we use Laplace transform and have

$$
(s I-A)^{-1}=\left[\begin{array}{ccc}
s & -1 & 0  \tag{7}\\
0 & s & -1 \\
0 & 0 & s+\alpha
\end{array}\right]^{-1}=\left[\begin{array}{ccc}
\frac{1}{s} & \frac{1}{s^{2}} & \frac{1}{s^{2}(s+\alpha)} \\
0 & \frac{1}{s} & \frac{1}{s(s+\alpha)} \\
0 & 0 & \frac{1}{s+\alpha}
\end{array}\right]
$$

The matrix exponential $e^{A \mathrm{th}_{i}}$ can be gotten by the inverse Laplace transform as

$$
A_{d}\left(t_{i-1}\right)=\left[\begin{array}{ccc}
1 & \mathrm{th}_{i} & \frac{\alpha \mathrm{th}_{i}-1+e^{-\alpha \mathrm{th}}}{\alpha_{i}}  \tag{8}\\
\alpha^{2} \\
0 & 1 & \frac{1-e^{-\alpha \mathrm{th}_{i}}}{\alpha} \\
0 & 0 & e^{-\alpha \mathrm{th}_{i}}
\end{array}\right]
$$

and by the similar approach, we can get the system parameter

$$
U_{d}\left(t_{i-1}\right)=\left[\begin{array}{c}
\frac{1}{\alpha}\left(-\mathrm{th}_{i}+\frac{\alpha \cdot \mathrm{th}_{i}^{2}}{2}+\frac{1-e^{-\alpha \cdot \mathrm{th}_{i}}}{\alpha}\right)  \tag{9}\\
\mathrm{th}_{i}-\frac{1-e^{-\alpha \cdot \mathrm{th}_{i}}}{\alpha} \\
1-e^{-\alpha \cdot \mathrm{th}_{i}}
\end{array}\right]
$$

and the variance of $w_{d}\left(t_{i-1}\right)$ as

$$
\begin{align*}
Q_{d}\left(t_{i-1}\right) & =E\left[w_{d}\left(t_{i-1}\right) w_{d}^{T}\left(t_{i-1}\right)\right] \\
& =2 \alpha \delta_{\alpha}^{2}\left[\begin{array}{lll}
q_{11} & q_{12} & q_{13} \\
q_{12} & q_{22} & q_{23} \\
q_{13} & q_{23} & q_{33}
\end{array}\right] \tag{10}
\end{align*}
$$

with the parameters described as

$$
\begin{align*}
& q_{11}=\frac{1}{2 \alpha^{5}}[ 1-e^{-2 \alpha \cdot \mathrm{th}_{i}}+2 \alpha \cdot \mathrm{th}_{i} \\
&\left.+\frac{2 \alpha^{3} \mathrm{th}_{i}^{3}}{3}-2 \alpha^{2} \mathrm{th}_{i}^{2}-4 \alpha \cdot \mathrm{th}_{i} e^{-\alpha \cdot \mathrm{th}_{i}}\right] \\
& q_{12}=\frac{1}{2 \alpha^{4}}\left[e^{-2 \alpha \cdot \mathrm{th}_{i}}+1-2 e^{-\alpha \cdot \mathrm{th}_{i}}\right. \\
&\left.+2 \alpha \cdot \mathrm{th}_{i} e^{-\alpha \cdot \mathrm{th}_{i}}-2 \alpha \cdot \mathrm{th}_{i}+\alpha^{2} \mathrm{th}_{i}^{2}\right]  \tag{11}\\
& q_{13}= \frac{1}{2 \alpha^{3}}\left[1-e^{-2 \alpha \cdot \mathrm{th}_{i}}-2 \alpha \cdot \mathrm{th}_{i} e^{-\alpha \cdot \mathrm{th}_{i}}\right] \\
& q_{22}=\frac{1}{2 \alpha^{3}}\left[4 e^{-\alpha \cdot \mathrm{th}_{i}}-3-e^{-2 \alpha \cdot \mathrm{th}_{i}}+2 \alpha \cdot \mathrm{th}_{i}\right] \\
& q_{23}= \frac{1}{2 \alpha^{2}}\left[e^{-2 \alpha \cdot \mathrm{th}_{i}}+1-2 \alpha \cdot \mathrm{th}_{i}\right] \\
& q_{33}=\frac{1}{2 \alpha}\left[1-e^{-2 \alpha \cdot \mathrm{th}_{i}}\right]
\end{align*}
$$

Then, we get the discrete state-space model of the tracking system as

$$
\begin{gather*}
x\left(t_{i}\right)=A_{d}\left(t_{i-1}\right) x\left(t_{i-1}\right)+U_{d}\left(t_{i-1}\right) \bar{a}\left(t_{i-1}\right)+w_{d}\left(t_{i-1}\right) \\
z\left(t_{i}\right)=H\left(t_{i}\right) x\left(t_{i}\right)+v\left(t_{i}\right) \tag{12}
\end{gather*}
$$

where $x=[x, \dot{x}, \ddot{x}]^{T}$ is the state of the system to be estimated and whose initial mean and covariance are known as $x_{0}$ and $P_{0}, w_{d}\left(t_{i}\right)$ and $v\left(t_{i}\right)$ are white noise with zero mean and independent of the initial state $x_{0}, z\left(t_{i}\right)$ is the measurement vector, $H\left(t_{i}\right)$ is measurement matrices, and $v\left(t_{i}\right)$ is measurement noise with known variance $R$. Until now, the irregular sampling is turned to the varying-parameter system. We can see the same sampling interval is just a particular case of the random sampling problem. Therefore, the model of the randomly sampling tracking is a general one.
2.2. System Parameters Estimation. Here, we assume the maneuver frequency $\alpha$ and the variance of the acceleration $\delta_{a}^{2}$ are not constant but variable and expressed as $\alpha_{i}$ and $\delta_{a i}^{2}$. From the processing model of (12), we have the discrete time equation of the acceleration as

$$
\begin{equation*}
\ddot{x}\left(t_{i}\right)=\beta_{i} \ddot{x}\left(t_{i-1}\right)+\left(1-\beta_{i}\right) \bar{a}\left(t_{i-1}\right)+w^{a}\left(t_{i-1}\right), \tag{13}
\end{equation*}
$$

where $\beta_{i}=e^{-\alpha_{i} \mathrm{th}_{i}}$ and $w^{a}\left(t_{i-1}\right)$ is a zero mean white noise sequence with the variance

$$
\begin{equation*}
\delta_{a w i}^{2}=\delta_{a i}^{2}\left(1-\beta_{i}^{2}\right) \tag{14}
\end{equation*}
$$

$\alpha_{i}$ is the maneuver frequency at the sampling time $t_{i} \cdot \bar{a}\left(t_{i-1}\right)$ is the mean of one interval, so we have $\bar{a}\left(t_{i}\right)=\bar{a}\left(t_{i-1}\right)$. Set $a\left(t_{i}\right)=\ddot{x}\left(t_{i}\right)-\bar{a}\left(t_{i}\right)$; then we can obtain

$$
\begin{equation*}
a\left(t_{i}\right)=\beta_{i} a\left(t_{i-1}\right)+w^{a}\left(t_{i-1}\right) \tag{15}
\end{equation*}
$$

Consider the estimation of acceleration $\widehat{a}\left(t_{i}\right)$ is a random process; we have

$$
\begin{equation*}
\bar{a}\left(t_{i-1}\right)=\frac{1}{i} \sum_{i=0}^{i-1} \widehat{\ddot{x}}\left(t_{i}\right) \tag{16}
\end{equation*}
$$

where $i$ is the number of data. For a first-order stationary Markov process (15), we have the statistics relation between the autocorrelation functions $r(0), r(1)$ with the parameters $\beta_{i}$ and $\delta_{a w i}^{2}$ by the Yule-Walker method [18]

$$
\begin{gather*}
r_{i}(0)=\frac{1}{i} \sum_{i=0}^{i-1} \widehat{a}\left(t_{i}\right) \widehat{a}\left(t_{i}\right) \\
r_{i}(1)=\frac{1}{i} \sum_{i=1}^{i-1} \widehat{a}\left(t_{i}\right) \widehat{a}\left(t_{i-1}\right)  \tag{17}\\
\beta_{i}=\frac{r_{i}(1)}{r_{i}(0)} \\
\delta_{a w i}^{2}=r_{i}(0)-\beta_{i} r_{i}(1)
\end{gather*}
$$

Next, we can get $\alpha_{i}$ and $\delta_{a i}^{2}$ by $\delta_{a i}^{2}=\delta_{a w i}^{2} /\left(1-\beta_{i}^{2}\right), \alpha_{i}=$ $\ln \beta_{i} /-$ th $_{i}$, and then get the system parameters $A_{d}\left(t_{i-1}\right)$, $U_{d}\left(t_{i-1}\right)$, and $Q_{d}\left(t_{i-1}\right)$ in process function (12).
2.3. Algorithm Summary. Now we summarize the closedloop estimation algorithm for the randomly sampled measurements as follows.
(1) Initialization $(i=0)$. Consider

$$
\begin{gather*}
\hat{x}\left(t_{0} \mid t_{0}\right)=x_{0} \\
P\left(t_{0} \mid t_{0}\right)=P_{0}, \alpha_{0}, \delta_{a 0}^{2}, \bar{a}\left(t_{0}\right),  \tag{18}\\
r_{0}\left(t_{0}\right)=\ddot{x}_{0} \cdot \ddot{x}_{0}, \quad r_{0}\left(t_{1}\right)=\ddot{x}_{0} .
\end{gather*}
$$

(2) Recursion $(i:=i+1)$
(a) System update: set $\mathrm{th}_{i}=t_{i}-t_{i-1}$ and the system parameter as

$$
\left.\begin{array}{c}
\widehat{A}_{d}\left(t_{i-1}\right)=\left[\begin{array}{ccc}
1 & \mathrm{th}_{i} & \frac{\alpha_{i} \mathrm{th}_{i}-1+e^{-\alpha_{i} t \mathrm{th}_{i}}}{\alpha_{i}^{2}} \\
0 & 1 & \frac{1-e^{-\alpha_{i} \mathrm{th}_{i}}}{\alpha_{i}} \\
0 & 0 & e^{-\alpha_{i} \mathrm{th}_{i}}
\end{array}\right] \\
\widehat{U}_{d}\left(t_{i-1}\right)=\left[\begin{array}{c}
\frac{1}{\alpha_{i}}\left(-\mathrm{th}_{i}+\frac{\alpha_{i} \cdot \mathrm{th}_{i}^{2}}{2}+\frac{1-e^{-\alpha_{i} \cdot \mathrm{th}_{i}}}{\alpha_{i}}\right.
\end{array}\right)  \tag{20}\\
\mathrm{th}_{i}-\frac{1-e^{-\alpha_{i} \cdot \mathrm{th}_{i}}}{\alpha_{i}} \\
1-e^{-\alpha_{i} \cdot \mathrm{th}_{i}}
\end{array}\right]
$$

and the variance of the $w_{d}\left(t_{i-1}\right)$ as

$$
\begin{align*}
\widehat{Q}_{d}\left(t_{i-1}\right) & =E\left[w_{d}\left(t_{i-1}\right) w_{d}^{T}\left(t_{i-1}\right)\right] \\
& =2 \alpha_{i} \delta_{\alpha i}^{2}\left[\begin{array}{lll}
q_{11} & q_{12} & q_{13} \\
q_{12} & q_{22} & q_{23} \\
q_{13} & q_{23} & q_{33}
\end{array}\right] \tag{21}
\end{align*}
$$

with parameters described as

$$
\begin{aligned}
& \begin{aligned}
q_{11}=\frac{1}{2 \alpha_{i}^{5}} & {[ } \\
& 1-e^{-2 \alpha_{i} \cdot \mathrm{th}_{i}}+2 \alpha_{i} \cdot \mathrm{th}_{i} \\
& \left.+\frac{2 \alpha_{i}^{3} \mathrm{th}_{i}^{3}}{3}-2 \alpha_{i}^{2} \mathrm{th}_{i}^{2}-4 \alpha_{i} \cdot \mathrm{th}_{i} e^{-\alpha \cdot \mathrm{th}_{i}}\right] \\
q_{12}=\frac{1}{2 \alpha_{i}^{4}}[ & e^{-2 \alpha_{i} \cdot \mathrm{th}_{i}}+1-2 e^{-\alpha_{i} \cdot \mathrm{th}_{i}} \\
& \left.+2 \alpha_{i} \cdot \mathrm{th}_{i} e^{-\alpha_{i} \cdot t \mathrm{th}_{i}}-2 \alpha_{i} \cdot \mathrm{th}_{i}+\alpha_{i}^{2} \mathrm{th}_{i}^{2}\right]
\end{aligned} \\
& q_{13}=\frac{1}{2 \alpha_{i}^{3}}\left[1-e^{-2 \alpha_{i} \cdot \mathrm{th}_{i}}-2 \alpha_{i} \cdot \mathrm{th}_{i} e^{-\alpha_{i} \cdot \mathrm{th}_{i}}\right] \\
& q_{22}=\frac{1}{2 \alpha_{i}^{3}}\left[4 e^{-\alpha_{i} \cdot \mathrm{th}_{i}}-3-e^{-2 \alpha_{i} \cdot \mathrm{th}_{i}}+2 \alpha_{i} \cdot \mathrm{th}_{i}\right] \\
& q_{23}= \frac{1}{2 \alpha_{i}^{2}}\left[e^{-2 \alpha_{i} \cdot \mathrm{th}_{i}}+1-2 \alpha_{i} \cdot \mathrm{th}_{i}\right] \\
& q_{33}=\frac{1}{2 \alpha_{i}}\left[1-e^{-2 \alpha_{i} \cdot \mathrm{th}_{i}}\right] .
\end{aligned}
$$

(b) State prediction: consider

$$
\begin{align*}
& \widehat{x}\left(t_{i} \mid t_{i-1}\right) \\
& \quad=\widehat{A}_{d}\left(t_{i-1}\right) \widehat{x}\left(t_{i-1} \mid t_{i-1}\right)+\widehat{U}_{d}\left(t_{i-1}\right) \bar{a}\left(t_{i-1}\right)  \tag{23}\\
& P\left(t_{i} \mid t_{i-1}\right) \\
& \quad=\widehat{A}_{d}\left(t_{i-1}\right) P\left(t_{i-1} \mid t_{i-1}\right) \widehat{A}_{d}^{T}\left(t_{i-1}\right)+\widehat{Q}_{d}\left(t_{i-1}\right) .
\end{align*}
$$

(c) State update: consider

$$
\begin{align*}
& \hat{x}\left(t_{i} \mid t_{i}\right)  \tag{24}\\
& \quad=\widehat{x}\left(t_{i} \mid t_{i-1}\right)+K\left(t_{i}\right)\left[z\left(t_{i}\right)-H\left(t_{i}\right) \hat{x}\left(t_{i} \mid t_{i-1}\right)\right] \\
& K\left(t_{i}\right) \\
& \quad=P\left(t_{i} \mid t_{i-1}\right) H^{T}\left(t_{i}\right)  \tag{25}\\
& \quad \times\left[H\left(t_{i}\right) P\left(t_{i} \mid t_{i-1}\right) H^{T}\left(t_{i}\right)+R\left(t_{i}\right)\right]^{-1} \\
& \quad P\left(t_{i} \mid t_{i}\right)=\left[I-K\left(t_{i}\right) H\left(t_{i}\right)\right] P\left(t_{i} \mid t_{i-1}\right) \tag{26}
\end{align*}
$$

(d) Parameter adaptation: the mean of the acceleration

$$
\begin{equation*}
\bar{a}\left(t_{i-1}\right)=\frac{1}{i} \sum_{i=0}^{i-1} \widehat{\ddot{x}}\left(t_{i} \mid t_{i}\right) \tag{27}
\end{equation*}
$$

When $i \leq K_{0}$, the maneuver frequency $\alpha_{i}$ is set to $\alpha_{0}$ and the covariance of the noise $\delta_{a i}^{2}$ is gotten by the following:

$$
\delta_{\alpha i}^{2}= \begin{cases}\frac{4-\pi}{\pi}\left[a_{M}-\widehat{\ddot{x}}\left(t_{i} \mid t_{i}\right)\right]^{2} & \text { when } \widehat{\ddot{x}}\left(t_{i} \mid t_{i}\right)>0  \tag{28}\\ \frac{4-\pi}{\pi}\left[\widehat{\ddot{x}}\left(t_{i} \mid t_{i}\right)-a_{-M}\right]^{2} & \text { when } \widehat{\ddot{x}}\left(t_{i} \mid t_{i}\right)<0 \\ \text { a small positive constant } & \text { when } \widehat{\ddot{x}}\left(t_{i} \mid t_{i}\right)=0 .\end{cases}
$$

When $i>K_{0}$, the parameter is updated by the following

$$
\begin{gather*}
\widehat{a}\left(t_{i}\right)=\widehat{\hat{x}}\left(t_{i} \mid t_{i}\right)-\bar{a}\left(t_{i}\right)  \tag{29}\\
r_{i}(1)=r_{i-1}(1)+\frac{1}{i}\left[\widehat{a}\left(t_{i}\right) \widehat{a}\left(t_{i-1}\right)-r_{i-1}(1)\right]  \tag{30}\\
r_{i}(0)=r_{i-1}(0)+\frac{1}{i}\left[\widehat{a}\left(t_{i}\right) \widehat{a}\left(t_{i}\right)-r_{i-1}(0)\right]  \tag{31}\\
\beta_{i}=\frac{r_{i}(1)}{r_{i}(0)} \quad \delta_{a w i}^{2}=r_{i}(0)-\beta_{i} r_{i}(1)  \tag{32}\\
\delta_{a i}^{2}=\frac{\delta_{a w i}^{2}}{1-\beta_{i}^{2}} \quad \alpha_{i}=\frac{\ln \beta_{i}}{-\mathrm{th}_{i}} . \tag{33}
\end{gather*}
$$

The irregular sampling time $t_{i-1}, t_{i}$ and the interval th ${ }_{i}$ reflect in the time-varying parameters of the system, so we can conclude that the Kalman filter shown in (23)-(33) based on system (12) with system parameters (19)-(22) can obtain the same estimation performance as regular sampling Kalman filter.


Figure 1: The video with simple background and one target.


Figure 2: The measurement of maneuvering target got from the video.

## 3. Proof of the Convergence

Based on the closed-loop estimation algorithm (18)-(33), we can see that the parameter used to estimate state is an estimated one and similarly the estimated states to calculate parameters $\alpha_{i}$ and $\delta_{a i}^{2}$ have estimation errors too. Therefore, it is important to guarantee the convergence of the estimation of the states and parameters.

From (27), (29), and (33), we know if the estimation $\widehat{\ddot{x}}\left(t_{i} \mid\right.$ $t_{i}$ ) increased suddenly, $\widehat{a}\left(t_{i}\right)$ will increase greatly because the mean changes less than $\widehat{\ddot{x}}\left(t_{i} \mid t_{i}\right)$, and $\delta_{a w i}^{2}$ becomes large too. Then, a very large positive $\delta_{a i}^{2}$ will be obtained, and $\widehat{Q}_{d}\left(t_{i}\right)$ will also contain a large number of elements (here, we call it a big matrix). From the Riccati equation of Kalman filter

$$
\begin{align*}
& P\left(t_{i+1} \mid t_{i}\right) \\
&=\widehat{A}_{d}\left(t_{i}\right)\left\{P\left(t_{i} \mid t_{i-1}\right)-P\left(t_{i} \mid t_{i-1}\right) H^{T}\left(t_{i}\right)\right. \\
& \times\left[H\left(t_{i}\right) P\left(t_{i} \mid t_{i-1}\right) H^{T}\left(t_{i}\right)+R\left(t_{i}\right)\right]^{-1}  \tag{34}\\
&\left.\times H\left(t_{i}\right) P\left(t_{i} \mid t_{i-1}\right)\right\} \widehat{A}_{d}^{T}\left(t_{i}\right)+\widehat{Q}_{d}\left(t_{i}\right),
\end{align*}
$$

we find that $P\left(t_{i+1} \mid t_{i}\right)$ will be a big matrix if $\widehat{\mathrm{Q}}_{d}\left(t_{i}\right)$ is a big one, and $K\left(t_{i+1}\right)$ will increase greatly. As a result, the esti-


Figure 3: The real trajectory and the estimation trajectory.
mation state $\widehat{x}\left(t_{i} \mid t_{i}\right)=\widehat{x}\left(t_{i} \mid t_{i-1}\right)+K\left(t_{i}\right)\left[z\left(t_{i}\right)-H\left(t_{i}\right) \widehat{x}\left(t_{i} \mid\right.\right.$ $\left.t_{i-1}\right)$ ] will be a big matrix too. This trend results in positive feedback loops, which means $\widehat{x}\left(t_{i} \mid t_{i}\right)$ will become larger and larger, and finally, divergence. We give the following theorem to guarantee the algorithm convergence.

Theorem 1. The estimation $\widehat{x}\left(t_{i+1} \mid t_{i+1}\right)$ is bounded if the variance of the target acceleration $\delta_{a i}^{2}$ has an upper bound; that is, there is a positive $\delta_{0}^{2}$ satisfying $\delta_{a i}^{2} \leq \delta_{0}^{2}$.

Proof. We firstly consider maneuvering frequency $\alpha_{i}$. From (19), (21), and (22), we know if $\alpha_{i} \rightarrow 0$ and $\delta_{a i}^{2} \leq \delta_{0}^{2}$, the target has the constant acceleration maneuvering, and the system model is the constant acceleration model with the parameter as follows

$$
\begin{gather*}
\widehat{A}_{d}\left(t_{i-1}\right) \longrightarrow \bar{A}_{d}\left(t_{i-1}\right)=\left[\begin{array}{ccc}
1 & \mathrm{th}_{i} & \frac{\mathrm{th}_{i}^{2}}{2} \\
0 & 1 & \mathrm{th}_{i} \\
0 & 0 & 1
\end{array}\right] \\
\widehat{Q}_{d}\left(t_{i-1}\right) \longrightarrow \bar{Q}_{d}\left(t_{i-1}\right)=\delta_{\alpha i}^{2}\left[\begin{array}{ccc}
\frac{\mathrm{th}_{i}^{5}}{20} & \frac{\mathrm{th}_{i}^{4}}{8} & \frac{\mathrm{th}_{i}^{3}}{6} \\
\frac{\mathrm{th}_{i}^{4}}{8} & \frac{\mathrm{th}_{i}^{3}}{3} & \frac{\mathrm{th}_{i}^{2}}{2} \\
\frac{\mathrm{th}_{i}^{3}}{6} & \frac{\mathrm{th}_{i}^{2}}{2} & \mathrm{th}_{i}
\end{array}\right] . \tag{35}
\end{gather*}
$$

If $\alpha_{i} \rightarrow \infty$ and $\delta_{a i}^{2} \leq \delta_{0}^{2}$, we can get the system parameter matrix such as $\widehat{A}_{d}\left(t_{i-1}\right) \rightarrow \overline{\bar{A}}_{d}\left(t_{i-1}\right)=\left[\begin{array}{cccc}1 & \text { th } & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{array}\right]$ and $\widehat{Q}_{d}\left(t_{i-1}\right) \rightarrow \overline{\bar{Q}}_{d}\left(t_{i-1}\right)=\delta_{\alpha i}^{2}\left[\begin{array}{lll}0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1\end{array}\right]$. Therefore, we can see that $\widehat{A}_{d}\left(t_{i-1}\right)$ and $\widehat{\mathrm{Q}}_{d}\left(t_{i-1}\right)$ are the monotonic matrix with finite value elements.


Figure 4: The estimations of horizontal and longitudinal axis.


Figure 5: The location estimation errors.

Then, we consider the solution of Riccati equation (34) on the condition that the system parameter matrix has errors, such as $\widehat{A}_{d}=\widetilde{A}_{d}+\Delta A_{d}$ and $\widehat{\mathrm{Q}}_{d}=\widetilde{\mathrm{Q}}_{d}+\Delta \mathrm{Q}_{d}$, where $\widetilde{A}_{d}$ and $\widetilde{\mathrm{Q}}_{d}$ are the actual system parameters and $\Delta A_{d}$ and $\Delta Q_{d}$ are the errors of the system parameter. Unlike the research about the uncertainty system, here we do not know the actual system matrices $\widetilde{A}_{d}$ and $\widetilde{Q}_{d}$, but we can know the upper bound of the system parameters $\widehat{A}_{d}\left(t_{i-1}\right)$ and $\widehat{Q}_{d}\left(t_{i-1}\right)$, when $\delta_{a i}^{2} \leq \delta_{0}^{2}$, such as

$$
\begin{gather*}
A_{\text {upper }}\left(t_{i-1}\right)=\bar{A}_{d}\left(t_{i-1}\right)=\left[\begin{array}{ccc}
1 & \mathrm{th}_{i} & \frac{\mathrm{th}_{i}^{2}}{2} \\
0 & 1 & \mathrm{th}_{i} \\
0 & 0 & 1
\end{array}\right], \\
Q_{\text {upper }}\left(t_{i-1}\right)=\delta_{0}^{2}\left[\begin{array}{ccc}
\frac{\mathrm{th}_{i}^{5}}{20} & \frac{\mathrm{th}_{i}^{4}}{8} & \frac{\mathrm{th}_{i}^{3}}{6} \\
\frac{\mathrm{th}_{i}^{4}}{8} & \frac{\mathrm{th}_{i}^{3}}{3} & \frac{\mathrm{th}_{i}^{2}}{2} \\
\frac{\mathrm{th}_{i}^{3}}{6} & \frac{\mathrm{th}_{i}^{2}}{2} & \mathrm{th}_{i}
\end{array}\right] . \tag{36}
\end{gather*}
$$

The perturbed discrete algebraic Riccati equation is as follows:

$$
\begin{align*}
P= & \widehat{A}_{d}\left(t_{i}\right) P \widehat{A}_{d}^{T}\left(t_{i}\right) \\
& -\widehat{A}_{d}\left(t_{i}\right) P H^{T}\left(t_{i}\right)\left[H\left(t_{i}\right) P H^{T}\left(t_{i}\right)+R\left(t_{i}\right)\right]^{-1}  \tag{37}\\
& \times H\left(t_{i}\right) P \widehat{A}_{d}^{T}\left(t_{i}\right)+\widehat{Q}_{d}\left(t_{i}\right) .
\end{align*}
$$

We know that (37) is equal to

$$
\begin{align*}
P= & \widehat{A}_{d}\left(t_{i}\right)\left(P^{-1}+H^{T}\left(t_{i}\right) R\left(t_{i}\right) H\left(t_{i}\right)\right)^{-1}  \tag{38}\\
& \times \widehat{A}_{d}^{T}\left(t_{i}\right)+\widehat{Q}_{d}\left(t_{i}\right) .
\end{align*}
$$

Then, for any vector $s$, we have

$$
\begin{align*}
& s^{T} P s \\
& \qquad \begin{aligned}
= & s^{T}[ \\
& \widehat{A}_{d}\left(t_{i}\right)\left(P^{-1}+H^{T}\left(t_{i}\right) R\left(t_{i}\right) H\left(t_{i}\right)\right)^{-1} \\
& \left.\times \widehat{A}_{d}^{T}\left(t_{i}\right)+\widehat{Q}_{d}\left(t_{i}\right)\right] s \\
\leq & \lambda_{1}\left(P^{-1}+H^{T}\left(t_{i}\right) R\left(t_{i}\right) H\left(t_{i}\right)\right)^{-1} s^{T} \widehat{A}_{d}\left(t_{i}\right) \widehat{A}_{d}^{T}\left(t_{i}\right) s \\
& +s^{T} \widehat{Q}_{d}\left(t_{i}\right) s,
\end{aligned}
\end{align*}
$$



Figure 6: The relation between $\mathrm{RMSE}_{2 \mathrm{D}}$ and IRrate.
where $\lambda_{1}(X)$ is the maximum eigenvalue. By the relation of vector eigenvalue $\lambda_{m}\left(X^{-1}\right)=\lambda_{M-m+1}^{-1}(X)$, where $\lambda_{1}(X) \geq$ $\lambda_{2}(X) \geq \cdots \geq \lambda_{M}(X)$, we have

$$
\begin{equation*}
s^{T} P s \leq \frac{s^{T} \widehat{A}_{d}\left(t_{i}\right) \widehat{A}_{d}^{T}\left(t_{i}\right) s}{\lambda_{M}\left(P^{-1}+H^{T}\left(t_{i}\right) R\left(t_{i}\right) H\left(t_{i}\right)\right)}+s^{T} \widehat{Q}_{d}\left(t_{i}\right) s \tag{40}
\end{equation*}
$$

That is,

$$
\begin{equation*}
P \leq \frac{\widehat{A}_{d}\left(t_{i}\right) \widehat{A}_{d}^{T}\left(t_{i}\right)}{\lambda_{M}\left(P^{-1}+H^{T}\left(t_{i}\right) R\left(t_{i}\right) H\left(t_{i}\right)\right)}+\widehat{Q}_{d}\left(t_{i}\right) \tag{41}
\end{equation*}
$$

We have

$$
\begin{align*}
& \lambda_{M}\left(P^{-1}+H^{T}\left(t_{i}\right) R\left(t_{i}\right) H\left(t_{i}\right)\right) \\
& \quad \geq \frac{1}{\lambda_{1}(P)}+\lambda_{M}\left(H^{T}\left(t_{i}\right) R\left(t_{i}\right) H\left(t_{i}\right)\right) . \tag{42}
\end{align*}
$$

Then, by (41) and (42), we have

$$
\begin{align*}
P & \leq \frac{\widehat{A}_{d}\left(t_{i}\right) \widehat{A}_{d}^{T}\left(t_{i}\right)}{\left(1 / \lambda_{1}(P)\right)+\lambda_{M}\left(H^{T}\left(t_{i}\right) R\left(t_{i}\right) H\left(t_{i}\right)\right)}+\widehat{Q}_{d}\left(t_{i}\right) \\
& =\frac{\lambda_{1}(P) \widehat{A}_{d}\left(t_{i}\right) \widehat{A}_{d}^{T}\left(t_{i}\right)}{1+\lambda_{1}(P) \lambda_{M}\left(H^{T}\left(t_{i}\right) R\left(t_{i}\right) H\left(t_{i}\right)\right)}+\widehat{Q}_{d}\left(t_{i}\right) \\
& \leq \frac{\lambda_{1}(P) A_{\text {upper }}\left(t_{i}\right) A_{\mathrm{upper}}^{T}\left(t_{i}\right)}{1+\lambda_{1}(P) \lambda_{M}\left(H^{T}\left(t_{i}\right) R\left(t_{i}\right) H\left(t_{i}\right)\right)}+Q_{\mathrm{upper}}\left(t_{i}\right) . \tag{43}
\end{align*}
$$

Next, by the relation of Hermite matrix and its eigenvalue, we have

$$
\begin{align*}
\lambda_{1}(P) \leq & \frac{\lambda_{1}(P) \lambda_{1}\left(A_{\text {upper }}\left(t_{i}\right) A_{\text {upper }}^{T}\left(t_{i}\right)\right)}{1+\lambda_{1}(P) \lambda_{M}\left(H^{T}\left(t_{i}\right) R\left(t_{i}\right) H\left(t_{i}\right)\right)}  \tag{44}\\
& +\lambda_{1}\left(Q_{\text {upper }}\left(t_{i}\right)\right)
\end{align*}
$$

Then, we have

$$
\begin{align*}
& \lambda_{1}^{2}(P) \lambda_{M}\left(H^{T}\left(t_{i}\right) R\left(t_{i}\right) H\left(t_{i}\right)\right) \\
& +\lambda_{1}(P)\left[1-\lambda_{1}\left(A_{\mathrm{upper}}\left(t_{i}\right) A_{\mathrm{upper}}^{T}\left(t_{i}\right)\right)\right. \\
& \left.\quad-\lambda_{M}\left(H^{T}\left(t_{i}\right) R\left(t_{i}\right) H\left(t_{i}\right)\right) \lambda_{1}\left(Q_{\mathrm{upper}}\left(t_{i}\right)\right)\right] \\
& \quad-\lambda_{1}\left(Q_{\mathrm{upper}}\left(t_{i}\right)\right) \leq 0 \tag{45}
\end{align*}
$$

Assume that $\lambda_{M}\left(H^{T}\left(t_{i}\right) R\left(t_{i}\right) H\left(t_{i}\right)\right)>0$ and set

$$
\begin{align*}
& 1-\lambda_{1}\left(A_{\text {upper }}\left(t_{i}\right) A_{\text {upper }}^{T}\left(t_{i}\right)\right) \\
& \quad-\lambda_{M}\left(H^{T}\left(t_{i}\right) R\left(t_{i}\right) H\left(t_{i}\right)\right) \lambda_{1}\left(Q_{\text {upper }}\left(t_{i}\right)\right)=p_{1} \tag{46}
\end{align*}
$$

We have the following solution of (45):
$\lambda_{1}(P)$

$$
\begin{equation*}
\leq \frac{-p_{1}+\sqrt{p_{1}^{2}+4 \lambda_{M}\left(H^{T}\left(t_{i}\right) R\left(t_{i}\right) H\left(t_{i}\right)\right) \lambda_{1}\left(Q_{\mathrm{upper}}\left(t_{i}\right)\right)}}{2 \lambda_{M}\left(H^{T}\left(t_{i}\right) R\left(t_{i}\right) H\left(t_{i}\right)\right)} \tag{47}
\end{equation*}
$$

Therefore, we can conclude that the maximum eigenvalue of estimation covariance $P$ has the upper bound shown as (47) if $\delta_{a i}^{2} \leq \delta_{0}^{2}$.

If one step predictive covariance is bounded, that is, $\left|P\left(t_{i} \mid t_{i-1}\right)\right| \leq P_{0}$, then we know $P\left(t_{i+1} \mid t_{i}\right)$ must be bounded by (47) with the fact that $\left|\widehat{Q}_{d}\left(t_{i}\right)\right| \leq \mathrm{Q}_{0}$. And based on (25), we know $K\left(t_{i+1}\right)$ must be a bounded matrix and $\widehat{x}\left(t_{i+1} \mid t_{i+1}\right)$ must be bounded too.

## 4. Simulations and Experiments

4.1. The Estimation by Different Extraction Rate and Irregular Rate. The method here is applied to a two-dimensional


- The real trajectory - $\theta$ - By current model
$-\nabla-$ By CV model -+- By IMM
$-\Delta$ - By CA model - - - By adaptive model
-*- By Singer model
(a)

(c)


| ○ The real trajectory | $-ө-$ By current model |
| :--- | :--- |
| $-\nabla-$ By CV model | -+- By IMM |
| $-\triangleq-$ By CA model | $-\boxminus-$ By adaptive model |
| --- By Singer model |  |

(e)


- The real trajectory - - - By current model
$-\nabla-$ By CV model -+- By IMM
$-\Delta-$ By CA model - - - By adaptive model
-*- By Singer model
(b)

- The real trajectory - - - By current model

| --- | By CV model |
| :--- | :--- |
| $-\Delta-$ | -+- By IMM |
| $-\Delta A$ model | $-\boxminus-$ |
| $-*$ By adaptive model |  |

(d)


Figure 7: The tracking results for videos.


| $\rightarrow$ By CV model | $\bullet$ By current model II |
| :--- | :--- |
| $\rightarrow$ By CA model | $\bullet$ By current model III |
| $\rightarrow$ By Singer model | $\bullet$ By IMM |
| $\rightarrow$ By current model I | - By adaptive model |

Figure 8: $\mathrm{RMSE}_{2 \mathrm{D}}$ under different EXrate and IRrate.


Figure 9: The tracking results in number $1,27,40,65,74,97,128,129$, $158,181,189$, and 226 frames.
planar video tracking. Here, as a tracking problem, we just use the simple background and one target. The video gotten by the Image Capture Test Bed is shown in Figure 1.

We control the car maneuvering on the test bed and catch the images of target movement by a stationary camera. For every image of the video, the target is extracted based on the color and then we get the measurement data of maneuvering target on the Image Capture Test Bed like Figure 2.

We know that the camera catches the image under the same interval, and that will produce large amounts of image data. If we can use some of images in the video for tracking, the image storage and computation cost will greatly reduce. But "using some of images" means that the measurements
no longer have the same sampling interval. Here, define the Extraction Rate as

## EXrate

$$
\begin{equation*}
=\frac{\text { extracted number of images from the video }}{\text { total number of images in the video }} \times 100 \% \tag{48}
\end{equation*}
$$

to describe the image compression rate. And define the Irregular Rate to measure the sampling interval as

$$
\begin{equation*}
\text { IRrate }=\frac{\sum_{i=1}^{N}\left|\mathrm{th}_{i}-\sum_{i=1}^{N} \mathrm{th}_{i}\right|}{N} \tag{49}
\end{equation*}
$$

The state for the target in the 2D space is $x(k)=$ $\left[\begin{array}{lllll}x(k) & \dot{x}(k) & \ddot{x}(k) & y(k) \quad \dot{y}(k) & \ddot{y}(k)\end{array}\right]$. The initial state estimate $x_{0}$ and covariance $P_{0}$ are assumed to be $x_{0}=$ $\left[\begin{array}{cccccc}x(0) & 0 & 0 & y(0) & 0 & 0\end{array}\right]^{T}$ and $P_{0}=\operatorname{diag}(10,10,10,10,10,10)$.

We extract 243 images from a video with 491 images where EXrate $=49.49 \%$ and IRrate $=0.1043$ and by the algorithm developed with the initial parameters $\alpha_{0}=1 / 20$, $\delta_{a 0}^{2}=10, \bar{a}_{0}=0, \alpha_{M}=3, K_{0}=3$, we get the estimation of trajectory with estimation covariance 10.0881 along the horizontal axis and 8.1660 along the vertical axis, shown in Figure 3. The estimation trajectories of horizontal and longitudinal axis is shown in Figure 4 and the estimation error are shown in Figure 5.

To illustrate how the irregular rate affects estimation performance, the algorithm is used to estimate the target

Table 1: The different irregular rate for 10 cases.

| Case 1 | Case 2 | Case 3 | Case 4 | Case 5 | Case 6 | Case 7 | Case 8 | Case 9 | Case 10 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0.01 | 0.06 | 0.08 | 0.09 | 0.10 | 0.13 | 0.14 | 0.14 | 0.18 | 0.19 |

trajectory under different Irregular Rate (shown in Table 1) with the same Extraction Rate, EXrate $=49.86 \%$. The RMSE position is defined as $\mathrm{RMSE}_{2 \mathrm{D}}=\sqrt{\mathrm{RMSE}_{H}^{2}+\mathrm{RMSE}_{L}^{2}}$, where $\mathrm{RMSE}_{H}$ and $\mathrm{RMSE}_{L}$ are the root-mean square errors (RMSE) of position for horizontal and longitudinal axis, respectively. The relation between $\mathrm{RMSE}_{2 \mathrm{D}}$ and IRrate is shown in Figure 6. We can see that the Irregular Rate affects the estimation performance very little. The Irregular Rate changes 21 times almost from 0.0088 in Case 1 to 0.1928 in Case 10, but RMSE $_{2 \text { D }}$ is about 14 for all IRrate. We can conclude that IRrate does not affect the tracking performance, when with the same EXrate.
4.2. The Performance with Different Models. Next, we compare the model developed here with other dynamics model, such as CV model [12], CA model [12], Singer model [13], current model [8], and IMM [14]. We set the process noise covariance as $Q=1$ for the CV and CA model and $\sigma_{w}^{2}=1$ and $\alpha=1 / 20$ for Singer model. Because the current model is very sensitive to the priori parameters, we give several system parameters such as $\alpha=1 / 30, \alpha_{\max }=3$ (current model I), $\alpha=$ $1 / 20, \alpha_{\max }=30$ (current model II), and $\alpha=1 / 20, \alpha_{\max }=3$ (current model III). After 100 Monte Carlo simulation runs, RMSE $_{2 \mathrm{D}}$ are calculated. For different trajectory with different EXrate and IRrate, the estimation results are shown in Figures 7(a)-7(f), where in order to show clearly, we use the black "O" to describe the actual trajectory at the sampling time in Figures 7(e) and 7(f).

Table 2 and Figure 8 show RMSE $_{2 \mathrm{D}}$ under the different IRrate and EXrate. We can see that the model here can get the better estimation performance than CV, CA, Singer model, current model, and IMM for almost all EXrate and IRrate. We also note that the current model needs the right parameter, or else the performance will become worse.

We note that in Figure 7(f), the tracking error of the developed model is larger than current models II, III, and IMM, even CA. We find that there is a big estimation error at 5th second. The reason is that there are not enough data gotten to update the parameter at $K_{0}=4$. Therefore, the estimation error is bigger. But we also note that the estimation error declined quickly, so the developed model has a strong advantage for the long trajectory tracking comparing the other models.

Another fact we also noticed is that though IRrate almost does not affect the tracking performance, it is obvious that low EXrate can decline the tracking performance. This is because the lower EXrate means less measured data gotten and less useful information that can be provided; therefore, the estimate is more inaccurate.

As to the sampling interval th ${ }_{i}$, the lower EXrate means larger th ${ }_{i}$. If the sampling interval th $h_{i}$ is large enough to break

Shannon Sampling Theorem, the estimation performance will decline.
4.3. The Estimation of Video Target. At last, we use the developed method to track a target in real scene. In order to decrease the calculation cost, we select some frames from the video according to the characteristics of the movement. That is, if we find that the target is stationary or moves slowly, then we discard these frames. We use a threshold to test whether a target makes a big maneuver or not. Obviously, a large threshold can make the calculation cost lower, but lower EXrate will make the performance decrease too.

So the threshold should be carefully selected to balance the calculation cost and performance. Here, we select 95 frames from 245 frames; EXrate and IRrate are $38.77 \%$ and 0.1367 , respectively. Figure 9 gives the tracking results of number $1,27,40,65,74,97,128,129,158,181,189$, and 226 frames in the video. The estimation of target is marked by "black" dot. The estimation covariance of $\mathrm{RMSE}_{2 \mathrm{D}}$ as 1.034 mm is obtained (the tracking area is $300 * 300 \mathrm{~mm}^{2}$ ).

## 5. Conclusions

The main contribution of this paper is to model the realtime system dynamics at the random sampling points. (1) By calculating the matrix exponential with inverse Laplace transform, the irregular sampling interval is transformed to time-varying parameters matrix of the system. (2) Based on the statistics relation between the autocorrelation function and the covariance of Markov random processing, the system model with online parameter is developed. The proof and the experimental results show that the developed method can get good tracking performance.

As an example, the developed method is used for the video tracking problem. According to the motion characteristics of the target, some frames are selected for the tracking purpose. The tracking results show that good tracking performance is obtained by a smaller amount of calculation.

## Disclosure

The authors declare that they have no financial or personal relationships with other people or organizations that can inappropriately influence their work and there is no professional or other personal interest of any nature in any product, service, and/or company that could be construed as influencing the position presented in this paper.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.
Table 2: The estimation covariance and irregular rate in Figure 7.


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# A Data-Driven Control Design Approach for Freeway Traffic Ramp Metering with Virtual Reference Feedback Tuning 

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#### Abstract

ALINEA is a simple, efficient, and easily implemented ramp metering strategy. Virtual reference feedback tuning (VRFT) is most suitable for many practical systems since it is a "one-shot" data-driven control design methodology. This paper presents an application of VRFT to a ramp metering problem of freeway traffic system. When there is not enough prior knowledge of the controlled system to select a proper parameter of ALINEA, the VRFT approach is used to optimize the ALINEA's parameter by only using a batch of input and output data collected from the freeway traffic system. The extensive simulations are built on both the macroscopic MATLAB platform and the microscopic PARAMICS platform to show the effectiveness and applicability of the proposed data-driven controller tuning approach.


## 1. Introduction

Freeway traffic control has become an important area in the field of traffic engineering and intelligent transportation systems due to the rapid expansion in worldwide development of freeway infrastructure and traffic demand. The frequent occurrence of congestions on freeway during rush hours, which may be caused by traffic demand being greater than capacity, traffic accidents, road works or weather, and so forth, leads to travel time delay, inefficient utilization of the freeway infrastructure, and decreasing traffic safety. Thus, freeway traffic control methods should be developed to prevent traffic jam and utilize the freeway infrastructure efficiently. In general, there are three typical freeway traffic control methods, ramp metering, mainline speed control, and corridor control.

Among these methods, ramp metering is the most popular one [1, 2]. Ramp metering is implemented by means of traffic lights, which is used to meter the number of entering vehicles and prevent traffic volume from exceeding freeway capacity. Ramp metering, when properly applied, is an effective way to ease freeway congestion and improve the efficiency of freeway utilization. From the viewpoint of
system control, it is a typical set-point problem and numerous model based control methods have been exploited, such as numerical methods [1,3], linearization method [4], one-step ahead prediction, and multiple prediction adaptive control [5]. However, as the freeway traffic system is expanded to be larger and larger, its accurate mathematical model may be difficult to be built. Therefore, it is desirable to develop a control method that is less dependent on the model accuracy.

In the field of control theory, several control methods for systems with exogenous disturbances and/or model uncertainties have been explored extensively. In [6], a new model transformation of discrete-time fuzzy systems with time-varying delays is analyzed and applied to dynamic output feedback controller design. In [7], a dissipativity based sliding mode control (SMC) is proposed for continuoustime switched stochastic systems with an external disturbance/ uncertainty. In [8], a sliding mode control (SMC) is proposed for Markovian jump singular time-delay systems. In [9], a stable robust model predictive controller with hard input constraints is designed for a multivariable system whose model is inaccurate. In [10], a fuzzy sliding model control method is presented for a class of
nonlinear systems with structured and unstructured uncertainties. Although the robust performance to disturbances can be achieved with these control methods, the nominal model or system structure is still required for the controller design.

More recently, several data-driven control methods, which focus on designing controller merely using measured input and output data of a plant, are found in the datadriven control field, such as PID control [11], model-free adaptive control [12-14], iterative learning control [15-19], unfalsified control [20], virtual reference feedback tuning [21-24], and iterative feedback tuning [25-27]. Note that ALINEA-a simple, efficient, and easily implemented ramp metering strategy-is a typical PI-type feedback regulator based on mainstream measurements of occupancy downstream of the ramp [28, 29]. Apparently, it is also a data-driven control strategy in nature without including any modeling information of the freeway system, but only depending on the input and output measurements. However, it is worth pointing out that successful implementation of ALINEA depends on four parameters [30]: the update cycle and the feedback gain of ALINEA controller, the feedback gain, the location, and the predefined optimal occupancy of freeway downstream of the merge. In fact, it is difficult to select an optimal feedback gain by trial-and-error method for a practical freeway traffic system if there is not enough prior knowledge of the controlled system.

In [30], an iterative feedback tuning (IFT) method is proposed to optimize the parameter of the ALINEA controller. The parameter of ALINEA controller is tuned iteratively only by using the input and output (I/O) data without any information of the plant model. However, IFT requires many experiments on the plant for data collection and suffers from local minima problems if it is not suitably initialized [24]. In contrast, VRFT [21-24] is a "one-shot" data-driven controller tuning method: one collects a batch of data from the plant and the procedure returns a controller, without requiring iterations and/or further accesses to the plant for experiments. VRFT formulates the controller tuning problem as a controller parameter identification problem by introducing virtual reference signal. VRFT is suitable for many practical applications since the minimization is conducted in one shot.

In this paper, VRFT method is applied to optimize the parameter of the ALINEA controller in the presence of the modeling uncertainties and exogenous disturbances, as an alternative to the difficult task of fine-tuning ALINEA in realworld testing. The ALINEA's parameter is tuned directly by using the measured input and output (I/O) data without any prior knowledge of the freeway traffic system. The effectiveness of the proposed data-driven controller tuning method is verified by simulations built on macroscopic MALAB and microscopic PARAMICS platforms.

The paper is organized as follows. Section 2 is the problem formulation, where a discretized macroscopic traffic mode is introduced. Section 3 describes VRFT approach for ALINEA. Simulation results with MATLAB and PARAMICS platforms are provided in Section 4. Finally, Section 5 concludes this paper.


Figure 1: A freeway segment subdivided into sections.

## 2. Problem Formulation

2.1. Macroscopic Traffic Model. The space and time discretized traffic flow model used in this section was proposed by Papageorgiou in 1989. It divides a freeway into several segments, and each segment contains one on-ramp and one off-ramp only, as shown in Figure 1.

The mathematical formulation of discretized traffic flow model is given as follows:

$$
\begin{align*}
& \rho_{i}(k+1)=\rho_{i}(k)+ \frac{T}{L_{i}}\left[q_{i-1}(k)-q_{i}(k)+r_{i}(k)-s_{i}(k)\right]  \tag{1}\\
& q_{i}(k)=\rho_{i}(k) v_{i}(k),  \tag{2}\\
& v_{i}(k+1)= v_{i}(k)+\frac{T}{\tau}\left[V\left(\rho_{i}(k)\right)-v_{i}(k)\right] \\
&+\frac{T}{L_{i}} v_{i}(k)\left[v_{i-1}(k)-v_{i}(k)\right]  \tag{3}\\
&-\frac{\gamma T}{\tau L_{i}} \frac{\left[\rho_{i+1}(k)-\rho_{i}(k)\right]}{\left[\rho_{i}(k)+\kappa\right]}, \\
& V\left(\rho_{i}(k)\right)= v_{\text {free }}\left(1-\left[\frac{\rho_{i}(k)}{\rho_{\mathrm{jam}}}\right]^{l}\right)^{m}, \tag{4}
\end{align*}
$$

where $T$ is the sample time interval in hour. $k$ is the $k$ th time interval, $i \in\{1, \ldots, N\}$ is the $i$ th section of a freeway, and $N$ is the total section number. Model variables are listed as follows:
$\rho_{i}(k)$ : density in section $i$ at time $k T$ (veh/lane/km);
$v_{i}(k)$ : space mean speed in section $i$ at time $k T(\mathrm{~km} / \mathrm{h})$;
$q_{i}(k)$ : traffic flow leaving section $i$ and entering section $i+1$ at time $k T$ (veh/h);
$r_{i}(k)$ : on-ramp traffic volume for section $i$ at time $k T$ (veh/h);
$s_{i}(k)$ : off-ramp traffic volume for section $i$ at time $k T$ (veh/h), which is regarded as an unknown disturbance;
$L_{i}$ : length of freeway in section $i(\mathrm{~km})$;
$v_{\text {free }}(\mathrm{km} / \mathrm{h})$ and $\rho_{\text {jam }}(\mathrm{veh} / \mathrm{lane} / \mathrm{km})$ : the free speed and the maximum possible density per lane, respectively. They are two important parameters in traffic flow model, since their accuracy affects the accuracy of traffic flow model;
$\tau, \gamma, \kappa, l$, and $m$ : constant parameters which reflect particular characteristics of a given traffic system and
depend upon the freeway geometry, vehicle characteristics, drivers' behaviors, and so forth.
Equations (1)-(4) constitute the macroscopic traffic model. Equation (1) is the well-known conservation equation, (2) is the flow equation, (3) is the empirical dynamic speed equation, and (4) represents the density-dependent equilibrium speed.
2.2. Boundary Conditions. We assume that the traffic flow rate entering section 1 during the time period $k T$ and $(k+1) T$ is $q_{0}(k)$ and the mean speed of the traffic entering section 1 is equal to the mean speed of section 1 ; that is, $v_{0}(k)=v_{1}(k)$. We also assume that the mean speed and traffic density of the traffic exiting section $N+1$ are equal to those of section $N$; that is, $v_{N+1}(k)=v_{N}(k), \rho_{N+1}(k)=\rho_{N}(k)$. Boundary conditions can be summarized as follows:

$$
\begin{align*}
\rho_{0}(k) & =\frac{q_{0}(k)}{v_{1}(k)}, \\
v_{0}(k) & =v_{1}(k),  \tag{5}\\
\rho_{N+1}(k) & =\rho_{N}(k), \\
v_{N+1}(k) & =v_{N}(k), \quad \forall k .
\end{align*}
$$

2.3. Control Objective. For the traffic system, the control objective is to seek an appropriate on-ramp traffic volume $r_{i}(k)$ such that the traffic density $\rho_{i}(k)$ tracks the desired traffic density $\rho_{d}$. It is worth to point out that the offramp traffic volume $s_{i}(k)$ is an uncontrollable variable and is regarded as exogenous disturbance here. Obviously, even though the freeway model is known, it is difficult to design a proper control law using the traditional model-based control approaches such as optimal control and adaptive control because of the strong nonlinearity and uncertainties in the freeway traffic flow model.

For the simplicity of formulation, the section index $i$ is omitted in the following equations.

## 3. Virtual Reference Feedback Tuning for ALINEA Controller

3.1. ALINEA Controller. Reactive ramp metering strategies are employed at a tactical level, that is, in the aim of keeping the freeway traffic conditions close to prespecified set values, based on real-time measurements. The occupancy strategy is based on the same philosophy as the demand-capacity strategy, but it relies on occupancy-based estimation of the freeway flow measurement upstream of the ramp, which may, under certain conditions, reduce the corresponding implementation cost. Since the concept of occupancy and traffic density is similar and has a linear proportion between them, in this paper, we use the traffic density instead of the occupancy to design the ALINEA controller as follows [30]:

$$
\begin{equation*}
r(k)=r(k-1)+\Theta\left[\rho_{d}-\rho(k)\right] \tag{6}
\end{equation*}
$$

where $\rho_{d}$ is the desired traffic density and $\Theta$ is the feedback gain of ALINEA controller.


Figure 2: The closed-loop freeway traffic system.
3.2. VRFT Approach. Consider the freeway traffic system. As shown in Figure 2, it is a classical one-degree-of-freedom control system, where $P$ is the freeway traffic system and C is the ALINEA controller. $\rho_{d}, \rho, r, d$, and $e$ are the reference traffic density, the traffic density, the ramp metering volume, disturbance, and the difference between $\rho_{d}$ and $\rho$, respectively. The closed-loop freeway traffic system is described as

$$
\begin{gather*}
\rho(k)=P(z) r(k), \\
r(k)=\mathrm{C}(z, \Theta)\left(\rho_{d}-\rho(k)\right), \tag{7}
\end{gather*}
$$

where $z$ is the one-step ahead shift operator and $\Theta$ is the controller parameter.

The transfer function of the closed-loop system can be rewritten as

$$
\begin{equation*}
\frac{P(z) \mathrm{C}(z, \Theta)}{1+P(z) \mathrm{C}(z, \Theta)} \tag{8}
\end{equation*}
$$

For an unknown freeway traffic system $P(z)$, the control objective is to find an optimal controller parameter $\Theta_{\text {opt }}$ by using a batch of the measured input/output data so that the freeway traffic system behavior approximates as much as possible to that of a given invertible reference model $M(z)$, where $z$ is the one-step ahead shift operator [21-24]. This can be achieved by minimizing the following model-reference criterion:

$$
\begin{equation*}
J(\Theta)=\left\|\left(\frac{P(z) \mathrm{C}(z, \Theta)}{1+P(z) \mathrm{C}(z, \Theta)}-M(z)\right) W(z)\right\|_{2}^{2} \tag{9}
\end{equation*}
$$

where $W(z)$ is a weighting function.
It is difficult to calculate the derivative of the criterion (9) with respect to controller parameter $\Theta$ if the freeway traffic system $P(z)$ is unknown. To address this issue, one can introduce a virtual reference density signal $\rho_{\text {vir }}(k)$ such that

$$
\begin{equation*}
\rho(k)=M(z) \rho_{\mathrm{vir}}(k) \tag{10}
\end{equation*}
$$

where $\rho_{\text {vir }}(k)$ does not exist in reality and was not used in the generation of $\rho(k)$.

Since $M(z)$ is a given invertible reference model, (10) is rewritten as

$$
\begin{equation*}
\rho_{\mathrm{vir}}(k)=M(z)^{-1} \rho(k), \tag{11}
\end{equation*}
$$

where $M(z)^{-1}$ is the inversion of $M(z)$.
Equation (10) implies that $\rho(k)$ is the desired density of the freeway system if the reference density signal is set as $\rho_{\mathrm{vir}}(k)$ and the corresponding virtual tracking error is $e_{\mathrm{vir}}(k)=\rho_{\mathrm{vir}}(k)-\rho(k)$. On the other hand, even though the freeway system is unknown, when the freeway traffic system

TABLE 1: Initial values and parameters associated with the traffic model $[18,19]$.

| Section | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 12 |  |  |  |  |  |  |  |  |  |  |  |
| $\rho_{i}(0)$ | 30 | 30 | 30 | 30 | 30 | 30 | 30 | 30 | 30 | 30 | 30 |
| $v_{i}(0)$ | 50 | 50 | 50 | 50 | 50 | 50 | 50 | 50 | 50 | 50 | 50 |
| Parameters | $v_{\text {free }}$ | $\rho_{\text {jam }}$ | $l$ | $m$ | $\kappa$ | $\tau$ | $T$ | $\gamma$ | $q_{0}(k)$ | $r_{i}(0)$ | $\alpha$ |
|  | $80 \mathrm{~km} / \mathrm{h}$ | 80 veh/lane $/ \mathrm{km}$ | 1.8 | 1.7 | $13 \mathrm{veh} / \mathrm{km}$ | 0.01 h | 0.00417 h | $35 \mathrm{~km}^{2} / \mathrm{h}$ | $1500 \mathrm{veh} / \mathrm{h}$ | $0 \mathrm{veh} / \mathrm{h}$ | 0.95 |

is fed by $r(k)$ (the actually measured ramp metering volume), it generates $\rho(k)$ (the corresponding measured output signal). Thus, if the reference signal is set to be the virtual reference density signal $\rho_{\mathrm{vir}}(k)$ and the corresponding virtual tracking error is $e_{\text {vir }}(k)$, a good controller must generate the ramp metering signal $r(k)$. Since both the signals $e_{\text {vir }}(k)$ and $r(k)$ are available, the control objective (9) can be transformed into the following standard identification problem:

$$
\begin{equation*}
J_{\mathrm{VRFT}}(\Theta)=\left\|L(z)\left(\mathrm{C}(z, \Theta) e_{\mathrm{vir}}-r\right)\right\|^{2} \tag{12}
\end{equation*}
$$

where $L(z)$ is a suitable filter.
Remark 1. For a practical control problem, the "ideal controller" is usually a complex nonlinear system and it does not belong to the given controller class. In [24], the filter $L(z)$ is introduced to deal with this problem. Minimizing $J_{\mathrm{VRFT}}(\Theta)$ with the filter will generate a "nearly minimizer" of $J(\Theta)$. It is proved that minimizing $J_{\mathrm{VRFT}}(\Theta)$ is equivalent to minimizing the second-order expansion of $J(\Theta)$ in a constrained sense.

The procedure of VRFT for ALINEA controller is summarized as follows:
(1) collect a batch of input/output data collected from the plant, expressed as $\left\{(r(k), \rho(k))_{k=1: N}\right\}$, where $N$ denotes the number of the input/output data pairs;
(2) calculate the virtual reference density signal $\left\{\left(\rho_{\text {vir }}(k)\right)_{k=1: N}\right\}$ for a given invertible $M(z)$ and the measured density signal $\left\{\rho(k)_{k=1: N}\right\}$ according to (11);
(3) calculate the virtual error signal $\left\{\left(e_{\text {vir }}(k)\right)_{k=1: N}\right\}$ according to the following equation:

$$
\begin{equation*}
e_{\mathrm{vir}}(k)=\rho_{\mathrm{vir}}(k)-\rho(k) ; \tag{13}
\end{equation*}
$$

(4) filter the signals $e_{\mathrm{vir}}(k)$ and $r(k)$ with a suitable filter $L(z)$, obtaining $e_{L}(k)=L(z) e_{\mathrm{vir}}(k)$ and $r_{L}(k)=$ $L(z) r(k)$;
(5) estimate the controller parameter

$$
\begin{equation*}
\widehat{\Theta}_{\mathrm{opt}}=\arg \min _{\Theta} J_{\mathrm{VRFT}}^{N}(\Theta), \tag{14}
\end{equation*}
$$

where

$$
\begin{equation*}
J_{\mathrm{VRFT}}^{N}(\Theta)=\frac{1}{N} \sum_{i=1}^{N}\left(r_{L}(i)-\mathrm{C}(z, \Theta) e_{L}(i)\right)^{2} \tag{15}
\end{equation*}
$$

Remark 2. As suggested in [24], the filter can be designed as follows:

$$
\begin{equation*}
L(z)=\left(1-M(z) z^{-1}\right)\left(\left.\frac{\partial P(z)}{\partial u}\right|_{u}\right) \tag{16}
\end{equation*}
$$

where $\partial P(z) /\left.\partial u\right|_{u}$ is linear and time varying and it can be estimated, for example, via forgetting factor identification techniques.

## 4. Illustrative Examples

MATLAB and PARAMICS are widely used in the area of traffic management and academic research for evaluation and validation. In this section, in order to evaluate the proposed VRFT-tuned ALINEA, two simulations are carried out on MATLAB and PARAMICS platforms, respectively, where the macroscopic traffic flow model (1)-(4) is simulated on MATLAB platform, and the microscopic traffic flow model is simulated on PARAMICS platform. Both simulations show the effectiveness of the proposed VRFT-tuned ALINEA.

### 4.1. Performance Evaluation with MATLAB Platform

4.1.1. Network Configuration. Consider a long segment of freeway that is divided uniformly into 12 sections. The length of each section is 0.5 km . The initial traffic volume entering section 1 is $1500 \mathrm{veh} / \mathrm{h}$. The desired density is $\rho_{d}(t)=$ $30 \mathrm{veh} / \mathrm{lane} / \mathrm{km}$. The initial density and mean speed of each section are shown in Table 1 and the parameters used in the macroscopic traffic model are also listed in Table 1.

There exist an on-ramp with known traffic demands in section 3 and an off-ramp with unknown exiting traffic flow in section 8. The traffic demand pattern (on-ramp) and the outflow pattern (off-ramp) are shown in Figure 3. They were chosen to simulate a traffic scenario during rush hour. Note that the queuing demands actually impose a constraint on the control inputs of ramp metering; that is, the on-ramp volumes cannot exceed the current demands plus the existing waiting queues at on-ramp 3 at time $k$; thus

$$
\begin{equation*}
r_{3}(k) \leq d_{3}(k)+l_{3}(k) \tag{17}
\end{equation*}
$$

where $l_{3}(k)$ denotes the length (in vehicles) of a possibly existing waiting queue at time $k$ at 3 rd on-ramp and $d_{3}(t)$ is the demand flow at time $k$ at 3rd on-ramp (veh/h).

On the other hand, the waiting queue is the accumulation of the difference between the demand and actual on-ramp; that is,

$$
\begin{equation*}
l_{3}(k+1)=l_{3}(k)+T\left(d_{3}(k)-r_{3}(k)\right) . \tag{18}
\end{equation*}
$$



Figure 3: Traffic demand in on-ramp 3 and exiting flow of off-ramp 8.


Figure 4: The open-loop I/O signals measured on the system.
4.1.2. Simulation and Results. Using the VRFT method presented in Section 3, the ALINEA controller can be straightforwardly designed using the I/O data measured on the freeway traffic system. Specifically, the control input signal used for open-loop excitation is a pseudorandom binary sequence signal sampled at 15 seconds. The length of the data vector is 256 (corresponding to 1.068 hour of data acquisition). The I/O signals measured on the system are displayed in Figure 4. According to the characteristics of the measured input signal, the filter for VRFT method is simply set to be $L(z)=1$.

The VRFT toolbox for MATLAB 6 Release 13 [31] is used to tune the controller parameter $\Theta$. For a given first-order reference model $M(z)=0.9 z^{-1} /\left(1-0.1 z^{-1}\right)$, a filter $L(z)=$ 1 , a weighting function $W(z)=1$, and a batch of measured input and output data, VRFT toolbox gives the optimal feedback gain $\Theta_{\text {opt }}=50.75$.


Figure 5: Control performance of VRFT-tuned ALINEA.

The control performance of ALINEA controller with feedback gain $\Theta_{\text {opt }}=50.75$ is quite well despite the unknown disturbance in off-ramp 8, as shown in Figure 5, where Figure 5(a) shows the density profile in section 3 and Figure 5(b) shows the entering flow in on-ramp 3. The simulation results illustrate that a proper ALINEA's parameter is obtained by using the VRFT method presented in this paper.
4.2. Performance Evaluation in PARAMICS Platform. Existing microscopic traffic simulation platforms have distinguished features, and the fundamental model is commonly the car following model, which makes simulations very similar to each other. In this paper, we adopt PARAMICS simulation platform, which is widely used in the area of traffic management and academic research for evaluation and validation.
4.2.1. Freeway Network. A single lane freeway link with 14 mainline sections, 1 on-ramp, and 1 off-ramp is considered. The on-ramp used to implement metering or flow control is connected to section 3 at the beginning and the off-ramp


Figure 6: Freeway simulation model.

Table 2: PARAMICS O-D table.

|  | Zone 1 | Zone 2 | Zone 3 | Zone 4 | Total |
| :--- | :---: | :---: | :---: | :---: | :---: |
| Zone 1 | 0 | 0 | 250 | 1250 | $\mathbf{1 5 0 0}$ |
| Zone 2 | 0 | 0 | 50 | 250 | $\mathbf{3 0 0}$ |
| Zone 3 | 0 | 0 | 0 | 0 | $\mathbf{0}$ |
| Zone 4 | 0 | 0 | 0 | 0 | $\mathbf{0}$ |
| Total | $\mathbf{0}$ | $\mathbf{0}$ | $\mathbf{3 0 0}$ | $\mathbf{1 5 0 0}$ | $\mathbf{1 8 0 0}$ |

is connected to section 8 at the end. As shown in Figure 6, vehicles enter into the network from two defined zones, Zone 1 and Zone 2 , at the beginning of the freeway mainline and onramp section, respectively, and will have their destinations to be either Zone 3 or Zone 4, defined at the end of off-ramp and the mainline.

In Table 2, Zone 1 and Zone 2 are used for origins to release vehicles into the network, and meanwhile Zone 3 and Zone 4 are used as destination for these vehicles. In the table, the number specified is the total number of vehicles expected to make a trip starting from the zone corresponding to the row to the zone corresponding to the column. The release rate in PARAMICS of traffic flow is specified in profile files. The duration time is divided uniformly into time intervals and a specified percentage of vehicles from the total demand are expected to be released from each origin zone during each time interval; additionally the release probability is subject to random process. In this paper the time interval length is set to be 3 minutes which divides the simulation duration of 1 hour into 20 intervals.
4.2.2. Network Configuration. The key parameters for traffic model and simulation are provided in Table 3. In Table 3, duration is the length of the simulation; time step is the number of discrete simulation intervals that are simulated per second; demand factor specifies the dynamic demand for the current simulation ranging from 0 to $200 \%$ of the current global demand; orientation specifies the side of the carriageway that vehicles travel upon (right-/left-hand drive); units specify the unit convention for display in PARAMICS (USA/UK/Metric); control time step specifies the length of time interval for updating of control signal. The rest of parameters are commonly used parameters in the field of control; therefore further explanations are omitted.
4.2.3. Simulation and Results. As for a realistic implementation in the PARAMICS microscopic simulation platform where metered vehicles can only be integer numbers, so a revised ALINEA law of (6) is given as follows:

$$
\begin{equation*}
r(k)=r(k-1)+\operatorname{INT}\left(\Theta\left(\rho_{d}-\rho(k)\right)\right) \tag{19}
\end{equation*}
$$

where the desired density $\rho_{d}$ is set to be $30 \mathrm{veh} / \mathrm{lane} / \mathrm{km}$.

Table 3: Parameters for simulation.

| Duration (HH:MM:SS) | 1:00:00 |
| :--- | :---: |
| Time step (second) | 2 |
| Control time step (second) | 30 |
| Demand factor (\%) | 100 |
| Section length (m) | 500 |
| Orientation | Left-hand drive |
| Units | Metric units |


(a)

(b)

Figure 7: The open-loop I/O signals measured on the system.

The control input signal used for open-loop excitation is a pseudorandom binary sequence signal sampled at 30 sec . The length of the data vector is 100 (corresponding to 50 min of data acquisition). The I/O signals measured on the system are displayed in Figure 7. According to the characteristics of the measured input signal, the filter for VRFT method is set to be $L(z)=\left(0.9 z^{-1}-1.1006 z^{-2}+0.089625 z^{-3}-0.19983 z^{-4}+\right.$ $\left.0.10872 z^{-5}+0.2021 z^{-6}\right) /\left(1-0.2 z^{-1}+0.01 z^{-2}\right)$. For a given first-order reference model $M(z)=0.9 z^{-1} /\left(1-0.1 z^{-1}\right)$, a weighting function $W(z)=1$, and a batch of measured input and output data, VRFT toolbox gives the optimal feedback gain $\Theta_{\mathrm{opt}}=44.95$. The simulation result is shown in Figure 8. It is clear that the density tracking performance using the VRFT-tuned ALINEA is satisfactory.

To further evaluate the controller's performance, we define average absolute difference between density and desired density as the performance index:

$$
\begin{equation*}
J=\frac{\sum_{k=T_{1}}^{k=T_{2}}\left|\rho_{d}-\rho(k)\right|}{T_{2}-T_{1}} \tag{20}
\end{equation*}
$$

where $T_{1}=20$ and $T_{1}=120$ are the beginning and ending time instances that decide the traffic period for evaluation. The final performance index is $J=1.5357$.


Figure 8: Mainline density resulting from VRFT-tuned ALINEA.

## 5. Conclusion

In this paper, a "one-shot" data-driven controller tuning method, VRFT, is applied to tune the parameters in ALINEA controller. It can easily find an optimal feedback gain. The main feature of VRFT is that the method aims at minimizing a cost function by using a batch of input and output data collected from the controlled plant. ALINEA controller tuned by using VRFT method is evaluated on macroscopic MATLAB and microscopic PARAMICS platforms, respectively. The simulation results show the effectiveness of the datadriven tuning approach. It is noted that the parameters, such as the vehicle characteristics, drivers' behaviors, are time varying naturally in the practical traffic network. Therefore, the extension of the VRFT method to time-varying freeway systems will be explored in our future work. Other effective control methods [6-10] to deal with large model uncertainties and exogenous disturbances will also be explored further with applications in freeway traffic systems.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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# Research on Gear Shifting Process without Disengaging Clutch for a Parallel Hybrid Electric Vehicle Equipped with AMT 

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#### Abstract

Dynamic models of a single-shaft parallel hybrid electric vehicle (HEV) equipped with automated mechanical transmission (AMT) were described in different working stages during a gear shifting process without disengaging clutch. Parameters affecting the gear shifting time, components life, and gear shifting jerk in different transient states during a gear shifting process were deeply analyzed. The mathematical models considering the detailed synchronizer working process which can explain the gear shifting failure, long time gear shifting, and frequent synchronizer failure phenomenon in HEV were derived. Dynamic coordinated control strategy of the engine, motor, and actuators in different transient states considering the detailed working stages of synchronizer in a gear shifting process of a HEV is for the first time innovatively proposed according to the state of art references. Bench test and real road test results show that the proposed control strategy can improve the gear shifting quality in all its evaluation indexes significantly.


## 1. Introduction

Auto gearshift can help to improve the driving comfort, reduce the friction of clutch and synchronizer, and achieve a better handling of driving even in a complex environment. For instance, drivers having little driving experience may have a bad driving performance or a dangerous accident due to their unskilled operating in a manual gear shifting process [1]. Compared with other common automatic shift technologies, including AT, CVT, and DCT, AMT has lower cost and higher transmission efficiency. Furthermore, it can undertake modification on the traditional transmissions. Therefore, it is used more widely in passenger cars, especially in the hybrid and pure electric vehicles [2].

The main concern of this paper is a single-shaft parallel hybrid electric vehicle equipped with AMT. The structure diagram of the HEV powertrain adopted in this paper is shown in Figure 1. The system is composed of a diesel engine ( 172 kW ), a permanent magnetic motor (PMSM 75 kW ), a clutch, and a 5 -speed automated mechanical gearbox. To obtain a smooth, quick, and successful gearshift, the engine,
motor, and actuator of the gearbox should be coordinately, controlled very well.

Most of the gearshift control strategies in traditional vehicles equipped with AMT disengage the clutch during a gear shifting process [3-5], which can introduce longtime power interruption and friction of the clutch. A few researchers proposed gearshift control strategies without disengaging the clutch using active engine control [6, 7], but the conditions have not been significantly improved due to the slow response of engine. The introduction of driving motor on the HEV gives chance to achieve a fast and smooth gearshift by its active control due to its good response [810].

## 2. Gear Shifting Process without Disengaging Clutch in the Hybrid Electric Vehicle

The gear shifting process is shown as in Figure 2. Typical gear shifting process without disengaging clutch in a parallel hybrid electric vehicle can be divided into the following

Table 1: Parameters of the researched HEV.

| $J_{e}$ | $J_{m}$ | $J_{\mathrm{ci}}$ | $J_{\mathrm{co}}$ | $J_{i}$ | $J_{o}$ | $T_{f \text {-static }}$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| $1.2 \mathrm{~kg} \cdot \mathrm{~m}^{2}$ | $0.24 \mathrm{~kg} \cdot \mathrm{~m}^{2}$ | $0.3 \mathrm{~kg} \cdot \mathrm{~m}^{2}$ | $0.028 \mathrm{~kg} \cdot \mathrm{~m}^{2}$ | $0.09 \mathrm{~kg} \cdot \mathrm{~m}^{2}$ | $0.015 \mathrm{~kg} \cdot \mathrm{~m}^{2}$ | $80 \mathrm{~N} \cdot \mathrm{~m}$ |



Figure 1: Configuration of the researched single-shaft parallel hybrid electric vehicle.
detailed stages: (1) unload of the power sources before a gearshift, (2) switch to neutral gear, (3) active speed synchronization by the motor, (4) mechanical speed synchronization by the synchronizer, (5) selection and engagement the new gears and (6) torque recovery.

The control in stage (1) is mainly aimed at reducing the transmitted torque between the engaged gears close to zero, which means that the power sources are not output driving torque and at the meantime are not driven by the output shaft. Several problems will come out when trying to disengage the gears transmitting large torque as follows.
(a) It is difficult to disengage the gears or damage the teeth surface of the gears due to the contact pressure between the meshing teeth.
(b) It introduces torsional vibration of the output shaft and affects the comfort of the vehicle due to the step change of torque.
(c) It increases the difficulty of active synchronization due to the introduced vibration of output shaft speed.
A gearshift when incompletely unloaded is shown in Figure 3. The main control difficulty in stage (1) is that there is no sensor to measure the transmission torque between the meshing teeth, so it is easy to unload the power sources incompletely or excessively and introduce reverse torque exerted to the input shaft. This phenomenon becomes more frequent when the driver needs a snap acceleration or a quick slowdown; it seriously affects the gear shifting quality.

Stag (3) is the active speed synchronization period; it needs to adjust the speed of the input shaft to a desired speed quickly without overshoot and oscillation; the response of active speed synchronization driven by the PMSM is much faster than that driven by engine [11].

In real driving cycles, stages (4) and (5) which were conduct at the same time were also crucial to obtain a high quality
gear shifting process. These two stages in HEV are quite different from those in the traditional vehicles. Rotational inertia of input components of the gearbox in a gearshift without disengaging clutch on a HEV is about 30 times larger than that of those on a traditional vehicle in a gearshift with disengaging clutch, and there is at least a fivefold friction torque increase which can be seen from Table 1.

We also found that there were frequent phenomena of gear engaging failure or long time gear engaging after synchronization, especially when the road condition changed or there was a braking. The synchronizer wear was serious and the obtained gearshift quality was quite poor. To the best of our knowledge, there is no relative research on this problem.

## 3. Evaluation Indexes of Gear Shifting Quality

3.1. The Gear Shifting Time. The gear shifting time of the single-shaft parallel hybrid electric vehicle includes the unload time $t_{\text {unload }}$, time of disengaging the gear $t_{\text {diseng }}$, time of active synchronization $t_{\text {PMSM }}$, time of mechanical synchronization and gear engaging $t_{\text {eng }}$, and time of torque recovery $t_{\text {recer }}$. The gear shifting time will affect the dynamics and comfort performance of the HEV.

$$
\begin{equation*}
t_{\text {shift }}=t_{\text {unload }}+t_{\text {diseng }}+t_{\text {PMSM }}+t_{\text {eng }}+t_{\text {recv }} \tag{1}
\end{equation*}
$$

3.2. The Friction Work. The friction work can reflect the wear of the friction pair; it is an important index of durability. The smaller the friction work is the longer the life of synchronizer can be obtained.

The friction work $L$ is given as

$$
\begin{equation*}
L=\int_{0}^{t_{s}} T_{s}\left|\omega_{o}-\frac{\omega_{i}}{i_{g}}\right| d t \tag{2}
\end{equation*}
$$

where $t_{s}$ is the friction time, $T_{s}$ is the friction torque, $\omega_{i}, \omega_{o}$ are the angular speed of input part and output part of the friction pairs, respectively [12].
3.3. The Gear Shifting Jerk. The jerk is derivative of longitudinal acceleration; it can reflect the oscillation of driving torque and can be expressed as

$$
\begin{equation*}
j=\frac{d^{2} v}{d t^{2}}=\frac{r}{i_{0}} \frac{d^{2} \omega_{o}}{d t^{2}} \tag{3}
\end{equation*}
$$

where $t$ is time, $v$ is the vehicle speed, $i_{0}$ is the gear ratio of the main retarder, and $r$ is the radius of the driving wheel.


Figure 2: Gear shifting process without disengaging clutch of the HEV.

To calculate the jerk equation (3) can be discretized as

$$
\begin{align*}
j(n) & =\frac{r}{i_{0}} \frac{\left(\omega_{n+2}-\omega_{n+1}\right)-\left(\omega_{n+1}-\omega_{n}\right)}{\Delta t^{2}} \\
& =\frac{2 \pi r}{60 i_{0}} \frac{\left(n_{n+2}-n_{n+1}\right)-\left(n_{n+1}-n_{n}\right)}{\Delta t^{2}} \tag{4}
\end{align*}
$$

where $n$ is the angular speed of the output shaft. The ride comfort of the vehicle becomes worse than the increase of the jerk. For the quantitative criteria of jerk, the recommended value of Germany is $|j| \leq 10 \mathrm{~m} / \mathrm{s}^{3}$, the one of former Soviet union is: $|j| \leq 31.36 \mathrm{~m} / \mathrm{s}^{3}$, and the one of China is $|j| \leq$ $17.64 \mathrm{~m} / \mathrm{s}^{3}$ [13].

## 4. Power Transmission System Modeling

The sketch of power transmission system for the researched single-shaft parallel hybrid electric vehicle is shown in Figure 4. It should be able to find out the main factors affecting the gear shifting quality in stages (1), (4), and (5). The rotational inertia of clutch is equaled to the permanent magnet synchronous motor due to the fact that there is no clutch disengaging in the gear shifting process. To analyze the torsional vibration of shafts between the driving motor and engine, the torsion between gearbox and final drive is taken into account. The torsion inside the gearbox is ignored.

When the system is in a certain gear, the differential equations are

$$
\begin{align*}
& \left(T_{e}+T_{m}-T_{e m}-T_{f}\right) i_{g}-T_{o w}-\frac{T_{r}}{i_{o}}  \tag{5}\\
& \quad=\left(J_{e} \dot{\omega}_{e}+J_{m+c} \dot{\omega}_{m}+J_{i} \dot{\omega}_{i}\right) i_{g}+J_{o} \dot{\omega}_{o}+J_{w} \dot{\omega}_{w}
\end{align*}
$$

When the system is in neutral gear, the differential equations are

$$
\begin{align*}
T_{e}+T_{m}-T_{e m}-T_{f} & =J_{e} \dot{\omega}_{e}+J_{m+c} \dot{\omega}_{m}+J_{i} \dot{\omega}_{i} \\
-T_{o w}-\frac{T_{r}}{i_{o}} & =J_{o} \dot{\omega}_{o}+J_{w} \dot{\omega}_{w} \tag{6}
\end{align*}
$$

When the system transforms from neutral gear to a certain gear, the differential equations are

$$
\begin{gathered}
T_{e}+T_{m}-T_{e m}-T_{f}-\frac{T_{s}}{i_{g}}=J_{e} \dot{\omega}_{e}+J_{m+c} \dot{\omega}_{m}+J_{i} \dot{\omega}_{i} \\
T_{s}-T_{o v}-\frac{T_{r}}{i_{o}}=J_{o} \dot{\omega}_{o}+J_{w} \dot{\omega}_{w}
\end{gathered}
$$

where

$$
\begin{gather*}
T_{e m}=k_{e m}\left(\theta_{e}-\theta_{m}\right)+c\left(\omega_{e}-\omega_{m}\right) \\
T_{o v}=k_{o v}\left(\theta_{o}-\theta_{v}\right)+c\left(\omega_{o}-\omega_{v}\right) \\
\dot{\theta}_{x}=\omega_{x}, \quad x=(e, m, o, v)  \tag{8}\\
\frac{T_{r}}{r_{w}}=G\left(f_{\text {road }} \cos \theta+\sin \theta\right)+\frac{C_{D} A v_{a}^{2}}{21.15},
\end{gather*}
$$



Figure 3: A gearshift when incompletely unloaded.


Figure 4: Sketch of the power transmission system.
where $T_{e}$ is the output torque of engine, $T_{m}$ is the output torque of driven motor, $T_{e m}$ is the viscoelastic torque of the shaft between motor and engine, $T_{f}$ is the rotating friction torque of transmission input side (including the friction torque of engine, motor, and input shaft) when the clutch is engaged, $T_{o v}$ is the rotating friction torque between transmission output shaft and final drive, $T_{r}$ is the resisting torque of road, $T_{s}$ is the friction torque of synchronizer, $r_{w}$ is the radius of wheel, $i_{g}$ is the transmission ratio, $i_{o}$ is


Figure 5: Engine MAP.
the final drive ratio, $\omega_{e}$ is the engine speed, $\omega_{m}$ is the motor speed, $\omega_{i}$ is the transmission input shaft speed, $\omega_{0}$ is the transmission output shaft speed, $\theta_{e}$ is the engine rotation angle, $\theta_{m}$ is the motor rotation angle, $\theta_{o}$ is the rotation angle of transmission output shaft, $\theta_{v}$ is the rotation angle of final drive, $G$ is the vehicle gravity, $f_{\text {road }}$ is the rolling resistance coefficient, $\theta$ is the road slope angle, $C_{D}$ is the air resistance coefficient, $A$ is the face area $\left(\mathrm{m}^{2}\right), v_{a}$ is the relative velocity ( $\mathrm{km} / \mathrm{h}$ ), which represents wind-free velocity, $J_{i}$ is the equivalent rotational inertia of first-speed gear and secondspeed gear in transmission which is converted to the input end of synchronizer, $J_{e+c}$ is the equivalent rotational inertia of engine crankshaft and clutch pressure plate, and $J_{m+c}$ is the equivalent rotational inertia of motor and driven part of clutch.

## 5. Components Modeling

There are two working states of the transmission, in a certain gear or in neutral gear. When it is in neutral gear, $T_{s}=0$. This paper focuses on the research of the switching process between states 1 and 2. It involves the dynamic process of each component.
5.1. Engine Modeling. The engine torque $T_{e}$ can be obtained from the engine MAP as follows:

$$
\begin{equation*}
T_{e}=f\left(\alpha_{e}, n_{e}\right) \tag{9}
\end{equation*}
$$

where $\alpha_{e}$ is the engine throttle opening and $n_{e}$ is the engine speed. The engine MAP is shown in Figure 5.
5.2. PMSM Modeling. The transient value of three-phase voltage and current of the motor can be ignored because the time constant of motor torque is generally very small, so the driving motor model can be simplified. Considering the demand torque $T_{m, \text { req }}$, the maximum allowable drive and brake torque of motor with current rotating speed $T_{m \text {, dis }}$ and $T_{m, \mathrm{chg}}$, and the maximum allowable torque of motor when


Figure 6: Motor MAP.
the power battery discharges and recharges $T_{b, \text { dis }}$ and $T_{b, \text { chg }}$, the motor torque can be represented as

$$
T_{m}=\left\{\begin{array}{cc}
\max \left(T_{m, \text { req }}, T_{m, \text { dis }}, T_{b, \text { dis }}\right) & \text { when } T_{m, \text { req }}>0  \tag{10}\\
\max \left(T_{m, \text { req }}, T_{m, \text { chg }}, T_{b, \text { chg }}\right) & \text { when } T_{m, \text { req }}<0 .
\end{array}\right.
$$

The motor MAP is shown in Figure 6.
5.3. Gear Shifting Actuator Modeling. The gear shifting actuator is driven by brushless DC motor (BLDCM). In a gear shifting process, parameters including the displacement of actuator and gear shifting force should be under control. The displacement should be tuned to a certain position rapidly and accurately. The output force of the BLDCM also should be accurately controlled.

The mathematical model and torque characteristics of BLDCM can be analyzed with the mode of 3-phase 6-state. At any moment, only two phases work and the remaining one shuts down. The motor adopts $Y$ connection with nonsalient pole. Based on the physical structure, we assume that the three phases of stator are completely symmetrical, the airgap magnetic field produced by rotor is square wave, the back electromotive force of 3-phase winding is trapezoidal wave, the armature reaction of stator winding and eddy current loss are ignored, and the magnetic circuit is not saturated. So the balance equation of the 3-phase stator voltage is given as

$$
\begin{equation*}
u_{k}=R_{s} i_{k}+e_{k}+\left(L_{s}-L_{m}\right) \frac{d i_{k}}{d t}, \quad k=a, b, c \tag{11}
\end{equation*}
$$

Leaving out the mechanical loss of rotor and assuming the electromagnetic power can totally translate to the kinetic energy of rotor, the electromagnetic torque of motor is

$$
\begin{equation*}
T_{\mathrm{eg}}=\frac{\left(e_{a} i_{a}+e_{b} i_{b}+e_{c} i_{c}\right)}{\omega_{\mathrm{mech}}} \tag{12}
\end{equation*}
$$

The kinetic equation of BLDCM is

$$
\begin{equation*}
T_{\mathrm{eg}}=J \frac{d \omega_{m}}{d t}+T_{L}+B \omega_{\mathrm{mech}} \tag{13}
\end{equation*}
$$

The speed of BLDCM $n$ can be represented as

$$
\begin{equation*}
n=\frac{60 \omega_{\text {mech }}}{2 \pi} \tag{14}
\end{equation*}
$$

The axial force acting on shift sleeve $F_{x}$ can be represented as

$$
\begin{equation*}
F_{x}=T_{\mathrm{eg}} i_{m} \eta_{m} \tag{15}
\end{equation*}
$$

where $i_{a}, i_{b}, i_{c}$ are phase currents, $u_{a}, u_{b}, u_{c}$ are phase voltages, $R_{s}$ is the phase resistance of stator, $e_{a}, e_{b}, e_{c}$ are phase potential, $L_{s}$ is the self-inductance of stator phase, $L_{m}$ is the mutual-inductor of stator phases, $T_{L}$ is the load torque of motor, $B$ is the damping coefficient, $\omega_{\text {mech }}$ is the angular speed of rotor, $J$ is the rotational inertia of motor, $i_{m}$ is the speed ratio between motor and sleeve operating mechanism, and $\eta_{m}$ is the mechanical efficiency between motor and sleeve operating mechanism.
5.4. Synchronizer Modeling. Synchronizer is one of the most important parts of transmission. The working process of synchronizer directly affects the gear shifting quality [14]. To derive the optimal coordinated control strategy of each part, the operating process of synchronizer is divided into five stages.
5.4.1. First Free Fly. This stage actually includes two processes. First, the sleeve moves forward axially without resistance and pushes the synchro ring to the target gear. The resistance axial force is very small in this stage. Second, the force transfers from sleeve to the sliders, then to the cone surfaces of synchro ring. The spring of slider here is used to keep the balance. When the force transferred increases to a certain extent, the limiting mechanism will lose its balance. So the sliders are pushed into grooves of the synchro hub by the radial pressure, and the sleeve can continue sliding forward. Then the spline teeth of sleeve is contacted to the spline teeth of synchro ring, and the axial force transfers from sleeve to the synchro ring. The schematic diagram of this stage is shown as Figure 7.

The kinetic equation is

$$
\begin{equation*}
F_{x}-F_{f}-F_{r}=m_{s} \dot{a}_{s}, \tag{16}
\end{equation*}
$$

where $m_{s}$ is the mass of sleeve, $\dot{a}_{s}$ is the acceleration of sleeve, $F_{f}$ is the sliding resistance when sleeve moves, and $F_{r}$ is the force acted on sleeve from limiting mechanism.
5.4.2. Angular Velocity Synchronization. The sleeve stops sliding, and its axial velocity becomes zero. The friction between mesh teeth of sleeve and mesh teeth of synchro ring starts to increase while the kinetic energy of sleeve and the target gear begin to decrease. The angular velocity difference between them decreases gradually to zero. The schematic diagram of this stage is shown in Figure 8.


Figure 7: First free fly.


Figure 8: Angular velocity synchronization.

In the sleeve deceleration stage one has

$$
\begin{equation*}
F_{j} t=m_{s} v_{s}, \tag{17}
\end{equation*}
$$

where $F_{j}$ is the impact force acted on meshing teeth of synchro ring from sleeve, $t$ is the sleeve deceleration time, and $v_{s}$ is the axial velocity of sleeve under the shifting force.

In the sliding friction stage one has

$$
\begin{align*}
& T_{s}=F_{x} * f * \frac{r}{\sin \alpha}, \\
& P_{s}=T_{s}\left|\omega_{o}-\frac{\omega_{i}}{i_{g}}\right| \tag{18}
\end{align*}
$$

where $F_{x}$ is the axial force of sleeve, $f$ is the friction coefficient, $r$ is the mean effective cone radius, $\alpha$ is half of the cone angle, and $P_{s}$ is the friction power.
5.4.3. Turning the Synchro Ring. When the two parts of synchronizer have been synchronized, the driving part rotates under the action of the sleeve teeth (the driven part connects rigidly to the vehicle body, so it cannot be turned). A low angular velocity differential between sleeve and synchro ring occurs with the turning of synchro ring. Then the synchronization is broken up and the locking state of synchro ring


Figure 9: Turning the synchro ring.
disappears. The schematic diagram of this stage is shown in Figure 9.

Before the turning one has

$$
\begin{equation*}
\omega_{r 0}=\omega_{o} \tag{19}
\end{equation*}
$$

During the turning process one has

$$
\begin{gather*}
F_{x} \frac{1-\mu \tan \beta}{\mu+\tan \beta} r_{s}-T_{\mathrm{loss}}-T_{i} i_{g}-T_{f}=J_{r} \dot{\omega}_{r}  \tag{20}\\
J_{r}=J_{e+c}+J_{m+c}+J_{i}  \tag{21}\\
\omega_{r 1}=\omega_{r 0}+\dot{\omega}_{r} t  \tag{22}\\
\omega_{r 1} \neq \omega_{o} \tag{23}
\end{gather*}
$$

where $\omega_{r 0}$ is the initial angular speed of synchronizer driving part before rotation, $\omega_{o}$ is the speed of output shaft, $\omega_{r 1}$ is the input speed after rotation, $\beta$ is the angle of roof shape at the interlock gearing, $\mu$ is the friction coefficient at roof-shaped edge, $r_{s}$ is the mean effective radius on interlock gearing, $T_{\text {loss }}$ is the moment of losses, $T_{i}$ is the input torque, $J_{r}$ is the equivalent inertia to the synchronizer input end, and $\dot{\omega}_{r}$ is the angular acceleration of synchronizer input part.
5.4.4. Turning the Target Gear. This stage starts from the separation of synchro ring and cone surface of the target gear. The driving part of the synchronizer including the synchro ring rotates under the action of the sleeve teeth. It ends at the beginning of eventual slide. The schematic diagram of this stage is shown in Figure 10.

During this stage one has

$$
\begin{gather*}
F_{x} \frac{1-\mu \tan \beta}{\mu+\tan \beta} r_{s}-T_{\text {loss }}-T_{i} i_{g}-T_{f}=\left(J_{r}+J_{s}\right) \dot{\omega}_{r}  \tag{24}\\
\omega_{r 2}=\omega_{r 1}+\dot{\omega}_{r} t
\end{gather*}
$$

where $J_{s}$ is the rotational inertia of synchro ring.
5.4.5. Final Free Fly. The sleeve engages with the target gear under the action of the gear shifting force. The kinetic


Figure 10: Turning the target gear.


Figure 11: Final free fly.
equation is similar to the first stage. The schematic diagram of this stage is shown in Figure 11. Consider

$$
\begin{gather*}
F_{x}-F_{f}-F_{r}=m_{s} \dot{a}_{s}, \\
F_{f}=\frac{\mu_{g} T_{i} i_{g}}{r_{s}}, \tag{25}
\end{gather*}
$$

where $\mu_{g}$ is the axial dynamic friction coefficient between sleeve and spline teeth of constant mesh gear and $T_{i}$ is the transferred torque. We can get the conclusion from the derived kinetic equations that in the gear shifting process without disengaging clutch for a parallel hybrid vehicle equipped with AMT, the inertia of input shaft, the speed control precision of driving motor, and the input torque of transmission can directly affect the synchronization time and friction work. The control of shifting force is also very important. An ideal control of each component in each stage can improve the quality of gear shifting and increase the life of synchronizer.

## 6. Coordinated Control of Each Component in Each Stage of a Gear Shifting Process

6.1. Switch to Neutral Gear. The necessary condition of switching to neutral gear is

$$
\begin{equation*}
F_{x}>F_{f}=\frac{\mu_{g} T_{i} i_{g}}{r_{s}} \tag{26}
\end{equation*}
$$

where the input torque of the power sources is

$$
\begin{equation*}
T_{i}=T_{e}+T_{m}-T_{f}-T_{e m} \tag{27}
\end{equation*}
$$

Obviously, the reasonable control of gear shifting force and the input torque is the precondition to disengage the gears smoothly. And the basic rules are to increase the gear shifting force $F_{x}$ and decrease the input torque $T_{i}$.

But the force to disengage the gears cannot increase infinitely, and the exceeding force might damage the surface of meshing teeth; therefore, it should be controlled within the stipulated limit. The key to disengage the gears smoothly is to control the input torque close to zero. In order to realize this control, we measured and got the input torque characteristics curve of the power sources as well as the friction torque characteristics curve of the driving part, then established the actual net output of torque model.

In this paper, we control the engine torque to zero and the friction torque of the input part of the system is compensated by the electrical motor to achieve a zero net output torque of the power system; the control law is represented as

$$
\begin{equation*}
T_{i}=f\left(\alpha_{e}, T_{m}, \omega_{i}\right) \tag{28}
\end{equation*}
$$

Another problem is the coordinated torque control of the two power sources during the unloading phase. The control parameters during the unloading phase involved the engine throttle opening $\alpha_{e}$ and motor torque $T_{m}$. Firstly, the torque $T_{i}$ should be adjusted linearly instead of step change. Secondly, the gear shifting time should be as short as possible. Finally, the input torque $T_{i}$ should be controlled close to zero.

The engine throttle opening is controlled linearly to zero without step change. The step change might cause the shaft torsional vibration between the engine and the driving motor; it also might influence the driving motor.

Bench test showed that if the unloading time of the engine is 0.8 s when the initial throttle opening is $100 \%$, the shaft torsional vibration between the engine and motor can be negligible. When the hybrid power system starts to unload linearly from the maximum torque 1200 Nm and the unload speed is $15 \mathrm{Nm} / 10 \mathrm{~ms}$, the shaft torsional vibration of the transmission system can be negligible as well. So 0.8 s is the maximum unloading time. The PMSM can respond much faster than engine; it can be controlled to compensate the resistance torque and adjust the input torque $T_{i}$ to zero linearly during the process of the engine unloading.


Figure 12: The diagram block of gearshift BLDCM controller.

The target throttle opening during the phase of engine unloading is

$$
\begin{array}{r}
\alpha_{i+1}=\alpha_{i}-1 \quad\left(i_{\min }=1, i_{\max }=\alpha_{e_{\text {init }}}\right)  \tag{29}\\
\text { if }\left(\alpha_{i}<1\right) \alpha_{i}=0
\end{array}
$$

where $\alpha_{i}$ is the throttle opening percentage of the CAN protocol, its physical values range from 0 to 100 , and its resolution is 0.1 and $i_{\text {max }}$ is the total unload steps equal to the initial physical values of $\alpha_{i}$ before unloading. The time interval between $\alpha_{i+1}$ and $\alpha_{i}$ is 10 ms , which is transmitted to the engine control unit (ECU) by transmission control unit (TCU) via CAN bus.

Without considering the effect of temperature, the relationship between friction torque and rotation speed of input shaft when the clutch is engaged can be given as

$$
\begin{equation*}
T_{f}=f\left(n_{i}\right) \tag{30}
\end{equation*}
$$

The target control torque of the motor is

$$
\begin{equation*}
T_{m}(i+1)=T_{m_{\mathrm{init}}}+\frac{\left(T_{f}-T_{m_{\mathrm{init}}}\right)}{i_{\max }} \tag{31}
\end{equation*}
$$

where $i_{\text {max }}$ is the total unload step and its value is equal to the initial physical value of $\alpha_{i}$ before unloading.

When $T_{i} \in(-15 \mathrm{Nm}, 15 \mathrm{Nm})$, the BLDCM can start to drive the gearshift actuator to disengage the gears. To get an accurate control position of the gearshift actuator and a smooth gearshift, the displacement of the actuator is chosen as feedback of the BLDCM control. The closed loop proportional-derivative (PD) control is chosen as the control strategy of gear selecting and shifting motor; the relationship between input $e(t)$ and output $u(t)$ after discretization is given as

$$
\begin{equation*}
u(k)=K_{p} e(k)+K_{d}[e(k)-e(k-1)] \tag{32}
\end{equation*}
$$

where $K_{d}, K_{p}$ are the differential coefficient and proportional coefficient of the controller, respectively; $e(k)$ is the deviation of current detecting position and the desired position; $u(k)$ is the output value of the controller after $k$ times sampling and calculating. The diagram block gearshift BLDCM controller is shown in Figure 12.

### 6.2. Switch to a Certain Gear

6.2.1. Active Speed Synchronous Control Driving by PMSM. In this stage, the gearbox is in neutral gear, the driving motor is controlled in speed control mode, and its speed should be adjusted to

$$
\begin{equation*}
n_{m_{\mathrm{aim}}}=n_{o} \cdot i_{g} . \tag{33}
\end{equation*}
$$

When $n_{m} \in\left(n_{m_{\text {aim }}}-20, n_{m_{\text {aim }}}+20\right)$, the system starts to execute the next action. In addition, to reduce the entire gear shifting time further, the gear selecting actuator can be simultaneously controlled during the process of speed adjusting.

### 6.2.2. Mechanical Synchronization and Gear Engaging Process.

 The PMSM should be controlled to follow the speed of output shaft, after the active speed synchronous control, the gearshift actuator starts to engage the target gears.(1) First Free Fly. This stage should be finished quickly and need a larger gear shifting force, but the force cannot be infinitely increased because excessive shift force may cause large impact force $F_{j}$, and reduce the life of synchro ring and sleeve.
(2) Angular Velocity Synchronization. In this stage, the BLDCM torque is directly controlled to an allowed maximum value to finish the mechanical synchronization process as soon as possible. The driving motor is still controlled in speed mode following the speed of output shaft in this stage, which could eliminate the rotation speed difference caused by the road condition or braking action timely; it can help to reduce the sliding friction work and the synchronization time.
(3)Turning the Synchro Ring. The control of this phase is the key to engage the gears successfully and quickly; we can know from (20) that to turn the synchro ring and unlock the synchronizer, the gearshift torque should overcome the torque loss $T_{\text {loss }}$, the input torque of power sources $T_{i} i_{g}$, and the friction torque of transmission input side $T_{f}$.

Real road test also confirmed that it was difficult to overcome the resistance torque $T_{i}$ (approximately 100 Nm ) depending on the gearshift torque which is limited using a fixed threshold. The target gears cannot be engaged for quite a long time after finishing the rotation speed synchronization process. However, it is obvious that the input torque of power sources $T_{i} i_{g}$ is also controllable in this stage. The engine is working in the idle state now, which can be considered as the load of transmission system. The driving motor is in control phase of following the speed of output shaft. Based on this point of view we proposed a control method called "actively unlock the synchronizer control" after finishing the speed synchronization.

The specific control method is to control the driving motor has a jagged speed fluctuation ranging from -10 rpm to 10 rpm after the mechanical synchronization, which can make the sleeve teeth wriggle between teeth of synchro ring, so the synchro ring can be turned with the help of active control of

PMSM and gear shifting force. The basic principle is shown in Figure 13.
(4) Turning the Target Gear. This stage is almost the same as the last one except that the synchro ring is unlocked. The actual wriggled components are still in the sleeve side; the control algorithm can be learned from the previous stage.
(5) Final Free Fly. In this stage, the gear shift actuator is controlled to the desired position and the engaging process is finished.
6.3. Torque Recovering. After a successful gear shifting, the torque of the engine and driving motor should be controlled to the desired value of the driver separately without large torsional vibration and long time power interrupt. The control algorithm is shown in Figure 14.

## 7. Bench Test and Road Test

The block diagram of the overall gear shifting control algorithm for a single-shaft parallel HEV equipped with AMT without disengaging clutch is shown in Figure 15.

In the bench the eddy current dynamometer works in the load following mode that can simulate the actual driving test cycle conditions. The monitor system of dynamometer can display the output torque of power sources in real time, and it can detect the jerk of output shaft when engagement or disengagement happens. The main purpose of the bench test is to verify and modify the control strategy. Figure 16 is a picture of the test bench.

After plenty of bench test, 1000 kilometers real road test with a real HEV was conducted. Figure 17 shows the test HEV.

Figure 18 represents some exemplary curves of gear shifting process in road test. when stage (1) is the unload stage, the engine is unloading exactly to maintain its idle state, and the driving motor is unloading to the torque that can just overcome the friction torque of the power input side; this stage costs about 0.3 s . Then, the gears are disengaged within 0.2 s . In stage (3), the driving motor is controlled in speed mode to finish the active synchronization process which costs about 0.4 s . In stage (4), the target gears are successfully and quickly engaged with the small speed oscillation driven by the motor after the mechanical synchronization; this stage costs about 0.2 s . The whole gear shifting process costs about 1 s .

Figure 19 shows that the torque of engine and driving motor transits to the driver's desired torque smoothly when the demand torque is recovering after a gearshift. And this stage costs about 0.5 s .

## 8. Conclusions

In this paper, mathematical models of every component for a HEV are described, derivation and analysis of the gear shifting dynamics model in every working process of the synchronizer is also given. Innovative control strategies including dynamic coordinated control in powertrain unload


Figure 13: The algorithm to unlock the synchronizer actively.


Figure 14: The algorithm of recovering torque.

Table 2: The statistical data of evaluation indexes in road test.

| Stage | 1 | 2 | 3 | 1 | 2 | 3 | 1 | 2 | 3 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Input shaft speed $(\mathrm{r} / \mathrm{min})$ | 1560 | - | 907 | 1049 | - | 799 | 1675 | - | 1279 |
| Maximum jerk $\left(\mathrm{m} / \mathrm{s}^{3}\right)$ | 11.5 | - | 5 | 12.3 | - | 5.4 | 9 | - | 5.4 |
| Stage time $(\mathrm{ms})$ | 329 | 488 | 549 | 368 | 414 | 560 | 366 | 480 | 569 |
| Total time $(\mathrm{ms})$ |  | 1353 |  |  | 1321 |  |  | 1412 |  |



Figure 15: Gear shifting control algorithm for a single-shaft parallel HEW without disengaging clutch.


Figure 16: The test bench.


Figure 17: The test HEV.
process and active unlock control of the synchronizer on a HEV are proposed.

By the dynamic coordinated control of the engine, motor and actuators in unloading process, the net input torque of the gearbox can be controlled more closely to zero, and it is easy to disengage the gears smoothly and quickly. By the dynamic coordinated control of the engine, motor, and actuators in every working stage of the synchronizer during a gear shifting process, the gears can be controlled to engage more quickly and successfully with less friction.

According to the statistical data of the evaluation indexes in Table 2, the whole gear shifting time is controlled within 1.5 s . And the time consuming for stage 1 (disengaging the gears) and stage 2 (recovering the torque) takes about $50 \%$ of the whole gear shifting time, which is determined by the gearshift points and the driver's demand of torque. The power interrupt time is controlled within 0.5 s and the dynamic property of the HEV is ensured. The jerk of gear shifting without disengaging clutch is controlled within $15 \mathrm{~m} / \mathrm{s}^{3}$ and the ride comfort of the HEV is ensured.


Figure 18: Experiment curves of a gear shifting process.

The proposed gear shifting control strategy can also be applied into other kinds of pure electric or hybrid electric vehicles with similar gear box to significantly improve the gear shifting quality in all its evaluation indexes. Control strategy using more advanced control technologies [15, 16] will be proposed to improve the gear shifting quality further in the near future.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.


Figure 19: The torque recovering process.

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# A Novel Emergent State Control Law for an Integrated Helicopter/Turboshaft Engine System 

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#### Abstract

A two-layer robust control scheme is proposed to get a better response ability for emergency maneuvers of helicopter. Note that the power used in ascending flight is the main coupling between helicopter and its turboshaft engines; therefore vertical flight control is separated from conventional helicopter control loops and combined with fuel flow and turbine bleeding to new control loops denoted as an inner layer, whereas the mission level flight control is as the out layer. A conclusion in global asymptotically tracking for devising this new scheme is firstly derived from a Generalized Gronwall-Bellman approach. Due to this integrated designing, not only is the helicopter better controlled, but also much better power rapid tracking is realized for engines. Simulations are conducted to validate the new scheme in emergent ascending and descending flights, and the results illustrate that the response time of the closed-loop system is dramatically reduced when compared to the traditional one. Moreover, the presented system also has better dynamic performance under inferences.


## 1. Introduction

Since modern aircrafts are highly coupled with their engines, the propulsion system has to be integrated with the flight control system. With the recent development of computer control system, it is a feasible deal with the control problem on modern aircrafts such as STOVL (short taking-off and vertical landing) vehicles [1-3] and helicopters by integrating flight and propulsion systems. Recently, the concept of IFPC (Integrated Flight and Propulsion Control design) has drawn tremendously attention by NASA (National Aeronautics and Space Administration) Glenn research center in developing an autonomous flight/prolusion system [4, 5]. In helicopter design, the coupling between controllability and propulsion system is mainly from torque variations, which is predominately resulted from the direct mechanical linkages between helicopter and its onboard engine. If the propulsion system cannot rapidly counteract these torque variations from the helicopter, the considerably varying rotor speed will have a dramatical effect on the responsiveness of helicopters
[6-10]. Such circumstance requires a carefully consideration in control system design.

The IFPC problem is an extremely extensive concept in both conventional fixed wing aircrafts and helicopters. The present research will specifically focus on emergency state control. Supported by the famous projects of IHPTET (Integrated High Performance Turbine Engine Technology) and VAATE (Versatile Affordable Advanced Turbine Engines), numerous researches, which focused on fast response control under some emergent conditions, had been initialed by NASA. For emergent conditions such as post-stall flight and forced landing caused by control surfaces failure, the effective control variable (thrust or torque) is individually supplied by engines [11-13]. In order to guarantee a safe landing or correcting angular regulation for aircrafts, the engines have to be operated in an unusual way to enlarge the thrust and response rate. Helicopters and their engines can also encounter the similar conditions, which requires a fast response ability [14, 15]. In 1990s, the Advanced Propulsion System Engine Control (APSEC) project [16] applied a novel


Figure 1: The coordinate systems for helicopter flight dynamics.
control method by using fuel flow and compressor guided vanes to regulate the engine's output power, which resulted in a considerable improvement in the agility of helicopters. In the earlier years of this century, for enhancing static and dynamic performance of the integrated system, American armies led an integrated helicopter/engine control program in which a Sikorsky Black Hawk helicopter was selected as the platform [6]. In this program, on the basis of compressor guided vanes regulation with a look-up table, it also showed more feasible to gain a faster response capability in combat modes. Certainly, these air fluid control based approaches have small negative influence on the compressor stall margin. Otherwise, another way by turbine bleeding can also be utilized to devise a fast response control reported in [14], in which an integrated control scheme, implemented by fuel flow, turbine bleeding, and rotor control angles, is developed by aid of LQR (Linear Quadratic Regulator) method. However, such method described does not provide how to realize asymptotically tracking and has not been fully validated over the entire envelop.

In this paper a novel two-layer method is proposed for helicopter's emergent control, so as to promote performance in maneuver ability. This method is an improved one from that reported in [17], where only bounded stability for a Generalized Gronwall-Bellman Lemma approach is investigated for aero engines. Whereas, a conclusion concerning asymptotically tracking is further proposed in our paper. Meanwhile, an UH-60 helicopter with an onboard T700 engine model is employed as the simulation platform. The proposed strategy is verified in terms of robustness in the whole envelope.

The paper is organized as follows. Section 2 discusses the simulation platform of an integrated helicopter and engine system, which is needed in verifying the proposed control
scheme. In Section 3, the design method is introduced for emergency flight state. Finally, Section 4 demonstrates the validations by two cases for the new two-layer emergency state control law. For convenience, variables and their annotation are listed in Nomenclature section.

