Abstract and Applied Analysis

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Guest Editors: Gonglin Yuan, Gaohang Yu, Neculai Andrei, Yunhai Xiao, and Li Zhang

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Editorial Board

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Editorial

Nonlinear Analysis: Optimization Methods, Convergence Theory, and Applications

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Nonlinear analysis has been used in many practical application fields, such as nonlinear fitting, economics, optimization, convergence, engineering, hydrodynamics, parameter estimating, function approximating, and elasticity. There are many achievements on nonlinear analysis that have been obtained by authors. However, there still exist lots of challenging problems, such as the large-scale problems, fast algorithm, and convergence, since the complex of the nonlinear object function on its variables cannot be obviously determined in many cases. So the research and application space of nonlinear analysis are broad.

The issue invites investigators to contribute original research articles as well as review articles that will help in understanding the important new developments in nonlinear analysis and its applications with a particular emphasis on the following potential topics. There exist many special topics including the nonlinear analysis: optimization, variation analysis, economical models, fixed point theory, numerical methods, convergence, nonlinear equations, semidefinite programming, polynomial optimization, tensor computation, image processing, and so forth.

The research papers are welcome with new ideas or good numerical experiments. (1) New methods for nonlinear analysis are encouraged, such as the new formulas on conjugate gradient methods, quasi-Newton methods, limited memory quasi-Newton method, trust region methods, and SQP methods; convergence results of algorithms are established which is needed. (2) Numerical experiments should be done to improve the theory idea: for unconstrained optimization problems, the CUTEr problems should be tested [1, 2] in Table 1. For nonlinear equations problems, there are many problems [3–7] that are listed in Table 2.

We hope that readers of this special issue will find not only convergence results and updated reviews on the common nonlinear analysis, but also important open problems to be resolved such as new formulas in optimization methods, new algorithms for variation analysis and new models for economic problems. Moreover, large-scale problems in nonlinear equations, semidefinite programming, and image processing are tested to turn out the performance of the new methods.

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Li Zhang

Table 1

Problems names	Character
ARGLINA, ARGLINB, ARGLINC, BDQRTIC, BROWNAL, BROYDN7D, BRYBND CHAINWOO, CHNROSNB, COSINE, CRAGGLVY, CURLY10, CURLY20, DIXMAANA, DIXMAANB, DIXMAANC, DIXMAAND, DIXMAANE, DIXMAANF, DIXMAANG, DIXMAANH DIXMAANI, DIXMAANJ, DIXMAANL, DIXON3DQ, DQDRTIC, DQRTIC, EDENSCH EG2, ENGVAL1, ERRINROS, EXTROSNB, FLETCBV2, FLETCHCR, FREUROTH GENHUMPS, GENROSE, INDEF, LIARWHD, MANCINO, MSQRTALS, MSQRTBLS NONCVXU2, NONDIA, NONDQUAR, PENALTY1, PENALTY2, POWELLSG POWER, QUARTC, SCHMVETT, SENSORS, SINQUAD, SPARSINE, SPARSQUR SPMSRTLS, SROSENBR, TESTQUAD, TOINTGSS, TQUARTIC, TRIDIA VARDIM, VAREIGVL, and WOODS	Academic
DECONVU, FMINSRF2, FMINSURF, MOREBV, TOINTGOR, and TOINTQOR	Modeling

Table 2

Functions names	Optimization value
Exponential function 1, exponential function 2, trigonometric function, singular function, logarithmic function, Broyden tridiagonal function, trigexp function, strictly convex function 1, linear function-full rank, penalty function, variable dimensioned function, tridiagonal system, five-diagonal system, extended Freudenstein and Roth function, discrete boundary value problem, Troesch problem, and so forth	0

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Research Article

A Smoothing Inexact Newton Method for Nonlinear Complementarity Problems

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A smoothing inexact Newton method is presented for solving nonlinear complementarity problems. Different from the existing exact methods, the associated subproblems are not necessary to be exactly solved to obtain the search directions. Under suitable assumptions, global convergence and superlinear convergence are established for the developed inexact algorithm, which are extensions of the exact case. On the one hand, results of numerical experiments indicate that our algorithm is effective for the benchmark test problems available in the literature. On the other hand, suitable choice of inexact parameters can improve the numerical performance of the developed algorithm.

1. Introduction

In the study of equilibria problems from economics, engineering, and management sciences, a complementarity problem (CP) often appears as the prominent mathematical model of the equilibria problems. Thus, it is the most practical interest to develop a robust and efficient algorithm for solving CP in the past decades (see the very recently published book [1] and the references therein). In this paper, we consider a nonlinear complementarity problem (denoted by NCP(F), for short): find a vector $x \in \mathbb{R}^n$ such that

$$x \geq 0, \quad F(x) \geq 0, \quad x^T F(x) = 0, \tag{1}$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable function. Due to the extensive applications, NCP(F) has attracted great attention of researchers (see, e.g., [2, 3] and the references therein). On the one hand, there have been many theoretical results on the existence of solutions and their structural properties. On the other hand, many attempts have been made to develop implementable algorithms for the solution of NCP(F).

A popular way to solve the NCP(F) is to reformulate (1) to a nonsmooth equation via an NCP-function. Function $\phi : \mathbb{R}^2 \to \mathbb{R}$ is said to be the NCP-function if

$$\phi(a, b) = 0 \Longleftrightarrow a \geq 0, \ b \geq 0, \ ab = 0. \tag{2}$$

Define $\Phi : \mathbb{R}^n \to \mathbb{R}^n$, given by

$$\Phi(x) = \begin{pmatrix} \phi(x_1, F_1(x)) \\ \phi(x_2, F_2(x)) \\ \vdots \\ \phi(x_n, F_n(x)) \end{pmatrix}. \tag{3}$$

Then, problem (1) is equivalent to

$$\Phi(x) = 0. \tag{4}$$

Thus, any efficient algorithm for solving (4) can be directly applied to find the solution of problem (1).

Smoothing method is a fundamental approach to solve the nonsmooth equation (4). In this connection, one can see, for example, [4–16]. The basic idea of this method is to construct a smooth function to approximate Φ. Let $\mu > 0$

be a given smoothing parameter. We define a continuously differentiable function $\Phi_\mu : \mathbb{R}^n \to \mathbb{R}^n$ such that for any $\mu > 0$ and $x \in \mathbb{R}^n$ there holds

$$\left\| \Phi(x) - \Phi_\mu(x) \right\| \longrightarrow 0, \quad \text{as } \mu \longrightarrow 0. \tag{5}$$

Then, problem (4) is approximated by a smooth equation:

$$\Phi_\mu(x) = 0. \tag{6}$$

Let $\{\mu^k\}$ be a given positive sequence which tends to 0. Then, we can obtain an approximate solution of (4) by solving (6) with $\mu = \mu^k$.

Recently, there are many different smoothing functions being employed to smooth the problem (4). Among them, the Fischer-Burmeiste function and the minimum function are two popular ones, which are defined by

$$\phi(a,b) = a + b - \sqrt{a^2 + b^2}, \quad \forall (a,b) \in \mathbb{R}^2, \tag{7}$$

$$\phi(a,b) = \min\{a,b\}, \quad \forall (a,b) \in \mathbb{R}^2, \tag{8}$$

respectively. With the 2-norm of (a,b) in the Fischer-Burmeiste function being replaced by a more general p-norm with $p \in (1,\infty)$, Chen and Pan proposed a family of new NCP-function in [6]. By combining the Fischer-Burmeiste function and the minimum function, Liu and Wu presented a smoothing function in [11] as follows:

$$\phi_\theta(a,b) = a + b - \sqrt{\theta(a-b)^2 + (1-\theta)(a^2+b^2)}, \quad \theta \in [0,1] \quad \forall (a,b) \in \mathbb{R}^2. \tag{9}$$

In [13], a symmetric perturbed Fischer-Burmeister function is constructed:

$$\phi(\mu,a,b) = (1+\mu)(a+b) - \sqrt{(a+\mu b)^2 + (b+\mu a)^2 + \mu^2}, \quad \forall (\mu,a,b) \in \mathbb{R}^3. \tag{10}$$

Very recetly, in [15], a more general smoothing function with the p-norm ($p \in (1,\infty)$) was presented. It is shown that for the nonmonotone smoothing Newton algorithm developed in [14] the numerical performance of algorithm is greatly improved if $p = 1.1$.

In this paper, we first write (8) as

$$\min\{a,b\} = \frac{1}{2}(a + b - |a-b|), \quad \forall (a,b) \in \mathbb{R}^2, \tag{11}$$

then, we intend to investigate a new smoothing method of $|\cdot|$, and in virtue of this new method, we will design a smoothing inexact Newton algorithm to solve the obtained smooth equations. Since an inexact parameter at each iteration is admissive to obtain an inexact Newton search direction, the developed algorithm is more helpful to numerical computation than the similar ones available in the literature. By suitably choosing a sequence of inexact parameters in advance, numerical performance of the developed algorithm in this paper can be improved. On the other hand, without the assumption of strict complementarity, we can establish the theory of convergence for our algorithm, which is weaker than that in the existing results.

The rest of this paper is organized as follows. In Section 2, we study a smoothing method of the absolute function. Section 3 is devoted to development of a smoothing inexact Newton algorithm to solve the nonlinear complementarity problem. In Section 4, the global convergence and the superlinear convergence are established. Numerical results are reported in Section 5. Some final remarks are made in Section 6.

The following notions will be used throughout this paper. For any vector or matrix A, A^T denotes the transposition of A. $\mathbb{R}^n$ denotes the space of n-dimensional real vectors. $\mathbb{R}^n_+$ and $\mathbb{R}^n_{++}$ denote the nonnegative and the positive subspaces in $\mathbb{R}^n$, respectively. For any vector $v \in \mathbb{R}^n$, $\mathrm{diag}\{v_i : i \in N\}$ denotes a diagonal matrix, whose ith diagonal element is v_i and $\mathrm{vec}\{v_i : i \in N\}$ the vector v, N represents the set $\{1,2,\ldots,n\}$. I represents the identity matrix with a suitable dimension. $\|\cdot\|$ stands for the 2-norm. For any $\alpha, \beta \in \mathbb{R}_{++}$, $\alpha = O(\beta)$ and $\alpha = o(\beta)$ represent that α/β is uniformly bounded and that α/β tends to zero as $\beta \to 0$, respectively.

2. Smoothing the Absolute Function

In this section, we will study a smoothing method of the absolute function.

We first present a function $\varphi_\mu : R \to R$, given by

$$\varphi_\mu(t) = \frac{2}{\pi} \arctan\left(\frac{t}{\mu}\right). \tag{12}$$

It is clear that

$$\lim_{\mu \to 0^+} \varphi_\mu(t) = \begin{cases} 1, & \text{if } t > 0, \\ 0, & \text{if } t = 0, \\ -1, & \text{if } t < 0. \end{cases} \tag{13}$$

Note that the generalized derivative of the absolution function $|\cdot|$ is calculated by

$$\mathrm{sign}(t) = \begin{cases} 1, & \text{if } t > 0, \\ [-1,1], & \text{if } t = 0, \\ -1, & \text{if } t < 0. \end{cases} \tag{14}$$

We can conclude that, except for $t = 0$, $\varphi_\mu(t)$ is a good approximation to the generalized derivative of $|t|$ with a sufficiently small μ. Actually, the following result was proved in [17].

Proposition 1. *For any given constant $\alpha > 0$, there is a constant $M_\alpha > 0$ independent of μ and t such that*

$$\begin{aligned} 0 \le \mathrm{sign}(t) - \varphi_\mu(t) \le M_\alpha \mu, \quad \forall t : t \ge \alpha,\ \forall \mu > 0, \\ 0 \le \varphi_\mu(t) - \mathrm{sign}(t) \le M_\alpha \mu, \quad \forall t : t \le -\alpha,\ \forall \mu > 0. \end{aligned} \tag{15}$$

By Proposition 1, to obtain an approximation of $|t|$, we calculate the integral of φ_μ:

$$\begin{aligned}\int \varphi_\mu(t)\,dt &= \frac{2}{\pi}\int \arctan\left(\frac{t}{\mu}\right)dt \\ &= t\varphi_\mu(t) - \frac{1}{\pi}\mu\ln\left(1+\frac{t^2}{\mu^2}\right) \triangleq \phi_\mu(t).\end{aligned} \tag{16}$$

Then, it is natural that we use $\phi_\mu(t)$ to approximate $|t|$. Actually, we have the following result (see [17]).

Proposition 2. *(1) For any $\mu > 0$, it holds that*

$$|t| \le \phi_\mu(t), \quad \forall t \in R. \tag{17}$$

The above inequality holds strictly for all $t \neq 0$.

(2) For any $t \in R^n$, $\lim_{\mu\to 0^+}\phi_\mu(t) = |t|$.

(3) $\lim_{\mu\to 0^+}\operatorname{dist}(\phi'_\mu(t), \partial h(t)) = 0$, where $h(t) = |t|$ and $\operatorname{dist}(v, S)$ denotes the distance from the point v to the set S.

3. A Smoothing Inexact Newton Algorithm for NCP(F)

In this section, we will develop a smoothing inexact Newton algorithm for solving a smooth equation obtained by reformulating the NCP(F).

Since

$$\min\{x_i, F_i(x)\} = \frac{1}{2}\left(x_i + F_i(x) - |x_i - F_i(x)|\right) \triangleq \phi_i(x), \quad (i \in N), \tag{18}$$

we construct an approximation of $\phi_i(x)$ by Proposition 2, defined by

$$\begin{aligned}\phi_i(\mu, x) &= \frac{1}{2}\left(x_i + F_i(x) - \phi_\mu(x_i - F_i(x))\right) \\ &= \frac{1}{2}\left(x_i + F_i(x) - \frac{2}{\pi}(x_i - F_i(x))\arctan\left(\frac{x_i - F_i(x)}{\mu}\right)\right. \\ &\qquad \left. + \frac{1}{\pi}\mu\ln\left(1 + \frac{(x_i - F_i(x))^2}{\mu^2}\right)\right).\end{aligned} \tag{19}$$

Define $\Phi_\mu : R^n \to R^n$, given by

$$\Phi_\mu(x) = \begin{pmatrix}\phi_1(\mu, x) \\ \vdots \\ \phi_n(\mu, x)\end{pmatrix}. \tag{20}$$

Then, $\Phi(x) = 0$ is approximated by a smooth equation:

$$\Phi_\mu(x) = 0. \tag{21}$$

Remark 3. The above smoothing method has been used to deal with NCP(F) in [17]. Then, by solving a generalized Newton equation:

$$\nabla\Phi_\mu(x^k)^T d + \Phi(x^k) = 0, \tag{22}$$

so as to obtain a search direction d at k-iteration for the developed algorithm in [17]. Different from the standard Newton method, $\Phi(x^k)$ is employed to replace $\Phi_\mu(x^k)$ in (22).

Taking into account the advantage of the standard smoothing Newton method (see, e.g., [12, 15, 16, 18]) in adjusting the the value of smoothing parameter automatically, we further transform problem (21) into a smooth optimization problem.

Denote $z = (\mu, x) \in \mathbb{R}_{++} \times \mathbb{R}^n$. We define a function $H : \mathbb{R}_{++} \times \mathbb{R}^n \to \mathbb{R}_{++} \times \mathbb{R}^n$ by

$$H(z) = \begin{pmatrix}\mu \\ \Phi(z)\end{pmatrix} \tag{23}$$

with $\Phi(z) = (\phi_1(z), \phi_2(z), \dots, \phi_n(z))^T$. Then, corresponding to any solution x^* of (21), $z^* = (0, x^*)$ is an optimal solution of the following minimization problem:

$$\min \Psi(z) \triangleq \|H(z)\|^2 = \mu^2 + \|\Phi(z)\|^2. \tag{24}$$

Conversely, if $z^* = (\mu^*, x^*)$ is an optimal solution of problem (24) with $\Psi(z^*) = 0$, then, x^* solves the system of (21).

Next, we focus on developing an efficient algorithm to solve problem (24). Before presentation of such an algorithm, we further investigate the properties of problem (24). The following definitions are useful to describe the properties of H.

Definition 4. A matrix $M \in \mathbb{R}^{n\times n}$ is said to be a P_0 matrix if all principal minors of M are nonnegative.

Definition 5. A function $F : \mathbb{R}^n \to \mathbb{R}^n$ is said to be a P_0 function if for all $x, y \in \mathbb{R}^n$ with $x \neq y$, there holds that

$$\max_{i\in N}(x_i - y_i)[F_i(x) - F_i(y)] \ge 0. \tag{25}$$

Definition 6 (see [19, 20]). Suppose that $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ is a locally Lipschitz continuous function, which has the generalized Jacobian $\partial\Psi(x)$ in the sense of Clarke [21], it is said to be semismooth (or strongly semismooth) at a point $x \in \mathbb{R}^n$ if and only if for any $V \in \partial\Psi(x + h)$, as $h \to 0$,

$$\begin{aligned}\left\|Vh - \Psi'(x, h)\right\| &= o(\|h\|), \quad \left(\text{or } O\left(\|h\|^2\right)\right), \\ \left\|\Psi(x+h) - \Psi(x) - \Psi'(x, h)\right\| &= o(\|h\|), \quad \left(\text{or } O\left(\|h\|^2\right)\right).\end{aligned} \tag{26}$$

We now prove the following results.

Lemma 7. *Let $z = (\mu, x)$ and H be defined by (23). Then, consider the following:*

(i) *H is continuously differentiable at any $z = (\mu, x) \in \mathbb{R}_{++} \times \mathbb{R}^n$ with its Jacobian matrix*

$$H'(z) = \begin{pmatrix} 1 & 0 \\ A(z) & B(z) \end{pmatrix}, \tag{27}$$

where

$$\begin{aligned} A(z) &= \frac{1}{2}\operatorname{vec}\left\{\frac{1}{\pi}\ln\left(1 + \frac{(x_i - F_i(x))^2}{\mu^2}\right) : i \in N\right\}, \\ B(z) &= \frac{1}{2}\left(I - D(z) + (I + D(z))F'(x)\right), \\ D(z) &= \frac{2}{\pi}\operatorname{diag}\left\{\arctan\left(\frac{x_i - F_i(x)}{\mu}\right) : i \in N\right\}. \end{aligned} \tag{28}$$

Furthermore, if F is a P_0-function, then, H' is nonsingular for any $\mu > 0$.

(ii) *H is locally Lipschitz continuous and semismooth on $\mathbb{R}^{n+1}$.*

Proof. (i) Since Φ is continuous differentiable at any $z = (\mu, x) \in \mathbb{R}_{++} \times \mathbb{R}^n$, then H is continuous differentiable. For any $\mu > 0$, by straightforward calculation, it yields (27) from the definition of H.

Note that, for all $i \in N$, $-1 < D(z)_{ii} < 1$. It is clear that $I - D(z)$ and $I + D(z)$ are two positive diagonal matrices. Since F is a P_0-function, F' is also a P_0-matrix for all $x \in \mathbb{R}^n$. Thus, the principal minors of the matrix $(I + D(z))F'(x)$ are nonnegative. By Definition 4, we know that the matrix $(I + D(z))F'(x)$ is a P_0-matrix. From Theorem 3.3 in [7], it follows that the matrix $B(z)$ is nonsingular. Then, it is concluded that the matrix $H'(z)$ is nonsingular.

(ii) It is clear that H is locally Lipschitz continuous and semismooth on $\mathbb{R}^{n+1}$. The proof is completed. □

With the properties of H in Lemma 7, we first present an algorithm to solve problem (24) similar to the idea in [18, 22–25].

Algorithm 8 (a smoothing inexact Newton method).

Step 0. Choose constants $\delta, \gamma \in (0, 1)$, $\sigma \in (0, 1/2)$, $\mu_0 > 0$ such that $\mu_0\gamma < 1$. Given an initial point $x^0 \in \mathbb{R}^n$, choose a sequence $\{\theta_k\} \subset \mathbb{R}_{++}$ such that $\theta_k \in (0, 1 - \mu_0\gamma)$. Set $z^0 := (\mu_0, x^0)$ and $k := 0$.

Step 1. If $\|H(z^k)\| = 0$, then the algorithm stops. Otherwise, compute

$$\begin{aligned} \beta(z^k) &:= \gamma \min\left\{1, \Psi(z^k)\right\}, \\ h(z^k) &:= \left(\mu_0\beta(z^k), \theta_k\Phi(z^k)\right). \end{aligned} \tag{29}$$

Step 2. Compute $\Delta z^k := (\Delta\mu^k, \Delta x^k) \in R \times R^n$ by

$$H(z^k) + H'(z^k)\Delta z^k = h(z^k). \tag{30}$$

Step 3. Set $\alpha_k := \delta^{m_k}$, where m_k is the smallest nonnegative integer m such that

$$\Psi(z^k + \delta^m\Delta z^k) \le \left[1 - 2\sigma(1 - \mu_0\gamma - \theta_k)\delta^m\right]\Psi(z^k). \tag{31}$$

Step 4. Set $z^{k+1} := z^k + \alpha_k\Delta z^k$ and $k := k + 1$. Return to Step 1.

Remark 9. Similar to the idea in [26], we develop Algorithm 8 by incorporating an inexact parameter θ_k at each iteration to obtain an inexact Newton direction of search in (30). Generally, we choose a sequence $\{\theta_k\}$ in advance, such that $\lim_{k\to\infty}\theta_k = 0$. Suitable choice of $\{\theta_k\}$ can be used to improve the numerical performance of Algorithm 8 by generating an inexact Newton direction Δz^k in Step 2 of Algorithm 8. The difference between Algorithm 8 and that developed in [26] lies in the distinct smoothing method. In [26], instead of the smoothing function (19), the Fischer-Burmeister function is adopted.

On the other hand, without the assumption of strict complementarity, we will establish the theory of global and local superlinearly convergences for Algorithm 8 in Section 4 under weaker conditions than the existing results.

If $\theta_k \equiv 0$, then, Algorithm 8 reduces to a smoothing Newton algorithm, which is similar to that developed in [18]. However, the definition of h in this paper is different from that in [18].

Denote

$$\Omega := \left\{z = (\mu, x) \in \mathbb{R}^{n+1} : \mu \ge \mu_0\beta(z)\right\}. \tag{32}$$

The following result shows that Algorithm 8 is well-defined.

Theorem 10. *Suppose that F is a continuous differentiable P_0-function.*

(1) *For the system of linear equations (30) in the unknown variable Δz^k, there exists a unique solution.*

(2) *In finitely many back-tracking steps, α_k in Step 3 of Algorithm 8 is obtained to satisfy (31).*

(3) *Let $\{z^k\}$ be the sequence generated by Algorithm 8. Then, for all $k > 0$, $z^k \in \Omega$.*

Proof. We prove the first result.

Since F is a continuously differentiable P_0-function, it follows from Lemma 7 that the matrix H' is nonsingular at z^k as $\mu_k > 0$. It implies that the system of linear equations (30) in the unknown variable Δz^k has a unique solution. Thus, Step 2 of Algorithm 8 is well-defined.

We now prove the second result.

By (30), we have

$$\Delta\mu_k = -\mu_k + \mu_0\beta(z^k). \tag{33}$$

From the definitions of $\Psi(z^k)$ and $\beta(z^k)$, it follows that, for all $k \geq 0$,

$$\begin{aligned} \beta\left(z^k\right) &\leq \gamma \Psi\left(z^k\right)^{1/2}, \\ \mu_k &\leq \Psi\left(z^k\right)^{1/2}. \end{aligned} \tag{34}$$

Thus, for any $\alpha \in (0,1)$

$$\begin{aligned} \left(\mu_k + \alpha\Delta\mu_k\right)^2 &= \left[(1-\alpha)\mu_k + \alpha\mu_0\beta\left(z^k\right)\right]^2 \\ &= (1-\alpha)^2\mu_k^2 + 2(1-\alpha)\alpha\mu_0\mu_k\beta\left(z^k\right) \\ &\quad + \alpha^2\mu_0^2\beta\left(z^k\right)^2 \\ &\leq (1-\alpha)^2\mu_k^2 + 2\alpha\mu_0\gamma\Psi\left(z^k\right) + O\left(\alpha^2\right). \end{aligned} \tag{35}$$

Denote

$$\varphi(\alpha) = \Phi\left(z^k + \alpha\Delta z_k\right) - \Phi\left(z^k\right) - \alpha\Phi'\left(z^k\right)\Delta z^k. \tag{36}$$

Since Φ is continuous differentiable at $z \in \mathbb{R}_{++} \times \mathbb{R}^n$, then $\|\varphi(\alpha)\| = o(\alpha)$; we conclude from (36) that

$$\begin{aligned} \left\|\Phi(z^k + \alpha\Delta z^k)\right\|^2 &= \left\|\Phi(z^k) + \alpha\Phi'(z^k)\Delta z^k + \varphi(\alpha)\right\|^2 \\ &= \left\|(1-\alpha+\alpha\theta_k)\Phi(z^k) + \varphi(\alpha)\right\|^2 \\ &\leq \left(1-\alpha+\alpha\theta_k\right)^2\left\|\Phi(z^k)\right\|^2 + o(\alpha). \end{aligned} \tag{37}$$

It yields

$$\begin{aligned} &\Psi\left(z^k + \alpha\Delta z_k\right) \\ &\quad = \mu_{k+1}^2 + \left\|\Phi(z^k + \alpha\Delta z_k)\right\|^2 \\ &\quad \leq (1-\alpha)^2\mu_k^2 + 2\alpha\mu_0\gamma\Psi\left(z^k\right) \\ &\qquad + \left(1-\alpha+\alpha\theta_k\right)^2\left\|\Phi(z^k)\right\|^2 + o(\alpha) + O\left(\alpha^2\right) \\ &\quad \leq \left(1-\alpha+\alpha\theta_k\right)^2\Psi\left(z^k\right) + 2\alpha\mu_0\gamma\Psi\left(z^k\right) + o(\alpha) \\ &\quad \leq \Psi\left(z^k\right) - 2\alpha\left(1-\theta_k\right)\Psi\left(z^k\right) \\ &\qquad + \alpha^2\left(1-\theta_k\right)^2\Psi\left(z^k\right) + 2\alpha\mu_0\gamma\Psi\left(z^k\right) + o(\alpha) \\ &\quad \leq \left[1 - 2\left(1-\mu_0\gamma-\theta_k\right)\alpha\right]\Psi\left(z^k\right) + o(\alpha). \end{aligned} \tag{38}$$

Since $\theta_k < 1 - \mu_0\gamma$, there exists a constant $\overline{\alpha} \in (0,1)$ such that, for any $\alpha \in (0, \overline{\alpha}]$ and $\sigma \in (0,1)$, there holds that

$$\Psi\left(z^k + \alpha\Delta z^k\right) \leq \left[1 - 2\sigma\left(1-\mu_0\gamma-\theta_k\right)\alpha\right]\Psi\left(z^k\right). \tag{39}$$

This demonstrates that Step 3 of Algorithm 8 is well-defined at each iteration.

Finally, we prove $z^k \in \Omega$ for all $k > 0$.

It is clear that $\mu_0\beta(z^0) \leq \mu_0\gamma \leq \mu_0$. In other words, $z^0 \in \Omega$. Suppose that $z^k \in \Omega$ as $k \geq 1$. Then, $\mu_k \geq \mu_0\beta(z^k)$. By (31), we get $\Psi(z^k) \geq \Psi(z^{k+1})$; then $\beta(z^k) \geq \beta(z^{k+1})$. By (33), we have

$$\begin{aligned} \mu_{k+1} &= (1-\alpha)\mu_k + \alpha\mu_0\beta\left(z^k\right) \\ &\geq (1-\alpha)\mu_0\beta\left(z^k\right) + \alpha\mu_0\beta\left(z^k\right) \\ &\geq \mu_0\beta\left(z^k\right) \geq \mu_0\beta\left(z^{k+1}\right). \end{aligned} \tag{40}$$

The last inequality implies that the desired result holds for $k+1$. By mathematical induction method, we concluded that $z^k \in \Omega$ for all $k > 0$.

We have completed the proof of Theorem 10. □

Remark 11. By Theorem 10, we know that Algorithm 8 is well-defined, and either it stops in finitely many steps or generates an infinite sequence $\{z^k = (\mu_k, x^k)\}$ with $\mu \in \mathbb{R}_{++}$ and $z^k \in \Omega$ for all $k \geq 0$. In the subsequent section, we will analyze the convergence of this sequence.

4. Convergence

In this section, we will establish the global convergence and the superlinear convergence for Algorithm 8.

We first prove the following result.

