Abstract and Applied Analysis

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Editorial Board

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Editorial

Recent Results on Fixed Point Approximations and Applications

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The aim of this special issue is to promote research and its applications in the area of nonlinear functional analysis and applications. It will reflect theoretical research and advanced applications. One of the most important and significant areas is fixed point theory being very rich, interesting, and extremely applicable area of mathematics and mathematical sciences.

In the last three decades, the problems of nonlinear analysis with its relation to fixed point theory have emerged as a rapidly growing area of research because of its applications in differential equation, KKM theory, nonlinear ergodic theory, game theory, optimization problem, control theory, and so on. Also, the iterative methods for finding the approximate solutions of fixed point problems, variational inequality problems, equilibrium problems, optimization problems, split feasibility problems, operator equations and inclusion problems, amenability of semigroup, and convergence of iterative approximations are very important and useful.

Acknowledgments

As guest editors for this special issue, we wish to thank all those who submitted manuscripts for publication and many mathematicians who served as the reviewers. We hope that all the papers which are published in this special issue can be motivated for the development of research works of the researchers.

Jong Kyu Kim
Poom Kumam
Xiaolong Qin
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Research Article

Approximating Iterations for Nonexpansive and Maximal Monotone Operators

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We present two algorithms for finding a zero of the sum of two monotone operators and a fixed point of a nonexpansive operator in Hilbert spaces. We show that these two algorithms converge strongly to the minimum norm common element of the zero of the sum of two monotone operators and the fixed point of a nonexpansive operator.

1. Introduction

Throughout, we assume that $\mathscr{H}$ is a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. Let $\mathscr{C}\subset\mathscr{H}$ be a nonempty closed convex set.

Definition 1. An operator $\mathbb{S}:\mathscr{C}\rightarrow\mathscr{C}$ is said to be *nonexpansive* if

$$\|\mathbb{S}u-\mathbb{S}v\|\leq\|u-v\| \tag{1}$$

for all $u,v\in\mathscr{C}$.

We denote by $\mathrm{Fix}(\mathbb{S})$ the set of fixed points of $\mathbb{S}$.

Definition 2. An operator $\mathbb{A}:\mathscr{C}\rightarrow\mathscr{H}$ is said to be ξ*-inverse strong monotone* if

$$\langle\mathbb{A}u-\mathbb{A}v,u-v\rangle\geq\xi\|\mathbb{A}u-\mathbb{A}v\|^2 \tag{2}$$

for some $\xi>0$ and for all $u,v\in\mathscr{C}$.

It is known that if $\mathbb{A}$ is ξ-inverse strong monotone, then $\mathbb{A}$ is $1/\xi$-lipschitz, that is,

$$\|\mathbb{A}u-\mathbb{A}v\|\leq\frac{1}{\xi}\|u-v\|, \tag{3}$$

for all $u,v\in\mathscr{C}$. Furthermore,

$$\begin{aligned}&\|(I-\delta\mathbb{A})u-(I-\delta\mathbb{A})v\|^2\\&\quad\leq\|u-v\|^2+\delta(\delta-2\xi)\|\mathbb{A}u-\mathbb{A}v\|^2,\quad\forall u,v\in\mathscr{C}.\end{aligned} \tag{4}$$

In particular, if $\delta\in(0,2\xi)$, then $I-\delta\mathbb{A}$ is nonexpansive.

Let $\mathbb{B}:\mathscr{H}\rightarrow 2^{\mathscr{H}}$ be a set-valued operator. The effective domain of $\mathbb{B}$ is denoted by $\mathrm{dom}(\mathbb{B})$, that is, $\mathrm{dom}(\mathbb{B})=\{x\in\mathscr{H}:\mathbb{B}x\neq\emptyset\}$.

Definition 3. A multivalued operator $\mathbb{B}$ is said to be a *monotone* on $\mathscr{H}$ if and only if

$$\langle x-y,u-v\rangle\geq 0 \tag{5}$$

for all $x,y\in\mathrm{dom}(\mathbb{B})$, $u\in\mathbb{B}x$, and $v\in\mathbb{B}y$.

A monotone operator $\mathbb{B}$ on $\mathscr{H}$ is said to be *maximal* if and only if its graph is not strictly contained in the graph of any other monotone operator on $\mathscr{H}$. We denote by $\mathbb{B}^{-1}0$ the set of the zero points of $\mathbb{B}$, that is, $\mathbb{B}^{-1}0=\{x\in\mathscr{H}:0\in\mathbb{B}x\}$.

For $\lambda>0$, we define a single-valued operator

$$J_\lambda^{\mathbb{B}}=(I+\lambda\mathbb{B})^{-1}:\mathscr{H}\longrightarrow\mathrm{dom}(\mathbb{B}), \tag{6}$$

which is called the resolvent of $\mathbb{B}$ for λ. It is known that the resolvent $J_\lambda^{\mathbb{B}}$ is firmly nonexpansive, that is,

$$\left\| J_\lambda^{\mathbb{B}} u - J_\lambda^{\mathbb{B}} v \right\|^2 \leq \left\langle J_\lambda^{\mathbb{B}} u - J_\lambda^{\mathbb{B}} v, u - v \right\rangle, \tag{7}$$

for all $u, v \in \mathscr{C}$ and $\mathbb{B}^{-1}0 = \mathrm{Fix}(J_\lambda^{\mathbb{B}})$ for all $\lambda > 0$.

In the present paper, we consider the variational inclusion of finding a zero $x \in \mathscr{H}$ of the sum of two monotone operators $\mathbb{A}$ and $\mathbb{B}$ such that

$$0 \in \mathbb{A}(x) + \mathbb{B}(x), \tag{8}$$

where $\mathbb{A} : \mathscr{H} \to \mathscr{H}$ is a single-valued operator and $\mathbb{B} : \mathscr{H} \to 2^{\mathscr{H}}$ is a set-valued operator. The set of solutions of problem (8) is denoted by $(\mathbb{A} + \mathbb{B})^{-1}(0)$.

Special Cases. (i) If $\mathscr{H} = \mathbb{R}^m$, then problem (8) becomes the generalized equation introduced by Robinson [1].

(ii) If $\mathbb{A} = 0$, then problem (8) becomes the inclusion problem introduced by Rockafellar [2].

It is known that (8) provides a convenient framework for the unified study of optimal solutions in many optimization related areas including mathematical programming, complementarity, variational inequalities, optimal control, mathematical economics, equilibria, and game theory. Also various types of variational inclusions problems have been extended and generalized. For related work, please see [3–20].

Zhang et al. [21] introduced the following iterative algorithm for finding a common element of the set of solutions to the problem (8) and the set of fixed points of a nonexpansive operator:

$$x_{n+1} = \alpha_n x_0 + (1 - \alpha_n) \mathbb{S} J_\lambda^{\mathbb{B}} (x_n - \lambda \mathbb{A} x_n), \tag{9}$$

where $\mathbb{S} : \mathscr{C} \to \mathscr{C}$ is a nonexpansive operator. Under some mild conditions, they prove that the sequence $\{x_n\}$ converges strongly to $x^* \in \mathrm{Fix}(\mathbb{S}) \cap (\mathbb{A} + \mathbb{B})^{-1}(0)$.

Recently, Takahashi et al. [22] introduced another iterative algorithm for finding a zero of the sum of two monotone operators and a fixed point of a nonexpansive operator

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) \mathbb{S} \left(\alpha_n x_0 + (1 - \alpha_n) J_{\lambda_n}^{\mathbb{B}} (x_n - \lambda_n \mathbb{A} x_n) \right) \tag{10}$$

for all $n \geq 0$. Under some assumptions, they proved that the sequence $\{x_n\}$ converges strongly to a point of $\mathrm{Fix}(\mathbb{S}) \cap (\mathbb{A} + \mathbb{B})^{-1}(0)$.

Motivated and inspired by (9) and (10), in the present paper, we suggest two algorithms

$$x_t = J_\lambda^{\mathbb{B}} \left((1 - t) \mathbb{S} x_t - \lambda \mathbb{A} \mathbb{S} x_t \right), \quad t \in (0, 1), \tag{11}$$

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) J_{\lambda_n}^{\mathbb{B}} \left((1 - \alpha_n) \mathbb{S} x_n - \lambda_n \mathbb{A} \mathbb{S} x_n \right), \quad n \geq 0. \tag{12}$$

It is obvious that (12) is very different from (9) and (10). Furthermore, we prove that both (11) and (12) converge strongly to the minimum norm element in $\mathrm{Fix}(\mathbb{S}) \cap (\mathbb{A} + \mathbb{B})^{-1}0$. It should be pointed out that we do not use the metric projection in (11) and (12).

2. Lemmas

In this section, we collect several useful lemmas for our next section.

First, the following resolvent equality is well known.

Lemma 4. *For $\lambda > 0$ and $\lambda^\dagger > 0$, one has*

$$J_\lambda^{\mathbb{B}} u = J_{\lambda^\dagger}^{\mathbb{B}} \left(\frac{\lambda^\dagger}{\lambda} u + \left(1 - \frac{\lambda^\dagger}{\lambda} \right) J_\lambda^{\mathbb{B}} u \right), \quad \forall u \in \mathscr{H}. \tag{13}$$

Lemma 5 (see [23]). *Let $\mathscr{C} \subset \mathscr{H}$ be a closed convex set. Let $\mathbb{S} : \mathscr{C} \to \mathscr{C}$ be a nonexpansive operator. Then* $\mathrm{Fix}(\mathbb{S})$ *is a closed convex subset of $\mathscr{C}$ and the operator $I - \mathbb{S}$ is demiclosed at* 0.

Lemma 6 (see [24]). *Let $\mathscr{X}$ be a Banach space. Let $\{u_n\} \subset \mathscr{X}$ and $\{v_n\} \subset \mathscr{X}$ be two bounded sequences. Let the sequence $\{\zeta_n\} \subset (0, 1)$ satisfy $0 < \liminf_{n\to\infty} \zeta_n \leq \limsup_{n\to\infty} \zeta_n < 1$. Suppose $u_{n+1} = (1 - \zeta_n) v_n + \zeta_n u_n$ for all $n \geq 0$ and $\limsup_{n\to\infty} (\|v_{n+1} - v_n\| - \|u_{n+1} - u_n\|) \leq 0$. Then $\lim_{n\to\infty} \|u_n - v_n\| = 0$.*

Lemma 7 (see [25]). *Let $\{\sigma_n\} \subset [0, \infty)$, $\{\gamma_n\} \subset (0, 1)$, and $\{\delta_n\} \subset \mathbb{R}$ be three sequences satisfying*

$$\sigma_{n+1} \leq (1 - \gamma_n) \sigma_n + \delta_n \gamma_n. \tag{14}$$

If $\sum_{n=1}^\infty \gamma_n = \infty$ and $\limsup_{n\to\infty} \delta_n \leq 0$ (or $\sum_{n=1}^\infty |\delta_n \gamma_n| < \infty$), then $\lim_{n\to\infty} \sigma_n = 0$.

3. Strong Convergence Results

Let $\mathscr{C} \subset \mathscr{H}$ be a nonempty closed convex set. Let $\mathbb{A} : \mathscr{C} \to \mathscr{H}$ be a ϱ-inverse strong monotone operator. Let $\mathbb{B}$ be a maximal monotone operator on $\mathscr{H}$ such that $\mathrm{dom}(\mathbb{B}) \subset \mathscr{C}$. Let $\mathbb{S} : \mathscr{C} \to \mathscr{C}$ be a nonexpansive operator.

Pick up a constant $\tau \in (0, 2\varrho)$. For any $t \in (0, (2\varrho - \tau)/2\varrho)$, we define an operator

$$\psi(x) = J_\tau^{\mathbb{B}} \left((1 - t) \mathbb{S} - \tau \mathbb{A} \mathbb{S} \right) x, \tag{15}$$

for all $x \in \mathscr{C}$.

Since $J_\tau^{\mathbb{B}}$, $\mathbb{S}$, and $I - \tau\mathbb{A}/(1 - t)$ are nonexpansive, we have

$$\begin{aligned} \|\psi(x) - \psi(y)\| &= \Big\| J_\tau^{\mathbb{B}} \left((1 - t) \left(I - \frac{\tau}{1 - t} \mathbb{A} \right) \mathbb{S} x \right) \\ &\quad - J_\tau^{\mathbb{B}} \left((1 - t) \left(I - \frac{\tau}{1 - t} \mathbb{A} \right) \mathbb{S} y \right) \Big\| \\ &\leq (1 - t) \Big\| \left(I - \frac{\tau}{1 - t} \mathbb{A} \right) \mathbb{S} x \\ &\quad - \left(I - \frac{\tau}{1 - t} \mathbb{A} \right) \mathbb{S} y \Big\| \\ &\leq (1 - t) \|x - y\|, \end{aligned} \tag{16}$$

for any $x, y \in \mathscr{C}$. Hence ψ is a contraction on $\mathscr{C}$. We use x_t to denote the unique fixed point of ψ in $\mathscr{C}$. Thus, $\{x_t\}$ satisfies the fixed point equation

$$x_t = J_\tau^{\mathbb{B}} \left((1 - t) \mathbb{S} x_t - \tau \mathbb{A} \mathbb{S} x_t \right). \tag{17}$$

Next, we give the convergence analysis of (17).

Theorem 8. *Assume that* $\mathrm{Fix}(\mathbb{S}) \cap (\mathbb{A}+\mathbb{B})^{-1}0 \neq \emptyset$. *Then* $\{x_t\}$ *defined by* (17) *converges strongly, as* $t \to 0+$, *to the minimum norm element in* $\mathrm{Fix}(\mathbb{S}) \cap (\mathbb{A}+\mathbb{B})^{-1}(0)$.

Proof. Choose any $z \in \mathrm{Fix}(\mathbb{S}) \cap (\mathbb{A}+\mathbb{B})^{-1}(0)$. It is obvious that $z = \mathbb{S}z = J_\tau^{\mathbb{B}}(z - \tau\mathbb{A}z)$ for all $\tau > 0$. So, we have

$$z = \mathbb{S}z = J_\tau^{\mathbb{B}}(z - \tau\mathbb{A}z) = J_\tau^{\mathbb{B}}\left(tz + (1-t)\left(I - \frac{\tau}{1-t}\mathbb{A}\right)\mathbb{S}z\right) \tag{18}$$

for all $t \in (0,1)$.

From (17), we have

$$\begin{aligned}
\|x_t - z\| &= \left\| J_\tau^{\mathbb{B}}\left((1-t)\left(I - \frac{\tau}{1-t}\mathbb{A}\right)\mathbb{S}x_t\right) - z\right\| \\
&= \left\| J_\tau^{\mathbb{B}}\left((1-t)\left(\mathbb{S}x_t - \frac{\tau}{1-t}\mathbb{A}\mathbb{S}x_t\right)\right)\right. \\
&\quad \left. - J_\tau^{\mathbb{B}}\left(tz + (1-t)\left(\mathbb{S}z - \frac{\tau}{1-t}\mathbb{A}\mathbb{S}z\right)\right)\right\| \\
&\leq \left\|(1-t)\left(\mathbb{S}x_t - \frac{\tau}{1-t}\mathbb{A}\mathbb{S}x_t\right)\right. \\
&\quad \left. - \left(tz + (1-t)\left(\mathbb{S}z - \frac{\tau}{1-t}\mathbb{A}\mathbb{S}z\right)\right)\right\| \\
&= \left\|(1-t)\left(\left(\mathbb{S}x_t - \frac{\tau}{1-t}\mathbb{A}\mathbb{S}x_t\right)\right.\right. \\
&\quad \left.\left. - \left(\mathbb{S}z - \frac{\tau}{1-t}\mathbb{A}\mathbb{S}z\right)\right) - tz\right\| \\
&\leq (1-t)\left\|\left(I - \frac{\tau}{1-t}\mathbb{A}\right)\mathbb{S}x_t - \left(I - \frac{\tau}{1-t}\mathbb{A}\right)\mathbb{S}z\right\| \\
&\quad + t\|z\| \\
&\leq (1-t)\|x_t - z\| + t\|z\|.
\end{aligned} \tag{19}$$

Hence, we get

$$\|x_t - z\| \leq \|z\|. \tag{20}$$

Thus, $\{x_t\}$ is bounded.

By (4) and (19), we derive

$$\begin{aligned}
\|x_t - z\|^2 &\leq \left\|(1-t)\left(\left(\mathbb{S}x_t - \frac{\tau}{1-t}\mathbb{A}\mathbb{S}x_t\right)\right.\right. \\
&\quad \left.\left. - \left(\mathbb{S}z - \frac{\tau}{1-t}\mathbb{A}\mathbb{S}z\right)\right) + t(-z)\right\|^2 \\
&\leq (1-t)\left\|\left(\mathbb{S}x_t - \frac{\tau}{1-t}\mathbb{A}\mathbb{S}x_t\right)\right. \\
&\quad \left. - \left(\mathbb{S}z - \frac{\tau}{1-t}\mathbb{A}\mathbb{S}z\right)\right\|^2 + t\|z\|^2 \\
&= (1-t)\left\|(\mathbb{S}x_t - \mathbb{S}z) - \frac{\tau}{1-t}(\mathbb{A}\mathbb{S}x_t - \mathbb{A}\mathbb{S}z)\right\|^2 \\
&\quad + t\|z\|^2 \\
&= (1-t)\left(\|\mathbb{S}x_t - \mathbb{S}z\|^2 - \frac{2\tau}{1-t}\right. \\
&\quad \times \langle \mathbb{A}\mathbb{S}x_t - \mathbb{A}\mathbb{S}z, \mathbb{S}x_t - \mathbb{S}z\rangle \\
&\quad \left. + \frac{\tau^2}{(1-t)^2}\|\mathbb{A}\mathbb{S}x_t - \mathbb{A}\mathbb{S}z\|^2\right) + t\|z\|^2 \\
&\leq (1-t)\left(\|\mathbb{S}x_t - \mathbb{S}z\|^2 - \frac{2\varrho\tau}{1-t}\|\mathbb{A}\mathbb{S}x_t - \mathbb{A}\mathbb{S}z\|^2\right. \\
&\quad \left. + \frac{\tau^2}{(1-t)^2}\|\mathbb{A}\mathbb{S}x_t - \mathbb{A}\mathbb{S}z\|^2\right) + t\|z\|^2 \\
&= (1-t)\left(\|\mathbb{S}x_t - \mathbb{S}z\|^2 + \frac{\tau}{(1-t)^2}(\tau - 2(1-t)\varrho)\right. \\
&\quad \left. \times \|\mathbb{A}\mathbb{S}x_t - \mathbb{A}\mathbb{S}z\|^2\right) + t\|z\|^2 \\
&\leq (1-t)\|x_t - z\|^2 + \frac{\tau}{1-t}(\tau - 2(1-t)\varrho) \\
&\quad \times \|\mathbb{A}\mathbb{S}x_t - \mathbb{A}\mathbb{S}z\|^2 + t\|z\|^2.
\end{aligned} \tag{21}$$

So,

$$\begin{aligned}
&\frac{\tau}{1-t}(2(1-t)\varrho - \tau)\|\mathbb{A}\mathbb{S}x_t - \mathbb{A}z\|^2 \\
&\quad \leq t\|z\|^2 - t\|x_t - z\|^2 \longrightarrow 0.
\end{aligned} \tag{22}$$

Since $2(1-t)\varrho - \tau > 0$ for all $t \in (0, 1 - \tau/2\varrho)$, we obtain

$$\lim_{t \to 0+}\|\mathbb{A}\mathbb{S}x_t - \mathbb{A}z\| = 0. \tag{23}$$

Using the firm nonexpansivity of $J_\tau^{\mathbb{B}}$, we have

$$\begin{aligned}
\|x_t - z\|^2 &= \left\|J_\tau^{\mathbb{B}}((1-t)\mathbb{S}x_t - \tau\mathbb{A}\mathbb{S}x_t) - z\right\|^2 \\
&= \left\|J_\tau^{\mathbb{B}}((1-t)\mathbb{S}x_t - \tau\mathbb{A}\mathbb{S}x_t) - J_\tau^{\mathbb{B}}(z - \tau\mathbb{A}z)\right\|^2 \\
&\leq \langle (1-t)\mathbb{S}x_t - \tau\mathbb{A}\mathbb{S}x_t - (z - \tau\mathbb{A}z), x_t - z\rangle \\
&= \frac{1}{2}\left(\|(1-t)\mathbb{S}x_t - \tau\mathbb{A}\mathbb{S}x_t - (z - \tau\mathbb{A}z)\|^2\right. \\
&\quad + \|x_t - z\|^2 \\
&\quad \left. - \|(1-t)\mathbb{S}x_t - \tau(\mathbb{A}\mathbb{S}x_t - \tau\mathbb{A}z) - x_t\|^2\right).
\end{aligned} \tag{24}$$

Note that

$$\begin{aligned}
&\|(1-t)\mathbb{S}x_t - \tau\mathbb{A}\mathbb{S}x_t - (z - \tau\mathbb{A}z)\|^2 \\
&\quad = \left\|(1-t)\left(\left(\mathbb{S}x_t - \frac{\tau}{1-t}\mathbb{A}\mathbb{S}x_t\right)\right.\right. \\
&\qquad \left.\left. - \left(\mathbb{S}z - \frac{\tau}{1-t}\mathbb{A}\mathbb{S}z\right)\right) + t(-z)\right\|^2
\end{aligned}$$

$$\begin{aligned}&\leq (1-t)\left\|\left(\mathbb{S}x_t-\frac{\tau}{1-t}\mathbb{A}\mathbb{S}x_t\right)\right.\\&\qquad\left.-\left(\mathbb{S}z-\frac{\tau}{1-t}\mathbb{A}\mathbb{S}z\right)\right\|^2+t\|z\|^2\\&\leq (1-t)\left\|x_t-z\right\|^2+t\|z\|^2.\end{aligned}\tag{25}$$

Thus,

$$\begin{aligned}\left\|x_t-z\right\|^2\leq\frac{1}{2}&\left((1-t)\left\|x_t-z\right\|^2+t\|z\|^2+\left\|x_t-z\right\|^2\right.\\&\left.-\left\|(1-t)\,\mathbb{S}x_t-\tau\left(\mathbb{A}\mathbb{S}x_t-\mathbb{A}z\right)-x_t\right\|^2\right).\end{aligned}\tag{26}$$

It follows that

$$\begin{aligned}\left\|x_t-z\right\|^2&\leq (1-t)\left\|x_t-z\right\|^2+t\|z\|^2\\&\quad-\left\|(1-t)\,\mathbb{S}x_t-x_t-\tau\left(\mathbb{A}\mathbb{S}x_t-\mathbb{A}z\right)\right\|^2\\&=(1-t)\left\|x_t-z\right\|^2+t\|z\|^2-\left\|(1-t)\,\mathbb{S}x_t-x_t\right\|^2\\&\quad+2\tau\left\langle (1-t)\,\mathbb{S}x_t-x_t,\mathbb{A}\mathbb{S}x_t-\mathbb{A}z\right\rangle\\&\quad-\tau^2\left\|\mathbb{A}\mathbb{S}x_t-\mathbb{A}z\right\|^2\\&\leq (1-t)\left\|x_t-z\right\|^2+t\|z\|^2-\left\|(1-t)\,\mathbb{S}x_t-x_t\right\|^2\\&\quad+2\tau\left\|(1-t)\,\mathbb{S}x_t-x_t\right\|\left\|\mathbb{A}\mathbb{S}x_t-\mathbb{A}z\right\|.\end{aligned}\tag{27}$$

Hence,

$$\begin{aligned}&\left\|(1-t)\,\mathbb{S}x_t-x_t\right\|^2\\&\quad\leq t\|z\|^2+2\tau\left\|(1-t)\,\mathbb{S}x_t-x_t\right\|\left\|\mathbb{A}\mathbb{S}x_t-\mathbb{A}z\right\|.\end{aligned}\tag{28}$$

This together with (23) implies that

$$\lim_{t\to 0+}\left\|(1-t)\,\mathbb{S}x_t-x_t\right\|=0.\tag{29}$$

So,

$$\lim_{t\to 0+}\left\|x_t-\mathbb{S}x_t\right\|=0.\tag{30}$$

By (19), we have

$$\begin{aligned}\left\|x_t-z\right\|^2&\leq\left\|(1-t)\left(\left(\mathbb{S}x_t-\frac{\tau}{1-t}\mathbb{A}\mathbb{S}x_t\right)\right.\right.\\&\qquad\left.\left.-\left(z-\frac{\tau}{1-t}\mathbb{A}z\right)\right)+t\,(-z)\right\|^2\\&=(1-t)^2\left\|\left(\mathbb{S}x_t-\frac{\tau}{1-t}\mathbb{A}\mathbb{S}x_t\right)\right.\\&\qquad\left.-\left(z-\frac{\tau}{1-t}\mathbb{A}z\right)\right\|^2\\&\quad+2t\,(1-t)\left\langle -z,\left(\mathbb{S}x_t-\frac{\tau}{1-t}\mathbb{A}\mathbb{S}x_t\right)\right.\\&\qquad\left.-\left(z-\frac{\tau}{1-t}\mathbb{A}z\right)\right\rangle+t^2\|z\|^2\\&\leq (1-t)^2\left\|x_t-z\right\|^2+2t\,(1-t)\\&\quad\times\left\langle -z,\mathbb{S}x_t-\frac{\tau}{1-t}\left(\mathbb{A}\mathbb{S}x_t-\mathbb{A}\mathbb{S}z\right)-z\right\rangle\\&\quad+t^2\|z\|^2.\end{aligned}\tag{31}$$

It follows that

$$\begin{aligned}\left\|x_t-z\right\|^2&\leq\left\langle -z,\mathbb{S}x_t-\frac{\tau}{1-t}\left(\mathbb{A}\mathbb{S}x_t-\mathbb{A}z\right)-z\right\rangle\\&\quad+\frac{t}{2}\left(\|z\|^2+\left\|x_t-z\right\|^2\right)\\&\quad+t\,\|z\|\left\|\mathbb{S}x_t-\frac{\tau}{1-t}\left(\mathbb{A}\mathbb{S}x_t-\mathbb{A}z\right)-z\right\|\\&\leq\left\langle -z,\mathbb{S}x_t-z\right\rangle+\left(t+\left\|\mathbb{A}\mathbb{S}x_t-\mathbb{A}z\right\|\right)M,\end{aligned}\tag{32}$$

where M is some constant such that

$$\begin{aligned}\sup_{t\in(0,(2\varrho-\tau)/2\varrho)}&\left\{\frac{1}{2}\left(\|z\|^2+\left\|x_t-z\right\|^2\right),\right.\\&\left.\|z\|\left\|\mathbb{S}x_t-\frac{\tau}{1-t}\left(\mathbb{A}\mathbb{S}x_t-\mathbb{A}z\right)-z\right\|\right\}\leq M.\end{aligned}\tag{33}$$

Now we show that $\{x_t\}$ is relatively norm-compact as $t\to 0+$. Assume $\{t_n\}\subset(0,(2\varrho-\tau)/2\varrho)$ such that $t_n\to 0+$ as $n\to\infty$. Put $x_n:=x_{t_n}$. From (32), we have

$$\left\|x_n-z\right\|^2\leq\left\langle -z,\mathbb{S}x_n-z\right\rangle+\left(t_n+\left\|\mathbb{A}\mathbb{S}x_n-\mathbb{A}z\right\|\right)M.\tag{34}$$

Since $\{x_n\}$ is bounded, without loss of generality, we may assume that $x_{n_j}\rightharpoonup\tilde{x}\in C$. Hence, $x_{n_j}-(\tau/(1-t_{n_j}))(\mathbb{A}\mathbb{S}x_{n_j}-\mathbb{A}z)\rightharpoonup\tilde{x}$ because of $\|\mathbb{A}\mathbb{S}x_n-\mathbb{A}z\|\to 0$ by (23). From (30), we have

$$\lim_{n\to\infty}\left\|x_n-\mathbb{S}x_n\right\|=0.\tag{35}$$

By Lemma 5 and (35), we deduce $\tilde{x}\in\mathrm{Fix}(\mathbb{S})$.

Next, we show that $\tilde{x}\in(\mathbb{A}+\mathbb{B})^{-1}0$. Let $v\in\mathbb{B}u$. Note that $x_n=J_\tau^{\mathbb{B}}((1-t_n)\mathbb{S}x_n-\tau\mathbb{A}\mathbb{S}x_n)$ for all n. Then, we have

$$\left(1-t_n\right)\mathbb{S}x_n-\tau\mathbb{A}\mathbb{S}x_n\in\left(I+\tau\mathbb{B}\right)x_n.\tag{36}$$

So,

$$\frac{1-t_n}{\tau}\mathbb{S}x_n-\mathbb{A}\mathbb{S}x_n-\frac{x_n}{\tau}\in\mathbb{B}x_n.\tag{37}$$

Since $\mathbb{B}$ is monotone, we have, for $(u, v) \in \mathbb{B}$,

$$\begin{aligned} &\left\langle \frac{t_n \gamma f(x_n)}{\tau} + \frac{1-t_n}{\tau}\mathbb{S}x_n - \mathbb{A}\mathbb{S}x_n - \frac{x_n}{\tau} - v, x_n - u \right\rangle \geq 0 \\ &\Longrightarrow \langle (1-t_n)\mathbb{S}x_n - \tau\mathbb{A}\mathbb{S}x_n - x_n - \tau v, x_n - u\rangle \geq 0 \\ &\Longrightarrow \langle \mathbb{A}\mathbb{S}x_n + v, x_n - u\rangle \\ &\qquad \leq \frac{1}{\tau}\langle \mathbb{S}x_n - x_n, x_n - u\rangle - \frac{t_n}{\tau}\langle \mathbb{S}x_n, x_n - u\rangle \\ &\Longrightarrow \langle \mathbb{A}\mathbb{S}\tilde{x} + v, x_n - u\rangle \\ &\qquad \leq \frac{1}{\tau}\langle \mathbb{S}x_n - x_n, x_n - u\rangle - \frac{t_n}{\tau}\langle \mathbb{S}x_n, x_n - u\rangle \\ &\qquad\quad + \langle \mathbb{A}\mathbb{S}\tilde{x} - \mathbb{A}\mathbb{S}x_n, x_n - u\rangle \\ &\Longrightarrow \langle \mathbb{A}\mathbb{S}\tilde{x} + v, x_n - u\rangle \\ &\qquad \leq \frac{1}{\tau}\|\mathbb{S}x_n - x_n\|\|x_n - u\| + \frac{t_n}{\tau}\|\mathbb{S}x_n\|\|x_n - u\| \\ &\qquad\quad + \|\mathbb{A}\mathbb{S}\tilde{x} - \mathbb{A}\mathbb{S}x_n\|\|x_n - u\|. \end{aligned} \tag{38}$$

It follows that

$$\begin{aligned} \langle \mathbb{A}\mathbb{S}\tilde{x} + v, \tilde{x} - u\rangle \leq{}& \frac{1}{\tau}\left\|\mathbb{S}x_{n_j} - x_{n_j}\right\|\left\|x_{n_j} - u\right\| \\ &+ \frac{t_{n_j}}{\tau}\left\|\mathbb{S}x_{n_j}\right\|\left\|x_{n_j} - u\right\| \\ &+ \left\|\mathbb{A}\mathbb{S}\tilde{x} - \mathbb{A}\mathbb{S}x_{n_j}\right\|\left\|x_{n_j} - u\right\| \\ &+ \left\langle \mathbb{A}\mathbb{S}\tilde{x} + v, \tilde{x} - x_{n_j}\right\rangle. \end{aligned} \tag{39}$$

Since

$$\left\langle x_{n_j} - \tilde{x}, \mathbb{A}\mathbb{S}x_{n_j} - \mathbb{A}\mathbb{S}\tilde{x}\right\rangle \geq \varrho\left\|\mathbb{A}\mathbb{S}x_{n_j} - \mathbb{A}\mathbb{S}\tilde{x}\right\|^2, \tag{40}$$

$\mathbb{A}\mathbb{S}x_{n_j} \to \mathbb{A}\mathbb{S}z$, and $x_{n_j} \rightharpoonup \tilde{x}$, we have $\mathbb{A}\mathbb{S}x_{n_j} \to \mathbb{A}\mathbb{S}\tilde{x}$. We also observe that $t_n \to 0$ and $\|\mathbb{S}x_n - x_n\| \to 0$. Then, from (39), we derive

$$\langle \mathbb{A}\mathbb{S}\tilde{x} + v, \tilde{x} - u\rangle \leq 0. \tag{41}$$

That is, $\langle -\mathbb{A}\tilde{x} - v, \tilde{x} - u\rangle \geq 0$. Since $\mathbb{B}$ is maximal monotone, we have $-\mathbb{A}\tilde{x} \in \mathbb{B}\tilde{x}$. This shows that $0 \in (\mathbb{A}+\mathbb{B})\tilde{x}$. Hence, we have $\tilde{x} \in \mathrm{Fix}(\mathbb{S}) \cap (\mathbb{A}+\mathbb{B})^{-1}0$. Therefore, we can substitute $\tilde{x}$ for z in (34) to get

$$\|x_n - \tilde{x}\|^2 \leq \langle -\tilde{x}, \mathbb{S}x_n - \tilde{x}\rangle + (t_n + \|\mathbb{A}\mathbb{S}x_n - \mathbb{A}\tilde{x}\|)M. \tag{42}$$

Consequently, the weak convergence of $\{x_n\}$ to $\tilde{x}$ actually implies that $x_n \to \tilde{x}$. This has proved the relative norm-compactness of the net $\{x_t\}$ as $t \to 0+$.

From (34), we get

$$\|\tilde{x} - z\|^2 \leq \langle -z, \tilde{x} - z\rangle, \quad \forall z \in \mathrm{Fix}(\mathbb{S}) \cap (\mathbb{A}+\mathbb{B})^{-1}0. \tag{43}$$

That is,

$$\langle \tilde{x}, \tilde{x} - z\rangle \leq 0, \quad \forall z \in \mathrm{Fix}(\mathbb{S}) \cap (\mathbb{A}+\mathbb{B})^{-1}0. \tag{44}$$

It follows that

$$\|\tilde{x}\| \leq \|z\|, \quad \forall z \in \mathrm{Fix}(\mathbb{S}) \cap (\mathbb{A}+\mathbb{B})^{-1}0. \tag{45}$$

It is obvious that $\tilde{x} = \mathrm{proj}_{\mathrm{Fix}(\mathbb{S})\cap(\mathbb{A}+\mathbb{B})^{-1}0}(0)$ by (44). This denotes that the entire net $\{x_t\}$ converges to $\tilde{x}$. This completes the proof. □

Next, we present another algorithm.

Algorithm 9. For given $x_0 \in \mathscr{C}$, define a sequence $\{x_n\} \subset \mathscr{C}$ iteratively by

$$x_{n+1} = \varsigma_n x_n + (1-\varsigma_n) J_{\tau_n}^{\mathbb{B}}((1-\varrho_n)\mathbb{S}x_n - \tau_n\mathbb{A}\mathbb{S}x_n), \quad \forall n \geq 0, \tag{46}$$

where $\{\tau_n\} \subset (0, 2\varrho)$, $\{\varrho_n\} \subset (0,1)$, and $\{\varsigma_n\} \subset (0,1)$.

Theorem 10. *Suppose that* $\mathrm{Fix}(\mathbb{S}) \cap (\mathbb{A}+\mathbb{B})^{-1}0 \neq \emptyset$. *Assume that the following conditions are satisfied:*

(i) $\lim_{n\to\infty}\varrho_n = 0$ *and* $\sum_n \varrho_n = \infty$;

(ii) $0 < \liminf_{n\to\infty}\varsigma_n \leq \limsup_{n\to\infty}\varsigma_n < 1$;

(iii) $a(1-\varrho_n) \leq \tau_n \leq b(1-\varrho_n)$, *where* $[a,b] \subset (0, 2\varrho)$ *and* $\lim_{n\to\infty}(\tau_{n+1} - \tau_n) = 0$.

Then $\{x_n\}$ *generated by* (46) *converges strongly to a point* $\tilde{x} = \mathrm{proj}_{\mathrm{Fix}(\mathbb{S})\cap(\mathbb{A}+\mathbb{B})^{-1}(0)}(0)$ *which is the minimum norm element in* $\mathrm{Fix}(\mathbb{S}) \cap (\mathbb{A}+\mathbb{B})^{-1}(0)$.

Proof. Let $z \in \mathrm{Fix}(\mathbb{S}) \cap (\mathbb{A}+\mathbb{B})^{-1}(0)$. We have $z = J_{\tau_n}^{\mathbb{B}}(z - \tau_n\mathbb{A}z) = J_{\tau_n}^{\mathbb{B}}(\varrho_n z + (1-\varrho_n)(z - \tau_n\mathbb{A}z/(1-\varrho_n)))$ for all $n \geq 0$. Since $J_{\tau_n}^{\mathbb{B}}$, $\mathbb{S}$, and $I - \tau_n\mathbb{A}/(1-\varrho_n)$ are nonexpansive, we have

$$\begin{aligned} &\left\|J_{\tau_n}^{\mathbb{B}}((1-\varrho_n)\mathbb{S}x_n - \tau_n\mathbb{A}\mathbb{S}x_n) - z\right\| \\ &= \left\|J_{\tau_n}^{\mathbb{B}}\left((1-\varrho_n)\left(\mathbb{S}x_n - \frac{\tau_n}{1-\varrho_n}\mathbb{A}\mathbb{S}x_n\right)\right) - J_{\tau_n}^{\mathbb{B}}\left(\varrho_n z + (1-\varrho_n)\left(z - \frac{\tau_n}{1-\varrho_n}\mathbb{A}z\right)\right)\right\| \\ &\leq \left\|\left((1-\varrho_n)\left(\mathbb{S}x_n - \frac{\tau_n}{1-\varrho_n}\mathbb{A}\mathbb{S}x_n\right)\right) - \left(\varrho_n z + (1-\varrho_n)\left(z - \frac{\tau_n}{1-\varrho_n}\mathbb{A}z\right)\right)\right\| \\ &= \left\|(1-\varrho_n)\left(\mathbb{S}x_n - \frac{\tau_n}{1-\varrho_n}\mathbb{A}\mathbb{S}x_n - \left(z - \frac{\tau_n}{1-\varrho_n}\mathbb{A}z\right)\right) + \varrho_n(-z)\right\| \\ &\leq (1-\varrho_n)\|x_n - z\| + \varrho_n\|z\|. \end{aligned} \tag{47}$$

Thus,

$$\begin{aligned}\|x_{n+1}-z\| &\le \varsigma_n \|x_n-z\| + (1-\varsigma_n)(1-\varrho_n)\|x_n-z\| \\ &\quad + (1-\varsigma_n)\varrho_n\|z\| \\ &= [1-\varrho_n(1-\varsigma_n)]\|x_n-z\| + (1-\varsigma_n)\varrho_n\|z\|. \end{aligned} \tag{48}$$

By induction, we have

$$\|x_{n+1}-z\| \le \max\{\|x_0-z\|, \|z\|\}. \tag{49}$$

Therefore, $\{x_n\}$ is bounded.

From (4) and (47), we derive

$$\begin{aligned}&\left\|(1-\varrho_n)\left(\left(\mathbb{S}x_n - \frac{\tau_n}{1-\varrho_n}\mathbb{A}\mathbb{S}x_n\right) - \left(z - \frac{\tau_n}{1-\varrho_n}\mathbb{A}z\right)\right)\right. \\ &\qquad \left. + \varrho_n(-z)\right\|^2 \\ &\le (1-\varrho_n)\left\|\left(\mathbb{S}x_n - \frac{\tau_n}{1-\varrho_n}\mathbb{A}\mathbb{S}x_n\right) - \left(z - \frac{\tau_n}{1-\varrho_n}\mathbb{A}z\right)\right\|^2 \\ &\qquad + \varrho_n\|z\|^2 \\ &= (1-\varrho_n)\left\|(\mathbb{S}x_n - z) - \frac{\tau_n}{1-\varrho_n}(\mathbb{A}\mathbb{S}x_n - Az)\right\|^2 + \varrho_n\|z\|^2 \\ &= (1-\varrho_n)\left(\|\mathbb{S}x_n - z\|^2 - \frac{2\tau_n}{1-\varrho_n}\langle \mathbb{A}\mathbb{S}x_n - \mathbb{A}z, \mathbb{S}x_n - z\rangle\right. \\ &\qquad \left. + \frac{\tau_n^2}{(1-\varrho_n)^2}\|\mathbb{A}\mathbb{S}x_n - \mathbb{A}z\|^2\right) + \varrho_n\|z\|^2 \\ &\le (1-\varrho_n)\left(\|x_n - z\|^2 - \frac{2\varrho\tau_n}{1-\varrho_n}\|\mathbb{A}\mathbb{S}x_n - \mathbb{A}z\|^2\right. \\ &\qquad \left. + \frac{\tau_n^2}{(1-\varrho_n)^2}\|\mathbb{A}\mathbb{S}x_n - \mathbb{A}z\|^2\right) + \varrho_n\|z\|^2 \\ &= (1-\varrho_n)\left(\|x_n - z\|^2 + \frac{\tau_n}{(1-\varrho_n)^2}(\tau_n - 2(1-\varrho_n)\varrho)\right. \\ &\qquad \left. \times \|\mathbb{A}\mathbb{S}x_n - \mathbb{A}z\|^2\right) + \varrho_n\|z\|^2. \end{aligned} \tag{50}$$

Set $u_n = (1-\varrho_n)\mathbb{S}x_n - \tau_n\mathbb{A}\mathbb{S}x_n$ for all $n \ge 0$. Since $\tau_n - 2(1-\varrho_n)\varrho \le 0$ for all $n \ge 0$, we obtain

$$\begin{aligned}&\|J^{\mathbb{B}}_{\tau_n}u_n - z\|^2 \\ &\quad \le (1-\varrho_n)\left(\|x_n - z\|^2 + \frac{\tau_n}{(1-\varrho_n)^2}(\tau_n - 2(1-\varrho_n)\varrho)\right. \\ &\qquad \left. \times \|\mathbb{A}\mathbb{S}x_n - \mathbb{A}z\|^2\right) + \varrho_n\|z\|^2. \end{aligned} \tag{51}$$

From (46), we have

$$\begin{aligned}\|x_{n+1}-z\|^2 &= \left\|\varsigma_n(x_n - z) + (1-\varsigma_n)\left(J^{\mathbb{B}}_{\tau_n}u_n - z\right)\right\|^2 \\ &\le \varsigma_n\|x_n - z\|^2 + (1-\varsigma_n)\left\|J^{\mathbb{B}}_{\tau_n}u_n - z\right\|^2. \end{aligned} \tag{52}$$

Set $y_n = J^{\mathbb{B}}_{\tau_n}((1-\varrho_n)\mathbb{S}x_n - \tau_n\mathbb{A}\mathbb{S}x_n)$ for all $n \ge 0$. Then $x_{n+1} = \varsigma_n x_n + (1-\varsigma_n)y_n$ for all $n \ge 0$. Next, we estimate $\|x_{n+1} - x_n\|$. In fact, we have

$$\begin{aligned}\|y_{n+1} - y_n\| &= \left\|J^{\mathbb{B}}_{\tau_{n+1}}u_{n+1} - J^{\mathbb{B}}_{\tau_n}u_n\right\| \\ &\le \left\|J^{\mathbb{B}}_{\tau_{n+1}}u_{n+1} - J^{\mathbb{B}}_{\tau_{n+1}}u_n\right\| + \left\|J^{\mathbb{B}}_{\tau_{n+1}}u_n - J^{\mathbb{B}}_{\tau_n}u_n\right\| \\ &\le \|((1-\varrho_{n+1})\mathbb{S}x_{n+1} - \tau_{n+1}\mathbb{A}\mathbb{S}x_{n+1}) \\ &\qquad - ((1-\varrho_n)\mathbb{S}x_n - \tau_n\mathbb{A}\mathbb{S}x_n)\| \\ &\qquad + \left\|J^{\mathbb{B}}_{\tau_{n+1}}u_n - J^{\mathbb{B}}_{\tau_n}u_n\right\| \\ &= \|(I - \tau_{n+1}\mathbb{A})\mathbb{S}x_{n+1} - (I - \tau_{n+1}\mathbb{A})\mathbb{S}x_n \\ &\qquad + (\tau_n - \tau_{n+1})\mathbb{A}\mathbb{S}x_n + \varrho_n\mathbb{S}x_n - \varrho_{n+1}\mathbb{S}x_{n+1}\| \\ &\qquad + \left\|J^{\mathbb{B}}_{\tau_{n+1}}u_n - J^{\mathbb{B}}_{\tau_n}u_n\right\| \\ &\le \|(I - \tau_{n+1}\mathbb{A})\mathbb{S}x_{n+1} - (I - \tau_{n+1}\mathbb{A})\mathbb{S}x_n\| \\ &\qquad + |\tau_{n+1} - \tau_n|\,\|\mathbb{A}\mathbb{S}x_n\| + \varrho_n\|\mathbb{S}x_n\| \\ &\qquad + \varrho_{n+1}\|\mathbb{S}x_{n+1}\| + \left\|J^{\mathbb{B}}_{\tau_{n+1}}u_n - J^{\mathbb{B}}_{\tau_n}u_n\right\|. \end{aligned} \tag{53}$$

Since $I - \tau_{n+1}\mathbb{A}$ is nonexpansive for $\tau_{n+1} \in (0, 2\varrho)$, we have

$$\begin{aligned}&\|(I - \tau_{n+1}\mathbb{A})\mathbb{S}x_{n+1} - (I - \tau_{n+1}\mathbb{A})\mathbb{S}x_n\| \\ &\quad \le \|\mathbb{S}x_{n+1} - \mathbb{S}x_n\| \le \|x_{n+1} - x_n\|. \end{aligned} \tag{54}$$

From (13), we have

$$J^{\mathbb{B}}_{\tau_{n+1}}u_n = J^{\mathbb{B}}_{\tau_n}\left(\frac{\tau_n}{\tau_{n+1}}u_n + \left(1 - \frac{\tau_n}{\tau_{n+1}}\right)J^{\mathbb{B}}_{\tau_{n+1}}u_n\right). \tag{55}$$

It follows that

$$\begin{aligned}&\left\|J^{\mathbb{B}}_{\tau_{n+1}}u_n - J^{\mathbb{B}}_{\tau_n}u_n\right\| \\ &\quad = \left\|J^{\mathbb{B}}_{\tau_n}\left(\frac{\tau_n}{\tau_{n+1}}u_n + \left(1 - \frac{\tau_n}{\tau_{n+1}}\right)J^{\mathbb{B}}_{\tau_{n+1}}u_n\right) - J^{\mathbb{B}}_{\tau_n}u_n\right\| \\ &\quad \le \left\|\left(\frac{\tau_n}{\tau_{n+1}}u_n + \left(1 - \frac{\tau_n}{\tau_{n+1}}\right)J^{\mathbb{B}}_{\tau_{n+1}}u_n\right) - u_n\right\| \\ &\quad \le \frac{|\tau_{n+1} - \tau_n|}{\tau_{n+1}}\left\|u_n - J^{\mathbb{B}}_{\tau_{n+1}}u_n\right\|. \end{aligned} \tag{56}$$

So,

$$\begin{aligned}\|y_{n+1}-y_n\| \le{}& \|x_{n+1}-x_n\| + |\tau_{n+1}-\tau_n| \, \|\mathbb{A}\mathbb{S}x_n\| \\ &+ \varrho_n \|\mathbb{S}x_n\| \\ &+ \varrho_{n+1} \|\mathbb{S}x_{n+1}\| + \frac{|\tau_{n+1}-\tau_n|}{\tau_{n+1}} \left\|u_n - J^{\mathbb{B}}_{\tau_{n+1}} u_n\right\|.\end{aligned} \tag{57}$$

Then,

$$\begin{aligned}&\|y_{n+1}-y_n\| - \|x_{n+1}-x_n\| \\ &\quad\le |\tau_{n+1}-\tau_n| \, \|\mathbb{A}\mathbb{S}x_n\| + \varrho_n \|\mathbb{S}x_n\| \\ &\qquad+ \varrho_{n+1} \|\mathbb{S}x_{n+1}\| + \frac{|\tau_{n+1}-\tau_n|}{\tau_{n+1}} \left\|u_n - J^{\mathbb{B}}_{\tau_{n+1}} u_n\right\|.\end{aligned} \tag{58}$$

Since $\varrho_n \to 0$, $\tau_{n+1}-\tau_n \to 0$ and $\underline{\lim}_{n\to\infty}\tau_n > 0$, we obtain

$$\limsup_{n\to\infty} \left(\|y_{n+1}-y_n\| - \|x_{n+1}-x_n\|\right) \le 0. \tag{59}$$

By Lemma 6, we get

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{60}$$

Consequently, we obtain

$$\lim_{n\to\infty} \|x_{n+1}-x_n\| = \lim_{n\to\infty} (1-\varsigma_n) \|y_n - x_n\| = 0. \tag{61}$$

From (51) and (52), we have

$$\begin{aligned}\|x_{n+1}-z\|^2 \le{}& \varsigma_n\|x_n-z\|^2 + (1-\varsigma_n)\left\|J^{\mathbb{B}}_{\tau_n}u_n - z\right\|^2 \\ \le{}& (1-\varsigma_n)(1-\varrho_n) \\ &\times \left(\|x_n-z\|^2 + \frac{\tau_n}{(1-\varrho_n)^2}(\tau_n - 2(1-\varrho_n)\varrho) \right. \\ &\left. \times \|\mathbb{A}\mathbb{S}x_n - \mathbb{A}z\|^2\right) \\ &+ (1-\varsigma_n)\varrho_n\|z\|^2 + \varsigma_n\|x_n-z\|^2 \\ ={}& [1-(1-\varsigma_n)\varrho_n]\|x_n-z\|^2 \\ &+ \frac{(1-\varsigma_n)\tau_n}{1-\varrho_n}(\tau_n - 2(1-\varrho_n)\varrho)\|\mathbb{A}\mathbb{S}x_n - \mathbb{A}z\|^2 \\ &+ (1-\varsigma_n)\varrho_n\|z\|^2 \\ \le{}& \|x_n-z\|^2 + \frac{(1-\varsigma_n)\tau_n}{1-\varrho_n}(\tau_n - 2(1-\varrho_n)\varrho) \\ &\times \|\mathbb{A}\mathbb{S}x_n - \mathbb{A}z\|^2 + (1-\varsigma_n)\varrho_n\|z\|^2.\end{aligned} \tag{62}$$

Then, we obtain

$$\begin{aligned}&\frac{(1-\varsigma_n)\tau_n}{(1-\varrho_n)}(2(1-\varrho_n)\varrho - \tau_n)\|\mathbb{A}\mathbb{S}x_n - \mathbb{A}z\|^2 \\ &\quad\le \|x_n-z\|^2 - \|x_{n+1}-z\|^2 + (1-\varsigma_n)\varrho_n\|z\|^2 \\ &\quad\le (\|x_n-z\| - \|x_{n+1}-z\|)\|x_{n+1}-x_n\| \\ &\qquad+ (1-\varsigma_n)\varrho_n\|z\|^2.\end{aligned} \tag{63}$$

Since $\lim_{n\to\infty}\varrho_n = 0$, $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$, and $\underline{\lim}_{n\to\infty}((1-\varsigma_n)\tau_n/(1-\varrho_n))(2(1-\varrho_n)\varrho - \tau_n) > 0$, we have

$$\lim_{n\to\infty} \|\mathbb{A}\mathbb{S}x_n - \mathbb{A}z\| = 0. \tag{64}$$

Next, we show $\|x_n - \mathbb{S}x_n\| \to 0$. By using the firm nonexpansivity of $J^{\mathbb{B}}_{\tau_n}$, we have

$$\begin{aligned}&\left\|J^{\mathbb{B}}_{\tau_n}u_n - z\right\|^2 \\ &\quad= \left\|J^{\mathbb{B}}_{\tau_n}((1-\varrho_n)\mathbb{S}x_n - \tau_n\mathbb{A}\mathbb{S}x_n) - J^{\mathbb{B}}_{\tau_n}(z-\tau_n\mathbb{A}z)\right\|^2 \\ &\quad\le \left\langle (1-\varrho_n)\mathbb{S}x_n - \tau_n\mathbb{A}\mathbb{S}x_n - (z-\tau_n\mathbb{A}z), J^{\mathbb{B}}_{\tau_n}u_n - z\right\rangle \\ &\quad= \frac{1}{2}\left(\|(1-\varrho_n)\mathbb{S}x_n - \tau_n\mathbb{A}\mathbb{S}x_n - (z-\tau_n\mathbb{A}z)\|^2\right. \\ &\qquad+ \left\|J^{\mathbb{B}}_{\tau_n}u_n - z\right\|^2 \\ &\qquad\left.- \left\|(1-\varrho_n)\mathbb{S}x_n - \tau_n(\mathbb{A}\mathbb{S}x_n - \mathbb{A}z) - J^{\mathbb{B}}_{\tau_n}u_n\right\|^2\right).\end{aligned} \tag{65}$$

Observe that

$$\begin{aligned}&\|(1-\varrho_n)\mathbb{S}x_n - \tau_n\mathbb{A}\mathbb{S}x_n - (z-\tau_n\mathbb{A}z)\|^2 \\ &\quad= \left\|(1-\varrho_n)\left(\mathbb{S}x_n - \frac{\tau_n}{1-\varrho_n}\mathbb{A}\mathbb{S}x_n \right.\right. \\ &\qquad\left.\left. -\left(z - \frac{\tau_n}{1-\varrho_n}\mathbb{A}z\right)\right) + \varrho_n(-z)\right\|^2 \\ &\quad\le (1-\varrho_n)\left\|\mathbb{S}x_n - \frac{\tau_n}{1-\varrho_n}\mathbb{A}\mathbb{S}x_n \right. \\ &\qquad\left. -\left(z - \frac{\tau_n}{1-\varrho_n}\mathbb{A}z\right)\right\|^2 + \varrho_n\|z\|^2 \\ &\quad\le (1-\varrho_n)\|x_n-z\|^2 + \varrho_n\|z\|^2.\end{aligned} \tag{66}$$

Hence,

$$\begin{aligned}\left\|J^{\mathbb{B}}_{\tau_n}u_n - z\right\|^2 \le{}& \frac{1}{2}\left((1-\varrho_n)\|x_n-z\|^2 + \varrho_n\|z\|^2 + \left\|J^{\mathbb{B}}_{\tau_n}u_n - z\right\|^2\right. \\ &\left. - \left\|(1-\varrho_n)\mathbb{S}x_n - J^{\mathbb{B}}_{\tau_n}u_n - \tau_n(\mathbb{A}\mathbb{S}x_n - \mathbb{A}z)\right\|^2\right).\end{aligned} \tag{67}$$

It follows that

$$\begin{aligned}
\left\|J_{\tau_n}^{\mathbb{B}}u_n - z\right\|^2 &\le (1-\varrho_n)\left\|x_n - z\right\|^2 + \varrho_n\|z\|^2 \\
&\quad - \left\|(1-\varrho_n)\mathbb{S}x_n - J_{\tau_n}^{\mathbb{B}}u_n - \tau_n(\mathbb{A}\mathbb{S}x_n - \mathbb{A}z)\right\|^2 \\
&= (1-\varrho_n)\left\|x_n - z\right\|^2 + \varrho_n\|z\|^2 \\
&\quad - \left\|(1-\varrho_n)\mathbb{S}x_n - J_{\tau_n}^{\mathbb{B}}u_n\right\|^2 \\
&\quad + 2\tau_n\left\langle(1-\varrho_n)\mathbb{S}x_n - J_{\tau_n}^{\mathbb{B}}u_n, \mathbb{A}\mathbb{S}x_n - \mathbb{A}z\right\rangle \\
&\quad - \tau_n^2\left\|\mathbb{A}\mathbb{S}x_n - \mathbb{A}z\right\|^2 \\
&\le (1-\varrho_n)\left\|x_n - z\right\|^2 + \varrho_n\|z\|^2 \\
&\quad - \left\|(1-\varrho_n)\mathbb{S}x_n - J_{\tau_n}^{\mathbb{B}}u_n\right\|^2 \\
&\quad + 2\tau_n\left\|(1-\varrho_n)\mathbb{S}x_n - J_{\tau_n}^{\mathbb{B}}u_n\right\|\left\|\mathbb{A}\mathbb{S}x_n - \mathbb{A}z\right\|.
\end{aligned} \tag{68}$$

This together with (52) implies that

$$\begin{aligned}
\left\|x_{n+1} - z\right\|^2 &\le \varsigma_n\left\|x_n - z\right\|^2 + (1-\varsigma_n)(1-\varrho_n)\left\|x_n - z\right\|^2 \\
&\quad + (1-\varsigma_n)\varrho_n\|z\|^2 \\
&\quad - (1-\varsigma_n)\left\|(1-\varrho_n)\mathbb{S}x_n - J_{\tau_n}^{\mathbb{B}}u_n\right\|^2 \\
&\quad + 2\tau_n(1-\varsigma_n)\left\|(1-\varrho_n)\mathbb{S}x_n - J_{\tau_n}^{\mathbb{B}}u_n\right\| \\
&\quad \times\left\|\mathbb{A}\mathbb{S}x_n - \mathbb{A}z\right\| \\
&= \left[1-(1-\varsigma_n)\varrho_n\right]\left\|x_n - z\right\|^2 + (1-\varsigma_n)\varrho_n\|z\|^2 \\
&\quad - (1-\varsigma_n)\left\|(1-\varrho_n)\mathbb{S}x_n - J_{\tau_n}^{\mathbb{B}}u_n\right\|^2 \\
&\quad + 2\tau_n(1-\varsigma_n)\left\|(1-\varrho_n)\mathbb{S}x_n - J_{\tau_n}^{\mathbb{B}}u_n\right\| \\
&\quad \times\left\|\mathbb{A}\mathbb{S}x_n - \mathbb{A}z\right\|.
\end{aligned} \tag{69}$$

Hence,

$$\begin{aligned}
&(1-\varsigma_n)\left\|(1-\varrho_n)\mathbb{S}x_n - J_{\tau_n}^{\mathbb{B}}u_n\right\|^2 \\
&\quad\le \left\|x_n - z\right\|^2 - \left\|x_{n+1} - z\right\|^2 - (1-\varsigma_n)\varrho_n\left\|x_n - z\right\|^2 \\
&\qquad + (1-\varsigma_n)\varrho_n\|z\|^2 + 2\tau_n(1-\varsigma_n) \\
&\qquad \times\left\|(1-\varrho_n)\mathbb{S}x_n - J_{\tau_n}^{\mathbb{B}}u_n\right\|\left\|\mathbb{A}\mathbb{S}x_n - \mathbb{A}z\right\| \\
&\quad\le \left(\left\|x_n - z\right\| + \left\|x_{n+1} - z\right\|\right)\left\|x_{n+1} - x_n\right\| \\
&\qquad + (1-\varsigma_n)\varrho_n\|z\|^2 + 2\tau_n(1-\varsigma_n) \\
&\qquad \times\left\|(1-\varrho_n)\mathbb{S}x_n - J_{\tau_n}^{\mathbb{B}}u_n\right\|\left\|\mathbb{A}\mathbb{S}x_n - \mathbb{A}z\right\|.
\end{aligned} \tag{70}$$

Since $\overline{\lim}_{n\to\infty}\varsigma_n < 1$, $\|x_{n+1} - x_n\| \to 0$, $\varrho_n \to 0$, and $\|\mathbb{A}\mathbb{S}x_n - \mathbb{A}z\| \to 0$ (by (60)), we deduce

$$\lim_{n\to\infty}\left\|(1-\varrho_n)\mathbb{S}x_n - J_{\tau_n}^{\mathbb{B}}u_n\right\| = 0. \tag{71}$$

This indicates that

$$\lim_{n\to\infty}\left\|\mathbb{S}x_n - J_{\tau_n}^{\mathbb{B}}u_n\right\| = 0. \tag{72}$$

Combining (60) and (72), we get

$$\lim_{n\to\infty}\left\|x_n - \mathbb{S}x_n\right\| = 0. \tag{73}$$

Put $\tilde{x} = \lim_{t\to 0+}x_t = \operatorname{proj}_{\operatorname{Fix}(\mathbb{S})\cap(\mathbb{A}+\mathbb{B})^{-1}(0)}(0)$, where x_t is the net defined by (17). We will finally show that $x_n \to \tilde{x}$.

Set $v_n = x_n - (\tau_n/(1-\varrho_n))(\mathbb{A}\mathbb{S}x_n - \mathbb{A}\tilde{x})$ for all n. Take $z = \tilde{x}$ in (64) to get $\|\mathbb{A}\mathbb{S}x_n - \mathbb{A}\tilde{x}\| \to 0$. First, we prove $\overline{\lim}_{n\to\infty}\langle -\tilde{x}, \mathbb{S}x_n - \tilde{x}\rangle \le 0$. We take a subsequence $\{\mathbb{S}x_{n_i}\}$ of $\{\mathbb{S}x_n\}$ such that

$$\overline{\lim_{n\to\infty}}\left\langle -\tilde{x}, \mathbb{S}x_n - \tilde{x}\right\rangle = \lim_{i\to\infty}\left\langle -\tilde{x}, \mathbb{S}x_{n_i} - \tilde{x}\right\rangle. \tag{74}$$

It is clear that $\{\mathbb{S}x_{n_i}\}$ is bounded due to the boundedness of $\{\mathbb{S}x_n\}$ and $\|\mathbb{A}\mathbb{S}x_n - \mathbb{A}\tilde{x}\| \to 0$. Then, there exists a subsequence $\{\mathbb{S}x_{n_{i_j}}\}$ of $\{\mathbb{S}x_{n_i}\}$ which converges weakly to some point $w \in \mathscr{C}$. Hence, $\{x_{n_{i_j}}\}$ and $\{y_{n_{i_j}}\}$ also converge weakly to w because of $\|\mathbb{S}x_{n_{i_j}} - x_{n_{i_j}}\| \to 0$ and $\|x_{n_{i_j}} - y_{n_{i_j}}\| \to 0$. By the demiclosedness principle of the nonexpansive mapping (see Lemma 5) and (73), we deduce $w \in \operatorname{Fix}(\mathbb{S})$. Furthermore, by similar argument as that of Theorem 8, we can show that w is also in $(\mathbb{A}+\mathbb{B})^{-1}(0)$. Hence, we have $w \in \operatorname{Fix}(\mathbb{S}) \cap (\mathbb{A}+\mathbb{B})^{-1}(0)$. This implies that

$$\begin{aligned}
\overline{\lim_{n\to\infty}}\left\langle -\tilde{x}, \mathbb{S}x_n - \tilde{x}\right\rangle &= \lim_{j\to\infty}\left\langle -\tilde{x}, \mathbb{S}x_{n_{i_j}} - \tilde{x}\right\rangle \\
&= \left\langle -\tilde{x}, w - \tilde{x}\right\rangle.
\end{aligned} \tag{75}$$

Note that $\tilde{x} = \operatorname{proj}_{\operatorname{Fix}(\mathbb{S})\cap(\mathbb{A}+\mathbb{B})^{-1}(0)}(0)$. Then, $\langle -\tilde{x}, w - \tilde{x}\rangle \le 0$, $w \in \operatorname{Fix}(\mathbb{S}) \cap (\mathbb{A}+\mathbb{B})^{-1}(0)$. Therefore,

$$\overline{\lim_{n\to\infty}}\left\langle -\tilde{x}, \mathbb{S}x_n - \tilde{x}\right\rangle \le 0. \tag{76}$$

From (46), we have

$$\begin{aligned}
&\left\|x_{n+1} - \tilde{x}\right\|^2 \\
&\quad\le \varsigma_n\left\|x_n - \tilde{x}\right\|^2 + (1-\varsigma_n)\left\|J_{\tau_n}^{\mathbb{B}}u_n - \tilde{x}\right\|^2 \\
&\quad= \varsigma_n\left\|x_n - \tilde{x}\right\|^2 + (1-\varsigma_n)\left\|J_{\tau_n}^{\mathbb{B}}u_n - J_{\tau_n}^{\mathbb{B}}(\tilde{x} - \tau_n A\tilde{x})\right\|^2 \\
&\quad\le \varsigma_n\left\|x_n - \tilde{x}\right\|^2 + (1-\varsigma_n)\left\|u_n - (\tilde{x} - \tau_n\mathbb{A}\tilde{x})\right\|^2
\end{aligned}$$

$$
\begin{aligned}
&= \varsigma_n \|x_n - \tilde{x}\|^2 + (1-\varsigma_n) \\
&\quad \times \|(1-\varrho_n)\mathbb{S}x_n - \tau_n \mathbb{A}\mathbb{S}x_n - (\tilde{x} - \tau_n \mathbb{A}\tilde{x})\|^2 \\
&= (1-\varsigma_n) \left\|(1-\varrho_n)\left(\left(\mathbb{S}x_n - \frac{\tau_n}{1-\varrho_n}\mathbb{A}\mathbb{S}x_n\right)\right.\right. \\
&\qquad\qquad \left.\left. - \left(\tilde{x} - \frac{\tau_n}{1-\varrho_n}\mathbb{A}\tilde{x}\right)\right)\right. \\
&\qquad\qquad \left. + \varrho_n(-\tilde{x})\right\|^2 + \varsigma_n \|x_n - \tilde{x}\|^2 \\
&= \varsigma_n \|x_n - \tilde{x}\|^2 + (1-\varsigma_n) \\
&\quad \times \left((1-\varrho_n)^2 \left\|\left(\mathbb{S}x_n - \frac{\tau_n}{1-\varrho_n}\mathbb{A}\mathbb{S}x_n\right)\right.\right. \\
&\qquad\qquad \left. - \left(\tilde{x} - \frac{\tau_n}{1-\varrho_n}\mathbb{A}\tilde{x}\right)\right\|^2 \\
&\qquad + 2\varrho_n(1-\varrho_n)\left\langle -\tilde{x}, \left(\mathbb{S}x_n - \frac{\tau_n}{1-\varrho_n}\mathbb{A}\mathbb{S}x_n\right)\right. \\
&\qquad\qquad \left.\left. - \left(\tilde{x} - \frac{\tau_n}{1-\varrho_n}\mathbb{A}\tilde{x}\right)\right\rangle + \varrho_n^2 \|\tilde{x}\|^2\right) \\
&\le \varsigma_n \|x_n - \tilde{x}\|^2 + (1-\varsigma_n) \\
&\quad \times \left((1-\varrho_n)^2 \|x_n - \tilde{x}\|^2 + 2\varrho_n \tau_n \langle -\tilde{x}, \mathbb{A}\mathbb{S}x_n - \mathbb{A}\tilde{x}\rangle\right. \\
&\qquad \left. + 2\varrho_n(1-\varrho_n)\langle -\tilde{x}, \mathbb{S}x_n - \tilde{x}\rangle + \varrho_n^2 \|\tilde{x}\|^2\right) \\
&\le \varsigma_n \|x_n - \tilde{x}\|^2 + (1-\varsigma_n) \\
&\quad \times \left((1-\varrho_n)^2 \|x_n - \tilde{x}\|^2 + 2\varrho_n \tau_n \|\tilde{x}\| \|\mathbb{A}\mathbb{S}x_n - \mathbb{A}\tilde{x}\|\right. \\
&\qquad \left. + 2\varrho_n(1-\varrho_n)\langle -\tilde{x}, \mathbb{S}x_n - \tilde{x}\rangle + \varrho_n^2 \|\tilde{x}\|^2\right) \\
&\le [1 - 2(1-\varsigma_n)\varrho_n] \|x_n - \tilde{x}\|^2 \\
&\quad + 2\varrho_n(1-\varsigma_n)\tau_n \|\tilde{x}\| \|\mathbb{A}\mathbb{S}x_n - \mathbb{A}\tilde{x}\| \\
&\quad + 2\varrho_n(1-\varsigma_n)(1-\varrho_n)\langle -\tilde{x}, \mathbb{S}x_n - \tilde{x}\rangle \\
&\quad + (1-\varsigma_n)\varrho_n^2 \left(\|\tilde{x}\|^2 + \|x_n - \tilde{x}\|^2\right) \\
&= [1 - 2(1-\varsigma_n)\varrho_n] \|x_n - \tilde{x}\|^2 \\
&\quad + 2(1-\varsigma_n)\varrho_n \left\{\tau_n \|\tilde{x}\| \|\mathbb{A}\mathbb{S}x_n - \mathbb{A}\tilde{x}\|\right. \\
&\qquad\qquad + (1-\varrho_n)\langle -\tilde{x}, \mathbb{S}x_n - \tilde{x}\rangle \\
&\qquad\qquad \left. + \varrho_n \left(\|\tilde{x}\|^2 + \|x_n - \tilde{x}\|^2\right)\right\}.
\end{aligned} \tag{77}
$$

It is clear that $\sum_n 2(1-\varsigma_n)\varrho_n = \infty$ and

$$
\limsup_{n\to\infty} \left\{\tau_n \|\tilde{x}\| \|\mathbb{A}\mathbb{S}x_n - \mathbb{A}\tilde{x}\| + (1-\varrho_n)\right. \\
\left. \times \langle -\tilde{x}, \mathbb{S}x_n - \tilde{x}\rangle + \varrho_n \left(\|\tilde{x}\|^2 + \|x_n - \tilde{x}\|^2\right)\right\} \le 0. \tag{78}
$$

By Lemma 7, we conclude that $x_n \to \tilde{x}$. This completes the proof. □

Corollary 11. *Suppose that $(\mathbb{A}+\mathbb{B})^{-1}(0) \neq \emptyset$. Let τ be a constant satisfying $a \le \tau \le b$, where $[a,b] \subset (0, 2\varrho)$. For $t \in (0, 1-\tau/(2\varrho))$, let $\{x_t\} \subset \mathscr{C}$ be a net generated by*

$$
x_t = J_\tau^{\mathbb{B}}((1-t)x_t - \tau \mathbb{A}x_t). \tag{79}
$$

Then the net $\{x_t\}$ converges strongly, as $t \to 0+$, to a point $\tilde{x} = \mathrm{proj}_{(\mathbb{A}+\mathbb{B})^{-1}(0)}(0)$ which is the minimum norm element in $(\mathbb{A}+\mathbb{B})^{-1}(0)$.

Corollary 12. *Suppose that $(\mathbb{A}+\mathbb{B})^{-1}(0) \neq \emptyset$. For given $x_0 \in \mathscr{C}$, let $\{x_n\} \subset \mathbb{C}$ be a sequence generated by*

$$
x_{n+1} = \varsigma_n x_n + (1-\varsigma_n) J_{\tau_n}^{\mathbb{B}}((1-\varrho_n)x_n - \tau_n \mathbb{A}x_n) \tag{80}
$$

for all $n \ge 0$, where $\{\tau_n\} \subset (0, 2\varrho)$, $\{\varrho_n\} \subset (0,1)$, and $\{\varsigma_n\} \subset (0,1)$ satisfy

(i) $\lim_{n\to\infty}\varrho_n = 0$ *and* $\sum_n \varrho_n = \infty$;

(ii) $0 < \liminf_{n\to\infty}\varsigma_n \le \limsup_{n\to\infty}\varsigma_n < 1$;

(iii) $a(1-\varrho_n) \le \tau_n \le b(1-\varrho_n)$, *where* $[a,b] \subset (0, 2\varrho)$ *and* $\lim_{n\to\infty}(\tau_{n+1} - \tau_n) = 0$.

Then $\{x_n\}$ converges strongly to a point $\tilde{x} = \mathrm{proj}_{(\mathbb{A}+\mathbb{B})^{-1}(0)}(0)$ which is the minimum norm element in $(\mathbb{A}+\mathbb{B})^{-1}(0)$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

The Best Approximation Theorems and Fixed Point Theorems for Discontinuous Increasing Mappings in Banach Spaces

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We prove that Fan's theorem is true for discontinuous increasing mappings f in a real partially ordered reflexive, strictly convex, and smooth Banach space X. The main tools of analysis are the variational characterizations of the generalized projection operator and order-theoretic fixed point theory. Moreover, we get some properties of the generalized projection operator in Banach spaces. As applications of our best approximation theorems, the fixed point theorems for non-self-maps are established and proved under some conditions. Our results are generalizations and improvements of the recent results obtained by many authors.

1. Introduction

Let X be a real Banach space with the dual space X^* and $C \subset X$ a nonempty subset of X. The set-valued mapping $P_C : X \to C$,

$$P_C(x) = \left\{ z \in C : \|x - z\| = \inf_{y \in C} \|x - y\| \right\}, \qquad (1)$$

is called the metric projection operator from X onto C. It is well known that the metric projection operator P_C plays an important role in nonlinear functional analysis, optimization theory, fixed point theory, nonlinear programming, game theory, variational inequality, complementarity problems, and so forth.

In 1994, Alber [1] introduced the generalized projections $\pi_C : X^* \to C$ and $\Pi_C : X \to C$ from Hilbert spaces to uniformly convex and uniformly smooth Banach spaces and studied their properties in detail. In [2], Li extended the generalized projection operator π_C from uniformly convex and uniformly smooth Banach spaces to reflexive Banach spaces and studied some properties of the generalized projection operator with applications to solving the variational inequality in Banach spaces. Recently, Isac [3] and Nishimura and Ok [4] studied the order-theoretic approach towards establishing the solvability of variational inequality on a Hilbert lattice X which is based on the fact that the metric projection operator P_C is order-preserving if only if C is a sublattice of X. Very recently, Li and Ok [5] obtained the generalized projection operator π_C is order-preserving in partially ordered Banach spaces.

Motivated and inspired by the above mentioned work, in this paper, we get the continuous property of generalized projection operator Π_C and increasing characterizations of Π_C in a partially ordered reflexive, strict convex, and smooth Banach space. Further, we consider the following Fan's approximation theorem (Theorem 2 in [6]) through the variational characterization of Π_C. The normed space version of the theorem is as follows.

Theorem 1. *Let C be a nonempty compact convex set in a normed linear space X. If f is a continuous map from C into X, then there exists a point u in C such that*

$$\|u - f(u)\| = d(f(u), C). \qquad (2)$$

The point u is called a best approximation point of f in C.

Fan's theorem has been of great importance in nonlinear analysis, approximation theory, game theory, and minimax theorems. Various aspects of this theorem have been studied by many authors under different assumptions. For some related works, refer to [7–21] and the references therein.

In this paper, we obtain the existence of minimum best approximation point and maximum best approximation point in order interval. As an applications of our best approximation theorems, the fixed point theorems for non-self-maps are established under some conditions which do not need to require any continuous and compact conditions on f.

The content of the present work can be summarized as follows. In Section 2, we review the definition of the generalized projection operator in Banach spaces and its basic properties. We also show some definitions in the partially ordered Banach space and some fundamental results for our theorems. In Section 3, we obtain the properties of the generalized projection operator in the partially ordered Banach space under some assumption. And we combine these results with an order-theoretic fixed point theorem to provide some of the best approximation theorems. Section 4 provides an application of these best approximation theorems to fixed point theory.

2. Preliminaries

2.1. The Partial Order. Suppose that X is a real Banach space and P is a nonempty closed convex cone of X. By θ we denote the zero element of X. We define a partial order $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. We will write $x \prec y$ if $x \preceq y$ and $x \neq y$.

The cone P is called normal if there is a number $K > 0$, such that for all $x, y \in X$, $0 \preceq x \preceq y$ implies $\|x\| \le K\|y\|$. The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}$ is a sequence such that $x_1 \preceq x_2 \preceq \cdots \preceq y$ for some $y \in X$, then there is $x \in X$ such that $\lim_{n\to\infty}\|x_n - x\| = 0$. Equivalently, the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. It has been proved in Theorem 1.2.1 in [22] that every regular cone is normal.

A cone P is called minihedral, if each two-element set $\{x, y\}$ has a least upper bound $\sup\{x, y\}$. Equivalently, the cone P is minihedral if and only if each two-element set $\{x, y\}$ has a greatest lower bound $\inf\{x, y\}$. As is convenient, we denote $\sup\{x, y\}$ as $x \vee y$ and $\inf\{x, y\}$ as $x \wedge y$. And if $\sup M$ exists for every nonempty and bounded from above $M \subset X$, we say the cone P is a strongly minihedral cone. If M is a nonempty subset of X which contains $x \vee y$ and $x \wedge y$ for every $x, y \in M$, then M is said to be subminihedral.

Let $(X, \preceq)$ be a real partially ordered Banach space. Given $u_0, v_0 \in X$ such that $u_0 \prec v_0$, the set $[u_0, v_0] = \{z \in X : u_0 \preceq z \preceq v_0\}$ is called ordered interval. If the cone P is minihedral, it is easy to see that $[u_0, v_0]$ is a subminihedral set of X.

Definition 2 (see [5]). For any partially ordered spaces $(X, \preceq_X)$ and $(Y, \preceq_Y)$, we say that a map $F : X \to Y$ is order-preserving if

$$x \preceq_X y \text{ implies } F(x) \preceq_Y F(y). \tag{3}$$

Definition 3 (see [23]). Let $(X, \preceq)$ be a partially ordered space and $D \subset X$ is convex; we say that a map $F : D \to X$ is convex if

$$F(tx + (1-t)y) \preceq tF(x) + (1-t)F(y), \quad \forall x, y \in D, \quad x \preceq y, \quad 0 \le t \le 1. \tag{4}$$

2.2. Order-Dual. Let $(X, \preceq)$ be a real partially ordered Banach space whose (topological) dual we denote by X^* and P a cone in X. Recall that $P^* = \{\phi \in X^* : \phi(x) \ge 0,\ \forall x \in P\}$ is called the dual cone of P. The dual of $\preceq$ is the partial order $\preceq^*$ on X^* defined as follows:

$$\phi \preceq^* \varphi \quad \text{iff } \varphi - \phi \in P^*. \tag{5}$$

If P is a minihedral cone, it is well known that P^* is a minihedral cone in X^*. We now show that $x \in P$ if and only if $\langle \varphi, x\rangle \ge 0$ for every $\varphi \in P^*$ (see [24, Proposition 1.4.2]).

We denote by $(H, \|\cdot\|_1)$ a Hilbert space H whose norm $\|\cdot\|_1$ satisfies

$$|x| \preceq |y| \text{ implies } \|x\|_1 \le \|y\|_1, \quad \forall x, y \in H, \tag{6}$$

where $|x|$ is defined by $|x| = x \vee (-x)$ for each $x \in H$.

2.3. The Generalized Projection Operator. Let X be a real Banach space with the dual X^*. We denote by J the normalized duality mapping from X to 2^{X^*} defined by

$$Jx = \{x^* \in X^* : \langle x^*, x\rangle = \|x^*\|\|x\|,\ \|x\| = \|x^*\|\}, \tag{7}$$

for all $x \in X$, where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing between X^* and X. See [1] for basic characterizations of the normalized duality mapping.

Recall that a Banach space X has the Kadec-Klee property, if for any sequence $\{x_n\} \subset X$ and $x \in X$ with $x_n \rightharpoonup x$ (weak convergence) and $\|x_n\| \to \|x\|$, then $\|x_n - x\| \to 0$, as $n \to \infty$. It is well known that if X is a uniformly convex Banach space, then X has the Kadec-Klee property.

Let X be a reflexive, strictly convex, and smooth Banach space and C a nonempty closed convex subset of X. Consider the Lyapunov functional defined by

$$W(x, y) = \|x\|^2 - 2\langle Jx, y\rangle + \|y\|^2, \quad \forall x, y \in X. \tag{8}$$

Following Alber [1], the generalized projection $\Pi_C : X \to C$ is a map that assigns to an arbitrary point $x \in X$ the minimum point of the functional $W(x, y)$; that is, $\Pi_C(x) = \hat{x}$, where $\hat{x} \in C$ is the solution to the minimization problem:

$$W(x, \hat{x}) = \inf_{y\in C} W(x, y); \tag{9}$$

existence and uniqueness of the operator Π_C follow from the properties of the functional $W(x, y)$ and strict monotonicity of the mapping J. It is obvious from the definition of functional W that

$$(\|x\| - \|y\|)^2 \le W(x, y) \le (\|x\| + \|y\|)^2, \quad \forall x, y \in X. \tag{10}$$

If X is a Hilbert space, then $W(x, y) = (\|x - y\|)^2$ and $\Pi_C = P_C$.

If X is a reflexive, strictly convex, and smooth Banach space, then for $x, y \in X, W(x, y) = 0$ if and only if $x = y$. It is sufficient to show that if $W(x, y) = 0$ then $x = y$. From (10), we have $\|x\| = \|y\|$. This implies that $\langle Jx, y\rangle = \|y\|^2 = \|Jx\|^2$. From the definition of J, one has $Jx = Jy$. Therefore, we have $x = y$. See [25, 26] for more details.

In [1], the generalized projection operators on arbitrary convex closed sets C satisfy the following property.

The point $\Pi_C(x) = \hat{x}$ is a generalized projection of x on $C \subset X$ if and only if the following inequality is satisfied:

$$\langle Jx - J\hat{x}, \hat{x} - y\rangle \geq 0, \quad \forall y \in C. \tag{11}$$

We denote $d_W(x, C) = \inf\{W(x, y) : y \in C\}$, where $x \in X$ and W is Lyapunov functional in X.

3. Best Approximation Theorems

First we give the following properties of the generalized projection operators.

Lemma 4 (see [27]). *Let $(X, \preceq)$ be a real partially ordered reflexive, strictly convex, and smooth Banach space with respect to a minihedral cone P. Suppose P^* is the dual cone of P. The following statements are equivalent:*

(H_1) the normalized duality mapping J is order-preserving;

(H_2) $\|Jx \wedge Jy\|^2 + \|Jx \vee Jy\|^2 \leq \|x\|^2 + \|y\|^2, \forall x, y \in X, x \preceq y$.

Lemma 5 (see [27]). *Let $(X, \preceq)$ be a real partially ordered reflexive, strictly convex, and smooth Banach space with respect to a minihedral cone P and satisfy condition (H_2). Suppose that C is closed convex subminihedral set of X. Moreover, C satisfies the condition:*

(H_3) $\|x \wedge y\|^2 + \|x \vee y\|^2 \leq \|x\|^2 + \|y\|^2, \ \forall x, y \in C$.

Then, Π_C is increasing.

Remark 6. The minihedral cones of many Banach spaces satisfy (H_3). For example, if $p \geq 2$, every subminihedral set M of $(\ell^p, \preceq)$ (here partial order $\preceq$ is defined coordinatewise) such that $x \succeq \theta, \ \forall x \in M$, then M satisfies (H_3); if $p \geq 2$, every subminihedral set M of $(R^{n,p}, \preceq)$ (here $\preceq$ stands again for the coordinatewise ordering), such that $x \succeq \theta, \ \forall x \in M$, then M satisfies (H_3). See [5] for more details.

Lemma 7. *If X is a uniformly convex and smooth Banach space and C is a nonempty, closed, and convex subset of X, then the generalized projection operator $\Pi_C : X \to C$ is continuous.*

Proof. Since X is a uniformly convex and smooth Banach space, Π_C is single valued. Suppose $x_n \to x$, as $n \to \infty$, and suppose $\Pi_C(x_n) = \hat{x}_n$ $(n = 1, 2, 3, \ldots)$, and $\Pi_C(x) = \hat{x}$. From the inequalities

$$\begin{aligned}(\|x_n\| - \|\hat{x}_n\|)^2 &\leq W(x_n, \hat{x}_n) \\ &\leq W(x_n, \hat{x}) \\ &\leq (\|x_n\| + \|\hat{x}\|)^2\end{aligned} \tag{12}$$

and the hypothesis that $x_n \to x$, as $n \to \infty$, it yields $\{\hat{x}_n\}$ is a bounded subset of X. Since X is reflexive, there exists a subsequence of $\{\hat{x}_n\}$; without loss of the generality, we may assume it is itself, such that $\{\hat{x}_n\}$ converges weakly to x'. From the properties of weakly convergence, we have $\|x'\| \leq \liminf_{n\to\infty} \|\hat{x}_n\|$. Moreover, $W(x, \hat{x}) \leq W(x, \hat{x}_n)$ and $W(x_n, \hat{x}_n) \leq W(x_n, \hat{x})$, which implies $W(x, \hat{x}_n) \to W(x, \hat{x})$, as $n \to \infty$. Now we have

$$\begin{aligned}W(x, x') &= \|x\|^2 - 2\langle Jx, x'\rangle + \|x'\|^2 \\ &= \lim_{n\to\infty}\left(\|x\|^2 - 2\langle Jx, \hat{x}_n\rangle + \|x'\|^2\right) \\ &\leq \liminf_{n\to\infty}\left(\|x\|^2 - 2\langle Jx, \hat{x}_n\rangle + \|\hat{x}_n\|^2\right) \\ &= \liminf_{n\to\infty} W(x, \hat{x}_n) \\ &= \lim_{n\to\infty} W(x, \hat{x}_n) \\ &= \inf_{y\in C} W(x, y).\end{aligned} \tag{13}$$

Thus we have $x' = \hat{x}$.

For any $\lambda \in [0, 1]$, one has $\lambda\hat{x} + (1-\lambda)\hat{x}_n \in C$. From the inequality $W(x, \hat{x}) \leq W(x, \lambda\hat{x} + (1-\lambda)\hat{x}_n)$, we have

$$\begin{aligned}&\|x\|^2 - 2\langle Jx, \hat{x}\rangle + \|\hat{x}\|^2 \leq \|x\|^2 \\ &\quad \leq \|x\|^2 - 2\langle Jx, \lambda\hat{x} + (1-\lambda)\hat{x}_n\rangle + \|\lambda\hat{x} + (1-\lambda)\hat{x}_n\|^2.\end{aligned} \tag{14}$$

Therefore,

$$2\langle Jx, (1-\lambda)(\hat{x}_n - \hat{x})\rangle \leq \|\lambda\hat{x} + (1-\lambda)\hat{x}_n\|^2 - \|\hat{x}\|^2. \tag{15}$$

Similar to the above argument, from inequality $W(x_n, \hat{x}_n) \leq W(x_n, \hat{x})$, we obtain

$$2\langle Jx_n, \hat{x} - \hat{x}_n\rangle \leq \|\hat{x}\|^2 - \|\hat{x}_n\|^2. \tag{16}$$

Adding the above two inequalities side by side, we obtain

$$\begin{aligned}&2\langle Jx - Jx_n, \hat{x}_n - \hat{x}\rangle \\ &\quad \leq \|\lambda\hat{x} + (1-\lambda)\hat{x}_n\|^2 \\ &\qquad - \|\hat{x}_n\|^2 + 2\lambda\langle Jx, \hat{x}_n - \hat{x}\rangle \\ &\quad \leq \lambda^2\|\hat{x}\|^2 \\ &\qquad + 2\lambda(1-\lambda)\|\hat{x}\|\|\hat{x}_n\| + (1-\lambda)^2\|\hat{x}_n\|^2\end{aligned}$$

$$\begin{aligned}&-\|\widehat{x}_n\|^2+2\lambda\langle Jx,\widehat{x}_n-\widehat{x}\rangle\\ &\le\lambda^2\|\widehat{x}\|^2+\lambda(1-\lambda)\left(\|\widehat{x}\|^2+\|\widehat{x}_n\|^2\right)\\ &\quad+(1-\lambda)^2\|\widehat{x}_n\|^2-\|\widehat{x}_n\|^2+2\lambda\langle Jx,\widehat{x}_n-\widehat{x}\rangle\\ &=\lambda\left(\|\widehat{x}\|^2-\|\widehat{x}_n\|^2\right)+2\lambda\langle Jx,\widehat{x}_n-\widehat{x}\rangle.\end{aligned}\tag{17}$$

So

$$2\langle Jx-Jx_n,\widehat{x}-\widehat{x}_n\rangle\ge\lambda\left(\|\widehat{x}_n\|^2-\|\widehat{x}\|^2\right)+2\lambda\langle Jx,\widehat{x}-\widehat{x}_n\rangle.\tag{18}$$

If we use the inequalities $W(x,\widehat{x})\le W(x,\widehat{x}_n)$ and $W(x_n,\widehat{x}_n)\le W(x_n,\lambda\widehat{x}+(1-\lambda)\widehat{x}_n)$, similar to the above argument, we obtain

$$\begin{aligned}2\langle Jx-Jx_n,\widehat{x}-\widehat{x}_n\rangle&\ge(1-\lambda)\left(\|\widehat{x}\|^2-\|\widehat{x}_n\|^2\right)\\ &\quad+2(1-\lambda)\langle Jx_n,\widehat{x}_n-\widehat{x}\rangle.\end{aligned}\tag{19}$$

In (18) and (19), taking $\lambda=1/2$, we have

$$\begin{aligned}&4\langle Jx-Jx_n,\widehat{x}-\widehat{x}_n\rangle\ge\left(\|\widehat{x}_n\|^2-\|\widehat{x}\|^2\right)+2\langle Jx,\widehat{x}-\widehat{x}_n\rangle,\\ &4\langle Jx-Jx_n,\widehat{x}-\widehat{x}_n\rangle\ge\left(\|\widehat{x}\|^2-\|\widehat{x}_n\|^2\right)+2\langle Jx_n,\widehat{x}_n-\widehat{x}\rangle.\end{aligned}\tag{20}$$

From the conditions that $x_n\to x$, as $n\to\infty$ and X is a smooth Banach space, we have $Jx_n\to Jx$, as $n\to\infty$. Using $\widehat{x}_n\rightharpoonup\widehat{x}$, as $n\to\infty$ and combining (20), it yields $\|\widehat{x}_n\|\to\|\widehat{x}\|$, as $n\to\infty$. Since X is a uniformly convex Banach space, then X has the Kadec-Klee property. Therefore, we obtain $\widehat{x}_n\to\widehat{x}$, as $n\to\infty$. Thus this lemma is proved. □

Lemma 8. *Let $(X,\preceq)$ be a real partially ordered reflexive, strictly convex, and smooth Banach space with respect to P and satisfy condition (H_2). Suppose that P is a minihedral cone and satisfies the condition:*

(H_4) $\|x\wedge y\|^2+\|x\vee y\|^2\le\|x\|^2+\|y\|^2$, $\forall x\in X$, $y\in P$.

Then, Π_P is increasing, and $\Pi_P(x+y)\preceq\Pi_P(x)+\Pi_P(y)$, $\forall x,y\in X$.

Proof. Since (H_4) implies (H_3) and P is subminihedral, from Lemma 5, Π_P is increasing. Next, we prove $x\preceq\Pi_P(x)$, $\forall x\in X$. To derive a contradiction, assume that there exists x_0 which does not satisfy $x_0\preceq\Pi_P(x_0)$; that is, $x_0\wedge\Pi_P(x_0)\ne x_0$ and $x_0\vee\Pi_P(x_0)\ne\Pi_P(x_0)$. Then we have

$$W(x_0,\Pi_P(x_0))<W(x_0,x_0\vee\Pi_P(x_0));\tag{21}$$

that is,

$$\begin{aligned}&\|x_0\|^2-2\langle Jx_0,\Pi_P(x_0)\rangle+\|\Pi_P(x_0)\|^2\\ &\quad<\|x_0\|^2-2\langle Jx_0,x_0\vee\Pi_P(x_0)\rangle+\|x_0\vee\Pi_P(x_0)\|^2.\end{aligned}\tag{22}$$

Hence,

$$\begin{aligned}&2\langle Jx_0,x_0\vee\Pi_P(x_0)-\Pi_P(x_0)\rangle\\ &\quad<\|x_0\vee\Pi_P(x_0)\|^2-\|\Pi_P(x_0)\|^2.\end{aligned}\tag{23}$$

As $x_0\wedge\Pi_P(x_0)\ne x_0$, we have

$$\begin{aligned}W(x_0,x_0\wedge\Pi_P(x_0))&=\|x_0\|^2-2\langle Jx_0,x_0\wedge\Pi_P(x_0)\rangle\\ &\quad+\|x_0\wedge\Pi_P(x_0)\|^2>0,\end{aligned}\tag{24}$$

and then,

$$2\langle Jx_0,x_0\wedge\Pi_P(x_0)\rangle<\|x_0\|^2+\|x_0\wedge\Pi_P(x_0)\|^2.\tag{25}$$

Since $x_0\wedge\Pi_P(x_0)+x_0\vee\Pi_P(x_0)=x_0+\Pi_P(x_0)$, from (23) and (25), we have

$$\begin{aligned}2\langle Jx_0,x_0\rangle&<\|x_0\|^2+\|x_0\wedge\Pi_P(x_0)\|^2\\ &\quad+\|x_0\vee\Pi_P(x_0)\|^2-\|\Pi_P(x_0)\|^2.\end{aligned}\tag{26}$$

And hence $\|x_0\wedge\Pi_P(x_0)\|^2+\|x_0\vee\Pi_P(x_0)\|^2-\|\Pi_P(x_0)\|^2-\|x_0\|^2>0$. This contradicts (H_4). Thus, $x\preceq\Pi_P(x)$, $\forall x\in X$. And hence,

$$x+y\preceq\Pi_P(x)+\Pi_P(y),\quad\forall x,y\in X.\tag{27}$$

As Π_P is increasing, we have

$$\Pi_P(x+y)\preceq\Pi_P(x)+\Pi_P(y),\quad\forall x,y\in X.\tag{28}$$

The assertion is proved. □

Lemma 9. *Let $(X,\preceq)$ be a real partially ordered reflexive, strictly convex, and smooth Banach space with respect to a minihedral cone P and satisfy condition (H_2). Suppose $u_0,v_0\in X$ with $u_0\prec v_0$ and the following condition is satisfied:*

(H_5) $\|x\wedge y\|^2+\|x\vee y\|^2\le\|x\|^2+\|y\|^2$, $\forall x\in X$, $y\in[u_0,v_0]$.

Then, $\Pi_{[u_0,v_0]}$ is increasing, and

$$\begin{aligned}&\Pi_{[u_0,v_0]}(tx+(1-t)y)\preceq t\Pi_{[u_0,v_0]}(x)+(1-t)\Pi_{[u_0,v_0]}(y)\\ &\qquad\forall t\in[0,1],\quad\forall x,y\preceq v_0.\end{aligned}\tag{29}$$

Proof. Following a similar argument as in the proof of Lemma 8, we obtain that $\Pi_{[u_0,v_0]}$ is increasing and $x\preceq\Pi_{[u_0,v_0]}(x)$, $\forall x\preceq v_0$. And hence,

$$\begin{aligned}&tx+(1-t)y\preceq t\Pi_{[u_0,v_0]}(x)+(1-t)\Pi_{[u_0,v_0]}(y),\\ &\qquad\forall t\in[0,1],\quad x,y\preceq v_0.\end{aligned}\tag{30}$$

As $\Pi_{[u_0,v_0]}$ is increasing and $t\Pi_{[u_0,v_0]}(x)+(1-t)\Pi_{[u_0,v_0]}(y)\in[u_0,v_0]$, we have

$$\Pi_{[u_0,v_0]}(tx+(1-t)y)\preceq t\Pi_{[u_0,v_0]}(x)+(1-t)\Pi_{[u_0,v_0]}(y).\tag{31}$$

The proof is completed. □

Remark 10. If $(H, \|\cdot\|_1)$ is a partially ordered Hilbert space with respect to P and P a minihedral cone, (H_4) and (H_5) are satisfied.

From the above properties of the generalized projection operators and order-theoretic fixed point theorems, we can obtain the following best approximation theorems.

Theorem 11. *Let $(X, \preceq)$ be a real partially ordered uniformly convex and smooth Banach space with respect to a minihedral cone P and satisfy condition (H_2). Suppose that $f : [u_0, v_0] \to X$ is an increasing map. Moreover, $[u_0, v_0]$ satisfies the condition (H_3) and $f([u_0, v_0])$ is relatively compact. Then, f has a minimum best approximation point x_* and a maximum best approximation point x^* with respect to $W(x, y)$ in $[u_0, v_0]$, such that*

$$\begin{aligned} u_0 \preceq u_1 \preceq \cdots \preceq u_n \preceq \cdots \preceq x_* \preceq x^* \\ \preceq \cdots \preceq v_n \preceq \cdots \preceq v_1 \preceq v_0, \end{aligned} \tag{32}$$

where $u_n = \Pi_{[u_0,v_0]}(f(u_{n-1}))$, $v_n = \Pi_{[u_0,v_0]}(f(v_{n-1}))$ $(n = 1, 2, 3, \ldots)$.

Proof. Define $F : [u_0, v_0] \to [u_0, v_0]$ by $F(x) = \Pi_{[u_0,v_0]}(f(x))$. From Lemma 5, we get F is increasing. It is easy to see $u_0 \preceq F(u_0)$ and $F(v_0) \preceq v_0$. By Lemma 7, we know $\Pi_{[u_0,v_0]}$ is continuous and $F([u_0, v_0])$ is relatively compact. Thus F satisfies all conditions of Theorem 2.1.4 in [22]. Then, F has a minimum fixed point x_* and a maximum fixed point x^* and satisfies (32). Now we consider $F(x_*) = x_*$, $F(x^*) = x^*$; that is, $\Pi_{[u_0,v_0]}(f(x_*)) = x_*$ and $\Pi_{[u_0,v_0]}(f(x^*)) = x^*$. By the definition of $\Pi_{[u_0,v_0]}$, we get

$$\begin{aligned} W(f(x_*), x_*) &= \inf_{y \in [u_0,v_0]} W(f(x_*), y) \\ &= d_W(f(x_*), [u_0, v_0]), \\ W(f(x^*), x^*) &= \inf_{y \in [u_0,v_0]} W(f(x^*), y) \\ &= d_W(f(x^*), [u_0, v_0]). \end{aligned} \tag{33}$$

The assertion is proved. □

Theorem 12. *Let $(X, \preceq)$ be a real partially ordered reflexive, strictly convex, and smooth Banach space with respect to a normal and minihedral cone P and satisfy condition (H_2). Suppose that $f : [u_0, v_0] \to X$ is an increasing map. Moreover, $[u_0, v_0]$ satisfies the condition (H_3). Then, f has a minimum best approximation point x_* and a maximum best approximation point x^* with respect to $W(x, y)$ in $[u_0, v_0]$. Moreover, if $u_n = \Pi_{[u_0,v_0]}(f(u_{n-1}))$, $v_n = \Pi_{[u_0,v_0]}(f(v_{n-1}))$ $(n = 1, 2, 3, \ldots)$, (32) holds.*

Proof. Define $F : [u_0, v_0] \to [u_0, v_0]$ by $F(x) = \Pi_{[u_0,v_0]}(f(x))$. From Lemma 5, we get F is increasing. It is easy to see $u_0 \preceq F(u_0)$ and $F(v_0) \preceq v_0$. Since X is reflexive and P is normal, P is regular. Thus F satisfies all conditions of Theorem 3.1.4 in [23]. Then, F has a minimum fixed point x_* and a maximum fixed point x^* and satisfies (32). By the definition of $\Pi_{[u_0,v_0]}$, the assertion is proved. □

Remark 13. In the above Theorem 11, f is discontinuous map. And in Theorem 12, f is discontinuous map and has no compact conditions.

Example 14. Let $(X, \preceq) = (\ell^2, \preceq)$. Here $\preceq$ stands for the coordinatewise ordering. It is easy to prove that all conditions in Theorem 12 hold. Given $u_0, v_0 \in \ell^2$ such that $u_0 \prec v_0$. Then, every increasing $f : [u_0, v_0] \to \ell^2$ has a minimum best approximation point and a maximum best approximation point with respect to $W(x, y)$ in $[u_0, v_0]$.

Theorem 15. *Let $(X, \preceq)$ be a real partially ordered reflexive, strictly convex, and smooth Banach space with respect to P. If $u_0 \prec v_0$ and the following conditions are satisfied,*

(i) *P is a normal, minihedral cone with satisfying (H_2) and (H_5);*
(ii) *$f : [u_0, v_0] \to X$ is an increasing and convex map;*
(iii) *there exists a $0 < \varepsilon < 1$ such that $f(v_0) \preceq \varepsilon u_0 + (1-\varepsilon) v_0$,*

then, f has a unique approximation point $\hat{x}$ with respect to $W(x, y)$ in $[u_0, v_0]$. Moreover, if we take $x_n = \Pi_{[u_0,v_0]}(f(x_{n-1}))$ $(n = 1, 2, 3, \ldots)$ for $\forall x_0 \in [u_0, v_0]$,

$$\|x_n - \hat{x}\| \longrightarrow 0 \quad (n \longrightarrow \infty), \tag{34}$$

$$\|x_n - \hat{x}\| \le M (1-\varepsilon)^n \quad (n = 1, 2, 3, \ldots), \tag{35}$$

where $M > 0$ has nothing to do with x_0.

Proof. Define $F : [u_0, v_0] \to [u_0, v_0]$ by $F(x) = \Pi_{[u_0,v_0]}(f(x))$. Since f is convex and $\Pi_{[u_0,v_0]}$ is increasing, for $\forall t \in [0, 1]$, we have

$$\begin{aligned} F(tx + (1-t)y) &= \Pi_{[u_0,v_0]}(f(tx + (1-t)y)) \\ &\preceq \Pi_{[u_0,v_0]}(tf(x) + (1-t)f(y)). \end{aligned} \tag{36}$$

Using Lemma 9 and $f(x) \preceq f(v_0) \preceq v_0$, we obtain

$$\begin{aligned} F(tx + (1-t)y) &\preceq t\Pi_{[u_0,v_0]}(f(x)) + (1-t)\Pi_{[u_0,v_0]}(f(y)) \\ &= tF(x) + (1-t)F(y). \end{aligned} \tag{37}$$

Thus F is convex. And $F(v_0) \preceq \varepsilon u_0 + (1-\varepsilon) v_0$. Thus F satisfies all conditions of Theorem 3.1.6 in [23]. Then, F has a unique fixed point $\hat{x}$ and satisfies (35). By the definition of $\Pi_{[u_0,v_0]}$, the assertion is proved. □

4. Fixed Point Theorems

In this section, we will prove some new fixed point theorems for non-self-maps by using results of Section 3.

Theorem 16. *Let $(X, \preceq)$ be a real partially ordered uniformly convex and smooth Banach space with respect to a minihedral cone P and satisfy condition (H_2). Suppose that $f : [u_0, v_0] \to X$ is an increasing map and $f([u_0, v_0])$ is relative compact. Moreover, $[u_0, v_0]$ satisfies the condition (H_3) and*

$$\left|\operatorname{co}\{x, f(x)\} \cap [u_0, v_0]\right| \ge 2, \quad \forall x \in [u_0, v_0]. \tag{38}$$

Then, f has at least one fixed point in $[u_0, v_0]$.

Proof. By Theorem 11, f has at least one best approximation point $\widehat{x}$ in $[u_0, v_0]$; that is, $\Pi_{[u_0,v_0]}(f(\widehat{x})) = \widehat{x}$. From (11), we have

$$\langle J(f(\widehat{x})) - J\widehat{x}, \widehat{x} - y\rangle \geq 0, \quad \forall y \in [u_0, v_0]. \tag{39}$$

We may use (38) to find a $\lambda \in (0,1]$ such that $(1-\lambda)\widehat{x}+\lambda f(\widehat{x}) \in [u_0, v_0]$, and hence

$$\langle J(f(\widehat{x})) - J\widehat{x}, \widehat{x} - [(1-\lambda)\widehat{x} + \lambda f(\widehat{x})]\rangle \geq 0; \tag{40}$$

that is,

$$\langle J(f(\widehat{x})) - J\widehat{x}, \widehat{x} - f(\widehat{x})\rangle \geq 0. \tag{41}$$

Moreover,

$$\begin{aligned}
&\langle J(f(\widehat{x})) - J\widehat{x}, f(\widehat{x}) - \widehat{x}\rangle \\
&\quad = \|f(\widehat{x})\|^2 - \langle J(f(\widehat{x})), \widehat{x}\rangle - \langle J\widehat{x}, f(\widehat{x})\rangle + \|\widehat{x}\|^2 \\
&\quad \geq \|f(\widehat{x})\|^2 - 2\|f(\widehat{x})\|\|\widehat{x}\| + \|\widehat{x}\|^2 \\
&\quad = (\|f(\widehat{x})\| - \|\widehat{x}\|)^2 \geq 0.
\end{aligned} \tag{42}$$

So we conclude that $\langle J(f(\widehat{x})) - J\widehat{x}, f(\widehat{x}) - \widehat{x}\rangle = 0$. It follows that $\|f(\widehat{x})\| = \|\widehat{x}\|$. Moreover, as $\langle J(f(\widehat{x})), \widehat{x}\rangle \leq \|f(\widehat{x})\|\|\widehat{x}\|$, and the inequality above must hold as an equality. We have $\langle J(f(\widehat{x})), \widehat{x}\rangle = \|f(\widehat{x})\|\|\widehat{x}\|$. Therefore, $J(f(\widehat{x})) = J\widehat{x}$. And thus $f(\widehat{x}) = \widehat{x}$. The assertion is proved. □

Following a similar argument as in the proof of Theorem 16, we can obtain the following fixed point theorems.

Theorem 17. *Let $(X, \preceq)$ be a real partially ordered uniformly convex and smooth Banach space with respect to P and satisfy condition (H_2). Suppose that P is a normal, minihedral cone and $f : [u_0, v_0] \to X$ is an increasing map. Moreover, $[u_0, v_0]$ satisfies the condition (H_3) and (38). Then, f has at least one fixed point in $[u_0, v_0]$.*

Example 18. Let $(X, \preceq) = (L^2(\Omega), \preceq)$, the space of measurable functions which are the 2nd power summable on Ω. Endow $L^2(\Omega)$ with the following norm and the cone P:

$$\begin{gathered}
\|x\| = \left(\int_\Omega |x(t)|^2 \, d\mu\right)^{1/2}, \\
P = \left\{x \in L^2(\Omega) : x(t) \geq 0, \forall \text{a.e. } t \in \Omega\right\}.
\end{gathered} \tag{43}$$

Given $u_0, v_0 \in L^2(\Omega)$ such that $u_0 \prec v_0$. It is easy to see that $(L^2(\Omega), \preceq)$ satisfies (H_2) and (H_3) holds in $[u_0, v_0]$. Thus, by Theorem 17, every increasing $f : [u_0, v_0] \to L^2(\Omega)$ satisfying (38) has at least one fixed point in $[u_0, v_0]$.

Theorem 19. *Let $(X, \preceq)$ be a real partially ordered reflexive, strictly convex, and smooth Banach space with respect to P. If $u_0 \prec v_0$ and the following conditions are satisfied,*

(i) *P is a normal, minihedral cone with satisfying (H_2), (H_5) and (38);*

(ii) *$f : [u_0, v_0] \to X$ is an increasing and convex map;*

(iii) *there exists $0 < \varepsilon < 1$ such that $f(v_0) \preceq \varepsilon u_0 + (1-\varepsilon)v_0$,*

then, f has a unique fixed point $\widehat{x}$ in $[u_0, v_0]$. Moreover, if we take $x_n = \Pi_{[u_0,v_0]}(f(x_{n-1}))$ $(n = 1, 2, 3, \ldots)$ for $\forall x_0 \in [u_0, v_0]$,

$$\begin{gathered}
\|x_n - \widehat{x}\| \longrightarrow 0 \quad (n \longrightarrow \infty), \\
\|x_n - \widehat{x}\| \leq M(1-\varepsilon)^n \quad (n = 1, 2, 3, \ldots),
\end{gathered} \tag{44}$$

where $M > 0$ has nothing to do with x_0.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Fixed Points Results for α-Admissible Mapping of Integral Type on Generalized Metric Spaces

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We introduce generalized (α, ψ)-contractive mappings of integral type in the context of generalized metric spaces. The results of this paper generalize and improve several results on the topic in literature.

1. Introduction and Preliminaries

In fixed point theory, one of the interesting research trends is to investigate the existence and uniqueness of certain mappings in the various abstract spaces. As a result of this approach, the notion of metric has been extended in several ways to get distinct abstract spaces. Among all, we mention the concept of generalized metric space that was introduced by Branciari [1] in 2001. The notion of generalized metric can be considered as a natural extension of the concept of a metric since it is obtained by replacing the the triangle inequality condition by a weaker condition, namely, quadrilateral inequality. Branciari [1] proved Banach's fixed point theorem in such a space. For more details, the reader can refer to [2–21].

At this point, we emphasize why the generalized metric space is interesting. Although the definitions of metric and generalized metric are very close to each other, the topology of the corresponding spaces is very different. In particular, a generalized metric may or may not be continuous. Furthermore, a convergent sequence in generalized metric spaces need not be Cauchy. Besides them, we cannot guarantee that a generalized metric space is Hausdorff, and hence the uniqueness of limits cannot be provided easily.

On the other hand, a notion of α-admissible mappings was defined by Samet et al. [22]. By using this notion, the authors introduced $\alpha - \psi$ contractive mappings and investigated the existence and uniqueness of a fixed point of such mappings in the context of metric space. Their results have attracted several authors since they are very interesting and that several existing fixed point theorems listed as consequences of the main result of this paper [22]. The approaches used in this paper have been extended and improved by a number of authors to get similar results in different settings; see, for example, [13, 15, 23–26].

The aim of this paper is to examine the existence and uniqueness of fixed points of α-admissible mappings of integral type in the setting of generalized metric spaces. We also underline that the phrase "a generalized metric" has been used for distinct notions since all such concepts generalize the notion of metric. For this reason, when we mention a "generalized metric" we mean the distance function introduced by Branciari [1]. It is evident that any metric space is a generalized metric space but the converse is not true [1].

For the sake of completeness, we recall some basic definitions and notations and fundamental results that will be used in the sequel.

$\mathbb{N}$ and $\mathbb{R}^+$ denote the set of positive integers and the set of nonnegative reals, respectively. Let Ψ be the family of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(i) ψ is upper semicontinuous;

(ii) $(\psi^n(t))_{n\in\mathbb{N}}$ converges to 0 as $n \to \infty$ for all $t > 0$;

(iii) $\psi(t) < t$, for any $t > 0$.

In the following, we recall the notion of a generalized metric space.

Definition 1 (see [1]). Let X be a nonempty set and let as $d : X\times X \to [0,\infty]$ satisfy the following conditions for all $x, y \in X$ and all distinct $u, v \in X$ each of which is different from x and y. Consider

$$\begin{aligned} &\text{(GMS1)} \quad d(x,y)=0 \text{ if and only if } x=y \\ &\text{(GMS2)} \quad d(x,y)=d(y,x) \\ &\text{(GMS3)} \quad d(x,y)\le d(x,u)+d(u,v)+d(v,y). \end{aligned} \tag{1}$$

Then, the map d is called a generalized metric and abbreviated as GMS. Here, the pair (X, d) is called a generalized metric space.

In the above definition, if d satisfies only (GMS1) and (GMS2), then it is called a semimetric (see, e.g., [27]).

The concepts of convergence, Cauchy sequence, completeness, and continuity on a GMS are defined below.

Definition 2.

(1) A sequence $\{x_n\}$ in a GMS (X, d) is GMS convergent to a limit x if and only if $d(x_n, x) \to 0$ as $n \to \infty$.

(2) A sequence $\{x_n\}$ in a GMS (X, d) is GMS Cauchy if and only if for every $\varepsilon > 0$ there exists positive integer $N(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for all $n > m > N(\varepsilon)$.

(3) A GMS (X, d) is called complete if every GMS Cauchy sequence in X is GMS convergent.

(4) A mapping $T : (X, d) \to (X, d)$ is continuous if for any sequence $\{x_n\}$ in X for which $\lim_{n\to\infty} d(x_n, x) = 0$, we have $\lim_{n\to\infty} d(Tx_n, Tx) = 0$.

The following assumption was suggested by Wilson [27] to replace the triangle inequality with the weakened condition.

(W): for each pair of (distinct) points u, v, there is a number $r_{u,v} > 0$ such that for every $z \in X$

$$r_{u,v} < d(u,z) + d(z,v). \tag{2}$$

Proposition 3 (see [28]). *In a semimetric space, the assumption* (W) *is equivalent to the assertion that limits are unique.*

Proposition 4 (see [28]). *Suppose that* $\{x_n\}$ *is a Cauchy sequence in a GMS* (X, d) *with* $\lim_{n\to\infty} d(x_n, u) = 0$, *where* $u \in X$. *Then* $\lim_{n\to\infty} d(x_n, z) = d(u, z)$ *for all* $z \in X$. *In particular, the sequence* $\{x_n\}$ *does not converge to* z *if* $z \neq u$.

The following concepts were defined by Samet et al. [22].

Definition 5 (see [22]). For a nonempty set X, let $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$ be mappings. We say that T is α-admissible if for all $x, y \in X$, one has

$$\alpha(x,y)\ge 1 \Longrightarrow \alpha(Tx,Ty)\ge 1. \tag{3}$$

In what follows we recall the notion of a $\alpha-\psi$ contractive mapping.

Definition 6 (see [22]). Let (X, d) be a metric space and let $T : X \to X$ be a given mapping. One says that T is a $\alpha-\psi$ contractive mapping if there exist two functions $\alpha : X\times X \to [0,\infty)$ and a certain ψ such that

$$\alpha(x,y)\, d(Tx,Ty) \le \psi(d(x,y)), \quad \forall x,y\in X. \tag{4}$$

Notice that any contractive mapping, that is a mapping satisfying the Banach contraction, is a $\alpha-\psi$ contractive mapping with $\alpha(x, y) = 1$ for all $x, y \in X$ and $\psi(t) = kt$, $k \in (0, 1)$.

Inspired by the results of Samet et al. [22], Karapınar [13] gave the analog of the notion of a $\alpha-\psi$ contractive mapping in the context of generalized metric spaces as follows.

Definition 7. Let (X, d) be a generalized metric space and let $T : X \to X$ be a given mapping. One says that T is a $\alpha-\psi$ contractive mapping if there exist two functions $\alpha : X\times X \to [0,\infty)$ and a certain ψ such that

$$\alpha(x,y)\, d(Tx,Ty) \le \psi(d(x,y)), \quad \forall x,y\in X. \tag{5}$$

Let (X, d) be a generalized metric space. A sequence $\{x_n\}$ is called regular if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \ge 1$ for all n and $x_n \to x \in X$ as $n \to \infty$; then $\alpha(x_n, x) \ge 1$ for all n.

Karapınar [13] also stated the following fixed point theorems.

Theorem 8. *Let* (X, d) *be a complete generalized metric space and let* $T : X \to X$ *be a* $\alpha-\psi$ *contractive mapping. Suppose that*

(i) T *is* α*-admissible;*

(ii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \ge 1$ *and* $\alpha(x_0, T^2x_0) \ge 1$;

(iii) *either* T *is continuous or* $\{x_n\}$ *is regular.*

Then there exists a $u \in X$ *such that* $Tu = u$.

For the uniqueness, an additional condition was considered.

(U): for all $x, y \in \mathrm{Fix}(T)$, one has $\alpha(x, y) \ge 1$, where $\mathrm{Fix}(T)$ denotes the set of fixed points of T.

Theorem 9. *Adding condition* (U) *to the hypotheses of Theorem 8, one obtains that* u *is the unique fixed point of* T.

As an alternative condition for the uniqueness of a fixed point of a $\alpha-\psi$ contractive mapping, one will consider the following hypothesis.

(H): for all $x, y \in \mathrm{Fix}(T)$, there exists $z \in X$ such that $\alpha(x, z) \ge 1$ and $\alpha(y, z) \ge 1$.

Theorem 10. *Adding conditions* (H) *and* (W) *to the hypotheses of Theorem 8, one obtains that* u *is the unique fixed point of* T.

Corollary 11. *Adding condition* (H) *to the hypotheses of Theorem 8 and assuming that* (X, d) *is Hausdorff, one obtains that* u *is the unique fixed point of* T.

2. Main Results

In this section, we will present our main results. For this purpose, we first define the following class of functions: $\Phi = \{\varphi : \varphi : \mathbb{R}^+ \to \mathbb{R}\}$ such that φ is nonnegative, Lebesgue integrable and satisfies

$$\int_0^{\epsilon} \varphi(t)\, dt > 0 \quad \text{for each } \epsilon > 0. \tag{6}$$

Definition 12 (see [29]). One says that $\phi \in \Phi$ is an integral subadditive if for each $a, b > 0$, one has

$$\int_0^{a+b} \phi(t)\, dt \leq \int_0^{a} \phi(t)\, dt + \int_0^{b} \phi(t)\, dt. \tag{7}$$

One denotes by Φ_s the class of all integral subadditive functions $\phi \in \Phi$.

Example 13 (see [29]). Let $\phi_1(t) = (1/2)(t+1)^{-1/2}$ for all $t \geq 0$, $\phi_2(t) = (2/3)(t+1)^{-1/3}$ for all $t \geq 0$, and $\phi_3(t) = e^{-t}$ for all $t \geq 0$. Then $\phi_i \in \Phi_s$, where $i = 1, 2, 3$.

In what follows we introduce notions of generalized α–ψ-contractive type mappings of integral type I and type II.

Definition 14. Let (X, d) be a generalized metric space and let $T : X \to X$ be a given mapping. One says that T is generalized α–ψ-contractive type mappings of integral type I if there exist two functions $\alpha : X \times X \to [0, +\infty)$ and $\psi \in \Psi$ such that for each $x, y \in X$

$$\alpha(x, y) \int_0^{d(Tx,Ty)} \varphi(t)\, dt \leq \psi\left(\int_0^{M(x,y)} \varphi(t)\, dt\right), \tag{8}$$

where $\varphi \in \Phi_s$ and

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}. \tag{9}$$

Definition 15. Let (X, d) be a generalized metric space and let $T : X \to X$ be a given mapping. One says that T is generalized $\alpha - \psi$-contractive type mappings of integral type II if there exist two functions $\alpha : X \times X \to [0, +\infty)$ and $\psi \in \Psi$ such that for each $x, y \in X$

$$\alpha(x, y) \int_0^{d(Tx,Ty)} \varphi(t)\, dt \leq \psi\left(\int_0^{N(x,y)} \varphi(t)\, dt\right), \tag{10}$$

where $\varphi \in \Phi_s$ and

$$N(x, y) = \max\left\{d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}\right\}. \tag{11}$$

Now, we state our first fixed point result.

Theorem 16. *Let* (X, d) *be a complete generalized metric space and let* $T : X \to X$ *be a generalized* $\alpha - \psi$*-contractive type mappings of integral type I. Suppose that*

(i) T *is* α*-admissible;*

(ii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$ *and* $\alpha(x_0, T^2x_0) \geq 1$*;*

(iii) T *is continuous.*

Then there exists a $u \in X$ *such that* $Tu = u$.

Proof. Regarding assumption (ii), we guarantee that there exists a point $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$. Starting this initial value $x_0 \in X$, we define an iterative sequence $\{x_n\}$ in X as follows:

$$x_{n+1} = Tx_n = T^{n+1}x_0 \quad \forall n \geq 0. \tag{12}$$

Notice that if $x_{n_0} = x_{n_0+1}$ for some n_0, then the proof is completed in this case. Indeed, we have $u = x_{n_0} = x_{n_0+1} = Tx_{n_0} = Tu$. As a consequence of this observation, throughout the proof, we assume that

$$x_n \neq x_{n+1} \quad \forall n. \tag{13}$$

It is evident that

$$\begin{aligned} &\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 \\ &\qquad \Longrightarrow \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1, \end{aligned} \tag{14}$$

since T is α-admissible. Recursively, we find that

$$\alpha(x_n, x_{n+1}) \geq 1, \quad \forall n = 0, 1, \ldots. \tag{15}$$

By repeating the same arguments, used above, we also derive that

$$\begin{aligned} &\alpha(x_0, x_2) = \alpha(x_0, T^2x_0) \geq 1 \\ &\qquad \Longrightarrow \alpha(Tx_0, Tx_2) = \alpha(x_1, x_3) \geq 1. \end{aligned} \tag{16}$$

From the previous inequalities, we conclude that

$$\alpha(x_n, x_{n+2}) \geq 1, \quad \forall n = 0, 1, \ldots. \tag{17}$$

We divide the proofs into 4 steps.

Step 1. We show that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{18}$$

By taking (8) and (15) into account, we obtain that

$$\begin{aligned} \int_0^{d(x_n,x_{n+1})} \varphi(t)\, dt &= \int_0^{d(Tx_{n-1},Tx_n)} \varphi(t)\, dt \\ &\leq \alpha(x_{n-1}, x_n) \int_0^{d(Tx_{n-1},Tx_n)} \varphi(t)\, dt \\ &\leq \psi\left(\int_0^{M(x_{n-1},x_n)} \varphi(t)\, dt\right), \end{aligned} \tag{19}$$

for all $n \geq 1$, where

$$\begin{aligned} &M(x_{n-1}, x_n) \\ &\quad = \max\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)\} \\ &\quad = \max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \\ &\quad = \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}. \end{aligned} \tag{20}$$

If we have $M(x_{n-1}, x_n) = d(x_n, x_{n+1})$ for some $n \in \mathbb{N}$, then inequality (19) turns into

$$\begin{aligned} \int_0^{d(x_n,x_{n+1})} \varphi(t)\,dt &\leq \psi\left(\int_0^{M(x_{n-1},x_n)} \varphi(t)\,dt\right) \\ &= \psi\left(\int_0^{d(x_n,x_{n+1})} \varphi(t)\,dt\right) \\ &< \int_0^{d(x_n,x_{n+1})} \varphi(t)\,dt, \end{aligned} \tag{21}$$

by regarding the property (iii) of the auxiliary function ψ. This is a contradiction. Consequently, we have $M(x_{n-1}, x_n) = d(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$ and (19) becomes

$$\int_0^{d(x_n,x_{n+1})} \varphi(t)\,dt \leq \psi\left(\int_0^{d(x_{n-1},x_n)} \varphi(t)\,dt\right) \quad \forall n \in \mathbb{N}. \tag{22}$$

This yields that

$$\int_0^{d(x_n,x_{n+1})} \varphi(t)\,dt < \int_0^{d(x_{n-1},x_n)} \varphi(t)\,dt \quad \forall n \in \mathbb{N}, \tag{23}$$

by recalling the property (iii) of the auxiliary function ψ. Due to (22), we find that

$$\int_0^{d(x_n,x_{n+1})} \varphi(t)\,dt \leq \psi^n\left(\int_0^{d(x_0,x_1)} \varphi(t)\,dt\right), \quad \forall n \in \mathbb{N}. \tag{24}$$

By property of ψ again, we deduce that

$$\lim_{n\to\infty} \int_0^{d(x_n,x_{n+1})} \varphi(t)\,dt = 0, \tag{25}$$

and hence

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{26}$$

Step 2. We show that

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0. \tag{27}$$

Combining (8) and (17), we conclude that

$$\begin{aligned} \int_0^{d(x_n,x_{n+2})} \varphi(t)\,dt &= \int_0^{d(Tx_{n-1},Tx_{n+1})} \varphi(t)\,dt \\ &\leq \alpha(x_{n-1}, x_{n+1}) \int_0^{d(Tx_{n-1},Tx_{n+1})} \varphi(t)\,dt \\ &\leq \psi\left(\int_0^{M(x_{n-1},x_{n+1})} \varphi(t)\,dt\right), \end{aligned} \tag{28}$$

for all $n \geq 1$, where

$$\begin{aligned} &M(x_{n-1}, x_{n+1}) \\ &\quad = \max\{d(x_{n-1}, x_{n+1}), d(x_{n-1}, Tx_{n-1}), d(x_{n+1}, Tx_{n+1})\} \\ &\quad = \max\{d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n), d(x_{n+1}, x_{n+2})\}. \end{aligned} \tag{29}$$

By (23), we have

$$\begin{aligned} M(x_{n-1}, x_{n+1}) &= \max\{d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n)\} \\ &= \max\{e_n, d_n\}, \end{aligned} \tag{30}$$

where $e_n = d(x_n, x_{n+2})$ and $d_n = d(x_n, x_{n+1})$. Thus, inequality (28) can be considered as

$$\begin{aligned} \int_0^{e_n} \varphi(t)\,dt = \int_0^{d(x_n,x_{n+2})} \varphi(t)\,dt &\leq \psi\left(\int_0^{M(x_{n-1},x_{n+1})} \varphi(t)\,dt\right) \\ &= \psi\left(\int_0^{\max\{e_{n-1},d_{n-1}\}} \varphi(t)\,dt\right) \quad \forall n \in \mathbb{N}. \end{aligned} \tag{31}$$

On the other hand, by (23)

$$\int_0^{d_n} \varphi(t)\,dt \leq \int_0^{d_{n-1}} \varphi(t)\,dt \leq \int_0^{\max\{e_{n-1},d_{n-1}\}} \varphi(t)\,dt. \tag{32}$$

Therefore,

$$\int_0^{\max\{e_n,d_n\}} \varphi(t)\,dt \leq \int_0^{\max\{e_{n-1},d_{n-1}\}} \varphi(t)\,dt \quad \forall n \in \mathbb{N}. \tag{33}$$

Then, the sequence $\{\int_0^{\max\{e_n,d_n\}} \varphi(t)dt\}$ is monotone nonincreasing, so it converges to some $t \geq 0$. Assume that $L > 0$. Now, by (18)

$$\begin{aligned} \limsup_{n\to\infty} \int_0^{e_n} \varphi(t)\,dt &= \limsup_{n\to\infty} \int_0^{\max\{e_n,d_n\}} \varphi(t)\,dt \\ &= \lim_{n\to\infty} \int_0^{\max\{e_n,d_n\}} \varphi(t)\,dt = L. \end{aligned} \tag{34}$$

Taking $n \to \infty$ in (31)

$$\begin{aligned} L &= \limsup_{n\to\infty} \int_0^{e_n} \varphi(t)\,dt \\ &\leq \limsup_{n\to\infty} \psi\left(\int_0^{\max\{e_{n-1},d_{n-1}\}} \varphi(t)\,dt\right) \\ &\leq \psi\left(\lim_{n\to\infty} \int_0^{\max\{e_{n-1},d_{n-1}\}} \varphi(t)\,dt\right) = \psi(L) < L, \end{aligned} \tag{35}$$

which is a contradiction; that is, (27) is proved.

Step 3. We will prove that

$$x_n \neq x_m \quad \forall n \neq m. \tag{36}$$

We argue by contradiction. Suppose that $x_n = x_m$ for some $m, n \in \mathbb{N}$ with $m \neq n$. Since $d(x_p, x_{p+1}) > 0$ for each $p \in \mathbb{N}$, so without loss of generality, assume that $m > n+1$. Consider now

$$\begin{aligned}\int_0^{d(x_n,x_{n+1})}\varphi(t)\,dt &= \int_0^{d(x_n,Tx_n)}\varphi(t)\,dt \\ &= \int_0^{d(x_m,Tx_m)}\varphi(t)\,dt \\ &= \int_0^{d(Tx_{m-1},Tx_m)}\varphi(t)\,dt \\ &\le \alpha(x_{m-1},x_m)\int_0^{d(Tx_{m-1},Tx_m)}\varphi(t)\,dt \\ &\le \psi\left(\int_0^{M(x_{m-1},x_m)}\varphi(t)\,dt\right),\end{aligned} \tag{37}$$

where

$$\begin{aligned}&M(x_{m-1},x_m)\\ &\quad= \max\{d(x_{m-1},x_m),d(x_{m-1},Tx_{m-1}),d(x_m,Tx_m)\}\\ &\quad= \max\{d(x_{m-1},x_m),d(x_{m-1},x_m),d(x_m,x_{m+1})\}\\ &\quad= \max\{d(x_{m-1},x_m),d(x_m,x_{m+1})\}.\end{aligned} \tag{38}$$

If $M(x_{m-1},x_m) = d(x_{m-1},x_m)$, then from (37) we get that

$$\begin{aligned}\int_0^{d(x_n,x_{n+1})}\varphi(t)\,dt &= \int_0^{d(x_n,Tx_n)}\varphi(t)\,dt \\ &= \int_0^{d(x_m,Tx_m)}\varphi(t)\,dt \\ &= \int_0^{d(x_m,x_{m+1})}\varphi(t)\,dt \\ &\le \alpha(x_m,x_{m+1})\int_0^{d(Tx_{m-1},Tx_m)}\varphi(t)\,dt \\ &\le \psi\left(\int_0^{M(x_{m-1},x_m)}\varphi(t)\,dt\right) \\ &= \psi\left(\int_0^{d(x_{m-1},x_m)}\varphi(t)\,dt\right) \\ &\le \psi^{m-n}\left(\int_0^{d(x_n,x_{n+1})}\varphi(t)\,dt\right).\end{aligned} \tag{39}$$

If $M(x_{m-1},x_m) = d(x_m,x_{m+1})$, inequality (37) becomes

$$\begin{aligned}\int_0^{d(x_n,x_{n+1})}\varphi(t)\,dt &= \int_0^{d(x_n,Tx_n)}\varphi(t)\,dt \\ &= \int_0^{d(x_m,Tx_m)}\varphi(t)\,dt \\ &= \int_0^{d(Tx_{m-1},Tx_m)}\varphi(t)\,dt \\ &\le \alpha(x_{m-1},x_m)\int_0^{d(Tx_{m-1},Tx_m)}\varphi(t)\,dt \\ &\le \psi\left(\int_0^{M(x_{m-1},x_m)}\varphi(t)\,dt\right) \\ &= \psi\left(\int_0^{d(x_m,x_{m+1})}\varphi(t)\,dt\right) \\ &\le \psi^{m-n+1}\left(\int_0^{d(x_n,x_{n+1})}\varphi(t)\,dt\right).\end{aligned} \tag{40}$$

Due to a property of ψ, inequalities (39) and (40) together yield that

$$\begin{aligned}\int_0^{d(x_n,x_{n+1})}\varphi(t)\,dt &\le \psi^{m-n}\left(\int_0^{d(x_n,x_{n+1})}\varphi(t)\,dt\right) \\ &< \int_0^{d(x_n,x_{n+1})}\varphi(t)\,dt,\end{aligned} \tag{41}$$

$$\begin{aligned}\int_0^{d(x_n,x_{n+1})}\varphi(t)\,dt &\le \psi^{m-n+1}\left(\int_0^{d(x_n,x_{n+1})}\varphi(t)\,dt\right) \\ &< \int_0^{d(x_n,x_{n+1})}\varphi(t)\,dt,\end{aligned} \tag{42}$$

respectively. In each case, there is a contradiction.

Step 4. We will prove that $\{x_n\}$ is a Cauchy sequence; that is,

$$\operatorname*{Lim}_{n\to\infty}\int_0^{d(x_n,x_{n+k})}\varphi(t)\,dt = 0 \quad \forall k\in\mathbb{N}. \tag{43}$$

The cases $k = 1$ and $k = 2$ are proved, respectively, by (18) and (27). Now, take $k \ge 3$ arbitrary. It is sufficient to examine two cases.

Case (I). Suppose that $k = 2m+1$ where $m \ge 1$. Then, by using step 3 and the quadrilateral inequality together with (24), we find

$$\begin{aligned}&\int_0^{d(x_n,x_{n+k})}\varphi(t)\,dt\\ &\quad= \int_0^{d(x_n,x_{n+2m+1})}\varphi(t)\,dt\\ &\quad\le \int_0^{d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+\cdots+d(x_{n+2m},x_{n+2m+1})}\varphi(t)\,dt\end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^{d(x_n,x_{n+1})} \varphi(t) + \int_0^{d(x_{n+1},x_{n+2})} \varphi(t) \\
&\quad + \cdots + \int_0^{d(x_{n+2m},x_{n+2m+1})} \varphi(t)\,dt \\
&\leq \sum_{p=n}^{n+2m} \psi^p \left(\int_0^{d(x_0,x_1)} \varphi(t)\,dt \right) \\
&\leq \sum_{p=n}^{+\infty} \psi^p \left(\int_0^{d(x_0,x_1)} \varphi(t)\,dt \right) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{44}
$$

Case (II). Suppose that $k = 2m$ where $m \geq 2$. Again, by applying the quadrilateral inequality and step 3 together with (24), we find

$$
\begin{aligned}
&\int_0^{d(x_n,x_{n+k})} \varphi(t)\,dt \\
&\quad = \int_0^{d(x_n,x_{n+2m})} \varphi(t)\,dt \\
&\quad \leq \int_0^{d(x_n,x_{n+2})+d(x_{n+2},x_{n+3})+\cdots+d(x_{n+2m-1},x_{n+2m})} \varphi(t)\,dt \\
&\quad \leq \int_0^{d(x_n,x_{n+2})} \varphi(t)\,dt + \int_0^{d(x_{n+2},x_{n+3})} \varphi(t)\,dt \\
&\quad\quad + \cdots + \int_0^{d(x_{n+2m-1},x_{n+2m})} \varphi(t)\,dt \\
&\quad \leq \int_0^{d(x_n,x_{n+2})} \varphi(t)\,dt + \sum_{p=n+2}^{n+2m-1} \psi^p \left(\int_0^{d(x_0,x_1)} \varphi(t)\,dt \right) \\
&\quad \leq \int_0^{d(x_n,x_{n+2})} \varphi(t)\,dt \\
&\quad\quad + \sum_{p=n}^{+\infty} \psi^p \left(\int_0^{d(x_0,x_1)} \varphi(t)\,dt \right) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{45}
$$

By combining expressions (44) and (45), we have

$$\lim_{n\to\infty} \int_0^{d(x_n,x_{n+k})} \varphi(t)\,dt = 0 \quad \forall k \geq 3. \tag{46}$$

Hence, we have

$$\lim_{n\to\infty} d(x_n, x_{n+k}) = 0 \quad \forall k \geq 3. \tag{47}$$

We conclude that $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is complete, there exists $u \in X$ such that

$$\lim_{n\to\infty} d(x_n, u) = 0. \tag{48}$$

Since T is continuous, we obtain from (48) that

$$\lim_{n\to\infty} d(x_{n+1}, Tu) = \lim_{n\to\infty} d(Tx_n, Tu) = 0; \tag{49}$$

that is, $\lim_{n\to\infty} x_{n+1} = Tu$. Taking Proposition 4 into account, we conclude that $Tu = u$; that is, u is a fixed point of T. □

The following result is deduced from Theorem 16 due to the obvious inequality $N(x, y) \leq M(x, y)$.

Theorem 17. *Let (X, d) be a complete generalized metric space and let $T : X \to X$ be generalized $\alpha-\psi$-contractive type mappings of integral type II. Suppose that*

(i) *T is α-admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$;*
(iii) *T is continuous.*

Then there exists a $u \in X$ such that $Tu = u$.

Theorem 16 remains true if we replace the continuity hypothesis by the following property.

If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all k.

This statement is given as follows.

Theorem 18. *Let (X, d) be a complete generalized metric space and let $T : X \to X$ be generalized $\alpha - \psi$-contractive type mappings of integral type I. Suppose that*

(i) *T is α-admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$;*
(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all k.*

Then, there exists $u \in X$ such that $Tu = u$.

Proof. Following the lines in the proof of Theorem 8, we deduce that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \geq 0$ is Cauchy and converges to some $u \in X$. In view of Proposition 4,

$$\lim_{k\to\infty} d(x_{n(k)+1}, Tu) = d(u, Tu). \tag{50}$$

By using the method of *reductio ad absurdum*, we will show that $Tu = u$. Suppose, on the contrary, that $Tu \neq u$; that is, $d(Tu, u) > 0$. From (15) and condition (iii), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, u) \geq 1$ for all k.

By applying (8), we find that

$$
\begin{aligned}
\int_0^{d(x_{n(k)+1},Tu)} \varphi(t)\,dt &\leq \alpha(x_{n(k)}, u) \int_0^{d(Tx_{n(k)},Tu)} \varphi(t)\,dt \\
&\leq \psi\left(\int_0^{M(x_{n(k)},u)} \varphi(t)\,dt \right),
\end{aligned} \tag{51}
$$

where

$$
\begin{aligned}
&M(x_{n(k)}, u) \\
&\quad = \max\{d(x_{n(k)}, u), d(x_{n(k)}, Tx_{n(k)}), d(u, Tu)\} \\
&\quad = \max\{d(x_{n(k)}, u), d(x_{n(k)}, x_{n(k)+1}), d(u, Tu)\}.
\end{aligned} \tag{52}
$$

By (18) and (50), we obtain

$$\operatorname*{Lim}_{k\to\infty}\int_0^{M(x_{n(k)},u)}\varphi(t)\,dt=\int_0^{d(u,Tu)}\varphi(t)\,dt. \tag{53}$$

Since ψ is upper semicontinuous, by letting $k\to\infty$ in (51) we derive that

$$\int_0^{d(u,Tu)}\varphi(t)\,dt\le\psi\left(\int_0^{d(u,Tu)}\varphi(t)\,dt\right)<\int_0^{d(u,Tu)}\varphi(t)\,dt. \tag{54}$$

This is a contradiction. Hence, we obtain that u is a fixed point of T; that is, $Tu=u$. □

In the following, the hypothesis of upper semicontinuity of ψ is not required. Similar to Theorem 18, for the generalized $\alpha-\psi$ contractive mappings of type II, we have the following.

Theorem 19. *Let (X,d) be a complete generalized metric space and let $T:X\to X$ be generalized $\alpha-\psi$-contractive type mappings of integral type II. Suppose that*

(i) *T is α-admissible;*

(ii) *there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\ge 1$ and $\alpha(x_0,T^2x_0)\ge 1$;*

(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1})\ge 1$ for all n and $x_n\to x\in X$ as $n\to\infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)},x)\ge 1$ for all k.*

Then, there exists $u\in X$ such that $Tu=u$.

Proof. Following the proof of Theorem 17 (which is the same as Theorem 16), we know that the sequence $\{x_n\}$ defined by $x_{n+1}=Tx_n$ for all $n\ge 0$ is Cauchy and converges to some $u\in X$. Similarly, in view of Proposition 4,

$$\lim_{k\to\infty}d\left(x_{n(k)+1},Tu\right)=d(u,Tu). \tag{55}$$

We will show that $Tu=u$. Suppose, on the contrary, that $Tu\ne u$. From (15) and condition (iii), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)},u)\ge 1$ for all k. By applying (10), for all k, we get that

$$\begin{aligned}\int_0^{d(x_{n(k)+1},Tu)}\varphi(t)\,dt&\le\alpha\left(x_{n(k)},u\right)\int_0^{d(Tx_{n(k)},Tu)}\varphi(t)\,dt\\&\le\psi\left(\int_0^{N(x_{n(k)},u)}\varphi(t)\,dt\right),\end{aligned} \tag{56}$$

where

$$\begin{aligned}&N\left(x_{n(k)},u\right)\\&=\max\left\{d\left(x_{n(k)},u\right),\frac{d\left(x_{n(k)},Tx_{n(k)}\right)+d(u,Tu)}{2}\right\}.\end{aligned} \tag{57}$$

Letting $k\to\infty$ in (56), we have

$$\lim_{k\to\infty}\int_0^{N(x_{n(k)},u)}\varphi(t)\,dt=\int_0^{d(u,Tu)/2}\varphi(t)\,dt. \tag{58}$$

From (58), for k large enough, we have $N(x_{n(k)},u)>0$, which implies that

$$\psi\left(\int_0^{N(x_{n(k)},u)}\varphi(t)\,dt\right)<\int_0^{N(x_{n(k)},u)}\varphi(t)\,dt. \tag{59}$$

Thus, from (56) and (58), we have

$$\int_0^{d(u,Tu)}\varphi(t)\,dt\le\int_0^{d(u,Tu)/2}\varphi(t)\,dt, \tag{60}$$

which is a contradiction. Hence, we obtain that u is a fixed point of T; that is, $Tu=u$. □

Theorem 20. *Adding condition (U) to the hypotheses of Theorem 16 (resp., Theorem 18), one obtains that u is the unique fixed point of T.*

Proof. By using the method of *reductio ad absurdum*, we will show that u is the unique fixed point of T. Let v be another fixed point of T with $v\ne u$. By hypothesis (U),

$$1\le\alpha(u,v)=\alpha(Tu,Tv). \tag{61}$$

Now, due to (8), we have

$$\begin{aligned}\int_0^{d(u,v)}\varphi(t)\,dt&\le\alpha(u,v)\int_0^{d(u,v)}\varphi(t)\,dt\\&=\alpha(Tu,Tv)\int_0^{d(Tu,Tv)}\varphi(t)\,dt\\&\le\psi\left(\int_0^{M(u,v)}\varphi(t)\,dt\right)\\&=\psi\left(\int_0^{\max\{d(u,v),d(u,Tu),d(v,Tv)\}}\varphi(t)\,dt\right)\\&=\psi\left(\int_0^{d(u,v)}\varphi(t)\,dt\right)<\int_0^{d(u,v)}\varphi(t)\,dt\end{aligned} \tag{62}$$

which is a contradiction. Hence, $u=v$. □

Theorem 21. *Adding condition (U) to the hypotheses of Theorem 17 (resp., Theorem 19), one obtains that u is the unique fixed point of T.*

Proof. As in Theorem 20, we use the method of *reductio ad absurdum* to show that u is the unique fixed point of T. Suppose, on the contrary, that v is another fixed point of T with $v\ne u$. It is evident that $1\le\alpha(u,v)=\alpha(Tu,Tv)$.

Now, due to (10), we have

$$\begin{aligned}\int_0^{d(u,v)}\varphi(t)\,dt &\le \alpha(u,v)\int_0^{d(u,v)}\varphi(t)\,dt\\ &= \alpha(Tu,Tv)\int_0^{d(Tu,Tv)}\varphi(t)\,dt\\ &\le \psi\left(\int_0^{N(u,v)}\varphi(t)\,dt\right)\\ &= \psi\left(\int_0^{\max\{d(u,v),(d(u,Tu)+d(v,Tv))/2\}}\varphi(t)\,dt\right)\\ &= \psi\left(\int_0^{d(u,v)}\varphi(t)\,dt\right) < \int_0^{d(u,v)}\varphi(t)\,dt\end{aligned} \tag{63}$$

which is a contradiction. Hence, $u = v$. □

For the uniqueness of a fixed point of a generalized $\alpha - \psi$ contractive mapping, we will consider the following hypotheses suggested in [11].

(H1): for all $x, y \in \text{Fix}(T)$, there exists z in X such that $\alpha(x,z) \ge 1$ and $\alpha(y,z) \ge 1$.

(H2): let $x, y \in \text{Fix}(T)$. If there exists $\{z_n\}$ in X such that $\alpha(x,z_n) \ge 1$ and $\alpha(y,z_n) \ge 1$, then

$$d(z_n,z_{n+1}) \le \inf\{d(x,z_n), d(y,z_n)\} \quad \forall n \in \mathbb{N}. \tag{64}$$

Theorem 22. *Adding conditions* $(H1)$, $(H2)$, *and* (W) *to the hypotheses of Theorem 16 (resp., Theorem 18), one obtains that* u *is the unique fixed point of* T.