## 2. Simulation Platform

In order to verify the proposed control law, a detailed helicopter/engines system model is required. On the basis of the data and modeling approaches provided and validated in [18-20], an UH-60/T700 system model was built, and for more details about this model one can refer to [21, 22]. The model consists of three major parts: main rotor, air frame, and engine models, as can be seen in Figure 1. The earth, airframe and rotor hub fixed coordinate systems are denoted as E-Frame, A-Frame, and H-Frame, respectively.
2.1. Main Rotor Model. The main rotor of UH-60 helicopter is a single rotor type and can be modelled throughout blade element theory. The relative lift and drag coefficients for blade segments are provided with verified wind tunnel test data [19, 20]. Through this model, the flapping and lag dynamics, which are the main motions of the main rotor, can be simulated accurately. Furthermore, all the moment and thrust of the rotor, which are responsible for helicopter motions, can also be instantaneously calculated.
2.2. Airframe Model. The airframe is composed by fuselage, horizontal tail, vertical tail, and tail rotor (see Figure 1). The fuselage is modeled on the basis of wind tunnel test data in wide ranges for high angles of attack and sideslip. The horizontal and vertical tail, are treated as aerodynamic
disks with lift and drag coefficients from look-up tables as a function of attack angles. And the tail rotor model is numerically represented by linearized Bailey theory. For the common case where only the 6 rigid body degrees of freedom
are taken into account, the dynamics of helicopter can be expressed as

$$
\begin{equation*}
\dot{\mathbf{X}}_{H}=f\left(\mathbf{X}_{H}, \mathbf{U}_{H}\right) \tag{1}
\end{equation*}
$$

where the elaborated form is described as

$$
\begin{align*}
& \dot{u}=\frac{\left(X_{\mathrm{SUM}}-G \sin \theta\right) g}{G}+r v-q w, \\
& \dot{v}=\frac{\left(Y_{\mathrm{SUM}}-G \cos \theta \sin \varphi\right) g}{G}+p w-r v, \\
& \dot{w}=\frac{\left(Z_{\mathrm{SUM}}+G \cos \theta \cos \varphi\right) g}{G}+q u-p v, \\
& \dot{p}=\frac{I_{Z}}{I_{X} I_{X}-I_{X Z}^{2}}\left\{L_{\mathrm{SUM}}-\left(I_{Z}-I_{Y}\right) q r+I_{X Z} p q\right\}+\frac{I_{X Z}}{I_{X} I_{Z}-I_{X Z}^{2}}\left\{N_{\mathrm{SUM}}-\left(I_{Y}-I_{X}\right) p q-I_{X Z} r q\right\}, \\
& \dot{q}=\frac{1}{I_{Y}}\left\{M_{\mathrm{SUM}}-\left(I_{X}-I_{Z}\right) p r+I_{X Z}\left(r^{2}-p^{2}\right)\right\}, \\
& \dot{r}=\frac{I_{X}}{I_{X} I_{Z}-I_{X Z}^{2}}\left\{N_{\mathrm{SUM}}-\left(I_{Y}-I_{X}\right) p q-I_{X Z} r q\right\}+\frac{I_{X Z}}{I_{X} I_{Z}-I_{X Z}^{2}}\left\{L_{\mathrm{SUM}}-\left(I_{Z}-I_{Y}\right) q r+I_{X Z} p q\right\}, \\
& \dot{\theta}=57.3(q \cos \phi-r \sin \phi), \\
& \dot{\psi}=\frac{57.3(r \cos \phi+q \sin \phi)}{\cos \theta}, \\
& \dot{\varphi}=57.3(p+\dot{\psi} \sin \theta), \\
& {\left[\begin{array}{c}
\dot{V}_{x} \\
\dot{V}_{y} \\
\dot{V}_{z}
\end{array}\right]=T\left[\begin{array}{c}
\dot{u} \\
\dot{v} \\
\dot{w}
\end{array}\right],} \\
& \mathbf{T}=\left[\begin{array}{ccc}
\cos \psi \cos \theta & \cos \psi \sin \theta \sin \phi-\sin \psi \cos \phi & \cos \psi \sin \theta \cos \phi+\sin \psi \sin \phi \\
\sin \psi \cos \theta & \sin \psi \sin \theta \sin \phi+\cos \psi \cos \phi & \sin \psi \sin \theta \cos \phi-\cos \psi \sin \phi \\
-\sin \theta & \cos \theta \sin \phi & \cos \theta \cos \phi
\end{array}\right], \tag{2}
\end{align*}
$$

where $\mathbf{X}_{H}=\left[\begin{array}{llllllll}V_{x} & V_{y} & V_{z} & p & q & r & \varphi & \theta\end{array}\right]^{T}, \mathbf{U}_{H}=$ $\left[\begin{array}{llll}\theta_{0} & A_{1 c} & B_{1 s} & \theta_{t}\end{array}\right]^{T}$ are defined as state and control vector accordingly. Obviously, the dynamic system (1) can be temporally solved by some integral methods such as Runge-Kutta algorithm. Key parameters for airframe dynamics are given in Table 1.
2.3. Engine Model. T700 engines can supply power to the helicopter for various flights. The engine (see Figure 2) is a two-shaft type consisting of axis compressor, centrifugal compressor, combustion chamber, gas turbine, power turbine, and exhaust nozzle. The engine dynamics are described

TAbLE 1: Modeling parameters for airframe motion.

| Parameter | Value |
| :--- | :---: |
| $I_{X}$ | $6316.8 \mathrm{kgm}^{2}$ |
| $I_{Y}$ | $52216.0 \mathrm{kgm}^{2}$ |
| $I_{Z}$ | $49889.0 \mathrm{kgm}^{2}$ |
| $I_{X Z}$ | $2551.7 \mathrm{kgm}^{2}$ |
| $G$ | 73961.0 N |

through component level method addressed in [23]. The dynamics of the engine can be formulated as

$$
\begin{equation*}
\dot{\mathbf{X}}_{E}=f\left(\mathbf{X}, \mathbf{U}_{E}\right) \tag{3}
\end{equation*}
$$



Figure 2: T700 engine structure.
where the detailed expression is

$$
\begin{gather*}
\dot{N}_{p}=\frac{(\pi / 30)^{2}\left(P W_{\mathrm{out}}-P W_{p}\right)}{\left(J_{p} \cdot N_{p}\right)}, \\
\dot{N}_{g}=\frac{(\pi / 30)^{2}\left(P W_{g}-P W_{c}\right)}{\left(J_{g} \cdot N_{g}\right)}, \\
\dot{P}_{4}=\frac{k_{4} R T_{4}}{V_{g}\left(m_{a 3}+W_{f}-m_{g 4}\right)}, \\
\dot{P}_{45}=\frac{k_{45} R T_{45}}{V_{p}\left(m_{g 4}-m_{g 45}-W_{g_{\text {out }}}\right)}, \\
\dot{P}_{5}=\frac{k_{5} R T_{5}}{V_{\mathrm{nz}}\left(m_{g 45}-m_{g 5}\right)}, \tag{4}
\end{gather*}
$$

where $\mathbf{X}_{E}=\left[\begin{array}{lllll}N_{g} & N_{p} & P_{4} & P_{45} & P_{5}\end{array}\right]^{T}, \mathbf{U}_{E}=\left[\begin{array}{lll}\theta_{0} & W_{f} & W_{g_{\text {out }}}\end{array}\right]^{T}$ are defined as state and control vector of engine, respectively. $R$ represents gas constant scalar, and $k_{4}, k_{45}$, and $k_{5}$ are denoted as relative adiabatic exponents in different position along the engine flow path. Key modeling parameters for engine dynamics are presented in Table 2.

## 3. Design for Fast Response Control Law in Emergence Flight

For emergent flight normally with a low forward velocity, in which most of power demand comes from vertical flight channel, thus it is possible and necessary to have the vertical flight integrated with engine control loop. In the inner layer of our novel scheme, it can be expressed as an integrated

Table 2: Modeling parameters for engine dynamics.

| Parameter | Value |
| :--- | :---: |
| $J_{p}$ | $0.064 \mathrm{kgm}^{2}$ |
| $J_{g}$ | $0.085 \mathrm{kgm}^{2}$ |



Figure 3: Structure of the two-layer control for integrated helicopter/engine system.
helicopter vertical flight/turboshaft controller, which is a 3-3 input and output structure, depicted in Figure 3. The control laws for other flight channels, like forward, sideward, and turn flight, are integrated as out layer control loop.

Obviously, the new scheme is devised differently from the traditional way in which control systems for engines and helicopter are often designed separately. Aiming at weakening the complex dynamic couplings the dynamics of engines and helicopter are taken into account as an integrated one, guaranteeing more feasible and applicable controllers.

Note that this new control must embody some necessary aspects in a maneuver flight as (1) the power demand of flight reflected by vertical velocity; (2) stability for power transmission guaranteed by keeping power turbine speed constant; (3) fast regulation of gas turbine speed to get a rapid power supply of engines. Considering a much better robustness and adaptive capability for the whole envelope, a novel control law for nonlinear plants is proposed and the followed structure is chosen to design the fast response controller for the integrated helicopter and engine system (see Figure 3).
3.1. Principle of the Proposed Multivariable Robust Control Law. The followed formulations can be employed to describe a nonlinear dynamic model for a helicopter or its engine working in a wider envelop as

$$
\begin{align*}
& \dot{\mathbf{x}}=A \mathbf{x}+\mathrm{G}_{1}(\mathbf{x})+\mathbf{B}_{1} \mathbf{u}+\mathbf{B}_{2} \mathbf{w},  \tag{5}\\
& \mathbf{y}=\mathbf{C x}+\mathrm{G}_{2}(\mathbf{x})+\mathrm{D}_{1} \mathbf{u}+\mathrm{D}_{11} \mathbf{w}
\end{align*}
$$

where $\mathbf{x}, \mathbf{y}, \mathbf{u}$, and $\mathbf{w}$ are denoted as the state, output, control, and disturbance vectors, respectively, and $\mathbf{A}, \mathbf{B}_{1}, \mathbf{C}, \mathbf{D}_{1}$, $\mathbf{B}_{2}$, and $\mathbf{D}_{11}$ are the system matrices relatively. $\mathbf{G}_{1}(\mathbf{x})$ and $\mathbf{G}_{2}(\mathbf{x})$ are defined as nonlinear error functions between the nonlinear plant and its simplified linear one.

The control aim is that system output $\mathbf{y}$ is capable of asymptotically tracking the command signal cmd such that

$$
\begin{equation*}
\lim _{\mathbf{t} \rightarrow \infty}\|\mathrm{e}\|=\lim _{\mathbf{t} \rightarrow \infty}\|\mathrm{cmd}-\mathbf{y}\|=\mathbf{0} \tag{6}
\end{equation*}
$$

where $\mathbf{e}=\mathbf{c m d}-\mathbf{y}$ is defined as output error.
Furthermore, if using $\overline{\mathbf{x}}=\left[\begin{array}{c}\int_{0}^{t} \text { ed } \\ \text { ed }\end{array}\right]$ as an argument vector and providing that $\mathbf{c m d}$ is a set point command, system (5) is reformulated as

$$
\begin{gather*}
\dot{\overline{\mathbf{x}}}=\overline{\mathbf{A}} \overline{\mathbf{x}}+\overline{\mathbf{G}}_{1}(\mathbf{x})+\overline{\mathbf{B}}_{1} \overline{\mathbf{u}}^{+} \overline{\mathbf{B}}_{2} \overline{\mathbf{w}}+\overline{\mathbf{B}}_{3} \widetilde{\mathbf{w}}, \\
\mathbf{z}_{1}=\overline{\mathbf{y}}=\overline{\mathbf{C}}_{1} \overline{\mathbf{x}}+\overline{\mathbf{G}}_{2}(\mathbf{x})+\overline{\mathbf{D}}_{1} \overline{\mathbf{u}}+\overline{\mathbf{D}}_{11} \overline{\mathbf{w}},  \tag{7}\\
\mathbf{z}_{2}=\overline{\mathbf{C}}_{2} \overline{\mathbf{x}}+\overline{\mathbf{D}}_{2} \overline{\mathbf{u}}
\end{gather*}
$$

where $\overline{\mathbf{G}}_{1}(\mathbf{x})=\left[\begin{array}{c}\mathbf{G}_{1}(\mathbf{x}) \\ -\mathbf{G}_{2}(\mathbf{x})\end{array}\right], \overline{\mathbf{G}}_{2}(\mathbf{x})=\mathbf{G}_{2}(\mathbf{x}), \overline{\mathbf{u}}=\mathbf{u}, \overline{\mathbf{w}}=\mathbf{w}, \overline{\mathbf{y}}=\mathbf{y}$, $\overline{\mathbf{A}}=\left[\begin{array}{cc}\mathbf{A} & 0 \\ -\mathbf{C} & 0\end{array}\right], \overline{\mathbf{B}}_{1}=\left[\begin{array}{c}\mathbf{B}_{1} \\ -\mathrm{D}_{1}\end{array}\right], \overline{\mathbf{B}}_{2}=\left[\begin{array}{c}\mathbf{B}_{2} \\ -\mathrm{D}_{11}\end{array}\right], \overline{\mathrm{C}}_{1}=\left[\begin{array}{ll}\mathrm{C} & 0\end{array}\right], \overline{\mathbf{D}}_{1}=\mathrm{D}_{1}$, and $\overline{\mathbf{D}}_{11}=\mathbf{D}_{11}$.

Assuming that a feedback control law is given as $\overline{\mathbf{u}}=$ $\overline{\mathbf{K}} \overline{\mathbf{x}}$, a theorem for convergent performance about tracking problems can be gotten as follows.

## Theorem 1. If the following conditions are held as

(A) there exists an integer $q \geq 1$ such that $\left\|\overline{\mathbf{G}}_{1}(\mathbf{x})\right\|=$ $\left\|\left[\begin{array}{c}\mathbf{G}_{1}(\mathbf{x}) \\ \mathbf{G}_{2}(\mathbf{x})\end{array}\right]\right\| \leq \gamma\|\mathbf{x}\|^{q}$,
(B) all the eigenvalues of $\overline{\mathbf{A}}+\overline{\mathbf{B}}_{1} \overline{\mathbf{K}}_{1}$ have a strictly negative real part,
(C) the initial state $\overline{\mathbf{x}}_{0}$ satisfies $\left\|\overline{\mathbf{x}}_{0}\right\|^{q-1}<|\lambda| / \gamma M^{q}$, where the constants $M>0$ and $\lambda<0$ are determined by $\left\|e^{\overline{\mathrm{A}}+\overline{\mathrm{BK}}}\right\|<M e^{\lambda t}, \forall t>0$.

Then, a globally convergent tracking of $\lim _{\mathbf{t} \rightarrow \infty}\|\mathbf{c m d}-\mathbf{y}\|=$ $\lim _{\mathbf{t} \rightarrow \infty}\|\mathrm{e}\|=\mathbf{0}$ will be realized.

Proof. Based on the Generalized Gronwall-Bellman lemma from [17], if the above three conditions (A)-(C) were all held for system (7), the state $\overline{\mathbf{x}}=\left[\int_{0}^{t^{\mathbf{x}}}\right.$ ed $\left.\tau\right]$ is bounded by

$$
\begin{equation*}
\|\overline{\mathbf{x}}\|<\frac{M\left\|x_{0}\right\| e^{\lambda t}}{\left(1-\gamma M^{q}\left\|x_{0}\right\|^{q-1} /|\lambda|\right)^{1 /(q-1)}} \tag{8}
\end{equation*}
$$

Fortunately for general engines and helicopters [17], condition (A) is held such that there exists an integer $q \geq 1$ such that $\left\|\overline{\mathbf{G}}_{1}(\mathbf{x})\right\|=\left\|\left[\begin{array}{c}\mathbf{G}_{1}(\mathbf{x}) \\ \mathbf{G}_{2}(\mathbf{x})\end{array}\right]\right\| \leq \gamma\|\mathbf{x}\|^{q}$.

Condition (B) can be satisfied by some feedback control design methods; here Lemma 2 in the following is introduced to meet this condition.

For helicopters and engines, due to some physical constraints as speed up and burn out limits condition (C) also can be easily checked such that $\left\|\overline{\mathbf{x}}_{0}\right\|^{q-1}<|\lambda| / \gamma M^{q}$.

Hence $\left\|\int_{0}^{t} \mathrm{e} d \tau\right\|<+\infty$, we also know a fact that $e \in L_{2}$ from Lemma 2. Based on the famous Barbalat's Lemma [24] a finite limit can be gotten as

$$
\begin{equation*}
\lim _{\mathbf{t} \rightarrow \infty}\|\mathbf{e}\|=\lim _{\mathbf{t} \rightarrow \infty}\|\mathbf{c m d}-\mathbf{y}\|=\mathbf{0} \tag{9}
\end{equation*}
$$

As discussed in Theorem 1, the following lemma is used to meet condition (3) and get the proper feedback control law.

Thus, consider the linear dynamic part for system (5) as

$$
\begin{gather*}
\dot{\mathbf{x}}=\mathbf{A x}+\mathbf{B}_{1} \mathbf{u}+\mathrm{B}_{2} \mathbf{w}, \\
\mathbf{y}=\mathbf{C} \mathbf{x}+\mathrm{D}_{1} \mathbf{u}+\mathrm{D}_{11} \mathbf{w},  \tag{10}\\
\dot{\mathbf{x}}=\overline{\mathbf{A}} \overline{\mathbf{x}}+\overline{\mathbf{B}}_{1} \overline{\mathbf{u}}+\overline{\mathbf{B}}_{2} \overline{\mathbf{w}}  \tag{11}\\
\mathbf{z}_{1}=\overline{\mathbf{y}}=\overline{\mathbf{C}}_{1} \overline{\mathbf{x}}+\overline{\mathbf{D}}_{1} \overline{\mathbf{u}}+\overline{\mathbf{D}}_{11} \overline{\mathbf{w}} .
\end{gather*}
$$

In order to evaluate a controllable output, a new virtual output is defined as

$$
\begin{equation*}
\mathrm{z}_{2}=\overline{\mathrm{C}}_{2} \overline{\mathrm{x}}+\overline{\mathrm{D}}_{2} \overline{\mathbf{u}}, \quad \overline{\mathrm{C}}_{2}=\binom{\Lambda^{1 / 2}}{0}, \quad \overline{\mathrm{D}}_{2}=\binom{0}{\mathbf{R}^{1 / 2}} \tag{12}
\end{equation*}
$$

where two weighted matrices are $\boldsymbol{\Lambda}^{\mathrm{T}}=\boldsymbol{\Lambda}>\mathbf{0}$ and $\mathbf{R}^{\mathrm{T}}=\mathbf{R}>$ 0.

For the augmented system (11), a $H_{2} / H_{\infty}$ robust control method [25-27] can be applied to get the state feedback controller $\overline{\mathbf{K}}$, which yields the transfer function matrix from $\overline{\mathbf{w}}$ to $z_{1}$ as $\left\|\mathrm{T}_{\overline{\mathbf{w}} \mathbf{z}_{1}}\right\|_{\infty}<\gamma$. Moreover, let the quadratic performance index $\mathbf{J}=\int_{0}^{\infty}\left(\overline{\mathbf{x}}^{\mathbf{T}} \boldsymbol{\Lambda} \overline{\mathbf{x}}+\overline{\mathbf{u}}^{\mathbf{T}} \mathbf{R} \overline{\mathbf{u}}\right) \mathbf{d t}$ be as small as possible.

The object of the above problem can be solved by Lemma 2. For further analysis, the system (11) can be converted into the form as

$$
\begin{gather*}
\dot{\mathbf{x}}=\overline{\mathrm{A}} \overline{\mathbf{x}}+\overline{\mathbf{B}}_{1} \overline{\mathbf{u}}^{2}+\overline{\mathbf{B}}_{2} \overline{\mathbf{w}}+\overline{\mathbf{B}}_{3} \widetilde{\mathbf{w}}, \\
\mathbf{z}_{1}=\overline{\mathrm{C}}_{1} \overline{\mathbf{x}}+\overline{\mathrm{D}}_{1} \overline{\mathbf{u}}_{2}+\overline{\mathrm{D}}_{11} \overline{\mathbf{w}},  \tag{13}\\
\mathbf{z}_{2}=\overline{\mathbf{C}}_{2} \overline{\mathbf{x}}+\overline{\mathrm{D}}_{2} \overline{\mathbf{u}},
\end{gather*}
$$

where $\widetilde{\mathbf{w}}$ stands for a virtual disturbance, $\|\widetilde{\mathbf{w}}\|_{2}<\gamma_{\mathbf{w}} \in$ $R^{+}$holds, and $\overline{\mathbf{B}}_{3}$ is a proper dimensional matrix yielding mathematical solution for this problem. To proceed, a lemma about system (10) is introduced here.

Lemma 2. For system (13) and a specific scalar $\gamma_{1}>0$, provided that the followed Linear Inequality Matrices are held, $\min \gamma_{2}$,

$$
\begin{gather*}
{\left[\begin{array}{ccc}
\overline{\mathbf{A}} \mathbf{X}+\overline{\mathbf{B}}_{1} \mathbf{W}+\left(\overline{\mathbf{A}} \mathbf{X}+\overline{\mathbf{B}}_{1} \mathbf{W}\right)^{\mathbf{T}} & \overline{\mathbf{B}}_{2} & \left(\overline{\mathbf{C}}_{1} \mathbf{X}+\overline{\mathbf{D}}_{1} \mathbf{W}\right)^{\mathrm{T}} \\
\overline{\mathbf{B}}_{2}^{\mathrm{T}} & -\boldsymbol{\gamma}_{1} \mathbf{I} & \mathbf{D}_{11}^{\mathrm{T}} \\
\overline{\mathbf{C}}_{1} \mathbf{X}+\overline{\mathbf{D}}_{1} \mathbf{W} & \mathbf{D}_{11} & -\boldsymbol{\gamma}_{1} \mathbf{I}
\end{array}\right]<\mathbf{0},} \\
\overline{\mathbf{A}} \mathbf{X}+\overline{\mathbf{B}}_{1} \mathbf{W}+\left(\overline{\mathbf{A}} \mathbf{X}+\overline{\mathbf{B}}_{1} \mathbf{W}\right)^{\mathbf{T}}+\overline{\mathbf{B}}_{3} \overline{\mathbf{B}}_{3}^{\mathrm{T}}<\mathbf{0}, \\
{\left[\begin{array}{cc}
-\mathbf{Z} & \overline{\mathbf{C}}_{2} \mathbf{X}+\overline{\mathbf{D}}_{2} \mathbf{W} \\
\left(\overline{\mathbf{C}}_{2} \mathbf{X}+\overline{\mathbf{D}}_{2} \mathbf{W}\right)^{\mathbf{T}} & -\mathbf{X}
\end{array}\right]<\mathbf{0},} \\
\operatorname{Trace}(\mathbf{Z})<\gamma_{2} . \tag{14}
\end{gather*}
$$

Moreover, if there are optimal solutions of $\mathbf{X}, \mathbf{Z}$, and $\mathbf{W}$ for the above LMIs problem, $\overline{\mathbf{u}}=\overline{\mathbf{K}} \overline{\mathbf{x}}=\mathbf{W}(\mathbf{X})^{-1} \overline{\mathbf{x}}$ is the $\mathrm{H}_{2} / \mathrm{H}_{\mathrm{o}}$ controller for system (5).

Furthermore, an equivalent form called quasi-PID (Proportional Integration Difference) is often used as (see [28])

$$
\begin{equation*}
\mathbf{u}=\mathrm{K}_{\mathrm{X}} \mathbf{x}+\mathrm{K}_{\mathrm{e}} \int_{0}^{\mathrm{t}} \mathrm{ed} \tau \tag{15}
\end{equation*}
$$

Proof.
(a) The first LMI in expression (14) guarantees the performance index $H_{\infty}$ yield $\left\|\mathbf{T}_{\overline{\mathbf{w}}}^{1} 10, ~\right\|_{\infty}$.
(b) Due to

$$
\begin{align*}
\mathbf{J} & =\int_{0}^{\infty}\left(\overline{\mathbf{x}}(\mathbf{t})^{\mathrm{T}} \Lambda \overline{\mathbf{x}}(\mathbf{t})+\overline{\mathbf{u}}(\mathbf{t})^{\mathrm{T}} \mathbf{R} \overline{\mathbf{u}}(\mathbf{t})\right) \mathbf{d t} \\
& =\int_{0}^{\infty} \mathbf{z}_{2}(\mathbf{t})^{\mathrm{T}} \mathbf{z}_{2}(\mathbf{t}) \mathbf{d t}=\left\|\mathbf{z}_{2}(\mathbf{t})\right\|_{2} . \tag{16}
\end{align*}
$$

Assuming that sensitive function of closed-loop for system (11) is $\mathbf{T}_{\widetilde{w} z_{2}}(s)$, the flowed inequality can be deduced as

$$
\begin{equation*}
\left\|\mathbf{z}_{2}\right\|_{2}=\left\|\mathbf{T}_{\mathbf{z}_{2} \widetilde{\mathbf{w}}} \widetilde{\mathbf{w}}\right\|_{2}<\left\|\mathbf{T}_{\mathbf{z}_{2} \widetilde{\mathbf{w}}}\right\|_{2} \cdot\|\widetilde{\mathbf{w}}\|_{2}<\gamma_{w} \cdot\|\widetilde{\mathbf{w}}\|_{2} \tag{17}
\end{equation*}
$$

and this says that $\mathbf{z}_{2} \in L_{2}$.
And a further deduction can be gotten as

$$
\begin{equation*}
\mathbf{x}, \mathbf{y}, \mathbf{e}, \mathbf{u} \in L_{2} . \tag{18}
\end{equation*}
$$

Thus, the control problem for system (10) can be transferred into a $H_{2} / H_{\infty}$ optimization problem as follows:

$$
\begin{gather*}
\min \gamma_{2} \\
\left\|\mathbf{T}_{z_{2} \widetilde{w}}\right\|_{2}<\gamma_{2} . \tag{19}
\end{gather*}
$$

Therefore, combined with the conclusions in (a) and (b), the proof for Lemma 2 is completed.
3.2. Two-Layer Robust Control Law for Helicopter's Emergency State. As presented above, for helicopters, a feasible design approach is integrated airframe and engine system control method, so the coupling between them should be well treated. In particular, in the emergency state, we propose a two-layer control law, in which the outer layer is designed for flight control and the inner layer is for engine fast response control. The key problem, or way to deal with couplings, is that vertical control input is calculated in the inner layer. The reason is that the engine has the most influence on the vertical channel, when extra control power, like turbine bleeding, is added in emergency state. The design steps are as follows.
(a) For helicopter, the out layer or flight control system, which is a four-loop control, including forward, sideward, climbing, and yaw flight, can be acquired based on Theorem 1 as

$$
\begin{align*}
\mathbf{U}_{H} & =\left[\begin{array}{l}
u_{H 1} \\
u_{H 2} \\
u_{H 3} \\
u_{H 4}
\end{array}\right]=\left[\begin{array}{c}
\theta_{0} \\
A_{1 s} \\
B_{1 c} \\
\theta_{t}
\end{array}\right]  \tag{20}\\
& =\mathbf{K}_{H X} \mathbf{X}_{H}+\mathbf{K}_{H e} \int_{\mathbf{0}}^{\mathbf{t}} \mathbf{e}_{H} \mathbf{d} \boldsymbol{\tau} .
\end{align*}
$$

(b) For engines, the inner layer control also can be designed by Theorem 1. As discussed in the front sections, this new scheme is a control structure in which fuel flow incorporates with turbine bleeding and rotor collective control to track the demand power from helicopter rapidly. The scheme is expressed as

$$
\begin{align*}
\mathbf{U}_{E} & =\left[\begin{array}{l}
u_{E 1} \\
u_{E 2} \\
u_{E 3}
\end{array}\right]=\left[\begin{array}{c}
\theta_{0}^{\prime} \\
W_{f} \\
W_{g_{\text {out }}}
\end{array}\right]  \tag{21}\\
& =\mathbf{K}_{E x} \mathbf{X}_{E}+\mathbf{K}_{E e} \int_{0}^{\mathbf{t}} \mathbf{e}_{E} \mathbf{d} \boldsymbol{\tau} .
\end{align*}
$$

(c) Now, it is easy to find that both (15) and (20) have the main rotor collective input. Since the main coupling item is $V_{Z}$ channel, we choose the control input $\theta_{0}^{\prime}$ in (20) as the final controller's output. So the two-layer control law turns into the form

$$
\begin{align*}
& \mathbf{U}_{H}=\left[\begin{array}{l}
u_{E 1} \\
u_{H 2} \\
u_{H 3} \\
u_{H 4}
\end{array}\right]=\left[\begin{array}{c}
\theta_{0}^{\prime} \\
A_{1 s} \\
B_{1 c} \\
\theta_{t}
\end{array}\right],  \tag{22}\\
& \mathbf{U}_{E}=\left[\begin{array}{l}
u_{E 2} \\
u_{E 3}
\end{array}\right]=\left[\begin{array}{c}
W_{f} \\
W_{g_{\text {out }}}
\end{array}\right] .
\end{align*}
$$

Remark 3. For out layer or flight control, it means imposing an extra disturbance on system input for replacing input $u_{H 1}$ by $u_{E 1}$. Therefore, provided that the closed-loop for helicopter has margin in terms of antidisturbance, it would still keep static and dynamic performance to some extent. Certainly this layer is designed based on Theorem 1, such that good robustness and anti-disturbance ability.

For inner layer or engine control loops, a similar conclusion may be drawn in terms of robustness and antidisturbance ability. In this case, the demanding power variations, not only decided by vertical climbing but also forward and sideslip flight, can be looked at as an additional system disturbance. Of course, due to its robust design the inner layer can also tolerate this kind of disturbance in this situation.
3.3. Out Layer Control in Emergency Flight. The out layer or flight control system for UH-60 helicopter is implemented in this section, and the system state, control input, and system output are introduced, respectively, as follows:

$$
\begin{aligned}
& \text { state vector for helicopter is } \mathbf{x}_{H}= \\
& \left.\left[\begin{array}{llllllll}
V_{x} & V_{y} & V_{z} & p & q & r & \phi & \psi
\end{array}\right]\right]^{T}
\end{aligned}
$$

$$
\begin{aligned}
& \text { control input vector is } \mathbf{U}_{H}=\left[\begin{array}{llll}
\theta_{0} & A_{1 c} & B_{1 s} & \theta_{t}
\end{array}\right]^{T} \text {, } \\
& \text { output vector is } \mathbf{y}_{H}=\left[\begin{array}{llll}
V_{x} & V_{y} & V_{z} & \psi
\end{array}\right]^{T} \text {, } \\
& \text { command signal is } \mathbf{c m d}_{H}= \\
& {\left[\begin{array}{lllll}
V_{x \mathrm{cmd}} & V_{y \mathrm{cmd}} & V_{z \mathrm{cmd}} & \psi_{\mathrm{cmd}}
\end{array}\right]^{T} \text {, }}
\end{aligned}
$$

output error vector is $\mathbf{e}_{H}=\mathbf{c m d}_{H}-\mathbf{y}_{H}$,
disturbance is $w_{H}=\Omega_{R}$.

In the hover flight state as $H=0 \mathrm{~m}$ and $V_{x}=V_{y}=$ $V_{z}=0 \mathrm{~m} / \mathrm{s}$, system matrices for the helicopter can be easily identified by perturbation methods [29] as follows:

$$
\mathbf{A}_{H 1}=\left[\begin{array}{ccccc}
-0.009920 & 0.000789 & 0.019326 & -0.259361 & 5.508862 \\
-0.006006 & -0.074958 & -0.012146 & -5.614851 & -0.242581 \\
0.015424 & -0.009135 & -0.377051 & -0.249239 & 0.415087 \\
-0.022955 & -0.168659 & -0.030995 & -4.149794 & 0.419433 \\
-0.000618 & 0.004887 & 0.019240 & -0.032284 & -0.601711 \\
0.001020 & 0.022554 & 0.018359 & -0.121840 & -0.091497 \\
0.000000 & 0.000000 & 0.000000 & 1.000000 & 0.000000 \\
0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 \\
0.000000 & 0.000000 & 0.000000 & 0.000000 & 1.000000
\end{array}\right]
$$

$$
\mathbf{A}_{H 2}=\left[\begin{array}{cccc}
0.189433 & 0.000000 & 0.000000 & -9.766719  \tag{23}\\
-0.188853 & 9.766719 & 0.000000 & 0.031043 \\
-0.061903 & 0.455786 & 0.000000 & -0.665203 \\
0.157180 & 0.000000 & 0.000000 & 0.000000 \\
-0.074104 & 0.000000 & 0.000000 & 0.000000 \\
-0.307293 & 0.000000 & 0.000000 & 0.000000 \\
0.068109 & 0.000000 & 0.000000 & 0.000000 \\
1.002317 & 0.000000 & 0.000000 & 0.000000 \\
0.000000 & 0.000000 & 0.000000 & 0.000000
\end{array}\right]
$$

$$
\begin{aligned}
& \mathbf{A}_{H}=\left[\begin{array}{ll}
\mathbf{A}_{H 1} & \mathbf{A}_{H 2}
\end{array}\right], \\
& \mathbf{B}_{H 1}=\left[\begin{array}{cccc}
0.112449 & -0.019934 & 0.191121 & -0.000000 \\
-0.033941 & 0.197670 & 0.022143 & 0.063477 \\
-1.628770 & 0.002718 & 0.004722 & -0.023100 \\
-0.166929 & 1.076946 & 0.190533 & 0.146018 \\
0.067651 & 0.025108 & -0.139675 & -0.031818 \\
0.227011 & 0.015675 & 0.001580 & -0.091496 \\
0.000000 & 0.000000 & 0.000000 & 0.000000 \\
0.000000 & 0.000000 & 0.000000 & 0.000000 \\
0.000000 & 0.000000 & 0.000000 & 0.000000
\end{array}\right], \\
& \mathbf{B}_{H 2}=\left[\begin{array}{llllllll}
-0.017279 & 0.005176 & 0.266392 & 0.015865 & -0.011623 & -0.007481 & 0.0 & 0.0 \\
0.0
\end{array}\right]^{T}, \\
& \mathbf{C}_{H}=\left[\begin{array}{lllllllll}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right], \quad \mathbf{D}_{H 1}=\mathbf{O}_{3 \times 3}, \quad \mathbf{D}_{H 11}=\mathbf{O}_{3 \times 1} .
\end{aligned}
$$

By trial and error, the weighted matrices are chosen as

$$
\begin{gather*}
\boldsymbol{\Lambda}=\operatorname{diag}\left(\left[\begin{array}{lllllllllllll}
1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 0.9 & 0.9 & 0.9 & 1.5
\end{array}\right]\right), \\
\mathbf{R}=\operatorname{diag}\left(\left[\begin{array}{llllll}
1.0 & 1.0 & 1.0 & 1.5
\end{array}\right]\right) \tag{24}
\end{gather*}
$$

And the scalar $\gamma_{1}=40$.

$$
\begin{gather*}
\mathbf{K}_{H x}=\left[\begin{array}{ccccccccc}
-0.4504 & -0.0772 & -1.0713 & -0.0865 & 0.7182 & 1.9362 & -1.1411 & 1.7660 & 6.5718 \\
-0.5341 & 2.2872 & 0.1233 & 2.5210 & 2.1920 & 2.4541 & 27.8091 & 1.1174 & 7.7398 \\
2.9540 & 0.3435 & 0.0242 & 0.3770 & -12.6501 & 1.8986 & 4.8150 & 0.7553 & -42.5235 \\
0.2852 & 0.1526 & -0.8469 & 0.2251 & -0.8352 & -6.3462 & 1.1669 & -4.0489 & -4.0343
\end{array}\right], \\
\mathbf{K}_{H e}=\left[\begin{array}{ccccc}
0.1086 & 0.0488 & 0.8277 & -0.5784 \\
0.1567 & -0.9179 & -0.0592 & -0.2213 \\
-0.9243 & -0.1294 & 0.0392 & -0.2139 \\
-0.0790 & -0.1598 & 0.3741 & 0.8449
\end{array}\right] . \tag{25}
\end{gather*}
$$

Thus, the out layer control law for the integrated helicopter and engine system is expressed as

$$
\begin{equation*}
\mathbf{U}_{H}=\mathbf{K}_{H x} \mathbf{x}_{H}+\mathbf{K}_{H e} \int_{\mathbf{0}}^{\mathbf{t}} \mathbf{e}_{H} \mathbf{d} \boldsymbol{\tau} \tag{26}
\end{equation*}
$$

3.4. Design for Inner Loop Control in Emergency Flight. For the integrated helicopter and engine system, the system state, control input, and system output are introduced, respectively, as follows:
state vector is $\mathbf{x}_{E}=\left[\begin{array}{lll}V_{z} & N_{p} & N_{g}\end{array}\right]^{T}$,
control input vector is $\mathbf{U}_{E}=\left[\begin{array}{lll}\theta_{0}^{\prime} & W_{f} & W_{g_{\text {out }}}\end{array}\right]^{T}$,
output vector is $\mathbf{y}_{E}=\left[\begin{array}{lll}V_{z} & N_{p} & N_{g}\end{array}\right]^{T}$,
command signal is $\mathbf{c m d}_{E}=\left[\begin{array}{lll}V_{z \mathrm{cmd}} & N_{p \mathrm{cmd}} & N_{g \mathrm{cmd}}\end{array}\right]^{T}$,
output error vector is $\mathbf{e}_{E}=\mathbf{c m d}_{E}-\mathbf{y}_{E}$,
disturbance is $w=Q_{H}$.
In the relative engine state ( $N_{p}=100 \%, N_{g}=88.6 \%$ ) for the above hover state, system matrices can also be fitted by small perturbation method [30] as

$$
\begin{aligned}
\mathbf{A}_{E} & =\left[\begin{array}{ccc}
-0.239124 & -0.254130 & -0.270133 \\
2.691120 & -0.165848 & -0.468932 \\
-0.012819 & -0.002656 & -1.693173
\end{array}\right], \\
\mathbf{B}_{E 1} & =\left[\begin{array}{lll}
-0.256693 & -0.016399 & -0.038242 \\
-0.228991 & -0.546528 & -0.492449 \\
-0.001338 & -0.269171 & -0.107159
\end{array}\right],
\end{aligned}
$$

$$
\begin{gather*}
\mathbf{B}_{E 2}=\left[\begin{array}{lll}
-0.017279 & -0.011623 & -0.007481
\end{array}\right]^{T} \\
\mathbf{C}_{E}=\mathbf{I}_{3 \times 3}, \quad \mathbf{D}_{E 1}=\mathbf{O}_{3 \times 3}, \quad \mathbf{D}_{E 11}=\mathbf{O}_{3 \times 1} \tag{27}
\end{gather*}
$$

By trial and error, the weighted matrices are chosen as

$$
\begin{gather*}
\boldsymbol{\Lambda}=\operatorname{diag}\left(\left[\begin{array}{llllll}
1.0 & 0.9 & 0.76 & 0.8 & 0.45 & 0.76
\end{array}\right]\right),  \tag{28}\\
\mathbf{R}=\operatorname{diag}\left(\left[\begin{array}{lll}
1.1 & 0.9 & 1.2
\end{array}\right]\right)
\end{gather*}
$$

And the scalar $\gamma_{1}=40$.
Also, using the design method described in Section 3.1, we can acquire the following controller gains:

$$
\begin{align*}
& \mathbf{K}_{E e}=\left[\begin{array}{ccc}
0.7257 & 0.6687 & -0.1614 \\
0.3082 & -0.5263 & -0.7918 \\
-0.6150 & 0.5253 & -0.5880
\end{array}\right], \\
& \mathbf{K}_{E x}=\left[\begin{array}{ccc}
-2.6069 & -0.5658 & -0.0655 \\
0.7030 & 0.8282 & 0.3956 \\
-0.5780 & -0.9703 & 0.5477
\end{array}\right] . \tag{29}
\end{align*}
$$

Thus, the inner controller for the integrated helicopter and engine system is expressed as

$$
\begin{equation*}
\mathbf{U}_{E}=\mathbf{K}_{E x} \mathbf{x}_{E}+\mathbf{K}_{E e} \int_{0}^{\mathbf{t}} \mathbf{e}_{E} \mathbf{d} \boldsymbol{\tau} \tag{30}
\end{equation*}
$$

3.5. Two-Layer Control Law for Integrated UH-60/T700 Engine System. Based on the description in Section 3.2,


Figure 4: Cascade PID control for engines.
the two-layer control law for the integrated UH-60/T700 engine system is followed by

$$
\begin{align*}
& \mathbf{U}_{H}=\left[\begin{array}{l}
u_{E 1} \\
u_{H 2} \\
u_{H 3} \\
u_{H 4}
\end{array}\right]=\left[\begin{array}{c}
\theta_{0}^{\prime} \\
A_{1 s} \\
B_{1 c} \\
\theta_{t}
\end{array}\right],  \tag{31}\\
& \mathbf{U}_{E}=\left[\begin{array}{l}
u_{E 2} \\
u_{E 3}
\end{array}\right]=\left[\begin{array}{c}
W_{f} \\
W_{g_{\text {out }}}
\end{array}\right] .
\end{align*}
$$

## 4. Validations and Discussions

To validate the feasibility of the proposed control law, some rapid ascent and descent flight tasks are simulated and compared with the conventional cascade PID control. For simplicity, the novel two-layer emergent integrated system control is labeled in short as TLESC here. Figure 4 depicts the block diagram of the conventional PID method.

The cascade PID control law is formulated as

$$
\begin{gather*}
W_{f}=k_{p 2} e_{2}+k_{i 2} \int_{0}^{t} e_{2} d t+k_{\theta_{0}} \frac{d \theta_{0}}{d t} \\
e_{1}=N_{p \mathrm{cmd}}-N_{p}, \quad e_{2}=N_{g \mathrm{cmd}}-N_{g}  \tag{32}\\
N_{g \mathrm{cmd}}=k_{p 1} e_{1}+k_{d 1} \frac{d e_{1}}{d t},
\end{gather*}
$$

where $k_{p 1}, k_{i 1}$ are the relative parameters for outer loop, $k_{p 2}, k_{i 2}$ are the parameters for inner loop, $e_{1}, e_{2}$ are denoted as errors for the two feedback loops, and $K_{\theta_{0}}$ is collective feed forward gain. And all the parameters for the PID control are well modulated and verified over the entire envelope.

Two testing cases are demonstrated as follows.
4.1. The First Testing Case. In this simulation case, the helicopter is initialed from a hover state with a low height $H=100 \mathrm{~m}$ and low forward velocity $V_{x}=8 \mathrm{~m} / \mathrm{s}$, and the relative engine states are power turbine speed $N_{p}=100 \%$ and gas turbine speed $N_{g}=87.88 \%$. At $t \stackrel{p}{=} 0 \mathrm{sec}$, a rapid climbing task (or bop up) began, and the command signals for the inner layer are preset as $V_{z \mathrm{cmd}}=4 \mathrm{~m} / \mathrm{s}$, $N_{\text {pcmd }}=100 \%$, and $N_{\text {gcmd }}=92.88 \%$. For the purpose of clarifying more clearly, all the parameters related to inner layer control are presented as deviations using a notation $\delta$. Then, conditions in Theorem 1 should be firstly checked. For inner layer or engine control, the parameters in condition (A)
are modulated as $\gamma_{E}=0.25, q_{E}=2$. Next, condition (B) is easily qualified by the above control law, and $M_{E}=1, \lambda_{E}=$ $\lambda_{\text {min }}\left(\overline{\mathbf{A}}_{E}+\overline{\mathbf{B}}_{E} \overline{\mathbf{K}}_{E}\right)=-1.7257$ would be gotten based on the formulation $\left\|e^{\overline{\mathbf{A}}_{E}+\overline{\mathbf{B}}_{E} \overline{\mathbf{K}}_{E}}\right\|<M_{E} e^{\lambda_{E} t}$. There upon for initial condition can be quantified as follows:

$$
\begin{equation*}
\left\|\overline{\mathbf{x}}_{0}\right\|<\frac{\left|\lambda_{E}\right|}{\gamma_{E} M_{E}^{q_{E}}}=\frac{1.72}{0.25}=6.88 . \tag{33}
\end{equation*}
$$

Figures 5(a), 5(b), and 5(c) depict that the tracking responses of the three channels of ascending velocity $V_{z}$, power turbine speed $N_{p}$, and gas turbine speed $N_{g}$, and the initial states for them satisfy the formulation $\left\|\overline{\mathbf{x}}_{0}\right\|^{q_{E}-1}<$ $\left|\lambda_{E}\right| / \gamma_{E} M_{E}{ }^{q_{E}}$. So conditions of Theorem 1 for this case are fulfilled. Time histories of control inputs are displayed in Figures 5(d), 5(e), and 5(f), respectively, which are rotor collective angle $\theta_{0}$, fuel flow $W_{f}$, and turbine bleeding gas flow $W_{\text {gout }}$. As can be seen clearly from these figures, when the proposed method is utilized to execute the flight task, it takes about 2.0 seconds for the helicopter to track the command signals asymptotically. Otherwise when using the PID method, the transient time of this process is about 10.0 seconds. Therefore, the TLESC enhances greatly the dynamic performance in the climbing task and significantly reduces the tracking time. Figures $5(\mathrm{~g}), 5(\mathrm{~h})$, and $5(\mathrm{i})$ give the time histories of power supplying to helicopter $H_{P P}$, total temperature of gas turbine outlet $T_{45}$, and stall margin of compressor $\mathrm{SM}_{\mathrm{C}} . H_{P P}$ changes are explanations for the convergent time to track command signals, and faster $H_{P P}$ changes means faster response to helicopter flight variations. In the transient process, the gas turbine outlet temperature (less than 1000 K ) and stall margin (more than $10 \%$ ) are both in permit ranges. In Figure $5(\mathrm{~g})$, it is shown that turbine bleeding can significantly influence the change rate of output power and bring a rapid change of $T_{45}$ (see Figure 5(h)). Furthermore, an interesting phenomenon can be observed in Figure 5(i); that is, when using this new method $\mathrm{SM}_{C}$ has an increasing trend in the whole process due to a reduction of total pressure in gas turbine outlet. Obviously, Figure 5(b) indicates that when the TLESC law is used, not only is the response time significantly reduced but also the $N_{p}$ variation in transient process is reduced from $1.87 \%$ under PID to $0.46 \%$. Therefore the antidisturbance capability of closedloop system is much more improved by the TLESC law. For the out layer or flight control, the simulation results are also provided here.

Similarly, for out layer or flight control the parameters in condition $(\mathrm{A})$ are modulated as $\gamma_{H}=0.9, q_{H}=2$. Next, $M_{H}=$ $1, \lambda_{H}=\lambda_{\text {min }}\left(\overline{\mathbf{A}}_{H}+\overline{\mathbf{B}}_{H} \overline{\mathbf{K}}_{H}\right)=-4.4776$ would be acquired based on the formulation $\left\|e^{\overline{\mathbf{A}}+\overline{\mathrm{BK}}}\right\|<M e^{\lambda t}$. Thus, the initial state condition of $\left\|\overline{\mathbf{x}}_{0}\right\|^{q_{H}-1}<\left|\lambda_{H}\right| / \gamma_{H} M_{H}{ }^{q_{H}}$ can be quantified as follows:

$$
\begin{equation*}
\left\|\overline{\mathbf{x}}_{0}\right\|<\frac{\left|\lambda_{H}\right|}{\gamma_{H} M_{H}{ }^{q_{H}}}=\frac{4.4776}{0.9}=4.975 . \tag{34}
\end{equation*}
$$

Figures 5(k), 5(l), 5(m), 5(n), and 5(o) depict time changes of forward flight velocity $V_{x}$, sideward flight velocity $V_{y}$, yaw angle $\psi$, lateral cyclic pitch $A_{1 c}$, longitudinal cyclic pitch $B_{1 s}$

(a) Time responses of tracking $V_{z \mathrm{cmd}}$

$-\delta \theta_{0}$-TLESC
$\cdots-\delta \theta_{0}$-PID
(d) Time histories of $\theta_{0}$

(g) Time histories of $H_{P P}$


$$
\begin{aligned}
& -V_{x} \text {-TLESC } \\
& \ldots-V_{x} \text {-PID }
\end{aligned}
$$

(j) Time histories of $V_{x}$

(b) Time responses of tracking $N_{p \mathrm{cmd}}$

$-\delta W_{f}$-TLESC
-. $\delta W_{f}$-PID
(e) Time histories of $W_{f}$

(h) Time histories of $T_{45}$

$-V_{y}$-TLESC
$-V_{y}$-PID
(k) Time histories of $V_{y}$

(c) Time responses of tracking $N_{g \mathrm{cmd}}$

$-\delta W_{\text {gout }}$-TLESC
.-. $\delta W_{\text {gout }}$-PID
(f) Time histories of $W_{\text {gout }}$

(i) Time histories of $\mathrm{SM}_{C}$

(l) Time histories of $\psi$

Figure 5: Continued.


Figure 5: Simulation results for helicopter rapid climbing.
and tail rotor collective angle $\theta_{t}$. It can be found that all the states of flight control are within the range of $\left\|\overline{\mathbf{x}}_{0}\right\|<4.975$.
4.2. The Second Case. In order to verify the robustness of this new TLESC method over the entire envelope, another rapid descent flight demonstration (Figure 6) is also conducted, which is triggered from a hover state of $H=400 \mathrm{~m}, V_{x}=$ $45 \mathrm{~m} / \mathrm{s}$, and $V_{z}=0 \mathrm{~m} / \mathrm{s}$, and the relative engine states are $N_{g}=82.17 \%$ and $N_{p}=100 \%$. Depicted in Figure 6, when at $t=0 \mathrm{~s}$ the command signals are preset as $V_{z \mathrm{cmd}}=-3 \mathrm{~m} / \mathrm{s}$, $N_{\text {pcmd }}=100 \%$, and $N_{\text {gcmd }}=80.17 \%$.

The tracking responses of $V_{z}, N_{g}$, and $N_{p}$ are shown in Figures 6(a), 6(b), and 6(c), whereas the time histories of control variables are demonstrated in Figures 6(d), 6(e), and $6(f)$, respectively. As can be seen clearly from these results, for the closed-loop system based on the new control law, the convergent time for tracking $V_{z \mathrm{cmd}}$ is about 1.6 seconds. On the contrary, when using the conventional PID control, the tracking time is about 18.2 seconds, much slower than the previous one. Thus the closed system constructed by TLESC control has better asymptotically tracking performance. Moreover, Figure 6(b) shows a slight smaller droop of $N_{p}$ under the TLESC method than PID. Furthermore, as can be seen from Figure 6(g), the TLESC control is capable of regulating output power more rapidly, which enhances the engine response to helicopter. Figure 6(i) presents a similar increasing trend in $\mathrm{SM}_{C}$ as happened in the first simulation case, and the mechanism is the same as analyzed in Figure 6. For the out layer or flight control loops, the simulation results are also provided.
4.3. Discussions. Turbine bleeding is added as an extra control parameter in the new control scheme; thereby it has the potential to regulate output power of engine in mechanism. Next, by the aid of the proposed control method, the TLESC method, incorporating with turbine bleeding, fuel flow, and rotor collective control, is developed to reach the control object of faster response for engines.

Of course, this rapid control for power demand also brings some negative effects especially in other flight
channels. As discussed in Section 3.2, the two-layer control significantly reduces the response time in vertical channel, while it also adds extra disturbance to other flight channels as shown in Figures 5 and 6. Nevertheless, the most significant consideration in emergency state is the rapid escaping motion, when the helicopter is close to or fleeing away some obstacle in vertical orientation. Hence, the negative influence can be omitted to a great extent due to the profit in response time.

## 5. Conclusions

A two-layer robust control law, augmented by turbine bleeding, is proposed to implement a feasible emergency state control for an integrated helicopter flight/engine system. Based on the integrated Hawk helicopter/T700 engines model, necessary applications are provided for the integrated system undergoing rapid climbing and decent tasks, in order to verify the feasibility and robustness of this new control method for nonlinear plants. Moreover, the simulation results are compared to conventional control laws. Simulation results show that the closed-loop system, designed by this proposed control law, has better dynamic and static performance in wider envelope and can asymptotically track the command signals more rapidly.

## Nomenclature

| $W_{f}$ : | Main fuel flow (kg/s) |
| :---: | :---: |
| $X_{\text {SUM }}, Y_{\text {SUM }}, Z_{\text {SUM }}$ : | Summed forces for all components of helicopter with respect to A-Frame (N) |
| $W_{g_{\text {out }}}$ : | Turbine bleeding gas flow ratio (-) |
| $L_{\text {SUM }}, M_{\text {SUM }}, N_{\text {SUM }}$ : | Summed moments for all components of helicopter with respect to A-Frame ( $\mathrm{N} \cdot \mathrm{m}$ ) |
| $N_{g}, N_{p}, N_{R}:$ | Revolution speed of gas turbine, power turbine, and rotor ( $\mathrm{r} / \mathrm{min}$ or \% for simplicity) |
| $I_{X}, I_{Y}, I_{Z}$ : | Moment of inertia about $X, Y$, and $Z$ axis with respect to A-Frame (kg.m²) |



$$
\begin{aligned}
& -\delta V_{z} \text {-TLESC } \\
& --\delta V_{z} \text {-PID }
\end{aligned}
$$

(a) Time responses of tracking $V_{z \mathrm{cmd}}$

(d) Time changes of $\theta_{0}$

(g) Time changes of $H_{P P}$


$$
\begin{aligned}
& -V_{x} \text {-TLESC } \\
& --V_{x} \text {-PID }
\end{aligned}
$$

(j) Time changes of $V_{x}$

(b) Time responses of tracking $N_{p \mathrm{cmd}}$

(e) Time changes of $W_{f}$

(h) Time changes of $T_{45}$

(k) Time changes of $V_{y}$

(c) Time responses of tracking $N_{g c m d}$

(f) Time changes of $W_{\text {gout }}$

(i) Time changes of $\mathrm{SM}_{C}$

(l) Time changes of $\psi$

Figure 6: Continued.


Figure 6: Simulation results for helicopter rapid decent.

| $P_{44}, P_{45}, P_{5}$ : |
| :---: |
| $I_{x Z}:$ |
| $V_{g}, V_{p} V_{\mathrm{nz}}:$ |
| $\begin{aligned} & G: \\ & T_{44}, T_{45}, T_{5}: \end{aligned}$ |
|  |  |
|  |

total pressure of gas turbine outlet, power turbine outlet, and nozzle outlet ( Pa )
Moment of inertia product about the crossing axis with respect to A-Frame (kg.m²) typical volumes of gas turbine, power turbine, and nozzle $\left(\mathrm{m}^{3}\right)$
weight of helicopter ( N )
Gas total temperature in gas turbine outlet, power turbine outlet, and nozzle outlet (K) gravity constant ( $9.8 \mathrm{~m} / \mathrm{s}^{2}$ )
Power needed by helicopter, power supplied from power turbine, power supplied from gas turbine, and that needed for compressor (kw)
$\Psi, \phi, \Theta: \quad$ Yaw angle, roll angle, and pitch angle of helicopter ( ${ }^{\circ}$ )
$m_{g 44}, m_{g 45}, m_{g 5}, m_{a 3}:$
Gas flow in gas turbine outlet, power turbine outlet, nozzle outlet, and compressor outlet air flow ( $\mathrm{kg} / \mathrm{s}$ )
$\theta_{0}, A_{1 c}, B_{1 s}, \theta_{t}: \quad$ Rotor collective angle, lateral cyclic pitch, and longitudinal cyclic pitch ( ${ }^{\circ}$ )
$H: \quad$ Flight altitude
$J_{R}: \quad$ Moment of inertia of rotor ( $\mathrm{kg} \cdot \mathrm{m}^{2}$ )
$J_{p}, J_{g}:$
$J_{\mathrm{GB}}:$
$\mathrm{SM}_{\mathrm{C}}$ :
$J_{\mathrm{TL}}$ :
$Q_{p}$ :
Power turbine moment of initial, power turbine moment of initial ( $\mathrm{kg} \cdot \mathrm{m}^{2}$ )
Moment of inertia of gearbox (kg.m ${ }^{2}$ )
Stall margin of compressor Moment of inertia of tail rotor (kg.m ${ }^{2}$ )
Output torque of power tur- bine $(\mathrm{N} \cdot \mathrm{m})$

| $J_{E}$ : | Moment of inertia of engine $\left(\mathrm{kg} \cdot \mathrm{m}^{2}\right)$ |
| :---: | :---: |
| $Q_{H}$ : | Torque of helicopter ( $\mathrm{N} \cdot \mathrm{m}$ ) |
| $J_{\text {acc }}$ : | Moment of inertia of other accessories ( $\mathrm{kg} \cdot \mathrm{m}^{2}$ ) |
| E-Frame: | An earth fixed coordinate system |
| $\Omega_{R}$ : | Rotor speed (rad/s) |
| $X_{E}, Y_{E}, Z_{E}$ : | $X$ axis, $Y$ axis, and $Z$ axis in EFrame |
| $\Omega_{E}$ : | Engine speed (rad/s) |
| A-Frame: | An airframe fixed coordinate system |
| $\Omega_{\mathrm{GB}}$ : | Gearbox output shaft speed (rad/s) |
| $X, Y, Z$ : | $X$ axis, $Y$ axis, and $Z$ axis in AFrame |
| $\Omega_{\text {TR }}$ : | Tail rotor speed (rad/s) |
| H-Frame: | A rotor hub fixed coordinate system |
| Cmd: | Command signal |
| $X_{H}, Y_{H}, Z_{H}$ : | $X$ axis, $Y$ axis, and $Z$ axis in H Frame |
| Subscript $H$ : | Helicopter |
| $V_{x}, V_{y}, V_{z}$ : | Velocities with respect to EFrame (m/s) |
| Subscript $E$ : | Engine |
| $u, v, w$ : | Velocities with respect to AFrame (m/s) |
| Subscript 1, 2, 3, 4, 44, 45, 5: | engine inlet, compressor inlet, combustion chamber inlet, gas turbine inlet, gas turbine outlet, power turbine outlet, and exhaust nozzle outlet |
| $p, q, r$ : | Angular rate about $X$-axis, $Y$ axis, and $Z$-axis with respect to A-Frame (rad/s). |

## Conflict of Interests

The authors declare no conflict of interests.

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## Review Article

# Modeling and Recognizing Driver Behavior Based on Driving Data: A Survey 

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#### Abstract

In recent years, modeling and recognizing driver behavior have become crucial to understanding intelligence transport systems, human-vehicle systems, and intelligent vehicle systems. A wide range of both mathematical identification methods and modeling methods of driver behavior are presented from the control point of view in this paper based on the driving data, such as the brake/throttle pedal position and the steering wheel angle, among others. Subsequently, the driver's characteristics derived from the driver model are embedded into the advanced driver assistance systems, and the evaluation and verification of vehicle systems based on the driver model are described.


## 1. Introduction

Modeling and recognizing human driving behavior have been of interest to researchers from many different disciplines like psychology, physiology, and ergonomics for more than half a century. Great progress has been made from the numerous specific studies on the various aspects of human physiology and psychology by capturing biological data. Driver model research has been made from the perspective of vehicle dynamics application [1] and human factors. Output parameters of driver models are usually steering wheel angle/torque, acceleration or brake pedal position/pressure, and the gear shift position. Driver model can be applied to (1) vehicle dynamics [1] including vehicle component design, vehicle dynamics analysis, overall vehicle stability analysis, and design of onboard controls; (2) intelligent transport systems (ITS) [2-4] including simulation of traffic flow based on the control theory models of driver behavior and modeling driver's risk taking behavior (3) driverless vehicle systems $[5,6]$. This paper aims to present the methods of recognizing driver's characteristics or modeling driver's driving behavior/skill/state from the perspective of driving data in detail, such as vehicle velocity/acceleration, throttle/brake position, and lateral acceleration.

It is commonly known that driving a car is a complex and dynamic task requiring drivers not only to make accurate perceptions and cognitions about information pertaining to the driver's own driving skill, driver state, vehicle performance, and traffic, but also to process all these information at a high rate of speed. Hence, Liu and Salvucci [7] have pointed out that driver models should take into account the characteristics of both high-level cognitive processing and low-level operation controlling.

Modeling human driving behavior and recognizing driver characteristics are necessary to relieve the driver's workload and improve the reliability and amenity of active vehicle safety systems, for example, collision detection and avoidance systems, and road departure warning systems. However, these active safety systems were designed based on an average of driver performance and rarely takes the individual driver's characteristics into consideration. Thus, even though average drivers can benefit from these systems, individual or special groups of drivers such as novices or the elderly might not be able to take advantage of them as effectively. If the characteristics of driver behavior can be accurately recognized and applied to dynamic vehicle systems, the vehicle might be personalized and therefore made intelligent.

Recognizing driver characteristics is by itself not a simple task with the other requirements of active vehicle safety and comfort of vehicle adding to the complexity. Many active safety systems, such as the automatic braking system (ABS), lane departure warning system, acceleration slip regulation (ASR), and various human-friendly vehicle control systems like adaptive cruise control system, lane-keeping assistant system, have been invented over the years. Information about the driver's driving skill can be used to adapt vehicle control parameters to facilitate the specific driver's needs in terms of vehicle performance and active safety [8]. According to the different objectives set by the various tasks which can be regarded as these actions performed with the help of those functions such as steering, speed control, gear shifting, interpreting the road ahead, and navigation, driving skill can be defined in many ways $[9,10]$. Different tasks require different driving skills. To win a race, excellent driving skills are required. On the other hand, to drive a car from point $A$ to $B$, the driver only needs to obey traffic rules with minimal skills involved.

Generally speaking, modeling and recognizing the driver behavior or driving skill/state can be classified into four steps.
(i) Modeling Driver Behavior. The model structure can be established and parameters of the driver model can be identified based on human driving behavior, which might be classified roughly into three cases: parameter identification, nonparameter identification, and semiparameter identification.
(ii) Recognizing the Characteristics of Driver Behavior. After driver model has been determined, the driver behavior or driver's driving skill should be characterized. Here, many driving tasks or situations (such as car following, lane change, collision avoidance, etc.) are described with numerous mathematical methods adopted.
(iii) Evaluating and Verifying Based on the Driver Model. The objectives of identification followed by the modeling of driver behavior are meant to improve the performance of vehicle dynamics and to design more intelligent driver systems. Therefore, the efficiency of the driver model needs to be evaluated and verified, especially in the field of handling quality and driver assistance systems.
(iv) Embedding Driver Characteristics into the Advanced Vehicle Systems. Producing more intelligent vehicle-driver systems is always the engineer's ultimate goal during the design process. Consequently, driver assistance systems that can timely and accurately detect and predict the driver's attention and seamlessly integrate with the driver's characteristics are crucial.

Based on this, the following sections have been arranged in the order of the aforementioned points.

## 2. Identification of Driver Model

Human driving behavior is extremely complex and contains the human characteristics of nonlinearity, uncertainty,
randomness, and so forth. Recently, a large number of articles about modeling driver behavior or recognizing driver model have been published [11-13] from the control point of view. Driver modeling is the simplification of the human driver with logical graphic and equation and so forth and can represent the basic characteristics of human driver like time delay and physical characteristics. Generally speaking, the goal of the driver model is to accurately imitate the driver while accomplishing some assigned tasks, which include two basic parts: longitudinal control (e.g., speed) and lateral control (e.g., steering angle).

The driver model has uncertainty and nonlinear characteristics, but when it comes to certain driving tasks like car following, the structures of the driver model can be determined. According to the certainty and uncertainty of the driver's model structure, the identification methods can be roughly categorized into three aspects from the perspective of pattern identification: parameter identification, nonparameter identification, and semiparameter identification.
2.1. Parameter Identification. During driving, car-following behavior is not uncommon. For example, when a driver is driving a car during rush hour on a highway, the driver may attempt to adjust the vehicle's velocity and its distance by a compromise between the urge to minimize trip duration and to maximize safety. Therefore, car-following models need to be developed in order to enhance traffic safety, and a great deal of car-following models (i.e., Gazis Herman Rothery (GHR) model, safety distance or collision avoidance models (CA), linear (Helly) models, psychophysical or action point models (AP), and fuzzy logic-based models) are presented and discussed in detail in [14]. However, the question of how to recognize the parameters of these driver models from the perspective of system identification under the condition that their structures have been established still remains open.

After one model structure has been prescribed, that is, the model can be shown by a function, and the number of parameters might be finite and fixed, then, the parameter identification techniques can be used to figure out the parameters of the model based on the experiment or simulation data. To address the issue of uncertainty in the driver model structure, Chen and Ulsoy [15-17] have conducted many studies in relation to (driver) model uncertainty including structured uncertainty (e.g., parametric uncertainty) and unstructured uncertainty (e.g., additive uncertainty due to unmodeled dynamics). In [16], while considering the uncertainty both within individual driver and across different drivers, the uncertainty modeling of driver steering control behavior is addressed, and the driver model is treated as a black box, wherein the input and output are lateral deviation from the centerline of the road ( $y_{\mathrm{dev}}$ ) and the steering wheel angle ( $\delta$ ), respectively. Chen and Ulsoy pointed out that the driver model structure considers the uncertainty characteristics, but model selection is dependent on the real driver behavior and the examination of experiment data [15], thus allowing some unstructured or uncertain aspect of driver behavior to be replaced by specific structuralization elements [18] as follows.
(i) Permissibility or Admissibility. It can also be called the complacency component of driver behavior [15, 19]. One example is the driver keeping his steering command constant when the required change in steering angle is small. This characteristic is illustrated by Figure 1(a).
(ii) Physical Limitations. Most information collected depends on the vision, vestibular, tactile, and auditory perceptions of the driver. However, by reason of human limitations (Figure 1(b)), the driver might be insensitive to some subtle changes. For instance, the driver may be unable to perceive changes in speed when the linear acceleration of the vehicle is lower than 0.005 g [18].
(iii) Transport Time Delay. There is a possibility of response time being different between individuals; for example, aged drivers might spend more time to brake or steer the wheel than younger drivers. The delayed in obtaining the information before starting to action can be replaced by Figure 1(c).

In [13], two new mathematical models (i.e., an optimal controller model and the "look-ahead model") of driver behavior in a single-lane car following situations were developed and identified using the Fletcher-Powell-Davidon (FPD) algorithm by Burnham et al. in 1974.

In many studies of parameter identification or driver behavior modeling, the ARM [19-21], NARMAX [15], and ARMAX [17] methods are usually selected to establish driver model structures to determine the parameters of the prescribed driver model by using experiment or simulation data.

In [22], to design an ACC controller suitable for driver behavior characteristics, three drivers' longitudinal behavior models including linear regression models, state-space models using subspace-based identification, and behavioral models are identified and implemented by using the collected data with the inputs being the space headway and velocity and its differential, the outputs being throttle angle and brake pressure.

In [23], to develop a driver model of curve driving, both the driver's steering control law (Figure 2(a)) and the vehicle-driver model (Figure 2(b)) are presented, wherein three parameters are used for characterizing the driver behaviors, namely, aim point distance $L_{a}$, response delay $T_{k}$, and steering angle $W$. With the aim of recognizing the parameters of driver model, two targets are prescribed: (1) to decrease the lateral deviation between the actual vehicle position and the driver's desired path as much as possible, and (2) to enable the path realized by the driver-vehicle model to be as similar as possible to the path realized by a real driver. With the aforementioned targets in mind, two simulative scenarios have been designed: the double-lane change maneuver (designed by the Standard No. ISO/TR 3888: 1975) and the driving reaction to wind gust. From the paper, we know that the driver model can be identified and this method can be used to research and evaluate the stability of the driver-vehicle systems, as well as make a combination between the vehicle dynamics properties and the individual driver characteristics. Similar to [23], to describe driver behavior more accurately and to simplify driver models, Saleh
et al. [24] developed a cybernetic driver model of lane keeping from the control point of view by adopting the visual, haptic, and kinaesthetic perception and neuromuscular dynamics. The inputs of the driver model are near/far angles, steering angle, and steering force feedback, with the output being steering wheel torque. Subsequently, the driver model can be presented by the state-space structure using the following equation with $\tau_{p}$ being the input delay:

$$
\begin{gather*}
\dot{x}(t)=A x(t)+B u\left(t-\tau_{p}\right), \quad x\left(t_{0}\right)=x_{0}  \tag{1}\\
y(t)=C x(t)+D u(t)
\end{gather*}
$$

The newly developed cybernetic driver model of lane keeping is simpler and can be easily embedded into the driver assistance system.

However, due to the complexity and uncertainty of driving situations, structures of driver model and targets of driver's choice might vary. Thus, with the aim of building a driver model applicable to a wide range of situations, further research about driver model, advanced mathematical methods, and advanced control theory could be done. For example, more nonlinear mathematical models can be used for characterizing the nonlinear driving behavior.
2.2. Nonparameter Identification. If parts of the driver model structures are uncertain and unstructured and cannot be replaced by the abovementioned elements, then these parts should be treated as a black box and identified by using the nonparametric system identification techniques, such as frequency response analysis (FRS) [25], spectral analysis, and estimating the disturbance spectrum.

In the case of nonparameter identification, the Fourier Coefficient Method (FCM) [8, 26] has been used to recognize driver behavior and driving skill.

With the purpose of characterizing and recognizing a driver's limit-maneuver handling behavior, the discrete fourier transform coefficients (DFTC) of steering wheel angle are treated as the discriminant features in [8], and the $N$-point DFTC of steering wheel angle is given as

$$
\begin{equation*}
X_{k}=\sum_{n=0}^{N-1} x_{i} e^{(-j(2 \pi i / N) k n)}, \quad k=0,1,2 \ldots, N-1 \tag{2}
\end{equation*}
$$

In [25], to recognize the parameters of a multiloop carfollowing model structure (Figure 3) that has only one direct forcing function, the driver transfer function can be identified by using frequency domain identification (FDI) methods.
$U(\omega)$ and $V(\omega)$ are relative velocity and acceleration pedal position, respectively, and can be collected during the experiment. The transfer function $H_{d}^{l}$ of a driver's carfollowing behavior can be identified by the spectral analysis techniques:

$$
\begin{equation*}
\widehat{H}_{d}^{l}(\omega)=-\frac{\widehat{S}_{d u}(\omega)}{\widehat{S}_{d v}(\omega)} \tag{3}
\end{equation*}
$$

To find a realistic control theoretic visual driver model of curve driving, the model structure should make the model

(a)

(b)

(c)

FIgure 1: The structuralization elements of driver characteristics: (a) the permissibility, (b) the limitations, and (c) the transport time delay $[15,18]$.

(b)

Figure 2: The diagram representing driver-vehicle steering control systems: (a) driver's steering control law, and (b) the vehicle-driver model [23].
parameters to be identified and estimated as accurately as possible. In [26], many models were evaluated and simulated and, if possible, frequency response function was identified using two system identification methods, namely, FCM and ARMAX method.

A car-following model was developed and identified by Wakita et al. [27, 28] using collected driver's behavior signals such as the positions of throttle/brake pedal and vehicle velocity collected via the driving simulator, as well as two different identification models and features. One


Figure 3: The modified driver model suitable for FDI with only one forcing function $D(\omega)$ [25].
is the stimulus-response model (physical dynamic model) assuming that individual driver's personality can be directly characterized by using model parameters; the other is the nonparametric model based on the statistical pattern techniques. Comparing the parametric models with the nonparametric models, the results show that (1) nonparametric models are better than the parametric models, (2) driver's signals (e.g., gas pedal pressure, brake pedal pressure, and steering angle) are more efficient than the environment and vehicle signals (e.g., velocity, acceleration, and engine speed). The nonparametric model shows promising result in [28].

Based on Wakita's conclusions, a nonparametric model with a Gaussian mixture model (GMM) was developed and identified by Miyajima et al. [29], based on the cepstral features of individual driver by using the spectral analysis of driving signals like gas and brake pedal pressures. In this model, the GMM is used to characterize the distributions of features vectors of cepstral coefficient of each driver with the expectation maximization algorithm adopted to estimate the parameters.

It is well known that driving situations (e.g., traffic factors and driver state) are not invariable when driving. For this reason, the driver's model structures and parameters cannot be prescribed. To characterize driving behavior in the case of steady-state and transient car following, a new nonparameter identification model combining the conditional evolving theory with the probabilistic model is developed by Filev et al. [30].
2.3. Semiparameter Identification. Although parameter and nonparameter identification methods have their own merits of identifying different systems as well as having their own operation range. For instance, the former method might have less stringent input or output requirements, but it needs to select a set of candidate driver models which require a known forcing function; however, the latter method suitable for the nonparameter model and black-box model only takes into account the relation between input and output, but ignores the inner state-variables.

To overcome the disadvantages and inherit the merits of both of them, a concept called semiparameter identification is proposed in this part. For example, the nonparameter and parameter identification method are combined together [25] to realize the objective that using only one forcing function to recognize the multiloop model. This method can be treated as semiparameter identification for the entire system identification.

The flow diagram of driver's model identification including parameter identification, nonparameter identifications, and semiparameter identification can be illustrated by Figure 4.

As seen from the above mentioned cases, any type of driver behavior can be modeled and identified using the parameter or/and nonparameter identification techniques, and most of them are based on the linear invariable and offline model. In [11], a real-time identification method of driver's steering manipulation model has been proposed and validated by using driving simulator experiments and the actual driving tests. To exploit the vehicle sensors utter mostly, the yaw rate, the steering angle, and vehicle's velocity are used as collected data, because these sensors have already been installed in a production vehicle.

From the perspective of lateral driving, to control the parameters of steering and lane-keeping behavior effectively, as well as to distinguish the variations in driving performance, in [12], this paper investigates the abilities of two common driver models. One model is based on the human driver visual perception with the input being the deviation angles between vehicle heading and the directions of experimentally determined preview two-points; the other is based on the lane-keeping task with adopted the vehicle lateral deviations and steering wheel angle as input and output, respectively. The preview point model and the lateral offset model can be denoted by $G_{\text {lat }}(q, \theta)$ and $G_{\text {pre }}(q, \theta)$, respectively, where $G_{\text {lat }}(q, \theta)$ and $G_{\text {pre }}(q, \theta)$ can be described as a second-order rational functions of $q$ and $\theta$ :

$$
\begin{equation*}
G(q, \theta)=\frac{B(q, \theta)}{A(q, \theta)}=\frac{b_{1} q^{-1}+b_{2} q^{-2}}{1+a_{1} q^{-1}+a_{2} q^{-2}} \tag{4}
\end{equation*}
$$

With identification and validation of the two models, Hermannstädter and Yang [12] have made final conclusions that the output error models is superior to the ARX model in simulation, but this method cannot distinguish the induced driver behavior distinctly. The author pointed out that there might be two reasons resulting in this. On one hand, the second-order model is too simple to correctly describe the characteristics of driver behavior; on the other hand, the uncertainty or unstructured factors (e.g., the signal noise and the nonlinear elements of driver) might make great influence on fidelity and accuracy of the driver model. Therefore, the advanced mathematical methods, such as the stochastic, nonlinear, and fuzzy theories, should be taken into account to develop driver model more accurately.


Figure 4: The flow diagram of driver model identification.

## 3. Identification of Driver Behavior and Skill

Lots of vehicle dynamic systems and vehicle control systems are designed by engineers, and they generally put their emphasis and interest on the vehicle itself. Recently, highperformance vehicle cannot meet the needs of customers who require more human-friendly vehicle. Thus, human driving skill and characteristics need to be embedded into the vehicle dynamic systems to improve the vehicle's drivability, maneuverability, and fuel economy.

Characterizing driver behavior and skill exactly is crucial to simulating driver behavior and optimizing driver-vehicleenvironment systems. In order to recognize the characteristics of driver behavior and skill, numerous advanced approaches had been applied, as well as more advanced information collecting and processing technology have been introduced into modeling and recognition of driver behavior and skill. In [31], the computer-aided tools including the instrumented vehicles (IVs) and driving simulators (DSs). have been developed.

Human driving characteristics are presented by Macadam [18] from the control perspective in terms of human behavior activities, such as driver distraction, side-tasking, and driver
impairments. The author pointed out that humans encompass the characteristics of nonlinearity, time delay, and limitation. Some physical limitations are presented in the realm of human factors by Macadam as followed.
(i) Human Time Delay and Threshold Limitations. Humans can be treated as a nonlinear system with time delay and sense limitations. Time delay consists of dead time resulting from the information processing in the central nervous system and the lag due to the nature of the muscular system, which are different for individual drivers.
(ii) Visual Characteristics. Vision system could not capture the velocity and position information accurately due to the jumplike saccadic response of the eyeball.
(iii) Motion Influences. Due to the influence of vestibular, experience and/or skill level may also play a crucial role in a human-vehicle system.
(iv) Auditory Information. Auditory information may be more useful under high workload conditions, and in general it can be treated as redundant information.
(v) Tactile and Haptic Information. Tactile and haptic information (e.g., steering wheel torque, the pedals position) conveyed through the steering wheel and throttle or acceleration pedals, but the fidelity of the information has threshold limitations.

Based on the assumption that a driver remains in control most of the time, a method to characterize and evaluate the specified driving skill was developed on the basis of path tracking driving skill by Erséus [9] in driving simulator tests. In these tests, four scenarios are designed as follows.
(i) Curved Cone Track Scenario. According to [32-34], we know that road width and curve radius have great influence on driver's speeds choice that can be treated as driver characteristics. The goal of this scenario was to investigate driver behavior with a focus on the variation of different driver's ability to steer the vehicle, that is, path tracking skill.
(ii) Avoidance Maneuver Scenario. This scenario was used to evaluate the relationship between driver skill and many objective vehicle parameters measured in the moving base simulator at VTI.
(iii) Driver Response Scenario. This scenario was designed and evaluated for the investigation of driver-vehicle characteristics when following a movable reference line, that is, line jump scenario.
(iv) Curving Road Scenario. Curving road scenario was designed in order to evaluate objective parameters of driver's driving skill when he or she drives on a normal curving road.

Erséus thought that there are some limitations in [9]. (1) This research just focused on the path tracking driving skill of drivers holding a Swedish driving license. (2) All the results are concluded based on the research about behavior
characteristics of the group, not the individual driver. (3) The scenarios designed are too few to demonstrate that whether tracking driving skill is the same with other driving skills or not. Thus, it is necessary to make further research about other types and characteristics of individual/group driver's driving skill in the same scenario, but different scenario parameters.

Angkititrakul et al. [35] developed a stochastic driverbehavior model that can characterize individual driver better than universal models in both short-term and long-term predictions by using the observed driving data based on Gaussian mixture model (GMM). Nevertheless, there are some disadvantages with the GMM, mass data should be collected and processed in-time in order to establish individual driver models more accurately. Then, to recognize individual and general driver's characteristics, Angkititrakul et al. [36] presented an improved driver-behavior model which involved both of them, and the patterns of individual and general driver styles are modeled by using Dirichlet process mixture model (DPM) and GMM, respectively. The result shows that the integrated model can better represent both observed and unobserved individual driver's behaviors.

It is commonly known that driving behavior/skill/styles are influenced by numerous factors, such as the driver's physiology/psychology, driving environment, and traffic conditions. The precise-driver model should take these factors into consideration, but it is not practical to collect and process so much data in-time. In addition, driver's model structures and parameters are uncertain in most common conditions. Therefore, a stochastic evolving real-time identification method was introduced in [30], and a new driver model was developed under the steady-state and transient car-following situation. The results show that this driver model is able to characterize driver's dynamic behaviors effectively in the uncertain driving situations.

Lin et al. [37] propose key parameters in a dynamic driver model to characterize driving skill. The general overview of [37] is illustrated in Figure 5. In this approach, the driver model is dynamic so as to mimic human driver outputs, and using an extensive set of driver model parameters the driver skill level is categorized into three levels: lower, typical, and expert. This model-based approach depends heavily on the validity and fidelity of the mathematical driver model.

In [8], Zhang et al. compared the utility of various pattern-recognition algorithms, including multilayer perception artificial neural networks (MLP-ANNs), decision tree, and support vector machines (SVMs), based on the coefficients of the discrete Fourier transform (DFT) of the sensor information (e.g., steering wheel angle, yaw, and lateral acceleration) getting from the driving simulator. The experiment results show that the DFT coefficients of the steering wheel angle not only can be applicable to discriminating the expert drivers from typical or low-skilled drivers, but also could be used as the discriminant features. Zhang et al. addressed this problem in their proposed pattern-recognition approach [8] for driving skill characterization. This approach is based on the theory that there are strong correlations between the driver's behavior and vehicle response. In this approach (Figure 6), the driver-modeling step is skipped and the relationship between driver's overt behavior and
the driver's driving skill is directly build. While both of these proposed an approach for driving skill characterization in different driving courses based on the driving simulator, it is hard to explain which course parameters (e.g., the course curve radius) could characterize driver's driving skill best.

In [19], a recognition method of steering behavior (e.g., lane keeping, lane changing) was presented and a new arithmetic was developed to improve the awareness of driving safety by using sequential labeling method based on boosting framework. To develop a discrimination model of lane change behavior recognition algorithm (boosting algorithm), four features are focused on: velocity, steering wheeling angle, moving variance, and moving standard deviation. One main result of these experiment data is that the threshold values of lane keeping and lane change behavior are different depending on the vehicle's velocity even if the moving standard deviation of steering wheel angle is the same value.

In [20], Pilutti and Ulsoy presented an online identification approach of driver state that is a desirable element of many proposed active safety systems. In this approach, an ARX model is allowed to describe the relationship between vehicle lateral position $(y)$ and steering wheel angular position $(\delta)$ based on driving lane-keeping task. In the model, $y$ and $\delta$ are the input and output, respectively, with an ARX structure as the candidate model structure:

$$
\begin{equation*}
A(q) y(t)=B(q) u(t-n k) \tag{5}
\end{equation*}
$$

where $y(t)$ is the driver model steering position output ( $\delta$ ), and $u(t-n k)$ is the delayed driver model input. Then, in [38, 39], the approach was applicable to collecting data from 12 2-h highway driving runs conducted in a full-vehicle driving simulator. In particular, in [38], the authors pointed out that there were five aspects of shortages in this approach: model structure inadequacy, nonlinear effects, poor model fits, trends masked by variations in parameters, and alternative approach.

Driver behavior encompasses the characteristics of dynamic, randomness, and nonlinearity, as well as obeying certain distribution. The complex mapping from sensory input to driver's action output might be strongly nonlinearity in nature; hence, the traditional control methods like PID control are unable to simulate human-driver-vehicle system actually. To overcome this problem and to improve the validity and fidelity of driver model, most of stochastic, nonlinear, and fuzzy theories (e.g., Hidden Markov Model system (HMMs), Hierarchical Hidden Markov Model (HHMM), autoregressive HMM (AR-HMM), nonlinear regression models, and the neural networks and fuzzy systems) have been used to recognize and predict driver behavior (see [40-45]).

Rich in mathematical structure, HMMs are powerful parametric models which have been applied extensively in the area of stochastic signal processing. To overcome (1) dynamic and/or (2) stochastic of driver model in nature, Pentland and Andrew [43], Nechyba and Xu [44] propose a driver model using HMM, and then the fidelity of the driver model is verified.

Drivers collect information of the vehicle or environment such as vehicle position, road profile, and pedestrians mainly


Figure 5: Model-based approach for driver skill characterization [37].
via visual system during driving. Macadam [18] has shown that vision ranks the top among the primary sensory (vision, vestibular and kinesthetic, tactile, and auditory) channels used in driving environment. With the visual information as a key consideration, Liu and Salvucci [7] described the application of Markov Dynamic Models (MDMs) in the field of modeling and prediction of driver behavior based on the assumption that driver's visual scanning behavior can be treated as another source of driver's state information.

As stated by aforementioned driver characteristics, it is obvious that driver-vehicle systems have the same characteristics as humans in nature. In recent years, the vehicles installed active safety systems are not uncommon in modern car, but most of them are designed by engineers who rarely take into account the human factor during the design process. Thus, modeling human-vehicle systems allowing for clarifying the relation between driver and driver assistance systems can facilitate the operating mode transitions. Kuge et al. [45] have proposed a recognition method of driver behavior by adopting HMMs to characterize and detect driving maneuvers, and then it was applied to the framework of a driver's behavior cognitive model. The authors put emphasis on information processing models of human drivers with using them to detect and recognize model-based HMMs. This paper demonstrates that (1) HMMs can be used to recognize the frame of driver model based on driver's lane change behavior; (2) an active vehicle safety system embedded with driver model can be developed. Although HMMs have some advantages of both recognizing certain driver behavior and mapping the relation between driver behavior/state/skill and vehicle responses (e.g., yaw, yaw rate, vehicle velocity, and acceleration), some questions still remain open as to its validity of general application in research.

Sekizawa et al. [21] pointed out that stochastic and nonlinear characteristics of the human driver could be expressed as much as possible by the abovementioned models, but there are two shortages in them. (1) The aforementioned models are often too complicated to recognize model parameters


Figure 6: Pattern-recognition approach.
rapidly and accurately, and (2) this, in return, makes it impossible to understand the physical behavior; it is often found that a driver appropriately switches between certain simple primitive skills instead of adopting a complex nonlinear control law. To formally address these shortages, in [21], modeling and recognition of driver behavior were developed based on a stochastic switched autoregressive exogenous (SSARX) model (Figure 7). The SS-ARX model is applied to characterizing driver's collision avoidance behavior at the instance when the preceding vehicle is brought to a sudden halt and the examinee is looking away from the road.

With the aim to simulate driver's collision avoidance behavior, three kinds of driving information, such as range between cars, range rate, and lateral displacement between cars, are collected, and the output value is also specified as steering amount. This experiment result shows that each driver's characteristic is unique; in particular, large variations are observed between driver behaviors with respect to the lateral displacement between cars and steering profiles.

Compared to the HMM or neural network model (NNM), the SS-ARX model has some advantages over both of them. First, the SS-ARX model can provide the information with extraction of driving primitives, but the HMM cannot. Second, when it comes to the discrete modes, the SS-ARX may show unique advantages over the standard HMM. Third, NNM can obtain the parameters of driver characteristics; however, the significance of them is not clear. Therefore the SS-ARX model can present the part from the control perspective with the switched controlled mechanism.

Since the piecewise polynomial (PWP) model or piecewise linear (PWL) model [46] includes both continuous behavior given by polynomials and discrete logical conditions, it can be regarded as a class of hybrid dynamical system (HDS). A modeling strategy of human driving behavior based


Figure 7: The SS-ARX model (three modes) [21].
on the controller switching model with focus on driver's collision avoidance maneuver was presented by Kim et al. [47, 48]. This model was expressed by PWP or PWL model, and the driving data (acceleration, braking, steering, etc.) are collected by using a three-dimensional driving simulator (3D-DS) based on CAVE. In this model, the driver's collision avoidance maneuvers are divided into four piecewise modes: the first period of avoidance, the second period of avoidance, the first period of recovery, and the second period of recovery. Subsequently, the parameters of every piecewise mode were identified using the mixed integer linear programming (MILP) techniques.

In [49], Michon held that driver model tasks should probably best be further classified into three hierarchies of skill from the driver's control perspective.
(1) Strategical (Planning) Level. This level can be treated as decision-making level from long-term perspective, such as the choice of trips goals, route, and driving model. For instance, taking cosiness and fuel-saving into consideration, drivers will choose to drive on roads in good traffic conditions.
(2) Tactical (Maneuvering) Level. In this level, the controlled action patterns, such as obstacle avoidance and overtaking, should be considered, and this level only takes up a few seconds.
(3) Operational (Control) Level. This level is defined from the point of view of control including steering, braking, and accelerating control. The automatic action patterns can be derived from this level.

Based on the three levels, a hybrid dynamical system (HDS) [50] with two parts (i.e., decision making and motioncontrol) included was proposed and designed by Kiencke et al. in 1998. The HDS can be illustrated by Figure 8, which consists of (continuous) primitive and (discrete) switching driving operations [51, 52]. Since the HDS was proposed, most researches of HDS (see, [47, 53-60]) have been made. In [51], a driving behavior model was developed based on the HDS, and a piecewise ARX (PWARX) model was established using driver's sensory information (e.g., the range between vehicles, range rate, and time derivative of the area of the back of the preceding vehicle) and the output of driver behavior, such as pedal operation. The parameters appearing in the primitive (continuous) and switching (discrete) operations can precisely be identified by using the PWARX model, allowing for the developed model to be used to design the advanced driver assistance systems that can switch between multimodels [53] in the HDS. However, the PWARX model cannot distinguish the overlapping modes explicitly, which is the first step to recognize driver model.

To address the issue mentioned in [51], Okuda et al. [61] have proposed a probability-weighted ARX (PrWARX) model, wherein the probabilistic weighting was given a crucial consideration. The difference between PWARX and PrWARX is that the deterministic partition in the PWARX model is replaced by softmax function:

$$
\begin{equation*}
P_{i}=\frac{e^{\eta_{j}^{T} \cdot \varphi_{k}}}{\sum_{j=1}^{s} e^{T_{j}^{T} \cdot \varphi_{k}}}, \quad \eta_{s}=0 \tag{6}
\end{equation*}
$$

where $\eta_{i}, i=1,2 \ldots, s-1$ is used to represent the probabilistic partition between regions corresponding to each mode. By introducing the probability-weighted concept, the decision entropy can be defined and applied to describe the vagueness in the switching operation, as well as being used as a verification index of the model.

In [62], an approach to recognize driver's driving manipulation skill is presented based on the HDS model. Different from the previous ones, HDS is treated as a hinging hyperplane autoregressive exogenous (HHARX) model in which each continuous submodel deals with its related manipulation model, and meanwhile, discrete model can switch between all submodels. Then, the parameters of HDS were identified by using a mixed-integer linear programming (MILP) method. Lastly, the identification model is embedded into a microcontroller to design an automatic driving system [63] with real-time image collection and processing.

Generally speaking, driving can be considered as a dynamic behavior, and the parameters of driver model might change with driving conditions and driver's psychologi$\mathrm{cal} / \mathrm{ph} y$ siological state. Hsiao [64] thought that the previous methods, such as ARX, ARMAX, and HMM, are mostly based on the linear time invariant (LTI) system, which can only be an approximation of driver behavior for a short time. Therefore, the LTI system may not recognize the timevariant parameters precisely. In response to this issue, a time-varying system identification method (i.e., time-varying ARX) has been developed by Tesheng Hsiao using maximum


FIGURE 8: Optional multimodels of hybrid dynamical systems (HDS) containing the (continuous) primitive driving operations and their (discrete) switching.
a posteriori estimation, and the model can be described as in the following equation:

$$
\begin{equation*}
y(k)=-\sum_{i=1}^{n_{a}} a_{i}(k) y(k-i)+\sum_{j=0}^{n_{b}} b_{j}(k) u\left(k-j-n_{k}\right)+\varepsilon(k), \tag{7}
\end{equation*}
$$

where $y(k)$ and $u(k)$ are output and input sequences, respectively, $\varepsilon(k)$ is the process noise, and $a_{i}(k), b_{i}(k)$ are the system parameters required to be identified.

HDS can clearly describe driver model or driving task, and one of its crucial issues is how to recognize the distinct state of driving operation from driver behavior and to determine the number of the state. In [51], the hierarchical clustering method was applicable to estimating the number of state, and then a stochastic piecewise affine (PWA) model was developed by Okamoto et al. [52].

## 4. Evaluation and Verification Based on Driver Model

This section addresses the evaluation and verification of vehicle handling qualities and advanced driver assistance systems based on the driver model from the control perspective.
4.1. Handling Qualities Evaluation. The concept of handling quality is first explained via the field of aerospace engineering [65] and is later applied to land vehicle design and evaluation. Harper [65] discusses the assessment of handling quality characteristics since handling quality deals with more than one element. Accordingly, an airplane and its pilot were represented in order to assess handling quality. In the case of driving a car, the car and its driver are incorporated into the system in the same manner. Therefore, when evaluating the handling quality, the driver should be taken into consideration. Conventionally, handling quality evaluation can be classified into objective handling quality evaluation and subjective handling quality evaluation which is made by observing the dynamic characteristics of the automobile and driver, respectively. A subjective method based on the driver's
comments can be used to evaluate the handling quality in a relatively precise way, but this method requires many actual driving tests that include every design parameter. Furthermore, these evaluations might differ between different test drivers. Especially while evaluating each driver's limitmaneuver handling behavior, this is very dangerous for the test-takers.

Since the 1990s, researchers began studying driver steering dynamics models that could replace test drivers. In [66], a control theoretic model of driver steering dynamics is developed and demonstrated to be able to produce driver/vehicle steering responses and compares favorably with those obtained from driver simulations. Using the theoretical model of driver steering dynamics, engineers who may not be experts in manual control are enabled to evaluate handling or maneuverability.

Driving style, behavior, and skill might vary for different drivers causing evaluation criteria to be diverse and in turn sharply reduce confidence of the handling quality indexes. Therefore, it is imperative to define driver characteristics as a criterion for handling objective evaluation. Reference [67] states that a parametric driver model for ISO lane change simulation was developed using the decreasing parameters dispersion method by Carlol et al.

Similar to [67], while making a subjective evaluation of handling quality, a multiloop structure of closed-loop drivervehicle systems including a multi-input driver model was developed by Horiuchi et al. [68]. In this driver model, the inputs are the lateral position error $y_{e}$ and the yaw angle $\varphi$, with output being the steering angle $\delta$. To characterize driver dynamics accurately, the three essential factors of time lag $e^{-\tau s} /\left(T_{1} s+1\right)$, predictive action $\left(T_{L} s+1\right)$, and proportional action $K$ have been taken into consideration. The driver model can be described by the following equation:

$$
\begin{equation*}
\delta=y_{e}\left[K_{y}\left(T_{L y} s\right)+1\right]-\psi\left[K_{\psi}\left(T_{L \psi} s+1\right) \frac{e^{-\tau s}}{T_{1} s+1}\right] . \tag{8}
\end{equation*}
$$

Subsequently, an analytical approach to subjectively rate handling quality of actively controlled vehicles is discussed and applied to the evaluation of the handling quality of fourwheel steering vehicles. Here, the three principle factors of
the objective function, that is, the task performance $J_{1}$, the driver's mental workload $J_{2}$, and his or her physical workload $J_{3}$, are considered. The results show that this method not only can be applicable to the prediction of the subjective evaluation of handling quality, but is also able to characterize driver-vehicle systems.

By analyzing the driver's characteristics and incorporating them into the closed-loop road-vehicle-driver test system (Figure 9), a cost function of the handling quality that can be used to estimate the handling quality analytically from the vehicle's dynamics is constructed [69]. Three crucial characteristics of driver handling are obtained with the driver-in-the-loop system by Miura et al. and are (1) the driver model's response to the yaw rate has a strong connection with the evaluation of handling quality, (2) different road conditions such as road radius and profile, could result in the drivers having different frequency responses, and (3) there are no notable differences in how each driver operates between driver and vehicle dynamics even though automobile dynamics are different. Based on the aforementioned characteristics, a benchmark driver model has been built and is applied to the handling quality rates in the absence of the driving experiment.

This approach uses two driver models (i.e., transfer function, $\left.H_{1}(s), H_{2}(s)\right)$ to represent the human control with the inputs $e_{y}$ and $e_{r}$ and the outputs $\delta_{f 1}$ and $\delta_{f 2}$. In Figure 9, $H_{1}(s)$ and $H_{2}(s)$ are the steering responses to the lateral displacement $y$ and yaw rate $r$, respectively. Correspondingly, $e_{y}, e_{r}$ are the deviations between the reference value and the actual value; $P(s)$ is the transfer function of a simple fourwheel nonlinear passenger automobile model applied as a vehicle model.

Because drivers can adapt themselves to the handling characteristics of the vehicle during a driving maneuver, their steering behavior reflects the vehicle's handling characteristics and plays an important role in the evaluation of handling quality. In [70], an approach to evaluate vehicle-handling quality based on steering characteristics is presented; wherein the steering characteristics were identified by a simple driver model using the relationship between the time histories of steering behavior and vehicle motion during lane change. The study presented a closed-loop driver-vehicle system (Figure 10), with $H(s)$ and $P(s)$ representing the driver model and the vehicle model correspondingly. A resulting transfer function of steering angle to lateral position error during a lane change is used as the driver model $H(s)$ with the deviation of lateral displacement $\Delta y=y_{0}-y$ as the input and the steering angle $\delta_{h}$ as the output and can be described as

$$
\begin{equation*}
H(s)=G_{h} \frac{1+\tau_{h} s}{1+T_{h} s} \tag{9}
\end{equation*}
$$

where the driver steering parameters $G_{h}, \tau_{h}$, and $T_{h}$ are steady-state gain, the derivative term of differential control, and the time constant of the first-order lag, respectively.
4.2. Evaluation and Verification of Driver Assistance Systems. Similar to the evaluation of handling quality, it is necessary to evaluate and verify the DAS with the human driver in
the closed loop. Reference [71] presents two newly developed driver models, which are applied to evaluating the impact of ACC vehicles on traffic flow and the effect of a vehicle stability control (VSC) system on possible vehicle roll prevention. One of driver models is the modified Gipps model which is used for evaluating the ACC, whereas the other model is applied to the evaluation of active safety systems based on the adaptive plant inversion concept. In its accompanying paper [72], a longitudinal human driver model (i.e., the modified Gipps model) used for performance evaluation of the ACC system on highway traffic from the microscopic and macroscopic traffic perspective was developed and simulated by Lee and Peng.

The majority of DASs can release drivers from some secondary tasks during driving and improve on safety, comfort, and performance. Notwithstanding, the driver might be confused, annoyed, and distracted, if he or she is sensitive to the monitoring and the excessive detection or frequent warning of DASs. For this reason, a driver model used to evaluate DSAs that can precisely mimic human driving is required so as to consider mistakes committed by human drivers. To achieve this, in [73, 74], an errorable driver model (Figure 11) was developed to evaluate both the collision warning and collision avoidance algorithms. The errorable driver model can generate both nominal (error-free) and devious (with error) behavior like in humans. Three common driver mistakes, namely, human perceptual limitations, distractions, and time delay were considered in establishing this errorable driver model.

Numerous driver models can be applied to the vehicle design process and to the evaluation or verification of active safety systems. Most of them are designed with the average (general) or atypical driver in mind and thus are unable to represent individual characteristics. This is illustrated by the fact that the DASs assessed by the universal driver model and its results show that the DASs are suited most for the common driver, but not the individual. In the case of an anxious and impulsive driver, the DASs might give warnings too frequently resulting in the driver getting bored with the DASs and accordingly decrease his or her situational awareness, comfort, and safety and ultimately increase the driver's workload. Hence, a driver model that can effectively characterize individual driving behavior, skill, and styles should be further developed and applied to the evaluation and verification of active safety systems.

## 5. The Advanced Vehicle System Embedded with Driver Characteristics

The ideas previously discussed show that recognizing the characteristics of driver's driving behavior/skill/state are vital to the drivers' safety, vehicle design, fuel efficiency, and vehicle ergonomics. Driving is a complex task which should be executed with several nonlinear subsystems such as the human driver, surrounding vehicles, driving environment, and electronic control systems. It is widely known that a normal/experienced driver can adapt to different vehicle systems and/or driving environments in a short period of


Figure 9: Driver-in-the-loop system [69].


Figure 10: Driver-vehicle system.
time by adjusting his or her nerve (neuromuscular) units. The human driver can develop an optimal route by taking into account long or short periods and change the throttle/brake position and steering wheel angle according to the collected real-time information and instantaneously adjust the vehicle's position or velocity and decrease the deviation between reality and expectation. In other words, after perceiving and processing the driving situation, the driver can select process control rules suitable for the situation that allow him or her to manipulate the vehicle's controls in a manner that satisfies the driver's control objectives even in different scenarios.

Most research on how drivers adjust various dynamic vehicle systems to adapt with their driving environments was conducted from the standpoint of psychology and physiology. To lighten the workload of drivers and in turn reduce the occurrences of traffic accidents, the idea of assisting drivers on the road was proposed for which the various driver assistance systems were subsequently developed. Even though the DAS can alleviate the driver's workload, there is still a chance of it having a negative effect on the driver-vehicle systems due to the disharmony or adverse interactions between the driver and the assistance systems under certain conditions; in conventional car systems, most of the DASs are electronic control systems with invariant design parameters. As it is not hard to imagine a novice driver benefitting more from the early intervention of a power-assisted steering system than an experienced driver [8], the intelligent driving assistance system (IDAS) or the advanced driver assistance system (ADAS) was developed. In the IDAS or ADAS, driver characteristics are embedded into both the longitudinal control and lateral control. In [75], Fancher et al. researched on driver-vehicle coordination before proposing the human-centered vehicle
system concept which has resulted in many similar theories and models following soon after.
5.1. Longitudinal Control Based on Driver Characteristics. With the rapid development of vehicle and traffic technology, car following has become the most prominent driving task, especially while driving on highways or urban roads during rush hour, with the aim of maintaining a safe and comfortable car-following state for the purposes of mitigating the workload of drivers, reducing the occurrences of traffic accidents, and increasing traffic flow rate. Several DASs have been developed based on driver characteristics from the control perspective such as the adaptive cruise control (ACC), stop and go (S\&G), and forward collision warning/avoidance (FCW/FCA).

Vadeby [76] has studied relative collision safety models together with driver characteristics for ten years. The change in consumer demand is reflected in the main design objectives shifting from power and performance to safety, comfort, and intelligence. Accordingly, researchers have focused more on human characteristics and designed "human-centered" automation or operations that account for the driver's expectations and automation goals. Despite being a sound theory, it is still difficult to describe driving behavior to arrive at a unique optimal multiattribute method for solving problems including designing a "human-centered" controller with these multiple attributes in mind. To address the issue, Goodrich and Boer [77] developed a systematic method that uses a multiattribute breakdown of human and automation goals which has been subsequently applied to the design of human-centered collision and accident avoidance systems (CAAS).

Similar to [78], the human-centered DASs including ACC and FCW were also presented by Fancher et al. [75]. To be human centered, DASs need to take vehicle dynamics into account and match them with the driver's physiological, psychological, and other attributes. Various aspects of human-centered DASs were also discussed, namely, the looming effect, ruler-based and skill-based behavior, the utilization of desired dynamics in controlling the driving process, and braking rules or collision-warning rulers. After


Figure 11: The diagram of the errorable driver model [73, 74].
which, the collected field-test data and on-road data were used to evaluate and verify the human-centered DASs. This paper makes great progress in explaining when and why a driver makes a braking action. The author also points out that answering the following basic questions about braking might facilitate the development of human-centered DASs. (1) When should braking occur or not occur? (2) What is the control objective when driving? (3) How do drivers do it?

In $[78,80]$, to develop and verify a new automatic advanced vehicle system integrated with human driver characteristics, an instrumented vehicle test bed called the Laboratory for Intelligent and Safe Automobiles-Q45 (LISAQ) was designed, after which, a collaborative approach for developing human-centered DAS (e.g., ACC) was proposed by McCall et al.

Though longitudinal control systems (ACC, FCW, etc.) can be designed with driver characteristics taken into account, some issues still exist. For example, owing to the finite number of techniques and the incompatibility of subsystems, excessive, inaccurate, or contradictory information might be transmitted, causing advanced vehicle systems to disturb, distract, or even overwhelm the driver. In [81], Zheng and McDonald raised the question on whether "drivers' expectations can be matched". They rightly pointed out that at present, no existing ACC system can deal with the full range of complex traffic situations in practice and that humans should act only as a "monitor" in ACC equipped drivervehicle systems. When a driver's expectations are breached, for instance, he or she will make interferential actions between his or her intentions and the ACC's capability. The best DAS system is not the one most capable of following traffic but the system that considers both comfort and safety characteristics the most. In order to improve the compatibility between ACC performance and driver's expectations, a large number of situations have been tested by changing variables systematically such as the parameters of the ACC algorithms, traffic scenarios, and time-headway settings. The results reveal that an appropriate ACC setting capable of meeting a driver's expectations can be found and that the ACC setting most adept in a range of traffic conditions may not necessarily be the most user-friendly.

Road conditions such as road profile and road friction coefficient might have great influence on the driver's characteristics and the driver-vehicle systems. To deal with the problem of rear-end crashes of moving and parked vehicles, Nakaoka et al. [82] conducted further research on forward collision warning systems (FCWs) that took into account road conditions (dry and wet) and individual driver characteristics. Time to collision (TTC) is usually used to evaluate the severity of a forward collision as the host vehicle approaches another vehicle from the front and can be calculated by

$$
\begin{equation*}
\mathrm{TTC}=\frac{X_{l}-X_{f}}{V_{l}-V_{f}} \tag{10}
\end{equation*}
$$

where $X_{l}$ and $X_{f}$ are the positions and $V_{l}$ and $V_{f}$ are the velocities of the leading and host vehicles, respectively. The formula states that the TTC is related to both the host and leading vehicles; therefore, using it to characterize individual drivers may be unreasonable. In [82], the timing of a driver's braking reaction time is used as a proxy for hazardous level instead of the traditional TTC.

Most driver behavior models applied to the automobile are limited to the single driving task, such as lane keeping and car following. In [79], in order to integrate individual driver characteristics into driver assistance, the state transition feature for individuals are taken into account, more specifically, the five categories involved in longitudinal driving situations of car following, braking, free following, decelerating, and stopping. Longitudinal vehicle dynamics driving data such as acceleration and braking was collected. By classifying the longitudinal driving situations into five parts and adopting the boosting sequential labeling method, the framework for driver-vehicle-environment (Figure 12) can be modeled allowing for the characterization of the driver state, followed by the design of advanced and personalized driver assistance systems.

Reference [83] describes a longitudinal driver model designed to embed individual driver characteristics into an advanced driver assistance system that is mainly applied to simulating throttle and braking operations. In this driver model, a generic model is developed based on the driving data


FIGURE 12: State flow diagram of the driver model with longitudinal driver-vehicle dynamics [79].
(e.g., the longitudinal velocity, acceleration, throttle/braking pedal pressure, and relative distance/velocity) collected under real traffic conditions. Time headway (THW) and time to collision (TTC) were then selected to analyze the longitudinal driving behavior. Subsequently, the parameters of the driver model were determined and identified in-time by using the recursive least-square (RLS) self-learning algorithm with a forgetting factor. Finally, an advanced automatic control systems (Figure 13) embedded with driver characteristics was designed and verified.
5.2. Lateral Control Based on Driver Characteristics. This subsection discusses lateral control systems based on driver characteristics. When the host vehicle is executing a following or tracking task on the straight way, the longitudinal control can be treated as the primary action to ensure that the distance between the host vehicle and the leading vehicle is safe enough. However, lateral control should be given great consideration in the following scenarios.
(1) The First Case Is Curve Driving. Before driving onto a curved road from a straight path or from one curved path to another, the driver usually decreases the vehicle's velocity before steering the wheel to track the curve road.
(2) The Second Is Lane Change. For instance, when a driver wants to overtake on the highway, lateral control is a vital for guaranteeing the host vehicle's safety in traffic flow.
(3) The Third Situation Is the Avoidance of Collisions. If the leading vehicle makes abrupt stop or if a person suddenly appears in front of a fast moving vehicle, the driver will most likely initiate some form of action to prevent a collision from happening, be it quickly steering the vehicle away or slamming on the brakes.

As seen from the abovementioned cases, steering wheel control is crucial for keeping the driver safe. In order to improve road tracking performance and relieve a driver's workload, the electric power steering (EPS) embedded with driver's characteristics has been designed. In [84, 85],
the steering assistance systems (Figure 14) for driver characteristics are presented with two controllers (guidance and steering) designed based on the gain scheduled control theory by Fujiwara and Adachi.

In the steering assistance systems, $\Delta D$ depicts driver characteristics. The steering wheel angle $\theta_{S}$ and the vehicle's steering torque $T_{S}$ are considered as the characteristics of a driver's operation.

Driving an automobile can be considered as a closed-loop control task executed by the human driver [1]. By modeling and recognizing driver behavior based on driving data, the characteristics of human driving behavior/skill/state can be used for the advanced controller design of vehicle dynamics systems embedded with the driver's characteristics [79, 8385]. However, human driving behavior encompasses the characteristics of randomness and uncertainty; hence, traditional feedback control systems based on driver characteristics may not completely represent the driver-vehicle system. To accurately simulate the driver-vehicle systems as well as aid in the development of more advanced driver-vehicle control systems, other methods of advanced controller design might be adopted, such as the robust static output feedback control (SOF) [86], $H_{\infty}$ step tracking control [87], robust $H_{\infty}$ slidingmode control (SMC) [88], and the networked predictive control [89].

When driving from one curve onto another especially at high speeds, the vehicle might reach its handling limit and could lead to a rollover accident. With vehicle stability control as a key consideration, an intelligent personal minder (IPM) system (Figure 15) has been developed [90]. This vehicle control system contains an IPM system that is able to provide timely and clear advisory information to the driver. When certain parameter values of the vehicle are close to the vehicle's handling limit or the defining relative handling limit $\operatorname{margin} h_{\text {env }}=\min \left\{h_{\mathrm{OS}}, h_{\mathrm{US}}, h_{\mathrm{TCS}}, h_{\mathrm{ABS}}, h_{\mathrm{SSRA}}\right\}$, the IPM will provide a timely warning to the driver, where $h_{\mathrm{OS}}$ and $h_{\mathrm{US}}$ are vehicle's yaw handling limit margin during oversteer and understeer situation respectively, $h_{\mathrm{TCS}}$ and $h_{\mathrm{ABS}}$ are the ABS and tracking handling limit margins, respectively, and $h_{\text {SSRA }}$ is the vehicle's sideslip handling limit margin. According to the $h_{\text {env }}$, a driver's adaptive styles in handling the vehicle under


Figure 13: The control strategy of the driver assistance system [83].


Figure 14: The system configuration of a driver-support-system embedded with driver characteristics [84, 85].
various driving conditions can be characterized in real-time with the control parameters adjusted correspondingly to the different driving styles; notwithstanding, this method should only be applied to long term rather than short term advisory. As driving skills can be used as a short term reference for adjusting the parameters of the DAS or ESC, it needs to be defined.