Lemma 12. *Let Φ_μ be defined by (20). If F is a P_0-function, then, for any $\mu > 0$, Φ_μ is coercive in x. That is,*

$$\lim_{\|x\| \to \infty} \left\|\Phi_\mu(x)\right\| \longrightarrow +\infty. \tag{41}$$

Proof. As $\|x\| \to \infty$, there exists a vector sequence $\{x^k\}$ which is unbounded. Then, there is a component $i_0 \in \{1, 2, \ldots, n\}$ such that $\{x^k_{i_0}\}$ is unbounded.

Define an index set $J = \{i \in N : \{x^k_i\}$ is unbounded$\}$. Then, J is a nonempty set. Without loss of generality, we assume that $\{|x^k_j|\} \to +\infty$, for all $j \in J$.

Let the sequence $\{\widehat{x}^k\}$ be defined by

$$\widehat{x}^k_i = \begin{cases} 0, & \text{if } i \in J, \\ x^k_i, & \text{if } i \notin J, \end{cases} \qquad i \in N. \tag{42}$$

Then, it is clear that $\{\widehat{x}^k\}$ is bounded. Since F is a P_0-function, by Definition 5, we have

$$\begin{aligned} 0 &\leq \max_{i \in N}\left\{\left(x^k_i - \widehat{x}^k_i\right)\left[F_i\left(x^k\right) - F_i\left(\widehat{x}^k\right)\right]\right\} \\ &= \max_{i \in J}\left\{x^k_i\left[F_i\left(x^k\right) - F_i\left(\widehat{x}^k\right)\right]\right\} \\ &= x^k_j\left[F_j\left(x^k\right) - F_j\left(\widehat{x}^k\right)\right], \end{aligned} \tag{43}$$

where j is one of the indices at which the max is attained. Since $j \in J$, and j can be supposed to be independent of k, we know $|x^k_j| \to +\infty$ as $k \to +\infty$.

Next, we continue the proof in the following six directions.

Case 1 ($x_j^k \to +\infty$ and $x_j^k - F_j(x^k) \to +\infty$). Since $\{F_j(\hat{x}^k), k \in N\}$ is bounded by the continuity of F_j and the definition of $\hat{x}^k$, we know that $F_j(x^k) \nrightarrow -\infty$ from (43). Thus,

$$\ln\left(1 + \frac{\left(x_j^k - F_j\left(x^k\right)\right)^2}{\mu^2}\right) \longrightarrow +\infty. \tag{44}$$

It yields

$$\begin{aligned}
&\left|\phi_j\left(\mu, x^k\right)\right| \\
&\quad = \frac{1}{2}\left| x_j^k + F_j\left(x^k\right) \right. \\
&\qquad - \frac{2}{\pi}\left(x_j^k - F_j\left(x^k\right)\right)\arctan\left(\frac{x_j^k - F_j\left(x^k\right)}{\mu}\right) \\
&\qquad \left. + \frac{1}{\pi}\mu\ln\left(1 + \frac{\left(x_j^k - F_j\left(x^k\right)\right)^2}{\mu^2}\right)\right| \longrightarrow +\infty.
\end{aligned} \tag{45}$$

Case 2 ($x_j^k \to -\infty$ and $x_j^k - F_j(x^k) \to +\infty$). It is clear that

$$\begin{gathered}
F_j\left(x^k\right) - x_j^k \longrightarrow -\infty, \\
\ln\left(1 + \frac{\left(x_j^k - F_j\left(x^k\right)\right)^2}{\mu^2}\right) \longrightarrow +\infty.
\end{gathered} \tag{46}$$

In virtue of

$$\lim_{u \to -\infty} \frac{\ln\left(1 + u^2\right)}{u} = 0, \tag{47}$$

we obtain

$$\begin{aligned}
&\left|\phi_j\left(\mu, x^k\right)\right| \\
&\quad = \frac{1}{2}\left| x_j^k + F_j\left(x^k\right) \right. \\
&\qquad - \frac{2}{\pi}\left(x_j^k - F_j\left(x^k\right)\right)\arctan\left(\frac{x_j^k - F_j\left(x^k\right)}{\mu}\right) \\
&\qquad \left. + \frac{1}{\pi}\mu\ln\left(1 + \frac{\left(x_j^k - F_j\left(x^k\right)\right)^2}{\mu^2}\right)\right| \\
&\quad = \left|\left(F_j\left(x^k\right) - x_j^k\right) \right. \\
&\qquad \left. + \frac{\mu}{2\pi}\ln\left(1 + \frac{\left(x_j^k - F_j\left(x^k\right)\right)^2}{\mu^2}\right) + x_j^k\right| \longrightarrow +\infty.
\end{aligned} \tag{48}$$

Case 3 ($x_j^k \to +\infty$ and $x_j^k - F_j(x^k) \to -\infty$). In the same reason as in Case 1, $F_j(x^k) \nrightarrow -\infty$. Thus,

$$\ln\left(1 + \frac{\left(x_j^k - F_j\left(x^k\right)\right)^2}{\mu^2}\right) \longrightarrow +\infty. \tag{49}$$

It yields

$$\begin{aligned}
&\left|\phi_j\left(\mu, x^k\right)\right| \\
&\quad = \frac{1}{2}\left| x_j^k + F_j\left(x^k\right) \right. \\
&\qquad - \frac{2}{\pi}\left(x_j^k - F_j\left(x^k\right)\right)\arctan\left(\frac{x_j^k - F_j\left(x^k\right)}{\mu}\right) \\
&\qquad \left. + \frac{1}{\pi}\mu\ln\left(1 + \frac{\left(x_j^k - F_j\left(x^k\right)\right)^2}{\mu^2}\right)\right| \\
&\quad = \left| x_j^k + \frac{\mu}{2\pi}\ln\left(1 + \frac{\left(x_j^k - F_j\left(x^k\right)\right)^2}{\mu^2}\right)\right| \longrightarrow +\infty.
\end{aligned} \tag{50}$$

Case 4 ($x_j^k \to -\infty$ and $x_j^k - F_j(x^k) \to -\infty$). Similar to Case 2, we can obtain

$$\begin{aligned}
&\left|\phi_j\left(\mu, x^k\right)\right| \\
&\quad = \frac{1}{2}\left| x_j^k + F_j\left(x^k\right) \right. \\
&\qquad - \frac{2}{\pi}\left(x_j^k - F_j\left(x^k\right)\right)\arctan\left(\frac{x_j^k - F_j\left(x^k\right)}{\mu}\right) \\
&\qquad \left. + \frac{1}{\pi}\mu\ln\left(1 + \frac{\left(x_j^k - F_j\left(x^k\right)\right)^2}{\mu^2}\right)\right| \\
&\quad = \left| x_j^k + \frac{\mu}{2\pi}\ln\left(1 + \frac{\left(x_j^k - F_j\left(x^k\right)\right)^2}{\mu^2}\right)\right| \\
&\quad = \left|\left(x_j^k - F_j\left(x^k\right)\right) + \frac{\mu}{2\pi}\ln\left(1 + \frac{\left(x_j^k - F_j\left(x^k\right)\right)^2}{\mu^2}\right) \right. \\
&\qquad \left. + F_j\left(x^k\right)\right| \longrightarrow +\infty.
\end{aligned} \tag{51}$$

Case 5 ($x_j^k \to +\infty$ and $x_j^k - F_j(x^k)$ is bounded). On the one hand, it is clear that

$$w = -\frac{2}{\pi}\left(x_j^k - F_j\left(x^k\right)\right)\arctan\left(\frac{x_j^k - F_j\left(x^k\right)}{\mu}\right) + \frac{1}{\pi}\mu\ln\left(1 + \frac{\left(x_j^k - F_j\left(x^k\right)\right)^2}{\mu^2}\right) \quad (52)$$

is bounded. On the other hand, $x_j^k \to +\infty$ and $x_j^k - F_j(x^k)$ is bounded; we know $F_j(x^k) \to +\infty$. Thus, $x_j^k + F_j(x^k) \to +\infty$. It yields

$$\left|\phi_j\left(\mu, x^k\right)\right| = \frac{1}{2}\left|x_j^k + F_j\left(x^k\right) + w\right| \longrightarrow +\infty. \quad (53)$$

Case 6 ($x_j^k \to -\infty$ and $x_j^k - F_j(x^k)$ is bounded). Similar to Case 5, it is easy to prove that $|\phi_j(\mu, x^k)| \to +\infty$.

The proof is completed. □

Remark 13. By Lemma 12, we can remove the assumption that the level set of the merit function is bounded. In addition, different from [13, 22, 27], the result of Lemma 12 is obtained in this paper for the nonsymmetric smoothing function.

Before statement of main results, we need the following assumption.

Assumption 14. The solution set S of NCP(F) (1) is nonempty and bounded.

Remark 15. Assumption 14 is a relatively weak condition to ensure the convergence of Algorithm 8. For example, in [26], it is assumed that the level sets

$$L\left(z^0\right) = \left\{z \in R^{n+1},\ \Phi(z) \le \Phi\left(z_0\right)\right\} \quad (54)$$

are bounded to prove the convergence of algorithm. Up to our knowledge, for the Fischer-Burmeister smoothing function, (54) is proved to be true under the condition that F in NCP(F) (1) is a uniform P-function.

However, with our smoothing method, we can prove that (54) holds. Since the proof is only involved with the condition that F is a P_0-function, Assumption 14 is weaker than that in [26].

With Lemma 12 and Assumption 14, we are now in a position to establish the convergence theory for Algorithm 8.

Theorem 16. *Let $\{z^k = (\mu_k, x^k)\}$ be the iteration sequence generated by Algorithm 8. Under Assumption 14, the following statements are true.*

(i) *$\{\Psi(z^k)\}$ and $\{\mu_k\}$ generated by Algorithm 8 are two monotonically decreasing and bounded sequences, whose limits are 0.*

(ii) *Any accumulation point of $\{z^k\}$ is a solution of (24).*

(iii) *Under Assumption 14, $\{z^k\}$ has at least one accumulation point $z^* = (\mu_*, x^*)$ with $H(z^*) = 0$ and $x^* \in S$.*

Proof. (i) From Steps 2 and 3 of Algorithm 8, it is clear that $\{\Psi(z^k)\}$, $\{\beta(z^k)\}$, and $\{\mu_k\}$ are three monotonically decreasing and bounded sequences.

(ii) By Lemma 12, we conclude that the sequence $\{z^k\}$ is bounded. Then, without loss of generality, we suppose that as $k \to \infty$, there exists z^* such that

$$z^k \longrightarrow z^*, \quad \beta\left(z^k\right) \longrightarrow \beta_*, \quad \Psi\left(z^k\right) \longrightarrow \Psi_*, \quad \mu^k \longrightarrow \mu_*. \quad (55)$$

If $\Psi_* > 0$, then, by the definition of $\beta(z^k)$, $\beta_* > 0$ and $\mu_* > 0$. From Lemma 7, it follows that $H'(z^*)$ is nonsingular. Thus, there exist a closed neighborhood $N(z^*)$ and a constant $\overline{\alpha} \in (0, 1]$, such that, for any $z \in N(z^*)$ and nonnegative integer m satisfying $\delta^m \in (0, \overline{\alpha}]$, the following inequality holds true:

$$\Psi\left(z^k + \delta^m \Delta z^k\right) \le \left[1 - 2\sigma\left(1 - \mu_0\gamma - \theta_k\right)\delta^m\right]\Psi\left(z^k\right). \quad (56)$$

If k is large enough such that $m^k \le m$ and $\delta^{m^k} \ge \delta^m$, then,

$$\begin{aligned}\Psi\left(z^{k+1}\right) &\le \left[1 - 2\sigma\left(1 - \mu_0\gamma - \theta_k\right)\delta^{m^k}\right]\Psi\left(z^k\right) \\ &\le \left[1 - 2\sigma\left(1 - \mu_0\gamma - \theta_k\right)\delta^m\right]\Psi\left(z^k\right).\end{aligned} \quad (57)$$

Therefore, as $k \to \infty$, it follows from $\Psi_* > 0$ that

$$2\sigma\left(1 - \mu_0\gamma - \theta_k\right) \le 0. \quad (58)$$

It contradicts $(1 - \mu_0\gamma - \theta_k) > 0$. We conclude that $\Psi(z^k) \to 0$ and $\mu_k \to 0$.

(iii) By Assumption 14, we know that $\Phi^{-1}(0)$ is nonempty and bounded. Thus, $\{z^k\}$ has at least one accumulation point $z^* = (\mu_*, x^*)$ with $H(z^*) = 0$ and $x^* \in S$. □

Theorem 17. *Suppose that Assumption 14 is satisfied and $z^* = (\mu_*, x^*)$ is an accumulation point of the sequence $\{z^k\}$ generated by Algorithm 8. If all $V \in \partial H(z^*)$ are nonsingular, then, $\{z^k\}$ converges to z^* superlinearly; that is, $\|z^{k+1} - z^*\| = o(\|z^k - z^*\|)$. Moreover, $\mu_{k+1} = o(\mu_k)$.*

Proof. By Theorem 16, we have $H(z^*) = 0$ and $x^* \in S$. Because all $V \in \partial H(z^*)$ are nonsingular, it follows that for all z^k sufficiently close to z^*,

$$\left\|H'\left(z^k\right)^{-1}\right\| = O(1). \quad (59)$$

From Lemma 7, it follows that $H(\cdot)$ is semismooth at z^*. Hence, for all z^k sufficiently close to z^*, we have

$$\left\|H\left(z^k\right) - H\left(z^*\right) - H'\left(z^k\right)\left(z^k - z^*\right)\right\| = o\left(\left\|z^k - z^*\right\|\right). \quad (60)$$

On the other hand, Lemma 7 implies that $H(\cdot)$ is locally Lipschitz continuous near z^*. Therefore, for all z^k sufficiently close to z^*, we have

$$\left\|H\left(z^k\right)\right\| = O\left(\left\|z^k - z^*\right\|\right). \tag{61}$$

Since $\lim_{k\to\infty}\theta_k = 0$, it is concluded that $\theta_k\|H(z^k)\| = o\|z^k - z^*\|$. Thus, by the definitions of $h(z)$ and $\beta(z)$, we have

$$\begin{aligned}\left\|h\left(z^k\right)\right\| &\le \left|\mu_0\beta\left(z^k\right)\right| + \left\|\theta_k\Phi\left(z^k\right)\right\| \\ &\le \mu_0\gamma\Psi\left(z^k\right) + \theta_k\left\|H\left(z^k\right)\right\| \\ &= o\left(\left\|z^k - z^*\right\|\right).\end{aligned} \tag{62}$$

Then, in view of (59), (60), and (62), it is obtained that

$$\begin{aligned}&\left\|z^k + \Delta z^k - z^*\right\| \\ &\quad = \left\|z^k + H'\left(z^k\right)^{-1}\left[-H\left(z^k\right) + h\left(z^k\right)\right] - z^*\right\| \\ &\quad \le \left\|H'\left(z^k\right)^{-1}\right\| \\ &\quad\quad \cdot \left(\left\|H\left(z^k\right) - H\left(z^*\right) - H'\left(z^k\right)\left(z^k - z^*\right)\right\| + \left\|h\left(z^k\right)\right\|\right) \\ &\quad = o\left(\left\|z^k - z^*\right\|\right).\end{aligned} \tag{63}$$

On the other hand, from (61), it follows that

$$\begin{aligned}&\Psi\left(z^k + \Delta z^k\right) \\ &\quad = \left\|H\left(z^k + \Delta z^k\right)\right\|^2 \\ &\quad = O\left(\left\|z^k + \Delta z^k - z^*\right\|^2\right) = o\left(\left\|z^k - z^*\right\|^2\right) \\ &\quad = o\left(\left\|H\left(z^k\right)\right\|^2\right) = o\left(\Psi\left(z^k\right)\right).\end{aligned} \tag{64}$$

Thus, as z^k sufficiently close to z^*, we have $z^{k+1} = z^k + \Delta z^k$. It yields

$$\mu_{k+1} = \mu_k + \Delta\mu_k = \mu_0\gamma\left\|H\left(z^k\right)\right\|^2. \tag{65}$$

In virtue of (65), we obtain

$$\frac{\mu_{k+1}}{\mu_k} = \frac{\left\|H\left(z^k\right)\right\|^2}{\left\|H\left(z^{k-1}\right)\right\|^2} = \frac{o\left(\Psi\left(z^{k-1}\right)\right)}{\Psi\left(z^{k-1}\right)}. \tag{66}$$

As z^k sufficiently close to z^*, we know $\mu_{k+1} = o(\mu_k)$. The proof has been completed. □

5. Numerical Experiments

In this section, we test the numerical performance of Algorithm 8 for solving benchmark test problems in NCP.

Algorithm 8 is implemented in MATLAB2008a on a PC 2.00 GHZ CPU with 2.00 GB RAM with the operation system of Windows 7. Throughout the experiments, the parameters in Algorithm 8 are chosen as follows:

$$\mu_0 = 0.01, \quad \sigma = 0.25, \quad \delta = 0.85, \quad \gamma = 0.2, \quad \theta_k = \frac{1}{4^{k+1}}. \tag{67}$$

We use $\|H(z^k)\| < 10^{-8}$ as the termination criterion.

Numerical results are reported in Tables 1–9, where we use the following denotations for conciseness:

IT: the number of iterations,

ST: the initial point x^0,

CT: the CPU time depleted by the algorithm,

x^*: a solution of the NCP,

x^k: the final value of x,

o: zero vector with n dimension,

e: unit vector with n dimension,

F: The algorithm fails to get a solution.

The test problems are from the literature (see, e.g., [22, 27, 28]).

Problem 1. In problem (1), $x \in \mathbb{R}^3$ and $F(x) : \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$F(x) = \begin{pmatrix} x_2 \\ x_3 \\ -x_2 + x_3 + 1 \end{pmatrix}. \tag{68}$$

This problem has infinitely many solutions $(0, \lambda, 0)$, where $\lambda \in [0, 1]$. The test results are listed in Table 1 by using different initial points.

Problem 2 (modified Mathiesen problem). In problem (1), $x \in \mathbb{R}^4$ and $F(x) : \mathbb{R}^4 \to \mathbb{R}^4$ is given by

$$F(x) = \begin{pmatrix} -x_2 + x_3 + x_4 \\ x_1 - \dfrac{(4.5x_3 + 2.7x_4)}{(x_2 + 1)} \\ 5 - x_1 - \dfrac{(0.5x_3 + 0.3x_4)}{(x_3 + 1)} \\ 3 - x_1 \end{pmatrix}. \tag{69}$$

This problem has infinitely many solutions $(\lambda, 0, 0, 0)$, where $\lambda \in [0, 3]$. The solutions are degenerate for $\lambda = 0$ or $\lambda = 3$ and nondegenerate for $\lambda \in (0, 3)$. With different starting points, we report results in Table 2.

Problem 3 (Kojima-Shindo problem). In problem (1), $x \in \mathbb{R}^4$ and $F(x) : \mathbb{R}^4 \to \mathbb{R}^4$ is given by

$$F(x) = \begin{pmatrix} 3x_1^2 + 2x_1x_2 + 2x_2^2 + x_3 + 3x_4 - 6 \\ 2x_1^2 + x_1 + x_2^2 + 10x_3 + 2x_4 - 2 \\ 3x_1^2 + x_1x_2 + 2x_2^2 + 2x_3 + 9x_4 - 9 \\ x_1^2 + x_2^2 + 2x_3 + 3x_4 - 3 \end{pmatrix}. \tag{70}$$

Table 1: Numerical results of Problem 1.

ST	IT	x^k	$\lvert x^T y\rvert$	$\lVert H^k\rVert$	CPU
o	4	$(0, 0.0000, 0)$	$4.83E-09$	$8.38E-09$	0.085
e	5	$(0, 0.4998, 0)$	$3.44E-11$	$3.97E-11$	0.103
$10e$	6	$(0, 1.0000, 0)$	$1.05E-11$	$9.09E-12$	0.124
$-e$	5	$(0, 0.0023, 0)$	$2.09E-09$	$3.61E-09$	0.105
$-10e$	6	$(0, 0.0071, 0)$	$2.49E-11$	$4.28E-11$	0.124
$-100e$	6	$(0, 0.0399, 0)$	$1.50E-09$	$2.50E-09$	0.125

Table 2: Numerical results of Problem 2.

ST	IT	x^k	$\lvert x^T y\rvert$	$\lVert H^k\rVert$	CPU
e	5	$(0.37, 0, 3, 0)$	$1.27E-08$	$3.91E-09$	0.304
o	5	$(0, 0, 0, 0)$	$6.59E-11$	$3.81E-11$	0.292
$-2e$	6	$(0.002, 0, 0, 0)$	$1.32E-08$	$3.31E-09$	0.356
$-11e$	6	$(1.85, 0, 0, 0)$	$1.06E-08$	$2.64E-09$	0.416
$-20e$	6	$(0.05, 0, 0, 0)$	$1.38E-08$	$3.46E-09$	0.366

Table 3: Numerical results of Problem 3.

ST	IT	x^k	$\lvert x^T y\rvert$	$\lVert H^k\rVert$	CPU
e	6	x_2	$2.99E-08$	$1.42E-09$	0.679
$10e$	7	x_2	$3.54E-08$	$1.77E-09$	0.729
10^2e	8	x_2	$3.02E-08$	$1.54E-09$	0.877
10^3e	9	x_1	$1.07E-08$	$9.99E-09$	0.965

Table 4: Solution of Problem 4 with random initial points.

ST	IT	$\lvert x^T y\rvert$	$\lVert H^k\rVert$	CPU
e	18	$8.52E-10$	$1.49E-10$	5.539
o	22	$2.50E-10$	$4.38E-11$	6.538
$-e$	38	$3.41E-09$	$5.96E-10$	11.306
rand(5, 1)	16	$4.20E-10$	$7.35E-11$	4.796

Table 5: Numerical results of Problem 5.

ST	IT	$\lvert x^T y\rvert$	$\lVert H^k\rVert$	CPU
o	6	$1.38E-08$	$4.97E-09$	0.858
e	7	$5.21E-09$	$1.89E-09$	1.109
$-e$	7	$1.32E-08$	$4.77E-09$	1.003
$-10e$	7	$1.32E-08$	$4.77E-09$	0.990
-10^2e	8	$1.32E-08$	$4.76E-09$	1.142
-10^3e	8	$1.32E-08$	$4.76E-09$	1.079
-10^4e	9	$1.40E-08$	$2.47E-09$	1.268

Table 6: Numerical results of Problem 6.

ST	IT	$\lvert x^T y\rvert$	$\lVert H^k\rVert$	CPU
e	7	$4.01E-11$	$1.39E-11$	0.248
$10e$	9	$8.61E-09$	$2.97E-09$	0.358
10^2e	10	$1.13E-09$	$3.91E-10$	0.316
10^3e	9	$2.45E-09$	$8.48E-10$	0.359
$-e$	8	$2.46E-11$	$8.51E-12$	0.294
$-10e$	15	$9.46E-12$	$3.28E-12$	0.504
-10^3e	24	$8.96E-10$	$3.10E-10$	0.759

Table 7: Numerical results of Problem 7.

n	IT	$\lvert x^T y\rvert$	$\lVert H^k\rVert$	CPU
128	8	$1.30E-10$	$1.48E-09$	0.040
256	8	$1.15E-10$	$1.90E-09$	0.155
512	9	$1.55E-11$	$3.50E-10$	1.010
800	10	$1.99E-11$	$6.44E-10$	2.701
1000	10	$1.58E-11$	$4.99E-10$	4.738

Table 8: Effect of inexact parameter in Problem 3.

ST	θ_k	IT	$\lvert x^T y\rvert$	$\lVert H^k\rVert$	CPU
	0	6	$7.95E-09$	$3.89E-10$	0.70
	0.2	6	$1.46E-08$	$7.27E-10$	0.66
e	0.4	7	$1.30E-09$	$8.41E-11$	0.73
	0.6	9	$4.77E-09$	$4.60E-10$	0.91
	0.8	13	$7.14E-08$	$4.05E-09$	1.33
	0	F	F	F	F
	0.2	8	$2.97E-08$	$1.52E-09$	0.92
$10^2 e$	0.4	10	$4.25E-12$	$1.71E-12$	1.12
	0.6	10	$8.98E-09$	$1.43E-09$	1.01
	0.8	15	$3.66E-08$	$6.99E-09$	1.57

This problem has one degenerate solution $x_1 = (\sqrt{6}/2, 0, 0, 1/2)^T$ and one nondegenerate solution $x_2 = (1, 0, 3, 0)^T$. We use different initial points and the test results are listed in Table 3.

Problem 4. In problem (1), $x \in \mathbb{R}^5$ and $F(x) = (F_1(x), \dots, F_5(x))^T$ where

$$F_i(x) = 2\exp\left(\sum_{i=1}^{5}(x_i - i + 2)^2\right)(x_i - i + 2), \quad i = 1, \dots, 5. \tag{71}$$

This problem has one solution $(0, 0, 1, 2, 3)$. We use different starting points and the last initial point x^0 is randomly generated whose elements are in the interval $(0, 1)$. The test results are listed in Table 4.

Problem 5. In problem (1), $x \in \mathbb{R}^7$ and $F(x) : \mathbb{R}^7 \to \mathbb{R}^7$ is given by

$$F(x) = \begin{pmatrix} 2x_1 - x_3 + x_5 + 3x_6 - 1 \\ x_2 + 2x_5 + x_6 - x_7 - 3 \\ -x_1 + 2x_3 + x_4 + x_5 + 2x_6 - 4x_7 + 1 \\ x_3 + x_4 + x_5 - x_6 - 1 \\ -x_1 - 2x_2 - x_3 - x_4 + 5 \\ -3x_1 - x_2 - 2x_3 + x_4 + 4 \\ x_2 + 4x_3 - 1.5 \end{pmatrix}. \tag{72}$$

The test results are listed in Table 5 by using different initial points.

Problem 6. In problem (1), $x \in \mathbb{R}^4$ and $F(x) : \mathbb{R}^4 \to \mathbb{R}^4$ is given by

$$F(x) = \begin{pmatrix} x_1^3 - 8 \\ x_2 - x_3 + x_2^3 + 3 \\ x_2 + x_3 + 2x_3^3 - 3 \\ x_4 + 2x_4^3 \end{pmatrix}. \tag{73}$$

In this problem, $F(x)$ is a P_0-function. It has only one solution $(2, 0, 1, 0)$. With different initial points, the results are listed in Table 6.

Problem 7. In problem (1), $x \in \mathbb{R}^n$ and $F(x) = Mx + q$ with

$$\begin{gathered} [M]_{ii} = 4(i-1) + 1, \quad i = 1, 2, \dots, n, \\ [M]_{ij} = [M]_{ii} + 1, \quad i = 1, 2, \dots, n-1, \; j = i+1, \dots, n, \\ [M]_{ji} = [M]_{jj} + 1, \quad j = 1, 2, \dots, n-1, \; i = j+1, \dots, n, \\ q = (-1, -1, \dots, -1)^T. \end{gathered} \tag{74}$$

This problem has only solutions $x^* = (1, 0, \dots, 0)^T$. From the initial point $x^0 = (1, 1, \dots, 1)^T$, we solve this problem with different dimensions. The test results are listed in Table 7.

In the end of this section, we intend to test the effect of the inexact parameter θ_k on the efficiency of Algorithm 8.

In Tables 8 and 9, for Problems 3 (not a P_0-function) and 6, we take different values of θ_k, $\theta_k = 0, 0.2, 0.4, 0.6, 0.8$, and implement Algorithm 8 to find the solutions of Problems 3 and 6, respectively.

Table 9: Effect of inexact parameter in Problem 6.

ST	θ_0	IT	$\|x^T y\|$	$\|H^k\|$	CPU
e	0	6	$2.01E-09$	$6.78E-10$	0.22
	0.2	6	$1.18E-08$	$4.06E-09$	0.22
	0.4	7	$7.50E-11$	$4.66E-11$	0.27
	0.6	9	$1.32E-11$	$1.99E-10$	0.27
	0.8	13	$9.29E-09$	$7.16E-09$	0.39
$10e$	0	F	F	F	F
	0.2	8	$9.13E-09$	$3.15E-09$	0.28
	0.4	10	$9.03E-11$	$3.11E-11$	0.34
	0.6	11	$7.19E-12$	$4.06E-11$	0.36
	0.8	14	$8.25E-09$	$9.51E-09$	0.44

From Tables 8 and 9, it is revealed that, for $\theta_k = 0$ (which corresponds to the smoothing exact Newton method), Algorithm 8 may fail for some initial points. On the other hand, a suitable value of inexact parameter may greatly improve the efficiency of Algorithm 8.

From the numerical results, we conclude as follows:

(1) In Tables 1–7, the choice of initial point only incurs weak impact on the CPU time and the iteration number of Algorithm 8. It indicates that the developed algorithm in this paper is robust even if for the randomly generated initial point.