Proof. We will show that u is the unique fixed point of T, by using the method of *reductio ad absurdum*. Let v be another fixed point of T with $v \ne u$; that is, $d(u,v) > 0$. Due to (H1), there exists $z \in X$ such that

$$\alpha(u,z) \ge 1, \qquad \alpha(v,z) \ge 1. \tag{65}$$

Since T is α-admissible, from (65), we have

$$\alpha(u,T^n z) \ge 1, \quad \alpha(v,T^n z) \ge 1, \quad \forall n. \tag{66}$$

Define the sequence $\{z_n\}$ in X by $z_{n+1} = Tz_n$ for all $n \ge 0$ and $z_0 = z$. From (66), for all n, we have

$$\begin{aligned}\int_0^{d(u,z_{n+1})}\varphi(t)\,dt &= \int_0^{d(Tu,Tz_n)}\varphi(t)\,dt\\ &\le \alpha(u,z_n)\int_0^{d(Tu,Tz_n)}\varphi(t)\,dt\\ &\le \psi\left(\int_0^{M(u,z_n)}\varphi(t)\,dt\right),\end{aligned} \tag{67}$$

where

$$\begin{aligned}M(u,z_n) &= \max\{d(u,z_n), d(u,Tu), d(z_n,Tz_n)\}\\ &= \max\{d(u,z_n), d(z_n,z_{n+1})\}.\end{aligned} \tag{68}$$

By (H2), we get

$$M(u,z_n) = d(u,z_n) \quad \forall n. \tag{69}$$

Iteratively, by using inequality (67), we get that

$$\int_0^{d(u,z_n)}\varphi(t)\,dt \le \psi^n\left(\int_0^{d(u,z_0)}\varphi(t)\,dt\right), \tag{70}$$

for all n. Letting $n \to \infty$ in the above inequality, we obtain

$$\lim_{n\to\infty}\int_0^{d(z_n,u)}\varphi(t)\,dt = 0, \tag{71}$$

and hence

$$\lim_{n\to\infty} d(z_n,u) = 0. \tag{72}$$

Similarly, one can show that

$$\lim_{n\to\infty} d(z_n,v) = 0. \tag{73}$$

Regarding (W), there exists $r_{u,v} > 0$ such that for all n

$$r_{u,v} < d(u,z_n) + d(v,z_n), \tag{74}$$

and hence

$$\int_0^{r_{u,v}}\varphi(t)\,dt < \int_0^{d(u,z_n)+d(v,z_n)}\varphi(t)\,dt. \tag{75}$$

From (71) and (73), by passing $n \to \infty$, it follows that $r_{u,v} = 0$, which is a contradiction. Thus, we proved that u is the unique fixed point of T. □

Theorem 23. *Adding conditions* $(H1)$, $(H2)$, *and* (W) *to the hypotheses of Theorem 17 (resp., Theorem 19), one obtains that* u *is the unique fixed point of* T.

Proof. Suppose that v is another fixed point of T and $u \ne v$. From (H1), there exists $z \in X$ such that

$$\alpha(u,z) \ge 1, \qquad \alpha(v,z) \ge 1. \tag{76}$$

Since T is α-admissible, from (76), we have

$$\alpha(u,T^n z) \ge 1, \quad \alpha(v,T^n z) \ge 1, \quad \forall n. \tag{77}$$

Define the sequence $\{z_n\}$ in X by $z_{n+1} = Tz_n$ for all $n \ge 0$ and $z_0 = z$. From (77), for all n, we have

$$\begin{aligned}\int_0^{d(u,z_{n+1})}\varphi(t)\,dt &= \int_0^{d(Tu,Tz_n)}\varphi(t)\,dt\\ &\le \alpha(u,z_n)\int_0^{d(Tu,Tz_n)}\varphi(t)\,dt\\ &\le \psi\left(\int_0^{N(u,z_n)}\varphi(t)\,dt\right),\end{aligned} \tag{78}$$

where

$$N(u,z_n) = \max\left\{d(u,z_n), \frac{d(u,Tu)+d(z_n,Tz_n)}{2}\right\} = \max\left\{d(u,z_n), \frac{d(z_n,z_{n+1})}{2}\right\}. \quad (79)$$

By (H2), we get

$$N(u,z_n) = d(u,z_n) \quad \forall n. \quad (80)$$

Iteratively, by using inequality (78), we get that

$$\int_0^{d(u,z_n)} \varphi(t)\,dt \le \psi^n\left(\int_0^{d(u,z_0)} \varphi(t)\,dt\right), \quad (81)$$

for all n. Letting $n \to \infty$ in the above inequality, we obtain

$$\lim_{n\to\infty} \int_0^{d(z_n,u)} \varphi(t)\,dt = 0, \quad (82)$$

and hence

$$\lim_{n\to\infty} d(z_n,u) = 0. \quad (83)$$

Analogously, one can show that

$$\lim_{n\to\infty} d(z_n,v) = 0. \quad (84)$$

Similarly, regarding (W) together with (83) and (84), it follows that $u = v$. Thus we proved that u is the unique fixed point of T. □

It is known that Hausdorffness property implies the uniqueness of the limit, so the (W) condition in Theorem 22 (resp., Theorem 23) can be replaced by Hausdorff property. Then, the proof of the following result is clear and hence it is omitted.

Corollary 24. *Adding conditions* $(H1)$ *and* $(H2)$ *to the hypotheses of Theorem 16 (resp., Theorems 18, 17, and 19) and assuming that* (X,d) *is Hausdorff, one obtains that* u *is the unique fixed point of* T.

3. Consequences

In what follows we introduce the notion of $\alpha-\psi$-contractive type mappings of integral type.

Definition 25 (Karapınar, [14]). Let (X,d) be a generalized metric space and let $T : X \to X$ be a given mapping. One says that T is an $\alpha-\psi$-contractive mapping of integral type if there exist two functions $\alpha : X\times X \to [0,+\infty)$ and $\psi \in \Psi$ such that for each $x, y \in X$

$$\alpha(x,y)\int_0^{d(Tx,Ty)} \varphi(t)\,dt \le \psi\left(\int_0^{d(x,y)} \varphi(t)\,dt\right), \quad (85)$$

where $\varphi \in \Phi_s$.

Now, we state the following fixed point theorem.

Theorem 26 (Karapınar, [14]). *Let* (X,d) *be a complete generalized metric space and let* $T : X \to X$ *be an* $\alpha-\psi$ *contractive mapping of integral type. Suppose that*

(i) *T is α-admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \ge 1$ and $\alpha(x_0,T^2x_0) \ge 1$;*
(iii) *either T is continuous or $\{x_n\}$ is regular.*

Then there exists a $u \in X$ *such that* $Tu = u$.

Proof. The proof is verbatim of the proofs of Theorems 16 and 18. □

Theorem 27 (Karapınar, [14]). *Adding condition* (U) *to the hypotheses of Theorem 26, one obtains that u is the unique fixed point of* T.

Proof. The proof is verbatim of the proofs of Theorem 20. □

Remark 28. The uniqueness condition (U) in Theorem 27 can be replaced with alternative criteria (H1), (H2), and (W) as in Theorems 22 and 23.

Corollary 29. *Let* (X,d) *be a complete generalized metric space and let* $T : X \to X$ *be a continuous mapping. Suppose that there exists a function* $\psi \in \Psi$ *such that*

$$\int_0^{d(Tx,Ty)} \varphi(t)\,dt \le \psi\left(\int_0^{M(x,y)} \varphi(t)\,dt\right), \quad (86)$$

for all $x, y \in X$, *where* $\varphi \in \Phi_s$ *and*

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}. \quad (87)$$

Then T *has a unique fixed point.*

Proof. Let $\alpha : X\times X \to [0,\infty)$ be the mapping defined by $\alpha(x,y) = 1$, for all $x, y \in X$. Then T is an $\alpha-\psi$-contraction mapping of integral type I. It is clear that all conditions of Theorem 20 are satisfied. Hence, T has a unique fixed point. □

Corollary 30. *Let* (X,d) *be a complete generalized metric space and let* $T : X \to X$ *be a continuous mapping. Suppose that there exists a function* $\psi \in \Psi$ *such that*

$$\int_0^{d(Tx,Ty)} \varphi(t)\,dt \le \psi\left(\int_0^{N(x,y)} \varphi(t)\,dt\right), \quad (88)$$

for all $x, y \in X$, *where* $\varphi \in \Phi_s$ *and*

$$N(x,y) = \max\left\{d(x,y), \frac{d(x,Tx)+d(y,Ty)}{2}\right\}. \quad (89)$$

Then T *has a unique fixed point.*

Proof. As in the corollary, it is sufficient to define $\alpha : X\times X \to [0,\infty)$ such that $\alpha(x,y) = 1$, for all $x, y \in X$. Then, evidently, T is an $\alpha-\psi$-contraction mapping of integral type II. Hence, all conditions of Theorem 21 are fulfilled. So, T has a unique fixed point. □

The following fixed point theorems follow immediately from Corollary 29 by taking $\psi(t) = \lambda t$, where $\lambda \in (0,1)$.

Corollary 31. *Let (X,d) be a complete generalized metric space and let $T : X \to X$ be a continuous mapping. Suppose that there exists a constant $\lambda \in (0,1)$ such that*

$$\int_0^{d(Tx,Ty)} \varphi(t)\,dt \le \lambda \int_0^{M(x,y)} \varphi(t)\,dt, \tag{90}$$

for all $x, y \in X$, where $\varphi \in \Phi_s$ and

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}. \tag{91}$$

Then T has a unique fixed point.

By taking $\psi(t) = \lambda t$, where $\lambda \in (0,1)$, in Corollary 30, we derive the following result.

Corollary 32. *Let (X,d) be a complete generalized metric space and let $T : X \to X$ be a continuous mapping. Suppose that there exists a constant $\lambda \in (0,1)$ such that*

$$\int_0^{d(Tx,Ty)} \varphi(t)\,dt \le \lambda \int_0^{N(x,y)} \varphi(t)\,dt, \tag{92}$$

for all $x, y \in X$, where $\varphi \in \Phi_s$ and

$$N(x,y) = \max\left\{d(x,y), \frac{d(x,Tx) + d(y,Ty)}{2}\right\}. \tag{93}$$

Then T has a unique fixed point.

Corollary 33 (cf. [11]). *Let (X,d) be a complete generalized metric space and let $T : X \to X$ be a continuous mapping. Suppose that there exists a function $\psi \in \Psi$ such that*

$$d(Tx,Ty) \le \psi(M(x,y)), \tag{94}$$

for all $x, y \in X$. Then T has a unique fixed point.

Proof. Let $\alpha : X \times X \to [0,\infty)$ be the mapping defined by $\alpha(x,y) = 1$, for all $x, y \in X$. Then T is an $\alpha - \psi$-contraction mapping. It is evident that all conditions of Theorem 8 are satisfied. Hence, T has a unique fixed point. □

The following fixed point theorems follow immediately from Corollary 33 by taking $\psi(t) = \lambda t$, where $\lambda \in (0,1)$.

Corollary 34 (see e.g. [11]). *Let (X,d) be a complete generalized metric space and let $T : X \to X$ be a continuous mapping. Suppose that there exists a constant $\lambda \in (0,1)$ such that*

$$d(Tx,Ty) \le \lambda N(x,y), \tag{95}$$

for all $x, y \in X$. Then T has a unique fixed point.

Now, we will show that many existing results in the literature can be deduced easily from our obtained results. The following theorems are the main results of Aydi et al. [11].

Theorem 35 (Aydi et al. [11]). *Let (X,d) be a complete generalized metric space and let $T : X \to X$ be a generalized $\alpha - \psi$ contractive mapping of type I. Suppose that*

(i) *T is α-admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \ge 1$ and $\alpha(x_0,T^2x_0) \ge 1$;*

(iii) *either T is continuous or if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \ge 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)},x) \ge 1$ for all k.*

Then there exists a $u \in X$ such that $Tu = u$.

Proof. It is sufficient to take $\varphi(t) = 1$ in Theorems 16 and 18. □

Theorem 36 (Aydi et al. [11]). *Let (X,d) be a complete generalized metric space and let $T : X \to X$ be a generalized $\alpha - \psi$ contractive mapping of type II. Suppose that*

(i) *T is α-admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \ge 1$ and $\alpha(x_0,T^2x_0) \ge 1$;*

(iii) *either T is continuous or if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \ge 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)},x) \ge 1$ for all k.*

Then there exists a $u \in X$ such that $Tu = u$.

Proof. If we take $\varphi(t) = 1$ in Theorems 17 and 19, then the proof follows immediately. □

Theorem 37 (Aydi et al. [11]). *Adding condition (U) to the hypotheses of Theorem 35 (resp., Theorem 36), one obtains that u is the unique fixed point of T.*

Proof. Let $\varphi(t) = 1$ in Theorems 20 and 21. □

Remark 38. Notice that all consequences and corollaries of Aydi et al. [11] can be added here since their main results are corollaries of the main results of this paper. To avoid the repetition, we do not want to state them here but we underline this fact.

Example 39. Let $X = [0,1]$ and $A = \{1/n : n \in \mathbb{N}\}$. We define the distance function $d : X \times X \to [0,\infty)$ as follows:

$$\begin{aligned}
&d(x,y) = 0 \quad \text{if } x = y,\\
&d(x,y) = d(y,x) \quad \forall x, y,\\
&d\left(\frac{1}{2},\frac{1}{3}\right) = d\left(\frac{1}{4},\frac{1}{5}\right) = \frac{1}{5},\\
&d\left(\frac{1}{2},\frac{1}{5}\right) = d\left(\frac{1}{3},\frac{1}{4}\right) = \frac{2}{5},\\
&d\left(\frac{1}{2},\frac{1}{4}\right) = d\left(\frac{1}{3},\frac{1}{5}\right) = 1,\\
&d(x,y) = |x-y| \quad \text{otherwise.}
\end{aligned} \tag{96}$$

It is clear that (X,d) is a generalized metric space. Notice also that d is not a metric since

$$1 = d\left(\frac{1}{2},\frac{1}{4}\right) > d\left(\frac{1}{2},\frac{1}{3}\right) + d\left(\frac{1}{3},\frac{1}{4}\right) = \frac{3}{5}. \tag{97}$$

We define $T : X \rightarrow X$ as $Tx = 1 - x$. Furthermore, let $\psi : [0,\infty) \rightarrow [0,\infty)$ be defined as $\psi(t) = t/3$ and $\varphi(t) = 1$. Now, we define $\alpha X \times X \rightarrow [0,\infty)$ as follows:

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x = y, \\ 5x & \text{if } x, y \in \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\right\} \text{ with } x \neq y, \\ 0 & \text{otherwise.} \end{cases} \quad (98)$$

Hence, all conditions of Theorem 20 are satisfied and $x = 1/2$ is a unique fixed point of T.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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Research Article

Convergence Theorems of Common Elements for Pseudocontractive Mappings and Monotone Mappings

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An algorithm for treating pseudocontractive mappings and monotone mappings is proposed. Convergence analysis of algorithm is investigated in the framework of Hilbert spaces.

1. Introduction

The motivation for common element problem is mainly due to its possible applications to mathematical modeling of concrete complex problems. The common element problems include mini-max problems, complementarily problems, equilibrium problems, common fixed point problems, and variational inequalities as special cases; see [1–7] and the references therein. It is well-known that the convex feasibility problem is a special case of the common zero (fixed) points of nonlinear mappings. And many important problems have reformulations which require finding zero points, for instance, evolution equations, complementarily problems, mini-max problems, and variational inequalities and optimization. For studying zero points of monotone mappings, the most well-known algorithm is the proximal point algorithm; see [8, 9] and the references therein. Regularization methods recently have been investigated for treating zero points of monotone mappings; see [2, 5, 6, 9] and references therein.

In 2010, Takahashi et al. [6] studied zero point problems of the sum of two monotone mappings and fixed point problems of a nonexpansive mapping based on the following iterative algorithm:

$$\begin{aligned} x_1 &= x \in C, \\ y_n &= \alpha_n x + (1-\alpha_n) J_{r_n}(I - r_n A) x_n, \\ x_{n+1} &= \beta_n x_n + (1-\beta_n) T y_n, \quad \forall n \ge 1, \end{aligned} \tag{1}$$

where C is a nonempty closed convex subset of a real Hilbert space H, $\{\alpha_n\}$ and $\{\beta_n\}$ are real number sequences in $(0,1)$, $\{r_n\}$ is a positive sequence, $T : C \to C$ is a nonexpansive mapping, $A : C \to H$ is an inverse strongly monotone mapping, $B : H \to 2^H$ is a maximal monotone mapping, and $J_{r_n} = (I + r_n B)^{-1}$, where I is the identity mapping. They proved that the sequence $\{x_n\}$ generated in (1) converges strongly to some $z \in F(T) \cap (A+B)^{-1}(0)$ provided that the control sequences satisfy some restrictions, where $F(T)$ is the set fixed points of T.

In 2014, Shahzad and Zegeye [5] considered an iterative method for a common point of fixed points of Lipschitzian pseudocontractive mappings and zeros of sum of two monotone mappings based on the projection method in a real Hilbert space. To be more precise, they investigated the following algorithm:

$$\begin{aligned} x_0 &\in C, \\ y_n &= (1-\beta_n) x_n + \beta_n T x_n, \\ x_{n+1} & \\ &= P_C\left[(1-\alpha_n)\left(\theta_n x_n + \delta_n T y_n + \gamma_n J_{r_n}(I - \lambda_n A) x_n\right)\right], \end{aligned} \tag{2}$$

where C is a nonempty closed convex subset of a real Hilbert space H, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\theta_n\}$, $\{\delta_n\}$, and $\{\gamma_n\}$ are real number sequences in $(0,1)$, $\{r_n\}$ is a positive sequence, $T : C \to C$ is a Lipschitzian pseudocontractive mapping, $A : C \to H$ is an inverse strongly monotone mapping, $B : H \to 2^H$

is a maximal monotone mapping, and $J_{r_n} = (I + r_nB)^{-1}$. They proved that the sequence $\{x_n\}$ generated in (2) converges strongly to the minimum-norm point $x \in F(T) \cap (A+B)^{-1}(0)$ provided that the control sequences satisfy some restrictions.

In this paper, we are concerned with the problem of finding a common element in the intersection $F(T_1) \cap F(T_2) \cap (A + B)^{-1}(0)$, where $F(T_i)$ denotes the fixed point set of the pseudocontractive mapping T_i, $i = 1, 2$, and $(A + B)^{-1}(0)$ denotes the zero point set of the sum of an inverse strongly monotone mapping A and a maximal monotone mapping B. Applications to a common element of the set of common fixed points of Lipschitzian pseudocontractive mappings and solutions of variational inequality for α-inverse strongly monotone mappings are included. Our theorems improve and extend those announced by Shahzad and Zegeye [5], Takahashi et al. [6], and other authors with the related interest.

2. Preliminaries

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H and let P_C be the metric projection from H onto C. Let $T : C \to C$ be a mapping. In this paper, we use $F(T)$ to denote the fixed point set of T; that is, $F(T) = \{x \in C : x = Tx\}$.

Recall that T is nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C. \tag{3}$$

T is said to be a γ-strictly pseudocontractive mapping if there exists $\gamma \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \gamma \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C. \tag{4}$$

Note that the class of γ-strictly pseudocontractive mappings includes the class of nonexpansive mappings as a special case. T is said to be a pseudocontractive mapping if

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C. \tag{5}$$

We note that inequalities (4) and (5) can be equivalently written as

$$\langle x - y, Tx - Ty \rangle \le \|x - y\|^2 - \gamma \|(I - T)x - (I - T)y\|^2 \tag{6}$$

for some $\gamma > 0$ and

$$\langle x - y, Tx - Ty \rangle \le \|x - y\|^2, \quad \forall x, y \in C, \tag{7}$$

respectively. Note that the class of γ-strictly pseudocontractive mappings is contained in the class of pseudocontractive mappings. We note that the inclusion is proper. We remark that T is a γ-strictly pseudocontractive mapping if and only if $I - T$ is a γ-inverse strongly monotone mapping and T is a pseudocontractive mapping if and only if $I - T$ is a monotone mapping.

Let $A : C \to H$ be a mapping and $A^{-1}0$ stands for the zero point set of A; that is, $A^{-1}0 = \{x \in C : Ax = 0\}$. Recall that A is said to be monotone if

$$\langle Ax - Ay, x - y \rangle \ge 0, \quad \forall x, y \in C. \tag{8}$$

A is said to be α-inverse strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \ge \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{9}$$

It is not hard to see that α-inverse strongly monotone mappings are Lipschitz continuous with constant $L = 1/\alpha$; that is, $\|Ax - Ay\| \le (1/\alpha)\|x - y\|$ for all $x, y \in C$.

Recall that the classical variational inequality, denoted by $\mathrm{VI}(C, A)$, is to find $u \in C$ such that

$$\langle Au, v - u \rangle \ge 0, \quad \forall v \in C. \tag{10}$$

A multivalued mapping $B : H \to 2^H$ with the domain $D(B) = \{x \in H : Bx \ne \phi\}$ and the range $R(B) = \{Bx : x \in D(B)\}$ is said to be monotone if, for $x_1 \in D(B)$, $x_2 \in D(B)$, $y_1 \in Bx_1$, and $y_2 \in Bx_2$, we have $\langle x_1 - x_2, y_1 - y_2 \rangle \ge 0$. A monotone mapping B is said to be maximal if its graph $G(B) = \{(x, y) : y \in Bx\}$ is not properly contained in the graph of any other monotone mapping. Let $B : H \to 2^H$ be a maximal monotone mapping. Then we can define, for each $\lambda > 0$, a nonexpansive single-valued mapping $J_\lambda : H \to H$ by $J_\lambda = (I + \lambda B)^{-1}$. It is called the resolvent of B. We know that $B^{-1}0 = F(J_\lambda)$ for all $\lambda > 0$ and J_λ is firmly nonexpansive.

Lemma 1. *Let H be a real Hilbert space. Then, for any given $x, y \in H$, the following inequality holds:*

$$\|x + y\|^2 \le \|x\|^2 + 2 \langle y, x + y \rangle. \tag{11}$$

Lemma 2 (see [10]). *Let C be a convex subset of a real Hilbert space H. Let $x \in H$. Then $x_0 = P_C x$ if and only if*

$$\langle z - x_0, x - x_0 \rangle \le 0, \quad \forall z \in C. \tag{12}$$

Lemma 3 (see [2]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a mapping and let $B : H \to 2^H$ be a maximal monotone mapping. Then $F(J_\lambda(I - \lambda A)) = (A + B)^{-1}0$.*

Lemma 4 (see [11]). *Let H be a Hilbert space. Let $B_1 : D(B_1) \subseteq H \to 2^H$ and let $B_2 : D(B_2) \subseteq H \to 2^H$ be maximal monotone mappings. Suppose that $D(A) \cap \mathrm{int}(D(B)) \ne \phi$. Then $B_1 + B_2$ is a maximal monotone mapping.*

Lemma 5 (see [4]). *Let $\{a_n\}$ be a sequence of real numbers. Assume that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all sufficiently large numbers $k \in \mathbb{N}$:*

$$a_{m_k} \le a_{m_k+1}, \qquad a_k \le a_{m_k+1}. \tag{13}$$

Lemma 6 (see [12]). *Let H be a real Hilbert space. Then, for all $x_i \in H$ and $\alpha_i \in [0,1]$ for $i = 1, 2, \ldots, n$ such that $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$, the following equality holds:*

$$\begin{aligned}\|\alpha_0 x_0 + \alpha_1 x_1 + \cdots + \alpha_n x_n\|^2 \\ = \sum_{i=0}^{n} \alpha_i \|x_i\|^2 - \sum_{0 \le i,j \le n} \alpha_i, \alpha_j \|x_i - x_j\|^2.\end{aligned} \tag{14}$$

Lemma 7 (see [7]). *Let C be a nonempty closed convex subset of a real Hilbert Hilbert and let $T : C \to C$ be a continuous pseudocontractive mapping. Then*

(i) *$F(T)$ is a closed convex subset of C;*

(ii) *$(I-T)$ is demiclosed at zero; that is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$ and $Tx_n - x_n \to 0$ as $n \to \infty$, then $x = Tx$.*

Lemma 8 (see [13]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \le (1-\alpha_n) a_n + \alpha_n \delta_n, \quad n \ge n_0, \tag{15}$$

where $\{\alpha_n\} \subset (0,1)$ and $\{\delta_n\} \subset \mathbb{R}$ satisfy the following conditions: $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\limsup_{n\to\infty} \delta_n \le 0$. Then $\lim_{n\to\infty} a_n = 0$.

3. Main Results

Theorem 9. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T_1, T_2 : C \to C$ be Lipschitzian pseudocontractive mappings with Lipschitz constants L_1 and L_2, respectively. Let $A : C \to H$ be an α-inverse strongly monotone mapping and let B be a maximal monotone mapping such that the domain of B is subset of C. Assume that $\mathcal{F} = F(T_1) \cap F(T_2) \cap (A+B)^{-1}0 \neq \phi$. Let $J_{\lambda_n} = (I + \lambda_n B)^{-1}$, where $\{\lambda_n\}$ is a positive real number sequence. Given $x_1, u \in C$, let $\{x_n\}$ be the sequence generated by the following algorithm:*

$$\begin{aligned} z_n &= (1-c_n) x_n + c_n T_2 x_n, \\ y_n &= (1-\beta_n) x_n + \beta_n T_1 x_n, \\ x_{n+1} &= P_C \Big[\alpha_n u + (1-\alpha_n) \\ &\quad \times \big(\theta_n x_n + \delta_n T_1 y_n + \gamma_n T_2 z_n \\ &\quad + \xi_n J_{\lambda_n} (I - \lambda_n A) x_n\big)\Big]. \end{aligned} \tag{16}$$

Assume that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{c_n\}$, $\{\theta_n\}$, $\{\delta_n\}$, $\{\gamma_n\}$, $\{\xi_n\}$, and $\{\lambda_n\}$ satisfy the following restrictions:

(a) *$0 < a < \lambda_n < b < 2\alpha$;*

(b) *$0 < c \le \theta_n, \delta_n, \gamma_n, \xi_n \le d < 1$ and $\theta_n + \delta_n + \gamma_n + \xi_n = 1$;*

(c) *$0 < \alpha_n < e < 1$, $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;*

(d) *$\delta_n + \gamma_n + \xi_n \le \beta_n$, $c_n \le \beta < 1/(\sqrt{1+L^2}+1)$, for all $n \ge 1$,*

for some real numbers $a, b, c, d, e, \beta > 0$, where $L = \max\{L_1, L_2\}$. Then $\{x_n\}$ converges strongly to some point $\overline{x}$, where $\overline{x} = P_{\mathcal{F}} u$.

Proof. First, we show that $I - \lambda_n A$ is nonexpansive.

Indeed, we have

$$\begin{aligned} &\|(I-\lambda_n A)x - (I-\lambda_n A)y\|^2 \\ &\quad = \|x-y\|^2 - 2\lambda_n \langle x-y, Ax - Ay\rangle \\ &\qquad + \lambda_n^2 \|Ax - Ay\|^2 \\ &\quad \le \|x-y\|^2 - \lambda_n (2\alpha - \lambda_n) \|Ax - Ay\|^2. \end{aligned} \tag{17}$$

It follows from restriction (a) that $I - \lambda_n A$ is nonexpansive.

Let $p \in \mathcal{F}$. It follows from (5), (16), and Lemmas 3 and 6 that

$$\begin{aligned} &\|x_{n+1} - p\|^2 \\ &= \Big\|P_C \big[\alpha_n u + (1-\alpha_n) \\ &\qquad \times \big(\theta_n x_n + \delta_n T_1 y_n + \gamma_n T_2 z_n \\ &\qquad + \xi_n J_{\lambda_n} (I - \lambda_n A) x_n\big)\big] - p\Big\|^2 \\ &\le \Big\|\alpha_n u + (1-\alpha_n) \\ &\qquad \times \big(\theta_n x_n + \delta_n T_1 y_n + \gamma_n T_2 z_n \\ &\qquad + \xi_n J_{\lambda_n} (I - \lambda_n A) x_n\big) - p\Big\|^2 \\ &\le \alpha_n \|u-p\|^2 + (1-\alpha_n) \\ &\quad \times \Big\|\theta_n (x_n - p) + \delta_n (T_1 y_n - p) \\ &\qquad + \gamma_n (T_2 z_n - p) \\ &\qquad + \xi_n \big(J_{\lambda_n} (I - \lambda_n A) x_n - p\big)\Big\|^2 \\ &\le \alpha_n \|u-p\|^2 + (1-\alpha_n) \\ &\quad \times \Big[\theta_n \|x_n - p\|^2 + \delta_n \|T_1 y_n - p\|^2 \\ &\qquad + \gamma_n \|T_2 z_n - p\|^2 \\ &\qquad + \xi_n \big\|J_{\lambda_n} (I - \lambda_n A) x_n - p\big\|^2\Big] \\ &\quad - (1-\alpha_n) \theta_n \delta_n \|T_1 y_n - x_n\|^2 \\ &\quad - (1-\alpha_n) \theta_n \gamma_n \|T_2 z_n - x_n\|^2 \\ &\quad - (1-\alpha_n) \theta_n \xi_n \big\|J_{\lambda_n} (I - \lambda_n A) x_n - x_n\big\|^2 \end{aligned}$$

$$\begin{aligned}&\leq \alpha_n\|u-p\|^2+(1-\alpha_n)(\theta_n+\xi_n)\|x_n-p\|^2\\&\quad+(1-\alpha_n)\delta_n\left[\|y_n-p\|^2+\|y_n-T_1y_n\|^2\right]\\&\quad+(1-\alpha_n)\gamma_n\left[\|z_n-p\|^2+\|z_n-T_2z_n\|^2\right]\\&\quad-(1-\alpha_n)\theta_n\delta_n\|T_1y_n-x_n\|^2\\&\quad-(1-\alpha_n)\theta_n\gamma_n\|T_2z_n-x_n\|^2\\&\quad-(1-\alpha_n)\theta_n\xi_n\|J_{\lambda_n}(I-\lambda_nA)x_n-x_n\|^2.\end{aligned} \tag{18}$$

It follows from (5), (16), and Lemma 6 that

$$\begin{aligned}&\|z_n-p\|^2\\&\quad=\|(1-c_n)(x_n-p)+c_n(T_2x_n-p)\|^2\\&\quad=(1-c_n)\|x_n-p\|^2\\&\qquad+c_n\|T_2x_n-p\|^2-c_n(1-c_n)\|x_n-T_2x_n\|^2\\&\quad\leq(1-c_n)\|x_n-p\|^2\\&\qquad+c_n\left[\|x_n-p\|+\|x_n-T_2x_n\|^2\right]\\&\qquad-c_n(1-c_n)\|x_n-T_2x_n\|^2\\&\quad=\|x_n-p\|^2+c_n^2\|x_n-T_2x_n\|^2,\\&\|y_n-p\|^2\\&\quad=\|(1-\beta_n)(x_n-p)+\beta_n(T_1x_n-p)\|^2\\&\quad\leq\|x_n-p\|^2+\beta_n^2\|x_n-T_1x_n\|^2.\end{aligned} \tag{19}$$

Similarly, we have that

$$\begin{aligned}&\|y_n-T_1y_n\|^2\\&\quad=\|(1-\beta_n)(x_n-T_1y_n)+\beta_n(T_1x_n-T_1y_n)\|^2\\&\quad=(1-\beta_n)\|x_n-T_1y_n\|^2\\&\qquad+\beta_n\|T_1x_n-T_1y_n\|^2\\&\qquad-\beta_n(1-\beta_n)\|x_n-T_1x_n\|^2\\&\quad\leq(1-\beta_n)\|x_n-T_1y_n\|^2+\beta_nL^2\|x_n-y_n\|^2\\&\qquad-\beta_n(1-\beta_n)\|x_n-T_1x_n\|^2\\&\quad=(1-\beta_n)\|x_n-T_1y_n\|^2\\&\qquad-\beta_n\left(1-\beta_n^2L^2-\beta_n\right)\|x_n-T_1x_n\|^2.\\&\|z_n-T_2z_n\|^2\\&\quad=\|(1-c_n)(x_n-T_2z_n)+c_n(T_2x_n-T_2z_n)\|^2\\&\quad\leq(1-c_n)\|x_n-T_2z_n\|^2\\&\qquad-c_n\left(1-c_n^2L^2-c_n\right)\|x_n-T_2x_n\|^2.\end{aligned} \tag{20}$$

Substituting (19) and (20) into (18), we obtain that

$$\begin{aligned}&\|x_{n+1}-p\|^2\\&\quad\leq\alpha_n\|u-p\|^2\\&\quad+(1-\alpha_n)(\theta_n+\xi_n)\|x_n-p\|^2\\&\quad+(1-\alpha_n)\delta_n\left[\|x_n-p\|^2+\beta_n^2\|x_n-T_1x_n\|^2\right]\\&\quad+(1-\alpha_n)\delta_n\left[(1-\beta_n)\|x_n-T_1y_n\|^2\right.\\&\qquad\qquad\left.-\beta_n\left(1-\beta_n^2L^2-\beta_n\right)\|x_n-T_1x_n\|^2\right]\\&\quad+(1-\alpha_n)\gamma_n\left[\|x_n-p\|^2+c_n^2\|x_n-T_2x_n\|^2\right]\\&\quad+(1-\alpha_n)\gamma_n\left[(1-c_n)\|x_n-T_2z_n\|^2\right.\\&\qquad\qquad\left.-c_n\left(1-c_n^2L^2-c_n\right)\|x_n-T_2x_n\|^2\right]\\&\quad-(1-\alpha_n)\theta_n\delta_n\|T_1y_n-x_n\|^2\\&\quad-(1-\alpha_n)\theta_n\gamma_n\|T_2z_n-x_n\|^2\\&\quad-(1-\alpha_n)\theta_n\xi_n\|J_{\lambda_n}(I-\lambda_nA)x_n-x_n\|^2\\&\quad=\alpha_n\|u-p\|^2+(1-\alpha_n)\|x_n-p\|^2\\&\quad-(1-\alpha_n)\delta_n\beta_n\left(1-2\beta_n-\beta_n^2L^2\right)\|x_n-T_1x_n\|^2\\&\quad-(1-\alpha_n)\gamma_nc_n\left(1-2c_n-c_n^2L^2\right)\|x_n-T_2x_n\|^2\\&\quad+(1-\alpha_n)\delta_n(\delta_n+\gamma_n+\xi_n-\beta_n)\|T_1y_n-x_n\|^2\\&\quad+(1-\alpha_n)\gamma_n(\delta_n+\gamma_n+\xi_n-c_n)\|T_2z_n-x_n\|^2\\&\quad-(1-\alpha_n)\theta_n\xi_n\|J_{\lambda_n}(I-\lambda_nA)x_n-x_n\|^2.\end{aligned} \tag{21}$$

In view of restriction (d), we find that

$$\begin{gathered}1-2\beta_n-\beta_n^2L^2\geq1-2\beta-\beta^2L^2>0,\\1-2c_n-c_n^2L^2\geq1-2\beta-\beta^2L^2>0,\\\delta_n+\gamma_n+\xi_n-\beta_n\leq0,\\\delta_n+\gamma_n+\xi_n-c_n\leq0,\end{gathered} \tag{22}$$

for all $n\geq1$. It follows from (21) and (22) that

$$\|x_{n+1}-p\|^2\leq\alpha_n\|u-p\|^2+(1-\alpha_n)\|x_n-p\|^2. \tag{23}$$

Putting $M=\max\{\|u-p\|^2,\|x_1-p\|^2\}$, we find that $\|x_n-p\|^2\leq M$ for all $n\geq1$.

Indeed, it is clear that $\|x_2 - p\|^2 \le M$. Suppose that $\|x_m - p\| \le M$ for some positive integer m. It follows that

$$\begin{aligned}\|x_{m+1} - p\|^2 &\le \alpha_m \|u - p\|^2 + (1-\alpha_m)\|x_m - p\|^2 \\ &\le \alpha_m M + (1-\alpha_m) M \\ &= M.\end{aligned} \tag{24}$$

This finds that $\{x_n\}$ is bounded and hence $\{y_n\}$ and $\{z_n\}$ are bounded.

Let $w_n = \theta_n x_n + \delta_n T_1 y_n + \gamma_n T_2 z_n + \xi_n J_{\lambda_n}(I - \lambda_n A)x_n$. Then we see that $x_{n+1} = P_C(\alpha_n u + (1-\alpha_n)w_n)$. Put $\overline{x} = P_{\mathcal{F}} u$. Using (16), (19), and (20) and Lemmas 1 and 6, we find that

$$\begin{aligned}
&\|x_{n+1} - \overline{x}\|^2 \\
&\quad \le \|\alpha_n(u - \overline{x}) + (1-\alpha_n)(w_n - \overline{x})\|^2 \\
&\le (1-\alpha_n)\|w_n - \overline{x}\|^2 \\
&\quad + 2\alpha_n \langle u - \overline{x}, \alpha_n(u - \overline{x}) + (1-\alpha_n)(w_n - \overline{x})\rangle \\
&\le (1-\alpha_n)\theta_n \|x_n - \overline{x}\|^2 \\
&\quad + (1-\alpha_n)\delta_n \|T_1 y_n - \overline{x}\|^2 \\
&\quad + (1-\alpha_n)\gamma_n \|T_2 z_n - \overline{x}\|^2 \\
&\quad + (1-\alpha_n)\xi_n \left\|J_{\lambda_n}(I - \lambda_n A)x_n - \overline{x}\right\|^2 \\
&\quad - (1-\alpha_n)\theta_n\delta_n \|T_1 y_n - x_n\|^2 \\
&\quad - (1-\alpha_n)\theta_n\gamma_n \|T_2 z_n - x_n\|^2 \\
&\quad - (1-\alpha_n)\theta_n\xi_n \left\|J_{\lambda_n}(I - \lambda_n A)x_n - x_n\right\|^2 \\
&\quad + 2\alpha_n^2 \|u - \overline{x}\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\langle u - \overline{x}, w_n - \overline{x}\rangle \\
&\le (1-\alpha_n)\theta_n \|x_n - \overline{x}\|^2 \\
&\quad + (1-\alpha_n)\delta_n \left[\|y_n - \overline{x}\|^2 + \|y_n - T_1 y_n\|^2\right] \\
&\quad + (1-\alpha_n)\gamma_n \left[\|z_n - \overline{x}\|^2 + \|z_n - T_2 z_n\|^2\right] \\
&\quad + (1-\alpha_n)\xi_n \|x_n - \overline{x}\|^2 \\
&\quad - (1-\alpha_n)\theta_n\delta_n \|T_1 y_n - x_n\|^2 \\
&\quad - (1-\alpha_n)\theta_n\gamma_n \|T_2 z_n - x_n\|^2 \\
&\quad - (1-\alpha_n)\theta_n\xi_n \left\|J_{\lambda_n}(I - \lambda_n A)x_n - x_n\right\|^2 \\
&\quad + 2\alpha_n^2 \|u - \overline{x}\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\langle u - \overline{x}, w_n - \overline{x}\rangle \\
&\le (1-\alpha_n)(\theta_n + \xi_n)\|x_n - \overline{x}\|^2 \\
&\quad + (1-\alpha_n)\delta_n \left[\|x_n - \overline{x}\|^2 + \beta_n^2 \|x_n - T_1 x_n\|^2\right] \\
&\quad + (1-\alpha_n)\delta_n \left[(1-\beta_n)\|x_n - T_1 y_n\|^2 \right. \\
&\qquad\qquad \left. -\beta_n\left(1 - \beta_n^2 L^2 - \beta_n\right)\|x_n - T_1 x_n\|^2\right] \\
&\quad + (1-\alpha_n)\gamma_n \left[\|x_n - \overline{x}\|^2 + c_n^2 \|x_n - T_2 x_n\|^2\right] \\
&\quad + (1-\alpha_n)\gamma_n \left[(1-c_n)\|x_n - T_2 z_n\|^2 \right. \\
&\qquad\qquad \left. -c_n\left(1 - c_n^2 L^2 - c_n\right)\|x_n - T_2 x_n\|^2\right] \\
&\quad - (1-\alpha_n)\theta_n\delta_n \|T_1 y_n - x_n\|^2 \\
&\quad - (1-\alpha_n)\theta_n\gamma_n \|T_2 z_n - x_n\|^2 \\
&\quad - (1-\alpha_n)\theta_n\xi_n \left\|J_{\lambda_n}(I - \lambda_n A)x_n - x_n\right\|^2 \\
&\quad + 2\alpha_n^2 \|u - \overline{x}\|^2 + 2\alpha_n(1-\alpha_n)\langle u - \overline{x}, w_n - \overline{x}\rangle,
\end{aligned} \tag{25}$$

which implies from (22) that

$$\begin{aligned}
&\|x_{n+1} - \overline{x}\|^2 \\
&\quad \le (1-\alpha_n)\|x_n - \overline{x}\|^2 \\
&\quad - (1-\alpha_n)\delta_n\beta_n\left(1 - 2\beta_n - \beta_n^2 L^2\right)\|x_n - T_1 x_n\|^2 \\
&\quad + (1-\alpha_n)\delta_n(\delta_n + \xi_n + \gamma_n - \beta_n)\|T_1 y_n - x_n\|^2 \\
&\quad - (1-\alpha_n)\gamma_n c_n\left(1 - 2c_n - c_n^2 L^2\right)\|x_n - T_2 x_n\|^2 \\
&\quad + (1-\alpha_n)\gamma_n(\delta_n + \xi_n + \gamma_n - c_n)\|T_2 z_n - x_n\| \\
&\quad - (1-\alpha_n)\theta_n\xi_n \left\|J_{\lambda_n}(I - \lambda_n A)x_n - x_n\right\|^2 \\
&\quad + 2\alpha_n^2 \|u - \overline{x}\|^2 + 2\alpha_n(1-\alpha_n)\langle u - \overline{x}, w_n - \overline{x}\rangle \\
&\le (1-\alpha_n)\|x_n - \overline{x}\|^2 + 2\alpha_n^2 \|u - \overline{x}\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\langle u - \overline{x}, w_n - \overline{x}\rangle.
\end{aligned} \tag{26}$$

Now we consider two cases.

Case 1. Suppose that there exists $n_0 \in N$ such that $\{\|x_n - \overline{x}\|\}$ is decreasing for all $n \ge n_0$. Then we get that $\{\|x_n - \overline{x}\|\}$ is convergent. It follows from (22) and (26) that

$$\begin{gathered} x_n - T_1 x_n \longrightarrow 0, \qquad x_n - T_2 x_n \longrightarrow 0, \\ x_n - J_{\lambda_n}(I - \lambda_n A)x_n \longrightarrow 0, \end{gathered} \tag{27}$$

as $n \to \infty$. Also we obtain from (27) that

$$\begin{gathered} \|y_n - x_n\| = \beta_n \|x_n - T_1 x_n\| \longrightarrow 0, \\ \|z_n - x_n\| = c_n \|x_n - T_2 x_n\| \longrightarrow 0, \end{gathered} \tag{28}$$

as $n \to \infty$. In view of the Lipschitz continuity of T_1, T_2 and (27) and (28), we find that

$$\begin{aligned}\|T_1 y_n - x_n\| &\le \|T_1 y_n - T_1 x_n\| + \|T_1 x_n - x_n\| \\ &\le L\|y_n - x_n\| + \|T_1 x_n - x_n\| \\ &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty,\end{aligned} \tag{29}$$

$$\begin{aligned}\|T_2 z_n - x_n\| &\le \|T_2 z_n - T_2 x_n\| + \|T_2 x_n - x_n\| \\ &\le L\|z_n - x_n\| + \|T_2 x_n - x_n\| \\ &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{30}$$

It follows from (27), (29), and (30) that

$$\begin{aligned}\|w_n - x_n\| &\le \delta_n \|T_1 y_n - x_n\| + \gamma_n \|T_2 z_n - x_n\| \\ &\quad + \xi_n \left\|J_{\lambda_n}(I - \lambda_n A) x_n - x_n\right\| \\ &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{31}$$

Since $\{w_n\}$ is a bounded subset of H, we can choose a subsequence $\{w_{n_i}\}$ of $\{w_n\}$ such that $w_{n_i} \rightharpoonup w$ and

$$\limsup_{n\to\infty} \langle u - \overline{x}, w_n - \overline{x}\rangle = \lim_{i\to\infty} \left\langle u - \overline{x}, w_{n_i} - \overline{x}\right\rangle. \tag{32}$$

It follows from (31) that $x_{n_i} \rightharpoonup w$. By (27) and Lemma 7, we obtain that $w \in F(T_1)$ and $w \in F(T_2)$.

Next, we show that $w \in (A+B)^{-1}0$.

Notice that

$$\begin{aligned}&\left\|J_{\lambda_n}(I - \lambda_n A)x_n - p\right\|^2 \\ &= \left\|J_{\lambda_n}(I - \lambda_n A)x_n - J_{\lambda_n}(I - \lambda_n A)p\right\|^2 \\ &\le \left\|(I - \lambda_n A)x_n - (I - \lambda_n A)p\right\|^2 \\ &= \|x_n - p\|^2 - 2\lambda_n \langle x_n - p, Ax_n - Ap\rangle \\ &\quad + \lambda_n^2 \|Ax_n - Ap\|^2 \\ &\le \|x_n - p\|^2 - 2\alpha\lambda_n \|Ax_n - Ap\|^2 + \lambda_n^2 \|Ax_n - Ap\|^2 \\ &= \|x_n - p\|^2 - \lambda_n (2\alpha - \lambda_n)\|Ax_n - Ap\|^2.\end{aligned} \tag{33}$$

It follows from (27) that

$$\begin{aligned}&\lambda_n (2\alpha - \lambda_n)\|Ax_n - Ap\|^2 \\ &\le \|x_n - p\|^2 - \left\|J_{\lambda_n}(I - \lambda_n A)x_n - p\right\| \\ &= \left(\|x_n - p\| + \left\|J_{\lambda_n}(I - \lambda_n A)x_n - p\right\|\right) \\ &\quad \times \left\|x_n - J_{\lambda_n}(I - \lambda_n A)x_n\right\| \\ &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{34}$$

Hence we get

$$\lim_{n\to\infty} \|Ax_n - Ap\|^2 = 0. \tag{35}$$

Putting $h_n = J_{\lambda_n}(I - \lambda_n A)x_n$, we find that $((x_{n_i} - h_{n_i})/\lambda_{n_i}) - Ax_{n_i} \in Bh_{n_i}$. Since B is monotone, we get that, for any $(u, v) \in G(B)$,

$$\left\langle h_{n_i} - u, \frac{x_{n_i} - h_{n_i}}{\lambda_{n_i}} - Ax_{n_i} - v\right\rangle \ge 0, \tag{36}$$

where $G(B) = \{(x, w) \in H \times H : x \in D(B), w \in Bx\}$. Since $\langle x_{n_i} - w, Ax_{n_i} - Aw\rangle \ge \alpha\|Ax_{n_i} - Aw\|^2$, $x_{n_i} \rightharpoonup w$, and $Ax_{n_i} \to Ap$ as $i \to \infty$, we have $Ax_{n_i} \to Aw$. Thus, letting $i \to \infty$, we obtain from (27) and (36) that $\langle w - u, -Aw - v\rangle \ge 0$. This means $-Aw \in Bw$, that is, $0 \in (A+B)w$. Hence we get $w \in (A+B)^{-1}0$. This implies from Lemma 2 that

$$\begin{aligned}\limsup_{n\to\infty} \langle u - \overline{x}, w_n - \overline{x}\rangle &= \lim_{i\to\infty} \left\langle u - \overline{x}, w_{n_i} - \overline{x}\right\rangle \\ &= \langle u - \overline{x}, w - \overline{x}\rangle \\ &\le 0.\end{aligned} \tag{37}$$

On the other hand, we have from (26) that

$$\begin{aligned}\|x_{n+1} - \overline{x}\|^2 &\le (1 - \alpha_n)\|x_n - \overline{x}\|^2 \\ &\quad + \alpha_n \left(2\alpha_n \|u - \overline{x}\|^2 \right. \\ &\quad \left. + (1 - \alpha_n)\langle u - \overline{x}, w_n - \overline{x}\rangle\right).\end{aligned} \tag{38}$$

From Lemma 8 and (37), we find that $\lim_{n\to\infty}\|x_n - \overline{x}\| = 0$.

Case 2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\left\|x_{n_i} - \overline{x}\right\| < \left\|x_{n_i+1} - \overline{x}\right\|, \tag{39}$$

for all $i \in \mathbb{N}$. By Lemma 5, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and

$$\left\|x_{m_k} - \overline{x}\right\| \le \left\|x_{m_k+1} - \overline{x}\right\|, \qquad \left\|x_k - \overline{x}\right\| \le \left\|x_{m_k+1} - \overline{x}\right\|, \tag{40}$$

for all $k \in \mathbb{N}$. From (22) and (26), we have $x_{m_k} - T_1 x_{m_k} \to 0$, $x_{m_k} - T_2 x_{m_k} \to 0$, and $x_{m_k} - J_{\lambda_{m_k}}(I - \lambda_{m_k} A)x_{m_k} \to 0$ as $k \to \infty$. Thus, like in Case 1, we obtain $w_{m_k} - x_{m_k} \to 0$ and

$$\limsup_{k\to\infty} \left\langle u - \overline{x}, w_{m_k} - \overline{x}\right\rangle \le 0. \tag{41}$$

From (26) and (40), we have

$$\begin{aligned}&\alpha_{m_k}\left\|x_{m_k} - \overline{x}\right\|^2 \\ &\quad \le \left\|x_{m_k} - \overline{x}\right\|^2 - \left\|x_{m_k+1} - \overline{x}\right\|^2 \\ &\quad + 2\alpha_{m_k}\left(\alpha_{m_k}\|u - \overline{x}\|^2 + \left(1 - \alpha_{m_k}\right)\left\langle u - \overline{x}, w_{m_k} - \overline{x}\right\rangle\right) \\ &\le 2\alpha_{m_k}\left(\alpha_{m_k}\|u - \overline{x}\|^2 + \left(1 - \alpha_{m_k}\right)\left\langle u - \overline{x}, w_{m_k} - \overline{x}\right\rangle\right).\end{aligned} \tag{42}$$

Applying (41) and $\alpha_{m_k} > 0$, we have $\|x_{m_k} - \overline{x}\| \to 0$ as $k \to \infty$. It implies that $\|x_{m_k+1} - \overline{x}\| \to 0$ as $k \to \infty$. By (40), we have $x_k \to \overline{x}$ as $k \to \infty$.

Therefore, from the above two cases, we can conclude that the sequence $\{x_n\}$ converges strongly to $\overline{x} = P_{\mathcal{F}}u$. This completes the proof. □

From Lemma 4, we have the following result.

Corollary 10. *Let C be a nonempty closed convex subset of a real Hilbert space H such that $\operatorname{int}(C) \neq \phi$. Let $T_1, T_2 : C \to C$ be Lipschitzian pseudocontractive mappings with Lipschitz constants L_1 and L_2, respectively. Let $B_1 : D(B_1) \to 2^H$ and $B_2 : D(B_2) \to 2^H$ be maximal monotone mappings such that $D(B_1) \cap \operatorname{int}(D(B_2)) \neq \phi$. Assume that $\mathcal{F} = F(T_1) \cap F(T_2) \cap (B_1 + B_2)^{-1}(0) \neq \phi$. Let $J_{\lambda_n} = (I + \lambda_n(B_1 + B_2))^{-1}$, where $\{\lambda_n\}$ is a positive real number sequence. Given $x_1, u \in C$, let $\{x_n\}$ be the sequence generated by the following algorithm:*

$$\begin{aligned} z_n &= (1 - c_n)x_n + c_nT_2x_n, \\ y_n &= (1 - \beta_n)x_n + \beta_nT_1x_n, \\ x_{n+1} &= P_C\left[\alpha_nu + (1 - \alpha_n)\right. \\ &\quad \left. \times\left(\theta_nx_n + \delta_nT_1y_n + \gamma_nT_2z_n + \xi_nJ_{\lambda_n}x_n\right)\right], \\ &\qquad \forall n \geq 1. \end{aligned} \tag{43}$$

Assume that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{c_n\}$, $\{\theta_n\}$, $\{\delta_n\}$, $\{\gamma_n\}$, $\{\xi_n\}$, and $\{\lambda_n\}$ satisfy the following restrictions:

(a) *$0 < a < \lambda_n < b < 1$;*

(b) *$0 < c \leq \theta_n, \delta_n, \gamma_n, \xi_n \leq d < 1$, and $\theta_n + \delta_n + \gamma_n + \xi_n = 1$;*

(c) *$0 < \alpha_n < e < 1$, $\lim_{n\to\infty}\alpha_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$;*

(d) *$\delta_n + \gamma_n + \xi_n \leq \beta_n$, $c_n \leq \beta < 1/(\sqrt{1 + L^2} + 1)$, for all $n \geq 1$,*

for some real numbers $a, b, c, d, e > 0$, where $L = \max\{L_1, L_2\}$. Then $\{x_n\}$ converges strongly to some point $\overline{x}$, where $\overline{x} = P_{\mathcal{F}}u$.

Remark 11. If $T_1 = T$, $T_2 = I$ (the identity mapping), and $u = 0$, then Theorem 9 reduces to Theorem 3.1 of Shahzad and Zegeye [6]. Thus, Theorem 9 covers Theorem 3.1 of Shahzad and Zegeye [6] as a special case.

4. Applications

In this section, we will consider equilibrium problems and variational inequalities.

Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. Recall the following equilibrium problem: find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{44}$$

We use $\mathrm{EP}(F)$ to denote the solution set of the equilibrium problem. To study the equilibrium problems, we assume that F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, that is, $F(x, y) + F(y, x) \geq 0$, for all $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\limsup_{t\downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y); \tag{45}$$

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

Lemma 12 (see [1]). *Let C be a nonempty closed convex subset of a real Hilbert space and let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4). Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C. \tag{46}$$

Further, define

$$T_r(x) = \left\{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C\right\}, \tag{47}$$

for all $r > 0$ and $x \in H$. Then the following hold:

(a) *T_r is single-valued;*

(b) *T_r is firmly nonexpansive; that is, for any $x, y \in H$,*

$$\|T_rx - T_ry\|^2 \leq \langle T_rx - T_ry, x - y\rangle; \tag{48}$$

(c) *$F(T_r) = \mathrm{EP}(F)$;*

(d) *$\mathrm{EP}(F)$ is closed and convex.*

Lemma 13 (see [13]). *Let C be a nonempty closed convex subset of a real Hilbert space H, let F be a bifunction from $C\times C$ to $\mathbb{R}$ which satisfies (A1)–(A4), and let A_F be a multivalued mapping of H into itself defined by*

$$A_Fx = \begin{cases} \{z \in H : F(x, y) \geq \langle y - x, z\rangle, \forall y \in C\}, & x \in C, \\ \phi, & x \notin C. \end{cases} \tag{49}$$

Then A_F is a maximal monotone mapping with the domain $D(A_F) \subset C$, $\mathrm{EP}(F) = A_F^{-1}0$, and

$$T_r(x) = (I + rA_F)^{-1}x, \quad \forall x \in H,\ r > 0, \tag{50}$$

where T_r is defined as in (47).

Now we consider an equilibrium problem. Using Lemmas 12 and 13, the following result holds.

Theorem 14. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T_1, T_2 : C \to C$ be Lipschitzian pseudocontractive mappings with Lipschitz constants L_1 and L_2, respectively. Assume that $\mathcal{F} = F(T_1) \cap F(T_2) \cap \mathrm{EP}(F) \neq \phi$.*

Given $x_1, u \in C$, let $\{x_n\}$ be the sequence generated by the following algorithm:

$$\begin{aligned} z_n &= (1-c_n)x_n + c_n T_2 x_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T_1 x_n, \\ u_n \in C \quad &\text{such that } F(u_n, v) + \frac{1}{r_n}\langle v-u_n, u_n - x_n\rangle \geq 0, \\ &\qquad \forall v \in C, \\ x_{n+1} &= P_C[\alpha_n u + (1-\alpha_n) \\ &\quad \times (\theta_n x_n + \delta_n T_1 y_n + \gamma_n T_2 z_n + \xi_n u_n)], \\ &\qquad \forall n \geq 1. \end{aligned} \tag{51}$$

Assume that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{c_n\}$, $\{\theta_n\}$, $\{\delta_n\}$, $\{\gamma_n\}$, $\{\xi_n\}$, and $\{r_n\}$ satisfy the following restrictions:

(a) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty}\|r_{n+1} - r_n\| = 0$;

(b) $0 < c \leq \theta_n, \delta_n, \gamma_n, \xi_n \leq d < 1$, *and* $\theta_n + \delta_n + \gamma_n + \xi_n = 1$;

(c) $0 < \alpha_n < e < 1$, $\lim_{n\to\infty}\alpha_n = 0$, *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(d) $\delta_n + \gamma_n + \xi_n \leq \beta_n$, $c_n < \beta < 1/(\sqrt{1+L^2}+1)$, *for all* $n \geq 1$,

for some real numbers $c, d, e > 0$, where $L = \max\{L_1, L_2\}$. Then $\{x_n\}$ converges strongly to some point $\overline{x}$, where $\overline{x} = P_{\mathcal{F}}u$.

Let $f : H \to (-\infty, +\infty]$ be a proper convex lower semicontinuous function. Then the subdifferential of ∂f of f is defined as follows:

$$\partial f(x) = \{y \in H : f(z) \geq f(x) + \langle z - x, y\rangle,\ z \in H\}, \qquad \forall x \in H. \tag{52}$$

From Rockafellar [14], we find that ∂f is maximal monotone. It is easy to verify that $0 \in \partial f(x)$ if and only if $f(x) = \min_{y\in H} f(y)$. Let I_C be the indicator function of C; that is,

$$I_C(x) = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C. \end{cases} \tag{53}$$

Then $I_C : H \to (-\infty, +\infty]$ is a proper convex lower semicontinuous function and ∂I_C is a maximal monotone mapping.

Lemma 15 (see [6]). *Let C be a nonempty closed convex subset of a real Hilbert space H, let P_C be the metric projection from H onto C, and let ∂I_C be the subdifferential of I_C, where I_C is the indicator function of C and let $J_\lambda = (I + \lambda\partial I_C)^{-1}$. Then*

$$y = J_\lambda x \Longleftrightarrow y = P_C x, \quad x \in H,\ y \in C. \tag{54}$$

Now we consider a variational inequality problem.

Theorem 16. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T_1, T_2 : C \to C$ be Lipschitzian pseudocontractive mappings with Lipschitz constants L_1 and L_2, respectively. Let $A : C \to H$ be an α-inverse strongly monotone mapping. Assume that $F(T_1) \cap F(T_2) \cap \mathrm{VI}(C, A) \neq \phi$. Given $x_1, u \in C$, let $\{x_n\}$ be the sequence generated by the following algorithm:*

$$\begin{aligned} z_n &= (1-c_n)x_n + c_n T_2 x_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T_1 x_n, \\ x_{n+1} &= P_C[\alpha_n u + (1-\alpha_n) \\ &\quad \times (\theta_n x_n + \delta_n T_1 y_n + \gamma_n T_2 z_n \\ &\quad + \xi_n P_C(I - \lambda_n A)x_n)], \quad \forall n \geq 1. \end{aligned} \tag{55}$$

Assume that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{c_n\}$, $\{\theta_n\}$, $\{\delta_n\}$, $\{\gamma_n\}$, $\{\xi_n\}$, and $\{\lambda_n\}$ satisfy the following restrictions:

(a) $0 < a < \lambda_n < b < 2\alpha$;

(b) $0 < c \leq \theta_n, \delta_n, \gamma_n, \xi_n \leq d < 1$, *and* $\theta_n + \delta_n + \gamma_n + \xi_n = 1$;

(c) $0 < \alpha_n < e < 1$, $\lim_{n\to\infty}\alpha_n = 0$, *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(d) $\delta_n + \gamma_n + \xi_n \leq \beta_n$, $c_n < \beta < 1/(\sqrt{1+L^2}+1)$, *for all* $n \geq 1$,

for some real numbers $a, b, c, d, e > 0$, where $L = \max\{L_1, L_2\}$. Then $\{x_n\}$ converges strongly to some point $\overline{x}$, where $\overline{x} = P_{\mathcal{F}}u$.

Proof. Put $B = \partial I_C$ in Theorem 9. Then we get that

$$\begin{aligned} x \in (A + \partial I_C)^{-1}0 &\Longleftrightarrow 0 \in Ax + \partial I_C x \\ &\Longleftrightarrow -Ax \in \partial I_C x \\ &\Longleftrightarrow \langle Ax, y - x\rangle \geq 0 \\ &\Longleftrightarrow x \in \mathrm{VI}(C, A). \end{aligned} \tag{56}$$

From Lemma 15, we can conclude the desired conclusion immediately. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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Research Article

Completion of a Dislocated Metric Space

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We provide a construction for the completion of a dislocated metric space (abbreviated d-metric space); we also prove that the completion of the metric associated with a d-metric coincides with the metric associated with the completion of the d-metric.

1. Introduction

Completion of a metric space via Cauchy sequences can be achieved because of certain convergence properties enjoyed by the metric and the property that convergent sequences are Cauchy sequences. Lack of some of these properties in weaker forms of metric spaces comes in the way of completion process in the above lines. In semimetric spaces several new ways of completeness were invented, for example, Cauchy completeness, McAuley notions of strong and weak completeness [1], Moore completeness [2], and so on. Moshokoa [3] introduced the notion of convergence completeness for semimetric spaces and discussed completion on these lines.

For d-metric spaces adoption of Van der-Waerdens completion process through Cauchy sequences is possible but is not routine, the difficulty being the mischief created by the isolated points. Here we show how to overcome this problem.

In his study of programming languages, Hitzler [4] associated a metric d' with every d-metric by defining

$$d'(a,b) = \begin{cases} d(a,b), & \text{if } a \neq b \\ 0, & \text{if } a = b. \end{cases} \tag{1}$$

We establish that the metric associated with the completion of a d-metric is the completion of the metric associated with d.

We recall [4] where a distance function on a set X is said to be a d-metric on X if

(i) $d(x,y) = d(y,x)$;

(ii) $d(x,y) = 0 \Rightarrow x = y$;

(iii) $d(x,z) \le d(x,y) + d(y,z)$ for all x, y, z in X.

If d is a d-metric on X then (X,d) is called a d-metric space. Many authors (see, e.g. [5–9]) have studied fixed point theorems in d-metric spaces but topology and topological aspects on this space are discussed by Sarma and Kumari [10].

The class $\mathscr{B} = \{V_\epsilon(x)/x \in X \text{ and } \epsilon > 0\}$ is an open base for the topology $\mathfrak{J}_d$ induced by d, where $V_\epsilon(x) = \{y/d(x,y) < \epsilon\} \cup \{x\}$. In what follows whenever we talk about topological properties of a d-metric space, we refer to the topology $\mathfrak{J}_d$.

In [11], the authors highlighted some convergence properties and covers a huge range of implications and nonimplications among them. By using these convergence axioms many authors (see, e.g. [12–15]) have proved fixed point theorems in certain spaces.

The presence of the triangle inequality lends the Haussdorff property for d and some nice properties to (X,d). In particular (X,d) satisfies properties C_1 through C_5:

C_1: $\lim d(x_n, y_n) = 0 = \lim d(x_n, x) \Rightarrow \lim d(y_n, x) = 0$;

C_2: $\lim d(x_n, x) = 0 = \lim d(y_n, x) \Rightarrow \lim d(x_n, y_n) = 0$;

C_3: $\lim d(x_n, y_n) = 0 = \lim d(y_n, z_n) \Rightarrow \lim d(x_n, z_n) = 0$;

C_4: $\lim d(x_n, x) = 0 \Rightarrow \lim d(x_n, y) = d(x, y)$;

C_5: $\lim d(x_n, x) = \lim d(x_n, y) = 0 \Rightarrow x = y$; for all $x, y \in X$.

Above mentioned convergence axioms can be found in [11]. If the triangular inequality is deleted from the axioms on d then it is difficult to define the concept of completion of the resulting distance space. In such an amorphous space, even constant sequences may fail to converge. This and related difficulties compel us to retain the triangle inequality in the discussion of completeness.

Definition 1. Let (X, d) and (Y, d') be distance spaces. A map $f : X \to Y$ is called an *isodistance* if for any $x, y \in X$ one has $d\{f(x), f(y)\} = d'(x, y)$.

2. Completion

In what follows, d is a d-metric on a nonempty set X. A complete d-metric space is a d-metric space in which every Cauchy sequence converges. "Cauchy sequences" in d-metric spaces are defined exactly as in metric spaces.

Lemma 2. *x is an isolated point of X if and only if $X = x$ or $\inf_{y \neq x} d(x, y) > 0$.*

Proof. Suppose x is an isolated point of X. Then there exists $r > 0$ such that $y \neq x \Rightarrow d(x, y) > r \Rightarrow X = \{x\}$ or $\inf_{y \neq x} d(x, y) \geq r > 0$. Conversely suppose $X = \{x\}$ or $\inf_{y \neq x} d(x, y) > 0$. If $X = \{x\}$, then clearly x is an isolated point of X. If $X \neq \{x\}$, then $\inf_{y \neq x} d(x, y) = r > 0$ which implies that $d(x, y) \geq r > r/2$ for all $y \neq x$. Hence x is an isolated point of X. □

Corollary 3. *If $d(x, x) > 0$, then x is an isolated point of X.*

Proof. If $y \neq x$, then $d(x, x) \leq d(x, y) + d(y, x) = 2d(x, y)$ and so $(1/2)d(x, x) \leq d(x, y)$ for all y in X. So $X = \{x\}$ or $\inf_{y \neq x} d(x, y) \geq (1/2)d(x, x) > 0$. □

Theorem 4. *Let (X, d) be a d-metric space. Then there exists a complete d-metric space (X^*, d^*) and an isodistance $T : (X, d) \to (X^*, d^*)$ such that $T(X)$ is dense in X^*.*

Proof. Let I be the collection of isolated points of X and let $J = X - I$. Let $\overline{I}$ be the collection of sequences in X which are ultimately a constant element lying in I and $\overline{J}$ denote the class of Cauchy sequences in J. We define relations R_I and R_J, respectively, on $\overline{I}$ and $\overline{J}$ as follows.

If $(x_n)(y_n)$ are sequences in $\overline{I}$ then $(x_n)R_I(y_n)$ iff the ultimately constant value of (x_n) coincides with that of (y_n).

If $(x_n)(y_n)$ are sequences in $\overline{J}$ then $(x_n)R_J(y_n)$ iff $\lim_{n\to\infty} d(x_n, y_n) = 0$. Clearly R_I is an equivalence relation. We verify that R_J is an equivalence relation. Suppose $(x_n) \in \overline{J}$ and $\epsilon > 0$. Since (x_n) is a Cauchy sequence in J, $d(x_n, x_n) = 0$ and hence R_J is reflexive.

Suppose $(x_n)R_J(y_n)$ for $(x_n), (y_n) \in \overline{J}$. Then $\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(y_n, x_n) = 0$. Hence R_J is symmetric.

If $(x_n), (y_n), (z_n) \in \overline{J}$, $(x_n)R_J(y_n)$ and $(y_n)R_J(z_n)$. If $\epsilon > 0$, then there exists an integer n_1 such that $d(x_n, y_n) < \epsilon/2$ and $d(y_n, z_n) < \epsilon/2$, if $n > n_1$. Consider

$$d(x_n, z_n) \leq d(x_n, y_n) + d(y_n, z_n) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \text{if } n > n_1. \tag{2}$$

This proves that R_J is transitive and hence an equivalence relation. Let $\overline{X} = \overline{I} \cup \overline{J}$. Then $\sim = R_I \cup R_J$ is an equivalence relation on $\overline{X}$.

Let X^* denote $\overline{X}/\sim$. If $(x_n) \in \overline{X}$, $[(x_n)]$ denotes the equivalence class in X^* containing the sequence (x_n).

If $x \in X$ let (x) be the constant sequence (x_n) where $x_n = x$, $\forall n$ and $\widehat{x} = [(x)]$, the equivalence class containing (x).

If $(x_n) \in \overline{J}$, $(y_n) \in \overline{J}$, it follows from the triangle inequality that $|d(x_n, y_n) - d(x_m, y_m)| \leq d(x_n, x_m) + d(y_n, y_m)$. Since $(x_n), (y_n)$ are Cauchy sequences, given that $\epsilon > 0$, there exists a positive integer n_0 such that $d(x_n, x_m) < \epsilon/2$ and $d(y_n, y_m) < \epsilon/2$ for all $n, m \geq n_0$.

This implies that $|d(x_n, y_n) - d(x_m, y_m)| < \epsilon$ proving that $(d(x_n, y_n))$ is a Cauchy sequence of real numbers. By the completeness of R this sequence converges. The definition of R_J makes it obvious that $\lim_{n\to\infty} d(x_n, y_n)$ is independent of the choice of the representative sequences $(x_n), (y_n)$, respectively, from the classes $[(x_n)], [(y_n)]$.

We can prove similarly if $x \in X$ and $(y_n) \in \overline{J}$, $(z_n) \in \overline{J}$, $\lim d(x, y_n)$, $\lim d(x, z_n)$ exists and or equal. Provided (y_n) and (z_n) belong to the same equivalence class.

We define $d^* : X^* \times X^* \to [0, \infty)$ as follows:

$d^*([(x_n)], [(y_n)]) = d(x, y)$ if $(x_n), (y_n) \in \overline{I}$ and x, y are, respectively, the ultimately constant terms of $(x_n), (y_n)$.

$d^*([(x_n)], [(y_n)]) = \lim_{n\to\infty} d(x, y_n)$ if $(x_n) \in \overline{I}$, $(y_n) \in \overline{J}$ and $x_n = x$ eventually.

If $(x_n) \in \overline{J}$, $(y_n) \in \overline{I}$, then define $d^*([(x_n)], [(y_n)]) = d^*([(Y_n)], [(X_n)])$.

If $(x_n) \in \overline{J}$, $(y_n) \in \overline{J}$, then define $d^*([(x_n)], [(y_n)]) = \lim_{n\to\infty} d(x_n, y_n)$.

Verification That d^* Is a d-Metric on X^*. Clearly $d^*(x^*, y^*) \geq 0$ and $d^*(x^*, y^*) = d^*(y^*, x^*)$ for $x^*, y^* \in X^*$.

Suppose $d^*(x^*, y^*) = 0$. Let $(x_n) \in x^*$ and $(y_n) \in y^*$. We first see that $(x_n), (y_n)$ either are both in $\overline{I}$ or are both in $\overline{J}$.

Suppose, on the contrary, $(x_n) \in \overline{I}$ and $(y_n) \in \overline{J}$. Let x be the ultimately constant value of (x_n). Consider

$$\begin{aligned} &0 \leq d(x, x) \leq 2d(x, y_n) \quad \forall n, \\ &\Longrightarrow 0 = d^*(x^*, y^*) = \lim_{n\to\infty} d(x, y_n). \end{aligned} \tag{3}$$

Hence $0 \leq d(x, x) \leq \lim_{n\to\infty} 2d(x, y_n) = 0$, contrary to the fact that $x \in I$.

Suppose $x^*, y^* \in \overline{I}$, $(x_n) \in x^*$, and $(y_n) \in y^*$ with a, b the ultimately constant values of (x_n) and (y_n), respectively.

Then $d^*(x^*, y^*) = 0 \Rightarrow d(a, b) = 0 \Rightarrow a = b \Rightarrow (x_n) \sim (y_n) \Rightarrow x^* = y^*$.

Suppose $x^*, y^* \in \bar{J}$, $(x_n) \in x^*$ and $(y_n) \in y^*$. Consider

$$\begin{aligned} d^*(x^*, y^*) = 0 &\Longrightarrow \lim_{n\to\infty} d(x_n, y_n) = 0 \\ &\Longrightarrow (x_n) \sim (y_n) \\ &\Longrightarrow x^* = y^*. \end{aligned} \tag{4}$$

Verification of the triangular inequality is routine.

Embedding of X in X^*. Define $T : X \to X^*$ by $T(x) = \hat{x}$. It is clear that T is an isodistance. We now verify that $T(x)$ is dense in X^*. Let $[(x_n)] \in X^*$ and $\epsilon > 0$.

Case (i) ($(x_n) \in \bar{I}$). In this case let "a" be the ultimately constant value of (x_n).

Then by the definition of $T, \hat{a} = [(x_n)] \in T(X)$.

Then $\hat{a} = [(x_n)]$. Thus $[(x_n)] \in T(X)$ in this case.

Case (ii) ($(x_n) \in \bar{J}$). There exists a positive integer n_0 such that $d(x_n, x_m) < \epsilon$ if $n, m \geq n_0$. Let $x_{n_0} = a$. Then since $a \in J$, $d(a, a) = 0$,

$$d^*([(x)], \hat{a}) = \lim_{n\to\infty} d(x_n, a) \leq \epsilon. \tag{5}$$

Hence $T(X)$ is dense in X^*.

(X^*, d^*) Is Complete. Let (x_n^*) be a Cauchy sequence in X^*, and $\epsilon > 0$. There exists n_0 such that $n \geq m \geq n_0$ implies $d^*(x_n^*, x_m^*) < \epsilon/3$.

There is no harm in assuming that $n_0 > \epsilon/3$. Since $T(X)$ is dense in X^*, for each positive integer n, there exists z_n in X such that $d(x_n^*, \hat{z}_n) < 1/n$.

Hence

$$\begin{aligned} d^*(\hat{z}_n, \hat{z}_m) &\leq d^*(\hat{z}_n, x_n^*) + d^*(x_n^*, x_m^*) + d^*(x_m^*, \hat{z}_m) \\ &< \frac{1}{n} + \frac{1}{m} + \frac{\epsilon}{3} \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \quad \text{if } n, m \geq n_0. \end{aligned} \tag{6}$$

Hence $(\hat{z}_n)$ is a Cauchy sequence in $T(X)$. Since T is an isodistance, (z_n) is a Cauchy sequence in X.

Moreover, $d(z_n, z_m) = d^*(\hat{z}_n, \hat{z}_m) < \epsilon$, if $n \geq m \geq n_0$.

Let z^* denote $[(z_n)]$, by the triangle inequality:

$< (1/n) + \lim_m d(z_n, z_m) < (2\epsilon/3) < \epsilon$ for $n \geq n_0$;

$d^*(x_n^*, z^*) \leq d^*(x_n^*, \hat{z}_n) + d^*(\hat{z}_n, z^*)$;

$\Rightarrow \lim_m d^*(x_n^*, z^*) = 0$ proving that (X^*, d^*) is complete.

□

Definition 5. Let (X, d) and (X_1, d_1) be d-metric spaces. (X_1, d_1) is said to be a completion of (X, d) if

(i) (X_1, d_1) is complete;

(ii) there is an isodistance $T : (X, d) \to (X_1, d_1)$ such that $T(X)$ is dense in X_1.

Note. If (X, d) is a complete metric space then its completion is (X, d) itself.

Lemma 6. *Let (X, d) be a d-metric space and let (X_1, d_1) be a completion of (X, d). Let $T : X \to X_1$ be isodistance embedding X in X_1 with $T(X)$ dense in X_1. Then a point y of X_1 is an isolated point if and only if $y = T(X)$ for some isolated point x of X_1.*

Proof. Suppose y is an isolated point of X_1. If y is not in $T(X)$, then since $T(X)$ is dense in X_1, there exists a sequence $T(x_n)$ in $T(X)$ such that $\lim_{n\to\infty} d(T(x_n), y) = 0$.

By Lemma 2, it follows that y is not an isolated point of X_1, a contradiction so that $y = T(X)$ for some $x \in X$. Hence Tx is an isolated point of X_1 and hence that of $T(X)$. Since X and $T(X)$ are isometric, x is an isolated point of X.

Conversely, suppose x is an isolated point of X. If $T(X)$ is not an isolated point of X_1, then for each positive integer k, there exists x_k in X_1 such that $0 < d_1(x_k, T(x)) \leq 1/2k$. Since $x_k \in X_1$, either $x_k \in T(X)$ or there exists y_n in $T(X)$ such that $0 < d_1((y_k), x_k) < d_1(x_k, T(x))$.

Now

$$\begin{aligned} 0 < d_1(y_k, T(x)) &\leq d_1((y_k), x_k) + d_1(x_k, T(x)) \\ &\leq \frac{1}{2k} + \frac{1}{2k} = \frac{1}{k}. \end{aligned} \tag{7}$$

Also $y_k \neq x$ since $d_1((y_k), x_k) < d_1(x_k, T(x))$.

Hence $0 < d_1(y_k, T(x)) < 1/k$ which, by Lemma 2, contradicts the fact that $T(x)$ is an isolated point of $T(X)$. □

Theorem 7. *Let (X, d) be a d-metric space, (X_1, d_1) and (X_2, d_2) completion of (X, d), and $T_i : (X, d) \to (X_i, d_i)$ $(i = 1, 2)$ isometrics such that $T_i(x)$ is dense in X_i. Then there exists an isodistance $T : (X_1, d_1) \xrightarrow{\text{on to}} (X_2, d_2)$ such that following diagram is commutative.*

Proof. Consider the following:

$$\begin{tikzcd} (X, d) \arrow[r, "T_1"] \arrow[dr, "T_2"'] & (X_1, d_1) \arrow[d, "T"] \\ & (X_2, d_2) \end{tikzcd} \tag{8}$$

Definition of T. If $x \in X_1$ and x is an isolated point of X_1, then $T_1^{-1}(x)$ is an isolated point of X; hence $T_2(T_1^{-1}(x))$ is an isolated point of X_2.

Define $T(x) = T_2(T_1^{-1}(x))$. If $x \in X_1$ and is not an isolated point, there exists a sequence (z_n) in X such that $\{T_1 z_n\}$ converges to x in (X_1, d_1).

Since T_1 is an isodistance and $\{T_1 z_n\}$ is convergent and hence a Cauchy sequence, it follows that $\{z_n\}$ is a Cauchy sequence in X. Since T_2 is an isodistance and $\{z_n\}$ is a Cauchy sequence, it follows that $\{T_2 z_n\}$ is a Cauchy sequence in (X_2, d_2). Since (X_2, d_2) is complete, there exists $z \in X_2$ such that $\lim d_2(T_2 z_n, z) = 0$. Clearly this z is independent of the choice of the sequence $\{z_n\}$ in X.

Define $T(x) = z$. Clearly $TT_1 = T_2$ and bijection.

T Is an Isodistance. If $x, y \in X$, $T(T_1(x)) = T_2(x)$ and $T(T_1(y)) = T_2(y)$.

So $d_2(T(T_1(x)), T(T_1(y))) = d_2((T_2(x)), T_2(y)) = d_2(x, y) = d_1((T_1(x)), T_1(y))$.

If $x, y \in X_1 - X$ and $x = \lim T_1 x_n$, $y = \lim T_1 y_n$ where $x_n, y_n \in X$, then

$$\begin{aligned} d_2(Tx, Ty) &= d_2(\lim T_2 x_n, \lim T_2 y_n) \\ &= \lim d_2(\lim T_2 x_n, \lim T_2 y_n) \\ &= \lim d(x_n, y_n) \\ &= d_1(\lim T_1 x_n, \lim T_1 y_n) \\ &= d_1(x, y). \end{aligned} \tag{9}$$

The arguments for the cases when $x \in X_1 - X$ and $y \in X$ or $x \in X$ and $y \in X_1 - X$ are similar. Hence T is an isodistance. Interchanging the places of X_1 and X_2, we get in a similar way an isodistance $S : X_2 \to X_1$ such that $ST_2 = T_1$.

Since $ST_2 = T_1$ and $TT_1 = T_2$, we have $TST_2 = TT_1$ and $STT_1 = ST_2 = T_1$.

Since $T(X)$ is dense in X_1 and $T_2(x)$ in X_2, we get $TS =$ identity on X_1 and ST is identity on X_2.

Hence S and T are bijections. □

3. Completion of the Metric Associated with a d-Metric

If d is a d-metric on X then d' is a metric on X if d' is defined by $d'(x, y) = d(x, y)$ when $x \neq y$ and $d'(x, y) = 0$ for all x, y in X.

Suppose $(\overline{X}, \overline{d})$ is the completion of (X, d); then $\overline{d}$ gives rise to a metric $\overline{d}'$ defined by $\overline{d}'(x, y) = \overline{d}(x, y)$ for all $x, y \in \overline{X}$ and $\overline{d}(x, x) = 0$ for all $x, y \in \overline{X}$.

Also, the metric space (X, d') has a metric space (X_0, d_0) as its completion. In this section, we prove that the metric spaces $(\overline{X}, \overline{d}')$ and (X_0, d_0) are isometric.

Definition 8. Let X, d be a d-metric space. Define ρ on $X \times X$ by

$$\rho(x, y) = \begin{cases} d(x, y), & \text{if } x \neq y \\ 0, & \text{if } x = y. \end{cases} \tag{10}$$

ρ is a metric on X and is called the metric associated with d.

Clearly $0 \le \rho(x, y) \le d(x, y)$ $\forall x, y$ and $d(x, x) \le 2\rho(x, y)$ whenever $x \neq y$. If $s \in \{\rho, d\}$, $r > 0$ and $x \in X$.

Write $\mathscr{B}_r^s(x) = \{y / s(x, y) < r\}$. Then $\mathscr{B}_r^\rho(x) = \mathscr{B}_r^d(x) \cup \{x\}$ and $\mathscr{V}_r^s(x) = \mathscr{B}_r^s(x) \cup \{x\}$.

The collection $\{\mathscr{V}_r^d(x) / x \in X, r > 0\}$ and $\mathscr{V}_r^\rho(x) = \{\mathscr{V}_r^\rho(x) / x \in X, r > 0\}$ generate the same topology on X. However, convergent sequences in X are not necessarily the same since constant sequences are convergent sequences with respect to ρ, while this holds with respect to d for x with $d(x, x) = 0$ only.

Existence of points with positive self-distance leads to unpleasantness in the extension of the concept of continuity in metric spaces as well. This is evident from the following.

Example 9. Let d be a d-metric on a set X which is not a metric. So that the set $A = \{x / d(x, x) \neq 0\}$ is nonempty. If ρ is a metric associated with d then the identity map $i : (X, \rho) \to (X, d)$ is continuous in the usual sense. But if $x \in A$, the constant sequence (x) converges in (X, ρ) while it does not converge in (X, d).

If $(X, d), (Y, \rho)$ we call $f : X \to Y$ sequentially d-continuous if $\lim d(x_n, x) = 0 \Rightarrow \lim \rho(f(x_n), f(x)) = 0$.

If $s \in \{\rho, d\}$ and $\{x_n\}$ is a sequence in X, we say that X is s-Cauchy sequence or simply s-Cauchy if $\{x_n\}$ is a Cauchy sequence in (X, s).

Proposition 10. $\lim \rho(x_n, x) = 0 \Leftrightarrow$ *either*

(i) $x_n = x$ *eventually or*

(ii) (x_n) *can be split into subsequences* (y_n) *and* (z_n) *where* $y_n = x$ *for every* n, $z_n \neq x$ *for any* n *and* $\lim d(z_n, x) = 0$.

Proof. Routine. □

Proposition 11. *If a sequence* (x_n) *in* X *is* d*-Cauchy then* (x_n) *is* ρ*-Cauchy. Conversely if* (x_n) *is* ρ*-Cauchy and is not eventually constant, then* (x_n) *is* d*-Cauchy.*

Proof. Since $0 \le \rho(x_n, x_m) \le d(x_n, x_m)$, d-Cauchy $\Rightarrow$ ρ-Cauchy.

Conversely suppose that (x_n) is ρ-Cauchy, given $\epsilon > 0$ $\exists$ $\mathscr{N}(\epsilon)$ such that $\rho(x_n, x_m) < \epsilon$ if $n \ge \mathscr{N}(\epsilon)$ and $m \ge \mathscr{N}(\epsilon)$. So if $m \ge \mathscr{N}(\epsilon)$, $n \ge \mathscr{N}(\epsilon)$, and $x_n \neq x_m$, then $d(x_m, x_n) < \epsilon$.

Since (x_n) is not eventually constant and $n \ge \mathscr{N}(\epsilon)$, there exists $m \ge \mathscr{N}(\epsilon)$ such that $x_m \neq x_n$. Then

$$\begin{aligned} d(x_n, x_n) &\le d(x_n, x_m)\, d(x_m, x_n) \\ &= 2d(x_m, x_n) \\ &= 2d(x_n, x_n) \\ &< 2\epsilon. \end{aligned} \tag{11}$$

Thus if (x_n) is not eventually constant then for all $n \ge \mathscr{N}(\epsilon)$ and $m \ge \mathscr{N}(\epsilon)$, $d(x_n, x_m) < 2\epsilon$. Hence (x_n) is d-Cauchy. □

Example 12. Let $X = (0, \infty)$ and $d(x, y) = x + y$; then

$$\rho(x, y) = \begin{cases} x + y, & \text{if } x \neq y \\ 0, & \text{if } x = y. \end{cases} \tag{12}$$

If (x_n) is any eventually nonconstant sequence in $(0, \infty)$, then (x_n) is d-Cauchy if and only if $\forall \epsilon > o$ there exists $\mathscr{N}(\epsilon)$ such that $x_n + x_m < \epsilon$ for $n \ge m \ge \mathscr{N}(\epsilon)$. This implies that $\lim x_n = 0$.

However, if $\lim x_n = 0$, then $\forall \epsilon > 0 \ \exists \ \mathcal{N}(\epsilon)$, such that $x_n < \epsilon/2$ for $m \geq \mathcal{N}(\epsilon)$, $n \geq \mathcal{N}(\epsilon)$.

Hence $x_n + x_m < \epsilon$ for $m \geq \mathcal{N}(\epsilon)$, $n \geq \mathcal{N}(\epsilon)$. However, constant sequences are not d-Cauchy but ρ-Cauchy.

Theorem 13. *Let (X, d) be a metric space, ρ the metric associated with d on X, (X^*, d^*) the completion of (X, d), and ρ^* the metric associated with d^* on X^*. Then (X^*, ρ^*) is the completion of (X, ρ). In particular if (X, d) is a complete metric space then (X, ρ) is a complete metric space. We prove that*

(i) *X is dense in (X^*, ρ^*);*

(ii) *every ρ^*-Cauchy sequence in X^* is ρ^*-convergent.*

Proof of (i). Let $x^* \in X^* - X$. Then there exists a sequence (x_n) in X such that $\lim d^*(x_n^*, x^*) = 0$ since $x_n \in X$, $x_n \neq x^*$ $\forall n$. So that $\lim \rho^*(x_n^*, x^*) = 0$.

This implies that X is dense in (X^*, ρ^*). □

Proof of (ii). Let (x_n^*) be ρ^*-Cauchy in X^*. If (x_n^*) is eventually constant, then there exist N and $x^* \in X^*$ such that $x_n^* = x^*$ for $n \geq N$.

In this case $\lim \rho^*(x_n^*, x^*) = 0$ for $n \geq N$; hence (x_n^*) is ρ^*-convergent.

Suppose (x_n^*) is not eventually constant. Then (x_n^*) is a d^*-Cauchy sequence. Since (X^*, d^*) is complete, there exists $x^* \in X^*$ such that $\lim d^*(x_n^*, x^*) = 0$. Since $0 \leq \rho^*(x_n^*, x^*) \leq d^*(x_n^*, x^*) = 0$, $\lim \rho^*(x_n^*, x^*) = 0$.

Hence (x_n^*) is ρ^*-convergent to x^*. This completes the proof of (ii). □

Disclosure

I. Ramabhadra Sarma is a retired professor from Acharya Nagarjuna University, Andhra Pradesh, India.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Common Fixed Point Theorems for Probabilistic Nearly Densifying Mappings

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The aim of this paper is to prove some coincidence and common fixed point theorems for probabilistic nearly densifying mappings in complete Menger spaces. Our results improve the results of Chamola et al. (1991), Dimri and Pant (2002), and Pant et al. (2004) and extend the results of Khan and Liu (1997) in the framework of probabilistic settings.

1. Introduction and Preliminaries

Banach contraction mapping principle is one of the most interesting and useful tools in applied mathematics. In recent years many generalizations of Banach contraction mapping principle have appeared. The notion of probabilistic metric spaces (in short PM-spaces) is a probabilistic generalization of metric spaces which are appropriate to carry out the study of those situations wherein distances are measured in the sense of distribution functions rather than nonnegative real numbers. The study of PM-spaces was initiated by Menger [1]. Since then, Schweizer and Sklar [2] enriched this concept and provided a new impetus by proving some fundamental results on this theme. The first result on fixed point theory in PM-spaces was given by Sehgal and Bharucha-Reid [3] wherein the notion of probabilistic contraction was introduced as a generalization of the classical Banach fixed point principle in terms of probabilistic settings. Some recent fixed point results can be studied in [4–7].

Kuratowski [8] introduced the notion of measure of noncompactness of a bounded subset of a metric space. Further, this study was carried on by Furi and Vignoli [9]. They introduced the notion of densifying (also called condensing) mapping in terms of Kuratowski's measure of noncompactness and obtained some fixed point theorems. Following Furi and Vignoli [9], a number of mathematicians worked on densifying mappings and proved some metrical fixed point theorem (cf. [10–14]). As a generalization of Kuratowski's measure of noncompactness, Bocsan and Constantin [15] introduced the notion of Kuratowski's measure of noncompactness in PM-spaces. Subsequently, Bocşan [16] studied the notion of probabilistic densifying mappings. Later, Hadžić [17], Tan [18], Chamola et al. [19], Dimri and Pant [20], Pant et al. [21], Pant et al. [22], and Singh and Pant [23] proved some results for such mappings. In [24], Ganguly et al. introduced the notion of probabilistic nearly densifying mappings and proved some interesting results in this setting.

The aim of this paper is to prove some coincidence and common fixed point theorems for certain classes of nearly densifying mappings in complete Menger spaces. First, we give some topological definitions and terminology defined in [8, 15–17].

Definition 1. A semigroup G is said to be left reversible if for any $r, s \in G$ there exist $a, b \in G$ such that $ra = sb$.

It is easy to see that the notion of left reversibility is equivalent to the statement that any two right ideals of G have nonempty intersection.

Definition 2. Let G be a family of self-mappings in X. A subset Y of X is called G-invariant if $gY \subseteq Y$ for all $g \in G$.

Definition 3. Let G^* be the semigroup generated by G under composition $*$. Clearly, $G^* \supseteq \{g^n : n \geq 0\}$ for any $g \in G$ and $G^*(u) = \{u\} \cup \{gu : g \in G^*\}$ for $u \in X$.

We restate the notion of probabilistic diameter for the sake of quick reference.

Definition 4. Let A be a nonempty subset of X. A function $D_A(\cdot)$ defined by

$$D_A(x) = \sup_{y<x} \left\{ \inf_{u,v \in A} F_{u,v}(y) \right\} \tag{1}$$

is called probabilistic diameter of A. A is said to be bounded if

$$\sup_{x \in R} D_A(x) = 1. \tag{2}$$

The following definition is due to Bocsan and Constantin [15].

Definition 5. For a probabilistic bounded subset A of X, $\alpha_A(x)$ defined by $\alpha_A(x) = \sup\{\varepsilon \geq 0 : \exists$ a finite cover $\mathscr{A}$ of A such that $D_S(x) \geq \varepsilon$ for all $S \in \mathscr{A}\}$ is called Kuratowski's function.

The following properties of Kuratowski's functions are proved in [8]:

(a) $\alpha_A \in \mathfrak{I}$, the set of distribution functions;

(b) $\alpha_A(x) \geq D_A(x)$;

(c) if $\phi \neq A \subset B \subset X$, then $\alpha_A(x) \geq \alpha_B(x)$;

(d) $\alpha_{A \cup B}(x) = \min\{\alpha_A(x), \alpha_B(x)\}$;

(e) let $\overline{A}$ be the closure of A in the (ε, λ)-topology on X; then

$$\alpha_{\overline{A}}(x) = \alpha_A(x); \tag{3}$$

(f) A is probabilistic precompact (totally bounded) if $\alpha_A = H$,

where H denotes the specific distribution function defined by

$$H(x) = \begin{cases} 0, & x \leq 0; \\ 1, & x > 0. \end{cases} \tag{4}$$

Definition 6. Let $(X, \mathscr{F})$ be a PM-space. A continuous mapping f of X into X is called a probabilistic densifying mapping if and only if, for every subset A of X, $\alpha_A < H$ implies $\alpha_{f(A)} > \alpha_A$.

Definition 7. A self-mapping $f : X \to X$ is probabilistic nearly densifying if $\alpha_{f(A)} > \alpha_A$, whenever $\alpha_A < H$, $A \subset H$, and A is f-invariant.

Definition 8. Suppose $\phi : [0, \infty) \to [0, \infty)$ is an upper semicontinuous function with $\phi(0) = 0$ and $\phi(t) < t$ for all $t > 0$.

2. Main Results

First, we prove some fixed point theorems for probabilistic nearly densifying mappings in Menger spaces.

Theorem 9. *Let P, Q, and R be three continuous and nearly densifying self-mappings on a complete Menger space $(X, \mathscr{F}, *)$ such that $\sup x * x = 1$ and R commutes with P and Q. If, for all $x < 1$, $u, v \in X$, the following conditions are satisfied:*

$$\begin{aligned} \phi_1(Pu, Qv) > \min \Bigg\{ & \phi_2(Ru, Rv), \phi_2(Ru, Pu), \\ & \phi_1(Rv, Qv), \frac{\phi_2(Ru, Pu)\, \phi_1(Rv, Qv)}{\phi_2(Ru, Rv)} \Bigg\} \\ & \text{for } Ru \neq Rv, \quad Pu \neq Qv; \end{aligned} \tag{5}$$

$$\begin{aligned} \phi_2(Qu, Pv) > \min \Bigg\{ & \phi_1(Ru, Rv), \phi_1(Ru, Qu), \phi_2(Rv, Pv), \\ & \frac{\phi_1(Ru, Qu)\, \phi_2(Rv, Pv)}{\phi_1(Ru, Rv)} \Bigg\}, \\ & \text{for } Ru \neq Rv, \quad Qu \neq Pv, \end{aligned} \tag{6}$$

where ϕ_1 and ϕ_2 are real valued mappings from $X \times X$ to ς, the collection of all distribution functions, with either ϕ_1 or ϕ_2 being upper semicontinuous (u.s.c.) and $\phi_1(u, u) = \phi_2(u, u) = 1$ for all $u \in X$.

Further, if, for some $u_0 \in X$, $G(u_0) = \{P^i Q^j R^k u_0 : i = 0, 1, 2, \ldots; j = 0, 1, 2, \ldots; k = 0, 1, 2, \ldots\}$ is bounded, then P and R or Q and R have a coincidence point.

Proof. For $u_0 \in X$, let $A = G(u_0)$ and $S = \{PQR\}$.

Then $A = \{u_0\} \cup P(A) \cup Q(A) \cup R(A)$.

If $\alpha_A < H$, then

$$\begin{aligned} \alpha_A &= \alpha_{\{u_0\} \cup P(A) \cup Q(A) \cup R(A)} \\ &= \min\{\alpha_{P(A)}, \alpha_{Q(A)}, \alpha_{R(A)}\} > \alpha_A, \end{aligned} \tag{7}$$

a contradiction. It implies that $\overline{A}$ is precompact.

Let $B = \bigcap_{n=0}^{\infty} (PQR)^n (\overline{A})$.

Then it is easy to see that $SB = B$ and B is nonempty compact subset of A. By the continuity of P, Q, and R, it follows that $P\overline{A} \subset \overline{A}$, $Q\overline{A} \subset \overline{A}$, and $R\overline{A} \subset \overline{A}$. Further, it is clear that $P(\overline{B}) \subset \overline{B}$, $Q(\overline{B}) \subset \overline{B}$, and $R(\overline{B}) \subset \overline{B}$.

Note that

$$\begin{gathered} R(B) = \bigcap_{n=0}^{\infty} R(PQR)^n \left(\overline{A}\right) \subset \bigcap_{n=0}^{\infty} (PQR)^n R\left(\overline{A}\right) \subset B, \\ B = PQR(B) = RPQ(B) \subset RP(B) \subset R(B), \end{gathered} \tag{8}$$

which implies $R(B) = B$ or $R^2(B) = B$.

Now, assume that ϕ_1 is upper semicontinuous. Then the function $T : B \to \mathfrak{I}$, defined by $T(u) = \phi_1(Ru, Qu)$, is u.s.c. So T assumes its maximal value at some point p in B. Clearly,

$p \in R^2(B)$, so there is a $w \in B$ such that $p = R^2(w)$. Suppose that neither P and R nor Q and R have a coincidence point. Then

$$\begin{aligned}
&T(PQ(w)) \\
&\quad = \phi_1(RPQ(w), QPQ(w)) \\
&\quad = \phi_1(PRQ(w), QPQ(w)) \text{ by } (5), \\
&\quad > \min\Big\{\phi_2\left(R^2Q(w), RPQ(w)\right), \\
&\qquad \phi_2\left(R^2Q(w), PRQ(w)\right), \phi_1(RPQ(w), QPQ(w)), \\
&\qquad \frac{\phi_2\left(R^2Q(w), PRQ(w)\right)\phi_1(RPQ(w), QPQ(w))}{\phi_2(R^2Q(w), RPQ(w))}\Big\} \\
&\quad = \phi_2\left(QR^2(w), PRQ(w)\right), \text{by } (6), \\
&\quad > \min\Big\{\phi_1\left(RR^2(w), R^2Q(w)\right), \phi_1\left(RR^2(w), QR^2(w)\right), \\
&\qquad \phi_2\left(R^2Q(w), PRQ(w)\right), \\
&\qquad \frac{\phi_1\left(RR^2(w), QR^2(w)\right)\phi_2\left(R^2Q(w), PRQ(w)\right)}{\phi_1(RR^2(w), R^2Q(w))}\Big\} \\
&\quad = \phi_1\left(RR^2(w), QR^2(w)\right) = \phi_1(Rp, Qp) = T(p),
\end{aligned} \tag{9}$$

a contradiction to the selection of p. Hence, P and R or Q and R must have a coincidence point.

The same result holds good if ϕ_2 is upper semicontinuous. This completes the proof of the theorem. □

Remark 10. The above theorem extends the results of Khan and Liu [25, Theorem 3.1 and Corollary 3.3] to PM-spaces.

Theorem 11. *Let X, P, Q, and R be as in Theorem 9. Further, let P, Q, and R satisfying (5) and (6) have a coincidence point w; then Rw is a unique common fixed point of P, Q, and R.*

Proof. We have $Pw = Qw = Rw$. By commutativity of R with P and Q, $PR(w) = RP(w) = RR(w)$ and $QR(w) = RQ(w) = RR(w)$, or $PR(w) = RR(w) = QR(w)$.

Now let $R^2w \neq Rw$; then by (5) and (6), we have

$$\begin{aligned}
&\phi_1\left(R^2w, Rw\right) \\
&\quad = \phi_1(PRw, Qw) \\
&\quad > \min\Big\{\phi_2\left(R^2w, Rw\right), \phi_2\left(R^2w, PRw\right), \\
&\qquad \phi_1(Rw, Qw), \frac{\phi_2\left(R^2w, PRw\right)\phi_1(Rw, Qw)}{\phi_2(R^2w, Rw)}\Big\} \\
&\quad = \phi_2\left(R^2w, PRw\right) = \phi_2(QRw, Pw) \\
&\quad > \min\Big\{\phi_1\left(R^2w, Rw\right), \phi_1\left(R^2w, QRw\right), \\
&\qquad \phi_2(Rw, Pw), \frac{\phi_1\left(R^2w, QRw\right)\phi_2(Rw, Pw)}{\phi_1(R^2w, Rw)}\Big\} \\
&\quad = \phi_1\left(R^2w, Rw\right),
\end{aligned} \tag{10}$$

which is a contradiction. Hence, $R^2w = Rw$. Thus, Rw is a fixed point of R. Thus, $Rw = R(Rw) = P(Rw) = Q(Rw)$. Therefore, Rw is a common fixed point of P, Q, and R.

The uniqueness of Rw as a common fixed point of P, Q, and R follows from (5) and (6). □

Theorem 12. *Let f and g be commuting, continuous, and nearly densifying self-mappings on a complete Menger space X satisfying*

$$\phi(gu, gv) > \min\{\phi(fu, fv), \phi(fu, gu), \phi(fv, gv)\} \tag{11}$$

for $fu \neq fv$, $gu \neq gv$, and $u, v \in X$, where $\phi : X \times X \to \zeta$ is u.s.c. and $\phi(u, u,) = 1$, $u \in X$. If, for some u_0 in X, $G(u_0) = \{f^i g^j u_0 : i = 0, 1, 2, \ldots; j = 0, 1, 2, \ldots\}$ is bounded, then f and g have a unique common fixed point.

Proof. Let $A = G(u_0)$. Since f and g are commuting and continuous, we have $f(\overline{A}) \subseteq \overline{A}$ and $g(\overline{A}) \subseteq \overline{A}$ and then $A = \{u_0\} \cup f(A) \cup g(A)$.

If $\alpha_A < H$, then

$$\begin{aligned}
\alpha_A &= \alpha_{\{u_0\}\cup f(A)\cup g(A)} \\
&= \min\left\{\alpha_{f(A)}, \alpha_{g(A)}\right\} > \alpha_A,
\end{aligned} \tag{12}$$

which is a contradiction. It implies that $\overline{A}$ is precompact.

Now define $B = \bigcap_{n=0}^{\infty}(fg)^n(\overline{A})$. Since $\{(fg)^n A\}$ is a decreasing sequence of nonempty compact subset of A, it follows that B is nonempty set such that $f(\overline{B}) \subset \overline{B}$, $g(\overline{B}) \subset \overline{B}$.

Suppose that $u \in B$; then $u \in (fg)^{n+1}\overline{A}$ for all n. Hence, there exists $\{x_n\} \subseteq (fg)^n\overline{A}$. Since $(fg)^n\overline{A}$ is compact and closed for all n, f and g are continuous and nearly densifying; therefore, there exists a point $p \in (fg)^n\overline{A}$ for all n so that $fg(p) = u$. Hence, $u \in f(B)$ and $u \in g(B)$. Thus, we have

$$f(B) = B = g(B). \tag{13}$$

Let us define a real valued function ψ on B by $\psi(u) = \phi(fu, gu)$. It is u.s.c. and hence attains its maximum at some point $p \in B$. Then there exists a $w \in B$ such that $p = fw$.

Suppose that there is no point u in X such that $fu = gu$; then we have by (11)

$$\begin{aligned}\psi(gw) &= \phi(fgw, ggw) = \phi(gfw, ggw) \\ &> \min\{\phi(f^2w, fgw), \phi(f^2w, gfw), \phi(fgw, ggw)\} \\ &= \min\{\phi(f^2w, fgw), \phi(fgw, ggw)\} \\ &= \phi(f^2w, fgw) = \phi(fp, gp) = \psi(p),\end{aligned} \tag{14}$$

which is a contradiction to the selection of p. Hence, there exists a $w_0 \in B$ such that $fw_0 = gw_0$ or $f^2w_0 = fgw_0 = gfw_0$.

Suppose $f^2w_0 \neq fw_0$; then we have

$$\begin{aligned}\phi(f^2w_0, fw_0) &= \phi(gfw_0, gw_0) \\ &> \min\{\phi(f^2w_0, fw_0), \phi(f^2w_0, gfw_0), \phi(fw_0, gw_0)\} \\ &= \phi(f^2w_0, fw_0),\end{aligned} \tag{15}$$

which is a contradiction. Hence, $f^2w_0 = gfw_0 = fw_0$. Therefore, fw_0 is common fixed point of f and g. Now we will prove the uniqueness of fw_0. Let w be the other fixed point of f and g; then, by (11), we have

$$\begin{aligned}\phi(w, fw_0) &= \phi(gw, fgw_0) = \phi(gw, gfw_0) \\ &> \min\{\phi(fw, f^2w_0), \phi(fw, gw), \phi(f^2w_0, gfw_0)\} \\ &= \phi(fw, f^2w_0) = \phi(w, fw_0), \text{ a contradiction.}\end{aligned} \tag{16}$$

Hence, fw_0 is unique. This completes the proof of the theorem. □

Remark 13. Theorems 9, 11, and 12 improve the result of Chamola et al. [19], Dimri and Pant [20], Ganguly et al. [24], and Pant et al. [21] under more natural conditions.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' Contribution

All authors contributed equally to this paper. The guidance of Aeshah Hassan Zakri is very important and she helped in revising the paper according to reviewers reports.

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Research Article

Fixed Point Theorems for an Elastic Nonlinear Mapping in Banach Spaces

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Let E be a smooth Banach space with a norm $\|\cdot\|$. Let $V(x,y) = \|x\|^2 + \|y\|^2 - 2\langle x, Jy\rangle$ for any $x, y \in E$, where $\langle\cdot,\cdot\rangle$ stands for the duality pair and J is the normalized duality mapping. We define a V-strongly nonexpansive mapping by $V(\cdot,\cdot)$. This nonlinear mapping is nonexpansive in a Hilbert space. However, we show that there exists a V-strongly nonexpansive mapping with fixed points which is not nonexpansive in a Banach space. In this paper, we show a weak convergence theorem and strong convergence theorems for fixed points of this elastic nonlinear mapping and give the existence theorem.

1. Introduction

Let E be a smooth Banach space with a norm $\|\cdot\|$ and let E^* be the dual space of E. We denote by $\langle\cdot,\cdot\rangle$ a duality pair on $E\times E^*$ and let J be the normalized duality mapping on E. It is well known that J is a continuous single-valued mapping in a smooth Banach space and a one-to-one mapping in a strictly convex Banach space (cf. [1]). We define a mapping $V : E\times E \to \mathbb{R}$ by $V(x,y) = \|x\|^2 + \|y\|^2 - 2\langle x, Jy\rangle$ for all $x, y \in E$, where $\mathbb{R}$ is a set of real numbers. It is obvious that $V(x,y) \geq (\|x\| - \|y\|)^2 \geq 0$. Let any $y \in E$ be fixed, and then $V(\cdot, y)$ is a convex function because of convexity of $\|\cdot\|^2$. Many nonlinear mappings which are defined by using $V(\cdot,\cdot)$ are studied (see [2–4]). We also defined a nonlinear mapping which is called a V-strongly nonexpansive mapping in [5] as follows.

Definition 1. Let C be a nonempty subset of a smooth Banach space E. A mapping $T : C \to E$ is called V-strongly nonexpansive if there exists a constant $\lambda > 0$ such that for all $x, y \in C$

$$V(Tx, Ty) \leq V(x,y) - \lambda V((I-T)x, (I-T)y), \quad (1)$$

where I is the identity mapping on E.

From this definition, it is obvious that the identity mapping I is also a V-strongly nonexpansive mapping. In a Hilbert space, it is trivial that this mapping is nonexpansive since $V(x,y) = \|x-y\|^2$ and that any firmly nonexpansive mapping is a V-strongly nonexpansive mapping with $\lambda = 1$ (see [5]). Moreover, we showed that if there exists a fixed point of a V-strongly nonexpansive mapping T, then T is strongly nonexpansive with a Bregman distance in [5]. However, in Banach spaces, as we give an example in the later section, we find that there exists a V-strongly nonexpansive mapping with fixed points which is not nonexpansive. We should point out that a guarantee of continuity of the V-strongly nonexpansive mappings has not been given in a generalized Banach space yet.

In this paper, we prove a weak convergence theorem and strong convergence theorems for finding fixed points of a V-strongly nonexpansive mapping in Banach spaces and show the existence theorem of fixed point with a dissipative property.

2. Preliminaries

In this section, at first we show the relationship between a V-strongly nonexpansive mapping and other nonlinear mappings, in a Hilbert space. Secondly, we state some properties of V-strongly nonexpansive mappings in a Banach space and give an example of a V-strongly nonexpansive mapping which is not a quasinonexpansive mapping in a Banach space although T has fixed points. We finally show some lemmas which are necessary in order to prove our theorems.

Let C be a subset of a Banach space E and let $T : C \to E$ be a mapping. Then a point p in the closure of C is said to be an asymptotically fixed point of T if C contains a sequence $\{x_n\}$ which converges weakly to p and the sequence $\{x_n - Tx_n\}$ converges strongly to 0. $\widehat{F}(T)$ denotes the set of asymptotically fixed points of T. In [6], Reich introduced a strongly nonexpansive mapping which is defined by using the Bregman distance $D(\cdot,\cdot)$.

Definition 2. Let E be a Banach space. The Bregman distance corresponding to a function $f : E \to \mathbb{R}$ is defined by

$$D(x, y) = f(x) - f(y) - f'(y)(x - y), \tag{2}$$

where f is Gâteaux differentiable and $f'(x)$ stands for the derivative of f at the point x. Let C be a nonempty subset of E. We say that the mapping $T : C \to E$ is strongly nonexpansive if $\widehat{F}(T) \neq \emptyset$ and

$$D(p, Tx) \le D(p, x) \quad \forall p \in \widehat{F}(T) \ \ x \in C, \tag{3}$$

and if it holds that $\lim_{n\to\infty} D(Tx_n, x_n) = 0$ for a bounded sequence $\{x_n\}$ such that $\lim_{n\to\infty}(D(p, x_n) - D(p, Tx_n)) = 0$ for any $p \in \widehat{F}(T)$.

Taking the function $\|\cdot\|^2$ as the convex, continuous, and Gâteaux differentiable function f, we obtain the fact that the Bregman distance $D(\cdot,\cdot)$ coincides with $V(\cdot,\cdot)$. In particular, in a Hilbert space, it is trivial that $D(x, y) = V(x, y) = \|x - y\|^2$.

Proposition 3 (see [5]). *In a Hilbert space, a V-strongly nonexpansive mapping with $\widehat{F}(T) \neq \emptyset$ is strongly nonexpansive.*

Next we recall two mappings of other nonlinear mappings (cf. [6–9]). A firmly nonexpansive mapping and an α-inverse strongly monotone mapping are defined as follows.

Definition 4. Let C be a nonempty, closed, and convex subset of a Banach space E. A mapping $T : C \to E$ is said to be firmly nonexpansive if

$$\|Tx - Ty\|^2 \le \langle x - y, j\rangle \tag{4}$$

for all $x, y \in C$ and some $j \in J(Tx - Ty)$.

It is trivial that a firmly nonexpansive mapping is nonexpansive.

Definition 5. Let H be a Hilbert space. A mapping $T : C \to H$ is said to be α-inverse strongly monotone if

$$\alpha \|Tx - Ty\|^2 \le \langle x - y, Tx - Ty\rangle \tag{5}$$

for all $x, y \in C$.

The relation among firmly nonexpansive mappings, α-inverse strongly monotone mappings and V-strongly nonexpansive mappings is shown in the following proposition.

Proposition 6 (see [5]). *In a Hilbert space, the following hold.*

(a) *A firmly nonexpansive mapping is V-strongly nonexpansive with $\lambda = 1$.*

(b) *Let A be an α-inverse strongly monotone mapping for $\alpha > 1/2$; then $S = (I - A)$ is V-strongly nonexpansive with $(2\alpha - 1)$.*

The above (b) is obvious by showing that, for all $x, y \in H$,

$$\langle Sx - Sy, x - y\rangle \le \|x - y\|^2 - \alpha \|(I - S)x - (I - S)y\|^2. \tag{6}$$

We will introduce some properties of V-strongly nonexpansive mappings in [5].

Proposition 7 (see [5]). *In a smooth Banach space E, the following hold.*

(a) *For $c \in (-1, 1]$, $T = cI$ is V-strongly nonexpansive. For $c = 1$, $T = I$ is V-strongly nonexpansive for any $\lambda > 0$. For $c \in (-1, 1)$, $T = cI$ is V-strongly nonexpansive for any $\lambda \in (0, (1 + c)/(1 - c)]$.*

(b) *If T is V-strongly nonexpansive with λ, then, for any $\alpha \in [-1, 1]$ with $\alpha \neq 0$, αT is also V-strongly nonexpansive with $\alpha^2\lambda$.*

(c) *If T is V-strongly nonexpansive with $\lambda \ge 1$, then $A = I - T$ is V-strongly nonexpansive with λ^{-1}.*

(d) *Suppose that T is V-strongly nonexpansive with λ and that $\alpha \in [-1, 1]$ satisfies $\alpha^2\lambda \ge 1$. Then $(I - \alpha T)$ is V-strongly nonexpansive with $(\alpha^2\lambda)^{-1}$. Moreover, if $T_\alpha = I - \alpha T$, then*

$$V(T_\alpha x, T_\alpha y) \le V(x, y) - \lambda^{-1} V(Tx, Ty). \tag{7}$$

Now we give an example of a V-strongly nonexpansive mapping in a Banach space.

Example 8 (see [10]). Let $1 < p, q < \infty$ such that $1/p + 1/q = 1$. Let $E = \mathbb{R} \times \mathbb{R}$ be a real Banach space with a norm $\|\cdot\|_p$ defined by

$$\|x\|_p = \{|x_1|^p + |x_2|^p\}^{1/p} \quad \forall x = (x_1, x_2) \in E. \tag{8}$$

Then E is smooth, and the normalized duality mapping J is single-valued. J is given by

$$Jx = \|x\|_p^{2-p} \left(x_1 |x_1|^{p-2}, x_2 |x_2|^{p-2}\right) \in l^q(\mathbb{R} \times \mathbb{R}) \qquad \forall x = (x_1, x_2) \in E. \tag{9}$$

Hence, we have for $x, y \in E$ that

$$\begin{aligned} V(x, y) &= \|x\|_p^2 + \|y\|_p^2 - 2\langle x, Jy\rangle \\ &= \|x\|_p^2 + \|y\|_p^2 - 2\|y\|_p^{2-p} \\ &\quad \cdot \left\{x_1 y_1 |y_1|^{p-2} + x_2 y_2 |y_2|^{p-2}\right\}. \end{aligned} \tag{10}$$

We define a mapping $T : E \to E$ as follows:

$$Tx = \begin{cases} x & \text{if } \|x\|_p \leq 1, \\ \dfrac{1}{\|x\|_p} x & \text{if } \|x\|_p > 1. \end{cases} \tag{11}$$

In a case of $p = 1$, we have shown that the mapping T defined by (11) is a V-strongly nonexpansive mapping (see [5]). We will show that T is V-strongly nonexpansive with any $\lambda \leq 1$, for $p > 1$.

Proposition 9. *Suppose that T is defined by the formula (11) under the above situation. Then, T is a V-strongly nonexpansive mapping with any $\lambda \leq 1$.*

Proof. Case (a): suppose that $x, y \in E$ with $\|x\|_p \leq 1$ and $\|y\|_p > 1$.

Since $Ty = ((Ty)_1, (Ty)_2) = (y_1\|y\|_p^{-1}, y_2\|y\|_p^{-1})$, we have that

$$\begin{aligned} V(Tx, Ty) = V(x, Ty) &= \|x\|_p^2 + \|Ty\|_p^2 - 2\|Ty\|_p^{2-p} \\ &\quad \cdot \left\{x_1 (Ty)_1 \left|(Ty)_1\right|^{p-2} + x_2 (Ty)_2 \left|(Ty)_2\right|^{p-2}\right\} \\ &= \|x\|_p^2 + 1 - 2\|y\|_p^{1-p} \\ &\quad \cdot \left\{x_1 y_1 |y_1|^{p-2} + x_2 y_2 |y_2|^{p-2}\right\}. \end{aligned} \tag{12}$$

Since

$$y - Ty = \left(\frac{\|y\|_p - 1}{\|y\|_p} y_1, \frac{\|y\|_p - 1}{\|y\|_p} y_2\right), \tag{13}$$

we have that

$$\begin{aligned} V(x - Tx, y - Ty) = V(0, y - Ty) &= \|y - Ty\|_p^2 \\ &= \left\{\frac{(\|y\|_p - 1)}{\|y\|_p} \|y\|_p\right\}^2 \\ &= \left(\|y\|_p - 1\right)^2. \end{aligned} \tag{14}$$

Hence, we obtain that

$$\begin{aligned} &V(x, y) - V(Tx, Ty) - \lambda V(x - Tx, y - Ty) \\ &\quad = \|x\|_p^2 + \|y\|_p^2 - 2\|y\|_p^{2-p}\left\{x_1 y_1 |y_1|^{p-2} + x_2 y_2 |y_2|^{p-2}\right\} \\ &\qquad - \|x\|_p^2 - 1 + 2\|y\|_p^{1-p}\left\{x_1 y_1 |y_1|^{p-2} + x_2 y_2 |y_2|^{p-2}\right\} \\ &\qquad - \lambda\left(\|y\|_p - 1\right)^2 \\ &\quad = \|y\|_p^2 - 1 - 2\|y\|_p^{1-p}\left(\|y\|_p - 1\right) \\ &\qquad \cdot \left\{x_1 y_1 |y_1|^{p-2} + x_2 y_2 |y_2|^{p-2}\right\} - \lambda\left(\|y\|_p - 1\right)^2 \\ &\quad \geq \left(\|y\|_p - 1\right)\left\{\left(\|y\|_p + 1\right) - 2\|y\|_p^{1-p}\right. \\ &\qquad \cdot \left(|x_1||y_1|^{p-1} + |x_2||y_2|^{p-1}\right) \\ &\qquad \left. - \lambda\left(\|y\|_p - 1\right)\right\}. \end{aligned} \tag{15}$$

Hölder's inequality implies that

$$\begin{aligned} |x_1||y_1|^{p-1} + |x_2||y_2|^{p-1} &\leq \|x\|_p \left\{\left(|y_1|^{p-1}\right)^q + \left(|y_2|^{p-1}\right)^q\right\}^{1/q} \\ &= \|x\|_p \left(|y_1|^p + |y_2|^p\right)^{1/q} \\ &= \|x\|_p \|y\|_p^{p-1}. \end{aligned} \tag{16}$$

Therefore, we obtain that

$$\begin{aligned} &V(x, y) - V(Tx, Ty) - \lambda V(x - Tx, y - Ty) \\ &\quad \geq \left(\|y\|_p - 1\right) \\ &\qquad \cdot \left\{\|y\|_p + 1 - 2\|y\|_p^{1-p}\|x\|_p\|y\|_p^{p-1} - \lambda\|y\|_p + \lambda\right\} \\ &\quad = \left(\|y\|_p - 1\right)\left\{\|y\|_p + 1 - 2\|x\|_p - \lambda\|y\|_p + \lambda\right\} \\ &\quad \geq \left(\|y\|_p - 1\right)\left\{(1 - \lambda)\|y\|_p + 1 - 2 + \lambda\right\} \\ &\quad = \left(\|y\|_p - 1\right)\left\{(1 - \lambda)\left(\|y\|_p - 1\right)\right\} \\ &\quad = (1 - \lambda)\left(\|y\|_p - 1\right)^2 \geq 0, \quad \text{for any } \lambda \in [0, 1]. \end{aligned} \tag{17}$$

That is, the inequality (1) holds.

Case (b): suppose that $x, y \in E$ with $\|x\|_p \geq 1$ and $\|y\|_p \leq 1$.

Then we have that

$$\begin{aligned} V(Tx, Ty) &= V(Tx, y) \\ &= 1 + \|y\|_p^2 - 2\|x\|_p^{-1}\|y\|_p^{2-p} \\ &\quad \cdot \left\{x_1 y_1 |y_1|^{p-2} + x_2 y_2 |y_2|^{p-2}\right\}, \end{aligned} \tag{18}$$

$$V(x - Tx, y - Ty) = V\left(\frac{\left(\|x\|_p - 1\right)}{\|x\|_p} x, 0\right) = \left(\|x\|_p - 1\right)^2. \tag{19}$$

Hence, we have that

$$\begin{aligned} &V(x, y) - V(Tx, Ty) - \lambda V(x - Tx, y - Ty) \\ &\quad = \|x\|_p^2 + \|y\|_p^2 - 2\|y\|_p^{2-p} \\ &\qquad \cdot \left\{x_1 y_1 |y_1|^{p-2} + x_2 y_2 |y_2|^{p-2}\right\} - 1 - \|y\|_p^2 \\ &\qquad + 2\|y\|_p^{2-p}\|x\|_p^{-1}\left\{x_1 y_1 |y_1|^{p-2} + x_2 y_2 |y_2|^{p-2}\right\} \\ &\qquad - \lambda\left(\|x\|_p - 1\right)^2 \end{aligned}$$

$$
\begin{aligned}
&\geq \|x\|_p^2 - 1 - 2\left\|y\right\|_p^{2-p} \\
&\quad \cdot \left\{\left|x_1\right|\left|y_1\right|^{p-1} + \left|x_2\right|\left|y_2\right|^{p-1}\right\}\left(1 - \|x\|_p^{-1}\right) \\
&\quad - \lambda\left(1 - \|x\|_p\right)^2 .
\end{aligned} \tag{20}
$$

As (a), we obtain from Hölder's inequality that

$$
\begin{aligned}
&V(x,y) - V(Tx,Ty) - \lambda V(x - Tx, y - Ty) \\
&\quad \geq \|x\|_p^2 - 1 - 2\|x\|_p\left\|y\right\|_p^{2-p}\left\|y\right\|_p^{p-1} \\
&\qquad \cdot \left(1 - \|x\|_p^{-1}\right) - \lambda\left(\|x\|_p - 1\right)^2 \\
&\quad = \left(\|x\|_p - 1\right)\left(\|x\|_p + 1\right) - 2\left\|y\right\|_p\left(\|x\|_p - 1\right) \\
&\qquad - \lambda\left(\|x\|_p - 1\right)^2 \\
&\quad = \left(\|x\|_p - 1\right)\left\{\|x\|_p + 1 - 2\left\|y\right\|_p - \lambda\|x\|_p + \lambda\right\} \\
&\quad \geq \left(\|x\|_p - 1\right)(1-\lambda)\left(\|x\|_p - 1\right) \\
&\quad = (1-\lambda)\left(\|x\|_p - 1\right)^2 \geq 0, \quad \text{for any } \lambda \in [0,1] .
\end{aligned} \tag{21}
$$

That is, the inequality (1) holds.

Case (c): suppose that $x, y \in E$ with $\|x\|_p, \|y\|_p \geq 1$. Then we have that

$$
\begin{aligned}
&V(Tx,Ty) \\
&\quad = 1 + 1 - 2\left\langle \|x\|_p^{-1}(x_1,x_2), \right. \\
&\qquad\qquad \left. \left\|y\right\|_p^{1-p}\left(y_1\left|y_1\right|^{p-2}, y_2\left|y_2\right|^{p-2}\right)\right\rangle \\
&\quad = 2 - 2\|x\|_p^{-1}\left\|y\right\|_p^{1-p}\left\{x_1y_1\left|y_1\right|^{p-2} + x_2y_2\left|y_2\right|^{p-2}\right\}, \\
&V(x - Tx, y - Ty) \\
&\quad = V\left(\frac{\|x\|_p - 1}{\|x\|_p}x, \frac{\left\|y\right\|_p - 1}{\left\|y\right\|_p}y\right) \\
&\quad = \left(\|x\|_p - 1\right)^2 + \left(\left\|y\right\|_p - 1\right)^2 \\
&\qquad - 2\left(\|x\|_p - 1\right)\left(\left\|y\right\|_p - 1\right)\|x\|_p^{-1}\left\|y\right\|_p^{-1}\left\|y\right\|_p^{2-p} \\
&\qquad \cdot \left\langle (x_1,x_2), \left(\left|y_1\right|^{p-2}y_1, \left|y_2\right|^{p-2}y_2\right)\right\rangle \\
&\quad = \left(\|x\|_p - 1\right)^2 + \left(\left\|y\right\|_p - 1\right)^2 \\
&\qquad - 2\left(\|x\|_p - 1\right)\left(\left\|y\right\|_p - 1\right)\|x\|_p^{-1}\left\|y\right\|_p^{1-p} \\
&\qquad \cdot \left\{x_1y_1\left|y_1\right|^{p-2} + x_2y_2\left|y_2\right|^{p-2}\right\}.
\end{aligned} \tag{22}
$$

Hence, we have that

$$
\begin{aligned}
&V(x,y) - V(Tx,Ty) - \lambda V(x - Tx, y - Ty) \\
&\quad = \|x\|_p^2 + \left\|y\right\|_p^2 - 2\left\|y\right\|_p^{2-p}\left\{x_1y_1\left|y_1\right|^{p-2} + x_2y_2\left|y_2\right|^{p-2}\right\} \\
&\qquad - 2 + 2\|x\|_p^{-1}\left\|y\right\|_p^{1-p}\left\{x_1y_1\left|y_1\right|^{p-2} + x_2y_2\left|y_2\right|^{p-2}\right\} \\
&\qquad - \lambda\left(\|x\|_p - 1\right)^2 - \lambda\left(\left\|y\right\|_p - 1\right)^2 \\
&\qquad + 2\lambda\left(\|x\|_p - 1\right)\left(\left\|y\right\|_p - 1\right)\|x\|_p^{-1}\left\|y\right\|_p^{1-p} \\
&\qquad \cdot \left\{x_1y_1\left|y_1\right|^{p-2} + x_2y_2\left|y_2\right|^{p-2}\right\} \\
&\quad = \|x\|_p^2 + \left\|y\right\|_p^2 - 2 - \lambda\left(\|x\|_p - 1\right)^2 - \lambda\left(\left\|y\right\|_p - 1\right)^2 \\
&\qquad - 2\|x\|_p^{-1}\left\|y\right\|_p^{1-p}\left\{x_1y_1\left|y_1\right|^{p-2} + x_2y_2\left|y_2\right|^{p-2}\right\} \\
&\qquad \cdot \left\{\|x\|_p\left\|y\right\|_p - 1 - \lambda\left(\|x\|_p - 1\right)\left(\left\|y\right\|_p - 1\right)\right\}.
\end{aligned} \tag{23}
$$

It is obvious that

$$
\|x\|_p\left\|y\right\|_p - 1 - \lambda\left(\|x\|_p - 1\right)\left(\left\|y\right\|_p - 1\right) \geq 0 \tag{24}
$$

for any $\lambda \in [0,1]$ and $\|x\|_p, \|y\|_p \geq 1$. Thus, we have from Hölder's inequality that

$$
\begin{aligned}
&V(x,y) - V(Tx,Ty) - \lambda V(x - Tx, y - Ty) \\
&\quad \geq \|x\|_p^2 + \left\|y\right\|_p^2 - 2 - \lambda\left(\|x\|_p - 1\right)^2 - \lambda\left(\left\|y\right\|_p - 1\right)^2 \\
&\qquad - 2\|x\|_p^{-1}\left\|y\right\|_p^{1-p}\|x\|_p\left\|y\right\|_p^{p-1} \\
&\qquad \cdot \left\{\|x\|_p\left\|y\right\|_p - 1 - \lambda\left(\|x\|_p - 1\right)\left(\left\|y\right\|_p - 1\right)\right\} \\
&\quad = \|x\|_p^2 + \left\|y\right\|_p^2 - 2 - \lambda\left(\|x\|_p - 1\right)^2 - \lambda\left(\left\|y\right\|_p - 1\right)^2 \\
&\qquad - 2\left\{\|x\|_p\left\|y\right\|_p - 1 - \lambda\left(\|x\|_p - 1\right)\left(\left\|y\right\|_p - 1\right)\right\} \\
&\quad = \|x\|_p^2 + \left\|y\right\|_p^2 - 2 - \lambda \\
&\qquad \cdot \left\{\|x\|_p^2 - 2\|x\|_p + 1 + \left\|y\right\|_p^2 - 2\left\|y\right\|_p + 1\right\} \\
&\qquad - 2\|x\|_p\left\|y\right\|_p + 2 + 2\lambda\left\{\|x\|_p\left\|y\right\|_p - \|x\|_p - \left\|y\right\|_p + 1\right\} \\
&\quad = \left(\|x\|_p - \left\|y\right\|_p\right)^2 - \lambda\left(\|x\|_p - \left\|y\right\|_p\right)^2 \\
&\quad = (1-\lambda)\left(\|x\|_p - \left\|y\right\|_p\right)^2 \geq 0, \quad \text{for any } \lambda \in [0,1] .
\end{aligned} \tag{25}
$$

That is, the inequality (1) holds.

It is clear that if $\|x\|_p, \|y\|_p \leq 1$ then inequality (1) holds. Therefore, from Cases (a), (b), and (c), we obtain the conclusion that T is V-strongly nonexpansive for any $\lambda \in (0,1]$. □

Remark 10. When $p = 1$, we have given the result in [5]. When $p = 2$, we already know that E is a Hilbert space and a V-strongly nonexpansive mapping T is nonexpansive.

Theorem 11. *There exists a V-strongly nonexpansive mapping T with a nonempty subset of fixed points such that T is not nonexpansive for some Banach space.*

Proof. It is enough to show that the V-strongly nonexpansive mapping which is given in the previous proposition is not nonexpansive.

Let $x = (0, 1) \in E$. Suppose that $y = (y_1, y_2)$ satisfies that $\|y\|_p^p = |y_1|^p + |y_2|^p > 1$ and $0 < y_1, y_2 < 1$. Then $Ty = \|y\|_p^{-1} y$. Let $h = (y_2/y_1)$ and $t = \|y\|_p^{-1} y_1 - y_1$. We have that $t < 0$ and $\|y\|_p^{-1} y_2 - y_2 = ht < 0$. Then we obtain that $Ty = (\|y\|_p^{-1} y_1, \|y\|_p^{-1} hy_1)$. Then, we have that

$$\begin{aligned} \|Tx - Ty\|_p^p &= \left\|\left(-\|y\|_p^{-1} y_1, 1 - \|y\|_p^{-1} hy_1\right)\right\|^p \\ &= \left|-\|y\|_p^{-1} y_1\right|^p + \left|1 - \|y\|_p^{-1} hy_1\right|^p \\ &= \left(\|y\|_p^{-1} y_1\right)^p + \left(1 - \|y\|_p^{-1} hy_1\right)^p \\ &= (y_1 + t)^p + (1 - h(y_1 + t))^p, \end{aligned} \tag{26}$$

and since $\|x - y\|_p^p = y_1^p + (1 - hy_1)^p$, we have that

$$\begin{aligned} &\|Tx - Ty\|_p^p - \|x - y\|_p^p \\ &\quad = (y_1 + t)^p - y_1^p + (1 - h(y_1 + t))^p - (1 - hy_1)^p. \end{aligned} \tag{27}$$

Therefore, we will show that

$$\begin{aligned} &\|Tx - Ty\|_p^p - \|x - y\|_p^p > 0 \\ &\Longleftrightarrow (y_1 + t)^p - y_1^p + (1 - h(y_1 + t))^p - (1 - hy_1)^p > 0 \\ &\Longleftrightarrow \left\{(y_1 + t)^p - y_1^p\right\} t^{-1} \\ &\qquad + \left\{(1 - h(y_1 + t))^p - (1 - hy_1)^p\right\} t^{-1} < 0, \end{aligned} \tag{28}$$

since $t < 0$. Let h be fixed. As $\|y\|_p^p = y_1^p + (hy_1)^p \to 1$, $t = \|y\|_p^{-1} y_1 - y_1 \to 0$. Thus, we have for a sufficiently small $|t|$ that

$$\begin{aligned} &\left\{(y_1 + t)^p - y_1^p\right\} t^{-1} \\ &\quad + \left\{(1 - h(y_1 + t))^p - (1 - hy_1)^p\right\} t^{-1} < 0 \\ &\Longleftrightarrow py_1^{p-1} - ph(1 - hy_1)^{p-1} < 0. \end{aligned} \tag{29}$$

It is trivial that

$$\begin{aligned} py_1^{p-1} - ph(1 - hy_1)^{p-1} < 0 &\Longleftrightarrow y_1^{p-1} < h(1 - hy_1)^{p-1} \\ &\Longleftrightarrow y_1^p < y_2(1 - y_2)^{p-1}. \end{aligned} \tag{30}$$

Let $p = 3/2$. For $y = (0.2, 0.95)$, we have that

$$y_1^p = (0.2)^{3/2} < 0.95(0.05)^{1/2} = y_2(1 - y_2)^{p-1}. \tag{31}$$

We obtain that $\|y\|_p^p = (0.2)^{3/2} + (0.95)^{3/2} > 1$ and that

$$\begin{aligned} \|Tx - Ty\|_p^p &= \|y\|_p^{-p} \left\{(0.2)^{3/2} + \left(\|y\|_p - 0.95\right)^{3/2}\right\} \\ &> (0.2)^{3/2} + (0.05)^{3/2} = \|x - y\|_p^p. \end{aligned} \tag{32}$$

Therefore, we obtain the conclusion. □

We remark that the symbols $x_n \to u$ and $x_n \rightharpoonup u$ mean that $\{x_n\}$ converges strongly and weakly to u, respectively. We will introduce the following important lemmas for proofs of our theorems.

Lemma 12. *(a) For all $x, y, z \in E$,*

$$\begin{aligned} V(x, y) &\le V(x, y) + V(y, z) \\ &= V(x, z) - 2\langle x - y, Jy - Jz\rangle. \end{aligned} \tag{33}$$

(b) Let $\{x_n\}$ be a sequence in E such that there exists $\lim_{n\to\infty} V(x_n, p) < \infty$ for some $p \in E$; then $\{x_n\}$ is bounded.

Lemma 13 (see [3]). *Let E be a smooth and uniformly convex Banach space and C a nonempty, convex, and closed subset of E. Suppose that $T : C \to E$ satisfies*

$$V(Tx, Ty) \le V(x, y) \quad \forall x, y \in C. \tag{34}$$

If a weakly convergent sequence $\{z_n\}_{n\ge1} \subset C$ satisfies that $\lim_{n\to\infty} V(Tz_n, z_n) = 0$, it holds that $z_n \rightharpoonup z \in F(T)$.

Theorem 14 (see [1, 11]). *Let Y be a compact subset of a topological vector space E and let X be a convex subset of Y. Let $A : X \to 2^Y$ be an operator such that, for each $y \in Y$, $A^{-1}y$ is convex. Suppose that $B : X \to 2^Y$ satisfies the following:*

(1) *$Bx \subset Ax$ for each $x \in X$,*

(2) *$B^{-1}y \ne \emptyset$ for each $y \in Y$,*

(3) *Bx is open for each $x \in X$.*

Then there exists a point $x_0 \in X$ such that $x_0 \in Ax_0$.

Lemma 15 (see [12]). *Let $s > 0$ and let E be a Banach space. Then E is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \to [0, \infty)$, $g(0) = 0$, such that*

$$\|x + y\|^2 \ge \|x\|^2 + 2\langle y, j\rangle + g(\|y\|) \tag{35}$$

for all $x, y \in \{z \in E : \|z\| \le s\}$ and $j \in Jx$.

Lemma 16 (see [13]). *Let E be a smooth and uniformly convex Banach space. Then, there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and, for each real number $r > 0$,*

$$0 \le g(\|x - y\|) \le V(x, y) \tag{36}$$

for all $x, y \in B_r = \{z \in E : \|z\| \le r\}$.

Lemma 17 (see [13]). *Let E be a smooth and uniformly convex Banach space and $\{y_n\}$ and $\{z_n\}$ in E. If $\lim_{n\to\infty} V(y_n, z_n) = 0$ and either $\{y_n\}$ or $\{z_n\}$ is bounded, then $\{y_n - z_n\} \to 0$.*

3. Main Results

In this section, we prove a weak convergence theorem and strong convergence theorems for finding fixed points of a V-strongly nonexpansive mapping T in Banach spaces, and then we show the existence theorem for fixed points of T with a dissipative property (cf. [10]).

Theorem 18. *Let E be a smooth and uniformly convex Banach space and C a nonempty, closed, and convex subset of E. Suppose that a mapping $T : C \to C$ is V-strongly nonexpansive with λ and that $F(T) \neq \emptyset$. One defines a Mann iterative sequence $\{x_n\}$ as follows: for any $x_1 \in C$ and $n \geq 1$,*

$$x_{n+1} = \beta_n x_n + (1-\beta_n) T x_n, \tag{37}$$

where $\{\beta_n\} \subset (0,1)$ and $\lim_{n\to\infty}\beta_n = 0$. Then $x_n \rightharpoonup p_0$ for some $p_0 \in F(T)$.

Proof. Suppose that $p \in F(T)$. Then we have from the convexity of V that

$$\begin{aligned} V(x_{n+1}, p) &= V(\beta_n x_n + (1-\beta_n) T x_n, p) \\ &\leq \beta_n V(x_n, p) + (1-\beta_n) V(T x_n, p) \\ &= \beta_n V(x_n, p) + (1-\beta_n) V(T x_n, T p). \end{aligned} \tag{38}$$

Since T is V-strongly nonexpansive with λ, we have that

$$\begin{aligned} &V(x_{n+1}, p) \\ &\quad\leq \beta_n V(x_n, p) + (1-\beta_n) \\ &\qquad \cdot \{V(x_n, p) - \lambda V((I-T)x_n, (I-T)p)\} \\ &\quad= V(x_n, p) - (1-\beta_n)\lambda V(x_n - T x_n, 0) \\ &\quad\leq V(x_n, p). \end{aligned} \tag{39}$$

Hence, we have $\lim_{n\to\infty} V(x_n, p) = \alpha < \infty$. From Lemma 12 (b), $\{x_n\}$ is bounded. Furthermore, we have that

$$(1-\beta_n)\lambda V(x_n - T x_n, 0) \leq V(x_n, p) - V(x_{n+1}, p). \tag{40}$$

Since $\lim_{n\to\infty}\beta_n = \lim_{n\to\infty}\{V(x_n, p) - V(x_{n+1}, p)\} = 0$, we obtain that

$$\lim_{n\to\infty} V(x_n - T x_n, 0) = \lim_{n\to\infty} \|x_n - T x_n\|^2 = 0. \tag{41}$$

This means that $\{x_n - T x_n\}$ converges strongly to 0. Hence, $\{T x_n\}$ is also bounded, and there exists $M > 0$ such that $\|x_n\|, \|T x_n\| \leq M - \|p\|$ for all $n \geq 1$.

On the other hand, we have from Lemma 12 (a) that

$$\begin{aligned} 0 &\leq V(x_n, T x_n) \\ &= V(x_n, p) - V(T x_n, p) - 2\langle x_n - T x_n, J T x_n - J p\rangle \\ &\leq V(x_n, p) - V(T x_n, p) + 2\|x_n - T x_n\|(\|T x_n\| + \|p\|) \\ &\leq V(x_n, p) - V(T x_n, p) + 2M\|x_n - T x_n\| \\ &= \|x_n\|^2 - \|T x_n\|^2 - 2\langle x_n - T x_n, J p\rangle + 2M\|x_n - T x_n\| \\ &= (\|x_n\| - \|T x_n\|)(\|x_n\| + \|T x_n\|) \\ &\quad - 2\langle x_n - T x_n, J p\rangle + 2M\|x_n - T x_n\| \\ &\leq \|x_n - T x_n\|(\|x_n\| + \|T x_n\| + 2M) - 2\langle x_n - T x_n, J p\rangle. \end{aligned} \tag{42}$$

Hence, we obtain that $\lim_{n\to\infty} V(x_n, T x_n) = \lim_{n\to\infty} V(T x_n, x_n) = 0$. From Lemma 13, there exists a point $p_0 \in F(T)$ such that $x_n \rightharpoonup p_0$ and $T x_n \rightharpoonup p_0$. □

The duality mapping J of a Banach space E with Gâteaux differentiable norm is said to be weakly sequentially continuous if $x_n \rightharpoonup x$ in E implies that $\{J x_n\}$ converges weak star to $J x$ in E^* (cf. [14]). This happens, for example, if E is a Hilbert space, or finite-dimensional and smooth, or l^p if $1 < p < \infty$ (cf. [15]). Next we prove a strong convergence theorem.

Theorem 19. *Let E be a reflexive, smooth, and strictly convex Banach space. Suppose that the duality mapping J of E is weakly sequentially continuous. Suppose that C is a nonempty, closed, and convex subset of E, $T : C \to C$ is V-strongly nonexpansive with λ, and $F(T) \neq \emptyset$. One defines a Mann iterative sequence $\{x_n\}$ as follows: for any $x_1 \in C$ and $n \geq 1$,*

$$x_{n+1} = \beta_n x_n + (1-\beta_n) T x_n, \tag{43}$$

where $\{\beta_n\} \subset (0,1)$ and $\lim_{n\to\infty}\beta_n = 0$. If T satisfies that

$$\langle x, J T x\rangle \leq 0 \quad \forall x \in C, \tag{44}$$

then $x_n \to p_0$ and $T x_n \to p_0$ for some $p_0 \in F(T)$.

Proof. As in the proof of Theorem 18, we obtain that $\lim_{n\to\infty} V(x_n, T x_n) = 0$ and $x_n \rightharpoonup p_0$ and $T x_n \rightharpoonup p_0$ for some $p_0 \in F(T)$. Furthermore, from Lemma 12 (a), we have that

$$\begin{aligned} 0 &\leq V(x_n, p_0) + V(p_0, T x_n) \\ &= V(x_n, T x_n) - 2\langle x_n - p_0, J p_0 - J T x_n\rangle \\ &= V(x_n, T x_n) - 2\langle x_n - p_0, J p_0\rangle \\ &\quad + 2\langle x_n, J T x_n\rangle - 2\langle p_0, J T x_n\rangle. \end{aligned} \tag{45}$$

Hence, the assumptions imply that

$$V(x_n, p_0) \longrightarrow 0, \quad V(p_0, T x_n) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{46}$$

From Lemma 17, we have the conclusion that $x_n \to p_0$ and $T x_n \to p_0$. □

Condition (44) is a definition of a linear dissipative mapping T (cf. [16]). Moreover, we give a definition of a J-dissipative mapping for nonlinear mappings in a Banach space.

Definition 20. Let J be a single-valued duality mapping on E and let C be a nonempty subset of E. Then a mapping $T : C \to E$ is called J-dissipative if it holds that

$$\langle x - y, JTx - JTy \rangle \le 0 \tag{47}$$

for all $x, y \in C$.

In a Hilbert space, such a mapping T is called dissipative. In Banach spaces, we remark that the J-dissipative mapping is not equal to the dissipative mapping (cf. [17]). Next we give a characterization of J-dissipative mappings by using $V(\cdot,\cdot)$.

Theorem 21. *Let E be a smooth Banach space, C a nonempty subset of E, and $T : C \to E$ a mapping. Then, the following are equivalent.*

(a) *T is J-dissipative.*

(b) *For all $x, y \in C$,*

$$V(x, Ty) + V(y, Tx) \le V(x, Tx) + V(y, Ty). \tag{48}$$

Proof. For any $x, y \in C$,

$$\langle x - y, JTx - JTy \rangle \le 0 \tag{49}$$

is equal to

$$\begin{aligned} &-2\langle x, JTy\rangle - 2\langle y, JTx\rangle \le -2\langle x, JTx\rangle - 2\langle y, JTy\rangle, \\ &-2\langle x, JTy\rangle - 2\langle y, JTx\rangle + \|x\|^2 + \|Ty\|^2 + \|y\|^2 + \|Tx\|^2 \\ &\quad \le -2\langle x, JTx\rangle - 2\langle y, JTy\rangle + \|x\|^2 + \|Tx\|^2 \\ &\qquad + \|y\|^2 + \|Ty\|^2. \end{aligned} \tag{50}$$

From the definition of V, this inequality is equivalent to

$$V(x, Ty) + V(y, Tx) \le V(x, Tx) + V(y, Ty). \tag{51}$$

□

Furthermore, we have the following result by this theorem.

Lemma 22. *Suppose that E is a smooth and strictly convex Banach space and that $C \subset E$ is a nonempty convex subset. Assume that a mapping $T : C \to E$ is J-dissipative. If there are fixed points of T, then $F(T)$ is singleton.*

Proof. Assume that there exist p_0 and q_0 such that $Tp_0 = p_0$ and $Tq_0 = q_0$. Since T is J-dissipative, we have by Theorem 21 that

$$\begin{aligned} 0 &\le V(p_0, Tq_0) + V(q_0, Tp_0) \\ &\le V(p_0, Tp_0) + V(q_0, Tq_0) \\ &= V(p_0, p_0) + V(q_0, q_0) = 0. \end{aligned} \tag{52}$$

Thus, we have that $V(p_0, q_0) = V(q_0, p_0) = 0$. This implies that

$$\begin{gathered} 0 \le (\|p_0\| - \|q_0\|)^2 \le V(p_0, q_0) = 0, \\ \|p_0\| = \|q_0\|. \end{gathered} \tag{53}$$

Furthermore, we have

$$\begin{aligned} V(p_0, q_0) &= \|p_0\|^2 + \|q_0\|^2 - 2\langle p_0, Jq_0\rangle \\ &= \|p_0\|^2 + \|p_0\|^2 - 2\langle p_0, Jq_0\rangle = 0, \end{aligned} \tag{54}$$

and we have $\|p_0\|^2 = \langle p_0, Jq_0\rangle$. Since E is strictly convex and J is one-to-one, we obtain that $p_0 = q_0$. □

We give a result before proving an existence theorem for fixed points.