Similar to [91], for the purpose of improving the electronic system's intelligence and flexibility thus allowing it to better recognize the driver's expectations or intentions, three modes to characterize the driver's behavior or styles (aggressive and cautious behavior) were discussed based on the driver-in-the-loop system (Figure 16) established by Filev et al. [91].
(i) Characterizing Unstructured Driver Behavior. From the long-term perspective, longitudinal vehicle control might be primarily affected by the driver's behavior or driving styles, but not the vehicle dynamics response. It is usually conditionally applied to the unconstrained driver by detecting the brake pedal position or the rate change followed by making the corresponding adjustments to the variables of the vehicle; this is positive to the vehicle's fuel efficiency and acceleration performance. Statistical mathematics and probability theory are usually adopted in the process of characterizing unstructured driver behavior.
(ii) Characterizing Semistructured Driver Behavior. In this method, part signals of the electronic control systems can be
used as the feedback information with the driving task being constrained, for example, car following, double lane change, and collision avoidance. The fuzzy control theory might be used in the course of recognizing driver behavior.
(iii) Characterizing Structured Driver Behavior. In order to fully utilize the vehicle's dynamic feedback information from the control theory point of view, the driving task should be described in more detail with the driver treated as a controller in the driver-in-the-loop system. To elaborate, the demanded safety distance between the host vehicle and the leading vehicle, the relative velocity of vehicles, and the TTC for car following might be used as the controller inputs. Since the driver models are structured under some fixed driving conditions, the classical control theory methods can be applied to the modeling or recognition driver behavior and skills.

Although numerous mathematical methods and concepts can be used to establish driver model to seamlessly coordinate driver and the electronic control system (ESC, DAS, ACC, IPM, etc.), further research needs to be made as there are still some issues left unaddressed.
(1) A System with Lower Order and Higher Accuracy. To design more intelligent and human-friendly vehicle's dynamic systems embedded with driver characteristics, driver model structures with greater accuracy but at the same time minimalistic and capable of both high-level cognitive processing and low-level operative control should be developed.


Figure 15: The block diagram of a vehicle control system augmented with an intelligent personal minder [90].


Figure 16: The driver-in-the-loop system configuration [91].
(2) Traffic Factors Should Be Considered. The various research studies we have discussed in this paper were only of humanvehicle interaction/cooperation and did not consider driving environment. When a driver is driving, the driver, the vehicle, and traffic factors can be treated as a closed-loop system, in which the traffic factors (e.g., weather condition, pedestrian behavior, the traffic light, and road conditions) have great influence on a driver's behavior. Therefore, we recommend that the relationships of vehicle to vehicle (V2V), vehicle to infrastructure (V2I), and vehicle to the external environment (V2E) to also be explored and studied further.
(3) Seamless Coordination between Vehicle and Driver Characteristics. The intelligence, reliability, and comfort provided by the seamless coordination between vehicle and driver characteristics can be improved on by introducing advanced mathematical methods, control theories, and system identification methods into the modeling and recognition of driver behavior and skills. Even though the concepts of "human-centered" and "driver-aware" vehicle systems have been proposed, the advanced ESC might impede on the driver's expectations and control due to inaccurate system identification as well as the influence of unstructured uncertainty models (e.g., additional uncertainty due to unmodeled dynamics).

## 6. Conclusions

The reviewed articles reveal a wide range of mathematical methods of modeling and recognition of driver characteristics which can be used to improve vehicle's dynamics performance, decrease driver's workload, and develop more intelligent driver assistance systems. Modeling and recognition of driver behavior/skill/state have great important significance in many fields, such as active safety systems, intelligence transport systems, and smart car systems. Most of the researches proposed in this paper have their focus on driver assistance systems and active safety systems, and their goals can be classified roughly as follows: (1) to put driver model in simulators with aim to evaluate and verify driver assistance systems; (2) to put driver model in simulators in order to recognize driver's driving behavior/skill/state in certain task (e.g., lane change, collision avoidance, and haste braking); (3) to be concern on about whether the human driver can fit to the vehicle systems or not by modeling and analyzing driver behavior; (4) to characterize driver's driving skill/state/styles with driver model.

This paper shows that numerous driver models have been developed from different perspectives by using various identification methods, and the characteristic parameters of driver's driving behavior/skill/state are not the same for
different driving tasks and situations. Some methods may seem to be more efficient than others for certain driving tasks and situations described in this paper, but these methods might be not the best for other driving tasks. Therefore, some issues still exist, for example, which parameters and driving situations are more sensitive to driver's driving behavior/skill/state, that is, how to characterize the human driver more exactly, easily, and quickly? Hence, further research may be conducted as in the following aspects.
(1) The driver model which can accurately describe all the individual driver's behaviors for different driving tasks and situations should be developed.
(2) The advanced mathematical methods which can precisely and quickly characterize driver behavior and skill may be introduced and developed.
(3) The advanced control theory should be seamlessly coordinated with driver characteristics.

In addition, driver's psychological and physiological factors which are rarely discussed in this paper are crucial to driver models; hence, more and more researchers may show interests in modeling and recognition of driver behavior in the future.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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# Nonlinear Analysis of Cable Vibration of a Multispan Cable-Stayed Bridge under Transverse Excitation 

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#### Abstract

The nonlinear vibrations of cable in a multispan cable-stayed bridge subjected to transverse excitation are investigated. The MECS (multielements cable system) model, where multielements per cable stay are used, is built up and used to analyze the model properties of the multispan cable-stayed bridges. Then, a simplified two-degrees-of-freedom (2-DOFs) model, where the tower or the deck is reduced to a beam, is proposed to analyze the nonlinear dynamic behaviors of the beam and cable. The results of MECS model analysis show that the main tower in the multispan cable-stayed bridge is prone to the transverse vibration, and the local vibration of cables only has a little impact on the frequency values of the global modes. The results of simplified model analysis show that the energy can be transformed between the modes of the beam and cable when the nature frequencies of them are very close. On the other hand, with the transverse excitation changing, the cable can exhibit richer quasi-periodic or chaotic motions due to the nonlinear terms caused by the coupled mode between the beam and cable.


## 1. Introduction

Recently, as use of multispan cable-stayed bridges increases continuously, the real local vibration features of inclined cables and coupled oscillation characteristics between the cable and the bridge is becoming a new topic in the design process. The stabilization of the central towers under extreme wind or seismic vibration is a key issue, since they cannot be anchored to an outer fixed support. One solution is to use the stabilizing cables which run from the top of the central towers to a location on the deck near the side towers, such as the Ting-Kau Bridge in Hong Kong. Another solution is to increase the stiffness of the central towers itself, and most multispan cable-stayed bridges adopt this way, such as the Maracaibo Bridge in Venezuela, the Millau Bridge in France, the Mezcala Bridge in Mexico, the Dong-Ting Lake Bridge, the Yi-Ling Bridge, and the Bin-Zhou Bridge in China.

Ni et al. [1] investigated the effect of stabilizing cables on the seismic response of Ting Kau Bridge based on
a validated 3D finite-element model. He et al. [2] investigated the influences of cable local modes and stabilizing cables for the Dong-Ting Lake Bridge on seismic excitation. Liu [3] investigated the seismic performance of rigid system, floating system, and passive energy dissipation system for the Bin-Zhou Bridge under two different earthquake records. In order to evaluate the stability of the main towers of Millau Bridge, Okamoto and Nakamura [4] proposed a new hybrid high tower and investigated their static and seismic behaviors. Most of the above studies focused on the stabilization of central towers, and little attention has been devoted to investigating the effects of cable vibration on the properties of tower or deck under transverse excitation.

On the other hand, all the above researchers used the finite-element (FE) model to investigate the stabilization of towers by taking the specific project as a paradigm. From the perspective of the coupled vibration between cable and tower (or deck), Fujino et al. [5] presented a 3-DOFs analytical model to investigate the auto-parametric interaction
behavior of cable and beam structure, the case which under a random excitation is firstly studied by Xia et al. [6, 7]. The results show that the horizontal motions of the cable and beam are excited due to the autoparametric nonlinear coupling behavior under some cases. Caetano et al. [8, 9] studied the dynamic interactions between the cable and deck or tower in cable-stayed bridges by physical modelling and experimental testing. Georgakis and Taylor [10, 11] presented an alternative cable-deck model to investigate the nonlinear dynamics of an inclined cable which both induced by sinusoidal and stochastic support excitations. The different cabledeck interactions of the Guadiana Bridge under environmental excitations were investigated by Caetano et al. [12], using the vibration data acquisitions and a refined finite element model. However, few studies focus on the nonlinear vibration of the cable-tower (or beam) coupled system subjected to transverse excitation.

The objective of this paper is to study the nonlinear vibrations of cable in a multispan cable-stayed bridge subjected to transverse excitation. Taking the Bin-Zhou cable-stayed bridge (BZB) as a paradigm, the MECS model, where multielements per cable stay are used, is firstly built up and used to investigate the effects of the cable's vibration on the properties of the BZB. Then, the possibility of the transverse resonance between the local model (cable) and global model (tower or deck) is analyzed. Based on the results of the above studies, a simplified 2-DOFs model, where the tower or the deck is reduced to a tower (or beam), is proposed to analyze the nonlinear coupled vibration of the cable and tower (or beam). The coupled relationship is completely from the dynamic interactions between the cable and tower (or beam). After that, the equations of motion are solved by using Galerkin's method for the spatial problems and the method of multiple time scales for temporal problems. The stability of the steadystate solution is examined. Finally, the nonlinear behavior of the cable and tower (or beam) is analyzed by using the time histories, phase portraits, and Poincare maps with the transverse excitation changing.

## 2. Vibration Properties of the Multispan Cable-Stayed Bridge

2.1. Outline of the Bin-Zhou Cable-Stayed Bridge. The BZB, as shown in Figure 1, is a three-tower cable-stayed bridge with two main spans of 300 m and two-side spans of 84 m each. The bridge deck is separated into two carriageways with a width of 13.75 m each. There are two minor towers with heights of 75.78 m in both sides and a main tower with a height of 125.28 m in the center. It has a unique feature that there is no horizontal component connecting the towers. The carriageways consist of two longitudinal pre-stressed concrete box girders with a length of 767.1 m and a height of 3 m . Along the deck edges, there are cross-girders at 7 m and 6 m intervals for the main span and two side spans, respectively. The locations of cables are consistent with the cross-girders. The bridge deck is fixed at main towers, but linked at the minor towers and two end abutments. There are 200 main stay cables anchored to the towers with intervals of

2 m (partial 2.5 m ). Shown as Figure 1, from left to right the cables anchored to the left minor tower, the main tower and the right minor tower are named as $\mathrm{A} 1 \sim \mathrm{~A} 12, \mathrm{~N} 1 \sim \mathrm{~N} 26$ and J1~J12 respectively.

The 3-D FE model of the BZB is established in ANSYS, as shown in Figure 2. The bridge members, girders, towers, and deck are simulated by spatial beam or shell elements with six DOFs at each node, respectively. In order to reflect the influence of cable vibration, the cables are simulated by spatial spar elements with 3-DOFs at each node; each cable is divided by a length of 6 m . As a result, the FE model involves 7696 nodes and 7514 elements.
2.2. Modal Analysis. Based on the aforementioned criterions, the natural vibration frequencies and global modes are examined by using OECS (one-element cable system) and MECS models. The frequencies and mode shapes of the first 10 modes of OECS model and the first 300 modes of MESC model are computed. Among the latter ones, the maximum frequency is up to 1.598 Hz , and over $85 \%$ of modes are pure cable local vibration modes. This property is similar to the Ting Kau Bridge [1, 13].

According to the same global mode shapes, Table 1 shows a comparison of the natural frequencies of the OECS and MESC models. From Table 1, the first mode of the BZB is predominantly vertical bending, which differs from most of the cable-stayed bridges. This phenomenon can be attributed to the semifloating system adopted in the BZB. The vibration frequencies of the first 10 global modes of MECS model have minor difference to the OECS model and have a max difference of $3.24 \%$ in 5 th mode and $-0.476 \%$ in 2nd mode. Moreover, the increase in frequencies is related to the deck and the decrease is related to the towers. These results differ from the conclusion given by Wang et al. [14]. The tower modes also have a larger proportion in the first 10 global modes of the BZB, and most of them are the lateral vibrations, as seen in Figure 3. It may be explained by the fact that the lack of the horizontal component between the towers makes the overall performance of the BZB very poor in the transversal direction. Compared to the OECS model, as seen in Figure 3, the MECS model can offer all the vibration modes of the BZB including the global coupled modes and the local modes of cable stays.

### 2.3. Possibility of the Transverse Resonance between the Local

 Model and Global Model. Table 2 shows the relationship between the natural frequencies of the global modes and part of the cables in the BZB. It is seen that the transverse coupled vibration between the tower and cable due to the $1: 2$ or $1: 1$ internal resonance mechanism are possibly to occur. For example, the natural frequencies of cables N23, N24, N25, N 26 , and $\mathrm{N} 13, \mathrm{~N} 14, \mathrm{~N} 15$ are in the one times or two times vicinity of the natural frequency of the 2nd lateral bending mode, respectively. On the other hand, the minor tower and cables also occur the transverse coupled vibration, such as the 7 th, 8 th, 9 th, and 10th lateral bending modes for the cables A9 and J8.

## 3. Simplified Model

3.1. Equations of Motion. In this section, the coupled vibrations between the cable and tower of the BZB under transverse excitation have been investigated by a simplified

2-DOFs model, in which the tower is reduced to a beam. The simplified model consists of a beam and a cable, each of them fixed at one end and attached to the other end as considered in Figure 4. The beam is considered as an Euler-Bernoulli beam and the cable is simulated ignoring the bending, torsional,


Figure 2: FE model of the Bin-Zhou Bridge.


Figure 3: The lateral mode shapes of the MESC model.

TABLE 1: Comparison of natural frequencies for the first ten global modes.

| Mode order | OESC (Hz) | MESC (Hz) | Structural component | Description (MESC) |
| :--- | :---: | :---: | :---: | :---: |
| 1 | 0.2725 | 0.2756 | Deck | Antisymmetric vertical bending |
| 2 | 0.2939 | 0.2925 | Main Tower | Lateral bending |
| 3 | 0.2961 | 0.2947 | Main Tower | Longitudinal bending |
| 4 | 0.4429 | 0.4470 | Deck | Torsional bending |
| 5 | 0.5424 | 0.5600 | Deck | Symmetric vertical bending |
| $6(41)$ | 0.6592 | 0.6686 | Deck | Antisymmetric vertical bending |
| $7(42)$ | 0.6737 | 0.6714 | Minor tower 1,2 | Lateral + longitudinal bending |
| $8(56)$ | 0.6748 | 0.6728 | Minor tower 1,2 | Longitudinal bending |
| $9(57)$ | 0.6996 | 0.6972 | Minor tower 1,2 | Lateral bending |
| $10(69)$ | 0.6996 | 0.6972 | Minor tower 1,2 | Longitudinal + lateral bending |

[^1]

Figure 4: The simplified model.

TABLE 2: Cables which may take place of the transverse coupled vibration between the global mode and cable.

| Mode number (OECS) | Natural frequency (Hz) | Cable number | Natural frequency (Hz) | Ratio |
| :--- | :---: | :---: | :---: | :---: |
| 2nd |  | N 23 | 0.313 | $0.939(1: 1)$ |
|  |  | N 24 | 0.310 | $0.948(1: 1)$ |
|  | 0.2939 | N 25 | 0.289 | $1.017(1: 1)$ |
|  |  | N 26 | 0.271 | $1.085(1: 1)$ |
|  |  | N 13 | 0.648 | $0.454(1: 2)$ |
| 7th, 8th | N 14 | 0.554 | $0.531(1: 2)$ |  |
|  |  | N 15 | 0.536 | $0.548(1: 2)$ |
| 9th, 10th | A9 | 1.392 | $0.484(1: 2)$ |  |
|  | 0.6737 | J 8 | 1.373 | $0.491(1: 2)$ |

and shear rigidities behavior. In addition, the beam and cable are assumed to be homogeneous and oscillate transversely only in outplane.

Considering the previous assumptions, the equations of motion for the simplified model are obtained by using the extended Hamilton principle:

$$
\begin{gather*}
m_{b} \ddot{w}_{b}+\xi_{b} \dot{w}_{b}+E_{b} I_{b} w_{b}^{\prime \prime \prime \prime}+N w_{b}^{\prime \prime}=P_{b} \cos \left(\Omega_{b} t\right)  \tag{1a}\\
m_{c} \ddot{w}_{c}+\xi_{c} \dot{w}_{c}-\left[H w_{c}^{\prime}+E_{c} A_{c} w_{c}^{\prime} e(t)\right]^{\prime}=P_{c} \cos \left(\Omega_{c} t\right)  \tag{1b}\\
N=\left(H+E_{c} A_{c} e(t)\right) \sin (\theta) \\
e(t)=\frac{1}{l_{c}} \int_{0}^{l_{c}} w_{c}^{\prime 2} d x_{c} \tag{2}
\end{gather*}
$$

where the symbols $m_{b}$ and $m_{c}$ are the mass per unit length of the beam and cable, respectively; $I_{b}$ and $I_{c}$ are the lengths of the beam and cable, respectively; $E_{b} I_{b}$ and $E_{c} A_{c}$ are the bending and axial stiffness of the beam and cable,
respectively; $E_{b}, E_{c}$ are the Young's modulus of the material; $I_{b}$ and $A_{c}$ are the moment of area and cross-sectional area, respectively; $N$ is the axial compressive load; $H$ is the initial tension of the cable; $\xi_{b}$ and $\xi_{c}$ are the damping coefficient of the beam and cable, respectively; $\Omega_{b}, \Omega_{c}, P_{b}$, and $P_{c}$ are the frequency and amplitude of the transverse loads, respectively; $w_{b}$ is the beam transverse displacement at location $x_{b}$; and $w_{c}$ is the transverse displacement of the cable at location $x_{c}$. The overdot indicates the differentiation with respect to the time $t$; the prime indicates the differentiation with respect to the coordinate $x$.

The associated geometric and relevant mechanical boundary conditions of the beam and cable are, respectively, written as

$$
\begin{gather*}
w_{b}(0, t)=w_{c}(0, t), \quad w_{b}\left(l_{b}, t\right)=0, \quad w_{c}\left(l_{c}, t\right)=0 \\
w_{b}^{\prime}\left(l_{b}, t\right)=0, \quad w_{b}^{\prime \prime}(0, t)=0 \\
E_{b} I_{b} w_{b}^{\prime \prime \prime}(0, t)=\lambda w_{b}(0, t) \tag{3}
\end{gather*}
$$

where the $\lambda=E_{c} A_{c} / l_{c}$ denotes the out-of-plane supported stiffness of the beam caused by the cable.

For convenience, a set of new variables and parameters are defined as

$$
\begin{gather*}
\tau=\omega t, \quad \bar{x}_{c}=\frac{x_{c}}{l_{c}}, \quad \bar{x}_{b}=\frac{x_{b}}{l_{b}}, \quad \bar{w}_{c}=\frac{w_{c}}{l_{c}}, \\
\bar{w}_{b}=\frac{w_{b}}{l_{b}}, \quad \bar{\xi}_{b}=\frac{\xi_{b}}{m_{b} \omega}, \quad \bar{\xi}_{c}=\frac{\xi_{c}}{m_{c} \omega}, \\
\bar{P}_{b}=\frac{P_{b}}{m_{b} l_{b} \omega^{2}}, \quad \bar{P}_{c}=\frac{P_{c}}{m_{c} l_{c} \omega^{2}}, \quad \rho=\frac{m_{c}}{m_{b}},  \tag{4}\\
\mu=\frac{E_{c} A_{c}}{H}, \quad \chi=\frac{E_{b} I_{b}}{l_{b}^{2} E_{c} A_{c}}, \\
\beta_{b}^{4}=\frac{\omega^{2} m_{b} l_{b}^{4}}{E_{b} I_{b}}, \quad \beta_{c}^{2}=\frac{\omega^{2} m_{c} l_{c}^{2}}{H},
\end{gather*}
$$

where $\omega$ is the natural frequency of the simplified model outplane. In the nondimensional form, (1a)-(3) become

$$
\begin{gather*}
\ddot{w}_{b}+\xi_{b} \dot{w}_{b}+\frac{1}{\beta_{b}^{4}} w_{b}^{\prime \prime \prime \prime}+\frac{\rho w_{b}^{\prime \prime}}{\beta_{c}^{2} \sin (\theta)}+\frac{\rho \mu e(t)}{\beta_{c}^{2} \sin (\theta)} w_{b}^{\prime \prime}  \tag{5a}\\
=P_{b} \cos \left(\Omega_{b} t\right), \\
\ddot{w}_{c}+\xi_{c} \dot{w}_{c}-\frac{1}{\beta_{c}^{2}} w_{c}^{\prime \prime}-\frac{\mu e(t)}{\beta_{c}^{2}} w_{c}^{\prime \prime}=P_{c} \cos \left(\Omega_{c} t\right)  \tag{5b}\\
e(t)=\int_{0}^{1} \frac{1}{2} w_{c}^{\prime 2} d x \\
w_{b}(0, t) \sin (\theta)=w_{c}(0, t), \quad w_{b}(1, t)=0 \\
w_{c}(1, t)=0, \quad w_{b}^{\prime}(1, t)=0, \quad w_{b}^{\prime \prime}(0, t)=0  \tag{6}\\
\chi w_{b}^{\prime \prime \prime}(0, t)=\left(\frac{1}{\mu}+e(t)\right) w_{c}^{\prime}(0)
\end{gather*}
$$

In this paper, only the first-order modes of the beam and cable are considered. Therefore, based on the research $[15,16]$, the transverse displacements $w_{b}(x, t)$ and $w_{c}(x, t)$ are approximated by the transverse modes of the first order as follows:

$$
\begin{gather*}
w_{b}(x, t)=\psi_{1}(x) p_{1}(t)  \tag{7a}\\
w_{c}(x, t)=\psi_{1}(x) p_{1}(t) \sin (\theta)+\psi_{2}(x) p_{2}(t), \tag{7b}
\end{gather*}
$$

where $\varphi_{1}(x)$ and $\varphi_{2}(x)$ are the mode shapes of the beam and cable, respectively. Both of them have the following form:

$$
\begin{gather*}
\psi_{1}(x)=A_{1} \sin \left(\beta_{b} x\right)+A_{2} \cos \left(\beta_{b} x\right)  \tag{8a}\\
+A_{3} \sinh \left(\beta_{b} x\right)+A_{4} \cosh \left(\beta_{b} x\right), \\
\psi_{2}(x)=\sin (\pi x) . \tag{8b}
\end{gather*}
$$

Using the Galerkin approach, substituting (7a) and (7b) into (5a) and (5b), the nonlinear governing equations of
motion with 2-DOFs for the simplified model are obtained as follows:

$$
\begin{align*}
& \ddot{p}_{1}(t)+\xi_{1} \dot{p}_{1}(t)+a_{1} p_{1}(t)+a_{122} p_{1}(t) p_{2}(t)^{2} \\
& \quad+a_{112} p_{1}(t)^{2} p_{2}(t)+a_{111} p_{1}(t)^{3}=f_{b} \cos \left(\Omega_{b} t\right),  \tag{9a}\\
& \ddot{p}_{2}(t)+\xi_{2} \dot{p}_{2}(t)+d_{1} \ddot{p}_{1}(t)+d_{2} \dot{p}_{1}(t)+b_{2} p_{2}(t) \\
& +b_{112} p_{1}(t)^{2} p_{2}(t)+b_{122} p_{1}(t) p_{2}(t)^{2}+b_{222} p_{2}(t)^{3} \\
& +b_{1} p_{1}(t)+b_{111} p_{1}(t)^{3}=f_{c} \cos \left(\Omega_{c} t\right),  \tag{9b}\\
& e(t)=\frac{1}{2} p_{2}(t)^{2} I_{4}+p_{1}(t) p_{2}(t) \sin (\theta) I_{5}  \tag{10}\\
& \quad+\frac{1}{2} p_{1}(t)^{2} \sin (\theta)^{2} I_{6},
\end{align*}
$$

where $f_{b}$ and $f_{c}$ are the amplitude of the harmonic functions and $a_{i}, b_{i}, d_{i}, a_{i j}, b_{i j}, a_{i j k}$, and $b_{i j k}$ are the Galerkin coefficients of the simplified model, respectively. All the coefficients are defined in the Appendix.
3.2. Perturbation Analysis. The multiple scales perturbation method [17] is applied to (9a) and (9b) to obtain an approximation solution of the model. To make the nonlinear terms weak, one can substitute $a_{i j k}, b_{i j k}$, and $c_{i j k}$ with $\varepsilon a_{i j k}, \varepsilon b_{i j k}$, and $\varepsilon c_{i j k}$. Then, (9a) and (9b) can be rewritten as

$$
\begin{align*}
\ddot{p}_{1}(t) & +\varepsilon \xi_{1} \dot{p}_{1}(t)+\omega_{1}^{2} p_{1}(t)+\varepsilon a_{122} p_{1}(t) p_{2}(t)^{2} \\
& +\varepsilon a_{112} p_{1}(t)^{2} p_{2}(t)+\varepsilon a_{111} p_{1}(t)^{3}  \tag{11a}\\
& -\varepsilon f_{b} \cos \left(\Omega_{b} t\right)=0, \\
\ddot{p}_{2}(t) & +\varepsilon \xi_{2} \dot{p}_{2}(t)+\varepsilon d_{1} \ddot{p}_{1}(t)+\varepsilon d_{2} \dot{p}_{1}(t)+\omega_{2}^{2} p_{2}(t) \\
& +\varepsilon b_{112} p_{1}(t)^{2} p_{2}(t)+\varepsilon b_{122} p_{1}(t) p_{2}(t)^{2}  \tag{11b}\\
& +\varepsilon b_{222} p_{2}(t)^{3}+\varepsilon b_{1} p_{1}(t)+\varepsilon b_{111} p_{1}(t)^{3}=0 .
\end{align*}
$$

Considering the fact there is $1: 1$ internal resonance between the modes of the beam and cable, primary resonance for the beam and autoparametric resonance for the cable, simultaneously, therefore, the resonant relations are represented as

$$
\begin{equation*}
\Omega=\omega_{1}+\varepsilon \sigma_{1}, \quad \omega_{2}=\omega_{1} . \tag{12}
\end{equation*}
$$

Assuming the first-order approximation solution of (11a) and (11b) in the form

$$
\begin{align*}
& p_{1}(t)=p_{10}\left(T_{0}, T_{1}\right)+\varepsilon p_{11}\left(T_{0}, T_{1}\right),  \tag{13a}\\
& p_{2}(t)=p_{20}\left(T_{0}, T_{1}\right)+\varepsilon p_{21}\left(T_{0}, T_{1}\right), \tag{13b}
\end{align*}
$$

the time derivatives become

$$
\begin{align*}
& \frac{d}{d t}=\frac{\partial}{\partial T_{0}} \frac{\partial T_{0}}{\partial t}+\frac{\partial}{\partial T_{1}} \frac{\partial T_{1}}{\partial t}+\cdots=D_{0}+\varepsilon D_{1}+\cdots  \tag{14a}\\
& \frac{d^{2}}{d t^{2}}=\left(D_{0}+\varepsilon D_{1}+\cdots\right)^{2}=D_{0}^{2}+2 \varepsilon D_{0} D_{1}+\cdots \tag{14b}
\end{align*}
$$

where $T_{n}=\varepsilon^{n} t(n=0,1)$ are the fast and slow time scales, respectively.

Substituting (13a), (13b), (14a), and (14b) into (11a) and (11b) and equating the powers of $\varepsilon$

$$
\begin{align*}
& D_{1,1}\left(p_{10}\right)+\omega_{1}^{2} p_{10}=0  \tag{15a}\\
& D_{1,1}\left(p_{20}\right)+\omega_{2}^{2} p_{20}=0 \tag{15b}
\end{align*}
$$

$$
\begin{align*}
D_{1,1} & \left(p_{11}\right)+\omega_{1}^{2} p_{11}+a_{112} p_{10}^{2} p_{20}+2 D_{1,2}\left(p_{10}\right) \\
& +\xi_{1} D_{1}\left(p_{10}\right)+a_{122} p_{10} p_{20}^{2}+a_{111} p_{10}^{3}  \tag{16a}\\
& -\cos (\Omega t) f_{b}=0 \\
D_{1,1} & \left(p_{21}\right)+\omega_{2}^{2} p_{21}+2 D_{1,2}\left(p_{20}\right)+\xi_{2} D_{1}\left(p_{20}\right) \\
& +b_{112} p_{10}^{2} p_{20}+b_{122} p_{10} p_{20}^{2}+b_{111} p_{10}^{3}  \tag{16b}\\
& +d_{1} D_{1,1}\left(p_{10}\right)+d_{2} D_{1}\left(p_{10}\right) \\
& +b_{1} p_{10}+b_{222} p_{20}^{3}=0 .
\end{align*}
$$

The general solution of (15a) and (15b) can be expressed in the form

$$
\begin{align*}
& p_{10}=A_{1}\left(T_{1}\right) \exp \left(\omega_{1} T_{0}\right)+\bar{A}_{1}\left(T_{1}\right) \exp \left(-\omega_{1} T_{0}\right)  \tag{17a}\\
& p_{20}=A_{2}\left(T_{1}\right) \exp \left(\omega_{2} T_{0}\right)+\bar{A}_{2}\left(T_{1}\right) \exp \left(-\omega_{2} T_{0}\right) \tag{17b}
\end{align*}
$$

where $A_{1}$ and $A_{2}$ are complex functions and $\overline{A_{1}}$ and $\overline{A_{2}}$ denote complex conjugate terms, respectively.

Substituting (17a) and (17b) and (12), into (16a) and (16b) and setting the coefficients of the secular terms to zero yield the solvability conditions as

$$
\begin{align*}
& 2 I D_{1}\left(A_{1}\right) \omega_{1}+a_{112} \overline{A_{2}} A_{1}^{2}+a_{122} \overline{A_{1}} A_{2}^{2}+3 a_{111} \overline{A_{1}} A_{1}^{2} \\
& \quad-\frac{1}{2} f_{b} \exp \left(I T_{0} \varepsilon \sigma_{1}\right)+2 a_{112} A_{1} \overline{A_{1}} A_{2}  \tag{18a}\\
& \quad+I \xi_{1} \omega_{1} A_{1}+2 a_{122} A_{1} A_{2} \overline{A_{2}}=0 \\
& I D_{2} \omega_{1} A_{1}+2 b_{122} A_{1} A_{2} \overline{A_{2}}+2 I D_{2}\left(A_{2}\right) \omega_{1}+b_{112} A_{1}^{2} \overline{A_{2}} \\
& \quad+b_{1} A_{1}+I \xi_{2} \omega_{1} A_{2}+2 b_{112} A_{1} \overline{A_{1}} A_{2} \\
& \quad+b_{122} A_{1} A_{2}^{2}+3 b_{222} A_{2}^{2} A_{2} \\
& \quad+3 b_{111} A_{1}^{2} A_{1}-d_{1} \omega_{1}^{2} A_{1}=0 . \tag{18b}
\end{align*}
$$

Let

$$
\begin{equation*}
A_{k}=\frac{1}{2}\left(x_{k}+i y_{k}\right) \quad(k=1,2) \tag{19}
\end{equation*}
$$

Substituting (19) into (18a) and (18b) and then separating the real and imaginary parts, the modulation equations obtained in the Cartesian form are as follows:

$$
\begin{align*}
& \frac{d}{d t} x_{1}(t)=-\frac{3 y_{1}^{3}+3 y_{1} x_{1}^{2}}{8 \omega_{1}} a_{111} \\
& +\frac{\left(x_{1} x_{2}-3 y_{1} y_{2}\right) \sqrt{x_{1}^{2}+y_{1}^{2}}}{8 \omega_{1}} a_{112}  \tag{20a}\\
& +\frac{\left(2 x_{1} x_{2} y_{2}-y_{1} x_{2}^{2}-3 y_{1} y_{2}^{2}\right)}{8 \omega_{1}} a_{122} \\
& +\frac{f_{b}+2 y_{1} \sigma_{1} \omega_{1}-x_{1} \xi_{1} \omega_{1}}{2 \omega_{1}}, \\
& \frac{d}{d t} y_{1}(t)=-\frac{3 x_{1}^{3}+3 x_{1} y_{1}^{2}}{8 \omega_{1}} a_{111} \\
& +\frac{\left(y_{1} x_{2}+3 x_{1} y_{2}\right) \sqrt{x_{1}^{2}+y_{1}^{2}}}{8 \omega_{1}} a_{112}  \tag{20b}\\
& +\frac{\left(2 y_{1} x_{2} y_{2}+x_{1} x_{2}^{2}+3 x_{1} y_{2}^{2}\right)}{8 \omega_{1}} a_{122} \\
& +\frac{-2 x_{1} \sigma_{1} \omega_{1}-y_{1} \xi_{1} \omega_{1}}{2 \omega_{1}}, \\
& \frac{d}{d t} x_{1}(t)=-\frac{\left(3 y_{1}^{2}+3 x_{1}^{2}\right) \sqrt{x_{1}^{2}+y_{1}^{2}}}{8 \omega_{1}} b_{111} \\
& -\frac{\left(3 x_{1}^{2} y_{2}+3 y_{1}^{2} y_{2}\right)}{8 \omega_{1}} b_{112} \\
& -\frac{\left(x_{2}^{2}+3 y_{2}^{2}\right) \sqrt{x_{1}^{2}+y_{1}^{2}}}{8 \omega_{1}} b_{122}-\frac{3 x_{2}^{2} y_{2}+3 y_{2}^{3}}{8 \omega_{1}} b_{222} \\
& -\frac{\left(b_{1}-d_{1} \omega_{1}^{2}\right) \sqrt{x_{1}^{2}+y_{1}^{2}}+x_{2} \xi_{2} \omega_{1}-2 y_{2} \sigma_{1} \omega_{1}}{2 \omega_{1}} \\
& -\frac{y_{2}}{x_{1}^{2}+y_{1}^{2}}\left(x_{1} \frac{d}{d t} y_{1}(t)+y_{1} \frac{d}{d t} x_{1}(t)\right),  \tag{20c}\\
& \frac{d}{d t} y_{1}(t)=\frac{\left(x_{1}^{2} x_{2}+y_{1}^{2} x_{2}\right)}{8 \omega_{1}} b_{112}+\frac{x_{2} y_{2} \sqrt{x_{1}^{2}+y_{1}^{2}}}{4 \omega_{1}} b_{122} \\
& +\frac{3 x_{2} y_{2}^{2}+3 x_{2}^{3}}{8 \omega_{1}} b_{222} \\
& -\frac{d_{2} \omega_{1} \sqrt{x_{1}^{2}+y_{1}^{2}}+y_{2} \xi_{2} \omega_{1}+2 x_{2} \sigma_{1} \omega_{1}}{2 \omega_{1}} \\
& +\frac{x_{2}}{x_{1}^{2}+y_{1}^{2}}\left(x_{1} \frac{d}{d t} y_{1}(t)+y_{1} \frac{d}{d t} x_{1}(t)\right) \text {. }
\end{align*}
$$

(20d)

It is seen that the (20a), (20b), (20c), and (20d) describe a nonlinear dynamic system, indicating that the transverse vibrations of the cable and tower are nonlinear, even though the beam model considered is linear and neglects the cable's sag effect. The nonlinear terms are only caused by the coupled behaviors between the modes of the beam and cable. It can be demonstrated that the simplified model shows the chaotic motion and period motion with the perturbation force $f_{b}$ changing.
3.3. Numerical Results and Discussion. In fact, there are many internal resonance forms between the modes of a beam and a cable when they act as an overall structure, such as two-to-one, one-to-one, and one-to-two. This study only focuses on the nonlinear vibrations of the simplified model in the one-to-one internal resonance case, taking into consideration the primary resonance to the beam. The parameters of the simplified model are selected as follows: $m_{c}=67.37 \mathrm{~kg} / \mathrm{m}$, $E_{c}=2.10 e 11 \mathrm{~N} / \mathrm{m}^{2}, A_{c}=8.58 e-3 \mathrm{~m}^{2}, l_{c}=152.70 \mathrm{~m}$, $H=609.00 \mathrm{kN}, m_{b}=1.34 e 5 \mathrm{~kg} / \mathrm{m}, l_{b}=76.81 \mathrm{~m}, \theta=30.20^{\circ}$, and $E_{b} I_{b}=8.29 \mathrm{e} 11 \mathrm{~N} \cdot \mathrm{~m}^{2}$. Therefore, the nondimension parameters are calculated as follows: $\rho=0.001, \chi=0.068$, $\mu=2959.3, \xi_{1}=0.02, \xi_{2}=0.001$, and $f_{c}=0$. The amplitude of the beam $\left(f_{b}\right)$ is chosen as a controlling parameter. The time histories, phase portraits, and Poincare maps are plotted to analyze the nonlinear dynamical motion of the simplified model. From the numerical simulations, the coupling motion between the beam and cable can be clearly found.

Figure 5 shows the nonlinear behavior of the simplified model at the force amplitude, $f_{b}=0.005$, involving time histories, phase portraits, and Poincare maps. As seen in Figures 5(a) and 5(b), the amplitude history of the beam varies with a particular period, while the cable is disorders and with many subharmonics. However, there are still energy transformation occurred. In order to identify the characteristic of these motions in the time histories, the phase portraits have been calculated. As seen in Figures 5(c) and 5(d), the phase portraits of the beam and cable exhibit periodicity and nonperiodicity, respectively, which are consistent with the description in Figures 5(a) and 5(b). Since the phase portraits cannot provide enough information to determine the onset for chaotic motion, they only are used to distinguish whether the model is periodic or non-periodic. Therefore, the Poincare maps have been further calculated. As seen in Figure 5(e), the Poincare map of the beam exhibits many irregular points, confirming that the motion is chaos. On the other hand, Figure 5(f) exhibits that the return points in the Poincare map form a closed curve. Generally, this shape indicates that this motion is periodic [18]. However, since the motion of the simplified model is coupled, the beam's motion takes on chaos characteristic, and the motion of the cable can be confirmed as chaos motion.

Figure 6 shows the nonlinear behavior of the simplified model at the force amplitude, $f_{b}=0.01$. As shown in Figures 6(a) and 6(b), the varying amplitudes of time history of beam and cable exhibit particular period while the cable has a few subharmonics. The phenomenon of energy transformation between them can be clearly found. As seen in Figures 6(c)
and 6(d), the phase portraits of the beam and cable both exhibit periodicity and are further confirmed by Figures 6(e) and 6(f). These phenomena indicate that the motions of the beam and cable are both quasi-periodic.

When the forcing amplitude is increased to $f_{b}=0.1$, the nonlinear behavior of the simplified model changes to chaotic motions, as shown in Figure 7. The Poincare maps given in Figures 7(e) and 7(f) demonstrate clearly that chaotic motions exist in the simplified model again. It can be observed that the time-histories and phase portraits represented by Figures 7(a), 7(b), 7(c), and 7(d) are very similar to that of Figure 6; only the complexity of these graphics increased. On the other hand, compared to Figures $5(a), 6(a)$, and $7(a)$, the soften phenomenon in the simplified model tends to be more apparent with the force amplitude $\left(f_{b}\right)$ increasing.

## 4. Conclusions

The frequency values of the Bin-Zhou cable-stayed bridge (BZB) influenced by the cable vibration have been investigated by two FE models in this study. One is the OECS model in which one single element per cable stay is used and the other is MECS model, where multi-elements per cable stay are used. The nonlinear behaviors of the cable vibration of BZB have also been examined by a simplified model, where the tower is simplified as a beam. The motions of the simplified model are utilized by Galerkin's method to truncate a two DOFs nonlinear coupled model. Based on above numerical experiments, some conclusions are summarized as follows.
(1) The local vibration of the cables only has a small impact on the frequency values of the BZB but increase for the deck and decrease for the towers.
(2) The results from the modal analysis also show that the modes of the towers are main component of the BZB in the first 10 modes, and most of them are the lateral vibrations. Compared to the OESC model, the MECS model not only offers global modes of the bridge, but also exhibits the local vibration of the cables.
(3) The results from the simplified model analysis show that the coupled system exhibits quasi-periodic and chaotic motion with the forcing amplitude changing, even though the beam model is linear and the cable model neglects the cable's sag effect. The simplified model also exhibits soften behavior with the forcing amplitude increasing.
(4) The energy transform can be found due to the $1: 1$ internal resonant between the modes of the beam and cable.


Figure 5: Nonlinear response of the simplified model when the $f_{b}=0.005$ ( 98.3 KN ): (a) Time history; (b) Local magnification of (a); (c) Phase portrait of the beam; (d) Phase portrait of the cable; (e) Poincare map of the beam; (f) Poincare map of the cable; (continued line beam, dashed line cable; $p_{10}=0.001, p_{20}=0.032, D p_{10}=0.001, D p_{20}=-0.01$ ).


Figure 6: Nonlinear response of the simplified model when the $f_{b}=0.01$ ( 197.46 kN ): (a) Time history; (b) Local magnification of (a); (c) Phase portrait of the beam; (d) Phase portrait of the cable; (e) Poincare map of the beam; (f) Poincare map of the cable; (continued line beam, dashed line cable; $p_{10}=0.001, p_{20}=0.032, D p_{10}=0.001, D p_{20}=-0.01$ ).


Figure 7: Nonlinear response of the simplified model when the $f_{b}=0.1$ ( 1974.58 kN ): (a) Time history; (b) Local magnification of (a); (c) Phase portrait of the beam; (d) Phase portrait of the cable; (e) Poincare map of the beam; (f) Poincare map of the cable; (continued line beam, dashed line cable; $p_{10}=0.001, p_{20}=0.032, D p_{10}=0.001, D p_{20}=-0.01$ ).

## Appendix

One has

$$
\begin{align*}
& I_{1}=\int_{0}^{1} D\left(\psi_{2}\right)(x)^{2} d x, \\
& I_{2}=\int_{0}^{1} D\left(\psi_{2}\right)(x) D\left(\psi_{1}\right)(x) d x, \\
& I_{3}=\int_{0}^{1} D\left(\psi_{1}\right)(x)^{2} d x, \quad I_{4}=\int_{0}^{1} \psi_{1}(x)^{2} d x, \\
& I_{5}=\int_{0}^{1} \psi_{1}(x) D^{(4)}\left(\psi_{1}\right)(x) d x, \\
& I_{6}=\int_{0}^{1} \psi_{1}(x) D^{(2)}\left(\psi_{1}\right)(x) d x, \\
& I_{7}=\int_{0}^{1} \psi_{1}(x) P_{w b}(x) d x, \quad I_{8}=\int_{0}^{1} \psi_{2}(x)^{2} d x, \\
& I_{9}=\int_{0}^{1} \psi_{1}(x) \psi_{2}(x) d x, \\
& I_{10}=\int_{0}^{1} \psi_{2}(x) D^{(2)}\left(\psi_{2}\right)(x) d x \text {, } \\
& I_{11}=\int_{0}^{1} \psi_{2}(x) D^{(2)}\left(\psi_{1}\right)(x) d x \text {, } \\
& I_{12}=\int_{0}^{1} \psi_{2}(x) P_{w c}(x) d x, \\
& a_{111}=\frac{1}{2} \frac{\rho \mu \sin (\theta) I_{6} I_{3}}{\beta_{c}^{2} I_{4}}, \quad a_{112}=\frac{\rho \mu I_{6} I_{2}}{\beta_{c}^{2} I_{4}}, \\
& a_{122}=\frac{1}{2} \frac{\rho \mu I_{6} I_{1}}{\beta_{c}^{2} \sin (\theta) I_{4}}, \\
& a_{1}=\frac{I_{5}}{\beta_{b}^{4} I_{4}}+\frac{\rho I_{6}}{\beta_{c}^{2} \sin (\theta) I_{4}}, \quad f_{b}=\frac{I_{7}}{I_{4}}, \\
& b_{222}=-\frac{1}{2} \frac{\mu I_{10} I_{1}}{I_{8} \beta_{c}^{2}}, \\
& b_{122}=-\frac{\mu \sin (\theta) I_{10} I_{2}}{I_{8} \beta_{c}^{2}}-\frac{1}{2} \frac{\mu \sin (\theta) I_{11} I_{1}}{I_{8} \beta_{c}^{2}}, \\
& b_{111}=-\frac{1}{2} \frac{\mu \sin (\theta)^{3} I_{11} I_{3}}{I_{8} \beta_{c}^{2}}, \\
& b_{112}=-\frac{1}{2} \frac{\mu \sin (\theta)^{2} I_{10} I_{3}}{I_{8} \beta_{c}^{2}}-\frac{\mu \sin (\theta)^{2} I_{11} I_{2}}{I_{8} \beta_{c}^{2}}, \\
& b_{2}=-\frac{I_{10}}{I_{8} \beta_{c}^{2}}, \quad b_{1}=-\frac{\sin (\theta) I_{11}}{I_{8} \beta_{c}^{2}}, \\
& d_{1}=\frac{\sin (\theta) I_{9}}{I_{8}}, \quad d_{2}=\frac{\xi_{2} \sin (\theta) I_{9}}{I_{8}}, \\
& f_{c}=\frac{I_{12}}{I_{8}} . \tag{A.1}
\end{align*}
$$

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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# Performance Analysis of Adaptive Neuro Fuzzy Inference System Control for MEMS Navigation System 

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#### Abstract

Characterized by small volume, low cost, and low power, MEMS inertial sensors are widely concerned and applied in navigation research, environmental monitoring, military, and so on. Notably in indoor and pedestrian navigation, its easily portable feature seems particularly indispensable and important. However, MEMS inertial sensor has inborn low precision and is impressionable and sometimes goes against accurate navigation or even becomes seriously unstable when working for a period of time and the initial alignment and calibration are invalid. A thought of adaptive neuro fuzzy inference system (ANFIS) is relied on, and an assistive control modulated method is presented in this paper, which is newly designed to improve the inertial sensor performance by black box control and inference. The repeatability and long-time tendency of the MEMS sensors are tested and analyzed by ALLAN method. The parameters of ANFIS models are trained using reasonable fuzzy control strategy, with high-precision navigation system for reference as well as MEMS sensor property. The MEMS error nonlinearity is measured and modulated through the peculiarity of the fuzzy control convergence, to enhance the MEMS function and the whole MEMS system property. Performance of the proposed model has been experimentally verified using low-cost MEMS inertial sensors, and the MEMS output error is well compensated. The test results indicate that ANFIS system trained by high-precision navigation system can efficiently provide corrections to MEMS output and meet the requirement on navigation performance.


## 1. Introduction

Characterized by small volume, low cost, and low power, MEMS inertial sensors are widely concerned and applied to navigation research, environmental monitoring, military and so on. Notably in unmanned air systems, its easily portable feature seems particularly indispensable and important.

However, MEMS inertial sensor HAS inborn low precision, and impressionable. It sometimes goes against accurate navigation or even becomes seriously unstable when working for a period of time, and more worse the initial alignment and calibration are invalid. According to this, many scholars use the Kalman filter method to compensate and correct the MEMS error. By this way, the MEMS system performance can be improved, but its effect is not good, as discussed in [1,2].

A thought of adaptive neurofuzzy inference system (ANFIS) is relied on. Compared with fuzzy inference system and artificial neural network, as discussed in [3-6],
adaptive neurofuzzy inference system (ANFIS) not only has advantages of the two methods but also makes up for their shortcomings. On one hand, it has effective self-learning mechanism and achieves self-learning function. On the other hand, it has a variety of neural networks, optimizes the control rules, and expresses the reasoning-function like human brain. It makes the system develop towards adaptive, selforganizing and self-learning, as discussed in [7-11].

As Integrated avionics system for vehicle is composed of different kinds of sensors to change the separated state, and achieve complementary, mutual backup and integrated usage information, where the system includes MEMS inertial measurement unit (MEMS-IMU) and high-precision IMU. According to this, ANFIS is newly designed to use reference IMU to improve MEMS-IMU performance by black box control and inference. The MEMS sensor error is measured and modulated through the peculiarity of the fuzzy control convergence, to enhance the MEMS function and the whole

MEMS system property. Performance of the proposed model is experimentally verified using low-cost MEMS inertial sensors, to meet the requirement on navigation performance.

## 2. Analysis of MEMS-IMU Property

2.1. Error Modelling of MEMS Inertial Sensor. As to error of MEMS-IMU outputs, constant error, scale factor error and installation error, are considered as the main composition of the IMU error. What is more, random error is also inevitable and impacts the output accuracy. According to this, and ignoring more than one-order small amount, the accelerometer model is built as follows, as discussed by the author in [12]:

$$
\begin{align*}
a_{m} & =\left(I+K_{a}\right)\left(I+\theta_{a}\right) a+\varepsilon_{a}+\nabla a  \tag{1}\\
& \approx\left(I+K_{a}+\theta_{a}\right) a+\varepsilon_{a}+\nabla a .
\end{align*}
$$

Then, the accelerometer error model is

$$
\begin{equation*}
\Delta a=\nabla a+K_{a} a+\theta_{a} a+\varepsilon_{a} . \tag{2}
\end{equation*}
$$

In calibration, the random error $\varepsilon_{a}$ is mainly affected by temperature. So $\varepsilon_{a}$ may be expressed as

$$
\begin{align*}
\varepsilon_{a} & =\varepsilon_{a T}+\varepsilon_{a a} \\
& =a_{a T} * \Delta T+b_{a T} *(\Delta T)^{2}+c_{a T} *(\Delta T)^{3}+\varepsilon_{a a} . \tag{3}
\end{align*}
$$

Similarly, the gyroerror model is

$$
\begin{equation*}
\Delta \omega=\nabla \omega+K_{\omega} \omega+\theta_{\omega} \omega+\varepsilon_{\omega} . \tag{4}
\end{equation*}
$$

And, the $\varepsilon_{\omega}$ is expressed as

$$
\begin{align*}
\varepsilon_{\omega} & =\varepsilon_{\omega T}+\varepsilon_{\omega \omega} \\
& =a_{\omega T} * \Delta T+b_{\omega T} *(\Delta T)^{2}+c_{\omega T} *(\Delta T)^{3}+\varepsilon_{\omega \omega} \tag{5}
\end{align*}
$$

where $\Delta \omega$ and $\Delta a$ are IMU error; $\nabla \omega$ and $\nabla a$ are IMU constant drift; $K_{\omega}$ and $K_{a}$ are the scale factor error matrix; $\theta_{\omega}$ and $\theta_{a}$ are the alignment matrix; $\Delta T$ is temperature variation; $a_{\omega T}, b_{\omega T}$, and $c_{\omega T}$ are the gyroerror coefficients affected by temperature; $a_{a T}, b_{a T}$, and $c_{a T}$ are the accelerometer error coefficients affected by temperature; $\varepsilon_{\omega \omega}$ and $\varepsilon_{a a}$ are the random error.

### 2.2. Experimental Analysis and Compensation of MEMS

 Inertial Sensor. The principle of rotating SINS is elaborated as follows. At the beginning, the rotating coordinate $(s)$ is coincident with the body coordinate (b), and $o x_{s}$ represents the real $x$-axis of gyro- and accelerometer; $o y_{s}$ represents the real $y$-axis of gyro-, and accelerometer; $o z_{s}$ is coincident with the $z$-axis of $b$ coordinate and vertical with $o x_{s}, o y_{s}$. The effective IMU outputs are received from the real IMU outputs in the process of coordinate conversion. The rotating angle is$$
\begin{equation*}
\alpha=\Omega \cdot t . \tag{6}
\end{equation*}
$$

The coordinate transformation matrix from $s$ to $b$ is

$$
\begin{align*}
C_{s}^{b} & =\left[\begin{array}{ccc}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{array}\right]  \tag{7}\\
& =\left[\begin{array}{ccc}
\cos (\Omega t) & -\sin (\Omega t) & 0 \\
\sin (\Omega t) & \cos (\Omega t) & 0 \\
0 & 0 & 1
\end{array}\right] .
\end{align*}
$$

Then, the gyroerror is

$$
\begin{align*}
\Delta \omega^{b}= & C_{s}^{b} \Delta \omega^{s}=C_{s}^{b}\left(\nabla \omega+K_{\omega} \omega+\theta_{\omega} \omega+\varepsilon_{\omega}\right) \\
= & C_{s}^{b}\left(\left[\begin{array}{l}
\nabla \omega_{x} \\
\nabla \omega_{y} \\
\nabla \omega_{z}
\end{array}\right]+\left[\begin{array}{lll}
K_{\omega x} & & \\
& K_{\omega y} & \\
& & K_{\omega z}
\end{array}\right]\left[\begin{array}{l}
\omega_{x} \\
\omega_{y} \\
\omega_{z}
\end{array}\right]\right.  \tag{8}\\
& \left.+\left[\begin{array}{ccc}
-\theta_{\omega y z} & \theta_{\omega x z} & -\theta_{\omega x y} \\
\theta_{\omega z y} & -\theta_{\omega z z} & \\
\theta_{\omega y x}
\end{array}\right]\left[\begin{array}{l}
\omega_{x} \\
\omega_{y} \\
\omega_{z}
\end{array}\right]+\varepsilon_{\omega}\right) .
\end{align*}
$$

Through analysis item by item,

$$
\begin{align*}
C_{s}^{b} \nabla \omega & =\left[\begin{array}{ccc}
\cos (\Omega t) & -\sin (\Omega t) & 0 \\
\sin (\Omega t) & \cos (\Omega t) & 0 \\
0 & 0 & 1
\end{array}\right]\left[\begin{array}{l}
\nabla \omega_{x} \\
\nabla \omega_{y} \\
\nabla \omega_{z}
\end{array}\right] \\
& =\left[\begin{array}{c}
\nabla \omega_{x} \cos (\Omega t)-\nabla \omega_{y} \sin (\Omega t) \\
\nabla \omega_{x} \sin (\Omega t)+\nabla \omega_{y} \cos (\Omega t) \\
\nabla \omega_{z}
\end{array}\right] . \tag{9}
\end{align*}
$$

In (9), it is indicated that such constant errors as $\nabla \omega_{x}, \nabla \omega_{y}$, and $\nabla \omega_{z}$ may be modulated by periodic rotation, and the error impact will be smaller.

However, as to error coefficients $K_{\omega}$ and $\theta_{\omega}$, the impact may be partly modulated. The $x$-axis error caused by $K_{\omega}$ and $\theta_{\omega}$ in $b$ coordinate is

$$
\begin{align*}
C_{s}^{b}\left(K_{\omega} \omega+\theta_{\omega} \omega\right)= & C_{s}^{b}\left(\left[\begin{array}{lll}
K_{\omega x} & & \\
& K_{\omega y} & \\
& & K_{\omega z}
\end{array}\right]\left[\begin{array}{l}
\omega_{x} \\
\omega_{y} \\
\omega_{z}
\end{array}\right]\right. \\
& \left.+\left[\begin{array}{ccc}
-\theta_{\omega y z} & & \theta_{\omega x z} \\
\theta_{\omega z} & -\theta_{\omega x y} \\
\theta_{\omega z z}
\end{array}\right]\left[\begin{array}{l}
\omega_{x} \\
\omega_{y} \\
\omega_{z}
\end{array}\right]\right) \\
= & {\left[\begin{array}{ccc}
\cos (\Omega t) & -\sin (\Omega t) & 0 \\
\sin (\Omega t) & \cos (\Omega t) & 0 \\
0 & 0 & 1
\end{array}\right] } \\
& \times\left[\begin{array}{cc}
K_{\omega x} \omega_{x}+\theta_{\omega x z} \omega_{y}-\theta_{\omega x y} \omega_{z} \\
-\theta_{\omega y z} \omega_{x}+K_{\omega y} \omega_{y}+\theta_{\omega y x} \omega_{z} \\
\theta_{\omega z y} \omega_{x}-\theta_{\omega z z} \omega_{y}+K_{\omega z} \omega_{z}
\end{array}\right] . \tag{10}
\end{align*}
$$

Set $K_{\omega x} \omega_{x}+\theta_{\omega x z} \omega_{y}-\theta_{\omega x y} \omega_{z}$ as an example; if $K_{\omega x} \omega_{x}+\theta_{\omega x z} \omega_{y}-$ $\theta_{\omega x y} \omega_{z}$ can be expressed as $a+f\left(x_{i}\right), i=1,2,3$, the error impact caused by $a$ may be modulated to zero, where a is the inductive constant and $f\left(x_{i}\right), i=1,2,3$ is the variable part.

Table 1: Means and deviation of MEMS gyro- and accelerometer.

|  | $x$-gyro ( $\left.{ }^{\circ} / \mathrm{s}\right)$ | $y$-gyro ( $\left.{ }^{\circ} / \mathrm{s}\right)$ | $z$-gyro $\left({ }^{\circ} / \mathrm{s}\right)$ | $x$-acce $(\mathrm{m} / \mathrm{s} / \mathrm{s})$ | $y$-acce $(\mathrm{m} / \mathrm{s} / \mathrm{s})$ | $z$-acce $(\mathrm{m} / \mathrm{s} / \mathrm{s})$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| MEMS-IMU raw data |  |  |  |  |  |  |
| Mean | 4.25 | 9.86 | 9.71 | 0.193 | 0.211 | 0.205 |
| Standard deviation | 0.35 | 0.55 | 0.51 | 0.038 | 0.036 | 0.053 |
| MEMS-IMU equivalent data |  |  |  |  |  | 0.002 |
| Mean | -0.106 | -0.019 | 10.012 | 0.013 | 0.074 |  |
| Standard deviation | 7.903 | 7.969 | 4.063 | 0.281 | 0.194 | 0.099 |



Figure 1: Gyro output changed with temperature.

The accelerometer error is

$$
\begin{equation*}
\Delta f^{b}=C_{s}^{b} \Delta f^{s}=C_{s}^{b}\left(\nabla f+K_{f} f+\theta_{f} f+\varepsilon_{f}\right) \tag{11}
\end{equation*}
$$

The error analysis and modulation are similar to those of gyros.

It is clearly shown from MEMS-IMU raw data in Table 1 that means and deviations of MEMS-IMU are bigger than those of normal situations. Through rotation modulation, mean data indicates that nonstochastic errors of MEMS-IMU are effectively and quickly improved when calculating navigation parameters. However, due to rotation movement by cosine function, the standard deviation is bigger than that of law data.

Moreover, the MEMS outputs are easily affected by environment. Generally, the temperature in use is -40 to $70^{\circ} \mathrm{C}$, and the internal structure of MEMS device may change under different temperature conditions. In such situation, the measured capacitance output of the inertial sensor is deviated from the normal value, and the output of the inertial sensor is inevitably composed of real value and error. Figures 1 and 2 are separately curves of gyro- and accelerometer values changed with temperature.

As the MEMS performance is unstable, the calibration is not always effective due to its temperature impact. When there is $5^{\circ}$ temperature rise, the output will be $10 \%$ changed according to the raw data at room temperature.


Figure 2: Accelerometer output changed with temperature.

## 3. ANFIS Identification and Compensation Program for MEMS-IMU

3.1. ANFIS Structure. ANFIS is a class of adaptive networks, and it makes the integration of the advantages of neural network and fuzzy inference system. Detailed mathematical progress of ANFIS is as follows. Firstly, it maps the input data by adjusting the shape and parameters of the membership function. Secondly, it remaps the data from the input space to the output space by the membership function of the output variables. During this process, the least squares algorithm is used to adjust ANFIS conclusion parameters, and the gradient descent algorithm is used to adjust ANFIS premise parameters, where the channel for adjusting conclusion parameters is called forward channel and the channel for adjusting the premise parameters is called backward channel, as discussed in [13-16].

In ANFIS structure, as to two input parameters $x, u$ and one output $y$, together with first-order Sugeno fuzzy model, there are two if-then fuzzy rules as follows

The equivalent ANFIS structure is as shown in Figure 3.
In the first layer, each knot is an adaptive knot with function. For example,

$$
\begin{equation*}
O_{1, i}=A_{i}(x), \quad i=1,2 \tag{12}
\end{equation*}
$$

where $x$ is input of knot $I$ and $A$ is the relevant fuzzy language identification.


Figure 3: Equivalent ANFIS structure.

In the second layer, each knot is a fixed knot identified by $\Pi$. The output is the product of all input signals and represents the firing strength of a rule as

$$
\begin{equation*}
O_{2, i}=A_{i}(x) \ldots . \tag{13}
\end{equation*}
$$

In the third layer, the ratio of firing strength for rule $i$ and the whole firing strength for all the rules is calculated, which means that the firing strength of each rule is normalized as

$$
\begin{equation*}
O_{3, i}=\bar{\omega}=\frac{\omega_{i}}{\sum \omega} . \tag{14}
\end{equation*}
$$

In the fourth layer, each knot has its own consequent node and output. The $i$ th knot is an adaptive knot with function

$$
\begin{equation*}
O_{4, i}=\overline{\omega_{i}} f_{i}=\overline{\omega_{i}}\left(p_{i} x+q_{i} y+r_{i}\right) \tag{15}
\end{equation*}
$$

where $\overline{\omega_{i}}$ is normalized motive force transformed from layer 3 and $p 1, q 1, r 1$ are parameters. The parameters in this layer are called conclusion parameters.

In the fifth layer, each knot is a fixed knot identified with $\sum$, and it calculates the output sum by all the signals

$$
\begin{equation*}
O_{5, i}=\sum \overline{\omega_{i}} f_{i}=\frac{\sum \overline{\omega_{i}} f_{i}}{\sum \overline{\omega_{i}}} \tag{16}
\end{equation*}
$$

When ANFIS is trained by reduplicative iteration, the pending characteristic parameters are adjusted dynamically to make sure the accuracy identified by ANFIS meets the requirement. The premise parameters and conclusion parameters are received, so that the ANFIS system is determined, as discussed in [17-19].
3.2. ANFIS System Training. As to inputs of ANFIS, the outputs of MEMS-IMU and standard IMU are segmented into different training data spaces. The initial structure of ANFIS is preset with 5 layers, and ANFIS uses the rule firing strength $\omega_{j}(j=1, \ldots, M(K))$ as a criterion for each rule, where $M(k)$ is the number of clusters at $K$. At time $K$, the separate input training data is $x_{i}(i=1, \ldots, n)$, and $n$ is the numbers of inputs.

If $\omega_{j}\left(x_{i}\right) \leq \alpha$ th, the rule is updated into a new one, and $\widetilde{M}=M(k)+1$. Random $\alpha$ th $\in(0.1,0.5)$ is a preset threshold,
it values $\omega_{j}$ and provides reference for new cluster generated in ANFIS. If $\alpha$ th is smaller, the number of clusters is larger, and the fuzzy system is more complicated.

When $\widetilde{M}$ is updated, it means that new cluster is generated, and needs to rebuild fuzzy system. The selected cluster centre may be set a bigger weight coefficient. This progress is repeatedly taken with inputs data, until the ANFIS meets the accuracy requirements.

ANFIS's parameters include premise parameters and consequent parameters. They are updated based on gradient descent learning algorithm. When the premise parameters are fixed, the least squares algorithm is applied to calculate the consequent data.

This is the whole progress of ANFIS(i) generation. In order to speed up the ANFIS progress and enhance the precision, all the data are separated into different parts for training and checking. The prior ANFIS structure is valued by the checking data and set as premise structure for next system generation, up to the expected modulated effect. The proposed ANFIS training process is shown as in Figure 4.
3.3. MEMS-IMU Error Compensation Based ANFIS. In integrated avionics system for vehicle, synthesis of multiple sensors is a trend; different kinds of sensors may change the separated state and achieve complementary, mutual-backup and integrated usage information provided by each sensor. Through integrated control and management of multisensors, the sensor system may have higher performance level than that of any single sensor.

Generally, as a backup system, MEMS inertial navigation system has low performance by comparison. In order to improve it, high-precision INS output is applied to help correcting MEMS-IMU outputs by ANFIS method. Highprecision INS outputs are set as standard information and separated into different spaces together with MEMS-IMU outputs. The different data spaces are divided into training parts and checking parts. The 3D outputs of MEMS gyros and accelerometers are trained objects and inputs of ANFIS. ANFIS works in update mode; the structure is built, valued, and screened continuously, until the optimal structure meets accuracy demands. The ANFIS building program for MEMSIMU is shown as in Figure 5.

When standard IMU outputs are lacked, ANFIS structure switches to the correction mode. The identified and trained error corrector by ANFIS is applied to modify the law MEMSIMU output and help provide higher-precision MEMS-IMU outputs.

## 4. Experiment and Analysis

The MEMS-IMU used in this paper is MPU6050. The MEMS gyro drift is $4.25\left({ }^{\circ} / \mathrm{s}\right)$, and random is $0.35\left({ }^{\circ} / \mathrm{s}\right)$; MEMS accelerometer bias is $0.193 \mathrm{~m} / \mathrm{s} / \mathrm{s}$, and random is $0.038 \mathrm{~m} / \mathrm{s} / \mathrm{s}$. The output frequency of MEMS-IMU is 50 Hz . As to standard IMU, the gyro drift is $0.003(\% / \mathrm{s})$, and random is $0.003(\% / \mathrm{s})$; the accelerometer bias is $0.043 \mathrm{~m} / \mathrm{s} / \mathrm{s}$, random is $0.017 \mathrm{~m} / \mathrm{s} / \mathrm{s}$, and the output frequency is 50 Hz too.


Figure 4: ANFIS training-process program.


Figure 5: ANFIS building program for MEMS-IMU.

Table 2: Bias of IMU data.

|  | $x$-gyro | $y$-gyro | $z$-gyro | $x$-acce | $y$-acce | $z$-acce |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| MEMS-IMU data | 4.251 | -9.86 | -9.71 | 0.193 | 0.211 | 0.187 |
| Standard IMU data | 0.001 | -0.003 | 0.002 | -0.043 | 0.014 | 0.026 |
| Modified IIMU data | 0.001 | -0.003 | 0.002 | -0.043 | 0.015 | 0.027 |

Table 3: Random of IMU data.

|  | $x$-gyro | $y$-gyro | $z$-gyro | $x$-acce | $y$-acce | $z$-acce |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| MEMS-IMU data | 0.352 | 0.553 | 0.516 | 0.038 | 0.036 | 0.053 |
| Standard IMU data | 0.003 | 0.003 | 0.004 | 0.065 | 0.046 | 0.017 |
| Modified IIMU data | 0.002 | 0.041 | 0.003 | 0.121 | 0.027 | 0.012 |

In order to testify the effectiveness of ANFIS modulation for MEMS-IMU performance under the auxiliary of highprecision standard IMU, both MEMS-IMU and standard IMU start simultaneously; then corresponding data is sent for training and testing. From Figures 6, 7, and 8, it is clear that MEMS-IMU data is modified, and the precision is highly improved. Tables 2 and 3 show the specific improved level of IMU data by drift and random quantity. Simultaneously, compared with random of each standard gyro, that of tested gyro in Table 2 also indicates that the tested data will be


Figure 6: Comparison of each $x$-gyro data.
modified towards the standard data, and not only that, but also stability and smoothness of the modified data will be better than those of standard data. Where, blue line with ring


Figure 7: Comparison of each $y$-gyro data.


Figure 8: Comparison of each $z$-gyro data.
represents modified MEMS-IMU data by ANFIS, red line represents MEMS-IMU data to be tested, and black line with star represents standard IMU data with high precision.

To be sure, the effect of modified accelerometer data is not as good as that of gyro, just because the standard accelerometer precision is not so high, and the modified effect has nothing to do with ANFIS structure. In other words, the performance level of standard data may be one of the main influence factors to decide the performance level of the modified data.

## 5. Conclusions

MEMS-IMU is characterized by its small size, low cost, and easy integration, so it has a wild range of applications.

However, low precision is a stumbling block to MEMS sensor, and sometimes it cannot meet the performance requirements. Nowadays, integrated avionics system for vehicle composes different kinds of sensors to change the separated state and achieve complementary, mutual backup and integrated usage information provided by each sensor. So, high-accuracy IMU may be selected to modify MEMS-IMU performance by high-level outputs.

ANFIS system combines the advantages of neural network and fuzzy control methods, and it is suitable for controlling objects with characters of fuzziness, uncertainty, nonlinearity, and time varying. As to MEMS-IMU property, ANFIS reference program is improved, the reference structure is continuously built for good modification performance, and MEMS-IMU performance is corrected. The experimental results show that ANFIS structure is much closer to the accurate model by multiple establishments, and MEMS-IMU outputs are high-level corrected by rules. Simultaneously, the MEMS-IMU performance is of smoothness by ANFIS modulation. In a word, the updated ANFIS program proposed in this paper is helpful to modify and improve the MEMS-IMU property and is a great reference to enhance MEMS sensor performance.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Nonlinear Dynamics of Controlled Synchronizations of Manipulator System 

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#### Abstract

The nonlinear dynamics of the manipulator system which is controlled to achieve the synchronization motions is investigated in the paper. Firstly, the control strategies and modeling approaches of the manipulator system are given, in which the synchronization goal is defined by both synchronization errors and its derivatives. The synchronization controllers applied on the manipulator system include neuron synchronization controller, improved OPCL synchronization controller, and MRAC-PD synchronization controller. Then, an improved adaptive synchronized control strategy is proposed in order to estimate online the unknown structure parameters and state variables of the manipulator system and to realize the needed synchronous compensation. Furthermore, a robust adaptive synchronization controller is also researched to guarantee the dynamic stability of the system. Finally, the stability of motion synchronizations of the manipulator system possessing nonlinear component is discussed, together with the effect of control parameters and joint friction and others. Some typical motions such as motion bifurcations and the loss of synchronization of it are obtained and illustrated as periodic, multiperiodic, and/or chaotic motion patterns.


## 1. Introduction

A manipulator system can be viewed as a highly nonlinear, strong coupling, time-varying, and multivariable dynamic one. In some particular sites, synchronization motions are needed for the manipulator system to achieve expected tasks. The synchronization control strategies applied to the manipulator system are important to be reasonablly designed. So, the nonlinear dynamics of the controlled manipulator system in synchronization should be discovered specifically. The main purpose of the nonlinear dynamic research on manipulator system under synchronization control is to design a synchronous controller which can guarantee the synchronous stable characteristic. The controller should meet the requirement of trajectory tracking control accuracy. One of the special advantages of synchronization control is that it can keep the specified kinematics relationship of manipulator system in an easy way.

Controlled synchronization of manipulator system is also valuable in the mechanical research and engineering applications. Different from the traditional synergic control
(i.e., coordination control associated with robot task assignment) and the coordinated control (i.e., force and position compliance control of robot), the controlled synchronization focuses on inertial characteristics, motion stiffness, and the rigid-flexible coupling characteristics of the manipulator system. The most important is to understand the dynamic behavior of the controlled system in synchronization. The synchronous control also needs to explore the synchronous tragedy and its stability and robustness when the manipulator system is different; in particular some nonlinearity and/or rigid-flexible coupling effort are concerned. Both the trajectory tracking error and synchronization error of manipulator system would converge to zero when using a controlled synchronization method, which is also useful to optimize the transient process of robot motion trajectory. More complicated motion patterns can be realized when using synchronization control, no matter the same or different structure, rigid or flexible links or joints of a manipulator system. In practices, the synchronization control strategies are also adopted to maintain more regular motions of the multiple industrial robots (such as for assembling, spraying,
transporting, welding, etc.) and improve their trajectory tracking accuracies.

Nowadays, nonlinear dynamics of controlled synchronization of manipulator system is one of the important interests in the fields of machinery dynamics and nonlinear sciences [1-6]. The theory of self-synchronization, namely, vibration synchronization, was thoroughly and systematically studied by many researchers including Blekhman [2] and Wen et al. [7]. Self-synchronizations are used widely in designing vibratory machines and bring people remarkable economic benefits. Study on controlled synchronization and generalized synchronization of mechanical systems was extended during the past decades [7, 8]. Recently, the theory of controlled synchronization is used to improve the control ability and control accuracy for many complicated mechanical systems, such as multiaxis machining tools, multirobotic coordination, and trajectory tracking of robots. Some representative achievements include the following. Koren [9] proposed a controlled synchronization strategy of a multiaxis machining tool where the cross-coupling control is used to achieve the multiaxis tracking synchronization and the synchronization errors are used to define the coordination ability of the machine. The research group leading by RodriguezAngeles and Nijmeijer [10] and Nijmeijer [11, 12] carried out external synchronization and internal synchronous controls of multirobot systems by using feedback controls. They proposed two new adaptive synchronization control methods to achieve the P-R-R planar parallel manipulators with uncertain parameters to guarantee the required trajectory tracking accuracy. Until now, many control strategies were also explored to achieve synchronizations of mechanical systems [13-20], including the controlled synchronization of speed of electromechanical systems consisting of double motors or multimotor running in a constant velocity ratio [7].

In addition, controlled synchronization of manipulators is critical in theoretical and engineering fields of mechanical system. The involved theoretical and technological results enclose the synchronous mechanisms, synchronization control strategies, chaotic synchronization controls, and so on. Many improved controlled synchronization methods are developed these years [21, 22]. With the deepening study of nonlinear behaviors in many control domains [23], the nonlinear behavior of the manipulator due to the strong coupling has attracted more and more interest. For the controlled synchronization of manipulator system, the authors achieved many tasks including the nonlinear dynamic modeling, control strategies and nonlinear behavior of the controlled synchronization of manipulator systems, to satisfy the dynamic design and vibration suppression of manipulators [24-28].

The main contents of the paper mainly focus on the comparison of the controllers based on the early works and further describe the nonlinear behavior of the manipulator under controlling. In Section 2, modeling methods of the nonlinear dynamics of manipulator system are introduced. Some new synchronized control strategies are proposed, including neuron synchronization controller, improved OPCL synchronization controller, and MRACPD synchronization controller. Then, in Section 3, dynamic stability of the controlled synchronization of manipulator

Table 1: Structure parameter values of the 2-DOF master manipulator.

| Link $i$ | $I_{z z}\left(\mathrm{kgm}^{2}\right)$ | $m_{i}(\mathrm{~kg})$ | $L_{c i}(\mathrm{~m})$ | $L_{i}(\mathrm{~m})$ |
| :--- | :---: | :---: | :---: | :---: |
| 1 | 0.083 | 1 | 0.5 | 1 |
| 2 | 0.33 | 1 | 1 | 2 |

Table 2: Structure parameter values of the 3-DOF master manipulator.

| Link $i$ | $I_{z z}\left(\mathrm{kgm}^{2}\right)$ | $m_{i}(\mathrm{~kg})$ | $L_{c i}(\mathrm{~m})$ | $L_{i}(\mathrm{~m})$ |
| :--- | :---: | :---: | :---: | :---: |
| 0 | - | - | - | 0.09 |
| 1 | 0.05 | 1.738 | 0.072 | 0.144 |
| 2 | 0.05 | 1.738 | 0.072 | 0.144 |
| 3 | 0.12 | 2.529 | 0.098 | 0.241 |

system and the dynamic stability of robust adaptive synchronized controller are discussed where the unknown constructor parameters, loading variables, and external disturbance of the manipulators are involved. In Section 4, the influences of the control parameters and joint friction on the nonlinear behavior of the synchronized system are described, and the typical processes of the bifurcation and the loss of synchronism are also illustrated. Finally, some conclusions are given in Section 5.

## 2. Dynamic Models of Manipulator System and Synchronization Control Methods

Two planar manipulators are shown in Figure 1, and the corresponding parameters are shown in Tables 1 and 2, respectively. Based on Newton-Euler formula, their dynamic equations are derived as follows:

$$
\begin{gather*}
\mathbf{F}_{i}^{i-1}=\mathbf{F}_{i}^{i+1}+m_{i} \dot{\mathbf{v}}_{c i}-m_{i} \mathbf{g} \\
\mathbf{M}_{i}^{i-1}=\mathbf{M}_{i}^{i+1}-\mathbf{r}_{i, c i} \times \mathbf{F}_{i}^{i+1}+\mathbf{r}_{i-1, c i}  \tag{1}\\
\times \mathbf{F}_{i}^{i-1}+\mathbf{I}_{i} \dot{\omega}_{i}+\omega_{i} \times \mathbf{I}_{i} \omega_{i},
\end{gather*}
$$

where $\mathbf{F}_{i}^{i-1}$ is the force of link $i+1$ on link $i, m_{i}$ is the mass of link $i, \mathbf{g}$ is the gravity vector, $\mathbf{M}_{i}^{i-1}$ is the torque of link $i+1$ on link $i, \mathbf{r}_{i-1, c i}$ is the vector from the coordinate origin $o_{i-1}$ adhering to joint $i$ to the center of mass $c_{i}, \mathbf{I}_{i}$ is the moment of inertia of link $i$ on mass center $c_{i}$.

Assume that the velocity of $c_{i}$ is $\mathbf{v}_{c i}$, where $c_{i}$ is the center of mass of link $i$, and the acceleration of $c_{i}$ is $\dot{\mathbf{v}}_{c i}$. The link $i$ rotates freely around $c_{i}$ with angular velocity $\omega_{i}$ and angular acceleration $\dot{\omega}_{i}$.

Define $\mathbf{q}$ as the joint variable, where $\mathbf{q}=\left\{\begin{array}{ll}\theta_{1} & \theta_{2}\end{array}\right\}^{T}$ for the 2-DOF manipulator and $\mathbf{q}=\left\{\begin{array}{lll}\theta_{1} & \theta_{2} & \theta_{3}\end{array}\right\}^{T}$ for the 3-DOF manipulator. The differential equation of motion of the manipulator is established including joint friction $\mathbf{f}(\dot{\mathbf{q}})$ and joint rigid $K$ as follows:

$$
\begin{equation*}
\mathbf{M}(\mathbf{q}) \ddot{\mathbf{q}}+\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{q}}+\mathbf{g}(\mathbf{q})=\tau-\mathbf{f}(\dot{\mathbf{q}})-K\left(\mathbf{q}-\mathbf{q}_{0}\right), \tag{2}
\end{equation*}
$$



Figure 1: Schematic of 2 planar manipulators.


Figure 2: The configuration of controlled synchronization manipulators.
where the inertial force is determined by acceleration and Coriolis force, centrifugal force, and gravity load are also involved in it.

The above theory is applied in the synchronization of planar manipulators as a prototype, in which the master manipulator has the same topological mechanism with that of the slave one. The synchronization control schematic diagram for the master-slave manipulators is shown in Figure 2.

A general dynamic equation of master-slave manipulator is addressed as follows. The master system is

$$
\begin{equation*}
\dot{y}=\mathbf{f}(y) . \tag{3}
\end{equation*}
$$

And the slave system is

$$
\begin{equation*}
\dot{x}=\mathbf{g}(x)+\mathbf{u}(\mathbf{x}, \mathbf{y}), \tag{4}
\end{equation*}
$$

where the synchronization controller is $\mathbf{u}(\mathbf{x}, \mathbf{y})$ :

$$
\begin{equation*}
\mathbf{u}(\mathbf{x}, \mathbf{y})=-\mathbf{g}(\mathbf{x})+\mathbf{f}(\mathbf{y})+\mathbf{h}(\mathbf{x}, \mathbf{y}) . \tag{5}
\end{equation*}
$$

The designing goal of synchronization controller is that the synchronization error of manipulators and its derivative will converge to be zero; that is, $\mathbf{e}(t) \rightarrow 0$ and $\dot{\mathbf{e}}(t) \rightarrow 0$ when $t \rightarrow \infty$. Then, the derivative of synchronization error is

$$
\begin{equation*}
\dot{\mathbf{e}}=\dot{\mathbf{x}}-\dot{\mathbf{y}}=\mathbf{g}(\mathbf{x})+\mathbf{u}(\mathbf{x}, \mathbf{y})-\mathbf{f}(\mathbf{y})=\mathbf{h}(\mathbf{x}, \mathbf{y}) . \tag{6}
\end{equation*}
$$

According to the Lyapunov stability theory, several new type synchronization controllers can be proposed based on the feedback control strategy as introduced as follows.
2.1. Neural Synchronization Controller. For the neural synchronization controller which consisted of two reciprocal inhibition neurons, its state equations are given as follows:
$\delta_{1} \dot{y}_{1}=c-y_{1}-b v_{1}-a x_{2}-\max (0, u)$

$$
\delta_{2} \dot{v}_{1}=x_{1}-v_{1}
$$

$\delta_{1} \dot{y}_{2}=c-y_{2}-b v_{2}-a x_{1}-\max (0,-u)$

$$
\delta_{2} \dot{v}_{2}=x_{2}-v_{2}
$$

$$
\begin{equation*}
x_{1}=\max \left(0, y_{1}\right) \quad x_{2}=\max \left(0, y_{2}\right) \quad x_{\text {out }}=x_{1}-x_{2} \tag{7}
\end{equation*}
$$

where $\delta_{1}$ and $\delta_{2}$ are two time constants, $a$ and $b$ are the weights of mutual inhibition and self-inhibition, respectively, $c$ is the excitatory tonic input, $y_{1}$ and $y_{2}$ are the membrane potential, $v_{1}$ and $v_{2}$ are the self-inhibition, and $u$ and $x_{\text {out }}$ are the input and output of control system, respectively.


Figure 3: The motion behavior of small regular swing of a 2-link manipulator.

Apply the controller on the 2-DOF manipulator in Figure 1(a). When the two links begin to swing, the first link gets energy from the second link by using the neural synchronization controller. The whole system will come into rhythmic swing state under well-tuned controller parameter of $\theta_{2}$, which is shown in Figure 3.

### 2.2. Improved OPCL (Open-Plus-Close-Loop) Synchronization

 Controller. An improved OPCL controller is designed to achieve synchronization motions based on chaos control method which consists of an amplifier and a limiter. This control method is proved to be asymptotically stable based on the Lyapunov theory given suitable control parameters. The controlled synchronizations of both the small swing and giant rotating motions of a 2-link manipulator are achieved based on the proposed improved OPCL controller. The controlled system is linearized in the neighborhood of the goal value via Taylor expansion as follows:$$
\begin{aligned}
\ddot{\mathbf{q}}_{s}= & \mathbf{F}\left(\mathbf{q}_{s}, \dot{\mathbf{q}}_{s}, t\right)=\mathbf{F}\left(\mathbf{q}_{m}-\mathbf{e}, \dot{\mathbf{q}}_{m}-\dot{\mathbf{e}}, t\right) \\
= & \mathbf{F}\left(\mathbf{q}_{m}, \dot{\mathbf{q}}_{m}, t\right)-\left(\frac{\partial \mathbf{F}\left(\mathbf{q}_{m}, \dot{\mathbf{q}}_{m}, t\right)}{\partial \mathbf{q}_{m}}\right) \mathbf{e} \\
& -\left(\frac{\partial \mathbf{F}\left(\mathbf{q}_{m}, \dot{\mathbf{q}}_{m}, t\right)}{\partial \dot{\mathbf{q}}_{m}}\right) \dot{\mathbf{e}}+o^{2}\left(\mathbf{q}_{m}, \dot{\mathbf{q}}_{m}\right) \\
= & \mathbf{F}\left(\mathbf{q}_{m}, \dot{\mathbf{q}}_{m}, t\right)+\mathbf{J}_{\mathbf{q} m} \mathbf{e}+\mathbf{J}_{\dot{\mathbf{q}} m} \dot{\mathbf{e}}+o^{2}\left(\mathbf{q}_{m}, \dot{\mathbf{q}}_{m}\right)
\end{aligned}
$$

where $\mathbf{J}_{\mathbf{q} m}$ and $\mathbf{J}_{\mathbf{q} m}$ are Jacobian matrices of $\mathbf{F}\left(\mathbf{q}_{m}, \dot{\mathbf{q}}_{m}, t\right)$, with respect to $\mathbf{q}_{m}, \dot{\mathbf{q}}_{m}$, respectively. The improved OPCL controller for the system is designed as

$$
\begin{equation*}
\mathbf{U}=\ddot{\mathbf{q}}_{s}-\mathbf{F}\left(\mathbf{q}_{m}, \dot{\mathbf{q}}_{m}, t\right)-\mathbf{J}_{\mathbf{q} m} \mathbf{e}-\mathbf{J}_{\dot{\mathbf{q}} m} \dot{\mathbf{e}}+A \dot{\mathbf{e}}+B \mathbf{e} \tag{9}
\end{equation*}
$$

where the term of $\ddot{\mathbf{q}}_{s}-\mathbf{F}\left(\mathbf{q}_{m}, \dot{\mathbf{q}}_{m}, t\right)$ is the open-loop part and the term of $-\mathbf{J}_{\mathbf{q} m} \mathbf{e}-\mathbf{J}_{\dot{\mathbf{q}} m} \dot{\mathbf{e}}+A \dot{\mathbf{e}}+B \mathbf{e}$ is the closed-loop part. The coefficient matrices of $A$ and $B$ are diagonal.
2.3. MARC-PD Synchronization Controller Based on PD Gains. In order to achieve the controlled synchronization (either ender motion or trajectory tracing synchronization) of manipulator system which moves in high speed together with changing loads, an improved model reference adaption control with PD gain (viz. MRAC-PD controller) is proposed to realize the desired motion, namely, synchronization.

After its global stability of the synchronization based on MRAC-PD method is proved, the effects due to the variation of control parameters have been investigated by numerical simulations.

For example, an improved MARC-PD controller is applied on a two-link manipulator to obtain the synchronization motions including small swing and giant rotation.

The principle of synchronization of ender trajectory tracing based on MRAC-PD controller is introduced as follows.

The model of controlled manipulator system is defined as

$$
\begin{equation*}
\ddot{\mathbf{q}}_{s}+\mathbf{M}_{s}^{-1}\left(\mathbf{q}_{s}\right) \mathbf{C}_{s}\left(\mathbf{q}_{s}, \dot{\mathbf{q}}_{s}\right) \dot{\mathbf{q}}_{s}+\mathbf{M}_{s}^{-1}\left(\mathbf{q}_{s}\right) \mathbf{g}_{s}\left(\mathbf{q}_{s}\right)=\mathbf{M}_{s}^{-1}\left(\mathbf{q}_{s}\right) \tau_{s} . \tag{10}
\end{equation*}
$$



FIGURE 4: The synchronous motion patterns of small swing of 2-link manipulator using MRAC-PD.

The synchronous error is defined as

$$
\begin{equation*}
\mathbf{e}=\mathbf{q}_{m}-\mathbf{q}_{s} \tag{11}
\end{equation*}
$$

and the dynamics equation of it is deduced as

$$
\begin{align*}
\ddot{\mathbf{e}}+2 \varsigma \omega_{n} \dot{\mathbf{e}}+\omega_{n} \mathbf{e}= & \omega_{n}^{2} \mathbf{q}_{m}-\mathbf{M}_{s}^{-1}\left(\mathbf{q}_{s}\right) \boldsymbol{\tau}_{s} \\
& +\left(\mathbf{M}_{s}^{-1}\left(\mathbf{q}_{s}\right) \mathbf{C}_{s}\left(\mathbf{q}_{s}, \dot{\mathbf{q}}_{s}\right)-2 \varsigma \omega_{n}\right) \dot{\mathbf{q}}_{s}  \tag{12}\\
& +\mathbf{M}_{s}^{-1}\left(\mathbf{q}_{s}\right) \mathbf{g}_{s}\left(\mathbf{q}_{s}\right)-\omega_{n}^{2} \mathbf{q}_{s}
\end{align*}
$$

Assuming the state vector of errors $\boldsymbol{\varepsilon}=[\mathbf{e}, \dot{\mathbf{e}}]^{T}$, the differential equation of it can be deduced as

$$
\dot{\boldsymbol{\varepsilon}}=\mathbf{A} \boldsymbol{\varepsilon}-\left[\begin{array}{c}
0  \tag{13}\\
\mathbf{M}_{s}^{-1}\left(\mathbf{q}_{s}\right)
\end{array}\right] \boldsymbol{\tau}_{s}+\left[\begin{array}{l}
0 \\
\Delta
\end{array}\right]
$$

where $\Delta$ is the model error.
It is known that, if the real parts of the eigenvalue of the coefficient matrix A are negative, the synchronous error shown in (12) is proved to be zero in asymptotic stability.

The control item $\hat{e}$ according to the PD gain is defined as

$$
\begin{equation*}
\widehat{e}=p_{1} e+p_{2} \dot{e} \tag{14}
\end{equation*}
$$

Taking the stability of Lyapunov function of the controlled system into account, the MRAC-PD controller is defined as follows:

$$
\begin{equation*}
\tau_{s}=k_{0} q_{m}(t)+k_{1} q_{s}(t)+k_{2} \dot{q}_{s}(t) \tag{15}
\end{equation*}
$$

where the coefficients are defined as follows: $\dot{k}_{0}=\lambda_{0} \widehat{e} q_{m}(t)$, $\dot{k}_{1}=\lambda_{1} \hat{e} q_{s}(t)$, and $\dot{k}_{2}=\lambda_{2} \hat{e} q_{s}(t)$.

Apply the controller on the 2-DOF manipulator in Figure 1(a). The results are shown in Figure 4, where the phase space trajectories of $q_{s 1}, q_{s 2}$ indicate the obtained unchangeable patterns during the motion synchronizations of small swing.

## 3. Dynamics Stability of Controlled Synchronization of Manipulator under Complicated Conditions

In order to eliminate the effectiveness by uncertainty structure parameters, friction compensation, unknown load, and flexible joint, the on-time estimation method of structure parameters and state variables of manipulator is built to achieve the synchronization of manipulator system. That is, the stability of controlled synchronization of manipulator for the casings of disturbance, uncertainty parameter, and unknown structure is improved as enhancing the robust function of synchronization controller.

### 3.1. Dynamics of Controlled Synchronization of Manipulator

 System Based on Estimation of Structure Parameters. Under the condition of structure parameters of the slave manipulator in a master-slave system being unknown, the exact expression of $\mathbf{g}(\mathbf{x})$ should be estimated in the process of synchronization control. In this case, the synchronization controller $\mathbf{u}(\mathbf{x}, \mathbf{y})$ is defined as the following three terms:$$
\begin{equation*}
\mathbf{u}(\mathbf{x}, \mathbf{y})=-\widehat{\mathbf{g}}(\mathbf{x})+\mathbf{f}(\mathbf{y})+\mathbf{h}(\mathbf{x}, \mathbf{y}) \tag{16}
\end{equation*}
$$

where $\widehat{\mathbf{g}}(\mathbf{x})$ is the estimator of $\mathbf{g}(x), \mathbf{f}(\mathbf{y})$ is the master function, and $\mathbf{h}(\mathbf{x}, \mathbf{y})$ is a special control term.

In order to obtain $\widehat{\mathbf{g}}(\mathbf{x})$, some assumptions are used and $\widehat{\mathbf{g}}(\mathbf{x})$ is expressed linearly as follows:

$$
\begin{equation*}
\widehat{\mathbf{g}}(\mathbf{x})=\widehat{\mathbf{M}}(\mathbf{x})+\mathbf{X}(x) \widehat{\mathbf{N}}+\mathbf{C}_{\text {const }} \tag{17}
\end{equation*}
$$

where $\widehat{\mathbf{M}}(\mathbf{x})$ is the estimation of nonlinear item, $\widehat{\mathbf{N}}$ is the estimator of the linear term containing unknown structure parameters, $\mathbf{X}(x)$ is the linear term containing state variables, and $\mathbf{C}$ is a constant.

Then, the synchronization error is rewritten as follows:

$$
\begin{align*}
\widetilde{\mathbf{g}}(\mathbf{x}) & =\widehat{\mathbf{g}}(\mathbf{x})-\mathbf{g}(\mathbf{x}) \\
& =(\widehat{\mathbf{M}}(\mathbf{x})-\mathbf{M}(\mathbf{x}))+\mathbf{X}(x)(\widehat{\mathbf{N}}(\mathbf{x})-\mathbf{N}(\mathbf{x}))  \tag{18}\\
& =\widetilde{\mathbf{M}}(\mathbf{x})+\mathbf{X}(\mathbf{x}) \widetilde{\mathbf{N}}(\mathbf{x}),
\end{align*}
$$

where $\widetilde{\mathbf{g}}(\mathbf{x})$ is the estimation error of $\mathbf{g}(\mathbf{x}), \widetilde{\mathbf{M}}(\mathbf{x})$ is the estimation error of nonlinear item, and $\widetilde{\mathbf{N}}(\mathbf{x})$ is the estimation error of linear item affected by the system parameter.

In particular, when nonlinear item $\mathbf{M}(\mathbf{x})=0$, then

$$
\begin{equation*}
\widetilde{\mathbf{g}}(\mathbf{x})=\mathbf{X}(\mathbf{x}) \widetilde{\mathbf{N}}(\mathbf{x}) . \tag{19}
\end{equation*}
$$

So the synchronization error equation is defined as

$$
\begin{align*}
\dot{\mathbf{e}}=\dot{\mathbf{x}}-\dot{\mathbf{y}} & =\mathbf{g}(\mathbf{x})+\mathbf{u}(\mathbf{x}, \mathbf{y})-\mathbf{f}(\mathbf{y}) \\
& =\mathbf{g}(\mathbf{x})-\widehat{\mathbf{g}}(\mathbf{x})+\mathbf{h}(\mathbf{x}, \mathbf{y})  \tag{20}\\
& =-\widetilde{\mathbf{g}}(\mathbf{x})+\mathbf{h}(\mathbf{x}, \mathbf{y})=-\mathbf{X}(\mathbf{x}) \widetilde{\mathbf{N}}(\mathbf{x})+\mathbf{h}(\mathbf{x}, \mathbf{y}) .
\end{align*}
$$

Considering the contribution of the estimation error $\widetilde{\mathbf{g}}(\mathbf{x})$, the adaptive law according to the Lyapunov stability theory is defined as follows:

$$
\begin{equation*}
\frac{d \widehat{\mathbf{N}}}{d t}=-\mathbf{X}^{T} \mathbf{e} \tag{21}
\end{equation*}
$$

If needed, some reasonable synchronization controllers can also be designed to realize the stable controlled synchronization under the situations of both the linear and the nonlinear coupling parameters being unknown.

The above theory is applied in the synchronization of two 3-DOF planar manipulators in Figure 1(b). Each joint of the slave will trace the corresponding joint trajectory of the master in a synchronization way. The synchronization control schematic diagram for the master-slave manipulators is shown as follows.

According to the characteristics of (2), the dynamic equation can be linearized and the adaptive method can guarantee the synchronization stability of the system with unknown slave parameters. Then, (2) can be rewritten as

$$
\begin{equation*}
\mathbf{M}_{s}\left(\mathbf{q}_{s}\right) \ddot{\mathbf{q}}_{m}+\mathbf{C}_{s}\left(\mathbf{q}_{s}, \dot{\mathbf{q}}_{s}\right) \dot{\mathbf{q}}_{m}+\mathbf{g}_{s}\left(\mathbf{q}_{s}\right)=\boldsymbol{\Phi}\left(\mathbf{q}_{s}, \dot{\mathbf{q}}_{s}, \dot{\mathbf{q}}_{m}, \ddot{\mathbf{q}}_{m}\right) \mathbf{P} \tag{22}
\end{equation*}
$$

where $\mathbf{P} \in R_{17 \times 1}, \Phi\left(\mathbf{q}_{s}, \dot{\mathbf{q}}_{s}, \dot{\mathbf{q}}_{m}, \ddot{\mathbf{q}}_{m}\right) \in R_{3 \times 17} . \mathbf{P}$ is a vector which contains all the constant parameters besides the angular information at each joint, and $\boldsymbol{\Phi}\left(\mathbf{q}_{s}, \dot{\mathbf{q}}_{s}, \dot{\mathbf{q}}_{m}, \ddot{\mathbf{q}}_{m}\right)$ does not contain any inertial characteristics and constant parameters of the manipulator.

Let $\dot{\mathbf{q}}_{r}=\dot{\mathbf{q}}_{m}-r \mathbf{e}$ and $\dot{\mathbf{e}}_{r}=\dot{\mathbf{e}}+r \mathbf{e}(r>0)$, and at last the novel controller for synchronization of the master-slave manipulators corresponding to (19) and (21) can be written as follows:

$$
\begin{gather*}
\boldsymbol{\tau}_{s}=\boldsymbol{\Phi}\left(\mathbf{q}_{s}, \dot{\mathbf{q}}_{s}, \dot{\mathbf{q}}_{r}, \ddot{\mathbf{q}}_{r}\right) \widehat{\mathbf{P}}-\mathbf{K}_{p} \mathbf{e}-\mathbf{K}_{v} \dot{\mathbf{e}}  \tag{23}\\
\dot{\hat{\mathbf{P}}}=-\mathbf{A \Phi}\left(\mathbf{q}_{s}, \dot{\mathbf{q}}_{s}, \dot{\mathbf{q}}_{r}, \ddot{\mathbf{q}}_{r}\right)^{T} \dot{\mathbf{e}}_{r}, \tag{24}
\end{gather*}
$$



Figure 5: Ender trajectories of the master-slave manipulators.
where $\widehat{\mathbf{P}}$ is the estimation of the parameter vector $\mathbf{P}$, and it is continuously adjusting according to the synchronization error. Through the adaptive law in (24), the adaptive controller can effectively control the manipulator.

The simulation results of the synchronization of two 3DOF planar manipulators based on the proposed control method are plotted in Figure 5.

### 3.2. Dynamics Stability of Manipulator Based on Robust

 Self-Adaptive Synchronization Control. Considering the unknown model error and possible disturbance, the dynamic equation of a manipulator is defined as$$
\begin{equation*}
\boldsymbol{\tau}=\mathbf{M}(\mathbf{q}) \ddot{\mathbf{q}}+\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{q}}+\mathbf{g}(\mathbf{q})+\Delta(\mathbf{q}, \dot{\mathbf{q}}), \tag{25}
\end{equation*}
$$

where $\Delta(\mathbf{q}, \dot{\mathbf{q}})$ is the model error. A robust controller of synchronous trajectory tracing is designed to guarantee the tracing error which is defined as $e(t)=q(t)-q_{d}(t)$ to be asymptotically limited when the model error is limited or to be asymptotically zero if the model error is zero.

Given an auxiliary signal, $s=\dot{e}+\delta e$, and that $\delta>0$ is a stability constant, the error equation of the controlled synchronization system is derived as follows:

$$
\begin{equation*}
\mathbf{M}(\mathbf{q}) \dot{\mathbf{s}}=-\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) s+\phi-\Delta(\mathbf{q}, \dot{\mathbf{q}})+\mathbf{u} . \tag{26}
\end{equation*}
$$

The robust condition of a synchronization controller is that the positive definite function $\rho(e, \dot{e})$ must be defined for any $\Delta(q, \dot{q})$ to meet the following inequality constraint:

$$
\begin{equation*}
\|\Delta(q, \dot{q})\| \leq \rho(e, \dot{e}) \tag{27}
\end{equation*}
$$

The proposed synchronization controller consists of two parts: one is the feed forward controller and the other is the feedback controller; they are as follows:

$$
\begin{equation*}
u_{1}=-\mathbf{K} s-\phi, \quad u_{2}=-v, \tag{28}
\end{equation*}
$$

where $\mathbf{K}$ is a positive feed forward coefficient, $\phi=\mathbf{M}(q) \delta \dot{e}+$ $\mathbf{C}(q, \dot{q}) \delta e$, and $v=s \rho^{2}(e, \dot{e}) /(\|s\| \rho(e, \dot{e})+\varepsilon)$ in which $\varepsilon>0$ is a definite constant.


Figure 6: Poincare maps of the joint motions of 2-link manipulator under different OPCL control parameters.

## 4. Nonlinear Dynamic Behavior of the Controlled Synchronous Manipulator System

From the viewpoint of the theories of nonlinear dynamics, bifurcations, and chaos, bifurcations and possible chaos could appear in the controlled synchronization processes of the manipulator system. The possible motion patterns of it include single periodic, multiple periodic, quasiperiodic, and chaotic. While the motion patterns and also the dynamic characteristics of manipulator under different synchronization controls will be greatly affected by the unavoidable joint frictions and especially designed control parameters.

### 4.1. Complicated Motions of a 2-Link Manipulator under OPCL

 Synchronization Control. Changing of control parameters can affect the synchronization motions such as small swing and giant rotating of a 2-link manipulator under OPCL control greatly and induce different motions of single periodic, multiple periodic, quasiperiodic, and chaotic ones. The transition processes of the two kinds of synchronization motions are also determined by the OPCL parameters.Assume that the structure parameter values of the 2 -link manipulator in Figure 1(a) are unchangeable, and the two joint angular trajectories are expected as harmonic ones; the obtained Poincare map of the joint angles is shown in Figure 6 when the feed forward coefficients of $\mathbf{A}$ and $\mathbf{B}$ change. For the chaotic case, the calculated Lyapunov exponents of the two joint angles are positive; that is, the biggest Lyapunov exponents of them are 0.5496 and 0.1431 , respectively.
4.2. Nonlinear Behavior of a Controlled Synchronous Manipulator considering Joint Friction. The unavoidable joint frictions will greatly affect the synchronization of the 2-link manipulator under OPCL control. Based on the Stribeck force model of joint friction, the influences of the viscous friction coefficient, static friction force, and Coulomb friction force on the synchronization motions of the 2-link manipulator under OPCL control are compared. The possible motions of it can be single periodic, multiple periodic, quasiperiodic, and chaotic if the joint viscous frictions are changed. Just


Figure 7: Bifurcations of joint angular motion along viscous frictions.
as shown in Figure 7, the joint angular motion bifurcation happens along the viscous friction changing from 0 to 4.15.

When the value of viscous friction is constant, the static friction and Coulomb friction will also affect the motions of manipulator too. Letting viscous friction $f_{v}=4.1$, different static friction coefficients of joint angular lead to quasiperiodic and chaotic motions of the manipulator, respectively, as shown in Figure 8.

## 5. Conclusions

The nonlinear dynamics of the manipulator system which is controlled to achieve synchronization motions is investigated in the paper. Firstly, the modeling approach of manipulator together with the corresponding synchronization control strategies is stated in detail. The dynamic phenomena of swing motions of a two-link manipulator controlled by a neural controller have been described thoroughly. The motion characteristics of two kinds of synchronization motions are simulated. An improved OPCL control method is proposed


Figure 8: Joint angular motions with different viscous frictions.
to achieve synchronization motions of both small swing and giant rotating for a two-link manipulator too. MRACPD synchronization controller is proposed to achieve more accurate trajectory tracing and synchronous motions for 3-DOF manipulator system under the conditions of high operating speed and unknown structure parameters.

The dynamic stability of controlled synchronization of manipulator system is also explained. An estimation based synchronization control method of manipulator system is investigated to eliminate the influences of unknown structure parameters, joint friction, and unknown load. The new synchronization method can improve the robustness of the manipulator system. The controlled synchronization stability is also improved even in case of disturbance, uncertainty parameter, and unknown structure.

Some complicated nonlinear behavior of the controlled synchronizations of manipulator is investigated including multiperiodic motions and bifurcation. Along the changing of the control parameters, viscous friction and static friction, the synchronous manipulator can present single-periodic, multiperiodic, quasiperiodic, and chaotic motions.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Differentiable Families of Planar Bimodal Linear Control Systems 

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#### Abstract

We consider bimodal linear control systems consisting of two subsystems acting on each side of a given hyperplane, assuming continuity along it. For a differentiable family of planar bimodal linear control systems, we obtain its stratification diagram and, if controllability holds for each value of the parameters, we construct a differentiable family of feedbacks which stabilizes both subsystems for each value of the parameters.


## 1. Introduction

Piecewise linear control systems (in particular, the bimodal ones: see, for example, [1-3]) have attracted the interest of the researchers in recent years, as a special class of switched systems (see, e.g., [4-6]), by their wide range of applications, as well as by the possible theoretical approaches, even in the planar case (see, e.g., [7]).

Bimodal linear control systems (BLCS) consist of two subsystems acting on each side of a given hyperplane, assuming continuity along the separating hyperplane. These systems present a complex dynamical behaviour, even for low dimensions, as has been shown in several works. For example, in [8], it is proved that a planar bimodal linear system is stable if each subsystem is stable, but this does not hold for a bimodal linear system with three-state variables. On the other hand, since typically the number of state variables of systems describing elementary circuits is two or three (see [9]), we devote a special attention to the planar case. Here we tackle two problems concerning parameterized families of planar BLCS.

Firstly, obtaining of its stratification diagram with regard to the natural equivalence relation is defined by change of basis in the state space (which preserve the hyperplanes parallel to the separating one). Previously, it is necessary to list the possible equivalence classes and to obtain a complete set of classifying invariant parameters (Theorem 6).

By the way, Arnold's theory allows us to restrict this study to the so-called "miniversal" deformation families. Indeed, the equivalence classes are just the orbits of a certain group action, so that they are differentiable manifolds and Arnold's machinery is applicable. Moreover we remark that by joining the orbits according to the discrete classifying invariants one obtains differentiable "strata" (each one formed by the union of classes differing only on continuous classifying invariants). We list the dimension of each orbit and the corresponding strata (Proposition 7). As an application of the previous results, we present the unobservable bifurcation diagram of a miniversal deformation (Example 10).

Secondly we consider parameterized families of controllable BLCS. It is known (see [3]) that for each value of the parameter there is a feedback which stabilizes the corresponding system. Thus, we lead to the quite general question of whether pointwise solvability implies the existence of a nicely parameterized solution [10]. This parameterized family of pointwise stabilizers may not be differentiable (not even continuous). Our results allow constructing a differentiable family of feedbacks which stabilizes the corresponding system for each value of the parameter (Theorem 15).

We point out that when dealing with parameterized families of BLCS, the nongeneric case of unobservable ones appears in a natural way. See, for example, the circuit modeling the Fitzghugh-Nagumo equations in [9], where the unobservable case appears if $R_{3}=R_{4}$.

Finally, notice that, as in previous works concerning single control systems, we will use geometrical techniques: reducing bases, stratifications, and miniversal deformations In this sense, we expect that the geometrical approach in [11] could be translated to BLCS in the future.

Throughout the paper, $\mathbb{R}$ will denote the set of real numbers and $M_{n \times m}(\mathbb{R})$ the set of matrices having $n$ rows and $m$ columns and entries in $\mathbb{R}$ (in the case where $n=m$, we will simply write $M_{n}(\mathbb{R})$ ).

## 2. Planar Bimodal Linear Control Systems

Let us consider a bimodal linear control system (BLCS) given by

$$
\begin{align*}
& \dot{x}(t)=A_{1} x(t)+B_{1} u(t), \\
& y(t)=C x(t), \quad \text { if } y(t) \leq 0,  \tag{1}\\
& \begin{array}{c}
\dot{x}(t)=A_{2} x(t)+B_{2} u(t), \quad \text { if } y(t) \geq 0, \\
y(t)=C x(t),
\end{array}
\end{align*}
$$

where $A_{1}, A_{2} \in M_{n}(\mathbb{R}) ; B_{1}, B_{2} \in M_{n \times 1}(\mathbb{R}) ; C \in M_{1 \times n}(\mathbb{R})$. One assumes that the dynamics is continuous along the separating hyperplane $H=\left\{x \in \mathbb{R}^{n}: C x=0\right\}$; that is to say, both subsystems coincide for $y(t)=0$.

By means of a linear change in the state variable $x(t)$, one can consider $C=\left(\begin{array}{llll}1 & 0 & \cdots\end{array}\right) \in M_{1 \times n}(\mathbb{R})$. Hence $H=\{x \in$ $\left.\mathbb{R}^{n}: x_{1}=0\right\}$ and continuity along $H$ is equivalent to

$$
\begin{equation*}
B_{2}=B_{1}, \quad A_{2} e_{i}=A_{1} e_{i}, \quad 2 \leq i \leq n . \tag{2}
\end{equation*}
$$

We will write from now on $B=B_{1}=B_{2}$.
Definition 1. In the above conditions, one says that the triple of matrices $\left(A_{1}, A_{2}, B\right)$ defines a bimodal linear control system (BLCS). Throughout the paper, $\mathscr{X}$ will denote the set of these triples:

$$
\begin{align*}
\mathcal{X}=\{ & \left(A_{1}, A_{2}, B\right) \in M_{n}(\mathbb{R}) \times M_{n}(\mathbb{R})  \tag{3}\\
& \left.\times M_{n \times 1}(\mathbb{R}) \mid A_{2} e_{i}=A_{1} e_{i}, 2 \leq i \leq n\right\}
\end{align*}
$$

which is obviously a $\left(n^{2}+2 n\right)$-differentiable manifold.
The system is called observable if

$$
\operatorname{rank}\left(\begin{array}{c}
C  \tag{4}\\
C A_{i} \\
\cdots \\
C A_{i}^{n-1}
\end{array}\right)=n, \quad i=1,2
$$

A natural goal is simplifying the matrices $A_{1}, A_{2}$, and $B$ by means of changes in the variables $x(t)$ which preserve the qualitative behavior of the system. So, one considers linear changes in the state variables space preserving the hyperplanes $x_{1}(t)=k$.

Definition 2. One calls admissible basis changes those given by the matrices

$$
\begin{align*}
& \mathcal{S}:=\left\{S \in G l_{n}(\mathbb{R}) \left\lvert\, S=\left(\begin{array}{cc}
1 & 0 \\
U & T
\end{array}\right)\right.,\right. \\
&\left.T \in G l_{n-1}(\mathbb{R}), U \in M_{n \times 1}(\mathbb{R})\right\} . \tag{5}
\end{align*}
$$

Then, $\left(A_{1}, A_{2}, B\right),\left(A_{1}^{\prime}, A_{2}^{\prime}, B^{\prime}\right) \in X$ are said to be equivalent if there exists a matrix $S \in \mathcal{S}$ (representing an admissible basis change) such that $\left(A_{1}^{\prime}, A_{2}^{\prime}, B^{\prime}\right)=$ $\left(S^{-1} A_{1} S, S^{-1} A_{2} S, S^{-1} B\right)$.

Notice that the matrix $C$ is not involved in this definition since $C S=C$ for any $S \in \mathcal{S}$.

When considering canonical forms, it is necessary that the coefficients appearing in them as well as the conditions used to distinguish the different types do not depend on the admissible basis which one considers; that is to say, they are preserved under admissible basis changes $S \in \mathcal{S}$. It is wellknown that $\operatorname{tr} A_{1}, \operatorname{tr} A_{2}$, $\operatorname{det} A_{1}$, and $\operatorname{det} A_{2}$ are invariant under any basis change $S \in G l_{n}(\mathbb{R})$. We focus on the additional invariants when only admissible basis changes $S \in$ $\mathcal{S}$ are considered.

Definition 3. A real number (resp., a property) associated with a triple $\left(A_{1}, A_{2}, B\right)$ is called $\mathcal{S}$-invariant if it is preserved by admissible basis changes; that is to say, it has the same value (resp., it is also true) for any other triple $\left(A_{1}^{\prime}, A_{2}^{\prime}, B^{\prime}\right) \mathcal{S}$ equivalent to the given one.