(2) From the results in Tables 8 and 9, the inexact parameter θ_k may play critical role in improve the numerical performance of Algorithm 8.

6. Final Remarks

In this paper, a smoothing inexact Newton method has been proposed for solving nonlinear complementarity problems based on a new smoothing function. Then, an implementable algorithm was developed. Under a suitable assumption, the global convergence and the superlinear convergence have been established for the algorithm. Results of numerical experiments indicate that our algorithm is effective for the benchmark test problems available in the literature.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

A Note on Continuity of Solution Set for Parametric Weak Vector Equilibrium Problems

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We consider the parametric weak vector equilibrium problem. By using a weaker assumption of Peng and Chang (2014), the sufficient conditions for continuity of the solution mappings to a parametric weak vector equilibrium problem are established. Examples are provided to illustrate the essentialness of imposed assumptions. As advantages of the results, we derive the continuity of solution mappings for vector optimization problems.

1. Introduction

It is well known that the vector equilibrium problem provides a unified model of several classes of problems, including vector variational inequality problems, vector complementarity problems, vector optimization problems, and vector saddle point problems. There are many papers which have discussed the existence results for different types of vector equilibrium problems (see [1–3] and references therein).

In 2008 Gong [4] studied parametric vector equilibrium problems. Based on a scalarization representation of the solution mapping and the property involving the union of a family of lower semicontinuous set-valued mappings of Cheng and Zhu [5], they established the sufficient conditions for the continuity of the solution set mapping for the mixed parametric monotone weak vector equilibrium problems in topological vector spaces. In the same year, Gong and Yao [6] discussed the lower semicontinuity of the efficient solution mappings to a parametric strong vector equilibrium problem with C-strict monotonicity of a vector-valued function, by using a scalarization method and density result. In 2009, Xu and Li [7] presented a new proof of lower semicontinuity of the set of efficient solutions to a parametric strong vector equilibrium problem, which is different from the one used in [6]. In 2010, Chen and Li [8] discussed and improved the lower semicontinuity and continuity results of efficient solution mappings to a parametric strong vector equilibrium problem in [4, 6], without the uniform compactness assumption. By virtue of the scalarization technique, [4, 6–8] have discussed the lower semicontinuity, in the case that ξ-efficient solution set is a singleton. However, in practical, the ξ-solution set may not be singleton but a general set. Recently, by using a weak assumption, Peng and Chang [9] discussed the lower semicontinuity of solution maps for parametric weak vector equilibrium problem under the case that the ξ-efficient solution mapping may not be single-valued as follows. Unfortunately, the results obtained in the corresponding papers [4, 6–9] cannot be used in the case of vector optimization problems. Hence, in this paper, we study the lower semicontinuity of the set of efficient solutions for parametric weak vector equilibrium problems when the ξ-efficient solution set is a general set. Moreover, our theorems can apply for vector optimization problems.

The structure of the paper is as follows. Section 2 presents the efficient solutions to parametric weak vector equilibrium problems and materials used in the rest of this paper. We establish, in Section 3, a sufficient condition for the continuity of the efficient solution mappings. We give some examples to illustrate that our main results are different from the corresponding ones in the literature. Section 4 is reserved for an application of the main result to a weak vector optimization problem.

2. Preliminaries

Throughout this paper, if not otherwise specified, X, Y will denote two real Hausdorff topological vector spaces, and Z a real topological space, and M a nonempty subset of Z. Let Y^* be the topological dual space of Y. Let $C \subset Y$ be a pointed, closed, and convex cone with $\operatorname{int} C \neq \emptyset$. Let

$$C^* := \{\xi \in Y^* : \xi(y) \geq 0,\ \forall y \in C\} \tag{1}$$

be the dual cone of C. Denote the quasi-interior of C^* by $C^\natural$; that is,

$$C^\natural := \{\xi \in Y^* : \xi(y) > 0,\ \forall y \in C \setminus \{0\}\}. \tag{2}$$

Since $\operatorname{int} C \neq \emptyset$, the dual cone C^* of C has a weak* compact base. Let $e \in \operatorname{int} C$. Then,

$$B_e^* := \{\xi \in C^* : \xi(e) = 1\} \tag{3}$$

is a weak* compact base of C^*.

Let $N(\mu_0) \subset M$ be neighborhoods of considered points μ_0. Let $A : M \rightrightarrows X$ be a set-valued mapping and let $f : X \times X \times M \to Y$ be a vector-valued mapping.

For each $\mu \in N(\mu_0)$, we consider the following parametric weak vector equilibrium problem (PWVEP): find $x \in A(\mu)$, such that

$$f(x, y, \mu) \notin -\operatorname{int} C, \quad \forall y \in A(\mu). \tag{4}$$

Let $S(\mu)$ be the efficient solution set of (4); that is,

$$S(\mu) := \{x \in A(\mu) : f(x, y, \mu) \notin -\operatorname{int} C,\ \forall y \in A(\mu)\}. \tag{5}$$

For each $\xi \in C^* \setminus \{0\}$ and $\mu \in N(\mu_0)$, let $S_\xi(\mu)$ denote the set of ξ-efficient solution set to (4); that is,

$$S_\xi(\mu) := \{x \in A(\mu) : \xi(f(x, y, \mu)) \geq 0,\ \forall y \in A(\mu)\}. \tag{6}$$

Throughout this paper, we always assume $S(\mu) \neq \emptyset$ for all $\mu \in \Lambda$. Now, we recall the definition of semicontinuity of set-valued mappings. Let Λ and X be two topological spaces, $F : \Lambda \to 2^X$ a set-valued mapping, and $\overline{\lambda} \in \Lambda$.

Definition 1 (see [10]). (i) F is said to be lower semicontinuous (l.s.c.) at $\overline{\lambda}$ if, for any open set U satisfying $U \cap F(\overline{\lambda}) \neq \emptyset$, there exists $\delta > 0$ such that $F(\lambda) \cap U \neq \emptyset$, for all $\lambda \in B(\overline{\lambda}, \delta)$.

(ii) F is said to be upper semicontinuous (u.s.c) at $\overline{\lambda}$ if, for any open set U satisfying $F(\overline{\lambda}) \subset U$, there exists $\delta > 0$ such that $F(\lambda) \subset U$, for all $\lambda \in B(\overline{\lambda}, \delta)$.

Proposition 2 (see [11, 12]). *(i) F is l.s.c. at $\overline{\lambda}$ if and only if, for any net $\{\lambda_\alpha\} \subset \Lambda$ with $\lambda_\alpha \to \overline{\lambda}$ and any $\overline{x} \in F(\overline{\lambda})$, there exists $x_\alpha \in F(\lambda_\alpha)$ such that $x_\alpha \to \overline{x}$.*

(ii) If F has compact values (i.e., $F(\lambda)$ is a compact set for each $\lambda \in \Lambda$), then F is u.s.c. at $\overline{\lambda}$ if and only if, for any net $\{\lambda_\alpha\} \subset \Lambda$ with $\lambda_\alpha \to \overline{\lambda}$ and for any $x_\alpha \in F(\lambda_\alpha)$, there exists $\overline{x} \in F(\overline{\lambda})$ and a subnet $\{x_\beta\}$ of $\{x_\alpha\}$ such that $x_\beta \to \overline{x}$.

Definition 3. Let X and Y be two vector spaces. Let B be a nonempty subset of X. A vector-valued function $\varphi : B \to Y$ is said to be

(a) *C-strictly convex* on a convex subset K of B, if

$$t\varphi(x) + (1-t)\varphi(y) \in \varphi(tx + (1-t)y) + \operatorname{int} C, \quad \forall x, y \in K \quad \text{with } x \neq y, \quad \forall t \in (0,1); \tag{7}$$

(b) *C-convex* on a convex subset K of B, if

$$t\varphi(x) + (1-t)\varphi(y) \in \varphi(tx + (1-t)y) + C, \quad \forall x, y \in K, \quad \forall t \in [0,1]; \tag{8}$$

(c) *C-convexlike* on convex subset K of B, if, for any x_1, $x_2 \in K$ and any $t \in [0,1]$, there exist $x_3 \in K$ such that $tf(x, x_1, \mu) + (1-t)f(x, x_2, \mu) \in f(x, x_3, \mu) + C$.

Obviously, we get that

$$(a) \Longrightarrow (b) \Longrightarrow (c). \tag{9}$$

Next, we recall the definitions of monotonicity which are in common use in review literature.

Definition 4. Let X and Y be two vector spaces. Let B be a nonempty subset of X. A bifunction $f : B \times B \to Y$ is said to be

(i) *monotone* on subset K of B, if

$$f(x, y) + f(y, x) \in -C, \quad \forall x, y \in K; \tag{10}$$

(ii) *strictly monotone* on subset K of B, if f is monotone and

$$f(x, y) + f(y, x) \in -\operatorname{int} C, \quad \forall x, y \in K,\ x \neq y. \tag{11}$$

Remark 5. It is clear that (ii) implies (i) but the converse is not true. An easy example is that $f(x, y) = g(y) - g(x)$ for all $x, y \in B$ where $g : B \to Y$; we see that $f(x, y) + f(y, x) = (g(y) - g(x)) + (g(x) - g(y)) = 0_Y \notin -\operatorname{int} C$ for all $x, y \in K$.

Now, we collect two vital lemmas.

Lemma 6 (see [13]). *Suppose that for each $\mu \in M$ and $x \in A(\mu)$, $f(x, A(\mu), \mu) + C$ is a convex set; then*

$$S(\mu) = \bigcup_{\xi \in C^* \setminus \{0\}} S_\xi(\mu). \tag{12}$$

Lemma 7 (see [14, Theorem 2, p. 114]). *The union $\Gamma = \cup_{i \in I} \Gamma_i$ of a family of l.s.c. set-valued mappings Γ_i from a topological space X to a topological space Y is also l.s.c. set-valued mapping from X to Y is also l.s.c. set-valued mapping from X to Y, where I is an index set.*

3. Main Results

In this section, we present the continuity of the efficient solution mapping to PWVEP.

Theorem 8. *Let $\mu_0 \in M$ be a considered point for (PWVEP). Suppose that the following conditions are satisfied:*

(i) *$A(\cdot)$ is continuous with nonempty compact convex values at μ_0;*

(ii) *$f(\cdot,\cdot,\cdot)$ is continuous on $B \times B \times M$;*

(iii) *$f(\cdot,\cdot,\mu_0)$ is monotone on $A(\mu_0)$;*

(iv) *$f(x,\cdot,\mu_0)$ is C-strictly convex on $A(\mu_0)$.*

Then, for each $\xi \in C^* \setminus \{0\}$, $S_\xi(\cdot)$ is continuous on μ_0.

Proof. We first prove that $S(\cdot)$ is lower semicontinuous at μ_0. Suppose the contrary that there exists a $\xi' \in C^* \setminus \{0\}$ such that $S_{\xi'}(\cdot)$ is not l.s.c. at μ_0. Then there exists a net $\{\mu_\alpha\}$ with $\mu_\alpha \to \mu_0$ and $x_0 \in S(\mu_0)$ such that for any $x_\alpha \in S_{\xi'}(\mu_\alpha)$, $x_\alpha \nrightarrow x_0$. Since $x_0 \in S_{\xi'}(\mu_0)$, we have $x_0 \in A(\mu_0)$ and

$$\xi'(f(x_0,y,\mu_0)) \geq 0, \quad \forall y \in A(\mu_0). \tag{13}$$

By the lower semicontinuity of $A(\cdot)$ at μ_0, there exists a net $\{\overline{x}_\alpha\} \subset A(\mu_\alpha)$ such that $\overline{x}_\alpha \to x_0$.

For any $y_\alpha \in S(\mu_\alpha)$, by the upper semicontinuity and compactness of $A(\cdot)$ at μ_0, we get that there exists $y_0 \in A(\mu_0)$ and a subsequence $\{y_{\alpha_i}\}$ of $\{y_\alpha\}$ such that $y_{\alpha_i} \to y_0$, denoted by $\{y_i\}$. We have

$$\xi'(f(y_i,x_i,\mu_i)) \geq 0 \quad \forall i, \; \xi'(f(x_0,y_0,\mu_0)) \geq 0. \tag{14}$$

By continuity of ξ' and $f(\cdot,\cdot,\cdot)$ on $B \times B \times M$, we get that

$$\xi'(f(y_0,x_0,\mu_0)) \geq 0. \tag{15}$$

We want to show that $x_0 = y_0$. Assume that $x_0 \neq y_0$, then by strict convexity of $f(x,\cdot,\mu_0)$ and linearity of ξ' imply that

$$\begin{aligned} 0 &\leq \xi'\left(f\left(x_0, \frac{1}{2}x_0 + \frac{1}{2}y_0, \mu_0\right)\right) \\ &< \frac{1}{2}\xi'(f(x_0,y_0,\mu_0)) + \frac{1}{2}\xi'(f(y_0,x_0,\mu_0)), \\ 0 &\leq \xi'\left(f\left(y_0, \frac{1}{2}x_0 + \frac{1}{2}y_0, \mu_0\right)\right) \\ &< \frac{1}{2}\xi'(f(y_0,x_0,\mu_0)) + \frac{1}{2}\xi'(f(y_0,y_0,\mu_0)). \end{aligned} \tag{16}$$

Monotonicity assumption of $f(\cdot,\cdot,\mu_0)$ implies that

$$\begin{aligned} 0 &< \frac{1}{2}\xi'(f(x_0,x_0,\mu_0)) + \frac{1}{2}\xi'(f(y_0,x_0,\mu_0)) \\ &\leq \frac{1}{2}\xi'(f(y_0,x_0,\mu_0)), \\ 0 &< \frac{1}{2}\xi'(f(y_0,x_0,\mu_0)) + \frac{1}{2}\xi'(f(y_0,y_0,\mu_0)) \\ &\leq \frac{1}{2}\xi'(f(y_0,x_0,\mu_0)). \end{aligned} \tag{17}$$

This implies that

$$0 < \xi'(f(y_0,x_0,\mu_0)), \tag{18}$$

$$0 < \xi'(f(y_0,x_0,\mu_0)). \tag{19}$$

Adding (18) and (19), it follows from linearity of ξ' and monotonicity of f that

$$0 < \xi'(f(x_0,y_0,\mu_0) + f(y_0,x_0,\mu_0)) \leq 0. \tag{20}$$

This is impossible by the contradiction assumption. This proof is complete. □

Before comparing our result with the result of [9], we first recall that result as follows.

Theorem 9 (see [9, Theorem 3.1]). *Let $\mu_0 \in M$ be a considered point for (PWVEP). Suppose that the following conditions are satisfied:*

(i) *$A(\cdot)$ is a mapping with nonempty compact convex valued and continuous at μ_0;*

(ii) *for each $\mu \in M$, $(x,y) \mapsto f(x,y,\mu)$ is continuous on $B \times B$;*

(iii) *for any $x, y \in A(\mu_0)$, if $x \neq y$, then $f(x,y,\mu_0) + f(y,x,\mu_0) \in -\operatorname{int} C$.*

Then, for each $\xi \in C^* \setminus \{0\}$, $S_\xi(\cdot)$ is l.s.c. at μ_0.

Remark 10. In [9], they assumed the condition of C-strict monotonicity (or called C-strongly monotone in [6, 7]) at the considered point μ_0. In the case, the ξ-solution set may be a general set, but not a singleton. Unfortunately, that result of [9] cannot be used in the case of vector optimization problems. Theorem 8 discusses the lower semicontinuity of the ξ-solution mappings. Compared with Theorem 3.1 of [9], assumption (iii) of Theorem 8 is relaxed from assumption (iii) in Theorem 3.1 in [9]. An advantage Theorem 8 is that it works for vector optimization problems. However, in some situations Theorem 8 is applicable while Theorem 3.1 in [9] is not, as shown by the following example.

Example 11. Let $X = Z = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $M = [1,2]$ be a subset of Z. Let $\mu_0 = 1 \in M$ be a considered point for (PVEP). Let $A : M \to X$ be a mapping defined by $A(\mu) = [1,2]$ and let $f : X \times X \times M \to Y$ be a mapping defined by

$$f(x,y,\mu) = \left(x\left(\mu y^2 - x^2\right), \mu\left(y^2 - x^2\right)\right). \tag{21}$$

It is clear that f is monotone on $A(\mu_0)$, but not satisfied condition (iii) in Theorem 3.1 of [9]. Indeed, for each $x, y \in A(\mu_0) = [1,2]$, we have

$$\begin{aligned} &f(x,y,\mu_0) + f(y,x,\mu_0) \\ &\quad = \left(x\left(y^2 - x^2\right), y^2 - x^2\right) + \left(y\left(x^2 - y^2\right), x^2 - y^2\right) \\ &\quad = \left(-\left(x^3 - x^2y - xy^2 + y^3\right), 0\right). \end{aligned} \tag{22}$$

Also, $f(x,\cdot,\mu_0)$ satisfy C-strictly convex on $A(\mu_0)$. Indeed, for any $t \in (0,1)$ and $y_1, y_2 \in A(\mu_0)$, we have

$$\begin{aligned} & tf(x,y_1,\mu_0) + (1-t) f(x,y_2,\mu_0) \\ & \quad - f(x,ty_1+(1-t)y_2,\mu_0) \\ & \quad = t\left(x\left(y_1^2-x^2\right), y_1^2-x^2\right) \\ & \qquad + (1-t)\left(x\left(y_2^2-x^2\right), y_2^2-x^2\right) \\ & \qquad - \left(x\left((ty_1+(1-t)y_2)^2-x^2\right),\right. \\ & \qquad \left.(ty_1+(1-t)y_2)^2-x^2\right) \\ & \quad = \left(x\left(ty_1^2+(1-t)y_2^2\right) - x(ty_1+(1-t)y_2)^2,\right. \\ & \qquad \left. ty_1^2+(1-t)y_2^2-(ty_1+(1-t)y_2)^2\right) \in \operatorname{int} C. \end{aligned} \tag{23}$$

Let $\bar{\xi} = (1,0) \in C^* \setminus \{0\}$. We directly compute that $S_{\bar{\xi}}(\mu) = [1, \sqrt{\mu}]$, for each $\mu \in M$. Thus, we can easily get that $S_{\bar{\xi}}(\mu') = [1, \sqrt{\mu'}]$ is a general set-valued one for each $\mu' \in N(\mu_0) \cap M \setminus \{\mu_0\}$ (where $N(\mu_0)$ is any neighborhood of μ_0), but not a singleton. Moreover, by Theorem 8, we can get that $S_{\bar{\xi}}(\cdot)$ is l.s.c. at μ_0.

However, to relax the condition (iii) in [9], we add the condition of strict convexity of f. The following example illustrates that the strict convexity of f is needed.

Example 12. Let $X = Z = \mathbb{R}, Y = \mathbb{R}^2, C = \mathbb{R}^2_+, M = [0,1]$ be a subset of Z. Let $\mu_0 = 1 \in M$ be a considered point for (PVEP). Let $A : M \to X$ be a mapping defined by $A(\mu) = [0,1]$ and let $f : X \times X \times M \to Y$ be a mapping defined by

$$f(x,y,\mu) = (\mu x(y-x), \mu(y-x)). \tag{24}$$

It is clear that f is monotone on $A(\mu_0)$ and $f(x,\cdot,\mu_0)$ also does not satisfy C-strictly convex on $A(\mu_0)$. Let $\bar{\xi} = (1,0) \in C^* \setminus \{0\}$. It follows from direct computation that

$$S(\mu) = \begin{cases} [0,1], & \text{if } \mu = 0, \\ \{0\}, & \text{if } \mu \in (0,1]. \end{cases} \tag{25}$$

Clearly, we see that $S(\cdot)$ is not l.s.c. at μ_0. Hence, the assumed strict convexity of f is essential.

Theorem 13. *Let $\mu_0 \in M$ be a considered point. Suppose that the following conditions are satisfied:*

(i) *$A(\cdot)$ is continuous with nonempty compact convex values at μ_0;*

(ii) *$f(\cdot,\cdot,\cdot)$ is continuous on $B \times B \times M$;*

(iii) *$f(\cdot,\cdot,\mu_0)$ is monotone on $A(\mu_0)$;*

(iv) *for each $x \in A(\mu_0)$, $f(x,\cdot,\mu_0)$ is C-strictly convex on $A(\mu_0)$;*

(v) *for each $\mu \in M$ and $x \in A(\mu)$, $f(x,\cdot,\mu)$ is C-convexlike on $A(\mu)$.*

Then, $S(\cdot)$ is l.s.c. at μ_0.

Proof. Since, for each $\mu \in M$ and for each $x \in A(\mu)$, $f(x,\cdot,\mu)$ is C-convexlike on $A(\mu)$, $F(x, A(\mu), \mu) + C$ is convex. It follows from Lemma 6 that

$$S(\mu) = \bigcup_{\xi \in C^* \setminus \{0\}} S_\xi(\mu). \tag{26}$$

By Theorem 8, for each $\xi \in C^* \setminus \{0_{Y*}\}$, $S_\xi(\cdot)$ is l.s.c. at μ_0. Therefore, by Lemma 7 it implies that $S(\cdot)$ is l.s.c. at μ_0. This completes the proof. □

Now, we give an example to illustrate that our result improves that of [9].

Example 14. Let $X = Z = \mathbb{R}, Y = \mathbb{R}^2, C = \mathbb{R}^2_+, M = [1,2]$ be a subset of Z. Let $\mu_0 = 1 \in M$ be a considered point for (PVEP). Let $A : M \to X$ be a mapping defined by $A(\mu) = [\mu, 6]$ and let $f : X \times X \times M \to Y$ be a mapping defined by

$$f(x,y,\mu) = \left(x\left(\mu y^2 - x^2\right), \mu\left(y^2 - x^2\right)\right). \tag{27}$$

It is clear that f is monotone on $A(\mu_0)$, but not satisfied C-strict monotone on $A(\mu_0)$. Also, $f(x,\cdot,\mu_0)$ satisfy C-strictly convex on $A(\mu_0)$. It follows from direct computation that $S(\mu) = [\mu, \mu\sqrt{\mu}]$, for each $\mu \in M$. Thus, we can easily get that $S_{\bar{\xi}}(\mu') = [\mu', \mu'\sqrt{\mu'}]$ is a general set-valued one for each $\mu' \in N(\mu_0) \cap M \setminus \{\mu_0\}$ (where $N(\mu_0)$ is any neighborhood of μ_0), but not a singleton. Moreover, by Theorem 13, we can get that $S_{\bar{\xi}}(\cdot)$ is l.s.c. at μ_0.

Theorem 15. *Let $\mu_0 \in M$ be a considered point. Suppose that the following conditions are satisfied:*

(i) *$A(\cdot)$ is continuous with nonempty compact convex values at μ_0;*

(ii) *$f(\cdot,\cdot,\cdot)$ is continuous on $B \times B \times M$.*

Then, $S(\cdot)$ is u.s.c. at μ_0.

Proof. Suppose the contrary that $S(\cdot)$ is not upper semicontinuous at μ_0. Then, there exist an open neighborhood U of $S(\mu_0)$ and a net $\{\mu_\alpha : \alpha \in \Lambda\}$ converging to μ_0 such that

$$S(\mu_\alpha) \nsubseteq U, \quad \forall \alpha \in \Lambda. \tag{28}$$

Then there exists some $x_\alpha \in S(\mu_\alpha)$ such that

$$x_\alpha \notin U, \quad \forall \alpha \in \Lambda. \tag{29}$$

Since $x_\alpha \in S(\mu_\alpha)$, we have $x_\alpha \in A(\mu_\alpha)$. By the assumption, $A(\cdot)$ is u.s.c. with compact valued at μ_0, then we have that there exists subnet $\{x_{\alpha_\beta}\}$ such that $x_{\alpha_\beta} \to x^*$.

We will show that $x^* \in S(\mu_0)$; suppose the contrary that $x^* \notin S(\mu_0)$. Then there exists $y^* \in A(\mu_0)$ such that

$$f(x^*, y^*, \mu_0) \in -\operatorname{int} C. \tag{30}$$

Since $A(\cdot)$ is l.s.c. at μ_0 and $y^* \in A(\mu_0)$ and $\mu_\alpha \to \mu_0$, we have that there exists $y_\alpha \in A(\mu_\alpha)$ such that $y_\alpha \to y^*$. It follows from $y_\alpha \in A(\mu_\alpha)$ that

$$f(x_\alpha, y_\alpha, \mu_\alpha) \notin -\operatorname{int} C \quad \forall \alpha. \tag{31}$$

By (ii) it implies that $f(x^*, y^*, \mu_0) \notin -\operatorname{int} C$, which leads to a contradiction with (30). Hence, we have $x^* \in S(\mu_0) \subseteq U$.

Since $x_\alpha \to x^*$ and U is an open set, there exists some $\alpha_0 \in \Lambda$ such that

$$x_\alpha \in U, \quad \forall \alpha \geq \alpha_0, \tag{32}$$

which leads to contradiction with (29). Thus $S(\cdot)$ is u.s.c. at μ_0. □

The following theorem is directly obtained from Theorems 13 and 15.

Theorem 16. *Let $\mu_0 \in M$ be a considered point. Suppose that the following conditions are satisfied:*

(i) *$A(\cdot)$ is continuous with nonempty compact convex values at μ_0;*

(ii) *$f(\cdot,\cdot,\cdot)$ is continuous on $B \times B \times M$;*

(iii) *for each $x \in A(\mu_0)$, $f(x,\cdot,\mu_0)$ is C-strictly convex on $A(\mu_0)$;*

(iv) *for each $\mu \in M$ and $x \in A(\mu)$, $f(x,\cdot,\mu)$ is C-convexlike on $A(\mu)$;*

(v) *$f(\cdot,\cdot,\mu_0)$ is monotone on $A(\mu_0)$.*

Then, $S(\cdot)$ is continuous at μ_0.

4. Vector Optimization Problem

Since the parametric weak vector equilibrium problem (PWVEP) contains the parametric weak vector optimization problems, we can derive from Theorem 17 direct consequences. We denote the ordering induced by C as follows:

$$\begin{aligned} x \leq y \quad &\text{iff } y - x \in C; \\ x < y \quad &\text{iff } y - x \in \operatorname{int} C. \end{aligned} \tag{33}$$

The ordering $\geq$ and the ordering $>$ are defined similarly. Let $g : X \times M \to Y$ be a vector-valued mapping. For each $\mu \in M$, consider the problem of parametric weak optimization problem (PWVOP) finding $x_0 \in A(\mu)$ such that

$$g(y,\mu) - g(x_0,\mu) \notin -\operatorname{int} C, \quad \forall y \in A(\mu). \tag{34}$$

Setting $f(x,y,\mu) = g(y,\mu) - g(x,\mu)$, PWVEP becomes a special case of PWVOP.

For each $\mu \in M$, the efficient solution set of (34) is denoted by

$$\begin{aligned} S^{OP}(\mu) := \{x \in A(\mu) : g(y,\mu) \\ -g(x_0,\mu) \notin -\operatorname{int} C, \forall y \in A(\mu)\}. \end{aligned} \tag{35}$$

The ξ-efficient solution set of (34) is

$$\begin{aligned} S_\xi^{OP}(\mu) := \{x \in A(\mu) : \xi(g(y,\mu)) \\ \geq \xi(g(x_0,\mu)), \forall y \in A(\mu)\}. \end{aligned} \tag{36}$$

We directly obtain the following theorem from Theorem 16.

Theorem 17. *Let $\mu_0 \in M$ be a considered point. Suppose that the following conditions are satisfied:*

(i) *$A(\cdot)$ is continuous with nonempty compact convex values at μ_0;*

(ii) *$g(\cdot,\cdot)$ is continuous on $B \times M$;*

(iii) *for each $x \in A(\mu_0)$, $g(\cdot,\mu_0)$ is C-strictly convex on $A(\mu_0)$;*

(iv) *for each $\mu \in M$ and $x \in A(\mu)$, $g(\cdot,\mu)$ is C-convexlike on $A(\mu)$.*

Then, $S(\cdot)$ is continuous at μ_0.

The following example illustrates that the strict convexity cannot be dropped.

Example 18. Let $X = Z = \mathbb{R}$, $Y = \mathbb{R}$, $C = [0,+\infty)$, $M = [0,1]$ be a subset of Z. Let $\mu_0 = 0 \in M$ be a considered point for PWVOP. Let $A : M \to X$ be a mapping defined by $A(\mu) = [0,1]$ and let $g : X \times M \to \mathbb{R}$ be a mapping defined by

$$g(x,\mu) = (\mu x, \mu x). \tag{37}$$

It is clear that g does not satisfy C-strictly convex on $A(\mu_0)$. It follows from direct computation that

$$S^{OP}(\mu) = \begin{cases} [0,1], & \text{if } \mu = 0, \\ \{0\}, & \text{if } \mu \in (0,1]. \end{cases} \tag{38}$$

Clearly, we see that $S(\cdot)$ is not l.s.c. at μ_0. Hence, the assumed strict convexity of g is essential.