Theorem 23 (see [10]). *Let E be a smooth and uniformly convex Banach space, and let $T : E \to E$ be a V-strongly nonexpansive mapping with λ. Then, one has that*

$$\lim_{\|x-y\|\to 0} \|Tx - Ty\| = 0, \tag{55}$$

for $\|x\|, \|y\|, \|Tx\|, \|Ty\| \le r$, where $r > 0$.

Proof. Since T is a V-strongly nonexpansive with λ, we have

$$\begin{aligned} 0 &\le V(Tx, Ty) + \lambda V(x - Tx, y - Ty) \\ &\le V(x, y) \\ &= \|x\|^2 + \|y\|^2 - 2\langle x, Jy\rangle \\ &= \|x\|^2 - \|y\|^2 - 2\langle x - y, Jy\rangle \\ &\le \|x - y\| (\|x\| + \|y\| + 2\|y\|) \\ &= \|x - y\| (\|x\| + 3\|y\|), \quad \text{for any } x, y \in E. \end{aligned} \tag{56}$$

Thus, we obtain, for x, y with $\|x\|, \|y\| \le r$,

$$\begin{gathered} V(Tx, Ty) \longrightarrow 0, \\ V(x - Tx, y - Ty) \longrightarrow 0 \quad \text{as } \|x - y\| \longrightarrow 0. \end{gathered} \tag{57}$$

From Lemma 16, we have that

$$0 \le g(\|Tx - Ty\|) \le V(Tx, Ty). \tag{58}$$

Therefore, we have from (57) that $\lim_{\|x-y\|\to 0} g(\|Tx - Ty\|) = 0$. From the definition of g, we obtain that

$$\lim_{\|x-y\|\to 0} \|Tx - Ty\| = 0. \tag{59}$$

□

Remark 24. If $x \in E$ satisfies that $\|Tx\| < r_0$ for $r_0 > 0$, the (57) implies that $\|Ty\| < r_0 + 1$ for y in the neighborhood of x.

We will prove the following existence theorem by using Theorem 14.

Theorem 25. *Let E be a reflexive, strictly convex, and smooth Banach space and C a nonempty, bounded, closed, and convex subset of E. Suppose $T : C \to C$ is a V-strongly nonexpansive and J-dissipative mapping. Then, there exists a unique fixed point of T.*

Proof. At first, we will show that there exists $y_0 \in C$ such that

$$\{x \in C : V(x, Tx) < V(y_0, Tx)\} = \emptyset. \tag{60}$$

Assume that, for all $y \in C$,

$$\{x \in C : V(x, Tx) < V(y, Tx)\} \neq \emptyset. \tag{61}$$

Let $Ax = \{y \in C : V(x, Ty) < V(y, Ty)\}$ and $Bx = \{y \in C : V(x, Tx) < V(y, Tx)\}$ for all $x \in C$. Then, from the assumption, $B^{-1}y$ is nonempty for all $y \in C$. Since T is J-dissipative, Theorem 21 implies that

$$V(x, Ty) - V(y, Ty) \leq V(x, Tx) - V(y, Tx) \tag{62}$$

for all $y \in Bx$. This means that $Bx \subset Ax$ for any $x \in C$. For any $y \in C$, let $v_j \in A^{-1}y$ with $j \in \{1, 2, \ldots, n\}$, and suppose that $v = \sum_{j=1}^{n} \alpha_j v_j$ and $\sum_{j=1}^{n} \alpha_j = 1$ with $\alpha_j > 0$. From the convexity of V, we have

$$\begin{aligned} V(v, Ty) &= V\left(\sum_{j=1}^{n} \alpha_j v_j, Ty\right) \leq \sum_{j=1}^{n} \alpha_j V(v_j, Ty) \\ &\leq \sum_{j=1}^{n} \alpha_j V(y, Ty) = V(y, Ty). \end{aligned} \tag{63}$$

Thus, we obtain that $A^{-1}y$ is convex for all $y \in C$. Since it is obvious that Bx is open for each $x \in C$, Theorem 14 implies that there exists a point $x_0 \in C$ such that $x_0 \in Ax_0$. This means that

$$V(x_0, Tx_0) < V(x_0, Tx_0). \tag{64}$$

This is a contradiction. Thus, we have for some $y_0 \in C$ that

$$\{x \in C : V(x, Tx) < V(y_0, Tx)\} = \emptyset. \tag{65}$$

This means that there exists $y_0 \in C$ such that

$$V(y_0, Tx) \leq V(x, Tx) \tag{66}$$

for all $x \in C$.

Furthermore, we will show $V(y_0, Ty_0) \leq V(x, Ty_0)$ for all $x \in C$ if y_0 satisfies (66). Let $y_t = (1-t)y_0 + tx$ for any $t \in (0, 1)$ and $x \in C$. Since C is convex, then $y_t \in C$. Thus, we obtain that

$$\begin{aligned} V(y_0, Ty_t) &\leq V(y_t, Ty_t) \\ &= V((1-t)y_0 + tx, Ty_t). \end{aligned} \tag{67}$$

From the convexity of $V(\cdot, y)$ for $y \in C$,

$$V(y_0, Ty_t) \leq (1-t)V(y_0, Ty_t) + tV(x, Ty_t) \tag{68}$$

and we have $V(y_0, Ty_t) \leq V(x, Ty_t)$. From the definition of $V(\cdot, \cdot)$, we have that

$$\begin{aligned} &|V(x, Ty_t) - V(x, Ty_0)| \\ &\quad = \left| \|Ty_t\|^2 - \|Ty_0\|^2 - 2\langle x, JTy_t - JTy_0\rangle \right| \\ &\quad \leq (\|Ty_t\| + \|Ty_0\|)\|Ty_t - Ty_0\| + 2\|x\|\|JTy_t - JTy_0\|. \end{aligned} \tag{69}$$

Therefore, we have, by Theorem 23 and the continuity of J on a smooth Banach space, that $\lim_{t\to 0+} V(x, Ty_t) = V(x, Ty_0)$ and

$$\begin{aligned} V(y_0, Ty_0) &= \lim_{t\to 0+} V(y_0, Ty_t) \\ &\leq \lim_{t\to 0+} V(x, Ty_t) = V(x, Ty_0) \end{aligned} \tag{70}$$

for all $x \in C$. Letting $x = Ty_0$, we have that

$$V(y_0, Ty_0) \leq V(Ty_0, Ty_0) = 0. \tag{71}$$

Hence, $V(y_0, Ty_0) = 0$. This implies that

$$\|y_0\|^2 + \|Ty_0\|^2 = 2\langle y_0, JTy_0\rangle \leq 2\|y_0\|\|Ty_0\|, \tag{72}$$

and then we obtain that

$$(\|y_0\| - \|Ty_0\|)^2 \leq 0. \tag{73}$$

Thus, we have $\|y_0\| = \|Ty_0\|$ and we have by (72) that $\|y_0\|^2 = \langle y_0, JTy_0\rangle$. Since J is one-to-one on a strictly convex Banach space, $JTy_0 = Jy_0$ implies that $Ty_0 = y_0$. Therefore, we have the conclusion. □

Finally, we will prove a strong convergence theorem for finding fixed points of a V-strongly nonexpansive mapping T in a Banach space, without the assumption that $F(T) \neq \emptyset$.

Theorem 26. *Let E be a smooth and uniformly convex Banach space, and let C be a nonempty, compact, and convex subset of E. Suppose that $T : C \to C$ is J-dissipative and V-strongly nonexpansive with λ. One defines a Mann iterative sequence $\{x_n\}$ as follows: for any $x_1 \in C$ and $n \geq 1$,*

$$x_{n+1} = \beta_n x_n + (1-\beta_n)Tx_n, \tag{74}$$

where $\{\beta_n\} \subset (0, 1)$ and $\lim_{n\to\infty}\beta_n = 0$. Then, there exists a unique fixed point $p_0 \in C$ such that $x_n \to p_0$ and $Tx_n \to p_0$.

Proof. From Theorem 25, we have that $F(T) \neq \emptyset$. As in the proof of Theorem 18, we obtain that $\lim_{n\to\infty} V(x_n, Tx_n) = 0$ and that there exists a point $p_0 \in F(T)$ such that $x_n \rightharpoonup p_0$ and $Tx_n \rightharpoonup p_0$. Since T is J-dissipative, Theorem 21 implies that

$$0 \leq V(x_n, Tp_0) + V(p_0, Tx_n) \leq V(x_n, Tx_n) + V(p_0, Tp_0). \tag{75}$$

From $Tp_0 = p_0$, we have for $n \geq 1$ that

$$\begin{aligned} 0 &\leq V(x_n, p_0) + V(p_0, Tx_n) \\ &\leq V(x_n, Tx_n) + V(p_0, p_0) = V(x_n, Tx_n). \end{aligned} \quad (76)$$

Since $\lim_{n\to\infty} V(x_n, Tx_n) = 0$, we have that

$$\lim_{n\to\infty} V(x_n, p_0) = \lim_{n\to\infty} V(p_0, Tx_n) = 0. \quad (77)$$

By Lemma 17, we obtain that $x_n \to p_0$ and $Tx_n \to p_0$. We have the conclusion. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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Research Article

Quasi-Triangular Spaces, Pompeiu-Hausdorff Quasi-Distances, and Periodic and Fixed Point Theorems of Banach and Nadler Types

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Let $C = \{C_\alpha\}_{\alpha\in\mathscr{A}} \in [1;\infty)^{\mathscr{A}}$, $\mathscr{A}$-index set. A quasi-triangular space $(X, \mathscr{P}_{C;\mathscr{A}})$ is a set X with family $\mathscr{P}_{C;\mathscr{A}} = \{p_\alpha : X^2 \to [0,\infty),\ \alpha \in \mathscr{A}\}$ satisfying $\forall_{\alpha\in\mathscr{A}}\ \forall_{u,v,w\in X}\ \{p_\alpha(u,w) \le C_\alpha[p_\alpha(u,v) + p_\alpha(v,w)]\}$. For any $\mathscr{P}_{C;\mathscr{A}}$, a left (right) family $\mathscr{J}_{C;\mathscr{A}}$ generated by $\mathscr{P}_{C;\mathscr{A}}$ is defined to be $\mathscr{J}_{C;\mathscr{A}} = \{J_\alpha : X^2 \to [0,\infty),\ \alpha \in \mathscr{A}\}$, where $\forall_{\alpha\in\mathscr{A}}\ \forall_{u,v,w\in X}\ \{J_\alpha(u,w) \le C_\alpha[J_\alpha(u,v) + J_\alpha(v,w)]\}$ and furthermore the property $\forall_{\alpha\in\mathscr{A}}\ \{\lim_{m\to\infty} p_\alpha(w_m,u_m) = 0\}$ ($\forall_{\alpha\in\mathscr{A}}\ \{\lim_{m\to\infty} p_\alpha(u_m,w_m) = 0\}$) holds whenever two sequences $(u_m : m \in \mathbb{N})$ and $(w_m : m \in \mathbb{N})$ in X satisfy $\forall_{\alpha\in\mathscr{A}}\ \{\lim_{m\to\infty}\sup_{n>m} J_\alpha(u_m,u_n) = 0$ and $\lim_{m\to\infty} J_\alpha(w_m,u_m) = 0\}$ ($\forall_{\alpha\in\mathscr{A}}\ \{\lim_{m\to\infty}\sup_{n>m} J_\alpha(u_n,u_m) = 0$ and $\lim_{m\to\infty} J_\alpha(u_m,w_m) = 0\}$). In $(X, \mathscr{P}_{C;\mathscr{A}})$, using the left (right) families $\mathscr{J}_{C;\mathscr{A}}$ generated by $\mathscr{P}_{C;\mathscr{A}}$ ($\mathscr{P}_{C;\mathscr{A}}$ is a special case of $\mathscr{J}_{C;\mathscr{A}}$), we construct three types of Pompeiu-Hausdorff left (right) quasi-distances on 2^X; for each type we construct of left (right) set-valued quasi-contraction $T : X \to 2^X$, and we prove the convergence, existence, and periodic point theorem for such quasi-contractions. We also construct two types of left (right) single-valued quasi-contractions $T : X \to X$ and we prove the convergence, existence, approximation, uniqueness, periodic point, and fixed point theorem for such quasi-contractions. $(X, \mathscr{P}_{C;\mathscr{A}})$ generalize ultra quasi-triangular and partiall quasi-triangular spaces (in particular, generalize metric, ultra metric, quasi-metric, ultra quasi-metric, b-metric, partial metric, partial b-metric, pseudometric, quasi-pseudometric, ultra quasi-pseudometric, partial quasi-pseudometric, topological, uniform, quasi-uniform, gauge, ultra gauge, partial gauge, quasi-gauge, ultra quasi-gauge, and partial quasi-gauge spaces).

1. Introduction

The *set-valued dynamic system* is defined as a pair (X,T), where X is a certain space and T is a set-valued map $T : X \to 2^X$; here 2^X denotes the family of all nonempty subsets of the space X. For $m \in \{0\} \cup \mathbb{N}$, we define $T^{[m]} = T \circ T \circ \cdots \circ T$ (m-times) and $T^{[0]} = I_X$ (an identity map on X). By Fix(T) and Per(T) we denote the sets of all *fixed points* and *periodic points* of T, respectively; that is, $\text{Fix}(T) = \{w \in X : w \in T(w)\}$ and $\text{Per}(T) = \{w \in X : w \in T^{[k]}(w)$ for some $k \in \mathbb{N}\}$. A *dynamic process* or a *trajectory starting at* $w^0 \in X$ or a *motion* of the system (X,T) at w^0 is a sequence $(w^m : m \in \{0\} \cup \mathbb{N})$ defined by $w^m \in T(w^{m-1})$ for $m \in \mathbb{N}$ (see, [1–4]).

Recall that *a single-valued dynamic system* is defined as a pair (X,T), where X is a certain space and T is a single-valued map $T : X \to X$; that is, $\forall_{x\in X}\ \{T(x) \in X\}$. By Fix$(T)$ and Per(T) we denote the sets of all *fixed points* and *periodic points* of T, respectively; that is, $\text{Fix}(T) = \{w \in X : w = T(w)\}$ and $\text{Per}(T) = \{w \in X : w = T^{[k]}(w)$ for some $k \in \mathbb{N}\}$. For each $w^0 \in X$, a sequence $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ is called a *Picard iteration starting at* w^0 of the system (X,T).

Let X be a (nonempty) set. A *distance* on X is a map $p : X^2 \to [0;\infty)$. The set X, together with distances on X, is called *distance spaces*.

The following distance spaces are important for several reasons.

Definition 1. Let X be a (nonempty) set, and let $p : X^2 \to [0;\infty)$.

(A) (X, p) is called metric if (i) $\forall_{u,w\in X}$ $\{p(u,w) = 0$ iff $u = w\}$, (ii) $\forall_{u,w\in X}$ $\{p(u,w) = p(w,u)\}$, and (iii) $\forall_{u,v,w\in X}$ $\{p(u,w) \le p(u,v) + p(v,w)\}$.

(B) (See [5]) (X, p) is called *ultra metric* if (i) $\forall_{u,w\in X}$ $\{p(u,w) = 0$ iff $u = w\}$, (ii) $\forall_{u,w\in X}$ $\{p(u,w) = p(w,u)\}$, and (iii) $\forall_{u,v,w\in X}$ $\{p(u,w) \le \max\{p(u,v), p(v,w)\}\}$.

(C) (See [6, 7]) (X, p) is called *b-metric* with parameter $C \in [1;\infty)$ if (i) $\forall_{u,w\in X}$ $\{p(u,w) = 0$ iff $u = w\}$, (ii) $\forall_{u,w\in X}$ $\{p(u,w) = p(w,u)\}$, and (iii) $\forall_{u,v,w\in X}$ $\{p(u,w) \le C[p(u,v) + p(v,w)]\}$.

(D) (See [8]) (X, p) is called *partial metric* if (i) $\forall_{u,w\in X}$ $\{u = w$ iff $p(u,u) = p(u,w) = p(w,w)\}$, (ii) $\forall_{u,w\in X}$ $\{p(u,u) \le p(u,w)\}$, (iii) $\forall_{u,w\in X}$ $\{p(u,w) = p(w,u)\}$, and (iv) $\forall_{u,v,w\in X}$ $\{p(u,w) \le p(u,v) + p(v,w) - p(v,v)\}$.

(E) (See [9]) (X, p) is called *partial b-metric* with parameter $C \in [1;\infty)$ if (i) $\forall_{u,w\in X}$ $\{u = w$ iff $p(u,u) = p(u,w) = p(w,w)\}$, (ii) $\forall_{u,w\in X}$ $\{p(u,u) \le p(u,w)\}$, (iii) $\forall_{u,w\in X}$ $\{p(u,w) = p(w,u)\}$, and (iv) $\forall_{u,v,w\in X}$ $\{p(u,w) \le C[p(u,v) + p(v,w)] - p(v,v)\}$.

(F) (See [10]) (X, p) is called *quasi-metric* if (i) $\forall_{u,w\in X}$ $\{p(u,w) = 0$ iff $u = w\}$ and (ii) $\forall_{u,v,w\in X}$ $\{p(u,w) \le p(u,v) + p(v,w)\}$.

(G) (X, p) is called *ultra quasi-metric* if (i) $\forall_{u,w\in X}$ $\{p(u,w) = 0$ iff $u = w\}$ and (ii) $\forall_{u,v,w\in X}$ $\{p(u,w) \le \max\{p(u,v), p(v,w)\}\}$.

(H) The distance p is called *pseudometric* (or the *gauge*) on X if (i) $\forall_{u\in X}$ $\{p(u,u) = 0\}$, (ii) $\forall_{u,w\in X}$ $\{p(u,w) = p(w,u)\}$, and (iii) $\forall_{u,v,w\in X}$ $\{p(u,w) \le p(u,v) + p(v,w)\}$.

(I) The distance p is called *quasi-pseudometric* (or the *quasi-gauge*) on X if (i) $\forall_{u\in X}$ $\{p(u,u) = 0\}$ and (ii) $\forall_{u,v,w\in X}$ $\{p(u,w) \le p(u,v) + p(v,w)\}$.

(J) (See [11]) The distance p is called *ultra quasi-pseudometric* (or the *ultra quasi-gauge*) on X if (i) $\forall_{u\in X}$ $\{p(u,u) = 0\}$ and (ii) $\forall_{u,v,w\in X}$ $\{p(u,w) \le \max\{p(u,v), p(v,w)\}\}$.

Definition 2 (see [12]). Let X be a (nonempty) set, and let $\mathscr{A}$ be an index set.

(A) Each family $\mathscr{D} = \{d_\alpha : \alpha \in \mathscr{A}\}$ of pseudometrics $d_\alpha : X^2 \to [0,\infty)$, $\alpha \in \mathscr{A}$, is called gauge on X. The gauge $\mathscr{D} = \{d_\alpha : \alpha \in \mathscr{A}\}$ on X is called separating if $\forall_{u,w\in X}$ $\{u \ne w \Rightarrow \exists_{\alpha\in\mathscr{A}} \{d_\alpha(u,w) > 0\}\}$.

(B) Let the family $\mathscr{D} = \{d_\alpha : \alpha \in \mathscr{A}\}$ be separating gauge on X. The topology $\mathscr{T}(\mathscr{D})$ having as a subbase the family $\mathscr{B}(\mathscr{D}) = \{B(u, d_\alpha, \varepsilon_\alpha) : u \in X,\ \varepsilon_\alpha > 0,\ \alpha \in \mathscr{A}\}$ of all balls $B(u, d_\alpha, \varepsilon_\alpha) = \{v \in X : d_\alpha(u,v) < \varepsilon_\alpha\}$ with $u \in X$, $\varepsilon_\alpha > 0$, and $\alpha \in \mathscr{A}$ is called topology *induced* by $\mathscr{D}$ on X; the topology $\mathscr{T}(\mathscr{D})$ is Hausdorff.

(C) A topological space $(X, \mathscr{T})$ such that there is a separating gauge $\mathscr{D}$ on X with $\mathscr{T} = \mathscr{T}(\mathscr{D})$ is called a *gauge space* and is denoted by $(X, \mathscr{D})$.

Definition 3 (see [13]). Let X be a (nonempty) set, and let $\mathscr{A}$ be an index set.

(A) Each family $\mathscr{P} = \{p_\alpha,\ \alpha \in \mathscr{A}\}$ of quasi-pseudometrics $p_\alpha : X^2 \to [0,\infty)$, $\alpha \in \mathscr{A}$, is called *quasi-gauge* on X.

(B) Let the family $\mathscr{P} = \{p_\alpha : \alpha \in \mathscr{A}\}$ be quasi-gauge on X. The topology $\mathscr{T}(\mathscr{P})$ having as a subbase of the family $\mathscr{B}(\mathscr{P}) = \{B(u, p_\alpha, \varepsilon_\alpha) : u \in X,\ \varepsilon_\alpha > 0,\ \alpha \in \mathscr{A}\}$ of all balls $B(u, p_\alpha, \varepsilon_\alpha) = \{v \in X : p_\alpha(u,v) < \varepsilon_\alpha\}$ with $u \in X$, $\varepsilon_\alpha > 0$ and $\alpha \in \mathscr{A}$ is called topology *induced* by $\mathscr{P}$ on X.

(C) A topological space $(X, \mathscr{T})$ such that there is a quasi-gauge $\mathscr{P}$ on X with $\mathscr{T} = \mathscr{T}(\mathscr{P})$ is called *quasi-gauge space* and is denoted by $(X, \mathscr{P})$.

Remark 4 (see [13, Theorems 4.2 and 2.6]). Each quasi-uniform space and each topological space is the quasi-gauge space.

There is a growing literature concerning set-valued and single-valued dynamic systems in the above defined distance spaces. These studies contain also various extensions of the Banach [14] and Nadler [15, 16] theorems. Of course, there is a huge literature on this topic. For some such spaces and theorems in these spaces, see, for example, M. M. Deza and E. Deza [17], Kirk and Shahzad [18], and references therein.

Recall that the first convergence, existence, approximation, uniqueness, and fixed point result concerning single-valued contractions in complete metric spaces were obtained by Banach in 1922 [14].

Theorem 5 (see [14]). *Let (X, d) be a complete metric space. If $T : X \to X$ and*

$$\exists_{0\le\lambda<1}\ \forall_{x,y\in X}\ \{d(T(x), T(y)) \le \lambda d(x,y)\}, \tag{1}$$

then the following are true: (i) T has a unique fixed point w in X (i.e., there exists $w \in X$ such that $w = T(w)$ and $\mathrm{Fix}(T) = \{w\}$); and (ii) for each $w^0 \in X$, the sequence $(T^{[m]}(w^0) : m \in \mathbb{N})$ converges to w.

The *Pompeiu-Hausdorff metric* H^d on the class of all nonempty closed and bounded subsets $\mathscr{CB}(X)$ of the metric space (X, d) is defined as follows:

$$H^d(U,W) = \max\left\{\sup_{u\in U} d(u,W), \sup_{w\in W} d(w,U)\right\}, \quad U, W \in \mathscr{CB}(X), \tag{2}$$

where for each $x \in X$ and $V \in \mathscr{CB}(X)$, $d(x,V) = \inf_{v\in V} d(x,v)$. Using Pompeiu-Hausdorff metric new contractions were received by Nadler in 1967 and 1969 [15, 16] as a tool to study the existence of fixed points of set-valued maps in complete metric spaces.

Theorem 6 (see [15], [16, Theorem 5]). *Let (X, d) be a complete metric space. If $T : X \to \mathscr{CB}(X)$ and*

$$\exists_{\lambda\in[0;1)}\ \forall_{x,y\in X}\ \{H^d(T(x), T(y)) \le \lambda d(x,y)\}, \tag{3}$$

then $\mathrm{Fix}(T) \ne \varnothing$ (i.e., there exists $w \in X$ such that $w \in T(w)$).

Markin [19, 20] gave a slighty defferent version of Theorem 6.

Our primary interest is to construct new very general distance spaces, deliver new contractive set-valued and single-valued dynamic systems in these distance spaces, present the new global methods for studying of these dynamic systems in these spaces, and prove new convergence, approximation, existence, uniqueness, periodic point, and fixed point theorems for such dynamic systems.

The goal of the present paper is to introduce and describe the *quasi-triangular spaces* $(X, \mathscr{P}_{C;\mathscr{A}})$ (Section 2) and more general *quasi-triangular spaces* $(X, \mathscr{P}_{C;\mathscr{A}})$ with *left* (*right*) *families* $\mathscr{J}_{C;\mathscr{A}}$ *generated by* $\mathscr{P}_{C;\mathscr{A}}$ (Sections 3–5). Moreover, we use new methods and adopt ideas of Pompeiu and Hausdorff (Section 7) (see [21] for an excellent introduction to these ideas), to establish in these spaces some versions of Banach and Nadler theorems (Sections 8 and 9). Here studied dynamic systems are *left* (*right*) $\mathscr{J}_{C;\mathscr{A}}$*-admissible* or *left* (*right*) $\mathscr{P}_{C;\mathscr{A}}$*-closed* (Section 6). Examples are provided (Sections 10–12) and concluding remarks are given (Section 13).

2. Quasi-Triangular Spaces $(X,\mathscr{P}_{C;\mathscr{A}})$

It is worth noticing that the distance spaces $(X, \mathscr{P}_{C;\mathscr{A}})$, introduced and described below, are not necessarily topological or Hausdorff or sequentially complete.

Definition 7. Let X be a (nonempty) set, let $\mathscr{A}$ be an index set, and let $C = \{C_\alpha\}_{\alpha\in\mathscr{A}} \in [1;\infty)^{\mathscr{A}}$.

(A) One says that a family $\mathscr{P}_{C;\mathscr{A}} = \{p_\alpha : X^2 \to [0,\infty),\ \alpha \in \mathscr{A}\}$ of distances is a *quasi-triangular family* on X if

$$\forall_{\alpha\in\mathscr{A}}\ \forall_{u,v,w\in X}\ \{p_\alpha(u,w) \le C_\alpha [p_\alpha(u,v) + p_\alpha(v,w)]\}. \quad (4)$$

A *quasi-triangular space* $(X, \mathscr{P}_{C;\mathscr{A}})$ is a set X together with the quasi-triangular family $\mathscr{P}_{C;\mathscr{A}}$ on X.

(B) Let $(X, \mathscr{P}_{C;\mathscr{A}})$ be the quasi-triangular space. One says that $\mathscr{P}_{C;\mathscr{A}}$ is *separating* if

$$\begin{aligned}&\forall_{u,w\in X}\ \{u \ne w \\ &\quad\Longrightarrow \exists_{\alpha\in\mathscr{A}}\ \{p_\alpha(u,w) > 0 \vee p_\alpha(w,u) > 0\}\}.\end{aligned} \quad (5)$$

(C) If $(X, \mathscr{P}_{C;\mathscr{A}})$ is an quasi-triangular space and $\forall_{\alpha\in\mathscr{A}}\ \forall_{u,w\in X}\ \{p_\alpha^{-1}(u,w) = p_\alpha(w,u)\}$, then $\forall_{\alpha\in\mathscr{A}}\ \forall_{u,v,w\in X}\ \{p_\alpha^{-1}(u,w) \le C_\alpha[p_\alpha^{-1}(u,v) + p_\alpha^{-1}(v,w)]\}$. One says that the quasi-triangular space $(X, \mathscr{P}_{C;\mathscr{A}}^{-1})$, $\mathscr{P}_{C;\mathscr{A}}^{-1} = \{p_\alpha^{-1} : X^2 \to [0,\infty),\ \alpha \in \mathscr{A}\}$, is the *conjugation* of $(X, \mathscr{P}_{C;\mathscr{A}})$.

Remark 8. In the spaces $(X, \mathscr{P}_{C;\mathscr{A}})$, in general, the distances $p_\alpha : X^2 \to [0,\infty)$, $\alpha \in \mathscr{A}$, do not vanish on the diagonal; they are asymmetric and do not satisfy triangle inequality (i.e., the properties $\forall_{\alpha\in\mathscr{A}}\ \forall_{u\in X}\ \{p_\alpha(u,u) = 0\}$ or $\forall_{\alpha\in\mathscr{A}}\ \forall_{u,w\in X}\ \{p_\alpha(u,w) = p_\alpha(w,u)\}$ or $\forall_{\alpha\in\mathscr{A}}\ \forall_{u,v,w\in X}\ \{p_\alpha(u,w) \le p_\alpha(u,v) + p_\alpha(v,w)\}$ do not necessarily hold); see Section 10.

Definition 9. Let X be a (nonempty) set, let $\mathscr{A}$ be an index set, and let $C = \{C_\alpha\}_{\alpha\in\mathscr{A}} \in [1;\infty)^{\mathscr{A}}$.

(A) One says that a family $\mathscr{L}_{C;\mathscr{A}} = \{l_\alpha : X^2 \to [0,\infty),\ \alpha \in \mathscr{A}\}$ of distances on X is a *ultra quasi-triangular family* if

$$\begin{aligned}&\forall_{\alpha\in\mathscr{A}}\ \forall_{u,v,w\in X}\ \{l_\alpha(u,w) \\ &\quad\le C_\alpha \max\{l_\alpha(u,v), l_\alpha(v,w)\}\}.\end{aligned} \quad (6)$$

An *ultra quasi-triangular space* $(X, \mathscr{L}_{C;\mathscr{A}})$ is a set X together with the ultra quasi-triangular family $\mathscr{L}_{C;\mathscr{A}}$ on X.

(B) One says that a family $\mathscr{S}_{C;\mathscr{A}} = \{s_\alpha : X^2 \to [0,\infty),\ \alpha \in \mathscr{A}\}$ of distances on X is a *partial quasi-triangular family* if

$$\begin{aligned}&\forall_{\alpha\in\mathscr{A}}\ \forall_{u,v,w\in X}\ \{s_\alpha(u,w) \le C_\alpha [s_\alpha(u,v) + s_\alpha(v,w)] \\ &\quad - s_\alpha(v,v)\}.\end{aligned} \quad (7)$$

A *partial quasi-triangular space* $(X, \mathscr{S}_{C;\mathscr{A}})$ is a set X together with the partial quasi-triangular family $\mathscr{S}_{C;\mathscr{A}}$ on X.

Remark 10. It is worth noticing that quasi-triangular spaces generalize ultra quasi-triangular and partial quasi-triangular spaces (in particular, generalize metric, ultra metric, quasi-metric, ultra quasi-metric, b-metric, partial metric, partial b-metric, pseudometric, quasi-pseudometric, ultra quasi-pseudometric, partial quasi-pseudometric, topological, uniform, quasi-uniform, gauge, ultra gauge, partial gauge, quasi-gauge, ultra quasi-gauge, and partial quasi-gauge spaces).

3. Left (Right) Families $\mathscr{J}_{C;\mathscr{A}}$ Generated by $\mathscr{P}_{C;\mathscr{A}}$ in Quasi-Triangular Spaces $(X,\mathscr{P}_{C;\mathscr{A}})$

In the metric spaces (X, d) there are several types of distances (determined by d) which generalize metrics d. First these distances were introduced by Tataru [22]. More general concepts of distances in metric spaces (X, d) which generalize d, of this sort, are given by Kada et al. [23] (w-distances), Lin and Du [24] (τ-functions), Suzuki [25] (τ-distances), and Ume [26] (u-distance). Distances in uniform spaces were given by Vályi [27]. In the appearing literature, these distances and their generalizations in other spaces provide efficient tools to study various problems of fixed point theory; see, for example, [28–30] and references therein. In this paper we also generalize these ideas.

Let $\mathscr{P}_{C;\mathscr{A}}$ be the quasi-triangular family on X. It is natural to define the notions of *left* (*right*) *families* $\mathscr{J}_{C;\mathscr{A}}$ *generated by* $\mathscr{P}_{C;\mathscr{A}}$ which provide new structures on X.

Definition 11. Let $(X, \mathscr{P}_{C;\mathscr{A}})$ be the quasi-triangular space.

(A) The family $\mathscr{J}_{C;\mathscr{A}} = \{J_\alpha : \alpha \in \mathscr{A}\}$ of distances $J_\alpha : X^2 \to [0,\infty)$, $\alpha \in \mathscr{A}$, is said to be a *left* (*right*) *family generated by* $\mathscr{P}_{C;\mathscr{A}}$ if

($\mathscr{J}$1) $\forall_{\alpha\in\mathscr{A}}\ \forall_{u,v,w\in X}\ \{J_\alpha(u,w) \le C_\alpha[J_\alpha(u,v) + J_\alpha(v,w)]\}$; and furthermore.

($\mathscr{J}2$) For any sequences $(u_m : m \in \mathbb{N})$ and $(w_m : m \in \mathbb{N})$ in X satisfying

$$\forall_{\alpha\in\mathscr{A}} \left\{\lim_{m\to\infty} \sup_{n>m} J_\alpha(u_m, u_n) = 0\right\}, \tag{8}$$

$$\left(\forall_{\alpha\in\mathscr{A}} \left\{\lim_{m\to\infty} \sup_{n>m} J_\alpha(u_n, u_m) = 0\right\}\right), \tag{9}$$

$$\forall_{\alpha\in\mathscr{A}} \left\{\lim_{m\to\infty} J_\alpha(w_m, u_m) = 0\right\}, \tag{10}$$

$$\left(\forall_{\alpha\in\mathscr{A}} \left\{\lim_{m\to\infty} J_\alpha(u_m, w_m) = 0\right\}\right), \tag{11}$$

the following holds

$$\forall_{\alpha\in\mathscr{A}} \left\{\lim_{m\to\infty} p_\alpha(w_m, u_m) = 0\right\}, \tag{12}$$

$$\left(\forall_{\alpha\in\mathscr{A}} \left\{\lim_{m\to\infty} p_\alpha(u_m, w_m) = 0\right\}\right). \tag{13}$$

(B) $\mathbb{J}^L_{(X,\mathscr{P}_{C;\mathscr{A}})}$ ($\mathbb{J}^R_{(X,\mathscr{P}_{C;\mathscr{A}})}$) is the set of all left (right) families $\mathscr{J}_{C;\mathscr{A}}$ on X generated by $\mathscr{P}_{C;\mathscr{A}}$.

Remark 12. From Definition 11 if follows that $\mathscr{P}_{C;\mathscr{A}} \in \mathbb{J}^L_{(X,\mathscr{P}_{C;\mathscr{A}})} \cap \mathbb{J}^R_{(X,\mathscr{P}_{C;\mathscr{A}})}$. Moreover, there are families $\mathscr{J}_{C;\mathscr{A}} \in \mathbb{J}^L_{(X,\mathscr{P}_{C;\mathscr{A}})}$ and $\mathscr{J}_{C;\mathscr{A}} \in \mathbb{J}^R_{(X,\mathscr{P}_{C;\mathscr{A}})}$ such that the distances J_α, $\alpha \in \mathscr{A}$, do not vanish on the diagonal, are asymmetric, and are quasi-triangular and thus are not metric, ultra metric, quasi-metric, ultra quasi-metric, b-metric, partial metric, partial b-metric, pseudometric (gauge), quasi-pseudometric (quasi-gauge), and ultra quasi-pseudometric (ultra quasi-gauge).

4. Relations between $\mathscr{J}_{C;\mathscr{A}}$ and $\mathscr{P}_{C;\mathscr{A}}$

Remark 13. The following result shows that Definition 11 is correct and that $\mathbb{J}^L_{(X,\mathscr{P}_{C;\mathscr{A}})} \setminus \{\mathscr{P}_{C;\mathscr{A}}\} \neq \varnothing$ and $\mathbb{J}^R_{(X,\mathscr{P}_{C;\mathscr{A}})} \setminus \{\mathscr{P}_{C;\mathscr{A}}\} \neq \varnothing$.

Theorem 14. *Let $(X, \mathscr{P}_{C;\mathscr{A}})$ be the quasi-triangular space. Let $E \subset X$ be a set containing at least two different points and let $\{\mu_\alpha\}_{\alpha\in\mathscr{A}} \in (0;\infty)^{\mathscr{A}}$ where*

$$\forall_{\alpha\in\mathscr{A}} \left\{\mu_\alpha \geq \frac{\delta_\alpha(E)}{2C_\alpha}\right\},$$
$$\forall_{\alpha\in\mathscr{A}} \left\{\delta_\alpha(E) = \sup\{p_\alpha(u, w) : u, w \in E\}\right\}. \tag{14}$$

If $\mathscr{J}_{C;\mathscr{A}} = \{J_\alpha : \alpha \in \mathscr{A}\}$ where, for each $\alpha \in \mathscr{A}$, the distance $J_\alpha : X^2 \to [0,\infty)$ is defined by

$$J_\alpha(u, w) = \begin{cases} p_\alpha(u, w) & \text{if } E \cap \{u, w\} = \{u, w\} \\ \mu_\alpha & \text{if } E \cap \{u, w\} \neq \{u, w\}, \end{cases} \tag{15}$$

then $\mathscr{J}_{C;\mathscr{A}}$ is left and right family generated by $\mathscr{P}_{C;\mathscr{A}}$.

Proof. Indeed, we see that condition ($\mathscr{J}1$) does not hold only if there exist some $\alpha_0 \in \mathscr{A}$ and $u_0, v_0, w_0 \in X$ such that

$$J_{\alpha_0}(u_0, w_0) > C_{\alpha_0}\left[J_{\alpha_0}(u_0, v_0) + J_{\alpha_0}(v_0, w_0)\right]. \tag{16}$$

Then (15) implies $\{u_0, v_0, w_0\} \cap E \neq \{u_0, v_0, w_0\}$ and the following Cases 1–4 hold.

Case 1. If $\{u_0, w_0\} \subset E$, then $v_0 \notin E$ and, by (16) and (15), $p_{\alpha_0}(u_0, w_0) > 2C_{\alpha_0}\mu_{\alpha_0}$. Therefore, by (14), $p_{\alpha_0}(u_0, w_0) > 2C_{\alpha_0}\mu_{\alpha_0} \geq \delta_{\alpha_0}(E)$. This is impossible.

Case 2. If $u_0 \in E$ and $w_0 \notin E$, then (16) and (15) give $\mu_{\alpha_0} > C_{\alpha_0}[p_{\alpha_0}(u_0, v_0) + \mu_{\alpha_0}] \geq C_{\alpha_0}\mu_{\alpha_0}$ whenever $v_0 \in E$ or $\mu_{\alpha_0} > C_{\alpha_0}[\mu_{\alpha_0} + \mu_{\alpha_0}] = 2C_{\alpha_0}\mu_{\alpha_0}$ whenever $v_0 \notin E$. This is impossible.

Case 3. If $u_0 \notin E$ and $w_0 \in E$, then (16) and (15) give $\mu_{\alpha_0} > C_{\alpha_0}[\mu_{\alpha_0} + p_{\alpha_0}(v_0, w_0)] \geq C_{\alpha_0}\mu_{\alpha_0}$ whenever $v_0 \in E$ or $\mu_{\alpha_0} > C_{\alpha_0}[\mu_{\alpha_0} + \mu_{\alpha_0}] = 2C_{\alpha_0}\mu_{\alpha_0}$ whenever $v_0 \notin E$. This is impossible.

Case 4. If $u_0 \notin E$ and $w_0 \notin E$, then (16) and (15) give $\mu_{\alpha_0} > C_{\alpha_0}[\mu_{\alpha_0} + \mu_{\alpha_0}] = 2C_{\alpha_0}\mu_{\alpha_0}$ for $v_0 \in X$. This is impossible.

Therefore, $\forall_{\alpha\in\mathscr{A}}\ \forall_{u,v,w\in X}\ \{J_\alpha(u, w) \leq C_\alpha[J_\alpha(u, v) + J_\alpha(v, w)]\}$; that is, the condition ($\mathscr{J}1$) holds.

Assume now that the sequences $(u_m : m \in \mathbb{N})$ and $(w_m : m \in \mathbb{N})$ in X satisfy (8) and (10). Then (12) holds. Indeed, (10) implies

$$\forall_{\alpha\in\mathscr{A}}\ \forall_{0<\varepsilon<\mu_\alpha}\ \exists_{m_0=m_0(\alpha)\in\mathbb{N}}\ \forall_{m\geq m_0}\ \left\{J_\alpha(w_m, u_m) < \varepsilon\right\}. \tag{17}$$

Denoting $m' = \min\{m_0(\alpha) : \alpha \in \mathscr{A}\}$, we see, by (17) and (15), that $\forall_{m\geq m'}\ \{E \cap \{w_m, u_m\} = \{w_m, u_m\}\}$. Then, in view of Definition 11(A), (15), and (17), this implies $\forall_{\alpha\in\mathscr{A}}\ \forall_{0<\varepsilon<\mu_\alpha}\ \exists_{m'\in\mathbb{N}}\ \forall_{m\geq m'}\ \{p_\alpha(w_m, u_m) = J_\alpha(w_m, u_m) < \varepsilon\}$. Hence we obtain that the sequences $(u_m : m \in \mathbb{N})$ and $(w_m : m \in \mathbb{N})$ satisfy (12). Thus we see that $\mathscr{J}_{C;\mathscr{A}}$ is left family generated by $\mathscr{P}_{C;\mathscr{A}}$.

In a similar way, we show that (13) holds if $(u_m : m \in \mathbb{N})$ and $(w_m : m \in \mathbb{N})$ in X satisfy (9) and (11). Therefore, $\mathscr{J}_{C;\mathscr{A}}$ is right family generated by $\mathscr{P}_{C;\mathscr{A}}$. We proved that $\mathscr{J}_{C;\mathscr{A}} \in \mathbb{J}^L_{(X,\mathscr{P}_{C;\mathscr{A}})} \cap \mathbb{J}^R_{(X,\mathscr{P}_{C;\mathscr{A}})}$ holds. □

The following is interesting in respect to its use.

Theorem 15. *Let $(X, \mathscr{P}_{C;\mathscr{A}})$ be the quasi-triangular space, and let $\mathscr{J}_{C;\mathscr{A}}$ be the left (right) family generated by $\mathscr{P}_{C;\mathscr{A}}$. If $\mathscr{P}_{C;\mathscr{A}}$ is separating on X (i.e., (5) holds), then $\mathscr{J}_{C;\mathscr{A}}$ is separating on X; that is,*

$$\forall_{u,w\in X}\ \{u \neq w \Longrightarrow \exists_{\alpha\in\mathscr{A}}\ \{J_\alpha(u, w) > 0 \vee J_\alpha(w, u) > 0\}\} \tag{18}$$

holds.

Proof. We begin by supposing that $u_0, w_0 \in X$, $u_0 \neq w_0$, and $\forall_{\alpha\in\mathscr{A}}\ \{J_\alpha(u_0, w_0) = 0 \wedge J_\alpha(w_0, u_0) = 0\}$. Then ($\mathscr{J}1$) implies $\forall_{\alpha\in\mathscr{A}}\ \{J_\alpha(u_0, u_0) \leq C_\alpha[J_\alpha(u_0, w_0) + J_\alpha(w_0, u_0)] = 0\}$ or, equivalently, $\forall_{\alpha\in\mathscr{A}}\ \{J_\alpha(u_0, u_0) = J_\alpha(w_0, u_0) = 0\}$ and $\forall_{\alpha\in\mathscr{A}}\ \{J_\alpha(u_0, u_0) = J_\alpha(u_0, w_0) = 0\}$. Assuming that $u_m = u_0$ and $w_m = w_0$, $m \in \mathbb{N}$, we conclude that $\forall_{\alpha\in\mathscr{A}}\ \{\lim_{m\to\infty}\sup_{n>m} J_\alpha(u_m, u_n) = \lim_{m\to\infty} J_\alpha(w_m, u_m) = 0\}$ and $\forall_{\alpha\in\mathscr{A}}\ \{\lim_{m\to\infty}\sup_{n>m} J_\alpha(u_n, u_m) = \lim_{m\to\infty} J_\alpha(u_m, w_m) = 0\}$. Therefore, it is not hard to see that (8)–(11) hold and, by ($\mathscr{J}2$), the above considerations lead to the following conclusion: $u_0 \neq w_0 \wedge \forall_{\alpha\in\mathscr{A}} \{\lim_{m\to\infty} p_\alpha(w_m, u_m) = \lim_{m\to\infty} p_\alpha(u_m, w_m) = 0\}$ or, equivalently, $u_0 \neq w_0 \wedge \forall_{\alpha\in\mathscr{A}} \{p_\alpha(w_0, u_0) = p_\alpha(u_0, w_0) = 0\}$. However, $\mathscr{P}_{C;\mathscr{A}}$ is separating. A contradiction. Therefore, $\mathscr{J}_{C;\mathscr{A}}$ is separating. □

5. Left (Right) $\mathscr{J}_{C;\mathscr{A}}$-Convergences and Left (Right) $\mathscr{J}_{C;\mathscr{A}}$-Sequentially Completeness

Definition 16. Let $(X, \mathscr{P}_{C;\mathscr{A}})$ be the quasi-triangular space, and let $\mathscr{J}_{C;\mathscr{A}}$ be the left (right) family generated by $\mathscr{P}_{C;\mathscr{A}}$.

(A) One says that a sequence $(u_m : m \in \mathbb{N}) \subset X$ is *left (right) $\mathscr{J}_{C;\mathscr{A}}$-Cauchy sequence* if $\forall_{\alpha\in\mathscr{A}} \{\lim_{m\to\infty} \sup_{n>m} J_\alpha(u_m, u_n) = 0\}$ $(\forall_{\alpha\in\mathscr{A}} \{\lim_{m\to\infty} \sup_{n>m} J_\alpha(u_n, u_m) = 0\})$.

(B) Let $u \in X$ and let $(u_m : m \in \mathbb{N}) \subset X$. One says that the sequence $(u_m : m \in \mathbb{N})$ is *left (right) $\mathscr{J}_{C;\mathscr{A}}$-convergent to* u if $u \in \mathrm{LIM}^{L-\mathscr{J}_{C;\mathscr{A}}}_{(u_m:m\in\mathbb{N})} \neq \varnothing$ $(u \in \mathrm{LIM}^{R-\mathscr{J}_{C;\mathscr{A}}}_{(u_m:m\in\mathbb{N})} \neq \varnothing)$ where

$$\begin{aligned} &\mathrm{LIM}^{L-\mathscr{J}_{C;\mathscr{A}}}_{(u_m:m\in\mathbb{N})} = \Big\{x \\ &\quad \in X : \forall_{\alpha\in\mathscr{A}} \Big\{\lim_{m\to\infty} J_\alpha(x, u_m) = 0\Big\}\Big\}, \\ &\Big(\mathrm{LIM}^{R-\mathscr{J}_{C;\mathscr{A}}}_{(u_m:m\in\mathbb{N})} \\ &\quad = \Big\{x \in X : \forall_{\alpha\in\mathscr{A}} \Big\{\lim_{m\to\infty} J_\alpha(u_m, x) = 0\Big\}\Big\}\Big). \end{aligned} \tag{19}$$

(C) One says that a sequence $(u_m : m \in \mathbb{N}) \subset X$ is *left (right) $\mathscr{J}_{C;\mathscr{A}}$-convergent in* X if $\mathrm{LIM}^{L-\mathscr{J}_{C;\mathscr{A}}}_{(u_m:m\in\mathbb{N})} \neq \varnothing$ $(\mathrm{LIM}^{R-\mathscr{J}_{C;\mathscr{A}}}_{(u_m:m\in\mathbb{N})} \neq \varnothing)$.

(D) If every left (right) $\mathscr{J}_{C;\mathscr{A}}$-Cauchy sequence $(u_m : m \in \mathbb{N}) \subset X$ is left (right) $\mathscr{J}_{C;\mathscr{A}}$-convergent in X (i.e., $\mathrm{LIM}^{L-\mathscr{J}_{C;\mathscr{A}}}_{(u_m:m\in\mathbb{N})} \neq \varnothing$ $(\mathrm{LIM}^{R-\mathscr{J}_{C;\mathscr{A}}}_{(u_m:m\in\mathbb{N})} \neq \varnothing)$), then $(X, \mathscr{P}_{C;\mathscr{A}})$ is called *left (right) $\mathscr{J}_{C;\mathscr{A}}$-sequentially complete*.

Remark 17. The structures on X determined by left (right) families $\mathscr{J}_{C;\mathscr{A}}$ generated by $\mathscr{P}_{C;\mathscr{A}}$ are more general than the structure on X determined by $\mathscr{P}_{C;\mathscr{A}}$; see Remark 34.

Remark 18. Let $(X, \mathscr{P}_{C;\mathscr{A}})$ be the quasi-triangular space. It is clear that if $(u_m : m \in \mathbb{N})$ is left (right) $\mathscr{P}_{C;\mathscr{A}}$-convergent in X, then $\mathrm{LIM}^{L-\mathscr{P}_{C;\mathscr{A}}}_{(u_m:m\in\mathbb{N})} \subset \mathrm{LIM}^{L-\mathscr{P}_{C;\mathscr{A}}}_{(v_m:m\in\mathbb{N})}$ $(\mathrm{LIM}^{R-\mathscr{P}_{C;\mathscr{A}}}_{(u_m:m\in\mathbb{N})} \subset \mathrm{LIM}^{R-\mathscr{P}_{C;\mathscr{A}}}_{(v_m:m\in\mathbb{N})})$ for each subsequence $(v_m : m \in \mathbb{N})$ of $(u_m : m \in \mathbb{N})$.

Definition 19. One says that $(X, \mathscr{P}_{C;\mathscr{A}})$ is *left (right) Hausdorff* if for each left (right) $\mathscr{P}_{C;\mathscr{A}}$-convergent in X sequence $(u_m : m \in \mathbb{N})$ the set $\mathrm{LIM}^{L-\mathscr{P}_{C;\mathscr{A}}}_{(u_m:m\in\mathbb{N})}$ $(\mathrm{LIM}^{R-\mathscr{P}_{C;\mathscr{A}}}_{(u_m:m\in\mathbb{N})})$ is a singleton.

6. Left (Right) $\mathscr{J}_{C;\mathscr{A}}$-Admissible and Left (Right) $\mathscr{P}_{C;\mathscr{A}}$-Closed Set-Valued Maps

The following terminologies will be much used in the sequel.

Definition 20. Let $(X, \mathscr{P}_{C;\mathscr{A}})$ be the quasi-triangular space, and let $\mathscr{J}_{C;\mathscr{A}}$ be the left (right) family generated by $\mathscr{P}_{C;\mathscr{A}}$. Let (X, T) be the set-valued dynamic system, $T : X \to 2^X$.

(A) Given $w^0 \in X$, One says that (X, T) is *left (right) $\mathscr{J}_{C;\mathscr{A}}$-admissible in* w^0 if, for each dynamic processes $(w^m : m \in \{0\} \cup \mathbb{N})$ starting at w^0, $\forall_{m\in\{0\}\cup\mathbb{N}} \{w^{m+1} \in T(w^m)\}$, $\mathrm{LIM}^{L-\mathscr{J}_{C;\mathscr{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})} \neq \varnothing$ $(\mathrm{LIM}^{R-\mathscr{J}_{C;\mathscr{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})} \neq \varnothing)$ whenever

$$\begin{aligned} &\forall_{\alpha\in\mathscr{A}} \Big\{\lim_{m\to\infty} \sup_{n>m} J_\alpha(w^m, w^n) = 0\Big\} \\ &\Big(\forall_{\alpha\in\mathscr{A}} \Big\{\lim_{m\to\infty} \sup_{n>m} J_\alpha(w^n, w^m) = 0\Big\}\Big). \end{aligned} \tag{20}$$

(B) One says that (X, T) is *left (right) $\mathscr{J}_{C;\mathscr{A}}$-admissible on* X, if (X, T) is *left (right) $\mathscr{J}_{C;\mathscr{A}}$-admissible* in each point $w^0 \in X$.

Remark 21. Let $(X, \mathscr{P}_{C;\mathscr{A}})$ be the quasi-triangular space and let $\mathscr{J}_{C;\mathscr{A}}$ be the left (right) family generated by $\mathscr{P}_{C;\mathscr{A}}$. Let (X, T) be the set-valued dynamic system on X. If $(X, \mathscr{P}_{C;\mathscr{A}})$ is left (right) $\mathscr{J}_{C;\mathscr{A}}$-sequentially complete, then (X, T) is left (right) $\mathscr{J}_{C;\mathscr{A}}$-admissible on X but the converse not necessarily holds.

We can define also the following generalization of continuity.

Definition 22. Let $(X, \mathscr{P}_{C;\mathscr{A}})$ be the quasi-triangular space. Let (X, T) be the set-valued dynamic system, $T : X \to 2^X$, and let $k \in \mathbb{N}$. The set-valued dynamic system $(X, T^{[k]})$ is said to be a *left (right) $\mathscr{P}_{C;\mathscr{A}}$-closed* on X if for every sequence $(x_m : m \in \mathbb{N})$ in $T^{[k]}(X)$, left (right) $\mathscr{P}_{C;\mathscr{A}}$-converging in X (thus $\mathrm{LIM}^{L-\mathscr{P}_{C;\mathscr{A}}}_{(x_m:m\in\mathbb{N})} \neq \varnothing$ $(\mathrm{LIM}^{R-\mathscr{P}_{C;\mathscr{A}}}_{(x_m:m\in\mathbb{N})} \neq \varnothing)$) and having subsequences $(v_m : m \in \mathbb{N})$ and $(u_m : m \in \mathbb{N})$ satisfying $\forall_{m\in\mathbb{N}} \{v_m \in T^{[k]}(u_m)\}$, the following property holds: there exists $x \in \mathrm{LIM}^{L-\mathscr{P}_{C;\mathscr{A}}}_{(x_m:m\in\mathbb{N})}$ $(x \in \mathrm{LIM}^{R-\mathscr{P}_{C;\mathscr{A}}}_{(x_m:m\in\mathbb{N})})$ such that $x \in T^{[k]}(x)$ $(x \in T^{[k]}(x))$.

7. Left (Right) Pompeiu-Hausdorff Quasi-Distances and Left (Right) Set-Valued Quasi-Contractions

In this section, in the quasi-triangular spaces $(X, \mathscr{P}_{C;\mathscr{A}})$, using left (right) families $\mathscr{J}_{C;\mathscr{A}}$ generated by $\mathscr{P}_{C;\mathscr{A}}$, we define three types of left (right) Pompeiu-Hausdorff quasi-distances on 2^X, and for each type a left (right) set-valued quasi-contraction $T : X \to 2^X$ is constructed.

Definition 23. Let $(X, \mathscr{P}_{C;\mathscr{A}})$ be the quasi-triangular space, and let $\mathscr{J}_{C;\mathscr{A}}$ be the left (right) family generated by $\mathscr{P}_{C;\mathscr{A}}$. Let $\lambda = \{\lambda_\alpha\}_{\alpha\in\mathscr{A}} \in [0; 1)^{\mathscr{A}}$, let (X, T) be a set-valued dynamic system, $T : X \to 2^X$, and let $\eta \in \{1, 2, 3\}$. Let

$$\begin{aligned} &\forall_{\alpha\in\mathscr{A}}\ \forall_{x\in X}\ \forall_{V\in 2^X}\ \{J_\alpha(x, V) = \inf\{J_\alpha(x, v) : v \in V\} \\ &\quad \wedge J_\alpha(V, x) = \inf\{J_\alpha(v, x) : v \in V\}\}. \end{aligned} \tag{21}$$

(A) Let $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})}$. If

$$\begin{aligned}
&\forall_{\alpha\in\mathcal{A}}\ \forall_{U,W\in 2^X}\ \Big\{D^{L-\mathcal{J}_{C;\mathcal{A}}}_{1;2^X;\alpha}(U,W) \\
&\quad = \max\Big\{\sup_{u\in U} J_\alpha(u,W), \sup_{w\in W} J_\alpha(U,w)\Big\}\Big\}, \\
&\forall_{\alpha\in\mathcal{A}}\ \forall_{U,W\in 2^X}\ \Big\{D^{L-\mathcal{J}_{C;\mathcal{A}}}_{2;2^X;\alpha}(U,W) \\
&\quad = \max\Big\{\sup_{u\in U} J_\alpha(u,W), \sup_{w\in W} J_\alpha(w,U)\Big\}\Big\}, \\
&\forall_{\alpha\in\mathcal{A}}\ \forall_{U,W\in 2^X}\ \Big\{D^{L-\mathcal{J}_{C;\mathcal{A}}}_{3;2^X;\alpha}(U,W) = \sup_{u\in U} J_\alpha(u,W)\Big\},
\end{aligned} \tag{22}$$

then a family $\mathcal{D}^{L-\mathcal{J}_{C;\mathcal{A}}}_{\eta;2^X} = \{D^{L-\mathcal{J}_{C;\mathcal{A}}}_{\eta;2^X;\alpha}, \alpha \in \mathcal{A}\}$ is said to be *left $\mathcal{D}^{L-\mathcal{J}_{C;\mathcal{A}}}_{\eta;2^X}$-quasi-distance on 2^X*.

If

$$\begin{aligned}
&\forall_{\alpha\in\mathcal{A}}\ \forall_{x,y\in X}\ \Big\{C_\alpha \cdot D^{L-\mathcal{J}_{C;\mathcal{A}}}_{\eta;2^X;\alpha}(T(x),T(y)) \\
&\quad \le \lambda_\alpha J_\alpha(x,y)\Big\},
\end{aligned} \tag{23}$$

then we say that (X,T) is a *left $(\mathcal{D}^{L-\mathcal{J}_{C;\mathcal{A}}}_{\eta;2^X}, \lambda)$-quasi-contraction on X*.

(B) Let $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^R_{(X,\mathcal{P}_{C;\mathcal{A}})}$. If

$$\begin{aligned}
&\forall_{\alpha\in\mathcal{A}}\ \forall_{U,W\in 2^X}\ \Big\{D^{R-\mathcal{J}_{C;\mathcal{A}}}_{1;2^X;\alpha}(U,W) \\
&\quad = \max\Big\{\sup_{u\in U} J_\alpha(u,W), \sup_{w\in W} J_\alpha(U,w)\Big\}\Big\}, \\
&\forall_{\alpha\in\mathcal{A}}\ \forall_{U,W\in 2^X}\ \Big\{D^{R-\mathcal{J}_{C;\mathcal{A}}}_{2;2^X;\alpha}(U,W) \\
&\quad = \max\Big\{\sup_{u\in U} J_\alpha(u,W), \sup_{w\in W} J_\alpha(w,U)\Big\}\Big\}, \\
&\forall_{\alpha\in\mathcal{A}}\ \forall_{U,W\in 2^X}\ \Big\{D^{R-\mathcal{J}_{C;\mathcal{A}}}_{3;2^X;\alpha}(U,W) = \sup_{u\in U} J_\alpha(u,W)\Big\},
\end{aligned} \tag{24}$$

then a family $\mathcal{D}^{R-\mathcal{J}_{C;\mathcal{A}}}_{\eta;2^X} = \{D^{R-\mathcal{J}_{C;\mathcal{A}}}_{\eta;2^X;\alpha},\ \alpha \in \mathcal{A}\}$ is said to be *right $\mathcal{D}^{R-\mathcal{J}_{C;\mathcal{A}}}_{\eta;2^X}$-quasi-distance on 2^X*.

If

$$\begin{aligned}
&\forall_{\alpha\in\mathcal{A}}\ \forall_{x,y\in X}\ \Big\{C_\alpha \cdot D^{R-\mathcal{J}_{C;\mathcal{A}}}_{\eta;2^X;\alpha}(T(x),T(y)) \\
&\quad \le \lambda_\alpha J_\alpha(x,y)\Big\},
\end{aligned} \tag{25}$$

then we say that (X,T) is a *right $(\mathcal{D}^{R-\mathcal{J}_{C;\mathcal{A}}}_{\eta;2^X}, \lambda)$-quasi-contraction on X*.

Remark 24. Observe that $\mathcal{D}^{L-\mathcal{J}_{C;\mathcal{A}}}_{\eta;2^X}$ and $\mathcal{D}^{R-\mathcal{J}_{C;\mathcal{A}}}_{\eta;2^X}$ extend (2). Quasi-contractions (23) and (25) extend (3).

Remark 25. Each $(\mathcal{D}^{L-\mathcal{J}_{C;\mathcal{A}}}_{\eta;2^X}, \lambda)$-quasi-contraction ($(\mathcal{D}^{R-\mathcal{J}_{C;\mathcal{A}}}_{\eta;2^X}, \lambda)$-quasi-contraction), $\eta \in \{1,2\}$, is $(\mathcal{D}^{L-\mathcal{J}_{C;\mathcal{A}}}_{3;2^X}, \lambda)$-quasi-contraction ($(\mathcal{D}^{R-\mathcal{J}_{C;\mathcal{A}}}_{3;2^X}, \lambda)$-quasi-contraction) but the converse does not necessarily hold.

8. Convergence, Existence, Approximation, and Periodic Point Theorem of Nadler Type for Left (Right) Set-Valued Quasi-Contractions

The following result extends Theorem 6 to spaces $(X, \mathcal{P}_{C;\mathcal{A}})$.

Theorem 26. *Let $(X, \mathcal{P}_{C;\mathcal{A}})$ be the quasi-triangular space, and let (X,T) be the set-valued dynamic system, $T : X \to 2^X$. Let $\eta \in \{1,2,3\}$, and let $\lambda = \{\lambda_\alpha\}_{\alpha\in\mathcal{A}} \in [0;1)^{\mathcal{A}}$.*

Assume that there exist a left (right) family $\mathcal{J}_{C;\mathcal{A}}$ generated by $\mathcal{P}_{C;\mathcal{A}}$ and a point $w^0 \in X$ with the following properties.

(A1) (X,T) is left $(\mathcal{D}^{L-\mathcal{J}_{C;\mathcal{A}}}_{\eta;2^X}, \lambda)$-quasi-contraction (right $(\mathcal{D}^{R-\mathcal{J}_{C;\mathcal{A}}}_{\eta;2^X}, \lambda)$-quasi-contraction) on X.

(A2) (X,T) is left (right) $\mathcal{J}_{C;\mathcal{A}}$-admissible in w^0.

(A3) For every $x \in X$ and for every $\beta = \{\beta_\alpha\}_{\alpha\in\mathcal{A}} \in (0;\infty)^{\mathcal{A}}$ there exists $y \in T(x)$ such that

$$\forall_{\alpha\in\mathcal{A}}\ \{J_\alpha(x,y) < J_\alpha(x,T(x)) + \beta_\alpha\}, \tag{26}$$

$$(\forall_{\alpha\in\mathcal{A}}\ \{J_\alpha(y,x) < J_\alpha(T(x),x) + \beta_\alpha\}). \tag{27}$$

Then the following hold.

(B1) There exist a dynamic process $(w^m : m \in \{0\}\cup\mathbb{N})$ of the system (X,T) starting at w^0, $\forall_{m\in\{0\}\cup\mathbb{N}}\ \{w^{m+1} \in T(w^m)\}$, and a point $w \in X$ such that $(w^m : m \in \{0\}\cup\mathbb{N})$ is left (right) $\mathcal{P}_{C;\mathcal{A}}$-convergent to w.

(B2) If the set-valued dynamic system $(X, T^{[k]})$ is left (right) $\mathcal{P}_{C;\mathcal{A}}$-closed on X for some $k \in \mathbb{N}$, then $\mathrm{Fix}(T^{[k]}) \neq \varnothing$ and there exist a dynamic process $(w^m : m \in \{0\}\cup\mathbb{N})$ of the system (X,T) starting at w^0, $\forall_{m\in\{0\}\cup\mathbb{N}}\ \{w^{m+1} \in T(w^m)\}$, and a point $w \in \mathrm{Fix}(T^{[k]})$ such that $(w^m : m \in \{0\}\cup\mathbb{N})$ is left (right) $\mathcal{P}_{C;\mathcal{A}}$-convergent to w.

Proof. We prove only the case when $\mathcal{J}_{C;\mathcal{A}}$ is a left family generated by $\mathcal{P}_{C;\mathcal{A}}$, (X,T) is left $\mathcal{J}_{C;\mathcal{A}}$-admissible in a point $w^0 \in X$, and $(X, T^{[k]})$ is left $\mathcal{P}_{C;\mathcal{A}}$-closed on X. The case of "right" will be omitted, since the reasoning is based on the analogous technique.

Part 1. Assume that (A1)–(A3) hold.

By (21) and the fact that $J_\alpha : X^2 \to [0;\infty)$, $\alpha \in \mathcal{A}$, we choose

$$r = \{r_\alpha\}_{\alpha\in\mathcal{A}} \in (0;\infty)^{\mathcal{A}} \tag{28}$$

such that

$$\forall_{\alpha\in\mathscr{A}} \left\{J_\alpha\left(w^0, T\left(w^0\right)\right) < \left(1-\frac{\lambda_\alpha}{C_\alpha}\right) r_\alpha\right\}. \quad (29)$$

Put

$$\forall_{\alpha\in\mathscr{A}} \left\{\beta_\alpha^{(0)} = \left(1-\frac{\lambda_\alpha}{C_\alpha}\right) r_\alpha - J_\alpha\left(w^0, T\left(w^0\right)\right)\right\}. \quad (30)$$

In view of (28) and (29) this implies $\beta^{(0)} = \{\beta_\alpha^{(0)}\}_{\alpha\in\mathscr{A}} \in (0;\infty)^{\mathscr{A}}$ and we apply (26) to find $w^1 \in T(w^0)$ such that

$$\forall_{\alpha\in\mathscr{A}} \left\{J_\alpha\left(w^0, w^1\right) < J_\alpha\left(w^0, T\left(w^0\right)\right) + \beta_\alpha^{(0)}\right\}. \quad (31)$$

We see from (30) and (31) that

$$\forall_{\alpha\in\mathscr{A}} \left\{J_\alpha\left(w^0, w^1\right) < \left(1-\frac{\lambda_\alpha}{C_\alpha}\right) r_\alpha\right\}. \quad (32)$$

Put now

$$\forall_{\alpha\in\mathscr{A}} \left\{\beta_\alpha^{(1)} = \left(\frac{\lambda_\alpha}{C_\alpha}\right)\left[\left(1-\frac{\lambda_\alpha}{C_\alpha}\right) r_\alpha - J_\alpha\left(w^0, w^1\right)\right]\right\}. \quad (33)$$

Then, in view of (32), we get $\beta^{(1)} = \{\beta_\alpha^{(1)}\}_{\alpha\in\mathscr{A}} \in (0;\infty)^{\mathscr{A}}$ and applying again (26) we find $w^2 \in T(w^1)$ such that

$$\forall_{\alpha\in\mathscr{A}} \left\{J_\alpha\left(w^1, w^2\right) < J_\alpha\left(w^1, T\left(w^1\right)\right) + \beta_\alpha^{(1)}\right\}. \quad (34)$$

Observe that

$$\forall_{\alpha\in\mathscr{A}} \left\{J_\alpha\left(w^1, w^2\right) < \left(\frac{\lambda_\alpha}{C_\alpha}\right)\left(1-\frac{\lambda_\alpha}{C_\alpha}\right) r_\alpha\right\}. \quad (35)$$

Indeed, from (34), Definition 23(A), and using (33), in the event that $\eta = 1$ or $\eta = 2$ or $\eta = 3$, we get

$$\begin{aligned}\forall_{\alpha\in\mathscr{A}} \Big\{J_\alpha\left(w^1, w^2\right) &< J_\alpha\left(w^1, T\left(w^1\right)\right) + \beta_\alpha^{(1)} \\ &\le \sup\left\{J_\alpha\left(u, T\left(w^1\right)\right) : u \in T\left(w^0\right)\right\} + \beta_\alpha^{(1)} \\ &\le D_{\eta;2^X;\alpha}^{L-\mathscr{J}_{C;\mathscr{A}}}\left(T\left(w^0\right), T\left(w^1\right)\right) + \beta_\alpha^{(1)} \\ &\le \left(\frac{\lambda_\alpha}{C_\alpha}\right) J_\alpha\left(w^0, w^1\right) + \beta_\alpha^{(1)} \\ &= \left(\frac{\lambda_\alpha}{C_\alpha}\right)\left(1-\frac{\lambda_\alpha}{C_\alpha}\right) r_\alpha\Big\}.\end{aligned} \quad (36)$$

Thus (35) holds.

Next define

$$\begin{aligned}\forall_{\alpha\in\mathscr{A}} \Big\{\beta_\alpha^{(2)} \\ = \left(\frac{\lambda_\alpha}{C_\alpha}\right)\left[\left(\frac{\lambda_\alpha}{C_\alpha}\right)\left(1-\frac{\lambda_\alpha}{C_\alpha}\right) r_\alpha - J_\alpha\left(w^1, w^2\right)\right]\Big\}.\end{aligned} \quad (37)$$

Then, in view of (35), $\beta^{(2)} = \{\beta_\alpha^{(2)}\}_{\alpha\in\mathscr{A}} \in (0;\infty)^{\mathscr{A}}$. Applying (26) in this situation, we conclude that there exists $w^3 \in T(w^2)$ such that

$$\forall_{\alpha\in\mathscr{A}} \left\{J_\alpha\left(w^2, w^3\right) < J_\alpha\left(w^2, T\left(w^2\right)\right) + \beta_\alpha^{(2)}\right\}. \quad (38)$$

We seek to show that

$$\forall_{\alpha\in\mathscr{A}} \left\{J_\alpha\left(w^2, w^3\right) < \left(\frac{\lambda_\alpha}{C_\alpha}\right)^2\left(1-\frac{\lambda_\alpha}{C_\alpha}\right) r_\alpha\right\}. \quad (39)$$

By (38), Definition 23(A), and using (37), in the event that $\eta = 1$ or $\eta = 2$ or $\eta = 3$, it follows that

$$\begin{aligned}\forall_{\alpha\in\mathscr{A}} \Big\{J_\alpha\left(w^2, w^3\right) &< J_\alpha\left(w^2, T\left(w^2\right)\right) + \beta_\alpha^{(2)} \\ &\le \sup_{u\in T(w^1)} J_\alpha\left(u, T\left(w^2\right)\right) + \beta_\alpha^{(2)} \\ &\le D_{\eta;2^X;\alpha}^{L-\mathscr{J}_{C;\mathscr{A}}}\left(T\left(w^1\right), T\left(w^2\right)\right) + \beta_\alpha^{(2)} \\ &\le \left(\frac{\lambda_\alpha}{C_\alpha}\right) J_\alpha\left(w^1, w^2\right) + \beta_\alpha^{(2)} \\ &= \left(\frac{\lambda_\alpha}{C_\alpha}\right)^2\left(1-\frac{\lambda_\alpha}{C_\alpha}\right) r_\alpha\Big\}.\end{aligned} \quad (40)$$

Thus (39) holds.

Proceeding as before, using Definition 23(A), we get that there exists a sequence $(w^m : m \in \mathbb{N})$ in X satisfying

$$\forall_{m\in\mathbb{N}} \left\{w^{m+1} \in T\left(w^m\right)\right\} \quad (41)$$

and for calculational purposes, upon letting $\forall_{m\in\mathbb{N}} \{\beta^{(m)} = \{\beta_\alpha^{(m)}\}_{\alpha\in\mathscr{A}}\}$ where

$$\begin{aligned}\forall_{\alpha\in\mathscr{A}} \forall_{m\in\mathbb{N}} \Big\{\beta_\alpha^{(m)} = \left(\frac{\lambda_\alpha}{C_\alpha}\right) \\ \cdot\left[\left(\frac{\lambda_\alpha}{C_\alpha}\right)^{m-1}\left(1-\frac{\lambda_\alpha}{C_\alpha}\right) r_\alpha - J_\alpha\left(w^{m-1}, w^m\right)\right]\Big\}\end{aligned} \quad (42)$$

we observe that $\forall_{m\in\mathbb{N}} \{\beta^{(m)} \in (0;\infty)^{\mathscr{A}}\}$,

$$\begin{aligned}\forall_{\alpha\in\mathscr{A}} \forall_{m\in\mathbb{N}} \Big\{J_\alpha\left(w^m, w^{m+1}\right) < J_\alpha\left(w^m, T\left(w^m\right)\right) \\ + \beta_\alpha^{(m)}\Big\},\end{aligned} \quad (43)$$

$$\begin{aligned}\forall_{\alpha\in\mathscr{A}} \forall_{m\in\mathbb{N}} \Big\{J_\alpha\left(w^m, w^{m+1}\right) \\ < \left(\frac{\lambda_\alpha}{C_\alpha}\right)^m\left(1-\frac{\lambda_\alpha}{C_\alpha}\right) r_\alpha\Big\}.\end{aligned} \quad (44)$$

Let now $m < n$. Using ($\mathcal{J}1$), we get

$$\forall_{\alpha\in\mathcal{A}} \left\{ J_\alpha(w^m, w^n) \leqslant C_\alpha J_\alpha(w^m, w^{m+1}) + C_\alpha^2 J_\alpha(w^{m+1}, w^{m+2}) + \cdots + C_\alpha^{n-m-1} J_\alpha(w^{n-2}, w^{n-1}) + C_\alpha^{n-m-1} J_\alpha(w^{n-1}, w^n) = \sum_{j=0}^{n-m-2} C_\alpha^{j+1} J_\alpha(w^{m+j}, w^{m+j+1}) + C_\alpha^{n-m-1} J_\alpha(w^{n-1}, w^n) \right\}. \tag{45}$$

Hence, by (44), for each $\alpha \in \mathcal{A}$,

$$J_\alpha(w^m, w^n) < \left(1 - \frac{\lambda_\alpha}{C_\alpha}\right) \cdot r_\alpha \left[\sum_{j=0}^{n-m-2} C_\alpha^{j+1} \left(\frac{\lambda_\alpha}{C_\alpha}\right)^{m+j} + C_\alpha^{n-m-1} \left(\frac{\lambda_\alpha}{C_\alpha}\right)^{n-2} \right] = \left(1 - \frac{\lambda_\alpha}{C_\alpha}\right) \cdot r_\alpha \left[C_\alpha \left(\frac{\lambda_\alpha}{C_\alpha}\right)^m \sum_{j=0}^{n-m-2} \lambda_\alpha^j + \left(\frac{C_\alpha}{\lambda_\alpha^2}\right) \frac{\lambda_\alpha^n}{C_\alpha^m} \right]. \tag{46}$$

This and (41) mean that

$$\exists_{(w^m : m\in\mathbb{N})} \forall_{m\in\{0\}\cup\mathbb{N}} \left\{ w^{m+1} \in T(w^m) \right\} \tag{47}$$

and since $m < n$ implies $\lambda_\alpha^n \leq \lambda_\alpha^m$,

$$\forall_{\alpha\in\mathcal{A}} \left\{ \lim_{m\to\infty} \sup_{n>m} J_\alpha(w^m, w^n) \leq \lim_{m\to\infty} \sup_{n>m} \left(1 - \frac{\lambda_\alpha}{C_\alpha}\right) \cdot r_\alpha \left[C_\alpha \left(\frac{\lambda_\alpha}{C_\alpha}\right)^m (1-\lambda_\alpha)^{-1} + \left(\frac{C_\alpha}{\lambda_\alpha^2}\right) \frac{\lambda_\alpha^n}{C_\alpha^m} \right] \leq \lim_{m\to\infty} \left(1 - \frac{\lambda_\alpha}{C_\alpha}\right) r_\alpha \left[C_\alpha \left(\frac{\lambda_\alpha}{C_\alpha}\right)^m (1-\lambda_\alpha)^{-1} + \left(\frac{C_\alpha}{\lambda_\alpha^2}\right) \left(\frac{\lambda_\alpha}{C_\alpha}\right)^m \right] = 0 \right\}. \tag{48}$$

Now, since (X, T) is left $\mathcal{J}_{C;\mathcal{A}}$-admissible in $w^0 \in X$, by Definition 20(A), properties (47) and (48) imply that there exists $w \in X$ such that

$$\forall_{\alpha\in\mathcal{A}} \left\{ \lim_{m\to\infty} J_\alpha(w, w^m) = 0 \right\}. \tag{49}$$

Next, defining $u_m = w^m$ and $w_m = w$ for $m \in \mathbb{N}$, by (48) and (49) we see that conditions (8) and (10) hold for the sequences $(u_m : m \in \mathbb{N})$ and $(w_m : m \in \mathbb{N})$ in X. Consequently, by ($\mathcal{J}2$), we get (12) which implies that

$$\forall_{\alpha\in\mathcal{A}} \left\{ \lim_{m\to\infty} p_\alpha(w, w^m) = \lim_{m\to\infty} p_\alpha(w_m, u_m) = 0 \right\} \tag{50}$$

and so in particular we see that $w \in \mathrm{LIM}^{L-\mathcal{P}_{C;\mathcal{A}}}_{(w^m:m\in\mathbb{N})}$.

Part 2. Assume that (A1)–(A3) hold and that, for some $k \in \mathbb{N}$, $(X, T^{[k]})$ is left $\mathcal{P}_{C;\mathcal{A}}$-closed on X.

By Part 1, $\mathrm{LIM}^{L-\mathcal{P}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})} \neq \varnothing$ and since, by (47), $w^{(m+1)k} \in T^{[k]}(w^{mk})$ for $m \in \{0\} \cup \mathbb{N}$, thus defining $(x_m = w^{m-1+k} : m \in \mathbb{N})$, we see that $(x_m : m \in \mathbb{N}) \subset T^{[k]}(X)$, $\mathrm{LIM}^{L-\mathcal{P}_{C;\mathcal{A}}}_{(x_m:m\in\{0\}\cup\mathbb{N})} = \mathrm{LIM}^{L-\mathcal{P}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})} \neq \varnothing$, the sequences $(v_m = w^{(m+1)k} : m \in \mathbb{N}) \subset T^{[k]}(X)$ and $(u_m = w^{mk} : m \in \mathbb{N}) \subset T^{[k]}(X)$ satisfy $\forall_{m\in\mathbb{N}} \{v_m \in T^{[k]}(u_m)\}$ and, as subsequences of $(x_m : m \in \{0\} \cup \mathbb{N})$, are left $\mathcal{P}_{C;\mathcal{A}}$-converging to each point of the set $\mathrm{LIM}^{L-\mathcal{P}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})}$. Moreover, by Remark 18, $\mathrm{LIM}^{L-\mathcal{P}_{C;\mathcal{A}}}_{(w^m:m\in\mathbb{N})} \subset \mathrm{LIM}^{L-\mathcal{P}_{C;\mathcal{A}}}_{(v_m:m\in\mathbb{N})}$ and $\mathrm{LIM}^{L-\mathcal{P}_{C;\mathcal{A}}}_{(w^m:m\in\mathbb{N})} \subset \mathrm{LIM}^{L-\mathcal{P}_{C;\mathcal{A}}}_{(u_m:m\in\mathbb{N})}$. By the above and by Definition 22, since $T^{[k]}$ is left $\mathcal{P}_{C;\mathcal{A}}$-closed, we conclude that there exist $w \in \mathrm{LIM}^{L-\mathcal{P}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})} = \mathrm{LIM}^{L-\mathcal{P}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})}$ such that $w \in T^{[k]}(w)$.

Part 3. The result now follows at once from Parts 1 and 2. □

9. Theorem of Banach Type in Quasi-Triangular Spaces $(X, \mathcal{P}_{C;\mathcal{A}})$

In this section, in the quasi-triangular spaces $(X, \mathcal{P}_{C;\mathcal{A}})$, using left (right) families $\mathcal{J}_{C;\mathcal{A}}$ generated by $\mathcal{P}_{C;\mathcal{A}}$, we construct two types of left (right) single-valued quasi-contractions $T : X \to X$, and convergence, existence, approximation, uniqueness, periodic point, and fixed point theorem for such quasi-contractions is also proved.

The following Definition 27 can be stated as a single-valued version of Definition 23.

Definition 27. Let $(X, \mathcal{P}_{C;\mathcal{A}})$ be the quasi-triangular space, and let $\mathcal{J}_{C;\mathcal{A}}$ be the left (right) family generated by $\mathcal{P}_{C;\mathcal{A}}$. Let (X, T) be the single-valued dynamic system, let $\lambda = \{\lambda_\alpha\}_{\alpha\in\mathcal{A}} \in [0;1)^{\mathcal{A}}$, and let $\eta \in \{1, 2\}$.

(A) If $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^L_{(X,\mathcal{P}_{C;\mathcal{A}})}$, then we define the *left $\mathcal{D}^{L-\mathcal{J}_{C;\mathcal{A}}}_{X,\eta}$-quasi-distance* on X by $\mathcal{D}^{L-\mathcal{J}_{C;\mathcal{A}}}_{X,\eta} = \{D^{L-\mathcal{J}_{C;\mathcal{A}}}_{\eta;X;\alpha} : X^2 \to [0;\infty), \alpha \in \mathcal{A}\}$ where

$$\forall_{\alpha\in\mathcal{A}} \forall_{u,w\in X} \left\{ D^{L-\mathcal{J}_{C;\mathcal{A}}}_{1;X;\alpha}(u, w) = \max\{J_\alpha(u, w), J_\alpha(w, u)\} \right\}, \qquad \forall_{\alpha\in\mathcal{A}} \forall_{u,w\in X} \left\{ D^{L-\mathcal{J}_{C;\mathcal{A}}}_{2;X;\alpha}(u, w) = J_\alpha(u, w) \right\}. \tag{51}$$

One says that (X, T) is *left* $(\mathscr{D}_{X,\eta}^{L-\mathscr{J}_{C;\mathscr{A}}}, \lambda)$*-quasi-contraction* on X if

$$\forall_{\alpha\in\mathscr{A}}\ \forall_{x,y\in X}\ \left\{C_\alpha \cdot D_{\eta;X;\alpha}^{L-\mathscr{J}_{C;\mathscr{A}}}(T(x), T(y)) \leq \lambda_\alpha J_\alpha(x, y)\right\}. \tag{52}$$

(B) If $\mathscr{J}_{C;\mathscr{A}} \in \mathbb{J}_{(X,\mathscr{P}_{C;\mathscr{A}})}^{R}$, then one defines the *right* $\mathscr{D}_{X,\eta}^{R-\mathscr{J}_{C;\mathscr{A}}}$*-quasi-distance* on X by $\mathscr{D}_{X,\eta}^{R-\mathscr{J}_{C;\mathscr{A}}} = \{D_{\eta;X;\alpha}^{R-\mathscr{J}_{C;\mathscr{A}}} : X^2 \to [0;\infty),\ \alpha \in \mathscr{A}\}$ where

$$\begin{aligned}&\forall_{\alpha\in\mathscr{A}}\ \forall_{u,w\in X}\ \left\{D_{1;X;\alpha}^{R-\mathscr{J}_{C;\mathscr{A}}}(u, w) = \max\{J_\alpha(u, w), J_\alpha(w, u)\}\right\},\\ &\forall_{\alpha\in\mathscr{A}}\ \forall_{u,w\in X}\ \left\{D_{2;X;\alpha}^{R-\mathscr{J}_{C;\mathscr{A}}}(u, w) = J_\alpha(u, w)\right\}.\end{aligned} \tag{53}$$

One says that (X, T) is *right* $(\mathscr{D}_{X,\eta}^{R-\mathscr{J}_{C;\mathscr{A}}}, \lambda)$*-quasi-contraction* on X if

$$\forall_{\alpha\in\mathscr{A}}\ \forall_{x,y\in X}\ \left\{C_\alpha \cdot D_{\eta;X;\alpha}^{R-\mathscr{J}_{C;\mathscr{A}}}(T(x), T(y)) \leq \lambda_\alpha J_\alpha(x, y)\right\}. \tag{54}$$

Remark 28. Observe that (52) and (54) extend (1).

The following terminologies will be much used in the sequel.

Definition 29. Let $(X, \mathscr{P}_{C;\mathscr{A}})$ be the quasi-triangular space, and let $\mathscr{J}_{C;\mathscr{A}}$ be the left (right) family generated by $\mathscr{P}_{C;\mathscr{A}}$. Let (X, T) be the single-valued dynamic system, $T : X \to X$.

(A) Given $w^0 \in X$, One says that (X, T) is *left (right)* $\mathscr{J}_{C;\mathscr{A}}$*-admissible in* w^0 if, for the sequence $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$, $\mathrm{LIM}_{(w^m:m\in\{0\}\cup\mathbb{N})}^{L-\mathscr{J}_{C;\mathscr{A}}} \neq \varnothing$ $(\mathrm{LIM}_{(w^m:m\in\{0\}\cup\mathbb{N})}^{R-\mathscr{J}_{C;\mathscr{A}}} \neq \varnothing)$ whenever

$$\forall_{\alpha\in\mathscr{A}}\ \left\{\lim_{m\to\infty}\sup_{n>m} J_\alpha(w^m, w^n) = 0\right\} \quad \left(\forall_{\alpha\in\mathscr{A}}\ \left\{\lim_{m\to\infty}\sup_{n>m} J_\alpha(w^n, w^m) = 0\right\}\right). \tag{55}$$

(B) We say that (X, T) is *left (right)* $\mathscr{J}_{C;\mathscr{A}}$*-admissible on* X, if (X, T) is *left (right)* $\mathscr{J}_{C;\mathscr{A}}$*-admissible* in each point $w^0 \in X$.

Remark 30. Let $(X, \mathscr{P}_{C;\mathscr{A}})$ be the quasi-triangular space, and let $\mathscr{J}_{C;\mathscr{A}}$ be the left (right) family generated by $\mathscr{P}_{C;\mathscr{A}}$. Let (X, T) be the single-valued dynamic system on X. If $(X, \mathscr{P}_{C;\mathscr{A}})$ is left (right) $\mathscr{J}_{C;\mathscr{A}}$-sequentially complete, then (X, T) is left (right) $\mathscr{J}_{C;\mathscr{A}}$-admissible on X.

We can define the following generalization of continuity.

Definition 31. Let $(X, \mathscr{P}_{C;\mathscr{A}})$ be the quasi-triangular space. Let (X, T) be the single-valued dynamic system, $T : X \to X$, and let $k \in \mathbb{N}$. The single-valued dynamic system $(X, T^{[k]})$ is said to be a *left (right)* $\mathscr{P}_{C;\mathscr{A}}$*-closed on* X if for each sequence $(x_m : m \in \mathbb{N})$ in $T^{[k]}(X)$, left (right) $\mathscr{P}_{C;\mathscr{A}}$-converging in X (thus $\mathrm{LIM}_{(x_m:m\in\mathbb{N})}^{L-\mathscr{P}_{C;\mathscr{A}}} \neq \varnothing$ $(\mathrm{LIM}_{(x_m:m\in\mathbb{N})}^{R-\mathscr{P}_{C;\mathscr{A}}} \neq \varnothing)$) and having subsequences $(v_m : m \in \mathbb{N})$ and $(u_m : m \in \mathbb{N})$ satisfying $\forall_{m\in\mathbb{N}}\ \{v_m = T^{[k]}(u_m)\}$; the following property holds: there exists $x \in \mathrm{LIM}_{(x_m:m\in\mathbb{N})}^{L-\mathscr{P}_{C;\mathscr{A}}}$ $(x \in \mathrm{LIM}_{(x_m:m\in\mathbb{N})}^{R-\mathscr{P}_{C;\mathscr{A}}})$ such that $x = T^{[k]}(x)$ $(x = T^{[k]}(x))$.

The following result extends Theorem 5 to spaces $(X, \mathscr{P}_{C;\mathscr{A}})$.

Theorem 32. *Let $(X, \mathscr{P}_{C;\mathscr{A}})$ be the quasi-triangular space, and let (X, T) be the single-valued dynamic system, $T : X \to 2^X$. Let $\eta \in \{1, 2\}$, and let $\lambda = \{\lambda_\alpha\}_{\alpha\in\mathscr{A}} \in [0; 1)^{\mathscr{A}}$.*

Assume that there exist a left (right) family $\mathscr{J}_{C;\mathscr{A}}$ generated by $\mathscr{P}_{C;\mathscr{A}}$ and a point $w^0 \in X$ with the following properties.

(A1) (X, T) is left $(\mathscr{D}_{X,\eta}^{L-\mathscr{J}_{C;\mathscr{A}}}, \lambda)$-quasi-contraction (right $(\mathscr{D}_{X,\eta}^{R-\mathscr{J}_{C;\mathscr{A}}}, \lambda)$-quasi-contraction) on X.

(A2) (X, T) is left (right) $\mathscr{J}_{C;\mathscr{A}}$-admissible in a point $w^0 \in X$.

Then the following hold.

(B1) There exists a point $w \in X$ such that the sequence $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ starting at w^0 is left (right) $\mathscr{P}_{C;\mathscr{A}}$-convergent to w.

(B2) If the single-valued dynamic system $(X, T^{[k]})$ is left (right) $\mathscr{P}_{C;\mathscr{A}}$-closed on X for some $k \in \mathbb{N}$, then $\mathrm{Fix}(T^{[k]}) \neq \varnothing$, there exists a point $w \in \mathrm{Fix}(T^{[k]})$ such that the sequence $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ starting at w^0 is left (right) $\mathscr{P}_{C;\mathscr{A}}$-convergent to w, and

$$\forall_{\alpha\in\mathscr{A}}\ \forall_{v\in\mathrm{Fix}(T^{[k]})}\ \{J_\alpha(v, T(v)) = J_\alpha(T(v), v) = 0\}. \tag{56}$$

(B3) If the family $\mathscr{P}_{C;\mathscr{A}} = \{p_\alpha,\ \alpha \in \mathscr{A}\}$ is separating on X and if the single-valued dynamic system $(X, T^{[k]})$ is left (right) $\mathscr{P}_{C;\mathscr{A}}$-closed on X for some $k \in \mathbb{N}$, then there exists a point $w \in X$ such that

$$\mathrm{Fix}\left(T^{[k]}\right) = \mathrm{Fix}(T) = \{w\}, \tag{57}$$

the sequence $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ starting at w^0 is left (right) $\mathscr{P}_{C;\mathscr{A}}$- convergent to w, and

$$\forall_{\alpha\in\mathscr{A}}\ \{J_\alpha(w, w) = 0\}. \tag{58}$$

Proof. By Theorem 26, we prove only (56)–(58) and only in the case of "left." We omit the proof in the case of "right," which is based on the analogous technique.

Part 1. Property (56) holds. Suppose that $\exists_{\alpha_0\in\mathscr{A}}\ \exists_{v\in\mathrm{Fix}(T^{[k]})}\ \{J_{\alpha_0}(v, T(v)) > 0\}$. Of course, $v = T^{[k]}(v) = T^{[2k]}(v)$, $T(v) = T^{[2k]}(T(v))$ and, for $\eta \in \{1, 2\}$, by Definition 27(A),

$$\begin{aligned}0 &< J_{\alpha_0}(v, T(v)) = J_{\alpha_0}\left(T^{[2k]}(v), T^{[2k]}(T(v))\right)\\ &\leq D_{\eta;X;\alpha_0}^{L-\mathscr{J}_{C;\mathscr{A}}}\left(T^{[2k]}(v), T^{[2k]}(T(v))\right)\end{aligned}$$

$$\leq \left(\frac{\lambda_{\alpha_0}}{C_{\alpha_0}}\right) J_{\alpha_0}\left(T^{[2k-1]}(v), T^{[2k-1]}(T(v))\right)$$
$$\leq \left(\frac{\lambda_{\alpha_0}}{C_{\alpha_0}}\right) \cdot D^{L-\mathcal{J}_{C;\mathcal{A}}}_{\eta;X;\alpha_0}\left(T^{[2k-1]}(v), T^{[2k-1]}(T(v))\right)$$
$$\leq \left(\frac{\lambda_{\alpha_0}}{C_{\alpha_0}}\right)^2 J_{\alpha_0}\left(T^{[2k-2]}(v), T^{[2k-2]}(T(v))\right) \leq \cdots$$
$$\leq \left(\frac{\lambda_{\alpha_0}}{C_{\alpha_0}}\right)^{2k} J_{\alpha_0}(v, T(v)) < J_{\alpha_0}(v, T(v)), \tag{59}$$

which is impossible. Therefore,

$$\forall_{\alpha\in\mathcal{A}} \forall_{v\in \mathrm{Fix}(T^{[k]})} \left\{J_\alpha(v, T(v)) = 0\right\}. \tag{60}$$

Suppose now that $\exists_{\alpha_0\in\mathcal{A}} \exists_{v\in\mathrm{Fix}(T^{[k]})} \{J_{\alpha_0}(T(v), v) > 0\}$. Then, by Definition 27(A) and property (60), using the fact that $v = T^{[k]}(v) = T^{[2k]}(v)$, we get, for $\eta \in \{1, 2\}$, that

$$\begin{aligned}
0 &< J_{\alpha_0}(T(v), v) = J_{\alpha_0}\left(T^{[k+1]}(v), T^{[2k]}(v)\right) \\
&\leq \sum_{m=1}^{k-2} C^m_{\alpha_0} J_{\alpha_0}\left(T^{[k+m]}(v), T^{[k+m+1]}(v)\right) \\
&\quad + C^{k-2}_{\alpha_0} J_{\alpha_0}\left(T^{[2k-1]}(v), T^{[2k]}(v)\right) \\
&\leq \sum_{m=1}^{k-2} C^m_{\alpha_0} \cdot D^{L-\mathcal{J}_{C;\mathcal{A}}}_{\eta;X;\alpha_0}\left(T^{[k+m]}(v), T^{[k+m+1]}(v)\right) \\
&\quad + C^{k-2}_{\alpha_0} \cdot D^{L-\mathcal{J}_{C;\mathcal{A}}}_{\eta;X;\alpha_0}\left(T^{[2k-1]}(v), T^{[2k]}(v)\right) \\
&\leq \sum_{m=1}^{k-2} C^m_{\alpha_0} \left(\frac{\lambda_{\alpha_0}}{C_{\alpha_0}}\right)^{k+m} J_{\alpha_0}(v, T(v)) \\
&\quad + C^{k-2}_{\alpha_0} \left(\frac{\lambda_{\alpha_0}}{C_{\alpha_0}}\right)^{2k-1} J_{\alpha_0}(v, T(v)) = 0,
\end{aligned} \tag{61}$$

which is impossible. Therefore,

$$\forall_{\alpha\in\mathcal{A}} \forall_{v\in\mathrm{Fix}(T^{[k]})} \left\{J_\alpha(T(v), v) = 0\right\}. \tag{62}$$

We see that (56) is a consequence of (60) and (62).

Part 2. Properties (57) and (58) hold. We first observe that

$$\forall_{v\in\mathrm{Fix}(T^{[k]})} \{T(v) = v\}; \tag{63}$$

in other words, $\mathrm{Fix}(T^{[k]}) = \mathrm{Fix}(T)$. In fact, if $v \in \mathrm{Fix}(T^{[k]})$ and $T(v) \neq v$, then, since the family $\mathscr{P}_{C;\mathcal{A}} = \{p_\alpha,\ \alpha \in \mathcal{A}\}$ is separating on X, we get that $T(v) \neq v \Rightarrow \exists_{\alpha\in\mathcal{A}} \{p_\alpha(T(v), v) > 0 \vee p_\alpha(v, T(v)) > 0\}$. In view of Theorem 15 this implies $T(v) \neq v \Rightarrow \exists_{\alpha\in\mathcal{A}} \{J_\alpha(T(v), v) > 0 \vee J_\alpha(v, T(v)) > 0\}$. However, by property (56), this is impossible.

Next we see that $\forall_{v\in\mathrm{Fix}(T^{[k]})=\mathrm{Fix}(T)} \{J_\alpha(v, v) = 0\}$. In fact, by Definition 11(A) and property (56), we conclude that $\forall_{\alpha\in\mathcal{A}} \forall_{v\in\mathrm{Fix}(T^{[k]})} \{J_\alpha(v, v) \leq C_\alpha[J_\alpha(v, T(v)) + J_\alpha(T(v), v)] = 0\}$.

Finally, suppose that $u, w \in \mathrm{Fix}(T)$ and $u \neq w$. Then, since the family $\mathscr{P}_{C;\mathcal{A}} = \{p_\alpha,\ \alpha \in \mathcal{A}\}$ is separating on X, we get $\exists_{\alpha_0\in\mathcal{A}} \{p_{\alpha_0}(u, w) > 0 \vee p_{\alpha_0}(w, u) > 0\}$. By applying Theorem 15, this implies $\exists_{\alpha_0\in\mathcal{A}} \{J_{\alpha_0}(u, w) > 0 \vee J_{\alpha_0}(w, u) > 0\}$. Consequently, for $\eta \in \{1, 2\}$, by Definition 27(A), we conclude that

$$\begin{aligned}
&\exists_{\alpha_0\in\mathcal{A}} \left\{\left[J_{\alpha_0}(u, w) > 0,\ J_{\alpha_0}(u, w)\right.\right. \\
&\quad = J_{\alpha_0}(T(u), T(w)) \leq D^{L-\mathcal{J}_{C;\mathcal{A}}}_{\eta;X;\alpha_0}(T(u), T(w)) \\
&\quad \left.\leq \left(\frac{\lambda_{\alpha_0}}{C_{\alpha_0}}\right) J_{\alpha_0}(u, w) < J_{\alpha_0}(u, w)\right] \text{ or } \left[J_{\alpha_0}(w, u)\right. \\
&\quad > 0,\ J_{\alpha_0}(w, u) = J_{\alpha_0}(T(w), T(u)) \\
&\quad \leq D^{L-\mathcal{J}_{C;\mathcal{A}}}_{\eta;X;\alpha_0}(T(w), T(u)) \leq \left(\frac{\lambda_{\alpha_0}}{C_{\alpha_0}}\right) J_{\alpha_0}(w, u) \\
&\quad \left.\left.< J_{\alpha_0}(w, u)\right]\right\},
\end{aligned} \tag{64}$$

which is impossible. This gives that $\mathrm{Fix}(T)$ is a singleton.

Thus (57) and (58) hold. □

10. Examples of Spaces $(X, \mathscr{P}_{C;\mathcal{A}})$

Example 1. Let $X = [0; 6]$, $\gamma \geq 81$ and let $p : X^2 \to [0; \infty)$ be of the form

$$p(u, w) = \begin{cases} 0 & \text{if } u \geq w,\ \{u, w\} \cap (0; 6) = \{u, w\}, \\ (w-u)^4 & \text{if } u < w,\ \{u, w\} \cap (0; 6) = \{u, w\}, \\ \gamma & \text{if } \{u, w\} \cap (0; 6) \neq \{u, w\}. \end{cases} \tag{65}$$

(1) $(X, \mathscr{P}_{\{8\};\{1\}})$, $\mathscr{P}_{\{8\};\{1\}} = \{p\}$, *is the quasi-triangular space*. In fact,

$$\forall_{u,v,w\in X} \{p(u, w) \leq 8[p(u, v) + p(v, w)]\}. \tag{66}$$

Inequality (66) is a consequence of Cases 1–6.

Case 1. If $u, v, w \in (0; 6)$ and $v \leq u < w$, then $p(u, v) = 0$ and $w - u \leq w - v$. This gives $p(u, w) = (w - u)^4 \leq (w - v)^4 < 8(w - v)^4 = 8[p(u, v) + p(v, w)]$.

Case 2. If $u, v, w \in (0; 6)$, $u < w$ and $u \leq v \leq w$, then $p(u, w) = (w - u)^4$ and $f(v_0) = \min_{u\leq v\leq w} f(v) = (w - u)^4$ where, for $u \leq v \leq w$, $f(v) = 8[p(u, v) + p(v, w)] = 8[(v-u)^4 + (w-v)^4]$ and $v_0 = (u + w)/2$.

Case 3. $\sup_{u,w\in(0;6);u<w} p(u, w) = \sup_{u,w\in(0;6);u<w} (w - u)^4 = 6^4 = 1296$ and $\sup_{u,w\in(0;6);u<w} \min_{u\leq v\leq w} 8[p(u, v) + p(v, w)] = \sup_{u,w\in(0;6);u<w} \min_{u\leq v\leq w} 8[(v - u)^4 + (w - v)^4] = 8[(3 - 0)^4 + (6 - 3)^4] = 8[81 + 81] = 1296$.

Case 4. If $u, v, w \in (0; 6)$ and $u < w \le v$, then $p(v, w) = 0$ and $w - u \le v - u$. This gives $p(u, w) = (w - u)^4 \le (v - u)^4 < 8(v - u)^4 = 8[p(u, v) + p(v, w)]$.

Case 5. If $u, w \in (0; 6)$, $u < w$ and $v \in \{0, 6\}$, then $p(u, w) \le 1296 \le 8[p(u, v) + p(v, w)] = 8[\gamma + \gamma]$.

Case 6. If $\{u, w\} \cap (0; 6) \ne \{u, w\}$, then $\forall_{v \in X}\ \{p(u, w) = \gamma < 8\gamma \le 8[p(u, v) + p(v, w)]\}$.

(2) $\mathscr{P}_{\{8\};\{1\}} = \{p\}$ *is asymmetric.* Indeed, we have that $0 = p(5, 1) \ne p(1, 5) = 256$. Therefore, condition $\forall_{u,w \in X}\ \{p(u, w) = p(w, u)\}$ does not hold.

(3) $\mathscr{P}_{\{8\};\{1\}} = \{p\}$ *does not vanish on the diagonal.* Indeed, if $u \in \{0, 6\}$, then $p(u, u) = \gamma \ne 0$. Therefore, the condition $\forall_{u \in X}\ \{p(u, u) = 0\}$ does not hold.

(4) *For the constant sequence of the form* $(u_m = 3 : m \in \mathbb{N}) \subset X$ *the sets* $\mathrm{LIM}^{L-\mathscr{P}_{\{8\};\{1\}}}_{(u_m:m\in\mathbb{N})}$ *and* $\mathrm{LIM}^{R-\mathscr{P}_{\{8\};\{1\}}}_{(u_m:m\in\mathbb{N})}$ *are not singletons.* Indeed, by (65), Remark 12, and Definition 16(B), we have that $\mathrm{LIM}^{L-\mathscr{P}_{\{8\};\{1\}}}_{(u_m:m\in\mathbb{N})} = [3; 6]$ and $\mathrm{LIM}^{R-\mathscr{P}_{\{8\};\{1\}}}_{(u_m:m\in\mathbb{N})} = [0; 3]$.

Example 2. Let X be a set (nonempty), $A \subset X$, $A \ne \varnothing$, $A \ne X$, $\gamma > 0$, and let $p : X^2 \to [0; \infty)$ be of the form

$$p(u, w) = \begin{cases} 0 & \text{if } A \cap \{u, w\} = \{u, w\}, \\ \gamma & \text{if } A \cap \{u, w\} \ne \{u, w\}. \end{cases} \tag{67}$$

(1) *A pair* $(X, \mathscr{P}_{\{1\};\{1\}})$, $\mathscr{P}_{\{1\};\{1\}} = \{p\}$, *is the quasi-triangular space.* Indeed, formula (67) yields $\forall_{u,v,w \in X}\ \{q(u, w) \le q(u, v) + q(v, w)\}$. Otherwise, $\exists_{u_0,v_0,w_0 \in X}\ \{q(u_0, w_0) > q(u_0, v_0) + q(v_0, w_0)\}$. It is clear that then $q(u_0, w_0) = \gamma$, $q(u_0, v_0) = 0$, and $q(v_0, w_0) = 0$. From this we see that $A \cap \{u_0, w_0\} \ne \{u_0, w_0\}$, $A \cap \{u_0, v_0\} = \{u_0, v_0\}$, and $A \cap \{v_0, w_0\} = \{v_0, w_0\}$. This is impossible.

(2) $\mathscr{P}_{\{1\};\{1\}} = \{p\}$ *does not vanish on the diagonal.* Indeed, if $u \in X \setminus A$, then $p(u, u) = \gamma \ne 0$. Therefore, the condition $\forall_{u \in X}\ \{p(u, u) = 0\}$ does not hold.

(3) $\mathscr{P}_{\{1\};\{1\}} = \{p\}$ *is symmetric.* This follows from (67).

(4) *We observe that* $\mathrm{LIM}^{L-\mathscr{P}_{\{1\};\{1\}}}_{(u_m:m\in\mathbb{N})} = \mathrm{LIM}^{R-\mathscr{P}_{\{1\};\{1\}}}_{(u_m:m\in\mathbb{N})} = A$ *for each sequence* $(u_m : m \in \mathbb{N}) \subset A$. We conclude this from (67).

Example 3. Let $X = [0; 6]$ and let $p : X^2 \to [0; \infty)$ be of the form

$$p(u, w) = \begin{cases} 0 & \text{if } u \ge w, \\ (w - u)^3 & \text{if } u < w. \end{cases} \tag{68}$$

(1) $(X, \mathscr{P}_{\{4\};\{1\}})$, $\mathscr{P}_{\{4\};\{1\}} = \{p\}$, *is the quasi-triangular space.* In fact, $\forall_{u,v,w \in X}\ \{q(u, w) \le 4[q(u, v) + q(v, w)]\}$ holds. This is a consequence of Cases 1–3.

Case 1. If $v \le u < w$, then $p(u, v) = 0$, $w - u \le w - v$, and, consequently, $p(u, w) = (w - u)^3 \le (w - v)^3 < 4(w - v)^3 = 4p(v, w) = 4[p(u, v) + p(v, w)]$.

Case 2. If $u < w$ and $u \le v \le w$, then $q(u, w) = (w - u)^3$ and $f(v_0) = \min_{u \leqslant v \leqslant w} f(v) = (w - u)^3$ where $v_0 = (u + w)/2$ is a minimum point of the map $f(v) = 4[p(u, v) + p(v, w)] = 4(w - u)[w^2 + wu + u^2 + 3v^2 - 3v(w + u)]$.

Case 3. If $u < w \le v$, then $p(v, w) = 0$ and, consequently, $p(u, w) = (w - u)^3 \le (v - u)^3 < 4(v - u)^3 = 4p(u, v) = 4[p(u, v) + p(v, w)]$.

(2) $\mathscr{P}_{\{4\};\{1\}} = \{p\}$ *is asymmetric.* Indeed, we have that $0 = p(6, 0) \ne p(0, 6) = 216$. Therefore, condition $\forall_{u,w \in X}\ \{p(u, w) = p(w, u)\}$ does not hold.

(3) $\mathscr{P}_{\{4\};\{1\}} = \{p\}$ *vanishes on the diagonal.* In fact, by (68), it is clear that $\forall_{u \in X}\ \{p(u, u) = 0\}$.

(4) *We observe that* $\mathrm{LIM}^{L-\mathscr{P}_{\{4\};\{1\}}}_{(u_m:m\in\mathbb{N})} = [2; 6]$ and $\mathrm{LIM}^{R-\mathscr{P}_{\{4\};\{1\}}}_{(u_m:m\in\mathbb{N})} = [1; 2]$ *for sequence* $(u_m = 2 : m \in \mathbb{N})$. We conclude this from (68).

Example 4. Let $X = [0; 6]$ and let $\mathscr{P}_{\{2\};\{1\}} = \{p\}$ where $p : X^2 \to [0; \infty)$ is of the form

$$p(u, w) = \begin{cases} 0 & \text{if } u \ge w, \\ (u - w)^2 & \text{if } u < w. \end{cases} \tag{69}$$

Let

$$E = [0; 3) \cup (3; 6] \tag{70}$$

and let $\mu \ge 36/4$ and $\mathscr{J}_{\{2\};\{1\}} = \{J\}$ where $J : X^2 \to [0; \infty)$ is of the form

$$J(u, w) = \begin{cases} p(u, w) & \text{if } E \cap \{u, w\} = \{u, w\}, \\ \mu & \text{if } E \cap \{u, w\} \ne \{u, w\}. \end{cases} \tag{71}$$

(1) $\mathscr{J}_{\{2\};\{1\}}$ *is* not *symmetric.* In fact, by (69)–(71), $J(0, 6) = 36$ and $J(6, 0) = 0$.

(2) $\mathscr{J}_{\{2\};\{1\}} = \{J\} \in \mathbb{J}^L_{(X,\mathscr{P}_{\{2\};\{1\}})} \cap \mathbb{J}^R_{(X,\mathscr{P}_{\{2\};\{1\}})}$. See Theorem 14.

Remark 33. By Examples 1–4 it follows that the distances p defined by (65) and (67)–(69) and J defined by (70) and (71) are not metrics, ultra metrics, quasi-metrics, ultra quasi-metrics, b-metrics, partial metrics, partial b-metrics, pseudometrics (gauges), quasi-pseudometrics (quasi-gauges), and ultra quasi-pseudometrics (ultra quasi-gauges).

11. Examples Illustrating Theorem 26

Example 1. Let $X = [0; 6]$, let $\gamma > 2048$ be arbitrary and fixed, and, for $u, w \in X$, let

$$p(u, w) = \begin{cases} 0 & \text{if } u \ge w, \quad \{u, w\} \cap (0; 6) = \{u, w\}, \\ (w - u)^4 & \text{if } u < w, \quad \{u, w\} \cap (0; 6) = \{u, w\}, \\ \gamma & \text{if } \{u, w\} \cap (0; 6) \ne \{u, w\}. \end{cases} \tag{72}$$

Define the set-valued dynamic system (X, T) by

$$T(u) = \begin{cases} [1; 2] & \text{if } u \in [0; 3) \cup (4; 6], \\ (4; 6) & \text{if } u \in [3; 4]. \end{cases} \tag{73}$$

Let

$$E = [0; 3) \cup (4; 6] \tag{74}$$

and let $J : X \times X \to [0; \infty)$ be of the form

$$J(u, w) = \begin{cases} p(u, w) & \text{if } E \cap \{u, w\} = \{u, w\}, \\ \gamma & \text{if } E \cap \{u, w\} \neq \{u, w\}. \end{cases} \tag{75}$$

(1) $(X, \mathcal{P}_{\{8\};\{1\}})$, *where* $\mathcal{P}_{\{8\};\{1\}} = \{p\}$, *is the quasi-triangular space, and* $\mathcal{J}_{\{8\};\{1\}} = \{J\}$ *is the left and right family generated by* $\mathcal{P}_{\{8\};\{1\}}$. This is a consequence of Definitions 7 and 11, Example 1, and Theorem 14; we see that $\gamma = \mu > 81$.

(2) (X, T) *is a* $(D = \mathcal{D}^{L-\mathcal{J}_{\{8\};\{1\}}}_{1;2^X} = \mathcal{D}^{R-\mathcal{J}_{\{8\};\{1\}}}_{1;2^X}$, $\lambda \in [2048/\gamma; 1))$*-quasi-contraction on* X; that is, $\forall_{\lambda \in [2048/\gamma;1)}\ \forall_{x,y \in X}\ \{8 \cdot D(T(x), T(y)) \leq \lambda J(x, y)\}$ *where*

$$D(U, W) = \max\left\{\sup_{u \in U} J(u, W), \sup_{w \in W} J(U, w)\right\}, \quad U, W \in 2^X. \tag{76}$$

Indeed, we see that this follows from (73)–(76) and from Cases 1–4 below.

Case 1. If $x, y \in [0; 3) \cup (4; 6]$, then $T(x) = T(y) = [1; 2] = U \subset E$ and $\sup_{u \in U}\{\inf_{w \in U} J(u, w)\} = \sup_{u \in U}\{J(u, u) = p(u, u) = 0\} = 0$. Thus $4D(T(x), T(y)) = 0 \leq \lambda J(x, y)$.

Case 2. If $x \in [0; 3) \cup (4; 6]$ and $y \in [3; 4]$, then $T(x) = [1; 2] = U \subset E$, $T(y) = (4; 6) = W \subset E$, and $\sup_{u \in U}\{\inf_{w \in W} J(u, w)\} = \sup_{u \in U}\{\inf_{w \in W}(w - u)^4\} = \sup_{u \in U}(4 - u)^4 = 81$ and $\sup_{w \in W}\{\inf_{u \in U} J(u, w)\} = \sup_{w \in W}\{\inf_{u \in U}(w - u)^4\} = \sup_{w \in W}(w - 2)^4 = 256$. Thus $8D(T(x), T(y)) = 2048$. On the other hand, $y \notin E$ which gives $J(x, y) = \gamma$. Therefore, $8D(T(x), T(y)) \leq \lambda J(x, y)$ whenever $2048 \leq \lambda\gamma$. This gives $2048/\gamma \leq \lambda < 1$ whenever $\gamma > \max\{2048; 81\}$.

Case 3. If $x \in [3; 4]$ and $y \in [0; 3) \cup (4; 6]$, then $T(x) = (4; 6) = U \subset E$ and $T(y) = [1; 2] = W \subset E$. Hence we obtain $\sup_{u \in U}\{\inf_{w \in W} J(u, w)\} = \sup_{u \in U}\{\inf_{w \in W} p(u, w)\} = \sup_{w \in W}\{\inf_{u \in U} J(u, w)\} = \sup_{w \in W}\{\inf_{u \in U} p(u, w)\} = 0$. Therefore, $8D(T(x), T(y)) = 0 \leq \lambda J(x, y)$.

Case 4. If $x, y \in [3; 4]$, then $T(x) = T(y) = (4; 6) = U \subset E$. Therefore $4D_1(T(x), T(y)) = 0 \leq \lambda J(x, y)$.

(3) *Property (26) holds; that is,* $\forall_{x \in X}\ \forall_{\beta \in (0;\infty)}\ \exists_{y \in T(x)}\ \{J(x, y) < J(x, T(x)) + \beta\}$. Indeed, this follows from Cases 1–4 below.

Case 1. If $x_0 = 0$ and $y_0 = 1 \in T(x_0) = [1; 2]$, then $J(x_0, y_0) = \gamma$, $J(x_0, T(x_0)) = \inf_{w \in [1;2]} J(x_0, w) = \gamma$, and $\forall_{\beta \in (0;\infty)}\ \{J(x_0, y_0) < J(x_0, T(x_0)) + \beta\}$.

Case 2. If $x_0 \in (0; 1]$ and $y_0 = 1 \in T(x_0) = [1; 2]$, then $J(x_0, y_0) = 1 - x_0$, $J(x_0, T(x_0)) = \inf_{w \in [1;2]} J(x_0, w) = 1 - x_0$, and $\forall_{\beta \in (0;\infty)}\ \{J(x_0, y_0) < J(x_0, T(x_0)) + \beta\}$.

Case 3. If $x_0 \in (1; 3) \cup (4; 6)$ and $y_0 = 1 \in T(x_0) = [1; 2]$, then $J(x_0, y_0) = 0$, $J(x_0, T(x_0)) = 0$, and $\forall_{\beta \in (0;\infty)}\ \{J(x_0, y_0) < J(x_0, T(x_0)) + \beta\}$.

Case 4. If $x_0 \in [3; 4]$ and $y_0 \in T(x_0) = (4; 6)$, then $J(x_0, y_0) = \gamma$, $p(x_0, T(x_0)) = \gamma$, and $\forall_{\beta \in (0;\infty)}\ \{J(x_0, y_0) < J(x_0, T(x_0)) + \beta\}$.

Case 5. If $x_0 = 6$ and $y_0 \in T(x_0) = [1; 2]$, then $J(x_0, y_0) = \gamma$, $p(x_0, T(x_0)) = \gamma$, and $\forall_{\beta \in (0;\infty)}\ \{J(x_0, y_0) < J(x_0, T(x_0)) + \beta\}$.

(4) (X, T) *is left and right* $\mathcal{J}_{\{8\};\{1\}}$*-admissible in each point* $w^0 \in X$. In fact, if $w^0 \in X$ and $(w^m : m \in \{0\} \cup \mathbb{N})$ are such that $\forall_{m \in \{0\} \cup \mathbb{N}}\ \{w^{m+1} \in T(w^m)\}$ and $\lim_{m \to \infty} \sup_{n > m} J(w^m, w^n) = 0$ $(\lim_{m \to \infty} \sup_{n > m} J(w^n, w^m) = 0)$, then $\forall_{m \geq 2}\ \{w^m \in [1; 2]\}$ and, consequently, by (72), $\forall_{w \in [2;6) \subset X}\ \{\lim_{m \to \infty} p(w, w^m) = 0\}$ $(\forall_{w \in (0;1] \subset X}\ \{\lim_{m \to \infty} p(w^m, w) = 0\})$. Hence, by (75) and (76), we get $\forall_{w \in [2;3) \cup (4;6) \subset X}\ \{\lim_{m \to \infty} J(w, w^m) = 0\}$ $(\forall_{w \in (0;1] \subset X}\ \{\lim_{m \to \infty} J(w^m, w) = 0\})$.

(5) (X, T) *is a left and right* $\mathcal{P}_{\{8\};\{1\}}$*-closed on* X. Indeed, let $(x_m : m \in \mathbb{N}) \subset T(X)$ be a left (right) $\mathcal{P}_{\{8\};\{1\}}$-converging sequence in X (thus $\mathrm{LIM}^{L-\mathcal{P}_{\{8\};\{1\}}}_{(x_m : m \in \mathbb{N})} \neq \varnothing$ $(\mathrm{LIM}^{R-\mathcal{P}_{\{8\};\{1\}}}_{(x_m : m \in \mathbb{N})} \neq \varnothing)$) and having subsequences $(v_m : m \in \mathbb{N})$ and $(u_m : m \in \mathbb{N})$ satisfying $\forall_{m \in \mathbb{N}}\ \{v_m \in T(u_m)\}$. Then $\forall_{m \geq 2}\ \{x_m \in [1; 2]\}$, $2 \in T(2)$ and $2 \in \mathrm{LIM}^{L-\mathcal{P}_{\{8\};\{1\}}}_{(x_m : m \in \mathbb{N})}$ $(1 \in T(1)$ and $1 \in \mathrm{LIM}^{R-\mathcal{P}_{\{8\};\{1\}}}_{(x_m : m \in \mathbb{N})})$.

(6) *All assumptions of Theorem 26 are satisfied.* This follows from (1)–(5) in Example 1.

We conclude that $\mathrm{Fix}(T) = [1; 2]$ and we have shown the following.

Claim A. $2 \in T(2)$ and $2 \in \mathrm{LIM}^{L-\mathcal{P}_{\{8\};\{1\}}}_{(w^m : m \in \{0\} \cup \mathbb{N})}$ for each $w^0 \in X$ and for each dynamic process $(w^m : m \in \{0\} \cup \mathbb{N})$ of the system (X, T).

Claim B. $1 \in T(1)$ and $1 \in \mathrm{LIM}^{R-\mathcal{P}_{\{8\};\{1\}}}_{(w^m : m \in \{0\} \cup \mathbb{N})}$ for each $w^0 \in X$ and for each dynamic process $(w^m : m \in \{0\} \cup \mathbb{N})$ of the system (X, T).

Example 2. Let X, $\mathcal{P}_{\{8\};\{1\}} = \{p\}$, and (X, T) be such as in Example 1.

(1) *For each* $\lambda \in [0; 1)$, *condition* $\forall_{x,y \in X}\ \{8D(T(x), T(y)) \leq \lambda p(x, y)\}$, *where* $D(U, W) = \max\{\sup_{u \in U} p(u, W), \sup_{w \in W} p(U, w)\}$, $U, W \in 2^X$, *does not hold*. Suppose that $\exists_{\lambda_0 \in [0;1)}\ \forall_{x,y \in X}\ \{8D(T(x), T(y)) \leq \lambda_0 p(x, y)\}$. Letting $x_0 = 2$ and $y_0 = 3$, it can be shown that $p(x_0, y_0) = 1$, $T(x_0) = [1; 2] = U$, $T(y_0) = (4; 6) = W$, $\sup_{u \in [1;2]} p(u, (4; 6)) = \sup_{u \in [1;2]}(4 - u)^4 = 3^4 = 81$, and $\sup_{w \in (4;6)} p([1; 2], w) = \sup_{w \in (4;6)}(w - 2)^4 = 4^4 = 256$. Therefore $2048 = 8D(T(x_0), T(y_0)) = 8\max\{81; 256\} \leq \lambda_0 p(x_0, y_0) = \lambda_0$, which is absurd.

Remark 34. We make the following remarks about Examples 1 and 2 and Theorem 26: (a) By Example 1, we observe that we may apply Theorem 26 for set-valued dynamic systems (X, T) in the left and right quasi-triangular space $(X, \mathcal{P}_{C;\mathcal{A}})$ with left and right family $\mathcal{J}_{C;\mathcal{A}}$ generated by $\mathcal{P}_{C;\mathcal{A}}$ where $\mathcal{J}_{C;\mathcal{A}} \neq \mathcal{P}_{C;\mathcal{A}}$. (b) By Example 2, we note, however, that we do not apply Theorem 26 in the quasi-triangular space $(X, \mathcal{P}_{C;\mathcal{A}})$ when $\mathcal{J}_{C;\mathcal{A}} = \mathcal{P}_{C;\mathcal{A}}$. (c) From (a) and (b) it follows that, in Theorem 26, the existence of left (right) families $\mathcal{J}_{C;\mathcal{A}}$ generated by $\mathcal{P}_{C;\mathcal{A}}$ and such that $\mathcal{J}_{C;\mathcal{A}} \neq \mathcal{P}_{C;\mathcal{A}}$ are essential.

Example 3. Let $X = (0;6)$, $\gamma > 0$, and

$$A = A_1 \cup A_2, \quad A_1 = (0;2], \ A_2 = [4;6). \tag{77}$$

Let $p : X^2 \to [0;\infty)$ be of the form

$$p(u,w) = \begin{cases} 0 & \text{if } A \cap \{u,w\} = \{u,w\}, \\ \gamma & \text{if } A \cap \{u,w\} \neq \{u,w\}, \end{cases} \tag{78}$$

and let $\mathcal{J}_{\{1\};\{1\}} = \mathcal{P}_{\{1\};\{1\}} = \{p\}$. Define the set-valued dynamic system (X,T) by

$$T(u) = \begin{cases} A_2 & \text{for } u \in (0;3), \\ A & \text{for } u = 3, \\ A_1 & \text{for } u \in (3;6). \end{cases} \tag{79}$$

(1) *$(X, \mathcal{P}_{\{1\};\{1\}})$ is quasi-triangular space.* See Example 2, Section 11.

(2) *(X,T) is a $(\mathcal{D}^{L-\mathcal{P}_{\{1\};\{1\}}}_{1;2^X}, \lambda \in [0;1))$-quasi-contraction on X*; that is, $\forall_{\lambda\in[0;1)} \forall_{x,y\in X} \{D^{L-\mathcal{P}_{\{1\};\{1\}}}_{1;2^X}(T(x),T(y)) \leq \lambda p(x,y)\}$. Indeed, if $x, y \in X$, then, by (77)–(79), $T(x), T(y) \subset A$ and $\max\{\sup_{u\in T(x)} p(u,T(y)), \sup_{w\in T(y)} p(T(x),w)\} = 0$.

(3) *Property (16) holds; that is,* $\forall_{x\in X} \forall_{\beta\in(0;\infty)} \exists_{y\in T(x)} \{p(x,y) < p(x,T(x)) + \beta\}$. Indeed, this follows from Cases 1–3 below.

Case 1. Let $x_0 \in (0;3)$ and $\beta \in (0;\infty)$ be arbitrary and fixed. If $y_0 \in T(x_0) = A_2$, then, by (78),

$$p(x_0,y_0) = p(x_0,T(x_0)) = \begin{cases} 0 & \text{if } x_0 \in A_1, \\ \gamma & \text{for } x_0 \in (0;3) \setminus A_1. \end{cases} \tag{80}$$

Therefore, $p(x_0,y_0) < p(x_0,T(x_0)) + \beta$.

Case 2. Let $x_0 = 3$ and let $\beta \in (0;\infty)$ be arbitrary and fixed. If $y_0 \in T(x_0) = A$, then, by (78), $p(x_0,y_0) = p(x_0,T(x_0)) = \gamma$. Therefore, $p(x_0,y_0) < p(x_0,T(x_0)) + \beta$.

Case 3. Let $x_0 \in (3;6)$ and $\beta \in (0;\infty)$ be arbitrary and fixed. If $y_0 \in T(x_0) = A_1$, then, by (78),

$$p(x_0,y_0) = p(x_0,T(x_0)) = \begin{cases} 0 & \text{if } x_0 \in A_2, \\ \gamma & \text{for } x_0 \in (3;6) \setminus A_2. \end{cases} \tag{81}$$

Therefore, $p(x_0,y_0) < p(x_0,T(x_0)) + \beta$.

(4) *(X,T) is left and right $\mathcal{P}_{\{1\};\{1\}}$-admissible in X.* Assuming that $w^0 \in X$ is arbitrary and fixed we prove that if the dynamic process $(w^m : m \in \{0\} \cup \mathbb{N})$ of (X,T) starting at w^0 is such that $\lim_{m\to\infty} \sup_{n>m} p(w^m,w^n) = 0$, then $\exists_{w\in X} \{\lim_{m\to\infty} p(w,w^m) = 0\}$. Indeed, if $w^0 \in X$, then, by (79), $\forall_{m\geq 1} \{w^m \in T(w^{m-1}) \subset A\}$ and, by (78), we immediately get $A = \mathrm{LIM}^{L-\mathcal{P}_{\{1\};\{1\}}}_{(w^m : m\in\{0\}\cup\mathbb{N})} = \mathrm{LIM}^{R-\mathcal{P}_{\{1\};\{1\}}}_{(w^m : m\in\{0\}\cup\mathbb{N})}$.

(5) *Set-valued dynamic system $(X, T^{[2]})$ is a left and right $\mathcal{P}_{\{1\};\{1\}}$-closed on X.* Indeed, if $(x_m : m \in \mathbb{N}) \subset T^{[2]}(X) = A$ is a left or right $\mathcal{P}_{\{1\};\{1\}}$-converging sequence in X and having subsequences $(v_m : m \in \mathbb{N})$ and $(u_m : m \in \mathbb{N})$ satisfying $\forall_{m\in\mathbb{N}} \{v_m \in T(u_m)\}$, then, by (77)–(79), we have that $\exists_{m_0\in\mathbb{N}} \forall_{m\geq m_0} \{x_m \in A\}$, $A = \mathrm{LIM}^{L-\mathcal{P}_{\{1\};\{1\}}}_{(x_m : m\in\mathbb{N})} = \mathrm{LIM}^{R-\mathcal{P}_{\{1\};\{1\}}}_{(x_m : m\in\{0\}\cup\mathbb{N})}$, and $\mathrm{Fix}(T^{[2]}) = A$.

(6) *For $(X, \mathcal{P}_{\{1\};\{1\}})$, $\mathcal{P}_{\{1\};\{1\}} = \{p\}$, $\mathcal{J}_{\{1\};\{1\}} = \mathcal{P}_{\{1\};\{1\}}$, and (X,T) defined by (77)–(79), all assumptions of Theorem 26 are satisfied.* This follows from (1)–(5) in Example 3.

We conclude that $\mathrm{Fix}(T^{[2]}) = A$ and we claim that if $w^0 \in X$, $w^1 \in T(w^0)$, and $w^2 = u \in T(w^1)$ are arbitrary and fixed, and $\forall_{m\geq 3} \{w^m = u\}$, then sequence $(w^m : m \in \{0\} \cup \mathbb{N})$ is a dynamic process of T starting at w^0 and left and right $\mathcal{P}_{\{1\};\{1\}}$-converging to each point of A. We observe also that $\mathrm{Fix}(T) = \emptyset$.

Example 4. Let $X = [0;6]$ and let $\mathcal{P}_{\{2\};\{1\}} = \{p\}$ where $p : X^2 \to [0;\infty)$ is of the form

$$p(u,w) = \begin{cases} 0 & \text{if } u \geq w, \\ (u-w)^2 & \text{if } u < w. \end{cases} \tag{82}$$

Define the set-valued dynamic system (X,T) by

$$T(u) = ([0;3) \cup (3;6]) \setminus \{u\} \quad \text{for } u \in [0;6]. \tag{83}$$

Let

$$E = [0;3) \cup (3;6] \tag{84}$$

and let $\mu \geq 36/4$ and $\mathcal{J}_{\{2\};\{1\}} = \{J\}$ where $J : X^2 \to [0;\infty)$ is of the form

$$J(u,w) = \begin{cases} p(u,w) & \text{if } E \cap \{u,w\} = \{u,w\}, \\ \mu & \text{if } E \cap \{u,w\} \neq \{u,w\}. \end{cases} \tag{85}$$

(1) *$\mathcal{J}_{\{2\};\{1\}}$ is not symmetric.* In fact, by (82), (84), and (85), $J(0,6) = 36$ and $J(6,0) = 0$.

(2) $\mathcal{J}_{\{2\};\{1\}} = \{J\} \in \mathbb{J}^L_{(X,\mathcal{P}_{\{2\};\{1\}})} \cap \mathbb{J}^R_{(X,\mathcal{P}_{\{2\};\{1\}})}$. See Theorem 14.

(3) *(X,T) is a $(D = \mathcal{D}^{L-\mathcal{J}_{\{2\};\{1\}}}_{1;2^X}, \lambda \in [0;1))$-contraction on X; that is, $\forall_{x,y\in X} \{2 \cdot D(T(x),T(y)) \leq \lambda J(x,y)\}$ where $\lambda \in [0;1)$ and*

$$D(U,W) = \max\left\{\sup_{u\in U} J(u,W), \sup_{w\in W} J(U,w)\right\}, \quad U, W \in 2^X. \tag{86}$$

Indeed, we see that this follows from (1), (2) in Example 4, and from Cases 1–4 below.

Case 1. Let $x, y \in [0;3) \cup (3;6]$. Then $x, y \in E$, $T(x) = ([0;3) \cup (3;6]) \setminus \{x\} = U \subset E$, and $T(y) = ([0;3) \cup (3;6]) \setminus \{y\} = W \subset E$. If $u \in U$, then we have $W = W^u \cup W_u$ and

$$\inf_{w \in W} J(u,w) \leq \begin{cases} \inf_{w \in W^u} q(u,w) = 0 & \text{if } W^u = \{w \in W : u \geq w\} \neq \varnothing, \\ \inf_{w \in W_u} (u-w)^2 = 0 & \text{if } W_u = \{w \in W : u < w\} \neq \varnothing \end{cases} \quad (87)$$

and if $w \in W$, then we have $U = U^w \cup U_w$ and

$$\inf_{u \in U} J(u,w) \leq \begin{cases} \inf_{u \in U^w} q(u,w) = 0 & \text{if } U^u = \{u \in U : u \geq w\} \neq \varnothing, \\ \inf_{u \in U_w} (u-w)^2 = 0 & \text{if } U_u = \{u \in U : u < w\} \neq \varnothing. \end{cases} \quad (88)$$

By (86), $2D(T(x), T(y)) = 0 \leq \lambda J(x,y)$.

Case 2. If $x = y = 3$, then $J(x,y) = \mu$ and $T(x) = T(y) = [0;3) \cup (3;6] = U \subset E$. Therefore, $2D(T(x), T(y)) = 2D(U,U) = 0 \leq \lambda J(x,y)$.

Case 3. If $x \in [0;3) \cup (3;6]$ and $y = 3$, then $x \in E$, $y \notin E$, $J(x,y) = \mu$, $T(x) = ([0;3) \cup (3;6]) \setminus \{x\} = U \subset E$, and $T(y) = [0;3) \cup (3;6] = W \subset E$. We see that $\sup_{u \in U}\{\inf_{w \in W} J(u,w)\} = 0$ since if $u \in U$, then also $w = u \in W$ and $\inf_{w \in W} J(u,w) = q(u,u) = 0$. Next, we see that $\sup_{w \in W}\{\inf_{u \in U} J(u,w)\} = 0$ since if $w \in W$, then $U = U^w \cup U_w$ and

$$\inf_{u \in U} J(u,w) \leq \begin{cases} \inf_{u \in U^w} q(u,w) = 0 & \text{if } U^u = \{u \in U : u \geq w\} \neq \varnothing, \\ \inf_{u \in U_w} (u-w)^2 = 0 & \text{if } U_u = \{u \in U : u < w\} \neq \varnothing. \end{cases} \quad (89)$$

Thus $2D(T(x), T(y)) = 0 \leqslant \lambda J(x,y)$.

Case 4. If $x = 3$ and $y \in [0;3) \cup (3;6]$, then $x \notin E$, $y \in E$, $J(x,y) = \mu$, $T(x) = [0;3) \cup (3;6] = U \subset E$, $T(y) = ([0;3) \cup (3;6]) \setminus \{y\} = W \subset E$, and $\sup_{u \in U}\{\inf_{w \in W} J(u,w)\} = 0$ since, for $u \in U$,

$$\inf_{w \in W} J(u,w) \leq \begin{cases} \inf_{w \in W^u} q(u,w) = 0 & \text{if } W^u = \{w \in W : u \geq w\} \neq \varnothing, \\ \inf_{w \in W_u} (u-w)^2 = 0 & \text{if } W_u = \{w \in W : u < w\} \neq \varnothing \end{cases} \quad (90)$$

and $\sup_{w \in W}\{\inf_{u \in U} J(u,w)\} = 0$ since $\inf_{u \in U} J(u,w) = J(w,w) = 0$ for $w \in W$. Thus $2D(T(x), T(y)) = 0 \leq \lambda J(x,y)$.

(4) *Property (26) holds; that is,* $\forall_{x \in X}\ \forall_{\gamma \in (0;\infty)} \exists_{y \in T(x)}\ \{J(x,y) < J(x, T(x)) + \gamma\}$. Indeed, this follows from Cases 1–3 below.

Case 1. Let $x_0 \in [0;3)$ and $\gamma \in (0;\infty)$ be arbitrary and fixed. If $y_0 \in T(x_0) = ([0;3) \cup (3;6]) \setminus \{x_0\} = W$ is such that $x_0 < y_0 < 3$, then $J(x_0, y_0) = (x_0 - y_0)^2$ and $J(x_0, T(x_0)) = \inf_{w \in W} J(x_0, w) = 0$ since

$$\inf_{w \in W} J(x_0, w) \leq \begin{cases} \inf_{w \in W^{x_0}} q(x_0, w) = 0 & \text{if } W^{x_0} = \{w \in W : x_0 \geq w\} \neq \varnothing, \\ \inf_{w \in W_{x_0}} (x_0 - w)^2 = 0 & \text{if } W_{x_0} = \{w \in W : x_0 < w\} \neq \varnothing. \end{cases} \quad (91)$$

Then we see that $J(x_0, y_0) = (x_0 - y_0)^2 < \gamma$ implies $y_0 < x_0 + \gamma^{1/2}$. From this we conclude that if $y_0 \in (x_0; \min\{3, x_0 + \gamma^{1/2}\})$, then $J(x_0, y_0) < J(x_0, T(x_0)) + \gamma$.

Case 2. Let $x_0 = 3$. Assume that $y_0 \in T(x_0) = [0;3) \cup (3;6]$ is arbitrary and fixed. Then $J(x_0, y_0) = \mu$, $J(x_0, T(x_0)) = \inf_{w \in [0;3) \cup (3;6]} J(x_0, w) = \mu$ and, for each $\gamma \in (0;\infty)$, $J(x_0, y_0) < J(x_0, T(x_0)) + \gamma$.

Case 3. Let $x_0 \in (3;6]$ and $\gamma \in (0;\infty)$ be arbitrary and fixed. If $y_0 \in T(x_0) = ([0;3) \cup (3;6]) \setminus \{x_0\} = W$ is such that $3 < y_0 < x_0$, then $J(x_0, y_0) = 0$ and, analogously as in Case 1, we get $J(x_0, T(x_0)) = \inf_{w \in W} J(x_0, w) = 0$. Therefore, $J(x_0, y_0) < J(x_0, T(x_0)) + \gamma$.

(5) (X, T) *is left* $\mathcal{J}_{\{2\};\{1\}}$*-admissible in* X. Assuming that $w^0 \in X$ is arbitrary and fixed we prove that if the dynamic process $(w^m : m \in \{0\} \cup \mathbb{N})$ of (X,T) starting at w^0 is such that $\lim_{m \to \infty} \sup_{n > m} J(w^m, w^n) = 0$, then $\exists_{w \in X}\ \{\lim_{m \to \infty} J(w, w^m) = 0\}$. We consider the following cases.

Case 1. If $w^0 \in [0;3) \cup (3;6]$, then $w^1 \in T(w^0) = ([0;3) \cup (3;6]) \setminus \{w^0\}$ and $\forall_{m \geq 2}\ \{w^m \in T(w^{m-1}) \subset [0;3) \cup (3;6]\}$ and using (82) we immediately get $6 \in \mathrm{LIM}^{L-\mathcal{J}_{\{2\};\{1\}}}_{(w^m : m \in \{0\} \cup \mathbb{N})}$.

Case 2. If $w^0 = 3$, then $w^1 \in T(w^0) = [0;3) \cup (3;6]$, $w^2 \in T(w^1) = ([0;3) \cup (3;6]) \setminus \{w^1\}$, and $\forall_{m \geqslant 3}\ \{w^m \in T(w^{m-1}) \subset [0;3) \cup (3;6]\}$ and using (82) we also immediately get $6 \in \mathrm{LIM}^{L-\mathcal{J}_{\{2\};\{1\}}}_{(w^m : m \in \{0\} \cup \mathbb{N})}$.

This shows that $6 \in \mathrm{LIM}^{L-\mathcal{J}_{\{2\};\{1\}}}_{(w^m : m \in \{0\} \cup \mathbb{N})}$ for each $w^0 \in X$ and for each dynamic process $(w^m : m \in \{0\} \cup \mathbb{N})$ of the system (X,T); we see that here property $\lim_{m \to \infty} \sup_{n > m} J(w^m, w^n) = 0$ of $(w^m : m \in \{0\} \cup \mathbb{N})$ is not required.

(6) *Set-valued dynamic system* $(X, T^{[2]})$ *is a left* $\mathcal{P}_{\{2\};\{1\}}$*-quasi-closed on* X. Indeed, if $(x_m : m \in \mathbb{N}) \subset T^{[2]}(X) = [0;3) \cup (3;6]$ is a left $\mathcal{P}_{\{2\};\{1\}}$-converging sequence in X and having subsequences $(v_m : m \in \mathbb{N})$ and $(u_m : m \in \mathbb{N})$ satisfying $\forall_{m \in \mathbb{N}}\ \{v_m \in T(u_m)\}$, then, by (83), we have that $\exists_{m_0 \in \mathbb{N}}\ \forall_{m \geq m_0}\ \{x_m \in [0;3) \cup (3;6]\}$. Therefore, in particular, $6 \in \mathrm{LIM}^{L-\mathcal{P}_{\{2\};\{1\}}}_{(x_m : m \in \mathbb{N})}$ and $6 \in T^{[2]}(6)$.

(7) *For* $\mathcal{P}_{\{2\};\{1\}} = \{p\}$, $\mathcal{J}_{\{2\};\{1\}} = \{J\}$, *and* (X,T) *defined by (82)–(85), all assumptions of Theorem 26 in the case of "left" are satisfied*. This follows from (1)–(6) in Example 4.

We conclude that $\mathrm{Fix}(T^{[2]}) = [0;3) \cup (3;6]$ and we claim that $6 \in T^{[2]}(6)$ and that $6 \in \mathrm{LIM}^{L-\mathcal{P}_{\{2\};\{1\}}}_{(w^m : m \in \{0\} \cup \mathbb{N})}$ for each $w^0 \in X$

and for each dynamic process $(w^m : m \in \{0\} \cup \mathbb{N})$ of the system (X, T). We observe also that $\mathrm{Fix}(T) = \varnothing$.

12. Example Illustrating Theorem 32

Example 1. Let $X = (0; 6)$, A, and $\mathcal{J}_{\{1\};\{1\}} = \mathcal{P}_{\{1\};\{1\}} = \{p\}$ be as in Example 3. Define the single-valued dynamic system (X, T) by

$$T(u) = \begin{cases} 4 & \text{for } u \in (0; 3), \\ 2 & \text{for } u \in [3; 6). \end{cases} \tag{92}$$

(1) (X, T) *is a* $(\mathcal{D}_{1;X}^{L-\mathcal{P}_{\{1\};\{1\}}}, \lambda \in [0; 1))$*-quasi-contraction on* X; that is, $\forall_{\lambda \in [0;1)} \forall_{x,y \in X} \{D_{1;X}^{L-\mathcal{P}_{\{1\};\{1\}}}(T(x), T(y)) = \max\{p(T(x), T(y)), p(T(y), T(x))\} \leq \lambda p(x, y)\}$ *and* $\mathrm{Fix}(T) = \varnothing$. Indeed, we see that if $x, y \in X$, then $T(x), T(y) \in A$ and, by (77) and (78), $D_{1;X}^{L-\mathcal{P}_{\{1\};\{1\}}}(T(x), T(y)) = 0 \leq \lambda p(x, y)$.

(2) (X, T) *is left and right* $\mathcal{P}_{\{1\};\{1\}}$*-admissible in* X. Assume that $w^0 \in X$ is arbitrary and fixed, $(w^m : m \in \{0\} \cup \mathbb{N})$ satisfies $\forall_{m \in \{0\} \cup \mathbb{N}} \{w^{m+1} = T(w^m)\}$, and $\lim_{m \to \infty} \sup_{n > m} p(w^m, w^n) = 0$. Then, by (92) and (78), we have $\forall_{m \in \mathbb{N}} \{w^m \in A\}$. This gives $A = \mathrm{LIM}_{(w^m : m \in \{0\} \cup \mathbb{N})}^{L-\mathcal{P}_{\{1\};\{1\}}} = \mathrm{LIM}_{(w^m : m \in \{0\} \cup \mathbb{N})}^{R-\mathcal{P}_{\{1\};\{1\}}}$.

(3) *Single-valued dynamic system* $(X, T^{[2]})$ *is a left and right* $\mathcal{P}_{\{1\};\{1\}}$*-closed on* X. Indeed, if $(x_m : m \in \mathbb{N}) \subset T^{[2]}(X) = \{2, 4\}$ is a left $\mathcal{P}_{\{1\};\{1\}}$-converging sequence in X and having subsequences $(v_m : m \in \mathbb{N})$ and $(u_m : m \in \mathbb{N})$ satisfying $\forall_{m \in \mathbb{N}} \{v_m = T^{[2]}(u_m)\}$, then, by (77), (78), and (92), we have that $A = \mathrm{LIM}_{(x_m : m \in \mathbb{N})}^{L-\mathcal{P}_{\{1\};\{1\}}}$. In particular, $2 = T^{[2]}(2) \in \mathrm{LIM}_{(x_m : m \in \mathbb{N})}^{L-\mathcal{P}_{\{1\};\{1\}}}$ and $4 = T^{[2]}(4) \in \mathrm{LIM}_{(x_m : m \in \mathbb{N})}^{L-\mathcal{P}_{\{1\};\{1\}}}$.

(4) *Property* (56) *holds*. Indeed, we have $\forall_{v \in \mathrm{Fix}(T^{[2]}) = \{2,4\}} \{p(v, T(v)) = p(T(v), v) = 0\}$ since $T(2) = 4$, $T(4) = 2$, and $T(\{2, 4\}) = \{2, 4\} \subset A$.

(5) $\mathcal{P}_{\{1\};\{1\}} = \{p\}$ *is not separating on* X. Indeed, if $u, w \in X / A$, then $p(u, w) = p(w, u) = \gamma > 0$.

(6) *For* $\mathcal{P}_{\{1\};\{1\}} = \{p\}$, (X, T), *and* $\mathcal{J}_{\{1\};\{1\}} = \mathcal{P}_{\{1\};\{1\}}$ *defined by (77), (78), and (79) parts (B1) and (B2) of Theorem 32 hold but part (B3) of Theorem 32 does not hold.* This follows from (1)–(5) in Example 1.