For example, it is obvious that they are $S$-invariant: the top coefficient $b_{1}$ in $B$, the matrix $C$, and the condition of ( $A_{1}, A_{2}, B$ ) being observable.

We introduce another $\mathcal{S}$-invariant that will be used under additional hypotheses.

## Definition 4. Given a triple

$$
A_{1}=\left(\begin{array}{ll}
a_{1} & a_{3}  \tag{6}\\
a_{2} & a_{4}
\end{array}\right), \quad A_{2}=\left(\begin{array}{ll}
\gamma_{1} & a_{3} \\
\gamma_{2} & a_{4}
\end{array}\right), \quad B=\binom{b_{1}}{b_{2}}
$$

one writes

$$
\begin{gather*}
\Delta_{0}=\operatorname{det}\left(\begin{array}{ll}
a_{3} & b_{1} \\
a_{4} & b_{2}
\end{array}\right)=a_{3} b_{2}-a_{4} b_{1}, \\
\Delta_{12}=a_{2}\left(a_{4}-\gamma_{1}\right)-\gamma_{2}\left(a_{4}-a_{1}\right),  \tag{7}\\
\Delta_{1}=b_{1} a_{2}+\left(a_{4}-a_{1}\right) b_{2}, \\
\Delta_{2}=b_{1} \gamma_{2}+\left(a_{4}-\gamma_{1}\right) b_{2} .
\end{gather*}
$$

Lemma 5. The above triple is unobservable if and only if $a_{3}=$ 0 . In this case one has
(1) $\operatorname{det}\left(\binom{a_{1}-\gamma_{1}}{a_{2}-\gamma_{2}} \left\lvert\, A_{i}\binom{a_{1}-\gamma_{1}}{a_{2}-\gamma_{2}}\right.\right)=\left(a_{1}-\gamma_{1}\right) \Delta_{12}, i=1,2$,
$\operatorname{det}\left(B \mid A_{i} B\right)=b_{1} \Delta_{i}, i=1,2 ;$
(2) the action of $S \in \mathcal{S}$ transforms $\Delta_{1}, \Delta_{2}$, and $\Delta_{12}$, respectively, into

$$
\begin{equation*}
\frac{1}{\operatorname{det} S} \Delta_{1}, \quad \frac{1}{\operatorname{det} S} \Delta_{2}, \quad \frac{1}{\operatorname{det} S} \Delta_{12} \tag{8}
\end{equation*}
$$

In particular, it is $\mathcal{S}$-invariant the sign (positive, negative, or zero):

$$
\begin{equation*}
\operatorname{sign}\left(\Delta_{1} \Delta_{2}\right) \tag{9}
\end{equation*}
$$

Proof. Clearly,

$$
\binom{C}{C A_{1}}=\left(\begin{array}{cc}
1 & 0  \tag{*}\\
a_{1} & a_{3}
\end{array}\right), \quad\binom{C}{C A_{2}}=\left(\begin{array}{cc}
1 & 0 \\
\gamma_{1} & a_{3}
\end{array}\right)
$$

do not have maximal rank when $a_{3}=0$. Then
(1) it is a straightforward computation;
(2) if $a_{3}=0$, then $a_{1}$ and $\gamma_{1}$ are eigenvalues of $A_{1}$ and $A_{2}$.

The action of $S$ transforms the matrices in (*) into their left product by $S^{-1}$.

Theorem 6. With the above notation:
(1) Table 1 summarizes some $\mathcal{S}$-invariant numbers and properties, as well as the hypotheses for each one;
(2) Table 2 lists the possible canonical forms and the classification criteria.

Proof. (i) Concerning Table 1
(1) the $\mathcal{S}$-action on $A_{1}$ and $B$ can be formulated as

$$
\begin{align*}
& S^{-1}\left(A_{1}, B\right)\left(\frac{S \mid 0}{0 \mid 1}\right) \\
& \quad=\left(\begin{array}{ll}
1 & 0 \\
u & t
\end{array}\right)^{-1}\left(\begin{array}{lll}
a_{1} & a_{3} & b_{1} \\
a_{2} & a_{4} & b_{2}
\end{array}\right)\left(\begin{array}{lll}
1 & 0 & 0 \\
u & t & 0 \\
0 & 0 & 1
\end{array}\right)  \tag{10}\\
& \quad=\left(\begin{array}{l}
* \\
*
\end{array} \left\lvert\,\left(\begin{array}{ll}
1 & 0 \\
u & t
\end{array}\right)^{-1}\left(\begin{array}{ll}
a_{3} & b_{1} \\
a_{4} & b_{2}
\end{array}\right)\left(\begin{array}{ll}
t & 0 \\
0 & 1
\end{array}\right)\right.\right) .
\end{align*}
$$

Therefore, $\Delta_{0}$ is $\mathcal{S}$-invariant:

$$
\operatorname{det}\left(\left(\begin{array}{ll}
1 & 0  \tag{11}\\
u & t
\end{array}\right)^{-1}\left(\begin{array}{ll}
a_{3} & b_{1} \\
a_{4} & b_{2}
\end{array}\right)\left(\begin{array}{ll}
t & 0 \\
0 & 1
\end{array}\right)\right)=\operatorname{det}\left(\begin{array}{ll}
a_{3} & b_{1} \\
a_{4} & b_{2}
\end{array}\right)
$$

We have seen that $a_{3} \neq 0$ if and only if

$$
\begin{equation*}
\operatorname{rank}\binom{C}{C A_{i}}=2, \quad i=1,2 \tag{12}
\end{equation*}
$$

which is $\mathcal{S}$-invariant.
(2) If $b_{1}=0$, then

$$
S^{-1}\binom{0}{b_{2}}=\frac{1}{t}\left(\begin{array}{cc}
t & 0  \tag{13}\\
-u & 1
\end{array}\right)\binom{0}{b_{2}}=\frac{1}{t}\binom{0}{b_{2}} .
$$

(3) If $a_{3}=0$, then $a_{1}, a_{4}$, and $\gamma_{1}$ are the eigenvalues of $A_{1}$ and $A_{2}$. Then

$$
\begin{equation*}
\operatorname{Ker}\left(A_{1}-a_{1} I\right)=\operatorname{Ker}\left(A_{2}-\gamma_{1} I\right) \tag{14}
\end{equation*}
$$

if and only if

$$
\operatorname{rank}\left(\begin{array}{ll}
a_{2} & a_{4}-a_{1}  \tag{15}\\
\gamma_{2} & a_{4}-\gamma_{1}
\end{array}\right)=1
$$

or, equivalently,

$$
0=\operatorname{det}\left(\begin{array}{ll}
a_{2} & a_{4}-a_{1}  \tag{16}\\
\gamma_{2} & a_{4}-\gamma_{1}
\end{array}\right)=\Delta_{12}
$$

In a similar way, $\Delta_{1}=0$ if and only if

$$
\begin{equation*}
\binom{b_{1}}{b_{2}} \in \operatorname{Ker}\left(A_{1}-a_{1} I\right) \tag{17}
\end{equation*}
$$

and $\Delta_{2}=0$ if and only if

$$
\begin{equation*}
\binom{b_{1}}{b_{2}} \in \operatorname{Ker}\left(A_{2}-\gamma_{1} I\right) \tag{18}
\end{equation*}
$$

(3') This case follows from (3) and the above lemma.
(4) Clearly, if $a_{3}=0$ and $a_{1}=a_{4}$, then $a_{2}=0$ if and only if $A_{1}$ diagonalizes.
(4) Analogously than (4) for $\gamma_{2}=0$.
(5) Returning to the formulation in (1):

$$
\left(\begin{array}{cc}
1 & 0  \tag{19}\\
u & t
\end{array}\right)^{-1}\left(\begin{array}{ccc}
a_{1} & 0 & 0 \\
a_{2} & a_{1} & b_{2}
\end{array}\right)\left(\begin{array}{ccc}
1 & 0 & 0 \\
u & t & 0 \\
0 & 0 & 1
\end{array}\right)=\left(\begin{array}{ccc}
a_{1} & 0 & 0 \\
\frac{a_{2}}{t} & a_{1} & \frac{b_{2}}{t}
\end{array}\right)
$$

(5') Analogously than (5) for $\gamma_{1}=a_{4}$.
(ii) This case follows from [12], bearing in mind the $\mathcal{S}$ invariants in (i).

## 3. Stratification Diagrams

In the previous section we have partitioned the set of BLCS into equivalence classes, characterized by the reduced forms in Theorem 6. In order to study, for example, the changes when a BLCS is perturbed, some natural questions arise about the geometric structure of this equivalence partition. We will see in a moment that each equivalence class is a manifold, as well as the "strata" obtained by joining the classes that differ only in the continuous classification parameters. Their dimensions are listed in Proposition 7.

Concerning perturbations, small changes in the coefficients of the matrices defining the system may give rise to

TABLE 1: $\mathcal{S}$-Invariant numbers.

|  | Hypotheses | Numbers | Properties |
| :--- | :---: | :---: | :---: |
| $(1)$ |  | $\Delta_{0}, b_{1}$ | $a_{3}=0$ |
| $(2)$ | $b_{1}=0$ |  | $b_{2}=0$ |
| $(3)$ | $a_{3}=0$ | $a_{1}, \gamma_{1}, a_{4}$ | $\Delta_{12}=0, \Delta_{1}=0, \Delta_{2}=0$ |
| $\left(3^{\prime}\right)$ | $a_{3}=0, \Delta_{12} \neq 0$ | $\Delta_{1} / \Delta_{12}, \Delta_{2} / \Delta_{12}$ |  |
| $(4)$ | $a_{3}=0, a_{1}=a_{4}$ |  | $a_{2}=0$ |
| $\left(4^{\prime}\right)$ | $a_{3}=0, \gamma_{1}=a_{4}$ | $b_{2} / a_{2}$ | $\gamma_{2}=0$ |
| $(5)$ | $b_{1}=0, a_{3}=0, a_{1}=a_{4}$ | $b_{2} / \gamma_{2}$ |  |
| $\left(5^{\prime}\right)$ | $b_{1}=0, a_{3}=0, \gamma_{1}=a_{4}$ |  |  |

TAble 2: Canonical forms.

| Label | Classification criteria | Canonical forms |
| :---: | :---: | :---: |
| CF1 | $a_{3} \neq 0$ | $\left(\begin{array}{cc}\operatorname{tr} A_{1} & 1 \\ -\operatorname{det} A_{1} & 0\end{array}\right),\left(\begin{array}{cc}\operatorname{tr} A_{2} & 1 \\ -\operatorname{det} A_{2} & 0\end{array}\right),\binom{b_{1}}{\Delta_{0}}$ |
| CF2 | $a_{3}=0, a_{1} \neq a_{4}, \gamma_{1} \neq a_{4}, \Delta_{12} \neq 0$ | $\left(\begin{array}{cc}a_{1} & 0 \\ 0 & a_{4}\end{array}\right),\left(\begin{array}{cc}\gamma_{1} & 0 \\ 1 & a_{4}\end{array}\right),\binom{b_{1}}{-\Delta_{1} / \Delta_{12}}$ |
| CF3 | $a_{3}=0, a_{1} \neq a_{4}, \gamma_{1} \neq a_{4}, \Delta_{12}=0, \Delta_{1} \neq 0$ | $\left(\begin{array}{cc}a_{1} & 0 \\ 0 & a_{4}\end{array}\right),\left(\begin{array}{cc}\gamma_{1} & 0 \\ 0 & a_{4}\end{array}\right),\binom{b_{1}}{1}$ |
| CF4 | $a_{3}=0, a_{1} \neq a_{4}, \gamma_{1} \neq a_{4}, \Delta_{12}=0, \Delta_{1}=0$ | $\left(\begin{array}{cc}a_{1} & 0 \\ 0 & a_{4}\end{array}\right),\left(\begin{array}{cc}\gamma_{1} & 0 \\ 0 & a_{4}\end{array}\right),\binom{b_{1}}{0}$ |
| CF5 | $a_{3}=0, a_{1}=a_{4}, \gamma_{1} \neq a_{4}, a_{2} \neq 0$ | $\left(\begin{array}{cc}a_{4} & 0 \\ 1 & a_{4}\end{array}\right),\left(\begin{array}{cc}\gamma_{1} & 0 \\ 0 & a_{4}\end{array}\right),\binom{b_{1}}{\Delta_{2} / \Delta_{12}}$ |
| CF5 ${ }^{\prime}$ | $a_{3}=0, a_{1} \neq a_{4}, \gamma_{1}=a_{4}, \gamma_{2} \neq 0$ | $\left(\begin{array}{cc}a_{1} & 0 \\ 0 & a_{4}\end{array}\right),\left(\begin{array}{cc}a_{4} & 0 \\ 1 & a_{4}\end{array}\right),\binom{b_{1}}{-\Delta_{1} / \Delta_{12}}$ |
| CF6 | $a_{3}=0, a_{1}=a_{4}, \gamma_{1} \neq a_{4}, a_{2}=0, \Delta_{2} \neq 0$ | $\left(\begin{array}{cc}a_{4} & 0 \\ 0 & a_{4}\end{array}\right),\left(\begin{array}{cc}\gamma_{1} & 0 \\ 0 & a_{4}\end{array}\right),\binom{b_{1}}{1}$ |
| CF6 ${ }^{\prime}$ | $a_{3}=0, a_{1} \neq a_{4}, \gamma_{1}=a_{4}, \gamma_{2}=0, \Delta_{1} \neq 0$ | $\left(\begin{array}{cc}a_{1} & 0 \\ 0 & a_{4}\end{array}\right),\left(\begin{array}{cc}a_{4} & 0 \\ 0 & a_{4}\end{array}\right),\binom{b_{1}}{1}$ |
| CF7 | $a_{3}=0, a_{1}=a_{4}, \gamma_{1} \neq a_{4}, a_{2}=0, \Delta_{2}=0$ | $\left(\begin{array}{cc}a_{4} & 0 \\ 0 & a_{4}\end{array}\right),\left(\begin{array}{cc}\gamma_{1} & 0 \\ 0 & a_{4}\end{array}\right),\binom{b_{1}}{0}$ |
| CF7 ${ }^{\prime}$ | $a_{3}=0, a_{1} \neq a_{4}, \gamma_{1}=a_{4}, \gamma_{2}=0, \Delta_{1}=0$ | $\left(\begin{array}{cc}a_{1} & 0 \\ 0 & a_{4}\end{array}\right),\left(\begin{array}{cc}a_{4} & 0 \\ 0 & a_{4}\end{array}\right),\binom{b_{1}}{0}$ |
| CF8 | $a_{3}=0, a_{1}=a_{4}=\gamma_{1}, a_{2} \neq 0, \gamma_{2} \neq 0, b_{1} \neq 0$ | $\left(\begin{array}{cc}a_{4} & 0 \\ 1 & a_{4}\end{array}\right),\left(\begin{array}{cc}a_{4} & 0 \\ \gamma_{2} / a_{2} & a_{4}\end{array}\right),\binom{b_{1}}{0}$ |
| CF9 | $a_{3}=0, a_{1}=a_{4}=\gamma_{1}, a_{2} \neq 0, \gamma_{2} \neq 0, b_{1}=0$ | $\left(\begin{array}{cc}a_{4} & 0 \\ 1 & a_{4}\end{array}\right),\left(\begin{array}{cc}a_{4} & 0 \\ \gamma_{2} / a_{2} & a_{4}\end{array}\right),\binom{0}{b_{2} / a_{2}}$ |
| CF10 | $a_{3}=0, a_{1}=a_{4}=\gamma_{1}, a_{2} \neq 0, \gamma_{2}=0, b_{1} \neq 0$ | $\left(\begin{array}{cc}a_{4} & 0 \\ 1 & a_{4}\end{array}\right),\left(\begin{array}{cc}a_{4} & 0 \\ 0 & a_{4}\end{array}\right),\binom{b_{1}}{0}$ |
| CF10 ${ }^{\prime}$ | $a_{3}=0, a_{1}=a_{4}=\gamma_{1}, a_{2}=0, \gamma_{2} \neq 0, b_{1} \neq 0$ | $\left(\begin{array}{cc}a_{4} & 0 \\ 0 & a_{4}\end{array}\right),\left(\begin{array}{cc}a_{4} & 0 \\ 1 & a_{4}\end{array}\right),\binom{b_{1}}{0}$ |
| CF11 | $a_{3}=0, a_{1}=a_{4}=\gamma_{1}, a_{2} \neq 0, \gamma_{2}=0, b_{1}=0$ | $\left(\begin{array}{cc}a_{4} & 0 \\ 1 & a_{4}\end{array}\right),\left(\begin{array}{cc}a_{4} & 0 \\ 0 & a_{4}\end{array}\right),\binom{0}{b_{2} / a_{2}}$ |
| CF11 ${ }^{\prime}$ | $a_{3}=0, a_{1}=a_{4}=\gamma_{1}, a_{2}=0, \gamma_{2} \neq 0, b_{1}=0$ | $\left(\begin{array}{cc}a_{4} & 0 \\ 0 & a_{4}\end{array}\right),\left(\begin{array}{cc}a_{4} & 0 \\ 1 & a_{4}\end{array}\right),\binom{0}{b_{2} / \gamma_{2}}$ |
| CF12 | $a_{3}=0, a_{1}=a_{4}=\gamma_{1}, a_{2}=0, \gamma_{2}=0, b_{1} \neq 0$ | $\left(\begin{array}{cc}a_{4} & 0 \\ 0 & a_{4}\end{array}\right),\left(\begin{array}{cc}a_{4} & 0 \\ 0 & a_{4}\end{array}\right),\binom{b_{1}}{0}$ |
| CF13 | $a_{3}=0, a_{1}=a_{4}=\gamma_{1}, a_{2}=0, \gamma_{2}=0, b_{1}=0, b_{2} \neq 0$ | $\left(\begin{array}{cc}a_{4} & 0 \\ 0 & a_{4}\end{array}\right),\left(\begin{array}{cc}a_{4} & 0 \\ 0 & a_{4}\end{array}\right),\binom{0}{1}$ |
| CF14 | $a_{3}=0, a_{1}=a_{4}=\gamma_{1}, a_{2}=0, \gamma_{2}=0, b_{1}=0, b_{2}=0$ | $\left(\begin{array}{cc}a_{4} & 0 \\ 0 & a_{4}\end{array}\right),\left(\begin{array}{cc}a_{4} & 0 \\ 0 & a_{4}\end{array}\right),\binom{0}{0}$ |

Table 3: Dimension of orbits and strata for each case in Table 2.

| Canonical form | Dimension of <br> the orbit | Dimension of <br> the stratum |
| :--- | :---: | :---: |
| CF1 | 2 | 8 |
| CF2 | 2 | 7 |
| CF3 | 2 | 6 |
| CF4 | 1 | 5 |
| CF5, CF5' | 2 | 6 |
| CF6, CF6 $^{\prime}$ | 2 | 5 |
| CF7, CF7 $^{\prime}$ | 1 | 4 |
| CF8 | 2 | 5 |
| CF9 | 1 | 4 |
| CF10, CF10 | 2 | 4 |
| CF11, CF11 | 1 | 3 |
| CF12 | 1 | 3 |
| CF13 | 1 | 2 |
| CF14 | 0 | 1 |

nonequivalent systems. Then in order to explain the behavior of the system under small perturbations, it is necessary to know the nearby equivalence classes. The "stratification diagram" ("bifurcation diagram" in Arnold's terminology) of a parameterized family of systems is the partition of the parameter space according to the equivalence class. Arnold's theory [13] shows that they are induced by the ones of the so-called versal deformations. In particular, the miniversal deformations are the simplest parameterized families which provide all the information about which equivalence classes are near a given one, that is to say, which canonical forms appear when the given one is perturbed.

The starting point is that the above equivalence classes are actually the orbits with regard to the action of the Lie group $\mathcal{S}$ on the differentiable manifold $\mathscr{X}$ :

$$
\begin{equation*}
\alpha: \mathcal{S} \times \mathscr{X} \longrightarrow \mathscr{X} \tag{20}
\end{equation*}
$$

defined by

$$
\begin{equation*}
\alpha(S, \mathscr{X})=\left(S^{-1} A_{1} S, S^{-1} A_{2} S, S^{-1} B\right) \tag{21}
\end{equation*}
$$

Given any triple of matrices $\left(A_{1}, A_{2}, B\right) \in \mathscr{X}$, we will denote by $\mathcal{O}\left(A_{1}, A_{2}, B\right)$ its orbit (or equivalence class).

As an application of the closed orbit lemma (see [14]), we deduce that equivalence classes are differentiable manifolds. Namely, any equivalence class is a locally closed differentiable submanifold of $\mathscr{X}$ and its boundary is a union of equivalence classes or orbits of strictly lower dimension. In particular, equivalence classes or orbits of minimal dimension are closed.

Moreover, orbits having the same discrete classification parameters (but differing in the continuous ones) can be joined in a finite number of "strata" which in our case are as well differentiable manifolds (see [12]). For the commodity of the reader we adapt the results there.

Proposition 7. Table 3 lists the dimensions of the orbits (i.e., the equivalence classes) and the corresponding strata (i.e., the
union of the orbits of the same type when the parameters appearing in the canonical form vary).

To illustrate the above considerations and as an application of the results in Section 2, we will present (see Figure 1) the unobservable bifurcation diagram of the miniversal deformation of a system of type CF10'. The main definitions and results about deformations and versality can be found in [13, 15]. Here we rewrite them down, adapted to our particular case.

Definition 8. A deformation of $\left(A_{1}, A_{2}, B\right) \in \mathscr{X}$ is a differentiable $\operatorname{map} \varphi: U \rightarrow \mathscr{X}$, with $U$ an open neighbourhood of the origin $\mathbb{R}^{d}$, such that $\varphi(0)=\left(A_{1}, A_{2}, B\right)$.

A deformation $\varphi: U \rightarrow \mathcal{X}$ of $\left(A_{1}, A_{2}, B\right)$ is called versal at 0 if, for any other deformation of $\left(A_{1}, A_{2}, B\right), \psi$ : $V \rightarrow \mathcal{X}$, there exists a neighbourhood $V^{\prime} \subseteq V$ with $0 \in V^{\prime}$, a differentiable map $\gamma: V^{\prime} \rightarrow U$ with $\gamma(0)=0$, and a deformation of the identity $I \in \mathcal{S}, \theta: V^{\prime} \rightarrow \mathcal{S}$, such that $\psi(\mu)=\alpha(\theta(\mu), \varphi(\gamma(\mu)))$ for all $\mu \in V^{\prime}$.

A versal deformation with minimal number of parameters $d$ is called miniversal deformation.

A miniversal deformation can be obtained from the normal space to the orbit with regard to some scalar product.

Proposition 9 (see [16]). We consider the following scalar product in $\mathfrak{X}$ :

$$
\begin{align*}
& \left\langle\left(A_{1}, A_{2}, B\right),\left(A_{1}^{\prime}, A_{2}^{\prime}, B^{\prime}\right)\right\rangle  \tag{22}\\
& \quad=\operatorname{tr}\left(A_{1}^{t} A_{1}^{\prime}\right)+\operatorname{tr}\left(A_{2}^{t} A_{2}^{\prime}\right)+\operatorname{tr}\left(B^{t} B^{\prime}\right)
\end{align*}
$$

(i) The normal space to the orbit of $\left(A_{1}, A_{2}, B\right)$ at $\left(A_{1}, A_{2}\right.$, B) $N_{\left(A_{1}, A_{2}, B\right)} \mathcal{O}\left(A_{1}, A_{2}, B\right) \cap \mathscr{X}$ is the vector subspace consisting of triples $\left(X_{1}, X_{2}, Y\right) \in \mathscr{X}$ such that

$$
\begin{equation*}
A_{1} X_{1}^{t}-X_{1}^{t} A_{1}+A_{2} X_{2}^{t}-X_{2}^{t} A_{2}-B Y^{t} \in \mathbb{A} \tag{23}
\end{equation*}
$$

where $\mathbb{A}$ is the set

$$
\begin{equation*}
\mathbb{A}=\left\{M=\left(m_{i}^{j}\right) \mid m_{i}^{j}=0,2 \leq i \leq n, 1 \leq j \leq n\right\} . \tag{24}
\end{equation*}
$$

(ii) Then the mapping

$$
\begin{gather*}
\mathbb{R}^{d} \longrightarrow \mathscr{X} \\
\left(\eta_{1}, \ldots, \eta_{d}\right) \longrightarrow\left(A_{1}, A_{2}, B\right)+\eta_{1} V_{1}+\cdots+\eta_{d} V_{d} \tag{25}
\end{gather*}
$$

where $\left\{V_{1}, \ldots, V_{d}\right\}$ is any basis of the vector space $N_{\left(A_{1}, A_{2}, B\right)} \mathcal{O}\left(A_{1}, A_{2}, B\right)$, is a miniversal deformation of $\left(A_{1}, A_{2}, B\right)$.

Normal spaces of two equivalent triples can be obtained one from the other. Thus, it is always possible to restrict ourselves to the case where the triple is in its canonical form.

Here, as an application of the previous results, we present the unobservable bifurcation diagram of a miniversal deformation: Figure 1 shows the geometrical configuration of the unobservable strata near a given system of type CF10'.

Example 10. Consider a bimodal linear dynamical system of type CF10' whose canonical form is

$$
A_{1}=\left(\begin{array}{cc}
a_{4} & 0  \tag{26}\\
0 & a_{4}
\end{array}\right), \quad A_{2}=\left(\begin{array}{cc}
a_{4} & 0 \\
1 & a_{4}
\end{array}\right), \quad B=\binom{b_{1}}{0} .
$$

Then, $N_{\left(A_{1}, A_{2}, B\right)} \mathcal{O}\left(A_{1}, A_{2}, B\right) \cap \mathscr{X}$ is the vector subspace consisting of triples $\left(X_{1}, X_{2}, Y\right) \in \mathscr{X}$

$$
X_{1}=\left(\begin{array}{ll}
x_{1} & x_{3}  \tag{27}\\
x_{2} & x_{4}
\end{array}\right), \quad X_{2}=\left(\begin{array}{ll}
x_{5} & x_{3} \\
x_{6} & x_{4}
\end{array}\right), \quad Y=\binom{y_{1}}{y_{2}}
$$

such that

$$
\begin{gather*}
x_{6}=0 \\
a_{4} x_{5}+b_{1} y_{2}=0 \tag{28}
\end{gather*}
$$

Moreover, parameter $x_{3}$ must be zero to avoid observable perturbations and parameters $x_{4}, y_{1}$ give orbits in the initial stratum.

Then the unobservable perturbations in the normal space to the stratum of $\left(A_{1}, A_{2}, B\right)$ are parameterized by

$$
\begin{align*}
& \varphi\left(x_{1}, x_{2}, x_{5}\right) \\
& \quad=\left(\left(\begin{array}{cc}
a_{4}+x_{1} & 0 \\
x_{2} & a_{4}
\end{array}\right),\left(\begin{array}{cc}
a_{4}+x_{5} & 0 \\
1 & a_{4}
\end{array}\right),\binom{b_{1}}{-\frac{a_{4}}{b_{1}} x_{5}}\right) . \tag{29}
\end{align*}
$$

We denote by $E_{i}$ the set of all triples of matrices having canonical form of type (CFi), $i=1, \ldots, 14$.

Clearly, if only $x_{1}$ (resp., $x_{2}$ ) is nonzero, it lies in $E 5^{\prime}$ (resp., $E 8)$. But for only $x_{5}$, the strata $E 6$ and $E 7$ are possible in principle, depending on the value of $\Delta_{2}$. In our case

$$
\begin{align*}
\Delta_{2} & =b_{1} \gamma_{2}+\left(a_{4}-\gamma_{1}\right) b_{2}=b_{1}+\left(-x_{5}\right)\left(-\frac{a_{4}}{b_{1}} x_{5}\right)  \tag{30}\\
& =\frac{1}{b_{1}}\left(b_{1}^{2}+a_{4} x_{5}^{2}\right) .
\end{align*}
$$

Hence, it belongs to $E 7$ for $x_{5}^{2}=-b_{1}^{2} / a_{4}$, and to $E 6$ otherwise.

In a similar way, if $x_{1}, x_{5} \neq 0$ only $E 2, E 3$, and $E 4$ are possible. We have $\Delta_{0}=-x_{2} x_{5}+x_{1}$. Hence, $x_{2}=0$ implies $\Delta_{0} \neq 0$, which corresponds to $E 2$. If $x_{2} \neq 0$, it gives again E2 except on the hyperbolic paraboloid $x_{1}=x_{2} x_{5}$. When it happens,

$$
\begin{equation*}
\Delta_{1}=b_{1} x_{2}+x_{1} \frac{a_{4}}{b_{1}} x_{5}=\frac{x_{2}}{b_{1}}\left(b_{1}^{2}+a_{4} x_{5}^{2}\right) \tag{31}
\end{equation*}
$$

Hence, it lies in $E 4$ for $x_{5}^{2}=-b_{1}^{2} / a_{4}$, and in $E 3$ otherwise.
Finally, it is straightforward that one obtains $E 5$ for $x_{1}=$ $0, x_{2}, x_{5} \neq 0$, and $E 5^{\prime}$ for $x_{5}=0, x_{1}, x_{2} \neq 0$. In summary (see Figure 1),
(i) if $x_{2}, x_{5}=0, x_{1} \neq 0$, then $\varphi\left(x_{1}, x_{2}, x_{5}\right) \in E 5^{\prime}$;
(ii) if $x_{1}, x_{5}=0, x_{2} \neq 0$, then $\varphi\left(x_{1}, x_{2}, x_{5}\right) \in E 8$;


Figure 1: Stratification diagram.
(iii) if $x_{1}, x_{2}=0, x_{5}^{2}=-b_{1}^{2} / a_{4}$, then $\varphi\left(x_{1}, x_{2}, x_{5}\right) \in E 7$;
(iv) if $x_{1}, x_{2}=0, x_{5} \neq 0, x_{5}^{2} \neq-b_{1}^{2} / a_{4}$, then $\varphi\left(x_{1}, x_{2}, x_{5}\right) \in$ E6;
(v) if $x_{5}=0, x_{1}, x_{2} \neq 0$, then $\varphi\left(x_{1}, x_{2}, x_{5}\right) \in E 5^{\prime}$;
(vi) if $x_{2}=0, x_{1}, x_{5} \neq 0$, then $\varphi\left(x_{1}, x_{2}, x_{5}\right) \in E 2$;
(vii) if $x_{1}=0, x_{2}, x_{5} \neq 0$, then $\varphi\left(x_{1}, x_{2}, x_{5}\right) \in E 5$;
(viii) if $x_{1}, x_{2}, x_{5} \neq 0, x_{1}=x_{2} x_{5}, x_{5}^{2}=-b_{1}^{2} / a_{4}$, then $\varphi\left(x_{1}, x_{2}, x_{5}\right) \in E 4 ;$
(ix) if $x_{1}, x_{2}, x_{5} \neq 0, x_{1}=x_{2} x_{5}, x_{5}^{2} \neq-b_{1}^{2} / a_{4}$, then $\varphi\left(x_{1}, x_{2}, x_{5}\right) \in E 3 ;$
(x) if $x_{1}, x_{2}, x_{5} \neq 0, x_{1} \neq x_{2} x_{5}$, then $\varphi\left(x_{1}, x_{2}, x_{5}\right) \in E 2$.

## 4. Controllability and Families of Stabilizers

We have seen that in a differentiable family of BLCS different equivalence classes can appear for different values of the parameters. Let us see that, however, some global treatments are possible. Indeed, we will prove that a differentiable family of stabilizers exists if each BLCS in the given family is controllable.

The notion of controllability of a single system is extended to bimodal ones in a natural way.

Definition 11. A BLCS is (completely) controllable if for any pair of states $\left(x_{0}, x_{f}\right)$ there exists a locally integrable input $u$ such that the solution $x^{x_{0}, u}$ passes through $x_{f}$; that is, $x^{x_{0}, u}(T)=x_{f}$ for some $T>0$.

A well-known remarkable fact is that a single linear system $\dot{x}=A x+B u$ is controllable if and only if its "controllability matrix" ( $B \quad A B \cdots A^{n-1} B$ ) has maximal rank. For planar BLCS we recall the characterization of controllability of planar BLCS obtained in [1] for observable systems and generalized in [17] to unobservable ones.

Proposition 12. Let one consider a planar BLCS defined by $\left(A_{1}, A_{2}, B\right)$. One writes $C_{1}, C_{2}$ the controllability matrices of both subsystems

$$
C_{1}=\left(\begin{array}{ll}
B & A_{1} B
\end{array}\right), \quad C_{2}=\left(\begin{array}{ll}
B & A_{2} B \tag{32}
\end{array}\right) .
$$

Then, it is controllable if and only if

$$
\begin{equation*}
\operatorname{det} C_{1} \operatorname{det} C_{2}>0 \tag{33}
\end{equation*}
$$

Remark 13. (1) Notice that, in particular, both subsystems must be controllable, but it is not a sufficient condition.
(2) Whereas for single systems the subset of controllable ones is open and dense, the above proposition shows that it is not for BLCS systems: controllability is an open, but not generic, property.

If the control function is a so-called "feedback" of the type $u(t)=f(x(t))$, one obtains a new dynamical system ("in closed loop"). For single linear systems $\dot{x}=A x+B u$, a feedback $u=F x$ gives $\dot{x}=(A+B F) x$. A remarkable fact is that it is stable for some suitable $F$, provided that the initial control system is controllable.

As a natural generalization, in [3] any controllable BLCS is proved to be feedback stabilizable. Hence, if a differentiable parameterized family $\left(A_{1}(s), A_{2}(s), B(s)\right)$ is pointwise controllable (observable or not), then it is also pointwise stabilizable; that is to say, for any $s \in \mathbb{R}$ there is a common feedback $F(s)$ such that both closed-loop systems $A_{1}(s)+$ $B(s) F(s), A_{2}(s)+B(s) F(s)$ are stable. However, the family $F(s)$ may not be differentiable (not even continuous). Here we prove that differentiable families of stabilizer feedbacks exist for $n=2$.

As we have pointed out in the Introduction, the unobservable case appears generically in parameterized families of bimodal systems. A typical case is considered in the following example. As an application of the above proposition, we characterize when this family is pointwise controllable.

Example 14. Let us consider the parameterized family of planar BLCS

$$
A_{1}(s)=\left(\begin{array}{cc}
a_{1} & s  \tag{34}\\
a_{2} & a_{4}
\end{array}\right), \quad A_{2}(s)=\left(\begin{array}{cc}
\gamma_{1} & s \\
\gamma_{2} & a_{4}
\end{array}\right), \quad B=\binom{b_{1}}{b_{2}}
$$

where $s \in \mathbb{R}$. Obviously, the systems defined by these matrices are observable except for $s=0$. Let us see that the family is pointwise controllable (i.e., for any $s \in \mathbb{R}$ the corresponding system is controllable) if and only if $b_{1} \neq 0$ and
(i) $a_{2} \gamma_{2}>0$, if $b_{2}=0$,
(ii) $\operatorname{det}\left(\begin{array}{ll}a_{1} & b_{1} \\ a_{2} & b_{2}\end{array}\right)=\operatorname{det}\left(\begin{array}{ll}\gamma_{1} & b_{1} \\ \gamma_{2} & b_{2}\end{array}\right)$, otherwise.

From Proposition 12, for any $s \in \mathbb{R}$ (including the case where $s=0$ ) the corresponding system is controllable if and only if

$$
\begin{align*}
& \left(b_{1}^{2} a_{2}+b_{1} b_{2} a_{4}-b_{1} b_{2} a_{1}-b_{2}^{2} s\right) \\
& \quad \times\left(b_{1}^{2} \gamma_{2}+b_{1} b_{2} a_{4}-b_{1} b_{2} \gamma_{1}-b_{2}^{2} s\right)>0 \tag{35}
\end{align*}
$$

In particular $b_{1} \neq 0$ (it suffices to take $s=0$ ).
If $b_{2}=0$, the above inequality is

$$
\begin{equation*}
\left(b_{1}^{2} a_{2}\right)\left(b_{1}^{2} \gamma_{2}\right)>0 \tag{36}
\end{equation*}
$$

that is to say,

$$
\begin{equation*}
a_{2} \gamma_{2}>0 \tag{37}
\end{equation*}
$$

Assume now $b_{2} \neq 0$. In general, two polynomials of degree 1 have the same sign at any point if and only if they have the same root and the slopes have the same sign. In our case both slopes are $-b_{2}^{2}$, so that the above inequality holds if and only if

$$
\begin{equation*}
\frac{b_{1}^{2} a_{2}+b_{1} b_{2} a_{4}-b_{1} b_{2} a_{1}}{b_{2}^{2}}=\frac{b_{1}^{2} \gamma_{2}+b_{1} b_{2} a_{4}-b_{1} b_{2} \gamma_{1}}{b_{2}^{2}} \tag{38}
\end{equation*}
$$

which is equivalent (recall $b_{1} \neq 0$ ) to

$$
\begin{equation*}
b_{1} a_{2}-b_{2} a_{1}=b_{1} \gamma_{2}-b_{2} \gamma_{1} \tag{39}
\end{equation*}
$$

Finally, we prove the existence of differentiable families of stabilizers for differentiable families of planar controllable bimodal systems.

Theorem 15. Let

$$
\begin{equation*}
\left(A_{1}(s), A_{2}(s), B(s)\right), \quad s \in \mathbb{R} \tag{40}
\end{equation*}
$$

be a differentiable family of planar BLCS. If it is pointwise controllable, then there is a differentiable family of feedbacks $F(s), s \in \mathbb{R}$, such that

$$
\begin{equation*}
A_{1}(s)+B(s) F(s), \quad A_{2}(s)+B(s) F(s) \tag{41}
\end{equation*}
$$

are stable for any $s \in \mathbb{R}$.
More explicitly, if

$$
\begin{align*}
& A_{1}(s)=\left(\begin{array}{ll}
a_{1} & a_{3} \\
a_{2} & a_{4}
\end{array}\right), \quad A_{2}(s)=\left(\begin{array}{ll}
\gamma_{1} & a_{3} \\
\gamma_{2} & a_{4}
\end{array}\right), \\
& B(s)=\binom{b_{1}}{b_{2}}, \\
& C_{1}=\left(B(s) A_{1}(s) B(s)\right), \quad C_{2}=\left(B(s) A_{2}(s) B(s)\right), \tag{42}
\end{align*}
$$

where all the coefficients are assumed to be differentiably depending on $s \in \mathbb{R}$, one can take

$$
\begin{align*}
F & =\left(\begin{array}{ll}
f_{1} & f_{2}
\end{array}\right) \\
& =\left(\begin{array}{ll}
x & y
\end{array}\right) \frac{1}{\operatorname{det} C_{1}}\binom{\operatorname{det}\left(\begin{array}{ll}
b_{1} & a_{1} \\
b_{2} & a_{2}
\end{array}\right) \operatorname{det}\left(\begin{array}{ll}
b_{1} & a_{3} \\
b_{2} & a_{4}
\end{array}\right)}{-b_{2}} \tag{43}
\end{align*}
$$

with

$$
\begin{gather*}
x<-\operatorname{tr} A_{1},-\operatorname{tr} A_{2}, \\
y<\operatorname{det} A_{1}, \\
y<-\operatorname{det}\left(\begin{array}{ll}
b_{1} & \gamma_{1}-a_{1} \\
b_{2} & \gamma_{2}-a_{2}
\end{array}\right) \operatorname{det}\left(\begin{array}{ll}
b_{1} & a_{3} \\
b_{2} & a_{4}
\end{array}\right) \frac{1}{\operatorname{det} C_{2}} x  \tag{44}\\
+\operatorname{det} A_{2} \frac{\operatorname{det} C_{1}}{\operatorname{det} C_{2}} .
\end{gather*}
$$

Proof. By hypothesis, we assume

$$
\begin{equation*}
\operatorname{det} C_{1} \operatorname{det} C_{2}>0 \tag{45}
\end{equation*}
$$

for any $s \in \mathbb{R}$.
We look for $F(s)=\left(\begin{array}{ll}f_{1} & f_{2}\end{array}\right)$ where again we assume the coefficients depending on $s \in \mathbb{R}$, such that the eigenvalues of

$$
\left(\begin{array}{ll}
a_{1}+b_{1} f_{1} & a_{3}+b_{1} f_{2}  \tag{46}\\
a_{2}+b_{2} f_{1} & a_{4}+b_{2} f_{2}
\end{array}\right), \quad\left(\begin{array}{ll}
\gamma_{1}+b_{1} f_{1} & a_{3}+b_{1} f_{2} \\
\gamma_{2}+b_{2} f_{1} & a_{4}+b_{2} f_{2}
\end{array}\right)
$$

have negative real part for any $s \in \mathbb{R}$ or, equivalently, the matrices have negative trace and positive determinant; that is to say,

$$
\begin{gather*}
b_{1} f_{1}+b_{2} f_{2}<-a_{1}-a_{4}, \\
b_{1} f_{1}+b_{2} f_{2}<-\gamma_{1}-a_{4}, \\
f_{1}\left(a_{3} b_{2}-a_{4} b_{1}\right)+f_{2}\left(a_{2} b_{1}-a_{1} b_{2}\right)<a_{1} a_{4}-a_{2} a_{3},  \tag{47}\\
f_{1}\left(a_{3} b_{2}-a_{4} b_{1}\right)+f_{2}\left(\gamma_{2} b_{1}-\gamma_{1} b_{2}\right)<\gamma_{1} a_{4}-\gamma_{2} a_{3} .
\end{gather*}
$$

We change the variables $\left(f_{1}, f_{2}\right)$ by $(x, y)$ defined by

$$
\begin{gather*}
x=b_{1} f_{1}+b_{2} f_{2}  \tag{48}\\
y=\left(b_{2} a_{3}-b_{1} a_{4}\right) f_{1}+\left(b_{1} a_{2}-b_{2} a_{1}\right) f_{2}
\end{gather*}
$$

which is a change of variables, because (by hypothesis)

$$
\operatorname{det}\left(\begin{array}{ll}
b_{1} & b_{2} a_{3}-b_{1} a_{4}  \tag{49}\\
b_{2} & b_{1} a_{2}-b_{2} a_{1}
\end{array}\right)=\operatorname{det} C_{1} \neq 0
$$

Then

$$
\left(\begin{array}{ll}
f_{1} & f_{2}
\end{array}\right)=\left(\begin{array}{ll}
x & y
\end{array}\right) \frac{1}{\operatorname{det} C_{1}}\left(\begin{array}{cc}
\operatorname{det}\left(\begin{array}{ll}
b_{1} & a_{1} \\
b_{2} & a_{2}
\end{array}\right) & \operatorname{det}\left(\begin{array}{ll}
b_{1} & a_{3} \\
b_{2} & a_{4}
\end{array}\right)  \tag{50}\\
-b_{2} & -b_{1}
\end{array}\right)
$$

With this change of variables, the desired inequalities become

$$
\begin{gathered}
x<-a_{1}-a_{4}=-\operatorname{tr} A_{1}, \\
x<-\gamma_{1}-a_{4}=-\operatorname{tr} A_{2}, \\
y<a_{1} a_{4}-a_{2} a_{3}=\operatorname{det} A_{1}, \\
\left(a_{3} b_{2}-a_{4} b_{1}\right) \frac{\left(b_{1} a_{2}-b_{2} a_{1}\right) x-b_{2} y}{\operatorname{det} C_{1}} \\
-\left(\gamma_{2} b_{1}-\gamma_{1} b_{2}\right) \frac{\left(b_{2} a_{3}-b_{1} a_{4}\right) x-b_{1} y}{\operatorname{det} C_{1}} \\
<\gamma_{1} a_{4}-\gamma_{2} a_{3} .
\end{gathered}
$$

It is straightforward that the last inequality can be rewritten:

$$
\operatorname{det}\left(\begin{array}{ll}
b_{1} & \gamma_{1}-a_{1}  \tag{52}\\
b_{2} & \gamma_{2}-a_{2}
\end{array}\right) \operatorname{det}\left(\begin{array}{ll}
b_{1} & a_{3} \\
b_{2} & a_{4}
\end{array}\right) \frac{1}{\operatorname{det} C_{1}} x+y \frac{\operatorname{det} C_{2}}{\operatorname{det} C_{1}}<\operatorname{det} A_{2} .
$$

Example 16. For the family in Example 14, when $b_{1} \neq 0, b_{2}=0$, $a_{2} \gamma_{2}>0$, differentiable families of feedbacks are given by

$$
\begin{gather*}
\left(\begin{array}{ll}
f_{1} & f_{2}
\end{array}\right)=\left(\begin{array}{ll}
x & y
\end{array}\right) \frac{1}{b_{1} a_{2}}\left(\begin{array}{cc}
a_{2} & -a_{4} \\
0 & -1
\end{array}\right), \\
x<-a_{1}-a_{4},-\gamma_{1}-a_{4},  \tag{53}\\
y<a_{1} a_{4}-a_{2} s, \\
y<\left(\frac{a_{2}}{\gamma_{2}}-1\right) a_{4} x+\frac{a_{2}}{\gamma_{2}} \gamma_{1} a_{4}-a_{2} s .
\end{gather*}
$$

For example,

$$
\begin{gather*}
x=\min \left\{-a_{1}-a_{4},-\gamma_{1}-a_{4}\right\}-1 \equiv \alpha, \\
y=\beta-\frac{a_{2}^{2}}{4} s^{2},  \tag{54}\\
\beta=\min \left\{a_{1} a_{4},\left(\frac{a_{2}}{\gamma_{2}}-1\right) a_{4} \alpha+\frac{a_{2}}{\gamma_{2}} \gamma_{1} a_{4}\right\}-1 .
\end{gather*}
$$

That is,

$$
\begin{equation*}
F(s)=\left(\frac{\alpha}{b_{1}}-\frac{a_{4} \alpha}{b_{1} a_{2}}-\frac{\beta}{b_{1} a_{2}}+\frac{a_{2}}{4 b_{1}} s^{2}\right) . \tag{55}
\end{equation*}
$$

## 5. Conclusion

In this work we consider planar bimodal linear control systems (BLCS) consisting of two subsystems acting on each side of a given hyperplane, assuming continuity along it. The set of BLCS is partitioned into equivalence classes by reducing each triple of matrices by means of a suitable change of basis. For a differentiable family of such systems (for example, perturbations of a given one) we study its stratification diagram, that is to say, the different equivalence types appearing for different values of the parameters. On the other hand, in spite of these different classes (even nonobservable ones), if pointwise controllability holds, we construct a differentiable family of feedbacks which stabilizes both subsystems for each value of the parameters.

Some extensions of this work could be the application of the same techniques to tackle the case of piecewise linear control systems composed of a different partition of the state space, for example, the ones composed of three regions, with the same subsystem acting on the outer ones (see, for example, [9]). Another possible work would be the extension to bimodal linear systems with three-state variables, starting from our results in [17].

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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# Recursive Subspace Identification of AUV Dynamic Model under General Noise Assumption 

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#### Abstract

A recursive subspace identification algorithm for autonomous underwater vehicles (AUVs) is proposed in this paper. Due to the advantages at handling nonlinearities and couplings, the AUV model investigated here is for the first time constructed as a Hammerstein model with nonlinear feedback in the linear part. To better take the environment and sensor noises into consideration, the identification problem is concerned as an errors-in-variables (EIV) one which means that the identification procedure is under general noise assumption. In order to make the algorithm recursively, propagator method (PM) based subspace approach is extended into EIV framework to form the recursive identification method called PM-EIV algorithm. With several identification experiments carried out by the AUV simulation platform, the proposed algorithm demonstrates its effectiveness and feasibility.


## 1. Introduction

In recent years, autonomous underwater vehicles (AUVs) have attracted increasing attentions due to their remarkable features such as high agility, excellent convenience and low cost in applications of underwater explorations and developments. However, contradictions lay between more and more complicated missions for AUV and the control and navigation systems that are not accurate enough. System identification methods have provided an alternative way other than traditional expensive instruments dependent approaches to improve the abilities of autonomous systems in various aspects [1-5]. As a result, a variety of researches have been put forward to identify the ordinary differential model of AUVs for model based control and navigation. Rentschler and coworkers [1] have demonstrated an iterative procedure to revise the model and controller of Odyssey III AUV to obtain better flight performances. Nonlinear observers based identification algorithm with sliding mode observer and EKF is also proposed for designation of nonlinear controller in [2]. For a more robust navigation system in case of sensor fault, Hegrenaes and Hallingstad [3] have used least squares algorithm to estimate both sea current disturbances and model
parameters of the HUGIN 4500 AUV. However, due to the complexity of AUV system, many nonlinearities and coupled terms exist in ordinary differential equations that make the identification of whole pack of hydrodynamic coefficients quite complicated and time consuming. As a consequence, AUV differential model usually need to be simplified through eliminations of nonlinear and coupled terms before identification process. For example, in the research of Tiano et al. [6], a set of decoupled AUV models concerning different degree of freedom were set up and the yaw dynamics of the Hammerhead AUV was identified according to observer Kalman filter identification (OKID) method. However, nonlinearities and couplings are two significant features being researched in the area of AUV, so as it is said in [6], construction of MIMO coupled AUV model is a rather important and challenging modeling issue.

In this paper, compared with the traditional differential equation used in AUV modeling, a Hammerstein model which consists of a static nonlinear part and a dynamic linear one is adopted in order to deal with nonlinear and linear property of AUV system separately. Due to the particular characteristics of AUV system, the Hammerstein model has to be modified with a static nonlinear feedback part added
on the linear part. As illustrated in Figure 1, one remarkable benefit of Hammerstein system is that it possesses advantages of linear MIMO system and can approximate nonlinear and coupled terms of AUV to a large extent at the same time. To the best of the authors' acknowledge, this is the first time that Hammerstein model is applied in the area of AUV modeling. Because of the conveniences brought by Hammerstein system, extensive attentions has been paid to obtain system parameters from input/output data. Cai and Bai [7] proposed a method to make the identification of parametric Hammerstein system a linear problem through regarding the average squared error cost function as the inner product between the true but unknown parameter vector and its estimations. But this method was discussed under the assumption that Hammerstein system only consists of single input and single output. As to MIMO Hammerstein system identification, a nonparametric algorithm based on stochastic approximation approach is proposed in [8]. However, nonlinear MIMO Hammerstein identification problem concerned in this paper is still challenging.

One group of widely studied MIMO system identification algorithms for Hammerstein system is subspace identification methods [9], which mainly include three different branches: numerical algorithms for state-space subspace system identification (N4SID), MIMO output-error state-space model identification (MOESP) and canonical variate analysis (CVA). Compared with other identification algorithms, subspace identification methods are more attractive due to several advantages [10]. For example, subspace identification methods can circumvent the complicated parameterization procedure for MIMO system of prediction error methods [11]. What is more, subspace identification methods do not require nonlinear searches in the parameter space based on computational tools such as QR factorization and singular value decomposition (SVD) [12]. Since the Hammerstein AUV model is a parametric one with multiinputs and multioutputs, MIMO MOESP algorithm is adopted as the theoretical basis for further investigation in this paper.

In addition, instead of regarding the identification procedure in ideal situations, practical engineering circumstances need to be taken into consideration. A widely studied one is that the general noise assumption has to be made which means "the measured input is corrupted by a white measurement noise while the measured output is corrupted by the sum of a white measurement noise and a term due to a white process noise" [13]. Another practical situation which need to be considered is that different oceanic environment may lead to different hydrodynamic coefficients in which case off-line identification results of Hammerstein AUV model will bring errors to control and navigation systems [14]. Besides, it is often the case that the structure of AUV usually has to be modified mildly in order to fulfill various tasks in practice. Therefore, recursive identification methods which can adjust model parameters online become rather significant and attractive in applications of such as adaptive control, model-aided navigation and so on.

As a result, in order to fulfill the situations concerned above, the identification of Hammerstein AUV model is regarded as an errors-in-variables (EIV) problem in this


Figure 1: Hammerstein AUV Model.
paper and we are going to make the MOESP method recursively under EIV framework so, as to carry out recursive identification algorithm for the Hammerstein AUV model. One major obstacle in recursive subspace identification is the increasing computational complexity of SVD [15, 16]. In previous studies, projection approximation subspace tracking (PAST) algorithm designed by Yang [17] was widely used to update the SVD. However, approximation in the algorithm will bring slight difference between identified model and the original one. In addition, IV-PAST algorithm [18] and gradient type subspace tracking method [19] are also developed to estimate the signal subspace. In [20], an instrumental variable propagator method (IVPM) based recursive identification algorithm was proposed. Nevertheless, the paper only studied the method within past input (PI)/past output (PO) MOESP framework. So in this paper, combining with Hammerstein AUV identification problem described above, IVPM method is extended into EIV framework and the PM-EIV algorithm for recursive subspace identification of errors-in-variables problem is derived. Compared with previous algorithms mentioned, the PM-EIV algorithm is more suitable to handle the MIMO AUV Hammerstein model identification problems under errors-in-variables framework recursively. As a matter of fact, since the Hammerstein model constructed can be viewed as a generalized one for mechanical systems, the proposed PM-EIV can be extended to identification of other systems as well.

The remainder of this paper is organized as follows. In Section 2, a Hammerstein AUV model with nonlinear feedback is formed with proper transformation from an ordinary differential one. Then the linearization process of nonlinear part is presented. In Section 3, the MOESP identification method is described with no consideration about the system noise. After that, a recursive subspace method called PM-EIV is derived under the general noise assumption. Identification and validation of the Hammerstein AUV model are presented in Section 4. At last, conclusions are made in Section 5.

Some notations used in this paper are followed. The superscript $(\cdot)^{T}$ denotes the transposition operator. $E(\cdot)$ is the expectation operator. $\mathbf{R}^{m \times n}$ represents the set of $m \times n$ real matrices. $M^{\perp}$ is the orthogonal complement of $M$.

## 2. AUV Modeling

In this section, a Hammerstein AUV model is formed after introducing the ordinary differential one. Then in order to make the Hammerstein AUV model suitable for subspace
identification method, linearization of the nonlinear part is brought out.
2.1. Hammerstein AUV Model. According to [21], a coupled nonlinear ordinary differential model for AUV based on Newtonian mechanics can be described as the following equation:

$$
\begin{equation*}
M \dot{\nu}+C(\nu) v+D(\nu) \nu=\tau \tag{1}
\end{equation*}
$$

where $v \in \mathbf{R}^{6}$ represents the state vector of AUV, $M \in \mathbf{R}^{6 \times 6}$ consists of inertia matrix and add mass matrix, $D(\nu) \in \mathbf{R}^{6 \times 6}$ is the linear and quadratic damping matrix, $C(\nu) \in \mathbf{R}^{6 \times 6}$ is the coriolis and centripetal matrix, $\tau \in \mathbf{R}^{6}$ represents the forces and moments acted on the vehicle.

The coupled and nonlinear terms in matrix $D(\nu)$ and $C(\nu)$ make the identification process quite complicated and time consuming. To obtain Hammerstein model of AUV, simplifications and transformations have to be made. First, some coupled terms with little influences are eliminated. Second, the nonlinear terms are separated from the system and the system is divided into static nonlinear input part and nonlinear feedback part. Then, remaining nonlinear and coupled terms constitute the nonlinear feedback part. At last, the remaining part of the model can be described as a linear MIMO state-space model. After those steps, a Hammerstein AUV model with nonlinear feedback part can be formed as in Figure 1. $d(k), v(k)$, and $\omega(k)$ are process noise, input measurement noise, and output measurement noise respectively.

According to the system structure above, define the nonlinear input function as $f(\cdot)$, nonlinear feedback as $g(\cdot)$, so discrete time MIMO state-space equations can be constructed as below.

$$
\begin{gather*}
x(k+1)=A x(k)+B_{1} f(u(k))+B_{2} g(y(k))+d(k), \\
y(k)=C x(k)+D_{1} f(u(k))+D_{2} g(y(k)),  \tag{2}\\
\tilde{y}(k)=y(k)+\omega(k), \\
\widetilde{u}(k)=u(k)+v(k),
\end{gather*}
$$

where $x(k) \in R^{6}$ represent the system states, $u(k) \in R^{3}$ are the system inputs including thruster revolution, rudder deflection, and elevator deflection, $y(k) \in R^{6}$ is output vector, consists of $u, v, w, p, q, r$ that represents the surge velocity, sway velocity, heave velocity, roll rate, pitch rate, yaw rate respectively. $A, B_{1}, B_{2}, C, D_{1}$, and $D_{2}$ are the corresponding linear subsystem matrices with appropriate dimensions. $\widetilde{u}(k), \widetilde{y}(k)$ are corrupted system inputs and outputs.

Many identification methods have focused on handling output measurement noise and process noise. In fact, input measurement noise is inevitable in any engineering processes including practical identification experiments. So in this paper, $d(k), v(k)$, and $\omega(k)$ are considered under the general noise assumption.
2.2. Linearization of $A U V$ Model. According to the Section 2.1, AUV model can be described as Hammerstein one with nonlinear feedback. In order to identify the
parameters of corresponding matrices, nonlinearity in the equations needs to be linearly parameterized so that recursive subspace identification methods can be applied. One traditional approach to approximate nonlinearity of the system is linear combination of basic functions. In Lovera [12], Tchebiceff polynomials are chosen to linearize nonlinearities. However, as the system studied in this paper contains nonlinear feedback part, Tchebiceff polynomials approximation of $g(\cdot)$ will influence the identifiability of system matrix. So in this paper, truncated Fourier series described in Luo and Leonessa [22], also known as trigonometric polynomials are adopted to linearize the nonlinear functions $f(\cdot)$ and $g(\cdot)$. Define the basic function as follows:

$$
\begin{gather*}
\varphi_{k}(x)=1, \quad k=0 \\
\varphi_{k}(x)=\left[\cos \left(\frac{k \pi\left(x-x_{m}\right)}{x_{d}}\right) \sin \left(\frac{k \pi\left(x-x_{m}\right)}{x_{d}}\right)\right]^{T}  \tag{3}\\
k \geq 1
\end{gather*}
$$

Then nonlinear function can be approximated by following equation:

$$
\begin{equation*}
F(x)=w_{o}+\sum_{i=1}^{N} w_{i} \varphi_{i}(x) \tag{4}
\end{equation*}
$$

where $x_{m}=\left(x_{\text {max }}+x_{\text {min }}\right) / 2, x_{d}=\left(x_{\text {max }}-x_{\text {min }}\right) / 2$, and $w_{i}=\left[\begin{array}{ll}w_{i \cos } & w_{i \sin }\end{array}\right]$. Since the Hammerstein AUV model is MIMO, define $\Phi(x)=\left[\begin{array}{llll}\varphi_{0}^{T}(x) & \varphi_{1}^{T}(x) & \cdots & \varphi_{N}^{T}(x)\end{array}\right]^{T}, W=$ $\left[\begin{array}{llll}w_{0} & w_{1} & \cdots & w_{N}\end{array}\right]$; then $f(\cdot), g(\cdot)$ can be expressed as:

$$
\begin{align*}
f(u)= & {\left[\begin{array}{llll}
W_{u 1}^{T} & W_{u 2}^{T} & \cdots & W_{u r}^{T}
\end{array}\right]^{T} } \\
& \cdot\left[\begin{array}{llll}
\Phi^{T}\left(u_{1}\right) & \Phi^{T}\left(u_{2}\right) & \cdots & \Phi^{T}\left(u_{m}\right)
\end{array}\right]^{T} \\
g(y)= & {\left[\begin{array}{llll}
W_{y 1}^{T} & W_{y 2}^{T} & \cdots & W_{y s}^{T}
\end{array}\right]^{T} }  \tag{5}\\
& \cdot\left[\begin{array}{llll}
\Phi^{T}\left(y_{1}\right) & \Phi^{T}\left(y_{2}\right) & \cdots & \Phi^{T}\left(y_{l}\right)
\end{array}\right]^{T}
\end{align*}
$$

where $m=3, l=6$, and corresponding coefficient vector $W_{u i} \in R^{1 \times 3(2 N+1)}$ and $W_{y i} \in R^{1 \times 6(2 N+1)}$. A simplified expression for nonlinear part can be presented as follows:

$$
\begin{align*}
& f(u)=K \xi(u) \\
& g(y)=P \zeta(y) \tag{6}
\end{align*}
$$

with
definitions
that
$\xi(u) \quad \Delta \quad\left[\begin{array}{llll}\Phi^{T}\left(u_{1}\right) & \Phi^{T}\left(u_{2}\right) & \cdots & \Phi^{T}\left(u_{m}\right)\end{array}\right]^{T}, \quad K \quad \stackrel{\Delta}{=}$ $\left[\begin{array}{llll}W_{u 1}^{T} & W_{u 2}^{T} & \cdots & W_{u r}^{T}\end{array}\right]^{T}$. The equation for $g(y)$ can also be formed in a similar principle. Then a new state-space model with no nonlinearity can be described as follow:

$$
\begin{gather*}
x(k+1)=A x(k)+M_{1} \xi(u(k))+M_{2} \zeta(y(k))+d(k), \\
y(k)=C x(k)+N_{1} \xi(u(k))+N_{2} \zeta(y(k)), \tag{7}
\end{gather*}
$$

where $M_{1}=B_{1} K \in R^{n \times 3(2 N+1)}, M_{2}=B_{2} P \quad \epsilon$ $R^{n \times 6(2 N+1)}, N_{1}=D_{1} K \in R^{l \times 3(2 N+1)}$, and $N_{2}=D_{2} P \in$ $R^{l \times 6(2 N+1)}$ represent new coefficients matrices, $\xi(u(k)) \in$ $R^{3(2 N+1) \times 1}, \zeta(y(k)) \in R^{6(2 N+1) \times 1}$, respectively represent the input vector and the feedback vector. Further, define $M=$ $\left[\begin{array}{ll}M_{1} & M_{2}\end{array}\right], N=\left[\begin{array}{ll}N_{1} & N_{2}\end{array}\right], e(k)=\left[\begin{array}{ll}\xi^{T}(u(k)) & \zeta^{T}(y(k))\end{array}\right]^{T}$ as the system matrix and input vector and the Hammerstein AUV model can be presented in classical form:

$$
\begin{gather*}
x(k+1)=A x(k)+M e(k)+d(k),  \tag{8}\\
y(k)=C x(k)+N e(k) .
\end{gather*}
$$

By now, a Hammerstein AUV model is established and has been linearized based on trigonometric polynomials with a description as in (8) being obtained. In the following section, a recursive subspace method to identify the system under EIV frame will be derived.

## 3. Recursive Subspace Identification Method

In this section, the PM-EIV subspace identification method for the Hammerstein AUV model will be proposed. To better illustrate the method, the MOESP algorithm have to be introduced first to lay the foundation so that the PM-EIV method can be derived. Before that, several assumptions have to be made.

Assumption 1. The difference between the nonlinear functions $f(\cdot), g(\cdot)$ and their linear approximations respectively, can be neglected.

Assumption 2. Persistent exciting condition is satisfied according to the definition from Ljung [23].

Assumption 3. The system noise sequences $d(k), v(k)$, and $\omega(k)$ are white noises independent from each other.
3.1. MOESP Identification Method. The fundamental feature of MOESP method is to estimate the extended observer matrix of the system based on input/output data. Then system matrices can be obtained through applying least-squares algorithms. To give a brief introduction of MOESP method, system noises are assumed to be zero in this section, that is $d(k) \equiv 0, v(k) \equiv 0$ and $\omega(k) \equiv 0$. According to Verhaegen [24], The following equations need to be formed firstly.

$$
\begin{equation*}
Y_{j, i, N}=\Gamma_{i} X_{j, N}+H_{i} E_{j, i, N} \tag{9}
\end{equation*}
$$

with definitions of $Y_{j, i, N}, X_{j, N}$, and $E_{j, i, N}$ as follows:

$$
Y_{j, i, N}=\left[\begin{array}{ccc}
y_{j} & \cdots & y_{j+N-1} \\
\vdots & \ddots & \vdots \\
y_{j+i-1} & \cdots & y_{j+N+i-2}
\end{array}\right]
$$

$$
\begin{align*}
E_{j, i, N} & =\left[\begin{array}{ccc}
e_{j} & \cdots & e_{j+N-1} \\
\vdots & \ddots & \vdots \\
e_{j+i-1} & \cdots & e_{j+N+i-2}
\end{array}\right], \\
X_{j, N} & =\left[\begin{array}{llll}
x_{j} & x_{j+1} & \cdots & x_{j+N-1}
\end{array}\right] . \tag{10}
\end{align*}
$$

Therefore, extended observer matrix $\Gamma_{i}$ and low triangular block Toeplitz matrix $H_{i}$ have the following structures:

$$
\begin{align*}
\Gamma_{i} & =\left[\begin{array}{llll}
C^{T} & (C A)^{T} & \cdots & \left(C A^{i-1}\right)^{T}
\end{array}\right]^{T} \\
H_{i} & =\left[\begin{array}{cccc}
N & 0 & \cdots & 0 \\
C M & N & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
C A^{i-2} M & C A^{i-3} M & \cdots & N
\end{array}\right] \tag{11}
\end{align*}
$$

Input/output data matrix is formed and factorized by RQ factorization. The following equation can be acquired:

$$
\left[\begin{array}{c}
E_{j, i, N}  \tag{12}\\
Y_{j, i, N}
\end{array}\right]=\left[\begin{array}{cc}
R_{11} & 0 \\
R_{21} & R_{22}
\end{array}\right]\left[\begin{array}{l}
Q_{1} \\
Q_{2}
\end{array}\right] .
$$

Combining (12) with (9), we can obtain that:

$$
\Gamma_{i} X_{j, N}=\left[\begin{array}{ll}
R_{21}-H_{i} R_{11} & R_{22}
\end{array}\right]\left[\begin{array}{l}
Q_{1}  \tag{13}\\
Q_{2}
\end{array}\right] .
$$

It can be derived that the column space of $\Gamma_{i}$ and the column space of $R_{22}$ are equal. After carrying out SVD of $R_{22}$ as in (14), columns of $U_{n}$ can be regarded as a basis for $\Gamma_{i}$ :

$$
R_{22}=\left[\begin{array}{ll}
U_{n} & U_{n}^{\perp}
\end{array}\right]\left[\begin{array}{cc}
S_{1} & 0  \tag{14}\\
o & S_{2}
\end{array}\right]\left[\begin{array}{c}
V_{n}^{T} \\
\left(V_{n}^{\perp}\right)^{T}
\end{array}\right] .
$$

Based on the definition of extended observer matrix $\Gamma_{i}$, transformed system matrices $A_{T}, C_{T}$ can be easily calculated with $U_{n}$ :

$$
\begin{gather*}
U_{n}^{(1)} A_{T}=U_{n}^{(2)}, \\
C_{T}=U_{n}(1: l,:) . \tag{15}
\end{gather*}
$$

And $M_{T}, N$ can be acquired by least-squares solutions for.

$$
\begin{equation*}
\left(U_{n}^{\perp}\right)^{T} H_{i}-\left(U_{n}^{\perp}\right)^{T} R_{21} R_{11}^{-1}=0 . \tag{16}
\end{equation*}
$$

3.2. Errors-in-Variables Problem. Because the identification problem of the Hammerstein AUV model is an EIV one, a more practical subspace identification method handling EIV problem is introduced in this section which takes disturbances $d(k), v(k), \omega(k)$ into consideration based on the basic MOESP algorithm above. In this case, (9) needs to be modified as:

$$
\begin{equation*}
\widetilde{Y}_{j, i, N}=\Gamma_{i} X_{j, N}+H_{i} \widetilde{E}_{j, i, N}-H_{i} V_{j, i, N}^{\prime}+G_{i} P_{j, i, N}+W_{j, i, N}, \tag{17}
\end{equation*}
$$

where $P_{j, i, N}, W_{j, i, N}$ are block Hankel matrices of noises $d(k)$ and $\omega(k) . G_{i}$ is defined as:

$$
G_{i}=\left[\begin{array}{cccc}
0 & 0 & \cdots & 0  \tag{18}\\
C & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
C A^{i-2} & C A^{i-3} & \cdots & 0
\end{array}\right]
$$

Notice that $V_{j, i, N}^{\prime}$ is not only made of $v(k)$, output measurement noise $\omega(k)$ is also involved due to the linearization of nonlinear feedback part. This brings the closed-loop problem into the identification process. One significant obstacle resulting from closed-loop situation is the interference between input signals and output measurement noise which violates the following equation.

$$
\begin{equation*}
E\left[u_{k} w_{j}^{T}\right]=0 \quad \text { for }(j<k) \tag{19}
\end{equation*}
$$

It means that input signals are no longer unrelated to the past noises. However, problems studied in this paper only need the condition that future noise is unrelated to past input. That is:

$$
\begin{equation*}
E\left[u_{k} w_{j}^{T}\right]=0 \quad \text { for }(j \geq k) \tag{20}
\end{equation*}
$$

So, in order to eliminate the influence from the noises, instrumental variables can still be formed as past input and output signals. Then following relation can be obtained:

$$
\begin{align*}
& \widetilde{Y}_{i+1, i, N}\left[\begin{array}{ll}
\widetilde{E}_{1, i, N}^{T} & \widetilde{Y}_{1, i, N}^{T}
\end{array}\right] \\
& =\quad  \tag{21}\\
& \Gamma_{i} X_{i+1, N}\left[\begin{array}{ll}
\widetilde{E}_{1, i, N}^{T} & \widetilde{Y}_{1, i, N}^{T}
\end{array}\right] \\
&  \tag{22}\\
& \quad+H_{i} \widetilde{E}_{i+1, i, N}\left[\begin{array}{ll}
\widetilde{E}_{1, i, N}^{T} & \widetilde{Y}_{1, i, N}^{T}
\end{array}\right], \\
& {\left[\begin{array}{cc}
\widetilde{E}_{i+1, i, N} \widetilde{E}_{1, i, N}^{T} & \widetilde{E}_{i+1, i, N} \widetilde{Y}_{1, i, N}^{T} \\
\widetilde{Y}_{i+1, i, N} \widetilde{E}_{1, i, N}^{T} & \widetilde{Y}_{i+1, i, N} \widetilde{Y}_{1, i, N}^{T}
\end{array}\right]} \\
& = \\
& =\left[\begin{array}{ll}
R_{11}(t) & 0 \\
R_{21}(t) & R_{22}(t)
\end{array}\right]\left[\begin{array}{l}
Q_{1}(t) \\
Q_{2}(t)
\end{array}\right] .
\end{align*}
$$

According to Theorem 3 of Chou and Verhaegen [13], the column space of $\Gamma_{i}$ can be consistently estimated from $R_{22} . A_{T}, M_{T}, C_{T}$, and $N$ can be acquired based on OE_PIV algorithm. To save the space, more details can be found in [13].
3.3. PM-EIV Subspace Identification. In the above sections, subspace identification method for AUV Hammerstein system has been developed under general noise assumption. As mentioned in Section 1, recursive method for AUV identification can be much more suitable due to the properties of oceanic environment and tasks. So in this section, identification method will be modified recursively. One of the key problems handled in recursive identification is the recursive update of SVD process in order to reduce the computational burden which comes with increasing input/output data. In [20], propagator method used in array signal processing area is introduced for recursive subspace identification under

PI/PO MOESP schemes. Considering about the EIV problem in the identification of AUV model, propagator method will be extended into EIV framework in this paper for the first time and the resulting algorithm will be called PM-EIV subspace identification method. An important step in PM subspace method is the calculation of observer vector, defined as.

$$
\begin{equation*}
z_{i}(t+1)=\widetilde{y}_{i}(t+1)-H_{i}(t+1) \widetilde{e}_{i}(t+1), \tag{23}
\end{equation*}
$$

where $t=j+N-1 ; \tilde{y}_{i}(t+1), \widetilde{e}_{i}(t+1)$ are new updated output/input data vectors respectively.
3.3.1. Update of Observer Vector $z_{i}$ in EIV Framework. In this section, RQ factorization method is adopted for updating of observe vector. Based on (22) in Section 3.2, update of the data matrix can be expressed as follows:

$$
\begin{align*}
& {\left[\begin{array}{cc}
\widetilde{E}_{i+1, i, N+1} \widetilde{E}_{1, i, N+1}^{T} & \widetilde{E}_{i+1, i, N+1} \widetilde{Y}_{1, i, N+1}^{T} \\
\widetilde{Y}_{i+1, i, N+1} \widetilde{E}_{1, i, N+1}^{T} & \widetilde{Y}_{i+1, i, N+1} \widetilde{Y}_{1, i, N+1}^{T}
\end{array}\right]}  \tag{24}\\
& \quad=\left[\begin{array}{lc}
R_{11}(t+1) & 0 \\
R_{21}(t+1) & R_{22}(t+1)
\end{array}\right]\left[\begin{array}{l}
Q_{1}(t+1) \\
Q_{2}(t+1)
\end{array}\right]
\end{align*}
$$

where $\widetilde{E}_{i+1, i, N+1}=\left[\widetilde{E}_{i+1, i, N} \quad \widetilde{e}_{i}(t+1)\right], \widetilde{E}_{1, i, N+1}$, and $\widetilde{Y}_{i+1, i, N+1}$, $\widetilde{Y}_{1, i, N+1}$ are formed in similar manner. A transformation of (24) is as follows:

$$
\begin{align*}
& {\left[\begin{array}{cc}
\widetilde{E}_{i+1, i, N} & \widetilde{e}_{i}(t+1) \\
\widetilde{Y}_{i+1, i, N} & \widetilde{y}_{i}(t+1)
\end{array}\right]\left[\begin{array}{cc}
\widetilde{E}_{1, i, N}^{T} & \widetilde{Y}_{1, i, N}^{T} \\
\widetilde{e}_{i}(N+1)^{T} & \widetilde{y}_{i}(N+1)^{T}
\end{array}\right]} \\
& \quad=\left[\begin{array}{ccc}
R_{11}(t) & 0 & \widetilde{e}_{i}(t+1) \phi(N+1) \\
R_{21}(t) & R_{22}(t) & \tilde{y}_{i}(t+1) \phi(N+1)
\end{array}\right]\left[\begin{array}{c}
Q_{1}(t) \\
Q_{2}(t) \\
I
\end{array}\right] \tag{25}
\end{align*}
$$

where $\phi(N+1)=\left[\widetilde{e}_{i}(N+1)^{T} \widetilde{y}_{i}(N+1)^{T}\right]$ denotes the update instrumental variables.

Then, given rotation can be implemented to eliminate the $\widetilde{e}_{i}(t+1) \phi(N+1)$ in the above equation in order to make a lower triangle form.

$$
\begin{align*}
& {\left[\begin{array}{ccc}
R_{11}(t) & 0 & \widetilde{e}_{i}(t+1) \phi(N+1) \\
R_{21}(t) & R_{22}(t) & \widetilde{y}_{i}(t+1) \phi(N+1)
\end{array}\right] \operatorname{Giv}(t+1)} \\
& \quad=\left[\begin{array}{ccc}
R_{11}(t+1) & 0 & 0 \\
R_{21}(t+1) & R_{22}(t) & \widehat{z}_{i}(t+1)
\end{array}\right] \tag{26}
\end{align*}
$$

So, the following relation can be obtained:
$R_{22}(t+1) R_{22}(t+1)^{T}=R_{22}(t) R_{22}(t)^{T}+\widehat{z}_{i}(t+1) \widehat{z}_{i}^{T}(t+1)$.

In addition, the following equation holds:

$$
\left.\begin{array}{rl}
R_{22} & (t+1) Q_{2}(t+1) \\
& =\left(\widetilde{Y}_{i+1, i, N+1}-\widehat{H}_{i} \widetilde{E}_{i+1, i, N+1}\right)\left[\widetilde{E}_{1, i, N+1}^{T}\right. \\
\quad & \widetilde{Y}_{1, i, N+1}^{T}
\end{array}\right] .\left[\begin{array}{ll} 
 \tag{28}\\
& (t) Q_{2}(t)+z_{i}(t+1)\left[\widetilde{e}_{i}(N+1)^{T}\right. \\
\widetilde{y}_{i}(N+1)^{T}
\end{array}\right] .
$$

Combining (27) and (28), relation between $z_{i}(t+1)$ and $\widehat{z}_{i}(t+1)$ can be found:

$$
\begin{align*}
& \widehat{z}_{i}(t+1) \widehat{z}_{i}(t+1)^{T} \\
&= R_{22}(t) Q_{2}(t) \phi(N+1)^{T} z_{i}(t+1)^{T} \\
&+z_{i}(t+1) \phi(N+1)\left(R_{22}(t) Q_{2}(t)\right)^{T}  \tag{29}\\
&+z_{i}(t+1) \phi(N+1) \phi(N+1)^{T} z_{i}(t+1)^{T} .
\end{align*}
$$

Assuming there is a matrix $K(t+1)$ satisfies the following equation:

$$
\begin{equation*}
K(t+1) K(t+1)^{T}=\widehat{z}_{i}(t+1) \widehat{z}_{i}(t+1)^{T}+R_{22}(t) R_{22}(t)^{T} . \tag{30}
\end{equation*}
$$

Then observer vector $z_{i}$ can be updated according to (31):

$$
\begin{gather*}
z_{i}(t+1)\left[e_{i}^{T}(N+1) y_{i}^{T}(N+1)\right] \\
=\left(K(t+1)-R_{22}(t) Q_{2}(t)\right)  \tag{31}\\
z_{i}(t+1)=K_{\phi}^{-1}\left(K(t+1)-R_{22}(t) Q_{2}(t)\right) \phi_{N}^{T},
\end{gather*}
$$

where $K_{\phi}=\phi_{N} \phi_{N}^{T}$ is the coefficient related to instrumental variables, and notice that $K(t+1)$ is not necessary to be square.
3.3.2. Estimation of Observer Matrix $\Gamma_{i}$. Since the observer vector $z_{i}$ can be obtained in EIV scheme. Then, extended observer matrix $\Gamma_{i}$ can be found through propagator method. Observer matrix $\Gamma_{i}$ can be expressed in the following form.

$$
\Gamma_{i}=\left[\begin{array}{l}
\Gamma_{i 1}  \tag{32}\\
\Gamma_{i 2}
\end{array}\right]=\left[\begin{array}{c}
I \\
P_{m}^{T}
\end{array}\right] \Gamma_{i 1}=Q_{s} \Gamma_{i 1} .
$$

Since the matrix $\Gamma_{i 1} \in \mathbf{R}^{n \times n}$ has full rank, column space of $\Gamma_{i}$ equals that of $Q_{s}$. Combining with (23), observer vector $z_{i}$ can be divided as:

$$
z_{i}(t+1)=Q_{s} \Gamma_{i 1} x(t+1)+b_{i}(t+1)=\left[\begin{array}{c}
z_{i 1}(t+1)  \tag{33}\\
z_{i 2}(t+1)
\end{array}\right] .
$$

Without consideration of noise term $b_{i}(t+1)$, it can be easily established that $z_{i 2}=P_{m}^{T} z_{i 1}$. Then the $P_{m}^{T}$ can be solved with least-squares methods. However, existence of noise term will lead to a biased estimation of $P_{m}^{T}$. So the IVPM algorithm proposed by Mercere [20] is adopted to estimate $P_{m}^{T}$. A suitable variable $\gamma \in R^{n \times 1}$ needs to be found with no correlation with system noises. According to Section 3.2, past system input date may satisfy the condition.


Figure 2: Information flow of AUV simulation platform.

Finally, recursive estimation of $P_{m}^{T}$ can be acquired in the following RLS form:

$$
\begin{align*}
& K_{p}(t+1) \\
& \quad=\gamma^{T}(t+1) R(t)\left(1+\gamma^{T}(t+1) R(t) z_{i 1}(t+1)\right)^{-1}, \\
& P_{m}^{T}(t+1) \\
& \quad=P_{m}^{T}(t)+\left[z_{i 2}(t+1)-P_{m}^{T}(t) z_{i 1}(t+1)\right] K_{p}(t+1), \\
& \quad R(t+1)=R(t)-R(t) z_{i 1}(t+1) K_{p}(t+1), \tag{34}
\end{align*}
$$

where $R(t)=E\left[z_{i 1}(t) \gamma^{T}(t)\right]^{-1}$.
With the estimation of observer matrix $\Gamma_{i}$, algorithms mentioned in Section 3.2 can be applied here to estimate the system matrices $A_{T}, M_{T}, C_{T}$, and $N$. Therefore, propagator based subspace identification method in errors-in-variables scheme (PM-EIV) is derived. Based on the IVPM method, the algorithm is more suitable for the identification of Hammerstein AUV model proposed in Section 2. In the following, the identification algorithm will be verified through experiments carried out by the AUV simulation platform.

## 4. Simulations and Results

In this section, simulation experiments are carried out to evaluate the performance of the PM-EIV algorithm proposed in this paper through identifying the Hammerstein AUV model based on data from the AUV simulation platform shown as in Figures 2-3. After a brief introduction of the AUV simulation platform, two typical identification cases are investigated. One is the identification of AUV model based on the MOESP method without consideration of noise. The other one is the verification of PM-EIV algorithm under general noise assumption. Finally, to be more practical,


Figure 3: AUV simulation platform.
model identification based on data from a closed-loop path following simulation experiment is performed.
4.1. AUV Simulation Platform. The basic structure of AUV simulation platform is depicted as in Figure 3. The whole system is connected through Ethernet and responsibilities of five main components are introduced here.
(C1) Surface Interface Computer (SIC). Surface interface software is running on SIC which is in charge of deploying missions for AUV and monitoring the states of the system. The software also allows for manual intervention in case of emergency.
(C2) Mission Control Computer (MiC). Mission Management software developed in QNX real-time system is the core of MiC. MiC is mainly responsible for path plan according to missions from SIC and fault diagnosis of the system.
(C3) Motion Control Computer (MoC). Motion control software running on MoC aims at controlling the states such as heading, speed and depth of AUV based on the preplanned paths from MiC. In field experiments, MoC is also in charge of navigation of the system.
(C4) Model Computer (MC). MC is the host for mathematic models of AUV, thruster and control surface. The mathematic AUV model is a validated model with full coefficients obtained from water-tank experiments.
(C5) 3D Simulation Computer (3DC). 3DC provides an approach for visual simulation relied on vir-tual reality technology. Vega Prime is used to construct the virtual oceanic environment and AUV model is developed in Multigen Creator.

A more detailed process flow is described in Figure 2.
4.2. Case 1: Identification without Consideration of Noise. In an ideal situation, process noise and measurement noises can be ignored, so that the ordinary MOESP algorithm can be applied to identify the Hammerstein AUV model described as in Figure 1. This case aims at testifying the reasonability of the


Figure 4: Prediction of surge speed.

Hammerstein model for an AUV system. All system input signals are chosen as sinusoid curves with different amplitudes and periods and the value of $N$ in (4) is set to be 4 . System identification results based on O-MOESP algorithm can be obtained. Figures 4, 5, and 6 shows the prediction errors between the outputs of the identified model and the original outputs. Three main system outputs surge velocity, pitch rate, and yaw rate, that play important roles in control and navigation of underactuated AUV which are considered here. From Figures $4-6$, it can be concluded that even though identification errors exist, Hammerstein model constructed in Section 2 can act as a suitable structure for AUV dynamics.
4.3. Case 2: Identification under EIV Framework. In this case, general noises are added on the model computer in the AUV simulation platform. So the identification problem becomes an EIV one which can be solved recursively by the PM-EIV algorithm proposed in this paper. System inputs of the simulation platform are still sinusoid curves and the value of $N$ is chosen to be 6. Since the PM-EIV algorithm is a recursive one,


Figure 5: Prediction of yaw rate.


Figure 6: Prediction of pitch rate.
a relative small amount of data is used to acquire an initial value of the model at the beginning. Identification results are shown in Figures 7, 8, and 9. Based on the Figures 7-9, it is reliable to conclude that the PM-EIV algorithm is effective and feasible in identifying the Hammerstein AUV model under general noise assumption.

Remark. Through the above two different identification cases under different situations, it has been approved that Hammerstein model proposed in Section 2 is capable of representing the AUV dynamic system and the PM-EIV method is able to identify the Hammerstein model recursively under general noise assumption. It is also interesting to notice that steady state prediction errors in Case 2 are decreased comparison with those in Case 1, it is also interesting to notice that steady state prediction errors in Case 2 are decreased comparing with those in Case 1 due to the increase of $N$ adopted.


Figure 7: Prediction of surge speed.


Figure 8: Prediction of system pitch rate.


Figure 9: Prediction of system yaw rate.


Figure 10: Simulation results: (a) horizontal projection of the trajectory; (b) 3D trajectory of AUV.


Figure 11: Identification result.
4.4. Identification from Closed-Loop Simulation. In the above simulations, AUV model is identified using data from open loop control of surge speed, heading and yaw. However, in practical applications, data for model identification is usually collected from field experiments with close loop control for specific preplanned paths. So in this section, AUV identification process is carried out based on data from a closed-loop simulation experiment. To be more pellucid, the feasibility of PM-EIV algorithm is not illustrated by predicting the surge speed, pitch rate and yaw rate, but shown by predicting the trajectory of AUV in the test. Figures 10 and 11 are the simulation results from the AUV simulation platform. The preplanned path is a circle with an origin at $(500,500) \mathrm{m}$ and radius of 300 m . The depth command is 10 m . Simulation results have shown that AUV can follow the preplanned circle very well and the diving process is stable and fast.

Then identification of the Hammerstein AUV model is based on the inputs/outputs of this experiment. Noises are also considered. Figure 11 has shown the identification result
in a different point of view. It can be seen that prediction error between the identified model of AUV and the actual trajectory of AUV is large at the beginning because the recursive identification procedure needs time to converge. Therefore, a position calibration operation is carried out when the identification results have converged at simulation time 350 s. The Cali-Point in Figure 11 is where the calibration is implemented. The green line indicates the distance between Cali-Point and expected point. After the calibration, the trajectory of the identified Hammerstein AUV model can follow the trajectory of AUV with satisfying accuracy.

## 5. Conclusions

In this paper, a recursive subspace identification algorithm PM-EIV is derived under general noise assumption. subspace identification algorithm based on propagator method is extended into EIV framework is extended into EIV framework. In order to implement the method on identification of AUV model with consideration about nonlinearities and couplings at the same time, a Hammerstein AUV model is constructed for the first time. Three simulation experiments under different conditions are carried out to verify the feasibility of the model and the effectiveness of the proposed algorithm.

In the future study, system noises will not be restricted to white noise and noise models related with oceanic environment and sensors will be introduced to make the algorithm more practical.

## Conflict of Interests

The authors, Zheping Yan, Di Wu, Jiajia Zhou, and Lichao Hao, declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Formation Control of Second-Order Multiagent Systems with Time-Varying Delays 

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#### Abstract

A formation control problem for second-order multiagent systems with time-varying delays is considered. First, a leader-following consensus protocol is proposed for theoretical preparation. With the help of Lyapunov-Krasovskii functional, a sufficient condition under this protocol is derived for stability of the multiagent systems. Then, the protocol is extended to the formation control based on a multiple leaders' architecture. It is shown that the agents will attain the expected formation. Finally, some simulations are provided to demonstrate the effectiveness of our theoretical results.


## 1. Introduction

Recent years have witnessed a rapidly growing interest in coordinated control of multiagent systems due to its broad applications in various disciplines [1-12]. As one of the important topics in this field, formation control has attracted great attention. Generally speaking, the main objective in formation control is to design appropriate protocol and algorithm such that the agents can achieve and preserve a predefined geometrical shape, such as a chain or a wedge. Potential applications of formation control include lots of cooperative tasks such as surveillance, exploration, search and rescue, transporting large objects, and control of arrays of satellites.

In the literature, researchers have proposed numerous approaches for the multiagent systems to achieve the anticipated formation, roughly categorized as behavior-based strategy [13], virtual structure method [14], and leader-following approach [15-21], to name a few. For instance, Xiao et al. [16] developed a formation framework of multiple leaders and applied a class of nonlinear consensus protocols to the formation control. Under the proposed framework, all the agents in the first-order systems could reach the expected formation. An adaptive formation control approach, in the absence of the velocity information of the leader, was proposed in [17]. Besides, the authors in [18] investigated the leader-following
formation control problems for nonlinear systems under fixed and switching topologies. The above works, however, did not take into account the effects of time delays.

Owing to the finite speed of information transmission and processing, time delays are inevitable in multiagent systems. In particular, one type of time delays is communication delays, whose effects on multiagent systems have been addressed by many researchers [3-9]. For formation control with time delays, Luo et al. [19] gave a sufficient condition of formation control of multiagent systems by using Lyapunov stability theory. Also, Rezaee and Abdollahi [20] provided a motion synchronization strategy with time delays. Note that the time delays in both papers were assumed to be constant. In reality, it is more practical when the timevarying delays are accounted for. Lu et al. [21] studied the formation control of second-order multiagent systems with time-varying delays, where the time delays existed only in the transmission of position information between neighbors. For second-order multiagent systems, it is worthwhile to mention that the information exchanged between neighbors may include velocity information as well as position information.

Motivated by the above analysis, we consider a leaderfollowing formation control problem for second-order multiagent systems, with time-varying delays existing in the transmission of both velocity and position. Here, we adopt the
formation framework proposed in [16]. More specifically, the formation information is divided into two parts: the global one and the local one, where the former determines the geometric pattern of the desired formation and the latter decides the relative information of agents with respect to their neighbors. Only a small number of agents called leaders have access to the global information, and the other agents called followers regulate their states according to the local information.

An outline of this paper is as follows. Section 2 provides some preliminary notions of graph theory and formulates the formation control problem. By utilizing Lyapunov-Krasovskii functional, Section 3 presents the main results under the proposed control protocol. Then several simulations are illustrated in Section 4. Finally, the conclusion is drawn in Section 5.

The following notations will be used throughout this paper. Given a matrix, the superscripts " $T$ " and " -1 " stand for its transposition and inverse, respectively; $\Lambda(\cdot)$ and $\|\cdot\|$ denote the set of all eigenvalues and the spectral norm of the matrix, respectively. Let $I_{n}$ be an $n \times n$ identity matrix, $\mathbf{1}_{n}=[1 \cdots 1]^{T} \in \mathbb{R}^{n}$, and $0_{n \times n}$ represents an $n \times n$ zero matrix. And $\operatorname{diag}\left\{b_{1}, \ldots, b_{n}\right\}$ denotes a diagonal matrix with diagonal elements being $b_{1}, \ldots, b_{n}$. For a complex number $\mu \in \mathbb{C}$, $\operatorname{Re}(\mu), \operatorname{Im}(\mu)$, and $|\mu|$ are its real part, imaginary part, and modulus, respectively. $\otimes$ denotes the Kronecker product.

## 2. Preliminaries and Problem Formulation

Let $\mathscr{G}=(\mathscr{V}, \mathscr{E}, A)$ be a weighted directed graph with the set of nodes $\mathscr{V}=\left\{v_{1}, v_{2}, \ldots, v_{n}\right\}$, set of edges $\mathscr{E} \subseteq \mathscr{V} \times \mathscr{V}$, and a weighted adjacency matrix $A=\left[a_{i j}\right]$ with nonnegative elements $a_{i j}$. The node indexes belong to a finite index set $\mathscr{J}=\{1,2, \ldots, n\}$. An edge in $\mathscr{E}$ is denoted by $e_{j i}=\left(v_{j}, v_{i}\right)$, where $e_{j i} \in \mathscr{E}$ if and only if $a_{i j} \neq 0$. In this case, we say node $j$ is a neighbor of node $i$ and denote the neighbors of node $i$ by $\mathcal{N}_{i}=\left\{j \in \mathscr{V}:\left(v_{j}, v_{i}\right) \in \mathscr{E}\right\}$. Moreover, we assume $a_{i i}=0$ for all $i \in \mathscr{V}$. Let the Laplacian matrix $L=\left[l_{i j}\right] \in \mathbb{R}^{n \times n}$ associated with $A$ be defined as $l_{i i}=\sum_{j=1, j \neq i}^{n} a_{i j}$ and $l_{i j}=-a_{i j}$.

A directed path in directed graph $\mathscr{G}$ from $v_{i_{1}}$ to $v_{i_{k}}$ is a sequence of edges of the form $\left(v_{i_{1}}, v_{i_{2}}\right),\left(v_{i_{2}}, v_{i_{3}}\right), \ldots,\left(v_{i_{k-1}}, v_{i_{k}}\right)$, where $v_{i_{j}} \in \mathscr{V}$ for $j=1,2, \ldots, k$. A directed graph is called strongly connected if any two distinct nodes of the graph can be connected via a directed path that follows the edges of the graph. A directed tree is a directed graph, where every node, except one special node without any parent, which is called the root, has exactly one parent, and the root can be connected to any other nodes through paths. A spanning tree of a digraph is a directed tree formed by graph edges that connect all the nodes of the graph.

For the $n$-agent system considered in this paper, suppose that there are $m(m \leq n)$ leaders and $n-m$ followers. For convenience, we use $\mathscr{R}=\{1,2, \ldots, m\}$ and $\mathscr{F}=\{m+$ $1, m+2, \ldots, n\}$ to denote, respectively, the leader set and the follower set. Further, the interaction topology among agents is modeled by a direct graph $\mathscr{G}$.

The dynamics of the $n$ autonomous agents are given by

$$
\begin{equation*}
\dot{x}_{i}=v_{i}, \quad \dot{v}_{i}=u_{i}, \quad i=1,2, \ldots, n \tag{1}
\end{equation*}
$$

where $x_{i}, v_{i}, u_{i} \in \mathbb{R}^{p}$ denote the position, velocity and control input of agent $i$, respectively.

Using the formation framework in [16], the formation information is divided into two independent parts: the global one and the local one. The global information, which determines the geometric pattern of the expected formation, is represented by a time-dependent column vector $F=\left[f_{1}^{T}, f_{2}^{T}, \ldots, f_{m}^{T}\right]^{T} \in \mathbb{R}^{2 m p}$ with $f_{i}=\left[f_{i}^{x^{T}}, f_{i}^{v^{T}}\right]^{T} \in$ $\mathbb{R}^{2 p}(i \in \mathscr{R})$. The local formation information is denoted by a time-independent nonnegative matrix $W=\left[W_{m+1}^{T}, \ldots, W_{n}^{T}\right]^{T} \in \mathbb{R}^{(n-m) \times m}$ with unit entry sum for each row $W_{i}=\left(w_{i}^{1}, \ldots, w_{i}^{m}\right) \in \mathbb{R}^{1 \times m}(i \in \mathscr{F})$.

Denote $x^{\mathscr{R}}=\left(x_{1}^{T}, x_{2}^{T}, \ldots, x_{m}^{T}\right)^{T}$ and $v^{\mathscr{R}}=\left(v_{1}^{T}, v_{2}^{T}, \ldots\right.$, $\left.v_{m}^{T}\right)^{T}$; we have the following.

Definition 1. We say that system (1) solves the formation problem if there exists a $\mathbb{R}^{2 p}$-valued function $f_{c}=$ $\left[f_{c}^{x}(t)^{T}, f_{c}^{v}(t)^{T}\right]$ with $f_{c}^{x}(t) \in \mathbb{R}^{p}, f_{c}^{v}(t) \in \mathbb{R}^{p}$, and $\dot{f}_{c}^{x}(t)=$ $f_{c}^{v}(t)$ such that $x_{i} \rightarrow f_{i}^{x}(t)+f_{c}^{x}(t), v_{i} \rightarrow f_{i}^{v}(t)+f_{c}^{v}(t)$ for $i \in \mathscr{R}$ and $x_{i} \rightarrow\left(W_{i} \otimes I_{p}\right) x^{\mathscr{R}}, v_{i} \rightarrow\left(W_{i} \otimes I_{p}\right) v^{\mathscr{R}}$ for $i \in \mathscr{F}$ as $t \rightarrow \infty$. In particular, the formation problem is called a timeinvariant formation (TIF) problem, a time-varying formation (TVF) problem, and a time-varying formation for trajectory tracking (TVFT) problem if $\dot{f}_{i}^{x}=\dot{f}_{c}^{x}=f_{i}^{v}=f_{c}^{v}=0$, $\dot{f}_{i}^{x}=f_{i}^{v} \neq 0$ with $\dot{f}_{c}^{x}=f_{c}^{v}=0$ and $\dot{f}_{i}^{x}=f_{i}^{v} \neq 0$ with $\dot{f}_{c}^{x}=f_{c}^{v} \neq 0$, respectively.

The column vector $f_{c}(t)$, which specifies the state of the formation and may be dependent on initial states or may be an external input, used to guide the group of agents to track a prescribed trajectory. The vector $F$ defines the basic frame of the anticipated formation formed by the leaders, and the nonnegative matrix $W$ specifies the local-state restrictions of followers with respect to their leader neighbors. Since each row entry sum of $W$ equals 1 , the followers should lie in the convex region covered by the leaders.

## 3. Main Results

3.1. Leader-Following Consensus. For the better understanding of the formation control problem, we first consider a leader-following consensus problem. The multiagent systems solve a leader-following consensus asymptotically if $\lim _{t \rightarrow \infty}\left\|x_{i}-f_{0}^{x}\right\|=0$ and $\lim _{t \rightarrow \infty}\left\|v_{i}-f_{0}^{v}\right\|=0$ for all $i=$ $1,2, \ldots, n$. Under this circumstance, all the agents can obtain the acceleration $\dot{f}_{0}^{v}$, but only the root agent can obtain the difference between its state and the formation information, and hence it can pin the other leaders to attain the anticipated formation. Define a matrix $B \in \mathbb{R}^{n \times n}$ as $B=\operatorname{diag}\left\{b_{1}, \ldots, b_{n}\right\}$, where $b_{i}>0$ if the interaction topology $\mathscr{G}$ has a spanning tree rooted at $i, 1 \leq i \leq n$ and $b_{i}=0$ otherwise.

The time-varying delay $\tau(t)$ is the time delay for information communicated between agents at time $t$. Owing to the communication time delays, each agent cannot instantly get the information from others. With time-varying delays
existing in the transmission of both velocity and position, we now provide the following consensus protocol:

$$
\begin{align*}
& u_{i}(t) \\
& \qquad \begin{aligned}
&=\dot{f}_{0}^{v}+\sum_{j \in \mathcal{N}_{i}} a_{i j}\left[\left(x_{j}(t-\tau(t))-x_{i}(t-\tau(t))\right)\right. \\
&\left.+k\left(v_{j}(t-\tau(t))-v_{i}(t-\tau(t))\right)\right] \\
&+b_{i}\left[\left(f_{0}^{x}(t-\tau(t))-x_{i}(t-\tau(t))\right)\right. \\
&+\left.k\left(f_{0}^{v}(t-\tau(t))-v_{i}(t-\tau(t))\right)\right] \\
& i=1,2, \ldots, n
\end{aligned}
\end{align*}
$$

where the control parameter $k>0, A=\left(a_{i j}\right)_{n \times n}$ is the adjacency matrix corresponding to the graph $\mathscr{G}, b_{i}>0$ if the interaction topology $\mathscr{G}$ has a spanning tree rooted at $i$ and $b_{i}=0$ otherwise, and the time-varying delay $\tau(t)$ is a continuously differentiable function with

$$
\begin{equation*}
\tau(t)<d_{1}, \quad \dot{\tau}(t) \leq d_{2}<1 \tag{3}
\end{equation*}
$$

Denote $x=\left(x_{1}^{T}, x_{2}^{T}, \ldots, x_{n}^{T}\right)^{T}, v=\left(v_{1}^{T}, v_{2}^{T}, \ldots, v_{n}^{T}\right)^{T}$; then, system (1) with protocol (2) can be written in a matrix form:

$$
\begin{gather*}
\dot{x}=v \\
\dot{v}=  \tag{4}\\
\dot{f}_{0}^{v} \mathbf{1}_{n}-(L+B) \otimes I_{p} x(t-\tau) \\
-k(L+B) \otimes I_{p} v(t-\tau) \\
+B\left[f_{0}^{x}(t-\tau) \cdot \mathbf{1}_{n}+k f_{0}^{v}(t-\tau) \cdot \mathbf{1}_{n}\right]
\end{gather*}
$$

where $L$ is the Laplacian matrix associated with $\mathscr{G}, B=$ $\operatorname{diag}\left\{b_{1}, \ldots, b_{n}\right\}, b_{i}>0$ if the interaction topology $\mathscr{G}$ has a spanning tree rooted at $i$ and $b_{i}=0$ otherwise.

Let $\bar{x}=x-f_{0}^{x} \mathbf{1}_{n}, \bar{v}=v-f_{0}^{v} \mathbf{1}_{n}$; then, (4) can be rewritten as

$$
\dot{\bar{x}}=\bar{v}
$$

$$
\begin{equation*}
\dot{\bar{v}}=-(L+B) \otimes I_{p} \bar{x}(t-\tau)-k(L+B) \otimes I_{p} \bar{v}(t-\tau) . \tag{5}
\end{equation*}
$$

Taking $\varepsilon=\left(\bar{x}^{T}, \bar{v}^{T}\right)^{T}$, (5) can be equally expressed as

$$
\begin{equation*}
\dot{\varepsilon}=Y \varepsilon(t)+Z \varepsilon(t-\tau) \tag{6}
\end{equation*}
$$

where

$$
\begin{gather*}
Y=\left(\begin{array}{cc}
0_{n \times n} & I_{n} \\
0_{n \times n} & 0_{n \times n}
\end{array}\right) \otimes I_{p} \\
Z=\left(\begin{array}{cc}
0_{n \times n} & 0_{n \times n} \\
-H & -k H
\end{array}\right) \otimes I_{p}  \tag{7}\\
H=L+B
\end{gather*}
$$

To proceed further, we need the following lemmas.
Lemma 2 (see [22]). Given a complex-coefficient polynomial

$$
\begin{equation*}
r(z)=z^{2}+(a+i b) z+c+i d \tag{8}
\end{equation*}
$$

where $a, b, c, d \in \mathbb{R}, r(z)$ is Hurwitz stable if and only if $a>0$ and $a b d+a^{2} c-d^{2}>0$.

Lemma 3. The matrix $F=Y+Z=\left(\begin{array}{cc}0_{n \times n} & I_{n} \\ -H & -k H\end{array}\right) \otimes I_{p}$ is Hurwitz stable if and only if $H$ is positive stable and $k>$ $\max _{\lambda \in \Lambda(H)}(|\operatorname{Im}(\lambda)| /(\sqrt{|\operatorname{Re}(\lambda)|} \cdot|\lambda|))$.

Proof. Let $z$ be an eigenvalue of $F$. Then one has

$$
\begin{align*}
\operatorname{det}\left(z I_{2 n p}-F\right) & =\operatorname{det}\left(\left[\begin{array}{cc}
z I_{n} & -I_{n} \\
H & z I_{n}+k H
\end{array}\right] \otimes I_{p}\right) \\
& =\operatorname{det}\left(\left(z\left(z I_{n}+k H\right)+H\right) \otimes I_{p}\right)  \tag{9}\\
& =\prod_{i=1}^{n}\left(z^{2}+k \lambda_{i} z+\lambda_{i}\right)^{p}
\end{align*}
$$

Clearly, the Hurwitz stability of matrix $F$ is equivalent to that of the polynomial: $r(z)=z^{2}+k \lambda_{i} z+\lambda_{i}$, where $\lambda_{i}=\operatorname{Re}\left(\lambda_{i}\right)+$ $i \operatorname{Im}\left(\lambda_{i}\right)$ is the $i$ th eigenvalue of $H$. It follows from Lemma 2 that $r(z)$ is Hurwitz stable if and only if $\operatorname{Re}\left(\lambda_{i}\right)>0$ and $k>$ $\left|\operatorname{Im}\left(\lambda_{i}\right)\right| /\left(\sqrt{\left|\operatorname{Re}\left(\lambda_{i}\right)\right|} \cdot\left|\lambda_{i}\right|\right), \lambda_{i} \in \Lambda(H)$.

Lemma 4 (see [4]). The matrix $H=L+B$ is positive stable if $\mathscr{G}$ has a spanning tree.

Now we give the main result of this subsection.
Theorem 5. For system (6), take

$$
\begin{equation*}
k>k_{0}=\max _{\lambda \in \Lambda(H)} \frac{|\operatorname{Im}(\lambda)|}{\sqrt{|\operatorname{Re}(\lambda)|} \cdot|\lambda|} \tag{10}
\end{equation*}
$$

and suppose that

$$
\begin{equation*}
0<\tau<\tau_{0}=\frac{1}{2\left\|Z^{T} Z\right\| /\left(1-d_{2}\right)+\left\|2 Y^{T} Y+P Z Z^{T} P\right\|} \tag{11}
\end{equation*}
$$

Then, the consensus is reached asymptotically if the graph $\mathscr{G}$ has a spanning tree.

Proof. Since $\mathscr{G}$ has a spanning tree, $H$ is positive stable based on Lemma 4. It follows from Lemma 3 that $F$ is Hurwitz stable. Therefore, there exists a positive definite matrix $P \in$ $\mathbb{R}^{2 n p \times 2 n p}$ such that

$$
\begin{equation*}
P F+F^{T} P=-I_{2 n p} \tag{12}
\end{equation*}
$$

To analyze the convergence of system (6), we define a Lyapunov-Krasovskii functional

$$
\begin{align*}
V(\varepsilon)= & \varepsilon^{T}(t) P \varepsilon(t)+\int_{t-\tau}^{T} \varepsilon^{T}(t) S \varepsilon(t) d s \\
& +\int_{-\tau}^{0} \int_{t+\theta}^{t} \dot{\varepsilon}^{T}(s) R \dot{\varepsilon}(s) d s d \theta \tag{13}
\end{align*}
$$

where $S=\beta I_{2 n p}$ and $R=I_{2 n p}$.
Calculating $\dot{V}(\varepsilon)$ along the solution of (6), we have

$$
\begin{align*}
\dot{V}(\varepsilon)= & \varepsilon^{T}(t)\left(Y^{T} P+P Y\right) \varepsilon(t)+2 \varepsilon^{T}(t) P Z \varepsilon(t-\tau) \\
& +\varepsilon^{T}(t) \beta \varepsilon(t)-\beta(1-\dot{\tau}) \varepsilon^{T}(t-\tau) \varepsilon(t-\tau)  \tag{14}\\
& -\int_{t-\tau}^{t} \dot{\varepsilon}^{T}(\theta) \dot{\varepsilon}(\theta) d \theta+\tau \dot{\varepsilon}^{T}(t) \dot{\varepsilon}(t)
\end{align*}
$$

Due to the fact that $\varepsilon(t-\tau)=\varepsilon(t)-\int_{t-\tau}^{t} \dot{\varepsilon}(s) d s$ and $2 a^{T} b \leq a^{T} \Psi^{-1} a+b^{T} \Psi b$ holds for any appropriate positive definite matrix $\Psi$, we can obtain that

$$
\begin{align*}
& 2 \varepsilon^{T}(t) P Z \varepsilon(t-\tau) \\
&=\varepsilon^{T}(t)\left(Z^{T} P+P Z\right) \varepsilon(t)+2 \int_{t-\tau}^{t}\left(-Z^{T} P \varepsilon(t)\right)^{T} \dot{\varepsilon}(s) d s \\
& \quad \leq \varepsilon^{T}(t)\left(Z^{T} P+P Z\right) \varepsilon(t)+\tau \varepsilon^{T}(t) P Z Z^{T} P \varepsilon(t) \\
&+\int_{t-\tau}^{t} \dot{\varepsilon}^{T}(s) \dot{\varepsilon}(s) d s . \tag{15}
\end{align*}
$$

Similarly,

$$
\begin{align*}
\tau \dot{\varepsilon}^{T}(t) \dot{\varepsilon}(t)= & \tau[Y \varepsilon(t)+Z \varepsilon(t-\tau)]^{T}[Y \varepsilon(t)+Z \varepsilon(t-\tau)] \\
= & \tau\left[\varepsilon^{T}(t) Y^{T} Y \varepsilon(t)+\varepsilon^{T}(t-\tau) Z^{T} Z \varepsilon(t-\tau)\right] \\
& +\varepsilon^{T}(t) Y^{T} Z \varepsilon(t-\tau)+\varepsilon^{T}(t-\tau) Z^{T} Y \varepsilon(t) \\
\leq & 2 \tau\left[\varepsilon^{T}(t) Y^{T} Y \varepsilon(t)+\varepsilon^{T}(t-\tau) Z^{T} Z \varepsilon(t-\tau)\right] . \tag{16}
\end{align*}
$$

Substituting (15) and (16) into (14) leads to

$$
\begin{align*}
\dot{V}(\varepsilon) \leq & \varepsilon^{T}(t)\left[-\left(P F+F^{T} P\right)+\beta I_{2 n p}\right. \\
& \left.+\tau\left(2 Y^{T} Y+P Z Z^{T} P\right)\right] \varepsilon(t) \\
& +\varepsilon^{T}(t-\tau)\left[-(1-\dot{\tau}) \beta I_{2 n p}+2 \tau Z^{T} Z\right] \varepsilon(t-\tau) \\
\leq & \varepsilon^{T}(t)\left(-1+\beta+\tau\left\|2 Y^{T} Y+P Z Z^{T} P\right\|\right) \varepsilon(t) \\
+ & \varepsilon^{T}(t-\tau)\left[-\left(1-d_{2}\right) \beta+2 \tau\left\|Z^{T} Z\right\|\right] \varepsilon(t-\tau) . \tag{17}
\end{align*}
$$

Consequently, a sufficient condition for $\dot{V}(\varepsilon)<0$ is

$$
\begin{gather*}
-1+\beta+\tau\left\|2 Y^{T} Y+P Z Z^{T} P\right\|<0 \\
-\left(1-d_{2}\right) \beta+2 \tau\left\|Z^{T} Z\right\|<0 \tag{18}
\end{gather*}
$$

From (18), we can obtain that

$$
\begin{equation*}
\tau<\tau_{0}=\frac{1}{2\left\|Z^{T} Z\right\| /\left(1-d_{2}\right)+\left\|2 Y^{T} Y+P Z Z^{T} P\right\|} \tag{19}
\end{equation*}
$$

Therefore, by Lyapunov-Krasovskii Theorem (see [23]), the error system (6) is uniformly asymptotically stable. Namely, the consensus is reached asymptotically.