5. Conclusions

In this paper, we study the lower semicontinuity of the set of efficient solutions for parametric weak vector equilibrium problems when the ξ-efficient solution set is a general set. Moreover, our theorems can apply for vector optimization problems.

Conflict of Interests

The authors declare that they have no competing interests.

Authors' Contribution

All authors read and approved the final paper.

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Research Article

On Newton-Kantorovich Method for Solving the Nonlinear Operator Equation

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We develop the Newton-Kantorovich method to solve the system of 2×2 nonlinear Volterra integral equations where the unknown function is in logarithmic form. A new majorant function is introduced which leads to the increment of the convergence interval. The existence and uniqueness of approximate solution are proved and a numerical example is provided to show the validation of the method.

1. Introduction

Nonlinear phenomenon appears in many scientific areas such as physics, fluid mechanics, population models, chemical kinetics, economic systems, and medicine and can be modeled by system of nonlinear integral equations. The difficulty lies in finding the exact solution for such system. Alternatively, the approximate or numerical solutions can be sought. One of the well known approximate method is Newton-Kantorovich method which reduces the nonlinear into sequence of linear integral equations. The the approximate solution is then obtained by processing the convergent sequence. In 1939, Kantorovich [1] presented an iterative method for functional equation in Banach space and derived the convergence theorem for Newton method. In 1948, Kantorovich [2] proved a semilocal convergence theorem for Newton method in Banach space, later known as the Newton-Kantorovich method. Uko and Argyros [3] proved a weak Kantorovich-type theorem which gives the same conclusion under the weaker conditions. Shen and Li [4] have established the Kantorovich-type convergence criterion for inexact Newton methods, assuming that the first derivative of an operator satisfies the Lipschitz condition. Argyros [5] provided a sufficient condition for the semilocal convergence of Newton's method to a locally unique solution of a nonlinear operator equation. Saberi-Nadjafi and Heidari [6] introduced a combination of the Newton-Kantorovich and quadrature methods to solve the nonlinear integral equation of Urysohn type in the systematic procedure. Ezquerro et al. [7] studied the nonlinear integral equation of mixed Hammerstein type using Newton-Kantorovich method with majorant principle. Ezquerro et al. [8] provided the semilocal convergence of Newton method in Banach space under a modification of the classic conditions of Kantorovich. There are many methods of solving the system of nonlinear integral equations, for example, product integration method [9], Adomian method [10], RBF network method [11], biorthogonal system method [12], Chebyshev wavelets method [13], analytical method [14], reproducing kernel method [15], step method [16], and single term Wlash series [17]. In 2003, Boikov and Tynda [18] implemented the Newton-Kantorovich method to the following system:

$$\begin{gathered} x(t)-\int_{y(t)}^{t} h(t,\tau)\, g(\tau)\, x(\tau)\, d\tau=0, \\ \int_{y(t)}^{t} k(t,\tau)\left[1-g(\tau)\right] x(\tau)\, d\tau=f(t), \end{gathered} \tag{1}$$

where $0 < t_0 \le t \le T$, $y(t) < t$, and the functions $h(t,\tau)$, $k(t,\tau) \in C_{[t_0,T]\times[t_0,T]}$, $f(t)$, $g(t) \in C_{[t_0,T]}$, and $(0 < g(t) < 1)$. In 2010, Eshkuvatov et al. [19] used the Newton-Kantorovich hypothesis to solve the system of nonlinear Volterra integral equation of the form

$$\begin{gathered} x(t)-\int_{y(t)}^{t} h(t,\tau)\, x^2(\tau)\, d\tau=0, \\ \int_{y(t)}^{t} k(t,\tau)\, x^2(\tau)\, d\tau=f(t), \end{gathered} \tag{2}$$

where $x(t)$ and $y(t)$ are unknown functions defined on $[t_0,\infty)$, $t_0 > 0$, and $h(t,\tau), k(t,\tau) \in C_{[t_0,\infty]\times[t_0,\infty]}$, $f(t) \in C_{[t_0,\infty]}$. In 2010, Eshkuvatov et al. [20] developed the modified Newton-Kantorovich to obtain an approximate solution of system with the form

$$\begin{gathered} x(t)-\int_{y(t)}^{t} H(t,\tau)\, x^n(\tau)\, d\tau=0, \\ \int_{y(t)}^{t} K(t,\tau)\, x^n(\tau)\, d\tau=f(t), \end{gathered} \tag{3}$$

where $0 < t_0 \le t \le T$, $y(t) < t$, and the functions $H(t,\tau), K(t,\tau) \in C_{[t_0,\infty]\times[t_0,\infty]}$, $f(t) \in C_{[t_0,\infty]}$, and the unknown functions $x(t) \in C_{[t_0,\infty]}$, $y(t) \in C^1_{[t_0,\infty]}$, $y(t) < t$.

In this paper, we consider the systems of nonlinear integral equation of the form

$$\begin{gathered} x(t)-\int_{y(t)}^{t} h(t,\tau)\log|x(\tau)|\, d\tau=g(t), \\ \int_{y(t)}^{t} k(t,\tau)\log|x(\tau)|\, d\tau=f(t), \end{gathered} \tag{4}$$

where $0 < t_0 \le t \le T$, $y(t) < t$, $x(t) \neq 0$, $h(t,\tau), h_\tau(t,\tau)$, $k(t,\tau), k_\tau(t,\tau) \in C(D)$ and the unknown functions $x(t) \in C[t_0,T]$, $y(t) \in C^1[t_0,T]$ to be determined, and $D = [t_0,T] \times [t_0,T]$.

The paper is organized as follows, in Section 2, Newton-Kantorovich method for the system of integral equations (4) is presented. Section 3 deals with mixed method followed by discretizations. In Section 4, the rate of convergence of the method is investigated. Lastly, Section 5 demonstrates the numerical example to verify the validity and accuracy of the proposed method, followed by the conclusion in Section 6.

2. Newton-Kantorovich Method for the System

Let us rewrite the system of nonlinear Volterra integral equation (4) in the operator form

$$P(X)=\left(P_1(X), P_2(X)\right)=0, \tag{5}$$

where $X = (x(t), y(t))$ and

$$\begin{aligned} P_1(X) &= x(t)-\int_{y(t)}^{t} h(t,\tau)\log|x(\tau)|\, d\tau-g(t), \\ P_2(X) &= \int_{y(t)}^{t} k(t,\tau)\log|x(\tau)|\, d\tau-f(t). \end{aligned} \tag{6}$$

To solve (5) we use initial iteration of Newton-Kantorovich method which is of the form

$$P'(X_0)(X-X_0)+P(X_0)=0, \tag{7}$$

where $X_0 = (x_0(t), y_0(t))$ is the initial guess and $x_0(t)$ and $y_0(t)$ can be any continuous functions provided that $t_0 < y(t) < t$ and $x(t) \neq 0$.

The Frechet derivative of $P(X)$ at the point X_0 is defined as

$$\begin{aligned} &P'(X_0)\,X \\ &= \Big(\lim_{s\to 0}\frac{1}{s}\left[P_1(X_0+sX)-P_1(X)\right], \\ &\qquad \lim_{s\to 0}\frac{1}{s}\left[P_2(X_0+sX)-P_2(X)\right]\Big) \\ &= \Big(\lim_{s\to 0}\frac{1}{s}\left[P_1(x_0+sx, y_0+sy)-P_1(x_0,y_0)\right], \\ &\qquad \lim_{s\to 0}\frac{1}{s}\left[P_2(x_0+sx, y_0+sy)-P_2(x_0,y_0)\right]\Big) \\ &= \Big(\lim_{s\to 0}\Big[\frac{\partial P_1(x_0,y_0)}{\partial x}sx+\frac{\partial P_1(x_0,y_0)}{\partial y}sy \\ &\qquad +\frac{1}{2}\Big(\frac{\partial^2 P_1}{\partial x^2}(x_0+\theta sx, y_0+\delta sy)\, s^2x^2 \\ &\qquad +2\frac{\partial^2 P_1}{\partial x\partial y}(x_0+\theta sx, y_0+\delta sy)\, s^2xy \\ &\qquad +\frac{\partial^2 P_1}{\partial y^2}(x_0+\theta sx, y_0+\delta sy)\, sy^2\Big)\Big], \end{aligned}$$

$$\begin{aligned}
\lim_{s\to 0}\frac{1}{s}\Bigg[&\frac{\partial P_2}{\partial x}(x_0,y_0)\,sx+\frac{\partial P_2}{\partial y}(x_0,y_0)\,sy\\
&+\frac{1}{2}\Bigg(\frac{\partial^2 P_2}{\partial x^2}(x_0+\theta sx,y_0+\delta sy)\,s^2x^2\\
&\quad+2\frac{\partial^2 P_2}{\partial x\partial y}(x_0+\theta sx,y_0+\delta sy)\,s^2xy\\
&\quad+\frac{\partial^2 P_2}{\partial y^2}(x_0+\theta sx,y_0+\delta sy)\,sy^2\Bigg)\Bigg]\Bigg)\\
=\Bigg(&\frac{\partial P_1(x_0,y_0)}{\partial x}x+\frac{\partial P_1(x_0,y_0)}{\partial y}y,\\
&\frac{\partial P_2(x_0,y_0)}{\partial x}x+\frac{\partial P_2(x_0,y_0)}{\partial y}y\Bigg).
\end{aligned}\tag{8}$$

Hence,

$$P'(X_0)\,X=\begin{pmatrix}\left.\dfrac{\partial P_1}{\partial x}\right|_{(x_0,y_0)} & \left.\dfrac{\partial P_1}{\partial y}\right|_{(x_0,y_0)}\\ \left.\dfrac{\partial P_2}{\partial x}\right|_{(x_0,y_0)} & \left.\dfrac{\partial P_2}{\partial y}\right|_{(x_0,y_0)}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}.\tag{9}$$

From (7) and (9) it follows that

$$\begin{aligned}
&\left.\frac{\partial P_1}{\partial x}\right|_{(x_0,y_0)}(\Delta x(t))+\left.\frac{\partial P_1}{\partial y}\right|_{(x_0,y_0)}(\Delta y(t))\\
&\quad=-P_1(x_0(t),y_0(t)),\\
&\left.\frac{\partial P_2}{\partial x}\right|_{(x_0,y_0)}(\Delta x(t))+\left.\frac{\partial P_2}{\partial y}\right|_{(x_0,y_0)}(\Delta y(t))\\
&\quad=-P_2(x_0(t),y_0(t)),
\end{aligned}\tag{10}$$

where $\Delta x(t)=x_1(t)-x_0(t)$, $\Delta y(t)=y_1(t)-y_0(t)$, and $(x_0(t), y_0(t))$ is the initial given functions. To solve (10) with respect to Δx and Δy we need to compute all partial derivatives:

$$\begin{aligned}
\left.\frac{\partial P_1}{\partial x}\right|_{(x_0,y_0)}&=\lim_{s\to 0}\frac{1}{s}\left(P_1(x_0+sx,y_0)-P_1(x_0,y_0)\right)\\
&=\lim_{s\to 0}\frac{1}{s}\Bigg[sx(t)\\
&\qquad-\int_{y_0(t)}^{t}h(t,\tau)\left(\log\left|x_0(\tau)+sx(\tau)\right|\right.\\
&\qquad\left.-\log\left|x_0(\tau)\right|\right)d\tau\Bigg]\\
&=x(t)-\int_{y_0(t)}^{t}h(t,\tau)\frac{x(\tau)}{x_0(\tau)}d\tau,
\end{aligned}$$

$$\begin{aligned}
\left.\frac{\partial P_1}{\partial y}\right|_{(x_0,y_0)}&=\lim_{s\to 0}\frac{1}{s}\left(P_1(x_0,y_0+sy)-P_1(x_0,y_0)\right)\\
&=\lim_{s\to 0}\frac{1}{s}\left[\int_{y_0(t)}^{y_0(t)+sy(t)}h(t,\tau)\log\left|(x_0(\tau))\right|d\tau\right]\\
&=h(t,y_0(t))\log\left|x_0(y_0(t))\right|y(t),
\end{aligned}\tag{11}$$

and in the same manner we obtain

$$\begin{aligned}
\left.\frac{\partial P_2}{\partial x}\right|_{(x_0,y_0)}&=\int_{y_0(t)}^{t}k(t,\tau)\frac{x(\tau)}{x_0(\tau)}d\tau,\\
\left.\frac{\partial P_2}{\partial y}\right|_{(x_0,y_0)}&=-k(t,y_0(t))\log\left|x_0(y_0(t))\right|y(t).
\end{aligned}\tag{12}$$

So that from (10)–(12) it follows that

$$\begin{aligned}
&\Delta x(t)-\int_{y_0(t)}^{t}h(t,\tau)\frac{\Delta x(\tau)}{x_0(\tau)}d\tau\\
&\quad+h(t,y_0(t))\log\left|x_0(y_0(t))\right|\Delta y(t)\\
&\quad=\int_{y_0(t)}^{t}h(t,\tau)\log\left|x_0(\tau)\right|d\tau-x_0(t)+g(t),\\
&\int_{y_0(t)}^{t}k(t,\tau)\frac{\Delta x(\tau)}{x_0(\tau)}d\tau\\
&\quad-k(t,y_0(t))\log\left|x_0(y_0(t))\right|\Delta y(t)\\
&\quad=-\int_{y_0(t)}^{t}k(t,\tau)\log\left|x_0(\tau)\right|d\tau+f(t).
\end{aligned}\tag{13}$$

Equation (13) is a linear, and, by solving it for Δx and Δy, we obtain $(x_1(t), y_1(t))$. By continuing this process, a sequence of approximate solution $(x_m(t), y_m(t))$ can be evaluated from

$$P'(X_0)\,\Delta X_m+P(X_m)=0,\tag{14}$$

which is equivalent to the system

$$\begin{aligned}
&\Delta x_m(t)-\int_{y_0(t)}^{t}h(t,\tau)\frac{\Delta x_m(\tau)}{x_0(\tau)}d\tau\\
&\quad+h(t,y_0(t))\log\left|x_0(y_0(t))\right|\Delta y_m(t)\\
&\quad=\int_{y_0(t)}^{t}h(t,\tau)\log\left|x_0(\tau)\right|d\tau-x_0(t)+g(t),\\
&\int_{y_0(t)}^{t}k(t,\tau)\frac{\Delta x_m(\tau)}{x_0(\tau)}d\tau\\
&\quad-k(t,y_0(t))\log\left|x_0(y_0(t))\right|\Delta y_m(t)\\
&\quad=-\int_{y_0(t)}^{t}k(t,\tau)\log\left|x_0(\tau)\right|d\tau+f(t),
\end{aligned}\tag{15}$$

where $\Delta x_m(t)=x_m(t)-x_{m-1}(t)$ and $\Delta y_m(t)=y_m(t)-y_{m-1}(t)$, $m=1,2,3,\ldots$.

Thus, one should solve a system of two linear Volterra integral equations to find each successive approximation. Let us eliminate $\Delta y(t)$ from the system (13) by finding the expression of $\Delta y(t)$ from the first equation of this system and substitute it in the second equation to yield

$$\begin{aligned}\Delta y(t) &= \frac{1}{H(t)}\left[\int_{y_0(t)}^{t} h(t,\tau)\left[\frac{\Delta x(\tau)}{x_0(\tau)} + \log|x_0(\tau)|\right]d\tau \right. \\ &\qquad \left. - [\Delta x(t) + x_0(t) - g(t)]\right], \\ G(t)&\left[\int_{y_0(t)}^{t} h(t,\tau)\left[\frac{\Delta x(\tau)}{x_0(\tau)} + \log|x_0(\tau)|\right]d\tau \right. \\ &\qquad \left. - [\Delta x(t) + x_0(t) - g(t)]\right] \\ &= \int_{y_0(t)}^{t} k(t,\tau)\frac{\Delta x(\tau)}{x_0(\tau)}d\tau \\ &\quad - \int_{y_0(t)}^{t} k(t,\tau)\log|x_0(\tau)|\,d\tau + f(t),\end{aligned} \tag{16}$$

where $G(t) = k(t, y_0(t))/h(t, y_0(t))$ and $H(t) = 1/[h(t, y_0(t))\log|x_0(y_0(t))|]$, and the second equation of (16) yields

$$\Delta x(t) - \int_{y_0(t)}^{t} k_1(t,\tau)\frac{\Delta x(\tau)}{x_0(\tau)}\tau = F_0(t), \tag{17}$$

where

$$\begin{aligned}k_1(t,\tau) &= h(t,\tau) - \frac{k(t,\tau)}{G(t)}, \\ G(t) &= \frac{k(t,y_0(t))}{h(t,y_0(t))}, \quad k(t,y_0(t)) \neq 0\ \forall t \in [t_0, T], \\ F_0(t) &= \int_{y_0(t)}^{t} k_1(t,\tau)\log|x_0(\tau)|\,d\tau - x_0(t) + g(t) + \frac{f(t)}{G(t)}.\end{aligned} \tag{18}$$

In an analogous way, $\Delta y_m(t)$ and $\Delta x_m(t)$ can be written in the form

$$\begin{aligned}\Delta y_m(t) &= \frac{1}{H(t)}\left[\int_{y_0(t)}^{t} h(t,\tau)\frac{\Delta x_m(\tau)}{x_0(\tau)}d\tau \right. \\ &\qquad + \int_{y_{m-1}(t)}^{t} h(t,\tau)\log|x_{m-1}(\tau)|\,d\tau \\ &\qquad \left. - \Delta x_m(t) - x_{m-1}(t) + g(t)\right],\end{aligned} \tag{19}$$

$$\Delta x_m(t) - \int_{y_0(t)}^{t} k_1(t,\tau)\frac{\Delta x_m(\tau)}{x_0(\tau)}d\tau = F_{m-1}(t), \tag{20}$$

where

$$\begin{aligned}F_{m-1}(t) &= \int_{y_{m-1}(t)}^{t} k_1(t,\tau)\log|x_{m-1}(\tau)|\,d\tau - x_{m-1}(t) \\ &\quad + g(t) + \frac{f(t)}{G(t)}.\end{aligned} \tag{21}$$

3. The Mixed Method (Simpson and Trapezoidal) for Approximate Solution

At each step of the iterative process we have to find the solution of (18) and (20) on the closed interval $[t_0, T]$. To do this the grid (ω) of points $t_i = t_0 + ih$, $i = 1, 2, 3, \ldots, 2N$, $h = (T - t_0)/2N$ is introduced, and by the collocation method with mixed rule we require that the approximate solution satisfies (18) and (20). Hence

$$\Delta x_m(t_0) = -x_{m-1}(t_0) + g(t_0) + \frac{f(t_0)}{G(t_0)}, \tag{22}$$

$$\begin{aligned}\Delta x_m(t_{2i}) &- \int_{y_0(t_{2i})}^{t_{2i}} k_1(t_{2i},\tau)\frac{\Delta x_m(\tau)}{x_0(\tau)}d\tau \\ &= F_{m-1}(t_{2i}), \quad i = 1, 2, \ldots, N.\end{aligned} \tag{23}$$

On the grid (ω) we set $v_{2i} = y_0(t_{2i})$, suct that

$$t_{v_{2i}} = \begin{cases} t_{v_{2i}}, & t_0 \leq y_0(t_{2i}) < t_{2i-2}, \\ t_{2i}, & t_{2i-2} \leq y_0(t_{2i}) < t_{2i}. \end{cases} \tag{24}$$

Consequently, the system (23) can be written in the form

$$\begin{aligned}\Delta x_m(t_{2i}) &- \int_{y_0(t_{2i})}^{t_{v_{2i}}} k_1(t_{2i},\tau)\frac{\Delta x_m(\tau)}{x_0(\tau)}d\tau \\ &- \sum_{j=v_{2i}}^{i-1}\int_{t_{2j}}^{t_{2j+2}} k_1(t_{2i},\tau)\frac{\Delta x_m(\tau)}{x_0(\tau)}d\tau \\ &= F_{m-1}(t_{2i}), \quad i = 1, 2, \ldots, N.\end{aligned} \tag{25}$$

By computing the integral in (26) using tapezoidal formula on the first integrals and Simpson formula on the second integral, we consider two cases.

Case 1. When $v_{2i} \neq 2i$, $i = 1, 2, \ldots, N$, then

$$\Delta x_m(t_{2i}) = \frac{F_{m-1}(t_{2i}) + A(i) + B(i) + C(i)}{1 - ((t_{2i} - t_{2i-2})/6 \cdot x_0(t_{2i}))\,k_1(t_{2i}, t_{2i})}, \tag{26}$$

where

$$
\begin{aligned}
A(i) ={}& 0.5\left(t_{v_{2i}} - y_0(t_{2i})\right) \\
&\times\left[k_1\left(t_{2i}, t_{v_{2i}}\right)\frac{\Delta x_m\left(t_{v_{2i}}\right)}{x_0\left(t_{v_{2i}}\right)} + k_1\left(t_{2i}, y_0(t_{2i})\right)\right. \\
&\quad\times\frac{\Delta x_m\left(t_{v_{2i}}\right)\left(t_{v_{2i}} - y_0(t_{2i})\right)}{\left(t_{v_{2i}} - t_{v_{2i-2}}\right)\left(x_0\left(y_0(t_{2i})\right)\right)} \\
&\quad + k_1\left(t_{2i}, y_0(t_{2i})\right) \\
&\quad\left.\times\frac{\Delta x_m\left(t_{v_{2i-2}}\right)\left(y_0(t_{2i}) - t_{v_{2i-2}}\right)}{\left(t_{v_{2i}} - t_{v_{2i-2}}\right)\left(x_0\left(y_0(t_{2i})\right)\right)}\right], \\
B(i) ={}& \sum_{j=v_{2i}}^{i-2}\frac{\left(t_{2j+2} - t_{2j}\right)}{6} \\
&\times\left[k_1\left(t_{2i}, t_{2j}\right)\frac{\Delta x_m\left(t_{2j}\right)}{x_0\left(t_{2j}\right)}\right. \\
&\quad + 4k_1\left(t_{2i}, t_{2j+1}\right)\frac{\Delta x_m\left(t_{2j+1}\right)}{x_0\left(t_{2j+1}\right)} \\
&\quad\left. + k_1\left(t_{2i}, t_{2j+2}\right)\frac{\Delta x_m\left(t_{2j+2}\right)}{x_0\left(t_{2j+2}\right)}\right], \\
C(i) ={}& \frac{\left(t_{2i} - t_{2i-2}\right)}{6}\left[k_1\left(t_{2i}, t_{2i-2}\right)\frac{\Delta x_m\left(t_{2i-2}\right)}{x_0\left(t_{2i-2}\right)}\right. \\
&\qquad\left. + 4k_1\left(t_{2i}, t_{2i-1}\right)\frac{\Delta x_m\left(t_{2i-1}\right)}{x_0\left(t_{2i-1}\right)}\right].
\end{aligned}
\tag{27}
$$

Case 2. When $v_{2i} = 2i$, $i = 1, 2, \ldots, N$, then

$$
\Delta x_m\left(t_{2i}\right) = \frac{D_1(i)}{D_2(i)}, \tag{28}
$$

where

$$
\begin{aligned}
D_1(i) ={}& F_{m-1}\left(t_{2i}\right) + 0.5k_1\left(t_{2i}, y_0(t_{2i})\right) \\
&\times\left[\frac{\Delta x_m\left(t_{2i-2}\right)}{x_0\left(y_0(t_{2i})\right)}\frac{\left(t_{2i} - y_0(t_{2i})\right)\left(y_0(t_{2i}) - t_{2i-2}\right)}{t_{2i} - t_{2i-2}}\right], \\
D_2(i) ={}& \left[1 - 0.5\left(t_{2i} - y_0(t_2)\right)\frac{k_1\left(t_{2i}, t_{2i}\right)}{x_0\left(t_{2i}\right)}\right. \\
&\quad\left. - 0.5k_1\left(t_{2i}, y_0(t_{2i})\right)\frac{\left(t_{2i} - y_0(t_{2i})\right)^2}{x_0\left(y_0(t_{2i})\right)\left(t_{2i} - t_{2i-2}\right)}\right].
\end{aligned}
\tag{29}
$$

Also, to compute $\Delta y_m(t)$ on the grid (ω), (18) can be represented in the form

$$
\begin{aligned}
\Delta y_m\left(t_{2i}\right) ={}& \frac{1}{H\left(t_{2i}\right)} \\
&\times\left[\int_{y_0(t_{2i})}^{t_{2i}} h\left(t_{2i}, \tau\right)\frac{\Delta x_m(\tau)}{x_0(\tau)}d\tau\right. \\
&\quad + \int_{y_{m-1}(t_{2i})}^{t_{2i}} h\left(t_{2i}, \tau\right)\log\left|x_{m-1}(\tau)\right| d\tau \\
&\quad\left. - \Delta x_m\left(t_{2i}\right) - x_{m-1}\left(t_{2i}\right) + g\left(t_{2i}\right)\right].
\end{aligned}
\tag{30}
$$

Let us set $v_{2i} = y_0(t_{2i})$ and $u_{2i} = y_{m-1}(t_{2i})$ and

$$
\begin{aligned}
t_{v_{2i}} &= \begin{cases} t_{2i}, & t_{2i-2} \le y_0\left(t_{2i}\right) < t_{2i}, \\ t_{v_{2i}}, & t_0 \le y_0\left(t_{2i}\right) < t_{2i-2}, \end{cases} \\
t_{u_{2i}} &= \begin{cases} t_{2i}, & t_{2i-2} \le y_{m-1}\left(t_{2i}\right) < t_{2i}, \\ t_{u_{2i}}, & t_0 \le y_{m-1}\left(t_{2i}\right) < t_{2i-2}. \end{cases}
\end{aligned}
\tag{31}
$$

Then (30) can be written as

$$
\begin{aligned}
\Delta y_m\left(t_{2i}\right) ={}& \frac{1}{H\left(t_{2i}\right)} \\
&\times\left[\int_{y_0(t_{2i})}^{t_{v_{2i}}} h\left(t_{2i}, \tau\right)\frac{\Delta x_m(\tau)}{x_0(\tau)}d\tau\right. \\
&\quad + \sum_{j=v_{2i}}^{i-1}\int_{t_{2j}}^{t_{2j+2}} h\left(t_{2i}, \tau\right)\frac{\Delta x_m(\tau)}{x_0(\tau)}d\tau \\
&\quad + \int_{y_{m-1}(t_{2i})}^{t_{u_{2i}}} h\left(t_{2i}, \tau\right)\log\left|x_{m-1}(\tau)\right| d\tau \\
&\quad + \sum_{j=u_{2i}}^{i-1}\int_{t_{2j}}^{t_{2j+2}} h\left(t_{2i}, \tau\right)\log\left|x_{m-1}(\tau)\right| d\tau \\
&\quad\left. - \Delta x_m\left(t_{2i}\right) - x_{m-1}\left(t_{2i}\right) + g\left(t_{2i}\right)\right],
\end{aligned}
\tag{32}
$$

and by applying mixed formula for (32) we obtain the following four cases.