13. Concluding Remarks

Remark 1. In Theorems 5 and 6 the following play an important role: (i) Distances d and H^d, as metrics, satisfy conditions (A) of Definition 1 on X and $\mathcal{CB}(X)$, respectively. (ii) (X, d) and $(\mathcal{CB}(X), H^d)$, as metric spaces, are topological and Hausdorff spaces and the completeness of (X, d) implies completeness of $(\mathcal{CB}(X), H^d)$. (iii) The continuity of d and H^d on $X \times X$ and $\mathcal{CB}(X) \times \mathcal{CB}(X)$, respectively; (iv) The continuity of maps $T : (X, d) \to (X, d)$ and $T : (X, d) \to (\mathcal{CB}(X), H^d)$ (as consequences of contractive properties defined in (1) and (3), resp.); (v) In Theorem 6 the assumption that, for each $x \in X$, $T(x) \in \mathcal{CB}(X)$.

Remark 2. Conclusions in Theorems 5 and 6 concern only fixed points but not periodic points; this is a consequence of separability of spaces (X, d) and $(\mathcal{CB}(X), H^d)$ and also continuity of T.

Remark 3. In Theorems 26 and 32, properties concening the spaces and maps such as mentioned above generally need not hold, since spaces $(X, \mathcal{P}_{C;\mathcal{A}})$ with left (right) families $\mathcal{J}_{C;\mathcal{A}}$ generated by $\mathcal{P}_{C;\mathcal{A}}$ are very general, which is an obstruction to use Nadler's and Banach's reasoning. Theorems 26 and 32 show how to rectify this situation and are obtained without restrictively required assumptions and with conclusions more profound as in the well known results of this sort existing in the literature.

Conflict of Interests

The author declares that he has no conflict of interests regarding the publication of this paper.

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Research Article

New Approach to Fractal Approximation of Vector-Functions

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This paper introduces new approach to approximation of continuous vector-functions and vector sequences by fractal interpolation vector-functions which are multidimensional generalization of fractal interpolation functions. Best values of fractal interpolation vector-functions parameters are found. We give schemes of approximation of some sets of data and consider examples of approximation of smooth curves with different conditions.

1. Introduction

It is well known that interpolation and approximation are an important tool for interpretation of some complicated data. But there are multitudes of interpolation methods using several families of functions: polynomial, exponential, rational, trigonometric, and splines to name a few. Still it should be noted that all these conventional nonrecursive methods produce interpolants that are differentiable a number of times except possibly at a finite set of points. But, in many situations, we deal with irregular forms, which can not be approximate with desired precision. Fractal approximation became a suitable tool for that purpose. This tool was developed and studied in [1–3].

We know that such curves as coastlines, price graphs, encephalograms, and many others are fractals since their Hausdorff-Besicovitch dimension is greater than unity. To approximate them, we use fractal interpolation curves [1] and their generalizations [4] instead of canonical smooth functions (polynomials and splines).

This paper is multidimensional generalization of [5]. In Section 2, we consider fractal interpolation vector-functions which depend on several matrices of parameters. Example of such functions is given. In Section 3, we set the optimization problem for approximation of vector-function from L_2 by fractal approximation vector-functions. We find best values of matrix parameters by means of matrix differential calculus. Section 4 illustrates some examples.

2. Fractal Interpolation Vector-Functions

Let $[a,b] \subset \mathbb{R}$ be a nonempty interval; let $1 < N \in \mathbb{N}$ and $\{(t_n, \mathbf{x}_n) \in [a,b] \times \mathbb{R}^M \mid a = t_0 < t_1 < \cdots < t_{N-1} < t_N = b\}$ be the interpolation points. For all $n = \overline{1,N}$, consider affine transformation

$$A_n : \mathbb{R}^{M+1} \longrightarrow \mathbb{R}^{M+1},$$
$$A_n \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} := \begin{pmatrix} a_n & \mathbf{0} \\ \mathbf{c}_n & \mathbf{D}_n \end{pmatrix} \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} + \begin{pmatrix} e_n \\ \mathbf{f}_n \end{pmatrix}. \quad (1)$$

Henceforth, small bold letters denote columns (rows) of length M and big bold letters denote matrices of $M \times M$.

Require that for all n the following conditions hold true:

$$A_n (t_0, \mathbf{x}_0) = (t_{n-1}, \mathbf{x}_{n-1}), \qquad A_n (t_N, \mathbf{x}_N) = (t_n, \mathbf{x}_n). \quad (2)$$

Then,

$$\begin{aligned} a_n t_0 + e_n &= t_{n-1}, \\ a_n t_N + e_n &= t_n, \\ \mathbf{c}_n t_0 + \mathbf{D}_n \mathbf{x}_0 + \mathbf{f}_n &= \mathbf{x}_{n-1}, \\ \mathbf{c}_n t_N + \mathbf{D}_n \mathbf{x}_N + \mathbf{f}_n &= \mathbf{x}_n. \end{aligned} \tag{3}$$

Solving the system, we have

$$\begin{aligned} a_n &= \frac{t_n - t_{n-1}}{b-a}, \\ e_n &= \frac{b t_{n-1} - a t_n}{b-a}, \\ \mathbf{c}_n &= \frac{\mathbf{x}_n - \mathbf{x}_{n-1} - \mathbf{D}_n (\mathbf{x}_N - \mathbf{x}_0)}{b-a}, \\ \mathbf{f}_n &= \frac{b\mathbf{x}_{n-1} - a\mathbf{x}_n - \mathbf{D}_n (b\mathbf{x}_0 - a\mathbf{x}_N)}{b-a}, \end{aligned} \tag{4}$$

where matrices $\{\mathbf{D}_n\}_{n=1}^N$ are considered as parameters.

Remark 1. Notice that $\sum_{n=1}^N a_n = 1$.

Also notice that for all n operator A_n takes straight segment between $(t_0, \mathbf{x}_0)$ and $(t_N, \mathbf{x}_N)$ to straight segment which connects points of interpolation $(t_{n-1}, \mathbf{x}_{n-1})$ and $(t_n, \mathbf{x}_n)$.

Let $\mathscr{K}$ be a space of nonempty compact subsets of $\mathbb{R}^{M+1}$, with Hausdorff metric. Define the Hutchinson operator [6]

$$\Phi : \mathscr{K} \longrightarrow \mathscr{K}, \quad \Phi(E) = \bigcup_{n=1}^N A_n(E). \tag{5}$$

By the condition (2) Hutchinson operator Φ takes a graph of any continuous vector-function on segment $[a,b]$ to a graph of a continuous vector-function on the same segment. Thus, Φ can be treated as operator on the space of continuous vector-functions $(C[a,b])^M$.

For all $n = \overline{1,N}$, denote

$$\begin{aligned} p_n &: [a,b] \longrightarrow [t_{n-1}, t_n], \quad p_n(t) := a_n t + e_n, \\ \mathbf{q}_n &: [a,b] \longrightarrow \mathbb{R}^M, \quad \mathbf{q}_n(t) := \mathbf{c}_n t + \mathbf{f}_n. \end{aligned} \tag{6}$$

In (1), substitute $\mathbf{x}$ to vector-function $\mathbf{g}(t)$. We have that Φ acts on $(C[a,b])^M$ according to

$$\begin{aligned} &(\Phi\mathbf{g})(t) \\ &= \sum_{n=1}^N \left(\left(\mathbf{q}_n \circ p_n^{-1}\right)(t) + \mathbf{D}_n \left(\mathbf{g} \circ p_n^{-1}\right)(t) \right) \chi_{[t_{n-1},t_n]}(t). \end{aligned} \tag{7}$$

Suppose that we consider all matrices $\mathbf{D}_n$ as linear operators on $\mathbb{R}^M$. Furthermore, they are contractive mappings; that is, constant $c \in [0,1)$ exists such that for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^M$ and $n = \overline{1,N}$ we have

$$\left|\mathbf{D}_n(\mathbf{v}) - \mathbf{D}_n(\mathbf{w})\right| \le c\,|\mathbf{v} - \mathbf{w}|. \tag{8}$$

Then, from (7), it follows that operator Φ is contraction with contraction coefficient c on Banach space $((C[a,b])^M, \|\cdot\|_\infty)$, where $\|\mathbf{g}(t) - \mathbf{h}(t)\|_\infty := \sup\{t \in [a,b] : |\mathbf{g}(t) - \mathbf{h}(t)|\}$. By the fixed-point theorem, there exists unique vector-function $\mathbf{g}^\star \in (C[a,b])^M$ such that $\Phi\mathbf{g}^\star = \mathbf{g}^\star$ and for all $\mathbf{g} \in (C[a,b])^M$ we have

$$\lim_{k\to\infty} \left\|\Phi^k(\mathbf{g}) - \mathbf{g}^\star\right\|_\infty = 0. \tag{9}$$

Function $\mathbf{g}^\star$ is called fractal interpolation vector-function. It is easy to notice that if $\mathbf{g} \in (C[a,b])^M$, $\mathbf{g}(t_0) = \mathbf{x}_0$, and $\mathbf{g}(t_N) = \mathbf{x}_N$, then $\Phi(\mathbf{g})$ passes through points of interpolation. In this case functions $\Phi^k(\mathbf{g})$ are called prefractal interpolation vector-functions of order k.

Example 2. Figure 1 shows fractal interpolation vector-function of plane. Here $t_0 = -1$, $t_1 = 0$, and $t_2 = 1$ and $x_0 = (1,-1)$, $x_1 = (0,0)$, and $x_2 = (1,1)$. Values of matrices $\mathbf{D}_1$ and $\mathbf{D}_2$ are

$$\begin{pmatrix} -\frac{1}{4} & \frac{3}{4} \\ -\frac{1}{4} & \frac{1}{2} \end{pmatrix}, \quad \begin{pmatrix} -\frac{1}{4} & -\frac{3}{4} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}. \tag{10}$$

3. Approximation

Henceforth, we assume that for all $n = \overline{1,N}$ linear operator $\mathbf{D}_n$ is contractive mapping with contraction coefficient $c \in [0,1)$. We approximate vector-function $\mathbf{g} \in (C[a,b])^M$ by fractal interpolation vector-function $\mathbf{g}^\star$ constructed on points of interpolation $\{(t_n, \mathbf{x}_n)\}_{n=0}^N$. Thus, we need to fit matrix parameters $\mathbf{D}_n$ to minimize the distance between $\mathbf{g}$ and $\mathbf{g}^\star$.

We use methods that have been developed for fractal image compression [7]. Denote Banach space of square integrated vector-functions on segment as $(L_2^M[a,b], \|\cdot\|_2)$, where norm $\|\cdot\|_2$ defines

$$\|\mathbf{g}\|_2 = \sqrt{\int_a^b |\mathbf{g}(t)|^2 \,dt}. \tag{11}$$

Then from (7) and (8) and Remark 1 it follows that for all $\mathbf{g}, \mathbf{h} \in L_2^M[a,b]$

$$\begin{aligned} &\|\Phi\mathbf{g} - \Phi\mathbf{h}\|_2^2 \\ &= \int_a^b |\Phi\mathbf{g} - \Phi\mathbf{h}|^2 \,dt \\ &= \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \left|\mathbf{D}_n \circ (\mathbf{g} - \mathbf{h}) \circ u_n^{-1}(t)\right|^2 dt \\ &= \sum_{n=1}^N a_n \int_a^b \left|\mathbf{D}_n \circ (\mathbf{g} - \mathbf{h})(t)\right|^2 dt \\ &\le \sum_{n=1}^N a_n c^2 \int_a^b |(\mathbf{g} - \mathbf{h})(t)|^2 \,dt = c^2 \|\mathbf{g} - \mathbf{h}\|_2^2. \end{aligned} \tag{12}$$

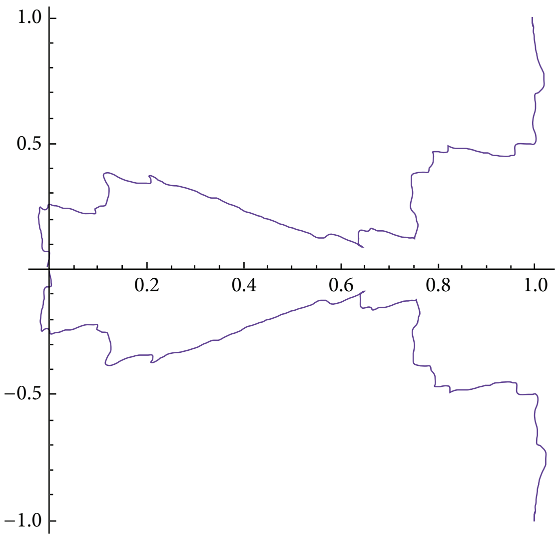


FIGURE 1: Fractal interpolation vector-function $\mathbf{g}^{\star}$.

Thus, $\Phi : L_2^M[a,b] \rightarrow L_2^M[a,b]$ is a contractive operator and $\mathbf{g}^{\star}$ is its fixed point.

Instead of minimizing $\|\mathbf{g}-\mathbf{g}^{\star}\|_2$ we minimize $\|\mathbf{g}-\Phi\mathbf{g}\|_2$ that makes the problem of optimization much easier. The collage theorem provides validity of such approach [8].

Theorem 3. *Let (X,d) be complete metric space and $T : X \rightarrow X$ is contractive mapping with contraction coefficient $c \in [0,1)$ and fixed point $x^{\star}$. Then*

$$d(x,x^{\star}) \leq \frac{d(x,T(x))}{1-c} \tag{13}$$

for all $x \in X$.

Considering (4) and (6), rewrite (7)

$$(\Phi\mathbf{g})(t) = \sum_{n=1}^{N} \left(\mathbf{u}_n(t) + \mathbf{D}_n\left(\mathbf{g}\circ w_n(t) - \mathbf{v}_n(t)\right)\right)\chi_{[t_{n-1},t_n]}(t), \tag{14}$$

where

$$\begin{aligned} \mathbf{u}_n(t) &= \frac{(\mathbf{x}_n-\mathbf{x}_{n-1})\,t + (t_n\mathbf{x}_{n-1}-t_{n-1}\mathbf{x}_n)}{t_n-t_{n-1}}, \\ \mathbf{v}_n(t) &= \frac{(\mathbf{x}_N-\mathbf{x}_0)\,t + (t_n\mathbf{x}_0-t_{n-1}\mathbf{x}_N)}{t_n-t_{n-1}}, \\ w_n(t) &= \frac{(b-a)\,t + (t_n a-t_{n-1}b)}{t_n-t_{n-1}}. \end{aligned} \tag{15}$$

Thus, we minimize the functional

$$\begin{aligned} &\|\mathbf{g}-\Phi\mathbf{g}\|_2^2 \\ &= \sum_{n=1}^{N}\int_{t_{n-1}}^{t_n} \left|\mathbf{g}(t)-\mathbf{u}_n(t)-\mathbf{D}_n\left(\mathbf{g}\circ w_n(t)-\mathbf{v}_n(t)\right)\right|^2 \mathrm{d}t. \end{aligned} \tag{16}$$

Lemma 4. *Let $\mathbf{f},\mathbf{h} \in L_2^M[a,b]$ be square integrated vector-functions. Suppose that matrix $\int_a^b \mathbf{h}\mathbf{h}^T \mathrm{d}t$ is nondegenerated. Matrix integration is implied to be componentwise. Then, the functional*

$$\Psi : \mathbb{R}^{M\times M} \longrightarrow \mathbb{R}, \quad \Psi(\mathbf{X}) = \int_a^b |\mathbf{f}-\mathbf{X}\mathbf{h}|^2\, \mathrm{d}t \tag{17}$$

reaches its minimum in $\mathbf{X} = \int_a^b \mathbf{f}\mathbf{h}^T \mathrm{d}t\,(\int_a^b \mathbf{h}\mathbf{h}^T \mathrm{d}t)^{-1}$.

Proof. To prove it, we use matrix differential calculus [9].

Consider

$$\begin{aligned} \mathrm{d}\Psi(\mathbf{X},\mathbf{U}) &= \mathrm{d}\left(\int_a^b (\mathbf{f}-\mathbf{X}\mathbf{h})^T(\mathbf{f}-\mathbf{X}\mathbf{h})\,\mathrm{d}t\right)\mathbf{U} \\ &= \mathrm{d}\left(\int_a^b \left(\mathbf{f}^T\mathbf{f}-\mathbf{h}^T\mathbf{X}^T\mathbf{f}-\mathbf{f}^T\mathbf{X}\mathbf{h}+\mathbf{h}^T\mathbf{X}^T\mathbf{X}\mathbf{h}\right)\mathrm{d}t\right)\mathbf{U} \\ &= \int_a^b \left(-\mathbf{h}^T\mathbf{U}^T\mathbf{f}-\mathbf{f}^T\mathbf{U}\mathbf{h}+\mathbf{h}^T\mathbf{U}^T\mathbf{X}\mathbf{h}+\mathbf{h}^T\mathbf{X}^T\mathbf{U}\mathbf{h}\right)\mathrm{d}t \\ &= 2\int_a^b \left(-\mathbf{h}^T\mathbf{U}^T\mathbf{f}+\mathbf{h}^T\mathbf{U}^T\mathbf{X}\mathbf{h}\right)\mathrm{d}t. \end{aligned} \tag{18}$$

Necessary condition of existence of functional Ψ extremum is $\mathrm{d}\Psi(\mathbf{X},\mathbf{U}) = 0$ for all $\mathbf{U} \in \mathbb{R}^{M\times M}$. Since there is U-linearity of functional $\mathrm{d}\Psi(\mathbf{X},\mathbf{U})$, it is sufficient to prove $\mathrm{d}\Psi(\mathbf{X},\mathbf{U}) = 0$ only for matrices $\mathbf{U}$ that consist of M^2-1 zeros and one unity. Therefore, we have M^2 expressions for finding coefficients of matrix $\mathbf{X}$. In matrix form these expressions are as follows:

$$\int_a^b \mathbf{f}\mathbf{h}^T \mathrm{d}t = \int_a^b \mathbf{X}\mathbf{h}\mathbf{h}^T \mathrm{d}t, \tag{19}$$

from which

$$\mathbf{X} = \int_a^b \mathbf{f}\mathbf{h}^T \mathrm{d}t\left(\int_a^b \mathbf{h}\mathbf{h}^T \mathrm{d}t\right)^{-1}. \tag{20}$$

Hence,

$$\mathrm{d}^2\Psi(\mathbf{X},\mathbf{U}) = 2\int_a^b \mathbf{h}^T\mathbf{U}^T\mathbf{U}\mathbf{h}\,\mathrm{d}t = 2\int_a^b |\mathbf{U}\mathbf{h}|^2\,\mathrm{d}t \geq 0, \tag{21}$$

and then functional Ψ is convex one. Thus, the value $\mathbf{X}$ is absolute minimum of Ψ. □

From Lemma 4, it follows that functional (16) reaches minimum when

$$\begin{aligned} \mathbf{D}_n = &\int_{t_{n-1}}^{t_n} \left(\mathbf{g}(t)-\mathbf{u}_n(t)\right)\left(\mathbf{g}\circ w_n(t)-\mathbf{v}_n(t)\right)^T \mathrm{d}t \\ &\cdot\left(\int_{t_{n-1}}^{t_n} \left(\mathbf{g}\circ w_n(t)-\mathbf{v}_n(t)\right)\left(\mathbf{g}\circ w_n(t)-\mathbf{v}_n(t)\right)^T \mathrm{d}t\right)^{-1}. \end{aligned} \tag{22}$$

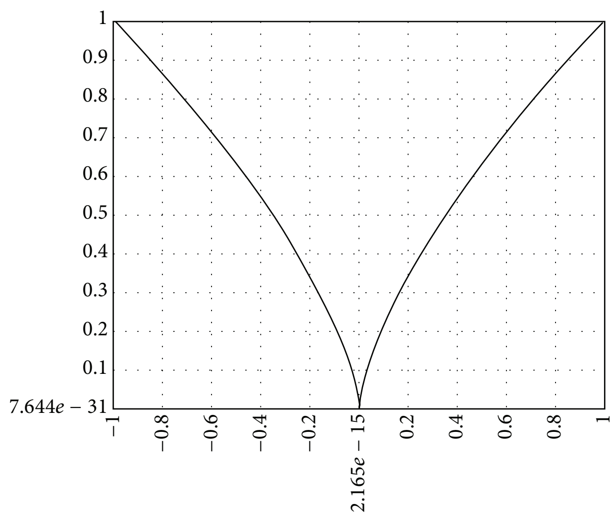


FIGURE 2: Vector-function $\mathbf{g}(t) = (t^2, t^3)$ and fractal interpolation vector-function $\mathbf{g}^\star$ completely identical.

Example 5. Let us approximate vector-function $\mathbf{g}(t) = (t^2, t^3)$ on segment $[-1, 1]$ by the fractal interpolation vector-function constructed on values of $\mathbf{g}(t)$ in points $t_0 = -1$, $t_1 = 0$, and $t_2 = 1$ and $x_0 = (1, -1)$, $x_1 = (0, 0)$, and $x_2 = (1, 1)$ (see Figure 2). Then,

$$\begin{gathered} a_1 = a_2 = \frac{1}{2}, \\ e_1 = -\frac{1}{2}, \qquad e_2 = \frac{1}{2}, \\ \mathbf{u}_1 = (-t, t), \qquad \mathbf{u}_2 = (t, t), \\ \mathbf{v}_1 = (1, 1 + 2t), \qquad \mathbf{v}_2 = (1, -1 + 2t), \\ w_1 = 1 + 2t, \qquad w_2 = -1 + 2t^2. \end{gathered} \tag{23}$$

Calculate $\mathbf{D}_1, \mathbf{D}_2$ according to formula (22) as follows:

$$\mathbf{D}_1 = \begin{pmatrix} \frac{1}{4} & 0 \\ -\frac{3}{8} & \frac{1}{8} \end{pmatrix} \qquad \mathbf{D}_2 = \begin{pmatrix} \frac{1}{4} & 0 \\ \frac{3}{8} & \frac{1}{8} \end{pmatrix}. \tag{24}$$

Apply affine transformations from (1) to vector $\{t, t^2, t^3\}$

$$\begin{aligned} A_1 \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix} &= \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & \frac{1}{4} & 0 \\ \frac{3}{8} & -\frac{3}{8} & \frac{1}{8} \end{pmatrix} \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix} + \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{4} \\ -\frac{1}{8} \end{pmatrix} \\ &= \begin{pmatrix} \frac{t}{2} - \frac{1}{2} \\ \frac{t^2}{4} - \frac{t}{2} + \frac{1}{4} \\ \frac{t^3}{8} - \frac{3t^2}{8} + \frac{3t}{8} - \frac{1}{8} \end{pmatrix} \\ &= \begin{pmatrix} \frac{t-1}{2} \\ \left(\frac{t-1}{2}\right)^2 \\ \left(\frac{t-1}{2}\right)^3 \end{pmatrix}, \\ A_2 \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix} &= \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{pmatrix} \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \\ \frac{1}{8} \end{pmatrix} \\ &= \begin{pmatrix} \frac{t}{2} + \frac{1}{2} \\ \frac{t^2}{4} + \frac{t}{2} + \frac{1}{4} \\ \frac{t^3}{8} + \frac{3t^2}{8} + \frac{3t}{8} + \frac{1}{8} \end{pmatrix} \\ &= \begin{pmatrix} \frac{t+1}{2} \\ \left(\frac{t+1}{2}\right)^2 \\ \left(\frac{t+1}{2}\right)^3 \end{pmatrix}. \end{aligned} \tag{25}$$

Thus, $\Phi(\mathbf{g}) = \mathbf{g}$ and $\mathbf{g} = \mathbf{g}^\star$.

4. Discretization and Results

In this section, we approximate discrete data $Z = \{(z_m, \mathbf{w}_m)\}_{k=0}^{K}$, $a = z_0 < z_1 < \cdots < z_K = b$ by fractal interpolation vector-function $\mathbf{g}^\star$ constructed on points of interpolation $X = \{(t_i, \mathbf{x}_i)\}_{i=0}^{N}$, $a = t_0 < t_1 < \cdots < t_N = b$, $N \ll K$. Assume that $X \subset Z$. We fit matrix parameters $\mathbf{D}_n$ to minimize functional

$$\sum_{k=0}^{K} \left|\mathbf{w}_k - \mathbf{g}^\star(z_k)\right|^2. \tag{26}$$

It is necessary to use results of previous section. Approximate Z by constant piecewise vector-function $\mathbf{g} : [a, b] \to \mathbb{R}^M$. More precisely $\mathbf{g}(z) = \mathbf{w}_k$, where $(z_k, \mathbf{w}_k) \in Z$ z_k is the nearest approximation neighbor of z. By substituting integrals in (22) to discretization points sums we obtain

$$\begin{aligned} \mathbf{D}_n &= \left(\sum_{z_k \in [t_{n-1}, t_n]} (\mathbf{g}(z_k) - \mathbf{u}_n(z_k)) (\mathbf{g} \circ w_n(z_k) - \mathbf{v}_n(z_k))^T \right) \\ &\quad \cdot \left(\sum_{z_k \in [t_{n-1}, t_n]} (\mathbf{g} \circ w_n(z_k) - \mathbf{v}_n(z_k)) \right. \end{aligned}$$

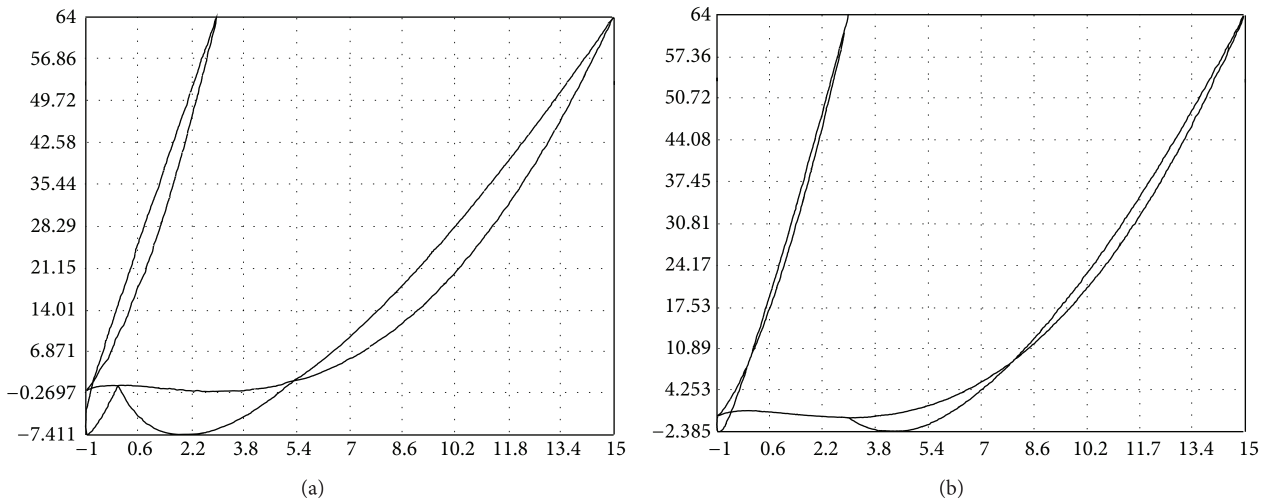


FIGURE 3: Approximation of vector-function $\mathbf{g}(t) = (t(t-2), (t-1)^2(t+1)^2)$ by fractal interpolation function $\mathbf{g}^\star$ with three (a) and four (b) points of interpolation correspondingly.

$$\left. \cdot \left(\mathbf{g} \circ w_n(z_k) - \mathbf{v}_n(z_k)\right)^T \right)^{-1}, \quad n = \overline{1,N}. \tag{27}$$

It is sufficient to apply (1) for constructing fractal interpolation vector-function after we find $\mathbf{D}_n$.

Consider several examples of approximation of discrete data.

Example 6. Let us approximate vector-function $\mathbf{g}(t) = (t(t-2), (t-1)^2(t+1)^2)$, where $t \in [-3,3]$. Figure 3 shows the results. Here, we have two pictures; the first one illustrates initial vector-function and its approximation with 3 points and the second one with 4 points, where two functions are nearly identical.

In this case affine transformations (1) have the following form:

$$A_1\begin{pmatrix} t \\ x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} 0.5 & 0 & 0 \\ -0.8842 & 0.0943 & -0.1045 \\ -0.1038 & -0.0530 & 0.2287 \end{pmatrix}\begin{pmatrix} t \\ x^1 \\ x^2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0.7320 \\ 5.3989 \end{pmatrix},$$
$$A_2\begin{pmatrix} t \\ x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} 0.5 & 0 & 0 \\ 3.4602 & 0.7065 & -1.5549 \\ -0.4554 & 0.1504 & -0.2847 \end{pmatrix}\begin{pmatrix} t \\ x^1 \\ x^2 \end{pmatrix} + \begin{pmatrix} 0.75 \\ 10.8875 \\ 8.1844 \end{pmatrix}. \tag{28}$$

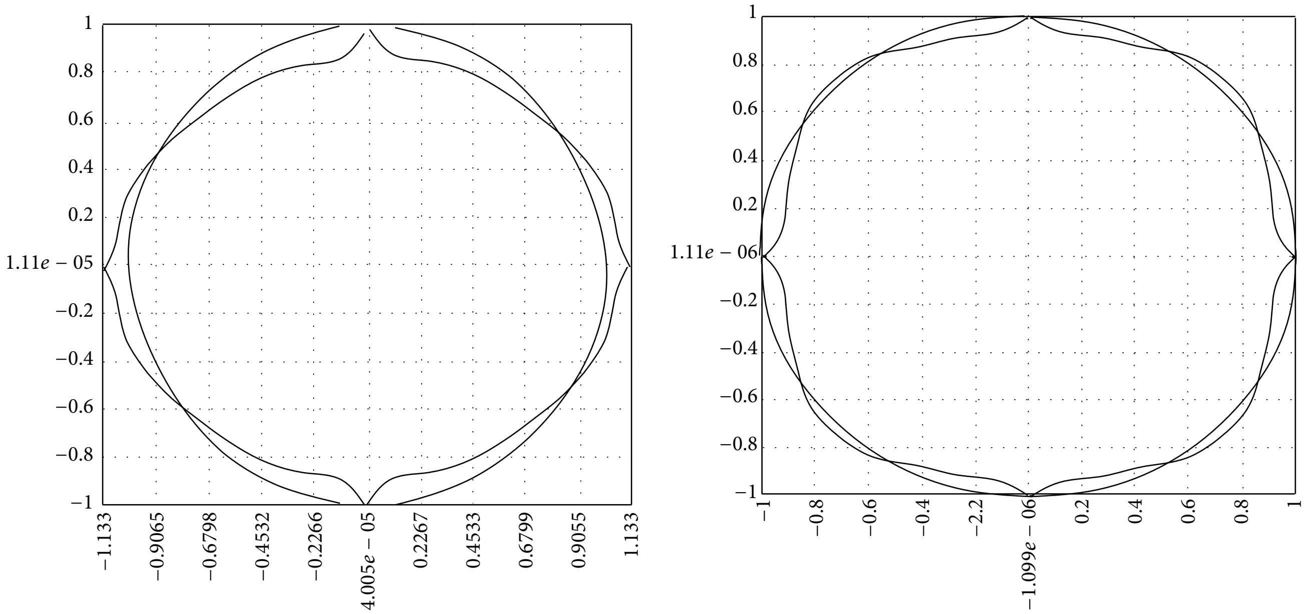
Remark 7. Vectors $\mathbf{c}_n$ in matrices of affine transformations (1) equal $\mathbf{0}$ (like in previous example). It means that fractal interpolation vector-function can be treated as attractor of classical affine IFS in $\mathbb{R}^M$.

Example 8. Next example is devoted to a circle $\mathbf{g}(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$. Figure 4 shows the results. Here we also have two pictures; the first one illustrates initial vector-function and its approximation with 3 points and the second one with 5 points.

In this case affine transformations (1) have the following form:

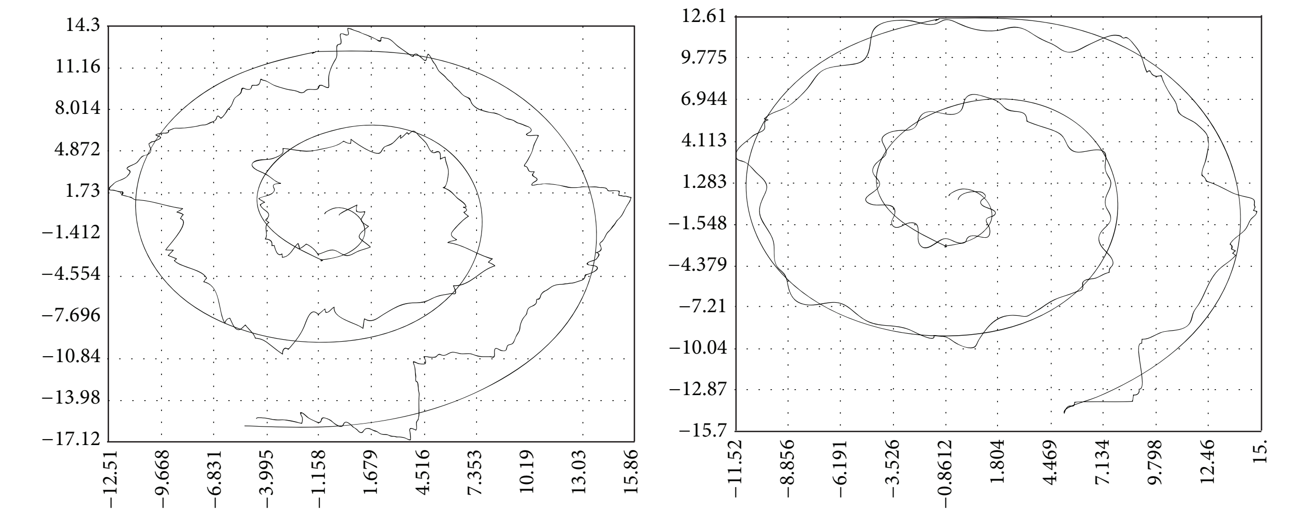
$$A_1\begin{pmatrix} t \\ x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} 0.5 & 0 & 0 \\ -0.3180 & 0.0006 & 0.2128 \\ -0.0013 & -0.5686 & -0.0038 \end{pmatrix}\begin{pmatrix} t \\ x^1 \\ x^2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0.9993 \\ 0.5686 \end{pmatrix},$$
$$A_2\begin{pmatrix} t \\ x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} 0.5 & 0 & 0 \\ 0.3181 & -0.0053 & -0.2128 \\ 0.0040 & 0.5686 & 0.0020 \end{pmatrix}\begin{pmatrix} t \\ x^1 \\ x^2 \end{pmatrix} + \begin{pmatrix} 3.151 \\ -0.9945 \\ -0.5770 \end{pmatrix}. \tag{29}$$

Example 9. Spiral of Archimedes $\mathbf{g}(t) = (t\cos t, t\sin t)$, $t \in [0, 5\pi]$, where the scheme is equal to the examples above, but here we use far more points of interpolation, as illustrated in Figure 5.



(a) (b)

Figure 4: Approximation of vector-function $\mathbf{g}(t) = (\cos t, \sin t)$ by fractal interpolation function $\mathbf{g}^{\star}$ with three (a) and five (b) points of interpolation correspondingly.



(a) (b)

Figure 5: Approximation of vector-function $\mathbf{g}(t) = (t\cos t, t\sin t)$, $t \in [0, 5\pi]$, by fractal interpolation function $\mathbf{g}^{\star}$ with twelve (a) and seventeen (b) points of interpolation correspondingly.

Example 10. Figure 6 shows approximation of vector-function $\mathbf{g}(t) = (\cos(1.5t), \sin(t))$, $t \in [0, 12\pi]$, by fractal interpolation vector-function with sixteen points of interpolation.

Example 11. The example illustrates approximation of graph of Weierstrass function $\omega(x) = \sum_{n=0}^{\infty} (1/2)^n \cos(2\pi 4^n x)$ (Figure 7) by fractal interpolation vector-function.

This example is taken from [10], where fractal approximation is used for approximate calculation of box dimension of fractal curves.

5. Conclusion

In this paper, we have introduced new effective method of approximation of continuous vector-functions and vector

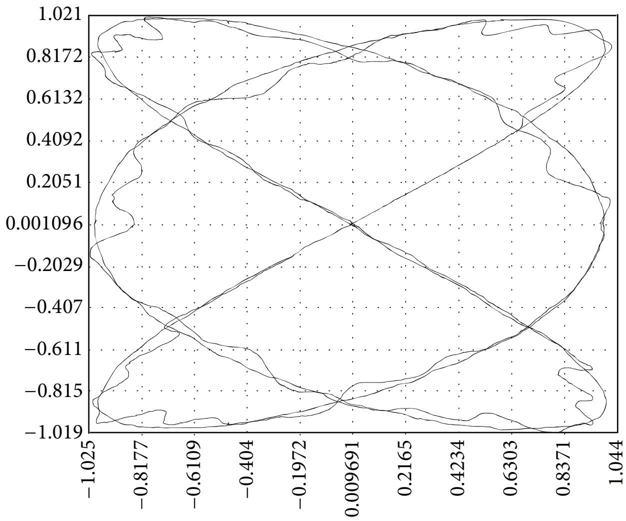


Figure 6: Approximation of vector-function $\mathbf{g}(t) = (\cos(1.5t), \sin(t))$ by fractal interpolation function $\mathbf{g}^{\star}$.

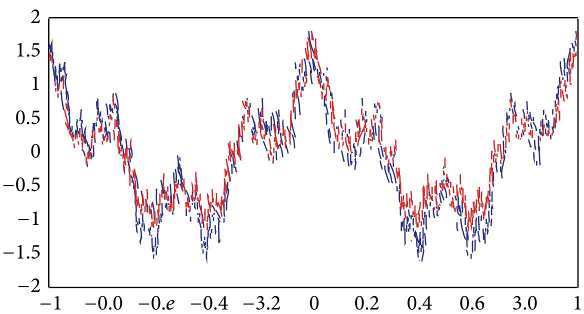


Figure 7: Weierstrass function (blue one) and approximating vector-function (red one).

sequences by fractal interpolation vector-functions, which are affine transformations with matrix parameters. Parameter fitting was a crucial part of approximation process. We have found appropriate parameter values of fractal interpolation vector-functions and illustrate it with several examples of different types of discrete data.

We assume that fractal approximation is highly promising computational tool for different types of data and it can be used in many ways, even in interdisciplinary fields, with a quite high precision that allows us to apply fractal approximation methods to a wide variety of curves, smooth and nonsmooth alike.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Strong Convergence Theorems for Mixed Equilibrium Problem and Asymptotically I-Nonexpansive Mapping in Banach Spaces

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This paper aims to use a hybrid algorithm for finding a common element of a fixed point problem for a finite family of asymptotically nonexpansive mappings and the set solutions of mixed equilibrium problem in uniformly smooth and uniformly convex Banach space. Then, we prove some strong convergence theorems of the proposed hybrid algorithm to a common element of the above two sets under some suitable conditions.

1. Introduction

Let E be a Banach space with norm $\|\cdot\|$. Let C be a nonempty closed convex subset of E and E^* denoted the dual space of E. Let $B : C \to E^*$ be a nonlinear mapping and $\mathscr{H}$ a bifunction from $C \times C$ to R, where R denotes the set of numbers. The generalized equilibrium problem is to find $x \in C$ such that

$$\mathscr{H}(x, y) + \langle Bx, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1}$$

The set of solution of (1) is denoted by $\mathrm{GEP}(\mathscr{H}, B)$, that is,

$$\mathrm{GEP}(\mathscr{H}, B) := \{x \in C, \mathscr{H}(x, y) + \langle Bx, y - x\rangle \geq 0, \forall y \in C\}. \tag{2}$$

In this paper, we are interested in solving the generalized equilibrium problem with those $\mathscr{H}$ given by

$$\mathscr{H}(x, y) = \mathscr{F}(x, y) + \mathscr{G}(x, y), \tag{3}$$

where $\mathscr{F}, \mathscr{G} : C \times C \to R$ are two bifunctions satisfying the following special properties (f_1)–(f_4), (g_1)–(g_3) and (H):

(f_1) $\mathscr{F}(x, x) = 0$, for all $x \in C$;

(f_2) $\mathscr{F}$ is maximal monotone;

(f_3) for all $x, y, z \in C$, we have $\limsup_{t\to 0^+}(\mathscr{F}(tz + (1-t)x, y)) \leq \mathscr{F}(x, y)$;

(f_4) for all $x \in C$, the function $y \mapsto \mathscr{F}(x, y)$ is convex and weakly lower semicontinuous;

(g_1) $\mathscr{G}(x, x) = 0$, for all $x \in C$;

(g_2) $\mathscr{G}$ is monotone and maximal monotone, and weakly upper semicontinuous in the first variable;

(g_3) $\mathscr{G}$ is convex in the second variable;

(H) for fixed $\lambda > 0$ and $x \in C$, there exist a bounded set $K \subset C$ and $a \in K$ such that

$$-\mathscr{F}(a, z) + \mathscr{G}(z, a) + \frac{1}{\lambda}\langle a - z, z - x\rangle < 0, \quad \forall z \in C \setminus K. \tag{4}$$

This is the well-know generalized mixed equilibrium problem, that is, to find an x in C such that

$$\mathscr{F}(x, y) + \mathscr{G}(x, y) + \langle Bx, y - x\rangle \geq 0, \quad \forall y \in C. \tag{5}$$

The solution set of (5) is denoted by $\mathrm{GMEP}(\mathscr{F}, \mathscr{G}, B)$, that is,

$$\mathrm{GMEP}(\mathscr{F}, \mathscr{G}, B) := \{x \in C, \mathscr{F}(x, y) + \mathscr{G}(x, y) + \langle Bx, y - x\rangle \geq 0, \forall y \in C\}. \tag{6}$$

If $B \equiv 0$, problem (5) reduces into mixed equilibrium problem for $\mathscr{F}$ and $\mathscr{G}$, denoted by $\mathrm{MEP}(\mathscr{F}, \mathscr{G})$, which is to find $x \in C$ such that (3).

If $\mathcal{G} = 0$ and $B \equiv 0$, reduces into equilibrium problem for $\mathcal{F}$, denoted by EP($\mathcal{F}$), which is to find $x \in C$ such that

$$\mathcal{F}(x, y) \geq 0, \quad \forall y \in C. \tag{7}$$

Mixed equilibrium problems are suitable and common format for investigation of various applied problems arising in economics, mathematical physics, transportation, communication systems, engineering, and other fields. Moreover, equilibrium problems are closely related with other general problems in nonlinear analysis, such as fixed points, game theory, variational inequality, and optimization problems. Recently, many authors studied a great number of iterative methods for solving a common element of the set of fixed points for a nonexpansive mapping and the set of solutions to a mixed equilibrium problem in the setting of Hilbert space and uniformly smooth and uniformly convex Banach space, respectively (please see, e.g., [1–11] and the references therein).

Let E be a real Banach space with norm $\|\cdot\|$, let C be a nonempty closed convex subset of E, and let J be the normalized duality mapping from E into E^* given by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\| \|x^*\|, \|x\| = \|x^*\|\}, \quad \forall x \in E, \tag{8}$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ the generalized duality pairing between E and E^*. It is easily known that if E^* is uniformly convex, then J is uniformly continuous on bounded subsets of E.

Consider the functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E. \tag{9}$$

It is obvious from the definition of ϕ that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2, \quad \forall x, y \in E. \tag{10}$$

On the other hand, in a Hilbert space H, (9) reduced to $\phi(x, y) = \|x - y\|^2$. Following Alber [12], the generalized projection $\Pi_C : E \to C$ is defined by

$$\Pi_C(x) = \inf_{y \in C} \phi(y, x), \tag{11}$$

where is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$.

In 2011, Kim [13] considered the following shrinking projection methods to obtain a convergence theorem, and these methods were introduced in [14] for quasi-φ-nonexpansive mappings in a uniformly convex and uniformly smooth Banach space.

Theorem 1 (see [13]). *Let E be a uniformly smooth and strictly convex Banach space which has the Kadec-Klee property and C a nonempty closed convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (f_1)–(f_4) and $T : C \to C$ a closed and asymptotically quasi-φ-nonexpansive mapping. Assume that T is asymptotically regular on C and $\digamma = F_{ix}(T) \bigcap EF(f)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{aligned}
&\forall x_0 \in E, \quad C_1 = C, \quad x_1 = \prod_{C_1} x_0, \\
&y_n = J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J T^n x_n), \\
&u_n \in \text{such that } f(u_n, x) + \frac{1}{r_n}\langle x - u_n, Ju_n - Jy_n \rangle \geq 0, \\
&\qquad \forall x \in C, \\
&C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) + (k_n - 1) M_n\}, \\
&x_{n+1} = \prod_{C_{n+1}} x_0,
\end{aligned} \tag{12}$$

where $M_n = \sup\{\phi(z, x_n) : z \in \digamma\}$ for each $n \geq 1$, $\{\alpha_n\}$ is a real sequence in $[0, 1]$ such that $\liminf_{n \to \infty} \alpha_n(1 - \alpha_n) > 0$, and $\{r_n\}$ is a real sequence in $[a, \infty)$, where a is some positive real number and J is the duality mapping on E. Then the sequence $\{x_n\}$ converges strongly to $\prod_\digamma x_0$, where $\prod_\digamma$ is the generalized projection from E onto $\digamma$.

Motivated and inspired by the researches going on in this direction (i.e., [4–11, 13–16]), the purpose of this paper is to use the following hybrid algorithm for finding a common element of the set of solutions to a mixed equilibrium problem and the set of the set of common fixed points for a finite family of asymptotically nonexpansive mappings in a uniformly smooth and uniformly convex Banach space.

Algorithm 2. Let

$$\begin{aligned}
&u_n \in C \text{ such that} \\
&\mathcal{F}(u_n, y) + \mathcal{G}(u_n, y) \\
&\quad \leq \frac{1}{r_n}\langle y - u_n, Ju_n - Jx_n \rangle, \quad \forall y \in C, \\
&y_n = \beta_n x_n + (1 - \beta_n) T^n u_n, \\
&x_{n+1} = \alpha_n(x_n) + (1 - \alpha_n) I^n y_n, \\
&\qquad \forall n \geq 1.
\end{aligned} \tag{13}$$

Consequently, under suitable conditions, we show that iterative algorithms converge strongly to a solution of some optimization problem. Note that our methods do not use any projection.

2. Preliminaries

Let $T : C \to C$ be a mapping. Denote by $F_{ix}(T)$ the set of fixed points of T, that is, $F_{ix}(T) = \{x \in C : Tx = x\}$. Throughout this paper, we always assume that $F_{ix}(T) \neq \emptyset$. Now we need the following known definitions.

Definition 3. A mapping $T : C \to C$ is said to be

(1) nonexpansive, if $\|Tx-Ty\| \le \|x-y\|$, for all $x, y \in C$;

(2) asymptotically nonexpansive, if there exists a sequence $\{\lambda_n\} \subset [1,\infty)$ with $\lim_{n\to\infty}\lambda_n = 1$ such that $\|T^n x - T^n y\| \le \lambda_n \|x-y\|$, for all $x, y \in C$ and $n \in N$;

(3) quasi-nonexpansive, $\|Tx-p\| \le \|x-p\|$, for all $x \in C$ and $p \in F_{ix}(T)$;

(4) asymptotically quasi-nonexpansive, if there exists a sequence $\{\mu_n\} \subset [1,\infty)$ with $\lim_{n\to\infty}\mu_n = 1$ such that $\|T^n x - p\| \le \mu_n \|x-p\|$, for all $x, y \in C$, $p \in F_{ix}(T)$ and $n \in N$.

There are many concepts which generalize a notion of nonexpansive mapping. In 2004, Shahzad [17] introduced the following concepts about I-nonexpansivity of a mapping T.

Definition 4. Let $T : C \to C$ and $I : C \to C$ be two mappings of a nonempty subset C, a real normal linear space E. Then T is said to be

(i) I-nonexpansive, if $\|Tx-Ty\| \le \|Ix-Iy\|$, for all x, $y \in C$;

(ii) asymptotically I-nonexpansive, if there exists a sequence $\{\lambda_n\} \subset [1,\infty)$ with $\lim_{n\to\infty}\lambda_n = 1$ such that $\|T^n x - T^n y\| \le \lambda_n \|I^n x - I^n y\|$, for all $x, y \in C$ and $n \ge 1$;

(iii) asymptotically quasi-I-nonexpansive, if there exists a sequence $\{\mu_n\} \subset [1,\infty)$ with $\lim_{n\to\infty}\mu_n = 1$ such that $\|T^n x - p\| \le \mu_n \|I^n x - p\|$, for all $x, y \in C$, $p \in F_{ix}(T) \bigcap F_{ix}(I)$ and $n \ge 1$.

Lemma 5 (see [4]). *Assume that $\psi : K \to R$ is convex, $x_0 \in$ $\mathrm{core}_K C$, $\psi(x_0) \le 0$, and $\psi(y) \ge 0$, for all $y \in C$. Then $\psi(y) \ge 0$, for all $y \in K$.*

Lemma 6 (see [18]). *Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space E, and let T be a relatively nonexpansive mapping from C into itself. Then $F_{ix}(T)$ is closed and convex.*

Lemma 7 (see [19]). *Let $\{a_n\}$, $\{b_n\}$ and $\{\sigma_n\}$ be sequences of nonnegative real sequences satisfying the following conditions: for all $n \ge 1$*

(1) $a_n \le a_n + b_n$,

(2) $a_n \le (1+\sigma_n)a_n + b_n$,

where $\sum_{n=1}^{\infty}\sigma_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$. Then $\lim_{n\to\infty} a_n$ exists.

Lemma 8 (see [20]). *Let E be a uniformly convex Banach space. Then, for each $r > 0$, there exists a strictly increasing, continuous, and convex function $h : [0,2r] \to R$ such that $h(0) = 0$ and*

$$\|tx + (1-t)y\|^2 \le t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)h\left(\|x-y\|^2\right), \tag{14}$$

for $\forall x, y \in B_r$, $t \in [0,1]$, where $B_r = \{z \in E : \|z\| \le r\}$.

Lemma 9 (see [21]). *Let E be a uniformly convex Banach space and let b, c be two constants with $0 < b < c < 1$. Suppose that $\{t_n\}$ is a sequence in $[b,c]$ and $\{x_n\}$, $\{y_n\}$ are two sequence in E such that*

$$\lim_{n\to\infty}\|t_n x_n + (1-t_n)y_n\| = d, \qquad \limsup_{n\to\infty}\|x_n\| \le d, \qquad \limsup_{n\to\infty}\|y_n\| \le d \tag{15}$$

holds some $d \ge 0$. Then $\lim\|x_n - y_n\| = 0$.

Definition 10 (see [22]). The mappings $T, I : C \to C$ are said to be satisfying condition (A) if there is a nondecreasing function $\mathfrak{f} : [0,\infty) \to [0,\infty)$ with $\mathfrak{f}(0) = 0$, $\mathfrak{f}(r) > 0$ for each $r \in [0,\infty)$ such that $(1/2)(\|x-Tx\| + \|x-Ix\|) \ge \mathfrak{f}(d(x,\Omega))$ for all $x \in C$, where $d(x,\Omega) = \inf\{\|x-p\| : p \in \Omega = F_{ix}(T)\bigcap F_{ix}(I)\}$.

Lemma 11 (see [23]). *Let E be a uniformly convex Banach space satisfying the Opial's condition, C a nonempty closed subset of E, and $T : C \to C$ an asymptotically nonexpansive mapping. If the sequence $\{x_n\} \subset C$ is a weakly convergent sequence with the weak limit p and if $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, then $Tp = p$.*

3. Main Results

Theorem 12. *Let E be a smooth, strictly convex, and reflexive Banach space, and let C be a nonempty closed convex subset of E. Let $\mathcal{F}, \mathcal{G} : C \times C \to R$ be two bifunctions which satisfy the conditions (f_1)–(f_4), (g_1)–(g_3), and (H). Then for every $x^* \in E^*$, there exists a unique point $z \in C$ such that*

$$0 \le \mathcal{F}(z,y) + \mathcal{G}(z,y) + \frac{1}{r}\langle y-z, Jz - Jx^*\rangle, \quad \forall y \in C. \tag{16}$$

The proof goes over the following three steps.

Proof.
Step 1. There exists point $z \in C$ such that

$$\mathcal{F}(y,z) \le \mathcal{G}(z,y) + \frac{1}{r}\langle y-z, Jz - Jx^*\rangle, \quad \forall y \in C. \tag{17}$$

Consider the closed sets

$$T_r(y) = \left\{z \in C \mid \mathcal{F}(y,z) \le \mathcal{G}(z,y) + \frac{1}{r}\langle y-z, Jz - Jx^*\rangle, y \in C\right\}. \tag{18}$$

We will show that $\bigcap_{y\in C} T_r(y) \ne \emptyset$. Let y_i, $i \in N$, be a finite subset of C. Let $I \subset N$ be nonempty. Let for all $\xi \in \mathrm{conv}\{y_i \mid i \in N\}$. Then

$$\xi = \sum_{i\in I}\mu_i y_i \quad \text{with } \mu_i \ge 0 \ (i \in I), \sum_{i\in I}\mu_i = 1. \tag{19}$$

Assume, for contradiction, that

$$-\mathcal{F}(y_i,\xi) + \mathcal{G}(\xi,y_i) + \frac{1}{r}\langle y_i - \xi, J\xi - Jx^*\rangle < 0, \quad \forall i \in N. \tag{20}$$

By the convexity of $\mathcal{F}$ and $\mathcal{G}$ and the monotonicity of $\mathcal{F}$, we obtain that

$$\begin{aligned}0 &= \mathcal{F}(\xi,\xi) + \mathcal{G}(\xi,\xi) + \frac{1}{r}\langle \xi-\xi, J\xi - Jx^*\rangle \\ &\le \sum_{i\in I}\mu_i\mathcal{F}(\xi,y_i) + \sum_{i\in I}\mu_i\mathcal{G}(\xi,y_i) \\ &\quad + \frac{1}{r}\sum_{i\in I}\mu_i\langle y_i-\xi, J\xi-Jx^*\rangle \\ &\le -\sum_{i\in I}\mu_i\mathcal{F}(y_i,\xi) + \sum_{i\in I}\mu_i\mathcal{G}(\xi,y_i) \\ &\quad + \frac{1}{r}\sum_{i\in I}\mu_i\langle y_i-\xi, J\xi-Jx^*\rangle \\ &= \sum_{i\in I}\mu_i\Big[-\mathcal{F}(y_i,\xi)+\mathcal{G}(\xi,y_i) \\ &\qquad + \frac{1}{r}\langle y_i-\xi, J\xi-Jx^*\rangle\Big] < 0,\end{aligned} \tag{21}$$

and that is absurd. Hence (20) cannot be true. and we have $\mathcal{F}(y_i,\xi) \le \mathcal{G}(\xi,y_i)+(1/r)\langle y_i-\xi, J\xi-Jx^*\rangle$ for some $i\in I$. Thus $\xi\in\bigcap_{y\in C}T_r(y_i)$ for some $i\in N$. Since for all $\xi\in\operatorname{conv}\{y_i \mid i\in N\}$, it follows that

$$\operatorname{conv}\{y_i \mid i\in N\} \subset \{T_r(y_i) \mid i\in N\}. \tag{22}$$

By the sets $T_r(y_i)$ being closed, it follows form the standard version of the KKM-Theorem that

$$\bigcap_{i\in N} T_r(y_i) \ne \emptyset. \tag{23}$$

In other words, any finite subfamily of the family $T_r(y)_{y\in C}$ has nonempty intersection. Since these sets are closed subsets of the compact set C, it follows that the entire family has nonempty intersection. Hence

$$\bigcap_{y\in C} T_r(y) \ne \emptyset. \tag{24}$$

Step 2. For every $x^*\in E^*$, the following statement are equivalent:

(i) $z\in C$, $\mathcal{F}(y,z)\le\mathcal{G}(z,y)+\langle y-z, Jz-Jx^*\rangle$, for all $y\in C$,

(ii) $z\in C$, $0\le\mathcal{F}(z,y)+\mathcal{G}(z,y)+\langle y-z, Jz-Jx^*\rangle$, for all $y\in C$.

Case 1. Let (ii) hold; since $\mathcal{F}$ is monotone, one has

$$\mathcal{F}(z,y) \le -\mathcal{F}(y,z). \tag{25}$$

Hence (i) follows.

Case 2. Let (i) hold, for t with $0<t\le 1$ and $y\in C$, and let

$$x_t = ty + (1-t)z. \tag{26}$$

Then $x_t\in C$, and from (i), $\mathcal{F}(x_t,z)\le\mathcal{G}(z,x_t)+\langle x_t-z, Jz-Jx^*\rangle$. By the properties of $\mathcal{F}$ and $\mathcal{G}$, it follows then, for all $0<t\le 1$,

$$\begin{aligned}0 &= \mathcal{F}(x_t,x_t)+\mathcal{G}(x_t,x_t)+\langle x_t-x_t, Jz-Jx^*\rangle \\ &\le t\mathcal{F}(x_t,y)+(1-t)\mathcal{F}(x_t,z) \\ &\quad + t\mathcal{G}(x_t,y)+(1-t)\mathcal{G}(x_t,z) \\ &\le \mathcal{F}(x_t,y)+\mathcal{G}(x_t,y).\end{aligned} \tag{27}$$

Let $t\to 0$ and thereby $x_t\to z$ and using the hemicontinuity of $\mathcal{F}$ we obtain in the limit

$$0\le\mathcal{F}(z,y)+\mathcal{G}(z,y)+\langle y-z, Jz-Jx^*\rangle. \tag{28}$$

Step 3. Take $\psi(\cdot)=\mathcal{F}(z,\cdot)+\mathcal{G}(z,\cdot)+\langle\cdot-z, Jz-Jx^*\rangle$. Then the function $\psi(\cdot)$ is convex and $\psi(y)\ge 0$, for all $y\in C$. If $z\in\operatorname{core}_K C$, then set $x_0=z$. If $z\in C\setminus\operatorname{core}_K C$, then set $x_0=a$, where a is as in assumption H for $x=z$. In both cases $x_0\in\operatorname{core}_K C$, and $\psi(x_0)\le 0$. Hence it follows from the Lemma 5 that

$$\begin{aligned}&\psi(y)\ge 0 \quad \forall y\in C, \\ &\text{that is, } \mathcal{F}(z,y)+\mathcal{G}(z,y)+\langle y-z, Jz-Jx^*\rangle \ge 0, \\ &\qquad\qquad \forall y\in K.\end{aligned} \tag{29}$$

□

Corollary 13. *Let E be a smooth, strictly convex, and reflexive Banach space, and let C be a nonempty closed convex subset of E. Let $\mathcal{F}$, $\mathcal{G}$: $C\times C\to R$ be two bifunctions which satisfy the following conditions: (f_1)–(f_4), (g_1)–(g_3), and (H) in Theorem 12. There for every $x^*\in E$ and $r>0$, there exists a unique point $z_r\in C$ such that*

$$0\le\mathcal{F}(z_r,y)+\mathcal{G}(z_r,y)+\frac{1}{r}\langle y-z_r, Jz_r-Jx^*\rangle, \quad \forall y\in C. \tag{30}$$

Proof. Let $x\in E$ and $r>0$ be given. Note that functions $r\mathcal{F}$ and $r\mathcal{G}$ also satisfy the conditions (f_1)–(f_4) and (g_1)–(g_3). Therefore, for $Jx^*\in E^*$, there exists a unique point $z_r\in C$ such that

$$r\mathcal{F}(z_r,y)+r\mathcal{G}(z_r,y)+\langle y-z_r, Jz_r-Jx^*\rangle\ge 0, \quad \forall y\in C. \tag{31}$$

This completes the proof. □

Under the same assumptions in Corollary 13, for every $r>0$, we may define a single-valued mapping $S_r : E\to C$ as follows:

$$\begin{aligned}S_r(x) = \Big\{z\in C \mid 0 &\le \mathcal{F}(z,y)+\mathcal{G}(z,y) \\ &+\frac{1}{r}\langle y-z, Jz-Jx\rangle, \forall y\in C\Big\},\end{aligned} \tag{32}$$

for $x\in E$, which is called the resolvent of $\mathcal{F}$ and $\mathcal{G}$ for r.

Theorem 14. *Let E be a smooth, strictly convex, and reflexive Banach space, and let C be a nonempty closed convex subset of E. Let $\mathcal{F}, \mathcal{G} : C \times C \to R$ be two bifunctions which satisfy conditions (f_1)–(f_4), (g_1)–(g_3), and (H). For $r > 0$ and $x \in E$, define a mapping S_r in (32). Then, the following hold:*

(a) *S_r is single-valued;*

(b) *S_r is a firmly nonexpansive mapping, that is,*

$$\langle S_r x - S_r y, JS_r x - JS_r y\rangle \le \langle S_r x - S_r y, Jx - Jy\rangle, \quad \forall x, y \in E; \tag{33}$$

(c) *$F_{ix}(S_r) = \mathrm{MEP}(\mathcal{F}, \mathcal{G})$;*

(d) *$\mathrm{MEP}(\mathcal{F}, \mathcal{G})$ is closed and convex;*

(e) *$\phi(p, S_r x) + \phi(S_r x, x) \le \phi(p, x)$.*

Proof. We divide the proof into several steps.

Step 1 (S_r is single-valued). Indeed, for $x \in C$ and $r > 0$, let $z_1, z_2 \in S_r x$. Then

$$\begin{aligned}&\mathcal{F}(z_1, z_2) + \mathcal{G}(z_1, z_2) + \frac{1}{r}\langle z_2 - z_1, Jz_1 - Jx\rangle \ge 0,\\ &\mathcal{F}(z_2, z_1) + \mathcal{G}(z_2, z_1) + \frac{1}{r}\langle z_1 - z_2, Jz_1 - Jx\rangle \ge 0.\end{aligned} \tag{34}$$

Adding the two inequalities, we obtain

$$\begin{aligned}&\mathcal{F}(z_1, z_2) + \mathcal{F}(z_2, z_1) + \mathcal{G}(z_1, z_2) + \mathcal{G}(z_2, z_1)\\ &\quad + \frac{1}{r}\langle z_1 - z_2, Jz_1 - Jz_2\rangle \ge 0.\end{aligned} \tag{35}$$

From (f_2), (g_2), and $r > 0$, we obtain

$$\frac{1}{r}\langle z_1 - z_2, Jz_1 - Jz_2\rangle \ge 0. \tag{36}$$

Since E is strictly convex, we obtain

$$z_1 = z_2. \tag{37}$$

Step 2 (S_r is a firmly nonexpansive mapping). For $x, y \in C$, we obtain

$$\begin{aligned}&\mathcal{F}(S_r x, S_r y) + \mathcal{G}(S_r x, S_r y) + \frac{1}{r}\langle S_r y - S_r x, JS_r x - Jx\rangle \ge 0,\\ &\mathcal{F}(S_r y, S_r x) + \mathcal{G}(S_r y, S_r x) + \frac{1}{r}\langle S_r x - S_r y, JS_r y - Jy\rangle \ge 0.\end{aligned} \tag{38}$$

Adding the two inequalities, we obtain

$$\begin{aligned}&\mathcal{F}(S_r x, S_r y) + \mathcal{F}(S_r y, S_r x) + \mathcal{G}(S_r x, S_r y) + \mathcal{G}(S_r y, S_r x)\\ &\quad + \frac{1}{r}\langle S_r y - S_r x, JS_r x - JS_r y - Jx + Jy\rangle \ge 0.\end{aligned} \tag{39}$$

From (f_2), (g_2), and $r > 0$, we obtain

$$\langle S_r y - S_r x, JS_r x - JS_r y - Jx + Jy\rangle \ge 0. \tag{40}$$

Therefore, we have

$$\langle S_r x - S_r y, JS_r x - JS_r y\rangle \le \langle S_r x - S_r y, Jx - Jy\rangle. \tag{41}$$

Step 3 ($F_{ix}(S_r) = \mathrm{MEP}(\mathcal{F}, \mathcal{G})$). Indeed, we obtain the following equation:

$$\begin{aligned}u \in F_{ix}(S_r) &\Longleftrightarrow u = S_r u\\ &\Longleftrightarrow \mathcal{F}(u, y) + \mathcal{G}(u, y)\\ &\qquad + \frac{1}{r}\langle y - u, Ju - Ju\rangle \ge 0, \quad \forall y \in C,\\ &\Longleftrightarrow \mathcal{F}(u, y) + \mathcal{G}(u, y), \quad \forall y \in C,\\ &\Longleftrightarrow u \in \mathrm{MEP}(\mathcal{F}, \mathcal{G}).\end{aligned} \tag{42}$$

Step 4 ($\mathrm{MEP}(\mathcal{F}, \mathcal{G})$ is closed and convex). From (c), we have $\mathrm{MEP}(\mathcal{F}, \mathcal{G}) = F_{ix}(S_r)$, and from (b), we obtain

$$\langle S_r x - S_r y, JS_r x - JS_r y\rangle \le \langle S_r x - S_r y, Jx - Jy\rangle, \quad x, y \in C. \tag{43}$$

Moreover, we obtain

$$\begin{aligned}&\phi(S_r x, S_r y) + \phi(S_r y, S_r x)\\ &\quad = 2\|S_r x\|^2 - 2\langle S_r x, JS_r y\rangle\\ &\qquad - 2\langle S_r y, JS_r x\rangle + 2\|S_r y\|^2\\ &\quad = 2\langle S_r x, S_r x - JS_r y\rangle\\ &\qquad + 2\langle S_r y, S_r y - JS_r x\rangle\\ &\quad = 2\langle S_r x - S_r y, S_r x - JS_r y\rangle,\\ &\phi(S_r x, y) + \phi(S_r y, x) - \phi(S_r x, x) - \phi(S_r y, y)\\ &\quad = \|S_r x\|^2 - 2\langle S_r x, Jy\rangle + \|y\|^2\\ &\qquad + \|S_r y\|^2 - 2\langle S_r y, Jx\rangle + \|x\|^2\\ &\qquad - \|S_r x\|^2 + 2\langle S_r x, Jx\rangle - \|y\|^2\\ &\qquad - \|S_r y\|^2 + 2\langle S_r y, Jy\rangle - \|x\|^2\\ &\quad = 2\langle S_r x, Jx - Jy\rangle + 2\langle S_r y, Jy - Jx\rangle\\ &\quad = 2\langle S_r x - S_r y, Jx - Jy\rangle.\end{aligned} \tag{44}$$

Hence, we obtain

$$\begin{aligned}&\phi(S_r x, S_r y) + \phi(S_r y, S_r x)\\ &\quad \le \phi(S_r x, y) + \phi(S_r y, x) - \phi(S_r x, x) - \phi(S_r y, y).\end{aligned} \tag{45}$$

So we get

$$\begin{aligned}\phi(S_r x, S_r y) + \phi(S_r y, S_r x)\\ \leq \phi(S_r x, y) + \phi(S_r y, x).\end{aligned} \tag{46}$$

Taking $y = u \in F_{ix}(S_r)$, we obtain

$$\phi(u, S_r x) \leq \phi(u, x). \tag{47}$$

Next, we show that $\widehat{F}_{ix}(S_r) = \mathrm{MEP}(\mathcal{F}, \mathcal{G})$. Let $p \in \widehat{F}_{ix}(S_r)$. Then, there exists the sequence of $\{z_n \in E\}$ such that $z_n \rightharpoonup p$ and $\lim_{n\to\infty}(z_n - S_r z_n) = 0$. Moreover, we obtain $S_r z_n \rightharpoonup p$. Hence we have $p \in C$. Since J is uniformly continuous on bounded sets, we obtain

$$\lim_{n\to\infty} \|J z_n - J S_r z_n\| = 0. \tag{48}$$

Form the definition of S_r, we obtain

$$\begin{aligned}\mathcal{F}(S_r z_n, y) + \mathcal{G}(S_r z_n, y)\\ + \frac{1}{r}\langle y - S_r z_n, J S_r z_n - J z_n\rangle \geq 0.\end{aligned} \tag{49}$$

Since the monotone of the $\mathcal{F}$, we have

$$\begin{aligned}\mathcal{G}(S_r z_n, y) + \frac{1}{r}\langle y - S_r z_n, J S_r z_n - J z_n\rangle\\ \geq -\mathcal{F}(S_r z_n, y) = \mathcal{F}(y, S_r z_n).\end{aligned} \tag{50}$$

According to (48) and $z_n \rightharpoonup p$ and form (f_3) and (g_2), we obtain

$$\mathcal{F}(y, p) \leq \mathcal{G}(p, y), \quad \forall y \in C. \tag{51}$$

For t with $0 < t \leq 1$ and $y \in H$, let $x_t = ty + (1-t)p$; then by the convexity of $\mathcal{F}$ and $\mathcal{G}$ we have

$$\begin{aligned}0 &= \mathcal{F}(x_t, x_t) + \mathcal{G}(x_t, x_t)\\ &\leq t\mathcal{F}(x_t, y) + (1-t)\mathcal{F}(x_t, p)\\ &\quad + t\mathcal{G}(x_t, y) + (1-t)\mathcal{G}(x_t, \omega)\\ &\leq t\mathcal{F}(x_t, y) + t\mathcal{G}(x_t, y).\end{aligned} \tag{52}$$

Passing $t \to 0^+$ and by (f_1) and (g_1), we have $0 \leq \mathcal{F}(p, y) + \mathcal{G}(p, y)$ for all $y \in H$. Therefore, $p \in \mathrm{MEP}(\mathcal{F}, \mathcal{G})$. So, we get $F_{ix}(S_r) = \mathrm{MEP}(\mathcal{F}, \mathcal{G}) = \widehat{F}_{ix}(S_r)$. Therefore, we have that S_r is a relatively nonexpansive mapping. From Lemma 6, then $F_{ix}(S_r) = \mathrm{MEP}(\mathcal{F}, \mathcal{G})$ is closed and convex.

Step 5 ($\phi(p, S_r x) + \phi(S_r x, x) \leq \phi(p, x)$). From (b) and (45), for each x, $y \in E$, we obtain

$$\begin{aligned}\phi(S_r x, S_r y) + \phi(S_r y, S_r x)\\ \leq \phi(S_r x, y) + \phi(S_r y, x) - \phi(S_r x, x) - \phi(S_r y, y).\end{aligned} \tag{53}$$

Letting $y = p \in F_{ix}(S_r)$, we obtain

$$\phi(p, S_r x) + \phi(S_r x, x) \leq \phi(p, x). \tag{54}$$

□

If $\mathcal{G}(x, y) = \psi(x) - \psi(y)$ and form Theorems 12 and 14, we obtain the following corollary.

Corollary 15 (see [24]). *Let E be a smooth, strictly convex, and reflexive Banach space, and C be a nonempty closed convex subset of E. Let $\mathcal{F} : C \times C \to R$ be a bifunctions which satisfy conditions (f_1)–(f_4). Let $\psi : C \to R$ be a lower semicontinuous and convex function. For $r > 0$ and $x \in E$. Then, the following hold:*

(i) $0 \leq \mathcal{F}(z, y) + \psi(y) - \psi(z) + (1/r)\langle y - z, Jz - Jx^*\rangle$, *for all $y \in C$.*

(ii) *If we define a mapping $S_r : E \to C$ as follows:*

$$\begin{aligned}S_r(x) = \Big\{x \in C \mid \leq 0 \leq F(z, y) + \psi(y)\\ -\psi(z) + \frac{1}{r}\langle y - z, Jz - Jx\rangle, y \in C\Big\},\end{aligned} \tag{55}$$

and the mapping S_r has the following properties:

(a) *S_r is single-valued;*

(b) *S_r is a firmly nonexpansive mapping, that is,*

$$\begin{aligned}\langle S_r z - S_r y, J S_r z - J S_r y\rangle \leq \langle S_r z - S_r y, Jz - Jy\rangle,\\ \forall z, y \in E;\end{aligned} \tag{56}$$

(c) $F_{ix}(S_r) = \mathrm{MEP}(\mathcal{F}, \psi)$;

(d) $\mathrm{MEP}(\mathcal{F}, \psi)$ *is closed and convex;*

(e) $\phi(p, S_r z) + \phi(S_r z, z) \leq \phi(p, z)$.

4. Strong Convergence Theorems

In this section, we introduce a new iterative scheme for finding a common element of the set of solutions of the mixed equilibrium problems and the set of fixed points for I-asymptotically nonexpansive mapping in Banach spaces.

Theorem 16. *Let E be uniformly smooth and uniformly convex Banach space, and let C be a nonempty closed convex subset of E. Let $\mathcal{F}$, $\mathcal{G} : C \times C \to R$ be two bifunctions which satisfy the conditions (f_1)–(f_4), (g_1)–(g_3), and (H), and let T be I-asymptotically nonexpansive self-mapping of C with sequences $\{s_n\} \subset [0, \infty\}$ such that $\sum_{n=1}^{\infty} s_n < \infty$, and let I be asymptotically nonexpansive self-mapping of C with sequences $\{t_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} t_n < \infty$, and $\Omega = F_{ix}(I) \bigcap F_{ix}(T) \bigcap \mathrm{MEP}(\mathcal{F}, \mathcal{G}) \neq \emptyset$. For an initial point $x_0 \in C$, generate a sequence $\{x_n\}$ by*

$$\begin{aligned}&u_n \in C\\ &\text{such that } \mathcal{F}(u_n, y) + \mathcal{G}(u_n, y) \leq \frac{1}{r_n}\langle y - u_n, J u_n - J x_n\rangle,\\ &\qquad \forall y \in C,\\ &y_n = \beta_n x_n + (1-\beta_n) T^n u_n,\\ &x_{n+1} = \alpha_n(x_n) + (1-\alpha_n) I^n y_n, \quad \forall n \geq 1,\end{aligned} \tag{57}$$

where $\{\alpha_n\}$ *is a sequence in* $[0,1]$, $\{\beta_n\} \subset [a,b]$ *for some* a, $b \in (0,1)$ *and* $\{r_n\} \subset [d,+\infty)$ *for* $d > 0$. *If the following conditions are satisfied:*

(i) $\sum_{n=0}^{\infty} \alpha_n < \infty$;

(ii) $\sum_{n=0}^{\infty} \beta_n < \infty$;

(iii) $\liminf_{n\to\infty} r_n > 0$,

then the sequence $\{x_n\}$ *generated by* (57) *converges strongly to a fixed point in* Ω *if and only if*

$$\liminf_{n\to\infty} d(x_n,\Omega) = 0. \tag{58}$$

Proof. We divide the proof into several steps.

Step 1 (The sequence $\{x_n\}$ is bounded). Let $u_n = T_{r_n} x_n$. Since T is a I-asymptotically nonexpansive mapping, it follows from and Theorem 14 that $\Omega := F_{ix}(T) \bigcap F_{ix}(I) \bigcap \mathrm{MEP}(\mathcal{F},\mathcal{G})$ is nonempty closed convex subset E and for each $p \in \Omega$.

$$\begin{aligned}
&\|x_{n+1} - p\| \\
&\quad = \|\alpha_n (x_n) + (1-\alpha_n) I^n y_n - p\| \\
&\quad = \alpha_n \|x_n - p\| + (1-\alpha_n) \|I^n y_n - p\| \\
&\quad \le \alpha_n \|x_n - p\| + (1-\alpha_n)(1+t_n) \|y_n - p\| .
\end{aligned} \tag{59}$$

Again from (57), we obtain that

$$\begin{aligned}
\|y_n - p\| &= \|\beta_n x_n + (1-\beta_n) T^n u_n - p\| \\
&\le \beta_n \|x_n - p\| + (1-\beta_n) \|T^n u_n - p\| \\
&\le \beta_n \|x_n - p\| + (1-\beta_n) \\
&\quad \times (1+s_n) \|I^n u_n - p\| \\
&\le \beta_n \|x_n - p\| + (1-\beta_n) \\
&\quad \times (1+s_n)(1+t_n) \|T_r x_n - p\| \\
&\le \beta_n \|x_n - p\| + (1-\beta_n) \\
&\quad \times (1+s_n)(1+t_n) \|x_n - p\| \\
&= [1 + (1-\beta_n)(s_n + t_n + s_n t_n)] \|x_n - p\| .
\end{aligned} \tag{60}$$

From (59) and (60), we obtain

$$\begin{aligned}
&\|x_{n+1} - p\| \\
&\quad \le \alpha_n \|x_n - p\| + (1-\alpha_n)(1+t_n) \\
&\qquad \times [1 + (1-\beta_n)(s_n + t_n + s_n t_n)] \|x_n - p\| \\
&\quad \le \alpha_n \|x_n - p\| + (1-\alpha_n)(1+t_n) \\
&\qquad \times [1 + (1-\beta_n)(s_n + t_n + s_n t_n)] \|x_n - p\| \\
&\quad \le (1+\rho_n) \|x_n - p\| ,
\end{aligned} \tag{61}$$

where

$$\begin{aligned}
\rho_n &= (1-\alpha_n)(1-\beta_n)(s_n + t_n + s_n t_n) \\
&\quad + (1-\alpha_n) t_n + (1-\alpha_n)(1-\beta_n) \\
&\quad \times t_n (s_n + t_n + s_n t_n) .
\end{aligned} \tag{62}$$

Moreover since $\{\alpha_n\} \subset [0,1]$, $\{\beta_n\} \subset [a,b]$ for some a, $b \in (0,1)$, $\sum_{n=1}^{\infty} s_n < \infty$, and $\sum_{n=1}^{\infty} t_n < \infty$, it follow that $\sum_{n=1}^{\infty} \rho_n < \infty$. Form (60) and, by Lemma 7, we obtain that the limit of $\{\|x_n - p\|\}$ exists for each $p \in \Omega$. This implies that $\{\|x_n - p\|\}$ is bounded and so are $\{x_n\}$, $\{y_n\}$, $\{u_n\}$, $\{I^n y_n\}$, and $\{T^n u_n\}$; on the other hand, we obtain that $d(x_{n+1},\Omega) \le (1+\rho_n) d(x_n,\Omega)$. Then by Lemma 7, $\lim_{n\to\infty} d(x_n,\Omega)$ exists and, by assumption $\liminf_{n\to\infty} d(x_n,\Omega) = 0$, we obtain

$$\lim_{n\to\infty} d(x_n,\Omega) = 0. \tag{63}$$

Step 2 ($\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$). Taking lim sup on both sides in the above inequality,

$$\lim_{n\to\infty} \|y_n - p\| = d. \tag{64}$$

Since I^n is asymptotically nonexpansive self-mappings of C, we can get that $\|I^n y_n - p\| \le (1+t_n)\|y_n - p\|$, which on taking $\limsup_{n\to\infty}$ and using (64), we obtain

$$\limsup_{n\to\infty} \|I^n y_n - p\| \le d. \tag{65}$$

Further,

$$\lim_{n\to\infty} \|x_{n+1} - p\| \le d. \tag{66}$$

That means that

$$\begin{aligned}
&\lim_{n\to\infty} \|\alpha_n (x_n) + (1-\alpha_n) I^n y_n - p\| \le d, \\
&\lim_{n\to\infty} \alpha_n \|(x_n) - p\| + (1-\alpha_n) \|I^n y_n - p\| \le d.
\end{aligned} \tag{67}$$

It follows from Lemma 9 that

$$\lim_{n\to\infty} \|I^n y_n - x_n\| = 0. \tag{68}$$

Moveover,

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|(1-\alpha_n)(I^n y_n - x_n)\| \\
&= (1-\alpha_n) \|I^n y_n - x_n\| .
\end{aligned} \tag{69}$$

Thus, from (68), we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{70}$$

Step 3 ($\lim_{n\to\infty} \|x_n - T^n u_n\| = 0$). Use (57) again, and Lemma 8 that for $r = \sup_{n\ge1}\{\|x_n\|, \|u_n\|, \|T^n u_n\|\}$, there exists a strictly increasing, continuous and convex function $h : [1,2] \to R$ that $h(0) = 0$ and

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad \le \alpha_n \|x_n - p\|^2 + (1-\alpha_n) \|I^n y_n - p\|^2 \\
&\qquad - \alpha_n (1-\alpha_n) h\left(\|x_n - I^n y_n\|^2\right) \\
&\quad \le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)(1+t_n) \|y_n - p\|^2 \\
&\qquad - \alpha_n (1-\alpha_n) h\left(\|x_n - I^n y_n\|^2\right)
\end{aligned}$$

$$\begin{aligned}
&\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)(1+t_n)\\
&\quad\times\left(\beta_n\|x_n-p\|^2+(1-\beta_n)\|T^nu_n-p\|^2\right.\\
&\qquad\left.-\beta_n(1-\beta_n)h\left(\|x_n-T^nu_n\|^2\right)\right)\\
&\quad-\alpha_n(1-\alpha_n)h\left(\|x_n-I^ny_n\|^2\right)\\
&\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)(1+t_n)\\
&\quad\times\left(\beta_n\|x_n-p\|^2+(1-\beta_n)(1+s_n)\|u_n-p\|^2\right.\\
&\qquad\left.-\beta_n(1-\beta_n)h\left(\|x_n-T^nu_n\|^2\right)\right)\\
&\quad-\alpha_n(1-\alpha_n)h\left(\|x_n-I^ny_n\|^2\right)\\
&\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)\\
&\quad\times(1+t_n)\left(\beta_n\|x_n-p\|^2\right.\\
&\qquad+(1-\beta_n)(1+s_n)\|x_n-p\|^2\\
&\qquad\left.-\beta_n(1-\beta_n)h\left(\|x_n-T^nu_n\|^2\right)\right)\\
&\quad-\alpha_n(1-\alpha_n)h\left(\|x_n-I^ny_n\|^2\right)\\
&\le (1+\rho_n)\|x_n-p\|^2-(1-\alpha_n)\\
&\quad\times(1+t_n)\beta_n(1-\beta_n)h\left(\|x_n-T^nu_n\|^2\right)\\
&\quad-\alpha_n(1-\alpha_n)h\left(\|x_n-I^ny_n\|^2\right),
\end{aligned}\tag{71}$$

where

$$\begin{aligned}
\rho_n&=(1-\alpha_n)(1-\beta_n)\\
&\quad\times(s_n+t_n+s_nt_n)+(1-\alpha_n)t_n+(1-\alpha_n)\\
&\quad\times(1-\beta_n)t_n(s_n+t_n+s_nt_n).
\end{aligned}\tag{72}$$

From the discuss of the Step 1, we can easily know that $\sum_{n=1}^{\infty}\rho_n<\infty$. On the other hand, by (71) and the bounded sequence of $\{x_n\}$, we obtain that

$$\begin{aligned}
&(1-\alpha_n)(1+t_n)\beta_n(1-\beta_n)h\left(\|x_n-T^nu_n\|\right)\\
&\quad\le\phi(x_n,p)-\phi(x_{n+1},p)+\rho_n\phi(x_n,p).
\end{aligned}\tag{73}$$

From $\lim_{n\to\infty}h(\|x_n-T^nu_n\|)=0$, (73) and the property of h, we have

$$\lim_{n\to\infty}\|x_n-T^nu_n\|=0.\tag{74}$$

The same as the proof of (74), we can easily obtain that

$$\lim_{n\to\infty}\|x_n-I^ny_n\|=0.\tag{75}$$

From (57), we obtain that

$$\|y_n-x_n\|\le(1-\beta_n)\|x_n-T^nu_n\|.\tag{76}$$

It follows that

$$\lim_{n\to\infty}\|x_n-y_n\|=0.\tag{77}$$

Step 4 ($\lim_{n\to\infty}\|x_n-u_n\|=0$). Let $p\in\Omega=F_{ix}(I)\bigcap F_{ix}(T)\bigcap \mathrm{MEP}(\mathcal{F},\mathcal{G})$. Then, from (59) and (60), it follows that

$$\begin{aligned}
\|u_{n+1}-p\|&=\left\|T_{r_{n+1}}x_{n+1}-p\right\|\\
&\le\|x_{n+1}-p\|\\
&\le\|\alpha_nx_n+(1-\alpha_n)I^ny_n-p\|\\
&\le\alpha_n\|x_n-p\|+(1-\alpha_n)\\
&\quad\times(1+t_n)\|y_n-p\|\\
&\le\alpha_n\|x_n-p\|+(1-\alpha_n)(1+t_n)\\
&\quad\times[\beta_n\|x_n-p\|+(1-\beta_n)\\
&\qquad\times(1+s_n)(1+t_n)\|u_n-p\|]\\
&\le[\alpha_n+(1-\alpha_n)(1+t_n)\beta_n]\\
&\quad\times\|x_n-p\|+(1-\alpha_n)(1-\beta_n)\\
&\quad\times(1+s_n)(1+t_n)^2\|u_n-p\|\\
&\le M_1\|x_n-p\|+(1+M_2)\|u_n-p\|,
\end{aligned}\tag{78}$$

where

$$\begin{aligned}
M_1&=\alpha_n+(1-\alpha_n)(1+t_n)\beta_n,\\
M_2&=[t_n(2+t_n)(1+s_n)+s_n](\alpha_n\beta_n-\alpha_n-\beta_n)\\
&\quad+(t_n(2+t_n)(1+s_n)+s_n)\\
&\quad+(\alpha_n\beta_n-\alpha_n-\beta_n).
\end{aligned}\tag{79}$$

Moreover since $\{\alpha_n\}\subset[0,1]$, $\{\beta_n\}\subset[a,b]$ for some $a,b\in(0,1)$, $\sum_{n=1}^{\infty}s_n<\infty$ and $\sum_{n=1}^{\infty}t_n<\infty$, we can easily claim that $\sum_{n=1}^{\infty}M_1<\infty$ and $\sum_{n=1}^{\infty}M_2<\infty$. By Lemma 7, we obtain that $\lim_{n\to\infty}\|u_n-p\|$ exists and from Theorem 14(b) and (78), we have

$$\begin{aligned}
\|x_n-u_n\|&\le\|x_n-p\|-\left\|T_{r_n}x_n-p\right\|\\
&=\|x_n-p\|-\|u_n-p\|\\
&\le M_1\|x_{n-1}-p\|+(1+M_2)\\
&\quad\times\|u_{n-1}-p\|-\|u_n-p\|\\
&\le M_1\|x_{n-1}-p\|+M_2\|u_{n-1}-p\|\\
&\quad+\|u_{n-1}-p\|-\|u_n-p\|.
\end{aligned}\tag{80}$$

Thus, since $\{u_n\}$ converges, $\sum_{n=1}^{\infty}M_1<\infty$ and $\sum_{n=1}^{\infty}M_2<\infty$ and $\{x_n\}$ is bounded, it follows form Lemma 7 that

$$\lim_{n\to\infty}\|x_n-u_n\|=0.\tag{81}$$

Step 5 ($\lim_{n\to\infty}\|u_n - T^n u_n\| = 0$). By using the triangle inequality, we have

$$\|T^n u_n - u_n\| \le \|T^n u_n - x_n\| + \|x_n - u_n\|. \tag{82}$$

Thus, from (74) and (81), we obtain that

$$\lim_{n\to\infty}\|u_n - T^n u_n\| = 0. \tag{83}$$

Step 6 ($\lim_{n\to\infty}\|x_n - T^n x_n\| = \lim_{n\to\infty}\|x_n - I^n x_n\| = 0$). By using the triangle inequality again, we obtain

$$\begin{aligned}\|x_n - T^n x_n\| &\le \|x_n - T^n u_n\| + \|T^n x_n - T^n u_n\| \\ &\le \|x_n - T^n u_n\| + (1+s_n)\|x_n - u_n\|.\end{aligned} \tag{84}$$

From (74) and (81), we have

$$\lim_{n\to\infty}\|x_n - T^n x_n\| = 0. \tag{85}$$

From (57), we have

$$\begin{aligned}\|x_n - I^n x_n\| &\le \|x_n - I^n y_n\| + \|I^n x_n - I^n y_n\| \\ &\le \|x_n - I^n y_n\| + (1+t_n)\|x_n - y_n\| \\ &\le \|x_n - I^n y_n\| + (1+t_n) \\ &\quad \times (1-\alpha_n)\|x_n - I^n y_n\|.\end{aligned} \tag{86}$$

From $\{\alpha_n\} \subset [0,1]$, $\sum_{n=1}^{\infty} t_n < \infty$, and (68), we obtain

$$\lim_{n\to\infty}\|x_n - I^n x_n\| = 0. \tag{87}$$

Step 7 ($x^* \in \Omega = F_{ix}(I)\bigcap F_{ix}(T)\bigcap \mathrm{MEP}(\mathcal{F},\mathcal{G})$). Since $\{u_n\}$ is bounded, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\{u_{n_k}\}$ converges weakly to $x^* \in C$ when $x^* = J^{-1}p^*$ for some $p^* \in J(C)$. From (61), we have that $\{x_{n_k}\}$ converges weakly to $x^* \in C$ and, by (77), we also have that $\{y_{n_k}\}$ converges weakly to $x^* \in C$. Also, by (85), (87), and Lemma 11, we obtain that $x^* \in F_{ix}(I)\bigcap F_{ix}(T)$.

Next, we show that $x^* \in \mathrm{MEP}(\mathcal{F},\mathcal{G})$; that is, $Jx^* = p \in J(\mathrm{MEP}(\mathcal{F},\mathcal{G}))$. Since J is uniformly norm-to-norm continuous on bounded subset of E, it follows from (61) that

$$\lim_{n\to\infty}\|Jx_n - Ju_n\| = 0. \tag{88}$$

From the assumption $r_n \in [d,\infty)$, one sees

$$\lim_{n\to\infty}\frac{\|Jx_n - Ju_n\|}{r_n} = 0. \tag{89}$$

Since $\{x_n\}$ is bounded and so is $\{Jx_n\}$, there exists a subsequence $\{Jx_{n_k}\}$ of $\{Jx_n\}$ such that $\{Jx_n \rightharpoonup p^*\}$. Since $\{u_n\}$ is bounded, by (89), we also obtain $\{Ju_n \rightharpoonup p^*\}$. Noticing that $u_n = T_{r_n}x_n$, we obtain

$$\begin{gathered}\mathcal{F}(u_n,y) \le \mathcal{G}(y,u_n) + \frac{1}{r_n}\langle y-u_n, Ju_n - Jx_n\rangle, \quad y\in C, \\ \mathcal{F}(u_{n_k},y) \le \mathcal{G}(y,u_{n_k}) + \left\langle y-u_{n_k}, \frac{Ju_{n_k}-Jx_{n_k}}{r_n}\right\rangle, \\ y \in C.\end{gathered} \tag{90}$$

According to (89), we obtain $\lim_{k\to\infty}(\|Jx_{n_k} - Ju_{n_k}\|/r_{n_k}) = 0$. Then, by the conditions of (f_2) and (h_2), we obtain

$$\begin{aligned}\frac{1}{r_n}\|y-u_n\|\|Jx_n - Ju_n\| &\ge \langle y-u_n, Ju_n - Jx_n\rangle \\ &\ge -\mathcal{F}(u_n,y) + \mathcal{G}(y,u_n) \\ &\ge \mathcal{F}(y,u_n) + \mathcal{G}(y,u_n).\end{aligned} \tag{91}$$

Since $(1/r_n)\|Jx_n - Ju_n\| \rightharpoonup 0$ and $\{Ju_n \rightharpoonup p^*\}$, we obtain

$$\mathcal{F}(y,p^*) + \mathcal{G}(y,p^*) \le 0. \tag{92}$$

For t with $0 < t \le 1$ and $y \in E$, let $y_t = ty + (1-t)p^*$, we obtain

$$\mathcal{F}(y_t,p^*) + \mathcal{G}(y_t,p^*) \le 0. \tag{93}$$

So, from the conditions of (f_1), (f_3), (h_1), and (h_3), we have

$$\begin{aligned}0 &= \mathcal{F}(y_t,y_t) + \mathcal{G}(y_t,y_t) \\ &\le t\mathcal{F}(y_t,y) + (1-t)\mathcal{F}(y_t,p^*) \\ &\quad + t\mathcal{G}(y_t,y) + (1-t)\mathcal{G}(y_t,p^*) \\ &\le \mathcal{F}(y_t,y) + \mathcal{G}(y_t,y).\end{aligned} \tag{94}$$

Consequently

$$\mathcal{F}(y_t,y) + \mathcal{G}(y_t,y) \ge 0 \tag{95}$$

by (f_2) and (h_2), as $t \to 0$, and we obtain $p^* \in \mathrm{MEP}(\mathcal{F},\mathcal{G})$.

Step 8 (The sequence of $\{x_n\}$ converges strongly to a common Ω). From Step 1 and (61), for all $p \in \Omega$, $\|x_{n+1} - p\| \le (1+\rho_n)\|x_n - p\|$ for $n \ge 1$ with $\sum_{n=1}^{\infty}\rho_n < \infty$. This implies that $d(x_{n+1} - \Omega) \le (1+\rho_n)d(x_n - \Omega)$. Then by Lemma 7, $\lim_{n\to\infty}d(x_{n+1} - \Omega)$ exists. Also by Step 6, $\lim_{n\to\infty}\|x_n - T^n x_n\| = \|x_n - I^n x_n\| = 0$, and by the condition (A) in Definition 10 which guarantees that $\lim_{n\to\infty}\mathfrak{f}(d(x_{n+1} - \Omega)) = 0$. Since $\mathfrak{f}$ is a nondecreasing function and $\mathfrak{f}(0) = 0$, it follows that $\lim_{n\to\infty}d(x_n - \Omega)$. Form (81), we obtain

$$\|x_n - x_{n+m}\| \le \|x_n - u_n\| + \|x_{n+m} - u_{n+m}\|. \tag{96}$$

We know that $\{x_n\}$ is Cauchy sequence in C for all numbers m, n. This implies that $\{x_n\}$ converges strongly to $p \in \Omega$. This completes the proof. □

If T is an asymptotically quasi-nonexpansive self-mapping in Theorem 16, we easily obtain the following corollary.

Corollary 17. *Let E be uniformly smooth and uniformly convex Banach space, and let C be a nonempty closed convex subset of E. Let $\mathcal{F}, \mathcal{G} : C\times C \to R$ be two bifunctions which satisfy the conditions (f_1)–(f_4), (g_1)–(g_3), and (H), and let T be asymptotically quasi-nonexpansive self-mapping of C with sequences $\{s_n\} \subset [0,\infty\}$ such that $\sum_{n=1}^{\infty} s_n < \infty$, and let I be* *identity self-mapping of C, and $\Omega = F_{ix}(T) \bigcap \mathrm{MEP}(\mathcal{F}, \mathcal{G}) \neq \emptyset$. For an initial point $x_0 \in C$, generate a sequence $\{x_n\}$ by*

$$\begin{aligned} & u_n \in C \\ & \text{such that} \\ & \mathcal{F}(u_n, y) + \mathcal{G}(u_n, y) \\ & \quad \leq \frac{1}{r_n} \langle y - u_n, Ju_n - Jx_n \rangle, \quad \forall y \in C, \\ & y_n = \beta_n x_n + (1 - \beta_n) T^n u_n, \\ & x_{n+1} = \alpha_n x_n + (1 - \alpha_n) y_n, \quad \forall n \geq 1, \end{aligned} \tag{97}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$, $\{\beta_n\} \subset [a, b]$ for some $a, b \in (0, 1)$ and $\{r_n\} \subset [d, +\infty)$ for $d > 0$. If the following conditions are satisfied:

(i) $\sum_{n=0}^{\infty} \alpha_n < \infty$;

(ii) $\sum_{n=0}^{\infty} \beta_n < \infty$;

(iii) $\liminf_{n \to \infty} r_n > 0$,

then the sequence $\{x_n\}$ generated by (97) converges strongly to a fixed point in Ω if and only if $\liminf_{n\to\infty} d(x_n, \Omega) = 0$.

5. Numerical Example

In this section, we introduce an example of numerical test to illustrate the algorithms given in Corollary 17.

Example 1. Let $E = R$, $C = [-2000, 2000]$. The mixed equilibrium problem is to find $x \in C$ such that

$$\mathcal{F}(x, y) + \mathcal{G}(x, y) \geq 0, \quad \forall y \in C, \tag{98}$$

where we define $\mathcal{F}(x, y) = -3x^2 + 2xy + y^2$ and $\mathcal{G}(x, y) = x^2 + 3xy - 4y^2$.

Now, we can easily know that $\mathcal{F}$ and $\mathcal{G}$ satisfy the conditions (f_1)–(f_4), (g_1)–(g_3), and (H) as follows:

(f_1) $\mathcal{F}(x, x) = -3x^2 + 2xx + x^2 = 0$ for all $x \in [-2000, 2000]$;

(f_2) $\mathcal{F}(x, y) + \mathcal{F}(y, x) = -2(x - y)^2 \leq 0$ for all $x, y \in [-2000, 2000]$;

(f_3) for all $x, y, z \in [-2000, 2000]$,

$$\begin{aligned} & \limsup_{t \to 0^+} \mathcal{F}(x + t(z - x), y) \\ & \quad = \limsup_{t \to 0^+} - 3(x + t(z - x))^2 \\ & \quad\quad + 2x + t(z - x) y + y^2 \\ & \quad = -3x^2 + 2xy + y^2 \\ & \quad \leq \mathcal{F}(x, y); \end{aligned} \tag{99}$$

(f_4) for each $x \in [-2000, 2000]$, $\theta(y) = \mathcal{F}(x, y) = -3x^2 + 2xy + y^2$ is convex and weakly lower semicontinuous.

(g_1) $\mathcal{G}(x, x) = x^2 + 3xx - 4x^2 = 0$ for each $x \in [-2000, 2000]$;

(g_2) $\mathcal{G}(x, y) + \mathcal{G}(y, x) = -3(x - y)^2 \leq 0$ for all $x, y \in [-2000, 2000]$, and weakly upper semicontinuous in first variable;

(g_3) for each $x \in [-2000, 2000]$, $\theta(y) = \mathcal{G}(x, y) = x^2 + 3xy - 4y^2$ is convex.

Next, we find the formula of $S_r x$. From Theorem 14, we can claim that $S_r x$ is single-valued, for any $y \in C$, $r > 0$,

$$\begin{aligned} & \mathcal{F}(x, y) + \mathcal{G}(x, y) + \frac{1}{r} \langle x - z, y - x \rangle \\ & \quad \Longleftrightarrow -3ry^2 + (5rx + x - z) y \\ & \quad\quad + xz - 2rx^2 - x^2 \geq 0. \end{aligned} \tag{100}$$

Let $M(y) = -3ry^2 + (5rx + x - z)y + xz - 2rx^2 - x^2$. Then $M(y)$ is a quadratic function of y with coefficients $a = -3r$, $b = 5rx + x - z$, and $c = xz - 2rx^2 - x^2$. So its discriminant $\Delta = b^2 - 4ac$ is

$$\begin{aligned} \Delta &= (5rx + x - z)^2 - 4(-3r)\left(xz - 2rx^2 - x^2\right) \\ &= ((r + 1)x - z)^2. \end{aligned} \tag{101}$$

According to $M(y) \geq 0$ for all $y \in C$, form $\Delta \leq 0$, that is

$$((r + 1)x - z)^2 \leq 0. \tag{102}$$

Therefore, it follows that

$$x = \frac{z}{r + 1} \tag{103}$$

and so

$$S_r z = \frac{z}{r + 1}. \tag{104}$$

Now, let $C = [-1/\pi, 1/\pi]$ and $|k| < 1$, and define a mapping $T : C \to C$ by

$$T(x) = \begin{cases} kx \sin \dfrac{1}{x}, & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases} \tag{105}$$

for all $x \in C$. From the example in [25–27], we can easily know that T is an asymptotically quasi-nonexpansive mapping; furthermore $F_{ix}(T) = \{0\}$.

According to Theorem 14, we obtain

$$F_{ix}(S_r) = \mathrm{MEP}(\mathcal{F}, \mathcal{G}) = 0, \qquad F_{ix}(T) = 0, \tag{106}$$

and so $\Omega = 0$. Therefore, all the assumptions in Corollary 17 are satisfied. we can obtain the following numerical algorithms.

Algorithm 18. Let $r_n = 1$, $\alpha_n = 1/n^2$, and $\beta_n = 1/2n^2$. It is claim to check that

$$\begin{gathered} \sum_{n=0}^{\infty} \alpha_n < \infty, \qquad \sum_{n=0}^{\infty} \beta_n < \infty, \\ \liminf_{n \to \infty} r_n = 1. \end{gathered} \tag{107}$$

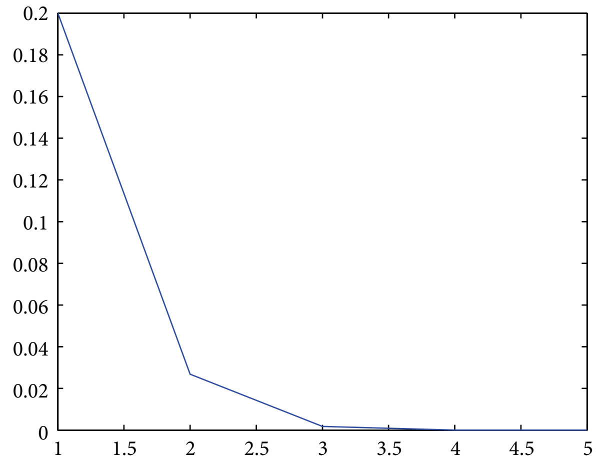

FIGURE 1: Convergence of iterative sequence $\{x_n\}$.

For an initial value $x_0 = 0.2$ and $k = 0.5$, let the sequences $\{u_n\}$ and $\{x_n\}$ generate by

$$T(x) = \frac{1}{2} x \sin \frac{1}{x},$$
$$u_n = S_r(x_n) = \frac{1}{2} x_n, \tag{108}$$
$$x_{n+1} = \frac{1+n^2}{2n^4} x_n + \frac{(1-n)(1-2n^2)}{4n^4} T^n x_n, \quad \forall n \geq 1.$$

Then, by the Corollary 17, the sequence $\{x_n\}$ converges to a solution of Example 1. Let $\|x_{n+1} - x_n\| \leq 10^{-5}$ and x^* be the fixed point of the Algorithm 18. Using the software of MATLAB, we generated a sequence $\{x_n\}$ convergence to $x^* = x_7 = 0$ as shown in Figure 1.

Hence the sequence x_n converges strongly to solve Example 1.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Convergence Axioms on Dislocated Symmetric Spaces

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Dislocated symmetric spaces are introduced, and implications and nonimplications among various kinds of convergence axioms are derived.

1. Introduction

A metric space is a special kind of topological space. In a metric space, topological properties are characterized by means of sequences. Sequences are not sufficient in topological spaces for such purposes. It is natural to try to find classes intermediate between those of topological spaces and those of metric spaces in which members sequences play a predominant part in deciding their topological properties. A galaxy of mathematicians consisting of such luminaries as Frechet [1], Chittenden [2], Frink [3], Wilson [4], Niemytzki [5], and Aranđelović and Kečkić [6] have made important contributions in this area. The basic definition needed by most of these studies is that of a symmetric space. If X is a nonempty set, a function $: X \times X \rightarrow R^+$ is called a dislocated symmetric on X if $d(x, y) = 0$ implies that $x = y$ and $d(x, y) = d(y, x)$ for all $x, y \in X$. A dislocated symmetric (simply d-symmetric) on X is called symmetric on X if $d(x, x) = 0$ for all x in X. The names dislocated symmetric space and symmetric space have expected meanings. Obviously, a symmetric space that satisfies the triangle inequality is a metric space. Since the aim of our study is to find how sequential properties and topological properties influence each other, we collect various properties of sequences that have been shown in the literature to have a bearing on the problem under study. In what follows "d" denotes a dislocated distance on a nonempty set X. x_n, y_n, x, y, and so forth are elements of X and C_i for $1 \leq i \leq 5$ and W_i for $1 \leq i \leq 3$ indicate properties of sequences in (X, d). Consider

C_1: $\lim d(x_n, y_n) = 0 = \lim d(x_n, x) \Rightarrow \lim d(y_n, x) = 0$,

C_2: $\lim d(x_n, x) = 0 = \lim d(y_n, x) \Rightarrow \lim d(x_n, y_n) = 0$,

C_3: $\lim d(x_n, y_n) = 0 = \lim d(y_n, z_n) \Rightarrow \lim d(x_n, z_n) = 0$.

A space in which C_1 is satisfied is called coherent by Pitcher and Chittenden [7]. Niemytzki [5] proved that a coherent symmetric space (X, d) is metrizable, and in fact there is a metric ρ on X such that (X, d) and (X, ρ) have identical topologies and also that $\lim d(x_n, x) = 0$ if and only if $\lim \rho(x_n, x) = 0$.

Cho et al. [8] have introduced

C_4: $\lim d(x_n, x) = 0 \Rightarrow \lim d(x_n, y) = d(x, y)$ for all y in X,

C_5: $\lim d(x_n, x) = \lim d(x_n, y) = 0 \Rightarrow x = y$.

The following properties were introduced by Wilson [4]:

W_1: for each pair of distinct points a, b in X there corresponds a positive number $r = r(a, b)$ such that $r < \inf_{c \in X} d(a, c) + d(b, c)$,

W_2: for each $a \in X$, for each $k > 0$, there corresponds a positive number $r = r(a, k)$ such that if b is a point of X such that $d(a, b) \geq k$ and c is any point of X then $d(a, c) + d(c, b) \geq r$,

W_3: for each positive number k there is a positive number $r = r(k)$ such that $d(a, c) + d(c, b) \geq r$ for all c in X and all a, b in X with $d(a, b) \geq k$.

2. Implications among the Axioms

Proposition 1. *In a d-symmetric space* (X, d), $C_3 \Rightarrow C_1 \Rightarrow C_5$, $C_3 \Rightarrow C_2$, *and* $C_4 \Rightarrow C_5$.

Proof. Assume that C_3 holds in (X, d) and let $\lim d(x_n, y_n) = 0$ and $\lim d(x_n, x) = 0$. Put $z_n = x\ \forall n$ so that

$$\begin{aligned}\lim d(x_n, z_n) &= \lim d(x_n, x) = 0\\ &= \lim d(x_n, y_n) = \lim d(y_n, x_n).\end{aligned} \tag{1}$$

By C_3, $\lim d(y_n, z_n) = 0$; that is, $\lim d(y_n, x) = 0$.

Hence

$$C_3 \Rightarrow C_1. \tag{2}$$

Assume that C_1 holds in (X, d) and let $\lim d(x_n, x) = 0$ and $\lim d(x_n, y) = 0$. Put $y_n = y\ \forall n$; then

$$\lim d(x_n, y_n) = \lim d(x_n, x) = 0. \tag{3}$$

By C_1, $\lim d(y_n, x) = 0$; that is, $\lim d(y, x) = 0$.

Consider $\lim d(x, y) = 0$; this implies that $x = y$. Hence C_5 holds. Thus

$$C_1 \Longrightarrow C_5. \tag{4}$$

Assume that C_3 holds and let $\lim d(x_n, x) = 0$ and $\lim d(y_n, x) = 0$.

Put $z_n = x\ \forall n$; then $\lim d(x_n, z_n) = \lim d(z_n, y_n) = 0$.

By C_3, $\lim d(x_n, y_n) = 0$. Hence

$$C_3 \Longrightarrow C_2. \tag{5}$$

Assume that C_4 holds and let $\lim d(x_n, x) = 0$ and $\lim d(x_n, y) = 0$.

By C_4, $\lim d(x_n, y) = d(x, y)$. Hence $d(x, y) = 0$. Hence $x = y$. □

The following proposition explains the relationship between Wilson's axioms [4] W_1, W_2, and W_3 and the C_i's.

Proposition 2. *Let* (X, d) *be a d-symmetric space; then*

$$(i)\ W_1 \Leftrightarrow C_5,\quad (ii)\ W_2 \Leftrightarrow C_1,\ \text{and}\ (iii)\ W_3 \Leftrightarrow C_3.$$

Proof. (i) Assume W_1. Suppose $\lim d(a, x_n) = \lim d(b, x_n) = 0$ but $a \neq b$.

Then

$$\lim \{d(a, x_n) + d(b, x_n)\} = 0 \quad \text{but } a \neq b. \tag{6}$$

By

$$W_1\ \exists\ r > 0 \ni \forall x, \quad d(a, x) + d(b, x) \geq r, \tag{7}$$

equations (6) and (7) are contradictory. Hence $a = b$. Thus $W_1 \Rightarrow C_5$.

Suppose that W_1 fails. Then there exist $a \neq b$ in X such that for every n there corresponds x_n in X such that $d(a, x_n) + d(b, x_n) < 1/n$:

$$\Longrightarrow \lim d(a, x_n) = \lim d(b, x_n) = 0 \quad \text{but } a \neq b. \tag{8}$$

Thus if W_1 fails then C_5 fails. That is, $C_5 \Rightarrow W_1$. Hence $W_1 \Leftrightarrow C_5$.

(ii) Assume W_2. Then for each $a \in X$ and each $k > 0$ there corresponds $r > 0$ such that, for all $b \in X$ with $d(a, b) \geq k$ and $\forall x \in X$, $d(a, x) + d(b, x) \geq r$.

Suppose that C_1 fails. There exist $a \in X$, $\{b_n\}$, and $\{c_n\}$ in X such that $\lim d(a, b_n) = \lim d(b_n, c_n) = 0$ but $\lim d(a, c_n) \neq 0$.

Since $\lim d(a, c_n) \neq 0$ there exists $k > 0$ and a subsequence (c_{n_k}) such that

$$d(a, c_{n_k}) > k \quad \forall n_k. \tag{9}$$

Since

$$d(a, c_{n_k}) > k, \qquad d(a, b_{n_k}) + d(b_{n_k}, c_{n_k}) \geq r, \tag{10}$$

this implies that $\lim\{d(a, b_n) + d(b_n, c_n)\} \neq 0$, a contradiction.

Conversely assume that W_2 fails. Then there exist $a \in X$ and $k > 0$ such that $\forall n > 0\ \exists\ b_n \in X$ and $c_n \in X$ such that

$$d(a, b_n) \geq k \quad \text{but } d(a, c_n) + d(b_n, c_n) < \frac{1}{n}. \tag{11}$$

This implies that $\lim d(a, c_n) = \lim d(b_n, c_n) = 0$ but $\lim d(a, b_n) \neq 0$.

Hence C_1 fails.

(iii) Assume W_3. Suppose that C_3 fails. Then there exist sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ in X such that $\lim d(a_n, b_n) = \lim d(b_n, c_n) = 0$ but $\lim d(a_n, c_n) \neq 0$.

Since W_3 holds, $\forall k > 0$ there corresponds $r > 0$ such that for all a, b with

$$d(a, b) \geq k, \qquad d(a, c) + d(b, c) \geq r \quad \forall c. \tag{12}$$

Since $\lim d(a_n, c_n) \neq 0$ there exists a positive number ϵ and a subsequence of positive integers $\{n_k\}$ such that $d(a_{n_k}, c_{n_k}) > \epsilon$. Choose r_1 corresponding to ϵ so that

$$d(a_{n_k}, b_{n_k}) + d(b_{n_k}, c_{n_k}) \geq r_1. \tag{13}$$

Thus

$$\lim \{d(a_{n_k}, b_{n_k}) + d(b_{n_k}, c_{n_k})\} \neq 0. \tag{14}$$

This contradicts the assumption that $\lim d(a_n, b_n) = \lim d(b_n, c_n) = 0$.

Hence

$$W_3 \Longrightarrow C_3. \tag{15}$$

Assume that W_3 fails.

Then there exists $k > 0$ such that, $\forall$ positive integer n, there exist a_n, b_n, and c_n with

$$d(a_n, b_n) \geq k \quad \text{but } d(a_n, c_n) + d(b_n, c_n) < \frac{1}{n}. \tag{16}$$

Hence

$$\lim d(a_n, b_n) \neq 0 \quad \text{but } \lim d(a_n, c_n) = \lim d(c_n, b_n) = 0. \tag{17}$$

Hence C_3 fails.
Hence

$$C_3 \Longrightarrow W_3. \tag{18}$$

This completes the proof of the proposition. □

We introduce the following.

Axiom C. Every convergent sequence satisfies Cauchy criterion. That is, if (x_n) is a sequence in X, $x \in X$ and $\lim d(x_n, x) = 0$; then given $\epsilon > 0\ \exists\ N(\epsilon) \in \mathbf{N}$ such that $d(x_n, x_m) < \epsilon$ whenever $m, n \geq N(\epsilon)$ we have the following.

Proposition 3. *In a d-symmetric space (X, d), $C_1 \Rightarrow C \Rightarrow C_2$.*

Proof. For $C_1 \Rightarrow C$, suppose that a sequence (x_n) in (X, d) is convergent to x but does not satisfy Cauchy criterion. Then $\exists r > 0$ such that for every positive integer k there correspond integers m_k, n_k such that

$$m_{k+1} > n_{k+1} > m_k > n_k, \quad d\left(x_{m_k}, x_{n_k}\right) > \gamma \quad \forall k. \tag{19}$$

Let

$$y_k = x_{m_k}, \quad z_k = x_{n_k} \quad \forall k. \tag{20}$$

Then

$$\lim d\left(y_k, x\right) = 0, \qquad \lim d\left(z_k, x\right) = 0. \tag{21}$$

But $\lim d(y_k, z_k) \neq 0$; this contradicts C_1. □

Proof. For $C \Rightarrow C_2$, suppose that $\lim d(x_n, x) = \lim d(y_n, x) = 0$.

Let (z_n) be the sequence defined by $z_{2n-1} = x_n$ and $z_{2n} = y_n$. Then $\lim d(z_n, x) = 0$. Hence (z_n) satisfies Cauchy criterion.

Given $\epsilon > 0\ \exists N(\epsilon) \in \mathbf{N}$ such that $d(z_n, z_m) < \epsilon$ for $m, n \geq N(\epsilon)$:

$$\Rightarrow d(z_{2n-1}, z_{2n}) < \epsilon \quad \text{for } n \geq N(\epsilon),$$

$$\Rightarrow \lim d(x_n, y_n) < \epsilon \quad \text{for } n \geq N(\epsilon),$$

$$\Rightarrow \lim d(x_n, y_n) = 0.$$

□

3. Examples for Nonimplications

Example 4. A d-symmetric space in which the triangular inequality fails and C_1 through C_5 hold.

Let $X = [0, 1]$. Define d on $X \times X$ as follows:

$$d(x, y) = \begin{cases} x + y & \text{if } x \neq y, \\ 1 & \text{if } x = y \neq 0, \\ 0 & \text{if } x = y = 0. \end{cases} \tag{22}$$

Clearly d is a d-symmetric space. d does not satisfy the triangular inequality since $d(0.1, 0.2) + d(0.2, 0.1) = 0.6 < 1 = d(0.1, 0.1)$.

We show that C_1 through C_5 holds. We first show that $\lim d(x_n, x) = 0$ iff $x = 0$ and $\lim x_n = 0$ in R.

If $x \neq 0$ then $\lim d(x_n, x) = x_n + x \geq x > 0$. Hence $\lim d(x_n, x) \geq x > 0$.

If $x = 0$ then $\lim d(x_n, 0) = 0$ or x_n. Hence $\lim d(x_n, x) = 0 \Leftrightarrow \lim x_n = 0$ in R.

Now we show that $\lim d(x_n, y_n) = 0$ if and only if $\lim x_n = \lim y_n = 0$ in R.

Consider $\lim d(x_n, y_n) = 0 \Rightarrow d(x_n, y_n) < 1/2$ for large n:

$\Rightarrow d(x_n, y_n) = x_n + y_n$ or 0 for large n,

$\Rightarrow$ either $x_n = y_n = 0$ or $d(x_n, y_n) = x_n + y_n$ for large n,

$\Rightarrow \lim x_n = \lim y_n = 0$ in R.

Conversely if $\lim x_n = \lim y_n = 0$ in R then $\lim d(x_n, y_n) = 0$ or $x_n + y_n$ for large n.

Hence $\lim d(x_n, y_n) = 0$.

Verification of validity of C_1 through C_5 is done as follows.

C_1: let $\lim d(x_n, y_n) = 0$ and $\lim d(x_n, x) = 0$; then $\lim x_n = \lim y_n = 0$ in R and $x = 0$.

Hence $d(y_n, x) = d(y_n, 0) = y_n$ or 0. This implies that $\lim d(y_n, x) = 0$.

C_2: let $d(x_n, x) = d(y_n, x) = 0$. Then $x = 0$ and $\lim x_n = \lim y_n = 0$ in R.

Hence $\lim d(y_n, x_n) = 0$.

C_3: let $d(x_n, y_n) = d(y_n, z_n) = 0$; then $\lim x_n = \lim y_n = \lim z_n = 0$ in R.

Hence $\lim d(x_n, z_n) = 0$.

C_4: let $\lim d(x_n, x) = 0$. Then $x = 0$ and $\lim x_n = 0$.

If $y = 0$, $0 \leq d(x_n, y) \leq x_n$. Hence $\lim d(x_n, y) = 0 = d(x, y)$.

If $y \neq 0$, $d(x_n, y) = x_n + y$. Hence $\lim d(x_n, y) = y = 0 + y = d(x, y)$.

C_5: let $\lim d(x_n, x) = 0$ and $\lim d(x_n, y) = 0$.

Then $x = 0$, $y = 0$ and $\lim x_n = 0$. Hence $x = y$.

Example 5. A d-symmetric space (X, d) in which C_1 [hence C_5] holds while C_j does not hold for $j = 2, 3, 4$.

Let $X = [0, \infty)$. Define d on $X \times X$ as follows:

$$d(x, y) = \begin{cases} x + y & \text{if } x \neq 0 \neq y, \\ \dfrac{1}{x} & \text{if } x \neq 0 = y, \\ \dfrac{1}{y} & \text{if } x = 0 \neq y, \\ 0 & \text{if } x = 0 = y. \end{cases} \tag{23}$$

Clearly (X, d) is a d-symmetric space. We show that C_1, C_5 hold.

Let $\lim d(x_n, x) = 0 = \lim d(x_n, y_n)$.

If $x \neq 0$, $d(x_n, x) > x$ if $x_n \neq 0$.

$$= \frac{1}{x} \text{ if } x_n = 0. \tag{24}$$

This implies that

$$\lim d(x_n, x) \geq \min\left\{x, \frac{1}{x}\right\} > 0. \tag{25}$$

Thus $\lim d(x_n, x) = 0 \Rightarrow x = 0$ and (x_n) can be split into two subsequences $({x_n}^{(1)}), ({x_n}^{(2)})$, where $({x_n}^{(1)}) = 0\ \forall n$, $({x_n}^{(2)}) \neq 0$ for every n and if $({x_n}^{(2)})$ is infinite subsequence $\lim({x_n}^{(2)}) = \infty$. We consider the case where both $({x_n}^{(1)})$ and $({x_n}^{(2)})$ are infinite sequences as when one is a finite sequence the same proof works with minor modifications. Consider

$$\begin{aligned} \lim d(x_n, y_n) = 0 \Longrightarrow{} & \lim d\left({x_n}^{(1)}, {y_n}^{(1)}\right) \\ & = \lim d\left({x_n}^{(2)}, {y_n}^{(2)}\right) = 0. \end{aligned} \tag{26}$$

If we show that ${y_n}^{(2)}$ cannot be positive for infinitely many n, it will follow that $\lim d({x_n}^{(2)}, {y_n}^{(2)}) = \lim d({x_n}^{(2)}, 0) = 0$ so that $\lim d(0, y_n) = 0$. Hence C_1 holds.

If ${y_n}^{(2)} \neq 0$ for infinitely many n, say $\{{y_{n_k}}^{(2)}\}$ is the infinite subsequence of $\{{y_n}^{(2)}\}$ with ${y_{n_k}}^{(2)} \neq 0\ \forall n_k$, then $d({x_{n_k}}^{(2)}, {y_{n_k}}^{(2)}) = {x_{n_k}}^{(2)} + {y_{n_k}}^{(2)} > {x_{n_k}}^{(2)}$ so that $\lim d({x_{n_k}}^{(2)}, {y_{n_k}}^{(2)}) \geq \lim {x_{n_k}}^{(2)} \geq \infty$ contradicting the assumption that $\lim d(x_n, y_n) = 0$. Thus C_1 holds. Since $C_1 \Rightarrow C_5$, C_5 holds.

C_2 does not hold since $d(n, 0) = 1/n$ while $d(n, n) = 2n\ \forall n$ so that $\lim d(n, n) \neq 0$.

C_3 does not hold since $\lim d(n, 0) = \lim d(0, n)$ while $\lim d(n, n) = \infty$.

C_4 does not hold since $\lim d(n, 0) = 0$ but $\lim d(n, 2) = \infty$ while $d(0, 2) = 1/2$.

Example 6. A d-symmetric space (X, d) in which C_2 holds but C_1, C_3 , C_4, and C_5 fail.

Let $X = [0, 1] \cup \{2\}$. Define d on $X \times X$ as follows:

$$d(x, y) = \begin{cases} x + y & \text{if } 0 \leq x \leq 1,\ 0 \leq y \leq 1 \\ x & \text{if } 0 \leq x \leq 1,\ y = 2 \\ y & \text{if } x = 2,\ 0 \leq y \leq 1 \\ 1 & \text{if } \begin{cases} x = 2,\ y \in \{0, 2\} \\ \quad \text{or} \\ x \in \{0, 2\},\ y = 2. \end{cases} \end{cases} \tag{27}$$

Clearly (X, d) is a d-symmetric space.

We first show that if $\{x_n\}$ in X converges to x in (X, d) then $x \in \{0, 2\}$.

Suppose that $x \neq 0$ and $x \neq 2$; then $x \in (0, 1]$:

$\Rightarrow \lim d(x_n, x) = 0 = x_n + x$ or x,

$\Rightarrow \lim d(x_n, x) \geq x > 0$,

$\Rightarrow \lim d(x_n, x) \neq 0$.

Hence if $\lim d(x_n, x) = 0$ then $x \in \{0, 2\}$.

C_1 fails: $x_n = 1/n$, $y_n = 2$, and $x = 0$;

$$\begin{gathered} d(x_n, y_n) = \frac{1}{n}, \qquad d(x_n, x) = \frac{1}{n}, \qquad d(y_n, x) = 1 \\ \Longrightarrow \lim d(x_n, y_n) = 0 = d(x_n, x) \quad \text{but } \lim d(y_n, x) \neq 0. \end{gathered} \tag{28}$$

C_2 holds: suppose that $\lim d(x_n, x) = \lim d(y_n, x) = 0$; then $x \in \{0, 2\}$.

Case 1. If $x = 2$, $\lim d(x_n, x) \rightarrow 0 \Rightarrow d(x_n, x) = x_n$ eventually and $\lim x_n = 0$ in R. Hence $\exists N \in \mathbf{N} \ni x_n < 1$ and $y_n < 1$ for $n \geq N$.

Here $d(x_n, y_n) = x_n + y_n$. This implies that $\lim d(x_n, y_n) = 0$.

Case 2. If $x = 0$,

$$d(x_n, 0) = \begin{cases} 1 & \text{if } x_n = 2 \text{ or } 0, \\ x_n & \text{if } 0 \leq x_n \leq 1. \end{cases} \tag{29}$$

If $\lim d(x_n, 0) = 0$, $d(x_n, 0) = x_n$ eventually and $\lim x_n = 0$ in R.

Similarly $d(y_n, 0) = y_n$ eventually and $\lim y_n = 0$ in R. As in Case 1 it follows that

$$\lim d(x_n, y_n) = \lim(x_n + y_n) = 0. \tag{30}$$

Thus C_2 holds.

C_3 fails since $C_3 \Rightarrow C_1$.

C_5 fails: let $x_n = 1/n$, $x = 0$, and $y = 2$

$$\lim d(x_n, 0) = \lim\left(\frac{1}{n}\right) = 0 = \lim d(x_n, 2) \tag{31}$$

C_4 fails since $C_4 \Rightarrow C_5$.

Example 7. A d-symmetric space (X,d) in which C_4 holds but C_1 fails.

Let $X = N \cup \{0\}$. Define d on $X \times X$ as follows:

$$d(m,n) = d(n,m) \quad \forall m,n \in X,$$
$$d(0,n) = \begin{cases} \dfrac{1}{n} & \text{if } n \text{ is odd,} \\ 1 & \text{if } n \text{ is even,} \end{cases}$$
$$d(0,0) = 0,$$
$$d(m,n) = \begin{cases} \left|\dfrac{1}{m} - \dfrac{1}{n}\right| & \text{if } m+n \text{ is even} \\ & \text{or } m+n \text{ is odd and } |m-n| = 1, \\ 1 & \text{if } m+n \text{ is odd and } |m-n| > 2. \end{cases} \tag{32}$$

If $\{x_n\}$ in X and $\lim d(x_n,0) = 0$ then x_n is eventually odd.

If $x \neq 0, d(x_n,x)$ cannot be 1 so $x_n + x$ is even or odd and $|x_n - x| = 1$.

But in this case $d(x_n,x) = |1/x_n - 1/x|$ so that $d(x_n,x) \neq 0$.

Thus $d(x_n,x) = 0 \Leftrightarrow x = 0$ and x_n is eventually odd.

If m is a fixed even integer and x_n is odd, $x_n + m$ is odd and eventually >2.

So

$$\lim d(x_n,m) = 1 = d(0,m). \tag{33}$$

If m is a fixed odd integer and x_n is odd, $x_n + m$ is even.

So $d(x_n,m) = |1/m - 1/x_n|$ so that $\lim d(x_n,0) = 0 \Rightarrow \lim d(x_n,m) = d(0,m)$.

If $m=0$ and x_n is odd eventually

$$d(x_n,0) = \frac{1}{n} \quad \text{so } \lim d(x_n,m) = \lim \frac{1}{n} = 0 = d(0,m). \tag{34}$$

If $m = 0$ and $x_n = 0$ eventually

$$d(x_n,0) = \frac{1}{n} \quad \text{so } \lim d(x_n,m) = \lim \frac{1}{n} = 0 = d(0,m). \tag{35}$$

Hence C_4 holds in (X,d).

C_1 does not hold: let $x_n = 2n-1$ and $y_n = 2n$:

$$d(x_n,0) = \frac{1}{2n-1}, \qquad d(x_n,y_n) = \frac{1}{2n-1} - \frac{1}{2n}, \qquad d(y_n,0) = 1. \tag{36}$$

Hence $d(x_n,0) = d(x_n,y_n) = 0$ and $d(y_n,0) \neq 0$.

Example 8. A d-symmetric space (X,d) in which C_3 holds but C_4 does not hold.

Let $X = [0,1] \cup \{2\}$. Define d on $X \times X$ as follows:

$$d(x,y) = \begin{cases} x+y & \text{if } 0 \le x \neq y \le 1, \\ 1 & \text{if } x = y \neq 0 \text{ or } x = y = 2 \\ & \text{or } x \in (0,1] \text{ and } y = 2, \\ 2 & \text{if } x = 0 \ \& \ y = 2 \text{ or } x = 2 \text{ and } y = 0, \\ 0 & \text{if } x = y = 0. \end{cases} \tag{37}$$

Clearly (X,d) is a d-symmetric space which is not a symmetric space.

We first show that if $\{x_n\}$ converges to x in (X,d) then $x \in \{0,2\}$.

Suppose that $0 \neq x \neq 2$; then $x \in (0,1]$:

$$\Longrightarrow d(x,x_n) = \begin{cases} x + x_n & \text{if } 0 < x \neq x_n \le 1, \\ 1 & \text{if } x = x_n \neq 0 \\ & \text{or } x_n = 2 \text{ or } x \in (0,1] \\ & \text{and } x_n = 2. \end{cases} \tag{38}$$

Since $\lim d(x,x_n) = 0 \ \exists N \ni d(x,x_n) < 1$ for $n \ge N$

$\Rightarrow d(x,x_n) = x + x_n \ge x$ for $n \ge N$,

$\Rightarrow \lim d(x,x_n) \neq 0$, a contradiction.

We now show that $\lim d(x_n,y_n) = 0$ if and only if $\lim x_n = \lim y_n = 0$. Consider

$$\begin{aligned} &\lim d(x_n,y_n) \\ &= 0 \Longrightarrow \exists N \in \mathbf{N} \ni d(x_n,y_n) < 1 \quad \text{for } n \ge N \\ &\Longrightarrow \lim d(x_n,y_n) = x_n + y_n \quad \text{or} \quad 0 \quad \text{for } n \ge N \\ &\Longrightarrow \text{either } x_n = y_n = 0 \quad \text{or} \quad d(x_n,y_n) = x_n + y_n \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } n \ge N \\ &\Longrightarrow \lim x_n = \lim y_n = 0. \end{aligned} \tag{39}$$

Conversely if $\lim x_n = \lim y_n = 0$ then $\exists N \in \mathbf{N} \ni x_n < 1, y_n < 1$ for $n \ge N \Rightarrow \lim d(x_n,y_n) = 0$ or $x_n + y_n$ for large n.

Hence $d(x_n,y_n) = 0$.

Thus $d(x_n,y_n) = 0$ if and only if $\lim x_n = \lim y_n = 0$.

As a consequence we have

$$\lim d(x_n,y_n) = 0 = \lim d(y_n,z_n) \Longrightarrow \lim d(x_n,z_n) = 0. \tag{40}$$

Hence C_3 holds in (X,d).

C_4 fails: $x_n = 1/(n+1)$ for $n \ge 1$:

$$d(x_n,0) = \frac{1}{n+1} \Longrightarrow \lim d(x_n,0) = 0,$$
$$d(x_n,2) = 1 \quad \forall n \Longrightarrow \lim d(x_n,2) = 1 \quad \text{but } d(0,2) = 2. \tag{41}$$

Example 9. A d-symmetric space (X,d) in which C_4 holds but C_2, C_3 fail to hold.

Let $X = N \cup \{0, \infty\}$. Define d on $X \times X$ as follows:

$$d(m,\infty) = d(\infty, m) = 1 \quad \text{if } m \in X,$$
$$d(m,0) = d(0,m) = \frac{1}{m} \quad \text{if } m \in N, \tag{42}$$
$$d(0,0) = 0.$$

If $m, n \in N$,

$$d(m,n) = \begin{cases} \left|\dfrac{1}{m} - \dfrac{1}{n}\right| & \text{if } |m-n| \geq 2, \\ 1 & \text{if } |m-n| \leq 1. \end{cases} \tag{43}$$

Clearly (X, d) is a d-symmetric space which is not a symmetric space.

We show that if $\lim d(x_n, x) = 0$ then $x = 0$ and $\{x_n\}$ consists of two subsequences $\{y_n\}$ and $\{z_n\}$, one of which may possibly be finite, where $y_n = 0\ \forall n$ and $0 \neq z_n \in N\ \forall n$ and $\lim(1/z_n) = 0$ (in case $\{z_n\}$ is an infinite sequence).

To prove this we first note that $\lim d(x_n, x) = 0 \Rightarrow x \neq \infty$ and $x_n \neq \infty$ eventually.

If $x \in N$, $d(x_n, x) = 1/x$ or 1 or $|1/x_n - 1/x|$.

Hence $\lim d(x_n, x) = 0 \Rightarrow x \notin N$; hence $x = 0$.

Further $d(x_n, 0) = 0$ or $1/x_n$. Consequently $\{x_n\}$ may be split into two sequences $\{y_n\}$ and $\{z_n\}$ as described above.

We show that C_4 holds. Assume that $\lim d(x_n, x) = 0$. Then $x = 0$.

Let $m \in N$ and $y_n = 0\ \forall n$. Then $d(y_n, m) = d(o, m) = 1/m$.

So $\lim d(y_n, m) = d(o, m)$.

If $z_n \neq 0\ \forall n$, and $\lim(1/z_n) = 0$ the $d(z_n, m) = |1/z_n - 1/m|$ for $n > m$ so that $\lim d(z_n, m) = 1/m = d(0, m)$.

Thus if $m \in N$ and $\lim d(x_n, x) = 0$ then $\lim d(x_n, m) = d(x, m)$.

Clearly this holds when $m = \infty$ or $m = 0$ as well.

Hence C_4 holds.

C_2 does not hold: let $x_n = x$, $y_n = n+1$, and $x = 0$:

$$d(x_n, x) = d(n, 0) = \frac{1}{n}, \quad \text{hence } \lim d(x_n, x) = 0,$$
$$d(y_n, x) = d(n+1, 0) = \frac{1}{n+1}, \quad \text{hence } \lim d(y_n, x) = 0,$$
$$\lim d(x_n, y_n) = \lim d(n, n+1) \Longrightarrow \lim d(x_n, y_n) \neq 0. \tag{44}$$

C_3 does not hold:

$$x_n = n, \qquad y_n = n+2, \qquad z_n = x_n,$$
$$d(x_n, y_n) = d(n, n+2) = \left|\frac{1}{n+2} - \frac{1}{n}\right| = \frac{1}{n} - \frac{1}{n+2},$$
$$d(y_n, z_n) = d(x_n, y_n) = \frac{1}{n} - \frac{1}{n+2},$$
$$\lim d(x_n, z_n) = \lim d(n, n) = 1,$$
$$\lim d(x_n, y_n) = \lim d(y_n, z_n) = 0 \quad \text{but } \lim d(x_n, z_n) = 1. \tag{45}$$

C_5 holds since $C_4 \Rightarrow C_5$.

Remarks. From this example we can conclude that

(1) C_5 does not imply C_2 as otherwise, since $C_4 \Rightarrow C_5$ it would follow that $C_4 \Rightarrow C_2$ which does not hold as is evident from the above example,

(2) in a d-symmetric space, convergent sequences are necessarily Cauchy sequences.

Example 10. A d-symmetric space (X, d) in which C_4 holds but C_2, C_3 fail to hold.

Let $X = N \cup \{0\}$. Define d on $X \times X$ as follows:

$$d(x,y) = d(y,x) = 1 \quad \text{for every } x, y \in X,$$
$$d(2m, 0) = 1,$$
$$d(2m-1, 0) = \frac{1}{2m-1} \quad \forall m,$$
$$d(0,0) = 0,$$
$$d(m,n) = \begin{cases} \dfrac{1}{m} + \dfrac{1}{n} & \text{if } m+n \text{ is even or } |m-n| = 1, \\ 1 & \text{if } m+n \text{ is odd and } |m-n| > 2. \end{cases} \tag{46}$$

Clearly (X, d) is a d-symmetric space.

We first characterize all convergent sequences in (X, d).

Suppose that $\lim d(x_n, x) = 0$. We show that $x = 0$.

If x is odd and x_n is even $d(x_n, x) = 1$ if $x_n > x + 2$.

So $\lim d(x_n, x) \neq 0$. Thus x_n is even for at most finitely many n.

We may thus assume that x_n is odd $\forall n$.

The $d(x_n, x) = 1/x_n + 1/x$ so that $d(x_n, x) \geq 1/x > 0$.

Hence x cannot be odd. Now suppose that $x > 0$ and x is even.

Then $d(x_n, x) = 1$ if $x_n = 0$ if x_n is odd and $|x_n - x| >$ 2while $d(x_n, x) = 1/x_n + 1/x$ if $x_n + x$ is even or $|x_n - x| = 1$. In all cases $\lim d(x_n, x) \neq 0$.

Hence the only possibility is $x = 0$.

We now show that the following are equivalent.

(a) $\lim d(x_n, x) = 0$ in R,

(b) there exists a positive integer N such that x_n is positive and even, only if $n < N$.

Assumption (b): x_n is odd or zero if $n \geq N$ so that $\lim d(x_n, 0) = \lim(1/x_n) = 0$.

Hence (b)$\Rightarrow$(a).

Assumption (a): since $d(2m, 0) = 1$ for $m \in N$, it follows that at most finitely many terms of $\{x_n\}$ can be even. This proves (b). Thus $\lim d(x_n, x) = 0 \Leftrightarrow x = 0$ and $\exists N \in \mathbf{N} \ni x_n$ is "0" or odd for $n \geq N$.

Consequently C_5 holds.

C_1 does not hold: let $x_n = 2n+1$, $y_n = 2n$ and $x = 0$;

$$\lim d(x_n, x) = \lim \frac{1}{2n+1} = 0,$$
$$\lim d(x_n, y_n) = \lim \frac{1}{2n+1} + \frac{1}{2n} = 0. \tag{47}$$

But $\lim d(y_n, x) = 1$ since $\lim d(2n, 0) = 1\ \forall n$.

C_2 holds: assume that $\lim d(x_n, x) = 0 = \lim d(y_n, x)$.

Then $x = 0$ and then there exists N such that x_n is "0" or odd and $y_n = 0$ or odd for $n \geq N$ and $\lim(1/x_n) = \lim(1/y_n) = 0$.

If $x_n = y_n = 0$, $d(x_n, y_n) = 0$.

If $x_n = 0$, y_n is odd, $d(x_n, y_n) = 1/y_n$.

If $y_n = 0$, x_n is odd, $d(x_n, y_n) = 1/x_n$.

If x_n is odd and y_n is odd, $d(x_n, y_n) = 1/x_n + 1/y_n$.

Consequently $\lim d(x_n, y_n) = 0$.

C_3 does not hold: let $x_n = 0$, $y_n = 2n + 1$, and $z_n = 2n$:

$$d(x_n, y_n) = \frac{1}{2n+1}, \qquad d(y_n, z_n) = \frac{1}{2n+1} + \frac{1}{2n}, \qquad d(x_n, z_n) = 1 \tag{48}$$

so that $\lim d(x_n, y_n) = \lim d(y_n, z_n) = 0$ but $\lim d(x_n, z_n) = 1$.

C_4 does not hold: let $x_n = 2n + 1$, $x = 0$, and $y = 3$:

$$\lim d(x_n, 0) = \lim \frac{1}{2n+1} = 0, \qquad \lim d(x_n, 3) = 1, \qquad \lim d(0, 3) = \frac{1}{3}. \tag{49}$$

Example 11. The following example shows that there exist symmetric spaces in which C does not hold.

Let $X = \{0, 1/2, 1/3, 1/4, \ldots\}$.

Define $d(x, x) = 0$, $d(x, y) = d(y, x)$

$$d\left(\frac{1}{n}, 0\right) = d\left(0, \frac{1}{n}\right) = \frac{1}{n} \quad \forall n \text{ in } N, \qquad d\left(\frac{1}{n}, \frac{1}{m}\right) = 1 \quad \forall n, m \text{ in } N. \tag{50}$$

Then (X, d) is a symmetric space; $\{1/n\}$ converges to 0 but is not a Cauchy sequence.

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Professor I. Ramabhadra Sarma is a retired professor from Acharya Nagarjuna University.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Stable Perturbed Iterative Algorithms for Solving New General Systems of Nonlinear Generalized Variational Inclusion in Banach Spaces

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We introduce and study a new general system of nonlinear variational inclusions involving generalized m-accretive mappings in Banach space. By using the resolvent operator technique associated with generalized m-accretive mappings due to Huang and Fang, we prove the existence theorem of the solution for this variational inclusion system in uniformly smooth Banach space, and discuss convergence and stability of a class of new perturbed iterative algorithms for solving the inclusion system in Banach spaces. Our results presented in this paper may be viewed as an refinement and improvement of the previously known results.

1. Introduction

Let m be a given positive integer, for any $i \in \{1, 2, \ldots, m\}$, X_i a real Banach space with dual space X_i^*. X_i, X_i^* all endowed with the norm $\|\cdot\|$, and $\langle\cdot,\cdot\rangle$ the dual pair between X_i and X_i^* (as matter of convenience). Let 2^{X_i} denote the family of all the nonempty subsets of X_i, $\eta_i : X_i \times X_i \to X_i^*$, $N_i : X_1 \times X_2 \times \cdots \times X_m \to X_i$ single-valued mappings, and $M_i : X_i \to 2^{X_i}$ generalized m-accretive mapping for $i = 1, 2, \ldots, m$. In this paper, we consider the following new general system for nonlinear variational inclusion involving generalized m-accretive mappings. Find $(x_1^*, x_2^*, \ldots, x_m^*) \in X_1 \times X_2 \times \cdots \times X_m$ such that

$$0 \in N_i(x_1^*, x_2^*, \ldots, x_m^*) + M_i(x_i^*) \tag{1}$$

for all $i = 1, 2, \ldots, m$. Some special cases of the problem (1) had been studied by many authors. See, for example, [1–34] and the reference therein. Here, we mention some of them as follows.

Case 1. The problem (1) with $X_i = \mathscr{H}_i$ $(i = 1, 2, \ldots, m)$, the Hilbert spaces, was introduced and studied as general system of monotone nonlinear variational inclusions problems by Peng and Zhao [29].

If $J_q^{-1}\eta_i(x_i^1, x_i^2) = x_i^1 - x_i^2$ and $M_i = \partial\varphi_i$, $\varphi_i : X_i \to (-\infty, +\infty]$ is proper, convex, and lower semi-continuous functional on X_i, and $\partial\varphi_i$ denote the subdifferential operators of the φ_i for $i = 1, 2, \ldots, m$, then the problem (1) is equivalent to finding $(x_1^*, x_2^*, \ldots, x_m^*) \in X_1 \times X_2 \times \ldots \times X_m$ such that

$$\begin{aligned}&\langle N_i(x_1^*, x_2^*, \ldots, x_m^*), j(x_i - x_i^*)\rangle \\ &\quad \geq \rho_i(\varphi_i(x_i^*) - \varphi_i(x_i)), \quad \forall x_i \in X_i.\end{aligned} \tag{2}$$

When $X_i = X$, 2-uniformly smooth Banach space with the smooth constant K, C is a nonempty closed convex subset of X, $N_i(x_1, x_2, \ldots, x_m) = \rho_i A_i(x_{i+1}) + x_i - x_{i+1}$, where $A_i : C \to X$ and $\rho_i > 0$ and $x_{m+1} = x_1$ for $i = 1, 2, \ldots, m$; the problem (2) reduces to the following system of finding $(x_1^*, x_2^*, \ldots, x_m^*) \in C \times C \times \cdots \times C$ such that

$$\begin{aligned}&\langle \rho_i A_i(x_{i+1}^*) + x_i^* - x_{i+1}^*, j(x - x_i^*)\rangle \\ &\quad \geq \rho_i(\varphi_i(x_i^*) - \varphi_i(x)), \quad \forall x \in X.\end{aligned} \tag{3}$$

Further, in the problem (3), when φ_i is the indicator function of a nonempty closed convex set C, in X defined by

$$\varphi_i(y) = \begin{cases} 0, & y \in C, \\ +\infty, & y \notin C, \end{cases} \tag{4}$$

then the system (3) reduces to finding $(x_1^*, x_2^*, \ldots, x_m^*) \in C \times C \times \cdots \times C$ such that

$$\begin{gathered} \langle \rho_1 A_1 x_2^* + x_1^* - x_2^*, j(x - x_1^*) \rangle \geq 0, \quad \forall x \in C, \\ \langle \rho_2 A_2 x_3^* + x_2^* - x_3^*, j(x - x_2^*) \rangle \geq 0, \quad \forall x \in C, \\ \langle \rho_3 A_3 x_4^* + x_3^* - x_4^*, j(x - x_3^*) \rangle \geq 0, \quad \forall x \in C, \\ \cdots \\ \langle \rho_m A_m x_1^* + x_m^* - x_1^*, j(x - x_m^*) \rangle \geq 0, \quad \forall x \in C, \end{gathered} \tag{5}$$

which was introduced and studied by Zhu et al. [34].