Remark 6. It can be seen that many zoom techniques are applied during the derivation of $\tau_{0}$, which result in a conservative estimation of $\tau_{0}$.
3.2. Time-Varying Formation for Trajectory Tracking. In this subsection, we consider the case that the agents form a time-varying formation as they track the desired trajectory. More specifically, the desired trajectory of the formation, represented by $f_{c}=\left[f_{c}^{x}(t)^{T}, f_{c}^{v}(t)^{T}\right]$, is assumed to be determined by the following equation:

$$
\begin{equation*}
\dot{f}_{c}^{x}=f_{c}^{v}=g\left(t, f_{c}^{x}\right) \tag{20}
\end{equation*}
$$

Also make the following assumptions about the multiagent system:
(A1) the local interaction topology of the leaders has a spanning tree, and the leaders' dynamic is unaffected by the followers;
(A2) the root agent in the local interaction topology of the leaders is able to access the reference trajectory;
(A3) in addition to the local information from their neighbors, the leaders can also obtain the global information $F$;
(A4) in addition to the local information from their follower neighbors, each follower can also get the local formation $W$ directly or indirectly from the leaders.
Using the following control protocol:

$$
\begin{gathered}
\dot{x}_{i}=v_{i}, \quad i=1,2, \ldots, n, \\
\dot{v}_{i}=\dot{f}_{i}^{v}+\dot{g}\left(t, f_{c}^{x}\right) \\
+\sum_{j=1}^{m} a_{i j}\left\{\left[\left(x_{j}(t-\tau)-f_{j}^{x}(t-\tau)\right)\right.\right. \\
\left.-\left(x_{i}(t-\tau)-f_{i}^{x}(t-\tau)\right)\right] \\
+k\left[\left(v_{j}(t-\tau)-f_{j}^{v}(t-\tau)\right)\right. \\
\left.\left.-\left(v_{i}(t-\tau)-f_{i}^{v}(t-\tau)\right)\right]\right\} \\
\dot{v}_{i}=\sum_{k=1}^{m} w_{i}^{k} \dot{v}_{k} \quad \\
\left.+k\left(v_{i}(t-\tau)-f_{i}^{v}(t-\tau)-g(t-\tau)\right)\right] \\
+\sum_{j=1}^{n} a_{i j}\left\{\left[(t-\tau)-f_{i}^{x}(t-\tau)-f_{c}^{x}(t-\tau)\right)\right. \\
\left.x_{j}(t-\tau)-\sum_{k=1}^{m} w_{j}^{k} x_{k}(t-\tau)\right) \\
\\
\left.\quad-\left(x_{i}(t-\tau)-\sum_{k=1}^{m} w_{i}^{k} x_{k}(t-\tau)\right)\right] \\
\\
+k\left[\left(v_{j}(t-\tau)-\sum_{k=1}^{m} w_{j}^{k} v_{k}(t-\tau)\right)\right.
\end{gathered}
$$

$$
\begin{align*}
-b_{i}\left[\left(x_{i}(t-\tau)-\sum_{k=1}^{m} w_{i}^{k} x_{k}(t-\tau)\right)\right. \\
\left.+k\left(v_{i}(t-\tau)-\sum_{k=1}^{m} w_{i}^{k} v_{k}(t-\tau)\right)\right], \\
i \in \mathscr{F}, \tag{21}
\end{align*}
$$

where $k>0, A=\left(a_{i j}\right)_{n \times n}$ is the adjacency matrix corresponding to the graph $\mathscr{G}$, and $b_{i}>0$ if there is a spanning tree rooted at $i$ in the graph $\mathscr{G}$ and $b_{i}=0$ otherwise, we have the following result.

Theorem 7. Suppose that the graph $\mathscr{G}$ has a spanning tree. Take

$$
\begin{gather*}
k>k_{0}=\max _{\lambda \in \Lambda(H)} \frac{|\operatorname{Im}(\lambda)|}{\sqrt{|\operatorname{Re}(\lambda)|} \cdot|\lambda|}, \\
0<\tau<\tau_{0}=\frac{1}{2\left\|Z^{T} Z\right\| /\left(1-d_{2}\right)+\left\|2 Y^{T} Y+P Z Z^{T} P\right\|} . \tag{22}
\end{gather*}
$$

With protocol (21), the multiagent systems attain a timevarying formation for trajectory tracking (TVFT) under assumptions (A1)-(A4).

Proof. Let $\widetilde{x}_{i}=x_{i}-f_{i}^{x}-f_{c}^{x}, \widetilde{v}_{i}=v_{i}-f_{i}^{v}-g\left(t, f_{c}^{x}\right)$ for $i \in \mathscr{R}$, and $\widetilde{x}_{i}=x_{i}-\sum_{k=1}^{m} w_{i}^{k} x_{k}, \widetilde{v}_{i}=v_{i}-\sum_{k=1}^{m} w_{i}^{k} v_{k}$ for $i \in \mathscr{F}$. Then we can rewrite protocol (21) as

$$
\begin{gather*}
\dot{\tilde{x}}_{i}=\widetilde{v}_{i} \\
\dot{\tilde{v}}_{i}=\sum_{j=1}^{n} a_{i j}\left[\left(\widetilde{x}_{j}(t-\tau)-\widetilde{x}_{i}(t-\tau)\right)\right. \\
\left.\quad+k\left(\widetilde{v}_{j}(t-\tau)-\widetilde{v}_{i}(t-\tau)\right)\right] \\
-b_{i}\left(\widetilde{x}_{i}(t-\tau)+k \widetilde{v}_{i}(t-\tau)\right), \quad i=1,2, \ldots, n \tag{23}
\end{gather*}
$$

Denote $\widetilde{x}=\left(\widetilde{x}_{1}^{T}, \widetilde{x}_{2}^{T}, \ldots, \widetilde{x}_{n}^{T}\right)$ and $\widetilde{v}=\left(\widetilde{v}_{1}^{T}, \widetilde{v}_{2}^{T}, \ldots, \widetilde{v}_{n}^{T}\right) ;(23)$ can be expressed in a matrix form:

$$
\begin{gather*}
\dot{\tilde{x}}=\widetilde{v} \\
\dot{\tilde{v}}=-(L+B) \otimes I_{p} \tilde{x}(t-\tau)-k(L+B) \otimes I_{p} \widetilde{v}(t-\tau) . \tag{24}
\end{gather*}
$$

It follows from Theorem 5 that $\widetilde{x}_{i}$ and $\widetilde{v}_{i}$ converge to zero asymptotically, and equally, $x_{i} \rightarrow f_{i}^{x}+f_{c}^{x}, v_{i} \rightarrow f_{i}^{v}+f_{c}^{v}$ for $i \in \mathscr{R}$ and $x_{i} \rightarrow\left(W_{i} \otimes I_{p}\right) x^{\mathscr{R}}, v_{i} \rightarrow\left(W_{i} \otimes I_{p}\right) v^{\mathscr{R}}$ for $i \in \mathscr{F}$ as $t \rightarrow \infty$.


Figure 1: The interaction topology.


Figure 2: The trajectories of some agents in TIF.

Remark 8. Obviously, Theorem 7 still holds if $\dot{f}_{i}^{x}=f_{i}^{v}=0$, or $\dot{f}_{i}^{x}=f_{i}^{v} \neq 0$ and $\dot{f}_{c}^{x}=f_{c}^{v}=g\left(t, f_{c}^{x}\right)=0$. In the form case, the multiagent systems attain a time-invariant formation (TIF). In the latter case, the multiagent systems attain a time-varying formation.

## 4. Simulations

In this section, to illustrate our theoretical results derived in the above section, we will provide several examples. Consider a multiagent system consisting of 25 agents moving in a plane (i.e., $p=2$ ), with the direct interaction topology described in Figure 1. The expected formation is a hexagram. Assume that the first 7 agents are leaders with the spanning tree rooted at agent 7. For simplicity, let $a_{i j}=1$ if agent $j$ is a neighbor of agent $i$ and $a_{i j}=0$ otherwise. With simple calculations, we can obtain that $k_{0}=0$ and $\tau_{0}=0.007$. Take $k=2, \tau=$ $0.0065|\cos t|$, and $d_{2}=0.05$.


Figure 3: The formation states of agents in TVFT.

Consider

$$
\begin{align*}
& f_{i}^{x}=R_{i}(t)\binom{\cos \left(\frac{\pi}{2}+\frac{2(i-1)}{3} \pi\right)}{\sin \left(\frac{\pi}{2}+\frac{2(i-1)}{3} \pi\right)}, \quad i=1,2, \ldots, 6, \\
& R_{i}(t) \equiv 5(\mathrm{TIF}), \\
& f_{i}^{x}=(0,0)^{T}, \quad i=7, \\
& \dot{f}_{i}^{x}=f_{i}^{v}, \quad i=1,2, \ldots, 25, \\
& f_{c}^{x}=4\binom{\sin 3 t}{\sin 3 t}, \quad f_{c}^{v}=\dot{f}_{c}^{x}, \\
& R_{i}(t)=6 \sin t(\mathrm{TVFT}), \\
& w_{24}^{1}=w_{24}^{6}=w_{24}^{7}=w_{3 i}^{i-1}=w_{3 i}^{i-2}=w_{3 i}^{7}=\frac{1}{3}, \\
& i=3,4, \ldots, 7, \\
& w_{8}^{1}=w_{25}^{1}=w_{3 i+1}^{i-1}=w_{3 i+2}^{i-1}=\frac{2}{3}, \quad i=3,4, \ldots, 7, \\
& w_{3 i+1}^{i-2}=w_{3 i+2}^{i}=w_{3 i-1}^{7}=w_{3 i+1}^{7}=\frac{1}{6}, \quad i=3,4, \ldots, 6, \\
& w_{8}^{2}=w_{3 i+1}^{i-2}=w_{3 i-1}^{7}=w_{3 i+1}^{7}=\frac{1}{6}, \quad i=7,8, \\
& w_{i}^{j}=0, \quad \text { otherwise } . \tag{25}
\end{align*}
$$

Figure 2 shows the trajectories of some agents in timeinvariant formation (TIF), where the initial states of the agents are randomly generated in a given bounded region. From Figure 2, we can see that the agents attain the vertexes of a hexagram. In other words, the expected time-invariant formation is reached. For the trajectory tracking formation, Figure 3 illustrates the formation states of agents. Since the expected trajectory of the formation $f_{c}^{x}$ satisfies the equation in (25), the formation in Figure 3 changes with time as it moves along a sinusoidal curve. Furthermore, the position and velocity errors of agents in TVFT with time delays


Figure 4: The position errors of agents in TVFT.
existing only in the transmission of position are shown in Figures 4(a) and 5(a), respectively, and the position and velocity errors of agents in TVFT with time delays existing in the transmission of both position and velocity are shown in Figures 4(b) and 5(b), respectively. It can be seen that all of the errors converge to zero ultimately, while the errors in the latter figures converge to zero faster than those in the former figures.

## 5. Conclusion

This paper investigates the formation control problem for second-order multiagent systems with time-varying delays. We first consider a leader-following consensus problem. By employing Lyapunov-Krasovskii functional, we prove that the multiagent systems can reach consensus. Then, under a special multiple leaders' framework, we apply the protocol to the formation control, and derive a sufficient condition for the system to achieve prescribed formation. Moreover, several


Figure 5: The velocity errors of agents in TVFT.
numerical simulations are shown to verify the theoretical analysis.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Gearbox Low-Noise Design Method Based on Panel Acoustic Contribution 

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#### Abstract

This paper presents a comprehensive procedure to calculate steady dynamic response and generated noise radiation from a gear reducer. In this process, the dynamic model of the cylindrical gear transmission system is built in consideration of the timevarying mesh stiffness, gear errors, and bearing supporting, while the data of dynamic bearing force is obtained through solving the model. Furthermore, taking the data of bearing force as the excitation, the gearbox vibrations and noise radiation are calculated by numerical simulation, and then the time history of node dynamic response, noise spectrum, and resonance frequency range of the gearbox are obtained. At last, the gearbox panel acoustic contribution at the resonance frequency range is calculated. According to the conclusions of the gearbox panel acoustic contribution analyses and the mode shapes, two gearbox stiffness improving plans is researched. By contrastive analysis of gearbox noise radiation, the effectiveness of the improving plans are verified. The study provides useful theoretical guideline to the gearbox design.


## 1. Introduction

With many advantages, that is, high efficiency, tight structure, stable speed ratio, and so forth, gear train has been widely used in many industrial fields. When the gear reducer is running, the gearbox vibration is generated, due to the effect of the gear pair dynamic mesh force, which not only affects the stability of the transmission system but also generates noise. In addition, excessive noise produced by a reducer causes crew fatigue, strained communication, and possible hearing damage. In order to ensure a quiet, smooth, and safe operation of a gear transmission system, it is necessary to understand mechanisms of the dynamic response and the noise radiation of the gear reducer; meanwhile, their reduction is highly desired.

With increasing demand for quieter gear systems, a large amount of work was reported in the literatures on analyzing the vibration and noise of the gearbox. Abbes et al. built the gearbox vibroacoustic system by using a three-dimensional finite-element (FE) approach, and the acoustic response of the system was evaluated [1]. Velex and Maatar computed the dynamic responses to mesh stiffness variations for numerical gears [2]. Their results showed the impact of mesh stiffness
variation on dynamic response and tooth loads. Dion et al. developed an experimental and numerical study of dynamic phenomena involving gear impacts with one loose gear inside an automotive gearbox [3]. Barthod et al. dealt with the rattle noise, caused by the fluctuation of the engine torque under special conditions, which could cause multiple impacts inside the gearbox [4]. Kato et al. simulated the vibration and noise radiation of a single-stage gearbox by combining finite-element (FE) vibration analysis with boundary element noise analysis [5]. The results of this analysis were well agreed with the corresponding measured data. Spur and helical gears were tested in the NASA gear-noise rig to compare the noise produced by different gear designs [6]. The useful conclusions about the effect of the gear design parameters on gearbox radiated noise were got. Choy et al. presented method to predict both the vibration and noise generated by a gear transmission system under normal operation conditions [7], and the application of the method is demonstrated by comparing the numerical and experimental results for the gear noise test rig. Yanyan and Zhen confirmed that the gear pair is the main excitation of the gear reducer and reduced the gearbox noise through matching the precision grade and stiffness of the gears [8]. Kahraman and Blankenship
investigated contact ratio effects experimentally using a back-to-back gearbox rig. The dynamic transmission error (DTE) amplitudes of spur gear pairs with varying contact ratios were measured. The measurements were performed for excitation at and around the torsional natural frequency of the gear pair. The gear mesh frequency was used as a form of torsional excitation, with the limitation that excitation is dependent on rotational speed [9]. Kostić and Ognjanović found that the noise emission of gear units (gearboxes) depends both on the disturbances (gear meshing, bearing operation, etc.) and on the insulating capabilities and modal behavior of the housing. Natural vibrations of the housing walls can be prevented or intensified depending on design parameters [10]. Tuma reviews practical techniques and procedures employed to quiet gearboxes and transmission units [11]. With the complexity of the gearbox structure and the gear excitation, excessive simplification has been made in most of the previous research; meanwhile no effective method to reducing vibration and noise is found.

In this study, we present a comprehensive procedure to predict the noise radiation of the gear reducer. In the procedure, the 4 -DOF dynamic model is built, and then taking the bearing force as the excitation, vibrations and noise radiation of the gearbox are researched. According to the results of the panel acoustic contribution analysis on the resonant frequency band of the gearbox and the mode shapes, effective methods to reducing vibration and noise are suggested.

## 2. Analysis Procedure of Noise Radiating

Gear errors and fluctuations in mesh stiffness can cause excitation during gear meshing; this excitation propagates from the gear shafts to the bearing and excites the gearbox and generations reducer noise which is radiated from the surface of the gearbox. In order to concern about both gear transmission system dynamic characteristics and gearbox dynamic characteristics, an excellent prediction method of gearbox noise radiation is proposed.

As illustrated in Figure 1, the developed method consists of three separate steps: dynamic bearing force calculation by solving the gear transmission system dynamic model, gearbox vibration analysis by using finite element method (FEM), and boundary element analysis (BEA) of the sound field. In this method, a commercial software, LMS.Virtual.lab, is used to analyze the sound radiation for the gear reducer. The input data are fundamental performance parameters of the gear reducer, which consist of the gearbox shape, material, gear error, bearing stiffness, and so on. The output data are vibrations and noise analysis results, which consist of dynamic responses, frequency spectrum for noise, panel acoustic contribution, and so on. The low-noise gearbox is designed according to the conclusion of panel acoustic contribution and gearbox dynamic characteristics.

## 3. Gearbox Excitation Calculation

In a power transmission gear system, the gear pair assembly remains one of the major noise and vibration sources in

Table 1: The gear system parameters.

| Power <br> $(\mathrm{Kw})$ | Gear <br> ratio | Module <br> $(\mathrm{mm})$ | Pressure <br> angle $(\mathrm{deg})$ | Face width <br> $(\mathrm{mm})$ | Rotational <br> speed $(\mathrm{r} / \mathrm{min})$ |
| :--- | :---: | :---: | :---: | :---: | :---: |
| 10 | $20 / 80$ | 3.0 | 20 | 60 | 1000 |

the system. The vibrations of the gear transmission system are generated due to the fluctuation of the dynamic meshing force, which is affected by the time-varying mesh stiffness and errors.
3.1. Gear Time-Varying Mesh Stiffness. The gear system parameters are given in Table 1. The variation of the mesh stiffness for the gear pair is obtained by using static finite element analysis, in which FEM Contact Algorithm is adopted.

The FE-model of the gear pair and boundary conditions is shown in Figure 2. During the calculation of the time-varying mesh stiffness, the driven gear is fixed, the torque $T$ is applied on the driving gear, and contact constraint is applied between the engaged tooth of the driving gear and the driven gear.

As a result of the tooth Hertzian contact deformations and tooth bending deformations, the driving gear will revolved a small angle $\theta$ on its centre. The small angle $\theta$ is obtained by solving the gear pair FE-model; then the total deformation of the meshing line is defined as

$$
\begin{equation*}
u=\theta \cdot R_{b}, \tag{1}
\end{equation*}
$$

where $R_{b}$ is the base radius of the driving gear.
So the mesh stiffness at this position is represented by the equation

$$
\begin{equation*}
K=\frac{T}{\left(R_{b} u\right)} \tag{2}
\end{equation*}
$$

where $T$ is the torque.
Since the gear rotation is continuous, the gear meshing stiffness is periodic at the mesh frequency, a complete mesh cycle is divided into several steps, and the rotatory angle and position of the gears at every step can be calculated according to gear mesh theory. Then the calculation of the mesh stiffness is repeated at every gear engaging position. The time-varying mesh stiffness function is formed by cubic spline interpolation, as shown in Figure 3(a). As the number of tooth pairs in contact changes, abrupt changes in the gear pair stiffness occur (the mesh of spur gears with two tooth pairs in contact is roughly twice as stiff as when one tooth pair is in contact).
3.2. Gear Errors. Vibrations of gear pairs are largely affected by the amplitude and phase of deviations of the tooth profile from the true involute one, which is induced by gear manufacturing and installing errors. Meanwhile, with the effect of the gear errors on the instantaneous contact ratio, the collision and impact occur while the gear pair is running [12]. As a result, gear errors must be included in the gear transmission system model. Generally, the deviations are assumed to be small enough so that tooth contacts remain on


FIGURE 1: Analysis procedure of gear reducer noise radiation.


Figure 2: Gear pair finite element model.
the theoretical line of action [2]. Error function, representing the sum of pitch, profile, pressure angle, and run out errors, is supposed as displacement excitations along the tooth profile as a sine wave in the model. The harmonic function is used to simulate the gear error variation which is shown in Figure 3(b). The error function is written as

$$
\begin{equation*}
e(t)=e_{r} \sin \left(\frac{w t}{T_{m}}+\phi\right) \tag{3}
\end{equation*}
$$

where $e_{r}$ is error amplitude, $T_{m}$ is the mesh cycle, $w$ is the angular velocity of the driving gear, and $\phi$ is the phase angle.
3.3. Gear Transmission Systems Dynamics Model. The proposed dynamic model of the gear pair is shown in Figure 4, which represents the driving gear (subscript $p$ ) meshing with driven gear (subscript $g$ ). The following assumptions are made in the model formulation.
(a) The deflection of the shaft is neglected, because the span of the bearings is little.
(b) Shaft mass and inertia are lumped at the gears.
(c) The bodies representing the two gear bodies are assumed to be rigid disks [13].
(d) The gear-shaft connections were assumed to be rigid, ignoring the stiffness of the connections and any consequent relative torsional motion between the shaft and gear hub.
(e) Gear pairs mesh flexibility and other parts flexibility are included in the form of a linear spring. The gear mesh stiffness is time variant; the support stiffness is constant.

Namely, $K_{p y}$ and $K_{g y}$ denote bearing stiffness of the driving gear and driven gear and $K_{m}$ denotes time-varying mesh stiffness. The angular displacements $\theta_{p}$ and $\theta_{g}$ of the driving gear and the driven gear are in the reversed direction; in the same time, the transverse displacements $Y_{p}$ and $Y_{g}$ in the direction of meshing line are considered.

The angular displacements and the transverse displacements of the gears will affect meshing state of the gear


Figure 3: The transmission system excitation.
pair, so the displacement is transformed to action line. The displacement on the action line is written as

$$
\begin{equation*}
\bar{y}_{p}=R_{p} \theta_{p}-y_{p}, \quad \bar{y}_{g}=y_{g}-R_{g} \theta_{g}, \tag{4}
\end{equation*}
$$

where $R_{p}$ is the base circle radius of the driving gear and $R_{g}$ is the base circle radius of the driven gear.

The mesh force and damping force of the gear pairs are written as

$$
\begin{align*}
& F_{k}=k_{m}\left(\bar{y}_{p}+\bar{y}_{g}-e\right)=k_{m}\left(R_{p} \theta_{p}-y_{p}+y_{g}-R_{g} \theta_{g}-e\right), \\
& F_{c}=c_{m}\left(\dot{\bar{y}}_{p}+\dot{\bar{y}}_{g}-e\right)=c_{m}\left(R_{p} \dot{\theta}_{p}-\dot{y}_{p}+\dot{y}_{g}-R_{g} \dot{\theta}_{g}-\dot{e}\right), \tag{5}
\end{align*}
$$

where $e$ is the gear error, $R_{p}$ is the base circle radius of the driving gear, $R_{g}$ is the base circle radius of the driven gear, and $c_{m}$ is mesh damping coefficient of gear pair $c_{m}=$ $2 \xi \sqrt{k_{m} /\left(1 / m_{p}+1 / m_{g}\right)}, m_{p}$ is the mass of the driving gear and $m_{g}$ is mass driven gear of the driven gear. The range of the damping ratio $\xi$ is $0.03 \sim 0.1$.

Therefore, dynamic mesh force $F_{p g}$ is defined as

$$
\begin{equation*}
F_{p g}=F_{k}+F_{c} . \tag{6}
\end{equation*}
$$

Similarly, supporting spring force (bearing force) is defined as

$$
\begin{equation*}
F_{p y}=k_{p y} y_{p}, \quad F_{g y}=k_{g y} y_{g} . \tag{7}
\end{equation*}
$$

According to the Newton mechanics law, the following differential equations of the gear system are set up, which contains


Figure 4: The dynamic model of the system.
the effects of time-varying mesh stiffness and error excitation. Total number of degree of freedom for the model is 4:

$$
\begin{gather*}
m_{p} \ddot{y}_{p}+c_{p y} \dot{y}_{p}+k_{p y} y_{p}=F_{p g} \\
I_{p} \ddot{\theta}_{p}=T_{p}-F_{p g} R_{p}  \tag{8}\\
m_{g} \ddot{y}_{g}+c_{g y} \dot{y}_{g}+k_{g y} y_{g}=-F_{p g}, \\
I_{g} \ddot{\theta}_{g}=T_{g}-F_{p g} R_{g} .
\end{gather*}
$$

Here, $T_{p}$ is the input torque. $T_{g}$ is the load torque. $I_{p}$ is the rotational inertia of driving gear. $I_{g}$ is the rotational inertia of driven gear.

Equation (6) in conjunction with (8) yields 4 coupled homogeneous ordinary differential equations in the form

$$
\begin{align*}
& m_{p} \ddot{y}_{p}+c_{p y} \dot{y}_{p}+k_{p y} y_{p} \\
&= c_{m}\left(R_{p} \dot{\theta}_{p}-\dot{y}_{p}+\dot{y}_{g}-R_{g} \dot{\theta}_{g}-\dot{e}\right) \\
&+k_{m}\left(R_{p} \theta_{p}-y_{p}+y_{g}-R_{g} \theta_{g}-e\right), \\
& I_{p} \ddot{\theta}_{p}=- {\left[c_{m}\left(R_{p} \dot{\theta}_{p}-\dot{y}_{p}+\dot{y}_{g}-R_{g} \dot{\theta}_{g}-\dot{e}\right)\right.} \\
&\left.+k_{m}\left(R_{p} \theta_{p}-y_{p}+y_{g}-R_{g} \theta_{g}-e\right)\right] R_{p}+T_{p}, \\
& m_{g} \ddot{y}_{g}+c_{g y} \dot{y}_{g}+k_{g y} y_{g} \\
&=-c_{m}\left(R_{p} \dot{\theta}_{p}-\dot{y}_{p}+\dot{y}_{g}-R_{g} \dot{\theta}_{g}-\dot{e}\right) \\
&-k_{m}\left(R_{p} \theta_{p}-y_{p}+y_{g}-R_{g} \theta_{g}-e\right), \\
& I_{g} \ddot{\theta}_{g}=[ c_{m}\left(R_{p} \dot{\theta}_{p}-\dot{y}_{p}+\dot{y}_{g}-R_{g} \dot{\theta}_{g}-\dot{e}\right) \\
&\left.+k_{m}\left(R_{p} \theta_{p}-y_{p}+y_{g}-R_{g} \theta_{g}-e\right)\right] R_{g}+T_{g} . \tag{9}
\end{align*}
$$

In the mathematical model, the angular displacements $\theta_{p}$ and $\theta_{g}$ are independent variables. In order to solve the equations, the angular displacements should be transformed into an independent variable. Therefore, transmission error is lead into the model, defined $y_{p g}=R_{p} \theta_{p}-R_{g} \theta_{g}$; then rigid body displacement is removed; the model can be written as

$$
\begin{align*}
& m_{p} \ddot{y}_{p}+c_{p y} \dot{y}_{p}-c_{m}\left(\dot{y}_{p g}-\dot{y}_{p}+\dot{y}_{g}\right)+k_{p y} y_{p} \\
& \quad-k_{m}\left(y_{p g}-y_{p}+y_{g}\right)=-c_{m} \dot{e}-k_{m} e \\
& m_{g} \ddot{y}_{g}+c_{g y} \dot{y}_{g}+c_{m}\left(\dot{y}_{p g}-\dot{y}_{p}+\dot{y}_{g}\right)+k_{g y} y_{g} \\
& \quad+k_{m}\left(y_{p g}-y_{p}+y_{g}\right)=c_{m} \dot{e}+k_{m} e  \tag{10}\\
& m_{g p} \ddot{y}_{p g}+c_{m}\left(\dot{y}_{p g}-\dot{y}_{p}+\dot{y}_{g}\right)+k_{m}\left(y_{p g}-y_{p}+y_{g}\right) \\
& \quad=c_{m} \dot{e}+k_{m} e-\frac{F_{g} m_{p g}}{\bar{m}_{g}}+\frac{F_{p} m_{p g}}{\bar{m}_{p}},
\end{align*}
$$

where $m_{p g}$ is equivalent mass of the gear pair, $m_{p g}=\bar{m}_{p} \bar{m}_{g} /$ $\left(\bar{m}_{p}+\bar{m}_{g}\right), \bar{m}_{p}=I_{p} / R_{p}^{2}, \bar{m}_{g}=I_{g} / R_{g}^{2}, F_{p}=T_{p} / R_{p}$, and $F_{g}=$ $T_{g} / R_{g}$.

The equation of motion is given in the matrix form as

$$
\begin{equation*}
[M]\{\ddot{X}\}+[C]\{\dot{X}\}+K(t)\{X\}=\{P(t)\} \tag{11}
\end{equation*}
$$

where $M$ is the mass matrix, $C$ is the damping matrix, $K(t)$ is the stiffness matrix, $X$ is the vector of the displacement, and $P(t)$ is the vector of the load. The mass matrix, the damping
matrix, the stiffness matrix, and load vector are given, respectively, as

$$
\begin{gather*}
{[M]=\left[\begin{array}{ccc}
m_{p} & & 0 \\
0 & m_{g} & m_{p g}
\end{array}\right],} \\
{[C]=\left[\begin{array}{ccc}
c_{p y}+c_{m} & -c_{m} & -c_{m} \\
-c_{m} & c_{g y}+c_{m} & c_{m} \\
c_{m} & -c_{m} & -c_{m}
\end{array}\right],} \\
{[K]=\left[\begin{array}{ccc}
k_{p y}+k_{m} & -k_{m} & -k_{m} \\
-k_{m} & k_{g y}+k_{m} & k_{m} \\
k_{m} & -k_{m} & -k_{m}
\end{array}\right],}  \tag{12}\\
-c_{m} \dot{e}-k_{m} e \\
c_{m} \dot{e}+k_{m} e \\
\{P\}=\left\{\begin{array}{cc}
F_{g} m_{p g} & F_{p} m_{p g} \\
-c_{m} \dot{e}-k_{m} e+\frac{\bar{m}_{g}}{\bar{m}_{p}}
\end{array}\right\} .
\end{gather*}
$$

3.4. Dynamic Bearing Force. Equation (11) is solved by using the Newmark time integration method. The Newmark method is a generalization of the linear acceleration method [14]. This latter method assumes that the acceleration varies linearly within the interval $(t+\Delta t)$. This give

$$
\begin{gather*}
\left\{\ddot{x}_{n}\right\}=\left\{\ddot{x}_{t}\right\}+\frac{1}{\Delta t}\left(\left\{\ddot{x}_{t+\Delta t}\right\}-\left\{\ddot{x}_{t}\right\}\right), \\
\left\{\dot{x}_{t+\Delta t}\right\}=\left\{\dot{x}_{t}\right\}+\left[(1-\delta)\left\{\dot{x}_{t}\right\}+\delta\left\{x_{t+\Delta t}\right\}\right] \Delta t, \\
\left\{x_{t+\Delta t}\right\}=\left\{x_{t}\right\}+\left\{\dot{x}_{t}\right\} \Delta t+\left[\left(\frac{1}{2}-\beta\right)\left\{\ddot{x}_{t}\right\}+\beta\left\{x_{t+\Delta t}\right\}\right] \Delta t^{2} . \tag{13}
\end{gather*}
$$

The response at time $t+\Delta t$ is obtained by evaluating the equation of motion at time $t+\Delta t$. The Newmark method is, therefore, an implicit method.

The Newmark method is unconditionally stable provided

$$
\begin{equation*}
\delta \geq 0.5, \quad \beta \geq \frac{1}{4}(\delta+0.5)^{2} \tag{14}
\end{equation*}
$$

One can find that $\delta \geq 0.5$ and $\beta=0.5$ lead to acceptable results for most of problems, $\delta \geq 0.5$ and $\beta=0.5$ are always used in this paper for simplification.

The dynamic bearing force is shown in Figure 5. Dynamic bearing force presents periodic fluctuations and the major components at 4 times, 5 times, and 6 times the mesh frequency $(333 \mathrm{~Hz})$.

## 4. Analysis of Gearbox Vibration and Noise Radiaton

4.1. Gearbox FE-Model. The gear reducer model is shown in Figure 6. In order to predict the noise of the transmission system during operation, vibration of the gearbox must


Figure 5: The dynamic bearing force.
be accurately computed. The finite element model of the realistic character gearbox is built up by using the commercial software ANSYS and shown in Figure 7. The model consisted of 146238 elements and 38634 nodes. The material of the gearbox is cast steel, whose elastic modulus $E=207 \mathrm{GPa}$, Poisson ratio $v=0.3$, and density $\rho=7800 \mathrm{Kg} / \mathrm{m}^{3}$. The bolt holes in the bottom of the gearbox are fixed, due to the gearbox connected with the base through the holes. For the convenience of dynamic load applying, a node is created in the center of the bearing bore; then the center node is coupled with the node on the inside surface of the bearing bore and the dynamic load is applied.
4.2. Gearbox Vibration Modal Analysis. The Lanczos method is used in the modal analysis of the gearbox. Eight modes in the frequency range 0 to 3000 Hz , shown in Figure 8, are chosen to represent the vibration of the gearbox. The vibration of the bottom half gearbox is not as intense as the upper half, because there are bolt constraints and the support of the stiffeners on the bottom of the gearbox.
4.3. Studies of Gearbox Dynamic Response. During the process of dynamic response solution, the dynamic load which


Figure 6: The gear box three-dimensional model.


Figure 7: The gear box FE-model.
is acted on the bearing should be transformed into discrete impact load; then structure response is computed under the impact load step by step, until it achieves steady state.

For the modal superposition method is used in the dynamic response calculation; all the modes which are influential to dynamic response should be calculated; otherwise the result will not be accurate due to the absence of modes. Nearly 200 vibrational modes are used in the calculation; the maximum natural frequency is 20000 Hz .

Figure 9 is the time domain dynamic response signal and corresponding frequency spectra of the signal for the node on the gearbox top surface. Figure 9(a) shows the node dynamic response (displacement) at operating speed of 1000 rpm . Note that the largest amplitude of the response is $1.6 \mu \mathrm{~m}$.

Figure 9(b) shows the frequency components of the response. The major vibration components occur at 3 times, 4 times, and 5 times the mesh frequency $(333 \mathrm{~Hz})$. The very large amplitude in the frequency components at the range of $1550 \mathrm{~Hz} \sim 1700 \mathrm{~Hz}$ is due to the fact that the gearbox forth natural frequency is near the 5 times $(1650 \mathrm{~Hz})$ the mesh frequency, and the mode shape is twisting of the upper half of the gearbox. The fundamental and the 2 times mesh frequency component are substantially smaller due to the lack of any gearbox natural frequencies near 333 Hz and 666 Hz .


Mode 1: frequency, 675.8 Hz


Mode 2: frequency, 1339.8 Hz

Mode 3: frequency, 1519.7 Hz



Mode 5: frequency, 2375.3 Hz


Mode 6: frequency, 2729.2 Hz

Mode 4: frequency, 1662 Hz

nent


Mode 7: frequency, 2783.3 Hz


Mode 8: frequency, 2885.3 Hz

Figure 8: The mode shapes of the gearbox.


Figure 9: The gear box dynamic response.
4.4. Gearbox Frequency Spectrum for Noise. The frequency responses corresponding to the calculated vibration velocities of the surface of the gearbox are inputted into the BEM to analyze the distribution of sound-pressure levels around the gearbox. In order to ensure that the vibration data transmission is correct, both the BE-model and FE-model are meshed in the same way, where the nodes of the two models are mutually corresponding.

Hemispherical sound field is defined outside of the gearbox and is shown in Figure 6. Three representative field points are selected in the sound field; they are located at the top of the gearbox (field point a) and the left and right sides of the gearbox (field points $b$ and $c$ ). The frequency
response of sound pressure level is shown in Figure 10. The noise of the field point at top gearbox is lower than right and left side, due to the fact that most of the mode shapes are swing or torsional vibrations that make normal vibration of the gearbox side plate more violent than top surface. The amplitudes and general shapes of the curves b and $c$ are similar; the frequency components of considerable magnitudes are observed at the mesh frequency of 333 Hz and doubling frequency with three very large components at the 2 times mesh frequency, 4 times mesh frequency, and 5 times mesh frequency; the largest peak value is 50 dB . When the frequency of excitation is larger than 1665 Hz , the sound pressure level is decreased as the frequency increases.


Figure 11: The waterfall of the gearbox dynamic response.


Figure 12: The waterfall of the gearbox noise.

## 5. The Effect of Rotation Speed on the Vibration and Noise

With change of the rotational speed, not only the gear pair meshing state will change, but also the frequency of the various harmonics will change at the same time. In order to research the effect of the rotation speed, the dynamic bearing force, gearbox vibration, and noise radiation are calculated when the rotation speed is within the range of $500 \mathrm{r} / \mathrm{min}$ and $3000 \mathrm{r} / \mathrm{min}$.

### 5.1. The Effect of Rotation Speed on the Gearbox Dynamic

 Response. Increasing the input speed steadily from $500 \mathrm{r} / \mathrm{min}$ to $3000 \mathrm{r} / \mathrm{min}$, a family of vibration and noise spectrum is obtained. Thus, two waterfall diagrams have been created, as shown in Figure 11 ( $f_{g}$ denotes mesh frequency; $f_{b 2}$ and $f_{b 4}$ denote the second and forth nature frequency of the gearbox).The spectral map illustrates how the various harmonics fall along radial lines and can, thus, be separated from


Figure 13: The gear box panel definition. View hole lid (1); roof panel (2); side plates of the upper half of the gearbox $(3,4)$; bearing cover $(5,6,7$, and 8$)$; side plates of the bottom half of the gearbox $(9,10,11$, $12,13$, and 14$)$; front and rear panels $(15,16)$.
the constant frequency components due to excessive amplification by a structural resonance. The excitation consists of harmonic components whose frequency is a multiple of the corresponding gear's rotational speed, so the major components of response fall along radial lines. Meanwhile, the gearbox produces violent vibration near 1664 Hz in different rotational speed, since the 2 times the mesh frequency in $2500 \mathrm{r} / \mathrm{min}, 3$ times the mesh frequency in $1600 \mathrm{r} / \mathrm{min}, 4$ times the mesh frequency in $1250 \mathrm{r} / \mathrm{min}$, and 5 times the mesh frequency in $1000 \mathrm{r} / \mathrm{min}$ are equal to the fourth natural frequency; the same phenomenon occurs near the second and third natural frequency. So it means that the second, third, and fourth natural frequency are sensitive to the dynamic bearing force.
5.2. The Effect of Rotation Speed on the Gearbox Noise Radiation. The gearbox noise spectral map in dB is shown in Figure 12. Note that the frequency components of the gearbox noise spectrum are not intense at low speed, as the rotational speed increases, and noise radiation was gradually strengthened. The distribution of sound pressure and dynamic response are consistent under various speeds; the resonant frequency band is produced at the range of $670 \mathrm{~Hz}, 1300-1700 \mathrm{~Hz}$, and $3000-4000 \mathrm{~Hz}$, which are near the gearbox natural frequency. So in order to reduce the gearbox vibration and noise radiation, the vibration at the resonant frequency band should be reduced during the gearbox design stage.

## 6. Gearbox Improvement

6.1. Gearbox Panel Acoustic Contribution. The vibration and noise of the gearbox are sensitive to the shape and structure of its housing. It is necessary to determine the noise contribution of each panel in the resonance region, which provide forceful basis as the gearbox structure is improved.

In order to quantify the noise proportion of each plate to the whole structure, we introduce the concept of panel acoustic contribution coefficient, which is the ratio of the noise


Figure 14: The gear box panel contribution.


Figure 15: Improved gearbox.
pressure produced by vibration of the panels to the overall noise pressure:

$$
\begin{equation*}
D_{e}=R_{e} \frac{P_{e} P^{*}}{|P|^{2}} \tag{15}
\end{equation*}
$$

where $P^{*}$ is the conjugate complex number of the sound pressure for the point and $R$ is its real part.

If the phase difference between the panel sound pressure and the overall sound pressure is less than $90^{\circ}$, the overall sound pressure will increase with the raise of the panel vibration velocity, and the contribution coefficient is defined as positive; otherwise, it is negative. The radiating noise can be reduced effectively if vibration of the panels whose acoustic contributions are positive and values are large can be reduced.

Each closed surface of the gearbox is defined as a panel, and the part whose radiation area is too small is neglected, such as the area of corner cutting. The whole gearbox outer surface is divided into 16 panels, as shown in Figure 13.

The panel acoustic contribution coefficient is shown in Figure 14, noted that the contributions of the roof panel and the front and rear panels are greater than other panels when the excitation frequency is 1665 Hz . When the excitation frequency is 667 Hz , the contributions of the roof and side plates of the upper half of the gearbox are bigger. Further analysis indicates that the noise is mainly caused by the panels of the upper half of the gearbox. Reducing vibrant intensity of

Table 2: The gear box natural frequencies (Hz).

| Mode steps | Original model | Plan 1 | Plan 2 |
| :--- | :---: | :---: | :---: |
| 1st | 716.5 | 791.2 | 770.7 |
| 2nd | 1518.8 | 1684.4 | 1631.8 |
| 3rd | 1682.3 | 1847.2 | 1800.1 |
| 4th | 1843.7 | 2138.1 | 2001.4 |
| 5th | 2459 | 2887 | 2765.3 |
| 6th | 2881.6 | 3178.4 | 3010.8 |
| 7th | 2887.3 | 3431.6 | 3259.4 |
| 8th | 3043.9 | 3434.3 | 3276.9 |

the panels $2,3,4,15$, and 16 is important to noise control of the gear reducer.
6.2. Gearbox Improvement. In order to reduce the intensity of vibration of the upper half of the gearbox and make the gearbox natural frequency avoid the 2 times and 5 times the mesh frequency, two low-noise design plans are proposed. The first plan increases the thickness of the side plates of the bottom half of the gearbox with 4 mm . Another one, the gearbox stiffness was strengthened by using ribs on the side plates, as Figure 15 shown. The natural frequency is shown in Table 2.


Figure 16: The frequency spectrum for noise.

The frequency-noise spectra of the gearbox pre- and postimprovement are shown in Figure 16, where curves a, b, and c represent the distribution of sound pressure level for the original model and the improved models, respectively. As can be seen, the differences in mesh frequency doubling are considerable. The noise of improved gearbox is reduced obviously when the extinction frequency is below 1700 Hz . The sound pressure level is reduced about 12 dB at 665 Hz and about 9 dB at 1332 Hz in magnitude.

With comparative analysis of the two low-noise design plans, the effect of increasing gearbox thickness and ribs on reducing noise and vibration is almost the same, but plan 1 will increase more weight and take up more interior room, so it is more realizable to provide stiffening ribs on the gearbox.

## 7. Conclusions and Summary

A procedure for predicting the vibration and noise of gear reducer is developed, in which both gear transmission system dynamic characteristics and gearbox dynamic characteristics are considered. The dynamic bearing force is taken as the excitation; the gearbox vibrations and noise radiating are calculated by using FEM/BEM. The resonant frequency band of the gearbox is obtained. Then the low-noise gearbox was designed based on the result of modal analysis and acoustic panel contribution analysis. It is available to reduce the noise radiation of the gearbox through increasing the structural stiffness of the gearbox and reducing the vibration of the panels whose acoustic contribution coefficients are positive and values are large.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this article.

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# An Analytical Tire Model with Flexible Carcass for Combined Slips 

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#### Abstract

The tire mechanical characteristics under combined cornering and braking/driving situations have significant effects on vehicle directional controls. The objective of this paper is to present an analytical tire model with flexible carcass for combined slip situations, which can describe tire behavior well and can also be used for studying vehicle dynamics. The tire forces and moments come mainly from the shear stress and sliding friction at the tread-road interface. In order to describe complicated tire characteristics and tire-road friction, some key factors are considered in this model: arbitrary pressure distribution; translational, bending, and twisting compliance of the carcass; dynamic friction coefficient; anisotropic stiffness properties. The analytical tire model can describe tire forces and moments accurately under combined slip conditions. Some important properties induced by flexible carcass can also be reflected. The structural parameters of a tire can be identified from tire measurements and the computational results using the analytical model show good agreement with test data.


## 1. Introduction

Tires are the only components of a road vehicle to directly contact with the road surface, and the forces and moments generated in the contact patch have significant effects on the vehicle performance [1-4]. Hence a huge number of tire models have been proposed for use in calculating the forces and moments at the tire-road interface [5-13].

The empirical models are often employed for the vehicle simulation and control, which rely basically on curve-fitted experimental data and can provide a good representation of experimental data for longitudinal force, lateral force, and aligning moment. However, empirical models would become quite complicated without theoretical support while considering all kinds of operating variables, such as slip angle, inclination angle, slip ratio, vertical load, inflation pressure, road friction, and rolling speed. The analytical models, which usually include carcass model (beam, string or rigid) with elastic tread elements, such as Brush model, Fiala model and String model, establish the relationship between tire structure
parameters and tire behavior [3]. So it can be employed to derive some useful qualitative conclusions for understanding tire properties, but most of these models are either relatively simple or more complicated, which limit their practical use. In this paper, we do not see the carcass as an actual beam or string, and the carcass deformation is described with relatively simple and general forms, composed by translational, bending, and twisting deformations. Besides, the arbitrary pressure distribution, dynamic friction coefficient, and anisotropic stiffness properties are also considered. Consequently, the analytical model would become more suitable for application and also appropriate for analyzing tire properties in detail.

In this paper, the key factors for developing the analytical tire model are firstly discussed; then, considering all these factors, the analytical tire model with flexible carcass for combined slips is introduced. By employing the model, effects of carcass compliance on tire properties are discussed, which are valuable for understanding tire properties; at the end, the analytical model is validated by test data.


Figure 1: Tire axis system.

## 2. Tire Axis System and Slip Ratios

The tire axis system is shown in Figure 1. The positive direction of the $X$-axis and the $Y$-axis is coincident with tire revolution direction (not wheel traveling direction); the $Z$ axis is perpendicular to road plane and upward. The wheel traveling speed is denoted as $V, \alpha$ is the slip angle, and $\gamma$ is inclination angle. The figure shows all the forces and moments associated with a wheel.

The longitudinal and lateral slip ratios are defined in the unified form [11]

$$
\begin{gather*}
S_{x}=\frac{-V_{s x}}{\Omega R_{e}}=-\frac{V \cos \alpha-\Omega R_{e}}{\Omega R_{e}} \quad S_{x} \in(-\infty,+\infty), \\
S_{y}=\frac{-V_{s y}}{\Omega R_{e}}=-\frac{V \sin \alpha}{\Omega R_{e}} \quad S_{y} \in(-\infty,+\infty), \tag{1}
\end{gather*}
$$

where $\Omega$ is the angular speed, $R_{e}$ is the effective rolling radius and $V_{s x}$ and $V_{s y}$ are the longitudinal and lateral sliding speeds of tire with respect to road surface.

Usually, the longitudinal slip ratio used in tire force test is defined as

$$
\begin{equation*}
\kappa=\frac{-V_{s x}}{V \cos \alpha} . \tag{2}
\end{equation*}
$$

The relationship between $S_{x}$ and $\kappa$ can be obtained easily:

$$
\begin{equation*}
S_{x}=\frac{\kappa}{1+\kappa} \tag{3}
\end{equation*}
$$

## 3. Key Factors for Analytical Tire Model

3.1. Arbitrary Pressure Distribution. Contact pressure distribution over contact patch strongly influences tire behaviors. Here, the contact patch is assumed to be rectangular in shape, and the contact pressure distribution is assumed to be uniform in lateral direction and of arbitrary form in circumferential direction to represent different kinds of pressure distributions.

The contact pressure $q_{z}$ along the contact patch length $2 a$ is expressed as an arbitrary form as

$$
\begin{equation*}
q_{z}(x)=\frac{F_{z}}{2 a} \cdot \eta(u) \tag{4}
\end{equation*}
$$

where $F_{z}$ is the tire vertical load; $u=x / a$ is the relative coordinate. $\eta(u)$ is the normalized pressure distribution function and is expressed as

$$
\begin{gather*}
\eta(u)=A \cdot\left(1-u^{2 n}\right) \cdot\left(1+\lambda \cdot u^{2 n}\right) \cdot(1-B \cdot u) \\
A=\frac{(2 n+1)(4 n+1)}{2 n(4 n+1+\lambda)}  \tag{5}\\
B=-\frac{3(2 n+3)(4 n+3)(4 n+1+\lambda)}{(2 n+1)(4 n+1)(4 n+3+3 \lambda)} \cdot \frac{\Delta}{a}
\end{gather*}
$$

where $n, \lambda$, and $\Delta$ are the parameters which determine the shape of pressure distribution. With these three parameters, (5) can be employed to express arbitrary pressure distribution over contact patch, as shown in Figure 2.
3.2. Carcass Structure Parameters. The deformation of carcass has an important effect on the tire properties under combined slip conditions. In this paper, three tire carcass stiffness parameters for tire lateral carcass deformation are introduced, that is, tire carcass lateral translation stiffness $K_{c y}$, carcass bending stiffness $K_{c b}$, and carcass twisting stiffness $N_{\theta}$. So, the carcass deformation includes lateral translating part $y_{c 0}$, bending part $y_{c b}$, and twisting part $y_{\theta}$, as shown in Figure 3. The carcass longitudinal deformation is assumed to include longitudinal translating part $x_{c 0}$, and the longitudinal translation stiffness is $K_{c x}$.

The lateral translating deformation of carcass can be calculated as

$$
\begin{equation*}
y_{c 0}=\frac{F_{y}}{K_{c y 0}} \tag{6}
\end{equation*}
$$

where $K_{c y}$ is the lateral translation stiffness of carcass; $F_{y}$ is the lateral force.

The lateral bending deformation of carcass can be calculated as

$$
\begin{equation*}
y_{c b}(x)=\frac{F_{y}}{K_{c b}} \cdot \xi\left(\frac{x}{a}\right) \tag{7}
\end{equation*}
$$

where $K_{c b}$ is the carcass bending stiffness; $\xi(x / a)$ is the general function of carcass bending deformation.

The zero-order moment and first-order moment of $\xi(u)$ are expressed as

$$
\begin{gather*}
D_{0}(u)=\int_{u}^{1} \xi(u) d u \\
D_{1}(u)=\int_{u}^{1} u \cdot \xi(u) d u . \tag{8}
\end{gather*}
$$

The twisting deformation of carcass is calculated by

$$
\begin{gather*}
y_{\theta}(x)=\theta \cdot x \\
\theta=\frac{M_{z}}{N_{\theta}} \tag{9}
\end{gather*}
$$




$$
\lambda=1
$$

$$
---n=3
$$


(b)

Figure 2: Pressure distribution.


Figure 3: Tire lateral carcass deformation.
where $\theta$ is the carcass twisting angle; $N_{\theta}$ is the carcass twisting stiffness; $M_{z}$ is the aligning moment.

The longitudinal translating deformation of carcass is expressed as

$$
\begin{equation*}
x_{c 0}=\frac{F_{x}}{K_{c x 0}} \tag{10}
\end{equation*}
$$

where $K_{c x 0}$ is the longitudinal translation stiffness of carcass; $F_{x}$ is the longitudinal force.
3.3. Dynamic Friction Coefficient. Friction coefficient used in the analytical tire model is the dynamic friction coefficient, which considers the significant influence of slip speed. The expression is as follows [14]:

$$
\begin{align*}
& \mu_{d} \\
& =\mu_{s}+\left(\mu_{m}-\mu_{s}\right)  \tag{11}\\
& \quad \cdot \exp \left(-\mu_{h}^{2} \cdot \log ^{2}\left(\left|\frac{V_{s}}{V_{s m}}\right|+N \cdot \exp \left(-\left|\frac{V_{s}}{V_{s m}}\right|\right)\right)\right),
\end{align*}
$$

where $\mu_{0}, \mu_{s}, \mu_{h}$, and $V_{s m}$ are friction characteristic parameters; $N$ (usually $N=0.8$ ) is a factor to make the friction coefficient increase slightly around the origin; $V_{s}$ is the slip speed between the road and tire.
3.4. Anisotropic Stiffness Properties. The anisotropy of tire slip stiffness will arouse the difference of tire shear stress direction in adhesion region and sliding region, which are expressed as [15]

$$
\begin{gather*}
\text { adhesion : } \theta_{\mathrm{ad}}=\arctan \left(\frac{\left(K_{y} S_{y}\right)}{\left(K_{x} S_{x}\right)}\right)  \tag{12}\\
\text { sliding : } \theta_{s}=\arctan \left(\frac{S_{y}}{S_{x}}\right)
\end{gather*}
$$

where $K_{x}$ and $K_{y}$ represent the longitudinal slip and cornering stiffness, respectively.

## 4. Analytical Tire Forces and Moments Model

4.1. Tire Forces and Moments without Sliding. In order to obtain the shear force in the contact patch, the deformations of the tread and carcass along the $X$ and $Y$ axes must be known firstly. The longitudinal and lateral deformations of the tread and carcass, under combined slip condition, are shown in Figure 4. In this figure, XOY is a coordinate system before the carcass is deformed, and xoy is a relative coordinate system for describing the tread deformation and bending and twisting deformation of carcass. $x_{c 0}$ and $y_{c 0}$ are the longitudinal and lateral translating deformation of carcass. "ABC" is the contact line of the contact patch under combined slip condition. $V$ is the wheel traveling speed; $\alpha$ is the slip angle and $\theta$ is the carcass twisting angle. In general case, the whole length of contact patch, $2 a$, is divided into two parts, the adhesion region " $A B$ " and the sliding region " $B C$," by the initial sliding point "B."


Figure 4: Deformation of carcass and tread element under combined slip condition.
$P_{c} P_{t}$ in the figure represents the tread element after rolling for a period of time $t$. The upper point of tread element, $P_{c}$, is attached to the belt of tire. Its coordinate could be written as

$$
\begin{gather*}
x_{p c}=x \\
y_{p c}=y_{\theta}(x)+y_{c b}(x) . \tag{13}
\end{gather*}
$$

The lower point of tread element, $P_{t}$, is contacted with the ground. Its coordinate could be written as

$$
\begin{gather*}
x_{p t}=x-\left(V t \cos \alpha-V_{r} t\right)=x+(a-x) S_{x},  \tag{14}\\
y_{p t}=-V t \sin \alpha+a \theta=a \theta+(a-x) S_{y} .
\end{gather*}
$$

Therefore, the longitudinal and lateral deformations of tread element is

$$
\begin{gather*}
\Delta x=x_{p t}-x_{p c}=(a-x) S_{x} \\
\Delta y=y_{p t}-y_{p c}=(a-x)\left(\frac{M_{z}}{N_{\theta}}+S_{y}\right)-\frac{F_{y}}{K_{c b}} \xi\left(\frac{x}{a}\right) . \tag{15}
\end{gather*}
$$

The main source of anisotropy is due to different tire structural flexibility in lateral and longitudinal direction, whereas tread anisotropy is present but comparatively small. In this paper, the stiffness of tread, denoted as $k_{t}$, is considered to be isotropic. The shear stresses of tread element in $X$ and $Y$ directions can be expressed as

$$
\begin{gather*}
q_{x}=k_{t} \cdot \Delta x=k_{t} \cdot(a-x) S_{x} \\
q_{y}=k_{t} \cdot \Delta y=k_{t} \cdot\left[(a-x)\left(\frac{M_{z}}{N_{\theta}}+S_{y}\right)-\frac{F_{y}}{K_{c b}} \xi\left(\frac{x}{a}\right)\right] . \tag{16}
\end{gather*}
$$

The shear force in the directions $X$ and $Y$ can be determined as follows:

$$
\begin{gather*}
F_{x}=\int_{-a}^{a} q_{x} d x \\
F_{y}=\int_{-a}^{a} q_{y} d x \\
M_{z}=-\int_{-a}^{a} q_{x} \cdot y_{p c} d x+\int_{-a}^{a} q_{y} \cdot x_{p c} d x-F_{x} \cdot y_{c 0}+F_{y} \cdot x_{c 0} . \tag{17}
\end{gather*}
$$

Considering the previous equations (6)~(10), (13), and (16), the forces and moments could be written as

$$
\begin{gather*}
F_{x}=2 a^{2} k_{t} S_{x}, \\
F_{y}=\frac{3}{a} \varepsilon_{\theta} M_{z}+2 a^{2} k_{t} S_{y}-\varepsilon_{b} F_{y}, \\
M_{z}=-a \varepsilon_{b} S_{x} F_{y}+\varepsilon_{\theta} S_{x} M_{z}+\frac{1}{2} a \varepsilon_{b} S_{x} F_{y} D_{1}(-1)-\varepsilon_{\theta} M_{z} \\
-\frac{2}{3} a^{3} k_{t} S_{y}-\frac{1}{2} a \varepsilon_{b} F_{y} D_{1}(-1)+\left(\frac{1}{K_{c x 0}}-\frac{1}{K_{c y 0}}\right) F_{x} F_{y}, \tag{18}
\end{gather*}
$$

where the bending characteristic ratio and twisting characteristic ratio have been introduced and defined by

$$
\begin{align*}
& \varepsilon_{b}=\frac{2 a k_{t}}{K_{c b}} \\
& \varepsilon_{\theta}=\frac{2}{3} \frac{a^{3} k_{t}}{N_{\theta}} \tag{19}
\end{align*}
$$

Solving (18) to obtain the explicit expression of $F_{y}$ and $M_{z}$, (18) becomes

$$
\begin{align*}
& F_{x}=K_{x} S_{x} \\
& F_{y}=K_{y} S_{y}  \tag{20}\\
& M_{z}=K_{m} S_{y}
\end{align*}
$$

with

$$
\begin{gather*}
K_{x}=2 a^{2} k_{t} \\
K_{y}=\frac{a\left(1+\varepsilon_{\theta}-\varepsilon_{\theta} S_{x}\right) K_{y 0}-3 \varepsilon_{\theta} K_{m 0}}{a\left(1+\varepsilon_{b}\right)\left(1+\varepsilon_{\theta}-\varepsilon_{\theta} S_{x}\right)+3 a \varepsilon_{b} \varepsilon_{\theta} S_{x}-3 \varepsilon_{\theta} c F_{x}}  \tag{21}\\
K_{m}=\frac{a}{3 \varepsilon_{\theta}}\left[\left(1+\varepsilon_{b}\right) K_{y}-K_{y 0}\right]
\end{gather*}
$$

where $K_{x}$ is the longitudinal slip stiffness, $K_{y}$ is the cornering stiffness, and $K_{m}$ is the aligning stiffness with flexible carcass. $K_{y 0}$ and $K_{m 0}$ are the cornering stiffness and aligning stiffness, respectively, when the carcass is assumed to be rigid, which are expressed as

$$
\begin{equation*}
K_{y 0}=2 a^{2} k_{t}, \quad K_{m 0}=\frac{2}{3} a^{3} k_{t} \tag{22}
\end{equation*}
$$

where $c$ in (21) is the translating compliance coefficient that is defined by

$$
\begin{equation*}
c=\frac{1}{K_{c x 0}}-\frac{1}{K_{c y 0}} . \tag{23}
\end{equation*}
$$

4.2. Tire Forces and Moments in General Case with Sliding Region. Considering that the sliding region might exist, the shear force in this area should be expressed as $q(x)=\mu q_{z}(x)$, based on which the coordinate of initial sliding point, $x=x_{c}$, or the relative coordinate $u_{c}=x_{c} / a$, can be solved. The coordinate of initial sliding point will satisfy the following equation:

$$
\begin{equation*}
q=\sqrt{q_{x}^{2}+q_{y}^{2}}=\frac{\mu F_{z}}{2 a} \cdot \eta\left(u_{c}\right) \tag{24}
\end{equation*}
$$

Integrating through both adhesion and sliding regions, the tire forces and moments can be derived as

$$
\begin{aligned}
F_{x}= & \int_{x_{c}}^{a} k_{t} \cdot \Delta x d x+\int_{-a}^{x_{c}} \frac{\mu F_{z}}{2 a} \cdot \eta\left(\frac{x}{a}\right) \cdot \theta_{s x} d x \\
F_{y}= & \int_{x_{c}}^{a} k_{t} \cdot \Delta y d x+\int_{-a}^{x_{c}} \frac{\mu F_{z}}{2 a} \cdot \eta\left(\frac{x}{a}\right) \cdot \theta_{s y} d x \\
M_{z}= & -\int_{x_{c}}^{a} k_{t} \cdot \Delta x \cdot y_{p c} d x+\int_{x_{c}}^{a} k_{t} \cdot \Delta y \cdot x_{p c} d x \\
& -\int_{-a}^{x_{c}} \frac{\mu F_{z}}{2 a} \cdot \eta\left(\frac{x}{a}\right) \cdot \theta_{s x} \cdot y_{p c} d x \\
& +\int_{-a}^{x_{c}} \frac{\mu F_{z}}{2 a} \cdot \eta\left(\frac{x}{a}\right) \cdot \theta_{s y} \cdot x_{p c} d x \\
& -F_{x} \cdot y_{c 0}+F_{y} \cdot x_{c 0}
\end{aligned}
$$

where $\theta_{s x}$ and $\theta_{s y}$ are, respectively, the longitudinal and lateral components of shear stress direction $\theta_{s}$ in sliding region.

Substituted with (6)~(12), (13), and (15), (25) becomes

$$
\begin{align*}
& F_{x}= B_{6} \cdot K_{x} S_{x}+B_{3} \cdot \mu F_{z} \theta_{s x} \\
& F_{y}= {\left[\left(P_{1} B_{6} \cdot K_{y 0}+P_{4} B_{7} \cdot K_{m 0}\right) S_{y}\right.} \\
&\left.+\left(P_{1} B_{3}+P_{4} B_{4} a\right) \mu F_{z} \theta_{s y}\right] \\
& \times\left(P_{2} P_{4}+P_{1} P_{3}\right)^{-1}, \\
& M_{z}=\left[\left(-P_{2} B_{6} \cdot K_{y 0}+P_{3} B_{7} \cdot K_{m 0}\right) S_{y}\right. \\
&\left.+\left(-P_{2} B_{3}+P_{3} B_{4} a\right) \mu F_{z} \theta_{s y}\right] \\
& \times\left(P_{2} P_{4}+P_{1} P_{3}\right)^{-1} \tag{26}
\end{align*}
$$

with

$$
\begin{gather*}
B_{1}=\frac{1}{2} \varepsilon_{b} D_{0}\left(u_{c}\right), \quad B_{2}=\frac{1}{2} \varepsilon_{b} D_{1}\left(u_{c}\right), \\
B_{3}=\frac{1}{2} m_{0}\left(u_{c}\right), \\
B_{4}=\frac{1}{2} m_{1}\left(u_{c}\right), \quad B_{5}=\frac{1}{2} m_{D}\left(u_{c}\right), \\
B_{6}=\frac{\left(1-u_{c}\right)^{2}}{4}, \quad B_{7}=\frac{1}{4}\left(1-3 u_{c}^{2}+2 u_{c}^{3}\right),  \tag{27}\\
P_{1}=1-B_{7} \varepsilon_{\theta}+B_{7} \varepsilon_{\theta} S_{x}+B_{4} \mu F_{z} \theta_{s x} a \frac{1}{N_{\theta}}, \\
P_{2}=\left(B_{2}+B_{1} S_{x}-B_{2} S_{x}\right) a+B_{5} \mu F_{z} \theta_{s x} \frac{1}{K_{c b}}-c F_{x}, \\
P_{3}=1+B_{1}, \quad P_{4}=\frac{3}{a} B_{6} \varepsilon_{\theta},
\end{gather*}
$$

where $m_{0}\left(u_{c}\right), m_{1}\left(u_{c}\right)$, and $m_{D}\left(u_{c}\right)$ are defined by

$$
\begin{gather*}
m_{0}\left(u_{c}\right)=\int_{-1}^{u_{c}} \eta(u) d u, \quad m_{1}\left(u_{c}\right)=\int_{-1}^{u_{c}} u \eta(u) d u, \\
m_{D}\left(u_{c}\right)=\int_{-1}^{u_{c}} \xi(u) \eta(u) d u . \tag{28}
\end{gather*}
$$

## 5. Simulation Analysis and Experiment Validation

According to the analytical tire model, the tire forces and moments under combined slip conditions can be simulated and the effects of carcass structure parameters can be analyzed. Furthermore, the analytical tire model can also be used in vehicle dynamics simulation by identifying the model parameters with test data.


Figure 5: Variation of cornering stiffness under combined slip condition.
5.1. Effect of Carcass Flexible on Tire Cornering Stiffness under Combined Slip Condition. First of all, the cornering stiffness under pure side slip condition could be calculated by assuming $S_{x}=0$ in (21),

$$
\begin{equation*}
K_{y \text { pure }}=K_{y 0} \cdot \frac{1}{\left(1+\varepsilon_{b}\right)\left(1+\varepsilon_{\theta}\right)} \tag{29}
\end{equation*}
$$

It can be seen that the bending characteristic ratio $\varepsilon_{b}$ and the twisting characteristic ratio $\varepsilon_{\theta}$ have important influence on cornering stiffness. With the increase of $\varepsilon_{b}$ or $\varepsilon_{\theta}$, the cornering stiffness will decrease obviously.

Then, the variation of cornering stiffness under combined slip condition is also our concern, which often has important influence on tire characteristics. Divided by $K_{y p u r e}$, the normalized cornering stiffness could be derived as

$$
\frac{K_{y}}{K_{y \text { pure }}}=\frac{\left(1-\varepsilon_{\theta} S_{x}\right)\left(1+\varepsilon_{b}\right)\left(1+\varepsilon_{\theta}\right)}{\left(1+\varepsilon_{b}\right)\left(1+\varepsilon_{\theta}-\varepsilon_{\theta} S_{x}\right)+3 \varepsilon_{b} \varepsilon_{\theta} S_{x}-(3 / a) \cdot \varepsilon_{\theta} c F_{x}} .
$$

It can be seen that the normalized cornering stiffness is a function of $S_{x}$, which means that the longitudinal slip ratio (or longitudinal force) will have influence on the tire cornering stiffness. Figure 5 shows the relationship between the normalized cornering stiffness and longitudinal slip ratio. It indicates that the cornering stiffness will increase with the action of braking force does not and decrease when acting driving force.

### 5.2. Effect of Carcass Compliance on Aligning Moment under

 Combined Slip Condition. The carcass translating compliance has a significant influence on aligning moment for a braked or driven wheel. In Figure 6, $c$ is the translating compliance coefficient defined in (23). As shown in Figure 6, the curve becomes more and more asymmetric when $c$ increases (absolute value) and $M_{z}$ changes its sign in the braking half of

Figure 6: Influence of carcass compliance on aligning moment under combined slip condition.

$-F_{z}=3000 \mathrm{~N}$
Figure 7: Longitudinal force characteristics.
the diagram. These phenomena correspond reasonably well with the experimental results given in the previous scholars' researches [2].

### 5.3. Simulation Results of Tire Forces and Moments under

 Combined Slip Condition. In this part, a number of example results of using the analytical tire model have been presented, as shown in Figures 7-11. Some important characteristics have been reflected.(1) Dynamic friction coefficient: Figure 7 shows the longitudinal force under combined slip conditions. With the increase of slip angle $\alpha$, the longitudinal force will decrease because of the limitation of friction


Figure 8: Lateral force characteristics.


Figure 9: $F_{y}$ versus $F_{x}$ characteristics.
force between the tire and the road. More important, because we adopt the dynamic friction coefficient as shown in (11), the sliding velocity dependent friction coefficient can be seen obviously in Figure 7.
(2) The variation of cornering stiffness: Figure 8 shows the lateral force for combined slip and Figure 9 shows the $F_{y}-F_{x}$ characteristics, which indicate the decrease of lateral force with the increase of longitudinal slip ratio. Besides, it can be seen that the lateral force has an increase to some extent when the wheel is braked slightly, which is mainly due to the increase of cornering stiffness.
(3) The asymmetry of aligning moment: Figure 10 shows the aligning moment for combined slip. It is obvious


Figure 10: Aligning moment characteristics ( $M_{z}$ versus $F_{x}$ ).


Figure 11: Resultant force direction.
that the curve is asymmetric and $M_{z}$ changes its sign in the braking half of the diagram.
(4) Anisotropic stiffness properties: the anisotropy of tire slip stiffness will cause the variation of resultant force direction under different combined slip conditions. Figure 11 shows the relationship between the resultant force direction and the slip direction. It can be seen that the resultant force direction changes gradually from adhesion direction to slip direction, which is reasonably and coincides with the test data provided in [15].

Some simulated results using the Brush model are shown in Figures 12 and 13. Figure 12 is the $F_{y}-F_{x}$ characteristics with different slip angles under combined slip conditions, and


Figure 12: $F_{y}$ versus $F_{x}$ characteristics (Brush model).


Figure 13: $M_{z}$ versus $F_{x}$ characteristics (Brush model).

Figure 13 shows the $M_{z}-F_{x}$ characteristics. Comparing the results of analytical tire model with flexible carcass proposed in this paper, it is obvious that the Brush model have the capacity to express the declining friction coefficient and the influence of carcass flexibility, which will lead to severe deviation for tire characteristics, especially for the aligning moments. The dynamic friction coefficient will cause the curve of $F_{y}-F_{x}$ characteristics to turn inward, as shown in Figure 9. The flexible carcass will make the curve of $M_{z}-F_{x}$ characteristics asymmetric, as shown in Figure 10.
5.4. Experiment Validation. Test data for the P245/65R17 specification at different slip angles are used to validate the analytical tire model. Figure 14 shows the tire shear forces under combined slip conditions. The vertical load is 4000 N


Figure 14: The comparison between the analytical tire model and test data.
and both the longitudinal and lateral forces are tested when the tire is rolling under combined tire cornering and braking conditions. From Figure 14, it can be seen that the analytical tire model can describe the tire characteristics accurately under combined slip conditions.

## 6. Conclusions

This paper presented an analytical tire model with flexible carcass for combined slips. With employing the model, the effects of carcass structure parameters on tire properties are discussed. The simulation results and experiment validation of longitudinal forces, lateral forces, and aligning moments under combined slip conditions are also provided. Some conclusions are summarized as follows.

Firstly, arbitrary pressure distribution, translational, bending, and twisting compliance of the carcass, dynamic friction coefficient, and anisotropic stiffness properties are the key factors for developing the analytical tire model.

Secondly, the carcass compliance has a great influence on tire cornering stiffness. With the increase of bending characteristic ratio $\varepsilon_{b}$ or twisting characteristic ratio $\varepsilon_{\theta}$, the cornering stiffness will decrease obviously. Moreover, the cornering stiffness is also influenced by the braking force or driving force and thus leads to the increase of lateral force when tire has a slight braking.

Thirdly, the carcass translating compliance has a significant influence on aligning moment under combined slip conditions. The curve of $M_{z}$ becomes more and more asymmetric when translating compliance coefficient $c$ increases (absolute value) and $M_{z}$ changes its sign in the braking half of the diagram.

Fourthly, the dynamic friction coefficient can express the friction delay with sliding velocity effectively; the anisotropic stiffness properties can express the resultant force direction
reasonably, which changes gradually from adhesion direction to slip direction.

Finally, considering all these key factors, the analytical tire model is capable of describing all kinds of tire properties reasonably and accurately. The model parameters can also be identified from tire measurements and the computational results using the analytical model show good agreement with test data.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Integral Terminal Sliding Mode Control for a Class of Nonaffine Nonlinear Systems with Uncertainty 

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#### Abstract

This paper is concerned with an integral terminal sliding mode tracking control for a class of uncertain nonaffine nonlinear systems. Firstly, the nonaffine nonlinear systems is approximated to facilitate the desired control design via a novel dynamic modeling technique. Next, for the unmeasured disturbance of nonlinear systems, integral terminal sliding mode disturbance observer is presented. The developed disturbance observer can guarantee the disturbance approximation error to converge to zero in the finite time. Subsequently, based on approximated nonlinear model and the designed disturbance observer, the integral terminal sliding mode tracking control is presented for nonaffine nonlinear systems with uncertainty. Different from traditional terminal slidingmode control, this paper accomplishes finite convergence time for nonaffine nonlinear systems and avoids the singular problem in the controller design. Furthermore, the control system is forced to start on the terminal sliding hyperplane, so that the reaching time of the sliding modes is eliminated. Finally, two numerical simulation results are given to illustrate the effectiveness of the proposed method.


## 1. Introduction

In recent years, there has been significant progress in the area of designing controllers for nonlinear systems [1-10]. Most of the controllers developed in this context have the common assumption that the system to be controlled is affine; that is, the plant is linear according to the input variables, and the nonlinearities are linearly parameterized by unknown parameters. However, developing a systematic synthesis method for nonaffine nonlinear systems is still a challenging problem for now.

Sliding-mode control (SMC) is a well-known efficient control scheme which has been widely applied for both linear and nonlinear systems [ $8,10-16]$. This control is also considered as an effective approach for control of the systems with uncertainties. However, the main disadvantage of SMC scheme is that the system states cannot reach the equilibrium point in finite time. So a new control scheme called terminal sliding-mode control (TSMC) is proposed to overcome this drawback utilizing nonlinear sliding surface. Nonlinear switching hyperplanes in TSMC can improve the transient performance substantially. Besides, compared
with the conventional SMC with linear sliding manifold, TSMC offers some superior properties such as faster, finite time convergence and higher control precision. However, there exists an intrinsic singular problem in TSMC due to using fractional power functions as the sliding hyperplane [17-23]. Thus, overcoming the singular problem becomes a considerable topic for TSMC. Unfortunately, most of nonsingular TSMC methods [22,23] are available only for affine nonlinear systems, especially robotic manipulators received the most attention. Nevertheless, these nonsingular TSMC schemes lack strict theoretical analysis and are only suitable for affine nonlinear systems. It is a worthwhile note that only a few researches discuss the control of nonaffine nonlinear systems even if allowing singularity in the TSMC. Therefore, nonsingular TSMC of nonaffine nonlinear systems needs to be investigated by a new technique.

The uncertainty is inherent in practical systems. Designing controller capability of handing uncertainty is of practical interest and is academically challenging. Neural networks (NNs) have been proposed recently as an adaptive controller for nonlinear systems. By the use of their universal approximation capability, the adaptive controller based on
neural networks can be designed without significant prior knowledge of the system dynamics [24-28]. Although it is widely accepted that the parameterized neural network is capable of approximating linear or nonlinear mapping by adequately choosing network structures and training methods, a challenging problem for designers is to select an appropriate structure for balancing the number of rules and the approximation accuracy. If the network size is chosen too small, it is impossible to assure the approximation to converge to an acceptable level due to the limited nodes. On the other hand, if overdetermined nodes are given, the computational burden is huge and the waste of computation resource implies its impracticality for real-world applications.

Based on the above works, this paper is to develop an integral terminal sliding mode control approach for a class of nonaffine nonlinear systems with uncertainty, parameters perturbation, and external disturbances. The organization of this paper is as follows. Following the introduction, the problem formulation is described briefly, some assumptions which will play a basic role in our analysis are introduced in Section 2. To facilitate the desired control design, a novel dynamic modeling technique has been proposed for the nonaffine nonlinear systems in Section 3. The integral terminal sliding mode disturbance observer is presented in Section 4. Section 5 proposes integral terminal sliding mode control based on disturbance observer, and then integral terminal sliding mode control is designed for uncertain nonlinear systems with control singularity. Simulation studies are shown in Section 6 to demonstrate the effectiveness of our proposed approaches. Finally, conclusions are drawn in Section 7.

## 2. Problem Formulation

Consider a class of uncertain nonaffine nonlinear systems that can be expressed in the following form:

$$
\begin{gather*}
\dot{x}(t)=f_{1}(x(t), u(t))+\Delta f_{1}(x(t), u(t))+d(t),  \tag{1}\\
y(t)=x(t)
\end{gather*}
$$

where $x(t)=\left[x_{1}(t), \ldots, x_{n}(t)\right] \in \mathbb{R}^{n}$ is the state vector of the system in the normal form which is assumed available for measurement, $y(t) \in \mathbb{R}^{n}$ is the output vector, $u(t) \in \mathbb{R}^{n}$ is control input vector, and $f_{1}(x(t), u(t)) \in \mathbb{R}^{n}$ is known smooth vector fields. $\Delta f_{1}(x(t), u(t))$ is assumed to be continuous of $x(t)$ denoting the the system uncertainty, which contains structural and modeling error. $d(t) \in \mathbb{R}^{n}$ is external disturbance.

After combining the uncertainty and disturbance together, the nonlinear system (1) can be rewritten as

$$
\begin{gather*}
\dot{x}(t)=f_{1}(x(t), u(t))+D(x(t), u(t), t),  \tag{2}\\
y(t)=x(t)
\end{gather*}
$$

where $D(x, u, t)=\Delta f_{1}(x(t), u(t))+d(t)$ is called compound disturbance.

To achieve the proposed control objective, the following assumptions are required.

Assumption 1. There exist known positive constants $\check{\epsilon}$ such that for all $t \in R^{+},\|D(x, u, t)\| \leqslant \check{\epsilon}$.

Assumption 2. Consider $\|\Delta u\| \in[0, \delta]$ and $0 \leqslant\|\partial f / \partial u\| \leqslant \check{\delta}$, where $\delta$ and $\check{\delta}$ are two finite positive constants.

Remark 3. In many actual process control systems and flight control systems, $\|\Delta u(t)\| \in[0, \delta]$ is a physical restriction of many practical systems because their states and outputs (actuators) cannot change too fast because of system "inertia." So, Assumption 2 is reasonable.

In this paper, the control objective is to design the disturbance-observer-based integral terminal sliding mode tracking control and make the system output follow a given desired output of the nonlinear system in the presence of uncertainty and external disturbance. For the desired tracking signal $y_{d}$, the proposed integral terminal sliding mode control must ensure that all closed-loop signals are convergent in the finite time.

## 3. Novel Nonaffine Nonlinear Approximation

The problem of controlling the plants characterized by models that are nonaffine in the control input vector is a difficult one. Especially for the tracking control, the liberalization may result in the design of sufficiently accurate controllers in the case of stabilization around the operating point, in the case of tracking of desired trajectories the problem becomes much more difficult, because the linearized model is timevarying. Hence, there is a clear need for the development of systematic control design techniques for nonlinear models that are nonaffine in $u$ and that are suitable for the case of tracking of desired trajectories.

For the nonaffine nonlinear model (2), the Taylor expansion of the nonlinear function $f_{1}(x(t), u(t))$ with respect to $u(t)$ around $u(t-\tau)$ can result in

$$
\begin{equation*}
\dot{x}=f_{1}(x, u(t-\tau))+g(x, u(t-\tau)) \Delta u+O(\cdot)+D \tag{3}
\end{equation*}
$$

where $\Delta u=u-u(t-\tau), g(x, u(t-\tau))=$ $\left.\left(\partial f_{1}(x, u) / \partial u\right)\right|_{u=u(t-\tau)}$, and the remainder $O(\cdot)=$ $[\Delta u]^{T} f_{d d} \Delta u / 2$ is bounded by

$$
\begin{equation*}
\|O(\cdot)\| \leqslant \frac{r_{p}\|\Delta u\|^{2}}{2} \tag{4}
\end{equation*}
$$

where $f_{d d}=\left.\left(\partial^{2} f_{1}(x, u) / \partial^{2} u\right)\right|_{u=\zeta}$ and $\zeta$ is a point between $u$ and $u(t-\tau)$. Let $0 \leq\left\|f_{d d}\right\| \leq r_{p}$ with $r_{p}$ as a finite positive number. The parameter $\tau>0$ is the updating input. It may be chosen as the sampling-time in a sampled-data control system or as an integer multiple of the sampling-time. A better choice of the parameter $\tau$ is the sampling because a larger $\tau$ may lead to an inaccurate approximation when the system function $f_{1}(x, u)$ varies quickly.

Equation (3) can be representation as the following form:

$$
\begin{equation*}
\dot{x}=f(x, u(t-\tau))+g(x, u(t-\tau)) u+O(\cdot)+D(x, u, t) \tag{5}
\end{equation*}
$$

where $f(x, u(t-\tau))=f_{1}(x, u(t-\tau))-f_{d}(x, u(t-\tau)) u(t-\tau)$. Convenient for the following statements, $u(t-\tau)$ is defined as $v(t)$; then (3) can be described as follows:

$$
\begin{equation*}
\dot{x}=f(x, v)+g(x, v) u+O(\cdot)+D(x, u, t) . \tag{6}
\end{equation*}
$$

Remark 4. In Assumption 2, $\|\Delta u\|$ should not be too large in order to limit the approximation error of model (3) for a computed $u(t)$. Therefore, to approximation accuracy, Assumption 2 must be satisfied. The significance of the Assumption 2 has been explained in Remark 3.

Remark 5. The traditional model simplification method does not global. It can be seen that (6) is a time-varying simplified model. The method which is proposed in this subsection can achieve the global approximation for nonlinear systems (2). So the proposed simplified model method can effectively solve the tracking control problem using affine nonlinear control strategy, such as sliding mode control, outputfeedback control, and backstepping control.

Remark 6. By (3) and Assumption 2, it can be seen that $u(t-\tau)$ should be around the input $u$. If the time-delay $\tau$ is selected too large, the precision of approximation of simplified model will be reduced. So the selection of $\tau$ often requires experience. Theoretically, the smaller the $\tau$ the better precision of global approximation, the best precision of global approximation if $\tau=0$. But $u$ is control law to be solved, so it is unable to be realized. In order to obtain exact time-varying trim point, here, further improvement of above proposed method is given as follows. Considering lag property of the filtering as

$$
\begin{equation*}
\dot{v}=-k_{\zeta} v+k_{\zeta} u \tag{7}
\end{equation*}
$$

then $\lim _{\lambda \rightarrow \infty} v=u$. This is a very good solution to the problem that $u(t-\tau)$ may not be around $u$. Here, $k_{\zeta} \rightarrow$ $\infty$ is only a rigorous expression for mathematics meanings, in general, $k_{\zeta} \in[5,50]$. The filter (7) is not unique. The filtering $v$ can be completely replaced by other filtering equation, such as higher-order differentiator [30] and integral filter [31].

## 4. Sign Integral Terminal Sliding Mode Disturbance Observer

In this section, the design process of the sign integral terminal sliding mode disturbance observer will be given. Firstly, the following auxiliary sign integral terminal sliding mode vector $s_{z}$ is introduced:

$$
\begin{equation*}
s_{z}=e_{z}+\vartheta e_{z i} \tag{8}
\end{equation*}
$$

where $\vartheta>0$ is a design parameter, $e_{z}=x-z=\left[e_{z 1}, \ldots, e_{z n}\right]^{T}$ is auxiliary error, $z$ and $e_{z i}$ are given by

$$
\begin{gather*}
\dot{z}=f_{1}(x, u)+\widehat{D}  \tag{9}\\
\dot{e}_{z i}=\operatorname{sign}\left(e_{z}\right) \tag{10}
\end{gather*}
$$

where $e_{z i}$ is the integration of $\operatorname{sign}\left(e_{z}\right)$ and has the initial value $-e_{z}(0) / \mathcal{Y}$; and $\operatorname{sign}\left(e_{z i}\right)=\left[\operatorname{sign}\left(e_{z 1}\right), \ldots\right.$, $\left.\operatorname{sign}\left(e_{z n}\right)\right]^{T}$ for $e_{z i}(i=1, \ldots, n)$ being the $i$ th element of the auxiliary error vector $e_{z}$.

If $s_{z}(t)$ can keep at zero, such that $e_{z}(t)=-\vartheta e_{z i}(t)$, then dynamics (10) will be

$$
\begin{equation*}
\dot{e}_{z i}(t)=\operatorname{sign}\left(e_{z}\right)(t) . \tag{11}
\end{equation*}
$$

Therefore, $e_{z i}(t)$ will converge to zero in the finite time $T_{f}$ :

$$
\begin{equation*}
T_{f}=\frac{\left\|e_{z}(0)\right\|}{\mathfrak{\vartheta}} \tag{12}
\end{equation*}
$$

Note that when the auxiliary sign integral terminal sliding vector $s_{z}$ satisfies $s_{z}(t)=0$, the convergence of the $e_{z}(t)$ is accomplished in the same time finite time (12) due to the fact that $e_{z}(t)=-\vartheta e_{z i}(t)$.

Next, to keep the system on the sign integral terminal sliding surface $s_{z}(t)=0$, the sliding mode disturbance estimate $\widehat{D}$ will be set to

$$
\begin{equation*}
\widehat{D}=k_{z} s_{z}+\kappa \operatorname{sign}\left(s_{z}\right)+\vartheta \operatorname{sign}\left(e_{z}\right), \tag{13}
\end{equation*}
$$

where $k_{z}>0$ and $\kappa>\check{\epsilon}>0$ are design parameters.
Theorem 7. Considering the uncertain nonaffine nonlinear system (1) and supposing that Assumption 1 is available, the sign integral terminal sliding mode observer is designed according to (8)-(13). Then, auxiliary errors $e_{z}(t)$ and $e_{z i}(t)$ are guaranteed with finite-time convergence stability.

Proof. Based on (8)-(13), the sign integral terminal sliding mode dynamic equation (8) also can be expressed by

$$
\begin{align*}
\dot{s}_{z} & =\dot{x}-\dot{z}+\vartheta \dot{e}_{z i} \\
& =f_{1}(x, u)+D-f_{1}(x, u)-\widehat{D}-\vartheta \operatorname{sign}\left(e_{z}\right)+\vartheta \dot{e}_{z i}  \tag{14}\\
& =D-k_{z} s_{z}-\kappa \operatorname{sign}\left(s_{z}\right) .
\end{align*}
$$

Choose the Lyapunov function candidate:

$$
\begin{equation*}
V_{s}=\frac{1}{2} s_{z}^{T} s_{z} \tag{15}
\end{equation*}
$$

The time derivative of $V_{s}$ along the trajectories of the equation in (14) is

$$
\begin{align*}
\dot{V}_{s} & =s_{z}^{T} \dot{s}_{z}=s_{z}^{T}\left(D-k_{z} s_{z}-\kappa \operatorname{sign}\left(s_{z}\right)\right) \\
& \leqslant-k_{z} s_{z}^{T} s_{z}+\left\|s_{z}\right\| \check{\epsilon}-\kappa s_{z}^{T} \operatorname{sign}\left(s_{z}\right) \tag{16}
\end{align*}
$$

Under Assumption 1, considering design parameter $\kappa>\check{\epsilon}$, that is, $\kappa>\|D\|$, (16) can be modified as

$$
\begin{equation*}
\dot{V}_{s} \leqslant-k_{z} s_{z}^{T} s_{z}=-2 k_{z} V_{s} \tag{17}
\end{equation*}
$$

From (17), we can get the conclusion that if $s_{z} \neq 0$, then $\dot{V}_{s}<0$ is true. Thus, the auxiliary sliding vector $s_{z}$ of the sign integral terminal sliding mode disturbance observer is always kept on the surface $s_{z}(t)=0$. At the result, the auxiliary errors $e_{z}(t)$ and $e_{z i}(t)$ are guaranteed with finite-time convergence stability. This ends the proof.

Remark 8. Comparing with the existing results [14], the proposed sign integral terminal sliding mode disturbance observer can guarantee the disturbance estimate error to converge to zero in the finite time. In addition, the advantages of proposed sign integral terminal sliding mode will be explained in Remarks 11 and 12.
Remark 9. It is worth noting that the known upper boundary of the dynamic error is required in the design of disturbance observer. However, upper boundary $\check{\epsilon}$ is difficult to be obtained in practice. So, the adaptive gain $\kappa$ in (13) is considered. There are many research results on adaptive gain [16]. For simple convenience, the detail is omitted.

## 5. Fractional Integral Terminal Sliding Mode Control

In this section, we develop the tracking control scheme for the case where all states are available using fractional integral terminal sliding mode control approach. Before the discussion, the tracking error is defined as $e(t)=y(t)-$ $y_{d}(t)$. Instead of using a linear sliding function, we introduce fractional integral terminal sliding mode below. First, the fractional integral terminal sliding mode function is defined as follows:

$$
\begin{gather*}
s(t)=e(t)+\breve{\alpha} e_{I}(t),  \tag{18}\\
\dot{e}_{I}(t)=e^{q / p}(t),
\end{gather*}
$$

where $\breve{\alpha}>0 ; q$ and $p$ are positive odd integers with $p>q$; $e_{I}=\int_{0}^{t} e^{q / p}(\tau) d \tau$ is the integration of the $q / p$-fractional power of the tracking error $e$ with the initial value $-e(0) / \breve{\alpha}$; the nonlinear $e^{q / p}$ is obtained by the operation $e_{0}^{q / p}=$ $\left[e_{1}^{q / p} \operatorname{sign}\left(e_{1}\right), \ldots, e_{n}^{q / p} \operatorname{sign}\left(e_{n}\right)\right]^{T}$.

If the surface $s(t)=0$, based on definition (18), the fractional integral terminal sliding mode function can be expressed in the form

$$
\begin{equation*}
s(t)=e(t)+\breve{\alpha} \int_{0}^{t} e^{q / p}(\tau) d \tau \tag{19}
\end{equation*}
$$

At the same time, the integrator $e_{I}(t)$ can be modified as

$$
\begin{equation*}
\dot{e}_{I}=-\breve{\alpha}^{q / p} e_{I}^{q / p}(t) \tag{20}
\end{equation*}
$$

From solving the error dynamic equation (20), the convergence time of $e_{I}$ is obtained as follows:

$$
\begin{equation*}
T_{f}=\frac{\left\|e_{I}\right\|^{1-q / p}}{\alpha^{q / p}(1-q / p)}=\frac{\|e(0)\|^{1-q / p}}{\breve{\alpha}(1-q / p)} . \tag{21}
\end{equation*}
$$

Meanwhile, the time spent for the convergence of the tracking error $e(t)$ is also $T_{f}$.

Next, to keep the system on the integral terminal sliding surface $s(t)=0$, we need to design control input vector $u$ for the system (2). Considering the time-varying simplified model (6), we modify the time derivative of $s(t)$ along the dynamics (6) as

$$
\begin{equation*}
\dot{s}(t)=\dot{e}+\breve{\alpha} e_{I}=f+g u+O(\cdot)+D-\dot{y}_{d}+\breve{\alpha} \dot{e}_{I} \tag{22}
\end{equation*}
$$

where $f, g$, and $O(\cdot)$ are defined in Section 3 .

According to Assumption 2, we will first consider the case when the control gain is nonsingular; that is, $|g| \neq 0$. Following that, our focus will be on the control design in the case when the control gain is singular; that is, $|g|=0$.
5.1. Case of Nonsingular Control Gain. In this subsection, we assume that $|g| \neq 0$ for the nonaffine nonlinear systems (2) with simplified model (6). Then, we consider the following the control vector $u$ as

$$
\begin{equation*}
u=g^{-1}\left(-k s-f-\widehat{D}-\breve{\alpha} \dot{e}_{I}+\dot{y}_{d}-r_{s}\right) \tag{23}
\end{equation*}
$$

where $k>0$ is a design parameter; $\widehat{D}$ is defined in (13); to restrain the dynamic error $O(\cdot)$ from (6), the robust term $r_{s}$ is designed as

$$
r_{s}= \begin{cases}\frac{\varsigma s}{\|s\|}, & \|s\| \neq 0  \tag{24}\\ 0, & s=0\end{cases}
$$

where $\varsigma>\|O(\cdot)\|$ is a design constant.
The above design procedure of the terminal sliding mode control can be summarized in the following theorem, which contains the results for disturbance-observer-based terminal sliding mode tracking control of uncertain nonaffine systems with external disturbance.

Theorem 10. Considering the uncertain nonaffine system (1) with the external disturbance and assuming that Assumptions 1 and 2 are available, nonaffine nonlinear approximation is given as (6) and the terminal sliding mode disturbance observer is designed as (8)-(13). If the proposed terminal sliding mode tracking control and the robust term are chosen as (23) and (24), then all signals of the closed-loop system are convergent in the finite time (21).

Proof. Choose the Lyapunov function candidate:

$$
\begin{equation*}
V=\frac{1}{2} s^{T} s \tag{25}
\end{equation*}
$$

Under Assumption 2, substituting (23) and (24) into (22), the time derivative of $V$ along the trajectories of (22) is

$$
\begin{align*}
& \dot{V}= s^{T} \dot{s}=s^{T}\left(f+g u+D+O(\cdot)-\dot{y}_{d}+\breve{\alpha} \dot{e}_{I}\right) \\
&= s^{T}\left(f-k s-f-\widehat{D}-\breve{\alpha} \dot{e}_{I}+\dot{y}_{d}\right. \\
&\left.\quad-r_{s}+D+O(\cdot)-\dot{y}_{d}+\breve{\alpha} \dot{e}_{I}\right)  \tag{26}\\
& \leqslant-k s^{T} s+\|s\|\|O(\cdot)\|-\varsigma\|s\| \\
& \leqslant \leqslant k s^{T} s=-2 k V .
\end{align*}
$$

Since $s(0)=0$ and (26), the system is always kept on the fractional integral terminal sliding surface $s(0)=0$. As a result, the tracking error $e_{I}(t)$ and error $e(t)$ converge to zero in finite time (21). This concludes the proof.

Remark 11. The characteristics of the proposed fractional integral terminal sliding mode control including (1) the finite convergence time can be easily adjusted according to (21); (2) the singular problem does not occur on the control law in contrast to traditional TSMC; (3) the system starts on the sliding mode surface $s=0$; that is, fast response is obtained.

Remark 12. Aside from the characteristics in Remark 11, the convergence time of the fractional integral terminal sliding mode control is calculable and analyzable in contrast to the high-order SMC. In comparison, the dynamic SMC only assures asymptotic stability.

Remark 13. In order to reduce chattering which is caused by discontinuous sign function, $\varsigma s /\|s\|$ in robust term (24) can be replaced by the continuous function $r_{s}$ defined by

$$
\begin{equation*}
r_{s}=\frac{\varsigma s}{\|s\|+\epsilon} \tag{27}
\end{equation*}
$$

with $\epsilon=\epsilon_{0}+\epsilon_{1}\|e\|$, where $\epsilon_{0}$ and $\epsilon_{1}$ are two positive constants.
5.2. Case of Singular Control Gain. In Section 5.1, we assume that $|g(x)| \neq 0$ for the simplified model (6). However, there exists the feasibility of $|g(x)|=0$ at a moment in the practical system which leads to the control singularity. Thus, we propose the fractional integral terminal sliding mode control for the simplified model (6) with control singularity case in this subsection. Considering the control singularity, the control input vector $u$ is given by

$$
\begin{equation*}
u=g(x, v)\left(g^{T}(x, v) g(x, v)+\lambda\right)^{-1} \check{u} \tag{28}
\end{equation*}
$$

where $\lambda>0$ is a design constant and $\check{u}$ will be given later.
It is clear that

$$
\begin{align*}
& g^{T}(x, v) g(x, v)\left(g^{T}(x, v) g(x, v)+\lambda\right)^{-1}  \tag{29}\\
& \quad=1-\lambda\left(g^{T}(x, v) g(x, v)+\lambda\right)^{-1}
\end{align*}
$$

Substituting (28) and (29) into (6), we obtain

$$
\begin{equation*}
\dot{x}=f(x, v)+\check{u}+D-\left(g^{T}(x, v) g(x, v)+\lambda\right)^{-1} \check{u}+O(\cdot) . \tag{30}
\end{equation*}
$$

According to (30), the compound disturbance can be modified as

$$
\begin{equation*}
\bar{D}=D-\left(g^{T}(x, v) g(x, v)+\lambda\right)^{-1} \check{u}+O(\cdot) . \tag{31}
\end{equation*}
$$

Due to the unknown compound disturbance $\bar{D}$, the sign integral terminal sliding mode disturbance observer needs to be developed to estimate it. Thus, the similar auxiliary sliding mode is expressed:

$$
\begin{align*}
& s_{z}=e_{z}+a_{0} e_{z i}  \tag{32}\\
& \dot{z}=f+\check{u}+\widehat{\bar{D}}  \tag{33}\\
& \dot{e}_{z i}=\operatorname{sign}\left(e_{z i}\right) \tag{34}
\end{align*}
$$

where $s_{z}, e_{z}, e_{z i}$, and $z$ are defined in Section 4. Based on (32)-(34), the sign integral terminal sliding mode disturbance estimate $\widehat{\bar{D}}$ is given by

$$
\begin{equation*}
\widehat{\bar{D}}=k_{z} s_{z}+\kappa \operatorname{sign}\left(s_{z}\right)+\vartheta \operatorname{sign}\left(e_{z}\right), \tag{35}
\end{equation*}
$$

where $\widehat{\bar{D}}$ is the estimate of compound disturbance $\bar{D}$.
Based on the sign integral terminal sliding mode disturbance observer, the fractional integral terminal sliding mode tracking control is designed as

$$
\begin{equation*}
\check{u}=-k s-f-\widehat{\bar{D}}-\breve{\alpha} e^{q / p}+\dot{y}_{d} \tag{36}
\end{equation*}
$$

where the parameters $k, \breve{\alpha}, q$, and $p$ are defined in (23) and (18), respectively.

The above design procedure and analysis can be summarized in the following theorem, which contains the results for the simplified model (6) with the control singularity case.

Theorem 14. Considering the uncertain nonaffine system (1) with the external disturbance and supposing that Assumptions 1 and 2 are available, nonaffine nonlinear approximation is given as (6) and the sign integral terminal sliding mode disturbance observer is designed as (32)-(35). If the proposed fractional integral terminal sliding mode control law is chosen as (36), then sliding mode surface will always keep at $s(t)=0$.

Proof. Considering the time-varying simplified model (6) with singular control gain, we modify the time derivative of $s(t)$ along the dynamics (30) as

$$
\begin{equation*}
\dot{s}(t)=\dot{e}+\breve{\alpha} e_{I}=f+g u+O(\cdot)+D-\dot{y}_{d}+\breve{\alpha} \dot{e}_{I} \tag{37}
\end{equation*}
$$

Choose the Lyapunov function candidate:

$$
\begin{equation*}
V=\frac{1}{2} s^{T} s \tag{38}
\end{equation*}
$$

Substituting (36) into (37), the time derivative of $V$ along the trajectories of the equation in (37) is

$$
\begin{align*}
\dot{V}= & s^{T} \dot{s}=s^{T}\left(f+\breve{u}+\bar{D}-\dot{y}_{d}+\breve{\alpha} \dot{e}_{I}\right) \\
= & s^{T}\left(f-k s-f-k_{z} s_{z}-\widehat{\bar{D}}\right. \\
& \left.\quad-\breve{\alpha} e^{q / p}+\dot{y}_{d}+\bar{D}-\dot{y}_{d}+\breve{\alpha} \dot{e}_{I}\right)  \tag{39}\\
\leqslant & -k s^{T} s=-2 k V .
\end{align*}
$$

According to $s(0)=0$ and (39), we can know that the system is always kept on the fractional integral terminal sliding surface $s(0)=0$. So, the tracking error $e_{I}(t)$ and error $e(t)$ converge to the equilibrium point in the finite time (21). This concludes the proof.


Figure 1: State response by the proposed approximation.


Figure 2: State response using the approach developed in [29].

Remark 15. From (31), the integrated effect of control singularity is treated as a part of the external disturbance which is approximated using the sign integral terminal sliding mode disturbance observer (33)-(35). Although the uncertain nonlinear system (1) has the feasibility of control singularity, Lyapunov analysis shows that the system is asymptotically
convergent in the finite time under the proposed disturbance-observer-based fractional integral terminal sliding mode control. In general, the design parameter $k_{z}$ should be chosen as a large positive constant to guarantee the design requirement of the proposed sliding mode disturbance observer.


Figure 3: States $V, \chi$, and $\gamma$ follow desired command $V_{d}, \chi_{d}$, and $\gamma_{d}$ for near space vehicle system with coordinated turn.

## 6. Simulation Results

6.1. Simulation Example for Duffing-Holmes System. To verify the validity of the proposed nonaffine nonlinear approximation in Section 3, consider Duffing-Holmes system

$$
\begin{gather*}
\dot{x}_{1}=x_{2} \\
\dot{x}_{2}=-p_{1} x_{1}-p_{2} x_{2}-x_{1}^{3}+h\left(x_{2}, u\right) \tag{40}
\end{gather*}
$$

where $p_{1}=0.2 \sin (10 t) ; p_{2}=0.2(1+\cos (5 t)) ; h\left(x_{2}, u\right)=u^{3}+$ $\left(2+\cos \left(x_{2}\right)\right) u$ is a nonaffine control term.

The initial conditions are chosen as $\left[x_{1}(0), x_{2}(0)\right]^{T}=0$, and the simulation time is chosen as $t=30 \mathrm{~s}$. we define control input value $u$ as

$$
u(t)= \begin{cases}0.1 \sin (2 t), & t \leqslant 7  \tag{41}\\ \sin (2 t), & 7<t \leqslant 30\end{cases}
$$

As compared with the existing approximate method, we adopt the method of [29] at the local working point $u=0$ for nonaffine nonlinear systems (2). Then, we get the affine nonlinear approximation as follows:

$$
\dot{x}_{1}=x_{2},
$$

$$
\begin{equation*}
\dot{x}_{2}=-p_{1} x_{1}-p_{2} x_{2}-x_{1}^{3}+\left(2+\cos \left(x_{2}\right)\right) u . \tag{42}
\end{equation*}
$$

By using the proposed approximation method in Section 3, the affine nonlinear approximation is

$$
\begin{gather*}
\dot{x}_{1}=x_{2}, \\
\dot{x}_{2}=-p_{1} x_{1}-p_{2} x_{2}-x_{1}^{3}-2 \zeta^{3}+3 \zeta^{2} u+\left(2+\cos \left(x_{2}\right)\right) u, \\
\dot{\zeta}=-k_{\zeta} \zeta+k_{\zeta} u, \quad \zeta(0)=0 . \tag{43}
\end{gather*}
$$

The designed constant in (43) is chosen as $k_{\zeta}=100$. Simulation results are shown in Figures 1 and 2, respectively.

From Figures 1 and 2, it is observed that two methods have the same approximation during working point $u=0$. On the other hand, when the working point stay away from $u=0$, changes. Therefore, from the global approximation, the proposed method is better than the method of [29].

### 6.2. Simulation Example for 6DOF Near Space Vehicle Dynam-

 ics with Coordinated Turn. To verify the validity of the fractional integral terminal sliding mode control, the differential equations governing the near space vehicle (NSV) dynamics with coordinated turn are given by$$
\begin{gather*}
\dot{x}=V \cos \gamma \cos \chi \\
\dot{y}=V \cos \gamma \sin \chi, \\
\dot{z}=V \sin \gamma \\
\dot{\chi}=f_{\chi}=\frac{1}{M V \cos \gamma}(L \sin \mu+Y \cos \mu+T \sin \mu \sin \alpha), \\
\dot{\gamma}=f_{\gamma}=\frac{1}{M V}(T \cos \alpha-D-M g \sin \gamma)  \tag{48}\\
(L \cos \mu-Y \sin \mu \\
-M g \cos \gamma+T \cos \mu \sin \alpha),
\end{gather*}
$$

where the three position variables $(x, y, z)$ in the inertial frame, airspeed $(V)$, fight path angle $(\gamma)$, and flight path heading $(\chi)$ are the six state variables; thrust $(T)$, attack angle $(\alpha)$, and roll angle $(\mu)$ are the control variables; the drag force $(D)$, lift force $(L)$, and lateral force $(Y)$ are expressed as follows:

$$
\begin{equation*}
D=\widehat{q} S C_{D}, \quad L=\widehat{q} S C_{L}, \quad Y=\widehat{q} S C_{Y}, \tag{50}
\end{equation*}
$$

where $\hat{q}=0.5 \rho V^{2}$ is dynamic pressure, $S$ is reference area, $C_{D}=C_{D \alpha}+C_{D \delta_{e}} \delta_{e}+C_{D \delta_{a}} \delta_{a}+C_{D \delta_{r}} \delta_{r}+C_{D \delta_{c}} \delta_{c}, C_{L}=C_{L \alpha}+$ $C_{L \delta_{e}} \delta_{e}+C_{L \delta_{a}} \delta_{a}+C_{L \delta_{c}} \delta_{c}$, and $C_{Y}=C_{Y \beta} \beta+C_{Y \delta_{e}} \delta_{e}+C_{Y \delta_{\alpha}} \delta_{\alpha}+$ $C_{Y \delta_{r}} \delta_{r}$. In this paper, we will assume that the parameters $C_{D \alpha}$, $C_{D \delta_{e}}, C_{D \delta_{a}}, C_{D \delta_{r}}, C_{D \delta_{c}} \delta_{c}, C_{L \alpha}, C_{L \delta_{e}}, C_{L \delta_{a}}, C_{L \delta_{c}}, C_{Y \beta}, C_{Y \delta_{e}}$, $C_{Y \delta_{\alpha}}$, and $C_{Y \delta_{r}}$ are uncertain, while the description of the variables are shown in [32].

In this paper, we will focus on the model of rate dynamics, that is, (47)-(49). So, to put the above equations in the form of (1), we define $x=\left[x_{1}, x_{2}, x_{3}\right]^{T}=[V, \chi, \gamma]^{T}, u=$ $\left[u_{1}, u_{2}, u_{3}\right]^{T}=[T, \alpha, \mu]^{T}$. Hence the rate equations become

$$
\begin{equation*}
\dot{x}=f(x, u), \tag{51}
\end{equation*}
$$

where $f=\left[f_{v}, f_{\chi}, f_{\gamma}\right]^{T}$ is defined in (47)-(49). By using the proposed approximated model in Section 3, we have

$$
\begin{gather*}
\dot{x}=\bar{f}+g u_{\zeta}+O(\cdot),  \tag{52}\\
u_{\zeta}=-k_{\zeta} u_{\zeta}+k_{\zeta} u, \quad u_{\zeta}(0)=0, \tag{53}
\end{gather*}
$$

where $\bar{f}=\breve{f}\left(x, u_{\zeta}\right)-g_{p} u_{\zeta}, u_{\zeta}=\left[u_{\zeta 1}, u_{\zeta 2}, u_{\zeta 3}\right]^{T}, k_{\zeta}$ is $s$ design parameter, $\breve{f}\left(x, u_{\zeta}\right)=\left[\breve{f}_{v}, \breve{f}_{\chi}, \breve{f}_{\gamma}\right], \breve{f}_{v}, \breve{f}_{\chi}, \breve{f}_{\gamma}$, and $g_{p}$ are expressed as

$$
\left.\begin{array}{c}
\breve{f}_{v}=\frac{1}{M}\left(u_{\zeta 1} \cos x_{2}-D-M g \sin x_{3}\right), \\
\breve{f}_{\chi}=\frac{1}{M x_{1} \cos x_{3}}\left(L \sin u_{\zeta 3}+Y \cos u_{\zeta 3}+T \sin u_{\zeta 3} \sin u_{\zeta 2}\right), \\
\breve{f}_{\gamma}=\frac{1}{M x_{1}}\left(L \cos u_{\zeta 3}-Y \sin u_{\zeta 3}-M g \cos x_{3}+u_{\zeta 1} \cos u_{\zeta 3} \sin u_{\zeta 2}\right), \\
g_{p}=\left[\begin{array}{ccc}
\frac{\cos u_{\zeta 2}}{M} & -\frac{u_{\zeta 1} \sin u_{\zeta 2}}{M} \\
\frac{1}{M u_{\zeta 1} \cos x_{3}} \sin u_{\zeta 2} \sin u_{\zeta 3} & 1 \\
\frac{1}{M u_{\zeta 1}} \sin u_{\zeta 2} \cos u_{\zeta 3} & \frac{1}{M u_{\zeta 1}} u_{\zeta 1} \cos x_{\zeta 2} \cos u_{\zeta 3} & -\frac{1}{M u_{\zeta 1}} u_{\zeta 1} \sin u_{\zeta 2} \sin u_{\zeta 3} \sin u_{\zeta 3}
\end{array}\right] \tag{54}
\end{array}\right] .
$$



Figure 4: Control input for near space vehicle system with coordinated turn.

The initial state conditions are arbitrarily chosen as $M=$ $136080 \mathrm{Kg}, V(0)=3000 \mathrm{~m} / \mathrm{s}, \chi(0)=\gamma(0)=0 \mathrm{deg}, T(0)=$ $240 \mathrm{KN}, \alpha(0)=1 \mathrm{deg}$, and $\mu(0)=0 \mathrm{deg}$.

The desired command is considered as $V_{d}=3000 \mathrm{~m} / \mathrm{s}$, $\chi_{d}=8 \mathrm{deg}$, and $\gamma_{d}=0 \mathrm{deg}$. In order to ensure the smoothness of airspeed change, we choose the filter as (7), in which the parameter is chosen as $k_{\zeta}=0.03$.

To estimate the uncertainty, we apply integral terminal sliding mode disturbance observer in Section 4 and set the parameters as $k_{z}=5, \mathcal{\vartheta}=2$, and $\kappa=0.5$. Furthermore, to demonstrate the effectiveness of the proposed integral terminal sliding mode control, the design parameters are chosen as $\breve{\alpha}=1.5, p=5, q=3, k=2$, and $\varsigma=0.3$. The tracking results are shown in Figures 3 and 4. Although, there exists uncertain in the system, the tracking performance is still satisfactory and tracking error converges to zero quickly.

From these simulation results of two cases, we can obtain that the proposed method is valid. And the developed sign integral terminal sliding mode disturbance observer can modify the control performance of the fractional integral terminal sliding mode control.

## 7. Conclusions

In this paper, the disturbance-observer-based terminal sliding mode tracking control has been proposed for a class of uncertain nonaffine nonlinear systems. To design tracking controller, an on-line approximation has been proposed for a class of nonaffine nonlinear systems. To improve the ability of the disturbance attenuation and system performance robustness, the sign integral terminal sliding mode disturbance observer has been developed to approximate the system disturbance in the finite time. Based on the output of the disturbance observer, the disturbance-observer-based fractional integral terminal sliding mode tracking control has been presented for the uncertain nonlinear system with the timevarying external disturbance. By innovating the fractional error integration, finite-time convergence of tracking errors and integral errors is achieved without singular problem. Furthermore, the finite convergence time is easily calculated in contrast to the traditional high-order sliding mode control. The stability of the closed-loop system has been proved using rigorous Lyapunov analysis. Finally, simulation results have been used to illustrate the effectiveness of the proposed robust
terminal sliding mode tracking control scheme. In addition, based on the proposed approach, how to relax Assumption 1 is our future works. At the same time, fault-tolerant control for a class of nonaffine nonlinear systems is also our future works.