Case 1. When $v_{2i} \neq 2i$ and $u_{2i} \neq 2i$, we have

$$\begin{aligned}
&\Delta y_m(t_{2i}) \\
&= \frac{1}{H(t_{2i})} \\
&\quad \times \Bigg[0.5\left(t_{v_{2i}} - y_0(t_{2i})\right) \\
&\qquad \times \left(h\left(t_{2i}, t_{v_{2i}}\right) \frac{\Delta x_m\left(t_{v_{2i}}\right)}{x_0\left(t_{v_{2i}}\right)} \right. \\
&\qquad\qquad \left. + h(t_{2i}, y_0(t_{2i})) \frac{\Delta x_m(y_0(t_{2i}))}{x_0(y_0(t_{2i}))} \right) \\
&\qquad + \sum_{j=v_{2i}}^{i-1} \frac{\left(t_{2j+2} - t_{2j}\right)}{6} \\
&\qquad\quad \times \left(h\left(t_{2i}, t_{2j}\right) \frac{\Delta x_m\left(t_{2j}\right)}{x_0\left(t_{2j}\right)} \right. \\
&\qquad\qquad + 4h\left(t_{2i}, t_{2j+1}\right) \frac{\Delta x_m\left(t_{2j+1}\right)}{x_0\left(t_{2j+1}\right)} \\
&\qquad\qquad \left. + h\left(t_{2i}, t_{2j+2}\right) \frac{\Delta x_m\left(t_{2j+2}\right)}{x_0\left(t_{2j+2}\right)} \right) \\
&\qquad + 0.5\left(t_{u_{2i}} - y_{m-1}(t_{2i})\right) \\
&\qquad \times \left(h\left(t_{2i}, t_{u_{2i}}\right) \log\left|x_{m-1}\left(t_{u_{2i}}\right)\right| \right. \\
&\qquad\qquad \left. + h(t_{2i}, y_{m-1}(t_{2i})) \log\left|(x_{m-1}(y_{m-1}(t_{2i})))\right| \right) \\
&\qquad + \sum_{j=u_{2i}}^{i-1} \frac{\left(t_{2j+2} - t_{2j}\right)}{6} \\
&\qquad\quad \times \left(h\left(t_{2i}, t_{2j}\right) \log\left(x_{m-1}\left(t_{2j}\right)\right) \right. \\
&\qquad\qquad + 4h\left(t_{2i}, t_{2j+1}\right) \log\left|x_{m-1}\left(t_{2j+1}\right)\right| \\
&\qquad\qquad \left. + h\left(t_{2i}, t_{2j+2}\right) \log\left|x_{m-1}\left(t_{2j+2}\right)\right| \right) \\
&\qquad - \Delta x_m(t_{2i}) - x_{m-1}(t_{2i}) + g(t_{2i}) \Bigg].
\end{aligned} \tag{33}$$

Case 2. If $v_{2i} = 2i$ and $u_{2i} \neq 2i$, then

$$\begin{aligned}
&\Delta y_m(t_{2i}) \\
&= \frac{1}{H(t_{2i})} \\
&\quad \times \Bigg[0.5(t_{2i} - y_0(t_{2i})) \\
&\qquad \times \left(h(t_{2i}, t_{2i}) \frac{\Delta x_m\left(t_{v_{2i}}\right)}{x_0(t_{2i})} \right. \\
&\qquad\qquad \left. + h(t_{2i}, y_0(t_{2i})) \frac{\Delta x_m(y_0(t_{2i}))}{x_0(y_0(t_{2i}))} \right) \\
&\qquad + 0.5\left(t_{u_{2i}} - y_{m-1}(t_{2i})\right) \\
&\qquad \times \left(h\left(t_{2i}, t_{u_{2i}}\right) \log\left|x_{m-1}\left(t_{u_{2i}}\right)\right| \right. \\
&\qquad\qquad \left. + h(t_{2i}, y_{m-1}(t_{2i})) \log\left|x_{m-1}(y_{m-1}(t_{2i}))\right| \right) \\
&\qquad + \sum_{j=u_{2i}}^{i-1} \frac{\left(t_{2j+2} - t_{2j}\right)}{6} \\
&\qquad\quad \times \left(h\left(t_{2i}, t_{2j}\right) \log\left|x_{m-1}\left(t_{2j}\right)\right| \right. \\
&\qquad\qquad + 4h\left(t_{2i}, t_{2j+1}\right) \log\left|x_{m-1}\left(t_{2j+1}\right)\right| \\
&\qquad\qquad \left. + h\left(t_{2i}, t_{2j+2}\right) \log\left|x_{m-1}\left(t_{2j+2}\right)\right| \right) \\
&\qquad - \Delta x_m(t_{2i}) - x_{m-1}(t_{2i}) + g(t_{2i}) \Bigg].
\end{aligned} \tag{34}$$

Case 3. When $v_{2i} \neq 2i$ and $u_{2i} = 2i$, we get

$$\begin{aligned}
&\Delta y_m(t_{2i}) \\
&= \frac{1}{H(t_{2i})} \\
&\quad \times \Bigg[0.5\left(t_{v_{2i}} - y_0(t_{2i})\right) \\
&\qquad \times \left(h\left(t_{2i}, t_{v_{2i}}\right) \frac{\Delta x_m\left(t_{v_{2i}}\right)}{x_0\left(t_{v_{2i}}\right)} \right. \\
&\qquad\qquad \left. + h(t_{2i}, y_0(t_{2i})) \frac{\Delta x_m(y_0(t_{2i}))}{x_0(y_0(t_{2i}))} \right)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=v_{2i}}^{i-1}\frac{\left(t_{2j+2}-t_{2j}\right)}{6}\\
&\quad\times\left(h\left(t_{2i},t_{2j}\right)\frac{\Delta x_m\left(t_{2j}\right)}{x_0\left(t_{2j}\right)}\right.\\
&\qquad+4h\left(t_{2i},t_{2j+1}\right)\frac{\Delta x_m\left(t_{2j+1}\right)}{x_0\left(t_{2j+1}\right)}\\
&\qquad\left.+h\left(t_{2i},t_{2j+2}\right)\frac{\Delta x_m\left(t_{2j+2}\right)}{x_0\left(t_{2j+2}\right)}\right)\\
&+0.5\left(t_{2i}-y_{m-1}\left(t_{2i}\right)\right)\\
&\times\left(h\left(t_{2i},t_{2i}\right)\log\left|x_{m-1}\left(t_{2i}\right)\right|\right.\\
&\qquad\left.+h\left(t_{2i},y_{m-1}\left(t_{2i}\right)\right)\log\left|x_{m-1}\left(y_{m-1}\left(t_{2i}\right)\right)\right|\right)\\
&\left.-\Delta x_m\left(t_{2i}\right)-x_{m-1}\left(t_{2i}\right)+g\left(t_{2i}\right)\right].
\end{aligned}\tag{35}
$$

Case 4. If $v_{2i}=2i$ and $u_{2i}=2i$, then

$$
\begin{aligned}
&\Delta y_m\left(t_{2i}\right)\\
&=\frac{1}{H\left(t_{2i}\right)}\\
&\times\left[0.5\left(t_{2i}-y_0\left(t_{2i}\right)\right)\right.\\
&\quad\times\left(h\left(t_{2i},t_{2i}\right)\frac{\Delta x_m\left(t_{2i}\right)}{x_0\left(t_{2i}\right)}\right.\\
&\qquad\left.+h\left(t_{2i},y_0\left(t_{2i}\right)\right)\frac{\Delta x_m\left(y_0\left(t_{2i}\right)\right)}{x_0\left(y_0\left(t_{2i}\right)\right)}\right)\\
&\quad+0.5\left(t_{2i}-y_{m-1}\left(t_{2i}\right)\right)\\
&\quad\times\left(h\left(t_{2i},t_{2i}\right)\log\left|x_{m-1}\left(t_{2i}\right)\right|\right.\\
&\qquad\left.+h\left(t_{2i},y_{m-1}\left(t_{2i}\right)\right)\log\left|x_{m-1}\left(y_{m-1}\left(t_{2i}\right)\right)\right|\right)\\
&\quad\left.-\Delta x_m\left(t_{2i}\right)-x_{m-1}\left(t_{2i}\right)+g\left(t_{2i}\right)\right].
\end{aligned}\tag{36}
$$

Thus, (32) can be computed by one of (33)–(36) according to the cases.

4. The Convergence Analysis of the Method

On the basis of general theorems of Newton-Kantorovich method [21, Chapter XVIII] for the convergence, we state the following theorem regarding the successive approximations described by (18)–(20).

First, consider the following classes of functions:

(i) $C_{[t_0,T]}$ the set of all continuous functions $f(t)$ defined on the interval $[t_0,T]$,

(ii) $C_{[t_0,T]\times[t_0,T]}$ the set of all continuous functions $\psi(t,\tau)$ defined on the region $[t_0,T]\times[t_0,T]$,

(iii) $\overline{C}=\{X : X=(x(t),y(t)) : x(t),y(t)\in C_{[t_0,T]}\}$,

(iv) $C^{<}_{[t_0,T]}=\{y(t)\in C^1_{[t_0,T]} : y(t)<t\}$.

And define the following norms

$$
\begin{gathered}
\|x\|=\max_{t\in[t_0,T]}|x(t)|,\\
\|\Delta X\|_{\overline{C}}=\max\left\{\|\Delta x\|_{C_{[t_0,T]}},\|\Delta y\|_{C_{[t_0,T]}}\right\},\\
\|X\|_{C^1}=\max\left\{\|x\|_{C_{[t_0,T]}},\left\|x'\right\|_{C_{[t_0,T]}}\right\},\\
\left\|\overline{X}\right\|_{\overline{C}}=\max\left\{\|\overline{x}\|_{C_{[t_0,T]}},\|\overline{y}\|_{C_{[t_0,T]}}\right\}\\
\|h(t,\tau)\|=H_1,\qquad\left\|h'_\tau(t,\tau)\right\|=H'_1,\\
\|k(t,\tau)\|=H_2,\qquad\left\|k'_\tau(t,\tau)\right\|=H'_2,\\
\left\|\frac{1}{x_0}\right\|=\max_{t\in[t_0,T]}\left|\frac{1}{x_0(t)}\right|=c_1,\\
\left\|\frac{1}{x_0^2}\right\|=\max_{t\in[t_0,T]}\left|\frac{1}{x_0^2(t)}\right|=c_2,\\
\left\|\frac{1}{G(t)}\right\|=\max_{t\in[t_0,T]}\left|\frac{1}{G(t)}\right|=c_3,\\
\|x_0\|=\max_{t\in[t_0,T]}\left|x_0(t)\right|=H_3,\\
\left\|x'_0\right\|=\max_{t\in[t_0,T]}\left|x'_0(t)\right|=H'_3,\\
\min_{t\in[t_0,T]}\left|y_0(t)\right|=H_4,\\
\|\log\|=\max_{t\in[t_0,T]}\left|\log(x(t))\right|=H_5,\\
\|g\|=\max_{t\in[t_0,T]}\left|g(t)\right|=H_6,\\
\|f\|=\max_{t\in[t_0,T]}\left|f(t)\right|=H_7.
\end{gathered}\tag{37}
$$

Let

$$
\begin{aligned}
\eta_1=\max\Big\{&H_1c_2\left(T-H_4\right),H_1c_1,H'_1H_5+H_1H'_3c_1,\\
&H_2c_2\left(T-H_4\right),H_2c_1,H'_2H_5+H_2H'_3c_1\Big\}.
\end{aligned}\tag{38}
$$

Let us consider real valued function

$$\psi(t) = K(t - t_0)^2 - (1 + K\eta)(t - t_0) + \eta, \tag{39}$$

where $K > 0$ and η are nonnegative real coefficients.

Theorem 1. *Assume that the operator $P(X) = 0$ in (5) is defined in $\Omega = \{X \in C([t_0, T]) : \|X - X_0\| \leq R\}$ and has continuous second derivative in closed ball $\Omega_0 = \{X \in C([t_0, T]) : \|X - X_0\| \leq r\}$ where $T = t_0 + r \leq t_0 + R$. Suppose the following conditions are satisfied:*

(1) $\|\Gamma_0 P(X_0)\| \leq \eta/(1 + K\eta)$,

(2) $\|\Gamma_0 P''(X)\| \leq 2K/(1+K\eta)$, *when* $\|X - X_0\| \leq t - t_0 \leq r$,

where K and η as in (39). Then the function $\psi(t)$ defined by (39) majorizes the operator $P(X)$.

Proof. Let us rewrite (5) and (39) in the form

$$t = \phi(t), \quad \phi(t) = t + c_0\psi(t), \tag{40}$$

$$X = S(X), \quad S(X) = X - \Gamma_0 P(X), \tag{41}$$

where $c_0 = -1/\psi'(t_0) = 1/(1 + K\eta)$ and $\Gamma_0 = [P'(X_0)]^{-1}$.

Let us show that (40) and (41) satisfy the majorizing conditions [21, Theorem 1, page 525]. In fact

$$\|S(X_0) - X_0\| = \|-\Gamma_0 P(X_0)\| \leq \frac{\eta}{1 + K\eta} = \phi(t_0) - t_0, \tag{42}$$

and for the $\|X - X_0\| \leq t - t_0$ with the Remark in [21, Remark 1, page 504] we have

$$\begin{aligned}
\|S'(X)\| &= \|S'(X) - S'(X_0)\| \\
&\leq \int_{X_0}^{X} \|S''(X)\| dX = \int_{X_0}^{X} \|\Gamma_0 P''(X)\| dX \\
&\leq \int_{t_0}^{t} c_0\psi''(\tau) d\tau = \int_{t_0}^{t} \frac{2K}{1 + K\eta} d\tau \\
&= \frac{2K}{1 + K\eta}(t - t_0) = \phi'(t).
\end{aligned} \tag{43}$$

Hence $\psi(t) = 0$ is a majorant function of $P(X) = 0$. □

Theorem 2. *Let the functions $f(t), g(t) \in C_{[t_0,T]}$, $x_0(t) \in C^1[t_0, T]$, $x_0(y_0(t)) \neq 0$, $x_0^2(t) \neq 0$, and the kernels $h(t, \tau), k(t, \tau) \in C^1_{[t_0,T]\times[t_0,T]}$ and $(x_0(t), y_0(t)) \in \Omega_0$; then*

(1) *the system (7) has unique solution in the interval $[t_0, T]$; that is, there exists Γ_0, and* $\|\Gamma_0\| \leq \sum_{j=1}^{\infty} (c_1 H_1 + c_1 c_3 H_2)^j ((T - H_4)^{j-1}/(j - 1)!) = \eta_2$,

(2) $\|\Delta X\| \leq \eta/(1 + K\eta)$,

(3) $\|P''(X)\| \leq \eta_1$,

(4) $\eta > 1/K$ *and* $r < \eta + t_0$,

where K and η as in (39). Then the system (4) has unique solution X^* in the closed ball Ω_0 and the sequence $X_m(t) = (x_m(t), y_m(t))$, $m \geq 0$ of successive approximations

$$\begin{aligned}
\Delta y_m(t) = \frac{1}{H(t)} \Bigg[&\int_{y_0(t)}^{t} h(t, \tau) \frac{\Delta x_m(\tau)}{x_0(\tau)} d\tau \\
&+ \int_{y_{m-1}(t)}^{t} h(t, \tau) \log|x_{m-1}(\tau)| d\tau \\
&- \Delta x_m(t) - x_{m-1}(t) + g(t) \Bigg],
\end{aligned} \tag{44}$$

$$\Delta x_m(t) - \int_{y_0(t)}^{t} k_1(t, \tau) \frac{\Delta x_m(\tau)}{x_0(\tau)} d\tau = F_{m-1}(t),$$

where $\Delta x_m(t) = x_m(t) - x_{m-1}(t)$ and $\Delta y_m(t) = y_m(t) - y_{m-1}(t)$, $m = 2, 3, \ldots$, and X_m converge to the solution X^*. The rate of convergence is given by

$$\|X^* - X_m\| \leq \left(\frac{2}{1 + K\eta}\right)^m \left(\frac{1}{K}\right). \tag{45}$$

Proof. It is shown that (7) is reduced to (17). Since (17) is a linear Volterra integral equation of 2nd kind with respect to $\Delta x(t)$ and since $k(t, y_0(t)) \neq 0$, $\forall t \in [t_0, T]$ which implies that the kernel $k_1(t, \tau)$ defined by (18) is continues it follows that (17) has a unique solution which can be obtained by the method of successive approximations. Then the function $\Delta y(t)$ is uniquely determined from (16). Hence the existence of Γ_0 is archived.

To verify that Γ_0 is bounded we need to establish the resolvent kernel $\Gamma_0(t, \tau)$ of (17), so we assume the integral operator U from $C[t_0, T] \to C[t_0, T]$ is given by

$$Z = U(\Delta x), \quad Z(t) = \int_{y_0(t)}^{t} k_2(t, \tau) \Delta x(\tau) d\tau, \tag{46}$$

where $k_2(t, \tau) = k_1(t, \tau)/x_0(\tau)$, and $k_1(t, \tau)$ is defined in (18).

Due to (46), (17) can be written as

$$\Delta x - U(\Delta x) = F_0. \tag{47}$$

The solution Δx^* of (47) is expressed in terms of F_0 by means of the formula

$$\Delta x^* = F_0 + B(F_0), \tag{48}$$

where B is an integral operator and can be expanded as a series in powers of U [21, Theorem 1, page 378]:

$$B(F_0) = U(F_0) + U^2(F_0) + \cdots + U^n(F_0) + \cdots, \tag{49}$$

and it is known that the powers of U are also integral operators. In fact

$$Z_n = U^n, \quad Z_n(t) = \int_{y_0(t)}^{t} k_2^{(n)}(t, \tau) \Delta x(\tau) d\tau, \quad (n = 1, 2, \ldots), \tag{50}$$

where $k_2^{(n)}$ is the iterated kernel.

Substituting (50) into (48) we obtain an expression for the solution of (47):

$$\Delta x^* = F_0(t) + \sum_{j=1}^{\infty} \int_{y_0(t)}^{t} k_2^{(j)}(t,\tau) F_0(\tau)\, d\tau. \tag{51}$$

Next, we show that the series in (51) is convergent uniformly for all $t \in [t_0, T]$. Since

$$\begin{aligned} |k_2(t,\tau)| &= \left|\frac{k_1(t,\tau)}{x_0(\tau)}\right| \\ &\le \left|\frac{h(t,\tau)}{x_0(\tau)}\right| + \left|\frac{k(t,\tau)}{x_0(\tau) G(t)}\right| \le c_1 H_1 + c_1 c_3 H_2. \end{aligned} \tag{52}$$

Let $M = c_1 H_1 + c_1 c_3 H_2$; then by mathematical induction we get

$$\begin{aligned} \left|k_2^{(2)}(t,\tau)\right| &\le \int_{y_0(t)}^{t} \left|k_2(t,u) k_2(u,\tau)\right| du \le \frac{M^2 (t - H_4)}{(1)!}, \\ \left|k_2^{(3)}(t,\tau)\right| &\le \int_{y_0(t)}^{t} \left|k_2(t,u) k_2^{(2)}(u,\tau)\right| du \le \frac{M^3 (t - H_4)^2}{(2)!}, \\ &\vdots \\ \left|k_2^{(n)}(t,\tau)\right| &\le \int_{y_0(t)}^{t} \left|k_2(t,u) k_2^{(n-1)}(u,\tau)\right| du \\ &\le \frac{M^n (t - H_4)^{n-1}}{(n-1)!}, \\ & \qquad (n = 1, 2, \ldots); \end{aligned} \tag{53}$$

then

$$\|U^n\| = \max_{t \in [t_0, T]} \int_{y_0(t)}^{t} \left|k_2^{(n)}(t,\tau)\right| d\tau \le \frac{M^n (T - H_4)^{(n-1)}}{(n-1)!}. \tag{54}$$

Therefore the nth root test of the sequence yields

$$\sqrt[n]{\|U^n\|} \le \frac{M (T - H_4)^{1-1/n}}{\sqrt[n]{(n-1)!}} \xrightarrow[n \to \infty]{} 0. \tag{55}$$

Hence $\rho = 1/\lim_{n \to \infty} \sqrt[n]{\|U^n\|} = \infty$ and a Volterra integral equations (17) has no characteristic values. Since the series in (51) converges uniformly (48) can be written in terms of resolvent kernel of (17):

$$\Delta x^* = F_0 + \int_{y_0(t)}^{t} \Gamma_0(t,\tau) F_0(\tau)\, d\tau, \tag{56}$$

where

$$\Gamma_0(t,\tau) = \sum_{j=1}^{\infty} k_2^{(j)}(t,\tau). \tag{57}$$

Since the series in (57) is convergent we obtain

$$\|\Gamma_0\| = \|B(F_0)\| \le \sum_{j=1}^{\infty} \left\|U^j\right\| \le \sum_{j=1}^{\infty} M^j \frac{(T - H)^{j-1}}{(j-1)!} \le \eta_2. \tag{58}$$

To establish the validity of second condition, let us represent operator equation

$$P(X) = 0, \tag{59}$$

as in (41) and its the successive approximations is

$$X_{n+1} = S(X_n), \quad (n = 0, 1, 2, \ldots). \tag{60}$$

For initial guess X_0 we have

$$S(X_0) = X_0 - \Gamma_0 P(X_0). \tag{61}$$

From second condition of (Theorem 1) we have

$$\begin{aligned} \|\Gamma_0 P(X_0)\| &= \|S(X_0) - X_0\| \\ &= \|X_1 - X_0\| = \|\Delta X\| \le \frac{\eta}{1 + K\eta}. \end{aligned} \tag{62}$$

In addition, we need to show that $\|P''(X)\| \le \eta_1$ for all $X \in \Omega_0$ where η_1 is defined in (38). It is known that the second derivative $P''(X_0)(X, \overline{X})$ of the nonlinear operator $P(X)$ is described by 3-dimensional array $P''(X_0) X \overline{X} = (D_1, D_2)(X, \overline{X})$, which is called bilinear operator; that is, $P''(X_0)(X \overline{X}) = B(X_0, X, \overline{X})$ where

$$\begin{aligned} & P''(X_0)(X, \overline{X}) \\ &= \lim_{s \to 0} \frac{1}{s} \left[P'(x_0 + s\overline{X}) - P'(X_0)\right] \\ &= \left\{ \lim_{s \to 0} \frac{1}{s} \left[\left(\frac{\partial P_1}{\partial x}(x_0 + s\overline{x}, y_0 + s\overline{y}) - \frac{\partial P_1}{\partial x}(x_0, y_0) \right) x \right.\right. \\ & \qquad \left. + \left(\frac{\partial P_1}{\partial y}(x_0 + s\overline{x}, y_0 + s\overline{y}) - \frac{\partial P_1}{\partial y}(x_0, y_0) \right) y \right], \\ & \quad \lim_{s \to 0} \frac{1}{s} \left[\left(\frac{\partial P_2}{\partial x}(x_0 + s\overline{x}, y_0 + s\overline{y}) - \frac{\partial P_2}{\partial x}(x_0, y_0) \right) x \right. \\ & \qquad \left.\left. + \left(\frac{\partial P_2}{\partial y}(x_0 + s\overline{x}, y_0 + s\overline{y}) - \frac{\partial P_2}{\partial y}(x_0, y_0) \right) y \right] \right\} \end{aligned}$$

$$
\begin{aligned}
= \Bigg\{ & \lim_{s\to 0}\frac{1}{s}\Bigg[\Bigg(\frac{\partial^2 P_1}{\partial x^2}(x_0,y_0)s\overline{x}+\frac{\partial^2 P_1}{\partial y\partial x}(x_0,y_0)s\overline{y} \\
&\quad +\frac{1}{2}\Bigg(\frac{\partial^3 P_1}{\partial x^3}(x_0+\theta s\overline{x},y_0+\delta s\overline{y})s^2\overline{x}^2 \\
&\qquad +2\frac{\partial^3 P_1}{\partial x^2\partial y}(x_0+\theta s\overline{x},y_0+\delta s\overline{y})s^2\overline{x}\,\overline{y} \\
&\qquad +\frac{\partial^3 P_1}{\partial y^2\partial x}(x_0+\theta s\overline{x},y_0+\delta s\overline{y})s^2\overline{y}\Bigg)\Bigg)x \\
&\quad +\Bigg(\frac{\partial^2 P_1}{\partial x\partial y}(x_0,y_0)s\overline{x}+\frac{\partial^2 P_1}{\partial y^2}(x_0,y_0)s\overline{y} \\
&\quad +\frac{1}{2}\Bigg(\frac{\partial^3 P_1}{\partial x^2\partial y}(x_0+\theta s\overline{x},y_0+\delta s\overline{y})s^2\overline{x}^2 \\
&\qquad +2\frac{\partial^3 P_1}{\partial x\partial y^2}(x_0+\theta s\overline{x},y_0+\delta s\overline{y})s^2\overline{x}\,\overline{y} \\
&\qquad +\frac{\partial^3 P_1}{\partial y^3}(x_0+\theta s\overline{x},\delta s\overline{y})s^2\overline{y}^2\Bigg)\Bigg)y\Bigg], \\
& \lim_{s\to 0}\frac{1}{s}\Bigg[\Bigg(\frac{\partial^2 P_2}{\partial x^2}(x_0,y_0)s\overline{x}+\frac{\partial^2 P_2}{\partial y\partial x}(x_0,y_0)s\overline{y} \\
&\quad +\frac{1}{2}\Bigg(\frac{\partial^3 P_2}{\partial x^3}(x_0+\theta s\overline{x},y_0+\delta s\overline{y})s^2\overline{x}^2 \\
&\qquad +2\frac{\partial^3 P_2}{\partial x^2\partial y}(x_0+\theta s\overline{x},y_0+\delta s\overline{y})s^2\overline{x}\,\overline{y} \\
&\qquad +\frac{\partial^3 P_2}{\partial y^2\partial x}(x_0+\theta s\overline{x},y_0+\delta s\overline{y})s^2\overline{y}\Bigg)\Bigg)x \\
&\quad +\Bigg(\frac{\partial^2 P_2}{\partial x\partial y}(x_0,y_0)s\overline{x}+\frac{\partial^2 P_2}{\partial y^2}(x_0,y_0)s\overline{y} \\
&\quad +\frac{1}{2}\Bigg(\frac{\partial^3 P_2}{\partial x^2\partial y}(x_0+\theta s\overline{x},y_0+\delta s\overline{y})s^2\overline{x}^2 \\
&\qquad +2\frac{\partial^3 P_2}{\partial x\partial y^2}(x_0+\theta s\overline{x},y_0+\delta s\overline{y})s^2\overline{x}\,\overline{y} \\
&\qquad +\frac{\partial^3 P_2}{\partial y^3}(x_0+\theta s\overline{x},\delta s\overline{y})s^2\overline{y}^2\Bigg)\Bigg)y\Bigg]\Bigg\} \\
= \Bigg(& \frac{\partial^2 P_1}{\partial x^2}(x_0,y_0)\overline{x}x+\frac{\partial^2 P_1}{\partial y\partial x}(x_0,y_0)\overline{y}x \\
&+\frac{\partial^2 P_1}{\partial x\partial y}(x_0,y_0)\overline{x}y+\frac{\partial^2 P_1}{\partial y^2}(x_0,y_0)\overline{y}x, \\
&\frac{\partial^2 P_2}{\partial x^2}(x_0,y_0)\overline{x}x+\frac{\partial^2 P_2}{\partial y\partial x}(x_0,y_0)\overline{y}x \\
&+\frac{\partial^2 P_2}{\partial x\partial y}(x_0,y_0)\overline{x}y+\frac{\partial^2 P_2}{\partial y^2}(x_0,y_0)\overline{y}x\Bigg),
\end{aligned}
\tag{63}
$$

where $\theta,\delta\in(0,1)$, so we have

$$
P''(X_0)(X,\overline{X})=\begin{pmatrix}D_1 & D_2\end{pmatrix}\begin{pmatrix}\overline{x}\\ \overline{y}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}, \tag{64}
$$

where

$$
\begin{aligned}
D_1&=\begin{pmatrix}\left.\dfrac{\partial^2 P_1}{\partial x^2}\right|_{(x_0,y_0)} & \left.\dfrac{\partial^2 P_1}{\partial y\partial x}\right|_{(x_0,y_0)}\\[2ex] \left.\dfrac{\partial^2 P_1}{\partial x\partial y}\right|_{(x_0,y_0)} & \left.\dfrac{\partial^2 P_1}{\partial y^2}\right|_{(x_0,y_0)}\end{pmatrix},\\
D_2&=\begin{pmatrix}\left.\dfrac{\partial^2 P_2}{\partial x^2}\right|_{(x_0,y_0)} & \left.\dfrac{\partial^2 P_2}{\partial y\partial x}\right|_{(x_0,y_0)}\\[2ex] \left.\dfrac{\partial^2 P_2}{\partial x\partial y}\right|_{(x_0,y_0)} & \left.\dfrac{\partial^2 P_2}{\partial y^2}\right|_{(x_0,y_0)}\end{pmatrix}.
\end{aligned}
\tag{65}
$$

Then the norms of every components of D_1 and D_2 have the estimate

$$
\begin{aligned}
\left\|\frac{\partial^2 P_1}{\partial x^2}\right\| &= \max_{\|X\|\le 1,\|\overline{X}\|\le 1}\left|\int_{y_0(t)}^{t} h(t,\tau)\frac{x(\tau)}{x_0^2(\tau)}\overline{x}(\tau)\,d\tau\right| \\
&\le H_1c_2(T-H_4), \\
\left\|\frac{\partial^2 P_1}{\partial x\partial y}\right\| &= \max_{\|X\|\le 1,\|\overline{X}\|\le 1}\left|h(t,y_0(t))\frac{x(y_0(t))}{x_0(y_0(t))}\overline{y}(t)\right|\le H_1c_1, \\
\left\|\frac{\partial^2 P_1}{\partial y\partial x}\right\| &= \max_{\|X\|\le 1,\|\overline{X}\|\le 1}\left|h(t,y_0(t))\frac{\overline{x}(y_0(t))}{x_0(y_0(t))}y(t)\right|\le H_1c_1, \\
\left\|\frac{\partial^2 P_1}{\partial y^2}\right\| &= \max_{\|X\|\le 1,\|\overline{X}\|\le 1}\Bigg|\Bigg[h'_\tau(t,y_0(t))\log\left|x_0(y_0(t))\right| \\
&\qquad + h(t,y_0(t))\frac{x'_0(y_0(t))}{x_0(y_0(t))}\Bigg] \\
&\qquad \times y(t)\,\overline{y}(t)\Bigg| \\
&\le H'_1H_5+H_1H'_3c_1, \\
\left\|\frac{\partial^2 P_2}{\partial x^2}\right\| &= \max_{\|X\|\le 1,\|\overline{X}\|\le 1}\left|\int_{y_0(t)}^{t} k(t,\tau)\frac{x(\tau)}{x_0^2(\tau)}\overline{x}(\tau)\,d\tau\right| \\
&\le H_2c_2(T-H_4), \\
\left\|\frac{\partial^2 P_2}{\partial x\partial y}\right\| &= \max_{\|X\|\le 1,\|\overline{X}\|\le 1}\left|-k(t,y_0(t))\frac{x(y_0(t))}{x_0(y_0(t))}\overline{y}(t)\right| \\
&\le H_2c_1, \\
\left\|\frac{\partial^2 P_2}{\partial y\partial x}\right\| &= \max_{\|X\|\le 1,\|\overline{X}\|\le 1}\left|-k(t,y_0(t))\frac{\overline{x}(y_0(t))}{x_0(y_0(t))}y(t)\right| \\
&\le H_2c_1,
\end{aligned}
$$

$$\left\|\frac{\partial^2 P_2}{\partial y^2}\right\| = \max_{\|X\|\le 1, \|\overline{X}\|\le 1} \left| \left[k'_\tau (t, y_0(t)) \log |x_0(y_0(t))| + k(t, y_0(t)) \frac{x'_0(y_0(t))}{x_0(y_0(t))} \right] \times y(t)\, \overline{y}(t) \right| \le H'_2 H_5 + H_2 H'_3 c_1. \tag{66}$$

Therefore, all the second derivatives exist and are bounded:

$$\left\|P''(X)\right\| \le \eta_1. \tag{67}$$

Since $\psi(t)$ majorizes operator $P(X)$ and utilizing the second condition of (Theorem 1) we get

$$\left\|\Gamma_0 P''(X)\right\| \le \frac{2K}{1+K\eta}. \tag{68}$$

Let us consider the discriminant of equation $\psi(t) = 0$:

$$D = K^2\eta^2 - 2K\eta + 1 = (k\eta - 1)^2, \tag{69}$$

and the two roots of $\psi(t) = 0$ are $r_1 = 1/K + t_0$ and $r_2 = \eta + t_0$; therefore, when $r_1 < r < r_2$ implies

$$\psi(r) \le 0, \tag{70}$$

then under the assumption of the fourth condition, that is, $1/K + t_0$ is the unique solution of $\psi(t) = 0$ in $[t_0, T]$ and the condition in (70) [21, Theorem 4, page 530] implies that X^* is the unique solution of operator equation (5) [21, Theorem 6, page 532] and

$$\left\|X^* - X_0\right\| \le t^* - t_0, \tag{71}$$

where t^* is the unique solution of $\psi(t) = 0$ in $[t_0, r]$.