Case 2. If $m = 3$, then the system (3) is equivalent to finding $(x_1^*, x_2^*, x_3^*) \in C \times C \times C$ such that

$$\begin{aligned} &\langle \rho_1 A_1 x_2^* + x_1^* - x_2^*, j(x - x_1^*) \rangle \\ &\qquad \geq \rho_1 (\varphi_1(x_1^*) - \varphi_1(x)), \quad \forall x \in C, \\ &\langle \rho_2 A_2 x_3^* + x_2^* - x_3^*, j(x - x_2^*) \rangle \\ &\qquad \geq \rho_2 (\varphi_2(x_2^*) - \varphi_2(x)), \quad \forall x \in C, \\ &\langle \rho_3 A_3 x_1^* + x_3^* - x_1^*, j(x - x_3^*) \rangle \\ &\qquad \geq \rho_3 (\varphi_3(x_3^*) - \varphi_3(x)), \quad \forall x \in C. \end{aligned} \tag{6}$$

It is easy to see that the mathematical model studied by Saewan and Kumam [31] is a variant of (6).

Case 3. If $m = 2$, then the problem (1) reduces to find $(x^*, y^*) \in X_1 \times X_2$ such that

$$0 \in N_1(x^*, y^*) + M_1(x^*), \qquad 0 \in N_2(x^*, y^*) + M_2(y^*). \tag{7}$$

Problem (7) is called a system of strongly nonlinear quasi-variational inclusion involving generalized m-accretive mappings, it is considered and studied by Lan [19]. There are many special cases of the problems (7) that can be found in [3, 7, 12–14, 17, 20, 28, 30] and the references cited therein.

Case 4. If $m = 1$ and $X_1 = \mathscr{H}$, then the problem (1) reduces to finding $x^* \in \mathscr{H}$ such that

$$0 \in N(x^*) + M(x^*), \tag{8}$$

which was introduced and studied by Fang and Huang [8]. We remark that for appropriate and suitable choices of positive integer m, the mappings η_i, N_i, and M_i, and the spaces X_i for $i = 1, 2, \ldots, m$, one can know that the problem (1) includes a number of general class of variational character known problems, including minimization or maximization (whether constraint or not) of functions and minimax problems et al. as special cases. For more details, see [1–34] and the reference therein.

On the other hand, many authors discussed stability of the iterative sequence generated by the algorithm for solving the problems that they studied. Lan [19] introduced the notion of S-stable or stable with respect to S. Moreover, Agarwal et al. [1, 2], Jin [16], Kazmi and Bhat [18], and Lan and Kim [21] constructed some stability under suitable conditions, respectively.

Motivated and inspired by the above works, the main purpose of this paper is to introduce and study the new general system of nonlinear variational inclusions (1) involving generalized m-accretive mapping in uniformly smooth Banach spaces. By using the resolvent operator technique for generalized m-accretive, we prove the existence theorem of the solution for this kind of system of variational inclusions in Banach spaces and discuss the convergence and stability of a new perturbed iterative algorithm for solving this general system of nonlinear variational inclusions in Banach spaces.

2. Preliminaries

In order to get the main results of the paper, we need the following concepts and lemmas. Let X be a real Banach space with dual space X^*, $\langle \cdot, \cdot \rangle$ the dual pair between X and X^*, and 2^X denote the family of all the nonempty subsets of X. The generalized duality mapping $J_q : X \to 2^{X^*}$ is defined by

$$J_q(x) = \left\{ f^* \in X^* : \langle x, f^* \rangle = \|x\|^q, \|f^*\| = \|x\|^{q-1} \right\}, \quad \forall x \in X, \tag{9}$$

where $q > 1$ is a constant. In particular, J_2 is the usual normalized duality mapping. It is known that if X^* is strictly convex or X is a uniformly smooth Banach space, then J_q is single-valued (see [33]), and if $X = \mathscr{H}$, the Hilbert space, then J_2 becomes the identity mapping on $\mathscr{H}$. We will denote the single-valued duality mapping by j_q.

In order to construct convergence and stability for researching the problem (1), we need to be using the following definition and lemma.

Definition 1. Let X_i be Banach spaces, and let $N_i : X_1 \times X_2 \times \cdots \times X_m \to X_i$ be single mappings for $(i = 1, 2, \ldots, m)$. Then N_i is said to be

(i) σ_j-strongly accretive with respect to jth argument if for any $(x_1, \ldots, x_{j-1}, x_j^1, x_{j+1}, \ldots, x_m)$, $(x_1, \ldots, x_{j-1}, x_j^2, x_{j+1}, \ldots, x_m) \in X_1 \times X_2 \times \cdots \times X_m$, there exists $j_{q_j}(x_j^1 - x_j^2) \in J_{q_j}(x_j^1 - x_j^2)$, such that

$$\begin{aligned} &\Big\langle N_i(x_1, \ldots, x_{j-1}, x_j^1, x_{j+1}, \ldots, x_m) \\ &\quad - N_i(x_1, \ldots, x_{j-1}, x_j^2, x_{j+1}, \ldots, x_m), \\ &\quad j_{q_j}(x_j^1 - x_j^2) \Big\rangle \geq \sigma_j \left\| x_j^1 - x_j^2 \right\|^{q_j}, \end{aligned} \tag{10}$$

where $q_j > 1$ is a constant;

(ii) $(\zeta_{i1},\ldots,\zeta_{ij},\ldots,\zeta_{im})$-Lipschitz continuous if there exists constants $\zeta_{i1} > 0,\ldots,\zeta_{ij} > 0,\ldots,\zeta_{im} > 0$, such that

$$\left\|N_i(x_1,\ldots,x_j,\ldots,x_m) - N_i(y_1,\ldots,y_j,\ldots,y_m)\right\| \le \sum_{j=1}^{m}\zeta_{ij}\left\|x_j - y_j\right\|, \tag{11}$$

for all $x_j, y_j \in X_j$ and $j = 1, 2\ldots, m$.

Remark 2. When $X_i = \mathscr{H}_i$ $(i = 1,2,\ldots,m)$, $\mathscr{H}_i$ is different or the same as Hilbert spaces, (i) and (ii) in Definition 1 reduce to strongly monotonicity with respect to jth argument of N_i and $(\zeta_{i1},\ldots,\zeta_{ij},\ldots,\zeta_{im})$-Lipschitz continuity of N_i, respectively (see [29]).

Definition 3. Let $\eta : X\times X \to X^*$ be single-valued mapping. Then set-valued mapping $M : X \to 2^X$ is said to be

(i) accretive if

$$\left\langle u - v, J_q(x-y)\right\rangle \ge 0, \quad \forall x,y \in X,\ u \in M(x),\ v \in M(y); \tag{12}$$

(ii) η-accretive if

$$\langle u - v, \eta(x,y)\rangle \ge 0, \quad \forall x,y \in X,\ u \in M(x),\ v \in M(y); \tag{13}$$

(iii) m-accretive if M is accretive and $(I+\rho M)(X) = X$ for all $\rho > 0$, where I denotes the identity operator on X;

(iv) generalized m-accretive if M is η-accretive and $(I+\rho M)(X) = X$ for all $\rho > 0$.

Remark 4. When $X = X^* = \mathscr{H}$, (i)–(iv) of Definition 3 reduce to the definitions of monotone operators, η-monotone operators, classical maximal monotone operators, and maximal η-monotone operators; if $\eta(x,y) = J_2(x-y)$, then (ii) and (iv) of Definition 3 reduce to the definitions of accretive and m-accretive of uniformly smooth Banach spaces (see [10, 11]).

Definition 5. The mapping $\eta : X\times X \to X^*$ is said to be

(i) δ-strongly monotone if there exists a constant $\delta > 0$ such that

$$\left\langle x^1 - x^2, \eta\left(x^1,x^2\right)\right\rangle \ge \delta\left\|x^1 - x^2\right\|^2, \quad \forall x^1, x^2 \in X; \tag{14}$$

(ii) τ-Lipschitz continuous if there exists a constant $\tau > 0$ such that

$$\left\|\eta\left(x^1,x^2\right)\right\| \le \tau\left\|x^1 - x^2\right\|, \quad \forall x^1, x^2 \in X. \tag{15}$$

In [10], Huang and Fang show that for any $\rho_i > 0$, inverse mapping $(I+\rho_i M_i)^{-1}$ is single-valued, if $\eta_i : X_i\times X_i \to X_i^*$ is strict monotone and $M_i : X_i \to 2^{X_i}$ is generalized m-accretive mapping, where I is the identity mapping. Based on this fact, Huang and Fang [10] gave the following definition.

Definition 6. Let $\eta_i : X_i\times X_i \to X_i^*$ be strictly monotone mapping, and let $M_i : X_i \to 2^{X_i}$ be generalized m-accretive mapping. Then the resolvent $J_{M_i}^{\rho_i}$ for M_i is defined as follows:

$$J_{M_i}^{\rho_i}(x_i) = (I+\rho_i M_i)^{-1}(x_i), \quad \forall x_i \in X_i, \tag{16}$$

where $\rho_i > 0$ is a constant and I denotes the identity mapping on X_i for $i = 1,2,\ldots,m$.

Lemma 7 (see [10, 11]). *Let $\eta_i : X_i\times X_i \to X_i^*$ be τ_i-Lipschitz continuous and δ_i-strongly monotone, and let $M_i : X_i \to 2^X_i$ be generalized m-accretive mapping. Then for any $\rho_i > 0$, the resolvent operator $J_{M_i}^{\rho_i}$ for M_i is τ_i/δ_i-Lipschitz continuous; that is,*

$$\left\|J_{M_i}^{\rho_i}(x_i) - J_{M_i}^{\rho_i}(y_i)\right\| \le \frac{\tau_i}{\delta_i}\left\|x_i - y_i\right\|, \quad \forall x_i, y_i \in X_i,\ i = 1,2,\ldots,m. \tag{17}$$

The modules of smoothness is a measure, it is depicted geometric structure of the underlying Banach space. The modules of smoothness of Banach space X are the function $\rho_X : [0,+\infty) \to [0,+\infty)$ defined by

$$\rho_X(t) = \sup\left\{\frac{1}{2}\left(\|x+y\| + \|x-y\|\right) - 1 : \|x\| \le 1, \|y\| \le t\right\}. \tag{18}$$

A Banach space X is called uniformly smooth if $\lim_{t\to 0}(\rho_X(t)/t) = 0$. X is called q-uniformly smooth if there exists a constant $c > 0$ such that $\rho_X(t) \le ct^q$, where $q > 1$ is a real number.

Remark that J_q is single-valued if X is uniformly smooth, and Hilbert space and L_p (or l_p) $(2 \le p < +\infty)$ spaces are 2-uniformly smooth Banach spaces. In what follows, we will denote the single-valued generalized duality mapping by j_q.

In the study of characteristic inequalities in q-uniformly smooth Banach spaces, Xu [35] proved the following result.

Lemma 8. *Let $q > 1$ be a given real number and let X be a real uniformly smooth Banach space. Then X is q-uniformly smooth if and only if there exists a constant $c_q > 0$ such that for all $x, y \in X$, $j_q(x) \in J_q(x)$, there holds the following inequality:*

$$\|x+y\|^q \le \|x\|^q + q\left\langle y, j_q(x)\right\rangle + c_q\|y\|^q. \tag{19}$$

Definition 9. Let S be a self-map of X, $x_0 \in X$, and let $x_{n+1} = h(S,x_n)$ define an iteration procedure which yields a sequence of points $\{x_n\}_{n=0}^{\infty}$ in X. Suppose that $\{x \in X : Sx = x\} \ne \emptyset$ and $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point x^* of S. Let $\{u_n\} \subset X$ and let $\epsilon_n = \|u_{n+1} - h(S,u_n)\|$. If $\lim_{n\to\infty}\epsilon_n = 0$ implies that $u_n \to x^*$, then the iteration procedure defined by $x_{n+1} = h(S,x_n)$ is said to be S-stable or stable with respect to S.

Lemma 10 (see [36]). *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be three nonnegative real sequences satisfying the following condition: there exists a natural number n_0 such that*

$$a_{n+1} \le (1-t_n)a_n + b_n t_n + c_n, \quad \forall n \ge n_0, \tag{20}$$

where $t_n \in [0,1]$, $\sum_{n=0}^{\infty} t_n = +\infty$, $\lim_{n\to\infty} b_n = 0$, and $\sum_{n=0}^{\infty} c_n < +\infty$. Then a_n converges to 0 as $n \to \infty$.

3. Existence Theorem

In this section, we will give the existence theorem of the problem (1). The solvability of the problem (1) depends on the equivalence between (1) and the problem of finding the fixed point of the associated generalized resolvent operator. It follows from the definition of generalized resolvent operator $J_{M_i}^{\rho_i}$ $(i = 1, 2, \ldots, m)$ that we can obtain the following conclusion.

Lemma 11. *Let $\eta_i : X_i \times X_i \to X_i^*$, $N_i : X_1 \times X_2 \times \cdots \times X_m \to X_i$ single-valued mappings, and $M_i : X_i \to 2^{X_i}$ generalized m-accretive mapping for $(i = 1, 2, \ldots, m)$. Then the following statements are mutually equivalent.*

(i) *An element $(x_1^*, x_2^*, \ldots, x_m^*) \in X_1 \times X_2 \times \cdots \times X_m$ is a solution to the problem* (1).

(ii) *There is an $(x_1^*, x_2^*, \ldots, x_m^*) \in X_1 \times X_2 \times \cdots \times X_m$ such that*

$$x_i^* = J_{M_i}^{\rho_i}\left[x_i^* - \rho_i N_i\left(x_1^*, \ldots, x_{i-1}^*, x_i^*, x_{i+1}^*, \ldots, x_m^*\right)\right], \quad (21)$$

where $J_{M_i}^{\rho_i} = (I + \rho_i M_i)^{-1}$, and $\rho_i > 0$ is constants for all $i = 1, 2, \ldots, m$.

(iii) *For any given constants $\rho_i > 0$, the map $F : X_1 \times X_2 \times \cdots \times X_m \to X_1 \times X_2 \times \cdots \times X_m$ is defined by*

$$\begin{aligned} F&\left(u_1, u_2, \ldots, u_m\right) \\ &= \left(P_{\rho_1}\left(u_1, u_2, \ldots, u_m\right), \ldots, P_{\rho_i}\left(u_1, u_2, \ldots, u_m\right), \ldots, \right. \\ &\qquad \left. P_{\rho_m}\left(u_1, u_2, \ldots, u_m\right)\right) \end{aligned} \quad (22)$$

for all $u_i \in X_i$ and $i = 1, 2, \ldots, m$, has a fixed point $(x_1^*, x_2^*, \ldots, x_m^*) \in X_1 \times X_2 \times \cdots \times X_m$, where maps $P_{\rho_i} : X_1 \times X_2 \times \cdots \times X_m \to X_i$ are defined by

$$\begin{aligned} P_{\rho_i}&\left(u_1, u_2, \ldots, u_m\right) \\ &= J_{M_i}^{\rho_i}\left[u_i - \rho_i N_i\left(u_1, \ldots, u_{i-1}, u_i, u_{i+1}, \ldots, u_m\right)\right] \end{aligned} \quad (23)$$

for $u_i \in X_i$ and $i = 1, 2, \ldots, m$.

Proof. We first prove that (i) $\Leftrightarrow$ (ii). Let $(x_1^*, x_2^*, \ldots, x_m^*) \in X_1 \times X_2 \times \cdots \times X_m$ satisfy the relation in (ii). Then, the definition of resolvent operator $J_{M_i}^{\rho_i}$ implies that this equality holds if and only if

$$x_i^* - \rho_i N_i\left(x_1^*, \ldots, x_{i-1}^*, x_i^*, x_{i+1}^*, \ldots, x_m^*\right) \in \left(I + \rho_i M_i\right)\left(x_i^*\right) \quad (24)$$

for $i = 1, 2, \ldots, m$; that is

$$0 \in N_i\left(x_1^*, x_2^*, \ldots, x_m^*\right) + M_i\left(x_i^*\right), \quad (25)$$

where $i = 1, 2, \ldots, m$. Thus $(x_1^*, x_2^*, \ldots, x_m^*) \in X_1 \times X_2 \times \cdots \times X_m$ is the solution of the problem (1).

Next, we show (ii) $\Leftrightarrow$ (iii). If $(x_1^*, x_2^*, \ldots, x_m^*) \in X_1 \times X_2 \times \cdots \times X_m$ satisfy following relation:

$$x_i^* = J_{M_i}^{\rho_i}\left[x_i^* - \rho_i N_i\left(x_1^*, \ldots, x_{i-1}^*, x_i^*, x_{i+1}^*, \ldots, x_m^*\right)\right], \quad (26)$$

then, for any $i = 1, 2, \ldots, m$, it follows from

$$\begin{aligned} P_{\rho_i}&\left(x_1^*, x_2^*, \ldots, x_m^*\right) \\ &= J_{M_i}^{\rho_i}\left[x_i^* - \rho_i N_i\left(x_1^*, \ldots, x_{i-1}^*, x_i^*, x_{i+1}^*, \ldots, x_m^*\right)\right] \end{aligned} \quad (27)$$

that

$$P_{\rho_i}\left(x_1^*, x_2^*, \ldots, x_m^*\right) = x_i^*. \quad (28)$$

Hence, $(x_1^*, x_2^*, \ldots, x_m^*) \in X_1 \times X_2 \times \cdots \times X_m$ is a fixed point of the mapping

$$\begin{aligned} F&\left(u_1, u_2, \ldots, u_m\right) \\ &= \left(P_{\rho_1}\left(u_1, u_2, \ldots, u_m\right), \ldots, P_{\rho_i}\left(u_1, u_2, \ldots, u_m\right), \ldots, \right. \\ &\qquad \left. P_{\rho_m}\left(u_1, u_2, \ldots, u_m\right)\right). \end{aligned} \quad (29)$$

Conversely, if $(x_1^*, x_2^*, \ldots, x_m^*) \in X_1 \times X_2 \times \cdots \times X_m$ is a fixed point of the mapping $F : X_1 \times X_2 \times \cdots \times X_m \to X_1 \times X_2 \times \cdots \times X_m$, then

$$P_{\rho_i}\left(x_1^*, x_2^*, \ldots, x_m^*\right) = x_i^* \quad (30)$$

for $i = 1, 2, \ldots, m$. Hence, from

$$\begin{aligned} P_{\rho_i}&\left(x_1^*, x_2^*, \ldots, x_m^*\right) \\ &= J_{M_i}^{\rho_i}\left[x_i^* - \rho_i N_i\left(x_1^*, \ldots, x_{i-1}^*, x_i^*, x_{i+1}^*, \ldots, x_m^*\right)\right], \end{aligned} \quad (31)$$

we have

$$x_i^* = J_{M_i}^{\rho_i}\left[x_i^* - \rho_i N_i\left(x_1^*, \ldots, x_{i-1}^*, x_i^*, x_{i+1}^*, \ldots, x_m^*\right)\right] \quad (32)$$

for $i = 1, 2, \ldots, m$. Therefore $(x_1^*, x_2^*, \ldots, x_m^*) \in X_1 \times X_2 \times \cdots \times X_m$ satisfy the relation of (ii). □

Theorem 12. *Let X_i be a real q_i-uniformly smooth Banach space with $q_i > 1$ and let $\eta_i : X_i \times X_i \to X_i^*$ be τ_i-Lipschitz continuous and δ_i-strongly monotone for any $i = 1, 2, \ldots, m$. Suppose that $M_i : X_i \to 2^{X_i}$ is generalized m-accretive mapping, and $N_i : X_1 \times X_2 \times \cdots \times X_m \to X_i$ is σ_i-strongly accretive in the ith argument and $(\zeta_{i1}, \ldots, \zeta_{ii}, \ldots, \zeta_{im})$-Lipschitz continuous for $i = 1, 2, \ldots, m$. If*

$$\frac{\tau_j}{\delta_j}\sqrt[q_j]{1 - q_j\rho_j\sigma_j + c_{q_j}\rho_j^{q_j}\zeta_{jj}^{q_j}} + \sum_{i=1, i\neq j}^{m}\frac{\zeta_{ij}\rho_i\tau_i}{\delta_i} < 1, \quad (33)$$

where c_{q_j} is the constants as in Lemma 8 for $j = 1, 2, \ldots, m$, then problem (1) has a unique solution $(x_1^*, x_2^*, \ldots, x_m^*) \in X_1 \times X_2 \times \cdots \times X_m$.

Proof. For any given $\rho_i > 0$ and $i = 1, 2, \ldots, m$, we first define $P_{\rho_i} : X_1 \times X_2 \times \cdots \times X_m \to X_i$ as follows:

$$P_{\rho_i}(u_1, u_2, \ldots, u_m) = J_{M_i}^{\rho_i}[u_i - \rho_i N_i(u_1, u_2, \ldots, u_m)], \quad (34)$$

for all $u_i \in X_i$. Now define $\|\cdot\|_*$ on $X_1 \times X_2 \times \cdots \times X_m$ by

$$\begin{aligned} &\|(u_1, u_2, \ldots, u_m)\|_* = \sum_{i=1}^{m} \|u_i\|, \\ &\forall (u_1, u_2, \ldots, u_m) \in X_1 \times X_2 \times \cdots \times X_m. \end{aligned} \quad (35)$$

It is easy to see that $(X_1 \times X_2 \times \cdots \times X_m, \|\cdot\|_*)$ is a Banach space. In fact

(i) $\|(u_1, u_2, \ldots, u_m)\|_* = \sum_{i=1}^{m} \|u_i\| \geq 0$, the negative being satisfied;

(ii) for all real number α,

$$\begin{aligned} \|\alpha(u_1, u_2, \ldots, u_m)\|_* &= \|(\alpha u_1, \alpha u_2, \ldots, \alpha u_m)\|_* \\ &= \sum_{i=1}^{m} \|\alpha u_i\| = \sum_{i=1}^{m} |\alpha| \|u_i\| \\ &= |\alpha| \sum_{i=1}^{m} \|u_i\| = |\alpha| \|(u_1, u_2, \ldots, u_m)\|_* \end{aligned} \quad (36)$$

homogeneity being satisfied;

(iii) for all $(u_1, u_2, \ldots, u_m), (v_1, v_2, \ldots, v_m) \in X_1 \times X_2 \times \cdots \times X_m$,

$$\begin{aligned} &\|(u_1, u_2, \ldots, u_m) + (v_1, v_2, \ldots, v_m)\|_* \\ &= \|(u_1 + v_1, u_2 + v_2, \ldots, u_m + v_m)\|_* \\ &= \sum_{i=1}^{m} \|u_i + v_i\| \leq \sum_{i=1}^{m} (\|u_i\| + \|v_i\|) \\ &= \sum_{i=1}^{m} \|u_i\| + \sum_{i=1}^{m} \|v_i\| \\ &= \|(u_1, u_2, \ldots, u_m)\|_* + \|(v_1, v_2, \ldots, v_m)\|_*, \end{aligned} \quad (37)$$

the triangle inequality being satisfied;

(iv) let $\|(u_1, u_2, \ldots, u_m)\|_* = 0$; that is, $\sum_{i=1}^{m} \|u_i\| = 0$; this implies that $\|u_i\| = 0$ $(i = 1, 2, \ldots, m)$; thus $u_i = 0$ $(i = 1, 2, \ldots, m)$; we get $\|\cdot\|_*$ is a norm on the $X_1 \times X_2 \times \cdots \times X_m$;

(v) let $(u_1^n, u_2^n, \ldots, u_m^n) \in X_1 \times X_2 \times \cdots \times X_m$ is Cauchy sequence; that is, for $\forall \epsilon > 0$, there exists a positive integer N; let $n > N$; we have

$$\begin{aligned} &\|(u_1^{n+p}, u_2^{n+p}, \ldots, u_m^{n+p}) - (u_1^n, u_2^n, \ldots, u_m^n)\|_* \\ &\quad = \sum_{i=1}^{m} \|u_i^{n+p} - u_i^n\| < \epsilon. \end{aligned} \quad (38)$$

Thus, for all $i \in \{1, 2, 3, \ldots, m\}$, we have $\|u_i^{n+p} - u_i^n\| < \epsilon$ $(n > N, p = 1, 2, 3, \ldots,)$; that is, $\{u_i^n\} \subset X_i$ is also Cauchy sequence; thus $\lim_{n \to \infty} u_i^n = u_i \in X_i$ for $i = 1, 2, \ldots, m$; we get $(u_1, u_2, \ldots, u_m) \in X_1 \times X_2 \times \cdots \times X_m$ and $(u_1, u_2, \ldots, u_m)$ is a cluster point on the $(X_1 \times X_2 \times \cdots \times X_m, \|\cdot\|_*)$; we claim $(X_1 \times X_2 \times \cdots \times X_m, \|\cdot\|_*)$ is a Banach space.

Now, by (34), for any given $\rho_i > 0$, define mapping $F : X_1 \times X_2 \times \cdots \times X_m \to X_1 \times X_2 \times \cdots \times X_m$ by

$$\begin{aligned} &F(u_1, u_2, \ldots, u_m) \\ &\quad = (P_{\rho_1}(u_1, u_2, \ldots, u_m), \ldots, P_{\rho_i}(u_1, u_2, \ldots, u_m), \ldots, \\ &\qquad P_{\rho_m}(u_1, u_2, \ldots, u_m)), \end{aligned} \quad (39)$$

where $u_i \in X_i$ for $i = 1, 2, \ldots, m$.

In the sequel, we prove that F is a contractive mapping on the $(X_1 \times X_2 \times \cdots \times X_m, \|\cdot\|_*)$. In fact, for any $u_i, v_i \in X_i$ and $i = 1, 2, \ldots, m$, it follows from (34) and Lemma 7 that

$$\begin{aligned} &\|P_{\rho_i}(u_1, u_2, \ldots, u_m) - P_{\rho_i}(v_1, v_2, \ldots, v_m)\| \\ &= \|J_{M_i}^{\rho_i}[u_i - \rho_i N_i(u_1, u_2, \ldots, u_m)] \\ &\qquad - J_{M_i}^{\rho_i}[v_i - \rho_i N_i(v_1, v_2, \ldots, v_m)]\| \\ &\leq \frac{\tau_i}{\delta_i} \|u_i - v_i - \rho_i (N_i(u_1, u_2, \ldots, u_m) - N_i(v_1, v_2, \ldots, v_m))\| \\ &\leq \frac{\tau_i}{\delta_i} \|u_i - v_i \\ &\qquad - \rho_i (N_i(u_1, \ldots, u_{i-1}, u_i, u_{i+1}, \ldots, u_m) \\ &\qquad\qquad - N_i(u_1, \ldots, u_{i-1}, v_i, u_{i+1}, \ldots, u_m))\| \\ &\quad + \frac{\rho_i \tau_i}{\delta_i} \|N_i(u_1, \ldots, u_{i-1}, v_i, u_{i+1}, \ldots, u_m) \\ &\qquad\qquad - N_i(v_1, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_m)\|. \end{aligned} \quad (40)$$

By assumptions and Lemma 8, we have

$$\begin{aligned} &\|u_i - v_i - \rho_i (N_i(u_1, \ldots, u_{i-1}, u_i, u_{i+1}, \ldots, u_m) \\ &\qquad - N_i(u_1, \ldots, u_{i-1}, v_i, u_{i+1}, \ldots, u_m))\|^{q_i} \\ &\leq \|u_i - v_i\|^{q_i} \\ &\quad + c_{q_i} \rho_i^{q_i} \|N_i(u_1, \ldots, u_{i-1}, u_i, u_{i+1}, \ldots, u_m) \\ &\qquad - N_i(u_1, \ldots, u_{i-1}, v_i, u_{i+1}, \ldots, u_m)\|^{q_i} \\ &\quad - q_i \rho_i \langle N_i(u_1, \ldots, u_{i-1}, u_i, u_{i+1}, \ldots, u_m) \\ &\qquad - N_i(u_1, \ldots, u_{i-1}, v_i, u_{i+1}, \ldots, u_m), j_{q_i}(u_i - v_i) \rangle \\ &\leq (1 - q_i \rho_i \sigma_i + c_{q_i} \rho_i^{q_i} \zeta_{ii}^{q_i}) \|u_i - v_i\|^{q_i}, \end{aligned}$$

$$\begin{aligned}&\left\|N_i\left(u_1,\ldots,u_{i-1},v_i,u_{i+1},\ldots,u_m\right)\right.\\&\quad\left.-N_i\left(v_1,\ldots,v_{i-1},v_i,v_{i+1},\ldots,v_m\right)\right\|\\&\leq\sum_{j=1,j\neq i}^{m}\zeta_{ij}\left\|u_j-v_j\right\|.\end{aligned}\tag{41}$$

From (40)-(41), we obtain

$$\begin{aligned}&\left\|P_{\rho_i}\left(u_1,u_2,\ldots,u_m\right)-P_{\rho_i}\left(v_1,v_2,\ldots,v_m\right)\right\|\\&\quad\leq\frac{\rho_i\tau_i}{\delta_i}\sum_{j=1,j\neq i}^{m}\zeta_{ij}\left\|u_j-v_j\right\|\\&\qquad+\frac{\tau_i}{\delta_i}\sqrt[q_i]{1-q_i\rho_i\sigma_i+c_{q_i}\rho_i^{q_i}\zeta_{ii}^{q_i}}\left\|u_i-v_i\right\|\end{aligned}\tag{42}$$

for $i=1,2,\ldots,m$. Equation (42) implies that

$$\begin{aligned}&\sum_{j=1}^{m}\left\|P_{\rho_j}\left(u_1,u_2,\ldots,u_m\right)-P_{\rho_j}\left(v_1,v_2,\ldots,v_m\right)\right\|\\&\quad=\sum_{i=1}^{m}\left\|P_{\rho_i}\left(u_1,u_2,\ldots,u_m\right)-P_{\rho_i}\left(v_1,v_2,\ldots,v_m\right)\right\|\\&\quad\leq\sum_{i=1}^{m}\left(\frac{\tau_i}{\delta_i}\sqrt[q_i]{1-q_i\rho_i\sigma_i+c_{q_i}\rho_i^{q_i}\zeta_{ii}^{q_i}}\left\|u_i-v_i\right\|\right.\\&\qquad\left.+\frac{\rho_i\tau_i}{\delta_i}\sum_{j=1,j\neq i}^{m}\zeta_{ij}\left\|u_j-v_j\right\|\right)\\&\quad\leq\sum_{i=1}^{m}\frac{\tau_i}{\delta_i}\sqrt[q_i]{1-q_i\rho_i\sigma_i+c_{q_i}\rho_i^{q_i}\zeta_{ii}^{q_i}}\left\|u_i-v_i\right\|\\&\qquad+\sum_{i=1}^{m}\frac{\rho_i\tau_i}{\delta_i}\sum_{j=1,j\neq i}^{m}\zeta_{ij}\left\|u_j-v_j\right\|\\&\quad=\sum_{j=1}^{m}\frac{\tau_j}{\delta_j}\sqrt[q_j]{1-q_j\rho_j\sigma_j+c_{q_j}\rho_j^{q_j}\zeta_{jj}^{q_j}}\left\|u_j-v_j\right\|\\&\qquad+\sum_{j=1}^{m}\left(\sum_{i=1,i\neq j}^{m}\frac{\rho_i\tau_i\zeta_{ij}}{\delta_i}\right)\left\|u_j-v_j\right\|\\&\quad=\sum_{j=1}^{m}\left[\frac{\tau_j}{\delta_j}\sqrt[q_j]{1-q_j\rho_j\sigma_j+c_{q_j}\rho_j^{q_j}\zeta_{jj}^{q_j}}+\sum_{i=1,i\neq j}^{m}\frac{\rho_i\tau_i\zeta_{ij}}{\delta_i}\right]\\&\qquad\times\left\|u_j-v_j\right\|\\&\quad\leq k\sum_{j=1}^{m}\left\|u_j-v_j\right\|,\end{aligned}\tag{43}$$

where $k = \max_{1\leq j\leq m}\{(\tau_j/\delta_j)\sqrt[q_j]{1-q_j\rho_j\sigma_j+c_{q_j}\rho_j^{q_j}\zeta_{jj}^{q_j}} + \sum_{i=1,i\neq j}^{m}(\zeta_{ij}\rho_i\tau_i/\delta_i)\}$. By (33), we know that $0\leq k<1$. It follows from (43) that

$$\begin{aligned}&\left\|F(u_1,u_2,\ldots,u_m)-F(v_1,v_2,\ldots,v_m)\right\|_*\\&\quad\leq k\left\|(u_1,u_2,\ldots,u_m)-(v_1,v_2,\ldots,v_m)\right\|_*.\end{aligned}\tag{44}$$

This proves that $F: X_1\times X_2\times\cdots\times X_m\to X_1\times X_2\times\cdots\times X_m$ is a contraction mapping. Hence, there exists a unique $(x_1^*,x_2^*,\ldots,x_m^*)\in X_1\times X_2\times\cdots\times X_m$ such that

$$F\left((x_1^*,x_2^*,\ldots,x_m^*)\right)=(x_1^*,x_2^*,\ldots,x_m^*);\tag{45}$$

that is, $P_{\rho_i}(x_1^*,x_2^*,\ldots,x_m^*)=x_i^*$ for $i=1,2,\ldots,m$; that is,

$$x_i^*=J_{M_i}^{\rho_i}\left[x_i^*-\rho_iN_i\left(x_1^*,x_2^*,\ldots,x_m^*\right)\right].\tag{46}$$

By Lemma 11, $(x_1^*,x_2^*,\ldots,x_m^*)$ is the unique solution of problem (1). This completes the proof. □

Remark 13. If $m=2$, then Theorem 12 reduces to Theorem 3.2 of Lan [19].

Corollary 14. *Let $\mathscr{H}_i$ be real Hilbert space and $\eta_i:\mathscr{H}_i\times\mathscr{H}_i\to\mathscr{H}_i$ be τ_i-Lipschitz continuous and δ_i-strongly monotone for any $i=1,2,\ldots,m$. Suppose that $M_i:\mathscr{H}_i\to 2^{\mathscr{H}_i}$ is maximal η_i-monotone mapping, $N_i:\mathscr{H}_1\times\mathscr{H}_2\times\cdots\times\mathscr{H}_m\to\mathscr{H}_i$ is σ_i-strongly monotone in the ith argument, and $(\zeta_{i1},\ldots,\zeta_{ii},\ldots,\zeta_{im})$-Lipschitz continuous for $i=1,2,\ldots,m$. If*

$$\frac{\tau_j}{\delta_j}\sqrt[2]{1-2\rho_j\sigma_j+\rho_j^2\zeta_{jj}^2}+\sum_{i=1,i\neq j}^{m}\frac{\zeta_{ij}\rho_i\tau_i}{\delta_i}<1,\tag{47}$$

then problem (1) has a unique solution $(x_1^*,x_2^*,\ldots,x_m^*)\in\mathscr{H}_1\times\mathscr{H}_2\times\cdots\times\mathscr{H}_m$.

Corollary 15. *Let $\mathscr{H}_i$ be real Hilbert space for any $i=1,2,\ldots,m$. Suppose that $\varphi_i:\mathscr{H}_i\to(-\infty,+\infty]$ is proper, convex, and lower semicontinuous functional on $\mathscr{H}_i$ and $N_i:\mathscr{H}_1\times\mathscr{H}_2\times\cdots\times\mathscr{H}_m\to\mathscr{H}_i$ is σ_i-strongly monotone in the ith argument and $(\zeta_{i1},\ldots,\zeta_{ii},\ldots,\zeta_{im})$-Lipschitz continuous for $i=1,2,\ldots,m$. If*

$$\sqrt[2]{1-2\rho_j\sigma_j+\rho_j^2\zeta_{jj}^2}+\sum_{i=1,i\neq j}^{m}\rho_i\zeta_{ij}<1,\tag{48}$$

then problem (2) has a unique solution $(x_1^*,x_2^*,\ldots,x_m^*)\in\mathscr{H}_1\times\mathscr{H}_2\times\cdots\times\mathscr{H}_m$.

4. Perturbed Iterative Algorithms

In this section, by using Definition 9 and Lemma 10, we construct a new perturbed iterative algorithm with mixed errors for solving problem (1) and prove the convergence and stability of the iterative sequence generated by the algorithm.

Algorithm 16. Let $\eta_i: X_i\times X_i\to X_i^*$ and $N_i: X_1\times X_2\times\cdots\times X_m\to X_i$ be single-valued mappings and let $M_i: X_i\to 2^{X_i}$

be generalized m-accretive mapping for $i = 1, 2, \ldots, m$. For any given initial point $(x_1^0, x_2^0, \ldots, x_m^0) \in X_1 \times X_2 \times \cdots \times X_m$, the perturbed iterative sequence $\{(x_1^n, x_2^n, \ldots, x_m^n)\}$ for problem (1) is defined by

$$x_i^{n+1} = (1 - \alpha_n) x_i^n + \alpha_n J_{M_i}^{\rho_i} [x_i^n - \rho_i N_i (x_1^n, x_2^n, \ldots, x_m^n)] + \alpha_n u_i^n + w_i^n, \tag{49}$$

where $n \geq 0$, $i = 1, 2, \ldots, m$, $\{\alpha_n\}$ is a sequence in $[0, 1]$, and $\{u_i^n\}, \{w_i^n\} \subset X_i$ are errors to take into account a possible inexact computation of the resolvent operator point satisfying the following conditions:

(i) $u_i^n = u_i'^n + u_i''^n$;

(ii) $\lim_{n \to \infty} \|u_i'^n\| = 0$;

(iii) $\sum_{n=0}^{\infty} \|u_i''^n\| < +\infty$, $\sum_{n=0}^{\infty} \|w_i^n\| < +\infty$ for $i = 1, 2, \ldots, m$.

Let $\{(z_1^n, z_2^n, \ldots, z_m^n)\}$ be any sequence in $X_1 \times X_2 \times \cdots \times X_m$ and define $\{(\epsilon_1^n, \epsilon_2^n, \ldots, \epsilon_m^n)\}$ by

$$\epsilon_i^n = \left\| z_i^{n+1} - \left\{ (1 - \alpha_n) z_i^n + \alpha_n J_{M_i}^{\rho_i} [z_i^n - \rho_i N_i (z_1^n, z_2^n, \ldots, z_m^n)] + \alpha_n u_i^n + w_i^n \right\} \right\| \tag{50}$$

for $i = 1, 2, \ldots, m$.

Algorithm 17. Let $\eta_i : \mathscr{H}_i \times \mathscr{H}_i \to \mathscr{H}_i$ and $N_i : \mathscr{H}_1 \times \mathscr{H}_2 \times \cdots \times \mathscr{H}_m \to \mathscr{H}_i$ be single-valued mappings and let $M_i : \mathscr{H}_i \to 2^{\mathscr{H}_i}$ be maximal η_i-monotone mapping for $i = 1, 2, \ldots, m$. For any given initial point $(x_1^0, x_2^0, \ldots, x_m^0) \in \mathscr{H}_1 \times \mathscr{H}_2 \times \cdots \times \mathscr{H}_m$, the perturbed iterative sequence $\{(x_1^n, x_2^n, \ldots, x_m^n)\}$ for problem (1) is defined by

$$x_i^{n+1} = (1 - \alpha_n) x_i^n + \alpha_n J_{M_i}^{\rho_i} [x_i^n - \rho_i N_i (x_1^n, x_2^n, \ldots, x_m^n)] + \alpha_n u_i^n + w_i^n, \tag{51}$$

where $n \geq 0$, $i = 1, 2, \ldots, m$, $\{\alpha_n\}$ is a sequence in $[0, 1]$, and $\{u_i^n\}, \{w_i^n\} \subset X_i$ are errors to take into account a possible inexact computation of the resolvent operator point satisfying the following conditions:

(i) $\lim_{n \to \infty} \|u_i^n\| = 0$;

(ii) $\sum_{n=0}^{\infty} \|w_i^n\| < +\infty$ for $i = 1, 2, \ldots, m$.

Let $\{(z_1^n, z_2^n, \ldots, z_m^n)\}$ be any sequence in $\mathscr{H}_1 \times \mathscr{H}_2 \times \cdots \times \mathscr{H}_m$ and define $\{(\epsilon_1^n, \epsilon_2^n, \ldots, \epsilon_m^n)\}$ by

$$\epsilon_i^n = \left\| z_i^{n+1} - \left\{ (1 - \alpha_n) z_i^n + \alpha_n J_{M_i}^{\rho_i} [z_i^n - \rho_i N_i (z_1^n, z_2^n, \ldots, z_m^n)] + \alpha_n u_i^n + w_i^n \right\} \right\| \tag{52}$$

for $i = 1, 2, \ldots, m$.

Algorithm 18. Let $N_i : \mathscr{H}_1 \times \mathscr{H}_2 \times \cdots \times \mathscr{H}_m \to \mathscr{H}_i$ be single-valued mappings and $\varphi_i : \mathscr{H}_i \to (-\infty, +\infty]$ is proper, convex, and lower semi-continuous functional on $\mathscr{H}_i$ for $i = 1, 2, \ldots, m$. For any given initial point $(x_1^0, x_2^0, \ldots, x_m^0) \in \mathscr{H}_1 \times \mathscr{H}_2 \times \cdots \times \mathscr{H}_m$, the perturbed iterative sequence $\{(x_1^n, x_2^n, \ldots, x_m^n)\}$ for problem (2) is defined by

$$x_i^{n+1} = (1 - \alpha_n) x_i^n + \alpha_n J_{\varphi_i} [x_i^n - \rho_i N_i (x_1^n, x_2^n, \ldots, x_m^n)] + w_i^n, \tag{53}$$

where $n \geq 0$, $i = 1, 2, \ldots, m$, $\{\alpha_n\}$ is a sequence in $[0, 1]$, $\{w_i^n\} \subset X_i$ are errors to take into account a possible inexact computation of the resolvent operator point satisfying the condition $\sum_{n=0}^{\infty} \|w_i^n\| < +\infty$ for $i = 1, 2, \ldots, m$. Let $\{(z_1^n, z_2^n, \ldots, z_m^n)\}$ be any sequence in $\mathscr{H}_1 \times \mathscr{H}_2 \times \cdots \times \mathscr{H}_m$ and define $\{(\epsilon_1^n, \epsilon_2^n, \ldots, \epsilon_m^n)\}$ by

$$\epsilon_i^n = \left\| z_i^{n+1} - \left\{ (1 - \alpha_n) z_i^n + \alpha_n J_{\varphi_i} [z_i^n - \rho_i N_i (z_1^n, z_2^n, \ldots, z_m^n)] + w_i^n \right\} \right\| \tag{54}$$

for $i = 1, 2, \ldots, m$.

Remark 19. If $m = 2$, then Algorithm 16 reduces to Algorithm 4.3 of Lan [19].

Next we will show the convergence and stability of Algorithm 16.

Theorem 20. *Suppose that X_i, η_i, N_i, and M_i $(i = 1, 2, \ldots, m)$ are the same as in Theorem 12. If $\sum_{n=0}^{\infty} \alpha_n = +\infty$ and condition* (33) *holds, then the perturbed iterative sequence $\{(x_1^n, x_2^n, \ldots, x_m^n)\}$ defined by Algorithm 16 converges strongly to the unique solution $(x_1^*, x_2^*, \ldots, x_m^*) \in X_1 \times X_2 \times \cdots \times X_m$ of the problem* (1). *Moreover, if there exists $a \in (0, \alpha_n]$ for all $n \geq 0$, then*

$$\lim_{n \to \infty} (z_1^n, z_2^n, \ldots, z_m^n) = (x_1^*, x_2^*, \ldots, x_m^*) \tag{55}$$

if and only if

$$\lim_{n \to \infty} (\epsilon_1^n, \epsilon_2^n, \ldots, \epsilon_m^n) = \underbrace{(0, 0, \ldots, 0)}_{m}, \tag{56}$$

where $(\epsilon_1^n, \epsilon_2^n, \ldots, \epsilon_m^n)$ is defined by (50).

Proof. From Theorem 12, we know that problem (1) has a unique solution

$$(x_1^*, x_2^*, \ldots, x_m^*) \in X_1 \times X_2 \times \cdots \times X_m. \tag{57}$$

It follows from (49) and the proof of (42) in Theorem 12 that, for $i = 1, 2, \ldots, m$,

$$\begin{aligned} &\left\|x_i^{n+1} - x_i^*\right\| \\ &\leq (1-\alpha_n)\left\|x_i^n - x_i^*\right\| \\ &\quad + \alpha_n \left\{ \frac{\tau_i}{\delta_i} \sqrt[q_i]{1 - q_i\rho_i\sigma_i + c_{q_i}\rho_i^{q_i}\zeta_{ii}^{q_i}} \left\|x_i^n - x_i^*\right\| \right. \\ &\qquad \left. + \frac{\rho_i\tau_i}{\delta_i}\sum_{j=1, j\neq i}^{m} \zeta_{ij}\left\|x_j^n - x_j^*\right\| \right\} \\ &\quad + \alpha_n\left\|u_i'^n\right\| + \left(\left\|u_i''^n\right\| + \left\|w_i^n\right\|\right). \end{aligned} \tag{58}$$

It follows from (58), we have

$$\begin{aligned} &\sum_{i=1}^{m}\left\|x_i^{n+1} - x_i^*\right\| \\ &\leq (1-\alpha_n)\sum_{i=1}^{m}\left\|x_i^n - x_i^*\right\| \\ &\quad + \alpha_n\sum_{j=1}^{m}\left[\frac{\tau_j}{\delta_j}\sqrt[q_j]{1 - q_j\rho_j\sigma_j + c_{q_j}\rho_j^{q_j}\zeta_{jj}^{q_j}} + \sum_{i=1, i\neq j}^{m}\frac{\rho_i\tau_i\zeta_{ij}}{\delta_i}\right] \\ &\quad \times\left\|x_j^n - x_j^*\right\| + \alpha_n\sum_{i=1}^{m}\left\|u_i'^n\right\| + \sum_{i=1}^{m}\left\|u_i''^n\right\| + \sum_{i=1}^{m}\left\|w_i^n\right\| \\ &= (1-\alpha_n)\sum_{j=1}^{m}\left\|x_j^n - x_j^*\right\| \\ &\quad + \alpha_n\sum_{j=1}^{m}\left[\frac{\tau_j}{\delta_j}\sqrt[q_j]{1 - q_j\rho_j\sigma_j + c_{q_j}\rho_j^{q_j}\zeta_{jj}^{q_j}} + \sum_{i=1, i\neq j}^{m}\frac{\rho_i\tau_i\zeta_{ij}}{\delta_i}\right] \\ &\quad \times\left\|x_j^n - x_j^*\right\| + \alpha_n\sum_{j=1}^{m}\left\|u_j'^n\right\| + \sum_{j=1}^{m}\left\|u_j''^n\right\| + \sum_{j=1}^{m}\left\|w_j^n\right\| \\ &\leq \left[1 - \alpha_n(1-k)\right]\sum_{j=1}^{m}\left\|x_j^n - x_j^*\right\| \\ &\quad + \alpha_n(1-k)\cdot\frac{1}{1-k}\sum_{j=1}^{m}\left\|u_j'^n\right\| \\ &\quad + \left(\sum_{j=1}^{m}\left\|u_j''^n\right\| + \sum_{j=1}^{m}\left\|w_j^n\right\|\right), \end{aligned} \tag{59}$$

where k is the same as in (43). Letting $t_n = \alpha_n(1-k) \in [0,1]$, $b_n = (1/(1-k))\sum_{j=1}^{m}\|u_j'^n\|$, and $c_n = \sum_{j=1}^{m}\|u_j''^n\| + \sum_{j=1}^{m}\|w_j^n\|$ $(n \geq 0)$, then it follows from $\sum_{n=0}^{\infty}\alpha_n = +\infty$ and (i)–(iii) of Algorithm 16 that

$$\begin{gathered} \sum_{n=0}^{\infty}t_n = +\infty, \qquad \lim_{n\to\infty}b_n = \frac{1}{1-k}\sum_{j=1}^{m}\left(\lim_{n\to\infty}\left\|u_j'^n\right\|\right) = 0, \\ \sum_{n=0}^{\infty}c_n = \sum_{j=1}^{m}\sum_{n=0}^{\infty}\left\|u_j''^n\right\| + \sum_{j=1}^{m}\sum_{n=0}^{\infty}\left\|w_j^n\right\| < +\infty. \end{gathered} \tag{60}$$

Setting $a_n = \sum_{j=1}^{m}\|x_j^n - x_j^*\|$, then (59) can be rewritten as

$$a_{n+1} \leq (1-t_n)a_n + b_nt_n + c_n, \quad n = 0, 1, 2, \ldots. \tag{61}$$

It follows from Lemma 10 that $\lim_{n\to\infty}a_n = 0$; that is,

$$\lim_{n\to\infty}\sum_{j=1}^{m}\left\|x_j^n - x_j^*\right\| = 0; \tag{62}$$

thus

$$x_j^n \longrightarrow x_j^* \quad (n\longrightarrow\infty), \; (j = 1, 2, \ldots, m). \tag{63}$$

Hence, we know that the sequence $\{(x_1^n, x_2^n, \ldots, x_m^n)\}$ converges strongly to the unique solution $(x_1^*, x_2^*, \ldots, x_m^*)$ of the problem (1).

Now we prove the second conclusion. By (50), now we know

$$\begin{aligned} &\left\|z_i^{n+1} - x_i^*\right\| \\ &\quad\leq \left\|(1-\alpha_n)z_i^n + \alpha_nJ_{M_i}^{\rho_i}\left[z_i^n - \rho_iN_i(z_1^n, z_2^n, \ldots, z_m^n)\right]\right. \\ &\qquad\left. + \alpha_nu_i^n + w_i^n - x_i^*\right\| + \epsilon_i^n, \end{aligned} \tag{64}$$

where $i = 1, 2, \ldots, m$. As the proof of inequality (59), we have

$$\begin{aligned} &\sum_{j=1}^{m}\left\|(1-\alpha_n)z_j^n + \alpha_nJ_{M_j}^{\rho_j}\left[z_j^n - \rho_jN_j(z_1^n, z_2^n, \ldots, z_m^n)\right]\right. \\ &\qquad\left. + \alpha_nu_j^n + w_j^n - x_j^*\right\| \\ &\leq \left[1 - \alpha_n(1-k)\right]\sum_{j=1}^{m}\left\|z_j^n - x_j^*\right\| \\ &\quad + \alpha_n(1-k)\cdot\frac{1}{1-k}\sum_{j=1}^{m}\left\|u_j'^n\right\| \\ &\quad + \left(\sum_{j=1}^{m}\left\|u_j''^n\right\| + \sum_{j=1}^{m}\left\|w_j^n\right\|\right). \end{aligned} \tag{65}$$

Since $0 < a \leq \alpha_n$ $(n = 0, 1, 2, \ldots,)$, it follows from (64) and (65) that

$$\begin{aligned}\sum_{j=1}^{m} \left\|z_j^{n+1} - x_j^*\right\| &\leq [1 - \alpha_n (1 - k)] \sum_{j=1}^{m} \left\|z_j^n - x_j^*\right\| \\ &\quad + \alpha_n (1 - k) \cdot \frac{1}{1 - k} \left(\sum_{j=1}^{m} \left\|u_j'^n\right\| + \frac{1}{a} \sum_{j=1}^{m} \epsilon_j^n \right) \\ &\quad + \left(\sum_{j=1}^{m} \left\|u_j''^n\right\| + \sum_{j=1}^{m} \left\|w_j^n\right\| \right).\end{aligned} \tag{66}$$

Suppose that $\lim_{n\to\infty}(\epsilon_1^n, \epsilon_2^n, \ldots, \epsilon_m^n) = \underbrace{(0, 0, \ldots, 0)}_{m}$. Letting $b_n' = (1/(1-k))(\sum_{j=1}^m \|u_j'^n\| + (1/a) \sum_{j=1}^m \epsilon_j^n)$ and $a_n' = \sum_{j=1}^m \|z_j^n - x_j^*\|$, then (66) implies that

$$a_{n+1}' \leq (1 - t_n) a_n' + b_n' t_n + c_n, \quad n = 0, 1, 2, \ldots, \tag{67}$$

where t_n, c_n are the same as previously. Since $\lim_{n\to\infty} \|u_j'^n\| = 0$ and $\lim_{n\to\infty} \epsilon_j^n = 0$ $(j = 1, 2, \ldots, m)$,

$$\lim_{n\to\infty} b_n' = \frac{1}{1 - k} \left[\sum_{j=1}^{m} \left(\lim_{n\to\infty} \left\|u_j'^n\right\| \right) + \frac{1}{a} \sum_{j=1}^{m} \left(\lim_{n\to\infty} \epsilon_j^n \right) \right] = 0. \tag{68}$$

It again follows from Lemma 10, we have $\lim_{n\to\infty} a_n' = 0$ and so

$$\lim_{n\to\infty} (z_1^n, z_2^n, \ldots, z_m^n) = (x_1^*, x_2^*, \ldots, x_m^*). \tag{69}$$

Conversely, if $\lim_{n\to\infty}(z_1^n, z_2^n, \ldots, z_m^n) = (x_1^*, x_2^*, \ldots, x_m^*)$, it follows from (50), then we get

$$\begin{aligned}\epsilon_i^n \leq& \left\|z_i^{n+1} - x_i^*\right\| \\ &+ \left\|(1 - \alpha_n) z_i^n + \alpha_n J_{M_j}^{\rho_i} \left[z_i^n - \rho_i N_i (z_1^n, z_2^n, \ldots, z_m^n)\right] \right. \\ &\left. + \alpha_n u_i^n + w_i^n - x_i^*\right\|, \quad \forall i = 1, 2, \ldots, m.\end{aligned} \tag{70}$$

Combining (65) with (70), we have

$$\begin{aligned}\sum_{i=1}^{m} \epsilon_i^n \leq& \sum_{i=1}^{m} \left\|z_i^{n+1} - x_i^*\right\| + [1 - \alpha_n (1 - k)] \sum_{j=1}^{m} \left\|z_j^n - x_j^*\right\| \\ &+ \alpha_n (1 - k) \cdot \frac{1}{1 - k} \sum_{j=1}^{m} \left\|u_j'^n\right\| \\ &+ \left(\sum_{j=1}^{m} \left\|u_j''^n\right\| + \sum_{j=1}^{m} \left\|w_j^n\right\| \right) \longrightarrow 0 \quad (n \longrightarrow \infty).\end{aligned} \tag{71}$$

This completes the proof. □

Corollary 21. *Suppose that $\mathscr{H}_i$, η_i, N_i, and M_i $(i = 1, 2, \ldots, m)$ are the same as in Corollary 14. If $\sum_{n=0}^{\infty} \alpha_n = +\infty$ and condition (47) holds, then the perturbed iterative sequence $\{(x_1^n, x_2^n, \ldots, x_m^n)\}$ defined by Algorithm 17 converges strongly to the unique solution $(x_1^*, x_2^*, \ldots, x_m^*) \in \mathscr{H}_1 \times \mathscr{H}_2 \times \cdots \times \mathscr{H}_m$ of the problem (1). Moreover, if there exists $a \in (0, \alpha_n]$ for all $n \geq 0$, then $\lim_{n\to\infty}(z_1^n, z_2^n, \ldots, z_m^n) = (x_1^*, x_2^*, \ldots, x_m^*)$ if and only if*

$$\lim_{n\to\infty} (\epsilon_1^n, \epsilon_2^n, \ldots, \epsilon_m^n) = \underbrace{(0, 0, \ldots, 0)}_{m}, \tag{72}$$

where $(\epsilon_1^n, \epsilon_2^n, \ldots, \epsilon_m^n)$ is defined by (50).

Corollary 22. *Assume that $\mathscr{H}_i$, N_i, and φ_i $(i = 1, 2, \ldots, m)$ are the same as in Corollary 15. If $\sum_{n=0}^{\infty} \alpha_n = +\infty$ and condition (48) holds, then the perturbed iterative sequence $\{(x_1^n, x_2^n, \ldots, x_m^n)\}$ defined by Algorithm 18 converges strongly to the unique solution $(x_1^*, x_2^*, \ldots, x_m^*) \in \mathscr{H}_1 \times \mathscr{H}_2 \times \cdots \times \mathscr{H}_m$ of the problem (2). Moreover, if there exists $a \in (0, \alpha_n]$ for all $n \geq 0$, then $\lim_{n\to\infty}(z_1^n, z_2^n, \ldots, z_m^n) = (x_1^*, x_2^*, \ldots, x_m^*)$ if and only if*

$$\lim_{n\to\infty} (\epsilon_1^n, \epsilon_2^n, \ldots, \epsilon_m^n) = \underbrace{(0, 0, \ldots, 0)}_{m}, \tag{73}$$

where $(\epsilon_1^n, \epsilon_2^n, \ldots, \epsilon_m^n)$ is defined by (54).

Remark 23. If $m = 2$, then Theorem 20 reduces to Theorem 4.3 of Lan [19]. Further, one can easily see that our results presented in this paper may be viewed as an refinement and improvement of the previously known results.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Steepest-Descent Approach to Triple Hierarchical Constrained Optimization Problems

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We introduce and analyze a hybrid steepest-descent algorithm by combining Korpelevich's extragradient method, the steepest-descent method, and the averaged mapping approach to the gradient-projection algorithm. It is proven that under appropriate assumptions, the proposed algorithm converges strongly to the unique solution of a triple hierarchical constrained optimization problem (THCOP) over the common fixed point set of finitely many nonexpansive mappings, with constraints of finitely many generalized mixed equilibrium problems (GMEPs), finitely many variational inclusions, and a convex minimization problem (CMP) in a real Hilbert space.

1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$; let C be a nonempty closed convex subset of H and let P_C be the metric projection of H onto C. Let $S : C \to H$ be a nonlinear mapping on C. We denote by $\mathrm{Fix}(S)$ the set of fixed points of S and by $\mathbf{R}$ the set of all real numbers. A mapping $S : C \to H$ is called L-Lipschitz continuous if there exists a constant $L > 0$ such that

$$\|Sx - Sy\| \le L\|x - y\|, \quad \forall x, y \in C. \tag{1}$$

In particular, if $L = 1$ then S is called a nonexpansive mapping; if $L \in (0, 1)$ then S is called a contraction.

Let $A : C \to H$ be a nonlinear mapping on C. The classical variational inequality problem (VIP) [1] is to find a point $x \in C$ such that

$$\langle Ax, y - x\rangle \ge 0, \quad \forall y \in C. \tag{2}$$

The solution set of VIP (2) is denoted by $\mathrm{VI}(C, A)$.

In 1976, Korpelevich [2] proposed an iterative algorithm for solving the VIP (2) in Euclidean space $\mathbf{R}^n$:

$$\begin{aligned} y_n &= P_C(x_n - \tau A x_n), \\ x_{n+1} &= P_C(x_n - \tau A y_n), \\ &\forall n \ge 0, \end{aligned} \tag{3}$$

with $\tau > 0$ a given number, which is known as the extragradient method. See, for example, [3–7] and the references therein.

Let $\varphi : C \to \mathbf{R}$ be a real-valued function; let $A : H \to H$ be a nonlinear mapping and let $\Theta : C \times C \to \mathbf{R}$ be a bifunction. In 2008, Peng and Yao [8] introduced the following generalized mixed equilibrium problem (GMEP) of finding $x \in C$ such that

$$\Theta(x, y) + \varphi(y) - \varphi(x) + \langle Ax, y - x\rangle \ge 0, \quad \forall y \in C. \tag{4}$$

We denote the set of solutions of GMEP (4) by $\mathrm{GMEP}(\Theta, \varphi, A)$.

In [8], Peng and Yao assumed that $\Theta : C \times C \to \mathbf{R}$ is a bifunction satisfying conditions (A1)–(A4) and $\varphi : C \to \mathbf{R}$ is a lower semicontinuous and convex function with restriction (B1) or (B2), where

(A1) $\Theta(x, x) = 0$ for all $x \in C$;

(A2) Θ is monotone; that is, $\Theta(x, y) + \Theta(y, x) \le 0$ for any $x, y \in C$;

(A3) Θ is upper-hemicontinuous; that is, for each $x, y, z \in C$,

$$\limsup_{t \to 0^+} \Theta(tz + (1-t)x, y) \le \Theta(x, y); \tag{5}$$

(A4) $\Theta(x, \cdot)$ is convex and lower semicontinuous for each $x \in C$;

(B1) for each $x \in H$ and $r > 0$, there exists a bounded subset $D_x \subset C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$\Theta(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0; \tag{6}$$

(B2) C is a bounded set.

Given a positive number $r > 0$. Let $T_r^{(\Theta,\varphi)} : H \to C$ be the solution set of the auxiliary mixed equilibrium problem; that is, for each $x \in H$,

$$T_r^{(\Theta,\varphi)}(x) := \Big\{ y \in C : \Theta(y, z) + \varphi(z) - \varphi(y) + \frac{1}{r}\langle y - x, z - y\rangle \ge 0, \forall z \in C \Big\}. \tag{7}$$

Let $f : C \to \mathbf{R}$ be a convex and continuously Fréchet differentiable functional. Consider the convex minimization problem (CMP) of minimizing f over the constraint set C:

$$\min_{x \in C} f(x) \tag{8}$$

(assuming the existence of minimizers). We denote by Γ the set of minimizers of CMP (8).

On the other hand, let B be a single-valued mapping of C into H and R be a set-valued mapping with $D(R) = C$. Considering the following variational inclusion, find a point $x \in C$ such that

$$0 \in Bx + Rx. \tag{9}$$

We denote by $I(B, R)$ the solution set of the variational inclusion (9). Let a set-valued mapping $R : D(R) \subset H \to 2^H$ be maximal monotone. We define the resolvent operator $J_{R,\lambda} : H \to \overline{D(R)}$ associated with R and λ as follows:

$$J_{R,\lambda} = (I + \lambda R)^{-1}, \quad \forall x \in H, \tag{10}$$

where λ is a positive number.

Let S and T be two nonexpansive mappings. In 2009, Yao et al. [9] considered the following hierarchical VIP: find hierarchically a fixed point of T, which is a solution to the VIP for monotone mapping $I - S$; namely, find $\tilde{x} \in \mathrm{Fix}(T)$ such that

$$\langle (I - S)\tilde{x}, p - \tilde{x}\rangle \ge 0, \quad \forall p \in \mathrm{Fix}(T). \tag{11}$$

The solution set of the hierarchical VIP (11) is denoted by Λ. It is not hard to check that solving the hierarchical VIP (11) is equivalent to the fixed point problem of the composite mapping $P_{\mathrm{Fix}(T)}S$; that is, find $\tilde{x} \in C$ such that $\tilde{x} = P_{\mathrm{Fix}(T)}S\tilde{x}$. The authors [9] introduced and analyzed the following iterative algorithm for solving the hierarchical VIP (11):

$$\begin{aligned} y_n &= \beta_n S x_n + (1 - \beta_n) x_n, \\ x_{n+1} &= \alpha_n V x_n + (1 - \alpha_n) T y_n, \\ &\forall n \ge 0. \end{aligned} \tag{12}$$

In this paper, we introduce and study the following triple hierarchical constrained optimization problem (THCOP) with constraints of the CMP (8), finitely many GMEPs and finitely many variational inclusions.

Problem I. Let M, N, and K be three positive integers. Assume that

(i) $f : C \to \mathbf{R}$ is a convex and continuously Fréchet differentiable functional with L-Lipschitz continuous gradient ∇f, $S_i : H \to H$ is a nonexpansive mapping, and $A_j : H \to H$ is ζ_j-inverse-strongly monotone for $i = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, K$;

(ii) $\widetilde{A}_1 : H \to H$ is α-inverse strongly monotone and $\widetilde{A}_2 : H \to H$ is β-strongly monotone and κ-Lipschitz continuous;

(iii) Θ_j is a bifunctions from $C \times C$ to $\mathbf{R}$ satisfying (A1)–(A4), and $\varphi_j : C \to \mathbf{R}$ is a lower semicontinuous and convex functional with restriction (B1) or (B2) for $j = 1, 2, \ldots, K$;

(iv) $R_k : C \to 2^H$ is a maximal monotone mapping and $B_k : C \to H$ is η_k-inverse strongly monotone for $k = 1, 2, \ldots, M$;

(v) $\mathrm{VI}(\cap_{i=i}^N \mathrm{Fix}(S_i), \widetilde{A}_1) \ne \emptyset$ with $(\cap_{i=i}^N \mathrm{Fix}(S_i)) \subset (\cap_{j=1}^K \mathrm{GMEP}(\Theta_j, \varphi_j, A_j)) \cap (\cap_{k=1}^M I(B_k, R_k)) \cap \Gamma$.

Then the objective is to

$$\begin{aligned} &\text{find } x^* \in \mathrm{VI}\left(\mathrm{VI}\left(\bigcap_{i=i}^N \mathrm{Fix}(S_i), \widetilde{A}_1\right), \widetilde{A}_2\right) \\ &\quad := \left\{ x^* \in \mathrm{VI}\left(\bigcap_{i=i}^N \mathrm{Fix}(S_i), \widetilde{A}_1\right) : \langle \widetilde{A}_2 x^*, v - x^*\rangle \right. \\ &\qquad \left. \ge 0, \forall v \in \mathrm{VI}\left(\bigcap_{i=i}^N \mathrm{Fix}(S_i), \widetilde{A}_1\right)\right\}. \end{aligned} \tag{13}$$

Motivated and inspired by the above facts, we introduce and analyze a hybrid iterative algorithm via Korpelevich's extragradient method, the steepest-descent method, and the gradient-projection algorithm obtained by the averaged mapping approach. It is proven that under mild conditions, the proposed algorithm converges strongly to a unique element of $\mathrm{VI}(\mathrm{VI}(\cap_{i=1}^{N}\mathrm{Fix}(S_i)), \widetilde{A}_1), \widetilde{A}_2)$ with $(\cap_{i=1}^{N}\mathrm{Fix}(S_i)) \subset (\cap_{j=1}^{K}\mathrm{GMEP}(\Theta_j,\varphi_j,A_j)) \cap (\cap_{k=1}^{M} I(B_k,R_k)) \cap \Gamma$, that is, the unique solution of the THCOP (13). In this paper, the results we acquired improve and extend the existing results found in this field.

2. Preliminaries

Throughout this paper, we assume that H is a real Hilbert space of which inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges strongly to x. Moreover, we use $\omega_w(x_n)$ to denote the weak ω-limit set of the sequence $\{x_n\}$; that is,

$$\omega_w(x_n) := \left\{x \in H : x_{n_i} \rightharpoonup x \text{ for some subsequence } \left\{x_{n_i}\right\} \text{ of } \{x_n\}\right\}. \tag{14}$$

Definition 1. A mapping $A : C \to H$ is called

(i) monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C; \tag{15}$$

(ii) η-strongly monotone if there exists a constant $\eta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \eta\|x - y\|^2, \quad \forall x, y \in C; \tag{16}$$

(iii) ζ-inverse-strongly monotone if there exists a constant $\zeta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \zeta\|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{17}$$

It is obvious that if A is ζ-inverse-strongly monotone, then A is monotone and $1/\zeta$-Lipschitz continuous. Moreover, we also have that, for all $u, v \in C$ and $\lambda > 0$,

$$\begin{aligned} &\|(I - \lambda A)u - (I - \lambda A)v\|^2 \\ &\quad \leq \|u - v\|^2 + \lambda(\lambda - 2\zeta)\|Au - Av\|^2. \end{aligned} \tag{18}$$

So, if $\lambda \leq 2\zeta$, then $I - \lambda A$ is a nonexpansive mapping from C to H.

The metric projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in H$, the unique point $P_C x \in C$, satisfying the property

$$\|x - P_C x\| = \inf_{y\in C}\|x - y\| =: d(x, C). \tag{19}$$

Some important properties of projections are gathered in the following proposition.

Proposition 2. *For given $x \in H$ and $z \in C$:*

(i) $z = P_C x \Leftrightarrow \langle x - z, y - z\rangle \leq 0,\ \forall y \in C$;

(ii) $z = P_C x \Leftrightarrow \|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2,\ \forall y \in C$;

(iii) $\langle P_C x - P_C y, x - y\rangle \geq \|P_C x - P_C y\|^2,\ \forall y \in H$. *(This implies that P_C is nonexpansive and monotone.)*

Next we list some elementary conclusions for the mixed equilibrium problem where $\mathrm{MEP}(\Theta,\varphi)$ is the solution set.

Proposition 3 (see [10]). *Assume that $\Theta : C \times C \to \mathbf{R}$ satisfies (A1)–(A4) and let $\varphi : C \to \mathbf{R}$ be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. For $r > 0$ and $x \in H$, define a mapping $T_r^{(\Theta,\varphi)} : H \to C$ as follows:*

$$\begin{aligned} T_r^{(\Theta,\varphi)}(x) = \Big\{ z \in C : \Theta(z, y) + \varphi(y) - \varphi(z) \\ + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C \Big\} \end{aligned} \tag{20}$$

for all $x \in H$. Then the following hold:

(i) *for each $x \in H, T_r^{(\Theta,\varphi)}(x)$ is nonempty and single-valued;*

(ii) *$T_r^{(\Theta,\varphi)}$ is firmly nonexpansive; that is, for any $x, y \in H$,*

$$\left\|T_r^{(\Theta,\varphi)}x - T_r^{(\Theta,\varphi)}y\right\|^2 \leq \left\langle T_r^{(\Theta,\varphi)}x - T_r^{(\Theta,\varphi)}y, x - y\right\rangle; \tag{21}$$

(iii) $\mathrm{Fix}(T_r^{(\Theta,\varphi)}) = \mathrm{MEP}(\Theta,\varphi)$;

(iv) *$\mathrm{MEP}(\Theta,\varphi)$ is closed and convex;*

(v) $\|T_s^{(\Theta,\varphi)}x - T_t^{(\Theta,\varphi)}x\|^2 \leq (s - t)/s\langle T_s^{(\Theta,\varphi)}x - T_t^{(\Theta,\varphi)}x, T_s^{(\Theta,\varphi)}x - x\rangle$ *for all $s, t > 0$ and $x \in H$.*

In the following, we recall some facts and tools in a real Hilbert space H.

Lemma 4. *Let X be a real inner product space. Then there holds the following inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in X. \tag{22}$$

Lemma 5. *Let H be a real Hilbert space. Then the following hold:*

(a) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle$ *for all $x, y \in H$;*

(b) $\|\lambda x + \mu y\|^2 = \lambda\|x\|^2 + \mu\|y\|^2 - \lambda\mu\|x - y\|^2$ *for all $x, y \in H$ and $\lambda, \mu \in [0, 1]$ with $\lambda + \mu = 1$;*

(c) *if $\{x_n\}$ is a sequence in H such that $x_n \rightharpoonup x$, it follows that*

$$\begin{aligned} &\limsup_{n\to\infty}\|x_n - y\|^2 \\ &\quad = \limsup_{n\to\infty}\|x_n - x\|^2 + \|x - y\|^2, \quad \forall y \in H. \end{aligned} \tag{23}$$

Definition 6. A mapping $T : H \to H$ is said to be an averaged mapping if it can be written as the average of the identity I and a nonexpansive mapping; that is,

$$T \equiv (1-\alpha) I + \alpha S, \tag{24}$$

where $\alpha \in (0,1)$ and $S : H \to H$ is nonexpansive. More precisely, when the last equality holds, we say that T is α-averaged. Thus firmly nonexpansive mappings (particularly, projections) are $1/2$-averaged mappings.

Lemma 7 (see [11]). *Let $T : H \to H$ be a given mapping.*

(i) *T is nonexpansive if and only if the complement $I - T$ is $1/2$-ism.*

(ii) *If T is ν-ism, then for $\gamma > 0$, γT is ν/γ-ism.*

(iii) *T is averaged if and only if the complement $I-T$ is ν-ism for some $\nu > 1/2$. Indeed, for $\alpha \in (0,1)$, T is α-averaged if and only if $I - T$ is $1/2\alpha$-ism.*

Lemma 8 (see [11]). *Let $S, T, V : H \to H$ be given operators.*

(i) *If $T = (1-\alpha)S + \alpha V$ for some $\alpha \in (0,1)$ and if S is averaged and V is nonexpansive, then T is averaged.*

(ii) *T is firmly nonexpansive if and only if the complement $I - T$ is firmly nonexpansive.*

(iii) *If $T = (1-\alpha)S + \alpha V$ for some $\alpha \in (0,1)$ and if S is firmly nonexpansive and V is nonexpansive, then T is averaged.*

(iv) *The composite of finitely many averaged mappings is averaged. That is, if each of the mappings $\{T_i\}_{i=1}^N$ is averaged, then so is the composite $T_1 \cdots T_N$. In particular, if T_1 is α_1-averaged and T_2 is α_2-averaged, where $\alpha_1, \alpha_2 \in (0,1)$, then the composite $T_1 T_2$ is α-averaged, where $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$.*

(v) *If the mappings $\{T_i\}_{i=1}^N$ are averaged and have a common fixed point, then*

$$\bigcap_{i=1}^{N} \operatorname{Fix}(T_i) = \operatorname{Fix}(T_1 T_2 \cdots T_N). \tag{25}$$

The notation $\operatorname{Fix}(T)$ *denotes the set of all fixed points of the mapping T; that is,* $\operatorname{Fix}(T) = \{x \in H : Tx = x\}$.

Let $f : C \to \mathbf{R}$ be a convex functional with L-Lipschitz continuous gradient ∇f. It is well known that the gradient-projection algorithm (GPA) generates a sequence $\{x_n\}$ determined by the gradient ∇f and the metric projection P_C:

$$x_{n+1} := P_C(x_n - \lambda \nabla f(x_n)), \quad \forall n \geq 0, \tag{26}$$

or more generally,

$$x_{n+1} := P_C(x_n - \lambda_n \nabla f(x_n)), \quad \forall n \geq 0, \tag{27}$$

where, in both (26) and (27), the initial guess x_0 is taken from C arbitrarily, and the parameters λ or λ_n are positive real numbers. The convergence of algorithms (26) and (27) depends on the behavior of the gradient ∇f.

Lemma 9 (see [12, Demiclosedness principle]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let T be a nonexpansive self-mapping on C. Then $I-T$ is demiclosed. That is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I-T)x_n\}$ strongly converges to some y, it follows that $(I-T)x = y$. Here I is the identity operator of H.*

Lemma 10. *Let $A : C \to H$ be a monotone mapping. In the context of the variational inequality problem the characterization of the projection (see* Proposition 2(i)*) implies*

$$u \in \mathrm{VI}(C, A) \Longleftrightarrow u = P_C(u - \lambda A u), \quad \lambda > 0. \tag{28}$$

Let C be a nonempty closed convex subset of a real Hilbert space H. We introduce some notations. Let λ be a number in $(0,1]$ and let $\mu > 0$. Associating with a nonexpansive mapping $T : C \to H$, we define the mapping $T^\lambda : C \to H$ by

$$T^\lambda x := Tx - \lambda\mu F(Tx), \quad \forall x \in C, \tag{29}$$

where $F : H \to H$ is an operator such that, for some positive constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strongly monotone on H; that is, F satisfies the conditions:

$$\|Fx - Fy\| \leq \kappa \|x - y\|, \qquad \langle Fx - Fy, x - y\rangle \geq \eta \|x - y\|^2 \tag{30}$$

for all $x, y \in H$.

Lemma 11 (see [13, Lemma 3.1]). *T^λ is a contraction provided by $0 < \mu < 2\eta/\kappa^2$; that is,*

$$\left\|T^\lambda x - T^\lambda y\right\| \leq (1 - \lambda\tau)\|x - y\|, \quad \forall x, y \in C, \tag{31}$$

where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0,1]$.

Lemma 12 (see [13]). *Let $\{s_n\}$ be a sequence of nonnegative numbers satisfying the conditions*

$$s_{n+1} \leq (1-\alpha_n) s_n + \alpha_n \beta_n, \quad \forall n \geq 1, \tag{32}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers such that

(i) *$\{\alpha_n\} \subset [0,1]$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, or equivalently,*

$$\prod_{n=1}^{\infty} (1-\alpha_n) := \lim_{n\to\infty} \prod_{k=1}^{n} (1-\alpha_k) = 0; \tag{33}$$

(ii) *$\limsup_{n\to\infty} \beta_n \leq 0$, or $\sum_{n=1}^{\infty} |\alpha_n \beta_n| < \infty$.*

Then $\lim_{n\to\infty} s_n = 0$.

Recall that a Banach space X is said to satisfy Opial's property [12] if, for any given sequence $\{x_n\} \subset X$ which converges weakly to an element $x \in X$, there holds the inequality

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \quad \forall y \in X, \ y \neq x. \tag{34}$$

It is well known that every Hilbert space H satisfies Opial's property in [12].

Finally, recall that a set-valued mapping $T : D(T) \subset H \to 2^H$ is called monotone if for all $x, y \in D(T)$, $f \in Tx$, and $g \in Ty$ imply

$$\langle f - g, x - y \rangle \geq 0. \tag{35}$$

A set-valued mapping T is called maximal monotone if T is monotone and $(I + \lambda T)D(T) = H$ for each $\lambda > 0$, where I is the identity mapping of H. We denote by $G(T)$ the graph of T. It is known that a monotone mapping T is maximal if and only if, for $(x, f) \in H \times H$, $\langle f - g, x - y \rangle \geq 0$, for every $(y, g) \in G(T)$, implies $f \in Tx$. Let $A : C \to H$ be a monotone, k-Lipschitz-continuous mapping and let $N_C v$ be the normal cone to C at $v \in C$; that is,

$$N_C v = \{u \in H : \langle v - p, u \rangle \geq 0, \ \forall p \in C\}. \tag{36}$$

Define

$$\widetilde{T} v = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases} \tag{37}$$

Then, $\widetilde{T}$ is maximal monotone such that

$$0 \in \widetilde{T} v \Longleftrightarrow v \in \mathrm{VI}(C, A). \tag{38}$$

Let $R : D(R) \subset H \to 2^H$ be a maximal monotone mapping. Let $\lambda, \mu > 0$ be two positive numbers.

Lemma 13 (see [14]). *There holds the resolvent identity*

$$J_{R,\lambda} x = J_{R,\mu} \left(\frac{\mu}{\lambda} x + \left(1 - \frac{\mu}{\lambda}\right) J_{R,\lambda} x \right), \quad \forall x \in H. \tag{39}$$

For $\lambda, \mu > 0$, there holds the following relation that

$$\begin{aligned} \left\| J_{R,\lambda} x - J_{R,\mu} y \right\| &\leq \|x - y\| + |\lambda - \mu| \\ &\quad \times \left(\frac{1}{\lambda} \left\| J_{R,\lambda} x - y \right\| + \frac{1}{\mu} \left\| x - J_{R,\mu} y \right\| \right), \\ &\qquad \forall x, y \in H. \end{aligned} \tag{40}$$

Based on Huang [15], there holds the following property for the resolvent operator $J_{R,\lambda} : H \to \overline{D(R)}$.

Lemma 14. *$J_{R,\lambda}$ is single-valued and firmly nonexpansive; that is,*

$$\langle J_{R,\lambda} x - J_{R,\lambda} y, x - y \rangle \geq \left\| J_{R,\lambda} x - J_{R,\lambda} y \right\|^2, \quad \forall x, y \in H. \tag{41}$$

Consequently, $J_{R,\lambda}$ is nonexpansive and monotone.

Lemma 15 (see [16]). *Let R be a maximal monotone mapping with $D(R) = C$. Then for any given $\lambda > 0$, $u \in C$ is a solution of problem* (10) *if and only if $u \in C$ satisfies*

$$u = J_{R,\lambda} (u - \lambda B u). \tag{42}$$

Lemma 16 (see [17]). *Let R be a maximal monotone mapping with $D(R) = C$ and let $B : C \to H$ be a strongly monotone, continuous, and single-valued mapping. Then, for each $z \in H$, the equation $z \in (B + \lambda R)x$ has a unique solution x_λ for $\lambda > 0$.*

Lemma 17 (see [16]). *Let R be a maximal monotone mapping with $D(R) = C$ and let $B : C \to H$ be a monotone, continuous, and single-valued mapping. Then $(I + \lambda(R + B))C = H$ for each $\lambda > 0$. In this case, $R + B$ is maximal monotone.*

3. Main Results

In this section, we will introduce and analyze a hybrid steepest-descent algorithm for finding a solution of the THCOP (13) with constraints of several problems: the CMP (8), finitely many GMEPs, and finitely many variational inclusions in a real Hilbert space. This algorithm is based on Korpelevich's extragradient method, the steepest-descent method, and the averaged mapping approach to the gradient-projection algorithm. We prove the strong convergence of the proposed algorithm to a unique solution of THCOP (13) under suitable conditions. Throughout this paper, let $\{S_i\}_{i=1}^N$ be N nonexpansive mappings $S_i : H \to H$ with $N \geq 1$ an integer. We write $S_{[k]} := S_{k \bmod N}$, for integer $k \geq 1$, with the mod function taking values in the set $\{1, 2, \ldots, N\}$ (i.e., if $k = jN + q$ for some integers $j \geq 0$ and $0 \leq q < N$, then $T_{[k]} = N$ if $q = 0$ and $T_{[k]} = q$ if $1 \leq q < N$).

The following is to state and prove the main result in this paper.

Theorem 18. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $f : C \to \mathbf{R}$ be a convex and continuously Fréchet differentiable functional with L-Lipschitz continuous gradient ∇f. Let $M, N, K \geq 1$ be three integers. Let Θ_j be a bifunctions from $C \times C$ to $\mathbf{R}$ satisfying (A1)–(A4), $\varphi_j : C \to \mathbf{R}$ a lower semicontinuous and convex functional with restriction (B1) or (B2), and $A_j : H \to H \zeta_j$-inverse-strongly monotone for $j = 1, 2, \ldots, K$. Let $R_k : C \to 2^H$ be a maximal monotone mapping and let $B_k : C \to H$ be η_k-inverse strongly monotone for $k = 1, 2, \ldots, M$. Let $\{S_i\}_{i=1}^N$ be a finite family of nonexpansive mappings on H. Let $\widetilde{A}_1 : H \to H$ be α-inverse strongly monotone and let $\widetilde{A}_2 : H \to H$ be β-strongly monotone and κ-Lipschitz continuous. Assume that $\mathrm{VI}(\cap_{i=1}^N \mathrm{Fix}(S_i), \widetilde{A}_1) \neq \emptyset$ with $(\cap_{i=1}^N \mathrm{Fix}(S_i)) \subset (\cap_{j=1}^K \mathrm{GMEP}(\Theta_j, \varphi_j, A_j)) \cap (\cap_{k=1}^M I(B_k, R_k)) \cap \Gamma$. Let $\mu \in (0, 2\beta/\kappa^2)$, $\{\alpha_n\}_{n=0}^\infty \subset (0, 1]$, $\{\rho_n\}_{n=0}^\infty \subset (0, 2\alpha]$, $\{\lambda_{k,n}\}_{n=0}^\infty \subset [a_k, b_k] \subset (0, 2\eta_k)$, and $\{r_{j,n}\}_{n=0}^\infty \subset [c_j, d_j] \subset (0, 2\zeta_j)$ where*

$j \in \{1, 2, \ldots, K\}$ *and* $k \in \{1, 2, \ldots, M\}$. *For arbitrarily given* $x_0 \in H$, *let* $\{x_n\}$ *be a sequence generated by*

$$\begin{aligned} u_n &= T_{r_{K,n}}^{(\Theta_K,\varphi_K)} (I - r_{K,n}A_K) T_{r_{K-1,n}}^{(\Theta_{K-1},\varphi_{K-1})} (I - r_{K-1,n}A_{K-1}) \\ &\quad \cdots T_{r_{1,n}}^{(\Theta_1,\varphi_1)} (I - r_{1,n}A_1) x_n, \\ v_n &= J_{R_M,\lambda_{M,n}} (I - \lambda_{M,n}B_M) J_{R_{M-1},\lambda_{M-1,n}} (I - \lambda_{M-1,n}B_{M-1}) \\ &\quad \cdots J_{R_1,\lambda_{1,n}} (I - \lambda_{1,n}B_1) u_n, \\ y_n &= S_{[n+1]} (I - \rho_n \widetilde{A}_1) T_n v_n, \\ x_{n+1} &= y_n - \mu\alpha_n \widetilde{A}_2 y_n, \quad \forall n \ge 0, \end{aligned} \tag{43}$$

where $P_C(I-\lambda_n\nabla f) = s_nI+(1-s_n)T_n$ *(here* T_n *is nonexpansive and* $s_n := s_n(\lambda_n) = (2 - \lambda_nL)/4 \in (0, 1/2)$ *for each* $\lambda_n \in (0, 2/L)$*). Assume that*

$$\begin{aligned} \bigcap_{i=1}^{N} \mathrm{Fix}(S_i) &= \mathrm{Fix}(S_1S_2\cdots S_N) \\ &= \mathrm{Fix}(S_NS_1\cdots S_{N-1}) \\ &= \cdots = \mathrm{Fix}(S_2S_3\cdots S_NS_1) \end{aligned} \tag{44}$$

and that the following conditions are satisfied:

(i) $\lim_{n\to\infty}\alpha_n = 0, \sum_{n=0}^{\infty}\alpha_n = \infty$ *and* $\rho_n \le \alpha_n$ *for all* $n \ge 0$;

(ii) $\lim_{n\to\infty}(|\alpha_n - \alpha_{n+N}|/\alpha_{n+N}) = 0$ *or* $\sum_{n=0}^{\infty}|\alpha_n - \alpha_{n+N}| < \infty$;

(iii) $\lim_{n\to\infty}(|s_n - s_{n+N}|/\alpha_{n+N}) = 0$ *or* $\sum_{n=0}^{\infty}|s_n - s_{n+N}| < \infty$;

(iv) $\lim_{n\to\infty}(|\rho_n - \rho_{n+N}|/\rho_{n+N}) = 0$ *or* $\sum_{n=0}^{\infty}|\rho_n - \rho_{n+N}| < \infty$;

(v) $\lim_{n\to\infty}(|\lambda_{k,n} - \lambda_{k,n+N}|/(\alpha_{n+N})) = 0$ *or* $\sum_{n=0}^{\infty}|\lambda_{k,n} - \lambda_{k,n+N}| < \infty$ *for* $k = 1, 2, \ldots, M$;

(vi) $\lim_{n\to\infty}(|r_{j,n} - r_{j,n+N}|/(\alpha_{n+N})) = 0$ *or* $\sum_{n=0}^{\infty}|r_{j,n} - r_{j,n+N}| < \infty$ *for* $j = 1, 2, \ldots, K$.

Then the following hold:

(a) $\{x_n\}_{n=0}^{\infty}$ *is bounded;*

(b) $\lim_{n\to\infty}\|x_n - x_{n+N}\| = 0$;

(c) $\lim_{n\to\infty}\|x_n - S_{[n+N]}\cdots S_{[n+1]}x_n\| = 0$ *provided* $\lim_{n\to\infty}(\|x_n - y_n\| + \|T_nv_n - v_n\|) = 0$;

(d) $\{x_n\}_{n=0}^{\infty}$ *converges strongly to the unique element of* $\mathrm{VI}(\mathrm{VI}(\cap_{i=1}^{N}\mathrm{Fix}(S_i), \widetilde{A}_1), \widetilde{A}_2)$ *provided* $\|x_n - y_n\| + \|T_nv_n - v_n\| = o(\rho_n)$.

Proof. Let $\{x^*\} = \mathrm{VI}(\mathrm{VI}(\Omega, \widetilde{A}_1), \widetilde{A}_2)$. Since ∇f is L-Lipschitzian, it follows that ∇f is $1/L$-ism. By Lemma 7(ii), we know that for $\lambda > 0$, $\lambda\nabla f$ is $1/\lambda L$-ism. So by Lemma 7(iii), we deduce that $I - \lambda\nabla f$ is $\lambda L/2$-averaged. Now since the projection P_C is $1/2$-averaged, it is easy to see from Lemma 8(iv) that the composite $P_C(I - \lambda\nabla f)$ is $(2 + \lambda L)/4$-averaged for $\lambda \in (0, 2/L)$. Hence we obtain that, for each $n \ge 0$, $P_C(I - \lambda_n\nabla f)$ is $(2 + \lambda_nL)/4$-averaged for each $\lambda_n \in (0, 2/L)$. Therefore, we can write

$$\begin{aligned} P_C(I - \lambda_n\nabla f) &= \frac{2 - \lambda_nL}{4}I + \frac{2 + \lambda_nL}{4}T_n \\ &= s_nI + (1 - s_n)T_n, \end{aligned} \tag{45}$$

where T_n is nonexpansive and $s_n := s_n(\lambda_n) = (2 - \lambda_nL)/4 \in (0, 1/2)$ for each $\lambda_n \in (0, 2/L)$. Since $\widetilde{A}_2$ is κ-Lipschitz continuous, we get

$$\left\|\widetilde{A}_2y_n - \widetilde{A}_2x^*\right\| \le \kappa\left\|y_n - x^*\right\|, \quad \forall n \ge 0. \tag{46}$$

Putting $z_n = (I - \rho_n\widetilde{A}_1)T_nv_n$, for all $n \ge 0$, we have

$$\begin{aligned} x_{n+1} &= y_n - \mu\alpha_n\widetilde{A}_2y_n \\ &= S_{[n+1]}z_n - \mu\alpha_n\widetilde{A}_2S_{[n+1]}z_n \\ &= S_{[n+1]}^{\alpha_n}z_n, \quad \forall n \ge 0. \end{aligned} \tag{47}$$

Put

$$\begin{aligned} \Delta_n^j &= T_{r_{j,n}}^{(\Theta_j,\varphi_j)} (I - r_{j,n}A_j) T_{r_{j-1,n}}^{(\Theta_{j-1},\varphi_{j-1})} (I - r_{j-1,n}A_{j-1}) \\ &\quad \cdots T_{r_{1,n}}^{(\Theta_1,\varphi_1)} (I - r_{1,n}A_1) x_n \end{aligned} \tag{48}$$

for all $j \in \{1, 2, \ldots, K\}$ and $n \ge 0$,

$$\begin{aligned} \Lambda_n^k &= J_{R_k,\lambda_{k,n}} (I - \lambda_{k,n}B_k) J_{R_{k-1},\lambda_{k-1,n}} (I - \lambda_{k-1,n}B_{k-1}) \\ &\quad \cdots J_{R_1,\lambda_{1,n}} (I - \lambda_{1,n}B_1) \end{aligned} \tag{49}$$

for all $k \in \{1, 2, \ldots, M\}$, $\Delta_n^0 = I$, and $\Lambda_n^0 = I$, where I is the identity mapping on H. Then we have that $u_n = \Delta_n^Kx_n$ and $v_n = \Lambda_n^Mu_n$.

We divide the rest of the proof into several steps.

Step 1. We prove that $\{x_n\}$ is bounded.

Indeed, utilizing (18) and Proposition 3(ii), we have

$$\begin{aligned} &\left\|u_n - x^*\right\| \\ &= \Big\|T_{r_{K,n}}^{(\Theta_K,\varphi_K)} (I - r_{K,n}B_K) \Delta_n^{K-1}x_n \\ &\qquad - T_{r_{K,n}}^{(\Theta_K,\varphi_K)} (I - r_{K,n}B_K) \Delta_n^{K-1}x^*\Big\| \\ &\le \left\|(I - r_{K,n}B_K) \Delta_n^{K-1}x_n - (I - r_{K,n}B_K) \Delta_n^{K-1}x^*\right\| \\ &\le \left\|\Delta_n^{K-1}x_n - \Delta_n^{K-1}x^*\right\| \\ &\quad \vdots \\ &\le \left\|\Delta_n^0x_n - \Delta_n^0x^*\right\| \\ &= \left\|x_n - x^*\right\|. \end{aligned} \tag{50}$$

Utilizing (18) and Lemma 14 we have

$$\begin{aligned}
&\|v_n - x^*\| \\
&= \left\| J_{R_M,\lambda_{M,n}} (I - \lambda_{M,n} A_M) \Lambda_n^{M-1} u_n \right. \\
&\qquad \left. - J_{R_M,\lambda_{M,n}} (I - \lambda_{M,n} A_M) \Lambda_n^{M-1} x^* \right\| \\
&\le \left\| (I - \lambda_{M,n} A_M) \Lambda_n^{M-1} u_n - (I - \lambda_{M,n} A_M) \Lambda_n^{M-1} x^* \right\| \\
&\le \left\| \Lambda_n^{M-1} u_n - \Lambda_n^{M-1} x^* \right\| \\
&\ \ \vdots \\
&\le \left\| \Lambda_n^{0} u_n - \Lambda_n^{0} x^* \right\| \\
&= \|u_n - x^*\|.
\end{aligned} \tag{51}$$

Combining (50) and (51), we have

$$\|v_n - x^*\| \le \|x_n - x^*\|. \tag{52}$$

Since $\widetilde{A}_1$ is α-inverse strongly monotone and $\{\rho_n\}_{n=0}^{\infty} \subset (0, 2\alpha]$, we have

$$\begin{aligned}
&\left\| T_n v_n - x^* - \rho_n (\widetilde{A}_1 T_n v_n - \widetilde{A}_1 x^*) \right\|^2 \\
&= \|T_n v_n - x^*\|^2 \\
&\quad - 2\rho_n \left\langle \widetilde{A}_1 T_n v_n - \widetilde{A}_1 x^*, T_n v_n - x^* \right\rangle \\
&\quad + \rho_n^2 \left\| \widetilde{A}_1 T_n v_n - \widetilde{A}_1 x^* \right\|^2 \\
&\le \|T_n v_n - x^*\|^2 - \rho_n (2\alpha - \rho_n) \left\| \widetilde{A}_1 T_n v_n - \widetilde{A}_1 x^* \right\|^2 \\
&\le \|T_n v_n - x^*\|^2 \\
&\le \|v_n - x^*\|^2.
\end{aligned} \tag{53}$$

Utilizing Lemma 11, we deduce from (52), $\rho_n \le \alpha_n$, and $S_{[n+1]}^{\alpha_n} x^* = x^* - \alpha_n \mu \widetilde{A}_2 x^*$ that for all $n \ge 0$

$$\begin{aligned}
&\|x_{n+1} - x^*\| \\
&= \left\| S_{[n+1]}^{\alpha_n} z_n - x^* \right\| \\
&\le \left\| S_{[n+1]}^{\alpha_n} z_n - S_{[n+1]}^{\alpha_n} x^* \right\| + \left\| S_{[n+1]}^{\alpha_n} x^* - x^* \right\| \\
&\le (1 - \alpha_n \tau) \|z_n - x^*\| + \alpha_n \mu \left\| \widetilde{A}_2 x^* \right\| \\
&= (1 - \alpha_n \tau) \left\| (I - \rho_n \widetilde{A}_1) T_n v_n - x^* \right\| + \alpha_n \mu \left\| \widetilde{A}_2 x^* \right\| \\
&= (1 - \alpha_n \tau) \left\| T_n v_n - x^* - \rho_n (\widetilde{A}_1 T_n v_n - \widetilde{A}_1 x^*) - \rho_n \widetilde{A}_1 x^* \right\| \\
&\quad + \alpha_n \mu \left\| \widetilde{A}_2 x^* \right\| \\
&\le (1 - \alpha_n \tau) \left[\left\| T_n v_n - x^* - \rho_n (\widetilde{A}_1 T_n v_n - \widetilde{A}_1 x^*) \right\| \right. \\
&\qquad \left. + \rho_n \left\| \widetilde{A}_1 x^* \right\| \right] + \alpha_n \mu \left\| \widetilde{A}_2 x^* \right\| \\
&\le (1 - \alpha_n \tau) \left[\|v_n - x^*\| + \rho_n \left\| \widetilde{A}_1 x^* \right\| \right] + \alpha_n \mu \left\| \widetilde{A}_2 x^* \right\| \\
&\le (1 - \alpha_n \tau) \left[\|x_n - x^*\| + \rho_n \left\| \widetilde{A}_1 x^* \right\| \right] + \alpha_n \mu \left\| \widetilde{A}_2 x^* \right\| \\
&\le (1 - \alpha_n \tau) \|x_n - x^*\| + \rho_n \left\| \widetilde{A}_1 x^* \right\| + \alpha_n \mu \left\| \widetilde{A}_2 x^* \right\| \\
&\le (1 - \alpha_n \tau) \|x_n - x^*\| + \alpha_n \left\| \widetilde{A}_1 x^* \right\| \\
&\quad + \alpha_n \mu \left\| \widetilde{A}_2 x^* \right\| \\
&= (1 - \alpha_n \tau) \|x_n - x^*\| \\
&\quad + \alpha_n \tau \frac{\left\| \widetilde{A}_1 x^* \right\| + \mu \left\| \widetilde{A}_2 x^* \right\|}{\tau} \\
&\le \max \left\{ \|x_n - x^*\|, \frac{\left\| \widetilde{A}_1 x^* \right\| + \mu \left\| \widetilde{A}_2 x^* \right\|}{\tau} \right\},
\end{aligned} \tag{54}$$

where $\tau = 1 - \sqrt{1 - \mu(2\beta - \mu\kappa^2)}$. So, by induction we obtain

$$\begin{aligned}
&\|x_n - x^*\| \\
&\le \max \left\{ \|x_0 - x^*\|, \frac{\left\| \widetilde{A}_1 x^* \right\| + \mu \left\| \widetilde{A}_2 x^* \right\|}{\tau} \right\}, \quad \forall n \ge 0.
\end{aligned} \tag{55}$$

Hence $\{x_n\}_{n=0}^{\infty}$ is bounded. Since $\widetilde{A}_1 : H \to H$ is α-inverse strongly monotone, it is known that $\widetilde{A}_1$ is $1/\alpha$-Lipschitz continuous. Thus, from (52), we get

$$\begin{aligned}
\left\| \widetilde{A}_1 T_n v_n - \widetilde{A}_1 x^* \right\| &\le \frac{1}{\alpha} \|T_n v_n - x^*\| \le \frac{1}{\alpha} \|v_n - x^*\| \\
&\le \frac{1}{\alpha} \|x_n - x^*\|, \quad \forall n \ge 0.
\end{aligned} \tag{56}$$

Consequently, the boundedness of $\{x_n\}$ ensures the boundedness of $\{v_n\}, \{T_n v_n\}$, and $\{\widetilde{A}_1 T_n v_n\}$. From $y_n = S_{[n+1]}(I - \rho_n \widetilde{A}_1) T_n v_n$ and the nonexpansivity of $S_{[n+1]}$, it follows that $\{y_n\}$ is bounded. Since $\widetilde{A}_2$ is κ-Lipschitz continuous, $\{\widetilde{A}_2 y_n\}$ is also bounded.

Step 2. We prove that $\lim_{n \to \infty} \|x_n - x_{n+N}\| = 0$.

Indeed, utilizing (18) and (40), we obtain that

$$\begin{aligned}
&\left\|v_{n+N}-v_n\right\| \\
&=\left\|\Lambda_{n+N}^M u_{n+N}-\Lambda_n^M u_n\right\| \\
&=\Big\|J_{R_M,\lambda_{M,n+N}}\left(I-\lambda_{M,n+N}B_M\right)\Lambda_{n+N}^{M-1}u_{n+N} \\
&\qquad -J_{R_M,\lambda_{M,n}}\left(I-\lambda_{M,n}B_M\right)\Lambda_n^{M-1}u_n\Big\| \\
&\leq\Big\|J_{R_M,\lambda_{M,n+N}}\left(I-\lambda_{M,n+N}B_M\right)\Lambda_{n+N}^{M-1}u_{n+N} \\
&\qquad -J_{R_M,\lambda_{M,n+N}}\left(I-\lambda_{M,n}B_M\right)\Lambda_{n+N}^{M-1}u_{n+N}\Big\| \\
&\quad +\Big\|J_{R_M,\lambda_{M,n+N}}\left(I-\lambda_{M,n}B_M\right)\Lambda_{n+N}^{M-1}u_{n+N} \\
&\qquad -J_{R_M,\lambda_{M,n}}\left(I-\lambda_{M,n}B_M\right)\Lambda_n^{M-1}u_n\Big\| \\
&\leq\left\|\left(I-\lambda_{M,n+N}B_M\right)\Lambda_{n+N}^{M-1}u_{n+N}-\left(I-\lambda_{M,n}B_M\right)\Lambda_{n+N}^{M-1}u_{n+N}\right\| \\
&\quad +\left\|\left(I-\lambda_{M,n}B_M\right)\Lambda_{n+N}^{M-1}u_{n+N}-\left(I-\lambda_{M,n}B_M\right)\Lambda_n^{M-1}u_n\right\| \\
&\quad +\left|\lambda_{M,n+N}-\lambda_{M,n}\right| \\
&\quad \times\Bigg(\frac{1}{\lambda_{M,n+N}}\Big\|J_{R_M,\lambda_{M,n+N}}\left(I-\lambda_{M,n}B_M\right)\Lambda_{n+N}^{M-1}u_{n+N} \\
&\qquad\qquad -\left(I-\lambda_{M,n}B_M\right)\Lambda_n^{M-1}u_n\Big\| \\
&\qquad +\frac{1}{\lambda_{M,n}}\Big\|\left(I-\lambda_{M,n}B_M\right)\Lambda_{n+N}^{M-1}u_{n+N} \\
&\qquad\qquad -J_{R_M,\lambda_{M,n}}\left(I-\lambda_{M,n}B_M\right)\Lambda_n^{M-1}u_n\Big\|\Bigg) \\
&\leq\left|\lambda_{M,n+N}-\lambda_{M,n}\right|\left(\left\|B_M\Lambda_{n+N}^{M-1}u_{n+N}\right\|+\widetilde{M}\right) \\
&\quad +\left\|\Lambda_{n+N}^{M-1}u_{n+N}-\Lambda_n^{M-1}u_n\right\| \\
&\leq\left|\lambda_{M,n+N}-\lambda_{M,n}\right|\left(\left\|B_M\Lambda_{n+N}^{M-1}u_{n+N}\right\|+\widetilde{M}\right) \\
&\quad +\left|\lambda_{M-1,n+N}-\lambda_{M-1,n}\right|\left(\left\|B_{M-1}\Lambda_{n+N}^{M-2}u_{n+N}\right\|+\widetilde{M}\right) \\
&\quad +\left\|\Lambda_{n+N}^{M-2}u_{n+N}-\Lambda_n^{M-2}u_n\right\| \\
&\ \ \vdots \\
&\leq\left|\lambda_{M,n+N}-\lambda_{M,n}\right|\left(\left\|B_M\Lambda_{n+N}^{M-1}u_{n+N}\right\|+\widetilde{M}\right) \\
&\quad +\left|\lambda_{M-1,n+N}-\lambda_{M-1,n}\right|\left(\left\|B_{M-1}\Lambda_{n+N}^{M-2}u_{n+N}\right\|+\widetilde{M}\right) \\
&\quad +\cdots+\left|\lambda_{1,n+N}-\lambda_{1,n}\right|\left(\left\|B_1\Lambda_{n+N}^{0}u_{n+N}\right\|+\widetilde{M}\right) \\
&\quad +\left\|\Lambda_{n+N}^{0}u_{n+N}-\Lambda_n^{0}u_n\right\| \\
&\leq\widetilde{M}_0\sum_{k=1}^{M}\left|\lambda_{k,n+N}-\lambda_{k,n}\right|+\left\|u_{n+N}-u_n\right\|,
\end{aligned} \tag{57}$$

where

$$\begin{aligned}
\sup_{n\geq 0,1\leq i\leq M}\Bigg\{&\frac{1}{\lambda_{i,n+N}}\Big\|J_{R_i,\lambda_{i,n+N}}\left(I-\lambda_{i,n}B_i\right)\Lambda_{n+N}^{i-1}u_{n+N} \\
&\qquad -\left(I-\lambda_{i,n}B_i\right)\Lambda_n^{i-1}u_n\Big\| \\
&+\frac{1}{\lambda_{i,n}}\Big\|\left(I-\lambda_{i,n}B_i\right)\Lambda_{n+N}^{i-1}u_{n+N} \\
&\qquad -J_{R_i,\lambda_{i,n}}\left(I-\lambda_{i,n}B_i\right)\Lambda_n^{i-1}u_n\Big\|\Bigg\}\leq\widetilde{M},
\end{aligned} \tag{58}$$

for some $\widetilde{M}>0$ and $\sup_{n\geq 0}\{\sum_{k=1}^{M}\|B_k\Lambda_{n+N}^{k-1}u_{n+N}\|+\widetilde{M}\}\leq\widetilde{M}_0$ for some $\widetilde{M}_0>0$.

Furthermore, since ∇f is $1/L$-ism, $P_C(I-\lambda_n\nabla f)$ is nonexpansive for $\lambda_n\in(0,2/L)$. So, it follows that

$$\begin{aligned}
&\left\|P_C\left(I-\lambda_{n+N}\nabla f\right)v_n\right\| \\
&\quad\leq\left\|P_C\left(I-\lambda_{n+N}\nabla f\right)v_n-x^*\right\|+\left\|x^*\right\| \\
&\quad=\left\|P_C\left(I-\lambda_{n+N}\nabla f\right)v_n-P_C\left(I-\lambda_{n+N}\nabla f\right)x^*\right\|+\left\|x^*\right\| \\
&\quad\leq\left\|v_n-x^*\right\|+\left\|x^*\right\| \\
&\quad\leq\left\|v_n\right\|+2\left\|x^*\right\|.
\end{aligned} \tag{59}$$

With the boundedness of $\{v_n\}$, this implies that $\{P_C(I-\lambda_{n+N}\nabla f)v_n\}$ is bounded. Also, observe that

$$\begin{aligned}
&\left\|T_{n+N}v_n-T_nv_n\right\| \\
&=\Bigg\|\frac{4P_C\left(I-\lambda_{n+N}\nabla f\right)-\left(2-\lambda_{n+N}L\right)I}{2+\lambda_{n+N}L}v_n \\
&\qquad -\frac{4P_C\left(I-\lambda_n\nabla f\right)-\left(2-\lambda_nL\right)I}{2+\lambda_nL}v_n\Bigg\| \\
&\leq\left\|\frac{4P_C\left(I-\lambda_{n+N}\nabla f\right)}{2+\lambda_{n+N}L}v_n-\frac{4P_C\left(I-\lambda_n\nabla f\right)}{2+\lambda_nL}v_n\right\| \\
&\quad +\left\|\frac{2-\lambda_nL}{2+\lambda_nL}v_n-\frac{2-\lambda_{n+N}L}{2+\lambda_{n+N}L}v_n\right\| \\
&=\Big\|\left(4\left(2+\lambda_nL\right)P_C\left(I-\lambda_{n+N}\nabla f\right)v_n\right. \\
&\qquad \left.-4\left(2+\lambda_{n+N}L\right)P_C\left(I-\lambda_n\nabla f\right)v_n\right) \\
&\qquad \times\left((2+\lambda_{n+N}L)(2+\lambda_nL)\right)^{-1}\Big\| \\
&\quad +\frac{4L\left|\lambda_{n+N}-\lambda_n\right|}{\left(2+\lambda_{n+N}L\right)\left(2+\lambda_nL\right)}\left\|v_n\right\|
\end{aligned}$$

$$\begin{aligned}
&= \Big\| \left(4L\left(\lambda_n - \lambda_{n+N}\right) P_C\left(I - \lambda_{n+N}\nabla f\right) v_n + 4\left(2 + \lambda_{n+N}L\right)\right. \\
&\qquad \times \left(P_C\left(I - \lambda_{n+N}\nabla f\right) v_n - P_C\left(I - \lambda_n \nabla f\right) v_n\right)\Big) \\
&\qquad \times \left((2 + \lambda_{n+N}L)(2 + \lambda_n L)\right)^{-1} \Big\| \\
&\quad + \frac{4L\left|\lambda_{n+N} - \lambda_n\right|}{\left(2 + \lambda_{n+N}L\right)\left(2 + \lambda_n L\right)} \left\|v_n\right\| \\
&\le \frac{4L\left|\lambda_n - \lambda_{n+N}\right| \left\|P_C\left(I - \lambda_{n+N}\nabla f\right) v_n\right\|}{\left(2 + \lambda_{n+N}L\right)\left(2 + \lambda_n L\right)} \\
&\quad + \Big(4\left(2 + \lambda_{n+N}L\right) \\
&\qquad \times \left\|P_C\left(I - \lambda_{n+N}\nabla f\right) v_n - P_C\left(I - \lambda_n \nabla f\right) v_n\right\|\Big) \\
&\quad \times \left((2 + \lambda_{n+N}L)(2 + \lambda_n L)\right)^{-1} \\
&\quad + \frac{4L\left|\lambda_{n+N} - \lambda_n\right|}{\left(2 + \lambda_{n+N}L\right)\left(2 + \lambda_n L\right)} \left\|v_n\right\| \\
&\le \left|\lambda_{n+N} - \lambda_n\right| \Big[L\left\|P_C\left(I - \lambda_{n+N}\nabla f\right) v_n\right\| \\
&\qquad\qquad + 4\left\|\nabla f\left(v_n\right)\right\| + L\left\|v_n\right\|\Big] \\
&\le \widetilde{M}_1 \left|\lambda_{n+N} - \lambda_n\right|,
\end{aligned} \tag{60}$$

where $\sup_{n\ge 0}\{L\|P_C(I-\lambda_{n+N}\nabla f)v_n\|+4\|\nabla f(v_n)\|+L\|v_n\|\} \le \widetilde{M}_1$ for some $\widetilde{M}_1 > 0$. Thus, we conclude from (57) and (60) that

$$\begin{aligned}
&\left\|T_{n+N}v_{n+N} - T_n v_n\right\| \\
&\quad \le \left\|T_{n+N}v_{n+N} - T_{n+N}v_n\right\| + \left\|T_{n+N}v_n - T_n v_n\right\| \\
&\quad \le \left\|v_{n+N} - v_n\right\| + \widetilde{M}_1 \left|\lambda_{n+N} - \lambda_n\right| \\
&\quad \le \left\|v_{n+N} - v_n\right\| + \frac{4\widetilde{M}_1}{L} \left|s_{n+N} - s_n\right| \\
&\quad \le \widetilde{M}_0 \sum_{k=1}^{M} \left|\lambda_{k,n+N} - \lambda_{k,n}\right| + \left\|u_{n+N} - u_n\right\| \\
&\qquad + \frac{4\widetilde{M}_1}{L} \left|s_{n+N} - s_n\right|.
\end{aligned} \tag{61}$$

Also, utilizing Proposition 3(ii), (v), we deduce that

$$\begin{aligned}
&\left\|u_{n+N} - u_n\right\| \\
&= \left\|\Delta_{n+N}^{K} x_{n+N} - \Delta_n^K x_n\right\| \\
&= \Big\|T_{r_{K,n+N}}^{(\Theta_K,\varphi_K)}\left(I - r_{K,n+N}A_K\right)\Delta_{n+N}^{K-1}x_{n+N} \\
&\qquad - T_{r_{K,n}}^{(\Theta_K,\varphi_K)}\left(I - r_{K,n}A_K\right)\Delta_n^{K-1}x_n\Big\| \\
&\le \Big\|T_{r_{K,n+N}}^{(\Theta_K,\varphi_K)}\left(I - r_{K,n+N}A_K\right)\Delta_{n+N}^{K-1}x_{n+N} \\
&\qquad - T_{r_{K,n}}^{(\Theta_K,\varphi_K)}\left(I - r_{K,n}A_K\right)\Delta_{n+N}^{K-1}x_{n+N}\Big\| \\
&\quad + \Big\|T_{r_{K,n}}^{(\Theta_K,\varphi_K)}\left(I - r_{K,n}A_K\right)\Delta_{n+N}^{K-1}x_{n+N} \\
&\qquad - T_{r_{K,n}}^{(\Theta_K,\varphi_K)}\left(I - r_{K,n}A_K\right)\Delta_n^{K-1}x_n\Big\| \\
&\le \Big\|T_{r_{K,n+N}}^{(\Theta_K,\varphi_K)}\left(I - r_{K,n+N}A_K\right)\Delta_{n+N}^{K-1}x_{n+N} \\
&\qquad - T_{r_{K,n}}^{(\Theta_K,\varphi_K)}\left(I - r_{K,n+N}A_K\right)\Delta_{n+N}^{K-1}x_{n+N}\Big\| \\
&\quad + \Big\|T_{r_{K,n}}^{(\Theta_K,\varphi_K)}\left(I - r_{K,n+N}A_K\right)\Delta_{n+N}^{K-1}x_{n+N} \\
&\qquad - T_{r_{K,n}}^{(\Theta_K,\varphi_K)}\left(I - r_{K,n}A_K\right)\Delta_{n+N}^{K-1}x_{n+N}\Big\| \\
&\quad + \left\|\left(I - r_{K,n}A_K\right)\Delta_{n+N}^{K-1}x_{n+N} - \left(I - r_{K,n}A_K\right)\Delta_n^{K-1}x_n\right\| \\
&\le \frac{\left|r_{K,n+N} - r_{K,n}\right|}{r_{K,n+N}} \\
&\quad \times \Big\|T_{r_{K,n+N}}^{(\Theta_K,\varphi_K)}\left(I - r_{K,n+N}A_K\right)\Delta_{n+N}^{K-1}x_{n+N} \\
&\qquad - \left(I - r_{K,n+N}A_K\right)\Delta_{n+N}^{K-1}x_{n+N}\Big\| \\
&\quad + \left|r_{K,n+N} - r_{K,n}\right| \left\|A_K\Delta_{n+N}^{K-1}x_{n+N}\right\| \\
&\quad + \left\|\Delta_{n+N}^{K-1}x_{n+N} - \Delta_n^{K-1}x_n\right\| \\
&= \left|r_{K,n+N} - r_{K,n}\right| \Big[\left\|A_K\Delta_{n+N}^{K-1}x_{n+N}\right\| + \frac{1}{r_{K,n+N}} \\
&\qquad\qquad \times \Big\|T_{r_{K,n+N}}^{(\Theta_K,\varphi_K)}\left(I - r_{K,n+N}A_K\right)\Delta_{n+N}^{K-1}x_{n+N} \\
&\qquad\qquad\qquad - \left(I - r_{K,n+N}A_K\right)\Delta_{n+N}^{K-1}x_{n+N}\Big\|\Big] \\
&\quad + \left\|\Delta_{n+N}^{K-1}x_{n+N} - \Delta_n^{K-1}x_n\right\| \\
&\ \ \vdots \\
&\le \left|r_{K,n+N} - r_{K,n}\right| \Big[\left\|A_K\Delta_{n+N}^{K-1}x_{n+N}\right\| + \frac{1}{r_{K,n+N}} \\
&\qquad\qquad \times \Big\|T_{r_{K,n+N}}^{(\Theta_K,\varphi_K)}\left(I - r_{K,n+N}A_K\right)\Delta_{n+N}^{K-1}x_{n+N} \\
&\qquad\qquad\qquad - \left(I - r_{K,n+N}A_K\right)\Delta_{n+N}^{K-1}x_{n+N}\Big\|\Big] \\
&\quad + \cdots + \left|r_{1,n+N} - r_{1,n}\right| \\
&\quad \times \Big[\left\|A_1\Delta_{n+N}^{0}x_{n+N}\right\| + \frac{1}{r_{1,n+N}} \\
&\qquad \times \Big\|T_{r_{1,n+N}}^{(\Theta_1,\varphi_1)}\left(I - r_{1,n+N}A_1\right)\Delta_{n+N}^{0}x_{n+N} \\
&\qquad\qquad - \left(I - r_{1,n+N}A_1\right)\Delta_{n+N}^{0}x_{n+N}\Big\|\Big]
\end{aligned}$$

$$+\left\|\Delta^0_{n+N}x_{n+N}-\Delta^0_n x_n\right\|$$
$$\le \widetilde{M}_2\sum_{j=1}^{K}\left|r_{j,n+N}-r_{j,n}\right|+\left\|x_{n+N}-x_n\right\|, \tag{62}$$

where $\widetilde{M}_2>0$ is a constant such that for each $n\ge 0$

$$\sum_{j=1}^{K}\left[\left\|A_j\Delta^{j-1}_{n+N}x_{n+N}\right\| + \frac{1}{r_{j,n+N}}\left\|T^{(\Theta_j,\varphi_j)}_{r_{j,n+N}}\left(I-r_{j,n+N}A_j\right)\Delta^{j-1}_{n+N}x_{n+N} - \left(I-r_{j,n+N}A_j\right)\Delta^{j-1}_{n+N}x_{n+N}\right\|\right]\le \widetilde{M}_2. \tag{63}$$

Therefore, it follows from (18), (61), (62), and $\{\rho_n\}_{n=0}^{\infty}\subset(0,2\alpha]$ that

$$\begin{aligned}
&\left\|z_{n+N}-z_n\right\|\\
&=\left\|\left(T_{n+N}v_{n+N}-\rho_{n+N}\widetilde{A}_1T_{n+N}v_{n+N}\right)-\left(T_nv_n-\rho_n\widetilde{A}_1T_nv_n\right)\right\|\\
&\le\left\|\left(T_{n+N}v_{n+N}-\rho_{n+N}\widetilde{A}_1T_{n+N}v_{n+N}\right)\right.\\
&\qquad\left.-\left(T_nv_n-\rho_{n+N}\widetilde{A}_1T_nv_n\right)\right\|\\
&\quad+\left\|\left(T_nv_n-\rho_{n+N}\widetilde{A}_1T_nv_n\right)-\left(T_nv_n-\rho_n\widetilde{A}_1T_nv_n\right)\right\|\\
&\le\left\|T_{n+N}v_{n+N}-T_nv_n\right\|+\left|\rho_{n+N}-\rho_n\right|\left\|\widetilde{A}_1T_nv_n\right\|\\
&\le\widetilde{M}_0\sum_{k=1}^{M}\left|\lambda_{k,n+N}-\lambda_{k,n}\right|+\left\|u_{n+N}-u_n\right\|\\
&\quad+\frac{4\widetilde{M}_1}{L}\left|s_{n+N}-s_n\right|+\left|\rho_{n+N}-\rho_n\right|\left\|\widetilde{A}_1T_nv_n\right\|\\
&\le\widetilde{M}_0\sum_{k=1}^{M}\left|\lambda_{k,n+N}-\lambda_{k,n}\right|+\widetilde{M}_2\sum_{j=1}^{K}\left|r_{j,n+N}-r_{j,n}\right|\\
&\quad+\left\|x_{n+N}-x_n\right\|+\frac{4\widetilde{M}_1}{L}\left|s_{n+N}-s_n\right|+\left|\rho_{n+N}-\rho_n\right|\left\|\widetilde{A}_1T_nv_n\right\|.
\end{aligned}\tag{64}$$

From Lemma 11 and (64), it is found that

$$\begin{aligned}
&\left\|x_{n+N}-x_n\right\|\\
&=\left\|y_{n+N-1}-\mu\alpha_{n+N-1}\widetilde{A}_2y_{n+N-1}\right.\\
&\qquad\left.-\left(y_{n-1}-\mu\alpha_{n-1}\widetilde{A}_2y_{n-1}\right)\right\|\\
&=\left\|S^{\alpha_{n+N-1}}_{[n+N]}z_{n+N-1}-S^{\alpha_{n-1}}_{[n]}z_{n-1}\right\|\\
&\le\left\|S^{\alpha_{n+N-1}}_{[n+N]}z_{n+N-1}-S^{\alpha_{n+N-1}}_{[n+N]}z_{n-1}\right\|\\
&\quad+\left\|S^{\alpha_{n+N-1}}_{[n+N]}z_{n-1}-S^{\alpha_{n-1}}_{[n]}z_{n-1}\right\|\\
&\le\left(1-\alpha_{n+N-1}\tau\right)\left\|z_{n+N-1}-z_{n-1}\right\|\\
&\quad+\mu\left|\alpha_{n+N-1}-\alpha_{n-1}\right|\left\|\widetilde{A}_2S_{[n]}z_{n-1}\right\|\\
&=\left(1-\alpha_{n+N-1}\tau\right)\left\|z_{n+N-1}-z_{n-1}\right\|\\
&\quad+\mu\left|\alpha_{n+N-1}-\alpha_{n-1}\right|\left\|\widetilde{A}_2y_{n-1}\right\|\\
&\le\left(1-\alpha_{n+N-1}\tau\right)\\
&\quad\times\left[\widetilde{M}_0\sum_{k=1}^{M}\left|\lambda_{k,n+N-1}-\lambda_{k,n-1}\right|\right.\\
&\qquad+\widetilde{M}_2\sum_{j=1}^{K}\left|r_{j,n+N-1}-r_{j,n-1}\right|+\left\|x_{n+N-1}-x_{n-1}\right\|\\
&\qquad+\frac{4\widetilde{M}_1}{L}\left|s_{n+N-1}-s_{n-1}\right|\\
&\qquad\left.+\left|\rho_{n+N-1}-\rho_{n-1}\right|\left\|\widetilde{A}_1T_{n-1}v_{n-1}\right\|\right]\\
&\quad+\mu\left|\alpha_{n+N-1}-\alpha_{n-1}\right|\left\|\widetilde{A}_2y_{n-1}\right\|\\
&\le\left(1-\alpha_{n+N-1}\tau\right)\left\|x_{n+N-1}-x_{n-1}\right\|\\
&\quad+\widetilde{M}_0\sum_{k=1}^{M}\left|\lambda_{k,n+N-1}-\lambda_{k,n-1}\right|+\widetilde{M}_2\sum_{j=1}^{K}\left|r_{j,n+N-1}-r_{j,n-1}\right|\\
&\quad+\frac{4\widetilde{M}_1}{L}\left|s_{n+N-1}-s_{n-1}\right|+\left|\rho_{n+N-1}-\rho_{n-1}\right|\left\|\widetilde{A}_1T_{n-1}v_{n-1}\right\|\\
&\quad+\mu\left|\alpha_{n+N-1}-\alpha_{n-1}\right|\left\|\widetilde{A}_2y_{n-1}\right\|\\
&\le\left(1-\alpha_{n+N-1}\tau\right)\left\|x_{n+N-1}-x_{n-1}\right\|\\
&\quad+\widetilde{M}_3\left(\sum_{k=1}^{M}\left|\lambda_{k,n+N-1}-\lambda_{k,n-1}\right|+\sum_{j=1}^{K}\left|r_{j,n+N-1}-r_{j,n-1}\right|\right.\\
&\qquad+\left|s_{n+N-1}-s_{n-1}\right|+\left|\rho_{n+N-1}-\rho_{n-1}\right|\\
&\qquad\left.+\left|\alpha_{n+N-1}-\alpha_{n-1}\right|\right),
\end{aligned}\tag{65}$$

where $\sup_{n\ge0}\{\widetilde{M}_0+4\widetilde{M}_1/L+\widetilde{M}_2+\|\widetilde{A}_1T_nv_n\|+\mu\|\widetilde{A}_2y_n\|\}\le\widetilde{M}_3$ for some $\widetilde{M}_3>0$. Applying Lemma 12 to (65) we obtain from conditions (i)–(vi) that

$$\lim_{n\to\infty}\left\|x_{n+N}-x_n\right\|=0. \tag{66}$$

Step 3. We prove that $\lim_{n\to\infty}\|x_n-S_{[n+N]}\cdots S_{[n+1]}x_n\|=0$ provided $\lim_{n\to\infty}(\|x_n-y_n\|+\|T_nv_n-v_n\|)=0$.

Indeed, from $\|x_{n+1}-y_n\|=\mu\alpha_n\|\widetilde{A}_2y_n\|\le\alpha_n\widetilde{M}_3$ and condition (i), we get $\lim_{n\to\infty}\|x_{n+1}-y_n\|=0$. Now, let us show that $\|u_n-x_n\|\to0$, $\|v_n-u_n\|\to0$ and $\|x_n-T_nv_n\|\to0$

as $n \to \infty$. As a matter of fact, utilizing Lemma 4, we get from (43)

$$\begin{aligned} \|y_n - x^*\|^2 &= \left\|S_{[n+1]}(T_n v_n - \rho_n \widetilde{A}_1 T_n v_n) - x^*\right\|^2 \\ &\le \left\|T_n v_n - x^* - \rho_n \widetilde{A}_1 T_n v_n\right\|^2 \\ &\le \|T_n v_n - x^*\|^2 - 2\rho_n \left\langle \widetilde{A}_1 T_n v_n, z_n - x^* \right\rangle \\ &\le \|v_n - x^*\|^2 + 2\rho_n \left\|\widetilde{A}_1 T_n v_n\right\| \|z_n - x^*\|. \end{aligned} \tag{67}$$

Observe that

$$\begin{aligned} &\left\|\Delta_n^j x_n - x^*\right\|^2 \\ &= \left\|T_{r_{j,n}}^{(\Theta_j,\varphi_j)}(I - r_{j,n}A_j)\Delta_n^{j-1}x_n - T_{r_{j,n}}^{(\Theta_j,\varphi_j)}(I - r_{j,n}A_j)x^*\right\|^2 \\ &\le \left\|(I - r_{j,n}A_j)\Delta_n^{j-1}x_n - (I - r_{j,n}A_j)x^*\right\|^2 \\ &\le \left\|\Delta_n^{j-1}x_n - x^*\right\|^2 + r_{j,n}\left(r_{j,n} - 2\zeta_j\right)\left\|A_j\Delta_n^{j-1}x_n - A_jx^*\right\|^2 \\ &\le \|x_n - x^*\|^2 + r_{j,n}\left(r_{j,n} - 2\zeta_j\right)\left\|A_j\Delta_n^{j-1}x_n - A_jx^*\right\|^2, \\ &\left\|\Lambda_n^k u_n - x^*\right\|^2 \\ &= \left\|J_{R_k,\lambda_{k,n}}(I - \lambda_{k,n}B_k)\Lambda_n^{k-1}u_n - J_{R_k,\lambda_{k,n}}(I - \lambda_{k,n}B_k)x^*\right\|^2 \\ &\le \left\|(I - \lambda_{k,n}B_k)\Lambda_n^{k-1}u_n - (I - \lambda_{k,n}B_k)x^*\right\|^2 \\ &\le \left\|\Lambda_n^{k-1}u_n - x^*\right\|^2 + \lambda_{k,n}\left(\lambda_{k,n} - 2\eta_k\right)\left\|B_k\Lambda_n^{k-1}u_n - B_kx^*\right\|^2 \\ &\le \|u_n - x^*\|^2 + \lambda_{k,n}\left(\lambda_{k,n} - 2\eta_k\right)\left\|B_k\Lambda_n^{k-1}u_n - B_kx^*\right\|^2 \\ &\le \|x_n - x^*\|^2 + \lambda_{k,n}\left(\lambda_{k,n} - 2\eta_k\right)\left\|B_k\Lambda_n^{k-1}u_n - B_kx^*\right\|^2, \end{aligned} \tag{68}$$

for $j \in \{1, 2, \ldots, K\}$ and $k \in \{1, 2, \ldots, M\}$. Combining (67)-(68), we get

$$\begin{aligned} &\|y_n - x^*\|^2 \\ &\le \|v_n - x^*\|^2 + 2\rho_n \left\|\widetilde{A}_1 T_n v_n\right\| \|z_n - x^*\| \\ &\le \left\|\Lambda_n^k u_n - x^*\right\|^2 + 2\rho_n \left\|\widetilde{A}_1 T_n v_n\right\| \|z_n - x^*\| \\ &\le \|u_n - x^*\|^2 + \lambda_{k,n}\left(\lambda_{k,n} - 2\eta_k\right)\left\|B_k\Lambda_n^{k-1}u_n - B_kx^*\right\|^2 \\ &\quad + 2\rho_n \left\|\widetilde{A}_1 T_n v_n\right\| \|z_n - x^*\| \\ &\le \left\|\Delta_n^j x_n - x^*\right\|^2 + \lambda_{k,n}\left(\lambda_{k,n} - 2\eta_k\right)\left\|B_k\Lambda_n^{k-1}u_n - B_kx^*\right\|^2 \\ &\quad + 2\rho_n \left\|\widetilde{A}_1 T_n v_n\right\| \|z_n - x^*\| \\ &\le \|x_n - x^*\|^2 + r_{j,n}\left(r_{j,n} - 2\zeta_j\right)\left\|A_j\Delta_n^{j-1}x_n - A_jx^*\right\|^2 \\ &\quad + \lambda_{k,n}\left(\lambda_{k,n} - 2\eta_k\right)\left\|B_k\Lambda_n^{k-1}u_n - B_kx^*\right\|^2 \\ &\quad + 2\rho_n \left\|\widetilde{A}_1 T_n v_n\right\| \|z_n - x^*\|, \end{aligned} \tag{69}$$

which immediately yields

$$\begin{aligned} &r_{j,n}\left(2\zeta_j - r_{j,n}\right)\left\|A_j\Delta_n^{j-1}x_n - A_jx^*\right\|^2 \\ &\quad + \lambda_{k,n}\left(2\eta_k - \lambda_{k,n}\right)\left\|B_k\Lambda_n^{k-1}u_n - B_kx^*\right\|^2 \\ &\le \|x_n - x^*\|^2 - \|y_n - x^*\|^2 + 2\rho_n \left\|\widetilde{A}_1 T_n v_n\right\| \|z_n - x^*\| \\ &\le \|x_n - y_n\| \left(\|x_n - x^*\| + \|y_n - x^*\|\right) \\ &\quad + 2\rho_n \left\|\widetilde{A}_1 T_n v_n\right\| \|z_n - x^*\|. \end{aligned} \tag{70}$$

Since $\{\lambda_{k,n}\}_{n=0}^{\infty} \subset [a_k, b_k] \subset (0, 2\eta_k)$ and $\{r_{j,n}\}_{n=0}^{\infty} \subset [c_j, d_j] \subset (0, 2\zeta_j)$ for $j = 1, 2, \ldots, K$ and $k = 1, 2, \ldots, M$ and $\{x_n\}, \{y_n\}, \{\widetilde{A}_1 T_n v_n\}$ and $\{z_n\}$ are bounded sequences, we deduce from $\rho_n \to 0$ and $\|x_n - y_n\| \to 0$ that

$$\begin{aligned} &\lim_{n\to\infty}\left\|A_j\Delta_n^{j-1}x_n - A_jx^*\right\| = 0, \\ &\lim_{n\to\infty}\left\|B_k\Lambda_n^{k-1}u_n - B_kx^*\right\| = 0, \end{aligned} \tag{71}$$

for all $j \in \{1, 2, \ldots, K\}$ and $k \in \{1, 2, \ldots, M\}$.

Furthermore, by Proposition 3(ii) and Lemma 5(a), we have

$$\begin{aligned} &\left\|\Delta_n^j x_n - x^*\right\|^2 \\ &= \left\|T_{r_{j,n}}^{(\Theta_j,\varphi_j)}(I - r_{j,n}A_j)\Delta_n^{j-1}x_n - T_{r_{j,n}}^{(\Theta_j,\varphi_j)}(I - r_{j,n}A_j)x^*\right\|^2 \\ &\le \left\langle \left(I - r_{j,n}A_j\right)\Delta_n^{j-1}x_n - \left(I - r_{j,n}A_j\right)x^*, \Delta_n^j x_n - x^* \right\rangle \\ &= \frac{1}{2} \\ &\quad \times \left(\left\|(I - r_{j,n}A_j)\Delta_n^{j-1}x_n - (I - r_{j,n}A_j)x^*\right\|^2\right. \\ &\qquad + \left\|\Delta_n^j x_n - x^*\right\|^2 \\ &\qquad \left. - \left\|(I - r_{j,n}A_j)\Delta_n^{j-1}x_n - (I - r_{j,n}A_j)x^* - (\Delta_n^j x_n - x^*)\right\|^2\right) \\ &\le \frac{1}{2}\left(\left\|\Delta_n^{j-1}x_n - x^*\right\|^2 + \left\|\Delta_n^j x_n - x^*\right\|^2\right. \\ &\qquad \left. - \left\|\Delta_n^{j-1}x_n - \Delta_n^j x_n - r_{j,n}\left(A_j\Delta_n^{j-1}x_n - A_jx^*\right)\right\|^2\right), \end{aligned} \tag{72}$$

which implies that

$$\begin{aligned}
&\left\|\Delta_n^j x_n - x^*\right\|^2 \\
&\leq \left\|\Delta_n^{j-1} x_n - x^*\right\|^2 \\
&\quad - \left\|\Delta_n^{j-1} x_n - \Delta_n^j x_n - r_{j,n}(A_j\Delta_n^{j-1} x_n - A_j x^*)\right\|^2 \\
&= \left\|\Delta_n^{j-1} x_n - x^*\right\|^2 - \left\|\Delta_n^{j-1} x_n - \Delta_n^j x_n\right\|^2 \\
&\quad - r_{j,n}^2\left\|A_j\Delta_n^{j-1} x_n - A_j x^*\right\|^2 \\
&\quad + 2r_{j,n}\left\langle \Delta_n^{j-1} x_n - \Delta_n^j x_n, A_j\Delta_n^{j-1} x_n - A_j x^*\right\rangle \\
&\leq \left\|\Delta_n^{j-1} x_n - x^*\right\|^2 - \left\|\Delta_n^{j-1} x_n - \Delta_n^j x_n\right\|^2 \\
&\quad + 2r_{j,n}\left\|\Delta_n^{j-1} x_n - \Delta_n^j x_n\right\|\left\|A_j\Delta_n^{j-1} x_n - A_j x^*\right\| \\
&\leq \left\|x_n - x^*\right\|^2 - \left\|\Delta_n^{j-1} x_n - \Delta_n^j x_n\right\|^2 \\
&\quad + 2r_{j,n}\left\|\Delta_n^{j-1} x_n - \Delta_n^j x_n\right\|\left\|A_j\Delta_n^{j-1} x_n - A_j x^*\right\|.
\end{aligned} \tag{73}$$

By Lemma 5(a) and Lemma 14, we obtain

$$\begin{aligned}
&\left\|\Lambda_n^k u_n - x^*\right\|^2 \\
&= \left\|J_{R_k,\lambda_{k,n}}(I-\lambda_{k,n}B_k)\Lambda_n^{k-1}u_n - J_{R_k,\lambda_{k,n}}(I-\lambda_{k,n}B_k)x^*\right\|^2 \\
&\leq \left\langle (I-\lambda_{k,n}B_k)\Lambda_n^{k-1}u_n - (I-\lambda_{k,n}B_k)x^*, \Lambda_n^k u_n - x^*\right\rangle \\
&= \frac{1}{2}\Big(\left\|(I-\lambda_{k,n}B_k)\Lambda_n^{k-1}u_n - (I-\lambda_{k,n}B_k)x^*\right\|^2 \\
&\qquad + \left\|\Lambda_n^k u_n - x^*\right\|^2 \\
&\qquad - \big\|(I-\lambda_{k,n}B_k)\Lambda_n^{k-1}u_n \\
&\qquad\quad - (I-\lambda_{k,n}B_k)x^* - \left(\Lambda_n^k u_n - x^*\right)\big\|^2\Big) \\
&\leq \frac{1}{2}\Big(\left\|\Lambda_n^{k-1}u_n - x^*\right\|^2 + \left\|\Lambda_n^k u_n - x^*\right\|^2 \\
&\qquad - \left\|\Lambda_n^{k-1}u_n - \Lambda_n^k u_n - \lambda_{k,n}(B_k\Lambda_n^{k-1}u_n - B_k x^*)\right\|^2\Big) \\
&\leq \frac{1}{2}\Big(\left\|u_n - x^*\right\|^2 + \left\|\Lambda_n^k u_n - x^*\right\|^2 \\
&\qquad - \left\|\Lambda_n^{k-1}u_n - \Lambda_n^k u_n - \lambda_{k,n}(B_k\Lambda_n^{k-1}u_n - B_k x^*)\right\|^2\Big) \\
&\leq \frac{1}{2}\Big(\left\|x_n - x^*\right\|^2 + \left\|\Lambda_n^k u_n - x^*\right\|^2 \\
&\qquad - \left\|\Lambda_n^{k-1}u_n - \Lambda_n^k u_n - \lambda_{k,n}(B_k\Lambda_n^{k-1}u_n - B_k x^*)\right\|^2\Big),
\end{aligned} \tag{74}$$

which immediately leads to

$$\begin{aligned}
&\left\|\Lambda_n^k u_n - x^*\right\|^2 \\
&\leq \left\|x_n - x^*\right\|^2 \\
&\quad - \left\|\Lambda_n^{k-1}u_n - \Lambda_n^k u_n - \lambda_{k,n}(B_k\Lambda_n^{k-1}u_n - B_k x^*)\right\|^2 \\
&= \left\|x_n - x^*\right\|^2 - \left\|\Lambda_n^{k-1}u_n - \Lambda_n^k u_n\right\|^2 \\
&\quad - \lambda_{k,n}^2\left\|B_k\Lambda_n^{k-1}u_n - B_k x^*\right\|^2 \\
&\quad + 2\lambda_{k,n}\left\langle \Lambda_n^{k-1}u_n - \Lambda_n^k u_n, B_k\Lambda_n^{k-1}u_n - B_k x^*\right\rangle \\
&\leq \left\|x_n - x^*\right\|^2 - \left\|\Lambda_n^{k-1}u_n - \Lambda_n^k u_n\right\|^2 \\
&\quad + 2\lambda_{k,n}\left\|\Lambda_n^{k-1}u_n - \Lambda_n^k u_n\right\|\left\|B_k\Lambda_n^{k-1}u_n - B_k x^*\right\|.
\end{aligned} \tag{75}$$

Combining (67) and (75) we conclude that

$$\begin{aligned}
\left\|y_n - x^*\right\|^2 &\leq \left\|v_n - x^*\right\|^2 + 2\rho_n\left\|\widetilde{A}_1 T_n v_n\right\|\left\|z_n - x^*\right\| \\
&\leq \left\|\Lambda_n^k u_n - x^*\right\|^2 + 2\rho_n\left\|\widetilde{A}_1 T_n v_n\right\|\left\|z_n - x^*\right\| \\
&\leq \left\|x_n - x^*\right\|^2 - \left\|\Lambda_n^{k-1}u_n - \Lambda_n^k u_n\right\|^2 \\
&\quad + 2\lambda_{k,n}\left\|\Lambda_n^{k-1}u_n - \Lambda_n^k u_n\right\|\left\|B_k\Lambda_n^{k-1}u_n - B_k x^*\right\| \\
&\quad + 2\rho_n\left\|\widetilde{A}_1 T_n v_n\right\|\left\|z_n - x^*\right\|,
\end{aligned} \tag{76}$$

which yields

$$\begin{aligned}
&\left\|\Lambda_n^{k-1}u_n - \Lambda_n^k u_n\right\|^2 \\
&\leq \left\|x_n - x^*\right\|^2 - \left\|y_n - x^*\right\|^2 \\
&\quad + 2\lambda_{k,n}\left\|\Lambda_n^{k-1}u_n - \Lambda_n^k u_n\right\|\left\|B_k\Lambda_n^{k-1}u_n - B_k x^*\right\| \\
&\quad + 2\rho_n\left\|\widetilde{A}_1 T_n v_n\right\|\left\|z_n - x^*\right\| \\
&\leq \left\|x_n - y_n\right\|\left(\left\|x_n - x^*\right\| + \left\|y_n - x^*\right\|\right) \\
&\quad + 2\lambda_{k,n}\left\|\Lambda_n^{k-1}u_n - \Lambda_n^k u_n\right\|\left\|B_k\Lambda_n^{k-1}u_n - B_k x^*\right\| \\
&\quad + 2\rho_n\left\|\widetilde{A}_1 T_n v_n\right\|\left\|z_n - x^*\right\|.
\end{aligned} \tag{77}$$

Since $\{\lambda_{k,n}\}_{n=0}^{\infty} \subset [a_k, b_k] \subset (0, 2\eta_k)$ for $k = 1, 2, \ldots, M$ and $\{u_n\}, \{x_n\}, \{y_n\}, \{\widetilde{A}_1 T_n v_n\}$ and $\{z_n\}$ are bounded sequences, we deduce from (71), $\rho_n \to 0$, and $\|x_n - y_n\| \to 0$ that

$$\lim_{n\to\infty}\left\|\Lambda_n^{k-1}u_n - \Lambda_n^k u_n\right\| = 0, \quad \forall k \in \{1, 2, \ldots, M\}. \tag{78}$$

Also, combining (51), (67), and (73), we deduce that

$$\begin{aligned}\|y_n - x^*\|^2 &\le \|v_n - x^*\|^2 + 2\rho_n \|\widetilde{A}_1 T_n v_n\| \|z_n - x^*\| \\ &\le \|u_n - x^*\|^2 + 2\rho_n \|\widetilde{A}_1 T_n v_n\| \|z_n - x^*\| \\ &\le \|\Delta_n^j x_n - x^*\|^2 + 2\rho_n \|\widetilde{A}_1 T_n v_n\| \|z_n - x^*\| \\ &\le \|x_n - x^*\|^2 - \|\Delta_n^{j-1} x_n - \Delta_n^j x_n\|^2 \\ &\quad + 2r_{j,n} \|\Delta_n^{j-1} x_n - \Delta_n^j x_n\| \|A_j \Delta_n^{j-1} x_n - A_j x^*\| \\ &\quad + 2\rho_n \|\widetilde{A}_1 T_n v_n\| \|z_n - x^*\|, \end{aligned} \tag{79}$$

which leads to

$$\begin{aligned}&\|\Delta_n^{j-1} x_n - \Delta_n^j x_n\|^2 \\ &\quad\le \|x_n - x^*\|^2 - \|y_n - x^*\|^2 \\ &\qquad + 2r_{j,n} \|\Delta_n^{j-1} x_n - \Delta_n^j x_n\| \|A_j \Delta_n^{j-1} x_n - A_j x^*\| \\ &\qquad + 2\rho_n \|\widetilde{A}_1 T_n v_n\| \|z_n - x^*\| \\ &\quad\le \|x_n - y_n\| (\|x_n - x^*\| + \|y_n - x^*\|) \\ &\qquad + 2r_{j,n} \|\Delta_n^{j-1} x_n - \Delta_n^j x_n\| \|A_j \Delta_n^{j-1} x_n - A_j x^*\| \\ &\qquad + 2\rho_n \|\widetilde{A}_1 T_n v_n\| \|z_n - x^*\|. \end{aligned} \tag{80}$$

Since $\{r_{j,n}\}_{n=0}^{\infty} \subset [c_j, d_j] \subset (0, 2\zeta_j)$ for $j = 1, 2, \ldots, K$ and $\{x_n\}, \{y_n\}, \{\widetilde{A}_1 T_n v_n\}$ and $\{z_n\}$ are bounded sequences, we conclude from (71), $\rho_n \to 0$, and $\|x_n - y_n\| \to 0$ that

$$\lim_{n\to\infty} \|\Delta_n^{j-1} x_n - \Delta_n^j x_n\| = 0, \quad \forall j \in \{1, 2, \ldots, K\}. \tag{81}$$

Hence from (78) and (81) we get

$$\begin{aligned}\|x_n - u_n\| &= \|\Delta_n^0 x_n - \Delta_n^K x_n\| \\ &\le \|\Delta_n^0 x_n - \Delta_n^1 x_n\| + \|\Delta_n^1 x_n - \Delta_n^2 x_n\| + \cdots \\ &\quad + \|\Delta_n^{K-1} x_n - \Delta_n^K x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \end{aligned} \tag{82}$$

$$\begin{aligned}\|u_n - v_n\| &= \|\Lambda_n^0 u_n - \Lambda_n^M u_n\| \\ &\le \|\Lambda_n^0 u_n - \Lambda_n^1 u_n\| + \|\Lambda_n^1 u_n - \Lambda_n^2 u_n\| + \cdots \\ &\quad + \|\Lambda_n^{M-1} u_n - \Lambda_n^M u_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \end{aligned} \tag{83}$$

respectively. Thus, from (82) and (83), we obtain

$$\|x_n - v_n\| \le \|x_n - u_n\| + \|u_n - v_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{84}$$

together with $\|v_n - T_n v_n\| \to 0$, which implies that

$$\lim_{n\to\infty} \|x_n - T_n v_n\| = 0. \tag{85}$$

On the other hand, we observe that the following relation holds:

$$\begin{aligned}x_{n+N} - x_n &= x_{n+N} - S_{[n+N]} (I - \rho_{n+N-1} \widetilde{A}_1) T_{n+N-1} v_{n+N-1} \\ &\quad + S_{[n+N]} (I - \rho_{n+N-1} \widetilde{A}_1) T_{n+N-1} v_{n+N-1} \\ &\quad - S_{[n+N]} S_{[n+N-1]} (I - \rho_{n+N-2} \widetilde{A}_1) T_{n+N-2} v_{n+N-2} \\ &\quad + \cdots + S_{[n+N]} \cdots S_{[n+2]} (I - \rho_{n+1} \widetilde{A}_1) T_{n+1} v_{n+1} \\ &\quad - S_{[n+N]} \cdots S_{[n+1]} (I - \rho_n \widetilde{A}_1) T_n v_n \\ &\quad + S_{[n+N]} \cdots S_{[n+1]} (I - \rho_n \widetilde{A}_1) T_n v_n - x_n. \end{aligned} \tag{86}$$

Since $\|x_{n+1} - y_n\| \to 0$ and $\rho_n \to 0$ as $n \to \infty$, from the nonexpansivity of each S_i $(i = 1, 2, \ldots, N)$ and boundedness of $\{\widetilde{A}_1 T_n v_n\}$ it follows from (85) that as $n \to \infty$ we have

$$\begin{aligned}&\|x_{n+N} - S_{[n+N]} (I - \rho_{n+N-1} \widetilde{A}_1) T_{n+N-1} v_{n+N-1}\| \\ &\quad = \|x_{n+N} - y_{n+N-1}\| \longrightarrow 0, \\ &\|S_{[n+N]} (I - \rho_{n+N-1} \widetilde{A}_1) T_{n+N-1} v_{n+N-1} \\ &\quad - S_{[n+N]} S_{[n+N-1]} (I - \rho_{n+N-2} \widetilde{A}_1) T_{n+N-2} v_{n+N-2}\| \\ &\le \|(I - \rho_{n+N-1} \widetilde{A}_1) T_{n+N-1} v_{n+N-1} \\ &\qquad - S_{[n+N-1]} (I - \rho_{n+N-2} \widetilde{A}_1) T_{n+N-2} v_{n+N-2}\| \\ &\le \|T_{n+N-1} v_{n+N-1} \\ &\qquad - S_{[n+N-1]} (I - \rho_{n+N-2} \widetilde{A}_1) T_{n+N-2} v_{n+N-2}\| \\ &\quad + \rho_{n+N-1} \|\widetilde{A}_1 T_{n+N-1} v_{n+N-1}\| \\ &\le \|T_{n+N-1} v_{n+N-1} - x_{n+N-1}\| \\ &\quad + \|x_{n+N-1} - S_{[n+N-1]} (I - \rho_{n+N-2} \widetilde{A}_1) T_{n+N-2} v_{n+N-2}\| \\ &\quad + \rho_{n+N-1} \|\widetilde{A}_1 T_{n+N-1} v_{n+N-1}\| \\ &= \|T_{n+N-1} v_{n+N-1} - x_{n+N-1}\| + \|x_{n+N-1} - y_{n+N-2}\| \\ &\quad + \rho_{n+N-1} \|\widetilde{A}_1 T_{n+N-1} v_{n+N-1}\| \longrightarrow 0, \\ &\qquad\vdots \\ &\|S_{[n+N]} \cdots S_{[n+2]} (I - \rho_{n+1} \widetilde{A}_1) T_{n+1} v_{n+1} \\ &\quad - S_{[n+N]} \cdots S_{[n+1]} (I - \rho_n \widetilde{A}_1) T_n v_n\| \\ &\le \|(I - \rho_{n+1} \widetilde{A}_1) T_{n+1} v_{n+1} - S_{[n+1]} (I - \rho_n \widetilde{A}_1) T_n v_n\| \\ &\le \|T_{n+1} v_{n+1} - S_{[n+1]} (I - \rho_n \widetilde{A}_1) T_n v_n\| \end{aligned}$$

$$
\begin{aligned}
&+ \rho_{n+1} \left\| \widetilde{A}_1 \mathrm{T}_{n+1} v_{n+1} \right\| \\
&\leq \left\| T_{n+1} v_{n+1} - x_{n+1} \right\| + \left\| x_{n+1} - S_{[n+1]} \left(I - \rho_n \widetilde{A}_1 \right) T_n v_n \right\| \\
&+ \rho_{n+1} \left\| \widetilde{A}_1 T_{n+1} v_{n+1} \right\| \\
&= \left\| T_{n+1} v_{n+1} - x_{n+1} \right\| + \left\| x_{n+1} - y_n \right\| \\
&+ \rho_{n+1} \left\| \widetilde{A}_1 T_{n+1} v_{n+1} \right\| \longrightarrow 0.
\end{aligned} \tag{87}
$$

Therefore, from (66) and (86), we obtain

$$
\lim_{n\to\infty} \left\| S_{[n+N]} \cdots S_{[n+1]} \left(I - \rho_n \widetilde{A}_1 \right) T_n v_n - x_n \right\| = 0. \tag{88}
$$

So, it follows that

$$
\begin{aligned}
&\left\| S_{[n+N]} \cdots S_{[n+1]} \left(I - \rho_n \widetilde{A}_1 \right) x_n - x_n \right\| \\
&\leq \left\| S_{[n+N]} \cdots S_{[n+1]} \left(I - \rho_n \widetilde{A}_1 \right) x_n \right. \\
&\quad \left. - S_{[n+N]} \cdots S_{[n+1]} \left(I - \rho_n \widetilde{A}_1 \right) T_n v_n \right\| \\
&+ \left\| S_{[n+N]} \cdots S_{[n+1]} \left(I - \rho_n \widetilde{A}_1 \right) T_n v_n - x_n \right\| \longrightarrow 0.
\end{aligned} \tag{89}
$$

Observe that

$$
\begin{aligned}
&\left\| S_{[n+N]} \cdots S_{[n+1]} x_n - x_n \right\| \\
&\leq \left\| S_{[n+N]} \cdots S_{[n+1]} x_n - S_{[n+N]} \cdots S_{[n+1]} \left(x_n - \rho_n \widetilde{A}_1 x_n \right) \right\| \\
&+ \left\| S_{[n+N]} \cdots S_{[n+1]} \left(x_n - \rho_n \widetilde{A}_1 x_n \right) - x_n \right\| \\
&\leq \rho_n \left\| \widetilde{A}_1 x_n \right\| + \left\| S_{[n+N]} \cdots S_{[n+1]} \left(x_n - \rho_n \widetilde{A}_1 x_n \right) - x_n \right\| \\
&\longrightarrow 0 \quad (n \longrightarrow \infty).
\end{aligned} \tag{90}
$$

That is,

$$
\lim_{n\to\infty} \left\| S_{[n+N]} \cdots S_{[n+1]} x_n - x_n \right\| = 0. \tag{91}
$$

Step 4. We prove that $\limsup_{n\to\infty} \langle \widetilde{A}_1 x^*, x^* - x_n \rangle \leq 0$ provided $\lim_{n\to\infty} (\|x_n - y_n\| + \|T_n v_n - v_n\|) = 0$.