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# Dynamic Characteristics Study with Multidegree-of-Freedom Coupling in TBM Cutterhead System Based on Complex Factors 

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#### Abstract

A multidegree-of-freedom coupling dynamic model, which contains a joint cutterhead, an inner ring gear, a support shield body, and pinions, is established, considering the external stochastic excitations, time-varying meshing stiffness, transmission errors, clearance, and so forth. Based on the parameters of an actual project and the strong impact of external excitations, the modal properties and dynamic responses are analyzed, and the cutterhead joint surface loads are obtained and treated by rain flow count. Numerical results indicate that the low natural frequencies are 57 Hz and 61 Hz , and natural vibration modes are pinionsmotors rotational mode and translational-overturning coupled mode of cutterhead with inner ring gear correspondingly. Besides, the axial and radial amplitude of dynamic responses are 0.55 mm and 0.25 mm , respectively. The frequencies of radial, torsional, and overturning vibrations are predominantly concentrated in 112 Hz and 120 Hz , which indicates that the vibration responses of cutterhead are mainly affected by the external excitations. Finally, as the rain-flow counting results have shown, the standard deviation of the cutterhead joint surface loads in each direction increases by 12-15 times, compared with that of the external excitations; therefore inertia effect should be considered in cutterhead design. The proposed research lays a foundation for dynamic performance optimization and fatigue crack growth life assessment of cutterhead structure.


## 1. Introduction

As a key component of the full face rock tunnel boring machine (TBM), the cutterhead plays the functions of crushing rock, stabilizing excavated opening, and so on, which affects the boring performance and efficiency of the whole machine [1]. Due to complicated geological conditions and variable tunneling parameters, cutterhead endures multipoint random impact loads, in many projects, for example, the Qinling tunnel, Dahuofang tunnel, and Zhongtianshan tunnel [2-4]. As a result, some engineering faults may appear, such as severe vibration, abnormal wears of cutting tools, cracking of cutterhead panel, and the seal failure of main bearing, which put forward high design requirements for structural strength, reliability, and fatigue life of cutterhead. Therefore, aiming at absorbance of dynamic impact loads, high reliability, high fatigue life, and superior static and dynamic characteristics, the research on coupled nonlinearity dynamical characteristics of TBM cutterhead system with
random impact loads provides an important theoretical value and practical significance.

For a long time, a great number of studies about the TBM cutterhead system design have been carried out. Samuel and Seow [5] studied the variations of cutter forces during field testing, and the instantaneous forces are compared to global machine performance. Zhang et al. [6] proposed the testing methods using the results, which are obtained in the measured cutter forces on a boring machine during field boring in a hard rock laboratory. According to Rostami [7], the distribution of cutters is critical to balance performance of cutterhead, and the methods of cutterhead system design for the hard rock TBM were studied based on different performance factors. A model of the specific energy [8, 9] was created for evaluating the energy consumption of tunnel machine. Then the model was conducted using the testing data from a subway project, so as to provide optimal ranges of the cutting depth per revolution. Furthermore, Xia et al. [10, 11] used the discrete element method to study the influence of
cutterhead performance under various geological conditions and cutter spaces, and the results were compared with a cutting test. Similar works present as follows. Sun et al. [1215] studied the simulation process of rock fragmentation with multicutters and space design. Moreover, based on the genetic algorithm and collaborative evolutionary, the method of disc cutters' plane layout design was proposed, as well as the optimal design for master parameter of cutterhead structure and support ribs were processed. In addition, a theoretical dynamic model of shield TBM has been established by Zhang [16], considering the interactions among the redundantly driven revolving system, hydraulic propulsion system, shield body, and geologic condition, to study the dynamic characteristics of revolving system under comprehensive geological conditions. Some related literature on multidegree-of-freedom coupling dynamic model have been published recently. Sakanushi et al. [17] and Yamada et al. [18] proposed the control characteristics and a design procedure of a two-degree-of-freedom simple repetitive control system which can specify the input-output characteristic and the disturbance attenuation characteristic separately, and a numerical example and an application in motors were presented to illustrate the effectiveness, which showed that the control system was expected to be used in practical applications. A six-degree-of-freedom fully-coupled plant model used in underwater vehicles was reported in [19], and the comparative experimental evaluations were estimated experimentally from data obtained in free-motion vehicle trials, which might be able to predict the performance of underwater vehicles.

As mentioned above, scholars have studied rock fragmentation mechanism, force models of disc cutters, stochastic loads of excavation, and disc cutters' plane layout about the TBM cutterhead system, by using the methods of similar model experiment, numerical simulation, and field test. In addition, many design methods and research achievements about multidegree-of-freedom coupling dynamic model have been obtained, which are of valuable reference to our study. However, the problem of vibration in split type of cutterhead with heavy random loads has not been previously performed. Moreover, the dynamic model in [16] did not consider the radial freedom of the cutterhead because of the small loads in the shield TBM system, which cannot be employed in the hard rock TBM system. The design methods in TBM cutterhead were presented in [12-15] to optimize the structure parameter, but the cutters' loads were static nominal load, ignoring the dynamic characteristic. These former models about cutterhead system may not be close to the actual conditions. Therefore, for the first time, this paper studies the dynamic characteristics of split-cutterhead system with multidegree-of-freedom coupling, comprehensively considering the time-varying external excitations, time-varying meshing stiffness, transmission errors, clearance, and bearing stiffness. Compared with conventional design methods and models, the proposed research is more effective to solve the complex project problems, which provides an effective foundation for dynamic performance optimization and fatigue crack propagation life assessment of cutterhead structure.


Figure 1: Components of TBM cutterhead system. 1: Cutterhead piece. 2: Main bearing. 3: Pinion. 4: Coupling. 5: Motor. 6: Reducer.


Figure 2: The structure of TBM split cutterhead.

## 2. The Split-Cutterhead System of Coupled Dynamics Model Influenced by Complicated Factors

In the TBM cutterhead system, the multiple pinions are driven by variable-frequency motors via planetary gear reducers and couplings. Then, the pinions drive an inner ring gear clockwise by redundant control, and the inner ring gear and flange are fixed with bolts, so as to drive the cutterhead. The TBM cutterhead system with various components is shown in Figure 1; here, we just present only one motor driving system. Meanwhile, the structure of TBM split-cutterhead is shown in Figure 2.

In this paper, a dynamic mathematic model of TBM cutterhead system is established by using lumped-parameter method, which is shown in Figure 3, where $k_{m p Q}$ and $k_{r L Q}$ are the torsional stiffness of gear shaft connections and cutterhead shaft connection, respectively, $k_{\text {eqy }}, k_{\text {eqL }}, k_{\text {eqr }}$, $k_{\text {eqz }}, k_{\text {eqd }}$, and $k_{\text {eqdz }}$ are the equivalent radial stiffness and axial stiffness of cutterhead, inner ring gear, and support shield body, respectively, $k_{\text {eqp }}$ and $k_{(t)}$ are the support equivalent stiffness of pinions and time-varying meshing


Figure 3: Continued.


Figure 3: Multidegree-of-freedom coupling dynamic model of TBM cutterhead system.
stiffness between inner ring gear and pinions, respectively, and $k_{L \zeta_{i}}, k_{L \eta i}$, and $k_{L i}$ are the equivalent tangential stiffness, radial stiffness, and axial stiffness of each cutterhead piece, respectively. $C_{x x x}$ ( $x x x$ denotes the subscript) is the damping coefficient of corresponding stiffness. $T_{p i}, T_{L}, F_{X}, F_{Y}, F_{L}, M_{X}$, and $F_{L i}$ represent the input torque, load torque, lateral force, longitudinal force, and axial thrust, transverse overturning moment of cutterhead center block, as well as axial force of each cutterhead piece, respectively. The backlash between the inner ring gear and each pinion is expressed by $b_{p}$ and meshing errors are expressed by $\varepsilon$. Besides, $r_{b r}$ and $r_{b p}$, respectively, denote the base circle radius of inner ring gear and pinions.

In the dynamic model of cutterhead system abovementioned, the moving coordinate systems are used for convenient modeling. The coordinates are illustrated in Figure 3, where $X, Y$, and $Z$, respectively, denote horizontal direction, vertical direction, and tunneling direction, and cutterhead, inner ring gear and support shield body are in this coordinate. Additionally, $\zeta$ and $\eta$ are the radial and tangential directions
of cutterhead pieces, which are rotating with cutterhead. Also, $H_{p}$ and $V_{p}$ are the radial and tangential directions of pinions. We use various subscripts for the purpose of distinguishing conveniently. The cutterhead pieces are specified by the subscript of $i$, and cutterhead center block, inner ring gear, and support shield body are $L$, $r$, and $d$, respectively, while pinions and motors are $p$ and $m$.

The system in Figure 3 has $(27+4 N)$ degree of freedoms, where $N$ is the number of pinions. The generalized vibration displacement is $X=\left(\zeta_{i}, \eta_{i}, Z_{i}, X_{L}, Y_{L}, Z_{L}, \theta_{L x}, \theta_{L y}\right.$, $\left.\theta_{L}, X_{r}, Y_{r}, Z_{r}, \theta_{x}, \theta_{y}, \theta_{r}, X_{d}, Y_{d}, Z_{d}, H_{p j}, V_{p j}, \theta_{p j}, \theta_{m j}\right)^{T}$, where $\zeta_{i}, \eta_{i}$, and $Z_{i}$ are the translational vibration displacements of each cutterhead piece. $X_{L}, Y_{L}$, and $Z_{L}$ are the translational vibration displacements, $\theta_{L x}$ and $\theta_{L y}$ are the bending vibration displacements around $X$ and $Y$ directions, and $\theta_{L}$ is the torsional vibration displacement in center block. $X_{r}, Y_{r}, Z_{r}$, $\theta_{x}, \theta_{y}$, and $\theta_{r}$ are corresponding vibration displacements of the inner ring gear. $X_{d}, Y_{d}$, and $Z_{d}$ are the translational vibration displacements of the support shield body. $H_{p j}$ and $V_{p j}$ are the transverse and longitudinal vibration displacements
of the mass center of each pinion, respectively. $\theta_{p j}$ is the torsional vibration displacements of each pinion and $\theta_{m j}$ is the torsional vibration displacements of each motor.

Because of the low speed of cutterhead, Coriolis accelerations and centripetal accelerations are ignored in published models. The differential equations of each component are as follows, based on the Lagrange's equation.
(1) For motors,

$$
\begin{equation*}
I_{m j} \ddot{\theta}_{m j}+C_{m p \mathrm{Q}}\left(\dot{\theta}_{m j}-\dot{\theta}_{p j}\right)+k_{m p \mathrm{Q}}\left(\theta_{m j}-\theta_{p j}\right)=T_{p j} \tag{1}
\end{equation*}
$$

(2) For pinions,

$$
\begin{gather*}
I_{p j} \ddot{\theta}_{p j}+\left(F_{p r j}+D_{p r j}\right) r_{b p}+C_{m p Q}\left(\dot{\theta}_{p j}-\dot{\theta}_{m j}\right) \\
+k_{m p Q}\left(\theta_{p j}-\theta_{m j}\right)=0, \\
m_{p j} \ddot{H}_{p j}+\left(F_{p r j}+D_{p r j}\right) \cos \alpha+C_{e q p j} \dot{H}_{p j}+k_{e q p j} H_{p j}=0, \\
m_{p j} \ddot{V}_{p j}+\left(F_{p r j}+D_{p r j}\right) \sin \alpha+C_{e q p j} \dot{V}_{p j}+k_{e q p j} V_{p j}=0 . \tag{2}
\end{gather*}
$$

Thus, $x_{p r j}$ is the relative dynamic displacement between each pinion and the inner gear ring along with the internal meshing line is expressed as

$$
\begin{align*}
x_{p r j}= & r_{b p} \theta_{p j}-r_{b r} \theta_{r}+V_{p j} \sin \alpha+H_{p j} \cos \alpha \\
& +X_{r} \sin \left(\alpha+\varphi_{j}\right)-Y_{r} \cos \left(\alpha+\varphi_{j}\right)-\varepsilon_{p r j}(t) \tag{3}
\end{align*}
$$

where $\varepsilon_{p r j}(t)(j=1-N)$ is the system error excitations, shown in part $2, \alpha$ is meshing angle, and $\varphi_{j}$ is the azimuth angle of each pinion, shown in Figure 2(c).

Then, the dynamic meshing loads can be calculated as

$$
\begin{equation*}
F_{p r j}=k_{p r j}(t) \cdot f\left(x_{p r j}, b_{p r j}\right), \tag{4}
\end{equation*}
$$

where $b_{p r j}$ is half of the backlash among inner ring gear and pinions, $k_{p r j}(t)$ is time-varying meshing stiffness, and $f(x, b)$ is the nonlinear function, which can be expressed as

$$
f(x, b)= \begin{cases}x-b, & x>b  \tag{5}\\ 0, & -b \leq x \leq b \\ x+b, & x<-b\end{cases}
$$

Similarly, the forces of meshing damping are expressed as

$$
\begin{equation*}
D_{p r j}=C_{p r j} \cdot \dot{x}_{p r j} . \tag{6}
\end{equation*}
$$

(3) For inner ring gear

$$
\begin{aligned}
m_{r} \ddot{X}_{r} & +\sum_{j=1}^{N}\left(F_{p r j}+D_{p r j}\right) \sin \left(\varphi_{j}+\alpha\right)+C_{e q r}\left(\dot{X}_{r}-\dot{X}_{d}\right) \\
& +C_{e q x}\left(\dot{X}_{r}-\dot{X}_{L}\right)+k_{e q r}\left(X_{r}-X_{d}\right) \\
& +k_{e q x}\left(X_{r}-X_{L}\right)=0
\end{aligned}
$$

$$
\begin{align*}
& m_{r} \ddot{Y}_{r}-\sum_{j=1}^{N}\left(F_{p r j}+D_{p r j}\right) \cos \left(\varphi_{j}+\alpha\right)+C_{e q r}\left(\dot{Y}_{r}-\dot{Y}_{d}\right) \\
& + \\
& +C_{e q y}\left(\dot{Y}_{r}-\dot{Y}_{L}\right)+k_{e q r}\left(Y_{r}-Y_{d}\right) \\
& + \\
& k_{e q y}\left(Y_{r}-Y_{L}\right)=0, \\
& m_{r} \ddot{Z}_{r}+\sum_{i=1}^{4}\left[C_{e q L i}\left(\dot{Z}_{r i}-\dot{Z}_{L i}\right)+C_{e q z i}\left(\dot{Z}_{r i}-\dot{Z}_{d}\right)\right. \\
& \left.\quad+k_{e q L i}\left(Z_{r i}-Z_{L i}\right)+k_{e q z i}\left(Z_{r i}-Z_{d}\right)\right]=0, \\
& I_{r x} \ddot{\theta}_{x}+r_{r}\left[C_{e q L 1}\left(\dot{Z}_{r 1}-\dot{Z}_{L 1}\right)-C_{e q L 3}\left(\dot{Z}_{r 3}-\dot{Z}_{L 3}\right)\right. \\
& \left.\quad+C_{e q z 1}\left(\dot{Z}_{r 1}-\dot{Z}_{d}\right)-C_{e q z 3}\left(\dot{Z}_{r 3}-\dot{Z}_{d}\right)\right] \\
& +r_{r}\left[k_{e q L 1}\left(Z_{r 1}-Z_{L 1}\right)-k_{e q L 3}\left(Z_{r 3}-Z_{L 3}\right)\right. \\
& \left.\quad+k_{e q z 1}\left(Z_{r 1}-Z_{d}\right)-k_{e q z 3}\left(Z_{r 3}-Z_{d}\right)\right]=0, \\
& I_{r y} \ddot{\theta}_{y}+r_{r}\left[C_{e q L 2}\left(\dot{Z}_{r 2}-\dot{Z}_{L 3}\right)-C_{e q L 4}\left(\dot{Z}_{r 4}-\dot{Z}_{L 4}\right)\right. \\
& \\
& \left.\quad+C_{e q z 2}\left(\dot{Z}_{r 2}-\dot{Z}_{d}\right)-C_{e q z 4}\left(\dot{Z}_{r 4}-\dot{Z}_{d}\right)\right] \\
& \quad+r_{r}\left[k_{e q L 2}\left(Z_{r 2}-Z_{L 2}\right)-k_{e q L 4}\left(Z_{r 4}-Z_{L 4}\right)\right.  \tag{7}\\
& \\
& \left.\quad+k_{e q z 2}\left(Z_{r 2}-Z_{d}\right)-k_{e q z 4}\left(Z_{r 4}-Z_{d}\right)\right]=0, \\
& I_{r} \ddot{\theta}_{r}-\sum_{j=1}^{N}\left(F_{p r j}+D_{p r j}\right) r_{b r}+C_{r L Q}\left(\dot{\theta}_{r}-\dot{\theta}_{L}\right) \\
& +k_{r L Q}\left(\theta_{r}-\theta_{L}\right)=0 .
\end{align*}
$$

(4) For cutterhead center block:

$$
\begin{gathered}
m_{L} \ddot{X}_{L}-\sum_{i=1}^{4}\left[\left(k_{L \zeta i} \delta_{1 \zeta i}+C_{L \zeta i} \dot{\delta}_{L \zeta i}\right) \sin \left(\omega t+\varphi_{i}\right)\right. \\
\left.+\left(k_{L \eta i} \delta_{L \eta i}+C_{L \eta i} \dot{\delta}_{L \eta i}\right) \cos \left(\omega t+\varphi_{i}\right)\right] \\
+C_{e q x}\left(\dot{X}_{L}-\dot{X}_{r}\right)+k_{e q x}\left(X_{L}-X_{r}\right)=F_{X}, \\
m_{L} \ddot{Y}_{L}+\sum_{i=1}^{4}\left[\left(k_{L \zeta i} \delta_{1 \zeta i}+C_{L \zeta i} \dot{\delta}_{1 \zeta i}\right) \cos \left(\omega t+\varphi_{i}\right)\right. \\
\left.\quad-\left(k_{L \eta i} \delta_{1 \eta i}+C_{L \eta i} \dot{\delta}_{1 \eta i}\right) \sin \left(\omega t+\varphi_{i}\right)\right] \\
+C_{e q y}\left(\dot{Y}_{L}-\dot{Y}_{r}\right)+k_{e q y}\left(Y_{L}-Y_{r}\right)=F_{Y}, \\
m_{L} \ddot{Z}_{L}+\sum_{i=1}^{4}\left[C_{L i}\left(\dot{Z}_{L i}-\dot{Z}_{i}\right)+k_{L i}\left(Z_{L i}-Z_{i}\right)\right. \\
\\
\left.+C_{e q L i}\left(\dot{Z}_{L i}-\dot{Z}_{r i}\right)+k_{e q L i}\left(Z_{L i}-Z_{r i}\right)\right]=F_{L}, \\
I_{L x} \ddot{\theta}_{L x}+a_{L}\left[C_{L 1}\left(\dot{Z}_{L 1}-\dot{Z}_{1}\right)-C_{L 3}\left(\dot{Z}_{L 3}-\dot{Z}_{3}\right)\right. \\
\left.\quad+k_{L 1}\left(Z_{L 1}-Z_{1}\right)-k_{L 3}\left(Z_{L 3}-Z_{3}\right)\right]
\end{gathered}
$$

$$
\begin{gather*}
+r_{r}\left[C_{e q L 1}\left(\dot{Z}_{L 1}-\dot{Z}_{r 1}\right)-C_{e q L 3}\left(\dot{Z}_{L 3}-\dot{Z}_{r 3}\right)\right. \\
\left.+k_{e q L 1}\left(Z_{L 1}-Z_{r 1}\right)-k_{e q L 3}\left(Z_{L 3}-Z_{r 3}\right)\right]=M_{X} \\
I_{L y} \ddot{\theta}_{L y}+a_{L}\left[C_{L 2}\left(\dot{Z}_{L 2}-\dot{Z}_{2}\right)-C_{L 4}\left(\dot{Z}_{L 4}-\dot{Z}_{4}\right)\right. \\
\\
\left.\quad+k_{L 2}\left(Z_{L 2}-Z_{2}\right)-k_{L 4}\left(Z_{L 4}-Z_{4}\right)\right] \\
+r_{r}\left[C_{e q L 2}\left(\dot{Z}_{L 2}-\dot{Z}_{r 2}\right)-C_{e q L 4}\left(\dot{Z}_{L 4}-\dot{Z}_{r 4}\right)\right. \\
 \tag{8}\\
\left.+k_{e q L 2}\left(Z_{L 2}-Z_{r 2}\right)-k_{e q L 4}\left(Z_{L 4}-Z_{r 4}\right)\right]=M_{Y}, \\
I_{L} \ddot{\theta}_{L}+C_{r L Q}\left(\dot{\theta}_{L}-\dot{\theta}_{r}\right)+k_{r L Q}\left(\theta_{L}-\theta_{r}\right)=-T_{L} .
\end{gather*}
$$

(5) For support shield body,

$$
\begin{align*}
m_{d} \ddot{X}_{d} & +C_{e q r}\left(\dot{X}_{d}-\dot{X}_{r}\right)+C_{e q d} \dot{X}_{d} \\
& +k_{e q r}\left(X_{d}-X_{r}\right)+k_{e q d} X_{d}=0, \\
m_{d} \ddot{Y}_{d} & +C_{e q r}\left(\dot{Y}_{d}-\dot{Y}_{r}\right)+C_{e q d} \dot{Y}_{d} \\
& +k_{e q r}\left(Y_{d}-Y_{r}\right)+k_{e q d} Y_{d}=0, \\
m_{d} \ddot{Z}_{d} & +C_{e q z 1}\left(\dot{Z}_{d}-\dot{Z}_{r 1}\right)+C_{e q z 2}\left(\dot{Z}_{d}-\dot{Z}_{r 2}\right)  \tag{9}\\
& +C_{e q z 3}\left(\dot{Z}_{d}-\dot{Z}_{r 3}\right)+C_{e q z 4}\left(\dot{Z}_{d}-\dot{Z}_{r 4}\right) \\
& +C_{e q d z} \dot{Z}_{d}+k_{e q z 1}\left(Z_{d}-Z_{r 1}\right) \\
& +k_{e q z 2}\left(Z_{d}-Z_{r 2}\right)+k_{e q z 3}\left(Z_{d}-Z_{r 3}\right) \\
& +k_{e q z 4}\left(Z_{d}-Z_{r 4}\right)+k_{e q d z} Z_{d}=0 .
\end{align*}
$$

(6) For cutterhead pieces,

$$
\begin{gather*}
m_{i} \ddot{\zeta}_{i}+k_{L \zeta i} \delta_{L \zeta i}+C_{L \zeta i} \dot{\delta}_{L \zeta i}=F_{\zeta i}, \\
m_{i} \ddot{\eta}_{i}+k_{L \eta i} \delta_{L \eta i}+C_{L \eta i} \dot{\delta}_{L \eta i}=F_{\eta i},  \tag{10}\\
m_{i} \ddot{Z}_{i}+k_{L i}\left(Z_{i}-Z_{L i}\right)+C_{L i}\left(\dot{Z}_{i}-\dot{Z}_{L i}\right)=F_{L i},
\end{gather*}
$$

where $F_{\zeta i}, F_{\eta i}$, and $F_{L i}$ are the tangential force, normal force, and axial force of each cutterhead piece, respectively.

The $\delta_{L \zeta i}$ and $\delta_{L \eta i}(i=1-4)$ in (10) are the tangential and radial relative deformations between each cutterhead piece and central block, expressed as follows:

$$
\begin{align*}
& \delta_{L \zeta i}=\zeta_{i}-X_{L} \sin \varphi_{i}+Y_{L} \cos \varphi_{i}, \\
& \delta_{L \eta i}=\eta_{i}-X_{L} \cos \varphi_{i}-Y_{L} \sin \varphi_{i}, \tag{11}
\end{align*}
$$

where $\varphi_{i}$ is the azimuth angle of each cutterhead piece and four pieces are evenly distributed, $\varphi_{i}=\pi(i-1) / 2(i=1-4)$.

As discussed above, assembling the system equations in matrix form yields

$$
\begin{equation*}
M \ddot{X}+C \dot{X}+K X=F \tag{12}
\end{equation*}
$$

where $M$ is the mass matrix, $C$ and $K$, respectively, represent the damping matrix and stiffness matrix, and $F$ is the incentive force vector.

## 3. Dynamic Excitations of TBM Cutterhead System

There are two types of excitations in the dynamic model, which are external and internal excitations, influencing the dynamic characteristics of the cutterhead system. The external excitations depend on the time variability of the parameters of disc cutters' layout, geological conditions, tunneling parameters, and so forth. Meanwhile, the internal excitations are affected by the time-varying meshing stiffness, comprehensive accumulated errors, bearing stiffness, and so on.
3.1. External Excitations of TBM Cutterhead System. A 3D simulation model with multicutters is established, under typical geological conditions, based on the procedure of LSDYNA, due to complicated geological environments. Then, the dynamic loads between disc cutters and surrounding rock are obtained and modified reference to the field data $[5,6]$. Consequently, the total loads of the TBM cutterhead system are calculated by summing the individual force contributions of each cutter, which can be provided to external excitations of the dynamic model.

Theoretically speaking, the disc cutters suffer normal forces $F_{v}$, tangential forces $F_{r}$, and side forces $F_{s}$ when the cutterhead turns, as shown in Figure 4, where $\rho$ is the radius from the center of the cutterhead, $\theta$ is the position angle of the cutter, and $\beta$ is the tilt angle of the gauge cutter. For the convenience of loads calculation, the assumptions are formulated as follows.
(i) The total loads of cutterhead equal to the resultant force of each disc cutter, ignoring the losses in transfer process.
(ii) Considering the complexity of actual rock breaking loads, the mean normal force is equal to nominal load of the disc cutter, and it is 0.15 times the mean tangential force and 0.1 times the mean side force $[5,6]$.

Deduced the relationship between the loads of cutterhead and cutters, the formulae of loads in each cutterhead piece are presented by the following expressions.
(1) Axial Forces. The axial forces are equal to the resultant force in $Z$ direction of each cutter, which can be expressed as

$$
\begin{equation*}
F_{L}=\sum_{i=1}^{n} F_{v i}+\sum_{j=1}^{m} F_{v j}+\sum_{k=1}^{p}\left(F_{v k} \cos \beta_{k}+F_{s k} \sin \beta_{k}\right) \tag{13}
\end{equation*}
$$

where $F_{v t}(t=i, j, k)$ are the normal force of the center cutter, normal cutter, and gauge cutter, respectively, $F_{s k}$ is the side force of the $k$ th gauge cutter, $n, m$, and $p$ are the number of the center cutter, normal cutter, and gauge cutter, respectively, and $\beta_{k}$ is the tilt angle of the $k$ th gauge cutter.


Figure 4: Forces acting on a normal cutter and a gauge cutter [14].

Also, $F_{L i}$ can be calculated by (13), just with different number of disc cutters.
(2) Radial Forces. The radial forces can be decomposed into lateral force $F_{X}$ and longitudinal force $F_{Y}$ as

$$
\begin{align*}
& F_{X}=F_{v x \Sigma}+F_{r x \Sigma}+F_{s x \Sigma} \\
& F_{Y}=F_{v y \Sigma}+F_{r y \Sigma}+F_{s y \Sigma} \tag{14}
\end{align*}
$$

where $F_{v x \Sigma}, F_{r x \Sigma}$, and $F_{s x \Sigma}$, respectively, represent the resultant force of normal forces, tangential forces, and side forces in $X$ direction, and the resultant forces in $Y$ direction are $F_{v y \Sigma}$, $F_{r y \Sigma}$, and $F_{s y \Sigma}$, correspondingly.

In the same way, $F_{\zeta i}$ and $F_{\eta i}$ are calculated by (14).
(3) Overturning Moments. Similarly, the overturning moments can also be decomposed into two directions, which load on the central block. Consider

$$
\begin{align*}
& M_{X}=\sum_{i=1}^{n} F_{v i} l_{i x}+\sum_{j=1}^{m} F_{v j} l_{j x}, \\
& M_{Y}=\sum_{i=1}^{n} F_{v i} l_{i y}+\sum_{j=1}^{m} F_{v j} l_{j y}, \tag{15}
\end{align*}
$$

where $l_{t x}$ and $l_{t y}(t=i, j)$ represent the distance to $X$ and $Y$ axis of the $t$ th cutter.
(4) Torques. The load torque is generated by the tangential forces, which is equal to the resultant torque of each tangential force around $Z$ axial. Consider

$$
\begin{equation*}
T_{L}=\sum_{i=1}^{n} F_{r i} \rho_{i}+\sum_{j=1}^{m} F_{r j} \rho_{j}+\sum_{k=1}^{p} F_{r k} \rho_{k} \tag{16}
\end{equation*}
$$

The input torque $T_{p i}$ is equal to $1 / N$ times the load torque without considering power loss.

### 3.2. Internal Excitations of TBM Cutterhead System

3.2.1. Time-Varying Meshing Stiffness of Gear Pairs. For spur gears, the meshing stiffness shows obvious nature of the step
period, which will change suddenly when the coincidence degree is not an integer, leading to the generation of the dynamic excitation force [20]. According to GB3480-1997, the peak and the average of meshing stiffness can be obtained. Then, the meshing stiffness variation can be modeled by a series of square wave functions. Finally, it is expanded in Fourier series without higher-order terms as follows:

$$
\begin{equation*}
k(t)=k_{m}+\sum_{i=1}^{n}\left[k_{p i} \sin (i \omega t)+k_{c i} \cos (i \omega t)\right] \tag{17}
\end{equation*}
$$

where $\omega$ is meshing frequency, $n$ is the harmonic number of the meshing stiffness, $k_{m}$ is mean meshing stiffness, and $k_{p i}$ and $k_{c i}$ represent the amplitude of the $i$ th order sinusoidal and cosine magnitude, respectively.
3.2.2. Error Excitations of System. The error excitations can be modeled by harmonic functions [21]. The main errors considered in this study are as follows: manufacturing error $E_{r}$, installation error $A_{r}$, tooth thickness deviation $\varepsilon_{r}$, and profile error $\delta_{r}$ in the inner ring gear, with the corresponding parameters in the pinions $A_{p j}, \varepsilon_{p j}$, and $\delta_{p j}$, as well as the eccentric error $E_{p j}$. Accordingly, the equivalent accumulated error caused by the aforementioned errors, along with the internal meshing line takes the form

$$
\begin{align*}
\varepsilon_{p r j}(t)= & E_{r} \sin \left(\omega_{r} t+\alpha+\varphi_{j}-\beta_{r}\right)+A_{r} \sin \left(\alpha+\varphi_{j}-\gamma_{r}\right) \\
& +E_{p j} \sin \left(\omega_{p} t+\alpha-\beta_{p i}\right)+A_{p j} \sin \left(\alpha-\gamma_{p j}\right) \\
& +\varepsilon_{r}+\varepsilon_{p j}+\sigma_{r}+\sigma_{p j} \quad(j=1-8) \tag{18}
\end{align*}
$$

where $\alpha$ is meshing angle, $\beta$ and $\gamma$ are the phase angles of errors, and $\omega_{p}$ and $\omega_{r}$ represent the angular velocity of pinion and ring gear, respectively.
3.2.3. Support Equivalent Stiffness of Pinions. Each pinion bears the radial load, so the support equivalent stiffness is calculated by the following empirical formula [22]:

$$
\begin{equation*}
K_{r}=0.34 \times 10^{4} F_{r}^{0.1} Z^{0.9} l^{0.8}(\cos \beta)^{1.9} \tag{19}
\end{equation*}
$$

where $F_{r}$ is radial load, $Z$ is the number of rollers, $l$ is the effective contact length of rollers, and $\beta$ is the contact angle of rollers.
3.2.4. Torsional Stiffness of Shaft Connections. According to Mechanics of Materials, the torsional stiffness of shaft connections can be expressed as

$$
\begin{equation*}
k_{Q}=\frac{G I_{p}}{L} \tag{20}
\end{equation*}
$$

where $I_{p}$ is the polar moment of inertia, $G$ is shear modulus, and $L$ is the equivalent length of shaft connections.
3.2.5. Stiffness of the Main Bearing. Since the positive thrust rollers and negative thrust rollers belong to radial-thrust bearing, the axial stiffness is defined as [23]

$$
\begin{gather*}
F_{a}=K_{n} Z(\sin \Psi)^{n+1} \delta_{a}^{n} \\
K_{a}=\frac{d F_{a}}{d \delta_{a}}=n K_{n} Z(\sin \Psi)^{n+1} \delta_{a}^{n-1},  \tag{21}\\
K_{n}=2.89 \times 10^{4} l_{e}^{0.82} D_{w}^{0.11}
\end{gather*}
$$

where $F_{a}$ is the axial load of the bearing, $n$ is equal to 1.11 about the roller bearing, $K_{n}$ is stiffness coefficient, $Z$ is the number of rollers, $\Psi$ is the contact angle of rollers, $\delta_{a}$ is the axial displacement of the bearing, $l_{e}$ is the effective contact length of rollers, and $D_{w}$ is the effective diameter of rollers.
3.2.6. Meshing Damping Coefficient. The meshing damping coefficient is calculated by the following empirical formula [20]:

$$
\begin{equation*}
C_{p r}=2 \varsigma \sqrt{\frac{k(t) m_{1} m_{2}}{\left(m_{1}+m_{2}\right)}} \tag{22}
\end{equation*}
$$

where $\varsigma$ is meshing damping ratio, which is equal to $0.03-0.17$, with the value of 0.03 in this study.
3.2.7. Torsional Damping Coefficient of Shaft Connections. The torsional damping coefficient of shaft connections is expressed as [20]

$$
\begin{equation*}
C_{\mathrm{Q}}=2 \xi_{\mathrm{Q}} \sqrt{\frac{k_{\mathrm{Q}} I_{m} I_{p}}{\left(I_{m}+I_{p}\right)}}, \tag{23}
\end{equation*}
$$

where $\xi_{Q}$ is torsional damping ratio, which is equal to $0.005-$ 0.075 , with the value of 0.005 in this study.
3.2.8. Other Stiffness and Damping Coefficient. The stiffness of other components, such as cutterhead and support shield body, is calculated by using the finite element method (FEM). And the damping coefficient is calculated by the following formula [24]:

$$
\begin{equation*}
c=2 \xi \sqrt{m_{e} k_{e}} \tag{24}
\end{equation*}
$$

where $\xi$ is damping ratio, which is equal to $0.02-0.05$, when the steel is in elastic stage, with the value of 0.02 in this study, and $m_{e}$ and $k_{e}$ is equivalent mass and equivalent stiffness, respectively.

## 4. Solution of the Dynamic Model

There are not good theoretical methods for solving the dynamic equation (12), because of the characteristics of multifactor coupling, strong nonlinear, time-varying, and randomness of the parameters. Thus, an interpolating algorithm of scattered data is presented, using the fourth-order RungeKutta integration method, to solve the dynamic system with an approximate solution. A flowchart of the solution is shown in Figure 5.

## 5. Engineering Project Example

Taking the cutterhead system of the hard rock TBM of a water tunnel project as a background, an application instance is presented. The relative parameters are as follows: (1) cutterhead geometry: the cutterhead diameter $D=8.53 \mathrm{~m}$, the mass of cutterhead $M=200 \mathrm{t}$, the cutter penetration $P=$ 10 mm , the center cutter number $n=4$, the normal cutter number $m=34$, the gauge cutter number $p=12$, and the disc cutters' plane layout is shown in Figure 6; (2) parameters of the driving system: the driving power $W=3440 \mathrm{~kW}$, the angular velocity of cutterhead $\omega=5.6 \mathrm{r} / \mathrm{min}=0.5861 \mathrm{rad} / \mathrm{s}$, the number of pinions $N=8$, the tooth number of inner ring gear $Z_{r}=174$, the tooth number of each pinion $Z_{p}=14$, the module $m=22$, and the error excitations can be obtained according to Mechanical Design Handbook [24]; (3) rock physical properties: the typical rock is mainly granite gneiss geology, and for detailed mechanical parameters see [12].
5.1. Calculation of External Excitations. Based on the field data $[5,6]$ and simulation's loads of rock breaking under the action of multicutters, combined with the above assumptions, the load-time histories of three types of cutters (center cutters, normal cutters, and gauge cutters) can be obtained, partially shown in Figure 7.

As can be seen in Figure 7, the loads of cutters composed of a series of impact loads show obvious nature of the step period and increase with the cutting depth of cutters in a cycle of rock breaking, which is identical with the engineering experience.

According to formulas (13)-(16), we can obtain the external excitations of cutterhead system, using Fast Fourier Transform (FFT) algorithm to generate the spectral responses. Due to space limitation, this paper only presents the frequency responses of radial force, axial force, and torque in center block, as shown in Figure 8.

Figure 8 shows that there are lots of frequency components in the spectral responses, mainly in $0-10 \mathrm{~Hz}$, which is in agreement with the conclusion in [5, 6]. Moreover, the remaining energy is concentrated in these typical frequency ranges, such as $100-120 \mathrm{~Hz}, 230-260 \mathrm{~Hz}, 340-390 \mathrm{~Hz}$, and $680-700 \mathrm{~Hz}$.


FIgURE 5: Solution flowchart of the dynamic model.


Figure 6: Layout of cutters.
5.2. Analysis of Modal Properties. The free vibration of the linear, time-invariant representation is considered, ignoring the damping coefficient and external excitations in formula (12). Since there is 59 degrees of freedom in the dynamic system, we can obtain 59 natural frequencies and mode

Table 1: Natural frequencies and vibration of the cutterhead system.

| Vibration modes | Natural frequencies $/ \mathrm{Hz}$ |
| :--- | :---: |
| Rigid mode | $f_{1}=0$ |
| Rotational vibration of pinions | $f_{2-9}=57$ |
| and motors | $f_{10}=61, f_{11}=70$, |
| Translational and overturning | $f_{12}=114, f_{13,14}=120$, |
| coupled vibration of cutterhead | $f_{15}=124$ |
| and inner ring gear |  |

shapes, and the lowest fifteen natural frequencies are listed in Table 1.

The vibration modes with normalization are illustrated in Figure 9, where the freedom number 1-59 is the corresponding degree of freedom, respectively.

The main conclusions obtained through Table 1 and Figure 9 are as follows.
(1) The first natural vibration mode is rigid mode, with the rigid motion of inelastic deformation, which keeps the constant transmission ratio to each rotational part.


Figure 7: Loads time history of cutters.
(2) The amplitudes of free vibration are mainly in the middle order modes, and the low and high order modes are relatively smaller.
(3) The lowest fifteen natural vibration modes are mainly rotational vibration of pinions and motors and translational and overturning coupled vibration around arbitrary axis of cutterhead and inner ring gear, which is consistent with the engineering example in [25]. In addition, the low natural frequencies are 57 Hz and 61 Hz , which are greater than the rotation frequency of pinions ( 1.16 Hz ) and meshing frequency ( 16.24 Hz ) of internal excitations. Nevertheless, the resonance of the cutterhead system may be inevitable with the
current parameters, due to the wide frequency of external excitations.
5.3. Dynamic Responses and Analysis. With the first 2 s of external excitations, we can obtain the dynamic responses of cutterhead, as shown in Figures 10 and 11.

From the dynamic results, Figure 10 can draw the following conclusions.
(1) The vibration amplitudes of cutterhead pieces in each direction are less than 1 mm , which shows that the stiffness of cutterhead system is relatively high to resist the impact of external excitations.


FIGURE 8: Frequency spectrum results of external excitations.


Figure 9: Vibration modes of the cutterhead system.
(2) The vibration in each direction is similar to the variation of external excitations, which is influenced greatly.
(3) The magnitude of amplitude in each cutterhead piece is identical, where the axial vibration is maximal, with the amplitude being up to about 0.55 mm . And the maximum amplitude of radial direction is close to 0.25 mm . It is shown that although the radial force is
much less than axial force, the radial stiffness is also relatively lower, which can explain the cause of identical magnitude in each direction. However, the radial vibrations have stronger influence on the cutterhead driving system, which may cause some engineering problems, such as seal failure and abnormal wears of the bearing raceway.

Similar conclusions may be obtained from Figure 11 as follows.
(1) The vibration regularity of center block is consistent with cutterhead pieces, and the amplitude in each direction is slightly smaller, with the maximum amplitude of 0.48 mm in axial direction and 0.2 mm in radial directions. These responses of translational vibration provide input conditions for calculating the joint surface loads of cutterhead.
(2) The maximum angular amplitude around $X$ and $Y$ axes is about 0.016 mrad , while the torsional amplitude around $Z$ axis is about 0.065 mrad , which has a great effect on cutters' wear and weld of cutterhead support ribs, and it may cause abnormal wears of the disc cutters, weld cracking, and other engineering problems.


Figure 10: Dynamic responses of cutterhead pieces.

The comparison of Figure 10 with Figure 11 shows that the translational vibration responses between cutterhead pieces and center block are analogous, and the amplitudes are almost the same. The reason is that each cutterhead piece is attached to the center block, and the connection stiffness is relatively large compared to the external excitations. That is, each cutterhead piece vibrates with the center block, which illustrates that the amplitude of each cutterhead piece is slightly larger than that of the center block. The translational responses of Figures 10 and 11 certificate this regularity.

Thus, as the results mentioned above, it is illustrated that the proposed model and method are effective and correct.

The dynamic responses of cutterhead system are not only related to the time domain but also affected by the frequency; thereby the Fast Fourier Transform (FFT) algorithm is used to generate the frequency responses, as shown in Figures 12 and 13.

As may be seen from the spectral analysis in Figures 12 and 13 , the following can be found.
(1) Under the influence of time-varying internal and external excitations, the main frequencies of dynamic responses are as follows: $100-120 \mathrm{~Hz}, 224 \mathrm{~Hz}, 236 \mathrm{~Hz}$,

390 Hz , and 693 Hz , which are consistent with external excitations. It is indicated that the vibration type of cutterhead belongs to forced vibration; in other words, the dynamic responses are influenced by the external excitations more greatly.
(2) The frequencies of radial, torsional, and overturning vibrations are concentrated in 112 Hz and 120 Hz , which are in good agreement with the natural frequencies of the translational and overturning coupled vibration mode (listed in Table 1). It is suggested that the two frequencies should be mainly considered when cutterhead structure is designed and the boring parameters are matched, avoiding the resonance phenomenon due to the unreasonable selection of system parameters.

The comparison of frequency responses with the natural frequencies and external excitations shows that the frequencies are basically identical, which can further demonstrate the effectiveness of the proposed model and method.

Substituting the dynamic responses of cutterhead into (11), one can obtain tangential and radial relative deformations between each cutterhead piece and central block.


Figure 11: Dynamic responses of the cutterhead center block.

Multiplying the corresponding equivalent support stiffness, the loads of cutterhead joint surface are gained, and, using the rain flow method, one can obtain the load range distributions, and the results are shown in Figure 14.

From the above-mentioned results, the distribution types and characteristic values can be estimated and tested by $K-S$ test. It is observed that the cutterhead joint surface loads approximately obey normal distribution, and the distribution parameters are listed in Table 2.

From the data in Figure 14 and Table 2, the following conclusions can be drawn.
(1) The cutterhead joint surface loads change rapidly with a considerable discrete degree, under the influence of complex factors.
(2) The mean of joint surface loads are highly consistent with the external excitations (minus represents the direction), while the standard deviation in each direction increases by $12-15$ times. It is indicated that the obtained simulation results are correct from the results of mean joint surface loads, and the inertial effect should be considered for the structure design of cutterhead, combined with dynamic analysis.


Figure 12: Frequency responses of the cutterhead pieces.

Table 2: Distribution statistics of the cutterhead joint surface loads.

|  | Joint surface loads |  |  | External excitations |
| :--- | :---: | :---: | :---: | :---: |
|  | Mean $/ \mathrm{kN}$ | Standard deviation/kN | Mean $/ \mathrm{kN}$ | 139.66 |
| Standard deviation/kN |  |  |  |  |
| Tangential | -139.69 | 93.25 | 106.32 | 7.40 |
| Normal | -106.27 | 130.03 | 1410.55 | 11.04 |
| Axial | 1410.55 | 915.48 |  | 60.00 |

These simulation results can provide boundary conditions for dynamic performance optimization and crack propagation of the cutterhead structure.

## 6. Conclusions

In this paper, a multidegree-of-freedom coupling dynamic model is presented for the TBM cutterhead system. Based on the parameters of an actual project and the cutters' forces, the structured modal properties and dynamic responses are analyzed. The main results are summarized as follows.
(1) The lowest fifteen natural vibration modes of the cutterhead system are classified as rigid mode, rotational
vibration modes of pinions and motors, and translational and overturning coupled vibration modes of cutterhead and inner ring gear, and the corresponding natural frequency is 57 Hz and 61 Hz , which is greater than rotation frequency of pinions and meshing frequency of internal excitations. However, the resonance of the cutterhead system may be inevitable due to the overlap frequencies between natural frequencies and external excitations.
(2) The vibration responses of cutterhead are similar to the variation of external excitations, with the identical magnitude of amplitude in each translational direction, where the axial amplitude is about 0.55 mm ,


Figure 13: Frequency responses of the cutterhead center block.
the radial amplitude is close to 0.25 mm , the angular amplitude around $X$ and $Y$ axes is about 0.016 mrad , and the torsional amplitude is almost 0.065 mrad . The results may provide reference for the design of cutterhead driving system and weld strength check of support ribs.
(3) The frequencies of dynamic responses are predominantly concentrated in $100-120 \mathrm{~Hz}, 224 \mathrm{~Hz}, 236 \mathrm{~Hz}$, 390 Hz , and 693 Hz . And it is suggested that the two frequencies of 112 Hz and 120 Hz should be avoided, while carrying out the structural design of cutterhead and matching the boring parameters.
(4) Considering the influence of internal and external excitations, it is shown that the cutterhead joint surface loads change rapidly with large amplitudes, as
well as complex nonlinear characteristics. As the rain flow results have shown, the standard deviation in each direction increases by $12-15$ times. It is indicated that the amplification effect of dynamic loads should be mainly considered in cutterhead structural design, so as to lay a foundation for dynamic optimization and fatigue life assessment of the cutterhead structure.

There are some further topics that should be studied, although we have obtained many effective results about TBM cutterhead system. In the next stage, we will study the parameter influence laws about dynamic characteristics, estimate the fatigue life of the cutterhead based on the joint surface loads, and carry out the field test and vibration experiment in the near further.


Figure 14: Time-varying histories and statistics results of the cutterhead joint surface loads.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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# Direct Surge Margin Control for Aeroengines Based on Improved SVR Machine and LQR Method 

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#### Abstract

A novel scheme of high stability engine control (HISTEC) on the basis of an improved linear quadratic regulator (ILQR), called direct surge margin control, is derived for super-maneuver flights. Direct surge margin control, which is different from conventional control scheme, puts surge margin into the engine closed-loop system and takes surge margin as controlled variable directly. In this way, direct surge margin control can exploit potential performance of engine more effectively with a decrease of engine stability margin which usually happened in super-maneuver flights. For conquering the difficulty that aeroengine surge margin is undetectable, an approach based on improved support vector regression (SVR) machine is proposed to construct a surge margin prediction model. The surge margin modeling contains two parts: a baseline model under no inlet distortion states and the calculation for surge margin loss under supermaneuvering flight conditions. The previous one is developed using neural network method, the inputs of which are selected by a weighted feature selection algorithm. Considering the hysteresis between pilot input and angle of attack output, an online scrolling window least square support vector regression (LSSVR) method is employed to firstly estimate inlet distortion index and further compute surge margin loss via some empirical look-up tables.


## 1. Introduction

To begin with, we will provide a brief background on high stability control for aeroengines, which is an aerodynamic concept different from the stability concept in control theory and an unstable state means that the fan or compressor of an engine goes into a surge state. Super-maneuverability is one of the essential techniques of modern fighters. Yet, in a post-stall state, when engines work under the conditions of high angle of attack, the inlet distortion becomes severe, leading to unstable operations such as weakened engine performance, reduced steady operating surge margin, and even surge in serious conditions. For this issue, national aeronautics and space administration (NASA) initialed and led a famous high stability engine control (HISTEC) research project in 1993, in which an engine stability control in supermaneuver flights was highlighted, and the main idea is that the pressure ratio of an F-100 turbofan engine could be adaptively controlled to regulate cooperating working point
or the surge margin of its fan and compressor when an F-15 ACTIVE (advanced control technology for integrated vehicles) aircraft entered into super-maneuver states. Thus, sufficient stability margin of engines can be ensured in super maneuver state [1-3]. Similarly, Wang et al. [4, 5] proposed an engine stability control law through the recovery of surge margin by compensating losses of the fan pressure ratio when a severe inlet distortion occurs. The literatures $[6,7]$ reported a direct surge margin control, in which a baseline and its loss for surge margin in severe distorted states need to compute, but the calculation for the baseline just employs a simpler definition that related to the fan pressure ratio; as a result it is impossible to reflect the influence in air flow. Although these above two schemes are able to ensure the engine stability, since the surge margin cannot be measured directly, it may be enlarged a lot in the maneuver process for the engine potential not to be fully exploited, resulting in much smaller engine thrust which cannot help the aircraft to accomplish the fast and right modulations in angular position.

Around this interesting topic of aeroengine stability control, a lot of articles, which concerns active component control, have been reported [8-12]. In recent years, an approach of more realistic active stall/surge control has been proposed as an effective approach to realize high stability control [13-15]. This technology mainly takes advantage of the flow characteristics which is denoted as stall inception in high frequency phase of compressor and implements a specific mathematical model between pressure correlation and surge margin. Thus, when the engine enters into a severe distortion state, some emergency fuel control with high frequency can be applied to the engine system so that the temperature and pressure of the combustor will drop quickly; therefore the stall margin will be restored to the safe area. Essentially, this control approach is within the scope of limit protection control near the surge border, the defections of which are still found in a worse coordination with the main control loop, as well as an insufficient exploration in performance potentials.

Obviously, those above methods have a comment prominent drawback which is that surge margin cannot be measured directly or be estimated accurately. Therefore, if the real-time accurate prediction of engine surge margin could be achieved, it can be used as the direct virtual parameter for accurate control, so that a fixed surge margin or a certain distance, between the cooperating work line and the surge border line in the compressor, can be always kept in super-maneuver flights. If this imagination could be realized, engines might sustain a more efficient and stable operation in super maneuver states. Near recently from a NASA report [16], it is right an expected solution to engine high stability control to apply some nonlinear methods like artificial mapping and compound kalman filter [17].

As said above, the establishment of engine surge margin prediction model has always been a challenge in the field of surge control. Our research is carried out on a twospool mixing exhaust turbofan engine with afterburning, the power of which lies in the same class of F-100 engine. First, through a novel feature selection algorithm, the most suitable measurable variables with the strongest correlation with surge margin are selected as inputs of a surge margin baseline model. Furthermore, based on a BP neural network, a surge margin baseline model is set up, which is able to predict the surge margin in undistorted states. In addition, an online scrolling window LSSVR model (OSW-LSSVR), which has a time series of the angle of attack and relative rudder angles as its inputs, is proposed and designed for the real-time prediction of inlet distortion index in distortion states. Due to a definite relationship between inlet distortion index and surge margin loss, the surge margin loss can be estimated using the OSW-LSSVR model. Ultimately, with the combination of the output of surge margin baseline model and the loss in distortion states, the surge margin under engine inlet distortion states can be accurately estimated. As for the design of high stability engine control, an improved linear quadratic regulation (ILQR) robust control is chosen here due to its good robustness for adapting a large variation in engine power states and envelope points [18].

## 2. Main Idea for Direct Surge Margin Control

2.1. A Novel Direct Surge Margin Control. An integrated aircraft and engine dynamic model depicted in Figure 1 is introduced to implement some necessary validations [6, 7]. This comprehensive model is composed by a dynamic aircraft model and its flight controller, a component-level turbofan engine model and its controller which will be designed as the proposed controller, and a conventional one for comparisons. For the aircraft model, the dynamics in longitudinal plane is only employed for simplicity, which can simulate level flight, climbing, accelerating, descending and super maneuver tasks. For the engine model, $A_{8}$ and $W_{\mathrm{fb}}$ are selected as the control parameter, whereas $S_{\mathrm{mf}}$ and $N_{c}$ are the relative controlled variables. Therefore, the engine command $r$ includes the references value of $S_{\mathrm{mf}}$ and $N_{c}$, and $e$ is a deviation vector with respect to command signal. A data set about the relationship between $\alpha$ and $\mathrm{DC}_{60}$ that is a type of distortion index are originated from flight test data of a F/A18A aircraft in the Flight Research Center in NASA Dryden [19]. As discussed above, if the surge margin for an engine can be predicted accurately, a direct engine surge margin control, or a high stability engine control, easily can be further realized. In our research, a novel scheme is proposed (see Figure 1), where $k$ represents the current moment and $\mathbf{S t}$ is a sequential engine state values. The main idea is as follows.
(a) Considering the hysteresis between pilot input and rudder output, the current manipulations will affect the outputs in $d$-step time delay (or $\tau$ seconds in continuous sense) and the prediction for $\alpha$ is very important in a super maneuver task in which $\alpha$ changes very violently [1-3] to have a great impact on the surge margin. Therefore, two parameters are needed to control this high stability control scheme. One is $\widetilde{S}_{\mathrm{mf}}(k+d)$, that is a predicted surge margin $\widetilde{S}_{\mathrm{mf}}$ at the $d$-step ahead time, to adapt the variation in $S_{\mathrm{mf}}$ due to the changes in $\alpha$. The other need to control is $N_{c}$ which regulates the power of the engine to supply enough thrust $F$.
(b) So when a super maneuver of the fighter is carried out, how to estimate $\widetilde{S}_{\mathrm{mf}}(k+d)$ is a core problem in this scheme. For calculating $\widetilde{S}_{\mathrm{mf}}(k+d)$, a time series of angle of $\alpha$ and the elevator $\delta_{z}$ are taken as inputs of the angle of attack prediction model to predict the $d$-step ahead value $\widetilde{\alpha}(k+d)$. Necessary engine state values in St series are obtained through the relative sensors to mapping the baseline value of $S_{\mathrm{mf}}$. After that, the $\widetilde{\alpha}(k+d)$ and St sequence are gathered as the inputs of fan surge margin prediction model; consequently the predicted surge margin $\widetilde{S}_{\mathrm{mf}}(k+d)$ would be gotten in real time. For showing clearly, an equivalent pilot input delay system is labeled with a dotted rectangle in Figure 1.

Remark 1. In our research, the simulation step is preset as 20 ms ; so $d=\tau / 0.02=0.5 / 0.02=25$ is predefined here based on [1-3].


Figure 1: Structure for direct engine surge margin control.
2.2. Necessity to Estimate Surge Margin. For a turbofan engine, a parameter or a concept is defined as surge margin to show a physical meaning of a relative distance from the cooperate working point to the surge border (see Figure 2), and the following equation is generally used to represent the surge margin of a fan, and similar for a compressor [20]:

$$
\begin{equation*}
S_{m}=\frac{\pi_{s} / W_{\mathrm{cor}, \mathrm{~s}}-\pi_{o} / W_{\mathrm{cor}, o}}{\pi_{o} / W_{\mathrm{cor}, o}}=\frac{\pi_{s}}{\pi_{o}} \cdot \frac{W_{\mathrm{cor}, o}}{W_{\mathrm{cor}, \mathrm{~s}}}-1 \tag{1}
\end{equation*}
$$

When engine inlet distortions happen due to high attack angle states, the surge border of the engine will be shifted down or the operation line will be shifted up, leading to a surge margin loss. Take the example of the shifting-down of surge border, and we have:

$$
\begin{align*}
\Delta S_{m}= & S_{m}-\widetilde{S}_{m} \\
= & \frac{\pi_{s} / W_{\mathrm{cor}, \mathrm{~s}}-\pi_{o} / W_{\mathrm{cor}, o}}{\pi_{o} / W_{\mathrm{cor}, o}}  \tag{2}\\
& -\frac{\pi_{l} / W_{\mathrm{cor}, \mathrm{~s}}-\pi_{o} / W_{\mathrm{cor}, o}}{\pi_{o} / W_{\mathrm{cor}, o}}=\frac{\pi_{s} / W_{\mathrm{cor}, \mathrm{~s}}-\pi_{1} / W_{\mathrm{cor}, l}}{\pi_{o} / W_{\mathrm{cor}, o \mathrm{o}}}
\end{align*}
$$

where $\Delta S_{m}$ shows surge margin loss and $\widetilde{S}_{m}$ represents the surge margin when entering distortion; the subscript 1 shows the lowered stable boundary value.

As can be seen from the above equations, $S_{m}$ and $\Delta S_{m}$ are parameters that cannot be measured. Under distorted states, if the undistorted $S_{m}$ and the surge margin loss $\Delta S_{m}$ might be accurately predicted, the distorted surge margin $\widetilde{S}_{m}$ will be expressed as

$$
\begin{equation*}
\widetilde{S}_{m}=S_{m}-\Delta S_{m} . \tag{3}
\end{equation*}
$$

For calculating $\Delta S_{m}$, an indirect parameter, called pressure ratio loss of the cooperating work point, is usually


Figure 2: Surge margin change in a fan characteristic map.
utilized according to the [21,22]. Due to the shifting down of the boundary line, the pressure ratio loss $\Delta P_{\mathrm{rs}}$ is expressed as

$$
\begin{equation*}
\Delta P_{\mathrm{rs}}=\frac{\pi_{s} / W_{\mathrm{cor}, \mathrm{~s}}-\pi_{1} / W_{\mathrm{cor}, l}}{\pi_{s} / W_{\mathrm{cor}, s}} \tag{4}
\end{equation*}
$$

With the combination of (1) and (2) and the definition of $\Delta P_{\mathrm{rs}}$, the surge margin loss is easily expressed as

$$
\begin{align*}
\Delta S_{m} & =\frac{\pi_{s} / W_{\mathrm{cor}, \mathrm{~s}}-\pi_{1} / W_{\mathrm{cor}, l}}{\pi_{o} / W_{\mathrm{cor}, o}} \\
& =\frac{\pi_{s} / W_{\mathrm{cor}, \mathrm{~s}}-\pi_{1} / W_{\mathrm{cor}, l}}{\pi_{s} / W_{\mathrm{cor}, s}} \frac{\pi_{s} / W_{\mathrm{cor}, s}}{\pi_{o} / W_{\mathrm{cor}, o}}  \tag{5}\\
& =\Delta P_{\mathrm{rs}}\left(1+S_{m}\right) .
\end{align*}
$$

2.3. Surge Margin Prediction Model. Based on the current and historical information of input parameters, dynamic


Figure 3: Engine surge margin modeling in distortion states.
solutions for the baseline value $S_{m}$ and the loss value $\Delta S_{m}$ can be acquired or estimated by a novel engine surge margin model for super-maneuver flight, which consists of a surge margin baseline model and surge margin loss model. The modeling process is shown in Figure 3 and described as follows.
2.3.1. Surge Margin Baseline Model. In the range of supermaneuver flight envelope (height of $3 \sim 7 \mathrm{~km}$, march number of $0.3 \sim 0.7$ ), a surge margin baseline model based on a BP neural network, which is under no inlet distortion of the fighter, is trained to predict surge margin baseline value. Consider a lot of and redundant measurable parameters correlated with surge margin, a feature selection algorithm, for choosing adaptable engine sensors to estimate surge margin, is proposed by a weighted LSSVR method. With a contribution criterion of each parameter to surge margin, a group of most related and affordable parameters are selected as the model inputs.
2.3.2. Surge Margin Modeling in Distortion States. On the basis of surge margin baseline model, surge margin modeling in distortion states can be implemented in the following steps.

First, an online scrolling window LSSVR model (OSWLSSVR) is proposed to predict the $d$-step ahead value $\widetilde{\alpha}(k+d)$. The OSW-LSSVR model is capable of updating the prediction model based on the time series of angle of attack and elevators to adapt significant changes in power states and the extensive envelope range.

Second, the predicted $\widetilde{\alpha}(k+d)$ is used to identify $\mathrm{DC}_{60}$ at the $d$ steps ahead moment [21], and a certain relationship [19, 22], some empirical curves or look-up tables, between $\mathrm{DC}_{60}$ and the pressure ratio loss $\Delta P_{\mathrm{rs}}$, is employed to get the relative value at the $d$ steps ahead moment. It follows that

$$
\begin{gather*}
\mathrm{D}_{60}(k+d)=\mathrm{DC}_{60}\left(M_{a}, \tilde{\alpha}(k+d)\right),  \tag{6}\\
\Delta \widetilde{P}_{\mathrm{rs}}(k+d)=\left(\mathrm{D}_{60}(k+d)\right) \tag{7}
\end{gather*}
$$

Next, from (5) engine surge margin loss at $k+d$ moment is estimated as

$$
\begin{equation*}
\Delta \widetilde{S}_{m}(k+d) \approx \Delta \widetilde{P}_{\mathrm{rs}}(k+d)\left(1+S_{m}(k)\right) \tag{8}
\end{equation*}
$$

Finally, the surge margin in distortion states at $k+d$ moment can be obtained as

$$
\begin{equation*}
\widetilde{S}_{m}(k+d)=S_{m}(k)-\Delta \widetilde{S}_{m}(k+d) \tag{9}
\end{equation*}
$$

Remark 2. $S_{m}(k)$ is used to calculate $\Delta S_{m}(k+d)$ in (8) if $d$ steps are too short for phugoid dynamics in flight motions to influence the $S_{m}$ baseline model. The detailed designation and validation for surge margin baseline model, angle of attack prediction model and direct engine surge margin control are, respectively, clarified as follows.

## 3. How to Set Up a Surge Margin Baseline Model

As discussed above, the surge margin baseline modeling should firstly identify the model inputs. Based on the LSSVR algorithm [23,24], a weighted feature selection algorithm is proposed so as to screen a group of most valuable parameters that can be measured as inputs of the baseline model. Then, a neural network with nonlinear mapping capability is used for the surge margin baseline model design. Three parts, containing the preliminary LSSVR algorithm, surge margin feature selection and baseline model, are deduced and stated respectively as follows.
3.1. LSSVR Algorithm. To solve a nonlinear regression problem for a training data set $\left\{\left(\mathbf{x}_{i}, y_{i}\right)\right\}_{i=1, \ldots, M}$, where $\mathbf{x}_{i} \in R^{m}$ is the input with $m$-dimension and $\mathbf{y}_{i} \in R$ is its corresponding output.

A least square support vector regression (LSSVR) for this training data set can be transformed to an optimization problem as

$$
\begin{array}{ll}
\min _{\mathbf{w}, e_{i}} J\left(\mathbf{w}, e_{i}\right)=\frac{1}{2} \mathbf{w}^{T} \mathbf{w}+\frac{\gamma}{2} \sum_{i=1}^{M} e_{i}^{2}  \tag{10}\\
\text { s.t. } & y_{i}=\mathbf{w}^{T} \varphi\left(\mathbf{x}_{i}\right)+b+e_{i}, \quad i=1, \ldots, M,
\end{array}
$$

where $\mathbf{w}$ represents the model complexity, $b$ is the offset, $\mathbf{e}=$ $\left[e_{1}, e_{2}, \ldots, e_{M}\right]$ represents prediction residual vector, $\gamma \in R^{+}$ is a regularization parameter, and $\varphi(\cdot)$ is a nonlinear mapping which can transform the input data into a high-dimensional

Table 1: Effects by feature selection for fan surge margin model.

| Order | 1 | 2 | 3 | 4 | 5 | Others |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| Sensors | $N_{f}$ | $W_{\mathrm{fb}}$ | $T_{22}$ | $P_{3}$ | $P_{2}$ | $\ldots$ |
| $\Upsilon(i)$ | 9.2179 | 1.5517 | 1.5298 | 1.1396 | 1.1178 |  |

feature space. In order to solve the above problem, a Lagrange function without constraints can be constructed as

$$
\begin{equation*}
L\left(\mathbf{w}, b, e_{i}, \alpha_{i}\right)=J-\sum_{i=1}^{M} \alpha_{i}\left\{\mathbf{w}^{T} \varphi\left(\mathbf{x}_{i}\right)+b+e_{i}-y_{i}\right\} \tag{11}
\end{equation*}
$$

where $\alpha_{i}$ is Lagrange multiplier.
Through Karush-Kuhn-Tucker (KKT) derivation, a Wolfe dual optimization problem for (10) can be expressed as

$$
\begin{gather*}
\min _{b, \alpha_{i}}\left\{L\left(b, \alpha_{i}\right)=\left[\frac{1}{2} \sum_{i, j=1}^{M} \alpha_{i} \alpha_{j} k\left(\mathbf{x}_{i}, \mathbf{x}_{j}\right)+\frac{1}{2 \gamma} \sum_{i=1}^{M} \alpha_{i}^{2}\right.\right. \\
\left.\left.-\sum_{i=1}^{M} \alpha_{i} y_{i}+b \sum_{i=1}^{M} \alpha_{i}\right]\right\} \tag{12}
\end{gather*}
$$

where $k\left(x_{i}, x_{j}\right)$ is a kernel function chosen as $\exp \left\{-\| x_{i}-\right.$ $\left.x_{j} \|^{2} / 2 \nu^{2}\right\}$.

After that, (12) can be reformulated for convenience:
$\min \left\{L(b, \boldsymbol{\alpha})=\left[\frac{1}{2}\left[\begin{array}{ll}\boldsymbol{\alpha}^{T} & b\end{array}\right]\left[\begin{array}{ll}\mathbf{K} & \mathbf{1} \\ \mathbf{1}^{T} & 0\end{array}\right]\left[\begin{array}{l}\boldsymbol{\alpha} \\ b\end{array}\right]-\left[\begin{array}{ll}\boldsymbol{\alpha}^{T} & b\end{array}\right]\left[\begin{array}{l}\mathbf{Y} \\ 0\end{array}\right]\right]\right\}$.

Then the optimal solution for (14) is easily obtained as

$$
\left[\begin{array}{l}
\boldsymbol{\alpha}^{*}  \tag{14}\\
b^{*}
\end{array}\right]=\left[\begin{array}{cc}
\mathbf{K} & \mathbf{1} \\
\mathbf{1}^{T} & 0
\end{array}\right]^{-1}\left[\begin{array}{l}
\mathbf{Y} \\
0
\end{array}\right]=\mathbf{R}\left[\begin{array}{l}
\mathbf{Y} \\
0
\end{array}\right]
$$

where $\mathbf{Y}=\left[y_{1}, y_{2}, \ldots, y_{M}\right]^{T}, \boldsymbol{\alpha}=\left[\alpha_{1}, \alpha_{2}, \ldots, \alpha_{M}\right]^{T}, \mathbf{1}=$ $\left[1_{1}, 1_{2}, \ldots, 1_{M}\right]^{T}$, and $\mathbf{K}$ is the kernel matrix in which

$$
\begin{array}{r}
\mathbf{K}_{i j}=k\left(x_{i}, x_{j}\right)+\frac{\delta_{i j}}{\gamma}=\varphi^{T}\left(x_{i}\right) \varphi\left(x_{j}\right) \\
\text { with } \delta_{i j}= \begin{cases}1, & i=j \\
0, & i \neq j\end{cases} \tag{15}
\end{array}
$$

After obtaining the solution $\boldsymbol{\alpha}$ and $b$ by (14), for any new testing sample $\mathbf{x} \in R^{m}$, a regression or a predictor for $\mathbf{x}$ is gotten as

$$
\begin{equation*}
f(\mathbf{x})=\sum_{i=1}^{M} \alpha_{i}^{*} k\left(\mathbf{x}_{i}, \mathbf{x}\right)+b^{*} \tag{16}
\end{equation*}
$$

3.2. Feature Selection for Different Sensors. An aeroengine is not allowable to install too many sensors; otherwise it will degrade overall performance and increase computational complexity of the control systems. Therefore, a problem must
be solved before setting up a surge margin baseline model; that is, which measurable variables will be selected as input variables for the model. In this section, an improved criterion for ranking variables is induced below [25, 26].

Considering that each measurable component in $\mathbf{x}$ has different affordable fractions such as maintainability and price for variant sensors, a weighted matrix is necessary to be introduced here to quantify the influence of the affordable aspects for each parameter. Therefore, every sample or input should be transformed into the following expression:

$$
\begin{equation*}
\mathbf{x}^{\prime}=\Lambda \mathbf{x} \tag{17}
\end{equation*}
$$

where $\Lambda=\Lambda^{T}>0$ is defined as an affordable weighted matrix and proper dimension, and the problem (16) is changed as follows:

$$
\min \left\{L(b, \boldsymbol{\alpha})=\left[\frac{1}{2}\left[\begin{array}{ll}
\boldsymbol{\alpha}^{T} & b
\end{array}\right]\left[\begin{array}{ll}
\mathbf{K}^{\prime} & \mathbf{1}  \tag{18}\\
\mathbf{1}^{T} & 0
\end{array}\right]\left[\begin{array}{l}
\boldsymbol{\alpha} \\
b
\end{array}\right]-\left[\begin{array}{ll}
\boldsymbol{\alpha}^{T} & b
\end{array}\right]\left[\begin{array}{l}
\mathbf{Y} \\
0
\end{array}\right]\right]\right\}
$$

where $\mathbf{K}^{\prime}$ is a weighted kernel matrix in which

$$
\mathbf{K}_{i j}^{\prime}=k\left(x_{i}^{\prime}, x_{j}^{\prime}\right)+\frac{\delta_{i j}}{\gamma} \quad \text { with } \delta_{i j}= \begin{cases}1, & i=j  \tag{19}\\ 0, & i \neq j\end{cases}
$$

So from (14), the optimal value $L^{*}$ of (17) is easily gotten as

$$
\begin{align*}
L^{*} & =-\frac{1}{2}\left[\begin{array}{ll}
\mathbf{Y}^{T} & 0
\end{array}\right]\left[\begin{array}{ll}
\mathbf{K}^{\prime} & \mathbf{1} \\
\mathbf{1}^{T} & 0
\end{array}\right]^{-1}\left[\begin{array}{l}
\mathbf{Y} \\
0
\end{array}\right]  \tag{20}\\
& =-\frac{1}{2}\left[\begin{array}{ll}
\mathbf{Y}^{T} & 0
\end{array}\right]\left[\begin{array}{l}
\boldsymbol{\alpha}^{* \prime} \\
b^{* \prime}
\end{array}\right]=-\frac{1}{2} \mathbf{Y}^{T} \boldsymbol{\alpha}^{* \prime}
\end{align*}
$$

In the calculation process, if the optimal value of $L, L^{*}$ is obtained, the surge margin prediction has the best precision. If the $i$ th variable is removed, $\bar{L}^{*}(i)=-1 / 2 \mathbf{Y}^{T} \overline{\boldsymbol{\alpha}}(i)$, where $\overline{\boldsymbol{\alpha}}(i)$ is the solution of (18) without the $i$ th input variable. Thus, a criterion for ranking variables can be put forward:

$$
\begin{equation*}
\Upsilon(i)=\frac{\bar{L}^{*}}{L^{*}}=\frac{\mathbf{Y}^{T} \overline{\boldsymbol{\alpha}}^{* \prime}(i)}{\mathbf{Y}^{T} \boldsymbol{\alpha}^{* \prime}} \tag{21}
\end{equation*}
$$

If the value of $\Upsilon(i)$ is smaller than the value of $\Upsilon(j)(j \neq i)$, the $i$ th variable is considered to make less contribution to the optimal value $\bar{L}^{*}$ than the $j$ th variable. As for the aeroengine, there are more than 20 measurable variables for the selection. Then, each $\Upsilon(i)$ for different sensors should be computed based on (21), so all the variables can be ranked and some of them might be selected as the most contributors to surge margin baseline model. At last, the inputs St of $S_{\mathrm{mf}}$ baseline model are chosen as $N_{f}, W_{\mathrm{fb}}, T_{22}, P_{2}$, and $P_{3}$ in Table 1, and the ones of $S_{\mathrm{mc}}$ baseline model are $N_{c}, T_{3}, \pi_{c}, T_{22}$, and $P_{3}$ (see Table 2).

TABLE 2: Effects by feature selection for compressor surge margin model.

| Order | 1 | 2 | 3 | 5 | Others |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| Sensors | $N_{c}$ | $T_{3}$ | $\pi_{c}$ | $T_{22}$ | $P_{3}$ | $\ldots$ |
| $\Upsilon(i)$ | 12.3796 | 2.1351 | 1.4773 | 1.3526 | 1.2584 | $\ldots$ |



Figure 4: The topological structure for BP model.
3.3. $S_{m}$ Baseline Model. After the determination for the input parameters of the surge margin baseline model, a threelayer BP neural network [27, 28], shown in Figure 4, can be employed for the model designing, and its mathematical expression is

$$
\begin{equation*}
\mathbf{Y}=\mathrm{NN}(\mathbf{X}) . \tag{22}
\end{equation*}
$$

Figure 4 shows the topological structure of the three-layer BP neural network, in which there are $p$ input points for input layer, $q$ hidden points for hidden layer, and $r$ out layer points. $w_{i j}$ is denoted as connecting weighted coefficient between input and hidden layers, and $w_{j k}$ denoted as connecting weighted coefficient between the input and hidden layers. BP model can be described as follows.

For the hidden layer, the followed equations can be acquired for the $j$ th point:

$$
\begin{equation*}
H(j)=f_{H}\left(\operatorname{net}_{j}\right), \quad \operatorname{net}_{j}=\sum_{i=1}^{p} w_{i j} X(i) \tag{23}
\end{equation*}
$$

where $f_{H}(\cdot)$ is denoted as an excitation function for hidden layer, $i=1$ to $p$ and $j=1$ to $q$.

And for the output layer, the following expressions can also be gotten for the $m$ th point:

$$
\begin{equation*}
\operatorname{net}_{m}=\sum_{j=1}^{q} w_{j m} H(j), \quad Y(m)=g_{\mathrm{O}}\left(\text { net }_{m}\right), \tag{24}
\end{equation*}
$$

where $g_{\mathrm{O}}(\cdot)$ is denoted as an excitation function for output layer and $m=1$ to $r . f(\cdot)$ and $g(\cdot)$ are both chosen as sigmoid functions.

If an evaluation function is defined as $E=$ $1 / 2 \sum_{m=1}^{r}(d(m)-Y(m))^{2}$, where $d(m)$ is an expected output for the network, the adjusting rule for those weighted coefficients is deduced as follows:

$$
\begin{equation*}
\Delta w_{j k}=-\eta \frac{\partial E}{\partial w_{j k}}, \quad \Delta w_{i j}=-\eta \frac{\partial E}{\partial w_{i j}} \tag{25}
\end{equation*}
$$

where the scalar $\eta>0$ is a factor about convergent rate. For more details about BP algorithm, one can refer to [26].

Overfitting often faces designers of intelligent identification methods like neural network, and it means that a more complex structure does not mean a more accurate mapping for all the inputs. So, the number of middle layer neurons needs to modulate with care to acquire the better choice, when the number of middle layer neurons in the $S_{\mathrm{mf}}$ model is adjusted as 8,14 , and 16 , and the relative test error is calculated as $7.58 \mathrm{e}^{-3}, 6.38 \mathrm{e}^{-3}$, and $6.68 \mathrm{e}^{-3}$ separately in Figure 5. Similarly, the number in the $S_{\mathrm{mc}}$ model is modulated as 7,12 , and 13 , and the relative test error is $6.73 \mathrm{e}^{-4}, 5.69 \mathrm{e}^{-4}$, and $5.79 \mathrm{e}^{-4}$, respectively (see Figure 6). So we choose 14 as the best number of middle layer neurons for the $S_{\mathrm{mf}}$ model, and 12 for the $S_{\mathrm{mc}}$ model.

Figure 7 presents a comparison between the predicted and real $S_{\mathrm{mf}}$ and $S_{\mathrm{mc}}$ in Matlab environment. The relative testing errors are within $5 \%$ and $1 \%$, respectively, indicating surge margin models have satisfactory prediction precision. Moreover, for testing the accuracy and real-time ability in VC++ environment which is our destination simulation environment, Figure 8 illustrates the simulation results between the outputs from real nonlinear plant and its prediction model for comparisons, and these results discover that the


Figure 5: Precision test of different middle layer neurons of surge margin model.


Figure 6: Precision test of different middle layer neurons of surge margin model.


Figure 7: Precision test for surge margin prediction model in Matlab environment.


Figure 8: Precision test of surge margin prediction model in VC++ environment.
prediction model is capable of tracking the relative surge margin satisfactorily (dynamic error within $5 \%$ for $S_{\mathrm{mf}}$ and $1 \%$ for, $S_{\mathrm{mc}}$ ), where "pre" means the predicted one.

## 4. How to Set Up an Angle of Attack Prediction Model

As discussed above, the inlet distorted index caused by flight angle of attack determines the loss of the surge margin. For adapting the changes of angle of attack that cannot be predefined due to time delay, an OSW-LSSVR model is proposed to estimate the $d$ step ahead value and it can pull new samples
into an online training set, called a scrolling window, to adapt large variations in engine states and envelope points. The OSW-LSSVR algorithm and angle of attack prediction model are, respectively, described as follows.
4.1. OSW-LSSVR Algorithm. In order to realize an online LSSVR, after obtaining a new training data pair $\left(\mathbf{x}_{n}, \mathbf{y}_{n}\right)$, a new predictor is needed to be reconstructed according to (14). For reducing the computational complexity to solve (14) directly, an iterative strategy $[29,30]$ is adopted here.

A concept $P$ is firstly introduced as an index set, and corresponding training data of which are utilized to construct
the predictor at $n$th iteration. $|\cdot|$ represents the cardinality. And the corresponding training data set or scrolling window set is defined as

$$
\begin{equation*}
\mathbf{x}^{P}=\left\{\mathbf{x}_{i}^{P}\right\}, \quad|P| \leq \bar{l} \tag{26}
\end{equation*}
$$

where $l \in N$ is denoted as the storage capacity of training samples in the scrolling window.

So for the $n$th iteration, it follows that

$$
\mathbf{R}^{n}=\left[\begin{array}{cc}
\mathbf{K}_{P P} & \mathbf{k}_{P n}  \tag{27}\\
\mathbf{k}_{P n}^{T} & k_{n n}
\end{array}\right]^{-1}
$$

where $\mathbf{K}_{P P}$ is a relative weighted kernel matrix in which

$$
\begin{gather*}
\mathbf{K}_{P P, i j}=k\left(x_{i}^{P}, x_{j}^{P}\right)+\frac{\delta_{i j}}{\gamma} \quad \text { with } \delta_{i j}=\left\{\begin{array}{cc}
1, & i=j, \\
0, & i \neq j,
\end{array}\right. \\
k_{n n}=k\left(\mathbf{x}_{n}, \mathbf{x}_{n}\right)+\frac{1}{\gamma}, \quad \mathbf{k}_{P n}=\left(\begin{array}{c}
k\left(\mathbf{x}_{1}^{P}, \mathbf{x}_{n}\right)+\frac{1}{\gamma} \\
\vdots \\
k\left(\mathbf{x}_{i}^{P}, \mathbf{x}_{n}\right)+\frac{1}{\gamma} \\
\vdots \\
k\left(\mathbf{x}_{|P|}^{P}, \mathbf{x}_{n}\right)+\frac{1}{\gamma}
\end{array}\right) \tag{28}
\end{gather*}
$$

Then, $\mathbf{R}^{n}$ can be obtained as

$$
\mathbf{R}^{n}=\left[\begin{array}{cc}
\mathbf{R}^{n-1} & \mathbf{0}  \tag{29}\\
\mathbf{0}^{T} & 0
\end{array}\right]+\lambda\left[\begin{array}{c}
\boldsymbol{\beta} \\
-1
\end{array}\right]\left[\begin{array}{ll}
\boldsymbol{\beta}^{T} & -1
\end{array}\right]
$$

where $\boldsymbol{\beta}=\mathbf{R}^{n-1}\left[\begin{array}{c}1 \\ \mathbf{k}_{P_{n}}\end{array}\right], \lambda=\left(k_{n n}-\left[\begin{array}{ll}1 & \mathbf{k}_{P n}^{T}\end{array}\right]^{T} \boldsymbol{\beta}^{T}\right)^{-1}, \mathbf{x}_{1}^{P} \in \mathbf{x}^{P}$.
Assuming that $\mathbf{a}$ and $b$ at the $n-1$ th iteration are computed with the equation $\left[\begin{array}{c}\boldsymbol{\alpha}_{P}^{n-1} \\ b^{n-1}\end{array}\right]=\mathbf{R}^{n-1}\left[\begin{array}{c}\mathbf{Y}_{P} \\ 0\end{array}\right]$, the formulation of computing a and $b$ at the $n$th iteration can be expressed as

$$
\begin{align*}
{\left[\begin{array}{l}
\boldsymbol{\alpha}_{P}^{n} \\
\alpha_{n} \\
b^{n}
\end{array}\right]=} & \mathbf{R}^{n}\left[\begin{array}{l}
\mathbf{Y}_{P} \\
y_{n} \\
0
\end{array}\right]=\left[\begin{array}{c}
\mathbf{R}^{n-1}\left[\begin{array}{c}
\mathbf{Y}_{P} \\
0
\end{array}\right] \\
0
\end{array}\right] \\
& +\lambda\left(\boldsymbol{\beta}^{T}\left[\begin{array}{c}
\mathbf{Y}_{P} \\
0
\end{array}\right]-y_{n}\right)\left[\begin{array}{c}
\boldsymbol{\beta} \\
-1
\end{array}\right] \\
= & {\left[\begin{array}{c}
\boldsymbol{\alpha}_{P}^{n-1} \\
b^{n-1} \\
0
\end{array}\right]+\lambda\left(\boldsymbol{\beta}^{T}\left[\begin{array}{c}
\mathbf{Y}_{P} \\
0
\end{array}\right]-y_{n}\right)\left[\begin{array}{c}
\boldsymbol{\beta} \\
-1
\end{array}\right] . } \tag{30}
\end{align*}
$$

From (29) and (30), R, a, and $b$ can be efficiently updated, and the predictor at the $n$th irritation is

$$
\begin{equation*}
f^{n}(\mathbf{x})=\sum_{i \in P} \alpha_{P, i}^{n} k\left(\mathbf{x}_{i}^{P}, \mathbf{x}\right)+b^{n} \tag{31}
\end{equation*}
$$

The $n$th predictor in (31) is utilized to estimate the output value of $\mathbf{x}_{n}$, denoted as $\tilde{y}_{n}$. If a criterion followed is defined as:

$$
\begin{equation*}
\left|y_{n}-\tilde{y}_{n}\right|<\varepsilon, \tag{32}
\end{equation*}
$$

where $\varepsilon$ is a threshold which can control the tradeoff between the prediction accuracy and the parsimoniousness.

Then, $\mathbf{x}_{n}$ will be discarded. Otherwise, it will be chosen as a new support vector to the scrolling window.

Remark 3. Different from the above online LSSVR proposed in [30] in which only considering current samples. For meeting a $d$ step ahead prediction model, some special improvement must be made and described as follows.
(a) As the output value in the future moment is unknown, the criterion proposed above cannot be realized when directly using $|\alpha(k+d)-\widetilde{\alpha}(k+d)|<\varepsilon$. Provided that the current attack angle has closer features with the one needing to predict at the $d$-step ahead moment, the criterion at $k+d$ moment is replaced with the one at current moment. In other words, $\mid \alpha(k)-$ $\widetilde{\alpha}(k) \mid<\varepsilon$ is taken as the criterion of prediction model. For showing the precision more clearly, a relative deviation $e(k)$ is taken as a new threshold rule as follows:

$$
\begin{equation*}
e(k)=\left|\frac{\alpha(k)-\widetilde{\alpha}(k)}{\alpha(k)}\right| \times 100 \%<\varepsilon^{\prime} \tag{33}
\end{equation*}
$$

(b) For enhancing the real-time ability for the online LSSVR, the $\bar{l}$ for a scrolling window should be kept constant, but in [30] the value is not limited. Meanwhile, a simple but efficient judge logic is adopted to determine whether the new sample or support vector can be accepted as the support vector. If the number of training samples is still not more than $\bar{l}, \mathbf{x}_{n}$ will be taken as support vector directly when $\mathbf{x}_{n}$ is added into the scrolling window. Otherwise, $\mathbf{x}_{\min }^{P}$ would be given away before $\mathbf{x}_{n}$ is put into the scrolling window, where

$$
\begin{equation*}
\mathbf{x}_{\min }^{P}=x_{i}^{P} \mid \min \left(\left|y_{i}^{P}-\widehat{y}_{i}^{P}\right|<\varepsilon\right) \tag{34}
\end{equation*}
$$

(c) As $d$ step ahead information is impossible to get, some offline samples need to be gathered in advance for starting the online training process. In this study case, as said above not less than 25 samples are needed.
4.2. Angle of Attack Prediction Model. With the above OSWLSSVR algorithm, the online angle of attack prediction model can be devised, and a 3-order prediction model with OSWLSSVR is set up as follows:

$$
\begin{equation*}
\mathbf{Y}^{\prime}(k)=\operatorname{AOA}\left(\mathbf{X}^{\prime}(k-d)\right) \tag{35}
\end{equation*}
$$

where $\mathbf{X}^{\prime}=\left[\alpha(k-d-2), \alpha(k-1-d), \alpha(k-d), \delta_{z}(k-d)\right]$, and $\mathbf{Y}^{\prime}=[\widetilde{\alpha}(k), \widetilde{\alpha}(k-1)]$.

Thus, for predicting the value, the $k+d$ moment is easily gotten as

$$
\begin{equation*}
\mathbf{Y}^{\prime}(k+d)=\operatorname{AOA}\left(\mathbf{X}^{\prime}(k)\right) \tag{36}
\end{equation*}
$$

where $\mathbf{X}^{\prime}(k)=\left[\alpha(k-2), \alpha(k-1), \alpha(k), \delta_{z}(k)\right]$, and $\mathbf{Y}^{\prime}(k+d)=$ $[\widetilde{\alpha}(k+d), \widetilde{\alpha}(k+d-1)]$.


Figure 9: Model precision test at $H=4 \mathrm{~km}, M_{a}=0.5$.


Figure 10: Model precision test at $H=5 \mathrm{~km}, M_{a}=0.55$.

As discussed above, the threshold determination is conducted through (33) to decide whether the scrolling window should be updated. If $e(k)$ is smaller than $\varepsilon^{\prime}$, it means that the prediction has a satisfactory precision, and $\alpha(k+d)$ can be accurately predicted without updating the scrolling window and vice versa. Modulated by trial and error, parameters in the OSW-LSSVR model are as follows.

Rolling window capacity $\bar{l}=80$, threshold value $\varepsilon^{\prime}=$ 0.0048 , kernel parameter $v=1.6$, and penalty factor $\gamma=$ $2^{24}$. Where the selection principle of $\bar{l}$ is to keep the rolling window capacity as small as possible, so as to optimize the instantaneity of the online model.

As shown in Figures 9 and 10, in the envelop point of $H=$ $4 \mathrm{~km}, M_{a}=0.5$, and $H=5 \mathrm{~km}, M_{a}=0.55$, the angle of attack prediction model is verified, respectively. For simulating the dynamic change of angle of attack in super maneuver flight, the elevator is sufficiently excited to get a variation range of angle of attack from $-10^{\circ}$ to $+50^{\circ}$. The correlation curves are shown in Figures 9(a) and 10(a), including about 1990 groups of samples for the actual and predicted one. The model can quickly and accurately predict the angle of attack at $k+d$ moment. And also seen clearly from Figures 9(b) and 10(b), the relative testing errors are within $1 \%$, which means a much better prediction effect compared to the effects reported in the literatures $[6,7]$.

## 5. Direct Engine Surge Margin Control

Based on the above surge margin prediction model, a direct surge margin control can be implemented, and the structure of it has been shown in Figure 2. A quasi-PID or ILQR method [31] is used to design this novel controller. Here, we just consider the surge margin of the fan as an application. And the ILQR robust control method, direct surge margin controller design, and the digital simulation are, respectively, introduced as follows.
5.1. ILQR Robust Control. Provided is that an engine dynamics with surge margin estimation can be formulated as follows:

$$
\begin{gather*}
\dot{\mathrm{x}}=\mathrm{Ax}+\mathrm{G}_{1}(\mathrm{x})+\mathrm{Bu},  \tag{37}\\
\mathrm{y}=\mathbf{C x}+\mathrm{G}_{2}(\mathbf{x})+\boldsymbol{\omega},
\end{gather*}
$$

where $\mathbf{x}, \mathbf{y}$, and $\mathbf{u}$ are denoted as system state, output, and input vector, respectively, and $G_{1}(\mathbf{x}), G_{2}(\mathbf{x})$ are defined as nonlinear compensation items for system state and output dynamics accordingly. Note that $G_{1}(\mathbf{x})$ and $G_{2}(\mathbf{x})$ represent differences between a nonlinear engine model and its linear one, due to variations in states and flight conditions, and $\boldsymbol{\omega}$


Figure 11: ILQR control structure.
represents the estimation and measured errors produced in surge margin prediction.

Take the augmented state vector as $\overline{\mathbf{x}}=\left[\begin{array}{ll}\mathbf{x}^{T} & \int_{o}^{t} \mathbf{e}^{T} d \tau\end{array}\right]^{T}=$ $\left[\mathbf{x}^{T} \quad \int_{o}^{t}(\mathbf{r}-\mathbf{y})^{T} d \tau\right]^{T}$, while $\mathbf{r}$ is a set-point command (37) which is easily transformed to

$$
\begin{align*}
\dot{\overline{\mathbf{x}}} & =\overline{\mathbf{A}} \overline{\mathbf{x}}+\overline{\mathbf{G}}_{1}(\overline{\mathbf{x}})+\overline{\mathbf{B}} \overline{\mathbf{u}} \\
\overline{\mathbf{y}} & =\overline{\mathbf{C}} \overline{\mathbf{x}}+\overline{\mathbf{G}}_{2}(\overline{\mathbf{x}})+\overline{\boldsymbol{w}} \tag{38}
\end{align*}
$$

where $\overline{\mathbf{u}}=\mathbf{u}, \overline{\mathbf{A}}=\left[\begin{array}{cc}\mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0\end{array}\right], \overline{\mathbf{B}}=\left[\begin{array}{c}\mathbf{B} \\ -\mathbf{D}\end{array}\right], \overline{\mathbf{C}}=\left[\begin{array}{ll}C & 0\end{array}\right], \overline{\mathbf{G}}_{1}(\overline{\mathbf{x}})=$ $\left[\begin{array}{c}\mathbf{G}_{1}(x) \\ -\mathbf{G}_{2}(x)\end{array}\right], \overline{\mathbf{G}}_{2}(\overline{\mathbf{x}})=\mathbf{G}_{2}(\mathbf{x})$, and $\overline{\boldsymbol{\omega}}=\boldsymbol{\omega}$.

Generally for aeroengines, a bounded condition is guaranteed as [32]

$$
\left\|\left[\begin{array}{c}
\mathbf{G}_{1}(x)  \tag{39}\\
\mathbf{G}_{2}(x)+\omega
\end{array}\right]\right\|_{\infty}<+\infty
$$

Based on the lemma proposed in [31], for (38) a relative LQR regulator can be easily designed so that all states can converge to a bounded area, that is,

$$
\begin{equation*}
\|\overline{\mathbf{x}}\|_{\infty}=\left\|\left[\mathbf{x}^{T} \int_{0}^{t} \mathbf{e}^{T} d \tau\right]^{T}\right\|_{\infty}<+\infty \tag{40}
\end{equation*}
$$

And it means that $\mathbf{e} \in L_{\infty}$. Furthermore, $\mathbf{e} \in L_{2}$ and $\mathbf{e} \in C^{1}$ can also be guaranteed for engines. Therefore based on Barbalat lemma, the output can track the command with limited conditions followed by

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \mathbf{e}=\lim _{t \rightarrow \infty}(\mathbf{r}-\mathbf{y})=\mathbf{0}, \quad \text { when } \mathbf{E}(\boldsymbol{\omega})=\mathbf{0} \tag{41}
\end{equation*}
$$

Otherwise,

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \mathbf{e}=\lim _{t \rightarrow \infty}(\mathbf{r}-\mathbf{y})<\sigma, \quad \text { when }|\mathbf{E}(\boldsymbol{\omega})|=\sigma \tag{42}
\end{equation*}
$$

Consider a well-known LQR problem for the system (37) as

$$
\min _{\overline{\mathbf{u}}}\left\{\left\|\begin{array}{l}
\mathbf{Q}^{1 / 2} \overline{\mathbf{x}} \tag{43}
\end{array}\right\|^{1 / 2} \overline{\mathbf{u}} \|_{2}=\int_{0}^{\infty}\left(\overline{\mathbf{x}}^{T} \mathbf{Q} \overline{\mathbf{x}}+\overline{\mathbf{u}}^{T} \overline{\mathbf{u}}\right) d t\right\}
$$

where $\mathbf{Q}=\mathbf{Q}^{T} \geq \mathbf{0}$ and $\mathbf{R}=\mathbf{R}^{T}>\mathbf{0}$ are proper dimension weighted matrices. Thus, an ILQR controller can be derived
as $\overline{\mathbf{u}}=\overline{\mathbf{K}} \overline{\mathbf{x}}$ where $\overline{\mathbf{K}}=\mathbf{R}^{-1} \overline{\mathbf{B}}^{T} \mathbf{P}$ and $\mathbf{P}$ satisfies the following Riccati equation

$$
\begin{equation*}
\overline{\mathbf{A}}^{T} \mathbf{P}+\mathbf{P} \overline{\mathbf{A}}-\mathbf{P} \overline{\mathbf{B}} \mathbf{R}^{-1} \overline{\mathbf{B}}^{T} \mathbf{P}+\mathbf{Q}=\mathbf{0} \tag{44}
\end{equation*}
$$

$\overline{\mathbf{K}}$ can be divided by $\mathbf{x}$ and $\int_{0}^{t} \mathbf{e} d \tau$ as a block matrix $\overline{\mathbf{K}}=$ $\left[\begin{array}{ll}\mathbf{K}_{x} & \mathbf{K}_{e}\end{array}\right]$; so we obtain the following quasi-PID control type [33]:

$$
\mathbf{u}=\overline{\mathbf{K}} \overline{\mathbf{x}}=\left[\begin{array}{ll}
\mathbf{K}_{x} & \mathbf{K}_{e}
\end{array}\right]\left[\begin{array}{c}
\mathbf{x}  \tag{45}\\
\int_{0}^{t} \mathbf{e} d \tau
\end{array}\right]=\mathbf{K}_{x} \mathbf{x}+\mathbf{K}_{e} \int_{0}^{t} \mathbf{e} d \tau
$$

And the structure of an ILQR controller is shown in Figure 11.
Remark 4. As well known, LQR method has better robustness with infinite magnitude margin and phase margin over $60^{\circ}$. However, LQR controllers cannot have the ability to eliminate the steady state error in control problems to trace some command signals. For nonlinear plants such as aeroengines with significant nonlinearity and wide variant states range, simple robust linear methods like LQR is not able to maintain a good controllability [34,35]. So an improved type of the above ILQR method is utilized here. In an ILQR control design, the tracing errors are augmented into the state vector so as to realize a convergence to the references signals. So the ILQR control method can be regarded as an improved LQR type; therefore it has not only has robustness to adapt large variation of engine power states and envelope changes, but also has capability to eliminate the steady state error when engine command signals change.
5.2. Design of Direct Surge Margin Control. Firstly, at the design point of $H=0 \mathrm{~km}, M_{a}=0$, and $\mathrm{Pla}=70^{\circ}$, a statespace model of the engine dynamics is established. And the actuators dynamics of $\mathbf{u}=\left[W_{\mathrm{fb}}, A_{8}\right]^{T}$ is also considered in it. Thus, a nonlinear model augmented with the actuator is gotten as follows:

$$
\begin{align*}
& \dot{\mathbf{x}}=A x+G_{1}(x)+B u \\
& y=C x+G_{2}(x)+\omega \tag{46}
\end{align*}
$$

where $\mathbf{x}=\left[N_{f}, N_{c}, W_{\mathrm{fb}}, A_{8}\right]^{T}, \mathbf{u}=\left[W_{\mathrm{fb}}, A_{8}\right]^{T}$, and $\mathbf{y}=\left[N_{f}, S_{\mathrm{mf}}\right]^{T}$. Through some conventional identification


Figure 12: Effects of the ILQR control at the design point.
method, the system matrices in (44) can be obtained as follows:

$$
\begin{gather*}
\mathbf{A}=\left[\begin{array}{cccc}
-2.3264 & 1.3350 & 1.4160 & 0.9059 \\
0.3056 & -3.9053 & 0.1613 & 0.5929 \\
0 & 0 & -10 & 0 \\
0 & 0 & 0 & -5
\end{array}\right], \\
\mathbf{B}=\left[\begin{array}{cc}
0 & 0 \\
0 & 0 \\
10 & 0 \\
0 & 5
\end{array}\right],  \tag{47}\\
\mathbf{C}=\left[\begin{array}{cccc}
0.0121 & 0.9597 & -0.0042 & 0.0017 \\
8.3219 & -13.7904 & 4.1979 & -0.6527
\end{array}\right] .
\end{gather*}
$$

For reducing the number of parameters needing to modulate, positive diagonal matrices are chosen for $\mathbf{Q}=$ $\mathbf{Q}^{T} \geq \mathbf{0}$ and $\mathbf{R}=\mathbf{R}^{T}>\mathbf{0}$, in which the former is related to output performance of the controlled system, and the latter is related to control input performance of the controlled system. The ranges of two matrices are roughly given to meet that of a stabilized closed-loop system. Then, they can be modulated by trial and error according to the dynamic and static performance of the closed system. Ultimately, the two
weighted matrices are selected as $\mathbf{Q}=\operatorname{diag}[1,1,0.1,0.2,10,2]$ and $\mathbf{R}=\operatorname{diag}[0.5,0.5]$, where $\operatorname{diag}[]$ means diagonal matrices:

$$
\begin{gather*}
\mathbf{K}_{e}=\left[\begin{array}{ccc}
29.5095 & 12.0353 \\
7.2756 & -0.3396
\end{array}\right] \\
\mathbf{K}_{x}=\left[\begin{array}{cccc}
-26.7581 & 8.2457 & -5.8700 & 0.2909 \\
-0.6300 & -2.7712 & 0.0010 & -0.4108
\end{array}\right] . \tag{48}
\end{gather*}
$$

The control effects of the above controller are verified firstly in the design point, as shown in Figure 12, and the commands are set as $S_{\mathrm{mf}}=20 \%, N_{c}=99 \%$. Figures 12(a) and 12(b) present the response curves of $S_{\mathrm{mf}}$ and $N_{c}$, and Figures 12(c) and 12(d) show the response curves of $W_{\mathrm{fb}}$ and $A_{8}$. As can be seen from these figures, when a step command signal is given at $t=5 \mathrm{~s}$, the closed system can response quickly to reach the control objective within 4 seconds. Besides, there are no static errors for two controlled variables (see Figures 12(a) and 12(b)).

For validating the robustness of this ILQR controller, some test results are also given in other envelope points in the scope of the super maneuver envelope range. It can be seen clearly that at the envelope point of $(H=4 \mathrm{Km}$, $\left.M_{a}=0.6\right)$ and $\left(H=5 \mathrm{Km}, M_{a}=0.5\right)$, the closed system designed by ILQR method still has excellent dynamic and static performance. Figures 13(a) and 13(b) show a fast, small


Figure 13: Control effects in other envelope points.
overshoot, and convergent tracking with respect to command signals, whereas Figures 13(c) and 13(d) illustrate acceptable and feasible control input performances. These results give a proof in good robustness of this control method, and it is qualified for application in the direct surge margin control under variant power states and at different envelope point.

## 6. Control Simulations and Validations

To validate effectiveness of the new method, some supermaneuver flight tasks are simulated with a flight condition in low altitudes and velocities, and for maintaining a stable and efficient operation, the fan surge margin command is usually expected to be $15 \%$. The direct surge margin control is compared to the conventional control as shown in Figures 14 and 15. The conventional engine controller is a bivariate controller designed by ILQR, and the control inputs are $A_{8}$ and $W_{\mathrm{fb}}$, and the controlled variables are $\pi_{t}$ and $N_{c}$. Figures 14(a) and 15(a) depict the response curves of fan surge margin $S_{\mathrm{mf}}$. "-no" indicates the conventional control response, "-DSC" represents the response of the direct surge margin control, and "-pre" represents the estimation of the angle of attack. Figures 14(b) and 15(b) illustrate different
curves of the angle of attack $\alpha$, the elevator angle $\delta_{z}$, inlet distorted index $\mathrm{DC}_{60}$, fan and compressor surge margin loss, compressor surge margin $S_{\mathrm{mc}}$, relative corrected speed $N_{f}$, relative corrected speed $N_{c}$, main fuel flow $W_{\mathrm{fb}}$, and throat area of nozzle $A_{8}$.

Figure 14 demonstrates the simulation results under the condition of $H=4 \mathrm{~km}, M_{a}=0.6$. At $t=0 \mathrm{~s}$, the conventional control and the direct surge margin control are, respectively, adopted to begin a high attack angle task. Then, the elevator $\delta_{z}$ is pulled along with some predefined path file (a negative degree means an upward shift of the elevator). When the elevator shifts downwards, the angle of attack $\alpha$ is increased, so $\mathrm{DC}_{60}$ is accordingly increased and vice versa, and the variation of $\mathrm{DC}_{60}$ is monotonically related with the surge margin loss of fan and compressor. Moreover, from the attack angle response curves, one can see that a real time and precisely prediction for $\alpha$ at $d$ steps ahead moment is verified here.

As also can be observed from Figure 14(a), while using the conventional control, $S_{\mathrm{mf}}$ changes very violently during the super maneuver flight, and in some time even less than $10 \%$ indicting that the cooperating work point is very close to the surge border. So, it fails to meet the safe requirements for engine in the super maneuver flight. On the contrary, the


Figure 14: Effects of direct surge margin control at $H=4 \mathrm{~km}, M_{a}=0.6$.
novel control can keep the surge margin $S_{\mathrm{mf}}$ always around $15 \%$, that is, an ideal engine stability control effect. In order to keep the surge margin at $15 \%$ overall the super maneuver flight, a combined regulation is applied via $W_{\mathrm{fb}}$ and $A_{8}$. Since a direct correlation [1-3] between $S_{\mathrm{mf}}$ and $A_{8}$, the modulation of $A_{8}$ serves as a more important role. So one can obviously observe that the bigger is the $A_{8}$, the larger is the surge margin. But for the changes of $W_{\mathrm{fb}}$, a bit similar tendency is found either in the new control or the conventional one, as
the main physical contribution of it is the proper controlling of $N_{c}$, which can ensure that there is no great thrust loss in the super maneuver flight.

For validating the robustness of this new control law, simulation results at $H=5 \mathrm{~km}, M_{a}=0.5$ are also given in Figure 15. Similarly through a fast adjustment of elevator angle $\delta_{z}$, a maneuvering flight is originated from an level flight. Directly using the predicted surge margin, Figure 15 clearly presents that the new control scheme can successfully


$$
\begin{aligned}
& \because S_{\mathrm{mf}} \text {-DSC } \\
& \sim S_{\mathrm{mf}} \text {-no }
\end{aligned}
$$

(a) Response curves of fan surge margin







$\rightarrow N_{\mathrm{f}}$-DSC
$-N_{\mathrm{f}}$-no

- $N_{c}$-DSC
$\because S_{m-}-$ DSC
$-S_{m c}-n o$
$-N_{\mathrm{c}}$-no

$\because W_{\mathrm{fb}}-$ DSC
$\simeq W_{\mathrm{fb}}-$ no


$$
\begin{aligned}
& \because A_{8} \text {-DSC } \\
& \backsim A_{8} \text {-no }
\end{aligned}
$$

(b) Response curves of each parameter of the engine

Figure 15: Effects of direct surge margin control at $H=5 \mathrm{~km}, M_{a}=0.5$.
achieve a real-time adjustment of the cooperating point of the engine. In the entire super maneuver mission, the fan surge margin is controlled slightly around $15 \%$; that is to say, the cooperating work point can always keep a safe distance from the surge border. Nevertheless, when using that conventional method, since surge margin is not well considered or better estimated, safety margin of the engine cannot be sustained in the large maneuver flight.

## 7. Conclusions

Considering the special stability need in super-maneuver flights, a novel direct surge margin controller is proposed and designed based on a surge margin prediction model and a ILQR robust control method which is capable of guaranteeing a good robustness and performance in a large variation of power states and flight conditions for the closed-loop system.

In terms of the surge margin prediction model, a weighted surge margin feature selection algorithm based on LSSVR is proposed to get a group of optimal measurable inputs for its baseline model, and an angle of attack model with an online scrolling window LSSVR method is proposed to estimate the inlet distortion index in super maneuver states. Furthermore, the surge margin in distortion states could be computed in real time by aid of the surge margin baseline and angle of attack prediction.

All these contributions are verified to effectively realize a high stability control with the flight conditions of low altitude, low speed, and high angle of attack.

## Nomenclature

## Variables

$S_{m}: \quad$ Surge margin/\%
$H$ : Flight height/km
$M_{a}$ : March number/-
Pla: Power lever angle/ ${ }^{\circ}$
$W_{\mathrm{fb}}$ : Main fuel flow $/ \mathrm{kg} \cdot \mathrm{s}^{-1}$
$W_{\text {cor }}$ : Corrected air mass flow
$A_{8}$ : Nozzle throat area $/ \mathrm{m}^{2}$
$P: \quad$ Total pressure/Pa
$T: \quad$ Total temperature/K
$N_{f}$ : Relative fan corrected speed/\%
$N_{c}$ : Relative compressor corrected speed/\%
$\tau$ : Delay time/s
$\pi$ : Pressure ratio/-
$\alpha$ : Angle of attack/ ${ }^{\circ}$
$\delta_{z}: \quad$ Elevator angle/ ${ }^{\circ}$
$\mathrm{DC}_{60}$ : Inlet distortion index/-
$F$ : Engine thrust/N.

## Subscripts

$f$ : Fan
c: Compressor
$t$ : Turbine
$s$ : A value on surge border
$o$ : A value on cooperation working point
2: Fan inlet cross-section
22: Fan outlet cross-section
3: Compressor outlet cross-section
8: Nozzle throat cross-section.

## Conflict of Interests

The authors declare no conflict of interests.

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# Electricity Price Forecast Using Combined Models with Adaptive Weights Selected and Errors Calibrated by Hidden Markov Model 

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#### Abstract

A combined forecast with weights adaptively selected and errors calibrated by Hidden Markov model (HMM) is proposed to model the day-ahead electricity price. Firstly several single models were built to forecast the electricity price separately. Then the validation errors from every individual model were transformed into two discrete sequences: an emission sequence and a state sequence to build the HMM, obtaining a transmission matrix and an emission matrix, representing the forecasting ability state of the individual models. The combining weights of the individual models were decided by the state transmission matrixes in HMM and the best predict sample ratio of each individual among all the models in the validation set. The individual forecasts were averaged to get the combining forecast with the weights obtained above. The residuals of combining forecast were calibrated by the possible error calculated by the emission matrix of HMM. A case study of day-ahead electricity market of Pennsylvania-New Jersey-Maryland (PJM), USA, suggests that the proposed method outperforms individual techniques of price forecasting, such as support vector machine (SVM), generalized regression neural networks (GRNN), day-ahead modeling, and self-organized map (SOM) similar days modeling.


## 1. Introduction

Since the 1990s, the monopoly vertically integrated utilities of electric power industries around the world have been deregulated into competitive markets, aiming to break monopoly and increase operation efficiency. It is crucial for all participants in the market to predict the electricity price with high accuracy. Their bid actions depend on the forecasting and their benefits therefore are affected by the forecasting; thus price forecasting draws great interests.

Electricity price is affected by various uncertainties, such as power load, weather, and bidders' expectations. These influential factors interact and have an intricate impact on price. Electricity price is more volatile than load with unexpected spikes (unusual prices), high frequency, and multiple seasonality (e.g., daily and weekly periodicity). So it is more difficult to be predicted than power load. There are primarily two categories of electricity price forecasting modeling, time series modeling, and artificial intelligence (AI) modeling.

Time series modeling forecasts future price with available historical prices by mining the relation information contained in the data, such as autoregressive moving average (ARMA), generalized autoregressive conditional heteroscedasticity (GARCH). Contreras et al. [1] used an ARMA model to forecast next-day electricity prices for mainland Spain and Californian markets. A novel technique was proposed to forecast day-ahead electricity prices based on wavelet transform and ARIMA models in [2]. A more robust time series modeling, GARCH model, was developed to forecast day-ahead electricity prices in [3, 4]. Time series modeling tries to mine the information contained in previous data however pays less attention to external influence leading to undesirable forecasting for the unstable characteristic of prices.

AI modeling usually exploits more circumstance influence factors than time series modeling and thus presents more desirable results. Artificial neural networks (ANNs) were developed to forecast electricity prices and showed
better performance than time series modeling [5, 6]. An ANN modeling based on similar days was proposed to forecast day-ahead electricity prices in Pennsylvania-New JerseyMaryland (PJM) market [7]. A technique with combining the Probability Neural Network (PNN) and Orthogonal Experimental Design (OED) was developed in [8] showing better performance than its counterparts.

Limited by the complexity of AI model, information contained in the historical prices is not made full use of. A hybrid model with support vector machines (SVM) to capture the nonlinear patterns and ARIMA to solve the residuals regression estimation problems was proposed in [9] showing the great potential of hybrid modeling. Another hybrid model combining SVM and GARCH was developed in [10] to forecast the day-ahead price of the PJM market.