To show the rate of convergence let us write the equation $\psi(t) = 0$ in a same form as in (40) then its successive approximation is

$$t_{m+1} = \phi(t_m), \quad m = 0, 1, 2, \ldots. \tag{72}$$

To estimate the difference between t^* and successive approximation t_m:

$$t^* - t_m = \phi(t^*) - \phi(t_{m-1}) = \phi'(\widetilde{t_m})(t^* - t_{m-1}), \tag{73}$$

where $\widetilde{t_m} \in (t_{m-1}, t^*)$ and

$$\phi'(t) = 1 + c_0\psi'(t) = \frac{2K}{1+K\eta}(t - t_0); \tag{74}$$

therefore

$$\begin{aligned} \phi'(\widetilde{t_m}) &= \frac{2K}{1+K\eta}(\widetilde{t_m} - t_0) \\ &\le \frac{2K}{1+K\eta}(t^* - t_0) = \frac{2}{1+K\eta}; \end{aligned} \tag{75}$$

then

$$\begin{aligned} t^* - t_m &\le \frac{2}{1+K\eta}(t^* - t_{m-1}), \\ t^* - t_{m-1} &\le \frac{2}{1+K\eta}(t^* - t_{m-2}); \\ &\vdots \\ t^* - t_1 &\le \frac{2}{1+K\eta}(t^* - t_0), \end{aligned} \tag{76}$$

consequently,

$$t^* - t_m \le \left(\frac{2}{1+K\eta}\right)^m \frac{1}{K}; \tag{77}$$

it implies

$$\left\|X^* - X_m\right\| \le (t^* - t_m) \le \left(\frac{2}{1+K\eta}\right)^m \frac{1}{K}. \tag{78}$$

□

5. Numerical Example

Consider the system of nonlinear equation

$$\begin{aligned} x(t) - \int_{y(t)}^{t} t\tau \log(|x(\tau)|)\, d\tau &= e^t - \frac{t^2}{3}, \\ \int_{y(t)}^{t} \tau \log(|x(\tau)|)\, d\tau &= \frac{t}{3}, \quad t \in [10, 15]. \end{aligned} \tag{79}$$

The exact solution is

$$\begin{aligned} x^*(t) &= e^t, \\ y^*(t) &= \sqrt[3]{t^3 - t}, \end{aligned} \tag{80}$$

and the initial guesses are

$$\begin{aligned} x_0(t) &= e^{10}(t - 9), \\ y_0(t) &= 0.6t + 4. \end{aligned} \tag{81}$$

Table 1 shows that $x_m(t)$ coincides with the exact $x^*(t)$ from the first iteration whereas only six iterations are needed for $y_m(t)$ to be very close to $y^*(t)$. Notations used here are as follows: N is the number of nodes, m is the number of iterations, and $\epsilon_x = \max_{t\in[10,15]}|x_m(t) - x^*(t)|$ and $\epsilon_y = \max_{t\in[10,15]}|y_m(t) - y^*(t)|$.

6. Conclusion

In this paper, the Newton-Kantorovich method is developed to solve the system of nonlinear Volterra integral equations which contains logarithmic function. We have introduced a new majorant function that leads to the increment of range of convergence of successive approximation process. A new theorem is stated based on the general theorems of Kantorovich. Numerical example is given to show the validation of the method. Table 1 shows that the proposed method is in good agreement with the theoretical findings.

Table 1: Numerical results for (79).

$N = 20,\ h = 0.25$		
m	ϵ_x	ϵ_y
1	0.00	0.0029
2	0.00	$4.3597E-006$
3	0.00	$3.1061E-008$
4	0.00	$1.0140E-009$
5	0.00	$1.2541E-010$
6	0.00	$3.9968E-011$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Notes on Convergence Properties for a Smoothing-Regularization Approach to Mathematical Programs with Vanishing Constraints

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We give some improved convergence results about the smoothing-regularization approach to mathematical programs with vanishing constraints (MPVC for short), which is proposed in Achtziger et al. (2013). We show that the Mangasarian-Fromovitz constraints qualification for the smoothing-regularization problem still holds under the VC-MFCQ (see Definition 5) which is weaker than the VC-LICQ (see Definition 7) and the condition of asymptotic nondegeneracy. We also analyze the convergence behavior of the smoothing-regularization method and prove that any accumulation point of a sequence of stationary points for the smoothing-regularization problem is still strongly-stationary under the VC-MFCQ and the condition of asymptotic nondegeneracy.

1. Introduction

We consider the following mathematical program with vanishing constraints:

$$\begin{aligned} \min \quad & f(z) \\ \text{s.t.} \quad & g_i(z) \le 0, \quad i = 1, 2, \ldots, m; \\ & h_j(z) = 0, \quad j = 1, 2, \ldots, p; \\ & H_i(z) \ge 0, \quad i = 1, 2, \ldots, l; \\ & G_i(z) H_i(z) \le 0, \quad i = 1, 2, \ldots, l, \end{aligned} \tag{1}$$

where $f : R^n \to R$, $g : R^n \to R^m$, $h : R^n \to R^p$ and $G, H : R^n \to R^l$ are all continuously differentiable functions.

The MPVC was firstly introduced to the mathematical community in [1]. It plays an important role in some fields such as optimization topology design problems in mechanical structures [1] and robot path-finding problems with logic communication constraints in robot motion planning [2]. The major difficulty in solving problem (1) is that it does not satisfy some standard constraint qualifications at the feasible points so that the standard optimization methods are likely to fail for this problem. The MPVC has attracted much attention in the recent years. Several theoretical properties and different numerical approaches for MPVC can be found in [1–12]. Very recently, in [3], the authors have proposed a smoothing-regularization approach to mathematical programs with vanishing constraints. Their basic idea is to reformulate the characteristic constraints of the MPVC via a nonsmooth function and to eventually smooth it and regularize the feasible set with the aid of a certain smoothing and regularization parameter $\varepsilon > 0$ such that the resulting problem is more tractable and coincides with the original program for $\varepsilon = 0$. Under the VC-LICQ and the condition of asymptotic nondegeneracy, the convergence behaviors of a sequence of stationary points of the smoothing-regularized problems have been investigated.

In this note, we give some improved convergence results about the smoothing-regularization approach to mathematical programs with vanishing constraints in [3]. We show that these properties still hold under the weaker VC-MFCQ and the condition of asymptotic nondegeneracy. The smoothing-regularization problems satisfy the standard MFCQ, which guarantees the existence of Lagrange multipliers at local minima; the sequence of multipliers is bounded, and the limit point is still strongly-stationary.

The rest of the note is organized as follows. In Section 2, we review some concepts of the nonlinear programming and the MPVC and present the smoothing-regularization method for (1), which is proposed in [3]. In Section 3, we give the improved convergent properties. We close with some final remarks in Section 4.

For convenience of discussion, some notations to be used in this paper are given. The ith component of G will be denoted by G_i; X denotes the feasible set of problem (1). For a function $g : R^n \to R^m$ and a given vector $\alpha \in R^n$, we use $I_g(z) = \{i : g_i(z) = 0\}$ and $\text{supp}(\alpha) = \{i : \alpha_i \neq 0\}$ to denote the active index set of g at z and the support of α, respectively.

2. Preliminaries

Firstly, we will introduce some definitions about the following optimization problem:

$$\begin{aligned} \min \quad & f(z) \\ \text{s.t.} \quad & g(z) \leq 0, \\ & h(z) = 0, \end{aligned} \tag{2}$$

where $f : R^n \to R$, $g : R^n \to R^m$, $h : R^n \to R^p$ are all continuously differentiable functions. F denotes the feasible set of problem (2).

Definition 1. A point $\bar{z} \in F$ is called a stationary point if there are multipliers λ, μ such that $(\bar{z}, \lambda, \mu)$ is a KKT point of (2); that is, the multipliers satisfy $\lambda \in R^m_+$ and $\mu \in R^p$ with $\lambda_i g_i(\bar{z}) = 0$ for all $i = 1, 2, \ldots, m$, and

$$\nabla f(\bar{z}) + \sum_{i=1}^{m} \lambda_i \nabla g_i(\bar{z}) + \sum_{i=1}^{p} \mu_i \nabla h_i(\bar{z}) = 0. \tag{3}$$

Definition 2. A feasible point $\bar{z}$ of (2) is said to satisfy the Mangasarian-Fromovitz constraint qualification (MFCQ for short) if the gradients $\{\nabla h_i(\bar{z}) \mid i = 1, 2, \ldots, p\}$ are linearly independent and there is a $d \in R^n$ such that

$$\begin{aligned} \nabla g_i(\bar{z})^T d < 0 \quad & \left(i \in I_g(\bar{z})\right), \\ \nabla h_i(\bar{z})^T d = 0 \quad & \left(i = 1, 2, \ldots, p\right). \end{aligned} \tag{4}$$

Definition 3 (see [13]). A finite set of vectors $\{a_i \mid i \in I_1\} \cup \{b_i \mid i \in I_2\}$ is said to be positive-linearly dependent if there exists $(\alpha, \beta) \neq 0$ such that

$$\sum_{i \in I_1} \alpha_i a_i + \sum_{i \in I_2} \beta_i b_i = 0, \quad \alpha_i \geq 0, \ \forall i \in I_1. \tag{5}$$

If the above system only has a solution $(\alpha, \beta) = 0$, we say that these vectors are positive-linearly independent.

By using Motzkin's theorem of the alternatives in [14], we can obtain the following property.

Lemma 4. *A point $\bar{z} \in F$ satisfies the MFCQ if and only if the gradients*

$$\left\{\nabla g_i(\bar{z}) \mid i \in I_g(\bar{z})\right\} \cup \left\{\nabla h_i(\bar{z}) \mid i = 1, 2, \ldots, p\right\} \tag{6}$$

are positive-linearly independent.

Now, we borrow notations from mathematical programs with complementarity constraints to define the following sets of active constraints in an arbitrary $\bar{z} \in X$ as follows:

$$\begin{aligned} I_+(\bar{z}) &= \left\{i \mid H_i(\bar{z}) > 0\right\}, \\ I_0(\bar{z}) &= \left\{i \mid H_i(\bar{z}) = 0\right\}, \\ I_{+0}(\bar{z}) &= \left\{i \mid H_i(\bar{z}) > 0, G_i(\bar{z}) = 0\right\}, \\ I_{+-}(\bar{z}) &= \left\{i \mid H_i(\bar{z}) > 0, G_i(\bar{z}) < 0\right\}, \\ I_{0+}(\bar{z}) &= \left\{i \mid H_i(\bar{z}) = 0, G_i(\bar{z}) > 0\right\}, \\ I_{00}(\bar{z}) &= \left\{i \mid H_i(\bar{z}) = 0, G_i(\bar{z}) = 0\right\}, \\ I_{0-}(\bar{z}) &= \left\{i \mid H_i(\bar{z}) = 0, G_i(\bar{z}) < 0\right\}. \end{aligned} \tag{7}$$

Definition 5 (see [1]). A feasible point $\bar{z}$ for (1) satisfies the vanishing constraints Mangasarian-Fromovitz constraints qualification (VC-MFCQ for short) if

$$\begin{aligned} \nabla h_i(\bar{z}) \quad & (i = 1, 2, \ldots, p), \\ \nabla H_i(\bar{z}) \quad & (i \in I_{0+}(\bar{z}) \cup I_{00}(\bar{z})) \end{aligned} \tag{8}$$

are linearly independent and there exists a vector $d \in R^n$ such that

$$\begin{aligned} \nabla h_i(\bar{z})^T d = 0 \quad & (i = 1, 2, \ldots, p), \\ \nabla H_i(\bar{z})^T d = 0 \quad & (i \in I_{0+}(\bar{z}) \cup I_{00}(\bar{z})), \\ \nabla g_i(\bar{z})^T d < 0 \quad & \left(i \in I_g(\bar{z})\right), \\ \nabla H_i(\bar{z})^T d > 0 \quad & (i \in I_{0-}(\bar{z})), \\ \nabla G_i(\bar{z})^T d < 0 \quad & (i \in I_{+0}(\bar{z})). \end{aligned} \tag{9}$$

Similar to Lemma 4, we can also deduce the following result.

Lemma 6. *A point $\bar{z} \in X$ satisfies the VC-MFCQ if and only if the gradients*

$$\begin{aligned} & \left\{\nabla g_i(\bar{z}) \mid i \in I_g(\bar{z})\right\} \\ & \quad \cup \left\{\nabla h_i(\bar{z}) \mid i = 1, 2, \ldots, p\right\} \\ & \quad \cup \left\{-\nabla H_i(\bar{z}) \mid i \in I_{0-}(\bar{z})\right\} \\ & \quad \cup \left\{\nabla G_i(\bar{z}) \mid i \in I_{+0}(\bar{z})\right\} \\ & \quad \cup \left\{\nabla H_i(\bar{z}) \mid i \in I_{00}(\bar{z}) \cup I_{0+}(\bar{z})\right\} \end{aligned} \tag{10}$$

are positive-linearly independent. In other words, the MPVC at $\bar{z}$ satisfies the VC-MFCQ if and only if there does not exist a vector $(\lambda_{I_g(\bar{z})}, \mu, \alpha_{I_{0-}(\bar{z})}, \alpha_{I_{00}(\bar{z}) \cup I_{0+}(\bar{z})}, \beta_{I_{+0}(\bar{z})}) \neq 0$ with $\lambda_i \geq 0$

for all $i \in I_g(\bar{z})$, $\alpha_i \geq 0$ *for all* $i \in I_{0-}(\bar{z})$, *and* $\beta_i \geq 0$ *for all* $i \in I_{+0}(\bar{z})$ *such that*

$$\sum_{i\in I_g(\bar{z})} \lambda_i \nabla g_i(\bar{z}) + \sum_{i=1}^{l} \mu_i \nabla h_i(\bar{z}) - \sum_{i\in I_{0-}(\bar{z})} \alpha_i \nabla H_i(\bar{z}) + \sum_{i\in I_{00}(\bar{z})\cup I_{0+}(\bar{z})} \alpha_i \nabla H_i(\bar{z}) + \sum_{i\in I_{+0}(\bar{z})} \beta_i \nabla G_i(\bar{z}) = 0 \tag{11}$$

holds true.

Definition 7 (see [1]). A feasible point $\bar{z}$ for (1) satisfies the vanishing linear independence constraints qualification (VC-LICQ for short) if and only if

$$\begin{aligned} &\nabla h_i(\bar{z}) \quad (i = 1, 2, \ldots, p), \\ &\nabla g_i(\bar{z}) \quad (i \in I_g(\bar{z})), \\ &\nabla G_i(\bar{z}) \quad (i \in I_{+0}(\bar{z})), \\ &\nabla H_i(\bar{z}) \quad (i \in I_0(\bar{z})) \end{aligned} \tag{12}$$

are linearly independent.

Remark 8. It is easy to see that the VC-LICQ implies the VC-MFCQ. Moreover, the VC-LICQ (VC-MFCQ) is weaker than the MPVC-LICQ (MPVC-MFCQ) (See [7]).

Definition 9. Let $\bar{z}$ be a feasible point for the problem (1), then

(a) $\bar{z}$ is said to be weak-stationary if there exist multiplier vectors $\bar{\lambda} \in R^m$, $\bar{\mu} \in R^p$, and $\bar{u}$, $\bar{v} \in R^l$ such that

$$\begin{aligned} &\nabla f(\bar{z}) + \nabla g(\bar{z})^T \bar{\lambda} + \nabla h(\bar{z})^T \bar{\mu} - \nabla H(\bar{z})^T \bar{v} + \nabla G(\bar{z})^T \bar{u} = 0, \\ &\bar{\lambda} \geq 0, \qquad \bar{z} \in X, \qquad \bar{\lambda}^T g(\bar{z}) = 0, \\ &\bar{v}_i = 0 \quad (i \in I_+(\bar{z})), \\ &\bar{v}_i \geq 0 \quad (i \in I_{0-}(\bar{z})), \\ &\bar{v}_i \quad \text{free} \quad (i \in I_{0+}(\bar{z}) \cup I_{00}(\bar{z})), \\ &\bar{u}_i = 0 \quad (i \in I_{+-}(\bar{z}) \cup I_{0-}(\bar{z}) \cup I_{0+}(\bar{z})), \\ &\bar{u}_i \geq 0 \quad (i \in I_{+0}(\bar{z}) \cup I_{00}(\bar{z})). \end{aligned} \tag{13}$$

(b) $\bar{z}$ is said to be strongly-stationary, if it is weak-stationary and

$$\bar{u}_i = 0, \quad \bar{v}_i \geq 0, \quad i \in I_{00}(\bar{z}). \tag{14}$$

Finally, we give the smoothing-regularization method of Problem (1), which is proposed in [3]. According to [3], with the help of a positive parameter, the MPVC (1) is approximated by the following smoothing-regularization problem:

$$\begin{aligned} \min \ & f(z) \\ \text{s.t.} \ & g(z) \leq 0, \quad h(z) = 0, \\ & r_\varepsilon(z) \leq \varepsilon, \end{aligned} \tag{15}$$

where

$$\begin{aligned} r_\varepsilon(z) &= \begin{pmatrix} r_{\varepsilon,1}(z) \\ \vdots \\ r_{\varepsilon,l}(z) \end{pmatrix}, \\ r_{\varepsilon,i}(z) &= \frac{1}{2}\Big(G_i(z) H_i(z) + \sqrt{(G_i(z) H_i(z))^2 + \varepsilon^2} + \sqrt{(H_i(z))^2 + \varepsilon^2} - H_i(z)\Big). \end{aligned} \tag{16}$$

In order to give our improved convergence analysis, the following concept of asymptotic nondegeneracy is necessary.

Definition 10 (see [3]). Let $\bar{z}$ be feasible for the MPVC (1). Then a sequence $\{z_k\}$ of feasible points for (15) converging to $\bar{z}$ as $\varepsilon_k \downarrow 0$ is called asymptotically nondegenerate, if any accumulation point of $\{\nabla r_{\varepsilon_k,i}(z_k)\}$ is different from 0 for each $i \in I_{+0}(\bar{z}) \cup I_0(\bar{z})$.

3. Some Improved Convergence Properties

In this section, we will consider the improved convergence properties of a sequence of stationary points for the smoothing-regularization problem (15). Firstly, we discuss the constraint qualification of (15).

For convenience of discussion, we give the following notations:

$$\begin{aligned} a_{\varepsilon,i}(z) &= H_i(z) + \frac{G_i(z) H_i(z)^2}{\sqrt{G_i(z)^2 H_i(z)^2 + \varepsilon^2}}, \\ b_{\varepsilon,i}(z) &= G_i(z) + \frac{G_i(z)^2 H_i(z)}{\sqrt{G_i(z)^2 H_i(z)^2 + \varepsilon^2}} + \frac{H_i(z)}{\sqrt{H_i(z)^2 + \varepsilon^2}} - 1, \\ I_{r_\varepsilon}(z) &= \{i : r_{\varepsilon,i}(z) = \varepsilon\}. \end{aligned} \tag{17}$$

To show that the Mangasarian-Fromovitz constraints qualification for the problem (15) holds under some conditions, the following lemma palys a very important role.

Lemma 11. *Let* $\bar{z}$ *be feasible for* (1) *such that the VC-MFCQ is satisfied at* $\bar{z}$ *and the sequence* $\{z_k\}$ *of feasible points for* (15) *converging to $\bar{z}$ as $\varepsilon_k \downarrow 0$ is asymptotically nondegenerate. Then, for sufficiently large k, the set of vectors*

$$\begin{gathered}
\nabla g_i(z_k), \quad i \in I_g(\bar{z}), \\
\nabla h_i(z_k), \quad i = 1, 2, \ldots, p, \\
-\left(a_{\varepsilon_k,i}(z_k)\nabla G_i(z_k) + b_{\varepsilon_k,i}(z_k)\nabla H_i(z_k)\right), \\
i \in I_{r_{\varepsilon_k}}(z_k) \cap I_{0-}(\bar{z}), \\
a_{\varepsilon_k,i}(z_k)\nabla G_i(z_k) + b_{\varepsilon_k,i}(z_k)\nabla H_i(z_k), \\
i \in I_{r_{\varepsilon_k}}(z_k) \cap I_{+0}(\bar{z}), \\
a_{\varepsilon_k,i}(z_k)\nabla G_i(z_k) + b_{\varepsilon_k,i}(z_k)\nabla H_i(z_k), \\
i \in I_{r_{\varepsilon_k}}(z_k) \cap \left(I_{0+}(\bar{z}) \cup I_{00}(\bar{z})\right)
\end{gathered} \tag{18}$$

are positive-linearly independent.

Proof. Since g, h, G, H are all continuous, for sufficiently large k, we have

$$I_g(z_k) \subseteq I_g(\bar{z}), \qquad I_h(z_k) \subseteq I_h(\bar{z}). \tag{19}$$

Because the VC-MFCQ holds, the gradients

$$\begin{aligned}
&\left\{\nabla g_i(\bar{z}) \mid i \in I_g(\bar{z})\right\} \\
&\quad \cup \left\{\nabla h_i(\bar{z}) \mid i = 1, 2, \ldots, p\right\} \\
&\quad \cup \left\{-\nabla H_i(\bar{z}) \mid i \in I_{0-}(\bar{z})\right\} \\
&\quad \cup \left\{\nabla G_i(\bar{z}) \mid i \in I_{+0}(\bar{z})\right\} \\
&\quad \cup \left\{\nabla H_i(\bar{z}) \mid i \in I_{00}(\bar{z}) \cup I_{0+}(\bar{z})\right\}
\end{aligned} \tag{20}$$

are positive-linearly independent by Lemma 6, taking into account that

$$\begin{gathered}
\left(I_{r_{\varepsilon_k}}(z_k) \cap I_{0-}(\bar{z})\right) \subseteq I_{0-}(\bar{z}), \\
\left(I_{r_{\varepsilon_k}}(z_k) \cap I_{+0}(\bar{z})\right) \subseteq I_{+0}(\bar{z}), \\
\left(I_{r_{\varepsilon_k}}(z_k) \cap I_{0+}(\bar{z})\right) \cup \left(I_{r_{\varepsilon_k}}(z_k) \cap I_{00}(\bar{z})\right) \\
\subseteq I_{00}(\bar{z}) \cup I_{0+}(\bar{z}).
\end{gathered} \tag{21}$$

In view of the condition of asymptotic nondegeneracy, we know that $a_{\varepsilon_k,i}(z_k) \neq 0$, $b_{\varepsilon_k,i}(z_k) \approx 0$ for all $i \in I_{+0}(\bar{z})$ and $a_{\varepsilon_k,i}(z_k) \approx 0$, $b_{\varepsilon_k,i}(z_k) \neq 0$ for $i \in I_0$ for all sufficiently large k. Similar to the proof of Proposition 2.2 in [15], we know that the set of vectors

$$\begin{gathered}
\nabla g_i(z_k), \quad i \in I_g(\bar{z}), \\
\nabla h_i(z_k), \quad i = 1, 2, \ldots, p, \\
-\left(a_{\varepsilon_k,i}(z_k)\nabla G_i(z_k) + b_{\varepsilon_k,i}(z_k)\nabla H_i(z_k)\right), \\
i \in I_{r_{\varepsilon_k}}(z_k) \cap I_{0-}(\bar{z}), \\
a_{\varepsilon_k,i}(z_k)\nabla G_i(z_k) + b_{\varepsilon_k,i}(z_k)\nabla H_i(z_k), \\
i \in I_{r_{\varepsilon_k}}(z_k) \cap I_{+0}(\bar{z}), \\
a_{\varepsilon_k,i}(z_k)\nabla G_i(z_k) + b_{\varepsilon_k,i}(z_k)\nabla H_i(z_k), \\
i \in I_{r_{\varepsilon_k}}(z_k) \cap \left(I_{0+}(\bar{z}) \cup I_{00}(\bar{z})\right)
\end{gathered} \tag{22}$$

are positive-linearly independent for all sufficiently large k. The proof is completed. □

Based on the above lemma, we can show the following theorem.

Theorem 12. *Let $\bar{z}$ be feasible for* (1) *such that the VC-MFCQ is satisfied at $\bar{z}$ and the sequence $\{z_k\}$ of feasible points for* (15) *converging to $\bar{z}$ as $\varepsilon_k \downarrow 0$ is asymptotically nondegenerate. Then, for sufficiently large k, Problem* (15) *satisfies the standard MFCQ at z_k.*

Proof. Taking Lemma 11 into account, we know that the set of vectors

$$\begin{gathered}
\nabla g_i(z_k), \quad i \in I_g(\bar{z}), \\
\nabla h_i(z_k), \quad i = 1, 2, \ldots, p, \\
-\left(a_{\varepsilon_k,i}(z_k)\nabla G_i(z_k) + b_{\varepsilon_k,i}(z_k)\nabla H_i(z_k)\right), \\
i \in I_{r_{\varepsilon_k}}(z_k) \cap I_{0-}(\bar{z}), \\
a_{\varepsilon_k,i}(z_k)\nabla G_i(z_k) + b_{\varepsilon_k,i}(z_k)\nabla H_i(z_k), \\
i \in I_{r_{\varepsilon_k}}(z_k) \cap I_{+0}(\bar{z}), \\
a_{\varepsilon_k,i}(z_k)\nabla G_i(z_k) + b_{\varepsilon_k,i}(z_k)\nabla H_i(z_k), \\
i \in I_{r_{\varepsilon_k}}(z_k) \cap \left(I_{0+}(\bar{z}) \cup I_{00}(\bar{z})\right)
\end{gathered} \tag{23}$$

are positive-linearly independent for sufficiently large k.