Indeed, choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$
\limsup_{n\to\infty} \left\langle \widetilde{A}_1 x^*, x^* - x_n \right\rangle = \lim_{i\to\infty} \left\langle \widetilde{A}_1 x^*, x^* - x_{n_i} \right\rangle. \tag{92}
$$

The boundedness of $\{x_{n_i}\}$ implies the existence of a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ and a point $\widehat{x} \in H$ such that $x_{n_{i_j}} \rightharpoonup \widehat{x}$. We may assume without loss of generality that $x_{n_i} \rightharpoonup \widehat{x}$; that is,

$$
\begin{aligned}
\limsup_{n\to\infty} \left\langle \widetilde{A}_1 x^*, x^* - x_n \right\rangle &= \lim_{i\to\infty} \left\langle \widetilde{A}_1 x^*, x^* - x_{n_i} \right\rangle \\
&= \left\langle \widetilde{A}_1 x^*, x^* - \widehat{x} \right\rangle.
\end{aligned} \tag{93}
$$

First, we can readily see that $\widehat{x} \in \cap_{i=1}^N \mathrm{Fix}(S_i)$. Since the pool of mappings $\{S_i : i \leq i \leq N\}$ is finite, we may further assume (passing to a further subsequence if necessary) that, for some integer $l \in \{1, 2, \ldots, N\}$,

$$
S_{[n_i]} \equiv S_l, \quad \forall i \geq 1. \tag{94}
$$

Then, it follows from (91) that

$$
x_{n_i} - S_{[i+N]} \cdots S_{[i+1]} x_n - x_{n_i} \longrightarrow 0. \tag{95}
$$

Hence, by Lemma 9, we conclude that

$$
\widehat{x} \in \mathrm{Fix} \left(S_{[i+N]} \cdots S_{[i+1]} \right). \tag{96}
$$

Together with the assumption

$$
\begin{aligned}
\bigcap_{i=1}^N \mathrm{Fix}\left(S_i\right) &= \mathrm{Fix}\left(S_1 S_2 \cdots S_N\right) \\
&= \mathrm{Fix}\left(S_N S_1 \cdots S_{N-1}\right) \\
&= \cdots = \mathrm{Fix}\left(S_2 S_3 \cdots S_N S_1\right),
\end{aligned} \tag{97}
$$

this implies that $\widehat{x} \in \cap_{i=1}^N \mathrm{Fix}(S_i)$. Now, since

$$
x^* \in \mathrm{VI}\left(\bigcap_{i=i}^N \mathrm{Fix}\left(S_i\right), \widetilde{A}_1 \right), \tag{98}
$$

we obtain

$$
\begin{aligned}
\limsup_{n\to\infty} \left\langle \widetilde{A}_1 x^*, x^* - x_n \right\rangle &= \lim_{i\to\infty} \left\langle \widetilde{A}_1 x^*, x^* - x_{n_i} \right\rangle \\
&= \left\langle \widetilde{A}_1 x^*, x^* - \widehat{x} \right\rangle \leq 0.
\end{aligned} \tag{99}
$$

Step 5. We prove that $\lim_{n\to\infty} \|x_n - x^*\| = 0$ provided $\|x_n - y_n\| + \|T_n v_n - v_n\| = o(\rho_n)$.

Indeed, first of all, let us show that $\limsup_{n\to\infty} \langle \widetilde{A}_1 x^*, x^* - x_n \rangle \leq 0$. We choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$
\limsup_{n\to\infty} \left\langle \widetilde{A}_2 x^*, x^* - x_n \right\rangle = \lim_{k\to\infty} \left\langle \widetilde{A}_2 x^*, x^* - x_{n_k} \right\rangle. \tag{100}
$$

The boundedness of $\{x_{n_k}\}$ implies that there is a subsequence of $\{x_{n_k}\}$ which converges weakly to a point $\overline{x} \in H$. Without loss of generality, we may assume that $x_{n_k} \rightharpoonup \overline{x}$; that is,

$$
\begin{aligned}
\limsup_{n\to\infty} \left\langle \widetilde{A}_2 x^*, x^* - x_n \right\rangle &= \lim_{k\to\infty} \left\langle \widetilde{A}_2 x^*, x^* - x_{n_k} \right\rangle \\
&= \left\langle \widetilde{A}_2 x^*, x^* - \overline{x} \right\rangle.
\end{aligned} \tag{101}
$$

Repeating the same argument as in the proof of $\widehat{x} \in \cap_{i=1}^N \mathrm{Fix}(S_i)$, we have $\overline{x} \in \cap_{i=1}^N \mathrm{Fix}(S_i)$. Let $p \in \cap_{i=1}^N \mathrm{Fix}(S_i)$ be fixed arbitrarily. Note that $\cap_{i=i}^N \mathrm{Fix}(S_i) \subset \cap_{j=1}^K \mathrm{GMEP}(\Theta_j, \varphi_j, A_j) \cap \cap_{k=1}^M I(B_k, R_k) \cap \Gamma$. Then, it follows from the nonexpansivity of each

S_i $(i = 1, 2, \ldots, N)$ and monotonicity of $\widetilde{A}_1$ that, for all $n \geq 0$,

$$\begin{aligned}\|y_n - p\|^2 &= \left\|S_{[n+1]}(I - \rho_n \widetilde{A}_1) T_n v_n - S_{[n+1]} p\right\|^2 \\ &\leq \left\|(T_n v_n - p) - \rho_n \widetilde{A}_1 T_n v_n\right\|^2 \\ &= \|T_n v_n - p\|^2 + 2\rho_n \left\langle \widetilde{A}_1 T_n v_n, p - T_n v_n \right\rangle \\ &\quad + \rho_n^2 \left\|\widetilde{A}_1 T_n v_n\right\|^2 \\ &= \|T_n v_n - p\|^2 + 2\rho_n \left\langle \widetilde{A}_1 T_n v_n - \widetilde{A}_1 p, p - T_n v_n \right\rangle \\ &\quad + 2\rho_n \left\langle \widetilde{A}_1 p, p - T_n v_n \right\rangle + \rho_n^2 \left\|\widetilde{A}_1 T_n v_n\right\|^2 \\ &\leq \|v_n - p\|^2 + 2\rho_n \left\langle \widetilde{A}_1 p, p - T_n v_n \right\rangle \\ &\quad + \rho_n^2 \left\|\widetilde{A}_1 T_n v_n\right\|^2 \\ &\leq \|x_n - p\|^2 + 2\rho_n \left\langle \widetilde{A}_1 p, p - T_n v_n \right\rangle + \rho_n^2 \widetilde{M}_3^2,\end{aligned} \tag{102}$$

which implies that

$$\begin{aligned}&\lim_{n\to\infty} \left\langle \widetilde{A}_1 p, p - T_n v_n \right\rangle \\ &\quad \leq \lim_{n\to\infty} \frac{1}{2\rho_n} \left[\|x_n - p\|^2 - \|y_n - p\|^2 + \rho_n^2 \widetilde{M}_3^2\right] \\ &\quad \leq \lim_{n\to\infty} \frac{\|x_n - y_n\|}{2\rho_n} \left(\|x_n - p\| + \|y_n - p\|\right) + \lim_{n\to\infty} \frac{\rho_n}{2} \widetilde{M}_3^2.\end{aligned} \tag{103}$$

So, from $\|x_n - y_n\| = o(\rho_n)$ and the boundedness of $\{x_n\}$ and $\{y_n\}$, we get

$$\limsup_{n\to\infty} \left\langle \widetilde{A}_1 p, p - T_n v_n \right\rangle \leq 0, \tag{104}$$

together with (85), which implies that

$$\begin{aligned}&\left\langle \widetilde{A}_1 p, p - \overline{x} \right\rangle \\ &\quad = \lim_{k\to\infty} \left\langle \widetilde{A}_1 p, p - x_{n_k} \right\rangle \\ &\quad \leq \limsup_{n\to\infty} \left\langle \widetilde{A}_1 p, p - x_n \right\rangle \\ &\quad \leq \limsup_{n\to\infty} \left(\left\langle \widetilde{A}_1 p, p - T_n v_n \right\rangle + \left\langle \widetilde{A}_1 p, T_n v_n - x_n \right\rangle\right) \\ &\quad \leq \limsup_{n\to\infty} \left\langle \widetilde{A}_1 p, p - T_n v_n \right\rangle \\ &\quad \leq 0.\end{aligned} \tag{105}$$

Thus, we have

$$\left\langle \widetilde{A}_1 p, p - \overline{x} \right\rangle \leq 0, \quad \forall p \in \bigcap_{i=1}^{N} \operatorname{Fix}(S_i). \tag{106}$$

Since $\widetilde{A}_1$ is monotone and $1/\alpha$-Lipschitz continuous, in terms of Minty's lemma [12], we deduce that $\overline{x} \in \mathrm{VI}(\cap_{i=1}^{N} \operatorname{Fix}(S_i), \widetilde{A}_1)$. Therefore, from $\{x^*\} = \mathrm{VI}(\mathrm{VI}(\Omega, \widetilde{A}_1), \widetilde{A}_2)$, we have

$$\begin{aligned}\limsup_{n\to\infty} \left\langle \widetilde{A}_2 x^*, x^* - x_n \right\rangle &= \lim_{k\to\infty} \left\langle \widetilde{A}_2 x^*, x^* - x_{n_k} \right\rangle \\ &= \left\langle \widetilde{A}_2 x^*, x^* - \overline{x} \right\rangle \leq 0.\end{aligned} \tag{107}$$

Finally, let us show that $\|x_n - x^*\| \to 0$ as $n \to \infty$. By utilizing Lemma 11, we deduce from (52) and $S_{[n+1]}^{\alpha_n} x^* = x^* - \alpha_n \mu \widetilde{A}_2 x^*$ that for all $n \geq 0$

$$\begin{aligned}&\|x_{n+1} - x^*\| \\ &= \left\|S_{[n+1]}^{\alpha_n} z_n - x^*\right\|^2 \\ &= \left\|S_{[n+1]}^{\alpha_n} z_n - S_{[n+1]}^{\alpha_n} x^* + S_{[n+1]}^{\alpha_n} x^* - x^*\right\|^2 \\ &\leq \left\|S_{[n+1]}^{\alpha_n} z_n - S_{[n+1]}^{\alpha_n} x^*\right\|^2 + 2\left\langle S_{[n+1]}^{\alpha_n} x^* - x^*, x_{n+1} - x^* \right\rangle \\ &\leq (1 - \alpha_n \tau) \|z_n - x^*\|^2 - 2\alpha_n \mu \left\langle \widetilde{A}_2 x^*, x_{n+1} - x^* \right\rangle \\ &= (1 - \alpha_n \tau) \left\|T_n v_n - x^* - \rho_n \widetilde{A}_1 T_n v_n\right\|^2 \\ &\quad - 2\alpha_n \mu \left\langle \widetilde{A}_2 x^*, x_{n+1} - x^* \right\rangle \\ &= (1 - \alpha_n \tau) \left[\|T_n v_n - x^*\|^2 \right. \\ &\qquad \left. + 2\rho_n \left\langle \widetilde{A}_1 T_n v_n, x^* - T_n v_n \right\rangle + \rho_n^2 \left\|\widetilde{A}_1 T_n v_n\right\|^2\right] \\ &\quad - 2\alpha_n \mu \left\langle \widetilde{A}_2 x^*, x_{n+1} - x^* \right\rangle \\ &= (1 - \alpha_n \tau) \left[\|T_n v_n - x^*\|^2 \right. \\ &\qquad + 2\rho_n \left\langle \widetilde{A}_1 T_n v_n - \widetilde{A}_1 x^*, x^* - T_n v_n \right\rangle \\ &\qquad \left. + 2\rho_n \left\langle \widetilde{A}_1 x^*, x^* - T_n v_n \right\rangle + \rho_n^2 \left\|\widetilde{A}_1 T_n v_n\right\|^2\right] \\ &\quad - 2\alpha_n \mu \left\langle \widetilde{A}_2 x^*, x_{n+1} - x^* \right\rangle \\ &\leq (1 - \alpha_n \tau) \left[\|v_n - x^*\|^2 + 2\rho_n \left\langle \widetilde{A}_1 x^*, x^* - T_n v_n \right\rangle \right. \\ &\qquad \left. + \rho_n^2 \left\|\widetilde{A}_1 T_n v_n\right\|^2\right] \\ &\quad - 2\alpha_n \mu \left\langle \widetilde{A}_2 x^*, x_{n+1} - x^* \right\rangle \\ &\leq (1 - \alpha_n \tau) \left[\|x_n - x^*\|^2 + 2\rho_n \left\langle \widetilde{A}_1 x^*, x^* - T_n v_n \right\rangle + \rho_n^2 \widetilde{M}_3^2\right] \\ &\quad - 2\alpha_n \mu \left\langle \widetilde{A}_2 x^*, x_{n+1} - x^* \right\rangle \\ &\leq (1 - \alpha_n \tau) \|x_n - x^*\|^2 \\ &\quad + 2\rho_n (1 - \alpha_n \tau) \left\langle \widetilde{A}_1 x^*, x^* - T_n v_n \right\rangle + \rho_n^2 \widetilde{M}_3^2 \\ &\quad - 2\alpha_n \mu \left\langle \widetilde{A}_2 x^*, x_{n+1} - x^* \right\rangle\end{aligned}$$

$$= (1-\alpha_n\tau)\|x_n - x^*\|^2 + \alpha_n\tau \cdot \frac{1}{\tau}\left[2\frac{\rho_n}{\alpha_n}(1-\alpha_n\tau)\left\langle \widetilde{A}_1x^*, x^* - T_nv_n\right\rangle + \frac{\rho_n^2}{\alpha_n}\widetilde{M}_3^2 + 2\mu\left\langle \widetilde{A}_2x^*, x^* - x_{n+1}\right\rangle\right]. \tag{108}$$

Since $\sum_{n=0}^{\infty}\alpha_n = \infty, \rho_n \le \alpha_n$ for all $n \ge 0$ and $\alpha_n \to 0$ as $n \to \infty$, we obtain, from (107) and (104) with $p = x^*$, that $\sum_{n=0}^{\infty}\alpha_n\tau = \infty, 2(\rho_n/\alpha_n)(1-\alpha_n\tau) \le 2$, and

$$\limsup_{n\to\infty}\frac{1}{\tau}\left[2\frac{\rho_n}{\alpha_n}(1-\alpha_n\tau)\left\langle \widetilde{A}_1x^*, x^* - T_nv_n\right\rangle + \frac{\rho_n^2}{\alpha_n}\widetilde{M}_3^2 + 2\mu\left\langle \widetilde{A}_2x^*, x^* - x_{n+1}\right\rangle\right] \le 0. \tag{109}$$

Applying Lemma 12 to (108), we infer that

$$\lim_{n\to\infty}\|x_n - x^*\| = 0. \tag{110}$$

This completes the proof. □

In Theorem 18, putting $f(x) \equiv 0, \forall x \in C$, we obtain that $\Gamma = C$ and $T_n = I$ which is the identity mapping of C. Hence Theorem 18 reduces to the following.

Corollary 19. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $M, N, K \ge 1$ be three integers. Let Θ_j be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (A1)–(A4), $\varphi_j : C \to \mathbf{R}$ a lower semicontinuous and convex functional with the restriction (B1) or (B2), and $A_j : H \to H$ ζ_j-inverse strongly monotone for $j = 1,2,\ldots,K$. Let $R_k : C \to 2^H$ be a maximal monotone mapping and let $B_k : C \to H$ be η_k-inverse strongly monotone for $k = 1,2,\ldots,M$. Let $\{S_i\}_{i=1}^N$ be a finite family of nonexpansive mappings on H. Let $\widetilde{A}_1 : H \to H$ be α-inverse strongly monotone and let $\widetilde{A}_2 : H \to H$ be β-strongly monotone and κ-Lipschitz continuous. Assume that $\mathrm{VI}(\cap_{i=1}^N \mathrm{Fix}(S_i), \widetilde{A}_1) \neq \emptyset$ with $(\cap_{i=1}^N \mathrm{Fix}(S_i)) \subset (\cap_{j=1}^K \mathrm{GMEP}(\Theta_j, \varphi_j, A_j)) \cap (\cap_{k=1}^M I(B_k, R_k))$. Let $\mu \in (0, 2\beta/\kappa^2)$, $\{\alpha_n\}_{n=0}^{\infty} \subset (0,1]$, $\{\rho_n\}_{n=0}^{\infty} \subset (0, 2\alpha]$, $\{\lambda_{k,n}\}_{n=0}^{\infty} \subset [a_k, b_k] \subset (0, 2\eta_k)$, and $\{r_{j,n}\}_{n=0}^{\infty} \subset [c_j, d_j] \subset (0, 2\zeta_j)$ where $j \in \{1,2,\ldots,K\}$ and $k \in \{1,2,\ldots,M\}$. For arbitrarily given $x_0 \in H$, let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned} u_n &= T_{r_{K,n}}^{(\Theta_K,\varphi_K)}(I - r_{K,n}A_K)T_{r_{K-1,n}}^{(\Theta_{K-1},\varphi_{K-1})}(I - r_{K-1,n}A_{K-1}) \cdots T_{r_{1,n}}^{(\Theta_1,\varphi_1)}(I - r_{1,n}A_1)x_n,\\ v_n &= J_{R_M,\lambda_{M,n}}(I - \lambda_{M,n}B_M)J_{R_{M-1},\lambda_{M-1,n}}(I - \lambda_{M-1,n}B_{M-1}) \cdots J_{R_1,\lambda_{1,n}}(I - \lambda_{1,n}B_1)u_n,\\ y_n &= S_{[n+1]}(I - \rho_n\widetilde{A}_1)v_n,\\ x_{n+1} &= y_n - \mu\alpha_n\widetilde{A}_2y_n, \quad \forall n \ge 0. \end{aligned} \tag{111}$$

Assume that

$$\bigcap_{i=1}^N \mathrm{Fix}(S_i) = \mathrm{Fix}(S_1S_2\cdots S_N) = \mathrm{Fix}(S_NS_1\cdots S_{N-1}) = \cdots = \mathrm{Fix}(S_2S_3\cdots S_NS_1) \tag{112}$$

and that the following conditions are satisfied:

(i) *$\lim_{n\to\infty}\alpha_n = 0, \sum_{n=0}^{\infty}\alpha_n = \infty$ and $\rho_n \le \alpha_n$ for all $n \ge 0$;*

(ii) *$\lim_{n\to\infty}(|\alpha_n - \alpha_{n+N}|/(\alpha_{n+N})) = 0$ or $\sum_{n=0}^{\infty}|\alpha_n - \alpha_{n+N}| < \infty$;*

(iii) *$\lim_{n\to\infty}(|\rho_n - \rho_{n+N}|/(\rho_{n+N})) = 0$ or $\sum_{n=0}^{\infty}|\rho_n - \rho_{n+N}| < \infty$;*

(iv) *$\lim_{n\to\infty}(|\lambda_{k,n} - \lambda_{k,n+N}|/(\alpha_n + N)) = 0$ or $\sum_{n=0}^{\infty}|\lambda_{k,n} - \lambda_{k,n+N}| < \infty$ for $k = 1,2,\ldots,M$;*

(v) *$\lim_{n\to\infty}(|r_{j,n} - r_{j,n+N}|/(\alpha_n + N)) = 0$ or $\sum_{n=0}^{\infty}|r_{j,n} - r_{j,n+N}| < \infty$ for $j = 1,2,\ldots,K$.*

Then the following hold:

(a) *$\{x_n\}_{n=0}^{\infty}$ is bounded;*

(b) *$\lim_{n\to\infty}\|x_n - x_{n+N}\| = 0$;*

(c) *$\lim_{n\to\infty}\|x_n - S_{[n+N]}\cdots S_{[n+1]}x_n\| = 0$ provided $\|x_n - y_n\| \to 0$ $(n \to \infty)$;*

(d) *$\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique element of $\mathrm{VI}(\mathrm{VI}(\cap_{i=1}^N \mathrm{Fix}(S_i), \widetilde{A}_1), \widetilde{A}_2)$ provided $\|x_n - y_n\| = o(\rho_n)$.*

In Corollary 19, putting $K = 1$ and $M = 2$, we obtain the following.

Corollary 20. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $N \ge 1$ be an integer. Let Θ be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (A1)–(A4), $\varphi : C \to \mathbf{R}$ a lower semicontinuous and convex functional with the restriction (B1) or (B2), and $A : H \to H$ ζ-inverse strongly monotone. Let $R_k : C \to 2^H$ be a maximal monotone mapping and let $B_k : C \to H$ be η_k-inverse strongly monotone for $k = 1,2$. Let $\{S_i\}_{i=1}^N$ be a finite family of nonexpansive mappings on H. Let $\widetilde{A}_1 : H \to H$ be α-inverse strongly monotone and let $\widetilde{A}_2 : H \to H$ be β-strongly monotone and κ-Lipschitz continuous. Assume that $\mathrm{VI}(\cap_{i=1}^N \mathrm{Fix}(S_i), \widetilde{A}_1) \neq \emptyset$ with $(\cap_{i=1}^N \mathrm{Fix}(S_i)) \subset \mathrm{GMEP}(\Theta, \varphi, A) \cap I(B_2, R_2) \cap I(B_1, R_1)$. Let $\mu \in (0, 2\beta/\kappa^2)$, $\{\alpha_n\}_{n=0}^{\infty} \subset (0,1]$, $\{\rho_n\}_{n=0}^{\infty} \subset (0, 2\alpha]$, $\{\lambda_{k,n}\}_{n=0}^{\infty} \subset [a_k, b_k] \subset (0, 2\eta_k)$, and $\{r_n\}_{n=0}^{\infty} \subset [c, d] \subset (0, 2\zeta)$*

for $k=1,2$. For arbitrarily given $x_0 \in H$, let $\{x_n\}$ be a sequence generated by

$$\begin{gathered}
\Theta(u_n, y)+\varphi(y)-\varphi(u_n)+\langle Ax_n, y-u_n\rangle \\
+\frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq 0, \quad \forall y \in C, \\
v_n = J_{R_2,\lambda_{2,n}}(I-\lambda_{2,n}B_2) J_{R_1,\lambda_{1,n}}(I-\lambda_{1,n}B_1) u_n, \\
y_n = S_{[n+1]}(I-\rho_n \widetilde{A}_1) v_n, \\
x_{n+1} = y_n - \mu\alpha_n \widetilde{A}_2 y_n, \quad \forall n \geq 0.
\end{gathered} \tag{113}$$

Assume that

$$\begin{aligned}
\bigcap_{i=1}^{N} \operatorname{Fix}(S_i) &= \operatorname{Fix}(S_1 S_2 \cdots S_N) \\
&= \operatorname{Fix}(S_N S_1 \cdots S_{N-1}) \\
&= \cdots = \operatorname{Fix}(S_2 S_3 \cdots S_N S_1)
\end{aligned} \tag{114}$$

and that the following conditions are satisfied:

(i) $\lim_{n\to\infty}\alpha_n = 0, \sum_{n=0}^{\infty}\alpha_n = \infty$ *and* $\rho_n \leq \alpha_n$ *for all* $n \geq 0$;

(ii) $\lim_{n\to\infty}(|\alpha_n-\alpha_{n+N}|/(\alpha_{n+N})) = 0$ *or* $\sum_{n=0}^{\infty}|\alpha_n-\alpha_{n+N}| < \infty$;

(iii) $\lim_{n\to\infty}(|\rho_n-\rho_{n+N}|/(\rho_{n+N})) = 0$ *or* $\sum_{n=0}^{\infty}|\rho_n-\rho_{n+N}| < \infty$;

(iv) $\lim_{n\to\infty}(|\lambda_{k,n}-\lambda_{k,n+N}|/(\alpha_{n+N})) = 0$ *or* $\sum_{n=0}^{\infty}|\lambda_{k,n}-\lambda_{k,n+N}| < \infty$ *for* $k=1,2$;

(v) $\lim_{n\to\infty}(|r_n-r_{n+N}|/(\alpha_{n+N})) = 0$ *or* $\sum_{n=0}^{\infty}|r_n-r_{n+N}| < \infty$.

Then the following hold:

(a) $\{x_n\}_{n=0}^{\infty}$ *is bounded;*

(b) $\lim_{n\to\infty}\|x_n - x_{n+N}\| = 0$;

(c) $\lim_{n\to\infty}\|x_n - S_{[n+N]}\cdots S_{[n+1]}x_n\| = 0$ *provided* $\|x_n - y_n\| \to 0$ $(n\to\infty)$;

(d) $\{x_n\}_{n=0}^{\infty}$ *converges strongly to the unique element of* $\mathrm{VI}(\mathrm{VI}(\cap_{i=1}^{N}\operatorname{Fix}(S_i), \widetilde{A}_1), \widetilde{A}_2)$ *provided* $\|x_n - y_n\| = o(\rho_n)$.

In Theorem 18, putting $K=1$ and $M=2$, we obtain the following.

Corollary 21. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $f : C \to \mathbf{R}$ be a convex functional with L-Lipschitz continuous gradient ∇f. Let $N \geq 1$ be an integer. Let Θ be a bifunction from $C\times C$ to $\mathbf{R}$ satisfying (A1)–(A4), $\varphi : C \to \mathbf{R}$ a lower semicontinuous and convex functional with the restriction (B1) or (B2), and $A : H \to H$ ζ-inverse-strongly monotone. Let $R_k : C \to 2^H$ be a maximal monotone mapping and let $B_k : C \to H$ be η_k-inverse strongly monotone for $k=1,2$. Let $\{S_i\}_{i=1}^{N}$ be a finite family of nonexpansive mappings on H. Let $\widetilde{A}_1 : H \to H$ be α-inverse strongly monotone and let $\widetilde{A}_2 : H \to H$ be β-strongly monotone and κ-Lipschitz continuous. Assume that $\mathrm{VI}(\cap_{i=1}^{N}\operatorname{Fix}(S_i), \widetilde{A}_1) \neq \emptyset$ with $\cap_{i=1}^{N}\operatorname{Fix}(S_i) \subset \mathrm{GMEP}(\Theta,\varphi,A) \cap I(B_2,R_2)\cap I(B_1,R_1)\cap\Gamma$. Let $\mu \in (0, 2\beta/\kappa^2)$, $\{\alpha_n\}_{n=0}^{\infty} \subset (0,1]$, $\{\rho_n\}_{n=0}^{\infty} \subset (0,2\alpha]$, $\{\lambda_{k,n}\}_{n=0}^{\infty} \subset [a_k,b_k] \subset (0,2\eta_k)$, and $\{r_n\}_{n=0}^{\infty} \subset [c,d] \subset (0,2\zeta)$ for $k=1,2$. For arbitrarily given $x_0 \in H$, let $\{x_n\}$ be a sequence generated by*

$$\begin{gathered}
\Theta(u_n, y)+\varphi(y)-\varphi(u_n)+\langle Ax_n, y-u_n\rangle \\
+\frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq 0, \quad \forall y \in C, \\
v_n = J_{R_2,\lambda_{2,n}}(I-\lambda_{2,n}B_2) J_{R_1,\lambda_{1,n}}(I-\lambda_{1,n}B_1) u_n, \\
y_n = S_{[n+1]}(I-\rho_n \widetilde{A}_1) T_n v_n, \\
x_{n+1} = y_n - \mu\alpha_n \widetilde{A}_2 y_n, \quad \forall n \geq 0,
\end{gathered} \tag{115}$$

where $P_C(I-\lambda_n\nabla f) = s_n I + (1-s_n)T_n$ (here T_n is nonexpansive and $s_n := s_n(\lambda_n) = (2-\lambda_n L)/4 \in (0,1/2)$ for each $\lambda_n \in (0,2/L)$). Assume that

$$\begin{aligned}
\bigcap_{i=1}^{N} \operatorname{Fix}(S_i) &= \operatorname{Fix}(S_1 S_2 \cdots S_N) \\
&= \operatorname{Fix}(S_N S_1 \cdots S_{N-1}) \\
&= \cdots = \operatorname{Fix}(S_2 S_3 \cdots S_N S_1)
\end{aligned} \tag{116}$$

and that the following conditions are satisfied:

(i) $\lim_{n\to\infty}\alpha_n = 0, \sum_{n=0}^{\infty}\alpha_n = \infty$ *and* $\rho_n \leq \alpha_n$ *for all* $n \geq 0$;

(ii) $\lim_{n\to\infty}(|\alpha_n-\alpha_{n+N}|/(\alpha_{n+N})) = 0$ *or* $\sum_{n=0}^{\infty}|\alpha_n-\alpha_{n+N}| < \infty$;

(iii) $\lim_{n\to\infty}(|s_n-s_{n+N}|/(\alpha_{n+N})) = 0$ *or* $\sum_{n=0}^{\infty}|s_n-s_{n+N}| < \infty$;

(iv) $\lim_{n\to\infty}(|\rho_n-\rho_{n+N}|/(\rho_{n+N})) = 0$ *or* $\sum_{n=0}^{\infty}|\rho_n-\rho_{n+N}| < \infty$;

(v) $\lim_{n\to\infty}(|\lambda_{k,n}-\lambda_{k,n+N}|/(\alpha_n+N)) = 0$ *or* $\sum_{n=0}^{\infty}|\lambda_{k,n}-\lambda_{k,n+N}| < \infty$ *for* $k=1,2$;

(vi) $\lim_{n\to\infty}(|r_n-r_{n+N}|/(\alpha_n+N)) = 0$ *or* $\sum_{n=0}^{\infty}|r_n-r_{n+N}| < \infty$.

Then the following hold:

(a) $\{x_n\}_{n=0}^{\infty}$ *is bounded;*

(b) $\lim_{n\to\infty}\|x_n - x_{n+N}\| = 0$;

(c) $\lim_{n\to\infty}\|x_n - S_{[n+N]}\cdots S_{[n+1]}x_n\| = 0$ *provided* $\lim_{n\to\infty}(\|x_n - y_n\| + \|T_n v_n - v_n\|) = 0$;

(d) $\{x_n\}_{n=0}^{\infty}$ *converges strongly to the unique element of* $\mathrm{VI}(\mathrm{VI}(\cap_{i=1}^{N}\operatorname{Fix}(S_i), \widetilde{A}_1), \widetilde{A}_2)$ *provided* $\|x_n - y_n\| + \|T_n v_n - v_n\| = o(\rho_n)$.

In Theorem 18, putting $K=1$ and $M=1$, we obtain the following.

Corollary 22. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $f : C \to \mathbf{R}$ be a convex functional with L-Lipschitz continuous gradient ∇f. Let $N \geq 1$ be an integer. Let Θ be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (A1)–(A4), $\varphi : C \to \mathbf{R}$ a lower semicontinuous and convex functional with the restriction (B1) or (B2), and $A : H \to H$ ζ-inverse-strongly monotone. Let $R : C \to 2^H$ be a maximal monotone mapping and let $B : C \to H$ be η-inverse strongly monotone. Let $\{S_i\}_{i=1}^N$ be a finite family of nonexpansive mappings on H. Let $\widetilde{A}_1 : H \to H$ be α-inverse strongly monotone and let $\widetilde{A}_2 : H \to H$ be β-strongly monotone and κ-Lipschitz continuous. Assume that $\mathrm{VI}(\cap_{i=1}^N \mathrm{Fix}(S_i), \widetilde{A}_1) \neq \emptyset$ with $(\cap_{i=1}^N \mathrm{Fix}(S_i)) \subset \mathrm{GMEP}(\Theta, \varphi, A) \cap I(B, R) \cap \Gamma$. Let $\mu \in (0, 2\beta/\kappa^2)$, $\{\alpha_n\}_{n=0}^\infty \subset (0, 1]$, $\{\rho_n\}_{n=0}^\infty \subset (0, 2\alpha]$, $\{\mu_n\}_{n=0}^\infty \subset [a, b] \subset (0, 2\eta)$, and $\{r_n\}_{n=0}^\infty \subset [c, d] \subset (0, 2\zeta)$. For arbitrarily given $x_0 \in H$, let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned} &\Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Ax_n, y - u_n\rangle \\ &\quad + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ &v_n = J_{R,\mu_n}(I - \mu_n B)u_n, \\ &y_n = S_{[n+1]}\left(I - \rho_n \widetilde{A}_1\right)T_n v_n, \\ &x_{n+1} = y_n - \mu\alpha_n \widetilde{A}_2 y_n, \quad \forall n \geq 0, \end{aligned} \tag{117}$$

where $P_C(I - \lambda_n \nabla f) = s_n I + (1 - s_n)T_n$ (here T_n is nonexpansive and $s_n := s_n(\lambda_n) = (2 - \lambda_n L)/4 \in (0, 1/2)$ for each $\lambda_n \in (0, 2/L)$). Assume that

$$\begin{aligned} \bigcap_{i=1}^N \mathrm{Fix}(S_i) &= \mathrm{Fix}(S_1 S_2 \cdots S_N) \\ &= \mathrm{Fix}(S_N S_1 \cdots S_{N-1}) \\ &= \cdots = \mathrm{Fix}(S_2 S_3 \cdots S_N S_1) \end{aligned} \tag{118}$$

and that the following conditions are satisfied:

(i) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=0}^\infty \alpha_n = \infty$ *and* $\rho_n \leq \alpha_n$ *for all* $n \geq 0$;

(ii) $\lim_{n\to\infty}(|\alpha_n - \alpha_{n+N}|/(\alpha_{n+N})) = 0$ *or* $\sum_{n=0}^\infty |\alpha_n - \alpha_{n+N}| < \infty$;

(iii) $\lim_{n\to\infty}(|s_n - s_{n+N}|/(\alpha_{n+N})) = 0$ *or* $\sum_{n=0}^\infty |s_n - s_{n+N}| < \infty$;

(iv) $\lim_{n\to\infty}(|\rho_n - \rho_{n+N}|/(\rho_{n+N})) = 0$ *or* $\sum_{n=0}^\infty |\rho_n - \rho_{n+N}| < \infty$;

(v) $\lim_{n\to\infty}(|\mu_n - \mu_{n+N}|/(\alpha_n + N)) = 0$ *or* $\sum_{n=0}^\infty |\mu_n - \mu_{n+N}| < \infty$;

(vi) $\lim_{n\to\infty}(|r_n - r_{n+N}|/(\alpha_n + N)) = 0$ *or* $\sum_{n=0}^\infty |r_n - r_{n+N}| < \infty$.

Then the following hold:

(a) $\{x_n\}_{n=0}^\infty$ *is bounded;*

(b) $\lim_{n\to\infty}\|x_n - x_{n+N}\| = 0$;

(c) $\lim_{n\to\infty}\|x_n - S_{[n+N]}\cdots S_{[n+1]}x_n\| = 0$ *provided* $\lim_{n\to\infty}(\|x_n - y_n\| + \|T_n v_n - v_n\|) = 0$;

(d) $\{x_n\}_{n=0}^\infty$ *converges strongly to the unique element of* $\mathrm{VI}(\mathrm{VI}(\cap_{i=1}^N \mathrm{Fix}(S_i), \widetilde{A}_1), \widetilde{A}_2)$ *provided* $\|x_n - y_n\| + \|T_n v_n - v_n\| = o(\rho_n)$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Strong Convergence of a Unified General Iteration for *k*-Strictly Pseudononspreading Mapping in Hilbert Spaces

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We introduce a unified general iterative method to approximate a fixed point of k-strictly pseudononspreading mapping. Under some suitable conditions, we prove that the iterative sequence generated by the proposed method converges strongly to a fixed point of a k-strictly pseudononspreading mapping with an idea of mean convergence, which also solves a class of variational inequalities as an optimality condition for a minimization problem. The results presented in this paper may be viewed as a refinement and as important generalizations of the previously known results announced by many other authors.

1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Recall that a mapping $T : C \to C$ is said to be k-strict pseudocontractive if there exists a constant $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C. \tag{1}$$

If $k = 0$, T is said to be nonexpansive mapping; that is,

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C. \tag{2}$$

The set of fixed points of T is denoted by $F(T)$; that is, $F(T) = \{x \in C : Tx = x\}$. Recall also that a mapping $T : C \to C$ is said to be nonspreading if

$$2\|Tx - Ty\|^2 \le \|Tx - y\|^2 + \|x - Ty\|^2, \quad \forall x, y \in C. \tag{3}$$

It is shown in the study by Iemoto and Takahashi [1] that (3) is equivalent to

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle, \quad \forall x, y \in C. \tag{4}$$

Observe that every nonspreading mapping is quasinonexpansive; that is, $\|Tx - p\| \le \|x - p\|$ for all $x \in C$ and all $p \in F(T)$. Following the terminology of Browder and Petryshyn [2], a mapping $T : C \to C$ is called k-strictly pseudononspreading if there exists a constant $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2 + 2\langle x - Tx, y - Ty\rangle, \quad \forall x, y \in C. \tag{5}$$

Clearly, every nonspreading mapping is k-strictly pseudononspreading, but the converse is not true. This shows that the class of k-strictly pseudononspreading mappings is more general than the class of nonspreading mappings. Moreover, we remark also that the class of k-strictly pseudononspreading mappings is independent of the class of k-strict pseudocontractions.

Fixed point problem of nonlinear mappings recently becomes an attractive subject because of its application in solving variational inequalities and equilibrium problems arising in various fields of pure and applied sciences. Moreover, various iterative schemes and methods have been developed for finding fixed points of nonlinear mappings. It is worth mentioning that iterative methods for nonexpansive and nonspreading mappings have been extensively investigated. However, iterative methods for strict pseudo-contractions are far less developed than those for nonexpansive mappings though Browder and Petryshyn [2] initiated their work in 1967; the reason is probably that the second term appearing in the right-hand side of (1) impedes the convergence analysis for iterative algorithms used to find a fixed point of the strict pseudo-contraction. This case is aggravated by adding another inner product to the right-hand side of (5) for k-strictly pseudononspreading mapping; see, for example, [3–13] and the references therein. On the other hand, k-strictly pseudononspreading mappings have more powerful applications than nonexpansive mappings do in solving mean ergodic problems; see, for example, [14, 15]. Therefore, it is interesting to develop the effective numerical methods for approximating fixed point of k-strictly pseudononspreading mapping.

In 2006, Marino and Xu [10] introduced a general iterative method and proved that, for a given $x_0 \in H$, the sequence $\{x_n\}$ generated by

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B) T x_n, \quad \forall n \geq 1, \tag{6}$$

where T is a self-nonexpansive mapping on H, f is a contraction of H into itself, $\{\alpha_n\} \subseteq (0,1)$ satisfies certain conditions, and B is a strongly positive bounded linear operator on H, converges strongly to $x^* \in F(T)$, which is the unique solution of the following variational inequality:

$$\langle (B - \gamma f) x^*, x^* - w \rangle \leq 0, \quad \forall w \in F(T), \tag{7}$$

and is also the optimality condition of problem $\min_{x \in C}(1/2)\langle Bx, x\rangle - h(x)$, where h is a potential function for γf (i.e., $h'(x) = \gamma f(x), \forall x \in H$). Thereafter, the general iterative method is used to find a common element of the set of fixed point problems and the set of solutions of variational inequalities and equilibrium problems (see, e.g., [11–13]).

Recently, Kurokawa and Takahashi [14] obtained a weak mean ergodic theorem for nonspreading mappings in Hilbert spaces. Furthermore, they proved a strong convergence theorem using an idea of mean convergence. In 2011, Osilike and Isiogugu [15] introduced a more general k-strictly pseudononspreading mapping and considered the following iterative schemes:

$$\begin{aligned} x_{n+1} &= \alpha_n u + (1 - \alpha_n) z_n, \\ z_n &= \frac{1}{n} \sum_{k=0}^{n-1} T_\beta^k x_n, \quad n \geq 1, \end{aligned} \tag{8}$$

where auxiliary mapping $T_\beta = \beta I + (1 - \beta) T$. They proved that the sequences $\{x_n\}$ and $\{z_n\}$ converge strongly to $P_{F(T)} u$, which is the metric projection of H onto $F(T)$. Moreover, they considered the following Halpern type iterative scheme:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) T_\beta x_n, \quad n \geq 1. \tag{9}$$

They also proved that $\{x_n\}$ generated by (9) converges strongly to $q \in F(T)$ under some suitable conditions and hence resolved in the affirmative the open problem raised by Kurokawa and Takahashi [14] in their final remark for the case where the mapping T is averaged.

In 2013, Kangtunyakarn [16] further studied variational inequalities and fixed point problem of k-strictly pseudononspreading mapping T by modifying the auxiliary mapping with projection technique. To be more precise, he introduced the following iterative scheme:

$$x_{n+1} = \alpha_n u + \beta_n P_C [I - \lambda_n (I - T)] x_n + \gamma_n S x_n, \quad n \geq 1, \tag{10}$$

where $\alpha_n, \beta_n, \gamma_n \in (0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$ and $\beta_n \in [c, d] \subset (0,1)$ and S is a nonexpansive mapping generated by a finite family of defining operators, whose fixed point problems are equivalent to variational inequalities. Moreover, under some suitable conditions, he proved that the sequence $\{x_n\}$ converges strongly to $P_\Omega u$, where Ω is the intersection of the set of fixed point problems and the set of solutions for variational inequalities.

Inspired and motivated by research going on in this area, we introduce a modified general iterative method for k-strictly pseudononspreading mapping, which is defined in the following way:

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + [(1 - \beta_n) I - \alpha_n B] T_{\lambda_n} x_n, \quad n \geq 1, \tag{11}$$

where $T_{\lambda_n} = P_C[I - \lambda_n (I - T)]$ with $\lambda_n \in (0,1)$ and sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $[0,1]$. Note that, if $\beta_n = 0$, scheme (11) reduces to general iterative method (6), which is mainly due to Marino and Xu [10]. If $\beta_n = 0$, $\gamma = 1$, and $B = I$, scheme (11) reduces to viscosity approximate method introduced by Moudafi [17] and developed by Inchan [18], which also extends the Halpern type results of [19, 20] with an idea of mean convergence for k-strictly pseudononspreading mapping.

Our purpose is not only to modify the general iterative method (6) and projection method (10) to the case of a k-strictly pseudononspreading mapping, but also to establish a new strong convergence theorem with an idea of mean convergence for a k-strictly pseudononspreading mapping, which also solves a class of variational inequalities as an optimality condition for a minimization problem. Our main results presented in this paper improve and extend the corresponding results of [10, 14–17] and many others.

2. Preliminaries

Let C be a nonempty closed convex subset of real Hilbert H space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. For every point $x \in H$, there exists a unique nearest point in C, denoted by P_C, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C. \tag{12}$$

Then P_C is called the metric projection of H onto C. It is well known that P_C is a nonexpansive mapping and the following inequality holds:

$$\langle x - u, u - y \rangle \geq 0, \quad \forall y \in C, \tag{13}$$

if and only if $u = P_C x$ for given $x \in H$ and $u \in C$.

Let A be a mapping from C into H. The normal variational inequality problem is to find a point $u \in C$ such that

$$\langle Au, v - u \rangle \geq 0, \quad \forall v \in C. \tag{14}$$

The set of all solutions of the variational inequality is denoted by $VI(C, A)$. Note that $u \in VI(C, A)$ if and only if $u = P_C(I - \lambda A)u$ for some $\lambda > 0$.

Recall that an operator B is strongly positive if there exists a constant $\overline{\gamma} > 0$ with the property

$$\langle Bx, x \rangle \geq \overline{\gamma}\|x\|^2, \quad \forall x \in H. \tag{15}$$

Recall also that an operator $f : C \to C$ is a contraction, if there exists a constant $\rho \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq \rho \|x - y\|, \quad \forall x, y \in C. \tag{16}$$

In order to prove our main results, we need the following lemmas and propositions.

Lemma 1. *Let H be a real Hilbert space. There hold the following well-known results:*

(i) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, (x + y) \rangle$, $\forall x, y \in H$;

(ii) $\|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2$, $t \in [0, 1]$, $\forall x, y \in H$.

Lemma 2 (see [6]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in Banach space E and let $\{\beta_n\}$ be a sequence in $[0, 1]$ such that $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$. Suppose $x_{n+1} = \beta_n x_n + (1 - \beta_n)z_n$ and*

$$\limsup_{n\to\infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0, \quad \forall n \geq 0. \tag{17}$$

Then $\lim_{n\to\infty}\|z_n - x_n\| = 0$.

Lemma 3 (see [10]). *Let B be a strongly positive linear bounded operator on a Hilbert space H with a coefficient $\overline{\gamma} > 0$ and $0 < \varrho < \|B\|^{-1}$. Then $\|I - \varrho B\| \leq 1 - \varrho\overline{\gamma}$.*

Lemma 4 (see [10]). *Let C be a nonempty closed convex subset of a Hilbert space H. Assume that $f : C \to C$ is a contraction with a coefficient $\rho \in (0, 1)$ and B is a strongly positive linear bounded operator with a coefficient $\overline{\gamma} > 0$. Then, for $0 < \gamma < \overline{\gamma}/\rho$,*

$$\langle x - y, (B - \gamma f)x - (B - \gamma f)y \rangle \geq (\overline{\gamma} - \gamma\rho)\|x - y\|^2, \quad \forall x, y \in H. \tag{18}$$

That is, $B - \gamma f$ is strongly monotone with coefficient $\overline{\gamma} - \gamma\rho$.

Lemma 5 (see [15]). *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T : C \to C$ be a k-strictly pseudononspreading mapping. Then $I - T$ is demiclosed at zero.*

Lemma 6 (see [15]). *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T : C \to C$ be a k-strictly pseudononspreading mapping. If $F(T) \neq \emptyset$, then it is closed and convex.*

Lemma 7 (see [16]). *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T : C \to C$ be a k-strictly pseudononspreading mapping with $F(T) \neq \emptyset$. Then $F(T) = VI(C, (I - T))$.*

Lemma 8 (see [21]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n\delta_n, \quad n \geq 0, \tag{19}$$

where $\{\gamma_n\}$ is a sequence in (0,1) and $\{\delta_n\}$ is a sequence such that

(i) $\sum_{n=1}^{\infty}\gamma_n = \infty$;

(ii) $\limsup_{n\to\infty}\delta_n \leq 0$ *or* $\sum_{n=1}^{\infty}|\gamma_n\delta_n| < \infty$.

Then $\lim_{n\to\infty}a_n = 0$.

3. Main Results

Theorem 9. *Let C be a nonempty closed convex subset of a Hilbert space H and let $T : C \to C$ be a k-strictly pseudononspreading mapping such that $F(T) \neq \emptyset$. Let $f : C \to C$ be a contraction with a coefficient $\rho \in (0, 1)$ and let B be a strongly positive bounded linear operator with $\overline{\gamma} > 0$. For a given point $x_0 \in C$ and $0 < \gamma < \overline{\gamma}/\rho$, assume that $\alpha_n, \beta_n, \lambda_n \in [0, 1]$ satisfying the following conditions:*

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(iii) $\lambda_n \in (0, 1 - k)$ *and* $\lim_{n\to\infty}\lambda_n = 0$.

Then the sequence $\{x_n\}$ generated by (11) converges strongly to $q \in F(T)$, which is the unique solution of the following variational inequality:

$$\langle (B - \gamma f)q, q - w \rangle \leq 0, \quad \forall w \in F(T). \tag{20}$$

Proof. First, we show that sequences $\{x_n\}$ and $\{Tx_n\}$ are bounded. Indeed, from the property of k-strictly pseudononspreading mapping defined on T and $p \in F(T)$, we have

$$\begin{aligned}
&\|Tx_n - Tp\|^2 \\
&\quad = \|(x_n - p) - [(I - T)x_n - (I - T)p]\|^2 \\
&\quad = \|x_n - p\|^2 - 2\langle x_n - p, (I - T)x_n \rangle + \|(I - T)x_n\|^2 \\
&\quad \leq \|x_n - p\|^2 + k\|(I - T)x_n - (I - T)p\|^2 \\
&\qquad + 2\langle (I - T)x_n, (I - T)p \rangle \\
&\quad = \|x_n - p\|^2 + k\|(I - T)x_n\|^2,
\end{aligned} \tag{21}$$

which implies that

$$(1-k)\|(I-T)x_n\|^2 \le 2\langle x_n - p, (I-T)x_n\rangle. \quad (22)$$

From $T_{\lambda_n} = P_C[I - \lambda_n(I-T)]$ and (22), we obtain

$$\begin{aligned}
&\|T_{\lambda_n}x_n - p\|^2 \\
&\quad \le \|(x_n - p) - \lambda_n[(I-T)x_n - (I-T)p]\|^2 \\
&\quad = \|x_n - p\|^2 - 2\lambda_n\langle x_n - p, (I-T)x_n\rangle + \lambda_n^2\|(I-T)x_n\|^2 \\
&\quad \le \|x_n - p\|^2 - \lambda_n(1-k)\|(I-T)x_n\|^2 + \lambda_n^2\|(I-T)x_n\|^2 \\
&\quad = \|x_n - p\|^2 - \lambda_n[(1-k) - \lambda_n]\|(I-T)x_n\|^2 \le \|x_n - p\|^2.
\end{aligned} \quad (23)$$

By (i) and Lemma 3, we have that $(1-\beta_n)I - \alpha_n B$ is positive and $\|(1-\beta_n)I - \alpha_n B\| \le 1 - \beta_n - \alpha_n\overline{\gamma}$ for all $n \ge 1$ (see, i.e., [8]). It follows from (11) and (23) that

$$\begin{aligned}
&\|x_{n+1} - p\| \\
&\quad = \|\alpha_n(\gamma f(x_n) - Bp) + \beta_n(x_n - p) \\
&\qquad + [(1-\beta_n)I - \alpha_n B](T_{\lambda_n}x_n - p)\| \\
&\quad \le \alpha_n\|\gamma f(x_n) - Bp\| \\
&\qquad + \beta_n\|x_n - p\| + (1 - \beta_n - \alpha_n\overline{\gamma})\|T_{\lambda_n}x_n - p\| \\
&\quad \le \alpha_n\gamma\|f(x_n) - f(p)\| + \alpha_n\|\gamma f(p) - Bp\| \\
&\qquad + \beta_n\|x_n - p\| + (1 - \beta_n - \alpha_n\overline{\gamma})\|x_n - p\| \\
&\quad \le [1 - \alpha_n(\overline{\gamma} - \gamma\rho)]\|x_n - p\| + \alpha_n\|\gamma f(p) - Bp\|.
\end{aligned} \quad (24)$$

By induction, we have that

$$\|x_n - p\| \le \max\left\{\|x_0 - p\|, \frac{1}{\overline{\gamma} - \gamma\rho}\|\gamma f(p) - Bp\|\right\}. \quad (25)$$

Therefore, $\{x_n\}$ is bounded and so is $\{T_{\lambda_n}x_n\}$. On the other hand, we estimate

$$\begin{aligned}
&\|Tx_n - p\|^2 \\
&\quad \le \|x_n - p\|^2 + k\|(I-T)x_n - (I-T)p\|^2 \\
&\qquad + 2\langle x_n - Tx_n, p - Tp\rangle \\
&\quad = \|x_n - p\|^2 + k\|(x_n - p) - (Tx_n - p)\|^2 \\
&\quad = \|x_n - p\|^2 + k(\|x_n - p\|^2 - 2\langle x_n - p, Tx_n - p\rangle \\
&\qquad + \|Tx_n - p\|^2),
\end{aligned} \quad (26)$$

which implies that

$$\begin{aligned}
(1-k)\|Tx_n - p\|^2 \le (1+k)\|x_n - p\|^2 \\
+ 2k\|x_n - p\|\|Tx_n - p\|.
\end{aligned} \quad (27)$$

From (27), we can obtain

$$\begin{aligned}
0 &\ge (1-k)\|Tx_n - p\|^2 \\
&\quad - (1+k)\|x_n - p\|^2 - 2k\|x_n - p\|\|Tx_n - p\| \\
&= (1-k)(\|Tx_n - p\|^2 + \|x_n - p\|\|Tx_n - p\|) \\
&\quad - (1+k)(\|x_n - p\|^2 + \|x_n - p\|\|Tx_n - p\|) \\
&= (1-k)\|Tx_n - p\|(\|Tx_n - p\| + \|x_n - p\|) \\
&\quad - (1+k)\|x_n - p\|(\|x_n - p\| + \|Tx_n - p\|).
\end{aligned} \quad (28)$$

It follows that

$$\|Tx_n - p\| \le \frac{1+k}{1-k}\|x_n - p\|. \quad (29)$$

Combining (25) and (29), we conclude that $\{Tx_n\}$ is bounded.

Next, we will show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. To do this, define a sequence $\{z_n\}$ by

$$x_{n+1} = \beta_n x_n + (1-\beta_n)z_n, \quad n \ge 1. \quad (30)$$

Observe that

$$\begin{aligned}
&z_{n+1} - z_n \\
&\quad = \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1-\beta_n} \\
&\quad = \frac{\alpha_{n+1}\gamma f(x_{n+1}) + [(1-\beta_{n+1})I - \alpha_{n+1}B]w_{n+1}}{1-\beta_{n+1}} \\
&\qquad - \frac{\alpha_n\gamma f(x_n) + [(1-\beta_n)I - \alpha_n B]w_n}{1-\beta_n} \\
&\quad = \frac{\alpha_{n+1}}{1-\beta_{n+1}}[\gamma f(x_{n+1}) - Bw_{n+1}] + (w_{n+1} - w_n) \\
&\qquad - \frac{\alpha_n}{1-\beta_n}[\gamma f(x_n) - Bw_n],
\end{aligned} \quad (31)$$

where $w_n = T_{\lambda_n}x_n$, and

$$\begin{aligned}
&\|w_{n+1} - w_n\| \\
&\quad \le \|(I - \lambda_{n+1}(I-T))x_{n+1} - (I - \lambda_n(I-T))x_n\| \\
&\quad = \|x_{n+1} - x_n - \lambda_{n+1}(I-T)x_{n+1} + \lambda_n(I-T)x_n\| \\
&\quad \le \|x_{n+1} - x_n\| + \lambda_{n+1}\|(I-T)x_{n+1} - (I-T)x_n\| \\
&\qquad + |\lambda_{n+1} - \lambda_n|\|(I-T)x_n\|.
\end{aligned} \quad (32)$$

From (31) and (32), we obtain

$$\begin{aligned}
&\|z_{n+1}-z_n\| \\
&\quad\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(x_{n+1})-Bw_{n+1}\| \\
&\qquad+\|w_{n+1}-w_n\|+\frac{\alpha_n}{1-\beta_n}\|\gamma f(x_n)-Bw_n\| \\
&\quad\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(x_{n+1})-Bw_{n+1}\| \\
&\qquad+\|x_{n+1}-x_n\|+\frac{\alpha_n}{1-\beta_n}\|\gamma f(x_n)-Bw_n\| \\
&\qquad+\lambda_{n+1}\|(I-T)x_{n+1}-(I-T)x_n\| \\
&\qquad+|\lambda_{n+1}-\lambda_n|\,\|(I-T)x_n\|.
\end{aligned}\tag{33}$$

It follows from conditions (i)–(iii) and Lemma 2 that

$$\lim_{n\to\infty}\|z_n-x_n\|=0. \tag{34}$$

From (30) and (34) and condition (ii), we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\beta_n)\|z_n-x_n\|=0. \tag{35}$$

Moreover, note that $w_n=T_{\lambda_n}x_n$ and

$$\begin{aligned}
&\|x_n-w_n\| \\
&\quad\le\|x_n-x_{n+1}\|+\|x_{n+1}-w_n\| \\
&\quad=\|x_n-x_{n+1}\| \\
&\qquad+\|\alpha_n\gamma f(x_n)+\beta_n x_n+[(1-\beta_n)I-\alpha_nB]w_n-w_n\| \\
&\quad\le\|x_n-x_{n+1}\|+\alpha_n\|\gamma f(x_n)-Bw_n\|+\beta_n\|x_n-w_n\|,
\end{aligned}\tag{36}$$

which implies that

$$\|x_n-w_n\|\le\frac{1}{1-\beta_n}\|x_n-x_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|\gamma f(x_n)-Bw_n\|. \tag{37}$$

Combining conditions (i)-(ii) and (35), we obtain

$$\lim_{n\to\infty}\|x_n-w_n\|=\lim_{n\to\infty}\|x_n-T_{\lambda_n}x_n\|=0. \tag{38}$$

That is,

$$\lim_{n\to\infty}\|x_n-P_C[I-\lambda_n(I-T)]x_n\|=0. \tag{39}$$

Next, we will prove that $\limsup_{n\to\infty}\langle\gamma f(q)-Bq,x_n-q\rangle\le 0$, where $q=P_{F(T)}(I-B+\gamma f)q$. To show this inequality, take a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\begin{aligned}
&\limsup_{n\to\infty}\langle\gamma f(q)-Bq,x_n-q\rangle \\
&\quad=\lim_{j\to\infty}\left\langle\gamma f(q)-Bq,x_{n_j}-q\right\rangle.
\end{aligned}\tag{40}$$

Without loss of generality, we may assume that $\{x_{n_j}\}$ converges weakly to w; that is, $x_{n_j}\rightharpoonup w$ as $j\to\infty$, where $w\in C$. We will show that $w\in F(T)$. From Lemmas 5 and 7, we have $F(T)=F(T_{\lambda_{n_j}})=F(P_C[I-\lambda_{n_j}(I-T)])$. Assume that $w\ne P_C[I-\lambda_{n_j}(I-T)]w$. By condition (iii), (38), and Opial's property, we obtain

$$\begin{aligned}
&\liminf_{j\to\infty}\left\|x_{n_j}-w\right\| \\
&\quad<\liminf_{j\to\infty}\left\|x_{n_j}-P_C\left[I-\lambda_{n_j}(I-T)\right]w\right\| \\
&\quad\le\liminf_{j\to\infty}\Big(\left\|x_{n_j}-T_{\lambda_{n_j}}x_{n_j}\right\| \\
&\qquad\qquad+\Big\|P_C\left[I-\lambda_{n_j}(I-T)\right]x_{n_j} \\
&\qquad\qquad\qquad-P_C\left[I-\lambda_{n_j}(I-T)\right]w\Big\|\Big) \\
&\quad\le\liminf_{j\to\infty}\Big(\left\|x_{n_j}-T_{\lambda_{n_j}}x_{n_j}\right\| \\
&\qquad\qquad+\left\|x_{n_j}-w\right\| \\
&\qquad\qquad+\lambda_{n_j}\left\|(I-T)x_{n_j}-(I-T)w\right\|\Big) \\
&\quad\le\liminf_{j\to\infty}\left\|x_{n_j}-w\right\|.
\end{aligned}\tag{41}$$

This is a contradiction. Then $w\in F(T)$. Since $x_{n_j}\rightharpoonup w$ as $j\to\infty$, we have

$$\begin{aligned}
&\limsup_{n\to\infty}\langle\gamma f(q)-Bq,x_n-q\rangle \\
&\quad=\lim_{j\to\infty}\left\langle\gamma f(q)-Bq,x_{n_j}-q\right\rangle \\
&\quad=\langle\gamma f(q)-Bq,w-q\rangle\le 0.
\end{aligned}\tag{42}$$

On the other hand, we will show the uniqueness of a solution of the variational inequality

$$\langle(B-\gamma f)x,x-w\rangle\le 0,\quad w\in F(T). \tag{43}$$

Suppose $q\in F(T)$ and $\widehat{q}\in F(T)$ both are solutions to (43); then

$$\begin{aligned}
&\langle(B-\gamma f)q,q-\widehat{q}\rangle\le 0, \\
&\langle(B-\gamma f)\widehat{q},\widehat{q}-q\rangle\le 0.
\end{aligned}\tag{44}$$

Adding up (44), we get

$$\langle(B-\gamma f)q-(B-\gamma f)\widehat{q},q-\widehat{q}\rangle\le 0. \tag{45}$$

From Lemma 4, the strong monotonicity of $B-\gamma f$, we obtain $q=\widehat{q}$ and the uniqueness is proved.

Finally, we show that $\{x_n\}$ converges strongly to q as $n \to \infty$. From (11), (23), and Lemma 1, we have (note that $w_n = T_{\lambda_n} x_n$)

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \\
&\quad = \langle \alpha_n \gamma f(x_n) + \beta_n x_n \\
&\qquad + [(1-\beta_n) I - \alpha_n B] w_n - q, x_{n+1} - q \rangle \\
&\quad = \alpha_n \langle \gamma f(x_n) - Bq, x_{n+1} - q \rangle \\
&\qquad + \langle [(1-\beta_n) I - \alpha_n B](w_n - q), x_{n+1} - q \rangle \\
&\qquad + \beta_n \langle x_n - q, x_{n+1} - q \rangle \\
&\quad \le \alpha_n \gamma \langle f(x_n) - f(q), x_{n+1} - q \rangle \\
&\qquad + \alpha_n \langle \gamma f(q) - Bq, x_{n+1} - q \rangle \\
&\qquad + \beta_n \|x_n - q\| \|x_{n+1} - q\| \\
&\qquad + (1 - \beta_n - \alpha_n \overline{\gamma}) \|w_n - q\| \|x_{n+1} - q\| \\
&\quad \le \alpha_n \gamma \rho \|x_n - q\| \|x_{n+1} - q\| \\
&\qquad + \alpha_n \langle \gamma f(q) - Bq, x_{n+1} - q \rangle \\
&\qquad + (1 - \alpha_n \overline{\gamma}) \|x_n - q\| \|x_{n+1} - q\| \\
&\quad = [1 - (\overline{\gamma} - \gamma\rho)\alpha_n] \|x_n - q\| \|x_{n+1} - q\| \\
&\qquad + \alpha_n \langle \gamma f(q) - Bq, x_{n+1} - q \rangle \\
&\quad \le \frac{1 - (\overline{\gamma} - \gamma\rho)\alpha_n}{2} \left(\|x_n - q\|^2 + \|x_{n+1} - q\|^2 \right) \\
&\qquad + \alpha_n \langle \gamma f(q) - Bq, x_{n+1} - q \rangle \\
&\quad \le \frac{1 - (\overline{\gamma} - \gamma\rho)\alpha_n}{2} \|x_n - q\|^2 + \frac{1}{2} \|x_{n+1} - q\|^2 \\
&\qquad + \alpha_n \langle \gamma f(q) - Bq, x_{n+1} - q \rangle .
\end{aligned} \tag{46}$$

It follows that

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\le [1 - (\overline{\gamma} - \gamma\rho)\alpha_n] \|x_n - q\|^2 \\
&\quad + 2\alpha_n \langle \gamma f(q) - Bq, x_{n+1} - q \rangle .
\end{aligned} \tag{47}$$

Together with $0 < \gamma < \overline{\gamma}/\rho$, condition (i), and (42), we can arrive at the desired conclusion $\lim_{n\to\infty} \|x_n - q\| = 0$ by Lemma 8. This completes the proof. □

Theorem 10. *Let C be a nonempty closed convex subset of a Hilbert space H and let $T : C \to C$ be a k-strictly pseudononspreading mapping such that $F(T) \neq \emptyset$. Let $f : C \to C$ be a contraction with a coefficient $\rho \in (0,1)$. Let $\{x_n\}$ be a sequence generated by $x_0 \in C$ in the following manner:*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n) T_{\lambda_n} x_n, \quad n \ge 1, \tag{48}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\lambda_n\}$ are sequences in (0,1) satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;
(iii) $\lambda_n \in (0, 1-k)$ *and* $\lim_{n\to\infty} \lambda_n = 0$.

Then the sequence $\{x_n\}$ converges strongly to $q \in F(T)$, which is the unique solution of the following variational inequality:

$$\langle (I - f) q, q - w \rangle \le 0, \quad \forall\, w \in F(T). \tag{49}$$

Proof. Putting $B = I$ and $\gamma = 1$, general iterative scheme (11) reduces to viscosity iteration (48). The desired conclusion follows immediately from Theorem 9. This completes the proof. □

Theorem 11. *Let C be a nonempty closed convex subset of a Hilbert space H and let $T : C \to C$ be a nonspreading mapping (or quasinonexpansive) such that $F(T) \neq \emptyset$. Let $f : C \to C$ be a contraction with a coefficient $\rho \in (0,1)$ and let B be a strongly positive bounded linear operator with $\overline{\gamma} > 0$ and $0 < \gamma < \overline{\gamma}/\rho$. Let $\{x_n\}$ be a sequence generated by $x_0 \in C$ in the following manner:*

$$\begin{aligned}
x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + [(1-\beta_n) I - \alpha_n B] T_{\lambda_n} x_n, \\
n \ge 1,
\end{aligned} \tag{50}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in (0,1) satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;
(iii) $\lambda_n \in (0, 1)$ *and* $\lim_{n\to\infty} \lambda_n = 0$.

Then the sequence $\{x_n\}$ converges strongly to $q \in F(T)$, which is the unique solution of the following variational inequality:

$$\langle (B - \gamma f) q, q - w \rangle \le 0, \quad \forall\, w \in F(T). \tag{51}$$

Proof. Clearly, every nonspreading mapping T is 0-strictly pseudononspreading, which is also quasinonexpansive. Therefore, the desired conclusion follows immediately from Theorem 9. This completes the proof. □

Remark 12. Theorems 9 and 10 extend the Halpern type methods of [14, 15] and viscosity methods of Moudafi [17] to more general unified general iterative methods for k-strictly pseudononspreading mapping, which also solves a class of variational inequalities related to an optimality problem.

Remark 13. Theorems 9 and 10 improve and extend the main results of Kangtunyakarn [16] for k-strictly pseudononspreading mapping in different directions.

Remark 14. The auxiliary mapping T_β of Osilike and Isiogugu [15] is generalized to the averaged mapping T_{λ_n} presented in scheme (11) with variable coefficient and projection operator based on the equivalence between variational inequality and fixed point problem.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Best Proximity Point for α-ψ-Proximal Contractive Multimaps

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We extend the notions of α-ψ-proximal contraction and α-proximal admissibility to multivalued maps and then using these notions we obtain some best proximity point theorems for multivalued mappings. Our results extend some recent results by Jleli and those contained therein. Some examples are constructed to show the generality of our results.

1. Introduction and Preliminaries

Samet et al. [1] introduced the notion of α-ψ-contractive type mappings and proved some fixed point theorems for such mappings in the frame work of complete metric spaces. Karapınar and Samet [2] generalized α-ψ-contractive type mappings and obtained some fixed point theorems for generalized α-ψ-contractive type mapping. Some interesting multivalued generalizations of α-ψ-contractive type mappings are available in [3–12]. Recently, Jleli and Samet [13] introduced the notion of α-ψ-proximal contractive type mappings and proved some best proximity point theorems. Many authors obtained best proximity point theorems in different setting; see, for example, [13–35]. Abkar and Gbeleh [16] and Al-Thagafi and Shahzad [18, 20] investigated best proximity points for multivalued mappings. The purpose of this paper is to extend the results of Jleli and Samet [13] for nonself multivalued mappings. To demonstrate generality of our main result we have constructed some examples.

Let (X,d) be a metric space. For $A,B \subset X$, we use the following notations: $\operatorname{dist}(A,B) = \inf\{d(a,b) : a \in A, b \in B\}$, $D(x,B) = \inf\{d(x,b) : b \in B\}$, $A_0 = \{a \in A : d(a,b) = \operatorname{dist}(A,B)$ for some $b \in B\}$, $B_0 = \{b \in B : d(a,b) = \operatorname{dist}(A,B)$ for some $a \in A\}$, $2^X \setminus \emptyset$ is the set of all nonempty subsets of X, $CL(X)$ is the set of all nonempty closed subsets of X, and $K(X)$ is the set of all nonempty compact subsets of X. For every $A,B \in CL(X)$, let

$$H(A,B) = \begin{cases} \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\right\} & \text{if the maximum exists;} \\ \infty & \text{otherwise.} \end{cases} \tag{1}$$

Such a map H is called the generalized Hausdorff metric induced by d. A point $x^* \in X$ is said to be the best proximity point of a mapping $T : A \to B$ if $d(x^*, Tx^*) = \operatorname{dist}(A,B)$. When $A = B$, the best proximity point reduces to fixed point of the mapping T.

Definition 1 (see [28]). Let (A,B) be a pair of nonempty subsets of a metric space (X,d) with $A_0 \neq \emptyset$. Then the pair (A,B) is said to have the weak P-property if and only if, for any $x_1, x_2 \in A$ and $y_1, y_2 \in B$,

$$\begin{aligned} d(x_1,y_1) &= \operatorname{dist}(A,B) \\ d(x_2,y_2) &= \operatorname{dist}(A,B) \end{aligned} \Longrightarrow d(x_1,x_2) \le d(y_1,y_2). \tag{2}$$

Example 2. Let $X = \{(0,1),(1,0),(0,3),(3,0)\}$, endowed with the usual metric d. Let $A = \{(0,1),(1,0)\}$ and $B = \{(0,3),(3,0)\}$. Then for

$$\begin{aligned} d((0,1),(0,3)) &= \operatorname{dist}(A,B), \\ d((1,0),(3,0)) &= \operatorname{dist}(A,B), \end{aligned} \tag{3}$$

we have

$$d((0,1),(1,0)) < d((0,3),(3,0)). \tag{4}$$

Also, $A_0 \neq \emptyset$. Thus, the pair (A,B) satisfies weak P-property.

Definition 3 (see [13]). Let $T : A \to B$ and $\alpha : A \times A \to [0,\infty)$. We say that T is an α-proximal admissible if

$$\left.\begin{array}{l} \alpha(x_1,x_2) \geq 1 \\ d(u_1,Tx_1) = \operatorname{dist}(A,B) \\ d(u_2,Tx_2) = \operatorname{dist}(A,B) \end{array}\right\} \Longrightarrow \alpha(u_1,u_2) \geq 1, \tag{5}$$

where $x_1, x_2, u_1, u_2 \in A$.

Example 4. Let $X = \mathbb{R}\times\mathbb{R}$, endowed with the usual metric d. Let a be any fixed positive real number, $A = \{(a,y) : y \in \mathbb{R}\}$ and $B = \{(0,y) : y \in \mathbb{R}\}$. Define $T : A \to B$ by

$$T(a,y) = \begin{cases} \left(0, \dfrac{y}{4}\right) & \text{if } y \geq 0 \\ (0,4y) & \text{if } y < 0. \end{cases} \tag{6}$$

Define $\alpha : A \times A \to [0,\infty)$ by

$$\alpha((a,x),(a,y)) = \begin{cases} 2 & \text{if } x,y \geq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Let $w_1 = (a,y_1)$, $w_2 = (a,y_2)$, $w_3 = (a,y_3)$, and $w_4 = (a,y_4)$ be arbitrary points from A satisfying

$$\alpha(w_1,w_2) = 2, \tag{8}$$

$$\begin{aligned} d(w_3,Tw_1) &= a = \operatorname{dist}(A,B), \\ d(w_4,Tw_2) &= a = \operatorname{dist}(A,B). \end{aligned} \tag{9}$$

It follows from (8) that $y_1, y_2 \geq 0$. Further, from (9), $y_3 = y_1/4$ and $y_4 = y_2/4$, which implies that $y_3, y_4 \geq 0$. Hence, $\alpha(w_3,w_4) = 2$. Therefore, T is an α-proximal admissible map.

Let Ψ denote the set of all functions ψ: $[0,\infty) \to [0,\infty)$ satisfying the following properties:

(a) ψ is monotone nondecreasing;
(b) $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0$.

Definition 5 (see [13]). A nonself mapping $T : A \to B$ is said to be an α-ψ-proximal contraction, if

$$\alpha(x,y)\, d(Tx,Ty) \leq \psi(d(x,y)) \quad \forall x,y \in A, \tag{10}$$

where $\alpha : A \times A \to [0,\infty)$ and $\psi \in \Psi$.

Example 6. Let us consider Example 4 again with $\psi(t) = t/2$ for each $t \geq 0$. Then it is easy to see that, for each $w_1, w_2 \in A$, we have

$$\alpha(w_1,w_2)\, d(Tw_1,Tw_2) \leq \frac{1}{2}\left|w_1 - w_2\right| = \psi(d(w_1,w_2)). \tag{11}$$

Thus, T is an α-ψ-proximal contraction.

The following are main results of [13].

Theorem 7 (see [13], Theorem 3.1). *Let A and B be two nonempty closed subsets of a complete metric space (X,d) such that A_0 is nonempty. Let $\alpha : A \times A \to [0,\infty)$ and $\psi \in \Psi$. Suppose that $T : A \to B$ be a mappings satisfying the following conditions:*

(i) *$T(A_0) \subseteq B_0$ and (A,B) satisfies the P-property;*

(ii) *T is an α-proximal admissible;*

(iii) *there exist elements $x_0, x_1 \in A_0$ such that*

$$d(x_1,Tx_0) = \operatorname{dist}(A,B), \quad \alpha(x_0,x_1) \geq 1; \tag{12}$$

(iv) *T is a continuous α-ψ-proximal contraction.*

Then there exists an element $x^* \in A_0$ such that $d(x^*,Tx^*) = \operatorname{dist}(A,B)$.

(C) If $\{x_n\}$ is a sequence in A such that $\alpha(x_n,x_{n+1}) \geq 1$ for all n and $x_n \to x \in A$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k},x) \geq 1$ for all k.

Theorem 8 (see [13], Theorem 3.2). *Let A and B be two nonempty closed subsets of a complete metric space (X,d) such that A_0 is nonempty. Let $\alpha : A \times A \to [0,\infty)$ and $\psi \in \Psi$. Suppose that $T : A \to B$ is a mapping satisfying the following conditions:*

(i) *$T(A_0) \subseteq B_0$ and (A,B) satisfies the P-property;*

(ii) *T is an α-proximal admissible;*

(iii) *there exist elements x_0 and $x_1 \in A_0$ such that*

$$d(x_1,Tx_0) = \operatorname{dist}(A,B), \quad \alpha(x_0,x_1) \geq 1; \tag{13}$$

(iv) *property (C) holds and T is an α-ψ-proximal contraction.*

Then there exists an element $x^* \in A_0$ such that $d(x^*,Tx^*) = \operatorname{dist}(A,B)$.

Definition 9 (see [16]). An element $x^* \in A$ is said to be the best proximity point of a multivalued nonself mapping T, if $D(x^*,Tx^*) = \operatorname{dist}(A,B)$.

2. Main Result

We start this section by introducing following definition.

Definition 10. Let A and B be two nonempty subsets of a metric space (X, d). A mapping $T : A \to 2^B \setminus \emptyset$ is called α-proximal admissible if there exists a mapping $\alpha : A \times A \to [0, \infty)$ such that

$$\left.\begin{array}{l}\alpha(x_1, x_2) \geq 1\\ d(u_1, y_1) = \operatorname{dist}(A, B)\\ d(u_2, y_2) = \operatorname{dist}(A, B)\end{array}\right\} \Longrightarrow \alpha(u_1, u_2) \geq 1, \tag{14}$$

where $x_1, x_2, u_1, u_2 \in A$, $y_1 \in Tx_1$, and $y_2 \in Tx_2$.

Definition 11. Let A and B be two nonempty subsets of a metric space (X, d). A mapping $T : A \to CL(B)$ is said to be an α-ψ-proximal contraction, if there exist two functions $\psi \in \Psi$ and $\alpha : A \times A \to [0, \infty)$ such that

$$\alpha(x, y) H(Tx, Ty) \leq \psi(d(x, y)), \quad \forall x, y \in A. \tag{15}$$

Lemma 12 (see [5]). *Let (X, d) be a metric space and $B \in CL(X)$. Then for each $x \in X$ with $d(x, B) > 0$ and $q > 1$, there exists an element $b \in B$ such that*

$$d(x, b) < qd(x, B). \tag{16}$$

Now we are in position to state and prove our first result.

Theorem 13. *Let A and B be two nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Let $\alpha : A \times A \to [0, \infty)$ and $\psi \in \Psi$ be a strictly increasing map. Suppose that $T : A \to CL(B)$ is a mapping satisfying the following conditions:*

(i) *$Tx \subseteq B_0$ for each $x \in A_0$ and (A, B) satisfies the weak P-property;*

(ii) *T is an α-proximal admissible;*

(iii) *there exist elements $x_0, x_1 \in A_0$ and $y_1 \in Tx_0$ such that*

$$d(x_1, y_1) = \operatorname{dist}(A, B), \quad \alpha(x_0, x_1) \geq 1; \tag{17}$$

(iv) *T is a continuous α-ψ-proximal contraction.*

Then there exists an element $x^* \in A_0$ such that $D(x^*, Tx^*) = \operatorname{dist}(A, B)$.

Proof. From condition (iii), there exist elements $x_0, x_1 \in A_0$ and $y_1 \in Tx_0$ such that

$$d(x_1, y_1) = \operatorname{dist}(A, B), \quad \alpha(x_0, x_1) \geq 1. \tag{18}$$

Assume that $y_1 \notin Tx_1$; for otherwise x_1 is the best proximity point. From condition (iv), we have

$$\begin{aligned}0 &< d(y_1, Tx_1) \leq H(Tx_0, Tx_1)\\ &\leq \alpha(x_0, x_1) H(Tx_0, Tx_1) \leq \psi(d(x_0, x_1)).\end{aligned} \tag{19}$$

For $q > 1$, it follows from Lemma 12 that there exists $y_2 \in Tx_1$ such that

$$0 < d(y_1, y_2) < qd(y_1, Tx_1). \tag{20}$$

From (19) and (20), we have

$$0 < d(y_1, y_2) < qd(y_1, Tx_1) \leq q\psi(d(x_0, x_1)). \tag{21}$$

As $y_2 \in Tx_1 \subseteq B_0$, there exists $x_2 \neq x_1 \in A_0$ such that

$$d(x_2, y_2) = \operatorname{dist}(A, B); \tag{22}$$

for otherwise x_1 is the best proximity point. As (A, B) satisfies the weak P-property, from (18) and (22), we have

$$0 < d(x_1, x_2) \leq d(y_1, y_2). \tag{23}$$

From (21) and (23), we have

$$0 < d(x_1, x_2) < qd(y_1, Tx_1) \leq q\psi(d(x_0, x_1)). \tag{24}$$

Since ψ is strictly increasing, we have $\psi(d(x_1, x_2)) < \psi(q\psi(d(x_0, x_1)))$. Put $q_1 = \psi(q\psi(d(x_0, x_1)))/\psi(d(x_1, x_2))$. Also, we have $\alpha(x_0, x_1) \geq 1$, $d(x_1, y_1) = \operatorname{dist}(A, B)$, and $d(x_2, y_2) = \operatorname{dist}(A, B)$. Since T is an α-proximal admissible, then $\alpha(x_1, x_2) \geq 1$. Thus we have

$$d(x_2, y_2) = \operatorname{dist}(A, B), \quad \alpha(x_1, x_2) \geq 1. \tag{25}$$

Assume that $y_2 \notin Tx_2$; for otherwise x_2 is the best proximity point. From condition (iv), we have

$$\begin{aligned}0 &< d(y_2, Tx_2) \leq H(Tx_1, Tx_2)\\ &\leq \alpha(x_1, x_2) H(Tx_1, Tx_2) \leq \psi(d(x_1, x_2)).\end{aligned} \tag{26}$$

For $q_1 > 1$, it follows from Lemma 12 that there exists $y_3 \in Tx_2$ such that

$$0 < d(y_2, y_3) < q_1 d(y_2, Tx_2). \tag{27}$$

From (26) and (27), we have

$$\begin{aligned}0 &< d(y_2, y_3) < q_1 d(y_2, Tx_2) \leq q_1 \psi(d(x_1, x_2))\\ &= \psi(q\psi(d(x_0, x_1))).\end{aligned} \tag{28}$$

As $y_3 \in Tx_2 \subseteq B_0$, there exists $x_3 \neq x_2 \in A_0$ such that

$$d(x_3, y_3) = \operatorname{dist}(A, B); \tag{29}$$

for otherwise x_2 is the best proximity point. As (A, B) satisfies the weak P-property, from (25) and (29), we have

$$0 < d(x_2, x_3) \leq d(y_2, y_3). \tag{30}$$

From (28) and (30), we have

$$\begin{aligned}0 &< d(x_2, x_3) < q_1 d(y_2, Tx_2) \leq q_1 \psi(d(x_1, x_2))\\ &= \psi(q\psi(d(x_0, x_1))).\end{aligned} \tag{31}$$

Since ψ is strictly increasing, we have $\psi(d(x_2,x_3)) < \psi^2(q\psi(d(x_0,x_1)))$. Put $q_2 = \psi^2(q\psi(d(x_0,x_1)))/\psi(d(x_2,x_3))$. Also, we have $\alpha(x_1,x_2) \geq 1$, $d(x_2,y_2) = \operatorname{dist}(A,B)$, and $d(x_3,y_3) = \operatorname{dist}(A,B)$. Since T is an α-proximal admissible then $\alpha(x_2,x_3) \geq 1$. Thus, we have

$$d(x_3,y_3) = \operatorname{dist}(A,B), \quad \alpha(x_2,x_3) \geq 1. \tag{32}$$

Continuing in the same way, we get sequences $\{x_n\}$ in A_0 and $\{y_n\}$ in B_0, where $y_n \in Tx_{n-1}$ for each $n \in \mathbb{N}$ such that

$$d(x_{n+1},y_{n+1}) = \operatorname{dist}(A,B), \quad \alpha(x_n,x_{n+1}) \geq 1, \tag{33}$$

$$d(y_{n+1},y_{n+2}) < \psi^n(q\psi(d(x_0,x_1))). \tag{34}$$

As $y_{n+2} \in Tx_{n+1} \subseteq B_0$, there exists $x_{n+2} \neq x_{n+1} \in A_0$ such that

$$d(x_{n+2},y_{n+2}) = \operatorname{dist}(A,B). \tag{35}$$

Since (A,B) satisfies the weak P-property form (33) and (35), we have $d(x_{n+1},x_{n+2}) \leq d(y_{n+1},y_{n+2})$. Then from (34), we have

$$d(x_{n+1},x_{n+2}) < \psi^n(q\psi(d(x_0,x_1))). \tag{36}$$

For $n > m$ we have

$$d(x_n,x_m) \leq \sum_{i=n}^{m-1} d(x_i,x_{i+1}) < \sum_{i=n}^{m-1} \psi^{i-1}(q\psi(d(x_0,x_1))). \tag{37}$$

Hence, $\{x_n\}$ is a Cauchy sequence in A. Similarly, we show that $\{y_n\}$ is a Cauchy sequence in B. Since A and B are closed subsets of a complete metric space, there exist x^* in A and y^* in B such that $x_n \to x^*$ and $y_n \to y^*$ as $n \to \infty$. By (35), we conclude that $d(x^*,y^*) = \operatorname{dist}(A,B)$ as $n \to \infty$. Since T is continuous and $y_n \in Tx_{n-1}$, we have $y^* \in Tx^*$. Hence, $\operatorname{dist}(A,B) \leq D(x^*,Tx^*) \leq d(x^*,y^*) = \operatorname{dist}(A,B)$. Therefore, x^* is the best proximity point of the mapping T. □

Theorem 14. *Let A and B be two nonempty closed subsets of a complete metric space (X,d) such that A_0 is nonempty. Let $\alpha : A \times A \to [0,\infty)$ and $T : A \to K(B)$ be mappings satisfying the following conditions:*

(i) *$Tx \subseteq B_0$ for each $x \in A_0$ and (A,B) satisfies the weak P-property;*

(ii) *T is an α-proximal admissible;*

(iii) *there exist elements $x_0, x_1 \in A_0$ and $y_1 \in Tx_0$ such that*

$$d(x_1,y_1) = \operatorname{dist}(A,B), \quad \alpha(x_0,x_1) \geq 1; \tag{38}$$

(iv) *T is a continuous α-ψ-proximal contraction.*

Then there exists an element $x^* \in A_0$ such that $D(x^*,Tx^*) =$ $\operatorname{dist}(A,B)$.

Theorem 15. *Let A and B be two nonempty closed subsets of a complete metric space (X,d) such that A_0 is nonempty. Let $\alpha : A \times A \to [0,\infty)$ and $\psi \in \Psi$ be a strictly increasing map. Suppose that $T : A \to CL(B)$ is a mapping satisfying the following conditions:*

(i) *$Tx \subseteq B_0$ for each $x \in A_0$ and (A,B) satisfies the weak P-property;*

(ii) *T is an α-proximal admissible;*

(iii) *there exist elements $x_0, x_1 \in A_0$ and $y_1 \in Tx_0$ such that*

$$d(x_1,y_1) = \operatorname{dist}(A,B), \quad \alpha(x_0,x_1) \geq 1; \tag{39}$$

(iv) *property (C) holds and T is an α-ψ-proximal contraction.*

Then there exists an element $x^* \in A_0$ such that $D(x^*,Tx^*) =$ $\operatorname{dist}(A,B)$.

Proof. Following the proof of Theorem 13, there exist Cauchy sequences $\{x_n\}$ in A and $\{y_n\}$ in B such that (33) holds and $x_n \to x^* \in A$ and $y_n \to y^* \in B$ as $n \to \infty$. From the condition (C), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k},x^*) \geq 1$ for all k. Since T is an α-ψ-proximal contraction, we have

$$\begin{aligned} H(Tx_{n_k},Tx^*) &\leq \alpha(x_{n_k},x^*) H(Tx_{n_k},Tx^*) \\ &\leq \psi(d(x_{n_k},x^*)), \quad \forall k. \end{aligned} \tag{40}$$

Letting $k \to \infty$ in the above inequality, we get $Tx_{n_k} \to Tx^*$. By continuity of the metric d, we have

$$d(x^*,y^*) = \lim_{k\to\infty} d(x_{n_k+1},y_{n_k+1}) = \operatorname{dist}(A,B). \tag{41}$$

Since $y_{n_k+1} \in T_{x_{n_k}}$, $y_{n_k} \to y^*$, and $Tx_{n_k} \to Tx^*$, then $y^* \in Tx^*$. Hence, $\operatorname{dist}(A,B) \leq D(x^*,Tx^*) \leq d(x^*,y^*) = \operatorname{dist}(A,B)$. Therefore, x^* is the best proximity point of the mapping T. □

Theorem 16. *Let A and B be two nonempty closed subsets of a complete metric space (X,d) such that A_0 is nonempty. Let $\alpha : A \times A \to [0,\infty)$ and $T : A \to K(B)$ be mappings satisfying the following conditions:*

(i) *$Tx \subseteq B_0$ for each $x \in A_0$ and (A,B) satisfies the weak P-property;*

(ii) *T is an α-proximal admissible;*

(iii) *there exist elements $x_0, x_1 \in A_0$ and $y_1 \in Tx_0$ such that*

$$d(x_1,y_1) = \operatorname{dist}(A,B), \quad \alpha(x_0,x_1) \geq 1; \tag{42}$$

(iv) *property (C) holds and T is an α-ψ-proximal contraction.*

Then there exists an element $x^* \in A_0$ such that $D(x^*,Tx^*) =$ $\operatorname{dist}(A,B)$.

Example 17. Let $X = [0, \infty) \times [0, \infty)$ be endowed with the usual metric d. Suppose that $A = \{(1/2, x) : 0 \le x < \infty\}$ and $B = \{(0, x) : 0 \le x < \infty\}$. Define $T : A \to CL(B)$ by

$$T\left(\frac{1}{2}, a\right) = \begin{cases} \left\{\left(0, \dfrac{x}{2}\right) : 0 \le x \le a\right\} & \text{if } a \le 1 \\ \left\{\left(0, x^2\right) : 0 \le x \le a^2\right\} & \text{if } a > 1, \end{cases} \tag{43}$$

and $\alpha : A \times A \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in \left\{\left(\dfrac{1}{2}, a\right) : 0 \le a \le 1\right\} \\ 0 & \text{otherwise.} \end{cases} \tag{44}$$

Let $\psi(t) = t/2$ for all $t \ge 0$. Notice that $A_0 = A$, $B_0 = B$, and $Tx \subseteq B_0$ for each $x \in A_0$. Also, the pair (A, B) satisfies the weak P-property. Let $x_0, x_1 \in \{(1/2, x) : 0 \le x \le 1\}$; then $Tx_0, Tx_1 \subseteq \{(0, x/2) : 0 \le x \le 1\}$. Consider $y_1 \in Tx_0$, $y_2 \in Tx_1$, and $u_1, u_2 \in A$ such that $d(u_1, y_1) = \operatorname{dist}(A, B)$ and $d(u_2, y_2) = \operatorname{dist}(A, B)$. Then we have $u_1, u_2 \in \{(1/2, x) : 0 \le x \le 1/2\}$. Hence, T is an α-proximal admissible map. For $x_0 = (1/2, 1) \in A_0$ and $y_1 = (0, 1/2) \in Tx_0$ in B_0, we have $x_1 = (1/2, 1/2) \in A_0$ such that $d(x_1, y_1) = \operatorname{dist}(A, B)$ and $\alpha(x_0, x_1) = 1$. If $x, y \in \{(1/2, a) : 0 \le a \le 1\}$, then we have

$$\alpha(x, y) H(Tx, Ty) = \frac{|x - y|}{2} = \frac{1}{2} d(x, y) = \psi(d(x, y)); \tag{45}$$

for otherwise

$$\alpha(x, y) H(Tx, Ty) \le \psi(d(x, y)). \tag{46}$$

Hence, T is an α-ψ-proximal contraction. Moreover, if $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x_{n+1}) = 1$ for all n and $x_n \to x \in A$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) = 1$ for all k. Therefore, all the conditions of Theorem 15 hold and T has the best proximity point.

Example 18. Let $X = [0, \infty) \times [0, \infty)$ be endowed with the usual metric d. Let $a > 1$ be any fixed real number, $A = \{(a, x) : 0 \le x < \infty\}$ and $B = \{(0, x) : 0 \le x < \infty\}$. Define $T : A \to CL(B)$ by

$$T(a, x) = \left\{\left(0, b^2\right) : 0 \le b \le x\right\} \tag{47}$$

and $\alpha : A \times A \to [0, \infty)$ by

$$\alpha((a, x), (a, y)) = \begin{cases} 1 & \text{if } x = y = 0 \\ \dfrac{1}{a(x + y)} & \text{otherwise.} \end{cases} \tag{48}$$

Let $\psi(t) = (1/a)t$ for all $t \ge 0$. Notice that $A_0 = A$, $B_0 = B$, and $Tx \subseteq B_0$ for each $x \in A_0$. If $w_1 = (a, y_1), w_2 = (a, y_2) \in A$ with either $y_1 \ne 0$ or $y_2 \ne 0$ or both are nonzero, we have

$$\begin{aligned} \alpha(w_1, w_2) H(Tw_1, Tw_2) &= \frac{1}{a(y_1 + y_2)} \left|(y_1)^2 - (y_2)^2\right| \\ &= \frac{1}{a} |y_1 - y_2| = \psi(d(w_1, w_2)); \end{aligned} \tag{49}$$

for otherwise

$$\alpha(w_1, w_2) H(Tw_1, Tw_2) = 0 = \psi(d(w_1, w_2)). \tag{50}$$

For $x_0 = (a, 1/2a) \in A_0$ and $y_1 = (0, 1/4a^2) \in Tx_0$ in B_0, we have $x_1 = (a, 1/4a^2) \in A_0$ such that $d(x_1, y_1) = a = \operatorname{dist}(A, B)$ and $\alpha(x_0, x_1) > 1$. Furthermore, it is easy to see that remaining conditions of Theorem 13 also hold. Thus, T has the best proximity point.

3. Consequences

From results of previous section, we immediately obtain the following results.

Corollary 19. *Let A and B be two nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Let $\alpha : A \times A \to [0, \infty)$ and $T : A \to B$ be mappings satisfying the following conditions:*

(i) *$T(A_0) \subseteq B_0$ and (A, B) satisfies the weak P-property;*
(ii) *T is an α-proximal admissible;*
(iii) *there exist elements x_0 and x_1 in A_0 such that*

$$d(x_1, Tx_0) = \operatorname{dist}(A, B), \quad \alpha(x_0, x_1) \ge 1; \tag{51}$$

(iv) *T is a continuous α-ψ-proximal contraction.*

Then there exists an element $x^* \in A_0$ such that $d(x^*, Tx^*) = \operatorname{dist}(A, B)$.

Corollary 20. *Let A and B be two nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Let $\alpha : A \times A \to [0, \infty)$ and $T : A \to B$ be mappings satisfying the following conditions:*

(i) *$T(A_0) \subseteq B_0$ and (A, B) satisfies the weak P-property;*
(ii) *T is an α-proximal admissible;*
(iii) *there exist elements x_0 and x_1 in A_0 such that*

$$d(x_1, Tx_0) = \operatorname{dist}(A, B), \quad \alpha(x_0, x_1) \ge 1; \tag{52}$$

(iv) *property (C) holds and T is an α-ψ-proximal contraction.*

Then there exists an element $x^* \in A_0$ such that $d(x^*, Tx^*) = \operatorname{dist}(A, B)$.

Remark 21. Note that Corollaries 19 and 20 generalize Theorems 7 and 8 in Section 1, respectively.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

A Fixed Point Theorem for Multivalued Mappings with δ-Distance

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We mainly study fixed point theorem for multivalued mappings with δ-distance using Wardowski's technique on complete metric space. Let (X, d) be a metric space and let $B(X)$ be a family of all nonempty bounded subsets of X. Define $\delta : B(X) \times B(X) \to \mathbb{R}$ by $\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}$. Considering δ-distance, it is proved that if (X, d) is a complete metric space and $T : X \to B(X)$ is a multivalued certain contraction, then T has a fixed point.

1. Introduction

Fixed point theory concern itself with a very basic mathematical setting. It is also well known that one of the fundamental and most useful results in fixed point theory is Banach fixed point theorem. This result has been extended in many directions for single and multivalued cases on a metric space X (see [1–9]). Fixed point theory for multivalued mappings is studied by both Pompeiu-Hausdorff metric H [10, 11], which is defined on $CB(X)$ (the family of all nonempty, closed, and bounded subsets of X), and δ-distance, which is defined on $B(X)$ (the family of all nonempty and bounded subsets of X). Using Pompeiu-Hausdorff metric, Nadler [12] introduced the concept of multivalued contraction mapping and show that such mapping has a fixed point on complete metric space. Then many authors focused on this direction [13–18]. On the other hand, Fisher [19] obtained different type of multivalued fixed point theorems defining δ-distance between two bounded subsets of a metric space X. We can find some results about this way in [20–23].

In this paper, we give some new multivalued fixed point results by considering the δ-distance. For this we use the recent technique, which was given by Wardowski [24]. For the sake of completeness,we will discuss its basic lines. Let $\mathcal{F}$ be the set of all functions $F : (0, \infty) \to \mathbb{R}$ satisfying the following conditions:

(F1) F is strictly increasing; that is, for all $\alpha, \beta \in (0, \infty)$ such that $\alpha < \beta$, $F(\alpha) < F(\beta)$.

(F2) For each sequence $\{a_n\}$ of positive numbers $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} F(a_n) = -\infty$.

(F3) There exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Definition 1 (see [24]). Let (X, d) be a metric space and let $T : X \to X$ be a mapping. Given $F \in \mathcal{F}$, we say that T is F-contraction, if there exists $\tau > 0$ such that

$$\begin{aligned} &x, y \in X, \\ &d(Tx, Ty) > 0 \Longrightarrow \tau + F(d(Tx, Ty)) \le F(d(x, y)). \end{aligned} \tag{1}$$

Taking different functions $F \in \mathcal{F}$ in (1), one gets a variety of F-contractions, some of them being already known in the literature. The following examples will certify this assertion.

Example 2 (see [24]). Let $F_1 : (0, \infty) \to \mathbb{R}$ be given by the formulae $F_1(\alpha) = \ln \alpha$. It is clear that $F_1 \in \mathcal{F}$. Then each self-mapping T on a metric space (X, d) satisfying (1) is an F_1-contraction such that

$$d(Tx, Ty) \le e^{-\tau} d(x, y), \quad \forall x, y \in X,\ Tx \ne Ty. \tag{2}$$

It is clear that for x, $y \in X$ such that $Tx = Ty$ the inequality $d(Tx, Ty) \le e^{-\tau} d(x, y)$ also holds. Therefore

T satisfies Banach contraction with $L = e^{-\tau}$; thus T is a contraction.

Example 3 (see [24]). Let $F_2 : (0,\infty) \to \mathbb{R}$ be given by the formulae $F_2(\alpha) = \alpha + \ln\alpha$. It is clear that $F_2 \in \mathcal{F}$. Then each self-mapping T on a metric space (X, d) satisfying (1) is an F_2-contraction such that

$$\frac{d(Tx,Ty)}{d(x,y)}e^{d(Tx,Ty)-d(x,y)} \le e^{-\tau}, \quad \forall x, y \in X,\ Tx \ne Ty. \tag{3}$$

We can find some different examples for the function F belonging to $\mathcal{F}$ in [24]. In addition, Wardowski concluded that every F-contraction T is a contractive mapping, that is,

$$d(Tx,Ty) < d(x,y), \quad \forall x, y \in X,\ Tx \ne Ty. \tag{4}$$

Thus, every F-contraction is a continuous mapping.

Also, Wardowski concluded that if $F_1, F_2 \in \mathcal{F}$ with $F_1(\alpha) \le F_2(\alpha)$ for all $\alpha > 0$ and $G = F_2 - F_1$ is nondecreasing, then every F_1-contraction T is an F_2-contraction.

He noted that, for the mappings $F_1(\alpha) = \ln\alpha$ and $F_2(\alpha) = \alpha + \ln\alpha$, $F_1 < F_2$ and a mapping $F_2 - F_1$ is strictly increasing. Hence, it was obtained that every Banach contraction satisfies the contractive condition (3). On the other side, [24, Example 2.5] shows that the mapping T is not an F_1-contraction (Banach contraction) but still is an F_2-contraction. Thus, the following theorem, which was given by Wardowski, is a proper generalization of Banach Contraction Principle.

Theorem 4 (see [24]). *Let (X, d) be a complete metric space and let $T : X \to X$ be an F-contraction. Then T has a unique fixed point in X.*

Following Wardowski, Mınak et al. [25] introduced the concept of Ćirić type generalized F-contraction. Let (X, d) be a metric space and let $T : X \to X$ be a mapping. Given $F \in \mathcal{F}$, we say that T is a Ćirić type generalized F-contraction if there exists $\tau > 0$ such that

$$\begin{gathered} x, y \in X, \\ d(Tx,Ty) > 0 \Longrightarrow \tau + F(d(Tx,Ty)) \le F(m(x,y)), \end{gathered} \tag{5}$$

where

$$\begin{aligned} m(x,y) = \max\Big\{ & d(x,y), d(x,Tx), d(y,Ty), \\ & \frac{1}{2}[d(x,Ty) + d(y,Tx)]\Big\}. \end{aligned} \tag{6}$$

Then the following theorem was given.

Theorem 5. *Let (X, d) be a complete metric space and let $T : X \to X$ be a Ćirić type generalized F-contraction. If T or F is continuous, then T has a unique fixed point in X.*

Considering the Pompeiu-Hausdorff metric H, both Theorems 4 and 5 were extended to multivalued cases in [26] and [27], respectively (see also [28, 29]). In this work, we give a fixed point result for multivalued mappings using the δ-distance. First recall some definitions and notations which are used in this paper.

Let (X, d) be a metric space. For $A, B \in B(X)$ we define

$$\begin{gathered} \delta(A,B) = \sup\{d(a,b) : a \in A, b \in B\}, \\ D(a,B) = \inf\{d(a,b) : b \in B\}. \end{gathered} \tag{7}$$

If $A = \{a\}$ we write $\delta(A, B) = \delta(a, B)$ and also if $B = \{b\}$, then $\delta(a, B) = d(a, b)$. It is easy to prove that for $A, B, C \in B(X)$

$$\begin{gathered} \delta(A,B) = \delta(B,A) \ge 0, \\ \delta(A,B) \le \delta(A,C) + \delta(C,B), \\ \delta(A,A) = \sup\{d(a,b) : a, b \in A\} = \operatorname{diam} A, \\ \delta(A,B) = 0, \quad \text{implies that } A = B = \{a\}. \end{gathered} \tag{8}$$

If $\{A_n\}$ is a sequence in $B(X)$, we say that $\{A_n\}$ converges to $A \subseteq X$ and write $A_n \to A$ if and only if

(i) $a \in A$ implies that $a_n \to a$ for some sequence $\{a_n\}$ with $a_n \in A_n$ for $n \in \mathbb{N}$,

(ii) for any $\varepsilon > 0$, $\exists m \in \mathbb{N}$ such that $A_n \subseteq A_\varepsilon$ for $n > m$, where

$$A_\varepsilon = \{x \in X : d(x,a) < \varepsilon \text{ for some } a \in A\}. \tag{9}$$

Lemma 6 (see [20]). *Suppose $\{A_n\}$ and $\{B_n\}$ are sequences in $B(X)$ and (X, d) is a complete metric space. If $A_n \to A \in B(X)$ and $B_n \to B \in B(X)$ then $\delta(A_n, B_n) \to \delta(A, B)$.*

Lemma 7 (see [20]). *If $\{A_n\}$ is a sequence of nonempty bounded subsets in the complete metric space (X, d) and if $\delta(A_n, y) \to 0$ for some $y \in X$, then $A_n \to \{y\}$.*

2. Main Result

In this section, we prove a fixed point theorem for multivalued mappings with δ-distance and give an illustrative example.

Definition 8. Let (X, d) be a metric space and let $T : X \to B(X)$ be a mapping. Then T is said to be a generalized multivalued F-contraction if $F \in \mathcal{F}$ and there exists $\tau > 0$ such that

$$\tau + F(\delta(Tx,Ty)) \le F(M(x,y)), \tag{10}$$

for all $x, y \in X$ with $\min\{\delta(Tx, Ty), d(x, y)\} > 0$, where

$$\begin{aligned} M(x,y) = \max\Big\{ & d(x,y), D(x,Tx), D(y,Ty), \\ & \frac{1}{2}[D(x,Ty) + D(y,Tx)]\Big\}. \end{aligned} \tag{11}$$

Theorem 9. *Let (X, d) be a complete metric space and let $T : X \to B(X)$ be a multivalued F-contraction. If F is continuous and Tx is closed for all $x \in X$, then T has a fixed point in X.*

Proof. Let $x_0 \in X$ be an arbitrary point and define a sequence $\{x_n\}$ in X as $x_{n+1} \in Tx_n$ for all $n \geq 0$. If there exists $n_0 \in \mathbb{N} \cup \{0\}$ for which $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point of T and so the proof is completed. Thus, suppose that, for every $n \in \mathbb{N} \cup \{0\}$, $x_n \neq x_{n+1}$. So $d(x_n, x_{n+1}) > 0$ and $\delta(Tx_{n-1}, Tx_n) > 0$ for all $n \in \mathbb{N}$. Then, we have from (10)

$$\begin{aligned}
&\tau + F(d(x_n, x_{n+1})) \\
&\leq \tau + F(\delta(Tx_{n-1}, Tx_n)) \\
&\leq F(M(x_{n-1}, x_n)) \\
&= F\left(\max\left\{\begin{matrix} d(x_{n-1}, x_n), D(x_{n-1}, Tx_{n-1}), D(x_n, Tx_n), \\ \frac{1}{2}[D(x_{n-1}, Tx_n) + D(x_n, Tx_{n-1})] \end{matrix}\right\}\right) \\
&\leq F(\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}) \\
&= F(d(x_{n-1}, x_n)),
\end{aligned} \tag{12}$$

and so

$$\begin{aligned}
F(d(x_n, x_{n+1})) &\leq F(d(x_{n-1}, x_n)) - \tau \\
&\leq F(d(x_{n-2}, x_{n-1})) - 2\tau \\
&\vdots \\
&\leq F(d(x_0, x_1)) - n\tau.
\end{aligned} \tag{13}$$

Denote $a_n = d(x_n, x_{n+1})$, for $n = 0, 1, 2, \ldots$. Then, $a_n > 0$ for all n and, using (10), the following holds:

$$F(a_n) \leq F(a_{n-1}) - \tau \leq F(a_{n-2}) - 2\tau \leq \cdots \leq F(a_0) - n\tau. \tag{14}$$

From (14), we get $\lim_{n\to\infty} F(a_n) = -\infty$. Thus, from (F2), we have

$$\lim_{n\to\infty} a_n = 0. \tag{15}$$

From (F3) there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} a_n^k F(a_n) = 0. \tag{16}$$

By (14), the following holds for all $n \in \mathbb{N}$:

$$a_n^k F(a_n) - a_n^k F(a_0) \leq -a_n^k n\tau \leq 0. \tag{17}$$

Letting $n \to \infty$ in (17), we obtain that

$$\lim_{n\to\infty} n a_n^k = 0. \tag{18}$$

From (18), there exits $n_1 \in \mathbb{N}$ such that $n a_n^k \leq 1$ for all $n \geq n_1$. So we have

$$a_n \leq \frac{1}{n^{1/k}}, \tag{19}$$

for all $n \geq n_1$. In order to show that $\{x_n\}$ is a Cauchy sequence consider $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. Using the triangular inequality for the metric and from (19), we have

$$\begin{aligned}
&d(x_n, x_m) \\
&\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\
&= a_n + a_{n+1} + \cdots + a_{m-1} \\
&= \sum_{i=n}^{m-1} a_i \\
&\leq \sum_{i=n}^{\infty} a_i \\
&\leq \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}.
\end{aligned} \tag{20}$$

By the convergence of the series $\sum_{i=1}^{\infty}(1/i^{1/k})$, we get $d(x_n, x_m) \to 0$ as $n \to \infty$. This yields that $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete metric space, the sequence $\{x_n\}$ converges to some point $z \in X$; that is, $\lim_{n\to\infty} x_n = z$. Now, suppose F is continuous. In this case, we claim that $z \in Tz$. Assume the contrary; that is, $z \notin Tz$. In this case, there exist an $n_0 \in \mathbb{N}$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $D(x_{n_k+1}, Tz) > 0$ for all $n_k \geq n_0$. (Otherwise, there exists $n_1 \in \mathbb{N}$ such that $x_n \in Tz$ for all $n \geq n_1$, which implies that $z \in Tz$. This is a contradiction, since $z \notin Tz$.) Since $D(x_{n_k+1}, Tz) > 0$ for all $n_k \geq n_0$, then we have

$$\begin{aligned}
&\tau + F(D(x_{n_k+1}, Tz)) \\
&\leq \tau + F(\delta(Tx_{n_k}, Tz)) \\
&\leq F(M(x_{n_k}, z)) \\
&\leq F\Big(\max\Big\{d(x_{n_k}, z), d(x_{n_k}, x_{n_k+1}), D(z, Tz), \\
&\qquad \frac{1}{2}[D(x_{n_k}, Tz) + d(z, x_{n_k+1})]\Big\}\Big).
\end{aligned} \tag{21}$$

Taking the limit $k \to \infty$ and using the continuity of F, we have $\tau + F(D(z, \overline{Tz})) \leq F(D(z, Tz))$, which is a contradiction. Thus, we get $z \in \overline{Tz} = Tz$. This completes the proof. □

Example 10. Let $X = \{0, 1, 2, 3, \ldots\}$ and $d(x, y) = \begin{cases} 0; & x=y \\ x+y; & x\neq y \end{cases}$. Then (X, d) is a complete metric space. Define $T : X \to B(X)$ by

$$Tx = \begin{cases} \{0\}; & x = 0 \\ \{0, 1, 2, 3, \ldots, x-1\}; & x \neq 0. \end{cases} \tag{22}$$

We claim that T is multivalued F-contraction with respect to $F(\alpha) = \alpha + \ln \alpha$ and $\tau = 1$. Because of the $\min\{\delta(Tx, Ty),$

$d(x, y)\} > 0$, we can consider the following cases while $x \neq y$ and $\{x, y\} \cap \{0, 1\}$ is empty or singleton.

Case 1. For $y = 0$ and $x > 1$, we have

$$\begin{aligned}\frac{\delta(Tx,Ty)}{M(x,y)}e^{\delta(Tx,Ty)-M(x,y)} &= \frac{x-1}{x}e^{x-1-x}\\ &= \frac{x-1}{x}e^{-1} < e^{-1}.\end{aligned} \tag{23}$$

Case 2. For $y = 1$ and $x > 1$, we have

$$\begin{aligned}\frac{\delta(Tx,Ty)}{M(x,y)}e^{\delta(Tx,Ty)-M(x,y)} &= \frac{x-1}{x}e^{x-1-x}\\ &= \frac{x-1}{x}e^{-1} < e^{-1}.\end{aligned} \tag{24}$$

Case 3. For $x > y > 1$, we have

$$\begin{aligned}\frac{\delta(Tx,Ty)}{M(x,y)}e^{\delta(Tx,Ty)-M(x,y)} &= \frac{x+y-2}{x+y}e^{x+y-2-x-y}\\ &= \frac{x+y-2}{x+y}e^{-2} < e^{-1}.\end{aligned} \tag{25}$$

This shows that T is multivalued F-contraction; therefore, all conditions of theorem are satisfied and so T has a fixed point in X.

On the other hand, for $y = 0$ and $x \neq 0$, since $\delta(Tx, Ty) = x - 1$ and $d(x, y) = x$, we get

$$\lim_{n\to\infty}\frac{\delta(Tx,Ty)}{M(x,y)} = \lim_{n\to\infty}\frac{x-1}{x} = 1; \tag{26}$$

then T does not satisfy

$$\delta(Tx,Ty) \leq \lambda M(x,y), \tag{27}$$

for $\lambda \in [0, 1)$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Iterative Algorithms for Mixed Equilibrium Problems, System of Quasi-Variational Inclusion, and Fixed Point Problem in Hilbert Spaces

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We introduce a new iterative algorithm for approximating a common element of the set of solutions for mixed equilibrium problems, the set of solutions of a system of quasi-variational inclusion, and the set of fixed points of an infinite family of nonexpansive mappings in a real Hilbert space. Strong convergence of the proposed iterative algorithm is obtained. Our results generalize, extend, and improve the results of Peng and Yao, 2009, Qin et al. 2010 and many authors.

1. Introduction

Throughout this paper, we assume that H is a real Hilbert space with inner product and norm denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. A mapping $T: C \to C$ is called *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$, $\forall x, y \in C$. They use $F(T)$ to denote the set of *fixed points* of T; that is, $F(T) = \{x \in C : Tx = x\}$. It is assumed throughout the paper that T is a nonexpansive mapping such that $F(T) \neq \emptyset$. Recall that a self-mapping $f : C \to C$ is a contraction on C if there exists a constant $\alpha \in [0, 1)$, and $x, y \in C$ such that $\|f(x) - f(y)\| \le \alpha\|x - y\|$.

Let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper extended real-valued function and let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. Ceng and Yao [1] considered the following *mixed equilibrium problem* for finding $x \in C$ such that

$$F(x, y) + \varphi(y) \ge \varphi(x), \quad \forall y \in C. \tag{1}$$

The set of solutions of (1) is denoted by $\mathrm{MEP}(F, \varphi)$. We see that x is a solution of problem (1) which implies that $x \in \operatorname{dom}\varphi = \{x \in C \mid \varphi(x) < +\infty\}$. If $\varphi \equiv 0$, then the mixed equilibrium problem (1) becomes the following *equilibrium problem* for finding $x \in C$ such that

$$F(x, y) \ge 0, \quad \forall y \in C. \tag{2}$$

The set of solutions of (2) is denoted by $\mathrm{EP}(F)$. The mixed equilibrium problems include fixed point problems, variational inequality problems, optimization problems, Nash equilibrium problems, and the equilibrium problem as special cases. Numerous problems in physics, optimization, and economics reduce to find a solution of (2). Some methods have been proposed to solve the equilibrium problem (see [2–14]).

Let $B : C \to H$ be a mapping. The *variational inequality problem,* denoted by $\mathrm{VI}(C, B)$, is for finding $x \in C$ such that

$$\langle Bx, y - x\rangle \ge 0, \tag{3}$$

for all $y \in C$. The variational inequality problem has been extensively studied in the literature. See, for example, [15, 16] and the references therein. A mapping B of C into H is called *monotone* if

$$\langle Bx - By, x - y\rangle \geq 0, \tag{4}$$

for all $x, y \in C$. B is called *β-inverse-strongly monotone* if there exists a positive real number $\beta > 0$ such that for all $x, y \in C$

$$\langle Bx - By, x - y\rangle \geq \beta\|Bx - By\|^2. \tag{5}$$

We consider a *system of quasi-variational inclusion* for finding $(x^*, y^*) \in H \times H$ such that

$$\begin{gathered} \theta \in x^* - y^* + \rho_1 (B_1 y^* + M_1 x^*), \\ \theta \in y^* - x^* + \rho_2 (B_2 x^* + M_2 y^*), \end{gathered} \tag{6}$$

where $B_i : H \to H$ and $M_i : H \to 2^H$ are nonlinear mappings for each $i = 1, 2$. The set of solutions of problem (6) is denoted by $\mathrm{SQVI}(B_1, M_1, B_2, M_2)$. As special cases of problem (6), we have the following.

(1) If $B_1 = B_2 = B$ and $M_1 = M_2 = M$, then problem (6) is reduced to (7) for finding $(x^*, y^*) \in H \times H$ such that

$$\begin{gathered} \theta \in x^* - y^* + \rho_1 (By^* + Mx^*), \\ \theta \in y^* - x^* + \rho_2 (Bx^* + My^*). \end{gathered} \tag{7}$$

(2) Further, if $x^* = y^*$, then problem (7) is reduced to (8) for finding $x^* \in H$ such that

$$\theta \in Bx^* + Mx^*, \tag{8}$$

where θ is the zero vector in H. The set of solutions of problem (8) is denoted by $I(B, M)$. A set-valued mapping $M : H \to 2^H$ is called *monotone* if for all $x, y \in H, f \in M(x)$ and $g \in M(y)$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping M is *maximal* if its graph $G(M) := \{(f, x) \in H \times H : f \in M(x)\}$ of M is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping M is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for all $(y, g) \in G(M)$ imply $f \in M(x)$. Let B be a monotone mapping of C into H and let $N_C\overline{y}$ be the *normal cone* to C at $\overline{y} \in C$; that is, $N_C\overline{y} = \{w \in H : \langle u - \overline{y}, w\rangle \leq 0, \forall u \in C\}$, and define

$$M\overline{y} = \begin{cases} B\overline{y} + N_C\overline{y}, & \overline{y} \in C; \\ \emptyset, & \overline{y} \notin C. \end{cases} \tag{9}$$

Then, M is the *maximal monotone* and $\theta \in M\overline{y}$ if and only if $\overline{y} \in \mathrm{VI}(C, B)$; see [17].

Let $M : H \to 2^H$ be a set-valued maximal monotone mapping; then, the single-valued mapping $J_{M,\lambda} : H \to H$ defined by

$$J_{M,\lambda} x^* = (I + \lambda M)^{-1} x^*, \quad x^* \in H \tag{10}$$

is called the *resolvent operator* associated with M, where λ is any positive number and I is the identity mapping. The following characterizes the resolvent operator.

(R1) The resolvent operator $J_{M,\lambda}$ is single-valued and nonexpansive for all $\lambda > 0$; that is,

$$\|J_{M,\lambda}(x) - J_{M,\lambda}(y)\| \leq \|x - y\|, \quad \forall x, y \in H,\ \forall \lambda > 0. \tag{11}$$

(R2) The resolvent operator $J_{M,\lambda}$ is 1-inverse-strongly monotone; see [18]; that is,

$$\begin{aligned} &\|J_{M,\lambda}(x) - J_{M,\lambda}(y)\|^2 \\ &\quad \leq \langle x - y, J_{M,\lambda}(x) - J_{M,\lambda}(y)\rangle, \quad \forall x, y \in H. \end{aligned} \tag{12}$$

(R3) The solution of problem (8) is a fixed point of the operator $J_{M,\lambda}(I - \lambda B)$ for all $\lambda > 0$; see also [19]; that is,

$$I(B, M) = F(J_{M,\lambda}(I - \lambda B)), \quad \forall \lambda > 0. \tag{13}$$

(R4) If $0 < \lambda \leq 2\beta$, then the mapping $J_{M,\lambda}(I - \lambda B) : H \to H$ is nonexpansive.

(R5) $I(B, M)$ is closed and convex.

Let A be a strongly positive linear bounded operator on H; that is, there exists a constant $\overline{\gamma} > 0$ with property

$$\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2, \quad \forall x \in H. \tag{14}$$

A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x \in F(T)} \frac{1}{2} \langle Ax, x\rangle - h(x), \tag{15}$$

where A is a strongly positive linear bounded operator and h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

In 2007, Plubtieng and Punpaeng [20] proposed the following iterative algorithm:

$$\begin{gathered} F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in H, \\ x_{n+1} = \epsilon_n \gamma f(x_n) + (I - \epsilon_n A) T u_n. \end{gathered} \tag{16}$$

They proved that if the sequences $\{\epsilon_n\}$ and $\{r_n\}$ of parameters satisfy appropriate conditions, then the sequences $\{x_n\}$ and $\{u_n\}$ both converge to the unique solution z of the variational inequality

$$\langle (A - \gamma f) z, x - z\rangle \geq 0, \quad \forall x \in F(T) \cap \mathrm{EP}(F), \tag{17}$$

which is the optimality condition for the minimization problem

$$\min_{x\in F(T)\cap \mathrm{EP}(F)} \frac{1}{2}\langle Ax, x\rangle - h(x), \tag{18}$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

In 2009, Peng and Yao [21] introduced an iterative algorithm based on extragradient method which solves the problem for finding a common element of the set of solutions of a mixed equilibrium problem, the set of fixed points of a family of finitely nonexpansive mappings, and the set of the variational inequality for a monotone, Lipschitz continuous mapping in a real Hilbert space. The sequences generated by $v \in C$ are

$$\begin{gathered} x_1 = x \in C, \\ F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \\ \forall y \in C, \\ y_n = P_C(u_n - \gamma_n B u_n), \\ x_{n+1} = \alpha_n v + \beta_n x_n + (1 - \alpha_n - \beta_n) W_n P_C(u_n - \lambda_n B y_n), \end{gathered} \tag{19}$$

for all $n \geq 1$, where W_n is W-mapping. They proved the strong convergence theorems under some mild conditions.

In 2010, Qin et al. [22] introduced an iterative method for finding solutions of a generalized equilibrium problem, the set of fixed points of a family of nonexpansive mappings, and the common variational inclusions. The sequences generated by $x_1 \in C$ and $\{x_n\}$ are a sequence generated by

$$\begin{gathered} F(u_n, y) + \langle A_3 x_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \\ \forall y \in C, \\ z_n = P_C(u_n - \lambda_n A_2 u_n), \\ y_n = P_C(z_n - \eta_n A_1 z_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n y_n, \quad \forall n \geq 1, \end{gathered} \tag{20}$$

where f is a contraction and A_i is inverse-strongly monotone mappings for $i = 1, 2, 3$ and W_n is called a W-mapping generated by $S_n, S_{n_1}, \ldots, S_1$ and $\gamma_n, \gamma_{n-1}, \ldots, \gamma_1$. They proved the strong convergence theorems under some mild conditions. Liou [23] introduced an algorithm for finding a common element of the set of solutions of a mixed equilibrium problem and the set of variational inclusion in a real Hilbert space. The sequences generated by $x_0 \in C$ are

$$\begin{aligned} &F(u_n, y) + \varphi(y) - \varphi(u_n) \\ &\quad + \frac{1}{r}\langle y - u_n, u_n - (x_n - rQx_n)\rangle \geq 0, \quad \forall y \in C, \end{aligned} \tag{21}$$

$$x_{n+1} = P_C[(1 - \alpha_n A) J_{M,\lambda}(u_n - \lambda B u_n)],$$

for all $n \geq 1$, where A is a strongly positive bounded linear operator and B, Q are inverse-strongly monotone. They proved the strong convergence theorems under some suitable conditions.

Next, Petrot et al. [24] introduced the new following iterative process for finding the set of solutions of quasi-variational inclusion problem and the set of fixed point of a nonexpansive mapping. The sequence is generated by

$$\begin{gathered} x_0 \in H, \quad \text{chosen arbitrary}, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S z_n, \\ z_n = J_{M,\lambda}(y_n - \lambda A y_n), \\ y_n = J_{M,\rho}(x_n - \rho A x_n), \end{gathered} \tag{22}$$

for all $n \in \mathbb{N} \cup \{0\}$, where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0, 1]$ and $\lambda \in (0, 2\alpha]$. They proved that $\{x_n\}$ generated by (22) converges strongly to z_0 which is the unique solution in $F(S) \cap I(A, M)$.

In 2011, Jitpeera and Kumam [25] introduced a shrinking projection method for finding the common element of the common fixed points of nonexpansive semigroups, the set of common fixed point for an infinite family, the set of solutions of a system of mixed equilibrium problems, and the set of solution of the variational inclusion problem. Let $\{x_n\}$, $\{y_n\}$, $\{v_n\}$, $\{z_n\}$, and $\{u_n\}$ be sequences generated by $x_0 \in C, C_1 = C$, $x_1 = P_{C_1}x_0$, $u_n \in C$, and

$$\begin{gathered} x_0 = x \in C \quad \text{chosen arbitrary}, \\ u_n = K^{F_N}_{r_{N,n}} K^{F_{N-1}}_{r_{N-1,n}} K^{F_{N-2}}_{r_{N-2,n}} \cdots K^{F_2}_{r_{2,n}} K^{F_1}_{r_{1,n}} x_n, \\ y_n = J_{M_2,\delta_n}(u_n - \delta_n B u_n), \\ v_n = J_{M_1,\lambda_n}(y_n - \lambda_n A y_n), \\ z_n = \alpha_n v_n + (1 - \alpha_n)\frac{1}{t_n}\int_0^{t_n} S(s) W_n v_n ds, \\ C_{n+1} = \Bigg\{ z \in C_n : \|z_n - z\|^2 \leq \|x_n - z\|^2 - \alpha_n(1 - \alpha_n) \\ \times \left\| v_n - \frac{1}{t_n}\int_0^{t_n} S(s) W_n v_n ds\right\|^2 \Bigg\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \in \mathbb{N}, \end{gathered} \tag{23}$$

where $K^{F_k}_{r_k} : C \to C$, $k = 1, 2, \ldots, N$. We proved the strong convergence theorem under certain appropriate conditions.

In this paper, motivated by the above results, we introduce a new iterative method for finding a common element of the set of solutions for mixed equilibrium problems, the set of solutions of a system of quasi-variational inclusions, and the set of fixed points of an infinite family of nonexpansive mappings in a real Hilbert space. Then, we prove strong convergence theorems which are connected with [5, 26–29]. Our results extend and improve the corresponding results of

Jitpeera and Kumam [25], Liou [23], Plubtieng and Punpaeng [20], Petrot et al. [24], Peng and Yao [21], Qin et al. [22], and some authors.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ and let C be a nonempty closed convex subset of H. Then,

$$\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle,$$
$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda) \times \|x-y\|^2, \quad \forall x, y \in H, \ \lambda \in [0,1]. \tag{24}$$

For every point $x \in H$, there exists a unique *nearest point* in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \le \|x - y\|, \quad \forall y \in C. \tag{25}$$

P_C is called the *metric projection* of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x-y, P_C x - P_C y\rangle \ge \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \tag{26}$$

Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x\rangle \le 0, \tag{27}$$

$$\|x-y\|^2 \ge \|x - P_C x\|^2 + \|y - P_C x\|^2, \quad \forall x \in H, \ y \in C. \tag{28}$$

Let B be a monotone mapping of C into H. In the context of the variational inequality problem, the characterization of projection (27) implies the following:

$$u \in \mathrm{VI}(C,B) \Longleftrightarrow u = P_C(u - \lambda Bu), \quad \lambda > 0. \tag{29}$$

It is also known that H satisfies the Opial condition [30]; that is, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \tag{30}$$

holds for every $y \in H$ with $x \ne y$.

For the infinite family of nonexpansive mappings of $T_1, T_2, \ldots$, and sequence $\{\lambda_i\}_{i=1}^{\infty}$ in $[0,1)$, see [31]; we define the mapping W_n of C into itself as follows:

$$\begin{aligned}
U_{n,0} &= I,\\
U_{n,1} &= \lambda_1 T_1 U_{n,0} + (1-\lambda_1) U_{n,0},\\
U_{n,2} &= \lambda_2 T_2 U_{n,1} + (1-\lambda_2) U_{n,1},\\
&\vdots\\
U_{n,N-1} &= \lambda_{N-1} T_{N-1} U_{n,N-2} + (1-\lambda_{N-1}) U_{n,N-2},\\
W_n = U_{n,N} &= \lambda_N T_N U_{n,N-1} + (1-\lambda_N) U_{n,N-1}.
\end{aligned} \tag{31}$$

Lemma 1 (Shimoji and Takahashi [32]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\mathcal{T} = \{T_i\}_{i=1}^{N}$ be a family of infinitely nonexpanxive mappings with $F(\mathcal{T}) = \bigcap_{i=1}^{\infty} F(T_i) \ne \emptyset$ and let $\{\lambda_i\}$ be a real sequence such that $0 < \lambda_i \le b < 1$ for every $i \ge 1$. Then*

(1) *W_n is nonexpansive and $F(W_n) = \bigcap_{i=1}^{n} F(T_i)$ for each $n \ge 1$;*

(2) *for each $x \in C$ and for each positive integer k, the limit $\lim_{n\to\infty} U_{n,k} x$ exists;*

(3) *the mapping $W : C \to C$ defined by $Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1} x$ is a nonexpansive mapping satisfying $F(W) = F(\mathcal{T})$ and it is called the W-mapping generated by $T_1, T_2, \ldots$, and $\lambda_1, \lambda_2, \ldots$;*

(4) *if K is any bounded subset of C, then $\lim_{n\to\infty} \sup_{x\in K} \|Wx - W_n x\| = 0$.*

For solving the mixed equilibrium problem, let us give the following assumptions for a bifunction $F : C \times C \to \mathbb{R}$ and a proper extended real-valued function $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ satisfies the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) F is monotone; that is, $F(x,y) + F(y,x) \le 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$, $\lim_{t\to 0} F(tz + (1-t)x, y) \le F(x,y)$;

(A4) for each $x \in C$, $y \mapsto F(x,y)$ is convex and lower semicontinuous;

(A5) for each $y \in C$, $x \mapsto F(x,y)$ is weakly upper semicontinuous;

(B1) for each $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$F(z, y_x) + \varphi(y_x) + \frac{1}{r}\langle y_x - z, z - x\rangle < \varphi(z); \tag{32}$$

(B2) C is a bounded set.

We need the following lemmas for proving our main results.

Lemma 2 (Peng and Yao [21]). *Let C be a nonempty closed convex subset of H. Let $F : C \times C \to \mathbb{R}$ be a bifunction that satisfies (A1)–(A5) and let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \Big\{ z \in C : F(z,y) + \varphi(y) + \frac{1}{r}\langle y - z, z - x\rangle \ge \varphi(z), \forall y \in C \Big\}, \tag{33}$$

for all $x \in H$. Then, the following hold:

(1) *for each $x \in H, T_r(x) \ne \emptyset$;*

(2) *T_r is single-valued;*

(3) T_r *is firmly nonexpansive; that is, for any* $x, y \in H$, $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle$;

(4) $F(T_r) = MEP(F, \varphi)$;

(5) $MEP(F, \varphi)$ *is closed and convex.*

Lemma 3 (Xu [33]). *Assume* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \alpha_n) a_n + \delta_n, \quad n \geq 0, \tag{34}$$

where $\{\alpha_n\}$ *is a sequence in* $(0, 1)$ *and* $\{\delta_n\}$ *is a sequence in* $\mathbb{R}$ *such that*

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(2) $\limsup_{n\to\infty}(\delta_n/\alpha_n) \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 4 (Suzuki [34]). *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space* X *and let* $\{\beta_n\}$ *be a sequence in* $[0, 1]$ *with* $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$. *Suppose* $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ *for all integers* $n \geq 0$ *and* $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. *Then,* $\lim_{n\to\infty}\|y_n - x_n\| = 0$.

Lemma 5 (Marino and Xu [35]). *Assume* A *is a strongly positive linear bounded operator on a Hilbert space* H *with coefficient* $\overline{\gamma} > 0$ *and* $0 < \rho \leq \|A\|^{-1}$. *Then,* $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.

Lemma 6. *For given* $x^*, y^* \in C \times C$, (x^*, y^*) *is a solution of problem* (6) *if and only if* x^* *is a fixed point of the mapping* $G : C \to C$ *defined by*

$$G(x) = J_{M_1,\lambda}\left[J_{M_2,\mu}(x - \mu E_2 x) - \lambda E_1 J_{M_2,\mu}(x - \mu E_2 x)\right], \quad \forall x \in C, \tag{35}$$

where $y^* = J_{M_2,\mu}(x - \mu E_2 x)$, λ, μ *are positive constants, and* $E_1, E_2 : C \to H$ *are two mappings.*

Proof.

$$\begin{aligned} \theta &\in x^* - y^* + \lambda(E_1 y^* + M_1 x^*), \\ \theta &\in y^* - x^* + \mu(E_2 x^* + M_2 y^*) \end{aligned} \tag{36}$$

$\Leftrightarrow$

$$\begin{aligned} x^* &= J_{M_1,\lambda}(y^* - \lambda E_1 y^*), \\ y^* &= J_{M_2,\mu}(x^* - \mu E_2 x^*) \end{aligned} \tag{37}$$

$\Leftrightarrow$

$$\begin{aligned} G(x^*) = J_{M_1,\lambda}&\left[J_{M_2,\mu}(x^* - \mu E_2 x^*)\right. \\ &\left. - \lambda E_1 J_{M_2,\mu}(x^* - \mu E_2 x^*)\right] = x^*. \end{aligned} \tag{38}$$

This completes the proof. □

Now, we prove the following lemmas which will be applied in the main theorem.

Lemma 7. *Let* $G : C \to C$ *be defined as in Lemma 6. If* $E_1, E_2 : C \to H$ *is* η_1, η_2*-inverse-strongly monotone and* $\lambda \in (0, 2\eta_1)$, *and* $\mu \in (0, 2\eta_2)$, *respectively, then* G *is nonexpansive.*

Proof. For any $x, y \in C$ and $\lambda \in (0, 2\eta_1)$, $\mu \in (0, 2\eta_2)$, we have

$$\begin{aligned} &\|G(x) - G(y)\|^2 \\ &= \left\| J_{M_1,\lambda}\left[J_{M_2,\mu}(x - \mu E_2 x) - \lambda E_1 J_{M_2,\mu}(x - \mu E_2 x)\right] \right. \\ &\qquad \left. - J_{M_1,\lambda}\left[J_{M_2,\mu}(y - \mu E_2 y) - \lambda E_1 J_{M_2,\mu}(y - \mu E_2 y)\right] \right\|^2 \\ &\leq \left\| \left[J_{M_2,\mu}(x - \mu E_2 x) - \lambda E_1 J_{M_2,\mu}(x - \mu E_2 x)\right] \right. \\ &\qquad \left. - \left[J_{M_2,\mu}(y - \mu E_2 y) - \lambda E_1 J_{M_2,\mu}(y - \mu E_2 y)\right] \right\|^2 \\ &= \left\| \left[J_{M_2,\mu}(x - \mu E_2 x) - J_{M_2,\mu}(y - \mu E_2 y)\right] \right. \\ &\qquad \left. - \lambda\left[E_1 J_{M_2,\mu}(x - \mu E_2 x) - E_1 J_{M_2,\mu}(y - \mu E_2 y)\right] \right\|^2 \\ &= \left\| J_{M_2,\mu}(x - \mu E_2 x) - J_{M_2,\mu}(y - \mu E_2 y) \right\|^2 \\ &\quad - 2\lambda \left\langle J_{M_2,\mu}(x - \mu E_2 x) - J_{M_2,\mu}(y - \mu E_2 y), \right. \\ &\qquad\qquad \left. E_1 J_{M_2,\mu}(x - \mu E_2 x) - E_1 J_{M_2,\mu}(y - \mu E_2 y) \right\rangle \\ &\quad + \lambda^2 \left\| E_1 J_{M_2,\mu}(x - \mu E_2 x) - E_1 J_{M_2,\mu}(y - \mu E_2 y) \right\|^2 \\ &\leq \left\| J_{M_2,\mu}(x - \mu E_2 x) - J_{M_2,\mu}(y - \mu E_2 y) \right\|^2 \\ &\quad - 2\lambda\eta_1 \left\| E_1 J_{M_2,\mu}(x - \mu E_2 x) - E_1 J_{M_2,\mu}(y - \mu E_2 y) \right\|^2 \\ &\quad + \lambda^2 \left\| E_1 J_{M_2,\mu}(x - \mu E_2 x) - E_1 J_{M_2,\mu}(y - \mu E_2 y) \right\|^2 \\ &= \left\| J_{M_2,\mu}(x - \mu E_2 x) - J_{M_2,\mu}(y - \mu E_2 y) \right\|^2 \\ &\quad + \lambda(\lambda - 2\eta_1) \left\| E_1 J_{M_2,\mu}(x - \mu E_2 x) - E_1 J_{M_2,\mu}(y - \mu E_2 y) \right\|^2 \\ &\leq \left\| J_{M_2,\mu}(x - \mu E_2 x) - J_{M_2,\mu}(y - \mu E_2 y) \right\|^2 \\ &\leq \left\| (x - \mu E_2 x) - (y - \mu E_2 y) \right\|^2 \\ &= \left\| (x - y) - \mu(E_2 x - E_2 y) \right\|^2 \\ &= \|x - y\|^2 - 2\mu\langle x - y, E_2 x - E_2 y\rangle + \mu^2\|E_2 x - E_2 y\|^2 \\ &\leq \|x - y\|^2 - 2\eta_2\mu\|E_2 x - E_2 y\|^2 + \mu^2\|E_2 x - E_2 y\|^2 \\ &= \|x - y\|^2 + \mu(\mu - 2\eta_2)\|E_2 x - E_2 y\|^2 \\ &\leq \|x - y\|^2. \end{aligned} \tag{39}$$

This shows that G is nonexpansive on C. □

3. Main Results

In this section, we show a strong convergence theorem for finding a common element of the set of solutions for mixed equilibrium problems, the set of solutions of a system of quasi-variational inclusion, and the set of fixed points of a infinite family of nonexpansive mappings in a real Hilbert space.

Theorem 8. *Let C be a nonempty closed convex subset of a real Hilbert Space H. Let F be a bifunction of $C \times C$ into real numbers $\mathbb{R}$ satisfying (A1)–(A5) and let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuos and convex function. Let $T_i : C \to C$ be nonexpansive mappings for all $i = 1, 2, 3, \ldots$, such that $\Theta := \cap_{i=1}^{\infty} F(T_i) \cap SQVI(B_1, M_1, B_2, M_2) \cap MEP(F, \varphi) \neq \emptyset$. Let f be a contraction of C into itself with coefficient $\alpha \in (0, 1)$ and let Q, E_1, E_2 be δ, η_1, η_2-inverse-strongly monotone mapping of C into H. Let A be a strongly positive bounded linear self-adjoint on H with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \overline{\gamma}/\alpha$, let $M_1, M_2 : H \to 2^H$ be a maximal monotone mapping. Assume that either B_1 or B_2 holds and let W_n be the W-mapping defined by* (31). *Let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{u_n\}$ be sequences generated by $x_0 \in C$, $u_n \in C$, and*

$$\begin{aligned} & F(u_n, y) + \varphi(y) - \varphi(u_n) \\ & \quad + \frac{1}{r} \langle y - u_n, u_n - (x_n - rQx_n) \rangle \geq 0, \quad \forall y \in C, \\ & \qquad z_n = J_{M_2,\mu}(u_n - \mu E_2 u_n), \\ & \qquad y_n = J_{M_1,\lambda}(z_n - \lambda E_1 z_n), \\ & x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n) I - \alpha_n A) W_n y_n, \\ & \qquad \forall n \geq 0, \end{aligned} \tag{40}$$

where $\{\alpha_n\}$ and $\{\beta_n\} \subset (0, 1)$, $\lambda \in (0, 2\eta_1)$, $\mu \in (0, 2\eta_2)$, and $r \in (0, 2\delta)$ satisfy the following conditions:

(C1) $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty} \alpha_n = 0$,

(C2) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

(C3) $\lim_{n\to\infty} |\lambda_{n,i} - \lambda_{n-1,i}| = 0$, $\forall i = 1, 2, \ldots, N$.

Then, $\{x_n\}$ converges strongly to $x^* \in \Theta$, where $x^* = P_\Theta(\gamma f + I - A)(x^*)$, P_Θ is the metric projection of H onto Θ and (x^*, y^*), where $y^* = J_{M_2,\mu}(x^* - \mu E_2 x^*)$ is solution to the problem (6).