Time series modeling and AI modeling have different weaknesses and strengthens in price forecasting since they place different emphasis on the exploitation method for the influence information of electricity price. Several predictions by different methods were suggested to combine to smooth the fluctuations which often occur in single model forecasting. The performance of the traditional combined forecast models relies on the combining weights of individual models, which usual are fixed and determined by historical performances of the models. Fixed weights are not always the best choice because the forecasting abilities of individual models vary along with the circumstance. Sometimes one model shows better performance, other times it does not. So it is necessary to select the combining weights of individual models according to their performance under certain circumstance. However it is a big challenge to analysis the circumstance and therefore to determine the proper combining weights of individual models under that circumstance. On the other hand, neither the single model nor the combining model can make full use of the influence information, and the modeling residuals usually contain information which have not be exploited by the models. It helps to improve the forecasting accuracy by analyzing the residual series and then to estimate the residual of next step [11].

A Markov chain is a random process which undergoes transitions from one state to another. It has an important character: the next state depends only on the current state and not on the sequence of events that preceded it. Markov chain can be used to analyze the performance of forecasting [1214]. A hidden Markov model (HMM) is a statistical Markov model in which the system being modeled is assumed to be a Markov process with unobserved (hidden) states. We can apply HMM to exploit the information contained in the forecasting error sequence. The forecasting errors can be treated as the observations of the HMM, and the forecasting abilities of a model under certain circumstance can be looked as the states [15]. In this paper a hybrid method consisting of a combining model with adaptive weights based circumstance and an error calibration technique was proposed to forecast the day-ahead electricity price. Several individual models were developed to forecast electricity price, respectively; then their performances under different circumstances were evaluated to build Hidden Markov models (HMMs). Together with the general past performance of the individual models,
the state sequences of the HMMs were proposed to decide the combining weights; the emission sequences of HMMs were exploited to calibrate the errors by the combining model.

The rest of the paper is organized as follows. In Section 2, we describe the fundamental of HMM and the principle for combined forecast by HMM; Section 3 demonstrates the approach of combined modeling and error calibration with HMM. Experiments of the proposed technique and compared methods are showed in Section 4. Finally, the conclusions are presented in Section 5.

## 2. Principle of Combined Forecast with Weights Selected by HMM

In this section the basic ideas of combined forecast with weights selected by HMM are discussed.
2.1. Principle of HMM. HMM can be regard as a dual random process, a sequence of emissions that can be seen, and the other invisible sequence of state in which the emissions are generated. There are two kinds of HMM, discrete HMM and continuous HMM. Here we discuss the former and apply it to build combining model. For simplicity and emphasis we just give a brief introduction of discrete HMM. More details of HMM principle and how HMM works can be read in [15, 16].

Discrete HMM can be described by series parameters of five dimensions: $\boldsymbol{\lambda}=(\mathbf{S}, \mathbf{O}, \mathbf{A}, \mathbf{B}, \boldsymbol{\pi})$, where
(1) S : a set of states where the observation locates, $S=\left\{S_{1}\right.$, $\left.\ldots, S_{n}\right\}$, and $n$ is the number of the states;
(2) $\mathbf{O}$ : a set of emissions or observations, $\mathbf{O}=\left\{O_{1}, \ldots\right.$, $\left.O_{m}\right\}$, and $m$ is the number of the potential observations (or emissions) in each state;
(3) A: a transition matrix which describes the probability of a transition from a given state to another state, $\mathbf{A}=$ $\left(a_{i j}\right)_{n \times n}$, and here, $a_{i j}=p\left(S_{t+1}=\theta_{j} / S_{t}=\theta_{i}\right)$;
(4) B: an emission matrix, whose $i, k$ entry gives the probability of emitting symbol $O_{k}$ given that the model is in state $S_{j} . \mathbf{B}=\left(b_{j k}\right)_{m \times n}$, where $b_{j k}=p\left(O_{t}=\right.$ $\left.v_{k} / S_{t}=\theta_{j}\right)$;
(5) $\pi$ : a vector of initial state distribution, $\pi=\left(\pi_{1}, \pi_{2}, \ldots\right.$, $\pi_{n}$ ).
HMM mainly aims to resolve three problems:
(1) to evaluate the most likely state path of a given sequence of emissions;
(2) to estimate transition and emission probabilities of a given sequence of emissions;
(3) to calculate the posterior probability that the model is in a particular state at any point in the sequence.
2.2. Combined Forecast by HMM. The basic idea of combined forecasting is to give a weighted sum of forecasting by different models to reduce the defects of individual modeling method. In this paper, we use HMM to determine the weights of combining models.

In electricity price forecasting, a sequence of errors generated from price modeling can be considered as a HMM process. The intervals in which the error of each forecasting locates form the sequence of observations or the emission sequence; the forecasting abilities of the individual models are regarded as the state of HMMs. The HMMs are built according to the validation errors of the individual models. Then the next states of the HMMs which depict the abilities of the individual models are used to decide the combining weights. The possible next emissions of individuals are averaged with combining weights and then used to calibrate the combined forecast.
2.3. Error Calibration by HMM. With the state probability vector of the next step assessed in Section 2.2 and the emission matrix $\mathbf{B}$ of HMM, the probabilities of emissions in the next step can be calculated. Since the different emissions present the range intervals where the error falls in as mentioned in Section 2.2, we can convert the emissions and their probabilities to expected value of forecasting error. Then the expected value is used for error calibration of the combined modeling.

## 3. Approach of Combined Forecasting and Error Calibration by HMM

This part depicts how to build a combined model with error calibration based on HMM techniques. As showed in Figure 1. Considering that the hourly prices in different hours shows great difference, we build 24 combining models to forecast the hourly prices one by one. For any hour price's modeling, the approach is the same, so we just take an hour as an example to show the modeling approach. The following 7 steps consist of the proposed method.

Step 1 (initiate). Including data pretreating and candidate models selection.

We cluster the experimental data into three sets: a training set, a validation set, and a test set. The first one is used to train models, the second one is used to tune models' parameters according to their performances, and the last one is applied to evaluate the modeling algorithms.
$N$ candidate models $\left(M_{1}, M_{2}, M_{3}, \ldots, M_{n}\right)$ are selected for combining forecast.

## Step 2. Individual modeling for combined forecast.

The following process is repeated for each individual model.

Substep 1. Build the individual model and calculate the validation error vector $\mathbf{e}$ and forecasting price $\mathbf{p}$.

We train the $M_{i}$ model with the training set, then tune the parameters in $M_{i}$ with the validation set, and after that test $M_{i}$ with the test set. In the above steps, we get the validation error vector $\mathbf{e}$ (see (1)) and forecasting price $\mathbf{p}$ (see (3)) separately

$$
\begin{equation*}
\mathbf{e}=\left(e_{i, 1}, e_{i, 2}, \ldots, e_{i, d}\right) \tag{1}
\end{equation*}
$$



Figure 1: Flow chart of electricity price forecasting by the proposed method.
where $e_{l, k}$ is validation error by $M_{i}$ for the $k$ th hour on the $l$ th day, and $d$ is the number of days in the validation set, and error $e_{i j}$ is calculated by

$$
\begin{equation*}
e_{i j}=\frac{\widehat{y}_{i j}-y_{i j}}{y_{i j}} \tag{2}
\end{equation*}
$$

where $\widehat{y}_{i j}$ is the forecast price and $y_{i j}$ is the actual price

$$
\begin{equation*}
\mathbf{p}=\left(p_{i, 1}, p_{i, 2}, \ldots, p_{i, D}\right) \tag{3}
\end{equation*}
$$

where $p_{l, k}$ is the forecasting price with $M_{i}$ for the $k$ th hour on the $l$ th day and $D$ is the number of days in the test set.

Substep 2. Calculate the emission sequence and the state sequence of $M_{i}$. In this step, for each model we transform the error sequence into discrete emission (observations) sequence and class the states according to the error in which denotes the forecasting abilities of the model.

Substep 3. Calculate the emission sequence and the state sequence of $M_{i}$.

In this step, we transform the error sequence into discrete emission sequence and obtain the state sequence according to the performance of modeling which denotes the forecasting abilities of the model.

As discrete HMMs are discussed here, the emission sequence needs to be discretized. Here we divide the range, in which $e$ spreads, into several intervals. Then marks the intervals where $e$ falls in with the emission values (elements of the emission set). Then we get the emission vector $\mathbf{s}^{e}$ according to the intervals in which each $e$ falls:

$$
\begin{equation*}
\mathbf{s}^{e}=\left(s_{i, 1}^{e}, s_{i, 2}^{e}, \ldots, s_{i, d}^{e}\right) \tag{4}
\end{equation*}
$$

where $s_{i, j}^{s}$ is the emission of the model $M_{i} ; s_{i, j}^{s} \in \mathbf{O}, \mathbf{O}$ is the emission set.

Then we begin to calculate the state matrix $\boldsymbol{s}^{s}$ according to certain criterions based on the model's performance. For simplicity, we just set three states to reveal the ability of the model forecasting, as follows: $U$ : the state of underestimate (when target price is significantly underestimated); $C$ : the state of proper prediction (when target price is estimated with acceptable accuracy), and $H$ : the state of overestimate (when target price is significantly overestimated):

$$
\begin{equation*}
\mathbf{s}^{s}=\left(s_{i, 1}^{s}, s_{i, 2}^{s}, \ldots, s_{i, d}^{s}\right) \tag{5}
\end{equation*}
$$

where $s_{i, j}^{s}$ is the state of the individual candidate in the $i$ th hour of the $j$ th day and $s_{i, j}^{s} \in \mathbf{S}, \mathbf{S}$ is the state set.

Substep 4. Estimate transition matrix A and emission matrix B.

In this step, the maximum likelihood estimate of the transition matrix $\mathbf{A}$ and emission matrix $\mathbf{B}$ are calculated with the known $\boldsymbol{s}^{e}$ and $\boldsymbol{s}^{s}$. The process can be easily accomplished with the function of hmmestimate in Matlab (those who interested in its theory see the following: Durbin, R., S. Eddy, A. Krogh, and G. Mitchison, Biological Sequence Analysis, Cambridge, UK: Cambridge University Press, 1998.), so we will not intend to give a detailed description here.

Given an initial state distribution, with the state sequence and the emission sequence, we can estimate the transition matrix $\mathbf{A}$ (see (6)) and the emission matrix $\mathbf{B}$ (see (7)) for HMM of the $i$ th hour:

$$
\mathbf{A}=\left[\begin{array}{ccc}
a_{11} & \cdots & a_{1 m}  \tag{6}\\
\cdots & a_{i j} & \cdots \\
a_{m 1} & \cdots & a_{m m}
\end{array}\right]=\left[\begin{array}{c}
\mathbf{a}_{1} \\
\cdots \\
\mathbf{a}_{m}
\end{array}\right]
$$

where $a_{i j}$ is the transition probability from the $i$ th state to the $j$ th state and $m$ is the number of states:

$$
\mathbf{B}=\left[\begin{array}{ccc}
b_{11} & \cdots & b_{1 k}  \tag{7}\\
& b_{i j} & \\
b_{m 1} & \cdots & b_{m k}
\end{array}\right]=\left[\begin{array}{l}
\mathbf{b}_{1} \\
\\
\mathbf{b}_{m}
\end{array}\right]
$$

where $b_{i j}$ is the probability of the $j$ th emitting symbol under the state $S_{i}$ and $k$ is the classes of the emissions.

Substep 5. Obtain the probabilities of the next state.
In Substep 4, we have obtained the transition matrix A and the emission matrix B. As discussed previously the matrix of $\mathbf{A}$ describes the probability of a transition from a given state to another state; matrix B gives the probability of emitting symbol under different states. So for a given state $s_{i, j}^{s}$ (suppose $s_{i, j}^{s}=n, 1 \leq n \leq m$ ) in the $i$ th hour on the $t$ th day, the probabilities of the next state (in the $i$ th hour on the $(t+1)$ th day) are the vector $\mathbf{a}_{n}$ in the transition matrix $\mathbf{A}$.

Step 3. Calculate probabilities of the next emission and estimate the possible error generated by the model.

The probabilities of the next state $\mathbf{a}_{i}$ obtained in Substep 4 are multiplied with the emission matrix to calculate the probabilities of the next emission.

The emission probabilities then are transformed to continuous possible error $g_{i}$ with the intervals defined in Substep 3.

Step 4. Calculate the combining weights of the next step.
In this step, the combining weights of the different models are settled. The number of samples under the proper state of each individual for the validation set is used to evaluate the abilities of these models, as shown by

$$
\begin{equation*}
c_{i}=\frac{n_{i}}{\sum_{i}^{m} n_{i}} \tag{8}
\end{equation*}
$$

where $c_{i}$ denotes the historical forecasting ability of $M_{i}$, and $n_{i}$ is the number of samples in state of $C$ by model $M_{i}$ among the validation set.

The proper forecasting probability $a_{i}$ in the vector $\mathbf{a}_{n}$ and the abilities $c_{i}$ are used to calculate the combining weights, as shown by

$$
\begin{equation*}
w_{i}=\frac{a_{i} \times c_{i}}{\sum_{i} a_{i} \times c_{i}} \tag{9}
\end{equation*}
$$

where $w_{i}$ is the combining weight of model $M_{i}$ in the next step and $a_{i}$ is the proper forecasting probability of the $M_{i}$.

Step 5. Get the combined forecast.
The forecasts from individual models are averaged with the weights obtained in Step 4 to get a combined forecast, as shown by

$$
\begin{equation*}
f=\sum_{i} f_{i} \times w_{i} \tag{10}
\end{equation*}
$$

where $f$ is the combined forecast and $f_{i}$ is the forecast by $M_{i}$.

Step 6. Calculate the expected value of forecasting error of the combined model.

The possible errors from individual models are averaged with the combining weights obtained in Step 5 to get possible error of combined forecast, as shown by

$$
\begin{equation*}
g=\sum_{i} g_{i} \times w_{i} \tag{11}
\end{equation*}
$$

where $g$ is the possible error of the combined forecast and $g_{i}$ is the possible error from $M_{i}$.

Step 7. Obtain the final forecast with expected value of error to calibrate the combined forecast, as shown by

$$
\begin{equation*}
p=f-g \tag{12}
\end{equation*}
$$

## 4. Numerical Results

4.1. Individual Models Building for Combined Forecast. The proposed method is validated on the day-ahead electricity market of PJM. Considering electricity price in summer is more volatile than in other seasons, we apply the method to forecast the hourly price of August 2010. The data in July serve as validation data, and date in June serve as training data. New prices and loads are appended in modeling process to accommodate the models to the new circumstance.

In order to consider the influences from different aspects, we select various methods as candidates for modeling. Here we choose intelligent algorithm modeling (SVM, generalized regression neural networks (GRNN)), time series modeling (GARCH and GARCHX), and two direct methods as modeling candidates. One direct method is day-ahead modeling, in which we take the hourly price of the previous day as the forecasting price. The other direct method is SOM similar days modeling, in which we find the similar days by SOM from the validation set and then average the hourly prices of the similar days as the forecasting price. For simplicity, we use $M_{1}$ to $M_{6}$ to represent the SVM, GRNN, GARCH, GARCHX, day-ahead modeling, and SOM similar days modeling, respectively.

The performance of intelligent algorithm is sensitive to the input, so the modeling data are pretreated to eliminate the scale effects before modeling. All the numeric data are scaled to $[0,1]$, as shown by

$$
\begin{equation*}
x_{i j}^{\prime}=\frac{\left(x_{i j}-x_{j}^{\min }\right)}{\left(x_{j}^{\max }-x_{j}^{\min }\right)} \tag{13}
\end{equation*}
$$

where $x_{i j}^{\prime}$ is the scaled value of the $j$ th attribute of the $i$ th sample, $x_{i j}$ is the raw sample value, $x_{j}^{\min }$ is the minimum of the $j$ th attribute of all the samples (all the data from 1 Jan. to July 31 ), and $x_{j}^{\max }$ is the maximum of the $j$ th attribute of all the samples (ditto).

The input for SVM, GRNN, and SOM model is $\left\{L_{i}, L_{i-24}, P_{i-24}, P_{i-48}, P_{i-168}, P_{i-192}, w_{i}, w_{i-24}\right\} . L_{i}$ is the forecast load of the target hour (it can be predicted day-ahead with high accuracy, so here we use the actual load as forecast

Table 1: Parameters of the candidate models and validation MAPE.

| Model | Parameter | Value of parameter | MAPE/\% |
| :--- | :---: | :---: | :---: |
| $M_{1}$ | $C, n_{u}, e, g$ | $200,0.2,0.00002,0.004$ | 8.32 |
| $M_{2}$ | $r$ | 0.05 | 11.10 |
| $M_{3}$ | $r, m, p, q$ | $2,1,1,1$ | 12.59 |
| $M_{4}$ | $r, m, p, q$ | $2,1,1,1$ | 12.04 |
| $M_{5}$ | - | - | 12.79 |
| $M_{6}$ | Layer dimension | $[4,4]$ | 12.38 |

load); $P_{i-24}$ is the price 24 hour previous to the target hour, and so on. $w_{i j}$ is a daily variable reflecting the price fluctuation with different day types. $w_{i j}$ can be calculated by

$$
\begin{align*}
w_{i j}=\frac{\sum_{k} p_{i j k}}{30}, \quad i=1,2, \ldots, 7, j & =1,2, \ldots, 24,  \tag{14}\\
k & =1,2, \ldots, 30,
\end{align*}
$$

where $p_{i j k}$ is the $j$ th hourly price on the $i$ th day of the $k$ th week in 2010. As data from August 1 to August 31 are applied to test the model, the prices of previous 30 weeks are used to calculate $w_{i j}$.

The exogenous variables in GARCHX are $w_{i j}$ and $w_{i, j-1}$.
The parameters of individual models and MAPE (for the validation set) are listed in Table 1. From Table 1, we can see that SVM outperforms the other models. The rest models have close results.

Figure 2 is the distribution of the validation error of the candidates. It can be seen that SVM outperforms other candidates obviously, and day-ahead modeling with right tailed is different with other candidates.

Table 2 is the correlation coefficient between the validation errors of candidates.

From Table 2, we can see that $M_{3}, M_{4}$, and $M_{5}$ show high correlation. Since each model has some contribution we mainly focus on the work of selection of combining weights and the error calibration, so all the six models are taken in as candidates for combined model.
4.2. Build HMM with Validation Errors of the Individual Models. In this part, HMM are built with the information of the validation error series of the selected models. As discussed in Step 2 of Section 3, the validation error sequence of $M_{i}$ is discretized to build HMM. Considering that most of the errors fall in the range $[-0.08,0.08$ ], we divide the range into 5 intervals, as shown by

$$
s_{i j}^{e}= \begin{cases}1 & e<-0.08  \tag{15}\\ 2 & -0.08 \leq e<-0.04 \\ 3 & -0.04 \leq e<0.04 \\ 4 & 0.04 \leq e<0.08 \\ 5 & e \geq 0.08\end{cases}
$$

With process detailedly described in Substeps 3 and 4 of Section 3, we get the transition matrix $\mathbf{A}$ and the emission matrix B for $M_{i}$.


Figure 2: Validation error distribution of the candidate models.
4.3. Selecting Weights for Combined Forecast. Suppose that we have obtained all the forecasting errors of the 6 models of the $i$ th hour on the $d$ th day, as well as the forecasting price of the $i$ th hour on the $(t+1)$ th day $(t>d)$. For each model, we separate the range where errors fall into three zones, $U, C$, and $H$, as discussed in Substep 3, indicating the forecasting ability of the model under certain circumstance. As shown by

$$
s_{i j}^{s}= \begin{cases}1 & -0.04>e  \tag{16}\\ 2 & -0.04 \leq e<0.04 \\ 3 & 0.04 \leq e\end{cases}
$$

Given $s_{t, i}^{s}$, the model state of the $i$ th hour on the $t$ th day, the state probabilities of the next day of the same hour can be easily obtained from the transition matrix $\mathbf{A}$. Then the probabilities and historical abilities of the individuals are multiplied to generate the combination weights, according to (9). The combined forecast can be obtained by multiplying the forecasting of different models and the corresponding probabilities of their probable states, according to (10).

### 4.4. Calibrate the Combined Forecast with the Possible Com-

 bining Error. In this step, the possible errors of the next step by different candidates are estimated by their emission matrix B as described in Substep 4 in Section 3; they are exploited to calibrate the combined forecast, according to (11).

Figure 3: Forecasting by different models.
4.5. Performance Comparison and Analysis. Table 3 shows the performance of the different modeling. It can be seen that the combination model significantly outperforms all the individual models. $M_{1}$ and $M_{2}$ also have the better forecasting than other individual models, just the same as the performance in the validation set.


Figure 4: Error calibration effects contrast.

Table 2: Correlation coefficients of validation error sequences between the candidate models.

|  | $M_{1}$ | $M_{2}$ | $M_{3}$ | $M_{4}$ | $M_{5}$ | $M_{6}$ | Mean |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $M_{1}$ | - | 0.63 | 0.52 | 0.47 | 0.34 | 0.66 | 0.44 |
| $M_{2}$ | 0.63 | - | 0.62 | 0.62 | 0.57 | 0.66 | 0.52 |
| $M_{3}$ | 0.52 | 0.62 | - | 0.91 | 0.78 | 0.62 | 0.57 |
| $M_{4}$ | 0.47 | 0.62 | 0.91 | - | 0.89 | 0.56 | 0.57 |
| $M_{5}$ | 0.34 | 0.57 | 0.78 | 0.89 | - | 0.46 | 0.51 |
| $M_{6}$ | 0.66 | 0.66 | 0.62 | 0.56 | 0.46 | - | 0.49 |
| Mean | 0.44 | 0.52 | 0.57 | 0.57 | 0.51 | 0.49 | - |

Table 3: Test MAPE of the candidate models.

| Model | $M_{1}$ | $M_{2}$ | $M_{3}$ | $M_{4}$ | $M_{5}$ | $M_{6}$ | Combination | Calibration |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| MAPE/\% | 9.46 | 9.78 | 13.83 | 11.30 | 11.87 | 11.70 | 9.03 | 8.85 |

Figure 3 is the comparison between the actual prices, forecasting by SVM and combined forecast with error calibration techniques. The forecasting by other models is not listed in the figure for simplicity since they are not as good as SVM. We can see from Figure 3 that most of the prices have been properly predicted by the combination model. Some extreme prices are too low or too high to model by both the two models. The hourly prices are overestimated by SVM, especially which have high prices on the previous day.

The effects of errors calibration is shown in Figure 4. The colors in Figure 4 show the value of the forecasting error: red color denotes big positive $e$ (high overestimation) and blue color denotes big negative $e$ (markedly underestimation), as the color bar lying on the right presents. We can see the most difference of the forecasting error from the black circle of the right part of Figure 4. The calibration reduces extreme errors (too high or too low) of the combined model; also it increases the number of proper forecasts, whose errors fall around zero, displayed by the grey color zone.

Figure 5 lists the error distributions by different models. The errors spread the range of $[-0.4,0.6]$. The minimum and maximum of errors reveal that some prices are not properly forecasted by some models. If we probe it deeply, we can find that SOM similar day modeling and day-ahead


Figure 5: Histogram of errors by the different models.
modeling have more extreme errors and SVM and GRNN have more desirable performance, but the calibration has the best performance. From Figure 5 we can see that all models tend to overestimate the price as the right tail in histogram shows. The errors in the combined model and calibration
model centralizes around zero, and near half of the prices (more than 300 points) are forecasted with relative error in the interval $[-0.05,0.05]$. SVM and GRNN also predict prices well, more than 250 errors fall in the [ $-0.05,0.05$ ]. Day-ahead modeling acts the worst; it has the least number of errors falling in the $[-0.05,0.05]$.

## 5. Conclusions

This paper proposes a comprehensive combined forecast technique for day-ahead price by HMM. Several models, SVM, GRNN, GARCH, GARCHX, day-ahead modeling, and SOM similar day modeling are selected as candidate models. The error distribution of each model is exploited to calculate the state of HMM and the intervals where minimum errors fall mark with emissions of HMM. Then the state sequence and emission sequence are used to estimate HMM. Given a state of current hour, the state probabilities of the combination modeling of next day can be obtained from transition matrix A. These probabilities are regarded as combination weights for the combined forecast. Then the HMM are used to calculate the weights of combined model and to calibrate the error of the combined model.

The combined forecast can adapt to the varieties of circumstance by changing its weights dynamically with HMM, and the error calibration technique helps to reduce the error generated by combined model. The case study to forecast summer prices in PJM market shows that the proposed method outperforms other comparison methods, including SVM.

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# Stability and $l_{1}$-Gain Control of Positive Switched Systems with Time-Varying Delays via Delta Operator Approach 

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#### Abstract

This paper investigates the problems of stability and $l_{1}$-gain controller design for positive switched systems with time-varying delays via delta operator approach. The purpose is to design a switching signal and a state feedback controller such that the resulting closedloop system is exponentially stable with $l_{1}$-gain performance. Based on the average dwell time approach, a sufficient condition for the existence of an $l_{1}$-gain controller for the considered system is established by constructing an appropriate copositive type LyapunovKrasovskii functional in delta domain. Moreover, the obtained conditions can unify some previously suggested relevant methods in the literature of both continuous- and discrete-time systems into the delta operator framework. Finally, a numerical example is presented to explicitly demonstrate the effectiveness and feasibility of the proposed method.


## 1. Introduction

Positive systems mean that their states and outputs are nonnegative whenever the initial conditions and inputs are nonnegative [1, 2]. A positive switched system consists of a family of positive subsystems and a switching signal, coordinating the operation of various subsystems to specify when and how the switching takes place among the subsystems. Recently, due to the broad applications in communication systems [3, 4], formation flying [5], viral mutation dynamics under drug treatment [2], and systems theories [6-10], positive systems have been highlighted and investigated by many researchers [11-14]. It has been shown that a linear copositive Lyapunov functional is powerful for the analysis and synthesis of positive systems [15-17].

The delta operator, a novel method with good finite word length performance under fast sampling rates, has drawn considerable interest in the past three decades. As we know, the standard shift operator was mostly adopted in the study of control theories for discrete-time systems. However, the dynamic response of a discrete system does not converge smoothly to its continuous counterpart when the sampling period tends to zero; namely, data are taken at high sampling rates. Until Goodwin et al. proposed a delta operator method
in [18] to take the place of the traditional shift operator, the above problem is avoided. It was shown that delta operator requires smaller word length when implemented in fixedpoint digital control processors than shift operator does [19]. The delta operator model can be regarded as a useful approach to deal with discrete-time systems under high sampling rates through the analysis methods of continuoustime systems [20-23]. Based on significant early investigations such as [24-26] studying the basic properties and performance of delta operator model, numerical properties and practical applications of delta operator model have been extensively investigated [27-29]. The delta operator is defined by

$$
\delta x(t)= \begin{cases}\frac{d x(t)}{d t}, & T=0  \tag{1}\\ \frac{(x(t+T)-x(t))}{T}, & T \neq 0\end{cases}
$$

where $T$ is a sampling period. When $T \rightarrow 0$, the delta operator model will approach the continuous system before discretization and reflect a quasicontinuous performance.

In real engineering, time delays are involved in many fields, such as mechanics, medicine, chemistry, biology,
physics, economics, engineering, and control theory [30-33]. The existence of time delay may give rise to the deterioration of system performance and instability. Many results have been reported for time-delay systems [34-39].

In addition, exogenous disturbances are commonly unavoidable in practical process, and the output will be inevitably affected by the disturbance in a system. Because of the peculiar nonnegative property of positive systems, the $l_{1}-$ gain (or $L_{1}$-gain) index [39] can characterize the disturbance rejection property, by means of which we can limit the effect of disturbance in a prescribed level. Some results on $l_{1}$-gain (or $L_{1}$-gain) analysis and control for positive systems have been reported in the literature [39, 40]. However, few results on the issue of $l_{1}$-gain performance for positive switched systems via delta operator approach are proposed, which motivates the current research.

In this paper, we focus our attention on investigating the stability and $l_{1}$-gain controller design for positive switched systems with time-varying delays via delta operator approach. The main contributions of this paper are fourfold. (1) The positive switched systems via delta operator approach are investigated for the first time. (2) By applying the average dwell time approach, sufficient conditions of exponential stability for positive switched delta operator systems are derived. Moreover, the results obtained can be applied to both continuous-time systems and discrete-time systems. (3) $l_{1}$-gain performance analysis of the underlying system is developed. (4) A state feedback controller design scheme is proposed such that the corresponding closed-loop system is exponentially stable with an $l_{1}$-gain performance.

The remainder of the paper is as follows. The problem formulation and some necessary lemmas are provided in Section 2. In Section 3, the issues of stability, $l_{1}$-gain performance analysis, and control of the underlying system are developed. A numerical example is presented to demonstrate the feasibility of the obtained results in Section 4. In Section 5 , concluding remarks are given.

Notations. $A \succeq 0(\preceq, \succ, \prec)$ means that all entries of matrix $A$ are nonnegative (nonpositive, positive, and negative); $A>$ $B(A \succeq B)$ means that $A-B \succ 0(A-B \succeq 0) ; A^{T}$ means the transpose of matrix $A ; R\left(R_{+}\right)$is the set of all real (positive real) numbers; $R^{n}\left(R_{+}^{n}\right)$ is an $n$-dimensional real (positive real) vector space; $R^{m \times n}$ is the set of all $m \times n$-dimensional real matrices; $Z_{+}$refers to the set of all positive integers; the vector 1 -norm is denoted by $\|x\|=\sum_{k=1}^{n}\left|x_{k}\right|$, where $x_{k}$ is the $k$ th element of $x \in R^{n} ; 1_{n} \in R^{n}$ denotes a column vector with $n$ rows containing only 1 entry; $l_{1}\left[k_{0}, \infty\right)$ is the space of absolute summable sequence on $\left[k_{0}, \infty\right)$; that is, we say $z:\left[k_{0}, \infty\right) \rightarrow R^{p}$ is in $l_{1}\left[k_{0}, \infty\right)$ if $\sum_{k=k_{0}}^{\infty}\|z(k)\|<\infty$.

## 2. Problem Formulation

Consider the following switched delta operator system with time-varying delays:

$$
\delta x(k)=A_{\sigma(k)} x(k)+A_{d \sigma(k)} x\left(k-d_{k}\right)+D_{\sigma(k)} w(k)
$$

$$
\begin{gather*}
x\left(k_{0}+\theta\right)=\varphi(\theta), \quad \theta=-\bar{d},-\bar{d}+1, \ldots, 0, \\
z(k)=C_{\sigma(k)} x(k)+E_{\sigma(k)} w(k), \tag{2}
\end{gather*}
$$

where $x(k) \in R^{n}$ denotes the state; $z(k) \in R^{l}$ is the controlled output; and $w(k) \in R^{w}$ is the disturbance input, which belongs to $l_{1}\left[k_{0}, \infty\right) . k$ means the time $t=k T$ and $T>0$ is the sampling period; $k_{0}$ is the initial time. $\sigma(k):\left[k_{0}, \infty\right) \rightarrow$ $\underline{m}=\{1,2, \ldots, m\}$ is the switching signal with $m$ representing the number of subsystems. $A_{p}, A_{d p}, C_{p}, D_{p}$, and $E_{p}$ are constant matrices with appropriate dimensions. $d_{k}$ denotes the time-varying discrete delay which satisfies $0 \leq \underline{d} \leq d_{k} \leq$ $\bar{d}$ for known integers $\underline{d}$ and $\bar{d} ;\{\varphi(\theta), \theta=-\bar{d},-\bar{d}+1, \ldots, 0\}$ is a given discrete vector-valued initial condition. The switch is assumed to only occur at the sampling time in this paper.

Remark 1. To illustrate the main advantage of delta operator systems directly, we consider a typical continuous system without time delays as follows:

$$
\begin{align*}
\dot{x}(t) & =A x(t)+D w(t)  \tag{3}\\
z(t) & =C x(t)+E w(t)
\end{align*}
$$

Using the traditional shift operator approach to discretize the system, the following discrete form in $z$-domain can be obtained $(k=0,1,2, \ldots)$ :

$$
\begin{gather*}
x((k+1) T)=A_{z} x(k T)+D_{z} w(k T), \\
z(k T)=C_{z} x(k T)+E_{z} w(k T), \tag{4}
\end{gather*}
$$

where $A_{z}=e^{A T}, D_{z}=\left(\int_{0}^{T} e^{A t} d t\right) D, C_{z}=C$, and $E_{z}=E$. When $T \rightarrow 0, \lim _{T \rightarrow 0} A_{z}=I$ and $\lim _{T \rightarrow 0} D_{z}=0$. The movement of the system poles towards stable boundary makes the system defective with the increase in the sampling rates. However, by utilizing the delta operator approach, we can obtain the following system expressed in delta domain:

$$
\begin{align*}
\delta x(k T) & =A_{\delta} x(k T)+D_{\delta} w(k T), \\
z(k T) & =C_{\delta} x(k T)+E_{\delta} w(k T), \tag{5}
\end{align*}
$$

where $A_{\delta}=\left(A_{z}-I\right) / T, D_{\delta}=D_{z} / T, C_{\delta}=C$, and $E_{\delta}=E$. When $T \rightarrow 0, \lim _{T \rightarrow 0} A_{\delta}=A$ and $\lim _{T \rightarrow 0} D_{z}=D$. It can be seen that the system matrices are the same as those of the original continuous system, alleviating the problems encountered with fast sampling.

Remark 2. Since a delta operator system can be regarded as a quasicontinuous system when $T \rightarrow 0$, the term $\delta x(k)$ can be utilized like $\dot{x}(t)$ in normal continuous-time systems.

Definition 3. System (2) is said to be positive if, for any initial conditions $\varphi(\theta) \succeq 0, \theta=-\bar{d},-\bar{d}+1, \ldots, 0$, any inputs $w(k) \geq 0$, and any switching signals $\sigma(k)$, the corresponding trajectories $x(k) \succeq 0$ and $z(k) \succeq 0$ hold for all $k \geq k_{0}$.

Remark 4. Definition 3 follows the general positivity definition of a positive system, which means that the state and
output are nonnegative whenever the initial condition and input are nonnegative $[1,2]$.

Lemma 5. System (2) is positive if and only if $\left(I+T A_{p}\right) \succeq 0$, $A_{d p} \succeq 0, D_{p} \succeq 0, C_{p} \succeq 0$, and $E_{p} \succeq 0$, for all $p \in \underline{m}$.

Proof. From the definition of delta operator $\delta$, the discrete form of system (2) can be obtained as follows:

$$
\begin{align*}
& x(k+1)=\left(I+T A_{\sigma(k)}\right) x(k)+T A_{d \sigma(k)} x\left(k-d_{k}\right) \\
&+T D_{\sigma(k)} w(k),  \tag{6}\\
& x\left(k_{0}+\right.\theta)=\varphi(\theta), \quad \theta=-\bar{d},-\bar{d}+1, \ldots, 0, \\
& z(k)=C_{\sigma(k)} x(k)+E_{\sigma(k)} w(k) .
\end{align*}
$$

Combining Lemma 2 in [41] and Lemma 1 in [42], one can obtain the remaining proof easily.

Remark 6. When $T \rightarrow 0$, system (2) degenerates to a general continuous-time positive switched system as follows:

$$
\begin{gather*}
\dot{x}(t)=A_{\sigma(t)} x(t)+A_{d \sigma(t)} x(t-d(t))+D_{\sigma(t)} w(t), \\
x\left(t_{0}+\theta\right)=\varphi(\theta), \quad \theta \in\left[-d_{2}, 0\right],  \tag{7}\\
z(t)=C_{\sigma(t)} x(t)+E_{\sigma(t)} w(t),
\end{gather*}
$$

where $d(t)$ denotes the time-varying delay which is everywhere time differentiable and satisfies $0 \leq d_{1} \leq d(t) \leq d_{2}$ and $\dot{d}(t) \leq d_{d}<1$ for known constants $d_{1}, d_{2}$, and $d_{d}$. Then according to [39], system (7) is positive if and only if $A_{p}$ are Metzler matrices, and $A_{d p} \succeq 0, C_{p} \succeq 0, D_{p} \succeq 0$, and $E_{p} \succeq 0$, for all $p \in \underline{m}$.

Remark 7. In the light of Lemma 2.1 of [43], it is clear that the $p$ th subsystem in system (2) is positive if and only if ( $I+$ $\left.T A_{p}\right) \succeq 0, \mathrm{~A}_{d p} \succeq 0, D_{p} \succeq 0, C_{p} \succeq 0$, and $E_{p} \succeq 0$, for all $p \in$ $\underline{m}$. Thus we can have an equivalent expression of Lemma 5: system (2) is positive under any switching signals if and only if it consists of a family of positive subsystems.

Definition 8 (see [44]). System (2) with $w(k)=0$ is said to be exponentially stable under $\sigma(k)$ if, for constants $\alpha>0$ and $\beta>0$, the solution $x(k)$ satisfies

$$
\begin{equation*}
\|x(k)\| \leq \alpha\left\|x\left(k_{0}\right)\right\|_{c} e^{-\beta\left(k-k_{0}\right)}, \quad \forall k \geq k_{0} \tag{8}
\end{equation*}
$$

where $\left\|x\left(k_{0}\right)\right\|_{c}=\sup _{-\bar{d} \leq \theta \leq 0}\left\|x\left(k_{0}+\theta\right)\right\|$.
Definition 9 (see [45]). For any switching signal $\sigma(k)$ and any $k_{2}>k_{1} \geq 0$, let $N_{\sigma}\left(k_{1}, k_{2}\right)$ denote the number of switches of $\sigma(k)$ over the interval $\left[k_{1}, k_{2}\right)$. For given $\tau_{a}>0$ and $N_{0} \geq 0$, if the inequality

$$
\begin{equation*}
N_{\sigma}\left(k_{1}, k_{2}\right) \leq N_{0}+\frac{k_{2}-k_{1}}{\tau_{a}} \tag{9}
\end{equation*}
$$

holds, then the positive constant $\tau_{a}$ is called an average dwell time and $N_{0}$ is called a chattering bound.

Without loss of generality, one chooses $N_{0}=0$ in this paper.

Definition 10. For $0<\alpha<1 / T$ and $\gamma>0$, system (2) is said to have a prescribed $l_{1}$-gain performance level $\gamma$ if there exists a switching signal $\sigma(k)$ such that the following conditions are satisfied:
(a) system (2) is exponentially stable when $w(k)=0$;
(b) under zero initial condition, that is, $\varphi(\theta)=0, \theta=-\bar{d}$, $-\bar{d}+1, \ldots, 0$, system (2) satisfies

$$
\begin{array}{r}
\sum_{k=k_{0}}^{\infty}(1-T \alpha)^{\left(k-k_{0}\right)}\|z(k)\| \leq \gamma \sum_{k=k_{0}}^{\infty}\|w(k)\|  \tag{10}\\
\forall w(k) \in l_{1}\left[k_{0}, \infty\right), \quad w(k) \neq 0
\end{array}
$$

Remark 11. In Definition 10, as proposed in [39], $l_{1}$-gain performance index $\gamma$ characterizes system's suppression to exogenous disturbances. The smaller the value of $\gamma$ is, the better the performance of the system is, that is, the lesser the effect of the disturbance input on the control output is.

The purposes of this paper are (1) to find a class of switching signals $\sigma(k)$ under which system (2) is exponentially stable and possesses an $l_{1}$-gain performance and (2) to determine a class of switching signals and a state feedback controller $u(k)=K_{\sigma(k)} x(k)$ for the following positive switched delta operator system with time-varying delays:

$$
\begin{gather*}
\delta x(k)=A_{\sigma(k)} x(k)+A_{d \sigma(k)} x\left(k-d_{k}\right) \\
\\
+B_{\sigma(k)} u(k)+D_{\sigma(k)} w(k),  \tag{11}\\
x\left(k_{0}+\theta\right)= \\
\psi(\theta), \quad \theta=-\bar{d},-\bar{d}+1, \ldots, 0, \\
z(k)=
\end{gather*}
$$

such that the resulting closed-loop system is exponentially stable with an $l_{1}$-gain performance.

## 3. Main Results

This section will focus on the problems of stability analysis and $l_{1}$-gain controller design for positive switched delta operator systems with time-varying delays.
3.1. Stability Analysis. First, we consider the following switched positive delta operator system:

$$
\begin{gather*}
\delta x(k)=A_{\sigma(k)} x(k)+A_{d \sigma(k)} x\left(k-d_{k}\right) \\
x\left(k_{0}+\theta\right)=\varphi(\theta), \quad \theta=-\bar{d},-\bar{d}+1, \ldots, 0 \tag{12}
\end{gather*}
$$

where $I+T A_{p} \succeq 0, A_{d p} \succeq 0$ for $p \in \underline{m}$, and $d_{k}$ is defined the same as system (2).

Sufficient conditions of exponential stability of system (12) are provided in the following theorem.

Theorem 12. Given a positive constant $0<\alpha<1 / T$, if there exist $v_{p}, v_{p}, \vartheta_{p} \in R_{+}^{n}$, such that, for all $p \in \underline{m}$,

$$
\begin{gather*}
A_{p}^{T} v_{p}+\alpha v_{p}+(1-T \alpha)(\bar{d}-\underline{d}+1) v_{p}+(1-T \alpha) \vartheta_{p} \preceq 0, \\
A_{d p}^{T} v_{p}-(1-T \alpha)^{\bar{d}+1} v_{p} \preceq 0, \tag{13}
\end{gather*}
$$

where $v_{p}=\left[v_{p 1}, v_{p 2}, \ldots, v_{p n}\right]^{T}, v_{p}=\left[v_{p 1}, v_{p 2}, \ldots, v_{p n}\right]^{T}$, and $\vartheta_{p}=\left[\vartheta_{p 1}, \vartheta_{p 2}, \ldots, \vartheta_{p n}\right]^{T}$, then system (12) is exponentially stable for any switching signals $\sigma(k)$ with average dwell time

$$
\begin{equation*}
\tau_{a}>\tau_{a}^{*}=-\frac{\ln \mu}{\ln (1-T \alpha)}, \tag{14}
\end{equation*}
$$

where $\mu \geq 1$ satisfies

$$
\begin{equation*}
v_{p} \preceq \mu v_{q}, \quad v_{p} \preceq \mu v_{q}, \quad \vartheta_{p} \leq \mu \vartheta_{q}, \quad \forall p, q \in \underline{m} . \tag{15}
\end{equation*}
$$

Furthermore, the state decay of system (12) is given by

$$
\begin{equation*}
\|x(k)\| \leq a b^{\left(k-k_{0}\right)}\left\|x\left(k_{0}\right)\right\|_{c}, \tag{16}
\end{equation*}
$$

where

$$
\begin{gather*}
a=\frac{\varepsilon_{2}}{\varepsilon_{1}}+\frac{T \varepsilon_{3} \bar{d}}{\varepsilon_{1}}+\frac{0.5 T \varepsilon_{3}(\bar{d}+\underline{d}-1)(\bar{d}-\underline{d})}{\varepsilon_{1}}+\frac{T \varepsilon_{4} \bar{d}}{\varepsilon_{1}} \\
b=\mu^{1 / \tau_{a}}(1-T \alpha) \\
\varepsilon_{1}=\min _{(r, p) \in \underline{n} \times \underline{m}}\left\{v_{p r}\right\}, \\
\varepsilon_{2}=\max _{(r, p) \in \underline{n} \times \underline{m}}\left\{v_{p r}\right\}  \tag{17}\\
\varepsilon_{3}=\max _{(r, p) \in \underline{n} \times \underline{m}}\left\{v_{p r}\right\} \\
\varepsilon_{4}=\max _{(r, p) \in \underline{n} \times \underline{m}}\left\{\vartheta_{p r}\right\}, \\
\left\|x\left(k_{0}\right)\right\|_{c}=\sup _{-\bar{d} \leq \theta \leq 0}\left\|x\left(k_{0}+\theta\right)\right\|, \quad \underline{n}=\{1,2, \ldots, n\}
\end{gather*}
$$

Proof. Choose the following piecewise copositive type Lyapunov functional for the $p$ th subsystem in system (12):

$$
\begin{align*}
V_{p}(k, x(k))= & V_{p 1}(k, x(k))+V_{p 2}(k, x(k))+V_{p 3}(k, x(k)) \\
& +V_{p 4}(k, x(k)), \tag{18}
\end{align*}
$$

where

$$
\begin{gather*}
V_{p 1}(k, x(k))=x^{T}(k) v_{p}, \\
V_{p 2}(k, x(k))=T \sum_{s=k-d_{k}}^{k-1}(1-T \alpha)^{k-s} x^{T}(s) v_{p}, \\
V_{p 3}(k, x(k))=T \sum_{l=-\bar{d}+1}^{-d} \sum_{s=k+l}^{k-1}(1-T \alpha)^{k-s} x^{T}(s) v_{p}, \\
V_{p 4}(k, x(k))=T \sum_{s=k-\bar{d}}^{k-1}(1-T \alpha)^{k-s} x^{T}(s) \vartheta_{p}, \quad \forall p \in \underline{m} . \tag{19}
\end{gather*}
$$

For simplicity, $V_{p}(k, x(k))$ is written as $V_{p}(k)$ (correspondingly, $V(k, x(k))$ is written as $V(k))$ in the later section of the paper.

The Lyapunov function in delta domain has the following form:

$$
\begin{aligned}
& \delta V_{p 1}(k, x(k))=\delta\left(x^{T}(k) v_{p}\right) \\
& =\left(\delta x^{T}(k)\right) v_{p} \\
& =x^{T}(k) A_{p}^{T} v_{p}+x^{T}\left(k-d_{k}\right) A_{d p}^{T} v_{p}, \\
& \delta V_{p 2}(k, x(k))=\frac{1}{T}\left[V_{p 2}(k+1)-V_{p 2}(k)\right] \\
& =\frac{1}{T}\left[T \sum_{s=k+1-d_{k+1}}^{(k+1)-1}(1-T \alpha)^{k+1-s} x^{T}(s) v_{p}\right. \\
& \left.-T \sum_{s=k-d_{k}}^{k-1}(1-T \alpha)^{k-s} x^{T}(s) v_{p}\right] \\
& \leq-T \alpha \sum_{s=k-d_{k}}^{k-1}(1-T \alpha)^{k-s} x^{T}(s) v_{p} \\
& +(1-T \alpha) x^{T}(k) v_{p} \\
& -(1-T \alpha)^{\bar{d}+1} x^{T}\left(k-d_{k}\right) v_{p} \\
& +\sum_{s=k+1-\bar{d}}^{k-d}(1-T \alpha)^{k+1-s} x^{T}(s) v_{p} \text {, } \\
& \delta V_{p 3}(k, x(k))=\frac{1}{T}\left[V_{p 3}(k+1)-V_{p 3}(k)\right] \\
& =\frac{1}{T}\left[T \sum_{l=-\bar{d}+1}^{-d} \sum_{s=k+1+l}^{(k+1)-1}(1-T \alpha)^{k+1-s} x^{T}(s) v_{p}\right. \\
& \left.-T \sum_{l=-\bar{d}+1}^{-\frac{d}{c}} \sum_{s=k+l}^{k-1}(1-T \alpha)^{k-s} x^{T}(s) v_{p}\right] \\
& =-T \alpha \sum_{l=-\bar{d}+1}^{-d} \sum_{s=k+l}^{k-1}(1-T \alpha)^{k-s} x^{T}(s) v_{p} \\
& +(1-T \alpha)(\bar{d}-\underline{d}) x^{T}(k) v_{p} \\
& -\sum_{s=k+1-\bar{d}}^{k-d}(1-T \alpha)^{k+1-s} x^{T}(s) v_{p} \text {, } \\
& \delta V_{p 4}(k, x(k))=\frac{1}{T}\left[V_{p 4}(k+1)-V_{p 4}(k)\right] \\
& =\frac{1}{T}\left[T \sum_{s=k+1-\bar{d}}^{(k+1)-1}(1-T \alpha)^{k+1-s} x^{T}(s) \vartheta_{p}\right. \\
& \left.-T \sum_{s=k-\bar{d}}^{k-1}(1-T \alpha)^{k-s} x^{T}(s) \vartheta_{p}\right]
\end{aligned}
$$

$$
\begin{align*}
= & -T \alpha \sum_{s=k-\bar{d}}^{k-1}(1-T \alpha)^{k-s} x^{T}(s) \vartheta_{p} \\
& +(1-T \alpha) x^{T}(k) \vartheta_{p} \\
& -(1-T \alpha)^{\bar{d}+1} x^{T}(k-\bar{d}) \vartheta_{p} \tag{20}
\end{align*}
$$

According to (20), we have

$$
\begin{align*}
& \delta V_{p}(k, x(k))+\alpha V_{p}(k, x(k)) \\
& \leq x^{T}(k) A_{p}^{T} v_{p}+x^{T}\left(k-d_{k}\right) A_{d p}^{T} v_{p}+\alpha x^{T}(k) v_{p} \\
&+(1-T \alpha) x^{T}(k) v_{p}-(1-T \alpha)^{\bar{d}+1} x^{T}\left(k-d_{k}\right) v_{p} \\
&+\sum_{s=k+1-\bar{d}}^{k-d}(1-T \alpha)^{k+1-s} x^{T}(s) v_{p} \\
& \quad+(1-T \alpha)(\bar{d}-\underline{d}) x^{T}(k) v_{p} \\
& \quad-\sum^{\sum^{k-d}}(1-\bar{d}-T \alpha)^{k+1-s} x^{T}(s) v_{p}  \tag{21}\\
& \quad+(1-T \alpha) x^{T}(k) \vartheta_{p}-(1-T \alpha)^{\bar{d}+1} x^{T}(k-\bar{d}) \vartheta_{p} \\
& \leq x^{T}(k)\left[A_{p}^{T} v_{p}+\alpha v_{p}+(1-T \alpha)\right. \\
&\left.\quad \times(\bar{d}-\underline{d}+1) v_{p}+(1-T \alpha) \vartheta_{p}\right] \\
& \quad+x^{T}\left(k-d_{k}\right)\left[A_{d p}^{T} v_{p}-(1-T \alpha)^{\bar{d}+1} v_{p}\right] .
\end{align*}
$$

From (13), we obtain

$$
\begin{align*}
\delta V_{p}(k)+\alpha V_{p}(k) & \leq 0 \\
& \Longrightarrow \delta V_{p}(k)=\frac{V_{p}(k+1)-V_{p}(k)}{T} \\
& \leq-\alpha V_{p}(k) \\
& \Longrightarrow V_{p}(k+1) \leq V_{p}(k)-T \alpha V_{p}(k) \\
& \Longrightarrow V_{p}(k+1) \leq(1-T \alpha) V_{p}(k) . \tag{22}
\end{align*}
$$

Let $k_{1}<\cdots<k_{g}$ denote the switching instants of $\sigma(k)$ over the interval $\left[k_{0}, k\right)$. Consider the following piecewise Lyapunov functional candidate for system (12):

$$
\begin{equation*}
V(k)=V_{\sigma(k)}(k) \tag{23}
\end{equation*}
$$

From (15) and (18), we obtain

$$
\begin{equation*}
V_{\sigma\left(k_{i}\right)}(k) \leq \mu V_{\sigma\left(k_{i}^{-}\right)}(k), \quad i=1,2, \ldots, g . \tag{24}
\end{equation*}
$$

Then, it follows from (22), (24), and the relation $N_{\sigma}\left(k_{0}, k\right) \leq$ $\left(k-k_{0}\right) / \tau_{a}$ that, for $\left[k_{i}, k_{i+1}\right)$,

$$
\begin{align*}
V_{\sigma(k)}(k) & =V_{\sigma\left(k_{i}\right)}(k) \leq(1-T \alpha)^{\left(k-k_{i}\right)} V_{\sigma\left(k_{i}\right)}\left(k_{i}\right) \\
& \leq \mu(1-T \alpha)^{\left(k-k_{i}\right)} V_{\sigma\left(k_{i}^{-}\right)}\left(k_{i}^{-}\right) \\
& \leq \mu(1-T \alpha)^{\left(k-k_{i}\right)+\left(k_{i}-k_{i-1}\right)} V_{\sigma\left(k_{i-1}\right)}\left(k_{i-1}\right) \\
& =\mu(1-T \alpha)^{\left(k-k_{i-1}\right)} V_{\sigma\left(k_{i-1}\right)}\left(k_{i-1}\right) \\
& =\mu^{N_{\sigma}\left(k_{i-1}, k\right)}(1-T \alpha)^{\left(k-k_{i-1}\right)} V_{\sigma\left(k_{i-1}\right)}\left(k_{i-1}\right)  \tag{25}\\
& \leq \cdots \\
& \leq \mu^{N_{\sigma}\left(k_{0}, k\right)}(1-T \alpha)^{\left(k-k_{0}\right)} V_{\sigma\left(k_{0}\right)}\left(k_{0}\right) \\
& \leq \mu^{\left(k-k_{0}\right) / \tau_{a}}(1-T \alpha)^{\left(k-k_{0}\right)} V_{\sigma\left(k_{0}\right)}\left(k_{0}\right) \\
& \leq\left(\mu^{1 / \tau_{a}}(1-T \alpha)\right)^{\left(k-k_{0}\right)} V_{\sigma\left(k_{0}\right)}\left(k_{0}\right) .
\end{align*}
$$

Considering the definition of $V_{\sigma(k)}(k), \varepsilon_{1}, \varepsilon_{2}, \varepsilon_{3}$, and $\varepsilon_{4}$ in Theorem 12, it yields that

$$
\begin{gather*}
V_{\sigma(k)}(k) \geq \varepsilon_{1}\|x(k)\|, \\
V_{\sigma\left(k_{0}\right)}\left(k_{0}\right) \leq\left(\varepsilon_{2}+T \varepsilon_{3} \bar{d}+0.5 T \varepsilon_{3}(\bar{d}+\underline{d}-1)\right. \\
\left.\times(\bar{d}-\underline{d})+T \varepsilon_{4} \bar{d}\right) \sup _{-\bar{d} \leq \theta \leq 0}\left\|x\left(k_{0}+\theta\right)\right\| . \tag{26}
\end{gather*}
$$

Combining (25)-(26), we obtain

$$
\begin{align*}
\|x(k)\| \leq & \left(\frac{\varepsilon_{2}}{\varepsilon_{1}}+\frac{T \varepsilon_{3} \bar{d}}{\varepsilon_{1}}+\frac{0.5 T \varepsilon_{3}(\bar{d}+\underline{d}-1)(\bar{d}-\underline{d})}{\varepsilon_{1}}+\frac{T \varepsilon_{4} \bar{d}}{\varepsilon_{1}}\right) \\
& \times\left(\mu^{1 / \tau_{a}}(1-T \alpha)\right)^{\left(k-k_{0}\right)}\left\|x\left(k_{0}\right)\right\|_{c} \tag{27}
\end{align*}
$$

where $\left\|x\left(k_{0}\right)\right\|_{c}=\sup _{-\bar{d} \leq \theta \leq 0}\left\|x\left(k_{0}+\theta\right)\right\|$.
Therefore, according to Definition 8, system (12) is exponentially stable for any switching signals $\sigma(k)$ with average dwell time (14).

This completes the proof.
Remark 13. When $\mu=1$ in (15), which leads to $v_{p}=v_{q}, v_{p}=$ $v_{q}, \vartheta_{p}=\vartheta_{q}$, for all $p, q \in \underline{m}$, and $\tau_{a}^{*}=0$ by (14), system (12) possesses a common copositive type Lyapunov-Krasovskii functional, and the switching signal can be arbitrary.

When $d_{k}=0$, system (12) can be represented by

$$
\begin{equation*}
\delta x(k)=\widetilde{A}_{\sigma(k)} x(k) \tag{28}
\end{equation*}
$$

where $\widetilde{A}_{p}=A_{p}+A_{d p}$ satisfies $I+T \widetilde{A}_{p} \succeq 0$, for all $p \in \underline{m}$. Then we have the following corollary.

Corollary 14. Given a positive constant $0<\alpha<1 / T$, if there exist $v_{p} \in R_{+}^{n}$, such that, for all $p \in \underline{m}$,

$$
\begin{equation*}
\widetilde{A}_{p}^{T} v_{p}+\alpha v_{p} \leq 0, \tag{29}
\end{equation*}
$$

then system (28) is exponentially stable for any switching signals $\sigma(k)$ with average dwell time (14), where $\mu \geq 1$ satisfies

$$
\begin{equation*}
v_{p} \leq \mu \nu_{q}, \quad \forall p, q \in \underline{m} . \tag{30}
\end{equation*}
$$

When the sampling period $T \rightarrow 0$, system (12) becomes a continuous-time system as follows:

$$
\begin{gather*}
\dot{x}(t)=A_{\sigma(t)} x(t)+A_{d \sigma(t)} x(t-d(t)),  \tag{31}\\
x\left(t_{0}+\theta\right)=\varphi(\theta), \quad \theta \in\left[-d_{2}, 0\right],
\end{gather*}
$$

where $A_{p}$ are Metzler matrices and $A_{d p} \geq 0$, for all $p \in \underline{m}$. $d(t)$ denotes the time-varying delay which satisfies $0 \leq d_{1} \leq$ $d(t) \leq d_{2}$ and $\dot{d}(t) \leq d_{d}<1$ for known constants $d_{1}, d_{2}$, and $d_{d}$.

We can obtain sufficient conditions of exponential stability of system (31) by Theorem 12.

Corollary 15. Given a positive constant $\alpha$, if there exist $v_{p}, v_{p}, \vartheta_{p} \in R_{+}^{n}$, such that, for all $p \in \underline{m}$,

$$
\begin{gather*}
A_{p}^{T} v_{p}+\alpha v_{p}+\left(d_{2}-d_{1}+1\right) v_{p}+\vartheta_{p} \leq 0 \\
A_{d p}^{T} v_{p}-\left(1-d_{d}\right) e^{-\alpha d_{2}} v_{p} \leq 0 \tag{32}
\end{gather*}
$$

then system (31) is exponentially stable for any switching signals $\sigma(t)$ with average dwell time

$$
\begin{equation*}
\tau_{a}>\tau_{a}^{*}=\frac{\ln \mu}{\alpha} \tag{33}
\end{equation*}
$$

where $\mu \geq 1$ satisfies (15).
Let $\bar{A}_{\sigma(k)}=A_{\sigma(k)}+I$. When the sampling period $T=1$, system (12) becomes a discrete-time system as follows:

$$
\begin{align*}
& x(k+1)=\bar{A}_{\sigma(k)} x(k)+A_{d \sigma(k)} x\left(k-d_{k}\right), \\
& x\left(k_{0}+\theta\right)=\varphi(\theta), \quad \theta=-\bar{d},-\bar{d}+1, \ldots, 0, \tag{34}
\end{align*}
$$

where $\bar{A}_{p} \geq 0$ and $A_{d p} \succeq 0$, for all $p \in \underline{m}$. One can obtain sufficient conditions of exponential stability of system (34) by Theorem 12.

Corollary 16. Given a positive constant $0<\alpha<1$, if there exist $v_{p}, v_{p}, \vartheta_{p} \in R_{+}^{n}$, such that, for all $p \in \underline{m}$,

$$
\begin{gather*}
\bar{A}_{p}^{T} v_{p}+\alpha v_{p}+(1-\alpha)(\bar{d}-\underline{d}+1) v_{p}+(1-\alpha) \vartheta_{p} \leq 0,  \tag{35}\\
A_{d p}^{T} v_{p}-(1-\alpha)^{\bar{d}+1} v_{p} \leq 0,
\end{gather*}
$$

then system (34) is exponentially stable for any switching signals $\sigma(k)$ with average dwell time

$$
\begin{equation*}
\tau_{a}>\tau_{a}^{*}=-\frac{\ln \mu}{\ln (1-\alpha)} \tag{36}
\end{equation*}
$$

where $\mu \geq 1$ satisfies (15).
3.2. $l_{1}$-Gain Analysis. The following theorem establishes sufficient conditions of exponential stability with $l_{1}$-gain property for system (2).

Theorem 17. For given positive constants $0<\alpha<1 / T$ and $\gamma$, if there exist $v_{p}, v_{p}, \vartheta_{p} \in R_{+}^{n}$, such that, for all $p \in \underline{m}$,

$$
\begin{gather*}
A_{p}^{T} v_{p}+\alpha v_{p}+(1-T \alpha)(\bar{d}-\underline{d}+1) v_{p} \\
+(1-T \alpha) \vartheta_{p}+\widetilde{c}_{p} \preceq 0  \tag{37}\\
A_{d p}^{T} v_{p}-(1-T \alpha)^{\bar{d}+1} v_{p} \preceq 0 \\
D_{p}^{T} v_{p}+\widetilde{e}_{p}-\gamma 1_{w} \leq 0 \tag{38}
\end{gather*}
$$

where $1_{w}=[\underbrace{1,1, \ldots, 1}_{w}]^{T}, \widetilde{c}_{p}=\left[\left\|c_{p 1}\right\|,\left\|c_{p 2}\right\|, \ldots,\left\|c_{p n}\right\|\right]^{T}, c_{p j}$ represents the $j$ th column of matrix $C_{p}, j \in \underline{n}=\{1,2, \ldots, n\}$, $\widetilde{e}_{p}=\left[\left\|e_{p 1}\right\|,\left\|e_{p 2}\right\|, \ldots,\left\|e_{p w}\right\|\right]^{T}$, and $e_{p j}$ represents the $j$ th column of matrix $E_{p}, j \in \underline{w}=\{1,2, \ldots, w\}$, then system (2) is exponentially stable with an $l_{1}$-gain performance for any switching signals $\sigma(k)$ with average dwell time (14), where $\mu \geq 1$ satisfies (15).

Proof. By Theorem 12, the exponential stability of system (2) with $w(k)=0$ is ensured if (37) holds. To show the weighted $l_{1}$-gain performance, we choose the Lyapunov functional (18). From (15), we have

$$
\begin{equation*}
V_{\sigma\left(k_{i}\right)}\left(k_{i}\right) \leq \mu V_{\sigma\left(k_{i}^{-}\right)}\left(k_{i}^{-}\right), \quad \forall i=1,2, \ldots . \tag{39}
\end{equation*}
$$

For any $k \in\left[k_{i}, k_{i+1}\right.$ ), noticing (37)-(38), we have

$$
\begin{equation*}
V(k) \leq(1-T \alpha)^{\left(k-k_{i}\right)} V_{\sigma\left(k_{i}\right)}\left(k_{i}\right)-\sum_{s=k_{i}}^{k}(1-T \alpha)^{(k-s)} \Lambda(s), \tag{40}
\end{equation*}
$$

where $\Lambda(s)=\|z(s)\|-\gamma\|w(s)\|$.
Combining (39) and (40) leads to

$$
\begin{aligned}
V(k) \leq & \mu(1-T \alpha)^{\left(k-k_{i}\right)} V_{\sigma\left(k_{i}^{-}\right)}\left(k_{i}^{-}\right)-\sum_{s=k_{i}}^{k}(1-T \alpha)^{(k-s)} \Lambda(s) \\
\leq & \mu(1-T \alpha)^{\left(k-k_{i-1}\right)} V_{\sigma\left(k_{i-1}\right)}\left(k_{i-1}\right) \\
& -\mu \sum_{s=k_{i-1}}^{k_{i}}(1-T \alpha)^{(k-s)} \Lambda(s) \\
& -\sum_{s=k_{i}}^{k}(1-T \alpha)^{(k-s)} \Lambda(s) \\
= & \mu^{N_{\sigma}\left(k_{i-1}, k\right)}(1-T \alpha)^{\left(k-k_{i-1}\right)} V_{\sigma\left(k_{i-1}\right)}\left(k_{i-1}\right) \\
& -\mu^{N_{\sigma}\left(k_{i-1}, k\right)} \sum_{s=k_{i-1}}^{k_{i}}(1-T \alpha)^{(k-s)} \Lambda(s) \\
& \quad-\sum_{s=k_{i}}^{k}(1-T \alpha)^{(k-s)} \Lambda(s) \\
\leq & \cdots
\end{aligned}
$$

$$
\begin{align*}
\leq & \mu^{N_{\sigma}\left(k_{0}, k\right)}(1-T \alpha)^{\left(k-k_{0}\right)} V_{\sigma\left(k_{0}\right)}\left(k_{0}\right) \\
& -\mu^{N_{\sigma}\left(k_{0}, k\right)} \sum_{s=k_{0}}^{k_{1}}(1-T \alpha)^{(k-s)} \Lambda(s) \\
& -\mu^{N_{\sigma}\left(k_{1}, k\right)} \sum_{s=k_{1}}^{k_{2}}(1-T \alpha)^{(k-s)} \Lambda(s)-\cdots \\
& -\sum_{s=k_{i}}^{k}(1-T \alpha)^{(k-s)} \Lambda(s) \\
= & \mu^{N_{\sigma}\left(k_{0}, k\right)}(1-T \alpha)^{\left(k-k_{0}\right)} V_{\sigma\left(k_{0}\right)}\left(k_{0}\right) \\
& -\sum_{s=k_{0}}^{k} \mu^{N_{\sigma}(s, k)}(1-T \alpha)^{(k-s)} \Lambda(s) \tag{41}
\end{align*}
$$

Under the zero initial condition, we obtain from (41) that

$$
\begin{equation*}
0 \leq-\sum_{s=k_{0}}^{k} \mu^{N_{\sigma}(s, k)}(1-T \alpha)^{(k-s)} \Lambda(s) \tag{42}
\end{equation*}
$$

namely,

$$
\begin{align*}
& \sum_{s=k_{0}}^{k} \mu^{N_{\sigma}(s, k)}(1-T \alpha)^{(k-s)}\|z(s)\|  \tag{43}\\
& \quad \leq \gamma \sum_{s=k_{0}}^{k} \mu^{N_{\sigma}(s, k)}(1-T \alpha)^{(k-s)}\|w(s)\| .
\end{align*}
$$

Multiplying both sides of (43) by $\mu^{-N_{\sigma}\left(k_{0}, k\right)}$ yields

$$
\begin{align*}
& \sum_{s=k_{0}}^{k} \mu^{-N_{\sigma}\left(k_{0}, s\right)}(1-T \alpha)^{(k-s)}\|z(s)\| \\
& \quad \leq \gamma \sum_{s=k_{0}}^{k} \mu^{-N_{\sigma}\left(k_{0}, s\right)}(1-T \alpha)^{(k-s)}\|w(s)\| \tag{44}
\end{align*}
$$

Noticing that $N_{\sigma(k)}\left(k_{0}, s\right) \leq\left(s-k_{0}\right) / \tau_{a}$, we have

$$
\begin{equation*}
\mu^{-N_{\sigma}\left(k_{0}, s\right)} \geq(1-T \alpha)^{\left(s-k_{0}\right)} \tag{45}
\end{equation*}
$$

Combining (44) and (45) leads to

$$
\begin{align*}
& \sum_{s=k_{0}}^{k}(1-T \alpha)^{\left(s-k_{0}\right)}(1-T \alpha)^{(k-s)}\|z(s)\|  \tag{46}\\
& \quad \leq \gamma \sum_{s=k_{0}}^{k}(1-T \alpha)^{(k-s)}\|w(s)\|
\end{align*}
$$

Summing both sides of (46) from $k=k_{0}$ to $\infty$ leads to

$$
\begin{equation*}
\sum_{k=k_{0}}^{\infty}(1-T \alpha)^{\left(k-k_{0}\right)}\|z(k)\| \leq \gamma \sum_{k=k_{0}}^{\infty}\|w(k)\| . \tag{47}
\end{equation*}
$$

From Definition 10, it can be concluded that system (2) is exponentially stable with a prescribed $l_{1}$-gain performance level $\gamma$.

This completes the proof.
Remark 18. When $\mu=1$ in Theorem 17, summing both sides of (44) from $k=k_{0}$ to $\infty$ leads to

$$
\begin{equation*}
\sum_{k=k_{0}}^{\infty}\|z(k)\| \leq \gamma \sum_{k=k_{0}}^{\infty}\|w(k)\| \tag{48}
\end{equation*}
$$

which gives the standard $l_{1}$-gain performance.
3.3. Controller Design. In this section, we are interested in designing a state feedback controller $u(k)=K_{\sigma(k)} x(k)$ for positive switched system (11) such that the corresponding closed-loop system

$$
\begin{align*}
\delta x(k)= & \left(A_{\sigma(k)}+B_{\sigma(k)} K_{\sigma(k)}\right) x(k) \\
& +A_{d \sigma(k)} x\left(k-d_{k}\right)+D_{\sigma(k)} w(k), \\
x\left(k_{0}+\theta\right)= & \varphi(\theta), \quad \theta=-\bar{d},-\bar{d}+1, \ldots, 0  \tag{49}\\
z(k)= & C_{\sigma(k)} x(k)+E_{\sigma(k)} w(k)
\end{align*}
$$

is exponentially stable with an $l_{1}$-gain performance.
Theorem 19. Considering system (11), for given positive scalars $0<\alpha<1 / T$ and $\gamma$, if there exist $\nu_{p}, v_{p}, \vartheta_{p} \in R_{+}^{n}$ and $g_{p} \in R^{n}$, such that, for all $p \in \underline{m}$,

$$
\begin{gather*}
A_{p}^{T} v_{p}+g_{p}+\alpha v_{p}+(1-T \alpha)(\bar{d}-\underline{d}+1) v_{p}  \tag{50}\\
+(1-T \alpha) \vartheta_{p}+\widetilde{c}_{p} \preceq 0 \\
A_{d p}^{T} v_{p}-(1-T \alpha)^{\bar{d}+1} v_{p} \preceq 0  \tag{51}\\
D_{p}^{T} v_{p}+\widetilde{e}_{p}-\gamma 1_{w} \preceq 0  \tag{52}\\
I+T\left(A_{p}+B_{p} K_{p}\right) \succeq 0 \tag{53}
\end{gather*}
$$

where $\tilde{c}_{p}$ and $\tilde{e}_{p}$ have been defined in Theorem 17 and $g_{p}=$ $K_{p}^{T} B_{p}^{T} v_{p}$, then the corresponding closed-loop system (49) is positive and exponentially stable with a prescribed $l_{1}$-gain performance level $\gamma$ for any switching signals $\sigma(k)$ with average dwell time (14), where $\mu \geq 1$ satisfies (15).

Proof. Denote $g_{p}=K_{p}^{T} B_{p}^{T} v_{p}$. Following the proof line of Theorem 17, one can exactly obtain Theorem 19. It is omitted here.

This completes the proof.
Consider the controller design of the following positive switched delta operator system without time delay:

$$
\begin{gather*}
\delta x(k)=\widetilde{A}_{\sigma(k)} x(k)+B_{\sigma(k)} u(k)+D_{\sigma(k)} w(k), \\
z(k)=C_{\sigma(k)} x(k)+E_{\sigma(k)} w(k), \tag{54}
\end{gather*}
$$

where $\widetilde{A}_{\sigma(k)}=A_{\sigma(k)}+A_{d \sigma(k)}$. Then we directly have the following corollary.

Corollary 20. Considering system (54), for given positive scalars $0<\alpha<1 / T$ and $\gamma$, if there exist $\nu_{p} \in R_{+}^{n}$ and $g_{p} \in R^{n}$, such that, for all $p \in \underline{m}$,

$$
\begin{gather*}
\widetilde{A}_{p}^{T} v_{p}+\alpha v_{p}+g_{p}+\widetilde{c}_{p} \leq 0, \\
D_{p}^{T} v_{p}+\widetilde{e}_{p}-\gamma 1_{w} \preceq 0,  \tag{55}\\
I+T\left(\widetilde{A}_{p}+B_{p} K_{p}\right) \succeq 0,
\end{gather*}
$$

where $\widetilde{c}_{p}$ and $\widetilde{e}_{p}$ have been defined in Theorem 17 and $g_{p}=$ $K_{p}^{T} B_{p}^{T} v_{p}$, then the corresponding closed-loop system is positive and exponentially stable with a prescribed $l_{1}$-gain performance level $\gamma$ for any switching signals $\sigma(k)$ with average dwell time (14), where $\mu \geq 1$ satisfies (30).

Based on Theorem 19, one is now in a position to present an effective algorithm for constructing the desired controller.

Algorithm 21. Consider the following.
Step 1. Input the matrices $A_{p}, A_{d p}, B_{p}, C_{p}, D_{p}$, and $E_{p}$.
Step 2. Choose the parameters $0<\alpha<1 / T$ and $\gamma>0$. By solving (50)-(52), one can obtain the solutions of $v_{p}, v_{p}, \vartheta_{p}$, and $g_{p}$.

Step 3. By the equation $g_{p}=K_{p}^{T} B_{p}^{T} v_{p}$ with the obtained $g_{p}$ and $v_{p}$, one can get the gain matrices $K_{p}$.

Step 4. Check condition (53) in Theorem 19. If it holds, go to Step 5; otherwise, adjust the parameter $\alpha$ and return to Step 2.

Step 5. Construct the feedback controller $u(k)=K_{\sigma(k)} x(k)$, where $K_{p}, p \in \underline{m}$, are the gain matrices.

## 4. Numerical Example

Consider positive switched delta operator system (11) consisting of two subsystems described by the following.

Subsystem 1:

$$
\begin{gather*}
A_{1}=\left[\begin{array}{cc}
1.8 & 4.5 \\
1.5 & -2.8
\end{array}\right], \quad A_{d 1}=\left[\begin{array}{ll}
0.5 & 0.0 \\
0.1 & 0.0
\end{array}\right], \quad B_{1}=\left[\begin{array}{l}
0.4 \\
0.1
\end{array}\right], \\
C_{1}=\left[\begin{array}{ll}
0.1 & 0.2
\end{array}\right], \quad D_{1}=\left[\begin{array}{l}
0.1 \\
0.2
\end{array}\right], \quad E_{1}=[0.1], \tag{56}
\end{gather*}
$$

Subsystem 2:

$$
\begin{gather*}
A_{2}=\left[\begin{array}{cc}
5.2 & 4.5 \\
3.8 & -1.8
\end{array}\right], \quad A_{d 2}=\left[\begin{array}{ll}
0.5 & 0.0 \\
0.1 & 0.0
\end{array}\right], \quad B_{2}=\left[\begin{array}{l}
0.5 \\
0.2
\end{array}\right], \\
C_{2}=\left[\begin{array}{ll}
0.2 & 0.3
\end{array}\right], \quad D_{2}=\left[\begin{array}{l}
0.2 \\
0.1
\end{array}\right], \quad E_{2}=[0.2], \tag{57}
\end{gather*}
$$



Figure 1: Switching signal.
and $\bar{d}=2, \underline{d}=0, \alpha=1.3, \gamma=2$, and $T=0.25$. Then, by solving (50)-(52) in Theorem 19, we can obtain the following solutions:

$$
\begin{array}{lll}
v_{1}=\left[\begin{array}{l}
9.6472 \\
2.5911
\end{array}\right], & v_{1}=\left[\begin{array}{c}
17.4916 \\
1.9008
\end{array}\right], & \vartheta_{1}=\left[\begin{array}{l}
0.7312 \\
1.4427
\end{array}\right], \\
v_{2}=\left[\begin{array}{l}
6.9694 \\
1.8534
\end{array}\right], & v_{2}=\left[\begin{array}{c}
12.6866 \\
1.3511
\end{array}\right], & \vartheta_{2}=\left[\begin{array}{l}
0.3727 \\
1.0628
\end{array}\right], \tag{58}
\end{array}
$$

and the state feedback gain matrices can be obtained as follows:

$$
\begin{align*}
K_{1} & =\left[\begin{array}{ll}
-14.2263 & -10.7904
\end{array}\right]  \tag{59}\\
K_{2} & =\left[\begin{array}{ll}
-18.2172 & -8.8405
\end{array}\right]
\end{align*}
$$

Obviously, condition (53) is satisfied.
According to (15), we have $\mu=2.0193$. Then from (14), we get $\tau_{a}>\tau_{a}^{*}=1.7880$. Choosing $\tau_{a}=2$, the simulation results are shown in Figures 1 and 2, where the initial conditions are $x(0)=\left[\begin{array}{cc}0.2 & 0.3\end{array}\right]^{T}$ and $x(k)=\left[\begin{array}{ll}0 & 0\end{array}\right]^{T}, k=-2,-1$, and the exogenous disturbance input is $w(k)=0.05 e^{-0.5 k}$ which belongs to $l_{1}[0, \infty)$. The switching signal with average dwell time $\tau_{a}=2$ is shown in Figure 1 and the state responses of the corresponding closed-loop system are given in Figure 2. From the simulation results, it can been seen that the closedloop system is exponentially stable with a prescribed $l_{1}$-gain performance level $\gamma=2$.

## 5. Conclusions

In this paper, the stability and $l_{1}$-gain controller design problems for positive switched systems with time-varying delays via delta operator approach have been investigated. By constructing a copositive type Lyapunov-Krasovskii functional and using the average dwell time approach, we proposed sufficient conditions of exponential stability and $l_{1}$ gain performance for the considered system. The desired


Figure 2: State responses of the closed-loop system.
state feedback $l_{1}$-gain controller was designed such that the corresponding closed-loop system is exponentially stable and satisfies an $l_{1}$-gain performance. Finally, a numerical example was presented to demonstrate the feasibility of the obtained results. In our future work, we will study the robust stabilization problem of positive switched systems with uncertainties and time-varying delays via delta operator approach.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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# Reaction Wheel Installation Deviation Compensation for Overactuated Spacecraft with Finite-Time Attitude Control 

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#### Abstract

A novel attitude tracking control scheme is presented for overactuated spacecraft to address the attitude stabilization problem in presence of reaction wheel installation deviation, external disturbance and uncertain mass of moment inertia. An adaptive sliding mode control technique is proposed to track the uncertainty. A Lyapunov-based analysis shows that the compensation control law can guarantee that the desired attitude trajectories are followed in finite-time. The key feature of the proposed control strategy is that it globally asymptotically stabilizes the system, even in the presence of reaction wheel installation deviation, external disturbances, and uncertain mass of moment inertia. The attitude track performance using the proposed finite-time compensation control is evaluated through a numerical example.


## 1. Introduction

In present, nearly all of the highly accurate slewing maneuvers necessitate the use of nonlinear differential equations for the kinematics and dynamics during the control system design [1]. However, the attitude tracking problem is further complicated by the external disturbance and uncertain mass of moment inertia. To address these issues, there have been several important developments in the design of feedback control laws for spacecraft maneuvering. A number of control design approaches using adaptive control [2,3], sliding mode control [4-7], $H_{\infty}[8,9]$, optimal control [10-13], and data driven control [14-16] have been proposed. However, few of them focus on the reaction wheel installation deviation that are of great theoretical and practical interest. In fact, the installation deviation is a widespread phenomenon, such as the actuator misalignment which is limited by the installation technique or generated by materials deforms the vehicle violent vibration during the launching process. In the area of actuator misalignment compensation, there currently exist few unified frameworks for the design of simple control structures.

Several solutions to actuator installation deviation have been presented in the literature [17-20]. In [17], the authors
presented a general adaptive tracking attitude controller design framework for a spacecraft subject to the actuator installation minor angle deviation. In [18], an adaptive attitude tracking method is proposed to compensate the actuator misalignment of nearly 15 degree. And in [19], a novel algorithm is employed precisely to estimate the information, such as installation angle of wheel and CMG alignments. And then the controller design can be on for the estimation information. Moreover, another recent paper in [20] proposed an adaptive control approach for satellite formation flying, in which backstepping technique is used to synthesize a controller to handle thrust magnitude error and misalignment. However, the torque is different between thruster and reaction wheel, one is literal and the other is time-variable. That is to say that this control strategy is not suitable for reaction wheel installation deviation compensation for overactuated spacecraft attitude control.

Treating the uncertain mass of moment inertia caused by the reaction wheel misalignment is another impossibly avoided problem. In practice, in order to ensure the reliability of on-orbit spacecraft operation, especially under high altitude sever external environment, overactuatation is widely employed to guarantee the control system reliability service. And finite-time is meanwhile necessary for time


Figure 1: Structure of the attitude compensation controller.
critical missions. As a result, more and more investigations also have focused on attitude control design with finite-time convergence. In [21-23], the finite-time control technique was applied to design an attitude controller. Feng et al. [24] proposed a terminal sliding mode controller to solve the singular problem for a second-order nonlinear dynamic system. A terminal sliding mode and the Chebyshev neural network were used in [25] to guarantee that the attitude manoeuvre was accomplished in finite time, even in the presence of an unknown inertia matrix, external disturbances, and control input constraints. Furthermore, two robust sliding mode controllers were proposed in [26] to realize attitude tracking in finite-time. Similar finite-time fault tolerant controllers for spacecrafts were investigated in [27-29].

This work focuses on developing a control scheme to perform attitude compensation for an overactuated spacecraft with reaction wheel installation deviation, external disturbances, and uncertain inertia parameters. More specifically, the attitude tracking error is required to be zero in finite time. The proposed approach is illustrated in Figure 1. The compensation control module is added to the output of the nominal controller to compensate for the reaction wheel misalignment, disturbances, and uncertain moment of inertia. The proposed scheme solves a difficult problem of reliable and high accuracy attitude tracking control in finite time that rejects external disturbances and, at the same time, compensates for actuator misalignment and system uncertainties so that the control objective is met.

The remainder of this paper is organized as follows. In Section 2, we summarize the mathematical model for the rigid spacecraft attitude and control problem. A compensation control solution with the misalignment, disturbance, and mass moment of inertia is presented in Section 3. Simulation results are presented in Section 4. Some conclusions are given in Section 5.

## 2. Mathematical Model and Problem Formulation

The notation adopted throughout this paper is introduced as follows. The symbol $\|\cdot\|$ denotes the standard Euclidean norm
or its induced norm; the symbol $\|\cdot\|_{\infty}$ denotes the infinite norm of a vector or matrix. For any given matrix $\mathbf{A} \in \mathbf{R}^{p \times q}$ with full row rank, $\mathbf{A}^{\dagger}$ denotes its pseudoinverse.
2.1. Dynamic Model of Rigid Spacecraft. Consider a rigid space system described by the following attitude kinematics and dynamics equations [30]:

$$
\begin{gather*}
\dot{\mathbf{q}}=\frac{1}{2}\left(\mathbf{q}^{\times}+q_{0} \mathbf{I}_{3}\right) \boldsymbol{\omega}  \tag{1}\\
\dot{q}_{0}=-\frac{1}{2} \mathbf{q}^{T} \boldsymbol{\omega}  \tag{2}\\
\mathbf{J} \dot{\boldsymbol{\omega}}+\boldsymbol{\omega}^{\times} \mathbf{J} \boldsymbol{\omega}=\mathbf{u}+\mathbf{d} \tag{3}
\end{gather*}
$$

where $\boldsymbol{\omega} \in \mathbf{R}^{3}$ is the angular velocity of a body-fixed reference frame expressed in the body-fixed reference frame, $\mathbf{J} \in$ $\mathbf{R}^{3 \times 3}$ (positive and definite) is the total inertia matrix of the spacecraft, $\mathbf{u}=\left[\begin{array}{lll}u_{1} & u_{2} & u_{3}\end{array}\right]^{T} \in \mathbf{R}^{3}$ denotes the combined control torque produced by the actuators, and $\mathbf{d}(t)=$ $\left[\begin{array}{lll}d_{1} & d_{2} & d_{3}\end{array}\right]^{T} \in \mathbf{R}^{3}$ denotes the external disturbance torque from the environment, which is assumed to be unknown but bounded; $q_{0}, \mathbf{q}$ are the scalar and vector components of the unit quaternion, respectively, with $\mathbf{q}=\left[\begin{array}{lll}q_{1} & q_{2} & q_{3}\end{array}\right]^{T} \in$ $\mathbf{R}^{3}$, satisfying the constraint $q_{0}^{2}+\mathbf{q}^{T} \mathbf{q}=1 ; \mathbf{I}_{3}$ represents the identity matrix with proper dimensions, and for $\forall \mathbf{a}=$ $\left[\begin{array}{lll}a_{1} & a_{2} & a_{3}\end{array}\right]^{T}, \mathbf{a}^{\times}$denotes a skew-symmetric matrix, more precisely,

$$
\mathbf{a}^{\times}=\left[\begin{array}{ccc}
0 & -a_{3} & a_{2}  \tag{4}\\
a_{3} & 0 & -a_{1} \\
-a_{2} & a_{1} & 0
\end{array}\right]
$$

### 2.2. Reaction Wheel Configuration with Installation Deviation.

 For orbiting spacecraft, loosely speaking, they have more than three reaction wheels aligned with the spacecraft body axes. However, in practice, the configuration of actuators will never be perfect; that is, to say, whether due to finite manufacturing tolerances or warping of the spacecraft structure duringlaunch, some alignment errors can always exist. Thus, in this section, the reaction wheels misalignment is taken into consideration; the faulty dynamics can be described by

$$
\begin{equation*}
\mathrm{J} \omega=-\omega^{\times} \mathrm{J} \omega+(\mathrm{D}+\Delta \mathrm{D}) \tau+\mathrm{d} \tag{5}
\end{equation*}
$$

where $\mathbf{D} \in \mathbf{R}^{3 \times N}$ denotes the actuator distribution matrix, $\Delta \mathbf{D}$ denotes the actuator distribution matrix induced by misalignment, and $\boldsymbol{\tau}=\left[\tau_{1}, \ldots, \tau_{N}\right]^{T} \in \mathbf{R}^{N}$ denotes the actual output torque of the $N$ reaction wheels.

Due to the rotation of the payload or the existence of the flywheel installation deviation, the moment of inertia $\mathbf{J}$ is uncertain but positive definite symmetric matrices and record $\mathbf{J}=\mathbf{J}_{0}+\Delta \mathbf{J}$, where $\mathbf{J}_{0}$ denotes the nominal rotational inertia and $\Delta \mathbf{J}$ denotes the uncertain rotational inertia. Here set $0<\|\Delta \mathrm{J}\| \leq\|\mathrm{J}\| \leq J_{\max }<\infty$ and $J_{\max }$ is a positive constant.
2.3. Attitude Tracking Model. Assume that the desired attitude to be followed is described with a desired frame $\mathscr{T}$ with respect to $\mathscr{F}$. It is specified by the desired unit quaternion $\mathbf{Q}_{\mathbf{d}}=\left(q_{d 0}, \mathbf{q}_{\mathbf{d}}^{T}\right) \in \mathbf{R} \times \mathbf{R}^{3}$. The desired angular velocity is denoted by $\boldsymbol{\omega}_{\mathbf{d}} \in \mathbf{R}^{3}$. Let the error quaternion $\mathbf{Q}_{\mathbf{e}}=\left(e_{0}, \mathbf{e}^{T}\right) \in$ $\mathbf{R} \times \mathbf{R}^{3}$ denote the attitude between $\mathscr{B}$ and $\mathscr{T}$, and let $\boldsymbol{\omega}_{\mathbf{e}} \in \mathbf{R}^{3}$ represent the corresponding error angular velocity. One has

$$
\begin{equation*}
\omega_{\mathrm{e}}=\omega-\widetilde{\mathbf{R}} \omega_{\mathrm{d}} \tag{6}
\end{equation*}
$$

where $\widetilde{\mathbf{R}} \in \mathbf{R}^{3 \times 3}$ denote the corresponding rotation matrix that brings $\mathscr{T}$ onto $\mathscr{B}$, and

$$
\begin{equation*}
\widetilde{\mathbf{R}}=\left(e_{0}^{2}-\mathbf{e}^{T} \mathbf{e}\right) \mathbf{I}_{3}+2 \mathbf{e e}^{T}-2 e_{0} \mathbf{e}^{\times} \tag{7}
\end{equation*}
$$

With (1)-(5), the attitude tracking error dynamics is given by:

$$
\begin{gather*}
\mathbf{J} \dot{\omega}_{\mathbf{e}}+\left(\dot{\omega}_{\mathbf{e}}+\widetilde{\mathbf{R}} \boldsymbol{\omega}_{\mathbf{d}}\right)^{\times} \mathbf{J}\left(\boldsymbol{\omega}_{\mathbf{e}}+\widetilde{\mathbf{R}} \omega_{\mathbf{d}}\right)-\mathbf{J}\left(\boldsymbol{\omega}_{\mathbf{e}}^{\times} \widetilde{\mathbf{R}} \boldsymbol{\omega}_{\mathbf{d}}-\widetilde{\mathbf{R}} \omega_{\mathbf{d}}\right)  \tag{8}\\
=(\mathbf{D}+\Delta \mathbf{D}) \mathbf{E}(t) \boldsymbol{\tau}(t)+\mathbf{d}(t), \\
\dot{e}_{0}=-\frac{1}{2} \mathbf{e}^{T} \omega_{\mathrm{e}},  \tag{9}\\
\dot{\mathbf{e}}=\frac{1}{2}\left(\mathbf{e}^{\times}+e_{0} \mathbf{I}_{3}\right) \boldsymbol{\omega}_{\mathbf{e}} . \tag{10}
\end{gather*}
$$

2.4. Control Objective. The control objective of this work can be stated as considering the uncertain attitude tracking system (8)-(10) and design a control law to guarantee that the attitude tracking error converges to zero in finite-time, even in the presence of actuator misalignment, uncertain inertia matrix, and external disturbance $\mathbf{d}(t)$.

We present now the main results of this study.

## 3. Finite-Time Attitude Compensation Control

For the proposed control approach shown in Figure 1, the nominal control power and the compensation control effort are presented in this section. First, a finite-time sliding mode surface is proposed. Then, based on the finite-time sliding
mode surface, a compensation controller is synthesized and added to the nominal controller to guarantee the global asymptotic stability of the resulting closed-loop attitude tracking system with finite-time convergence.
3.1. Finite-Time Sliding Mode Surface Design. We first introduce some lemmas which will be utilized in the subsequent control development and analysis.

Lemma 1 (see [31]). If $p \in(0,1)$, then the following inequality holds for any vector $\mathbf{x}=\left(x_{1}, x_{2}, \ldots, x_{n}\right)^{T} \in \mathbf{R}^{n}$ :

$$
\begin{equation*}
\sum_{i=1}^{n}\left|x_{i}\right|^{1+p} \geq\left(\sum_{i=1}^{n}\left|x_{i}\right|^{2}\right)^{(1+p) / 2} \tag{11}
\end{equation*}
$$

Lemma 2 (see [32]). Suppose that $\mathbf{V}(\mathbf{x})$ is a $C^{1}$ smooth positive-definite function such that

$$
\begin{equation*}
\dot{\mathbf{V}}(\mathbf{x})+\lambda_{1} \mathbf{V}(\mathbf{x})+\lambda_{2} \mathbf{V}^{\beta}(\mathbf{x}) \leq 0 \tag{12}
\end{equation*}
$$

where $\lambda_{1} \in \mathbf{R}^{+}, \lambda_{2} \in \mathbf{R}^{+}, \beta \in \mathbf{R}^{+}$and $0<\beta<1$. Then for any initial value $x(0)=x_{0}$, it follows that $\mathbf{V}(\mathbf{x}(t))=0$ for all the time $t \geq t_{F_{1}}$,

$$
\begin{equation*}
t_{F_{1}} \leq \frac{1}{\lambda_{1}(1-\beta)} \ln \frac{\lambda_{1} \mathbf{V}^{1-\beta}\left(x_{0}\right)+\lambda_{2}}{\lambda_{2}} \tag{13}
\end{equation*}
$$

To this end, in this work, a sliding mode surface is introduced as

$$
\begin{equation*}
\mathbf{s}=\boldsymbol{\omega}_{\mathbf{e}}+\mu_{1} \mathbf{e}+\mu_{2} \operatorname{sgn}(\mathbf{e})^{r} \tag{14}
\end{equation*}
$$

where $\mathbf{s}=\left[\begin{array}{lll}s_{1} & s_{2} & s_{3}\end{array}\right]^{T} \in \mathbf{R}^{3}, \mu_{1}>0, \mu_{2}>0$, and $0<\gamma<1$ are the design parameters and sgn $(\mathbf{e})$ is the sign function defined by

$$
\begin{equation*}
\operatorname{sgn}(\mathbf{e})^{r}=\left(\left|e_{1}\right|^{r} \operatorname{sgn}\left(e_{1}\right),\left|e_{2}\right|^{r} \operatorname{sgn}\left(e_{2}\right),\left|e_{3}\right|^{r} \operatorname{sgn}\left(e_{3}\right)\right)^{T} . \tag{15}
\end{equation*}
$$

Theorem 3. If an controller $\mathbf{u}(t)$ is appropriately designed to let the states reach the sliding surface $\mathbf{s}$, then it has $\mathbf{e}(t) \equiv 0$, $e_{0}(t) \equiv 1$, and $\omega_{\mathbf{e}}(t) \equiv 0$ for all the $t \geq t_{F 1}$.

Proof. From the sliding mode theory [33], it is known that once the state trajectories of the attitude tracking system reach the sliding surface, that is, $\mathbf{s}=\mathbf{0}$, it follows that

$$
\begin{equation*}
\omega_{\mathbf{e}}=-\mu_{1} \mathbf{e}-\mu_{2} \operatorname{sgn}(\mathbf{e})^{r} \tag{16}
\end{equation*}
$$

Consider a candidate Lyapunov function as

$$
\begin{equation*}
V_{1}=\left(1-e_{0}\right)^{2}+\mathbf{e}^{T} \mathbf{e} \tag{17}
\end{equation*}
$$

Because the inequality $\sum_{i=0}^{3}\left|e_{i}\right|^{r+1} \geq\left(\sum_{i=0}^{3}\left|e_{i}\right|^{2}\right)^{(r+1) / 2}$ holds for $0<r<1$, it is obtained from (15) that

$$
\begin{equation*}
\dot{V}_{1}=-2 \dot{e}_{0}=\mathbf{e}^{T} \boldsymbol{\omega}_{\mathbf{e}}=-\mu_{1} \mathbf{e}^{T} \mathbf{e}-\mu_{2} \sum_{i=1}^{3}\left|e_{i}\right|^{r+1} \tag{18}
\end{equation*}
$$

which implies that $\dot{V}_{1}=0$ if and only if $\mathbf{e}=0$. Thus, $V_{1}$ is really a Lyapunov function such that the signal $\mathbf{e}$ will converge to zero and, accordingly, $e_{0}$ tends to $\pm 1$ as $t \rightarrow \infty$ by using the constraint in (18). Note that the equilibrium point $\left(e_{0}, \mathbf{e}\right)=(-1, \mathbf{0})$ is not a stable equilibrium point [34]. Then, by Lemma 1, we obtain

$$
\begin{equation*}
\dot{V}_{1} \leq-\mu_{1} \mathbf{e}^{T} \mathbf{e}-\mu_{2}\left(\mathbf{e}^{T} \mathbf{e}\right)^{(r+1) / 2} \tag{19}
\end{equation*}
$$

Because $\mathbf{e}=(-1,0)^{T}$ is not the stable equilibrium point, the signal $\mathbf{e}$ will converge to zero. Thus, $\lim _{t \rightarrow \infty} e_{0}(t)=1$ can be obtained from the constraint $\mathbf{e}^{T} \mathbf{e}+e_{0}^{2}=1$. There exists a finite time $\bar{t} \geq 0$ such that $e_{0}(t)>0$ for $t \geq \bar{t}$. Then, for $t \geq \bar{t}$, one has

$$
\begin{align*}
\left(1-e_{0}\right)^{2} & =2 e_{0}^{2}+\mathbf{e}^{T} \mathbf{e}-2 e_{0}  \tag{20}\\
& =2 e_{0}\left(e_{0}-1\right)+\mathbf{e}^{T} \mathbf{e}-2 \leq \mathbf{e}^{T} \mathbf{e}
\end{align*}
$$

Then,

$$
\begin{equation*}
V_{1} \leq 2 \mathbf{e}^{T} \mathbf{e} \tag{21}
\end{equation*}
$$

Using (21), (19) can be further bounded by

$$
\begin{equation*}
\dot{V}_{1} \leq-\frac{1}{2} \mu_{1} V_{1}-\left(\frac{1}{2}\right)^{(r+1) / 2} \mu_{2} V_{1}^{(r+1) / 2} \tag{22}
\end{equation*}
$$

Using $0.5<(r+1) / 2<1$ and Lemma 2, one has $V_{1}(t) \equiv 0$ for all $t \geq t_{F 1}$. According to definition of $V_{1}(t)$ in (17), $e_{0}(t) \equiv$ $1, \mathbf{e}(t) \equiv 0$, and $\boldsymbol{\omega}_{\mathbf{e}}(t) \equiv 0$ for all $t \geq t_{F 1}$ are concluded. Thereby, the proof is completed here.
3.2. Attitude Compensation Controller Design. Considering the reaction wheel installation deviation and external disturbance, it is obtained from the sliding surface (14) that

$$
\begin{equation*}
\mathbf{J} \dot{\mathbf{s}}=\mathbf{D} \boldsymbol{\tau}+\Delta \mathbf{D} \boldsymbol{\tau}+\mathbf{L}-\beta \mathbf{q}_{e}-0.5 \mathbf{J} \mathbf{s} \tag{23}
\end{equation*}
$$

Because $\mathbf{J}$ is unknown but bounded, then $\mathbf{J}=0$ is established. Then, $\mathbf{L}$ can be represented to be bounded by $\|\mathbf{L}\| \leq \alpha_{0}+\alpha_{1}\|\boldsymbol{\omega}\|+\alpha_{2}\|\boldsymbol{\omega}\|^{2}$, where $\boldsymbol{\alpha}_{i}, i=1,2,3$ are positive constants [35].

In order to facilitate analysis and proof, firstly, define $\kappa=\lambda_{\text {min }}\left(\mathbf{D} \mathbf{D}^{T}\right), 3\|\Delta \mathbf{D}\|_{\infty}\left\|\mathbf{D}^{\dagger}\right\|_{\infty}=\varepsilon<1$, and $\mathbf{D}^{\dagger}$ is the pseudoinverse of $\mathbf{D}$. Now, we are ready to present the main result in Theorem 4.

Theorem 4. Considering the uncertainty attitude tracking dynamics described by (5) with actuator misalignment $\Delta \mathbf{D}$ and external disturbance torque $\mathbf{d}(t)$, design an attitude compensation control law as

$$
\begin{equation*}
\boldsymbol{\tau}=\boldsymbol{\tau}_{\mathrm{nom}}(t)+\boldsymbol{\tau}_{a d p}(t)+\boldsymbol{\tau}_{m i s}(t), \tag{24}
\end{equation*}
$$

where

$$
\begin{gather*}
\boldsymbol{\tau}_{\text {nom }}(t)=-k_{1} \beta \frac{\mathbf{D}^{T}\left\|\mathbf{q}_{e}\right\| \mathbf{s}}{\|\mathbf{s}\|}-K \frac{\mathbf{D}^{T} \mathbf{s}}{\|\mathbf{s}\|^{2}}  \tag{25}\\
\boldsymbol{\tau}_{a d p}(t)=\frac{\mathbf{D}^{T}\left(-\widehat{k}_{3}-\widehat{k}_{4}\|\boldsymbol{w}\|-\widehat{k}_{5}\|\boldsymbol{\omega}\|^{2}\right) \mathbf{s}}{\|\mathbf{s}\|}  \tag{26}\\
\boldsymbol{\tau}_{m i s}(t)=-\frac{\left(\widehat{\pi}_{1}-1\right) \varphi(t) \mathbf{D}^{\dagger} \mathbf{s}}{\|\mathbf{s}\|_{\infty}} \tag{27}
\end{gather*}
$$

where $K \in \mathbf{R}^{+}$is control parameter and $k_{1}$ is carefully chosen such that $k_{1} \kappa-1>0 ; \hat{\pi}_{1}$ is the estimate of $\pi_{1}=1 /(1-\delta) ; \widehat{k}_{3}$ is the estimate of $k_{3}=\alpha_{0} / \kappa ; \widehat{k}_{4}$ is the estimate of $k_{4}=\alpha_{1} / \kappa$; $\widehat{k}_{5}$ is the estimate of $k_{5}=\alpha_{2} / \kappa$. Moreover, $\dot{\hat{\pi}}_{1}, \widehat{k}_{i}, i=3,4,5$ are adaptively updated by $\dot{\hat{k}}_{2}=\left(\left\|\mathbf{s}^{T}\right\| / \ell_{4}\right) \hat{k}_{i}, \dot{\hat{k}}_{3}=\|\boldsymbol{\omega}\|\|\boldsymbol{s}\| / \ell_{5}$, $\dot{\hat{k}}_{4}=\|\boldsymbol{\omega}\|^{2}\|\boldsymbol{s}\| / \ell_{6}$, and $\dot{\bar{\pi}}_{1}=l_{1} \varphi(t)\|\boldsymbol{s}\|_{\infty}$, respectively. Then the system states reach the sliding mode surface $\mathbf{s}(t)=0$ in finitetime for any initial state $\mathbf{Q}(0)$ and $\boldsymbol{\omega}(0)$.

Proof. When $\mathbf{s} \neq \mathbf{0}$, consider a candidate Lyapunov function:

$$
\begin{equation*}
V_{2}=\frac{1}{2} \mathbf{s}^{T} \mathbf{J} \mathbf{s}+\frac{(1-\varepsilon)}{2 l_{1}} \widetilde{\pi}_{1}^{2}+\frac{1}{2} l_{3} \widetilde{k}_{3}^{2}+\frac{1}{2} l_{4} \widetilde{k}_{4}^{2}+\frac{1}{2} l_{5} \widetilde{k}_{5}^{2} \tag{28}
\end{equation*}
$$

where $\widetilde{\pi}_{1}=\pi_{1}-\widehat{\pi}_{1}, \widetilde{k}_{i}=k_{i}-\widehat{k}_{i}, i=3,4,5$, and $l_{1}, l_{3}, l_{4}, l_{5}$ are the positive constants.

Calculating the time-derivative of $V_{2}$, it yields

$$
\begin{align*}
\dot{V}_{2} \leq & \mathbf{s}^{T} \mathbf{J} \dot{\mathbf{s}}-\frac{(1-\varepsilon)}{l_{1}} \tilde{\pi} \dot{\hat{\pi}}-l_{3} \widetilde{k}_{3} \dot{\hat{k}}_{3}-l_{4} \widetilde{k}_{4} \dot{\hat{k}}_{4}-l_{5} \dot{\hat{k}}_{3} \\
= & \mathbf{s}\left(\mathbf{D} \boldsymbol{\tau}_{\text {nom }}-\beta \mathbf{q}_{e}\right)+\mathbf{s}\left(\mathbf{D} \boldsymbol{\tau}_{\text {adp }}+\mathbf{L}\right)+\mathbf{s}\left(\mathbf{D} \boldsymbol{\tau}_{\mathrm{mis}}+\Delta \mathbf{D} \boldsymbol{\tau}\right) \\
& -\frac{(1-\varepsilon)}{l_{1}} \tilde{\pi} \dot{\hat{\pi}}-l_{3} \widetilde{\mathrm{k}}_{3} \dot{\hat{k}}_{3}-l_{4} \widetilde{k}_{4} \dot{\hat{k}}_{4}-l_{5} \dot{\hat{k}}_{3} . \tag{29}
\end{align*}
$$

With (25), it follows that

$$
\begin{align*}
\mathbf{s}\left(\mathbf{D} \boldsymbol{\tau}_{\mathrm{nom}}-\beta \mathbf{q}_{e}\right) & \leq \mathbf{s} \mathbf{D} \boldsymbol{\tau}_{\mathrm{nom}}+\beta\left\|\mathbf{q}_{e}\right\|\|\mathbf{s}\| \\
& \leq-\kappa k_{1} \beta\|\mathbf{s}\|\left\|\mathbf{q}_{e}\right\|+\beta\left\|\mathbf{q}_{e}\right\|\|\mathbf{s}\|-K \kappa \\
& =-\left(k_{1} \kappa \beta-\beta\right)\left\|\mathbf{q}_{e}\right\|-\kappa K  \tag{30}\\
& \leq-\kappa K .
\end{align*}
$$

In the same way, with (26), the following equality is yielded:

$$
\begin{align*}
\mathbf{s}\left(\mathbf{D} \boldsymbol{\tau}_{\mathrm{adp}}+\mathbf{L}\right) \leq & \mathbf{s} \mathbf{D} \boldsymbol{\tau}_{\mathrm{adp}}+\|\boldsymbol{s}\|\|\mathbf{L}\| \\
\leq & -\kappa\left(\widehat{k}_{3}-\frac{\alpha_{0}}{\kappa}\right)\|\boldsymbol{s}\|-\kappa\left(\widehat{k}_{4}-\frac{\alpha_{1}}{\kappa}\right)\|\boldsymbol{s}\|\|\boldsymbol{\omega}\| \\
& -\kappa\left(\widehat{k}_{5}-\frac{\alpha_{2}}{\kappa}\right)\|\boldsymbol{s}\|\|\boldsymbol{\omega}\|^{2} \\
= & \kappa \widetilde{k}_{3}\|\boldsymbol{s}\|+\kappa \widetilde{k}_{4}\|\boldsymbol{s}\|\|\boldsymbol{\omega}\|+\kappa \widetilde{k}_{5}\|\boldsymbol{s}\|\|\boldsymbol{\omega}\|^{2} . \tag{31}
\end{align*}
$$

Define $\varphi(t)=\left\|\mathbf{D}^{\dagger}\left(\boldsymbol{\tau}_{\text {nom }}+\boldsymbol{\tau}_{\text {adp }}\right)\right\|_{\infty}$. If the choice of the control gains is such that $3\|\Delta \mathbf{D}\|_{\infty}\left\|\mathbf{D}^{-1}\right\|_{\infty}=\varepsilon<1$, using the inequality $\mathbf{x y} \leq 3\|\mathbf{x}\|_{\infty}\|\mathbf{y}\|_{\infty}>0$ for any vector $\mathbf{x}, \mathbf{y} \in \mathbf{R}^{n}$ and applying (29) lead to

$$
\begin{align*}
& \mathbf{s}\left(\mathbf{D} \boldsymbol{\tau}_{\text {mis }}+\Delta \mathbf{D} \boldsymbol{\tau}\right) \\
& \quad= \mathbf{s} \Delta \mathbf{D}\left(\boldsymbol{\tau}_{\text {nom }}+\boldsymbol{\tau}_{\text {adp }}+\boldsymbol{\tau}_{\text {mis }}\right)+\mathbf{s} \mathbf{D} \boldsymbol{\tau}_{\text {mis }} \\
& \leq 3\|\Delta \mathbf{D}\|_{\infty}\left\|\mathbf{D}^{\dagger}\right\|_{\infty} \varphi(t)\|\mathbf{s}\|_{\infty}+\mathbf{s} \Delta \mathbf{D} \boldsymbol{\tau}_{\text {mis }}+\mathbf{s D} \boldsymbol{\tau}_{\text {mis }} \\
& \leq \varepsilon \varphi(t)\|\mathbf{s}\|_{\infty}-\left(\hat{\pi}_{1}-1\right) \varphi(t)\|\mathbf{s}\|_{\infty}  \tag{32}\\
& \quad+\mathbf{s} \Delta \mathbf{D}\left[-\frac{\left(\hat{\pi}_{1}-1\right) \varphi(t) \mathbf{D}^{\dagger} \mathbf{s}}{\|\mathbf{s}\|_{\infty}}\right] \\
& \quad(1-\varepsilon) \widetilde{\pi}_{1} \varphi(t)\|\mathbf{s}\|_{\infty} .
\end{align*}
$$

Substituting (30)-(32) into (29), consequently, it follows that

$$
\begin{equation*}
\dot{V} \leq-\kappa K \tag{33}
\end{equation*}
$$

And then,

$$
\begin{align*}
& \int_{0}^{t} \dot{V}_{2}(\mu) d \mu \leq-\kappa K \int_{0}^{t} d \mu ; \quad \text { that is, }  \tag{34}\\
& V_{2}(t)-V_{2}(0) \leq-\kappa K t
\end{align*}
$$

Due to $V_{2}(t) \geq 0$, solving (34) leads to $V_{2}(t) \equiv 0$, for $t \geq$ $t_{F 2}$,

$$
\begin{equation*}
t_{F 2} \leq-\frac{V_{2}(t)-V_{2}(0)}{\kappa K} \tag{35}
\end{equation*}
$$

Then, it can be concluded that the system states reach the surface $\mathbf{s}(t)=\mathbf{0}$ in finite time. Thereby, the proof is completed here.

Remark 5. The controller (24) includes three parts: $\boldsymbol{\tau}_{\text {adp }}(t)$ is used to compensate for system uncertainty caused by the external disturbance and moment inertia, $\boldsymbol{\tau}_{\text {mis }}(t)$ is used to accommodate actuator misalignment, and $\boldsymbol{\tau}_{\text {nom }}(t)$ is the nominal control.

Theorem 6. Consider the attitude tracking system given by (5), (9), and (10). If the control scheme (24) is implemented, then the attitude tracking maneuver can be accomplished in a finite time $t_{F}=t_{F 1}+t_{F 2}$; that is, $\omega_{\mathbf{e}}(t) \equiv 0$ and $\mathbf{e}(t) \equiv 0$ are guaranteed for all the time $t \geq t_{F}$.

Proof. It is obtained from Theorem 4 that all the states of the attitude tracking system reach the sliding mode surface $\mathbf{s}(t)=0$ in finite-time $t_{F 2}$ and maintain the motion state on the slide mode surface. Furthermore, from Theorem 3, it is obtained that once the system state reaches the slide mode surface (14) the system state can reach the equilibrium point $\left(e_{0}, \mathbf{e}\right)=(1, \mathbf{0})$ in finite time $t_{F 1}$. Therefore, for any initial state $\mathbf{Q}(0)$ and $\boldsymbol{\omega}(0)$, the desired attitude trajectory can be followed in a finite time $t_{F}$; that $i s, \mathbf{e}(t) \equiv 0, e_{0}(t) \equiv 1$, and $\boldsymbol{\omega}_{\mathbf{e}}(t) \equiv 0$ are achieved for all the time $t \geq t_{F}$. Thereby, the proof is completed here.


Figure 2: Configuration of four reaction wheels.