We now prove that the standard MFCQ holds at z_k for Problem (15) for sufficiently large k. In view of Lemma 4, we have to show that

$$\begin{aligned}
0 = &\sum_{i \in I_g(z_k)} \lambda_i^k \nabla g_i(z_k) + \sum_{i=1}^{p} \mu_i^k \nabla h_i(z_k) \\
&+ \sum_{i=1}^{l} \gamma_i^k \left(a_{\varepsilon_k,i}(z_k)\nabla G_i(z_k) + b_{\varepsilon_k,i}(z_k)\nabla H_i(z_k)\right)
\end{aligned} \tag{24}$$

with $\mu^k \in R^p$ and $\lambda^k, \gamma^k \geq 0$ holds for the zero vector. To see this, we rewrite (24) as

$$\begin{aligned} 0 = &\sum_{i\in I_g(z_k)} \lambda_i^k \nabla g_i(z_k) + \sum_{i=1}^{p} \mu_i^k \nabla h_i(z_k) \\ &- \sum_{i\in I_{r_{\varepsilon_k}}(z_k)\cap I_{0-}(\bar{z})} \gamma_i^k \left(-a_{\varepsilon_k,i}(z_k)\nabla G_i(z_k) \right. \\ &\qquad \left. -b_{\varepsilon_k,i}(z_k)\nabla H_i(z_k)\right) \\ &+ \sum_{i\in I_{r_{\varepsilon_k}}(z_k)\cap I_{+0}(\bar{z})} \gamma_i^k \left(a_{\varepsilon_k,i}(z_k)\nabla G_i(z_k) \right. \\ &\qquad \left. + b_{\varepsilon_k,i}(z_k)\nabla H_i(z_k)\right) \\ &+ \sum_{i\in I_{r_{\varepsilon_k}}(z_k)\cap(I_{0+}(\bar{z})\cup I_{00}(\bar{z}))} \gamma_i^k \left(a_{\varepsilon_k,i}(z_k)\nabla G_i(z_k) \right. \\ &\qquad \left. + b_{\varepsilon_k,i}(z_k)\nabla H_i(z_k)\right). \end{aligned} \tag{25}$$

In view of the condition of asymptotic nondegeneracy, applying the positive linear independence of vectors from (23) to (25) and (19), one gets

$$\begin{aligned} \lambda_i^k &= 0 \quad (i \in I_g(z_k)), \\ \mu_i^k &= 0 \quad (i = 1,2,\ldots,p), \\ \gamma_i^k &= 0 \quad \left(i \in I_{r_{\varepsilon_k}}(z^k) \cap I_{0-}(\bar{z})\right), \\ \gamma_i^k &= 0 \quad \left(i \in I_{r_{\varepsilon_k}}(z_k) \cap I_{+0}(\bar{z})\right), \\ \gamma_i^k &= 0 \quad \left(i \in I_{r_{\varepsilon_k}}(z^k) \cap (I_{0+}(\bar{z}) \cup I_{00}(\bar{z}))\right). \end{aligned} \tag{26}$$

The proof is completed. □

Remark 13. In Theorem 12, by relaxing the condition of the VC-LICQ, we show that the VC-MFCQ and the condition of asymptotic nondegeneracy imply that the smoothing-regularization problems satisfy the standard MFCQ. Hence, Theorem 12 is an improved version of Lemma 5.6 in [3].

To establish the relations between the solutions of the original problem and those of the smoothing-regularization problem under the VC-MFCQ and the condition of asymptotic nondegeneracy, we give the following key lemma.

Lemma 14. *Let $\varepsilon_k > 0$ be convergent to zero. Suppose that $\{z_k\}$ is a sequence of stationary points of Problem* (15) *with $\varepsilon = \varepsilon_k$ and $(\lambda^k, \mu^k, \gamma^k)$ being the corresponding multiplier vectors. If $\bar{z}$ is an accumulation point of the sequence $\{z_k\}$ such that the VC-MFCQ holds at $\bar{z}$ and the condition of asymptotic nondegeneracy for $\{z_k\}$ is satisfied, then the sequence of multipliers $\{(\lambda^k, \mu^k, \gamma^k)\}$ is bounded.*

Proof. It follows from Theorem 12 that, for sufficiently large k, there exist lagrangian multiplier vectors $(\lambda^k, \mu^k, \gamma^k)$ such that

$$\begin{aligned} \nabla f(z_k) &+ \sum_{i=1}^{m} \lambda_i^k \nabla g_i(z_k) + \sum_{i=1}^{p} \mu_i^k \nabla h_i(z_k) \\ &+ \sum_{i=1}^{l} \gamma_i^k \nabla r_{\varepsilon_k,i}(z_k) = 0, \end{aligned} \tag{27}$$

$$\begin{aligned} \lambda_k \geq 0, &\qquad \operatorname{supp}(\lambda^k) \subseteq I_g(z_k), \\ \gamma_k \geq 0, &\qquad \operatorname{supp}(\gamma^k) \subseteq I_{r_{\varepsilon_k}}(z_k). \end{aligned} \tag{28}$$

From (27), we have

$$\begin{aligned} \nabla f(z_k) &+ \sum_{i\in \operatorname{supp}(\lambda_k)} \lambda_i^k \nabla g_i(z_k) \\ &+ \sum_{i\in \operatorname{supp}(\mu_k)} \mu_i^k \nabla h_i(z_k) \\ &+ \sum_{i\in \operatorname{supp}(\gamma_k)\cap I_{0-}(\bar{z})} \gamma_i^k a_{\varepsilon_k,i}(z_k)\nabla G_i(z_k) \\ &+ \sum_{i\in \operatorname{supp}(\gamma_k)\cap I_{0-}(\bar{z})} \gamma_i^k b_{\varepsilon_k,i}(z_k)\nabla H_i(z_k) \\ &+ \sum_{i\in \operatorname{supp}(\gamma_k)\cap I_{+0}(\bar{z})} \gamma_i^k a_{\varepsilon_k,i}(z_k)\nabla G_i(z_k) \\ &+ \sum_{i\in \operatorname{supp}(\gamma_k)\cap I_{+0}(\bar{z})} \gamma_i^k b_{\varepsilon_k,i}(z_k)\nabla H_i(z_k) \\ &+ \sum_{i\in \operatorname{supp}(\gamma_k)\cap(I_{0+}(\bar{z})\cup I_{00}(\bar{z}))} \gamma_i^k a_{\varepsilon_k,i}(z_k)\nabla G_i(z_k) \\ &+ \sum_{i\in \operatorname{supp}(\gamma_k)\cap(I_{0+}(\bar{z})\cup I_{00}(\bar{z}))} \gamma_i^k b_{\varepsilon_k,i}(z_k)\nabla H_i(z_k) = 0. \end{aligned} \tag{29}$$

We can define

$$\begin{aligned} \beta_i^k &= \begin{cases} -\gamma_i^k b_{\varepsilon_k,i}(z_k), & i \in \operatorname{supp}(\gamma^k) \cap I_{0-}(\bar{z}); \\ 0, & \text{otherwise}, \end{cases} \\ \bar{\gamma}_i^k &= \begin{cases} \gamma_i^k a_{\varepsilon_k,i}(z_k), & i \in \operatorname{supp}(\gamma^k) \cap I_{+0}(\bar{z}); \\ 0, & \text{otherwise}, \end{cases} \\ \bar{\nu}_i^k &= \begin{cases} \gamma_i^k b_{\varepsilon_k,i}(z_k), & i \in \operatorname{supp}(\gamma^k) \cap (I_{0+}(\bar{z}) \cup I_{00}(\bar{z})); \\ 0, & \text{otherwise}. \end{cases} \end{aligned} \tag{30}$$

Noting that with $\beta_i^k, \overline{\gamma}_i^k$, and $\overline{\nu}_i^k$, (27) can be rewritten as

$$\begin{aligned}
0 = {} & \nabla f(z_k) + \sum_{i=1}^{m} \lambda_i^k \nabla g_i(z_k) \\
& + \sum_{i=1}^{p} \mu_i^k \nabla h_i(z_k) + \sum_{i=1}^{l} \beta_i^k (-\nabla H_i(z_k)) \\
& + \sum_{i=1}^{l} \overline{\gamma}_i^k \nabla G_i(z_k) + \sum_{i=1}^{l} \overline{\nu}_i^k \nabla H_i(z_k) \\
& + \sum_{i \in \operatorname{supp}(\gamma_k) \cap (I_{0+}(\overline{z}) \cup I_{00}(\overline{z}))} \gamma_i^k a_{\varepsilon_k, i}(z_k) \nabla G_i(z_k) \\
& + \sum_{i \in \operatorname{supp}(\gamma_k) \cap I_{+0}(\overline{z})} \gamma_i^k b_{\varepsilon_k, i}(z_k) \nabla H_i(z_k) \\
& + \sum_{i \in \operatorname{supp}(\gamma_k) \cap I_{0-}(\overline{z})} \gamma_i^k a_{\varepsilon_k, i}(z_k) \nabla G_i(z_k) .
\end{aligned} \tag{31}$$

The following objective is to prove that the sequence $\{(\lambda^k, \mu^k, \beta^k, \overline{\gamma}^k, \overline{\nu}^k, \gamma^k_{I_{+0}(\overline{z}) \cup I_{0+}(\overline{z}) \cup I_{00}(\overline{z}) \cup I_{0-}(\overline{z})})\}$ is bounded.

Assume that the sequence $\{(\lambda^k, \mu^k, \beta^k, \overline{\gamma}^k, \overline{\nu}^k, \gamma^k_{I_{+0}(\overline{z}) \cup I_{0+}(\overline{z}) \cup I_{00}(\overline{z}) \cup I_{0-}(\overline{z})})\}$ is unbounded. Then, there exists a subset K such that

$$\begin{aligned}
& \left\| \left(\lambda^k, \mu^k, \beta^k, \overline{\gamma}^k, \overline{\nu}^k, \gamma^k_{I_{+0}(\overline{z}) \cup I_{0+}(\overline{z}) \cup I_{00}(\overline{z}) \cup I_{0-}(\overline{z})} \right)_K \right\| \\
& \quad \longrightarrow +\infty \quad (k \longrightarrow +\infty) .
\end{aligned} \tag{32}$$

So the corresponding normed sequence converges:

$$\begin{aligned}
& \frac{\left(\lambda^k, \mu^k, \beta^k, \overline{\gamma}^k, \overline{\nu}^k, \gamma^k_{I_{+0}(\overline{z}) \cup I_{0+}(\overline{z}) \cup I_{00}(\overline{z}) \cup I_{0-}(\overline{z})} \right)}{\left\| \left(\lambda^k, \mu^k, \beta^k, \overline{\gamma}^k, \overline{\nu}^k, \gamma^k_{I_{+0}(\overline{z}) \cup I_{0+}(\overline{z}) \cup I_{00}(\overline{z}) \cup I_{0-}(\overline{z})} \right) \right\|} \\
& \underset{k \in K}{\longrightarrow} \left(\lambda, \mu, \beta, \overline{\gamma}, \overline{\nu}, \gamma_{I_{+0}(\overline{z}) \cup I_{0+}(\overline{z}) \cup I_{00}(\overline{z}) \cup I_{0-}(\overline{z})} \right) \neq 0.
\end{aligned} \tag{33}$$

Combined with (31), it yields

$$\begin{aligned}
0 = {} & \sum_{i=1}^{m} \lambda_i \nabla g_i(\overline{z}) + \sum_{i=1}^{p} \mu_i \nabla h_i(\overline{z}) \\
& + \sum_{i=1}^{l} \beta_i (-\nabla H_i(\overline{z})) + \sum_{i=1}^{l} \overline{\gamma}_i \nabla G_i(\overline{z}) + \sum_{i=1}^{l} \overline{\nu}_i \nabla H_i(\overline{z}) ,
\end{aligned} \tag{34}$$

that is,

$$\begin{aligned}
0 = {} & \sum_{i \in \operatorname{supp}(\lambda)} \lambda_i \nabla g_i(\overline{z}) + \sum_{i \in \operatorname{supp}(\mu)} \mu_i \nabla h_i(\overline{z}) \\
& + \sum_{i \in \operatorname{supp}(\beta)} \beta_i (-\nabla H_i(\overline{z})) + \sum_{i \in \operatorname{supp}(\overline{\gamma})} \overline{\gamma}_i \nabla G_i(\overline{z}) \\
& + \sum_{i \in \operatorname{supp}(\overline{\nu})} \overline{\nu}_i \nabla H_i(\overline{z}) ,
\end{aligned} \tag{35}$$

where $\lambda \geq 0$ and, for all $k \in K$ being large enough,

$$\begin{aligned}
& \operatorname{supp}(\lambda) \subseteq I_g(z_k) \subseteq I_g(\overline{z}) , \\
& \operatorname{supp}(\beta) \subseteq \operatorname{supp}(\beta^k) \subseteq \left(\operatorname{supp}(\gamma^k) \cap I_{0-}(\overline{z}) \right) \subseteq I_{0-}(\overline{z}) , \\
& \operatorname{supp}(\overline{\gamma}) \subseteq \operatorname{supp}(\overline{\gamma}^k) \\
& \qquad \subseteq \left(\operatorname{supp}(\gamma^k) \cap I_{+0}(\overline{z}) \right) \subseteq I_{+0}(\overline{z}) , \\
& \operatorname{supp}(\overline{\nu}) \subseteq \operatorname{supp}(\overline{\nu}^k) \\
& \qquad \subseteq \left(\operatorname{supp}(\gamma^k) \cap (I_{00}(\overline{z}) \cup I_{0+}(\overline{z})) \right) \\
& \qquad \subseteq I_{00}(\overline{z}) \cup I_{0+}(\overline{z}) .
\end{aligned} \tag{36}$$

We can prove that $(\lambda, \mu, \beta, \overline{\gamma}, \overline{\nu}) \neq 0$. Actually, if $(\lambda, \mu, \beta, \overline{\gamma}, \overline{\nu}) = 0$, then, for at least one $i \in I_{+0}(\overline{z}) \cup I_{0+}(\overline{z}) \cup I_{00}(\overline{z}) \cup I_{0-}(\overline{z})$, $\gamma_i \neq 0$. Without loss of generality, assume that $\gamma_i \neq 0$ for an $i \in I_{+0}(\overline{z})$, then, for all k sufficiently large, $\gamma_i^k \neq 0$. Consequently, $\overline{\gamma}_i^k = \gamma_i^k a_{\varepsilon_k, i}(z_k)$ for those k. Taking into account the condition of asymptotic nondegeneracy, for $i \in I_{+0}(\overline{z})$, we have

$$\overline{\gamma}_i = \lim_{k \in K} \overline{\gamma}_i^k = \lim_{k \in K} \gamma_i^k a_{\varepsilon_k, i}(z_k) \neq 0, \tag{37}$$

which contradicts the assumption $\overline{\gamma} = 0$.

By Lemma 6, we know that $(\lambda, \mu, \beta, \overline{\gamma}, \overline{\nu}) \neq 0$ contradicts the fact that the VC-MFCQ holds at $\overline{z}$. Thus, the sequence $\{(\lambda^k, \mu^k, \beta^k, \overline{\gamma}^k, \overline{\nu}^k, \gamma^k_{I_{+0}(\overline{z}) \cup I_{0+}(\overline{z}) \cup I_{00}(\overline{z}) \cup I_{0-}(\overline{z})})\}$ is bounded.

Again, noting the condition of asymptotic nondegeneracy and the definitions of $\beta_i^k, \overline{\gamma}_i^k, \overline{\nu}_i^k$, we can prove that the sequence of multipliers $\{(\lambda^k, \mu^k, \gamma^k)\}$ are bounded. The proof is completed. □

Based on Lemma 14, similar to the proof of Theorem 5.3 in [3], we can obtain the following convergence result.

Theorem 15. *Let $\varepsilon_k > 0$ be convergent to zero. Suppose that $\{z_k\}$ is a sequence of stationary points of Problem* (15) *with $\varepsilon = \varepsilon_k$. If $\overline{z}$ is an accumulation point of the sequence $\{z_k\}$ such that the VC-MFCQ holds at $\overline{z}$ and the condition of asymptotic nondegeneracy for $\{z_k\}$ is satisfied, then $\overline{z}$ is a strongly-stationary point of Problem* (1).

Remark 16. In Theorem 15, by replacing the condition of the VC-LICQ, we prove that any accumulation point of stationary points for the smoothing-regularization problem is still strongly-stationary under the VC-MFCQ and the condition of asymptotic nondegeneracy. Hence, Theorem 15 includes Theorem 5.3 in [3] as a special case.

4. Concluding Remarks

In this note, we have shown that the VC-LICQ assumption can be replaced by the weaker VC-MFCQ condition in order to get the strong stationarity for the smoothing-regularization approach to mathematical programs with vanishing constraints, which is proposed in [3]. We have also shown that the VC-MFCQ implies that the smoothing-regularization problems satisfy the standard MFCQ. While it seems possible to prove that many other VC-tailored constraint qualifications imply that the corresponding standard constraint qualification holds for the smoothing-regularization problem, it is an open question whether one can further relax the VC-MFCQ assumption to get strong stationarity in the limit.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

A Limited Memory BFGS Method for Solving Large-Scale Symmetric Nonlinear Equations

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A limited memory BFGS (L-BFGS) algorithm is presented for solving large-scale symmetric nonlinear equations, where a line search technique without derivative information is used. The global convergence of the proposed algorithm is established under some suitable conditions. Numerical results show that the given method is competitive to those of the normal BFGS methods.

1. Introduction

Consider

$$h(x) = 0, \quad x \in \mathfrak{R}^n, \tag{1}$$

where $h : \mathfrak{R}^n \rightarrow \mathfrak{R}^n$ is continuously differentiable, the Jacobian $\nabla h(x)$ of g is symmetric for all $x \in \mathfrak{R}^n$, and n denotes the large-scale dimensions. It is not difficult to see that if g is the gradient mapping of some function $f : \mathfrak{R}^n \rightarrow \mathfrak{R}$, problem (1) is the first order necessary condition for the problem $\min_{x\in\mathfrak{R}^n} f(x)$. Furthermore, considering

$$\min f(u) \quad \text{s.t. } a(u) = 0, \tag{2}$$

where a is a vector-valued function, then the KKT conditions can be represented as the system (1) with $x = (u, v)$ and $h(u, v) = (\nabla f(u) + \nabla a(u)v, a(u))$, where v is the vector of Lagrange multipliers. The above two cases show that problem (1) can come from an unconstrained problem or an equality constrained optimization problem in theory. Moreover, there are other practical problems that can also take the form of (1), such as the discretized two-point boundary value problem, the saddle point problem, and the discretized elliptic boundary value problem (see Chapter 1 of [1] in detail). Let θ be the norm function $\theta(x) = (1/2)\|h(x)\|^2$; then problem (1) is equivalent to the following global optimization problem:

$$\min \theta(x) \quad x \in \mathfrak{R}^n, \tag{3}$$

where $\|\cdot\|$ is the Euclidean norm.

In this paper we will focus on the line search method for (1), where its normal iterative formula is defined by

$$x_{k+1} = x_k + \alpha_k d_k, \tag{4}$$

where d_k is the so-called search direction and α_k is a steplength along d_k. To begin with, we briefly review some methods for α_k.

(i) Normal Line Search (Brown and Saad [2]). The stepsize α_k is determined by

$$\theta(x_k + \alpha_k d_k) - \theta(x_k) \le \sigma\alpha_k \nabla\theta(x_k)^T d_k, \tag{5}$$

where $\sigma \in (0, 1)$ and $\nabla\theta(x_k) = \nabla h(x_k)h(x_k)$. The convergence is proved and some good results are obtained. We all know that the nonmonotone idea is more interesting than the normal technique in many cases. Then a nonmonotone line search technique based on this motivation is presented by Zhu [3].

(ii) Nonmonotone Line Search (Zhu [3]). The stepsize α_k is determined by

$$\theta(x_k + \alpha_k d_k) - \theta(x_{l(k)}) \le \sigma\alpha_k \nabla\theta(x_k)^T d_k, \tag{6}$$

$\theta(x_{l(k)}) = \max_{0\le j\le m(k)}\{\theta(x_{k-j})\}, m(0) = 0, m(k) = \min\{m(k-1) + 1, M\}$ (for $k \ge 1$), and M is a nonnegative integer. The global convergence and the superlinear convergence are established under mild conditions, respectively. It is not difficult to see that, for the above two line search techniques, the Jacobian matrix ∇h_k must be computed at each iteration, which obviously increase the workload and the CPU time consumed. In order to avoid this drawback, Yuan and Lu [4] presented a new backtracking inexact technique.

(iii) A New Line Search (Yuan and Lu [4]). The stepsize α_k is determined by

$$\|h(x_k + \alpha_k d_k)\|^2 \le \|h(x_k)\|^2 + \delta\alpha_k^2 h_k^T d_k, \tag{7}$$

where $\delta \in (0,1)$ and $h_k = h(x_k)$. They established the global convergence and the superlinear convergence. And the numerical tests showed that the new line search technique is more effective than those of the normal line search technique. However, these three line search techniques can not directly ensure the descent property of d_k. Thus more interesting line search techniques are studied.

(iv) Approximate Monotone Line Search (Li and Fukushima [5]). The stepsize α_k is determined by

$$\begin{aligned}\theta(x_k + \alpha_k d_k) - \theta(x_k) \le & -\delta_1\|\alpha_k d_k\|^2 \\ & - \delta_2\|\alpha_k h_k\|^2 + \epsilon_k\|h_k\|^2,\end{aligned} \tag{8}$$

where $\alpha_k = r^{i_k}$, $r \in (0,1)$, i_k is the smallest nonnegative integer i satisfying (8), $\delta_1 > 0$ and $\delta_2 > 0$ are constants, and ϵ_k is such that

$$\sum_{k=0}^{\infty}\epsilon_k < \infty. \tag{9}$$

The line search (8) can be rewritten as

$$\begin{aligned}\|h(x_k + \alpha d_k)\|^2 \le & (1+\epsilon_k)\|h_k\|^2 - \delta_1\|\alpha h_k\|^2 \\ & - \delta_2\|\alpha d_k\|^2;\end{aligned} \tag{10}$$

it is straightforward to see that as $\alpha \to 0$, the right-hand side of the above inequality goes to $(1+\epsilon_k)\|h_k\|^2$. Then it is not difficult to see that the sequence $\{x_k\}$ generated by one algorithm with line search (8) is approximately norm descent. In order to ensure the sequence $\{x_k\}$ is norm descent, Gu et al. [6] presented the following line search.

(v) Monotone Descent Line Search (Gu et al. [6]). The stepsize α_k is determined by

$$\begin{aligned}\|h(x_k + \alpha_k d_k)\|^2 - \|h_k\|^2 \le & -\delta_1\|\alpha_k h_k\|^2 \\ & - \delta_2\|\alpha_k d_k\|^2,\end{aligned} \tag{11}$$

where α_k, δ_1, and δ_2 are similar to (8).

In the following, we present some techniques for d_k.

(i) Newton Method. The search direction d_k is defined by

$$\nabla h(x_k)\, d_k = -h(x_k). \tag{12}$$

Newton method is one of the most effective methods since it normally requires a fewest number of function evaluations and is very good at handling ill-conditioning. However, its efficiency largely depends on the possibility to efficiently solve a linear system (12) which arises when computing. Moreover, the exact solution of the system (12) could be too burdensome or is not necessary when x_k is far from a solution [7]. Thus the quasi-Newton methods are proposed.

(ii) Quasi-Newton Method. The search direction d_k is defined by

$$B_k d_k + h_k = 0, \tag{13}$$

where B_k is the quasi-Newton update matrix. The quasi-Newton methods represent the basic approach underlying most of the Newton-type large-scale algorithms (see [3, 4, 8], etc.), where the famous BFGS method is one of the most effective quasi-Newton methods, generated by the following formula:

$$B_{k+1} = B_k - \frac{B_k s_k s_k^T B_k}{s_k^T B_k s_k} + \frac{y_k {y_k}^T}{{y_k}^T s_k}, \tag{14}$$

where $s_k = x_{k+1} - x_k$ and $y_k = h_{k+1} - h_k$ with $h_k = h(x_k)$ and $h_{k+1} = h(x_k + \alpha_k d_k)$. By (11) and (14), Yuan and Yao [9] proposed a BFGS method for nonlinear equations and some good results were obtained. Denote $H_k = B_k^{-1}$, and then (14) has the inverse update formula represented by

$$\begin{aligned}H_{k+1} &= H_k - \frac{{y_k}^T(s_k - H_k y_k)\, s_k s_k^T}{({y_k}^T s_k)^2} \\ &\quad + \frac{(s_k - H_k y_k)\, s_k^T + s_k(s_k - H_k y_k)^T}{({y_k}^T s_k)^2} \\ &= \left(I - \frac{s_k {y_k}^T}{{y_k}^T s_k}\right) H_k \left(I - \frac{y_k s_k^T}{{y_k}^T s_k}\right) + \frac{s_k s_k^T}{{y_k}^T s_k}.\end{aligned} \tag{15}$$

Unfortunately, both the Newton method and the quasi-Newton method require many space to store $n \times n$ matrix at every iteration, which will prevent the efficiency of the algorithm for problems, especially for large-scale problems. Therefore low storage matrix information method should be necessary.

(iii) Limited Memory Quasi-Newton Method. The search direction d_k is defined by

$$d_k = -H_k h_k, \tag{16}$$

where H_k is generated by limited memory quasi-Newton method, where the famous limited memory quasi-Newton method is the so-called limited memory BFGS method. The L-BFGS method is an adaptation of the BFGS method for large-scale problems (see [10] in detail), which often requires minimal storage and provides a fast rate of linear convergence. The L-BFGS method has the following form:

$$\begin{aligned} H_{k+1} &= V_k^T H_k V_k + \rho_k s_k s_k^T \\ &= V_k^T \left[V_{k-1}^T H_{k-1} V_{k-1} + \rho_{k-1} s_{k-1} s_{k-1}^T \right] V_k + \rho_k s_k s_k^T \\ &= \cdots \\ &= \left[V_k^T \cdots V_{k-\widetilde{m}+1}^T \right] H_{k-\widetilde{m}+1} \left[V_{k-\widetilde{m}+1} \cdots V_k \right] \\ &\quad + \rho_{k-\widetilde{m}+1} \left[V_{k-1}^T \cdots V_{k-\widetilde{m}+2}^T \right] \\ &\quad \times s_{k-\widetilde{m}+1} s_{k-\widetilde{m}+1}^T \left[V_{k-\widetilde{m}+2} \cdots V_{k-1} \right] + \cdots + \rho_k s_k s_k^T, \end{aligned} \tag{17}$$

where $\rho_k = 1/s_k^T y_k$, $V_k = I - \rho_k y_k s_k^T$, $\widetilde{m} > 0$ is an integer, and I is the unit matrix. Formula (17) shows that matrix H_k is obtained by updating the basic matrix $H_0 \widetilde{m}$ times using BFGS formula with the previous $\widetilde{m}$ iterations. By (17), together with (7) and (8), Yuan et al. [11] presented the L-BFGS method for nonlinear equations and got the global convergence. At present, there are many papers proposed for (1) (see [6, 12–15], etc.).

In order to effectively solve large-scale nonlinear equations and possess good theory property, based on the above discussions of α_k and d_k, we will combine (11) and (16) and present a L-BFGS method for (1) since (11) can make the norm function be descent and (16) need less low storage. The main attributes of the new algorithm are stated as follows.

(i) A L-BFGS method with (11) is presented.

(ii) The norm function is descent.

(iii) The global convergence is established under appropriate conditions.

(iv) Numerical results show that the given algorithm is more competitive than the normal algorithm for large-scale nonlinear equations.

This paper is organized as follows. In the next section, the backtracking inexact L-BFGS algorithm is stated. Section 3 will present the global convergence of the algorithm under some reasonable conditions. Numerical experiments are done to test the performance of the algorithms in Section 4.

2. Algorithms

This section will state the L-BFGS method in association with the new backtracking line search technique (11) for solving (1).

Algorithm 1.

Step 0. Choose an initial point $x_0 \in \mathfrak{R}^n$, an initial symmetric positive definite matrix $H_0 \in \mathfrak{R}^{n\times n}$, positive constants δ_1, δ_2, constants $r, \rho \in (0,1)$, and a positive integer m_1. Let $k := 0$.