Proof. Let $x^* \in \Theta$; that is $T_r(x^* - rQx^*) = J_{M_1,\lambda}[J_{M_2,\mu}(x^* - \mu B_2 x^*) - \lambda B_1 J_{M_2,\mu}(x^* - \mu B_2 x^*)] = T_i(x^*) = x^*$, $i \geq 1$. Putting $y^* = J_{M_2,\mu}(x^* - \mu E_2 x^*)$, one can see that $x^* = J_{M_1,\lambda}(y^* - \lambda B_1 y^*)$.

We divide our proofs into the following steps:

(1) sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{u_n\}$ are bounded;

(2) $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$;

(3) $\lim_{n\to\infty} \|Qx_n - Qx^*\| = 0$, $\lim_{n\to\infty} \|E_1 z_n - E_1 x^*\| = 0$ and $\lim_{n\to\infty} \|E_2 u_n - E_2 x^*\| = 0$;

(4) $\lim_{n\to\infty} \|x_n - Wx_n\| = 0$;

(5) $\limsup_{n\to\infty} \langle \gamma f(x^*) - Ax^*, x_n - x^* \rangle \leq 0$, where $x^* = P_\Theta(\gamma f + I - A)x^*$;

(6) $\lim_{n\to\infty} \|x_n - x^*\| = 0$.

Step 1. From conditions (C1) and (C2), we may assume that $\alpha_n \leq (1 - \beta_n)\|A\|^{-1}$. By the same argument as that in [9], we can deduce that $(1 - \beta_n)I - \alpha_n A$ is positive and $\|(1 - \beta_n)I - \alpha_n A\| \leq 1 - \beta_n - \alpha_n \overline{\gamma}$. For all $x, y \in C$ and $r \in (0, 2\delta)$. since Q is a δ-inverse-strongly monotone and B_1, B_2 are η_1, η_2-inverse-strongly monotone, we have

$$\begin{aligned} & \|(I - rQ)x - (I - rQ)y\|^2 \\ & \quad = \|(x - y) - r(Qx - Qy)\|^2 \\ & \quad = \|x - y\|^2 - 2r\langle x - y, Qx - Qy \rangle + r^2 \|Qx - Qy\|^2 \\ & \quad \leq \|x - y\|^2 - 2r\delta \|Qx - Qy\|^2 + r^2 \|Qx - Qy\|^2 \\ & \quad = \|x - y\|^2 + r(r - 2\delta) \|Qx - Qy\|^2 \\ & \quad \leq \|x - y\|^2. \end{aligned} \tag{41}$$

It follows that $\|(I - rQ)x - (I - rQ)y\| \leq \|x - y\|$; hence $I - rQ$ is nonexpansive.

In the same way, we conclude that $\|(I - \lambda E_1)x - (I - \lambda E_1)y\| \leq \|x - y\|$ and $\|(I - \mu E_2)x - (I - \mu E_2)y\| \leq \|x - y\|$; hence $I - \lambda E_1$, $I - \mu E_2$ are nonexpansive. Let $y_n = J_{M_1,\lambda}(z_n - \lambda E_1 z_n)$, $n \geq 0$. It follows that

$$\begin{aligned} \|y_n - x^*\| & = \|J_{M_1,\lambda}(z_n - \lambda E_1 z_n) - J_{M_1,\lambda}(y^* - \lambda E_1 y^*)\| \\ & \leq \|(z_n - \lambda E_1 z_n) - (y^* - \lambda E_1 y^*)\| \\ & \leq \|z_n - y^*\|, \\ \|z_n - y^*\| & = \|J_{M_2,\mu}(u_n - \mu E_2 u_n) - J_{M_2,\mu}(x^* - \mu E_2 x^*)\| \\ & \leq \|(u_n - \mu E_2 u_n) - (x^* - \mu E_2 x^*)\| \\ & \leq \|u_n - x^*\|. \end{aligned} \tag{42}$$

By Lemma 2, we have $u_n = T_r(x_n - rQx_n)$ for all $n \geq 0$, $\forall x, y \in C$. Then, for $r \in (0, 2\delta)$, we obtain

$$\begin{aligned} \|u_n - x^*\|^2 & = \|T_r(x_n - rQx_n) - T_r(x^* - rQx^*)\|^2 \\ & \leq \|(x_n - rQx_n) - (x^* - rQx^*)\|^2 \\ & \leq \|x_n - x^*\|^2 + r(r - 2\delta) \|Qx_n - Qx^*\|^2 \\ & \leq \|x_n - x^*\|^2. \end{aligned} \tag{43}$$

Hence, we have

$$\|y_n - x^*\| \leq \|x_n - x^*\|. \tag{44}$$

From (40) and (44), we deduce that

$$\begin{aligned}\|x_{n+1}-x^*\| &= \|\alpha_n(\gamma f(x_n)-Ax^*)+\beta_n(x_n-x^*)\\&\quad+((1-\beta_n)I-\alpha_nA)(W_ny_n-x^*)\|\\&\le \alpha_n\|\gamma f(x_n)-Ax^*\|+\beta_n\|x_n-x^*\|\\&\quad+(1-\beta_n-\alpha_n\overline{\gamma})\|y_n-x^*\|\\&\le \alpha_n\|\gamma f(x_n)-Ax^*\|+\beta_n\|x_n-x^*\|\\&\quad+(1-\beta_n-\alpha_n\overline{\gamma})\|x_n-x^*\|\\&\le \alpha_n\gamma\|f(x_n)-f(x^*)\|+\alpha_n\|\gamma f(x^*)-Ax^*\|\\&\quad+(1-\alpha_n\overline{\gamma})\|x_n-x^*\|\\&\le \alpha_n\gamma\alpha\|x_n-x^*\|+\alpha_n\|\gamma f(x^*)-Ax^*\|\\&\quad+(1-\alpha_n\overline{\gamma})\|x_n-x^*\|\\&=(1-\alpha_n(\overline{\gamma}-\gamma\alpha))\|x_n-x^*\|\\&\quad+\alpha_n(\overline{\gamma}-\gamma\alpha)\frac{\|\gamma f(x^*)-Ax^*\|}{(\overline{\gamma}-\gamma\alpha)}\\&\le \max\left\{\|x_n-x^*\|,\frac{\|\gamma f(x^*)-Ax^*\|}{(\overline{\gamma}-\gamma\alpha)}\right\}.\end{aligned}\tag{45}$$

It follows by mathematical induction that

$$\|x_{n+1}-x^*\|\le\max\left\{\|x_0-x^*\|,\frac{\|\gamma f(x^*)-Ax^*\|}{(\overline{\gamma}-\gamma\alpha)}\right\},\quad n\ge 0.\tag{46}$$

Hence, $\{x_n\}$ is bounded and also $\{u_n\}$, $\{z_n\}$, $\{y_n\}$, $\{W_ny_n\}$, $\{AW_ny_n\}$, and $\{fx_n\}$ are all bounded.

Step 2. We show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.

Putting $t_n=(x_{n+1}-\beta_nx_n)/(1-\beta_n)=(\alpha_n\gamma f(x_n)+((1-\beta_n)I-\alpha_nA)W_ny_n)/(1-\beta_n)$, we get $x_{n+1}=(1-\beta_n)t_n+\beta_nx_n$, $n\ge 1$. We note that

$$\begin{aligned}t_{n+1}-t_n&=\frac{\alpha_{n+1}\gamma f(x_{n+1})+((1-\beta_{n+1})I-\alpha_{n+1}A)W_{n+1}y_{n+1}}{1-\beta_{n+1}}\\&\quad-\frac{\alpha_n\gamma f(x_n)+((1-\beta_n)I-\alpha_nA)W_ny_n}{1-\beta_n}\\&=\frac{\alpha_{n+1}}{1-\beta_{n+1}}\gamma f(x_{n+1})-\frac{\alpha_n}{1-\beta_n}\gamma f(x_n)\\&\quad+W_{n+1}y_{n+1}-W_ny_n\\&\quad-\frac{\alpha_{n+1}}{1-\beta_{n+1}}AW_{n+1}y_{n+1}+\frac{\alpha_n}{1-\beta_n}AW_ny_n\\&=\frac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma f(x_{n+1})-AW_{n+1}y_{n+1})\\&\quad+\frac{\alpha_n}{1-\beta_n}(AW_ny_n-\gamma f(x_n))\\&\quad+W_{n+1}y_{n+1}-W_{n+1}y_n+W_{n+1}y_n-W_ny_n.\end{aligned}\tag{47}$$

It follows that

$$\begin{aligned}&\|t_{n+1}-t_n\|-\|x_{n+1}-x_n\|\\&\quad\le\frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f(x_{n+1})\|+\|AW_{n+1}y_{n+1}\|)\\&\qquad+\frac{\alpha_n}{1-\beta_n}(\|AW_ny_n\|+\|\gamma f(x_n)\|)\\&\qquad+\|W_{n+1}y_{n+1}-W_{n+1}y_n\|\\&\qquad+\|W_{n+1}y_n-W_ny_n\|-\|x_{n+1}-x_n\|\\&\quad\le\frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f(x_{n+1})\|+\|AW_{n+1}y_{n+1}\|)\\&\qquad+\frac{\alpha_n}{1-\beta_n}(\|AW_ny_n\|+\|\gamma f(x_n)\|)\\&\qquad+\|y_{n+1}-y_n\|+\|W_{n+1}y_n-W_ny_n\|\\&\qquad-\|x_{n+1}-x_n\|.\end{aligned}\tag{48}$$

By the definition of W_n,

$$\begin{aligned}&\|W_{n+1}y_n-W_ny_n\|\\&\quad=\|\lambda_{n+1,N}T_NU_{n+1,N-1}y_n+(1-\lambda_{n+1,N})y_n\\&\qquad-\lambda_{n,N}T_NU_{n,N-1}y_n-(1-\lambda_{n,N})y_n\|\\&\quad\le|\lambda_{n+1,N}-\lambda_{n,N}|\|y_n\|\\&\qquad+\|\lambda_{n+1,N}T_NU_{n+1,N-1}y_n-\lambda_{n,N}T_NU_{n,N-1}y_n\|\\&\quad\le|\lambda_{n+1,N}-\lambda_{n,N}|\|y_n\|\\&\qquad+\|\lambda_{n+1,N}(T_NU_{n+1,N-1}y_n-T_NU_{n,N-1}y_n)\|\\&\qquad+|\lambda_{n+1,N}-\lambda_{n,N}|\|T_NU_{n,N-1}y_n\|\\&\quad\le 2M|\lambda_{n+1,N}-\lambda_{n,N}|\\&\qquad+\lambda_{n+1,N}\|U_{n+1,N-1}y_n-U_{n,N-1}y_n\|,\end{aligned}\tag{49}$$

where M is an approximate constant such that $M\ge\max\{\sup_{n\ge1}\{\|y_n\|\},\sup_{n\ge1}\{\|T_mU_{n,m-1}y_n\|\}\mid m=1,2,\ldots,N\}$. Since $0<\lambda_{n_i}\le 1$ for all $n\ge 1$ and $i=1,2,\ldots,N$, we compute

$$\begin{aligned}&\|U_{n+1,N-1}y_n-U_{n,N-1}y_n\|\\&\quad=\|\lambda_{n+1,N-1}T_{N-1}U_{n+1,N-2}y_n+(1-\lambda_{n+1,N-1})y_n\\&\qquad-\lambda_{n,N-1}T_{N-1}U_{n,N-2}y_n-(1-\lambda_{n,N-1})y_n\|\end{aligned}$$

$$\begin{aligned}
&\leq |\lambda_{n+1,N-1}-\lambda_{n,N-1}|\,\|y_n\| \\
&\quad+\|\lambda_{n+1,N-1}T_{N-1}U_{n+1,N-2}y_n-\lambda_{n,N-1}T_{N-1}U_{n,N-2}y_n\| \\
&\leq |\lambda_{n+1,N-1}-\lambda_{n,N-1}|\,\|y_n\| \\
&\quad+\|\lambda_{n+1,N-1}(T_{N-1}U_{n+1,N-2}y_n-T_{N-1}U_{n,N-2}y_n)\| \\
&\quad+|\lambda_{n+1,N-1}-\lambda_{n,N-1}|\,\|T_{N-1}U_{n,N-2}y_n\| \\
&\leq 2M|\lambda_{n+1,N-1}-\lambda_{n,N-1}|+\|U_{n+1,N-2}y_n-U_{n,N-2}y_n\|.
\end{aligned} \tag{50}$$

It follows that

$$\begin{aligned}
&\|U_{n+1,N-1}y_n-U_{n,N-1}y_n\| \\
&\quad\leq 2M|\lambda_{n+1,N-1}-\lambda_{n,N-1}|+2M|\lambda_{n+1,N-2}-\lambda_{n,N-2}| \\
&\qquad+\|U_{n+1,N-3}y_n-U_{n,N-3}y_n\| \\
&\quad\leq 2M\sum_{i=2}^{N-1}|\lambda_{n+1,i}-\lambda_{n,i}|+\|U_{n+1,1}y_n-U_{n,1}y_n\| \\
&\quad= 2M\sum_{i=2}^{N-1}|\lambda_{n+1,i}-\lambda_{n,i}| \\
&\qquad\qquad+\|\lambda_{n+1,1}T_1y_n+(1-\lambda_{n+1,1})y_n \\
&\qquad\qquad\quad-\lambda_{n,1}T_1y_n-(1-\lambda_{n,1})y_n\| \\
&\quad\leq 2M\sum_{i=1}^{N-1}|\lambda_{n+1,i}-\lambda_{n,i}|.
\end{aligned} \tag{51}$$

Substituting (51) into (49),

$$\begin{aligned}
&\|W_{n+1}y_n-W_ny_n\| \\
&\quad\leq 2M|\lambda_{n+1,N}-\lambda_{n,N}|+2\lambda_{n+1,N}M\sum_{i=1}^{N-1}|\lambda_{n+1,i}-\lambda_{n,i}| \\
&\quad\leq 2M\sum_{i=1}^{N}|\lambda_{n+1,i}-\lambda_{n,i}|.
\end{aligned} \tag{52}$$

We note that

$$\begin{aligned}
&\|y_{n+1}-y_n\| \\
&\quad=\|J_{M_1,\lambda}(z_{n+1}-\lambda E_1z_{n+1})-J_{M_1,\lambda}(z_n-\lambda E_1z_n)\| \\
&\quad\leq\|(z_{n+1}-\lambda E_1z_{n+1})-(z_n-\lambda E_1z_n)\| \\
&\quad\leq\|z_{n+1}-z_n\| \\
&\quad=\|J_{M_2,\mu}(u_{n+1}-\mu E_2u_{n+1})-J_{M_2,\mu}(u_n-\mu E_2u_n)\| \\
&\quad\leq\|(u_{n+1}-\mu E_2u_{n+1})-(u_n-\mu E_2u_n)\| \\
&\quad\leq\|u_{n+1}-u_n\| \\
&\quad=\|T_r(x_{n+1}-rDx_{n+1})-T_r(x_n-rDx_n)\| \\
&\quad\leq\|(x_{n+1}-rDx_{n+1})-(x_n-rDx_n)\| \\
&\quad\leq\|x_{n+1}-x_n\|.
\end{aligned} \tag{53}$$

Applying (52) and (53) in (48), we get

$$\begin{aligned}
&\|t_{n+1}-t_n\|-\|x_{n+1}-x_n\| \\
&\quad\leq\frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma f(x_{n+1})\|+\|AW_{n+1}y_{n+1}\|) \\
&\qquad+\frac{\alpha_n}{1-\beta_n}(\|AW_ny_n\|+\|\gamma f(x_n)\|)+\|x_{n+1}-x_n\| \\
&\qquad+2M\sum_{i=1}^{N}|\lambda_{n+1,i}-\lambda_{n,i}|-\|x_{n+1}-x_n\|.
\end{aligned} \tag{54}$$

By conditions (C1)–(C3), imply that

$$\limsup_{n\to\infty}(\|t_{n+1}-t_n\|-\|x_{n+1}-x_n\|)\leq 0. \tag{55}$$

Hence, by Lemma 4, we obtain

$$\lim_{n\to\infty}\|t_n-x_n\|=0. \tag{56}$$

It follows that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\beta_n)\|t_n-x_n\|=0. \tag{57}$$

We obtain that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{58}$$

Step 3. We can rewrite (40) as $x_{n+1}=\alpha_n(\gamma f(x_n)-AW_ny_n)+\beta_n(x_n-W_ny_n)+W_ny_n$. We observe that

$$\begin{aligned}
\|x_n-W_ny_n\|&\leq\|x_n-x_{n+1}\|+\|x_{n+1}-W_ny_n\| \\
&\leq\|x_n-x_{n+1}\|+\alpha_n\|\gamma f(x_n)-AW_ny_n\| \\
&\quad+\beta_n\|x_n-W_ny_n\|;
\end{aligned} \tag{59}$$

it follows that

$$\begin{aligned}
&\|x_n-W_ny_n\| \\
&\quad\leq\frac{1}{1-\beta_n}\|x_n-x_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|\gamma f(x_n)-AW_ny_n\|.
\end{aligned} \tag{60}$$

By conditions (C1), (C2), and (58), imply that

$$\lim_{n\to\infty}\|x_n-W_ny_n\|=0. \tag{61}$$

From (42) and (43), we get

$$\begin{aligned}
&\|y_n - x^*\|^2 \\
&\quad = \|J_{M_1,\lambda}(z_n - \lambda E_1 z_n) - J_{M_1,\lambda}(x^* - \lambda E_1 x^*)\|^2 \\
&\quad \le \|(z_n - \lambda E_1 z_n) - (x^* - \lambda E_1 x^*)\|^2 \\
&\quad \le \|z_n - x^*\|^2 + \lambda(\lambda - 2\eta_1)\|E_1 z_n - E_1 x^*\|^2 \\
&\quad \le \|J_{M_2,\mu}(u_n - \mu E_2 u_n) - J_{M_2,\mu}(x^* - \mu E_2 x^*)\|^2 \\
&\qquad + \lambda(\lambda - 2\eta_1)\|E_1 z_n - E_1 x^*\|^2 \\
&\quad \le \|(u_n - \mu E_2 u_n) - (x^* - \mu E_2 x^*)\|^2 \\
&\qquad + \lambda(\lambda - 2\eta_1)\|E_1 z_n - E_1 x^*\|^2 \\
&\quad \le \|u_n - x^*\|^2 + \mu(\mu - 2\eta_2)\|E_2 u_n - E_2 x^*\|^2 \\
&\qquad + \lambda(\lambda - 2\eta_1)\|E_1 z_n - E_1 x^*\|^2 \\
&\quad \le \|x_n - x^*\|^2 + r(r - 2\delta)\|Q x_n - Q x^*\|^2 \\
&\qquad + \mu(\mu - 2\eta_2)\|E_2 u_n - E_2 x^*\|^2 \\
&\qquad + \lambda(\lambda - 2\eta_1)\|E_1 z_n - E_1 x^*\|^2.
\end{aligned} \tag{62}$$

By (40), we obtain

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\quad = \|\alpha_n(\gamma f(x_n) - A x^*) + \beta_n(x_n - W_n y_n) \\
&\qquad + (I - \alpha_n A)(W_n y_n - x^*)\|^2 \\
&\quad \le \|(I - \alpha_n A)(W_n y_n - x^*) + \beta_n(x_n - W_n y_n)\|^2 \\
&\qquad + 2\alpha_n\langle \gamma f(x_n) - A x^*, x_{n+1} - x^*\rangle \\
&\quad \le \|(I - \alpha_n A)(y_n - x^*) + \beta_n(x_n - W_n y_n)\|^2 \\
&\qquad + 2\alpha_n\|\gamma f(x_n) - A x^*\|\|x_{n+1} - x^*\| \\
&\quad = (1 - \alpha_n\overline{\gamma})^2\|y_n - x^*\|^2 + \beta_n^2\|x_n - W_n y_n\|^2 \\
&\qquad + 2(1 - \alpha_n\overline{\gamma})\beta_n\|y_n - x^*\|\|x_n - W_n y_n\| \\
&\qquad + 2\alpha_n\|\gamma f(x_n) - A x^*\|\|x_{n+1} - x^*\|.
\end{aligned} \tag{63}$$

Substituting (62) into (63), imply that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 \le{}& \|x_n - x^*\|^2 + r(r - 2\delta)\|Q x_n - Q x^*\|^2 \\
&+ \mu(\mu - 2\eta_2)\|E_2 u_n - E_2 x^*\|^2 \\
&+ \lambda(\lambda - 2\eta_1)\|E_1 z_n - E_1 x^*\|^2 \\
&+ \beta_n^2\|x_n - W_n y_n\|^2 \\
&+ 2(1 - \alpha_n\overline{\gamma})\beta_n\|y_n - x^*\|\|x_n - W_n y_n\| \\
&+ 2\alpha_n\|\gamma f(x_n) - A x^*\|\|x_{n+1} - x^*\|.
\end{aligned} \tag{64}$$

Thus,

$$\begin{aligned}
&r(2\delta - r)\|Q x_n - Q x^*\|^2 + \mu(2\eta_2 - \mu)\|E_2 u_n - E_2 x^*\|^2 \\
&\qquad + \lambda(2\eta_1 - \lambda)\|E_1 z_n - E_1 x^*\|^2 \\
&\quad \le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \beta_n^2\|x_n - W_n y_n\|^2 \\
&\qquad + 2(1 - \alpha_n\overline{\gamma})\beta_n\|y_n - x^*\|\|x_n - W_n y_n\| \\
&\qquad + 2\alpha_n\|\gamma f(x_n) - A x^*\|\|x_{n+1} - x^*\| \\
&\quad \le (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_{n+1} - x_n\| \\
&\qquad + \beta_n^2\|x_n - W_n y_n\|^2 \\
&\qquad + 2(1 - \alpha_n\overline{\gamma})\beta_n\|y_n - x^*\|\|x_n - W_n y_n\| \\
&\qquad + 2\alpha_n\|\gamma f(x_n) - A x^*\|\|x_{n+1} - x^*\|.
\end{aligned} \tag{65}$$

By conditions (C1), (C2), (58), and (61), we deduce immediately that

$$\begin{aligned}
\lim_{n\to\infty}\|Q x_n - Q x^*\| &= \lim_{n\to\infty}\|E_1 z_n - E_1 x^*\| \\
&= \lim_{n\to\infty}\|E_2 u_n - E_2 x^*\| = 0.
\end{aligned} \tag{66}$$

Step 4. We show that $\lim_{n\to\infty}\|x_n - W x_n\| = 0$. Since T_r is firmly nonexpansive, we have

$$\begin{aligned}
&\|u_n - x^*\|^2 \\
&\quad = \|T_r(x_n - rQx_n) - T_r(x^* - rQx^*)\|^2 \\
&\quad \le \langle (x_n - rQx_n) - (x^* - rQx^*), u_n - x^*\rangle \\
&\quad = \frac{1}{2}\left\{\|(x_n - rQx_n) - (x^* - rQx^*)\|^2 + \|u_n - x^*\|^2\right\} \\
&\qquad - \frac{1}{2}\left\{\|(x_n - rQx_n) - (x^* - rQx^*) - (u_n - x^*)\|^2\right\} \\
&\quad = \frac{1}{2}\left\{\|x_n - x^*\|^2 + \|u_n - x^*\|^2\right. \\
&\qquad\qquad \left. - \|(x_n - u_n) - r(Qx_n - Qx^*)\|^2\right\}
\end{aligned}$$

$$= \frac{1}{2}\left\{\left\|x_n - x^*\right\|^2 + \left\|u_n - x^*\right\|^2 - \left(\left\|x_n - u_n\right\|^2 + r^2\left\|Qx_n - Qx^*\right\|^2 - 2r\left\langle x_n - u_n, Qx_n - Qx^*\right\rangle\right)\right\}$$
$$\le \frac{1}{2}\left\{\left\|x_n - x^*\right\|^2 + \left\|u_n - x^*\right\|^2 - \left\|x_n - u_n\right\|^2 - r^2\left\|Qx_n - Qx^*\right\|^2 + 2r\left\|x_n - u_n\right\|\left\|Qx_n - Qx^*\right\|\right\}, \tag{67}$$

which implies that

$$\left\|u_n - x^*\right\|^2 \le \left\|x_n - x^*\right\|^2 - \left\|x_n - u_n\right\|^2 + 2r\left\|x_n - u_n\right\|\left\|Qx_n - Qx^*\right\|. \tag{68}$$

Since $J_{M_1,\lambda}$ is 1-inverse-strongly monotone, we have

$$\left\|y_n - x^*\right\|^2 = \left\|J_{M_1,\lambda}\left(z_n - \lambda E_1 z_n\right) - J_{M_1,\lambda}\left(x^* - \lambda E_1 x^*\right)\right\|^2$$
$$\le \left\langle\left(z_n - \lambda E_1 z_n\right) - \left(x^* - \lambda E_1 x^*\right), y_n - x^*\right\rangle$$
$$= \frac{1}{2}\left\{\left\|\left(z_n - \lambda E_1 z_n\right) - \left(x^* - \lambda E_1 x^*\right)\right\|^2 + \left\|y_n - x^*\right\|^2\right\} - \frac{1}{2}\left\{\left\|\left(z_n - \lambda E_1 z_n\right) - \left(x^* - \lambda E_1 x^*\right) - \left(y_n - x^*\right)\right\|^2\right\}$$
$$= \frac{1}{2}\left\{\left\|z_n - x^*\right\|^2 + \left\|y_n - x^*\right\|^2 - \left\|\left(z_n - y_n\right) - \lambda\left(E_1 z_n - E_1 x^*\right)\right\|^2\right\}$$
$$= \frac{1}{2}\left\{\left\|z_n - x^*\right\|^2 + \left\|y_n - x^*\right\|^2 - \left(\left\|z_n - y_n\right\|^2 + \lambda^2\left\|E_1 z_n - E_1 x^*\right\|^2 - 2\lambda\left\langle z_n - y_n, E_1 z_n - E_1 x^*\right\rangle\right)\right\}$$
$$\le \frac{1}{2}\left\{\left\|z_n - x^*\right\|^2 + \left\|y_n - x^*\right\|^2 - \left\|z_n - y_n\right\|^2 - \lambda^2\left\|E_1 z_n - E_1 x^*\right\|^2 + 2\lambda\left\|z_n - y_n\right\|\left\|E_1 z_n - E_1 x^*\right\|\right\}, \tag{69}$$

which implies that

$$\left\|y_n - x^*\right\|^2 \le \left\|z_n - x^*\right\|^2 - \left\|z_n - y_n\right\|^2 + 2\lambda\left\|z_n - y_n\right\|\left\|E_1 z_n - E_1 x^*\right\|. \tag{70}$$

In the same way with (70), we can get

$$\left\|z_n - x^*\right\|^2 \le \left\|u_n - x^*\right\|^2 - \left\|u_n - z_n\right\|^2 + 2\mu\left\|u_n - z_n\right\|\left\|E_2 u_n - E_2 x^*\right\|. \tag{71}$$

Substituting (71) into (70), imply that

$$\left\|y_n - x^*\right\|^2 \le \left\|u_n - x^*\right\|^2 - \left\|u_n - z_n\right\|^2 + 2\mu\left\|u_n - z_n\right\|\left\|E_2 u_n - E_2 x^*\right\| - \left\|z_n - y_n\right\|^2 + 2\lambda\left\|z_n - y_n\right\|\left\|E_1 z_n - E_1 x^*\right\|. \tag{72}$$

Again, substituting (68) into (72), we get

$$\left\|y_n - x^*\right\|^2 \le \left\{\left\|x_n - x^*\right\|^2 - \left\|x_n - u_n\right\|^2 + 2r\left\|x_n - u_n\right\|\left\|Qx_n - Qx^*\right\|\right\} - \left\|u_n - z_n\right\|^2 + 2\mu\left\|u_n - z_n\right\|\left\|E_2 u_n - E_2 x^*\right\| - \left\|z_n - y_n\right\|^2 + 2\lambda\left\|z_n - y_n\right\|\left\|E_1 z_n - E_1 x^*\right\|. \tag{73}$$

Substituting (73) into (63), imply that

$$\left\|x_{n+1} - x^*\right\|^2 \le \left(1 - \alpha_n\overline{\gamma}\right)^2\left\{\left\|x_n - x^*\right\|^2 - \left\|x_n - u_n\right\|^2 + 2r\left\|x_n - u_n\right\|\left\|Qx_n - Qx^*\right\| - \left\|u_n - z_n\right\|^2 + 2\mu\left\|u_n - z_n\right\|\left\|E_2 u_n - E_2 x^*\right\| - \left\|z_n - y_n\right\|^2 + 2\lambda\left\|z_n - y_n\right\|\left\|E_1 z_n - E_1 x^*\right\|\right\} + \beta_n^2\left\|x_n - W_n y_n\right\|^2 + 2\left(1 - \alpha_n\overline{\gamma}\right)\beta_n\left\|y_n - x^*\right\|\left\|x_n - W_n y_n\right\| + 2\alpha_n\left\|\gamma f\left(x_n\right) - Ax^*\right\|\left\|x_{n+1} - x^*\right\|. \tag{74}$$

Then, we derive

$$\left(1 - \alpha_n\overline{\gamma}\right)^2\left(\left\|x_n - u_n\right\|^2 + \left\|u_n - z_n\right\|^2 + \left\|z_n - y_n\right\|^2\right) \le \left\|x_n - x^*\right\|^2 - \left\|x_{n+1} - x^*\right\|^2 + 2r\left\|x_n - u_n\right\|\left\|Qx_n - Qx^*\right\| + 2\mu\left\|u_n - z_n\right\|\left\|E_2 u_n - E_2 x^*\right\| + 2\lambda\left\|z_n - y_n\right\|\left\|E_1 z_n - E_1 x^*\right\| + \beta_n^2\left\|x_n - W_n y_n\right\|^2 + 2\left(1 - \alpha_n\overline{\gamma}\right)\beta_n\left\|y_n - x^*\right\|\left\|x_n - W_n y_n\right\| + 2\alpha_n\left\|\gamma f\left(x_n\right) - Ax^*\right\|\left\|x_{n+1} - x^*\right\|$$

$$\begin{aligned}
&\leq (\|x_n - x^*\| + \|x_{n+1} - x^*\|) \|x_{n+1} - x_n\| \\
&\quad + 2r \|x_n - u_n\| \|Qx_n - Qx^*\| \\
&\quad + 2\mu \|u_n - z_n\| \|E_2 u_n - E_2 x^*\| \\
&\quad + 2\lambda \|z_n - y_n\| \|E_1 z_n - E_1 x^*\| + \beta_n^2 \|x_n - W_n y_n\|^2 \\
&\quad + 2(1 - \alpha_n \overline{\gamma}) \beta_n \|y_n - x^*\| \|x_n - W_n y_n\| \\
&\quad + 2\alpha_n \|\gamma f(x_n) - Ax^*\| \|x_{n+1} - x^*\|.
\end{aligned} \tag{75}$$

By conditions (C1), (C2), (58), (61), and (66), we obtain

$$\lim_{n\to\infty} \|x_n - u_n\| = \lim_{n\to\infty} \|u_n - z_n\| = \lim_{n\to\infty} \|z_n - y_n\| = 0. \tag{76}$$

Observe that

$$\begin{aligned}
\|W_n y_n - y_n\| \leq \|W_n y_n - x_n\| + \|x_n - u_n\| \\
+ \|u_n - z_n\| + \|z_n - y_n\|.
\end{aligned} \tag{77}$$

By (61) and (76), we have

$$\lim_{n\to\infty} \|W_n y_n - y_n\| = 0. \tag{78}$$

Note that

$$\|W y_n - y_n\| \leq \|W y_n - W_n y_n\| + \|W_n y_n - y_n\|. \tag{79}$$

From Lemma 1, we get

$$\lim_{n\to\infty} \|W y_n - W_n y_n\| = 0. \tag{80}$$

By (78) and (80), we have $\lim_{n\to\infty}\|Wy_n - y_n\| = 0$. It follows that $\lim_{n\to\infty}\|Wx_n - x_n\| = 0$.

Step 5. We show that $\limsup_{n\to\infty}\langle(\gamma f - A)z, x_n - z\rangle \leq 0$, where $z = P_\Theta(\gamma f + I - A)z$. It is easy to see that $P_\Theta(\gamma f + (I - A))$ is a contraction of H into itself. Indeed, since $0 < \gamma < \overline{\gamma}/\alpha$, we have

$$\begin{aligned}
&\|P_\Theta(\gamma f + (I - A))x - P_\Theta(\gamma f + (I - A))y\| \\
&\quad \leq \|(\gamma f + (I - A))x - (\gamma f + (I - A))y\| \\
&\quad \leq \gamma \|f(x) - f(y)\| + |I - A| \|x - y\| \\
&\quad \leq \gamma\alpha \|x - y\| + (1 - \overline{\gamma}) \|x - y\| \\
&\quad = (1 - \overline{\gamma} + \gamma\alpha) \|x - y\|.
\end{aligned} \tag{81}$$

Since H is complete, there exists a unique fixed point $z \in H$ such that $z = P_\Theta(\gamma f + I - A)(z)$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$, such that

$$\lim_{i\to\infty} \left\langle (A - \gamma f) z, z - x_{n_i} \right\rangle = \limsup_{n\to\infty} \left\langle (A - \gamma f) z, z - x_n \right\rangle. \tag{82}$$

Also, since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to $w \in C$. Without loss of generality, we can assume that $x_{n_i} \rightharpoonup w$. From $\|Wx_n - x_n\| \to 0$, we obtain $Wx_{n_i} \rightharpoonup w$. Then, by the demiclosed principle of nonexpansive mappings, we obtain $w \in \cap_{i=1}^{\infty} F(T_i)$.

Next, we show that $w \in \mathrm{MEP}(F, \varphi)$. Since $u_n = T_r(x_n - rQx_n)$, we obtain

$$\begin{aligned}
&F(u_n, y) + \varphi(y) - \varphi(u_n) \\
&\quad + \frac{1}{r} \langle y - u_n, u_n - (x_n - rQx_n) \rangle \geq 0, \quad \forall y \in C.
\end{aligned} \tag{83}$$

From (A2), we also have

$$\varphi(y) - \varphi(u_n) + \frac{1}{r} \langle y - u_n, u_n - (x_n - rQx_n) \rangle \geq F(y, u_n), \quad \forall y \in C, \tag{84}$$

and hence,

$$\begin{aligned}
&\varphi(y) - \varphi(u_{n_i}) + \left\langle y - u_{n_i}, \frac{u_{n_i} - (x_{n_i} - rQx_{n_i})}{r} \right\rangle \\
&\quad \geq F(y, u_{n_i}), \quad \forall y \in C.
\end{aligned} \tag{85}$$

For t with $0 < t \leq 1$ and $y \in H$, let $y_t = ty + (1 - t)w$. From (85) we have

$$\begin{aligned}
\left\langle y_t - u_{n_i}, Qy_t \right\rangle &\geq \left\langle y_t - u_{n_i}, Qy_t \right\rangle - \varphi(y_t) + \varphi(u_{n_i}) \\
&\quad - \left\langle y_t - u_{n_i}, \frac{u_{n_i} - (x_{n_i} - rQx_{n_i})}{r} \right\rangle \\
&\quad + F(y_t, u_{n_i}) \\
&= \left\langle y_t - u_{n_i}, Qy_t - Qu_{n_i} \right\rangle \\
&\quad + \left\langle y_t - u_{n_i}, Qu_{n_i} - Qx_{n_i} \right\rangle \\
&\quad - \varphi(y_t) + \varphi(u_{n_i}) \\
&\quad - \left\langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r} \right\rangle + F(y_t, u_{n_i}).
\end{aligned} \tag{86}$$

Since $\|u_{n_i} - x_{n_i}\| \to 0$, we have $\|Qu_{n_i} - Qx_{n_i}\| \to 0$. Further, from an inverse-strongly monotonicity of Q, we have $\langle y_t - u_{n_i}, Qy_t - Qu_{n_i}\rangle \geq 0$. So, from (A4), (A5), and the weakly lower semicontinuity of φ, $\langle u_{n_i} - x_{n_i}\rangle/r \to 0$ and $u_{n_i} \to w$ weakly, we have

$$\langle y_t - w, Qy_t \rangle \geq -\varphi(y_t) + \varphi(w) + F(y_t, w). \tag{87}$$

From (A1), (A4), and (87), we also have

$$\begin{aligned}0 &= F(y_t, y_t) + \varphi(y_t) - \varphi(y_t) \\ &\le tF(y_t, y) + (1-t)F(y_t, w) + t\varphi(y) \\ &\quad + (1-t)\varphi(w) - \varphi(y_t) \\ &= t(F(y_t, y) + \varphi(y) - \varphi(y_t)) \\ &\quad + (1-t)(F(y_t, w) + \varphi(w) - \varphi(y_t)) \\ &\le t(F(y_t, y) + \varphi(y) - \varphi(y_t)) + (1-t)\langle y_t - w, Qy_t\rangle \\ &= t(F(y_t, y) + \varphi(y) - \varphi(y_t)) + (1-t)t\langle y - w, Qy_t\rangle,\end{aligned} \tag{88}$$

and hence,

$$0 \le F(y_t, y) + \varphi(y) - \varphi(y_t) + (1-t)\langle y - w, Qy_t\rangle. \tag{89}$$

Letting $t \to 0$, we have, for each $y \in C$,

$$F(w, y) + \varphi(y) - \varphi(w) + \langle y - w, Qw\rangle \ge 0. \tag{90}$$

This implies that $w \in \mathrm{MEP}(F, \varphi)$.

Lastly, we show that $w \in \mathrm{SQVI}(B_1, M_1, B_2, M_2)$. Since $\|u_n - z_n\| \to 0$ and $\|z_n - y_n\| \to 0$ as $n \to \infty$, we get

$$\|u_n - y_n\| \le \|u_n - z_n\| + \|z_n - y_n\|, \tag{91}$$

we conclude that $\|u_n - y_n\| \to 0$ as $n \to \infty$. Moreover, by the nonexpansivity of G in Lemma 6, we have

$$\begin{aligned}&\|y_n - G(y_n)\| \\ &\quad = \Big\| J_{M_1,\lambda}\left[J_{M_2,\mu}(u_n - \mu E_2 u_n) - \lambda E_1 J_{M_2,\mu}(u_n - \mu E_2 u_n)\right] \\ &\qquad - G(y_n)\Big\| \\ &\quad = \|G(u_n) - G(y_n)\| \\ &\quad \le \|u_n - y_n\|.\end{aligned} \tag{92}$$

Thus, $\lim_{n\to\infty}\|y_n - G(y_n)\| = 0$. According to Lemma 7, we obtain that $w \in \mathrm{SQVI}(B_1, M_1, B_2, M_2)$. Hence, $w \in \Theta$. Since $z = P_\Theta(I - A + \gamma f)(z)$, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle(\gamma f - A)z, x_n - z\rangle &= \limsup_{i\to\infty}\left\langle(\gamma f - A)z, x_{n_i} - z\right\rangle \\ &= \langle(\gamma f - A)z, w - z\rangle \\ &\le 0.\end{aligned} \tag{93}$$

Step 6. We show that $\{x_n\}$ converges strongly to z; we compute that

$$\begin{aligned}&\|x_{n+1} - z\|^2 \\ &\quad = \|\alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)W_n y_n - z\|^2 \\ &\quad = \|\alpha_n(\gamma f(x_n) - Az) + \beta_n(x_n - z) \\ &\qquad + ((1-\beta_n)I - \alpha_n A)(W_n y_n - z)\|^2 \\ &\quad = \alpha_n^2\|\gamma f(x_n) - Az\|^2 \\ &\qquad + \|\beta_n(x_n - z) + ((1-\beta_n)I - \alpha_n A)(W_n y_n - z)\|^2 \\ &\qquad + 2\langle\beta_n(x_n - z) + ((1-\beta_n)I - \alpha_n A) \\ &\qquad\qquad \times (W_n y_n - z), \alpha_n(\gamma f(x_n) - Az)\rangle \\ &\quad \le \alpha_n^2\|\gamma f(x_n) - Az\|^2 \\ &\qquad + \{\beta_n\|x_n - z\| + (1 - \beta_n - \alpha_n\overline{\gamma})\|y_n - z\|\}^2 \\ &\qquad + 2\alpha_n\beta_n\langle x_n - z, \gamma f(x_n) - Az\rangle \\ &\qquad + 2\alpha_n(1 - \beta_n - \alpha_n\overline{\gamma})\langle W_n y_n - z, \gamma f(x_n) - Az\rangle \\ &\quad \le \alpha_n^2\|\gamma f(x_n) - Az\|^2 \\ &\qquad + \{\beta_n\|x_n - z\| + (1 - \beta_n - \alpha_n\overline{\gamma})\|x_n - z\|\}^2 \\ &\qquad + 2\alpha_n\beta_n\langle x_n - z, \gamma f(x_n) - \gamma f(z)\rangle \\ &\qquad + 2\alpha_n\beta_n\langle x_n - z, \gamma f(z) - Az\rangle \\ &\qquad + 2\alpha_n(1 - \beta_n - \alpha_n\overline{\gamma})\langle W_n y_n - z, \gamma f(x_n) - \gamma f(z)\rangle \\ &\qquad + 2\alpha_n(1 - \beta_n - \alpha_n\overline{\gamma})\langle W_n y_n - z, \gamma f(z) - Az\rangle \\ &\quad \le \alpha_n^2\|\gamma f(x_n) - Az\|^2 + (1 - \alpha_n\overline{\gamma})^2\|x_n - z\|^2 \\ &\qquad + 2\alpha_n\beta_n\gamma\|x_n - z\|\|f(x_n) - f(z)\| \\ &\qquad + 2\alpha_n\beta_n\langle x_n - z, \gamma f(z) - Az\rangle \\ &\qquad + 2\alpha_n(1 - \beta_n - \alpha_n\overline{\gamma})\gamma\|W_n y_n - z\|\|f(x_n) - f(z)\| \\ &\qquad + 2\alpha_n(1 - \beta_n - \alpha_n\overline{\gamma})\langle W_n y_n - z, \gamma f(z) - Az\rangle \\ &\quad \le \alpha_n^2\|\gamma f(x_n) - Az\|^2 + (1 - \alpha_n\overline{\gamma})^2\|x_n - z\|^2 \\ &\qquad + 2\alpha_n\beta_n\gamma\alpha\|x_n - z\|^2 \\ &\qquad + 2\alpha_n\beta_n\langle x_n - z, \gamma f(z) - Az\rangle \\ &\qquad + 2\alpha_n(1 - \beta_n - \alpha_n\overline{\gamma})\gamma\alpha\|x_n - z\|^2 \\ &\qquad + 2\alpha_n(1 - \beta_n - \alpha_n\overline{\gamma})\langle W_n y_n - z, \gamma f(z) - Az\rangle \\ &\quad = \alpha_n^2\|\gamma f(x_n) - Az\|^2\end{aligned}$$

$$
\begin{aligned}
&+\left(1-2\alpha_n\overline{\gamma}+\alpha_n^2\overline{\gamma}^2+2\alpha_n\gamma\alpha-2\alpha_n^2\overline{\gamma}\gamma\alpha\right)\\
&\quad\times\|x_n-z\|^2+2\alpha_n\beta_n\langle x_n-z,\gamma f(z)-Az\rangle\\
&\quad+2\alpha_n(1-\beta_n-\alpha_n\overline{\gamma})\langle W_ny_n-z,\gamma f(z)-Az\rangle\\
&\le\left\{1-\alpha_n\left(2\overline{\gamma}-\alpha_n\overline{\gamma}^2-2\gamma\alpha+2\alpha_n\overline{\gamma}\gamma\alpha\right)\right\}\|x_n-z\|^2\\
&\quad+\alpha_n^2\|\gamma f(x_n)-Az\|^2\\
&\quad+2\alpha_n\beta_n\langle x_n-z,\gamma f(z)-Az\rangle\\
&\quad+2\alpha_n(1-\beta_n-\alpha_n\overline{\gamma})\langle W_ny_n-z,\gamma f(z)-Az\rangle\\
&\le\left\{1-\alpha_n\left(2\overline{\gamma}-\alpha_n\overline{\gamma}^2-2\gamma\alpha+2\alpha_n\overline{\gamma}\gamma\alpha\right)\right\}\|x_n-z\|^2\\
&\quad+\alpha_n\sigma_n,
\end{aligned}
\tag{94}
$$

where $\sigma_n = \alpha_n\|\gamma f(x_n)-Az\|^2 + 2\beta_n\langle x_n-z,\gamma f(z)-Az\rangle + 2(1-\beta_n-\alpha_n\overline{\gamma})\langle W_ny_n-z,\gamma f(z)-Az\rangle$. It is easy to see that $\limsup_{n\to\infty}\sigma_n \le 0$. Applying Lemma 3 to (94), we conclude that $x_n \to z$. This completes the proof. □

Next, the following example shows that all conditions of Theorem 8 are satisfied.

Example 9. For instance, let $\alpha_n = 1/2(n+1)$, let $\beta_n = (2n+2)/2(2n)$, let $\lambda_n = n/(n+1)$. Then, we will show that the sequences $\{\alpha_n\}$ satisfy condition (C1). Indeed, we take $\alpha_n = 1/2(n+1)$; then, we have

$$
\begin{gathered}
\sum_{n=1}^{\infty}\alpha_n=\sum_{n=1}^{\infty}\frac{1}{2(n+1)}=\infty,\\
\lim_{n\to\infty}\alpha_n=\lim_{n\to\infty}\frac{1}{2(n+1)}=0.
\end{gathered}
\tag{95}
$$

We will show that the sequences $\{\beta_n\}$ satisfy condition (C2). Indeed, we set $\beta_n = (2n+2)/2(2n) = (1/2)+(1/2n)$. It is easy to see that $0<\liminf_{n\to\infty}\beta_n < \limsup_{n\to\infty}\beta_n < 1$.

Next, we will show the condition (C3) is satisfied. We take $\lambda_n = n/(n+1)$; then we compute

$$
\begin{aligned}
\lim_{n\to\infty}|\lambda_n-\lambda_{n-1}|&=\lim_{n\to\infty}\left|\frac{n}{n+1}-\frac{n-1}{(n-1)+1}\right|\\
&=\lim_{n\to\infty}\left|\frac{n(n)-(n-1)(n+1)}{(n+1)n}\right|\\
&=\lim_{n\to\infty}\left|\frac{n^2-n^2+1}{(n+1)n}\right|\\
&=\lim_{n\to\infty}\left|\frac{1}{n(n+1)}\right|.
\end{aligned}
\tag{96}
$$

Then, we have $\lim_{n\to\infty}|\lambda_{n+1}-\lambda_n| = 0$. The sequence $\{\lambda_n\}$ satisfies condition (C3).

Using Theorem 8, we obtain the following corollaries.

Corollary 10. *Let C be a nonempty closed convex subset of a real Hilbert Space H. Let F be a bifunction of $C\times C$ into real numbers $\mathbb{R}$ satisfying (A1)–(A5) and let $\varphi : C \to \mathbb{R}\cup\{+\infty\}$ be a proper lower semicontinuos and convex function. Let $T_i : C \to C$ be nonexpansive mappings for all $i=1,2,3,\ldots$, such that $\Theta := \cap_{i=1}^{\infty}F(T_i)\cap SQVI(B_1,M_1,B_2,M_2)\cap MEP(F,\varphi)\neq\emptyset$. Let f be a contraction of C into itself with coefficient $\alpha\in(0,1)$ and let Q,E_1,E_2 be δ,η_1,η_2-inverse-strongly monotone mapping of C into H. Let $M_1,M_2 : H \to 2^H$ be a maximal monotone mapping. Assume that either B_1 or B_2 holds and let W_n be the W-mapping defined by* (31). *Let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{u_n\}$ be sequences generated by $x_0\in C$, $u_n\in C$, and*

$$
\begin{gathered}
F(u_n,y)+\varphi(y)-\varphi(u_n)\\
+\frac{1}{r}\langle y-u_n,u_n-(x_n-rQx_n)\rangle\ge 0,\quad\forall y\in C,\\
z_n=J_{M_2,\mu}(u_n-\mu E_2u_n),\\
y_n=J_{M_1,\lambda}(z_n-\lambda E_1z_n),\\
x_{n+1}=\alpha_nf(x_n)+\beta_nx_n+(1-\beta_n-\alpha_n)W_ny_n,\quad\forall n\ge 0,
\end{gathered}
\tag{97}
$$

where $\{\alpha_n\}$ and $\{\beta_n\}\subset(0,1)$, $\lambda\in(0,2\eta_1)$, $\mu\in(0,2\eta_2)$, and $r\in(0,2\delta)$ satisfy the following conditions:

(C1) $\sum_{n=0}^{\infty}\alpha_n=\infty$ *and* $\lim_{n\to\infty}\alpha_n=0$,

(C2) $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$,

(C3) $\lim_{n\to\infty}|\lambda_{n,i}-\lambda_{n-1,i}|=0$, $\forall i=1,2,\ldots,N$.

Then, $\{x_n\}$ converges strongly to $x^*\in\Theta$, where $x^*=P_\Theta(f+I)(x^*)$, P_Θ is the metric projection of H onto Θ and (x^*,y^*), where $y^*=J_{M_2,\mu}(x^*-\mu E_2x^*)$ is solution to the problem (6).

Proof. Taking $\gamma\equiv 1$ and $A\equiv I$ in Theorem 8, we can conclude the desired conclusion easily. □

Corollary 11. *Let C be a nonempty closed convex subset of a real Hilbert Space H. Let F be a bifunction of $C\times C$ into real numbers $\mathbb{R}$ satisfying (A1)–(A5) and let $\varphi : C \to \mathbb{R}\cup\{+\infty\}$ be a proper lower semicontinuos and convex function. Let $T_i : C \to C$ be a nonexpansive mappings for all $i=1,2,3,\ldots$, such that $\Theta := \cap_{i=1}^{\infty}F(T_i)\cap SQVI(B_1,M_1,B_2,M_2)\cap MEP(F,\varphi)\neq\emptyset$. Let f be a contraction of C into itself with coefficient $\alpha\in(0,1)$ and let E_1,E_2 be η_1,η_2-inverse-strongly monotone mapping of C into H. Let A be strongly positive bounded linear self-adjoint on H with coefficient $\overline{\gamma}>0$ and $0<\gamma<\overline{\gamma}/\alpha$, let $M_1,M_2 : H\to 2^H$ be a maximal monotone mapping. Assume that either B_1 or B_2 holds and let W_n be the W-mapping defined* *by (31). Let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{u_n\}$ be sequences generated by $x_0 \in C$, $u_n \in C$, and*

$$\begin{aligned} & F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r}\langle y - u_n, u_n - x_n\rangle \geq 0, \\ & \qquad \forall y \in C, \\ & z_n = J_{M_2,\mu}(u_n - \mu E_2 u_n), \\ & y_n = J_{M_1,\lambda}(z_n - \lambda E_1 z_n), \\ & x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n) I - \alpha_n A) W_n y_n, \\ & \qquad \forall n \geq 0, \end{aligned} \tag{98}$$

where $\{\alpha_n\}$ and $\{\beta_n\} \subset (0,1)$, $\lambda \in (0, 2\eta_1)$, $\mu \in (0, 2\eta_2)$, and $r \in (0, \infty)$ satisfy the following conditions:

(C1) $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty}\alpha_n = 0$,

(C2) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$,

(C3) $\lim_{n\to\infty}|\lambda_{n,i} - \lambda_{n-1,i}| = 0$, $\forall i = 1, 2, \ldots, N$.

Then, $\{x_n\}$ converges strongly to $x^* \in \Theta$, where $x^* = P_\Theta(\gamma f + I - A)(x^*)$, P_Θ is the metric projection of H onto Θ and (x^*, y^*), where $y^* = J_{M_2,\mu}(x^* - \mu E_2 x^*)$ is solution to the problem (6).

Proof. Taking $Q \equiv 0$ in Theorem 8, we can conclude the desired conclusion easily. □

Corollary 12. *Let C be a nonempty closed convex subset of a real Hilbert Space H. Let F be a bifunction of $C \times C$ into real numbers $\mathbb{R}$ satisfying (A1)–(A5) and let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuos and convex function such that $\Theta := SQVI(B_1, M_1, B_2, M_2) \cap MEP(F, \varphi) \neq \emptyset$. Let f be a contraction of C into itself with coefficient $\alpha \in (0,1)$ and let Q, E_1, E_2 be δ, η_1, η_2-inverse-strongly monotone mapping of C into H. Let A be a strongly positive bounded linear self-adjoint on H with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \overline{\gamma}/\alpha$, let $M_1, M_2 : H \to 2^H$ be a maximal monotone mapping. Assume that either B_1 or B_2 holds, let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{u_n\}$ be sequences generated by $x_0 \in C$, $u_n \in C$, and*

$$\begin{aligned} & F(u_n, y) + \varphi(y) - \varphi(u_n) \\ & \quad + \frac{1}{r}\langle y - u_n, u_n - (x_n - rQx_n)\rangle \geq 0, \quad \forall y \in C, \\ & z_n = J_{M_2,\mu}(u_n - \mu E_2 u_n), \\ & y_n = J_{M_1,\lambda}(z_n - \lambda E_1 z_n), \\ & x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n) I - \alpha_n A) y_n, \quad \forall n \geq 0, \end{aligned} \tag{99}$$

where $\{\alpha_n\}$ and $\{\beta_n\} \subset (0,1)$, $\lambda \in (0, 2\eta_1)$, $\mu \in (0, 2\eta_2)$, and $r \in (0, 2\delta)$ satisfy the following conditions:

(C1) $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty}\alpha_n = 0$,

(C2) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$,

(C3) $\lim_{n\to\infty}|\lambda_{n,i} - \lambda_{n-1,i}| = 0$, $\forall i = 1, 2, \ldots, N$.

Then, $\{x_n\}$ converges strongly to $x^* \in \Theta$, where $x^* = P_\Theta(\gamma f + I - A)(x^*)$, P_Θ is the metric projection of H onto Θ and (x^*, y^*), where $y^* = J_{M_2,\mu}(x^* - \mu E_2 x^*)$ is solution to the problem (7), which is the unique solution of the variational inequality

$$\langle (\gamma f - A) x^*, x - x^* \rangle \geq 0, \quad \forall x \in \Theta, \tag{100}$$

Proof. Taking $W_n \equiv I$ in Theorem 8, we can conclude the desired conclusion easily. □

Corollary 13. *Let C be a nonempty closed convex subset of a real Hilbert Space H. Let F be a bifunction of $C \times C$ into real numbers $\mathbb{R}$ satisfying (A1)–(A5). Let $T_i : C \to C$ be nonexpansive mappings for all $i = 1, 2, 3, \ldots$, such that $\Theta := \cap_{i=1}^{\infty} F(T_i) \cap SQVI(B_1, M_1, B_2, M_2) \cap EP(F) \neq \emptyset$. Let f be a contraction of C into itself with coefficient $\alpha \in (0,1)$ and let Q, E_1, E_2 be δ, η_1, η_2-inverse-strongly monotone mapping of C into H. Let A be a strongly positive bounded linear self-adjoint on H with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \overline{\gamma}/\alpha$, let $M_1, M_2 : H \to 2^H$ be a maximal monotone mapping. Assume that either B_1 or B_2 holds and let W_n be the W-mapping defined by (31). Let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{u_n\}$ be sequences generated by $x_0 \in C$, $u_n \in C$, and*

$$\begin{aligned} & F(u_n, y) + \frac{1}{r}\langle y - u_n, u_n - (x_n - rQx_n)\rangle \geq 0, \\ & \qquad \forall y \in C, \\ & z_n = J_{M_2,\mu}(u_n - \mu E_2 u_n), \\ & y_n = J_{M_1,\lambda}(z_n - \lambda E_1 z_n), \\ & x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n) I - \alpha_n A) W_n y_n, \\ & \qquad \forall n \geq 0, \end{aligned} \tag{101}$$

where $\{\alpha_n\}$ and $\{\beta_n\} \subset (0,1)$, $\lambda \in (0, 2\eta_1)$, $\mu \in (0, 2\eta_2)$, and $r \in (0, 2\delta)$ satisfy the following conditions:

(C1) $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty}\alpha_n = 0$,

(C2) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$,

(C3) $\lim_{n\to\infty}|\lambda_{n,i} - \lambda_{n-1,i}| = 0$, $\forall i = 1, 2, \ldots, N$.

Then, $\{x_n\}$ converges strongly to $x^* \in \Theta$, where $x^* = P_\Theta(\gamma f + I - A)(x^*)$, P_Θ is the metric projection of H onto Θ and (x^*, y^*), where $y^* = J_{M_2,\mu}(x^* - \mu E_2 x^*)$ is solution to the problem (6).

Proof. Taking $\varphi \equiv 0$ in Theorem 8, we can conclude the desired conclusion easily. □

Corollary 14. *Let C be a nonempty closed convex subset of a real Hilbert Space H. Let F be a bifunction of $C \times C$ into real numbers $\mathbb{R}$ satisfying (A1)–(A5) such that $\Theta := SQVI(B_1, M_1, B_2, M_2) \cap EP(F) \neq \emptyset$. Let f be a contraction of C into itself with coefficient $\alpha \in (0,1)$ and let Q, E_1, E_2 be δ, η_1, η_2-inverse-strongly monotone mapping of C into H. Let A be a strongly positive bounded linear self-adjoint on H with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \overline{\gamma}/\alpha$, let $M_1, M_2 : H \to 2^H$ be a maximal monotone mapping. Assume that either B_1 or B_2*

holds, let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, *and* $\{u_n\}$ *be sequences generated by* $x_0 \in C$, $u_n \in C$, *and*

$$\begin{gathered} F(u_n, y) + \frac{1}{r}\langle y - u_n, u_n - (x_n - rQx_n)\rangle \geq 0, \\ \forall y \in C, \\ z_n = J_{M_2,\mu}(u_n - \mu E_2 u_n), \\ y_n = J_{M_1,\lambda}(z_n - \lambda E_1 z_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)y_n, \\ \forall n \geq 0, \end{gathered} \tag{102}$$

where $\{\alpha_n\}$ *and* $\{\beta_n\} \subset (0,1)$, $\lambda \in (0, 2\eta_1)$, $\mu \in (0, 2\eta_2)$, *and* $r \in (0, 2\delta)$ *satisfy the following conditions:*

(C1) $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty}\alpha_n = 0$,
(C2) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$,
(C3) $\lim_{n\to\infty}|\lambda_{n,i} - \lambda_{n-1,i}| = 0$, $\forall i = 1, 2, \ldots, N$.

Then, $\{x_n\}$ *converges strongly to* $x^* \in \Theta$, *where* $x^* = P_\Theta(\gamma f + I - A)(x^*)$, P_Θ *is the metric projection of* H *onto* Θ *and* (x^*, y^*), *where* $y^* = J_{M_2,\mu}(x^* - \mu E_2 x^*)$ *is solution to the problem* (6).

Proof. Taking $\varphi \equiv 0$ and $W_n \equiv I$ in Theorem 8, we can conclude the desired conclusion easily. □

Corollary 15. *Let* C *be a nonempty closed convex subset of a real Hilbert Space* H. *Let* F *be a bifunction of* $C \times C$ *into real numbers* $\mathbb{R}$ *satisfying (A1)–(A5) such that* $\Theta := SQVI(B_1, M_1, B_2, M_2) \cap EP(F) \neq \emptyset$. *Let* f *be a contraction of* C *into itself with coefficient* $\alpha \in (0,1)$ *and let* Q, E_1, E_2 *be* δ, η_1, η_2*-inverse-strongly monotone mapping of* C *into* H. *Let* $M_1, M_2 : H \to 2^H$ *be a maximal monotone mapping. Let* $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, *and* $\{u_n\}$ *be sequences generated by* $x_0 \in C$, $u_n \in C$, *and*

$$\begin{gathered} F(u_n, y) + \frac{1}{r}\langle y - u_n, u_n - (x_n - rQx_n)\rangle \geq 0, \\ \forall y \in C, \\ z_n = J_{M_2,\mu}(u_n - \mu E_2 u_n), \\ y_n = J_{M_1,\lambda}(z_n - \lambda E_1 z_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1-\beta_n-\alpha_n)y_n, \\ \forall n \geq 0, \end{gathered} \tag{103}$$

where $\{\alpha_n\}$ *and* $\{\beta_n\} \subset (0,1)$, $\lambda \in (0, 2\eta_1)$, $\mu \in (0, 2\eta_2)$, *and* $r \in (0, 2\delta)$ *satisfy the following conditions:*

(C1) $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty}\alpha_n = 0$,
(C2) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$,
(C3) $\lim_{n\to\infty}|\lambda_{n,i} - \lambda_{n-1,i}| = 0$, $\forall i = 1, 2, \ldots, N$.

Then, $\{x_n\}$ *converges strongly to* $x^* \in \Theta$, *where* $x^* = P_\Theta(f + I)(x^*)$, P_Θ *is the metric projection of* H *onto* Θ *and* (x^*, y^*), *where* $y^* = J_{M_2,\mu}(x^* - \mu E_2 x^*)$ *is solution to the problem* (6).

Proof. Taking $\gamma \equiv 1$, $A \equiv I$, $\varphi \equiv 0$, and $W_n \equiv I$ in Theorem 8, we can conclude the desired conclusion easily. □

Corollary 16. *Let* C *be a nonempty closed convex subset of a real Hilbert Space* H. *Let* F *be a bifunction of* $C \times C$ *into real numbers* $\mathbb{R}$ *satisfying (A1)–(A5) such that* $\Theta := SQVI(B_1, M_1, B_2, M_2) \cap EP(F) \neq \emptyset$. *Let* f *be a contraction of* C *into itself with coefficient* $\alpha \in (0,1)$ *and let* Q, E *be* δ, η*-inverse-strongly monotone mapping of* C *into* H. *Let* $M_1, M_2 : H \to 2^H$ *be a maximal monotone mapping. Let* $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, *and* $\{u_n\}$ *be sequences generated by* $x_0 \in C$, $u_n \in C$, *and*

$$\begin{gathered} F(u_n, y) + \frac{1}{r}\langle y - u_n, u_n - (x_n - rQx_n)\rangle \geq 0, \\ \forall y \in C, \\ z_n = J_{M_2,\mu}(u_n - \mu E u_n), \\ y_n = J_{M_1,\lambda}(z_n - \lambda E z_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1-\beta_n-\alpha_n)y_n, \\ \forall n \geq 0, \end{gathered} \tag{104}$$

where $\{\alpha_n\}$ *and* $\{\beta_n\} \subset (0,1)$, $\lambda \in (0, 2\eta)$, $\mu \in (0, 2\eta)$, *and* $r \in (0, 2\delta)$ *satisfy the following conditions:*

(C1) $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty}\alpha_n = 0$,
(C2) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$,
(C3) $\lim_{n\to\infty}|\lambda_{n,i} - \lambda_{n-1,i}| = 0$, $\forall i = 1, 2, \ldots, N$.

Then, $\{x_n\}$ *converges strongly to* $x^* \in \Theta$, *where* $x^* = P_\Theta(f + I)(x^*)$, P_Θ *is the metric projection of* H *onto* Θ *and* (x^*, y^*), *where* $y^* = J_{M_2,\mu}(x^* - \mu E x^*)$ *is solution to the problem* (6).

Proof. Taking $E_1 = E_2 = E$ in Corollary 15, we can conclude the desired conclusion easily. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

An Approach to Existence of Fixed Points of Generalized Contractive Multivalued Mappings of Integral Type via Admissible Mapping

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We investigate the existence of a fixed point of certain contractive multivalued mappings of integral type by using the admissible mapping. Our results generalize the several results on the topic in the literature involving Branciari, and Feng and Liu. We also construct some examples to illustrate our results.

1. Preliminaries and Introduction

Fixed point theory is one of the most celebrated research areas that has an application potential not only in nonlinear but also in several branches of mathematics. As a consequence of this fact, several fixed point results have been reported. It is not easy to know, manage, and use all results of this reich theory to get an application. To overcome such problems and clarify the literature, several authors have suggested a more general construction in a way that a number of existing results turn into a consequence of the proposed one. One of the examples of this trend is the investigations of fixed point of certain operator by using the α-admissible mapping introduced Samet et al. [1]. This paper has been appreciated by several authors and this trend has been supported by reporting several interesting results; see for example [2–12].

In this paper, we define (α^*, ψ)-contractive multivalued mappings of integral type and discuss the existence of a fixed point of such mappings. Our construction and hence results improve, extend, and generalize several results including Branciari [13] and Feng and Liu [14].

In what follows, we recall some basic definitions, notions, notations, and fundamental results for the sake of completeness. Let Ψ be a family of nondecreasing functions, $\psi : [0,\infty) \to [0,\infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0$, where ψ^n is the nth iterate of ψ. It is known that, for each $\psi \in \Psi$, we have $\psi(t) < t$ for all $t > 0$ and $\psi(0) = 0$ for $t = 0$ [1]. We denote by Φ the set of all Lebesgue integrable mappings, $\phi : [0,\infty) \to [0,\infty)$ which is summable on each compact subset of $[0,\infty)$ and $\int_0^{\epsilon} \phi(t)dt > 0$, for each $\epsilon > 0$.

Let (X, d) be a metric space. We denote by $N(X)$ the space of all nonempty subsets of X, by $B(X)$ the space of all nonempty bounded subsets of X, and by $CL(X)$ the space of all nonempty closed subsets of X. For $A \in N(X)$ and $x \in X$,

$$d(x, A) = \inf \{d(x, a) : a \in A\}. \tag{1}$$

For every $A, B \in B(X)$,

$$\delta(A, B) = \sup \{d(a, b) : a \in A,\ b \in B\}. \tag{2}$$

We denote $\delta(A, B)$ by $\delta(x, B)$ when $A = \{x\}$. If, for $x_0 \in X$, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that $x_n \in Gx_{n-1}$, then $O(G, x_0) = \{x_0, x_1, x_2, \ldots\}$ is said to be an orbit of $G : X \to CL(X)$ at x_0. A mapping $f : X \to \mathbb{R}$ is G orbitally lower semicontinuous at x, if $\{x_n\}$ is a sequence in $O(G, x_0)$ and $x_n \to x$ implies $f(x) \le \liminf_n f(x_n)$. Branciari [13] extended the Banach contraction principle [15] in the following way.

Theorem 1. *Let (X, d) be a complete metric space and let $G : X \to X$ be a mapping such that*

$$\int_0^{d(Tx,Ty)} \phi(t)\,dt \le c \int_0^{d(x,y)} \phi(t)\,dt \tag{3}$$

for each $x, y \in X$, where $c \in [0, 1)$ and $\phi \in \Phi$. Then G has a unique fixed point.

Since then many authors used integral type contractive conditions to prove fixed point theorems in different settings; see for example [12, 16–22]. Feng and Liu [14] extended the result of Branciari [13] to multivalued mappings as follows.

Theorem 2 (see [14]). *Let (X, d) be a complete metric space and let $G : X \to CL(X)$ be a mapping. Assume that for each $x \in X$ and $y \in Gx$, there exists $z \in Gy$ such that*

$$\int_0^{d(y,z)} \phi(t)\,dt \le \psi\left(\int_0^{d(x,y)} \phi(t)\,dt\right), \tag{4}$$

where $\psi \in \Psi$ and $\phi \in \Phi$. Then G has a fixed point in X provided $f(\xi) = d(\xi, G\xi)$ is lower semicontinuous, with $\xi \in X$.

Definition 3 (see [3]). Let (X, d) be a metric space and $\alpha : X \times X \to [0, \infty)$ be a mapping. A mapping $G : X \to CL(X)$ is α^*-admissible if $\alpha(x, y) \ge 1 \Rightarrow \alpha^*(Gx, Gy) \ge 1$, where $\alpha^*(Gx, Gy) = \inf\{\alpha(a, b) : a \in Gx, b \in Gy\}$.

Definition 4 (see [3]). Let (X, d) be a metric space. A mapping $G : X \to CL(X)$ is called α^*-ψ-contractive if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that

$$\alpha^*(Gx, Gy)\, H(Gx, Gy) \le \psi(d(x, y)) \tag{5}$$

for all $x, y \in X$.

Theorem 5 (see [3]). *Let (X, d) be a complete metric space, let $\alpha : X \times X \to [0, \infty)$ be a function, let $\psi \in \Psi$ be a strictly increasing map, and let G be a closed-valued α^*-admissible and α^*-ψ-contractive multifunction on X. Suppose that there exist $x_0 \in X$ and $x_1 \in Gx_0$ such that $\alpha(x_0, x_1) \ge 1$. Assume that if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \ge 1$ for all n and $x_n \to x$, then $\alpha(x_n, x) \ge 1$ for all n. Then G has a fixed point.*

Definition 6 (see [2]). Let (X, d) be a metric space and let $G : X \to CL(X)$ be a mapping. We say that G is a generalized (α^*, ψ)-contractive if there exists $\psi \in \Psi$ such that

$$\alpha^*(Gx, Gy)\, d(y, Gy) \le \psi(d(x, y)) \tag{6}$$

for each $x \in X$ and $y \in Gx$, where $\alpha^*(Gx, Gy) = \inf\{\alpha(a, b) : a \in Gx, b \in Gy\}$.

Theorem 7 (see [2]). *Let (X, d) be a complete metric space and let $G : X \to B(X)$ be a mapping such that for each $x \in X$ and $y \in Gx$, we have*

$$\alpha^*(Gx, Gy)\, \delta(y, Gy) \le \psi(d(x, y)), \tag{7}$$

where $\psi \in \Psi$. Assume that there exist $x_0 \in X$ and $x_1 \in Gx_0$ such that $\alpha(x_0, x_1) \ge 1$. Moreover G is an α^*-admissible mapping. Then there exists an orbit $\{x_n\}$ of G at x_0 and $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Moreover, $\{x\} = Gx$ if and only if $f(\xi) = \delta(\xi, G\xi)$ is lower semicontinuous at x.

2. Main Results

In this section, we state and proof our main results. We first give the definition of the following notion.

Definition 8. Let (X, d) be a metric space. We say that $G : X \to CL(X)$ is an integral type (α^*, ψ)-contractive mapping if there exist two functions $\psi \in \Psi$ and $\phi \in \Phi$ such that for each $x \in X$ and $y \in Gx$, there exists $z \in Gy$ satisfying

$$\int_0^{\alpha^*(Gx,Gy) d(y,z)} \phi(t)\,dt \le \psi\left(\int_0^{d(x,y)} \phi(t)\,dt\right), \tag{8}$$

where $\alpha^*(Gx, Gy) = \inf\{\alpha(a, b) : a \in Gx, b \in Gy\}$.

Example 9. Let $X = \mathbb{R}$ be endowed with the usual metric d. Define $G : X \to CL(X)$ by

$$Gx = \begin{cases} [x, \infty) & \text{if } x \ge 0 \\ (-\infty, 6x] & \text{if } x < 0, \end{cases} \tag{9}$$

and $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} x + y + 1 & \text{if } x, y \ge 0 \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

Take $\psi(t) = t/4$ and $\phi(t) = 2t$ for all $t \ge 0$. Then, for each $x \in X$ and $y \in Gx$, there exists $z \in Gy$ such that

$$\int_0^{\alpha^*(Gx,Gy) d(y,z)} \phi(t)\,dt \le \psi\left(\int_0^{d(x,y)} \phi(t)\,dt\right). \tag{11}$$

Hence G is an integral type (α^*, ψ)-contractive mapping. Note that (4) does not hold at $x = -2$.

Definition 10. We say that $\phi \in \Phi$ is an integral subadditive if, for each $a, b > 0$, we have

$$\int_0^{a+b} \phi(t)\,dt \le \int_0^{a} \phi(t)\,dt + \int_0^{b} \phi(t)\,dt. \tag{12}$$

We denote by Φ_s the class of all integral subadditive functions $\phi \in \Phi$.

Example 11. Let $\phi_1(t) = (1/2)(t + 1)^{-1/2}$ for all $t \ge 0$, $\phi_2(t) = (2/3)(t + 1)^{-1/3}$ for all $t \ge 0$, and $\phi_3(t) = e^{-t}$ for all $t \ge 0$. Then $\phi_i \in \Phi_s$, where $i = 1, 2, 3$.

Definition 12. Let (X, d) be a metric space. We say that $G : X \to CL(X)$ is a subintegral type (α^*, ψ)-contractive if there exist two functions $\psi \in \Psi$ and $\phi \in \Phi_s$ such that for each $x \in X$ and $y \in Gx$, there exists $z \in Gy$ satisfying

$$\int_0^{\alpha^*(Gx,Gy)d(y,z)} \phi(t)\,dt \le \psi\left(\int_0^{d(x,y)} \phi(t)\,dt\right), \quad (13)$$

where $\alpha^*(Gx, Gy) = \inf\{\alpha(a, b) : a \in Gx, b \in Gy\}$.

Example 13. Let $X = \mathbb{R}$ be endowed with the usual metric d. Define $G : X \to CL(X)$ by

$$Gx = \begin{cases} \left[\dfrac{x}{4}, \dfrac{x}{2}\right] & \text{if } x \ge 0, \\ [24x, 12x] & \text{if } x < 0, \end{cases} \quad (14)$$

and $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 2 & \text{if } x = y = 0, \\ 0 & \text{otherwise.} \end{cases} \quad (15)$$

Take $\psi(t) = t/3$ and $\phi(t) = (2/3)(t+1)^{-1/3}$ for all $t \ge 0$. Then, for each $x \in X$ and $y \in Gx$, there exists $z \in Gy$ such that

$$\int_0^{\alpha^*(Gx,Gy)d(y,z)} \phi(t)\,dt \le \psi\left(\int_0^{d(x,y)} \phi(t)\,dt\right). \quad (16)$$

Hence G is an subintegral type (α^*, ψ)-contractive mapping.

Theorem 14. *Let (X, d) be a complete metric space and let $G : X \to CL(X)$ be an α^*-admissible subintegral type (α^*, ψ)-contractive mapping. Assume that there exist $x_0 \in X$ and $x_1 \in Gx_0$ such that $\alpha(x_0, x_1) \ge 1$. Then there exists an orbit $\{x_n\}$ of G at x_0 and $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Moreover, x is a fixed point of G if and only if $f(\xi) = d(\xi, G\xi)$ is G orbitally lower semicontinuous at x.*

Proof. By the hypothesis, there exist $x_0 \in X$ and $x_1 \in Gx_0$ such that $\alpha(x_0, x_1) \ge 1$. Since G is α^*-admissible, then $\alpha^*(Gx_0, Gx_1) \ge 1$. For $x_0 \in X$ and $x_1 \in Gx_0$, there exists $x_2 \in Gx_1$ such that

$$\begin{aligned}\int_0^{d(x_1,x_2)} \phi(t)\,dt &\le \int_0^{\alpha^*(Gx_0,Gx_1)d(x_1,x_2)} \phi(t)\,dt \\ &\le \psi\left(\int_0^{d(x_0,x_1)} \phi(t)\,dt\right).\end{aligned} \quad (17)$$

Since ψ is nondecreasing, we have

$$\psi\left(\int_0^{d(x_1,x_2)} \phi(t)\,dt\right) \le \psi^2\left(\int_0^{d(x_0,x_1)} \phi(t)\,dt\right). \quad (18)$$

As $\alpha(x_1, x_2) \ge 1$ by α^*-admissibility of G, we have $\alpha^*(Gx_1, Gx_2) \ge 1$. For $x_1 \in X$ and $x_2 \in Gx_1$, there exists $x_3 \in Gx_2$ such that

$$\begin{aligned}\int_0^{d(x_2,x_3)} \phi(t)\,dt &\le \int_0^{\alpha^*(Gx_1,Gx_2)d(x_2,x_3)} \phi(t)\,dt \\ &\le \psi\left(\int_0^{d(x_1,x_2)} \phi(t)\,dt\right) \\ &\le \psi^2\left(\int_0^{d(x_0,x_1)} \phi(t)\,dt\right).\end{aligned} \quad (19)$$

Since ψ is nondecreasing, we have

$$\psi\left(\int_0^{d(x_2,x_3)} \phi(t)\,dt\right) \le \psi^3\left(\int_0^{d(x_0,x_1)} \phi(t)\,dt\right). \quad (20)$$

By continuing the same process, we get a sequence $\{x_n\}$ in X such that $x_n \in Gx_{n-1}$, $\alpha(x_{n-1}, x_n) \ge 1$, and

$$\int_0^{d(x_n,x_{n+1})} \phi(t)\,dt \le \psi^n\left(\int_0^{d(x_0,x_1)} \phi(t)\,dt\right), \quad \text{for each } n \in \mathbb{N}. \quad (21)$$

Letting $n \to \infty$ in above inequality, we have

$$\lim_{n\to\infty} \int_0^{d(x_n,x_{n+1})} \phi(t)\,dt = 0. \quad (22)$$

Also, we have

$$\lim_{n\to\infty} \int_0^{d(x_n,Gx_n)} \phi(t)\,dt = 0, \quad (23)$$

which implies that

$$\lim_{n\to\infty} d(x_n, Gx_n) = 0. \quad (24)$$

For any $n, p \in \mathbb{N}$, we have

$$d(x_n, x_{n+p}) \le \sum_{i=n}^{n+p-1} d(x_i, x_{i+1}). \quad (25)$$

Since $\phi \in \Phi_s$, it can be shown by induction that

$$\int_0^{d(x_n,x_{n+p})} \phi(t)\,dt \le \sum_{i=n}^{n+p-1} \int_0^{d(x_i,x_{i+1})} \phi(t)\,dt. \quad (26)$$

From (21) and (26), we have

$$\int_0^{d(x_n,x_{n+p})} \phi(t)\,dt \le \sum_{i=n}^{n+p-1} \psi^i\left(\int_0^{d(x_0,x_1)} \phi(t)\,dt\right). \quad (27)$$

Since $\psi \in \Psi$ it follows that $\{x_n\}$ is Cauchy sequence in X. As X is complete, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Suppose $f(\xi) = d(\xi, G\xi)$ is G orbitally lower semicontinuous at x^*; then

$$d(x^*, Gx^*) \le \liminf_n f(x_n) = \liminf_n d(x_n, Gx_n) = 0. \quad (28)$$

By closedness of G it follows that $x^* \in Gx^*$. Conversely, suppose that x^* is a fixed point of G then $f(x^*) = 0 \le \liminf_n f(x_n)$. □

Example 15. Let $X = \mathbb{R}$ be endowed with the usual metric d. Define $G : X \to CL(X)$ by

$$Gx = \begin{cases} [x, x+1] & \text{if } x \geq 0, \\ (-\infty, 6x] & \text{if } x < 0, \end{cases} \tag{29}$$

and $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} x + y + 1 & \text{if } x, y \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{30}$$

Take $\psi(t) = t/2$ and $\phi(t) = (1/2)(t+1)^{-1/2}$ for all $t \geq 0$. Then, for each $x \in X$ and $y \in Gx$, there exists $z \in Gy$ such that

$$\int_0^{\alpha^*(Gx,Gy)d(y,z)} \phi(t)\,dt \leq \psi\left(\int_0^{d(x,y)} \phi(t)\,dt\right). \tag{31}$$

Hence G is a subintegral type (α^*, ψ)-contractive mapping. Clearly, G is α^*-admissible. Also, we have $x_0 = 1$ and $x_1 = 2 \in Gx_0$ such that $\alpha(x_0, x_1) = 4$. Therefore, all the conditions of Theorem 14 are satisfied and G has infinitely many fixed points. Note that Theorem 2 in Section 1 is not applicable here. For example, take $x = -1$ and $y = -6$.

Definition 16. Let (X, d) be a metric space. We say that $G : X \to B(X)$ is an integral type (α^*, ψ, δ)-contractive mapping if there exist two functions $\psi \in \Psi$ and $\phi \in \Phi$ such that

$$\int_0^{\alpha^*(Gx,Gy)\delta(y,Gy)} \phi(t)\,dt \leq \psi\left(\int_0^{d(x,y)} \phi(t)\,dt\right) \tag{32}$$

for each $x \in X$ and $y \in Gx$, where $\alpha^*(Gx, Gy) = \inf\{\alpha(a, b) : a \in Gx, b \in Gy\}$.

Definition 17. Let (X, d) be a metric space. We say that $G : X \to B(X)$ is a subintegral type (α^*, ψ, δ)-contractive mapping if there exist two functions $\psi \in \Psi$ and $\phi \in \Phi_s$ such that

$$\int_0^{\alpha^*(Gx,Gy)\delta(y,Gy)} \phi(t)\,dt \leq \psi\left(\int_0^{d(x,y)} \phi(t)\,dt\right) \tag{33}$$

for each $x \in X$ and $y \in Gx$, where $\alpha^*(Gx, Gy) = \inf\{\alpha(a, b) : a \in Gx, b \in Gy\}$.

Theorem 18. *Let (X, d) be a complete metric space and let $G : X \to B(X)$ be an α^*-admissible subintegral type (α^*, ψ, δ)-contractive mapping. Assume that there exist $x_0 \in X$ and $x_1 \in Gx_0$ such that $\alpha(x_0, x_1) \geq 1$. Then there exists an orbit $\{x_n\}$ of G at x_0 and $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Moreover, $x \in X$ such that $\{x\} = Gx$ if and only if $f(\xi) = \delta(\xi, G\xi)$ is G orbitally lower semicontinuous at x.*

Proof. By the hypothesis, there exist $x_0 \in X$ and $x_1 \in Gx_0$ such that $\alpha(x_0, x_1) \geq 1$. Since G is α^*-admissible, then $\alpha^*(Gx_0, Gx_1) \geq 1$. For $x_0 \in X$ and $x_1 \in Gx_0$, we have

$$\int_0^{\alpha^*(Gx_0,Gx_1)\delta(x_1,Gx_1)} \phi(t)\,dt \leq \psi\left(\int_0^{d(x_0,x_1)} \phi(t)\,dt\right). \tag{34}$$

Since $Gx_1 \neq \emptyset$, then we have $x_2 \in Gx_1$ such that

$$\begin{aligned} \int_0^{d(x_1,x_2)} \phi(t)\,dt &\leq \int_0^{\alpha^*(Gx_0,Gx_1)\delta(x_1,Gx_1)} \phi(t)\,dt \\ &\leq \psi\left(\int_0^{d(x_0,x_1)} \phi(t)\,dt\right). \end{aligned} \tag{35}$$

Since ψ is nondecreasing, we have

$$\psi\left(\int_0^{d(x_1,x_2)} \phi(t)\,dt\right) \leq \psi^2\left(\int_0^{d(x_0,x_1)} \phi(t)\,dt\right). \tag{36}$$

As $\alpha(x_1, x_2) \geq 1$ by α^*-admissibility of G, we have $\alpha^*(Gx_1, Gx_2) \geq 1$. Thus, we have $x_3 \in Gx_2$ such that

$$\begin{aligned} \int_0^{d(x_2,x_3)} \phi(t)\,dt &\leq \int_0^{\alpha^*(Gx_1,Gx_2)\delta(x_2,Gx_2)} \phi(t)\,dt \\ &\leq \psi\left(\int_0^{d(x_1,x_2)} \phi(t)\,dt\right) \\ &\leq \psi^2\left(\int_0^{d(x_0,x_1)} \phi(t)\,dt\right). \end{aligned} \tag{37}$$

Since ψ is nondecreasing, we have

$$\psi\left(\int_0^{d(x_2,x_3)} \phi(t)\,dt\right) \leq \psi^3\left(\int_0^{d(x_0,x_1)} \phi(t)\,dt\right). \tag{38}$$

By continuing the same process, we get a sequence $\{x_n\}$ in X such that $x_n \in Gx_{n-1}$, $\alpha(x_{n-1}, x_n) \geq 1$, and

$$\begin{aligned} \int_0^{d(x_n,x_{n+1})} \phi(t)\,dt &\leq \int_0^{\delta(x_n,Gx_n)} \phi(t)\,dt \\ &\leq \psi^n\left(\int_0^{d(x_0,x_1)} \phi(t)\,dt\right), \\ &\qquad \text{for each } n \in \mathbb{N}. \end{aligned} \tag{39}$$

Letting $n \to \infty$ in above inequality, we have

$$\lim_{n\to\infty} \int_0^{\delta(x_n,Gx_n)} \phi(t)\,dt = 0, \tag{40}$$

which implies that

$$\lim_{n\to\infty} \delta(x_n, Gx_n) = 0. \tag{41}$$

For any $n, p \in \mathbb{N}$, we have

$$d(x_n, x_{n+p}) \leq \sum_{i=n}^{n+p-1} d(x_i, x_{i+1}). \tag{42}$$

Since $\phi \in \Phi_s$, it can be shown by induction that

$$\int_0^{d(x_n,x_{n+p})} \phi(t)\,dt \leq \sum_{i=n}^{n+p-1} \int_0^{d(x_i,x_{i+1})} \phi(t)\,dt. \tag{43}$$

From (39) and (43), we have

$$\int_0^{d(x_n,x_{n+p})} \phi(t)\,dt \leq \sum_{i=n}^{n+p-1} \psi^i \left(\int_0^{d(x_0,x_1)} \phi(t)\,dt \right). \quad (44)$$

Since $\psi \in \Psi$ it follows that $\{x_n\}$ is Cauchy sequence in X. As X is complete, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Suppose $f(\xi) = \delta(\xi, G\xi)$ is G orbitally lower semicontinuous at x^*; then

$$\delta(x^*, Gx^*) \leq \liminf_n f(x_n) = \liminf_n \delta(x_n, Gx_n) = 0. \quad (45)$$

Hence, $\{x^*\} = Gx^*$ because $\delta(A, B) = 0$ implies $A = B = \{a\}$. Conversely, suppose that $\{x^*\} = Gx^*$. Then $f(x^*) = 0 \leq \liminf_n f(x_n)$. □

Example 19. Let $X = \{1, 3, 5, 7, 9, \ldots\}$ be endowed with the usual metric d. Define $G : X \to B(X)$ by

$$Gx = \begin{cases} \{x-2, x+2\} & \text{if } x \neq 1, \\ \{1\} & \text{if } x = 1, \end{cases} \quad (46)$$

and $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x = y = 1, \\ \dfrac{1}{4} & \text{otherwise.} \end{cases} \quad (47)$$

Take $\psi(t) = t/2$ and $\phi(t) = (2/3)(t+1)^{-1/3}$ for all $t \geq 0$. Clearly, G is an α^*-admissible subintegral type (α^*, ψ, δ)-contractive mapping. Also, we have $x_0 = 1$ and $x_1 = 1 \in Gx_0$ such that $\alpha(x_0, x_1) = 1$. Therefore, all the conditions of Theorem 18 hold and G has fixed points.

Example 20. Let $X = \mathbb{R}$ be endowed with the usual metric d. Define $G : X \to B(X)$ by

$$Gx = \begin{cases} \{\lfloor x \rfloor, \lceil x \rceil\} & \text{if } x \geq 0, \\ \left(\dfrac{\lfloor x \rfloor}{4}, \dfrac{\lceil x \rceil}{2} \right) & \text{if } x < 0, \end{cases} \quad (48)$$

and $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \geq 0, \\ 0 & \text{otherwise.} \end{cases} \quad (49)$$

Take $\psi(t) = t/4$ and $\phi(t) = e^{-t}$ for all $t \geq 0$. Then it is easy to check that all the conditions of Theorem 18 hold. Therefore G has infinitely many fixed points.

Remark 21. Let $\phi(t) = 1$ for all $t \geq 0$; Theorem 18 reduces to Theorem 7 in Section 1.

Remark 22. Note that subadditivity of the integral was needed in the proofs of Theorems 14 and 18 in order to obtain inequalities (26) and (43). It is natural to ask wether the conclusions of Theorems 14 and 18 are valid if we replace subintegral contractive conditions (13) and (33) by integral contractive conditions (8) and (32), respectively. Looking at our proofs, we can say that it will be true if the inequalities (26) and (43) hold. Here we would like to mention that many authors (see for example [14, 23]) while proving the results on integral contractions have not assumed that the integral is subadditive but indeed they used the subadditivity of the integral in the proofs of their results while obtaining the inequalities comparable to inequalities (26) and (43).

3. Application

In this section, we obtain some fixed point results for partially ordered metric spaces, as consequences of aforementioned results. Moreover, we apply our result to prove the existence of solution for an integral equation.

Let A and B be subsets of a partially ordered set. We say that $A \preceq_r B$, if for each $a \in A$ and $b \in B$, we have $a \preceq b$.

Theorem 23. *Let $(X, \preceq, d)$ be a complete ordered metric space and let $G : X \to CL(X)$ be a mapping such that for each $x \in X$ and $y \in Gx$ with $x \preceq y$, there exists $z \in Gy$ satisfying*

$$\int_0^{d(y,z)} \phi(t)\,dt \leq \psi \left(\int_0^{d(x,y)} \phi(t)\,dt \right), \quad (50)$$

where $\psi \in \Psi$ and $\phi \in \Phi_s$. Assume that there exist $x_0 \in X$ and $x_1 \in Gx_0$ such that $x_0 \preceq x_1$. Also, assume that $x \preceq y$ implies $Gx \preceq_r Gy$. Then there exists an orbit $\{x_n\}$ of G at x_0 and $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Moreover, x is a fixed point of G if and only if $f(\xi) = d(\xi, G\xi)$ is G orbitally lower semicontinuous at x.

Proof. Define $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x \preceq y, \\ 0 & \text{otherwise.} \end{cases} \quad (51)$$

By using hypothesis of corollary and definition of α, we have $\alpha(x_0, x_1) = 1$. As $x \preceq y$ implies $Gx \preceq_r Gy$, by using the definitions of α and $\preceq_r$, we have that $\alpha(x, y) = 1$ implies $\alpha^*(Gx, Gy) = 1$. Moreover, it is easy to check that G is an integral type (α^*, ψ)-contractive mapping. Therefore, by Theorem 14, there exists an orbit $\{x_n\}$ of G at x_0 and $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Moreover, x is a fixed point of G if and only if $f(\xi) = d(\xi, G\xi)$ is G orbitally lower semicontinuous at x. □

Considering $G : X \to X$ and $\phi(t) = 1$ for each $t \geq 0$, Theorem 23 reduces to following result.

Corollary 24. *Let $(X, \preceq, d)$ be a complete ordered metric space and let $G : X \to X$ be a nondecreasing mapping such that, for each $x \in X$ with $x \preceq Gx$, we have*

$$d(Gx, G^2 x) \leq \psi(d(x, Gx)), \quad (52)$$

where $\psi \in \Psi$. Assume that there exists $x_0 \in X$ such that $x_0 \preceq Gx_0$. Then there exists an orbit $\{x_n\}$ of G at x_0 and $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Moreover, x is a fixed point of G if and only if $f(\xi) = d(\xi, G\xi)$ is G orbitally lower semicontinuous at x.

Consider an integral equation of the form

$$x(t) = \int_a^b K(t, s, x(s))\, ds, \quad t \in [a, b], \tag{53}$$

where $K : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing.

Theorem 25. *Assume that*

(i) *for $u, v \in C([a, b], \mathbb{R})$, with $u(t) \leq v(t)$ for each $t \in [a, b]$, we have*

$$|K(t, s, u(t)) - K(t, s, v(t))| \leq \frac{\psi(d(u, v))}{(b - a)} \tag{54}$$

for each $t, s \in [a, b]$, where $\psi \in \Psi$;

(ii) *for each $t, s \in [a, b]$, there exists $x_0 \in C([a, b], \mathbb{R})$ such that*

$$x_0(t) \leq \int_a^b K(t, s, x_0(s))\, ds. \tag{55}$$

Then there exists an iterative sequence $\{x_n\}$, starting from x_0, and $x \in C([a, b], \mathbb{R})$ such that $\lim_{n\to\infty} x_n = x$. Moreover, x is a solution of (53) *if and only if $f(\xi) = d(\xi, y)$ is lower semicontinuous at x, where $y(t) = \int_a^b K(t, s, \xi(s))ds$.*

Proof. It is easy to see that $X = C([a, b], \mathbb{R})$ is complete with respect to the metric $d(x, y) = \max_{t\in[a,b]} |x(t) - y(t)|$. We define partial ordering on X as follows: $x \preceq y$ if and only if $x(t) \leq y(t)$ for each $t \in [a, b]$. Define $G : X \to X$ by $Gx = y$, where $y(t) = \int_a^b K(t, s, x(s))ds$, for each $t, s \in [a, b]$. From (ii), we have $x_0 \preceq Gx_0$. For $x \in X$, let $Gx = y$ and $Gy = z$; that is, $y(t) = \int_a^b K(t, s, x(s))ds$ and $z(t) = \int_a^b K(t, s, y(s))ds$, for each $t, s \in [a, b]$. Then, for each $x \in X$ with $x \preceq Gx$, we have

$$\begin{aligned} d\left(Gx, G^2 x\right) &= \max_{t\in[a,b]} |y(t) - z(t)| \\ &= \max_{t\in[a,b]} \left| \int_a^b K(t, s, x(s))\, ds \right. \\ &\qquad \left. - \int_a^b K(t, s, y(s))\, ds \right| \\ &\leq \max_{t\in[a,b]} \int_a^b |K(t, s, x(s)) - K(t, s, y(s))|\, ds \\ &\leq \frac{\psi(d(x, Gx))}{(b - a)} (b - a). \end{aligned} \tag{56}$$

That is $d(Gx, G^2 x) \leq \psi(d(x, Gx))$, for each $x \in X$ with $x \preceq Gx$. Clearly, G is nondecreasing. Therefore, all conditions of Corollary 24 hold and the conclusions follow from Corollary 24. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' Contribution

All authors contributed equally and significantly in writing this paper. All authors read and approved the final paper.

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Research Article

Invariant Means and Reversible Semigroup of Relatively Nonexpansive Mappings in Banach Spaces

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The purpose of this paper is to study modified Halpern type and Ishikawa type iteration for a semigroup of relatively nonexpansive mappings $\mathfrak{I} = \{T(s) : s \in S\}$ on a nonempty closed convex subset C of a Banach space with respect to a sequence of asymptotically left invariant means $\{\mu_n\}$ defined on an appropriate invariant subspace of $l^\infty(S)$, where S is a semigroup. We prove that, given some mild conditions, we can generate iterative sequences which converge strongly to a common element of the set of fixed points $F(\mathfrak{I})$, where $F(\mathfrak{I}) = \bigcap\{F(T(s)) : s \in S\}$.

1. Introduction

Let E be a real Banach space with the topological dual E^* and let C be a closed and convex subset of E. A mapping T of C into itself is called *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for each $x, y \in C$.

Three classical iteration processes are often used to approximate a fixed point of a nonexpansive mapping. The first one is introduced by Halpern [1] and is defined as follows:

$$\begin{aligned} x_0 &= u \in C, \quad \text{chosen arbitrarily,} \\ x_{n+1} &= \alpha_n u + (1 - \alpha_n) T x_n, \quad \forall n \ge 1, \end{aligned} \tag{1}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$. He pointed out that the conditions $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$ are necessary in the sense that if the iteration (1) converges to a fixed point of T, then these conditions must be satisfied. The second iteration process is known as Mann's iteration process [2] which is defined as follows:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad \forall n \ge 1, \tag{2}$$

where the initial x_1 is taken in C arbitrary and the sequence $\{\alpha_n\}$ is in $[0, 1]$.

The third iteration process is referred to as Ishikawa's iteration process [3] which is defined as follows:

$$\begin{aligned} y_n &= \beta_n x_n + (1 - \beta_n) T x_n, \\ x_{n+1} &= \alpha_n x_n + (1 - \alpha_n) T y_n, \quad \forall n \ge 1, \end{aligned} \tag{3}$$

where the initial x_1 is taken in C arbitrary and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$.

In 2007, Lau et al. [4] proposed the following modification of Halpern's iteration (1) for amenable semigroups of nonexpansive mappings in a Banach space.

Theorem 1. *Let S be a left reversible semigroup and let $\mathfrak{I} = \{T(s) : s \in S\}$ be a representation of S as nonexpansive mappings from a compact convex subset C of a strictly convex and smooth Banach space E into C, let X be an amenable and $\mathfrak{I}$-stable subspace of $l^\infty(S)$, and let $\{\mu_n\}$ be a strongly left regular sequence of means on X. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ such that $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Let $x_1 = x \in C$ and let $\{x_n\}$ be the sequence defined by*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) T(\mu_n) x_n, \quad n \ge 2. \tag{4}$$

Then $\{x_n\}$ converges strongly to Px, where P denotes the unique sunny nonexpansive retraction of C onto $F(\mathfrak{I})$.

Let C be a closed and convex subset of E and let T be a mapping from C into itself. We denote by $F(T)$ the set of fixed points of T. Point p in C is said to be an *asymptotic fixed point* of T [5] if C contains a sequence $\{x_n\}$ which converges weakly to p such that the strong $\lim_{n\to\infty}(Tx_n - x_n) = 0$. The set of asymptotic fixed points of T will be denoted by $\widehat{F}(T)$. A mapping T from C into itself is called *relatively nonexpansive* [6–8], if $\widehat{F}(T) = F(T)$ and $\phi(p, Tx) \le \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. The asymptotic behavior of relatively nonexpansive mappings was studied in [6, 7, 9].

Recently, Kim [10] proved a strong convergence theorem for relatively nonexpansive mappings in a Banach space by using the shrinking method.

Theorem 2. *Let S be a left reversible semigroup and let $\mathfrak{I} = \{T(s) : s \in S\}$ be a representation of S as relatively nonexpansive mappings from a nonempty, closed, and convex subset C of a uniformly convex and uniformly smooth Banach space E into C with $F(\mathfrak{I}) \neq \emptyset$. Let X be a subspace of $l^\infty(S)$ and let $\{\mu_n\}$ be a asymptotically left invariant sequence of means on X. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ such that $0 < \alpha_n < 1$ and $\lim_{n\to\infty}\alpha_n = 0$. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\begin{gathered} x_0 \in C, \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}\left(\alpha_n J x_1 + (1-\alpha_n) J T_{\mu_n} x_n\right), \\ C_{n+1} = \{z \in C_n : \\ \phi(z, y_n) \le \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \ge 1. \end{gathered} \tag{5}$$

Then $\{x_n\}$ converges strongly to $\Pi_{F(\mathfrak{I})}x_1$, where $\Pi_{F(\mathfrak{I})}$ is the generalized projection from C onto $F(\mathfrak{I})$.

Let S be a semigroup. The purpose of this paper is to study modified Halpern type and Ishikawa type iterations for a semigroup of relatively nonexpansive mappings $\mathfrak{I} = \{T(s) : s \in S\}$ on a nonempty closed convex subset C of a Banach space with respect to a sequence of asymptotically left invariant means $\{\mu_n\}$ defined on an appropriate invariant subspace of $l^\infty(S)$. We prove that, given some mild conditions, we can generate iterative sequences which converge strongly to a common element of the set of fixed points $F(\mathfrak{I})$, where $F(\mathfrak{I}) = \bigcap\{F(T(s)) : s \in S\}$.

2. Preliminaries

A real Banach space E is said to be *strictly convex* if $\|(x+y)/2\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be *uniformly convex* if $\lim_{n\to\infty}\|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty}\|(x_n + y_n)/2\| = 1$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be *smooth* if

$$\lim_{t\to 0}\frac{\|x+ty\| - \|x\|}{t} \tag{6}$$

exists for each $x, y \in U$. It is said to be *uniformly smooth* if the limit is attained uniformly for $x, y \in E$.

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the dual space of E. Denote by $\langle\cdot,\cdot\rangle$ the duality product. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \left\{f^* \in E^* : \langle x, f^*\rangle = \|x\|^2 = \|f^*\|^2\right\}, \tag{7}$$

for $x \in E$. A Banach space E is said to have the Kadec-Klee property if a sequence $\{x_n\}$ of E satisfies that $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ and then $x_n \to x$, where $\rightharpoonup$ and $\to$ denote the weak convergence and the strong convergence, respectively.

We know the following:

(1) the duality mapping J is monotone, that is, $\langle x-y, x^* - y^*\rangle \ge 0$ whenever $x^* \in Jx$ and $y^* \in Jy$;

(2) if E is strictly convex, then J is one-to-one; that is, if $Jx \cap Jy$ is nonempty, then $x = y$;

(3) if E is strictly convex, then J is strictly monotone; that is, $x = y$ whenever $\langle x-y, x^* - y^*\rangle = 0$, $x^* \in Jx$ and $y^* \in Jy$;

(4) if E is uniformly convex, then E has the Kadec-Klee property;

(5) if E is uniformly convex, then E is reflexive and strictly convex;

(6) if E is smooth, then J is single-valued and norm-to-weak* continuous;

(7) if E is uniformly smooth, then J is uniformly norm-to-norm continuous on bounded subsets of E;

(8) if E is reflexive, then J is onto;

(9) if E is smooth and reflexive, then J is norm-to-weak continuous; that is, $Jx_n \rightharpoonup Jx$ whenever $x_n \to x$;

(10) if E is smooth, strictly convex, and reflexive, then J is single-valued, one-to-one and onto; in this case, the inverse mapping J^{-1} coincides with the duality mapping on E;

(11) if E^* is strictly convex, then J is single-valued;

(12) the norm of E^* is Fréchet differentiable if and only if E is strictly convex and reflexive Banach space which has the Kadec-Klee property.

For more details, see [11].

As well known, if C is a nonempty, closed, and convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive (see, the reference therein). This fact actually characterizes Hilbert spaces. Consequently, it is not true to more general Banach spaces. In this connection, Alber [12] introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection in Hilbert spaces. Consider the function defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \tag{8}$$

for $x, y \in E$. Observe that, in a Hilbert space H, (8) reduces to

$$\phi(x, y) = \|x - y\|^2, \tag{9}$$

for $x, y \in H$. The generalized projection $\Pi_C : E \to C$ is a mapping that assigns an arbitrary point $x \in E$ to the minimum point of the functional $\phi(x, y)$; that is, $\Pi_C x = \overline{x}$, where $\overline{x}$ is the solution to the minimization problem:

$$\phi(\overline{x}, x) = \inf_{y \in C} \phi(y, x). \tag{10}$$

The existence and uniqueness of the operator Π_C follow from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping J (see, e.g., [12, 13]). In a Hilbert space, $\Pi_C = P_C$. It is obvious from the definition of the function ϕ that

(ϕ_1) $(\|x\| - \|y\|)^2 \le \phi(x, y) \le (\|x\| + \|y\|)^2$ for all $x, y \in E$,

(ϕ_2) $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle$ for all $x, y, z \in E$,

(ϕ_3) $\phi(x, y) = \langle x, Jx - Jy\rangle + \langle y - x, Jy\rangle \le \|x\|\|Jx - Jy\| + \|y - x\|\|y\|$ for all $x, y \in E$,

(ϕ_4) if E is a reflexive, strictly convex, and smooth Banach space, then, for all $x, y \in E$,

$$\phi(x, y) = 0 \quad \text{iff } x = y. \tag{11}$$

For more details see [14].

Let S be a semigroup. We denote by $l^\infty(S)$ the Banach space of all bounded real-valued functionals on S with supremum norm. For each $s \in S$, we define the left and right translation operators $l(s)$ and $r(s)$ on $l^\infty(S)$ by

$$(l(s) f)(t) = f(st), \qquad (r(s) f)(t) = f(ts), \tag{12}$$

for each $t \in S$ and $f \in l^\infty(S)$, respectively. Let X be a subspace of $l^\infty(S)$ containing 1. An element μ in the dual space X^* of X is said to be a *mean* on X if $\|\mu\| = \mu(1) = 1$. For $s \in S$, we can define a point evaluation δ_s by $\delta_s(f) = f(s)$ for each $f \in X$. It is well known that μ is mean on X if and only if

$$\inf_{s \in S} f(s) \le \mu(f) \le \sup_{s \in S} f(s), \tag{13}$$

for each $f \in X$.

Let X be a translation invariant subspace of $l^\infty(S)$ (i.e., $l(s)X \subset X$ and $r(s)X \subset X$ for each $s \in S$) containing 1. Then a mean μ on X is said to be *left invariant* (resp., *right invariant*) if

$$\mu(l(s) f) = \mu(f), \quad (\text{resp., } \mu(r(s) f) = \mu(f)) \tag{14}$$

for each $s \in S$ and $f \in X$. A mean μ on X is said to be *invariant* if μ is both left and right invariant [15–19]. X is said to be *left* (resp., *right*) *amenable* if X has a left (resp., right) invariant mean. X is amenable if X is left and right amenable. We call a semigroup S *amenable* if X is amenable. Further, amenable semigroups include all commutative semigroups and solvable groups. However, the free group or semigroup of two generators is not left or right amenable (see [20–22]).

A net $\{\mu_\alpha\}$ of means on X is said to be *asymptotically left* (resp., *right*) *invariant* if

$$\lim_\alpha (\mu_\alpha(l(s) f) - \mu_\alpha(f)) = 0, \qquad \left(\text{resp., } \lim_\alpha (\mu_\alpha(r(s) f) - \mu_\alpha(f)) = 0\right), \tag{15}$$

for each $f \in X$ and $s \in S$, and it is said to be *left* (resp., *right*) *strongly asymptotically invariant* (or *strong regular*) if

$$\lim_\alpha \|l^*(s)\mu_\alpha - \mu_\alpha\| = 0, \qquad \left(\text{resp., } \lim_\alpha \|r^*(s)\mu_\alpha - \mu_\alpha\| = 0\right), \tag{16}$$

for each $s \in S$, where $l^*(s)$ and $r^*(s)$ are the adjoint operators of $l(s)$ and $r(s)$, respectively. Such nets were first studied by Day in [20] where they were called *weak* invariant* and *norm invariant*, respectively.

It is easy to see that if a semigroup S is left (resp., right) amenable, then the semigroup $S' = S \cup \{e\}$, where $es' = s'e = s'$ for all $s' \in S$, is also left (resp., right) amenable and converse.

From now on S denotes a semigroup with an identity e. S is called *left reversible* if any two right ideals of S have nonvoid intersection; that is, $aS \cap bS \ne \emptyset$ for $a, b \in S$. In this case, $(S, \preceq)$ is a directed system when the binary relation "$\preceq$" on S is defined by $a \preceq b$ if and only if $\{a\} \cup aS \supseteq \{b\} \cup bS$ for $a, b \in S$. It is easy to see that $t \preceq ts$ for all $t, s \in S$. Further, if $t \preceq s$ then $pt \preceq ps$ for all $p \in S$. The class of left reversible semigroup includes all groups and commutative semigroups. If a semigroup S is left amenable, then S is left reversible. But the converse is not true [23–28].

Let S be a semigroup and let C be a closed and convex subset of E. Let $F(T)$ denote the fixed point set of T. Then $\mathfrak{T} = \{T(s) : s \in S\}$ is called a *representation of S as relatively nonexpansive mappings on* C if $T(s)$ is relatively nonexpansive with $T(e) = I$ and $T(st) = T(s)T(t)$ for each $s, t \in S$. We denote by $F(\mathfrak{T})$ the set of common fixed points of $\{T(s) : s \in S\}$; that is,

$$F(\mathfrak{T}) = \bigcap_{s \in S} F(T(s)) = \bigcap_{s \in S} \{x \in C : T(s)x = x\}. \tag{17}$$

We know that if μ is a mean on X and if for each $x^* \in E^*$ the function $s \mapsto \langle T(s)x, x^*\rangle$ is contained in X and C is weakly compact, then there exists a unique point x_0 of E such that $\mu\langle T(\cdot)x, x^*\rangle = \langle x_0, x^*\rangle$ for each $x^* \in E^*$. We denote such a point x_0 by $T_\mu x$. Note that $T_\mu x$ is contained in the closure of the convex hull of $\{T(s)x : s \in S\}$ for each $x \in C$. Note that $T_\mu z = z$ for each $z \in F(\mathfrak{T})$; see [29–31].

3. Lemmas

We need the following lemmas for the proof of our main results.

Lemma 3 (see [9]). *Let E be a strictly convex and smooth Banach space, let C be a closed convex subset of E, and let T*

be a relatively nonexpansive mapping from C into itself. Then $F(T)$ is closed and convex.

Lemma 4 (see [12, 32]). *Let E be a reflexive, strictly convex, and smooth Banach space and let C be a nonempty, closed, and convex subset of E and $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \tag{18}$$

for all $y \in C$.

Lemma 5 (see [32]). *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$, $\{y_n\}$ be two sequences of E. If $\lim_{n\to\infty}\phi(x_n, y_n) = 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\lim_{n\to\infty}\|x_n - y_n\| = 0$.*

Lemma 6 (see [4, 33]). *Let μ be a left invariant mean on X. Then $F(\mathfrak{I}) = F(T_\mu) \cap C_a$, where C_a denotes the set of almost periodic elements in C; that is, all $x \in C$ such that $\{T(s)x : s \in S\}$ is relatively compact in the norm topology of E.*

Lemma 7 (cf. [4, 10]). *Let $\{\mu_n\}$ be an asymptotically left invariant sequence of means on X. If $z \in C_a$ and $\liminf_{n\to\infty}\|T_{\mu_n}z - z\| = 0$, then z is a common fixed point of $\mathfrak{I}$.*

4. Strong Convergence Theorems

In this section, we will establish two strong convergence theorems of various iterative sequences for finding common fixed point of relatively nonexpansive mappings in a uniformly convex and uniformly smooth Banach spaces (cf. [34–36]).

We begin with a strong convergence theorem of modified Halpern's type.

Theorem 8. *Let S be a left reversible semigroup and let $\mathfrak{I} = \{T(s) : s \in S\}$ be a representation of S as relatively nonexpansive mappings from a nonempty, closed, and convex subset C of a uniformly convex and uniformly smooth Banach space E into itself. Let X be a subspace of $l^\infty(S)$ and let $\{\mu_n\}$ be an asymptotically left invariant sequence of means on X. Let $\{\alpha_n\}$ be a sequence in $(0, 1)$ such that $\lim_{n\to\infty}\alpha_n = 0$. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\begin{gathered} x_0 \in C, \quad \textit{chosen arbitrarily}, \\ x_{n+1} = \Pi_C J^{-1}\left(\alpha_n J x_0 + (1-\alpha_n) J T_{\mu_n} x_n\right), \quad \forall n \geq 0. \end{gathered} \tag{19}$$

If the interior of $F(\mathfrak{I})$ is nonempty, then $\{x_n\}$ converges strongly to some common fixed point $F(\mathfrak{I})$.

Proof. We show first that the sequence $\{x_n\}$ converges strongly in C.

From Lemma 3, we know $F(T)$ is closed and convex. So, we can define the generalized projection Π_C onto $F(\mathfrak{I})$. Most of all, we have

$$\begin{aligned} \left\|T_{\mu_n} x_n\right\| &= \sup\left\{\left|\left\langle T_{\mu_n} x_n, x^*\right\rangle\right| : x^* \in E^*, \|x^*\| = 1\right\} \\ &= \sup\left\{\left|(\mu_n)_s \left\langle T(s) x_n, x^*\right\rangle\right| : x^* \in E^*, \|x^*\| = 1\right\} \\ &\leq \sup\left\{(\mu_n)_s \left(\|T(s) x_n\| \|x^*\|\right) : x^* \in E^*, \|x^*\| = 1\right\} \\ &= (\mu_n)_s \|T(s) x_n\|. \end{aligned} \tag{20}$$

Then, from the definition of relatively nonexpansive, we have

$$\begin{aligned} \phi\left(u, T_{\mu_n} x_n\right) &= \|u\|^2 - 2\left\langle u, J T_{\mu_n} x_n\right\rangle + \left\|T_{\mu_n} x_n\right\|^2 \\ &= \|u\|^2 - 2(\mu_n)_s \left\langle u, J T(s) x_n\right\rangle \\ &\quad + (\mu_n)_s \|T(s) x_n\|^2 \\ &= (\mu_n)_s \phi(u, T(s) x_n) \\ &\leq (\mu_n)_s \phi(u, x_n) = \phi(u, x_n), \end{aligned} \tag{21}$$

for all $u \in F(\mathfrak{I})$. From the convexity of $\|\cdot\|^2$ and (21), we get

$$\begin{aligned} &\phi(u, x_{n+1}) \\ &\quad = \phi\left(u, \Pi_C J^{-1}\left(\alpha_n J x_0 + (1-\alpha_n) J T_{\mu_n} x_n\right)\right) \\ &\quad \leq \phi\left(u, J^{-1}\left(\alpha_n J x_0 + (1-\alpha_n) J T_{\mu_n} x_n\right)\right) \\ &\quad = \|u\|^2 - 2\left\langle u, \alpha_n J x_0 + (1-\alpha_n) J T_{\mu_n} x_n\right\rangle \\ &\qquad + \left\|\alpha_n J x_0 + (1-\alpha_n) J T_{\mu_n} x_n\right\|^2 \\ &\quad \leq \|u\|^2 - 2\alpha_n \langle u, J x_0\rangle - 2(1-\alpha_n)\left\langle u, J T_{\mu_n} x_n\right\rangle \\ &\qquad + \alpha_n \|x_0\|^2 + (1-\alpha_n)\left\|T_{\mu_n} x_n\right\|^2 \\ &\quad = \alpha_n \phi(u, x_0) + (1-\alpha_n)\phi\left(u, T_{\mu_n} x_n\right) \\ &\quad \leq \alpha_n \phi(u, x_0) + (1-\alpha_n)\phi(u, x_n). \end{aligned} \tag{22}$$

So, we have

$$\begin{aligned} &(1-\alpha_n)\{\phi(u, x_{n+1}) - \phi(u, x_n)\} \\ &\quad \leq \alpha_n\{\phi(u, x_0) - \phi(u, x_{n+1})\} \\ &\quad \leq \alpha_n \phi(u, x_0). \end{aligned} \tag{23}$$

Since $\lim_{n\to\infty}\alpha_n = 0$, we obtain

$$\lim_{n\to\infty}\{\phi(u, x_{n+1}) - \phi(u, x_n)\} \leq 0. \tag{24}$$

Therefore $\{\phi(u, x_n)\}$ is bounded and $\lim_{n\to\infty}\phi(u, x_n)$ exists. Then $\{x_n\}$ is also bounded. This implies that $\{T_{\mu_n} x_n\}$ is bounded. Since the interior of $F(\mathfrak{I})$ is nonempty, there exist $p \in F(\mathfrak{I})$ and $r > 0$ such that

$$p + rq \in F(\mathfrak{I}), \tag{25}$$

whenever $\|q\| \leq 1$. By (ϕ_2), we have

$$\begin{aligned} \phi(u, x_n) &= \phi(u, x_{n+1}) + \phi(x_{n+1}, x_n) \\ &\quad + 2\langle u - x_{n+1}, J x_{n+1} - J x_n\rangle, \end{aligned} \tag{26}$$

for any $u \in F(\mathfrak{I})$. This implies

$$\langle x_{n+1} - u, Jx_n - Jx_{n+1} \rangle + \frac{1}{2}\phi(x_{n+1}, x_n) = \frac{1}{2}(\phi(u, x_n) - \phi(u, x_{n+1})). \tag{27}$$

Also, we have

$$\begin{aligned} &\langle x_{n+1} - p, Jx_n - Jx_{n+1} \rangle \\ &\quad = \langle x_{n+1} - (p + rq) + rq, Jx_n - Jx_{n+1} \rangle \\ &\quad = \langle x_{n+1} - (p + rq), Jx_n - Jx_{n+1} \rangle \\ &\qquad + r\langle q, Jx_n - Jx_{n+1} \rangle. \end{aligned} \tag{28}$$

On the other hand, by (24) and (25), we have that

$$\phi(p + rq, x_{n+1}) \le \phi(p + rq, x_n). \tag{29}$$

From (27), we get

$$\begin{aligned} 0 &\le \frac{1}{2}(\phi(p + rq, x_n) - \phi(p + rq, x_{n+1})) \\ &= \langle x_{n+1} - (p + rq), Jx_n - Jx_{n+1} \rangle \\ &\quad + \frac{1}{2}\phi(x_{n+1}, x_n) \\ &= \langle x_{n+1} - p, Jx_n - Jx_{n+1} \rangle \\ &\quad - r\langle q, Jx_n - Jx_{n+1} \rangle + \frac{1}{2}\phi(x_{n+1}, x_n). \end{aligned} \tag{30}$$

Then, by (27), we have

$$\begin{aligned} &r\langle q, Jx_n - Jx_{n+1} \rangle \\ &\quad \le \langle x_{n+1} - p, Jx_n - Jx_{n+1} \rangle + \frac{1}{2}\phi(x_{n+1}, x_n) \\ &\quad = \frac{1}{2}(\phi(p, x_n) - \phi(p, x_{n+1})), \end{aligned} \tag{31}$$

for $p \in F(\mathfrak{I})$. Hence

$$\langle q, Jx_n - Jx_{n+1} \rangle \le \frac{1}{2r}(\phi(p, x_n) - \phi(p, x_{n+1})). \tag{32}$$

Since q with $\|q\| \le 1$ is arbitrary, by (24), we have

$$\|Jx_n - Jx_{n+1}\| \le \frac{1}{2r}(\phi(p, x_n) - \phi(p, x_{n+1})). \tag{33}$$

So, we have

$$\begin{aligned} &\|Jx_{n+m} - Jx_n\| \\ &\quad = \|Jx_{n+m} - Jx_{n+m-1} + Jx_{n+m-1} \\ &\qquad - \cdots - Jx_{n+1} + Jx_{n+1} - Jx_n\| \\ &\quad \le \sum_{i=n}^{n+m-1} \|Jx_i - Jx_{i+1}\| \\ &\quad \le \frac{1}{2r}\sum_{i=n}^{n+m-1}(\phi(p, x_i) - \phi(p, x_{i+1})) \\ &\quad = \frac{1}{2r}(\phi(p, x_n) - \phi(p, x_{n+1})). \end{aligned} \tag{34}$$

We know that $\{\phi(p, x_n)\}$ converges. Hence, $\{Jx_n\}$ is a Cauchy sequence. Since E^* is complete, $\{Jx_n\}$ converges strongly to some point in E^*. Since E is uniformly convex, E^* has a Fréchet differentiable norm. Then J^{-1} is continuous on E^*. Hence $\{x_n\}$ converges strongly to some point v in C.

Now, we show that $v \in F(\mathfrak{I})$, where $v = \lim_{n\to\infty}\Pi_{F(\mathfrak{I})}x_n$.

By (33) and the convergence of $\{\phi(p, x_n)\}$, we have

$$\lim_{n\to\infty} \|Jx_n - Jx_{n+1}\| = 0. \tag{35}$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, it follows that

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{36}$$

Let $z_n = J^{-1}(\alpha_n Jx_0 + (1 - \alpha_n)JT_{\mu_n}x_n)$. Then, we have

$$\begin{aligned} &\|Jz_n - JT_{\mu_n}x_n\| \\ &\quad = \|\alpha_n Jx_0 + (1 - \alpha_n)JT_{\mu_n}x_n - JT_{\mu_n}x_n\| \\ &\quad = \alpha_n\|Jx_0 - JT_{\mu_n}x_n\|. \end{aligned} \tag{37}$$

Since $\lim_{n\to\infty}\alpha_n = 0$, we have

$$\lim_{n\to\infty} \|Jz_n - JT_{\mu_n}x_n\| = 0. \tag{38}$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, we get

$$\lim_{n\to\infty} \|z_n - T_{\mu_n}x_n\| = 0. \tag{39}$$

From $x_{n+1} = \Pi_C z_n$ and Lemma 4, we have

$$\begin{aligned} &\phi(T_{\mu_n}x_n, x_{n+1}) + \phi(x_{n+1}, z_n) \\ &\quad = \phi(T_{\mu_n}x_n, \Pi_C z_n) + \phi(\Pi_C z_n, z_n) \\ &\quad \le \phi(T_{\mu_n}x_n, z_n). \end{aligned} \tag{40}$$

Since

$$\begin{aligned}\phi\left(T_{\mu_n}x_n, z_n\right) &= \phi\left(T_{\mu_n}x_n, J^{-1}\left(\alpha_n Jx_0 + (1-\alpha_n) JT_{\mu_n}x_n\right)\right)\\ &= \left\|T_{\mu_n}x_n\right\|^2 - 2\left\langle T_{\mu_n}x_n, \alpha_n Jx_0 + (1-\alpha_n) JT_{\mu_n}x_n\right\rangle\\ &\quad + \left\|\alpha_n Jx_0 + (1-\alpha_n) JT_{\mu_n}x_n\right\|^2\\ &\le \left\|T_{\mu_n}x_n\right\|^2 - 2\alpha_n\left\langle T_{\mu_n}x_n, Jx_0\right\rangle\\ &\quad - 2(1-\alpha_n)\left\langle T_{\mu_n}x_n, JT_{\mu_n}x_n\right\rangle\\ &\quad + \alpha_n\|x_0\|^2 + (1-\alpha_n)\left\|T_{\mu_n}x_n\right\|^2\\ &= \alpha_n\phi\left(T_{\mu_n}x_n, x_0\right) + (1-\alpha_n)\phi\left(T_{\mu_n}x_n, T_{\mu_n}x_n\right)\\ &= \alpha_n\phi\left(T_{\mu_n}x_n, x_0\right)\end{aligned} \tag{41}$$

and $\lim_{n\to\infty}\alpha_n = 0$, we have

$$\lim_{n\to\infty}\phi\left(T_{\mu_n}x_n, z_n\right) = 0. \tag{42}$$

From (40), we get

$$\operatorname*{Lim}_{n\to\infty}\phi\left(T_{\mu_n}x_n, x_{n+1}\right) = \lim_{n\to\infty}\phi\left(x_{n+1}, z_n\right) = 0. \tag{43}$$

By Lemma 5, we obtain

$$\lim_{n\to\infty}\left\|T_{\mu_n}x_n - x_{n+1}\right\| = \lim_{n\to\infty}\left\|x_{n+1} - z_n\right\| = 0. \tag{44}$$

Since $\|x_n - T_{\mu_n}x_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\| + \|z_n - T_{\mu_n}x_n\|$, from (36), (39), and (44), we have

$$\lim_{n\to\infty}\left\|x_n - T_{\mu_n}x_n\right\| = 0. \tag{45}$$

From Lemma 7, we have $x_n \in F(\mathfrak{I})$. Since $F(\mathfrak{I})$ is closed and $\lim_{n\to\infty}x_n = v$, we have $v \in F(\mathfrak{I})$, where $v = \lim_{n\to\infty}\Pi_{F(\mathfrak{I})}x_n$. □

We now establish a convergence theorem of modified Ishikawa type.

Theorem 9. *Let S be a left reversible semigroup and let $\mathfrak{I} = \{T(s) : s \in S\}$ be a representation of S as relatively nonexpansive mappings from a nonempty, closed, and convex subset C of a uniformly convex and uniformly smooth Banach space E into itself. Let X be a subspace of $l^\infty(S)$ and let $\{\mu_n\}$ be an asymptotically left invariant sequence of means on X. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences of real numbers such that $\alpha_n, \beta_n \in (0,1)$ and $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\beta_n = 1$. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\begin{gathered}x_0 \in C, \quad \text{chosen arbitrarily},\\ y_n = J^{-1}\left(\beta_n Jx_n + (1-\beta_n) JT_{\mu_n}x_n\right),\\ x_{n+1} = \Pi_C J^{-1}\left(\alpha_n Jx_n + (1-\alpha_n) JT_{\mu_n}y_n\right), \quad \forall n \ge 0.\end{gathered} \tag{46}$$

If the interior of $F(\mathfrak{I})$ is nonempty, then $\{x_n\}$ converges strongly to some common fixed point $F(\mathfrak{I})$.

Proof. Firstly, we show that $\{x_n\}$ converges strongly in C.

From Lemma 3, we know $F(T)$ is closed and convex. So, we can define the generalized projection Π_C onto $F(\mathfrak{I})$. Let $u \in F(\mathfrak{I})$. From the definition of relatively nonexpansive and the convexity of $\|\cdot\|^2$, from (21), we have

$$\begin{aligned}\phi(u, y_n) &= \phi\left(u, J^{-1}\left(\beta_n Jx_n + (1-\beta_n) JT_{\mu_n}x_n\right)\right)\\ &\le \beta_n\phi(u, x_n) + (1-\beta_n)\phi\left(u, T_{\mu_n}x_n\right)\\ &\le \phi(u, x_n),\end{aligned} \tag{47}$$

for all $u \in F(\mathfrak{I})$. From (47), we obtain

$$\begin{aligned}\phi(u, x_{n+1}) &= \phi\left(u, \Pi_C J^{-1}\left(\alpha_n Jx_n + (1-\alpha_n) JT_{\mu_n}y_n\right)\right)\\ &\le \phi\left(u, J^{-1}\left(\alpha_n Jx_n + (1-\alpha_n) JT_{\mu_n}y_n\right)\right)\\ &\le \alpha_n\phi(u, x_n) + (1-\alpha_n)\phi\left(u, T_{\mu_n}y_n\right)\\ &\le \alpha_n\phi(u, x_n) + (1-\alpha_n)\phi(u, y_n)\\ &\le \phi(u, x_n).\end{aligned} \tag{48}$$

Hence, $\{\phi(u, x_n)\}$ is bounded and $\lim_{n\to\infty}\phi(u, x_n)$ exists. This implies that $\{x_n\}$, $\{T_{\mu_n}x_n\}$, and $\{y_n\}$ are bounded. Since the interior of $F(\mathfrak{I})$ is nonempty, similar to the proof of Theorem 8, we obtain that $\{x_n\}$ converges strongly to v in C.

Next, we show that $v \in F(\mathfrak{I})$, where $v = \lim_{n\to\infty}\Pi_{F(\mathfrak{I})}x_n$. Let

$$z_n = J^{-1}\left(\alpha_n Jx_n + (1-\alpha_n) JT_{\mu_n}y_n\right). \tag{49}$$

From Lemma 4, we have

$$\begin{aligned}\phi(x_n, x_{n+1}) + \phi(x_{n+1}, z_n) &= \phi(x_n, \Pi_C z_n) + \phi(\Pi_C z_n, z_n)\\ &\le \phi(x_n, z_n).\end{aligned} \tag{50}$$

Also,

$$\begin{aligned}\phi(x_n, z_n) &= \phi\left(x_n, J^{-1}\left(\alpha_n Jx_n + (1-\alpha_n) JT_{\mu_n}y_n\right)\right)\\ &\le \alpha_n\phi(x_n, x_n) + (1-\alpha_n)\phi\left(x_n, T_{\mu_n}y_n\right)\\ &\le \alpha_n\phi(x_n, x_n) + (1-\alpha_n)\phi(x_n, y_n)\\ &\le \phi(x_n, y_n),\end{aligned} \tag{51}$$

$$\begin{aligned}\|Jx_n - Jy_n\| &= \left\|Jx_n - \left(\beta_n Jx_n - (1-\beta_n) JT_{\mu_n}x_n\right)\right\|\\ &= (1-\beta_n)\left\|Jx_n - JT_{\mu_n}x_n\right\|.\end{aligned} \tag{52}$$

From $\lim_{n\to\infty}\beta_n = 1$ and (52), we have

$$\lim_{n\to\infty}\|Jx_n - Jy_n\| = 0. \tag{53}$$

Since J^{-1} is uniformly norm-to-norm continuous, we obtain

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \tag{54}$$

Hence,

$$\begin{aligned}\phi(x_n, y_n) &= \|x_n\|^2 - 2\langle x_n, Jy_n\rangle + \|y_n\|^2 \\ &= \|x_n\|^2 - 2\langle x_n, Jy_n - Jx_n\rangle \\ &\quad - 2\langle x_n, Jx_n\rangle + \|y_n\|^2 \\ &\le \|y_n\|^2 - \|x_n\|^2 \\ &\quad + 2\|x_n\|\|Jy_n - Jx_n\| \\ &\le \|y_n - x_n\|(\|y_n\| + \|x_n\|) \\ &\quad + 2\|x_n\|\|Jy_n - Jx_n\|.\end{aligned} \tag{55}$$

By (53) and (54), we have

$$\lim_{n\to\infty}\phi(x_n, y_n) = 0. \tag{56}$$

From (50) and (51), we obtain

$$\lim_{n\to\infty}\phi(x_n, x_{n+1}) = \lim_{n\to\infty}\phi(x_n, z_n) = 0. \tag{57}$$

From Lemma 5, we get

$$\lim_{n\to\infty}\|x_n - x_{n+1}\| = \lim_{n\to\infty}\|x_n - z_n\| = 0. \tag{58}$$

Since

$$\begin{aligned}\|Jz_n - JT_{\mu_n}y_n\| &= \|\alpha_n Jx_n + (1-\alpha_n)JT_{\mu_n}y_n - JT_{\mu_n}y_n\| \\ &= \alpha_n\|Jx_n - JT_{\mu_n}x_n\|\end{aligned} \tag{59}$$

and $\lim_{n\to\infty}\alpha_n = 0$, we have

$$\lim_{n\to\infty}\|Jz_n - JT_{\mu_n}y_n\| = 0. \tag{60}$$

Since J^{-1} is uniformly norm-to-norm continuous, we obtain

$$\lim_{n\to\infty}\|z_n - T_{\mu_n}y_n\| = 0. \tag{61}$$

Since $\lim_{n\to\infty}\|x_n - z_n\| = 0$ and J is uniformly norm-to-norm continuous,

$$\lim_{n\to\infty}\|Jx_n - Jz_n\| = 0. \tag{62}$$

By (46) and (49), we have

$$\begin{aligned}JT_{\mu_n}x_n &= \frac{1}{1-\beta_n}(Jy_n - \beta_n Jx_n), \\ JT_{\mu_n}y_n &= \frac{1}{1-\alpha_n}(Jz_n - \alpha_n Jx_n).\end{aligned} \tag{63}$$

From (63), we obtain

$$\begin{aligned}&\|JT_{\mu_n}x_n - JT_{\mu_n}y_n\| \\ &\quad = \left\|\frac{1}{1-\beta_n}(Jy_n - \beta_n Jx_n) - \frac{1}{1-\alpha_n}(Jz_n - \alpha_n Jx_n)\right\| \\ &\quad = \left\|Jy_n + \frac{\beta_n}{1-\beta_n}(Jy_n - Jx_n) - \left(Jz_n + \frac{\alpha_n}{1-\alpha_n}(Jz_n - Jx_n)\right)\right\| \\ &\quad \le \|Jy_n - Jx_n\| + \|Jx_n - Jz_n\| \\ &\qquad + \frac{\beta_n}{1-\beta_n}\|Jy_n - Jx_n\| + \frac{\alpha_n}{1-\alpha_n}\|Jz_n - Jx_n\| \\ &\quad = \frac{1}{1-\alpha_n}\|Jz_n - Jx_n\| + \frac{1}{1-\beta_n}\|Jy_n - Jx_n\|.\end{aligned} \tag{64}$$

Combining (53), (62), and (64), we get

$$\lim_{n\to\infty}\|JT_{\mu_n}x_n - JT_{\mu_n}y_n\| = 0. \tag{65}$$

Since J^{-1} is uniformly norm-to-norm continuous, we have

$$\lim_{n\to\infty}\|T_{\mu_n}x_n - T_{\mu_n}y_n\| = 0. \tag{66}$$

Since

$$\begin{aligned}\|x_n - T_{\mu_n}x_n\| &\le \|x_n - z_n\| + \|z_n - T_{\mu_n}y_n\| \\ &\quad + \|T_{\mu_n}y_n - T_{\mu_n}x_n\|,\end{aligned} \tag{67}$$

therefore, by (58), (61), (66), and (67), we obtain

$$\lim_{n\to\infty}\|x_n - T_{\mu_n}x_n\| = 0. \tag{68}$$

From Lemma 7, we have $x_n \in F(\mathfrak{I})$. Since $F(\mathfrak{I})$ is closed and $\lim_{n\to\infty}x_n = v$, we have $v \in F(\mathfrak{I})$, where $v = \lim_{n\to\infty}\Pi_{F(\mathfrak{I})}x_n$. □

If we set $\beta_n = 1$, then the iteration (46) reduces modified Mann type. Hence we obtain the following corollary.

Corollary 10. *Let S be a left reversible semigroup and let $\mathfrak{I} = \{T(s) : s \in S\}$ be a representation of S as relatively nonexpansive mappings from a nonempty, closed, and convex subset C of a uniformly convex and uniformly smooth Banach space E into itself. Let X be a subspace of $l^\infty(S)$ and let $\{\mu_n\}$ be an asymptotically left invariant sequence of means on X. Let $\{\alpha_n\}$ be a sequence of real number such that $\alpha_n \in (0,1)$ and $\lim_{n\to\infty}\alpha_n = 0$. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\begin{aligned}&x_0 \in C, \quad \textit{chosen arbitrarily}, \\ &x_{n+1} = \Pi_C J^{-1}\left(\alpha_n Jx_n + (1-\alpha_n)JT_{\mu_n}x_n\right), \quad \forall n \ge 0.\end{aligned} \tag{69}$$

If the interior of $F(\mathfrak{I})$ is nonempty, then $\{x_n\}$ converges strongly to some common fixed point $F(\mathfrak{I})$.

In a Hilbert space, J is the identity operator. Theorems 8 and 9 reduce to the following.

Corollary 11. *Let S be a left reversible semigroup and let $\mathfrak{I} = \{T(s) : s \in S\}$ be a representation of S as relatively nonexpansive mappings from a nonempty, closed, and convex subset C of a Hilbert space H into itself. Let X be a subspace of $l^{\infty}(S)$ and let $\{\mu_n\}$ be an asymptotically left invariant sequence of means on X. Let $\{\alpha_n\}$ be a sequence in $(0,1)$ such that $\lim_{n\to\infty}\alpha_n = 0$. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$
\begin{gathered}
x_0 \in C, \quad \text{chosen arbitrarily}, \\
x_{n+1} = P_C\left(\alpha_n x_0 + \left(1-\alpha_n\right) T_{\mu_n} x_n\right), \quad \forall n \geq 0.
\end{gathered} \tag{70}
$$

If the interior of $F(\mathfrak{I})$ is nonempty, then $\{x_n\}$ converges strongly to some common fixed point $F(\mathfrak{I})$, where P_C is a metric projection.

Corollary 12. *Let S be a left reversible semigroup and let $\mathfrak{I} = \{T(s) : s \in S\}$ be a representation of S as relatively nonexpansive mappings from a nonempty, closed, and convex subset C of a Hilbert space H into itself. Let X be a subspace of $l^{\infty}(S)$ and let $\{\mu_n\}$ be an asymptotically left invariant sequence of means on X. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences of real numbers such that $\alpha_n, \beta_n \in (0,1)$ and $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\beta_n = 1$. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$
\begin{gathered}
x_0 \in C, \quad \text{chosen arbitrarily}, \\
y_n = \beta_n x_n + \left(1-\beta_n\right) T_{\mu_n} x_n, \\
x_{n+1} = P_C\left(\alpha_n x_n + \left(1-\alpha_n\right) T_{\mu_n} y_n\right), \quad \forall n \geq 0.
\end{gathered} \tag{71}
$$

If the interior of $F(\mathfrak{I})$ is nonempty, then $\{x_n\}$ converges strongly to some common fixed point $F(\mathfrak{I})$, where P_C is a metric projection.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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Research Article

Some Common Fixed Point Results for Modified Subcompatible Maps and Related Invariant Approximation Results

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We improve the class of subcompatible self-maps used by (Akbar and Khan, 2009) by introducing a new class of noncommuting self-maps called modified subcompatible self-maps. For this new class, we establish some common fixed point results and obtain several invariant approximation results as applications. In support of the proved results, we also furnish some illustrative examples.

1. Introduction and Preliminaries

From the last five decades, fixed point theorems have been used in many instances in invariant approximation theory. The idea of applying fixed point theorems to approximation theory was initiated by Meinardus [1] where he employs a fixed point theorem of Schauder to establish the existence of an invariant approximation. Later on, Brosowski [2] used fixed point theory to establish some interesting results on invariant approximation in the setting of normed spaces and generalized Meinardus's results. Singh [3], Habiniak [4], Sahab et al. [5], and Jungck and Sessa [6] proved some similar results in the best approximation theory. Further, Al-Thagafi [7] extended these works and proved some invariant approximation results for commuting self-maps. Al-Thagafi results have been further extended by Hussain and Jungck [8], Shahzad [9–14] and O'Regan and Shahzad [15] to various class of noncommuting self-maps, in particular to R-subweakly commuting and R-subcommuting self-maps. Recently, Akbar and Khan [16] extended the work of [7–15] to more general noncommuting class, namely, the class of subcompatible self-maps.

In this paper, we improve the class of subcompatible self-maps used by Akbar and Khan [16] by introducing a new class of noncommuting self-maps called modified subcompatible self-maps which contain commuting, R-subcommuting, R-subweakly, commuting, and subcompatible maps as a proper subclass. For this new class, we establish some common fixed point results for some families of self-maps and obtain several invariant approximation results as applications. The proved results improve and extend the corresponding results of [3–8, 10–15].

Before going to the main work, we need some preliminaries which are as follows.

Definition 1. Let (X, d) be a metric space, M be a subset of X, and S and T be self-maps of M. Then the family $\{A_i : i \in \mathbb{N} \cup \{0\}\}$ of self-maps of M is called (S, T):

(i) *contraction* if there exists k, $0 \le k < 1$ such that for all $x, y \in M$,

$$d(A_0x, A_iy) \le kd(Sx, Ty), \quad \text{for each } i \in \mathbb{N}, \tag{1}$$

(ii) *nonexpansive* if for all $x, y \in M$,

$$d(A_0x, A_iy) \le d(Sx, Ty), \quad \text{for each } i \in \mathbb{N}. \tag{2}$$

In Definition 1, if we take $T = S$, then this family $\{A_i : i \in \mathbb{N} \cup \{0\}\}$ is called S-contraction (resp., S-nonexpansive).

Definition 2. Let M be a subset of a metric space (X, d) and S, T be self-maps of M. A point $x \in M$ is a coincidence point (common fixed point) of S and T if $Sx = Tx$ $(Sx = Tx = x)$.

The set of coincidence points of S and T is denoted by $C(S,T)$. The pair $\{S,T\}$ is called

(1) commuting if $STx = TSx$ for all $x \in M$;

(2) R-weakly commuting [17], provided there exists some positive real number R such that $d(STx, TSx) \leq Rd(Sx, Tx)$ for each $x \in M$;

(3) compatible [18] if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$ whenever $\{x_n\}$ is a sequence in M such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in M$;

(4) weakly compatible [19] if $STx = TSx$ for all $x \in C(S,T)$.

For a useful discussion on these classes, that is, the class of commuting, R-weakly commuting, compatible, and weakly compatible maps, see also [20].

Definition 3. Let X be a linear space and let M be a subset of X. The set M is said to be star-shaped if there exists at least one point $q \in M$ such that the line segment $[x, q]$ joining x to q is contained in M for all $x \in M$; that is, $kx + (1-k)q \in M$ for all $x \in M$, where $0 \leq k \leq 1$.

Definition 4. Let X be a linear space and let M be a subset of X. A self-map $A : M \to M$ is said to be

(i) *affine* [21] if M is convex and

$$A(kx + (1-k)y) = kA(x) + (1-k)A(y) \quad \forall x, y \in M, \quad k \in (0,1), \tag{3}$$

(ii) *q-affine* [21] if M is q-star-shaped and

$$A(kx + (1-k)q) = kA(x) + (1-k)q \quad \forall x \in M, \quad k \in (0,1). \tag{4}$$

Here we observe that if A is q-affine then $Aq = q$.

Remark 5. Every affine map A is q-affine if $Aq = q$ but its converse need not be true even if $Aq = q$, as shown by the following examples.

Example 6. Let $X = R$ and $M = [0,1]$. Let $A : M \to M$ be defined as

$$A(x) = \begin{cases} x & \text{if } 0 \leq x < \dfrac{1}{2} \\ 1-x & \text{if } \dfrac{1}{2} \leq x \leq 1. \end{cases} \tag{5}$$

Then A is q-affine for $q = 1/2$, while A is not affine because for $x = 3/5$, $y = 0$, and $k = 1/3$

$$A(kx + (1-k)y) = kA(x) + (1-k)A(y) \tag{6}$$

does not hold.

Example 7. Let $X = R^2$ and $\lambda \in R^+ = [0, \infty)$. Let $M = M_1 \cup M_2$, where

$$M_1 = \{(x,y) \in R^2 : (x,y) = (\lambda, 3\lambda)\}, \quad M_2 = \{(x,y) \in R^2 : (x,y) = (\lambda, \lambda)\}. \tag{7}$$

Then M is q-star-shaped for $q = (0,0)$. Define $A : M \to M$ as

$$A(x,y) = \begin{cases} (0,0) & \text{if } (x,y) \in M_1 \\ (x,y) & \text{if } (x,y) \in M_2. \end{cases} \tag{8}$$

Then A is q-affine for $q = (0,0)$ but A is not affine, because for $x = (1,3) \in M$, $y = (1,1) \in M$, and $k = 1/2$, $kx + (1-k)y \notin M$, though $kA(x) + (1-k)A(y) = (1/2, 1/2) \in M$.

Definition 8. Let M be a subset of a normed linear space $(X, \|\cdot\|)$. The set $B_M(p) = \{x \in M : \|x - p\| = \operatorname{dist}(p, M)\}$ is called the set of best approximants to $p \in X$ out of M, where $\operatorname{dist}(p, M) = \inf\{\|y - p\| : y \in M\}$.

Definition 9 (see [11]). Let M be a subset of a normed linear space X and let S and T be self-maps of M. Then the pair (S,T) is called R-subweakly commuting on M with respect to q if M is q-star-shaped with $q \in F(S)$ (where $F(S)$ denote the set of fixed point of S) and $\|STx - TSx\| \leq R \operatorname{dist}(Sx, [q, Tx])$ for all $x \in M$ and some $R > 0$.

Definition 10. Let X be a Banach space. A map $S : M \subseteq X \to X$ is said to be demiclosed at 0 whenever $\{x_n\}$ is a sequence in M such that x_n converges weakly to $x \in M$ and Sx_n converges strongly to $0 \in M$; then $0 = Sx$.

Definition 11. A Banach space X is said to satisfy Opial's condition whenever $\{x_n\}$ is a sequence in X such that x_n converges weakly to $x \in X$; then

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \quad \text{holds } \forall y \neq x. \tag{9}$$

Note that Hilbert and l^p $(1 < p < \infty)$ spaces satisfy Opial's condition.

2. Common Fixed Point for Modified Subcompatible Self-Maps

First we introduce the notion of modified subcompatible maps.

Definition 12. Let M be a q-star-shaped subset of a normed linear space X and let S and T be self-maps of M with $q \in F(S)$. Define $\Lambda_q(S,T) = \bigcup_{k\in(0,1)} \Lambda(S, T_k)$, where $T_k(x) = (1-k)q + kTx$ and $\Lambda(S, T_k) = \{\{x_n\} \subset M : \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} T_k x_n = t \in M\}$. Then S and T are called modified subcompatible if $\lim_{n\to\infty} \|STx_n - TSx_n\| = 0$ for all sequences $\{x_n\} \in \Lambda_q(S,T)$.

In the definition of subcompatible maps (see [16]), $\Lambda_q(S,T) = \bigcup_{k\in[0,1]} \Lambda(S, T_k)$, but here $k \in (0,1)$. The following examples reveal the impact of this and show that R-subweakly commuting maps and also subcompatible maps of [16] form a proper subclass of modified subcompatible maps.