## 4. Numerical Simulation Results

4.1. Reaction Wheel Configuration. To demonstrate the effectiveness and performance of the proposed compensation control scheme, numerical simulations have been carried out using the rigid spacecraft system (3) and (6) in conjunction with the developed compensation control law (24). The spacecraft is activated by four reaction wheels with a limited control torque $u_{\max }=0.1 \mathrm{~N} \cdot \mathrm{~m}$. The configuration of those four actuators is shown in Figure 2. $\boldsymbol{\alpha}_{i}=35.26^{\circ}$ and $\boldsymbol{\beta}_{i}=45^{\circ}$ are the nominal alignment angles, $i=1,2,3,4 . \Delta \boldsymbol{\alpha}_{i}$ and $\Delta \boldsymbol{\beta}_{i}$ are the misalignment angles.

With the configuration shown in Figure 2, the relation between the actual output torque of reaction wheel and the total torque acting on the spacecraft is to be calculated as

$$
\begin{align*}
\mathbf{u}(t)= & \tau_{1}\left(\begin{array}{c}
\cos \left(\alpha_{1}+\Delta \alpha_{1}\right) \sin \left(\beta_{1}+\Delta \beta_{1}\right) \\
-\sin \left(\alpha_{1}+\Delta \alpha_{1}\right) \\
\cos \left(\alpha_{1}+\Delta \alpha_{1}\right) \cos \left(\beta_{1}+\Delta \beta_{1}\right)
\end{array}\right) \\
& +\tau_{2}\left(\begin{array}{c}
-\cos \left(\alpha_{2}+\Delta \alpha_{2}\right) \cos \left(\beta_{2}+\Delta \beta_{2}\right) \\
-\sin \left(\alpha_{2}+\Delta \alpha_{2}\right) \\
\cos \left(\alpha_{2}+\Delta \alpha_{2}\right) \sin \left(\beta_{2}+\Delta \beta_{2}\right)
\end{array}\right) \\
& +\tau_{3}\left(\begin{array}{c}
-\cos \left(\alpha_{3}+\Delta \alpha_{3}\right) \sin \left(\beta_{3}+\Delta \beta_{3}\right) \\
-\sin \left(\alpha_{3}+\Delta \alpha_{3}\right) \\
-\cos \left(\alpha_{3}+\Delta \alpha_{3}\right) \cos \left(\beta_{3}+\Delta \beta_{3}\right)
\end{array}\right)  \tag{36}\\
& +\tau_{4}\left(\begin{array}{c}
\cos \left(\alpha_{4}+\Delta \alpha_{4}\right) \cos \left(\beta_{4}+\Delta \beta_{4}\right) \\
-\sin \left(\alpha_{4}+\Delta \alpha_{4}\right) \\
-\cos \left(\alpha_{4}+\Delta \alpha_{4}\right) \sin \left(\beta_{4}+\Delta \beta_{4}\right)
\end{array}\right)
\end{align*}
$$

Although the misalignment angles exist due to finitemanufacture technique and vehicle vibration, those angles
$\Delta \boldsymbol{\alpha}_{i}, \Delta \boldsymbol{\beta}_{i}(i=1,2,3,4)$ are small values. They can be approximated by

$$
\begin{align*}
& \cos \Delta \boldsymbol{\alpha}_{i} \approx 1, \quad \sin \Delta \boldsymbol{\alpha}_{i} \approx \Delta \boldsymbol{\alpha}_{i}, \quad \cos \Delta \boldsymbol{\beta}_{i} \approx 1, \\
& \sin \Delta \boldsymbol{\beta}_{i} \approx \Delta \boldsymbol{\beta}_{i}, \quad \sin \Delta \boldsymbol{\alpha}_{i} \sin \Delta \boldsymbol{\beta}_{i} \approx 0  \tag{37}\\
& (i=1,2,3,4) .
\end{align*}
$$

Hence, (36) can be re written as

$$
\begin{equation*}
\mathbf{u}(t)=\mathbf{D} \boldsymbol{\tau}(t)+\Delta \mathbf{D} \boldsymbol{\tau}(t) \tag{38}
\end{equation*}
$$

where $\mathbf{D}$ and $\Delta \mathbf{D}=\left(\Delta \mathbf{D}_{1}, \Delta \mathbf{D}_{2}, \Delta \mathbf{D}_{3}, \Delta \mathbf{D}_{4}\right)$ are calculated as

$$
\begin{align*}
& \mathbf{D}=\left(\begin{array}{cccc}
\cos \alpha_{1} \sin \beta_{1} & -\cos \alpha_{2} \sin \beta_{2} & -\cos \alpha_{3} \sin \beta_{3} & \cos \alpha_{4} \sin \beta_{4} \\
-\sin \alpha_{1} & -\sin \alpha_{2} & -\sin \alpha_{3} & -\sin \alpha_{4} \\
\cos \alpha_{1} \cos \beta_{1} & \cos \alpha_{2} \cos \beta_{2} & -\cos \alpha_{3} \cos \beta_{3} & -\cos \alpha_{4} \cos \beta_{4}
\end{array}\right) \\
& =\left(\begin{array}{rrrr}
\frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} \\
-\frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} \\
\frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3}
\end{array}\right), \\
& \Delta \mathbf{D}_{1}=\left(\begin{array}{c}
\Delta \beta_{1} \cos \alpha_{1} \cos \beta_{1}-\Delta \alpha_{1} \sin \alpha_{1} \sin \beta_{1} \\
-\Delta \alpha_{1} \cos \alpha_{1} \\
-\Delta \beta_{1} \cos \alpha_{1} \sin \beta_{1}-\Delta \beta_{1} \sin \alpha_{1} \cos \beta_{1}
\end{array}\right) \\
& \Delta \mathbf{D}_{2}=\left(\begin{array}{c}
-\Delta \beta_{2} \cos \alpha_{2} \cos \beta_{2}+\Delta \alpha_{2} \sin \alpha_{2} \sin \beta_{2} \\
-\Delta \alpha_{2} \cos \alpha_{2} \\
-\Delta \beta_{2} \cos \alpha_{2} \sin \beta_{2}-\Delta \beta_{2} \sin \alpha_{2} \cos \beta_{2}
\end{array}\right) \\
& \Delta \mathbf{D}_{3}=\left(\begin{array}{c}
-\Delta \beta_{3} \cos \alpha_{3} \cos \beta_{3}+\Delta \alpha_{3} \sin \alpha_{3} \sin \beta_{3} \\
-\Delta \alpha_{3} \cos \alpha_{3} \\
\Delta \beta_{3} \cos \alpha_{3} \sin \beta_{3}+\Delta \beta_{3} \sin \alpha_{3} \cos \beta_{3}
\end{array}\right) \\
& \Delta \mathbf{D}_{4}=\left(\begin{array}{c}
\Delta \beta_{4} \cos \alpha_{4} \cos \beta_{4}-\Delta \alpha_{4} \sin \alpha_{4} \sin \beta_{4} \\
-\Delta \alpha_{4} \cos \alpha_{4} \\
\Delta \beta_{4} \cos \alpha_{4} \sin \beta_{4}+\Delta \alpha_{4} \sin \alpha_{4} \cos \beta_{4}
\end{array}\right) . \tag{39}
\end{align*}
$$

Remark 7. Theorem 4 gives out the sufficient condition $3\|\Delta \mathbf{D}\|_{\infty}\left\|\mathbf{D}^{\dagger}\right\|_{\infty}=\varepsilon<1$ of efficacious processes on reaction wheel installation deviation $\Delta \mathbf{D}$ for guaranteeing the attitude controller (24). Particulary, according to the definition $\Delta \mathbf{D}$ and matrix norm, then $\|\Delta \mathbf{D}\|_{\infty} \leq \max _{i=1,2,3} \theta_{i}$, where

$$
\begin{aligned}
\theta_{1}= & \sum_{i=1}^{4}\left|\Delta \boldsymbol{\alpha}_{i} \sin \boldsymbol{\alpha}_{i} \sin \boldsymbol{\beta}_{i}\right|+\sum_{i=1}^{4}\left|\Delta \boldsymbol{\beta}_{i} \cos \boldsymbol{\alpha}_{i} \cos \boldsymbol{\beta}_{i}\right| \\
& =\frac{\sqrt{6}}{6} \sum_{i=1}^{4}\left|\Delta \boldsymbol{\alpha}_{i}\right|+\frac{\sqrt{3}}{3} \sum_{i=1}^{4}\left|\Delta \boldsymbol{\beta}_{i}\right|, \\
\theta_{2}= & \sum_{i=1}^{4}\left|\Delta \boldsymbol{\alpha}_{i} \cos \boldsymbol{\alpha}_{i}\right|=\frac{\sqrt{6}}{3} \sum_{i=1}^{4}\left|\Delta \boldsymbol{\alpha}_{i}\right|, \\
\theta_{3}= & \sum_{i=1}^{4}\left|\Delta \boldsymbol{\alpha}_{i} \sin \boldsymbol{\alpha}_{i} \cos \boldsymbol{\beta}_{i}\right|+\sum_{i=1}^{4}\left|\Delta \boldsymbol{\beta}_{i} \cos \boldsymbol{\alpha}_{i} \sin \boldsymbol{\beta}_{i}\right| \\
& =\frac{\sqrt{6}}{6} \sum_{i=1}^{4}\left|\Delta \boldsymbol{\alpha}_{i}\right|+\frac{\sqrt{3}}{3} \sum_{i=1}^{4}\left|\Delta \boldsymbol{\beta}_{i}\right| .
\end{aligned}
$$

On the other hand, inequality $3\|\Delta \mathbf{D}\|_{\infty}\left\|\mathbf{D}^{\dagger}\right\|_{\infty}=\varepsilon<1$ means that $\|\Delta \mathbf{D}\|_{\infty}=\varepsilon /\left(3\left\|\mathbf{D}^{\dagger}\right\|_{\infty}\right)<1 /\left(3\left\|\mathbf{D}^{\dagger}\right\|_{\infty}\right)$. Therefore, the establishment conditions $\max _{i=1,2,3,4} \theta_{i}<1 / 3\left\|\mathbf{D}^{\dagger}\right\|_{\infty}=$ 0.2566 rad of Theorem 4 from (40) can be obtained, that is to say, the installation deviation angle of any two reaction wheels is not larger than 0.2566 rad. Relying on this, this largest installation deviation angle, that is $\max _{i=1,2,3,4} \theta_{i}=14.7021^{\circ}$, of the reaction wheel installation structure is considered in this paper.
4.2. Simulation Results. The nominal inertia matrix of the considered spacecraft is specified by [36]

$$
\begin{gather*}
\mathbf{J}_{0}=\left(\begin{array}{ccc}
35 & 3 & -1.5 \\
3 & 28 & 2 \\
-1.5 & 2 & 30
\end{array}\right) \mathrm{kg} \cdot \mathrm{~m}^{2},  \tag{41}\\
\Delta \mathbf{J}=\left(1+e^{-0.1 t}+2 \vartheta(t-10)-4 \vartheta(t-20)\right) \operatorname{diag}(3,2,1) \tag{42}
\end{gather*}
$$

which incorporated into the model, where $\vartheta(\cdot)$ is defined as $\vartheta(t \geq 0)=1$ and $\vartheta(t<0)=0$. External disturbance $\mathbf{d}(t)$ is chosen as [35]

$$
\begin{equation*}
\mathbf{d}(t)=\left(\|\boldsymbol{\omega}\|^{2}+0.05\right)(\sin 0.8 t, \cos 0.5 t, \cos 0.3 t)^{T} \tag{43}
\end{equation*}
$$

The reaction wheel misalignment angle $\Delta \boldsymbol{\alpha}_{i}(i=1,2,3,4)$ can be selected randomly between $-4.5^{\circ} \sim+4.5^{\circ}$, and $\Delta \boldsymbol{\beta}_{i}(i=$ $1,2,3,4)$ can be selected randomly between $-5.5^{\circ} \sim+5.5^{\circ}$.

In this simulation, spacecraft initial parameter is set as follows: initial angular velocity $\boldsymbol{\omega}_{i}(0)=$ $\left[\begin{array}{ccc}0.1 & -0.1 & -0.05\end{array}\right]^{\circ} / s, i=1,2,3$; initial attitude quaternary $\mathbf{q}_{i}(0)=\left[\begin{array}{llll}0.181 & -0.287 & 0.792 & -0.524\end{array}\right]^{T}, i=0,1,2,3$; the corresponding initial roll angle, pitch angle, and yaw angle are set $0.2^{\circ},-0.4^{\circ}$ and $-0.3^{\circ}$ respectively. The control gains are selected by the following list: $\beta=0.7, K=0.05, \kappa=1.05$, $k_{i}=1.5(i=1,3,4,5)$, as $l_{i}=1.5(i=1,3,4,5)$; moreover, the initial of the adaptive update laws are $\dot{\widehat{\pi}}_{1}(0)=1.25$, $\dot{\hat{k}}_{3}(0)=0.68, \dot{\widehat{k}}_{4}(0)=0.42, \dot{\widehat{k}}_{5}(0)=0.22$.

To demonstrate the effectiveness of the proposed misalignment compensation and disturbance rejection scheme, a spacecraft is numerically simulated using the proposed control compensation strategy (24).

We see in Figures 3-8 the controller managed to stabilize the origin equilibrium point in 30 seconds with great pointing accuracy. Indeed, since the knowledge of spacecraft inertia parameters was not required and an implicit integral item was incorporated in the control law design, external disturbance effect on the attitude control performance can be compensated efficiently, and also great robustness to system uncertainties, such as misalignment, can be guaranteed.

We can see in Figures 3 and 4 the time responses of angle velocity and attitude angle; the proposed control scheme surely realized the high precision stable control in the presence of external disturbance, uncertain moment of inertia, and reaction wheel misalignment, and the pointing accuracy is superior to $0.01^{\circ}$; the attitude stable precision is superior to $0.001^{\circ} / \mathrm{s}$. Meanwhile, from Figures 5 and 6,



Figure 5: Time response of control torque.
Figure 3: Time responses of angular velocity.


Figure 4: Time response of satellite attitude angle.


Figure 6: Time response of control torque for reaction wheels.


Figure 7: Time responses of quaternion.
we can see that for compensating the misalignment and other uncertainties, the designed control command of control redundancy configuration for 4 reaction wheels $\tau$ is allocated to the three-actual-output torque $\mathbf{u}$, and then the purpose of compensation, attitude high precision control is realized. In addition, the finite-time control validity is shown in Figure 8. And from Figure 8, we can see that the spacecraft attitude control system status has realized the tracking control at $t_{F}=30.5$. Thereinto, the spacecraft attitude has arrived at slide mode surface $s$ at $t_{F 1}=25.9$; afterwards, the statuses converge to equilibrium point at $t_{F 2}=5.4$ under the normal control $\boldsymbol{\tau}_{\text {nom }}(t)$. The same validity of finite-time attitude compensation control strategy proposed in this paper can be further proved from the time response of the quaternion as shown in Figure 7.

From the above illustrated simulation results, it is shown that the proposed scheme can accomplish the attitude stabilization in finite-time in presence of time-varying external disturbances, uncertain inertial parameters, and even reaction wheel installation deviation.

## 5. Conclusions and Future Works

Considering the spacecraft issues about reaction misalignment, external disturbances, and parameters uncertainty, in this paper, a finite-time adaptive attitude compensation control has been proposed. A quantitative installation deviation angle analysis has been done and given out the value range of the reaction wheel misalignment angle. In the end the system stability and engineering practical value have been discussed from the perspective of theory and engineering.


Figure 8: Time response of sliding mode surface.

Numerical simulation of this novel control strategy was also presented to confirm the advantages and improvements over existing controllers. The case of actuator misalignment mentioned in Section 4 had only discussed for four reaction wheel configuration, but this compensation control scheme is suitable for more than that reaction wheel number. Moreover, the actuator faults have not been considered. The latter case should be as one of subjects for future research. Meanwhile, the method optimal control approach combined robust control $[37,38]$ also can be applied in this field.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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# Composite Antidisturbance Control for a Class of Nonlinear Stochastic Systems via Disturbance Observer 

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#### Abstract

The stabilization problem is investigated in this paper for a class of nonlinear systems with disturbances. The disturbances are supposed to be classified into two types. One type in the input channel is generated by an exogenous system, which can represent the constant or harmonic signals with unknown phase and magnitude. The other type is stochastic disturbance. Two kinds of nonlinear dynamics in the plants are considered, respectively, which correspond to the known and unknown functions. By integrating the disturbance observers with conventional control method, the first type of disturbances can be estimated and rejected. Simultaneously, the desired dynamic performances can be guaranteed. An example is given to show the effectiveness of the proposed scheme.


## 1. Introduction

Though the stochastic stabilization theory emerged in the 1960s [1], the progress has been slow. This was mainly due to a fundamental theory obstacle in the Lyapunov analysis; the Itô differentiation introduces not only the gradient but also the Hessian term of the Lyapunov function. Along with the advances in differential geometric theory [2] and the discovery of a simple constructive formula for Lyapunov stabilization [3], the stochastic stabilization problem was reexamined and some constructive results have been achieved. The stabilization of nonlinear stochastic systems was considered in the work of Florchinger [4-7], who, among other things, extended the concept of control Lyapunov functions and Sontag's stabilization formula to stochastic setting. Pan and Bașer [8] solved the stabilization problem for a class of strict-feedback systems representative of stabilization results for deterministic systems. Deng and Krstić [9] developed a simpler control strategy for strict-feedback systems and then extended the results on inverse optimal stabilization for general systems to the stochastic case. The authors of [10] considered the dissipativity analysis and dissipativity-based sliding mode control for a class of continuous-time switched stochastic systems and [11] designed multistep predictive
controller for a class of Markov jump convex polyhedron linear parameter time-varying systems with both constraints on input and output. The adaptive neural tracking control problem was the concern in [12] for a class of strict-feedback stochastic nonlinear systems with unknown dead zone. However, most of these results were focused on systems that only have one kind of disturbance-stochastic disturbance. In [13], Hinrichsen proposed the stochastic robust control method for systems with deterministic and stochastic disturbances, which enabled us to deal with a broader class of systems. However, only stability of the nominal system in the absence of deterministic disturbances was the concern in this approach, which means that the stability cannot be guaranteed in the presence of both deterministic and stochastic disturbances.

Disturbance-observer-based control (DOBC) approach, which is based on the idea of feed-forward compensation, appeared in the late 1980s and has attracted considerable attention in control theory literatures [14-17]. The controller design of this method can be accomplished in two steps. First, a disturbance observer is designed to estimate the deterministic disturbance and then compensate it. Second, feedback controller is designed to stabilize the nominal system without disturbance. DOBC approach has its roots in
many mechanical applications in the last two decades, in particular for linear systems [18-20]. Recently, some attempts have been made to establish theoretic justification of these DOBC applications and extend DOBC from linear systems to nonlinear systems [21, 22]. Besides, [23] designed output feedback controller for a class of Markovian jump repeated scalar nonlinear systems and [24] investigated the problem of composite DOBC and $H_{\infty}$ control for Markovian jump systems with nonlinearity and multiple disturbances.

This paper considers the application of DOBC approach to a class of nonlinear stochastic systems. The nonlinear dynamics are described by known and unknown nonlinear functions, respectively. And apart from the stochastic noises, the deterministic disturbance is supposed to be generated by an exogenous system as investigated in $[18,19]$, which is not confined to be bounded in norm [25]. By using the disturbance estimation, the DOBC strategy can be integrated with the conventional stabilization controllers to reject the deterministic disturbance and globally stabilize the closed-loop systems in probability. Finally, simulations on an A4D aircraft model show the effectiveness of the proposed approaches.

## 2. Problem Statement

The following MIMO stochastic system with nonlinearity is described as

$$
\begin{align*}
d x(t)= & \left\{A x(t)+F_{0} f(x(t), t)+B[u(t)+v(t)]\right\} d t  \tag{1}\\
& +A_{0} x(t) d \omega,
\end{align*}
$$

where $x(t) \in R^{n}, u(t) \in R^{m}$ and $v(t) \in R^{m}$, are the state, control input, and disturbance, respectively. $f(x(t), t)$ is a nonlinear vector function satisfying bounded condition as described in Assumption 1. $\omega$ is an $r$-dimensional standard Brownian motion defined on a complete probability space $(\Omega, \mathscr{F}, \mathscr{P})$ with $\Omega$ being a sample space, $\mathscr{F}$ being a $\sigma$-field, and $\mathscr{P}$ being the probability measure. $A, B, F_{0}$, and $A_{0}$ are given system matrices with corresponding dimensions.

Assumption 1. For any $x_{j}(t) \in R^{n}, j=1,2$, nonlinear function $f(x(t), t)$ satisfy

$$
\begin{align*}
f(0, t) & =0 \\
\left\|f\left(x_{1}(t), t\right)-f\left(x_{2}(t), t\right)\right\| & \leq\left\|U\left(x_{1}(t)-x_{2}(t)\right)\right\|, \tag{2}
\end{align*}
$$

where $U$ is a given constant weighting matrix.
Assumption 2. The disturbance $v(t)$ in the control input path is supposed to be generated by the following exogenous systems:

$$
\begin{equation*}
\dot{\omega}(t)=W \omega(t), \quad v(t)=V \omega(t), \tag{3}
\end{equation*}
$$

where $W \in R^{r \times r}, V \in R^{m \times r}$ are known constant weighting matrices.

Remark 3. In fact, many kinds of disturbances in engineering can be described by Assumption 2, for example, unknown constant and harmonics with unknown phase and magnitude.

The following assumption is the necessary condition for the DOBC problem.

Assumption 4. $(A, B)$ is controllable and $(W, B V)$ is observable.

In this paper, we suppose system states are available, which means that only the estimation of disturbance needs to be focused on. In this situation, the objective of DOBC is to design an observer for system (1) to estimate the unknown disturbance $v(t)$, and then construct a composite controller with the disturbance estimation and a conventional controller so that the disturbance can be rejected and the stability in probability of the resulting composite system can be guaranteed.

## 3. DOBC for the Case with <br> Known Nonlinearity

In this section, the nonlinearity function $f(x(t), t)$ is supposed to be given and Assumptions 1, 2, and 4 hold. Due to the fact that the states of system are available, the disturbance observer is designed as

$$
\begin{equation*}
\widehat{v}(t)=V \widehat{\omega}(t), \quad \widehat{\omega}(t)=s(t)-L x(t), \tag{4}
\end{equation*}
$$

where $\widehat{\omega}(t)$ is the estimation of $\omega(t)$ and $s(t)$ is the auxiliary variable generated by

$$
\begin{align*}
d s(t)=[ & (W+L B V)(s(t)-L x(t)) \\
& \left.+L\left(A x(t)+F_{0} f(x, t)+B u(t)\right)\right] d t . \tag{5}
\end{align*}
$$

Denote the estimation error as $e_{\omega}(t)=\omega(t)-\widehat{\omega}(t)$. Based on (1), (3), (4), and (5), the error dynamics satisfy

$$
\begin{equation*}
d e_{\omega}(t)=(W+L B V) e_{\omega} d t+L A_{0} x(t) d \omega . \tag{6}
\end{equation*}
$$

The objective of disturbance rejection can be achieved by designing the observer gain $L$ such that (6) satisfies the desired stability in probability.

For DOBC strategy, the controller is usually selected as [22, 26-28]

$$
\begin{equation*}
u(t)=-\widehat{v}(t)+K x(t), \tag{7}
\end{equation*}
$$

where $\widehat{v}(t)$ is to compensate the disturbance in control input path and $K$ is the conventional feedback gain needed to be determined later.

By substituting (7) into (1), the closed loop system can be written in the following form:

$$
\begin{align*}
d x(t)= & \left\{(A+B K) x(t)+F_{0} f(x(t), t)+B V e_{\omega}(t)\right\} d t  \tag{8}\\
& +A_{0} x(t) d \omega .
\end{align*}
$$

Combining (8) with (6), the composite system is described as

$$
\begin{equation*}
d \bar{x}(t)=\{\bar{A} \bar{x}(t)+F f(x, t)\} d t+\bar{A}_{0} \bar{x}(t) d \omega \tag{9}
\end{equation*}
$$

where

$$
\begin{gather*}
\bar{x}(t)=\left[\begin{array}{c}
x(t) \\
e_{\omega}(t)
\end{array}\right], \quad \bar{A}=\left[\begin{array}{cc}
A+B K & B V \\
0 & W+L B V
\end{array}\right] \\
F=\left[\begin{array}{c}
F_{0} \\
0
\end{array}\right], \quad \bar{A}_{0}=\left[\begin{array}{cc}
A_{0} & 0 \\
L A_{0} & 0
\end{array}\right] \tag{10}
\end{gather*}
$$

In the following, the objective is to design gain $L$ and $K$ such that system (9) is asymptotical stabilization in probability. For the convenience of research, the following lemma is presented.

Lemma 5 (see [29, 30]). For a given stochastic system

$$
\begin{equation*}
d x(t)=g(x, t) d t+h(x, t) d \omega, \quad x\left(t_{0}\right)=x_{0} \tag{11}
\end{equation*}
$$

if there exists function $V(x, t) \in C^{1,2}, \mu_{1}(\cdot), \mu_{2}(\cdot) \in \mathscr{K}_{\infty}$, constants $c_{1}>0$, and a nonnegative $M(x, t)$, such that

$$
\begin{equation*}
\mu_{1}(x) \leq V(x, t) \leq \mu_{2}(x), \quad \mathscr{L} V \leq-c_{1} M(x, t) \tag{12}
\end{equation*}
$$

then, one has the following.
(i) The equilibrium $x=0$ is globally stable in probability and the solution $x(t)$ satisfied $P\left\{\lim _{t \rightarrow \infty} M(x(t))=\right.$ $0\}=1$, when $g(0, t)=0, h(0, t)=0$, and $M(x, t)=$ $M(x)$ is continuous.
(ii) The equilibrium $x=0$ is globally asymptotically stable in probability, when $g(0, t)=0, h(0, t)=0$, and $M(x, t)$ is positive definite.
Here, the differential operator $\mathscr{L}$ for differentiable function $V(x, t)$ is defined as

$$
\begin{equation*}
\mathscr{L} V=\frac{\partial V}{\partial x} g(x, t)+\frac{1}{2} \operatorname{Tr}\left\{h(x, t)^{T} \frac{\partial^{2} V}{\partial x^{2}} h(x, t)\right\} . \tag{13}
\end{equation*}
$$

Theorem 6. Consider system (1) with disturbance (3) under Assumptions 1, 2, and 4. For some $\lambda>0$, if there exist $P_{1}=$ $P_{1}^{T}>0, R_{1}$ and constant $\beta>0$ satisfying

$$
\left[\begin{array}{cc}
\operatorname{sym}\left(P_{1} W+R_{1} B V\right) & V^{T} B^{T}  \tag{14}\\
B V & -\beta^{2} I
\end{array}\right]<0
$$

and $Q_{2}=Q_{2}^{T}>0, R_{2}$ satisfying

$$
\left[\begin{array}{cccc}
\Xi & Q_{2} A_{0}^{T} & Q_{2} A_{0}^{T} R_{1}^{T} & Q_{2} U^{T}  \tag{15}\\
A_{0} Q_{2} & -Q_{2} & 0 & 0 \\
R_{1} A_{0} Q_{2} & 0 & -P_{1} & 0 \\
U Q_{2} & 0 & 0 & -\lambda^{2} I
\end{array}\right]<0,
$$

where $\Xi=\operatorname{sym}\left(A Q_{2}+B R_{2}\right)+\lambda^{2} F_{0} F_{0}^{T}+\beta^{2} I$, then the closed loop system (8) under DOBC law (7) with gain $K=R_{2} Q_{2}^{-1}$ and observer (4) with gain $L=P_{1}^{-1} R_{1}$ are global asymptotical stabilization in probability.

Proof. Define

$$
\begin{gather*}
\Sigma_{1}\left(e_{\omega}, t\right)=e_{\omega}^{T} P_{1} e_{\omega} \\
\Sigma_{2}(x, t)=x^{T} P_{2} x+\frac{1}{\lambda^{2}} \int_{0}^{t}\left[\|U x(t)\|^{2}-\|f(x, t)\|^{2}\right] d \tau \tag{16}
\end{gather*}
$$

where $P_{2}^{-1}=Q_{2}$.

It is noted that for all $x$ and $e_{\omega}, \Sigma_{1} \geq 0, \Sigma_{2} \geq 0$. In addition, along with (6), (8), the Itó differential of $\Sigma_{1}, \Sigma_{2}$ satisfies

$$
\begin{align*}
\mathscr{L} \Sigma_{1}= & e_{\varpi}^{T}\left[P_{1}(W+L B V)+(W+L B V)^{T} P_{1}\right] e_{\emptyset} \\
& +x^{T} A_{0}^{T} L^{T} P_{1} L A_{0} x, \\
\mathscr{L} \Sigma_{2}= & x^{T}\left[P_{2}(A+B K)+(A+B K)^{T} P_{2}\right] x  \tag{17}\\
& +2 x^{T} P_{2} F_{0} f+2 x^{T} P_{2} B V e_{\omega}+\frac{1}{\lambda^{2}} x^{T} U^{T} U x \\
& +x^{T} A_{0}^{T} P_{2} A_{0} x-\frac{1}{\lambda^{2}} f^{T} f .
\end{align*}
$$

Via Young inequality, we get

$$
\begin{gather*}
2 x^{T} P_{2} F_{0} f \leq \lambda^{2} x^{T} P_{2} F_{0} F_{0}^{T} P_{2} x+\frac{1}{\lambda^{2}} f^{T} f, \\
2 x^{T} P_{2} B V e_{\omega} \leq \beta^{2} x^{T} P_{2} P_{2} x+\frac{1}{\beta^{2}} e_{\omega}^{T} V^{T} B^{T} B V e_{\omega} . \tag{18}
\end{gather*}
$$

Then we have

$$
\begin{align*}
\mathscr{L} \Sigma_{2} \leq x^{T}\{ & \operatorname{sym}\left[P_{2}(A+B K)\right]+A_{0}^{T} P_{2} A_{0}+\frac{1}{\lambda^{2}} U^{T} U \\
& \left.+\lambda^{2} P_{2} F_{0} F_{0}^{T} P_{2}+\beta^{2} P_{2} P_{2}\right\} x+\frac{1}{\beta^{2}} e_{\omega}^{T} V^{T} B^{T} B V e_{\omega} \tag{19}
\end{align*}
$$

A Lyapunov function candidate for (9) is chosen as $\Sigma\left(x, e_{\mathscr{Q}}, t\right)=\Sigma_{1}\left(e_{\mathscr{D}}, t\right)+\Sigma_{2}(x, t)$; hence, it is easy to get

$$
\begin{align*}
\mathscr{L} \Sigma \leq & e_{\circlearrowleft}^{T}\left\{\operatorname{sym}\left[P_{1}(W+L B V)\right]+\frac{1}{\beta^{2}} V^{T} B^{T} B V\right\} e_{\omega} \\
& +x^{T}\left\{\operatorname{sym}\left[P_{2}(A+B K)\right]+A_{0}^{T} P_{2} A_{0}+\frac{1}{\lambda^{2}} U^{T} U\right. \\
& \left.\quad+\lambda^{2} P_{2} F_{0} F_{0}^{T} P_{2}+\beta^{2} P_{2} P_{2}+A_{0}^{T} L^{T} P_{1} L A_{0}\right\} x \\
= & e_{\omega}^{T} \Pi_{1} e_{\omega}+x^{T} \Pi_{2} x, \tag{20}
\end{align*}
$$

where

$$
\begin{align*}
\Pi_{1} & =\operatorname{sym}\left[P_{1}(W+L B V)\right]+\frac{1}{\beta^{2}} V^{T} B^{T} B V, \\
\Pi_{2}= & \operatorname{sym}\left[P_{2}(A+B K)\right]+A_{0}^{T} P_{2} A_{0}+\frac{1}{\lambda^{2}} U^{T} U  \tag{21}\\
& +\lambda^{2} P_{2} F_{0} F_{0}^{T} P_{2}+\beta^{2} P_{2} P_{2}+A_{0}^{T} L^{T} P_{1} L A_{0} .
\end{align*}
$$

Using Lemma 5, it can be verified that system (9) is global asymptotical stabilization in probability if $\Pi_{1}<0$ and $\Pi_{2}<0$ hold.

Based on Schur complement, $\Pi_{1}<0$ is equivalent to $\Pi_{10}<0$, where

$$
\Pi_{10}=\left[\begin{array}{cc}
\operatorname{sym}\left[P_{1} W+R_{1} B V\right] & V^{T} B^{T}  \tag{22}\\
B V & -\beta^{2} I
\end{array}\right]
$$

and $\Pi_{2}<0$ is equivalent to $\Pi_{21}<0$, where

$$
\Pi_{21}=\left[\begin{array}{cccc}
\Xi_{0} & A_{0}^{T} & A_{0}^{T} L^{T} P_{1} & U^{T}  \tag{23}\\
* & -P_{2}^{-1} & 0 & 0 \\
* & * & -P_{1} & 0 \\
* & * & * & -\lambda^{2} I
\end{array}\right]
$$

and $\Xi_{0}=\operatorname{sym}\left[P_{2}(A+B K)\right]+\lambda^{2} P_{2} F_{0} F_{0}^{T} P_{2}+\beta^{2} P_{2} P_{2}$. In addition, * represents the corresponding elements in the symmetric matrix.
$\Pi_{21}$ is premultiplied and postmultiplied simultaneously by $\operatorname{diag}\left\{Q_{2}, I, I, I\right\}$; then it is equivalent to $\Pi_{20}$, where

$$
\Pi_{20}=\left[\begin{array}{cccc}
\Xi & Q_{2} A_{0}^{T} & Q_{2} A_{0}^{T} R_{1}^{T} & Q_{2} U^{T}  \tag{24}\\
* & -Q_{2} & 0 & 0 \\
* & * & -P_{1} & 0 \\
* & * & * & -\lambda^{2} I
\end{array}\right]
$$

and $\Xi=\operatorname{sym}\left[(A+B K) Q_{2}\right]+\lambda^{2} F_{0} F_{0}^{T}+\beta^{2} I$.
Thus, (14), (15) can be obtained.
On the other hand, (14), (15) hold, meaning that there exist $\alpha_{1}>0, \alpha_{2}>2$ such that $\Pi_{10}<-\alpha_{1} I, \Pi_{20}<-\alpha_{2} I$; that is, $\Pi_{1}<-\alpha_{1} I, \Pi_{2}<-\alpha_{2} I$. Hence, we have

$$
\begin{align*}
\mathscr{L} \Sigma & \leq e_{\omega}^{T} \Pi_{1} e_{\omega}+x^{T} \Pi_{2} x \\
& \leq-\alpha_{1}\left\|e_{\omega}\right\|^{2}-\alpha_{2}\|x\|^{2} \\
& \leq-\min \left\{\alpha_{1}, \alpha_{2}\right\}\left(\|x\|^{2}+\left\|e_{\omega}\right\|^{2}\right)  \tag{25}\\
& =-\min \left\{\alpha_{1}, \alpha_{2}\right\}\|\bar{x}\|^{2} .
\end{align*}
$$

Therefore, the closed-loop system (9) is global asymptotical stabilization in probability when the control gain is selected as $K=R_{2} Q_{2}^{-1}$ and observer gain is selected as $L=P_{1}^{-1} R_{1}$. The proof is completed.

## 4. DOBC for the Case with Unknown Nonlinearity

In this section, the nonlinear function $f(x(t), t)$ is supposed to be unknown, which means disturbance observer should be designed different from Section 3. In this case, the disturbance observer can be constructed as

$$
\widehat{v}(t)=V \widehat{\omega}(t), \quad \widehat{\omega}(t)=s(t)-L x(t),
$$

$$
\begin{equation*}
d s(t)=(W+L B V)(s(t)-L x(t))+L(A x(t)+B u(t)) d t \tag{26}
\end{equation*}
$$

Compared with (6), the estimation error $e_{\omega}(t)=\omega(t)-$ $\widehat{\omega}(t)$ satisfies

$$
\begin{equation*}
d e_{\varpi}(t)=\left[(W+L B V) e_{\varpi}+L F_{0} f(x, t)\right] d t+L A_{0} x(t) d \omega \tag{27}
\end{equation*}
$$

Thus, the composite system combined (8) with (27) is given by

$$
\begin{equation*}
d \bar{x}(t)=\{\bar{A} \bar{x}(t)+F f(x, t)\} d t+\bar{A}_{0} \bar{x}(t) d \omega \tag{28}
\end{equation*}
$$

where

$$
\begin{gather*}
\bar{x}(t)=\left[\begin{array}{c}
x(t) \\
e_{\omega}(t)
\end{array}\right], \quad \bar{A}=\left[\begin{array}{cc}
A+B K & B V \\
0 & W+L B V
\end{array}\right], \\
F=\left[\begin{array}{c}
F_{0} \\
L F_{0}
\end{array}\right], \quad \bar{A}_{0}=\left[\begin{array}{cc}
A_{0} & 0 \\
L A_{0} & 0
\end{array}\right] . \tag{29}
\end{gather*}
$$

The objective of this section is similar to Section 3, that is, to design gain $L$ and $K$ such that system (28) is asymptotically stable in probability.

Theorem 7. Consider system (1) with disturbance (3) under Assumptions 1, 2, and 4. For some $\lambda_{1}>0, \lambda_{2}>0$, if there exist $P_{1}=P_{1}^{T}>0, R_{1}$ and constant $\beta>0$ satisfying

$$
\left[\begin{array}{ccc}
\operatorname{sym}\left(P_{1} W+R_{1} B V\right) & R_{1} F_{0} & V^{T} B^{T}  \tag{30}\\
* & -\lambda_{1}^{-2} I & 0 \\
* & * & -\beta^{2} I
\end{array}\right]<0
$$

and $Q_{2}=Q_{2}^{T}>0, R_{2}$ satisfying

$$
\left[\begin{array}{ccccc}
\bar{\Xi} & Q_{2} A_{0}^{T} & Q_{2} A_{0}^{T} R_{1}^{T} & Q_{2} U^{T} & Q_{2} U^{T}  \tag{31}\\
* & -Q_{2} & 0 & 0 & 0 \\
* & * & -P_{1} & 0 & 0 \\
* & * & * & -\lambda_{1}^{2} I & 0 \\
* & * & * & * & -\lambda_{2}^{2} I
\end{array}\right]<0,
$$

where $\bar{\Xi}=\operatorname{sym}\left(A Q_{2}+B R_{2}\right)+\lambda_{2}^{2} F_{0} F_{0}^{T}+\beta^{2} I$, then the closed loop system (28) under DOBC law (7) with gain $K=R_{2} Q_{2}^{-1}$ and observer (26) with gain $L=P_{1}^{-1} R_{1}$ is global asymptotical stabilization in probability.

Proof. Let

$$
\begin{align*}
& \bar{\Sigma}_{1}\left(e_{\varpi}, t\right)=e_{\circlearrowleft}^{T} P_{1} e_{\varpi}+\frac{1}{\lambda_{1}^{2}} \int_{0}^{t}\left[\|U x(t)\|^{2}-\|f(x, t)\|^{2}\right] d \tau \\
& \bar{\Sigma}_{2}(x, t)=x^{T} P_{2} x+\frac{1}{\lambda_{2}^{2}} \int_{0}^{t}\left[\|U x(t)\|^{2}-\|f(x, t)\|^{2}\right] d \tau \tag{32}
\end{align*}
$$

where $P_{2}^{-1}=Q_{2}$. And a Lyapunov function candidate for (28) is chosen as $\bar{\Sigma}\left(x, e_{\omega}, t\right)=\bar{\Sigma}_{1}\left(e_{\omega}, t\right)+\bar{\Sigma}_{2}(x, t)$. The following proof procedure can be given similarly to that of the proof for Theorem 6.

## 5. Simulation Example

In [21], DOBC strategy was employed to a deterministic system of A4D aircraft with disturbance, and better system performance was obtained than some previous results [31]. However, stochastic noise should be considered when higher precision of system performance was required. In this section, the stochastic system that represents the longitudinal dynamics of A4D aircraft is considered, which is described as follows:

$$
\begin{align*}
d x(t)= & \left\{A x(t)+F_{0} f(x, t)+B[u(t)+v(t)]\right\} d t  \tag{33}\\
& +A_{0} x(t) d \omega
\end{align*}
$$



Figure 1: System performance using stochastic robust control strategy.
with the following coefficient:

$$
\begin{align*}
A & =\left[\begin{array}{cccc}
-0.0605 & 32.37 & 0 & 32.2 \\
-0.00014 & -1.475 & 1 & 0 \\
-0.0111 & -34.72 & -2.793 & 0 \\
0 & 0 & 1 & 0
\end{array}\right] \\
F_{0} & =\left[\begin{array}{l}
0 \\
0 \\
0 \\
60
\end{array}\right], \quad B=\left[\begin{array}{c}
0 \\
-0.1064 \\
-33.8 \\
0
\end{array}\right],  \tag{34}\\
A_{0} & =\left[\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 0.3 & 0 & 0.2 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.5
\end{array}\right]
\end{align*}
$$

The parameter matrices for disturbance $v(t)$ that are described by (3) are given by

$$
W=\left[\begin{array}{cc}
0 & 5  \tag{35}\\
-5 & 0
\end{array}\right], \quad V=\left[\begin{array}{ll}
25 & 10
\end{array}\right]
$$

Case 1 (with known nonlinearity). In this case, the nonlinear dynamic is supposed to be denoted by $f(x, t)=$ $\sin (10 \pi t) x_{2}(t)$ in the simulation. In order to satisfy Assumption $1, U$ is selected as $\operatorname{diag}\{0,1,0,0\}$. The initial value of state is taken to be $x(0)=[2,-2,3,0]$ and $\lambda$ is selected as 20. Based on Theorem 6, it can be solved that

$$
\begin{gathered}
L=\left[\begin{array}{llll}
0 & -0.0001 & 0.0019 & 0 \\
0 & -0.0001 & 0.0012 & 0
\end{array}\right], \\
K=\left[\begin{array}{llll}
0.2952 & 2.4053 & 0.3805 & 4.8333
\end{array}\right] .
\end{gathered}
$$



Figure 2: System performance using DOBC strategy for known nonlinearity.


Figure 3: Estimation error of disturbance for known nonlinearity.

When the stochastic robust control strategy is applied to (33), which was first given in [13], it can be shown from Figure 1 that asymptotical stabilization in probability cannot be guaranteed in the presence of disturbance $v(t)$. Figures 2 and 3 show the system response and estimation error of system disturbance for the case with known nonlinearity, respectively. The simulation results show that asymptotical stabilization can be achieved using the method proposed in this paper and that the proposed disturbance observer is fine and effective.

Case 2 (with unknown nonlinearity). When nonlinear term $f(x, t)$ is unknown, we assume $f(x, t)=r(t) x_{2}(t)$ in


Figure 4: System performance using DOBC strategy for unknown nonlinearity.


Figure 5: Estimation error of disturbance for unknown nonlinearity.
simulation, where $r(t)$ is supposed to be stochastic input that obeys uniform distribution. Based on Theorem 7, it can be solved that

$$
\begin{gather*}
L=\left[\begin{array}{llll}
0 & 0.0001 & 0.0011 & 0 \\
0 & 0.0001 & 0.0006 & 0
\end{array}\right]  \tag{37}\\
K=\left[\begin{array}{llll}
0.2209 & 0.9695 & 0.2536 & 3.1946
\end{array}\right] .
\end{gather*}
$$

It is clear from Figure 4 that all states of system converge to zero and estimation errors of disturbance also converge to zero as shown in Figure 5. As has been shown above, we can see that asymptotical stabilization in probability is guaranteed
and satisfied performance of the closed-loop control systems is achieved.

## 6. Conclusion

In this paper, the DOBC approach is investigated for a class of nonlinear systems with deterministic and stochastic disturbances. Feasible design procedures are proposed under different conditions to estimate and reject deterministic disturbance for the plants with known and unknown nonlinearity. Based on the estimation of disturbances, the composite control laws can guarantee the composite closed-loop systems to be global asymptotical stabilization in probability in the presence of disturbances. Simulation for an aircraft model shows the efficiency of the proposed algorithms.

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## Research Article

# Stator Flux Observer for Induction Motor Based on Tracking Differentiator 

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#### Abstract

Voltage model is commonly used in direct torque control (DTC) for flux observing of asynchronous motor. In order to improve low-speed and dynamic performance of the voltage model, a modified low-pass filter (LPF) algorithm is proposed. Firstly, the tracking differentiator is brought in to modulate the measured stator current, which suppresses the measurement noise, and then amplitude and phase compensation is made towards the stator electromotive force (EMF), after which the stator flux is obtained through a low-pass filter. This method can eliminate the dynamic error of flux filtered by LPF and improve low-speed performance. Experimental results demonstrate effectiveness and improved dynamic performance of such method.


## 1. Introduction

The direct torque control technology based on stator flux orientation has been widely used in high-performance induction motor control system. It has the advantages of simple structure, not being sensitive to motor parameters, and so forth [1, 2]. The key of achieving direct torque control of asynchronous motor effectively lies in the accurate obtaining of stator flux information; especially observing motor flux exactly at a low stator frequency is even a big issue of AC speed regulation [3-5].

In the aspect of robust estimator, scholars have done a lot of research work. Reference [6] considers the modeling and adaptive output tracking of an FCTFPM as a nonlinear system with unknown nonlinearities by utilizing HGO and RBF neural networks. A fuzzy reliable control strategy has been presented for the tracking problem of the longitudinal dynamics of FAHVs model with actuator or sensor faults and external disturbance. Based on the T-S fuzzy modeling technology, a T-S fuzzy model has been constructed to represent the nonlinear dynamics of the FAHVs [7]. Reference [8] presents a fault tolerant tracking controller for a VTOL aircraft flight in uncertain conditions. The considered system contains structured uncertainties which affect the mechanical parameters of the air vehicle. Reference [9]
investigates the energy-to-peak filtering problem for nonuniformly sampled nonlinear systems. The sampled nonlinear systems are modeled by T-S fuzzy systems under the discretetime framework. Reference [10] is devoted to the ammonia coverage ratio estimation problem in SCR systems. Reference [11] studies the state estimation problem for discrete-time systems subject to network-induced delay. By considering the occurrence probability for the delay, the exponential meansquare stability and the $H_{\infty}$ performance are exploited for the estimation error system. Reference [12] investigates the $H_{\infty}$ filtering problem for T-S fuzzy systems under the discretetime framework. By using Finsler's lemma, a new $H_{\infty}$ criterion for discrete-time fuzzy systems is obtained. With the partition technique, the Lyapunov weighting matrices and the parameters to be designed are decoupled.

Voltage model is the basic method of the following stator flux. It uses an ideal integrator described in formula (1). The algorithm of this model is simple which only need to know the stator resistance, which is why the flux observing method which is based on voltage model has always been given special importance [13, 14]. Consider

$$
\begin{equation*}
\psi_{s}=\int\left(\mathbf{u}_{s}-\mathbf{i}_{s} R_{s}\right) d t \tag{1}
\end{equation*}
$$

Although the voltage model is quite simple, there are still some problems in practical application [15]. (1) A small DC bias or drift in the current measurement channel will cause the integrator saturation. (2) The stator resistance variation in low stator frequency makes the stator flux amplitude and phase observations have a big error. (3) The initial value of the integral produces the dc bias in the observed flux amplitude.

In order to eliminate the effect of DC bias, it is proposed in [16] that the pure integrator should be replaced with low-pass filter, but amplitude and phase error of flux can be introduced to it. Adopting a programmable cascaded low-pass filter can overcome the effect of zero drift theoretically [17], but the cutoff frequency is of high precision. It is difficult to get the desired effect in practical application, and the dynamic performance is poor. Using improved PLL to observe the stator flux is proposed in [18], but this method needs to use the motor speed information based on permanent magnet synchronous motor, and the system has two convergence points meanwhile.

The biggest reason why flux observation is inaccurate is that the DC bias and unbalanced gain exist in current measurement channel; hence this paper reduces the current measurement interference by using tracking differentiator [19, 20] to filter the measuring stator current; it also restrains the DC component of stator current by using low-pass filter instead of pure integral, and it eliminates the amplitude attenuation and phase error brought by low-pass filter by using the back EMF compensation algorithm. Experimental results demonstrate effectiveness and improved dynamic performance of such method.

## 2. Traditional Method of Stator Flux Estimation

The voltage model for flux observation is obtained according to the stator voltage equation. Stator voltage equation and flux equation are expressed as

$$
\begin{gather*}
\mathbf{u}_{s}=R_{s} \mathbf{i}_{s}+\frac{d \psi_{s}}{d t}  \tag{2}\\
\psi_{s}=\int\left(\mathbf{u}_{s}-\mathbf{i}_{s} R_{s}\right) d t \tag{3}
\end{gather*}
$$

where $\mathbf{u}_{s}$ is the stator voltage, $R_{s}$ is the stator resistant, $\mathbf{i}_{s}$ is the stator current, and $\psi_{s}$ is the stator flux.

Equation (3) is called the U-I model or the voltage model of flux estimator. Since the formula contains a pure integrator, small DC bias can cause integral saturation which will result in flux estimation error. So it is usual to replace pure integral with the first-order low-pass filter in voltage model; namely, let the integral input signal go through a high-pass filter firstly to filtrate the DC component.

The stator flux has invariant amplitude in a steady state revolving in the synchronous frequency, which can be expressed as

$$
\begin{equation*}
\psi_{s}=\left|\psi_{s}\right| e^{j \omega_{e} t}=\left|\psi_{s}\right| \angle \omega_{e} t, \tag{4}
\end{equation*}
$$

where $\omega_{e}$ is the synchronous frequency of the motor.

Equations (2) and (4) become

$$
\begin{gather*}
\frac{d \psi_{s}}{d t}=j \omega_{e} * \psi_{s}=\mathbf{u}_{s}-\mathbf{i}_{s} R_{s},  \tag{5}\\
\psi_{s}=\frac{\mathbf{u}_{s}-\mathbf{i}_{s} R_{s}}{j \omega_{e}} .
\end{gather*}
$$

Equation (5) is the expression of actual stator flux.
When observing stator flux through low-pass filter, the following equation can be written:

$$
\begin{gather*}
\frac{d \widehat{\psi}_{s}}{d t}+\omega_{c} \widehat{\psi}_{s}=j \omega_{e} * \widehat{\psi}_{s}+\omega_{c} \widehat{\psi}_{s}=\mathbf{u}_{s}-\mathbf{i}_{s} R_{s},  \tag{6}\\
\widehat{\psi}_{s}=\frac{\mathbf{u}_{s}-\mathbf{i}_{s} R_{s}}{j \omega_{e}+\omega_{c}},
\end{gather*}
$$

where $\widehat{\psi}_{s}$ is the stator flux which is observed by low-pass filter and $\omega_{c}$ is the cutoff frequency of the low-pass filter.

The connection between the estimated stator flux $\widehat{\psi}_{s}$ and the actual stator flux $\psi_{s}$ can be concluded from (5) and (6) which can be expressed as

$$
\begin{align*}
\widehat{\psi}_{s} & =\frac{\omega_{e}}{\sqrt{\omega_{e}^{2}+\omega_{c}^{2}}} \psi_{s} \angle \theta  \tag{7}\\
\theta & =\frac{\pi}{2}-\arctan \frac{\omega_{e}}{\omega_{c}}
\end{align*}
$$

From (7), it is clear that errors of flux in the amplitude and the phase result from the replacement of pure integral with a low-pass filter. The higher the cutoff frequency is, the more serious the distortion is shown in the flux amplitude and phase.

According to the principle of the direct torque control, the flux error can affect the steady state and dynamic operation of asynchronous motor directly. Direct torque control selects the appropriate voltage vector according to the observed flux and torque. Meanwhile the low-pass filter cuts down the flux amplitude, so when the observing flux reached a given value, the actual flux has already been far beyond that, which leads to the saturation of motor magnetic field. Phase shift of the observed flux can influence the accurate selection of voltage vector as well as the control characteristic of the motor. Besides, the estimation of torque in direct torque control is also affected by flux value directly. Therefore it is essential to have amplitude and phase compensation for the result of lowpass filter. Traditional method of flux observing is shown in Figure 1.

## 3. Improved Method of Flux Observing

3.1. Modified Low-Pass Filter (LPF) Compensation Algorithm. Make amplitude and phase compensation for the result of the low-pass filter as follows:

$$
\begin{equation*}
\widehat{\psi}_{s} \mathbf{G}=\psi_{s}, \tag{8}
\end{equation*}
$$

where $\widehat{\psi}_{s}$ is the stator flux which is observed by low-pass filter, $\mathbf{G}$ is the penalty function, and $\psi_{s}$ is the stator flux.


Figure 1: Traditional method of flux observing.

According to (7), the penalty function $\mathbf{G}$ can be expressed as

$$
\begin{gather*}
\mathbf{G}=\frac{\sqrt{\omega_{e}^{2}+\omega_{c}^{2}}}{\omega_{e}} e^{j\left(\arctan \left(\omega_{e} / \omega_{c}-\pi / 2\right)\right)}=\frac{\sqrt{\omega_{e}^{2}+\omega_{c}^{2}}}{\omega_{e}} e^{j \rho\left(\omega_{e}\right)}, \\
e^{j \rho\left(\omega_{e}\right)}=\cos \left[\rho\left(\omega_{e}\right)\right]+j \sin \left[\rho\left(\omega_{e}\right)\right] \\
\cos \left[\rho\left(\omega_{e}\right)\right]=\frac{\omega_{e}}{\sqrt{\omega_{e}^{2}+\omega_{c}^{2}}}  \tag{9}\\
\sin \left[\rho\left(\omega_{e}\right)\right]=\frac{\omega_{c}}{\sqrt{\omega_{e}^{2}+\omega_{c}^{2}}}
\end{gather*}
$$

where $\omega_{e}$ is the synchronous frequency of the motor $\omega_{c}$ is the cutoff frequency of the low-pass filter.

By choosing appropriate cutoff frequency, this compensation algorithm can make the flux observer have a better ability in restraining DC drift and also have a strong ability of anti-interference. But this algorithm has a poor dynamic performance. The estimation of flux value will have big error when stator current frequency has a sudden change [21]. For this reason the sequence of applying the low-pass filtering algorithm and the flux compensation can be exchanged, making compensation for the back electromotive force firstly as follows:

$$
\begin{equation*}
\mathbf{E}_{s}=\mathbf{u}_{s}-\mathbf{i}_{s} R_{s}, \quad \widehat{\mathbf{E}}_{s} \mathbf{G}=\mathbf{E}_{s} . \tag{10}
\end{equation*}
$$

In the $\alpha-\beta$ coordinate system, back electromotive force components $e_{s \alpha}$ and $e_{s \beta}$ are at $\pi / 2$ space angle. Assuming that the stator flux is in counterclockwise rotation, it passes $\alpha$ axis firstly and then $\beta$ axis. Therefore in a constant flux control mode, $e_{s \alpha}$ and $e_{s \beta}$ have the same amplitude and different phase, which can be rewritten as

$$
\begin{equation*}
e_{s \alpha}=j e_{s \beta}, \quad e_{s \beta}=-j e_{s \alpha} . \tag{11}
\end{equation*}
$$

Combining (9)~(11) leads to the following expression:

$$
\begin{align*}
& e_{s \alpha}=\widehat{e}_{s \alpha}+\frac{\omega_{c}}{\omega_{e}} \widehat{e}_{s \beta}, \\
& e_{s \beta}=\widehat{e}_{s \beta}-\frac{\omega_{c}}{\omega_{e}} \widehat{e}_{s \alpha} . \tag{12}
\end{align*}
$$

Practice shows that the optimal cutoff frequency for lowpass filter should be $20 \% \sim 30 \%$ of the synchronous frequency $\omega_{e}$ [16]. It is hard to estimate synchronous frequency of the motor when it is running, whereas the stator current frequency can be obtained through the detected current signal, so it can replace the synchronous frequency. Therefore the cutoff frequency of low-pass filter can be calculated as follows:

$$
\begin{equation*}
\omega_{c}=\omega_{0}+k \omega_{i} \tag{13}
\end{equation*}
$$

where $\omega_{0}$ is the initial value. It ensures that when rotating speed is close to zero, the cutoff frequency will not be too low. $\omega_{i}$ is the stator current frequency and $k$ is coefficient of proportionality (0.2-0.3).

According to the above, in order to achieve this modified low-pass filter algorithm, it is necessary to settle on the stator current frequency $\omega_{i}$. The space situation of the stator current in the $\alpha-\beta$ coordinate system can be expressed as

$$
\begin{equation*}
\theta_{i}=\arctan \left(\frac{i_{s \beta}}{i_{s \alpha}}\right) \tag{14}
\end{equation*}
$$

where the stator current frequency can be obtained by differentiating $\theta_{i}$; the discretization formula is shown as follows:

$$
\begin{equation*}
\omega_{i(k)}=\frac{\theta_{(k)}-\theta_{(k-1)}}{\Delta T} \tag{15}
\end{equation*}
$$

3.2. Tracking Differentiator. Because of the measurement noise, the stator current can affect the precision of flux observation; therefore, stator current needs to be filtered. Tracking differentiator (TD) is the essential part of ADRC [22]. The initial purpose of TD is to rationally extract continuous signal and differential signal from discontinuous or band random noise signal when it comes to the practical engineering problems. After a further research on TD, discretization form of TD was proposed, making it easier for computer calculation and better in filtering.

TD discretization formula can be written as

$$
x_{1}(k+1)=x_{1}(k)+h * x_{2}(k)
$$

$$
\begin{align*}
& x_{2}(k+1)  \tag{16}\\
& \quad=x_{2}(k)+h * f s t\left(x_{1}(k)-v(k), x_{2}(k), r, h_{1}\right),
\end{align*}
$$



Figure 2: TD modeling in Simulink.


Figure 3: Filtering result of TD.
where $v(k)$ is input signal, $x_{1}$ is the tracking signal of $v(k)$, and $x_{2}$ is the derivative of $x_{1}$ which can be seen as the derivative of input signal. Consider

$$
\begin{align*}
& f s t\left(x_{1}(k)-v(k), x_{2}(k), r, h_{1}\right)=-r * \operatorname{sat}(g(k), \delta), \\
& \delta=h_{1} * r, \quad \delta_{1}=h_{1} * \delta, \\
& e(k)=x_{1}(k)-v(k), \\
& y(k)=e(k)+h_{1} * x_{2}(k), \\
& g(k)= \begin{cases}x_{2}(k)+\operatorname{sign}(y(k)) & \\
* \frac{\sqrt{8 r|y(k)|+\delta^{2}}-\delta}{2} & |y(k)| \geq \delta_{1} \\
x_{2}(k)+\frac{y(k)}{h_{1}} & |y(k)| \leq \delta_{1},\end{cases}  \tag{17}\\
& \operatorname{sat}(x, \delta)= \begin{cases}\operatorname{sign}(x) & |x| \geq \delta \\
\frac{x}{\delta} & |x| \leq \delta,\end{cases}
\end{align*}
$$

where $h$ is integration step and $r$ is a parameter that determines the tracking speed.

In order to validate the filtering performance of tracking differentiator, as shown in Figure 2, a Simulink simulation model is built with an interference input signal as follows:

$$
\begin{equation*}
v(t)=\sin t+d(t) \tag{18}
\end{equation*}
$$

where $d(t)$ is the uniformly distributed random disturbance signal with the amplitude $1 \%$ and is used to simulate the measurement noise of the current sampling.

The simulation results are shown in Figure 3; it can be seen that the tracking differentiator restores the contaminated original signal. Hence this paper tries to introduce tracking differentiator to filter the stator current.
3.3. Build Up Complete Flux Observing Model. Based on the above analysis, the complete illustrative diagram of stator flux observing model is shown in Figure 4. The current which is used to count the back electromotive force is filtered by tracking differentiator and then compensate the back electromotive force, and then get the stator flux through the low-pass filter. Use the stator flux signal to calculate angular frequency and cutoff frequency which is fed back to compensation algorithm and low-pass algorithm.

## 4. Experimental Verification

4.1. Experimental Platform. In order to verify the performance of the flux observing model in this paper, an experimental platform is established for the asynchronous motor direct torque control system, as shown in Figure 5. The experimental platform is powered by programmable DC supply; the development suite is the high-voltage motor control and PFC Development Suite v2.0 from TI Company; the MCU is TMS320F28335; parameter of the triphase asynchronous motor is shown in Table 1. The proposed method involves


Figure 4: Model of the proposed stator flux observer.


Figure 5: The experimental platform.

TABLE 1: Motor parameters.

| Rated value | Parameter value |
| :--- | :---: |
| Rated speed $1725 \mathrm{r} / \mathrm{min}$ | Stator resistance $11.05 \Omega$ |
| Rated power 184 W | Rotor resistance $6.11 \Omega$ |
| Rated voltage 220 V | Self-inductance 0.316423 H |
| Rated torque $1 \mathrm{~N} \cdot \mathrm{~m}$ | Mutual inductance 0.293939 H |
| Rated current 1.3 A | Number of pole-pairs 2 |

some division operations, requiring a higher speed processor. In this paper, we chose DSP28335, up to 150 MHz , which can meet the requirements.

In the control procedure of experiment, the control period is $100^{\mu s}$. The two flux observing methods are compared in the experiment; the result of the traditional method is shown in Figure 1, whereas the result of the method proposed in this paper is shown in Figure 4. Except the flux observing method, other conditions are all the same in this experiment.
4.2. Experimental Results. Figures 6 and 7 give the compared experimental waveforms of the two methods where the targeted motor speeds are both $150 \mathrm{r} / \mathrm{min}$. It can be seen from Figure 6 that when the targeted motor speed is higher,
the speed waveforms are basically the same of the two methods. From Figure 7 it can be seen that the current waveforms are basically the same of the two methods as well, but current harmonic wave is smaller in the method proposed in this paper, which illustrates that the use of tracking differentiator in filtering has improved the current fluctuation.

With the targeted speed getting slower, the traditional method can barely guarantee performance of the motor control; Figures 8 and 9 give the compared experimental waveforms of the two methods where the targeted motor speeds are both $50 \mathrm{r} / \mathrm{min}$. It can be seen from Figure 8 that, in the traditional method, the motor speed has a huge fluctuation; that is to say, the DTC has already become invalid; the motor speed is kept around the targeted one and has a small fluctuation by using the flux observing method of this paper. From the current waveforms shown in Figure 9, it is clear that there are a lot of current harmonic waves and the waveform also has distortions in the traditional method, but current waveform in the method of this paper is still in good state.

When the targeted speed is set to be $25 \mathrm{r} / \mathrm{min}$, the experimental result of DTC system using traditional method has a poor performance, the motor operates intermittently, and the control is totally invalid, whereas the DTC system based on flux observing model of this paper can still run smoothly. Using the method of this paper, the speed and the current waveforms which are shown in Figure 10 aimed at a speed of $25 \mathrm{r} / \mathrm{min}$. The experimental results show that the flux observation method proposed in this paper can improve the low-speed performance of DTC.

Under the experimental condition that the speed changes sharply from $150 \mathrm{r} / \mathrm{min}$ to $50 \mathrm{r} / \mathrm{min}$ in the 4.5 s , Figures 11 and 12 give the current waveforms of two methods. When the speed turns sharply, it is obvious that the load and current fluctuation are smaller in the method of this paper. When the targeted speed is $50 \mathrm{r} / \mathrm{min}$, under the experimental condition that the load torque changes sharply from $0 \mathrm{~N} \cdot \mathrm{~m}$ to $0.3 \mathrm{~N} \cdot \mathrm{~m}$ in the 4.5 s , Figures 13,14 , and 15 show the speed and stator current and torque waveforms of two methods. When the load torque changes sharply, the torque of traditional method has a huge fluctuation, whereas in the method of this paper the torque has a small fluctuation. Experimental results demonstrate the improved dynamic performance of such flux observing method mentioned in this paper.

## 5. Conclusion

To solve the stator flux observation problem of asynchronous motor, the observation scheme that compensates for back EMF firstly and then filters through low-pass filter is proposed; at the mean time tracking differentiator is used to filter the stator current and the simple voltage model is retained, and all the above lead to the improvement of the dynamic precision of flux observing. This scheme can improve the dynamic and low-speed performance of the DTC system of induction motors. The accuracy of flux observation is less influenced by the stator frequency mutation, and there is less current harmonic waves with efficiently restrained torque fluctuation.


Figure 6: Motor speed waveform when target speed is $150 \mathrm{r} / \mathrm{min}$.


Figure 7: Stator current waveform when target speed is $150 \mathrm{r} / \mathrm{min}$.


Figure 8: Motor speed waveform when target speed is $50 \mathrm{r} / \mathrm{min}$.


Figure 9: Stator current waveform when target speed is $50 \mathrm{r} / \mathrm{min}$.


Figure 10: Waveforms with the proposed method when target speed is $25 \mathrm{r} / \mathrm{min}$.


Figure 11: Motor speed waveform with speed step input from $150 \mathrm{r} / \mathrm{min}$ to $50 \mathrm{r} / \mathrm{min}$.


Figure 12: Stator current waveform with speed step input from $150 \mathrm{r} / \mathrm{min}$ to $50 \mathrm{r} / \mathrm{min}$.


Figure 13: Motor speed waveform with torque input from $0 \mathrm{~N} \cdot \mathrm{~m}$ to $0.3 \mathrm{~N} \cdot \mathrm{~m}$.


Figure 14: Stator current waveform with torque input from $0 \mathrm{~N} \cdot \mathrm{~m}$ to $0.3 \mathrm{~N} \cdot \mathrm{~m}$.


Figure 15: Torque waveform with torque input from $0 \mathrm{~N} \cdot \mathrm{~m}$ to $0.3 \mathrm{~N} \cdot \mathrm{~m}$.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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# RBF Neural Network of Sliding Mode Control for Time-Varying 2-DOF Parallel Manipulator System 

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#### Abstract

This paper presents a radial basis function (RBF) neural network control scheme for manipulators with actuator nonlinearities. The control scheme consists of a time-varying sliding mode control (TVSMC) and an RBF neural network compensator. Since the actuator nonlinearities are usually included in the manipulator driving motor, a compensator using RBF network is proposed to estimate the actuator nonlinearities and their upper boundaries. Subsequently, an RBF neural network controller that requires neither the evaluation of off-line dynamical model nor the time-consuming training process is given. In addition, Barbalat Lemma is introduced to help prove the stability of the closed control system. Considering the SMC controller and the RBF network compensator as the whole control scheme, the closed-loop system is proved to be uniformly ultimately bounded. The whole scheme provides a general procedure to control the manipulators with actuator nonlinearities. Simulation results verify the effectiveness of the designed scheme and the theoretical discussion.


## 1. Introduction

The past several decades have seen a rapid increase in parallel manipulators connected to the control system. It is well known that parallel manipulators with high rigidity are of generally higher accuracy and of lower error accumulation than similarly sized serial manipulators. Their closed kinematics structure allows them to obtain high structural stiffness and perform high-speed motions. The inertia of its mobile parts is reduced, since the actuators of a parallel manipulator are often fixed to its base and the end effectors can perform movements with higher accelerations. Adaptive tracking control for a class of nonlinear systems is given in [1-3], and in [4] a design of the sliding mode surface integral algorithm is proposed to inhibit the steady-state error and enhance the robustness. In $[5,6]$ sliding mode control (SMC) with low-pass filter is used to keep trajectory tracking accurately.

It is known that SMC has the intrinsic nature of robustness, good transient fast response, and insensitivity to the variation of plant parameters and external disturbances in [7]. Thus, the SMC is considered as an effective approach for
the control of many systems such as uncertain nonlinear systems in [8], discrete-time nonlinear systems in [9], and singular stochastic hybrid systems in [10-12]. The control process of SMC consists of two parts; one part is continuous and the other discontinuous. When the system reaches the sliding mode, the system with variable structure control is insensitive to the external disturbances and the variations of the plant parameters, and it has been widely applied to the manipulator system due to its operation characteristics for the sake of fastness, robustness, and stability in large load variations. All those merits are gotten at the cost of the chattering. Furthermore, in many time-variable systems, parameters and perturbation upper bounds are often uncertain, so switching control gain should be as high as possible to keep stable [1316]. The control algorithm in [17-20] reduces the reaching phase extremely and achieves better robustness than SMC. However, in [17] with disturbance observer based on SMC, the algorithm assumes that the disturbance is produced by a linear exogenous system, but in fact it is difficult to accurately predict the uncertainty and disturbance of the time-variable system. In [18-20], the adaptive sliding mode control (ASMC) algorithm is proposed with the switching


Figure 1: Structure of 2-DOF parallel manipulator.
gain excessive the SMC values, which brings the more serious chattering problems. So the appropriate switching gain is the key to the accurate control in time-variable system.

Motivated by the above observations, we propose a new approach to using RBF neural network to estimate the actuator nonlinearities and their upper boundaries for the switching gain. In this paper, we aim to solve these problems by focusing on the accurate tracking problem of the uncertain mechanical system. Our research begins with a time-variable SMC (TVSMC) algorithm with the dynamics of the manipulator. The main characteristics of such a TVSMC algorithm are discussed. To deal with the finite static error brought by the continuous approximation of the TVSMC algorithm, the RBF neural network is utilized, a novel Lyapunov function is introduced, and then a new time-variable stability criterion is presented. In the numerical simulation, the RBF TVSMC algorithm and the TVSMC algorithm are compared after being tested.

## 2. System Model

The 2-degree-of-freedom parallel manipulator (2-DOF parallel manipulator) is made up of three groups of two links in one platform, in which one group has a base of open chain mechanism, respectively, installed by the AC servo motor and a speed reducer drive shown in Figure 1 and coordinate in Figure 2. The 2-DOF parallel manipulator system can be described by Laugrange's equations [21, 22]:

$$
\begin{equation*}
J \ddot{\theta}+B \ddot{\theta}+W \dot{\theta}=U+U_{d} . \tag{1}
\end{equation*}
$$

In (1), $\theta \in R^{3}$ stands for a displacement angle of generalized coordinates, $J$ for the symmetric and positivedefinite inertia matrix, $B$ for a damping coefficient matrix, $W$ for a stiffness coefficient matrix, $U$ for a voltage vector of generalized control input, and $U_{d}$ for a damping voltage vector of external disturbance.

Defining $J=\widehat{J}+\Delta J, B=\widehat{B}+\Delta B$, and $W=\widehat{W}+\Delta W$, the superscript $(\wedge)$ stands for the nominal value, and the notation


Figure 2: Coordinate of 2-DOF parallel manipulator.
$(\Delta)$ represents the uncertainty. According to the structural feature, the effect caused by the model uncertainties can be merged into the disturbance term, which then can be regarded as the lumped disturbance in the following form:

$$
\begin{equation*}
d=U_{d}-\Delta W \dot{\theta}-\Delta B \ddot{\theta}-\Delta J \ddot{\theta} \tag{2}
\end{equation*}
$$

From (2), it is assumed that the lumped disturbance is bounded by a upper bound; that is, $\|d\|_{\infty} \leq d_{\max }$, where $d_{\text {max }} \in R^{+}$is a constant scalar and $\|\cdot\|_{\infty}$ is the infinite norm of a vector. $\dot{\bar{J}}-2 \widehat{B}$ is a skew-symmetric matrix (see details in [23, 24]).

So (1) can be rewritten as

$$
\begin{equation*}
\widehat{J} \ddot{\theta}+\widehat{B} \ddot{\theta}+\widehat{W} \dot{\theta}=U+d \tag{3}
\end{equation*}
$$

## 3. TVSMC Design and Parameter Setting

In this paper, we will address the tracking control problem of the system in (3). For the desired trajectory with the system states of $\theta_{d}, \dot{\theta}_{d}$, and $\ddot{\theta}_{d}$, a controller $u$ for the system is designed so that the system states $\theta_{d}, \dot{\theta}_{d}$, and $\ddot{\theta}_{d}$ can track the desired trajectory in the presence of parametric uncertainty and external disturbance.

For the given trajectory, the tracking error and tracking angle are defined as

$$
\begin{align*}
& E=\theta-\theta_{d}, \\
& \dot{E}=\dot{\theta}-\dot{\theta}_{d},  \tag{4}\\
& \ddot{E}=\ddot{\theta}-\ddot{\theta}_{d},
\end{align*}
$$

with the initial tracking error satisfying $E(0) \neq 0_{3 \times 1}$, $\dot{E}(0) \neq 0_{3 \times 1}$, and $\ddot{E}(0) \neq 0_{3 \times 1}$, where the subscript denotes the appropriate dimensions of the matrix.


Figure 3: Phase locus and sliding mode function.


Figure 4: Architecture of the RBF neural network.


Figure 5: The desired manipulator trajectory.

In order to derive the SMC algorithm, the switching surface can be chosen as

$$
S=\left[\begin{array}{lll}
s_{1} & s_{2} & s_{3} \tag{5}
\end{array}\right]^{T}=C E,
$$

where $C \in R^{3 \times 3}$ is the matrix with strictly positive every elements and $E=\left(\begin{array}{lll}E & \dot{E} & \ddot{E}\end{array}\right)^{T}$. The switching surface is then determined by $S=0_{3 \times 1}$, which is the desired dynamics.

The input of SMC algorithm is set as

$$
\begin{equation*}
U=u_{\mathrm{eq}}+u_{n} \tag{6}
\end{equation*}
$$



Figure 6: The desired joints trajectories.

In (6), $u_{\mathrm{eq}}$ is the equivalent control. The equivalent control of the ideal sliding mode is obtained on $S=0$ with $\dot{S}=0$; it can be gotten as

$$
\begin{gather*}
u_{\mathrm{eq}}=-\left(\widehat{J}\left(C_{1} \dot{E}+C_{2} \ddot{E}\right)-\widehat{B}\left(C_{1} E+C_{2} \dot{E}\right)\right.  \tag{7}\\
\left.+\left(\hat{J} \ddot{\theta}_{d}+\widehat{B} \ddot{\theta}_{d}+\widehat{W} \dot{\theta}_{d}\right)\right) .
\end{gather*}
$$

$u_{n}$ is the switching control:

$$
\begin{equation*}
u_{n}=-\gamma \cdot \operatorname{sign}(S) . \tag{8}
\end{equation*}
$$

$\gamma=\operatorname{diag}\left(\gamma_{1}, \gamma_{2}, \gamma_{3}\right) \in R^{3 \times 3}$ is the switching gain matrix with the elements $\gamma_{i}>d_{\text {max }}$, and the $\operatorname{sign}(S)=$ $\left(\operatorname{sign}\left(s_{1}\right) \operatorname{sign}\left(s_{2}\right) \operatorname{sign}\left(s_{3}\right)\right)^{T}$ presents the sign function.

Theorem 1. Considering the system in (3) under the lumped disturbance with an upper boundary of $d_{\text {max }}$, by adopting the time-varying sliding mode function in (5) and the corresponding input control in (6), if $\gamma_{i}>d_{\text {max }}$, then the controlled system is stable.


Figure 7: Control voltage generated by TVSMC.

Proof. Consider the following Lyapunov function:

$$
\begin{equation*}
V=\frac{1}{2} S^{T} \widehat{J} S \tag{9}
\end{equation*}
$$

Differentiating $V$ with respect to time yields

$$
\begin{align*}
\dot{V}= & S^{T} \widehat{J} \dot{S}+\frac{1}{2} S^{T} \dot{\hat{J}} S \\
= & S^{T} \widehat{J}\left(C_{1} \dot{E}+C_{2} \ddot{E}+\dddot{E}\right)+\frac{1}{2} S^{T} \dot{\hat{J}} S \\
= & S^{T}\left(U+d+\widehat{J}\left(C_{1} \dot{E}+C_{2} \ddot{E}\right)\right. \\
& \left.\quad-\widehat{B} \ddot{E}-\widehat{W} \dot{E}-\widehat{J} \ddot{\theta}_{d}-\widehat{B} \ddot{\theta}_{d}-\widehat{W} \dot{\theta}_{d}\right) \\
& +\frac{1}{2} S^{T} \dot{\widehat{J}}\left(C_{1} E+C_{2} \dot{E}+\ddot{E}\right) \tag{11}
\end{align*}
$$

$$
\begin{align*}
&=S^{T}\left(U+d+\widehat{J}\left(C_{1} \dot{E}+C_{2} \ddot{E}\right)+\widehat{B}\left(C_{1} E+C_{2} \dot{E}\right)\right. \\
&\left.-\left(\widehat{J} \ddot{\theta}_{d}+\widehat{B} \ddot{\theta}_{d}+\widehat{W} \dot{\theta}_{d}\right)\right) . \tag{10}
\end{align*}
$$

$U=u_{\text {eq }}+u_{n}$ is substituted in Lyapunov function, and for $\dot{J}-2 B$ is a skew-symmetric matrix such that $(1 / 2) S^{T}(\dot{\bar{J}}-$ $2 \widehat{B}) S=0$. The time derivative of $V$ will be

$$
\begin{aligned}
\dot{V} & =S^{T}(-\gamma \operatorname{sign}(S)+d) \\
& =\sum_{i=1}^{3}\left(-\gamma_{i} \cdot\left|s_{i}\right|+d_{i} s_{i}\right) \\
& \leq-\sum_{i=1}^{3}\left(\gamma_{i}-d_{\max }\right)\left|s_{i}\right|<0 .
\end{aligned}
$$



Figure 8: Control voltage generated by RBF TVSMC.

As $V$ is positively defined and $\dot{V}<0$, according to the Lyapunov stability theory, the TVSMC is stable. And Figure 3 is gotten.

## 4. RBF SMC Algorithm Design

From Figure 3(b), the process of SMC action is divided into two states: one is the reaching state and the other is the sliding state. Then reaching state will be joined to the sliding state inside the sliding tranche with bandwidth $2 \gamma$ surrounding the sliding surface, in a finite time limited by the switching frequency. The system adopts the dynamic of the surface and reaches the equilibrium point. In a short phrase, the switching gain depends on $\gamma$. Consider the following saturation function to replace the sign function to decrease chattering:

$$
\operatorname{sat}(S)= \begin{cases}\frac{S}{|S|_{\infty}+\delta}, & \left|s_{i}\right| \leq \delta  \tag{12}\\ \operatorname{sign}(S), & \text { otherwise }\end{cases}
$$

In (12) $\delta$ is tiny positive number and is also the boundary layer thickness, which can reduce the chattering if appropriately chosen.

On the other hand, radial basis function neural network based on controller design is one of the popular methods of high precision control. In the SMC based controller, the RBF NN is used to approximate the upper boundary of lumped disturbance (Figure 4).

The upper boundary of the lumped disturbance can be designed as

$$
\begin{equation*}
\gamma_{i}=\sum_{j=1}^{4} w_{i j} h_{i j} \tag{13}
\end{equation*}
$$

where $h_{i j}$ is the Gaussian radial basis function.
Here, further consideration will be given about adjusting the network weights. The weight adjustment is $A_{d}=$ $(1 / 2) E^{T} E$.


Figure 9: Trajectory tracking by TVSMC.

The RBF NN earning algorithm is

$$
\begin{align*}
\Delta w & =-\frac{\partial A_{d}}{\partial w}=-E^{T} \frac{\partial E}{\partial w} \\
& =-E^{T} \frac{\partial E}{\partial u} \frac{\partial u}{\partial \gamma} \frac{\partial \gamma}{\partial W}  \tag{14}\\
& \approx-E^{T} \operatorname{sign}\left(\frac{\partial \theta}{\partial u}\right) \frac{\partial u}{\partial \gamma} \frac{\partial \gamma}{\partial W},
\end{align*}
$$

where the value of $(\partial \theta / \partial u)$ can be substituted by coefficient of learning rate, $\theta$ is in direct proportion to $u$ in step response, $\operatorname{sign}(\partial \theta / \partial u)=1,(\partial u / \partial \gamma)=-\operatorname{sign}(S)$, and $(\partial \gamma / \partial W)=\phi$.

So $\Delta w$ can be described as follows:

$$
\begin{equation*}
\Delta w \approx E^{T} \operatorname{sign}(S) \phi=\left(C^{-1}\right)^{T}\left|S^{T}\right| \cdot \phi \tag{15}
\end{equation*}
$$

Theorem 2. Considering the system in (3) under the lumped disturbance with an upper boundary of $d_{\text {max }}$, by adopting the time-varying sliding mode function in (5) and the lumped disturbance identified by using the RBF neural network in (13), the controlled system is global asymptotic stable.

Proof. Let optimal network weights be $w^{*}$ and the estimated upper boundary value of RBF NN be $\widetilde{\gamma}$; then we get

$$
\begin{equation*}
\sum_{j=1}^{3} w_{j_{i}}^{*} h_{j}-\widetilde{\gamma}_{i}=\varepsilon, \quad|\varepsilon|<\varepsilon_{0} \tag{16}
\end{equation*}
$$



Figure 10: Trajectory tracking by RBF TVSMC.
where $\varepsilon$ is a tiny number and $\varepsilon_{0}$ is tiny positive number, and let $\widetilde{\gamma}_{i}-\left|d_{i}\right|>\varepsilon_{1}>\varepsilon_{0}, \dot{w}=\left(C^{-1}\right)^{T}\left|S^{T}\right| \phi$. A neural Lyapunov function can be designed as

$$
\begin{equation*}
V_{1}=\frac{1}{2} S^{T} \widehat{J} S+\frac{1}{2}\left(w^{*}-w\right)^{T} C^{T}\left(w^{*}-w\right) \tag{17}
\end{equation*}
$$

Differentiating $V_{1}$ with respect to time yields, we get

$$
\begin{equation*}
\dot{V}=S^{T} \hat{J} \dot{S}+\frac{1}{2} S^{T} \dot{\hat{J}} S-\left(w^{*}-w\right)^{T} C^{T} \dot{w} \tag{18}
\end{equation*}
$$

with $U=u_{\text {eq }}+u_{n}$ and simplifying

$$
\begin{aligned}
\dot{V}_{1}= & S^{T}(-\gamma \operatorname{sign}(S)+d)-\left(w^{*}-w\right)^{T} C^{T} \dot{w} \\
= & S^{T}(-\gamma \operatorname{sign}(S)+\widetilde{\gamma} \operatorname{sign}(S) \\
& \quad-\widetilde{\gamma} \operatorname{sign}(S)+d)-C^{T}\left(w^{*}-w\right)^{T} \dot{w} \\
\leq & -\left|S^{T}\right|\left(w^{T} \phi-w^{*^{T}} \phi+\varepsilon\right)
\end{aligned}
$$

$$
-\left|S^{T}\right|(\tilde{\gamma}-|d|)-\left|S^{T}\right|\left(w^{*}-w\right)^{T} \phi
$$

$$
=-\sum_{i=1}^{3}\left|s_{i}\right| \cdot\left(\sum_{j=1}^{6}\left(w_{i j}-w_{i j}^{*}\right) h_{j}\right)
$$



Figure 11: Angle displacement error responses controlled by TVSMC.

$$
\begin{align*}
& -\left|S^{T}\right| \varepsilon-\left|S^{T}\right|(\tilde{\gamma}-|d|) \\
& -\sum_{i=1}^{3}\left|s_{i}\right|\left(\sum_{j=1}^{6}\left(w_{i j}-w_{i j}^{*}\right) h_{j}\right) \\
= & -\left|S^{T}\right| \varepsilon-\left|S^{T}\right|(\tilde{\gamma}-|d|) \\
\leq & \left|S^{T}\right| \varepsilon_{0}-\left|S^{T}\right| \varepsilon_{1} \\
= & -\left|S^{T}\right|\left(\varepsilon_{1}-\varepsilon_{0}\right) \leq 0 . \tag{19}
\end{align*}
$$

Using integral transform in (19), the neural Lyapunov function can be gotten as

$$
\begin{align*}
V_{1}(t) & =V_{1}(0)+\int_{0}^{t} \dot{V}_{1} d \tau  \tag{20}\\
& \leq V_{1}(0)+\int_{0}^{t}-\left|S^{T}\right|\left(\varepsilon_{1}-\varepsilon_{0}\right) d \tau
\end{align*}
$$

By the Barbalat Lemma, the system is asymptotically stable.

## 5. Simulation Tests

The parameters of the 2-DOF parallel manipulator are listed as follows: joints reduction ratio is $40: 1, l_{11}=l_{12}=l_{21}=$ $l_{22}=l_{31}=l_{32}=244 \mathrm{~mm}$, back electromotive force constant is $0.04297 \mathrm{~V} /(\mathrm{s} / \mathrm{rad})$, torque constant is $3.41 \mathrm{~N} \cdot \mathrm{~m} / \mathrm{A}$,
resistance factor in driving side is $8.1 \times 10^{-5} \mathrm{~N} \cdot \mathrm{~m} /(\mathrm{rad} / \mathrm{s})$, winding resistance of electrical machine is $1.025 \Omega$, winding inductance of electrical machine is 0.03837 H , rotational inertia $J$ is $0.39 \mathrm{~kg} \cdot \mathrm{~m}^{2}$, and the uncertain inertia $\Delta J \leq$ $0.1 \mathrm{~kg} \cdot \mathrm{~m}^{2}$. The desired manipulator trajectory is the right triangle shown in Figure 5, and the desired joints trajectories are shown in Figure 6.

The trajectory tracking processes controlled by the TVSMC algorithm and the RBF TVSMC algorithm are compared in Figures 7-12 with the corresponding angle displacement in Figure 6.

According to the angle displacement responses in Figures 7 and 8, it can be seen that the maximum control voltage generated by TVSMC algorithms is 10 V , while that by RBF TVSMC algorithms is no more than 2.5 V . And the control voltage generated by the RBF TVSMC algorithms is more stable than the that by TVSMC algorithm. When we focus on the control accuracy, significant differences exist in steady-state regime of the closed-loop system as seen from the angle trajectory tracking responses in Figures 9 and 10. Figure 10 has more accurate trajectory tracking responses than Figure 9. According to angle displacement error responses of joint A1 in Figures 11 and 12, we can see that the angle error of joint stable at origin is within 40 s based on TVSMC algorithm, but based on RBF TVSMC algorithm it is not later than 3 s . The average error responses controlled by the TVSMC algorithm are about 0.3 rad , while for the RBF SMC algorithm, the average error is less than 0.03 rad , which proves the precision improvement of the RBF TVSMC algorithm.


Figure 12: Angle displacement error responses controlled by RBF TVSMC.

## 6. Conclusions

In this paper, the accurate tracking control problem of 2-DOF parallel manipulator in the presence of parameter variation and uncertain disturbance is investigated via the TVSMC technique. An effective method is provided for the parameter selection in the TVSMC framework. RBF NN based time-varying sliding mode control algorithm is proposed to address the global chattering problem and increase the control accuracy. Simulation results verify the effectiveness of the proposed algorithm.
(1) This paper provides a trajectory algorithm of RBF neural network based time-varying sliding mode control for 2-DOF parallel manipulator system. And a compensator using RBF network is proposed to estimate the actuator nonlinearities and eliminate their upper boundaries. So that an RBF neural network controller can work properly requiring neither the evaluation of off-line dynamical model nor the time-consuming training process. The other article, "The Implementation to Servomotor Based on RBF Neural Network Equivalent to Sliding Mode Variable Structure Control," provides a trajectory optimization algorithm for 2-DOF parallel manipulator system of servomotor which is
assumed as one linear system. The algorithm of input control is divided into two parts: one is the sliding mode control with the linear control and the other is the nonlinear control of output of the RBF replacing the switching input. (2) This paper is further research based on the other article, and it involves the three joints' trajectories but the other article just focuses on one joint.

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## Research Article

# Strategy of Starting Sensorless BLDCM with Inductance Method and EMF Integration 

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#### Abstract

In, conventional 3-stage start-up method of sensorless brushless direct current motor (BLDCM), the rotor is likely to jitter because rotor position cannot be obtained, and the motor is apt to lose step when it starts with load. These defects limit its use in engineering applications. In order to achieve smooth start in specific direction and guarantee start-up success rate with load, a start-up method based on improved inductance method and electromotive force (EMF) integration is proposed applying different voltage vectors according to rotor position interval judged by inductance method and determining integrator start-up time according to rotor initial position and the EMF. Experiments show that the method guarantees smooth acceleration and increases start-up success rate with load.


## 1. Introduction

BLDCM drive system has a wide range of applications in automobiles, such as the electric power steering system, the main drive system of small electric vehicle, and the new fuel pump systems of the traditional vehicle, and it is developing toward sensorless control. Comparing sensorless control with a position sensor control method, it has advantages of compact structure and high reliability because of saving installation space and reducing the sensor signal wires. So in recent years, the sensorless control of BLDCM has become a hot spot. At present, the main disadvantage limiting the application of sensorless BLDCM is that it is rather hard to start it directly. Now, a 3-stage start-up method [1] is a successful approach used in the start-up stage. The three stages are rotor preposition, acceleration, and switching. Among them, the rotor preposition mainly involves conducting the two phases with each other and forcing the rotor to rotate to the specified location. This method can be applied in cases where accuracy requirement is not high. For the sensorless BLDCM drive system used in the automobile, the size of the load cannot be predicted and the rotor is prevented from running freely at start-up. The existing start-up method cannot fit the
requirement and also cannot guarantee the start-up success rate under load $[2,3]$.

Over the years, in order to solve this problem, many start-up methods used for detecting the rotor position are proposed continually by researchers. The electrical inductance method based on stator core magnetic saturation effect is presented in [4-8], in which six short time pulses are used to locate and start up the rotor $[4,5]$. This method has advantages of simple hardware circuit design and easy controlling algorithm, whereby the rotor can be started without reversal and can be switched smoothly. The theory introduction, using sensorless BLDCM to determine the rotor position based on the inductance method, is proposed by [7]. But it does not offer the whole experiment process. In [8], the rotor position is estimated by using the change of the voltage of the nonconducting phase. This method is easy to be performed and does not need current sensors, but the precision of the position detection is not satisfied. References [ 9,10 ] use state observer and Kalman filtering to detect the motor rotor position. This method has a high degree of dependence on the motor's parameter and a huge calculated amount, which can be realized only by using high-speed DSP.

Reference [11] introduced a third harmonic detection method based on virtual neutral point electromotive force. This method can obtain accurate commutation signal and is easy to be controlled. But, the position of static rotor cannot be judged by this method, and this method can be realized only by using high speed DSP. Several other control methods are introduced separately in [12-15]. But, all of them just concern the start-up stage after rotor location and ignore the problem about the jitter and reversal of the rotor in the initial stage.

The new start-up method proposed in this paper is as follows: when the motor is static, it can be prelocated within the scope of $\pi / 6$ by using the improved inductance method and supplying power to the coil in a specified direction. In order to know the real rotor position, we should collect the EMF of the nonconducting phase on time and determine the time to start the EMF integrator. Applying the inductance method and the EMF integration to start the initial stage will prevent the motor from jittering and reversing, which results in a smooth start-up process and a rapid response.

## 2. Theory Application Analysis

EMF integration and stator core magnetic saturation effect are cited in this paper as basis of the theory.
2.1. EMF Integration. EMF integration focuses on comparing the EMF integration of the nonconducting phase with a threshold value. When the EMF integration of a phase reaches the threshold value, it is time to commutate this phase. The authors in $[1,14]$ merely provide the demonstration that the EMF integration has no relation to the rotate speed when the rotate speed is constant, but do not take into consideration the condition that the rotate speed is fluctuant in the start-up initial stage.
2.1.1. Demonstration for Independency between EMF Integration and the Rotate Speed in Start-Up Stage. Take a bipolar motor as an example; the instant EMF of the nonconducting phase can be recorded as

$$
\begin{equation*}
e_{\alpha}=N K_{w} \phi w(t) . \tag{1}
\end{equation*}
$$

In this equation, $e_{\alpha}$ is the instant EMF, $N$ is the number of turns of phase winding, $K_{w}$ is the distribution coefficient, $\phi$ is the instant air gap flux, and $\omega(t)$ is the electric angular speed.

At the beginning of the start-up, air gap flux can be approximated as trapezoidal wave. Supposing that the parameter of the motor is constant, the air gap flux merely relates to the angle. Using the $\pi / 6$ electrical angle after EMF zerocrossing point can decide the commutation moment, and the air gap flux reaches the maximum at this moment. Air gap flux can be written as

$$
\begin{equation*}
\phi=\phi_{p} \frac{6 \theta(t)}{\pi} . \tag{2}
\end{equation*}
$$

In this equation, $\theta(t)$ is the rotor angle and $\phi_{p}$ is the flux amplitude of each pole. The EMF integrator begins to work
at the moment the EMF crosses zero. The integrator's output voltage is

$$
\begin{equation*}
U=\int_{0}^{t} e_{\alpha} d t \tag{3}
\end{equation*}
$$

It can be deduced from (1)-(3) that

$$
\begin{equation*}
U=\int_{0}^{t} e_{\alpha} d t=\int_{0}^{t} N K_{w} \phi_{p} \frac{6 \theta(t)}{\pi} w(t) d t \tag{4}
\end{equation*}
$$

Based on the timely change of the motor's angle, the angular speed can be written as

$$
\begin{equation*}
w(t)=\frac{d \theta(t)}{d t} \tag{5}
\end{equation*}
$$

If $K_{e}=(6 / \pi) N K_{w} \phi_{p}$, it can be deduced from (4)-(5) that

$$
\begin{gather*}
U=\int_{0}^{t} K_{e} \theta(t) \frac{d \theta(t)}{d t} d t \\
U=\int_{0}^{\pi / 6} K_{e} \theta(t) d \theta(t)  \tag{6}\\
U=U_{0}=K_{e} \frac{\pi^{2}}{72} .
\end{gather*}
$$

In these equations, $U$ is the integrator's output voltage, $U_{0}$ is the threshold value, and $K_{e}$ is the EMF coefficient. When integrator's output voltage $U$ reaches the threshold value $U_{0}$, the integrator should stop the integration at once and outputs the commutation signal.
2.1.2. Significance of Applying EMF Integration in the StartUp Stage. From the demonstration in Section 2.1.1, it is clear that the value of EMF integration is independent of the rotate speed at the beginning of the start-up. For different motors, the time of commutation can be adjusted by changing the threshold value.

The traditional method needs to set the frequency of commutation before starting. The frequency is decided by the experiment method with the purpose that rotor can reestablish moment balance again when it is disturbed. Because this time cannot be changed in the start-up initial stage, the motor is likely to have commutation error or even start-up failure. Because the value of EMF integration is independent of the rotate speed, the time that reaches one integration value will change according to different loads and constant commutation frequency will also change with it. In this way, the motor can be started smoothly in certain range of load.

In the start-up initial stage, applying EMF integration will also face the condition of inaccurate detection of zerocrossing point, but this will not lead to a serious error in commutation because, in the EMF integration, the result of integration on numerical value is equal to the area that is included by the EMF wave in corresponding integrating range and time axis. From the integration principle and the EMF wave we can see that commutation error will be far less than the zero-crossing point detection error. Taking the EMF rising section of one phase as an example, as shown


Figure 1: Analysis of the error in the method of integrating EMF.
in Figure 1, zero-crossing point detection error is $t_{1}$ and commutation error is $t_{2}$; if we use the scheme of commutating at $\pi / 6$ after EMF zero-crossing point, then $t_{1}=t_{2}$. By contrast, because the integration threshold value is a constant value when adopting EMF integration to commutate, then

$$
\begin{gather*}
S_{1}=S_{2} \\
S_{1}=\frac{1}{2} t_{1} \frac{t_{1}}{t_{\text {inte }}} E_{\max }  \tag{7}\\
S_{2}=t_{2} E_{\max } \\
t_{2}=\frac{t_{1}}{2 t_{\text {inte }}} t_{1} .
\end{gather*}
$$

In these equations, $t_{1}$ is the zero-crossing point detection error, $t_{2}$ is the commutation error, $t_{\text {inte }}$ is the perfect integration section, $E_{\max }$ is the maximum EMF, $S_{1}$ is a zero error area, and $S_{2}$ is the commutation delay area.

In the actual operation, $t_{1}$ is far less than $t_{\text {inte }}$ and, as a result, $t_{2}$ is far less than $t_{1}$, which means that zero-crossing point detection error has little effect to the commutation point error.
2.2. Inductance Method Analysis. Using the inductance method to estimate the rotor position is based on the characteristic of the stator core magnetic saturation. In terms of the BLDCM, the flux produced by the stator winding and the permanent magnet flux affect the saturation degree of the stator core together. If the stator core is closer to the pole of the permanent magnet, the magnetization will be strengthened. If the combined flux has an effect to add magnetic force, the more saturate the magnetic field is, the less the winding inductance is and the larger the current is. If the combined flux has an effect to reduce magnetic force, the saturation degree of the magnetic field becomes less, the winding inductance gets larger, and the current becomes smaller.

In the inductance method, the time of duration of the voltage vector is an important parameter, because this method is exactly estimated based on the current response of stator winding. The current response relies on the time constant of the stator winding, and time constant $\tau$ can be


Figure 2: Current response curve.
expressed as $\tau=L / R$. From the equation, time constant is proportional to the inductance.

Figure 2 shows the response curve of current $i_{1}$ and $i_{2}$ when the motor rotor is in different positions, and $i_{1}$ and $i_{2}$ can be calculated as follows:

$$
\begin{align*}
& i_{1}(t)=\frac{V_{\mathrm{DC}}}{R_{\mathrm{eq}}}\left(1-e^{-\left(t / \tau_{1}\right)}\right) \\
& i_{2}(t)=\frac{V_{\mathrm{DC}}}{R_{\mathrm{eq}}}\left(1-e^{-\left(t / \tau_{2}\right)}\right) . \tag{8}
\end{align*}
$$

From these equations, $V_{\mathrm{DC}}$ is the DC side voltage, $R_{\text {eq }}$ is the equivalent resistance, and $\tau_{1}$ and $\tau_{2}$ are the time constants of equivalent circuit when the voltage is applied. As we can see, the bigger the time constant is, the smaller the current is. Because the time constant is small, the conduction time should be the one when the difference of current is the most obvious under a different voltage vector.

The current differential $\Delta_{i}$ of $i_{1}$ and $i_{2}$ can be calculated as follows in this way:

$$
\begin{equation*}
\Delta i(t)=i_{2}(t)-i_{1}(t)=\frac{V_{\mathrm{DC}}}{R_{\mathrm{eq}}}\left(e^{-\left(t / \tau_{1}\right)}-e^{-\left(t / \tau_{2}\right)}\right) . \tag{9}
\end{equation*}
$$

Because the time constant of the motor is small, it can be expressed as $\tau_{2}=\tau_{1}+\Delta, \Delta \approx 0$. Take the derivative of (9) with respect to time, and the point when the difference of the current is the biggest can be found. In terms of each voltage vector, conduction time can be expressed as

$$
\begin{equation*}
T_{s}=\lim _{\Delta \rightarrow 0} \frac{\ln \left(\tau_{1} /\left(\tau_{1}+\Delta\right)\right)}{\left(1 /\left(\tau_{1}+\Delta\right)\right)-\left(1 / \tau_{1}\right)}=\tau_{1} \tag{10}
\end{equation*}
$$

From the above, we can see that the preferable conduction time of voltage vector should be the numerical value similar to the time constant of the motor. But, if the time constant of the motor is too big, it can bring about a long conduction time that will force the motor to run. So, reasonable conduction time should be chosen to ensure the accuracy of preposition.

## 3. New Method for Start-Up

The inductance method can detect the rotor position in advance when the motor is static. As power is supplied to
calculated phases, this method can avoid the consequence of reversal and jitter which can affect the start-up accuracy of the motor. EMF integration can reflect the real angle of the rotor and help the motor start successfully with load. This paper is based on the two advantages above. Firstly, judge the rotor position based on inductance method and determine the integrator start-up time according to the rotor position, and then adjust the commutation frequency to smooth the startup process and eliminate jitter. By using this method, rapid response can also be achieved. Due to the simple arithmetic, those functions can all be realized by common MCU.
3.1. Identify the Initial Rotor Position. Conventional inductance method adopts two-to-two or three-to-three conducting methods, applying six-voltage vector in different direction continually and locating the rotor position in the scope of $\pi / 3$. When using the traditional method, there are two problems: one is that the rotor position cannot be determined precisely, the other is that applying six-voltage vectors at the same time will make the arbitration rules complicated and the programming code lengthy. The improved inductance method is adopted in this paper, which can determine the rotor position in the scope of $\pi / 6$ easily and has simple arbitration rules. It can be linked up with the EMF integration when it starts. The advantage of EMF integration will be more obvious in this way. Sampling quantity and calculated amount decrease and stability increases, which will make the improved effect of start-up more obvious.

The improved inductance method proposed in this paper is as follows: when the motor is static, utilize two-to-two conducting method to apply voltage pulse to the MOSFET. Assuming that the rotor position is shown in Figure 3, firstly apply two-voltage vectors $V_{\mathrm{AB}}$ (direction $C$ ) and $V_{\mathrm{BA}}$ (direction $Z$ ) in the opposite direction for a while. $I_{2}$ and $I_{5}$ are collected and stored as bus current separately. Because the magnetic saturation of the iron core is different, the sizes of $I_{2}$ and $I_{5}$ are also different. If $I_{2}>I_{5}$, from the analysis above, it can be known that the $N$ pole of the rotor is on the left side of line I-II. The rotor is located in the scope of $\pi$ for the first time. Then apply two-voltage vectors $V_{\mathrm{CB}}$ (direction $X$ ) and $V_{\mathrm{AC}}$ (direction $Y$ ) for a while and collect bus currents which can be stored as $I_{3}$ and $I_{1}$.

According to the arbitration rules in this paper, the rotor area can be judged in different conditions. The arbitration rules are as follows: if $I_{2}-I_{1}>0, a=1$ or $a=0$. If $I_{2}-I_{3}>0$, $b=1$ or $b=0$. Bring the results into the arbitration formula $y=2 a+b$. If $y=1$, the rotor is in area 1. If $y=3$, the rotor is in area 2. If $y=2$, the rotor is in area 3. The rotor position in the right side of I-II can be judged in the same way. In this way, the rotor position can be determined in the range of $\pi / 3$.

According to the assumption above, the rotor should be in area 1 . According to the result of location, apply $V_{\mathrm{BC}}$ again and collect bus current and store it as $I_{6}$. Now, there are three positions that the rotor may be in: $Y$ coordinate or a position shift from the $Y$ coordinate by a distance within the range of $\pi / 6$. The method in this paper is treated as the same condition of the rotor in $Y$ coordinate or above the $Y$ coordinate within the range $\pi / 6$. Comparing $I_{6}$ with $I_{2}$, the rotor position can be determined in the scope of $\pi / 6$. According to the assumption


Figure 3: The stator magnetomotive force (MMF) and rotor position.


Figure 4: Rotor position.
above, the result now should be $I_{2}<I_{6}$ and the $N$ pole of the rotor should be above the $Y$ coordinate within the range of $\pi / 6$. The judgment of the position of other rotor is similar.
3.2. EMF Integration Start-Up. According to the above, the rotor position is already determined in the range of $\pi / 6$ and the voltage vector can be applied to the rotor according to the torque maximized and predefined rotate direction. Two conditions will come across when the voltage vector is applied: the first is the EMF of nonconducting phase that has already crossed zero, the second is that EMF of nonconducting phase that has not crossed zero.

If the rotor position is in the shadow area in Figure 4(a) now, the power is supplied towards $Z$ direction, and the rotor position has not crossed zero, so the EMF of $C$ phase should be collected. When the voltage vector is $V_{\mathrm{BA}}$, the initial EMF of $C$ phase is minus. The electrical direction should be constant at this moment until the $C$ phase EMF appears zerocrossing point. This zero-crossing point is the common commutation zero-crossing point. Because of the undetermined load, in order to avoid the condition where commutation is too rapid and the rotor rotates in reverse with jitter because of the heavy load, the integrator should be started at the moment when the zero-crossing point is detected. The result of the integration is compared with the integration threshold value on time. If it overflows, the motor will be commutated at once and proceed into the acceleration process.

If the rotor position is in the shadow area in Figure 4(b) now and the power is supplied in A direction by utilizing two-to-two conducting method, now the rotor position is


Figure 5: Test bench.
between the zero-crossing point and commutation point. The integrator should be started with the initiator. The result of integration is compared with the integration threshold value on time. If it overflows, the motor will be commutated at once and proceed into the acceleration process.

The commutation error is smaller than the integration initial error, which has been demonstrated as above. As a result of it, the accuracy of zero-crossing point detection should not be worried about. When the integrator is detected to overflow and commutate, the frequency of energization and the speed of the rotor are in synchronization. The motor begins to be accelerated smoothly.

## 4. Experimental Verification

The 32-bit DSP 28335 is selected as the core to build an experimental platform to demonstrate the superiority of the start-up method proposed in this paper. The experimental platform is shown in Figure 5. The rated speed of the motor is $5000 \mathrm{r} / \mathrm{min}$. The experiment is divided into two steps: one is adding inertia load by adopting inertia disc, the other is adding static load on the load test bed.
4.1. Start-Up Method Application Example. In the scope of $\pi / 6$ above $Y$ coordinate, the current waveform and terminal voltage waveform of the rotor detected by the method proposed in this paper are shown in Figure 6.

As shown in the Figure 6(a), $I_{2}>I_{5}, I_{2}-I_{3}>0, I_{2}-I_{1}<0$, we can know that the $N$ pole of the rotor is in the location 1 , according to the arbitration rules proposed in this paper. Because $I_{6}>I_{2}$, the rotor should be above the $Y$ coordinate in the range of $\pi / 6$. After applying the voltage vector, the integrator is started after C EMF crosses zero. A terminal voltage waveform can be obtained as shown in Figure 6(b).

The current waveform of the rotor in the scope of $\pi / 6$ below $Y$ coordinate obtained by the method of this paper is shown in Figure 7, and the start-up terminal voltage waveform on the load test bed without load at this moment is also shown in it.

As shown in the Figure 7(a), $I_{2}>I_{5}, I_{2}-I_{3}>0, I_{2}-I_{1}<0$, we can know that the $N$ pole of the rotor is in the location 1 , according to the arbitration rules proposed in this paper. And, because $I_{6}<I_{2}$, the rotor should be in the range of $\pi / 6$ below the $Y$ coordinate. Start the integrator when applying


Figure 6: End voltage waveform and currents response under EMF non-zero crossing.
the voltage vector $V_{\mathrm{BC}}$, and terminal voltage waveform can be obtained as shown in Figure 7(b).

The rotor position can be known in advance by utilizing method proposed in this paper. The method can ensure that the start-up process has no reversal, no jitter, and commutation signal, and rotor synchronized quickly to get a better end voltage waveform when it starts.
4.2. Contrast Test. The advantage of the method proposed in this paper is that the rotor position can be determined in advance and a different start-up program can be executed according to the rotor position. Some problems such as jitter and reversal of the motor are avoided. The application of integration can also make the motor achieve a nice startup effect with heavy or light load. The applied range of the motor becomes wider and its start-up process becomes better. Compared with the start-up waveform obtained by the traditional method with different load, the advantage of the method proposed in this paper can be demonstrated.

The experimental results under inertia disk with the traditional method and the method proposed in this paper are shown in Figure 8. Comparing the two figures, it can be seen that when adding the inertia load, a severe jitter appears in the start-up stage using the traditional method and the smooth start-up of the motor is not realized in the commutation period displayed by the figure. The method of this paper determines the rotor position in the first period and it starts according to the actual load condition, which can make the start-up process more smooth without jitter.

Terminal voltage waveform with no load, rated torque, and twice the rated torque experimented on the load test bed


Figure 7: Terminal voltage waveform and currents response under EMF zero crossing.


Figure 8: End voltage waveform with two different methods under inertia disk.


Figure 9: Terminal voltage waveform with the traditional method and the method proposed in this paper under load.
with the traditional method and the method of this paper are shown in Figures 9, 10, and 11.

From the comparison of the three figures above, the motor performance that use traditional method has serious fluctuation and bad waveform both in no-load, fractional load, and overload. When controlling with the method proposed in this paper, the commutation signal and the rotor can be synchronized in the first commutation period, whether with fractional load or overload. And, the fluctuation is small and the start-up is smooth. From the figures above, the integration time changes with the load can be easily found.

Current waveforms obtained by two different start-up methods of the motor are shown in Figure 12.

As can be seen from the results above, the method proposed in this paper has an obvious advantage over the traditional method. Firstly, two sets of waveforms of the current and phase terminal voltage applying the method of this paper are shown in the example, while the rotor position in the beginning of the start is different. It shows that the inductance method can detect the position of the rotor fast and accurately and then judge when to apply the EMF. This can ensure that the process starts smoothly and steady, and also ensure that the motor commutes precisely. Experiments of the different rotor position in the start have been carried out to verify the reliability of this method. The results indicate that the method is credible. In the following contrast tests, the motor is started under different load. The comparison waveforms of phase terminal voltage show clearly that this method is superior


Figure 10: The end voltage waveform with the traditional method and the method proposed in this paper with rated torque.


Figure 11: End voltage waveform with traditional method and method proposed in this paper under twice the rated torque.


Figure 12: Comparison of the current of two different methods.
to the traditional method and the starting process is smooth and fast. The results demonstrate that this method can guarantee the startability when the motor is under different load. Due to the improved performance, this method can expand the application range of sensorless BLDC. At last, two sets of current waveform are given to show the improvement of this method. It can be seen from the figure, that this method can make the current achieve steadily fast and smoothly without distinct jitter. The comparison of the current waveform indicates that this method can alleviate the phenomenon of jitter in the start and smooth the starting process.

## 5. Conclusion

The advantages of the inductance method and the EMF integration are analyzed in detail in this paper. And, after combining these two methods, a new start-up method for sensorless BLDCM is proposed. The integrator start-up time is determined by implementing the improved inductance method and the new arbitration rules. The experiment demonstrates the advantages of the new start-up method proposed in this paper compared to the traditional one in start-up stage. The location of the rotor when the motor is static is realized, and the commutation signal and the rotor position can be synchronized at once. It increases the startup success rate with load and improves rapidity and stability of the current response.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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[^0]:    (i) Pump1 and Pump2 never operate simultaneously: P1. $P 2=0 ;$
    (ii) If Pump1 operates, Pump2 cannot operate: $P 1 \leq \overline{P 2}$;

[^1]:    Note. The numbers in brackets are the mode order of the MESC model.