Step 1. Stop if $\|h_k\| = 0$.

Step 2. Determine d_k by (16).

Step 3. If

$$\left\| h\left(x_k + d_k\right) \right\| \le \rho \left\| h\left(x_k\right) \right\|, \tag{18}$$

then take $\alpha_k = 1$ and go to Step 5. Otherwise go to Step 4.

Step 4. Let i_k be the smallest nonnegative integer i such that (11) holds for $\alpha = r^i$. Let $\alpha_k = r^{i_k}$.

Step 5. Let the next iterative be $x_{k+1} = x_k + \alpha_k d_k$.

Step 6. Let $\widetilde{m} = \min\{k+1, m_1\}$. Put $s_k = x_{k+1} - x_k = \alpha_k d_k$ and $y_k = h_{k+1} - h_k$. Update H_0 for $\widetilde{m}$ times to get H_{k+1} by (17).

Step 7. Let $k := k + 1$. Go to Step 1.

In the following, to conveniently analyze the global convergence, we assume that the algorithm updates B_k (the inverse of H_k) with the basically bounded and positive definite matrix B_0 (H_0's inverse). Then Algorithm 1 with B_k has the following steps.

Algorithm 2.

Step 2. Determine d_k by

$$B_k d_k = -h_k. \tag{19}$$

Step 6. Let $\widetilde{m} = \min\{k+1, m_1\}$. Put $s_k = x_{k+1} - x_k = \alpha_k d_k$ and $y_k = h_{k+1} - h_k$. Update B_0 for $\widetilde{m}$ times; that is, for $l = k - \widetilde{m} + 1, \ldots, k$ compute

$$B_k^{l+1} = B_k^l - \frac{B_k^l s_l s_l^T B_k^l}{s_l^T B_k^l s_l} + \frac{y_l y_l^T}{y_l^T s_l}, \tag{20}$$

where $s_l = x_{l+1} - x_l$, $y_l = h_{l+1} - h_l$, and $B_k^{k-\widetilde{m}+1} = B_0$ for all k.

Remark 3. Algorithms 1 and 2 are mathematically equivalent. Throughout this paper, Algorithm 2 is given only for the purpose of analysis, so we only discuss Algorithm 2 in theory. In the experiments, we implement Algorithm 1.

3. Global Convergence

Define the level set Ω by

$$\Omega = \left\{ x \mid \|h(x)\| \le \left\| h\left(x_0\right) \right\| \right\}. \tag{21}$$

In order to establish the global convergence of Algorithm 2, similar to [4, 11], we need the following assumptions.

Assumption A. g is continuously differentiable on an open convex set Ω_1 containing Ω. Moreover the Jacobian of g is symmetric, bounded, and positive definite on Ω_1; namely, there exist positive constants $M \ge m > 0$ satisfying

$$\begin{aligned} \|\nabla h(x)\| &\le M \quad \forall x \in \Omega_1, \\ m\|d\|^2 &\le d^T \nabla h(x)\, d \quad \forall x \in \Omega_1,\ d \in \mathfrak{R}^n. \end{aligned} \tag{22}$$

Assumption B. B_k is a good approximation to ∇g_k; that is,

$$\left\|\left(\nabla h_k - B_k\right) d_k\right\| \le \epsilon \left\|h_k\right\|, \tag{23}$$

where $\epsilon \in (0,1)$ is a small quantity.

Remark 4. Assumption A implies

$$\left\|y_k\right\| \le M \left\|s_k\right\|, \qquad {y_k}^T s_k \ge m\left\|s_k\right\|^2. \tag{24}$$

The relations in (24) can ensure that B_{k+1} generated by (20) inherits symmetric and positive definiteness of B_k. Thus, (19) has a unique solution for each k. Moreover, the following lemma holds.

Lemma 5 (see Theorem 2.1 in [16] or see Lemma 3.4 of [11]). *Let Assumption A hold and let $\{\alpha_k, d_k, x_{k+1}, g_{k+1}\}$ be generated by Algorithm 2. Then, for any $r_0 \in (0,1)$ and $k \ge 0$, there are positive constants β_j, $j = 1,2,3$; the following relations*

$$\beta_2\left\|s_i\right\|^2 \le s_i^T B_i s_i \le \beta_3\left\|s_i\right\|^2, \qquad \left\|B_i s_i\right\| \le \beta_1 \left\|s_i\right\| \tag{25}$$

hold for at least $\lceil r_0 k\rceil$ values of $i \in [0,k]$.

By Assumption B, similar to [4, 9, 11, 15], it is easy to get the following lemma.

Lemma 6. *Let Assumption B hold and let $\{\alpha_k, d_k, x_{k+1}, h_{k+1}\}$ be generated by Algorithm 2. Then d_k is a descent direction for $\theta(x)$ at x_k; that is, $\nabla\theta(x_k)^T d_k < 0$ holds.*

Based on the above lemma, by Assumption B, similar to Lemma 3.8 in [2], we can get the following lemma.

Lemma 7. *Let Assumption B hold and let $\{\alpha_k, d_k, x_{k+1}, h_{k+1}\}$ be generated by Algorithm 2. Then $\{x_k\} \subset \Omega$. Moreover, $\{\|h_k\|\}$ converges.*

Lemma 8. *Let Assumptions A and B hold. Then, in a finite number of backtracking steps, Algorithm 2 will produce an iterate $x_{k+1} = x_k + \alpha_k d_k$.*

Proof. It is sufficient for us to prove that the line search (11) is reasonable. By Lemma 3.8 in [2], we can deduce that, in a finite number of backtracking steps, α_k is such that

$$\begin{aligned}\left\|h(x_k + \alpha_k d_k)\right\|^2 - \left\|h\left(x_k\right)\right\|^2 &\le \delta\alpha_k h(x_k)^T \nabla h\left(x_k\right) d_k, \\ &\qquad \delta \in (0,1).\end{aligned} \tag{26}$$

By (19), we get

$$\begin{aligned}\nabla\theta(x_k)^T d_k &= h(x_k)^T \nabla h\left(x_k\right) d_k \\ &= h(x_k)^T \left[\left(\nabla h\left(x_k\right) - B_k\right) d_k - h\left(x_k\right)\right] \\ &= h(x_k)^T \left(\nabla h\left(x_k\right) - B_k\right) d_k - h(x_k)^T h\left(x_k\right).\end{aligned} \tag{27}$$

Thus

$$\begin{aligned}\nabla\theta(x_k)^T d_k + \left\|h_k\right\|^2 &\le h(x_k)^T \left(\nabla h\left(x_k\right) - B_k\right) d_k \\ &\le \left\|h\left(x_k\right)\right\| \left\|\left(\nabla h\left(x_k\right) - B_k\right) d_k\right\|.\end{aligned} \tag{28}$$

By Assumption B, we have

$$\begin{aligned}\nabla\theta(x_k)^T d_k &\le \left\|h\left(x_k\right)\right\| \left\|\left(\nabla h\left(x_k\right) - B_k\right) d_k\right\| - \left\|h\left(x_k\right)\right\|^2 \\ &\le -\left(1-\epsilon\right) \left\|h\left(x_k\right)\right\|^2.\end{aligned} \tag{29}$$

Using (19) again and $\alpha_k \le 1$, we obtain

$$\begin{aligned}\alpha_k h(x_k)^T \nabla h\left(x_k\right) d_k &\le -\alpha_k \left(1-\epsilon\right) \left\|h(x_k)\right\|^2 \\ &= -\frac{(1-\epsilon)}{2\alpha_k}\left\|\alpha_k h_k\right\|^2 - \frac{(1-\epsilon)}{2\alpha_k}\left\|\alpha_k B_k d_k\right\|^2 \\ &\le -\frac{(1-\epsilon)}{2}\left\|\alpha_k h_k\right\|^2 - \frac{\beta_2^2(1-\epsilon)}{2}\left\|\alpha_k d_k\right\|^2.\end{aligned} \tag{30}$$

Setting $\delta_1 \in (0, \delta((1-\epsilon)/2))$ and $\delta_2 \in (0, \delta(\beta_2^2(1-\epsilon)/2))$ implies (11). This completes the proof. □

Remark 9. The above lemma shows that Algorithm 2 is well defined. By a way similar to Lemma 3.2 and Corollary 3.4 in [5], it is not difficult to deduce that

$$\alpha_k \ge \frac{\beta_2 r}{\beta_1^2\sigma_1 + \sigma_2 + M^2} \tag{31}$$

holds; we do not prove it anymore. Now we establish the global convergence theorem.

Theorem 10. *Let Assumptions A and B hold. Then the sequence $\{x_k\}$ generated by Algorithm 2 converges to the unique solution x^* of* (1).

Proof. Lemma 7 implies that $\{\|h_k\|\}$ converges. If

$$\lim_{k\to\infty} \left\|h_k\right\| = 0, \tag{32}$$

then every accumulation point of $\{x_k\}$ is a solution of (1). Assumption A means that (1) has only one solution. Moreover, since Ω is bounded, $\{x_k\} \subseteq \Omega$ has at least one accumulation point. Therefore $\{x_k\}$ itself converges to the unique solution of (1). Therefore, it suffices to verify (32).

If (18) holds for infinitely many k's, then (32) is trivial. Otherwise, if (18) holds for only finitely many k's, we conclude that Step 3 is executed for all k sufficiently large. By (11), we have

$$\delta_1\left\|\alpha_k h_k\right\|^2 + \delta_2\left\|s_k\right\|^2 \le \left\|h_k\right\|^2 - \left\|h_{k+1}\right\|^2. \tag{33}$$

Since $\{\|h_k\|\}$ is bounded, by adding these inequalities, we get

$$\sum_{k=0}^{\infty}\left\|\alpha_k h_k\right\|^2 < \infty, \qquad \sum_{k=0}^{\infty}\left\|\alpha_k d_k\right\|^2 < \infty. \tag{34}$$

Then we have

$$\lim_{k\to\infty} \left\|\alpha_k h_k\right\| = 0, \tag{35}$$

which together with (31) implies (32). This completes the proof. □

4. Numerical Results

This section reports numerical results with Algorithm 1 and normal BFGS algorithm. The test problems with the associated initial guess x_0 are listed with

$$h(x) = (f_1(x), f_2(x), \ldots, f_n(x))^T. \tag{36}$$

Problem 1. Exponential function 1:

$$\begin{aligned} f_1(x) &= e^{x_1-1} - 1, \\ f_i(x) &= i\left(e^{x_i-1} - x_i\right), \quad i = 2, 3, \ldots, n. \end{aligned} \tag{37}$$

Initial guess: $x_0 = (1/n^2, 1/n^2, \ldots, 1/n^2)^T$.

Problem 2. Exponential function 2:

$$\begin{aligned} f_1(x) &= e^{x_1} - 1, \\ f_i(x) &= \frac{i}{10}\left(e^{x_i} + x_{i-1} - i\right), \quad i = 2, 3, \ldots, n. \end{aligned} \tag{38}$$

Initial guess: $x_0 = (1/n^2, 1/n^2, \ldots, 1/n^2)^T$.

Problem 3. Trigonometric function:

$$\begin{aligned} f_i(x) = 2&\left(n + i(1 - \cos x_i) - \sin x_i - \sum_{j=1}^{n} \cos x_j\right) \\ &\times (2\sin x_i - \cos x_i), \quad i = 1, 2, 3, \ldots, n. \end{aligned} \tag{39}$$

Initial guess: $x_0 = (101/100n, 101/100n, \ldots, 101/100n)^T$.

Problem 4. Singular function:

$$\begin{aligned} f_1(x) &= \frac{1}{3}x_1^3 + \frac{1}{2}x_2^2, \\ f_i(x) &= -\frac{1}{2}x_i^2 + \frac{i}{3}x_i^3 + \frac{1}{2}x_{i+1}^2, \quad i = 2, 3, \ldots, n-1, \\ f_n(x) &= -\frac{1}{2}x_n^2 + \frac{n}{3}x_n^3. \end{aligned} \tag{40}$$

Initial guess: $x_0 = (1, 1, \ldots, 1)^T$.

Problem 5. Logarithmic function:

$$f_i(x) = \ln(x_i + 1) - \frac{x_i}{n}, \quad i = 1, 2, 3, \ldots, n. \tag{41}$$

Initial guess: $x_0 = (1, 1, \ldots, 1)^T$.

Problem 6. Broyden tridiagonal function [17, pages 471-472]:

$$\begin{aligned} f_1(x) &= (3 - 0.5x_1)x_1 - 2x_2 + 1, \\ f_i(x) &= (3 - 0.5x_i)x_i - x_{i-1} + 2x_{i+1} + 1, \\ &\qquad i = 2, 3, \ldots, n-1, \\ f_n(x) &= (3 - 0.5x_n)x_n - x_{n-1} + 1. \end{aligned} \tag{42}$$

Initial guess: $x_0 = (-1, -1, \ldots, -1)^T$.

Problem 7. Trigexp function [17, page 473]:

$$\begin{aligned} f_1(x) &= 3x_1^3 + 2x_2 - 5 + \sin(x_1 - x_2)\sin(x_1 + x_2), \\ f_i(x) &= -x_{i-1}e^{x_{i-1}-x_i} + x_i\left(4 + 3x_i^2\right) + 2x_{i+1} \\ &\quad + \sin(x_i - x_{i+1})\sin(x_i + x_{i+1}) - 8, \\ &\qquad i = 2, 3, \ldots, n-1, \\ f_n(x) &= -x_{n-1}e^{x_{n-1}-x_n} + 4x_n - 3. \end{aligned} \tag{43}$$

Initial guess: $x_0 = (0, 0, \ldots, 0)^T$.

Problem 8. Strictly convex function 1 [18, page 29]: $h(x)$ is the gradient of $h(x) = \sum_{i=1}^{n}(e^{x_i} - x_i)$. Consider

$$f_i(x) = e^{x_i} - 1, \quad i = 1, 2, 3, \ldots, n. \tag{44}$$

Initial guess: $x_0 = (1/n, 2/n, \ldots, 1)^T$.

Problem 9. Linear function-full rank:

$$f_i(x) = x_i - \frac{2}{n}\sum_{j=1}^{n} x_j + 1. \tag{45}$$

Initial guess: $x_0 = (100, 100, \ldots, 100)^T$.

Problem 10. Penalty function:

$$\begin{aligned} f_i(x) &= \sqrt{10^{-5}}(x_i - 1), \quad i = 1, 2, 3, \ldots, n-1, \\ f_n(x) &= \left(\frac{1}{4n}\right)\sum_{j=1}^{n} x_j^2 - \frac{1}{4}. \end{aligned} \tag{46}$$

Initial guess: $x_0 = (1/3, 1/3, \ldots, 1/3)^T$.

Problem 11. Variable dimensioned function:

$$\begin{aligned} f_i(x) &= x_i - 1, \quad i = 1, 2, 3, \ldots, n-2, \\ f_{n-1}(x) &= \sum_{j=1}^{n-2} j\left(x_j - 1\right), \\ f_n(x) &= \left(\sum_{j=1}^{n-2} j\left(x_j - 1\right)\right)^2. \end{aligned} \tag{47}$$

Initial guess: $x_0 = (1 - (1/n), 1 - (2/n), \ldots, 0)^T$.

Problem 12. Tridiagonal system [19]:

$$\begin{aligned} f_1(x) &= 4\left(x_1 - x_2^2\right), \\ f_i(x) &= 8x_i\left(x_i^2 - x_{i-1}\right) - 2(1 - x_i) \\ &\quad + 4\left(x_i - x_{i+1}^2\right), \quad i = 2, 3, \ldots, n-1, \\ f_n(x) &= 8x_n\left(x_n^2 - x_{n-1}\right) - 2(1 - x_n). \end{aligned} \tag{48}$$

Initial guess: $x_0 = (12, 12, \ldots, 12)$.

Table 1

Nr.	Dim	Algorithm 1			Normal BFGS algorithm		
		NI/NG	GN	Time	NI/NG	GN	Time
Problem 1	500	24/25	$9.908338e-05$	$2.901619e+01$	23/87	NaN	$1.950013e+00$
	1000	9/10	$5.697414e-05$	$5.366434e+01$	23/87	NaN	$1.124767e+01$
	1500	9/10	$3.363290e-05$	$1.689803e+02$	NI > 1000	Inf	$8.094892e+01$
	2000	9/10	$2.706880e-05$	$3.880213e+02$	NI > 1000	Inf	$1.550962e+02$
Problem 2	500	8/16	$3.660478e-05$	$6.926444e+00$	60/390	NaN	$7.269647e+00$
	1000	8/16	$6.406401e-05$	$4.513109e+01$	40/314	NaN	$2.857938e+01$
	1500	9/17	$7.655321e-05$	$1.689959e+02$	33/258	NaN	$7.102726e+01$
	2000	9/17	$9.651851e-05$	$3.881617e+02$	29/226	NaN	$1.410717e+02$
Problem 3	500	18/33	$7.747640e-06$	$2.110694e+01$	NI > 1000	$5.905028e+04$	$1.220396e+02$
	1000	17/32	$8.381440e-05$	$1.270472e+02$	NI > 1000	$1.577282e+05$	$5.830069e+02$
	1500	17/32	$6.658002e-05$	$3.975217e+02$	NI > 1000	$2.929920e+05$	$1.645561e+03$
	2000	17/32	$5.689277e-05$	$9.154451e+02$	NI > 1000	$4.345902e+05$	$3.776363e+03$
Problem 4	500	809/3134	$9.614778e-05$	$1.113785e+03$	9/73	NaN	$1.107607e+00$
	1000	960/3894	$9.865549e-05$	$8.674373e+03$	8/65	NaN	$5.725237e+00$
	1500	197/695	$8.562547e-05$	$5.509394e+03$	8/65	NaN	$1.584970e+01$
	2000	220/676	$9.847745e-05$	$1.418851e+04$	8/65	NaN	$3.584903e+01$
Problem 5	500	6/7	$4.925073e-06$	$4.305628e+00$	6/7	$4.925073e-06$	$7.488048e-01$
	1000	6/7	$6.747222e-06$	$2.725337e+01$	6/7	$6.747222e-06$	$4.477229e+00$
	1500	6/7	$8.176417e-06$	$8.352294e+01$	6/7	$8.176417e-06$	$1.340049e+01$
	2000	6/7	$9.391335e-06$	$1.925364e+02$	6/7	$9.391335e-06$	$3.001459e+01$
Problem 6	500	96/97	$9.183793e-05$	$1.272812e+02$	65/66	$9.473359e-05$	$8.346053e+00$
	1000	17/18	$7.206980e-05$	$1.263140e+02$	63/64	$8.700852e-05$	$4.623870e+01$
	1500	17/18	$6.663689e-05$	$3.956965e+02$	63/64	$9.605270e-05$	$1.402917e+02$
	2000	17/18	$6.136239e-05$	$9.107026e+02$	64/65	$8.724048e-05$	$3.194588e+02$
Problem 7	500	14/15	$7.551200e-05$	$1.533490e+01$	52/53	$9.761790e-05$	$6.333641e+00$
	1000	15/16	$5.530847e-05$	$1.084051e+02$	55/63	$8.871231e-05$	$3.985826e+01$
	1500	14/15	$8.445286e-05$	$3.119708e+02$	60/68	$7.722045e-05$	$1.336617e+02$
	2000	15/16	$4.705617e-05$	$7.805354e+02$	13/63	NaN	$5.564556e+01$
Problem 8	500	6/7	$4.600878e-05$	$4.196427e+00$	6/7	$2.375490e-05$	$7.020045e-01$
	1000	6/7	$6.434846e-05$	$2.658257e+01$	6/7	$3.327487e-05$	$4.461629e+00$
	1500	6/7	$7.851881e-05$	$8.319533e+01$	6/7	$4.062329e-05$	$1.335369e+01$
	2000	6/7	$9.049762e-05$	$1.928484e+02$	6/7	$4.683284e-05$	$2.985859e+01$
Problem 9	500	2/10	$2.027911e-10$	$5.460035e-01$	NI > 1000	$4.514037e+117$	$1.037407e+01$
	1000	2/10	$2.666034e-10$	$2.698817e+00$	NI > 1000	$6.383813e+117$	$2.861058e+01$
	1500	2/10	$1.379742e-09$	$8.330453e+00$	NI > 1000	$7.818542e+117$	$5.319634e+01$
	2000	2/10	$1.701735e-09$	$1.942212e+01$	NI > 1000	$9.028075e+117$	$8.375694e+01$
Problem 10	500	435/2865	$5.154286e-05$	$5.901362e+02$	41/210	NaN	$3.806424e+00$
	1000	637/4215	$5.701191e-05$	$5.749432e+03$	67/425	NaN	$3.689424e+01$
	1500	303/1914	$4.003848e-05$	$8.526469e+03$	63/491	NaN	$1.146919e+02$
	2000	473/3281	$3.558910e-05$	$3.074095e+04$	64/499	NaN	$2.622845e+02$
Problem 11	500	1/2	$0.000000e+00$	$1.092007e-01$	1/2	$0.000000e+00$	$1.560010e-01$
	1000	1/2	$0.000000e+00$	$7.956051e-01$	1/2	$0.000000e+00$	$6.864044e-01$
	1500	1/2	$0.000000e+00$	$2.277615e+00$	1/2	$0.000000e+00$	$2.324415e+00$
	2000	1/2	$0.000000e+00$	$5.101233e+00$	1/2	$0.000000e+00$	$5.210433e+00$
Problem 12	500	260/800	$1.654459e-05$	$3.509398e+02$	8/51	NaN	$7.800050e-01$
	1000	324/1053	$8.997051e-05$	$2.902289e+03$	8/51	NaN	$5.226033e+00$
	1500	254/829	$1.257117e-05$	$7.123224e+03$	8/51	NaN	$1.583410e+01$
	2000	372/1353	$9.331122e-05$	$2.414117e+04$	8/51	NaN	$3.046700e+01$

Table 1: Continued.

Nr.	Dim	Algorithm 1 NI/NG	Algorithm 1 GN	Algorithm 1 Time	Normal BFGS algorithm NI/NG	Normal BFGS algorithm GN	Normal BFGS algorithm Time
Problem 13	500	96/209	$9.194537e-05$	$1.287632e+02$	10/74	NaN	$1.185608e+00$
	1000	53/89	$5.718492e-05$	$4.531049e+02$	10/74	NaN	$6.021639e+00$
	1500	NI > 1000	$1.180195e+10$	$2.835008e+04$	10/74	NaN	$1.790891e+01$
	2000	54/132	$3.919816e-05$	$3.337298e+03$	10/74	NaN	$4.031066e+01$
Problem 14	500	18/68	$5.826508e-05$	$2.076373e+01$	16/115	NaN	$6.240040e-01$
	1000	18/68	$8.239959e-05$	$1.357989e+02$	16/115	NaN	$3.354021e+00$
	1500	19/69	$5.002681e-05$	$4.532921e+02$	16/115	NaN	$9.765663e+00$
	2000	19/69	$5.774900e-05$	$1.042820e+03$	16/115	NaN	$2.148134e+01$
Problem 15	500	8/9	$6.870902e-05$	$6.910844e+00$	22/23	$8.899350e-05$	$2.667617e+00$
	1000	7/8	$7.203786e-05$	$3.561503e+01$	16/17	$9.828207e-05$	$1.174688e+01$
	1500	7/8	$4.809670e-05$	$1.122739e+02$	13/14	$9.654107e-05$	$2.903179e+01$
	2000	7/8	$3.609946e-05$	$2.583845e+02$	11/12	$9.465949e-05$	$5.519315e+01$
Problem 16	500	0/1	$0.000000e+00$	$0.000000e+00$	0/1	$0.000000e+00$	$0.000000e+00$
	1000	0/1	$0.000000e+00$	$0.000000e+00$	0/1	$0.000000e+00$	$0.000000e+00$
	1500	0/1	$0.000000e+00$	$1.560010e-02$	0/1	$0.000000e+00$	$0.000000e+00$
	2000	0/1	$0.000000e+00$	$1.560010e-02$	0/1	$0.000000e+00$	$1.560010e-02$

Problem 13. Five-diagonal system [19]:

$$\begin{aligned}
f_1(x) &= 4\left(x_1 - x_2^2\right) + x_2 - x_3^2,\\
f_2(x) &= 8x_2\left(x_2^2 - x_1\right) - 2\left(1 - x_2\right) + 4\left(x_2 - x_3^2\right) + x_3 - x_4^2,\\
f_i(x) &= 8x_i\left(x_i^2 - x_{i-1}\right) - 2\left(1 - x_i\right) + 4\left(x_i - x_{i+1}^2\right) + x_{i-1}^2\\
&\quad - x_{i-2} + x_{i+1} - x_{i+2}^2, \quad i = 3, 4, \ldots, n-2,\\
f_{n-1}(x) &= 8x_{n-1}\left(x_{n-1}^2 - x_{n-2}\right) - 2\left(1 - x_{n-1}\right)\\
&\quad + 4\left(x_{n-1} - x_n^2\right) + x_{n-2}^2 - x_{n-3},\\
f_n(x) &= 8x_n\left(x_n^2 - x_{n-1}\right) - 2\left(1 - x_n\right) + x_{n-1}^2 - x_{n-2}.
\end{aligned} \tag{49}$$

Initial guess: $x_0 = (-2, -2, \ldots, -2)$.

Problem 14. Extended Freudentein and Roth function (n is even) [20]: for $i = 1, 2, \ldots, n/2$

$$\begin{aligned}
f_{2i-1}(x) &= x_{2i-1} + \left(\left(5 - x_{2i}\right)x_{2i} - 2\right)x_{2i} - 13,\\
f_{2i}(x) &= x_{2i-1} + \left(\left(1 + x_{2i}\right)x_{2i} - 14\right)x_{2i} - 29.
\end{aligned} \tag{50}$$

Initial guess: $x_0 = (6, 3, 6, 3, \ldots, 6, 3)$.

Problem 15. Discrete boundry value problem [21]:

$$\begin{aligned}
f_1(x) &= 2x_1 + 0.5h^2(x_1 + t)^3 - x_2,\\
f_i(x) &= 2x_i + 0.5h^2(x_i + ti)^3 - x_{i-1} + x_{i+1},\\
&\qquad i = 2, 3, \ldots, n-1,\\
f_n(x) &= 2x_n + 0.5h^2(x_n + tn)^3 - x_{n-1},\\
&\qquad t = \frac{1}{n+1}.
\end{aligned} \tag{51}$$

Initial guess: $x_0 = (t(t-1), t(2t-1), \ldots, t(nt-1))$.

Problem 16. Troesch problem [22]:

$$\begin{aligned}
f_1(x) &= 2x_1 + \varrho h^2 \sin t\left(\varrho x_1\right) - x_2,\\
f_i(x) &= 2x_i + \varrho h^2 \sin t\left(\varrho x_i\right) - x_{i-1} - x_{i+1},\\
&\qquad i = 2, 3, \ldots, n-1,\\
f_n(x) &= 2x_n + \varrho h^2 \sin t\left(\varrho x_n\right) - x_{n-1},\\
&\qquad t = \frac{1}{n+1}, \quad \varrho = 10.
\end{aligned} \tag{52}$$

Initial guess: $x_0 = (0, 0, \ldots, 0)$.

In the experiments, the parameters in Algorithm 1 and the normal BFGS method were chosen as $r = 0.1$, $\rho = 0.5$, $m_1 = 6$, $\delta_1 = \delta_2 = 0.001$, H_0 and is the unit matrix. All codes were written in MATLAB r2013b and run on PC with

6600@2.40 GHz Core 2 CPU processor and 4.00 GB memory and Windows 7 operation system. We stopped the program when the condition $\|h(x)\| \leq 10^{-4}$ was satisfied. Since the line search cannot always ensure the descent condition $d_k^T h_k < 0$, uphill search direction may occur in the numerical experiments. In this case, the line search rule maybe fails. In order to avoid this case, the stepsize α_k will be accepted if the searching time is larger than eight in the inner circle for the test problems. We also stop this program if the iteration number arrived at 1000. The columns of the tables have the following meaning.

Dim: the dimension. NI: the total number of iterations.

NG: the number of the norm function evaluations. Time: the CPU time in second.

GN: the normal value of $\| h(x) \|$ when the program stops.

NaN: not-a-number, impling that the code fails to get a real value.

Inf: returning the IEEE arithmetic representation for positive infinity or infinity which is also produced by operations like dividing by zero.

From the numerical results in Table 1, it is not difficult to show that the proposed method is more successful than the normal BFGS method. We can see that there exist many problems which can not be successfully solved by the normal BFGS method. Moreover, the normal BFGS method fails to get real value for several problems. Then we can conclude that the presented method is more competitive than the normal BFGS method.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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