Example 13. Let $X = R$ with the usual norm and $M = [0, \infty)$. Define $S, T : M \to M$ by

$$S(x) = \begin{cases} \dfrac{x}{2}, & 0 \le x < 1 \\ 2x^2 - 1, & x \ge 1, \end{cases}$$
$$T(x) = \begin{cases} \dfrac{1}{2}, & 0 \le x < 1 \\ 4x - 3, & x \ge 1. \end{cases} \tag{10}$$

Then M is 1-star-shaped with $q = 1 \in F(S)$ and $\Lambda_q(S,T) = \{\{x_n\} : 1 \le x_n < \infty, \lim_{n\to\infty} x_n = 1\}$. Moreover, S and T are modified subcompatible but not subcompatible because for the sequence $\{1 - 1/n\}_{n\ge1}$, we have $\lim_{n\to\infty} S(x_n) = \lim_{n\to\infty} T_1(x_n) = 1/2$ and $\lim_{n\to\infty}\|ST(x_n) - TS(x_n)\| \ne 0$. Note that S and T are neither R-subweakly commuting nor R-subcommuting.

Example 14. Let $X = R$ with the usual norm and $M = [0, \infty)$. Define $S, T : M \to M$ by

$$S(x) = \begin{cases} \dfrac{1}{2}, & 0 \le x < 1 \\ x^2, & x \ge 1, \end{cases}$$
$$T(x) = \begin{cases} \dfrac{3}{2}, & 0 \le x < 1 \\ x, & x \ge 1. \end{cases} \tag{11}$$

Then M is $1/2$-star-shaped with $q = 1/2 \in F(S)$ and $\Lambda_q(S,T) = \phi$. Clearly S and T are modified subcompatible but not subcompatible because for any sequence $\{x_n\}_{0 \le x_n < 1}$, we have $\lim_{n\to\infty} S(x_n) = \lim_{n\to\infty} T_0(x_n) = 1/2$ and $\lim_{n\to\infty}\|ST(x_n) - TS(x_n)\| \ne 0$. Also, S and T are not R-subweakly commuting.

The following two examples show that the modified subcompatible self-maps and compatible self-maps are of different classes.

Example 15. Let $X = R$ with usual norm and $M = [1, \infty)$. Let $S, T : M \to M$ be defined by

$$S(x) = 6x - 5, \qquad T(x) = 3x^2 - 2, \tag{12}$$

for all $x \in M$. Then

$$\begin{aligned} \|T(x_n) - S(x_n)\| &= 3\left\|(x_n - 1)^2\right\| \longrightarrow 0 \\ &\qquad \text{iff } x_n \longrightarrow 1, \\ \|ST(x_n) - TS(x_n)\| &= 90\left\|(x_n - 1)^2\right\| \longrightarrow 0 \\ &\qquad \text{if } x_n \longrightarrow 1. \end{aligned} \tag{13}$$

Thus S and T are compatible. Obviously M is q-star-shaped with $q = 1$ and $Sq = q$. Note that for any sequence $\{x_n\}$ in M with $x_n \to 2$, we have

$$\|T_{2/3}(x_n) - S(x_n)\| = 2\|(x_n - 1)(x_n - 2)\| \longrightarrow 0. \tag{14}$$

However, $\lim_{n\to\infty}\|ST(x_n) - TS(x_n)\| \ne 0$. Thus S and T are not modified subcompatible maps. Hence, they are not R-subweakly commuting.

Example 16. Let $X = R$ with norm $\|x\| = |x|$ and $M = [0, \infty)$. Let $S, T : M \to M$ be defined by

$$S(x) = \begin{cases} x, & 0 \le x < 1 \\ 3, & x \ge 1, \end{cases}$$
$$T(x) = \begin{cases} 3 - 2x, & 0 \le x < 1 \\ 3, & x \ge 1, \end{cases} \tag{15}$$
$$\forall x \in M.$$

Then M is 3-star-shaped with $S(3) = 3$ and $\Lambda_q(S,T) = \{\{x_n\} : 1 \le x_n < \infty\}$. Clearly S and T are modified subcompatible. Moreover, for any sequence $\{x_n\}$ in $[0, 1)$ with $\lim_{n\to\infty} x_n = 1$, we have $\lim_{n\to\infty}\|T(x_n) - S(x_n)\| = 0$. However, $\lim_{n\to\infty}\|ST(x_n) - TS(x_n)\| \ne 0$. Thus S and T are not compatible.

The following general common fixed point result is a consequence of Theorem 5.1 of Jachymski [22], which will be needed in the sequel.

Theorem 17. *Let S and T be self-maps of a complete metric space (X, d) and either S or T is continuous. Suppose $\{A_i\}_{i\in\mathbb{N}\cup\{0\}}$ is a sequence of self-maps of X satisfying the following.*

(1) *$A_0(X) \subseteq T(X)$ and $A_i(X) \subseteq S(X)$ for each $i \in \mathbb{N}$.*

(2) *The pairs (A_0, S) and (A_i, T) are compatible for each $i \in \mathbb{N}$.*

(3) *For each $i \in \mathbb{N}$ and, for any $x, y \in M$,*

$$d(A_0x, A_iy) \le h \max M(x, y) \quad \text{for some } h \in (0, 1), \tag{16}$$

where

$$\begin{aligned} M(x, y) = \Big\{ & d(Sx, Ty), d(A_0x, Sx), \\ & d(A_iy, Ty), \\ & \frac{1}{2}\left[d(A_0x, Ty) + d(A_iy, Sx)\right]\Big\}; \end{aligned} \tag{17}$$

then there exists a unique point z in X such that $z = Sz = Tz = A_iz$, for each $i \in \mathbb{N} \cup \{0\}$.

The following result extends and improves [7, Theorem 2.2], [8, Theorem 2.2], [6, Theorem 6], and [13, Theorem 2.2].

Theorem 18. *Let M be a nonempty q-star-shaped subset of a normed space X and let S and T be continuous and q-affine* *self-maps of M. Let $\{A_i\}_{i\in\mathbb{N}\cup\{0\}}$ be a family of self-maps of M satisfying the following.*

(1) *$A_0(M) \subseteq T(M)$ and $A_i(M) \subseteq S(M)$ for each $i \in \mathbb{N}$.*

(2) *(A_0, S) and (A_i, T) are modified subcompatible for each $i \in \mathbb{N}$.*

(3) *For each $i \in \mathbb{N}$ and, for any $x, y \in M$*

$$\|A_0x - A_iy\| \leq \max M(x, y), \tag{18}$$

where

$$\begin{aligned} M(x, y) = \Big\{ &\|Sx - Ty\|, \operatorname{dist}(Sx, [A_0x, q]), \\ &\operatorname{dist}(Ty, [A_iy, q]), \\ &\frac{1}{2}[\operatorname{dist}(Sx, [A_iy, q]) \\ &\quad + \operatorname{dist}(Ty, [A_0x, q])] \Big\}; \end{aligned} \tag{19}$$

then all the A_i ($i \in \mathbb{N} \cup \{0\}$), S and T have a common fixed point provided one of the following conditions hold.

(a) *M is sequentially compact and A_i is continuous for each $i \in \mathbb{N} \cup \{0\}$.*

(b) *M is weakly compact, $(S - A_i)$ is demiclosed at 0 for each $i \in \mathbb{N} \cup \{0\}$, and X is complete.*

Proof. For each $i \in \mathbb{N} \cup \{0\}$, define $A_i^n : M \to M$ by

$$A_i^n x = (1 - k_n) q + k_n A_i x \tag{20}$$

for all $x \in M$ and a fixed sequence of real numbers k_n ($0 < k_n < 1$) converging to 1. Then, A_i^n is a self-map of M for each $i \in \mathbb{N} \cup \{0\}$ and for each $n \geq 1$.

Firstly, we prove $A_0^n(M) \subseteq T(M)$; for this let $y \in A_0^n(M)$, which implies $y = A_0^n x$ for some $x \in M$.

Now, by using (20)

$$\begin{aligned} y = A_0^n x &= (1 - k_n) q + k_n A_0 x \\ &= (1 - k_n) q + k_n Tz, \quad \text{for some } z \in M \\ &\Longrightarrow y \in T(M), \quad \text{as } T \text{ is } q\text{-affine}, M \text{ is } q\text{-star-shaped}. \end{aligned} \tag{21}$$

Hence $A_0^n(M) \subseteq T(M)$ for each $n \geq 1$.

Similarly, it can be shown that for each $i \in \mathbb{N}$ and each $n \geq 1$, $A_i^n(M) \subseteq S(M)$, as S is q-affine and M is q-star-shaped.

Now, we prove that for each $n \geq 1$, the pair (A_0^n, S) is compatible; for this let $\{x_m\} \subseteq M$ with $\lim_{m\to\infty} Sx_m = \lim_{m\to\infty} A_0^n x_m = t \in M$. Since the pair (A_0, S) is modified subcompatible, therefore, by the assumption of A_{0_k}, we have

$$\lim_{m\to\infty} A_{0_{k_n}} x_m = \lim_{m\to\infty} A_0^n x_m = t. \tag{22}$$

As the pair (A_0, S) is modified subcompatible and S is q-affine, therefore

$$\begin{aligned} &\lim_{m\to\infty} \|A_0^n S x_m - S A_0^n x_m\| \\ &\quad = k_n \lim_{m\to\infty} \|A_0 S x_m - S A_0 x_m\| = 0. \end{aligned} \tag{23}$$

Hence, the pair (A_0^n, S) is compatible for each n.

Similarly, we can prove that the pair (A_i^n, T) is compatible for each $i \in \mathbb{N}$ and each $n \geq 1$.

Also, using (18) and (20) we have

$$\begin{aligned} \|A_0^n x - A_i^n y\| &= k_n \|A_0 x - A_i y\| \\ &\leq k_n \max \Big\{ \|Sx - Ty\|, \operatorname{dist}(Sx, [A_0x, q]), \\ &\qquad \operatorname{dist}(Ty, [A_iy, q]), \\ &\qquad \frac{1}{2}[\operatorname{dist}(Sx, [A_iy, q]) \\ &\qquad\quad + \operatorname{dist}(Ty, [A_0x, q])] \Big\} \\ &\leq k_n \max \Big\{ \|Sx - Ty\|, \|Sx - A_0^n x\|, \\ &\qquad \|Ty - A_i^n y\|, \\ &\qquad \frac{1}{2}[\|Sx - A_i^n y\| \\ &\qquad\quad + \|Ty - A_0^n x\|] \Big\} \end{aligned} \tag{24}$$

for each $x, y \in M$ and $0 < k_n < 1$. By Theorem 17, for each $n \geq 1$, there exists $x_n \in M$ such that $x_n = Sx_n = Tx_n = A_i^n x_n$, for each $i \in \mathbb{N} \cup \{0\}$.

(a) As M is sequentially compact and $\{x_n\}$ is a sequence in M, so $\{x_n\}$ has a convergent subsequence $\{x_m\}$ such that $x_m \to z \in M$. Thus, by the continuity of S, T and all A_i ($i \in \mathbb{N}\cup\{0\}$), we can say that z is a common fixed point of S, T and all A_i ($i \in \mathbb{N}\cup\{0\}$). Thus $F(T)\cap F(S)\cap F(A_0) \cap (\bigcap_{i\in\mathbb{N}} F(A_i)) \neq \phi$.

(b) Since M is weakly compact, there is a subsequence $\{x_m\}$ of $\{x_n\}$ converging weakly to some $u \in M$. But, S and T being q-affine and continuous are weakly continuous, and the weak topology is Hausdorff, so u is a common fixed point of S and T. Again the set M is bounded, so $(S-A_i)(x_m) = x_m - x_m k_m^{-1} - q(1-k_m^{-1}) \to 0$ as $m \to \infty$. Now demiclosedness of $(S - A_i)$ at 0 gives that $(S - A_i)(u) = 0$ for each $i \in \mathbb{N} \cup \{0\}$, and hence $F(T) \cap F(S) \cap F(A_0) \cap (\bigcap_{i\in\mathbb{N}} F(A_i)) \neq \phi$. □

Theorem 19. *Let M be a nonempty q-star-shaped subset of a normed space X, and let S and T be continuous and q-affine self-maps of M. Let $\{A_i\}_{i\in\mathbb{N}\cup\{0\}}$ be a family of self-maps with $A_0(M) \subseteq T(M)$ and $A_i(M) \subseteq S(M)$ for each $i \in \mathbb{N}$. If the pairs (A_0, S) and (A_i, T) are modified subcompatible for each $i \in \mathbb{N}$*

and also the family $\{A_i\}_{i\in\mathbb{N}\cup\{0\}}$ *of maps is* (S,T)*-nonexpansive, then* $F(T)\cap F(S)\cap F(A_0)\cap(\bigcap_{i\in\mathbb{N}}F(A_i))\neq\phi$*, provided one of the following conditions hold.*

(a) *M is sequentially compact.*

(b) *M is weakly compact,* $(S-A_i)$ *is demiclosed at 0 for each* $i\in\mathbb{N}\cup\{0\}$*, and X is complete.*

(c) *M is weakly compact and X is a complete space satisfying Opial's condition.*

Proof. (a) The proof follows from Theorem 18(a).

(b) The proof follows from Theorem 18(b).

(c) Following the proof of Theorem 18(b), we have $Su = u = Tu$ and for each $i\in\mathbb{N}\cup\{0\}$, $\|Sx_m - A_ix_m\| \to 0$ as $m\to\infty$. Since the family $\{A_i\}_{i=0}^{\infty}$ is (S,T)-nonexpansive, therefore, for each $i\in\mathbb{N}$, we have $A_0u = A_iu$. Now we have to show that $Su = A_0u$. If not, then by Opial's condition of X and (S,T)-nonexpansiveness of the family $\{A_i\}_0^{\infty}$, we get

$$\begin{aligned}
\liminf_{m\to\infty}\|Sx_m - Tu\| &= \liminf_{m\to\infty}\|Sx_m - Su\| \\
&< \liminf_{m\to\infty}\|Sx_m - A_0u\| \\
&\le \liminf_{m\to\infty}\|Sx_m - A_ix_m\| \\
&\quad + \liminf_{m\to\infty}\|A_ix_m - A_0u\|, \\
&\qquad\qquad \text{where } i\in\mathbb{N} \\
&= \liminf_{m\to\infty}\|A_0u - A_ix_m\| \\
&\le \liminf_{m\to\infty}\|Su - Tx_m\| \\
&= \liminf_{m\to\infty}\|Tu - Sx_m\|,
\end{aligned} \tag{25}$$

which is a contradiction. Therefore, $Su = A_0u$ and, hence, $F(T)\cap F(S)\cap F(A_0)\cap(\bigcap_{i\in\mathbb{N}}F(A_i))\neq\phi$. □

In Theorems 18 and 19, if we take $A_i = A$ for each $i\in\mathbb{N}\cup\{0\}$, we obtain the following corollary which generalizes Theorems 2.2 and 2.3 of Hussain and Jungck [8], respectively.

Corollary 20. *Let M be a nonempty q-star-shaped subset of a normed space X, and let S and T be continuous and q-affine self-maps of M. Let A be a self-map of M satisfying the following.*

(1) $A(M)\subseteq S(M)\cap T(M)$.

(2) *The pairs* (A,S) *and* (A,T) *are modified subcompatible.*

(3) *For all* $x,y\in M$,

$$\|Ax - Ay\| \le \max\ M(x,y), \tag{26}$$

where

$$\begin{aligned}
M(x,y) = \Big\{&\|Sx - Ty\|, \operatorname{dist}(Sx,[Ax,q]), \\
&\operatorname{dist}(Ty,[Ay,q]), \\
&\frac{1}{2}\big[\operatorname{dist}(Sx,[Ay,q]) \\
&\quad + \operatorname{dist}(Ty,[Ax,q])\big]\Big\}.
\end{aligned} \tag{27}$$

Then S, T, and A have a common fixed point provided one of the following conditions hold.

(a) *M is sequentially compact and A is continuous.*

(b) *M is weakly compact,* $(S-A)$ *is demiclosed at 0, and X is complete.*

(c) *M is complete,* $\mathrm{cl}(A(M))$ *is compact, and A is continuous.*

Proof. (a) and (b) follow from Theorem 18 by taking $A_i = A$ for each $i\in\mathbb{N}\cup\{0\}$.

(c) Define $A^n\colon M\to M$ by

$$A^n x = (1-k_n)\,q + k_nAx. \tag{28}$$

As we have done in Theorem 18, for each $n\ge 1$, there exists $x_n\in M$ such that $x_n = Sx_n = Tx_n = A^nx_n$. Then, compactness of $\mathrm{cl}(A(M))$ implies that there exists a subsequence $\{Ax_m\}$ of $\{Ax_n\}$ such that $Ax_m\to z$ as $m\to\infty$. Then the definition of A^mx_m implies $x_m\to z$; thus, by continuity of A, S, and T, we can say that z is a common fixed point of A, S, and T. □

Corollary 21. *Let M be a nonempty q-star-shaped subset of a normed space X, and let S and T be continuous and q-affine self-maps of M. Let A be a self-map of M with* $A(M)\subseteq S(M)\cap T(M)$*. If the pairs* (A,S) *and* (A,T) *are modified subcompatible and also the map A is* (S,T)*-nonexpansive, then* $F(T)\cap F(S)\cap F(A)\neq\phi$*, provided one of the following conditions hold.*

(a) *M is sequentially compact.*

(b) *M is weakly compact,* $(S-A)$ *is demiclosed at 0, and X is complete.*

(c) *M is weakly compact and X is complete space satisfying Opial's condition.*

(d) *M is complete and* $\mathrm{cl}(A(M))$ *is compact.*

In Corollary 20(b), if we take $T = S$, then we obtain the following corollary as a generalization of Theorem 4 proved by Shahzad [12].

Corollary 22. *Let M be a nonempty weakly compact q-star-shaped subset of a Banach space X, and let A and S be self-maps of M. Suppose that S is q-affine and continuous, and* $A(M)\subseteq S(M)$*. If* $(S-A)$ *is demiclosed at 0, the pair* (A,S) *is modified subcompatible and satisfies*

$$\|Ax - Ay\| \le \max\ M(x,y), \tag{29}$$

where

$$M(x,y)=\Big\{\|Sx-Sy\|,\operatorname{dist}(Sx,[Ax,q]),\ \operatorname{dist}(Sy,[Ay,q]),\ \frac{1}{2}[\operatorname{dist}(Sx,[Ay,q])+\operatorname{dist}(Sy,[Ax,q])]\Big\}\tag{30}$$

for all $x, y \in M$*; then* $F(S)\cap F(A)\neq\phi$.

In Theorems 18 and 19, if we take $T = S$, then we obtain the following corollary.

Corollary 23. *Let M be a nonempty q-star-shaped subset of a normed space X. Suppose that S is continuous and is a q-affine self-map of M. Let $\{A_i\}_{i\in\mathbb{N}\cup\{0\}}$ be a family of self-maps of M satisfying the following.*

(1) $\bigcup_{i=0}^{\infty}A_i(M)\subseteq S(M)$ *and for each* $i\in\mathbb{N}\cup\{0\}$*, the pair* (A_i,S) *is modified subcompatible.*

(2) *For each* $i\in\mathbb{N}$ *and, for any* $x,y\in M$

$$\|A_0x-A_iy\|\le\max M(x,y),\tag{31}$$

where

$$M(x,y)=\Big\{\|Sx-Sy\|,\operatorname{dist}(Sx,[A_0x,q]),\ \operatorname{dist}(Sy,[A_iy,q]),\ \frac{1}{2}[\operatorname{dist}(Sx,[A_iy,q])+\operatorname{dist}(Sy,[A_0x,q])]\Big\};\tag{32}$$

then S and all the A_i $(i\in\mathbb{N}\cup\{0\})$ have a common fixed point provided one of the following conditions hold.

(a) *M is sequentially compact and A_i is continuous for each $i\in\mathbb{N}\cup\{0\}$.*

(b) *M is weakly compact, $(S-A_i)$ is demiclosed at 0 for each $i\in\mathbb{N}\cup\{0\}$, and X is complete.*

Corollary 24. *Let M be a nonempty q-star-shaped subset of a normed space X. Suppose that S is continuous and is a q-affine self-map of M. Let $\{A_i\}_{i\in\mathbb{N}\cup\{0\}}$ be a family of self-maps with $\bigcup_{i=0}^{\infty}A_i(M)\subseteq S(M)$ and the pairs (A_i,S) are modified subcompatible for each $i\in\mathbb{N}\cup\{0\}$. If this family $\{A_i\}_{i\in\mathbb{N}\cup\{0\}}$ of maps is S-nonexpansive then $F(S)\cap F(A_0)\cap(\bigcap_{i\in\mathbb{N}}F(A_i))\neq\phi$, provided one of the following conditions hold.*

(1) *M is sequentially compact.*

(2) *M is weakly compact, $(S-A_i)$ is demiclosed at 0 for each $i\in\mathbb{N}\cup\{0\}$, and X is complete.*

(3) *M is weakly compact and X is a complete space satisfying Opial's condition.*

3. Applications to Best Approximation

The following theorem extends and generalizes [5, Theorem 2], [8, Theorem 2.8], and main result of [3].

Theorem 25. *Let M be a subset of a normed space X and let $S, T, A_i : X\to X$ be mappings for each $i\in\mathbb{N}\cup\{0\}$ such that $u\in F(T)\cap F(S)\cap F(A_0)\cap(\bigcap_{i\in\mathbb{N}}F(A_i))$ for some $u\in X$ and for each $i\in\mathbb{N}\cup\{0\}$, $A_i(\partial M\cap M)\subseteq M$. Suppose that S and T are q-affine and continuous on $P_M(u)$ and also $P_M(u)$ is q-star-shaped and $S(P_M(u))=P_M(u)=T(P_M(u))$.*

Moreover, if

(1) *the pairs (A_0,S) and (A_i,T) are modified subcompatible for each $i\in\mathbb{N}$.*

(2) *for each $i\in\mathbb{N}$, and for all $x\in P_M(u)\bigcup\{u\}$,*

$$\|A_0x-A_iy\|\le\begin{cases}\|Sx-Tu\|, & \text{if } y=u\\ \max\Big\{\|Sx-Ty\|,\operatorname{dist}(Sx,[q,A_0x]),\ \operatorname{dist}(Ty,[q,A_iy]),\ \frac{1}{2}[\operatorname{dist}(Sx,[q,A_iy])+\operatorname{dist}(Ty,[q,A_0x])]\Big\} & \text{if } y\in P_M(u),\end{cases}\tag{33}$$

$$\|A_ix-A_0u\|\le\|A_0x-A_iu\|.\tag{34}$$

Then $P_M(u)\cap F(T)\cap F(S)\cap F(A_0)\cap(\bigcap_{i\in\mathbb{N}}F(A_i))\neq\phi$, provided one of the following conditions hold.

(a) *$P_M(u)$ is sequentially compact and A_i is continuous for each $i\in\mathbb{N}\cup\{0\}$.*

(b) *$P_M(u)$ is weakly compact, X is complete, and $(S-A_i)$ is demiclosed at 0 for each $i\in\mathbb{N}\cup\{0\}$.*

Proof. Let $x\in P_M(u)$. Then $\|x-u\|=d(u,M)$. Note that for any $k\in(0,1)$,

$$\|ku+(1-k)x-u\|=(1-k)\|x-u\|<d(u,M).\tag{35}$$

It follows that the line segment $\{ku+(1-k)x:0<k<1\}$ and the set M are disjoint. Thus, x is not interior of M and so $x\in\partial M\cap M$. As $A_i(\partial M\cap M)\subseteq M$ for each $i\in\mathbb{N}\cup\{0\}$, therefore, for each $i\in\mathbb{N}\cup\{0\}$, $A_ix\in M$. Now we have to show that $A_0x\in P_M(u)$ and for each $i\in\mathbb{N}$, $A_ix\in P_M(u)$. Since $Sx\in P_M(u)$, $u\in F(T)\cap F(S)\cap F(A_0)\cap(\bigcap_{i\in\mathbb{N}}F(A_i))$ and S, T, and A_i's satisfy (33); therefore, we have

$$\|A_0x-u\|=\|A_0x-A_iu\|\le\|Sx-Tu\|=\|Sx-u\|=d(u,M),\quad\text{where } i\in\mathbb{N}.\tag{36}$$

Then the definition of $P_M(u)$ implies

$$A_0x\in P_M(u).\tag{36a}$$

Again using (33) and (34), for each $i \in \mathbb{N}$, we have

$$\begin{aligned}\|A_i x - u\| &= \|A_i x - A_0 u\| \\ &\leq \|A_0 x - A_i u\| \leq \|Sx - Tu\| \\ &= \|Sx - u\| = d(u, M).\end{aligned} \tag{37}$$

This yields that

$$A_i x \in P_M(u), \quad \text{for each } i \in \mathbb{N}. \tag{37a}$$

Then combining (36a) and (37a), we get $A_i x \in P_M(u)$ for each $i \in \mathbb{N} \cup \{0\}$. Consequently, $A_i(P_M(u)) \subseteq P_M(u)$, for each $i \in \mathbb{N} \cup \{0\}$. Since $S(P_M(u)) = P_M(u) = T(P_M(u))$, therefore we have

$$\begin{gathered} A_0(P_M(u)) \subseteq S(P_M(u)), \\ A_i(P_M(u)) \subseteq T(P_M(u)), \\ \text{for each } i \in \mathbb{N}. \end{gathered} \tag{38}$$

Hence, by Theorem 18 $P_M(u) \cap F(T) \cap F(S) \cap F(A_0) \cap (\bigcap_{i\in\mathbb{N}} F(A_i)) \neq \phi$. □

The following corollary improves and extends [4, Theorem 8], [8, Corollary 2.9], and [10, Theorem 4].

Corollary 26. *Let M be a subset of a normed space X and let S, T, A_i: $X \to X$ be mappings for each $i \in \mathbb{N} \cup \{0\}$ such that $u \in F(T) \cap F(S) \cap F(A_0) \cap (\bigcap_{i\in\mathbb{N}} F(A_i))$ for some $u \in X$ and $A_i(\partial M \cap M) \subseteq M$ for each $i \in \mathbb{N} \cup \{0\}$. Suppose that S and T are q-affine and continuous on $P_M(u)$ and also $P_M(u)$ is q-star-shaped and $S(P_M(u)) = P_M(u) = T(P_M(u))$. If the pairs (A_0, S) and (A_i, T) are modified subcompatible for each $i \in \mathbb{N}$ and also the family $\{A_i\}_{i\in\mathbb{N}\cup\{0\}}$ of maps is (S, T)-nonexpansive, then $P_M(u) \cap F(T) \cap F(S) \cap F(A_0) \cap (\bigcap_{i\in\mathbb{N}} F(A_i)) \neq \phi$, provided one of the following conditions hold.*

(a) *$P_M(u)$ is sequentially compact.*

(b) *$P_M(u)$ is weakly compact, X is complete, and $(S - A_i)$ is demiclosed at 0 for each $i \in \mathbb{N} \cup \{0\}$.*

(c) *$P_M(u)$ is weakly compact and X is complete space satisfying Opial's condition.*

The following corollary generalizes [12, Theorem 5] and [8, Corollary 2.10].

Corollary 27. *Let M be a subset of a normed space X and let S, $A : X \to X$ be mappings such that $u \in F(A) \cap F(S)$ for some $u \in X$ and $A(\partial M \cap M) \subseteq M$. Suppose that S is q-affine and continuous on $P_M(u)$ and also $P_M(u)$ is q-star-shaped and $S(P_M(u)) = P_M(u)$. If the pair (A, S) is modified subcompatible and satisfies for all $x \in P_M(u) \cup \{u\}$*

$$\|Ax - Ay\| \leq \begin{cases} \|Sx - Su\|, & \text{if } y = u \\ \max\Big\{\|Sx - Sy\|, \operatorname{dist}(Sx, [q, Ax]), \\ \quad \operatorname{dist}(Sy, [q, Ay]), \\ \quad \frac{1}{2}[\operatorname{dist}(Sx, [q, Ay]) \\ \quad + \operatorname{dist}(Sy, [q, Ax])]\Big\} & \text{if } y \in P_M(u), \end{cases} \tag{39}$$

then $P_M(u) \cap F(S) \cap F(A) \neq \phi$, provided one of the following conditions hold.

(a) *$P_M(u)$ is sequentially compact.*

(b) *$P_M(u)$ is complete and $\mathrm{cl}(A(P_M(u)))$ is compact.*

(c) *$P_M(u)$ is weakly compact, X is complete, and $(S - A)$ is demiclosed at 0.*

4. Examples

Now, we present some examples which demonstrate the validity of the proved results.

Example 28. Let $X = R$ with usual norm $\|x\| = |x|$ and $M = [0, 1]$. Suppose A_0, $A_i : M \to M$ are defined as

$$\begin{aligned} A_0(x) &= 1, \quad \text{for } 0 \leq x \leq 1, \\ A_i(x) &= \frac{x+i}{i+1}, \quad \text{for each } i \in \mathbb{N}, 0 \leq x \leq 1 \end{aligned} \tag{40}$$

and also $S, T : M \to M$ are defined as

$$S(x) = \frac{x+1}{2}, \quad T(x) = x, \quad \text{for } 0 \leq x \leq 1. \tag{41}$$

Here $A_0(M) = \{1\}$, $T(M) = [0, 1]$, $S(M) = [1/2, 1]$, and $A_i(M) = [i/(i + 1), 1]$ for each $i \in \mathbb{N}$, so that $A_0(M) \subseteq T(M)$ and $A_i(M) \subseteq S(M)$ for each $i \in \mathbb{N}$. Besides M is compact and the pairs of mappings $\{A_0, S\}$ and $\{A_i, T\}$ are modified subcompatible for each $i \in \mathbb{N}$ and also the maps S and T are q-affine for $q = 1$. Further the mappings S, T, and A_i for each $i \in \mathbb{N} \cup \{0\}$ satisfy the inequality (18). Hence all the conditions of Theorem 18(a) are satisfied. Therefore S, T, and all A_i $(i \in \mathbb{N} \cup \{0\})$ have a common fixed point and $x = 1$ is such a unique common fixed point.

Remark 29. (1) In Example 28, if we define $A_0(x) = A_i(x) = S(x) = T(x) = x$ for all $x \in X \sim M$, then S, T, and all A_i $(i \in \mathbb{N} \cup \{0\})$ are self-maps of X and $u = 2 \in F(T) \cap F(S) \cap F(A_0) \cap (\bigcap_{i\in\mathbb{N}} F(A_i))$. Clearly, $P_M(u) = \{1\}$ is q-star-shaped and $S(P_M(u)) = P_M(u) = T(P_M(u))$. Therefore, all the conditions of Theorem 25 are satisfied and, hence, $P_M(u) \cap F(T) \cap F(S) \cap F(A_0) \cap (\bigcap_{i\in\mathbb{N}} F(A_i)) \neq \phi$. Here, $x = 1 \in P_M(u) \cap F(T) \cap F(S) \cap F(A_0) \cap (\bigcap_{i\in\mathbb{N}} F(A_i)) \neq \phi$.

(2) If inequality (18) in Theorem 18 is replaced with the weaker condition

$$\begin{aligned} &\|A_0x - A_iy\| \\ &\quad \le \max\Big\{\|Sx - Ty\|, \|Sx - A_0x\|, \\ &\qquad \|Ty - A_iy\|, \\ &\qquad \frac{1}{2}\left[\|Sx - A_iy\| + \|Ty - A_0x\|\right]\Big\}, \end{aligned} \tag{42}$$

for each $i \in \mathbb{N}$ and, for any $x, y \in M$. Then, Theorem 18 need not be true. This can be seen by the following example.

Example 30. Let $X = R$ with usual norm $\|x\| = |x|$ and $M = [0, 1]$. Suppose $A_0, A_i : M \to M$ are defined as

$$\begin{aligned} A_0(x) &= \frac{1}{2}, \quad \text{for } 0 \le x \le 1, \\ A_i(x) &= \frac{3}{4}, \quad \text{for each } i \in \mathbb{N}, 0 \le x \le 1 \end{aligned} \tag{43}$$

and also $S, T : M \to M$ are defined as

$$S(x) = \begin{cases} \frac{1}{2} & \text{if } 0 \le x < \frac{1}{2} \\ \frac{1}{2}x + \frac{1}{4} & \text{if } \frac{1}{2} \le x \le 1, \end{cases} \tag{44}$$

$$T(x) = \begin{cases} x & \text{if } 0 \le x < \frac{1}{2} \\ 1 - x & \text{if } \frac{1}{2} \le x \le 1. \end{cases} \tag{45}$$

Here $A_0(M) = \{1/2\}$, $T(M) = [0, 1/2]$, $S(M) = [1/2, 3/4]$, and $A_i(M) = \{3/4\}$ for each $i \in \mathbb{N}$, so that $A_0(M) \subseteq T(M)$ and $A_i(M) \subseteq S(M)$ for each $i \in \mathbb{N}$. Besides M is compact and the pairs of mappings $\{A_0, S\}$ and $\{A_i, T\}$ are modified subcompatible for each $i \in \mathbb{N}$ and also the maps S and T are q-affine for $q = 1/2$. Further, the mappings S, T, and A_i for each $i \in \mathbb{N} \cup \{0\}$ are continuous and satisfy the inequality (42). Note that $F(T) \cap F(S) \cap F(A_0) \cap (\bigcap_{i\in\mathbb{N}} F(A_i)) = \phi$.

Remark 31. Clearly mappings S, T, and A_i for each $i \in \mathbb{N} \cup \{0\}$ defined in Example 30 satisfy all of the conditions of Theorem 18(a) except the inequality (18) at $x = 1/2$, $y = 1/2$. Note that there is no common fixed point of S, T, and A_i for each $i \in \mathbb{N} \cup \{0\}$.

Example 32. Let $X = R$ with usual norm $\|x\| = |x|$ and $M = [0, 1]$. Suppose $T, S, A : M \to M$ are defined as

$$\begin{aligned} T(x) &= \begin{cases} x & \text{if } 0 \le x < \frac{1}{2} \\ 1 - x & \text{if } \frac{1}{2} \le x \le 1, \end{cases} \\ S(x) &= \begin{cases} x & \text{if } 0 \le x < \frac{1}{2} \\ \frac{1}{2}x + \frac{1}{4} & \text{if } \frac{1}{2} \le x \le 1, \end{cases} \\ A(x) &= \frac{1}{2}, \quad \text{for } 0 \le x \le 1. \end{aligned} \tag{46}$$

Here we observe that $A(M) = \{1/2\}$, $S(M) = [0, 3/4]$, and $T(M) = [0, 1/2]$ so that $A(M) \subseteq S(M) \cap T(M)$. Also, M is q-star-shaped and the maps S and T are q-affine with $q = 1/2$. We also observe that the pairs (A, S) and (A, T) are modified subcompatible and M is sequentially compact. Further, the mappings A, S, and T satisfy (26). Hence, the mappings A,S, and T satisfy all the conditions of Corollary 20(a) and $x = 1/2$ is the unique common fixed point of mappings A, S, and T.

Remark 33. In Example 32, S and T are not affine because for $x = 3/5$, $y = 0$, and $k = 1/3$, $S(kx + (1 - k)y) = kS(x) + (1 - k)S(y)$ and $T(kx + (1 - k)y) = kT(x) + (1 - k)T(y)$ do not hold. Therefore, Theorem 2.2 of Hussain and Jungck [8] cannot apply to Example 32; hence Corollary 20 is more general than Theorem 2.2 of [8].

Example 34. Take X, M, and S as in Example 32 and define

$$\begin{aligned} A(x) &= \frac{1}{4}, \quad \text{for } 0 \le x \le 1, \\ T(x) &= \begin{cases} \frac{1}{2} & \text{if } 0 \le x < \frac{1}{2} \\ 1 - x & \text{if } \frac{1}{2} \le x \le 1. \end{cases} \end{aligned} \tag{47}$$

Then all of the conditions of Corollary 20(a) are satisfied except that the pair (A, T) is modified subcompatible. Note that $F(T) \cap F(S) \cap F(A) = \phi$.

Remark 35. All results of the paper can be proved for Hausdorff locally convex spaces defined and studied by various authors (see [16, 23–27]).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

A Generalized System of Nonlinear Variational Inequalities in Banach Spaces

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We introduce a new generalized system of nonlinear variational inequality problems (GSNVIP) by using the generalized projection method. Moreover, we introduce an iterative scheme for finding a solution to this problem. Moreover, some existence and strong convergence theorems are established in uniformly smooth and strictly convex Banach spaces under suitable conditions. The results presented in the paper improve and extend some recent results.

1. Introduction

Variational inequality theory has become a very effective and powerful tool for studying a wide range of problems arising in pure and applied sciences which include work on differential equations, general equilibrium problems in economics and mechanics, control problems, and transportation. In 2005, Verma [1] introduced a general model for two-step projection methods and applied it to the approximation solvability of a system of nonlinear variational inequality problems in a Hilbert space. Based on the convergence of projection methods, Chang et al. [2] introduced and studied the approximate solvability of a generalized system for relaxed cocoercive nonlinear variational inequalities in Hilbert spaces (see, for instance, [3–5] and the references therein). Recently, Chang et al. [6] introduced a system of generalized nonlinear variational inequalities and an iterative scheme for finding a solution to a system of generalized nonlinear variational inequality problems by using the generalized projection method. Moreover, they proved some existence and strong convergence theorems in uniformly smooth and strictly convex Banach spaces.

In this paper, we introduce a generalized system of nonlinear variational inequality problems (GSNVIP) by using the generalized projection approach to introduce an iterative scheme for finding a solution to this problem. Finally, we prove some existence and strong convergence theorems in uniformly smooth and strictly convex Banach spaces under suitable conditions.

2. Preliminaries

Let E be a real Banach space with dual space E^*, $\langle \cdot, \cdot \rangle$ the dual pair between E and E^*, and K a nonempty closed convex subset of E. The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) = \left\{ f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2 \right\}, \quad \forall x \in E. \tag{1}$$

A Banach space E is said to be strictly convex if $\|x + y\|/2 < 1$ for all $x, y \in U = \{z \in E : \|z\| = 1\}$ with $x \neq y$. E is said to be uniformly convex if for each $\epsilon \in (0, 2]$ there exists $\delta > 0$ such that $\|x + y\|/2 < 1 - \delta$ for all $x, y \in U$ with $\|x - y\| \geq \epsilon$. E is said to be smooth if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}. \tag{2}$$

exists for all $x, y \in U$. E is said to be uniformly smooth if the above limit exists uniformly in $x, y \in U$.

Remark 1 (see [7]). (i) If E is a uniformly smooth Banach space, then the normalized duality mapping J is uniformly continuous on each bounded subset of E.

(ii) If E is a smooth, strictly convex and reflexive Banach space, then the normalized duality mapping $J : E \to 2^{E^*}$ is a single valued bijective mapping.

(iii) If E is a smooth, strictly convex and reflexive Banach space and $J^* : E^* \to E$ is the duality mapping in E^*, then $J^{-1} = J^*$, $JJ^* = I_{E^*}$, and $J^*J = I_E$.

(iv) If E is a strictly convex and reflexive Banach space, then J^{-1} is hemicontinuous; that is, J^{-1} is norm-weak-continuous.

(v) E is uniformly smooth if and only if E^* is uniformly convex.

(vi) If E is a uniformly smooth and strictly convex Banach space with the Kadec-Klee property (i.e., for any sequence $\{x_n\} \subset E$, if $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, then $x_n \to x$), then both the normalized duality mappings $J : E \to E^*$ and $J^* = J^{-1} : E^* \to E$ are continuous.

(vii) Each uniformly convex Banach space E has the Kadec-Klee property.

Assume that E is a smooth, strictly convex and reflexive Banach space and K is a nonempty closed convex subset of E; $\phi : E\times E \to \mathbb{R}^+ := [0, \infty)$ to denote the Lyapunov functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, y \in E. \tag{3}$$

Following Alber [8], the generalized projection $\prod_K : E \to K$ is defined by $\prod_K x = z$, where z is the unique solution to the minimization problem

$$\phi(z, x) = \min_{y\in K}\phi(y, x). \tag{4}$$

The existence and uniqueness of the mapping $\prod_K$ follow from the property of the function $\phi(x, y)$ and the strict monotonicity of the mapping J.

Lemma 2 (see [8]). *Let E be a smooth, strictly convex and reflexive Banach space and K a nonempty closed convex subset of E. Then the following conclusions hold:*

(a) *if $x \in E$ and $z \in K$, then*

$$z = \prod_K x \Longleftrightarrow \langle y - z, Jz - Jx\rangle \geq 0, \quad \forall y \in K; \tag{5}$$

(b) *$\prod_K$ is a continuous mapping from E onto K.*

Remark 3. If E is a real Hilbert space, then $J = I$ (identity mapping), $\phi(x, y) = \|x - y\|^2$, and $\prod_K$ is the metric projection P_K from E onto K.

Lemma 4 (see [9, 10]). *Let E be a uniformly convex Banach space, $r > 0$ a positive number, and $B_r(0) := \{x \in E : \|x\| \leq r\}$ a closed ball of E. Then, for any given finite subset $\{x_1, x_2, \ldots, x_N\} \subset B_r(0)$ and for any given positive numbers $\lambda_1, \lambda_2, \ldots, \lambda_N$ with $\sum_{n=1}^{N}\lambda_n = 1$, there exists a continuous, strictly increasing, and convex function $g : [0, 2r) \to [0, \infty)$ with $g(0) = 0$ such that for any $i, j \in \{1, 2, \ldots, N\}$ with $i < j$ the following holds:*

$$\left\|\sum_{n=1}^{N}\lambda_n x_n\right\|^2 \leq \sum_{n=1}^{N}\lambda_n\|x_n\|^2 - \lambda_i\lambda_j g\left(\left\|x_i - x_j\right\|\right). \tag{6}$$

Lemma 5 (see [11]). *Let E be a real reflexive, smooth, and strictly convex Banach space. Then the following inequality holds:*

$$\|f + g\|^2 \leq \|f\|^2 + 2\left\langle g, J^{-1}(f + g)\right\rangle, \quad \forall f, g \in E^*. \tag{7}$$

Lemma 6 (see [6]). *Let E be a real Banach space, K a nonempty closed convex subset of E with $0 \in K$, and $\prod_K : E \to K$ the generalized projection. Then for each $x \in E$, one has $\|\prod_K x\| \leq \|x\|$.*

3. Main Results

In this section, we assume that E is a real Banach space with dual space E^* and K is a nonempty closed convex subset of E. Let $T_1, \ldots, T_N : K^N \to E^*$ be nonlinear mappings and $f : K \to E$ a mapping. The generalized system of nonlinear variational inequality problems (GSNVIP) is to find $x_1^*, \ldots, x_N^*$ such that for all $x \in K$

$$\begin{gathered}\left\langle f(x) - f(x_1^*), T_1(x_2^*, x_3^*, \ldots, x_N^*, x_1^*)\right\rangle \geq 0,\\ \left\langle f(x) - f(x_2^*), T_2(x_3^*, x_4^*, \ldots, x_N^*, x_1^*, x_2^*)\right\rangle \geq 0,\\ \vdots\\ \left\langle f(x) - f(x_N^*), T_N(x_1^*, x_2^*, \ldots, x_{N-1}^*, x_N^*)\right\rangle \geq 0.\end{gathered} \tag{8}$$

If $N = 3$, $f = I$, and $T_1, T_2, T_3 : K^3 \to E^*$ are nonlinear mappings, then the generalized system of nonlinear variational inequality problems (GSNVIP) reduces to the following problem (see [6]) to find x_1^*, x_2^*, x_3^* such that, for all $x \in K$,

$$\begin{gathered}\left\langle x - x_1^*, T_1(x_2^*, x_3^*, x_1^*)\right\rangle \geq 0,\\ \left\langle x - x_2^*, T_2(x_3^*, x_1^*, x_2^*)\right\rangle \geq 0,\\ \left\langle x - x_3^*, T_3(x_1^*, x_2^*, x_3^*)\right\rangle \geq 0.\end{gathered} \tag{9}$$

If $N = 2$ and $T_1, T_2 : K^2 \to E^*$ are nonlinear mappings and $f : K \to E$ is a mapping, then the generalized system of nonlinear variational inequality problems (GSNVIP) reduces to the following problem to find x_1^*, x_2^* such that, for all $x \in K$,

$$\begin{gathered}\left\langle f(x) - f(x_1^*), T_1(x_2^*, x_1^*)\right\rangle \geq 0,\\ \left\langle f(x) - f(x_2^*), T_2(x_1^*, x_2^*)\right\rangle \geq 0.\end{gathered} \tag{10}$$

If $T, S : K^2 \to E^*$ are nonlinear mappings and $g, f : K \to E$ are two mappings. Define $T_1, T_2 : K^2 \to E^*$ by $T_1(x_1^*, x_2^*) = \rho_1 T(x_1^*, x_2^*) + g(x_2^*) - g(x_1^*)$ and $T_2(x_1^*, x_2^*) = \rho_2 S(x_1^*, x_2^*) + g(x_2^*) - g(x_1^*)$. Then the generalized system of nonlinear variational inequality problems (GSNVIP) reduces to the following problem to find $x_1^*, x_2^* \in K$ such that, for all $x \in K$,

$$\begin{aligned} &\left\langle f(x) - f(x_1^*), \rho_1 T(x_2^*, x_1^*) + g(x_2^*) - g(x_1^*)\right\rangle \geq 0, \\ &\left\langle f(x) - f(x_2^*), \rho_2 S(x_1^*, x_2^*) + g(x_2^*) - g(x_1^*)\right\rangle \geq 0, \end{aligned} \tag{11}$$

where ρ_1 and ρ_2 are two positive constants.

Lemma 7. *Let E be a smooth, strictly convex, and reflexive Banach space and K a nonempty closed convex subset of E. Let $T_1, \ldots, T_N : K^N \to E^*$ be mappings, $f : K \to K$ a bijective mapping, and $\rho_1, \ldots, \rho_N$ any positive real numbers. Then $(x_1^*, \ldots, x_N^*) \in K^N$ is a solution to problem (8) if and only if $(x_1^*, \ldots, x_N^*) \in K^N$ is a solution to the following system of operator equations:*

$$\begin{aligned} x_1^* &= f^{-1}\prod_K J^{-1}\left(Jf(x_1^*) - \rho_1 T_1(x_2^*, x_3^*, \ldots, x_N^*, x_1^*)\right), \\ x_2^* &= f^{-1}\prod_K J^{-1}\left(Jf(x_2^*) - \rho_2 T_2(x_3^*, x_4^*, \ldots, x_N^*, x_1^*, x_2^*)\right), \\ &\vdots \\ x_{N-1}^* &= f^{-1}\prod_K J^{-1}\left(Jf(x_{N-1}^*) - \rho_{N-1} T_{N-1}(x_N^*, x_1^*, x_2^*, \ldots, x_{N-2}^*, x_{N-1}^*)\right), \\ x_N^* &= f^{-1}\prod_K J^{-1}\left(Jf(x_N^*) - \rho_N T_N(x_1^*, x_2^*, \ldots, x_N^*)\right). \end{aligned} \tag{12}$$

Proof. By Lemma 2, we have that $(x_1^*, \ldots, x_N^*) \in K^N$ is a solution of problem (8),

$$\Longleftrightarrow \begin{cases} \left\langle f(x) - f(x_1^*), \rho_1 T_1(x_2^*, x_3^*, \ldots, x_N^*, x_1^*)\right\rangle \geq 0, \\ \left\langle f(x) - f(x_2^*), \rho_2 T_2(x_3^*, x_4^*, \ldots, x_N^*, x_1^*, x_2^*)\right\rangle \geq 0, \\ \quad\vdots \\ \left\langle f(x) - f(x_{N-1}^*), \rho_{N-1} T_{N-1}(x_N^*, x_1^*, x_2^*, \ldots, x_{N-2}^*, x_{N-1}^*)\right\rangle \geq 0, \\ \left\langle f(x) - f(x_N^*), \rho_N T_N(x_1^*, x_2^*, \ldots, x_N^*)\right\rangle \geq 0, \end{cases}$$

$$\Longleftrightarrow \begin{cases} \left\langle f(x) - f(x_1^*), Jf(x_1^*) - Jf(x_1^*) + \rho_1 T_1(x_2^*, x_3^*, \ldots, x_N^*, x_1^*)\right\rangle \geq 0, \\ \left\langle f(x) - f(x_2^*), Jf(x_2^*) - Jf(x_2^*) + \rho_2 T_2(x_3^*, x_4^*, \ldots, x_N^*, x_1^*, x_2^*)\right\rangle \geq 0, \\ \quad\vdots \\ \left\langle f(x) - f(x_{N-1}^*), Jf(x_{N-1}^*) - Jf(x_{N-1}^*) + \rho_{N-1} T_{N-1}(x_N^*, x_1^*, x_2^*, \ldots, x_{N-2}^*, x_{N-1}^*)\right\rangle \geq 0, \\ \left\langle f(x) - f(x_N^*), Jf(x_N^*) - Jf(x_N^*) + \rho_N T_N(x_1^*, x_2^*, \ldots, x_N^*)\right\rangle \geq 0, \end{cases}$$

$$\Longleftrightarrow \begin{cases} \left\langle f(x) - f(x_1^*), Jf(x_1^*) - J\left(J^{-1}\left(Jf(x_1^*) - \rho_1 T_1(x_2^*, x_3^*, \ldots, x_N^*, x_1^*)\right)\right)\right\rangle \geq 0, \\ \left\langle f(x) - f(x_2^*), Jf(x_2^*) - J\left(J^{-1}\left(Jf(x_2^*) - \rho_2 T_2(x_3^*, x_4^*, \ldots, x_N^*, x_1^*, x_2^*)\right)\right)\right\rangle \geq 0, \\ \quad\vdots \\ \left\langle f(x) - f(x_{N-1}^*), Jf(x_{N-1}^*) - J\left(J^{-1}\left(Jf(x_{N-1}^*) - \rho_{N-1} T_{N-1} \times (x_N^*, x_1^*, x_2^*, \ldots, x_{N-2}^*, x_{N-1}^*)\right)\right)\right\rangle \geq 0, \\ \left\langle f(x) - f(x_N^*), Jf(x_N^*) - J\left(J^{-1}\left(Jf(x_N^*) - \rho_N T_N(x_1^*, x_2^*, \ldots, x_N^*)\right)\right)\right\rangle \geq 0, \end{cases} \tag{13}$$

for all $x \in K$,

$$\Longleftrightarrow \begin{cases} f(x_1^*) = \prod_K J^{-1}\left(Jf(x_1^*) - \rho_1 T_1(x_2^*, x_3^*, \ldots, x_N^*, x_1^*)\right), \\ f(x_2^*) = \prod_K J^{-1}\left(Jf(x_2^*) - \rho_2 T_2(x_3^*, x_4^*, \ldots, x_N^*, x_1^*, x_2^*)\right), \\ \quad\vdots \\ f(x_{N-1}^*) = \prod_K J^{-1}\left(Jf(x_{N-1}^*) - \rho_{N-1} T_{N-1} \times (x_N^*, x_1^*, x_2^*, \ldots, x_{N-2}^*, x_{N-1}^*)\right), \\ f(x_N^*) = \prod_K J^{-1}\left(Jf(x_N^*) - \rho_N T_N(x_1^*, x_2^*, \ldots, x_N^*)\right), \end{cases} \tag{14}$$

for any $\rho_1 > 0, \ldots, \rho_N > 0$,

$$\Longleftrightarrow \begin{cases} x_1^* \\ \quad = f^{-1}\prod_K J^{-1}\left(Jf\left(x_1^*\right) \right. \\ \qquad \left. - \rho_1 T_1\left(x_2^*, x_3^*, \ldots, x_N^*, x_1^*\right)\right), \\ x_2^* \\ \quad = f^{-1}\prod_K J^{-1}\left(Jf\left(x_2^*\right) \right. \\ \qquad \left. -\rho_2 T_2\left(x_3^*, x_4^*, \ldots, x_N^*, x_1^*, x_2^*\right)\right), \\ \qquad \vdots \\ x_{N-1}^* \\ \quad = f^{-1}\prod_K J^{-1}\left(Jf\left(x_{N-1}^*\right) - \rho_{N-1} T_{N-1} \right. \\ \qquad \left. \times\left(x_N^*, x_1^*, x_2^*, \ldots, x_{N-2}^*, x_{N-1}^*\right)\right), \\ x_N^* \\ \quad = f^{-1}\prod_K J^{-1}\left(Jf\left(x_N^*\right) \right. \\ \qquad \left. - \rho_N T_N\left(x_1^*, x_2^*, \ldots, x_N^*\right)\right). \end{cases} \tag{15}$$

□

Algorithm 8. For any given initial points $x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(N)} \in K$, compute the sequences $\{x_n^{(1)}\}, \{x_n^{(2)}\}, \ldots, \{x_n^{(N)}\}$ by the iterative processes

$$\begin{aligned} x_{n+1}^{(N)} &= f^{-1} \\ &\quad \times\left(J^{-1}\left(\left(1-\alpha_n^{(N)}\right) Jf\left(x_n^{(N)}\right) + \alpha_n^{(N)} J \right.\right. \\ &\quad \times\left(\prod_K J^{-1}\left(Jf\left(x_n^{(N)}\right) - \rho_N T_N \right.\right. \\ &\quad \left.\left.\left.\left. \times\left(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(N)}\right)\right)\right)\right)\right), \\ x_{n+1}^{(N-1)} &= f^{-1} \\ &\quad \times\left(J^{-1}\left(\left(1-\alpha_n^{(N-1)}\right) Jf\left(x_n^{(N-1)}\right) + \alpha_n^{(N-1)} J \right.\right. \\ &\quad \times\left(\prod_K J^{-1}\left(Jf\left(x_n^{(N-1)}\right) - \rho_{N-1} T_{N-1} \right.\right. \\ &\quad \times\left(x_{n+1}^{(N)}, x_n^{(1)}, x_n^{(2)}, \ldots, \right. \\ &\quad \left.\left.\left.\left.\left. x_n^{(N-2)}, x_n^{(N-1)}\right)\right)\right)\right)\right), \\ &\vdots \\ x_{n+1}^{(2)} &= f^{-1} \\ &\quad \times\left(J^{-1}\left(\left(1-\alpha_n^{(2)}\right) Jf\left(x_n^{(2)}\right) + \alpha_n^{(2)} J \right.\right. \\ &\quad \times\left(\prod_K J^{-1}\left(Jf\left(x_n^{(2)}\right) - \rho_2 T_2 \right.\right. \\ &\quad \times\left(x_{n+1}^{(3)}, x_{n+1}^{(4)}, \ldots, x_{n+1}^{(N)}, \right. \\ &\quad \left.\left.\left.\left.\left. x_n^{(1)}, x_n^{(2)}\right)\right)\right)\right)\right), \\ x_{n+1}^{(1)} &= f^{-1} \\ &\quad \times\left(J^{-1}\left(\left(1-\alpha_n^{(1)}\right) Jf\left(x_n^{(1)}\right) + \alpha_n^{(1)} J \right.\right. \\ &\quad \times\left(\prod_K J^{-1}\left(Jf\left(x_n^{(1)}\right) - \rho_1 T_1 \right.\right. \\ &\quad \times\left(x_{n+1}^{(2)}, x_{n+1}^{(3)}, \ldots, \right. \\ &\quad \left.\left.\left.\left.\left. x_{n+1}^{(N)}, x_n^{(1)}\right)\right)\right)\right)\right), \\ &\qquad n \geq 0, \end{aligned} \tag{16}$$

where $\prod_K$ is the generalized projection and $\{\alpha_n^{(1)}\}, \{\alpha_n^{(2)}\}, \ldots, \{\alpha_n^{(N)}\}$ are sequences in $[0, 1]$.

Theorem 9. *Let E be a real uniformly smooth and strictly convex Banach space with Kadec-Klee property and K a nonempty closed and convex subset of E with $\theta \in K$. Let $f : K \to K$ be an isometry mapping, $T_1, \ldots, T_N : K^N \to E^*$ continuous mappings, and $\{\alpha_n^{(1)}\}, \{\alpha_n^{(2)}\}, \ldots, \{\alpha_n^{(N)}\}$ the sequences in (a, b) with $0 < a < b < 1$ satisfying the following conditions:*

(i) *there exist a compact subset $C \subset E^*$ and constants $\rho_1 > 0, \rho_2 > 0, \ldots, \rho_N > 0$ such that*

$$\begin{aligned} &\left(J(K) - \rho_N T_N\left(K^N\right)\right) \cup \left(J(K) - \rho_{N-1} T_{N-1}\left(K^N\right)\right) \\ &\quad \cup \cdots \cup \left(J(K) - \rho_1 T_1\left(K^N\right)\right) \subset C, \end{aligned} \tag{17}$$

where $J(x_1, x_2, \ldots, x_N) = Jx_N$, *for all* $(x_1, x_2, \ldots, x_N) \in K^N$, *and*

$$\begin{aligned}
&\left\langle T_1(x_1, x_2, \ldots, x_N),\right.\\
&\qquad \left. J^{-1}\left(Jx_N - \rho_1 T_1(x_1, x_2, \ldots, x_N)\right)\right\rangle \geq 0,\\
&\left\langle T_2(x_1, x_2, \ldots, x_N),\right.\\
&\qquad \left. J^{-1}\left(Jx_N - \rho_2 T_2(x_1, x_2, \ldots, x_N)\right)\right\rangle \geq 0,\\
&\qquad\qquad \vdots\\
&\left\langle T_N(x_1, x_2, \ldots, x_N),\right.\\
&\qquad \left. J^{-1}\left(Jx_N - \rho_N T_N(x_1, x_2, \ldots, x_N)\right)\right\rangle \geq 0,
\end{aligned} \tag{18}$$

for all $x_1, x_2, \ldots, x_N \in K$;

(ii) $\lim_{n\to\infty}\alpha_n^{(1)} = d_1 \in (a,b)$, $\lim_{n\to\infty}\alpha_n^{(2)} = d_2 \in (a,b)$, $\ldots$, $\lim_{n\to\infty}\alpha_n^{(N)} = d_N \in (a,b)$. *Let* $\{x_n^{(1)}\}, \{x_n^{(2)}\}, \ldots, \{x_n^{(N)}\}$ *be the sequences defined by* (16). *Then the problem* (8) *has a solution* $(x_1^*, x_2^*, \ldots, x_N^*) \in K^N$ *and the sequences* $\{x_n^{(1)}\}, \{x_n^{(2)}\}, \ldots, \{x_n^{(N)}\}$ *converge strongly to* $x_1^*, x_2^*, \ldots, x_N^*$, *respectively.*

Proof.
Step 1. We first show that the sequences $\{x_n^{(1)}\}, \{x_n^{(2)}\}, \ldots, \{x_n^{(N)}\}$ are bounded in K. It follows from Lemma 5 where J is bijective and condition (18) that

$$\begin{aligned}
&\left\|Jf\left(x_n^{(N)}\right) - \rho_N T_N\left(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(N)}\right)\right\|^2\\
&\quad \leq \left\|Jf\left(x_n^{(N)}\right)\right\|^2\\
&\qquad - 2\rho_N \left\langle T_N\left(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(N)}\right),\right.\\
&\qquad\qquad \left. J^{-1}\left(Jf\left(x_n^{(N)}\right) - \rho_N T_N\left(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(N)}\right)\right)\right\rangle\\
&\quad \leq \left\|Jf\left(x_n^{(N)}\right)\right\|^2 = \left\|f\left(x_n^{(N)}\right)\right\|^2.
\end{aligned} \tag{19}$$

Similarly, we note that

$$\begin{aligned}
&\left\|Jf\left(x_n^{(N-1)}\right) - \rho_{N-1} T_{N-1}\left(x_{n+1}^{(N)}, x_n^{(1)}, \ldots, x_n^{(N-2)}, x_n^{(N-1)}\right)\right\|^2\\
&\quad \leq \left\|f\left(x_n^{(N-1)}\right)\right\|^2,\\
&\left\|Jf\left(x_n^{(N-2)}\right)\right.\\
&\qquad \left. - \rho_{N-2} T_{N-2}\left(x_{n+1}^{(N-1)}, x_{n+1}^{(N)}, x_n^{(1)}, \ldots, x_n^{(N-3)}, x_n^{(N-2)}\right)\right\|^2\\
&\qquad \leq \left\|f\left(x_n^{(N-2)}\right)\right\|^2,\\
&\qquad\qquad \vdots\\
&\left\|Jf\left(x_n^{(2)}\right) - \rho_2 T_2\left(x_{n+1}^{(3)}, x_{n+1}^{(4)}, \ldots, x_{n+1}^{(N)}, x_n^{(1)}, x_n^{(2)}\right)\right\|^2\\
&\quad \leq \left\|f\left(x_n^{(2)}\right)\right\|^2,\\
&\left\|Jf\left(x_n^{(1)}\right) - \rho_1 T_1\left(x_{n+1}^{(2)}, x_{n+1}^{(3)}, \ldots, x_{n+1}^{(N)}, x_n^{(1)}\right)\right\|^2\\
&\quad \leq \left\|f\left(x_n^{(1)}\right)\right\|^2.
\end{aligned} \tag{20}$$

By Lemma 6, we obtain that

$$\begin{aligned}
&\left\|f\left(x_{n+1}^{(N)}\right)\right\|\\
&= \left\| ff^{-1} \right.\\
&\qquad \times \left(J^{-1}\left(\left(1-\alpha_n^{(N)}\right) Jf\left(x_n^{(N)}\right)\right.\right.\\
&\qquad\qquad + \alpha_n^{(N)} J\left(\prod_K J^{-1}\right.\\
&\qquad\qquad\qquad \times \left(Jf\left(x_n^{(N)}\right)\right.\\
&\qquad\qquad\qquad\quad - \rho_N T_N\\
&\qquad\qquad\qquad\quad \left.\left.\left.\left.\left. \times \left(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(N)}\right)\right)\right)\right)\right)\right\|\\
&= \left\| J^{-1}\left(\left(1-\alpha_n^{(N)}\right) Jf\left(x_n^{(N)}\right)\right.\right.\\
&\qquad + \alpha_n^{(N)} J\left(\prod_K J^{-1}\right.\\
&\qquad\qquad \times \left(Jf\left(x_n^{(N)}\right)\right.\\
&\qquad\qquad \left.\left.\left.\left. - \rho_N T_N\left(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(N)}\right)\right)\right)\right)\right\|
\end{aligned}$$

$$
\begin{aligned}
&= \Big\| \left(1-\alpha_n^{(N)}\right) Jf\left(x_n^{(N)}\right) \\
&\quad + \alpha_n^{(N)} J\Big(\prod_K J^{-1} \\
&\qquad \times \left(Jf\left(x_n^{(N)}\right) \right.\\
&\qquad\qquad \left. - \rho_N T_N\left(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)}\right)\right)\Big)\Big\| \\
&\le \left(1-\alpha_n^{(N)}\right) \left\| Jf\left(x_n^{(N)}\right)\right\| \\
&\quad + \alpha_n^{(N)} \Big\| J\Big(\prod_K J^{-1} \\
&\qquad \times \left(Jf\left(x_n^{(N)}\right) - \rho_N T_N\left(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)}\right)\right)\Big)\Big\| \\
&\le \left(1-\alpha_n^{(N)}\right) \left\| Jf\left(x_n^{(N)}\right)\right\| \\
&\quad + \alpha_n^{(N)} \left\| JJ^{-1}\left(Jf\left(x_n^{(N)}\right) - \rho_N T_N\left(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)}\right)\right)\right\| \\
&= \left(1-\alpha_n^{(N)}\right) \left\| f\left(x_n^{(N)}\right)\right\| \\
&\quad + \alpha_n^{(N)} \left\| Jf\left(x_n^{(N)}\right) - \rho_N T_N\left(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)}\right)\right\| \\
&\le \left(1-\alpha_n^{(N)}\right) \left\| f\left(x_n^{(N)}\right)\right\| + \alpha_n^{(N)} \left\| f\left(x_n^{(N)}\right)\right\| \\
&= \left\| f\left(x_n^{(N)}\right)\right\|.
\end{aligned}
\tag{21}
$$

Since f is an isometry mapping, we have $\|x_{n+1}^{(N)}\| \le \|x_n^{(N)}\|$. By the same argument method as given above, we have $\|x_{n+1}^{(N-1)}\| \le \|x_n^{(N-1)}\|, \dots, \|x_{n+1}^{(1)}\| \le \|x_n^{(1)}\|$. Therefore, we note that $\lim_{n\to\infty}\|x_n^{(1)}\|, \dots, \lim_{n\to\infty}\|x_n^{(N)}\|$ exist and hence the sequences $\{x_n^{(1)}\}, \{x_n^{(2)}\}, \dots, \{x_n^{(N)}\}$ are bounded in K.

Step 2. By Lemmas 4 and 6, where f is an isometry mapping and (19), it follows that there exists a continuous strictly increasing and convex function $g : [0, 2r) \to [0, \infty)$ with $g(0) = 0$ such that

$$
\begin{aligned}
&\left\| f(x_{n+1}^{(N)})\right\|^2 \\
&\le \left(1-\alpha_n^{(N)}\right) \left\| Jf\left(x_n^{(N)}\right)\right\|^2 \\
&\quad + \alpha_n^{(N)} \Big\| J\prod_K J^{-1}\left(Jf\left(x_n^{(N)}\right) \right.\\
&\qquad\qquad \left. - \rho_N T_N\left(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)}\right)\right)\Big\|^2 \\
&\quad - \left(1-\alpha_n^{(N)}\right)\alpha_n^{(N)} g \\
&\quad \times \Big(\Big\| Jf\left(x_n^{(N)}\right) \\
&\qquad + J\prod_K J^{-1}\left(Jf\left(x_n^{(N)}\right) \right.\\
&\qquad\qquad \left. - \rho_N T_N\left(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)}\right)\right)\Big\|\Big) \\
&\le \left(1-\alpha_n^{(N)}\right) \left\| f\left(x_n^{(N)}\right)\right\|^2 \\
&\quad + \alpha_n^{(N)} \left\| Jf\left(x_n^{(N)}\right) - \rho_N T_N\left(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)}\right)\right\|^2 \\
&\quad - \left(1-\alpha_n^{(N)}\right)\alpha_n^{(N)} \\
&\quad \times g\Big(\Big\| Jf\left(x_n^{(N)}\right) \\
&\qquad + J\prod_K J^{-1}\left(Jf\left(x_n^{(N)}\right) \right.\\
&\qquad\qquad \left. - \rho_N T_N\left(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)}\right)\right)\Big\|\Big) \\
&\le \left(1-\alpha_n^{(N)}\right) \left\| f\left(x_n^{(N)}\right)\right\|^2 + \alpha_n^{(N)} \left\| f\left(x_n^{(N)}\right)\right\|^2 \\
&\quad - \left(1-\alpha_n^{(N)}\right)\alpha_n^{(N)} \\
&\quad \times g\Big(\Big\| Jf\left(x_n^{(N)}\right) \\
&\qquad + J\prod_K J^{-1}\left(Jf\left(x_n^{(N)}\right) \right.\\
&\qquad\qquad \left. - \rho_N T_N\left(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)}\right)\right)\Big\|\Big) \\
&= \left\| f\left(x_n^{(N)}\right)\right\|^2 - \left(1-\alpha_n^{(N)}\right)\alpha_n^{(N)} \\
&\quad \times g\Big(\Big\| Jf\left(x_n^{(N)}\right) \\
&\qquad + J\prod_K J^{-1}\left(Jf\left(x_n^{(N)}\right) \right.\\
&\qquad\qquad \left. - \rho_N T_N\left(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)}\right)\right)\Big\|\Big).
\end{aligned}
\tag{22}
$$

This implies that

$$
\begin{aligned}
&\left(1-\alpha_n^{(N)}\right)\alpha_n^{(N)} g\Big(\Big\| Jf\left(x_n^{(1)}\right) \\
&\qquad + J\prod_K J^{-1}\Big(Jf\left(x_n^{(N)}\right) \\
&\qquad\qquad - \rho_N T_N \\
&\qquad\qquad \times \left(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)}\right)\Big)\Big\|\Big) \\
&\le \left\| f\left(x_n^{(N)}\right)\right\|^2 - \left\| f\left(x_{n+1}^{(N)}\right)\right\|^2.
\end{aligned}
\tag{23}
$$

Since $\{\|x_n^{(k)}\|\}$ converges for all $k = 1, 2, \ldots, N$, it follows by letting $n \to \infty$ in (23), condition (ii), and the property of g that

$$\left\| Jf\left(x_n^{(N)}\right) - J\prod_K J^{-1}\left(Jf\left(x_n^{(N)}\right) - \rho_N T_N\left(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(N)}\right)\right)\right\| \longrightarrow 0, \tag{24}$$

as $n \to \infty$. By (16) and (24), we have

$$\begin{aligned}\left\|Jf\left(x_{n+1}^{(N)}\right) - Jf\left(x_n^{(N)}\right)\right\| &= \alpha_n^{(N)} \left\|Jf\left(x_n^{(N)}\right) - J\prod_K J^{-1}\left(Jf\left(x_n^{(N)}\right) - \rho_N T_N\left(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(N)}\right)\right)\right\| \longrightarrow 0,\end{aligned} \tag{25}$$

as $n \to \infty$. Similarly, we can prove that

$$\begin{aligned}&\left\|Jf\left(x_{n+1}^{(N-1)}\right) - Jf\left(x_n^{(N-1)}\right)\right\| \\ &\quad = \alpha_n^{(N-1)} \left\|Jf\left(x_n^{(N-1)}\right) - J\prod_K J^{-1} \times \left(Jf\left(x_n^{(N-1)}\right) - \rho_{N-1} T_{N-1} \times \left(x_{n+1}^{(N)}, x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(N-2)}, x_n^{(N-1)}\right)\right)\right\| \longrightarrow 0, \\ &\qquad \vdots \\ &\left\|Jf\left(x_{n+1}^{(2)}\right) - Jf\left(x_n^{(2)}\right)\right\| \\ &\quad = \alpha_n^{(2)} \left\|Jf\left(x_n^{(2)}\right) - J\prod_K J^{-1} \times \left(Jf\left(x_n^{(2)}\right) - \rho_2 T_2 \times \left(x_{n+1}^{(3)}, x_{n+1}^{(4)}, \ldots, x_{n+1}^{(N)}, x_n^{(1)}, x_n^{(2)}\right)\right)\right\| \longrightarrow 0, \\ &\left\|Jf\left(x_{n+1}^{(1)}\right) - Jf\left(x_n^{(1)}\right)\right\| \\ &\quad = \alpha_n^{(1)} \left\|Jf\left(x_n^{(1)}\right) - J\prod_K J^{-1} \times \left(Jf\left(x_n^{(1)}\right) - \rho_1 T_1 \times \left(x_{n+1}^{(2)}, x_{n+1}^{(3)}, \ldots, x_{n+1}^{(N)}, x_n^{(1)}\right)\right)\right\| \longrightarrow 0,\end{aligned} \tag{26}$$

as $n \to \infty$.

Step 3. Since $\{x_n^{(1)}\}, \{x_n^{(2)}\}, \ldots, \{x_n^{(N)}\}$ are bounded and there exists a compact subset $C \subset E^*$ such that $(J(K) - \rho_N T_N(K^N)) \subset C$, there exists a subsequence $\{x_{n_{i(N)}}^{(N)}\}$ of $\{x_{n_j}^{(N)}\}$ such that

$$Jf\left(x_{n_{i(N)}}^{(N)}\right) - \rho_N T_N\left(x_{n_{i(N)}}^{(1)}, x_{n_{i(N)}}^{(2)}, \ldots, x_{n_{i(N)}}^{(N)}\right) \longrightarrow h_1 \in E^*. \tag{27}$$

Since E is uniformly smooth and strictly convex, it follows by Lemma 2 (b) and Remark 1 that $\prod_K$ and J^{-1} are continuous. Thus

$$\prod_K J^{-1}\left(Jf\left(x_{n_{i(N)}}^{(N)}\right) - \rho_N T_N \times \left(x_{n_{i(N)}}^{(1)}, x_{n_{i(N)}}^{(2)}, \ldots, x_{n_{i(N)}}^{(N)}\right)\right) \longrightarrow \prod_K J^{-1}(h_1) := f\left(x_N^*\right), \tag{28}$$

$$J\prod_K J^{-1}\left(Jf\left(x_{n_{i(N)}}^{(N)}\right) - \rho_N T_N\left(x_{n_{i(N)}}^{(1)}, x_{n_{i(N)}}^{(2)}, \ldots, x_{n_{i(N)}}^{(N)}\right)\right) \longrightarrow Jf\left(x_N^*\right). \tag{29}$$

From (24) and (29), we get

$$Jf\left(x_{n_{i(N)}}^{(N)}\right) \longrightarrow Jf\left(x_N^*\right) \quad \left(\text{as } n_{i(N)} \longrightarrow \infty\right). \tag{30}$$

By (25) and (30), we have

$$Jf\left(x_{n_{i(N)}+1}^{(N)}\right) \longrightarrow Jf\left(x_N^*\right) \quad \left(\text{as } n_{i(N)} \longrightarrow \infty\right). \tag{31}$$

Since E is strictly convex and reflexive, it follows by Remark 1 (iv) that J^{-1} is norm-weak-continuous. Therefore, from (30) and (31), we note that

$$f\left(x_{n_{i(N)}}^{(N)}\right) \rightharpoonup f\left(x_N^*\right), \qquad f\left(x_{n_{i(N)}+1}^{(N)}\right) \rightharpoonup f\left(x_N^*\right) \tag{32}$$

and

$$\begin{aligned}\left\|f\left(x_{n_{i(N)}}^{(N)}\right)\right\| &\longrightarrow \left\|f\left(x_N^*\right)\right\|, \\ \left\|f\left(x_{n_{i(N)}+1}^{(N)}\right)\right\| &\longrightarrow \left\|f\left(x_N^*\right)\right\| \\ &\left(\text{as } n_{i(N)} \longrightarrow \infty\right).\end{aligned} \tag{33}$$

By the Kadec-Klee property, we have

$$f\left(x_{n_{i(N)}}^{(N)}\right) \longrightarrow f\left(x_N^*\right), \qquad f\left(x_{n_{i(N)}+1}^{(N)}\right) \longrightarrow f\left(x_N^*\right) \quad \left(\text{as } n_{i(N)} \rightarrow \infty\right). \tag{34}$$

Since f^{-1} is continuous, it implies that $\{x_{n_{i(N)}}^{(N)}\}$ is a subsequence of $\{x_{n_j}^{(N)}\}$ such that $x_{n_{i(N)}}^{(N)} \rightarrow x_N^* \in E$. Therefore $x_n^{(N)} \rightarrow x_N^*$ as $n \rightarrow \infty$. So, it follows from (16), (30), (34), and condition (ii) that

$$\begin{aligned} Jf\left(x_N^*\right) &= \lim_{n\rightarrow\infty} Jf\left(x_{n+1}^{(N)}\right) \\ &= \lim_{n\rightarrow\infty}\left\{\left(1-\alpha_n^{(N)}\right)Jf\left(x_n^{(N)}\right)+\alpha_n^{(N)} f^{-1}J\prod_K J^{-1} \times\left(Jf\left(x_n^{(N)}\right)-\rho_N T_N\left(x_n^{(1)},x_n^{(2)},\ldots,x_n^{(N)}\right)\right)\right\} \\ &= \left(1-d_N\right)Jf\left(x_N^*\right) + d_N J\prod_K J^{-1}\left(Jf\left(x_N^*\right)-\rho_N T_N\left(x_1^*,x_2^*,\ldots,x_N^*\right)\right). \end{aligned} \tag{35}$$

Since f is a bijective mapping, we obtain that

$$x_N^* = f^{-1}\prod_K J^{-1}\left(Jf\left(x_N^*\right)-\rho_N T_N\left(x_1^*,x_2^*,\ldots,x_N^*\right)\right). \tag{36}$$

Similarly, we can prove that for every subsequence $\{x_{n_j}^{(k)}\}$ of $\{x_n^{(k)}\}$ there exist a subsequence $\{x_{n_{i(k)}}^{(k)}\}$ of $\{x_{n_j}^{(k)}\}$ and $x_k^* \in E$ such that

$$f\left(x_{n_{i(k)}}^{(k)}\right) \longrightarrow f\left(x_k^*\right) \quad \left(\text{as } n_{i(k)} \longrightarrow \infty\right), \quad \forall k = 1,2,\ldots,N-1. \tag{37}$$

Since f^{-1} is a continuous mapping, we note that

$$x_{n_{i(k)}}^{(k)} \longrightarrow x_k^* \quad \left(\text{as } n_{i(k)} \longrightarrow \infty\right). \tag{38}$$

Hence $x_n^{(k)} \rightarrow x_k^* \in E$, for all $k = 1,2,\ldots,N-1$. Therefore, we have

$$\begin{aligned} x_{N-1}^* &= f^{-1}\prod_K J^{-1}\left(Jf\left(x_{N-1}^*\right) - \rho_{N-1}T_{N-1}\left(x_N^*,x_1^*,\ldots,x_{N-2}^*,x_{N-1}^*\right)\right) \\ &\vdots \\ x_1^* &= f^{-1}\prod_K J^{-1}\left(Jf\left(x_1^*\right) - \rho_1 T_1\left(x_2^*,x_3^*,\ldots,x_N^*,x_1^*\right)\right). \end{aligned} \tag{39}$$

By Lemma 7, we can conclude that $(x_1^*,x_2^*,\ldots,x_N^*)$ is a solution of (8) and $x_n^{(1)} \rightarrow x_1^*, x_n^{(2)} \rightarrow x_2^*, \ldots, x_n^{(N)} \rightarrow x_N^*$. □

Setting $N = 3$ and $f = I$ in Theorem 9, we immediately obtain the following result.

Corollary 10 (see [6]). *Let E be a real uniformly smooth and strictly convex Banach space with Kadec-Klee property and K a nonempty closed and convex subset of E with $\theta \in K$. Let $T_1,T_2,T_3 : K^3 \rightarrow E^*$ be continuous mappings and $\{\alpha_n^{(1)}\}$, $\{\alpha_n^{(2)}\}$, and $\{\alpha_n^{(3)}\}$ the sequences in (a,b) with $0 < a < b < 1$ satisfying the following conditions.*

(i) *There exist a compact subset $C \subset E^*$ and constants $\rho_1 > 0$, $\rho_2 > 0$, and $\rho_3 > 0$ such that*

$$\left(J(K)-\rho_3T_3\left(K^3\right)\right)\cup\left(J(K)-\rho_2T_2\left(K^3\right)\right) \cup\left(J(K)-\rho_1T_1\left(K^3\right)\right)\subset C, \tag{40}$$

where $J(x_1,x_2,x_3) = Jx_3$, for all $(x_1,x_2,x_3) \in K^3$, and

$$\begin{aligned} &\left\langle T_1\left(x_1,x_2,x_3\right), J^{-1}\left(Jx_3-\rho_1T_1\left(x_1,x_2,x_3\right)\right)\right\rangle \geq 0, \\ &\left\langle T_2\left(x_1,x_2,x_3\right), J^{-1}\left(Jx_3-\rho_2T_2\left(x_1,x_2,x_3\right)\right)\right\rangle \geq 0, \\ &\left\langle T_3\left(x_1,x_2,x_3\right), J^{-1}\left(Jx_3-\rho_3T_3\left(x_1,x_2,x_3\right)\right)\right\rangle \geq 0, \end{aligned} \tag{41}$$

for all $x_1,x_2,x_3 \in K$.

(ii) $\lim_{n\rightarrow\infty}\alpha_n^{(1)} = d_1 \in (a,b)$, $\lim_{n\rightarrow\infty}\alpha_n^{(2)} = d_2 \in (a,b)$, *and* $\lim_{n\rightarrow\infty}\alpha_n^{(3)} = d_3 \in (a,b)$. *Let $\{x_n^{(1)}\}$, $\{x_n^{(2)}\}$, and $\{x_n^{(3)}\}$ be the sequences defined by*

$$\begin{aligned} x_{n+1}^{(3)} &= J^{-1}\left(\left(1-\alpha_n^{(3)}\right)Jf\left(x_n^{(3)}\right)+\alpha_n^{(3)}J \times\left(\prod_K J^{-1}\left(Jf\left(x_n^{(3)}\right)-\rho_3T_3\left(x_n^{(1)},x_n^{(2)},x_n^{(3)}\right)\right)\right)\right), \\ x_{n+1}^{(2)} &= J^{-1}\left(\left(1-\alpha_n^{(2)}\right)Jf\left(x_n^{(2)}\right)+\alpha_n^{(2)}J \times\left(\prod_K J^{-1}\left(Jf\left(x_n^{(2)}\right)-\rho_2T_2\left(x_{n+1}^{(3)},x_n^{(1)},x_n^{(2)}\right)\right)\right)\right), \\ x_{n+1}^{(1)} &= J^{-1}\left(\left(1-\alpha_n^{(1)}\right)Jf\left(x_n^{(1)}\right)+\alpha_n^{(1)}J \times\left(\prod_K J^{-1}\left(Jf\left(x_n^{(1)}\right)-\rho_1T_1\left(x_{n+1}^{(2)},x_{n+1}^{(3)},x_n^{(1)}\right)\right)\right)\right), \\ & n \geq 0. \end{aligned} \tag{42}$$

Then the problem (9) has a solution $(x_1^*,x_2^*,x_3^*) \in K^3$ and the sequences $\{x_n^{(1)}\}$, $\{x_n^{(2)}\}$ and $\{x_n^{(3)}\}$ converge strongly to x_1^*, x_2^*, and x_3^*, respectively.

Setting E as a real Hilbert space in Theorem 9, we have the following result.

Corollary 11. *Let H be a real Hilbert space and K a nonempty closed and convex subset of H. Let $f : K \to K$ be an isometry mapping and $T_1, \ldots, T_N : K^N \to H$ continuous mappings and $\{\alpha_n^{(1)}\}, \{\alpha_n^{(2)}\}, \ldots, \{\alpha_n^{(N)}\}$ are sequences in (a, b) with $0 < a < b < 1$ satisfying the following conditions.*

(i) *There exist a compact subset $C \subset H$ and constants $\rho_1 > 0, \rho_2 > 0, \ldots, \rho_N > 0$ such that*

$$\begin{aligned}&\left(I(K) - \rho_N T_N\left(K^N\right)\right)\\&\quad\cup\left(I(K) - \rho_{N-1} T_{N-1}\left(K^N\right)\right)\\&\quad\cup\cdots\cup\left(I(K) - \rho_1 T_1\left(K^N\right)\right) \subset C,\end{aligned} \tag{43}$$

where $(x_1, x_2, \ldots, x_N) = x_N$, for all $(x_1, x_2, \ldots, x_N) \in K^N$, and

$$\begin{gathered}\langle T_1(x_1, x_2, \ldots, x_N), x_N - \rho_1 T_1(x_1, x_2, \ldots, x_N)\rangle \geq 0,\\ \langle T_2(x_1, x_2, \ldots, x_N), x_N - \rho_2 T_2(x_1, x_2, \ldots, x_N)\rangle \geq 0,\\ \vdots\\ \langle T_N(x_1, x_2, \ldots, x_N), x_N - \rho_N T_N(x_1, x_2, \ldots, x_N)\rangle \geq 0,\end{gathered} \tag{44}$$

for all $x_1, x_2, \ldots, x_N \in K$.

(ii) $\lim_{n\to\infty}\alpha_n^{(1)} = d_1 \in (a, b)$, $\lim_{n\to\infty}\alpha_n^{(2)} = d_2 \in (a, b), \ldots, \lim_{n\to\infty}\alpha_n^{(N)} = d_N \in (a, b)$. *Let $\{x_n^{(1)}\}, \{x_n^{(2)}\}, \ldots, \{x_n^{(N)}\}$ be the sequences defined by*

$$\begin{aligned}
x_{n+1}^{(N)} &= f^{-1}\left(\left(1 - \alpha_n^{(N)}\right) f\left(x_n^{(N)}\right) + \alpha_n^{(N)} P_K\right.\\
&\qquad \left.\times\left(f\left(x_n^{(N)}\right) - \rho_N T_N\left(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(N)}\right)\right)\right),\\
x_{n+1}^{(N-1)} &= f^{-1}\left(\left(1 - \alpha_n^{(N-1)}\right) f\left(x_n^{(N-1)}\right) + \alpha_n^{(N-1)} P_K\right.\\
&\qquad \times\left(f\left(x_n^{(N-1)}\right) - \rho_{N-1} T_{N-1}\right.\\
&\qquad\quad \left.\left.\times\left(x_{n+1}^{(N)}, x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(N-2)}, x_n^{(N-1)}\right)\right)\right),\\
&\ \ \vdots\\
x_{n+1}^{(2)} &= f^{-1}\left(\left(1 - \alpha_n^{(2)}\right) f\left(x_n^{(2)}\right) + \alpha_n^{(2)} P_K\right.\\
&\qquad \times\left(f\left(x_n^{(2)}\right) - \rho_2 T_2\right.\\
&\qquad\quad \left.\left.\times\left(x_{n+1}^{(3)}, x_{n+1}^{(4)}, \ldots, x_{n+1}^{(N)}, x_n^{(1)}, x_n^{(2)}\right)\right)\right),\\
x_{n+1}^{(1)} &= f^{-1}\left(\left(1 - \alpha_n^{(1)}\right) f\left(x_n^{(1)}\right)\right.\\
&\qquad + \alpha_n^{(1)} P_K\left(f\left(x_n^{(1)}\right) - \rho_1 T_1\right.\\
&\qquad\quad \left.\left.\times\left(x_{n+1}^{(2)}, x_{n+1}^{(3)}, \ldots, x_{n+1}^{(N)}, x_n^{(1)}\right)\right)\right),\\
&\qquad n \geq 0,
\end{aligned} \tag{45}$$

where P_K is a metric projection on H to K. Then the problem (8) has a solution $(x_1^*, x_2^*, \ldots, x_N^*) \in K^N$ and the sequences $\{x_n^{(1)}\}, \{x_n^{(2)}\}, \ldots, \{x_n^{(N)}\}$ converge strongly to $x_1^*, x_2^*, \ldots, x_N^*$, respectively.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

A Regularized Algorithm for the Proximal Split Feasibility Problem

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The proximal split feasibility problem has been studied. A regularized method has been presented for solving the proximal split feasibility problem. Strong convergence theorem is given.

1. Introduction

Throughout, we assume that $\mathscr{H}_1$ and $\mathscr{H}_2$ are two real Hilbert spaces, $f : \mathscr{H}_1 \to \mathscr{R} \cup \{+\infty\}$ and $g : \mathscr{H}_2 \to \mathscr{R} \cup \{+\infty\}$ are two proper, lower semicontinuous convex functions, and $A : \mathscr{H}_1 \to \mathscr{H}_2$ is a bounded linear operator.

In the present paper, we are devoted to solving the following minimization problem:

$$\min_{x^\dagger \in \mathscr{H}_1} \left\{ f\left(x^\dagger\right) + g_\lambda\left(Ax^\dagger\right)\right\}, \tag{1}$$

where g_λ stands for the Moreau-Yosida approximation of the function g of parameter λ; that is,

$$g_\lambda(u) = \min_{v \in \mathscr{H}_2} \left\{ g(v) + \frac{1}{2\lambda}\|u - v\|^2 \right\}. \tag{2}$$

Problem (1) includes the split feasibility problem as a special case. In fact, we choose f and g as the indicator functions of two nonempty closed convex sets $C \subset \mathscr{H}_1$ and $Q \in \mathscr{H}_2$; that is,

$$f\left(x^\dagger\right) = \delta_C\left(x^\dagger\right) = \begin{cases} 0, & \text{if } x^\dagger \in C, \\ +\infty, & \text{otherwise}, \end{cases}$$

$$g\left(x^\dagger\right) = \delta_Q\left(x^\dagger\right) = \begin{cases} 0, & \text{if } x^\dagger \in Q, \\ +\infty, & \text{otherwise}. \end{cases} \tag{3}$$

Then, problem (1) reduces to

$$\min_{x^\dagger \in \mathscr{H}_1} \left\{ \delta_C\left(x^\dagger\right) + \left(\delta_Q\right)_\lambda\left(Ax^\dagger\right)\right\}, \tag{4}$$

which equals

$$\min_{x^\dagger \in C} \left\{ \frac{1}{2\lambda}\left\|\left(I - \mathrm{proj}_Q\right)\left(Ax^\dagger\right)\right\|^2 \right\}. \tag{5}$$

Now we know that solving (5) is exactly to solve the following split feasibility problem of finding $x^\ddagger$ such that

$$x^\ddagger \in C, \qquad Ax^\ddagger \in Q, \tag{6}$$

provided $C \cap A^{-1}(Q) \neq \emptyset$.

The split feasibility problem in finite-dimensional Hilbert spaces was first introduced by Censor and Elfving [1] for modeling inverse problems which arise from phase retrievals and in medical image reconstruction. Recently, the split feasibility problem (6) has been studied extensively by many authors; see, for instance, [2–8].

In order to solve (6), one of the key ideas is to use fixed point technique according to $x^\dagger$ which solves (6) if and only if

$$x^\dagger = \text{proj}_C \left(I - \gamma A^* \left(I - \text{proj}_Q \right) A \right) x^\dagger. \tag{7}$$

Next, we will use this idea to solve (1). First, by the differentiability of the Yosida approximation g_λ, we have

$$\begin{aligned} \partial \left(f\left(x^\dagger\right) + g_\lambda\left(Ax^\dagger\right)\right) &= \partial f\left(x^\dagger\right) + A^* \nabla g_\lambda\left(Ax^\dagger\right) \\ &= \partial f\left(x^\dagger\right) + A^* \left(\frac{I - \text{prox}_{\lambda g}}{\lambda} \right)\left(Ax^\dagger\right), \end{aligned} \tag{8}$$

where $\partial f(x^\dagger)$ denotes the subdifferential of f at $x^\dagger$ and $\text{prox}_{\lambda g}(x^\dagger)$ is the proximal mapping of g. That is,

$$\begin{aligned} \partial f\left(x^\dagger\right) &= \left\{ x^* \in \mathscr{H}_1 : f\left(x^\ddagger\right) \geq f\left(x^\dagger\right) + \left\langle x^*, x^\ddagger - x^\dagger \right\rangle, \right. \\ &\qquad \left. \forall x^\ddagger \in \mathscr{H}_1 \right\}, \\ \text{prox}_{\lambda g}\left(x^\dagger\right) &= \arg \min_{x^\ddagger \in \mathscr{H}_2} \left\{ g\left(x^\ddagger\right) + \frac{1}{2\lambda} \left\| x^\ddagger - x^\dagger \right\|^2 \right\}. \end{aligned} \tag{9}$$

Note that the optimality condition of (8) is as follows:

$$0 \in \partial f\left(x^\dagger\right) + A^* \left(\frac{I - \text{prox}_{\lambda g}}{\lambda} \right)\left(Ax^\dagger\right), \tag{10}$$

which can be rewritten as

$$0 \in \mu\lambda \partial f\left(x^\dagger\right) + \mu A^* \left(I - \text{prox}_{\lambda g}\right)\left(Ax^\dagger\right), \tag{11}$$

which is equivalent to the fixed point equation

$$x^\dagger = \text{prox}_{\mu\lambda f}\left(x^\dagger - \mu A^* \left(I - \text{prox}_{\lambda g}\right)\right)\left(Ax^\dagger\right). \tag{12}$$

If $\arg\min f \cap A^{-1}(\arg\min g) \neq \emptyset$, then (1) is reduced to the following proximal split feasibility problem of finding $x^\dagger$ such that

$$x^\dagger \in \arg\min f, \qquad Ax^\dagger \in \arg\min g, \tag{13}$$

where

$$\begin{aligned} \arg\min f &= \left\{ x^* \in \mathscr{H}_1 : f\left(x^*\right) \leq f\left(x^\dagger\right), \forall x^\dagger \in \mathscr{H}_1 \right\}, \\ \arg\min g &= \left\{ x^\dagger \in \mathscr{H}_2 : g\left(x^\dagger\right) \leq g\left(x\right), \forall x \in \mathscr{H}_2 \right\}. \end{aligned} \tag{14}$$

In the sequel, we will use Γ to denote the solution set of (13).

Recently, in order to solve (13), Moudafi and Thakur [9] presented the following split proximal algorithm with a way of selecting the stepsizes such that its implementation does not need any prior information about the operator norm.

Split Proximal Algorithm

Step 1 (initialization).

$$x_0 \in \mathscr{H}_1. \tag{15}$$

Step 2. Assume that x_n has been constructed and $\theta(x_n) \neq \emptyset$. Then compute x_{n+1} via the manner

$$x_{n+1} = \text{prox}_{\mu_n \lambda f}\left[x_n - \mu_n A^* \left(I - \text{prox}_{\lambda g}\right) Ax_n \right], \quad \forall n \geq 0, \tag{16}$$

where the stepsize $\mu_n = \rho_n((h(x_n) + l(x_n))/\theta^2(x_n))$ in which $0 < \rho_n < 4$, $h(x_n) = (1/2)\|(I - \text{prox}_{\lambda g})Ax_n\|^2$, $l(x_n) = (1/2)\|(I - \text{prox}_{\mu_n \lambda f})x_n\|^2$ and $\theta(x_n) = \sqrt{\|\nabla h(x_n)\|^2 + \|\nabla l(x_n)\|^2}$.

If $\theta(x_n) = 0$, then $x_{n+1} = x_n$ is a solution of (13) and the iterative process stops; otherwise, we set $n := n + 1$ and go to (16).

Consequently, they demonstrated the following weak convergence of the above split proximal algorithm.

Theorem 1. *Suppose that $\Gamma \neq \emptyset$. Assume that the parameters satisfy the condition:*

$$\epsilon \leq \rho_n \leq \frac{4h\left(x_n\right)}{h\left(x_n\right) + l\left(x_n\right)} - \epsilon \quad \textit{for some } \epsilon > 0 \textit{ small enough.} \tag{17}$$

Then the sequence x_n weakly converges to a solution of (13).

Note that the proximal mapping of g is firmly nonexpansive, namely,

$$\begin{aligned} \left\langle \text{prox}_{\lambda g} x - \text{prox}_{\lambda g} y, x - y \right\rangle &\geq \left\| \text{prox}_{\lambda g} x - \text{prox}_{\lambda g} y \right\|^2, \\ &\qquad \forall x, y \in \mathscr{H}_2, \end{aligned} \tag{18}$$

and it is also the case for complement $I - \text{prox}_{\lambda g}$. Thus, $A^*(I - \text{prox}_{\lambda g})A$ is cocoercive with coefficient $1/\|A\|^2$ (recall that a mapping $B : \mathscr{H}_1 \to \mathscr{H}_1$ is said to be *cocoercive* if $\langle Bx - By, x - y\rangle \geq \alpha\|Bx - By\|^2$ for all $x, y \in \mathscr{H}_1$ and some $\alpha > 0$). If $\mu \in (0, 1/\|A\|^2)$, then $I - \mu A^*(I - \text{prox}_{\lambda g})A$ is nonexpansive. Hence, we need to regularize (16) such that the strong convergence is obtained. This is the main purpose of this paper. In the next section, we will collect some useful lemmas and in the last section we will present our algorithm and prove its strong convergence.

2. Lemmas

Lemma 2 (see [10]). *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq \left(1 - \alpha_n\right) a_n + \alpha_n \sigma_n + \delta_n, \quad n \geq 0, \tag{19}$$

where

(i) $\{\alpha_n\}_{n\in\mathbb{N}} \subset [0, 1]$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\limsup_{n\to\infty} \sigma_n \leq 0$;

(iii) $\sum_{n=1}^{\infty} \delta_n < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 3 (see [11]). *Let $\{\gamma_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that there exists a subsequence $\{\gamma_{n_i}\}_{i\in\mathbb{N}}$ of $\{\gamma_n\}_{n\in\mathbb{N}}$ such that $\gamma_{n_i} < \gamma_{n_i+1}$ for all $i \in \mathbb{N}$. Then, there exists a nondecreasing sequence $\{m_k\}_{k\in\mathbb{N}}$ of $\mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$\gamma_{m_k} \le \gamma_{m_k+1}, \qquad \gamma_k \le \gamma_{m_k+1}. \tag{20}$$

In fact, m_k is the largest number n in the set $\{1,\ldots,k\}$ such that the condition $\gamma_n < \gamma_{n+1}$ holds.

3. Main results

Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two real Hilbert spaces. Let $f : \mathscr{H}_1 \to \mathscr{R} \cup \{+\infty\}$ and $g : \mathscr{H}_2 \to \mathscr{R} \cup \{+\infty\}$ be two proper, lower semicontinuous convex functions and $A : \mathscr{H}_1 \to \mathscr{H}_2$ a bounded linear operator.

Now, we firstly introduce our algorithm.

Algorithm 4

Step 1 (initialization).

$$x_0 \in \mathscr{H}_1. \tag{21}$$

Step 2. Assume that x_n has been constructed. Set $h(x_n) = (1/2)\|(I-\mathrm{prox}_{\lambda g})Ax_n\|^2$, $l(x_n) = (1/2)\|(I-\mathrm{prox}_{\mu_n\lambda f})x_n\|^2$ and $\theta(x_n) = \sqrt{\|\nabla h(x_n)\|^2 + \|\nabla l(x_n)\|^2}$ for all $n \in \mathbb{N}$.

If $\theta(x_n) \ne \emptyset$, then compute x_{n+1} via the manner

$$\begin{aligned} x_{n+1} &= \mathrm{prox}_{\mu_n\lambda f}\left[\alpha_n u + (1-\alpha_n)x_n - \mu_n A^*\left(I-\mathrm{prox}_{\lambda g}\right)Ax_n\right], \\ &\qquad \forall n \ge 0, \end{aligned} \tag{22}$$

where $u \in \mathscr{H}_1$ is a fixed point and $\{\alpha_n\}_{n\in\mathbb{N}} \subset [0,1]$ is a real number sequence and μ_n is the stepsize satisfying $\mu_n = \rho_n((h(x_n)+l(x_n))/\theta^2(x_n))$ with $0 < \rho_n < 4$.

If $\theta(x_n) = 0$, then $x_{n+1} = x_n$ is a solution of (13) and the iterative process stops; otherwise, we set $n := n+1$ and go to (22).

Theorem 5. *Suppose that $\Gamma \ne \emptyset$. Assume that the parameters $\{\alpha_n\}$ and $\{\rho_n\}$ satisfy the conditions:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$;

(C2) $\sum_{n=0}^{\infty}\alpha_n = \infty$;

(C3) $\epsilon \le \rho_n \le (4h(x_n)/(h(x_n)+l(x_n))) - \epsilon$ *for some* $\epsilon > 0$ *small enough.*

Then the sequence x_n converges strongly to $\mathrm{proj}_\Gamma(u)$.

Proof. Let $x^* \in \Gamma$. Since minimizers of any function are exactly fixed points of its proximal mappings, we have $x^* = \mathrm{prox}_{\mu_n\lambda f}x^*$ and $Ax^* = \mathrm{prox}_{\lambda g}Ax^*$. By (22) and the nonexpansivity of $\mathrm{prox}_{\mu_n\lambda f}$, we derive

$$\begin{aligned} &\|x_{n+1}-x^*\|^2 \\ &= \left\|\mathrm{prox}_{\mu_n\lambda f}\left[\alpha_n u + (1-\alpha_n)x_n - \mu_n A^*\left(I-\mathrm{prox}_{\lambda g}\right)Ax_n\right]\right. \\ &\qquad \left. - \mathrm{prox}_{\mu_n\lambda f}x^*\right\|^2 \\ &\le \left\|\alpha_n u + (1-\alpha_n)x_n - \mu_n A^*\left(I-\mathrm{prox}_{\lambda g}\right)Ax_n - x^*\right\|^2 \\ &= \left\|\alpha_n(u-x^*) + (1-\alpha_n)\right. \\ &\qquad \left.\times\left[x_n - \frac{\mu_n}{1-\alpha_n}A^*\left(I-\mathrm{prox}_{\lambda g}\right)Ax_n - x^*\right]\right\|^2 \\ &\le \alpha_n\|u-x^*\|^2 + (1-\alpha_n) \\ &\qquad \times\left\|x_n - \frac{\mu_n}{1-\alpha_n}A^*(I-\mathrm{prox}_{\lambda g})Ax_n - x^*\right\|^2. \end{aligned} \tag{23}$$

Since $\mathrm{prox}_{\lambda g}$ is firmly nonexpansive, we deduce that $I-\mathrm{prox}_{\lambda g}$ is also firmly nonexpansive. Hence, we have

$$\begin{aligned} &\left\langle A^*\left(I-\mathrm{prox}_{\lambda g}\right)Ax_n, x_n - x^*\right\rangle \\ &\quad = \left\langle\left(I-\mathrm{prox}_{\lambda g}\right)Ax_n, Ax_n - Ax^*\right\rangle \\ &\quad = \left\langle\left(I-\mathrm{prox}_{\lambda g}\right)Ax_n - \left(I-\mathrm{prox}_{\lambda g}\right)Ax^*, Ax_n - Ax^*\right\rangle \\ &\quad \ge \left\|\left(I-\mathrm{prox}_{\lambda g}\right)Ax_n\right\|^2 = 2h(x_n). \end{aligned} \tag{24}$$

Note that $\nabla h(x_n) = A^*(I-\mathrm{prox}_{\lambda g})Ax_n$ and $\nabla l(x_n) = (I-\mathrm{prox}_{\mu_n\lambda f})x_n$. From (24), we obtain

$$\begin{aligned} &\left\|x_n - \frac{\mu_n}{1-\alpha_n}A^*(I-\mathrm{prox}_{\lambda g})Ax_n - x^*\right\|^2 \\ &\quad = \|x_n-x^*\|^2 + \frac{\mu_n^2}{(1-\alpha_n)^2}\left\|A^*(I-\mathrm{prox}_{\lambda g})Ax_n\right\|^2 \\ &\qquad - \frac{2\mu_n}{1-\alpha_n}\left\langle A^*\left(I-\mathrm{prox}_{\lambda g}\right)Ax_n, x_n-x^*\right\rangle \\ &\quad = \|x_n-x^*\|^2 + \frac{\mu_n^2}{(1-\alpha_n)^2}\|\nabla h(x_n)\|^2 \\ &\qquad - \frac{2\mu_n}{1-\alpha_n}\left\langle\nabla h(x_n), x_n-x^*\right\rangle \\ &\quad \le \|x_n-x^*\|^2 + \frac{\mu_n^2}{(1-\alpha_n)^2}\|\nabla h(x_n)\|^2 \\ &\qquad - \frac{4\mu_n h(x_n)}{1-\alpha_n} \end{aligned}$$

$$
\begin{aligned}
&= \|x_n - x^*\|^2 + \rho_n^2 \frac{(h(x_n) + l(x_n))^2}{(1-\alpha_n)^2 \theta^4(x_n)} \|\nabla h(x_n)\|^2 \\
&\quad - 4\rho_n \frac{h(x_n) + l(x_n)}{(1-\alpha_n)\theta^2(x_n)} h(x_n) \\
&\le \|x_n - x^*\|^2 + \rho_n^2 \frac{(h(x_n) + l(x_n))^2}{(1-\alpha_n)^2 \theta^2(x_n)} \\
&\quad - 4\rho_n \frac{(h(x_n) + l(x_n))^2}{(1-\alpha_n)\theta^2(x_n)} \frac{h(x_n)}{h(x_n) + l(x_n)} \\
&= \|x_n - x^*\|^2 - \rho_n \left(\frac{4h(x_n)}{h(x_n) + l(x_n)} - \frac{\rho_n}{1-\alpha_n} \right) \\
&\quad \times \frac{(h(x_n) + l(x_n))^2}{(1-\alpha_n)\theta^2(x_n)}.
\end{aligned}
\tag{25}
$$

By condition (C3), without loss of generality, we can assume that $(4h(x_n)/(h(x_n)+l(x_n))) - (\rho_n/(1-\alpha_n)) \ge 0$ for all $n \ge 0$. Thus, from (23) and (25), we obtain

$$
\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\le \alpha_n \|u - x^*\|^2 + (1-\alpha_n) \\
&\quad \times \left[\|x_n - x^*\|^2 \right. \\
&\qquad \left. - \rho_n \left(\frac{4h(x_n)}{h(x_n) + l(x_n)} - \frac{\rho_n}{1-\alpha_n} \right) \frac{(h(x_n) + l(x_n))^2}{(1-\alpha_n)\theta^2(x_n)} \right] \\
&= \alpha_n \|u - x^*\|^2 + (1-\alpha_n) \|x_n - x^*\|^2 \\
&\quad - \rho_n \left(\frac{4h(x_n)}{h(x_n) + l(x_n)} - \frac{\rho_n}{1-\alpha_n} \right) \frac{(h(x_n) + l(x_n))^2}{\theta^2(x_n)} \\
&\le \alpha_n \|u - x^*\|^2 + (1-\alpha_n) \|x_n - x^*\|^2 \\
&\le \max\left\{ \|u - x^*\|^2, \|x_n - x^*\|^2 \right\}.
\end{aligned}
\tag{26}
$$

Hence, $\{x_n\}$ is bounded.

Let $z = P_\Gamma u$. From (26), we deduce

$$
\begin{aligned}
0 &\le \rho_n \left(\frac{4h(x_n)}{h(x_n) + l(x_n)} - \frac{\rho_n}{1-\alpha_n} \right) \frac{(h(x_n) + l(x_n))^2}{\theta^2(x_n)} \\
&\le \alpha_n \|u - z\|^2 + (1-\alpha_n) \|x_n - z\|^2 - \|x_{n+1} - z\|^2.
\end{aligned}
\tag{27}
$$

We consider the following two cases.

Case 1. One has $\|x_{n+1} - z\| \le \|x_n - z\|$ for every $n \ge n_0$ large enough.

In this case, $\lim_{n\to\infty} \| x_n - z \|$ exists as finite and hence

$$
\lim_{n\to\infty} (\|x_{n+1} - z\| - \|x_n - z\|) = 0. \tag{28}
$$

This together with (27) implies that

$$
\rho_n \left(\frac{4h(x_n)}{h(x_n) + l(x_n)} - \frac{\rho_n}{1-\alpha_n} \right) \frac{(h(x_n) + l(x_n))^2}{\theta^2(x_n)} \longrightarrow 0. \tag{29}
$$

Since $\liminf_{n\to\infty} \rho_n((4h(x_n)/(h(x_n)+l(x_n))) - (\rho_n/(1-\alpha_n))) \ge 2\epsilon$ (by condition (C3)), we get

$$
\frac{(h(x_n) + l(x_n))^2}{\theta^2(x_n)} \longrightarrow 0. \tag{30}
$$

Noting that $\theta^2(x_n) = \|\nabla h(x_n)\|^2 + \|\nabla l(x_n)\|^2$ is bounded, we deduce immediately that

$$
\lim_{n\to\infty} (h(x_n) + l(x_n)) = 0. \tag{31}
$$

Therefore,

$$
\lim_{n\to\infty} h(x_n) = \lim_{n\to\infty} l(x_n) = 0. \tag{32}
$$

Next, we prove

$$
\limsup_{n\to\infty} \langle u - z, x_n - z \rangle \le 0. \tag{33}
$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ satisfying $x_{n_i} \rightharpoonup z^\dagger$ and

$$
\limsup_{n\to\infty} \langle u - z, x_n - z \rangle = \lim_{i\to\infty} \langle u - z, x_{n_i} - z \rangle. \tag{34}
$$

By the lower semicontinuity of h, we get

$$
0 \le h(z^\dagger) \le \liminf_{i\to\infty} h(x_{n_i}) = \lim_{n\to\infty} h(x_n) = 0. \tag{35}
$$

So,

$$
h(z^\dagger) = \frac{1}{2} \left\| \left(I - \operatorname{prox}_{\lambda g}\right) A z^\dagger \right\| = 0. \tag{36}
$$

That is, $Az^\dagger$ is a fixed point of the proximal mapping of g or equivalently $0 \in \partial g(Az^\dagger)$. In other words, $Az^\dagger$ is a minimizer of g.

Similarly, from the lower semicontinuity of l, we get

$$
0 \le l(z^\dagger) \le \liminf_{i\to\infty} l(x_{n_i}) = \lim_{n\to\infty} l(x_n) = 0. \tag{37}
$$

Therefore,

$$
l(z^\dagger) = \frac{1}{2} \left\| \left(I - \operatorname{prox}_{\mu_n \lambda f}\right) z^\dagger \right\| = 0. \tag{38}
$$

That is, $z^\dagger$ is a fixed point of the proximal mapping of f or equivalently $0 \in \partial f(z^\dagger)$. In other words, $z^\dagger$ is a minimizer of f. Hence, $z^\dagger \in \Gamma$. Therefore,

$$
\begin{aligned}
\limsup_{n\to\infty} \langle u - z, x_n - z \rangle &= \lim_{i\to\infty} \langle u - z, x_{n_i} - z \rangle \\
&= \langle u - z, z^\dagger - z \rangle \le 0.
\end{aligned}
\tag{39}
$$

From (22), we have

$$\begin{aligned}
&\left\|x_{n+1}-z\right\|^2 \\
&\leq \Big\|\alpha_n(u-z)+(1-\alpha_n) \\
&\qquad \times \left[x_n-\frac{\mu_n}{1-\alpha_n}A^*\left(I-\operatorname{prox}_{\lambda g}\right)Ax_n-z\right]\Big\|^2 \\
&=(1-\alpha_n)^2\left\|x_n-\frac{\mu_n}{1-\alpha_n}A^*(I-\operatorname{prox}_{\lambda g})Ax_n-z\right\|^2 \\
&\quad+\alpha_n^2\|u-z\|^2+2\alpha_n(1-\alpha_n) \\
&\quad\times\left\langle x_n-\frac{\mu_n}{1-\alpha_n}A^*\left(I-\operatorname{prox}_{\lambda g}\right)Ax_n-z,u-z\right\rangle \\
&\leq(1-\alpha_n)^2\left\|x_n-z\right\|^2+\alpha_n^2\|u-z\|^2 \\
&\quad+2\alpha_n(1-\alpha_n)\langle x_n-z,u-z\rangle \\
&\quad-2\alpha_n\mu_n\langle\nabla h(x_n),u-z\rangle \\
&\leq(1-\alpha_n)\left\|x_n-z\right\|^2 \\
&\quad+\alpha_n\Big(\alpha_n\|u-z\|^2+2(1-\alpha_n)\langle x_n-z,u-z\rangle \\
&\qquad\quad+2\mu_n\|\nabla h(x_n)\|\,\|u-z\|\Big).
\end{aligned} \tag{40}$$

Since ∇h is Lipschitz continuous with Lipschitzian constant $\|A\|^2$ and ∇l is nonexpansive, $\nabla h(u_n)$,$\nabla l(u_n)$, and $\theta^2(x_n) = \|\nabla h(x_n)\|^2 + \|\nabla l(x_n)\|^2$ are bounded. Note that $\mu_n\|\nabla h(x_n)\| = \rho_n((h(x_n)+l(x_n))/\theta^2(x_n))\|\nabla h(x_n)\|$. Thus, $\mu_n\|\nabla h(x_n)\| \to 0$ by (32). From Lemma 2, (39), and (40) we deduce that $x_n \to z$.

Case 2. There exists a subsequence $\{\|x_{n_j}-z\|\}$ of $\{\|x_n-z\|\}$ such that

$$\left\|x_{n_j}-z\right\| < \left\|x_{n_j+1}-z\right\|, \tag{41}$$

for all $j \geq 1$. By Lemma 3, there exists a strictly increasing sequence $\{m_k\}$ of positive integers such that $\lim_{k\to\infty}m_k = +\infty$ and the following properties are satisfied by all numbers $k \in \mathbb{N}$:

$$\left\|x_{m_k}-z\right\| \leq \left\|x_{m_{k+1}}-z\right\|, \qquad \left\|x_k-z\right\| \leq \left\|x_{m_{k+1}}-z\right\|. \tag{42}$$

Consequently,

$$\begin{aligned}
0 &\leq \lim_{k\to\infty}\left(\left\|x_{m_{k+1}}-z\right\|-\left\|x_{m_k}-z\right\|\right) \\
&\leq \limsup_{n\to\infty}\left(\left\|x_{n+1}-z\right\|-\left\|x_n-z\right\|\right) \\
&\leq \limsup_{n\to\infty}\left(\alpha_n\|u-z\|+(1-\alpha_n)\left\|x_n-z\right\|-\left\|x_n-z\right\|\right) \\
&= \limsup_{n\to\infty}\alpha_n\left(\|u-z\|-\left\|x_n-z\right\|\right)=0.
\end{aligned} \tag{43}$$

Hence,

$$\lim_{k\to\infty}\left(\left\|x_{m_{k+1}}-z\right\|-\left\|x_{m_k}-z\right\|\right)=0. \tag{44}$$

By a similar argument as that of Case 1, we can prove that

$$\begin{gathered}
\limsup_{k\to\infty}\left\langle u-z,x_{m_k}-z\right\rangle \leq 0, \\
\left\|x_{m_k+1}-z\right\|^2 \leq \left(1-\alpha_{m_k}\right)\left\|x_{m_k}-z\right\|^2+\alpha_{m_k}\sigma_{m_k},
\end{gathered} \tag{45}$$

where $\sigma_{m_k} = \alpha_{m_k}\|u-z\|^2 + 2(1-\alpha_{m_k})\langle x_{m_k}-z,u-z\rangle + 2\mu_{m_k}\|\nabla h(x_{m_k})\|\|u-z\|$.

In particular, we get

$$\begin{aligned}
&\alpha_{m_k}\left\|x_{m_k}-z\right\|^2 \\
&\leq \left\|x_{m_k}-z\right\|^2-\left\|x_{m_k+1}-z\right\|+\alpha_{m_k}\sigma_{m_k} \\
&\leq \alpha_{m_k}\sigma_{m_k}.
\end{aligned} \tag{46}$$

Then,

$$\limsup_{k\to\infty}\left\|x_{m_k}-z\right\|^2 \leq \limsup_{k\to\infty}\sigma_{m_k} \leq 0. \tag{47}$$

Thus, from (42) and (44), we conclude that

$$\limsup_{k\to\infty}\left\|x_k-z\right\| \leq \limsup_{k\to\infty}\left\|x_{m_k+1}-z\right\|=0. \tag{48}$$

Therefore, $x_n \to z$. This completes the proof. □

Remark 6. Note that problem (13) was considered, for example, in [12, 13]; however, the iterative methods proposed to solve it need to know a priori the norm of the bounded linear operator A.

Remark 7. We would like also to emphasize that by taking $f = \delta_C$, $g = \delta_Q$ the indicator functions of two nonempty closed convex sets C, Q of H_1 and H_2 respectively, our algorithm (22) reduces to

$$\begin{aligned}
&x_{n+1} \\
&\quad=\operatorname{proj}_C\left[\alpha_n u+(1-\alpha_n)x_n-\mu_n A^*\left(I-\operatorname{proj}_Q\right)Ax_n\right], \\
&\qquad\qquad\forall n\geq 0.
\end{aligned} \tag{49}$$

We observe that (49) is simpler than the one in [14].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

A New Iterative Method for the Set of Solutions of Equilibrium Problems and of Operator Equations with Inverse-Strongly Monotone Mappings

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The purpose of the paper is to present a new iteration method for finding a common element for the set of solutions of equilibrium problems and of operator equations with a finite family of λ_i-inverse-strongly monotone mappings in Hilbert spaces.

1. Introduction

Let H be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H, and let G be a bifunction from $C\times C$ into $(-\infty,+\infty)$. The equilibrium problem for G is to find $u^*\in C$ such that

$$G(u^*,v)\geq 0,\quad \forall v\in C. \tag{1}$$

The set of solutions of (1) is denoted by $\mathrm{EP}(G)$.

Equilibrium problem (1) includes the numerous problems in physics, optimization, economics, transportation, and engineering, as special cases.

Assume that the bifunction G satisfies the following standard properties.

Assumption A. Let $G: C\times C\rightarrow(-\infty,+\infty)$ be a bifunction satisfying the conditions (A1)–(A4):

(A1) $G(u,u)=0$, $\forall u\in C$;

(A2) $G(u,v)+G(v,u)\leq 0$, $\forall(u,v)\in C\times C$;

(A3) for each $u\in C$, $G(u,\cdot): C\rightarrow(-\infty,+\infty)$ is lower semicontinuous and convex;

(A4) $\overline{\lim}_{t\rightarrow+0}G((1-t)u+tz,v)\leq G(u,v)$, $\forall(u,z,v)\in C\times C\times C$.

Let $\{T_i\}$, $i=1,\ldots,N$, be a finite family of k_i-strictly pseudocontractive mappings from C into C with the set of fixed points $F(T_i)$; that is,

$$F(T_i)=\{x\in C: T_ix=x\}. \tag{2}$$

Assume that

$$\mathcal{S}:=\bigcap_{i=1}^{N}F(T_i)\cap\mathrm{EP}(G)\neq\emptyset. \tag{3}$$

The problem of finding an element

$$u^*\in\mathcal{S} \tag{4}$$

is studied intensively in [1–27].

Recall that a mapping T in H is said to be a k-strictly pseudocontractive mapping in the terminology of Browder and Petryshyn [28] if there exists a constant $0\leq k<1$ such that

$$\|Tx-Ty\|^2\leq\|x-y\|^2+k\|(I-T)x-(I-T)y\|^2, \tag{5}$$

for all $x,y\in D(T)$, the domain of T, where I is the identity operator in H. Clearly, if $k=0$, then T is nonexpansive; that is,

$$\|T(x)-T(y)\|\leq\|x-y\|. \tag{6}$$

We know that the class of k-strictly pseudocontractive mappings strictly includes the class of nonexpansive mappings.

In the case that $T_i \equiv I$, (4) is reduced to the equilibrium problem (1) and shown in [5, 23] to cover monotone inclusion problems, saddle point problems, variational inequality problems, minimization problems, Nash equilibria in noncooperative games, vector equilibrium problems, and certain fixed point problems (see also [29]). For finding approximative solutions of (1) there exist several methods: the regularization approach in [7, 9, 15, 24, 30, 31], the gap-function approach in [8, 15, 16, 18, 19], and the iterative procedure approach in [1–4, 6, 8, 11–14, 19–22, 32, 33].

In the case that $G \equiv 0$ and $N = 1$, (4) is a problem of finding a fixed point for a k-strictly pseudocontractive mapping in C and is given by Marino and Xu [17].

Theorem 1 (see [17]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a k-strictly pseudocontractive mapping for some $0 \leq k < 1$, and assume that*

$$F(T) \neq \emptyset. \tag{7}$$

Let $\{x_n\}$ be the sequence generated by the following algorithm:

$$\begin{aligned} & x_0 \in C, \\ & y_n = \alpha_n x_n + (1 - \alpha_n) T x_n, \\ & C_n = \{ z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 \\ & \qquad + (1 - \alpha_n)(k - \alpha_n) \|x_n - T x_n\|^2 \}, \\ & Q_n = \{ z \in C : \langle x_n - z,\ x_0 - x_n \rangle \geq 0 \}, \\ & x_{n+1} = P_{C_n \cap Q_n} x_0. \end{aligned} \tag{8}$$

Assume that the control sequence $\{\alpha_n\}$ is chosen so that $\alpha_n < 1$ for all n. Then $\{x_n\}$ converges strongly to $P_{F(T)} x_0$, the projection of x_0 onto $F(T)$.

For the case that $G \equiv 0$ and $N > 1$, (4) is a problem of finding a common fixed point for a finite family of k_i-strictly pseudocontractive mappings T_i in C and is studied in [27].

Let $x_0 \in C$ and $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ three sequences in $[0, 1]$ satisfying $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \geq 1$, and let $\{u_n\}$ be a sequence in C. Then the sequence $\{x_n\}$ generated by

$$\begin{aligned} & x_1 = \alpha_1 x_0 + \beta_1 T_1 x_1 + \gamma_1 u_1, \\ & x_2 = \alpha_2 x_1 + \beta_2 T_2 x_2 + \gamma_2 u_2, \\ & \vdots \\ & x_N = \alpha_N x_{N-1} + \beta_N T_N x_N + \gamma_N u_N, \\ & x_{N+1} = \alpha_{N+1} x_N + \beta_{N+1} T_1 x_{N+1} + \gamma_{N+1} u_{N+1}, \\ & \vdots \end{aligned} \tag{9}$$

is called the implicit iteration process with mean errors for a finite family of strictly pseudocontractive mappings $\{T_i\}_{i=1}^N$.

The scheme (9) can be expressed in the compact form as

$$x_n = \alpha_n x_{n-1} + \beta_n T_n x_n + \gamma_n u_n, \tag{10}$$

where $T_n = T_{n \bmod N}$.

Theorem 2 (see [27]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^N$ be a finite family of strictly pseudocontractive mappings of C into itself such that*

$$\bigcap_{i=1}^{N} F(T_i) \neq \emptyset. \tag{11}$$

Let $x_0 \in C$ and let $\{u_n\}$ be a bounded sequence in C; let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be three sequences in $[0, 1]$ satisfying the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n = 1$, $\forall n \geq 1$;

(ii) *there exist constants σ_1, σ_2 such that $0 < \sigma_1 \leq \beta_n \leq \sigma_2 < 1$, $\forall n \geq 1$;*

(iii) $\sum_{n=1}^{\infty} \gamma_n < \infty$.

Then the implicit iterative sequence $\{x_n\}$ defined by (9) converges weakly to a common fixed point of the mappings $\{T_i\}_{i=1}^N$. Moreover, if there exists $i_0 \in \{1, 2, \ldots, N\}$ such that T_{i_0} is demicompact, then $\{x_n\}$ converges strongly.

If G is an arbitrary bifunction satisfying Assumption A and $N = 1$, then (4) is a problem of finding a common element of the fixed point set for a k-strictly pseudocontractive mapping in C and of the solution set of equilibrium problem for G (see [26]).

Theorem 3 (see [26]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let G be a bifunction from $C \times C$ to $(-\infty, +\infty)$ satisfying Assumption A, and let T be a nonexpansive mapping of C into H such that*

$$F(T) \cap EP(G) \neq \emptyset. \tag{12}$$

Let f be a contraction of H into itself and let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in H$ and

$$\begin{aligned} & G(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\ & x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T u_n, \end{aligned} \tag{13}$$

for all $n \in \mathbf{N}$, where $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$ satisfy

$$\begin{gathered} \lim_{n \to \infty} \alpha_n = 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty, \qquad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \\ \liminf_{n \to \infty} r_n > 0, \qquad \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty. \end{gathered} \tag{14}$$

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in F(T) \cap EP(G)$, where

$$z = P_{F(T) \cap EP(G)} f(z). \tag{15}$$

Set $A_i = I - T_i$. Obviously, A_i are λ_i-inverse-strongly monotone; that is,

$$\langle A_i(x) - A_i(y), x - y\rangle \geq \lambda_i \|A_i(x) - A_i(y)\|^2, \quad \forall x, y \in D(A_i), \quad \lambda_i = \frac{1 - k_i}{2}. \tag{16}$$

From now on, let $\{A_i\}_{i=1}^N$ be a finite family of λ_i-inverse-strongly monotone mappings in H with $C \subset \bigcap_{i=1}^N D(A_i)$ and $\lambda_i > 0$, $i = 1, \ldots, N$. On the other hand, if there exists $i_0 \in \{1, 2, \ldots, N\}$ such that $\lambda_{i_0} > 1$, then A_{i_0} is a contraction; that is, $\|A_{i_0}(x) - A_{i_0}(y)\| \leq (1/\lambda_{i_0})\|x - y\|$ with $1/\lambda_{i_0} < 1$. And hence, A_{i_0} has only one solution and, consequently, the stated problem does not have sense. So, without loss of generality, assume that $0 < \lambda_i \leq 1$, $i = 1, \ldots, N$.

Set

$$S = \bigcap_{i=1}^N S_i, \tag{17}$$

where $S_i = \{x \in C : A_i(x) = 0\}$ is the solution set of A_i in C.

Assume that $\mathrm{EP}(G) \cap S \neq \emptyset$.

Our problem is to find an element

$$u^* \in \mathrm{EP}(G) \cap S. \tag{18}$$

Since the mapping $A = I - T$ is $(1/2)$-inverse-strongly monotone for each nonexpansive mapping T, the problem of finding an element $u^* \in C$, which is not only a solution of a variational inequality involving an inverse-strongly monotone mapping but also a fixed point of a nonexpansive mapping, is a particular case of (18).

For instance, the case that $G(u, v) \equiv \langle A(u), v - u\rangle$, where A is some inverse-strongly monotone mapping and $N = 1$, is studied in [25].

Theorem 4 (see [25]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\lambda > 0$. Let A be a λ-inverse-strongly monotone mapping of C into H, and let T be a nonexpansive mapping of C into itself such that*

$$F(T) \cap VI(C, A) \neq \emptyset, \tag{19}$$

where $VI(C, A)$ denotes the solution set of the following variational inequality: find $x^* \in C$ such that

$$\langle A(x^*), x - x^*\rangle \geq 0, \quad \forall x \in C. \tag{20}$$

Let $\{x_n\}$ be a sequence defined by

$$x_0 \in C, \quad x_{n+1} = \alpha_n x_n + (1 - \alpha_n) TP_C(x_n - \lambda_n A(x_n)), \tag{21}$$

for every $n = 0, 1, \ldots$, where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 2\lambda)$ and $\{\alpha_n\} \subset (c, d)$ for some $c, d \in (0, 1)$. Then, $\{x_n\}$ converges weakly to $z \in F(T) \cap VI(C, A)$, where

$$z = \lim_{n\to\infty} P_{F(T)\cap VI(C,A)} x_n. \tag{22}$$

The following theorem is an improvement of Theorem 4 for the case of nonself-mapping.

Theorem 5 (see [34]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be a λ-inverse-strongly monotone mapping of C into H, and let T be a nonexpansive nonself-mapping of C into H such that*

$$F(T) \cap VI(C, A) \neq \emptyset. \tag{23}$$

Suppose that $x_1 = x \in C$ and $\{x_n\}$ is given by

$$x_{n+1} = P_C(\alpha_n x + (1 - \alpha_n) TP_C(x_n - \lambda_n A(x_n))) \tag{24}$$

for every $n = 1, 2, \ldots$, where $\{\alpha_n\}$ is a sequence in $[0, 1)$ and $\{\lambda_n\}$ is a sequence in $[0, 2\alpha]$. If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < 2\alpha$,

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \quad \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty, \tag{25}$$

then $\{x_n\}$ converges strongly to $P_{F(T)\cap VI(C,A)} x$.

We know that λ-inverse-strongly monotone mapping is $(1/\lambda)$-Lipschitz continuous and monotone. Therefore, for the case that $G(u, v) \equiv \langle A(u), v - u\rangle$, where A is not inverse-strongly monotone, but Lipschitz continuous and monotone, Nadezhkina and Takahashi [35] prove the following theorem.

Theorem 6 (see [35]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be a monotone and k-Lipschitz continuous mapping of C into H, and let T be a nonexpansive mapping of C into itself such that*

$$F(T) \cap VI(C, A) \neq \emptyset. \tag{26}$$

Let $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be sequences generated by

$$\begin{aligned} x_0 &= x \in C, \\ y_n &= P_C(x_n - \lambda_n A(x_n)), \\ z_n &= P_C(x_n - \lambda_n A(y_n)), \\ C_n &= \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n &= \{z \in C : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} &= P_{C_n\cap Q_n} x \end{aligned} \tag{27}$$

for every $n = 0, 1, \ldots$, where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 1/k)$ and $\alpha_n \subset [0, c]$ for some $c \in [0, 1)$. Then the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converge strongly to $P_{F(T)\cap VI(C,A)} x$.

Some similar results are also considered in [36, 37].

Buong [38] introduced two new implicit iteration methods for solving problem (18).

We construct a regularization solution u_n of the following single equilibrium problem: find $u_n \in C$ such that

$$\mathcal{F}(u_n, v) \geq 0, \quad \forall v \in C, \tag{28}$$

where

$$\mathcal{F}(u,v) := G(u,v) + \sum_{i=1}^{N} \alpha_n^{\mu_i} G_i(u,v) + \alpha_n \langle u, v-u\rangle, \quad \alpha_n > 0, \tag{29}$$

$$G_i(u,v) = \langle A_i(u), v-u\rangle, \quad i = 1,\ldots,N,$$

$$0 < \mu_i < \mu_{i+1} < 1, \quad i = 2,\ldots,N-1,$$

and $\{\alpha_n\}$ is the positive sequence of regularization parameters that converges to 0, as $n \to +\infty$.

The first one is the following theorem.

Theorem 7 (see [38]). *For each $\alpha_n > 0$, problem* (28) *has a unique solution u_n such that*

(i) $\lim_{n\to+\infty} u_n = u^*$, $u^* \in EP(G) \cap S$, $\|u^*\| \le \|y\|$, $\forall y \in EP(G) \cap S$;

(ii)

$$\|u_n - u_m\| \le (\|u^*\| + dN) \frac{|\alpha_n - \alpha_m|}{\alpha_n}, \tag{30}$$

where d is a positive constant.

Next, we introduce the second result. Let $\{\widetilde{c}_n\}$ and $\{\gamma_n\}$ be some sequences of positive numbers, and let z_0 and z_1 be two arbitrary elements in C. Then, the sequence $\{z_n\}$ of iterations is defined by the following equilibrium problem: find $z_{n+1} \in C$ such that

$$\widetilde{c}_n \left(G(z_{n+1}, v) + \sum_{i=1}^{N} \alpha_n^{\mu_i} G_i(z_{n+1}, v) + \alpha_n \langle z_{n+1}, v - z_{n+1}\rangle \right) + \langle z_{n+1} - z_n, v - z_{n+1}\rangle - \gamma_n \langle z_n - z_{n-1}, v - z_{n+1}\rangle \ge 0, \quad \forall v \in C. \tag{31}$$

Theorem 8 (see [38]). *Assume that the parameters $\widetilde{c}_n$, γ_n, and α_n are chosen such that*

(i) $0 < c_0 < \widetilde{c}_n$, $0 \le \gamma_n < \gamma_0$,

(ii) $\sum_{n=1}^{\infty} b_n = +\infty$, $b_n = \widetilde{c}_n\alpha_n/(1+\widetilde{c}_n\alpha_n)$,

(iii) $\sum_{n=1}^{\infty} \gamma_n b_n^{-1} \|z_n - z_{n-1}\| < +\infty$,

(iv) $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}(|\alpha_n - \alpha_{n+1}|/\alpha_n b_n) = 0$.

Then, the sequence $\{z_n\}$ defined by (31) *converges strongly to the element u^*, as $n \to +\infty$.*

In this paper, we consider the new another iteration method: for an arbitrary element x_0 in H, the sequence $\{x_n\}$ of iterations is defined by finding $u_n \in C$ such that

$$G(u_n, y) + \langle u_n - x_n, y - u_n\rangle \ge 0, \quad \forall y \in C,$$

$$\begin{aligned} x_{n+1} &= P_C\left(x_n - \beta_n \left[x_n - u_n + \sum_{i=1}^{N} \alpha_n^{\mu_i} A_i(x_n) + \alpha_n x_n \right] \right) \\ &= P_C\left(x_n - \beta_n \left[\sum_{i=1}^{N} \alpha_n^{\mu_i} A_i(x_n) + (1+\alpha_n) x_n - u_n \right] \right), \end{aligned} \tag{32}$$

where P_C is the metric projection of H onto C and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers.

The strong convergence of the sequence $\{x_n\}$ defined by (32) is proved under some suitable conditions on $\{\alpha_n\}$ and $\{\beta_n\}$ in the next section.

2. Main Results

We formulate the following lemmas for the proof of our main theorems.

Lemma 9 (see [9]). *Let C be a nonempty closed convex subset of a real Hilbert space H and let G be a bifunction of $C \times C$ into $(-\infty, +\infty)$ satisfying Assumption A. Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$G(z,y) + \frac{1}{r}\langle z - x, y - z\rangle \ge 0, \quad \forall y \in C. \tag{33}$$

Lemma 10 (see [9]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Assume that $G : C \times C \to (-\infty, +\infty)$ satisfies Assumption A. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{ z \in C : G(z,y) + \frac{1}{r}\langle z - x, y - z\rangle \ge 0 \right\}, \quad \forall y \in C. \tag{34}$$

Then, the following statements hold:

(i) *T_r is single valued;*

(ii) *T_r is firmly nonexpansive; that is, for any $x, y \in H$,*

$$\|T_r(x) - T_r(y)\|^2 \le \langle T_r(x) - T_r(y), x - y\rangle; \tag{35}$$

(iii) *$F(T_r) = EP(G)$;*

(iv) *$EP(G)$ is closed and convex.*

Lemma 11 (see [36]). *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be the sequences of positive numbers satisfying the following conditions:*

(i) $a_{n+1} \le (1 - b_n)a_n + c_n$,

(ii) $\sum_{n=0}^{\infty} b_n = +\infty$, $b_n < 1$, $\lim_{n\to+\infty}(c_n/b_n) = 0$.

Then, $\lim_{n\to+\infty} a_n = 0$.

Lemma 12 (see [38]). *Let A be any inverse-strongly monotone mapping from C into H with the solution set $S_A := \{x \in C : A(x) = 0\}$, and let C_0 be a closed convex subset of C such that*

$$S_A \cap C_0 \neq \emptyset. \tag{36}$$

Then, the solution set of the following variational inequality

$$\langle A(\tilde{y}), x - \tilde{y}\rangle \geq 0, \quad \forall x \in C_0,\ \tilde{y} \in C_0, \tag{37}$$

is coincided with $S_A \cap C_0$.

From Lemma 9, we can consider the firmly nonexpansive mapping T_0 defined by

$$T_0(x) = \{z \in C : G(z, y) + \langle z - x, y - z\rangle \geq 0,\ \forall y \in C\}, \quad \forall x \in H. \tag{38}$$

From Lemma 10, we know that T_0 is nonexpansive. Consequently, $A_0 := I - T_0$ is $(1/2)$-inverse-strongly monotone. Let

$$S_0 := \{x \in C : A_0(x) = 0\}. \tag{39}$$

Then, $S_0 = \mathrm{EP}(G)$ and problem (18) are equivalent to finding

$$u^* \in S_0 \cap S. \tag{40}$$

Now, we construct a regularization solution y_n for (40) by solving the following variational inequality problem: find $y_n \in C$ such that

$$\left\langle \sum_{i=0}^{N} \alpha_n^{\mu_i} A_i(y_n) + \alpha_n y_n, v - y_n \right\rangle \geq 0, \quad \forall v \in C, \tag{41}$$
$$\mu_0 = 0 < \mu_1 < \cdots < \mu_N < 1,$$

where the positive regularization parameter $\alpha_n \to 0$, as $n \to +\infty$.

Now we are in a position to introduce and prove the main results.

Theorem 13. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let G be a bifunction from $C \times C$ to $(-\infty, +\infty)$ satisfying Assumption A and let $\{A_i\}_{i=1}^N$ be a finite family of λ_i-inverse-strongly monotone mappings in H with $C \subset \bigcap_{i=1}^N D(A_i)$ and $\lambda_i > 0$, $i = 1, \ldots, N$, such that*

$$EP(G) \cap S \neq \emptyset, \tag{42}$$

where $EP(G)$ denotes the set of solutions for (1) *and*

$$S = \bigcap_{i=1}^{N} S_i, \quad S_i = \{x \in C : A_i(x) = 0\}. \tag{43}$$

Then, for each $\alpha_n > 0$, problem (41) *has a unique solution y_n such that*

(i) $\lim_{n\to+\infty} y_n = u^*$, $u^* \in EP(G) \cap S$,

(ii) $\|u^*\| \leq \|y\|$, $\forall y \in EP(G) \cap S$,

(iii)

$$\|y_n - y_m\| \leq \frac{|\alpha_n - \alpha_m|}{\alpha_n}\left(\|u^*\| + dN\right), \tag{44}$$

where d is some positive constant.

Proof. From Lemma 12, we know that S_0 is the set of solutions for the following variational inequality problem: find $u^* \in C$ such that

$$\langle A_0(u^*), v - u^*\rangle \geq 0, \quad \forall v \in C. \tag{45}$$

If we define the new bifunction $G_0(u, v)$ by

$$G_0(u, v) = \langle A_0(u^*), v - u^*\rangle, \tag{46}$$

then problem (41) is the same as (28) with a new $G(u, v)$, and the proof for the theorem is a complete repetition of the proof for Theorem 2.1 in [38].

Set

$$L = \max\left\{2, \frac{1}{\lambda_i},\ i = 1, \ldots, N\right\}. \tag{47}$$

□

Theorem 14. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let G be a bifunction from $C \times C$ to $(-\infty, +\infty)$ satisfying Assumption A and let $\{A_i\}_{i=1}^N$ be a finite family of λ_i-inverse-strongly monotone mappings in H with $C \subset \bigcap_{i=1}^N D(A_i)$ and $\lambda_i > 0$, $i = 1, \ldots, N$, such that*

$$EP(G) \cap S \neq \emptyset, \tag{48}$$

where $EP(G)$ denotes the set of solutions for (1) *and*

$$S = \bigcap_{i=1}^{N} S_i, \quad S_i = \{x \in C : A_i(x) = 0\}. \tag{49}$$

Suppose that α_n, β_n satisfy the following conditions:

$$\begin{gathered}\alpha_n, \beta_n > 0\ (\alpha_n \leq 1), \qquad \lim_{n\to\infty} \alpha_n = 0, \\ \lim_{n\to\infty} \frac{|\alpha_n - \alpha_{n+1}|}{\alpha_n^2 \beta_n} = 0, \qquad \sum_{n=0}^{\infty} \alpha_n \beta_n = \infty, \\ \overline{\lim_{n\to\infty}} \beta_n \frac{(L(N+1) + \alpha_n)^2}{\alpha_n} < 1.\end{gathered} \tag{50}$$

Then, the sequence $\{x_n\}$ defined by (32) *converges strongly to $u^* \in EP(G) \cap S$; that is,*

$$\lim_{n\to\infty} x_n = u^* \in EP(G) \cap S. \tag{51}$$

Proof. Let y_n be the solution of (41). Then,

$$y_n = P_C\left(y_n - \beta_n\left[\sum_{i=0}^{N} \alpha_n^{\mu_i} A_i(y_n) + \alpha_n y_n\right]\right). \tag{52}$$

Set $\Delta_n = \|x_n - y_n\|$. Obviously,

$$\Delta_{n+1} = \|x_{n+1} - y_{n+1}\| \le \|x_{n+1} - y_n\| + \|y_{n+1} - y_n\|. \quad (53)$$

From the nonexpansivity of P_C, the monotone and Lipschitz continuous properties of A_i, $i = 0, \ldots, N$, (41), (52), and $y_n = T_0(x_n)$, we have

$$\begin{aligned}
&\|x_{n+1} - y_n\| \\
&\quad \le \left\| x_n - y_n - \beta_n \left[\sum_{i=0}^{N} \alpha_n^{\mu_i} \left(A_i(x_n) - A_i(y_n) \right) + \alpha_n (x_n - y_n) \right] \right\|,
\end{aligned}$$

$$\begin{aligned}
&\left\| x_n - y_n - \beta_n \left[\sum_{i=0}^{N} \alpha_n^{\mu_i} \left(A_i(x_n) - A_i(y_n) \right) + \alpha_n (x_n - y_n) \right] \right\|^2 \\
&\quad = \|x_n - y_n\|^2 \\
&\qquad + \beta_n^2 \left\| \left[\sum_{i=0}^{N} \alpha_n^{\mu_i} \left(A_i(x_n) - A_i(y_n) \right) + \alpha_n (x_n - y_n) \right] \right\|^2 \\
&\qquad - 2\beta_n \left\langle \sum_{i=0}^{N} \alpha_n^{\mu_i} \left(A_i(x_n) - A_i(y_n) \right) + \alpha_n (x_n - y_n), x_n - y_n \right\rangle \\
&\quad \le \|x_n - y_n\|^2 \left[1 - 2\beta_n \alpha_n + \beta_n^2 \left(2 + \sum_{i=1}^{N} \alpha_n^{\mu_i} \frac{1}{\lambda_i} + \alpha_n \right)^2 \right].
\end{aligned} \quad (54)$$

Thus,

$$\|x_{n+1} - y_n\| \le \Delta_n \left(1 - 2\beta_n \alpha_n + \beta_n^2 \left(L(N+1) + \alpha_n \right)^2 \right)^{1/2}. \quad (55)$$

Therefore,

$$\begin{aligned}
\Delta_{n+1} &\le \Delta_n \left(1 - 2\beta_n \alpha_n + \beta_n^2 \left(L(N+1) + \alpha_n \right)^2 \right)^{1/2} \\
&\quad + \frac{|\alpha_n - \alpha_{n+1}|}{\alpha_n} \left(\|u^*\| + dN \right) \\
&\le \Delta_n \left(1 - \alpha_n \beta_n \right)^{1/2} + \frac{|\alpha_n - \alpha_{n+1}|}{\alpha_n} \left(\|u^*\| + dN \right).
\end{aligned} \quad (56)$$

We note that, for $\varepsilon > 0$, $a > 0$, $b > 0$, the inequality

$$(a+b)^2 \le (1+\varepsilon) \left(a^2 + \frac{b^2}{\varepsilon} \right) \quad (57)$$

holds. Thus, applying inequality (57) for $\varepsilon = \alpha_n \beta_n / 2$, we obtain

$$\begin{aligned}
0 &\le \Delta_{n+1}^2 \\
&\le \Delta_n^2 \left(1 - \alpha_n \beta_n \right) \left(1 + \frac{1}{2} \alpha_n \beta_n \right) \\
&\quad + \left(\frac{\alpha_n - \alpha_{n+1}}{\alpha_n} \left(\|u^*\| + dN \right) \right)^2 \frac{2}{\alpha_n \beta_n} \left(1 + \frac{1}{2} \alpha_n \beta_n \right) \\
&= \Delta_n^2 \left(1 - \frac{1}{2} \alpha_n \beta_n - \frac{1}{2} (\alpha_n \beta_n)^2 \right) \\
&\quad + \left(\frac{\alpha_n - \alpha_{n+1}}{\alpha_n^2 \beta_n} \left(\|u^*\| + dN \right) \right)^2 2\alpha_n \beta_n \left(1 + \frac{1}{2} \alpha_n \beta_n \right).
\end{aligned} \quad (58)$$

Set

$$\begin{aligned}
b_n &= \alpha_n \beta_n \left(\frac{1}{2} + \frac{1}{2} \alpha_n \beta_n \right) \\
c_n &= \left(\frac{\alpha_n - \alpha_{n+1}}{\alpha_n^2 \beta_n} \left(\|u^*\| + dN \right) \right)^2 2\alpha_n \beta_n \left(1 + \frac{1}{2} \alpha_n \beta_n \right).
\end{aligned} \quad (59)$$

Then, it is not difficult to check that b_n and c_n satisfy the conditions in Lemma 11 for sufficiently large n. Hence, $\lim_{n \to +\infty} \Delta_n^2 = 0$. Since $\lim_{n \to \infty} y_n = u^*$, we have

$$\lim_{n \to \infty} x_n = u^* \in \mathrm{EP}(G) \cap S. \quad (60)$$

This completes the proof. □

Remark 15. The sequences $\alpha_n = (1+n)^{-p}$, $0 < p < 1/2$, and $\beta_n = \gamma_0 \alpha_n$ with

$$0 < \gamma_0 < \frac{1}{\left(L(N+1) + \alpha_0 \right)^2} \quad (61)$$

satisfy all the necessary conditions in Theorem 14.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' Contribution

The main idea of this paper was proposed by Jong Kyu Kim. Jong Kyu Kim and Nguyen Buong prepared the paper initially and performed all the steps of proof in this research. All authors read and approved the final paper.

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